COUNTING CONJUGACY CLASSES IN GROUPS WITH CONTRACTING ELEMENTS

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Abstract. In this paper, we derive an asymptotic formula for the number of conjugacy classes of elements in a class of statistically convex-cocompact actions with contracting elements. Denote by \( C(o, n) \) (resp. \( C'(o, n) \)) the set of (resp. primitive) conjugacy classes of algebraic length at most \( n \) for a basepoint \( o \). The main result is the following asymptotic formula:

\[
\# C(o, n) \asymp \# C'(o, n) \asymp \exp(\omega(G)n) n.
\]

A similar formula holds for conjugacy classes using stable length. As a consequence of the formulae, the conjugacy growth series is transcendental for all non-elementary relatively hyperbolic groups, graphical small cancellation groups with finite components. As by-product of the proof, we establish several useful properties for an exponentially generic set of elements. In particular, it yields a positive answer to a question of J. Maher that an exponentially generic elements in mapping class groups have their Teichmüller axis contained in the principal stratum.

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1. INTRODUCTION

1.1. Motivation. Let \( G \) be a countable group acting by isometries on a proper geodesic metric space \( (Y, d) \). Assume that the action of \( G \) on \( Y \) is proper, so that
for any basepoint \( o \in Y \), the set \( N(o,n) := \{ g \in G : d(o,go) \leq n \} \) is finite. For any subset \( X \subset G \), the growth rate \( \omega(X) \) is defined as follows:

\[
\omega(X) := \limsup_{n \to \infty} \frac{\log \#X \cap N(o,n)}{n}.
\]

The quantity \( \omega(X) \) does not depend on the basepoint \( o \). We assume that \( \omega(G) \) also called critical exponent for the action is finite in this paper.

The motivating example is the action of a finitely generated group \( G \) on its Cayley graph \((Y,d)\) with respect to a finite generating set \( S \). Here, \( d \) is the word metric and, by a subadditive inequality, the critical exponent is a true limit, which is the growth rate of \( G \) with respect to \( S \).

There has been considerable interest in studying the number of conjugacy classes of \( G \) with stable translation length at most \( n \) on \( Y \), in particular its asymptotics as \( n \to \infty \) (see below for precise definitions).

When \( Y \) is a negatively curved contractible manifold, the problem of counting conjugacy classes is equivalent to counting closed geodesics in the quotient \( Y/G \). Indeed the conjugacy class of a loxodromic element \( g \in G \) defines a closed geodesic on \( Y/G \) which is the image in \( Y/G \) of the translation axis of \( g \), and its stable length is precisely the length of the associated geodesic. The conjugacy class is called primitive if \( g \) is not a proper power of any element of \( G \), in which case the associated closed geodesic is primitive. In his 1970 thesis [42], Margulis established a precise asymptotic formula for the set \( \mathcal{C}'(n) \) of primitive closed geodesics with length less than \( n \) as follows

\[
\#\mathcal{C}'(n) \sim \frac{\exp(hn)}{hn}
\]

where \( h = \omega(\pi_1 M) \) is the topological entropy of geodesic flow on the compact manifold \( M \). Margulis' result has been generalized to various actions which display some hyperbolicity. An analogous formula for closed geodesics has been obtained for the following actions:

1. Quotients of \( \text{CAT}(-1) \) space by a geometrically finite groups of isometries admitting a finite Bowen-Margulis-Sullivan measure on the unit tangent bundle, [51 Théorème 5.2].
2. Compact rank-1 manifolds, [39].
3. The moduli space of closed orientable surfaces of genus \( \geq 2 \) endowed with the Teichmüller metric (which is the quotient of Teichmüller space by the action of the mapping class group), [23], [24].
4. Covers of the above moduli space associated to convex-cocompact subgroups of the above mapping class groups, [28].

Beyond the manifold setting, Coornaert and Knieper [17], [18] proved that for hyperbolic groups acting on their Cayley graphs, the number of primitive conjugacy classes of stable length at most \( n \) is up to a bounded multiplicative constant equal to \( e^{hn}/n \) where \( h \) is the exponential growth rate of the Cayley graph (see also Antolin and Ciobanu[2] for all conjugacy classes).

Motivated by the work of Margulis, Guba and Sapir [33] initiated a systematic study of conjugacy growth function in groups, namely the number \( \#\mathcal{C}(1,n) \) of conjugacy classes intersecting a ball centered at \( 1 \) of radius \( n \). Many examples of groups were found to have exponential conjugacy growth \( \#\mathcal{C}(1,n) \geq a^n \) for some \( a > 1 \), including non virtually solvable linear groups [11], non-elementary acylindrically
hyperbolic groups [37], and so on. We refer the reader to [33] and many related references therein.

1.2. Statement of main results. The goal of the present paper is to establish coarse multiplicative asymptotic formulae describing growth of conjugacy classes for a more general class of actions called statistically convex-cocompact actions with a contracting element, encompassing the above examples.

The notion of a contracting element plays a significant role in counting conjugacy classes. An element \( g \in G \) is called contracting if for some (or any) basepoint \( o \in Y \), the stable length defined by

\[
\tau(g) := \lim_{n \to \infty} \frac{d(o, g^n o)}{n}
\]

is positive and the subset \( \langle g \rangle \cdot o \) is contracting in \( Y \). Here a subset \( X \subset Y \) is contracting if it enjoys a certain property suggestive of negative curvature: namely that any metric ball \( B \) outside \( X \) has the uniformly bounded shortest projection to \( X \). We remark that \( \tau(g) > 0 \) if and only if the map \( n \in \mathbb{Z} \mapsto g^n o \in Y \) is a quasi-isometric embedding.

To make precise the notion of counting conjugacy classes, we now describe two ways to assign a length to a conjugacy class: stable length and algebraic length. By definition, \( \tau(g) \) does not depend on the basepoint, so it is a conjugacy invariant. Denote by \( [g] \) the set of elements conjugate to \( g \). We then have a well-defined function \( \tau([g]) \) on the set of conjugacy classes \( [g] \).

We shall fix a basepoint \( o \in Y \) throughout. The algebraic length \( \ell_o[g] \) of a conjugacy class is defined as

\[
\ell_o[g] := \inf \{ d(o, g' o) : g' \in [g] \}.
\]

This clearly depends on the choice of the basepoint. By the subadditivity, we see that

\[
\tau(g) = \inf \{ \frac{d(g^n o, o)}{n} : n \geq 1 \}.
\]

Thus, for every \( g \in G \), we have \( \tau(g) \leq \ell_o[g] \).

In many geometric examples, the stable length is realized by the algebraic length for a certain basepoint. This is the case when \( Y \) is the universal cover of a compact Riemannian manifold, a CAT(0) space, and in Examples [11].

In [57], the second-named author defined a class of statistically convex-cocompact actions, which is a dynamical generalization of convex-cocompact actions studied in many different settings. Given constants \( 0 \leq M_1 \leq M_2 \), let \( O_{M_1, M_2} \) be the set of elements \( g \in G \) such that there exists some geodesic \( \gamma \) between \( \gamma^- \in B(o, M_2) \) and \( \gamma^+ \in B(go, M_2) \) with the property that the interior of \( \gamma \) lies outside \( N_{M_1}(Go) \).

**Definition 1.1** (statistically convex-cocompact action). If there exist two positive constants \( M_1, M_2 > 0 \) such that \( \omega(O_{M_1, M_2}) < \omega(G) < \infty \), then the action of \( G \) on \( Y \) is called statistically convex-cocompact (SCC).

Among other results in [57], let us point out that a SCC action with a contracting element has the purely exponential growth (PEG) property

\[
\sharp N(o, n) \asymp \exp(\omega(G)n),
\]

where \( \asymp \) denotes the two sides differ by a multiplicative constant.

We say that an element \( g \in G \) is primitive if it cannot be written as a proper power \( g = g_0^n \) for \( |n| \geq 2 \) and \( g_0 \in G \). If \( g \) is a contracting element, we shall also consider a stronger notion of primitivity. Let \( E(g) \) be the maximal elementary
subgroup containing $g$. By a theorem of Stallings there exists a subgroup $E^+ < E(g)$ of index at most two and a finite group $F$ with the following exact sequence

$$1 \to F \to E^+ \to \mathbb{Z} \to 1.$$ 

Any element $h \in E^+$ in the preimage $\phi^{-1}(\pm 1)$ is called strongly primitive. A strongly primitive (contracting) element is primitive by Lemma 2.7, but the converse may not be true.

Our main theorem gives asymptotic formulae using algebraic length and stable length for primitive, as well as for all, conjugacy classes. Let

$$C(o,n) = \{ [g] : 0 \leq \ell_o[g] \leq n \},$$

$$C(n) = \{ [g] : 0 < \tau[g] \leq n \},$$

and $C'(o,n) \subset C(o,n)$ and $C'(n) \subset C(n)$ denote the subsets of primitive (or strongly primitive) ones respectively.

**Main Theorem.** Suppose that a non-elementary group $G$ admits a SCC action on a proper geodesic metric space $(Y,d)$ with a contracting element and $\omega(G) < \infty$. Let $o \in Y$ be a basepoint. There exists a constant $D = D(o) > 0$ such that the following statements hold:

1. $\sharp C(o,n) \asymp D \frac{\exp(\omega(G)n)}{n}$.
2. $\sharp(C(n) \cap C(o,n)) \asymp D \frac{\exp(\omega(G)n)}{n}$.
3. $\frac{\sharp(C'(n) \cap C(o,n))}{\sharp(C(n) \cap C(o,n))} \to 1$ and $\frac{\sharp C'(o,n)}{\sharp C(o,n)} \to 1$ exponentially fast.

From now on, a formula as above in (1) (2) will be referred to as a prime conjugacy growth formula.

**Remark.** Most of the forementioned results count conjugacy classes of bounded stable length. Our result also considers the algebraic length, which depends on the basepoint, in the setting of non-cocompact actions. We remark that the conjugacy growth formula for algebraic length is not a direct consequence of the one using stable length.

Regardless of the complicated formula in (2), we wish to explain some subtleties about counting conjugacy classes with respect to stable length.

For a general proper action, measuring conjugacy classes in stable length has the weakness that the conjugacy growth function might not be even defined as there may be infinitely many conjugacy classes with bounded stable length. For example, Conner [16] constructed examples of groups with stable lengths accumulating at 0 (the opposite is called translation discrete there). See [38] for examples with discrete spectrum of stable lengths.

Examples of relatively hyperbolic groups whose stable length spectrum is not discrete can be easily built by taking a free product of two groups with this property. This forces us to consider the formula with stable length more carefully in [Main Theorem]. From a geometric point of view, one may wonder whether these conjugacy classes coming from parabolic subgroups are not interesting since they degenerate on the quotient manifolds (e.g. geometrically finite hyperbolic manifolds). This perhaps motivates one to count only loxodromic conjugacy classes associated with closed geodesics. However, this still does not correct the formula:
there are examples of hyperbolic manifolds with fundamental groups satisfying the SCC condition but containing infinitely many closed geodesics with bounded length. See §8.3 for construction of such examples. Moreover, such examples could exist in the class of acylindrical actions on hyperbolic spaces. See Lemma 8.5.

Nevertheless, there are many new settings in which we can obtain a satisfactory formula with stable length.

1.3. Applications. We first consider the class of CAT(0) groups and its subclass of cubical groups. An isometry of a proper CAT(0) space is contracting with respect to the CAT(0) metric exactly when it has a rank-1 axis in the space, i.e.: a geodesic which does not bound a flat half-plane ([9, Theorem 5.4]). If the space is a CAT(0) cube complex, such an isometry is also contracting with respect to the $\ell^1$-metric on the 1-skeleton, which coincides with the standard word metric for a right-angled Artin or Coxeter group.

Corollary 1.2 (CAT(0) groups). 

1. A non-elementary group $G$ acting geometrically on a CAT(0) space $X$ with a rank-1 element satisfies the prime conjugacy growth formulae for algebraic length and stable length with respect to the CAT(0) metric on $X$.

2. A non-elementary group $G$ acting geometrically on a CAT(0) cube complex with a rank-1 element satisfies the prime conjugacy growth formulae for algebraic length and stable length with respect to the $\ell^1$ metric on the 1-skeleton.

In particular, this holds for Right angled Artin or Coxeter group with the standard word metric, provided that the group is not virtually a product of nontrivial groups.

Remark. As mentioned in Examples 1.1, the analogue of (1) for smooth manifolds of non-positive curvature is due to Knieper [39]. His method uses conformal densities on the boundary, whereas our methods are completely geometric and elementary and do not involve any measure theory.

The formula for algebraic length is immediate by Main Theorem. But for the stable length, it needs an additional ingredient that for every rank-1 element, its stable length coincides with algebraic length up to a uniform error. This might not be true in other classes of groups.

Given a relatively hyperbolic group $(G, \mathcal{P})$, a hyperbolic element is by definition an infinite order element not conjugated into any subgroup in $\mathcal{P}$. For a hyperbolic element, we note that the stable length coincides with algebraic length, up to a uniform error. We thus obtain the following corollary.

Corollary 1.3. Let $(G, \mathcal{P})$ be a relatively hyperbolic group. Then for the action on the Cayley graph, the prime conjugacy growth formulae holds for all conjugacy classes with algebraic length, and for conjugacy classes of contracting elements with stable length.

We note that the class of the cubical groups and relatively hyperbolic groups contains many groups which are not Gromov hyperbolic. We next consider the class of graphical small cancellation groups; this is a class containing many groups which are not relatively hyperbolic, see [32] for many examples. Graphical small cancellation groups contain contracting elements with respect to actions on their
Cayley graphs by [4, Theorem 5.1] and thus Main Theorem applies to count their conjugacy classes.

**Corollary 1.4.** The prime conjugacy growth formula for algebraic length holds for graphical small cancellation groups with finite components on the Cayley graph with respect to small cancellation presentation.

**Remark.** By [4, Lemma 5.3], some power of every contracting element preserves a geodesic. However, we do not know whether the stable length can be realized by algebraic length, without raising the element to a power. So a formula for stable length is not available.

Since the stable length of a pseudo-Anosov element coincides with the length of a closed geodesic, Main Theorem applies to count closed geodesics on certain covers of moduli space corresponding to subgroups acting by a SCC action on Teichmüller spaces. Convex-cocompact subgroups in the sense of Farb and Mosher [25] are obviously SCC, for which the first-named author [28] previously obtained a precise formula of closed geodesics. In [57, Proposition 6.6], examples of non-convex-cocompact SCC actions are constructed out of subgroups generated by disjoint Dehn twists and a sufficiently high power of a pseudo-Anosov element.

**Corollary 1.5.** The prime conjugacy growth formula holds for closed geodesics on the cover of the moduli space associated with subgroups in mapping class groups on Teichmüller space constructed in [57].

We remark that this corollary is not an immediate consequence of Main Theorem. It requires an additional fact about the subgroups \( \Gamma < Mod(S) \) at hand which is evident from the construction in [57], namely that the Teichmüller geodesic axis of any pseudo-Anosov \( g \in \Gamma \) is contained within a bounded distance of an orbit of \( \Gamma \). See Section 8 for more details.

We can conclude Corollary 1.5 from the following general statement. Compare with [51, Théorème 5.1.1] and examples in §8.3.

**Theorem 1.6.** Under the assumption of Main Theorem, assume in addition that every contracting element preserves a geodesic axis. Then the prime conjugacy growth formula with stable length holds for all contracting elements with axis intersecting a fixed finite neighborhood of the orbit \( G_0 \).

**Remark (moduli space).** On one hand, Hamenstädt [35] and Rafi [48] proved that there are closed geodesics outside every compact part of moduli space; on the other hand, Eskin and Mirzakhani [23] showed that the number of closed geodesics outside a certain compact part is exponentially small relative to the ones intersecting it.

If an analogue of the latter holds for the SCC covers of moduli space, then Main Theorem allows to count all closed geodesics on any SCC cover as in Corollary 1.5.

**Applications to conjugacy growth series.** There is some recent interest in understanding the complexity of the following formal conjugacy growth series for a basepoint \( o \in G \):

\[
P(z) = \sum_{[g] \in G} z^{\ell_o([g])} \in \mathbb{Z}[[z]],
\]

in particular whether it is rational, algebraic, or transcendental over \( \mathbb{Q}(z) \). One could similarly look at formal series obtained from counting primitive conjugacy
classes and using stable length. The same result stated below holds for them as well.

This series is naturally stated with respect to the action on the Cayley graph, where \( o \) is the identity and \( \ell_1[g] \) is the minimal length of elements in \( [g] \). It is well-known that, if counting \( N(1,n) \) instead of \( C(1,n) \), the formal growth series \( \sum_{g \in G} z^{d(1,g)} \) is rational for any hyperbolic group (cf. [12]). However, in [49] [50], Rivin computed the formal conjugacy growth series for free groups with word metric which turns out to be irrational.

Furthermore, Rivin [49] conjectured that the conjugacy growth series of a hyperbolic group is rational if and only if it is virtually cyclic. In [41], Ciobanu, Hermiller, Holt and Rees proved that a virtually cyclic group has rational conjugacy growth series. Later on, Antolin and Ciobanu [2] established the other direction of Rivin’s conjecture by showing that a non-elementary hyperbolic group has transcendental conjugacy growth series. The main ingredient is a prime conjugacy growth formula for all conjugacy classes in hyperbolic groups, which extends earlier work of Coornaert and Knieper [17], [18]. Hence, by the same reasoning, we obtain the following consequence.

**Theorem 1.7.** Let \( G \) be a non-elementary group acting on a proper geodesic space \((Y,d)\) with a contracting element. Then the conjugacy growth series is transcendental.

In [52], Sisto proved that if a group admits a proper action with a contracting element then it must be acylindrically hyperbolic in the sense of Osin [45]. In particular, we have the following corollary.

**Corollary 1.8.** Let \( G \) be a non-elementary group with a finite generating set \( S \). If \( G \) has a contracting element with respect to the action on the corresponding Cayley graph, then the conjugacy growth series is transcendental.

This confirms Rivin’s conjecture for a large subclass of acylindrically hyperbolic groups, including right-angled Artin/Coxeter groups, relatively hyperbolic groups and graphical small cancellation groups, etc.

In the class of relatively hyperbolic groups, the conclusion actually holds for every generating set. This is a direct generalization of the corresponding statement in [2] for hyperbolic groups.

**Corollary 1.9.** The conjugacy growth series of a non-elementary relatively hyperbolic group with respect to any finite generating set is transcendental.

To conclude the introduction of our results, we mention a result of independent interest which is a by-product of the proof of our main theorem.

Let \( E(g) \) be the maximal elementary subgroup of \( G \) containing \( g \). Define the set \( A(g) = E(g) \cdot o \) to be the coarse axis of a contracting element \( g \). We define a \( \langle g \rangle \) invariant subset \( A_R(g) \) of \( Y \) to be an \( R \)-stable axis of the element \( g \), if for any point \( x \in A_R(g) \), the ball \( B(x,R) \) intersects any bi-infinite geodesic \( \alpha \) which is contained a finite neighborhood of \( A(g) \) and \( d(x, gx) > 3R \). A simplified version of Theorem 3.10 is stated below.

**Theorem 1.10.** Assume that \( G \) admits a SCC action on a proper geodesic metric space \((Y,d)\). Fix a contracting element \( g \). Then there exist \( R > 0 \) depending on \( f \) such that for any \( 1 > \theta_1, \theta_2 > 0 \) and any integer \( m > 1 \), the set of elements \( g \) with \( n = d(o, go) \) satisfying
\( n \geq \tau[g] \geq (1 - \theta_1)n, \)
\( d(o, ax(g)) \leq \theta_2, \)
\( \) any bi-infinite geodesic which is contained in a finite neighborhood of \( Ax(g) \) contains an \( (\epsilon, f^n) \)-barrier (see Def. 2.10),
\( is exponentially generic.

Examining the axis of a pseudo-Anosov element in Teichmüller space, we derive an application to mapping class groups which gives a positive answer to the (first part of) the question posed by J. Maher in [20, Question 6.4]. By abuse of language, we denote below by \( ax(g) \) the Teichmüller axis of a pseudo-Anosov element of \( g \) both as a subset of Teichmüller space and its unit tangent bundle.

Recall that a pseudo-Anosov element is contracting with respect to Teichmüller metric, cf. [44]. Moreover, the action of \( MCG(S) \) on \( Teich(S) \) is statistically convex-cocompact (see [24, Theorem 1.7], [3, Section 10]).

Let \( PS \) be the principal stratum of quadratic differentials.

**Theorem 1.11.** Let the mapping class group \( G \) act on the Teichmüller space \((Y, d)\) endowed with Teichmüller metric. Then for any \( 0 < \theta_1, \theta_2 < 1 \), we have
\[ \{ g \in G : n \geq \tau[g] \geq \theta_1 n & d(o, ax(g)) \leq \theta_2 n & ax(g) \in PS \text{ where } n = d(o, go) \} \]
is exponentially generic.

**1.4. Outline of the proof of Main Theorem.** Recall that \( N(o, n) = \{ g : d(o, go) \leq n \} \) is the set of elements in a ball of radius \( n \). When the action is SCC, we have
\[ \sharp N(o, n) \asymp \exp(\omega(G)n). \]
Since \( \ell_o[g] \geq \tau[g] \) for any \( o \in Y \), the following relations are basic in our discussion:
\[ \mathcal{C}(o, n) \subset \mathcal{C}(n), \mathcal{C}(o, n) \subset N(o, n). \]

The key idea of the proof is to choose an exponentially generic set of contracting elements with locally uniform hyperbolic properties. Such properties are encapsulated in the following notion (See Def. 2.10 for a precise definition).

**Definition 1.12.** With a basepoint \( o \in Y \) fixed, an element \( h \in G \) is called \( (\epsilon, M, g) \)-barrier-free if there exists an \( (\epsilon, f) \)-barrier-free geodesic \( \gamma \) with \( \gamma_- \in B(o, M) \) and \( \gamma_+ \in B(ho, M) \): there exists no \( t \in G \) such that \( d(t \cdot o, \gamma), d(t \cdot ho, \gamma) \leq \epsilon. \)

In [57], the second-named author proved that for any \( f \in G \), the set of \( (\epsilon, M, f) \)-barrier-free elements is exponentially negligible for a constant \( M > 0 \) appearing in Definition 1.1. As fore-mentioned, a contracting element represents a sense of hyperbolicity in direction. Therefore, once a contracting element \( f \) is provided and fixed through out, the geodesic with \( (\epsilon, M, f) \)-barriers uniformly behaves like a geodesic in a Gromov-hyperbolic space when coming to the barriers. This often allows us to run arguments via hyperbolic geometry (e.g. see Proposition 3.6).

Furthermore, the set of minimal representatives in conjugacy classes of non-contracting elements are barrier-free, so exponentially negligible as well. We show that \( \mathcal{C}(o, n) \) has the growth rate \( \omega(G) \); it is sufficient to count conjugacy classes of contracting elements.

The next step is to compute the algebraic length and stable length of a contracting element. The algebraic length is a bit easier to estimate from the definition. However, giving a uniform way to estimate the stable length of every contracting
element seems to be hard, if not impossible. The solution is here that we can estimate the stable length for an exponentially generic set of contracting elements.

**Lemma 1.13** (Corollary 3.7, Stable length \(\simeq\) algebraic length). There exist an exponentially generic set \(\mathcal{G}\) of contracting elements and a constant \(D = D(f_1, f_2, o) > 0\) for two independent contracting elements \(f_1, f_2\) such that for each \(g \in \mathcal{G}\), the following holds

\[
0 \leq \ell_o[g] - \tau[g] \leq D.
\]

Therefore we do not need to distinguish between stable length and algebraic length, and so \(\sharp \mathcal{C}(o, n)\) and \(\sharp (\mathcal{C}(n) \cap \mathcal{C}(o, n))\) are coarsely equal.

1. **Upper bound on strongly primitive conjugacy classes.** To get the upper bound of \(\mathcal{C}(o, n)\), we follow a piece of argument in [18] which works for a cocompact action of a (hyperbolic) group. Namely, take a conjugacy class \([g]\) \(\in \mathcal{C}(o, n)\) and if \(g\) is strongly primitive and write \(g = s_1 \cdot s_2 \cdots s_n\), then by cyclic permutation we obtain \(n\) words in the same conjugacy class. If all cyclic permutations represent distinct elements, then the upper bound of \(\mathcal{C}(o, n)\) is obtained as follows:

\[
\sharp \mathcal{C}'(o, n) \cdot n \leq \sharp \mathcal{N}(o, n).
\]

However, this argument breaks down when the action is not cocompact. The reason is that since \([o, go]\) may have large proportion outside the orbit \(G_0\), there is no way to write \(g\) as a product of a number of elements linear in \(d(o, go)\). We overcome this by showing that geodesic segments associated to generic contracting elements spend a definite proportion of the time in \(N_M(G_0)\).

**Lemma 1.14** (Corollary 5.3, Thick contracting elements). There exists an exponentially generic set \(\mathcal{G}\) of contracting elements such that for each \(g \in \mathcal{G}\), we have

\[
\ell([o, go] \cap N_M(G_0)) \geq 0.9 \cdot \ell([o, go]).
\]

Thus, the upper bound on the number of strongly primitive conjugacy classes is obtained in Corollary [5.6].

2. **Strongly non-primitive ones are growth tight.** Counting strongly non-primitive conjugacy classes will require more effort. The idea however is simple: the number of non-primitive conjugacy classes is exponentially negligible, compared to the primitive ones. The proof of this result, Lemma 7.1, is intuitively clear, since a non-primitive element \(g\) is a proper power of a primitive element \(g_0\). An inspection of the argument shows a difficulty as follows.

Let \(h = h_0^m f\) be a non-strongly primitive element in \(E(g)\) for \(|m| \geq 2\) and \(f \in F\). Define a map \(\Pi\) sending \([h]\) to \([h_0]\). Clearly, the image of \(\Pi\) is exponentially negligible. But there is no reason that \(\Pi\) is uniformly finite to one, since the size of \(F\) can change. The following result fixes this issue.

**Lemma 1.15** (Lemma 6.6, Uniform kernel). There exist an exponentially generic set \(\mathcal{G}\) of contracting elements and an integer \(N > 0\) such that for each \(g \in \mathcal{G}\), we have

\[
1 \to F \to E^+(g) \xrightarrow{\phi} \mathbb{Z} \to 1
\]

and \(\sharp F \leq N\).

The proof of the lemma relies on the very recent work of Bestvina, Bromberg, Fujiwara and Sisto [8] improving the earlier work [7] so that the action on the projection complex is acylindrical hyperbolic. We then prove that an exponentially
generic set of elements act by loxodromic isometries on the projection complex. Then a result of Osin [45, Lemma 6.8] concludes the proof.

From this result, we show that non-primitive conjugacy classes are exponentially negligible (Lemma 7.1). Hence, the upper bound for all conjugacy classes is proved in Corollary 7.2.

3. Lower bound on conjugacy classes. The lower bound is by construction. By [57] (recalled in Lemma 4.1), there exists a maximal separated set $T$ in $A(o,n,\Delta)$ and a contracting element $f$ such that $T \cdot f$ consists of contracting elements and has the same cardinality as $T$. Following an argument of Coornaert and Knieper [17], we show that each conjugacy class $[g]$ contains at most $\theta n$ elements in $T \cdot f$ for some uniform number $\theta > 0$. Thus, we constructed at least $\exp(o(G)n)$ conjugacy classes (see Corollary 4.5). Finally, the lower bound for primitive conjugacy classes is a direct consequence of the growth tightness of non-primitive ones mentioned above (Corollary 7.4).

The structure of the paper is as follows. The preliminary §2 introduces the contracting property, and a class of periodic admissible paths. The core of §3 is Proposition 3.6 identifying the stable length with algebraic length. Along the way, the linear growth is proved in Theorem 1.10. Many conjugacy classes are constructed in §4, establishing the lower bound. Then §5 deals with non-cocompact actions and obtains the upper bound for primitive conjugacy classes. Section §6 addresses the issue of unbounded torsion. With previous ingredients in hand, the proof of the main theorem is completed in §7. The final §8 explains the applications to several specific classes of groups.

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2. Preliminaries

2.1. Notations and conventions. Let $(Y, d)$ be a proper geodesic metric space. Given a point $y \in Y$ and a closed subset $X \subset Y$, let $\pi_X(y)$ be the set of points $x$ in $X$ such that $d(y, x) = d(y, X)$. The projection of a subset $A \subset Y$ to $X$ is then $\pi_X(A) := \cup_{a \in A} \pi_X(a)$. Whenever talking about projection, we shall assume the closedness of the subset $X$ under consideration so that $\pi_X(A)$ is nonempty.

Denote $d_X(Z_1, Z_2) := \text{diam}(\pi_X(Z_1 \cup Z_2))$, which is the diameter of the projection of the union $Z_1 \cup Z_2$ to $X$. So $d_X^\kappa(\cdot, \cdot)$ satisfies the triangle inequality

$$d_X^\kappa(A, C) \leq d_X^\kappa(A, B) + d_X^\kappa(B, C).$$

We use $d_X(Z) := \text{diam}(\pi_X(Z))$ as well in the sequel.

We always consider a rectifiable path $\alpha$ in $Y$ with arc-length parametrization. Denote by $\ell(\alpha)$ the length of $\alpha$, and by $\alpha_-, \alpha_+$ the initial and terminal points of $\alpha$ respectively. Let $x, y \in \alpha$ be two points which are given by parametrization. Then $[x, y]_\alpha$ denotes a choice of a geodesic between $x, y \in Y$.

Entry and exit points. Given a property $(P)$, a point $z$ on $\alpha$ is called the entry point satisfying $(P)$ if $\ell([\alpha_-, z]_\alpha)$ is minimal among the points $z$ on $\alpha$ with the
property (P). The exit point satisfying (P) is defined similarly so that \( \ell([w, \alpha_+], \alpha) \) is minimal.

A path \( \alpha \) is called a \( c \)-quasi-geodesic for \( c \geq 1 \) if the following holds
\[
\ell(\beta) \leq c \cdot d(\beta_-, \beta_+) + c
\]
for any rectifiable subpath \( \beta \) of \( \alpha \).

Let \( \alpha, \beta \) be two paths in \( Y \). Denote by \( \alpha \cdot \beta \) (or simply \( \alpha \beta \)) the concatenated path provided that \( \alpha_+ = \beta_- \).

Let \( f, g \) be real-valued functions with domain understood in the context. Then \( f \prec_c g \) means that there is a constant \( C > 0 \) depending on parameters \( c_i \) such that \( f < Cg \). The symbols \( \succ_c \) and \( \asymp_c \) are defined analogously. For simplicity, we shall omit \( c_i \) if they are universal constants. We also denote \( f \asymp_c g \) if \( |f - g| \leq C \).

2.2. Contracting property.

**Definition 2.1 (Contracting subset).** Let \( QG \) denote a preferred collection of quasi-geodesics in \( Y \). For given \( C \geq 1 \), a subset \( X \) in \( Y \) is called \( C \)-contracting with respect to \( QG \) if for any quasi-geodesic \( \gamma \in QG \) with \( d(\gamma, X) > C \), we have
\[
d_X(\gamma) \leq C.
\]
A collection of \( C \)-contracting subsets is referred to as a \( C \)-contracting system (w.r.t. \( QG \)).

**Example 2.2.** We note the following examples in various contexts.

1. Quasi-geodesics and quasi-convex subsets are contracting with respect to the set of all quasi-geodesics in hyperbolic spaces.
2. Fully quasi-convex subgroups (and in particular, maximal parabolic subgroups) are contracting with respect to the set of all quasi-geodesics in relatively hyperbolic groups (see Proposition 8.2.4 in [31]).
3. The subgroup generated by a hyperbolic element is contracting with respect to the set of all quasi-geodesics in groups with non-trivial Floyd boundary. This is described in [56, Section 7].
4. Contracting segments in CAT(0)-spaces in the sense of in Bestvina and Fujiwara are contracting here with respect to the set of geodesics (see Corollary 3.4 in [9]).
5. The axis of any pseudo-Anosov element is contracting relative to geodesics by Minsky [44].
6. Any finite neighborhood of a contracting subset is still contracting with respect to the same \( QG \).

**Convention 2.3.** In view of Examples 2.2, the preferred collection \( QG \) in the sequel will always be the set of all geodesics in \( Y \).

We collect a few properties that will be used often later on. The proof is a straightforward application of the contracting property, and is left to the interested reader.

**Proposition 2.4.** Let \( X \) be a contracting set.

1. \( X \) is \( \sigma \)-quasi-convex for a function \( \sigma : \mathbb{R}_+ \to \mathbb{R}_+ \): given \( c \geq 1 \), any \( c \)-quasi-geodesic with endpoints in \( X \) lies in the neighborhood \( N_{\sigma(c)}(X) \).
2. Let \( Z \) be a set with finite Hausdorff distance to \( X \). Then \( Z \) is contracting.
In most cases, we are interested in a contracting system with the \( R \)-bounded intersection property for a function \( R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) so that
\[
\forall X \neq X' \in \mathbb{X} : \text{diam}(N_r(X) \cap N_r(X')) \leq R(r)
\]
for any \( r \geq 0 \). This property is, in fact, equivalent to the bounded intersection property of \( \mathbb{X} \): there exists a constant \( B > 0 \) such that the following holds
\[
d_{X'}(X) \leq B
\]
for \( X \neq X' \in \mathbb{X} \). See [57, Lemma 2.11] for a proof of equivalence.

Recall that \( G \) acts properly on a proper geodesic metric space \((Y, d)\). An element \( h \in G \) is called contracting if the orbit \( \langle h \rangle \cdot o \) is contracting, and the orbital map
\[
n \in \mathbb{Z} \to h^n o \in Y
\]
is a quasi-isometric embedding. Thus, any root or power of a contracting element is contracting. Note that the set of contracting elements is preserved under conjugacy.

Given a contracting element \( h \), we define a group
\[
E(h) := \{ g \in G : d_H(\langle h \rangle o, g(h)o) < \infty \}
\]
where \( d_H \) denotes the Hausdorff distance. By [57, Lemma 2.11], \( E(h) \) is the unique maximal elementary group containing \( \langle h \rangle \) as a finite-index subgroup. Moreover, it can be described as follows,
\[
E(h) = \{ g \in G : \exists n > 0, (gh^n g^{-1} = h^n) \lor (gh^n g^{-1} = h^{-n}) \}.
\]

In what follows, the contracting subset
\[
Ax(h) = \{ f \cdot o : f \in E(h) \}
\]
will be called the coarse axis of \( h \). Hence, the collection \( \{ gAx(h) : g \in G \} \) is a contracting system with bounded intersection (by [57, Lemma 2.11]). Compare with combinatorial axis in Definition 2.8 and stable axis in Definition 3.3.

Two contracting elements \( h, k \in G \) are independent if the collection of contracting sets \( \{ gAx(h), gAx(k) : g \in G \} \) has bounded intersection. Equivalently, they are independent if \( E(h) \) and \( E(k) \) are not conjugate in \( G \).

For \( i = 1, 2 \), two (oriented) geodesics \( \gamma_i : \mathbb{R} \to Y \) have the same orientation if
\[
d_H(\gamma_1([0, +\infty]), \gamma_2([0, +\infty])) < \infty.
\]
An element \( g \in G \) preserves the orientation of an oriented geodesic \( \alpha \) if \( \alpha \) and \( ga \) have the same orientation.

The following fact is elementary.

Lemma 2.5. For every contracting element \( g \in G \), there exists a bi-infinite geodesic \( \alpha \) in a finite neighborhood of the axis \( Ax(g) \). Moreover, the element \( g \) preserves the orientation of the geodesic \( \alpha \).

Proof. By Proposition 2.4, the first statement follows from the quasiconvexity of the contracting subset \( Ax(g) \) by a Cantor diagonal argument. To prove the moreover statement, fix an orientation of \( \alpha : \mathbb{R} \to Y \), and a basepoint \( o = \alpha(0) \in \alpha \). Let \( R > 0 \) be the Hausdorff distance between \( \alpha \) and \( Ax(f) \). Thus, for \( x_n = \alpha(n) \), there exists \( k_n \in \mathbb{Z} \) such that \( d(g^{k_n} o, x_n) \leq R \).

Suppose to the contrary that \( R' = d_H(\alpha([0, +\infty]), ga([0, -\infty])) < \infty \). Thus, for every \( n > 0 \), there exists \( m < 0 \) such that \( d(gx_n, x_m) \leq R' \). Then \( d(g^{k_n+1} o, g^{k_m} o) \leq 2R + R' \). Since the action is proper, we obtain that \( k_n - k_m = n_0 \) for infinitely many
n > 0, m < 0. However, this is a contradiction since \( d(g^{k_0}o, g^{k_0}o) \geq d(x_n, x_m) - 2R \)
tends \(\infty\) when \(|n|, |m| \to +\infty\). \(\square\)

Corollary 2.6. For a contracting element \( h \), let \( E^+(h) \) be the subgroup of orientation
preserving elements in \( E(h) \). Then \( E^+(h) = \{ g \in G : \exists n > 0, gh^n g^{-1} = h^n \} \) is of
index at most two and contains all contracting elements in \( E(h) \).

Remark. A similar statement appears in Corollary 6.6 in [19], where the space \( Y \)
is assumed to be \( \delta \)-hyperbolic.

Since \( E^+(h) \) is two-ended, a theorem of Stallings implies the following exact sequence
\[
1 \to F \to E^+(h) \xrightarrow{\phi} Z \to 1,
\]
where \( F \) is a finite group and \( Z \) is the group of integers. Any element \( g \in E^+(h) \)
in \( \phi^{-1}(\pm 1) \) is called strongly primitive.

Lemma 2.7. A strongly primitive contracting element is a primitive element.

Proof. Let \( g \in E(h) \) be a strongly primitive contracting element. If \( g \) is not prim-
itive, then \( g = g_0^k \) for some \( |k| \geq 2, g_0 \in G \). Since \( g \) is contracting, it is readily
checked that \( (g_0) \cdot o \) is a contracting quasi-geodesic, which implies that \( g_0 \) is a con-
tracting element. By definition of \( E(h) \), we have \( g_0 \in E(h) \) and then \( g_0 \in E^+(h) \)
by Lemma 2.5. We obtain now \( \phi(g) = k \cdot \phi(g_0) \neq \pm 1 \), contradicting the strong
primitivity of \( g \). \(\square\)

Remark. Note that the converse may not be true. However, a primitive but non-
strongly primitive contracting element \( g \in E(h) \) can be written as \( g_0^k f \) for \( |k| \geq 2 \)
and \( f \in F \setminus \{1\} \), where \( g_0 \) is strongly primitive.

2.3. Periodic admissible paths. In [56], the notion of an admissible path is
defined with respect to a contracting system \( \mathcal{X} \) in \( Y \). In the present paper, we shall
focus on a particular class of admissible paths, which will serve as coarse axes for
contracting elements.

Definition 2.8 (Periodic Admissible Path). Let \( D, \tau > 0 \) and \( \mathcal{X} \) be a contracting
system with bounded projection. Given an element \( g \in G \), a bi-infinite path \( \gamma =
\bigcup_{i \in \mathbb{Z}}(q_i, p_i) \) is called periodic \((g, D, \tau)\)-admissible path if the following hold
\begin{enumerate}
  
  \item each \( p_i \) is a geodesic of length at least \( D \) with two endpoints in \( X_i \in \mathcal{X} \),
  \item each \( q_i \) is a geodesic with \( \tau \)-bounded projection to \( X_i \) and \( X_{i-1} \):
  \[ \max\{d_{X_i}(q_i), d_{X_{i+1}}(q_i)\} \leq \tau, \]
  \item \( q_ip_i = g^i(q_0p_0) \) for \( i \in \mathbb{Z} \).
\end{enumerate}

The collection of \( X_i \in \mathcal{X} \) is called a combinatorial axis of the element \( g \), denoted
by \( \mathcal{X}(g) \).

Remark. The main difference with the definition in [56] is the third item which
gives the way to represent the admissible path periodically. For large \( D \gg 0 \), it is
easy to see that we must have \( X_i = g^iX_0 \) by bounded intersection of \( X_i \in \mathcal{X} \).

The collection \( \mathcal{X}(g) \) was called the saturation of \( \gamma \) in [57]. Here it can be thought
of as an axis with respect to the action of \( G \) on the projection complex built from
the collection \( \mathcal{X} \), where each \( X \in \mathcal{X} \) is collapsed to one vertex. Compare with a
similar notion in [7, Prop 3.26] and see Lemma 6.5.
Since a periodic admissible path is a special case of the more general notion of an admissible path, the results proved in \([56]\) and \([57]\) apply here. We summarize the properties of periodic admissible paths as follows.

**Proposition 2.9.** Let \(X\) be a \(C\)-contracting system with \(\mathcal{R}\)-bounded projection. For any \(\tau > 0\), there are constants \(D = D(\tau, C, \mathcal{R}), \epsilon = \epsilon(\tau, C, \mathcal{R}) > 0\) such that the following holds.

Let \(g \in G\) be an element admitting a periodic \((g, D, \tau)\)-admissible path \(\gamma = \bigcup_{i \in \mathbb{Z}} (q_i, p_i)\). Then

1. \(g\) is a contracting element.
2. For any \(i < j\) we have \(\pi_X(X_j) \subset B((p_i)_+, \epsilon)\) and \(\pi_X(X_i) \subset B((p_j)_-, \epsilon)\).
3. For any bi-infinite geodesic \(\alpha\) in a finite neighborhood of \(\gamma\), we have that \(\alpha\) intersects the \(\epsilon\)-neighborhood of \((p_i)_-\) and \((p_i)_+\).

**Proof.** By Proposition 2.9.2 in \([57]\) we know that \(\gamma\) is a contracting quasi-geodesic, so \(g\) is contracting by definition.

The assertion (3) was proved in \([57]\) where \(\gamma\) is a finite path with the same endpoints as \(\alpha\). This followed from \([56]\) Corollary 3.7 of the bounded projection to \(X = X_i\) from both sides: there exists \(B = B(\sigma, \mathcal{R}) > 0\) such that

\[
(5) \quad \max\{\pi_X(\beta_1), \pi_X(\beta_2)\} \leq B
\]

where \(\beta_1\) is the left one-sided infinite subpath of \(\gamma\) issuing from \((p_i)_-\) and \(\beta_2\) is the right one from \((p_i)_+\).

![Figure 1. Fellow travel periodic admissible paths.](image-url)

In the current setting, we first note \(\alpha \cap N_C(X) \neq \emptyset\). Indeed, if \(\alpha\) is disjoint from \(N_C(X)\), then \(d_X(\alpha) \leq C\) by contracting property of \(X\). Say for some \(R > 0\), \(\gamma\) stays in an \(R\)-neighborhood of \(\alpha\) by the assumption. Then we can take two points \(z_i \in \beta_i\) \((i = 1, 2)\) far from \(N_R(X)\) such that \(d(z_i, \alpha) \leq R\) and thus the ball centered at \(z_i\) of radius \(R\) misses \(X\). By the contracting property, \(d_X(B(z_i, R)) \leq C\) for each ball around \(z_i\) of radius \(R\). By a projection argument, we obtain

\[
\ell(p_i) \leq d_X(\alpha) + \sum_{i=1,2} (d_X(\beta_i) + d_X(B(z_i, R))) \leq 2B + 3C.
\]

This gives a contradiction if \(D > 2B + 3C\) is assumed. Hence \(\alpha \cap N_C(X) \neq \emptyset\) is proved. Let \(\alpha_1\) be the left geodesic ray in \(\alpha \setminus N_C(X)\) so we have \(d_X(\alpha_1) \leq C\) and \(d(\alpha_1)_-, X) \leq C\). Again a projection argument shows

\[
d((p_i)_-, \alpha_1) \leq d_X(\alpha_1) + d_X(\beta_1) + C \leq B + 2C.
\]

Set \(\epsilon = B + 2C\) completes the proof of (3).
The assertion (2) is an immediate consequence of (3). Indeed, let $\alpha$ be any geodesic segment from $X_j$ to $X_i$ for $i < j$. By (3), we have $\alpha \cap B((p_i)_+, \epsilon) \neq \emptyset$, so $\pi_{X_i}(X_j) \subset B((p_i)_+, \epsilon)$. □

2.4. Exponential negligibility of barrier-free elements. Recall that the notion of a statistically convex-cocompact action is given in Definition 1.1, so that

$$\omega(O_{M_1,M_2}) < \omega(G) < \infty$$

for some constants $M_1, M_2 > 0$. Here, $O_{M_1,M_2}$ is the set of elements $g \in G$ such that there exists some geodesic $\gamma$ between $\gamma_- \in B(o, M_2)$ and $\gamma_+ \in B(go, M_2)$ with the property that the interior of $\gamma$ lies outside $N_{M_1}(Go)$. In the applications under consideration, since $O_{M_2,M_2} \subset O_{M_1,M_2}$, we can assume that $M_1 = M_2$ and from now on, denote $O_M := O_{M,M}$ for an easy notation.

The next tool in our study is the (exponential) negligibility of a class of barrier-free elements. A set $X$ in $G$ is called generic if

$$\|X \cap N(o, n)\| \rightarrow 1,$$

as $n \rightarrow \infty$. It is called exponentially generic if

$$\|X \cap N(o, n)\| / \|N(o, n)\| \leq \exp(-\epsilon n),$$

for some $\epsilon > 0$ and all $n \gg 0$.

By definition, a subset $X$ of $G$ is called growth tight if $\omega(X) < \omega(G)$. If $G$ has purely exponential growth then the growth tightness of a subset is equivalent to its exponential negligibility.

We now recall a notion of barriers on a geodesic and an element from [57].

**Definition 2.10.** Fix constants $\epsilon, M > 0$ and a set $P$ in $G$.

1. Given $\epsilon > 0$ and $f \in P$, we say that a geodesic $\gamma$ contains an ($\epsilon, f$)-barrier if there exists an element $t \in G$ so that

$$\max\{d(t \cdot o, \gamma), d(t \cdot fo, \gamma)\} \leq \epsilon.$$

If there exists no $t \in G$ so that the above inequality holds, then $\gamma$ is called ($\epsilon, f$)-barrier-free.

Generally, $\gamma$ is called ($\epsilon, P$)-barrier-free if it is ($\epsilon, f$)-barrier-free for some $f \in P$. An obvious fact is that any subsegment of $\gamma$ is also ($\epsilon, P$)-barrier-free.

2. An element $g \in G$ is ($\epsilon, M, P$)-barrier-free if there exists an ($\epsilon, P$)-barrier-free geodesic between $B(o, M)$ and $B(go, M)$. Denote by $V_{\epsilon,M,P}$ the set of ($\epsilon, M, P$)-barrier-free elements in $G$.

**Remark.** By abuse of language and for simplicity, we shall say that the contracting subset $t \cdot Ax(f)$ (or even the element $t$ when $f$ is clear in context) is an ($\epsilon, f$)-barrier.

**Theorem 2.11.** [57, Thm. C & Cor. 4.5] Assume that a non-elementary group $G$ acts properly on a proper geodesic metric $Y$ with a contracting element. Then for any $M \gg 0$ there exists $\epsilon > 0$ such that the following properties hold for any element $f \in G$:

1. If the action is SCC, then the barrier-free set $V_{\epsilon,M,f}$ is exponentially negligible.

2. If the action has PEG, then the barrier-free set $V_{\epsilon,M,f}$ is negligible.
We remark that if the action is SCC, then it has purely exponential growth. However, the converse is not true: there exists examples of geometrically finite Kleinian groups on Hadamard manifolds with PEG actions but without the parabolic gap property. Indeed, M. Peigné [47] constructed a class of exotic Schottky groups acting geometrically finitely on a simply connected Hadamard manifold without parabolic gap property so that the corresponding Bowen-Margulis-Sullivan measure is finite. By Roblin’s work [51], the finiteness of Bowen-Margulis-Sullivan measure is equivalent to the purely exponential growth of the action.

3. Genericty properties of contracting elements

The goal of this section is to choose an exponentially generic set of contracting elements so that we can compute efficiently their stable length. We start by recalling some results from [58] about genericy of contracting elements.

Throughout this section, let $M > 0$ be the constant appearing in the definition of a statistically convex-cocompact action. Let $\epsilon = \epsilon(M) > 0$ given by Theorem 2.11 so that the barrier-free set $V_{\epsilon,M,f}$ is exponentially negligible for any $f \in G$. We shall omit $M$ in $V_{\epsilon,M,f}$ when it is clear from context.

3.1. Genericy of contracting elements. In [58], it is proved that contracting elements are (resp. exponentially) generic if the proper action is PEG (resp. SCC). The proof of this result replies on the following more general result. Recall that $V_{\epsilon,f}$ denotes the set of $(\epsilon,f)$-barrier-free elements.

**Theorem 3.1.** [58, Theorem 4.1] For each contracting element $f \in G$ the set of elements in $G$ conjugated into $V_{\epsilon,f}$ is exponentially negligible for SCC actions.

**Remark.** Parallel to Theorem 2.11 there is an additional statement that, when the action is PEG, the above set in the conclusion is negligible. This might be used to generalize Main Theorem to any proper PEG action.

With the fixed basepoint $o$ in mind, an element $g \in G$ in its conjugacy class is called minimal if $d(o,go) \leq d(o,ho)$ for any $h \in [g]$. The second named author proved in [58] Theorem 3.1 that non-contracting elements admit minimal conjugacy representatives in $V_{\epsilon,f,m}$ for some $m > 0$. Together with Theorem 3.1 this implies that non-contracting elements are exponentially negligible. As a consequence, in order to count conjugacy classes ordered by algebraic length, it suffices to consider contracting elements. Moreover, we can consider the set of elements admitting a minimal representative with an $(\epsilon,f)$-barrier. We record this observation into the following.

**Lemma 3.2.** For a SCC action, the set of non-contracting elements is exponentially negligible. Moreover, the set of elements which admit a minimal $(\epsilon,f)$-barrier-free representative is exponentially negligible.

3.2. Stable axis of contracting elements. One of the goals of this section is to show that for a generic set of conjugacy classes, the stable length and algebraic length differ only by a bounded amount. For that purpose, we shall introduce a notion of stable axis in order to facilitate the computation of stable length.

Given a contracting element $g$, the group $E(g)$ is the maximal elementary subgroup containing $g$ in $G$, and $\text{Ax}(g) = E(g) \cdot o$ is the (coarse) axis of $g$ (depending on the basepoint $o \in Y$).
We now introduce a finer notion of axis for a contracting element, which is a stable subset of the coarse axis \( \text{Ax}(g) \) in a sense that it belongs to the negatively curved part of \( \text{Ax}(g) \). For a pseudo-Anosov element on Teichmüller space this is the (possibly disconnected) subsegment of its translation axis contained over a fixed compact part of the moduli space.

**Definition 3.3 (Stable axis).** Let \( g \) be a contracting element. Given \( R > 0 \), a \( (g) \)-invariant subset \( \mathcal{A}_R(g) \subset Y \) is called a stable \( R \)-axis, if for any point \( x \in \mathcal{A}_R(g) \), the ball \( B(x, R) \) intersects any bi-infinite geodesic \( \alpha \) contained in a finite neighborhood of \( \text{Ax}(g) \), and \( d(x, gx) \geq 3R \).

The terminology of a stable axis is explained by the following lemma.

**Lemma 3.4.** Assume that a contracting element \( g \) admits a stable \( R \)-axis \( \mathcal{A}_R(g) \) for some \( R > 0 \). Then for any \( x \in \mathcal{A}_R(g) \), we have \( |\tau(g) - d(x,gx)| \leq 2R \).

**Proof.** Choose a reference point \( x_0 \in \mathcal{A}_R(g) \) and a geodesic \( \alpha \) within a finite Hausdorff distance of \( \text{Ax}(g) \). Since \( \text{Ax}(g) \) is \( (g) \)-invariant, we have that for each \( i \in \mathbb{Z} \), \( g^i\alpha \) stays in a finite neighborhood of \( \text{Ax}(g) \). By definition of the stable axis, we obtain that \( d(g^i x_0, \alpha) \leq R \). Denoting \( x_i = g^i x_0 \), there exists \( z_i \in \alpha \) such that \( d(x_i, z_i) \leq R \) and so \( d(z_i, z_{i+1}) - d(x_0, g x_0) \leq 2R \) for each \( i \). Since \( d(x_0, g x_0) \geq 3R \) and \( g \) preserves the orientation of \( \alpha \), we see that \( z_i \) are linearly ordered on \( \alpha \). This shows

\[
|d(z_0, z_n) - nd(x_0, gx_0)| \leq 2nR.
\]

Since \( \max\{d(z_0, x_0), d(z_n, g^n x_0)\} \leq R \), we have

\[
\tau(g) = \lim_{n \to \infty} \frac{d(g^n x_0, x_0)}{n} = \lim_{n \to \infty} \frac{d(z_n, z_0)}{n}.
\]

This gives \( |\tau(g) - d(x_0, gx_0)| \leq 2R \), completing the proof. \( \square \)

Let us now relate the stable axis to the combinatorial axis of a periodic admissible path (see Def. 2.8).

Fix a basepoint \( o \in Y \). In the next lemma, we consider a contracting element \( f \in G \). Let \( \mathcal{K} = \{g \text{Ax}(f) : g \in G\} \) the collection of \( C \)-contracting subsets with bounded intersection for a constant \( C > 0 \) by [77, Lemma 2.11].

**Lemma 3.5.** For any given \( \tau > 0 \), there exist \( D_0 = D_0(f, \tau), R = R(f, \tau) > 0 \) such that the following holds.

Consider a contracting element \( g \in G \setminus E(f) \) with a periodic \( (g, D, \tau) \)-admissible path \( \gamma = \cup_{i \in \mathbb{Z}} (q_i, p_i) \) for some \( D > D_0 \). Assume that the initial endpoint \( (p_0)_- \) of \( p_0 \) is the basepoint \( o \). Then

1. Let \( \mathcal{A}(g) \subset \mathcal{K} \) be the combinatorial axis of \( g \). The following union

\[
\mathcal{A}(g) := \cup \{\pi_X(Y) : X \neq Y \in \mathcal{K}\}
\]

consists of a stable \( R \)-axis of the element \( g \).

2. If \( g = g_0^k \) for \( k \geq 1 \), then

\[
|\tau(g_0^k) - \ell_o[g_0]| \leq R.
\]

**Proof.** (1). Since \( g^i \pi_X(Y) = \pi_{g^iX}(g^iY) \) and \( \mathcal{A}(g) \) is \( (g) \)-invariant, we have \( \mathcal{A}(g) \) is \( (g) \)-invariant.

Consider a bi-infinite geodesic \( \alpha \) in a finite neighborhood of \( \text{Ax}(g) \). If \( \epsilon > 0 \) is the constant given by Proposition 2.9 then any point \( x \in \mathcal{A}(g) \) is \( \epsilon \)-close to a
point \((p_i)−\) for some \(j \in \mathbb{Z}\) which is in turn \(\epsilon\)-close to \(\alpha\). Setting \(R = 6\epsilon\), we have \(d(x, \alpha) \leq 2\epsilon < R\).

By the definition of a periodic admissible path, we have \(g(p_i)− = (p_{i+1})−\) for every \(i \in \mathbb{Z}\). See a schematic picture \([1]\) of a periodic admissible path. We deduce from Proposition 2.9 that \((p_0)−\) is \(\epsilon\)-close to \([(q_0)−, (p_0)+]\), so by the periodicity we have

\[
d((p_i)−, (p_{i+1})−) − (\ell(p_0) + \ell(q_0))| \leq 2\epsilon
\]

for any \(i \in \mathbb{Z}\). If \(D > 3R + 4\epsilon\) is assumed, then the above inequality with \(\ell(p_0) > D\) implies for any \(i \in \mathbb{Z}\):

\[
d((p_i)−, g(p_i)−) \geq \ell(p_0) - 2\epsilon \geq 3R + 2\epsilon.
\]

Recalling \(d(x, p_j)− \leq \epsilon\), we have \(d(x, gx) \geq 3R\) and thus \(\mathcal{A}(g)\) is a stable \(R\)-axis.

As a consequence, the assertion (2) for \(k = 1\) (i.e. \(g = g_0\)) follows. Indeed, by Lemma 3.4 we have

\[
|\tau[g] - d(x, gx)| \leq 2R
\]

for any \(x \in \mathcal{A}(g)\). By the discussion above, \(x\) can be chosen to be \(\epsilon\)-close to the initial endpoint \(o = (p_0)−\) of \(p_0\). Since \(\tau[g] \leq \ell_0[g] \leq d(o, go)\) always holds, we obtain

\[
|\tau[g] - \ell_o[g]| \leq 3R.
\]

The full generality for \(k \geq 2\) of (2) shall be proved below by exhibiting a stable axis for \(g_0\) which contains \(\mathcal{A}(g)\).

(2). Any root of a contracting element is contracting. Thus, \(g_0\) is contracting with the same maximal elementary group \(E(g_0) = E(g)\) and the axis \(\text{Ax}(g_0) = \text{Ax}(g)\).

Observe that \(g_i^j \mathcal{A}(g) \cap g_i^j \mathcal{A}(g) = \emptyset\) for \(0 \leq i \neq j \leq k\). Indeed, note that \(\mathcal{A}(g) = \{g^k \text{Ax}(f) : k \in \mathbb{Z}\}\). Assume for the contrary that \(g_0^{-i} g_i^j \text{Ax}(f) = g_m \text{Ax}(f)\) for some \(m \neq i \in \mathbb{Z}\). Then \(g_0^{-i} g_i^j (l-m)\) lies in \(E(f)\) and is a non-trivial power of \(g_0\) for \(|i-j| \leq k-1\). Since \(E(g_0)\) is the unique maximal elementary group containing \(\langle g_0 \rangle\), we obtain that \(E(g) = E(g_0) = E(f)\). This however contradicts to the assumption that \(g \notin E(f)\).

Hence, \(\mathbb{B} := \cup_{i=0}^{k-1} g_0^i \mathcal{A}(g)\) is a disjoint union of \(k\) copies of \(\mathcal{A}(g)\). By the bounded intersection, there exists \(B > 0\) such that for any two \(X \neq X' \in \mathbb{B}\) we have

\[
\text{diam}(N_C(X) \cap N_C(X')) \leq B.
\]

Let \(\alpha\) be a geodesic in finite neighborhood of \(\text{Ax}(g_0)\). Here is the consequence of the above discussion.

**Claim.** If \(D > 2B + 2\epsilon\) is assumed, the intersections of \(\alpha\) with all \(X \in \mathbb{B}\) appear in a linear order so that any two distinct intersection have distance at least \(D - B - 2\epsilon\).

**Proof of the claim.** Note that \(\text{Ax}(g_0) = \text{Ax}(g)\) is \(\langle g_0 \rangle\)-invariant so \(g_0^{-i} \alpha\) lies in a finite neighborhood of \(\text{Ax}(g)\) for any \(i \in \mathbb{Z}\). We now apply Proposition 2.9(3) to get

\[
\text{diam}(N_C(X) \cap g_0^{-i} \alpha) \geq D - 2\epsilon
\]

for every \(X \in \mathcal{A}(g)\). Thus, for any \(X \in \mathbb{B}\), we have

\[
\text{diam}(N_C(X) \cap \alpha) \geq D - 2\epsilon
\]

The claim thus follows by combining (8) and (7). \(\square\)
We are ready to show that the following \((g_0)\)-invariant set

\[ \mathcal{A}(g_0) := \bigcup_{i=0}^{k-1} g_0^i \mathcal{A}(g) \]

as the union of \(k\) copies of \(\mathcal{A}(g)\) is a stable \(R\)-axis of the element \(g_0\). We noted above that if \(\alpha\) lies in a finite neighborhood of \(A x(g_0) = A x(g)\), then \(\alpha\) intersects the \(R\)-neighborhood of \(g_0^i \mathcal{A}(g)\) for each \(i\), and of thus \(\mathcal{A}(g_0)\). It thus remains to show \(d(x,g_0 x) \geq 3R\) for each \(x \in \mathcal{A}(g_0)\).

For concreteness, assume that \(g_0^{-i} x \in \mathcal{A}(g)\) for some \(0 \leq i \leq k - 1\). Set \(\beta := g_0^{-i} \alpha\). Then \(\beta\) lies in a finite neighborhood of \(A x(g)\). By Proposition 2.9 there exists an endpoint of \(p_j\) denoted by \(x_0 \in X_j\) such that \(d(g_0^{-i} x, x_0) \leq \epsilon\). On the other hand, \(X_j \in \hat{A}(g)\) and \(g_0 X_j \in \mathcal{B}\) so the claim above implies \(d(x_0, g_0 x_0) \geq D - 2\epsilon - B\). Thus, if \(D \geq 3R + B + 4\epsilon\) is assumed, then

\[ d(x, g_0 x) \geq d(x_0, g_0 x_0) - 2\epsilon \geq 3R. \]

Since \(\mathcal{A}(g)\) is included in \(\mathcal{A}(g_0)\) by definition, the inequality (6) follows similarly for \(g_0\). Resetting \(R := 3R\) finishes the proof of (2). \(\square\)

3.3. Stable length of generic contracting elements. The main result of this subsection is that a generic contracting element admits a periodic admissible path, so its stable length could be estimated by Lemma 3.5

**Proposition 3.6.** Fix a basepoint \(o \in Y\), and two independent contracting elements \(f_1, f_2 \in G\) such that \(d(o, f_i o) \gg 0\). There exists a constant \(R = R(o, f_1, f_2) > 0\) with the following property.

Let \(g \in G\) be a minimal conjugacy representative in \([g]\) so that \([o, go]\) contains at least one \((\epsilon, f_1)\)-barrier and one \((\epsilon, f_2)\)-barrier. Then

\[ |\tau[g] - d(o, go)| \leq R. \]

Moreover, for any \(g_0^k = g\) with \(k \in \mathbb{Z}\), we have \(|\tau[g_0] - d(o, g_0 o)| \leq R.\)

**Proof.** Assume that there exists an \((\epsilon, f_i)\)-barrier \(t_i\) for \(\alpha = [o, go]\) where \(i = 1, 2\). Let \(b_i = t_i \cdot A x(f_i)\). Since \(f_1, f_2\) are independent, we have that \(b_1\) and \(b_2\) have bounded intersection: for given \(\epsilon > 0\) there exists \(B = B(\epsilon) > 0\) such that

\[ \text{diam}(N_{\epsilon}(b_1) \cap N_{\epsilon}(b_2)) \leq B. \]

The proof is based on the following.

**Claim.** There exists \(b \in \{b_1, b_2\}\) with the following property:

Denote by \(x, y\) the entry and exit points of the geodesic \(\alpha\) in \(N_{\epsilon}(b)\). We have

\[ d(x, y) + d(y, gx) \geq d(o, go) - 2\epsilon, \]

\[ \text{diam}(N_{\epsilon}(b) \cap g[o, x]_{\alpha}) \leq B + 4\epsilon, \]

\[ \text{diam}(N_{\epsilon}(b) \cap g^{-1}[y, go]_{\alpha}) \leq B + 4\epsilon. \]

We will first complete the proof of the proposition assuming the **Claim.** To that end, we construct the concatenated path as follows:

\[ \gamma = \bigcup_{i \in \mathbb{Z}} g^i \cdot (x, y)_{\alpha} \cap [y, gx] \]

where \([x, y]_{\alpha}\) is the subsegment of \(\alpha\) between \(x\) and \(y\).

Denote \(D = \max\{d(o, f_i o) : i = 1, 2\} - 2\epsilon > 0\) and \(\sigma = 9\epsilon + B\). We now verify that \(\gamma\) is a periodic \((g, D, \sigma)\)-admissible path with associated contracting subsets \(\hat{A}(\gamma) = \{g^i N_{\epsilon}(b) : i \in \mathbb{Z}\}\) of bounded intersection.
Since \( \mathcal{K} := \{gAx(f_1), gAx(f_2) : g \in G \} \) has bounded intersection and \( d(x, y) \geq D \), according to Definition 2.8 of admissible paths, it thus remains to verify the following condition (BP):

\[
[y, gx] \text{ has } \sigma\text{-bounded projection to } N_\epsilon(b) \text{ and } N_\epsilon(gb).
\]

We only prove it for \( N_\epsilon(b) \); the case for \( N_\epsilon(gb) \) is symmetric.

Note that \( y \) is the exit point of \( \alpha \) in \( N_\epsilon(b) \) and since \( \epsilon > C \), we have \( [y, go] \cap N_C(b) = \emptyset \). The contracting property then implies \( d_b([y, go]) \leq C \). By [58, Lemma 6.1], we obtain that \( d_b(g(o, x)_\alpha) \leq \text{diam}(N_\epsilon(b) \cap g(o, x)_\alpha) + 4C \). By (10) from the Claim,

\[
d_b([y, gx]) \leq d_b([y, go]) + d_b(g(o, x)_\alpha) \leq 5C + B + 4\epsilon \leq \sigma.
\]

Hence, \( \gamma \) is a periodic \((g, D, \sigma)\)-admissible path. Choose \( d(o, f, o) \gg 0 \) for \( i = 1, 2 \) such that the constant \( D > D_0 \) satisfies Proposition 2.9. By Lemma 3.5, there exists \( R > 0 \) such that

\[
|\tau[g] - d(x, y) - d(y, gx)| \leq R.
\]

With (9), the estimates on stable length of \( g \) follows and the proposition is proved.

![Figure 2](https://example.com/image2.png)

**Figure 2.** Capping the angles by dotted geodesics gives the periodic admissible path \( \gamma \).

Therefore, it remains to prove the above Claim.

**Proof of the Claim.** We shall first prove (9) for any \( b \in \{b_1, b_2\} \). Denote by \( x, y \) the corresponding entry and exit points of the geodesic \( \alpha \) in \( N_\epsilon(b) \). Since \( b = tAx(f) \) for some \( f \in \{f_1, f_2\} \) and \( t \in G \), there exists \( 0 \neq k \in \mathbb{Z} \) such that \( d(y, tf^k o) \leq \epsilon \).

Let us look at the triangle \( \Delta(y, gx, go) \). The minimality of \( g \) in \([g]\) implies the second inequality:

\[
d(y, gy) \geq d(tf^k o, gt f^k o) - 2\epsilon \geq d(o, go) - 2\epsilon.
\]

Similarly, we have \( d(x, gx) \geq d(o, go) - 2\epsilon \).

Note that \( d(o, go) = d(o, x) + d(x, y) + d(y, go) \) and \( d(y, gy) \leq d(y, gx) + d(gx, gy) \).

From (13), we infer that

\[
d(y, go) + d(go, gx) \leq d(y, gx) + 2\epsilon
\]

Together \( d(o, go) = d(o, x) + d(x, y) + d(y, go) \), we obtain (9) for any \( b \in \{b_1, b_2\} \).

We are next going to prove (10) only; (11) is similar. By way of contradiction, assume that for any \( b \in \{b_1, b_2\} \) the following holds:

\[
\text{diam}(N_\epsilon(b) \cap g(o, x)_\alpha) > B + 6\epsilon
\]

By assumption, \( \alpha = [o, go] \) contains at least two distinct barriers. Up to exchanging notations, we choose \( b \neq \tilde{b} \in \{b_1, b_2\} \) such that \( \tilde{b} \) is on the right side of
shall need the following estimates to arrive at a contradiction: 

\[ d(y, go) \leq 4\epsilon. \]

Indeed, let \( w \) be the entry point of \([go, gx]\) in \( N_\epsilon(b) \). Since \( \epsilon > C \) is assumed, the contracting property implies \( d(y, w) \leq d_b([y, go]_\alpha) + d_b([go, w]_{ga}) \leq 2C \). Hence, for \( w \in [go, gx] \), we have

\[ d(y, gx) \leq d(w, gx) + d(y, w) \leq d(w, gx) + 2C \leq d(go, gx) + 2C. \]

By (14), we then obtain that

\[ d(y, go) \leq d(y, gx) - d(go, gx) + 2\epsilon \leq 2C + 2\epsilon \leq 4\epsilon. \]

The inequality in (16) is then proved.

Recall that \( b \) is chosen on the right side of \( b \). By \( B \)-bounded intersection of \( N_\epsilon(b) \) and \( N_\epsilon(b) \), we deduce from (15) that the corresponding exit points of \( \alpha \) in \( N_\epsilon(b) \) and \( N_\epsilon(b) \) have distance strictly greater than \( 4\epsilon \). Hence, \( d(y, go) > 4\epsilon \) contradicts (16).

Therefore, there exists \( b \in \{b_1, b_2\} \) such that (10) and (11) are true. The Claim is proved.

The proof of the proposition is complete.

We now record the main consequence of Proposition 3.6.

Corollary 3.7. There exist a number \( B > 0 \) and an exponentially generic set \( G \) of elements such that for each \( g \in G \), \( 0 \leq \ell_\alpha[g] - \tau[g] \leq B \). Moreover, for any \( g^k = g \) with \( k \in \mathbb{Z} \), we have \( |\tau[g_0] - \ell_\alpha[g_0]| \leq B \).

Proof. Only the exponential genericity of elements in Proposition 3.6 needs to be verified, which follows from Theorem 3.1.

Remark. Theorem 3.1 is only invoked here in this section, and the further results (Lemma 3.8, Theorem 3.10 below) about genericity do not rely on it.

3.4. Growth tightness of fractional barrier-free elements. The goal of the remaining subsections (independent of the proof of Main Theorem) is Theorem 3.10 which shall be used in Theorem 1.11 about generic Teichmüller geodesics.

Let \( \overrightarrow{\theta}(n) = (\theta_1(n), \theta_2(n)) : \mathbb{R}_{\geq 0} \to [0, 1] \times [0, 1] \) be a function such that \( \theta_1(n) \leq \theta_2(n) \). A \( \overrightarrow{\theta} \)-interval of a geodesic segment \( \alpha \), denoted by \( \alpha_{\overrightarrow{\theta}} \), is the closed subsegment so that the initial endpoint has a distance \( \theta_1(n)n \) to \( \alpha_- \) and the terminal endpoint has a distance \( \theta_2(n)n \) to \( \alpha_+ \) where \( n := \ell(\alpha) \). In most cases, we choose a constant function \( \overrightarrow{\theta}(n) = (\theta_1, \theta_2) \) for \( 0 \leq \theta_1 < \theta_2 \leq 1 \).

Lemma 3.8. Assume that the action \( G \acts Y \) is SCC. Let \( \overrightarrow{\theta}(n) := (\theta_1(n), \theta_2(n)) : \mathbb{R}_{\geq 0} \to [0, 1]^2 \) be a function such that \( \theta(n) = \theta_2(n) - \theta_1(n) \geq n^{-a} \) for some \( a \in (0, 1) \).

Let \( f \) be a contracting element. Then the set of elements \( g \in G \) for which the interval \([a, go]_{\overrightarrow{\theta}} \) does not contain an \((\epsilon, f)\)-barrier is negligible.

Moreover, when \( \liminf_{n \to 1} \theta(n) > 0 \), the above set is exponentially negligible.

Remark. The SCC assumption on actions is crucial to obtain the growth tightness in the “moreover” statement.

In its proof, we frequently use the following criterion for a set to be negligible.
Lemma 3.9. Assume that the action $G \act Y$ has purely exponential growth. Let $D > 0$ and $\theta : \mathbb{R}_{\geq 0} \to (0, 1]$ such that $\theta(n) \geq n^{-a}$ for some $1 > a > 0$. For a growth tight subset $Z \subset G$, the set of elements $g \in G$ satisfying the following two properties:

1. $g = g_1 g_2 g_3$ can be written as a product of three elements such that $|d(o, go) - \sum_{i=1,2,3} d(o, g_i o)| \leq D$ and
2. one of the three $g_i$’s belongs to $Z$ and has length bigger than $\theta(d(o, go)) \cdot d(o, go)$.

is negligible. Moreover, when $\liminf_{n \geq 1} \theta(n) > 0$, the above set is exponentially negligible.

Proof. Let us first consider the set of elements $g = g_1 g_2 g_3$, where $g_2$ satisfies the second property. Denote $n := d(o, go)$ and $\theta_n := \theta(n) \geq n^{-a}$ for some $a \in (0, 1)$. The number of these elements is upper bounded by

$$\sum_{0 \leq k+l \leq n+D \atop n \geq 1 \geq n \theta_n} \sharp N(o, k) \cdot \sharp N(o, l) \cap Z \cdot \sharp N(o, n + D - k - l).$$

(17)

Since the action has PEG, we have

$$\sharp N(o, i) \asymp \exp(\omega(G) i)$$

for $i \geq 0$. Since the set $Z$ is growth tight, there exists $0 < \omega_1 < \omega(G)$ such that

$$\sharp N(o, l) \cap Z \asymp \exp(\omega_1 l).$$

For $\epsilon := \omega(G) - \omega_1 > 0$, each summand in (17) takes proportion of $N(o, n)$ at most $\ll \exp(-\epsilon \cdot n \theta_n) \ll \exp(-\epsilon n^{1-a})$. Since there are at most $n^2$ summands and $n^2 \exp(-\epsilon n^{1-a}) \to 0$, we obtain that the set of elements $g$ with this property is negligible. When $g_1$ or $g_3$ satisfies the second property, an even simpler proof shows that the corresponding sets are negligible as well.

If $\theta_n$ is uniformly away from 0, then the above computation shows that the set under consideration is exponentially negligible. Thus the result is proved. \qed

We are ready to prove Lemma 3.8.

Proof of Lemma 3.8. Denote $\beta = \alpha \overline{\partial}$ the $\overline{\partial}$-interval of $\alpha := [o, go]$, and for simplicity, $\theta_2 = \theta_2(n), \theta_1 = \theta_1(n)$ where $n = d(o, go)$. Let $g \in G$ be an element so that $\beta$ does not contain an $(\epsilon, f)$-barrier. The proof proceeds by showing that $g$ belongs to a finite union of negligible sets.

First of all, we can assume that the entire geodesic $\alpha$ contains an $(\epsilon, f)$-barrier. Otherwise, $g$ belongs to the barrier-free set $\mathcal{V}_{\epsilon, f}$, which is exponentially negligible by Theorem 2.11.

Let $t \in G$ be any $(\epsilon, f)$-barrier such that $d(to, o), d(tfo, o) \leq \epsilon$. Let $x, y \in \alpha$ such that $d(to, x), d(tfo, y) \leq \epsilon$.

Claim. The segment $[x, y]_\alpha$ intersects $\beta$ in a diameter at most $\ell(\beta)/3$.

Proof of the Claim. Note

$$d(o, go) + 4 \epsilon \geq d(o, to) + d(to, tfo) + d(tfo, go).$$

Consider the growth tight set $Z := E(f)$, $D := 4 \epsilon$ and $\theta := (\theta_2 - \theta_1)/3$. If $\ell([x, y]_\alpha \cap \beta) \geq \ell(\beta)/3$, then the element $g = t \cdot f \cdot (t f)^{-1} g$ can be written as a product of three elements satisfying the properties of Lemma 3.9. Hence, the set of
these elements $g$ is negligible. So we can assume that the subpath $[x, y]_\alpha$ intersects $\beta$ in a diameter at most $\ell(\beta)/3$. \hfill \Box

By the Claim, $[x, y]_\alpha$ does not contain the middle point of $\beta$, so any barrier of $\alpha$ stays either on the left side or on the right side of $\beta$.

If one side, say the left side, of $\beta$ does not contain an $(\epsilon, f)$-barrier, then the right side of $\beta$ must contain at least one $(\epsilon, f)$-barrier since $\alpha$ contains one. Consider the left-most $(\epsilon, f)$-barrier $t_2 \in G$ with the entry point $w$ of $\alpha$ in $N_\epsilon(t_2 \text{Ax}(f))$. This implies that $[o, w]_\alpha$ is $(\epsilon, f)$-barrier-free and $\ell([o, w]_\alpha) \geq 2/3 \cdot \ell(\beta) \geq 2n\theta$. Recalling that $\text{Ax}(f) = E(f) o$, we choose an element $t_2 \in t_2 E(f)$ such that $d(w, t_2 o) \leq \epsilon$. We then write $g = t_2 \cdot (t_2^{-1}g)$ as a product of two elements with $t_2$ being $(\epsilon, f)$-barrier-free. Apply Lemma 3.9 with the growth right $Z := \mathcal{V}_{\epsilon, f}$, $D = 2\epsilon$ and $2\theta$. We thus see that, in this case, $g$ belongs to a negligible set as well.

We now turn to the case that each side of $\beta$ contains an $(\epsilon, f)$-barrier. Let $t_1 \in G$ be the right-most barrier on the left side of $\beta$. Thus, if we denote by $z \in \alpha$ the exit point in $N_\epsilon(t_1 \cdot \text{Ax}(f))$, we have $d(z, t_1 o) \leq \epsilon$ for some $t_1 \in t_1 E(f)$. Similarly, for the left-most barrier $t_2$ on the right side of $\beta$, we have $d(w, t_2 o) \leq \epsilon$ for the entry point $w \in \alpha$ in $N_\epsilon(t_2 \cdot \text{Ax}(f))$ and for some $t_2 \in t_2 E(f)$.

Recall that each barrier intersects $\beta$ in a segment of length less than $\ell(\beta)/3$, and $\beta$ contains no barrier by assumption. We conclude that $[z, w]_\alpha$ is $(\epsilon, f)$-barrier-free, and therefore so is the element $t_1^{-1}t_2$. Noting that $\ell([z, w]_\alpha) \geq \ell(\beta)/3 \geq n\theta$ and writing

$$g = t_1 \cdot (t_1^{-1}t_2) \cdot (t_2^{-1}t_1 g)$$

as a product of three elements, the set of such elements $g$ is negligible by using Lemma 3.9 again.

In summary, assuming that $\beta$ does not contain an $(\epsilon, f)$-barrier, we obtain that the set of these elements $g$ satisfying (1) and (2) is negligible.

When $\liminf_{n \to \infty} \theta(n) > 0$, Lemma 3.9 shows the above sets are exponentially negligible. Hence, the “moreover” statement is proved along the same lines. \hfill \Box

3.5. Generic elements with linear stable length. In this subsection, we prove that generic contracting elements have their stable length (at most) sub-linearly close to the radius and their axis sublinearly near to the basepoint.

Similar to that of Proposition 3.6, the proof consists in constructing periodic admissible paths but which is technically much simpler. We emphasize that the arguments here do not use Theorem 3.1, so this gives a new proof of the main result of [58] that exponentially generic elements are contracting.

Theorem 3.10. Assume that $G$ admits a SCC action with a contracting element on a proper geodesic metric space $(Y, d)$. There exist constants $\epsilon = \epsilon(f), R = R(f) > 0$ for any contracting element $f \in G$ such that the following holds.

Let $\theta_i : \mathbb{R}_{\geq 0} \to (0, 1]$ be such that $\theta_i(n) \geq n^{-a}$ and $a \in (0, 1)$ where $i = 1, 2$. Then for any integer $m > 0$, the set of elements $g$ with $n = d(o, go)$ satisfying the following properties is generic:

1. $n \geq \tau|g| \geq (1 - \theta_1(n))n$,
2. $d(o, \mathcal{A}_f(g)) \leq n\theta_2(n)$,
3. any bi-infinite geodesic in a finite neighborhood of $\text{Ax}(g)$ contains an $(\epsilon, f^m)$-barrier.
Moreover, if \( \lim \inf_{n \geq 1} \theta_i(n) > 0 \) for \( i = 1, 2 \), then the above set is exponentially generic.

By definition of a stable axis, we could replace \( \mathcal{A}_R(g) \) in the above property (2) with any bi-infinite geodesic in a finite neighborhood of \( Ax(f) \).

**Proof.** We shall assume \( \theta_1(\cdot) = \theta_2(\cdot) \) for ease of exposition. The general case follows by taking the intersection of two (exponentially) generic sets.

Fix any \( \theta : \mathbb{R}_{\geq 0} \to (0, 1/2] \) such that \( \theta(n) \geq n^{-a} \) for some \( a \in (0, 1) \). For given \( m > 0 \), we shall choose a big integer \( k = k(m) > 0 \) determined below (at the last paragraph). Let \( \mathcal{G} \) be the set of elements \( g \in G \) satisfying the following two conditions:

1. the \([\frac{\theta}{2}, \frac{\theta}{2}]\)-interval of \( \alpha := [o, go] \) contains an \((\epsilon, f^k)\)-barrier \( t_1 \in G \),
2. the \([1 - \frac{\theta}{2}, 1 - \frac{\epsilon}{4}]\)-interval of \( \alpha \) contains an \((\epsilon, f^k)\)-barrier \( t_2 \in G \),

where \( \theta_n := \theta(n) \) and \( n = d(o, go) \).

By Lemma 3.8 the set \( \mathcal{G} \) is generic, and when \( \lim \inf_{n \geq 1} \theta(n) > 0 \), this set is exponentially generic.

Denote \( b_1 = t_1 Ax(f) \) and \( b_2 = t_2 Ax(f) \). Up to ignoring a growth tight set, we can first assume that \( b_2 \cap NC(go) = \emptyset \) and \( b_1 \cap NC(g^{-1}o) = \emptyset \). Indeed, we shall prove the set of elements \( g \in G \) satisfying \( b_2 \cap NC(go) \neq \emptyset \) is growth tight; the other possibility is analogous.

![Figure 3](image-url) **Figure 3.** The relative position of the barriers \( g^{-1}b_2, b_1, b_2 \) and \( gb_1 \).

Let \( z \) be the entry point of \( \alpha \) in \( NC(g^{-1}b_2) \), so \( gz \) is the entry point of \( go \) in \( NC(b_2) \). Let \( w \) be the exit point of \( \alpha \) in \( NC(b_2) \). If two geodesics issuing from the same point intersect the \( C \)-neighborhood of a \( C \)-contracting set, we can deduce from the contracting property that their entry points are bounded above by a distance \( 4C \). Thus, \( d(gz, w) \leq 4C \).

Let us choose \( h \in G \) such that \( d(ho, z) \leq C \). Then \( |d(gho, ho) - d(z, w)| \leq d(gho, w) + d(z, ho) \leq C + d(gho, gz) + d(gz, w) \leq 6C \). Setting \( \hat{g} := h^{-1}gh \), we see that the element \( g = \hat{g}h^{-1} \) is an almost geodesic decomposition:

\[
(18) \quad d(o, go) \simeq_{24C} 2d(o, ho) + d(o, \hat{g}o).
\]

By the condition (1) as above, we see that \( d(o, ho) \geq d(o, z) - d(z, ho) \geq \theta_n/4 \cdot d(o, go) - C \). By Lemma 3.9 we deduce from (18) that the set of \( g \) with \( b_2 \cap NC(go) \neq \emptyset \) is growth tight.

From now on, let us assume that \( b_2 \cap NC(go) = \emptyset \) and \( b_1 \cap NC(g^{-1}o) = \emptyset \). We shall construct a periodic admissible path for the element \( g \).
Let \( x, y \) be the entry and exit points of \( \alpha \) in \( N_C(b_1) \) respectively. We now prove that a path \( \gamma \) constructed as follows is a \((g, D, C)\)-admissible path

\[
\gamma = \bigcup_{i \in \mathbb{Z}} g^i([x, y]_\alpha \cdot [y, gx]),
\]

where \( g^i[x, y]_\alpha \) are associated with contracting sets \( g^i N_C(b_1) \) with bounded intersection. By the definition of admissible path, it remains to show that \([y, gx] \) has a bounded \( C\)-projection to \( N_C(b_1) \) and \( N_C(g b_1) \). Since \([y, gx] \) is disjoint from them, the projection bounded by \( C \) follows by the contracting property. Thus, \( \gamma \) is a periodic \((g, D, C)\)-admissible path.

We now verify the first two assertions. Let \( R = R(f, C) \) be the constant given by Lemma \[3.5\]. By the construction of \( \gamma \), the collection \( \mathcal{X} \) of contracting sets \( g^i N_C(b_1) \) for \( i \in \mathbb{Z} \) is a combinatorial axis of \( g \). By Lemma \[3.5\], the stable axis \( \mathcal{A}_R(g) \) is given by the union of mutual projections of elements in \( \mathcal{X} \). By Proposition \[2.9\], the point \( x \in \gamma \) is \( \epsilon \)-close to some point in \( \mathcal{A}_R(g) \) and thus

\[
d(o, \mathcal{A}_R(g)) \leq d(o, x) + \epsilon \leq n\theta_n + \epsilon.
\]

From \(0\) in the proof of Lemma \[3.5\], we get

\[
\tau[g] \geq d(x, gx) - 3R \geq d(x, go) - d(o, x) \geq (1 - \theta_n)n - 3R.
\]

At last, we need to verify the assertion (3) that every geodesic \( \beta \) in \( \mathbb{A}_x(g) \) contains an \((\epsilon, f^m)\)-barrier for given \( m > 0 \). By Proposition \[2.9\], we obtain that \( d(x, \alpha), d(y, \alpha) \leq \epsilon \). Since \( x, y \) are the entry and exit points of \( \alpha \) in the \((\epsilon, f^k)\)-barrier \( b_1 \), we have \( d(x, y) \geq d(o, f^k o) - 2\epsilon \). Consider \( \mathbb{A}_x(f) \) as a \( C \)-contracting quasi-geodesic path. By Proposition \[2.4\](1), we can choose \( k \) as a large multiple of \( m \) such that there exist a constant \( \bar{\epsilon} = \epsilon(C) > 0 \) and a subpath from \( \mathbb{A}_x(f) \) labeled by \( f^m \) two endpoints in the \( \bar{\epsilon} \)-neighborhood of \([x, y]_\alpha \). By definition of barriers, \( \beta \) contains an \((\bar{\epsilon}, f^m)\)-barrier. Setting \( \epsilon = \max\{\bar{\epsilon}, \epsilon\} \), the assertion (3) follows. \( \square \)

4. Lower bound on conjugacy classes

Most of this section assumes only that \( G \ltimes \mathbb{Y} \) is a properly discontinuous action on a proper geodesic metric space with a contracting element. At the end of the section, we derive the lower bound in \textbf{Main Theorem} under the additional assumption action has purely exponential growth.

We first recall a result from \[57\] which holds for any proper action. This could be thought of as an analogue to an orbit closing lemma in Anosov flows. For any \( \Delta > 0 \), we define \( A(o, n, \Delta) = \{g \in G : |d(o, go) - n| \leq \Delta \} \).

**Lemma 4.1.** \[57\] Lemma 2.19 \( \) There exist a set \( F \) of three contracting elements and constants \( 0 < \theta < 1, \sigma, D, \Delta > 0 \) with the following property. For each \( n > 0 \), there exist a subset \( T \) of \( A(o, n, \Delta) \) and an element \( f \in F \) such that

1. \( zT \geq \theta \cdot zA(o, n, \Delta) \),
2. for all but finitely many \( f \in E(f) \), the map

\[
g \in T \mapsto fg \in f \cdot T
\]

is injective.
3. each \( fg \in fT \) is a contracting element so that the path

\[
\gamma(fg) := \bigcup_{i \in \mathbb{Z}} (fg)^i[0, f o][fo, f go]
\]

is a periodic \((f, D, \sigma)\)-admissible path.
(4) for any \( g \neq g' \in T \), two paths \( \gamma(fg) \) and \( \gamma(fg') \) have infinite Hausdorff distance.

**Sketch of proof.** By [57, Lemma 2.19], the assertions (2-4) hold for any \( R \)-separated set \( T \) of \( (o, n, \Delta) \) for a sufficiently large \( R > 0 \). The proof was to verify that the labeled path \( \gamma(fg) \) by \( fg \) as above is \((D, \sigma)\)-admissible, i.e. a periodic \((fg, D, \sigma)\)-admissible path in the terminology of this paper.

For \( g \neq g' \in T \), by Proposition 2.9 the sufficient \( R \)-separation implies that if \( \gamma(fg) \) and \( \gamma(fg') \) has finite Hausdorff distance, then \( g = g' \). So the injectivity of the above maps follows. Choosing a maximal net \( T \) then takes the major proportion of \( A(o, n, \Delta) \) as is requested by the assertion (1). The quantifier “for all but finitely many” thus follows for any \( f \) with \( d(o, fo) > D \) by Proposition 2.9.

Fix a choice \( f \in E(\hat{f}) \) satisfying the second statement of Lemma 4.1. Note that \( Ax(f) = Ax(\hat{f}) \) since \( E(f) = E(\hat{f}) \). Then the set \( \hat{T} := f \cdot T \) consists of contracting elements.

**Lemma 4.2.** There exists a constant \( R = R(f, \sigma, D, \Delta) > 0 \) such that for each element \( fg \in \hat{T} \), we have \(|\tau[fg] - n| \leq R \) and \(|\ell_o[fg] - n| \leq R \).

**Proof.** By Lemma 4.1(3), the element \( fg \) admits a periodic \((fg, D, \sigma)\)-admissible path \( \gamma \). Thus, the conclusion follows from Lemma 3.5.

The following result is from [17, Lemma 7.2], and a proof is given for completeness.

**Lemma 4.3.** Given a compact set \( K \subset Y \), there exists an integer \( C = C(K) > 0 \) such that for any geodesic segment \( \alpha \), we have

\[ \sharp\{g \in G : g\alpha \cap K \neq \emptyset\} \leq C \cdot \ell(\alpha). \]

**Proof.** We subdivide the geodesic \( \alpha \) into a maximal number of segments \( \alpha_i \) (0 \( \leq i \leq \lceil \ell(\alpha) \rceil \)) with length at most 1. Set \( C := \sharp\{g \in G : gK \cap N_1(K) \neq \emptyset\} \). Observe that for each \( \alpha_i \), we have

\[ \sharp\{g \in G : g\alpha_i \cap K \neq \emptyset\} \leq C. \]

Indeed, fix an element \( g_0 \) so that \( \alpha_i \cap g_0 K \neq \emptyset \). Thus for any element \( g \) from the left-hand set, we have \( g_0 K \cap N_1(K) \neq \emptyset \) and the above inequality follows. The conclusion thus follows.

Denote by \([\hat{T}]\) the set of conjugacy classes \([fg]\) where \( fg \in \hat{T} \). Recall that \( \hat{T} = fT \), where \( T \subset A(o, n, \Delta) \). We obtain the following lower bound.

**Lemma 4.4.** There exists a constant \( L = L(f, o, \Delta) > 0 \) such that each conjugacy class \([fg]\) in \([\hat{T}]\) contains at most \( Ln \) elements in \( \hat{T} \).

**Proof.** Set \( A := [fg] \cap \hat{T} \) and fix a choice \( fg \in A \). By Lemma 4.1 \( \gamma := \gamma(fg) \) is a periodic \((fg, D, \tau)\)-admissible path associated with contracting sets \((fg)^iAx(f)\) where \( i \in \mathbb{Z} \). It is clear that \( \langle fg \rangle \) acts on \( \gamma \) with a fundamental domain \( S := [o, fo][fo, fg o] \).

Similarly, for any \( fg' \in A \setminus \{fg\} \), we have a periodic \((fg', D, \tau)\)-admissible path \( \gamma' \) and \( S' := [o, fo][fo, fg' o] \) a fundamental domain for \( \langle fg' \rangle \) on \( \gamma' \).
Since \( h'g'g'h^{-1} = fg \) for some \( h' \in G \), it follows that \( h'\gamma' \) has a finite Hausdorff distance to \( \gamma \). By Proposition 2.9, there exists a constant \( \epsilon > 0 \) such that \( \gamma \) intersects the \( \epsilon \)-neighborhood of every segment \( (fg')[o,fo] \) of \( h'\gamma' \). Thus,

\[
N_{\epsilon}(S) \cap h'(fg')'[o,fo] \neq \emptyset.
\]

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5. Growth tightness of fractional barrier-free sets

In this and next sections, we assume that the action $G \curvearrowright Y$ is SCC and prepare two further ingredients to prove the upper bound on $\mathcal{C}(n)$. The first one, contained in this section, is required when the action is not cocompact. The second one in the next section will address the possibly unbounded kernel of the exact sequence $[4]$.

5.1. Growth tightness for fractional barrier-free sets. We begin with a generalization of barrier-free elements. See §3.4 for another formulation of this notion.

Similar results in Teichmüller spaces have been obtained in the work of Dowdall, Duchin and Masur [21]. However, none of these could be implied by the other, and our method uses very little information from the theory of Teichmüller spaces except that the action of the mapping class group on Teichmüller space is SCC with a contracting element.

**Definition 5.1.** Fix $\theta \in (0, 1]$, $\epsilon, M, L > 0$ and $P \subset G$. Let $g \in G$ be an element. If there exists a set $K$ of disjoint connected open subintervals $\alpha$ of length at least $L$ in $[o, go]$ such that each $\alpha$ is $(\epsilon, P)$-barrier-free with two endpoints $\partial \alpha \subset N_M(Go)$ and such that

$$\sum_{\alpha \in K} \ell(\alpha) \geq \theta d(o, go),$$

then $g$ is said to satisfy a $(\theta, L)$-fractional $(\epsilon, P)$-barrier-free property.

Denote by $V_{\theta, L, \epsilon, M, P}$ the set of elements $g \in G$ with the $(\theta, L)$-fractional $(\epsilon, P)$-barrier-free property.

We are actually interested in the set of elements of $g \in G$ so that $[o, go]$ spends $\theta$-percentage of time in $N_M(Go)$. Here, we prove a more general statement for potential future applications.

**Theorem 5.2.** Let $\epsilon, M > 0$ be given by Theorem 2.11. For any $0 < \theta \leq 1$, there exists $L = L(\theta) > 0$ such that $V_{\theta, L, \epsilon, M, P}$ is a growth tight set for any $P \subset G$.

**Proof.** Consider an element $g \in V_{\theta, L, \epsilon, M, P}$ so that $[o, go]$ cumulatively spends at least $\theta$-percentage of time in $N_M(Go)$ with each stay having length at least $L$ and each stay being $(\epsilon, P)$-barrier-free. Then the number $m = \frac{\theta}{2}K$ of stays is at most $\theta n/L$.
Let us denote the endpoints of the $i$-th interval in $\mathbb{K}$ by $x_i, y_i$ for $1 \leq i \leq m$ so that there exist elements $g_i, h_i \in G$ with the property that
\[ d(g_i o, x_i), \; d(h_i o, y_i) \leq M. \]
By definition, we have $g_i^{-1} h_i \in \mathcal{V}_{\epsilon, M, P}$. Let $\omega_1 > 0$ be the growth rate of $\mathcal{V}_{\epsilon, M, P}$, which by Theorem 2.11 is strictly less than $\omega(G)$. For some $\lambda_1 > 0$, we have
\[ \sharp \mathcal{V}_{\epsilon, M, P} \cap N(o, n) \leq \lambda_1 \exp(n\omega_1). \]
By Theorem B in [58], there exists $\lambda_2 > 0$ such that the following holds for any proper action with a contracting element:
\[ \sharp N(o, n) \leq \lambda_2 \exp(n\omega(G)) \]
for $n \geq 1$.

Keeping the positions at $x_i, y_i$ as fixed along a geodesic of length $n$, the number of elements $g$ is bounded above by
\[ \lambda^{2m} \cdot \exp(n(1 - \theta)\omega(G)) \cdot \exp(n\theta \omega_1) \]
where $\lambda > 1$ is a constant depending only on $\lambda_1$ and $M$. Indeed, the second factor comes from the product of the number of elements in balls of radius $d(g_i o, h_{i+1} o) \leq d(y_i, x_{i+1}) + 2M$, and the third factor is the product of barrier-free elements $g_i^{-1} h_i$ corresponding to $x_i, y_i$. The coefficient $\lambda^{2m}$ takes into account the constant $\lambda_1, M$ in $2m$ multiplications.

Fixing $m \leq n\theta / L$ and varying $x_i, y_i$, the number of configurations is at most $C_n^m$, bounded above by
\[ \left( \frac{e n}{m} \right)^m \leq \left( \frac{e L}{\theta} \right)^{n \theta / L}, \]
which follows from Stirling’s formula.

Let $\omega_2 = \omega(G) - \omega_1 > 0$. We deduce that $\sharp \mathcal{V}_{\epsilon, M, P}^{\theta, L}$ is bounded above by
\[ n\theta / L \cdot \left( \frac{e L}{\theta} \right)^{n \theta / L} \cdot \lambda^{2m} \cdot \exp(-n\theta \omega_2) \cdot \exp(n\omega(G)) \]
\[ \leq n\theta / L \cdot \left( \frac{e L}{\theta} \right)^{2m} \cdot \exp(-n\theta \omega_2) \cdot \exp(n\omega(G)) \]
For given $\theta$, if $L$ is large enough, the factor $(\frac{e L}{\theta} \lambda^2)^{n \theta / L}$ is little-o of $\exp(\sigma n)$ for some $\omega_2 > \sigma > 0$. Hence, the product of the first three factors is of order $\exp((\sigma - \omega_2) n)$. The growth tightness of $\mathcal{V}_{\epsilon, M, P}^{\theta, L}$ follows as desired. \hfill \Box

We now derive a corollary of Theorem 5.2 in a specific setting. Let $\theta \in (0, 1]$ and $L > 0$. Given $g \in G$, consider the set $\mathbb{K}$ of maximal connected components $\alpha$ of length at least $L$ in the intersection of $(o, g o)$ with the complement of $N_M(G o)$. Let $\mathcal{O}_{M}^{\theta, L}$ denote the set of elements $g$ with the following property:
\[ \sum_{\alpha \in \mathbb{K}} \ell(\alpha) \geq \theta d(o, g o). \]

\textbf{Corollary 5.3.} For any $\theta \in (0, 1]$ there exists $L = L(\theta) > 0$ such that $\mathcal{O}_{M}^{\theta, L}$ is a growth tight set.

\textbf{Proof.} Fix a contracting element $f \in G$. To apply Theorem 5.2 it suffices to verify that each component $\alpha$ in $\mathbb{K}$ defined as above is $(\epsilon, f^n)$-barrier-free for some $n \gg 0$. Indeed, if not, assume that $\alpha$ contains an $(\epsilon, f^n)$-barrier $t \in G$ for any large $n$. By the contracting property, we see that if $n$ is chosen so that $d(o, f^n o)$ is large enough,
then \( \alpha \) intersects a uniform \( C \)-neighborhood of the contracting set \( tAx(f) \), where \( C \) is the contraction constant of \( tAx(f) \). However, the component \( \alpha \) is disjoint from \( N_M( Go ) \) by definition, so we have a contradiction when \( M \geq C \). Hence, \( \alpha \) is \( (\epsilon,f^n) \)-barrier-free for some \( n \gg 0 \). The proof is complete. \( \square \)

5.2. Upper bound for strongly primitive conjugacy classes. As applications of previous results, we establish the desired upper bound on the number of strongly primitive conjugacy classes.

We fix a choice of constants \( \theta \in (0,1], L = L(\theta) > 0 \) so that Corollary 5.3 holds. Thus, there exists an exponentially large set of elements \( g \in G \) such that at least \( (1-\theta) \)-proportion of \([o,go]\) lies in the \((M+L/2)\)-neighbourhood of \( Go \). In other words, the cumulative stay of \([o,go]\) in \( N_{M+L/2}( Go ) \) takes up at least \((1-\theta)\)-percentage of the whole segment \([o,go]\).

**Lemma 5.4.** There exist an exponentially generic set \( G \) of elements and \( \theta,K > 0 \) with the following property. If \( g \in G \) is a strongly primitive contracting element then \([g]\) contains at least \( \theta \cdot \tau[g] \) elements in \( N(a,\tau[g] + K) \).

**Proof.** Let \( G \) be the set of elements \( g \in G \) such that \( g \) satisfies Corollary 5.3 and does not belong to \( O^\theta M(Go) \). Thus, \( G \) is exponentially generic since it is the intersection of two such sets.

Assume that \( g \) is strongly primitive and minimal in \([g]\). The next three paragraphs can be ignored, if the reader is interested only in the case of a co-compact action.

As discussed above, for \( g \notin O^\theta M(Go) \), at least \((1-\theta)\)-proportion of \([o,go]\) is contained in \( N_{M+L/2}(Go) \). We plot inductively a sequence of points with step of length at least \( 2M \) so that the total number of points is less than \((1-\theta)\)\(n/2M\).

Precisely, from left to right, choose \( x_0 = o \) to start. If the point on \([o,go]\) with an exact distance \( 2M \) from \( x_0 \) lies in \( N_{M+L/2}(Go) \), let \( x_1 \) be this point; otherwise choose \( x_1 \) to be the closest to \( x_0 \) among points in \( N_{M+L/2}(Go) \) satisfying \( d(x_1, x_0) \geq 2M \). We do this repeatedly so that \( d(x_{i+1}, x_i) \geq K \) and \( x_i \in N_{M+L/2}(Go) \), until the terminal \( go \) is within \( 2M \)-distance of \( x_{m-1} \). Finally, set \( x_m = go \).

Observe that \( m \geq (1-\theta)\)\(n/4M\). Indeed, the union of \( 2M \)-neighborhoods of \( x_i \)’s covers \([o,go]\) \( \cap N_{M+L/2}(Go) \) so the lower bound on \( m \) follows.

We have thus subdivided \([o,go]\) into segments of length at least \( 2M \). Thus, for some \( \theta' = \theta'(\theta) > 0 \), there exists \( m = \theta' n \) elements \( h_i \) such that \( d(h_i, x_i) \leq M + L/2 \) and \( d(h_i, x_{i+1}) \geq 2M \). We can then write \( g = s_1 s_2 \cdots s_m \) as a product of elements \( s_i = h_i^{-1} h_{i+1} \) for \( 1 \leq i \leq m \).

The remainder of the proof is to show that all \( m \) cyclic permutations of the word \( s_1 s_2 \cdots s_m \) give different elements \( g_i = s_{i+1} \cdots s_n s_1 \cdots s_i \) with length bounded by \( d(o, go) \leq d(o, go) + 2M \). Since the strong primitivity is a conjugacy invariant and \( g_i \) is conjugate to \( g \), we have that each \( g_i \) is strongly primitive.

First of all, we verify the upper bound:

\[
\begin{align*}
d(o, go) &\leq d(o, s_1 \cdots s_i o) + d(o, s_{i+1} \cdots s_n o) \\
&\leq d(o, x_i) + M + d(x_i, go) + M \\
&\leq d(o, go) + 2M.
\end{align*}
\]

Now it remains to prove that \( g_i \neq g_j \) for \( i \neq j \). To derive a contradiction, assume that \( i < j \) and \( g_i = g_j = h \). Denoting \( t = s_{i+1} \cdots s_j \), we have \( h t = g t = g_j = th \), so \( t \in E^+(h) \) by Corollary 2.6. The strong primitivity of \( h \) then allows to write
\[ t = h^k f, \text{ where } |k| > 1 \text{ and } f \] belongs to a finite normal subgroup \( F \) of \( E^+(h) \).

Thus, there exists some \( n > 0 \) such that \( t^n = h^{nk} \), so \( \tau[t] = |k| \tau[h] \) for \( |k| \geq 2 \).

The stable length is a conjugacy invariant, so \( \tau[h] = \tau[g] \). Since \( g \in G \) satisfies Proposition 3.6, there exists \( K_0 > 0 \) such that

\[ |\tau[g] - d(o, go)| \leq K_0 \]

yielding

\[ \tau[t] \geq |k|(d(o, go) - K_0). \]

Recalling that \( t = s_{i+1} \cdots s_j \), we obtain that

\[ \tau[t] \leq d(o, to) \leq d(h_i o, h_{j+1} o) \leq d(x_i, x_{j+1}) + 2M \]
\[ \leq d(o, go) - 2M + 2M \leq d(o, go). \]

This gives a contradiction when \( d(o, go) \) is large enough. Accordingly, all cyclic permutations \( g_i \) are distinct: there are at least \( m = \theta'n \) elements in \( N(o, d(o, go) + 2M) \), where \( \theta' \) is a uniform number. The lemma is thus proved.

We also need the following lemma.

**Lemma 5.5.** [18, Lemma 3.2] Let \( \omega > 0 \). Then there exists \( C > 0 \) such that

\[ \sum_{0 \leq i \leq n} \exp(i\omega) \cdot \frac{\exp(i\omega)}{i} \geq C \exp(n\omega) \frac{\exp(n\omega)}{n}. \]

Denote by \( C''(n) \) the set of conjugacy classes of strongly primitive contracting elements of stable length at most \( n \).

**Corollary 5.6.** Assume the action \( G \acts Y \) is SCC. Then

\[ \sharp C''(n) \cap C(o, n) \prec \frac{\exp(\omega(G)n)}{n}. \]

**Proof.** Define

\[ \mathcal{A}''(i, \Delta) = \{ [g] \in C(o, n) \cap C''(n) : |\tau[g] - i| \leq \Delta \} \]

for \( 0 \leq i \leq n, \Delta \geq 0 \). By Lemma 5.4, each class \([g] \in \mathcal{A}''(i, \Delta)\) of strongly primitive contracting elements contains at least \( \theta(i + \Delta) \) elements in \( N(o, i + K) \) for some uniform \( K > 0 \). Since \( \sharp N(o, i) \approx \exp(\omega(G)i) \), we obtain \( \sharp \mathcal{A}''(i, \Delta) \prec \frac{\exp(\omega(G)i)}{i} \).

The proof is completed by Lemma 5.5. \hfill \Box

### 6. Uniform Contracting Elements

We now discuss the second ingredient to deal with torsion elements in the kernel of the following exact sequence

\[ 1 \to F \to E^+(g) \to Z \to 1 \]

The main technical result of independent interest is that an exponentially large set of contracting elements have a uniform bound on the kernel \( F \). This shall be proved using Bestvina-Bromberg-Fujiwara’s construction of projection complex.

We point out that the results in this section is not necessary, if the action has no torsion elements or there exists a uniform bound on the size of finite subgroups. The latter condition is satisfied in large varieties of groups: hyperbolic groups, CAT(0) groups and mapping class groups, to point out a few. However, there exist examples of relatively hyperbolic groups which have finite subgroups of unbounded cardinality.
6.1. **Projection complex.** Let $\mathcal{X}$ be a collection of uniformly contracting sets with bounded intersection. By [52], $\mathcal{X}$ satisfies the projection complex axioms introduced in [7, Section 3.1]. We recall some of their results that will be useful for us.

For a constant $K > 0$, the projection complex $\mathcal{P}_K(\mathcal{X})$ is a graph obtained in the following way. They introduced a function $\tilde{d}_W(X_1, X_2)$ for every $X_2 \neq W \neq X_1 \in \mathcal{X}$, which is variant of $d_W(X_1, X_2) = \text{diam}(\pi_W(X_1 \cup X_2))$ so that

\[
\tilde{d}_W(X_1, X_2) \simeq_\theta \tilde{d}_W(X_1, X_2)
\]

where $\theta > 0$ depends only on $\mathcal{X}$. The vertex set consists of all elements in $\mathcal{X}$, and two distinct vertices $X_1, X_2$ are connected if and only if $\tilde{d}_W(X_1, X_2) < K$ for every $W \in \mathcal{X}$. Introduce the interval-like set

\[
\mathcal{X}_K(V, W) = \{ X \in \mathcal{X} : \tilde{d}_X(V, W) \geq K \}.
\]

Thus, two vertices $X_1, X_2$ are adjacent if and only if $\mathcal{X}_K(X_1, X_2) = \emptyset$. The basic result in [7] is that for a large $K > 0$, the projection complex $\mathcal{P}_K(\mathcal{X})$ is a quasi-tree of infinite diameter on which $G$ acts co-boundly.

It is proved in [7, Prop 3.7] that the interval $\mathcal{X}_K(V, W)$ gives a path in the projection complex between $V$ and $W$ which is not necessarily geodesic. However, if raising $K$ to a large amount, we have the following result.

**Lemma 6.1.** [7, Lemma 3.18] *For a sufficiently large $K'$ relative to $K$, all the elements in $\mathcal{X}_{K'}(V, W)$ indeed appear in any geodesic between $V, W$.***

The class of acylindrical actions has received great interest in recent years. See Osin [45] and references therein, for a survey of a rapidly growing body of studies of acylindrical hyperbolic groups.

**Definition 6.2.** An action of $G$ on a geodesic metric space $Y$ is acylindrical if for any $D > 0$ there exist $R, N > 0$ such that

\[
\tilde{z}(g \in G : d(x, gx) \leq D, d(y, gy) \leq D, d(x, y) > R) \leq N
\]

for any given $x, y \in Y$.

The action of $G$ on the projection complex in [7] was recently shown to be acylindrical for a variety of interesting examples, including a proper action with contracting elements.

**Theorem 6.3.** [8, Theorem 5.10] *There exists $K > 0$ such that $G$ acts acylindrically on the projection complex $\mathcal{P}_K(\mathcal{X})$ that is a quasi-tree.*

**Proof.** This is stated in [8, Theorem 5.10], as a consequence of [8, Theorem 3.9] applied in our setting. However, no explicit proof is given there to verify the following condition: for some fixed $N$ and $B$, for any $N$ distinct elements of any $\mathcal{X}_K(V, W)$ the common stabilizer is a finite subgroup of size at most $B$.

Indeed, any fixed integer $N \geq 2$ suffices. If $g$ lies in the stabilizer of a set of $N$ elements, say containing distinct $V = t_1A\mathbf{x}(f), W = t_2A\mathbf{x}(f) \in \mathcal{X}$, then a power $g^k$ fixes $V$ and $W$ for some $k = k(N)$. By definition of $A\mathbf{x}(f) = E(f) \cdot o$ and $E(f) \{2\}$, we have that $E(f)$ is the stabilizer of $A\mathbf{x}(f)$ and so $g^k \in t_1E(f)t_1^{-1} \cap t_2E(f)t_2^{-1}$. It suffices to see that $tE(f)t^{-1} \cap E(f)$ contains at most $B$ elements for any $t \in G \setminus E(f)$. Since the action is proper, there are only finitely many $tE(f)$ such that the intersection is non-trivial. For each $tE(f) \neq E(f)$, the intersection $tE(f)t^{-1} \cap E(f)$ is finite. So we obtained a uniform number $B_0$ bounding on $\sharp tE(f)t^{-1} \cap E(f)$ for any $t \notin E(f)$.

Thus, $B = kB_0$ is the desired constant in the above condition. \qed
Denote by $d_P$ the induced length metric on the graph $\mathcal{P}_K(\mathcal{X})$. The following criterion for an isometry to be loxodromic was obtained in [7, Lemma 3.22].

**Lemma 6.4.** Let $K'$ be the constant given in Lemma 6.1. If there exists $N > 0$ and $X \in \mathcal{X}$ such that $d_P(g^{-N}X, g^{N}X) > K'$, then $g$ acts as a loxodromic isometry on $\mathcal{P}_K(\mathcal{X})$.

6.2. **Generic elements act loxodromically on projection complex.** In this subsection, we fix two independent contracting elements $f_1, f_2$. Then the set $\mathcal{X} = \{gAx(f_i) : g \in G, i = 1, 2\}$ is a contracting system with bounded projection, for which we construct the projection complex $\mathcal{P}_K(\mathcal{X})$ with $K$ satisfying Theorem 6.3.

We are now ready to state the main result of this section.

**Lemma 6.5.** There exists an exponentially generic set $\mathcal{G}$ of elements which act by loxodromic isometries on the projection complex $\mathcal{P}_K(\mathcal{X})$.

**Proof.** Fix a large integer $m > 0$. Let $\mathcal{G}$ be the set of elements $g$ such that the minimal representative in $[g]$ contains at least one $(\epsilon, f_1^{m})$-barrier and one $(\epsilon, f_2^{m})$-barrier. By Theorem 3.1, this is an exponentially generic set.

We shall apply Lemma 6.4 to show that $g \in \mathcal{G}$ is a loxodromic isometry on $\mathcal{P}_K(\mathcal{X})$. Since loxodromic elements are preserved under conjugacy, we may assume that $g$ is a minimal element in $[g]$. Denote $\alpha = [o, go]$. In the proof of Proposition 3.6, it is proved that there exists a periodic $(g, D, \sigma)$-admissible path

$$\gamma = \bigcup_{i \in \mathbb{Z}} g^i ([x, y]_\alpha \cdot [y, gx])$$

defined in [12], where $x, y$ are the entry and exit points of $\alpha$ in $N_\epsilon (b)$, and $b$ is either an $(\epsilon, f_1^{m})$-barrier or an $(\epsilon, f_2^{m})$-barrier. The associated combinatorial axis is $\mathcal{A}(\gamma) = \{g^i N_\epsilon (b) : i \in \mathbb{Z}\}$ and let us denote $X = N_\epsilon (b) \in \mathcal{P}_K(\mathcal{X})$.

Consider the geodesic $[g^{-N}X, g^{N}X]$ in $\mathcal{P}_K(\mathcal{X})$. We choose $m$ big enough so that

$$D > \max \{d(o, f_i o) : i = 1, 2\} - 2\epsilon \gg K'$$

where the constant $K'$ is given by Lemma 6.1. By Proposition 2.9, we can derive that for each $gX$ with $|i| < N$, we have $d_{\gamma}g^{-N}X, g^{N}X) > K'$. By Lemma 6.1, we have that $g^{-N}X, g^{N}X) > K'$. This implies that the length of the geodesic $[g^{-N}X, g^{N}X]$ grows linearly as $N \to \infty$. Thus, the assumption of Lemma 6.4 is fulfilled, so $g$ has positive stable length in $\mathcal{P}_K(\mathcal{X})$. \hfill \Box

The following consequence will be useful in the next section.

**Lemma 6.6.** There exists an integer $N > 0$ such that for any $g \in \mathcal{G}$ (in Lemma 6.5), the following exact sequence

$$1 \to F \to E^+(g) \to \mathbb{Z} \to 1$$

where $\sharp F < N$.

**Remark.** The lemma is sharp in the sense that there is no uniform bound on the kernel for a certain sequence of hyperbolic (contracting) elements in Dunwoody’s inaccessible groups [1]. Note that, Dunwoody’s groups are infinitely-ended and thus relatively hyperbolic, so the prime conjugacy growth formula holds by Corollary 1.3.

**Proof.** The proof is due to [15, Lemma 6.8] and is short, so we include it for completeness. Pick any point $x \in Y$. Since $Y$ is a hyperbolic space, the diameter of the coarse center of $F \cdot x$ is uniformly bounded by a constant $D > 0$. Without loss of generality, assume that $x$ lies in the center so $d(x, fx) \leq D$ for any $f \in F$. 


Choose $h \in E^+$ such that $d(x, hx) \leq R$. Since $h^{-1}fh \in F$ and $d(fh, hx) = d(h^{-1}fh, x) \leq D$, the definition of acylindrical hyperbolicity implies that the cardinality of $F$ is bounded above by $N$. □

7. Primitive conjugacy classes are generic: end of the proof of Main Theorem

To derive the upper bound for all conjugacy classes, we will show that the non-primitive ones are exponentially negligible.

Lemma 7.1. Assume that the action is SCC. Then

$$\frac{\sharp(C(o,n) \setminus C^\prime(o,n))}{\sharp C(o,n)} \to 0,$$

exponentially quick as $n \to \infty$.

Proof. By Lemma 3.2, we can assume that all elements in $C(o,n)$ are contracting. By Lemma 6.6, assume further that for each $[g] \in C(o,n)$, there exists a uniform number $N > 0$ with the following exact sequence

$$1 \to F \to E^+(g) \xrightarrow{\phi} Z \to 1$$

where $\sharp F < N$. By Corollary 3.7, we can assume that

$$|\tau[g] - \ell_o[g]| \leq B$$

for a uniform constant $B > 0$.

By Lemma 2.7, a non-primitive contracting element is not strongly primitive. Thus, it suffices to bound elements which are not strongly primitive. Let $C''(o,n)$ be the set of conjugacy classes of strongly primitive elements with algebraic length at most $n$.

Let $[g] \in C(o,n) \setminus C''(o,n)$ be a non-strongly primitive element, so by definition, there is a strongly primitive element $g_0 \in E^+(g)$ so that $g = g_0^m$ for some $|m| \geq 2$. Note that there are at most $\sharp F$ choices of $g_0$, where $\sharp F < N$.

By Corollary 3.7,

$$\ell_o[g_0] \leq B + \frac{\tau[g]}{|m|} \leq B + n/|m|.$$

We define a map as follows

$$\Pi : C(o,n) \setminus C''(o,n) \to C''(o,n + B)$$

by $\Pi([g]) := [g_0] \in C''(o, \frac{n}{|m|} + B)$.

For $|m| \geq 2$, we obtain a constant $0 < \omega_1 < \omega(G)$ such that

$$\sum_{|m| \geq 2} \sharp C''(o, \frac{n}{|m|} + B) \leq \sum_{|m| \geq 2} \sharp N(o, \frac{n}{|m|} + B) \leq \exp(\omega_1 n).$$

Recall that for each $n \geq |m| \geq 2$, there are at most $N$ conjugacy classes $[g]$ such that $\Pi([g]) = [g_0]$ with $\phi(g) = m$. Hence,

$$\sharp C(o,n) \setminus C''(o,n) \leq N \exp(\omega(G)n).$$

Since $C''(o,n) \subset C'(o,n)$ and $\sharp C(o,n) > \frac{\exp(\omega(G)n)}{n}$ by Corollary 4.5, the conclusion follows. □

Combined with Corollary 5.6, we obtain the upper bound for all conjugacy classes.
Corollary 7.2. Assume that the action is SCC. Then
\[ \sharp(C(n) \cap C(o,n)) \leq \frac{\exp(\omega(G)n)}{n}. \]

Lemma 7.3. Assume that the action is SCC. Then
\[ \frac{\sharp(C(o,n) \setminus C'(n))}{\sharp(C(n) \cap C(o,n))} \to 0 \]
expONENTIALLY QUICK AS \( n \to \infty \).

Proof. Denote by \( C''(n) \) the set of conjugacy classes of strongly primitive elements with stable length at most \( n \). Noting that \( C''(o,n) \subset C''(n) \), the proof of Lemma 7.1 shows
\[ \frac{\sharp(C(o,n) \setminus C''(n))}{\sharp(C(n) \cap C(o,n))} \to 0 \]
exponentially quick. By Corollaries 4.5 and 7.2, we have \( \sharp(C(n) \cap C(o,n)) \succ \sharp C(o,n) \). We then obtain that \( C(o,n) \setminus C''(n) \) is exponentially small relative to \( C(n) \cap C(o,n) \). Thus, the conclusion follows.

By Corollary 4.5, we obtain the lower bound for primitive conjugacy classes.

Corollary 7.4. Assume that the action is SCC. Then
\[ \sharp C'(o,n) \geq \frac{\exp(\omega(G)n)}{n}. \]

Therefore, all assertions in Main Theorem are proved.

8. Applications

8.1. Application to Teichmüller space. See e.g. [43, 54] for background on Teichmüller theory.

Let \( S \) be a closed oriented surface of genus at least 2. The Teichmüller space \( \text{Teich}(S) \) is the space of marked conformal structures on \( S \). The Teichmüller metric is given by
\[ d(x,y) = \frac{1}{2} \inf_{f} \log K(f) \]
where \( K(f) \) denotes the quasiconformal constant and \( f \) varies over a given homotopy class. This makes \( \text{Teich}(S) \) into a proper unique geodesic metric space, and in fact a Finsler manifold. The unit (co)tangent bundle of \( \text{Teich}(S) \) may be identified with the space \( Q(S) \) of unit area holomorphic quadratic differentials. By slight abuse of notation, we consider a Teichmüller geodesic as a subset of both \( Q^1(S) \) and \( \text{Teich}(S) \).

The principal stratum \( Q_{top} \) consists of those quadratic differentials all of whose zeros are simple. The mapping class group \( \text{MCG}(S) \) acts on \( \text{Teich}(S) \) by isometries. An element \( f \in \text{MCG}(S) \) is contracting for this action if and only if it is pseudo-Anosov. In this case, it preserves an invariant Teichmüller geodesic: the axis \( \text{ax}(f) \) of \( f \). The stable length \( \tau(f) \) of \( f \) is the displacement of \( f \) along this axis.

Gadre and Maher prove the following [27, Proposition 2.7].

Proposition 8.1. Let \( f \in \text{MCG}(S) \) be a pseudo-Anosov mapping class such that \( \text{ax}(f) \) lies in the principal stratum. For each \( K > 0 \) there is an \( L = L(S, f) > 0 \) such that whenever \( \gamma \) is a bi-infinite Teichmüller geodesic with uniquely ergodic
vertical and horizontal measured foliations and containing points \(X_1, X_2\) in Teichmüller space with \(d(X_1, X_2) > L\) and \(d(X_i, ax(f)) < K\) for \(i = 1, 2\) it follows that \(\gamma\) also lies in the principal stratum.

We now prove Theorem \[\ref{thm:main} \] from the introduction.

**Theorem 8.2.** (Theorem \[\ref{thm:main} \]) Let \(o \in \text{Teich}(S)\) and \(\theta_1, \theta_2 \in (0, 1)\). The subset of \(g \in \text{MCG}(S)\) of pseudo-Anosov elements satisfying the following properties is exponentially generic with respect to the Teichmüller metric:

1. \(d(o, go) \geq \tau(g) > \theta_1 d(o, go)\)
2. \(d(o, ax(g)) < (1 - \theta_2) d(o, go)\)
3. \(ax(g)\) lies in the principal stratum

**Proof.** This essentially follows from Theorem \[\ref{thm:main} \] and Proposition \[\ref{prop:3} \]. The only point that requires clarification is that the set of elements satisfying (3) is exponentially generic. Indeed, assume without loss of generality that \(o \in ax(f)\). Theorem \[\ref{thm:main} \] (3) implies that for each \(n > 0\) the set \(A(f, n, K)\) of pseudo-Anosov \(g\) such that \(ax(g)\) contains an \((K, f^n)\)-barrier is exponentially generic. This means for each \(g \in A(f, n, K)\) there is an \(h \in \text{MCG}\) with \(d(o, h^{-1}ax(g)) < K\) and \(d(f^n o, h^{-1}ax(g)) < K\). Note, \(h^{-1}ax(g) = ax(h^{-1}gh)\). Thus \(ax(h^{-1}gh)\) contains \(X_1, X_2\) with \(d(X_1, o) < K\), \(d(X_1, f^n o) < K\). Hence, \(d(X_1, ax(f)) < K\) and \(d(X_1, X_2) > d(o, f^n o) - 2K = n\tau(f) - 2K\). For large enough \(n\) we have \(n\tau(f) - 2K\) is greater than the constant \(L = L(S, f)\) from Proposition \[\ref{prop:3} \] so that \(ax(h^{-1}gh)\) lies in the principal stratum. Since the principal stratum is \(\tilde{\text{MCG}}(S)\) invariant it follows that \(ax(g)\) lies in the principal stratum.

\[\Box\]

8.2. **New examples with prime conjugacy growth formulae.** In this subsection, we give some detail about examples appearing in Corollaries \[\ref{cor:prime} \] and Proposition \[\ref{prop:3} \].

We first consider the class of cubical groups. A group \(G\) is cubical if it admits a geometric action on a \(\text{CAT}(0)\) cube complex \(Y\). One the one hand, when \(Y\) is endowed with \(\text{CAT}(0)\) metric, an isometry is called rank-1 a geometric action on a \(\text{CAT}(0)\) cube complex \(Y\). One the one hand, it is very useful to study the cubical geometry when the one-skeleton of \(Y\) is equipped with the combinatorial metric. This is an \(\ell^1\)-metric, in contrast with \(\ell^2\)-metric induced from \(\text{CAT}(0)\) metric. The following result is certainly known to experts \[\ref{lem:prime} \].

**Lemma 8.3.** Let \(G \acts Y\) be a cubical group such that \(Y\) does not factor as a product of unbounded cube subcomplexes. Then \(G\) contains a contracting element with respect to the action on one-skeleton of \(Y\) with the \(\ell^1\)-metric. Moreover, every rank-1 element preserves an \(\ell^1\)-geodesic by translation.

**Proof.** Two disjoint hyperplanes \(H_1, H_2\) are called \(k\)-separated for \(k \geq 0\) if there are at most \(k\) hyperplanes intersecting both \(H_1\) and \(H_2\). An element \(g\) skewers \(H_1, H_2\) if it pushes one halfspace bounding by \(H_1\) into one bounding by \(H_1\). By \[\ref{thm:prime} \] Theorem 6.3], there exists a contracting isometry \(g\) in \(\ell^2\)-metric so that it skewers 0-separated hyperplanes. Such a hyperplane skewing involves no metrics at all and thus implies that \(g\) is contracting with respect to \(\ell^1\)-metric by \[\ref{cor:prime} \] Theorem 3.13]. The conclusion follows.

In fact, a contracting element in \(\ell^2\)-metric is exactly contracting in \(\ell^1\)-metric. It is easy to see that an element \(g\) is a rank-1 element in \(\ell^2\)-metric iff it skewers
a pair of \( k \)-separated hyperplanes for some \( k \geq 0 \). The direction “\( \leq \)” is given by [13, Lemma 6.2], and the other direction follows from [15, Lemma 4.6]. Then the proof is concluded by [29, Theorem 3.13] that the same hyperplane relation characterizes the contracting property in \( \ell^1 \)-metric. The “moreover” statement is proved by Haglund in [34], since a contracting element acts without inversions on cube complex \( Y \) by Lemma 2.5.

\[\square\]

**Right-angled Artin (Coxeter) groups.** The class of right-angled Artin groups (RAAGs) is presented by

\[ G = \langle V(\Gamma) | v_1v_2 = v_2v_1 \Leftrightarrow (v_1, v_2) \in E(\Gamma) \rangle \]

for a finite simplicial graph \( \Gamma \). See [14] and [40] for references on RAAGs. A RAAG acts properly and cocompactly on a CAT(0) cube complex called the Salvetti complex. In [5, Theorem 5.2], it is proved that if \( G \) is not a direct product, then it contains a rank-1 element.

The class of right-angled Coxeter groups (RACGs) is defined as in (21) with additional relations \( v^2 = 1 \) for each \( v \in V(\Gamma) \). A RACG also acts properly and cocompactly on a CAT(0) cube complex called the Davis complex. From [6, Proposition 2.11] and [15, Theorem 2.14], a RACG \( G \) is not virtually a direct product of groups if and only if it contains a rank-1 element.

Using the co-compact action, it is clear that if an isometry preserves a geodesic by translation, then the stable length coincides with algebraic length up to a uniform error. Together the assertions (1) (2) in Corollary 1.2 follow by Lemma 8.3.

**Relatively hyperbolic groups.** In a relatively hyperbolic group, hyperbolic elements are contracting with respect to the action on the Cayley graph, cf. [31], [30]. To count conjugacy classes of hyperbolic elements in stable length, we need the following fact.

**Lemma 8.4.** There exists a uniform constant \( B > 0 \) such that for every hyperbolic element \( h \), we have \( |\tau[h] - \ell_o[h]| \leq B \).

**Sketch of the proof.** We refer to [55] for related notions and [22, Proposition 7.8] for a similar argument. Let \( h_-, h_+ \) be the fixed points of \( h \) in the Bowditch boundary. By [55, Lemma 2.20], there exists a sequence of \((\varepsilon, R)\)-transitional points along any bi-infinite geodesic \( \gamma \) between \( h_-, h_+ \), where \( \varepsilon, R \) are uniform depending only on \( (G, P) \). Moreover, if pickup any such transitional point \( x \), then \( \langle h \rangle x \) is uniformly close to \( \gamma \). This implies that \( |\tau[h] - \ell_o[h]| \) is uniformly bounded by an argument in Lemma 3.4.

\[\square\]

By the prime conjugacy formula for hyperbolic elements with stable length follows and Corollary 1.3 is proved.

**SCC covers of moduli spaces.** We will now explain the conjugacy formula for certain cover of moduli spaces associated to subgroups constructed in [57, Proposition 6.6]. For the convenience of the reader, we briefly explain the construction.

The subgroup \( H < MCG(S) \) is generated by a free abelian group \( A \) generated by Dehn twists and a cyclic group generated by a pseudo-Anosov element \( p \). In [57, Proposition 6.6], it is proved that \( H \) is a free product \( A \ast \langle p \rangle \). In order to prove this, an admissible path in Teichmüller space is constructed for each word in the free product \( (A \ast \langle p \rangle) \setminus (A \cup \langle p \rangle) \). The admissible path travels alternatively along a sufficiently long segment of the (translates of) axis of \( p \). This description shows
that the word represents a pseudo-Anosov element in $H$, with the axis uniformly close to a translation axis of $p$. In other words, each closed geodesic on the cover $Teich(S)/H$ of moduli space intersects a finite neighborhood of the closed geodesic associated to $p$.

Since each pseudo-Anosov element in $H$ admits an axis uniformly close to the basepoint $o$, it follows that the stable length is within uniform additive error from the algebraic length. Thus, Corollary 1.5 now follows from the second item of Main Theorem. The same proof also gives the general theorem 1.6.

8.3. SCC actions with infinitely many closed geodesics of bounded length.

We now give a geometrically infinite hyperbolic 3-manifold with infinitely many closed geodesics of bounded length and with a SCC deck transformation group action. This is essentially due to Peigné in [46], where geometrically infinite examples with finite Bowen-Margulis-Sullivan measure are constructed as the Schottky product of two Kleinian groups. For the convenience of the reader, we give some details about the construction.

Let $H$ be an infinitely generated discrete group acting on the hyperbolic plane $H^2$ such that $\omega(H) \leq 1$ and $H^2/H$ contains infinitely many closed geodesics with bounded length. Such subgroups can be chosen as the fundamental group of an infinite regular cover of a compact hyperbolic surface.

Via the Poincaré extension, the group $H$ acts on $H^3$ with the limit set $\Lambda_H$ contained in a circle. We now take a discrete group $G$ of divergent type acting on $H^3$ with $\omega(G) > 1$ and the limit set $\Lambda_G$ disjoint from $\Lambda_H$. Note that $H$ (resp. $G$) acts properly discontinuously outside $\Lambda_H$ (resp. $\Lambda_G$). By taking sufficiently deep finite index subgroups, we can have the following property: any $h \in H$ maps $\Lambda_G$ into a small neighborhood of $\Lambda_H$ and the same for $g \in G$. Thus, $H$ and $G$ stay at a ping-pong position so they generate a free product $\Gamma = G * H$.

The (infinitely generated) $\Gamma$ acts on $H^3$ with infinitely many closed geodesics of bounded length on $H^3/G$. Since $G$ is of divergent type, we have $\omega(\Gamma) > \omega(G)$. Thus, by [57, Proposition 6.3], $\Gamma$ acts by a SCC action on $H^3$.

The above construction applies in higher dimension, but we do not know of a finitely generated example with these properties. By Ahlfors measure theorem, finitely generated examples cannot exist in dimension 3. (cf. [47, Introduction]).

Moreover, such examples could be produced in the acylindrical actions on (possibly non-proper) hyperbolic spaces.

At last, we would like to show that such phenomenon exists in the class of groups acting acylindrically on hyperbolic spaces (termed acylindrically hyperbolic groups by Osin [45]). This is pointed out by the referee that Watanabe [53, Theorem 1.3] constructed infinitely many pseudo-Anosov mapping classes with same stable length on curve graphs (since the stable length spectrum is discrete by Bowditch [10]).

Lemma 8.5. Let $G$ be a non-elementary group acting acylindrically on a hyperbolic space $X$ so that some infinite subgroup $H$ has bounded orbits. Then there exists infinitely conjugacy classes of loxodromic elements in $G$ with bounded stable length.

Proof. Fix a basepoint $x \in X$ so that $Hx$ has diameter at most $D > 0$. Let $R > 0$ be the constant in Definition 6.2. Let $f$ be a loxodromic element and $E(f)$ be the maximal elementary subgroup of $G$ containing $f$. Then the quasi-geodesic $\gamma := E(f)x$ gives the quasi-axis of $f$. The following is the key claim. There are
infinitely many distinct $E(f)$-cosets $g_n \cdot E(f)$ over $g_n \in H$ so that $g_n \gamma, g_m \gamma$ fellow travel within $D$-distance at most $R$ for any $g_n \neq g_m$. Indeed, if not, we choose a point $y \in g_n \gamma$ such that $d(x, y) > R$. The $R$-long fellow travel of $g_n \gamma$ and $g_1 \gamma$ implies that $d(g_n x, y), d(g_m y, y) \leq D$ holds for infinitely many $g_n$. This contradicts to the definition of acylindrical action.

Since $h_n \gamma$ and $\gamma$ diverge after the time $R$, it is easy exercise in hyperbolic geometry that there exists a big power $f$ of $f$ (relative to $R$) such that $h_n := \tilde{f} g_n$ are loxodromic elements for $n \geq 1$. Similar to periodic admissible paths, the quasi-axis denoted by $\gamma_n$ of $h_n$ has the fundamental domain $[o, f o][f o, f g_n o]$, where $[o, g_n o]$ has bounded length by $D$. Hence, all $h_n$ have uniformly bounded stable length.

It remains to see that $\{h_n\}$ are contained in infinitely many distinct conjugacy classes. In fact, we shall show that no two $h_n \neq h_m$ lie in the same conjugacy class. Indeed, if $h' h_n h'^{-1} = h_m$ for some $h' \in G$, consider the configuration of the axis $\gamma_m$ and $h' \gamma_n$ as in Figure 4 (with $g$ replaced with $g_n$, $f$ with $\tilde{f}$, etc). Since $d(x, y) \leq D$, we conclude that $h'$ must send subsegments $[x, \tilde{f} x]$ and $\tilde{f} g_n [x, \tilde{f} x]$ of $\gamma_n$ into a uniform neighborhood of $[x, \tilde{f} x]$ and $\tilde{f} g_m [x, \tilde{f} x]$ in $\gamma_m$. It is known that if the action is acylindrical, the set of translated quasi-axis $\{g' : g \in G\}$ has bounded intersection. This implies $h' \gamma = \gamma$ and $h' \tilde{f} g_n \gamma = \tilde{f} g_m \gamma$. Since $\tilde{f} \in E(f)$ and $\gamma = E(f) x$, we see $h' \in E(f)$ and thus $g_n E(f) = g_m E(f)$. This contradicts to the choice of $g_n$ in different $E(f)$-cosets. The proof is complete. \hfill \square

8.4. Transcendental growth series. Theorem 1.7 is a consequence of the prime conjugacy growth formula.

Define $A(n, \Delta) = \{ [g] \in C : |\ell_o[g] - n| \leq \Delta \}$ for $n, \Delta \geq 0$. Each conjugacy class $[g]$ is contained in a uniform number of annuli sets $A(n, \Delta)$ where $n > 0$. Hence, for fixed $\Delta > 1$, 

$$P(z) = \sum_{[g] \in G} z^{\ell_o[g]} \asymp_n \sum_{n \geq 0} z A(n, \Delta) z^n.$$ 

By Main Theorem the prime conjugacy growth formula holds for the annulus $z A(n, \Delta) \asymp_\Delta \exp(n \omega(G))/n$.

The proof of Theorem 1.7 follows immediately from the asymptotics of the coefficients of an algebraic growth series in [20, Theorem D]. See the proof of [2, Theorem 1.1] for relevant details.

REFERENCES

1. C. Abbott, Not all finitely generated groups have universal acylindrical actions, Proc. Amer. Math. Soc. 144 (2016), 4151–4155.
2. Y. Antolín and L. Ciobanu, Formal conjugacy growth in acylindrically hyperbolic groups, Int Math Res Notices 1 (2017), 121–157, arXiv:1508.06229.
3. G. Arzhantseva, C. Cashen, and J. Tao, Growth tight actions, Pacific Journal of Mathematics 278 (2015), 1–49.
4. Goulhara N. Arzhantseva, Christopher H. Cashen, Dominik Gruber, and David Hume, Negative curvature in graphical small cancellation groups, Groups Geom. Dyn. 13 (2019), no. 2, 579–632. MR 3950644
5. J. Behrstock and R. Charney, Divergence and quasimorphisms of right-angled artin groups, Mathematische Annalen 352 (2012), 339–356.
6. J. Behrstock, M. Hagen, and A. Sisto, Thickness, relative hyperbolicity, and randomness in coxeter groups, To appear in Algebr. Geom. Topol.
7. M. Bestvina, K. Bromberg, and K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publications mathématiques de l’IHÉS **122** (2015), no. 1, 1–64, arXiv:1006.1939.

8. M. Bestvina, K. Bromberg, and A. Sisto, *Acylindrical actions on projection complexes*, Enseign. Math. **65** (2019), no. 1-2, 1–32. MR 4057354

9. M. Bestvina and K. Fujiwara, *A characterization of higher rank symmetric spaces via bounded cohomology*, Geometric and Functional Analysis **19** (2009), no. 1, 11–40, arXiv:math/0702274.

10. B. Bowditch, *Tight geodesics in the curve complex*, Invent. Math. **171** (2008), no. 2, 281–300. MR 2367021

11. E. Breuillard, Y. de Cornulier, A. Lubotzky, and C. Meiri, *On conjugacy growth of linear groups*, Math. Proc. Cambridge Philos. Soc. **154** (2013), no. 2, 261–277.

12. J. Cannon, *The combinatorial structure of cocompact discrete hyperbolic groups*, Geom. Ded. (1984), no. 2, 123–148.

13. P. Caprace and M. Sageev, *Rank rigidity for CAT(0) cube complexes*, Geom. Funct. Anal. **21** (2011), 851–891.

14. R. Charney, *An introduction to right-angled artin groups*, Geometriae Dedicata **125** (2007), 141–158.

15. R. Charney and H. Sultan, *Contracting boundaries of CAT(0) spaces*, J Topology **8** (2015), no. 1, 93–117.

16. G. R. Conner, *Discreteness properties of translation numbers in solvable groups*, J. Group Theory **3** (2000), no. 1, 77–94.

17. M. Coornaert and G. Knieper, *Growth of conjugacy classes in gromov hyperbolic groups*, GAFA **12** (2002), 464–478.

18. F. Dahmani, V. Guirardel, and D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. **245** (2017), no. 1156, v+152. MR 3589159

19. K. Delp, D. Hoffoss, and J. Manning, *Problems in groups, geometry and 3-manifolds*, arXiv:1512.04620, 2015.

20. S. Dowdall, M. Duchin, and H. Masur, *Statistical hyperbolicity in Teichmüller space*, Geometric and Functional Analysis **24** (2014), no. 3, 748–795 (English).

21. M. Dussaule and I. Gekhtman, *Entropy and drift for word metric on relatively hyperbolic groups*, arXiv:1811.10849, 2018.

22. A. Eskin and M. Mirzakhani, *Counting closed geodesics in moduli space*, J. Mod. Dyn. **5** (2011), no. 1, 71–105.

23. Alex Eskin, Maryam Mirzakhani, and Kasra Rafi, *Counting closed geodesics in strata*, Invent. Math. **215** (2019), no. 2, 555–607. MR 3910070

24. V. Gadre and J. Maher, *The stratum of random mapping classes*, arxiv:1607.01281.

25. I. Gekhtman, *Dynamics of convex cocompact subgroups of mapping class groups*, Geometry & Topology **15** (2011), 599–616.

26. V. Gerasimov and L. Potyagailo, *Quasi-isometries and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. **15** (2013), 2115 – 2137.

27. A. Genevois, *Contracting isometries of cat(0) cube complexes and acylindrical hyperbolicity of diagram groups*, arXiv:1610.07791.

28. V. Gerasimov and L. Potyagailo, *Quasi-isometries and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. **15** (2013), 2115 – 2137.

29. V. Guaschi and J. Lafont, *On the conjugacy growth functions of groups*, Illinois J. Math. **54** (2010), 301–313.

30. D. Guaschi and T. Januszkiewicz, *Isometries of CAT(0) cube complexes are semi-simple*, 2007, preprint.

31. J. Y. Huang, *private communications*, 2016.
37. M. Hull and D. Osin, *Conjugacy growth of finitely generated groups*, Advances in Mathematics 235 (2013), no. 1, 361–389.
38. Ilya Kapovich, *Small cancellation groups and translation numbers*, Trans. Amer. Math. Soc. 349 (1997), no. 5, 1851–1875.
39. G. Knieper, *On the asymptotic geometry of nonpositively curved manifolds*, GAFA 7 (1997), 755–782.
40. T. Koberda, *Right-angled artin groups and their subgroups*, Course notes at Yale University.
41. L. Ciobanu, S. Hermiller, D. F. Holt, and S. Rees, *Conjugacy languages in groups*, Israel Journal of Mathematics 211 (2016), 311–347.
42. G. Margulis, *On some aspects of the theory of anosov flows*, Ph.D. thesis, 1970. 2003. Springer.
43. H. Masur, *Geometry of teichmüller space with the teichmüller metric*, http://math.uchicago.edu/~masur/chapter3.pdf.
44. Y. Minsky, *Quasi-projections in Teichmüller space*, J. Reine Angew. Math. 473 (1996), 121–136.
45. D. Osin, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016), no. 2, 851–888.
46. M. Peigné, *On the Patterson-Sullivan measure of some discrete groups of isometries*, Israel Journal of math. 133 (2003), no. 1, 77 – 88.
47. ______, *On some exotic Schottky groups*, Discrete and Continuous Dynamical Systems 118 (2011), no. 31, 559 – 579.
48. K. Rafi, *private communications*, 2022.
49. I. Rivin, *Some properties of the conjugacy class growth function*, Contemp. Math. 360 (2004), Amer. Math. Soc., Providence, RI.
50. ______, *Growth in free groups (and other stories) - twelve years later*, Illinois J. Math. 54 (2010), Amer. Math. Soc., Providence, RI.
51. T. Roblin, *Équidistribution et équidistribution en courbure négative*, no. 95, Mémoires de la SMF, 2003.
52. A. Sisto, *Contracting elements and random walks*, J. Reine Angew. Math. 742 (2018), 79–114. MR 3849623
53. Y. Watanabe, *Pseudo–anosov mapping classes from pure mapping classes*, Trans. Amer. Math. Soc. 373 (2020), no. 1, 419–434.
54. A. Wright, *Translation surfaces and their orbit closures: an introduction for a broad audience*, EMS Surv. Math. Sci. (2015), no. 1, 63–108.
55. W. Yang, *Patterson-Sullivan measures and growth of relatively hyperbolic groups*, Preprint, arXiv:1308.6326, accepted in Peking Mathematical Journal, 2013.
56. ______, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc 157 (2014), 297 – 319.
57. Wen-yuan Yang, *Statistically convex-cocompact actions of groups with contracting elements*, Int. Math. Res. Not. IMRN (2019), no. 23, 7259–7323. MR 409013.
58. ______, *Genericity of contracting elements in groups*, Math. Ann. 376 (2020), no. 3-4, 823–861. MR 4081104

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