Hyperbolic Unit Groups and Quaternion Algebras

S. O. Juriaans, I. B. S. Passi*, A. C. Souza Filho

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Abstract

We classify the quadratic extensions $K = \mathbb{Q}[\sqrt{d}]$ and the finite groups $G$ for which the group ring $\mathfrak{o}_{K}[G]$ of $G$ over the ring $\mathfrak{o}_{K}$ of integers of $K$ has the property that the group $\mathcal{U}_1(\mathfrak{o}_K[G])$ of units of augmentation 1 is hyperbolic. We also construct units in the $\mathbb{Z}$-order $\mathcal{H}(\mathfrak{o}_K)$ of the quaternion algebra $\mathcal{H}(K) = \left( \frac{-1}{K}, \frac{-1}{K} \right)$, when it is a division algebra.

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1 Introduction

The finite groups $G$ for which the unit group $\mathcal{U}(\mathbb{Z}[G])$ of the integral group ring $\mathbb{Z}[G]$ is hyperbolic, in the sense of M. Gromov [8], have been characterized in [13]. The main aim of this paper is to examine the hyperbolicity of the group $\mathcal{U}_1(\mathfrak{o}_K[G])$ of units of augmentation 1 in the group ring $\mathfrak{o}_{K}[G]$ of $G$ over the ring $\mathfrak{o}_{K}$ of integers of a quadratic extension $K = \mathbb{Q}[\sqrt{d}]$ of the field $\mathbb{Q}$ of rational numbers, where $d$ is a square-free integer $\neq 1$. Our main result (Theorem 4.7) provides a complete characterization of such group rings $\mathfrak{o}_K[G]$.

In the integral case the hyperbolic unit groups are either finite, hence have zero end, or have two or infinitely many ends (see [4, Theorem 1.8.32] and [13]); in fact, in this case, the hyperbolic boundary is either empty, or consists of two points, or is a Cantor set. In particular, the hyperbolic boundary is not a (connected) manifold. However, in the case we study here, it turns out that when the unit group is hyperbolic and non-abelian it has one end, and the hyperbolic boundary is a compact manifold of constant positive curvature. (See Remark after Theorem 4.7.)

Our investigation naturally leads us to study units in the order $\mathcal{H}(\mathfrak{o}_K)$ of the standard quaternion algebra $\mathcal{H}(K) = \left( \frac{-1}{K}, \frac{-1}{K} \right)$, when this algebra is a division algebra. We construct units, here called Pell and Gauss units, using solutions of certain diophantine quadratic equations. In particular, we exhibit units of norm $-1$ in $\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}); this construction, when combined with the deep work in [5], helps to provide a set of generators for the full unit group $\mathcal{U}(\mathcal{H}(\mathfrak{o}_{\mathbb{Q}[\sqrt{-7}]}))$.

* Senior Scientist, INSA.
The work reported in this paper corresponds to the first chapter of the third author’s PhD thesis [18], where analogous questions about finite semi-groups, see [10], and RA-loops, see [14], have also been studied.

2 Preliminaries

Let $\Gamma$ be a finitely generated group with identity $e$ and $S$ a finite symmetric set of generators of $\Gamma$, $e \notin S$. Consider the Cayley graph $G = G(\Gamma, S)$ of $\Gamma$ with respect to the generating set $S$ and $d = d_S$ the corresponding metric (see [4, chap. 1.1]). The induced metric on the vertex set $\Gamma$ of $G(\Gamma, S)$ is then the word metric: for $\gamma_1, \gamma_2 \in \Gamma$, $d(\gamma_1, \gamma_2)$ equals the least non-negative integer $n$ such that $\gamma_1^{-1} \gamma_2 = s_1 s_2 \ldots s_n$, $s_i \in S$. Recall that in a metric space $(X, d)$, the Gromov product $(y \cdot z)_x$ of elements $y, z \in X$ with respect to a given element $x \in X$ is defined to be

$$(y \cdot z)_x = \frac{1}{2}(d(y, x) + d(z, x) - d(y, z)),$$

and that the metric space $X$ is said to be hyperbolic if there exists $\delta \geq 0$ such that for all $w, x, y, z \in X$,

$$(x \cdot y)_w \geq \min\{ (x \cdot z)_w, (y \cdot z)_w \} - \delta.$$

The group $\Gamma$ is said to be hyperbolic if the Cayley graph $G$ with the metric $d_S$ is a hyperbolic metric space. This is a well-defined notion which depends only on the group $\Gamma$, and is independent of the chosen generating set $S$ (see [8]).

A map $f : X \rightarrow Y$ between topological spaces is said to be proper if $f^{-1}(C) \subseteq X$ is compact whenever $C \subseteq Y$ is compact. For a metric space $X$, two proper maps (rays) $r_1, r_2 : [0, \infty] \rightarrow X$ are defined to be equivalent if, for each compact set $C \subseteq X$, there exists $n \in \mathbb{N}$ such that $r_i([n, \infty]), i = 1, 2,$ are in the same path component of $X \setminus C$. Denote by $\text{end}(r)$ the equivalence class of the ray $r$, by $\text{End}(X)$ the set of the equivalence classes $\text{end}(r)$, and by $|\text{End}(X)|$ the cardinality of the set $\text{End}(X)$. The cardinality $|\text{End}(G, d_S)|$ for the Cayley graph $(G, d_S)$ of $\Gamma$ does not depend on the generating set $S$; we thus have the notion of the number of ends of the finitely generated group $\Gamma$ (see [4], [8]).

We next recall some standard results from the theory of hyperbolic groups:

1. Let $\Gamma$ be a group. If $\Gamma$ is hyperbolic, then $\mathbb{Z}^2 \rtimes \Gamma$, where $\mathbb{Z}^2$ denotes the free Abelian group of rank 2. [4, Corollary III.Γ 3.10(2)]

2. An infinite hyperbolic group contains an element of infinite order. [4, Proposition III.Γ 2.22]

3. If $\Gamma$ is hyperbolic, then there exists $n = n(\Gamma) \in \mathbb{N}$ such that $|H| \leq n$ for every torsion subgroup $H < \Gamma$. [4, Theorem III.Γ 3.2] and [7, Chapter 8, Corollaire 36]

These results will be used freely in the sequel. In view of (1) above, the following observation is quite useful.
Lemma 2.1. Let $A$ be a unital ring whose additive group is torsion free, and let $\theta_1, \theta_2 \in A$ be two 2-nilpotent commuting elements which are $\mathbb{Z}$-linearly independent. Then $\mathcal{U}(A)$ contains a subgroup isomorphic to $\mathbb{Z}^2$.

**Proof.** Set $u = 1 + \theta_1$ and $v = 1 + \theta_2$. It is clear that $u, v \in \mathcal{U}(A)$ and both have infinite order. If $1 \neq w \in \langle u \rangle \cap \langle v \rangle$ then there exists $i, j \in \mathbb{Z} \setminus \{0\}$, such that, $u^i = w = v^j$. Since $u^i = 1 + i\theta_1$ and $v^j = 1 + j\theta_2$, it follows that $i\theta_1 - j\theta_2 = 0$ and hence $\{\theta_1, \theta_2\}$ is $\mathbb{Z}$-linearly dependent, a contradiction. Hence $\mathbb{Z}^2 \simeq \langle u, v \rangle \subseteq \mathcal{U}(A)$. \hfill $\Box$

Let $C_n$ denote the cyclic group of order $n$, $S_3$ the symmetric group of degree 3, $D_4$ the dihedral group of order 8, and $Q_{12}$ the split extension $C_3 \rtimes C_4$. Let $K$ be an algebraic number field and $\mathfrak{o}_K$ its ring of integers. The analysis of the implication for torsion subgroups $G$ of a hyperbolic unit group $\mathcal{U}(\mathbb{Z}[I])$ leading to [13, Theorem 3] is easily seen to remain valid for torsion subgroups of hyperbolic unit groups $\mathcal{U}(\mathfrak{o}_K[I])$. We thus have the following:

**Theorem 2.2.** A torsion group $G$ of a hyperbolic unit group $\mathcal{U}(\mathfrak{o}_K[I])$ is isomorphic to one of the following groups:

1. $C_5, C_8, C_{12}$, an Abelian group of exponent dividing 4 or 6;

2. a Hamiltonian 2-group;

3. $S_3, D_4, Q_{12}, C_4 \rtimes C_4$.

We denote by $\mathcal{H}(K) = (\frac{a+b}{K})$ the generalized quaternion algebra over $K$: $\mathcal{H}(K) = K[i, j : i^2 = a, j^2 = b, ji = -ij = -k]$. The set $\{1, i, j, k\}$ is a $K$-basis of $\mathcal{H}(K)$. Such an algebra is a totally definite quaternion algebra if the field $K$ is totally real and $a, b$ are totally negative. If $a, b \in \mathfrak{o}_K$, then the set $\mathcal{H}(\mathfrak{o}_K)$, consisting of the $\mathfrak{o}_K$-linear combinations of the elements $1, i, j$ and $k$, is an $\mathfrak{o}_K$-algebra. We denote by $N$ the norm map $\mathcal{H}(K) \to K$, sending $x = x_1 + x_2 + x_3 + x_4$ to $N(x) = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2$.

Let $d \neq 1$ be a square-free integer, $K = \mathbb{Q}[\sqrt{d}]$. Let us recall the basic facts about the ring of integers $\mathfrak{o}_K$ (see, for example, [11], or [16]). Set

$$\vartheta = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \pmod{4} \\ (1 + \sqrt{d})/2, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

Then $\mathfrak{o}_K = \mathbb{Z}[\vartheta]$ and the elements $1, \vartheta$ constitute a $\mathbb{Z}$-basis of $\mathfrak{o}_K$. If $d < 0$, then

$$\mathcal{U}(\mathfrak{o}_K) = \begin{cases} \{\pm 1, \pm \vartheta\}, & \text{if } d = -1, \\ \{\pm 1, \pm \vartheta, \pm \vartheta^2\}, & \text{if } d = -3, \\ \{\pm 1\}, & \text{otherwise}. \end{cases} \quad (1)$$

If $d > 0$, then there exists a unique unit $\epsilon > 1$, called the fundamental unit, such that

$$\mathcal{U}(\mathfrak{o}_K) = \pm \langle \epsilon \rangle. \quad (2)$$
We need the following:

**Proposition 2.3.** Let $K = \mathbb{Q}[\sqrt{d}]$, with $d \neq 1$ a square-free integer, be a quadratic extension of $\mathbb{Q}$, and $u \in \mathcal{U}(\mathfrak{o}_K)$. Then $u^i \equiv 1 \pmod{2}$, where

$$i = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{8}, \\ 2 & \text{if } d \equiv 2, 3 \pmod{4}, \\ 3 & \text{if } d \equiv 5 \pmod{8}. \end{cases}$$

**Proof.** The assertion follows immediately on considering the prime factorization of the ideal $2\mathfrak{o}_K$, see [3, Theorem 1, p. 236].

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3 Abelian groups with hyperbolic unit groups

**Proposition 3.1.** Let $R$ be a unitary commutative ring, $C_2 = \langle g \rangle$. Then $u = a + (1 - a)g$, $a \in R \setminus \{0, 1\}$ is a non-trivial unit in $\mathcal{U}_1(R[C_2])$ if, and only if, $2a - 1 \in \mathcal{U}(R)$.

**Proof.** Let $C_2 = \langle g \rangle$ and suppose that $u = a + (1 - a)g$, $a \in R \setminus \{0, 1\}$ is a non-trivial unit in $R[C_2]$ having augmentation 1. Let $\rho : R[C_2] \rightarrow M_2(R)$ be the regular representation. Clearly $\rho(u) = \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}$. Since $u$ is a unit, it follows that $2a - 1 = \det \rho(u) \in \mathcal{U}(R)$.

Conversely, let $a \in R \setminus \{0, 1\}$ be such that $e = 2a - 1 \in \mathcal{U}(R)$. It is then easy to see that $u = a + (1 - a)g$ is a non-trivial unit in $R[C_2]$ with inverse $v = ae^{-1} + (1 - ae^{-1})g$. [9, Proposition I]

**Proposition 3.2.** The unit group $\mathcal{U}(\mathfrak{o}_K[C_2])$ is trivial if, and only if, $K = \mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$, i.e., $d < 0$.

**Proof.** It is clear from the description (1) of the unit group of $\mathfrak{o}_K$ that the equation

$$2a - 1 = u, \ a \in \mathfrak{o}_K \setminus \{0, 1\}, \ u \in \mathcal{U}(\mathfrak{o}_K) \tag{3}$$

does not have a solution when $K = \mathbb{Q}$ or $d < 0$.

Suppose $d > 1$ and $\epsilon$ is the fundamental unit in $\mathfrak{o}_K$. In this case we have $\mathcal{U}(\mathfrak{o}_K) = \pm \langle \epsilon \rangle$. By Proposition 2.3, $\epsilon^i \in 1 + 2\mathfrak{o}_K$ for some $i \in \{1, 2, 3\}$. Consequently the equation (3) has a solution and so, by Proposition 3.1, $\mathcal{U}(\mathfrak{o}_K[C_2])$ is non-trivial.
Theorem 3.3. Let $\mathfrak{o}_K$ be the ring of integers of a real quadratic extension $K = \mathbb{Q}[\sqrt{d}]$, $d > 1$ a square-free integer, $\epsilon > 1$ the fundamental unit of $\mathfrak{o}_K$ and $C_2 = \langle \epsilon \rangle$. Then

$$U_1(\mathfrak{o}_K[C_2]) \cong \langle \epsilon \rangle \times \langle \frac{1 + \epsilon^n}{2} + \frac{1 - \epsilon^n}{2} \rangle \cong C_2 \times \mathbb{Z},$$

where $n$ is the order of $\epsilon$ mod $2\mathfrak{o}_K$.

Proof. Let $u \in U_1(\mathfrak{o}_K[C_2])$ be a non-trivial unit. Then, there exists $a \in \mathfrak{o}_K$ such that, $2a - 1 = \pm \epsilon^m$ for some non-zero integer $m$. Since $n$ is the order of $\epsilon$ mod $2\mathfrak{o}_K$, $m = nq$ with $q \in \mathbb{Z}$. We thus have $u = a + (1 - a)g$

$$= \frac{1 + \epsilon^m}{2} + \frac{1 - \epsilon^m}{2}g
$$

$$= \frac{1 + \epsilon^n}{2} + \frac{1 - \epsilon^n}{2}g = \left(\frac{1 + \epsilon^m}{2} + \frac{1 - \epsilon^m}{2}g\right)^q, \text{ or } g \left(\frac{1 + \epsilon^m}{2} + \frac{1 - \epsilon^m}{2}g\right)^q.
$$

Hence $U_1(\mathfrak{o}_K[C_2]) \cong \langle \epsilon \rangle \times \langle \frac{1 + \epsilon^n}{2} + \frac{1 - \epsilon^n}{2} \rangle \cong C_2 \times \mathbb{Z}$. \qed

As an immediate consequence of the preceding analysis, we have:

Corollary 3.4. If $K$ is a quadratic extension of $\mathbb{Q}$, then $U_1(\mathfrak{o}_K[C_2])$ is a hyperbolic group.

Corollary 3.5. Let $G$ be a non-cyclic elementary Abelian 2-group. Then $U_1(\mathfrak{o}_K[G])$ is hyperbolic if, and only if, $\mathfrak{o}_K$ is imaginary.

Proof. Suppose $\mathfrak{o}_K$ is real. Since $G$ is not cyclic, there exist $g, h \in G$, $g \neq h$, $o(g) = o(h) = 2$. By Theorem 3.3, $U_1(\mathfrak{o}_K[\langle g \rangle]) \cong C_2 \times \mathbb{Z} \cong U_1(\mathfrak{o}_K[\langle h \rangle])$. Since $\langle g \rangle \cap \langle h \rangle = \{1\}, U_1(\mathfrak{o}_K[\langle g \rangle]) \cap U_1(\mathfrak{o}_K[\langle h \rangle]) = \{1\}$. Therefore $U_1(\mathfrak{o}_K)$ contains an Abelian group of rank 2, so it is not hyperbolic. Conversely, if $\mathfrak{o}_K$ is imaginary, then, proceeding by induction on the order $|G|$ of $G$, we can conclude that $U_1(\mathfrak{o}_K[G])$ is trivial, and hence is hyperbolic. \qed

For an Abelian group $G$, we denote by $r(G)$ its torsion-free rank. In order to study the hyperbolicity of $U_1(\mathfrak{o}_K[G])$, it is enough to determine the torsion-free rank $r(U_1(\mathfrak{o}_K[G]))$. Since $U(\mathfrak{o}_K[G]) \cong U(\mathfrak{o}_K) \times U_1(\mathfrak{o}_K[G])$, we have $r(U_1(\mathfrak{o}_K[G])) = r(U(\mathfrak{o}_K[G])) - r(U(\mathfrak{o}_K))$. If $K$ is an imaginary extension, then $r(U(\mathfrak{o}_K[G])) = r(U_1(\mathfrak{o}_K[G]))$, whereas if $K$ is a real quadratic extension, then $r(U(\mathfrak{o}_K[G])) = 1$, and therefore

$$r(U_1(\mathfrak{o}_K[G])) = r(U(\mathfrak{o}_K[G])) - 1.$$

We note that

$\mathbb{Q}[C_n] \cong \bigoplus_{d|n} \mathbb{Q}[\zeta_d]$. 

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where \( \zeta_d \) is a primitive \( d^{th} \) root of unity, and therefore, for any algebraic number field \( L \),

\[
L[C_n] \cong \bigoplus_{d|n} L \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_d].
\]

We say that two groups are commensurable with each other when they contain finite index subgroups isomorphic to each other. Since the unit group \( U_{oL[C_n]} \) is commensurable with \( U(\Lambda) \), where \( \Lambda = \bigoplus_{d|n} oL \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_d] \), we essentially need to compute the torsion-free rank of \( o_K \otimes_{\mathbb{Q}} [\zeta_d] \) for the needed cases.

**Proposition 3.6.** Let \( K = \mathbb{Q}[\sqrt{d}] \), with \( d \) a square-free integer \( \neq 1 \). The table below shows the torsion-free rank of the groups \( U_1(o_K[C_n]) \), \( n \in \{2, 3, 4, 5, 6, 8\} \).

| \( n \) | \( r(U_1(o_K[C_n])) \) | \( n \) | \( r(U_1(o_K[C_n])) \) |
|-----|-----------------|-----|-----------------|
| 2   | \( 0 \) if \( d < 0 \) | 3   | 1 if \( d < 0 \), \( d \neq -3 \) |
|     | 1 if \( d > 1 \) |     | 0 if \( d = -3 \) |
| 4   | 1 if \( d < -1 \) | 5   | 6 if \( d < 0 \) |
|     | 0 if \( d = -1 \) |     | 2 if \( d = 5 \) |
|     | 2 if \( d > 1 \) | 6   | 6 if \( d \in \mathbb{Z}^+ \setminus \{1, 5\} \) |
| 6   | 2 if \( d < -3 \) | 8   | 4 if \( d < -1 \) |
|     | 0 if \( d = -3 \) |     | 1 if \( d = -1 \) |
|     | 3 if \( d > 1 \) |     | 4 if \( d = 2 \) |

In all the cases, the computation is elementary and we omit the details.

**Theorem 3.7.** If \( K = \mathbb{Q}[\sqrt{d}] \), with \( d \) a square-free integer \( \neq 1 \), then

1. \( U_1(o_K[C_3]) \) is hyperbolic;
2. \( U_1(o_K[C_4]) \) is hyperbolic if, and only if, \( d < 0 \);
3. for an Abelian group \( G \) of exponent dividing \( n > 2 \), the group \( U_1(o_K[G]) \) is hyperbolic if, and only if, \( n = 4 \) and \( d = -1 \), or \( n = 6 \) and \( d = -3 \);
4. \( U_1(o_K[C_8]) \) is hyperbolic if, and only if, \( d = -1 \);
5. \( U_1(o_K[C_5]) \) is not hyperbolic.
Proof. The Proposition 3.6 gives us the torsion-free rank

\[ r := r(U_1(\mathfrak{o}_K[C_n])) \]

for \( n \in \{2, 3, 4, 5, 8\} \). The group \( U_1(\mathfrak{o}_K[C_n]) \) is hyperbolic if, and only if, \( r \in \{0, 1\} \). Thus, it only remains to consider the case (3).

Suppose \( n = 6 \) and \( U_1(\mathfrak{o}_K[G]) \) is hyperbolic. We, hence, have \( r \in \{0, 1\} \). If \( G \) is cyclic, then, by Proposition 3.6 we have \( d = -3 \). If \( G \) is not cyclic, then \( G \cong C_2^l \times C_3^m \), \( l, m \geq 1 \). Since \( \mathfrak{o}_K[C_3] \hookrightarrow \mathfrak{o}_K[G] \), it follows that \( d = -3 \).

Conversely, if \( n = 6 \) and \( d = -3 \) then, proceeding by induction on \( |G| \), it can be proved that \( U_1(\mathfrak{o}_K[G]) \) is hyperbolic.

The case \( n = 4 \) can be handled similarly. \( \square \)

Proposition 3.8. If \( K = \mathbb{Q}[\sqrt{d}] \), with \( d \) square-free integer \( \neq 1 \), then \( U_1(\mathfrak{o}_K[C_{12}]) \) is not hyperbolic.

Proof. Since \( K[C_{12}] \cong K \otimes_{\mathbb{Q}} [\mathbb{Q}[C_{12}]] \cong K \otimes_{\mathbb{Q}} \mathbb{Q}[C_3 \times C_4] \cong K[C_3 \times C_4] \), we have the immersions \( \mathfrak{o}_K[C_3] \hookrightarrow \mathfrak{o}_K[C_{12}] \) and \( \mathfrak{o}_K[C_4] \hookrightarrow \mathfrak{o}_K[C_{12}] \). Therefore, \( r(U_1(\mathfrak{o}_K[C_{12}]))) \geq r(U_1(\mathfrak{o}_K[C_3])) + r(U_1(\mathfrak{o}_K[C_4])) \).

Suppose \( U_1(\mathfrak{o}_K[C_{12}]) \) is hyperbolic. Then, since \( r(U_1(\mathfrak{o}_K[C_{12}])) < 2 \), we have, by the Proposition 3.6 \( d \in \{-3, -1\} \). We also have

\[ K[C_3 \times C_4] \cong (K[C_3])[C_4] \cong (K \oplus K[\sqrt{-3}])[C_4] \cong K[C_4] \oplus (K[\sqrt{-3}])[C_4] \cong 2K \oplus K[\sqrt{-3}] \oplus 2K[\sqrt{-3}] \oplus K[\sqrt{-3} + \sqrt{-1}] \]

Set \( L = \mathbb{Q}[\sqrt{-3} + \sqrt{-1}] \) and suppose \( d = -3 \). Then \( \mathfrak{o}_K[C_{12}] \hookrightarrow 4\mathfrak{o}_K \oplus 2\mathfrak{o}_L \) and \( r(U(\mathfrak{o}_L))) = 1 \). Thus \( r(U(\mathfrak{o}_K[C_{12}])) = 2 \), and we have a contradiction.

Analogously, for \( d = -1 \), \( \mathfrak{o}_K[C_{12}] \hookrightarrow 3\mathfrak{o}_K \oplus 3\mathfrak{o}_L \) and so \( r(U(\mathfrak{o}_K[C_{12}])) = 3 \). Since the extensions are non-real, we have that \( r(U_1(\mathfrak{o}_K[C_{12}])) = r(U(\mathfrak{o}_K[C_{12}])) \geq 2 \), and, hence, we again have a contradiction.

We conclude that \( U_1(\mathfrak{o}_K[C_{12}]) \) is not hyperbolic. \( \square \)

4 Non-Abelian groups with hyperbolic unit groups

Theorem 2.2 classifies the finite non-Abelian groups \( G \) for which the unit group \( U_1(\mathbb{Z}[G]) \) is hyperbolic. These groups are: \( S_3, D_4, Q_{12}, C_4 \times C_4 \), and the Hamiltonian 2-group, where \( Q_{12} = C_3 \times C_4 \), with \( C_4 \) acting non-trivially on \( C_3 \), and also on \( C_4 \) (see [13]).
E. Jespers, in [12], classified the finite groups $G$ which have a normal non-Abelian free complement in $U(\mathbb{Z}[G])$. The group algebra $\mathbb{Q}[G]$ of these groups has at most one matrix Wedderburn component which must be isomorphic to $M_2(\mathbb{Q})$.

**Lemma 4.1.** Let $G$ be a group and $K$ a quadratic extension. If $M_2(K)$ is a Wedderburn component of $K[G]$ then $\mathbb{Z}^2 \hookrightarrow U_1(o_K[G])$. In particular, $U_1(o_K[G])$ is not hyperbolic.

**Proof.** The ring $\Gamma = M_2(o_K)$ is a $\mathbb{Z}$-order in $M_2(K)$ and
\[
X = \{e_{12}, e_{12}\sqrt{d}\} \subset \Gamma
\]
is a set of commuting nilpotent elements of index 2, where $e_{ij}$ denotes the elementary matrix. The set $\{1, \sqrt{d}\}$ is a linearly independent set over $\mathbb{Q}$, and hence so is $X$. Therefore, by Lemma 2.1 $\mathbb{Z}^2 \hookrightarrow U_1(\Gamma) \subset U_1(o_K[G])$, and so, $U_1(o_K[G])$ is not hyperbolic. \hfill \Box

**Corollary 4.2.** If $G \in \{S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$ then $U_1(o_K[G])$ is not hyperbolic.

**Proof.** We have that $K[G] \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[G])$. For each of the groups under consideration, $M_2(\mathbb{Q})$ is a Wedderburn component of $\mathbb{Q}[G]$; it therefore follows that $M_2(K)$ is a Wedderburn component of $K[G]$. The preceding lemma implies that $U_1(o_K[G])$ is not hyperbolic. \hfill \Box

If $H$ is a non-Abelian Hamiltonian 2-group, then $H = E \times Q_8$, where $E$ is an elementary Abelian 2-group and $Q_8$ is the quaternion group of order 8. Since $Q_8$ contains a cyclic subgroup of order 4, it follows, by Theorem 3.7 that if $U_1(o_K[Q_8])$ is hyperbolic, then $o_K$ is not real.

**Proposition 4.3.** If $G$ is a Hamiltonian 2-group of order greater than 8, then $U_1(o_K[G])$ is not hyperbolic.

**Proof.** Let $G = E \times Q_8$ with $E$ elementary Abelian of order $2^n > 1$. We then have $K[G] = K[E \times Q_8] \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[E \times Q_8]) \cong K \otimes_{\mathbb{Q}} (\mathbb{Q}[E])[Q_8] \cong K \otimes_{\mathbb{Q}} (2^n\mathbb{Q})[Q_8] \cong (2^nK)[Q_8]$. If $d = -1$ it is well known that $KQ_8$ has a Wedderburn component isomorphic to $M_2(K)$ and hence, by Lemma 2.1 $U_1(o_KQ_8)$ is not hyperbolic. If $d < -1$, then, by Proposition 3.10 $r(U_1(o_K[C_4])) = 1$. Since $C_4$ is a subgroup of $Q_8$, it follows that $U_1((2^n o_K)[C_4])$ embeds into $U_1(o_K[G])$. Thus, since $U_1(\prod_{2^n} o_K[C_4])$ has rank $2^n \geq 2$, $U_1(o_K[G])$ is not hyperbolic. \hfill \Box

In view of the above Proposition, it follows that $Q_8$ is the only Hamiltonian 2-group for which $U_1(o_K[G])$ can possibly be hyperbolic, and in this case $o_K$ is the ring of integers of an imaginary extension. By Lemma 2.1 $K[Q_8]$ can not have a matrix ring as a Wedderburn component. Since $\mathbb{Q}[Q_8] \cong 4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})$, we have $K[Q_8] \cong K \otimes_{\mathbb{Q}} (4\mathbb{Q} \oplus \mathcal{H}(\mathbb{Q})) \cong 4K \oplus \mathcal{H}(K)$; hence $K[Q_8]$ must be a direct sum of division rings, or equivalently, has no non-zero nilpotent elements. In particular, $\mathcal{H}(K)$ is a division ring.
Theorem 4.4. Let $K = \mathbb{Q}[\sqrt{d}]$, with $d$ square-free integer $\neq 1$. Then $K[Q_8]$ is a direct sum of division rings if, and only if, one of the following holds:

(i) $d \equiv 1 \pmod{8}$;

(ii) $d \equiv 2, 3 \pmod{4}$, or $d \equiv 5 \pmod{8}$, and $d > 0$.

Proof. The assertion follows from [1, Theorem 2.3]; [3, Theorem 1, p. 236] and [17, Theorem 3.2].

Corollary 4.5. If $K = \mathbb{Q}[\sqrt{d}]$, where $d$ is a negative square-free integer, then the group $U_1(\mathfrak{o}_K[Q_8])$ is not hyperbolic if $d \not\equiv 1 \pmod{8}$.

Let $\mathbb{H} : \mathbb{C} \times ]0, \infty[ \subset H$ be the upper half-space model of three-dimensional hyperbolic space and $Iso(\mathbb{H})$ its group of isometries. In the quaternion algebra $\mathcal{H} := \mathcal{H}(-1, -1)$ over $\mathbb{R}$, with its usual basis, we may identify $\mathbb{H}$ with the subset $\{z + rj : z \in \mathbb{C}, r \in \mathbb{R}^+ \}$. The group $PSL(2, \mathbb{C})$ acts on $\mathbb{H}$ in the following way:

$$\varphi : PSL(2, \mathbb{C}) \times \mathbb{H} \rightarrow \mathbb{H} \quad (M, P) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} P := (aP + b)(cP + d)^{-1},$$

where $(cP + d)^{-1}$ is calculated in $\mathcal{H}$. Explicitly, $MP = M(z + r) = z^* + r^*j$, with

$$z^* = \frac{(az + b)(cz + d) + acr^2}{|cz + d|^2 + |c|^2r^2}, \quad \text{and} \quad r^* = \frac{r}{|cz + d|^2 + |c|^2r^2}.$$

Let $K$ be an algebraic number field and $\mathfrak{o}_K$ its ring of integers. Let

$$SL_1(\mathcal{H}(\mathfrak{o}_K)) := \{x \in \mathcal{H}(\mathfrak{o}_K) : N(x) = 1\},$$

where $N$ is the norm in $\mathcal{H}(K)$. Clearly the groups $U(\mathcal{H}(\mathfrak{o}_K))$ and $U(\mathfrak{o}_K) \times SL_1(\mathcal{H}(\mathfrak{o}_K))$ are commensurable. Consider the subfield $F = K[i] \subset \mathcal{H}(K)$ which is a maximal subfield in $\mathcal{H}(K)$. The inner automorphism $\sigma$,

$$\sigma : \mathcal{H}(K) \rightarrow \mathcal{H}(K) \quad x \mapsto jxj^{-1},$$

fixes $F$. The algebra $\mathcal{H}(K) = F \oplus Fj$ is a crossed product and embeds into $M_2(\mathbb{C})$ as follows:

$$\Psi : \mathcal{H}(K) \leftarrow M_2(\mathbb{C}) \quad x + yj \mapsto \begin{pmatrix} x & y \\ -\sigma(y) & \sigma(x) \end{pmatrix}. \quad (4)$$

This embedding enables us to view $SL_1(\mathcal{H}(\mathfrak{o}_K))$ and $SL_1(\mathcal{H}(K))$ as subgroups of $SL(2, \mathbb{C})$ and hence $SL_1(\mathcal{H}(K))$ acts on $\mathbb{H}$. 


**Proposition 4.6.** Let $K = \mathbb{Q}[\sqrt{d}]$, $d \equiv 1 \pmod{8}$ a square-free negative integer, and $\mathfrak{o}_K$ its ring of integers. Then $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ and $\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8])$ are hyperbolic groups.

**Proof.** Observe that $SL_1(\mathcal{H}(\mathfrak{o}_K))$ acts on the space $\mathbb{H}$ and, hence, is a discrete subgroup of $SL_2(\mathbb{C})$ (see [6, Theorem 10.1.2, p. 446]). The quotient space $Y := \mathbb{H}/SL_1(\mathcal{H}(\mathfrak{o}_K))$ is a Riemannian manifold of constant curvature $-1$ and, since $\mathbb{H}$ is simply connected, we have that $SL_1(\mathcal{H}(\mathfrak{o}_K)) \cong \pi_1(Y)$. Since $d \equiv 1 \pmod{8}$, $\mathcal{H}(K)$ is a division ring and, therefore, co-compact and $Y$ is compact (see [6, Theorem 10.1.2, item (3)]). Hence $SL_1(\mathcal{H}(\mathfrak{o}_K))$ is hyperbolic (see [2, Example 2.25.5]). Since $\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ and $\mathcal{U}(\mathfrak{o}_K) \times SL_1(\mathcal{H}(\mathfrak{o}_K))$ are commensurable and $\mathcal{U}(\mathfrak{o}_K) \cong \mathbb{Z}$, we conclude that $\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8])$ is hyperbolic. Since $\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8]) \cong \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathfrak{o}_K) \times \mathcal{U}(\mathcal{H}(\mathfrak{o}_K))$ and $\mathcal{U}(\mathfrak{o}_K) \cong C_2$, we conclude that $\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8])$ is hyperbolic.

Combining the results in the present and the preceding section, we have the following main result.

**Theorem 4.7.** Let $K = \mathbb{Q}[\sqrt{d}]$, with $d$ square-free integer $\neq 1$, and $G$ a finite group. Then $\mathcal{U}_1(\mathfrak{o}_K[G])$ is hyperbolic if, and only if, $G$ is one of the groups listed below and $\mathfrak{o}_K$ (or $K$) is determined by the corresponding value of $d$:

1. $G \in \{C_2, C_3\}$ and $d$ arbitrary;

2. $G$ is an Abelian group of exponent dividing $n$ for:
   - $n = 2$ and $d < 0$, or $n = 4$ and $d = -1$, or $n = 6$ and $d = -3$.

3. $G = C_4$ and $d < 0$.

4. $G = C_8$ and $d = -1$.

5. $G = Q_8$ and $d < 0$ and $d \equiv 1 \pmod{8}$.

**Remark.** If the group $\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8])$ is hyperbolic then the hyperbolic boundary $\partial(\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8])) \cong \mathbb{S}^2$, the Euclidean sphere of dimension 2, and $\text{Ends}(\mathcal{U}(\mathfrak{o}_K[\mathbb{Q}_8]))$ has one element (see [2, Example 2.25.5]). Note that if $\mathcal{U}(\mathbb{Z}[G])$ is an infinite non-Abelian hyperbolic group, then $\partial(\mathcal{U}(\mathbb{Z}[G]))$ is totally disconnected and is a Cantor set. So, in this case, $\mathcal{U}(\mathbb{Z}[G])$ has infinitely many ends and also is a virtually free group, ([13, Theorem 2] and [8, §3]). However, if $\mathcal{U}(\mathfrak{o}_K[G])$ is a non-Abelian hyperbolic group, then $\mathcal{U}(\mathfrak{o}_K[G])$ is an infinite group which is not virtually free, has one end and $\partial(\mathcal{U}(\mathbb{Z}[G]))$ is a smooth manifold.
5 Pell and Gauss Units

When the algebra $\mathcal{H}(K)$ is isomorphic to $M_2(K)$ it is known how to construct the unit group of a $\mathbb{Z}$-order up to a finite index. Nevertheless, if $\mathcal{H}(K)$ is a division ring, this is a highly non-trivial task; see [5], for example. In this section we study construction of units of $U(\mathcal{H}(\mathfrak{o}_K))$ in the case when the quaternion algebra $\mathcal{H}(K)$ is a division ring.

In the sequel, $K = \mathbb{Q}[\sqrt{-d}]$ is an imaginary quadratic extension with $d$ a square-free integer congruent to 7 (mod 8), and $\mathfrak{o}_K$ the ring of integers of the field $K$. Note that $s(K)$, the stufe of $K$, is 4, the quaternion algebra $\mathcal{H}(K)$ is a division ring and $U(\mathfrak{o}_K) = \{\pm 1\}$. Thus, if $u = u_1 + u_i + u_j + u_k k \in U(\mathcal{H}(\mathfrak{o}_K))$, then its norm $N(u) = u_1^2 + u_i^2 + u_j^2 + u_k^2 = \pm 1$; furthermore, if any of the coefficients $u_1, u_i, u_j, u_k$ is zero then $N(u) = 1$, $s(K)$ being 4.

The representation of $u$, given by (11), is

$$[u] := \Psi(u) = \begin{pmatrix} u_1 + u_i i & u_j + u_k i \\ -u_j + u_k i & u_1 - u_i i \end{pmatrix} \in M_2(\mathbb{C}).$$

Denote by $\chi_u$ the characteristic polynomial of $[u]$, and by $m_u$ its minimal polynomial. The degree $\partial(\chi_u)$ of $\chi_u$ is 2 and therefore $\partial(m_u) \leq 2$. If $\partial(m_u) = 1$ then $m_u(X) = X - z_0$, $z_0 \in \mathbb{C}$, and therefore $u = z_0$. Note that the characteristic polynomial is $\chi_u(X) = X^2 - \text{trace}([u])X + \text{det}([u])$, where $\text{trace}([u]) = u_1 + u_i i + \sigma(u_1 + u_i i) = 2u_1$ and $\text{det}([u]) = \pm 1$:

$$\chi_u(X) = X^2 - 2u_1 X \pm 1.$$

**Proposition 5.1.** Let $u = u_1 + u_i i + u_j j + u_k k \in U(\mathcal{H}(\mathfrak{o}_K))$. Then the following statements hold:

1. $u^2 = 2u_1 u - N(u)$.

2. If $N(u) = 1$, then $u$ is a torsion unit if, and only if, $u_1 \in \{-1, 0, 1\}$ and the order of $u$ is 1, 2, or 4.

3. If $N(u) = -1$, then order of $u$ is infinite.

**Proof.** (1) is obvious.

(2) Suppose $N(u) = 1$ and $u$ is a torsion unit of order $n$, say. If $X^2 - 2u_1 X + \eta(u) = (X - \zeta_1)(X - \zeta_2)$, then $\zeta_i, \ i = 1, 2$, are roots of unity and $\zeta_1 \zeta_2 = 1$. It follows that $2u_1 = \zeta_1 + \zeta_2$ is a real number. Since $u \in \mathcal{H}(\mathfrak{o}_K)$ and $\{1, \vartheta\}$ is an integral basis of $\mathfrak{o}_K$, it follows that $u_1 \in \mathbb{Z}$. From the equality $2u_1 = \zeta_1 + \zeta_2$, we have $2|u_1| = |\zeta_1 + \zeta_2| \leq 2$, and therefore $u_1 \in \{-1, 0, 1\}$. If $u_1 = 0$, then $u^2 = -1$ and therefore $o(u) = 4$. If $u_1 = \pm 1$, then $\chi_u(X) = X^2 \mp 2X + 1 = (X \mp 1)^2$, and therefore $0 = \chi_u(u) = (u \mp 1)^2 \in \mathcal{H}(K)$; hence $u = \pm 1$. 

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(3) If \( N(u) = -1 \), then \( u^2 = 2u_1 + 1 \), \( (u^2)_1 = 2u_1^2 + 1 \), \( \eta(u^2) = 1 \). If \( u \) were a torsion unit, then, by (2) above, \( (u^2)_1 \in \{-1, 0, 1\} \). If \( (u^2)_1 = 0 \), then \( 1/2 = -u_1^2 \in \mathfrak{o}_K \), which is not possible. If \( (u^2)_1 = 1 \), then \( u_1 = 0 \), and therefore \( u^2 = 1 \) yielding \( u = \pm 1 \) which is not the case, because \( N(u) = -1 \). Finally, if \( (u^2)_1 = -1 \), then \( u_1^2 = -1 \) which implies that \( \sqrt{-1} \in K \) which is also not the case, because \( \mathcal{H}(K) \) is a division ring. Hence \( u \in \mathcal{U}(\mathfrak{o}_K) \) is an element of infinite order.

Let \( \xi \neq \psi \) be elements of \( \{1, i, j, k\} \). Suppose

\[ u := m\sqrt{-d}\xi + p\psi, \quad p, m \in \mathbb{Z}, \]

is an element in \( \mathcal{H}(\mathfrak{o}_K) \) having norm 1. Then

\[ p^2 - m^2d = 1, \]

i.e., \( (p, m) \) is a solution of the Pell’s equation \( X^2 - dY^2 = 1 \). Let \( L := \mathbb{Q}[\sqrt{d}] \). Equation (6) implies that \( \epsilon = p + m\sqrt{d} \) is a unit in \( \mathfrak{o}_L \). Conversely, if \( \epsilon = p + m\sqrt{d} \) is a unit of norm 1 in \( \mathfrak{o}_L \), then, necessarily, \( p^2 - m^2d = 1 \), and, therefore, for any choice of \( \xi, \psi \) in \( \{1, i, j, k\} \), \( \xi \neq \psi \),

\[ m\sqrt{-d}\xi + p\psi \]

is a unit in \( \mathcal{H}(\mathfrak{o}_K) \). In particular,

\[ u(\epsilon, \psi) := p + m\sqrt{-d}\psi, \quad \psi \in \{i, j, k\}, \]

is a unit in \( \mathcal{H}(\mathfrak{o}_K) \).

With the notations as above, we have:

**Proposition 5.2.**

1. If \( \frac{1}{2} \notin supp(u) \), the support of \( u \), then \( u \) is a torsion unit.

2. If \( \epsilon = p + m\sqrt{d} \) is a unit in \( \mathfrak{o}_L \), then

\[ u^0(\epsilon, \psi) = u(\epsilon^n, \psi) \]

for all \( \psi \in \{i, j, k\} \) and \( n \in \mathbb{Z} \).

**Proof.** If \( 1 \notin supp(u) \), then \( u_1 = 0 \); therefore, by the Proposition 5.1, \( u \) is torsion unit.

Let \( \mu = A + B\sqrt{d} \) and \( \nu = C + D\sqrt{d} \), be units in \( \mathfrak{o}_L \). Then \( u(\mu, \psi) = A + B\sqrt{-d}\psi \) and \( u(\nu, \psi) = C + D\sqrt{-d}\psi \) are units in \( \mathcal{H}(\mathfrak{o}_K) \). We have

\[ \mu\nu = AC + dBD + (AD + BC)\sqrt{d}. \]

Also \( u(\mu, \psi)u(\nu, \psi) = (AC + dBD) + (AD + BC)\sqrt{-d}\psi = u(\mu\nu, \psi) \). It follows that we have \( u^0(\epsilon, \psi) = u(\epsilon^n, \psi) \) for all \( \psi \in \{i, j, k\} \) and \( n \in \mathbb{Z} \).

The units \( u(\epsilon, \psi) \) constructed above are called 2-Pell units.
Proposition 5.3. Let $L = \mathbb{Q}[\sqrt{2d}]$, $2d$ square-free, $\xi, \psi, \phi$ pairwise distinct elements in \{1, i, j, k\} and $p, m \in \mathbb{Z}$. Then the following are equivalent:

(i) $u := m\sqrt{-d}\xi + p\psi + (1 - p)\phi \in \mathcal{U}(\mathcal{O}_K)$.

(ii) $\epsilon := (2p - 1) + m\sqrt{2d} \in \mathcal{U}(\mathcal{O}_L)$.

Proof. If $u$ is a unit in $\mathcal{O}_K$ then $N(u) = -m^2d + p^2 + (1 - p)^2 = 1$, i.e., $2p^2 - 2p - m^2d = 0$, and thus $(2p - 1)^2 - m^22d = 1$. Consequently, $\epsilon = (2p - 1) + m\sqrt{2d}$ is invertible in $\mathcal{O}_L$. The steps being reversible, the equivalence of (i) and (ii) follows.

The units constructed above are called 3-Pell units. We shall next determine units of the form $u = m\sqrt{-d} + (m\sqrt{-d})i + pj + qk$, with $m, p, q \in \mathbb{Z}$ and $N(u) = -2m^2d + p^2 + q^2 = 1$. Set $p + q =: r$ and consider the equation

$$2p^2 - 2pr - 2m^2d + r^2 - 1 = 0.$$  \hfill (9)

Theorem 5.4. If $r = 1$, then equation (9) has a solution in $\mathbb{Z}$, and for each such solution, $u = m\sqrt{-d} + (m\sqrt{-d})i + pj + qk$ is a unit in $\mathcal{O}_K$ of norm 1.

Proof. Viewed as a quadratic equation in $p$, (9) has real roots

$$p = \frac{1 \pm \sqrt{1 + 4m^2d}}{2}.$$  

To obtain a solution in $\mathbb{Z}$, we need the argument under the radical to be a square; we thus need to solve the diophantine equation

$$X^2 - 4dY^2 = 1.$$  \hfill (10)

Let $\epsilon = x + y\sqrt{d}$, with $x, y \in \mathbb{Z}$, be a unit in $\mathcal{O}_K$ having infinite order. Replacing $\epsilon$ by $\epsilon^2$, if necessary, we can assume that $y$ is even. We then have $x^2 - y^2d = 1$, and so $x$ must be odd. Taking $m = y/2$ and $p = \frac{x + y}{2}$, we obtain a solution of (10) in $\mathbb{Z}$. Clearly, for such a solution, the element $u$ lies in $\mathcal{O}_K$ and has norm 1.

Using Gauss' result which states that a positive integer $n$ is a sum of three squares if, and only if, $n$ is not of the form $4^a(8b - 1)$, where $a \geq 0$ and $b \in \mathbb{Z}$, it is easy to see that, for every integer $m \equiv 2 \pmod{4}$, the integers $m^2d - 1$ and $m^2d + 1$ can be expressed as sums of three squares. We can thus construct units $u = m\sqrt{-d} + pi + qj + rk \in \mathcal{O}_K$ having prescribed norm 1 or $-1$; we call such units Gauss units.

Example. In [5], all units exhibited in $\mathcal{O}_K[\sqrt{-7}]$ are of norm 1. We present some units of norm $-1$ in this ring. The previous theorem guarantees the existence of integers $p, q, r$, such that

$$u = 6\sqrt{-7} + pi + qj + rk$$
is a unit of norm \(-1\). Indeed,

\[(p, q, r) \in \{(\pm 15, \pm 5, \pm 1), (\pm 13, \pm 9, \pm 1), (\pm 11, \pm 11, \pm 3)\},\]

and the triples obtained by permutation of coordinates, are all possible integral solutions. In [5], the authors have constructed a set \(S\) of generators of the group \(SL_1(\mathcal{H}(\mathbb{o}_{\mathbb{Q}(\sqrt{-2})}))\). If \(v_0\) is a unit of \(\mathcal{H}(\mathbb{o}_{\mathbb{Q}(\sqrt{-2})})\) having norm \(-1\), then clearly \((v_0, S) = \mathcal{U}(\mathcal{H}(\mathbb{o}_{\mathbb{Q}(\sqrt{-2})}))\). Thus, for example, taking \(v_0 = 6\sqrt{-7} + 15i + 5j + k\), we have

\[
\mathcal{U}(\mathcal{H}(\mathbb{o}_{\mathbb{Q}(\sqrt{-2})})) = (v_0, S).
\]

The set \(\{1, \frac{1 + \sqrt{-7}}{2}\}\) is an integral basis of \(R = \mathbb{Z}[\frac{1 + \sqrt{-7}}{2}]\). Consider units of the form

\[
\frac{m + \sqrt{-d}}{2} \pm (\frac{m - \sqrt{-d}}{2})i + pj
\]

These are neither Pell nor Gauss units. Those of norm \(\pm 1\), are solutions of the equation

\[
m^2 + 2p^2 = \pm 2 + d
\]

in \(\mathbb{Z}\). The main result of [5] states that if \(d = 7\) then the units of norm \(1\) of the above type, together with the trivial units \(i\) and \(j\), generate the group \(SL_1(\mathcal{H}(R))\).

For \(d \equiv 7 \pmod{8}\) there are no units of norm \(-1\) of the above type, since, in this case, the equation \(m^2 + 2p^2 = -2 + d\) has no solution in \(\mathbb{Z}\), as can be easily seen working modulo 8.

In case \(d \neq 7\), we give some more examples of negative norm units of the form

\[
\frac{m + \sqrt{-d}}{2} \pm (\frac{m - \sqrt{-d}}{2})i + pj
\]

If \(d = 15\) then the equation (12) becomes \(m^2 + 2p^2 = 17\); the pairs \((m, p) \in \{(3, 2), (3, -2), (-3, 2), (-3, -2)\}\) are its integral solutions. For \(m = 3\) either \(p = 2\) or \(p = -2\) and so there are 8 units. Each coefficient of \(u\) is distinct, hence for each solution \((m, p)\) there are 3! units with the same support, thus there are 36 different units for a given fixed support. By Proposition 5.1 all these units have infinite order if \(u_1 \notin \{-1, 0, 1\}\). If \(1 \in \text{supp}(u)\) then either \(\{i, j\} \subset \text{supp}(u)\) or \(\{i, k\} \subset \text{supp}(u)\), or \(\{j, k\} \subset \text{supp}(u)\). Therefore there are 108 of these units and, for example,

\[
\frac{3 + \sqrt{-15}}{2} + \frac{3 - \sqrt{-15}}{2} j - 2k
\]

is one of them.

If \(1 \notin \text{supp}(u)\) then \(u\) is a torsion unit, so there are 36 torsion units of this type. One of them is the unit

\[
\frac{-3 - \sqrt{-15}}{2} i + \frac{-3 + \sqrt{-15}}{2} j + 2k,
\]

of order 4.

For \(d = 31\) we obtain \(m^2 + 2p^2 = 33\) whose solutions in \(\mathbb{Z}\) are: \((m, p) \in \{(1, 4), (1, -4), (-1, 4), (-1, -4)\}\).

As another example of a unit of norm \(-1\) in a quaternion algebra, we may mention that, in \(\mathcal{H}(\mathbb{o}_{\mathbb{Q}(\sqrt{-15})})\), \(u = 5\sqrt{-23} + 23i + 6j + 3k\) is a unit of norm \(-1\).

We next exhibit some Gauss units of norm 1. For \(\mathcal{H}(\mathbb{o}_{\mathbb{Q}(\sqrt{-23})})\), there exist \(p, q, r\), such that \(u = 10\sqrt{-15} + pi + qj + rk\) is a unit of norm 1. In fact, \((36, 14, 3), (36, 13, 6), (32, 21, 6), (30, 24, 5)\)
are some of the possible choices for \((p, q, r)\). For \(\mathcal{H}(\mathbb{Q}(\sqrt{-23}))\), \(u = 2\sqrt{-23} + 8i + 5j + 2k\) is a unit of norm 1. It is interesting to note that \(u = 3588\sqrt{-23} + 12168i + 12167j\) is a Gauss unit, although \(4\) divides \(3588\).

We conclude with the following result:

**Theorem 5.5.** Let \(K = \mathbb{Q}(\sqrt{-d})\), \(0 < d \equiv 7 \pmod{8}\) and \(\mathfrak{o}_K\) the ring of integers of \(K\). If \(\epsilon = p + m\sqrt{d}\) is a unit in \(\mathbb{Z}(\sqrt{d})\), and \(x := u(\epsilon, \psi)\), \(y := u(\epsilon, \psi')\) are two 2-Pell units in \(\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))\), where \(\psi\) and \(\psi' \in \{i, j, k\}\) and \(\psi \neq \psi'\), then there exists a natural number \(m\) such that \(\langle x^m, y^m \rangle\) is a free group of rank 2.

**Proof.** By Proposition 4.6, \(\mathcal{U}(\mathcal{H}(\mathfrak{o}_K))\) is a hyperbolic group. In view of [4, Proposition III.1.3.20], there exists a natural \(m\), such that, \(\langle x^m, y^m \rangle\) is a free group of rank at most 2. However, Proposition 5.2 item (2) ensures that \(\langle x \rangle \cap \langle y \rangle = \{1\}\). Therefore, \(\langle x^m, y^m \rangle\) has rank at least 2, and hence \(\langle x^m, y^m \rangle\) is a free group of rank 2.

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Instituto de Matemática e Estatística, 
Universidade de São Paulo (IME-USP),
Caixa Postal 66281, São Paulo, 
CEP 05315-970 - Brasil
email – ostanley@ime.usp.br –

Centre for Advanced Study in Mathematics, 
Panjab University, 
Chandigarh 160014 - India
email – ibspassi@yahoo.co.in –

Escola de Artes, Ciências e Humanidades, 
Universidade de São Paulo (EACH-USP),
Rua Arlindo Béttio, 1000, Ermelindo Matarazzo, São Paulo, 
CEP 03828-000 - Brasil
email – acsouzafilho@usp.br –