A SIEGEL-WEIL FORMULA FOR \((U(1, 1), U(V))\) OVER A FUNCTION FIELD WITH \(\dim V\) GREATER THAN 2

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To the memory of Professor Linsheng Yin.

Abstract. We establish a Siegel-Weil formula for the dual pair \((U(1, 1), U(V))\) over a function field, where \(V\) is a hermitian space of dimension greater than 2.

Introduction

Let \(F\) be a function field in one variable over a finite field with \(\text{char}(F) \neq 2\), and let \(E\) be a quadratic field extension of \(F\). We denote by \(\mathbb{A}\) the adele ring of \(F\), and let \(\psi\) be a nontrivial character of \(F\). Let \(V\) be a non-degenerate hermitian space over \(E\), of dimension \(m \geq 3\), with hermitian form \((,): V \times V \to E\). Let \(H = U(V)\) be the associated unitary group.

In this paper, we establish a Siegel-Weil formula for the unitary dual pair \((U(1, 1), U(V))\). A particular form of this formula states that for any Schwartz-Bruhat function \(\Phi \in \mathcal{S}(V(\mathbb{A}))\), the following identity holds:

\[
\int_{H(F) \backslash H(\mathbb{A})} \left( \sum_{\xi \in V(F)} \Phi(h^{-1}\xi) \right) dh = 2\Phi(0) + 2 \sum_{b \in F} \int_{V(\mathbb{A})} \Phi(x)\psi(b(x, x)) dx.
\]

where \(dh\) is the Tamagawa measure on \(H(\mathbb{A})\) and \(dx\) is the self-dual Haar measure on \(V(\mathbb{A})\) with respect to \(\psi\).

This work is motivated by Haris’s pioneering work \[4\], in which he established a Siegel-Weil formula for orthogonal groups over a function field. In proving the formula, we use Haris’s results in \[4\] heavily.

The Siegel-Weil formulas over number fields have been well investigated (see for example \[9, 10, 11, 5, 6, 20, 3\] and the references cited therein). In contrast, the Siegel-Weil formulas over function fields are less studied and are only known in some special cases (see for example \[4, 10\]).

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This paper is dedicated to the memory of Professor Linsheng Yin. The author benefited greatly from Professor Yin’s guidance as a post-doc at the Tsinghua University. Professor Yin was an expert on function fields and I hope that it is suitable to dedicate this work to his memory.

1. Notations and statement of main result

Recall that $F$ is a function field in one variable over a finite field with $\text{char}(F) \neq 2$, and $E$ is a quadratic field extension of $F$. We denote the adele ring of $F$ by $\mathbb{A}_F$ and the adele ring of $E$ by $\mathbb{A}_E$.

Let $V = E^m$ be the space of column vectors over $E$ of dimension $m$, equipped with a non-degenerate hermitian form $(,): V \times V \to E$. We always assume $m \geq 3$.

Let $H = U(V)$ be the unitary group associated with $V$, given by

$$H(F) = \{ h \in GL_m(E) : (hx, hy) = (x, y), \forall x, y \in V \}.$$ 

Let $G = U(1, 1)$ be the quasi-split unitary group of rank 1, given by

$$G(F) = \{ g \in GL_2(E) : g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{g} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}. $$

Then $(G, H)$ forms a unitary dual pair.

The group $G$ has a Siegel parabolic subgroup $P = NM$, where $N$ is the unipotent radical and $M$ is the Levi subgroup. Precisely,

$$N(F) = \{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \}$$

and

$$M(F) = \{ m(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} : a \in E^\times \}. $$

Then we have the Bruhat decomposition $G(F) = P(F) \cup P(F)wN(F)$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For a place $v$ of $F$, let $F_v$ be the completion of $F$ at $v$ and let $O_v$ be the ring of integers of $F_v$. Let $K = \prod_v G(O_v)$. Then there is the Iwasawa decomposition

$$G(\mathbb{A}) = P(\mathbb{A}) \cdot K.$$ 

For $g \in G(\mathbb{A})$, we write $g = n(b)m(a)k$ with $n(b) \in N(\mathbb{A})$, $m(a) \in M(\mathbb{A})$ and $k \in K$, and we put

$$|a(g)| = |a|_{\mathbb{A}_E}.$$ 

Fix a nontrivial character $\psi : F \setminus \mathbb{A} \to \mathbb{C}$, and fix a Hecke character $\chi : E^\times \setminus \mathbb{A}_E^\times \to \mathbb{C}$ such that $\chi|_{\mathbb{A}_E^\times} = \epsilon$, where $\epsilon$ is the quadratic character associated to the quadratic extension $E/F$ by class field theory. Then there is an associated Weil representation $\omega = \omega_{\psi, \chi}$ of $G(\mathbb{A}) \times H(\mathbb{A})$ on the space $S(V(\mathbb{A}))$ of Schwartz-Bruhat functions on $V(\mathbb{A})$, given by

$$\omega(m(a))\Phi(x) = \chi(a)|a|_{\mathbb{A}_E} \Phi(xa),$$

$$\omega(n(b))\Phi(x) = \psi(b(x, x))\Phi(x),$$

$$\omega(w)\Phi(x) = \int_{V(\mathbb{A})} \Phi(y)\psi(tr_{E/F}(y, x))dy,$$

$$\omega(h)\Phi(x) = \Phi(h^{-1}x),$$
for $\Phi \in S(V(\mathbb{A}))$, $x \in V(\mathbb{A})$, $m(a) \in M(\mathbb{A})$, $n(b) \in N(\mathbb{A})$, and $h \in H(\mathbb{A})$, where $dy$ is the self-dual Haar measure on $V(\mathbb{A})$ with respect to the pairing on $V(\mathbb{A})$ given by $(x, y) \mapsto \psi(\text{tr}_{E/F}(x, y))$ (see [12] Lem. 4.1 for the local case and see [5] §1 for the number field case).

To each $\Phi \in S(V(\mathbb{A}))$, there are two associated objects, which are both functions on $G(\mathbb{A})$. One is the Siegel Eisenstein series and the other is the theta integral. The Siegel-Weil formula relates these two objects. Now we define them.

For $\Phi \in S(V(\mathbb{A}))$ and $s \in \mathbb{C}$, the Siegel Eisenstein series on $G(\mathbb{A})$ is defined by

$$E(g, s, \Phi) = \sum_{\gamma \in P(F) \setminus G(F)} f_{\Phi}^{(s)}(\gamma g), \quad \forall g \in G(\mathbb{A}),$$

where $f_{\Phi}^{(s)}(g) = |a(g)|^{s-s_0} \omega(g)\Phi(0)$ is a function on $G(\mathbb{A})$ and $s_0 = (m - 1)/2$.

On the other hand, for $\Phi \in S(V(\mathbb{A}))$, the theta integral on $G(\mathbb{A})$ is defined by

$$I(g, \Phi) = \int_{[H]} \Theta(\Phi)(g, h)dh, \quad \forall g \in G(\mathbb{A}),$$

where $\Theta(\Phi)(g, h) = \sum_{x \in V(F)} \omega(g, h)\Phi(x)$ is the theta kernel function, $[H] := H(F) \setminus H(\mathbb{A})$ is the quotient group, and $dh$ is the Tamagawa measure on $H(\mathbb{A})$.

Note that for a connected algebraic group $G$ over $F$, there is a canonical Haar measure on $G(\mathbb{A})$ called the Tamagawa measure (see [18, 19] and [14]), which is derived from a left invariant gauge form on $G$ (a gauge form on $G$ is a nonzero algebraic differential form on $G$ defined over $F$ and of degree $\dim G$). Note that Weil in [18, 19] defined the Tamagawa measure for any system of convergence factors for $G$, while Oesterlé in [14] specified a canonical system of convergence factors (i.e. the local Artin $L$-values $L_v(1, G)$).

Now we state the main result of this paper.

**Theorem 1.1.** Assume $m \geq 3$. Then for any $\Phi \in S(V(\mathbb{A}))$ and $g \in G(\mathbb{A})$, $E(g, s, \Phi)$ is holomorphic at $s_0 = (m - 1)/2$, $I(g, \Phi)$ is absolutely convergent, and

$$I(g, \Phi) = 2E(g, s_0, \Phi).$$

To prove this result we follow the method in [14] (also [7]). We give the main ideas below. First consider the case where $g = 1$ is the identity element in $G(\mathbb{A})$.

By Bruhat decomposition $G(F) = P(F) \cup P(F)wN(F)$, we have

$$E(1, s_0, \Phi) = \Phi(0) + \sum_{b \in F} \int_{V(\mathbb{A})} \Phi(x)\psi(b(x, x))dx.$$

On the other hand, by the orbit decomposition $V(F) = \{0\} \cup \bigcup_{a \in F} V_a(F)$ for the action of $H(F)$ on $V(F)$, where $V_a(F) = \{x \in V(F) : x \neq 0, q(x) = a\}$, we have

$$I(1, \Phi) = \tau(H)\Phi(0) + \sum_{a \in F} I_a \left( \sum_{\xi \in V_a(F)} \Phi(h^{-1}\xi) \right)dh,$$

where $\tau(H) = \text{Vol}([H], dh)$ is the Tamagawa number of $H$. In fact, $\tau(H) = 2$.

For $\Phi \in S(V(\mathbb{A}))$, define a function $F_{\Phi}^{*}$ on $\mathbb{A}$ by

$$F_{\Phi}^{*}(a) = \int_{V(\mathbb{A})} \Phi(x)\psi(a(x, x))dx.$$
Then
\[ E(1, s_0, \Phi) = \Phi(0) + \sum_{a \in F} F_\Phi(a). \]

Let \( F_\Phi \) be the Fourier transform of \( F_\Phi^* \) with respect to \( \psi \). Then we can show that
\[ F_\Phi(a) = \frac{1}{\tau(H_a)} \int_{|H|} \left( \sum_{\xi \in V_a(F)} \Phi(h^{-1}\xi) \right) dh, \]
where \( H_a \) is the stabilizer of \( H \) at any \( \xi_a \in V_a(F) \) when \( V_a(F) \neq \emptyset \) (if \( V_a(F) = \emptyset \), then \( F_\Phi(a) = 0 \)). In fact, we also have \( \tau(H_a) = 2 \). Then
\[ I(1, \Phi) = \tau(H)\Phi(0) + \sum_{a \in F} \tau(H_a) F_\Phi(a) = 2\Phi(0) + 2 \sum_{a \in F} F_\Phi(a). \]

Now consider the mapping \( q : V \to F \) given by \( q(x) = (x, x) \). Then \( (V, q) \) is a quadratic space over \( F \) of dimension \( 2n \). Consider the adelization \( q_\mathbb{A} : V(\mathbb{A}) \to \mathbb{A} \) given by \( q_\mathbb{A}(x) = (x, x) \) for \( x \in V(\mathbb{A}) \). A key result in this paper is that \( q_\mathbb{A} \) satisfies a so-called “condition (B)” defined by Weil in [18, Prop. 2], and hence we have
\[ \sum_{a \in F} F_\Phi^*(a) = \sum_{a \in F} F_\Phi(a), \]
and both series are absolutely convergent. So \( I(1, \Phi) = 2E(1, s_0, \Phi) \) and both are absolutely convergent.

In general, \( E(g, s_0, \Phi) = E(1, s_0, \omega(g)\Phi) \) and \( I(g, \Phi) = I(1, \omega(g)\Phi) \). The desired formula thus follows.

2. Proof of the formula

In this section, we give the details of the proof of the Siegel-Weil formula.

2.1. Some results of Weil. We first recall some results of Weil in [18].

**Proposition 2.1.** ([18, Prop. 1]) Let \( X \) and \( G \) be locally compact abelian groups, and let \( f : X \to G \) be a continuous mapping. Let \( G^* \) be the dual group of \( G \). Let \( dg \) be a Haar measure on \( G \), and let \( dg^* \) be the dual measure. Let \( dx \) be a Haar measure on \( X \). For \( \Phi \in \mathcal{S}(X) \) a Schwartz-Bruhat function on \( X \), define a function on \( G^* \) by
\[ F_\Phi^*(g^*) = \int_X \Phi(x)\langle f(x), g^* \rangle dx. \]

Suppose \( f \) satisfies the following condition:

(A) for \( \Phi \in \mathcal{S}(X) \), \( F_\Phi^* \) is integrable on \( G^* \) and the integral \( \int |F_\Phi^*|dg^* \) converges uniformly on every compact subset of \( \mathcal{S}(X) \).

Then for every \( g \in G \), there corresponds a measure \( \mu_g \) on \( X \), of support contained in \( f^{-1}\{ \{ g \} \} \), such that for every continuous function \( \Phi \) on \( X \) with compact support, the function on \( G \) defined by \( F_\Phi(g) = \int \Phi d\mu_g \) is continuous and satisfies \( \int F_\Phi dg = \int \Phi dx \). Moreover, the measures \( \mu_g \) are tempered; and for every \( \Phi \in \mathcal{S}(X) \), \( F_\Phi \) is continuous, belonging to \( L^1(G) \), satisfies \( \int F_\Phi dg = \int \Phi dx \), and is the Fourier transform of \( F_\Phi \).

Weil gave criteria for the above “condition (A)”. 
Proposition 2.2. ([18] Prop. 7 and the bottom of p. 8) Let G be a locally compact abelian group, \( \Gamma \) a discrete subgroup of G such that \( G/\Gamma \) is compact, and \( \Gamma^* \) the discrete subgroup of \( G^* \) which corresponds by duality to \( \Gamma \). Let X be a locally compact abelian group and \( f : X \to G \) a continuous mapping.

(i) Suppose \( f \) satisfies the following condition:

(B) for any \( \Phi \in \mathcal{S}(X) \) and \( g^* \in G^* \), the series

\[
\sum_{\gamma^* \in \Gamma^*} |F^*(g^* + \gamma^*)|
\]

is convergent, and is uniformly convergent on every compact subset of \( \mathcal{S}(X) \times G^* \).

Then \( f \) satisfies “condition (A)” of the above proposition. Moreover, if \( F_\Phi \) denotes the Fourier transform of the function \( F^*_\Phi \), then

\[
\sum_{\gamma \in \Gamma} F_\Phi(\gamma) = \sum_{\gamma^* \in \Gamma^*} F^*_\Phi(\gamma^*),
\]

the two series are absolutely convergent.

(ii) Suppose \( f \) satisfies the following two conditions:

\( (B_0) \) \( (\Phi, g^*) \to \Phi \to \Phi \) is a continuous mapping of \( \mathcal{S}(X) \times G^* \) into \( \mathcal{S}(X) \), where \( \Phi_{g^*} \) is given by \( \Phi_{g^*}(x) = \Phi(x)(f(x), g^*) \);

\( (B_1) \) the series \( \sum_{\gamma^* \in \Gamma^*} |F^*_\Phi(\gamma^*)| \) is uniformly convergent on every compact subset of \( \mathcal{S}(X) \).

Then \( f \) satisfies “condition (B)”, and all the above conclusions hold.

In the following, we will identify \( \mathbb{A} \) with its dual via the pairing \( \langle a, b \rangle \mapsto \psi(ab) \), and identify \( V(\mathbb{A}) \) with its dual via the pairing \( \langle x, y \rangle \mapsto \psi(\text{tr}_{E/F}(x, y)) \). For a place \( v \) of \( F \), we also identify \( F_v \) (resp. \( V(F_v) \)) with its dual via \( \psi_v \). We will always choose self-dual Haar measures on the groups \( \mathbb{A}, F_v, V(\mathbb{A}) \) and \( V(F_v) \).

We want to show that the mapping \( q_\mathbb{A} : V(\mathbb{A}) \to \mathbb{A} \) given by \( q_\mathbb{A}(x) = (x, x) \) satisfies “conditions \( (B_0) \) and \( (B_1) \)” and hence also satisfies “condition (B)” and “condition (A)”.

First we consider the local case.

2.2. The local case. Let \( v \) be a place of \( F \). Consider the mapping \( q_v : V(F_v) \to F_v \) given by \( q_v(x) = (x, x) \) for \( x \in V(F_v) \).

For \( \Phi \in \mathcal{S}(V(F_v)) \), define a function \( F^*_\Phi \) on \( F_v \) by

\[
F^*_\Phi(a) = \int_{V(F_v)} \Phi(x)\psi_v(aq_v(x))dx, \quad \forall a \in F_v.
\]

where \( dx \) is the self-dual Haar measure on \( V(F_v) \) with respect to \( \psi_v \).

Lemma 2.3. Let \( C \) be a compact subset of \( \mathcal{S}(V(F_v)) \). Then there exists a positive constant \( c \) such that

\[
|F^*_\Phi(a)| \leq c \cdot \max(1, |a|_v)^{-m}
\]

for all \( \Phi \in C \) and \( a \in F_v \), where \( |\cdot|_v \) is the absolute value on \( F_v \).

Proof. This follows from [4] Lem. 1] by noting that \( V(F_v) \) is of dimension \( 2m \geq 6 \) over \( F_v \) when regarded as a quadratic space over \( F_v \). \( \square \)

Proposition 2.4. The mapping \( q_v : V(F_v) \to F_v \) satisfies “condition (A)”.
Proof. By the above lemma, it suffices to show that \( \int_{F_v} \max(1, |a|_v)^{-m} da \) is convergent. Now
\[
\int_{F_v} \max(1, |a|_v)^{-m} da = \int_{O_v} \max(1, |a|_v)^{-m} da + \int_{F_v - O_v} \max(1, |a|_v)^{-m} da
\]
\[
= \text{Vol}(O_v) + \int_{|a|_v > 1} \max(1, |a|_v)^{-m} da
\]
\[
= \text{Vol}(O_v) + \int_{|a|_v > 1} |a|^{-m} da.
\]
But
\[
\int_{|a|_v > 1} |a|^{-m} da = \sum_{n \geq 1} \int_{|a|_v = q_v^n} q_v^{-mn} da = \sum_{n \geq 1} q_v^{-mn} \int_{|a|_v = q_v^n} da
\]
\[
= \text{Vol}(O_v^\times, da) \sum_{n \geq 1} q_v^{n(1-m)} < \infty
\]
since \( m \geq 3 \), where \( q_v \) is the cardinality of the residue field of \( O_v \). Thus
\[
\int_{F_v} \max(1, |a|_v)^{-m} da < \infty.
\]
\[\square\]

For \( \Phi \in \mathcal{S}(V(F_v)) \), let \( F_{\Phi} \) be the Fourier transform of \( F_{\Phi}^* \), i.e.
\[
F_{\Phi}(b) = \int_{F_v} F_{\Phi}^*(a) \psi(ab) da, \quad \forall b \in F_v,
\]
where \( da \) is the self-dual Haar measure on \( F_v \) with respect to \( \psi_v \).

Since \( q_v : V(F_v) \to F_v \) satisfies “condition (A)”, by Proposition 2.1, for any \( a \in F_v \), there is a measure \( \mu_a \) on \( V(F_v) \), of support contained in \( q_v^{-1}(\{a\}) \) such that
\[
F_{\Phi}(a) = \int_{V(F_v)} \Phi d\mu_a
\]
and
\[
\int_{F_v} F_{\Phi}(a) da = \int_{V(F_v)} \Phi(x) dx
\]
for all continuous functions \( \Phi \) on \( V(F_v) \) with compact support.

Since \( q_v^{-1}(\{a\}) = V_a(F_v) \cup \{0\} \) and \( \mu_a(\{0\}) = 0 \), we have
\[
F_{\Phi}(a) = \int_{q_v^{-1}(\{a\})} \Phi d\mu_a = \int_{V_a(F_v)} \Phi d\mu_a.
\]
Next we identity the measures \( \mu_a \). For \( a \in F_v \), let \( V_a(F_v) = \{ x \in V(F_v) : q_v(x) = a, x \neq 0 \} \); then there is a gauge form \( \theta_a(x) = (dx/d(q_v(x)))_a \) on \( V_a(F_v) \) (see [18, p. 12]). Moreover, by the discussions on [18, p. 13], the measure \( |\theta_a|_v \) on \( V_a(F_v) \) derived from \( \theta_a \) has the following property: for all continuous functions \( \Phi \) on \( V(F_v) \) with compact support, we have
\[
\int_{V(F_v)} \Phi(x) dx = \int_{F_v} \left( \int_{V_a(F_v)} \Phi |\theta_a|_v \right) da.
\]
Proposition 2.8. The mapping following Poisson formula.

□

desired result follows.

for all \( \Phi \in \text{"condition (A)"}. \)

Proof. We have seen that the equality holds for \( \Phi \) continuous and of compact support. Since the space of all the continuous functions \( \Phi \) on \( V(F_v) \) with compact support is dense in \( S(V(F_v)) \), the equality holds for all \( \Phi \in S(V(F_v)) \).

\[ \int_{V(F_v)} \Phi(x)dx = \int_{F_v} \left( \int_{V_\alpha(F_v)} \Phi d\mu_\alpha \right) da. \]

By the uniqueness of the measures \( \mu_\alpha \), we have \( \mu_\alpha = |\theta_\alpha|_v \) on \( V_\alpha(F_v) \).

Lemma 2.5. For \( \Phi \in S(V(F_v)) \), the Fourier transform \( F_\Phi \) of \( F_\Phi^* \) is given by

\[ F_\Phi(a) = \int_{V(F_v)} \Phi(a) \psi(\alpha a) dx. \]

Proof. We have shown that the equality holds for \( \Phi \) continuous and of compact support. Since the space of all the continuous functions \( \Phi \) on \( V(F_v) \) with compact support is dense in \( S(V(F_v)) \), the equality holds for all \( \Phi \in S(V(F_v)) \).

2.3. The global case. In this section, we show that the mapping \( q_\lambda : V(\mathbb{A}) \to \mathbb{A} \) given by \( q_\lambda(x) = (x, x) \) satisfies "condition (B)" and "condition (A)".

Recall that for \( \Phi \in S(V(\mathbb{A})) \), we have defined a function \( F_\Phi^* \) on \( \mathbb{A} \) by

\[ F_\Phi^*(a) = \int_{V(\mathbb{A})} \Phi(x) \psi(\alpha a(x)) dx. \]

Lemma 2.6. The mapping \( q_\lambda \) satisfies "condition (B)"; i.e., \( (\Phi, a) \mapsto \Phi(a) \) is a continuous mapping of \( S(V(\mathbb{A})) \times \mathbb{A} \) into \( S(V(\mathbb{A})) \), where \( \Phi(a) = \Phi(x) \psi(\alpha a(x)) \).

Proof. This is obvious.

\[ \text{Lemma 2.7. The mapping } q_\lambda \text{ satisfies "condition (B)";} \]

i.e. the series \( \sum_{a \in F} |F_\Phi^*(a)| \) is uniformly convergent on every compact subset of \( S(V(\mathbb{A})) \).

Proof. We follow the arguments on [4] p. 230 (see also [7] p. 190). Let \( C \) be a compact subset of \( S(V(\mathbb{A})) \). We want to show that the series \( \sum_{a \in F} |F_\Phi^*(a)| \) is uniformly convergent on \( C \).

Note that the adelic space \( V(\mathbb{A}) \) is the inductive limit of \( V_T = V_0^T \times V_1^T \), where \( V_T = \prod_{\nu \in T} V_\nu \) and \( V_1^T = \prod_{\nu \in T} V_1(F_\nu) \), as \( T \) runs over all the finite sets of places of \( F \), where \( V_0^T = V(O_v) \).

For the compact subset \( C \) of \( S(V(\mathbb{A})) \), there exists a finite set \( S \) of places of \( F \) and a compact subset \( C_1 \) of \( S(V_1^T) \), such that every \( \Phi \in C \) is of the form \( \Phi = \Phi_0 \otimes \Phi_1 \), where \( \Phi_0 \) is the characteristic function of \( V_0^S = \prod_{\nu \notin S} V_\nu^0 \) and \( \Phi_1 \in C_1 \).

By Lemma 2.3 and Fubini’s theorem, there is a positive constant \( c \) such that

\[ \sum_{a \in F} |F_\Phi^*(a)| \leq c \sum_{a \in F} \prod_{\nu \in S} \max(1, |\alpha|_v)^{-m} \]

for all \( \Phi \in C \). By [4] Prop. 1, the right hand side is convergent since \( m \geq 3 \). The desired result follows.

\[ \text{Consequently, by Proposition 2.2, we have} \]

Proposition 2.8. The mapping \( q_\lambda : V(\mathbb{A}) \to \mathbb{A} \) satisfies "condition (B)" and "condition (A)".

Let \( F_\Phi \) be the Fourier transform of \( F_\Phi^* \). Then by Proposition 2.2, we have the following Poisson formula.
Proposition 2.9. For \( \Phi \in \mathcal{S}(V(\mathbb{A})) \), we have
\[
\sum_{a \in F} F_\Phi^a(a) = \sum_{a \in F} F_\Phi(a),
\]
and both series are absolutely convergent.

Now we show that \( \sum_{a \in F} F_\Phi^a(a) \) is related to the Siegel Eisenstein series \( E(1, s_0, \Phi) \), where 1 is the identity element in the group \( G(\mathbb{A}) \).

Proposition 2.10. For \( \Phi \in \mathcal{S}(V(\mathbb{A})) \), we have
\[
E(1, s_0, \Phi) = \Phi(0) + \sum_{a \in F} F_\Phi^a(a),
\]
and the series on the right hand side is absolutely convergent.

Proof. By Bruhat decomposition \( G(F) = P(F) \cup P(F)wN(F) \), we have
\[
E(1, s_0, \Phi) = \sum_{\gamma \in P(F) \setminus G(F)} f_\Phi^{(s_0)}(\gamma) = \sum_{b \in F} f_\Phi^{(s_0)}(1) + \sum_{b \in F} f_\Phi^{(s_0)}(wn(b)).
\]
Note that \( f_\Phi^{(s_0)}(1) = \Phi(0) \) and \( f_\Phi^{(s_0)}(wn(b)) = F_\Phi^a(b) \). So
\[
E(1, s_0, \Phi) = \Phi(0) + \sum_{b \in F} F_\Phi^a(b).
\]
The absolute convergence follows from the above proposition.

Lemma 2.11. For any \( \Phi \in \mathcal{S}(V(\mathbb{A})) \) and \( g \in G(\mathbb{A}) \), the Siegel Eisenstein series \( E(g, s, \Phi) \) is holomorphic at \( s_0 \).

Proof. Note that \( E(g, s_0, \Phi) = E(1, s_0, \omega(g)\Phi) \) is absolutely convergent.

In general, \( E(g, s, \Phi) = f_\Phi^{(s)}(g) + \sum_{b \in F} f_\Phi^{(s)}(wn(b)g) \) is a series of holomorphic functions in \( s \). To show that \( E(g, s, \Phi) \) is holomorphic at \( s_0 \), by a theorem of Weierstrass ([I p. 177]), it suffices to show that the series converges uniformly on every compact neighborhood of \( s_0 \). If \( |s - s_0| \leq r \), then
\[
\sum_{b \in F} |f_\Phi^{(s)}(wn(b)g)| \leq \sum_{b \in F} |a(wn(b)g)|^r |\omega(wn(b)g)\Phi(0)|.
\]
Similar to [8 Lem. 6.7], we can show that there exists a positive constant \( C = C(g) \) such that
\[
|a(wn(b)g)| \leq C|a(g)|
\]
for all \( b \in F \). So for \( |s - s_0| \leq r \) we have
\[
\sum_{b \in F} |f_\Phi^{(s)}(wn(b)g)| \leq C^r|a(g)|^r \sum_{b \in F} |\omega(wn(b)g)\Phi(0)|.
\]
Note that \( \sum_{b \in F} \omega(wn(b)g)\Phi(0) = E(g, s_0, \Phi) - \omega(g)\Phi(0) \) is absolutely convergent. It follows that \( \sum_{b \in F} f_\Phi^{(s)}(wn(b)g) \) is uniformly convergent on the disc \( |s - s_0| \leq r \) by the Weierstrass M test. Thus \( E(g, s, \Phi) \) is holomorphic at \( s = s_0 \).

Next we show that \( \sum_{a \in F} F_\Phi(a) \) is related to the theta integral \( I(1, \Phi) \). For \( a \in F \), let \( V_a(F) = \{ x \in V(F) : x \neq 0, q(x) = a \} \). When \( V_a(F) \neq \emptyset \), choose \( \xi_a \in V_a(F) \) and let \( H_a \) be the stabilizer of \( H \) at \( \xi_a \), i.e. \( H_a = \{ h \in H : h\xi_a = \xi_a \} \). Thus \( V_a \) and \( H_a \) are algebraic groups over \( F \). By Witt’s theorem, \( V_a(F) \cong H_a(H) \setminus H(F) \). Moreover, by [2 Thm. A(ii)] and [19 Thm. 2.4.2], we have \( V_a(\mathbb{A}) \cong H_a(\mathbb{A}) \setminus H(\mathbb{A}) \).
We can describe the structure of $H_a$ as follows. We say a group is of type $U_n$ if it is the unitary group associated to a hermitian space of dimension $n$, and we say a group is of type $G_a$ if it is isomorphic to $G_a^r$ for some positive integer $r$.

**Lemma 2.12.** Assume $V_a(F) \neq \emptyset$.

(i) If $a \neq 0$, then $H_a$ is of type $U_{m-1}$.

(ii) If $a = 0$, then $H_a$ is isomorphic to the semidirect product of a group of type $U_{m-2}$ and a group $H'$, where $H'$ is the semidirect product of two groups of type $G_a$.

**Proof.** Similar to the proof of [19] Lem. 4.1.1. \qed

**Corollary 2.13.** Assume $V_a(F) \neq \emptyset$. Then (1) is a system of convergence factors for $V_a$. Moreover, $\tau(H) = \tau(H_a) = 2$ and $\tau(V_a) = 1$.

**Proof.** First note that (1) is a system of convergence factors for a group of type $G_a$, so $H_a$ has the same system of convergence factors as $H$ and (1) is a system of convergence factors for $V_a \cong H_a \setminus H$ by [19] Thm. 2.4.3.

Next note that the Tamagawa number of the unitary $H = U(V)$ is equal to 2: we know that $\tau(SU(V)) = 1$ (see [19] Thm. 4.4.1]) and $\tau(E^1) = 2$ (see [19] p. 65] and [13] Definition 4.7], where $E^1 = \{a \in E : a\bar{a} = 1\}$, by noting that the Artin $L$-function $L(s, E^1) = L(s, \epsilon)$; so $\tau(H) = \tau(SU(V)) \cdot \tau(E^1) = 2$ by [19] Corollary to Thm. 2.4.4]. Also note that the Tamagawa number of a group of type $G_a$ is equal to 1, so the Tamagawa number of $H_a$ is the same as that of a unitary group by [19] Corollary to Thm. 2.4.4] and hence $\tau(H_a) = 2$. \qed

Next we study $F_\Phi$. Since $q_\Lambda : V(\Lambda) \to \Lambda$ satisfies “condition (A)” by Proposition 2.1, for each $a \in F$, there is a positive measure $\mu_a$ on $V(\Lambda)$, of support contained in $q_\Lambda^{-1}\{\{a\}\}$, such that

$$F_\Phi(a) = \int_{V(\Lambda)} \Phi d\mu_a.$$  

Since $q_\Lambda^{-1}\{\{a\}\} = V_a(\Lambda) \cup \{0\}$ and $\mu_a(\{0\}) = 0$, we have

$$F_\Phi(a) = \int_{q_\Lambda^{-1}\{\{a\}\}} \Phi d\mu_a = \int_{V_a(\Lambda)} \Phi d\mu_a.$$  

Note that if $V_a(F) = \emptyset$, then $V_a(\Lambda) = \emptyset$ by the Hasse-Minkowski principle for quadratic forms (see [13] p. 223 or [13] p. 170]), and hence $F_\Phi(a) = 0$.

Next we identify the measures $\mu_a$. For $a \in F$, consider the gauge form $\theta_a(x) = (dx/dg(x))|_a$ on $V_a$ (see [18] p. 12]). Let $|\theta_a|_\Lambda$ be the Tamagawa measure on $V_a(\Lambda)$ derived from $\theta_a$.

**Lemma 2.14.** For all $\Phi \in \mathcal{S}(V(\Lambda))$, we have

$$F_\Phi(a) = \int_{V_a(\Lambda)} \Phi |\theta_a|_\Lambda, \quad \forall a \in F.$$  

**Proof.** We follow the arguments in the proof of [13] Lem. 3] (see also [13] p. 191]).

It suffices to show the equality for $\Phi$ restricted to a subset of $\mathcal{S}(V(\Lambda))$ which spans a dense subset of $\mathcal{S}(V(\Lambda))$.

Take $\Phi = \prod_v \Phi_v$, where each $\Phi_v \in \mathcal{S}(V(F_v))$ and $\Phi_v$ is the characteristic function of $V(O_v)$ for almost all $v$. Then $F_\Phi(a) = \prod_v F_{\Phi_v}(a)$, where $F_{\Phi_v}$ is the Fourier
transform of $\tilde{F}_{\Phi_{v}}^{*}$. By Lemma 2.5, $F_{\Phi_{v}}(a) = \int_{V_{v}(F_{v})} \Phi_{v}|_{\theta_{a}|_{v}}$. Moreover, since (1) is a system of convergence factors for $V_{a}$, $|\theta_{a}|_{A} = \prod_{v} |\theta_{a}|_{v}$. Thus

$$F_{\Phi}(a) = \prod_{v} F_{\Phi_{v}}(a) = \prod_{v} \int_{V_{v}(F_{v})} \Phi_{v}|_{\theta_{a}|_{v}} = \int_{V_{a}(A)} \Phi_{\theta_{a}|_{A}}.$$ 

\[ \square \]

For $a \in F$, let $D_{a}$ be the gauge form on $V_{a}$ which is the image of $\frac{\partial}{\partial_{\eta}}$ under the isomorphism $V_{a} \cong H_{a} \setminus H$, where $\eta$ (resp. $\eta_{a}$) is a left invariant gauge form on $H$ (resp. on $H_{a}$). Let $|D_{a}|_{A}$ be the Tamagawa measure on $V_{a}(A)$ derived from $D_{a}$.

**Lemma 2.15.** For $a \in F$ with $V_{a}(F) \neq \emptyset$, let $H_{a}$ be the stabilizer of $H$ at any element in $V_{a}(F)$. Then

$$\int_{[H]} \left( \sum_{\xi \in V_{a}(F)} \Phi(h^{-1}\xi) \right) dh = \tau(H_{a}) \int_{V_{a}(A)} \Phi|D_{a}|_{A}.$$ 

Note that if $V_{a}(F) = \emptyset$, then $V_{a}(A) = \emptyset$ and the two integrals above are all zero.

**Proof.** Assume $V_{a}(F) \neq \emptyset$. Choose $\xi_{a} \in V_{a}(F)$. Since $H_{a}(F) \setminus H(F) \cong V_{a}(F)$ via $h \mapsto h^{-1}\xi_{a}$, we have

$$\int_{[H]} \left( \sum_{\xi \in V_{a}(F)} \Phi(h^{-1}\xi) \right) dh = \int_{H_{a}(F) \setminus H(A)} \Phi(h^{-1}\xi_{a}) dh.$$ 

Now

$$\int_{H_{a}(F) \setminus H(A)} \Phi(h^{-1}\xi_{a}) dh = \int_{H_{a}(A) \setminus H(A)} \int_{H_{a}(F) \setminus H_{a}(A)} \Phi(h^{-1}h_{a}^{-1}\xi_{a}) d\tilde{h} \; dh_{a} = \tau(H_{a}) \int_{H_{a}(A) \setminus H(A)} \Phi(h^{-1}\xi_{a}) d\tilde{h}$$

$$= \tau(H_{a}) \int_{V_{a}(A)} \Phi|D_{a}|_{A},$$

where $dh_{a}$ is the Tamagawa measure on $H_{a}(A)$, $d\tilde{h} = \frac{dh}{dh_{a}}$ is the quotient measure on $H_{a}(A) \setminus H(A)$, and note that $|D_{a}|_{A}$ is the image measure of $dh$ under the isomorphism $V_{a}(A) \cong H_{a}(A) \setminus H(A)$. \[ \square \]

Note that $D_{a}$ and $\theta_{a}$ are both left invariant gauge forms on $V_{a}$, so $|D_{a}|_{A} = |\theta_{a}|_{A}$.

**Lemma 2.16.** For $a \in F$, we have

$$\int_{[H]} \left( \sum_{\xi \in V_{a}(F)} \Phi(h^{-1}\xi) \right) dh = 2F_{\Phi}(a).$$

By the discussions in §1, we thus have the following result.

**Proposition 2.17.** For $\Phi \in \mathcal{S}(V(A))$, we have

$$I(1, \Phi) = 2\Phi(0) + 2 \sum_{a \in F} F_{\Phi}(a),$$

and it is absolutely convergent.

Now we can finally show the Siegel-Weil formula.
Theorem 2.18. Assume $m \geq 3$. Then for any $\Phi \in S(V(\mathbb{A}))$ and $g \in G(\mathbb{A})$, $E(g, s, \Phi)$ is holomorphic at $s_0 = (m - 1)/2$, $I(g, \Phi)$ is absolutely convergent, and $I(g, \Phi) = 2E(g, s_0, \Phi)$.

Proof. First note that $E(g, s, \Phi)$ is holomorphic at $s_0 = (m - 1)/2$ by Lemma 2.11. Also note that $E(g, s_0, \Phi) = E(1, s_0, \omega(g)\Phi)$ and $I(g, \Phi) = I(1, \omega(g)\Phi)$, so it suffices to consider the case $g = 1$.

The equality $I(1, \Phi) = 2E(1, s_0, \Phi)$ follows from propositions 2.9, 2.10 and 2.17. □

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