Majorana representation of $A_6$ involving 3C-algebras

A. A. Ivanov

Abstract We study a possible Majorana representation $\mathcal{R}$ of the alternating group $A_6$ of degree 6 such that for some involutions $s$ and $t$ in $A_6$, generating a $D_6$-subgroup, the corresponding Majorana axes $a_s$ and $a_t$ generate a subalgebra of type 3C. We show that there exists at most one such representation $\mathcal{R}$ and that its dimension is at most 70. The representation $\mathcal{R}$ does not correspond to a subalgebra in the Monster algebra generated by a subset of the Majorana axes canonically indexed by the involutions of an $A_6$-subgroup in the Monster.

1 Majorana representations

A tuple

$$\mathcal{R} = (G, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)$$

is said to be a Majorana representation (of $G$) if the following conditions hold: $G$ is a finite group; $T$ is a set of involutions (elements of order 2) in $G$ which generates $G$ and is stable under conjugation by elements of $G$ (this means that $T$ is a generating union of conjugacy classes of involutions in $G$); $V$ is a real vector space equipped with an inner product $(\cdot, \cdot)$ and with a commutative non-associative algebra product $\cdot$, which associate with each other and satisfy the Norton inequality: $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$ for all $u, v \in V$;
\( \varphi : G \to GL(V) \)

is a homomorphism, whose image preserves \((,\)\) and \(\cdot\); and \(\psi\) is an an rule which injectively assigns to every \(t \in T\) a vector \(a_t = \psi(t)\) which is a Majorana axis in \((V, (,\), \()\) (cf. the definition in the next paragraph) and such that the Majorana involution associated with \(a_t\) coincides with \(\varphi(t)\). It is further assumed that \(V\) is generated by the elements \(a_t\) taken for all \(t \in T\), and \(\dim(V)\) is said to be the dimension of \(R\). It is assumed that \(\varphi\) and \(\psi\) commute in the sense that

\[
a_{g^{-1}tg} = (a_t)^{\psi(g)}
\]

for every \(g \in G\). For a subset \(X\) of vectors in \(V\) we denote the linear span of \(X\) in \(V\) and the algebra closure of \(X\) in \((V, (,\), \()\) by

\[
\langle X \rangle \text{ and } \langle\langle X \rangle\rangle,
\]

respectively. Thus \(\langle\langle X \rangle\rangle\) is the smallest subspace \(Y\) in \(V\) containing \(X\) such that \(y_1 \cdot y_2 \in Y\) whenever \(y_1, y_2 \in Y\).

By the definition in [1] a Majorana axis \(a\) in \((V, (,\), \()\) is an idempotent of length 1, whose adjoint operator

\[
ad_a : v \mapsto a \cdot v
\]

is semi-simple with spectrum \(\{1, 0, \frac{1}{4}, \frac{1}{32}\}\) (when talking about eigenvectors of \(a\) one really means eigenvectors of \(ad_a\)). The following conditions concerning the eigenspaces are imposed. The 1-eigenvectors of \(a\) are precisely the scalar multiples of \(a\). The Majorana involution \(\tau(a)\) associated with \(a\) is the linear transformation of \(V\) which negates every \(\frac{1}{32}\)-eigenvector and centralizes the other eigenvectors. By the above definition the Majorana involution is an automorphism of \((V, (,\), \()\). This condition is equivalent to the fusion rules involving \(\frac{1}{32}\)-eigenvectors: if \(v\) and \(u\) are \(\frac{1}{32}\)-eigenvectors of \(a\), and if \(x\) and \(y\) are \(\lambda\)- and \(\mu\)-eigenvectors for \(\lambda, \mu \in \{1, 0, \frac{1}{4}\}\), then \(v \cdot x\) is a \(\frac{1}{32}\)-eigenvector, and both \(v \cdot u\) and \(x \cdot y\) project to zero in the \(\frac{1}{32}\)-eigenspace. The remaining fusion rules are as follows: if \(\alpha_1\) and \(\alpha_2\) are 0-eigenvectors of \(a\), and \(\beta_1\) and \(\beta_2\) are \(\frac{1}{4}\)-eigenvectors of \(a\), then

\[
\alpha_1 \cdot \alpha_2, \; \beta_1 \cdot \beta_2 - (\beta_1 \cdot \beta_2, a) a \quad \text{and} \quad \alpha_1 \cdot \beta_1
\]

are \(\lambda\)-eigenvectors of \(a\), where \(\lambda = 0, 0\) and \(\frac{1}{4}\), respectively. The complete set of fusion rules can be read from Table 1 (where \(Sp = \{1, 0, \frac{1}{2}, \frac{1}{4}\}\) is the spectrum of \(a\)). The meaning of the fusion rules is the inclusion

\[
V^{(a)}_{\lambda} \cdot V^{(a)}_{\mu} \subseteq \bigoplus_{v \in Sp(\lambda, \mu)} V^{(a)}_{v}.
\]

where \(\lambda, \mu \in Sp\) and \(Sp(\mu, \lambda)\) is the \((\lambda, \mu)\)-entry in Table 1.
Sakuma’s theorem [9], together with Norton’s explicit description of the subalgebras in the Monster algebra generated by pairs of transposition axes [7], implies the following proposition, which constitutes the foundation of the Majorana theory.

**Proposition 1.1** [7,9] Let \( R = (G, T, V, (\cdot, \cdot), \varphi, \psi) \) be a Majorana representation. For two distinct involutions \( t_0 \) and \( t_1 \) in \( T \) put \( a_0 = \psi(t_0), a_1 = \psi(t_1), t_0 = \varphi(t_0), t_1 = \varphi(t_1), \rho = t_0t_1 \). Let \( D \) be the dihedral subgroup in \( G L(V) \) generated by \( t_0 \) and \( t_1 \), and let \( |D| = 2N \). Then the subalgebra \( Y = \langle \langle a_0, a_1 \rangle \rangle \) is isomorphic to one of the eight Norton–Sakuma algebras in Table 2 (more specifically to an algebra of type \( NX \), where \( N \) is as above and \( X \in \{A, B, C\} \)). For an integer \( i \) and \( \epsilon \in \{0, 1\} \) the vector \( a_{2i+\epsilon} \) is the image of \( a_i \) under the \( i \)-th power of \( \rho \) (so that \( \tau(a_{2i+\epsilon}) = \rho^{-i}\tau(\rho^\epsilon) \)), and the remaining vectors in the basis of \( Y \), given in the second column in Table 2, are centralized by \( D \). The kernel of the action of \( D \) on \( Y \) coincides with the centre of \( D \).

The shape of a Majorana representation \( R \) is a rule which specifies the type of the Norton–Sakuma subalgebra \( \langle \langle \psi(t_0), \psi(t_1) \rangle \rangle \) for every pair \( t_0, t_1 \) of involutions in \( T \). The rule must be stable under conjugation by the elements of \( G \) and must respect the embeddings of the algebras:

\[
2A \leftrightarrow 4B, \quad 2A \leftrightarrow 6A, \quad 2B \leftrightarrow 4A, \quad 3A \leftrightarrow 6A.
\]

The Monster algebra possesses important properties which we are included sometimes into the hypothesis:

(2A) the following conditions hold (where \( t_0, t_1, t_2 \in T \) and \( a_i = \psi(t_i) \) for \( 0 \leq i \leq 2 \):

(a) if \( t_0t_1t_2 = 1 \), then \( \langle \langle a_0, a_1 \rangle \rangle \) is of type 2A and \( a_2 = a_\rho := a_0 + a_1 - 8a_0 \cdot a_1 \);

(b) if \( \langle \langle a_0, a_1 \rangle \rangle \) is of type 2A, 4B or 6A, then \( t_0t_1, (t_0t_1)^2 \) or \( (t_0t_1)^3 \) belongs to \( T \), and \( \psi(t_0t_1), \psi((t_0t_1)^2) \) or \( \psi((t_0t_1)^3) \) coincides with \( a_\rho, a_\rho^2 \) or \( a_\rho^3 \).

(3A) the following condition holds (where \( t_0, t_1, t_2, t_3 \in T, a_i = \psi(t_i) \) for \( 0 \leq i \leq 3 \) if

(a) \( \langle t_0, t_1 \rangle \cong \langle t_2, t_3 \rangle \cong D_6 \);

(b) \( t_0t_1 = t_2t_3 \);

(c) both \( \langle \langle a_0, a_1 \rangle \rangle \) and \( \langle \langle a_2, a_3 \rangle \rangle \) have type 3A;

then the 3A-axial vectors \( u_{t_0t_0} \) and \( u_{t_2t_3} \) in the subalgebras in (c) are equal.
Table 2

| Type | Basis | Products and angles |
|------|-------|---------------------|
| 2A   | \(a_0, a_1, a_\rho\) | \(a_0 \cdot a_1 = \frac{1}{27}(a_0 + a_1 - a_\rho)\), \(a_0 \cdot a_\rho = \frac{1}{27}(a_0 + a_\rho - a_1)\) |
| 2B   | \(a_0, a_1\) | \(a_0 \cdot a_1 = 0, (a_0, a_1) = 0\) |
| 3A   | \(a_{-1}, a_0, a_1, a_\rho\) | \(a_0 \cdot a_1 = \frac{1}{25}(2a_0 + 2a_1 + a_{-1}) - \frac{33}{27} u_\rho\) |
|      | \(u_\rho\) | \(a_0 \cdot u_\rho = \frac{1}{27}(2a_0 - a_1 - a_{-1}) + \frac{5}{27} u_\rho\) |
| 3C   | \(a_{-1}, a_0, a_1\) | \(a_0 \cdot a_1 = \frac{1}{26}(a_0 + a_1 - a_{-1})\), \(a_0, a_1) = \frac{1}{26}\) |
| 4A   | \(a_{-1}, a_0, a_1, a_\rho\) | \(a_0 \cdot a_1 = \frac{1}{27}(3a_0 + 3a_1 + a_2 + a_{-1} - 3v_\rho)\) |
|      | \(a_2, v_\rho\) | \(a_0 \cdot v_\rho = \frac{1}{25}(5a_0 - 2a_1 - 2a_2 - 2a_{-1} + 3v_\rho)\) |
|      | \(v_\rho \cdot v_\rho = u_\rho\) | \(a_0 \cdot a_2 = 0\) |
| 4B   | \(a_{-1}, a_0, a_1, a_\rho^2\) | \(a_0 \cdot a_1 = \frac{1}{25}(a_0 + a_1 - a_{-1} - a_2 + a_\rho^2)\) |
| 5A   | \(a_{-2}, a_{-1}, a_0\) | \(a_0 \cdot a_1 = \frac{1}{25}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho\) |
|      | \(a_1, a_2, w_\rho\) | \(a_0 \cdot a_2 = \frac{1}{27}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho\) |
|      | \(a_0 \cdot w_\rho = \frac{1}{27}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{27} w_\rho\) |
|      | \(w_\rho \cdot w_\rho = u_\rho^2\) | \(u_\rho, w_\rho, a_\rho^2, a_\rho \) |
| 6A   | \(a_{-2}, a_{-1}, a_0\) | \(a_0 \cdot a_1 = \frac{1}{25}(2a_0 + 2a_1 - 2a_2 - a_{-1} - a_{-2}) - \frac{33}{27} u_\rho^2\) |
|      | \(a_{1,2,3}\) | \(a_0 \cdot a_2 = \frac{1}{25}(2a_0 + 2a_2 - a_{-1} - a_{-2}) - \frac{33}{27} u_\rho^2\) |
|      | \(a_\rho^3, u_\rho^2\) | \(a_0 \cdot u_\rho^2 = \frac{1}{27}(2a_0 - a_2 - a_\rho^3) + \frac{5}{27} u_\rho^2\) |
|      | \(a_0 \cdot a_3 = \frac{1}{27}(a_0 + a_3 - a_\rho^3)\), \(a_\rho^3 \cdot u_\rho^2 = 0\), \(a_\rho^3, u_\rho^2 = 0\) |
|      | \(a_0, a_1, a_2, a_3\) | \(a_0, a_2) = \frac{13}{25}\), \(a_0, a_2) = \frac{13}{25}\) |

2 The alternating group \(A_6\)

Let \(G \cong A_6\). Then \(G\) contains a single class of involutions, two classes of subgroups isomorphic to the alternating group \(A_5\) of degree 5 (say \(K_1\) and \(K_2\)), two classes of subgroups of order 3 (say \(H^{(1)}\) and \(H^{(2)}\)), one class of subgroups of order 5, and two classes of elements of order 5. Let \(K \in K_1\) be an \(A_5\)-subgroup in \(G\) and let \(\Omega = \{1, 2, 3, 4, 5, 6\}\) be the set of cosets of \(K\) in \(G\). We assume that the subgroups in \(H^{(1)}\) are generated by 3-cycles on \(\Omega\) and those in \(H^{(2)}\) by products of two commuting 3-cycles. Thus a Majorana representation of \(A_6\) satisfying conditions (2A) and (3A) has shape \((2A, 3X, 3Y, 4B, 5A)\), where
Majorana representation of $A_6$ involving 3C-algebras

$$\langle a_s, a_t \rangle \cong 3X \text{ or } 3Y \text{ if } \langle s, t \rangle \in H^{(1)} \text{ or } H^{(2)},$$

respectively. Since the classes $H^{(1)}$ and $H^{(2)}$ are fused in the automorphism group of $A_6$, a representation of shape $(2A, 3X, 3Y, 4B, 5A)$ twisted by a suitable automorphism of $A_6$ has shape $(2A, 3Y, 3X, 4B, 5A)$.

It was proved in [2] that $A_6$ possesses a unique representation of shape $(2A, 3A, 3A, 4B, 5A)$, which satisfies conditions (2A) and (3A). Here we prove the following.

**Theorem 1** The following assertions hold:

(i) there are no Majorana representations of $A_6$ of shape $(2A, 3A, 3C, 4B, 5A)$ satisfying condition (2A);

(ii) there exists at most one Majorana representation $\mathcal{R}^{CC}$ of $A_6$, satisfying condition (2A), whose shape is $(2A, 3C, 3C, 4B, 5A)$; the dimension of $\mathcal{R}^{CC}$ is at most 70.\(^1\)

3 Inner products

In this section by $\mathcal{R}^{AC}$ or $\mathcal{R}^{CC}$ we denote a representation satisfying the hypothesis in Theorem 1 (i) or (ii), so that

$$\langle A^2 \rangle = \langle A \cup U^{(1)} \cup W \rangle \text{ or } \langle A^2 \rangle = \langle A \cup W \rangle,$$

respectively, where $A = \{a_t \mid t^2 = 1\}$, $U^{(1)} = \{u_h \mid \langle h \rangle \in H^{(1)}\}$, $W = \{w_f \mid f^5 = 1\}$ with $u_{h^{-1}} = u_h$, $w_f = -w_{f^2} = -w_{f^3} = w_{f^4}$.

**Lemma 3.1** Let $\langle h \rangle, \langle k \rangle \in H^{(1)}$. If $[h, k] = 1$ then in the representation $\mathcal{R}^{AC}$ the inner product $(u_h, u_k)$ is negative.

**Proof** For $h = (1, 2, 3)$, $k = (4, 5, 6)$ and $t = (2, 3)(4, 5)$ the vector

$$\alpha_h^{(t)} = u_h - \frac{2 \cdot 5}{3^3} a_t + \frac{2^5}{3^3} (a_{hth^{-1}} + a_{h^{-1}th})$$

is a 0-eigenvector of $a_t$ in $\langle a_t, u_h \rangle \cong 3A$, and

$$\beta_k^{(t)} = u_k - \frac{2^3}{3^2 \cdot 5} a_t - \frac{2^5}{3^2 \cdot 5} (a_{kth^{-1}} + a_{k^{-1}th})$$

is a $\frac{1}{5}$-eigenvector of $a_t$ in $\langle a_t, u_k \rangle \cong 3A$. Expanding the orthogonality relation $(\alpha_h^{(t)}, \beta_k^{(t)}) = 0$ one gets

\(^1\) 70-dimensional representation $\mathcal{R}^{CC}$ was constructed by Ákos Seress (private communication of February 9, 2011).
Thus the Norton inequality condition fails under the hypothesis of Theorem 1 (i) and from now on we only deal with the representation \( \mathcal{R}^{CC} \) where all 3-elements are of type 3C. The Majorana representations of \( A_5 \) were classified in [5]. There only one such representation which satisfying the \( (2A) \) condition has shape \( (2A, 3C, 5A) \). Thus the restriction of \( \mathcal{R}^{CC} \) to an \( A_5 \)-subgroup is known.

Recall that \( \text{Out} (A_6) \) is elementary abelian of order 4, and if \( A_6 < X < \text{Aut} (A_6) \), then \( X \cong S_6, \ PGL_2(9) \) or \( M_{10} \). In \( S_6 \) all the elements of order 5 are conjugate, but there are still two classes of elements/subgroups of order 3; in \( PGL_2(9) \) there are two classes of elements of order 5, but all elements of order 3 are conjugate, while \( M_{10} \) enjoys both the conjugation properties: a single class of 5-elements and a single class of 3-elements.

The following lemma is a direct consequence of the Majorana axioms and the shape of \( \mathcal{R}^{CC} \).

**Lemma 3.2** If \( s \) and \( t \) are involutions in \( A_6 \), then \((a_s, a_t) = 1, \frac{1}{8}, \frac{1}{64}, \frac{1}{64} \) or \( \frac{3}{128} \) if \( o(st) = 1, 2, 3, 4 \) or 5, respectively. \( \square \)

**Lemma 3.3** Let \( r \) be an involution in \( A_6 \) and let \( f \) be an element of order 5 in \( A_6 \). Then the possible values of \((a_r, w_f)\) are described by Table 3, where \( f = (1, 2, 3, 4, 5) \).

**Proof** Since the restriction of \( \mathcal{R}^{CC} \) to an \( A_5 \)-subgroup is known, it is sufficient to justify the zero entries in the last column of the last two rows. For the sixth row consider \( t = (2, 5)(3, 4) \). Then \( t \) inverts \( f \) and generates with \( r \) a \( D_8 \)-subgroup. Therefore

\[
\beta^{(t)}_f = w_f + \frac{1}{2^7} (-a_{ftf^4} + a_{f^2tf^3} + a_{f^3tf^2} - a_{f^4tf})
\]

(which is a \( \frac{1}{4} \)-eigenvector of \( a_t \) in \( \langle a_t, w_f \rangle \cong 5A \)) and

\[
a^{(t)}_r = a_r + a_{rtt} - \frac{1}{2^5} a_t - \frac{1}{2^3} (a_{t(t)}^2 - a_{ttt})
\]

\( \Diamond \) Springer
Table 4 Order 5 subgroups in $A_6$

| (1, 2, 3, 4, 5) | (1, 4, 5, 2, 3) | (1, 2, 4, 5, 3) | (1, 2, 5, 3, 4) | (1, 2, 5, 4, 3) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| (6, 5, 3, 4, 2) | (6, 3, 5, 2, 4) | (6, 5, 4, 2, 3) | (6, 4, 5, 3, 2) | (6, 4, 5, 2, 3) |
| (6, 1, 4, 5, 3) | (6, 2, 5, 1, 4) | (6, 3, 1, 2, 5) | (6, 4, 2, 3, 1) |

(which is a 0-eigenvector of $a_t$ in $\langle a_t, a_r \rangle \cong 4B$) are perpendicular. Since $t$ stabilizes $w_f$ and swaps $a_r$ with $a_{trt}$, we have

$$(w_f, a_r) = (w_f, a_{trt}).$$

Now, making use of Lemma 3.1 and substituting the known entries from the last columns of the second and third rows in Table 3, we deduce the equality $(w_f, a_r) = 0$. For the seventh row the calculations are similar. □

Next we determine the inner products between $w_f$’s. Before doing that we describe the $A_6$-orbits on the pairs of its subgroups of order 5. The set of these subgroups forms a $6 \times 6$ grid with rows corresponding to the $A_5$-subgroups in $K_1$ and columns corresponding to the $A_5$-subgroups in $K_2$. Every subgroup of $F$ = $\langle f \rangle$ of order 5 in $A_6$ is contained in a unique subgroup $K_1(f) \in K_1$ and in a unique subgroup $K_2(f) \in K_2$, so that

$$K_1(f) \cap K_2(f) = N_{A_6}(F) \cong D_{10}.$$

The following (partially filled up) Table 4 contains generators of the subgroups of order 5 in two members of $K_1$ (the first and second rows) and in two members of $K_2$ (the first and the second columns). The generators are chosen to be $A_6$-conjugate, and the subgroup generated by the element in the $i$-th row and $j$-th column will be denoted by $F_{ij}$, so that $F_{11} = \langle (1, 2, 3, 4, 5) \rangle$.

**Lemma 3.4** Let $t$ be an involution in $A_6$. Then the following assertions hold:

(i) $t$ is contained in exactly two members of $K_1$ and in two members of $K_2$;
(ii) $t$ normalizes exactly four subgroups of order 5 in $A_6$;
(iii) if $F$ and $E$ are two distinct subgroups of order 5 normalized by $t$ then
   (a) there is an element $s \in A_6$ such that $E = s^{-1}F_s$ and $[s, t] = 1$;
   (b) $s^2 = 1$ if $E, F \cong A_5$ and $s^4 = 1 \neq s^2$ otherwise;
(iv) if $F$ is a subgroup of order 5 and $K$ is an $A_5$-subgroup not containing $F$, then $K$ contains a unique involution which normalizes $F$.

**Proof** These are all very elementary and easy to check, especially making use of the outer automorphism of $A_6$ which permutes $K_1$ and $K_2$. For instance, if $t = (2, 5)(3, 4)$, then the $A_5$-subgroups in (i) containing $t$ correspond to the first and the second row/column Table 4; the subgroups in (ii) normalized by $t$ are $F_{11}, F_{12}, F_{21}$ and $F_{22}$. Since
$N_{A_6}(t) \cong D_8$ and for $t \in A_5$ we have $N_{A_5}(t) \cong 2^2$, (iii) follows. The pair $\{F_{11}, F_{22}\}$ provides an example of two subgroups of order 5 normalized by a common involution, but not contained in a common $A_5$-subgroup. In (iv) if $F = F_{11}$ and $K$ corresponds to the second row (so that $K$ is the stabilizer in $A_6$ of $1 \in \Omega$) then $t = (2, 5)(3, 4)$ is the unique involution with the required properties.

**Lemma 3.5** Let $f$ and $e$ be elements of order 5 in $A_6$ contained in the same conjugacy class, such that $F = \langle f \rangle$ and $E = \langle e \rangle$ are distinct. Then the following assertions hold:

(i) $(w_f, w_f) = \frac{s^{3,7}}{2^{77}}$,

(ii) if $\langle F, E \rangle \cong A_5$, then $(w_f, w_e) = -\frac{s^{2,7}}{2^{77}}$;

(iii) if $\langle F, E \rangle = A_6$, then $(w_f, w_e) = 0$ whenever $f$ and $e$ are inverted by a common involution, and $(w_f, w_e) = \frac{s^{2,7}}{2^{77}}$, otherwise.

In order to prove the above lemma we will make use of the following restatement of Lemma 5.4 from [5].

**Lemma 3.6** Let $e$ be an element of order 5 in $K \cong A_5$ and $t$ be an involution in $K$ which does not normalize $\langle e \rangle$. Then

$$a_t \cdot w_e = (a_t, w_e) a_t - \frac{7}{2^7} (\beta^{(t)}_{f_1} + \beta^{(t)}_{f_2}) + \frac{1}{2^6} (w_e - w_{tet}),$$

where $f_1$ and $f_2$ are $K$-conjugates of $e$ inverted by $t$ and generating distinct subgroups, and for $d = f_i$ for $i = 1$ or $2$

$$\beta^{(t)}_d = w_d + \frac{1}{2^7} (-a_{d_1d_4} + a_{d_2d_3} + a_{d_3d_2} - a_{d_4d_1})$$

is a $\frac{1}{4}$-eigenvector of $a_t$ in $\langle a_t, w_f \rangle \cong 5A$. \hfill $\square$

An immediate consequence of Lemma 3.6 and the Majorana axioms is the following.

**Lemma 3.7** With $e$ and $t$ being as in Lemma 3.6 the projections of $w_e$ to $1$, $\frac{1}{32}$, $\frac{1}{4}$, and 0-eigenspaces of $a_t$ are equal to

$$\langle a_t, w_e \rangle a_t, \quad \frac{1}{2} (w_e - w_{tet}), \quad -\frac{7}{2^5} (\beta^{(t)}_{f_1} + \beta^{(t)}_{f_2}), \quad \text{and}$$

$$\varepsilon^{(t)}_e := \frac{1}{2} (w_e + w_{tet}) - (a_t, w_e) a_t + \frac{7}{2^5} (\beta^{(t)}_{f_1} + \beta^{(t)}_{f_2}),$$

respectively. \hfill $\square$

$\copyright$ Springer
Now suppose that \( f \not\in K \). Then by Lemma 3.4 (iv) we can assume without loss of generality that \( f \) is inverted by \( t \), which provides us with an algorithm of calculating \((w_f, w_e)\) from the orthogonality relation

\[
(e^{(t)}_e, \beta^{(t)}_f) = 0.
\]

In what follows we will demonstrate an easier method how to get \((w_f, w_e)\). At this stage we just summarize an important consequence of the orthogonality relation.

**Lemma 3.8** The inner product \((w_e, w_f)\) is the same whenever \( e \) and \( f \) are conjugate and not inverted by a common involution. \( \square \)

An important property of the Majorana representation of \( A_5 \) of shape \( (2A, 3C, 5A) \) is the following (cf. Lemma 5.7 (i) in [5]).

**Lemma 3.9** Let \( K^{(5)} \) be a set of conjugate representatives of the subgroups of order 5 in \( K \cong A_5 \). Then

\[
\sum_{f \in K^{(5)}} w_f = 0.
\]

\( \square \)

**Proof of Lemma 3.7.** The assertions (i) and (ii) follow from Lemmas 4.1 (vii) and 4.2 (ii) in [5], respectively. Let \( e \) and \( f \) be as in (iii). If \( t \) is an involution which simultaneously inverts \( e \) and \( f \), then the value of \((w_f, w_e)\) can be deduced from the orthogonality relation

\[
(e^{(t)}_e, \beta^{(t)}_f) = 0,
\]

where \( \beta^{(t)}_f \) is the \( \frac{1}{4} \)-eigenvector of \( a_t \) in \( \langle a_t, w_f \rangle \cong 5A \) defined in Lemma 3.6, while

\[
\alpha^{(t)}_e = w_e - \frac{3 \cdot 7}{2^{12}} a_t + \frac{7}{26} (a_{ete^4} + a_{e^4te})
\]

is a 0-eigenvector of \( a_t \) in \( \langle a_t, w_e \rangle \cong 5A \) (cf. Table 3 in [5]). Thus it only remains to handle the case when \( f \) and \( e \) are not inverted by a common involution. Let \( K \) be an \( A_5 \)-subgroup in \( A_6 \), let \( K^{(5)} \) be as in Lemma 3.9, and let \( e \) an element of order 5, which is not in \( K \), but conjugate in \( A_6 \) to members of \( K^{(5)} \). Then by Lemma 3.4 \( K^{(5)} \) contains a unique element contained in a common \( A_5 \)-subgroup with \( e \) and a unique element not in a common \( A_5 \)-subgroup with \( e \), but simultaneously with \( e \) inverted by an involution and four elements in the ‘general position’ with respect to \( e \). Thus if \( \lambda = (w_a, w_b) \) where \( a \) and \( b \) are in the general position, then by Lemmas 3.8 and 3.9 we obtain
\[
0 = \left( w_e, \sum_{f \in K^{(5)}} w_f \right) = -\frac{5^2 \cdot 7}{2^{19}} + 0 + 4 \lambda,
\]

which gives \( \lambda = \frac{5^2 \cdot 7}{2^{17}} \) as claimed. \( \square \)

4 Product closure

For a subgroup \( H \) in \( A_6 \) put \( A(H) = \{a_t \mid t \in H, t^2 = 1\}, \) \( W(H) = \{w_f \mid f \in H, f^5 = 1\}, \) and let \( X(H) \) denote the linear span of \( A(H) \cup W(H) \). Our goal is to prove that the algebra product \( \cdot \) is closed on \( X(A_6) \). Since \( \cdot \) is closed on \( X(K) \) for every \( A_5 \)-subgroup \( K \) in \( A_6 \), in order to achieve the goal it is sufficient to prove the following two assertions.

**Proposition 4.1** \( X(A_6) \) contains \( a_r \cdot w_f \) whenever \( \langle r, f \rangle = A_6 \).

**Proposition 4.2** \( X(A_6) \) contains \( w_f \cdot w_e \) whenever \( \langle f, e \rangle = A_6 \).

4.1 Proof of Proposition 4.1

Throughout the subsection we assume that \( f = (1, 2, 3, 4, 5) \) and \( r = (1, 6)(3, 5) \), so that the pair \( (f, r) \) corresponds to the sixth row of Table 3. The situation when the pair corresponds to the seventh row can be handled in a similar way.

There are five involutions in \( A_6 \) inverting \( f \). If \( s_j \) denotes the involution which inverts \( f \) and stabilizes the element \( j \in \Omega_1 \), then the information on the relationship between \( r \) and \( s_j \) can be read from the following table.

For every \( 1 \leq j \leq 5 \) the subalgebra \( Z_j = \langle \langle w_f, a_{s_j} \rangle \rangle \) is of type \( 5A \). Now for \( j \) being 4 or 5 we apply a variation of the resurrection principle (cf. Lemma 1.8 in [3]).

The vectors

\[
\alpha_f^{(s_j)} = w_f - \frac{3 \cdot 7}{2^{12}} a_{s_j} + \frac{7}{2^6} (a_{f s_j f^4} + a_{f^4 s_j f})
\]

and

\[
\beta_f^{(s_j)} = w_f + \frac{1}{2^7} (-a_{f s_j f^4} + a_{f^2 s_j f^3} + a_{f^3 s_j f^2} - a_{f^4 s_j f})
\]

are 0- and \( \frac{1}{4} \)-eigenvectors of \( a_{s_j} \) in \( Z_j \), while

\[
\alpha_r^{(s_j)} = (a_r + a_{s_j r s_j}) - \frac{1}{32} a_{s_j}
\]

is a 0-eigenvector of \( a_{s_j} \) in \( Y_j \cong 3C \).
Consider
\[
\alpha^{(s_j)} := \alpha_f^{(s_j)} \cdot \alpha_r^{(s_j)} = (a_r + a_{s_j s r s_j}) \cdot w_f + x,
\]
\[
\beta^{(s_j)} := \beta_f^{(s_j)} \cdot \alpha_r^{(s_j)} = (a_r + a_{s_j s r s_j}) \cdot w_f + y,
\]
which are 0- and \( \frac{1}{4} \)-eigenvectors of \( a_{s_j} \) and where the vectors \( x \) and \( y \) are contained in \( X(A_6) \), since \( X(A_6) \) contains the product \( a_t \cdot a_s \) for all \( t, s \in T \). By the resurrection principle Lemma 1.8 in [3] we have
\[
(a_r + a_{s_j s r s_j}) \cdot w_f = -[4a_{s_j} \cdot (x - y) + y].
\]
This gives the following (notice that \( w_f^{\phi(s_j)} = w_{f^{-1}} = w_f \)).

**Lemma 4.3** For \( j = 4 \) and \( 5 \) the vector \( a_r \cdot w_f \) and the negative of \( (a_r \cdot w_f)^{\phi(s_j)} = a_{s_j s r s_j} \cdot w_f \) are equal modulo \( X(A_6) \).

It is easy to prove the above lemma also for \( j = 1 \) and \( 3 \), but we don’t need it.

**Lemma 4.4** Modulo \( X(A_6) \) the vectors \( a_r \cdot w_f \) and \( (a_r \cdot w_f)^{\phi(f^2)} = a_{f^3 r f^2} \cdot w_f \) are equal.

**Proof** Multiplying \( s_4 \) by \( s_5 \) in \( A_6 \) we obtain \( f^2 \), and since \( (-1)(-1) = 1 \), the result follows.

The vector \( w_f - w_{r f r} \) is a \( \frac{1}{32} \)-eigenvector of \( a_r \):
\[
a_r \cdot (w_f - w_{r f r}) = \frac{1}{32} (w_f - w_{r f r}).
\]
Since the right hand side of the above equality in contained in \( X(A_6) \), this gives

**Lemma 4.5** The vectors \( a_r \cdot w_f \) and \( (a_r \cdot w_f)^{\phi(r)} = (a_r \cdot w_{r f r}) \) are equal modulo \( X(A_6) \).

Since \( \langle r, f \rangle = A_6 \), Lemmas 4.4 and 4.5 imply that
\[
(a_r \cdot w_f)^{\phi(g)} = a_r \cdot w_f \mod X(A_6)
\]
for every \( g \in A_6 \). Combining this equality taken for \( g = s_j \) with Lemma 4.3 we observer that \( a_r \cdot w_f \) is equal to its negative modulo \( X(A_6) \) which completes the proof of Proposition 4.1.
4.2 Proof of Proposition 4.2

The core of the argument here is similar to that in the previous subsection. Notice that $X(A_6)$ contains all the products $a_t \cdot a_s$ for $t, s \in T$ by definition and all the products $a_t \cdot w_f$ by Proposition 4.1 and since the product $\cdot$ is closed on $X(K)$ for every $A_5$-subgroup $K$ in $A_6$.

Let $f$ and $e$ be two conjugate elements of order 5 in $A_6$, generating the whole of $A_6$. Suppose first both $f$ and $e$ are inverted by an involution $t$. In this case the product $w_f \cdot w_e$ can be calculated by applying the resurrection principle to the 0-eigenvector

\[ \alpha_f^{(t)} = w_f - \frac{3 \cdot 7}{212} a_t + \frac{7}{26} (a_{ftf^4} + a_{f^4tf}) \]

of $a_t$ in $\langle a_t, w_f \rangle \cong S_5$ and the $\frac{1}{4}$-eigenvector

\[ \beta_e^{(t)} = w_f + \frac{1}{27} (-a_{ftf^4} + a_{f^3tf} + a_{f^4tf} - a_{f^4tf}) \]

of $a_t$ in $\langle a_t, w_e \rangle \cong S_5$. Therefore from now on we assume that $f$ and $e$ are not inverted by a common involution.

Let $K_i(e)$ be the unique $A_5$-subgroup in $K_i$ containing $e$ and let $t_i(f, e)$ be the unique involution in $K_i(e)$ which inverts $t$ (cf. Lemma 3.4 (iv)). Put

\[ t_1 = t_1(f, e), \quad t_2 = t_2(f, e), \quad t_3 = t_1(e, f), \quad t_4 = t_2(e, f). \]

**Lemma 4.6** The involutions $t_1, t_2, t_3$ and $t_4$ are pairwise distinct, and

\[ \langle t_1 t_2 \rangle = \langle f \rangle, \quad \langle t_3 t_4 \rangle = \langle e \rangle. \]

**Proof** Since $K_1(e) = K_1(e^{11})$ and $K_2(e) = K_2(e^{12})$, the equality $t_1 = t_2$ would immediately imply that $t_1$ inverts $e$. In a similar way one shows that $t_3 \neq t_4$. It is also clear that $t_j \neq t_k$ for $1 \leq j \leq 2$ and $3 \leq k \leq 4$ since $t_j$ inverts $f$, while $t_k$ inverts $e$, while $f$ and $e$ are chosen not to be inverted by a common involution. Since $t_1, t_2 \in N_{A_6}((f)) \cong D_{10}$ and $t_3, t_4 \in N_{A_6}((e)) \cong D_{10}$ the equalities in the assertion follow. \[ \square \]

For $i = 1$ or 2 let $\epsilon_e^{(t_i)}$ be the 0-eigenvector of $a_t$ defined in Lemma 3.7 and contained in the image of the restriction of $R^{CC}$ to $K_i(e) \cong A_5$. Let $\alpha_f^{(t_i)}$ and $\beta_f^{(t_i)}$ be the 0- and $\frac{1}{4}$-eigenvectors of $a_t$ as in the paragraph after Table 5 (with $s_j$ substituted by $t_i$) which are contained in $\langle a_t, w_f \rangle \cong S_5$. By the fusion rules we have

\[ \alpha := \alpha_f^{(t_i)} \cdot \epsilon_e^{(t_i)} = w_f \cdot (w_e + w_{t_i e t_i}) + u, \]

\[ \beta := \beta_f^{(t_i)} \cdot \epsilon_e^{(t_i)} = w_f \cdot (w_e + w_{t_i e t_i}) + v \]

are 0-and $\frac{1}{4}$-eigenvectors of $t_i$, where $u$ and $v$ are explicitly computable vectors from $X(A_6)$.\[ \square \]
Table 5 Neighbouring subalgebras

| j  | 4 | 5 | 1 | 3 | 2 |
|----|---|---|---|---|---|
| o(rs_j) | 3 | 3 | 4 | 4 | 5 |
| Y_i := ⟨⟨a_r, a_s_j⟩⟩ | 3C | 3C | 4B | 4B | 5A |

Lemma 4.7 The following assertions hold:

(i) for \( 1 \leq i \leq 4 \) the vector \( w_f \cdot w_e \) is equal to the negative of \( (w_f \cdot w_e)^{\psi(t_i)} = w_{t_i} f_{t_i} \cdot w_{t_i} e_{t_i} \) modulo \( X(A_6) \);

(ii) the vectors \( w_f \cdot w_e, (w_f \cdot w_e)^{\psi(f)} = w_f \cdot w_{f^{-1} e f} \) and \( (w_f \cdot w_e)^{\psi(e)} = w_{e^{-1} f e} \cdot w_e \) are equal modulo \( X(A_6) \).

Proof The assertion (i) for \( i = 1 \) or 2 follows from the resurrection principle applied to the eigenvectors \( \alpha \) and \( \beta \) defined just before the lemma. This assertion for \( i = 3 \) or 4 is obtained by exchanging the roles of \( f \) and \( e \). Now (ii) follows from (i) and Lemma 4.6.

Since \( \langle f, e \rangle = X(A_6) \), it follows from Lemma 4.7 (ii) that

\[
(w_f \cdot w_e)^{\psi(g)} \equiv w_f \cdot w_e \mod X(A_6)
\]

for every \( g \in A_6 \). Combining this equality for \( g = t_i \) with Lemma 4.7 (i) we conclude that \( w_f \cdot w_e \) equals to its negative modulo \( X(A_6) \) and Proposition 4.2 follows.

4.3 Dimension

By Propositions 4.1 and 4.2 the dimension of \( \mathcal{R}^{CC} \) equals to the rank of the Gram matrix \( \Gamma \) of the set \( A(A_6) \cup W(A_6) \) of size \( 45 + 36 = 81 \). We are certain that this matrix is singular. In fact, every \( A_5 \)-subgroup \( K \) in \( A_6 \) leads to a relation as in Lemma 3.9. There are 12 such subgroup, but there is at least one dependence among these twelve relations: the sums of the relations in Lemma 3.9 over \( K_1 \) and \( K_2 \) lead to the same equality

\[
\sum_{f \in X^{(5)}} w_f = 0,
\]

where \( X^{(5)} \) is a set of conjugate representatives of subgroups of order 5 in \( A_6 \). Therefore we have the following upper bound on the dimension of \( \mathcal{R}^{CC} \).

Proposition 4.8 The dimension of \( \mathcal{R}^{CC} \) is at most 70.\(^2\)

\(^2\) Igor Faradjev has computed the eigenvalues of the Gram matrix \( \Gamma \). The zero eigenvalue appears with multiplicity 11, which means that the upper bound is attained (private communication of January 4, 2011).
Open Access  This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

1. Ivanov, A.A.: The Monster Group and Majorana Involutions. Cambridge University Press, Cambridge (2009)
2. Ivanov, A.A.: On Majorana representations of $A_6$ and $A_7$. Commun. Math. Phys. (2011, To appear)
3. Ivanov, A.A., Pasechnik, D.V., Seress, Á., Shpectorov, S.: Majorana representations of the symmetric group of degree 4. J. Algebra 324, 2432–2463 (2010)
4. Ivanov, A.A., Shpectorov, S.: Majorana representations of $L_3(2)$. Adv. Geom. (submitted)
5. Ivanov, A.A., Seress, Á.: Majorana Representations of $A_5$. Math. Z (submitted)
6. Miyamoto, M.: Vertex operator algebras generated by two conformal vectors whose $\tau$-involutions generate $S_3$. J. Algebra 268, 653–671 (2003)
7. Norton, S.P.: The Monster algebra: some new formulae. In: Moonshine, the Monster and Related Topics. Contemp. Math., vol. 193, pp. 297–306. AMS, Providence (1996)
8. Norton, S.P.: Anatomy of the Monster I. In: The Atlas of Finite Groups: Ten Years On. LMS Lect. Notes Ser., vol. 249, pp. 198–214. Cambridge University Press, Cambridge (1998)
9. Sakuma, S.: 6-Transposition property of $\tau$-involutions of vertex operator algebras. Int. Math. Res. Notes 2007(9), 19 (2007)