Some Aspects of Spherical Symmetric Extremal Dyonic Black Holes in $4d$ $N = 1$ Supergravity

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ABSTRACT

In this paper we study several aspects of extremal spherical symmetric black hole solutions of four dimensional $N = 1$ supergravity coupled to vector and chiral multiplets with the scalar potential turned on. In the asymptotic region the complex scalars are fixed and regular which can be viewed as the critical points of the black hole and the scalar potential with vanishing scalar charges. It follows that the asymptotic geometries are of a constant and non-zero scalar curvature which further deform to the symmetric spaces, namely anti-de Sitter and de Sitter spaces. These spaces correspond to the near horizon geometries which are the product spaces of a two surface and the two sphere. We finally give some simple $\mathbb{C}^n$-models with both linear superpotential and gauge couplings.

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1 Introduction

Solitonic solutions such as black holes of $N \geq 2$ supergravity have been studied and developed over a decade, see for a review for example in [1]. The main interest of the study is due to the so-called attractor mechanism which was firstly discovered in four dimensional ungauged $N = 2$ supergravity by some authors [2, 3] and $N = 1$ supergravity without introducing the scalar potential [4]. The formalism is basically to find nondegenerate critical points of the black hole potential $V_{BH}$ with $V_{BH} > 0$ and particularly, all eigenvalues of the Hessian matrix of $V_{BH}$ at its nondegenerate critical points are strictly positive.

In this paper we particularly present some results of a particular class of black holes in four dimensional $N = 1$ supergravity coupled to vector and chiral multiplets in the presence of the scalar potential $V$. The black hole is non-supersymmetric and simply admits a spherical symmetry. Since the theory is coupled to vector and chiral multiplets, then it has electric, magnetic, and scalar charges [2, 5]. Such a black hole can be regarded as a solution of a set of the equations of motions such as the Einstein field equation, the gauge field and the scalar field equations of motions by varying the $N = 1$ supergravity action with respect to the metric, gauge fields, and scalar fields on the spherical symmetric metric.

Moreover, our main interest is to consider a special class of black holes, namely extremal black holes. The word “extreme” means that the two black hole horizons coincide on which the black hole potential extremizes at fixed values of scalars. Such a case has been considered for supersymmetric black holes in the context of four dimensional $N = 2$ supergravity [2, 5, 6, 7] whose asymptotic background is flat. While, in our case we study extremal non-supersymmetric black holes with curved asymptotic backgrounds, i.e. four dimensional anti-de Sitter ($AdS_4$) and four dimensional de Sitter ($dS_4$) in the $N = 1$ theory with non-zero $V$.

Let us mention the results as follows. Firstly, in the asymptotic region the scalars are frozen with respect the radial coordinate $r$ and further, can be viewed as the critical points of both the black hole potential $V_{BH}$ and the scalar potential $V$ with vanishing scalar charges in order to have a regular value of the scalars. The geometries are of a constant scalar curvature which further deform to the symmetric spaces, namely $AdS_4$ or $dS_4$ which is the boundaries as $r$ becomes much larger than the mass of the black hole and the value of $V_{BH}$ in the region.

Secondly, near the horizon the scalars are also frozen with respect to $r$ which are the critical points of so-called effective black hole potential $V_{eff}$ which is a function of both the black hole potential $V_{BH}$ and the scalar potential $V$. Here, the black hole geometries are the product of a two dimensional surface $M^{1,1}$ and the two-sphere $S^2$. This $M^{1,1}$ could be $AdS_2$ or $dS_2$ with different radii compared to $S^2$. Therefore, the near-horizon geometry is not conformally flat.

Finally, since the scalar charges vanish in the asymptotic region, then the frozen scalars should be identical in both regions in order to extremize the ADM mass of extremal black holes which has been considered in the asymptotic flat case [5]. Thus, if the radius of $AdS_2$ is less than the radius of $S^2$, then the asymptotic geometry is $AdS_4$. Whereas, we have the asymptotic $dS_4$ if the radius of $AdS_2$ is greater than the radius of $S^2$. However,

\[This non-supersymmetric class of black holes has also been studied in four dimensional $N = 2$ supergravity coupled to vector multiplets with FI terms [8] and recently in [9].\]
it is still unclear for $dS_2$ case, although the asymptotic geometry is $dS_4$.

The structure of the paper is as follows. Section 2 is devoted to give a review of $N = 1$ supergravity coupled to vector and chiral multiplets. Our convention here follows rather closely [10, 11]. Then, in Section 3 we derive the equations of motions of each field mentioned above. We split Section 4 into three parts: in the first part we derive the equations of motions on the spherical symmetric metric, while the rest two parts we discuss some properties of extremal black hole in the asymptotic region and near the horizon. Then, in Section 4.3 we show the existence of extremal dyonic Bertotti-Robinson-like black hole. We give some simple models, namely $\mathbb{C}^n$-models with both linear superpotential and gauge couplings in Section 5. Finally, our conclusions is in Section 6.

2 $N = 1$ Supergravity Coupled with Vector and Chiral Multiplets

In this section we review shortly four dimensional $N = 1$ supergravity coupled to arbitrary vector and chiral multiplets. Here, we assemble the terms which are useful our analysis in the paper. Interested reader can further read, for example, the references [10, 11]. The theory consists of a gravitational multiplet, $n_V$ vector and $n_C$ chiral multiplets. Here, we mention the field content of the multiplets: a gravitational multiplet $(e^a_\mu, \psi_\mu)$, a vector multiplet $(A_\Lambda^\mu, \lambda)$, and a chiral multiplet $(z, \chi)$ where $e^a_\mu$, $A_\mu$, and $z$ are a vierbein, a gauge field, and a complex scalar, respectively, while $\psi_\mu$, $\lambda$, and $\chi$ are the fermion fields. The bosonic sector of the Lagrangian can be written down as

$$L^{N=1} = -\frac{1}{2} R + \mathcal{R}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} + i \mathcal{I}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \tilde{F}^{\Sigma|\mu\nu} + g_{ij}(z, \bar{z}) \partial_\mu z^i \partial^\mu \bar{z}^j - V(z, \bar{z}),$$

(2.1)

where $i, j = 1, \ldots, n_C$, $\Lambda, \Sigma = 1, \ldots, n_V$, and $\mu, \nu = 0, \ldots, 3$. The quantity $R$ is the Ricci scalar of four dimensional spacetime, whereas $F^\Lambda_{\mu\nu}$ is an Abelian field strength of $A^\Lambda_\mu$, and $\tilde{F}_\mu^\lambda$ is a Hodge dual of $F^\lambda_{\mu\nu}$. Meanwhile, we have a Hodge-Kähler manifold $\mathcal{M}$ spanned by the complex scalars $(z, \bar{z})$ with metric $g_{ij}(z, \bar{z}) \equiv \partial_i \partial_j K(z, \bar{z})$ where $K(z, \bar{z})$ is a real function called the Kähler potential.

The gauge couplings $\mathcal{N}_{\Lambda\Sigma}$ are arbitrary holomorphic functions, while $\mathcal{R}_{\Lambda\Sigma}$ and $\mathcal{I}_{\Lambda\Sigma}$ are real and imaginary parts of $\mathcal{N}_{\Lambda\Sigma}$, respectively. Similar to the gauge couplings, the function $W(z)$ is also an arbitrary holomorphic function called holomorphic superpotential. The scalar potential $V(z, \bar{z})$ is real and given by

$$V(z, \bar{z}) = e^{K/M_P^2} \left( g_{ij} \nabla_i W \nabla_j \bar{W} - \frac{3}{M_P^2} W \bar{W} \right),$$

(2.2)

where $W$ is a holomorphic superpotential, $K \equiv K(z, \bar{z})$, and $\nabla_i W \equiv \partial_i W + (K_i/M_P^2) W$. In addition, the Lagrangian (2.1) has a supersymmetric invariance with respect to the

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3 We thank T. Kimura for the discussion of this point.

4 Here, we assume that there is no volume deformation of Kähler manifolds. Such a situation has also been considered for domain wall cases in [12]. On the other hand, some cases with volume deformation of Kähler manifolds have been studied in several references [13, 14, 15, 16, 17].
variation of fields up to three-fermion terms \[10, 11\]:

\[
\delta \psi = M_P \left( D_\nu \psi + \frac{1}{2} e^{K/2M_P^2} W \gamma_\nu \psi + \frac{i}{2M_P Q_\nu} \right),
\]

\[
\delta \lambda = \frac{1}{2} \left( \mathcal{F}^\Lambda_{\mu\nu} - i \tilde{\mathcal{F}}^\Lambda_{\mu\nu} \right) \gamma^\nu \lambda^\nu,
\]

\[
\delta \chi^i = i \partial_\nu z^i \gamma^\nu \psi^\dagger + N^i \psi^\dagger \psi^\star + i \partial_\nu \bar{\psi} \mu \lambda^\Lambda,
\]

\[
\delta e^a = -i M_P \left( \bar{\psi} \gamma^a \psi^\dagger + \bar{\psi} \gamma^a \psi^\dagger \right),
\]

\[
\delta A^\Lambda = i \left( \bar{\lambda}^\Lambda \gamma^\mu \psi^\dagger + \bar{\psi} \gamma^\mu \lambda^\Lambda \right),
\]

\[
\delta z^i = \bar{\chi}^i \psi^\dagger,
\]

where \(N^i \equiv e^{K/2M_P^2} g^i_\bar{j} \bar{W}_j\), \(g^{ij}\) is the inverse of \(g^i_\bar{j}\), and the \(U(1)\) connection \(Q_\nu \equiv -(K_\nu \partial_\nu \bar{z}^i - K_\nu \partial_\nu \bar{z}^i)\).

### 3 The Equations of Motions

Let us first discuss the equations of motions of the fields which can be obtained by varying the action related to the Lagrangian (2.1) with respect to \(g^\mu_\nu\), \(A^\Lambda_{\mu}\), and \(z^i\). Then, by setting all fermions vanish at the level of the equation of motions, we have three equations, namely the Einstein field equation

\[
R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = g^i_\bar{j} \left( \partial_\mu z^i \partial_\nu \bar{z}^j + \partial_\nu z^i \partial_\mu \bar{z}^j \right) - g^i_\bar{j} g^\mu_\nu \partial_\rho z^i \partial^\rho \bar{z}^j + 4 \mathcal{R}_\Lambda \mathcal{F}^\Lambda_{\mu\nu} \mathcal{F}^{\Sigma}_{\sigma\rho} - g^\mu_\nu \mathcal{R}_\Lambda \mathcal{F}^\Lambda_{\rho\sigma} \mathcal{F}^{\Sigma}_{\mu\nu} + g^\mu_\nu V,
\]

the gauge field equation of motion

\[
\partial_\nu \left( \varepsilon^{\mu\rho\sigma} \sqrt{-g} K_{\Lambda|\rho\sigma} \right) = 0,
\]

with

\[
K_{\Lambda|\rho\sigma} \equiv K_{\Lambda\Sigma} F^{\Sigma}_{\rho\sigma} - K_{\Lambda\Sigma} \tilde{F}^{\Sigma}_{\rho\sigma},
\]

are the electric field strengths, and the scalar field equation of motion

\[
\frac{g^{i_3} \sqrt{-g}}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{i\mu} \partial_\nu \bar{z}^j \right) + \partial_\mu g^{i_3} \partial_\nu \bar{z}^j \partial^\mu \bar{z}^k = \partial_\nu K_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma}_{\rho\sigma} \partial_\rho \bar{z}^j + \partial_\nu K_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \tilde{F}^{\Sigma}_{\mu\nu} - \partial_\nu V,
\]

where \(g \equiv \det(g^\mu_\nu)\). Additionally, there are the Bianchi identities

\[
\partial_\nu \left( \varepsilon^{\mu\rho\sigma} \sqrt{-g} F^\Lambda_{\rho\sigma} \right) = 0,
\]

from the definition of \(F^\Lambda_{\rho\sigma}\).

Before proceeding to the explicit model, we briefly point out the setup in this paper as follows. From (3.1) we obtain the scalar curvature

\[
R = 2g^{i_3} \partial_\mu z^i \partial^\mu \bar{z}^j - 4 V,
\]
whose dynamics are controlled by $\partial _\mu z$ and $z$ together with their complex conjugate. It is easily to see that the scalar curvature (3.6) becomes a constant if the scalar fields are fixed with respect to the spacetime coordinates, namely $\partial _\mu z^i = 0$. In this paper we particularly consider the case where such situations occur in the asymptotic and the near horizon regions. To achieve such a results, firstly, assuming that in the asymptotic region the scalar fields have to be frozen, namely $z^i_0$ and $\partial _\mu z^i = 0$, there exists a constant $\kappa$ such that the black hole has a constant scalar curvature where $\kappa$ is related to the potential $V$ evaluated at its critical point $(z_0, \bar{z}_0)$. Secondly, near the horizon we do similar way as before, namely the scalar fields are freezed to $z^i_h$ and $\partial _\mu z^i = 0$ and $\kappa$ is related to so called an effective potential evaluated at its critical point $(z_h, \bar{z}_h)$ (see section 4.3). In the rest of this paper we particularly construct the model in which the black holes are spherically symmetric and extremal.

4 Spherical Symmetric Extremal Black Holes

4.1 General Setup

Let us start to construct a black hole solution of the equations (3.1), (3.2), and (3.4). Our starting point is particularly to write the ansatz metric

$$ds^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - e^{C(r)} (d\theta^2 + \sin^2 \theta d\phi^2) ,$$

which is static and has a spherical symmetric. Among the functions $A(r)$, $B(r)$, and $C(r)$, only two of them are independent since one can generally employ the radial coordinate redefinition $r$ to absorb one of the three.

On the ansatz (4.1), the next step is to solve the gauge field equation of motions (3.2) together with the Bianchi identities (3.5). Simply taking a case where the field strength components $F^A_{01}(r)$ and $F^A_{23}(\theta)$ are nonzero, we obtain

$$F^A_{01} = \frac{1}{2} e^{\frac{1}{2}(A+B)-C} (R^{-1})^{\Lambda \Sigma} (\mathcal{I}_{\Sigma} g^\Gamma - q_{\Sigma}) ,$$

$$F^A_{23} = -\frac{1}{2} g^A \sin \theta ,$$

where $q_A$ and $g^A$ are the electric and magnetic charges, respectively [8]. Then, using (4.2) we have two sets of equations as follows. The first set of equations is coming from the Einstein field equation and the Maxwell equation, namely

$$-e^{-B} \left( C'' + \frac{3}{4} C'^2 - \frac{1}{2} C' B' \right) + e^{-C} = e^{-B} g_{ij} \bar{z}^j \bar{z}^i + V + e^{-2C} V_{BH} ,$$

$$-\frac{1}{2} \left( \frac{1}{2} C' + A' \right) + e^{B-C} = -g_{ij} \bar{z}^i \bar{z}^j + e^B \left( V + e^{-2C} V_{BH} \right) ,$$

$$-\frac{1}{2} e^{-B} \left( A'' + C'' + \frac{1}{2} (A' + C')(A' - B') + \frac{1}{2} C'^2 \right) = e^{-B} g_{ij} \bar{z}^i \bar{z}^j + V - e^{-2C} V_{BH} ,$$

where $\nu' \equiv dv/dr$, while the second equation is the scalar field equation of motions given by

$$g_{ij} \bar{z}^{ji} + \bar{\partial}_k g_{ij} \bar{z}^j \bar{z}^i \bar{z}^k + \frac{1}{2} (A' - B' + 2C') g_{ij} \bar{z}^i \bar{z}^j = e^B \left( e^{-2C} \partial_i V_{BH} + \partial_i V \right) ,$$

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where we have assumed that $z^i$ depend only on the radial coordinate $r$. The black hole potential $V_{\text{BH}}$ has the form

$$V_{\text{BH}} \equiv -\frac{1}{2} \left( g \, q \right) \mathcal{M} \left( \begin{array}{c} g \\ q \end{array} \right),$$

which is called the black hole potential

$$\mathcal{M} = \begin{pmatrix} \mathcal{R} + \mathcal{I} \mathcal{R}^{-1} \mathcal{I} & -\mathcal{I} \mathcal{R}^{-1} \\ -\mathcal{R}^{-1} \mathcal{I} & \mathcal{R}^{-1} \end{pmatrix}.$$  

The function $V$ is the scalar potential and in addition, $V_{\text{BH}}$ contains all charges, namely electric, magnetic, and scalar charges, with $V_{\text{BH}} \geq 0$.

It is worth mentioning that if the scalars $z$ are fixed for all $r$, then the black hole geometries indeed have a constant scalar curvature. This case is nothing but the Reissner-Nordström-(anti) de Sitter solution with magnetic charges.

In the next two sections we show that a regular solution of (4.3) and (4.4) indeed exists in particular regions, namely near asymptotic and near horizon regions. As we will see that around these regions the spacetimes have constant curvatures demanding that the complex scalar fields $z^i$ have to be fixed which can be viewed as critical points of potentials defined in the theory.

### 4.2 Black Hole Geometries Near Asymptotic Region

In this section we construct a special solution of (4.3) around $r \to +\infty$ in which the scalars $z$ are frozen and can be viewed as critical points of the black hole and the scalar potentials. Or in other words we restrict ourselves to a regular solution of (4.4) in this region. As we will see that around these regions the spacetimes have constant curvatures demanding that the complex scalar fields $z^i$ have to be fixed which can be viewed as critical points of potentials defined in the theory.

Our starting points is to take the condition

$$z'^i(r) \to 0, \quad z^i(r) \to z^i_0,$$

around the asymptotic region. We simply then set

$$C(r) = 2 \ln r,$$

since the ansatz metric admits only two independent functions among $A(r)$, $B(r)$, and $C(r)$. So, from (4.3) we find that the geometry of black holes has the form

$$ds^2 = \Delta dt^2 - \Delta^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$\Delta \equiv 1 - \frac{2M}{r} + \frac{V^0_{\text{BH}}}{r^2} - \frac{1}{3} V_0 r^2,$$

and

$$V^0_{\text{BH}} \equiv V_{\text{BH}}(z_0, \bar{z}_0), \quad V_0 \equiv V(z_0, \bar{z}_0).$$
The metric (4.9) has a constant scalar curvature but not Einstein describing a non-supersymmetric solution since the variations of the fermionic fields in (2.3) do not vanish. The form of (4.9) looks like Reissner-Nordström-(anti) de Sitter metric, and since we are dealing with asymptotic geometries, (4.10) must be strictly positive. Therefore, it does not poses any positive root in the region.

Let us make the above statements more detail in the case of $V_0 = 0$ and then defining
\[ \Delta_0 \equiv r^2 - 2Mr + V_{\text{BH}}^0. \] (4.12)

The first case is that if $M > 0$, then we have $M < (V_{\text{BH}}^0)^{1/2}$ and $V_{\text{BH}}^0 > 0$. This means that (4.12) does not have any root. On the other hand, if $M < 0$, then we have negative roots of (4.12).

Now let us turn to the scalar field equation of motions (4.4). In this region, taking into account $z_i' = 0$, (4.4) limits to
\[ \frac{g_{ij}^0}{r} \left( r \delta^i \right)'' = \Delta^{-1} \left( \frac{1}{r^4} (\partial_i V_{\text{BH}})_{z_0} + (\partial_i V)_{z_0} \right), \] (4.13)
which gives
\[ z'' = -\frac{\Sigma^i}{r^2} + \left( P'(r) \left( g^{ij} \partial_j V_{\text{BH}} \right)_{z_0} + Q'(r) \left( g^{ij} \partial_j V \right)_{z_0} \right), \]
\[ z' = z_0^i + \frac{\Sigma^i}{r} + \left( P(r) \left( g^{ij} \partial_j V_{\text{BH}} \right)_{z_0} + Q(r) \left( g^{ij} \partial_j V \right)_{z_0} \right), \] (4.14)
where $\Sigma^i$ are the scalar charges introduced in [5]. The functions $P(r)$ and $Q(r)$ are
\[ P(r) = \frac{1}{r} \int \left( \int \frac{\Delta^{-1}}{r^3} dr \right) dr, \]
\[ Q(r) = \int \left( \int r \Delta^{-1} dr \right) dr. \] (4.15)

Thus, in order to have a consistent picture with the condition (4.7) it should be then
\[ (\partial_i V_{\text{BH}})_{z_0} = 0, \]
\[ (\partial_i V)_{z_0} = 0, \]
\[ \Sigma' = 0. \] (4.16)

In other words, the moduli fields $z_0^i$ can be thought of as critical points of both the scalar and black hole potentials, namely defined by (2.2) and (1.5) respectively, describing vacua of the theory. Moreover, the first and the second conditions in (4.16) may prevent the scalar fields to be ill defined caused by (4.15) in the asymptotic region. This can be easily showed, for example, when the geometries are of zero scalar curvature, namely $V_0 = 0$, whose functions $P(r)$ and $Q(r)$ have the form
\[ P(r) = \frac{1}{V_{\text{BH}}^0} \ln r + ... , \]
\[ Q(r) = \frac{1}{6} r^2 + Mr + ... , \]
respectively, where the dots represent regular terms as \( r \to +\infty \).

It is important to notice that although we have obtained the above asymptotic geometries as spacetimes of constant scalar curvatures, they are however not the boundaries of the black holes because they indeed deform to the symmetric spaces, namely de Sitter \((dS_4)\) and anti-de Sitter \((AdS_4)\) for \( V_0 \neq 0 \) or flat \( \mathbb{R}^{1,3} \) for \( V_0 = 0 \), as \( r \) runs to \( r \gg |M| \) and \( r \gg V_{BH}^0 \) such that the second and the third terms in (4.10) are suppressed to zero. The scalar fields already have a regular value ensured by (4.16) in the region.

### 4.3 Black Hole Geometries Near the Horizon

This section is devoted to show the existence of Bertotti-Robinson-like geometries when the scalars are frozen near the horizon and can be viewed as critical points of an effective scalar potential which is similar to the case of \( N = 2 \) supergravity [8]. These black holes are related to the attractor model discussed, for example, in [1, 2, 3].

The first step is to freeze the complex scalars, \( z^i = 0 \) and correspondingly, the near horizon geometry of the metric (4.1) is a product of two surfaces \( M^{1,1} \times S^2 \), where \( M^{1,1} \) and \( S^2 \) are respectively two dimensional surfaces and two-spheres. The setup then implies that the functions in (4.1) are governed by

\[
\frac{1}{2} e^{-B} \left( A'' + \frac{1}{2} A'(A' - B') \right) = \ell , \\
C = \ln r_h ,
\]

(4.17) where \( r_h \equiv r_h(g,q) \) is the radius of \( S^2 \), while the first equation in (4.17) determine the geometry of \( M^{1,1} \) with \( \ell \equiv \ell(g,q) \).

Next, in this limit the equations in (4.3) and (4.4) reduce to

\[
\frac{1}{r_h^2} = \frac{1}{r_h^2} V_{BH}^h + V_h , \\
\ell = \frac{1}{r_h^2} V_{BH}^h - V_h ,
\]

\[
\left( \frac{1}{r_h^4} \frac{\partial V_{BH}}{\partial z^i} + \frac{\partial V}{\partial z^i} \right) (p_h) = 0 ,
\]

(4.18) and \( p_h \equiv (z_h, \bar{z}_h) \) where we have introduced

\[
V_{BH}^h \equiv V_{BH}(p_h) , \\
V_h \equiv V(p_h) ,
\]

(4.19)

\[
\lim_{r \to r_h} z^i \equiv z_h^i .
\]

A set of solutions of (4.18) is given by

\[
r_h^2 = V_{eff}^h , \\
\ell^{-1} = \frac{V_{eff}^h}{\sqrt{1 - 4V_{BH}^h V_h}} ,
\]

(4.20)

\[
\frac{\partial V_{eff}}{\partial z^i} (p_h) = 0 ,
\]
where
\[ V_{\text{eff}} \equiv 1 - \sqrt{1 - 4V_{\text{BH}}V} \]
is called the effective black hole potential \[8\] and
\[ V_{\text{eff}}^h \equiv V_{\text{eff}}(p_h) . \]
The last equation in (4.20) proves that the scalars \( z^i_h \) are indeed the critical points of \( V_{\text{eff}} \) in the scalar manifold \( M \) near the horizon and \( z^i_h \equiv z^i_h(g, q) \). In this case the black hole entropy simply takes the form \[18\]
\[ S = \frac{A_h}{4} = \pi r_h^2 = \pi V_{\text{eff}}^h . \] (4.23)
In addition, the positivity of the entropy (4.23) restricts \( r_h^2 > 0 \). All of the above results follow that one can get if \( B = \pm A \), then \( M^{1,1} \simeq \text{AdS}_2 \) for \( B = -A \) and \( M^{1,1} \simeq \text{dS}_2 \) for \( B = A \).

Let us relate these results to the results in the preceding subsection. Employing the Komar integral on curved backgrounds in \[19\] to our case, we then get the ADM mass \( M_{\text{ex}}(z_0(g, q)) \) for extremal black holes satisfying \[5\]
\[ \left( \frac{\partial M_{\text{ex}}}{\partial z} \right)_{z = z_h} = 0 , \] (4.24)
which implies \( \Sigma^i = 0 \) for every \( i \) or the last condition in (4.16). Thus, it is convenient to have
\[ z^i_0 = z^i_h , \quad \text{for every } i \] (4.25)
which implies that the scalar curvature (3.6) becomes
\[ R = -4V_0 = -4 \left( \frac{1}{r_h^2} - \ell \right) . \] (4.26)
As observed in \[8\], in the first case the spacetime is not conformally flat since \( r_a \neq r_h \) where \( r_a \equiv \ell^{-1/2} \) is the radius of \( \text{AdS}_2 \). If the asymptotic geometry is the spacetime of negative curvature \( (i.e. \text{AdS}_4) \), then \( r_a < r_h \). While for the case of \( r_a > r_h \), the asymptotic geometry has a positive curvature \( (i.e. \text{dS}_4) \) \[9\]. On the other hand, if \( M^{1,1} \simeq \text{dS}_2 \), then we only have the asymptotic geometry of a positive curvature \( (i.e. \text{dS}_4 \) which is however still unclear.

In the following we give in order some remarks. As mentioned in the previous section, the black hole potential \( V_{\text{BH}} \geq 0 \), while the scalar potential \( V \) is not necessarily positive. Therefore, the effective potential \( V_{\text{eff}} \) takes the real value with necessary condition
\[ V_{\text{BH}}V < \frac{1}{4} , \] (4.27)
while the regularity of the first order derivative of the effective black hole potential (4.21) forbids the equal sign. Moreover, in order to get a consistent picture the entropy (4.23) further demands that \( V_{\text{eff}} \) must be strictly positive at ground states. We also have
\[ \lim_{V \to 0} V_{\text{eff}} = V_{\text{BH}} , \]
\[ \lim_{V_{\text{BH}} \to 0^+} V_{\text{eff}} = 0 . \] (4.28)
From (4.18) and (4.20) we can directly see that the second equation in (4.28) is a singular case with vanishing entropy (4.23). Another singular model is when $M^{1,1}$ is flat Minkowski surface $\mathbb{R}^{1,1}$.

5 Simple $\mathbb{C}^{n_c}$-Models

In this section we consider some simple models on $\mathbb{C}^{n_c}$ whose Kähler potential has the form

$$K(z, \bar{z}) = |z|^2,$$

(5.1)

where $|z|^2 \equiv \delta_{ij} z^i \bar{z}^j$. Particularly, the gauge couplings and the superpotential have the form

$$N_{\Lambda \Sigma}(z) = (b_0 + b_i z^i ) \delta_{\Lambda \Sigma},$$

$$W(z) = a_0 + a_i z^i,$$

(5.2)

respectively, with $a_0, a_i, b_0, b_i \in \mathbb{R}$. The black hole potential and the scalar potential are given by

$$V_{\text{BH}}(x, y) = \left( b_0 + b_i x^i + \frac{(b_i y^i)^2}{(b_0 + b_i x^i)} \right) g^2 - \frac{2b_j y^j}{(b_0 + b_i x^i)} gq + \frac{q^2}{(b_0 + b_i x^i)},$$

$$V(x, y) = e^{(x^2 + y^2)/M_p^2} \left[ a^2 - \frac{3a_0^2}{M_p^2} - \frac{4a_0}{M_p^2} a_i x^i - \frac{1}{M_p^2} (a_i x^i)^2 + (a_i y^i)^2 \right] + \frac{1}{M_p^4} (x^2 + y^2) \left( (a_0 + a_i x^i)^2 + (a_i y^i)^2 \right),$$

(5.3)

respectively, where we have introduced coordinates $x^i, y^i \in \mathbb{R}$ such that $z^i = x^i + iy^i$ and defined some quantities

$$g^2 \equiv \delta_{\Lambda \Sigma} g^\Lambda g^\Sigma, \quad gg \equiv g^\Lambda q_\Lambda,$$

$$q^2 \equiv \delta^{\Lambda \Sigma} q_\Lambda q_\Sigma, \quad a^2 \equiv \delta^{ij} a_i a_j,$$

$$x^2 \equiv \delta_{ij} x^i x^j, \quad y^2 \equiv \delta_{ij} y^i y^j.$$  

(5.4)

We begin the construction by taking $b_i = 0$ for all $i$ which means for all $z$ we have $\partial_i N_{\Lambda \Sigma}(z) = 0$. This follows that the black hole potential $V_{\text{BH}}$ becomes

$$V_{\text{BH}}^0 = b_0 g^2 + \frac{q^2}{b_0},$$

(5.5)

which is positive with $b_0 > 0$. Firstly, we simply take $z_0 = 0$ and then, get

$$\partial_i V(0) = -\frac{4}{M_p^2} a_0 a_i = 0,$$

(5.6)

which can be split into two cases as follows. The first case is $a_i = 0$ for all $i$ and $a_0 \neq 0$. The scalar potential (2.2) then becomes

$$V(0) = -\frac{3a_0^2}{M_p^2},$$

(5.7)
which shows that the black hole has an anti-de Sitter background. Then, the effective potential (4.21) in this case is given by

\[ V_{\text{eff}}(0) = \frac{M_P^2}{6a_0^3} \left( \sqrt{1 + \frac{12a_0^2b_0}{M_P^4} \left( g^2 + \frac{q^2}{b_0^2} \right) - 1} \right), \]  

which is strictly positive. The Hessian matrix of the scalar potential (2.2) is simply

\[ \partial_i \partial_j V_{\text{eff}}(0) = -\frac{4a_0^2}{M_P^4} \delta_{ij} \frac{\partial V_{\text{eff}}}{\partial V}(0), \]  

with \( \partial V_{\text{eff}} / \partial V(0) > 0 \) showing that the model is not attractive.

The second case is \( a_i \neq 0 \) for some \( i \) and \( a_0 = 0 \) which follows that the scalar potential (2.2) is simply

\[ V(0) = a^2, \]  

ensuring that the black hole has a de Sitter background. The Hessian matrix of the effective potential (4.21) has the form

\[ \partial_i \partial_j V_{\text{eff}}(0) = \frac{2}{M_P^2} \left( a^2 \delta_{ij} - a_i a_j \right) \frac{\partial V_{\text{eff}}}{\partial V}(0). \]  

In this model there may exist an attractor if all eigenvalues of (5.11) are strictly positive but \( n_c \neq 1 \).

Finally, we consider a more general case, namely

\[ \partial_i V_{\text{BH}} = 0 \quad \text{and} \quad \partial_i N_{\Lambda \Sigma} \neq 0, \]
\[ \partial_i V(\tau) = 0. \]  

Here, we simply set \( n_c = 1 \), but \( n_v > 1 \). Moreover, \( a_1 = 0 \), while the other pre-coefficients are non-zero. After some steps, we find that the critical point is

\[ x_0 = -\frac{b_0}{b_1} + \frac{1}{b_1g^2} \sqrt{g^2q^2 - (gq)^2}, \]
\[ y_0 = \frac{gq}{b_1g^2}, \]  

with

\[ b_1 = \frac{1}{M_P\sqrt{2}} \left( (gq)^2g^{-4} + \left( -b_0 + g^{-2}\sqrt{g^2q^2 - (gq)^2} \right)^2 \right)^{1/2}. \]  

In the case at hand, the potentials in (5.3) have the form

\[ V_{\text{BH}}(x_0, y_0) = 2 \sqrt{g^2q^2 - (gq)^2}, \]
\[ V(x_0, y_0) = -\frac{e^2a_0^2}{M_P^2}, \]  

and thus, we have a dyonic black hole with anti-de Sitter background and positive black hole potential since \( g^2q^2 > (gq)^2 \). The effective potential (4.21) in this model has the form

\[ V_{\text{eff}}(x_0, y_0) = \frac{M_P^2}{e^2a_0^2} \left( \sqrt{1 + \frac{8e^2a_0^2}{M_P^4} \sqrt{g^2q^2 - (gq)^2} - 1} \right). \]  

The analysis of the Hessian matrix of (4.21) at \((x_0, y_0)\) shows that this model admits an attractor since \( \partial V_{\text{eff}} / \partial V(x_0, y_0) > 0 \) and \( \partial V_{\text{eff}} / \partial V_{\text{BH}}(x_0, y_0) > 0 \).
6 Conclusions

In the present paper we have considered several aspects of extremal dyonic black holes in four dimensional $N = 1$ supergravity that have electric and magnetic charges with curved asymptotic backgrounds, namely $dS_4$ and $AdS_4$. The black holes are particularly non-supersymmetric and spherical symmetric.

In the asymptotic region we set the scalars to be fixed, namely $z_i^0$, which can be viewed as the critical points of the black hole potential $V_{BH}$ and the scalar potential $V$ with vanishing the scalar charges $\Sigma^i$. Then, the black hole geometry tends to have a constant and non-zero scalar curvature which then deforms to the symmetric spaces, namely $AdS_4$ and $dS_4$.

At the horizon the ansatz metric (4.1) becomes a product of two surfaces at vacua defined in the last equation in (4.20), namely $M^{1,1} \times S^2$. These vacua correspond to the near-horizon limits of (4.3) and (4.4) where the scalars $z^i$ are frozen and can be regarded as critical points of $V_{\text{eff}}$ with additional condition $V_{\text{eff}}^h > 0$ coming from the positivity of the entropy (4.23). The surface $M^{1,1}$ is Einstein if $B = \pm A$ where $M^{1,1} \simeq AdS_2$ for $B = -A$ and $M^{1,1} \simeq dS_2$ for $B = A$. In general, this spacetime is not conformally flat since $\ell \neq r_h^{-2}$. In particular, if all the Hessian eigenvalues of $V_{\text{eff}}$ are strictly positive, then the critical point $p_h$ is an attractor. Note that we exclude the singularities of this model, namely $V_{BH} \to 0^+$ and the two dimensional surface $M^{1,1}$ is a flat Minkowskian.

Furthermore, the extremal condition (4.16) follows that we must identify $z_i^0 = z_i^h$ for every $i$ which has been observed previously for flat cases [5]. In other words, if the asymptotic geometry is $AdS_4$, then $r_a < r_h$ where $r_a \equiv \ell^{-1/2}$ is the radius of $AdS_2$. While for the case of $r_a > r_h$, the asymptotic geometry is $dS_4$ [9]. On the other hand, it is still not obvious for $M^{1,1} \simeq dS_2$ whose asymptotic geometry is also $dS_4$.

At the end, we have worked out $C^{mc}$ models in which the superpotential and the gauge couplings both have the linear forms. In the the first model where the ground state is simply the origin for $\partial_i N_{\Lambda \Sigma}(z) = 0$ case, we have a black hole which is asymptotically $AdS_4$ and not attractive $a_i = 0$ for all $i$ and $a_0 \neq 0$, whereas for $a_i \neq 0$ for some $i$ and $a_0 = 0$, the black hole has a de Sitter background and may have an attractor if all eigenvalues of (5.11) are strictly positive but $n_c \neq 1$. Secondly, for $\partial_i N_{\Lambda \Sigma}(z) \neq 0$ case, simply taking $n_c = 1$ we also obtain a black hole with anti-de Sitter background which is attractive.

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