1. Introduction

The following well-known theorem was first proven by Goldštein and Vodopyanov [8]; see also [19, 5, 9] and the recent extension to manifolds in [7]:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open set. If \( f \in W^{1,n}(\Omega, \mathbb{R}^n) \) and
\[
\text{Jac}(f) := \det(Df) > 0 \quad \text{a.e. in } \Omega,
\]
then \( f \) is continuous.

The strict inequality \( \text{Jac}(f) > 0 \) is necessary as the following counterexample shows:
Example 1.2. Let $B$ denote the unit ball in $\mathbb{R}^n$. Let $\tilde{f} \in W^{1,n}(B, \mathbb{R})$ be discontinuous, e.g. $\tilde{f}(x) := \log \log \frac{2}{|x|}$. Set

$$f(x) := (\tilde{f}(x), 0, \ldots, 0).$$

Clearly $f \in W^{1,n}(B, \mathbb{R}^n)$ and $\text{Jac}(f) = \det(Df) \equiv 0$. However, $f$ is still discontinuous.

The aim of this note is to give a reasonable extension to Theorem 1.1 to fractional Sobolev spaces $W^{s,p}(\Omega, \mathbb{R}^n)$, $s \in (0, 1)$. These are the spaces of maps $f \in L^p(\Omega, \mathbb{R}^n)$ with finite $W^{s,p}$-Gagliardo semi-norm

$$[f]_{W^{s,p}(\Omega)} := \left( \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} < \infty.$$ 

Clearly, for a pointwise definition of the Jacobian of $f$ to make sense, $f$ should be almost everywhere differentiable; however, as a distributional operator, the Jacobian also exists for maps in fractional Sobolev spaces $W^{s,p}$, where $s < 1$ is large enough. For the sake of presentation we restrict our attention to the critical scaling, that is to the Sobolev spaces $W^{s,n}_s, s \in (0, 1)$. The space $W^{s,p}_0(\Omega)$ denotes, as usual, the closure of $C_\infty(\Omega)$-functions in the $W^{s,p}$-norm.

Lemma 1.3. Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary, $n \geq 2$. For $s \in (\frac{n-1}{n}, 1)$ and $f \in W^{s,\frac{n}{s}}(\Omega)$ the Jacobian operator extends to a bounded linear operator on $W^{(1-s)n,\frac{1}{1-s}}_0(\Omega)$ in the following sense. The operator

$$\text{Jac}(f)[\varphi] := \lim_{k \to \infty} \int_\Omega \det(Df_k) \varphi_k$$

is well-defined for any $f_k \in C_\infty(\Omega)$ which is a smooth approximation of $f \in W^{s,\frac{n}{s}}(\Omega)$ and any $\varphi_k \in C_\infty(\Omega)$ which is a smooth approximation of $\varphi$ in $W^{(1-s)n,\frac{1}{1-s}}(\Omega)$.

We recall a proof of Lemma 1.3 in Section 2.

We will restrict our attention to the case $s \geq \frac{n}{n+1}$. This threshold appears in several situations on degree-type estimates in fractional Sobolev spaces; see, e.g., [6, 16]. It is exactly the case when (up to the boundary data) a map $f \in W^{s,\frac{n}{s}}$ can serve as a testfunction for its own Jacobian $\text{Jac}(f)$. Lemma 1.3 warrants the following definition for a distributional Jacobian.

Definition 1.4. Assume $s \geq \frac{n}{n+1}$ and $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain. Let $f \in W^{s,\frac{n}{s}}(\Omega, \mathbb{R}^n)$.

- We say $\text{Jac}(f) \geq 0$ in $\Omega$ if for any $\varphi \in W^{s,\frac{n}{s}}_0(\Omega), \varphi \geq 0$ a.e., there holds

$$\text{Jac}(f)[\varphi] \geq 0.$$ 

\footnote{The case $s = 1$ is also true (with $W^{0,\infty}$ replaced by $BMO$): it is the famous theorem by Coifman-Lions-Meyer-Semmes [3].}
• We say $\text{Jac}(f) > 0$ if $\text{Jac}(f) \geq 0$ and for any $\varphi \in W^{s,2}_0(\Omega)$, $\varphi \geq 0$ a.e.,
  $\text{Jac}(f)[\varphi] = 0$ implies that $\varphi \equiv 0$.

Our main result is the following version of Theorem 1.1 for fractional Sobolev spaces $W^{s,2}$.

**Theorem 1.5.** Let $f \in W^{s,2}(\Omega, \mathbb{R}^n)$, $s \geq \frac{n}{n+1}$, for some open and bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary.

If $\text{Jac}(f) > 0$ then $f$ is continuous.

By the counterexample, Example 1.2, there is no hope of getting Theorem 1.5 under merely the assumption $\text{Jac}(f) \geq 0$. However, as it is used for the planar Monge-Ampère equation, a curl-free condition is a remedy – similar properties are known, e.g. for $W^{1,n}$-maps, see [14, Lemma 2.1.], or $C^{0,\alpha}$, $\alpha > \frac{2}{3}$, see [13]. Namely we have

**Theorem 1.6.** Let $f = (f_1, f_2) \in W^{s,2}(\Omega, \mathbb{R}^2)$ for some open and bounded set $\Omega \subset \mathbb{R}^2$ with smooth boundary and for some $s \geq \frac{2}{3}$. If $\text{Jac}(f) \geq 0$ and if $\text{curl}(f) = 0$ in distributional sense, i.e.

$$\text{curl}(f)[\varphi] = -\int_{\Omega} f_2 \partial_1 \varphi - f_1 \partial_2 \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

then $f$ is continuous.

Along the way of proving Theorem 1.5 and Theorem 1.6 we obtain degree estimates for maps with signed Jacobian which are of independent interest.

If $f \in W^{s,2}(\Omega)$ then for any $x_0 \in \Omega$ we have $f \in W^{s,2}(\partial B(x_0, r)) \hookrightarrow C^{0,\frac{s}{n}}(\partial B(x_0, r))$ for almost every $0 < r < \text{dist}(x_0, \partial \Omega)$, by means of Sobolev embedding and Fubini’s theorem, Lemma 2.2. In particular, for any $p \in \mathbb{R}^n \setminus f(\partial B(x, r))$ the degree $\text{deg}(f, B(x, r), p)$ is well-defined as the Brouwer degree of the map $\frac{f - p}{|f - p|} : \partial B_r(x) \to S^{n-1}$ for almost every $r$, cf. [4].

We first observe that $f$ with non-negative Jacobian is monotone in the following sense:

**Proposition 1.7.** Let $f \in W^{s,2}(\Omega, \mathbb{R}^n)$ for $\Omega \subset \mathbb{R}^n$ open and $s \geq \frac{n}{n+1}$. Let $B(x, r) \subset B(x, R) \subset \Omega$ and assume that $f$ (in the trace sense) restricted to $\partial B(x, r)$ and $\partial B(x, R)$ is continuous.

If $\text{Jac}(f) \geq 0$ in $\Omega$, then for any $p \notin (f(\partial B(x, r)) \cup f(\partial B(x, R)))$ we have

$$\text{deg}(f, B(x, r), p) \leq \text{deg}(f, B(x, R), p).$$

We also have

**Proposition 1.8.** Let $f \in W^{s,2}(\Omega, \mathbb{R}^n)$ for $\Omega \subset \mathbb{R}^n$ open and $s \geq \frac{n}{n+1}$. Let $B(x, R) \subset \Omega$ and assume that $f$ is continuous on $\partial B(x, R)$.
If $\text{Jac}(f) \geq 0$, then for any $p \notin f(\partial B(x, R))$ we have
\[
\deg(f, B(x, R), p) \geq 0.
\]

Next, we obtain that if $f$ is continuous and the Jacobian of $f$ is positive then $f$ is sense-preserving:

**Proposition 1.9.** Let $f \in W^{s, \frac{n}{s}} \cap C^0(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ open, $s \geq \frac{n}{n+1}$.

If $\text{Jac}(f) > 0$ in $\Omega$ then for any ball $B(r) \subset \Omega$ if $p \in f(B(r)) \setminus f(\partial B(r))$ then $\deg(f, B(r), p) \geq 1$.

If the Jacobian is positive, the image of a ball $f(B(r))$ has an essential diameter comparable to the diameter of $f(\partial B(r))$. This will be the main ingredient towards the proof of Theorem 1.5.

**Proposition 1.10.** There exists some $\Lambda > 0$ depending only on the dimension such that the following holds. Let $f \in W^{s, \frac{n}{s}}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ open, and $s \geq \frac{n}{n+1}$. Assume that $\text{Jac}(f) > 0$ in $\Omega$. For any $B(r) \subset \subset \Omega$ such that $\int_{\partial B(r)}$ is continuous, we can find a ball $B(q, R) \subset \mathbb{R}^n$ with
\[
R \leq \Lambda \text{ diam} (f(\partial B(r))),
\]
and
\[
\{x \in B(r) : f(x) \notin B(q, R)\}
\]
is a null set.

The number $2R$ may be viewed as the “essential diameter” of $f(B(r))$.

The remainder of this paper is organized as follows. In Section 2 we refer to some needed results for Sobolev spaces. In Section 3 we prove the degree estimates for maps with signed Jacobians, namely Propositions 1.7, 1.8, 1.9, 1.10. In Section 4 we prove Theorem 1.5 and Theorem 1.6.

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## 2. Fractional Sobolev spaces

Lemma 1.3 was (essentially) proven in [17] as an extension of the ground-breaking paper [3], which showed that Jacobians of $W^{1,n}$-maps can be tested with BMO-maps. The proof in [17] uses Littlewood-Paley theory and paraproducts. In [2] Brezis and Nguyen gave a
simpler and more elegant proof of this result for $s = \frac{n}{n+1}$. We present here the following slight adaptation of their argument due to [12].

We restrict our attention to the \textit{a priori} estimates, from which the claim follows easily due to \textit{multi-linearity}.

\textbf{Proof of Lemma 1.3 (a priori estimates).} Let $\varphi \in C^\infty_c(\Omega)$ and $f \in C^\infty(\Omega)$. $\Omega$ is an extension domain, [11, 20], so we may assume that $f \in W^{s, n}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. Then

$$\int_{\Omega} \det(Df) \varphi = \int_{\mathbb{R}^n} \det(Df) \varphi.$$ 

Extend $f$ and $\varphi$ harmonically to $\mathbb{R}^{n+1}$, say to $F$ and $\Phi$ respectively. We write $(x, t) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. By Stokes’ theorem and Hölder’s inequality,

$$\left| \int_{\mathbb{R}^n} \det(Df) \varphi \right| \leq \left( \int_{\mathbb{R}^{n+1}} |t^{1-\frac{s}{p}} \sqrt{n} DF| \right)^s \left( \int_{\mathbb{R}^{n+1}} |t^{1-(1-s)-(1-s)n} D\Phi|^{\frac{1}{1-s}} \right)^{1-s}.$$ 

If $s \in \left(\frac{n}{n-1}, 1\right]$, then $(1-s)n \in (0, 1)$. Then, by trace estimates, see \textit{e.g.} [12, Proposition 10.2], we have

$$\left( \int_{\mathbb{R}^{n+1}} |t^{1-\frac{s}{p}} \sqrt{n} DF| \right)^s \approx [f]_{W^{s, \frac{n}{s}}(\Omega)}$$

and

$$\left( \int_{\mathbb{R}^{n+1}} |t^{1-(1-s)-(1-s)n} D\Phi|^{\frac{1}{1-s}} \right)^{1-s} \approx [\varphi]_{W^{(1-s)n, 1-s}(\Omega)}.$$ 

Here we also used the fact that $[f]_{W^{s, \frac{n}{s}}(\mathbb{R}^n)} \lesssim [f]_{W^{s, \frac{n}{s}}(\Omega)}$. This is because $\Omega$ is an extension domain; see [11, 20]. We conclude, because we have shown

$$\int_{\Omega} \det(Df) \varphi \lesssim [f]_{W^{s, \frac{n}{s}}(\Omega)} [\varphi]_{W^{(1-s)n, 1-s}(\Omega)}.$$ 

\hfill $\Box$

The ensuing result on trace operators will be useful for the subsequent developments. For detailed treatments we refer to [15, §2.4.2, Theorem 1], [1, Theorem 7.43, Remark 7.45] and [18, Lemma 36.1].

\textbf{Lemma 2.1 (Trace Theorem).} Let $\Omega \subset \mathbb{R}^n$ be either bounded or the complement of a bounded set, with smooth boundary. If $s \in (0, 1)$, $p \in (1, \infty)$ with $s - \frac{1}{p} > 0$, then the trace operator on $T = \int_{\partial \Omega}$ is a bounded, linear, surjective operator from $W^{s, p}(\Omega)$ to $W^{s-\frac{1}{p}, p}(\partial \Omega)$. The harmonic extension is a bounded linear right-inverse of $T$. 
The following is well-known for Sobolev functions in $W^{1,p}$ (it is essentially Fubini’s theorem):

**Lemma 2.2** (Restriction theorem). For $\Omega$ a smooth, bounded domain let $f \in W^{s,p}(\Omega)$. Fix $x_0 \in \Omega$. There exists a representative of $f$ such that for $\mathcal{L}^1$-almost every $r \in (0, \text{dist} (x_0, \partial\Omega))$ we have $f \in W^{s,p}(\partial B(x_0, r))$.

Moreover, for $\Omega = B(x_0, R)$ we have

$$\left( \int_0^R [f]_{W^{s,p}(\partial B(x_0, r))}^p \right)^{\frac{1}{p}} \lesssim [f]_{W^{s,p}(B(x_0, R))}.$$

**Proof.** As $\Omega$ is an extension domain, see [11, 20], we may assume that $\Omega = \mathbb{R}^n$ and $f \in W^{s,p}(\mathbb{R}^n)$ with $f \equiv 0$ outside a compact set. Denote by $F : \mathbb{R}^{n+1} \to \mathbb{R}$ the harmonic extension of $f$, and w.l.o.g. set $x_0 = 0$. Then (see [12, Proposition 10.2])

$$\left\| (x_{n+1})^{1-s} p^{-s} DF \right\|_{L^p(\mathbb{R}^{n+1})} \approx [f]_{W^{s,p}(\mathbb{R}^n)} < \infty.$$

By Fubini’s theorem, for $\mathcal{L}^1$-almost every $r > 0$,

$$\left\| (x_{n+1})^{1-s} p^{-s} DF \right\|_{L^p(\partial B(r) \times (0, \infty))} < \infty.$$

This implies that $f \in W^{s,p}(\partial B(r))$ for almost every $r > 0$.

The last claim also follows from Fubini’s theorem in $\mathbb{R}^{n+1}$:

$$\int_0^R [f]_{W^{s,p}(\partial B(x_0, r))}^p dr \lesssim \int_0^R \int_{\partial B(x_0, r) \times (0, \infty)} \left| (x_{n+1})^{1-s} p^{-s} DF \right|^p = \int_{B(x_0, R) \times (0, \infty)} \left| (x_{n+1})^{1-s} p^{-s} DF \right|^p.$$

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. For $s \in (0, 1)$, $p \in (1, \infty)$ such that $s - \frac{1}{p} > 0$:

1. If $f \in W^{s,p}(\Omega)$ and $g \in W^{s,p}(\mathbb{R}^n \setminus \Omega)$ with $f = g$ on $\partial \Omega$ in the trace sense. Then

$$h := \begin{cases} f & \text{in } \Omega \\ g & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

belongs to the Sobolev space and

$$[h]_{W^{s,p}(\mathbb{R}^n)} \leq C(\Omega) \left( [f]_{W^{s,p}(\Omega)} + [g]_{W^{s,p}(\mathbb{R}^n \setminus \Omega)} \right).$$

2. In particular, if $f \in W^{s,p}(\Omega)$ satisfies $f = 0$ on $\partial \Omega$ in the trace sense, then $f \in W^{s,p}_0(\Omega)$ and that

$$h := \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

belongs to $W^{s,p}(\mathbb{R}^n)$. 
Lemma 2.4. Let $B(R)$ be a ball in $\mathbb{R}^n$. Let $f \in W^{s, \frac{n}{s}}(B(R))$ for some $s \in (0, 1)$ and $f\big|_{\partial B(R)} \in C^0(\partial B(R))$. Then there exists an approximation $f_k \in C^\infty_c(\mathbb{R}^n)$ converging to $f$ in $W^{s, \frac{n}{s}}(B(R))$ and $f_k \to f$ uniformly on $\partial B(R)$.

Proof. W.l.o.g. $B(R) = B := B(0, 1)$.

By the trace theorem, Lemma 2.1, $f \in W^{s-\frac{n}{s}, \frac{n}{s}}(\partial B)$. Let $g$ be the harmonic extension of $f$ to $\mathbb{R}^n \setminus B$. Then $g \in W^{s, \frac{n}{s}}(\mathbb{R}^n \setminus B)$, again by Lemma 2.1. Also, since $f$ is continuous on $\partial B$, $g$ is also continuous. Set $h := \begin{cases} g & \text{in } \mathbb{R}^n \setminus B, \\ f & \text{in } B. \end{cases}$

By Lemma 2.3, $h \in W^{s, \frac{n}{s}}(\mathbb{R}^n)$ and $h$ is locally uniformly continuous on $\mathbb{R}^n \setminus B$. This last fact implies that $h_k(x) := h\left(\frac{k+1}{k}x\right)$ converges uniformly to $h$ on $\partial B$ as $k \to \infty$, and also in $W^{s, \frac{n}{s}}_{loc}(\mathbb{R}^n)$.

Now let us consider the standard mollification $f_\varepsilon := h_k * \eta_k$. For $\varepsilon$ small enough in comparison with $\frac{1}{k}$, $f_\varepsilon$ converges uniformly on $\partial B$ to $f$ and in $W^{s, \frac{n}{s}}(B)$). This completes the proof. \(\square\)

3. Degree of maps with signed Jacobians: Proof of Propositions 1.7, 1.8, 1.9, 1.10

Here and hereafter, without further specifications, a null set is understood with respect to the Lebesgue measure $\mathcal{L}^n$.

For continuous $f \in C^0(\partial B(r), \mathbb{R}^n)$ for some given ball $B(r) \subset \mathbb{R}^n$ and some point $p \in \mathbb{R}^n \setminus f(\partial B_r)$, the degree of $f(B(r))$ around this point $p$ is simply the number of times that $f(\partial B(r))$ winds around $p$, i.e.,

$$\deg(f, B_r, p) := \text{Brouwer degree of } \left( \psi := \frac{f - p}{|f - p|} : \partial B(r) \to \mathbb{S}^{n-1} \right).$$

We can approximate $f$ by smooth functions $f_\varepsilon : \partial B(r) \to \mathbb{S}^n$ which are uniformly close to $f$. Moreover, the Brouwer degree of $\psi = \frac{f - p}{|f - p|}$ is the same as that of $\psi_\varepsilon = \frac{f_\varepsilon - p}{|f_\varepsilon - p|}$ for $\varepsilon$ small enough, since maps that are uniformly close to each other have the same Brouwer degree.

For the smooth functions $\psi_\varepsilon$ we can compute the Brouwer degree from an integral formula: denote by $\omega \in C^\infty(\wedge^{n-1} \mathbb{R}^n)$ the standard volume form on $\mathbb{S}^{n-1}$:

$$\omega = \sum_{j=1}^n (-1)^{j-1} x^j \, dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots \wedge dx^n.$$
Then, for all \( \varepsilon \) small enough,

\[
\deg(f, B(r), p) = \deg(f_\varepsilon, B(r), p) = \int_{\partial B(r)} \psi_\varepsilon^*(\omega).
\]

If we extend \( \psi_\varepsilon \) from a map \( \partial B(r) \to \mathbb{S}^{n-1} \) to a map \( \psi_\varepsilon : B(r) \to \mathbb{R}^{n+1} \), then from Stokes’ theorem we may obtain:

\[
\deg(f, B(r), p) = \int_{B(r)} \psi_\varepsilon^*(d\omega) = C \int_{B(r)} \det(D\psi_\varepsilon).
\]

In the last equation we used the fact that \( d\omega = C \, dx^1 \wedge \ldots \wedge dx^n \). Most of our arguments below are based on choosing a suitable extension of \( \psi_\varepsilon \).

3.1. Monotonicity for non-negative Jacobian: Proof of Proposition 1.7, Proposition 1.8. We only give the proof of Proposition 1.7, the proof of Proposition 1.8 is almost verbatim (it is the “\( r = 0 \)” case).

**Proof of Proposition 1.7.** Recall that \( f \in C^0(\partial B(r)) \cap C^0(\partial B(R)) \) and that \( p \in \mathbb{R}^n \setminus (f(\partial B(r)) \cup f(\partial B(R))) \). We set

\[
c := \min \{ \text{dist} (f(\partial B(r)), p), \text{dist} (f(\partial B(R)), p) \}
\]

For this \( c > 0 \) let us take \( d = d_c \in C^{1,1}((0, \infty), (0, \infty)) \) as in Lemma A.1.

Let \( f_\varepsilon \) be the approximation in Lemma 2.4 and set (for \( \varepsilon \ll 1 \))

\[
\psi_\varepsilon := (f_\varepsilon - p) \, d(|f_\varepsilon - p|).
\]

Then, by Stokes’ theorem, we have

\[
\deg(f, B(R), p) - \deg(f, B(r), p) = \deg(f_\varepsilon, B(R), p) - \deg(f_\varepsilon, B(r), p)
\]

\[
= \int_{\partial B(R)} \psi_\varepsilon^*(\omega) - \int_{\partial B(r)} \psi_\varepsilon^*(\omega)
\]

\[
\quad = \int_{B(R) \setminus B(r)} \psi_\varepsilon^*(d\omega)
\]

\[
\quad = C \int_{B(R) \setminus B(r)} \det(D\psi_\varepsilon).
\]

Below we assume \( p = 0 \) for simplicity of notation. Observe that

\[
D\psi_\varepsilon = \left( d(|f_\varepsilon|) I_{n \times n} + \frac{d'(|f_\varepsilon|)}{|f_\varepsilon|} f_\varepsilon \otimes f_\varepsilon \right) Df_\varepsilon,
\]

so we have \( D\psi_\varepsilon = W(f_\varepsilon) Df_\varepsilon \), where

\[
W(v) := \left( d(|v|) I_{n \times n} + \frac{d'(|v|)}{|v|} v \otimes v \right).
\]

From the properties of \( d \) (see Lemma A.1), in particular, since \( d'(|v|) = 0 \) whenever \( |v| \) is small and \( |d'(|v|)| \approx |v|^{-2} \) whenever \( |v| \) is large, we have
(1) \( \sup_{v \in \mathbb{R}^n} |W(v)| < \infty \).

(2) \( W \in \text{Lip}(\mathbb{R}^n) \).

(3) \( \det(W(v)) \geq 0 \) for all \( v \in \mathbb{R}^n \).

Indeed, if \( v = 0 \), then \( d'(|v|) = 0 \) and hence \( \det(W(v)) = \det(d(0)I) \geq 0 \). Assume now \( v \neq 0 \). Since \( W(v) \) is symmetric, it suffices to show that the eigenvalues are nonnegative. Observe that \( v/|v| \) and any orthonormal basis of \( v^\perp \) are the eigenvectors of \( W(v) \).

In the former case, we compute

\[
W(v)\frac{v}{|v|} = d(|v|)\frac{v}{|v|} + d'(|v|)|v|\frac{v}{|v|} = (d(|v|) + d'(|v|)|v|) \frac{v}{|v|}.
\]

That is, the eigenvalue for the eigenvector \( v/|v| \) is non-negative.

In the latter case, given any \( o \in v^\perp \) with \( |o| = 1 \), one has

\[
W(v)o = d(|v|)o + 0.
\]

So the eigenvalue for any eigenvector \( o \) perpendicular to \( v \) is \( d(|v|) \geq 0 \).

Therefore, all the eigenvalues of \( W(v) \) are non-negative, thus \( \det(W(v)) \geq 0 \).

(4) \( \det(W(v)) = 0 \) whenever \( |v| > \frac{\epsilon}{2} \).

Indeed, this is because

\[
W(v)v = (d(|v|) + d'(|v|)|v|) v
\]

where, for \( |v| > \frac{\epsilon}{2} \), we have

\[
d(|v|) + d'(|v|)|v| = \frac{1}{|v|} - \frac{1}{|v|^2}|v| = 0.
\]

This shows that \( W(v) \) is non-invertible, so \( \det(W(v)) = 0 \).

Now, since \( W \) is Lipschitz (and globally bounded), \( \det(W(f_\epsilon)) \) lies uniformly in \( W^{s,\frac{n}{2}}(B(R)\setminus B(r)) \) and converges strongly in \( W^{s,\frac{n}{2}}(B(R)\setminus B(r)) \) to \( \det(W(f)) \).

On the other hand, for any fixed small enough \( \epsilon > 0 \) there exists a neighborhood of \( \partial(B(R)\setminus B(r)) \) where \( |f_\epsilon - p| > \frac{3\epsilon}{4} \) - this holds since \( |f_\epsilon - p| > \frac{4\epsilon}{5} \) on \( \partial(B(R)\setminus B(r)) \) for all \( \epsilon \) small enough, due to uniform convergence.

That is, for any small \( \epsilon > 0 \) there exists a neighborhood around \( \partial(B(R)\setminus B(r)) \) where \( \det(W(f_\epsilon)) \equiv 0 \).

This implies that \( \det(W(f_\epsilon)) \) and \( \det(W(f)) \) all lie in \( W^{s,\frac{n}{2}}_0(B(R)\setminus B(r)) \). By virtue of Lemma 2.3, we can extend these functions by zero to all of \( \Omega \), and they belong consequently to \( W^{s,\frac{n}{2}}_0(\Omega) \).

That is, we have shown that

\[
\deg(f, B(R), p) - \deg(f, B(r), p) = \lim_{\epsilon \to 0} \int_{\Omega} \det(Df_\epsilon) \det(W(f_\epsilon)) = \text{Jac}(f)[\det(W(f))].
\]
The right-hand side is nonnegative by assumption, and Proposition 1.7 is proven. □

3.2. Positive Jacobian implies sense-preserving: Proof of Proposition 1.9.

Proof of Proposition 1.9. The proof is very similar to that of Proposition 1.7. With the notation used therein, we have

$$\text{deg}(f, B(r), p) = \text{Jac}(f)[\det(W(f))]$$

where again

$$W(v) := \left( d(|v|) I_{n \times n} + \frac{d'(|v|)}{|v|} v \otimes v \right).$$

for $d$ taken from Lemma A.1 with $c := \frac{1}{2} \text{dist}(f(\partial B(r)), p)$.

As before, we have $\det(W(f)) \geq 0$. The assumption $\text{Jac}(f) > 0$ implies $\text{deg}(f, B(r), p) \geq 0$. It remains to show that if $\text{deg}(f, B(r), p) = 0$ then $p \not\in f(B(r))$, for the claim that $\text{deg}(f, B(r), p) \geq 1$ in $f(B(r))$ shall follow immediately.

So, assume that $p \not\in f(\partial B(r))$ and $\text{deg}(f, B(r), p) = 0$. From Definition 1.4 we see that $\text{Jac}(f) > 0$ readily implies $\det(W(f)) \equiv 0$. That is, one of the eigenvalues of $W(f)$ is zero. As computed in the proof of Proposition 1.7, the eigenvalues of $W(v)$ are

$$(d(|v|) + d'(|v|)|v|) \quad \text{and} \quad d(|v|).$$

Since $d(|v|) \neq 0$ for all $v$, $\det(W(f(x))) \equiv 0$ implies that necessarily

$$d(|f(x) - p|) + d'(|f(x) - p|)|f(x) - p| = 0 \quad \text{for all } x \in B(r).$$

By the properties of $d$ (see Lemma A.1), we deduce that $\inf_{B(r)} |f(x) - p| > 0$. Thus $p \not\in f(B(r))$ as claimed. □

3.3. Comparability of diameters: Proof of Proposition 1.10. The proof below is an adaptation from the argument in [19, 7]. Modifications are necessary due to the fact that we do not have a pointwise Jacobian.

Proof. Since $f$ is continuous on $\partial B(r)$, we can find a large ball $B(q, \rho)$ of radius $\rho := \text{diam} f(\partial B(r))$ such that $f(\partial B(r)) \subset B(q, \rho)$.

Take $\pi = \pi_\lambda$ from Lemma A.2 for $\lambda := 10\rho$.

Let $f_\varepsilon$ be the smooth approximation of $f$ from Lemma 2.4. For all small enough $\varepsilon > 0$ we have $f_\varepsilon(\partial B(r)) \subset B(q, 2\rho)$.

In particular if we set

$$g_\varepsilon := (f_\varepsilon - q) \pi(|f_\varepsilon - q|) + q$$
then \( g_\varepsilon = f_\varepsilon \) on \( \partial B(r) \). Consequently (by an integration by parts argument it is easy to see that the integral of the Jacobian of a map on a ball only depends on the boundary value of that map, [10, Lemma 4.7.2]),
\[
\int_{B(r)} \det(Df_\varepsilon) = \int_{B(r)} \det(Dg_\varepsilon).
\]
Computing \( Dg_\varepsilon \) similar as in the proof of Proposition 1.7, setting
\[
W(v) = \pi(|v|)I_{n \times n} + \frac{\pi'(|v|)}{|v|} v \otimes v,
\]
we obtain
\[
\int_{B(r)} \det(Df_\varepsilon) \left(1 - \det(W(f_\varepsilon - q))\right) = 0.
\]
As in the proof of Proposition 1.7, the map \( 1 - \det(W(f_\varepsilon - q)) \) belongs to \( W^{s, \frac{n}{2}}(B(r)) \) and converges strongly in that space to \( 1 - \det(W(f - q)) \).

Moreover, as in the proof of Proposition 1.7, we can compute
\[
\det(W(v)) = \pi(|v|)^{n-1} \left( \pi(|v|) + |v|\pi'(|v|) \right),
\]
and by the properties of \( \pi \), see Lemma A.2,
\[
1 - \det(W(f_\varepsilon - q)) \geq 0 \quad \text{a.e. in } B(r).
\]
Moreover, since \( \pi(|v|) \equiv 1 \) for \( |v| \leq 10\rho \), we have
\[
W(f_\varepsilon - q) \equiv I_{n \times n} \quad \text{close to } \partial B(r).
\]
That is,
\[
1 - \det(W(f_\varepsilon - q)) \equiv 0 \quad \text{close to } \partial B(r).
\]
By Lemma 2.1 and Lemma 2.3 we can thus again extend \( 1 - \det(W(f_\varepsilon - q)) \) and \( 1 - \det(W(f - q)) \) by zero to a \( W^{s, \frac{n}{2}}_{0}(\Omega) \)-function. Thus, we conclude that
\[
\text{Jac}(f) \left[1 - \det(W(f - q))\right] = 0.
\]
Since by assumption \( \text{Jac}(f) > 0 \) and \( 1 - \det(W(f - q)) \geq 0 \) a.e., we may infer (see Definition 1.4) that
\[
1 - \det(W(f - q)) \equiv 0.
\]
That is,
\[
\pi(|f - q|)^{n-1} \left( \pi(|f - q|) + |f - q|\pi'(f - q) \right) \equiv 1
\]
But by the properties of \( \pi \), see Lemma A.2, this implies
\[
|f(x) - q| < 2\lambda = 20\rho = 20 \text{diam}(f(\partial B(r))) \quad \text{a.e. in } x \in B(r).
\]
Therefore,
\[
\{x \in B(r) : |f(x) - q| \geq 20 \text{diam}(f(\partial B(r)))\} \quad \text{is a null set.}
\]
4. Continuity of maps with positive Jacobian: Proof of Theorem 1.5, 1.6

The proof of Theorem 1.5 crucially relies on the diameter estimates of Proposition 1.10. Once we have this, we adapt the argument in [19] to fractional Sobolev spaces in a more or less straightforward fashion, namely Theorem 1.5 is a corollary of the following statement.

**Proposition 4.1.** Let \( \Omega \subset \mathbb{R}^n \). Assume that \( f \in W^{s,\frac{n}{s}}(\Omega, \mathbb{R}^n) \), \( s \in (0,1) \) satisfies the following: for any \( x_0 \in \Omega \) and \( \mathcal{L}^1 \)-almost all radii \( 0 < r < \rho < \text{dist} (x_0, \Omega) \), there holds

\[
\text{osc} \frac{\partial B(x_0,r)}{\partial B(x_0,\rho)} f \leq \text{osc} \frac{\partial B(x_0,r)}{\partial B(x_0,\rho)} f.
\]

Then \( f \) is continuous. Moreover, for any ball \( B \supseteq \Omega \), \( s > 0 \), and \( x, y \in B \), we have

\[
|f(z) - f(y)|^p \leq \frac{1}{C(s,p,B) - \log(|x-y|)} [f]^\frac{n}{W^{s,\frac{n}{s}}(B)}.
\]

Observe that an easy extension of Proposition 4.1 holds for \( W^{s,p} \)-maps whenever \( s - n - 1 > 0 \).

**Proof of Theorem 1.5.** Fix \( x_0 \in \Omega \) and let \( R := \text{dist} (x_0, \partial \Omega) \). W.l.o.g. \( x_0 = 0 \). Let \( 0 < r < \rho < R \) such that \( f \) is continuous on \( \partial B(r) \) and \( \partial B(\rho) \). By Lemma 2.2 we know this happens for \( \mathcal{L}^1 \)-a.e. \( r \) and \( \rho \).

By Proposition 1.10, for almost any \( 0 < r < \rho < R \) we have the monotonicity

\[
(4.1) \quad \text{diam} (f(\partial B(r))) \leq \Lambda \text{diam} (f(\partial B(\rho))).
\]

Indeed, by Lemma 2.2, for almost any \( 0 < r < \rho < R \) the map \( f \) is continuous on \( \partial B(r) \) and \( \partial B(\rho) \). Thus

\[
\text{diam} (f(B(\rho))) \leq \Lambda \text{diam} (f(\partial B(\rho))),
\]

where \( \text{diam} (f(B(\rho))) \) is understood as

\[
\text{diam} (f(B(\rho))) = \inf \{\text{diam} (f(A)) : A \subset B(\rho), |B(\rho) \setminus A| = 0\}.
\]

The trace \( f|_{\partial B(\rho)} \) is \( H^{n-1} \)-a.e. attained by sequences of \( f|_{\partial B(\tilde{r})} \) as \( \tilde{r} \to r \). If \( A \) is as in the definition of \( \text{diam} \) above then for \( \mathcal{L}^1 \)-almost every \( \tilde{r} \) we have \( H^{n-1}(A \cap \partial B(\tilde{r})) = H^{n-1}(\partial B(\tilde{r})) \). So, we find a sequence \( r_i \to r \) with \( \partial H^{n-1}(A \cap \partial B(r_i)) = H^{n-1}(\partial B(r_i)) \) and \( f|_{\partial B(r_i)} \) converging \( H^{n-1} \)-a.e. to \( f|_{\partial B(r)} \). Thus, whenever \( \rho > r \),

\[
\text{diam} (f(\partial B(r))) \leq \text{diam} (f(B(\rho))).
\]

This establishes (4.1).

Now, we may deduce from (4.1) that, for almost any \( 0 < r < \rho < R \),

\[
\text{osc} \frac{\partial B(r)}{\partial B(\rho)} f \leq \Lambda \text{osc} \frac{\partial B(r)}{\partial B(\rho)} f.
\]

From here one concludes the continuity property with Proposition 4.1. \( \square \)
Proof of Proposition 4.1. By Sobolev embedding for $s > 0$,
\[
\text{osc}_{\partial B(\rho)} f \lesssim \rho^s f_{W^{s, \frac{2}{s}}(\partial B(\rho))}.
\]
Thus we find
\[
\left( \text{osc}_{\partial B(x,r)} f \right)^{\frac{n}{s}} \log(\frac{R}{r}) \leq \int_r^R \frac{1}{\rho} \left( \text{osc}_{\partial B(\rho)} f \right)^{\frac{n}{s}} d\rho \leq \int_r^R \left[ f \right]_{W^{s, \frac{2}{s}}(\partial B(\rho))} d\rho.
\]
In view of Lemma 2.2 we obtain
\[
\left( \text{osc}_{\partial B(x,r)} f \right)^{\frac{n}{s}} \leq \frac{1}{\log(\frac{R}{r})} \left[ f \right]_{W^{s, \frac{2}{s}}(B(R))}^{\frac{n}{s}}.
\]
This readily implies the claim. \qed

Theorem 1.6 also follows from Proposition 4.1, and additionally the following distortion argument.

Lemma 4.2. Let $f \in W^{s, \frac{2}{s}}(\Omega, \mathbb{R}^2)$, $\Omega \subset \mathbb{R}^2$ open, $s \geq \frac{2}{3}$. Assume that $\text{Jac}(f) \geq 0$ and $\text{curl}(f) := \partial_1 f^2 - \partial_2 f^1 = 0$ in distributional sense.

For $\delta \in \mathbb{R}\setminus\{0\}$ set $f_\delta(x_1, x_2) := f(x_1, x_2) + \delta(-x_2, x_1)^T$. Then $\text{Jac}(f_\delta) > 0$.

Proof. Let $f_\varepsilon$ be an approximation of $f$ in $W^{s, \frac{2}{s}}(\Omega)$. We have
\[
\det(Df_\varepsilon + \delta(-x_2, x_1)^T) = \det(Df_\varepsilon) + \delta^2 + \delta(\partial_1 f^2_\varepsilon - \partial_2 f^1_\varepsilon).
\]
That is, for any $\varphi \in C_0^\infty(\Omega),$
\[
\text{Jac}(f_\delta)[\varphi] = \lim_{\varepsilon \to 0} \int_\Omega \det(Df_\varepsilon + \delta(-x_2, x_1)^T) \varphi
\]
\[
= \text{Jac}(f)[\varphi] + \delta^2 \int_\Omega \varphi + \delta \lim_{\varepsilon \to 0} \int_\Omega (\partial_1 f^2_\varepsilon - \partial_2 f^1_\varepsilon) \varphi.
\]
Integrating by parts and the pointwise a.e. convergence of $f_\varepsilon$ to $f$ implies
\[
\int_\Omega (\partial_1 f^2_\varepsilon - \partial_2 f^1_\varepsilon) \varphi = -\text{curl}[f_\varepsilon](\varphi) \xrightarrow{\varepsilon \to 0} \text{curl}[f](\varphi) = 0.
\]
Thus,
\[
\text{Jac}(f_\delta)[\varphi] = \text{Jac}(f)[\varphi] + \delta^2 \int_\Omega \varphi.
\]
In particular if $\varphi \geq 0$ and $\text{Jac}(f_\delta)[\varphi] = 0$ we have $\varphi \equiv 0$, i.e. $\text{Jac}(f_\delta) > 0$. \qed

Proof of Theorem 1.6. By Lemma 4.2 and Theorem 1.5 we have $f_\delta$ is continuous, and indeed we have a estimate on the modulus of continuity of $f_\delta$ by Proposition 4.1. This estimate is uniform in $\delta$, and by Arzela-Ascoli we conclude that $f = \lim_{\delta \to 0} f_\delta$ still enjoys the same continuity estimate. \qed
Appendix A. Two functions

Lemma A.1. For any $c > 0$ there exists $d = d_c \in C^{1,1}([0, \infty), (0, \infty))$ such that

\[
\begin{cases}
d(t) + td'(t) \geq 0 & \forall t > 0 \\
d(t) = t^{-1} & \text{for } t > c/2 \\
d(t) \equiv d(0) & \text{for } t \approx 0 \\
d(t) + td'(t) > 0 & \text{for } t \approx 0.
\end{cases}
\]

Proof. Observe that if $d(t)$ satisfies the above assumptions for $c = 2$, then $d_c(t) := c^{-1}d(t/c)$ satisfies the assumptions for generic $c > 0$. So, w.l.o.g. $c = 2$.

Set

\[
d(t) := \begin{cases}
t^{-1} & t \geq 1 \\
-t^2 + t + 1 & t \in [\frac{1}{2}, 1] \\
\frac{5}{4} & t \in [0, \frac{1}{2}].
\end{cases}
\]

Observe that $\lim_{t\to 1^+} d(t) = \lim_{t\to 1^-} d(t) = 1$, and that $\lim_{t\to 1/2^+} d(t) = \lim_{t\to 1/2^-} d(t) = \frac{5}{4}$. Also,

\[
d'(t) := \begin{cases}
t^{-2} & t \geq 1 \\
-2t + 1 & t \in [\frac{1}{2}, 1] \\
0 & t \in [0, \frac{1}{2}].
\end{cases}
\]

In particular, $\lim_{t\to 1^+} d'(t) = \lim_{t\to 1^-} d'(t) = -1$ and $\lim_{t\to 1/2^+} d'(t) = \lim_{t\to 1/2^-} d'(t) = 0$. That is, $d \in C^{1,1}([0, \infty))$. The only thing left to check is that

\[
d(t) + td'(t) = \begin{cases}
0 & t \geq 1 \\
-3t^2 + 2t + 1 & t \in [\frac{1}{2}, 1] \\
\frac{5}{4} & t \in [0, \frac{1}{2}]
\end{cases}
\]

is non-negative. But this is immediate. \[\square\]

Lemma A.2. For any $n \in \mathbb{N}$ and any $\lambda > 0$ there exists $\pi \in C^{1,1}([0, \infty), (0, \infty))$ with the following properties:

\[
\begin{cases}
\pi(t)^{n-1}(\pi(t) + t\pi'(t)) \leq 1 & \forall t \geq 0 \\
\pi(t)^{n-1}(\pi(t) + t\pi'(t)) < 1 & \forall t \geq 2\lambda \\
\pi(t) \equiv 1 & \text{for } t \leq \lambda \\
\sup_t \pi(t) < C & \text{with } C \text{ independent of } \lambda \\
\sup_t \pi'(t) < C(\lambda).
\end{cases}
\]

Proof. Setting $\pi_\lambda(t) := \pi(t/\lambda)$ we can reduce to the case $\lambda = 1$, which we shall now consider.
Set \( r(t) := t^n \pi(t)^n \). Then the differential inequalities become

\[
\begin{align*}
  r'(t) &\leq nt^{n-1} \quad \forall t \geq 0 \\
  r'(t) &< nt^{n-1} \quad \forall t \geq 2 \\
  r(t) &= t^n \quad \text{for } t \leq 1
\end{align*}
\]

Clearly we can find a \( C^{1,1} \)-function \( r \) that satisfies these conditions, e.g.

\[
\begin{align*}
  r(t) &= t^n \quad \text{for } t \leq 1 \\
  r(t) &= t^n - a(t) \quad \text{for } 1 < t < 2 \\
  r(t) &= t^n - \frac{1}{2} t \quad \text{for } t \geq 2
\end{align*}
\]

where \( a(t) \) is any smooth non-decreasing function such that \( a(1) = a'(1) = 0, a(2) = 1, \) and \( a'(2) = \frac{1}{2} \).

Then \( p(t) := t^{-1} \sqrt{r(t)} \) is bounded, has derivatives bounded, and satisfies all the other assumptions as well. \( \square \)

**References**

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] H. Brezis and H. Nguyen. The Jacobian determinant revisited. *Invent. Math.*, 185(1):17–54, 2011.

[3] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)*, 72(3):247–286, 1993.

[4] I. Fonseca and W. Gangbo. *Degree theory in analysis and applications*, volume 2 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[5] I. Fonseca and W. Gangbo. Local invertibility of Sobolev functions. *SIAM J. Math. Anal.*, 26(2):280–304, 1995.

[6] P. Gladbach and H. Olbermann. Coarea formulae and chain rules for the Jacobian determinant in fractional Sobolev spaces. *arXiv:1903.07420*, 2019.

[7] P. Goldstein, P. Hajłasz, and M. R. Pakzad. Finite distortion Sobolev mappings between manifolds are continuous. *arXiv e-prints*, page arXiv:1705.05773, May 2017.

[8] V. M. Goldstéǐn and S. K. Vodopyanov. A test of the removability of sets for \( L^p \) spaces of quasiconformal and quasi-isomorphic mappings. *Sibirsk. Mat. Ž.*, 18(1):48–68, 237, 1977.

[9] P. Hajłasz and J. Malý. Approximation in Sobolev spaces of nonlinear expressions involving the gradient. *Ark. Mat.*, 40(2):245–274, 2002.

[10] T. Iwaniec and G. Martin. *Geometric function theory and non-linear analysis*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[11] A. Jonsson and H. Wallin. A Whitney extension theorem in \( L_p \) and Besov spaces. *Ann. Inst. Fourier (Grenoble)*, 28(1):vi, 139–192, 1978.

[12] E. Lenzmann and A. Schikorra. Sharp commutator estimates via harmonic extensions. *Nonlinear Analysis (accepted)*.

[13] M. Lewicka and M. R. Pakzad. Convex integration for the Monge-Ampère equation in two dimensions. *Anal. PDE*, 10(3):695–727, 2017.

[14] M. R. Pakzad. On the Sobolev space of isometric immersions. *J. Differential Geom.*, 66(1):47–69, 2004.
[15] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.

[16] A. Schikorra and J. Van Schaftingen. An estimate of the Hopf degree of fractional Sobolev mappings. *preprint, arXiv:1904.12549*, 2019.

[17] W. Sickel and A. Youssfi. The characterisation of the regularity of the Jacobian determinant in the framework of potential spaces. *J. London Math. Soc. (2)*, 59(1):287–310, 1999.

[18] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Berlin; UMI, Bologna, 2007.

[19] V. Šverák. Regularity properties of deformations with finite energy. *Arch. Rational Mech. Anal.*, 100(2):105–127, 1988.

[20] Y. Zhou. Fractional Sobolev extension and imbedding. *Trans. Amer. Math. Soc.*, 367(2):959–979, 2015.

(Siran Li) Department of Mathematics, Rice University, MS 136 P.O. Box 1892, Houston, Texas, 77251-1892, USA

Department of Mathematics, McGill University, Burnside Hall, 805 Sherbrooke Street West, Montreal, Quebec, H3A 0B9, Canada.

E-mail address: Siran.Li@rice.edu

(Armin Schikorra) Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA

E-mail address: armin@pitt.edu