PARIKH MOTIVATED STUDY ON REPETITIONS IN WORDS

GHAJENDRAN POOVANANDRAN, ADRIAN ATANASIU, AND WEN CHEAN TEH

Abstract. We introduce the notion of general prints of a word, which is substantialized by certain canonical decompositions, to study repetition in words. These associated decompositions, when applied recursively on a word, result in what we term as core prints of the word. The length of the path to attain a core print of a general word is scrutinized. This paper also studies the class of square-free ternary words with respect to the Parikh matrix mapping, which is an extension of the classical Parikh mapping. It is shown that there are only finitely many matrix-equivalence classes of ternary words such that all words in each class are square-free. Finally, we employ square-free morphisms to generate infinitely many pairs of square-free ternary words that share the same Parikh matrix.

1. Introduction

The word banana can be written as ba(na)$^2$. Repetition in words is among the prominent themes that are studied in combinatorics on words and formal language theory. The systematic study of repetition in words dates back to the works of Axel Thue at the dawn of the 20th century, where he first showed the existence of an infinitely long binary (respectively ternary) word that is cube-free (respectively square-free). In this paper, we explore new domains in studying repetition in words.

We newly introduce and study the notion of general prints of word. A general print of a word is obtained by rewriting, in a certain prescribed fashion, repeated factors in a word into a single occurrence of that factor. Out of all the possible ways to reduce a given word in such manner, we choose two natural ones—that is to commence from the right or the left of that word. This paper also studies square-free (i.e. repetition-free) words with respect to the notion of Parikh matrix mapping. The latter, which was introduced in [10], is a generalization of the classical Parikh mapping [12]. The Parikh matrix mapping is well-studied in the literature (for example, see [1, 7, 11, 13, 15, 18, 20]), particularly as a tool to deal with subword occurrences in words.

The remainder of this paper is structured as follows. Section 2 provides the basic terminology and preliminaries. Section 3 introduces and scrutinize the left and right general prints of a word. Furthermore, the notion of core prints of a word is proposed and studied—a core print of a word is obtained by recursively reducing a word to its corresponding general print until the resulting word is no longer reducible. Section 4 presents new results on square-free words pertaining...
to the Parikh matrix mapping, exclusively for the ternary alphabet. It is shown that there are infinitely many pairs of square-free ternary words which share the same Parikh matrix. Our conclusions follow after that.

2. Preliminaries

The set of all positive integers is denoted by \( \mathbb{Z}^+ \) and let \( \mathbb{N} = \mathbb{Z}^+ \cup \{0\} \). The cardinality of a set \( A \) is denoted by \( |A| \).

Suppose \( \Sigma \) is a finite and nonempty alphabet. The set of all words over \( \Sigma \) is denoted by \( \Sigma^* \) and \( \lambda \) is the unique empty word. Let \( \Sigma^+ = \Sigma^* \setminus \{\lambda\} \). If \( v, w \in \Sigma^* \), the concatenation of \( v \) and \( w \) is denoted by \( vw \). An ordered alphabet \( \Sigma = \{a_1, a_2, \ldots, a_s\} \) with an ordering on it. For example, if \( a_1 < a_2 < \cdots < a_s \), then we may write \( \Sigma = \{a_1 < a_2 < \cdots < a_s\} \). For convenience, we shall frequently abuse notation and use \( \Sigma \) to denote both the ordered alphabet and its underlying alphabet.

A word \( v \) is a scattered subword (or simply subword) of \( w \in \Sigma^* \) if and only if there exist \( x_1, x_2, \ldots, x_n, y_0, y_1, \ldots, y_n \in \Sigma^* \) (possibly empty) such that \( v = x_1 x_2 \cdots x_n \) and \( w = y_0 x_1 y_1 \cdots y_{n-1} x_n y_n \). If the letters in \( v \) occur contiguously in \( w \) (i.e. \( y_1 = y_2 = \ldots = y_{n-1} = \lambda \)), then \( v \) is a factor of \( w \). A word \( w \in \Sigma^* \) is square-free iff it does not contain any factor of the form \( u^2 \) for some \( u \in \Sigma^* \).

The number of occurrences of a word \( v \) as a subword of \( w \) is denoted by \( |w|_v \). Two occurrences of \( v \) are considered different if and only if they differ by at least one position of some letter. For example, \( |abab|_{ab} = 3 \) and \( |abcabc|_{abc} = 4 \). By convention, \( |w|_\lambda = 1 \) for all \( w \in \Sigma^* \).

For any integer \( n \geq 2 \), let \( \mathcal{M}_n \) denote the multiplicative monoid of \( n \times n \) upper triangular matrices with nonnegative integral entries and unit diagonal.

**Definition 2.1.** [10] Suppose \( \Sigma = \{a_1 < a_2 < \cdots < a_k\} \) is an ordered alphabet. The Parikh matrix mapping with respect to \( \Sigma \), denoted by \( \Psi_\Sigma \), is the morphism:

\[
\Psi_\Sigma : \Sigma^* \rightarrow \mathcal{M}_{k+1},
\]
defined such that for every integer \( 1 \leq q \leq k \), if \( \Psi_\Sigma(a_q) = (m_{i,j})_{1 \leq i, j \leq k+1} \), then

- \( m_{i,i} = 1 \) for all \( 1 \leq i \leq k+1 \);
- \( m_{q,q+1} = 1 \); and
- all other entries of the matrix \( \Psi_\Sigma(a_q) \) are zero.

Matrices of the form \( \Psi_\Sigma(w) \) for \( w \in \Sigma^* \) are termed as Parikh matrices.

**Theorem 2.2.** [10] Suppose \( \Sigma = \{a_1 < a_2 < \cdots < a_k\} \) is an ordered alphabet and \( w \in \Sigma^* \). The matrix \( \Psi_\Sigma(w) = (m_{i,j})_{1 \leq i, j \leq k+1} \) has the following properties:

- \( m_{i,j} = 0 \) for all \( 1 \leq j < i \leq k+1 \);
- \( m_{i,i} = 1 \) for all \( 1 \leq i \leq k+1 \);
- \( m_{i,j+1} = |w|_{a_ia_{i+1}\cdots a_j} \) for \( 1 \leq i \leq j \leq k \).

**Remark 2.3.** Suppose \( \Sigma = \{a < b < c\} \) and \( w \in \Sigma^* \). Then

\[
\Psi_\Sigma(w) = \begin{pmatrix}
1 & |w|_a & |w|_{ab} & |w|_{abc} \\
0 & 1 & |w|_b & |w|_{bc} \\
0 & 0 & 1 & |w|_c \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
Example 2.4. Suppose \( \Sigma = \{ a < b < c \} \) and \( w = abac \). Then
\[
\Psi_\Sigma(w) = \Psi_\Sigma(a)\Psi_\Sigma(b)\Psi_\Sigma(a)\Psi_\Sigma(c) = \Psi_\Sigma^M(a)\Psi_\Sigma^M(b)\Psi_\Sigma^M(a)\Psi_\Sigma^M(c).
\]
\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Definition 2.5. Suppose \( \Sigma \) is an ordered alphabet. Two words \( w, w' \in \Sigma^* \) are \( M \)-equivalent, denoted by \( w \equiv_M w' \), iff \( \Psi_\Sigma(w) = \Psi_\Sigma(w') \). A word \( w \in \Sigma^* \) is \( M \)-ambiguous iff it is \( M \)-equivalent to another distinct word. Otherwise, \( w \) is \( M \)-unambiguous. For any word \( w \in \Sigma^* \), we denote by \( C_w \) the set of all words that are \( M \)-equivalent to \( w \).

The following rewriting rules, introduced in [1], are elementary in deciding whether two words are \( M \)-equivalent. (The version provided here is stated exclusively for the ternary alphabet.) Suppose \( \Sigma = \{ a < b < c \} \) and \( w, w' \in \Sigma^* \).

Rule E1. If \( w = xacy \) and \( w' = xcay \) for some \( x, y \in \Sigma^* \), then \( w \equiv_M w' \).

Rule E2. If \( w = xabycb \) and \( w' = xbacyhb \) for some \( \alpha \in \{ a, c \} \), \( x, z \in \Sigma^* \) and \( y \in \{ a, b \}^* \), then \( w \equiv_M w' \).

Definition 2.6. Suppose \( \Sigma \) is an ordered alphabet and \( w, w' \in \Sigma^* \).

(1) We say that \( w \) is 1-equivalent to \( w' \) if and only if \( w' \) can be obtained from \( w \) by finitely many applications of Rule E1.

(2) We say that \( w \) is ME-equivalent to \( w' \) if and only if \( w' \) can be obtained from \( w \) by finitely many applications of Rule E1 and E2.

Example 2.7. Suppose \( \Sigma = \{ a < b < c \} \). Consider
\[
w = ababcbcbcb \rightarrow abacbcacbc \rightarrow abacbcacbc = w'.
\]
Thus \( w \) is ME-equivalent to \( w' \).

The following notion, introduced by Şerbanuţă in [17], is closely related to the central object of study in the next section.

Definition 2.8. Suppose \( \Sigma \) is an alphabet and \( w \in \Sigma^* \). Suppose \( w = a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} \) such that \( a_i \in \Sigma \) and \( p_i > 0 \) for all \( 1 \leq i \leq n \) with \( a_i \neq a_{i+1} \) for all \( 1 \leq i \leq n-1 \). The print of \( w \) is the word \( a_1a_2\cdots a_n \).

3. General Prints of a Word

In this section, we introduce and study the notion of general prints of a word. We first present a canonical decomposition of words which will serve as a basis for our study. Note that this decomposition was first introduced in [2] as a means to obtain the (right) Parikh normal form of a word. Hence we retain the notation used to denote the final product of this decomposition.

In this section, let \( \Sigma \) be a fixed alphabet with size at least two.

*The term elementary matrix equivalence (ME-equivalence) was first introduced in [16].
3.1. Parikh Normal Form

Definition 3.1 (Parikh Normal Form). Suppose \( w \in \Sigma^+ \).

- Define \( R_w = \{(u, v, n) \in \Sigma^+ \times \Sigma^+ \times \mathbb{Z}^+ \mid w = uv^n\} \).
- Define \( \tau_r(w) = \max\{n \in \mathbb{Z}^+ \mid (u, v, n) \in R_w \text{ for some } u \in \Sigma^+ \text{ and } v \in \Sigma^+ \} \).
- Define \( \theta_r(w) \) as follows:
  - if \( \tau_r(w) = 1 \), then \( \theta_r(w) \) is defined to be the minimum element of the following set:
    \[
    \{|v| \mid v \in \Sigma^+ \text{ and } w = uv \text{ for some } u \in \Sigma^+ \text{ with } \tau_r(u) \neq 1\},
    \]
    provided it is nonempty; otherwise \( \theta_r(w) = |w| \).
  - if \( \tau_r(w) > 1 \), then \( \theta_r(w) \) is defined to be the maximum element of the following set:
    \[
    \{|v| \mid v \in \Sigma^+ \text{ and } w = uv^{\tau_r(w)} \text{ for some } u \in \Sigma^+\}.
    \]
- Define \( \rho_r(w) = (u', v', \tau_r(w)) \) to be the unique triplet in \( R_w \) such that \( |v'| = \theta_r(w) \).
- Let \( w_0 = w \) and \((w_1, v_0, n_0) = \rho_r(w_0)\). For all integers \( i \geq 1 \) and while \( w_i \neq \lambda \), recursively define \((w_{i+1}, v_i, n_i) = \rho_r(w_i)\). Let \( k \geq 0 \) be the largest integer such that \( w_k \neq \lambda \).

We denote the form \( v_k^{n_k}v_{k-1}^{n_{k-1}} \cdots v_0^{n_0} \) of \( w \) by \( P_n(w) \).

Remark 3.2. The requirement \( v \in \Sigma^+ \) in the first item of Definition 3.1 eliminates the trivial decomposition of a word \( w \) into \( w = w\lambda^n \) at each stage as \( n \) does not have an upper bound in this case.

The following example illustrates the mechanism of the right decomposition of a word.

Example 3.3. Suppose \( \Sigma = \{a, b, c\} \). Consider the word \( w = abccbcabbabb \).

Starting from right to left, we first aim to decompose the word \( w \) such that the power of the right most component is the highest. In this case, the highest such power is two, where \( abccbcabbabb \) can be decomposed to either \( abccbcabbab^2 \) or \( abccbc(ab)^2 \). Since |\( ab \)| > |\( b \)|, our choice of decomposition would be \( abccbc(ab)^2 \).

Next, we look at the remaining part of \( w \), which is yet to be decomposed, that is \( abccbc \). Here, the highest power attainable on the right most component is one. Therefore, we decompose \( abccbc \) in a way that the right most component has the shortest length such that the remaining part can be decomposed to a power higher than one. That is to say, we decompose \( abccbc \) into \( abcc(bc)^1 \).

The part that remains to be decomposed now is \( abcc \). Continuing the process as in the first step, we decompose \( abcc \) into \( abc^3 \).

The final remaining part of \( w \) is \( ab \). The highest power attainable on the right most component here is one. Furthermore, there is no way for us to decompose \( ab \) such that there exists some remaining factor which can be decomposed to a power higher than one. Thus the final component is \( (ab)^1 \).

Therefore, we have \( P_n(w) = (ab)^1 c^3(bc)^1(ab)^2 \). Omitting the parentheses and power when the latter is one, we write \( P_n(w) = abc^3bc(ab)^2 \).
We define the left decomposition of a word \( w \) analogously to Definition 3.1 such that the decomposition commences from left to right. Furthermore, in a similar fashion, we denote the final product of the left decomposition of \( w \) by \( \text{Pn}_l(w) \).

**Example 3.4.** Suppose \( \Sigma = \{a, b, c\} \). Consider the word \( w = abababcbabc \). Then \( \text{Pn}_l(w) = (ab)^3a(cb)^2c \).

The following result holds directly by the definitions of the right and left decompositions of a word.

**Proposition 3.5.** Suppose \( w \in \Sigma^+ \). Let

\[
\text{Pn}_r(w) = v_k^{n_k} \cdots v_1^{n_1} v_0^{n_0} \quad \text{and} \quad \text{Pn}_l(w) = u_0^{m_0} u_1^{m_1} \cdots u_j^{m_j}
\]

for some integers \( j, k \in \mathbb{N} \), \( m_i \in \mathbb{Z}^+ \) \((0 \leq i \leq k)\) and \( n_i \in \mathbb{Z}^+ \) \((0 \leq i \leq k)\), and words \( u_i \in \Sigma^+ \) \((0 \leq i \leq k)\) and \( v_i \in \Sigma^+ \) \((0 \leq i \leq k)\). Then

- \( \text{Pn}_r(\text{mi}(w)) = [\text{mi}(u_j)]^{m_j} \cdots [\text{mi}(u_1)]^{m_1} [\text{mi}(u_0)]^{m_0} \);
- \( \text{Pn}_l(\text{mi}(w)) = [\text{mi}(v_0)]^{n_0} [\text{mi}(v_1)]^{n_1} \cdots [\text{mi}(v_k)]^{n_k} \).

### 3.2. General Prints

**Definition 3.6 (General Prints).** Suppose \( w \in \Sigma^+ \). Let

\[
\text{Pn}_r(w) = v_k^{n_k} \cdots v_1^{n_1} v_0^{n_0} \quad \text{and} \quad \text{Pn}_l(w) = u_0^{m_0} u_1^{m_1} \cdots u_j^{m_j}
\]

for some integers \( j, k \in \mathbb{N} \), \( m_i \in \mathbb{Z}^+ \) \((0 \leq i \leq j)\) and \( n_i \in \mathbb{Z}^+ \) \((0 \leq i \leq k)\), and words \( u_i \in \Sigma^+ \) \((0 \leq i \leq j)\) and \( v_i \in \Sigma^+ \) \((0 \leq i \leq k)\).

1. The **right** general print of \( w \), denoted by \( \text{gpr}_R(w) \), is the word \( v_k \cdots v_1 v_0 \).
2. The **left** general print of \( w \), denoted by \( \text{gpr}_L(w) \), is the word \( u_0 u_1 \cdots u_j \).

**Remark 3.7.** Express a word \( w \in \Sigma^+ \) in the form \( y_1^{n_1} y_2^{n_2} \cdots y_k^{n_k} \) such that \( y_i \in \Sigma^+ \) and \( n_i > 0 \) for all \( 1 \leq i \leq k \) with \( y_i \neq y_{i+1} \) for all \( 1 \leq i \leq k - 1 \). Informally, the word \( y_1 y_2 \cdots y_k \) can be regarded as a general print (associated to the decomposition) of \( w \). However, in this paper, we study only the ones as in Definition 3.6 as they are the two natural canonical forms. The notion of a general print of a word is in fact a generalization of the notion of the print (see Definition 2.8) of a word.

**Example 3.8.** Suppose \( \Sigma = \{a, b, c\} \) and consider the word \( w = cabccabc \). Then \( \text{Pn}_r(w) = \text{Pn}_l(w) = (cab)^2 \). Thus both the left and right general prints of \( w \) are the same, which is \( cabc \). On the other hand, the (Șerbănuță’s) print of \( w \) is \( cababc \).

The following assertion holds by the definitions and some simple observation.

**Remark 3.9.** For any word \( w \in \Sigma^+ \), if \( \text{mi}(w) = w \), then \( \text{gpr}_R(w) = \text{gpr}_L(w) \).

**Proposition 3.10.** For every \( w \in \Sigma^+ \), we have

1. \( \text{gpr}_R(w) = \text{mi}(\text{gpr}_L(\text{mi}(w))) \);
2. \( \text{gpr}_L(w) = \text{mi}(\text{gpr}_R(\text{mi}(w))) \).
Proof. It suffices to prove (1) as (2) follows immediately from (1).

Let \( P_n(w) = v_k^{n_k} \cdots v_1^{n_1} v_0^{n_0} \) for some integers \( k \in \mathbb{N} \), \( n_i \in \mathbb{Z}^+ (0 \leq i \leq k) \) and words \( v_i \in \Sigma^+ (0 \leq i \leq k) \). Then, by Proposition 3.5, we have \( P_n(mi(w)) = [mi(v_0)]^{n_0} [mi(v_1)]^{n_1} \cdots [mi(v_k)]^{n_k} \). Correspondingly, we have

\[
\text{gpr}_R(w) = v_k \cdots v_1 v_0 \quad \text{and} \quad \text{gpr}_L(mi(w)) = mi(v_0) mi(v_1) \cdots mi(v_k).
\]

It remains to see that

\[
\text{mi}(\text{gpr}_L(mi(w))) = \text{mi}(mi(v_0) mi(v_1) \cdots mi(v_k)) = \text{mi}(mi(v_0)) mi(mi(v_1)) \cdots mi(mi(v_k)) = v_k \cdots v_1 v_0 = \text{gpr}_R(w).
\]

\[\square\]

The following shows that for a general alphabet, in the case where the right and left general prints of a word are different, the respective lengths of the general prints can either be the same or different.

Example 3.11. Suppose \( a, b \in \Sigma \). Consider the words \( w = babaabaa \) and \( w' = babaa \). Then

- \( P_n(w) = ba(baa)^2 \) and \( P_n(w) = (ba)^2 aba^2 \), hence \( \text{gpr}_R(w) = babaa \) and \( \text{gpr}_L(w) = baaba \).
- \( P_n(w') = babaa^2 \) and \( P_n(w') = (ba)^2 a \), hence \( \text{gpr}_R(w') = baba \) and \( \text{gpr}_L(w') = baa \).

Question 3.12. Out of the all the possible general prints (see Remark 3.7) of a word \( w \in \Sigma^+ \), does \( \text{gpr}_R(w) \) or \( \text{gpr}_L(w) \) give you the one of the shortest length?

The answer is no. Suppose \( a, b, c \in \Sigma \). Consider the word \( w = abcbcbabcbcbca \). We have \( P_n(w) = a(abcbcbabcbcbca)^2 \) and \( P_n(w) = (abcbcbca)^2 a \). Therefore \( \text{gpr}_R(w) = \text{gpr}_L(w) = abcbcbca \). However, notice that another possible decomposition of \( w \) is \( abcbcbca \), which gives a shorter general print of \( w \), which is \( abcbcbca \).

The shortest general print of a word is however not necessarily unique. For instance, consider the word \( w = ababababcbcbabcbcbabcbcbabcbcbabcbcbabcbcb \). We have \( P_n(w) = a(ab)^2 (bc)^3 \) and \( P_n(w) = (ab)^3 (cb)^2 c \), hence \( \text{gpr}_R(w) = ababc \) and \( \text{gpr}_L(w) = abcbcbca \). It can be easily verified that both \( \text{gpr}_R(w) \) and \( \text{gpr}_L(w) \) are the shortest general prints of \( w \).

3.3. Core Prints

Definition 3.13 (Core Prints). Suppose \( w \in \Sigma^+ \). Let \( w_0 = w_0' = w \). For all integers \( i \geq 0 \), recursively define \( w_{i+1} = \text{gpr}_R(w_i) \) and \( w_{i+1}' = \text{gpr}_L(w_i) \). Let \( I \) (respectively \( I' \)) be the least nonnegative integer such that \( w_I = w_{I+1} \) (respectively \( w_{I'} = w_{I'+1} \)).

1. The right core print of \( w \), denoted by corepr\(_R\)(w), is the word \( w_I \).
2. The left core print of \( w \), denoted by corepr\(_L\)(w), is the word \( w_{I'} \).

Let \( l_R(w) \) and \( l_L(w) \) denote the integers \( I \) and \( I' \) respectively.
Remark 3.14. For every $w \in \Sigma^+$, the following are equivalent:

- $\text{gpr}_R(w) = w$;
- $\text{gpr}_L(w) = w$;
- $\text{corepr}_R(w) = w$;
- $\text{corepr}_L(w) = w$;
- $w$ is square-free.

Remark 3.15. Suppose $\Sigma = \{a, b\}$ and $w \in \Sigma^+$. Then $\text{corepr}_R(w), \text{corepr}_L(w) \in \{a, b, ab, ba, aba, bab\}$.

Theorem 3.16. Suppose $|\Sigma| = 2$. For every $w \in \Sigma^+$, we have $\text{corepr}_R(w) = \text{corepr}_L(w)$.

Proof. Let $\Sigma = \{a < b\}$. If $w$ is either $a^p$ or $b^p$ for some positive integers $p$, then the conclusion trivially holds.

Assume $w = ayb$ for some $y \in \Sigma^*$. Then, both $\text{corepr}_R(w)$ and $\text{corepr}_L(w)$ must start with a letter $a$ and end with a letter $b$. By Remark 3.15 this is only possible if $\text{corepr}_R(w) = ab = \text{corepr}_L(w)$. By similar argument, it can be shown that if $w = bya$ for some $y \in \Sigma^*$, then $\text{corepr}_R(w) = ba = \text{corepr}_L(w)$.

Assume $w = aya$ for some $y \in \Sigma^*$ with $|y|_a \geq 1$. Then, both $\text{corepr}_R(w)$ and $\text{corepr}_L(w)$ must start and end with a letter $a$ and contain at least one letter $b$ in between. By Remark 3.15 this is only possible if $\text{corepr}_R(w) = aba = \text{corepr}_L(w)$. By similar argument, it can be shown that if $w = byb$ for some $y \in \Sigma^*$ with $|y|_a \geq 1$, then $\text{corepr}_R(w) = bab = \text{corepr}_L(w)$. Thus our conclusion holds.

Theorem 3.16 however cannot be extended to cater for larger alphabets, as illustrated in the following example.

Example 3.17. Suppose $a, b, c \in \Sigma$. Consider the word $w = ababcabc$. We have

- $P_1(w) = a(babc)^2$, thus $w_1 = \text{gpr}_R(w) = ababc$;
- $P_2(w_1) = (ab)^2c$, thus $w_2 = \text{gpr}_R(w_1) = abc$;
- $P_3(w_2) = abc$, thus $w_3 = \text{gpr}_R(w_2) = abc$.

Therefore, $\text{corepr}_R(w) = abc \neq ababc = \text{corepr}_L(w)$.

The word in Example 3.17 which is of length nine, is in fact a counterexample of the shortest length. The other such words are listed below:

- $cabcba$ $abcabc$ $acacabc$ $acacabc$ $acbcac$ $acbcac$
- $bacabac$ $bacabac$ $bacabca$ $bacabca$ $bacabca$ $bacabca$
- $cabacab$ $cacabca$ $cababc$ $cababc$

Interestingly, there are only 12 such words out of $3^9 = 19683$ ternary words of length 9.

Every word over $\Sigma$ corresponds to a unique sequence of decompositions to attain the right (respectively left) core print of that word. We now introduce a function that captures, for every positive integer $n$, the maximal length of such sequences with respect to the set of all words over $\Sigma$ with length $n$. 
Definition 3.18. Suppose \( r \geq 2 \) is an integer. The \textit{core print characteristic function} of order \( r \) is the function \( \zeta_r : \mathbb{Z}^+ \to \mathbb{N} \) defined as
\[
\zeta_r(n) = \max\{k \in \mathbb{N} \mid l_R(w) = k \text{ for some } w \in \Sigma^* \text{ with } |w| = n \}
\]
where \( \Sigma \) is any alphabet with \( |\Sigma| = r \).

Remark 3.19. In general, for a word \( w \), the values of \( l_R(w) \) and \( l_L(w) \) may not be the same. However, by some simple analysis and Proposition 3.10 one could see that \( l_R(w) = l_L(\text{mi}(w)) \) for any \( w \in \Sigma^* \). Thus changing the condition \( l_R(w) = k \) in the definition of \( \zeta_r(n) \) to \( l_L(w) = k \) does not alter the function. The current choice is simply a matter of preference.

Appendix A exhausts the values of \( \zeta_2(n) \) for every integer \( 1 \leq n \leq 30 \). The following are the (only) words \( w \in \{a,b\}^* \) with length 30 such that \( l_R(w) = 6 = \zeta_2(30) \). (Meanwhile, there are 25924760 words \( w \) with length 30 such that \( l_R(w) = 5 \).)

\[
\text{abaababaaabaabbaabbbbaabbab, \ abaababaabaabbaabbaabbbbaabb, } \\
\text{abaababaabbaabbaabbaabbbbbaabb, \ abaababaabaabbaabbaabbaabbab, } \\
\text{abaababaabbaabbaabbaabbaabbabaab, \ abaababaabbaaabaabbaabbaabbab, } \\
\text{abaababaabbaabbaabbaabbaabbaabab, \ babbabbbababbbababbaabbaabbaabab, } \\
\text{babbabbbababbbababbbababbaabbaab, \ babbabbbababbbababbaabbaabbab, } \\
\text{babbababbaabbaabbaabbaabbaabbaab, \ babbababbaabbaabbaabbaabbaabbaab, } \\
\text{babbababbaabbaabbaabbaabbaabbaab, \ babbababbaabbaabbaabbaabbaabbaab. }
\]

For instance, one can see that for the first word in the above list, the path to attain its right core print is as follows:

Example 3.20. Let \( w_0 = \text{abaababaaabaabbaabbbbaabbab} \). We have
\[
\begin{align*}
\text{Pn}_r(w_0) &= ab(a)b^2a^3ba(aab)^2(baabb)^2ab, \text{ thus } w_1 = \text{gpr}_R(w_0) \\
&= abaabaabaababab \;
\end{align*}
\]
\[
\begin{align*}
\text{Pn}_r(w_1) &= ab(a)b^2a^2(ab)ba^2b, \text{ thus } w_2 = \text{gpr}_R(w_1) = abaababbbab; \\
\text{Pn}_r(w_2) &= a(baa)^2ba^2b, \text{ thus } w_3 = \text{gpr}_R(w_2) = ababab; \\
\text{Pn}_r(w_3) &= ab(a)b^2, \text{ thus } w_4 = \text{gpr}_R(w_3) = abab; \\
\text{Pn}_r(w_4) &= ab^2b, \text{ thus } w_5 = \text{gpr}_R(w_3) = abab; \\
\text{Pn}_r(w_5) &= (ab)^2, \text{ thus } w_6 = \text{gpr}_R(w_5) = ab. \\
\text{Pn}_r(w_6) &= ab, \text{ thus } w_7 = \text{gpr}_R(w_6) = ab.
\end{align*}
\]

Remark 3.21. For all integers \( r \geq 2 \) and \( n \geq 1 \), we have \( \zeta_{r+1}(n) \geq \zeta_r(n) \).

For the case of \( \zeta_3(n) \), we have computationally checked that \( \zeta_3(n) = \zeta_2(n) \) for every integer \( 1 \leq n \leq 14 \) but \( 5 = \zeta_3(15) \neq \zeta_2(15) = 4 \). An example of a word \( w \in \{a,b,c\}^* \) with length 15 such that \( l_R(w) = 5 \) is \( cbacacacacacacba \).

Appendix A also suggests the possibility that the function \( \zeta_2 \) is nondecreasing. In general, appending a letter to the right or left of a word \( w \) may reduce the value of \( l_R(w) \). The following shows an extreme-case example of this.

Example 3.22. Suppose \( \Sigma = \{a,b\} \) and consider the word \( w = \text{abaababbbabbb} \) of length 13. We have \( l_R(w) = 4 = \zeta_2(13) \). However, \( l_R(aw) = l_R(bw) = l_R(wa) = 3 \) and \( l_R(wb) = 2 \).
However, if the following more general assertion holds, then the monotonicity of the function $\zeta_2$ is implied directly.

**Conjecture 3.23.** For any word $w \in \Sigma^*$, a letter $x \in \Sigma$ can be inserted into $w$ to obtain a word $w'$ such that $l_R(w') \geq l_R(w)$.

4. **On Square-free Words and $M$-equivalence over the Ternary Alphabet**

A notion often investigated when dealing with the subject of repetition in words is square-freeness. In Section 3, we see that the square-free property of a word has direct consequences on the general prints and core prints of that word (see Remark 5.14).

We now investigate the class of square-free words over the ternary alphabet with respect to the Parikh matrix mapping. In particular, we present new results on square-free ternary words pertaining to the notion of $M$-equivalence.

**Lemma 4.1.** Suppose $\Sigma = \{a < b < c\}$ and $w \in \Sigma^*$ with $|w|_b = k \geq 3$. Assume every word in $C_w$ is square free. Write $w = u_0bu_1bu_2\cdots bu_k$ where $u_i \in \{a, c\}^*$ for every integer $0 \leq i \leq k$. Then, $u_i \in \{a, c\}$ for every integer $1 \leq i \leq k - 1$ and $u_0, u_k \in \{\lambda, a, c\}$ such that $u_i \neq u_{i+1}$ for every integer $0 \leq i \leq k - 1$.

**Proof.** Since every word in $C_w$ is square-free (by the hypothesis), clearly $w$ is square-free as well. To show that $u_i \neq u_{i+1}$ for every integer $0 \leq i \leq k - 1$, we argue by contradiction. Assume $u_m = u_{m+1}$ for some integer $0 \leq m \leq k - 1$. Notice that if $0 \leq m \leq k - 2$, then

$$w = yu_mbu_{m+1}by' = yu_mbu_mby' = y(u_mb)^2y'$$

for some $y, y' \in \Sigma^*$. On the other hand, if $m = k - 1$, then

$$w = u_0bu_1\cdots bu_{k-1}bu_k = u_0bu_1\cdots bu_kb_k = u_0bu_1\cdots (bu_k)^2.$$ 

In both cases, we have a contradiction as $w$ is square-free.

To prove the remaining part of the assertion, note that for every integer $1 \leq i \leq k - 1$, the word $u_i$ has to be nonempty. Otherwise, the square $b^2$ will be a factor in $w$. At the same time, for every integer $0 \leq i \leq k$, the word $u_i$ has to be square-free as well.

The only square-free words over the alphabet $\{a, c\}$ are $a, c, ac, ca, aca$ and $cac$. Assume $u_m = aca$ for some integer $0 \leq m \leq k$. Then $w = yacay'$ for some $y, y' \in \Sigma^*$. Observe that $w = yacay' \equiv_M yaacy' = ya^2cy'$, thus $ya^2cy' \in C_w$. However, this is impossible as every word in $C_w$ is square-free. Thus $u_i \neq aca$ for every integer $0 \leq i \leq k$. Similarly, it can be shown that $u_i \neq cac$ for every integer $0 \leq i \leq k$.

Assume $u_m = ac$ for some integer $0 \leq m \leq k$.

**Case 1.** $0 \leq m \leq k - 2$.

Then $w = yacbu_{m+1}by'$ for some $y, y' \in \Sigma^*$. Since $u_{m+1} \notin \{\lambda, u_m\}$, it follows that $u_{m+1} \in \{a, c, ca\}$. Observe that

(1) if $u_{m+1} = a$, then $w = yacbab' \equiv_M ycababy' = yc(ab)^2y'$;
(2) if $u_{m+1} = c$, then $w = yacbcby' = ya(cb)^2y'$;
(3) if $u_{m+1} = ca$, then $w = yacbcaby' \equiv_M yacbcaby' = y(cab)^2y'$.
Each case leads to a square word in $C_w$, thus a contradiction.

Case 2. $k - 1 \leq m \leq k$.
Then $w = ybu_{m-1}bacy'$ for some $y, y' \in \Sigma^*$ because $k \geq 3$. Since $u_{m-1} \notin \{\lambda, u_m\}$, it follows that $u_{m-1} \in \{a, c, ca\}$. From here, argue similarly as in Case 1.

Therefore, $u_i \neq ac$ for every integer $0 \leq i \leq k$. Similarly, it can be shown that $u_i \neq ca$ for every integer $0 \leq i \leq k$. Thus we conclude that $u_i \in \{a, c\}$ for all integers $1 \leq i \leq k - 1$ and $u_1, u_k \in \{\lambda, a, c\}$. \hfill \Box

Lemma 4.2. Suppose $\Sigma = \{a < b < c\}$ and $w \in \Sigma^*$. If $|w|_b > 4$, then there exists some word in $C_w$ that is not square-free.

Proof. We argue by contradiction. Assume that $|w|_b > 4$ and every word in $C_w$ is square-free. Since $|w|_b > 4$, it follows that $w = u_0buc_{i+1}bu_{i+2}...uc_{k-1}bu_k$ for some integer $k \geq 5$ and $u_i \in \{a, c\}$ ($0 \leq i \leq k$). By Lemma 4.1, it holds that $u_i \in \{a, c\}$ for every integer $1 \leq i \leq k - 1$ such that $u_i \neq u_{i+1}$ for every integer $1 \leq i \leq k - 1$.

Therefore, if $u_1 = a$, then $w = u_0babcbab...uc_{k-1}bu_k = u_0(babc)^2...uc_{k-1}bu_k$. On the other hand, if $u_1 = c$, then $w = u_0cbcbabc...uc_{k-1}bu_k = u_0(cbca)^2...uc_{k-1}bu_k$. In both cases, we have a contradiction as $w$ is square-free. Thus our conclusion holds. \hfill \Box

Let $\Sigma = \{a < b < c\}$. The following is an exhaustive list of every $M$-equivalence class over $\Sigma$ such that all words in it are square-free. The list can be verified by some simple analysis supported by Lemma 4.1 and Lemma 4.2.

$$
\{a\}, \{c\}, \{ac, ca\}, \{b\}, \{ab\}, \{ba\}, \{cb\}, \{bc\},\\
\{abc\}, \{cba\}, \{acb, cab\}, \{bac, bca\}, \{bac, cbca, cbac, cabac, cabca\},\\
\{acba, caba\}, \{acbc, cabc\}, \{abac, abca\}, \{cbac, cbca\},\\
\{abcba, cbaba\}, \{bacb, bcab\}, \{babc, cbac\},\{abaca, abcba\},\\
\{babcb, babc\}, \{babc, bab\}, \{abcba, cbac\}, \{babcba, babcb\}, \{babcbaba\},$$

One can see by the above list that the following holds.

Theorem 4.3. Suppose $\Sigma$ is an ordered alphabet with $|\Sigma| = 3$. There are only finitely many $M$-equivalence classes over $\Sigma$ such that all words in each class are square-free. Furthermore, every such class is a 1-equivalence class.

Remark 4.4. Note that in general, two distinct ternary square-free words that are $M$-equivalent need not be 1-equivalent, for example the words $abcbabcba$ and $bacabcbab$. These two words form one of the shortest-length pairs of square-free ternary words that are $M$-equivalent but not 1-equivalent (the only other such pair of words with length 10 is $bacababcba$ and $cbabcabac$).
Next, we show that for an arbitrary ternary ordered alphabet, there are infinitely many pairs of square-free words that are $M$-equivalent. However, we first need the following notion and known result.

**Definition 4.5.** Suppose $\Sigma$ is an alphabet and $k$ is a positive integer. The $k$-spectrum of a word $w \in \Sigma^*$ is the set $\{(u, |w|_u) \in \Sigma^* \times \mathbb{N} \mid |u| \leq k\}$.

The notion of $k$-spectrum was originally termed as $k$-deck in its introduction in [9]. Some examples of prominent works on $k$-spectrum are [5, 8, 9, 14].

**Theorem 4.6.** [19] Suppose $\Sigma, \Pi$ are alphabets, $\phi : \Sigma^* \rightarrow \Pi^*$ is a morphism, and $k$ is a positive integer. If two words $w, w' \in \Sigma^*$ have the same $k$-spectrum, then $\phi(w)$ and $\phi(w')$ have the same $k$-spectrum as well.

Suppose now $\Sigma$ is a ternary alphabet and $w, w' \in \Sigma^*$. By Remark 2.3, one can see that if $w$ and $w'$ have the same 3-spectrum, then $w$ and $w'$ have the same Parikh matrix (i.e. $w$ and $w'$ are $M$-equivalent) with respect to any ordered alphabet with underlying alphabet $\Sigma$. Therefore, Theorem 4.6 is desirably resourceful as it allows us to generate infinitely many pairs of ternary words that have the same 3-spectrum via an arbitrary morphism.

However, since our aim is to generate infinitely many pairs of $M$-equivalent ternary words that are square-free, the chosen morphism should preserve the square-freeness of the words throughout the (infinitely many) applications. Such morphisms are known as square-free morphisms and they are well-studied and presented in the literature (for example, [3, 4, 6]).

**Example 4.7.** Suppose $\Sigma = \{a, b, c\}$ and consider the square-free morphism $\phi : \Sigma^* \rightarrow \Sigma^*$ defined by:

$$
\phi(a) = abcbacbcabcbac; \quad \phi(b) = bcacbacbabcab; \quad \phi(c) = cabacbababcab.
$$

Let $w, w' \in \Sigma^*$ be words such that $w$ and $w'$ have the same 3-spectrum. For all integers $i > 0$, define $w_i = \phi(w_{i-1})$ and $w'_i = \phi(w'_{i-1})$. Then for all integers $i \geq 0$, the words $w_i$ and $w'_i$ are both square-free and they have the same 3-spectrum.

**Remark 4.8.** The morphism used in Example 4.7 is in fact a uniform square-free morphism, meaning that under the morphism, all letters have images of the same length (in this case, it is 13). This morphism is due to Leech [6].

Therefore, the final step is to find a pair of ternary square-free words having the same 3-spectrum. Surprisingly, the shortest length of such ternary words is 18. The following exhausts all pairs of such words:

$$
\{cabacbabcbacbcabcbac, abcacbcabcbabacab\}
\{acbacbcacbcabacab, bacabacbcacbcabca\}
\{cbacbabcbacbcabcb, abcbacbcacbcacab\}
\{bcacbabcabcabcb, abcbacbcacbcacab\}
\{bcacbacbcacbcab, cabcbacbababcabc\}
\{cbabcabacbcabcb, bacacbcacbcacab\}.
$$

By now, it is clear that the following result holds.
Theorem 4.9. Suppose $\Sigma$ is an alphabet with $|\Sigma| = 3$. There are infinitely many pairs of square-free words $w, w' \in \Sigma^*$ such that $w$ and $w'$ have the same 3-spectrum. Furthermore, every such pair of words $w$ and $w'$ are $M$-equivalent with respect to any ordered alphabet with underlying alphabet $\Sigma$.

The relation $ME$-equivalence is strictly stronger than $M$-equivalence in the sense that any two $ME$-equivalent words are $M$-equivalent but not vice versa. The shortest pairs of square-free ternary words that are $ME$-equivalent are of length 15 and as exhausted below:

$$\{cbacabacbabcb, abcacbcabcbacab\}$$
$$\{bacabcbabacabc, bacabcabcbacacba\}.$$

Finally, we end this section with a conjecture on the class of square-free ternary words with respect to $M$-unambiguity.

Definition 4.10. Suppose $\Sigma$ is an ordered alphabet with $|\Sigma| = 3$. A word $w \in \Sigma^*$ is square-free-$M$-unambiguous iff $w$ is not $M$-equivalent to any other distinct square-free word.

The table in Appendix B shows that up until length 60, the proportion of square-free-$M$-unambiguous words eventually decreases steadily. Thus, we conjecture the following:

Conjecture 4.11. Suppose $\Sigma$ is an ordered alphabet with $|\Sigma| \geq 3$. Then

$$\lim_{k \to \infty} \frac{|\{w \in \Sigma^* : w \text{ is square-free-$M$-unambiguous and } |w| = k\}|}{|\{w \in \Sigma^* : w \text{ is square-free and } |w| = k\}|} = 0.$$  

5. Conclusion

The introduction of the notion of general prints of a word opens up a potential direction in studying repetition in words. It is interesting to see what other canonical ways of decompositions can be proposed to attain a general print of a word.

The behavior of the function $\zeta_r$ is intriguing. The monotonicity of the function is suggested by Appendix A but it is yet to be conclusively determined. Some natural directions of research concerning the function $\zeta_r$ would be:

- to determine (for feasible range of $n$) the values $\zeta_r(n)$ for higher orders $r$,
- or estimate the values $\zeta_r(n)$ within bounds as tight as possible;
- to study the growth rate of $\zeta_r$.

Finally, Lemma 4.2 implies that every square-free word $w \in \{a < b < c\}^*$ with $|w|_b > 4$ is $M$-ambiguous. That is to say, there exists an upper bound on the length of $M$-unambiguous square-free words for the ternary alphabet (precisely, the upper bound is 7 as shown in the list before Theorem 4.3). For our future work, we aim to generalize Lemma 4.2 to obtain the corresponding upper bounds for larger alphabets.
Acknowledgement

The first and third authors gratefully acknowledge support for this research by a Research University Grant No. 1001/PMATHS/8011019 of Universiti Sains Malaysia.

References

[1] A. Atanasiu, R. Atanasiu, and I. Petre. Parikh matrices and amiable words. Theoret. Comput. Sci., 390(1):102–109, 2008.
[2] A. Atanasiu, G. Poovanandran, and W. C. Teh. Parikh matrices for powers of words. (Preprint).
[3] J. Berstel. Some recent results on squarefree words. In Annual Symposium on Theoretical Aspects of Computer Science, pages 14–25. Springer, 1984.
[4] M. Crochemore. Sharp characterizations of squarefree morphisms. Theoret. Comput. Sci., 18(2):221–226, 1982.
[5] M. Dudik and L. J. Schulman. Reconstruction from subsequences. J. Comp. Theory A, 103(2):337–348, 2003.
[6] J. Leech. A problem on strings of beads. Math. Gazette, 41(338):277–278, 1957.
[7] K. Mahalingam, S. Bera, and K. G. Subramanian. Properties of Parikh matrices of words obtained by an extension of a restricted shuffle operator. Internat. J. Found. Comput. Sci., 29(3):403–3413, 2018.
[8] J. Manuch. Characterization of a word by its subwords. In Developments in Language Theory, G. Rozenberg, et al. Ed., World Scientific Publ. Co., Singapore, pages 210–219, 2000.
[9] B. Manvel, A. Meyerowitz, A. Schwenk, K. Smith, and P. Stockmeyer. Reconstruction of sequences. Discrete Math., 94(3):209–219, 1991.
[10] A. Mateescu, A. Salomaa, K. Salomaa, and S. Yu. A sharpening of the Parikh mapping. Theor. Inform. Appl., 35(6):551–564, 2001.
[11] A. Mateescu, A. Salomaa, and S. Yu. Subword histories and Parikh matrices. J. Comput. System Sci., 68(1):1–21, 2004.
[12] R. J. Parikh. On context-free languages. J. Assoc. Comput. Mach., 13:570–581, 1966.
[13] G. Poovanandran and W. C. Teh. Elementary matrix equivalence and core transformation graphs for Parikh matrices. Discrete Appl. Math. (In press), 2018, doi: 10.1016/j.dam.2018.06.002.
[14] A. Salomaa. Connections between subwords and certain matrix mappings. Theoret. Comput. Sci., 340(2):188–203, 2005.
[15] A. Salomaa. Independence of certain quantities indicating subword occurrences. Theoret. Comput. Sci., 362(1):222–231, 2006.
[16] A. Salomaa. Criteria for the matrix equivalence of words. Theoret. Comput. Sci., 411(16):1818–1827, 2010.
[17] V. N. Şerbănuţă and T. F. Şerbănuţă. Injectivity of the Parikh matrix mappings revisited. Fund. Inform., 73(1):265–283, 2006.
[18] W. C. Teh. On core words and the Parikh matrix mapping. Internat. J. Found. Comput. Sci., 26(1):123–142, 2015.
[19] W. C. Teh. Separability of M-equivalent words by morphisms. Internat. J. Found. Comput. Sci., 27(1):39–52, 2016.
[20] W. C. Teh, A. Atanasiu, and G. Poovanandran. On strongly M-unambiguous prints and Şerbănuţă’s conjecture for Parikh matrices. Theoret. Comput. Sci., 719:86–93, 2018.
Appendix A. Values of $\zeta_2(n)$ for every integer $1 \leq n \leq 24$

$$N(n) = |\{w \in \Sigma^* \text{ with } |w| = n \text{ and } l_R(w) = \zeta_2(n)\}|$$

$$P(n) = \frac{N_u(n)}{|\{w \in \Sigma^* \text{ with } |w| = n\}|} \times 100\% \text{ (rounded to two decimal places)}$$

| n  | $\zeta_2(n)$ | $N(n)$ | $P(n)$ |
|----|--------------|--------|--------|
| 1  | 0            | 2      | 100.00 |
| 2  | 1            | 2      | 50.00  |
| 3  | 1            | 6      | 75.00  |
| 4  | 1            | 16     | 100.00 |
| 5  | 2            | 8      | 25.00  |
| 6  | 2            | 24     | 37.50  |
| 7  | 3            | 2      | 1.56   |
| 8  | 3            | 16     | 6.25   |
| 9  | 3            | 64     | 12.50  |
| 10 | 3            | 178    | 17.38  |
| 11 | 4            | 10     | 0.49   |
| 12 | 4            | 48     | 1.17   |
| 13 | 4            | 180    | 2.20   |
| 14 | 4            | 552    | 3.37   |
| 15 | 4            | 1642   | 5.01   |
| 16 | 4            | 4410   | 6.73   |
| 17 | 4            | 11286  | 8.61   |
| 18 | 5            | 24     | 0.01   |
| 19 | 5            | 266    | 0.05   |
| 20 | 5            | 1314   | 0.13   |
| 21 | 5            | 4996   | 0.24   |
| 22 | 5            | 16134  | 0.38   |
| 23 | 5            | 47214  | 0.56   |
| 24 | 5            | 128846 | 0.77   |
| 25 | 5            | 333068 | 0.99   |
| 26 | 5            | 830620 | 1.24   |
| 27 | 5            | 2015582| 1.50   |
| 28 | 5            | 4794990| 1.79   |
| 29 | 5            | 11225526| 2.09  |
| 30 | 6            | 18     | 0.00   |
### Appendix B. Proportion of Square-free-\(M\)-unambiguous Words

- \(\text{Sq}(k)\) = Number of square-free words of length \(k\)
- \(\text{Un}(k)\) = Number of square-free-\(M\)-unambiguous words of length \(k\)
- \(\text{Pr}(k) = \frac{\text{Sq}(k)}{\text{Un}(k)} \times 100\%\) (rounded to one decimal place)

| \(k\) | \(\text{Sq}(k)\) | \(\text{Un}(k)\) | \(\text{Pr}(k)\) |
|------|-----------------|-----------------|-----------------|
| 1    | 3               | 3               | 100.0           |
| 2    | 6               | 4               | 66.7            |
| 3    | 12              | 8               | 66.7            |
| 4    | 18              | 8               | 44.4            |
| 5    | 30              | 18              | 60.0            |
| 6    | 42              | 26              | 61.9            |
| 7    | 60              | 42              | 70.0            |
| 8    | 78              | 60              | 76.9            |
| 9    | 108             | 82              | 75.9            |
| 10   | 144             | 114             | 79.1            |
| 11   | 204             | 162             | 79.4            |
| 12   | 264             | 196             | 74.2            |
| 13   | 342             | 274             | 80.1            |
| 14   | 456             | 348             | 76.3            |
| 15   | 618             | 470             | 76.1            |
| 16   | 798             | 574             | 71.9            |
| 17   | 1044            | 780             | 74.7            |
| 18   | 1392            | 1004            | 72.1            |
| 19   | 1830            | 1296            | 70.8            |
| 20   | 2388            | 1650            | 69.1            |
| 21   | 3180            | 2232            | 70.1            |
| 22   | 4146            | 2848            | 68.7            |
| 23   | 5418            | 3670            | 67.7            |
| 24   | 7032            | 4818            | 68.5            |
| 25   | 9198            | 6242            | 67.9            |
| 26   | 11892           | 8024            | 67.5            |
| 27   | 15486           | 10308           | 66.5            |
| 28   | 20220           | 13222           | 65.4            |
| 29   | 26424           | 16850           | 63.8            |
| 30   | 34422           | 21578           | 62.7            |
School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Malaysia
E-mail address: p.ghajendran@gmail.com

Consulting Prof. at Faculty of Mathematics and Computer Science, Bucharest University, Str. Academiei 14, Bucharest 010014, Romania
E-mail address: aadrian@gmail.com

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Malaysia
E-mail address, Corresponding author: dasmenteh@usm.my