Perturbative QCD Analysis of $B$ to $\pi$ and $B$ to $\rho$ Transitions

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Abstract

We calculate the form factors of $B \to \pi$ and $B \to \rho$ heavy to light transition matrix elements by using the factorization formalism of perturbative QCD. We obtain them at $q^2 = 0$ and show their dependences on the parameter $\epsilon$ of the B meson distribution amplitude. We also obtain the form factors as functions of $q^2$ in the region $0 \leq q^2 \leq M_B^2/2$. The relations among the form factors are found in the limit of $m_{\pi}/M_B = 0$, $m_{\rho}/M_B = 0$, $M_b/M_B = 1$ and $(1 - x) \ll 1$.

PACS codes: 12.38.-t, 12.38.Bx, 13.20.He, 13.25.Hw

Key words: B meson decay, Perturbative QCD, Heavy to light transition, Form factors

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1. Introduction

CP-violation is one of the most important and mysterious phenomena in high energy physics, for which we have only the $K_L \to \pi\pi$ decay \[1\] and the charge asymmetry in the decay $K_L \to \pi^\pm l^\mp \nu$ \[2\] for more than 30 years. The mechanism of CP-violation through the complex phase of the Cabibbo-Kobayashi-Maskawa (CKM) \[3\] three family mixing matrix in the Weinberg-Salam model is presently the standard model for CP-violation. The B meson system offers many possibilities to investigate CP-violation \[4\], and the B-factories in KEK and SLAC are under construction for this purpose. In order to probe the CKM model precisely, it is crucial obtain the values of the CKM matrix elements accurately from B meson decays. For the decays involving $b \to c$ transition, we can apply the heavy quark symmetry and it is possible to determine $V_{cb}$ reliably through the heavy quark effective theory (HQET) \[5\]. However, for those involving $b \to u$ it is less likely that the heavy quark symmetry applies, and the determination of $V_{ub}$ has heavily relied on the models for the form factors.

The dynamical content of hadron decays is described by Lorentz invariant form factors of current matrix elements. The theoretical calculation of the form factors involving $b \to u$ transition is a difficult task, since it is concerned with the nonperturbative realm of QCD and we cannot apply the heavy quark symmetry. Recently there have been active investigations of the form factors of $B \to \pi$ and $B \to \rho$ by using quark model, QCD sum rule and lattice calculations \[6\]. CLEO has presented first experimental results of the branching ratios of $B \to \pi l\nu$ and $B \to \rho l\nu$ \[7\], which are still model dependent.

In this paper we will calculate the form factors of $B \to \pi$ ($F_0, F_1$) and $B \to \rho$ transitions ($V, A_0, A_1, A_2$) by using the method which Szczepaniak et al. employed for obtaining the $B \to \pi$ form factors \[8\]. This method is based on the meson theory of Brodsky and Lepage \[9\]. Ref. \[8\] noticed that in the case of a heavy
meson decaying into two lighter mesons the large momentum transfers are involved and the factorization formula of perturbative QCD (PQCD) for exclusive reactions becomes applicable: the amplitude can be written as a convolution of a hard-scattering quark-gluon amplitude $T_h$ and mason distribution amplitudes $\phi(x,Q^2)$ which describe the fractional longitudinal momentum distribution amplitude of the quark and antiquark in each meson.

In the present work we calculate the form factors at $q^2 = 0$, where $q^\mu$ is the difference between initial and final meson momenta, for various values of $\epsilon$ which is a parameter describing the sharpness of the initial heavy meson distribution amplitude. Therefore we show the $\epsilon$ dependences of the form factors, which in return can also be useful for understanding the structure of the B meson. Then we obtain the $q^2$ dependences of the form factors in the region $0 \leq q^2 \leq M_B^2/2$, since it can be considered that in this region the large momentum transfers are involved for the interaction between the quark and antiquark in the meson. We also obtain the relations among the form factors of $B \to \pi$ and $B \to \rho$ transitions in the limit of $m_\pi/M_B = 0$, $m_\rho/M_B = 0$, $M_b/M_B = 1$ and $(1 - x) << 1$. We think these relations are valuable for improving the knowledge of the heavy to light transition form factors.

In section 2 we study the form factors of $B \to \pi$, $F_0^{B\pi}(q^2)$ and $F_1^{B\pi}(q^2)$. In section 3, those of $B \to \rho$, $V^{B\rho}(q^2)$, $A_1^{B\rho}(q^2)$, $A_2^{B\rho}(q^2)$ and $A^{B\rho}(q^2)$, are calculated. We obtain in section 4 the form factors and the relations among them in the limit of $m_\pi/M_B = 0$, $m_\rho/M_B = 0$, $M_b/M_B = 1$ and $(1 - x) << 1$. We organize our results of the form factors and compare them with other existing calculations. Section 5 constitutes the conclusion.

2. Form Factors $F_0^{B\pi}(q^2)$ and $F_1^{B\pi}(q^2)$

From Lorentz invariance one finds the decomposition of the hadronic matrix
In (4) we use the distribution amplitude given by \[8, 9, 11, 12\] element in terms of hadronic form factors [10]:

\[< \pi^- (p_\pi) | V^\mu | B^0 (p_B) > = (p_B + p_\pi)^\mu f^\pi_+ (q^2) + (p_B - p_\pi)^\mu f^{\pi -}_- (q^2) = (r^\mu - \frac{m_B^2 - m_\pi^2}{q^2} q^\mu) F_1^{\pi^+} (q^2) + \frac{m_B^2 - m_\pi^2}{q^2} q^\mu F_0^{\pi^+} (q^2), \] (1)

where \(V^\mu = \bar{u} \gamma^\mu b\), \(q^\mu = (p_B - p_\pi)^\mu\), \(r^\mu = (p_B + p_\pi)^\mu\), and

\[F_1^{\pi^+} (q^2) = f^\pi_+ (q^2), \quad F_0^{\pi^+} (q^2) = f^{\pi -}_- (q^2) + \frac{q^2}{M_B^2 - m_\pi^2} f^{\pi -}_- (q^2). \] (2)

In the rest frame of the decay products, \(F_1\) and \(F_0\) correspond to \(1^-\) and \(0^+\) exchanges, respectively. At \(q^2 = 0\) we have the constraint

\[F_1^{\pi^+} (0) = F_0^{\pi^+} (0), \] (3)

since the hadronic matrix element in [11] is nonsingular at this kinematic point.

We calculate the \(B\) to \(\pi\) (heavy to light) transition matrix element by using the PQCD factorization of exclusive amplitudes at high momentum transfer and neglect all final state interactions [8]. To the first order in \(\alpha_s = \alpha_s (Q^2)\) we have

\[< \pi^- (p_\pi) | V^\mu | B^0 (p_B) > = \frac{8 \pi \alpha_s}{3} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B (x) \times \left[ \text{Tr} \left\{ \left( \frac{\bar{\gamma}_5 \gamma^\mu}{M_B^2} \right) \phi_B + g(x) M_B \gamma_5 \gamma^\nu \right\} \right] + \frac{\text{Tr} \left\{ \left( \frac{\bar{\gamma}_5 \gamma^\mu}{M_B^2} \right) \phi_B + g(x) M_B \gamma_5 \gamma^\nu \right\}}{(k_2^2 - M_b^2) Q^2} \phi_\pi (y), \] (4)

where \(Q^\mu = (1 - x) p_B^\mu - (1 - y) p_\pi^\mu\), \(k_1^\mu = -(1 - x) p_\pi^\mu + p_\pi^\mu\), \(k_2^\mu = p_B^\mu - (1 - y) p_\pi^\mu\), and

\[Q^2 = M_B^2 [- (1 - x)(1 - y) \left( 1 - \frac{q^2}{M_B^2} \right) + (1 - x)^2 + ((1 - y)^2 - (1 - x)(1 - y)) \frac{m_\pi^2}{M_B^2}], \]

\[k_1^2 = M_B^2 [- (1 - x)(1 - y) \left( 1 - \frac{q^2}{M_B^2} \right) + (1 - x)^2 + (1 - y)^2 \frac{m_\pi^2}{M_B^2}], \]

\[k_2^2 - M_b^2 = M_B^2 [- (1 - y) \left( 1 - \frac{q^2}{M_B^2} \right) + (1 - M_b^2 - (1 - y) \frac{m_\pi^2}{M_B^2}). \] (5)

In [8] we use the distribution amplitude given by \[8, 9, 11, 12\]

\[\phi_\pi (x) = \sqrt{\frac{3}{2}} f_\pi x (1 - x), \] (6)

\[\phi_B (x) = \frac{1}{2 \sqrt{6}} f_B \int_0^1 \varphi (x) dx, \quad \varphi (x) = \frac{x^2 (1 - x)^2}{[\epsilon^2 x + (1 - x)^2]^2}, \] (7)
whose integrals are related to the meson decay constant by

\[ \int_0^1 dx \phi_M(x) = \frac{1}{2\sqrt{6}} f_M. \]  

(8)

In the right hand sides of (7) and (8) there are extra factor \( \frac{1}{\sqrt{2}} \) compared with (8), since in this paper we adopt the convention of the meson decay constant given by

\[ <0|A^\mu|M(p)> = i f_M p^\mu \] in which \( f_\pi \equiv f_\pi^+ = 131.74 \pm 0.15 \text{ MeV} \) \[13\]. In (7) we took the upper limit of the integration over momentum fraction \( y \) of a quark in the light meson as \( 1 - \epsilon \), since the integration in the interval \( 1 - \epsilon \leq y \leq 1 \) corresponds to the Drell-Yan-West \[14\] end-point region. It gives only a small correction to the form factors, and this region is also expected to be suppressed by a Sudakov form factor \[8\].

After some calculations we have

\[ < \pi^- (p_\pi)|V^\mu|B^0(p_B) > = \frac{8\pi\alpha_s}{3} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x) \left[ \frac{K_a}{k_1^2 Q^2} + \frac{K_b}{(k_2^2 - M_b^2) Q^2} \right] \phi_\pi(y), \]  

(9)

where

\[ K_a = 4M_B^2 \{ r^\mu[ -(1 - x) \frac{q^2}{M_B} - x 2g \frac{m_\pi}{M_B} - x \frac{m_\pi^2}{M_B^2} ] \}
+ q^\mu[(1 - x)(2 - \frac{q^2}{M_B}) + (2 - x)2g \frac{m_\pi}{M_B} - x \frac{m_\pi^2}{M_B^2}] \}, \]

(10)

\[ K_b = 4M_B^2 \{ r^\mu[ (2g \frac{M_b}{M_B} - 1) + (1 - y)(1 - \frac{q^2}{M_B^2}) + (\frac{M_b}{M_B} - y2g) \frac{m_\pi}{M_B} ] \]
+ q^\mu[-(2g \frac{M_b}{M_B} - 1) - (1 - y)(1 - \frac{q^2}{M_B^2}) + (\frac{M_b}{M_B} - (2 - y)2g) \frac{m_\pi}{M_B} - 2(1 - y) \frac{m_\pi^2}{M_B^2}] \}.

Then from (11) and (9) we have

\[ F_1^{B\pi}(q^2) = \frac{8\pi\alpha_s}{3} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x) \left[ \frac{\bar{F}_a}{k_1^2 Q^2} + \frac{\bar{F}_b}{(k_2^2 - M_b^2) Q^2} \right] \phi_\pi(y), \]  

(11)

\[ F_1^a = 4M_B^2 \{ -(1 - x) \frac{q^2}{M_B^2} - x 2g \frac{m_\pi}{M_B} - x \frac{m_\pi^2}{M_B^2} \}, \]

\[ \bar{F}_1^a = 4M_B^2 \{ (2g \frac{M_b}{M_B} - 1) + (1 - y)(1 - \frac{q^2}{M_B^2}) + (\frac{M_b}{M_B} - y2g) \frac{m_\pi}{M_B} \}, \]
and

\[
F_{0}^{B \pi}(q^2) = \frac{8\pi\alpha_s}{3} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x) \left[ \frac{\bar{F}_0^a}{k_1^2 Q^2} + \frac{\bar{F}_0^b}{(k_2^2 - M_b^2)Q^2} \right] \phi_\pi(y),
\]

(12)

\[
\bar{F}_0^a = 4M_B^2\left[-(1-x)\frac{q^2}{M_B^2} - x2g\frac{m_\pi}{M_B^2} - x\frac{M_b^2}{M_B^2}\right]
\]

\[
+ \frac{q^2}{M_B^2 - m_\pi^2}(1-x)(2 - \frac{q^2}{M_B^2}) + (2-x)2g\frac{m_\pi}{M_B^2} - x\frac{M_b^2}{M_B^2}],
\]

\[
\bar{F}_0^b = 4M_B^2\left[(2g\frac{M_b}{M_B} - 1) + (1-y)(1 - \frac{q^2}{M_B^2}) + (\frac{M_b}{M_B} - y2g)\frac{m_\pi}{M_B^2}\right]
\]

\[
+ \frac{q^2}{M_B^2 - m_\pi^2}[-(2g\frac{M_b}{M_B} - 1) - (1-y)(1 - \frac{q^2}{M_B^2})]
\]

\[
+ (\frac{M_b}{M_B} - (2-y)2g\frac{m_\pi}{M_B^2} - 2(1-y)\frac{M_b^2}{M_B^2}].
\]

For \(m_\pi = 0\), we have

\[
Q^2 = M_B^2[-(1-x)(1-y)(1 - \frac{q^2}{M_B^2}) + (1-x)^2],
\]

\[
k_1^2 = M_B^2[-(1-x)(1 - \frac{q^2}{M_B^2}) + (1-x)^2],
\]

\[
k_2^2 - M_b^2 = M_B^2[-(1-y)(1 - \frac{q^2}{M_B^2}) + (1 - \frac{M_b^2}{M_B^2})],
\]

(13)

\[
\bar{F}_1^a = 4M_B^2\left[-(1-x)\frac{q^2}{M_B^2}\right],
\]

(14)

\[
\bar{F}_1^b = 4M_B^2[(2g\frac{M_b}{M_B} - 1) + (1-y)(1 - \frac{q^2}{M_B^2})],
\]

and

\[
\bar{F}_0^a = 4M_B^2\left[(1-x)\frac{q^2}{M_B^2}(1 - \frac{q^2}{M_B^2})\right],
\]

(15)

\[
\bar{F}_0^b = 4M_B^2\left[(2g\frac{M_b}{M_B} - 1) + (1-y)(1 - \frac{q^2}{M_B^2})\right](1 - \frac{q^2}{M_B^2}).
\]

Then, for \(m_\pi = 0\) and \(\frac{M_b}{M_B} = 1\), we have

\[
F_{1,0}^{B \pi}(q^2) = \frac{32\pi\alpha_s}{3M_B^2} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x) \phi_\pi(y) \frac{1}{(1-x)(1-y)^2} f_{1,0}.
\]

(16)
where

\[
f_1 = 2(1 - y) \frac{1}{1 - \frac{q^2}{M_B^2}} + [(2g - 1) - (1 - y)] \frac{1}{(1 - \frac{q^2}{M_B^2})^2},
\]

(17)

\[
f_0 = [(2g - 1) + (1 - y)] \frac{1}{1 - \frac{q^2}{M_B^2}}.
\]

(18)

3. Form Factors \( V^{B\rho}(q^2) \), \( A_1^{B\rho}(q^2) \), \( A_2^{B\rho}(q^2) \) and \( A^{B\rho}(q^2) \)

From Lorentz invariance one finds the decomposition of the hadronic matrix element in terms of hadronic form factors \([10]\):

\[
< \rho^- (p_{\rho}, \varepsilon) | (V - A)^\mu | B^0 (p_B) > = \frac{2V(q^2)}{M_B + m_\rho} i \varepsilon^{\mu \alpha \beta \gamma} \varepsilon^{* \alpha} p_B p_\rho^\beta
\]

(19)

\[
- (M_B + m_\rho) \varepsilon^\mu A_1(q^2) + \frac{(\varepsilon^* \cdot p_B)}{M_B + m_\rho} (p_B + p_\rho)^\mu A_2(q^2) - 2m_\rho \frac{(\varepsilon^* \cdot p_B)}{q^2} q^\mu A(q^2).
\]

The form factor \( A(q^2) \) can be written as

\[
A(q^2) = A_0(q^2) - A_3(q^2), \quad \text{where } A_3(q^2) = \frac{M_B + m_\rho}{2m_\rho} A_1(q^2) - \frac{M_B - m_\rho}{2m_\rho} A_2(q^2),
\]

(20)

and at \( q^2 = 0 \) we have the constraint

\[
A_0(0) = A_3(0).
\]

(21)

We calculate the \( B \) to \( \rho \) (heavy to light) transition matrix element by using the PQCD factorization of exclusive amplitudes at high momentum transfer and neglect all final state interactions \([8]\). To the first order in \( \alpha_s = \alpha_s(Q^2) \) we have

\[
< \rho^- (p_{\rho}, \varepsilon) | V^\mu | B^0 (p_B) > = \frac{8\pi \alpha_s}{3} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x)
\]

\[
\times \left[ \frac{\text{Tr}\{(\not\! p_{\rho} + m_\rho) \not\! \epsilon \gamma^\nu (J_1 \gamma^\mu (\not\! p_B + g(x)M_B) \gamma_5 \gamma_\nu)\}}{k_1^2 Q^2} \right.
\]

(22)

\[
+ \left. \frac{\text{Tr}\{(\not\! p_{\rho} + m_\rho) \not\! \epsilon \gamma^\mu (J_2 + M_\rho) \gamma^\nu (\not\! p_B + g(x)M_B) \gamma_5 \gamma_\nu\}}{(k_2^2 - M_\rho^2) Q^2} \right] \phi_\rho(y),
\]
where $V^\mu = \bar{u}\gamma^\mu b$, $Q^\mu = (1 - x)p_B^\mu - (1 - y)p_\rho^\mu$, $k_1^\mu = -(1 - x)p_B^\mu + p_\rho^\mu$, $k_2^\mu = p_B^\mu - (1 - y)p_\rho^\mu$, and $Q^2$, $k_1^2$ and $k_2^2 - M_b^2$ are given by (3) with $m_\pi$ replaced by $m_\rho$. In (22) we use the distribution amplitude of B meson given in (3) and that of $\rho$ meson given by [4, 8, 9, 11, 12].

\[
\phi_\rho(x) = \sqrt{\frac{3}{2}} f_\rho x (1 - x),
\]

where $<0|V^\mu|\rho(\varepsilon)> = f_\rho m_\rho \varepsilon^\mu$ in which $f_\rho \equiv f_{\rho^+} = 216 \pm 5$ MeV [13].

After some calculations we have

\[
<\rho^-(p_\rho, \varepsilon)|V^\mu|B^0(p_B)> = \frac{8\pi\alpha_s}{3} \int_0^1 dx \int_0^{1-\varepsilon} dy \phi_B(x) \left[ \frac{\bar{V}^a}{k_1^2 Q^2} + \frac{\bar{V}^b}{(k_2^2 - M_b^2)Q^2} \right] \phi_\rho(y),
\]

where

\[
\bar{V}^a = 8M_B \frac{m_\rho}{M_B} i\varepsilon_{\mu\alpha\beta\gamma} \varepsilon^{*\alpha} p_B^\beta p_\rho^\gamma,
\]

\[
\bar{V}^b = 8M_B (-(2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}) i\varepsilon_{\mu\alpha\beta\gamma} \varepsilon^{*\alpha} p_B^\beta p_\rho^\gamma,
\]

\[
\left[ \frac{\bar{V}^a}{k_1^2 Q^2} + \frac{\bar{V}^b}{(k_2^2 - M_b^2)Q^2} \right] = 8M_B i\varepsilon_{\mu\alpha\beta\gamma} \varepsilon^{*\alpha} p_B^\beta p_\rho^\gamma
\]

\[
\times \left[ \frac{1}{k_1^2 Q^2 M_B} + \frac{1}{(k_2^2 - M_b^2)Q^2} (-(2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}) \right].
\]

To the first order in $\alpha_s = \alpha_s(Q^2)$ we have

\[
<\rho^-(p_\rho, \varepsilon)|A^\mu|B^0(p_B)> = \frac{8\pi\alpha_s}{3} \int_0^1 dx \int_0^{1-\varepsilon} dy \phi_B(x) \times \left[ \frac{\text{Tr}\{(\not{\bar{p}} B_\rho + m_\rho)e\gamma^\mu (k_1 + M_b)(\not{\bar{p}} B + g(x)M_B)\gamma_5 \gamma_\nu\}}{k_1^2 Q^2} + \frac{\text{Tr}\{(\not{\bar{p}} B_\rho + m_\rho)e\gamma^\mu \gamma_5 (k_2 + M_b)\gamma_5 (\not{\bar{p}} B + g(x)M_B)\gamma_\nu\}}{(k_2^2 - M_b^2)Q^2} \right] \phi_\rho(y),
\]

where $A^\mu = \bar{u}\gamma^\mu \gamma_5 b$. After some calculations we have

\[
<\rho^-(p_\rho, \varepsilon)|A^\mu|B^0(p_B)> = \frac{8\pi\alpha_s}{3} \int_0^1 dx \int_0^{1-\varepsilon} dy \phi_B(x) \left[ \frac{\bar{A}^a}{k_1^2 Q^2} + \frac{\bar{A}^b}{(k_2^2 - M_b^2)Q^2} \right] \phi_\rho(y),
\]
where

\[
\begin{align*}
\vec{A}^a &= \varepsilon^{*a} M_B^2 4m_\rho (-1 - \frac{q^2}{M_B^2}) + 2(1 - x) - \frac{m_\rho^2}{M_B^2} \\
&+ (\varepsilon^* \cdot p_B) r^\mu 4m_\rho (1 - 2(1 - x)) \\
&+ (\varepsilon^* \cdot p_B) q^\mu 4m_\rho (-1 - 2(1 - x)), \\
\vec{A}^b &= \varepsilon^{*b} M_B^3 ((2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}) (1 - \frac{q^2}{M_B^2}) \\
&- 2(2g \frac{M_b}{M_B} - 1) \frac{m_\rho}{M_B} + ((2g - \frac{M_b}{M_B}) - 4g(1 - y)) \frac{m_\rho^2}{M_B^2} - (1 - y) \frac{m_\rho^3}{M_B^3} \\
&+ (\varepsilon^* \cdot p_B) r^\mu M_B^2 (-2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B} \\
&+ (\varepsilon^* \cdot p_B) q^\mu M_B^2 (-1)(-2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}),
\end{align*}
\]

\[
\left[ \frac{\vec{A}^a}{k_1^2 Q^2} + \frac{\vec{A}^b}{(k_2^2 - M_b^2) Q^2} \right] = 4M_B^3 \\
\times \left\{ \varepsilon^{*a} \left[ \frac{1}{k_1^2 Q^2} \varepsilon^{*b} M_B^2 (-1 - \frac{q^2}{M_B^2}) + 2(1 - x) - \frac{m_\rho^2}{M_B^2} \right] \\
+ \frac{1}{(k_2^2 - M_b^2) Q^2} ((2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}) (1 - \frac{q^2}{M_B^2}) \\
- 2(2g \frac{M_b}{M_B} - 1) \frac{m_\rho}{M_B} + ((2g - \frac{M_b}{M_B}) - 4g(1 - y)) \frac{m_\rho^2}{M_B^2} - (1 - y) \frac{m_\rho^3}{M_B^3} \right] \\
+ \frac{(\varepsilon^* \cdot p_B) r^\mu}{M_B^2} \left[ \frac{1}{k_1^2 Q^2} M_B^2 (1 - 2(1 - x)) \right] \\
+ \frac{1}{(k_2^2 - M_b^2) Q^2} ((-2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}) \right] \\
+ \frac{(\varepsilon^* \cdot p_B) q^\mu}{M_B^2} \left[ \frac{1}{k_1^2 Q^2} M_B^2 (-1 - 2(1 - x)) \right] \\
+ \frac{1}{(k_2^2 - M_b^2) Q^2} ((-2g - \frac{M_b}{M_B}) - (1 - y) \frac{m_\rho}{M_B}) \right}. \tag{30}
\]

For \( m_\rho = 0, Q^2, k_1^2 \) and \( k_2^2 - M_b^2 \) are given by \([13]\), and we have

\[
\left[ \frac{\vec{V}^a}{k_1^2 Q^2} + \frac{\vec{V}^b}{(k_2^2 - M_b^2) Q^2} \right] = \frac{\vec{V}^b}{(k_2^2 - M_b^2) Q^2} \\
= 8M_B i \varepsilon_{\mu \alpha \beta \gamma} \varepsilon^{*a} p_B^\beta p_B^\gamma \frac{1}{(k_2^2 - M_b^2) Q^2} (2g - \frac{M_b}{M_B})(-1), \tag{31}
\]
and

\[
\begin{align*}
\left[ \frac{\bar{A}^a}{k_1^2 Q^2} + \frac{\bar{A}^b}{(k_2^2 - M_b^2)Q^2} \right] &= \frac{\bar{A}^b}{(k_2^2 - M_b^2)Q^2} \\
&= 4M_B^2 \left( \frac{1}{(k_2^2 - M_b^2)Q^2} (2g - \frac{M_B}{M_b}) \right) \\
&\times \{ \varepsilon^\mu (1 - \frac{q^2}{M_B^2}) + (\varepsilon^\ast \cdot p_B) r^\mu (1) + \frac{(\varepsilon^\ast \cdot p_B) q^\mu}{M_B^2} \}. \quad (32)
\end{align*}
\]

4. Relations among Form Factors in the Limit of \( m_\pi / M_B = 0, \ m_\rho / M_B = 0, M_b / M_B = 1 \) and \( (1 - x) \ll 1 \)

In this section we study the form factors \( F_0^{B\pi}(q^2) \) and \( F_1^{B\pi}(q^2) \) of \( B \rightarrow \pi \), and \( V^{B\rho}(q^2), A_0^{B\rho}(q^2), A_1^{B\rho}(q^2) \) and \( A_2^{B\rho}(q^2) \) of \( B \rightarrow \rho \), in the limit of \( m_\pi / M_B = 0, m_\rho / M_B = 0, M_b / M_B = 1 \) and \( (1 - x) \ll 1 \). The approximations \( m_\pi / M_B = 0, m_\rho / M_B = 0 \) and \( M_b / M_B = 1 \) are reasonable ones, since the B meson mass is much larger than the masses of light masons or light quarks. \( (1 - x) \ll 1 \) is also a good approximation in the region \( 0 \leq q^2 \leq M_B^2 / 2 \) as can be seen from (5), since \( (1 - x) \) is roughly given by the ratio of light and \( b \) quark masses or roughly by the value of the parameter \( \epsilon \) in the B meson distribution amplitude (7).

From (16)-(19) and (31)-(32), we can organize the form factors as follows:

\[
F_i(q^2) = \frac{32\pi \alpha_s}{3M_B^2} \int_0^1 dx \int_0^{1-x} dy \phi_B(x) \phi_\pi(y) \frac{1}{(1-x)(1-y)^2} f_i, \quad (33)
\]

where \( F_i=F_0,F_1,V,A_0,A_1,A_2 \), and \( \phi_i(y) = \phi_\pi(y) \) for \( F_0 \) and \( F_1 \), and \( \phi_i(y) = \phi_\rho(y) \) for \( V, A_0, A_1 \) and \( A_2 \). In (33) \( f_i \) are given by

\[
\begin{align*}
f_0 &= [(2g - 1) + (1 - y)]^{1/2} , \\ f_1 &= 2(1-y)^{1/2} + [(2g - 1) - (1 - y)]^{1/2} , \\ v &= (-1)(2g - 1)^{1/2} ,
\end{align*}
\]

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\[
a_1 = (2g - 1) \frac{1}{z}, \tag{37}
\]
\[
a_2 = (2g - 1) \frac{1}{z^2}, \tag{38}
\]
\[
a = \frac{g^2}{2m_\rho M_B} (2g - 1) \frac{1}{z^2}, \tag{39}
\]

where \( z \equiv 1 - \frac{q^2}{M_B^2} \). By taking the terms up to the first order in \( m_\rho/M_B \) for \( a_1, a_2 \) and \( a \) in (29) and (30), we obtain from the relations (20):
\[
a_0 = (-1)[(2g - 1) + (1 - y)] \frac{1}{z^2}. \tag{40}
\]

From (33)–(40) we find the relations among the form factors:
\[
F_1(q^2) = F_0(q^2)(2 - \frac{1}{z}) + 2 \frac{f_\pi}{f_\rho} A_1(q^2)(-1 + \frac{1}{z}) \tag{41}
\]
\[
F_0(q^2) \frac{1}{z} = - \frac{f_\pi}{f_\rho} A_0(q^2) \tag{42}
\]
\[
A_1(q^2) \frac{1}{z} = A_2(q^2) = -V(q^2). \tag{43}
\]

At \( q^2 = 0 \), we have the following relations:
\[
F_1(0) = F_0(0) = - \frac{f_\pi}{f_\rho} A_0(0), \quad A_1(0) = A_2(0) = -V(0). \tag{44}
\]

Ball and Braun also obtained the second relation in (14) to their accuracy in their QCD sum rule calculation [13]. We calculate \( F_1(0) \) and \( A_1(0) \) from (33)–(34) and (37) for the values of the parameter \( \epsilon \) of the B meson distribution amplitude (7) in the range \( 0.01 \leq \epsilon \leq 0.10 \), and present the results in Table 1 and Fig. 1. In this calculation we took \( g = 1, \alpha_s = 0.38 \) [8], and \( f_B = 0.2 \) GeV. We find that \( F_1(0) \) and \( A_1(0) \) depend much on the value of \( \epsilon \). The commonly used value \( F_1(0) = 0.33 \) obtained by Wirbel et al. [10] in quark model corresponds to \( \epsilon = 0.022 \). In this way, the information of the form factor is related to the structure of the B meson, and those two help each other as clues for the understanding of mesons. For \( \epsilon = 0.022 \) we have
\[
F_1(0) = 0.33, \quad A_1(0) = 0.47 \quad \text{for} \; \epsilon = 0.022. \tag{45}
\]
The values of other form factors at \( q^2 = 0 \) can be given by the relations in (44). In Table 1 and Fig. 1 we also present the dependence of the ratio \( F_1(0)/A_1(0) \) on the parameter \( \epsilon \) in the range \( 0.01 \leq \epsilon \leq 0.10 \), and find that this ratio is much less dependent on \( \epsilon \). In Table 2, we compare our results of the form factor values at \( q^2 = 0 \) given by (44) and (45) with other existing results obtained by quark model, QCD sum rule and lattice calculations.

The \( q^2 \) dependences of the form factors are given by (33)−(40) and (41)−(43). We obtain them in the region \( 0 \leq q^2 \leq M_B^2/2 \), and present the results in Fig. 2. The formulas in (33)−(40) can be written as

\[
F_0(q^2) = (a + b) \frac{1}{z}, \quad F_1(q^2) = 2b \frac{1}{z} + (a - b) \frac{1}{z^2},
\]

\[
A_1(q^2) = a f_\pi \frac{1}{f_\pi z}, \quad A_2(q^2) = -V(q^2) = a f_\rho \frac{1}{f_\pi z^2}, \quad -A_0(q^2) = (a + b) f_\rho \frac{1}{f_\pi z^2},
\]

where

\[
a = \frac{32\pi \alpha_s}{3M_B^2} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x) \phi_\pi(y) \frac{2g - 1}{(1 - x)(1 - y)^2},
\]

\[
b = \frac{32\pi \alpha_s}{3M_B^2} \int_0^1 dx \int_0^{1-\epsilon} dy \phi_B(x) \phi_\pi(y) \frac{1}{(1 - x)(1 - y)}.
\]

We note that the expressions in (46), and also the relations in (11)−(13), are independent of the shapes of the distribution amplitudes \( \phi_B(x) \), \( \phi_\pi(y) \) and the value of the parameter \( \epsilon \). Their dependences appear in the values of \( a \) and \( b \) in (47). For the distribution amplitudes in (3), (4) and (23), and \( \epsilon = 0.022 \) which gives (45), we have \( a = 0.28 \) and \( b = 0.05 \). From (46) we find that \( F_0(q^2) \) and \( A_1(q^2) \) have the simple pole \( q^2 \) dependence, and \( A_2(q^2) \), \( V(q^2) \) and \( A_0(q^2) \) have the dipole \( q^2 \) dependence. \( F_1(q^2) \) has the mixture of the simple pole and dipole \( q^2 \) dependences, but the dipole \( q^2 \) dependence is dominant. These characters of the form factors are shown clearly in Fig. 2.
5. Conclusion

We calculated the form factors of $B \to \pi$ and $B \to \rho$ heavy to light transition matrix elements by using the factorization formalism of perturbative QCD. We obtained them at $q^2 = 0$ for the values of the parameter $\epsilon$ of the B meson distribution amplitude in the range $0.01 \leq \epsilon \leq 0.10$, and found that they depend much on the value of $\epsilon$ unless $g = 1/2$. We also obtained the $q^2$ dependences of the form factors in the region $0 \leq q^2 \leq M_B^2/2$, since we can consider that in this region the large momentum transfers are involved for the interaction between the quark and antiquark in the meson. The relations among the form factors are found in the limit of $m_\pi/M_B = 0$, $m_\rho/M_B = 0$, $M_b/M_B = 1$ and $(1 - x) \ll 1$. These conditions are reasonable ones since the B meson mass is much larger than the masses of light meson or light quarks, and $(1 - x)$ is roughly given by the ratio of light and $b$ quark masses.

For the heavy to heavy transitions like $B \to D$($^*\right)$, HQET can be applied and all the relevant form factors are expressed by the one Isgur-Wise function [5]. However, for heavy to light transitions like $B \to \pi$ and $B \to \rho$, we cannot apply HQET, and it is very important to understand the form factors of heavy to light transitions better. Improvements in this area of study are not only invaluable for the analyses of experimental data, for example, in the extraction of the CKM matrix elements from the experimental results of the B meson decay branching ratios, but also for the better understanding of the structures of mesons. Stech studied the form factors of heavy to light transitions in the latter context [16]. In the factorization formalism of perturbative QCD we obtained the relations among the $q^2$ dependent form factors in (41)-(43), and the relations (44) at $q^2 = 0$. The second relation $A_1(0) = A_2(0) = -V(0)$ in (44) was also obtained by Ball and Braun in their QCD sum rule calculation [15]. In the first relation $F_1(0) = F_0(0) = -\frac{f_\pi}{f_\rho}A_0(0)$ in (44), the first equality is a well-known relation as explained in (3), however,
the second equality is not a usual one. This relation $F_1(0) = -\frac{f_\rho}{f_\pi} A_0(0)$ can be checked by measuring the differential branching ratios $d\mathcal{B}(B^0 \to \pi^- l^+ \nu)/dq^2$ and $d\mathcal{B}(B^0 \to \rho^- l^+ \nu)/dq^2$ at $q^2 = 0$, which are given by

$$
\frac{d\mathcal{B}(B^0 \to \pi^- l^+ \nu)}{dq^2}\bigg|_{q^2=0} = \frac{G_F^2 M_B^3 |V_{ub}|^2}{192\pi^3 \Gamma_B} (1 - \frac{m_\pi^2}{M_B^2})^3 |F_1(0)|^2, \quad (48)
$$

$$
\frac{d\mathcal{B}(B^0 \to \rho^- l^+ \nu)}{dq^2}\bigg|_{q^2=0} = \frac{G_F^2 M_B^3 |V_{ub}|^2}{192\pi^3 \Gamma_B} (1 - \frac{m_\rho^2}{M_B^2})^3 |A_0(0)|^2. \quad (49)
$$

From (48) and (49) we have

$$
\frac{d\mathcal{B}(B^0 \to \pi^- l^+ \nu)/dq^2|_{q^2=0}}{d\mathcal{B}(B^0 \to \rho^- l^+ \nu)/dq^2|_{q^2=0}} = \frac{(1 - \frac{m_\pi^2}{M_B^2})^3 |F_1(0)|^2}{(1 - \frac{m_\rho^2}{M_B^2})^3 |A_0(0)|^2} = 1.06 \frac{|F_1(0)|^2}{|A_0(0)|^2}. \quad (50)
$$

CLEO reported \cite{7} $\mathcal{B}(B^0 \to \pi^- l^+ \nu) = (1.8 \pm 0.4 \pm 0.3 \pm 0.2) \times 10^{-4}$ and $\mathcal{B}(B^0 \to \rho^- l^+ \nu) = (2.5 \pm 0.4_{-0.7}^{+0.5} \pm 0.5) \times 10^{-4}$. Then we expect that the ratio in the left hand side of (50) will be measured in near future, which will provide the ratio $|F_1(0)|/|A_0(0)|$.

We obtained the expressions for the form factors given by (46) and the relations among the form factors given by (41)–(43). We note that they are independent of the shapes of the distribution amplitudes $\phi_B(x)$, $\phi_\pi(y)$ and the value of the parameter $\epsilon$. Their dependences appear only in the numerical values of $a$ and $b$ in (47). The formulas in (46) show that $F_0(q^2)$ and $A_1(q^2)$ have the simple pole $q^2$ dependence, and $A_2(q^2)$, $V(q^2)$ and $A_0(q^2)$ have the dipole $q^2$ dependence. $F_1(q^2)$ has the mixture of the simple pole and dipole $q^2$ dependences, but the dipole $q^2$ dependence is dominant. These results have been possible since in the case of the B meson decaying into $\pi$ or $\rho$ meson with $q^2$ in the range of $0 \leq q^2 \leq M_B^2/2$, large momentum transfers are involved, and the factorization formula of perturbative QCD for exclusive reactions becomes applicable. Therefore, the heavy to light decays possess their own characteristic and interesting properties whose deeper understandings are desirable.
Acknowledgements

The authors are grateful to Stanley J. Brodsky for helpful discussions and for reading the manuscript carefully. They are also thankful to Adam Szczepaniak for useful discussions. This work was supported in part by the Basic Science Research Institute Program, Ministry of Education, Project No. BSRI-97-2414, in part by Korea Science and Engineering Foundation through the SRC Program of SNU-CTP, and in part by Non-Directed-Research-Fund, Korea Research Foundation 1997.
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Table 1: The $\epsilon$ dependences of $F_{1 \pi}^B(0)$, $A_{1 \rho}^B(0)$ and $F_{1 \pi}^B(0)/A_{1 \rho}^B(0)$.

| $\epsilon$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 |
|------------|------|------|------|------|------|------|------|------|------|------|
| $F_{1 \pi}^B(0)$ | 0.853 | 0.374 | 0.229 | 0.161 | 0.098 | 0.081 | 0.068 | 0.058 | 0.051 |
| $A_{1 \rho}^B(0)$ | 1.232 | 0.526 | 0.317 | 0.220 | 0.165 | 0.130 | 0.088 | 0.075 | 0.065 |
| $F_{1 \pi}^B(0)/A_{1 \rho}^B(0)$ | 0.693 | 0.710 | 0.723 | 0.734 | 0.744 | 0.754 | 0.762 | 0.771 | 0.779 | 0.786 |

Table 2: The results of this work of the form factor values at $q^2 = 0$ obtained with $\epsilon = 0.022$, and other existing results obtained by quark model, QCD sum rule and lattice calculations.

| | $F_{1 \pi}^B(0)$ | $A_{1 \rho}^B(0)$ | $A_{2 \rho}^B(0)$ | $-V_{\rho}^B(0)$ |
|----------------|----------------|----------------|----------------|----------------|
| This work | 0.33 | 0.47 | 0.47 | 0.47 |
| (Quark Model) | | | | |
| WSB [10] | 0.33 | 0.28 | 0.28 | 0.33 |
| ISGW [17] | 0.09 | 0.02 | 0.27 |
| Jaus [18] | 0.27 | 0.24 | 0.35 |
| FGM [19] | 0.20±0.02 | 0.26±0.03 | 0.31±0.03 | 0.29±0.03 |
| Melikhov [20] | 0.29 | 0.17–0.18 | 0.155 | 0.215 |
| (QCD Sum Rule) | | | | |
| BKR [21] | 0.30 | — | — | — |
| KRWY [22] | 0.27 | — | — | — |
| (Lattice) | | | | |
| UKQCD [23] | 0.27±0.11 | 0.27±0.03 | 0.25±0.03 | 0.35±0.03 |
| GSS [24] | 0.43±0.19 | 0.28±0.03 | 0.46±0.23 | 0.65±0.15 |
| APE [25] | 0.35±0.08 | 0.24±0.12 | 0.27±0.80 | 0.53±0.31 |
| ELC [26] | 0.30±0.14±0.05 | 0.22±0.05 | 0.49±0.21±0.05 | 0.37±0.11 |
Figure Captions

Fig. 1. The $\epsilon$ dependences of $F_1(0)$, $A_1(0)$ and $F_1(0)/A_1(0)$.

Fig. 2. The $q^2$ dependences of the form factors. $F_0(q^2)$ and $A_1(q^2)$ have the simple pole dependence, and $A_2(q^2)$, $V(q^2)$ and $A_0(q^2)$ have the dipole dependence. $F_1(q^2)$ has the mixture of the simple pole and dipole dependences, but the dipole dependence is dominant.
