MODULE HOM-ALGEBRAS

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Abstract. We study a twisted version of module algebras called module Hom-algebras. It is shown that module algebras deform into module Hom-algebras via endomorphisms. As an example, we construct certain \( q \)-deformations of the usual \( sl(2) \)-action on the affine plane.

1. Introduction

Let \( H \) be a bialgebra, and let \( A \) be an algebra. An \( H \)-module algebra structure on \( A \) consists of an \( H \)-module structure such that the multiplication map on \( A \) becomes an \( H \)-module morphism. In other words, one has the module algebra axiom

\[
(x(ab)) = \sum (x'(a))(x''b)
\]

for \( x \in H \) and \( a, b \in A \), where \( \Delta(x) = \sum (x', x'') \) is the Sweedler’s notation for comultiplication. Module algebras arise often in algebraic topology, quantum groups, Lie and Hopf algebras theory, and group representations. For example, the singular mod \( p \) cohomology \( H^*(X; \mathbb{Z}/p) \) of a topological space \( X \) is an \( \mathbb{A}_p \)-module algebra, where \( \mathbb{A}_p \) is the Steenrod algebra associated to the prime \( p \). Likewise, the complex cobordism \( MU^*(X) \) of a topological space \( X \) is an \( S \)-module algebra, where \( S \) is the Landweber-Novikov algebra of stable cobordism operations.

The purpose of this paper is to study a Hom-algebra analogue of module algebras, in which (1.0.1) is twisted by a linear map. Hom-algebras were first defined for Lie algebras. A Hom-Lie algebra is a vector space \( L \) together with a bilinear skew-symmetric bracket \( [\cdot, \cdot] : L \otimes L \to L \) and a linear map \( \alpha : L \to L \) such that \( \alpha([x, y]) = [\alpha(x), \alpha(y)] \) for \( x, y \in L \) (multiplicativity) and that the following Hom-Jacobi identity holds:

\[
[[[x, y], z], \alpha(x)] + [[[z, x], \alpha(y)], [y, z], \alpha(x)] = 0.
\]

Hom-Lie algebras were introduced in \[10\] (without multiplicativity) to describe the structures on certain \( q \)-deformations of the Witt and the Virasoro algebras. Earlier precursors of Hom-Lie algebras can be found in \[16, 17\]. A Lie algebra is a Hom-Lie algebra with \( \alpha = 1d \). More generally, if \( L \) is a Lie algebra and \( \alpha \) is a Lie algebra endomorphism on \( L \), then \( L \) becomes a Hom-Lie algebra with the bracket \( [x, y]_\alpha = \alpha([x, y]) \).

Likewise, a Hom-associative algebra \( A \) has a bilinear map \( \mu : A \otimes A \to A \) and a linear map \( \alpha : A \to A \) such that \( \alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y)) \) and

\[
\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))
\]

for \( x, y, z \in A \). It is shown in \[11\] that a Hom-associative algebra \( (A, \mu, \alpha) \) gives rise to a Hom-Lie algebra \( (A, [-, -], \alpha) \) in which \( [x, y] = \mu(x, y) - \mu(y, x) \), i.e., the commutator bracket of \( \mu \). Conversely,

\[
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\]

\[
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\]

\[
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\]
given a Hom-Lie algebra \( L \), there is a universal enveloping Hom-associative algebra \( U(L) \) [19, 21]. Dualizing \((1.0.2)\), one can define Hom-coassociative coalgebra and Hom-bialgebra \([12, 14, 21]\). It is shown in \([21]\) that the universal enveloping Hom-associative algebra \( U(L) \) is a Hom-bialgebra. Given an algebra \( A \) and an algebra endomorphism \( \alpha \), one obtains a Hom-associative algebra structure on \( A \) with multiplication \( \mu_\alpha = \alpha \circ \mu \) \([20]\). The same procedure can be applied to coalgebras, bialgebras, and other kinds of algebraic structures, as was done in \([3, 14, 21]\), to obtain Hom-coassociative coalgebras, Hom-bialgebras, and other Hom-algebra structures.

Using \((1.0.2)\) as a model, one can define modules and their morphisms over a Hom-associative algebra. The precise definitions will be given in Section 3. Let \( H = (H, \mu_H, \Delta_H, \alpha_H) \) be a Hom-bialgebra and \( A = (A, \mu_A, \alpha_A) \) be a Hom-associative algebra. Then an \( H \)-module Hom-algebra structure on \( A \) consists of an \( H \)-module structure \( \rho: H \otimes A \to A \) on \( A \), \( \rho(x \otimes a) = xa \), such that the module Hom-algebra axiom
\[
\alpha^2_H(x)(ab) = \sum_{(x)} (x'a)(x''b)
\]
holds. Of course, if \( \alpha_H = Id_H \) and \( \alpha_A = Id_A \), then an \( H \)-module Hom-algebra is exactly an \( H \)-module algebra.

As in the case of module algebras, the module Hom-algebra axiom \((1.0.3)\) can be interpreted as a certain multiplication map being an \( H \)-module morphism. The following characterization of module Hom-algebras will be proved in Section 2. The symbol \( \tau_{H,A} \) denotes the twist isomorphism \( H \otimes A \to A \otimes H \) where \( \tau_{H,A}(x \otimes a) = a \otimes x \).

**Theorem 1.1.** Let \( H = (H, \mu_H, \Delta_H, \alpha_H) \) be a Hom-bialgebra, \( A = (A, \mu_A, \alpha_A) \) be a Hom-associative algebra, and \( \rho: H \otimes A \to A \) be an \( H \)-module structure on \( A \). Then the following statements hold.

1. The map
\[
\tilde{\rho} = \rho \circ (\alpha^2_H \otimes Id_A): H \otimes A \to A
\]
gives \( A \) another \( H \)-module structure.

2. The map
\[
\rho^2 = \rho^\otimes_2 \circ (Id_H \otimes \tau_{H,A} \otimes Id_A) \circ (\Delta_H \otimes Id_A^\otimes_2): H \otimes A^\otimes_2 \to A^\otimes_2
\]
gives \( A^\otimes_2 \) an \( H \)-module structure.

3. The map \( \rho \) gives \( A \) the structure of an \( H \)-module Hom-algebra if and only if \( \mu_A: A^\otimes_2 \to A \) is a morphism of \( H \)-modules, where in \( \mu_A \) we equip \( A^\otimes_2 \) and \( A \) with the \( H \)-module structures \((1.1.2)\) and \((1.1.3)\), respectively.

As we mentioned above, algebras, coalgebras, and bialgebras deform into the respective types of Hom-algebras via an endomorphism. The following result, which will be proved in Section 3, shows that module algebras deform into module Hom-algebras via an endomorphism. This result provides a large class of examples of module Hom-algebras.

**Theorem 1.2.** Let \( H = (H, \mu_H, \Delta_H) \) be a bialgebra and \( A = (A, \mu_A) \) be an \( H \)-module algebra via \( \rho: H \otimes A \to A \). Let \( \alpha_H: H \to H \) be a bialgebra endomorphism and \( \alpha_A: A \to A \) be an algebra endomorphism such that
\[
\alpha_A \circ \rho = \rho \circ (\alpha_H \otimes \alpha_A).
\]
Write \( H_\alpha \) for the Hom-bialgebra \((H, \mu_{\alpha,H} = \alpha_H \circ \mu_H, \Delta_{\alpha,H} = \Delta_H \circ \alpha_H, \alpha_H)\) and \( A_\alpha \) for the Hom-associative algebra \((A, \mu_{\alpha,A} = \alpha_A \circ \mu_A, \alpha_A)\). Then the map
\[
\rho_\alpha = \alpha_A \circ \rho: H \otimes A \to A
\]
gives the Hom-associative algebra \( A_\alpha \) the structure of an \( H_\alpha \)-module Hom-algebra.
We now describe some consequences of Theorem 1.2. In the context of the above Theorem, if \( \alpha_H = Id_H \), then we have the condition \( \alpha_A \circ \rho = \rho \circ (Id_H \otimes \alpha_A) \), which means exactly that \( \alpha_A \) is \( H \)-linear. Thus, using the same notations as above, we have the following special case.

**Corollary 1.3.** Let \( H = (H, \mu_H, \Delta_H) \) be a bialgebra, \( A = (A, \mu_A) \) be an \( H \)-module algebra via \( \rho : H \otimes A \to A \), and \( \alpha_A : A \to A \) be an algebra endomorphism that is also \( H \)-linear. Then the map \( \rho_\alpha \) gives the Hom-associative algebra \( A_\alpha \) the structure of an \( H \)-module Hom-algebra, where \( H \) denotes the Hom-bialgebra \( (H, \mu_H, \Delta_H, Id_H) \).

Examples that illustrate Corollary 1.3 will be given in Section 3.

Module algebras over the universal enveloping bialgebra \( U(L) \) of a Lie algebra \( L \) are important in the study of Lie algebras and quantum groups. For example, there is a \( U(sl(2)) \)-module algebra structure on the affine plane \( k[x, y] \) that captures all the finite dimensional simple \( sl(2) \)-modules [6]. The following result, which will be proved in Section 3, applies Theorem 1.2 in the context of the above Theorem, if \( \rho \) is \( H \)-linear. Thus, using the same notations as above, we have the following special case.

**Theorem 1.4.** Let \( L \) be a Lie algebra, \( A = (A, \mu_A) \) be a \( U(L) \)-module algebra via \( \rho : U(L) \otimes A \to A \), \( \alpha_L : L \to L \) be a Lie algebra endomorphism, and \( \alpha_A : A \to A \) be an algebra endomorphism. Suppose that \( \alpha_A(za) = \alpha_A(x)\alpha_A(a) \) for \( x \in L \) and \( a \in A \). Then the following statements hold.

1. There exists a unique bialgebra endomorphism extension \( \alpha_U : U(L) \to U(L) \) of \( \alpha_L \) such that
   \[
   \alpha_A(za) = \alpha_U(z)\alpha_A(a)
   \] (1.4.1)
   for \( z \in U(L) \) and \( a \in A \).
2. The map
   \[
   \rho_\alpha = \alpha_A \circ \rho : U(L) \otimes A \to A
   \]
gives the Hom-associative algebra \( A_\alpha = (A, \mu_{\alpha_A} = \alpha_A \circ \mu_A, \alpha_A) \) the structure of a \( U(L)_\alpha \)-module Hom-algebra. Here \( U(L)_\alpha \) is the Hom-bialgebra \( (U(L), \mu_{\alpha,U} = \alpha_U \circ \mu_U, \Delta_{\alpha,U} = \Delta_U \circ \alpha_U, \alpha_U) \).

Using Theorem 1.4, we will construct certain \( q \)-deformations of the \( U(sl(2)) \)-module algebra structure on the affine plane mentioned above.

We note that module Hom-algebras, and other kinds of Hom-algebras (e.g., Hom-(co)associative (co)algebras, Hom-Lie (co)algebbras, Hom-bialgebras [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]), and n-ary Hom-Nambu/Lie/associative algebras [15], are algebras over their respective (colored) PROPs [13]. In particular, their algebraic deformations, in the sense of Gerstenhaber, are governed by some \( L_\infty \)-deformation complexes, as shown in [4].

The rest of this paper is organized as follows. In Section 2 we state the relevant definitions and prove Theorem 1.1. In Section 3 we prove Theorem 1.2 and provide some examples to illustrate Corollary 1.3. In Section 4 we prove Theorem 1.4 and illustrate it with the case \( L = sl(2) \) and \( A = k[x, y] \) (Example 4.1).

## 2. Hom-algebra analogue of module algebra

Before we define module Hom-algebras and prove Theorem 1.1, let us first recall some basic definitions regarding Hom-modules, Hom-associative algebras, and Hom-bialgebras. The first two parts of Theorem 1.1 will be proved as Lemmas 2.5 and 2.8. The last part will be proved at the end of this section.
2.1. Conventions and notations. Throughout the rest of this paper, \( k \) denotes a field of any characteristic. The only exception is Example 1.1, where \( k \) is assumed to have characteristic 0. Vector spaces, tensor products, and linearity are all meant over \( k \) \( i.e., k \) characteristic. The only exception is Example 4.1, where notation for comultiplication: \( \Delta(x) = \sum(x) x' \otimes x'' \) [8].

Given two vector spaces \( V \) and \( W \), denote by \( \tau = \tau_{V,W} : V \otimes W \to W \otimes V \) the twist isomorphism, \( i.e., \tau(v \otimes w) = w \otimes v. \) For a coalgebra \( C \) with comultiplication \( \Delta : C \to C \otimes C \), we use Sweedler’s notation for comultiplication: \( \Delta(x) = \sum(x) x' \otimes x'' \) [8].

2.2. Hom-modules. A Hom-module is a pair \((V, \alpha)\) [11] in which \( V \) is a vector space and \( \alpha : V \to V \) is a linear map. A morphism \((V, \alpha_V) \to (W, \alpha_W)\) of Hom-modules is a linear map \( f : V \to W \) such that \( \alpha_W \circ f = f \circ \alpha_V \). We will often abbreviate a Hom-module \((V, \alpha)\) to \( V \). The tensor product of the Hom-modules \((V, \alpha_V)\) and \((W, \alpha_W)\) consists of the vector space \( V \otimes W \) and the linear self-map \( \alpha_V \otimes \alpha_W \).

2.3. Hom-associative algebras. A Hom-associative algebra [11, 20] is a triple \((A, \mu, \alpha)\) in which \((A, \alpha)\) is a Hom-module and \( \mu : A \otimes A \to A \) is a bilinear map such that

\[
\begin{align*}
&\text{(1)} \quad \alpha \circ \mu = \mu \circ \alpha \otimes \alpha^2 \quad \text{(multiplicativity)} \\
&\text{(2)} \quad \mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha) \quad \text{(Hom-associativity)}.
\end{align*}
\]

In what follows, we will also write \( \mu(a \otimes b) \) as \( ab \).

For example, an algebra \((A, \mu)\) can be regarded as a Hom-associative algebra \((A, \mu, Id_A)\). Conversely, let \( \alpha : A \to A \) be an algebra endomorphism of the algebra \((A, \mu)\). Define the new multiplication

\[
\mu_\alpha = \alpha \circ \mu : A \otimes^2 A \to A. \tag{2.3.1}
\]

One can check that \( A_\alpha = (A, \mu_\alpha, \alpha) \) is a Hom-associative algebra [20]. In fact, both \( \mu_\alpha \circ (\alpha \otimes \mu_\alpha) \) and \( \mu_\alpha \circ (\mu_\alpha \otimes \alpha) \), when applied to \( a \otimes b \otimes c \in A \otimes^3 \), are equal to \( \alpha^2(abc) \). So \( \mu_\alpha \) is Hom-associative. Multiplicativity of \( \alpha \) with respect to \( \mu_\alpha \) can be checked similarly.

Suppose that \((A, \mu_A, \alpha_A)\) and \((B, \mu_B, \alpha_B)\) are Hom-associative algebras. Their tensor product \( A \otimes B \) as a Hom-associative algebra is defined in the usual way, with \( \alpha_{A \otimes B} = \alpha_A \otimes \alpha_B \) and \( \mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (Id_A \otimes \tau_{B,A} \otimes Id_B) \).

A morphism \( f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B) \) of Hom-associative algebras is a morphism \( f : (A, \alpha_A) \to (B, \alpha_B) \) of the underlying Hom-modules such that \( f \circ \mu_A = \mu_B \circ f \otimes^2 \).

2.4. Modules over Hom-associative algebras. Let \((A, \mu_A, \alpha_A)\) be a Hom-associative algebra and \((M, \alpha_M)\) be a Hom-module. An \( A \)-module structure on \( M \) consists of a morphism \( \rho : A \otimes M \to M \) of Hom-modules, called the structure map, such that

\[
\rho \circ (\alpha_A \otimes \rho) = \rho \circ (\mu_A \otimes \alpha_M). \tag{2.4.1}
\]

We will also write \( \rho(a \otimes m) \) as \( am \) for \( a \in A \) and \( m \in M \). In this notation, \( \rho(a \otimes m) \) can be rewritten as

\[
\alpha_A(a)(bm) = (ab) \alpha_M(m) \tag{2.4.2}
\]

for \( a, b \in A \) and \( m \in M \). If \( M \) and \( N \) are \( A \)-modules, then a morphism of \( A \)-modules \( f : M \to N \) is a morphism of the underlying Hom-modules such that

\[
f \circ \rho_M = \rho_N \circ (Id_A \otimes f), \tag{2.4.3}
\]

i.e., \( f(am) = af(m) \).
The following Lemma will be needed when we give an alternative characterization of a module Hom-algebra. It proves the first part of Theorem 1.1.

**Lemma 2.5.** Let \((A, \mu_A, \alpha_A)\) be a Hom-associative algebra and \((M, \alpha_M)\) be an \(A\)-module with structure map \(\rho: A \otimes M \to M\). Define the map
\[
\tilde{\rho} = \rho \circ (\alpha_A^2 \otimes \text{Id}_M): A \otimes M \to M. 
\tag{2.5.1}
\]
Then \(\tilde{\rho}\) is the structure map of another \(A\)-module structure on \(M\).

**Proof.** The fact that \(\rho\) is a morphism of Hom-modules means that
\[
\alpha_M \circ \rho = \rho \circ (\alpha_A \otimes \alpha_M). 
\tag{2.5.2}
\]
To see that \(\tilde{\rho}\) is a morphism of Hom-modules, we compute as follows:
\[
\alpha_M \circ \tilde{\rho} = \alpha_M \circ \rho \circ (\alpha_A^2 \otimes \text{Id}_M)
= \rho \circ (\alpha_A \otimes \alpha_M) \circ (\alpha_A^2 \otimes \text{Id}_M) \quad \text{by (2.5.2)}
= \rho \circ (\alpha_A^2 \otimes \text{Id}_M) \circ (\alpha_A \otimes \alpha_M)
= \tilde{\rho} \circ (\alpha_A \otimes \alpha_M).
\]
To see that \(\tilde{\rho}\) satisfies (2.4.1) (with \(\tilde{\rho}\) in place of \(\rho\)), we compute as follows:
\[
\tilde{\rho} \circ (\alpha_A \otimes \tilde{\rho}) = \rho \circ (\alpha_A^2 \otimes \text{Id}_M) \circ (\alpha_A \otimes (\rho \circ (\alpha_A^2 \otimes \text{Id}_M)))
= \rho \circ (\alpha_A \otimes \rho) \circ (\alpha_A^2 \otimes \alpha_A^2 \otimes \text{Id}_M)
= \rho \circ (\mu_A \otimes \alpha_M) \circ (\alpha_A^2 \otimes \alpha_A^2 \otimes \text{Id}_M) \quad \text{by (2.4.1)}
= \rho \circ ((\alpha_A^2 \otimes \mu_A) \otimes \alpha_M) \quad \text{by multiplicativity of } \alpha_A
= \rho \circ (\alpha_A^2 \otimes \text{Id}_M) \circ (\mu_A \otimes \alpha_M)
= \tilde{\rho} \circ (\mu_A \otimes \alpha_M).
\]
We have shown that \(\tilde{\rho}\) is the structure map of an \(A\)-module structure on \(M\). \(\square\)

### 2.6. Hom-bialgebras.

**Definition 2.7.** A Hom-bialgebra is a quadruple \((H, \mu, \Delta, \alpha)\) in which:

1. \((H, \mu, \alpha)\) is a Hom-associative algebra.
2. The comultiplication \(\Delta: H \to H^{\otimes 2}\) is linear and is Hom-coassociative, in the sense that
\[
(\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta. 
\tag{2.7.1}
\]
3. \(\Delta\) is a morphism of Hom-associative algebras.

Note that \(\Delta\) being a morphism of Hom-associative algebras means that
\[
\Delta \circ \alpha = \alpha^{\otimes 2} \circ \Delta 
\tag{2.7.2}
\]
and
\[
\Delta \circ \mu = \mu^{\otimes 2} \circ (\text{Id}_H \otimes \tau \otimes \text{Id}_H) \circ \Delta^{\otimes 2}. 
\tag{2.7.3}
\]

The following Lemma will be needed when we give an alternative characterization of a module Hom-algebra. It proves the second part of Theorem 1.1. By an \(H\)-module, we mean a module over the Hom-associative algebra \((H, \mu, \alpha)\).
Lemma 2.8. Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra and \((M, \alpha_M)\) be an \(H\)-module with structure map \(\rho: H \otimes M \to M\). Define the map
\[
\rho^2 = \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) : H \otimes M \otimes M \to M.
\] (2.8.1)

Then \(\rho^2\) is the structure map of an \(H\)-module structure on \(M \otimes M\).

Proof. To see that \(\rho^2\) is a morphism of Hom-modules, we compute as follows:
\[
\begin{align*}
\alpha_M^2 \circ \rho^2 &= \alpha_M^2 \circ \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ (\alpha_H \otimes \rho^2) \\
&= \rho \circ (\alpha_H \otimes \rho^2) \\
&= \rho \circ (\alpha_H \otimes \rho^2) \\
&= \rho \circ (\alpha_H \otimes \rho^2).
\end{align*}
\]

To see that \(\rho^2\) satisfies (2.4.1) (with \(\rho^2, H, \text{and } M \otimes M\) in place of \(\rho, A, \text{and } M, \text{resp.}\)), we compute as follows, where some obvious subscripts have been left out:
\[
\begin{align*}
\rho^2 \circ (\alpha_H \otimes \rho^2) &= \rho^2 \circ \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ \rho^2 \circ (\Delta_H \otimes M) \circ (\Delta_M \otimes H) \circ (\Delta_H \otimes M) \\
&= \rho \circ (\alpha_H \otimes \rho^2) \\
&= \rho \circ (\alpha_H \otimes \rho^2) \\
&= \rho \circ (\alpha_H \otimes \rho^2) \\
&= \rho \circ (\alpha_H \otimes \rho^2).
\end{align*}
\]

We have shown that \(\rho^2\) is the structure map of an \(H\)-module structure on \(M \otimes M\).

2.9. Module Hom-algebra. Let \((H, \mu_H, \Delta_H, \alpha_H)\) be a Hom-bialgebra and \((A, \mu_A, \alpha_A)\) be a Hom-associative algebra. An \(H\)-module Hom-algebra structure on \(A\) consists of an \(H\)-module structure \(\rho: H \otimes A \to A\) on \(A\) such that
\[
\rho \circ (\alpha_H^2 \otimes \mu_A) = \mu_A \circ \rho^2. \tag{2.9.1}
\]

We call (2.9.1) the module Hom-algebra axiom. Here \(\rho^2: H \otimes A \otimes A \to A\) is the map (2.8.1) in Lemma 2.8.

If we write \(\rho(x \otimes a) = xa\) for \(x \in H\) and \(a \in A\), then (2.9.1) can be written as
\[
\alpha_H^2(x)(ab) = \sum_{(x)} (x')a(x''b) \tag{2.9.2}
\]
for \(x \in H\) and \(a, b \in A\). If \(\alpha_H^2 = Id_H\) (e.g., if \(\alpha_H = Id_H\)), then (2.9.2) reduces to the usual module algebra axiom
\[
x(ab) = \sum_{(x)} (x')a(x''b). \tag{2.9.3}
\]

In particular, module algebras are examples of module Hom-algebras in which \(\alpha = Id\) for both the bialgebra and the algebra involved.

We are now ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. The first two parts of the Theorem were proved in Lemma 2.3 and Lemma 2.8. By multiplicativity, \(\mu_A: A \otimes A \to A\) is a morphism of Hom-modules. Now equip \(A\) and \(A \otimes A\) with the
$H$-module structures $\tilde{\rho}$ (2.5.1) and $\rho^2$ (2.8.1), respectively. Then $\mu_A$ is a morphism of $H$-modules if and only if

$$\mu_A \circ \rho^2 = \tilde{\rho} \circ (\text{Id}_H \otimes \mu_A) \quad \text{by} \quad (2.4.3)$$

$$= \rho \circ (\alpha^2_H \otimes \text{Id}_A) \circ (\text{Id}_H \otimes \mu_A) \quad \text{by} \quad (2.5.1)$$

$$= \rho \circ (\alpha^2_H \otimes \mu_A).$$

The above equality is exactly the module Hom-algebra axiom (2.9.1), as desired.

3. Deforming module algebras into module Hom-algebras

The purpose of this section is to prove Theorem 1.2, and hence also Corollary 1.3. We will also provide some examples to illustrate Corollary 1.3.

Proof of Theorem 1.2. The Hom-associative algebra $A_\alpha = (A, \mu_{\alpha,A} = \alpha_A \circ \mu_A, \alpha_A)$ was discussed in §2.3. As for the Hom-bialgebra $H_\alpha$, dualizing the argument for $A_\alpha$, one can check that $\Delta_{\alpha,H} = \Delta_H \otimes \alpha_H$ is Hom-cosassiative (2.7.1). The conditions (2.7.2) and (2.7.3) follow from the assumptions that $\alpha_H$ is a bialgebra morphism and that $H$ is a bialgebra, respectively.

To show that $\rho_\alpha = \alpha_A \circ \rho$ gives the Hom-associative algebra $A_\alpha$, the proof of Theorem 1.2 and that (ii) the module Hom-algebra axiom (2.9.1) holds. The proof that $\rho_\alpha$ is the structure map of an $H$-module structure on $A$ is similar to the proofs of Lemmas 2.5 and 2.8, so we will leave it to the reader as an easy exercise.

It remains to prove the module Hom-algebra axiom (2.9.1) in this case, which states that

$$\rho_\alpha \circ (\alpha^2_H \otimes \mu_{\alpha,A}) = \mu_{\alpha,A} \circ \rho^2_{\alpha}.$$  

Let us first decipher the map $\rho^2_{\alpha}$. From (2.8.1) and $\rho_\alpha = \alpha_A \circ \rho$, we have

$$\rho^2_{\alpha} = \rho_\alpha \circ (\text{Id}_H \otimes \tau_{H,A} \otimes \text{Id}_A) \circ (\Delta_{\alpha,H} \otimes \text{Id}^\otimes_A)$$

$$= \alpha_A \circ \rho^2 \circ (\text{Id}_H \otimes \tau_{H,A} \otimes \text{Id}_A) \circ ((\Delta_H \circ \alpha_H) \otimes \text{Id}^\otimes_A) \quad \text{by} \quad (1.2.1) \text{ and } (2.7.2).$$

Since $A$ is an $H$-module algebra, it follows from (2.9.3) that

$$\mu_A \circ \rho^2 \circ (\text{Id}_H \otimes \tau_{H,A} \otimes \text{Id}_A) \circ ((\Delta_H \circ \alpha_H^2) \otimes \alpha^\otimes_A) = \rho \circ (\alpha^2_H \otimes (\mu_A \circ \alpha^\otimes_A)).$$  

Therefore, we have

$$\rho_\alpha \circ (\alpha^2_H \otimes \mu_{\alpha,A}) = \alpha_A \circ \rho \circ (\alpha^2_H \otimes (\mu_A \circ \alpha^\otimes_A)) \quad \text{by multiplicativity of } \alpha_A$$

$$= \alpha_A \circ \mu_A \circ \rho^2 \circ (\text{Id}_H \otimes \tau_{H,A} \otimes \text{Id}_A) \circ ((\Delta_H \circ \alpha_H^2) \otimes \alpha^\otimes_A) \quad \text{by} \quad (3.0.6)$$

$$= \mu_{\alpha,A} \circ \rho^2 \quad \text{by} \quad (3.0.7).$$

This proves (3.0.4), as desired.

We now give some examples that illustrate Corollary 1.3, which is the special case of Theorem 1.2 when $\alpha_H = \text{Id}_H$.

Example 3.1. Let $(A, \mu)$ be an associative algebra. Denote by $G$ the group of algebra automorphisms of $A$ and by $k[G]$ its group bialgebra, in which $\Delta(\varphi) = \varphi \otimes \varphi$ for $\varphi \in G$. It is easy to check that there is a $k[G]$-module algebra structure on $A$ whose structure map $\rho: k[G] \otimes A \to A$ is given by $\rho(\varphi \otimes a) = \varphi(a)$.
Suppose that $\alpha : A \to A$ is an algebra endomorphism such that $\alpha \circ \varphi = \varphi \circ \alpha$ for all $\varphi \in G$. For example, if $A$ is unital and $a \in A$ is invertible such that $\varphi(a) = a$ for all $\varphi \in G$, then $i_a \circ \varphi = \varphi \circ i_a$, where $i_a(b) = aba^{-1}$. Such a map $\alpha$ is clearly $k[G]$-linear. Therefore, by Corollary 1.3 there is a $k[G]$-module Hom-algebra structure

$$\rho_\alpha = \alpha \circ \rho : k[G] \otimes A \to A, \quad \rho_\alpha(\varphi \otimes a) = \alpha(\varphi(a))$$
onumber

on the Hom-associative algebra $A_\alpha = (A, \mu_{\alpha, A} = \alpha \circ \mu, \alpha)$. \hfill $\square$

**Example 3.2.** Fix a prime $p$, and let $X$ be a topological space. The singular mod $p$ cohomology $A = H^\ast(X; \mathbb{Z}/p)$ of $X$ is an $A_p$-module algebra, where $A_p$ is the Steenrod algebra associated to the prime $p$ \[3, 6\]. Now let $f : X \to X$ be a continuous self-map of $X$. Then the induced map $f^* : A \to A$ on mod $p$ cohomology is a map of $\mathbb{Z}/p$-algebras that respects the Steenrod operations, i.e., $f^*$ is $A_p$-linear. Therefore, by Corollary 1.3 there is an $A_p$-module Hom-algebra structure

$$\rho_f : A_p \otimes A \to A, \quad \rho_f(Sq^k \otimes x) = f^*(Sq^k(x))$$

on $A$, where $x \in A$ and $Sq^k \in A_p$ is the $k$th Steenrod operation of degree $2k(p-1)$ (resp. $k$) if $p$ is odd (resp. if $p = 2$).

Likewise, we can consider the complex cobordism $MU^\ast(X)$ of $X$, which is an $S$-module algebra, where $S$ is the Landweber-Novikov algebra \[3, 17\] of stable cobordism operations. The same considerations as in the previous paragraph, now with $A = MU^\ast(X)$ and $S$ instead of $A_p$, can be applied here. \hfill $\square$

4. **Twisted $sl(2)$-action on the affine plane**

The purposes of this section are to prove Theorem 1.4 and to use this Theorem to construct some $q$-deformations of the $sl(2)$-action on the affine plane.

**Proof of Theorem 1.4.** First note that we only need to prove the first part, since the second part follows from it and Theorem 1.2.

To prove the first part, observe that, if $\alpha_L$ can be extended to an algebra endomorphism $\alpha_U$ of the universal enveloping algebra $U(L)$, then $\alpha_U$ must be unique because $L$ generates $U(L)$ as an algebra. Thus, it remains to show that $\alpha_L$ can be extended to a bialgebra endomorphism $\alpha_U$ of $U(L)$ that satisfies (1.4.1).

We define $\alpha_U : U(L) \to U(L)$ by extending $\alpha_L$ linearly and multiplicatively with $\alpha_U(1) = 1$. To see that this $\alpha_U$ is a well-defined algebra endomorphism, let $x$ and $y$ be elements in $L$. Then we have

$$\alpha_U([x, y] - xy + yx) = \alpha_L([x, y]) - \alpha_L(x)\alpha_L(y) + \alpha_L(y)\alpha_L(x) = [\alpha_L(x), \alpha_L(y)] - \alpha_L(x)\alpha_L(y) + \alpha_L(y)\alpha_L(x) = 0,$$

showing that $\alpha_U$ is an algebra endomorphism of $U(L)$. To check that $\alpha_U$ is a bialgebra endomorphism, we must show that $\Delta_U \circ \alpha_U = \alpha_U^{\otimes 2} \circ \Delta_U$. Since both $\alpha_U$ and $\Delta_U$ are algebra endomorphisms of $U(L)$, it suffices to check this on $xy$ with $x, y \in L$. We compute as follows:

$$\Delta_U(\alpha_U(xy)) = \Delta_U(\alpha_U(x)) \Delta_U(\alpha_U(y))$$

$$= (\alpha_U(x) \otimes 1 + 1 \otimes \alpha_U(x))(\alpha_U(y) \otimes 1 + 1 \otimes \alpha_U(y))$$

$$= \alpha_U^{\otimes 2}(xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x)$$

$$= \alpha_U^{\otimes 2}(\Delta_U(xy)).$$

This proves that $\alpha_U$ is a bialgebra endomorphism of $U(L)$. 
Finally, to check (1.4.1), it suffices to check it when $z = xy$ for $x, y \in L$. We have

$$
\alpha_A((xy)a) = \alpha_A(x(ya)) = \alpha_L(x)\alpha_A(ya) = \alpha_L(x)(\alpha_L(y)\alpha_A(a))
$$

$$
= (\alpha_L(x)\alpha_L(y))\alpha_A(a) = \alpha_U(xy)\alpha_A(a),
$$

as desired. \(\square\)

In the following example, we use Theorem 1.4 to construct some $q$-deformations of the $sl(2)$-action on the affine plane.

**Example 4.1.** Let us first recall the usual, and the most important, $sl(2)$-action on the affine plane $A = k[x, y]$, where $k$ is assumed to have characteristic $0$. The reader may consult, e.g., [8, Chapter V], for the details. Denote by $sl(2)$ the Lie algebra (under the commutator bracket) of $2 \times 2$-matrices with entries in $k$ and trace $0$. It has a standard basis consisting of the matrices

$$
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

satisfying

$$
[X, Y] = Z, \quad [X, Z] = -2X, \quad [Y, Z] = 2Y.
$$

Let $H$ denote its universal enveloping bialgebra $U(sl(2))$. Then there is an $H$-module algebra structure on $A = k[x, y]$. Then is an $H$-module algebra determined by

$$
XP = x\frac{\partial P}{\partial y}, \quad YP = y\frac{\partial P}{\partial x}, \quad ZP = x\frac{\partial P}{\partial x} - y\frac{\partial P}{\partial y}, \quad (4.1.1)
$$

where $P \in A$ and $\partial/\partial x$ and $\partial/\partial y$ are the formal partial derivatives. What is special about this $H$-module algebra is that it captures all the finite dimensional simple $sl(2)$-modules. Indeed, for $n \geq 0$, let $A_n$ denote the $(n+1)$-dimensional subspace of $A$ consisting of the homogeneous polynomials of degree $n$. Then $A_n$ is a sub-$H$-module of $A$ that is isomorphic to the unique (up to isomorphism) simple $sl(2)$-module $V(n)$ of dimension $n+1$.

We now deform the above $H$-module algebra structure on $A$ into a module Hom-algebra using Theorem 1.4. Fix a non-zero scalar $q \in k$. Define an algebra endomorphism $\alpha_A: A \rightarrow A$ on the affine plane $A$ by setting

$$
\alpha_A(x) = q^2x \quad \text{and} \quad \alpha_A(y) = qy.
$$

Also define a linear map $\alpha_L: sl(2) \rightarrow sl(2)$ by setting

$$
\alpha_L(X) = qX, \quad \alpha_L(Y) = q^{-1}Y, \quad \alpha_L(Z) = Z.
$$

It is easy to check that $\alpha_L$ is a Lie algebra endomorphism on $sl(2)$. To use Theorem 1.4, it suffices to check that

$$
\alpha_A(WP) = \alpha_L(W)\alpha_A(P) \quad (4.1.2)
$$

for $P \in A = k[x, y]$ and $W \in \{X, Y, Z\} \subseteq sl(2)$. Note that

$$
\alpha_A(P) = P(q^2x, qy).
$$

For example, if $P$ is the monomial $x^iy^j$, then

$$
\alpha_A(P) = (q^2x)^i(qy)^j = q^{2i+j}x^iy^j = q^{2i+j}P.
$$

We check (4.1.2) for $W = X$; the other two cases ($W = Y$ and $W = Z$) are essentially identical. Using (4.1.1), we compute as follows:

$$
\alpha_A(XP) = q^2x \cdot \left(\frac{\partial P}{\partial y}\right)(q^2x, qy) = qx \cdot \left(\frac{\partial P}{\partial y}\right)(q^2x, qy) \cdot \frac{d(qy)}{dy}
$$

$$
= qx \cdot \frac{\partial(P(q^2x, qy))}{\partial y} = qX(P(q^2x, qy)) = \alpha_L(X)\alpha_A(P).
$$
Thus, Theorem 1.4 applies here with \( L = \text{sl}(2) \), \( A = k[x, y] \), and \( \rho \) being determined by (4.1.1). In other words, with the notations in Theorem 1.4, there is an \( H_{\alpha} \)-module Hom-algebra structure \( \rho_\alpha = \alpha_A \circ \rho : H \otimes A \to A \) on the Hom-associative algebra \( A_\alpha \). The structure map \( \rho_\alpha \) is determined by

\[
\rho_\alpha(X \otimes P) = q^2 x \cdot \left( \frac{\partial P}{\partial y} \right) (q^2 x, qy), \quad \rho_\alpha(Y \otimes P) = qy \cdot \left( \frac{\partial P}{\partial x} \right) (q^2 x, qy),
\]

\[
\rho_\alpha(Z \otimes P) = q^2 x \cdot \left( \frac{\partial P}{\partial x} \right) (q^2 x, qy) - qy \cdot \left( \frac{\partial P}{\partial y} \right) (q^2 x, qy).
\]

Of course, if \( q = 1 \in k \), then \( \alpha_A = Id_A, \alpha_L = Id_L \), and \( \rho_\alpha = \rho \) is the original structure map (4.1.1) of the \( H \)-module algebra \( A \).

\[ \square \]

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