Operadic Hochschild chain complex and free loop spaces.

by

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Abstract. We construct, for any algebra \( A \) over an operad \( O \), an Hochschild chain complex, \( C^\bullet(O, A) \) which is also an \( O \)-algebra. This Hochschild chain complex coincides with the usual one, whenever \( A \) is a commutative differential graded algebra. Let \( X \) be a simply connected space, \( N^\bullet(-) \) be the singular cochain functor, \( X^{S^1} \) be the free loop space, \( C_\infty \) be a cofibrant replacement of commutative operad and \( M_X \) a \( C_\infty \)-cofibrant model of \( X \). We prove that The operadic chain complex \( C^\bullet(C_\infty, M_X) \) is quasi-isomorphic to \( N^\bullet(X^{S^1}) \) as a \( C_\infty \)-algebra. In particular, for any prime field of coefficients this identifies the action of the large Steenrod algebra on the Hochschild homology \( H^\bullet(N^\bullet(X)) \) with the usual Steenrod operations on \( H^\bullet(X^{S^1}) \).

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Introduction.

As illustrated by the fundamental result of Mandell, \([20]\), \( E_\infty \)-algebras are the good tools for the study of the homotopy theory of topological spaces. Indeed, for a prime field of coefficients the homology of a \( E_\infty \)-algebra is an unstable algebra over the large Steenrod algebra. This last property originates with the previous works of Dold, \([9]\), May \([21]\) and has been recently extensively studied by the first author and Livernet \([8]\).

In this paper we develop an application of the homotopy theory of \( E_\infty \)-algebras to the study of the free loop space, \( X^{S^1} \) on a space \( X \). For this purpose, we consider the almost free operad, denoted \( C_\infty \), which is a cofibrant model of the commutative operad. A quotient of this operad, also denoted \( C_\infty \), has been studied by Kadeisvili, \([13]\), Kontsevich, \([16]\) and Getzler-Jones, \([10]\). It appears that the category of \( C_\infty \)-algebras is a closed model category which is very similar to the category of commutative and associative differential graded algebras.

It follows from Theorem C and Proposition 4.2-2:

Main Theorem: We denote by \( N^\bullet(-) \) the normalized singular cochain functor with coefficients in an arbitrary commutative ring \( lk \) and by \( H^\bullet(-) = H(N^\bullet(-)) \) the functor of singular cohomology. Let \( X \) be a 1-connected space, if each \( H^i(X) \) is finitely generated then there exist natural equivalences of \( C_\infty \)-algebras between \( N^\bullet(X^{S^1}) \) and the operadic Hochschild complex of the \( C_\infty \)-algebra \( N^\bullet(X) \).

An associative \( C_\infty \)-algebra is a particular case of strongly homotopy algebra (see 3.3). Our result is, in this special case, a substancial improvement of the results proved in \([3]\) and \([4]\) since the structure of unstable algebra over the Steenrod algebra but also all secondary cohomological operations are preserved \([8]\).

All the paper is devoted to the proof of this result. The required knowledge about
operads is presented in section 1. In section 2 we define the operad $\mathcal{C}_\infty$ in relation with the Barratt-Eccles operad, extensively studied by Berger and Fresse, [1]. In section 3, we construct for any operad $\mathcal{O}$ in the category of differential graded modules the operadic Hochschild complex of an $\mathcal{O}$-algebra, $A$. When $A$ is supposed to be an associative and commutative differential graded algebra we compare the operadic Hochschild complex of an $\mathcal{O}$-algebra with the usual Hochschild complex. We also study the particular case when $A$ is an almost free $\mathcal{O}$-algebra (Theorem B). In the last section we end the proof of our main result (Theorem C).

1. Backgrounds about algebras over an operad.

1.1 Notation. We denote by $\mathbb{k}$-$\text{GM}$ (resp. $\mathbb{k}$-$\text{DGM}$) the category of graded modules (resp. of differential graded modules). We also consider the forgetful functor:

$$\# : \mathbb{k}$-$\text{DGM} \rightarrow \mathbb{k}$-$\text{GM}, \ (V,d) \mapsto (V,d)_\# = V .$$

We are mainly concerned by the following categories :

$\Delta$, the simplicial category of finite ordered sets with objects $[n] = \{0,1,...n\}$ and non decreasing maps

$C^\Delta$, the category of cosimplicial objects and cosimplicial maps of $C$:

$$X = (\{X_n\}_{n \geq 0}, d^i, s^i)$$

$C^{\Delta^{op}}$, the category of simplicial objects and simplicial maps of $C$:

$$X = (\{X_n\}_{n \geq 0}, d_i, s_i)$$

1.2 Operads. Recall from [11], [10] and [17], that an operad $\mathcal{O}$ is defined in any symmetric monoidal category $C$ as a sequence of left $\mathbb{k}[\Sigma_i]$-modules (where $\Sigma_i$ is the symmetric group) $\mathcal{O}(i), i \geq 0$, with composition products

$$\mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \mathcal{O}(i_2) \otimes \ldots \otimes \mathcal{O}(i_n) \rightarrow \mathcal{O}(i_1 + i_2 + \ldots + i_n), \quad x_0 \otimes x_1 \otimes \ldots \otimes x_n \mapsto x_0(x_1, x_2, ..., x_n)$$

which are equivariant, associative and with a unit. Homomorphisms of operads are defined in an obvious way.

The category of operads is a closed model category [2], [12], where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.

The universal example of operad is the endomorphism operad. Let $(V,d_V)$ be a differential graded module, $\mathcal{E}nd_V$ is an operad in $\mathbb{k}$-$\text{DGM}$ such that:

$$\mathcal{E}nd_V(n) = \text{Hom}(V^\otimes n, V), \ n \geq 1 .$$

1.3 Algebras over an operad. Let $\mathcal{O}$ be an operad in $\mathbb{k}$-$\text{DGM}$. An $\mathcal{O}$-algebra, $(A,\rho)$, is a differential graded module, $A = (\{A_i\}_{i \in \mathbb{Z}}, d_A : A_i \rightarrow A_{i-1})$, with an operadic representation

$$\rho : \mathcal{O} \rightarrow \mathcal{E}nd_A .$$

determined by a sequence of maps differential graded $\mathbb{k}$-modules, called the evaluation product:

$$\tilde{\rho}_n : \mathcal{O}(n) \otimes A^\otimes n \rightarrow A, \quad \tilde{\rho}_n(x \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \rho_n(x)(a_1 \otimes a_2 \otimes \ldots \otimes a_n).$$

invariant under the action of $\Sigma_n$ and compatible with the composition product of $\mathcal{O}$.
O-algebras and homomorphisms of O-algebras is a category, denoted O-ALG. If O' is another operad in lk-DGM, a homomorphism of operads \( f : O \rightarrow O' \) induces a natural functor \( f^* : O'-\text{ALG} \rightarrow O-\text{ALG} \).

The free O-algebra generated by a differential graded module V is the differential graded module

\[
F(O, V) = \bigoplus_{k=0}^{\infty} O(k) \otimes_{\Sigma_k} V^{\otimes k},
\]

with evaluation products \( \tilde{\rho}_n : O(n) \otimes F(O, V)^{\otimes n} \rightarrow F(O, V) \) induced by the composition products of O. Any homomorphism \( f : V \rightarrow W \) of graded modules extends uniquely in homomorphism of graded modules \( F(O, f) : F(O, V) \rightarrow F(O, W) \). The functor \( F(O, -) : lk-DGM \rightarrow O-\text{ALG} \) is a left adjoint to the forgetful functor O-ALG \( \rightarrow lk-DGM \).

1.4 The associative operad and the commutative operad. For each \( n \geq 0 \), the canonical augmentation \( \epsilon_n : lk[\Sigma_n] \rightarrow lk \) of the group ring \( lk[\Sigma_n] \) defines a homomorphism of operads in lk-GM

\[
\epsilon : A \rightarrow C
\]

from the associative operad A to the commutative operad C such that \( A(n) = lk[\Sigma_n] \) and \( C(n) = lk \) with composition products given respectively by composition of permutations and multiplication in \( lk \). An A-algebra is an associative differential graded algebra while a C-algebra is a commutative differential graded algebra which is also, associative. Indeed, the representations \( \rho : A \rightarrow \mathcal{E}nd_A \) and \( C \rightarrow \mathcal{E}nd_A \) are the iterated products:

\[
lk[\Sigma_n] \otimes_{\Sigma_n} A^{\otimes n} = A^{\otimes n} \rightarrow A \text{ and } lk \otimes_{\Sigma_n} (A^{\otimes n}) = (A^{\otimes n})_{\Sigma_n} \rightarrow A.
\]

The free C-algebra (resp. a free A-algebra) generated by the differential graded module V is the differential graded module \( F(C, V) = \bigoplus_{n=0}^{\infty} lk \otimes_{\Sigma_n} V^{\otimes n} = \bigoplus_{n=0}^{\infty} (V^{\otimes n})_{\Sigma_n} = S(V) \) with graded commutative multiplication of the elements of V (resp. \( F(A, V) = \bigoplus_{n=0}^{\infty} lk[\Sigma_n] \otimes_{\Sigma_n} V^{\otimes n} = \bigoplus_{n=0}^{\infty} V^{\otimes n} = T(V) \) the usual tensor algebra on V).

1.5 Let A be an O-algebra. Then A is called almost free if \( A_# = F(O_#, V) \) for a graded module V. If the category of O-algebras is a closed model category (this is the case whenever O is a cofibrant operad) then any O-algebra admits a cofibrant model. An O-algebra is cofibrant if and only if it is a retract of an almost free O-algebra.

2. The operads BΣ and C∞.

2.1 Let us denote \( A = (\{A_n\}_{n \in \mathbb{N}}, d, s) \) a simplicial differential graded module with internal differential \( d_A : A_{n,q} \rightarrow A_{n,q-1} \). Then the total complex of the bicomplex

\[
\Delta_{p-1,q} \xrightarrow{\sum_{i=0}^{p-1} (-1)^i d_i} \Delta_{p,q} \xrightarrow{d_{p+1}} \Delta_{p,q+1}
\]

is denoted by \( \text{Tot}_* (A) \):

\[
\text{Tot}(A) = \{\text{Tot}_n(A)\}_{n \in \mathbb{Z}}, \quad d : \text{Tot}_n(A) \rightarrow \{\text{Tot}_{n-1}(A)\}
\]

\[
\text{Tot}_n(A) = \bigoplus_{p+q=n} \Delta_{p,q}, \quad dx = d_A x + (-1)^p \sum_{i=1}^{p} (-1)^i d_i x, \quad x \in \Delta_{p,q}.
\]
Let $D_n(A)$ be the subcomplex generated by degeneracies in $A_n$. The quotient complex $\text{Tot}_s(A)/D_n(A) := N_s(A)$ is the normalized differential graded module. The quotient map $\text{Tot}(A) \rightarrow N_s(A)$ is a chain equivalence, [19]-Theorem 6.1.

The singular chain complex of $X$ with coefficients in $\mathbb{k}$ is $C_*(X; \mathbb{k}) := C_*(S_*(X; \mathbb{k}))$ and the normalized chain complex is $N_s(X; \mathbb{k}) := N_s(S_*(X))$. The subcomplex of normalized cochain complex $N^*(X, \mathbb{k}) \subset \text{Hom}(C_*(X, \mathbb{Z}), \mathbb{k})$ is, in this paper, simply denoted $N^*(X)$. It follows from, [19]-Theorem 9.1 that $N^*$ is a contravariant functor from the category of pointed topological spaces to the category of augmented associative differential graded algebras.

The functor $N(-)$

$$l^k\text{-DGM}^{\Delta_{op}} \rightarrow l^k\text{-DGM}$$

is a monoidal functor. The functor $N$ transforms operads $O$ in $l^k\text{-DGM}^{\Delta_{op}}$ (resp. $O$-algebras in $l^k\text{-DGM}^{\Delta_{op}}$) into operads in $l^k\text{-DGM}$ (resp. algebras over an operad in $l^k\text{-DGM}$), [17]-page 51. For further use we need the slightly more general result.

2.2 Lemma. Let $A = \{A_{p,q}\}_{p,q\in \mathbb{N}}$ be a simplicial $O$-algebra. Then the total complex $\text{Tot}(A)$ is an $O$-algebra. Moreover, $N_s(A)$ is an $O$-algebra and the quotient map $\text{Tot}(A) \rightarrow N_s(A)$ is an equivalence of $O$-algebras.

Proof.

Let $A$ and $B$ be two simplicial graded modules and consider the shuffle product

$$sh : C_pA \otimes C_qB \rightarrow C_{p+q}(A \times B), a \otimes b \mapsto \sum_{\mu,\nu} (-1)^{\epsilon(\mu)} s_\mu a \times s_\nu b, \quad \begin{array}{l} a \in A_p \\ b \in B_q \end{array}$$

where the sum is taken over the $p+q$ shuffles $\mu_1 < \mu_2 < \ldots < \mu_p, \nu_1 < \nu_2 < \ldots < \nu_q$, $\mu_i, \nu_j \in \{1,2,\ldots,p+q\}$, $\epsilon(\mu)$ is the graded signature of the $(p,q)$-shuffle, $s_\mu = s_{\mu_p} \circ s_{\mu_{p-1}} \circ \ldots \circ s_1$, $s_\nu = s_{\nu_q} \circ s_{\nu_{q-1}} \circ \ldots \circ s_1$, [19]-Chapter 8. If we assume that $A$ and $B$ are two simplicial differential graded modules, one easily check that $sh$ commutes with the differentials so that we obtain:

$$sh : \text{Tot}A \otimes \text{Tot}B \rightarrow \text{Tot}(A \times B), \quad \text{and} \quad sh : N_sA \otimes N_sB \rightarrow N_{s+q}(A \times B).$$

Since, when $A = B$, the shuffle product is associative (and commutative) one defines the iterated shuffle product

$$sh^0 = id, \quad sh^{k+1} = (sh \otimes id) \circ sh^k, \quad k \geq 0$$

Let $\hat{\rho}_{n,k} : O(k) \otimes (A_n)^{\otimes k} \rightarrow A$ be an evaluation product of $A_n$. We consider $O$ as a constant simplicial module and we define the map $\hat{\rho}_k : O(k) \otimes (N_s(A))^\otimes k \rightarrow N_s(A)$ as the composite

$$O(k) \otimes (N_s(A))^\otimes k \rightarrow N_s(O(k)) \otimes N_s(A)^\otimes k \rightarrow N_s(O(k)) \otimes N_s(A^\otimes k) \rightarrow N_s(A).$$

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$$O(k) \otimes (N_s(A))^\otimes k \rightarrow N_s(O(k)) \otimes N_s(A)^\otimes k \rightarrow N_s(O(k)) \otimes N_s(A^\otimes k) \rightarrow N_s(A).$$
2.3 The operad $\mathcal{BE}$. The operad $\mathcal{BE}$ (also called the Barrat-Eccles operad, [1]-1.1.) is an operad in the category of differential graded $\mathbb{k}$-modules such that:

$$\mathcal{BE}(n) = N_* (W(\Sigma_n)) = \text{the normalized bar construction on } \Sigma_n.$$  

□

2.4 Some properties of $\mathcal{BE}$.

1) The operad $\mathcal{BE}$ is a resolution of the operad $\mathcal{C}$: the homomorphism of operads $\tau : \mathcal{BE} \rightarrow \mathcal{C}$ is defined by the augmentations of the bar resolution for each component.

2) $\mathcal{BE}$ is an $E_\infty$-operad. Recall that an operad $\mathcal{O} = \{ (\mathcal{O}(n))_i \}_{i \geq 0}$ in $\mathbb{k}$-DGM is an $E_\infty$-operad if each $\mathcal{O}(n)$ is an acyclic $\Sigma_n$-free module.

3) The natural map $\tau : \mathcal{A} \rightarrow \mathcal{C}$ factorises as $\mathcal{A} \rightarrow \mathcal{BE} \xrightarrow{\tau} \mathcal{C}$. In particular, $\mathcal{BE}$-algebras are associative algebras.

4) The operad $\mathcal{BE}$ is not cofibrant.

5) Berger and Fresse [1] have proved that the normalized singular cochains $N^* (-)$ is a functor from the category of topological spaces to the category of $\mathcal{BE}$-differential algebras.

6) If $\mathbb{k} = \mathbb{F}_p$ is the prime field of characteristic $p$ and $\mathcal{A}$ is a $\mathcal{BE}$-algebra then $H(\mathcal{A})$ is an unstable algebra over the big Steenrod algebra ([21], [8]). Indeed, consider the Standard small free resolution $\mathcal{W}$ (resp. $\mathcal{W}'$) of the cyclic group of order $p$; $\pi \subset \Sigma_p$ (resp. of $\Sigma_p$). Thus we obtain the homomorphism $\mathcal{W} \rightarrow \mathcal{W}' \rightarrow \mathcal{BE}(p)$ and the evaluation product $\tilde{\rho} : \mathcal{BE}(p) \otimes \mathcal{A}^p \rightarrow \mathcal{A}$ restricts to the structure map $\mathcal{W} \otimes \mathcal{A}^{\otimes p} \rightarrow \mathcal{A}$ considered by May, [21], in order to define “algebraic Steenrod operations”. Recall that the big Steenrod algebra, denoted $B_p$, is such that the quotient $B_p/(P^1 = id)$ is the usual Steenrod algebra, see [20]-theorem 1.4. In particular, Adem relations are satisfied ([21] and [8]).

2.5 The operad $\mathcal{C}_\infty$. Let $\mathcal{C}_\infty$ be a cofibrant replacement of $\mathcal{C}$. There exists a quasi-isomorphism of operads

$$\mathcal{C}_\infty \rightarrow \mathcal{BE}.$$  

In particular, by remarks 5) and 6) above, $N^*(X)$ is a $\mathcal{C}_\infty$-algebra and any quasi-isomorphism of $\mathcal{C}_\infty$-algebras $\mathcal{A} \rightarrow N^*(X)$ identifies the action of the large Steenrod operations on $H(\mathcal{A})$ to the usual action on $N^*(X)$.

By (1.5) any $\mathcal{C}_\infty$-algebra admits a cofibrant replacement which is an almost free $\mathcal{C}_\infty$-algebra, see [13] and [2].

3. The Operadic Hochschild chain complex.

3.1 Let us recall that the category of $\mathcal{O}$-algebras has all limits and colimits, [10]-Theorem 1.13. In particular, [20]-3, the coproduct of two $\mathcal{O}$-algebras $\mathcal{A}$ and $\mathcal{B}$ is an $\mathcal{O}$-algebra, denoted $\mathcal{A} \coprod \mathcal{B}$. Hereafter, we will use the following notation: $\mathcal{A} \rightarrowtail A \coprod B \twoheadrightarrow B$ for the natural inclusions and $A \coprod A \xrightarrow{\tau} A$ for the folding map. The symmetric group $\Sigma_n$ acts on $A \coprod \ldots \coprod A$ ($n$ terms) by permutations of factors. We denote by $\tau_n : A^{\coprod n} \rightarrow A^{\coprod n}$ the homomorphism corresponding to the cyclic permutation $(n, 1, 2, \ldots, n-1) \in \Sigma_n$. 

It is then easy to check that the $\hat{\rho}_n$ are composition products.
3.2 Let \( O \) be an operad in \( k\text{-DGM} \) and \( A \) be an \( O \)-algebra. We denote by \( A \) the simplicial \( O \)-algebra

\[
\Delta_n = A^\Pi_{n+1}, n \geq 0, \quad d_i : A^\Pi_{n+1} \to A^\Pi_{n}, \quad s_i : A^\Pi_{n} \to A^\Pi_{n+1}
\]

\[
d_i = \left\{ \begin{array}{ll}
id^\Pi_{n+1} \cup \nabla^\Pi_i \cup d^\Pi_{n-i} & \text{if } i = 0, 1, ..., n - 1 \\
\nabla^\Pi_i \cup d^\Pi_{n-i} \circ \tau_n & \text{if } i = n
\end{array} \right.
\]

The normalization \( N(A) \), (see 2.2), of the simplicial \( O \)-algebra \( A \) is an \( O \)-algebra, denoted \( C^d_O(A, A) \), and called the Hochschild chain complex of the \( O \)-algebra \( A \). The homology of the \( C^d_O(A, A) \) is the operadic Hochschild homology, denoted \( H^d_O(A, A) \).

3.3 If \((A, d_A)\) is any associative differential graded algebra supposed unital and augmented we denote by \( A \) the kernel of the augmentation. The (classical) Hochschild chain complex is defined as follows:

\[
\mathfrak{c}_* A = \{ \mathfrak{c}_k A \}_{k \geq 0}, \quad \mathfrak{c}_k A = A \otimes sA^\circ k,
\]

with \( A = k \oplus A \). A generator of \( \mathfrak{c}_k A \) is written \( a_0[s_{a_1}|s_{a_2}|...|s_{a_k}] \) if \( k > 0 \) and \( a[] \) if \( k = 0 \). We set \( \epsilon_i = |a_0| + |a_1| + |s_{a_2}| + ... + |s_{a_i}|, i \geq 1 \). The differential \( d = d^1 + d^2 \) is defined by:

\[
d^1 a_0[a_1|a_2|...|a_k] = da_0[a_1|a_2|...|a_k] - \sum_{i=1}^{k} (-1)^{\epsilon_i} a_0[a_1|...|da_i|...|a_k]
\]

\[
d^2 a_0[a_1|a_2|...|a_k] = (-1)^{\epsilon_0} a_0[a_1|a_2|...|a_k] + \sum_{i=2}^{k} (-1)^{\epsilon_i} a_0[a_1|...|a_{i-1}|a_i|...|a_k]
\]

\[
-(-1)^{|s|}[sa_k|s]a_0[a_1|...|a_{k-1}]
\]

Consider the shuffle map, 4.2.1, \( sh : \mathfrak{c}_* A \otimes \mathfrak{c}_* A \to \mathfrak{c}_*(A \otimes A) \) defined by:

\[
sh(a_0[a_1|a_2|...|a_n], b_0[b_1|b_2|...|b_m]) = (-1)^t \sum_{\sigma \in \Sigma_{n,m}} (-1)^{\epsilon(\sigma)} a_0 \otimes b_0[c_{\sigma(1)}|...|c_{\sigma(m+n)}]
\]

where \( t = |b_0|(|a_0| + ... + |a_n|), \quad \epsilon(\sigma) = \sum |c_{\sigma(i)}||c_{\sigma(m+j)}|, \quad \text{summed over all pairs } (i, m + j) \text{ such that } \sigma(m + j) < \sigma(i) \).

Clearly, \( sh \) induces a chain map still denoted \( sh : \mathfrak{c}_* A \otimes \mathfrak{c}_* A \to \mathfrak{c}_*(A \otimes A) \). Let \( HH_* (A) \) be the homology of \( \mathfrak{c}_* A \).

If \( A \) is commutative (in the graded sense) then the multiplication \( \mu_A : A \otimes A \to A \) is a homomorphism of differential graded algebras. Thus the composite \( \mathfrak{c}_* \mu_A \circ sh : \mathfrak{c}_* A \otimes \mathfrak{c}_* A \to \mathfrak{c}_* A \) defines a multiplication on \( \mathfrak{c}_* A \) which makes it into a commutative differential graded algebra 4.2.2.

If \( A \) is an associative (non commutative) differential graded algebra, there is no interesting product on \( \mathfrak{c}_*(A) \) while \( C^d_* (A, A) \) is naturally an associative differential graded algebra. Obviously, \( \mathfrak{c}_*(A)_# \neq C^d_* (A, A)_# \). N. Bitjong and second named author, 8, have proved that for any strongly homotopy commutative \( k\)-algebra \( A \), in the sense of 22, there is a well defined cup product on \( HH_* (A) \), (8-Theorem 1), which is induced from a non canonical product on \( \mathfrak{c}_*(A) \). In the formalism of operads, a strongly homotopy commutative \( k\)-algebra \( A \) is an associative \( \mathcal{B}_\infty \)-algebra, in the sense of 10-5.2, (see 3-Proposition 2).

The graded vector space \( C^d_* (\mathcal{B}_\infty, A) \) is not isomorphic to \( \mathfrak{c}_*(A) \). An interesting question is: Let \( A \) be a strongly homotopy commutative algebra \( A \) does \( H^d_* (\mathcal{B}_\infty, A) \cong HH_* (A) \) as commutative graded algebras?
Let $A$ be an associative (unital) differential graded algebra. There is classically associated to $A$ an other simplicial differential graded algebra, which we denote $\underline{A}$ and which is defined as follows (see [14]-Exemple 1.4):

$$\underline{A}_n = A^\otimes n + 1, n \geq 0, \quad d_i : A^\otimes n + 1 \to A^\otimes n, \quad s_i : A^\otimes n \to A^\otimes n + 1$$

$$d_i = \begin{cases} \text{id} \otimes A \otimes \mu_A \otimes \text{id}^\otimes n-i & \text{if } i = 0, 1, \ldots, n-1 \\ (\otimes \mu_A \otimes \text{id}^\otimes n) \circ \tau_n & \text{if } i = n \end{cases}, \quad s_i = \text{id}^\otimes i \otimes 1 \otimes \text{id}^\otimes n-i.$$

where $\mu_A$ denotes the multiplication of $A$ and $\tau_n$ the map $a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto (1)^{(\sum a_0 + \ldots + a_n)} a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$.

The complex $\tilde{c}(A) = N_* \underline{A}$ is the unreduced Hochschild chain complex. By [19]-Chapter X-Corollary 2.2 and Theorem 9.1, the maps $(\text{id} \otimes s^\otimes n)$ define a quasi-isomorphism

$$\tilde{c}(A) \to c_*(A).$$

If we assume that $A$ is commutative, $\tilde{c}_*(A)$ is also a differential graded algebra. The multiplication is the shuffle product defined in the same way that the shuffle product on $c_*(A)$. Therefore the quasi-isomorphism (1) is a homomorphism of differential graded algebras.

**Theorem A.** Assume that $A$ is associative commutative differential graded algebra. Then exists a natural quasi-isomorphism of commutative differential graded algebras

$$C^\partial_*(C, A) \to c_*(A).$$

**Remark.** If the $C$-algebra $A$ is considered as a $C_\infty$-algebra then, by naturality there is a surjective homomorphism of $C_\infty$-algebras

$$C^\partial_*(C_\infty, A) \to C^\partial_*(C, A).$$

Does this map induce an isomorphism in homology?

**Proof.** Let $B$ be a $C$-algebra. By universal property, there exists a natural isomorphism $\Phi_{A,B} : A \boxplus B \to A \otimes B$ of commutative differential graded algebras such that the following diagrams commute where we put $\Phi_{A,A} = \Phi_1$.

$$\begin{array}{cc}
\begin{array}{ccc}
A \boxplus A & \xrightarrow{\Sigma} & A \\
\Phi_1 & \downarrow & \\
A \otimes A & \xrightarrow{\mu} & A \\
\end{array}
& &
\begin{array}{ccc}
A \boxplus A & \xrightarrow{T} & A \boxplus A \\
\Phi_1 & \downarrow & \\
A \otimes A & \xrightarrow{T} & A \otimes A \\
\end{array}
\end{array}$$

where, $\mu$ denotes the usual product on $A \otimes A$ and $T$ the usual twisting map. The associativity properties permit iteration so that we obtain for any $n \geq 0$ an isomorphism $\Phi_n : A \boxplus_{n-1} \to A^\otimes n + 1$. These $\Phi_n$’s induce an isomorphism of simplicial differential graded modules $\underline{A} \to \underline{A}$ which in turn induces an isomorphism

$$C^\partial_*(A) = N_*(\underline{A}) \to N_*(\underline{A}) = \tilde{c}_*(A).$$
Composition of the isomorphism (2) with the quasi-isomorphism (1) gives the quasi-isomorphism

$$(3) \quad C^g_*(A) \to \mathfrak{e}_*(A).$$

On the other hand, the $\mathcal{C}$-algebra structure on $C^g_*(A)$ is such that the isomorphism (2) is an isomorphism of differential graded algebras. Thus (3) is a quasi-isomorphism of commutative differential graded algebras.

\[\square\]

3.6 An operad $C_\infty$ is a Hopf operad "up to homotopy" (for the notion of Hopf operad we refer to (1)-5.3), that is to say it has a diagonal which is not coassociative but only up to homotopy. In this case, the tensor product of two $C_\infty$-algebras $A$ and $B$ is a $C_\infty$-algebra $A \otimes B$ with underlying differential graded module being the tensor product of the underlying differential graded modules, denoted $A \otimes B$. Indeed, Hinich [13] has proved that if the $C_\infty$-algebras, $A$ and $B$ are cofibrant then there exists a natural quasi-isomorphism

$$\Phi_{A,B} : A \coprod B \to A \otimes B.$$

Let $f : A \to A'$ and $g : B \to B'$ be two homomorphisms of $C_\infty$-algebras. Then, by naturality of $\Phi_{A,B}$, we obtain the commutative diagram

$$
\begin{array}{ccc}
A \coprod B & \xrightarrow{f \coprod g} & A' \coprod B' \\
\Phi_{A,B} \downarrow & & \downarrow \Phi_{A',B'} \\
A \otimes B & \xrightarrow{f \otimes g} & A' \otimes B'
\end{array}
$$

If we assume that $A$ and $A'$ are cofibrant and that $f$ and $g$ are quasi-isomorphisms, then by [2]-Theorem 3.2, the homomorphism $f \coprod g$ is a quasi-isomorphism. Therefore, $f \otimes g$ is also a quasi-isomorphism.

3.7 Assume that $A = F(\mathcal{O}, V)$ is an almost free $\mathcal{O}$-algebra. Thus, we have the sequence of direct summands

$$\mathcal{A}_n \supset \mathcal{O}(k) \otimes \Sigma_k (V^\otimes n+1)^{\otimes k} \supset \mathcal{O}(k) \otimes \Sigma_k \left( V^{\otimes k_0} \otimes V^{\otimes k_1} \otimes \ldots \otimes V^{\otimes k_n} \right),$$

with $k = k_0 + k_1 + \ldots + k_n$. Therefore, an element of $\mathcal{A}_n$ is finite sum of elements of the form $x \otimes v_1 \otimes \ldots v_{k_0} \otimes v_{k_0+1} \otimes \ldots v_{k_0+k_1} \otimes \ldots \otimes v_{k_0+\ldots+k_{n-1}+1} \otimes v_k$ with $x \in \mathcal{O}(k)$ and $v_{k_0+\ldots+k_{n-1}+1} \otimes v_k \in V^{\otimes k}$ with usual convention $V^{\otimes 0} = k$. With this notation we obtain explicit formulas for the map $r$, $l$ and $\nabla$ defined in 3.1:

$$r : A_0 \to A_1, \quad r(x \otimes v_1 \otimes \ldots v_{k_0} = x \otimes v_1 \otimes \ldots v_{k_0} \otimes 1$$

$$l : A_0 \to A_1, \quad l(x \otimes v_1 \otimes \ldots v_{k_0} = x \otimes 1 \otimes v_1 \otimes \ldots v_{k_0}$$

$$\nabla : A_1 \to A_0, \quad \nabla(x \otimes v_1 \otimes \ldots v_{k_0} \otimes v_{k_0+1} \otimes \ldots v_{k_0+k_1}) = x \otimes v_1 \otimes \ldots v_{k_0} v_{k_0+1} \otimes \ldots v_{k_0+k_1}.$$

Therefore,

$$d_i : A_{n+1} \to A_n, \quad d_i \left( x \otimes v_1 \otimes \ldots v_{k_0} \otimes v_{k_0+1} \otimes \ldots v_{k_0+k_1} \otimes \ldots \otimes v_{k_0+\ldots+k_n-1+1} \otimes v_k \right) =$$

$$\begin{cases} x \otimes v_1 \otimes \ldots v_{k_0} \otimes \ldots \otimes v_{k_0+k_1+1} \otimes \ldots v_{k_0+\ldots+k_{n-1}+2} \otimes \ldots \otimes v_{k_0+\ldots+k_n-1+1} \otimes v_k & \text{if } i = 0, 1, \ldots, n-1 \\
= x \otimes v_{k_0+\ldots+k_{n-2}+1} \otimes \ldots v_{k_0+\ldots+k_{n-1}} & \text{if } i = n
\end{cases}$$

$$s_i : A_n \to A_{n+1}, \quad s_i \left( x \otimes v_1 \otimes \ldots v_{k_0} \otimes v_{k_0+1} \otimes \ldots v_{k_0+k_1} \otimes \ldots \otimes v_{k_0+\ldots+k_{n-1}+1} \otimes v_k \right) =$$

$$x \otimes v_1 \otimes \ldots v_{k_0} \otimes \ldots \otimes 1 \otimes \ldots \otimes v_{k_0+\ldots+k_{n-1}+1} \otimes \ldots v_{k_0+\ldots+k_{n-1}+1} \otimes \ldots v_{k_0+\ldots+k_{n-1}+1} \otimes v_k.$$
Let us denote by \( \mathcal{A}^+_n \) the submodule of the \( k \)-module \( \mathcal{A}_n \) generated by the elements of the form \( x \otimes v_1 \ldots v_{k_0} \otimes v_{k_0+1} \ldots v_{k_0+k_1} \otimes \ldots \otimes v_{k_0+\ldots+k_n} \) with \( x \in \mathcal{O}(k) \) and \( v_{k_0+\ldots+k_n} \in V^k \) such that each \( k_i > 0 \). The above formulas for \( d_i \) and \( s_i \) show that

a) the graded module \( \mathcal{A}^+_n \) is stable for the \( d_i \)'s but not stable for the \( s_i \)'s.

b) the submodule \( \mathcal{D}_n \) of \( \mathcal{A}_n \) generated by all degenerate elements (\( \mathcal{D}_0 = 0 \)) is exactly the submodule generated by the elements \( x \otimes v_1 \ldots v_{k_0} \otimes v_{k_0+1} \ldots v_{k_0+k_1} \otimes \ldots \otimes v_{k_0+\ldots+k_n} \) such that at least one \( k_i = 0 \).

Since \( C^5_\ast(\mathcal{O}, \mathcal{A}) = N(\mathcal{A}) = \text{Tot}(\mathcal{A})/D(\mathcal{A}) \) and since \( d_\ast(\mathcal{A}^+_n) \subset \mathcal{A}^+_n \) we have proved the first part of the next result.

**Theorem B.** Assume that \( A = F(\mathcal{O}, \mathcal{V}) \) is an almost free \( \mathcal{O} \)-algebra. Then the restriction of the natural chain equivalence \( \text{Tot} \mathcal{A} \to N_\ast \mathcal{A} \) to the total complex, \( \text{Tot}(\mathcal{A}^+) \) is an isomorphism of differential graded modules

\[
\text{Tot}(\mathcal{A}^+) \to C^5_\ast(\mathcal{O}, \mathcal{A}).
\]

Moreover, this homomorphism is an isomorphism of \( \mathcal{O} \)-algebras.

**End of proof.** Let us precise first that the evaluation products of the simplicial algebra \( \mathcal{A} \) are determined by the maps

\[
\mathcal{O}(k)_q \otimes \mathcal{A}^+_{p_1,q_1} \otimes \mathcal{A}^+_{p_2,q_2} \otimes \ldots \otimes \mathcal{A}^+_{p_k,q_k} \to \mathcal{A}^+_{p_1+p_2+\ldots+p_k,q_1+\ldots+q_k},
\]

which are explicitely given by the shuffle products and the evaluation product of \( \mathcal{O} \) (see proof of lemma 2.2). This implies that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{O}(n)_q \otimes \mathcal{A}^+_{p_1,q_1} \otimes \ldots \otimes \mathcal{A}^+_{p_n,q_n} & \to & \mathcal{O}(n)_q \otimes \mathcal{A}^+_{p_1,q_1} \otimes \ldots \otimes \mathcal{A}^+_{p_n,q_n} \\
\downarrow & & \downarrow \\
\mathcal{A}^+_{p_1+\ldots+p_n,q_1+\ldots+q_n} & \to & \mathcal{A}^+_{p_1+\ldots+p_n,q_1+\ldots+q_n}
\end{array}
\]

Therefore, each \( \mathcal{A}^+_n \) is a sub \( \mathcal{O} \)-algebra of \( \mathcal{A}_n \).

It results from the formula above that \( d_0 = d_1 : \mathcal{A}_1 \to \mathcal{A}_0 \) and that \( d_i \left( \bigoplus_{p>0,q \in \mathbb{Z}} \mathcal{A}^+_{p,q} \right) \subset \bigoplus_{p>0,q \in \mathbb{Z}} \mathcal{A}^+_{p,q} \). Thus we obtain:

**Proposition.** Assume that \( A = F(\mathcal{O}, \mathcal{V}) \) is an almost free \( \mathcal{O} \)-algebra. Then we have the natural splitting of \( \mathcal{O} \)-algebras

\[
\text{Tot}(\mathcal{A}^+) = (A, 0) \oplus \left( \bigoplus_{p>0,q \in \mathbb{Z}} \mathcal{A}^+_{p,q}, d \right).
\]

4. **Free loop space.**

4.1 Write \( \text{Top} \) (resp. \( \text{Costop} \)) for the category of topological spaces (resp. of cosimplicial topological spaces). The geometric realization of a cosimplicial set is the covariant functor

\[
||-|| : \text{Costop} \to \text{Top}, \quad Z \mapsto ||Z|| = \text{Costop}(\Delta, Z) \subset \prod_{n \geq 0} \text{Top}(\Delta^n, Z(n)),
\]
where $|Z|$ is equipped with the topology induced by this inclusion. Here $\triangle(n) = \Delta^n$ with the usual coface and codegeneracy maps. If $Z$ is any cosimplicial topological space, then $N^*Z$ is a simplicial set and $N^*\mathbb{Z}$ is a simplicial cochain complex with total complex $\text{Tot}(N^*Z)$:

$$(\text{Tot}_n(N^*Z) = \bigoplus_{p-q=n} N^q\mathbb{Z}(p), \quad Dx = \sum_{i=1}^p (-1)^i C^*(d_i) + (-1)^p \delta x, \ x \in N^*\mathbb{Z}(p).$$

The $d_i$ are the coface operators of $\mathbb{Z}$ and $\delta$ is the internal differential of $N^*(\mathbb{Z}(p))$. Recall that in general the natural map $\text{Tot}(N^*\mathbb{Z}) \to N^*(|Z|)$ is not a weak equivalence, \[3\]. It results from lemma 2.2 and 2.3-5 that the total complex $\text{Tot}(N^*(Z))$ is naturally a $C_\infty$-algebra and $\text{Tot}(N^*(\mathbb{Z})) \to N^*(|Z|)$ a homomorphism of $C_\infty$-algebras.

Hereafter we denote $N^*(N^*(\mathbb{Z}))$ the normalization of $\text{Tot}(N^*(\mathbb{Z}))$.

4.2 One of the interest for considering cosimplicial spaces is the following result, \[3\]. Proposition 5.1, (see also \[23\]-Corollary 1): If $\underline{\mathbb{L}}$ is a simplicial set and $T$ a topological space then the cosimplicial space $\mathbb{L}$ is such that there is a homeomorphism:

$$||T\mathbb{L}|| := \text{Costop}(\underline{\triangle}, T\underline{\mathbb{L}}) \cong \text{Top}(\{\mathbb{L}\}, T) = T\mathbb{L}.$$ 

In particular, if we consider the simplicial set $K$ defined as follows: $K(n) = \mathbb{Z}/(n+1)\mathbb{Z}$, and, if $\overline{k}^n$ denotes an element in $\mathbb{Z}/n\mathbb{Z}$, the face maps $d_i : K(n) \to K(n-1)$ with $0 \leq i \leq n - 1$ and the degeneracy maps $s_j : K(n) \to K(n+1)$ with $0 \leq j \leq n$ are:

$$d_i \overline{k}^{n+1} = \begin{cases} \overline{k}^n & \text{if } k \leq i \\ \overline{k} \frac{n+1}{k-1} & \text{if } k > i \end{cases} \quad s_j \overline{k}^{n+1} = \begin{cases} \overline{k}^n \frac{n+2}{k+1} & \text{if } k \leq j \\ \overline{k} \frac{n+1}{k} & \text{if } k > j. \end{cases}$$

and $d_n \overline{k}^{n+1} = \overline{k}^n$. The geometric realization of $K$, \[3\] (proposition 1.4), $|K|$ is homeomorphic to the circle $S^1$. Therefore, the cosimplicial model, $\underline{X}$, for the free loop space, used by Jones, \[14\],

$$\underline{X}(n) = \text{Map}(K(n), X) = X \times \ldots \times X \quad \text{(n+1)-times}$$

$$d_i(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_n), \quad 0 \leq i \leq n$$

$$d_{n+1}(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_n, x_0)$$

$$s_j(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_j, x_{j+2}, \ldots, x_n), \quad 0 \leq j \leq n.$$ 

is such that $||\underline{X}|| \cong \text{Top}(\underline{K}, X) = X^{S^1}$, ($\cong$ means homeomorphism). From lemma 2.2, we deduce then:

**Proposition.** If $X$ is simply connected, the natural map $\text{Tot}(N^*\underline{X}) \to N^*(X^{S^1})$ is a quasi-isomorphism of $C_\infty$-algebras.

4.3 Theorem C. Let $X$ be a simply connected space such that each $H^i(X)$ is finitely generated. Given an almost free model of the space $X$

$$\varphi_X : M_X = (F(C, V), d) \to N^*X$$

there exists a natural quasi-isomorphism of $C_\infty$-algebras

$$C^\partial_*(C_\infty, M_X) \to N^*(N^*(\underline{X}))$$.
Proof. Let $\varphi_X : M_X = (F(C, V), d) \to N^*(X)$ (resp. $\varphi_Y : M_Y = (F(C, V), d) \to N^*(Y)$) be a almost free model for the space $X$ (resp. for the space $Y$). By universal property, we obtain the commutative diagram

\[
\begin{array}{c}
M_X \xrightarrow{\iota} M_X \coprod M_Y \xleftarrow{\xi} M_Y \\
\sim \downarrow \varphi_X \hspace{1cm} \psi_{X,Y} \downarrow \hspace{1cm} \sim \downarrow \varphi_Y \\
N^*(X) \xrightarrow{N^*(pr_X)} N^*(X \times Y) \xleftarrow{N^*(pr_Y)} N^*(Y)
\end{array}
\]

where $pr_X$ and $pr_Y$ (resp. $i_X$ and $i_Y$) are the natural projections (resp. inclusions). We have also the following commutative diagrams

\[
\begin{array}{c}
M_X \coprod M_X \xrightarrow{\nabla} M_X \\
\psi_{X,X} \downarrow \hspace{1cm} \sim \downarrow \varphi_X \\
N^*(X \times X) \xrightarrow{N^*(\Delta_X)} N^*(X) \\
\psi_{X,X} \downarrow \hspace{1cm} \sim \downarrow \varphi_X \\
N^*(X \times Y) \xrightarrow{N^*(T_X)} N^*(X)
\end{array}
\]

where $\Delta_X$ is the diagonal and $T_X$ the topological interchange map.

By [20]-Lemma 5.2, we know that if each $H^i(X)$ is finitely generated then $\psi_{X,X}$ is a weak equivalence of differential graded module. It is now easy to prove that iteration furnishes a quasi-isomorphism of simplicial $\mathcal{C}_\infty$-algebras $M_X \to N^*(X)$.

\[\blacksquare\]

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