A HOMEOMORPHISM INVARIANT FOR SUBSTITUTION TILING SPACES

by

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Abstract

We derive a homeomorphism invariant for those tiling spaces which are made by rather general substitution rules on polygonal tiles, including those tilings, like the pinwheel, which contain tiles in infinitely many orientations. The invariant is a quotient of Čech cohomology, is easily computed directly from the substitution rule, and distinguishes many examples, including most pinwheel-like tiling spaces. We also introduce a module structure on cohomology which is very convenient as well as of intuitive value.

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1. Introduction.

In this paper we study the topology of substitution tiling spaces. Specifically, we are interested in ways in which the geometry of the substitution is reflected in the topology.

We follow a history of work where one studies the topology of spaces created with an underlying dynamical system. An early reference to the circle of ideas is Parry and Tuncel [PaT; chap. 4]. They show that the first cohomology group of the standard suspension space $\tilde{X}$ for a $\mathbb{Z}$-subshift $(X, T)$ is a useful invariant for topological conjugacy of $(X, T)$. Here, $\tilde{X} = (X \times [0, 1])/\sim$, where $(x, 1) \sim (Tx, 0)$.

Later, Herman, Putnam and Skau [HPS] associated a $C^*$-algebra with general minimal $\mathbb{Z}$-subshifts via the crossed product construction, in analogy with the way von Neumann algebras had been associated previously with measurable dynamics. For these systems the $K_0$-group of the $C^*$-algebra coincides with the first cohomology of the suspension space [HPS]. Giordano, Putnam and Skau [GPS] used the (ordered) $K$-groups of the $C^*$-algebra to characterize orbit equivalence for minimal $\mathbb{Z}$-subshifts.

In a similar vein Connes [Con] used Penrose tilings to motivate ideas on noncommutative topology. Specifically, he considered the quotient $X$ of the space of Penrose tilings by the equivalence relation of translation. Using the hierarchical structure of the Penrose tilings there is a natural homeomorphism between $X$ and a quotient $Y$ of sequences modulo another equivalence relation. As is common for equivalence relations coming from dynamics, the quotient topology of $Y$ is trivial. And so Connes considers a (noncommutative) $C^*$-algebra associated with the quotient $Y$. The $K$-theory of this $C^*$-algebra is his substitute for the (trivial) quotient topology.

This was pushed one step further by Kellendonk and Putnam [Ke1-4, KeP, AnP] in which examples of tiling dynamical systems such as the Penrose system are considered and their (unordered) $K$-theory is worked out, obtaining for instance cohomology groups for the spaces of tilings.

We consider substitution tiling systems in $\mathbb{R}^d$, such as the Penrose tilings in $\mathbb{R}^2$, as dynamical systems with $\mathbb{R}^d$ (translation) actions. These are natural generalizations of the suspensions of $\mathbb{Z}^d$ (substitution) subshifts. (We will restrict attention to $d = 2$; while most of our methods carry over to higher dimensions, there are complications which we wish to avoid.) As noted above, suspensions are sometimes introduced to get invariants to distinguish between subshifts, insofar as the spaces on which the subshifts act are all the same — Cantor sets. For tilings the natural objects already have $\mathbb{R}^d$ actions and different tiling spaces need not be homeomorphic. In particular, the cohomology groups of such spaces are invariants of topological conjugacy. For 1-dimensional tiling systems, Barge and Diamond [BD] worked out complete homeomorphism invariants.

Although in principle it should be possible to work out the cohomology groups for tiling spaces, in practice it can be difficult since the number of different tiles in interesting models (especially after collaring — see below), which gives the number of generators for the chain groups, is usually large, even infinite for examples such
as pinwheel tilings. New in this work is analysis of two aspects of the cohomology of substitution tiling spaces that are both easy to compute and informative about the topology of the spaces.

The first aspect is an order structure for the top dimensional Čech cohomology. For some history in a “noncommutative” setting see [GPS], but the idea is much older, appearing for instance in [SuW]. Fundamental to this paper as well is a representation of our tiling spaces as inverse limits, as discussed in [AnP], and in particular we use this to define a positive cone in cohomology. More specifically, we construct a homeomorphism from the top Čech cohomology group to the real numbers, and define the positive cone to be the preimage of the non-negative reals. The image of the map is closely related to the additive group of $\mathbb{Z}[1/\lambda]$, where $\lambda$ is the area stretching factor of the substitution. We show that the positive cone is a homeomorphism invariant, and that one can therefore extract invariant topological information from the easily computable stretching factor $\lambda$.

We prove results of this form for two classes of tiling spaces. For “fixed-orientation” tiling spaces (denoted $X_\times$ below), we need the additional assumption that tiles only appear in a finite number of distinct orientations in each tiling. For “all-orientation” tiling spaces (denoted $X_\phi$ below), this assumption is not needed. Our main result is

**Theorem 1.** Let $X$ and $Y$ be substitution tiling systems whose substitutions have area stretching factors $\lambda$ and $\lambda'$, respectively. Suppose that either i) $X$ and $Y$ have finite relative orientation groups and the fixed-orientation spaces $X_\times$ and $Y_\times$ are homeomorphic, or ii) the all-orientation spaces $X_\phi$ and $Y_\phi$ are homeomorphic. Then

1. If $\lambda$ is an integer so is $\lambda'$, and $\mathbb{Z}[1/\lambda] = \mathbb{Z}[1/\lambda']$ as subsets of $\mathbb{R}$. In particular, $\lambda$ and $\lambda'$ have the same prime factors, although not necessarily with the same multiplicities.

2. If $\lambda$ is not an integer then $\lambda$ and $\lambda'$ are irrational, and $\mathbb{Q}[\lambda] = \mathbb{Q}[\lambda']$ as subsets of $\mathbb{R}$.

(We will see in examples below that this distinguishes topologically between tiling spaces, such as that of the pinwheel and the $(2,3)$-pinwheel, which had not previously been known to be distinguishable even in the stronger sense of topological conjugacy, that is, including the natural $\mathbb{R}^2$ action of translations on the spaces.)

To prove this theorem we show that the order structure, although defined in terms of the (not necessarily invariant) inverse limit structure, is in fact invariant. The relations between $\lambda$ and $\lambda'$ then follow by algebraic arguments.

To describe the second aspect of our analysis we must discuss the unusual rotational properties of substitution tilings such as those of Penrose. No Penrose tiling is invariant under rotation by $2\pi/10$ about any point, but every translation invariant Borel probability measure on the space of Penrose tilings ($X_\times$ or $X_\phi$) is invariant under rotation by $2\pi/10$. We say $\mathbb{Z}_{10}$ is the “relative orientation group” of the Penrose tilings. This symmetry has been the source for most of the explosion of work on such tilings, in particular their use in modelling quasicrystals [Ra2]. (We
have already investigated the use of rotational symmetry as a conjugacy invariant [RaS].)

As we will see below, each element of the cyclic rotation group $\mathbb{Z}_{10}$ gives an automorphism of the space $X_x$ of Penrose tilings. As a result, the group $\mathbb{Z}_{10}$, and by extension its group ring $\mathbb{Z}[t]/(t^{10}-1)$, acts naturally on the (Čech) cohomology groups of $X_x$. This gives $H^*(X_x)$ the structure of a module over the group ring, and this module structure is quite useful. In particular, for any realization of $X_x$ as the inverse limit of simplicial complexes, as in [AnP], the (co)chain groups are themselves modules, and the boundary and substitution maps are equivariant. This greatly simplifies the calculations. For Penrose tiles for example [AnP], there are 4 types of tiles, each appearing in 10 different orientations. Without using the rotational symmetry the action of substitution on 2-chains would be given by a $40 \times 40$ matrix. Using the rotational symmetry we express it instead as a $4 \times 4$ matrix with entries in the group ring of $\mathbb{Z}_{10}$, which can be analyzed one irreducible representation at a time. The substitutions on 0- and 1-chains, and the boundary maps, are handled similarly. The result is a streamlined calculation that yields information beyond that obtained by older methods. Anderson and Putnam [AnP] computed $H^0(X_x) = \mathbb{Z}$, $H^1(X_x) = \mathbb{Z}^5$ and $H^2(X_x) = \mathbb{Z}^8$. We note that, as modules, a finite-index additive subgroup of $H^1(X_x)$ equals $\mathbb{Z}[t]/(t-1) \oplus \mathbb{Z}[t]/(t^4-t^3+t^2-t+1)$ while a finite index subgroup of $H^2(X_x)$ equals $(\mathbb{Z}[t]/(t-1))^2 \oplus (\mathbb{Z}[t]/(t+1))^2 \oplus \mathbb{Z}[t]/(t^4-t^3+t^2-t+1)$, where rotation by $2\pi/10$ is multiplication by $t$.

2. Tiling spaces and their associated inverse limits.

Before defining substitution tiling spaces in general we present some examples whose topological properties we will later consider.

A “chair” tiling of the plane, Fig. 1, can be made as follows. Consider the L-shaped tile of Fig. 2. Divide this tile (also called a “tile of level 0”) into four pieces as in Fig. 3 and rescale by a linear factor of 2 so that each piece is the same size as the original. This yields a collection of 4 tiles that we call a “tile of level 1”. Subdividing each of these tiles and rescaling gives a collection of 16 tiles that we call a tile of level 2. Repeating the process $n$ times gives a collection of $4^n$ tiles — a tile of level $n$. A “chair” tiling is a tiling of the plane with the property that every finite subcollection of tiles is congruent to a subset of a tile of some level. A chair tiling has only one type of tile, appearing in 4 different orientations. (For chair tilings, and in fact very generally for tilings made by such substitution rules, there are “matching rules” which provide a different method of construction [Goo, Ra2].)

Somewhat more complicated are the “Penrose” tilings, Fig. 4, which have 4 different types of triangular tiles, each appearing in 10 different orientations. The 4 tiles, and the rule for subdividing them, are shown in Fig. 5. (These tiles are not the familiar kites and darts. However, the space of substitution tilings developed from these 4 triangles is homeomorphic to the space of Penrose kite and dart tilings). Here the linear stretching factor is the golden mean $\tau = (1+\sqrt{5})/2$, and the area
Figure 1. A chair tiling
Figure 2. The chair tile

Figure 3. The chair substitution
Figure 4. A Penrose tiling
Figure 5. The Penrose substitution
Figure 6. A pinwheel tiling
Figure 7. The substitution for pinwheel tilings

Figure 8. The substitution for (2,3) pinwheel tilings
stretched factor is \( \tau^2 = (3 + \sqrt{5})/2 \).

A “pinwheel” tiling of the plane, Fig. 6 [Ra1], has tiles that appear in an infinite number of orientations. The two basic tiles, a 1-2-\(\sqrt{5}\) right triangle and its mirror image, are shown in Fig. 7, with their substitution rule.

Notice that at the center of a tile of level 1 there is a tile of level 0 similar to the level 1 tile but rotated by an angle \( \alpha = \tan^{-1}(1/2) \). Similarly, the center tile of a tile of level \( n \) is rotated by \( n\alpha \) relative to the tile. Since \( \alpha \) is an irrational multiple of \( \pi \), we see, using the fact that within a tile of level 2 the center there is a tile of level 0 similar to the level 2 tile, that this rotation never stops, and each tiling contains tiles in an infinite number of distinct orientations.

Finally, we consider the “(2,3)-pinwheel” tilings defined by the substitution of Fig. 8, whose tiles are 2-3-\(\sqrt{13}\) right triangles. (Such variants of the pinwheel are easily constructed for any integral legs \( m < n \).) Like the ordinary pinwheel, variant pinwheel tilings also necessarily have tiles in an infinite number of distinct orientations. In fact the relative orientation groups for all pinwheel tilings are algebraically isomorphic. Theorem 1 shows that the tiling spaces for the pinwheel and (2,3)-pinwheel are not homeomorphic.

In all the above cases it is easy to construct explicit examples of tilings. Pick a tile to include the origin of the plane. Embed this tile in a tile of level 1 (there are several ways to do this). Embed that tile of level 1 in a tile of level 2, embed that in a tile of level 3, and so on. The union of these tiles will cover an infinite region, typically – though not necessarily – the entire plane.

We now give a general definition of substitution tiling systems. Let \( \mathcal{A} \) be a nonempty finite collection of polygons in the plane. Let \( X(\mathcal{A}) \) be the set of all tilings of the plane by congruent copies, which we call tiles, of the elements of (the “alphabet”) \( \mathcal{A} \). We label the “types” of tiles by the elements of \( \mathcal{A} \). We endow \( X(\mathcal{A}) \) with the metric

\[
d(x, x') = \sup_n \frac{1}{n} m_H[B_n(\partial x), B_n(\partial x')],
\]

where \( B_n(\partial x) \) denotes the intersection of two sets: the closed disk \( B_n \) of radius \( n \) centered at the origin of the plane and the union \( \partial x \) of the boundaries of all tiles in \( x \). The Hausdorff metric \( m_H \) is defined as follows. Given two compact subsets \( P \) and \( Q \) of \( \mathbb{R}^2 \), \( m_H[P, Q] = \max\{\bar{d}(P, Q), \bar{d}(P, Q)\} \), where

\[
\bar{d}(P, Q) = \sup_{p \in P} \inf_{q \in Q} ||p - q||,
\]

with \( ||w|| \) denoting the usual Euclidean norm of \( w \).

Under this metric two tilings are close if they agree, up to a small Euclidean motion, on a large disk around the origin. The converse is also true for tiling systems whose tiles meet full edge to full edge (as we require) – closeness implies agreement, up to small Euclidean motion, on a large disk around the origin. Although the metric \( d \) depends on the location of the origin, the topology induced by \( d \) is translation invariant. A sequence of tilings converges in the metric \( d \) if and only if its restriction to every compact subset of \( \mathbb{R}^2 \) converges in \( m_H \). It is not hard to show [RaW] that
$X(\mathcal{A})$ is compact and that the natural action of the connected Euclidean group $G_E$ on $X(\mathcal{A})$, $(g, x) \in G_E \times X(\mathcal{A}) \rightarrow g[x] \in X(\mathcal{A})$, is continuous.

A “substitution tiling space” is a closed subset $X_\phi \subset X(\mathcal{A})$ satisfying some additional conditions. To understand these conditions we first need the notion of “patches”. A patch is a (finite or infinite) subset of an element $x \in X(\mathcal{A})$; the set of all patches for a given alphabet will be denoted by $W$. Next we need, as for the above examples, an auxiliary “substitution function” $\phi$, a map from $W$ to $W$, with the following properties:

1. There is some constant $c(\phi) > 1$ such that, for any $g \in G_E$ and $x \in X$, $\phi(g[x]) = \phi(g)[x]$, where $\phi(g)$ is the conjugate of $g$ by the similarity of Euclidean space consisting of stretching about the origin by $c(\phi)$.
2. For each tile $T \in \mathcal{A}$ and for each $n \geq 1$, the union of the tiles in $\phi^n T$ is congruent to $[c(\phi)]^n T$, and these tiles meet full edge to full edge.
3. For each tile $T \in \mathcal{A}$, $\phi T$ contains at least one tile of each type.
4. For each tile $T \in \mathcal{A}$ there is $n_T \geq 1$ such that $\phi^n T$ contains a tile of the same type as $T$ and parallel to it.
5. No tile $T \in \mathcal{A}$ has a nontrivial rotational symmetry.

Condition (2) is significant. It is satisfied by the pinwheel tilings only if we add additional vertices at midpoints of the legs of length 2, creating boundaries of 4 edges. (A similar alteration is needed for the chair tilings.) The tile of level $n$, $\phi^n T$, will be said to be of “type” $T$.

Condition (5), by contrast, is technical, and does not significantly limit the scope of this work. If a tile does have $n$-fold rotational symmetry, we can recover condition (5) by breaking the tile into $n$ congruent but asymmetric pieces, in a manner consistent with the subdivision rules.

**Definition 1.** For a given alphabet $\mathcal{A}$ of polygons and substitution function $\phi$ the “substitution tiling space” is the compact subspace $X_\phi \subset X(\mathcal{A})$ of those tilings $x$ such that every finite subpatch of $x$ is congruent to a subpatch of $\phi^n (T)$ for some $n > 0$ and $T \in \mathcal{A}$. We assume $x$ can be decomposed in one and only one way by tiles of level $n$, for any fixed $n \geq 0$ [Sol].

The above definition gives spaces of tilings in which the tiles can appear in arbitrary orientations. Although any fixed chair tiling, for example, has tiles in only four orientations, the space of all chair tilings also contains rotated versions of that tiling, and so contains tilings in which chairs appear in any orientation.

In many cases, especially when working with tilings whose tiles have only a finite number of orientations per tiling, it is convenient (and customary) to allow tiles only a minimal set of orientations. Instead of using the metric $d$, we define a metric $d'$ such that two tilings are close if (and only if) they agree on a large disk around the origin up to a small translation, rather than up to a small Euclidean motion. Specifically,

$$d'(x, x') = \inf\left\{ \frac{R + S}{1 + R + S} : B_{1+1/S}(g[\partial x]) = B_{1+1/S}(h[\partial x']) \right\}$$
where \( g, h \in \mathbb{R}^2, R > 0, S > 0, |g| < R, |h| < R \) and \( d'(x, x') \equiv 1 \) if the defining set is empty. We pick any one tiling \( x \in X_\phi \), and define \( X_x \subset X_\phi \) to be \( \{ x' \in X_\phi : d'(x, x') < 1 \} \). For tiling spaces with only a finite number of orientations per tiling, \( X_x \) consists of those tilings in which tiles only appear parallel to tiles in \( x \). We call \( X_x \) a “fixed-orientation” tiling space.

Finally, we consider the quotient of \( X_\phi \) by rotations about the origin and denote this space \( X_0 \). For any fixed tiling \( x \), \( X_0 \) is also the quotient of \( X_x \) by the “relative orientation group” \( G \) of the tiling \( x \), defined as follows: \( G \) is generated by the rotations that take a tile in \( x \) and turn it parallel to another tile of the same type in \( x \). In [RaS] it was shown that this group is the same for all tilings \( x \in X_\phi \). The topologies of these three spaces are closely related. The space \( X_\phi \) is a circle bundle over \( X_0 \) with some singular fibers; these fibers correspond to tilings with a discrete rotational symmetry about the origin.

Next we show how to give a substitution tiling space the structure of an inverse limit. We begin by doing this first for a fixed-orientation space, and we begin in particular with the example of the chair. Fix a chair tiling \( x \), and in it a tile \( T \). Each of the other tiles in \( x \) are rotated with respect to \( T \) by an element of the cyclic rotation group \( G_{ch} = \mathbb{Z}_4 \). We construct an (uncollared) chair complex \( \Sigma^{uc} \) as follows. We start with the topological disjoint sum \( \Sigma' \) of the elements of \( \{ g[T] : g \in G_{ch} \} \). We then identify those edges in \( \Sigma' \) which “meet” somewhere in \( x \), defining \( \Sigma^{uc} \). We consider a countable number of copies \( \Sigma_{j}^{uc} \) of \( \Sigma^{uc} \), indexed by the non-negative integers. We think of the tiles in \( \Sigma_{j}^{uc} \) as referring to level \( j \) tiles. The substitution \( \phi \) (whereby we think of each tile of level \( j + 1 \) as the union of four tiles of level \( j \)) then defines a map from \( \Sigma_{j+1}^{uc} \) to \( \Sigma_{j}^{uc} \) for any \( j \geq 0 \), and allows us to define the inverse limit \( \Sigma^{uc} \equiv \varprojlim \Sigma_{j}^{uc} \).

There is a map from the tiling space \( X_x \) to \( \Sigma^{uc} \) defined, on \( x \in X_x \), by determining precisely how the origin in \( x \) sits in each successive tile of higher and higher level. For many but not all substitution systems, this map is neither 1-1 nor onto. A point in the inverse limit precisely describes the hierarchy of tiles containing the origin. In some instances this union of tiles may not cover the entire plane; if not, it may be that there is more than one way to extend the tiling to cover the entire plane (in which case the map is not 1-1), or it may be that there is no way to extend it (in which case the map is not onto). For example, in a chair tiling the tile containing the origin might sit in the upper left corner of the tile of level 1 of Fig. 2, which sits in the same position in a corresponding tile of level 2, and so on. The union of these tiles covers only a quadrant, and this can be extended to a complete chair tiling of the plane in more than one way.

This problem is absent in substitution systems (such as the Penrose tilings [AnP]) which “force the border”, that is, for which there is some integer \( N \) such that, in every tiling, for every tile \( \phi^NT \) of level \( N \) the (level 0) tiles which abut this tile are completely determined by the type \( T \). From this it follows that, for any \( M \), the tiles of level \( M \) that abut a given tile of level \( N + M \) are also determined. An infinite-level tile, therefore, determines a tiling of the entire plane, and is consistent.
with a tiling of the entire plane, even if it does not itself cover the entire plane. The map from the space of tilings to the inverse limit is therefore 1-1 and onto.

It is easy to prove, but very useful, that for any substitution system we can extend the substitution in a simple way to a larger set of “collared” tiles producing a tiling space naturally homeomorphic to the original but now forcing the border. The new set of tiles consists of multiple marked versions of the original tiles, one for each way the original tile can be surrounded by tiles in a tiling. (For the chair one needs 14 versions of each of the original tiles; the pinwheel requires over 50 versions.) Any (fixed-orientation) substitution tiling space can thus be modelled as an inverse limit.

If the original tiles force the border, we let $\Sigma_x = \Sigma^{uc}$. If the original tiles do not force the border, let $\Sigma_x$ denote the complex constructed from collared tiles. Either way, there is a natural substitution map from $\Sigma_x$ to itself, and $X_x = \Sigma_x \leftarrow$.

The other two kinds of tiling spaces may also be constructed as inverse limits. To construct $X_0$ we consider the type but not orientation of the tiles of various levels that contain the origin. In the complex $\Sigma_0$ the basic cells are tiles $T$, where each type appears in only one orientation. As before, if it is possible for a tile $T_1$ to meet a tile $T_2$ then their common edge is identified, and it may be necessary to consider collared tiles. As before the substitution $\phi$ maps $\Sigma_0$ to itself and we consider the inverse limit. A point in the inverse limit is a consistent instruction: the origin sits at such-and-such a point in such-and-such a tile, which sits in a particular way in a tile of level 1, which sits in a particular way in a tile of level 2, and so on. Since we are using collared tiles (or tiles that force the border), this prescription defines a unique tiling of the entire plane, up to an overall rotation, i.e. a point in $X_0$.

The construction of $\Sigma_\phi$ is similar, except that the basic cells are products $S^1 \times T$, where the $T$ are as in the construction of $\Sigma_0$. If a tile $T_1$, rotated by an angle $\alpha$ (with respect to the standard orientation of $A$), can meet a tile $T_2$, rotated by an angle $\beta$, along a common edge $e$, then we identify $(\alpha, e) \subset S^1 \times T_1$ with $(\beta, e) \subset S^1 \times T_2$. It then follows that for any angle $\alpha'$, $(\alpha', e) \subset S^1 \times T_1$ is identified with $(\alpha' + \beta - \alpha, e) \subset S^1 \times T_2$. Once again, substitution maps $\Sigma_\phi$ to itself, and the inverse limit corresponds to instructions on how to build a tiling around the origin. Only now the instructions include information on how to orient each tile, and so defines a tiling uniquely, i.e. a point in $X_\phi$.

If the relative orientation group $G$ is finite then $\Sigma_\phi$, $\Sigma_x$ and $\Sigma_0$ are all compact simplicial complexes. This is clear for $\Sigma_x$ and $\Sigma_0$, as the cells, finite in number, are simplices that meet along common edges. The cells in $\Sigma_\phi$ are not simplices — they have the topology of $S^1$ — but can be divided into $n$ contractible pieces, meeting along common edges, where $n$ is the order of $G$.

If the relative orientation group is infinite then $\Sigma_0$ is a compact simplicial complex but $\Sigma_x$ is not compact (as it contains an infinite number of cells). The space $\Sigma_\phi$, while a compact CW complex, is not necessarily simplicial.

Finally, these complexes are closely related. Just as $X_0$ is the quotient of $X_x$ by $G$ and the quotient of $X_\phi$ by $S^1$, $\Sigma_0$ is the quotient of $\Sigma_x$ by $G$ and the quotient of $\Sigma_\phi$ by $S^1$. 

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3. Finite relative orientation group and \( P \)-positivity.

Here we define the “\( P \)-positive cone” of the top dimensional Čech cohomology group for substitution tiling spaces. In this section we assume our system has a finite relative orientation group, so the simplicial complex \( \Sigma_x \) is compact. We also restrict ourselves to fixed-orientation spaces \( X_x \). Later, we will investigate the role of rotations, and consider order structures on the top cohomologies of \( X_\phi \) and \( X_0 \) for tiling spaces with arbitrary relative orientation groups. (We continue to assume our tilings are 2-dimensional, although this construction applies equally well to other dimensions.)

As discussed above, the space \( X_x \) is an inverse limit of compact simplicial complexes \( \Sigma_x \) with the substitution (self-)map \( \phi \) between them. One can therefore compute the Čech cohomology of \( X_x \) by taking a direct limit of the Čech = simplicial cohomology of \( \Sigma_x \) under the map \( \phi^* \).

Recall that the direct limit of an Abelian group under a map \( M \) is the disjoint union of an infinite number of copies of the group, indexed by the non-negative integers, modulo the equivalence relation

\[(g, k) \sim (Mg, k + 1), \quad (3)\]

for all \( g \) and all \( k \), where \( (g, k) \) denotes the element \( g \) in the \( k \)-th copy of the group. The direct limit of the cohomology groups \( H^2(\Sigma_x) \) can be obtained by taking the direct limit of the cochain groups \( C^2(\Sigma_x) \) then moding out by the image of the coboundary map \( \delta \) from the direct limit of \( C^1(\Sigma_x) \). (The proof of this is simple diagram chasing.)

Since \( \Sigma_x \) is a finite simplicial complex, \( C_2(\Sigma_x) \), the space of simplicial chains, is generated by the tiles themselves and is isomorphic to \( \mathbb{Z}^n \) where \( n \) is the number of distinct tiles in \( \Sigma_x \). To compute the cohomology, each tile must be given an orientation. We define this orientation in the following way: Select an orientation for \( \mathbb{R}^2 \) and select a tiling \( x_0 \in X_x \). The translations comprise a free \( \mathbb{R}^2 \)-action on \( x_0 \). Each tile that appears in \( x_0 \) thus inherits an orientation from the orientation on \( \mathbb{R}^2 \). All such orientations are consistent and are independent of \( x_0 \). The projection map from \( X_x \) to \( \Sigma_x \) defines an orientation on tiles in \( \Sigma_x \). From here on, we refer to this as the “positive orientation” of the tiles in \( \Sigma_x \).

If all tiles at all levels are given the positive orientation, the induced map \( \phi_* : C_2(\Sigma_x) \rightarrow C_2(\Sigma_x) \) is an \( n \times n \) matrix with only non-negative entries. (The \( i, j \) entry in the matrix is the number of copies of tile type \( i \) which appear in a level 1 tile of type \( j \).) By assumption the matrix is primitive, that is, some power has all of its entries strictly positive. The cochain group \( C^2(\Sigma_x) \) is generated by the duals to the tiles. Relative to this basis, the pullback map \( \phi^* \) is described by the transpose of \( \phi_* \). Similar constructions apply to \( C^1 \), except that the elements of \( C_1 \) and \( C^1 \) are not naturally oriented, so the substitution matrix on \( C^1 \) may have negative entries.

Let \( \mathcal{C} \) denote the direct limit of the cochains \( C^2(\Sigma_x) \) under the map \( \phi^* \), and define \( \mathcal{C}_+ \) to be the semigroup of elements \( (c, k) \in \mathcal{C} \) which have the property that \( (\phi^*)^m c \in (\mathbb{Z}_+)^n \) for some \( m \), where \( n \) is the number of distinct tiles in \( \Sigma_x \). It is clear that \( \mathcal{C}_+ - \mathcal{C}_+ = \mathcal{C} \). Since the matrix \( \phi^* \) contains only non-negative entries, it
is also clear that \( C_+ + C_+ \subset C_+ \), \( C_+ \cap -C_+ = \{0\} \) and thus \((C, C_+)\) is an ordered group (in fact, a dimension group [Ell]).

Let \( H = H^2(X_x) \) denote the top Čech cohomology group for \( X_x \). We wish to define a positive cone on \( H \) by

\[
H_+ = \{ [c] \mid c \in C_+ \}
\]

where elements of \( C \) are equivalent if their difference is in \( \text{Im} \delta \). To see that \((H, H_+)\) is an ordered group, we introduce a function \( \mu : C \to \mathbb{R} \).

**Lemma 1.** There is a homomorphism \( \mu : C \to \mathbb{R} \) such that \( C_+ - \{0\} = \mu^{-1}(\mathbb{R}_+ - \{0\}) \).

Proof: Since the matrix \( \phi^* \) is primitive it has a Perron eigenvalue \( \lambda > 0 \) (of multiplicity 1). This is the area stretching factor of the substitution. (To see this, note that a vector whose entries are the areas of the various tiles is an eigenvector of the substitution matrix. Since all entries of this vector are positive this is the Perron eigenvalue. This eigenvalue is the square of the linear stretching factor \( c(\phi) \).) Let \( r \) denote a left eigenvector of \( \phi^* \). Define the map \( \mu : C \to \mathbb{R} \) by

\[
(v, k) \mapsto \lambda^{-k} r^t v.
\]

The map is well-defined since

\[
\mu(\phi^* v, k + 1) = \lambda^{-k-1} r^t \phi^* v = \lambda^{-k} r^t v.
\]

Suppose \((v, k) \in C_+ - \{0\}\) then \((\phi^*)^m v \in (\mathbb{Z}_+)^n\) for some \( m \). Since \( \phi^* \) is primitive, we may choose \( m \) such that all entries of \((\phi^*)^m v\) are positive. Since all entries of \( r \) are non-negative,

\[
\mu(v, k) = \lambda^{-k} r^t v = \lambda^{-k-m} r(\phi^*)^m v > 0.
\]

To prove the reverse direction, take a basis of \( \mathbb{C}^n \) which includes \( w_0 \), the right eigenvector for \( \lambda \), and \((n-1)\) vectors \( \{w_1, w_2, \ldots, w_{n-1}\} \), each \( w_i \in \text{Ker}(I\lambda_i - \phi^*)^l \) for some \( l \) and \( \lambda_i \neq \lambda \). We can and do choose \( w_0 \) with all its entries positive. Now \( r w_i = 0 \) since

\[
0 = r(I\lambda_i - \phi^*)^l w_i = (\lambda_i - \lambda)^l r w_i.
\]

Therefore if \( \mu(v, k) > 0 \) then writing \( v \) in the basis \( \{w_0, w_1, \ldots, w_{n-1}\} \) the coefficient \( c_0 \) of \( w_0 \) must be positive. Since \( \lambda \) exceeds the modulus of all other eigenvalues, as \( l \to \infty \), we have \( ||(\lambda^{-l}(\phi^*)^l v - c_0 w_0)|| \to 0 \). This implies \((\phi^*)^l v \in (\mathbb{Z}_+)^n\) for some \( l \).

Next we show that the map \( \mu : C \to \mathbb{R} \) induces a map from \( H \) to \( \mathbb{R} \); since \( H = C/\text{Im} \delta \), the following will show that this map is well-defined.

**Lemma 2.** \( \text{Im} \delta \subseteq \text{Ker} \mu \).

Proof: Let \( e \) be an edge and let \( f_e \in C^1(\Sigma_x) \) be the 1-cochain which assigns a value 1 to the edge \( e \) and 0 to all other edges. Then \( \delta(f_e) \) is a 2-cochain which acts on
the tiles as follows: let $T$ be an $m$th level tile. Then $\delta(f_e)(T)$ counts the number of occurrences of $e$ on the boundary $T$ with a $\pm$ sign depending upon the orientation. In particular, $|\delta(f_e)(T)|$ is less than or equal to the number of occurrences of $e$ on the boundary of the level $m$ tile $T$.

Now assume $\mu(\delta(f_e)) > 0$. Then there is an $l$ such that for all for any level 0 tile $T$, $(\phi^*)^l \delta(f_e)(T) > 0$. Furthermore, for fixed $T$ this quantity scales like $\lambda^l$ for sufficiently large $l$. This means that for level $l$ tiles there are at least $O(\lambda^l)$ occurrences of $e$ on the boundary. However, $\lambda$ is the area stretching factor, so there can only be $O(\lambda^{l/2})$ edges on the boundary of a level $l$ tile (since the perimeter scales like $\lambda^{l/2}$ and there is a minimum length to each edge). This is a contradiction.

A similar argument shows that $\mu(\delta(f_e))$ cannot be negative. Therefore the image of $\delta$ is a subgroup of the kernel of $\mu$ and $C_+ \cap \text{Im} \delta = \{0\}$. ■

Abusing notation, define $\mu : H \to \mathbb{R}$ by $\mu([x]) = \mu(x)$. Letting $H_+$ be the pre-image under $\mu$ of $\mathbb{R}_+$, $(H, H_+)$ is an ordered group since

1. $H_+ + H_+ \subset H_+$
2. $H_+ \cap -H_+ = \{0\}
3. H_+ - H_+ = H$ (this follows from the similar property of $C_+$).

4. Some relations between $X_x$, $X_0$ and $X_\delta$.

Before we prove the invariance of $P$-positivity and extend to arbitrary relative orientation groups, we need to understand the role of rotations in tilings. Let $G$ be the relative orientation group of a tiling and let $R$ be the associated group ring.

Finite relative orientation groups. We begin by assuming $G$ is finite. As previously indicated, $G$, and therefore $R$, acts naturally on the simplicial (co)chain groups $C^i(\Sigma_x)$ and $C_i(\Sigma_x)$, giving them the structure of modules over $R$. A finite index subgroup of each module can be written as a direct sum of irreducible representations of $G$. The substitution maps are equivariant and so are module homomorphisms, mapping each irreducible representation to itself. The (co)boundary maps are also equivariant, and so map a representation in $C^i$ (or $C_i$) to the corresponding representation in $C^{i+1}$ (or $C_{i-1}$). Thus the integer (co)homology groups, and in particular the top cohomology, are themselves finite extensions of finite quotients of direct sums of irreducible representations.

The need for finite extensions and finite quotients comes about as follows. When a compact group acts on $\mathbb{R}^n$, $\mathbb{R}^n$ splits up as the direct sum of irreducible representations of that group. However, the same observation does not apply to $\mathbb{Z}^n$. For example, if $\mathbb{Z}_2$ acts on $\mathbb{Z}^2$ by permuting the two coordinates, the irreducible representations are all multiples of $(1, 1)$ and all multiples of $(1, -1)$. The direct sum of these two representations gives all elements $(a, b)$ with $a + b$ even — an index 2 subgroup of $\mathbb{Z}^2$. Similarly, finite index subgroups of the kernels and images of the (co)boundary maps are direct sums of representations of $G$. Computing (co)homology one representation at a time is equivalent to taking a finite index subgroup of the kernel of $\delta$ and modding out by a finite index subgroup of the image of $\delta$. Correcting for our taking too small a numerator means doing a finite
extension. Correcting for the denominator involves taking a finite quotient.

Since the substitution map is equivariant, rotating an eigenvector gives another eigenvector with the same eigenvalue. Since the Perron eigenvectors are unique, with purely positive entries, they must be rotationally invariant and therefore belong to the trivial representation. Since the left Perron eigenvector is invariant, $\mu$ of a cochain is also rotationally invariant. But that implies that $\mu$ acting on each nontrivial representation is identically zero.

For example in the uncollared chair complex there is just one tile, appearing in four different orientations, and $C^2 \sim R = \mathbb{Z}[t]/(t^4 - 1)$ where $t$ represents rotation by $\pi/2$. The substitution map is given by the $1 \times 1$ matrix $2 + t + t^3$. In the trivial representation we set $t = 1$, so our matrix has Perron eigenvalue 4, with the Perron eigenvector being any multiple of an invariant element of $C^2$, i.e. of $(1 + t + t^2 + t^3)$.

This suggests a streamlined computation of the Perron eigenvalue and Perron eigenvector. Instead of working on all of $\Sigma_x$, work on the simplicial complex $\Sigma_0 = \Sigma_x/G$, whose (co)chain groups are precisely the trivial representations of $G$ within the (co)chain groups of $\Sigma_x$. $C^2(\Sigma_x) \otimes_\mathbb{Z} R$ equals $C^2(\Sigma_0) \otimes_\mathbb{Z} R$ plus the other representations that appear in $C^2(\Sigma_x)$. The Perron eigenvalue on $C^2(\Sigma_x) \otimes_\mathbb{R} R$ is precisely the Perron eigenvalue on $C^2(\Sigma_0) \otimes_\mathbb{R} R$, and the (left and right) Perron eigenvectors on $C^2(\Sigma_x) \otimes_\mathbb{R} R$ come from Perron eigenvectors on $C^2(\Sigma_0) \otimes_\mathbb{R} R$.

Since the (co)chain groups of $\Sigma_0$ are isomorphic to the invariant parts of the (co)chain groups of $\Sigma_x$, and since the (co)boundary and substitution maps are equivariant, we have

**Theorem 2.** Up to finite extensions, the (co)homology of $\Sigma_0$ is equal to the rotationally invariant part of the (co)homology of $\Sigma_x$, and the cohomology of $X_0$ is isomorphic to the rotationally invariant part of the cohomology of $X_x$.

We cannot make any statements about the homologies of $X_0$ and $X_x$ because homology, unlike cohomology, does not behave well under inverse limits.

Now $X_\phi$ is a circle fibration over $X_0$, with singular fibers corresponding to tilings that are symmetric about the origin. (These singular fibers have finite multiplicity, since any given tiling can only admit discrete rotational symmetry about the origin.) The existence of these singular fibers means that computing $\pi_1(X_\phi)$, $H^1(X_\phi)$ and $H^2(X_\phi)$ from corresponding data on $X_0$ can be complicated. However, computing the top cohomology $H^3(X_\phi)$ is quite easy. From the spectral sequence of the fibration, it follows immediately that $H^3(X_\phi)$ is isomorphic to $H^2(X_0)$ [BoT]. From this, and from the fact that $\Sigma_\phi$ is a circle fibration over $\Sigma_0$, we have

**Corollary 1.** Up to finite extensions, the top cohomology of $\Sigma_\phi$ is equal to the rotationally invariant part of the top cohomology of $\Sigma_x$. Up to finite extensions, the top cohomology of $X_\phi$ is isomorphic to the rotationally invariant part of the top cohomology of $X_x$.

Note also that the top-dimensional substitution matrix for $\Sigma_\phi$ is identical to the top-dimensional substitution matrix for $\Sigma_0$ — the $S^1$ factor just comes along for the ride. We may therefore compute our order structure on whichever of the
three spaces $\Sigma_x$, $\Sigma_0$ or $\Sigma_\phi$ is most convenient.

To illustrate these principles, we compute the cohomology of the Penrose tilings. Here the relative orientation group is $\mathbb{Z}_{10}$, with group ring $R = \mathbb{Z}[t]/(t^{10} - 1)$, where $t$ represents rotation by $2\pi/10$. There are 4 irreducible representations of $G$, corresponding to the factorization $t^{10} - 1 = (t - 1)(t + 1)(t^4 + t^3 + t^2 + t + 1)(t^4 - t^3 + t^2 - t + 1)$. The first two are one dimensional and $t$ acts by multiplication by $\pm 1$. The other two representations are 4 dimensional and correspond algebraically to $\mathbb{Z}[t]/(t^4 + t^3 + t^2 + t + 1)$ and $\mathbb{Z}[t]/(t^4 - t^3 + t^2 - t + 1)$.

In this tiling, there are 4 types of tiles, whose geometry and substitutions are shown in Fig. 9. Similarly, there are four kinds of edges, each in 10 orientations, and four kinds of vertices, which we label $\alpha, \beta, \gamma, \delta$, with the relations $\alpha = t\beta$, $\beta = t\alpha$, $\gamma = t\delta$ and $\delta = t\gamma$. The chain groups of $\Sigma_x$ are therefore $C_2 = C_1 = R^4$ and $C_0 = (\mathbb{Z}[t]/(t^2 - 1))^2$. The boundary maps are given by the matrices

$$
\partial_1 = \begin{pmatrix} 1 - t & -1 & -t & -1 \\ 0 & 1 & 1 & t \end{pmatrix}; \quad \partial_2 = \begin{pmatrix} -1 & t & t^4 & -t^7 \\ -1 & t^9 & -t & t^8 \\ 1 & -t^5 & 0 & 0 \\ 0 & 0 & 1 & -t^5 \end{pmatrix}, \quad 8)
$$

where we have taken $(\alpha, \gamma)$ as our basis for $C_0$. The coboundary maps are given by the transposes of these matrices, with $t$ replaced throughout by $t^{-1}$. The ranks of these matrices are easily computed by row reduction, one representation at a time:

1. If $t = 1$, then $\partial_2^*$ has rank 2 and $\partial_1^*$ has rank 1, so $H^0(\Sigma_x)$ and $H^1(\Sigma_x)$ contain one copy of this representation while $H^2(\Sigma_x)$ contains two.
2. If $t = -1$, then $\partial_2^*$ and $\partial_1^*$ have rank two, so $H^2(\Sigma_x)$ contains two copies of this representation while $H^1(\Sigma_x)$ and $H^0(\Sigma_x)$ contain none.
3. If $t^4 + t^3 + t^2 + t + 1 = 0$, then $\partial_2^*$ is zero, since $C_0$ (also $C_0^0$) does not contain this representation. $\partial_2^*$ has rank 4, so all cohomologies are trivial.
4. If $t^4 - t^3 + t^2 - t + 1 = 0$, then again $\partial_1^*$ is trivial. Now $\partial_2^*$ has rank 3, so $H^1(\Sigma_x)$ and $H^2(\Sigma_x)$ each contain this representation once.

Combining these results, we have that, up to finite extensions, $H^0(\Sigma_x) = \mathbb{Z}$, $H^1(\Sigma_x) = \mathbb{Z}[t]/(t-1) \oplus \mathbb{Z}[t]/(t^4 - t^3 + t^2 - t + 1)$ and $H^2(\Sigma_x) = (\mathbb{Z}[t]/(t - 1))^2 \oplus (\mathbb{Z}[t]/(t+1))^2 \oplus (\mathbb{Z}[t]/(t^4 - t^3 + t^2 - t + 1))$.

Next we look at the substitution matrices, which in dimensions 2 and 1 are

$$
\phi_2 = \begin{pmatrix} t^7 & 0 & 0 & t^4 \\ 0 & t^3 & t^6 & 0 \\ t^3 & 0 & t^4 & 1 \\ 0 & t^7 & 1 & t^6 \end{pmatrix}; \quad \phi_1 = \begin{pmatrix} 0 & 0 & 0 & t^8 \\ t^4 & 0 & 0 & 0 \\ -t^7 & 0 & 0 & 0 \\ 0 & -t^3 & 0 & -t^5 \end{pmatrix}, \quad 9)
$$

These matrices (and their pullbacks) are easily seen to be invertible in all representations, and so induce isomorphisms in cohomology. Therefore $H^*(X_x) = H^*(\Sigma_x)$.

We can also compute the cohomology of $\Sigma_0$ and $X_0$. The complex $\Sigma_0$ has four faces, four edges and two vertices. The boundary maps can be read off from 8) by replacing $t$ throughout by 1, and the substitution maps are similarly obtained from 9). In other words, the computation for $\Sigma_0$ and $X_0$ is precisely the same as the $t = 1$ part of the computation for $\Sigma_x$ and $X_x$, with the result that $H^2(X_0) = H^2(\Sigma_0) = \mathbb{Z}^2$.

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Figure 9. The Penrose substitution in detail
$H^1(X_0) = H^0(X_0) = H^1(\Sigma_0) = H^0(\Sigma_0) = \mathbb{Z}$. We therefore also obtain $H^3(X_\phi) = \mathbb{Z}^2$.

**Arbitrary relative orientation groups.** We have seen that the spaces $\Sigma_x$ and $X_x$ are generally more complicated than the spaces $\Sigma_0$ and $X_0$, or $\Sigma_\phi$ and $X_\phi$. In the case of finite relative orientation group this is a virtue, as the cohomology of $X_x$ gives us information beyond that of $X_\phi$. In the case of infinite relative orientation group, however, the topologies of $\Sigma_x$ and $X_x$ are too rich, and we shall see that the cohomologies (when nontrivial) are infinitely generated. This makes them too big to handle with the formalism of primitive matrices and Perron eigenvectors. Instead, we are forced to work on $\Sigma_0$ and $X_0$ (or $\Sigma_\phi$ and $X_\phi$) where the cohomology groups are finitely generated and we can define an order structure. This structure then pulls back to an ordering of the rotationally-invariant elements of the top cohomology of $X_x$.

If a tiling space has an infinite relative orientation group, then the simplicial complex $\Sigma_x$ is noncompact, but $\Sigma_0$ is still compact, and has a finite substitution matrix. In that case, the top homology of $\Sigma_x$, if nontrivial, is infinitely generated. This is seen as follows. Let $C$ be a boundaryless 2-chain on $\Sigma_x$. Since $C$ is a finite collection of tiles, only a finite number of orientations appear in $C$. By applying an appropriate rotation $g$, we obtain another boundaryless 2-chain $C' = g(C)$ whose tiles appear in a completely different set of orientations from those of $C$. Similarly, $C'' = g(C')$ is different from $C'$. If the element $g$ has infinite order, we can generate an infinite collection of boundaryless chains, all linearly independent in $C_2(\Sigma_x)$. Thus $H_2(\Sigma_x)$ is infinitely generated. A similar argument, involving the dual to a tile in $C$, shows that $H^2$ is also infinitely generated. Furthermore, since the dual to a direct sum is a direct product, $H^2$ contains elements represented by infinite sums of duals of tiles.

Finding the left Perron eigenvector is also a problem. The left Perron eigenvector of $\phi^*$ is the same as a right Perron eigenvector of $\phi$. This must be rotationally invariant, and is therefore a sum of an infinite number of tiles. Pairing this infinite sum with an element of $C_2$, which itself has an infinite number of terms, gives an infinite sum of real numbers that need not converge. Thus the $\mu$ map on $H^2(\Sigma_x)$, or on $C_2(\Sigma_x)$, is not well defined.

Working on $\Sigma_0$, however, we have no problem. The space $\Sigma_0$ is a compact simplicial complex, with finitely generated (co)chain complexes, and finitely generated (co)homology. The substitution matrix $\phi^*$ is finite and primitive, and has a Perron eigenvalue and Perron eigenvector exactly as before. On $H^2(\Sigma_0)$, and on the limit $H^2(X_0) (= H^3(X_\phi))$ the order structure defined above works exactly as in the case of finite relative orientation group.

**5. The invariance of positivity.**

Having defined the P-positive cone in the top Čech cohomology group of our tiling spaces, we wish to show that this positive cone is invariant (up to an overall sign) under homeomorphisms. (This is not immediate, since the inverse limit
structure that we used to define the positive cone is not invariant.)

In general, the Čech cohomology of a space is the direct limit of Čech cohomologies of open covers where the open covers generate the topology of the space (see [BoT] for details). In our situation we begin by considering a triangulation of $\Sigma_n$ and the open cover of $\Sigma_n$ given by “open stars” of the vertices of the triangulation. By the open star $(\text{star}(v))$ of a vertex $v$ we mean the collection of all simplices that touch $v$, but not including the opposite boundary simplex. We associate to this open cover a simplicial complex called the “nerve” of the open cover. The nerve of an open cover is constructed by including one vertex for each open set in the cover, then including a $k$-simplex between any $k$-vertices such that the intersection of the corresponding $k$ sets in the open cover is nonempty. The Čech cohomology of the cover is defined as the simplicial cohomology of the nerve of that cover. Since every vertex of the triangulation lies in exactly one open star and every $k$-simplex lies in exactly $k$ open stars, the nerve of the open star cover of $\Sigma_n$ is identical to the triangulation of $\Sigma_n$ as a simplicial complex. Thus the Čech cohomology of the open star cover of $\Sigma_n$ is canonically identified with the simplicial cohomology of the triangulation of $\Sigma_n$. (This argument shows that simplicial and Čech cohomology agree for all simplicial complexes.)

Consider triangulations of $\Sigma_n$, generated recursively, so that each triangle in the triangulation of $\Sigma_n$ maps by $\phi$ to a single triangle in the first barycentric subdivision of the triangulation of $\Sigma_{n-1}$. This triangulation of $\Sigma_n$ corresponds to an open cover of $\Sigma_n$, which pulls back by the natural projection $\pi_n : X_x \to \Sigma_n$ to give an open cover $A_n$ of $X_x$. That is, an open set in $A_n$ is the set of tilings in which the origin sits in a specified neighborhood in a tile of level $n$. These neighborhoods, as defined by the triangulations of $\Sigma_n$, are increasingly fine as $n \to \infty$ (even after accounting for the expansion due to $\phi$), so the sequence of open covers $A_n$ generates the topology of $X_x$.

The Čech cohomology of $X_x$ is the direct limit of the Čech cohomologies of the covers $A_n$ (hence the direct limit of the simplicial cohomologies of $\Sigma_n$) with a bonding map defined as follows. From their construction, $A_{n+1}$ refines $A_n$. Each element of $A_n$ (resp. $A_{n+1}$) is a set of the form $\pi_n^{-1}(\text{star}(v))$ where $v$ is a vertex in the triangulation of $\Sigma_n$ (resp. $\Sigma_{n+1}$). We define a vertex map from $\Sigma_{n+1}$ to $\Sigma_n$ by mapping each vertex $u$ in the triangulation of $\Sigma_{n+1}$ to a vertex $v$ in the triangulation of $\Sigma_n$ where $\pi_{n+1}^{-1}(\text{star}(u)) \subset \pi_n^{-1}(\text{star}(v))$. After extending this vertex map linearly to the interior of higher level simplices, the result is our bonding map $f_n : \Sigma_{n+1} \to \Sigma_n$. Although the vertex map, and therefore $f_n$, may not be uniquely defined, the induced chain map $f_n^*$ on cohomology is always the same [BoT].

To define positivity recall that we used an orientation in $\mathbb{R}^2$ to define an orientation of all 2-simplices in $\Sigma_n$. Assume $\alpha$ and $\beta$ are two such oriented 2-simplices which both occur within the same tile $T \in \Sigma_n$. Then the duals to $\alpha$ and $\beta$ are cohomologous. Any positive sum of these cohomology classes was declared to be P-positive in section 3.

To show that this notion of P-positivity is invariant we must consider the isomorphism of Čech cohomology induced by a homeomorphism $h$. Let $\{\Sigma_n\}$ and
Theorem 3. If $X_x$ and $X'_x$ are two fixed-orientation tiling spaces with finite relative orientation groups, and $h : X_x \to X'_x$ is a homeomorphism, then the isomorphism $h^* : H^2(X'_x) \to H^2(X_x)$ preserves the $P$-positive cones, up to an overall sign.

Proof: To see that the $P$-positive cone is preserved by $h^*$, first consider whether the map $h : X_x \to X'_x$ is orientation preserving or reversing on each path-component. That is, fix a tiling $x_0 \in X_x$ and $h(x_0) \in X'_x$. There is a 1:1 map $j : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(g(x_0)) = j(g)h(x_0)$ where $g \in \mathbb{R}^2$ acts by translation. We say $h$ is orientation preserving or reversing based on whether $j$ is orientation preserving or reversing as a homeomorphism of $\mathbb{R}^2$. Since the orbit of each point is dense, and since the action of $\mathbb{R}^2$ is continuous, whether $h$ is orientation preserving or not is independent of the choice of $x_0$. Without loss of generality, assume that $h$ is orientation preserving.

The goal is to show that every $P$-positive cohomology class on $X'$ pulls back to a $P$-positive cohomology class on $X$. But the $P$-positive classes on $X'$ are precisely the classes represented by positive linear combinations of basis cochains on $\Sigma'_n$, with $n$ arbitrarily large. So we take a basic elementary cochain on $\Sigma'_n$ (with $n$ large), pull it back to $X$, and try to show that it’s positive there. Without loss of generality we can assume that this cochain, which we call $1_\Delta$, is dual to a triangle $\Delta$ sitting in the middle of one of the tiles of $\Sigma'_n$. We then find an $m$ large enough so that the open cover induced by the partition of $\Sigma_m$ is a refinement of the open cover induced by the partition of $\Sigma'_n$. This refinement gives a vertex map $\rho$ from the nerve of the cover of $\Sigma_m$ (that is, $\Sigma_m$ itself) to the nerve of the cover of $\Sigma'_n$ (that is, $\Sigma'_n$ itself). We must show that $\alpha = \rho^*1_\Delta$ represents a $P$-positive cohomology class.

Now $\alpha$ itself is not necessarily a positive linear combination of elementary cochains. There will be some triangles $T$ on $\Sigma_m$ where $\alpha$ evaluates to $+1$ or $-1$, and others where it evaluates to $0$. We call the triangles where it evaluates to $\pm 1$ essential. Our task is to show that the positive essential triangles outnumber the negative ones in the following sense: there exists an $N > m$ such that every complete tile in $\Sigma_N$, mapped to $\Sigma_m$ and paired with $\alpha$, yields a positive number. Put another way, $\alpha$ evaluates positively on the image (in $\Sigma_m$) of every tile of level $N$ in a particular tiling in $X$.

Here we are using the identification between a particular tiling in $X$ (viewed as $\mathbb{R}^2$ with appropriate markers) and a path-component in $X$. In this identification, a point on a tiling corresponds to a translate of that tiling with that particular point situated at the origin. Call this space $P$ (for path-connected) and let $P'$ be the corresponding path-component of $X'$. The triangulation of $\Sigma_m$ induces...
a triangulation of $P$, with each vertex, edge, or triangle of $P$ mapping by $\pi_m$ to a corresponding vertex, edge, or triangle of $\Sigma_m$. There are two metrics on $P$, namely the Euclidean metric and the metric inherited from $X$. For short-to-moderate distances, these metrics behave essentially identically:

**Lemma 3.** There exist constants $\epsilon_1$, $\epsilon_2$, and $M$, with $0 < \epsilon_1 < M$ and $0 < \epsilon_2$, such that any two points in the same path component with Euclidean distance in $(\epsilon_1, M)$ have tiling-space distance greater than $\epsilon_2$.

Sketch of Proof: For our fixed set of polygonal tiles let $K$ be a lower bound on the distance between the midpoint of any edge of any tile in any tiling and any other edge in that tiling, and let $\alpha$ be a lower bound of the interior angles of any of the tiles. Let $N$ be such that any open ball of radius $N$ in any tiling contains at least one tile and all the tiles touching it. The result then follows if $M < \min\{K/2, \sin(\alpha/2)\}$ and $\epsilon_2 < \epsilon_1/N$. ■

**Corollary 2.** A path-connected set of tiling-space diameter $\epsilon_2$ or less has Euclidean diameter $\epsilon_1$ or less.

We return to the proof of Theorem 3.

Pick $\epsilon_{1,2}$ for $X$ as in the lemma. Since our homeomorphism $h$ is between compact spaces, $h^{-1}$ is uniformly continuous, so there is a $\delta$ for which all sets of (tiling-space) diameter $\delta$ or less in $X'$ get mapped to sets of (tiling-space) diameter $\epsilon_2$ or less in $X$. Then pick $n$ large enough that we can choose our triangle $\Delta$ to have Euclidean diameter $\delta$ or less. This implies that each path-connected component (PCC) of $h^{-1}(\Delta)$ in $X$ will have Euclidean diameter $\epsilon_1$ or less.

Our proof of positivity now proceeds in four steps:

1. Show that preimages in $P$ of essential triangles appear in disjoint and localized clusters, with each cluster associated to a specific PCC of $h^{-1}(\Delta)$.
2. Show that $\alpha$ applied to each complete cluster gives exactly $+1$.
3. Show that the number of complete clusters in a tile (of any level) is bounded from below by a multiple of the area of the tile.
4. Show that $\alpha$, applied to triangles whose clusters are only partially in the tile, is bounded in magnitude from above by a multiple of the perimeter of the tile. (This contribution need not be positive.)

Since volume scales faster than perimeter, $\alpha$ then evaluates positively on all tiles of sufficiently high level.

Step 1. If $T$ is an essential triangle, then its 3 vertices get sent by the bonding map to the 3 vertices of $\Delta$. But this means that the open star of each of the vertices of $T$ is contained in $h^{-1}$ of the open star of the corresponding vertex of $\Delta$. But then the intersection of these three open stars (namely $T$ itself) is contained in $h^{-1}$ of the intersection of the open stars of the vertices of $\Delta$ (namely $\Delta$ itself). Thus the open set $T$ is contained in the open set $h^{-1}(\Delta)$, and every PCC of $T$ must lie in a PCC of $h^{-1}(\Delta)$.

We define a cluster to be all the PCCs of essential triangles within a specific PCC of $h^{-1}(\Delta)$. The PCCs of $\Delta$ in $P'$ are disjoint (since $\Delta$ was chosen to lie in the
interior of a tile in $\Sigma'_n$) and have Euclidean diameter $\delta$ or less. This implies that the PCCs of $h^{-1}(\Delta)$ in $P$, and therefore the clusters, are disjoint and have Euclidean diameter $\epsilon_1$ or less.

Step 2. Let $D$ be a PCC of $h^{-1}(\Delta)$, and let $\Gamma$ be the sum of all the triangles that intersect $D$. $\Gamma$ is one of our clusters, plus a number of components of inessential triangles. We must show that $\alpha(\Gamma) = +1$. We do by showing that, as a simplicial chain, $\rho(\pi_m(\Gamma)) = \Delta$.

Let $\{v_i\}$ be the collection of all vertices in $\Gamma$. This is precisely the set of vertices whose stars contain points of $D$. If $v_i$ is one of the vertices in this collection then $\rho(\pi_m(v_i))$ must be one of the vertices, $d_1, d_2, d_3$, of $\Delta$. This follows since (star($v_i$)) contains a point in $D$, and the image of this point lies in exactly three open sets in the open cover of $\Sigma'_n$, namely the stars of the three vertices of $\Delta$. Since the cover induced by the triangulation of $\Sigma_m$ is a refinement of the cover induced by the triangulation of $\Sigma'_n$, the open set corresponding to $\pi_m(v_i)$ is contained in one (or more) of these three open sets, and in no others. Thus if $\gamma$ is a 2-simplex in $\Gamma$ then $\rho(\pi_n(\gamma))$ is either $\pm \Delta$ or zero. Therefore, $\rho(\pi_m(\Gamma)) = k\Delta$ for some integer $k$.

We will show that $k = 1$ by showing that the boundary of $\Gamma$ maps to $+1$ times the boundary of $\Delta$.

The boundary of $\Gamma$ is a closed loop in $X_x$. Let $x_1, x_2, x_3$ be the three vertices of $D$ (so that $\{h(x_i)\}$ are the 3 vertices of a PCC of $\Delta$). We may choose 3 distinguished vertices $v_1, v_2, v_3$ on the boundary of $\Gamma$ such that star($v_i$) contains $d_i$ for $i = 1, 2, 3$. Because $x_i$ is contained in a unique open set in $B_n$ (namely that corresponding to the $d_i$), we must have $\rho(\pi_n(v_i)) = d_i$ for all $i$. Now let $p_{12}$ denote the path from $x_1$ to $x_2$ along the boundary of $D$. There is a portion of the boundary of $\Gamma$ which forms a path from $v_1$ to $v_2$ with the property that the star of each vertex along this path intersects $p_{12}$. Since the only elements of $B_m$ that can contain points in $p_{12}$ are the open sets corresponding to $d_1$ and $d_2$, the simplicial map $\rho \circ \pi_m$ applied to this portion of the boundary of $\Gamma$ gives a path that begins at $d_1$, ends at $d_2$ and is completely contained in the edge between $d_1$ and $d_2$. The net output of the chain map $\rho \circ \pi_m$ applied to this path is therefore $+1$ times the edge from $d_1$ to $d_2$. Applying the same reasoning around the entire boundary of $\Gamma$ and recalling that the map $h$ is orientation-preserving, we see that $\rho \circ \pi_m$ applied to boundary of $\Gamma$ is exactly $\partial \Delta$, as required.

Step 3. Since each preimage of an essential triangle has fixed Euclidean area, and since each PCC of $h^{-1}(\Delta)$ has diameter $\epsilon_1$ or less, there is an upper bound to how many essential triangles can fit in a cluster. The number of clusters is then bounded from below by the number of PCCs of essential triangles, divided by the maximum number of triangles per cluster. But each essential triangle appears with prescribed positive density, so the number of clusters must scale with the area. Of these we must subtract off the partial clusters near the boundary, but this scales like the perimeter. See Step 4, below.

Step 4. Of the previously considered clusters, we must give special treatment to the ones within a distance $\epsilon_1$ of the boundary of the tile. These may lie only partially within the tile, and the contribution of the essential triangles within the
tile may not be positive. However, the area of this boundary region is at most $\epsilon_1$ times the perimeter of the tile. Since the area of each triangle is bounded from below, the number of essential triangles in this region is bounded by a multiple of the perimeter.

6. Extension to $X_\phi$ for arbitrary relative orientation groups.

The previous development was for the cohomology of $X_x$ in the case where the relative orientation group is finite. We now consider the Čech cohomology of $X_\phi$ with no restrictions on the relative orientation group.

In the case of arbitrary relative orientation group, P-positivity was defined for $H^2(X_0)$, using Perron eigenvectors and the substitution matrix, at the end of section 4. The natural isomorphisms between $H^3(X_\phi)$ and $H^2(X_0)$, and between $H^3(\Sigma_\phi)$ and $H^2(\Sigma_0)$, then give a consistent definition of P-positivity for $H^3(X_\phi)$. This notion of positivity can be rephrased directly in the language of Čech cohomology.

When working with $X_\phi$, our complexes $\Sigma_n$ are comprised of $S^1 \times$ tiles. Triangulations of $\Sigma_n$ then contain vertices, edges, triangles and tetrahedra. Correspondingly, our open covers, by stars of vertices, have single, double, triple and quadruple intersections. We orient the tetrahedra according to some global orientation on the 2-dimensional Euclidean group. The dual to any two 3-simplices within any $S^1 \times$ tile are cohomologous. We declare that the positive sum of any of the duals to any of these 3-simplices is positive. A cohomology class on $X_\phi$ is positive if it can be represented as the pullback from some $\Sigma_n$ of such a positive sum.

**Theorem 4.** If $X_\phi$ and $X'_\phi$ are two all-orientation tiling spaces, and $h : X_\phi \to X'_\phi$ is a homeomorphism, then the isomorphism $h^* : H^3(X'_\phi) \to H^3(X_x)$ preserves the P-positive cones, up to an overall sign.

Sketch of Proof: The proof of Theorem 3 carries over with only minor modification, as sketched below.

We first consider whether $h$ preserves or reverses the orientations of path components of the tilings spaces – that is, whether it preserves or reverses the orientations of the action of the Euclidean group. Without loss of generality, assume that $h$ is orientation-preserving.

If $\Delta$ is a 3-simplex arising from the triangulation of $\Sigma'_n$, with $n$ large, we choose $m$ such that the open cover induced by the triangulation of $\Sigma_m$ refines the open cover induced by the triangulation of $\Sigma'_n$, pulled back to $X_\phi$ by $h^{-1}$. We then define the cochain $\alpha = \rho^*1_\Delta$ on $\Sigma_n$, where $\rho : \Sigma_m \to \Sigma'_n$ is a vertex corresponding to the refinement. We call a tetrahedron on $\Sigma$ essential if $\alpha$ applied to that tetrahedron is nonzero. We must show that every tile of sufficiently high level, crossed with $S^1$, contains more preimages of positive essential tetrahedra than of negative essential tetrahedra.

As before we work on a particular path-component $P$ of $X_\phi$. $P$ is naturally identified, not with a single tiling, but with the unit tangent bundle of a single tiling as follows. A unit vector at a particular point in the tiling means that the tiling should be translated to put that point at the origin, and rotated so that the
vector points in the positive $x$-direction. This unit tangent bundle has a natural Riemannian metric that plays the same role as that played by the Euclidean metric in the proof of Theorem 3. In particular, analogs to Lemma 3 and Corollary 2 are easily proven.

The proof then proceeds in 4 steps, exactly as before. We show that preimages of essential tetrahedra appear in clusters, that $\alpha$ applied to $\pi_m$ of each complete cluster is $+1$, that the number of complete clusters in a tile of high level (crossed with $S^1$) scales as the area of the tile, and that the effect of the incomplete clusters is bounded by a multiple of the perimeter of the tile.

The only subtlety is in step 2, where previously we had made explicit use of the geometry of triangles. We define $D$ and $\Gamma$ as before, and show that each vertex in $\Gamma$ must be mapped by $\rho \circ \pi_m$ to a vertex of $\Delta$. Those vertices in $\Gamma$ whose open stars contain a vertex $x_i$ of $D$, must get mapped to the vertex $d_i$ of $\Delta$. Those whose open stars intersect the edge of $D$ that runs from $x_i$ to $x_j$ must get mapped to $d_i$ or $d_j$. Those that intersect the face whose corners are $x_i$, $x_j$ and $x_k$ must be mapped to $d_i$ or $d_j$ or $d_k$. These considerations, together with the fact that $\Gamma$ is contractible, imply that $\partial \Gamma$, viewed as a chain, is mapped to $+1$ times $\partial \Delta$, and hence that $\Gamma$ is mapped to exactly $+1$ times $\Delta$.

7. Algebraic consequences.

Below we examine some of the consequences of isomorphism of ordered groups obtained as direct limits. By Theorems 3 and 4 this will show that certain properties of the area stretching factor are homeomorphism invariants of $X_x$ for the finite orientation case, and are homeomorphism invariants of $X_\phi$ in all cases.

Our setting is as follows. We assume we have Abelian groups $H$ and $H'$, obtained as direct limits of finitely generated free Abelian groups by primitive maps $\phi^*$ with Perron eigenvalues $\lambda$ and $\lambda'$. An element of $H$ will be denoted $[(v, k)]$, meaning the class of the element $v$ in the $k$-th approximant to $H$. Of course, the application we have in mind is for $H$ to be the top Čech cohomology of a tiling space, for the free Abelian groups to be the free part of the top cohomology of $\Sigma_x$ or $\Sigma_{\phi}$, and for the primitive map to be induced by substitution.

Lemma 4. We can choose the left Perron eigenvector $r = (r_1, r_2, \ldots, r_n)$ of the primitive matrix $\phi^*$ so that each $r_i$ is a polynomial in $\lambda$ with integer coefficients.

Proof: To find $r$, we wish to solve the equation

$$r(I\lambda - \phi^*) = 0. \quad (11)$$

In order to solve this, we perform column operations until the matrix $\phi^*$ is in lower triangular form. Throughout this process, each entry of the matrix will be a quotient of integer polynomials in $\lambda$. Therefore there is a solution to the equation where each entry of $r$ is a quotient of integer polynomials in $\lambda$. Thus by rescaling we may obtain a row eigenvector such that each entry is an integer polynomial in $\lambda$. ■
**Integer Case:** In the case where \( \lambda \) is an integer, we will take the \( r \) from the previous lemma and scale by a factor of \( 1/g \) where \( g \) is the greatest common divisor of the entries of \( r \).

**Noninteger Case:** In the case where \( \lambda \) is not an integer (and therefore irrational), take the \( r \) from the previous lemma and scale by a factor of \( 1/\lambda \) where \( p \) is the maximal power of \( \lambda \) that occurs in the entries of \( r \). Now the row eigenvector is in \( \mathbb{Z}[1/\lambda] \).

We define the map \( \mu : H \to \mathbb{R} \) by \( \mu([v, k]) \mapsto \lambda^{-k} rv \) where \( r \) is fixed as above. In either the integer or noninteger case it is clear that the image of \( \mu \) is a subset of \( \mathbb{Z}[1/\lambda] \). Our positive cone \( H_+ \) is then the preimage of the positive real numbers.

Now we suppose that we have two such groups, \( H \) and \( H' \), with maps \( \mu \) and \( \mu' \) from each to \( \mathbb{R} \), constructed as above, with Perron eigenvalues \( \lambda \) and \( \lambda' \), and suppose that \( h : (H, H_+) \to (H', H'_+) \) is an order isomorphism. (We do not assume that \( h \) respects the direct limit structure, only that it respects the order structure.)

**Lemma 5.** Let \( x, y \in H \) with \( \mu(x) \) and \( \mu(y) \) both positive. Then

\[
\frac{\mu(x)}{\mu(y)} = \frac{\mu'(h(x))}{\mu'(h(y))}.
\]

Proof: Let \( a, b \) be positive integers. Then

\[
ax - by \in H_+ \iff \mu(ax - by) \geq 0 \\
\iff a\mu(x) - b\mu(y) \geq 0 \\
\iff \frac{\mu(x)}{\mu(y)} \geq \frac{b}{a}.
\]

Since \( h \) is an order isomorphism,

\[
ax - by \in H_+ \iff ah(x) - bh(y) \in H'_+ \\
\iff \frac{\mu'(h(x))}{\mu'(h(y))} \geq \frac{b}{a}.
\]

Therefore \( b/a \leq \mu(x)/\mu(y) \) if and only if \( b/a \leq \mu'(h(x))/\mu'(h(y)) \). Since this is true for any positive integers \( a \) and \( b \) it must be the case that

\[
\frac{\mu'(h(x))}{\mu'(h(y))} = \frac{\mu(x)}{\mu(y)}.
\]

**Lemma 6.** \( \mathbb{Q}[\lambda] \) and \( \mathbb{Q}[\lambda'] \) are identical as subsets of \( \mathbb{R} \).

Proof: First we claim that there exist \( p(x), q(x) \in \mathbb{Z}[x] \) such that

\[
\lambda = \frac{p(\lambda')}{q(\lambda')}.
\]
Choose \( x = [(v, k)] \in H \) such that \( \mu(x) > 0 \). Let \( y = [(v, k + 1)] \in H \). Then

\[
\frac{\mu'(h(x))}{\mu'(h(y))} = \frac{\mu(x)}{\mu(y)} = \lambda.
\]  

Since the image of \( \mu' \) is a subset of \( \mathbb{Z}[1/\lambda'] \), the claim follows.

Now let \( f(x) \) be the minimal monic polynomial in \( \mathbb{Z}[x] \) for the algebraic integer \( \lambda' \). Since \( q(\lambda') \neq 0 \), the polynomial \( f(x) \) does not divide \( q(x) \). Since \( f(x) \) is irreducible, the number 1 is a greatest common divisor of \( f(x) \) and \( q(x) \) in the ring \( \mathbb{Q}[x] \).

Since \( \mathbb{Q}[x] \) is a principal ideal domain, there exist polynomials \( a(x), b(x) \in \mathbb{Q}[x] \) such that

\[
a(x)q(x) + b(x)f(x) = 1.
\]  

This implies

\[
a(\lambda')q(\lambda') = 1.
\]  

Therefore,

\[
\lambda = a(\lambda')q(\lambda') \in \mathbb{Q}[\lambda'].
\]

which implies that as sets of real numbers,

\[
\mathbb{Q}[\lambda] \subseteq \mathbb{Q}[\lambda'].
\]

The other inclusion follows similarly.  

(Of course, the previous lemma is trivial in the case where \( \lambda, \lambda' \) are integers.)

**Lemma 7.** If \( \lambda \) is an integer then \( \lambda' \) is also an integer. Furthermore, \( \lambda \) and \( \lambda' \) have the same prime factors.

Proof: Assume \( \lambda \) is an integer and recall \( \mu([(v, k)]) \) is defined as \( \lambda^{-k}rv \) where \( r \) is an integer vector with relatively prime entries. Therefore there are integers \( v_1, v_2, \ldots, v_n \) such that

\[
\sum_{i=1}^{n} v_ir_i = 1.
\]

Define \( v \) to be the column vector \( (v_1, v_2, \ldots, v_n)^t \). Then \( \mu([(v, 1)]) = 1 \) and the number 1 is in the image of \( \mu' \).

From the previous lemma it is clear that \( \lambda' \) must also be an integer (it is a Perron eigenvalue which is either an integer or irrational). Thus for the same reason 1 is in the image of \( \mu' \).

Suppose \( y \in H' \) with \( \mu'(y) = 1 \). Let \( [(u, k)] = h^{-1}(y) \). Then

\[
\frac{\mu([(u, k + 1)])}{\mu([(u, k)])} = \frac{\mu'(h([(u, k + 1)]))}{\mu'(y)} = \frac{1}{\lambda} = \mu'(h([(u, k + 1)])).
\]  

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This implies $1/\lambda \in \mathbb{Z}[1/\lambda']$. Similarly, $1/\lambda' \in \mathbb{Z}[1/\lambda]$. Therefore, as subsets of $\mathbb{R}$,

$$\mathbb{Z}[1/\lambda] = \mathbb{Z}[1/\lambda'].$$

This is the case if and only if $\lambda$ and $\lambda'$ have the same prime factors. \qed

Combining these results with Theorems 3 and 4 then yields our main result, Theorem 1.

8. Conclusion.

We add here some remarks of a general nature. The basic objects of the above study are hierarchical tilings of the Euclidean plane, which are mainly of interest for their unusual symmetry (geometric) properties. Our analysis is an outgrowth of a traditional dynamical analysis, in which a space of tilings is equipped with a topological and Borel structure together with the action of Euclidean motions, the latter maintaining contact with geometry. It is notable that we have been ignoring the dynamical action, obtaining topological invariants of tiling spaces. This is a recent development, but not new to this paper, appearing as it has in noncommutative approaches. However a distinctive feature of this work is that our invariant is only sensible in the top (Čech) cohomology, which suggests to us that it might not be essentially topological; more specifically, it might be invariant under maps more general than homeomorphisms.

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