$H^s_x \times H^s_x$ scattering theory for a weighted gradient system of 3D radial defocusing NLS

Xianfa Song*
Department of Mathematics, School of Mathematics, Tianjin University, Tianjin, 300072, P. R. China
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Abstract
In this paper, using I-method, we establish $H^s_x \times H^s_x$ scattering theories for the following Cauchy problem

$$\begin{cases}
    i u_t + \Delta u = \lambda |v|^2 u, & x \in \mathbb{R}^3, t > 0, \\
    u(x,0) = u_0(x), & x \in \mathbb{R}^3.
\end{cases}$$

Here $\lambda > 0, \mu > 0, (u_0, v_0) \in H^s_x(\mathbb{R}^3) \times H^s_x(\mathbb{R}^3)$ and $\frac{1}{2} < s < 1$.

Keywords: Weighted(or essential) gradient system; Schrödinger equation; Scattering; I-method.

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1 Introduction

In this paper, we consider the following Cauchy problem:

$$\begin{cases}
    i u_t + \Delta u = \lambda |v|^2 u, & x \in \mathbb{R}^3, t > 0, \\
    u(x,0) = u_0(x), & x \in \mathbb{R}^3.
\end{cases}$$

Here $\lambda > 0, \mu > 0, (u_0, v_0) \in H^s_x(\mathbb{R}^3) \times H^s_x(\mathbb{R}^3)$ and $\frac{1}{2} < s < 1$. Model (1.1) often appears in condensed matter theory, in quantum mechanics, in nonlinear optics, in plasma physics and in the theory of Heisenberg ferromagnet and magnons. $\lambda > 0$ and $\mu > 0$ mean the nonlinearities are defocusing ones. An interesting topic on (1.1) is scattering phenomenon which we will study in this paper.

First, we would like to review some scattering results on the following Cauchy problem when $d = 3$

$$\begin{cases}
    i u_t + \Delta u = \lambda |u|^p u, & x \in \mathbb{R}^d, t \in \mathbb{R}, \\
    u(x,0) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}$$

where $p > 0$ and $\lambda \geq 0$, i.e., the equation in (1.2) has the defocusing nonlinearity. About the classical results on $L^2_x$, $\Sigma$ and $H^1_x$ scattering theories for (1.2), to see more details, we can refer to the books [5, 7, 14, 34] and the numerous references therein. Recently, different

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*E-mail: songxianfa2004@163.com(X.F. Song)
types of scattering results on (1.2) when \( d = 3 \) were established by many authors, we can see [1, 8, 10, 11, 13, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 30, 33].

We specially point out that the \( H^s_x \) scattering theory for (1.2) with \( p = 2 \) is also studied by many authors. Bourgain established the \( H^s_x \) scattering theory for (1.2) in [2] with \( p = 2 \) when \( d = 2 \) and initial data in \( H^s_x \) with \( s \geq \frac{5}{2} \). In [3], for (1.2) with \( p = 2 \) when \( d = 3 \) and radial initial data \( u_0 \), Bourgain proved global well-posedness and scattering for \( s \geq \frac{5}{2} \). In [3], for (1.2) with \( p = 2 \), through introducing an operator \( I : H^s_x(\mathbb{R}^d) \to H^1_x(\mathbb{R}^d) \) and tracking the change of \( E(Iw(t)) \), Colliander et. al. proved global well-posedness for the solution to (1.2) when \( d = 2 \) for \( s > \frac{5}{2} \) and when \( d = 3 \) for \( s > \frac{5}{2} \).

Later, the result when \( d = 3 \) was respectively extend to \( s > \frac{5}{2} \) by Colliander et. al. in [9], \( s > \frac{5}{2} \) by Dodson in [12], \( s > \frac{5}{2} \) by Su in [13]. Recently, using \( I \)-method, Dodson obtained the \( H^s_x \) scattering theory for (1.2) with \( p = 2 \) for \( s > \frac{5}{2} \).

There are some results on the scattering theory for a system of Schrödinger equations. For \([H^1_x]^N\)-scattering theory, we can refer to [4, 6, 9, 10, 29, 30, 31, 36, 37, 39] and see the details. In [31], Saanouni established the \([H^s_x]^N\)-scattering theory for \( \frac{5}{2} < s < 1 \). Very recently, in [32], we studied

\[
\begin{align*}
    iu_t + \Delta u &= \lambda |u|^\alpha |v|^{\beta+2}u, & iu_t + \Delta v &= \mu |u|^\alpha |v|^{\beta}v, & x \in \mathbb{R}^d, & t > 0, \\
    u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in \mathbb{R}^d.
\end{align*}
\]

Besides establishing the local well-posedness of the \( H^1_x \times H^1_x \), \( \Sigma \times \Sigma \) and \( H^s_x \times H^s_x \) solutions, we found that there exists a critical exponents line \( \alpha + \beta = 2 \) in the sense that (1.3) always possesses a unique bounded \( H^1_x \times H^1_x \)-solution for any initial data \((u_0, v_0) \in H^1_x(\mathbb{R}^3) \times H^1_x(\mathbb{R}^3)\) if \( \alpha + \beta \leq 2 \), while there exist some initial data \((u_0, v_0) \in H^1_x(\mathbb{R}^3) \times H^1_x(\mathbb{R}^3)\) such that it doesn’t have the unique global \( H^1_x \times H^1_x \)-solution if \( \alpha + \beta > 2 \) and \( \alpha = \beta \). We also established \( H^1_x \times H^1_x \) and \( \Sigma \times \Sigma \) scattering theories when \((\alpha, \beta)\) below the critical exponents line when \( d = 3 \), and \( H^s_x \times H^s_x \) scattering theory when \( d \geq 5 \), where \( s_c = \frac{d+2}{2} > 1 \).

As a special case of (1.3), (1.1) is a weighted gradient system of Schrödinger equations. Therefore, we can define the following weighted mass and energy

\[
M_\omega(u, v) = \mu \| u \|_{L^2_x} + \lambda \| v \|_{L^2_x}, \quad E_\omega(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} [\mu |\nabla u|^2 + \lambda |\nabla v|^2 + \lambda \mu |u|^2 |v|^2] dx.
\]

Naturally, we hope to establish the \( H^s_x \times H^s_x \) scattering theory for (1.1) and \( \frac{5}{2} < s < 1 \).

In fact, using \( I \)-method, we will establish \( H^s_x \times H^s_x \) scattering theory for (1.1) as follows.

**Theorem 1.1 (H^s_x \times H^s_x \text{ scattering theory for (1.1)})**. Assume that \( \lambda > 0, \mu > 0 \), \((u_0, v_0)\) radial and \((u_0, v_0) \in H^s_x(\mathbb{R}^3) \times H^s_x(\mathbb{R}^3), \frac{5}{2} < s < 1 \). Then the initial value problem (1.1) is globally well-posed and scattering, i.e., there exist \((u_+, v_+) \in H^s_x(\mathbb{R}^3) \times H^s_x(\mathbb{R}^3)\) and \((u_-, v_-) \in H^s_x(\mathbb{R}^3) \times H^s_x(\mathbb{R}^3)\) such that

\[
\begin{align*}
    \lim_{t \to +\infty} \| u(t) - e^{it\Delta} u_+ \|_{H^s_x(\mathbb{R}^3)} + \| v(t) - e^{it\Delta} v_+ \|_{H^s_x(\mathbb{R}^3)} &= 0, \\
    \lim_{t \to -\infty} \| u(t) - e^{it\Delta} u_- \|_{H^s_x(\mathbb{R}^3)} + \| v(t) - e^{it\Delta} v_- \|_{H^s_x(\mathbb{R}^3)} &= 0.
\end{align*}
\]

**Remark 1.2**: 1. Scattering for (1.1) corresponds to \( \| u \|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^3)} + \| v \|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^3)} < +\infty \).

2. If \( u_0(x) \equiv v_0(x) \equiv w_0(x) \in H^s_x(\mathbb{R}^3) \) and \( \lambda = \mu \), then (1.1) degrades into the scalar Schrödinger equation \( iu_t + \Delta w = \lambda |w|^2 w \) subject to \( w(x, 0) = w_0(x) \), and the conclusions of Theorem 1.3 meet with those of Theorem 1.2 in [14]. In this sense, our results cover those of Theorem 1.2 in [14].

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we will use \( I \)-method to establish \( H^s_x \times H^s_x \) scattering theory for (1.1).
2 Preliminaries

In this section, we will give some notations and useful lemmas which were mentioned in [14]. We use $L^q_t L^r_x(I \times \mathbb{R}^3)$ to denote the Banach space of functions $u : I \times \mathbb{R}^3 \rightarrow \mathbb{C}$ subject to the norm
\[
\|u\|_{L^q_t L^r_x(\mathbb{R}^3)} := \left( \int_I \left( \int_{\mathbb{R}^3} |f(x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}} < \infty
\]
for any spacetime slab $I \times \mathbb{R}^3$, with the usual modifications when $q$ or $r$ is infinity. Especially, we abbreviate $L^q_t L^r_x$ as $L^q_t$ if $q = r$.

The Fourier transform on $\mathbb{R}^3$ and the fractional differential operators $|\nabla|^s$ are defined by
\[
\hat{f}(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i x \cdot \xi} f(x) \, dx, \quad |\nabla|^s f(x) = |\xi|^s \hat{f}(\xi),
\]
and the homogeneous and inhomogeneous Sobolev norms are respectively defined by
\[
\|f\|_{H^s(\mathbb{R}^3)} := \||\nabla|^s f\|_{L^2(\mathbb{R}^3)} = \||\xi|^s \hat{f}(\xi)\|_{L^2(\mathbb{R}^3)}, \quad \|f\|_{H^s(\mathbb{R}^3)} := \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^2(\mathbb{R}^3)}.
\]

Let $e^{it\Delta}$ be the free Schrödinger propagator and the generated group of isometries $(\mathcal{J}(t))_{t \in \mathbb{R}}$. Then for $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,
\[
\mathcal{J}(t) f(x) := e^{it\Delta} f(x) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{i|x-y|^2/4t} f(y) \, dy, \quad t \neq 0,
\]
\[
e^{it\Delta} f(\xi) = e^{-4\pi^2 t|\xi|^2} \hat{f}(\xi),
\]
\[
\|e^{it\Delta} f\|_{L^q_t L^r_x(\mathbb{R}^3)} \lesssim |t|^{-\frac{n}{2} - \frac{3}{2}} \|f\|_{L^q_t L^r_x(\mathbb{R}^3)}, \quad t \neq 0,
\]
\[
\|e^{it\Delta} f\|_{L^q_t L^r_x(\mathbb{R}^3)} \lesssim |t|^{-\frac{n}{2} - \frac{1}{2} - \frac{1}{q}} \|f\|_{L^q_t L^r_x(\mathbb{R}^3)}, \quad t \neq 0.
\]

And the following Duhamel’s formula holds
\[
u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} (iu_t + \Delta u)(s) ds.
\]

We mention some definitions and estimates which also appeared in [14] below.

**Definition 2.1 (Strichartz space).** Let $\tilde{S}^0(I)$ be the Strichartz space
\[
\tilde{S}^0(I) = L^\infty_t L^2_x(I \times \mathbb{R}^3) \cap L^1_t L^6_x(I \times \mathbb{R}^3),
\]
and $\tilde{N}^0(I)$ be the dual
\[
\tilde{N}^0(I) = L^1_t L^2_x(I \times \mathbb{R}^3) + L^2_t L^\frac{6}{5}_x(I \times \mathbb{R}^3).
\]

And the following Strichartz estimates hold
\[
\|e^{it\Delta} u_0\|_{\tilde{S}^0(I \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)}, \quad \| \int_0^t e^{i(t-\tau)\Delta} F(\tau) \, d\tau \|_{\tilde{S}^0(I \times \mathbb{R}^3)} \lesssim \|F\|_{\tilde{N}^0(I \times \mathbb{R}^3)}.
\]
Definition 2.2 (Littlewood-Paley decomposition). Given a radial and decreasing function \( \psi \in C_0^\infty(\mathbb{R}^3) \) satisfying \( \psi(x) = 1 \) for \( |x| \leq 1 \) and \( \psi(x) = 0 \) for \( |x| > 2 \). Let
\[
\phi_j(x) = \psi(2^{-j}x) - \psi(2^{-j+1}x)
\] (2.10)
and \( P_j \) be the Fourier multiplier defined by
\[
\hat{P_j}f(\xi) = \phi_j(\xi)\hat{f}(\xi) \text{ for any } j.
\] (2.11)
This gives the following Littlewood-Paley decomposition
\[
f = \sum_{j=-\infty}^{\infty} P_j f \text{ at least in the } L^2 \text{ sense.} \tag{2.12}
\]
In convenience, we write
\[
P_{\leq N}u = \sum_{j:2^j \leq N} P_j u, \quad P_{> N}u = u - P_{\leq N}u.
\] (2.13)
We may abbreviate \( u_{\leq N} = P_{\leq N}u \) and \( u_{> N} = P_{> N}u \) in the sequels.

The following proposition gives some well-known properties for the Littlewood-Paley decomposition.

Proposition 2.3. For any \( 1 < p < +\infty \),
\[
\|f\|_{L^p_x(\mathbb{R}^3)} \sim_p \left( \sum_{j=-\infty}^{\infty} |P_j f|^2 \right)^{\frac{1}{2}} \|P_j f\|_{L^p_x(\mathbb{R}^3)},
\] (2.14)
\[
\|f\|_{L^p_x(\mathbb{R}^3)} \lesssim \|f\|_{H^s_x(\mathbb{R}^3)} \text{ for } s = 3\left(\frac{1}{2} - \frac{1}{p}\right). \tag{2.15}
\]
We also have for \( 2 \leq p \leq \infty \)
\[
\|P_j f\|_{L^p_x(\mathbb{R}^3)} \lesssim 2^{3j\left(\frac{3}{2} - \frac{1}{p}\right)} \|P_j f\|_{L^2_x(\mathbb{R}^3)},
\] (2.16)
and the radial Sobolev embedding
\[
\|\|\|\|\|_{L^\infty_x(\mathbb{R}^3)} \lesssim \|P_j f\|_{H^s_x(\mathbb{R}^3)}. \tag{2.17}
\]

Let \( \psi \) is the same \( \psi \) as in Definition 1 and \( \chi(x) = \psi(\frac{R}{x}) - \psi(x) \). Then
\[
\|\psi(\frac{x}{R})e^{it\Delta}(P_j u_0)\|_{L^2_{x,t}(I \times \mathbb{R}^3)} \lesssim 2^{-\frac{3}{2}j} R^\frac{3}{2} \|P_j u_0\|_{L^2_x(\mathbb{R}^3)},
\] (2.18)
\[
\int_{I} e^{-it\Delta}\psi(\frac{x}{R})(P_j f) dt \|_{L^2_x(\mathbb{R}^3)} \lesssim 2^{-\frac{3}{2}j} R^\frac{3}{2} \|\psi(\frac{x}{R})P_j f\|_{L^2_{x,t}(\mathbb{R}^3)},
\] (2.19)
and for any \( q < 2 \), if \( F \) is supported on \( |x| \leq R \),
\[
\|\|\|\|\|_{L^q_{x,t}(\mathbb{R}^3)} \lesssim R^{1-\frac{3}{2}} \|F\|_{L^q_x(\mathbb{R}^3)},
\] (2.20)
\[
\lesssim R^{1-\frac{3}{2}} \|F\|_{X_R(I \times \mathbb{R}^3)}, \tag{2.21}
\]
where
\[ \|F\|_{X_R(I \times \mathbb{R}^3)} = R^{\frac{1}{2} - 1} \|\psi(Rx)F\|_{L^p_tL^2_x(I \times \mathbb{R}^3)} + R^{\frac{1}{2} - 1} \sum_{j \geq 0} 2^j(1 - \frac{1}{2}) \|\chi(2^{-j}Rx)F\|_{L^p_tL^2_x(I \times \mathbb{R}^3)}. \] (2.22)

**Definition 2.4 (U^p_\Delta spaces).** Let \( 1 \leq p < \infty \) and \( U^p_\Delta \) be an atomic space whose atoms are piecewise solutions of the linear equation,
\[ u_j = \sum_k \chi_{[t_k,t_{k+1}]} e^{it\Delta} u_k, \quad \sum_k \|u_k\|^p_{L^2_x(\mathbb{R}^3)} = 1. \] (2.23)

Then for any \( 1 \leq p < \infty \),
\[ \|u\|_{U^p_\Delta} = \inf \{ \sum_j |c_j| : u = \sum_j u_j, \quad u_j \text{ are } U^p_\Delta \text{ atoms} \}. \] (2.24)

**Proposition 2.5.** If \( u \) solves
\[ iu_t + \Delta u = F_1 + F_2, \quad u(0,x) = u_0(x) \] (2.25)
on the interval \( 0 \in I \subset \mathbb{R} \), then for \( q < 2 \),
\[ \|\nabla|^{-\frac{1}{2}} u\|_{U^p_\Delta(I \times \mathbb{R}^3)} \lesssim_q \|\nabla|^{-\frac{1}{2}} u_0\|_{L^2_x(\mathbb{R}^3)} + \|F_1\|_{X_R(I \times \mathbb{R}^3)} + \|\nabla|^{-\frac{1}{2}} F_2\|_{L^p_tL^2_x(I \times \mathbb{R}^3)}. \] (2.26)

### 3 Scattering result on (1.1)

In this section, we will use the I-method to establish scattering result on (1.1).

**Definition 3.1 (I-operator).** Let \( I : H^s_x(\mathbb{R}^3) \rightarrow H^1_x(\mathbb{R}^3) \) be the Fourier multiplier
\[ \hat{I}f(\xi) = m_N(\xi)\hat{f}(\xi), \] (3.1)
where
\[ m_N(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N, \\ N + |\xi| & \text{if } |\xi| \geq 2N. \end{cases} \] (3.2)

We first give some results on the I-operator and the modified energy \( E(Iu(t), Iv(t)) \). By Sobolev embedding, we have
\[ E_w(Iu(t), Iv(t)) = \frac{1}{2} \int_{\mathbb{R}^3} [\mu|\nabla u|^2 + \lambda|\nabla v|^2 + \lambda \mu |u|^2 |v|^2] dx \]
\[ \lesssim \int_{\mathbb{R}^3} \|\nabla u\|^2 + |\nabla v|^2 + |u|^4 + |v|^4] dx \]
\[ \lesssim \|u\|^2_{H^1_x(\mathbb{R}^3)} + \|v\|^2_{H^1_x(\mathbb{R}^3)} + \|u\|^2_{H^2_x(\mathbb{R}^3)} \|v\|^2_{H^2_x(\mathbb{R}^3)}, \] (3.3)
and consequently
\[ E_w(Iu(0), Iv(0)) \lesssim C(\|u_0\|_{H^s_x}, \|v_0\|_{H^s_x}) N^{2(1-s)}. \] (3.4)
Meanwhile,
\[ \|u(t)\|_{L^2_x(\mathbb{R}^3)} + \|v(t)\|_{L^2_x(\mathbb{R}^3)} \lesssim E_w(Iu(t), Iv(t)) + M_w(Iu(t), Iv(t)). \] (3.5)

Here
\[ M_w(Iu(t), Iv(t)) = \mu \|u(t)\|_{L^2_x(\mathbb{R}^3)}^2 + \lambda \|v(t)\|_{L^2_x(\mathbb{R}^3)}^2. \]

Moreover, we can use the rescaling
\[ (u(t, x), v(t, x)) \rightarrow (u_a(t, x), v_a(t, x)) = a^\frac{7}{2}(u(a^2 t, a x), v(a^2 t, a x)) \] (3.6)
such that \( E_w(Iu(t), Iv(t)) \leq 1 \). And there exists \( a^{\frac{5}{2}} \sim C(\|u_0\|_{H^1_x(\mathbb{R}^3)}; \|v_0\|_{H^1_x(\mathbb{R}^3)}) N^{\frac{3}{4}} \) such that
\[ E_w(Iu_\lambda(0), Iv_\lambda(0)) \leq \frac{1}{2}, \] (3.7)
and
\[ \|u_\lambda(0)\|_{L^2_x(\mathbb{R}^3)} \lesssim \|u(0)\|_{H^1_x(\mathbb{R}^3)} N^{\frac{1}{4}} \|v(0)\|_{L^2_x(\mathbb{R}^3)}, \] (3.8)
\[ \|v_\lambda(0)\|_{L^2_x(\mathbb{R}^3)} \lesssim \|v(0)\|_{H^1_x(\mathbb{R}^3)} N^{\frac{1}{4}} \|v(0)\|_{L^2_x(\mathbb{R}^3)}. \] (3.9)

**Proposition 3.2 (Weight coupled interaction Morawetz estimate).** Assume that \((u, v)\) is a solution of (1.4) on some interval \( J \). Then
\[ \|u\|_{L^1_t L^{\infty}_x(J \times \mathbb{R}^3)} + \|v\|_{L^1_t L^{\infty}_x(J \times \mathbb{R}^3)} + \|u\|_{L^1_t L^2_x(J \times \mathbb{R}^3)} \lesssim \left[ \|u\|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)}^2 + \|v\|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)}^2 \right]. \] (3.10)

**Proof:** Similar to [32], we define the following weight-coupled interaction Morawetz potential:
\[ M_a^{\otimes 2}(t) = 2\mu^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla a(x, y) \nabla (\bar{u}(t, x) \bar{u}(t, y)) \nabla (u(t, x) u(t, y)) dx dy \]
\[ + 2\lambda^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla a(x, y) \nabla (\bar{v}(t, x) \bar{v}(t, y)) \nabla (v(t, x) v(t, y)) dx dy \]
\[ + 2\lambda u \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla a(x, y) \nabla (\bar{u}(t, x) \bar{v}(t, y)) \nabla (u(t, x) v(t, y)) dx dy \]
\[ + 2\lambda u \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla a(x, y) \nabla (\bar{v}(t, x) \bar{u}(t, y)) \nabla (v(t, x) u(t, y)) dx dy, \] (3.11)
where \( a(x, y) = |x - y|, \nabla = (\nabla_x, \nabla_y), x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^3 \). By the inequality (4.10) in [32], we have
\[ \int_I \int_{\mathbb{R}^3} \|u(t, x)\|^4 + \|v(t, x)\|^4 dt dx \lesssim |M_a^{\otimes 2}(t)| \]
\[ \lesssim \left[ \|u\|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)}^2 + \|v\|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)}^2 \right] \left[ \|u\|_{L^\infty_t H^\frac{1}{2}_x(J \times \mathbb{R}^3)}^2 + \|v\|_{L^\infty_t H^\frac{1}{2}_x(J \times \mathbb{R}^3)}^2 \right], \]
which implies (3.10) because \( |u|^2 |v|^2 \leq |u|^4 + |v|^4 \). \( \square \)

**Lemma 3.3.** If \( E(Iu(a_1), Iv(a_1)) \leq 1, J_t = [a_1, b_1], \) and \( \|u\|_{L^1_t L^{\infty}_x(J_t \times \mathbb{R}^3)} + \|v\|_{L^1_t L^{\infty}_x(J_t \times \mathbb{R}^3)} \leq \epsilon \) for some \( \epsilon > 0 \) sufficiently small, then
\[ \|\nabla Iu\|_{\dot{S}^0(J_t \times \mathbb{R}^3)} + \|\nabla Iv\|_{\dot{S}^0(J_t \times \mathbb{R}^3)} \lesssim 1. \] (3.12)
Proof: Let
\[ Z_I(t) = \| \nabla I u \|_{\dot{S}^0([a_t,b_t] \times \mathbb{R}^3)} + \| \nabla I v \|_{\dot{S}^0([a_t,b_t] \times \mathbb{R}^3)}. \]  
(3.13)

Applying \( \nabla I \) to the equations of (1.1), we have
\[
\begin{cases}
  i(\nabla I u)_t + \Delta(\nabla I u) = \lambda \nabla I (|u|^2 u), & i(\nabla I v)_t + \Delta(\nabla I v) = \mu \nabla I (|u|^2 v), \\
  \nabla I u(x,0) = \nabla I u_0, & \nabla I v(x,0) = \nabla I v_0.
\end{cases}
\]  
(3.14)

Using Strichartz estimates with \( q' = r' = \frac{10}{7} \), then applying a fractional Leibniz rule, we obtain
\[
Z_I(t) \lesssim \| \nabla I u_0 \|_{L^2(\mathbb{R}^3)} + \| \nabla I v_0 \|_{L^2(\mathbb{R}^3)} + \| \nabla I (|u|^2 u) \|_{L^{\frac{10}{7}}(\mathbb{R}^3)} + \| \nabla I (|u|^2 v) \|_{L^{\frac{10}{7}}(\mathbb{R}^3)}
\lesssim \left[ \| \nabla I u \|_{L^{\frac{10}{7}}(\mathbb{R}^3)} + \| \nabla I v \|_{L^{\frac{10}{7}}(\mathbb{R}^3)} \right] \left[ \| u \|_{L^{\frac{10}{7}}(\mathbb{R}^3)} + \| v \|_{L^{\frac{10}{7}}(\mathbb{R}^3)} \right]
+ \| \nabla I u_0 \|_{L^2(\mathbb{R}^3)} + \| \nabla I v_0 \|_{L^2(\mathbb{R}^3)}.
\]  
(3.15)

Since the \( L^{\frac{10}{7}}_{t,x} \) factors are bounded by \( Z_I(t) \), we only need to consider the bounds of \( \| u \|_{L^{\frac{10}{7}}_{t,x}(\mathbb{R}^3)} \) and \( \| v \|_{L^{\frac{10}{7}}_{t,x}(\mathbb{R}^3)} \). In fact, similar to the proof of (3.10) in [9], we can get
\[
\| u \|_{L^{\frac{10}{7}}_{t,x}(\mathbb{R}^3)} \lesssim \epsilon^\delta \left( \| \nabla I u \|_{\dot{S}^0([a_t,b_t] \times \mathbb{R}^3)} \right)^{\delta_1},
\]  
(3.16)

\[
\| v \|_{L^{\frac{10}{7}}_{t,x}(\mathbb{R}^3)} \lesssim \epsilon^\delta \left( \| \nabla I v \|_{\dot{S}^0([a_t,b_t] \times \mathbb{R}^3)} \right)^{\delta_2}.
\]  
(3.17)

for some \( \delta_1 > 0 \) and \( \delta_2 > 0 \).

By (3.15), (3.10) and (3.17), we have
\[
Z_I(t) \lesssim 1 + \epsilon^{\delta_3} \left( Z_I(t) \right)^{1+\delta_4} \quad \text{for some } \delta_3 > 0, \delta_4 > 0.
\]  
(3.18)

For \( \epsilon \) sufficiently small, (3.18) yields (3.12).\hfill \Box

(3.12) implies that
\[
\| \nabla I u \|_{L^2(J_t \times \mathbb{R}^3)} + \| \nabla I v \|_{L^2(J_t \times \mathbb{R}^3)} \lesssim 1.
\]  
(3.19)

**Proposition 3.4.** Let \( 0 < J_t \) be an interval such that \( E(Iu(t), Iv(t)) \leq 1 \) on \( J_t \). Then for \( N(s, \| u_0 \|_{H^s}, \| v_0 \|_{H^s}) \) sufficiently large,
\[
\| P_{> \frac{N}{s}} \nabla I u \|_{L^2(J_t \times \mathbb{R}^3)} + \| P_{> \frac{N}{s}} \nabla I v \|_{L^2(J_t \times \mathbb{R}^3)} \lesssim 1.
\]  
(3.20)

**Proof:** Decomposing
\[
|v|^2 u = \mathcal{O}(v_{> \frac{N}{s}}^2 u) + \mathcal{O}(u_{> \frac{N}{s}} u_{\leq \frac{N}{s}} v_{\leq \frac{N}{s}}) + \mathcal{O}(u_{\leq \frac{N}{s}} u_{\leq \frac{N}{s}} v_{\leq \frac{N}{s}} v_{\leq \frac{N}{s}}) + \mathcal{O}((v_{\leq \frac{N}{s}})^2 u_{> \frac{N}{s}})
+ \mathcal{O}(u_{\leq \frac{N}{s}}^2 u_{\leq \frac{N}{s}}),
\]  
(3.21)

\[
|u|^2 v = \mathcal{O}(u_{> \frac{N}{s}}^2 v) + \mathcal{O}(u_{> \frac{N}{s}} v_{> \frac{N}{s}} u_{\leq \frac{N}{s}}) + \mathcal{O}(u_{\leq \frac{N}{s}} v_{\leq \frac{N}{s}} u_{< \frac{N}{s}}) + \mathcal{O}((u_{\leq \frac{N}{s}})^2 v_{> \frac{N}{s}})
+ \mathcal{O}((u_{> \frac{N}{s}})^2 v_{\leq \frac{N}{s}}),
\]  
(3.22)
we can get

\[ P_{>M}(|v|^2u) = P_{>M}[\mathcal{O}((v_{\leq M}^2)^2u)] + P_{>M}[\mathcal{O}(v_{>M}u_{\leq M}v_{\leq M})] + P_{>M}[\mathcal{O}(v_{< M}u_{< M}v_{> M})] + P_{>M}[\mathcal{O}(v_{\leq M}^2u_{< M})], \]  

\[ P_{>M}(|u|^2v) = P_{>M}[\mathcal{O}((u_{\leq M}^2)^2v)] + P_{>M}[\mathcal{O}(u_{>M}v_{\leq M}u_{\leq M})] + P_{>M}[\mathcal{O}(u_{\leq M}v_{< M}u_{> M})] + P_{>M}[\mathcal{O}(u_{\leq M}^2v_{> M})], \]  

(3.23)

(3.24)

because \( P_{>M}[\mathcal{O}((v_{\leq M}^2)^2u_{\leq M})] = 0 \) and \( P_{>M}[\mathcal{O}((u_{\leq M}^2)^2v_{\leq M})] = 0 \). Since \( \nabla I \) is a Fourier multiplier whose symbol is increasing as \(|\xi| \nearrow \infty\), by the product rule and \( (2.26) \), we obtain

\[ \|\nabla IP_{>M}u(t)\|_{L^2_x(J \times \mathbb{R}^3)} \leq \|\nabla IP_{>M}u(0)\|_{L^2_x(\mathbb{R}^3)} + \|\nabla IP_{>M}[\mathcal{O}(v_{\leq M}^2)]u\|_{L^2_x(L^\infty_x(\mathbb{R}^3))} + \|\nabla IP_{>M}[\mathcal{O}(v_{> M}u_{\leq M})]v_{\leq M}\|_{L^2_x(L^\infty_x(\mathbb{R}^3))} + \|\nabla IP_{>M}[\mathcal{O}(v_{\leq M}^2u_{< M})]v_{< M}\|_{L^2_x(L^\infty_x(\mathbb{R}^3))} \]

(3.25)

if \( M \leq N \). Here all the \( L^2_x-L^\infty_x \) norms are on \( (J \times \mathbb{R}^3) \).

Choosing \( \delta(\epsilon) > 0 \) such that \((2 - \epsilon, \frac{6}{5} + \delta)\) is the dual of an admissible pair, using the properties of \( \nabla I \) and Hölder’s inequality, Berstein’s inequality, we have

\[ (I) \leq \|\nabla u\|_{L^\infty_t-L^2_x} \|\nabla IP_{> M}v\|_{L^2_tL^6_x} + \|\nabla IP_{> M}v\|_{L^2_tL^6_x} \|\nabla IP_{> M}u\|_{L^\infty_tL^6_x} \|\nabla IP_{> M}v\|_{L^2_tL^6_x} \|\nabla IP_{> M}u\|_{L^\infty_tL^6_x} \]

\[ \leq \|\nabla IP_{> M}v\|_{L^2_tL^6_x} (M^{-1}\|\nabla u\|_{L^\infty_t-L^2_x} + \|\nabla IP_{> M}v\|_{L^2_tL^6_x} + \|\nabla IP_{> M}u\|_{L^\infty_tL^6_x}) \]

\[ \leq M^{-1}N^{\frac{4(1-\epsilon)}{5(2-\epsilon)} - \frac{2\epsilon}{5(2-\epsilon)}} \|\nabla IP_{> M}v\|_{L^2(J \times \mathbb{R}^3)}. \]

(3.26)

Here all the norms are on \( (J \times \mathbb{R}^3) \), and we have use the following facts

\[ \|\nabla u\|_{L^\infty_t-L^2_x} \|\nabla IP_{> M}v\|_{L^2_tL^6_x} \leq s_n\|u_0\|_{H^s(\mathbb{R}^3)} \|v_0\|_{H^s(\mathbb{R}^3)} \]

\[ N^{\frac{4(1-\epsilon)}{5(2-\epsilon)} - \frac{2\epsilon}{5(2-\epsilon)}}, \]

(3.27)

\[ \|\nabla IP_{> M}v\|_{L^2_tL^6_x} \|\nabla IP_{> M}u\|_{L^\infty_tL^6_x} \leq s_n\|u_0\|_{H^s(\mathbb{R}^3)} \|v_0\|_{H^s(\mathbb{R}^3)} \]

\[ N^{-\frac{1}{2} + \frac{4(1-\epsilon)}{5(2-\epsilon)} - \frac{2\epsilon}{5(2-\epsilon)}}. \]

(3.28)

Similarly,

\[ (II) \leq M^{-1}N^{\frac{4(1-\epsilon)}{5(2-\epsilon)} - \frac{2\epsilon}{5(2-\epsilon)}} \|\nabla IP_{> M}u\|_{L^2(J \times \mathbb{R}^3)} + \|\nabla IP_{> M}v\|_{L^2(J \times \mathbb{R}^3)} \]

(3.29)

and

\[ (III) = \|\nabla [\mathcal{O}(v_{\leq M}^2)]\|_{L^2_x-L^\infty_x} \leq \|\nabla v_{\leq M}^2\|_{L^\infty_t-L^2_x} \|\nabla IP_{> M}v\|_{L^2_tL^6_x} \|\nabla IP_{> M}u\|_{L^\infty_tL^6_x} \]

\[ \leq M^{-1}N^{\frac{4(1-\epsilon)}{5(2-\epsilon)} - \frac{2\epsilon}{5(2-\epsilon)}} \|\nabla IP_{> M}u\|_{L^2(J \times \mathbb{R}^3)} + \|\nabla IP_{> M}v\|_{L^2(J \times \mathbb{R}^3)}, \]

(3.30)
Choosing bedding can get

Interpolating (3.33), by Strichartz estimates, using (3.8), (3.9) and (3.19), we have

Meanwhile, by (2.18) and (3.35),

Similarly,

The estimate for (VI) can be given by (3.36) and (3.37). We can obtain the estimate for (VII) similar to (3.36) and (3.37).

Combining (3.25), (3.26), (3.29), (3.30), (3.31) with the estimates for (VI) and (VII), we can get

for any s with \( \epsilon = \epsilon(s) \) sufficiently small, \( C_1(s) \epsilon < \frac{1}{4} \) and \( C_2(s) \epsilon < \frac{1}{4} \).
If \( M > C(s, \|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)}), \) \((3.38)\) and \((3.39)\) mean that

\[
\|\nabla IP_{\geq M} u\|_{L^2_x(J \times R^3)} + \|\nabla IP_{\geq M} v\|_{L^2_x(J \times R^3)} \\
\lesssim_{\|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)}} 1 + N^{-\frac{1}{2}} C(s, \|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)})^{-\frac{3}{2}} \\
\times \|[\nabla IP_{\geq M} u, v]\|_{L^2_x(J \times R^3)} + \|[\nabla IP_{\geq M} v, u]\|_{L^2_x(J \times R^3)}.
\]

\((3.40)\)

Meanwhile, recalling \((3.19)\), we have

\[
\|\nabla IP_{\geq M} u\|_{L^2_x(J \times R^3)} + \|\nabla IP_{\geq M} v\|_{L^2_x(J \times R^3)} \lesssim N^{\frac{3(1-s)}{2s-1}}.
\]

\((3.41)\)

By induction, for \( C(s, \|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)}) \) sufficiently large,

\[
\|\nabla IP_{\geq M} u\|_{L^2_x(J \times R^3)} + \|\nabla IP_{\geq M} v\|_{L^2_x(J \times R^3)} \lesssim_{\|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)}} 1 + N^{\frac{3(1-s)}{2s-1}} N^{-c \ln(N)}.
\]

\((3.42)\)

Choosing \( N \) sufficiently large such that \( c \ln(N) > \frac{3(1-s)}{4s-2} \), then using \((3.42)\), we get

\[
\|\nabla IP_{\geq M} u\|_{L^2_x(J \times R^3)} + \|\nabla IP_{\geq M} v\|_{L^2_x(J \times R^3)} \lesssim_{\|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)}} 1,
\]

which implies \((3.20)\).

We give a bound on the modified energy increment.

**Lemma 3.5.** Let \( N \) be sufficiently large such that

\[
\ln(N) \geq \frac{C_0(1-s)}{2s-1} + \ln(C(s, \|u_0\|_{H^s_x(R^3)}, \|v_0\|_{H^s_x(R^3)})),
\]

then

\[
\int_{J} \| \frac{d}{dt} E_w(Iu(t), Iv(t)) \| dt \lesssim \frac{1}{N^{1-}}.
\]

\((3.44)\)

**Proof:** Note that

\[
E_w(Iu, Iv) = \frac{1}{2} \int_{R^3} [\mu |\nabla Iv|^2 + \lambda |\nabla Iv|^2 - \lambda \mu |Iu|^2 |Iv|^2] dx
\]

and \( \Re((\overline{I_{u_t}})(iu_{u_t})) = 0 \). Then

\[
\frac{d}{dt} E_w(Iu(t), Iv(t)) = \Re \int_{R^3} \{ \mu |\nabla Iv|^2 (Iu) - \Delta Iv - I u_t ) + \lambda (|Iv|^2 - Iu_t)(|Iu|^2 (Iv) - \Delta Iv - iIv_t) \} dx
\]

\[
= \Re \int_{R^3} \lambda \mu \{ (|Iv|^2 (Iu) - I(|v|^2 u)) + (|Iv_t|^2 (Iv) - I(|u|^2 v)) \} dx.
\]

\((3.45)\)

Since \( I \) is a Fourier multiplier which is constant in time and \( \Delta \) commutes with \( I \), by the equations of \((1.1)\),

\[
\begin{cases} 
 iu_t + \Delta u = \lambda |v|^2 (Iu) + \lambda |Iv|^2 u - |v|^2 (Iu), & x \in R^3, t > 0, \\
 iv_t + \Delta v = \mu |Iu|^2 (Iv) + \mu |Iu|^2 v - |Iu|^2 (Iv), & x \in R^3, t > 0.
\end{cases}
\]

\((3.46)\)
Integrating the equations of (3.46) by parts, we have

\[
\begin{align*}
\frac{d}{dt} E_w(Iu(t), Iu(t)) &= -\mu <i\nabla u, \nabla (|Iv|^2(Iu) - I(|v|^2u))> -\lambda \mu <i|Iv|^2u, (|Iv|^2(Iu) - I(|v|^2u)) > \\
&\quad -\lambda <i\nabla v, \nabla (|Iv|^2(Iv) - I(|v|^2v))> -\lambda \mu <i|Iv|^2v, (|Iv|^2(Iv) - I(|v|^2v)) >.
\end{align*}
\]

(3.47)

Note that

\[
egin{align*}
|Iv|^2(Iu) - I(|v|^2u) &= (IP_{\leq \frac{N}{8}}v)^2(IP_{> \frac{N}{8}}u) - I((P_{\leq \frac{N}{8}}v)^2(P_{> \frac{N}{8}}u)) \\
&\quad + (IP_{> \frac{N}{8}}v)^2(IP_{\leq \frac{N}{8}}u) - I((P_{> \frac{N}{8}}v)^2(P_{\leq \frac{N}{8}}u)) \\
&\quad + 2(IP_{> \frac{N}{8}}v)(IP_{\leq \frac{N}{8}}v)(IP_{> \frac{N}{8}}u)(\frac{N}{8}) - 2I((P_{> \frac{N}{8}}v)(P_{\leq \frac{N}{8}}v)(P_{> \frac{N}{8}}u)) \\
&\quad + 2(IP_{> \frac{N}{8}}v)(IP_{\leq \frac{N}{8}}v)(IP_{\leq \frac{N}{8}}u) - 2I((P_{> \frac{N}{8}}v)(P_{\leq \frac{N}{8}}v)(P_{\leq \frac{N}{8}}u)) \\
&:= (1) + (2) + (3) + (4) + (5),
\end{align*}
\]

(3.48)

\[
egin{align*}
|Iu|^2(Iv) - I(|u|^2v) &= (IP_{\leq \frac{N}{8}}u)^2(IP_{> \frac{N}{8}}v) - I((P_{\leq \frac{N}{8}}u)^2(P_{> \frac{N}{8}}v)) \\
&\quad + (IP_{> \frac{N}{8}}u)^2(IP_{\leq \frac{N}{8}}v) - I((P_{> \frac{N}{8}}u)^2(P_{\leq \frac{N}{8}}v)) \\
&\quad + 2(IP_{> \frac{N}{8}}u)(IP_{\leq \frac{N}{8}}u)(IP_{> \frac{N}{8}}v)(\frac{N}{8}) - 2I((P_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}v)(P_{> \frac{N}{8}}v)) \\
&\quad + 2(IP_{> \frac{N}{8}}u)(IP_{\leq \frac{N}{8}}u)(IP_{\leq \frac{N}{8}}v) - 2I((P_{> \frac{N}{8}}u)(P_{\leq \frac{N}{8}}v)(P_{\leq \frac{N}{8}}v)) \\
&:= (6) + (7) + (8) + (9) + (10),
\end{align*}
\]

(3.49)

and

\[
|m(\xi_2 + \xi_3 + \xi_4) - m(\xi_2)| \lesssim \frac{|\xi_3| + |\xi_4|}{|\xi_2|}.
\]

(3.50)

By $E_w(Iu(t), Iu(t)) \leq 1$ and (3.20), we can obtain

\[
\begin{align*}
\int_j <i\nabla Iu, \nabla ((IP_{\leq \frac{N}{8}}v)^2(IP_{> \frac{N}{8}}u) - I((P_{\leq \frac{N}{8}}v)^2(P_{> \frac{N}{8}}u))) > dt \\
+ \int_j <i\nabla Iv, \nabla ((IP_{\leq \frac{N}{8}}u)^2(IP_{> \frac{N}{8}}v) - I((P_{\leq \frac{N}{8}}u)^2(P_{> \frac{N}{8}}v))) > dt \\
\lesssim \frac{1}{N} \left[ \|\nabla Iu\|_{L^2_t L^2_x} + \|\nabla Iv\|_{L^2_t L^2_x} \right] \left[ \|\nabla IP_{> \frac{N}{8}}u\|_{L^2_t L^6_x}^2 + \|\nabla IP_{> \frac{N}{8}}v\|_{L^2_t L^6_x}^2 \right] \\
\times \left[ \|Iu\|_{L^\infty_t L^2_x} + \|Iv\|_{L^\infty_t L^2_x} \right] \lesssim \frac{1}{N},
\end{align*}
\]

(3.51)

\[
\begin{align*}
\int_j <i\nabla Iu, \nabla ((IP_{\leq \frac{N}{8}}v)^2(IP_{\leq \frac{N}{8}}u) - I((P_{\leq \frac{N}{8}}v)^2(P_{\leq \frac{N}{8}}u))) > dt \\
+ \int_j <i\nabla Iv, \nabla ((IP_{> \frac{N}{8}}u)^2(IP_{\leq \frac{N}{8}}v) - I((P_{> \frac{N}{8}}u)^2(P_{\leq \frac{N}{8}}v))) > dt \\
\lesssim \left[ \|\nabla Iu\|_{L^2_t L^2_x} + \|\nabla Iv\|_{L^2_t L^2_x} \right] \left[ \|\nabla IP_{> \frac{N}{8}}u\|_{L^2_t L^6_x} + \|\nabla IP_{> \frac{N}{8}}v\|_{L^2_t L^6_x} \right] \\
\times \left[ \|IP_{> \frac{N}{8}}u\|_{L^\infty_t L^2_x} + \|IP_{\leq \frac{N}{8}}v\|_{L^\infty_t L^2_x} \right] \lesssim \frac{1}{N},
\end{align*}
\]

(3.52)
\[
\int_J < i\nabla u, \nabla ((P_\rho v)^2 (P_\lambda u) - I ((P_\rho v)^2 (P_\lambda u))) > dt
\]
\[
+ \int_J < i\nabla v, \nabla ((P_\rho v)^2 (P_\lambda u) - I ((P_\rho v)^2 (P_\lambda u))) > dt
\]
\[
\lesssim \|\nabla u\|_{L^\infty_t L^2_x}^2 + \|\nabla v\|_{L^\infty_t L^2_x}^2 \|\nabla IP_\rho v\|_{L^2_t L^\infty_x} \|\nabla IP_\lambda v\|_{L^2_t L^\infty_x} + \|\nabla IP_\rho v\|_{L^2_t L^\infty_x} + \|\nabla IP_\lambda v\|_{L^2_t L^\infty_x} \lesssim \frac{1}{N} ,
\]
(3.53)
\[
\int_J < i\nabla u, \nabla [2(IP_\rho v) (IP_\lambda u) - 2I ((P_\rho v)(P_\lambda u))] > dt
\]
\[
+ \int_J < i\nabla v, \nabla [2(IP_\rho v) (IP_\lambda u) - 2I ((P_\rho v)(P_\lambda u))] > dt
\]
\[
\lesssim \|\nabla u\|_{L^\infty_t L^2_x}^2 + \|\nabla v\|_{L^\infty_t L^2_x}^2 \|\nabla IP_\rho v\|_{L^2_t L^\infty_x} \|\nabla IP_\lambda v\|_{L^2_t L^\infty_x} + \|\nabla IP_\rho v\|_{L^2_t L^\infty_x} + \|\nabla IP_\lambda v\|_{L^2_t L^\infty_x} \lesssim \frac{1}{N} ,
\]
(3.54)
\[
\int_J < i\nabla u, \nabla [2(IP_\rho v) (IP_\lambda u) - 2I ((P_\rho v)(P_\lambda u))] > dt
\]
\[
+ \int_J < i\nabla v, \nabla [2(IP_\rho v) (IP_\lambda u) - 2I ((P_\rho v)(P_\lambda u))] > dt
\]
\[
\lesssim \frac{1}{N} \|\nabla u\|_{L^\infty_t L^2_x}^2 + \|\nabla v\|_{L^\infty_t L^2_x}^2 \|\nabla IP_\rho v\|_{L^2_t L^\infty_x} \|\nabla IP_\lambda v\|_{L^2_t L^\infty_x} \lesssim \frac{1}{N} .
\]
(3.55)

All the norms above are on \( J \times \mathbb{R}^3 \). (3.53) - (3.55) give the estimates for
\[
- \mu < i\nabla u, \nabla \|v\|^2 (Iu) > - \lambda < i\nabla v, \nabla \|u\|^2 (Iv) - I(|v|^2 u)) > .
\]

Now we consider the estimate for
\[
- \lambda \mu < i\|v\|^2 u), (\|v\|^2 (Iu) - I(|v|^2 u)) > - \lambda \mu < i\|u\|^2 v), (\|u\|^2 (Iv) - I(|u|^2 u)) > .
\]

As a matter of convenience, we denote
\[
I(|v|^2)u = (IP_\rho v)^2 (IP_\lambda u) + 2(IP_\rho v)(IP_\lambda u)(IP_\rho v)(IP_\lambda u) + (IP_\rho v)^2 (IP_\lambda u)
\]
\[
+ (IP_\rho v)^2 (IP_\lambda u) + 2(IP_\rho v)(IP_\lambda u)(IP_\rho v)(IP_\lambda u) + (IP_\rho v)^2 (IP_\lambda u)
\]
\[
:= (I) + (II) + (III) + (IV) + (V) + (VI),
\]
(3.56)
\[
I(|u|^2)v = (IP_\rho u)^2 (IP_\lambda v) + 2(IP_\rho u)(IP_\lambda v)(IP_\rho u)(IP_\lambda v) + (IP_\rho u)^2 (IP_\lambda v)
\]
\[
+ (IP_\rho u)^2 (IP_\lambda v) + 2(IP_\rho u)(IP_\lambda v)(IP_\rho u)(IP_\lambda v) + (IP_\rho u)^2 (IP_\lambda v)
\]
\[
:= (I') + (II') + (III') + (IV') + (V') + (VI').
\]
(3.57)

Then
\[
\int_J < i\|v|^2 u), (\|v|^2 (Iu) - I(|v|^2 u)) > dt
\]
\[
= \int_J (< (I), (1) > + < (I), (2) > + < (I), (3) > + < (I), (4) > + < (I), (5) >) dt
\]
\[
+ \int_J (< (II), (2) > + < (II), (3) > + < (II), (4) > + < (II), (5) >) dt
\]

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Using Sobolev embedding theorem, Bernstein's inequality, we have
\[
\| (IP_{\frac{\pi}{8}} v)^2 (IP_{\frac{n}{8}} u) \|_{L^2_{t,x}} \\
\lesssim \| |\nabla IP_{\frac{n}{8}} u| \|_{L^2_t L^6_x} + \| |\nabla IP_{\frac{n}{8}} v| | \|_{L^2_t L^6_x} [P_{\frac{n}{8}} u]_{L^6_t L^\infty} + P_{\frac{n}{8}} \| v \|_{L^6_t L^\infty} \| \frac{1}{N},
\]
and
\[
\int_j < (I), (2) > dt \lesssim \| (IP_{\frac{n}{8}} v)^2 (IP_{\frac{n}{8}} u) \|_{L^2_{t,x}} \lesssim \| P_{\frac{n}{8}} u \|_{L^2_t L^6_x} + [P_{\frac{n}{8}} v]_{L^6_t L^\infty} \| \frac{1}{N^2},
\]
\[
\int_j < (I), (3) > dt \lesssim \int_j [P_{\frac{n}{8}} u]^2 + [P_{\frac{n}{8}} v]^2 [P_{\frac{n}{8}} u]^2 + [P_{\frac{n}{8}} v]^2 [u]^2 + [v]^2 dt \lesssim \| P_{\frac{n}{8}} u \|_{L^2_t L^6_x}^2 + [P_{\frac{n}{8}} v]_{L^6_t L^\infty}^2 [P_{\frac{n}{8}} u]_{L^6_t L^\infty} + [P_{\frac{n}{8}} v]_{L^6_t L^\infty} \| \frac{1}{N^2},
\]
\[
\int_j < (VI), (3) > dt \lesssim \frac{1}{N^2} \| |\nabla IP_{\frac{n}{8}} u| \|_{L^2_t L^6_x} + [IP_{\frac{n}{8}} v]_{L^6_t L^\infty}^2 [IP_{\frac{n}{8}} u]_{L^6_t L^\infty} + [IP_{\frac{n}{8}} v]_{L^6_t L^\infty} \| \frac{1}{N^2}.
\]
All the norms above are on \( J \times \mathbb{R}^3 \). Similar to (5.61), we can get \( \int_j < (I), (4) > dt \lesssim \frac{1}{N^2} \).
Similar to (5.63), we can obtain \( \int_j < (VI), (5) > dt \lesssim \frac{1}{N^2} \). Similarly to (5.62), the estimates for other terms in (3.58) can be bounded by \( \frac{1}{N^2} \) because they all contain two \( P_{\frac{n}{8}} \) factors and two \( P_{\frac{m}{8}} \) factors. Putting all the results above together, we obtain the bound for \( \int_j < I(|u|^2 u), (|v|^2 v) - I(|u|^2 u) > dt \).

Similarly, the bound for \( \int_j < I(|u|^2 v), (|u|^2 v) - I(|u|^2 v) > dt \) can be established. Then substituting all the results into (3.47), we get (3.44). □

Recalling that
\[
\| u(t) \|_{L^2_t(L^\infty_{x})}^2 + \| v(t) \|_{L^2_t(L^\infty_{x})}^2 = \| u(0) \|_{L^2_t(L^\infty_{x})}^2 + \| v(0) \|_{L^2_t(L^\infty_{x})}^2,
\]
and
\[
\| u(t) \|_{H^2_t(L^\infty_{x})}^2 + \| v(t) \|_{H^2_t(L^\infty_{x})}^2 \lesssim E_{\infty}(Iu(t), Iv(t)) + \| u(0) \|_{L^2_t(L^\infty_{x})}^2 + \| v(0) \|_{L^2_t(L^\infty_{x})}^2,
\]
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by Lemma 5.10 and (3.44), we have
\[ \|u(t)\|_{H^s_x(\mathbb{R}^3)} + \|v(t)\|_{H^s_x(\mathbb{R}^3)} \lesssim C(s, \|u_0\|_{H^s_x(\mathbb{R}^3)}, \|v_0\|_{H^s_x(\mathbb{R}^3)}) \left( \|u_0\|_{H^2_x(\mathbb{R}^3)} + \|v_0\|_{H^2_x(\mathbb{R}^3)} \right). \] *(3.64)*

Let \((p, q)\) be a \(\frac{1}{2}\)-admissible pair which satisfies \(\frac{2}{p} = 3\left(\frac{1}{2} - \frac{1}{q} - \frac{1}{6}\right)\). Interpolating (3.10) with (3.64), we get a bounded on \(L^p_t L^q_x\). Now we can partition \(\mathbb{R}\) into finitely many pieces with \(\|u\|_{L^p_t L^q_x(J \times \mathbb{R}^3)} + \|v\|_{L^p_t L^q_x(J \times \mathbb{R}^3)} < \epsilon\) and use a perturbation argument to obtain \(\|u\|_{L^5_t L^6_x(\mathbb{R}^3)} + \|v\|_{L^5_t L^6_x(\mathbb{R}^3)} < +\infty\), which implies scattering. \(\Box\)

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