O(\(p^6\)) extension of the large–\(N_C\) partial wave dispersion relations

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Continuing our previous work [1], large–\(N_C\) techniques and partial wave dispersion relations are used to discuss \(\pi\pi\) scattering amplitudes. We get a set of predictions for \(O(\(p^6\))\) low-energy chiral perturbation theory couplings. They are provided in terms of the masses and decay widths of scalar and vector mesons.

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Introduction

Chiral perturbation theory (\(\chi\)PT) is a powerful tool in the study of low energy hadron physics. An important issue in \(\chi\)PT is the determination of the values of low energy constants (LECs), which are crucial to make predictions. In addition to an exhaustive phenomenological discussions about the LECs, Refs. [2] and [3] provided a deeper theoretical understanding. In these papers, the authors constructed a phenomenological lagrangian including the heavy resonances, which were then integrated out to predict the LECs at tree level in terms of the resonance couplings.

In a previous paper [1], we obtained a generalization of the \(KSRF\) relation [4], a new relation between resonance couplings and a prediction for the chiral constants \(L_2\) and \(L_3\) [3]:

\[
\frac{144\pi f^2 \Gamma_V}{M_V} + \frac{32\pi f^2 \Gamma_S}{M_S} = 1,
\]

\[
\frac{9\Gamma_V}{M_V} [\alpha_V + 6] + \frac{2\Gamma_S}{3M_S} [\alpha_S + 6] = 0,
\]

\[
L_2 = 12\pi f^4 \frac{\Gamma_V}{M_V},
\]

\[
L_3 = 4\pi f^4 \left( \frac{2\Gamma_S}{3M_S} - \frac{9\Gamma_V}{M_V} \right),
\]

where \(\Gamma_R\) and \(M_R\) stand, respectively, for the value of the \(R\) resonance width and mass in the chiral limit. The parameter \(\alpha_R\) is given by their \(O(m_R^2)\) correction in the ratio \(\frac{\Gamma_R}{\Gamma_R} = \frac{\Gamma_R}{\Gamma_R} \left[ 1 + \alpha_R \frac{m_R^2}{\Gamma_R} + O(\frac{m_R^4}{\Gamma_R}) \right].\)

No particular realization of the resonance lagrangian was considered in Ref. [1]. While in the lagrangian approach one has to pay attention to different realizations of the vector fields [3], all our analyses only rely on general properties like crossing symmetry and analyticity.

Chiral symmetry was incorporated by matching chiral perturbation theory (\(\chi\)PT) at low energies [2, 7, 8]. In Ref. [1], we found that the minimal resonance chiral theory lagrangian [2] was unable to fulfill the high-energy constraints for the partial wave \(\pi\pi\)-scattering amplitudes once the matching was taken up to order \(p^4\). Another interesting finding is that in large \(N_C\) limit the \([1,1]\) Padé approximation in SU(3) \(\chi\)PT for \(\pi\pi\) scatterings means to neglect the left hand cuts contribution completely [3], but the understanding to the latter is very important to accept the \(\sigma\) meson even in the non-linear realization of chiral symmetry [10]. However, in Ref. [1] the \(\pi\pi\) scattering was only matched up to \(O(\(p^6\))\) This paper is devoted to extending the discussion up to \(O(\(p^8\)).\)

Dispersive analysis

The \(\pi\pi\) scattering amplitude \(T(s, t, u)\) admits a decomposition into partial waves of definite angular momentum \(J [11].\)

\[
T(s, t, u) = \sum_J 32\pi (2J + 1) P_J(\cos \theta) T_J(s),
\]

where every \(T_J(s)\) accepts a once-subtracted dispersion relation of the form,

\[
T_J(s) - T_J(0) = \int_{-\infty}^{0} \frac{ds'}{s'} \text{Im} T_J(s') + \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'(s' - s)}.
\]

In general, we will work with amplitudes and partial-waves with definite isospin, \(T(s, t, u)^I\) and \(T_J^I(s)\), respectively. We however quite often in the following omit the indices \(I, J\) for simplicity when no confusion is caused.

At large–\(N_C\), the resonances become narrow-width states, allowing the recovering of the right-hand cut contribution in Eq. (3). In the previous paper [1], we have demonstrated that the PKU parametrization of S matrix [12] will give the same results in large \(N_C\) limit as Eq. (3). The \(s\)-channel exchange of a resonance \(R\) with proper quantum numbers \(IJ\) provides for \(s > 0\) the ab-
The imaginary part of \( T \) with \( \chi \)PT couplings and the \( \pi \pi \) scattering amplitude is determined by the function \( A(s, t, u) \),

\[
A \left[ \pi^a(p_1) + \pi^b(p_2) \rightarrow \pi^c(p_3) + \pi^d(p_4) \right] = \delta^{ab} \delta^{cd} A(s, t, u) + \delta^{ac} \delta^{bd} A(t, u, s) + \delta^{ad} \delta^{bc} A(u, t, s),
\]

which is given up to \( O(p^6) \) in Ref. \[13\]. Since we are interested in the \( m_\pi \) dependence of the amplitude, we express the amplitude explicitly in terms of LECs, momenta and masses:

\[
A(s, t, u) = \frac{s - m_\pi^2}{f_\pi^2} + \frac{16m_\pi^4}{f_4} \left( L_2 + L_3 + L_8 - \frac{1}{2} L_5 \right) - \frac{16m_\pi^2 s}{f_2} (L_2 + L_3) + \frac{2s^2}{f_2} (2L_3 + 3L_2) + \frac{2(t - u)^2}{f_4} L_2 - \frac{16m_\pi^6}{f_6} (-8L_5^2 + 32L_8L_5 - 32L_3^2) + \frac{m_\pi^6}{f_6} (r_1 + 2r_f) + \frac{m_\pi^4 s^2}{f_6} r_3 + \frac{m_\pi^2 (t - u)^2}{f_6} r_4 + \frac{s^3}{f_5} r_5 + \frac{s(t - u)^2}{f_5} r_6
\]

(12)

with \( s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2 = 4m_\pi^2 - s - t \), and where we have used the chiral expansion of the pion decay constant \( f_\pi \) up to \( O(p^6) \) \[12, 16\]:

\[
f_\pi = f \left[ 1 + \frac{4L_5m_\pi^2}{f_2} + \frac{(32L_5^2 - 64L_8L_5 + r_f)m_\pi^4}{f_4} + O(m_\pi^6) \right].
\]

(13)

In both expressions, only the leading terms in the \( 1/N_C \) expansion are kept. Following the notation in the former work \[1\], the large-\( N_C \) \( O(p^4) \) SU(2) LECs have been expressed in terms of SU(3) constants \[8\].

The isospin amplitudes are given by the combinations

\[
T(s, t, u)^{I=0} = 3A(s, t, u) + A(t, s, u) + A(u, s, t),
\]

\[
T(s, t, u)^{I=1} = A(t, s, u) - A(u, s, t),
\]

\[
T(s, t, u)^{I=2} = A(t, s, u) + A(u, s, t).
\]

(14)

Finally, in order to get amplitudes with definite angular momentum, one performs the partial wave projection,

\[
T(s, t, u)^I = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{s - 4m_\pi^2} \left( P_J(1 + \frac{2t}{s - 4m_\pi^2})T(s, t, u)^I dt \right). \]

(15)

This yields the \( \chi \)PT results for different partial-wave amplitudes up to \( O(p^6) \):

\[
\text{Im} T_j^{I,R}(s) = \pi \frac{M_{\pi R}}{\rho_R} \delta(s - M_{\pi R}^2), \quad (4)
\]

where \( \rho_R = \sqrt{\frac{M_{\pi R}^2 - 4m_\pi^2}{M_{\pi R}^2}} \) and the subscript \( R \) denote the different resonances.

Crossing symmetry relates the right to the left-hand cut through the expression \[11\],

\[
\text{Im} T_j^L(s) = (s - M_{\pi R}^2 + 4m_\pi^2)^{-1} \int_{-\infty}^{\infty} \frac{dM_{\pi R}^2}{M_{\pi R}^2 - s} \left( \frac{2s}{M_{\pi R}^2 - 4m_\pi^2} \right) P_J(1 + \frac{2t}{s - 4m_\pi^2}) \pi \frac{M_{\pi R}}{\rho_R} \Gamma_{\pi R}(s - M_{\pi R}^2),
\]

(7)

Putting the different imaginary parts together, it is then possible to calculate the right and left-hand cut integrals:

\[
T^R(s) = \frac{s}{\pi} \int_{-\infty}^{\infty} ds' \frac{\text{Im} T^R(s')}{s'(s' - s)},
\]

(8)

\[
T^L(s) = \frac{s}{\pi} \int_{-\infty}^{\infty} ds' \frac{\text{Im} T^L(s')}{s'(s' - s)},
\]

(9)

where these expressions only depend on the mass and width of the resonances. The precise results for \( T^R \) and \( t^R \), with \( R = S, V \), are given in Ref. \[1\].

We consider now the low energy limit where the \( \pi \pi \) scattering is described by \( \chi \)PT which determines the left-hand side of Eq. \[3\]. For convenience, the dispersion relation is rewritten in the way,

\[
T^{\chi PT}(s) - T^{\chi PT}(0) = T^R(s) + T^R(s),
\]

(10)

where the l.h.s. only contains \( \chi \)PT couplings and the r.h.s. only contains resonances parameters. Comparing the different terms of the chiral expansion on both sides, one gets the low-energy constants (LECs) in terms of parameters of resonances and some other useful relations.
1. $IJ = 00$ channel

$$l.h.s. = \frac{s}{16\pi f^2} - \frac{10L_2 + 5L_3}{3\pi f^4}m^2_s$$
$$- 3r_2 + 8r_3 + 32r_4 + 36r_5 + 4r_6 + 6r_f m^4_s$$
$$+ \frac{25L_2 + 11L_3 s^2 + 11r_4 + 18r_5 + 10r_6 m^2_s s^2}{24\pi f^4}$$
$$+ \frac{15r_5 - 5r_6 s^3}{192\pi f^6},$$  \hspace{1cm} (16)

2. $IJ = 11$ channel

$$l.h.s. = \frac{s}{96\pi f^2} + \frac{L_3}{6\pi f^4}m^2_s$$
$$+ \frac{5r_2 + 40r_3 - 80r_4 + 216r_5 - 24r_6 - 10r_f m^4_s}{480\pi f^6}$$
$$+ \frac{-L_3 s^2}{24\pi f^4} - \frac{5r_3 - 15r_4 + 54r_5 + 14r_6 m^2_s s^2}{480\pi f^6}$$
$$+ \frac{3r_5 + 3r_6 s^3}{320\pi f^6},$$  \hspace{1cm} (17)

3. $IJ = 20$ channel

$$l.h.s. = -\frac{s}{32\pi f^2} - \frac{8L_2 + L_3}{6\pi f^4}m^2_s$$
$$- \frac{3r_2 + 16r_3 + 40r_4 + 72r_5 + 56r_6 - 6r_f m^4_s}{96\pi f^6}$$
$$+ \frac{5L_2 + L_3 s^2 + r_3 + 7r_4 + 9r_5 + 17r_6 m^2_s s^2}{12\pi f^4}$$
$$- \frac{3r_5 + 11r_6 s^3}{192\pi f^6},$$  \hspace{1cm} (18)

where $l.h.s.$ means the left hand side of Eq. (10).

For the $r.h.s.$ of Eq. (10), a similar chiral expansion is performed up to $O(p^6)$:

1. $IJ = 00$ channel

$$T^{sR} = \frac{\Gamma_S}{M_S^3} s + \frac{2\Gamma_S}{M_S^3} m^2_s s + \frac{6\Gamma_S}{M_S^3} m^4_s s + \frac{\Gamma_S}{M_S^3} s^2$$
$$+ 2\frac{\Gamma_S}{M_S^3} m^2_s s^2 + \frac{\Gamma_S}{M_S^3} s^3 + O(p^8),$$  \hspace{1cm} (19)

$$T^{tR} = -\frac{\Gamma_S}{3M_S^3} s - \frac{22\Gamma_S}{9M_S^3} m^2_s s - \frac{122\Gamma_S}{9M_S^3} m^4_s s - \frac{9\Gamma_V}{2M_V^3} s$$
$$+ \frac{74\Gamma_V}{M_V^3} m^2_s s + \frac{446\Gamma_V}{M_V^3} m^4_s s + \frac{2\Gamma_S}{M_S^3} s^2 + \frac{22\Gamma_S}{9M_S^3} m^2_s s^2$$
$$- \frac{\Gamma_S}{6M_S^3} s^3 - \frac{4\Gamma_V}{M_V^3} m^2_s s^2 - \frac{46\Gamma_V}{M_V^3} m^4_s s^2 + \frac{5\Gamma_V}{2M_V^3} s^3 + O(p^8);$$  \hspace{1cm} (20)

2. $IJ = 11$ channel

$$T^{sR} = \frac{\Gamma_V}{M_V^3} s + \frac{2\Gamma_V}{M_V^3} m^2_s s + \frac{6\Gamma_V}{M_V^3} m^4_s s + \frac{\Gamma_V}{M_V^3} s^2$$
$$+ \frac{2\Gamma_V}{M_V^3} m^2_s s^2 + \frac{\Gamma_V}{M_V^3} s^3 + O(p^8),$$  \hspace{1cm} (21)

$$T^{tR} = \frac{\Gamma_S}{9M_S^3} s + \frac{10\Gamma_S}{9M_S^3} m^2_s s + \frac{326\Gamma_S}{45M_S^3} m^4_s s + \frac{\Gamma_V}{2M_V^3} s$$
$$+ \frac{\Gamma_V}{M_V^3} m^2_s s - \frac{37\Gamma_V}{5M_V^3} m^4_s s - \frac{\Gamma_S}{9M_S^3} s^2 - \frac{64\Gamma_S}{45M_S^3} m^2_s s^2$$
$$+ \frac{\Gamma_S}{10M_S^3} s^3 + \frac{\Gamma_V}{2M_V^3} s^2 + \frac{38\Gamma_V}{5M_V^3} m^2_s s^2 - \frac{11\Gamma_V}{20M_V^3} s^3 + O(p^8);$$  \hspace{1cm} (22)

3. $IJ = 20$ channel

$$T^{sR} = 0,$$  \hspace{1cm} (23)

$$T^{tR} = -\frac{\Gamma_S}{3M_S^3} s - \frac{22\Gamma_S}{9M_S^3} m^2_s s - \frac{122\Gamma_S}{9M_S^3} m^4_s s - \frac{9\Gamma_V}{2M_V^3} s$$
$$- \frac{37\Gamma_V}{M_V^3} m^2_s s - \frac{223\Gamma_V}{9M_V^3} m^4_s s + \frac{2\Gamma_S}{M_S^3} s^2 + \frac{22\Gamma_S}{9M_S^3} m^2_s s^2$$
$$- \frac{\Gamma_S}{6M_S^3} s^3 + \frac{2\Gamma_V}{M_V^3} s^2 + \frac{23\Gamma_V}{M_V^3} m^2_s s^2 - \frac{5\Gamma_V}{4M_V^3} s^3 + O(p^8);$$  \hspace{1cm} (24)

where only the lightest multiplet of vector and scalar resonances is taken into account, respectively denoted by the subscripts $V$ and $S$.

The masses $M_R$ and decay widths $\Gamma_R$ in Eqs. (19)–(24) denote the physical ones at large–$N_C$. They carry an implicit $m^2_s$ dependence that we parameterize in the form

$$\frac{\Gamma_R}{M_R} = \frac{\Gamma_R}{\overline{M}_R} \left[ 1 + \beta_R \frac{m^2_s}{\overline{M}_R} + O(m^4_s) \right],$$  \hspace{1cm} (25)

$$\frac{\Gamma_R}{M_R^3} = \frac{\Gamma_R}{\overline{M}_R^3} \left[ 1 + \alpha_R \frac{m^2_s}{\overline{M}_R^2} + \gamma_R \frac{m^4_s}{\overline{M}_R^4} + O(m^6_s) \right],$$  \hspace{1cm} (26)

where $\overline{M}_R$ and $\overline{\Gamma}_R$ are the chiral limit of $M_R$ and $\Gamma_R$, respectively. Notice that $\Gamma_R$ and $\overline{M}_R$ were denoted as $M_R^{(0)}$ and $\Gamma_R^{(0)}$ in Ref. (1).

After expanding the resonance contributions on the r.h.s. of Eq. (10) in powers of $s$ and $m^2_s$, it is possible to perform a matching with $\chi$PT. Ref. (1) was devoted to the analysis of the constraints derived from $\chi$PT at $O(p^2)$ and $O(p^4)$. The present work studies the relations that stem from the matching at $O(p^6)$.
1. $IJ = 00$ channel

$$
3r_2 - 8r_3 - 32r_4 - 36r_5 - 4r_6 - 6r_f =\frac{96\pi f^6}{192\pi f^6} \left( -\frac{68}{9} + \frac{4\beta_5}{9} + \frac{2\gamma_5}{3} \right) + \frac{\Gamma_S}{M_S} (446 + 74\beta_V + 9\gamma_V) ,
$$

(27)

$$
11r_3 + 17r_4 + 18r_5 + 10r_6 = \frac{96\pi f^6}{M_S^2} \left( \frac{40}{9} + \frac{11\beta_5}{9} \right) + \frac{\Gamma_V}{M_V} (-46 - 4\beta_V) ,
$$

(28)

$$
\frac{15r_5 - 5r_6}{192\pi f^6} = \frac{5\Gamma_S}{6M_S} + \frac{5\Gamma_V}{2M_V} .
$$

(29)

2. $IJ = 11$ channel

$$
5r_2 + 40r_3 - 80r_4 + 216r_5 - 24r_6 - 10r_f = \frac{480\pi f^6}{M_S^2} \left( \frac{326}{45} + \frac{10\beta_5}{9} + \frac{\gamma_5}{9} \right) + \frac{\Gamma_S}{M_S} \left( -\frac{7}{5} + 3\beta_V + \frac{3\gamma_V}{2} \right) ,
$$

(30)

$$
-5r_3 + 15r_4 - 54r_5 - 14r_6 = \frac{480\pi f^6}{M_S^2} \left( -\frac{\beta_5}{9} - \frac{64}{45} \right) + \frac{\Gamma_V}{M_V} \left( \frac{48}{5} + \frac{3\beta_V}{2} \right) ,
$$

(31)

$$
\frac{3r_5 + 3r_6}{320\pi f^6} = \frac{\Gamma_S}{10M_S} + \frac{9\Gamma_V}{20M_V} .
$$

(32)

3. $IJ = 20$ channel

$$
-3r_2 - 16r_3 - 40r_4 - 72r_5 - 56r_6 + 6r_f = \frac{96\pi f^2}{192\pi f^6} \left( \frac{122}{9} + \frac{22\beta_5}{9} + \frac{\gamma_5}{3} \right) - \frac{\Gamma_S}{M_S} \left( 223 + 37\beta_V + \frac{9\gamma_V}{2} \right) .
$$

(33)

$$
\frac{r_3 + 8r_4 + 9r_5 + 17r_6}{96\pi f^6} = \frac{\Gamma_S}{M_S} \left( \frac{22}{9} + \frac{2\beta_5}{9} \right) + \frac{\Gamma_V}{M_V} (23 + 2\beta_V) ,
$$

(34)

$$
\frac{-3r_5 - 11r_6}{192\pi f^6} = -\frac{\Gamma_S}{6M_S} - \frac{5\Gamma_V}{4M_V} .
$$

(35)

Eqs. (27), (30) and (33) refer to the matching of the terms $O(m_s^4)$. Eqs. (28), (31) and (34) correspond to the $O(m_s^2 s^2)$ terms. Eqs. (27), (30) and (33) provide the matching at $O(s^3)$.

It is remarkable that the system of nine equations for six unknowns $(r_i, i = f, 2...6)$ is actually compatible. The $O(s^3)$ relations determine $r_5$ and $r_6$. After that, it is then possible to extract $r_3$ and $r_4$ from the $O(m_s^2 s^2)$ equations. Finally, using these values, one can extract the combination $r_2 - 2r_f$ from the $O(m_s^4 s^2)$ constraints. The LECs always appear in this particular combination, avoiding an independent determination of $r_2$ and $r_f$. This yields the predictions:

$$
\frac{r_2 - 2r_f}{64\pi f^6} \left( 1 + \frac{\beta_5}{3} + \frac{\gamma_5}{6} \right) + \frac{\pi f^6}{M_V} (7584 + 1248\beta_V + 144\gamma_V) ,
$$

(36)

$$
\frac{r_3}{3M_S} - \frac{768\pi f^6}{M_V} (1 + \frac{3\beta_V}{32}) ,
$$

(37)

$$
\frac{r_4}{192\pi f^6} \left( 1 + \frac{\beta_V}{8} \right) ,
$$

(38)

$$
\frac{r_5}{32\pi f^6} \left( 1 + \beta_V \right) + \frac{36\pi f^6}{M_V} ,
$$

(39)

$$
\frac{r_6}{12\pi f^6} \frac{\Gamma_V}{M_V} .
$$

(40)

**An example of $O(p^6)$ coupling determination**

The authors of Ref. [13] provide an estimate of the $O(p^6)$ LECs $r_i$ in terms of resonances couplings, where they consider a phenomenological lagrangian including
one multiplet of vector and scalar resonances. The vector interaction is given by
\[
\mathcal{L}_V = -i \frac{g_V}{2\sqrt{2}} \langle \bar{V}_\mu [u^\mu, u'] \rangle + f_x \langle \bar{\chi}\{u^\mu, \chi_-\} \rangle ,
\]
and for the scalar,
\[
\mathcal{L}_S = c_d \langle S u^\mu u_\mu \rangle + c_m \langle S \chi_+ \rangle + \bar{c}_d S_1 \langle u^\mu u_\mu \rangle + \bar{c}_m S_1 \langle \chi_+ \rangle .
\]
where (...) is short for trace in flavour space and the tensors \(u^\mu, \chi_\pm\) introduce the chiral Goldstones. For further details on the notations, see Ref. \[13\] and references therein. At large--\(N_C\), the \(SU(3)\) singlet and octet states become degenerate and one has \(\bar{c}_d = c_d / \sqrt{3}, \bar{c}_m = c_m / \sqrt{3}, M_{S_1} = M_S [2]\). Using this lagrangian, the authors computed the contributions to the \(\pi\pi\) scattering from resonance exchanges and provided a set of values for the LECs \(r_i\) [13].

As an example of our method, we will rederive their result. In order to do that, in a first step, we will neglect the wave-function renormalizations \(Z_R\) and \(Z_{\pi}\), and only the resonance exchange contribution will be considered, as it was done in Ref. [13]. At large--\(N_C\), the meson wave functions get renormalized if there are tree-level tadpole diagrams that connect the scalar meson field to the vacuum [13]. After recovering the results in Ref. [13], we will compute the LECs including also the effect of \(Z_{\pi}\) and \(Z_R\) and their impact on the numerical estimates will be analyzed.

We need first to calculate the \(R \rightarrow \pi\pi\) decay widths corresponding to this lagrangian. Ignoring the wave-function renormalizations, one gets
\[
\Gamma_V = \frac{g_V^2 M_V^5 \rho_s^3}{48 \pi f^4} \left[ 1 + \frac{4 \sqrt{2} f_x}{g_V} \frac{m_\pi^2}{M_V^2} \right]^2 ,
\]
\[
\Gamma_S = \frac{3c_d^2 M_S^3 \rho_s}{16 \pi f^4} \left[ 1 + \frac{2(c_m - c_d)}{c_d} \frac{m_\pi^2}{M_S^2} \right]^2 ,
\]
where the subscript \(S\) denote the \(SU(2)\) singlet \(\sigma = \sqrt{\frac{2}{3}} S_0 - \sqrt{\frac{1}{3}} S_8 \sim \frac{1}{\sqrt{2}} (u u + d d)\). The large--\(N_C\) resonances masses are \(m_\pi\)-independent within this model, i.e., \(M_R = \bar{M}_R\).

With the above expressions of \(\Gamma_V\) and \(\Gamma_S\), we can get the parameters \(\alpha_R, \beta_R\) and \(\gamma_R\) defined in Eq. (25) and (26)
\[
\alpha_V = \beta_V = \frac{8\sqrt{2} f_x}{g_V} - 6 ,
\]
\[
\gamma_V = \frac{32 f_x^2 s}{g_V^2} - \frac{4 \sqrt{2} f_x}{g_V} + 6 ,
\]
\[
\alpha_S = \beta_S = \frac{4 c_m}{c_d} - 6 ,
\]
\[
\gamma_S = 10 - \frac{16 c_m}{c_d} + \frac{4 c_m^2}{c_d^2} .
\]

Using Eqs. (36)–(10), one gets the predictions on \(O(p^6)\) LECs in terms of the resonance large--\(N_C\) parameters \(g_V, f_x, c_d\) and \(c_m\):
\[
r_2 - 2r_f = 20 a_V + 16 b_V + 3 c_V + \frac{8 f_2 \left( c_m - c_d \right)^2}{M_S^4} ,
\]
\[
r_3 = -7 a_V - 3 b_V + \frac{8 f_2 c_d (c_m - c_d)}{M_S^4} ,
\]
\[
r_4 = a_V + b_V ,
\]
\[
r_5 = \frac{3 a_V}{4} + \frac{2 f_2^2 c_d^2}{M_S^2} ,
\]
\[
r_6 = \frac{1}{4} a_V ,
\]
with \(a_V \equiv \frac{g_V^2 f^2}{M_V^5}, b_V \equiv \frac{4 \sqrt{2} f_x g_V f^2}{M_V^5}, c_V \equiv \frac{32 f_x^2 s}{M_V^2}\). If one neglects the wave-function renormalization and the tadpole effects then the pion decay constant is given by \(f_\pi = f\) and therefore \(r_f = 0\). Taking this into account, we get a set of predictions for LECs \(r_2, \ldots, r_5\), in complete agreement with the results in Ref. [13].

However, all the former results ignored the effects of the scalar tadpole [14, 15]. The term \(c_m \langle S \chi_+ \rangle\) connects the scalar field to the vacuum, inducing a pion wave-function renormalization and a more complicated relation between \(m_\pi\) and the quark mass [14]. Thus, one has the large--\(N_C\) relations,
\[
Z_\pi = 1 - \frac{8 c_d c_m}{f^2} \frac{m_\pi^2}{M_S^2} + \frac{6 c_d c_m^3}{f^4} \frac{m_\pi^4}{M_S^4} + O(m_\pi^6) ,
\]
\[
2 B_0 \bar{m} = m_\pi^2 - \frac{8 c_m (c_d - c_m)}{f^2} \frac{m_\pi^4}{M_S^4} + O(m_\pi^6) ,
\]
with \(\bar{m}\) the \(u\) and \(d\) quark masses in the isospin limit. The expressions for \(r_i\) provided in Ref. [13] did not take this effect into account. Our results in Eqs. (36)–(10) are fully general and allow a simple implementation of this correction. Thus, one gets the corrected widths,
\[
\Gamma_V = \frac{g_V^2 M_V^5 \rho_s^3}{48 \pi f^4} \left[ 1 + \frac{4 \sqrt{2} f_x}{g_V} \frac{2 B_0 \bar{m}}{M_V^2} \right] ,
\]
\[
\Gamma_S = \frac{3c_d^2 M_S^3 \rho_s}{16 \pi f^4} \left[ 1 - \frac{2m_\pi^2}{M_S^2} + \frac{2c_m}{c_d} \frac{2 B_0 \bar{m}}{M_S^2} \right] ,
\]
with \(f_\pi = f Z_\pi^{-\frac{1}{2}} [14]\). The resonance masses remain \(m_\pi\)-independent. From this, one is able to recover the real
parameters that provide the LECs:
\[
\alpha_V = \beta_V = \frac{8\sqrt{2}f_\pi}{g_V} - 6 - \frac{16c_d c_m M_\pi^2}{f^2 M_S^2}, \quad (58)
\]
\[
\gamma_V = \frac{32f_\pi^2}{g_V^2} - \frac{48\sqrt{2}f_\pi}{g_V} \left[ 1 + \frac{4c_m(c_d + c_m) M_V^2}{f^2 M_S^2} \right] + 6 \left[ 1 + \frac{16c_d c_m M^2}{f^2 M_S^2} + \frac{32c_m c_d + 2c_m M^4}{f^2 M_S^2} \right], \quad (59)
\]
\[
\alpha_S = \beta_S = \frac{4c_m c_d}{c_d} - 6 - \frac{16c_d c_m}{f^2}, \quad (60)
\]
\[
\gamma_S = 10 \left[ 1 + \frac{48c_d c_m}{5f^2} + \frac{32c_m^2}{5f^4} \right] - \frac{16c_d}{c_d} \left[ 1 + \frac{2c_d c_m}{f^2} + \frac{8c_m^2}{f^4} \right] + \frac{4c_m^2}{c_d} \left[ 1 - \frac{8c_m c_d}{f^2} \right], \quad (61)
\]
Substituting these values in Eqs. (36)–(40), one recovers the value of \( r_0 \). The wave-function renormalization in Eq. (54) provides one can extract the corresponding \( O(p^4) \) LECs [2].

The wave-function renormalization in Eq. (54) provides the value of \( f_\pi \) in the resonance theory under consideration. Comparing this to the \( f_\pi \) expression in \( \chi \)PT from Eq. (13) and using the values of \( L_5 \) and \( L_8 \) from Eq. (62), one can extract the corresponding \( O(p^6) \) LEC in terms of the resonance couplings:

\[
r_f = -\frac{8c_d^2 c_m}{M_S^2}. \quad (63)
\]

We proceed now to a numerical comparison of our new calculation and the original results in Ref. [13], where one had
\[
r_2 = 1.3 \cdot 10^{-4}, \quad r_3 = -1.7 \cdot 10^{-4}, \quad r_4 = -1.0 \cdot 10^{-4}. \quad (64)
\]
This can be compared to our determinations
\[
r_2 = 18 \cdot 10^{-4}, \quad r_3 = 0.9 \cdot 10^{-4}, \quad r_4 = -1.9 \cdot 10^{-4}, \quad (65)
\]

where we took the same inputs used in Ref. [13] to extract the values of the LECs in Eq. (64), \( f = 93.2 \) MeV, \( g_V = 0.09, \ f_\pi = -0.03, \ M_V = M_\rho = 770 \) MeV, \( c_d = 32 \) MeV, \( c_m = 42 \) MeV, \( M_S = 983 \) MeV. The

kaon and eta contributions [13] have also been added in Eq. (66) in order to compare with Eq. (64). The impact of this modifications on the whole amplitude is not large since it is an \( O(p^6) \) effect.

Observing the scattering-lengths derived from Ref. [13], we get slight shifts on the values:
\[
\begin{align*}
\delta a^0_0 &= 0.004, \quad \delta b^0_0 = 0.004, \\
10 \cdot \delta a^0_2 &= -0.003, \quad 10 \cdot \delta b^0_2 = -0.017, \\
10 \cdot \delta a^0_4 &= 0.001, \quad 10 \cdot \delta b^0_4 = -0.003,
\end{align*}
\]
given in \( m_\pi \) units for the mass-dimension quantities. Although there are large variations in the \( O(p^6) \) LECs (especially \( r_2 \)), we verified that the effect on the global uncertainties in the current scattering-length determinations [15] is negligible. Nevertheless, the lack of control on the \( r_8 \) avoids any improvement of the errors beyond these values even if the accuracy in the remaining inputs is considerably increased. Hence, from our estimate in Eq. (66) we consider that it is hard to further decrease, for instance, \( \Delta a^0_0 \) below 0.004 unless our knowledge on the resonance parameters is adequately improved.

This exercise shows that the extraction of the these couplings requires of a very subtle analysis and a closer examination of the resonance lagrangian. The lagrangian in Eqs. (61) and (62) provides only a rough approximation and there can be more resonance contributions to the \( O(p^6) \) LECs beside the scalar tadpole [18]. These variations due to unheeded contributions just point out the level of theoretical uncertainty that comes into play from our ignorance of the resonance lagrangian.

We presented in this note a new method to calculate \( \chi \)PT low-energy constants in terms of resonance parameters in a model independent way, without relying on any particular form of the resonance lagrangian. This technique provides a convenient procedure of implementing the high and low-energy constraints and can be useful for future studies.

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