DYNAMIC EXTENSIONS FOR THE LOGIC OF KNOWING WHY WITH PUBLIC ANNOUNCEMENTS OF FORMULAS

NICHOLAS PISCHKE

Abstract. In this paper, we address the logic of knowing why under the extensions with ideas from dynamic epistemic logic, namely public announcements. Through the additional notions present in the knowing why context, we consider two possible variants, namely the extensions by (i): public announcements of a formula and by (ii): public announcements of reasons, although the deeper analysis of the latter is left for future work. Following to this, we consider another logical operator, the conditional knowing-why operator, for which we study the applications to the axiomatization of public announcements as well as the solely framework. At the end, we consider the logical expressivity of these different logics in comparison to each other. We show one of the main problems with the usual process of proving completeness of the higher logic through translation.

1. Introduction

The logic of knowing why, as introduced by Xu, Wang and Studer in [13], considers a synthesis of ideas emerging from justification logic together with the notions of epistemic logic to provide a framework for reasoning not only about knowing formulas but also knowing why in the concrete sense of knowing explanations for formulae in all concerned worlds of an agent. For this, the authors introduced a new modal operator \( K_y \) in extension to the basic epistemic framework. This synthesis, although (conceptually different) considered by others before in different realizations like e.g. in Fitting’s work [2], promises interesting applications, especially after an enhancement by dynamic notions, in a way as it was done for the core parts separately, like in classical dynamic epistemic logic and e.g. for justification logic by Renne in [4] and e.g. [5]. Although dynamic epistemic logic encompasses many different formalizations, we consider public announcement logic for the extension. While originally introduced by Plaza in his seminal work [3], we will mainly adapt, and refer to, the presentation in the monograph [7] and therefore also to the embedding in the whole class of dynamic epistemic logics.

As there are two different kinds of notions concerned, namely solely knowing and knowing why (or knowing reasons) we also can imagine two notions of public announcements from this context. Either publicly announcing that a formula has to hold or even publicly announcing why a formula has to hold. The second one could then follow the ideas and concepts presented by Eijck, Gattinger and Wang in [8]. We will only examine the first notions in this paper, i.e. an enhancement of the logic of knowing why with classical public announcement operators is considered. We then find that, in difference to usual enhancements by public announcement operators, the logic of knowing why bears some deeper semantical idiosyncrasies in cooperation with these notions, making a classical axiomatization impossible. We then introduce a relativized version of the knowing-why operator, following suggestions made in [13], to provide a possible workaround for these problems. As a main goal, we provide an axiomatization of this latter logic which is then proved sound and complete with respect to the basic model classes presented in the original paper.

We then consider expressivity comparisons between the logic using public announcements and the logic incorporating the relativized knowing why operator. Together with this, we address the problem of proving completeness in the standard sense of providing a translation for the other logics presented and discussed here.

2. Preliminaries

The main purpose of this section is to provide an overview of the work done about the logic of knowing why, by Xu et. al. in [13], to create a common ground on which the dynamic extensions and other modifications later take place. For the following, it’s assumed that the reader is familiar with the basic notions of propositional classical modal logic, Kripke-frames and the concepts of basic epistemic logic.

Key words and phrases. epistemic logic, logic of knowing why, dynamic logic, public announcement logic.
2.1. The logic of knowing why. The logic of knowing why, in the following denoted by ELKy in correspondence to the initial paper, is syntactically defined with the BNF

$$\mathcal{L}_{ELKy} : \phi ::= p \mid \neg \phi \mid (\phi \land \phi) \mid K_a \phi \mid K_y \phi$$

with \(p \in P\) and \(a \in A\). The sets \(P\) and \(A\) used here are respectively the sets for the atomic propositions, being countably infinite, and the countable set of agents. Obviously, \(\mathcal{L}_{ELKy}\) denotes the respective language as the set of well-formed formulae build from this BNF. All other general connectives like \(\to, \leftrightarrow, \land\) as well as \(\bot\) and \(\top\) are defined in a similar way to classical propositional logic. Following the general notations of epistemic logic, the dual of the modal operator \(K_a\) is defined as \(\neg K_a \neg \phi\). As it follows naturally, while \(K_a\) reads like the agent \(a\) knows ... is the case, the new modal operator \(K_y\) read like the agent \(a\) knows why ... is the case.

The semantics of the the logic ELKy were defined in a model-theoretic sense with following the approach of Fitting models for justification logic, i.e. a ELKy-model \(\mathfrak{M} = (W, E, \{R_a \mid a \in A\}, E, V)\) is defined over (i) a non-empty set of worlds \(W\), (ii) an own accessibility relation \(R_a \subseteq W \times W\) for each agent \(a \in A\), (iii) a basic evaluation function \(E : E \times \mathcal{L}_{ELKy} \to 2^W\) and (v) an admissible explanation function \(V : P \to 2^W\). A pointed version of a model, i.e. a combination of a model and a designated world is then denoted by \((\mathfrak{M}, w)\).

As it was defined in the initial paper, the set \(E\) holding possible explanations for formulae has to satisfy two conditions, namely (1) holding a designated explanation \(e\) and (2) being closed under the explanation-combination operator \(\cdot : E \times E \to E\), i.e. \(s, t \in E\) implying \((s \cdot t) \in E\). With \(E\) being independent of an agents view, it can be seen as an omnipresent domain of explanations.

The explanation function, relating the worlds \(w \in W\) to a formula \(\phi\) and an explanation \(s\) where \(s\) is perceived as an explanation of \(\phi\), also has to fulfill two conditions about its behavior, namely (1) whenever the formula is in a designated set \(\Lambda\), it holds that \(E(e, \phi) = W\) and (2) the function \(E\) is distributed over the \((MP)\) rule with \(E(s, \phi \to \psi) \cap E(t, \phi) \subseteq E((s \cdot t), \psi)\). This designated set \(\Lambda\) was defined by the authors as the tautology ground, simply a set consisting of valid formulae, which represent a fixed argumentation ground for all agents which are agreed self-evidently true.

The accessibility relations \(R_a\) for all agents are in the following required to be so called S5 relations, i.e. being (i) reflexive, that for all \(w, (w, w) \in R_a\), (ii) transitive, that for all \(w, u, v\), if \((w, u) \in R_a\) and \((u, v) \in R_a\), then \((w, v) \in R_a\) and (iii) symmetric, that for all \(w, u, v\), if \((w, u) \in R_a\) then \((u, w) \in R_a\). Although there are many other classes of models, defined over different restrictions of the accessibility relations, the main emphasis will be on those S5 models.

The local satisfiability, namely the validity of a formula in a specific world of a model, is then defined over the relation \(\models\) with

\((\mathfrak{M}, w) \models p \iff w \in V(p)\)
\((\mathfrak{M}, w) \models \neg \phi \iff (\mathfrak{M}, w) \not\models \phi\)
\((\mathfrak{M}, w) \models \phi \land \psi \iff (\mathfrak{M}, w) \models \phi\) and \((\mathfrak{M}, w) \models \psi\)
\((\mathfrak{M}, w) \models K_y \phi \iff \forall v \in W : (w, v) \in R_a\) implies \((\mathfrak{M}, v) \models \phi\)

for the classical operators from epistemic logic, and with
\((\mathfrak{M}, w) \models K_y \phi \iff (1): (\mathfrak{M}, w) \models K_y \phi\) and (2): \(\exists t \in E : \forall v \in W : (w, v) \in R_a\) implies \(v \in E(t, \phi)\)

for the new operator \(K_y\). The case of a formula being valid in all worlds \(w\) of a model \(\mathfrak{M}\) is simply denoted by \(\mathfrak{M} \models \phi\). In this initial paper, the authors proposed an axiomatic system \(\text{SKy}\), Fig. 1 for which they’ve proved soundness and completeness with respect to the class of all S5-ELKy-models as defined as above.

The notions for semantic deduction of a formula \(\phi\) from a set of formulas \(\Gamma\) in a specific model class \(\mathfrak{M}\), \(\Gamma \models_{\mathfrak{M}} \phi\), as well as the formal proof in a system \(\mathcal{S}\), \(\Gamma \vdash_{\mathcal{S}} \phi\), are defined as usual in the context of modal propositional logics.

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1Note, that at some occurrences throughout the paper, we will write e.g. \(\hat{R}\) for a set(or in a sense, vector) over the set of agents \(A\).

2Following from this, the special explanation \(e\) is called the self-evident explanation.

3Note, that the corresponding class of models is denoted by \(K_N\), while the basic class of models with no restrictions for the accessibility relations is simply denoted by \(K\).

4The presentation of the theorems and corresponding proofs are omitted here(s). Theorem 15 and Theorem 23 in [15].
The corresponding dual to this operator $\phi$ is denoted by $\langle \phi \rangle \psi$ and semantically defined by the following:

$$(\mathcal{M}, w) \models \langle \phi \rangle \psi \text{ if and only if } (\mathcal{M}, w) \models \phi \text{ and } (\mathcal{M}^\phi, w) \models \psi$$

Like for the classical modal-type operators, this dual corresponds to the notion of possibility. Following the classical way of axiomatizing public-announcement-type logics over reduction-style axioms, we’re considering the Hilbert-style calculus presented in [13] for the underlying logic $ELKy$ as shown in the preliminaries together with the axiomatization for propositional public announcement operators [3]. Therefore, we first consider the following additional axioms used there to axiomatize the basic public announcement logic $PA$:

\[
\begin{align*}
K_a(\phi \rightarrow \psi) & \rightarrow (K_a \phi \rightarrow K_a \psi) \\
K_ya(\phi \rightarrow \psi) & \rightarrow (K_ya \phi \rightarrow K_ya \psi) \\
K_a \phi & \rightarrow \phi \\
K_a \phi & \rightarrow K_a K_a \phi \\
\neg K_a \phi & \rightarrow K_a \neg K_a \phi \\
K_ya \phi & \rightarrow K_a K_ya \phi \\
K_ya \phi & \rightarrow K_a K_ya \phi \\
\end{align*}
\]

Table 1. The System $SKy$

| Axiom | Description |
|-------|-------------|
| PT    | Propositional axioms |
| K     | $K_a(\phi \rightarrow \psi) \rightarrow (K_a \phi \rightarrow K_a \psi)$ |
| Ky    | $K_ya(\phi \rightarrow \psi) \rightarrow (K_ya \phi \rightarrow K_ya \psi)$ |
| T     | $K_a \phi \rightarrow \phi$ |
| (4)   | $K_a \phi \rightarrow K_a K_a \phi$ |
| (5)   | $\neg K_a \phi \rightarrow K_a \neg K_a \phi$ |
| PS    | $K_ya \phi \rightarrow K_a K_ya \phi$ |
| (4YK) | $K_ya \phi \rightarrow K_a K_ya \phi$ |

As the main purpose of this paper, we will now consider the logic of knowing why under the extensions with the operator for public announcements of formulas. The basic notions and definitions about this new operator are then defined according to the common notions of classical public announcement logic $PA$. We will denote this new logic with $PAKY$ where $PA$ emphasizes the announcement of formulas additionally.

**Definition 1.** For a countable set of agents $A$, a countably infinite set of atomic propositions $P$, the language of the logic $PAKY$ is defined with

$$L_{PAKY} : \phi ::= p \mid \neg \phi \mid (\phi \land \phi) \mid K_a \phi \mid K_ya \phi \mid [\phi][\phi]$$

where $p \in P$ and $a \in A$.

Obviously, $PAKY$ is simply an augmentation of $ELKy$ by the public announcement operator $[]$ filled with a to-announce formulae $\phi$. Verbally, the construct $[\phi][\psi]$ translates to “after the public announcement of the formula $\phi$, it holds that $\psi$”. The semantics of this augmented logic are again defined over the same $S5$-relational models $\mathcal{M}$, being constructed as presented in the preliminaries. Following to this, the satisfaction relation $\models$ now gets extended for the handling of this new notion with

$$(\mathcal{M}, w) \models [\phi][\psi] \text{ if and only if } (\mathcal{M}, w) \models \phi \text{ implies } (\mathcal{M}[\phi], w) \models \psi$$

The construct $(\mathcal{M}^\phi, w)$ represents the pointed version of an extended(or updated) model, $\mathcal{M}[\phi]$, with the following construction.

**Definition 2.** The model $\mathcal{M}$ after the public announcement by $\phi$, $\mathcal{M}[\phi] = (W', E, \{R'_a \mid a \in A\}, E', V')$, is defined with

- $W' = \{ w \in W \mid (\mathcal{M}, w) \models \phi \}$
- $R'_a = R_a \cap (W' \times W')$
- $V'(p) = V(p) \cap W'$
- $E'(t, \phi) = E(t, \phi) \cap W'$

Note, that $E$ from the previous model is not changed in the “augmentation“ process. This comes from the fact that the set is simply the set of explanations not related to the set of worlds in any direct way (only over the function $E$). For the right hand definition of $W'$, we will write $[\phi][\mathcal{M}]$ in the following. One can also imagine the operator $|$ of the change of a model by a public announced formula $\phi$ as the function $\vdash : K_yS5 \times L_{PAKY} \rightarrow K_yS5$. The corresponding dual to this operator $[\phi]$ is denoted by $\langle \phi \rangle \psi$ and semantically defined by the following:

$$(\mathcal{M}, w) \models \langle \phi \rangle \psi \text{ if and only if } (\mathcal{M}, w) \models \phi \text{ and } (\mathcal{M}[\phi], w) \models \psi$$

Like for the classical modal-type operators, this dual corresponds to the notion of possibility. Following the classical way of axiomatizing public-announcement-type logics over reduction-style axioms, we’re considering the Hilbert-style calculus presented in [13] for the underlying logic $ELKy$ as shown in the preliminaries together with the axiomatization for propositional public announcement operators [3]. Therefore, we first consider the following additional axioms used there to axiomatize the basic public announcement logic $PA$.
into the formula, since modifying this matter equally with a reduction like in the axioms shown above, i.e. pushing the announcements further.

We sketch the model-theoretic considerations. Since there is no basic formula in the logic \( \text{ELKy} \) which is able to express this matter equally with a reduction like in the axioms shown above, i.e. pushing the announcements further into the formula, since modifying \( \psi \) results in some problems concerning the integrity of \( \mathcal{E} \) in the evaluation. We concretize this assumption through the following theorem.

**Theorem 1.** \( \text{PAFKy} \) is more expressive than \( \text{ELKy} \).

**Proof.** We sketch the model-theoretic considerations. Since \( \text{PAFKy} \) is an extension of \( \text{ELKy} \), it can’t be less expressive. Now, consider the following two models \( \mathfrak{M}_1, \mathfrak{M}_2 \).

For \( \mathfrak{M}_1 \), we consider a set of worlds containing two elements \( w_1, w_2 \), we assume that \( E \) contains two basic elements \( s, t \) besides the usual conditions, \( R_a \) is expected to be total for all agents \( a \). For \( \mathcal{E} \), we set \( \mathcal{E}(s,p) = \{ w_1 \}, \mathcal{E}(t,p) = \{ w_2 \} \) while the rest are set to \( \emptyset \). For the basic evaluation function \( V \), we finally set \( V(p) = \{ w_1, w_2 \} \) and \( V(q) = \{ w_1 \} \). Visually, this model may be imagined as the following:

\[
\begin{array}{c}
\mathfrak{M}_1 \\
\begin{array}{c}
\downarrow w_1 \quad w_2 \\
| \quad | \\
p, q & s : p \quad p, t : p \\
\downarrow a \\
\end{array}
\end{array}
\]

Suppose that \( \mathfrak{M}_1 \) is a submodel of \( \mathfrak{M}_2 \). Therefore, we will only mention the additional settings. We consider an augmented set of worlds by a third world \( w_3 \), together with a third contained basic explanation \( r \in E \). \( R_a \) is still considered to be total for all agents and we additionally require \( \mathcal{E}(r,p) = \{ w_3 \} \) and \( V(p) = \{ w_1, w_2, w_3 \} \), \( V(q) = \{ w_1, w_3 \} \). We may imagine this second model in a slightly reduced representation as:

\[
\begin{array}{c}
\mathfrak{M}_2 \\
\begin{array}{c}
\downarrow w_1 \quad w_2 \\
| \quad | \\
p, q & s : p \quad p, t : p \\
\downarrow a \\
\end{array}
\end{array}
\]

In \( \text{PAFKy} \), we may distinguish these model through the formula \([q]K_{y_a}p \) in the designed world \( w_1 \) contained in both models. For \( \mathfrak{M}_1 \), we find that the only world left after the announcement made in \( w_1 \) is \( w_1 \) itself which provides \( \mathfrak{M}_1, w_1 \models [q]K_{y_a}p \). For \( \mathfrak{M}_2 \), we find that \( W_2^q = \{ w_1, w_3 \} \) after the announcement. Through \( \bigcap_{t \in E} \mathcal{E}_2(t,p) = \emptyset \), we find that \( \not\exists t \in E : \forall v \in W_2^q : v \notin \mathcal{E}(t,p) \), i.e. \( \mathfrak{M}_2, w_1 \not\models [q]K_{y_a}p \).

To show that there exists no \( \phi \in \mathcal{L}_{\text{ELKy}} \) which can distinguish between the reasons from only \( w_1 \) and \( w_3 \), we suppose the opposite. If so, then \( \phi = q \rightarrow \Theta \) for some \( \Theta \in \mathcal{L}_{\text{ELKy}} \). By \( q \) used in the implication, we exclude the case of \( q \) being un-announceable in \( w_1 \). By the semantics of \([q]K_{y_a}p, \Theta \) would then have to model the behavior of \( \mathfrak{M}(q, w_1) \models K_{y_a}p \), i.e. the behavior of \( K_{y_a}p \) restricted to all \( q \)-worlds.
By a simple induction on the structure of $\Theta$, it can be seen that no such formula exists, since to express something about the reasons of $p$ in reachable worlds, the only possibility is to include $Ky_a p$ as some subformula which itself can not be limited to some subset of worlds over the use of $\neg, \wedge, K_a$ or $Ky_a (\text{which form the induction steps})$.

In the proof, it became apparent that although the sub-case $[\phi]K_a \psi$ is expressible over an adequate translation, we have no other possibility of expressing something concerning the explanation function $E$ than the operator $Ky_a \psi$ and for this purpose, there can be no modification of $\psi$ as it would mess with the before mentioned integrity of $E$ by changing the concerned formula for the existence of explanations, i.e. there is no way to restrict the application of $E$. Through this theorem, we can obviously not apply Plaza’s method from [2] to provide completeness over reduction.

3.1. A relativized knowing-why operator. To address this problem, we’re following the ideas of [2], where the authors used this concept of relativization for problems concerning common knowledge, and of [11], [12] from the context of non-classical epistemic logics, by relativizing the $Ky_a$ operator to a conditional one, namely $K_y'(\phi, \psi)$, with the following semantics related to the semantics of the construct shown above:

$$(\mathfrak{M}, w) \models K_y'(\phi, \psi) \text{ if } \exists \mathfrak{E} \in E : \forall v \in W \text{ such that } (v, w) \in R_a \text{ and } (\mathfrak{M}, v) \models \phi \text{ it is that }$$

$$\text{ (1): } v \in \mathcal{E}(t, \psi) \text{ and (2): } (\mathfrak{M}, v) \models \psi$$

This operator relates to “the agent a knows why $\psi$, under the condition $\phi$”. Clearly, the original, unary, operator $Ky_a \phi$ corresponds to $K_y'(\top, \phi)$.

Two new versions of the logics presented before, namely $ELKy^r$ and following to this $PAFKy^r$ are directly emerging from this, simply with

$$L_{ELKy^r} : \phi ::= p | \neg \phi | (\phi \land \psi) | K_a \phi | K_y'(\phi, \psi)$$

with $p \in \mathcal{P}$ and $a \in \mathcal{A}$ and $L_{PAFKy^r}$ being simply the augmentation of $ELKy^r$ with the notion of the $[\phi]$ operator as shown above. The class of models associated with this new logic, named $Ky^r S\delta$ for the $S\delta$-relational version, are structurally similar to the $KyS\delta$-models presented before, while the semantic evaluation of $Ky$ in $\models$ is exchanged with the definition for $K_y^r$.

Proposition 1. The following formulae are valid in the class of all $S\delta$-$ELKy^r$-models.

(i): $K_y'(\phi, \psi \rightarrow \chi) \rightarrow (K_y'(\phi, \psi) \rightarrow K_y'(\phi, \chi))$

(ii): $K_y'(\bot, \phi)$

(iii): $K_y'(\phi, \psi) \rightarrow K_a (\phi \rightarrow \psi)$

(iv): $K_y'(\phi, \psi) \rightarrow K_a Ky_a'(\phi, \psi)$

Proof. In the following proofs, let $\mathfrak{M}$ be an arbitrary model and $w \in W$ an arbitrary world.

(i): Suppose $(\mathfrak{M}, w) \models K_y'(\phi, \psi \rightarrow \chi)$ and $(\mathfrak{M}, w) \models K_y'(\phi, \psi)$. By the rules of $\models$, the first one translates to $\exists \mathfrak{E} \in E : \forall v \in W : (v, w) \in R_a$ and $(\mathfrak{M}, v) \models \phi$ implies $(\mathfrak{M}, v) \models \psi \rightarrow \chi$ and $v \in \mathcal{E}(t, \psi \rightarrow \chi)$. The second one translates to $\exists s \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, v) \models \phi$ implies $(\mathfrak{M}, v) \models \psi$ and $v \in \mathcal{E}(s, \psi)$. By meet over $W, R_a, E$ and $\phi$, we have: $\exists s, t \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, w) \models \phi$ implies $v \in \mathcal{E}(t, \phi \rightarrow \chi)$, $v \in \mathcal{E}(s, \psi)$ and $(\mathfrak{M}, v) \models \psi$ and $(\mathfrak{M}, v) \models \psi \rightarrow \chi$. From $(MP)$, we have implied that $(\mathfrak{M}, v) \models \chi$. With this, and $v \in \mathcal{E}(t, \phi \rightarrow \chi) \cap \mathcal{E}(s, \psi) \subseteq \mathcal{E}(s \cdot t, \chi)$, we have $(\mathfrak{M}, w) \models K_y'(\phi, \chi)$.

(ii): The formula translates to $\exists t \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, v) \models \bot$ implies $(\mathfrak{M}, v) \models \phi$ and $v \in \mathcal{E}(t, \psi)$. This obviously relates to the structure of the formula $\bot \rightarrow \psi \equiv \top$, which immediately implies $(\mathfrak{M}, w) \models K_y'(\bot, \psi)$.

(iii): Suppose $(\mathfrak{M}, w) \models K_y'(\phi, \psi)$. By this, we have $\exists t \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, v) \models \phi$ implies $(\mathfrak{M}, v) \models \psi$ and $v \in \mathcal{E}(t, \psi)$. By cutting out $E$, we have $\forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, v) \models \phi$ implies $(\mathfrak{M}, v) \models \psi$. This implies propositional, that $\forall v \in W : (w, v) \in R_a$ implies $(\mathfrak{M}, v) \models \phi$ implies $(\mathfrak{M}, v) \models \psi$, i.e. $(\mathfrak{M}, w) \models K_a (\phi \rightarrow \psi)$.

(iv): Suppose that $(\mathfrak{M}, w) \models K_y'(\phi, \psi)$. Now, we consider any $v \in W$ with $(w, v) \in R_a$ and $u \in W$ with $(v, u) \in R_a$. By the transitivity of the relations, we have $(w, u) \in R_a$ implied. By $(w, u) \in R_a$, we have $\exists t \in E : \forall u \in W : (w, u) \in R_a$ and $(\mathfrak{M}, u) \models \phi$ implies $(\mathfrak{M}, u) \models \psi$ and $u \in \mathcal{E}(t, \psi)$, therefore $(\mathfrak{M}, v) \models K_y'(\phi, \psi)$ by $(v, u) \in R_a$. From that and $(w, v) \in R_a$, we have $(\mathfrak{M}, w) \models K_a K_y'(\phi, \psi)$.

Remark 1. The formulas (i), (ii) and (iii) are obviously valid in the basic class of $ELKy^r$-models(without restrictions to the accessibility relation) as well.
We can even consider a stronger representation of the distribution of formulas in the $K y^\gamma$ operator by taking different premises into account.

**Proposition 2.** $K y^\gamma_\theta (\chi, \phi \to \psi) \land K y^\gamma_\theta (\theta, \phi) \to K y^\gamma_\theta (\chi \land \theta, \psi)$ is valid in $K y^\gamma S 5$.

**Proof.** Suppose $(\mathfrak{M}, w) \models K y^\gamma_\theta (\chi, \phi \to \psi)$ and $(\mathfrak{M}, w) \models K y^\gamma_\theta (\theta, \phi)$. The first one translates to $\exists t \in E : \forall v \in W : (w, v) \in R_0$ and $(\mathfrak{M}, v) \models \chi$ implies $(\mathfrak{M}, v) \models \phi \to v$ and $v \in E(t, \phi \to \psi)$, while the second translates to $\exists s \in E : \forall v \in W : (w, v) \in R_0$ and $(\mathfrak{M}, v) \models \theta$ implies $(\mathfrak{M}, v) \models \phi$ and $v \in E(s, \phi)$. By meet over $W$, $R_0$ and $E$, we have $\exists s, t \in E : \forall v \in W : (w, v) \in R_0$ and $(\mathfrak{M}, v) \models \chi \land \theta$ implies $(\mathfrak{M}, v) \models \phi \to v$ and $(\mathfrak{M}, v) \models \phi$ and $v \in E(t, \phi \to \psi) \cap E(s, \phi)$. By the laws of modus ponens, we have $(\mathfrak{M}, v) \models \psi$ for the latter and by the properties of $E$, we have $E(t, \phi \to \psi) \land E(s, \phi) \subseteq E(t \cdot s, \psi)$, therefore $(\mathfrak{M}, w) \models K y^\gamma_\theta (\chi \land \theta, \psi)$.

For illustrating this property of composition further, one can imagine this situation by the following world-announcement diagram shown in Fig. 2. Of course, the so labeled announcements shall be seen as the constraints in the operator $K y^\gamma_\theta$ and not as the usual announcements. Obviously, the transition from the initial considered set of worlds $W$ to the set after the announcement $[\chi \land \lambda]$ is transitive with the announcements $[\lambda]$ and $[\chi]$. Concerning the validity of the there shown formula, one can imagine the inner bars dividing the rectangle of worlds taking any shape in correspondence to the announcements, and by that intersecting with each other in all possible ways. Obviously, the there shown formulae are always valid in this intersection. If there is no intersection between these world-separators, i.e. no worlds captured by $[\chi \land \lambda]$, it automatically relates to the validity $K y^\gamma_\theta (\bot, \phi)$.

The validity (i) from Prop. 1 is obviously a special case of Prop. 2 with both $\chi$ and $\theta$ representing the same formula.

**3.2. Axiomatization of ELK $y^\gamma$.** As an axiomatization for the logic of $ELK y^\gamma$, we propose the here shown Hilbert-calculus (Fig. 3) as an axiomatic and rule based extensions to the system $SKY$ from the initial paper. In correspondence to the before-mentioned paper, we call this system $SKYR$.

The axiom $(E K y^\gamma R)$ provides, as mentioned before, a stronger version of the distribution of conclusions by modus ponens using different premises. The axiom $(D K y^\gamma R)$ defines the decomposition of the $K y^\gamma$ operator into a specific instantiation of the basic knowledge operator $K$ and axiom $(4 Y K R)$ provides the positive introspection of $K y^\gamma$ by the classical operator $K$. One may wonder about the axiom $(5 Y K R)$, the negative introspection of $K y^\gamma$ by the operator $K$, which is actually provable, exactly as its unconditioned companion $(5 Y K)$ was either in the basic system $S K Y$ (s. Proposition 11, [13]).

**Proposition 3.** $\neg K y^\gamma_\theta (\phi, \psi) \to K a \neg K y^\gamma_\theta (\phi, \psi)$ is provable in $SKYR$.

**Proof.** For the beginning, we take $K a K y^\gamma_\theta (\phi, \psi) \to K y^\gamma_\theta (\phi, \psi)$ as an instance of the axiom $(T)$. From this, we propose the contraposition $\neg K y^\gamma_\theta (\phi, \psi) \to \neg K a K y^\gamma_\theta (\phi, \psi)$. Following, we consider an instance of the axiom $(5)$ with $\neg K a K y^\gamma_\theta (\phi, \psi) \to K a \neg K a K y^\gamma_\theta (\phi, \psi)$ together with an instance of $(4 Y K R)$ with $K y^\gamma_\theta (\phi, \psi) \to K a K y^\gamma_\theta (\phi, \psi)$ and its contraposition $\neg K a K y^\gamma_\theta (\phi, \psi) \to \neg K y^\gamma_\theta (\phi, \psi)$. Following from the rule $(N K)$ and this last contraposition,
Completeness. The ELKy model being defined over all maximal consistent sets with the (later more explicitly defined) common canonical $S$. Proof. This theorem is easily obtained by considering Lem. 1, Prop. 1 and Prop. 2 together with the fact, that for every agent. Therefore, we also have $\exists$ that $(\text{Soundness of Theorem 2})$

Proof. Lemma 1. The rule (NKyR) is valid.

Proof. Suppose that $\phi \in \Lambda$ and consider an arbitrary formula $\psi$. Let $\mathfrak{M}$ be an arbitrary model. Since $\Lambda$ only contains tautologies, we have $(\mathfrak{M}, w) \models \phi$ for all $w \in W$. By the first rule regarding the behavior of $E$, we have that $\exists e \in E : \forall w \in W : w \in E(e, \phi)$. Therefore, we also have $\forall v \in W : (w, v) \in R_a$ implying these explanations for every agent. Therefore, we also have $\exists e \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, v) \models \psi$ imply $(\mathfrak{M}, v) \models \phi$ and $v \in E(t, \phi)$ since $t$ can always be at least $e$.

Theorem 2 (Soundness of $\text{SKYR}$ over $K_{\psi'} S_5$). The system $\text{SKYR}$ is sound with respect to the class of all $S_5 - ELKy'$-models.

Proof. This theorem is easily obtained by considering Lem. 1, Prop. 1 and Prop. 2 together with the fact, that the $\text{ELKy}'$-models are based on the standard Kripkean-$S_5$-models making the standard $S_5$ axioms valid.

3.2.1. Completeness. For the proof of completeness’ sake, we follow the common approach by considering a canonical model being defined over all maximal consistent sets with the (later more explicitly defined) common properties like truth in order to provide the framework for a somehow standard proof of a completeness theorem in the context of modal logics. For this, we first consider:

Definition 3 (Consistency). A set $\Gamma$ is called consistent (in a specific system), if $\Gamma \not\vdash \bot$. Otherwise, it is called inconsistent, $\Gamma \vdash \bot$. Following to this, a set is called maximal consistent if

(1) it is consistent, $\Gamma \not\vdash \bot$,
(2) it is maximal, $\exists \Gamma' \supset \Gamma : \Gamma' \not\vdash \bot$.

Following from these considerations, we now define a canonical model with one world for each maximal consistent set over the following definition.

Definition 4 (Canonical model for $\text{SKYR}$). The canonical model for $\text{SKYR}$ is defined as the tupel $\mathfrak{M}^c = \langle W^c, E^c, \{R_a^c \mid a \in A \}, E^c, V^c \rangle$.
Proposition 5. The canonical model $\mathfrak{M}^c$ is well-defined, given the conditions for $\text{ELKy}^r$ ($\text{ELKy}$)-models.

Proof. The proof is split into two parts, concerning on the one side the two conditions (1) and (2) for the explanation function $\mathcal{E}$ and on the other side the accessibility relations covered in $R^c$ being equivalence relations.

Claim: $\mathcal{E}^c$ satisfies the conditions (1) and (2) presented in the preliminaries for the general prototype of the function $\mathcal{E}$.

Proof (1): Concerning the first property, suppose $\phi \in \Delta$. Consider a world $w_T = \langle \Gamma, F, \vec{f} \rangle \in W^c$. By condition (i) concerning the worlds of the canonical model, we have $(e, (\phi, \psi)) \in F$ with any $\psi \in \mathcal{L}_{\text{ELKy}^r}$. By the definition of $\mathcal{E}^c$, we have therefore $w_T \in \mathcal{E}^c$. Since $w_T$ was arbitrary, we have $\mathcal{E}^c(e, \phi) = W^c$.

(2): Concerning the second property, suppose $w_T \in W^c$ as above with $w_T \in \mathcal{E}^c(s, \phi \rightarrow \psi) \cap \mathcal{E}^c(t, \phi)$. Therefore, there exists $\chi, \lambda \in \mathcal{L}_{\text{ELKy}^r}$ with $(s, (\chi, \phi \rightarrow \psi)) \in F$ and $(t, (\lambda, \phi)) \in F$. By condition (i) of $W^c$, we have $(s,t, (\chi \land \lambda, \psi)) \in F$. Therefore $w_T \in \mathcal{E}^c(s, t, \psi)$, implying $\mathcal{E}^c(s, \phi \rightarrow \psi) \cap \mathcal{E}^c(t, \phi) \subseteq \mathcal{E}^c(s, t, \psi)$.

Concerning the accessibility relations, we consider the following:

Claim: For any $a \in \mathcal{A}$, $R^c_a$ is (i): reflexive, (ii): transitive and (iii): symmetric.

Proof: In the following, we consider the three arbitrary worlds $w_T = \langle \Gamma, F, \vec{f} \rangle$, $w_\Delta = \langle \Delta, G, \vec{g} \rangle$ and $w_\Theta = \langle \Theta, H, \vec{h} \rangle$ together with an arbitrary agent $a \in \mathcal{A}$.

(i): Obviously, it hold for $w_T$, that we have $f_a = f_a$. Now, for all $K_a \phi \in \Gamma$, we have $\phi \in \Gamma$ by axioms $(T)$ and the deductive closure of max. consistent sets, therefore $(w_T, w_\Gamma) \in R^c_a$. The case for $K_a \phi$ is obvious either.
(ii): Suppose \((w_T, w_\Delta) \in R^c_a \) and \((w_\Delta, w_\Theta) \in R^c_a \). Now, consider \(Ky'_n(\phi, \psi), K_\alpha \chi \in \Gamma \). By the properties

of \(R^c_a \), we have \(f_a = g_a \) and \(g_a = h_a \), therefore \(f_a = h_a \). Following from Prop. 4 we have \(Ky'_{0}(\phi, \psi), K_\alpha \chi \in \Delta \). Now, by the definition of \(R^c_a \) we get \(Ky'_{\chi}(\phi, \psi), \chi \in \Theta \), therefore \((w_T, w_\Theta) \in R^c_a \).

(iii): Suppose \((w_T, w_\Delta) \in R^c_a \), then we first have \(f_a = g_a \). Now, consider \(Ky'_n(\phi, \psi), K_\alpha \chi \in \Delta \). By Prop. 4 we then have \(Ky'_{\chi}(\phi, \psi), K_\alpha \chi \in \Delta \). By axiom (1), we then have \(\chi \in \Gamma \), thus \((w_\Delta, w_T) \in R^c_a \).

With these two claims, the canonical model \(\mathfrak{M}^c \) from Def. 5 follows all conditions presented for \(ELKy^c \)-models.

In order to fully proof the functioning of the canonical model, we have left to show that \(W^c \) is not empty, and following [13], we provide a construction of the corresponding \(F \) and \(\vec{f} \) for a given \(\Gamma \), by that showing that there exists one world \(w_T \) for every maximal consistent set \(\Gamma \) in the language of \(\mathcal{L}_{ELKy^c} \).

Definition 5. Given any maximal consistent set \(\Gamma \), construct the corresponding world parts \(F^\Gamma \) and \(\vec{f}^\Gamma \) as the following:

For the set \(F^\Gamma \), consider as a base \(F^\Gamma_0 = \{(\phi, \psi), (\phi, \psi) \mid \exists a \in A : Ky'_n(\phi, \psi) \in \Gamma \} \cup \{(e, (\psi, \phi)) \mid \phi \in \Lambda \}

and as a recursion step \(F^\Gamma_{n+1} = F^\Gamma_n \cup \{(s \cdot t, (x \land \lambda, \psi)) \mid \exists (s, (\chi, \phi \rightarrow \psi)), (t, (\lambda, \phi)) \in F^\Gamma_n \} \). Then the set \(F^\Gamma \) is then defined as

\[ F^\Gamma = \bigcup_{n \in \mathbb{N}} F^\Gamma_n \]

For the set of functions \(\vec{f}^\Gamma \), define \(\forall a \in A : \vec{f}^\Gamma_a(\phi, \psi) = (\phi, \psi) \).

From this construction, we can now consider the following proposition providing the non-emptiness of \(W^c \).

Proposition 6. For any maximal consistent set \(\Gamma \), \(w_T = (\Gamma, F, \vec{f}) \in W^c \).

Proof. Since it is already supposed that \(\Gamma \) is maximal consistent and since \(\forall a \in A \), the function \(\vec{f}^\Gamma_a \) defined accordingly to Def. 5 fulfills the properties (i) - (iii) from the canonical model.

(i): Suppose \((s, (\chi, \phi \rightarrow \psi)), (t, (\lambda, \phi)) \in F^\Gamma_n \). By the construction of \(F^\Gamma \) over the union, we have that \(\exists k \in \mathbb{N} \) such that \((s, (\chi, \phi \rightarrow \psi)), (t, (\lambda, \phi)) \in F^\Gamma_k \subseteq F^\Gamma \). Therefore, by the construction of \(F^\Gamma \), it holds that \((s \cdot t, (\chi \land \lambda, \psi)) \in F^\Gamma_{k+1} \subseteq F^\Gamma \).

(ii): Suppose \(\phi \in \Lambda \). By the definition of \(F^\Gamma_0 \), we have \((e, (\psi, \phi)) \in F^\Gamma_0 \subseteq F^\Gamma \) for every \(\psi \).

(iii): Suppose \(Ky'_n(\phi, \psi) \in \Gamma \), then we have \((\phi, \psi), (\phi, \psi) \in F^\Gamma \subseteq F^\Gamma_n \). Since the definition of the function \(f_a \in \vec{f} \), we have \(\vec{f}^\Gamma_n(\phi, \psi) = (\phi, \psi) \), therefore \((\vec{f}^\Gamma_n(\phi, \psi), (\phi, \psi)) = ((\phi, \psi), (\phi, \psi)) \in F^\Gamma_0 \subseteq F^\Gamma \).

In the following, we will now reestablish the existence lemmas for both \(K_a \) and \(Ky'_n \) following the ideas of [13] in order provide the last necessary helps to consider the truth lemma. The key of both existence lemmas is to provide constructions of worlds related by an accessibility relation which refute either the formula itself or any possible explanation in some way, provided that the corresponding \(K_a \) or \(Ky'_n \) formula is not member of the to-speak set.

Lemma 2 (\(K_a \) existence lemma). For any \(w_T \in W^c \), if \(K_a \phi_1 \not\subseteq \Gamma \), there exists a world \(w_\Delta \) with \((w_T, w_\Delta) \in R^c_a \) such that \(\neg \phi_1 \in \Delta \).

Proof. Suppose \(K_a \phi_1 \not\subseteq \Gamma \), i.e. \(\neg K_a \phi_1 \in \Gamma \). For the desired properties, consider

\[ \Delta^- = \{ \neg \phi_1 \} \cup \{ Ky'_n(\phi, \psi) \mid Ky'_n(\phi, \psi) \in \Gamma \} \cup \{ \phi \mid K_a \phi \in \Gamma \} \]

and then define the world \(w_\Delta = (\Delta, G, G) \) as the following:

For the set \(\Delta \), consider the extension of \(\Delta^- \) to a maximal consistent set. For the set \(G \), consider as a base \(G_0 = F \cup \{(\phi, \psi), (\phi, \psi) \mid Ky'_n(\phi, \psi) \in \Delta \) and \(b \neq a \) and as a recursion step \(G_{n+1} = G_n \cup \{(s \cdot t, (\chi \land \lambda, \psi)) \mid (s, (\chi, \phi \rightarrow \psi)), (t, (\lambda, \phi)) \in G_n \} \). The set \(G \) is then defined as

\[ G = \bigcup_{n \in \mathbb{N}} G_n \]
For the set of functions $\bar{g}$, define $\forall b \in A$:

$$g_b(\phi, \psi) = \begin{cases} f_a(\phi, \psi), & \text{if } a = b \\ (\phi, \psi), & \text{otherwise} \end{cases}$$

Obviously, $\Delta^-$ can only get extended to a maximal consistent set if it is consistent in the first place, which still remains to be shown. This leads us to the following.

**Claim:** $\Delta^-$ is consistent.

*Proof:* Proof by contradiction, i.e. suppose that $\Delta^-$ is inconsistent. Then there exists some finite subset $\Theta \subseteq \Delta^- \setminus \{\neg \phi_1\}$ consisting of $m$ $K_y\phi$-instances and $n$ $\chi$-instances for each $\kappa \chi$ such that

$$\vdash_{\text{SKYR}} \bigwedge_{i=1}^{m} K_y\phi_i(\phi, \psi_i) \land \bigwedge_{j=1}^{n} \chi_j \rightarrow \phi_1$$

From (NK) and $\vdash_{\text{SKYR}} \bigwedge_{j=1}^{k} \phi_j \leftrightarrow \bigwedge_{j=1}^{k} \kappa \phi_j$ we infer

$$\vdash_{\text{SKYR}} \bigwedge_{i=1}^{m} K_y\phi_i(\phi, \psi_i) \land \bigwedge_{j=1}^{n} \kappa \chi_j \rightarrow \kappa \phi_1$$

For all $i$ with $1 \leq i \leq m$, we have $K_y\phi_i(\phi, \psi_i) \in \Gamma$ by the definition of $\Delta^-$. Since $\Gamma$ is said to be closed under deduction, we have $K_\kappa \phi_i(\phi, \psi_i) \in \Gamma$ by (4YKR). Also, for all $j$ with $1 \leq j \leq n$, we have $K_\kappa \chi_j \in \Gamma$ per definition of $\Delta^-$. From the second theorem in the calculus $\text{SKYR}$ from above and the fact that we have now all assumptions present in $\Gamma$, we infer $\kappa \phi_1 \in \Gamma$. This contradict our supposition of $\kappa \phi_1 \notin \Gamma$. \hfill $\blacksquare$

On the other hand, it is still left to show that the by that constructed world follows the conditions of the canonical model.

**Claim:** $w_{\Delta} \in \mathcal{W}^c$ if constructed from the above given premises.

*Proof:* Since $\Delta$ is said to be maximal consistent, we just need to show the condition (i) - (iii) of $\mathcal{W}^c$.

(i): Suppose $(s, (\chi, \phi \rightarrow \psi), (t, (\lambda, \phi)) \in G$. Therefore, $\exists \in \mathbb{N}$ such that $(s, (\chi, \phi \rightarrow \psi), (t, (\lambda, \phi)) \in G_k \subseteq G$. By construction of the successor for any $n$, we have $(s \cdot t, (\chi \land \lambda, \psi)) \in G_{k+1} \subseteq G$.

(ii): Suppose $\phi \in A$, thus $(e, (\psi, \phi)) \in F$. Since $w_{\Gamma}$, therefore $F$ is well-defined according to the conditions of $\mathcal{W}^c$. Therefore $(e, (\psi, \phi)) \in G_0$ by construction, i.e. $(e, (\psi, \phi)) \in G$.

(iii): Suppose $K_y\phi(\psi, \phi) \in \Delta$. If $b = a$, we have $f_a = g_b = g_a = f_a$, therefore $(g_b(\phi, \psi), (\phi, \psi)) = (f_a(\phi, \psi), (\phi, \psi)) \in F \subseteq G$. If $b \neq a$, we have $((\phi, \psi), (\phi, \psi)) \in G_0$ per definition. \hfill $\blacksquare$

Now, by construction of $w_{\Delta}$, we have $\neg \phi_1 \in \Delta$ and $(w_{\Gamma}, w_{\Delta}) \in R_{\mathcal{W}}^c$ for a given $w_{\Gamma}$ with $K_\kappa \phi_1 \notin \Gamma$. \hfill $\square$

**Lemma 3** ($K_y\phi$ existence lemma). For any world $w_{\Gamma} \in \mathcal{W}^c$, if $K_y\phi_1(\phi_2, \phi_2) \notin \Gamma$ then for any $(t, (\phi_1, \phi_2)) \in F$, there exists $w_{\Delta} \in \mathcal{W}^c$ with $(w_{\Gamma}, w_{\Delta}) \in R_{\mathcal{W}}^c$ such that $(t, (\phi_1, \phi_2)) \notin G$.

*Proof:* Initially suppose that $K_y\phi_1(\phi_2, \phi_2) \notin \Gamma$ and that $(t, (\phi_1, \phi_2)) \in F$(with $w_{\Gamma} = \langle \Gamma, F, \bar{f} \rangle \in \mathcal{W}^c$). Now, construct $w_{\Delta} = (\Delta, G, \bar{g})$ as follows:\footnote{For showing the existence of the mentioned world $w_{\Delta}$ with the desired properties, we recall another chain of rules providing a construction algorithm initially propose for the basic operator $K_y\phi$ in [13].}

For the maximal consistent set, consider

\begin{align*}
\Delta &= \Gamma \\
\text{Consider } \Psi &= \{(s, (\phi, \psi)) \mid (s, (\phi, \psi)) \in F \text{ and } K_y\phi(\phi, \psi) \notin \Gamma\} \\
\text{and } \Psi' &= \{(t \cdot s, (\phi, \psi)) \mid (s, (\phi, \psi)) \in \Psi\} \\
\text{and for the set } G, \text{consider as a base } G_0 &= (F \setminus \Psi) \cup \Psi' \\
\text{and as a recursion step } G_{n+1} &= G_n \cup \{(s \cdot r, (\chi \land \lambda, \psi)) \mid (s, (\chi, \phi \rightarrow \psi)), (r, (\lambda, \phi)) \in G_n\} \\
\text{The set } G \text{ is then defined as } G &= \bigcup_{n \in \mathbb{N}} G_n
\end{align*}

For the set of functions $\bar{g}$, define $\forall b \in A$

$$g_b(\phi, \psi) = \begin{cases} f_b(\phi, \psi), & \text{if } (f_b(\phi, \psi), (\phi, \psi)) \notin \Psi \\ (t \cdot f_b(\phi, \psi)), & \text{otherwise} \end{cases}$$
Since $\Delta = \Gamma$, we automatically have fulfilled the first condition of $W^c$. We now consider the three properties proposed in the definition of the canonical model for the set of worlds $W^c$.

(i): Suppose $\langle s, (x, \phi \rightarrow \psi) \rangle, \langle r, (\lambda, \phi) \rangle \in G$, therefore $\exists k \in \mathbb{N}$ such that $\langle s, (x, \phi \rightarrow \psi) \rangle, \langle r, (\lambda, \phi) \rangle \in G_k \subseteq G$. By construction of a corresponding $G_{k+1}$, we have $\langle s \cdot r, (x \wedge \lambda, \psi) \rangle \in G_{k+1} \subseteq G$.

(ii): Suppose $\phi \in \Lambda$, then $\langle e, (\psi, \phi) \rangle \in F$ and obviously $Ky^*_e(\psi, \phi) \in \Gamma$. From the latter, we have $\langle e, (\psi, \phi) \rangle \notin \Psi$, therefore $\langle e, (\psi, \phi) \rangle \in G_0 \subseteq G$.

Since the construction of $w_\Delta$ is now proven to follow the rules of $W^c$, we can now consider the two proposed conclusions of the lemma.

Claim: $(w_T, w_\Delta) \in R^c_\Psi$ for the given premises.

Proof: Since $\Delta = \Gamma$, we automatically have fulfilled the first condition of $R^c_\Psi$. For the second condition, $f_a = g_a$, we consider again that $\Delta = \Gamma$, therefore dom$(f_a) = \text{dom}(g_a)$. By that, for any $\langle \phi, \psi \rangle \in \text{dom}(f_a)$, we have $Ky^*_e(\phi, \psi) \in \Gamma$ and $\langle f_a(\phi, \psi), (\phi, \psi) \rangle \in F$. For (1), we then have $f_a(\phi, \psi) = g_a(\phi, \psi)$. Therefore, we have $\langle g_a(\phi, \psi), (\phi, \psi) \rangle \notin \Psi$. This concludes to $(g_a(\phi, \psi), (\phi, \psi)) \in G_0 \subseteq G$. For (2), we have $g_a(\phi, \psi) = (t, f_a(\phi, \psi))$ from the definition of $\bar{g}$. Following to this, we have $\langle (g_a(\phi, \psi), (\phi, \psi)) \in \Psi'$, therefore $\langle g_a(\phi, \psi), (\phi, \psi) \rangle \in G_0 \subseteq G$. $\blacksquare$

For the second property, we first consider the following helpful claim as an adaptation from [13], Claim 3, Lemma 21 into the context of conditions.

Claim: If $Ky^*_e(\theta, \psi) \notin \Gamma$ and $\langle s, (\theta, \psi) \rangle \in G_{n+1} \setminus G_n$, then $t$ is a proper sub-term of $s$.

Remark 2. A proper sub-term in the context of the set of explanations is defined as the following: Let $x, y, z \in E$ be explanation terms, $z$ is a proper sub-term of $x$, $x \succ z$, if $z$ occurs in $x$, for example $x = y \cdot z$. Obviously, the sub-term relation behaves monotonic over continuous extension like in the construction of the set $G$.

Proof: We follow the proof of the before mentioned claim in a conditioned manner. Suppose $Ky^*_e(\theta, \psi) \notin \Gamma$.

The second premise is established by induction on $n$:

**Claim (IB):** $n = 0$: Suppose $\langle s, (\theta, \psi) \rangle \in G_1 \setminus G_0$. By definition of $G$, there exists a $\langle r, (\chi, \phi \rightarrow \psi) \rangle, \langle u, (\lambda, \phi) \rangle \in G_0$ with $s = r \cdot u$ and $\theta = \chi \wedge \lambda$. Now, by the definition for the construction of $G$, since $\langle r, (\chi, \phi \rightarrow \psi) \rangle, \langle u, (\lambda, \phi) \rangle \in G_0$, we have either $\langle 1 \rangle: \langle r, (\chi, \phi \rightarrow \psi) \rangle \in \Psi'$, $\langle 2 \rangle: \langle u, (\lambda, \phi) \rangle \in \Psi'$ or $\langle 3 \rangle: \langle r, (\chi, \phi \rightarrow \psi) \rangle, \langle u, (\lambda, \phi) \rangle \notin \Psi'$, i.e. $\langle r, (\chi, \phi \rightarrow \psi) \rangle, \langle u, (\lambda, \phi) \rangle \in F \setminus \Psi$. For (1) and/or (2), it follows that either $r > t$ or $u > t$, therefore $s > t$. For (3), it follows that $\langle r, (\chi, \phi \rightarrow \psi) \rangle, (u, (\lambda, \phi)) \in F$, therefore $Ky^*_e(\chi, \phi \rightarrow \psi), Ky^*_e(\lambda, \phi) \in \Gamma$. Since $\Gamma$ is closed by deduction by being a maximal consistent set, we have $Ky^*_e(\chi, \lambda, \psi) = Ky^*_e(\theta, \psi) \in \Gamma$, which is a contradiction to the premises. Therefore, all possible paths lead to the conclusions.

The induction hypothesis is obviously established over the to-prove claim, therefore we continue with:

**Claim (IS):** $n > 0$: Suppose $\langle s, (\theta, \psi) \rangle \in G_{n+1} \setminus G_n$ for any $n > 0$. Then there exists $\langle r, (\chi, \phi \rightarrow \psi) \rangle, (u, (\lambda, \phi)) \in G_n$ as above. Now, since $\Gamma$ is closed by deduction and $Ky^*_e(\chi, \lambda, \psi) \notin \Gamma$, we have either $Ky^*_e(\chi, \phi \rightarrow \psi) \notin \Gamma$ or $Ky^*_e(\lambda, \phi) \notin \Gamma$. By (1) and/or (2), we can only be inside $G_n$ without $G_{n-1}$, i.e. $\langle r, (\chi, \phi \rightarrow \psi) \rangle \in G_n \setminus G_{n-1}$ or $(u, (\lambda, \phi)) \in G_n \setminus G_{n-1}$. At the end, we differ between the following cases:

1. $Ky^*_e(\chi, \phi \rightarrow \psi) \notin \Gamma$ and $\langle r, (\chi, \phi \rightarrow \psi) \rangle \in G_n \setminus G_{n-1}$: By these premises, we have automatically $r > t$, therefore $s > t$ by composition of $s$.

2. $Ky^*_e(\lambda, \phi) \notin \Gamma$ and $(u, (\lambda, \phi)) \in G_n \setminus G_{n-1}$: By these premises set, we have $u > t$, i.e. $s > t$. 

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By the construction of $r_kG$ be located in any $s$ for the logic of knowing-why, with a first look at public announcements. As the logic of knowing-why, with a first look at public announcements.

Claim: $(t, (\phi_1, \phi_2)) \notin G$ for the given premises of the lemma.

Proof: By the construction of $G$, it is enough to show that for all $k \in \mathbb{N}$, $(t, (\phi_1, \phi_2)) \notin G_k$. Since $K_y^\circ (\phi_1, \phi_2) \notin \Gamma$ but $(t, (\phi_1, \phi_2)) \in F$, we have $(t, (\phi_1, \phi_2)) \in \Psi$, therefore $(t, (\phi_1, \phi_2)) \notin G_0$. From the previous claim, we know that if $K_y^\circ (\phi_1, \phi_2) \notin \Gamma$, $t$ can never exist solely in any successor of a given set $G_k$, since it would always be a proper sub-term, therefore $(t, (\phi_1, \phi_2))$ can’t be added in any $G_k$, therefore $(t, (\phi_1, \phi_2)) \notin G$.

By that, all conclusions have been drawn from the premises using the claims.

Following from these considerations, we now propose the truth lemma corresponding to our canonical model $\mathfrak{M}^c$.

**Lemma 4 (Truth).** $(\mathfrak{M}^c, w_T) \models \phi$ if and only if $\phi \in \Gamma$ for all $\phi \in \mathcal{L}_{ELKy}^r$.

Proof. The proof is established by induction over the structure of the formula $\phi$.

For the induction base, we consider

$(\text{IB}): \phi = p$: By the definition of $V^c$, $p \in \Gamma$ if $w_T \in V^c(p)$, i.e. $(\mathfrak{M}^c, w_T) \models p$.

Following to this, we propose the induction hypothesis with:

$(\text{IH}):$ For every maximal consistent set $\Gamma$ and for a given formula $\phi$, the truth lemma holds.

For the induction step we now distinguish the following cases for the different structures of the formula $\phi$:

$(\text{IS}):$ (i): $\phi = \neg \psi$, (ii): $\phi = \psi \land \chi$, (iii): $\phi = K_a \psi$ and (iv): $\phi = K_y^\circ (\psi, \chi)$. In the following, let $w_T = (\Gamma, F, \hat{f})$ and $w_\Delta = (\Delta, G, \hat{g})$. As the first three cases are in some sense standard (for example Lemma 7.5. [2]), we will just focus on item (iv):

First suppose that $K_y^\circ (\psi, \chi) \in \Gamma$, then for all $w_\Delta \in W^c$ with $(w_T, w_\Delta) \in R^c_a$ we have $K_y^\circ (\psi, \chi) \in \Delta$ and additionally $f_a = g_a$. Now consider an arbitrary world $w_\Delta$ with $(w_T, w_\Delta) \in R^c_a$ and additionally $(\mathfrak{M}^c, w_\Delta) \models \psi$. By $(\text{IH})$, we have $\psi \in \Delta$ from the latter. Since $K_y^\circ (\psi, \chi) \in \Gamma$, we have $(f_a(\psi, \chi), (\psi, \chi)) \in F$ and since $K_y^\circ (\psi, \chi) \in \Delta$, we have $(g_a(\psi, \chi), (\psi, \chi)) \in G$. By that and $f_a = g_a$ there exists a $t = f_a(\psi, \chi) = g_a(\psi, \chi) \in E^c$ with $w_\Delta \in E^c(t, \chi)$. From $K_y^\circ (\psi, \chi) \in \Delta$, we have $K_a(\psi \rightarrow \chi) \in \Delta$ by $(DKyR)$, therefore $\psi \rightarrow \chi \in \Delta$ by $(T)$. Now, with $\psi \in \Delta$, we have $\chi \in \Delta$ by $(MP)$, i.e. $(\mathfrak{M}^c, w_\Delta) \models \chi$ by $(\text{IH})$. From this, we infer $(\mathfrak{M}^c, w_T) \models K_y^\circ (\psi, \chi)$.

On the other hand, suppose that $K_y^\circ (\psi, \chi) \notin \Gamma$, therefore $\neg K_y^\circ (\psi, \chi) \in \Gamma$. By this and $(DKyR)$, we consider either (i): $K_a(\psi \rightarrow \chi) \in \Gamma$ or (ii): $K_a(\psi \rightarrow \chi) \notin \Gamma$.

For (i), if $\exists t \in E^c$ such that $(t, (\psi, \chi)) \in F$, we have a world $w_\Delta$ such that $(w_T, w_\Delta) \in R^c_a$ and $(t, (\psi, \chi)) \notin G$ by Lem. 3. By the semantics of $K_y^\circ$, we have $(\mathfrak{M}^c, w_T) \models K_y^\circ (\psi, \chi)$.

For (ii), i.e. $\neg K_a(\psi \rightarrow \chi) \in \Gamma$, by Lem. 2 we have a world $w_\Delta$ with $(w_T, w_\Delta) \in R^c_a$ such that $\neg (\psi \rightarrow \psi) \in \Delta$, i.e. $\psi \rightarrow \chi \notin \Delta$. Therefore $\neg \chi \in \Delta$. Thus $(\mathfrak{M}^c, w_T) \models K_y^\circ (\psi, \chi)$.

**Theorem 3 (Completeness of $\mathfrak{S}_5$).** $\Gamma \models \phi \iff \phi$ implies $\Gamma \vdash \phi$.

Proof. Suppose $\Gamma \models \phi$, proof by contradiction. Suppose $\Gamma \not\models \phi$, then $\Gamma \cup \{\neg \phi\}$ is obviously consistent. Now, let $\Gamma'$ be the extensions of $\Gamma \cup \{\neg \phi\}$ to a maximal consistent set. By the definition of the canonical model, there exists a world for this set in $W^c$, namely $w_\Delta$. By Lem. 4 we have $(\mathfrak{M}^c, w_\Delta) \models \Gamma \cup \{\neg \phi\}$, thus $\Gamma \cup \{\neg \phi\}$ is satisfiable. Therefore, we have $\Gamma \not\models \phi$.

4. Expressivity comparisons

As it was said at the beginning of the paper, the initial motivation was to study the dynamic extensions for the logic of knowing-why, with a first look at public announcements. As the logic $\text{ELKy}^r$ was presented as some sort of workaround for problems concerning the logic $\text{PAFKy}$ (or more precise, concerning the classic style of axiomatization), we now make expressivity comparisons between those logics newly introduced in this
Before proceeding, it may additionally be notable that we find that $Ky_a^\phi(\phi, \psi)$, although suggested in the final section of [13], does not directly correspond to $[\phi]Ky_a\psi$. For this, one may easily imagine the following model $\mathfrak{M}(R_a$ is expected to be total although not displayed as such):

```
\begin{tikzpicture}
  \node (w1) at (0,0) {$w_1$};
  \node (q) at (1,1) {$q$};
  \node (p) at (1,2) {$p, q$};
  \node (s) at (0,1) {$s : p$};
  \node (a) at (1,3) {$a$};
  \node (w2) at (2,0) {$w_2$};
  \node (s') at (0,2) {$s' : p$};
  \node (a') at (1,4) {$a'$};
  \node (w3) at (2,1) {$w_3$};

  \path[->] (q) edge (p)
  (w1) edge (s)
  (s) edge (q)
  (p) edge (a)
  (w2) edge (s')
  (s') edge (q)
  (a) edge (p)
  (p) edge (w3)
  (w3) edge (a')
  (a') edge (q)

\end{tikzpicture}
```

Since $(\mathfrak{M}, w_1) \not\models p$, we have that $(\mathfrak{M}, w_1) \models [p]Ky_aq$ from the semantics of $[\cdot]$. At the same time, we find that $(\mathfrak{M}, w_1) \not\models Ky_a^\phi(p, q)$ as, although $w_1$ is not considered in the evaluation of the $\mathcal{E}$-clause because of the before mentioned condition, we still find that there does not exists a uniform $t \in E$ for the left-to-consider worlds $w_2$ and $w_3$. Therefore, we find that $(\mathfrak{M}, w_1) \not\models Ky_a^\phi(p, q)$.

The main difference exploited here is the missing $\phi$-implication in the semantical definition of $Ky_a^\phi(\phi, \psi)$. In the following argument though, it can also be seen that even a corresponding modification has no possibility in providing an adequate translation. For this, we introduce a second concept from the original paper [13].

**Definition 6** (Factivity Property). A model $\mathfrak{M}$ has the factivity property (is factive), if whenever $w \in \mathcal{E}(t, \phi)$, then $(\mathfrak{M}, w) \models \phi$.

Given a model $\mathfrak{M} = (W, E, \{R_a \mid a \in \mathcal{A}\}, \mathcal{E}, V)$, one may construct its factive companion $\mathfrak{M}^F = (W, E, \{R_a \mid a \in \mathcal{A}\}, \mathcal{E}^F, V)$ where

$$\mathcal{E}^F(t, \phi) = \mathcal{E}(t, \phi) \setminus \{w \in W \mid (\mathfrak{M}, w) \not\models \phi\}$$

Obviously, for a factive model $\mathfrak{M}$, $\mathfrak{M}$ and $\mathfrak{M}^F$ coincide. The following lemma now asserts that the ELKy-formulas are neutral in respect to factivity.

**Lemma 5** (Xu, Wang, Studer [13].) For any $\phi \in \mathcal{L}_{\text{ELKy}}$, model $\mathfrak{M}$ and $w \in W$, $(\mathfrak{M}, w) \models \phi$ if and only if $(\mathfrak{M}^F, w) \models \phi$.

We obtain the following generalization for ELKy$^\psi$.

**Lemma 6.** For any $\phi \in \mathcal{L}_{\text{ELKy}^\psi}$, model $\mathfrak{M}$ s.t. $w \in W$, $(\mathfrak{M}, w) \models \phi$ if and only if $(\mathfrak{M}^F, w) \models \phi$.

**Proof.** Proof by induction on the structure of formulas. We leave the classical propositional and modal cases considered, as $\mathfrak{M}$ is only possibly different from $\mathfrak{M}^F$ in the explanation function. Thus consider $Ky_a^\phi(\phi, \psi)$:

Suppose $(\mathfrak{M}, w) \models Ky_a^\phi(\phi, \psi)$, i.e. $\exists t \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}, v) \models \phi$ implies $(\mathfrak{M}, v) \models \psi$ and $v \in \mathcal{E}(t, \psi)$. Thus, for all those $v$, $v \not\in \{u \mid (\mathfrak{M}, u) \not\models \psi\}$, i.e. $v \in \mathcal{E}^F(t, \psi)$. By (IH), we have $(\mathfrak{M}^F, w) \models Ky_a^\phi(\phi, \psi)$.

Suppose otherwise that $(\mathfrak{M}^F, w) \models Ky_a^\phi(\phi, \psi)$, i.e. $\exists t \in E : \forall v \in W : (w, v) \in R_a$ and $(\mathfrak{M}^F, v) \models \phi$ implies $(\mathfrak{M}^F, v) \models \psi$ and $v \in \mathcal{E}^F(t, \psi)$. By $v \in \mathcal{E}^F(t, \psi)$, we automatically have $v \in \mathcal{E}(t, \psi)$ and by (IH), we obtain $(\mathfrak{M}, w) \models Ky_a^\phi(\phi, \psi)$. ⊓⊔

**Lemma 7.** There exists $\phi \in \mathcal{L}_{\text{PAFKy}}$, a model $\mathfrak{M}$ and $w \in W$ such that $(\mathfrak{M}, w) \models \phi$ but $(\mathfrak{M}^F, w) \not\models \phi$.

**Proof.** Consider the following model $\mathfrak{M}$ (reflexive arrows are not shown, but expected)

```
\begin{tikzpicture}
  \node (p) at (0,0) {$p, q$};
  \node (t) at (1,0) {$t : K_a q$};
  \node (a) at (0.5,1) {$a$};
  \node (1) at (-1,0) {1};
  \node (2) at (2,0) {2};
  \node (3) at (-1,-1) {3};

  \path[->] (p) edge (t)
  (t) edge (1)
  (p) edge (a)
  (a) edge (1)
  (p) edge (3)
  (3) edge (a)
  (a) edge (2)
\end{tikzpicture}
```
and its factive counterpart $\mathcal{M}^F$, where we loose all explanations for $K_aq$ as in every world $x$, by $(\mathcal{M}, 2) \not\models q$, we find that $(\mathcal{M}, x) \not\models K_aq$ as the relation $R_a$ is total among these worlds:

Consider the formula $\phi := [p]Ky_aK_aq$. We have $(\mathcal{M}, 1) \models \phi$. For this, first note that $(\mathcal{M}, 1) \models p$. Second, $\forall v \in W : (1, v) \in R_a$ and $(\mathcal{M}, v) \models p$ leaves us with worlds 1, 3. For those, we obtain both instances of $(\mathcal{M}[p, v] \models K_aq$ as $[[p]]_{\mathcal{M}} = \{1, 3\}$ and $(\mathcal{M}, v) \models q$. At last, note that $1, 3 \in E(t, K_aq)$. At the same time, we have $(\mathcal{M}^F, 1) \not\models \phi$ since, although similarly $[[p]]_{\mathcal{M}} = \{1, 3\}$, there exits no $t \in E$ such that $1, 3 \in E^F(t, K_aq)$ as $E^F(t, K_aq)$ has to be empty for every $t$. □

Theorem 4. PAFKy and ELKy$^r$ are not equally expressive.

Proof. Suppose that PAFKy and ELKy$^r$ are equally expressive. Thus, there exists a translation function $t$ from formulas of PAFKy to formulas of ELKy$^r$ such that $\phi \equiv \phi^t$ for any $\phi \in L_{PAFKy}$. Then the following diagram

\[
\begin{array}{ccc}
(\mathcal{M}, w) \models \phi & \quad & (\mathcal{M}^F, w) \models \phi \\
\downarrow t & \quad & \downarrow t \\
(\mathcal{M}, w) \models \phi^t & \quad & (\mathcal{M}^F, w) \models \phi^t \\
\end{array}
\]

may be completed at the top which is a contradiction to Lem. 7. □

Corollary 1. PAFKy$^r$ has greater expressivity than ELKy$^r$.

Proof. Since ELKy$^r$ is contained in PAFKy$^r$, it can’t be more expressive. By Thm. 4 and Lem. 7, there exists a formula $\phi \in L_{PAFKy}$ which can’t be expressed in ELKy$^r$. As PAFKy$^r$ contains PAFKy, we have greater expressivity. □

By this argument it can be seen that a relativized knowing-why operator does not suffice to imitate public announcement behavior in the logic of knowing why, thus leaving a rest of void it initially intended to fill. Although we leave further inspections for future work, we still want to advocate for the consideration of the use of non-standard semantics for public announcements, namely context-dependent semantics, which seems like a promising alternative way of dealing with the laid out problems.

4.1. Using non-standard semantics. The following semantic concepts for public announcement logic are mainly due to Wang in their current form, see [10] and [9]. For providing a small recollection of the basic notions of context-dependent semantics of the operator $[\phi]$, we consider another semantic relation $|=_{\rho}$, similar to the classical $|=_{V}$, but induced with a formula $\rho \in L_{PAFKy^r}$ providing a specific evaluation context. For the behavior of this new relation, one may consider the following (only the interesting cases are provided, i.e. the out-carrying of the relation over $\land, \neg$ is obviously following the old structure as the main difference lies in the many-world context).

\[
\begin{align*}
(\mathcal{M}, w) \models \phi & \iff (\mathcal{M}, w) |=_{\tau} \phi \\
(\mathcal{M}, w) \models_{\rho} p & \iff w \in V(p) \\
(\mathcal{M}, w) \models_{\rho} K_a \phi & \iff \forall v \in W : (w, v) \in R_a \text{ and } (\mathcal{M}, v) |=_{\tau} \rho \text{ implies } (\mathcal{M}, v) \models_{\rho} \phi \\
(\mathcal{M}, w) \models_{\rho} [\psi] \phi & \iff (\mathcal{M}, w) |=_{\tau} \phi \text{ implies } (\mathcal{M}, w) \models_{\rho \land \psi} \phi
\end{align*}
\]

As it was shown in [10], the composition axiom in this semantics turns out to be much simpler with

\[\phi[[\psi] \chi \leftrightarrow [\phi \land \psi] \chi\]
This new induced handling of compositions of public announcement should make it possible to find a simple reduction axiom like \( \phi \rightarrow \mathcal{K} \mathcal{A}(\phi, \psi, \chi) \) as a proposition. But on deeper insights, the whole idea of a conditionalized versions of the knowing-why operator may(or should) be completely unnecessary in this new context. The axiomatization and the more intensive study of this semantics in this context are left as future work.

Another use of different semantics may lie in restructuring the semantics of \( \mathcal{K} \mathcal{A} \) itself. Such a different semantic definition shall obviously provide a possibility for translation between the relativized versions and the versions incorporating the public announcements. This, for example, is enabled by the following semantics:

\[
(\mathfrak{M}, w) \models \mathcal{K} \mathcal{A}(\phi, \psi) \iff \exists E : \forall v \in W : (w, v) \in R_q \text{ and } (\mathfrak{M}, v) \models \phi \implies (\mathfrak{M}[w], w) \models \psi \text{ and } v \in \mathcal{E}(t, \psi)
\]

Obviously, through the enforcement of the \( \phi \)-restricted model only in the continuing evaluation of \( \phi \), we avoid the problems laid out before. From this, we may translate \( [\phi]\mathcal{K} \mathcal{A} \psi \) to \( \phi \rightarrow \mathcal{K} \mathcal{A}(\phi, \psi) \), although it seems to require a quite different approach of axiomatization. Additionally, the usual idea of introducing a relativized operator, i.e. leaving the context updated models, is discarded which doesn’t get along with the usual spirit.

5. Conclusions

In this paper, the logic of knowing why under the extension with public announcement operators for formulas was addressed. Through the difficulties arising with providing of a reduction-based axiomatic system, we considered another logical operator, namely the conditionalized version of the basic \( \mathcal{K} \mathcal{A} \)-operator to provide a partial workaround. Following to this, as the main result of this paper, we proved the new axiomatic system concerning the logic using this relativized operator as being sound and complete with respect to the basic \( \mathcal{S}5 \)-class of models following the definition of the initial paper \[13\]. A conditionalized version of the canonical model presented in \[13\] and \[11\], \[12\] is here provided in order to achieve these results. In the following section, we then considered the problem of expressivity between the different logics in discourse, where we found that \( \mathcal{E}L \mathcal{K} \mathcal{A} \mathcal{Y}^r \) does not fulfill its reduction promise as hoped and so lies as an intermediate among \( \mathcal{E}L \mathcal{K} \mathcal{Y} \), \( \mathcal{P}A \mathcal{F} \mathcal{K} \mathcal{Y} \) and \( \mathcal{P}A \mathcal{F} \mathcal{K} \mathcal{Y}^r \) concerning its expressive power.

The situation was at first sight similar to the problem of a reduction style axiomatization of common knowledge with public announcement operators(or similar dynamic notions), as examined in \[1\], \[6\]. The main difference and cause for problems to those approaches is the sensitivity of the argument of the knowing why operator, as the core syntactical structure of a formula is needed for the evaluation of the \( \mathcal{E} \)-function and we thus can’t use similar formulas, in the sense of being equal under satisfaction. These problems did not arise with previous attempts of adding public announcements to other non-standard epistemic logics, e.g. \[11\], \[12\], \[8\], as the non-classical operators there concerned objects disconnected from the set of well-formed formulas and the other classical parts of the model definition.

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Hoch-Weisers Str. 46, BUTZLACHER, 35510, HESSE, GERMANY
E-mail address: pischkenicholas@gmail.com