On Optimal Robustness to Adversarial Corruption in Online Decision Problems

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Abstract

This paper considers two fundamental sequential decision-making problems: the problem of prediction with expert advice and the multi-armed bandit problem. We focus on stochastic regimes in which an adversary may corrupt losses, and we investigate what level of robustness can be achieved against adversarial corruptions. The main contribution of this paper is to show that optimal robustness can be expressed by a square-root dependency on the amount of corruption. More precisely, we show that two classes of algorithms, anytime Hedge with decreasing learning rate and algorithms with second-order regret bounds, achieve $O\left(\frac{\log N}{\Delta} + \sqrt{\frac{C \log N}{\Delta}}\right)$-regret, where $N$, $\Delta$, and $C$ represent the number of experts, the gap parameter, and the corruption level, respectively. We further provide a matching lower bound, which means that this regret bound is tight up to a constant factor. For the multi-armed bandit problem, we also provide a nearly tight lower bound up to a logarithmic factor.

1 Introduction

In this work, we consider two fundamental sequential decision-making problems, the problem of prediction with expert advice (expert problem) and the multi-armed bandit (MAB) problem. In both problems, a player chooses probability vectors $p_t$ over a given action set $[N] = \{1, 2, \ldots, N\}$ in a sequential manner. More precisely, in each round $t$, a player chooses a probability vector $p_t \in [0, 1]^N$ over the action set, and then an environment chooses a loss vector $\ell_t \in [0, 1]^N$. After the player chooses $p_t$, the player observes $\ell_t$ in the expert problem. In an MAB problem, the player picks action $i_t \in [N]$ following $p_t$ and then observe $\ell_{ti_t}$. The goal of the player is to minimize the (pseudo-) regret $\bar{R}_T$ defined as

$$R_{T,i^*} = \sum_{t=1}^T \ell_t^T p_t - \sum_{t=1}^T \ell_{ti^*}, \quad \bar{R}_{T,i^*} = \mathbf{E} [R_{T,i^*}], \quad \bar{R}_T = \max_{i^* \in [N]} \bar{R}_{T,i^*}. \quad (1)$$

For such decision-making problems, two main types of environments have been studied: stochastic environments and an adversarial environments. In stochastic environments, the loss vectors are assumed to follow an unknown distribution, i.i.d. for all rounds. It is known that the difficulty of the problems can be characterized by the suboptimality gap parameter $\Delta > 0$, which denotes the minimum gap between the expected losses for the optimal action and for suboptimal actions. Given the parameter $\Delta$, mini-max optimal regret bounds can be expressed as $\Theta\left(\frac{\log N}{\Delta}\right)$ in the expert problem [Degene and Perchet, 2016, Mourtada and Gaïffas, 2019] and $\Theta\left(\frac{N \log T}{\Delta}\right)$ in the MAB problem [Auer et al., 2002a, Lai and Robbins, 1985, Lai, 1987]. In contrast to the stochastic model, the adversarial model does not assume any generative models for loss vector, but the loss at each round may behave adversarially depending on the choices of the player up until that round. The mini-max optimal regret bounds for the adversarial model are $\Theta(\sqrt{T \log N})$ in the expert problem.
To overcome this BOBW-property limitation, our work focuses on an intermediate (and comprehensive) regime between the stochastic and adversarial settings. More precisely, we consider the adversarial regime with a self-bounding constraint introduced by Zimmert and Seldin [2021]. As shown by them, this regime includes the stochastic regime with adversarial corruptions [Lykouris et al., 2018; Amir et al., 2020] as a special case, in which an adversary modifies the i.i.d. losses to the extent that the total amount of changes does not exceed $C$, which is an unknown parameter referred to as the corruption level.

For the expert problem, Gaillard et al. [2014] have shown that the decreasing Hedge algorithm achieves $O\left(\frac{K \log T}{N}\right)$-regret in the stochastic regime and $O\left(\sqrt{KT}\right)$-regret in the adversarial regime. One limitation of BOBW guarantees is, however, that it does not necessarily provide nontrivial regret bounds for a situation in which the stochastic and the adversarial regimes are mixed, i.e., an intermediate situation.

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As can be seen in the regret bounds, achievable performance differs greatly between the stochastic and adversarial regimes, which implies that the choice of the models and algorithms will matter in many practical applications. One promising solution to this challenge is to develop best-of-both-worlds (BOBW) algorithms, which perform (nearly) optimally in both stochastic and adversarial regimes. For the expert problem, Gaillard et al. [2014] provide an algorithm with a BOBW property, and Mourtada and Gaïffas [2019] have shown that the well-known Hedge algorithm with decreasing learning rate (decreasing Hedge) enjoys a BOBW property as well. For the MAB problem, the Tsallis-INF algorithm by Zimmert and Seldin [2021] has a BOBW property, i.e., achieves $O\left(\frac{K \log T}{N}\right)$-regret in the stochastic regime and $O\left(\sqrt{KT}\right)$-regret in the adversarial regime. One limitation of BOBW guarantees is, however, that it does not necessarily provide nontrivial regret bounds for a situation in which the stochastic and the adversarial regimes are mixed, i.e., an intermediate situation.

Table 1: Regret bounds in stochastic regimes with adversarial corruptions

| Problem setting         | Upper bound                                      | Lower bound                                      |
|-------------------------|--------------------------------------------------|--------------------------------------------------|
| Expert problem          | $O\left(\frac{K \log T}{N}\right)$ [Amir et al., 2020] | $\Omega\left(\frac{K \log T}{N}\right)$         |
| Multi-armed bandit       | $O\left(\frac{N \log T}{\Delta} + \sqrt{\frac{C N \log T}{\Delta}}\right)$ [Zimmert and Seldin, 2021] | $\Omega\left(\frac{N \log T}{\Delta} + \sqrt{\frac{C N \log T}{\Delta}}\right)$ [Theorem 6] |

Note that Amir et al. [2020] adopt a different definition of regret than in this paper. Details and notes for comparison are discussed in Remark 1.
Zimmert and Seldin [2021] achieves a nearly optimal regret bound up to an $O(\log T)$ factor in the adversarial regime with self-bounding constraints.

The regret bounds in Theorems 3 and 4 are smaller than the regret bound shown by Amir et al. [2020] for the stochastic regime with adversarial corruptions, especially when $C = \Omega(\log N)$, and they can be applied to more general problem settings of the adversarial regime with self-bounding constraints. Note here that this study and their study consider slightly different definitions of regret. More precisely, they define regret using losses without corruptions, while this study uses losses after corruption to define regret. In practice, appropriate definitions would vary depending on situations. For example, if each expert’s prediction is itself corrupted, the former definition seems appropriate. However, even after taking this difference in definitions into account, we can see that the regret bound in our work is, in a sense, stronger than theirs, as is discussed in Remark 1 of this paper. In particular, we would like to emphasize that the new bound of $O(\log N + \sqrt{\log T \log N})$ provides the first theoretical evidence implying that the corresponding algorithms are more robust than the naive Follow-the-Leader algorithm, against adversarial corruptions. On the other hand, we also note that the regret bound by Amir et al. [2020] is tight as long as the former regret definition is used.

This work shows the tight regret upper bounds for two types of known algorithms. The first, (Theorem 3), is the decreasing Hedge algorithm, which has been analyzed by Amir et al. [2020] and Mourtada and Gaïffas [2019] as well. The second (Theorem 4) is algorithms with second-order regret bounds Cesa-Bianchi et al. [2007], Gaillard et al. [2014], Hazan and Kale [2010], Steinhardt and Liang [2014], Luo and Schapire [2015]. It is worth mentioning that Gaillard et al. [2014] have shown that a kind of second-order regret bounds imply $O(\log N)$-regret in the stochastic regime. Theorem 4 in this work extends their analysis to a broader setting of the adversarial regime with self-bounding constraints. In the proof of Theorems 3 and 4 we follow a proof technique given by Zimmert and Seldin [2021] to exploit self-bounding constraints.

To show regret lower bounds in Theorems 5 and 6 we construct specific environments with corruptions that provide insight into effective attacks which would make learning fail. Our approach to corruption is to modify the losses so that the optimality gaps reduce. This approach achieves a mini-max lower bound in the expert problem (Theorem 5) and a nearly-tight lower bound in MAB up to a logarithmic factor in $T$ (Theorem 6). We conjecture that there is room for improvement in this lower bound for MAB under assumptions on consistent policies [Lai and Robbins, 1985], and that the upper bound by Zimmert and Seldin [2021] is tight up to a constant factor.

2 Related work

In the context of the expert problem, studies on stochastic settings seem to be more limited than those on the adversarial setting. De Rooij et al. [2014] have focused on the fact that the Follow-the-Leader (FTL) algorithm works well for a stochastic setting, and they have provided an algorithm that combines FTL and Hedge algorithms to achieve the best of both worlds. Gaillard et al. [2014] have provided an algorithm with a second-order regret bound depending on $V_{T,\tau} = \sum_{\tau=1}^{T} (\ell_{t}^\top p_t - \ell_{t+\tau})^2$ in place of $T$ and have shown that such an algorithm achieves $O(\frac{\log N}{\Delta})$-regret in the stochastic regime. Mourtada and Gaïffas [2019] have shown that a simpler Hedge algorithm with decreasing learning rates of $\eta_t = \Theta(\sqrt{\frac{\log N}{t}})$ enjoys a tight regret bound in the stochastic regime as well. This simple decreasing Hedge algorithm has been shown by Amir et al. [2020] to achieve $O(\frac{\log N}{\Delta} + C)$-regret in the stochastic regime with adversarial corruptions. For online linear optimization, a generalization of the expert problem, Huang et al. [2016] have shown that FTL achieves smaller regret in the stochastic setting and provided best-of-both-worlds algorithms via techniques reported by Sani et al. [2014].

For MAB, there are a number of studies on best-of-both-worlds algorithms Bubeck and Slivkins [2012], Zimmert and Seldin [2021], Seldin and Slivkins [2014], Seldin and Lugosi [2017], Pozdin and Lattimore [2020], Auer and Chiang [2016], Wei and Luo [2018], Zimmert et al. [2019], Lee et al. [2021], Ito [2021]. Among these, studies by Wei and Luo [2018], Zimmert and Seldin [2021], Zimmert et al. [2019] are closely related to this work. In their studies, gap-dependent regret bounds in the stochastic regime are derived via $\{p_t\}$-dependent regret bounds in the adversarial
We note that the environment in Definition 1 includes the adversarial setting as a special case since \( N \) player is given

This paper focuses on environments in the following regime:

The regime defined in Definition 1 includes the following examples:

Studies on online optimization algorithms robust to adversarial corruptions has been extended to a variety of models, including those for the multi-armed bandit [Lykouris et al., 2018, Gupta et al., 2019, Zimmert and Seldin, 2021, Hajiesmaili et al., 2020]. Gaussian process bandits [Bogunovic et al., 2020], Markov decision processes [Lykouris et al., 2019], the problem of prediction with expert advice [Amir et al., 2020], online linear optimization [Li et al., 2019], and linear bandits [Bogunovic et al., 2021, Lee et al., 2021]. There can be found the literature on effective attacks to bandit algorithms [Jun et al., 2018, Liu and Shroff, 2019] as well.

As summarized by Hajiesmaili et al. [2020], there can be found studies on two different models of adversarial corruptions: the oblivious corruption model and the targeted corruption model. In the former (e.g., in studies by Lykouris et al. [2018], Gupta et al. [2019], Bogunovic et al. [2020]) the attacker may corrupt the losses \( \ell_t \) after observing \((\ell_t, p_t)\) without knowing the chosen action \( i_t \) while, in the latter (e.g., in studies by Jun et al. [2018], Hajiesmaili et al. [2020], Liu and Shroff [2019], Bogunovic et al. [2021]), the attacker can choose corruption depending on \((\ell_t, p_t, i_t)\). We discuss the differences between these models in Section 3. This work mainly focuses on the oblivious corruption model for MAB problems. It is worth mentioning that Tsallis-INF [Zimmert and Seldin, 2021] works well in the oblivious corruption models, as is shown in Table 1 as well as achieving best-of-both-worlds.

3 Problem setting

A player is given \( N \) the number of actions. In each round \( t = 1, 2, \ldots \) the player chooses a probability vector \( p_t = (p_{t1}, p_{t2}, \ldots, p_{tN})^\top \in \{ p \in [0, 1]^N | \|p\|_1 = 1 \} \), and then the environment chooses a loss vector \( \ell_t = (\ell_{t1}, \ell_{t2}, \ldots, \ell_{tN})^\top \in [0, 1]^N \). In the expert problem, the player can observe all entries of \( \ell_t \) after outputting \( p_t \). By way of contrast, in MAB problem, the player picks \( i_t \) w.r.t. \( p_t \), i.e., choose \( i_t \) so that \( \text{Prob}(i_t = i | p_t) = p_{ti} \) and then observes \( \ell_{ti} \). The performance of the player is measured by means of the regret defined in (1).

Note that in MAB problems we have

\[
E \left[ \sum_{t=1}^{T} (\ell_{ti_t} - \ell_{ti^*}) \right] = E \left[ \sum_{t=1}^{T} (\ell^\top_t p_t - \ell_{ti^*}) \right] = \tilde{R}_{T, i^*}
\]

under the assumption that \( \ell_t \) is independent of \( i_t \) given \( p_t \).

This paper focuses on environments in the following regime:

**Definition 1** (Adversarial regime with a self-bounding constraint [Zimmert and Seldin, 2021]). We say that the environment is in an adversarial regime with a \((i^*, \Delta, C, T)\) self-bounding constraint if

\[
\tilde{R}_{T, i^*} = E \left[ \sum_{t=1}^{T} (\ell_t^\top p_t - \ell_{ti^*}) \right] \geq \Delta \cdot E \left[ \sum_{t=1}^{T} (1 - p_{ti^*}) \right] - C
\]

holds for any algorithms, where \( \Delta \in [0, 1] \) and \( C \geq 0 \).

In this paper, we deal with the situation in which the player is not given parameters \((i^*, \Delta, C, T)\). We note that the environment in Definition 1 includes the adversarial setting as a special case since (3) holds for any \( \Delta \in [0, 1] \) and \( i^* \) if \( C \geq 2T \).

The regime defined in Definition 1 includes the following examples:

**Example 1** (Stochastic regime). Suppose \( \ell_t \in [0, 1]^N \) follows an unknown distribution \( D \) over i.i.d. for \( t \in \{1, 2, \ldots \} \). Denote \( \mu = E_{\ell \sim D}[\ell] \), and let \( i^* = \arg\min_i \{ \mu_i \} \). Then the environment is in the adversarial regime with a self-bounding constraint (3) with \( C = 0 \). Note here that \( \Delta > 0 \) and implies that the optimal action is unique, i.e., \( \mu_i > m_{i^*} \) holds for any action \( i \in [N] \setminus \{i^*\} \).

**Example 2** (Stochastic regime with adversarial corruptions). Suppose \( \ell_t \in [0, 1]^N \) is given as follows: (i) a temporary loss \( \ell'_t \in [0, 1]^N \) is generated from an unknown distribution \( D \) (i.i.d. for \( t \)) (ii) an adversary corrupts \( \ell'_t \) after observing \( p_t \) to determine \( \ell_t \) subject to the constraint of
\[ \sum_{t=1}^{T} \| \mathbb{E}[\ell_t] - \mathbb{E}[\ell_t'] \|_\infty \leq C. \] As shown in [Zimmert and Seldin, 2021], this regime satisfies 3, i.e., is a special case of the adversarial regime with a self-bounding constraint.

**Remark 1.** For the stochastic regime with adversarial corruptions, different regret notions can be found in the literature. An alternative to the definition in (1) is the regret w.r.t. the losses without corruptions, i.e., \( R_{T,i}^0 = \sum_{t=1}^{T} (\ell_t^i - \ell_t^i') \). \( R_{T,i}^0 \) can also be defined in a similar way. In general, which metric will be appropriate depends on the situation of the application. For example, in the case of prediction with expert advice, if each expert’s prediction is itself corrupted, the player’s performance should be evaluated in terms of the regret \( R \) determined by the losses \( \ell_t \) after corruptions, not by \( \ell_t' \). In contrast, if only the observation of the player is corrupted, the performance should be evaluated in terms of \( R_{T,i}^0 \).

We can easily see that \( |R_{T,i}^0 - R_{T,i}^0| \leq 2C \). Amir et al. [2020] have shown a regret bound of \( \bar{R}_T = O\left( \frac{\log N}{\Delta} + C \right) \), which immediately implies \( \bar{R}_T = O\left( \frac{\log N}{\Delta} + C \right) \). Similarly, a regret bound of \( \bar{R}_T = O\left( \frac{\log N}{\Delta} + \sqrt{C \log N} \right) \) immediately implies \( \bar{R}_T = O\left( \frac{\log N}{\Delta} + C \right) \). In fact, from the AM-DM inequality, we have

\[
\frac{\log N}{\Delta} + \sqrt{C \log N} \leq \frac{\log N}{\Delta} + \frac{1}{2} \left( C + \frac{\log N}{\Delta} \right) = O\left( C + \frac{\log N}{\Delta} \right).
\]

We here note that the former bound of \( \bar{R}_T = O\left( \frac{\log N}{\Delta} + C \right) \) is properly stronger than the latter of \( \bar{R}_T = O\left( \frac{\log N}{\Delta} + C \right) \), as the latter does not necessarily implies the former.

**Remark 2.** In MAB, a targeted corruption model has been considered to be a variant of the model in Example 2. In this model, the adversary corrupts the losses after observing \( i_t \). In this case, the loss \( \ell_t \) after corruptions and \( i_t \) are dependent given \( p_t \), and hence (2) does not always hold.

## 4 Regret upper bound

### 4.1 Known regret bounds for adversarial regimes by hedge algorithms

The Hedge algorithm [Freund and Schapire, 1997] (also called the multiplicative weight update [Arora et al., 2012] or the weighted majority forecaster [Littlestone and Warmuth, 1994]) is known to be a mini-max optimal algorithm for the expert problem. In the Hedge algorithm, the probability vector \( p_t \) is defined as follows:

\[ w_{t_i} = \exp \left( - \eta_t \sum_{j=1}^{t-1} \ell_{j,i} \right), \quad p_t = \frac{w_t}{\|w_t\|_1}. \quad (4) \]

where \( \eta_t > 0 \) are learning rate parameters. If \( p_t \) is given by (4), the regret is bounded as follows:

**Lemma 1.** If \( \{p_t\}_{t=1}^{T} \) is given by (4) with decreasing learning rates (i.e., \( \eta_t \geq \eta_{t+1} \) for all \( t \)), for any \( \{\ell_t\}_{t=1}^{T} \) and \( i^* \in [N] \), the regret is bounded as

\[
R_{T,i^*} \leq \frac{\log N}{\eta_{T}} + \sum_{t=1}^{T} \left( \frac{1}{\eta_t} \sum_{i=1}^{N} p_{t,i} g(\eta_t(-\ell_{t,i} + \alpha_t)) \right) + \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) H(p_{t+1}), \quad (5)
\]

for any \( \{\alpha_t\}_{t=1}^{T} \subseteq \mathbb{R} \), where \( g \) and \( H \) are defined as

\[
g(x) = \exp(x) - x - 1, \quad H(p) = \sum_{i=1}^{N} p_i \log \frac{1}{p_i}. \quad (6)
\]

From this lemma, using \( g(x) \approx x^2/2 \) and \( H(p) \leq \log N \), we obtain the following regret bounds for adversarial settings.

**Theorem 1** (Theorem 2.3 in [Cesa-Bianchi and Lugosi, 2006]). If \( p_t \) is given by (4) with \( \eta_t = \sqrt{\frac{8 \log N}{T}} \), for any \( T, i^* \in [N] \) and \( \{\ell_t\}_{t=1}^{T} \subseteq [0,1]^N \), the regret is bounded as

\[
R_{T,i^*} \leq \sqrt{2T \log N + \log N/8}. \quad (7)
\]
Hedge algorithms with decreasing rates $\eta_t = \Theta(\sqrt{\frac{\log N}{t}})$, as in Theorem 1, are referred to as decreasing Hedge, e.g., in [Mourtada and Gaïffas, 2019]. Such algorithms are shown by Mourtada and Gaïffas [2019] to achieve $O(\sqrt{\frac{\log N}{T}})$-regret in stochastic regimes, and are also shown, by Amir et al. [2020], to achieve $O(\sqrt{\frac{\log N}{T} + C})$-regret in stochastic regimes with adversarial corruptions.

Besides such worst-case regret bounds as found in Theorem 1, various data-dependent regret bounds have been developed (see, e.g., [Steinhardt and Liang, 2014]). One remarkable example is that of the second-order bounds by Cesa-Bianchi et al. [2007], which depend on parameters $V_T$ defined as follows:

$$v_t = \sum_{i=1}^{N} p_i (\ell_{t,i} - \ell_t^*)^2, \quad V_T = \sum_{t=1}^{T} v_t$$

(8)

A regret bound depending on $V_T$ rather than $T$ can be achieved by the following adaptive learning rates:

**Theorem 2** (Theorem 5 in Cesa-Bianchi et al., 2007). If $p_t$ is given by (4) with $\eta_t = \min \bigg\{ \sqrt{\frac{2(\sqrt{2} - 1) \log N}{(x - 2) \log(x - 1)}}, 1 \bigg\}$, the regret is bounded as

$$R_{T,i^*} \leq 4 \sqrt{V_T \log N} + 2 \log N + 1/2$$

(9)

for any $T$, $i^*$ and $\{\ell_t\}_{t=1}^{T} \subseteq [0, 1]^N$.

As $V_T \leq T$ follows from the definition (8), the regret bound in (2) includes the worst-case regret bound of $R_{T,i^*} = O(\sqrt{T \log N})$. Further, as shown in Corollary 3 of Cesa-Bianchi et al. [2007], the bound in Theorem 2 implies $R_{T,i^*} = O(\sqrt{L_{T,i^*} (T - L_{T,i^*}) \log N})$, where $L_{T,i^*} = \sum_{t=1}^{T} \ell_{t,i^*}$. This means that the regret will be improved if the cumulative loss $L_{T,i^*}$ for optimal action $i^*$ is small or is close to $T$.

### 4.2 Refined regret bound for decreasing Hedge

This subsection shows that the algorithm described in Theorem 1 enjoys the following regret bound as well:

**Theorem 3.** If $p_t$ is given by (4) with $\eta_t = \sqrt{\frac{8 \log N}{t}}$, we have

$$\tilde{R}_{T,i^*} \leq 33 + \frac{100 \log N}{\Delta} + 10 \sqrt{\frac{C \log N}{\Delta}}$$

under the assumption that (3) holds.

**Proof.** Using the fact that $g(x) \leq \frac{(x-1)x^2}{2}$ for $x \leq 1$ and $H(p) \leq (1-p) (1 + \log N - \log (1 - p))$, from Lemma 1, we obtain

$$R_{T,i^*} \leq 32 \log N + \sqrt{8 \log N} \sum_{t=1}^{T} \frac{1 - p_{t,i^*}}{\sqrt{t}} \left(1 + \frac{1}{16 \log N} \log \frac{1}{1 - p_{t,i^*}}\right)$$

(11)

A complete proof for (11) can be found in the appendix. From (3) and (11), for any $\lambda > 0$, we have

$$\tilde{R}_{T,i^*} = (1 + \lambda) \tilde{R}_{T,i^*} - \lambda \tilde{R}_{T,i^*} \leq \mathbb{E} \left[ (1 + \lambda) \left( 32 \log N + \sqrt{8 \log N} \sum_{t=1}^{T} \frac{1 - p_{t,i^*}}{\sqrt{t}} \left(1 - \frac{\log(1 - p_{t,i^*})}{16 \log N}\right) \right) - \lambda \left( \Delta \sum_{t=1}^{T} (1 - p_{t,i^*}) - C \right) \right]$$

$$\leq 32 (1 + \lambda) \log N + \lambda C + \frac{1 + \lambda}{\sqrt{32 \log N}} \mathbb{E} \left[ \sum_{t=1}^{T} \frac{1 - p_{t,i^*}}{\sqrt{t}} \left(16 \log N - \frac{\lambda \Delta \sqrt{32 t \log N}}{1 + \lambda} - \log(1 - p_{t,i^*})\right) \right].$$
To bound the values of the expectation, we use the following inequality
\[
\sum_{t=1}^{T} \frac{x_t}{\sqrt{T}} (a - b\sqrt{T - \log x_t}) \leq \frac{2a^2 + 1}{b} + b
\] (12)
which holds for any \(a, b > 0, T\) and \(\{x_t\}_{t=1}^{T} \subseteq (0, 1)\). A proof of (12) is given in the appendix.

Combining the above two displayed inequalities with \(a = 16 \log N\) and \(b = \frac{\lambda \Delta + 2 \log N}{1 + \lambda}\) we obtain
\[
\bar{R}_{T_{i^*}} \leq 32(1 + \lambda) + \lambda C + \left(1 + \lambda^2 (16 \log N + 1)\right) + \Delta
\]
\[
= 33 + \frac{36 \log N}{\Delta} + \lambda \left(32 + C + \frac{18 \log N}{\Delta}\right) + \frac{1.18 \log N}{\Delta}.
\]
By choosing \(\lambda = \sqrt{\frac{16 \log N}{\Delta (32 + C) + 16 \log N}}\), we obtain
\[
\bar{R}_{T_{i^*}} \leq 33 + \frac{36 \log N}{\Delta} + 2 \sqrt{\left(32 + C + \frac{18 \log N}{\Delta}\right) \frac{18 \log N}{\Delta}}
\]
\[
\leq 33 + \frac{100 \log N}{\Delta} + 10 \sqrt{\frac{C \log N}{\Delta}},
\]
where the second inequality follows from \(\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}\) for \(x, y \geq 0\).

Combining Theorems 1 and 3, we can see that the decreasing Hedge with \(\eta_t = \sqrt{\frac{8 \log N}{t}}\) achieves
\[
\bar{R}_{T_{i^*}} = O(\min\{\frac{\log N}{t} + \sqrt{\frac{C \log N}{\Delta}}, \sqrt{T \log N}\})
\]
in the adversarial regime with self-bounding constraints. This bound will be shown to be tight up to a constant factor in Section 5.

4.3 Refined regret bound for adaptive Hedge

In this subsection, we show that a second-order regret bound as seen in Theorem 2 implies tight gap-dependent regret bounds in the adversarial regime with a self-bounding constraint.

We start from the observation that \(v_t\) defined in (8) satisfies \(v_t \leq (1 - p_{t^*})\) for any \(i^*\). In fact, we have \(v_t \leq \sum_{i=1}^{N} p_{ti} (\ell_{ti} - \alpha)^2\) for any \(\alpha \in \mathbb{R}\) as the right-hand side is minimized when \(\alpha = \ell_{t^*}^t p_{t^*}\), from which it follows that
\[
v_t \leq \sum_{i=1}^{N} p_{ti} (\ell_{ti} - \ell_{t^*})^2 = \sum_{i \in [N] \setminus \{i^*\}} p_{ti} (\ell_{ti} - \ell_{t^*})^2 \leq \sum_{i \in [N] \setminus \{i^*\}} p_{ti} = 1 - p_{t^*}.
\] (13)

Hence, the regret bound in Theorem 2 implies
\[
R_{T_{i^*}} \leq 4 \sqrt{\log N \sum_{t=1}^{T} (1 - p_{t^*}) + 2 \log N + 1/2}.
\] (14)

Such a regret bound depending on \(\sum_{t=1}^{T} (1 - p_{ti^*})\) leads to a tight gap-dependent regret bound, as shown in the following theorem:

**Theorem 4.** Suppose that the regret is bounded as
\[
R_{T_{i^*}} \leq \sqrt{A \sum_{t=1}^{T} (1 - p_{ti^*}) + B}.
\] (15)

Then, under the condition of (3), the pseudo-regret is bounded as
\[
\bar{R}_{T_{i^*}} \leq A \Delta + B + \sqrt{\frac{A(B + C)}{2\Delta}}.
\] (16)
Proof. From (13) and (3), for any $\lambda > 0$ we have
\[
\tilde{R}_{T_i^*} = (1 + \lambda)R_{T_i^*} - \lambda\tilde{R}_{T_i^*},
\]
\[
\begin{align*}
&\leq E \left[ (1 + \lambda) \left( A \sum_{t=1}^{T} (1 - p_{t,i^*}) + B \right) - \lambda \left( \Delta \sum_{t=1}^{T} (1 - p_{t,i^*}) - C \right) \right] \\
&\leq A(1 + \lambda)^2 + (1 + \lambda)B + \lambda C = \frac{A}{2\Delta} + B + \frac{A}{4\Delta} + \lambda \left( \frac{A}{4\Delta} + B + C \right),
\end{align*}
\]
where the second inequality follows from $a\sqrt{x} - bx = -b(\sqrt{x} - \frac{a}{2b})^2 + \frac{a^2}{4b} \leq \frac{a^2}{4b}$ for $a > 0, b \in \mathbb{R}$ and $x \geq 0$. By choosing $\lambda = \sqrt{\frac{A}{2\Delta(B+C)}}$, we obtain
\[
\tilde{R}_{T_i^*} \leq \frac{A}{2\Delta} + B + \sqrt{\frac{A(B+C)}{2\Delta}},
\]
where the second inequality follows from $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$.

Combining this theorem and (14), we obtain the following regret bound for the algorithm described in Theorem 2:

Corollary 1. If $p_i$ is chosen by (4) with $\eta_i = \min\{1, \sqrt{2(\sqrt{e} - 1) \log N}\}$, under the condition of (3), the pseudo regret is bounded as $\tilde{R}_{T_i^*} \leq 16\log N + 4\sqrt{\frac{(3\log N + C) \log N}{\Delta}} + 3\log N$.

Theorem 4 can be applied to algorithms other than the one in Theorem 2. One example is an algorithm by [Gaillard et al., 2014]. In Corollary 8 of their paper, a regret bound of $R_{T_i^*} \leq C_1\sqrt{\log N \sum_{t=1}^{T} (\ell_t^* - \bar{\ell}_t^*)^2 + C_2}$ is provided. Then, as it holds that $(\ell_t^* - \bar{\ell}_t^*)^2 \leq 1 - p_{t,i^*}$, we have (15) with appropriate $A$ and $B$, and consequently, we obtain a regret bound given in (16).

5 Regret lower bound

This section provides (nearly) tight lower bounds for the expert problem and the MAB problem in the adversarial regime with a self-bounding constraint. We start with describing the statement for the expert problem:

Theorem 5. For any $\Delta \in (0, 1/4)$, $N \geq 4$, $T \geq 4\log N$, $C \geq 0$, and for any algorithm for the expert problem, there exists an environment in the adversarial regime with a $(i^*, \Delta, N, C, T)$ self-bounding constraint for which the pseudo-regret is at least
\[
\tilde{R}_{T_i^*} = \Omega \left( \min \left\{ \frac{\log N}{\Delta}, \sqrt{\frac{C \log N}{\Delta}}, \sqrt{T \log N} \right\} \right).
\] (17)

To show this lower bound, we define a distribution $D_{\Delta, i^*}$ over $\{0, 1\}^N$ for $\Delta \in (0, 1/4)$ and $i^* \in [N]$, as follows: if $\ell \sim D_{\Delta, i^*}$, $\ell_{i^*}$ follows a Bernoulli distribution of parameter $1/2 - \Delta$, i.e., $\text{Prob}[\ell_{i^*} = 1] = 1/2 - \Delta$ and $\text{Prob}[\ell_{i^*} = 0] = 1/2 + \Delta$, and $\ell_i$ follows a Bernoulli distribution of parameter $1/2$ for $i \in [N] \setminus i^*$, independently. We then can employ the following lemma:

Lemma 2 (Proposition 2 in [Mourtada and Gaïffas, 2019]). For any algorithm and for any $\Delta \in (0, 1/4)$, $N \geq 4$ and $T \geq \frac{\log N}{\Delta^2}$, there exists $i^*$ such that $\tilde{R}_{T_i^*} \geq \frac{\log N}{\Delta^2}$ holds for $(\ell_i)_{i=1}^{T} \sim D_{\Delta, i^*}$.

Using this lower bound, we can show Theorem 5.

Proof of Theorem 5. We show lower bounds for the following four cases: (i) If $T \leq \frac{\log N}{\Delta^2}$, $\tilde{R}_{T_i^*} = \Omega(\sqrt{T \log N})$. (ii) If $\frac{C}{\Delta^2} \leq \frac{\log N}{\Delta^2} \leq T$, $\tilde{R}_{T_i^*} = \Omega(\log N / \Delta)$. (iii) If $\frac{\log N}{\Delta^2} \leq \frac{C}{\Delta^2} \leq T$, $\tilde{R}_{T_i^*} = \Omega(\sqrt{\frac{C \log N}{\Delta}})$. (iv) If $\frac{\log N}{\Delta^2} \leq T \leq \frac{C}{\Delta^2}$, $\tilde{R}_{T_i^*} = \Omega(\sqrt{T \log N})$. Combining all four cases of (i)-(iv), we obtain (17).
(i) Suppose \( T < \frac{\log N}{\Delta} \). Set \( \Delta' = \frac{\log N}{T} \). We then have \( T = \frac{\log N}{\Delta'} \) and \( \Delta < \Delta' \leq 1/4 \). If \( \ell_t \sim D_{\Delta',i} \) for all \( t \in [T] \), then the environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint for any \( C \geq 0 \), and the regret is bounded as \( R_{T,i^*} \geq \log N \frac{\log N}{256\Delta'} = \Omega(\sqrt{T \log N}) \) from Lemma 2.

(ii) Suppose \( \frac{\log N}{T} < C \leq 1 \). If \( \ell_t \sim D_{\Delta,i} \) for all \( t \in [T] \), the regret is bounded as \( R_{T,i^*} \geq \log N \frac{\log N}{C} \) for some \( i^* \) from Lemma 3. The environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding for any \( C \geq 0 \).

(iii) Assume \( \frac{\log N}{T} \leq C \leq T \). Define \( \Delta' = \sqrt{\frac{\Delta \log N}{C}} \leq \Delta \). We then have \( \frac{\log N}{\Delta'} = \sqrt{\frac{C \log N}{\Delta}} \). Let \( T' = \lceil \frac{\log N}{\Delta} \rceil = \lceil \frac{\Delta}{\Delta'} \rceil \leq T \). Consider an environment in which \( \ell_t \sim D_{\Delta',i} \) for \( t \in [T'] \) and \( \ell_t \sim D_{\Delta,i} \) for \( t \in [T'+1, T] \). Then from Lemma 2 there exists \( i^* \in [N] \) such that \( R_{T,i^*} = R_{T',i^*} \geq \frac{\log N}{\Delta'} = \Omega(\sqrt{\frac{C \log N}{\Delta}}) \). Further, we can show that the environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint. In fact, we have \( \sqrt{T' (\Delta - \Delta')} \leq \frac{C}{\Delta} (\Delta - \Delta') \leq C \).

(iv) Suppose \( \frac{\log N}{T} \leq T \leq \frac{\log N}{C} \). Consider \( \ell_t \sim D_{\Delta',i} \) for all \( t \in [T] \). Then the regret is bounded as \( R_{T,i^*} \geq \frac{\log N}{256\Delta'} = \Omega(\sqrt{T \log N}) \) for some \( i^* \), from Lemma 2. We can confirm that the environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint, as we have \( \Delta' T \leq \Delta T \leq C \), where the first and second inequalities follow from \( \frac{\log N}{\Delta} \leq T \) and \( T \leq \frac{\log N}{C} \), respectively.

Via a similar strategy to this proof, we can show the regret lower bound for the MAB problem as well:

**Theorem 6.** For any \( \Delta \in (0, 1/4) \), \( N \geq 4 \), \( T \geq 4 \log N \), \( C \geq 0 \), and for any multi-armed bandit algorithm, there exists an environment in the adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint for which the pseudo-regret is at least

\[
R_{T,i^*} = \Omega\left(\min\left\{ \frac{N}{\Delta} + \sqrt{\frac{CN}{\Delta}}, \sqrt{NT} \right\}\right).
\]

We can show this theorem by means of the following lemma:

**Lemma 3 (Auer et al., 2002b).** For any multi-armed bandit algorithm and for any \( \Delta \in (0, 1/4) \), \( N \geq 4 \) and \( T \geq \frac{N}{\Delta^2} \), there exists \( i^* \) such that \( R_{T,i^*} \geq \frac{N}{2\Delta^2} \) holds for \( (\ell_t)_{t=1}^{T} \sim D_{\Delta,i}^{T} \).

A complete proof of Theorem 6 can be found in the appendix.

### 6 Discussion

In this paper, we have shown \( O(R + \sqrt{CR}) \)-regret bounds for the expert problem, where \( R \) stands for the regret bounds for the environment without corruptions and \( C \) stands for the corruption level. From the matching lower bound, we can see that this \( O(\sqrt{CR}) \)-term characterizes the optimal robustness against the corruptions. One natural question is whether such an \( O(R + \sqrt{CR}) \)-type regret bounds can be found for other online decision problems, such as online linear optimization, online convex optimization, linear bandits, and convex bandits. Other than the algorithms for the expert problem, the Tsallis-INF algorithm by Zimmert and Seldin (2021) for the MAB problem is only concrete example that achieves \( O(R + \sqrt{CR}) \)-regret to our knowledge. What these algorithm have in common is that they use a framework of Follow-the-Regularized-Leader with decreasing learning rates and that they achieve the best-of-both-world simultaneously. As Amir et al. (2020) suggest, Online Mirror Descent algorithms does not have \( O(R + \sqrt{CR}) \)-regret bound, in contrast to Follow-the-Regularized-Leader. We believe that characterizing algorithms with \( O(R + \sqrt{CR}) \)-regret bounds is an important future work.

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References

I. Amir, I. Attias, T. Koren, Y. Mansour, and R. Livni. Prediction with corrupted expert advice. *Advances in Neural Information Processing Systems*, 33, 2020.

S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: A meta-algorithm and applications. *Theory of Computing*, 8(1):121–164, 2012.

J.-Y. Audibert and S. Bubeck. Minimax policies for adversarial and stochastic bandits. In *Conference on Learning Theory*, pages 217–226, 2009.

P. Auer and C.-K. Chiang. An algorithm with nearly optimal pseudo-regret for both stochastic and adversarial bandits. In *Conference on Learning Theory*, pages 116–120. PMLR, 2016.

P. Auer, N. Cesa-Bianchi, and P. Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47(2-3):235–256, 2002a.

P. Auer, N. Cesa-Bianchi, Y. Freund, and R. E. Schapire. The nonstochastic multiarmed bandit problem. *SIAM Journal on Computing*, 32(1):48–77, 2002b.

I. Bogunovic, A. Krause, and J. Scarlett. Corruption-tolerant gaussian process bandit optimization. In *International Conference on Artificial Intelligence and Statistics*, pages 1071–1081. PMLR, 2020.

I. Bogunovic, A. Losalka, A. Krause, and J. Scarlett. Stochastic linear bandits robust to adversarial attacks. In *International Conference on Artificial Intelligence and Statistics*, pages 991–999. PMLR, 2021.

S. Bubeck and A. Slivkins. The best of both worlds: Stochastic and adversarial bandits. In *Conference on Learning Theory*, pages 42.1–42.23, 2012.

N. Cesa-Bianchi and G. Lugosi. *Prediction, Learning, and Games*. Cambridge University Press, 2006.

N. Cesa-Bianchi, Y. Mansour, and G. Stoltz. Improved second-order bounds for prediction with expert advice. *Machine Learning*, 66(2):321–352, 2007.

S. De Rooij, T. Van Erven, P. D. Grünwald, and W. M. Koolen. Follow the leader if you can, hedge if you must. *The Journal of Machine Learning Research*, 15(1):1281–1316, 2014.

R. Degenne and V. Perchet. Anytime optimal algorithms in stochastic multi-armed bandits. In *International Conference on Machine Learning*, pages 1587–1595. PMLR, 2016.

Y. Freund and R. E. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of Computer and System Sciences*, 55(1):119–139, 1997.

P. Gaillard, G. Stoltz, and T. Van Erven. A second-order bound with excess losses. In *Conference on Learning Theory*, pages 176–196. PMLR, 2014.

A. Gupta, T. Koren, and K. Talwar. Better algorithms for stochastic bandits with adversarial corruptions. In *Conference on Learning Theory*, pages 1562–1578. PMLR, 2019.

M. Hajiesmaili, M. S. Talebi, J. Lui, W. S. Wong, et al. Adversarial bandits with corruptions: Regret lower bound and no-regret algorithm. *Advances in Neural Information Processing Systems*, 33, 2020.

E. Hazan and S. Kale. Extracting certainty from uncertainty: Regret bounded by variation in costs. *Machine learning*, 80(2-3):165–188, 2010.

R. Huang, T. Lattimore, A. György, and C. Szepesvári. Following the leader and fast rates in linear prediction: Curved constraint sets and other regularities. In *Advances in Neural Information Processing Systems*, pages 4970–4978, 2016.

S. Ito. Parameter-free multi-armed bandit algorithms with hybrid data-dependent regret bounds. In *Conference on Learning Theory*, pages 2552–2583. PMLR, 2021.
K.-S. Jun, L. Li, Y. Ma, and X. Zhu. Adversarial attacks on stochastic bandits. In Proceedings of the 32nd International Conference on Neural Information Processing Systems, pages 3644–3653, 2018.

T. L. Lai. Adaptive treatment allocation and the multi-armed bandit problem. Annals of Statistics, 15(3):1091–1114, 1987.

T. L. Lai and H. Robbins. Asymptotically efficient adaptive allocation rules. Advances in Applied Mathematics, 6(1):4–22, 1985.

T. Lattimore and C. Szepesvári. Bandit Algorithms. Cambridge University Press, 2020.

C.-W. Lee, H. Luo, C.-Y. Wei, M. Zhang, and X. Zhang. Achieving near instance-optimality and minimax-optimality in stochastic and adversarial linear bandits simultaneously. arXiv preprint arXiv:2102.05858, 2021.

Y. Li, E. Y. Lou, and L. Shan. Stochastic linear optimization with adversarial corruption. arXiv preprint arXiv:1909.02109, 2019.

N. Littlestone and M. Warmuth. The weighted majority algorithm. Information and Computation, 108(2):212–261, 1994.

F. Liu and N. Shroff. Data poisoning attacks on stochastic bandits. In International Conference on Machine Learning, pages 4042–4050. PMLR, 2019.

H. Luo and R. E. Schapire. Achieving all with no parameters: Adanormalhedge. In Conference on Learning Theory, pages 1286–1304. PMLR, 2015.

T. Lykouris, V. Mirrokni, and R. Paes Leme. Stochastic bandits robust to adversarial corruptions. In Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, pages 114–122, 2018.

T. Lykouris, M. Simchowitz, A. Slivkins, and W. Sun. Corruption robust exploration in episodic reinforcement learning. arXiv preprint arXiv:1911.08689, 2019.

S. Masoudian and Y. Seldin. Improved analysis of the tsallis-inf algorithm in stochastically constrained adversarial bandits and stochastic bandits with adversarial corruptions. In Conference on Learning Theory, pages 3330–3350. PMLR, 2021.

J. Mourtada and S. Gaïffas. On the optimality of the hedge algorithm in the stochastic regime. Journal of Machine Learning Research, 20:1–28, 2019.

R. Pogodin and T. Lattimore. On first-order bounds, variance and gap-dependent bounds for adversarial bandits. In Uncertainty in Artificial Intelligence, pages 894–904, 2020.

A. Sani, G. Neu, and A. Lazaric. Exploiting easy data in online optimization. In Advances in Neural Information Processing 27, 2014.

Y. Seldin and G. Lugosi. An improved parametrization and analysis of the exp3++ algorithm for stochastic and adversarial bandits. In Conference on Learning Theory, pages 1743–1759, 2017.

Y. Seldin and A. Slivkins. One practical algorithm for both stochastic and adversarial bandits. In International Conference on Machine Learning, pages 1287–1295, 2014.

J. Steinhardt and P. Liang. Adaptivity and optimism: An improved exponentiated gradient algorithm. In International Conference on Machine Learning, pages 1593–1601. PMLR, 2014.

C.-Y. Wei and H. Luo. More adaptive algorithms for adversarial bandits. In Conference On Learning Theory, pages 1263–1291, 2018.

J. Zimmert and Y. Seldin. Tsallis-inf: An optimal algorithm for stochastic and adversarial bandits. Journal of Machine Learning Research, 22(28):1–49, 2021.

J. Zimmert, H. Luo, and C.-Y. Wei. Beating stochastic and adversarial semi-bandits optimally and simultaneously. In International Conference on Machine Learning, pages 7683–7692. PMLR, 2019.
A Appendix

A.1 Proof of Lemma 1

The Hedge algorithm defined in (4) can be interpreted as a special case of follow-the-regularized leader (FTRL) methods, as follows:

\[
p_t \in \arg\min_{p \in [0,1]^N: \|p\|_1 = 1} \left\{ \sum_{j=1}^{t-1} \ell_j^T \mathbf{p} - \frac{1}{\eta_t} H(p) \right\}.
\]  

(19)

From a standard analysis of FTRL (see, e.g., Exercise 28.12 in the book by Lattimore and Szepesvári [2020], where we set \( F_t(x) = -\frac{1}{\eta_t} H(x) \)), we have

\[
R_T \leq \sum_{t=1}^{T} \left( \ell_t^T (p_t - p_{t+1}) - \frac{1}{\eta_t} KL(p_{t+1}, p_t) \right) + \frac{H(p_1)}{\eta_1} + \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) H(p_{t+1}),
\]  

(20)

where \( KL(p, q) \) represents the KL divergence defined by \( KL(p, q) = \sum_{i=1}^{N} (p_i \log \frac{p_i}{q_i} - p_i + q_i) \).

For any \( \alpha_t \in \mathbb{R} \), the first term in the right-hand side can be bounded as

\[
\ell_t^T (p_t - p_{t+1}) - \frac{1}{\eta_t} KL(p_{t+1}, p_t) = (\ell_t - \alpha_t \mathbf{1})^T (p_t - p_{t+1}) - \frac{1}{\eta_t} KL(p_{t+1}, p_t)
\]

\[
\leq \max_{p \in \mathbb{R}^{N}_0} \left\{ (\ell_t - \alpha_t \mathbf{1})^T (p_t - p) - \frac{1}{\eta_t} KL(p_{t+1}, p_t) \right\},
\]  

(21)

where \( \mathbf{1} \in \mathbb{R}^N \) represents the all-one vector, and the equality follows from the fact that \( \sum_{i=1}^{N} p_{t+1,i} = \sum_{i=1}^{N} p_{t,i} = 1 \). The maximum in (21) is attained by \( p = (p_{i,t} \exp(-\eta_t (\ell_{i,t} - \alpha_t)))_{i=1}^{N} \). In fact, the objective function is concave in \( p \) and its gradient can be expressed as

\[
\nabla_p \left( (\ell_t - \alpha_t \mathbf{1})^T (p_t - p) - \frac{1}{\eta_t} KL(p_{t+1}, p_t) \right) = -((\ell_t - \alpha_t \mathbf{1}) - \frac{1}{\eta_t} ((\log p_{i,t})_{i=1}^{N} - (\log p_{i,t})_{i=1}^{N}),
\]

which is the zero vector if and only if \( p = (p_{i,t} \exp(-\eta_t (\ell_{i,t} - \alpha_t)))_{i=1}^{N} \). By substituting this into the objective function, we have

\[
\max_{p \in \mathbb{R}^{N}_0} \left\{ (\ell_t - \alpha_t \mathbf{1})^T (p_t - p) - \frac{1}{\eta_t} KL(p_{t+1}, p_t) \right\}
\]

\[
= \sum_{i=1}^{N} (\ell_{i,t} - \alpha_t) p_{i,t} - \frac{1}{\eta_t} (p_{i,t} - p_{i,t} \exp(-\eta_t (\ell_{i,t} - \alpha_t)))
\]

\[
= \frac{1}{\eta_t} \sum_{i=1}^{N} p_{i,t} (\exp(-\eta_t (\ell_{i,t} - \alpha_t))) + \eta_t (\ell_{i,t} - \alpha_t) - 1
\]

\[
= \frac{1}{\eta_t} \sum_{i=1}^{N} g (\eta_t (\ell_{i,t} - \alpha_t)).
\]

Combining this with (20) and (21), and from \( H(p_1) = H(1/N) = \log N \), we obtain (5). \( \square \)

A.2 Proof of (11)

To show (11), we use the following upper bound on \( H(p) \):

Lemma 4. For any \( p \in [0,1]^N \) such that \( \|p\|_1 = 1 \) and \( i^* \in [N] \), we have

\[
H(p) \leq (1 - p_{i^*}) \left( 1 + \log \frac{N - 1}{1 - p_{i^*}} \right).
\]  

(22)

Proof. The value of \( H(p) \) can be expressed as

\[
H(p) = p_{i^*} \log \frac{1}{p_{i^*}} + \sum_{i \in [N] \setminus \{i^*\}} p_i \log \frac{1}{p_i}.
\]  

(23)
The first term can be bounded as
\[ p_i^* \log \frac{1}{p_i^*} = p_i^* \log \left( 1 + \frac{1 - p_i^*}{p_i^*} \right) \leq p_i^* \left( 1 - p_i^* \right) \leq 1 - p_i^*, \] (24)
where the inequality follows from \( \log(1 + x) \leq x \) that holds for any \( x > -1 \). When \( p_i^* \) is fixed, the second term of the right-hand side of (23) is maximized by setting \( p_i = \frac{1}{N} \) for all \( i \in [N] \setminus \{i^*\} \), and hence, its value can be bounded as
\[ \sum_{i \neq [N] \setminus \{i^*\}} p_i \log \frac{1}{p_i} \leq (N - 1) \frac{1 - p_i^*}{N - 1} \log \frac{N - 1}{1 - p_i^*} = (1 - p_i^*) \log \frac{N - 1}{1 - p_i^*}. \] (25)
Combining this with (23) and (24), we obtain (22). \( \Box \)

**Proof of (11).** Set \( T' = \lceil \log_2 N \rceil \). We then have \( \eta_t \leq 1 \) for \( t > T' \). From Lemma 1 with \( \alpha_t = 0 \) for \( t \leq T' \) and \( \alpha_t = \ell_t^* \) for \( t > T' \), we have
\[ R_{T^*} \leq \frac{\log N}{\eta_1} + \sum_{t=1}^{T'} \frac{1}{\eta_t} \sum_{i=1}^{N} p_i \log g(\eta_t \ell_t^*) + \sum_{t=T'+1}^{T} \frac{1}{\eta_t} \sum_{i=1}^{N} p_i \ell_t \log g(\eta_t (\ell_t^* - \ell_t)) + \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \log p_{t+1} \]
\[ \leq \frac{\log N}{\eta_1} + \sum_{t=1}^{T'} \eta_t (1 - p_i^*) + \sum_{t=T'+1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \left( 1 - p_i^* \right) \left( 1 + \log \frac{N - 1}{1 - p_i^*} \right) \]
\[ \leq \sqrt{\frac{\log N}{8}} + 8 \log N + \sqrt{8 \log N} (e - 2) \sum_{t=T'+1}^{T} \frac{1}{\sqrt{t}} + \frac{1}{2 \sqrt{8 \log N}} \sum_{t=1}^{T} \frac{1 - p_i^*}{\sqrt{t}} \left( 1 + \log \frac{N - 1}{1 - p_i^*} \right), \]
where the second inequality follows from \( g(y) \leq y \) for \( y \geq 0 \), \( g(-y) \leq (e - 2)y^2 \) for \( y \geq -1 \), and \( \ell_t \in [0, 1] \). The third inequality follows from (22) and the forth inequality follows from \( \eta_t = \sqrt{\log N} \). \( \Box \)

### A.3 Proof of (12)

We use the following lemma to bound the left-hand side of (12).

**Lemma 5.** For any \( c \in \mathbb{R} \), we have
\[ \max_{x \in (0, 1]} x(c - \log x) \leq \begin{cases} \exp(c - 1) & c \leq 1 \\ c & c > 1 \end{cases}. \] (26)

**Proof.** Define \( f : \mathbb{R}_{>0} \to \mathbb{R} \) by \( f(x) = x(c - \log x) \). The derivative of \( f \) can be expressed as
\[ f'(x) = (c - \log x) - 1, \] (27)
which implies that \( x = \exp(c - 1) \) is the unique stationary point of \( f \). As \( f(x) \) is concave in \( x \in \mathbb{R}_{>0} \), subject to the constraint of \( x \in (0, 1] \), \( f(x) \) is maximized by \( x = \min\{1, \exp(c - 1)\} \). By substituting this into \( f(x) = x(c - \log x) \), we obtain (26). \( \Box \)

**Proof of (12).** Set \( T' = \lceil \log \eta \rceil \). From (26), we have
\[ \sum_{t=1}^{T'} \frac{x_t}{\sqrt{t}} \left( a - b \sqrt{t} - \log x_t \right) = \sum_{t=1}^{T'} \frac{x_t}{\sqrt{t}} \left( a - b \sqrt{t} - \log x_t \right) + \sum_{t=T'+1}^{T} \frac{x_t}{\sqrt{t}} \left( a - b \sqrt{t} - \log x_t \right) \]
\[ \leq \sum_{t=1}^{T'} \frac{a - b \sqrt{t}}{\sqrt{t}} + \sum_{t=T'+1}^{T} \frac{1}{\sqrt{t}} \exp \left( a - b \sqrt{t} - 1 \right) \leq a \sum_{t=1}^{T'} \frac{1}{\sqrt{t}} + \exp(a - 1) \sum_{t=T'+1}^{T} \frac{\exp(-b \sqrt{t})}{\sqrt{t}}. \] (28)
The first term can be bounded by
\[
\sum_{t=1}^{T'} \frac{1}{\sqrt{t}} \leq 2 \sum_{t=1}^{T'} \frac{1}{\sqrt{t} + \sqrt{t - 1}} = 2 \sum_{t=1}^{T'} \left( \sqrt{t} - \sqrt{t - 1} \right) = 2\sqrt{T'} \leq 2\sqrt{\left( \frac{a}{b} \right)^2 + 1} \leq \frac{2a}{b} + \frac{b}{a}.
\] (29)

Further, as \(\exp(-b\sqrt{y})\) is convex in \(y \geq 0\), and as its derivative in \(y\) can be expressed as \(-\frac{b}{2\sqrt{y}}\exp(-b\sqrt{y})\), we have
\[
\exp(-b\sqrt{t - 1}) - \exp(-b\sqrt{t}) \geq \frac{b}{2\sqrt{t}} \exp(-b\sqrt{t}).
\] (30)

From this, we have
\[
\exp(a - 1) \sum_{t=T'+1}^{\infty} \frac{\exp(-b\sqrt{t})}{\sqrt{t}} \leq 2 \frac{\exp(a - 1)}{b} \sum_{t=T'+1}^{\infty} \left( \exp(-b\sqrt{t - 1}) - \exp(-b\sqrt{t}) \right) = \frac{2}{b} \exp(-b\sqrt{T'} + a - 1) \leq \frac{2}{b} \exp(-1) \leq \frac{1}{b},
\]
where the first inequality follows from (30) and the second inequality follows from \(T' \geq \left( \frac{a}{b} \right)^2\). Combining this with (28) and (29), we obtain (12). \(\square\)

### A.4 Proof of Theorem 6

We show lower bounds for the following four cases: (i) If \(T \leq \frac{N}{\Delta'}\), \(\bar{R}_{T,i^*} = \Omega(\sqrt{TN})\). (ii) If \(\frac{C}{\Delta} \leq \frac{N}{\Delta'} \leq T\), \(\bar{R}_{T,i^*} = \Omega(\frac{N}{\Delta'})\). (iii) If \(\frac{N}{\Delta'} \leq \frac{C}{\Delta} \leq T\), \(\bar{R}_{T,i^*} = \Omega\left(\sqrt{\frac{CN}{\Delta}}\right)\). (iv) If \(\frac{N}{\Delta^2} \leq T \leq \frac{C}{\Delta}\), \(\bar{R}_{T,i^*} = \Omega(\sqrt{TN})\). Combining all four cases of (i)–(iv), we obtain (13).

(i) Suppose \(T < \frac{N}{\Delta'}\). Set \(\Delta' = \frac{\sqrt{C}}{T}\). We then have \(T = \frac{N}{\Delta'}\) and \(\Delta < \Delta' \leq 1/4\). If \(\ell_t \sim \mathcal{D}_{\Delta,i^*}\) for all \(t \in [T]\), then the environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint for any \(C \geq 0\), and the regret is bounded as \(\bar{R}_{T,i^*} \geq \frac{N}{32\Delta} = \Omega(\sqrt{TN})\) from Lemma 3.

(ii) Suppose \(\frac{C}{\Delta} \leq \frac{N}{\Delta'} \leq T\). If \(\ell_t \sim \mathcal{D}_{\Delta,i^*}\) for all \(t \in [T]\), the regret is bounded as \(\bar{R}_{T,i^*} \geq \frac{N}{32\Delta}\) for some \(i^*\) from Lemma 3. The environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint for any \(C \geq 0\).

(iii) Suppose \(\frac{N}{\Delta^2} \leq \frac{C}{\Delta} \leq T\). Define \(\Delta' = \sqrt{\frac{\Delta N}{C}} \leq \Delta\). We then have \(\frac{C}{\Delta} = \sqrt{\frac{CN}{\Delta}}\). Let \(T' = \lceil \frac{N}{\Delta^2} \rceil = \lceil \frac{C}{\Delta} \rceil \leq T\). Consider an environment in which \(\ell_t \sim \mathcal{D}_{\Delta^*, i^*}\) for \(t \in [T']\) and \(\ell_t \sim \mathcal{D}_{\Delta^*, i^*}\) for \(t \in [T' + 1, T]\). Then from Lemma 3 there exists \(i^* \in [N]\) such that \(\bar{R}_{T,i^*} \geq \frac{N}{32\Delta^2} = \Omega\left(\sqrt{\frac{CN}{\Delta}}\right)\). Further, we can show that the environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint. In fact, we have \(T'\left(\Delta - \Delta'\right) \leq \frac{C}{\Delta}\left(\Delta - \Delta'\right) \leq C\).

(iv) Suppose \(\frac{N}{\Delta^2} \leq T \leq \frac{C}{\Delta}\). Set \(\Delta' = \frac{N}{T}\) and consider \(\ell_t \sim \mathcal{D}_{\Delta^*, i^*}\) for all \(t \in [T]\). Then the regret is bounded as \(\bar{R}_{T,i^*} \geq \frac{N}{32\Delta^2} = \Omega(\sqrt{TN})\) for some \(i^*\), from Lemma 3. We can confirm that the environment is in an adversarial regime with a \((i^*, \Delta, N, C, T)\) self-bounding constraint, as we have \(\Delta' T \leq \Delta T \leq C\), where the first and second inequalities follow from \(\frac{N}{\Delta^2} \leq T\) and \(T \leq \frac{C}{\Delta}\), respectively. \(\square\)