ON THE GLOBAL AND ∇-FILTRATION DIMENSIONS OF QUASI-HEREDITARY ALGEBRAS

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ABSTRACT. In this paper we consider how the ∇-, Δ- and global dimensions of a quasi-hereditary algebra are interrelated. We first consider quasi-hereditary algebras with simple preserving duality and such that if µ < λ then ∇.f.d.(L(µ)) < ∇.f.d.(L(λ)) where µ, λ are in the poset and L(µ), L(λ) are the corresponding simples. We show that in this case the global dimension of the algebra is twice its ∇-filtration dimension. We then consider more general quasi-hereditary algebras and look at how these dimensions are affected by the Ringel dual and by two forms of truncation. We restrict again to quasi-hereditary algebras with simple preserving duality and consider various orders on the poset compatible with quasi-hereditary structure and the ∇-, Δ- and injective dimensions of the simple and the costandard modules.

INTRODUCTION

Quasi-hereditary algebras were first introduced by Scott [19] in order to study highest weight categories in the representation theory of semisimple complex Lie algebras and algebraic groups, and many important results were proved by Cline, Parshall and Scott (see for example [3]). These algebras can be defined in the context of arbitrary finite-dimensional algebras, and they were studied from this point of view by Dlab and Ringel (see for example [5], [6]) and others. In particular, it turns out that quasi-hereditary algebras are quite common.

One important property of quasi-hereditary algebras is that they have finite global dimension. Furthermore, there is a natural concept of ∇-filtration dimension for representations of quasi-hereditary algebras. This can be considered as a generalisation of the notion of injective dimenions. This was introduced for algebraic groups by Friedlander and Parshall [10] (where they define the notion of good filtration dimension which equals our notion of ∇-filtration dimension of a module). Later work [15, 17], shows that the understanding of the ∇-filtration dimension gives a strong hold on homological properties.

For Schur algebras, the ∇-filtration dimension of simple modules and of Weyl modules and the projective dimensions of Weyl modules have nice relationships with the partial order of the weights.

1991 Mathematics Subject Classification. 16G99.
Key words and phrases. ∇-filtration dimension, good filtration dimension, global dimension, quasi-hereditary algebra, Ringel dual.
Moreover, since there is a duality fixing the simple modules, the combination of $\nabla$-filtration dimension and the dual concept of $\Delta$-filtration dimension, give us an exact relationship between the $\nabla$-filtration dimension and the injective dimension of a module (for the regular blocks).

In this paper, we investigate to what extent the interrelations which were observed for Schur algebras hold for arbitrary quasi-hereditary algebras which have a duality fixing the simple modules. These include the blocks of the category $\mathcal{O}$ defined by Bernštěín, Gel’fand and Gel’fand in [1].

As applications, we determine the $\nabla$-filtration dimension and the global dimension of the Ringel duals of Schur algebras $S(2, r)$. Furthermore, we show that a quasi-hereditary algebra with duality for which the $\nabla$-filtration dimension is strictly increasing as a function on the poset has global dimension twice its $\nabla$-filtration dimension. This applies in particular to the regular blocks of Schur algebras $S(n, r)$ with $p > n$ and regular blocks for category $\mathcal{O}$ ([17, theorem 4.7, section 7]). This proves a particular case of a conjecture of Caenepeel and Zhu [2] and Mazorchuk and Parker [13].

1. Preliminaries

**Definition 1.1.1.** Suppose $S$ is a finite-dimensional algebra over a field $k$. Let $L(\lambda)$ for $\lambda \in \Lambda^+$ be a full set of irreducible $S$-modules, and let $P(\lambda)$ be the projective cover of $L(\lambda)$. We fix a partial order $(\Lambda^+, \leq)$. We then define the **standard module** $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ with composition factors $L(\mu)$ such that $\mu \leq \lambda$.

Recall that $S$ is **quasi-hereditary** if for each $\lambda \in \Lambda^+$,

(i) the simple module $L(\lambda)$ occurs only once as a composition factor of $\Delta(\lambda)$, and

(ii) the projective $P(\lambda)$ has a filtration by standard modules where $\Delta(\lambda)$ occurs once, and if $\Delta(\mu)$ occurs then $\mu \geq \lambda$.

The **costandard modules** $\nabla(\lambda)$, are defined dually by replacing projective by injective modules and quotients by submodules.

We work with finite-dimensional $S$-modules. We write $\mathcal{F}(\Delta)$ for the class of $S$-modules which have a filtration where the sections are $\Delta(\mu)$ for various $\mu$, and similarly we write $\mathcal{F}(\nabla)$ for the class of $S$-modules which have a filtration where the sections are $\nabla(\mu)$ for various $\mu$.

We henceforth assume that $S$ is a quasihereditary algebra with poset $(\Lambda^+, \leq)$. Note that by the definition of a quasi-hereditary algebra all the projective $S$-modules belong to $\mathcal{F}(\Delta)$ and all the injective $S$-modules belong to $\mathcal{F}(\nabla)$. There are other ways of defining quasi-hereditary algebras, but they turn out to be equivalent. See [12] or [8 appendix] for a reasonably self-contained introduction to quasi-hereditary algebras.
We define \( \text{Ext}_S^i(-, -) \) in the usual way (using projective resolutions) on the category of \( S \)-modules. We will drop the subscript if it is clear which category we are working in.

**Definition 1.1.2.** Any \( S \)-module \( X \) has a \( \nabla \)-resolution, that is, there is an exact sequence

\[
0 \to X \to M_0 \to M_1 \to \cdots \to M_d \to 0
\]

with \( M_i \in \mathcal{F}(\nabla) \). We say that \( X \) has \( \nabla \)-filtration dimension \( d \), denoted \( \text{\( \nabla \)} \text{-f.d.}(X) = d \) if the following two equivalent conditions hold:

(i) \( X \) has a \( \nabla \)-resolution of length \( d \) but no \( \nabla \)-resolution of length smaller than \( d \);

(ii) \( \text{Ext}_S^i(\Delta(\lambda), X) = 0 \) for all \( i > d \) and all \( \lambda \in \Lambda^+ \), but there exists \( \lambda \in \Lambda^+ \) such that \( \text{Ext}_S^d(\Delta(\lambda), X) \neq 0 \).

(See [10, proposition 3.4] for a proof of the equivalence of (i) and (ii) where this property is known as the good filtration dimension of \( X \)).

Dually we have the notion of \( \Delta \)-filtration dimension. This is denoted as \( \Delta \text{-f.d.}(X) \). We also define for a quasi-hereditary algebra \( S \),

\[
\text{\( \nabla \)} \text{-f.d.}(S) = \sup\{\text{\( \nabla \)} \text{-f.d.}(M) \mid M \text{ an } S \text{-module}\}
\]

\[
\Delta \text{-f.d.}(S) = \sup\{\Delta \text{-f.d.}(M) \mid M \text{ an } S \text{-module}\}.
\]

But note that \( S \) considered as a left \( S \)-module is projective and hence we have \( \Delta \text{-f.d.}(S) = 0 \). Thus we will only use \( \Delta \text{-f.d.}(S) \) and \( \text{\( \nabla \)} \text{-f.d.}(S) \), which are both non-zero in general, as they are defined above.

Recall that \( \text{Ext}_S^i(\Delta(\mu), M) \) for a \( S \)-module \( M \) vanishes for all \( i > 0 \) and all \( \mu \in \Lambda^+ \) if and only if \( M \) has a \( \nabla \)-filtration. Thus if \( \nabla \text{-f.d.}(M) = 0 \) then \( M \in \mathcal{F}(\nabla) \) and so the \( \nabla \)-filtration dimension is a generalisation of this property.

We also use the notation \( i \text{-d.}(M) \) for the injective dimension of \( M \) and \( p \text{-d.}(M) \) for the projective dimension, as well as \( \text{gl.dim}(S) \) for the global dimension of \( S \).

We have the following important lemma.

**Lemma 1.1.3.** ([10, lemma 2.2].) For \( S \) a quasi-hereditary algebra, \( M, N \) \( S \)-modules and for \( i > \Delta \text{-f.d.}(M) + \text{\( \nabla \)} \text{-f.d.}(S) \) we have

\[
\text{Ext}_S^i(M, N) = 0.
\]

As a consequence we have \( \text{gl.dim}(S) \leq \text{\( \nabla \)} \text{-f.d.}(S) + \Delta \text{-f.d.}(S) \).

It is possible that different partial orders on the set \( \Lambda^+ \) lead to the same quasi-hereditary structure. (I.e. different partial orders may lead to the same standard and costandard modules.)
Once we have a given quasi-hereditary structure (i.e. we are given the standard and costandard modules) we can replace the given partial order by a different one which gives the same standard and costandard modules but is which more labels would be incomparable.

That is if $\lambda < \mu$ and $\lambda$ and $\mu$ are adjacent in the order (that is there is no $\nu \in \Lambda^+$ such that $\mu < \nu < \lambda$), but $L(\lambda)$ is not composition factor of $\Delta(\mu)$ nor of $\nabla(\mu)$ (and hence $\Delta(\lambda)$ is not a section of $I(\mu)$ by Brauer-Humphreys reciprocity), then we may safely remove this relation without affecting the standards or the costandards, since we still get the same modules by Definition 1.1.1.

We may continue removing relations in this fashion until we obtain some minimal partial order which still gives the original standards and costandards. Thus, we may assume that if $\mu < \lambda$ and $\mu$ and $\lambda$ are adjacent in the order then $L(\mu)$ is a composition factor of $\Delta(\lambda)$ or of $\nabla(\lambda)$.

Essentially we have replaced the original partial order by one that is generated by the preorder $\mu < \lambda$ if $L(\mu)$ occurs as a composition factor of $\nabla(\lambda)$ or of $\Delta(\lambda)$.

In this paper we will often assume that $S$ has a duality $^\circ$ fixing the simple modules. (Such a duality is sometimes known as strong duality.) For such an algebra, it then follows that the dual of the costandard module $\nabla(\lambda)^\circ$ is isomorphic to $\Delta(\lambda)$. It is also clear that $\text{Ext}^i_S(M, N) \cong \text{Ext}^i_S(N^\circ, M^\circ)$ for all $i \geq 0$, and hence that $\nabla.f.d.(M) = \Delta.f.d.(M^\circ)$ for $M, N \in \text{mod}(S)$.

2. The Global Dimension of $S$ with Duality

Let $S$ be a quasi-hereditary algebra with duality fixing the simple modules. Then we know that $\text{gl.dim}(S) \leq 2 \nabla.f.d.(S)$ (as $\Delta.f.d.(S) = \nabla.f.d.(S)$ using the remarks above and applying lemma 1.1.6). We ask whether equality holds. (This was originally conjectured for Schur algebras in [16] and for more general $S$ in [2] and [13].) For most of this section we will be assuming that $S$ satisfies a particular property which we will call strong property $A$. (We will weaken this condition slightly in section 5). That is:

$$\mu < \lambda \Rightarrow \nabla.f.d.(L(\mu)) < \nabla.f.d.(L(\lambda)).$$

Regular blocks of the Schur algebra satisfy this property as well as the regular blocks of category $O$. [17].

In the following we write $S \in S_n$ if $S$ is a quasi-hereditary algebra with duality fixing the simples, with an ordering on the simples such that strong property $A$ is satisfied and $\nabla.f.d.(S) = n$.

2.1. The case with $\nabla$-filtration dimension one. We first suppose that we have a quasi-hereditary algebra $S$ which belongs to $S_1$. We split the poset up into a disjoint union $\Lambda^+ = \Lambda_0^+ \cup \Lambda_1^+$ so that $\nabla.f.d.(L(\lambda_0)) = 0$ for $\lambda_0 \in \Lambda_0^+$ and $\nabla.f.d.(L(\lambda_1)) = 1$ for $\lambda_1 \in \Lambda_1^+$. In this case we know that

$$\text{Ext}^2_S(L(\lambda_1), L(\lambda_1)) \cong \text{Hom}_S(Q^\circ, Q) \neq 0.$$

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where \( Q = \nabla(\lambda_1)/L(\lambda_1) \) using [15, lemma 2.6]. So clearly the algebra has \( \text{gl. dim}(S) = 2 \). For the induction to come we will use the following.

**Lemma 2.1.1.** Let \( S \) be in \( S_1 \). Then for all \( S \)-modules \( Q \) with \( \nabla.\text{f.d.}(Q) = 1 \) we have

\[
\text{Ext}^2_S(Q^o, Q) \neq 0.
\]

**Proof.** We first note that the presence of strong property \( A \) gives us that if \( \lambda_1 \) and \( \lambda_2 \) are both in \( \Lambda_i^+ \) for \( i \in \{0, 1\} \) then \( \lambda_1 \) and \( \lambda_2 \) are incomparable and hence \( \text{Ext}^1_S(L(\lambda_1), L(\lambda_2)) \cong \text{Ext}^1_S(L(\lambda_2), L(\lambda_1)) \cong 0 \). This in particular implies that the injective hull of \( L(\lambda_i) \) is \( \nabla(\lambda_i) \) for \( \lambda_i \in \Lambda_i^+ \) and also that the quotient \( \nabla(\lambda_i)/L(\lambda_i) \) is a direct sum of simples \( L(\mu_j) \) with \( \mu_j \in \Lambda_0^+ \).

Case (1): Assume first that the socle of \( Q \) has only \( L(\lambda_i) \) with \( \lambda_i \in \Lambda_1^+ \). Then we have a (non-split) injective hull

\[
0 \to Q \to I(Q) = \bigoplus_i \nabla(\lambda_i) \to N \to 0
\]

and \( N \) is a direct sum of copies of \( L(\mu_j) \) with \( \mu_j \in \Lambda_1^+ \). Applying \( \text{Hom}_S(Q^o, -) \) gives

\[
\text{Ext}^1_S(Q^o, N) \cong \text{Ext}^2_S(Q^o, Q)
\]

But \( \text{Ext}^1_S(Q^o, N) \cong \text{Ext}^1(N^o, Q) = \text{Ext}^2_S(N, Q) \) since \( N \cong \bigoplus_j L(\mu_j) \) is self-dual. This latter Ext group is non-zero (consider the above exact sequence).

Case (2): Now suppose \( Q \) is arbitrary, then we have an exact sequence

\[
0 \to \bigoplus_j L(\mu_j) \to Q \to \bar{Q} \to 0
\]

where \( \mu_j \in \Lambda_0^+ \), \( \bar{Q} \neq 0 \) and has only \( L(\lambda_i) \) with \( \lambda_i \in \Lambda_1^+ \) in the socle. Now \( \nabla.\text{f.d.}(L(\mu_j)) = 0 \) and \( \nabla.\text{f.d.}(Q) = 1 \) hence \( \nabla.\text{f.d.}(\bar{Q}) = 1 \), using [15, lemma 2.5]. Using case (1) we know that \( \text{Ext}^2_S(\bar{Q}^o, \bar{Q}) \neq 0 \). We will show that there is an epimorphism from

\[
\text{Ext}^2_S(Q^o, \bar{Q}) \to \text{Ext}^2_S(\bar{Q}^o, \bar{Q})
\]

and this will be enough to show that the first Ext group is non-zero.

Apply \( \text{Hom}_S(Q^o, -) \) to the exact sequence for \( Q \), this gives an exact sequence

\[
\ldots \text{Ext}^2_S(Q^o, Q) \to \text{Ext}^2_S(Q^o, \bar{Q}) \to 0
\]

as \( \bigoplus_j L(\mu_j) \) has injective dimension \( \leq 2 \).

Now apply \( \text{Hom}_S(-, \bar{Q}) \) to the exact sequence

\[
0 \to \bar{Q}^o \to Q^o \to \bigoplus_j L(\mu_j) \to 0
\]

This gives an exact sequence

\[
\text{Ext}^2_S(Q^o, \bar{Q}) \to \text{Ext}^2_S(\bar{Q}^o, \bar{Q}) \to 0
\]
as $\bigoplus_j L(\mu_j)$ has projective dimension $\leq 2$. The composite of these two maps gives the desired epimorphism.

2.2. Assume now that $S$ is a quasi-hereditary algebra in $\mathcal{S}_n$. We know an algebra $S_1$ in $\mathcal{S}_1$ has global dimension $2$. Moreover for every $S_1$-module $Q$ with $\nabla \text{ f. d.}(Q) = 1$ we know $\text{Ext}_{S_1}^2(Q^o, Q) \neq 0$.

**Theorem 2.2.1.** An algebra $S$ in $\mathcal{S}_n$ has global dimension $2n$. Moreover for every $S$-module $Q$ with $\nabla \text{ f. d.}(Q) = n$ we have

$$\text{Ext}_{S}^{2n}(Q^o, Q) \neq 0.$$ 

**Proof.** We have already proved this for $n = 1$. We now assume inductively that an algebra $S_{n-1} \in \mathcal{S}_{n-1}$ has global dimension $2(n-1)$ and that for every $S_{n-1}$-module $Q$ with $\nabla \text{ f. d.}(Q) = n - 1$ we have

$$\text{Ext}_{S_{n-1}}^{2(n-1)}(Q^o, Q) \neq 0.$$ 

We first show that $\text{gl. dim}(S) = 2n$. We know that $\text{gl. dim}(S) \leq 2 \nabla \text{ f. d.}(S) = 2n$. So it is enough to show that $\text{Ext}_{S}^{2n}(L(\lambda), L(\lambda)) \neq 0$, for $\lambda$ with $\nabla \text{ f. d.}(L(\lambda)) = n$. We have an exact sequence

$$0 \rightarrow L(\lambda) \rightarrow \nabla(\lambda) \rightarrow Q \rightarrow 0$$

and $\nabla \text{ f. d.}(Q) = n - 1$. Let $S_{n-1}$ be the quotient $S/Se_1S$ of $S$ where $\Gamma = \{ \mu \mid \nabla \text{ f. d.}(L(\mu)) = n \}$, then $S_{n-1}$ belongs to $\mathcal{S}_{n-1}$. (See section 4.1 for more details about $S/Se_1S$.) Moreover $Q$ is an $S_{n-1}$-module, by the assumptions on $S$.

Applying $\text{Hom}_S(L(\lambda), -)$ to the above exact sequence and then $\text{Hom}_S(-, Q)$ to its $^o$-dual, gives us

$$\text{Ext}_{S}^{2n}(L(\lambda), L(\lambda)) \cong \text{Ext}_{S}^{2n-1}(L(\lambda), Q) \cong \text{Ext}_{S}^{2(n-1)}(Q^o, Q)$$

since $\nabla(\lambda)$ is injective. So by the inductive hypothesis we get that

$$\text{Ext}_{S}^{2n}(L(\lambda), L(\lambda)) \cong \text{Ext}_{S}^{2(n-1)}(Q^o, Q) \cong \text{Ext}_{S_{n-1}}^{2(n-1)}(Q^o, Q) \neq 0.$$ 

(For the last equality see section 4.1.)

Now let $Q$ be some $S$-module with $\nabla \text{ f. d.}(Q) = n$. We must show that $\text{Ext}_{S}^{2n}(Q^o, Q) \neq 0$. We note that the modules $\nabla(\lambda_i)$ with $\nabla \text{ f. d.}(L(\lambda_i)) = n$ must be injective. Also $Q$ must have at least one $L(\lambda)$ as a composition factor with $\nabla \text{ f. d.}(L(\lambda)) = n$.

Assume first that the socle of $Q$ is a direct sum of $L(\lambda_i)$’s with $\nabla \text{ f. d.}(L(\lambda_i)) = n$. Then we have the injective hull

$$0 \rightarrow Q \rightarrow I \rightarrow R \rightarrow 0$$
where $I \cong \oplus \nabla (\lambda_i)$. Moreover $\nabla \cdot f \cdot d. (R) = n - 1$ since $I$ is injective. We also know that $R$ is an $S_{n-1}$-module, $(S_{n-1}$ as before) by construction. So by the inductive hypothesis we know that $\text{Ext}^{2(n-1)}_{S_{n-1}}(R^o, R) \neq 0$. We now have

$$\text{Ext}^{2n}_{S}(Q^o, Q) \cong \text{Ext}^{2(n-1)}_{S}(R^o, R) \cong \text{Ext}^{2(n-1)}_{S_{n-1}}(R^o, R)$$

as before as $I$ is injective.

We now consider the general case. Let $U \subset Q$ be the largest submodule with no composition factors $L(\lambda_i)$ with $\nabla \cdot f \cdot d. (L(\lambda_i)) = n$, and let $V = Q/U$. Then the socle of $V$ has only composition factors $L(\lambda_i)$, note also that $V \neq 0$. Consider the exact sequence

$$(*) \quad 0 \to U \to Q \to V \to 0.$$ 

We know that $\nabla \cdot f \cdot d. (U) \leq n - 1$ since $U$ is an $S_{n-1}$-module. But $Q$ has $\nabla \cdot f \cdot d. (Q) = n$ and since $\nabla \cdot f \cdot d. (V) \leq n$ it follows that $\nabla \cdot f \cdot d. (V) = n$. So we know from the first case that $\text{Ext}^{2n}_{S}(V^o, V) \neq 0$. Therefore it is enough to show that there is an epimorphism from $\text{Ext}^{2n}_{S}(Q^o, Q)$ onto $\text{Ext}^{2n}_{S}(V^o, V)$.

Apply $\text{Hom}_S(Q^o, -)$ to the exact sequence $(*), this gives

$$\to \text{Ext}^{2n}_{S}(Q^o, Q) \xrightarrow{\phi} \text{Ext}^{2n}_{S}(Q^o, V) \to \text{Ext}^{2n+1}_{S}(Q^o, U).$$

The last term is zero since $S$ is known to have global dimension $2n$.

Next, apply $\text{Hom}_S(-, V)$ to the exact sequence

$$0 \to V^o \to Q^o \to U^o \to 0$$

which gives

$$\to \text{Ext}^{2n}_{S}(Q^o, V) \xrightarrow{\psi} \text{Ext}^{2n}_{S}(V^o, V) \to \text{Ext}^{2n+1}_{S}(U^o, V) = 0.$$

The composite $\psi \circ \phi$ gives the required epimorphism. \hfill $\square$

2.3. The previous section proves a special case of the conjecture of [2][13] that the global dimension of any quasi-hereditary algebra $S$ with simple preserving duality is twice its $\nabla$-filtration dimension. We suspect that a stronger property may be true. That is that one of the equivalent conditions of the following lemma hold.

**Lemma 2.3.1.** $\text{Ext}^{2i}(M^o, M) \neq 0$ for all $i \leq \nabla \cdot f \cdot d. (M)$ if and only if $\text{Ext}^{2}(M^o, M) \neq 0$ for all $M$ with $\nabla \cdot f \cdot d. (M) \neq 0$.

**Proof.** $(\Rightarrow)$ clear. $(\Leftarrow)$ Clearly $\text{Hom}(M^o, M) \neq 0$ for all $M \neq 0$ as the head of $M^o$ is isomorphic to the socle of $M$.

Now take an injective resolution for $M$ with $d = \nabla \cdot f \cdot d. (M) \neq 0$ (so $M$ is not injective).
We denote the images of the map $I_i \to I_{i+1}$ by $N_{i+1}$. We have $\nabla. f. d.(N_i) = \sup\{0, \nabla. f. d.(M) - i | i \in \mathbb{N}\}$ by dimension shifting.

Now suppose $i \leq \nabla. f. d.(M) = d$. By dimension shifting and duality we have that

$$\text{Ext}^2(M^\circ, M) \cong \text{Ext}_{i+1}^{i+1}(M^\circ, N_{i-1})$$
$$\cong \text{Ext}^{i+1}(N_{i-1}^\circ, M)$$
$$\cong \text{Ext}^2(N_{i-1}^\circ, N_{i-1})$$

which is non-zero as $i \leq d = \nabla. f. d.(M)$ and so $\nabla. f. d.(N_i - 1) = d - i + 1 \geq 1$. □

In a similar vein we have:

**Lemma 2.3.2.** $\text{Ext}^2(M^\circ, M) \neq 0$ for $d = \nabla. f. d.(M)$ if and only if $\text{Ext}^2(M^\circ, M) \neq 0$ for all $M$ with $\nabla. f. d.(M) = 1$.

Indeed, we have proved that the first condition of this lemma holds for our special case in the previous section. We will give another example of a quasi-hereditary algebra for which the first condition of this lemma holds in example 5.5.1.

3. $\nabla$-filtration and Global dimensions for Ringel duals

In this section we investigate the relationship between the $\nabla$-and $\Delta$-filtration dimensions for a quasi-hereditary algebra and its Ringel dual (as defined in [18]).

3.1. A tilting module is a module with both a $\nabla$-filtration and a $\Delta$-filtration. There is a unique indecomposable tilting module $T(\lambda)$ for each $\lambda \in \Lambda^+$ such that $L(\lambda)$ occurs only once and any other composition factor $L(\mu)$ of $T(\lambda)$ has $\mu < \lambda$. Every tilting module is a direct sum of $T(\mu_i)$ for some $\mu_i \in \Lambda^+$. A full tilting module $T$ is a tilting module for which for all $\mu \in \Lambda^+$, $T(\mu)$ is a direct summand. We take a full tilting module $T$ and form a Ringel dual $S' = \text{End}_S(T)^{\text{op}}$. A Ringel dual is also a quasi-hereditary algebra with poset $(\Lambda^+, \leq')$, where $\leq'$ is the opposite ordering to $\leq$ on $\Lambda^+$. We distinguish the standards, costandards etc. for a Ringel dual from that of the starting algebra by a prime. Different $T$ lead to different ‘Ringel duals’ but it is unique up to Morita equivalence. So we often say the Ringel dual. There is a left exact functor $F : S \to S'$ which takes a module $M$ to $\text{Hom}_S(T, M)$ regarded as an $S'$-module in the usual manner.

The following relationships hold between various modules for $S$ and $S'$. $\Delta'(\lambda) = F\nabla(\lambda)$, $P'(\lambda) = FT(\lambda)$ and $T'(\lambda) = FI(\lambda)$ for $\lambda \in \Lambda^+$. 
Proposition 3.1.1. We have the following equalities.

(i) $\Delta \text{. f. d.}(\nabla(\lambda)) = \text{p. d.}(\Delta'(\lambda))$

(ii) $\text{i. d.}(\nabla(\lambda)) = \nabla \text{. f. d.}(\Delta'(\lambda))$

(iii) $\text{p. d.}(\Delta(\lambda)) = \Delta \text{. f. d.}(\nabla(\lambda))$

(iv) $\nabla \text{. f. d.}(\Delta(\lambda)) = \text{i. d.}(\nabla(\lambda))$

Proof. (i). We take a minimal length tilting resolution for $\nabla(\lambda)$

$$0 \to T_d \to \cdots \to T_1 \to T_0 \to \nabla(\lambda) \to 0$$

using [8, proposition A4.4]. (That is we have a resolution of shortest possible length where each $T_i$ is a tilting module). Such a resolution is also a $\Delta$-resolution for $\nabla(\lambda)$ and hence $d \geq \Delta \text{. f. d.}(\nabla(\lambda))$. But if the resolution is minimal then $\text{Ext}^d(\nabla(\lambda), T_d) \neq 0$, thus $d \leq \Delta \text{. f. d.}(\nabla(\lambda))$ as $T_d \in F(\nabla)$. So $d = \Delta \text{. f. d.}(\nabla(\lambda))$. We also note that $\text{Ext}^d(\nabla(\lambda), T_d) \cong \text{Ext}^d(\Delta'(\lambda), FT_d) \neq 0$ using [8, proposition A4.8]. So $\text{p. d.}(\Delta'(\lambda)) \geq d$. We now form an projective resolution for $\Delta'(\lambda)$ using the fact that $F$ is exact on $F(\nabla)$ [8, statement (1)(i) preceding lemma A4.6].

$$0 \to P'_d \to \cdots \to P'_1 \to P'_0 \to \Delta'(\lambda) \to 0$$

where the $P'_i = FT_i$ are projective. Thus $\text{p. d.}(\Delta'(\lambda)) = d = \Delta \text{. f. d.}(\nabla(\lambda))$.

(ii). We similarly take a minimal length injective resolution for $\nabla(\lambda)$ and apply $F$ to get a minimal length tilting resolution for $\Delta'(\lambda)$. By a similar argument to that above we know that the length of a minimal tilting resolution for $\Delta'(\lambda)$ is the same as its $\nabla$-filtration dimension.

(iii) and (iv) follow by applying (i) and (ii) to the modules for $S'$ and using the fact that $S$ and $S''$ are Morita equivalent as quasi-hereditary algebras. □

Corollary 3.1.2. Let $S$ be a quasi-hereditary algebra, then

(i) $\nabla \text{. f. d.}(S) = \Delta \text{. f. d.}(S')$ and

(ii) $\Delta \text{. f. d.}(S) = \nabla \text{. f. d.}(S')$.

Proof. We prove the first statement, the second is similar. Since

$$\nabla \text{. f. d.}(S) = \sup\{\text{p. d.}(\Delta(\lambda)) \mid \lambda \in \Lambda^+\}$$

we have

$$\nabla \text{. f. d.}(S) = \sup\{\Delta \text{. f. d.}(\nabla'(\lambda)) \mid \lambda \in \Lambda^+\}$$

which equals $\Delta \text{. f. d.}(S')$ using [17, lemma 2.10]. □
If $S$ has a simple-preserving duality then so does its Ringel dual, (using [14, theorem 1] in the case where the induced automorphism is the identity map). Hence in this situation we have $\nabla \text{. f. d.}(S) = \Delta \text{. f. d.}(S) = \Delta \text{. f. d.}(S') = \nabla \text{. f. d.}(S')$.

**Example 3.1.3.** We can now write down various formulae for the $\nabla$-filtration dimensions of the simple modules for the regular blocks of the Schur algebras and their Ringel duals.

Recall that the simples for the Schur algebra, $S(n,r)$, as defined in [11], are indexed by the set of partitions of $r$ into less than or equal to $n$ parts, $\Lambda^+(n,r)$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is *regular* if $\lambda_i - \lambda_j \not\equiv i - j \pmod{p}$ for all $1 \leq i < j \leq n$. We say a block is regular if all the partitions in a block are regular. We define for $\lambda \in \Lambda^+(n,r)$

$$d(\lambda) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left\lfloor \frac{\lambda_i - \lambda_j - i + j - 1}{p} \right\rfloor.$$ 

Now if $\lambda$ is a partition in a regular block with maximal element $\mu$ we have $d(\lambda) = \nabla \text{. f. d.}(\Delta(\lambda)) = i. \ d.(\nabla(\lambda))$ and $d(\mu) - d(\lambda) = i. \ d.(\nabla(\lambda)) = \nabla \text{. f. d.}(\Delta'(\lambda))$ using [17] and proposition 3.1.1.

4. **Truncation properties**

In this section we investigate the behaviour of $\nabla$-filtration dimensions and injective dimensions under two forms of truncation. Let $S$ be a quasi-hereditary algebra with poset $\Lambda^+$. We fix a saturated subset $\Pi$ of $\Lambda^+$, that is $\Pi$ is a subset of $\Lambda^+$ with the property that if $\lambda \in \Pi$ and $\mu \in \Lambda^+$ then $\mu < \lambda$ implies $\mu \in \Pi$. We write $\Gamma$ for $\Lambda^+ \setminus \Pi$. Let $e_\Gamma := \sum_{\lambda \in \Gamma} e_\lambda$, an idempotent; where we use a fixed decomposition of 1 into a sum of orthogonal primitive idempotents, and where $e_\lambda S$ is a projective module isomorphic to $P(\lambda)$.

4.1. The algebra $S$ has a quotient $S/Se_\Gamma S$, denoted by $S(\Pi)$ in [8, A3.9]. It is quasi-hereditary with respect to $(\Pi, \leq)$ with standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$, the same as for $S$ when $\lambda \in \Pi$.

For $M, N$ in $\text{mod}(S(\Pi))$ considered in the natural way as a subcategory of $\text{mod}(S)$, we have $\text{Ext}_S^i(M, N) \cong \text{Ext}_S^i(M, N)$ (see [8, A3.3] or [8, Appendix]). We have the following: if $M \in \text{mod}(S(\Pi))$ and if $\text{Ext}_S^i(M, \nabla(\lambda)) \not\cong 0$ for any $i$, then $\lambda \in \Pi$. This can be seen by noting that if $\text{Ext}_S^i(M, \nabla(\lambda)) \not\cong 0$ then $M$ must contain a composition factor $L(\mu)$ with $\mu \geq \lambda$. But since $M \in \text{mod}(S(\Pi)), \mu \in \Pi$ and hence by saturation of $\Pi$, $\lambda$ is in $\Pi$.

Thus if $M \in \text{mod}(S(\Pi))$ and using the isomorphism $\text{Ext}_{S(\Pi)}^i(M, N) \cong \text{Ext}_S^i(M, N)$, $\Delta \text{. f. d.}(M)$ as an $S(\Pi)$-module is the same as $\Delta \text{. f. d.}(M)$ as an $S$-module and similarly for the $\nabla$-filtration dimensions. Hence if a module is unchanged by this form of truncation then its $\Delta$- and $\nabla$-filtration dimensions are also unchanged.

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Note however that if $M$ is an $S(\Pi)$-module then the injective and projective dimensions as a module for $S$ are usually larger than those as a module for $S(\Pi)$.

4.2. Let $e = e_\Gamma$ as before, then the algebra $eSe$ is also quasi-hereditary, with respect to $(\Gamma, \leq)$, and with standard modules $e\Delta(\lambda)$ and costandard modules $e\nabla(\lambda)$, for $\lambda \in \Gamma$ (see [8] A3.11] or [9] §1.6]). We note that $e\Delta(\mu) \neq 0$ if and only if $\mu \in \Gamma$. If $\mu$ is in $\Gamma$ and $N$ is an $S$-module then $\Ext^i_S(M, N) \cong \Ext^i_{eSe}(eM, eN)$ ([8] proposition A3.13]). Thus if $\mu \in \Gamma$ then the projective dimension of $\Delta(\mu)$ is unchanged under this form of truncation. But the $\Delta$- and $\nabla$-filtration dimensions usually are smaller for $eSe$ than for $S$.

4.3. The two types of truncation are related by Ringel duality, see [4] theorem 3.4.6] or [8] A4.9]. We have that $eSe$ is a Ringel dual of a quasi-hereditary quotient $S'/S'\epsilon S'$ where $S'$ is the Ringel dual of $S$ and $\epsilon = \epsilon_\Gamma$ is defined as $e_\Gamma$ but for $S'$. Note that $(\Gamma, \leq_{op})$ is a saturated subset of the poset $(\Lambda^+, \leq_{op})$ for $S'$. So the conclusions in 4.2 also follow from 4.1 together with Proposition 3.1.1.

Example 4.3.1. In the case of the Schur algebra $S(2, r)$ for $GL_2$ we can now completely describe the values of all the dimensions mentioned for $L(\lambda)$, $\nabla(\lambda)$ and $\Delta(\lambda)$ under the various forms of truncation and under Ringel duality. For $S(2, r)$ the poset $\Lambda^+(2, r)$ is totally ordered. Moreover once we split the poset into block components then the resulting order is a minimal one.

Let $S$ be a block of $S(2, r)$ or of its Ringel dual $S'(2, r)$ or of any algebra obtained from $S(2, r)$ or of $S'(2, r)$ by the two forms of truncation defined above.

Suppose $S$ has $n+1$ simple modules. Then the simple modules for $S$ can be labelled by the numbers $0, 1, 2, \ldots, n$ with the usual ordering. We have $\nabla.f.d.(L(i)) = \Delta.f.d.(L(i)) = \Delta.f.d.(\nabla(i)) = \nabla.f.d.(\Delta(i)) = i$ and $\i.d.(\nabla(i)) = \p.d.(\Delta(i)) = n + 1 - i$. We can also say that $\text{gl.dim}(S) = 2 \nabla.f.d.(S) = 2n$.

5. Relating the partial order and inequalities for the homological dimensions

5.1. Let $S$ be a quasi-hereditary algebra with a duality fixing the simple modules. We consider the following properties, that $S$ may not satisfy in general but which are motivated by properties of the Schur algebras.

We assume that the partial order $<$ is minimal in the sense of section 11 and that $S$ consists of one block (for $C$).

(A) For all $\lambda, \mu \in \Lambda^+$, if $\mu < \lambda$ then $\nabla.f.d.(L(\mu)) \leq \nabla.f.d.(L(\lambda))$

(B) For all $\lambda, \mu \in \Lambda^+$, if $\mu < \lambda$ then $\Delta.f.d.(\nabla(\mu)) \leq \Delta.f.d.(\nabla(\lambda))$
(C) For all $\lambda \in \Lambda^+$, $\Delta.\ f.d.(\nabla(\lambda)) = \nabla.\ f.d.(S) - i.\ d.(\nabla(\lambda))$.

(D) For all $\lambda \in \Lambda^+$, $\Delta.\ f.d.(\nabla(\lambda)) = \Delta.\ f.d.(L(\lambda))$

(E) For all $\lambda, \mu \in \Lambda^+$, if $\mu < \lambda$ then $i.\ d.(\nabla(\mu)) \geq i.\ d.(\nabla(\lambda))$.

5.2. Comparisons. First we list some easy observations.

- We have $A$ implies $D$; this follows by induction on $\leq$, using [15, 2.5]. Moreover, if $D$ holds then $A$ and $B$ are equivalent. So $A$ (and $D$) imply $B$.
- $A$ and $C$ imply $E$.
- If $C$ holds then $B$ and $E$ are equivalent.
- By proposition 3.1.1 and its corollary 3.1.2 we have:
  - $B$ holds for $S$ if and only if $E$ holds for $S'$ and
  - $C$ holds for $S$ if and only if $C$ holds for $S'$.

5.3. Examples. We now give a few examples which show that some of the reverse implications do not hold.

Example 5.3.1. This shows that $A$, $B$ and $D$ do not imply $E$ or $C$.

We know that $A$, $B$, $C$, $D$ and $E$ all hold for the blocks of the Schur algebra consisting of regular weights [17]. We also know that in the case $n = 3$ (the first value of $n$ for which there are primitive non-regular weights) that $A$, $B$ and $D$ hold but that $C$ and $E$ do not hold in general for the non-regular blocks.

As the representation theory for the Schur algebra is controlled by that of the Special linear group we now give an example for $\text{SL}_3$ where condition $E$ fails (and necessarily $C$ fails as well). We use the standard notation and terminology of algebraic groups as in [8].

Consider $S(3,6)$ for characteristic 2. (A similar example works for general characteristic.) The weights (in $\text{SL}_3$-notation) of the non–simple block are

$$(0,0), (3,0), (0,3), (2,2), (4,1), (6,0)$$

We observe that $(3,0) < (2,2)$, and we claim that $i.d.(\nabla(3,0)) = 1$ but $i.d.(\nabla(2,2)) = 2$.

First, the injective $I(2,2)$ has a $\nabla$-filtration with quotients $\nabla(2,2)$ and $\nabla(4,1)$ (only), this follows by reciprocity from the decomposition matrix. Moreover, $I(4,1)$ has $\nabla$-quotients $\nabla(4,1), \nabla(6,0)$ and $\nabla(6,0)$ is injective. This implies that $\nabla(2,2)$ has minimal injective resolution

$$0 \to \nabla(2,2) \to I(2,2) \to I(4,1) \to I(6,0) \to 0.$$ 

We claim now that $I(3,0)$ is isomorphic to the tilting module $T(4,1)$ and hence is also projective.
Now the tilting module $T(2, 1)$ is isomorphic to $T(1, 0) \otimes S$ and hence is isomorphic to the injective hull of $L(1, 0)$ as a $G_1$-module. Thus it is indecomposable as a $G_1$ module so by proposition 2.1, the tilting module $T(4, 1)$ is isomorphic to $T(1, 0) \otimes I$. The module $T(4, 1)$ has simple socle $L(3, 0)$ and its injective hull is $I(3, 0)$ and it follows that $T(4, 1) \cong I(3, 0)$ since both have the same $\nabla$-quotients.

This implies that $I(3, 0)/\nabla(3, 0)$ is indecomposable. (It has simple head $L(3, 0)$.) But $\text{Ext}^1(\nabla(4, 1), \nabla(2, 2)) \cong k$ and we know that $I(2, 2)$ is the non-split extension of $\nabla(4, 1)$ and $\nabla(2, 2)$, so $I(3, 0)/\nabla(3, 0) \cong I(2, 2)$. Thus $i.d.(\nabla(3, 0)) = 1$.

Note that since $B$ holds but $E$ does not, this means that in particular that $S(3, 6)$ is not isomorphic to its Ringel dual. Also note that the Ringel dual of $S(3, 6)$ has property $E$ but not $B$.

**Example 5.3.2.** This shows $B$ and $E$ do not imply $A$ or $C$ or $D$.

Let $S$ be the algebra $kQ/I$ where $Q$ is the quiver

\[
\begin{array}{c}
\bullet \\
0 \quad \alpha_0 \quad \beta_0 \quad \gamma_0 \\
1 \quad \alpha_1 \quad 2 \quad \beta_1 \quad 3
\end{array}
\]

with relations (composing on the right):

\[
\alpha_1\alpha_0 = 0, \quad \gamma_1\gamma_0 = 0, \quad \beta_1\alpha_1 = 0 = \beta_1\beta_0, \quad \alpha_0\beta_0 = 0.
\]

This is quasi-hereditary, with respect to the natural order on

\[
\Lambda^+ = \{0, 1, 2, 3\}
\]

The composition factors of the $\nabla(i)$ are as follows

| $\nabla(i)$ | $L(0)$ | $L(1)$ | $L(2)$ | $L(3)$ |
|-------------|--------|--------|--------|--------|
| $\nabla(0)$ | 1      |        |        |        |
| $\nabla(1)$ | 1      | 1      |        |        |
| $\nabla(2)$ | 0      | 1      | 1      |        |
| $\nabla(3)$ | 0      | 1      | 1      | 1      |

with $\nabla(3)/L(3) \cong \nabla(2)$. Then $2 < 3$ but $\nabla.f.d.(L(3)) = 1$ and $\nabla.f.d.(L(2)) = 2$. So this does not satisfy $A$. Properties $D$ and $C$ also fail. We have $\Delta.f.d.(\nabla(3)) = \Delta.f.d.(\nabla(2)) = 2$, but $i.d.(\nabla(2)) = 1$. It does satisfy $B$ and $E$.

**Example 5.3.3.** This shows that $D$ does not imply $A$ or $C$. 

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Let $S$ be the algebra $kQ/I$ where $Q$ is the quiver

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
0 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\alpha_0 & \delta_0 & \beta_0 & \gamma_0 \\
\alpha_1 & \delta_1 & \beta_1 & \gamma_1 \\
\epsilon_0 & \epsilon_1 & & \\
\end{array}
\]

with the following relations:

\[
\begin{align*}
\alpha_1 \delta_1 &= 0 = \alpha_1 \epsilon_0 = \alpha_1 \alpha_0, \\
\beta_1 \epsilon_1 &= \gamma_0 \delta_0, \\
\gamma_1 \gamma_0 &= 0 = \beta_1 \beta_0, \\
\delta_0 \epsilon_0 &= 0 = \delta_0 \epsilon_0 = \delta_0 \alpha_0, \\
\epsilon_0 \beta_0 &= \delta_1 \gamma_1, \\
\epsilon_1 \delta_1 &= 0 = \epsilon_1 \alpha_0 = \epsilon_1 \epsilon_0.
\end{align*}
\]

We take the natural order on the index set $\Lambda^+ = \{0, 1, 2, 3\}$. The structure of the standard modules is as follows.

\[
\begin{align*}
\text{rad}(\Delta(1)) &= \Delta(0)^2, \\
\text{rad}(\Delta(2)) &= \mathcal{U}(L(1), L(0)), \\
\text{rad}(\Delta(3)) &= \Delta(0) \oplus \Delta(2).
\end{align*}
\]

(writing $\mathcal{U}(-, -)$ for a uniserial module, listing composition factors starting at the top). We have a projective cover

\[
0 \to P(2) \to P(1) \to \Delta(1) \to 0
\]

We have $\Delta.f.d.(L(i)) = i$ for $i \leq 2$ and $\Delta.f.d.(L(3)) = 1$, hence property $A$ fails. We will now show that $\nabla.f.d.(\Delta(i)) = i$ for $i \leq 2$ and $\nabla.f.d.(\Delta(3)) = 1$ i.e. property $D$ holds.

This is clear for $i = 0, 1$. Consider $i = 2$. We have the exact sequence

\[
0 \to \mathcal{U}(1, 0) \to \Delta(2) \to L(2) \to 0.
\]

The kernel has $\nabla$-filtration dimension equal to one and the cokernel has $\nabla$-filtration dimension equal to two. It follows from the dual version of [15 lemma 2.5(i)] that $\nabla.f.d.(\Delta(2)) = 2$. But we will need the following more explicit information about the Ext groups.

**Lemma 5.3.4.** We have

\[
\begin{align*}
(1) \quad \text{Ext}^1(\Delta(1), \Delta(2)) &= k, & \text{Ext}^2(\Delta(1), \Delta(2)) &= 0; \\
(2) \quad \text{Ext}^1(\Delta(0), \Delta(2)) &= k^2, & \text{Ext}^2(\Delta(0), \Delta(2)) &= k.
\end{align*}
\]
Proof. (1) From the projective cover for ∆(1) (see above) we get Ext^i(∆(1), ∆(2)).

(2) We have Ext^2(∆(0), ∆(2)) ∼ Ext^2(∆(0), L(2)) using ∇. f. d.(rad(∆(2))) = 1. This latter Ext group is isomorphic to Ext^1(∆(0), Q) where Q is the quotient ∇(2)/L(2). This Ext group is one-dimensional as any non-split extension must have simple socle L(1) and hence must embed in ∇(1). Thus any non-split extension is isomorphic to ∇(1) (by dimensions).

Clearly Hom(∆(0), ∆(2)) = k, so we can now use the fact that

\[ \sum_i (-1)^i \text{Ext}^i(\Delta(0), \Delta(2)) = 0 \]

(see IN p.71] and that \( \text{v. f. d.}(\Delta(2)) = 2 \) to deduce that \( \text{Ext}^1(\Delta(0), \Delta(2)) = k^2 \). □

Now we will show that \( \text{v. f. d.}(\Delta(3)) = 1 \). That is we show that \( \text{Ext}^2(\Delta(i), \Delta(3)) = 0 \) for \( i \leq 2 \).

Apply Hom(\( \Delta(i) \), - ) to the exact sequence

\[ 0 \to \Delta(2) \oplus \Delta(0) \to \Delta(3) \xrightarrow{\beta} L(3) \to 0 \]

Consider the resulting long exact sequence, for \( i \leq 2 \). Noting that \( \text{Ext}^j(\Delta(i), \Delta(0)) = 0 \) for \( i \geq 1 \), Hom(\( \Delta(i), L(3) \)) = 0 and that \( \text{v. f. d.}(L(3)) = 1 \) this gives

\[ 0 \to \text{Ext}^1(\Delta(i), \Delta(2)) \to \text{Ext}^1(\Delta(i), \Delta(3)) \to \text{Ext}^1(\Delta(i), L(3)) \]

\[ \to \text{Ext}^2(\Delta(i), \Delta(2)) \to \text{Ext}^2(\Delta(i), \Delta(3)) \to 0. \]

Thus for \( i = 1 \) or 2 we have \( \text{Ext}^2(\Delta(i), \Delta(3)) = 0 \).

It remains to consider \( i = 0 \), we substitute the dimensions proved in the Lemma and get an exact sequence

\[ 0 \to k^2 \to \text{Ext}^1(\Delta(0), \Delta(3)) \xrightarrow{\beta^*} k \to \text{Ext}^2(\Delta(0), \Delta(3)) \to 0 \]

To complete the proof we will show that the map \( \beta^* \) is zero.

Take an element in \( \text{Ext}^1(\Delta(0), \Delta(3)) \), say \( \eta \), which is represented by

\[ 0 \to \Delta(3) \to V \xrightarrow{\pi} \Delta(0) \to 0 \]

then \( \beta^*(\eta) \) is represented by the push-out of \( \beta \). Suppose this is non-zero, then the middle term of the sequence \( \beta^*(\eta) \) must be uniserial with a simple top \( L(0) \) and therefore, the top of \( V \) must also be simple, isomorphic to \( L(0) \). So there is an epimorphism \( \psi : e_0S = P(0) \to V \). Now we will use the relation \( \delta_1\gamma_1 = e_0\delta_0 \) to derive a contradiction.

We have \( \psi(e_0) = 0 \) (since \( L(1) \) does not occur in the top of \( \text{rad}(V) \), i.e. of \( \Delta(3) \)). It follows that \( \psi(e_0\delta_0) = \psi(e_0)\delta_0 = 0 \). Therefore also \( \psi(\delta_1\gamma_1) = 0 \). But on the other hand, \( \psi(\delta_1) \) generates \( \text{ker}(\pi) = \Delta(3) \) and \( \psi(\delta_1\gamma_1) = \psi(\delta_1)\gamma_1 \) which spans \( \Delta(3)e_2 \) and is therefore non-zero, a contradiction.
5.4. We expect that $C$ is independent of any of the other conditions. It certainly is not implied by any of them. But to construct an example with $C$ but not $A$ seems to need rather a lot of technical detail and would require many simple modules.

In summary $B$, $D$ and $E$ are independent of each other and none of these imply $A$ or $C$, while $A$ implies $B$ and $D$ but not $C$ or $E$.

5.5. We now consider an algebra satisfying property $A$ and show that the first condition of lemma 2.3 holds for this algebra.

*Example 5.5.1.* In this example we consider a quasi-hereditary algebra $S$ with duality preserving the simples that satisfies property $A$. We show that $\operatorname{gl.dim}(S) = 2 \nabla. f. d.(S)$ for this algebra.

Let $S$ be a quasi-hereditary algebra with four simple modules, and assume the $\nabla$-filtration dimensions of the simples $L(0)$, $L(1)$, $L(2)$, $L(3)$ are respectively $0, 1, 1, 2$. Then the quasi-hereditary quotient $S_2$ of $S$ obtained by factoring out $Se_3S$, where $e_3S = P(3)$, must have $\nabla. f. d.(S_2) = 1 \leq \operatorname{gl.dim}(S_2) \leq 2 = 2 \nabla. f. d.(S_2)$.

We would like to show that $\operatorname{Ext}_S^4(L(3), L(3)) \neq 0$. To do this we need $\operatorname{Ext}_S^2(Q^\circ, Q) \cong \operatorname{Ext}_{S_2}^2(Q^\circ, Q) \neq 0$ where $Q$ is the quotient $\nabla(3)/L(3)$.

We now show that if $Q$ is any $S_2$-module with $\nabla. f. d.(Q) = 1$ then $\operatorname{Ext}_{S_2}^2(Q^\circ, Q) \neq 0$.

**Case 1** $L(2)$ is not a composition factor of $Q$. Let $S_1$ be the quasi-hereditary quotient of $S_2$ obtained by factoring $S_2e_2S_2$. (So that $S_1 \in S_1$ where $S_1$ is as in section 2.) Then $Q$ is an $S_1$-module with $\nabla. f. d.(Q) = 1$ and then by theorem 2.2.1 we know that $\operatorname{Ext}_{S_1}^2(Q^\circ, Q) \cong \operatorname{Ext}_{S_2}^2(Q^\circ, Q) \neq 0$.

**Case 2** The socle of $Q$ has only $L(2)$ as a composition factor. Then consider the injective hull

$$0 \to Q \to I \to R \to 0$$

where $I$ is isomorphic to a direct sum of copies of $\nabla(2)$. Note that $R \neq 0$. We have $\nabla. f. d.(R) = 0$ (by dimension shift). That is, $R$ has a $\nabla$-filtration. Moreover $R$ is an $S_1$-module. By dimension shift we get

$$\operatorname{Ext}_{S_2}^2(Q^\circ, Q) \cong \operatorname{Ext}_{S_1}^1(Q^\circ, R).$$

Now apply $\operatorname{Hom}_{S_2}(-, R)$ to the exact sequence

$$0 \to R^\circ \to I^\circ \to Q^\circ \to 0.$$

This gives the exact sequence

$$0 \to \operatorname{Hom}_{S_2}(Q^\circ, R) \to \operatorname{Hom}_{S_2}(I^\circ, R) \to \operatorname{Hom}_{S_2}(R^\circ, R) \to \operatorname{Ext}_{S_2}^1(Q^\circ, R) \to 0.$$
Since $L(2)$ is not a composition factor of $R$ and all the top composition factors of $I^\circ$ are $L(2)$ we deduce that $\text{Hom}_{S_2}(I^\circ, R) = 0$ and hence the required Ext-space is isomorphic to $\text{Hom}_{S_2}(R^\circ, R)$.

This is certainly non-zero.

**General Case.** Let $U$ be the largest submodule of $Q$ which does not have $L(2)$ as a composition factor, and let $V$ be the quotient. Then the socle of $V$ has only $L(2)$ as composition factors. Consider the exact sequence of $S_2$-modules

$$0 \to U \to Q \to V \to 0.$$ 

So each term has $\nabla$-filtration dimension at most one. Now $\nabla$ f.d.$(Q) = 1$, therefore at least one of $U$ and $V$ has $\nabla$-filtration dimension equal to one. If $\nabla$ f.d.$(V) = 1$ then proceed as in the previous proof, to show that there is a surjection

$$\text{Ext}^2_{S_2}(Q^\circ, Q) \to \text{Ext}^2_{S_2}(V^\circ, V)$$

the latter Ext group being non-zero by Case 2.

So assume now that $\nabla$ f.d.$(V) = 0$. Then actually $V$ is a direct sum of $\nabla(2)$'s and hence is injective. Using this we show that

$$\text{Ext}^2_{S_2}(Q^\circ, Q) \cong \text{Ext}^2_{S_2}(U^\circ, U).$$

But $U$ is as in Case 1 and hence this is non-zero.

Thus if property $A$ holds for a quasi-hereditary algebra with simple preserving duality then if the global dimension is not twice the $\nabla$-filtration dimension then we must have at least five simples.

**References**

1. I. N. Bernstein, I. M. Gel’fand, and S. I. Gel’fand, *A category of $g$-modules*, Funct. Anal. and Appl. **10** (1976), 87–92.

2. S. Caenepeel and B. Zhu, *On good filtration dimensions for standardly stratified algebras*, To appear in Comm. Alg.

3. E. Cline, B. J. Parshall, and L. L. Scott, *Finite dimensional algebras and highest weight categories*, J. reine angew. Math. **391** (1988), 85–99.

4. *Stratifying endomorphism algebras*, Memoirs American Math. Soc. **124** (1996), no. 591.

5. V. Dlab and C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. **33** (1989), no. 2, 280–291.

6. *The module theoretical approach to quasi-hereditary algebras*, Representations of algebras and related topics (Kyoto, 1990) (H. Tachikawa and S. Brenner, eds.), London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, Cambridge, 1992, pp. 200–224.

7. S. Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993), 39–60.

8. *The $q$-Schur Algebra*, London Math. Soc. Lecture Note Ser., vol. 253, Cambridge University Press, Cambridge, 1998.
[9] K. Erdmann, *Symmetric groups and quasi-hereditary algebras*, Finite-dimensional algebras and related topics (Ottawa, ON, 1992) (V. Dlab and L. L. Scott, eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht/Boston/London, 1994, pp. 123–161.

[10] E. M. Friedlander and B. J. Parshall, *Cohomology of Lie algebras and algebraic groups*, Amer. J. Math. 108 (1986), 235–253.

[11] J. A. Green, *Polynomial Representations of $GL_n$*, Lecture Notes in Mathematics, vol. 830, Springer–Verlag, Berlin/Heidelberg/New York, 1980.

[12] M. Klucznik and S. König, *Characteristic Tilting Modules over Quasi–hereditary Algebras*, unpublished notes, 1999.

[13] V. Mazorchuk and A. E. Parker, *On the relation between finitistic and good filtration dimensions*, To appear in Comm. Alg.

[14] G. J. McNinch, *Filtrations and positive characteristic Howe duality*, Math. Z. 235 (2000), 651–685.

[15] A. E. Parker, *The global dimension of Schur algebras for $GL_2$ and $GL_3$*, J. Algebra 241 (2001), 340–378.

[16] ——, *On the global dimension of Schur algebras and related algebras*, Ph.D. thesis, University of London, 2001.

[17] ——, *On the good filtration dimension of Weyl modules for a linear algebraic group*, J. reine angew. Math. (2003), 5–21.

[18] C. M. Ringel, *Tame algebras and integral quadratic form*, Lecture Notes in Mathematics, vol. 1099, Springer, 1984.

[19] L. L. Scott, *Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 271–281.

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