GRADED QUANTUM GROUPS

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ABSTRACT. Starting from a Hopf algebra endowed with an action of a group \( \pi \) by Hopf automorphisms, we construct (by a “twisted” double method) a quasitriangular Hopf \( \pi \)-coalgebra. This method allows us to obtain non-trivial examples of quasitriangular Hopf \( \pi \)-coalgebras for any finite group \( \pi \) and for infinite groups \( \pi \) such as \( \text{GL}_n(k) \). In particular, we define the graded quantum groups, which are Hopf \( \pi \)-coalgebras for \( \pi = \mathbb{C}[[h]] \) and generalize the Drinfeld-Jimbo quantum enveloping algebras.

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Introduction

The aim of the present paper is to construct examples of quasitriangular Hopf group-coalgebras. These algebraic structures were introduced by Turaev [Tur00].

Let \( \pi \) be a group. The category of representations of a quasitriangular Hopf \( \pi \)-coalgebra is a braided \( \pi \)-category. Such categories are used in [Tur00] to construct 3-dimensional homotopy quantum field theory with target an Eilenberg-Mac Lane space of type \( K(\pi,1) \). Moreover, Hopf \( \pi \)-coalgebras are used in [Vir01, Vir02b] to construct Hennings-type and Kuperberg-type invariants of flat \( \pi \)-bundles over link complements and over \( 3 \)-manifolds.

Let \( \pi \) be a group. A Hopf \( \pi \)-coalgebra is a family \( H = \{ H_\alpha \}_{\alpha \in \pi} \) of algebras (over a field \( k \)) endowed with a comultiplication \( \Delta = \{ \Delta_{\alpha,\beta} : H_\alpha \otimes H_\beta \}_{\alpha,\beta \in \pi} \), a counit \( \varepsilon : H_1 \to k \), and an antipode \( S = \{ S_\alpha : H_\alpha \to H_{\alpha^{-1}} \}_{\alpha \in \pi} \) which verify some compatibility conditions. A crossing for \( H \) is a family of algebra isomorphisms \( \varphi = \{ \varphi_{\beta} : H_\alpha \to H_{\beta^{-1}} \}_{\alpha,\beta \in \pi} \) which preserves the comultiplication and the counit, and which yields an action of \( \pi \) in the sense that \( \varphi_{\beta} \varphi_{\beta'} = \varphi_{\beta \beta'} \). A crossed Hopf \( \pi \)-coalgebra \( H \) is quasitriangular when it is endowed with an \( R \)-matrix \( R = \{ R_{\alpha,\beta} \in H_\alpha \otimes H_\beta \}_{\alpha,\beta \in \pi} \) verifying some axioms (involving the crossing \( \varphi \) ) which generalize...
the classical ones given in [Dri87]. The case π = 1 is the standard setting of Hopf algebras.

Starting from a crossed Hopf π-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \), Zunino [Zun02] constructed a double \( Z(H) = \{ Z(H)_\alpha \}_{\alpha \in \pi} \) of \( H \) which is a quasitriangular Hopf π-coalgebra in which \( H \) is embedded. One has that \( Z(H)_\alpha = H_\alpha \otimes (\oplus_{\beta \in \pi} H_\beta) \) as a vector space. Unfortunately, each component \( Z(H)_\alpha \) is infinite-dimensional (unless \( H_\beta = 0 \) for all but a finite number of \( \beta \in \pi \)).

To obtain non-trivial examples of quasitriangular Hopf π-coalgebras with finite-dimensional components, we restrict ourself to a less general situation: our initial data is not any crossed Hopf π-coalgebra but a Hopf algebra endowed with an action of π by Hopf algebra automorphisms. Remark indeed that the component \( H_1 \) of a Hopf π-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is a Hopf algebra and that a crossing for \( H \) induces an action of \( \pi \) on \( H_1 \) by Hopf automorphisms.

In this paper, starting from a Hopf algebra \( A \) endowed with an action \( \phi : \pi \rightarrow \text{Aut}_{\mathrm{Hopf}}(A) \) of a group \( \pi \) by Hopf automorphisms, we construct a quasitriangular Hopf π-coalgebra \( D(A, \phi) = \{ D(A, \phi_\alpha) \}_{\alpha \in \pi} \). The algebra \( D(A, \phi_\alpha) \) is constructed in a manner similar to the Drinfeld double (in particular \( D(A, \phi_\alpha) = A \otimes A^* \) as a vector space) except that its multiplication is “twisted” by the Hopf automorphism \( \phi_\alpha : A \rightarrow A \). The algebra \( D(A, \text{id}_A) \) is the usual Drinfeld double. In general, the algebras \( D(A, \phi_\alpha) \) and \( D(A, \phi_\beta) \) are not isomorphic when \( \alpha \neq \beta \).

This method allows us to define non-trivial examples of quasitriangular Hopf π-coalgebras for any finite group \( \pi \) and for infinite groups \( \pi \) such as \( \text{GL}_n(k) \). In particular, given a complex simple Lie algebra \( g \) of rank \( n \), we define the graded quantum groups \( \{ U_h^n(g) \}_{\alpha \in \mathbb{C}^n} \) and \( \{ U_h^n(g) \}_{\alpha \in \mathbb{C}[\hbar]^n} \) which are crossed Hopf group-coalgebras. They are obtained as quotients of \( D(U_q^+(b_+), \phi) \) and \( D(U_h^+(b_+), \phi') \), where \( b_+ \) denotes the Borel subalgebra of \( g \), \( \phi \) is an action of \( \mathbb{C}^n \) by Hopf automorphisms of \( U_q(b_+) \), and \( \phi' \) is an action of \( \mathbb{C}[\hbar]^n \) by Hopf automorphisms of \( U_h(b_+) \). Furthermore, the crossed Hopf \( \mathbb{C}[\hbar]^n \)-coalgebra \( \{ U_h^n(g) \}_{\alpha \in \mathbb{C}[\hbar]^n} \) is quasitriangular.

The paper is organized as follows. In Section 1 we review the basic definitions and properties of Hopf π-coalgebras. In Section 2 we define the twisted double of a Hopf algebra \( A \) endowed with an action of a group \( \pi \) by Hopf automorphisms. In Section 3 we explore the case \( A = k[G] \) where \( G \) is a finite group. In Section 4 we give an example of a quasitriangular Hopf \( \text{GL}_n(k) \)-coalgebra. Finally, we define the graded quantum groups in Sections 5 and 6.

Throughout this paper, we let \( \pi \) be a group (with neutral element 1) and \( k \) be a field.

1. HOPF GROUP-COALGEBRAS

In this section, we review some definitions and properties concerning Hopf group-coalgebras. For a detailed treatment of the theory of Hopf group-coalgebras, we refer to [Vir02a].

1.1. Hopf π-coalgebras. A Hopf π-coalgebra (over \( \mathbb{C} \)) is a family \( H = \{ H_\alpha \}_{\alpha \in \pi} \) of \( \mathbb{C} \)-algebras endowed with a family \( \Delta = \{ \Delta_{\alpha, \beta} : H_{\alpha \beta} \rightarrow H_\alpha \otimes H_\beta \}_{\alpha, \beta \in \pi} \) of algebra homomorphisms (the comultiplication) and an algebra homomorphism \( \varepsilon : H_1 \rightarrow \mathbb{C} \).
(the counti) such that, for all $\alpha, \beta, \gamma \in \pi$,

\begin{align}
(1.1) \quad & (\Delta_{\alpha, \beta} \otimes \text{id}_{H_\gamma}) \Delta_{\alpha, \beta, \gamma} = (\text{id}_{H_\alpha} \otimes \Delta_{\beta, \gamma}) \Delta_{\alpha, \beta, \gamma}, \\
(1.2) \quad & (\text{id}_{H_\alpha} \otimes \varepsilon) \Delta_{\alpha, 1} = \text{id}_{H_\alpha} = (\varepsilon \otimes \text{id}_{H_\alpha}) \Delta_{1, \alpha},
\end{align}

and with a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in \pi}$ of $k$-linear maps (the antipode) which verifies that, for all $\alpha \in \pi$,

\begin{equation}
(1.3) \quad m_\alpha (S_{\alpha^{-1}} \otimes \text{id}_{H_\alpha}) \Delta_{\alpha^{-1}, \alpha} = \varepsilon 1_\alpha = m_\alpha (\text{id}_{H_\alpha} \otimes S_{\alpha^{-1}}) \Delta_{\alpha, \alpha^{-1}},
\end{equation}

where $m_\alpha : H_\alpha \otimes H_\alpha \to H_\alpha$ and $1_\alpha \in H_\alpha$ denote respectively the multiplication and unit element of $H_\alpha$.

When $\pi = 1$, one recovers the usual notion of a Hopf algebra. In particular $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a Hopf algebra.

Remark that the notion of a Hopf $\pi$-coalgebra is not self-dual and that if $H = \{H_\alpha\}_{\alpha \in \pi}$ is a Hopf $\pi$-coalgebra, then $\{\alpha \in \pi \mid H_\alpha \neq 0\}$ is a subgroup of $\pi$.

A Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is said to be of finite type if, for all $\alpha \in \pi$, $H_\alpha$ is finite-dimensional (over $k$). Note that it does not mean that $\oplus_{\alpha \in \pi} H_\alpha$ is finite-dimensional (unless $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$).

The antipode of a Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is anti-multiplicative: each $S_\alpha : H_\alpha \to H_{\alpha^{-1}}$ is an anti-homomorphism of algebras, and anti-comultiplicative: \( \varepsilon S_1 = \varepsilon \) and $\Delta_{\alpha^{-1}, \alpha} S_{\alpha \beta} = \sigma_{H_{\alpha^{-1}, H_{\beta^{-1}}}} (S_\alpha \otimes S_{\beta}) \Delta_{\alpha, \beta}$ for any $\alpha, \beta \in \pi$, see [Vir02a, Lemma 1.1].

The antipode $S = \{S_\alpha\}_{\alpha \in \pi}$ of $H = \{H_\alpha\}_{\alpha \in \pi}$ is said to be bijective if each $S_\alpha$ is bijective. As for Hopf algebras, the antipode of a finite type Hopf $\pi$-coalgebra is always bijective (see [Vir02a, Corollary 3.7(a)]).

We extend the Sweedler notation for the comultiplication of a Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ in the following way: for any $\alpha, \beta \in \pi$ and $h \in H_{\alpha \beta}$, we write $\Delta_{\alpha, \beta}(h) = \sum_{(h)} h_{(1, \alpha)} \otimes h_{(2, \beta)} \in H_\alpha \otimes H_\beta$, or shortly, if we leave the summation implicit, $\Delta_{\alpha, \beta}(h) = h_{(1, \alpha)} \otimes h_{(2, \beta)}$. The coassociativity of $\Delta$ gives that, for any $\alpha, \beta, \gamma \in \pi$ and $h \in H_{\alpha \beta \gamma}$,

\[ h_{(1, \alpha \beta)} (1, \alpha) \otimes h_{(1, \alpha \beta)} (2, \beta) \otimes h_{(2, \gamma)} = h_{(1, \alpha)} \otimes h_{(2, \beta \gamma)} (1, \beta) \otimes h_{(2, \beta \gamma)} (2, \gamma). \]

This element of $H_\alpha \otimes H_\beta \otimes H_\gamma$ is written as $h_{(1, \alpha)} \otimes h_{(2, \beta)} \otimes h_{(3, \gamma)}$. By iterating the procedure, we define inductively $h_{(1, \alpha \beta \gamma \cdots n)} \otimes \cdots \otimes h_{(1, \alpha \beta \gamma \cdots n)}$ for any $h \in H_{\alpha \beta \gamma \cdots n}$.

1.2. **Crossed Hopf $\pi$-coalgebras.** A Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is said to be crossed if it is endowed with a family $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta \alpha^{-1}}\}_{\alpha, \beta \in \pi}$ of algebra isomorphisms (the crossing) such that, for all $\alpha, \beta, \gamma \in \pi$,

\begin{align}
(1.4) \quad & (\varphi_\beta \otimes \varphi_\beta) \Delta_{\alpha, \gamma} = \Delta_{\beta \alpha \beta^{-1}, \gamma} \varphi_\beta, \\
(1.5) \quad & \varepsilon \varphi_\beta = \varepsilon, \\
(1.6) \quad & \varphi_\alpha \varphi_\beta = \varphi_{\alpha \beta}.
\end{align}

It is easy to check that $\varphi_1|_{H_\alpha} = \text{id}_{H_\alpha}$ and $\varphi_\beta S_\alpha = S_{\beta \alpha^{-1}} \varphi_\beta$ for all $\alpha, \beta \in \pi$.

1.3. **Quasitriangular Hopf $\pi$-coalgebras.** A crossed Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is said to be quasitriangular if it is endowed with a family $R = \{R_{\alpha, \beta} \in H_\alpha \otimes H_\beta\}_{\alpha, \beta \in \pi}$ of invertible elements (the $R$-matrix) such that, for all $\alpha, \beta, \gamma \in \pi$,
and $x \in H_{\alpha \beta}$.

\begin{align}
R_{\alpha, \beta} \cdot \Delta_{\alpha, \beta}(x) &= \sigma_{\beta, \alpha}(\varphi_{\alpha^{-1}} \otimes \text{id}_{H_{\alpha}}) \Delta_{\alpha, \beta \alpha^{-1}, \alpha}(x) \cdot R_{\alpha, \beta}, \\
(\text{id}_{H_{\alpha}} \otimes \Delta_{\beta, \gamma})(R_{\alpha, \beta, \gamma}) &= (R_{\alpha, \gamma})_{13} \cdot (R_{\alpha, \beta})_{127}, \\
(\Delta_{\alpha, \beta} \otimes \text{id}_{H_{\alpha}})(R_{\alpha, \beta, \gamma}) &= [(\text{id}_{H_{\alpha}} \otimes \varphi_{\beta^{-1}})(R_{\alpha, \beta \gamma \beta^{-1}})]_{13} \cdot (R_{\alpha, \gamma})_{123}, \\
(\varphi_{\beta} \otimes \varphi_{\beta})(R_{\alpha, \gamma}) &= R_{\beta \alpha \beta^{-1}, \beta \gamma \beta^{-1}},
\end{align}

where $\sigma_{\beta, \alpha}$ denotes the flip map $H_{\beta} \otimes H_{\alpha} \rightarrow H_{\alpha} \otimes H_{\beta}$ and, for $k$-spaces $P, Q$ and $r = \sum j p_{j} \otimes q_{j} \in P \otimes Q$, we set $r_{12} = r \otimes 1_{\gamma} \in P \otimes Q \otimes H_{\gamma}$, $r_{23} = 1_{\alpha} \otimes r \in H_{\alpha} \otimes P \otimes Q$, and $r_{13} = \sum j p_{j} \otimes 1_{\beta} \otimes q_{j} \in P \otimes H_{\beta} \otimes Q$.

Note that $R_{1,1}$ is a (classical) $R$-matrix for the Hopf algebra $H_{1}$.

When $\pi$ is abelian and $\varphi$ is trivial (that is, $\varphi_{\beta}|_{H_{\alpha}} = \text{id}_{H_{\alpha}}$ for all $\alpha, \beta \in \pi$), one recovers the definition of a quasitriangular $\pi$-colored Hopf algebra given by Ohtsuki in [Oht93].

The $R$-matrix always verifies (see [Vir02a, Lemma 6.4]) that, for any $\alpha, \beta, \gamma \in \pi$,

\begin{align}
(\varepsilon \otimes \text{id}_{H_{\alpha}})(R_{1, \alpha}) &= 1_{\alpha} = (\text{id}_{H_{\alpha}} \otimes \varepsilon)(R_{\alpha, 1}), \\
(S_{\alpha^{-1}} \varphi_{\alpha} \otimes \text{id}_{H_{\alpha}})(R_{\alpha^{-1}, \beta}) &= R_{\alpha, \beta}^{-1} \\
(S_{\alpha} \otimes \text{id}_{H_{\alpha}})(R_{\alpha, \beta}) &= (\varphi_{\alpha} \otimes \text{id}_{H_{\beta}^{-1}})(R_{\alpha^{-1}, \beta^{-1}}),
\end{align}

and provides a solution of the $\pi$-colored Yang-Baxter equation:

\begin{align}
(R_{\beta, \gamma})_{\alpha 23} \cdot (R_{\alpha, \gamma})_{13} \cdot (R_{\alpha, \beta})_{127} \\
= (R_{\beta, \gamma})_{127} \cdot [(\text{id}_{H_{\alpha}} \otimes \varphi_{\beta^{-1}})(R_{\alpha, \beta \gamma \beta^{-1}})]_{13} \cdot (R_{\beta, \gamma})_{\alpha 23}.
\end{align}

### 1.4. Ribbon Hopf $\pi$-coalgebras

A quasitriangular Hopf $\pi$-coalgebra $H = \{H_{\alpha}\}_{\alpha \in \pi}$ is said to be ribbon if is it endowed with a family $\theta = \{\theta_{\alpha} \in H_{\alpha}\}_{\alpha \in \pi}$ of invertible elements (the twist) such that, for any $\alpha, \beta \in \pi$,

\begin{align}
\varphi_{\alpha}(x) &= \theta_{\alpha^{-1}} x \theta_{\alpha} \quad \text{for all } x \in H_{\alpha}, \\
S_{\alpha}(\theta_{\alpha}) &= \theta_{\alpha^{-1}}, \\
\varphi_{\beta}(\theta_{\alpha}) &= \theta_{\beta \alpha^{-1}}, \\
\Delta_{\alpha, \beta}(\theta_{\alpha}) &= (\theta_{\alpha} \otimes \theta_{\beta}) \cdot \sigma_{\beta, \alpha}((\varphi_{\alpha^{-1}} \otimes \text{id}_{H_{\alpha}})(R_{\alpha \beta \alpha^{-1}, \alpha})) \cdot R_{\alpha, \beta}.
\end{align}

Note that $\theta_{1}$ is a (classical) twist of the quasitriangular Hopf algebra $H_{1}$.

### 1.5. Hopf $\pi$-coideals

Let $H = \{H_{\alpha}\}_{\alpha \in \pi}$ be a Hopf $\pi$-coalgebra. A Hopf $\pi$-coideal of $H$ is a family $I = \{I_{\alpha}\}_{\alpha \in \pi}$, where each $I_{\alpha}$ is an ideal of $H_{\alpha}$, such that, for any $\alpha, \beta \in \pi$,

\begin{align}
\Delta_{\alpha, \beta}(I_{\alpha \beta}) &\subset I_{\alpha} \otimes H_{\beta} + H_{\alpha} \otimes I_{\beta}, \\
\varepsilon(I_{1}) &= 0, \\
S_{\alpha}(I_{\alpha}) &\subset I_{\alpha^{-1}}.
\end{align}

The quotient $\tilde{H} = \{\tilde{H}_{\alpha} = H_{\alpha}/I_{\alpha}\}_{\alpha \in \pi}$, endowed with the induced structure maps, is then a Hopf $\pi$-coalgebra. If $H$ is furthermore crossed, with a crossing $\varphi$ such that, for any $\alpha, \beta \in \pi$,

\begin{align}
\varphi_{\beta}(I_{\alpha}) &\subset I_{\alpha \beta \alpha^{-1}},
\end{align}

then so is $\tilde{H}$ (for the induced crossing).
2. Twisted double of Hopf algebras

In this section, we give a method (the twisted double) of defining a quasitriangular Hopf $\pi$-coalgebra from a Hopf algebra endowed with an action of a group $\pi$ by Hopf automorphisms.

2.1. Hopf pairings. Recall that a Hopf pairing between two Hopf algebras $A$ and $B$ (over $\mathbb{k}$) is a bilinear pairing $\sigma: A \times B \to \mathbb{k}$ such that, for all $a,a' \in A$ and $b,b' \in B$,

\begin{align}
(2.1) & \quad \sigma(a, bb') = \sigma(a, b)\sigma(a, b'), \\
(2.2) & \quad \sigma(aa', b) = \sigma(a, b_{(2)})\sigma(a_{(1)}, b_{(1)}), \\
(2.3) & \quad \sigma(a, 1) = \varepsilon(a) \quad \text{and} \quad \sigma(1, b) = \varepsilon(b).
\end{align}

Note that such a pairing always verifies that, for any $a \in A$ and $b \in B$,

\begin{equation}
(2.4) \quad \sigma(S(a), S(b)) = \sigma(a, b).
\end{equation}

(Since both $\sigma$ and $\sigma \circ (S \times S)$ are the inverse of $\sigma \circ (\text{id} \times S)$ in the algebra $\text{Hom}_\mathbb{k}(A \times B, \mathbb{k})$ endowed with the convolution product).

Let $\sigma: A \times B \to \mathbb{k}$ be a Hopf pairing. Its annihilator ideals are $I_A = \{a \in A \mid \sigma(a, b) = 0 \text{ for all } b \in B\}$ and $I_B = \{b \in B \mid \sigma(a, b) = 0 \text{ for all } a \in A\}$. It is easy to check that $I_A$ and $I_B$ are Hopf ideals of $A$ and $B$, respectively. Recall that $\sigma$ is said to be non-degenerate if $I_A$ and $I_B$ are both reduced to 0. A degenerate Hopf pairing $\sigma: A \times B \to \mathbb{k}$ induces (by passing to the quotients) a Hopf pairing $\bar{\sigma}: A/I_A \times B/I_B \to \mathbb{k}$ which is non-degenerate.

Most of Hopf algebras we shall consider in the sequel will be defined by generators and relations. The following provides us with a method of constructing Hopf pairings, see [Dae93, KRT97].

Let $A$ (resp. $B$) be a free algebra generated by elements $a_1, \ldots, a_p$ (resp. $b_1, \ldots, b_q$) over $\mathbb{k}$. Suppose that $A$ and $B$ have Hopf algebra structures such that each $\Delta(a_i)$ for $1 \leq i \leq p$ (resp. $\Delta(b_j)$ for $1 \leq i \leq q$) is a linear combination of tensors $a_r \otimes a_s$ (resp. $b_r \otimes b_s$). Given $pq$ scalars $\lambda_{i,j} \in \mathbb{k}$ with $1 \leq i \leq p$ and $1 \leq j \leq q$, there is a unique Hopf pairing $\sigma: A \times B \to \mathbb{k}$ such that $\sigma(a_i, b_j) = \lambda_{i,j}$.

Suppose now that $A$ (resp. $B$) is the algebra obtained as the quotient of $\tilde{A}$ (resp. $\tilde{B}$) by the ideal generated by elements $r_1, \ldots, r_m \in \tilde{A}$ (resp. $s_1, \ldots, s_n \in \tilde{B}$). Suppose also that the Hopf algebra structure in $\tilde{A}$ (resp. $\tilde{B}$) induces a Hopf algebra structure in $A$ (resp. $B$). Then a Hopf pairing $\sigma: A \times B \to \mathbb{k}$ induces a Hopf pairing $A \times B \to \mathbb{k}$ if and only if $\sigma(r_i, b_j) = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq q$, and $\sigma(a_i, s_j) = 0$ for all $1 \leq i \leq p$ and $1 \leq j \leq n$.

2.2. The twisted double construction.

Definition-Lemma 2.1. Let $\sigma: A \times B \to \mathbb{k}$ be Hopf pairing between two Hopf algebras $A$ and $B$. Let $\phi: A \to A$ be a Hopf algebra endomorphism of $A$. Set $D(A, B; \sigma, \phi) = A \otimes B$ as a $\mathbb{k}$-space. Then $D(A, B; \sigma, \phi)$ has a structure of an associative and unitary algebra given, for any $a, a' \in A$ and $b, b' \in B$, by

\begin{align}
(2.5) & \quad (a \otimes b) \cdot (a' \otimes b') = \sigma(\phi(a'_{(1)}), S(b_{(1)}))\sigma(a_{(1)}, b_{(1)}) aa'_{(2)} \otimes b_{(2)} b', \\
(2.6) & \quad 1_{D(A, B; \sigma, \phi)} = 1_A \otimes 1_B.
\end{align}
Moreover, the linear embeddings $A \hookrightarrow D(A, B; \sigma, \phi)$ and $B \hookrightarrow D(A, B; \sigma, \phi)$ defined by $a \mapsto a \otimes 1_B$ and $b \mapsto 1_A \otimes b$, respectively, are algebra morphisms.

Remarks 2.2. (a) Note that $D(A, B; \sigma, \text{id}_A)$ is the underlying algebra of the usual quantum double of $A$ and $B$ (obtained by using the Hopf pairing $\sigma$).

(b) If $\phi$ and $\phi'$ are different Hopf algebra endomorphisms of $A$, then the algebras $D(A, B; \sigma, \phi)$ and $D(A, B; \sigma, \phi')$ are not in general isomorphic, see Remark 4.2.

Proof. Let $a, a', a'' \in A$ and $b, b', b'' \in B$. Using the fact that $\sigma$ is a Hopf pairing and $\phi$ is a Hopf algebra endomorphism, we have that

\[
((a \otimes b) \cdot (a' \otimes b')) \cdot (a'' \otimes b'') \\
= \sigma(\phi(a'_{(1)}), S(b_{(1)})) \sigma(a'_{(3)}, b_{(3)}) \phi(\phi(a''_{(1)}), S(b_{(2)})) \\
\quad \cdot \sigma(a''_{(3)}, b_{(3)}b'_{(3)}) a a'' \otimes b b' \otimes b'' \\
= \sigma(\phi(a'_{(1)}), S(b_{(1)})) \sigma(a'_{(3)}, b_{(3)}) \phi(\phi(a''_{(1)}), S(b_{(2)})) \\
\quad \cdot \sigma(a''_{(3)}, b_{(3)}b'_{(3)}) a a'' \otimes b b' \otimes b'' \\
\quad \cdot \sigma(a''_{(3)}, b_{(3)}b'_{(3)}) a a'' \otimes b b' \otimes b''
\]

and

\[
(a \otimes b) \cdot ((a' \otimes b') \cdot (a'' \otimes b'')) \\
= \sigma(\phi(a'_{(1)}), S(b'_{(1)})) \sigma(a''_{(3)}, b'_{(3)}) \phi(\phi(a''_{(1)}), S(b'_{(2)})) \\
\quad \cdot \sigma(a''_{(3)}, b'_{(3)}b''_{(3)}) a a'' \otimes b b' \otimes b'' \\
= \sigma(\phi(a'_{(1)}), S(b'_{(1)})) \sigma(a''_{(3)}, b'_{(3)}) \phi(\phi(a''_{(1)}), S(b'_{(2)})) \\
\quad \cdot \sigma(a''_{(3)}, b'_{(3)}b''_{(3)}) a a'' \otimes b b' \otimes b'' \\
\quad \cdot \sigma(a''_{(3)}, b'_{(3)}b''_{(3)}) a a'' \otimes b b' \otimes b''
\]

Hence the product is associative. Finally, $1_A \otimes 1_B$ is the unit element since

\[
(a \otimes b) \cdot (1 \otimes 1) = \sigma(\phi(1), S(b_{(1)})) \sigma(1, b_{(3)}) a \otimes b_{(2)} = \varepsilon(S(b_{(1)})) \varepsilon(b_{(3)}) a \otimes b_{(2)} = a \otimes b,
\]

and

\[
(1 \otimes 1) \cdot (a \otimes b) = \sigma(\phi(a_{(1)}), S(1)) \sigma(a_{(3)}, 1) a_{(2)} \otimes b = \varepsilon(\phi(a_{(1)})) \varepsilon(a_{(3)}) a_{(2)} \otimes b = a \otimes b.
\]

Finally, for any $a, a' \in A$ and $b, b' \in B$, we have that

\[
(a \otimes 1) \cdot (a' \otimes 1) = \sigma(\phi(a'_{(1)}), S(1)) \sigma(a'_{(3)}, 1) a a'_{(2)} \otimes 1
\quad = \varepsilon(\phi(a'_{(1)})) \varepsilon(a'_{(3)}) a a'_{(2)} \otimes 1
\quad = a a' \otimes 1
\]

and

\[
(1 \otimes b) \cdot (1 \otimes b') = \sigma(\phi(1), S(b_{(1)})) \sigma(1, b_{(3)}) 1 \otimes b_{(2)} b'
\quad = \varepsilon(S(b_{(1)})) \varepsilon(b_{(3)}) 1 \otimes b_{(2)} b' = 1 \otimes bb'.
\]

Therefore $A \hookrightarrow D(A, B; \sigma, \phi)$ and $B \hookrightarrow D(A, B; \sigma, \phi)$ are algebra morphisms.

In the sequel, the group of Hopf automorphisms of a Hopf algebra $A$ will be denoted by $\text{Aut}_{\text{Hopf}}(A)$. 

Theorem 2.3. Let $\sigma: A \times B \to k$ be Hopf pairing between two Hopf algebras $A$ and $B$, and $\phi: \pi \to \text{Aut}_{\text{Hopf}}(A)$ be group homomorphism (that is, an action of $\pi$ on $A$ by Hopf automorphisms). Then the family of algebras $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_a)\}_{a \in \pi}$ (see Definition 2.4) has a structure of a Hopf $\pi$-coalgebra given, for any $a \in A$, $b \in B$, and $\alpha, \beta \in \pi$, by

\begin{align*}
\Delta_{\alpha, \beta}(a \otimes b) &= (\phi_{\beta}(a_{(1)}) \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}), \\
\epsilon(a \otimes b) &= \epsilon_A(a) \epsilon_B(b), \\
S_{\alpha}(a \otimes b) &= \sigma(\varphi_{\alpha}(a_{(1)}), b_{(1)}) \sigma(a_{(3)}, S(b_{(3)})) \varphi_{\alpha} S(a_{(2)}) \otimes S(b_{(2)}).
\end{align*}

Proof. The coassociativity follows directly from the coassociativity of the coproducts of $A$ and $B$ and the fact that $\phi_{\beta \gamma} = \phi_{\beta} \circ \phi_{\gamma}$. Axiom (2) is a direct consequence of $\epsilon_A \circ \varphi_{\alpha} = \epsilon_A$. Since $\varphi_1 = \text{id}_A$ and $D(A, B; \sigma, \text{id}_A)$ is underlying algebra of the usual quantum double of $A$ and $B$, the counit $\epsilon$ is multiplicative. Let us verify that $\Delta_{\alpha, \beta}$ is multiplicative. Let $a, a' \in A$ and $b, b' \in B$. On one hand we have:

\begin{align*}
\Delta_{\alpha, \beta}((a \otimes b) \cdot (a' \otimes b')) &= \sigma(\phi_{\alpha \beta}(a'_{(1)}), S(b_{(1)})) \sigma(a'_{(3)}, b_{(3)}) \Delta_{\alpha, \beta}(a_{(2)} \otimes b_{(2)} b') \\
&= \sigma(\phi_{\alpha \beta}(a'_{(1)}), S(b_{(1)})) \sigma(a_{(4)}, b_{(4)}) \phi_{\beta}(a_{(1)} a'_{(2)}) \otimes b_{(2)} b'_{(1)} \otimes a_{(2)} a'_{(3)} \otimes b_{(3)} b'_{(2)}.
\end{align*}

One the other hand,

\begin{align*}
\Delta_{\alpha, \beta}(a \otimes b) \cdot \Delta_{\alpha, \beta}(a' \otimes b') &= (\phi_{\beta}(a_{(1)}) \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}) \cdot (\phi_{\beta}(a'_{(1)}) \otimes b'_{(1)} \otimes a'_{(2)} \otimes b'_{(2)}) \\
&= \sigma(\phi_{\alpha \beta}(a'_{(1)}), S(b_{(1)})) \sigma(\phi_{\beta}(a'_{(3)}), b_{(3)}) \sigma(\phi_{\beta}(a_{(4)}), S(b_{(4)})) \sigma(a_{(6)}, b_{(6)}) \\
&= \sigma(\phi_{\alpha \beta}(a'_{(1)}), S(b_{(1)})) \sigma(\phi_{\beta}(a_{(3)}), S(b_{(4)})) \sigma(a_{(5)}, b_{(6)}) \\
&= \sigma(\phi_{\alpha \beta}(a'_{(1)}), S(b_{(1)})) \sigma(a_{(4)} a'_{(2)} \otimes b_{(2)} b'_{(1)} \otimes a_{(2)} a'_{(3)} \otimes b_{(3)} b'_{(2)}).
\end{align*}

Let us verify the first equality of (2). Let $a \in A$, $b \in B$, and $\alpha \in \pi$. Denote the multiplication in $D(A, B; \sigma, \phi_a)$ by $m_{\alpha}$. We have

\begin{align*}
m_{\alpha}(S_{\alpha^{-1}} \otimes \text{id}_{D(A, B; \sigma, \phi_a)}) \Delta_{\alpha^{-1}, \alpha}(a \otimes b)
&= \sigma(a_{(1)}, b_{(1)}) \sigma(\phi_{\alpha}(a_{(3)}), S(b_{(5)})) \sigma(\phi_{\alpha}(a_{(4)}), S(b_{(6)})) \sigma(a_{(6)}, S(b_{(2)})) \\
&= \sigma(a_{(1)}, b_{(1)}) \sigma(\phi_{\alpha}(a_{(3)}), S(b_{(5)})) S^{2}(b_{(4)}) \sigma(a_{(5)}, S(b_{(2)})) \\
&= \sigma(a_{(1)}, b_{(1)}) \sigma(a_{(4)}, S(b_{(2)})) S(a_{(2)} a_{(4)} \otimes S(b_{(3)} b_{(4)})) \\
&= \sigma(a_{(1)}, b_{(1)}) \sigma(a_{(2)}, S(b_{(2)})) 1 \otimes 1 \\
&= \epsilon(a) \epsilon(b) 1 \otimes 1.
\end{align*}

The second equality of (2) can be verified similarly. \hfill \Box

Let $\sigma: A \times B \to k$ be a Hopf pairing between two Hopf algebras $A$ and $B$, and $\phi: \pi \to \text{Aut}_{\text{Hopf}}(A)$ be an action of $\pi$ on $A$ by Hopf automorphisms. An action
ψ: π → Aut_{Hopf}(B) of π on B by Hopf automorphisms is said to be (σ, φ)-compatible if, for all a ∈ A, b ∈ B and β ∈ π,
\begin{equation}
\sigma(\phi_\beta(a), \psi_\beta(b)) = \sigma(a, b).
\end{equation}

**Lemma 2.4.** Let σ: A × B → k be a Hopf pairing between two Hopf algebras A and B, and φ: π → Aut_{Hopf}(A), ψ: π → Aut_{Hopf}(B) be two actions of π by Hopf automorphisms on A and B, respectively. Suppose that ψ is (σ, φ)-compatible. Then the Hopf π-coalgebra D(A, B; σ, φ) = {D(A, B; σ, φ_α)}_{α ∈ π} (see Theorem 2.3) admits a crossing φ given, for any a ∈ A, b ∈ B and β ∈ π, by
\begin{equation}
φ_\beta(a ⊗ b) = φ_\beta(a) ⊗ ψ_\beta(b).
\end{equation}

**Proof.** Let α, β ∈ π. We have that φ_β(1_A ⊙ 1_B) = φ_β(1_A) ⊙ ψ_β(1_B) = 1_A ⊙ 1_B, and, for any a, a' ∈ A and b, b' ∈ B,
\begin{align*}
φ_β(a ⊙ b) &· φ_β(a' ⊙ b') \\
= &\sigma(φ_{αβ−1}(φ_α(a'_{(1)})), S(ψ_β(b_{(1)}))) \sigma(φ_α(a'_{(3)}), ψ_β(b_{(3)})) \phi_β(a)φ_β(a'_{(2)}) ⊙ ψ_β(b_{(2)})ψ_β(b') \\
= &\sigma(φ_α(a'_{(1)}), S(b_{(1)})) \sigma(φ_α(a'_{(3)}), ψ_β(b_{(3)})) \phi_β(a)φ_β(a'_{(2)}) ⊙ ψ_β(b_{(2)})ψ_β(b') \\
= &\sigma(φ_α(a'_{(1)}), S(b_{(1)})) \sigma(α'_{(3)}, b_{(3)}) \phi_β(aa'_{(2)}) ⊙ ψ_β(b_{(2)}b') \\
= &φ_β((a ⊙ b) · (a' ⊙ b')).
\end{align*}
Moreover φ_β and ψ_β are bijective and so is φ_β. Therefore φ_β : D(A, B; σ, φ_α) → D(A, B; σ, φ_αβ−1) is an algebra isomorphism.

Let a ∈ A, b ∈ B and α, β, γ ∈ π. We have that
\begin{align*}
\Delta_{αβ,γ−1}(φ_β(a ⊙ b)) & = φ_βγ−1(φ_β(a_{(1)})) ⊙ ψ_β(b_{(1)}) ⊙ φ_β(a_{(2)}) ⊙ ψ_β(b_{(2)}) \\
& = φ_βγ−1φ_β(a_{(1)}) ⊙ ψ_β(b_{(1)}) ⊙ φ_β(a_{(2)}) ⊙ ψ_β(b_{(2)}) \\
& = φ_βφ_γ(a_{(1)}) ⊙ ψ_β(b_{(1)}) ⊙ φ_β(a_{(2)}) ⊙ ψ_β(b_{(2)}) \\
& = (φ_β ⊙ φ_β)Δ_{α,γ}(a ⊙ b),
\end{align*}
\begin{align*}
εφ_β(a ⊙ b) &= ε(φ_β(a)) ε(ψ_β(b)) = ε(a) ε(b) = ε(a ⊙ b),
\end{align*}
and
\begin{align*}
φ_αφ_β(a ⊙ b) &= φ_αφ_β(a) ⊙ ψ_αψ_β(b) = φ_αβ(a) ⊙ ψ_αβ(b) = φ_αβ(a ⊙ b).
\end{align*}
Therefore φ satisfies Axioms 1.1, 1.3 and 1.6.

**Corollary 2.5.** Let σ: A × B → k be a non-degenerate Hopf pairing and φ: π → Aut_{Hopf}(A) be an action of π on A by Hopf automorphisms. Then there exists a unique action φ^*: π → Aut_{Hopf}(B) which is (σ, φ)-compatible. It is characterized, for any a ∈ A, b ∈ B and β ∈ π, by
\begin{equation}
σ(a, φ_β^*(b)) = σ(φ_{β−1}(a), b).
\end{equation}
Consequently the Hopf π-coalgebra D(A, B; σ, φ) = {D(A, B; σ, φ_α)}_{α ∈ π} (see Theorem 2.3) is crossed with crossing defined by φ_β = φ_β ⊙ φ_β^* for any β ∈ π.

**Proof.** Let β ∈ π. Since σ is non-degenerate, (2.12) does define a linear map φ_β^*: B → B. Since σ is a Hopf pairing and φ_β−1 is a Hopf algebra isomorphism of A, the map φ_β^* is a Hopf algebra isomorphism of B. Moreover φ_β^* is an action since φ_β^* = id_B (because φ_1 = id_A) and, for any a ∈ A, b ∈ B and α, β ∈ π,
\begin{align*}
σ(a, φ_α^*(b)) &= σ(φ_{β−1α−1}(a), b) = σ(φ_{β−1φ_α−1}(a), b) \\
&= σ(φ_{α−1}(a), φ_β^*(b)) = σ(a, φ_αφ_β^*(b)).
\end{align*}
Theorem 2.6. Let $B$ and $\phi$ be Hopf algebras with finite dimensional homogeneous components and that $1_B$ is invertible in $B$. Suppose that $\sigma$ is non-degenerate and that $A$ (and so $B$) is finite dimensional. Then the crossed Hopf $\pi$-coalgebra $D(A,B;\sigma,\phi) = \{D(A,B;\sigma,\phi_{\alpha})\}_{\alpha \in \pi}$ (see Corollary 2.5) is quasitriangular with $R$-matrix given, for all $\alpha, \beta \in \pi$, by

$$\sum_{i,j} (e_i \otimes 1_B) \otimes (1_A \otimes f_i),$$

where $(e_i)_i$ and $(f_i)_i$ are bases of $A$ and $B$, respectively, such that $\sigma(e_i, f_j) = \delta_{i,j}$.

**Remarks.**

(a) The element $\sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i) \in A \otimes B \otimes A \otimes B$ is canonical, i.e., independent of the choices of the basis $(e_i)_i$ of $A$ and $(f_i)_i$ of $B$ such that $\sigma(e_i, f_j) = \delta_{i,j}$.

(b) The hypothesis $A$ is finite dimensional is to ensure that the sum $\sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ lies in $A \otimes B \otimes A \otimes B$. More generally, assume that $A$ and $B$ are graded Hopf algebras with finite dimensional homogeneous components and that $\sigma$ is compatible with the gradings. Then the quotient Hopf algebras $A/I_A$ and $B/I_B$ are also graded and can be identified via $\sigma$ with the duals of each other. Suppose also that the action of $\phi$ respects the grading so does the quotient $\phi : \pi \to \text{Aut}_\text{Hopf}(A/I_A)$. In this case, there exists a unique action $\pi \to \text{Aut}_\text{Hopf}(B/I_B)$ which is $(\bar{\sigma}, \bar{\phi})$-compatible, where $\bar{\phi} : A/I_A \times B/I_B \to k$ is the induced Hopf pairing. Then the Hopf $\pi$-coalgebra $D(A/I_A, B/I_B; \bar{\pi}, \bar{\phi})$ is quasitriangular by the same construction as in Theorem 2.6.

**Proof.** Fix basis $(e_i)_i$ of $A$ and $(f_i)_i$ of $B$ such that $\sigma(e_i, f_j) = \delta_{i,j}$ (such basis always exist since $\sigma$ is non-degenerate). Note that $x = \sum_i \sigma(x, f_i) x$ and $y = \sum_i \sigma(e_i, y) y$ for any $x \in A$ and $y \in B$.

Recall that, since $\sum_i e_i \otimes 1_B \otimes 1_A \otimes f_i$ is the $R$-matrix of the usual quantum double $D(A,B,\sigma,\text{id}_A)$, we have

$$\sum_{i,j} S(e_i) e_j \otimes f_i f_j = 1_A \otimes 1_B,$$

$$\sum_i e_i \otimes f_i(1) \otimes f_i(2) = \sum_{i,j} e_i e_j \otimes f_j \otimes f_i,$$

$$\sum_{i,j} e_i(1) \otimes e_i(2) \otimes f_i = \sum_{i,j} e_i \otimes e_j \otimes f_i f_j.$$

Let $\alpha, \beta \in \pi$. From (2.14) and since $A$ (resp. $B$) can be viewed as a subalgebra of $D(A,B;\sigma,\phi_{\alpha})$ (resp. $D(A,B;\sigma,\phi_{\beta})$) via $a \mapsto a \otimes 1_B$ (resp. $b \mapsto 1_A \otimes b$), we get that $R_{\alpha,\beta}$ is invertible in $D(A,B;\sigma,\phi_{\alpha}) \otimes D(A,B;\sigma,\phi_{\beta})$ with inverse

$$R_{\alpha,\beta}^{-1} = \sum_i S(e_i) \otimes 1_B \otimes 1_A \otimes f_i.$$
Let \( a \in A \), \( b \in B \) and \( \alpha, \beta, \gamma \in \pi \). For all \( x \in A \), we have that
\[
(id_{A \otimes B} \otimes \varnothing(x, \cdot))(R_{\alpha, \beta} \cdot \Delta_{\alpha, \beta}(a \otimes b)) = \sum_i \sigma(\phi_\beta(a_{(2)}), S(f_{i(1)})) \sigma(a_{(4)}, f_{i(3)}) \sigma(x, f_{i(2)}) \sigma(b_{(2)}) e_i \phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(3)}
\]
\[
= \sum_i \sigma(\phi_\beta S^{-1}(a_{(2)}), f_{i(1)}) \sigma(a_{(4)}, f_{i(3)}) \sigma(x_{(1)}, f_{i(2)}) \sigma(x_{(2)}, b_{(2)}) e_i \phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(3)}
\]
\[
= \sum_i \sigma(a_{(4)} x_{(1)} \phi_\beta S^{-1}(a_{(2)}), f_i) \sigma(x_{(2)}, b_{(2)}) e_i \phi_\beta(a_{(1)}) \otimes b_{(1)} \otimes a_{(3)}
\]
\[
= \sigma(x_{(2)}, b_{(2)}) a_{(4)} x_{(1)} \phi_\beta(S^{-1}(a_{(2)}) a_{(1)}) \otimes b_{(1)} \otimes a_{(3)}
\]
\[
= \sigma(x_{(2)}, b_{(2)}) a_{(2)} x_{(1)} b_{(1)} \otimes a_{(1)}
\]
and, since \( x_{(1)} \otimes x_{(2)} \otimes x_{(2)} \otimes x_{(2)} = \sum_i \sigma(x_{(2)}, f_i) x_{(1)} \otimes e_i(1) \otimes e_i(2) \otimes e_i(3), \)
\[
(id_{A \otimes B} \otimes \varnothing(x, \cdot))(\sigma_\beta \otimes \sigma_\beta)(\varnothing_\alpha \otimes \sigma_\beta(x, \cdot))(\varnothing_\alpha \otimes \varnothing_\beta)(\varnothing_\beta \otimes \varnothing_\beta)(R_{\alpha, \beta}) = \sum_i \sigma(\phi_\beta(e_{i(1)}), S(f_{i(1)})) \sigma(e_{i(3)}, b_{(2)}) \sigma(x, \phi_\beta^*(b_{(2)}) f_i) a_{(2)} e_i(2) \otimes b_{(3)} \otimes a_{(1)}
\]
\[
= \sum_i \sigma(\phi_\beta(e_{i(1)}), S(b_{(2)})) \sigma(e_{i(3)}, b_{(2)}) \sigma(x, \phi_\beta^*(b_{(2)}) f_i) a_{(2)} e_i(2) \otimes b_{(3)} \otimes a_{(1)}
\]
\[
= \sum_i \sigma(\phi_\beta(x_{(2)}), S(b_{(2)})) \sigma(e_{i(3)}, b_{(2)}) \sigma(\phi_\beta(x_{(1)}), b_{(1)}) \sigma(x_{(2)}, f_i) a_{(2)} e_i(2) \otimes b_{(3)} \otimes a_{(1)}
\]
\[
= \sigma(\phi_\beta(x_{(1)}), b_{(1)} S(b_{(2)})) \sigma(e_{i(3)}, b_{(2)}) a_{(2)} x_{(2)} \otimes b_{(3)} \otimes a_{(1)}
\]
\[
= \sigma(x_{(2)}, b_{(2)}) a_{(2)} x_{(1)} b_{(1)} \otimes a_{(1)}
\]
Hence, since \( \varnothing \) is non-degenerate, Axiom \ref{1.16} is satisfied.

Let us verify Axioms \ref{1.18} and \ref{1.19}. Let \( \alpha, \beta, \gamma \in \pi \). Since \( \phi^* \) is \( (\sigma, \phi) \)-compatible (by definition), the basis \( (\phi_\beta(e_i))_i \) of \( A \) and \( (\phi_\beta^*(f_i))_i \) of \( B \) satisfy \( \sigma(\phi_\beta(e_i), \phi_\beta^*(f_j)) = \sigma(e_i, f_j) = \delta_{i,j} \). Therefore we have that
\[
(\varnothing_\beta \otimes \varnothing_\beta)(R_{\alpha, \gamma}) = \sum_i \phi_\beta(e_i) \otimes 1_B \otimes 1_A \otimes \phi_\beta^*(f_j) = R_{\beta, \gamma}^{-1, \alpha} \otimes R_{\alpha, \gamma}^{-1, \alpha}.
\]

Finally, let us check Axioms \ref{1.18} and \ref{1.13}. Let \( \alpha, \beta, \gamma \in \pi \). Using \ref{2.44}, we have
\[
(id_{D(A, B; \sigma, \phi_\alpha)} \otimes \Delta_{\beta, \gamma})(R_{\alpha, \beta}) = \sum_i e_i \otimes 1_B \otimes 1_A \otimes f_{i(1)} \otimes 1_A \otimes f_{i(2)}
\]
\[
= \sum_i e_i e_j \otimes 1_B \otimes 1_A \otimes f_j \otimes 1_A \otimes f_i
\]
\[
= (R_{\alpha, \gamma})_{1, \beta} \otimes (R_{\alpha, \beta})_{2, \gamma}
\]
Using \ref{2.44} and \ref{2.44}, we have
\[
(\Delta_{\alpha, \beta} \otimes id_{D(A, B; \sigma, \phi_\gamma)})(R_{\alpha, \beta}) = \sum_i \phi_\beta(e_{i(1)}) \otimes 1_B \otimes e_{i(2)} \otimes 1_B \otimes 1_A \otimes f_i
\]
\[
= \sum_i \phi_\beta(e_{i(1)}) \otimes 1_B \otimes e_j \otimes 1_B \otimes 1_A \otimes f_i f_j
\]
\[
= [(\varnothing_\beta \otimes id_{D(A, B; \sigma, \phi_\gamma)})(R_{\beta, \gamma}^{-1})]_{1, \beta} \otimes [(R_{\beta, \gamma}^{-1})_{2, \gamma}]
\]
This completes the proof of the quasiternity of \( D(A, B; \sigma, \phi). \) \qed

The following corollary is a direct consequence of Corollary 2.5 and Theorem 2.6.
Corollary 2.8. Let $A$ be a finite-dimensional Hopf algebra and $\phi: \pi \to \text{Aut}_{\text{Hopf}}(A)$ be an action of $\pi$ on $A$ by Hopf algebras automorphisms. Recall that the duality bracket $\langle \cdot, \cdot \rangle_{A^*}\cdot$ is a non-degenerate Hopf pairing between $A$ and $A^{*\text{cop}}$. Then $D(A, A^{*\text{cop}}; \langle \cdot, \cdot \rangle_{A^*}\cdot, \phi)$ is a quasitriangular Hopf $\pi$-coalgebra.

Remark 2.9. Recall that the group of Hopf automorphisms of a finite-dimensional semisimple Hopf algebra $A$ over a field of characteristic 0 is finite (see [Rad90]). To obtain non-trivial examples of (quasitriangular) Hopf $\pi$-coalgebras for an infinite group $\pi$ by using the twisted double method, one has to consider non-semisimple Hopf algebras (at least in characteristic 0).

2.3. The $h$-adic case. In this subsection, we develop the $h$-adic variant of Hopf group-coalgebras. A technical argument for the need of $h$-adic Hopf group-coalgebras is that they are necessary for a mathematically rigorous treatment of $R$-matrices for quantize enveloping algebras endowed with a group action.

Recall that if $V$ is a vector space over $\mathbb{C}[[h]]$, the topology on $V$ for which the sets $\{h^nV + v \mid n \in \mathbb{N}\}$ are a neighborhood base of $v \in V$ is called the $h$-adic topology. If $V$ and $W$ are vector spaces over $\mathbb{C}[[h]]$, we shall denote by $V \hat{\otimes} W$ the completion of the tensor product space $V \otimes_{\mathbb{C}[[h]]} W$ in the $h$-adic topology. Let $V$ be a complex vector space. Then the set $V[[h]]$ of all formal power series $f = \sum_{n=0}^{\infty} v_n h^n$ with coefficients $v_n \in V$ is a vector space over $\mathbb{C}[[h]]$ which is complete in the $h$-adic topology. Furthermore, $V[[h]] \hat{\otimes} W[[h]] = (V \hat{\otimes} W)[[h]]$ for any complex vector spaces $V$ and $W$.

An $h$-adic algebra is a vector space $A$ over $\mathbb{C}[[h]]$ which is complete in the $h$-adic topology and endowed with a $\mathbb{C}[[h]]$-linear map $m: A \hat{\otimes} A \to A$ and an element $1 \in A$ satisfying $m(id_A \hat{\otimes} m) = m(m \hat{\otimes} id_A)$ and $m(a \hat{\otimes} 1) = a = m(1 \hat{\otimes} a)$ for all $a \in A$.

By an $h$-adic Hopf $\pi$-coalgebra, we shall mean a family $H = \{H_{\alpha}\}_{\alpha \in \pi}$ of $h$-adic algebras which is endowed with $h$-adic algebra homomorphisms $\Delta_{\alpha,\beta}: H_{\alpha} \to H_{\alpha} \hat{\otimes} H_{\beta}$ ($\alpha, \beta \in \pi$) and $\varepsilon: A \to \mathbb{C}[[h]]$ satisfying (1.1) and (1.2), and with $\mathbb{C}[[h]]$-linear maps $S_\alpha: H_{\alpha} \to H_{\alpha^{-1}}$ ($\alpha \in \pi$) satisfying (1.3). In the previous axioms, one has to replace the algebraic tensor products $\otimes$ by the $h$-adic completions $\hat{\otimes}$.

The notions of crossed and quasitriangular $h$-adic Hopf $\pi$-coalgebras can be defined similarly as in Sections 1.2 and 1.3. The definitions of Section 2 and Theorem 2.8 carry over almost verbatim to $h$-adic Hopf algebras. The only modifications are that $\sigma: A \hat{\otimes} B \to \mathbb{C}[[h]]$ is $\mathbb{C}[[h]]$-linear and that the algebra $D(A, B; \sigma, \phi)$, where $\phi$ is an $h$-adic Hopf endomorphism of $A$, is built over the completion $A \hat{\otimes} B$ of $A \otimes B$ in the $h$-adic topology. The reasoning of the proof of Theorem 2.6 give the following $h$-adic version.

Theorem 2.10. Let $\sigma: A \hat{\otimes} B \to \mathbb{C}[[h]]$ be an $h$-adic Hopf pairing between two $h$-adic Hopf algebras $A$ and $B$, and $\phi: \pi \to \text{Aut}_{\text{Hopf}}(A)$ be an action of $\pi$ on $A$ by $h$-adic Hopf automorphisms. Suppose that $\sigma$ is non-degenerate. Let $(e_i)_{\pi}$ and $(f_i)_{\pi}$ be dual basis of the vector spaces $A$ and $B$, respectively, with respect to the form $\sigma$. If $R_{\alpha,\beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ belongs to the $h$-adic completion $D(A, B; \sigma, \phi_\alpha) \hat{\otimes} D(A, B; \sigma, \phi_\beta)$, then $R = \{R_{\alpha,\beta}\}_{\alpha,\beta \in \pi}$ is a $R$-matrix of the crossed $h$-adic Hopf $\pi$-coalgebra $D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_\alpha)\}_{\alpha \in \pi}$.

3. The case of algebras of finite groups

Let $G$ be a finite group. In this section, we describe the Hopf $G$-coalgebras obtained by the twisted double method from the Hopf algebra $k[G]$. 

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Recall that the Hopf algebra structure of the (finite-dimensional) \( k \)-algebra \( k[G] \) of \( G \) is given by \( \Delta(g) = g \otimes g \), \( \varepsilon(g) = 1 \) and \( S(g) = g^{-1} \) for all \( g \in G \). The dual of \( k[G] \) is the Hopf algebra \( F(G) = k^G \) of functions \( G \to k \). It has a basis \( (e_g : G \to k)_{g \in G} \) defined by \( e_g(h) = \delta_{g,h} \) where \( \delta_{g,h} = 1 \) and \( \delta_{g,h} = 0 \) if \( g \neq h \). The structure maps of \( F(G) \) are given by \( e_g e_h = \delta_{g,h} e_g \), \( 1_{F(G)} = \sum_{g \in G} e_g \), \( \Delta(e_g) = \sum_{xy = g} e_x \otimes e_y \), \( \varepsilon(e_g) = \delta_{g,1} \) and \( S(e_g) = e_{g^{-1}} \) for any \( g, h \in G \).

Set \( \phi : k[G] \to \text{Aut}_{\text{Hopf}}(k[G]) \) defined by \( \phi_g(h) = ghg^{-1} \). It is a well defined group homomorphism (since any \( g \in G \) is group-like in \( k[G] \)). By Corollary 2.8, this data leads to a quasitriangular Hopf \( G \)-coalgebra \( D(k[G], F(G); \langle \cdot, \cdot \rangle_{k[G]} \times F(G), \phi) \) which will be denoted by \( D_G(G) = \{ D_\alpha(G) \}_{\alpha \in G} \).

Let us describe \( D_G(G) \) more precisely. For any \( \alpha \in G \), the algebra structure of \( D_\alpha(G) \), which is equal to \( k[G] \otimes F(G) \) as a \( k \)-space, is given by

\[
(g \otimes e_h) \cdot (g' \otimes e_{h'}) = \delta_{\alpha g', \alpha^{-1} h^{-1} g' h'} gg' \otimes e_{h'} \quad \text{for all } g, g', h, h' \in G,
\]

\[
1_{D_\alpha(G)} = \sum_{g \in G} 1 \otimes e_g.
\]

The structure maps of \( D_G(G) \) are given, for any \( \alpha, \beta \in G \) and \( g, h \in G \), by

\[
\Delta_{\alpha,\beta}(g \otimes e_h) = \sum_{xy = h} \beta g \beta^{-1} \otimes e_y \otimes g \otimes e_x,
\]

\[
\varepsilon(g \otimes e_h) = \delta_{h,1},
\]

\[
S_\alpha(g \otimes e_h) = \alpha g^{-1} \alpha^{-1} \otimes e_{\alpha^{-1} g^{-1} h^{-1}},
\]

\[
\varphi_\alpha(g \otimes e_h) = \alpha g^{-1} \otimes e_{\alpha^{-1}},
\]

The crossed Hopf \( G \)-coalgebra \( D_G(G) \) is quasitriangular and furthermore ribbon with \( R \)-matrix and twist given, for any \( \alpha, \beta \in G \), by

\[
R_{\alpha,\beta} = \sum_{g, h \in G} g \otimes e_h \otimes 1 \otimes e_g \quad \text{and} \quad \theta_\alpha = \sum_{g \in G} \alpha^{-1} g \alpha \otimes e_g.
\]

Note that \( \theta_\alpha^n = \sum_{g \in G} \alpha^{-n}(g \alpha)^n \otimes e_g \) for any \( n \in \mathbb{Z} \).

4. Example of a quasitriangular Hopf \( GL_n(k) \)-coalgebra

In this section, \( k \) is a field whose characteristic is not 2. Fix a positive integer \( n \). We use a (finite dimensional) Hopf algebra whose group of automorphisms is known to be the group \( GL_n(k) \) of invertible \( n \times n \)-matrices with coefficients in \( k \) (see [Rad90]) to derive an example of a quasitriangular Hopf \( GL_n(k) \)-coalgebra.

**Definition-Proposition 4.1.** For \( \alpha = (\alpha_{i,j}) \in GL_n(k) \), let \( A_n^\alpha \) be the \( \mathbb{C} \)-algebra generated \( g, x_1, \ldots, x_n, y_1, \ldots, y_n \), subject to the following relations

\[
(1) \quad g^2 = 1, \quad x_i^2 = \cdots = x_n^2 = 0, \quad gx_i = -x_i g, \quad x_ix_j = -x_j x_i,
\]

\[
(2) \quad y_i^2 = \cdots = y_n^2 = 0, \quad gy_i = -y_i g, \quad y_i y_j = -y_j y_i,
\]

\[
(3) \quad x_i y_j - y_j x_i = (\delta_{i,j} - \alpha_{i,j}) g,
\]

where \( 1 \leq i, j \leq n \). The family \( \mathcal{A}_n = \{ A_n^\alpha \}_{\alpha \in GL_n(k)} \) has a structure of a crossed Hopf \( GL_n(k) \)-coalgebra given, for any \( \alpha = (\alpha_{i,j}) \in GL_n(k) \), \( \beta = (\beta_{i,j}) \in GL_n(k) \),
and 1 ≤ i ≤ n, by
\[ \Delta_{\alpha,\beta}(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S_{\alpha}(g) = g, \]
(4.4)
\[ \Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^{n} \beta_{k,i} x_k \otimes g, \quad \varepsilon(x_i) = 0, \quad S_{\alpha}(x_i) = \sum_{k=1}^{n} \alpha_{k,i} g x_k, \]
(4.5)
\[ \Delta_{\alpha,\beta}(y_i) = y_i \otimes 1 + g \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S_{\alpha}(y_i) = -h y_i, \]
(4.6)
\[ \varphi_{\alpha}(g) = g, \quad \varphi_{\alpha}(x_i) = \sum_{k=1}^{n} \alpha_{k,i} x_k, \quad \varphi_{\alpha}(y_i) = \sum_{k=1}^{n} \alpha_{k,i} y_k, \]
(4.7)
where \((\hat{\alpha}_{i,j}) = \alpha^{-1}\). Moreover \(A_n\) is quasitriangular with \(R\)-matrix given, for any \(\alpha, \beta \in \text{GL}_n(k)\), by
\[ R_{\alpha,\beta} = \frac{1}{2} \sum_{S \subseteq [n]} x_S \otimes y_S + x_S \otimes g y_S + g x_S \otimes y_S + g x_S \otimes g y_S. \]

Here \([n] = \{1, \ldots, n\}\), \(x_0 = 1, y_0 = 1\), and, for a nonempty subset \(S\) of \([n]\), we let \(x_S = x_{i_1} \cdots x_{i_s}\) and \(y_S = y_{i_1} \cdots y_{i_s}\) where \(i_1 < \cdots < i_s\) are the elements of \(S\).

**Remark 4.2.** From relations (4.4), it can be shown that the algebras \(A_n^\alpha\) and \(A_n^\beta\) are in general not isomorphic when \(\alpha, \beta \in \text{GL}_n(k)\) are such that \(\alpha \neq \beta\).

**Proof.** Let \(A_n\) be the \(k\)-algebra generated by \(g, x_1, \ldots, x_n\) which satisfy the relations (4.1). The algebra \(A_n\) is 2\(^{n+1}\)-dimensional and has a Hopf algebra structure given by
\[ \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g, \]
\[ \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \varepsilon(x_i) = 0, \quad S(x_i) = g x_i. \]
Radford [Rad90] showed that the group of Hopf automorphisms of \(A_n\) is isomorphic to the group \(\text{GL}_n(k)\) of invertible \(n \times n\)-matrices with coefficients in \(k\). This group automorphism \(\phi: \text{GL}_n(k) \to \text{Aut}_{\text{Hopf}}(A_n)\) is given, for any \(\alpha = (\alpha_{i,j}) \in \text{GL}_n(k)\), by
\[ \phi_{\alpha}(g) = g \quad \text{and} \quad \phi_{\alpha}(x_i) = \sum_{k=1}^{n} \alpha_{k,i} x_k. \]
The Hopf algebra \(B_n = A_n^{\text{cop}}\) is the \(k\)-algebra generated by the symbols \(h, y_1, \ldots, y_n\) which satisfy the relations \(h^2 = 1\) and \(h y_i = 0\) and its Hopf algebra structure is given by
\[ \Delta(h) = h \otimes h, \quad \varepsilon(h) = 1, \quad S(h) = h, \]
\[ \Delta(y_i) = y_i \otimes 1 + h \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S(y_i) = -h y_i. \]
Let us denote the cardinality of a set \(T\) by \(|T|\). The elements \(g^k x_S\) (resp. \(h^k y_S\)), where \(k \in \{0, 1\}\) and \(S \subseteq [n]\), form a basis for \(A_n\) (resp. \(B_n\)). Since \(\Delta\) is multiplicative, it follows that
\[ \Delta(g^k x_S) = \sum_{T \subseteq S} \lambda_{T,S} g^{k+|T|} x_{S \setminus T} \quad \text{and} \]
\[ \Delta(h^k y_S) = \sum_{T \subseteq S} \lambda_{T,S} h^{k+|T|} y_{S \setminus T} \otimes h^k y_T, \]
where \(\lambda_{T,S} = \pm 1\) and \(\lambda_{\emptyset,S} = 1 = \lambda_{S,S}\).
By Section 2.1, there exists a (unique) Hopf pairing \( \sigma: A_n \times B_n \to k \) such that, for any \( 1 \leq i, j \leq n \),
\[
\sigma(g, h) = -1, \quad \sigma(g, y_j) = \sigma(x_i, h) = 0 \quad \text{and} \quad \sigma(x_i, y_j) = \delta_{i,j}.
\]
Using (118) and (119), one gets (by induction on \( |S| \)) that
\[
\sigma(g^k x_S, h^l y_T) = (-1)^{kl} \delta_{S,T}
\]
for any \( k, l \in \{0, 1\} \) and \( S, T \subseteq [n] \), where \( \delta_{S,S} = 1 \) and \( \delta_{S,T} \) if \( S \neq T \). Set \( z_0 = (1 + h)/2 \) and \( z_1 = (1 - h)/2 \). The elements \( z_k y_S \), where \( k \in \{0, 1\} \) and \( S \subseteq [n] \), form a basis for \( B_n \) such that
\[
(4.10) \quad \sigma(g^k x_S, z_l y_T) = \delta_{k,l} \delta_{S,T}
\]
for any \( k, l \in \{0, 1\} \) and \( S, T \subseteq [n] \). Therefore the pairing \( \sigma \) is non-degenerate. Note that this implies that \( A_n^* \cong A_n \) as a Hopf algebra.

By Theorem 2.6, we get a quasitriangular Hopf \( GL_n(k) \)-coalgebra \( D(A_n, B_n; \sigma, \phi) \). For any \( \alpha = (\alpha_{i,j}) \in GL_n(k) \), \( D(A_n, B_n; \sigma, \phi) \) is the algebra generated by \( g, h, x_1, \ldots, x_n, y_1, \ldots, y_n \), subject to the relations \( h^2 = 1 \), (4.10), and (4.11) and the following relations
\[
\sigma(g, h) = 1, \quad \sigma(g, y_j) = \sigma(x_i, h) = \sigma(1, y_j) = 0, \quad \text{relation (2.30) gives}
\]
\[
y_j x_i = \sigma(\phi(1), x_i, y_j) = \sigma(g, y_j) = \sigma(1, y_j) = 0, \quad \text{and relation (4.12) gives}
\]
\[
x_i y_j - y_j x_i = \delta_{i,j} h - \alpha_{i,j} g.
\]
Indeed \( D(A_n, B_n; \sigma, \phi) \) is the free algebra generated by the algebras \( A_n \) and \( B_n \) with cross relation (2.6). Further, it suffices to require the cross relations (2.6) for \( (1 \otimes b) \cdot (a \otimes 1) \) with \( a = g, x_i \) and \( b = h, y_j \). To simplify the notations, we identify \( a \) with \( a \otimes 1 \) and of \( b \) with \( 1 \otimes b \). (Recall that these natural maps \( A_n \to D(A_n, B_n; \sigma, \phi) \) and \( B_n \to D(A_n, B_n; \sigma, \phi) \) are algebra monomorphisms.) For example, let \( a = x_i \) and \( b = y_j \). Since \( \sigma(x_i, 1) = \sigma(g, y_j) = \sigma(x_i, h) = \sigma(1, y_j) = 0 \), relation (2.30) gives
\[
y_j x_i = \sigma(\phi_\alpha(1), x_i, y_j) = \sigma(g, y_j) = \sigma(1, y_j) = 0, \quad \text{and relation (4.12) gives}
\]
\[
x_i y_j - y_j x_i = \delta_{i,j} h - \alpha_{i,j} g.
\]
Inserting the values \( \sigma(g, 1) = \sigma(1, h) = 1, \quad \sigma(x_i, y_j) = \delta_{i,j} \), and \( \sigma(\phi_\alpha(x_i), y_j h) = -\alpha_{i,j} \), we get (4.12).

From Theorem 2.8, we obtain that the comultiplication \( \Delta_{\alpha, \beta} \), the counit \( \varepsilon \), the antipode \( S_\alpha \), and the crossing \( \varphi_\alpha \) of \( D(A_n, B_n; \sigma, \phi) \) are given by
\[
\Delta_{\alpha, \beta}(g) = g \otimes g, \quad \Delta_{\alpha, \beta}(h) = h \otimes h,
\]
\[
\Delta_{\alpha, \beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^n \beta_{k,i} x_k \otimes g, \quad \Delta_{\alpha, \beta}(y_i) = y_i \otimes 1 + h \otimes y_i,
\]
\[
\varepsilon(g) = \varepsilon(h) = 1, \quad \varepsilon(x_i) = \varepsilon(y_i) = 0, \quad S_\alpha(g) = g,
\]
\[
S_\alpha(h) = h, \quad S_\alpha(x_i) = \sum_{k=1}^n \alpha_{k,i} x_k, \quad S_\alpha(y_i) = -h y_i,
\]
\[
\varphi_\alpha(g) = g, \quad \varphi_\alpha(h) = h, \quad \varphi_\alpha(x_i) = \sum_{k=1}^n \alpha_{k,i} x_k, \quad \varphi_\alpha(y_i) = \sum_{k=1}^n \tilde{\alpha}_{k,i} y_k,
\]
where \( (\tilde{\alpha}_{i,j}) = \alpha^{-1} \).

For any \( \alpha \in GL_n(k) \), let \( I_\alpha \) be the ideal of \( D(A_n, B_n; \sigma, \phi) \) generated by \( g - h \). Using the above description of the structure maps of \( D(A_n, B_n; \sigma, \phi) \), we get that \( I = \{ I_\alpha \} \) for \( \alpha \in GL_n(k) \) is a crossed Hopf \( GL_n(k) \)-coideal of \( D(A_n, B_n; \sigma, \phi) \). The quotient \( D(A_n, B_n; \sigma, \phi)/I = \{ D(A_n, B_n; \sigma, \phi)/I_\alpha \} \) is precisely
\[ A_n = \{ A^\alpha_n \}_{\alpha \in \text{GL}_n(k)} \] and so the latter has a quasitriangular Hopf GL\(_n\)(k)-coalgebra structure which can be described by replacing \( h \) with \( q \) in Definition-Proposition 5.1.

Finally, the \( R \)-matrix of \( A_n \) is obtained as the image under the projection maps \( D(A_n, B_n; \sigma, \phi) \overset{p_\alpha}{\longrightarrow} D(A_n, B_n; \sigma, \phi)/I_\alpha = A^\alpha_n \) of the \( R \)-matrix of \( D(A_n, B_n; \sigma, \phi) \), that is, using (11.10),

\[
R_{\alpha, \beta} = \sum_{S \subseteq [n]} p_\alpha(x_S) \otimes p_\beta(z_0 y S) + p_\alpha(g x S) \otimes p_\beta(z_1 y S)
= \sum_{S \subseteq [n]} x_S \otimes \left( \frac{1+q}{2} \right) y S + g x S \otimes \left( \frac{1-q}{2} \right) y S
= \frac{1}{2} \sum_{S \subseteq [n]} x_S \otimes y S + x_S \otimes g y S + g x_S \otimes y S + g x_S \otimes g y S.
\]

\[ \square \]

5. Graded Quantum Groups

Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra of rank \( l \) with Cartan matrix \( (a_{i,j}) \). We let \( d_i \) be the coprime integers such that the matrix \( (d_i a_{i,j}) \) is symmetric. Let \( q \) be a fixed non-zero complex number and let \( q_i = q^{d_i} \). Suppose that \( q_i^2 \neq 1 \) for \( i = 1, 2, \ldots, l \).

**Definition-Proposition 5.1.** Set \( \pi = (\mathbb{C}^\ast)^l \). For \( \alpha = (\alpha_1, \ldots, \alpha_l) \in \pi \), let \( U^\alpha_q(\mathfrak{g}) \) be the \( \mathbb{C} \)-algebra generated by \( K_i^{\pm 1}, E_i, F_i, 1 \leq i \leq l \), subject to the following defining relations:

\[
\begin{align*}
(5.1) & \quad K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
(5.2) & \quad K_i E_j = q_i^{a_{i,j}} E_j K_i, \\
(5.3) & \quad K_i F_j = q_i^{-a_{i,j}} F_j K_i, \\
(5.4) & \quad E_i F_j - F_j E_i = \delta_{i,j} \alpha_i K_i - K_i^{-1}, \\
(5.5) & \quad \sum_{r=0}^{1-a_{i,j}} (-1)^r \left[ \frac{1-a_{i,j}}{r} \right]_{q_i} E_i^{1-a_{i,j}-r} E_j E_i^r = 0 \quad \text{if} \quad i \neq j, \\
(5.6) & \quad \sum_{r=0}^{1-a_{i,j}} (-1)^r \left[ \frac{1-a_{i,j}}{r} \right]_{q_i} F_i^{1-a_{i,j}-r} F_j F_i^r = 0 \quad \text{if} \quad i \neq j.
\end{align*}
\]

The family \( U^\pi_q(\mathfrak{g}) = \{ U^\alpha_q(\mathfrak{g}) \}_{\alpha \in \pi} \) has a structure of a crossed Hopf \( \pi \)-coalgebra given, for \( \alpha = (\alpha_1, \ldots, \alpha_l) \in \pi \), \( \beta = (\beta_1, \ldots, \beta_l) \in \pi \) and \( 1 \leq i \leq l \), by

\[
\begin{align*}
\Delta_{\alpha, \beta}(K_i) &= K_i \otimes K_i, \\
\Delta_{\alpha, \beta}(E_i) &= \beta_i E_i \otimes K_i + 1 \otimes E_i, \\
\Delta_{\alpha, \beta}(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \\
\varepsilon(K_i) &= 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \\
S_{\alpha}(K_i) &= K_i^{-1}, \quad S_{\alpha}(E_i) = -\alpha_i E_i K_i^{-1}, \quad S_{\alpha}(F_i) = -K_i F_i, \\
\varphi_{\alpha}(K_i) &= K_i, \quad \varphi_{\alpha}(E_i) = \alpha_i E_i, \quad \varphi_{\alpha}(F_i) = \alpha_i^{-1} F_i.
\end{align*}
\]
Remark 5.2. Note that \((U_q^1(g), \Delta_{1,1}, \varepsilon, S_1)\) is the usual quantum group \(U_q(g)\).

Proof. Let \(U_+\) be the \(\mathbb{C}\)-algebra generated by \(E_i, K_i^{\pm 1}, 1 \leq i \leq l\), subject to the relations \((5.1), (5.2)\) and \((5.3)\). Let \(U_-\) be the \(\mathbb{C}\)-algebra generated by \(F_i, K_i^{\pm 1}, 1 \leq i \leq l\), subject to the relations \((5.1), (5.3)\) and \((5.6)\) where one has to replace \(K_i\) with \(K_i'\). The algebras \(U_+\) and \(U_-\) have a Hopf algebra structure given by

\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \\
\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0, \quad S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \\
\Delta(K_i') = K_i' \otimes K_i', \quad \Delta(F_i) = F_i \otimes 1 + K_i^{\prime -1} \otimes F_i, \\
\varepsilon(K_i') = 1, \quad \varepsilon(F_i) = 0, \quad S(K_i) = K_i^{-1}, \quad S(F_i) = -K_i'F_i.
\]

Using the method described in Section 2.4 it can be verified that there exists a (unique) Hopf pairing \(\sigma: U_+ \times U_- \to \mathbb{C}\) such that

\[
\sigma(E_i, F_j) = \frac{\delta_{i,j}}{q_i - q_i^{-1}}, \quad \sigma(E_i, K_j') = \sigma(K_i, F_j) = 0, \quad \sigma(K_i, K_j') = q_i^{a_{i,j}} - q_j^{a_{j,i}}.
\]

Let \(\phi: \pi \to \text{Aut}_{\text{Hopf}}(U_+)\) and \(\psi: \pi \to \text{Aut}_{\text{Hopf}}(U_-)\) defined, for \(\beta = (\beta_1, \ldots, \beta_l) \in \pi\) and \(1 \leq i \leq l\), by

\[
\phi_\beta(K_i) = K_i, \quad \phi_\beta(E_i) = \beta_i E_i, \quad \psi_\beta(K_i') = K_i', \quad \psi_\beta(F_i) = \beta_i^{-1} F_i.
\]

It is straightforward to verify that \(\psi\) is \((\sigma, \phi)\)-compatible. By Lemma 2.4 we can consider the crossed Hopf \(\pi\)-coalgebra \(D(U_+, U_-; \sigma, \phi) = \{D(U_+, U_-; \sigma, \phi_\alpha)\}_{\alpha \in \pi}\).

Now, for any \(\alpha \in \pi\), \(D(U_+, U_-; \sigma, \phi_\alpha)\) is the algebra generated by \(K_i^{\pm 1}, K_i^{\prime \pm 1}, E_i, F_i\), where \(1 \leq i \leq l\), subject to the relations \((5.1), (5.2), (5.3)\), the relations \((5.1), (5.2), (5.6)\) where one has to replace \(K_i\) with \(K_i'\), and the following relations

\[
K_i K_j = K_j K_i, \quad K_i F_j q_i^{-a_{i,j}} F_j K_i, \quad K_j' E_j = q_i^{a_{i,j}} E_j K_j',
\]

\[
E_i F_j - F_j E_i = \delta_{i,j} \frac{\alpha_i K_i - K_i'}{q_i - q_i^{-1}}.
\]

Indeed \(D(U_+, U_-; \sigma, \phi_\alpha)\) is the free algebra generated by the algebras \(U_+\) and \(U_-\) with cross relation \((2.4)\). Further, it suffices to require the cross relations \((2.4)\) for \((1 \otimes b) \cdot (a \otimes 1)\) with \(a = K_i, E_i\) and \(b = K_i', F_i\). To simplify the notations, we identify of \(a\) with \(a \otimes 1\) and of \(b\) with \(1 \otimes b\) (recall that these natural maps \(U_+ \to D(U_+, U_-; \sigma, \phi_\alpha)\) and \(U_- \to D(U_+, U_-; \sigma, \phi_\alpha)\) are algebra monomorphisms). For example, let \(a = E_i\) and \(b = F_j\). Since \(\sigma(E_i, 1) = \sigma(K_i, F_j) = \sigma(E_i, K_j') = \sigma(1, F_j) = 0\), relation \((2.6)\) gives

\[
F_j E_i = \sigma(\alpha_i E_i, S(F_j)) \sigma(K_i, 1) K_i + \sigma(1, K_j') \sigma(K_i, 1) E_i F_j + \sigma(1, K_j') \sigma(E_i, F_j) K_j^{-1}.
\]

Inserting the values \(\sigma(K_i, 1) = \sigma(1, K_j') = 1\), \(\sigma(E_i, F_j) = \delta_{i,j}(q_i - q_i^{-1})^{-1}\) and \(\sigma(E_i, S(F_j)) = -\delta_{i,j}(q_i - q_i^{-1})^{-1}\), we get \((5.8)\).
From Theorem 6.1 we obtain that the comultiplication $\Delta_{\alpha,\beta}$, the counit $\varepsilon$, the antipode $S_{\alpha}$, and the crossing $\varphi_{\alpha}$ of $D(U_+, U_-; \sigma, \phi)$ are given, for $1 \leq i \leq l$, by

\begin{align*}
\Delta_{\alpha,\beta}(K_i) &= K_i \otimes K_i, \quad \Delta_{\alpha,\beta}(K'_i) = K'_i \otimes K'_i, \\
\varepsilon(K_i) &= \varepsilon(K'_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S_{\alpha}(K_i) = K_i^{-1}, \\
S_{\alpha}(K'_i) &= K'_i^{-1}, \quad S_{\alpha}(E_i) = -\alpha_i E_i K_i^{-1}, \quad S_{\alpha}(F_i) = -K'_i F_i, \\
\varphi_{\alpha}(K_i) &= K_i, \quad \varphi_{\alpha}(K'_i) = K'_i, \quad \varphi_{\alpha}(E_i) = \alpha_i E_i, \quad \varphi_{\alpha}(F_i) = \alpha_i^{-1} F_i.
\end{align*}

Finally, for any $\alpha \in \pi$, let $I_{\alpha}$ be the ideal of $D(U_+, U_-; \sigma, \phi)$ generated by $K_i - K'_i$ and $K_i^{-1} - K'_i^{-1}$, where $1 \leq i \leq l$. Using the above description of the structure maps of $D(U_+, U_-; \sigma, \phi)$, we get that $I = \{I_{\alpha}\}_{\alpha \in \pi}$ is a crossed Hopf $\pi$-coideal of $D(U_+, U_-; \sigma, \phi)$. The quotient $D(U_+, U_-; \sigma, \phi)/I = D(U_+, U_-; \sigma, \phi)/I_{\alpha \in \pi}$ is precisely $U^\pi_q(\mathfrak{g}) = \{U^\alpha_q(\mathfrak{g})\}_{\alpha \in \pi}$ and so the latter has a crossed Hopf $\pi$-coalgebra structure which can be described by replacing $K'_i$ with $K_i$ in (5.9)-(5.13).

6. $h$-adic graded quantum groups

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $l$ with Cartan matrix $(a_{i,j})$. We let $d_i$ be the coprime integers such that the matrix $(d_i a_{i,j})$ is symmetric.

**Definition-Proposition 6.1.** Set $\pi = \mathbb{C}[\hbar]^l$. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$, let $U^\alpha_q(\mathfrak{g})$ be the algebra over $\mathbb{C}[\hbar]$ topologically generated by the elements $H_i$, $E_i$, $F_i$, $1 \leq i \leq l$, subject to the following defining relations:

\begin{align*}
(H_i, H_j) &= 0, \\
(H_i, E_j) &= a_{i,j} E_j, \\
(H_i, F_j) &= -a_{i,j} F_j, \\
E_i E_j &= \sum_{r=0}^{1-a_{i,j}} (-1)^r \frac{1 - a_{i,j}}{r!} e^{d_i h} H_i^{1-a_{i,j}^{-r}} E_j E_i^r = 0 \quad (i \neq j), \\
F_i F_j &= \sum_{r=0}^{1-a_{i,j}} (-1)^r \frac{1 - a_{i,j}}{r!} e^{d_i h} F_j F_i^r = 0 \quad (i \neq j).
\end{align*}

The family $\{U^\alpha_q(\mathfrak{g})\}_{\alpha \in \pi}$ has a structure of a crossed Hopf $\pi$-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi$, by $\Delta_{\alpha,\beta}(H_i) = H_i \otimes 1 + 1 \otimes H_i, \Delta_{\alpha,\beta}(E_i) = e^{d_i h} H_i \otimes e^{d_i h} H_i + 1 \otimes E_i, \Delta_{\alpha,\beta}(F_i) = F_i \otimes 1 + e^{-d_i h} H_i \otimes F_i, \varepsilon(H_i) = \varepsilon(E_i) = \varepsilon(F_i) = 0, S_{\alpha}(H_i) = -H_i, S_{\alpha}(E_i) = -e^{d_i h} H_i e^{-d_i h} H_i, S_{\alpha}(F_i) = -e^{d_i h} H_i F_i, \varphi_{\alpha}(H_i) = H_i, \varphi_{\alpha}(E_i) = e^{d_i h} H_i, \varphi_{\alpha}(F_i) = e^{-d_i h} H_i.
Remarks 6.2. (a) \((U_0^0 (g), \Delta_{0,0}, \varepsilon, S_0)\) is the usual quantum group \(U_h (g)\).
(b) The element \(e^{d_H} - e^{-d_H} \in \mathbb{C}[h]\) is not invertible in \(\mathbb{C}[h]\), because the constant term is zero. But the expression on the right hand side of (6.4) is a formal power series \(\sigma_\alpha p_\alpha (H_i) h^n\) with certain polynomials \(p_\alpha (H_i)\), and so it is a well-defined element of the \(h\)-adic algebra generated by \(E_i, F_i, H_i\).

Proof. Let \(U_+\) be the \(h\)-adic algebra generated by \(H_i, E_i, 1 \leq i \leq l\), subject to the relations (6.1), (6.2) and (6.4). Let \(U_-\) be the \(h\)-adic algebra generated by \(H'_i, F_i, 1 \leq i \leq l\), subject to the relations (6.1), (6.5) and (6.6) where one has to replace \(H_i\) with \(H'_i\). The algebras \(U_+\) and \(U_-\) have a \(h\)-adic Hopf algebra structure given by

\[
\Delta (H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta (E_i) = E_i \otimes e^{d_H} H_i + 1 \otimes E_i, \\
\varepsilon (H_i) = \varepsilon (E_i) = 0, \quad S (H_i) = -H_i, \quad S (E_i) = -E_i e^{-d_H} H_i, \\
\Delta (H'_i) = H'_i \otimes 1 + 1 \otimes H'_i, \quad \Delta (F_i) = F_i \otimes 1 + e^{-d_H} H'_i \otimes F_i, \\
\varepsilon (H'_i) = \varepsilon (F_i) = 0, \quad S (H'_i) = -H'_i, \quad S (F_i) = -e^{-d_H} H'_i F_i.
\]

Let us consider the \(h\)-adic Hopf algebra \(\tilde{U}_- = \mathbb{C}[h][1 + hU_-]\). The elements \(\tilde{H}_i = h H'_i\) and \(\tilde{F}_i = h F_i\) belong to \(\tilde{U}_-\) and satisfy

\[
[\tilde{H}_i, \tilde{F}_j] = -h a_{ij} \tilde{F}_j, \quad \Delta (\tilde{H}_i) = \tilde{H}_i \otimes 1 + 1 \otimes \tilde{H}_i, \quad \Delta (\tilde{F}_i) = \tilde{F}_i \otimes 1 + e^{-d_H} \tilde{H}_i \otimes \tilde{F}_i. 
\]

The element \(e^{-d_H} H'_i = 1 + \sum_{k \geq 1} \frac{1}{k!} (-d_H)^k H_i^k\) is also in \(\tilde{U}_-\). Note that \(e^{-d_H} H'_i\) is not in the \(h\)-adic subalgebra of \(U_-\) generated by \(H'_i\).

Using the method described in Section 2.4 it can be verified that there exists a (unique) Hopf pairing \(\sigma : U_+ \times U_- \rightarrow \mathbb{C}[h]\) such that

\[
\sigma (H_i, \tilde{H}_j) = d_i^{-1} a_{ij}, \quad \sigma (H_i, \tilde{F}_j) = \sigma (E_i, \tilde{H}_j) = 0, \quad \sigma (E_i, \tilde{F}_j) = \frac{\delta_{i,j} h}{e^{d_H} - e^{-d_H}}.
\]

Let \(\phi : \pi \rightarrow \text{Aut}_{H\text{opf}} (U_+)\) and \(\psi : \pi \rightarrow \text{Aut}_{H\text{opf}} (\tilde{U}_-)\) defined, for \(\alpha = (\alpha_1, \ldots, \alpha_l) \in \pi\) and \(1 \leq i \leq l\), by

\[
\phi_\alpha (H_i) = H_i, \quad \phi_\alpha (E_i) = e^{d_H a_\alpha} E_i, \quad \psi_\alpha (\tilde{H}_i) = \tilde{H}_i, \quad \psi_\beta (\tilde{F}_i) = e^{-d_H a_\alpha} \tilde{F}_i.
\]

It is straightforward to verify that \(\psi\) is \((\sigma, \phi)\)-compatible. By the \(h\)-adic version of Lemma 2.4 we can consider the crossed \(h\)-adic Hopf \(\pi\)-coalgebra \(D (U_+, \tilde{U}_-; \sigma, \phi) = \{D (U_+, \tilde{U}_-; \pi, \phi_\alpha)\}_{\alpha \in \pi}\) whose structure can be explicitly described as in the proof of Proposition 6.3. For any \(\alpha \in \pi\), let \(I_\alpha\) be the \(h\)-adic ideal of \(D (U_+, \tilde{U}_-; \pi, \phi_\alpha)\) generated by \(H'_i - h H_i\) where \(1 \leq i \leq l\). Using the description of the structure maps of \(D (U_+, \tilde{U}_-; \pi, \phi_\alpha)\), we get that \(I = \{I_\alpha\}_{\alpha \in \pi}\) is a crossed \(h\)-adic Hopf \(\pi\)-coalgebra of \(D (U_+, \tilde{U}_-; \pi, \phi)\). The quotient \(D (U_+, \tilde{U}_-; \pi, \phi)/I = \{D (U_+, \tilde{U}_-; \pi, \phi_\alpha)/I_\alpha\}_{\alpha \in \pi}\) is precisely \(U_0^n (g) = \{U_0^n (h)\}_{\alpha \in \pi}\) and so the latter has a crossed \(h\)-adic Hopf \(\pi\)-coalgebra structure.

It is well-known (see, e.g., [K-S92]) that the Hopf pairing \(\sigma : U_+ \times \tilde{U}_- \rightarrow \mathbb{C}[h]\) is non-degenerate and that, if \((e_i, f_i)\) are dual basis of the vector spaces \(U_+\) and \(\tilde{U}_-\) with respect to the form \(\sigma\), then \(\sum_i (e_i \otimes 1) \otimes (1 \otimes f_i)\) belongs to the \(h\)-adic completion \(D (U_+, \tilde{U}_-; \pi, \phi_\alpha) \hat{\otimes} D (U_+, \tilde{U}_-; \pi, \phi_\beta)\). Therefore, by Theorem 2.10 the
crossed $h$-adic Hopf $\pi$-coalgebra $D(U_+, \tilde{U}_-; \sigma, \phi)$ is quasitriangular. Hence, as a quotient of $D(U_+, \tilde{U}_-; \sigma, \phi)$, $U_h^\pi(\mathfrak{g})$ is also quasitriangular.

For example, when $\mathfrak{g} = \mathfrak{sl}_2$ and so $\pi = \mathbb{C}[\lbrack h \rbrack]$, we have that the $R$-matrix of $U_h^{\mathbb{C}[\lbrack h \rbrack]}(\mathfrak{sl}_2)$ is given, for any $\alpha, \beta \in \mathbb{C}[\lbrack h \rbrack]$, by

$$R_{\alpha, \beta} = e^{h(H \otimes H)/2} \sum_{n=0}^{\infty} R_n(h) E^n \otimes F^n,$$

where $R_n(h) = q^n(n+1)/2 \frac{(1-q^{-2})^n}{[n]_q!}$ and $q = e^h$.

For $n \geq 1$, there exits a representation $\rho_n^\alpha : U_h^\pi(\mathfrak{sl}_2) \to \text{GL}(V_n^\alpha)$, where $V_n^\alpha = \mathbb{C}^n$ as a vector space, given on the standard basis $(e_i)_{1 \leq i \leq n}$ of $\mathbb{C}^n$ by

$$\rho_n^\alpha(H)e_i = (n - 2i + 1 - \frac{\alpha}{2})e_i,$$

$$\rho_n^\alpha(E)e_i = \begin{cases} e^\frac{\alpha n}{2} [n-i+1]_q e_{i-1} & \text{if } i > 1 \\ 0 & \text{if } i = 0 \end{cases},$$

$$\rho_n^\alpha(F)e_i = \begin{cases} [i]_q e_{i+1} & \text{if } i < n \\ 0 & \text{if } i = n. \end{cases}$$

Together with the quasitriangularity of $U_h^{\mathbb{C}[\lbrack h \rbrack]}(\mathfrak{sl}_2)$, this data leads in particular to a solution of the $\mathbb{C}[\lbrack h \rbrack]$-colored Yang-Baxter equation.

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