NEW PROOFS OF THEOREMS OF KATHRYN MANN

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To the memory of Akio Hattori

Abstract

We give a shorter proof of the following theorem of Kathryn Mann [M]: the identity component of the group of the compactly supported $C^r$ diffeomorphisms of $\mathbb{R}^n$ cannot admit a nontrivial $C^p$-action on $S^1$, provided $n \geq 2$, $r \neq n + 1$ and $p \geq 2$. We also give a new proof of another theorem of Mann [M]: any nontrivial homomorphism from the group of the orientation preserving $C^r$ diffeomorphisms of the circle to the group of $C^p$ diffeomorphisms of the circle is the conjugation of the standard inclusion by a $C^p$ diffeomorphism, if $r \geq p$, $r \neq 2$ and $p \neq 1$.

1. Introduction

É. Ghys [G] asked if the group of diffeomorphisms of a manifold admits a nontrivial action on a lower dimensional manifold. A break through towards this problem was obtained by Kathryn Mann [M] in the case where the target manifold is one dimensional. Let us denote by $\text{Diff}^r_0(\mathbb{R}^n)$ the identity component of the group of the compactly supported $C^r$ diffeomorphisms of $\mathbb{R}^n$, $r = 0, 1, \ldots, \infty$. She showed the following theorem.

**Theorem 1.** Assume $n \geq 2$, $r \neq n + 1$ and $p \geq 2$. Then any (abstract) homomorphism from $\text{Diff}^r_0(\mathbb{R}^n)$ to $\text{Diff}^p(S^1)$ is trivial.

The condition $r \neq n + 1$ is for the simplicity of the source group. The condition $p \geq 2$ is necessary since the proof is built upon a theorem of Kopell and Szekeres. Notice that by the fragmentation lemma, the same statement holds true if we replace $\mathbb{R}^n$ by any $n$-dimensional manifold, compact or not. One aim of this notes is to give a short proof of the above theorem. We also show the following result.

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Theorem 2. Assume $n \geq 2$, $r \neq n + 1$ and $p \geq 2$. Then any homomorphism from $\text{Diff}^r_{c}(\mathbb{R}^n)_0$ to $\text{Diff}^p(\mathbb{R})$ is trivial.

This is a generalization of a result of [M] for the target group $\text{Diff}^p((0,1))$.

Next we consider the case where the source manifold is one dimensional. We provide a shorter proof of the following theorem, also contained in [M]. Denote by $\text{Diff}^r_{c}(S^1)$ the group of the orientation preserving $C^r$ diffeomorphisms of $S^1$.

Theorem 3. Assume $r \geq p$, $r \neq 2$ and $p \neq 1$. Then any nontrivial homomorphism from $\text{Diff}^r_{c}(S^1)$ to $\text{Diff}^p(S^1)$ is the conjugation of the standard inclusion by a $C^p$ diffeomorphism.

In the above theorem, the case where $p = 0$ is new. We also have the following.

Theorem 4. Assume $p \neq 1$. Then any nontrivial homomorphism from $\text{PSL}(2,\mathbb{R})$ to $\text{Diff}^p(S^1)$ is the conjugation of the standard inclusion by a $C^p$ diffeomorphism.

As for Ghys’s question for target manifolds of dimension $> 1$, a satisfactory answer is obtained by S. Hurtado [H]. Some part of his argument is an induction on the dimension of the target manifold. It is based upon Theorems 1 and 3.

In [M], Theorems 1 and 3 are shown using the following result.

Theorem 1.1. Assume $r \geq 3$, $p \geq 2$ and $r \geq p$. Any nontrivial homomorphism $\Phi$ from $\text{Diff}^r_{c}((0,1))$ to $\text{Diff}^p((0,1))$ without interior global fixed point of the $\Phi$-action is the conjugation of the standard inclusion by a $C^p$ diffeomorphism of $(0,1)$.

Our proofs of Theorems 1 and 3 do not use Theorem 1.1. On the other hand, we would like to stress that Theorem 1.1 is more involved, and cannot be shown by the techniques of the present paper.

2. Theorem of Kopell and Szekeres

Our main tool for the proof of Theorems 1 and 2 is the following theorem due to Kopell [K] and Szekeres. (See 4.1.11 in [N2].) This forces us to assume $p \geq 2$ in these theorems.

Theorem 2.1. Let $p \geq 2$. Assume that $\psi \in \text{Diff}^p((0,1))$ admits no interior fixed point. Then there is a unique $C^1$ flow $\{\psi^t\}$ on $[0,1)$ such that $\psi = \psi^1$. Moreover any element $\phi$ of the centralizer $C(\psi)$ of $\psi$ in $\text{Diff}^p((0,1))$ can be written as $\phi = \psi^t$ for some $t \in \mathbb{R}$. 

\[ \text{Theorem 2.1. } \]
Corollary 2.2. Let $p \geq 2$ and $\psi \in \text{Diff}^p([0,1))$. Then $C(\psi) = C(\psi^2)$.  

Proof. Choose any element $g \in C(\psi^2)$. Let $J$ be the closure of any component of $[0,1) \setminus \text{Fix}(\psi)$. (Notice that $\text{Fix}(\psi^2) = \text{Fix}(\psi)$.) Then by Theorem 2.1, $g$ commutes with $\psi$ on $J$. Since $J$ is arbitrary, $g$ commutes with $\psi$ everywhere. \qed 

We also have the following result, whose proof is the same as above.

Corollary 2.3. Assume $p \geq 2$ and $\psi \in \text{Diff}^p_+(\mathbb{R})$ admits fixed points. Then the centralizers in $\text{Diff}^p_+(\mathbb{R})$ satisfy $C(\psi) = C(\psi^2)$. 

3. Commuting subgroups of Diff$^0_+(S^1)$ 

Another basic result needed for the proof is the following.

Proposition 3.1. Let $G_1$ and $G_2$ be simple nonabelian subgroups of Diff$^0_+(S^1)$. Assume that $G_2$ is conjugate to $G_1$ in Diff$^0_+(S^1)$ and that any element of $G_1$ commutes with any element of $G_2$. Then there is a global fixed point of $G_1$: $\text{Fix}(G_1) \neq \emptyset$. 

Proof. First of all, let us show that there is an element $g \in G_1 \setminus \{\text{id}\}$ such that $\text{Fix}(g)$ is nonempty. Assume the contrary. Consider the group $\tilde{G}_1$ formed by any lift of any element of $G_1$ to the universal covering space $\mathbb{R} \to S^1$. The canonical projection $\pi : \tilde{G}_1 \to G_1$ is a group homomorphism. Now $\tilde{G}_1$ acts freely on $\mathbb{R}$. A theorem of Hölder asserts that $\tilde{G}_1$ is abelian. See for example [N1]. Therefore $G_1 = \pi(\tilde{G}_1)$ would be abelian, contrary to the assumption of the proposition. 

Let $X_2 \subset S^1$ be a minimal set of $G_2$. The set $X_2$ is either a finite set, a Cantor set or the whole of $S^1$. If $X_2$ is a singleton, then $G_2$ admits a fixed point. Since $G_1$ is conjugate to $G_2$, we have $\text{Fix}(G_1) \neq \emptyset$, as is required. If $X_2$ is a finite set which is not a singleton, we get a nontrivial homomorphism from $G_2$ to a finite abelian group, contrary to the assumption. In the remaining case, it is well known, easy to show, that the minimal set is unique. That is, $X_2$ is contained in any nonempty $G_2$ invariant closed subset. 

Let $F_1$ be the subset of $G_1$ formed by the elements with nonempty fixed point set. Since $G_1$ and $G_2$ commutes, the fixed point set $\text{Fix}(g)$ of any element $g \in F_1$ is $G_2$ invariant. Then we have:

\begin{equation}
X_2 \subset \text{Fix}(g) \quad \text{for any } g \in F_1.
\end{equation}

This shows that $F_1$ is in fact a subgroup. By the very definition, $F_1$ is normal. Since $G_1$ is simple and $F_1$ is nontrivial, $F_1 = G_1$. Finally again by (3.1), $\text{Fix}(G_1) \neq \emptyset$. \qed
4. Proof of Theorem 1

Assume $n \geq 2$, $r \neq n + 1$ and $p \geq 2$. Let $\Phi : \text{Diff}_c^r(\mathbb{R}^n)_0 \to \text{Diff}^p(S^1)$ be a nontrivial homomorphism. Our purpose is to deduce a contradiction. Since $\text{Diff}_c^r(\mathbb{R}^n)_0$ is simple by the assumption $r \neq n + 1$, the map $\Phi$ is injective and its image is contained in the group of orientation preserving diffeomorphisms. Let $B_1$ and $B_2$ be disjoint open balls of radius 2 in $\mathbb{R}^n$. The group $G_i = \text{Diff}_c^r(B_i)_0$ is nonabelian and simple. Clearly $G_2$ is conjugate to $G_1$ in $\text{Diff}_c^r(\mathbb{R}^n)_0$.

Therefore $\Phi(G_i)$ satisfies all the conditions of Proposition 3.1. Thus $\Phi(G_1)$ has a fixed point, and one can identify $\Phi(G_1) \subset \text{Diff}^p([0,1))$. In view of Corollary 2.2 and the injectivity of $\Phi$, it is sufficient to construct an element $g \in G_1$ such that $C(g) \neq C(g^2)$. Let $B'_1$ be the concentric ball in $B_1$ of radius 1. Any element $g \in G_1$ which is an involution on $B'_1$ will do.

5. Proof of Theorem 2

Let $n \geq 2$, $r \neq n + 1$ and $p \geq 2$. Assume there is a nontrivial homomorphism $\Phi : \text{Diff}_c^r(\mathbb{R}^n)_0 \to \text{Diff}^p(\mathbb{R})$. By the simplicity of $\text{Diff}_c^r(\mathbb{R}^n)_0$, $\Phi$ is injective, with its image contained in the group of orientation preserving diffeomorphisms. In view of Corollary 2.3 and the last step of the previous section, it suffices to show that any element of the image of $\Phi$ has nonempty fixed point set. The rest of this section is devoted to its proof.

Assume for contradiction that there is an element $g' \in \text{Diff}_c^r(\mathbb{R}^n)_0$ such that Fix($\Phi(g')$) = $\emptyset$. Choose open balls $B_i$ ($i = 1, 2$) in $\mathbb{R}^n$ as in the previous section. Again let $G_i = \text{Diff}_c^r(B_i)_0$. There is a conjugate $g$ of $g'$ in $G_2$. Notice that Fix($\Phi(g)$) = $\emptyset$. Then $\Phi(G_2)$ has a cross section $I$ in $\mathbb{R}$, that is, $I$ is a compact interval such that any $\Phi(G_2)$ orbit hits $I$. Now we follow the proof of Proposition 6.1 in [DKNP], to show that there is a unique minimal set $X_2$ for $\Phi(G_2)$. Moreover we shall show that there is a nonempty $\Phi(G_2)$ invariant closed subset $X_2$ in $\mathbb{R}$ which has the property that any nonempty $\Phi(G_2)$ invariant closed subset contains $X_2$.

The proof goes as follows. Let $F$ be the family of nonempty $\Phi(G_2)$ invariant closed subsets of $\mathbb{R}$, and $F_I$ the family of nonempty closed subsets $Y$ in $I$ such that $\Phi(G_2)(Y) \cap I = Y$, where we denote

$$\Phi(G_2)(Y) = \bigcup_{g \in G_2} \Phi(g)(Y).$$

Define a map $\phi : F \to F_I$ by $\phi(X) = X \cap I$, and $\psi : F_I \to F$ by $\psi(Y) = \Phi(G_2)(Y)$. They satisfy $\psi \circ \phi = \phi \circ \psi = \text{id}$.

Let $\{Y_x\}$ be a totally ordered set in $F_I$. Then the intersection $\bigcap_x Y_x$ is nonempty. Let us show that it belongs to $F_I$, namely,

\begin{equation}
\Phi(G_2)(\bigcap_x Y_x) \cap I = \bigcap_x Y_x.
\end{equation}
For the inclusion \( < \), we have

\[
\Phi(\mathcal{G}_2) \left( \bigcap_x Y_x \right) \cap I = \left( \bigcap_x \Phi(\mathcal{G}_2)(Y_x) \right) \cap I = \bigcap_x \left( \Phi(\mathcal{G}_2)(Y_x) \cap I \right) = \bigcap_x Y_x.
\]

For the other inclusion, notice that

\[
\bigcap_x Y_x \subseteq \Phi(\mathcal{G}_2) \left( \bigcap_x Y_x \right) \quad \text{and} \quad \bigcap_x Y_x \subseteq I.
\]

Therefore by Zorn’s lemma, there is a minimal element \( Y_2 \) in \( \mathcal{F}_I \). The set \( Y_2 \) is not finite. In fact, if it is finite, the set \( X_2 = \psi(Y_2) \) in \( F \) is discrete, and there would be a nontrivial homomorphism from \( \Phi(\mathcal{G}_2) \) to \( \mathbb{Z} \), contrary to the fact that \( \mathcal{G}_2 \), (and hence \( \Phi(\mathcal{G}_2) \)) is simple.

Let \( \mathcal{F}_1 \) be the subset of \( \mathcal{G}_1 \) formed by the elements \( g \) such that \( \text{Fix}(\Phi(g)) \neq \emptyset \). Again by a theorem of Hölder, \( \mathcal{F}_1 \) contains a nontrivial element. Now we have

\[
(5.2) \quad X_2 \subseteq \text{Fix}(\Phi(g)) \quad \text{for any } g \in \mathcal{F}_1.
\]

This shows that \( \mathcal{F}_1 \) is a subgroup, normal by the definition. Since \( \mathcal{G}_1 \) is simple, \( \mathcal{F}_1 = \mathcal{G}_1 \). Finally again by (5.2), we have \( \text{Fix}(\Phi(\mathcal{G}_1)) \neq \emptyset \). This contradicts the fact that \( \Phi(\mathcal{G}_2) \), being conjugate to \( \Phi(\mathcal{G}_1) \), must also have a free element. The contradiction shows that any element of the image \( \Phi \) has nonempty fixed point set.

6. Proof of Theorem 3

We first prove Theorem 3 for \( p = 0 \). Assume \( r \neq 2 \) and let \( \Phi \) be a nontrivial homomorphism from \( F = \text{Diff}^r_+(S^1) \) to \( \text{Diff}^0(S^1) \). Since \( F \) is simple, \( \Phi \) is injective and the image of \( \Phi \) is contained in \( \text{Diff}^0_+(S^1) \). For any \( x \in S^1 \), denote by \( F_x \) the isotropy subgroup at \( x \).

**Proposition 6.1.** For any \( x \in S^1 \), the fixed point set \( \text{Fix}(\Phi(F_x)) \) is a singleton.
The proof uses Theorem 5.2 in [M'], which states as follows.

**Theorem 6.2.** Any nontrivial homomorphism from $\text{PSL}(2,\mathbb{R})$ to $\text{Diff}^0_+(S^1)$ is the conjugation of the standard inclusion by a homeomorphism $h$.

**Proof of Proposition 6.1.** Let $G_x = \text{Diff}^r_x(S^1 \setminus \{x\})$. First we shall show $\text{Fix}(\Phi(G_x)) \neq \emptyset$. Let $\{U_n\}_{n \in \mathbb{N}}$ be an increasing sequence of precompact open intervals in $S^1 \setminus \{x\}$ such that $\bigcup_n U_n = S^1 \setminus \{x\}$ and let $V_n$ be an open interval in $S^1 \setminus \{x\}$ disjoint from $U_n$. Then $\Phi(\text{Diff}^r_x(U_n))$ and $\Phi(\text{Diff}^r_x(V_n))$ satisfy the conditions of Proposition 3.1. Therefore $\text{Fix}(\Phi(\text{Diff}^r_x(U_n)))$ is nonempty. Since $\Phi(G_x) = \bigcup_n \Phi(\text{Diff}^r_x(U_n))$, we have $\text{Fix}(\Phi(G_x)) = \bigcap_n \text{Fix}(\Phi(\text{Diff}^r_x(U_n)))$. But the RHS is a decreasing intersection of nonempty compact subsets. Therefore we get $\text{Fix}(\Phi(G_x)) \neq \emptyset$.

Next let us show that $\text{Fix}(\Phi(G_x))$ is a singleton. Assume there are two distinct points $\xi_0$ and $\xi_1$ in $\text{Fix}(\Phi(G_x))$. By Theorem 6.2, there is a rotation $R \in \text{PSL}(2,\mathbb{R})$ such that $\Phi(R)(\xi_0) = \xi_1$. Then $\xi_1$ is left fixed both by $\Phi(G_x)$ and $\Phi(RG_xR^{-1}) = \Phi(G_{Rx})$, and hence by $\Phi(F)$, since $G_x$ and $G_{Rx}$ generate $F$. Especially $\Phi(\text{PSL}(2,\mathbb{R}))$ admits a global fixed point, contradicting Theorem 6.2.

Finally since $\Phi(G_x)$ is a normal subgroup of $\Phi(F_x)$ and since $\text{Fix}(\Phi(G_x))$ is a singleton, we have $\text{Fix}(\Phi(F_x)) = \text{Fix}(\Phi(G_x))$. □

Define a map $h' : S^1 \to S^1$ by $\{h'(x)\} = \text{Fix}(\Phi(F_x))$. Let us show that $h'$ coincides with a homeomorphism $h$ in Theorem 6.2. In fact, for any $x \in S^1$, consider a parabolic element $g \in \text{PSL}(2,\mathbb{R})$ such that $g(x) = x$. Then, by Theorem 6.2, $h(x)$ is the unique fixed point of $\Phi(g)$. Therefore $h(x) = h'(x)$.

Now for any $f \in F$, we have

$$\{h(f(x))\} = \text{Fix}(\Phi(F_{f(x)})) = \text{Fix}(\Phi(fF_xf^{-1})) = \text{Fix}(\Phi(f)\Phi(F_x)\Phi(f)^{-1})$$

$$= \Phi(f)\text{Fix}(\Phi(F_x)) = \Phi(f)\{h(x)\}.$$  
This shows that $h \circ f = \Phi(f) \circ h$ for any $f \in F$. Namely $\Phi$ is the conjugation by a homeomorphism $h$. This completes the proof for $p = 0$.

Let us consider the case $p \geq 2$. We follow an argument in [T], and show that the homeomorphism $h$ established above is in fact a $C^p$ diffeomorphism. Denote $H = \text{Diff}_p^r(S^1)$. First of all, since $h$ is locally monotone and thus of bounded variation, it is differentiable Lebesgue almost everywhere. But we have $h \circ R_t = f_t \circ h$ ($\forall t \in S^1$), where $R_t$ is the rotation by $t$ and $f_t \in H$. This shows that $h$ is differentiable everywhere, with nonvanishing derivative.

Let us show that for any point $x \in S^1$, $h$ is a $C^p$ diffeomorphism on some neighbourhood of $x$. Choose $f \in F$ such that $f(x) = x$ and $f^r(x) = \lambda \in (0,1)$. Then $h \circ f \circ h^{-1} \in H$ leaves $h(x)$ fixed and the derivative there is also $\lambda$. Notice that $f$ also belongs to $H$, since $r \geq p$. By the Sternberg linearization theorem (Theorem 2 of [S]), there is a $C^p$ diffeomorphism $k_1$ (resp. $k_2$) from a neighbourhood of $x$ (resp. $h(x)$) to a neighbourhood of $0$ in $\mathbb{R}$ such that $k_1 \circ f = L \circ k_1$ (resp. $k_2 \circ (h \circ f \circ h^{-1}) = L \circ k_2$), where $L$ is a linear map of $\mathbb{R}$.
with slope $\lambda$. (The assumption $p \neq 1$ is necessary for Theorem 2 of [S].) Then the composite $k_2 \circ h \circ k_1^{-1}$ is a local homeomorphism of $(\mathbb{R}, 0)$ commuting with $L$ and differentiable at $0$. It is easy to prove that $k_2 \circ h \circ k_1^{-1}$ is a linear map. (The linear contraction $L \times L$ of $\mathbb{R}^2$ leaves the graph of $k_2 \circ h \circ k_1^{-1}$ invariant.) Then $h$ is a $C^p$ diffeomorphism in a neighbourhood of $x$, as is required.

The proof of Theorem 4 is given by the same argument as above starting from the homeomorphism $h$ in Theorem 6.2.

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