FLUX AND FREUND-RUBIN SUPERPOTTENTIALS IN M-THEORY

NEIL LAMBERT\textsuperscript{1}
Dept. of Mathematics
King’s College
The Strand
London
WC2R 2LS, UK

Abstract

We discuss the effective action for weak $G_2$ compactifications of M-theory. The presence of fluxes acts as a source for the the axions and drives the Freund-Rubin parameter to zero. The result is a stable non-supersymmetric vacuum with a negative cosmological constant. We also give the superpotential which generates the effective potential and discuss a simple model which aims to incorporate the effects of supersymmetry breaking by the gauge sector.

\textsuperscript{1}lambert@mth.kcl.ac.uk
1 Introduction

Since the early days of string theory it has been clear that there is a critical need to obtain phenomenologically realistic vacua. Originally this meant obtaining low energy effective actions which agreed with the standard model or some supersymmetric/grand-unified generalization. However over the past decade or so advances in cosmology have emphasized the point that string theory must also account for a cosmological standard model, incorporating inflation and a positive cosmological constant. Even more recently, with the growing realization that string theory likely contains a huge number of vacua [1] - the so-called landscape - it has become important to gain an understanding of the full structure of all four-dimensional vacua. In other words we are not just interested in what we view as phenomenologically relevant but also what other scientists who live in other parts of some great multiverse would view as phenomenologically relevant.

One approach to finding such vacua has been to consider M-theory on singular special holonomy manifolds (for a recent review as well as a list of references see [2]). The contribution of such M theory vacua to the landscape has been recently studied in [3]. A subset of these constructions revisits the Freund-Rubin ansatz within the context of singular weak G$_2$ manifolds. This program was detailed in [4] and a few examples were discussed. A benefit of this approach is that compact weak G$_2$ manifolds are easier to construct than compact G$_2$ manifolds. It is also possible to construct stable, chargeless brane configurations in the compact space without the need to introduce orientifold planes. On the other hand the four dimensional vacuum state is necessarily anti-de Sitter space. In this paper we wish to study the low energy four-dimensional dynamics of these vacua, including internal fluxes from the M theory four-form.

Flux compactifications of M-theory to four dimensions, and in particular the role of the superpotential, have been studied from a variety of points of view (for example see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). In this paper we wish to consider the analysis of fluxes in compactifications of M-theory on weak G$_2$ manifolds. We will mainly follow the analysis of [8] for the case of G$_2$ compactification. Similar issues for weak G$_2$ compactifications have have recently been studied in [14].

A generic feature of anti-de Sitter space there is a mass splitting between scalar fields within the same chiral supermultiplet. Thus one cannot simply truncate to the massless modes and obtain a supersymmetric low energy effective action. Indeed it is not clear in general that there is a supersymmetric truncation of an arbitrary dimensional reduction involving a internal special holonomy Einstein manifold. We will use the term moduli here to refer to
any suitably light scalar field, which maybe massive or even tachyonic.

A related problem that arises in “compactifications” over an Einstein manifold is that the mass scale of the light Kaluza-Klein modes is of the same order of magnitude as the cosmological constant. Indeed Freund-Rubin solutions are perhaps more naturally thought of as adS duals to three-dimensional conformal field theories, rather than as traditional Kaluza-Klein models. This presents a critical problem for phenomenological models built out of a compactification on an Einstein space: one needs to somehow split the mass scales so that the Kaluza Klein modes can be made sufficiently heavy while making the cosmological constant small. One of the motivations for the work presented here is to explore mechanisms, in particular supersymmetry breaking, where such a split might be made.

The rest of this paper is organized as follows. In section two we discuss the four-dimensional low energy dynamics of the “light” Bosonic modes. In particular we derive an effective action for the metric and scalar fields. In section three we postulate the form of the superpotential in terms of geometrical data and show that it correctly reproduces the effective potential of section two. In section four we consider a simple model which attempts to incorporate the effect of supersymmetry breaking by gauge theory fields which are localized at conical singularities in the internal manifold. Finally in section five we close with a discussion and some comments.

2 The Effective Potential

Consider the Bosonic sector of the low energy effective action of M-theory

\[ S = \frac{1}{2\kappa^9} \int \sqrt{-g_{11}} R - \frac{1}{2} G \wedge \ast G - \frac{1}{6} C \wedge G \wedge G \]  

(1)

where \( G = dC \). We are interested in compactifications of the form

\[ g_{11} = V_0 \text{Vol}(X)^{-1} g_4(M) + g_7(X) \]  

(2)

where

\[ \text{Vol}(X) = \int_X \sqrt{g_7} \]  

(3)

is the volume modulus field and \( V_0 \) is the volume of \( X \) as measured in some solution that we wish to perturb about.

We will consider backgrounds that preserve four-dimensional \( N = 1 \) supersymmetry and hence we assume that \( X \) is a weak \( G_2 \) manifold. This means that there exists a spinor \( \eta \) on \( X \) such that

\[ \nabla_i \eta = \frac{i}{2} \lambda_\gamma \gamma_i \eta \]  

(4)
for some $\lambda_7$. This in turn implies that

$$R_{ij}(X) = 6 \lambda_7^2 g_{ij}(X)$$

(5)

From $\eta$ one can construct a three-form

$$\Phi = \frac{i}{3!} \bar{\eta} \gamma_{ijk} \eta dx^i \wedge dx^j \wedge dx^k$$

(6)

which satisfies

$$d_7 \Phi = 4 \lambda_7 \ast_7 \Phi , \quad d_7 \ast_7 \Phi = 0$$

(7)

The existence of a three-form that satisfies (7) provides an alternative definition of a weak $G_2$ manifold $X$.

To construct the four-dimensional effective action we follow [8] and expand about a configuration which we take to be a Freund-Rubin flux background with

$$G = \ast_4 M + G_X ,$$

(8)

where $\ast_4$ is the four-dimensional Hodge star associated to $g_4$ and $G_X$ has no components tangent to $M$. Here $M$ is the Freund-Rubin parameter, although it is important to keep in mind that it is not in general a constant, and $G_X$ is a topological flux, i.e. a harmonic four-form.

Let us now discuss the light fields that will be present in the effective action. There will be axion-like moduli $C^i$ from the periods of $C$ over three-cycles in $X$. As is usual for Kaluza-Klein theory the lightest such modes are in a one-to-one correspondence with harmonic three-forms on $X$.

We will also obtain scalar moduli $s^I$ from the the moduli of $X$ which preserve the existence of a weak $G_2$ form (7), including a possible rescaling of $\lambda_7$. These have been discussed in [14] and are in correspondence with three-forms $\varphi^I$ that satisfy

$$d_7 P_1 \varphi_1 = 4 \lambda_7 \ast_7 P_1 \varphi_1 , \quad d_7 P_2 \varphi_1 = -4 \lambda_7 \ast_7 P_2 \varphi_1 , \quad P_7 \varphi_1 = 0$$

(9)

where $P_n$ denotes the projection onto the $n$-dimensional representation of $G_2$.

In the $G_2$ holonomy case where $\lambda_7 = 0$ both these moduli are in a one-to-one correspondence with harmonic three-forms and together form the complex Bosonic scalar of a four-dimensional chiral supermultiplet. If $\lambda_7 \neq 0$ then this is not the case. Indeed in the classic Freund-Rubin example where $X = S^7$ there are no harmonic three-forms and yet there is a volume modulus which preserves the weak $G_2$ structure, up to a rescaling of $\lambda_7$.

Thus it follows that we should consider two types of complex moduli $z^i = C^i + is^i$ and $z^I = C^I + is^I$. The $C^i$ and $s^I$ are massless (at least
when the Chern-Simons and Flux terms are ignored) whereas their superpartners \( \tilde{s}^i \) and \( \tilde{C}^I \) will not be. This mass splitting between scalars within a supermultiplet is what one expects for chiral supermultiplets in anti-de Sitter space.

The \( \tilde{C}^I \) have a straightforward interpretation as massive Kaluza-Klein modes. In particular our full ansatz for the three-form \( C \) is

\[
C = \sum_i C^i \omega_i + \sum_I \tilde{C}^I \varphi_I + C_X + C_0
\]

(10)

here \( \omega_i \) are basis of harmonic three-forms on \( X \) and \( C_X + C_0 \) is the background \( C \)-field that gives rise to (8). The \( C^i \) and \( \tilde{C}^I \) are therefore massless and massive Kaluza-Klein modes respectively. Note that we have assumed that there are no one-cycles on \( X \) and we set the vector fields \( A \), which arise from harmonic two-forms on \( X \), to zero.

The precise role of the \( \tilde{s}^i \) moduli is less clear. They correspond to deformations of \( \Phi \) by harmonic three forms and hence they do not preserve the weak \( G_2 \) structure. Thus we are led to parameterize the weak \( G_2 \) form by

\[
\Phi = \sum_I s^I \varphi_I + \sum_i \tilde{s}^i \omega_i
\]

(11)

It should be kept in mind that we expand about a configuration with \( s^I \neq 0 \) and \( \tilde{s}^i = 0 \) and under a generic deformation involving the \( \tilde{s}^i \) the internal manifold \( X \) is no longer a weak \( G_2 \) manifold. In this section we will simplify our calculations by setting \( \tilde{s}^i = 0 \). We can readily deduce the kinetic terms for the \( \tilde{s}^i \) since they are related by supersymmetry to the kinetic terms for \( C^i \). Since the \( \tilde{C}^I \) are more massive than their superpartners, and the \( C^i \) are massless, we expect that the \( \tilde{s}^i \) are tachyonic. In the next section we will deduce the potential for \( \tilde{s}^i \) from supersymmetry and verify that this is the case.

Note that in the limit \( \lambda_7 \to 0 \) we recover two copies of the supermultiplets found in a \( G_2 \) compactification. In other words turning on \( \lambda_7 \) introduces a splitting between two identical copies of the same multiplet. This seems odd as one might have expected a smooth limit as \( \lambda_7 \to 0 \). The cause of this pathology can be traced to the fact that if \( \lambda_7 \neq 0 \) then the \( \omega_i \) and \( \varphi_I \) forms are orthogonal

\[
\int_X \omega_i \wedge \star \varphi_I \sim \frac{1}{\lambda_7} \int_X \omega_i \wedge d \varphi_I = 0
\]

(12)

but this is not the case if \( \lambda_7 = 0 \). Hence one can’t really think of weak \( G_2 \) manifolds that are close to being \( G_2 \) by taking \( \lambda_7 \) small. This shouldn’t be
surprising as by blowing up the volume we can send $\lambda_7 \to 0$ but this clearly does not change the manifold in any interesting way.

Our first step is to substitute this ansatz into the original action and obtain the following effective action in the Einstein frame:

$$S_{\text{eff}} = \frac{1}{\kappa_4^2} \int \sqrt{-g_4} \left( \frac{1}{2} R_4 - g_{i\bar{j}} \partial_\mu z^i \partial_\nu \bar{z}^j - g_{I\bar{J}} \partial_\mu z^I \partial_\nu \bar{z}^J - V_{\text{eff}} \right) + \mathcal{T} + \ldots$$

(13)

Here the ellipsis denotes terms involving vector fields and Fermions and

$$\kappa_4^2 = \frac{\kappa_0}{V_0}$$

(14)

is the four-dimensional Planck length. One can see that there is no cross kinetic term $g_{i\bar{j}}$ as this would imply a term $\partial_\mu C^i \partial_\nu \bar{C}^\bar{j}$ which vanishes since Kaluza-Klein modes of different levels are orthogonal. In particular we find

$$g_{i\bar{j}} = \frac{1}{4\text{Vol}(X)} \int_X \omega_i \wedge \ast_7 \omega_j$$

$$g_{I\bar{J}} = \frac{1}{4\text{Vol}(X)} \int_X \varphi_I \wedge \ast_7 \varphi_J$$

$$V_{\text{eff}} = 16 \lambda_7^2 \frac{V_0}{\text{Vol}(X)} \tilde{C}^I \tilde{C}^J g_{I\bar{J}} - \frac{21V_0 \lambda_7^2}{\text{Vol}(X)} - \frac{\text{Vol}(X)^3}{4 \cdot 4! V_0^3} G_0 \wedge \ast_4 G_0$$

$$+ \frac{1}{4 \text{Vol}(X)^2} \int_X G_X \wedge \ast_7 G_X$$

$$\mathcal{T} = - \frac{1}{4V_0} G_0 C^i \int_X \omega_i \wedge G_X - \frac{1}{4V_0} \tilde{C}^I \tilde{C}^J \int \varphi_I \wedge d\varphi_J$$

$$- \frac{1}{4V_0} G_0 \int X C_X \wedge G_X$$

(15)

Note that we have written $G_0$ instead of $\ast_4 M$ to emphasize that the Chern-Simons term is independent of the metric as well as to make the the metric dependence of the $G_0 \wedge \ast_4 G_0$ term more explicit. Note also that there is no linear term in $\mathcal{T}$ from the Kaluza-Klein axions $\tilde{C}^I$ since, if $\lambda_7 \neq 0$,

$$\int_X \varphi_I \wedge G_X = \int_X \ast_7 \varphi \wedge \ast_7 G_X \sim \frac{1}{\lambda_7} \int d\varphi \wedge \ast_7 G_X = 0$$

(16)

as we have taken $G_X$ to be harmonic.

First consider the case without fluxes but $\lambda_7 \neq 0$. If we vary the volume, say for example by rescaling $g_7 \to \Omega^2 g_7$, then it is clear that $\lambda_7 \to \Omega^{-1} \lambda_7$
and hence
\[ \text{Vol}(X) \frac{\partial}{\partial \text{Vol}(X)} \lambda_7 = -\frac{1}{7} \lambda_7 \]  
(17)

Similarly we note for future reference that
\[ \text{Vol}(X) \frac{\partial}{\partial \text{Vol}(X)} \int_X G_X \wedge \ast G_X = -\frac{1}{7} \int_X G_X \wedge \ast_7 G_X \]  
(18)

We find an extremum of \( V_{\text{eff}} \) at
\[ \text{Vol}(X) = \left( \frac{36 \lambda_7^2}{M^2} \right)^{\frac{1}{4}} V_0 , \quad V_{\text{eff}} = -5 \sqrt{6} \lambda_7^3 M^{\frac{3}{2}} \]  
(19)

This is not the Freund-Rubin solution. While this potential predicts the correct value for \( \text{Vol}(X) \) it does not reproduce the correct value for the cosmological constant. Thus we see that the effective action (13) is misleading, even at the classical level. We will see that the cause is that \( M \) is not a constant. However it can be integrated out using the equations of motion, in the form of the conservation of Page charge.

To see this in more detail we can proceed with our analysis in the presence of fluxes. We see from the effective action that there will be a tadpole for the massless axion fields coming from the Chern-Simons term in the potential
\[ \frac{\partial \mathcal{T}}{\partial C^i} = -\frac{1}{4 V_0} G_0 \int_X \omega_i \wedge G_X \]  
(20)

Indeed this prediction can be verified directly using the full eleven dimensional equation of motion \( d \ast G + \frac{1}{2} G \wedge G = 0 \). If we assume that \( G = G_0 + G_X \), \( i.e. \) the axions are constant, then projecting this equation onto the components tangent to \( M \times \Sigma \), where \( \Sigma \) is a four-cycle in \( X \) gives
\[ d \ast G_X + MG_X = 0 \]  
(21)

Thus \( G_X \) is exact and hence there can be no topologically non-trivial fluxes if \( M \neq 0 \). This forbids static solutions with both topologically non-trivial fluxes and a non-vanishing Freund-Rubin parameter \( M \). What we have seen is that turning on both a Freund-Rubin parameter and flux acts as a source for the axions.

A related observation is that \( M \) is no longer conserved in the presence of fluxes and non-trivial axions. Rather the conserved the object is the Page charge.

\[ I am grateful to B. Acharya and F. Denef for pointing this out. \]
charge

\[ P_0 = \int_X \star G + \frac{1}{2} C \wedge G \]
\[ = \frac{\text{Vol}(X)^3}{V_0^2} \star_4 G_0 + \frac{1}{2} C^i \int_X \omega_i \wedge G_X + \frac{1}{2} \int_X C_X \wedge G_X \]
\[ + \frac{1}{2} \bar{C}^I \bar{C}^J \int_X \varphi_I \wedge d \varphi_J \]

(22)

Hence \( M \) alone is not conserved. Therefore we should take into account the full \( C_0 \) equation of motion which will result in a time-dependent \( M = -\star_4 G_0 \). To proceed we write

\[ \star_4 G_0 = \bar{P}_0 - \frac{V_0}{\text{Vol}(X)^3} \left( \frac{1}{2} \frac{V_0^2}{\text{Vol}(X)^3} C^i \int_X \omega_i \wedge G_X - \frac{1}{2} \frac{V_0^2}{\text{Vol}(X)^3} \bar{C}^I \bar{C}^J \int_X \varphi_I \wedge d \varphi_J \right) \]

(23)

where we have redefined the Page charge to absorb a constant arising from the fluxes

\[ \bar{P}_0 = P_0 - \frac{1}{2} \int_X C_X \wedge G_X \]

(24)

We can now substitute (23) into the remaining equations of motion. This leads to a system of equations for the metric and moduli which arise from the action

\[ S_{\text{eff}} = \frac{1}{\kappa_4^2} \int \sqrt{-g_4} \left( \frac{1}{2} R_4 - g_{ij} \partial_{\mu} z^i \partial_{\mu} \bar{z}^j - g_{IJ} \partial_{\mu} z^I \partial_{\mu} \bar{z}^J - U_{\text{eff}} \right) \]

(25)

with

\[ U_{\text{eff}} = \frac{16 \lambda_2^2 V_0}{\text{Vol}(X)} \bar{C}^I \bar{C}^J g_{IJ} - \frac{21 V_0 \lambda_2^2}{4 \text{Vol}(X)^2} \int_X G_X \wedge \star_7 G_X \]
\[ + \frac{V_0}{4 \text{Vol}(X)^3} \left( \frac{1}{2} C^k \int_X \omega_k \wedge G_X + \frac{1}{2} \bar{C}^I \bar{C}^J \int_X \varphi_I \wedge d \varphi_J - \bar{P}_0 \right)^2 \]

(26)

Note that the final term within the brackets is proportional to \( M \). Note also that this action does not simply result from substituting (23) into the effective action (13), except if \( P_0 = 0 \). Thus we find that classically we can “integrate out” the Freund-Rubin parameter using its equation of motion, i.e. conservation of Page charge, and find an effective action for the remaining light fields involving \( U_{\text{eff}} \). The effective potential \( U_{\text{eff}} \) is a generalization of
that found in [8] to include \( \lambda_7 \) and a non-vanishing \( P_0 \). It also generalizes the result of [14] to include fluxes and the additional \( z^i \) moduli.

We now recover the correct Freund-Rubin solution if the fluxes vanish. It occurs at

\[
\text{Vol}(X) = \left( \frac{36\lambda_7^2}{M^2} \right)^{\frac{1}{4}} V_0, \quad U_{\text{eff}} = -2\sqrt{6}\lambda_7^3 M^{\frac{1}{2}}
\]  

(27)

and contrary to the maximum of \( V_{\text{eff}} \) found above this does have the correct value of the cosmological constant. Thus the correct effective action is the one containing \( U_{\text{eff}} \), i.e. with \( M \) integrated out in favour of \( P_0 \). This shows that, even at the classical level, the effective potential \( V_{\text{eff}} \) is incorrect and one needs to remove \( M \) by its equation of motion to obtain a valid effective potential for the scalar moduli alone.

The effective potential \( U_{\text{eff}} \) is bounded from below and has a global minimum corresponding to \( C^I = 0 \) and \( M = 0 \) with negative energy. Since the minimum is only one constraint we see that there will also be \( b_3 - 1 \) flat directions corresponding to axionic modes which are perpendicular to

\[
\int_X \omega_i \wedge G_X
\]  

(28)

Therefore if one starts out with a Freund-Rubin parameter \( M \) and topologically non-trivial fluxes then the system will evolve until \( M \) is driven to zero.

Finally we would like to comment that the Page charge is somewhat mysterious as, on the one hand it is quantized, and yet under a large gauge transformation \( C_X \rightarrow C_X + \Omega \) we see that

\[
P_0 \rightarrow P_0 + \frac{1}{2} \int_X \Omega \wedge G_X
\]  

(29)

Thus \( P_0 \) is only gauge invariant modulo an integer and in some cases can be set to zero (for comments on this see [8] and [15]). However this is not possible in the absence of fluxes, since in this case \( P_0 \) is gauge invariant. Indeed in the absence of fluxes we should be able to identify the Freund-Rubin solution as an extremum of \( U_{\text{eff}} \) and we have seen that this requires \( P_0 \neq 0 \) in order to stabilize \( \text{Vol}(X) \).

### 3 The Superpotential

We wish to cast the above potential in the generic form for \( N = 1 \) supergravity in terms of the Kahler potential \( K \) and a superpotential \( W \)

\[
U_{\text{eff}} = e^K \left( g_{ij} D_i W D_j \bar{W} + g^{IJ} D_I W D_J \bar{W} - 3 W \bar{W} \right)
\]  

(30)
with $D_i W = \partial_i W + \partial_i KW$ and $\partial_i = \partial / \partial z^i$ and similarly for $i \rightarrow I$. This will also allow us to deduce the dependence on the metric moduli $\tilde{s}^i$. For the case that $\lambda_7 = 0$ the superpotential was derived in [8] and for the case $G_X = 0$ it was derived in [14]. Here we wish extend these results to our case with both $\lambda_7$ and $G_X$ non-vanishing.

The general form for superpotentials in the presence of fluxes was first proposed in [5, 6]. For the $G_2$ case the superpotential was taken in [8] to be

$$W = \frac{1}{4V_0} \int_X \left( \frac{1}{2} C + i\Phi \right) \wedge G$$

(31)

however one can easily check that this is no longer holomorphic if $d\Phi \neq 0$. A natural choice for the generalization is (see also [7, 14, 16])

$$W = -\frac{1}{4V_0} P_0 + \frac{1}{8V_0} \int_X (C + i\Phi) \wedge d(C + i\Phi)$$

(32)

and we will show that this does indeed reproduce the effective potential $U_{eff}$, along with a suitable choice of Kahler potential. From this expression it is clear that $W$ is holomorphic

$$\delta W = \frac{1}{4V_0} \int_X (\delta C + i\delta \Phi) \wedge (C + i\Phi)$$

(33)

provided that we view $P_0$ as a constant, rather than given by the expression (22). This is reasonable as our effective action is only valid within a supersection sector where the Page charge is held fixed. Furthermore we see that

$$W = -\frac{1}{4V_0} P_0 + \frac{1}{8V_0} \int_X (C + i\Phi) \wedge G + \frac{i}{8V_0} \int_X (C + i\Phi) \wedge d\Phi$$

$$= -\frac{1}{4V_0} P_0 + \frac{1}{8V_0} \int_X \left( \frac{1}{2} C + i\Phi \right) \wedge G - \frac{1}{8V_0} \int_X \Phi \wedge d\Phi$$

(34)

so if $d\Phi = 0$ then we recover the original superpotential of [8], shifted by the addition of the constant $P_0$.

Our first step is to calculate $\partial_i W$ and $\partial_I W$. Here we find

$$\partial_i W = \frac{1}{4V_0} \int_X \omega_i \wedge G_X \quad \partial_I W = \frac{1}{4V_0} \tilde{C}^I \int_X \varphi_I \wedge d\varphi_J + \frac{i}{4V_0} \int_X \varphi_I \wedge d\Phi$$

(35)

Recall that we are using the Kaluza-Klein ansatz (10) as well as an expansion of $\Phi$ as in (11). Following similar arguments to those in [8] we can then see
that
\[
g^{\bar{i}j} \partial_i W \partial_j \bar{W} = \text{Vol}(X) \int_X G_X \wedge *G_X
\]
\[
g^{IJ} \partial_I W \partial_J \bar{W} = \frac{16 \text{Vol}(X)^2 \lambda^2}{V_0^2} \tilde{C}^{IJ} \tilde{C}^{IJ} g_{IJ} + \frac{1}{16V_0^2} \int_X \Phi_I \wedge d\Phi \int_X \Phi_J \wedge d\Phi
\]
\[
W = -\frac{1}{4V_0} \bar{P} + \frac{1}{8V_0} C^i \int_X \omega_i \wedge G_X + \frac{1}{8V_0} \tilde{C}^{iJ} \tilde{C}^{Ji} \int_X \varphi_I \wedge d\varphi_J
\]
\[
-\frac{1}{8V_0} \int_X \Phi \wedge d\Phi + \frac{i}{4V_0} \tilde{C}^{ij} \int_X \varphi_I \wedge d\Phi + \frac{i}{4V_0} \int_X \Phi \wedge G_X
\]
(36)

Next we turn our attention to the Kahler potential. We postulate that
\[
\text{Vol}(X) = \frac{1}{7} \int_X \Phi \wedge *\Phi
\]
(37)
which is the case if \( \bar{s}^i = 0 \) [14] or \( s^i = 0 \) [8]. Then the general arguments of [8] (see also [17, 18, 14]) show that
\[
K = -3 \ln \left( \frac{\text{Vol}(X)}{V_0} \right)
\]
(38)
is the correct Kahler potential. One can also see that
\[
\partial_i K = \frac{i}{2 \text{Vol}(X)} \int_X \omega_i \wedge *\Phi, \quad \partial_I K = \frac{i}{2 \text{Vol}(X)} \int_X \varphi_I \wedge *\Phi
\]
(39)
which allows us to evaluate
\[
g^{\bar{i}j} \partial_i K \partial_j \bar{K} + g^{IJ} \partial_I K \partial_J \bar{K} = 7
\]
\[
g^{\bar{i}j} \partial_i KW \partial_j \bar{W} = 2i \text{Im} W - \frac{i}{2V_0} W \tilde{C}^{iJ} \int_X \varphi_I \wedge d\Phi
\]
\[
g^{IJ} \partial_I KW \partial_J \bar{W} = \frac{1}{2V_0} W \int_X \Phi \wedge d\Phi + \frac{i}{2V_0} W \tilde{C}^{ij} \int_X \varphi_I \wedge d\Phi
\]
(40)

Putting this all together gives
\[
U_{\text{eff}} = \frac{16\lambda^2}{\text{Vol}(X)^2} \tilde{C}^{IJ} \tilde{C}^{IJ} g_{IJ} + \frac{V_0}{16\text{Vol}(X)^3} g^{IJ} \int_X \Phi_I \wedge d\Phi \int_X \Phi_J \wedge d\Phi
\]
\[
+ \frac{V_0}{4\text{Vol}(X)^2} \int_X G_X \wedge *G_X + \frac{4V_0^3}{\text{Vol}(X)^3} |W|^2
\]
The last line can be evaluated more explicitly with the help of the identity
\[ g_{I\bar{J}} s^I s^J + g_{ij} \tilde{s}^i \tilde{s}^j = \frac{7}{4} \] and leads to a potential for the \( \tilde{s}^i \). In this way we arrive at the final expression for \( U_{\text{eff}} \)

\[
U_{\text{eff}} = \frac{16 V_0^2}{\text{Vol}(X)^3} \tilde{C}^I \tilde{C}^J g_{I\bar{J}} + \frac{V_0}{4 \text{Vol}(X)^2} \int_X G_X \wedge *G_X
\]

\[
+ \frac{V_0}{4 \text{Vol}(X)^3} \left( \frac{1}{2} C^i \int_X \omega_i \wedge G_X + \frac{1}{2} \tilde{C}^I \tilde{C}^J \int_X \varphi_I \wedge d \varphi_J - \tilde{P}_0 \right)^2
\]

\[
+ \frac{16 \lambda_7^2 V_0}{4 \text{Vol}(X)^2} \int_X \Phi_I \wedge d \Phi_J - \frac{V_0}{16 \text{Vol}(X)^3} \left( \int_X \Phi \wedge d \Phi \right)^2
\]

(41)

Here we find exact agreement with the potential of the previous section if we set \( \tilde{s}^i = 0 \). We also see that the \( \tilde{s}^i \) are indeed tachyonic in the vacuum corresponding to a weak \( G_2 \) manifold, i.e. at \( \tilde{s}^i = 0 \). Note that supersymmetry, or more accurately the existence of the superpotential, implies that any tachyonic modes about a supersymmetric solution will satisfy the Breitenlohner-Freedman bound and do not represent an instability.

Next we can look for supersymmetric vacua. These are solutions of \( D_I W = D_I \tilde{W} = 0 \);

\[
0 = \frac{1}{4 V_0} \int_X \omega_i \wedge G_X + \frac{i W}{2 \text{Vol}(X)} \int_X \omega_i \wedge *\gamma \Phi
\]

\[
0 = \frac{1}{4 V_0} \tilde{C}^I \int_X \varphi_I \wedge d \varphi_J + \frac{i}{4 V_0} \int_X \varphi_I \wedge d \Phi + \frac{i W}{2 \text{Vol}(X)} \int_X \varphi_I \wedge *\gamma \Phi
\]

(43)

The first equation tells us that either \( \Re W = 0 \) or \( \int_X \omega_i \wedge *\gamma \Phi = 0 \).

In the former case we learn from the second equation that \( d \Phi = 0 \), i.e. \( s^I = 0 \) and \( \lambda_7 = 0 \). This further implies that \( \tilde{C}^I = 0 \). However a little
algebra shows that $G_X = 0$ and hence we have a $G_2$ compactification with $\tilde{P}_0 = W = 0$.

In the latter case $\tilde{s}^i = 0$ and the first equation gives $G_X = 0$. A little bit of algebra shows that $\tilde{C}^i = 0$, $\tilde{P}_0 = -6\lambda_7 \text{Vol}(X)$ and $W = -2\lambda_7 \text{Vol}(X) V_0^{-1}$. This is of course just the Freund-Rubin solution (27).

Thus in the presence of fluxes supersymmetry is broken and one is driven to a global minimum with $M = \tilde{C}^i = 0$ as we mentioned above. However we now see that the metric moduli have a minimum at $g_{ij}\tilde{s}^i\tilde{s}^j = \frac{1}{2}$ and hence $g_{IJ}\tilde{s}^I\tilde{s}^J = \frac{5}{4}$. In this case the volume and cosmological constant are fixed to

$$\text{Vol}(X) = \frac{\lambda^{-2}}{60} \int_X G_X \wedge \star_{7} G_X \quad U_{\text{eff}} = -600V_0\lambda_7^4 \left( \int_X G_X \wedge \star_{7} G_X \right)^{-1}$$

(44)

4 Modeling Supersymmetry Breaking

Even if we were able to find the standard model (or a supersymmetric/grand-unified generalization of it) in one of the compactifications discussed above things would still not be very realistic. In particular these vacua suffer from a negative vacuum energy density and ideally one would like to create vacua with a positive cosmological constant. Another problem here is that in a typical Freund-Rubin compactification the Kaluza-Klein scale is of the same order of magnitude as the cosmological constant. Thus it is of interest to separate these scales so that one can probe microscopic distances without exciting Kaluza-Klein modes.

In $G_2$ and weak $G_2$ compactifications the phenomenologically relevant non-Abelian gauge fields arise on certain codimension four surfaces $Q \subset X$ and the charged chiral Fermions arise at co-dimension seven conical singularities in $X$ that also sit on $Q$ (for a review see [2]). One method of breaking supersymmetry in such a scenario would be to imagine that one or more of the gauge theories located at the singularities dynamically breaks supersymmetry. This would manifest itself in a positive vacuum energy located at the singularity. Since we expect that such breaking effects are due to the Fermions this non-vanishing vacuum energy should be confined to points in $X$ where there is a conical singularity.

Therefore to model such effects in the effective action we could add a term

$$S_{\text{susy}} = -\sum_A \int d^4x \sqrt{-\tilde{g}}\Lambda_A$$

(45)

where $\Lambda_A$ is a positive vacuum energy and $A = 1, \ldots, n$ labels the singularity.
This vacuum energy may well depend on the moduli of $X$ however since comes from localized sources on $X$ we assume that it is independent of $\text{Vol}(X)$ and for simplicity we will also assume that it is independent of the axions. This results in an extra term appearing in $U_{\text{eff}}$

$$
U_{\text{eff}} = -\frac{21V_0\lambda_7^2}{\text{Vol}(X)} + \frac{V_0}{4\text{Vol}(X)^2} \int_X G_X \wedge \ast_7 G_X + \frac{\Lambda_{\text{susy}}V_0}{\text{Vol}(X)^2} 
+ \frac{V_0}{4\text{Vol}(X)^3} \left( \frac{1}{2} C^k \int_X \omega_k \wedge G_X + \frac{1}{2} \int_X C_X \wedge G_X - P_0 \right)^2 
- \frac{16\lambda_7^2V_0}{\text{Vol}(X)} \left( g_{ij} \bar{s}^i \bar{s}^j - (g_{ij} \bar{s}^i \bar{s}^j)^2 \right)
$$

(46)

where $\Lambda_{\text{susy}} = \sum_A \Lambda_A \kappa^9$ This has a very similar dependence on $\text{Vol}(X)$ as the flux term in $U_{\text{eff}}$.

Since there are tachyonic modes about the $\bar{s}^i = 0$ vacuum, and we are breaking supersymmetry, we cannot guarantee stability of the $\bar{s}^i = 0$ vacua (although imposing the Breitenlohner-Freedman bound would provide perturbative stability). Therefore we will consider vacua where $g_{ij} \bar{s}^i \bar{s}^j = \frac{1}{2}$. There are essentially two cases to consider. In the first case suppose that there is no flux, \textit{i.e.} pure Freund-Rubin. One then finds a minimum for $\text{Vol}(X)$

$$
\text{Vol}(X) = \sqrt{\frac{7P_0^2}{300\lambda_7^2} + \frac{7^2\Lambda_{\text{susy}}^2}{25^2 \cdot 9^2\lambda_7^4} + \frac{7\Lambda_{\text{susy}}}{9 \cdot 25\lambda_7^2}}
$$

$$
U_{\text{eff}} = -\frac{V_0}{\text{Vol}(X)^3} \left( \frac{5}{9} \Lambda_{\text{susy}} \text{Vol}(X) + \frac{1}{3} P_0^2 \right)
$$

(47)

In the second case we turn on the fluxes (which we have seen breaks supersymmetry even if $\Lambda_{\text{susy}} = 0$). The minimum will occur at $M = 0$ but now with

$$
\text{Vol}(X) = \frac{14}{9 \cdot 25\lambda_7^{-2}\Lambda_{\text{susy}}} + \frac{1}{60} \lambda_7^{-2} \int_X G_X \wedge \ast_7 G_X
$$

$$
U_{\text{eff}} = -\frac{V_0}{\text{Vol}(X)^{3/2}} \left( \frac{5}{9} \Lambda_{\text{susy}} + \frac{1}{6} \int_X G_X \wedge \ast_7 G_X \right)
$$

(48)

Thus we see that in this model turning on a positive $\Lambda_{\text{susy}}$ raises the volume of the internal space and lowers the magnitude of cosmological constant.
but it nevertheless remains negative. One sees that, although the supersymmetry breaking contribution to the vacuum energy density can be made arbitrarily large for any given volume, this is not sufficient to overcome the background negative energy density since one also finds that the volume is shifted and this in turn suppresses the contribution of $\Lambda_{\text{susy}}$ to the the vacuum energy. Note that one cannot make $U_{\text{eff}}$ small by tuning $\Lambda_{\text{susy}}$ to a negative value as this will cause the volume to become negative.

We would like to return to our question about how much we can separate the Kaluza-Klein and cosmological constant scales. For this we can choose $V_0$ to be the volume in the vacuum we wish to discuss (so that there is no spurious conformal factor in the metric ansatz (2)). One then sees that in all the vacua with $\Lambda_{\text{susy}} = 0$, $U \sim -\lambda_7^2$. Hence the four-dimensional cosmological constant is of the same order as $\lambda_7$, which in turn we expect to be the same order as the Kaluza-Klien scale, $\lambda_7 \sim \text{Vol}(X)^{-\frac{1}{7}}$. In order to break this relationship we need to somehow fine-tune the potential so as to ensure that the various terms that contribute to $U$ cancel to a high degree. Unfortunately turning on $\Lambda_{\text{susy}}$ does not seem to enable us to do this.

5 Comments

In this paper we discussed the low energy effective action for M-theory compactified on weak $G_2$ manifolds in the presence of topologically non-trivial fluxes. This required the introduction of two types of complex moduli, those associated to massless axions and those associated to massless metric deformations. The appearance of such fluxes leads to a decay of the Freund-Rubin parameter mediated by the production of the massless axion modes. However there is a non-supersymmetric global minimum of the effective potential in the presence of fluxes which is the end point of such a decay. Although it should be born in mind that in a cosmological setting such a decay between spacetimes with negative vacuum energies is likely to end in big crunch, rather than pure anti-de Sitter space (see [19]). It would be interesting to obtain a more explicit understanding of this non-supersymmetric solution.

We also presented the superpotential for the low energy supergravity and saw that the only supersymmetric solutions correspond to vanishing flux; either a pure $G_2$ compactification or a pure Freund-Rubin compactification. Lastly we discussed a simple model designed to incorporate the effect of supersymmetry breaking by fields localized at codimension seven singularities in the compact manifold. In this model the cosmological constant can be increased, but it always remains negative, nor does it enable us to separate the Kaluza Klein and cosmological scales. Since there is no Bose-Fermi
mass degeneracy in a supersymmetric anti-de Sitter vacuum one might hope for phenomenologically interesting supersymmetry breaking patterns to arise and it would be interesting to study in greater detail.

Finally we would also like to mention that there are Freund-Rubin solutions with topologically trivial fluxes. These have the form

$$G_X = \ast_7 \Phi, \quad M = -4\lambda_7$$

where $\Phi$ is the weak $G_2$ three-form. Such solutions were first constructed long ago [20]. Presumably these solutions correspond here to setting $G_X = 0$ and taking $P_0$ sufficiently negative so that there is a second extremum at a non-zero value of $\tilde{C}^I$. It would be interesting consider a similar analysis by expanding about such flux backgrounds.

**Acknowledgments**

I would like to thank B. Acharya, F. Denef, M. Douglas, R. Minasian, G. Moore and D. Tong for helpful discussions. This work was supported by a PPARC advanced Fellowship as well as the PPARC research grant PPA/G/O/2002/00475.

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