Swirling of buckled pre-stressed active elastic filaments in a fluid

Soheil Fatehiboroujeni† and Sachin Goyal‡
Department of Mechanical Engineering, University of California, Merced

Arvind Gopinath‡
Department of Bioengineering, University of California, Merced

Slender filaments clamped at both ends and pre-stressed into a buckled planar shape are known to exhibit stable flapping oscillations in a dissipative medium under nonconservative follower forces. However, beyond a critical pre-stress value clamped rods transition to an out-of-plane (bent and twisted) equilibrium shape in the absence of follower forces. We analyze the nonlinear three-dimensional spatiotemporal dynamics of these three-dimensional pre-stressed shapes under the action of follower forces. In this regime, we find emergence of swirling oscillations with two characteristic time-scales with the first time scale characterizing a purely rotational (swirling) motion around the end-to-end axis of the filament and the second time scale capturing the rate at which the direction of swirling oscillations are reversed. This reversal of swirling oscillations resembles relaxation oscillations where a sudden jump in torsional deformation is followed by a period of gradual decrease in net torsion leading to the next cycle of variations. Our work suggests means by which mechanical deformation mediated by non-conservative follower force may be used to generate oscillatory behavior that can pump or mix fluid at macroscales.

I. INTRODUCTION

Flagella and cilia are micro scale thin filaments found in many eukaryotic cells that play crucial functions such as locomotion [1], material transport [2], and mechanosensing [3] and exhibit a diversity of both planar and non-planar beating patterns [4–11]. In the limit of extremely low Reynolds number where viscous effects are dominant and inertial effects negligible, a reciprocal (back and forth) flapping can not generate propulsion or fluid transport; this so called scallop theorem thus restrains the shape changes that may be utilized to enable motion or propulsion. In nature, flagellar and ciliary motions use elasticity to break free of the constraints imposed by the scallop theorem and generate net fluid transport via active and complex beating patterns. Based on this observation, a growing number of studies have focused on using the competition between activity, dissipation and elasticity to mimic or replicate such dynamics. Recent experimental results have demonstrated that discrete filaments synthesized by surface micro-machining techniques, or by connecting a chain of magnetic or polar colloids also exhibit cilia-like oscillatory motion when actuated by external fields [12–16].

In previous work [17, 18] we used a continuum nonlinear computational rod model and studied the emergent spatio-temporal flapping (oscillatory) dynamics of rods subject to non-conservative follower forces under purely planar perturbations. We demonstrated how prescribing an initial compression that generates buckling slack furnishes the necessary degree of freedom for planar oscillations to emerge with dynamics being controlled using the rate of initial slack. Our main observations were that (a) the critical force for onset of flapping is approximately linearly dependent on pre-stress, and that (b) far from the critical point, the flapping frequency scales with intensity of the follower force with a power law relationship the exponent of which is a function of drag force as well as rod properties. We also finally explored how switching boundary constraints can be used to regulate the type of emergent response. These studies complement continuum as well as discrete analysis of the dynamics of stress-free filaments animated by follower forces [19, 20].

However in real systems, beyond a critical pre-stress value clamped rods transition to an out-of-plane (bent and twisted) equilibrium shape in the absence of follower forces. As shown in Figure 1 by increasing the initial compression beyond a certain threshold \(1 - \frac{L_{cc}}{L} \approx 0.6\) the planar buckled shapes become unstable and rod transitions to more energetically favorable configurations that allows part of the strain energy to be stored in torsional form. Analyzing the nonlinear three-dimensional spatiotemporal dynamics of these three-dimensional pre-stressed shapes under the action of follower forces is necessary to understand the filament dynamics in all its generality. In the current manuscript, we take the first steps towards this and focus on constrained (clamped-clamped) rods buckled in non-planar (bent and twisted) configurations. We find that instead of flapping oscillations the structure undergoes swirling (purely rotational) motion around the end-to-end axis under any non-zero distributed follower force. We also find that swirling oscillations undergo a sudden reversal of direction (i.e., flipping) periodically.

The organization of this article is as follows. We first review our formulation of a geometrically nonlinear elastic rod that is used to computationally analyze the dynamics of constrained (clamped-clamped) rods subject to distributed follower forces. We then analyze the trajec-
The continuum rod model that we use follows the Kirchhoff’s approach [21] assuming each cross-section of the rod to be rigid. The model is described in detail elsewhere [17, 22]. To briefly summarize, equilibrium equations (1) and (2), and the compatibility conditions (3) and (4) are given below:

\[ m \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega} \times \mathbf{v} \right) - (\frac{\partial \mathbf{f}}{\partial s} + \mathbf{\kappa} \times \mathbf{f}) - \mathbf{f}_e = 0 \]  

(1)

\[ \mathbf{I}_m \cdot \frac{\partial \mathbf{\omega}}{\partial t} - \mathbf{\omega} \times \mathbf{I}_m \cdot \mathbf{\omega} - (\frac{\partial \mathbf{q}}{\partial s} + \mathbf{\kappa} \times \mathbf{q}) + \mathbf{f} \times \mathbf{r} - \mathbf{q}_e = 0 \]  

(2)

\[ \frac{\partial \mathbf{r}}{\partial t} + \mathbf{\omega} \times \mathbf{r} - (\frac{\partial \mathbf{v}}{\partial s} + \mathbf{\kappa} \times \mathbf{v}) = 0 \]  

(3)

\[ \frac{\partial \mathbf{\kappa}}{\partial t} - (\frac{\partial \mathbf{\omega}}{\partial s} + \mathbf{\kappa} \times \mathbf{\omega}) = 0 \]  

(4)

Here \( s \) is the cross-section location along the rod, \( t \) is time, \( m(s) \) is the mass of the rod per unit length and tensor \( \mathbf{I}_m(s) \) is the moment of inertia per unit length in the body-fixed frame of reference. The centerline tangent vector is \( \mathbf{r}(s, t) \) and its variations along the length (\( \partial \mathbf{r} / \partial s \)) in body-fixed frame) capture shear and extension. In this paper, such assumptions are assumed to be zero to ensure in-extensibility and un-shearability, therefore \( \mathbf{r} \) becomes constant and collinear with the cross-sectional normal vector \( \mathbf{\hat{a}}_3 \) shown in Figure 1 part (a). The vectors \( \mathbf{f}_e \) and \( \mathbf{q}_e \) are the external distributed force and moment, respectively. They include the distributed follower force as well as interactions of the rod with the environment such as fluid drag. Note that the spatial and temporal derivatives in equations (1) - (4) are relative to the body-fixed frame \( \{ \mathbf{\hat{a}} \} \), which obviates the need of transforming body-fixed follower forces and drag to inertial frame in our formulation.

The unknown variables in equations (1) - (4) that we need to solve for are: the vector \( \mathbf{\kappa}(s, t) \) that captures two-axes bending and torsion, the vectors \( \mathbf{v}(s, t) \) and \( \mathbf{\omega}(s, t) \) that represent the translational and the angular velocities of each cross-section, respectively, and the vector \( \mathbf{f}(s, t) \) that represent internal shear force and tension. The internal moment vector \( \mathbf{q}(s, t) \) in the angular momentum equation (2) is related to \( \mathbf{\kappa}(s, t) \) by the linear constitutive law

\[ \mathbf{q}(s, t) = \mathbf{B} : \mathbf{\kappa} \]  

(5)

where the tensor \( \mathbf{B}(s) \) represents the bending and torsional stiffness of the rod. In the body-fixed frame that coincides with principal torsion-flexure axes, the stiffness tensor \( \mathbf{B} \) is expressed in matrix form (denoted by [\( \mathbf{B} \)]) as

\[ \begin{bmatrix} EI & 0 & 0 \\ 0 & EI & 0 \end{bmatrix} \]  

(6)
In equation (6), $E$ is the Young's modulus, $G$ is the shear modulus, and $I$ and $J$ are the second moments of cross-section area about the principal torsion-flexure axes.

The Generalized-a method is adopted to compute the numerical solution of this system, subjected to necessary and sufficient initial and boundary conditions. A detailed description of this numerical scheme applied to this formulation is given in a previous publication and we refer the reader to that [22]. We have validated this scheme by comparing our findings for the critical value of the follower force beyond in the planar cantilever (fixed-free) scenario [17].

To model fluid dissipation, we use a quadratic form for drag in our simulations. In previous work we have analyzed the effect of both linear (Stokesian) and quadratic (Morrison) drag on planar flapping of clamped rods. As we discuss later, the choice of drag model is inconsequential to the results of this paper. We thus use the equation for Morrison drag [23] that has the form

$$f_M = -\frac{1}{2} \rho f d \left( C_n |v \times t| t \times (v \times t) + \pi C_t (v \cdot t) |v \cdot t| t \right). \quad (7)$$

Here, $\rho f$ is the fluid density, $d$ is diameter of the rod, $t$ is the unit tangent vector along the rod's centerline and $C_n$ and $C_t$ are drag coefficients in the normal and tangential directions, respectively. The first term in equation (7) captures the components of the velocity vector along the normal and binormal directions to the rod local centerline, while the second term represents the projection of the velocity vector in tangential direction, $t$.

### Table I. Numerical values for the properties of the rod and drag coefficients used in the simulations.

| Quantity          | Symbol | Value          |
|-------------------|--------|----------------|
| Bending Stiffness | $EI$   | 29.231 N m²    |
| Torsional Stiffness | $GJ$   | 23.385 N m²    |
| Mass per unit length | $m$   | 0.2019 kg/m    |
| Length            | $L$    | 8 m            |
| Normal drag coefficient | $C_n$ | 0.1            |
| Tangential drag coefficient | $C_t$ | 0.01           |
| Surrounding fluid density | $\rho_f$ | 1000 kg/m³    |

### III. RESULTS AND DISCUSSION

In this section we analyze the response of pre-stressed rods with out-of-plane equilibrium shapes ($L_{ee} \leq 0.4L$) when subjected to a distributed follower force.

In all the simulations an initially straight cylindrical rod is used with the properties given in Table 1 chosen to represent a soft filament. The pre-stress is generated by moving one end of the rod relative to, and towards the other as shown in Figure 1 (a). The pre-stress values are determined and controlled by the end-to-end distance, $L_{ee}$.

#### A. Overview of Buckled Shapes in Absence of Active Force

Consider a cylindrical rod clamped at both ends preventing boundary motion in all directions but along the end-to-end axis. To achieve this we impose $\mathbf{v}(0, t) = \omega(0, t) = 0$ at one end while the other end at $s = L$ is allowed to slide only in axial direction satisfying $v_1(L, t) = v_2(L, t) = 0$ and $\omega(L, t) = 0$. By exerting a compressive force, $f_3$ at $s = L$ as well as three-dimensional perturbations along the length we observe the rod to bend and buckle in plane for $4\pi^2 EI/L^2 \leq f_3$ as can be predicted by a linear static analysis. Next, by prescribing the axial velocity, $v_3$ at $s = L$ we move two boundaries towards one another as a means to control pre-stress developing a slack as shown in Figure 1. Despite the static indeterminacy of this loading configuration the buckling slack allows for oscillatory motion to emerge under follower forces as we demonstrate in previous works. However, beyond a certain threshold of compression ($1 - L_{ee}/L \approx 0.6$) the planar buckled shapes become unstable and rod transitions to more energetically favorable configurations that allow part of the strain energy to be stored in torsional form, as is also shown in the literature [23, 24]. In this paper we explore the dynamics of rods with such three-dimensional base states corresponding to the interval $0.60 \leq 1 - L_{ee}/L \leq 0.675$. Nonetheless, we also briefly review the dynamics of clamped-clamped rods with planar base states animated by follower forces.

#### B. Flapping Motion of Planar Base States

As explained above, we can control the end-to-end distance by specifying proper boundary conditions. We then apply a uniformly distributed follower load of intensity $F$ along the tangential direction of the rod’s centerline, $t$ as shown in Figure 2.

For the range of end-to-end parameters with planar buckling equilibria, $1 - L_{ee}/L < 0.6$, using numerical analysis we have shown that beyond a critical value of the follower force, $F_{cr}$, the buckled shapes no longer maintain static equilibrium and flapping oscillations emerge in the plane of buckling [17, 18]. Figure 2 shows an example of such oscillations, spatio-temporal variations of curvature and angular velocity, and evolution of rod shapes during one cycle of flapping.

#### C. Swirling Motion with Periodic Reversal of Non-Planar Base States

For the range of parameters with non-planar buckling equilibria tested here ($0.60 \leq 1 - L_{ee}/L \leq 0.675$), an example is shown in Figure 3. For any non-zero follower force gives rise to a purely rotational oscillation which we call swirling motion. Moreover, we find that under a constant loading, swirling rates decrease.
FIG. 2. Distribution of angular velocity and curvature during one cycle of flapping is illustrated along the rod (left). Configurations of the oscillating rod with $L_{ee} = 0.9L$ when $|F| = 34N/m$ is depicted during one cycle of flapping.

FIG. 3. Third angle projections of the rod at stable post-buckling equilibrium with end-to-end distance fixed to $1 - L_{ee}/L = 0.6$. When rod is subject to a distributed follower force, $Ft$, a swirling oscillations about the end-to-end axis, $\hat{e}_3$ emerges. It is found that swirling motion periodically flips direction between positive and negative $\hat{e}_3$ directions. Net moment of the follower force $\mathbf{M} = (M_1, M_2, M_3)$ is calculated by the formula $\int_0^L (\mathbf{R}_c \times Ft) ds$ where $\mathbf{R}_c$ is the position vector of the cross-section at $s$ in the inertial frame $\{\hat{e}_i\}$ located at the middle point between two clamped ends.

gradually until an abrupt reversal of direction (or flipping) occurs, hence the second characteristic time-scale of the oscillations that we evaluate from simulations are called here the rate of flipping. In each flipping cycle, which takes about two orders of magnitude longer than a swirling cycle, total strain energy changes negligibly (in this case less than 0.1% per cycle) while bending energy increases significantly (about 4% per cycle) and the torsional energy decreases (about 200% per cycle)–torsion is slowly converted to bending. Eventually, as the net torsion approaches zero and the rod limits toward a planar shape, bending energy is suddenly discharged into torsional energy (total strain energy changes negligibly), and the direction of swirling is reversed. We would also
point out that in the base states prior to exerting the follower force, about 98% of strain energy is in the form of bending deformation while only 2% in torsional form. Hence, in the following segment we take a closer look at the variations of energy relative to the energy in the base state.

D. Energy Exchange During Oscillations

Figure 4 demonstrates energy variations upon exertion of a unit follower force to a rod buckled out-of-plane for $1 - L_{ce}/L = 0.60$ relative to the energy levels of the base state. During each rotational (swirling) cycle we observe gradual decrease in kinetic energy and simultaneously a gradual increase in total strain energy. This trend is "punctuated" with flipping phenomenon which results in a sudden increase in kinetic energy and a small but sudden decrease in strain energy. Moreover, the figure illustrates that in each flipping cycle an overall decrease in torsional energy that concurs with an increase in branding energy. Figure 4 also demonstrates graphically superimposed shapes during a full swirling cycle and during flipping.

Descriptions used to evaluate energy torsional energy, $E_T$, bending energy, $E_B$, total strain energy, $E_S$, and kinetic energy, $E_K$, in time with respect to the energy levels at the base state, $E_0$ are shown in part (a). Variations of energy dissipated by fluid drag, $W_{M}$, and work done by follower force, $W_{F}$ are given in part (b). As a measure of the swirling frequency, the component of the angular velocity about $\hat{a}_3$ (i.e., the tangential direction to the centerline curve of the rod) at the mid-span length is plotted in time for $|F| = 1 \text{ N/m}$ and $1 - L_{ce}/L = 0.60$ as well as superimposed shapes during an entire swirling cycle are depicted in (c). Evolution of angular velocity at the mid-span length during the flipping along with the rendition of the shapes is illustrated in part (d).

\[ E_s = \frac{1}{2} \int_0^L (B \cdot \kappa) \cdot \kappa ds = E_B + E_T \]
\[ = \frac{1}{2} \int_0^L EI (\kappa_1^2 + \kappa_2^2) ds + \frac{1}{2} \int_0^L GJ\kappa_3^2 ds \]
FIG. 5. Frequency of oscillations as a function of follower force density for the interval $0.325 \leq \frac{L_{ee}}{L} \leq 0.375$. The left hand plot shows the frequency of swirling oscillations about the end-to-end axis. The right hand plot demonstrates the frequency of flipping oscillations. The results reveal that for a fixed value of follower force by decreasing the end-to-end distance $L_{ee}$, the rate of swirling, $w_1$ increases while the rate of flipping, $w_2$ decreases.

$$E_K = \frac{1}{2} \int_0^L m(v \cdot v) ds$$ (9)

$$W_{fM} = \int_0^t \int_0^L (f_M \cdot v) ds d\tau$$ (10)

$$W_F = \int_0^t \int_0^L (Ft \cdot v) ds d\tau$$ (11)

E. Frequencies of Oscillations

Figure 5 shows the rates of both swirling and flipping oscillations as a function of force intensity and pre-stress. The results include three distinct pre-stress rates corresponding to $L_{ee}/L = \{0.325, 0.350, 0.375\}$. We find that frequency of swirling, $w_1$ linearly varies with follower force intensity in all cases tested. However, the flipping frequency, $w_2$ shown in the same figure is found to become insensitive to the follower force as the force intensity grows.

IV. SUMMARY

We have analyzed the oscillatory response of elastic filaments clamped at both ends and buckled to out-of-plane configurations under distributed follower forces using a geometrically nonlinear continuum rod model. We observe a swirling (purely rotational) motion around the end-to-end axis under any non-zero follower force. Moreover, a second characteristic time scale is observed during which the swirling motion undergoes reversal or flipping. We identified the force-frequency behavior as a function of pre-stress, measured by end-to-end distance, as well as the force density while keeping the inertia and elasticity constant. For the range of parameters examined here we identify a linear relationship between the force density and the swirling frequency. The flipping frequency is found to be sensitive to the force density only when forces are small, becoming independent of the force when intensity increases. We used a quadratic drag model which corresponds to high Reynolds number. Nonetheless, based on previous work on planar flapping with both linear and non-linear drag models, we expect the oscillations we observe here to change only quantitatively at low Reynolds number where viscous effects are dominant.

[1] D. Bray, Cell Movements: From Molecules To Motility (New York, NY: Garland Science, 2001).

[2] S. Nonaka, H. Shiratori, Y. Saijoh, and H. Hamada, Determination of left–right patterning of the mouse embryo by artificial nodal flow, Nature 418, 96 (2002).
[3] A. M. D. Malone, C. T. Anderson, P. Tummala, R. Y. Kwon, T. R. Johnston, T. Stearns, and C. R. Jacobs, Primary cilium mediate mechanosensing in bone cells by a calcium-independent mechanism, *Science* **104**, 13325 (2007).

[4] C. Brennen and H. Winet, Fluid mechanics of propulsion by cilia and flagella, *Annual Review of Fluid Mechanics* **9**, 339 (1977).

[5] F. Ling, H. Guo, and E. Kanso, Instability-driven oscillations of elastic microfilaments, *Journal of The Royal Society Interface* **15**, 20180594 (2018).

[6] Y. L. Zhang and D. M. Crothers, Statistical Mechanics of Sequence-Dependent Circular DNA and its Application for DNA Cyclization, *Biophysical Journal* **84**, 136 (2003).

[7] P. Satir and T. Matsuoka, Splitting the ciliary axoneme: Implications for a switch-point model of dynein arm activity in ciliary motion, *Cell Motility* **14**, 345 (2003).

[8] C. J. Brokaw, Thinking about flagellar oscillation, *Cell Motility* **66**, 425 (2018).

[9] D. M. Woolley, Flagellar oscillation: a commentary on proposed mechanisms, *Biological Reviews* **85**, 453 (2010).

[10] I. H. Riedel-Kruse, A. Hilfinger, J. Howard, and F. Jülicher, How molecular motors shape the flagellar beat, *HFSP Journal* **1**, 192 (2007).

[11] C. B. Lindemann and K. A. Lesich, Flagellar and ciliary beating: the proven and the possible, *123*, 519 (2010).

[12] S. Hanasoge, P. J. Hesketh, and A. Alexeev, Microfluidic pumping using artificial magnetic cilia, *Microsystems & Nanoengineering* **4**, 11 (2018).

[13] R. Dreyfus, J. Baudry, M. L. Roper, M. Ferriglier, H. A. Stone, and J. Bibette, Microscopic artificial swimmers, *Nature* **437**, 862 (2005).

[14] D. Nishiguchi, J. Iwasawa, H.-R. Jiang, and M. Sano, Flagellar dynamics of chains of active janus particles fueled by an AC electric field, *New Journal of Physics* **20**, 015002 (2018).

[15] A. E. Patteson, A. Gopinath, and P. E. Arratia, Active colloids in complex fluids, *Current Opinion in Colloid & Interface Science* **21**, 86 (2016).

[16] Y. Sasaki, Y. Takikawa, V. S. R. Jampini, H. Hoshikawa, T. Seto, C. Bahr, S. Herminghaus, Y. Hidaka, and H. Orihara, Colloidal caterpillars for cargo transportation., *Soft Matter* **10**, 8813 (2014).

[17] S. Fatehiboroujeni, A. Gopinath, and S. Goyal, Nonlinear oscillations induced by follower forces in prestressed clamped rods subjected to drag, *Journal of Computational and Nonlinear Dynamics* **13**, 121005 (2018).

[18] S. Fatehiboroujeni, A. Gopinath, and S. Goyal, Effect of Boundary Constraints on the Nonlinear Flapping of Filaments Animated by Follower Forces, *arXiv* e-prints , arXiv:1905.08421 (2019), arXiv:1905.08421 [physics.bio-ph].

[19] G. De Canio, E. Lauga, and R. E. Goldstein, Spontaneous oscillations of elastic filaments induced by molecular motors *14*, 10.1098/rsif.2017.0491 (2017).

[20] R. Chelakkot, A. Gopinath, L. Mahadevan, and M. F. Hagan, Flagellar dynamics of a connected chain of active, polar, brownian particles, *Journal of the Royal Society, Interface* **11**, 20130884 (2014).

[21] G. Kirchhoff, Über das gleichgewicht und die bewegung eines unendlich dunnen elastischen stabes, J. Reine Angew. Math. (Crelle) **56**, 285 (1859).

[22] S. Fatehiboroujeni, H. Palanthandalam-Madapusi, and S. Goyal, Computational rod model with user-defined nonlinear constitutive laws, *Journal of Computational and Nonlinear Dynamics* **13**, 101006 (2018).

[23] S. Goyal, N. C. Perkins, and C. L. Lee, Nonlinear Dynamics and Loop Formation in Kirchhoff Rods with Implications to the Mechanics of DNA and Cables, *Journal of Computational Physics* **209**, 371 (2005).

[24] G. van der Heijden, S. Neukirch, V. Goss, and J. Thompson, Instability and self-contact phenomena in the writhing of clamped rods, *International Journal of Mechanical Sciences* **45**, 161 (2003).