Power law scaling of early-stage forces during granular impact

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We experimentally and computationally study the early-stage forces during high-speed impacts into granular beds. Experiments consist of impacts into 2D assemblies of photoelastic disks of varying stiffness, and complimentary discrete-element simulations are performed in 2D and 3D. The peak force during the initial stages of impact and the time at which it occurs depend only on the intruder velocity, the elastic modulus of the grains, the mass density of the grains, and the intruder size according to power-law scaling forms that are not consistent with Poncelet models or granular shock theory. We find that the early-stage forces are inherently related to wave propagation, but the current theory of shocks in granular media cannot capture them, suggesting that a new theory is needed. The insensitivity of our results to many system details suggest that they may also apply to impacts into similar materials like foams and emulsions.

High-speed impact by an intruder into a granular bed is a ubiquitous process with broad relevance in many disciplines, including ballistics \cite{1-4}, robotics \cite{5,6}, astrophysics \cite{7}, and earth science \cite{8}. The forces exerted by the grains on the intruder are often described using Poncelet drag, which is dominated by a velocity-squared force with a nearly constant drag coefficient \cite{9,12}. However, as noted by Pica Ciamarra \textit{et al.} \cite{13} and many others \cite{8,12,14,18}, the initial impact forces are consistently larger than expected from Poncelet drag or similar drag models, which is sometimes attributed \cite{12} to a shock wave at impact \cite{19}. These large forces are short lived but may be most important for determining damage to the grains and projectile \cite{3,20} and for capturing the dynamics of a highly transient process like a jumping animal or robot \cite{5,6}. However, there is currently no theory that captures these initial forces. In this Letter, we use experiments and simulations to demonstrate that peak forces during the initial stages of granular impact obey simple, power-law scaling forms that depend only on the impact speed, the mass density and stiffness of the grains, and the intruder size. These scaling laws do not fit within the framework of any existing theory related to impact, including Poncelet models and the theory of shocks in granular media \cite{19,22,27}, implying that a new theory is needed for this process.

We uncover these scaling laws through experiments and simulations of impacts into granular beds with varying grain and intruder properties. Experiments involve circular intruders striking a collection of more than 10,000 photoelastic disks (3 mm thick) confined between two Plexiglas sheets (0.91 m \times 1.22 m \times 1.25 cm). These experiments have been used previously to study the microscopic origins of the Poncelet drag laws during the penetration regime \cite{15,16,21} as well as the speed and spatial structure of the shocks propagating away from the point of impact \cite{19}. Here, we focus on the intruder dynamics during the initial stages. Intruders are machined from bronze sheet into disks of diameters $D = 6.35, 12.7$, and 20.32 cm, with masses of $M = 0.062, 0.258$, and 0.671 kg, respectively. We also cut one circular intruder out of aluminum with diameter $D = 12.7$ cm, which has a smaller bulk density, leading to a mass that is 3.4 times smaller than the corresponding bronze intruder. Photoelastic particles are made from three different materials, which we denote soft, medium, and hard; further detail can be found in Ref. \cite{19}. For all particles, the force $f$ required to compress a particle by a distance $\delta$ is experimentally found to obey $f = E^*w\delta(\frac{\delta}{d})^\alpha$, where $\alpha \approx 1.4$, $w = 3$ mm is the particle thickness, $d$ is the particle diameter, and $E^*$ is an effective Young’s modulus. We find $E^* \approx 3$ MPa, 23 MPa, and 360 MPa for soft, medium, and hard particles, respectively. We use bidisperse mixtures: hard particles have $d = 4.3$ and 6 mm, and medium and soft particles have $d = 6$ and 9 mm.

Intruders are dropped from varying heights, yielding an impact velocity $v_0 \leq 6.6$ m/s. We record results with a Photron FASTCAM SA5 at frame rates of 10,000, 25,000, and 40,000 frames per second for soft, medium, and hard particles, respectively. Intruder trajectories are determined by tracking the position at each image using discrete differentiation combined with a low-pass filter to obtain the velocity and acceleration; sample images and trajectories are shown in Fig. 1. Since discrete differentiation of noisy data requires a low-pass filter, we cut off some high-frequency data in the intruder trajectories. This is particularly relevant for hard particles, where the force dynamics evolve on time scales faster than those that we can resolve in the data for the intruder acceleration. However, the photoelastic data can be calibrated to estimate the fast force dynamics, as described in Ref. \cite{21}. For soft and medium particles, the force curves from tracking and from calibrated photoelastic data agree well.

Figure 1 shows photoelastic images along with corresponding trajectory plots during impacts into soft and medium particles; we observe similar phenomenology for all experiments and simulations. We mark times 1-6 on the plots, which correspond with the images 1-6 above. Frames 1-3 in both sets of images show the buildup of
forces beneath the intruder, where frame 3 corresponds to a maximum force $F_{\text{max}}$ on the intruder at time $t_{\text{max}}$. After $t_{\text{max}}$, the shock wave continues to propagate down into the material, but the net force on the intruder begins to decrease. Boundaries do not affect our results: the boundary of the experiment is roughly 5 particle diameters below the bottom of the image, and the shock front in frame 3, corresponding to $t_{\text{max}}$, is still clearly above the lower boundary. We also verify that our results are independent of the boundaries using simulations.

Previous work [19] found that the shock propagation speeds are well described by a granular shock theory developed for 1D Hertzian bead chains [22–25] and then extended to 2D [26] and 3D [27] disordered Hertzian packings. This theory states that if stresses propagating in granular media are large compared with the prestress (as is true in the case of impact where the prestress vanishes at the surface), then they propagate at a shock speed $v_f$ that obeys power law scaling relationships with the maximum grain speed $v_0$ (set by the intruder or piston speed) and the peak pressure $P_{\text{max}}$ inside the shock. Specifically, $v_f \propto v_0^{\frac{2\alpha}{\alpha+1}} \propto (P_{\text{max}})^{\frac{\alpha-1}{\alpha+1}}$, or, by rearranging, $P_{\text{max}} \propto v_0^{\frac{2\alpha}{\alpha+1}}$. This yields $P_{\text{max}} \propto v_0^{6/5}$ for Hertzian grains (where $\alpha = 1.5$) and $P_{\text{max}} \propto v_0^{7/6}$ for the disks we use (with $\alpha \approx 1.4$). Thus, granular shock theory predicts that $F_{\text{max}} \propto P_{\text{max}} \propto v_0^{7/6}$, whereas a Poncelet model would predict $P_{\text{max}} \propto v_0^2$. While the Poncelet theory has no obvious prediction for $t_{\text{max}}$, the shock theory might suggest that $t_{\text{max}} \propto v_f^{-1} \propto v_0^{-1}$, or $t_{\text{max}} \propto v_0^{-1/6}$ if $\alpha \approx 1.4$.

Figure 2(a) shows three experimental trajectories for impacts into hard, medium, and soft particles at a similar impact velocity, $v_0 \approx 4.6$ m/s, along with an additional trajectory with medium particles and slower impact velocity, $v_0 \approx 2.7$ m/s. The velocity $v$, and net upward force $-Ma$ are plotted with respect to time after impact, where $v$ and $a$ are the first and second time derivatives of $z$, respectively, and $M$ is the intruder mass. We measure $F_{\text{max}} = -M(a + g)$ as the maximum force exerted by the grains on the intruder, where $g = 9.81$ m/s$^2$ is the gravitational constant. For the hardest particles, we know $F_{\text{max}}$ will be underestimated from only tracking the intruder, since the force dynamics are too fast to resolved in this manner [21]. Thus, we take the value...
obtained from tracking the intruder [the thicker, black line in Fig. 2(a)], and multiply it by a correction factor (roughly 5) that is the average ratio between the maximum of the calibrated photoelastic signal [the thinner, black line in Fig. 2(a)] and the data obtained by tracking the intruder across all experiments. Similar results are obtained by merely keeping the maximum value of the calibrated photoelastic data, but with significantly more scatter. For all particles, we obtain \( t_{\text{max}} \) by the time at which the intruder deceleration obtained from tracking its trajectory is maximum.

Figure 2(b) and (c) show \( F_{\text{max}} \) and \( t_{\text{max}} \) for circular intruders plotted as a function of initial impact velocity \( v_0 \) for different intruders and particle types. We note several features of this data. First, \( F_{\text{max}} \) and \( t_{\text{max}} \) both appear to depend on \( v_0 \) according to a power law. The solid black lines show \( F_{\text{max}} \propto v_0^{4/3} \) and \( t_{\text{max}} \propto v_0^{-2/3} \), and similar exponent values are found by performing linear fits to the logarithmic data shown. The exponent value of 4/3 for the force is not consistent with the predictions of the shock (7/6) or a Poncelet (2) models. Similarly, the exponent of \(-2/3\) is inconsistent with the shock theory prediction \((-1/6)\). Second, the magnitude of the peak force varies significantly with the intruder size (symbol color) and is less sensitive to the grain stiffness (circles, squares, and triangles represent soft, medium, and hard particles, respectively). The intruder mass density also plays a role, since aluminum intruders (nearly four times lighter) impacting soft particles exhibits weaker peak forces. However, simulations, discussed below, show that this dependence vanishes for heavy intruders. Finally, at small \( v_0 \), \( F_{\text{max}} \) appears to level off as expected, since \( F_{\text{max}} \approx mg \) for very slow impacts where the velocities and accelerations are small and the granular force roughly balances the gravitational force.

Complimentary numerical simulations are implemented using standard DEM techniques [28, 29] in C++ for 2D and LAMMPS [30] (http://lammps.sandia.gov) for 3D; further details are given in Supplemental Material. We prepare a static, gravitationally loaded bed of 10,000 grains in 2D and 100,000 grains in 3D, which we verify are large enough that system size does not affect our results. We then put an intruder just above the bed with downward velocity \( v_0 \), after which time it is free to accelerate due to forces from grains or gravity. We observe trajectories and phenomenology that are similar to those shown in Figs. 1 and 2(a) as well as throughout the literature [13, 17]. We measure \( F_{\text{max}} \) and \( t_{\text{max}} \) as a function of all system parameters. Beginning with 2D simulations of circular intruders impacting beds of frictional, circular grains, there are nine system parameters that can be varied: the intruder diameter \( D \), intruder mass \( M \), impact speed \( v_0 \), grain diameter \( d \), grain mass \( m \), and stiffness \( K = E^* w \), the intergran friction coefficient \( \mu \), the force law exponent \( \alpha \), and the gravitational constant \( g \). The intruder and grain masses \( M \) and \( m \) can also be expressed in terms of mass per area, i.e., \( 4M/\pi D^2 = \sigma_i = \rho_i w \) and \( 4m/\pi d^2 = \sigma_g = \rho_g w \), where \( \rho_i \) and \( \rho_g \) are the mass per volume and \( w \) is the out-of-plane thickness of the particles (\( w \) only has meaning in the experiments). Additionally, there are two output parameters, \( F_{\text{max}} \) and \( t_{\text{max}} \).

Figure 2(a,b) show that the power law scaling is nearly independent of both \( \mu \) and \( \alpha \). The lack of dependence on \( \alpha \) confirms that granular shock theory cannot explain the dependence of \( F_{\text{max}} \) and \( t_{\text{max}} \) with \( v_0 \). Figure 2(c,d) show that the power law scaling for \( F_{\text{max}} \) and \( t_{\text{max}} \) with \( v_0 \) is nearly independent of the intruder weight, set by \( g \) and \( \sigma_i \), particularly once \( \sigma_i/\sigma_g > 4 \). We also find our results do not explicitly depend on \( d \); this is implicitly shown in Fig. 2 which includes values of \( d \) that vary by an order of magnitude. This leaves six quantities: \( F_{\text{max}}, t_{\text{max}}, v_0, D, K, \) and \( \sigma_g \). These form three dimensionless groups, \( F_{\text{max}}/KD, t_{\text{max}}v_0/D, \) and \( v_0/v_b \), where \( v_b = \sqrt{K/\sigma_g} = \sqrt{E^*/\rho_g} \) is the bulk sound speed inside...
a grain as well as the velocity scale associated with any force pulse propagating along networks of grains [19, 20]. Figure 3(a,b) explicitly shows that all our data for 2D simulations collapse when $F_{\text{max}}/KD$ and $t_{\text{max}}v_b/D$ are plotted as a function of $v_0/v_b$.

Surprisingly, we find identical scaling behavior in 3D simulations in LAMMPS, where grain compression is governed by $f = E^*d^2(\alpha/\rho)^g$ (along with frictional and dissipative intergrain interactions; see Supplemental Material). $F_{\text{max}}$ and $t_{\text{max}}$ are constant for slow impact speeds and depend on $v_0$ according to a power-law form for faster impact speeds, with exponents 4/3 (for $F_{\text{max}}$) and $-2/3$ (for $t_{\text{max}}$). The behavior is again nearly independent of $\mu$ and $\alpha$, as well as $g$, $d$, and the mass per volume $\rho_i$ of the spherical intruder. The remaining six parameters are $F_{\text{max}}$, $t_{\text{max}}$, $v_0$, $E^*$, $D$, and $\rho_g$. These parameters form three dimensionless groups, $F_{\text{max}}/E^*D^2$, $t_{\text{max}}v_b/D$, and $v_0/v_b$, where $v_b = \sqrt{E^*/\rho_g}$. Figure 4(c,d) shows collapsed data in 3D frictional Hertzian simulations, which are nearly identical to the 2D frictionless Hookean results shown in Figure 4(a,b). Experimental results are also plotted in Figure 4(a,b). The experimental data are more scattered than the simulations; this is partly expected since force is not directly measured but inferred by tracking the intruder and, in the case of hard particles, using a calibrated photoelastic signal. Experimental impacts of large intruders into soft particles appear to deviate from the scaling, which could be due to the extreme particle deformation and collective stiffening during these impacts [19].

In conclusion, we observe a seemingly universal power law scaling for the peak forces and associated time scale during the early stage of impact, with $F_{\text{max}}/D \approx K(v_0/v_b)^{4/3}$ in 2D and $F_{\text{max}}/D^2 \approx E(v_0/v_b)^{4/3}$ in 3D, and $t_{\text{max}} \approx (D/v_b)(v_0/v_b)^{-2/3}$ in both 2D and 3D, independent of all other system parameters. This scaling is inherently elastic and thus connected to force propagation, but it is not consistent with granular shock theory. Additionally, the lack of dependence on $\rho_g$, shown in Figure 3(d), suggests that these results are not primarily a result of the intruder decelerating but the material beginning to flow. A new theory is needed to explain these scaling laws beyond Poncelet laws and granular shock theory. Additionally, the soft repulsive disks and spheres used in

FIG. 3. (a) $F_{\text{max}}/KD$ and (b) $t_{\text{max}}(v_b/D)$ are plotted as a function of $v_0/v_b$ for 2D simulations with varying grain-grain friction coefficient $\mu$ and force law exponent $\alpha$, showing that power law scaling for peak force and time is independent of $\mu$ and $\alpha$. (c) $F_{\text{max}}/KD$ is plotted as a function of $v_0/v_b$ for varying gravitational constants $g$ and intruder mass density $\rho_i$, showing that the power law scaling is independent of these quantities as well. (d) $F_{\text{max}}/KD$ is plotted as a function of $\rho_i/\rho_g$ for the largest value of $v_0$ shown in (c), showing that $\rho_i$ has little effect on the peak force, especially when $\rho_i > 4\rho_g$.

FIG. 4. Scaled plots of $F_{\text{max}}$ and $t_{\text{max}}$ versus $v_0$ for (a,b) 2D, frictionless grains with $\alpha = 1$, (c,d) 3D frictional grains with $\alpha = 1.5$, and (e,f) 2D, frictional grains from experiment, with $\alpha \approx 1.4$. The symbols in (e,f) are the same as Fig. 2(c,d); For simulations shown in (a-f), the symbol conventions are given in Supplemental Material.
the simulations are commonly used to model other soft, particulate media (like foams or emulsions), suggesting that this description may apply to a much broader group of materials.

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