Symplectic No-core Shell-model Approach to Intermediate-mass Nuclei

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We present a microscopic description of nuclei in an intermediate-mass region, including the proximity to the proton drip line, based on a no-core shell model with a schematic many-nucleon long-range interaction with no parameter adjustments. The outcome confirms the essential role played by the symplectic symmetry to inform the interaction and the winnowing of shell-model spaces. We show that it is imperative that model spaces be expanded well beyond the current limits up through fifteen major shells to accommodate particle excitations that appear critical to highly-deformed spatial structures and the convergence of associated observables.

I. INTRODUCTION

For intermediate-mass (‘sd-shell’) nuclei, which are currently inaccessible by standard ab initio no-core shell-model (NCSM) [1] calculations, symmetry-based considerations are essential. In particular, we employ the no-core symplectic (NCSpM) shell model for symmetry-preserving interactions [2] with Sp(3, R) the underpinning symmetry [3]. This symmetry is inherent to the dynamics of deformed nuclear systems [4–8]. The present study uses a schematic, but fully microscopic and physically motivated effective many-nucleon interaction, a choice that enables the use of group-theoretical methods with analytical expressions for Hamiltonian matrix elements, and which in turn makes large space solutions for sd-shell nuclei feasible.

Recently, we successfully applied the NCSpM model to the rotational and alpha-cluster substructures of $^{12}$C, including the Hoyle state (the second 0$^+$ state in $^{12}$C) and its rotational band [2], as well as of $^{8}$Be [9]. The symplectic model has been used previously to achieve a remarkable reproduction of enhanced $E2$ transition strengths in $^{20}$Ne without effective charges and with the use of a relatively simple symmetry-breaking valence-shell interaction [5]. In addition, it has been applied to $^{166}$Er using the Davidson potential [7].

The main objective of the present study is to offer qualitative results that can provide guidance for ab initio shell model approaches by informing key features of nuclear structure and the interaction, first on the physically relevant truncation of shell-model spaces, but also on the dominant deformation and particle-hole configurations. This is especially useful for the ab initio symmetry-adapted no-core shell model (SA-NCSM) [10], which will then bring forward, with the use of a realistic nucleon-nucleon interaction, an accurate reproduction and reliable prediction of energy spectra and associated transition rates that majorly impact modeling of stellar explosions and astrophysical processes.

In this study, we explore the ground-state (g.st.) rotational band of lower sd-shell nuclei, namely, $^{20}$O, $^{20,22,24}$Ne, $^{20,22}$Mg, and $^{24}$Si. These low-lying states are expected to be highly influenced by large deformation. This, together with the combinatorial growth in model space dimensionality with number of particles and the spaces in which they primarily reside, has hitherto precluded a no-core shell model description and typically, in this region, valence shell model or mean field approaches have been employed (e.g., [11–13]). Many of these nuclei are in close proximity to the proton drip line and are key to understanding, e.g., nova and X-ray bursts (see, e.g., [14]). In particular, properties of low-lying 2$^+$ and 4$^+$ states in isotopes as $^{20}$Mg and $^{24}$Si are required to predict $(p, \gamma)$ reaction rates that are expected to affect the light curve for X-ray bursts. As such unstable isotopes are very hard to make experimentally and state-of-the-art radioactive-beam measurements have only recently started to provide new information [14], theoretical predictions are valuable.

The present approach utilizes symmetry to reduce the dimensionality of the model space through a very structured winnowing of the basis states to physically relevant subspaces. Indeed, experimental evidence supports the fact that in this mass range, the dynamics favors a dominance of low spin and high deformation, which has been demonstrated by symmetry-guided theoretical studies [4–6] as well as through an ab initio study [8–15]. The latter exploits symplectic symmetry and its deformation-related SU(3) subgroup in an analysis of ab initio large-scale nuclear physics applications for $^{12}$C and $^{16}$O. The outcome of this study has revealed that typically only one or two symplectic many-body basis states (vertical cones) suffice to represent a large fraction – typically in excess of about 80% of the physics – as measured by projecting ab initio NCSM results onto a symmetry-adapted equivalent basis. Such a symplectic pattern has been also observed in an ab initio SA-NCSM study of $^{6}$Li, $^{6}$He, and $^{8}$Be [10]. These findings point to the relevance of the symplectic symmetry, first to the many-body nuclear wavefunctions, and then to the inter-nucleon interaction (as symplectic basis states appear not to mix strongly). The NCSpM builds upon these considerations, and here we offer solutions to sd-shell nuclei in the framework of a fully microscopic no-core shell model. This, in turn,
allows us to examine the role of currently inaccessible shell-model spaces, up through 15 major shells, and of associated particle excitations to these shells for a description of large deformation.

II. SYMMETRY-INFORMED APPROACH

We employ the no-core symplectic model (NCSpM), outlined in Refs. [2], with a novel interaction that is effectively realized by an exponential dependence on the quadrupole-quadrupole (Q,Q) two-body interaction, the physically relevant interaction of each particle with the total quadrupole moment of the nucleus system. This introduces simple but important many-body interactions that enter in a prescribed hierarchical way given in powers of a small parameter, the only adjustable parameter in the model. The model offers a microscopic no-core shell-model description of nuclei in terms of mixed deformed configurations and allows the inclusion of higher-energy particle excitations [2] that are currently inaccessible by ab initio shell models. It reduces to the successful Elliott model [16] in the limit of a single valence shell and a zero model parameter.

The underlying symmetry of the NCSpM is the symplectic Sp(3, R) group [3] and its embedded SU(3) subgroup [16]. The symplectic basis (detailed in [6]) utilized in NCSpM is related, via a unitary transformation, to the three-dimensional HO (m-scheme) many-body basis used in the NCSM (see the review [17]). The conventional NCSM basis spaces [1] are constructed using HO single-particle states and are characterized by the hΩ oscillator strength (or equivalently, the oscillator length b = √(ℏ/(mt)) for a nucleon mass m) as well as by the cutoff in total oscillator quanta, Nmax, above the lowest HO energy configuration for a given nucleus. Indeed, the NCSpM employed within a complete model space up through Nmax, will coincide with the NCSM for the same Nmax cutoff.

The important feature of the NCSpM model is its ability to down-select to the most relevant configurations, which are chosen among all possible Sp(3, R) irreducible representations (irreps) within an Nmax model space. The Sp(3, R) irreps divide the space into ‘vertical cones’ that are comprised of basis states of definite (λμ) quantum numbers of SU(3) linked to the intrinsic quadrupole deformation [18, 20]. E.g., the simplest cases, (0,0), (λ,0), and (0,μ), describe spherical, prolate, and oblate deformation, respectively, while a general nuclear state is typically a superposition of several hundred various (λμ) triaxial deformation configurations.

A. Symplectic Sp(3, R) group

The translationally invariant (intrinsic) symplectic generators can be written as SU(3) tensor operators in terms of the harmonic oscillator raising, \( b_{n_{\alpha}}^{(10)} = \frac{1}{\sqrt{2}} (X_{n_{\alpha}} - i P_{n_{\alpha}}) \), and lowering \( b_{0(1)}^{(01)} \) dimensionless operators (with X and P the lab-frame position and momenta), and \( \alpha = 1, 2, 3 \) for the three spatial directions,

\[
A_{L=2}^{(20)} = \frac{1}{\sqrt{2}} \sum_{i=1}^{A} \left[ b_{i}^{+} b_{i} \right]_{L=2}^{(20)} - \frac{1}{\sqrt{2A}} \sum_{s,t=1}^{A} \left[ b_{s}^{+} b_{t} \right]_{L=2}^{(1)} \]
\[
C_{L=2}^{(11)} = \sqrt{2} \sum_{i=1}^{A} \left[ b_{i}^{+} b_{i} \right]_{L=2}^{(11)} - \frac{\sqrt{2}}{A} \sum_{s,t=1}^{A} \left[ b_{s}^{+} b_{t} \right]_{L=2}^{(11)} ,
\]

together with \( B_{L=2}^{(02)} = (-) \xi^{-\eta} (A_{L=2}^{(20)})^{\dagger} (L = 0, 2) \) and \( H_{L=0}^{(00)} = \sqrt{3} \sum_{i} \left[ b_{i}^{+} b_{i} \right]_{L=0}^{(00)} - \frac{\sqrt{2}}{A} \sum_{s,t} \left[ b_{s}^{+} b_{t} \right]_{L=0}^{(00)} + \frac{2}{7} (A - 1) \), where the sums run over all A particles of the system. The eight operators \( C_{L=2}^{(11)} (L = 1, 2) \) generate the SU(3) subgroup of Sp(3, R). They realize the angular momentum operator:

\[
L_{1M} = C_{1M}^{(11)} , \quad M = 0, \pm 1 ,
\]

and the Elliott algebraic quadrupole moment tensor \( Q_{L=2}^{(11)} = \sqrt{3} C_{L=2}^{(11)} , M = 0, \pm 1, \pm 2 \). The mass quadrupole moment can be constructed in terms of the symplectic generators as,

\[
Q_{L=2} = \sqrt{3} (A_{L=2}^{(20)} + C_{L=2}^{(11)} + B_{L=2}^{(02)}).
\]

B. Symplectic basis

A many-body basis state of a symplectic irrep is labeled according to the group chain,

\[
\text{Sp}(3, \mathbb{R}) \supset U(3) \supset \text{SO}(3) \supset \text{SO}(2) \quad \sigma \quad \nu \quad \omega \quad \kappa \quad L \quad M
\]

and constructed by acting with symmetrically coupled polynomials in the symplectic raising operators, \( A_{L=2}^{(20)} \), on a unique symplectic bandhead configuration, \( |\sigma\rangle \),

\[
|\sigma_{\nu\omega\kappaLM}\rangle = \left[ \left( A_{L=2}^{(20)} \times A_{L=2}^{(20)} \cdots \times A_{L=2}^{(20)} \right) \right]^{n} |\sigma\rangle_{\nu\omega\kappaLM} ,
\]

where \( \sigma \equiv N_{\sigma} (\lambda_{\sigma} \mu_{\sigma}) \) labels the Sp(3, R) irrep, \( n \equiv N_{\eta} (\lambda_{\eta} \mu_{\eta}) \), \( \omega \equiv N_{\omega} (\lambda_{\omega} \mu_{\omega}) \), and \( N_{\omega} = N_{\sigma} + N_{\eta} \) is the total number of HO quanta (\( n \) and \( \kappa \) are multiplicity labels). This can be generalized to include spin, \( |\sigma_{\nu\omega\kappaLM}\rangle = \sum_{M=0}^{\infty} \langle LM; \gamma_{S_{\eta}} \rangle|\sigma_{\nu\omega\kappaLM}\rangle \), and also isospin. States within a symplectic irrep have the same spin (isospin) value, which is given by the spin \( S_{\sigma} \) (isospin \( T_{\sigma} \)) of the bandhead |\sigma; S_{\sigma}\rangle [17]. Symplectic basis states span the entire shell-mode space. \(^1\)

\(^1\) A complete set of labels includes additional quantum numbers ({\( \alpha \)} \( \sigma \)) that distinguish different bandheads with the same \( N_{\sigma} (\lambda_{\sigma} \mu_{\sigma}) \). Sp(3, R)-preserving Hamiltonians render energy spectra degenerate with respect to \( \{\alpha\} \). However, for all present calculations for g.s.t. rotational bands and associated observables, \( \{\alpha\} \) is unique (an only set).
The symplectic structure accommodates relevant particle-hole (p-h) configurations in a natural way (see also Fig. 1 of Ref. [2]). According to Eq. (1), the basis states of an Sp(3, R) irrep (vertical cone) are built over a bandhead |σ⟩ by 2ℏΩ 1p-1h (one particle raised by two shells) monopole (L = 0) or quadrupole (L = 2) excitations, realized by the first term in $A_{SM}^{(20)}$ of Eq. (1), together with a smaller 2ℏΩ 2p-2h correction for eliminating the spurious center-of-mass (CM) motion, realized by the second term in $A_{SM}^{(20)}$. The symplectic bandhead |σ⟩ is the lowest-weight Sp(3, R) state, which is defined by the usual requirement that the symplectic lowering operators annihilate it – in analogy to a |J, Mf = −J⟩ state for the case of the SU(2) group of angular momentum, that is, $J_− |J, M_f = −J⟩ = 0$. The bandhead, |σ; κσLσMσ⟩, is an SU(3)-coupled many-body state with a given nucleon distribution over the HO shells and while not utilized here, can be obtained in terms of the creation operators $a^\dagger_{(σ,0)}$, which create a particle in the HO shell $η = 0, 1, 2, \ldots$. E.g., for a 0ℏΩ bandhead, the nucleon distribution is a single configuration,

$$\left[a^\dagger_{(σ,0)} \times a^\dagger_{(η_2,0)} \times \cdots \times a^\dagger_{(η_A,0)}\right] = \left|\lambda_σ \mu_σ \right> \left<(λ_σ \mu_σ \right| \left|0\right>$$

with $N_σ = η_1 + η_2 + \cdots + η_A + \frac{3}{2}(A−1)$, such that $N_σℏΩ$ includes the HO zero-point energy and 3/2 is subtracted to ensure a proper treatment of the CM. To eliminate the spurious CM motion, the NCSpM also uses symplectic generators constructed in $r_i$ ($p_i$) particle position (momentum) coordinates relative to the CM. These generators are used to build the basis, the interaction, the many-particle kinetic energy operator, as well as to evaluate observables.

An example for the symplectic basis states follows for $^{24}$Mg. Its lowest HO-energy configuration is given by $N_σ = 62.5$ or 0ℏΩ, while the 4ℏΩ (200) symplectic irrep includes:

1. A bandhead ($N_σ = 0$) with $N_σ = 66.5$ (or 4ℏΩ) and $(λ_σ \mu_σ) = (200)$;

2. $N_σ = 2$ states with $N_σ = 68.5$ and $(λ_σ \mu_σ) = (220)$, (201), and (182);

3. and so forth for higher $N_σ$.

For each $(λ_σ \mu_σ)$, the quantum numbers $κ$, $L$ and $M$ are given by Elliott [10]. E.g., for (220), $κ = 0$, $L = 0, 2, 4, \ldots, 22$, and $M = −L, −L + 1, \ldots, L$.

### C. Symmetry-preserving interactions

We note that the NCSpM, as presented here, is limited to interactions that preserve the Sp(3, R) symmetry. This restriction facilitates the use of a group-theoretical apparatus and analytical expressions for the Hamiltonian matrix elements, which, in turn, makes it possible to incorporate large $N_{\text{max}}$ spaces in applications of the theory. *Ab initio* calculations lie beyond the scope of the current analysis, but the addition of symmetry-mixing terms in the interaction is feasible, and a logical extension of the theory to include such terms is under development. Sp(3, R)-symmetric Hamiltonians appear to be particularly suitable to capture the essential characteristics of the low-energy nuclear kinematics and dynamics. The reason is that important pieces of the inter-nucleon Hamiltonian of a quantum many-body system can be expressed in terms of the Sp(3, R) generators, which directly relate to the relative particle momentum and position coordinates, as well as straightforwardly account for the Pauli exclusion principle. Indeed, the many-particle kinetic energy ($\sum \frac{p_i^2}{2m}$), the HO potential ($\sum_i \frac{mΩ^2 r_i^2}{2}$), the mass quadrupole moment operator (Q), and the orbital momentum (L) are all elements of the Sp(3, R) ⊃ U(1) × SU(3) ⊃ SO(3) structure.

Hence, Sp(3, R)-preserving Hamiltonians can include the many-particle kinetic energy:

$$\frac{T}{ℏΩ} = \frac{1}{ℏΩ} \sum_i \frac{p_i^2}{2m} = \frac{1}{2} H_{00}^{(00)} - \sqrt{\frac{3}{8}} (A_{00}^{(20)} + B_{00}^{(02)}),$$

the HO potential:

$$\frac{V_{\text{HO}}}{ℏΩ} = \frac{1}{ℏΩ} \sum_i \frac{mΩ^2 r_i^2}{2} = \frac{1}{2} H_{00}^{(00)} + \sqrt{\frac{3}{8}} (A_{00}^{(20)} + B_{00}^{(02)}),$$

as well as terms dependent on L, see Eq. (3), and Q, see Eq. (4). These interactions have analytical matrix elements in the Sp(3, R) basis (6) and act within a symplectic vertical cone (σf = σi = σ). For example, for the dimensionless many-particle kinetic energy, $\frac{T}{ℏΩ}$, the matrix elements are given as:
where \( \langle \omega_i \kappa_i L_i M_i; (2.0)00 | \omega_f \kappa_f L_f M_f \rangle \) is an SU(3) Clebsch-Gordan coefficient. The matrix elements of the Sp(3, \( \mathbb{R} \)) generators, \( A^{(20)} \), \( B^{(02)} \), and \( C^{(11)} \), reduced with respect to SU(3), such as \( \langle \sigma_n \rho_i \omega_i | A^{(20)} | \sigma_i \rho_i \omega_i \rangle \), are known exactly [21-25] and the steps to calculate them are outlined in the appendix.

The simplest, Sp(3, \( \mathbb{R} \))-preserving Hamiltonian, besides the HO Hamiltonian \( (H_0) \), of importance to nuclear dynamics is [16 26]

\[
H_E = H_0 - \frac{\chi}{2} Q \cdot Q,
\]

with \( \chi \) being a coupling constant and

\[
H_0 = \sum_{i=1}^{A} \left( \frac{p_i^2}{2m} + \frac{m \omega_i^2 r_i^2}{2} \right)
\]

\[
\frac{1}{2} Q \cdot Q = \frac{1}{2} \sum_{i<j} q(i) \cdot q(j).
\]

Here \( q_{2M}(i) = \sqrt{16 \pi/5 \delta} r_i^2 Y_{2M}(\hat{r}_i) \) is the dimensionless single-particle mass quadrupole moment and \( r_i (p_i) \) is the particle position (momentum) coordinate relative to the CM. In the limit of a single, valence shell \( (N_n = 0) \), where SU(3) becomes the relevant symmetry, the Hamiltonian \( H_E \) was shown to effectively describe rotational features of light nuclei in the framework of the established Elliott model [16]. The success of such an effective nuclear interaction is not unexpected, as the spherical HO potential and the \( Q \cdot Q \) interaction directly follow from the second and third term, respectively, in the long-range expansion of any two-body central force, e.g., like the Yukawa radial dependence, \( V^{(2)} = \sum_{i<j} V(r_{ij}/a) = \sum_{i<j} (\xi_0 + \xi_2 r_{ij}^2/a^2 + \xi_4 r_{ij}^4/a^4 + \ldots) \) [27], for a range parameter \( a \). However, in multi-shell studies, the attractive \( Q \cdot Q \) term becomes ever stronger with increasing \( N_n \) and starts to dominate the dynamics. Hence, \( H_E \) yields unphysical solutions.

A successful extension to multiple shells has been achieved and applied to the \( ^{24}\text{Mg} \) g.s.t. rotational band [28], where an interaction given as a polynomial in \( Q \cdot Q, \langle Q \times Q \rangle \cdot Q, \) and \( (Q \cdot Q)^2 \), was employed.

Furthermore, in multi-shell studies, the \( Q \cdot Q - \langle Q \cdot Q \rangle_{N_n} \) interaction has been employed [29 30], where \( \langle Q \cdot Q \rangle_{N_n} \) is the average contribution of \( Q \cdot Q \) within the sub-space of \( N_n \) HO excitations, that is, the trace of \( Q \cdot Q \) divided by the space dimension for a fixed \( N_n \). This removes the large monopole contribution of the \( Q \cdot Q \) interaction, which, in turn, helps eliminate the considerable renormalization of the zero-point HO energy, while retaining the \( Q \cdot Q \)-driven behavior of the wavefunctions.

### D. No-core symplectic model with \( H_{\gamma} \) for intermediate-mass nuclei

We consider a novel effective many-nucleon interaction [2] suitable for large-\( N_{\text{max}} \) no-core shell models,

\[
H_{\gamma} = H_0 + \frac{\chi}{\gamma} \left( e^{-\gamma (Q \cdot Q - \langle Q \cdot Q \rangle)} - 1 \right),
\]

that addresses the limitations of the conventional \( H_E \), while retaining the \( H_E \) important features in the limit \( \gamma \to 0 \), where \( \gamma \) is a positive adjustable parameter. We take \( \chi = h \Omega / (4 \sqrt{\omega_{f,j} \omega_{i,j}}) \) with \( \omega_{f,j} \) the total HO quanta of the final (initial) many-body basis state. The decrease of \( \chi \) with \( N_{\omega} \), leading order in \( \lambda/N_{\omega} \), has been shown by Rowe [31] based on self-consistent arguments and used in an Sp(3, \( \mathbb{R} \))-based study of cluster-like states of \( ^{16}\text{O} \) [32].

Above all, the effective interaction [14] introduces hierarchical many-body interactions in a prescribed way (for \( \gamma \ll 1 \)). \( H_{\gamma} \) also ensures that the \( Q \cdot Q \) term tails off for large \( N_n \) eliminating its ever stronger attraction with increasing \( N_n \). Such an interaction directly ties to the \( Q \) polynomial considered in the above-mentioned study of Ref. [28]. Indeed, while higher-order terms in \( Q \cdot Q \) of Eq. [14] could be understood as a renormalization (as shown in Ref. [33]) of the \( \chi \) coupling constant of the NN interaction, \( -\frac{1}{2} \sum_{ij} q(i) \cdot q(j) \):

\[
\frac{\lambda}{2 \gamma} (e^{-\gamma Q \cdot Q} - 1) = -\frac{1}{2} \left[ \lambda \left( \sum_{k=0}^{\infty} \frac{(-\gamma)^k (Q \cdot Q)^k}{(k+1)!} \right) \right] Q \cdot Q,
\]

they become quickly negligible for a reasonably small \( \gamma \). E.g., we find that for \( ^{12}\text{C} \), besides \( Q \cdot Q \), only one term is sufficient for the ground-state band, while three terms are sufficient for the Hoyle-state band [2].

For the NCSpM calculations, we use the empirical estimate \( h \Omega \approx 41/A^{1/3} \), namely, \( h \Omega = 18\text{ MeV} \) for \( A = 12 \) (for analysis of \( ^{12}\text{C} \)) and \( h \Omega = 15\text{ MeV} \) for \( A = 20 \) and \( A = 22 \) isotopes. The \( h \Omega \) value, in turn, fixes the
Energy (MeV) Expt. NCSpM

FIG. 1: NCSpM energy spectrum of (a) $^{20}\text{Mg}$ and (c) $^{20}\text{O}$ using the $S_p = S_n = S = 0$ 48.5(12) Sp(3, R) irrep built over the most deformed $0\hbar\Omega$ bandhead, as well as of (b) $^{20}\text{Ne}$ using the $S_p = S_n = S = 0$ 48.5(80) Sp(3, R) irrep built over the most deformed $0\hbar\Omega$ bandhead. Experimental data (“Expt.”) is from [36]. $B(E2)$ transition rates are in W.u. units.

FIG. 2: $^{20}\text{Ne}$ low-lying states obtained by NCSpM for an $N_{\text{max}} = 12$ model space consisting of all possible $0\hbar\Omega$ 0p-0h symplectic bandheads (for $S = 0$ and $S = 1$; spin-2 states are not shown). The lowest-lying 48.5(80) Sp(3, R) irrep is selected for the $^{20}\text{Ne}$ calculations (Fig. 1b).

In particular, we focus on the g.st. rotational band of selected $A = 20, 22$ and 24 isotopes. We note that, for the g.st. band as opposed to cluster-driven excited rotational bands, comparatively lower $N_{\text{max}}$ values are necessary to achieve convergence of energies, $E2$ observables and radii, with $N_{\text{max}} = 12$ found to be sufficient for the present calculations.

Model spaces are down-selected based on findings of ab initio large-scale calculations for $^{12}\text{C}$ and $^{20}\text{O}$ that have revealed low-spin and high-deformation dominance [15], as well as the importance of symplectic irreps built over the most deformed $0\hbar\Omega$ bandhead (the leading SU(3) configuration) [8]. For example, the latter study has shown a preponderance of the $0\hbar\Omega$ (04) symplectic irrep in $^{12}\text{C}$, which is indeed the irrep built over the most deformed $0\hbar\Omega$ bandhead, that is, the spin-zero (04).

A. $^{20}\text{Ne}$ and $A = 20$ isotopes

We present calculations for the g.st. rotational band of $^{20}\text{Ne}$ together with the short-lived $^{20}\text{Mg}$ at the proton-drip line (with no measured energy spectrum) and its mirror nucleus, the neutron-rich $^{20}\text{O}$. They are indeed well described by the NCSpM in a $N_{\text{max}} = 12$ model space, where convergence of results is achieved (Fig. 1 and Table I).

For $^{20}\text{Mg}$ and $^{20}\text{O}$, the model space is down-selected to only one spin-zero symplectic irrep, (42), for $J^\pi = 0^+, 2^+$, and $4^+$, with 1299 basis states (fixed $M$). For $^{20}\text{Ne}$, the model space consists of the spin-zero symplectic irrep, (80), for $J^\pi = 0^+, 2^+, 4^+$, and $6^+$, with 1070 basis states. All these irreps are built over the most deformed 0p-0h bandhead and expand up through $N_{\text{max}} = 12$.

To show the significance of the symmetry-based selection and the important role of the most deformed $0\hbar\Omega$ bandhead together with the Sp(3, R) excitations thereof, we consider a model space for $^{20}\text{Ne}$ that consists of all symplectic irreps that start at $0\hbar\Omega$. The resulting NCSpM energy spectrum is displayed in Fig. 2 for $S = 0$ and $S = 1$. Indeed, no other $0^+$ state is found to lie below the $0^+$ of (80) for the $\gamma$ parameter used here ($S = 2$ symplectic bandheads render states higher than 8-9 MeV). The $0^+$ of (61) is as much as 5 MeV above the (80), while all other $0^+$ states lie at $\sim 10$ MeV and higher. This indicates that the (80) irrep expanded up through $N_{\text{max}} = 12$ is indeed suitable for a reasonable description of the ground state of $^{20}\text{Ne}$.

An important feature of the NCSpM is that it provides electric observables without the need for introducing effective charges. And while g.st. rotational energies converge comparatively quickly, at $N_{\text{max}} \approx 4$, we find that larger model spaces are needed to reproduce observables sensitive to enhanced collectivity (Fig. 3). For $N_{\text{max}} \approx 4$, observables, such as the $B(E2)$ transition strengths, electric quadrupole moments, and matter rms radii have realized only 60% of their total increase as compared to the $N_{\text{max}} = 0$ counterparts. Indeed, additional 40 – 50%

III. RESULTS AND DISCUSSIONS

The NSCPm utilizes Bahri’s symplectic computational code [34] that uses Draayer & Akiyama’s numerical SU(3) package [35]. The model has been successfully applied to the ground-state and Hoyle-state rotational bands in $^{12}\text{C}$ [2], where both rotational features and $\alpha$-cluster substructures have been described in the fully microscopic $N_{\text{max}} = 20$ no-core shell-model framework, as suggested by the reasonably close agreement of the model outcome with experiment and ab initio results in smaller spaces. The present study reveals that the model is also applicable to low-lying states of other light nuclei without any parameter adjustment, namely, we use $\gamma = 0.74 \times 10^{-4}$, the value obtained in the NCSpM analysis for $^{12}\text{C}$.
are needed for the $B(E2)$ strengths and 20% for the $Q$ moments to obtain converged values. To reach convergence and to avoid the use of effective charges, at least $N_{\text{max}} = 10$ is necessary, which is where results are also found to compare reasonably to experiment (Fig. 3 and Table I). This suggests that the model successfully reproduces observables that are informative of the state structure and the long-range behavior of the wavefunctions.

![FIG. 3: NCSpM observables for $^{20}$Ne using the 0p-0h 48.5(80) $\text{Sp}(3, \Xi)$ irrep as a function of the model space, $N_{\text{max}}$: (a) $B(E2; 2^+_1 \to 0^+_\text{gs})$ and $B(E2; 4^+_2 \to 2^+_1)$ transition strengths; (b) electric quadrupole moments for $2^+_1$ and $4^+_2$; and (c) the matter rms radius of the ground state.](image)

While the model parameter $\gamma$ has not been adjusted in the present study (but was significantly limited by the three lowest-lying $0^+$ states in $^{12}$C [1]), its value has a large effect upon observables under consideration. A typical dependence on this parameter for $sd$-shell nuclei is shown for $^{14}$Ne (Fig. 3). As the $\gamma$ value decreases for a given nucleus, thereby increasing the tendency of the high-$\hbar \Omega$ excitations to become energetically favor-
B. $A = 22$ and 24

We perform NCSpM $N_{\text{max}} = 12$ calculations with no parameter adjustment (using $h\Omega = 15$ MeV and $\gamma = 0.74 \times 10^{-4}$) for $^{22}\text{Mg}$ and $^{22}\text{Ne}$. For these nuclei, the $N_{\text{max}} = 12$ model space is down-selected to only one spin-zero symplectic irrep, $(8 \ 2)$, for $J^\pi = 0^+, 2^+, 4^+$ and $6^+$, with 2900 basis states (fixed $M$).

Calculations for $^{22}\text{Mg}$ and $^{22}\text{Ne}$ yield energy spectra in close agreement with experiment (Fig. 5). In addition, most of the $B(E2)$ transitions strengths (Fig. 5) as well as electric quadrupole moments and matter rms radii (Table I) predicted by the model fall within the experimental uncertainties where measurements or experimentally deduced values exist.

NCSpM $N_{\text{max}} = 12$ calculations (Fig. 3) are also presented here for the short-lived $^{24}\text{Si}$, even though a slightly larger $h\Omega$ value, 21 MeV, is used only in this case. This nucleus is difficult to measure and knowledge on its structure, including spin-parity assignments and deformation of states, is necessary. For comparison, we also study the mirror nucleus $^{24}\text{Ne}$, for which richer experimental data is available. For these nuclei, the model space is down-selected to only one spin-zero symplectic irrep, $(100)$, for $J^\pi = 0^+, 2^+, 4^+$ and $6^+$, with 1171 basis states (fixed $M$). The results, including energy spectra, $E2$ observables and radii, are found reasonable as compared to the available experiment (Fig. 6 and Table I). The $^{24}\text{Si}$ wavefunctions, calculated by NCSpM and independent of the choice for $h\Omega$, are found to be dominated by the $(100)$ state, $(120)$ $2h\Omega$- and $(140)$ $4h\Omega$-configurations (Fig. 7), thereby, as discussed in the following section, carrying considerably large prolate deformation. While there are 96 (or 274) basis states in the $(100)$ symplectic irrep for the $0^+$ (or $2^+$) state, only a few of them contribute to the wavefunction at a level greater than 0.1%, as shown in Fig. 7. We note that the slightly smaller radius calculated by the model for $^{24}\text{Ne}$ suggests that additional spin-zero and spin-one irreps besides the $(100)$ vertical cone are likely to influence the low-energy dynamics. However, they are expected to remain of secondary importance to $(100)$.

![Energy spectrum of $^{22}\text{Mg}$ and $^{22}\text{Ne}$](image1)

**FIG. 5:** NCSpM energy spectrum of $^{22}\text{Mg}$ and its mirror nucleus $^{22}\text{Ne}$ using the $S_p = S_n = S = 0$ and $S = 0$ basis states. Experimental data (“Expt.”) is from [38]. $B(E2)$ transition strengths are in W.u. units.

![Energy spectrum of $^{24}\text{Si}$ and $^{24}\text{Ne}$](image2)

**FIG. 6:** NCSpM energy spectrum of $^{24}\text{Si}$ and its mirror nucleus $^{24}\text{Ne}$ using the $S_p = S_n = S = 0$ basis states. Experimental data (“Expt.”) is from [39]. $B(E2)$ transition strengths are in W.u. units.

![Probability distribution for $^{24}\text{Si}$](image3)

**FIG. 7:** Probability distribution (in %) for the $0^+$ ground state and the lowest $2^+$ state of $^{24}\text{Si}$ ($^{24}\text{Ne}$). Only contributions greater than 0.1% are shown.

IV. DOMINANT DEFORMED CONFIGURATION

Examinations of the SU(3) content of the NCSpM wavefunctions bring forward important information on deformation and associated dominant configurations through the deformation-related ($\lambda_\omega, \mu_\omega$). This is based on the mapping [20] between the shell-model ($\lambda \mu$) SU(3) labels (microscopic) and the shape variables of the Bohr-Mottelson collective model [20], which provides a description of the nuclear surface in terms of the elongation $\beta > 0$ and the $0 \leq \gamma \leq \pi/2$ asymmetry parameter. Specifically, in the limit of large deformation, ($\lambda \ 0$) and ($0 \mu$) can be associated with a prolate ($\gamma = 0^\circ$) and oblate...
(γ = 60°) shapes, respectively, while larger λ (µ) values are linked to larger deformation, β.

![Diagram](image)

FIG. 8: NCSpM probability distribution (specified by the area of the circles) for the ground state of (a) 20Ne using the (8 0) Sp(3, R) irrep, and (b) 22Ne using the (8 2) Sp(3, R) irrep. The symplectic states are grouped according to their (λγ µ) SU(3) symmetry, which are mapped onto the (βγ) shape variables of the collective model (see text for further details).

It is then clear that the nuclei under consideration are highly deformed and prolate in their low-lying states, as manifested in Fig. 8 for selected nuclei (this feature is the least pronounced for 20O and 20Mg). This is also confirmed by the negative large electric quadruple moments (Table I) and enhanced B(E2) values. Fig. 8 further reveals that the most dominant modes are observed among the ones with high λ and low µ. Configurations in higher-ℏΩ model spaces tend to increase the deformation (larger β) and decrease nuclear triaxiality (smaller γ) as compared to the predominant 0ℏΩ configuration (the bandhead). These high-ℏΩ configurations (Nn = 6 and beyond), while contributing only slightly to the wavefunctions, bring in substantially large deformation, thereby becoming critical for the convergence of the observables shown in Fig. 3.

V. CONCLUSION

We carried forward a no-core symplectic NCSpM study with a schematic long-range many-nucleon interaction that showed how highly deformed structures in intermediate-mass nuclei emerged out of a no-core shell-model framework. While previously the NCSpM has been successfully employed in a study of the α-cluster driven Hoyle-state rotational band in 12C [2], which has been successfully employed in a study of the α-cluster driven Hoyle-state rotational band in 12C [2], which has

fixed the only adjustable parameter in the schematic interaction, here we show that the framework is extensible to low-lying states of other nuclei without any parameter adjustment. We focused on the g.st. rotational band of 20Ne (a nucleus multiple of an α particle), as well as of 22,24Ne, 20O and of the proton-rich 20,22Mg, and 24Si nuclei.

By utilizing the symplectic symmetry, we were able to accommodate model spaces up through 15 major HO shells and hence, to take into account essential high-ℏΩ particle excitations. These excitations were found key to the description of large deformation and the convergence of electric observables without effective charges. These configurations were included in the shell-model space by considering only one symplectic irrep (vertical cone) built on the most deformed spin-zero bandhead and extended to Nmax = 12. This further confirms the dominance of low-spin/high-deformation and the importance of the symplectic symmetry to the low-energy nuclear dynamics.

Most importantly, the NCSpM has allowed us to identify, from a no-core shell-model perspective, components of the inter-nucleon interaction and type of particle excitations that appear foremost responsible for unveiling the primary physics governing highly-deformed structures, starting with rotational and alpha-clustering features in the case of 12C [2] and 8Be [9], but also expanding to the region of the lower sd shell (A ≤ 24). Therefore, the NCSpM appears as a useful tool to inform properties of the inter-nucleon interaction and to suggest efficacious shell-model truncation strategies to be employed in ab initio studies.

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Appendix

The SU(3)-reduced matrix elements of the Sp(3, R) generators are analytically known [21, 23-25]. The steps to compute (σnf µωf ||A(20)||σni µωi), similarly for B(02) = (--ωM(A(20)M)†)1, are outlined in what follows:

1. Calculations of non-normalized (n ||A(20)||n) using Eq. (4.51) of Ref. [9] with n1 = Nn+2λn+µn, n2 = Nn−λn+2µn and n3 = Nn−λn−2µn associated with n1 = Nn,1(λn,1 µn,1) and n2, together with the notation, A(20) → a†.

2. Calculations of non-normalized
(\sigma n f \rho f \omega f | A^{(20)} | \sigma n i \rho i \omega i) \) from \( (n_f | A^{(20)} | n_i) \) using Eq. (4.50) of Ref. 6;

3. Calculations of \( (\sigma n f \rho f \omega f | A^{(20)} | \sigma n i \rho i \omega i) \) from the non-normalized reduced matrix elements (step 2) using the \( K \)-matrix approach. The present calculations utilize the full \( K \) matrix (exact calculations). However, in the multiplicity-free case \( (\rho_i^{\text{max}} = \rho_f^{\text{max}} = 1) \) or in the limit of large \( \sigma \), the normalization matrix reduces to normalization coefficients (a diagonal \( K \) matrix) given by Eq. (17) of Ref. 22.

For the \( C_{\text{SU}(3)}^{(1)} \) SU(3)-reduced matrix elements, see, e.g., Eq. (19) of Ref. 23. Using the reduced matrix elements of the \( \text{Sp}(3, \mathbb{R}) \) generators and the relation 4, the analytical formula for the \( Q \cdot Q \) matrix elements has been derived in Ref. 25.

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