A Three-parameter Family Of Involutions In The Riordan Group Defined By Orthogonal Polynomials

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Abstract

We show how to define, for every Riordan group element \((g(x), f(x))\), an involution in the Riordan group. More generally, we show that for every pseudo-involution \(P\) in the Riordan group, we can define a new involution beginning with an arbitrary element \((g(x), f(x))\) in the Riordan group. We then use this result to show that certain two-parameter families of orthogonal polynomials defined by a Riordan array can lead to involutions in the Riordan group, and we give an explicit form of these involutions.

1 Preliminaries

The Riordan group \(\mathcal{R}\) [3, 13, 14] is a subgroup of the group of lower-triangular invertible matrices, whose elements are defined algebraically. We work over a field of characteristic zero \(k\), which can be taken to be the field \(\mathbb{C}\) of complex numbers. In practice, many of the examples we shall give will have entries taken in the ring of integers \(\mathbb{Z}\). The group \(\mathcal{R}\) can be viewed as an abstract group, or as its matrix realization. We explain this now. We let

\[
F_0 = \{g(x) = g_0 + g_1x + g_2x^2 + \cdots \mid g_0 \neq 0, g_i \in k\},
\]

be the ring of invertible power series in the indeterminate \(x\) over \(k\). Similarly we let

\[
F_1 = \{f(x) = f_1x + f_2x^2 + \cdots \mid f_0 = 0, f_1 \neq 0, f_i \in k\}
\]

be the ring of composable power series in the indeterminate \(x\) over the field \(k\). Then as an abstract group, we have

\[
\mathcal{R} = \{(g(x), f(x)) \mid g \in F_0, f \in F_1\}.
\]

In fact, we have that

\[
\mathcal{R} = F_0 \rtimes F_1.
\]

The matrix representation of the group \(\mathcal{R}\) can be obtained by associating to the pair of power series \((g(x), f(x))\) the matrix \((a_{n,k})_{0 \leq n,k \leq \infty}\), with \(a_{n,k} \in k\), where

\[
a_{n,k} = [x^n]g(x)f(x)^k,
\]
with \([x^n]\) denoting the functional \([11]\) that extracts the coefficient of \(x^n\) is the power series to which it is applied. The fact that \(f(x) \in F_1\) ensures that this matrix is lower triangular. We use the term \textit{Riordan array} to denote the matrix element corresponding to the abstract element \((g(x), f(x))\).

The group structure on \(R\) is defined by the product

\[
(g(x), f(x)) \cdot (u(x), v(x)) = (g(x).u(f(x)), v(f(x))),
\]

and the inverse

\[
(g(x), f(x))^{-1} = \left( \frac{1}{g(\bar{f}(x))}, \bar{f}(x) \right),
\]

where \(\bar{f}(x) = f^{-1}(x)\) is the compositional inverse of \(f\). This means that \(\bar{f}\) is the solution \(u(x)\) of the equation \(f(u) = x\) which satisfies \(u(0) = 0\).

The identity element is \((1, x)\). The product in \(R\) corresponds to matrix multiplication in its matrix representation.

**Example 1.** Using the product rule, we have

\[
(g(x), f(x)) = (g(x), x) \cdot (1, f(x)),
\]

which corresponds to \(R = F_0 \rtimes F_1\).

**Example 2.** The Riordan group element \((\frac{1}{1-x}, \frac{x}{1-x})\) is represented by the binomial matrix (Pascal’s triangle) \((\binom{n}{k})_{0 \leq n, k \leq \infty}\).

In the rest of this note, we shall drop the distinction between abstract elements such as \((\frac{1}{1-x}, \frac{x}{1-x})\) and their representations (here, \((\binom{n}{k})_{0 \leq n, k \leq \infty}\)), and we shall use the most convenient form for the topic under discussion.

## 2 Involutions and pseudo-involutions in the Riordan group

By an \textit{involution} in the Riordan group \([1, 4, 5, 6, 7, 8, 9, 10, 12]\) we understand an element \((g(x), f(x)) \in R\) such that

\[
(g(x), f(x))^2 = (g(x), f(x)) \cdot (g(x), f(x)) = (g(x).g(f(x)), f(f(x))) = (1, x).
\]

We see immediately that for an involution \((g(x), f(x))\), we have

\[
\bar{f} = f
\]

and

\[
g(x) = \frac{1}{g(\bar{f}(x))} = \frac{1}{g(f(x))}.
\]

In particular, we have \((g(x), f(x))^{-1} = (g(x), f(x))\).
Example 3. The Riordan arrays $(1, x)$ and $(1, -x)$ are involutions.

Example 4. The signed binomial matrix $\left( \frac{1}{1-x}, \frac{-x}{1-x} \right)$ or $\left( \binom{n}{k}(-1)^k \right)$ is an involution in the Riordan group.

By a pseudo-involution in the Riordan group we shall understand an element $(g(x), f(x))$ such that

$$(g(x), f(x)) \cdot (1, -x) = (g(x), -f(x))$$

is an involution.

We see for instance that the binomial matrix $\left( \frac{1}{1-x}, \frac{x}{1-x} \right)$ is thus a pseudo-involution.

Example 5. The identity $(1, x)$ is trivially a pseudo-involution. This is because $(1, x) \cdot (1, -x) = (1, -x)$ is an involution.

Example 6. The set of elements

$$\text{Bin} = \left\{ \left( \frac{1}{1-\alpha x}, \frac{x}{1-\alpha x} \right) \mid \alpha \in k \right\}$$

is a subgroup of the Riordan group. All its elements are pseudo-involutions.

3 Orthogonal polynomials and Riordan arrays

By a family of orthogonal polynomials $P_n(x)$ we shall understand a sequence of polynomials $P_n(x)$ of exact degree $n, n \geq -1$, such that they satisfy a three-term recurrence

$$P_n(x) = (x - \alpha_n)P_{n-1}(x) - \beta_nP_{n-2}(x),$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$. If we let $P_n(x) = \sum_{i=0}^{n} a_{n,i}x^i$, then it is clear that the matrix $(a_{n,k})$ is lower triangular. The question then arises as to whether a Riordan array can be the coefficient matrix of a family of orthogonal polynomials. The answer is in the affirmative [2, 14]. We have that the Riordan array

$$\left( \frac{1-rx-sx^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2} \right)$$

is the coefficient array of the family of (constant coefficient) generalized Chebyshev polynomials $P_n(x)$ that satisfy the three-term recurrence

$$P_n(x) = (x - a)P_{n-1}(x) - bP_{n-2}(x),$$

with $P_0(x) = 1, P_1(x) = x - a - r$ and $P_2(x) = x^2 - (2a + r)x + a^2 + ar - b - s$. 

3
4 The main result

Our results concerning orthogonal polynomials will be a consequence of the following general results.

**Proposition 7.** We let $P$ denote an arbitrary pseudo-involution in the Riordan group. For arbitrary $(g(x), f(x)) \in \mathcal{F}_0 \times \mathcal{F}_1$, the Riordan array

$$(g(x), f(x))^{-1} \cdot P \cdot (g(-x), f(-x))$$

is an involution.

**Proof.** We have

$$(g(-x), f(-x)) = (1, -x) \cdot (g(x), f(x)).$$

Thus

$$(g(x), f(x))^{-1} \cdot P \cdot (g(-x), f(-x)) = (g(x), xg(x))^{-1} \cdot P \cdot (1, -x) \cdot (g, xg(x)).$$

Now $\mathcal{I} = P \cdot (1, -x)$ is an involution, and so

$$(g(x), f(x))^{-1} \cdot P \cdot (g(-x), f(-x)) = (g(x), f(x))^{-1} \cdot \mathcal{I} \cdot (g(x), f(x))$$

is the conjugate of an involution, which is again an involution. \hfill \square

**Example 8.** We take the example of

$$g(x) = c(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

the generating function of the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ A000108. We let $f(x) = xg(x)$, and we take $P = (1, x)$, the identity, which is both an involution and a pseudo-involution. Then we obtain the involution

$$((1 + xc(x))c(x), -x(1 + xc(x))c(x)).$$

**Example 9.** We let $(g(x), f(x)) = \left( \frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2} \right)$. Then

$$((g(x), f(x))^{-1} \cdot (g(-x), -f(-x)) = \left( \frac{1}{1-2rx}, \frac{-x}{1-2rx} \right),$$

an element of the subgroup $\text{Bin}$.

**Example 10.** The array product

$$\left( \frac{1}{1+x^2}, \frac{x(1-x)}{1+x^2} \right)^{-1} \cdot \left( \frac{1}{1+x^2}, \frac{-x(1+x)}{1+x^2} \right)$$

yields the involution

$$(1, -x(1+x)c(x(1+x))).$$
Example 11. The RNA matrix. Cameron and Nkwanta \cite{5} give the example of the RNA matrix. Here, we take \((g, f) = \left( \frac{1}{1+x^2}, \frac{x}{1+x^2} \right)^{-1}\), and \(P = \left( \frac{1}{1-x}, \frac{x}{1-x} \right)\). Thus the RNA involution matrix is given by the product
\[
\left( \frac{1}{1+x^2}, \frac{x}{1+x^2} \right) \cdot \left( \frac{1}{1-x}, \frac{x}{1-x} \right) \cdot \left( \frac{1}{1+x^2}, \frac{-x}{1+x^2} \right)^{-1}.
\]
The matrix begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 \\
2 & -3 & 3 & -1 & 0 & 0 & 0 \\
4 & -6 & 6 & -4 & 1 & 0 & 0 \\
8 & -13 & 13 & -10 & 5 & -1 & 0 \\
17 & -28 & 30 & -24 & 15 & -6 & 1
\end{pmatrix}.
\]
The unsigned version, which is a pseudo-involution, is A097724.

Example 12. Our final example of this section uses the matrix \((g(x), f(x)) = (M(x), xM(x))\) where \(M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}\) is the generating function of the Motzkin numbers A001006. Then the product \((M(x), xM(x))^{-1} \cdot (M(-x), -xM(-x))\) yields the involution
\[
\left( \frac{1}{(1+x)^2} c \left( \frac{x^2}{(1+x)^4} \right), \frac{-x}{(1+x)^2} c \left( \frac{x^2}{(1+x)^4} \right) \right).
\]
This involution begins
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 1 & 0 & 0 & 0 & 0 \\
-10 & -12 & -6 & -1 & 0 & 0 & 0 \\
28 & 36 & 24 & 8 & 1 & 0 & 0 \\
-82 & -112 & -86 & -40 & -10 & -1 & 0 \\
248 & 356 & 300 & 168 & 60 & 12 & 1
\end{pmatrix}.
\]

5 Orthogonal polynomials and involutions

We consider the Riordan array
\[
(g, f) = \left( \frac{1}{1+rx+sx^2}, \frac{x(1-tx)}{1+rx+sx^2} \right).
\]
Taking \(P = (1, x)\), this gives us the involution \((\tilde{g}(x), \tilde{f}(x))\) in the Riordan group where
\[
\tilde{g}(x) = \frac{s+tr+t}{s+t(t+1)-(r^2t+2rs-s+t^2)x+t(r-1)\sqrt{1-2(r+2t)x+(r^2-4s)x^2}}.
\]
and
\[ \tilde{f}(x) = \frac{t\sqrt{1 - 2(r + 2t)x + (r^2 - 4s)x^2} + (t^2 - s)x - t}{s + t^2 - r(rt + 2s)x + r\sqrt{1 - 2(r + 2t)x + (r^2 - 4s)x^2}}. \]

The Riordan array \( \left( \frac{1 + (r - s)x}{1 + (r + s)x}, \frac{x}{1 + (r + s)x} \right) \) is the coefficient array of the family of orthogonal polynomials \( P_n(x; r, s) \) that satisfy the 3-term recurrence
\[ P_n(x; r, s) = (x - (2r + s))P_{n-1}(x; r, s) - r(r + s)P_{n-2}(x; r, s), \]
with \( P_0(x; r, s) = 1 \) and \( P_1(x; r, s) = x - 2s \).

We then have the following proposition.

**Proposition 13.** For \( r, s \in \mathbb{Z} \), the Riordan array
\[ \left( \frac{1 - (r - 2t)x - (rt - s - t^2)x^2}{1 + (r + 2t)x + (rt + s + t^2)x^2}, \frac{x}{1 + (r + s)x} \right)^{-1} \cdot \left( 1, \frac{-x(1 + 2tx)}{1 - (r - 2t)x - (rt - s - t^2)x^2} \right) \]

is an involution in the Riordan group.

**Proof.** Evaluating the product, we find that it is equal to \((\tilde{g}(x), \tilde{f}(x))\). \(\square\)

Thus for each triple \((r, s, t) \in \mathbb{Z}^3\), we can associate to the family of orthogonal polynomials with coefficient array the Riordan array \( \left( \frac{1 + (r - 2t)x - (rt - s - t^2)x^2}{1 + (r + 2t)x + (rt + s + t^2)x^2}, \frac{x}{1 + (r + s)x} \right) \) an involution in the Riordan group.

We note that the generating function \( \tilde{g}(x) \) has the following generating function expression,
\[ \tilde{g}(x) = \frac{1}{1 - 2rx - \frac{2rtx^2}{1 - (r + 2t)x - \frac{(s + t(r + t))x^2}{1 - (r + 2t)x - \cdots}}}, \]

while the row sums of the involution \((\tilde{g}(x), \tilde{f}(x))\) have a generating function that can be expressed as the related continued fraction
\[ \frac{1}{1 - (2r - 1)x - \frac{2t(r - 1)x^2}{1 - (r + 2t)x - \frac{(s + t(r + t))x^2}{1 - (r + 2t)x - \cdots}}}. \]

**Example 14.** We let \( s = t(r - t), r = 1, t = \frac{1}{2} \). Then we obtain that
\[ \left( \frac{1}{(1 + x)^2}, \frac{x}{(1 + x)^2} \right)^{-1} \cdot (1, -x(1 + x)) = (c(x)^2, -xc(x)^3) \]
is an involution.

A variant of this is found by setting \( r = 2t \) and \( s = t^2 \) to obtain that
\[ \left( \frac{1}{(1 + 2tx)^2}, \frac{x}{(1 + 2tx)^2} \right)^{-1} \cdot (1, -x(1 + 2tx)) \]
is an involution. This shows that

\[(c(2tx)^2, -xct(2tx)^3)\]

is an involution.

**Corollary 15.** Given \(r, t \in \mathbb{Z}\), the Riordan array

\[
\left(\frac{1 + (t - r)x}{1 + (t + r)x}, \frac{x}{(1 + tx)(1 + (t + r)x)}\right)^{-1} \cdot \left(1, \frac{-x(1 + 2tx)}{(1 + tx)(1 + (t - r)x)}\right)
\]

is an involution in the Riordan group.

**Proof.** This is the case \(s = 0\) of the above proposition.

\[\square\]

**Example 16.** We take the case of \(r = t = 1\) to obtain the involution

\[
\left(\frac{1}{1 + 2x}, \frac{x}{(1 + x)(1 + 2x)}\right)^{-1} \cdot \left(1, \frac{-x(1 + 2x)}{1 + x}\right) = (S(x), -xS(x)^2),
\]

where \(S(x)\) denotes the generating function of the large Schröder numbers \(A006318\). This array begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 & 0 \\
6 & -6 & 1 & 0 & 0 & 0 & 0 \\
22 & -30 & 10 & -1 & 0 & 0 & 0 \\
90 & -146 & 70 & -14 & 1 & 0 & 0 \\
394 & -714 & 430 & -126 & 18 & -1 & 0 \\
1806 & -3534 & 2490 & -938 & 198 & -22 & 1
\end{pmatrix}.
\]

The unsigned matrix is \(A110098\).

We note that the \(B\)-sequence of this array is 4, 4, 4, . . . .

For the general case of the involution

\[
\left(\frac{1}{1 + 2rx}, \frac{x}{(1 + x)(1 + 2x)}\right)^{-1} \cdot \left(1, \frac{-x(1 + 2rx)}{1 + rx}\right)
\]

we observe that the \(B\)-sequence begins \(4r, 4r^2, 4r^3, \ldots\).

**Example 17.** We take the case of \(s = 0, r = 1, t = 2r = 2\). We find that the product

\[
\left(\frac{1 + x}{1 + 3x}, \frac{x}{(1 + 2x)(1 + 3x)}\right)^{-1} \cdot \left(1, \frac{-x(1 + 4x)}{(1 + x)(1 + 2x)}\right)
\]

gives the involution

\[
\left(\frac{3}{2 - x + \sqrt{1 - 10x + x^2}}, -\frac{1 - 5x + x^2 - (1 - x)\sqrt{1 - 10x + x^2}}{1 + 2x}\right).
\]

7
The corresponding pseudo-involution begins

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
8 & 8 & 1 & 0 & 0 & 0 & 0 \\
44 & 56 & 14 & 1 & 0 & 0 & 0 \\
284 & 404 & 140 & 20 & 1 & 0 & 0 \\
2012 & 3044 & 1268 & 260 & 26 & 1 & 0 \\
15140 & 23804 & 11132 & 2852 & 416 & 32 & 1
\end{pmatrix}.
\]

6 Conclusions

The central result of this note is that for any Riordan array \((g, f)\), and any pseudo-involution \(P\), we have an involution \(I(g, f, P)\) in the Riordan group, where

\[
I(g, f, P) = (g(x), f(x))^{-1} \cdot P \cdot (g(-x), f(-x)).
\]

We have investigated this in the specific case of the coefficient matrices of certain families of orthogonal polynomials that can be defined by Riordan arrays. Clearly, other families of Riordan arrays \((g(x), f(x))\) could be investigated. In our examples, we have confined our discussions to the cases \(P = (1, x)\) and \(P = \left(\frac{1}{1-x}, \frac{x}{1-x} \right)\). Again, it is clear that other choices of \(P\) may lead to other interesting examples. With \(I = I(g, f, P)\), we have expressions such as

\[
(g(x), f(x)) \cdot I = P \cdot (g(-x), f(-x))
\]

and

\[
(g(x), f(x)) = P \cdot (g(-x), f(-x)) \cdot I,
\]

which may warrant further consideration.

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