Gravitational corrections to the scattering of charged pions

B. Charneski,1,∗ A. C. Lehum,1,2,† and A. J. da Silva1,‡

1Instituto de Física, Universidade de São Paulo
Caixa Postal 66318, 05315-970, São Paulo, São Paulo, Brazil

2Escola de Ciências e Tecnologia, Universidade Federal do Rio Grande do Norte
Caixa Postal 1524, 59072-970, Natal, Rio Grande do Norte, Brazil

Abstract

In this letter we discuss the role of gravitational contributions to the running of the electric charge through the evaluation of the scattering amplitude of charged pions in scalar electrodynamics. Computing the quadratic divergent part of the S-matrix amplitude for two distinct scattering processes, we show that a single definition of the running of the electric charge, that render it asymptotically free, can be made for both processes. Our result agrees with earlier ones, based on the calculation of the effective action, which suggest that the running of gauge coupling constants are driven in the direction of asymptotic freedom, when quantum gravitational corrections are taken into account.

PACS numbers: 12.20.-m, 04.60.-m, 11.10.Hi

∗bruno@fma.if.usp.br
†andrelehum@ect.ufrn.br
‡ajsilva@fma.if.usp.br
The theory of gravitational field quantized for small fluctuations around a flat metric is nonrenormalizable \[1–3\], i.e., an infinity number of free parameters is required to absorb all kinds of new UV divergences which are generated as the order of perturbative calculations is increased. On the other hand, quantum effects due to gravitation at low energy, much below of the Planck scale \(M_P \approx 1.4 \times 10^{19} \text{ GeV}/c^2\) (a natural energy scale of quantum theory of gravitation), can be calculated, by incorporating its effects in the spirit of an effective field theory \[4\]. From this point of view, Robinson and Wilczek \[5\] proposed that quantum gravity corrections make gauge theories asymptotically free (i.e., the gauge coupling constant goes to zero in the limit of very high energy scales), even though the gauge coupling does not exhibit this property in the absence of gravity (as in Quantum Electrodynamics (QED)). The origin of this effect is the arising of a quadratic UV divergence, associated to the one graviton exchange, that could be absorbed in a gauge coupling constant redefinition.

After Robinson and Wilczek paper’s, Pietrykowski argued that this effect should be gauge-dependent \[6\] and therefore, without physical meaning. Considering the QED coupled to gravity and using the Vilkovisky–DeWitt method, which is a gauge-invariant and gauge-condition independent way of computing effective action, Toms reinforced that the quadratic divergences, due to the gravitational corrections, turn the electric charge asymptotically free \[7, 8\]. As stressed by him, these quadratic divergences can not be seen if a dimensional regularization (DR) procedure is used, because such a regularization ‘renormalizes’ quadratic divergences. It has also been argued, that asymptotic freedom in gauge theories could happen in the presence of a cosmological constant \[9\], through logarithmic divergences (which can be seen in DR) and is made possible by the presence of a second dimensionful parameter (\(\Lambda\)), besides \(M_P\). In addition, in non gauge theories, the Yukawa and \(\phi^4\) interactions was shown to share the same property of asymptotic freedom, due to gravitational corrections \[10\], in the presence of mass for the scalar and fermionic particles, an effect that vanishes when the masses are withdrawn.

On the other hand, the physical significance of the definition of the running coupling constants, as inferred from the effective action was questioned by Anber et. al. \[11\] and Donoghue \[12\]; as argued by them only a scattering matrix computation should give a real physical definition for running coupling constants. Through the computations of scattering amplitudes in the presence of quantum gravity, they have shown that any attempt to define a running Yukawa coupling would be process dependent, i.e., what appears like an asymptotic
freedom behavior in one process, in another would appear as an increase of the strength
of the coupling constant \([11]\). Thus, the asymptotic freedom induced by the gravitational
corrections should not be an universal effect.

In view of these controversial results, in a tentative to shed light on this problem, in the
present letter we study the scalar QED involving two different 'pions', coupled to gravity.
We evaluate the gravitational contribution to the running of the electric charge, through a
direct computation of the scattering amplitudes for two different processes involving the two
charged pions. Our results provide an example where the gravitational corrections admit an
'universal' definition of an asymptotically free running charge, for the two different processes
studied.

The effective model for pions with electromagnetic interaction (scalar QED) coupled to
the Einstein gravity is given by the following action

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{2}{\kappa^2} R - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} + g^{\mu\nu} (\partial_\mu + ieA_\mu) \phi_j (\partial_\nu - ieA_\nu) \phi_j^* - m^2 \phi_j^* \phi_j - \lambda (\phi_j^* \phi_j)^2 + \cdots \right\},
\]

(1)

where \(\kappa^2 = 32\pi G = 32\pi/M_P^2\) and \(G\) is the Newtonian gravitational constant. The dots stand
for all the higher order monomials in the effective action, induced by the nonrenormalizable
gravitational interactions \([4]\). The reason for considering two flavors of pions \((\phi_a\) and \(\phi_b\),
i.e., \(j = a, b\)) is to simplify the analysis of the scattering amplitudes. By studying scattering
of different particles we reduce the scattering amplitudes to only one channel, what reduces
the (high) number of graphs involved. The presence of a \(\lambda \phi^4\) term is necessary to guarantee
the renormalizability of the model when we withdraw the gravitational interaction. As we
will see it is also an important ingredient to absorb quadratic divergences of contact type
induced by gravitational corrections, in the processes that we study.

We will consider small fluctuations around flat metric, i.e.,

\[
g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},
\]

(2)

\[
g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_{\alpha\nu} + \mathcal{O}(\kappa^3),
\]

(3)

\[
\sqrt{-g} = 1 + \frac{1}{2\kappa} h - \frac{1}{4\kappa^2} h_{\alpha\beta} P^{\alpha\beta\mu\nu} h_{\mu\nu} + \mathcal{O}(\kappa^3),
\]

(4)

where \(\eta_{\mu\nu} = (+, -, -, -)\), \(P^{\alpha\beta\mu\nu} = \frac{1}{2} (\eta^{\alpha\mu} \eta^{\beta\nu} + \eta^{\alpha\nu} \eta^{\beta\mu} - \eta^{\alpha\beta} \eta^{\mu\nu})\) and \(h = \eta^{\mu\nu} h_{\mu\nu}\).

Employing the harmonic gauge-fixing function, \(G_\mu = \partial_\mu h - \frac{1}{2} \partial_\mu h\), by adding to the
Lagrangian the term \(G^2/2\) plus the corresponding Faddeev-Popov ghost terms, the graviton
propagator can be cast as
\[
\langle T h^{\alpha\beta}(p)h^{\mu\nu}(-p) \rangle = D^{\alpha\beta\mu\nu}(p) = -\frac{i}{p^2} D^{\alpha\beta\mu\nu}.
\] (5)

The propagators for the other fields are also obtained by the usual Faddeev-Popov method, resulting
\[
\langle T A^\mu(p)A^\nu(-p) \rangle = \Delta^{\mu\nu}(p) = \frac{i}{p^2} \left( \eta^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right),
\] (6)
\[
\langle T \phi^*_i(p)\phi_j(-p) \rangle = \Delta_{ij}(p) = \frac{i}{p^2 - m^2} \delta_{ij}.
\] (7)

As we will restrict to calculations of one loop order with only external pion legs, we will not need the ghost propagators.

Let us consider two scattering processes. The first one is \( (\pi^+_a + \pi^+_b \rightarrow \pi^+_a + \pi^+_b) \), containing the exchange of photons only in the t-channel. We will also consider the \( (\pi^+_a + \pi^-_a \rightarrow \pi^+_b + \pi^-_b) \) process, which contains photon propagators only in the s-channel. These choices will allow the comparison with the results obtained in [11].

The amplitude at tree level, Figure 1, plus (only) the quadratic divergent terms of the one-loop contribution to \( (\pi^+_a + \pi^+_b \rightarrow \pi^+_a + \pi^+_b) \), Figure 2, in the massless limit, is given by
\[
\mathcal{M}_{(\pi^+_a + \pi^+_b \rightarrow \pi^+_a + \pi^+_b)} = -\lambda + e^2 \frac{(p_1 + p_3) \cdot (p_2 + p_4)}{(p_4 - p_2)^2} + \frac{5\lambda \kappa^2}{32\pi^4} E^2
\]
\[
= \frac{e^2 \kappa^2}{64\pi^4} E^2 \frac{4(p_1 + p_3) \cdot (p_2 + p_4) - 9p_1 \cdot p_3 - 9p_2 \cdot p_4}{(p_4 - p_2)^2}
\]
\[
= \frac{s - u}{t} e^2 \left( 1 - \frac{\kappa^2}{16\pi^4} E^2 \right) - \frac{9e^2 \kappa^2}{64\pi^4} E^2 - \lambda + \frac{5\lambda \kappa^2}{32\pi^4} E^2.
\] (8)

The external momenta \( (p_1, p_2) \) and \( (p_3, p_4) \) are the incoming and outgoing momenta, respectively and \( E \) is a momentum cut-off regulator (as mentioned above, DR would kill these quadratic divergent contributions). We used FeynArts [13] to generate the topologies and create the Feynman amplitudes, and FeynCalc [14] to manipulate and simplify the expressions.

By choosing the renormalization point as \( s = -t = E_0^2 \) and \( u = 0 \), the quadratic divergences can absorbed by the definition of the electric charge and \( \lambda \), i.e., we can define the renormalized electric charge as
\[
e^2(E) = e^2 \left[ 1 - \frac{\kappa^2}{16\pi^4} (E^2 - E_0^2) \right],
\] (9)
where $E_0$ is the subtraction point. Taking $E$ differentially close to $E_0$, we obtain the following Callan-Zymanzik $\beta$ function

$$\beta(e) = E \frac{de(E)}{dE} = -\frac{e\kappa^2}{16\pi^4} E^2. \quad (10)$$

This $\beta$ function supports the idea that gravitational corrections should render gauge theories asymptotically free. In particular, our result corroborates the effective action calculations to the running of electric charge in Einstein-QED theory [7, 8].

The quadratic divergence that appears like a contact term is absorbed by a redefinition of $\lambda$, just as the logarithmic divergences present in diagrams of two photon exchange in the usual model at one-loop order, i.e.,

$$\lambda(E) = \lambda + \frac{9e^2\kappa^2}{64\pi^4}(E^2 - E_0^2) - \frac{5\lambda\kappa^2}{32\pi^4}(E^2 - E_0^2)$$

$$= \lambda - (10\lambda - 9e^2)\frac{\kappa^2}{64\pi^4}(E^2 - E_0^2). \quad (11)$$

The $\beta$ function of $\lambda(E)$, $\beta(\lambda) = -\frac{(10\lambda - 9e^2)E^2}{32\pi^4}$, depends on the difference $(10\lambda - 9e^2)$. Without electromagnetic interaction, $\beta(\lambda)$ is negative, so $\lambda(E)$ runs in direction to asymptotic freedom, in agreement with [15]. For $10\lambda < 9e^2$ this effect is destroyed.

The same conclusions are obtained analyzing the scattering ($\pi^+_a + \pi^-_a \rightarrow \pi^+_b + \pi^-_b$). The amplitude at tree level, Figure 1, plus the quadratic divergent part of the one-loop contribution, Figure 3 can be cast as

$$\mathcal{M}_{(\pi^+_a + \pi^-_a \rightarrow \pi^+_b + \pi^-_b)} = -\lambda + e^2\frac{(p_2 - p_1) \cdot (p_4 - p_3)}{(p_3 + p_4)^2} + \frac{5\lambda\kappa^2}{32\pi^4} E^2$$

$$-\frac{e^2\kappa^2}{64\pi^4} \frac{4(p_2 - p_1) \cdot (p_4 - p_3) + 9p_3 \cdot p_4 + 9p_1 \cdot p_2}{(p_3 + p_4)^2}$$

$$= \frac{u - t}{s} e^2 \left(1 - \frac{\kappa^2}{16\pi^4} E^2\right) - \frac{9e^2\kappa^2}{64\pi^4} E^2 - \lambda + \frac{5\lambda\kappa^2}{32\pi^4} E^2$$

$$= \frac{u - t}{s} e^2(E) - \lambda(E). \quad (12)$$

Note that kinematical crossing or mixing with higher order operators (e.g., $\partial^\mu \phi^*_a \partial_\mu \phi_a \phi^*_b \phi_b$) which could invalidate the definition of the running couplings [11] are not present in the quadratic dependence of the scattering amplitudes on the cut-off $E$.

In summary, we have evaluated the quadratic divergent part of the scattering matrix for charged pions, considering two different processes involving the exchange of one graviton. Our results suggest that quantum gravitational corrections change the running of the electric charge near the Planck scale, where it pass to exhibit an asymptotic freedom behavior,
corroborating the idea that gravitational corrections to the running of gauge couplings turn them asymptotically free. It is important to remark that our approach is based in a perturbative expansion which limits the validity of our conclusions to energies much lower than the Planck scale. But, paraphrasing Robinson and Wilczek, "it is entertaining to consider some consequences of extrapolating" \textsuperscript{[5]} these results to higher energy scales.

Another relevant question, no less controversial, is on the running of gravitational coupling constant $\kappa$, that have been studied in the context of asymptotic safety \textsuperscript{[16, 17]}. The study of scattering processes of gravitons must also give important support to the solution of these controversies. The study of the complete divergence structure of the present model including a renormalization group study of the gravitational coupling constant is currently in progress.

Acknowledgments. This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and Fundação de Apoio à Pesquisa do Rio Grande do Norte (FAPERN). The authors would like to thank M. Gomes for useful comments.

\leavevmode\newline[1] G. ‘t Hooft and M. J. G. Veltman, Annales Poincare Phys. Theor. A \textbf{20}, 69 (1974).
[2] S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. \textbf{32}, 245 (1974).
[3] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D \textbf{10}, 411 (1974).
[4] J. F. Donoghue, Phys. Rev. D \textbf{50}, 3874 (1994) [gr-qc/9405057].
[5] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. \textbf{96}, 231601 (2006) [hep-th/0509050].
[6] A. R. Pietrykowski, Phys. Rev. Lett. \textbf{98}, 061801 (2007) [hep-th/0606208].
[7] D. J. Toms, Nature \textbf{468}, 56 (2010) [arXiv:1010.0793 [hep-th]].
[8] D. J. Toms, Phys. Rev. D \textbf{84}, 084016 (2011).
[9] D. J. Toms, Phys. Rev. Lett. \textbf{101}, 131301 (2008) [arXiv:0809.3897 [hep-th]].
[10] A. Rodigast and T. Schuster, Phys. Rev. Lett. \textbf{104}, 081301 (2010) [arXiv:0908.2422 [hep-th]].
[11] M. M. Anber, J. F. Donoghue and M. El-Houssieny, Phys. Rev. D \textbf{83}, 124003 (2011) [arXiv:1011.3229 [hep-th]].
[12] J. F. Donoghue, AIP Conf. Proc. \textbf{1483}, 73 (2012) [arXiv:1209.3511 [gr-qc]].
[13] Thomas Hahn, Comp. Phys. Comm. \textbf{140}, 418 (2001).
[14] R. Mertig and M. Bahm and A. Denne, Comp. Phys. Comm. 64, 345 (1991).

[15] A. R. Pietrykowski, Phys. Rev. D 87, 024026 (2013) [arXiv:1210.0507 [hep-th]].

[16] S. Weinberg, In General Relativity: An Einstein Centenary Survey, ed. S.W. Hawking and W. Israel, pp.790-831; Cambridge University Press (1979).

[17] G. P. Vacca and O. Zanusso, Phys. Rev. Lett. 105, 231601 (2010) [arXiv:1009.1735 [hep-th]].

Figure 1. Feynman diagrams for the pions scattering amplitude at tree level. Diagrams (a) and (b) stand for \( (\pi^+_a + \pi^+_b \to \pi^+_a + \pi^+_b) \) process, while (c) and (d) for \( (\pi^+_a + \pi^-_a \to \pi^+_b + \pi^-_b) \). Dashed and wavy lines represent the pions and photons propagators, respectively.
Figure 2. Diagrams that contribute with quadratic divergences to the \((\pi_a^+ + \pi_b^+ \rightarrow \pi_a^+ + \pi_b^+)\) scattering amplitude at one-loop order up to \(O(\kappa^2)\). Continuous lines represent the graviton propagator.
Figure 3. Diagrams that contribute with quadratic divergences to the $(\pi^+ + \pi^- \rightarrow \pi^+_b + \pi^-_b)$ scattering amplitude at one-loop order up to $\mathcal{O}(\kappa^2)$. 