We propose a model of dynamical symmetry breaking, in which a new type of fundamental scalar fields of zero mass-dimension mediate the couplings of fermions to the gravitational field, represented here as a tetrad field in the same manner as Riemann-Cartan gravity. In our model, the tetrad couples to the standard model fermions non-minimally, and the very coupling coefficients are the fundamental scalar fields. There are exactly 36 scalar fields in the model, which are distinguishable by flavor indices on the fields. This is the precise number of zero dimension scalar fields that leads to a vanishing Weyl anomaly and a vanishing vacuum energy. Precisely the same number of these very same scalar fields is required for the coupling of all of the different standard model fermions to the vielbein field. At the same time their interaction with fermions gives rise to fermion mass terms, without the need to introduce a fundamental Higgs field. Within the proposed theory we construct a toy model that deals solely with the top and bottom quarks, and we demonstrate that their observable masses can appear in the action dynamically. Moreover, this mechanism allows for the top and bottom quarks to acquire distinctly different masses, as opposed to our previous, even simpler toy model that contained only the top quark.
1. INTRODUCTION

In this article we put forward a new model of dynamical symmetry breaking. We assume that the standard model (SM) fermions couple to a background gravitational field, which we represent as a tetrad field in the same formalism as the Palatini action [1]. The coupling of the fermions to the tetrad is the same non-minimal coupling as the Holst action [2] (where the coupling coefficients are constants) but with the coupling coefficients being scalar fields. The SM has 36 different fermion particles. Assuming that each one of the 36 fermion fields has a different coupling to the tetrad, then 36 different scalar fields make up the required coupling terms. Separately, the following recently discovered observation was published in [3]. The current SM includes the leptons, quarks, photon, $W^{\pm}$ and $Z$ bosons, gluons, and the Higgs boson. With this configuration of particles the Weyl anomaly and the vacuum energy are non-zero. By removing the Higgs boson (which has mass dimension one) and inserting 36 scalar fields with mass dimension zero, then not only does the Weyl anomaly vanish, but so does the vacuum energy. This is in striking agreement with the number of couplings required for the coupling of the tetrad field to the SM fermions. These observations motivated our original hypothesis, that adding 36 zero dimension scalar fields to the action could lead
to dynamical symmetry breaking, and in turn the emergence of fermion mass terms. We have shown that this mechanism does work in a simplified toy model containing just one quark \[4\]. Specifically, we showed that for the sector of the SM action containing just the top quark, dynamical symmetry breaking occurs, and the predicted top quark mass is very close to its experimentally observed mass. In this article we generalize this model from the case of identical masses for both quarks in the third generation, to the case of distinct mass terms for the top and bottom quarks.

The Higgs mechanism is how mass terms are ascribed to the $W^\pm$ and $Z$ bosons, and to the quarks and leptons. It was first conjectured between 1964 and 1967 in numerous papers, of which the most seminal are refs. \[5-10\], at the time when particle physicists were attempting to resolve the problem that the SM action had no mass terms for those particles. The Higgs mechanism is a $\phi^4$ theory containing an $SU(2)$ doublet of scalar fields with a non-zero vacuum expectation value (VEV), which later became known as the Higgs Boson. The $SU(2) \times U(1)$ gauge covariant derivative of the Higgs leads to mass terms for the $W^\pm$ and $Z$, but breaks the $SU(2) \times U(1)$ symmetry. In addition a Yukawa term comprising the Higgs coupled to the fermion fields leads to fermion mass terms, but breaks chiral symmetry. These masses are proportional to the VEV of the Higgs. If the VEV is non-zero these masses will be non-zero, and as a result the $SU(2) \times U(1)$ symmetry of the theory gets spontaneously broken down to $U(1)$, as originally observed by \[5, 8, 9, 11-13\]. The current standard electroweak theory was formulated \[14-19\] based on the Higgs mechanism, in which spontaneous $SU(2) \times U(1)$ symmetry breaking leads to non-zero masses for the $W^\pm$, $Z$, quarks and leptons. The Higgs Boson has been detected as a scalar excitation with a mass found at 125 GeV \[20-25\], consistent with theoretical predictions.

Open questions remain about the Higgs sector of the SM as eloquently summarized in \[26\], of which fall into the following categories. The Higgs Boson is merely inserted into the action by hand with no consideration of how it emerges in nature, except that without it particles would be devoid of mass. Quarks and leptons form matter. Bosons are the force carriers that govern interactions between matter. Is there an analogous role played by the Higgs Boson? It’s origins remain an enigma. The appearance of the Higgs in the action as a $\phi^4$ theory gives rise to self-interactions between the Higgs and itself. What is the nature of this interaction? The SM inclusive of the Higgs bears a hierarchy problem with vastly disparate energy scales in the theory. On the structure of the Higgs itself, is it a fundamental particle or composed of known fundamental particles itself. These questions and more prompted the pursuit of composite Higgs models.

One of the earliest composite Higgs model is based on two papers \[27, 28\] by Nambu Jona and Lasinio (NJL) in 1961, inspired by the theory of superconductivity by Bardeen, Cooper and Schrieffer (BCS) first published in \[29\] and subsequently given a mathematically elegant form in \[30, 32\]. Given that the idea behind the NJL model is replicated in so many later composite Higgs
models based on four-fermion interactions, including our model, it is informative to give a paragraph here with an overview of its origins. In BCS theory a pair of fermions join to form a condensate forming a type of superconductor. An energy gap emerges between the ground state and excited states of the condensate, due to a self-interaction between the two fermions. In \[27, 28\] an analogy is drawn between this energy gap of condensates, and the energy gap between the two energies of the two pairs of solutions in a four-component Dirac spinor. The analogy is as follows: just like the energy gap in a condensate is created by a self-interaction between fermions, so too the mass of a fermion arises due to an interaction between massless fermions. The NJL model inclusive of this interaction is based on a Lagrangian of the form

\[
L_{NJL} = -\bar{\psi}i\gamma^\mu \partial_\mu \psi + g_0[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]
\]

where \(\psi\) is a Dirac fermion and \(g_0\) a positive constant with dimensions \([\text{mass}]^{-2}\). The four-fermion terms \((\bar{\psi}\psi)^2\) are precisely analogous to similar four-fermion terms in BCS theory. Note that there is no bare mass to violate chiral symmetry. The NJL model assumes an unknown fermion self-energy function, \(\Sigma(m, g, \Lambda)\) where \(m\) is the observed fermion mass, \(g\) a coupling related to the bare coupling \(g_0\), and \(\Lambda\) a cut-off. Accordingly the full inverse propagator has the general form

\[
i\gamma \cdot p + m = i\gamma \cdot p + m_0 + \Sigma(m, g, \Lambda) = 0
\]

where \(m_0\) is the bare mass, giving rise to the mass gap equation \(m - m_0 = \Sigma(m, g, \Lambda)\). From the Lagrangian \(L_{NJL}\), an expression for \(\Sigma\) can be derived, leading to a gap equation of the form \(m = m_0 g_0 \int dp (p^2 + m^2 - i\epsilon)^{-1} F(p, \Lambda)\) where \(F(p, \Lambda)\) is a cut-off factor and \(\int dp(\ldots) = (2\pi)^{-4} \int d^4p(\ldots)\). In this way the chiral condensate \((\bar{\psi}\psi)\) in \(L_{NJL}\) leads to an effective mass term and hence spontaneously broken chiral symmetry. In the same year BCS theory was applied to the problem of dynamical symmetry breaking in the standard model \[33\] using a similar approach. A more pedagogical description of the NJL model can be found for example in refs. \[34, 36\].

Despite its elegance, the NJL model falls short of constituting a viable model. Firstly, the formation of chiral condensates at a certain energy means there is no confinement. Secondly, the NJL model is non-renormalizable in four dimensions, so at best it can only be an effective field theory, which needs UV completion. Still, it remains a highly useful starting point for informing subsequent composite Higgs models based on the notion of a four-fermion interaction. And importantly, a precise relation for the mass of the Higgs boson can be extracted from the NJL model, described below.

Nambu \[37\] discovered a precise relation between the masses of the boson excitation states of the theory, namely the Higgs bosons that emerge, and the mass of the fermion that forms the condensate, known as the Nambu sum rule. It has the form \(m_1^2 + m_2^2 + \cdots = 4m_f^2\) where \(m_i\) is the mass of each Higgs boson and \(m_f\) is the mass of the condensed fermion. A similar relation was also discussed earlier in the original article on the NJL approximation to QCD in \[27\]. It was pointed out in \[38\], based on the Nambu sum rule the possibility that the 125 GeV excitation discovered \[20, 25\], is not the only Higgs boson. Higgs bosons appear in top-quark condensation models, where the masses of
the bosonic excitations are related to the top quark mass by a sum rule \[38\] similar to the Nambu sum rule of the NJL model \[27\]. This sum rule suggests the existence of so called Nambu partners for the 125 GeV Higgs boson, whose masses can be predicted by the Nambu sum rule. For example, if there are only two states in the given channel, the mass of the Nambu partner is \( \sim 325 \) GeV. They together satisfy the Nambu sum rule \( m_1^2 + m_2^2 = 4m_t^2 \), where \( m_t \sim 175 \) GeV is the mass of the top quark. Based on this idea it has been suggested by Volovik and Zubkov \[38, 40\] that other scalar excitations corresponding to the Higgs Boson may exist, as well as the 125 GeV excitation discovered in \[20, 25\].

A specific model based on the Higgs scalar being a fermion condensate was put forward by Terazawa et. al. in \[41\], which was later made more precise \[42\] with the hypothesis that the Higgs is composed of a top-quark condensate (\( t\bar{t} \)). This was guided by the result that the mass of the Higgs scalar, as predicted by the Nambu sum rule, indicates that the Higgs scalar is close to the \( t\bar{t} \) threshold, and could behave like a \( t\bar{t} \) bound state. Later on in 1989, the original \( t\bar{t} \) model of Terazawa et. al. was revived by Miransky, Tanabashi, Yamawaki \[43, 44\], and later by Bardeen, Hill and Lindner \[45, 48\], who proposed the top-quark condensate as a means of electroweak symmetry breaking. In the \( t\bar{t} \) model the Higgs is composite at short distance scales. However in such \( t\bar{t} \) models the energy scale of new dynamics is assumed to be \( \sim 10^{15} \) GeV, corresponding to a Higgs mass of the order \( 2m_t \sim 350 \) GeV \[41, 43, 45, 48\], which is starkly different from experimental observations. In response to this problem the top seesaw model was put forward by Chivukula, Dobrescu, Georgi and Hill \[49\], but in addition to the top quark, an additional heavy fermion, \( \chi \) \[49, 50\] is required by the model.

In the 1970s the technicolor (TC) model was formulated \[51, 52\] to resolve issues arising from the SM with a fundamental Higgs boson (listed in the opening remarks), while at the same time predicting a Higgs mass in agreement with experiments. The TC model is a gauge theory of fermions with no elementary scalars, which also has dynamical electroweak symmetry breaking and flavor symmetry breaking. The Higgs, instead of being a fundamental scalar is composed of technifermions, a new class of fermions interacting via technicolor gauge bosons. This interaction is attractive, and hence, by analogy with BCS theory, generates fermion condensates. By itself the TC model doesn’t have mass terms for the quarks and leptons, so it was extended to the Extended Technicolor (ETC) model \[53, 56\], but the ETC model is inconsistent with experimental constraints on flavor changing neutral currents, and precision electroweak measurements. Later on the walking technicolor model \[57, 58\] was composed that resolves the above issues, but it yields the wrong value for the top-quark mass. For an in-depth review of the original TC model and its various derivatives the reader should consult Refs. \[48, 51, 52, 59, 60\], while for a more pedagogical introduction to the TC and ETC models see Refs. \[26, 61\]. A more recent alternative to the TC model can also be found in \[62, 64\].
Subsequently the partial compositeness model was conjectured, by Kaplan \[65\] initially. Therein each SM particle has a heavy partner that can mix with it. Like this the SM particles are linear combinations of elementary and composite states via a mixing angle. The elegance of the partial compositeness model is its simplicity, without the need for deviations beyond the standard model, thus relying on the known fundamental particles only \[66\]. More recently, among a variety of theoretical papers that appeared between December 2015 and August 2016, there are several that consider both the 125 GeV Higgs boson, and the hypothetical new heavier Higgs boson as composite due to the new strong interaction, as noticed for example in Refs. \[67, 68\]. In a different approach there are other papers devoted to the description of the composite nature of the heavier (750 GeV) Higgs boson only \[69–82\]. Other more modern composite Higgs models consist of the walking model, \[83, 84\], the ideal walking model \[85–88\], the technicolor scalar model \[89\], Sannino’s model of the generalized orientifold gauge theory approach to electroweak symmetry breaking \[90\], and the walking model in higher-dimensional SU\( (N) \) gauge theories of Dietrich and Sannino \[91, 92\]. A more comprehensive review of modern composite Higgs models can be found in \[93\].

We have discussed a number of composite Higgs models based on fermion condensates. It is reasonable to suppose that the 125 GeV Higgs, as well as the extra scalars suggested in \[38–40\], are composed of known SM fermions, due to some unknown interaction between them with a scale \( \Lambda \) above 1 TeV. If it exists, such an interaction would have specific properties that make it distinctly different from the conventional TC interactions \[26, 48, 94\]. Firstly, these interactions must occur at appropriate energy scales such there is no confinement, since otherwise they would confine quarks and leptons to extremely small regions of space \( \sim 1/\Lambda \) to the point that strong and weak interactions would take place on length scales unobservable. At the same time, these interactions must produce spontaneous symmetry breaking needed to make \( W \) and \( Z \) bosons massive, and spontaneous chiral symmetry breaking to yield massive fermions. A number of models with these properties have been thought of within the framework of topcolor models \[44–46, 48, 95–99\]. Other candidates considered are topcolor assisted technicolor models \[100–104\], which combines both technicolor and topcolor ingredients. Indeed in refs. \[38, 39, 62–64\] such a model was suggested in which chiral condensates appear comprising SM fermions, not necessarily just the \( t \bar{t} \) condensate. This particular configuration provides masses for both the \( W \), \( Z \) bosons, and for the fermions.

Quantum gravity in the first order formalism with either the Palatini action \[105, 106\] or the Holst action \[107\], shares a property with the above composite Higgs models: it leads to a four-fermion interaction. This four fermion interaction is now between spinor fields coupled in a minimal way to the torsion field \[1, 108\], but importantly this four-fermion interaction leads to fermion condensates of the type that could constitute the composite Higgs. A similar idea has already been suggested in Ref. \[109\], namely that the torsion field coupled in a non-minimal way to fermion fields
Starting with a theory of the SM coupled to gravity, the fermion and gauge fields are coupled to a background classical gravitational field, with a right-handed neutrino for each generation also coupled to the background field. This representation is a QFT on a classical background spacetime. The Weyl anomaly is a measure of the failure of the classically Weyl-invariant theory to define a Weyl-invariant quantum theory, and can be expressed in terms of the number of different fields present in the theory. In the SM there are \( n_0 = 4 \) ordinary real scalars in the usual complex Higgs doublet, \( n_{1/2} = 3 \times 16 = 48 \) Weyl spinors (16 per generation), \( n_1 = 8 + 3 + 1 = 12 \) gauge fields of \( SU(3) \times SU(2) \times U(1) \), and a single gravitational field \( n_2 = 1 \). With these combinations the Weyl anomaly is non-zero. However suppose that the Higgs and graviton fields are not fundamental fields but rather composite, such that their contributions to the vacuum energy and Weyl anomaly can be dropped, and \( n_0 = n_2 = 0 \). The implication is that by introducing \( n'_0 = 36 \) scalars with zero mass-dimension, then not only does the Weyl anomaly vanish, but also the vacuum energy vanishes as well.

This embodies our motivation for the starting point of this work: a model comprising Dirac fermions of the SM with a non-minimal coupling to gravity, that includes 36 fundamental scalar fields. The details of the construction of the theory is described in full in \( \S 4.1 \). We find that in this model, the theory admits mass terms for the fermions. By using the Schwinger-Dyson equation for the fermion self-energy function, a relationship is derived between the top-quark mass, the Planck mass (the ultraviolet cut-off point of the loop integral) and the strength of the inverse coupling between the fermion fields and the vielbein. From this relation an estimation for the mass of the top-quark can be extracted.

This paper is organized in the following way. In \( \S 2 \) we introduce the tetrad formalism, the Palatini action. We explain the origins of the form of the action of fermions coupled non-minimally to gravity and sketch the derivation. In \( \S 3 \) we expand on the definition of the Weyl anomaly, and show that it vanishes with zero dimension scalars added to the SM. We also summarize our previous toy model with just one quark. In \( \S 4 \) we construct the action of our model: zero dimension scalars coupled to gravity. We then focus in on the sector of the action containing the third generation of quarks coupled to the gravitational (tetrad) field through zero dimension scalar fields. In \( \S 6 \) we apply the Schwinger-Dyson approach to derive the mass gap equations for the top and bottom quarks and we present our numerical results for the masses and the couplings coefficients. In \( \S 7 \) we summarize our results and lay out our plans for the next piece of this research. Appendix \( \S A \) contains a list of definitions for the Dirac matrices in the conventions used throughout this paper, together with a number of useful identities. In Appendix \( \S B \) we give details of the Wick rotations used for simplifying the calculation of the one-loop contribution to the self-energy of the quarks (see Fig. 2). Finally in
Appendix §C gives the of how to evaluate the angular integrals in the one-loop diagram in Fig. 2.

Throughout this article lower case Latin letters \(a, b, c \cdots = 0, 1, 2, 3\) label internal Lorentz indices, Greek letters \(\mu, \nu, \cdots = 0, 1, 2, 3\) label spacetime indices, and \(\epsilon_{abcd}\) is the completely antisymmetric Levi-Civita symbol. Lorentz indices are raised and lowered by the Minkowski metric \(\eta_{ab}\) and spacetime indices are raised and lowered by a spacetime metric \(g_{\mu\nu}\). Metrics are assumed to be ‘mostly negative’, and specifically the Minkowski metric is \(\eta_{ab} = \text{diag}(1, -1, -1, -1)\).

2. THE VIELBEIN FORMALISM

The aim of this section is to offer a brief introduction to the notations and conventions of the tetrad formalism, and some insight into why it is chosen as a starting point for our model. The majority of this overview is based on [1]. Only the main results are given here. For more details of how these results are derived the interested reader should consult that reference.

In the vielbein formalism the metric \(g_{\mu\nu}(x)\) is expressed in terms of a tetrad field or vielbein field \(e^a_\mu(x)\) through the completeness relation

\[
g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab} \tag{2.1}
\]

where \(\eta_{ab} = \text{diag}(-1, 1, 1, 1)\) is the flat space Minkowski metric. The orthogonality relation,

\[
\eta_{ab} = e^\mu_\alpha(x)e^\nu_\beta(x)g_{\mu\nu}(x) \tag{2.2}
\]

where \(e^\mu_\alpha\) is the inverse tetrad, can be obtained directly from the completeness relation via the inverse conditions

\[
e^\mu_\alpha e^\alpha_\nu = \delta^\mu_\nu, \quad e^\mu_\alpha e^\beta_\mu = \delta^\beta_\alpha, \tag{2.3}
\]

where \(\delta^\beta_\alpha\) and \(\delta^\mu_\nu\) are kronecker delta functions.

The vielbein \(e^a_\mu(x)\) acts as a map from tangent space at a given point \(x\) to flat Minkowski space. In this regard the vielbein captures Einstein’s intuition that spacetime is locally flat. The metric is invariant under local \(SO(3,1)\) gauge transformations in the sense that (2.1) is invariant under

\[
e^a_\mu(x) \rightarrow \Lambda^a_\beta e^b_\mu(x). \tag{2.4}
\]

The action for the gravitational field itself is given by the Einstein-Hilbert (EH) action \(\int dx \sqrt{-g}R\), where \(R\) is the Ricci scalar and \(g\) is the determinant of the metric. Its variation with respect to the metric yields the Einstein equations. The action itself contains up to second order derivatives in the metric. As an attempt to simplify the field equations Palatini [105] proposed an action in which the metric and connection are independent variables, in order that the action will
have up to first order derivatives only, thus simplifying the equations of motion. The Palatini action is given in terms of the tetrad instead of the metric, and has the form

$$S[\epsilon, \omega] = \int dx \epsilon^\mu_a \epsilon^\nu_b F_{\mu\nu}^{ab}. \quad (2.5)$$

Here $\epsilon = \sqrt{-g}$ is the determinant of $\epsilon^a_\mu$, and $\omega^a_{\mu b}$ is a Lorentz connection that defines a covariant partial derivative on fields with Lorentz (a) indices as $D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} v_b$, and in general a gauge-covariant exterior derivative $D$ on $p$-forms. For example, for a one-form $T^a_\mu$ with a Lorentz index, $D_\mu T^a_\nu = \partial_\mu T^a_\nu + \omega^a_{[\mu c} \omega_{\nu] c} b$, or equally

$$F^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^b_c, \quad (2.6)$$

where the first term is an exterior derivative, and spacetime indices have been omitted in (2.6). The invariance of the action (2.5) with respect to variations of the connection $\omega$ yields

$$D_{[\mu} \epsilon^a_{\nu]} = 0. \quad (2.7)$$

Eq. (2.7) with spacetime indices omitted has the form $D e^a = de^a + \omega^a_{b c} v^b = 0$. The left-hand side is the definition of the torsion two-form. The equation of motion in (2.7) is the statement that the torsion vanishes.

Eq. (2.7) has a unique solution, denoted by $\omega = \omega[e]$, with the form

$$\omega^{ab}_{\mu}[e] = \epsilon^a_\alpha \nabla_\mu \epsilon^b_\beta \eta^{\alpha\beta}. \quad (2.8)$$

$\omega[e]$ is the torsion-free spin connection of the tetrad field $\epsilon$. It is analogous to to the metric compatibility condition $\nabla_\mu g_{\alpha\beta} = 0$ where $\nabla$ is the covariant spacetime derivative in GR, whose unique solution is a torsion-free metric.

Substitution of $\omega = \omega[e]$ in the curvature leads to $F^{ab}_{\alpha\beta}(\omega[e]) = R^{ab}_{\alpha\beta}$ where $R^{ab}_{\alpha\beta} = R^{[\mu}_{\alpha \rho} c_{\mu}^{a} e_{\nu]}^{b}$ and $R^{\alpha\beta}_{\mu\nu}$ is the ordinary Riemann tensor. The invariance of the action (2.5) under variations of $e$ leads to the vacuum Einstein equation. Eq. (2.7) is called the first Cartan equation while (2.6) is called the second Cartan equation.

An extension of the Palatini action called the Holst action bears the same equations of motion as (2.5). The Holst action has the form

$$S_{\text{Holst}}[\epsilon, \omega] = \int dx \epsilon^\mu_a \epsilon^\nu_b F_{\mu\nu}^{ab} - \frac{1}{\gamma} \int dx \epsilon^\mu_a \epsilon^\nu_b \ast F_{\mu\nu}^{ab}, \quad (2.9)$$

where $\ast F^{a b}_{\mu\nu} = \frac{1}{2} \epsilon^{a c}d_{\mu\nu}^{cd} F^{cd}_{\mu\nu}^{cd}$ is the Hodge dual, and the parameter $\gamma$ is the Immirzi parameter. The first term in (2.9) is identical to (2.5), while the second distinguishes the Holst action from
the Palatini action. The functional dependence of (2.9) on the tetrad is identical to (2.5), thus it’s variation with $e$ yields the Einstein equations in the same manner. Variation of (2.9) with $\omega$ leads to the same equation of motion as (2.7), with the solution $\omega = \omega[e]$. Substitution of $\omega = \omega[e]$ in (2.9) forces the second term to vanish due to the Bianchi identity $R_{[\alpha\beta\mu]\nu} = 0$.

The equation of motion (2.7) changes when fermions are introduced into the action, resulting in a solution to $\omega[e]$. In the presence of a fermion field, $\psi$ the action (2.5) becomes

$$S[e, \omega, \psi] = S[e, \omega] + \frac{i}{2} \int dx \left( \bar{\psi}\gamma^a e^\mu_a D_\mu \psi - \overline{D_\mu \psi} \gamma^a e^\mu_a \psi \right),$$

(2.10)

where $S[e, \omega]$ is given in (2.5), $\gamma^a$ are the Dirac matrices, and

$$D_\mu \psi = \partial_\mu \psi - \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_\nu \psi.$$

(2.11)

By varying $\psi$ in (2.10) the $SO(3,1)$ gauge-covariant Dirac equation is implied:

$$(i\gamma^a e^\mu_a D_\mu - m)\psi = 0.$$}

(2.12)

Upon varying (2.10) with $\omega$, the following equation of motion is obtained

$$D_\mu \left( e e_\mu^\nu J^{[\nu]}_b \right) = e J_{ab}$$

(2.13)

where the fermion current is given by

$$J_{ab}^\nu = \frac{1}{4} e_\nu^d \epsilon_{abc} J^c, \quad J^c \equiv \bar{\psi} \gamma_5 \gamma^c \psi.$$

(2.14)

It is immediately clear from (2.13) that the presence of a fermion field in the action introduces a torsion component in the connection arising from the fermion current, and thus the connection is no longer torsion free, that is, $D_\mu e^\mu_a \neq 0$. Eq. (2.13) relates the tetrad to the current, which is itself a bilinear function of the fermion field. This is the same manner in which the background metric $g$ in GR is specified in terms of matter fields present in the Lagrangian through the Einstein equations.

Because of (2.13), the second term in (2.9) no longer vanishes. The steps leading up the solution to (2.13) are given in [1]. The solution denoted $\omega[e, \psi]$ carries an explicit dependence on the fermion field $\psi$ itself, and has the form

$$\omega_\mu^{ab}[e, \psi] = -\frac{\gamma}{\gamma^2 + 1} \left( 2 e_\mu^{[a} J^{b]} - \gamma e^{ab}_{cd} J^c e_\mu^d \right).$$

(2.15)

The connection in (2.15) may be substituted back into the action (2.9). The key ingredient is that $\omega[e, \psi]$ contains fermion terms that couple back to the fermions via the covariant derivative. The full derivation is not given here, but can be found in [1]. The resulting form for the total action is

$$S[e, \psi] = S[e] + S_f[e, \psi] + S_I[e, \psi],$$

(2.16)
where the first two terms are the standard second–order tetrad action of general relativity with fermions,

\[
S[e] + S_f[e, \psi] = \int dx \, e^{\mu} a e^{\nu} b \, F_{\mu \nu}^{ab}[\omega[e]]
\]

\[+ \frac{i}{2} \int dx \, e \left( \bar{\psi} \gamma^\mu e^{\mu} a D_\mu [\omega[e]] - D_\mu [\omega[e]] \bar{\psi} \gamma^\mu e^{\mu} a \psi \right) \psi ,
\]

(2.17)

and the interaction term is

\[
S_I[e, \psi] = -\frac{\gamma^2}{\gamma^2 + 1} \int dx \, e \left( \bar{\psi} \gamma_5 \gamma^a \psi \right) \left( \bar{\psi} \gamma_5 \gamma^a \psi \right).
\]

(2.18)

The first term is the same second–order tetrad action of general relativity given by (2.5) with \( \omega = \omega[e] \), and the second term is the action of fermions coupled to gravity given in (2.10). The third term describes a four-fermion interaction mediated by a non-propagating torsion. An interaction of this form is well known: it is predicted by the Einstein-Cartan theory. Crucially, this is precisely the type of four-fermion interaction that could forge a composite Higgs model, as discussed in the introductory remarks above.

From now on spacetime indices are omitted, and the total action in (2.16) shall be written in the equivalent form [1, 114]

\[
S[e, \psi] = S[e] + S_f[e, \psi] + S_I[e, \psi]
\]

(2.19)

\[
= \int dx \, e^a \wedge e^b \wedge F^{cd} \epsilon_{abcd}
\]

\[+ \frac{1}{6} \int dx \, \theta^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd}
\]

\[+ \int dx \left( \bar{\psi} \gamma_5 \gamma^a \psi \right) \left( \bar{\psi} \gamma_5 \gamma^a \psi \right),
\]

(2.20)

where

\[
\theta^a \equiv \frac{i}{2} \left( \bar{\psi} \gamma^a D_\mu \psi - D_\mu \bar{\psi} \gamma^a \psi \right) dx^\mu.
\]

(2.21)

Eq. (2.19) is called the minimal coupling action.

A more general action is derived in [2] that gives rise to the same equations of motion has the same form as Eq. (2.19) but with

\[
\theta^a \equiv \frac{i}{2} \left( \bar{\psi} (1 + \xi) \gamma^a D_\mu \psi - D_\mu \bar{\psi} (1 + \xi^*) \gamma^a \psi \right) dx^\mu,
\]

(2.22)

where \( \xi \) is a complex constant. The resulting action is called the non-minimal coupling action.

3. THE WEYL ANOMALY AND THE ROLE OF DIMENSION-ZERO SCALAR FIELDS

Given a conformally invariant classical theory (sometimes referred to as a Weyl invariant classical theory), the corresponding quantum theory preserves this Weyl invariance if the Weyl anomaly
cancels. The Weyl anomaly itself is given by

$$\langle T_\mu^\mu \rangle = \frac{1}{360(4\pi)^2} \left( 3cC^2 - aE \right) + \zeta \Box R,$$

where

$$C^2 = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{3} R^2 \tag{3.2}$$

is the square of the Weyl curvature,

$$E = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2 \tag{3.3}$$

is the Gauss-Bonnet density, and the parameters $a$ and $c$ in (3.1) are given by [115]-[124]

$$a = n_0 + \frac{11}{2} n_{1/2} + 62n_1 + 1142n_2 \tag{3.4}$$

$$c = n_0 + 3n_{1/2} + 12n_1 + 522n_2 \tag{3.5}$$

Here, $n_0$ is the number of dimension-one real scalar fields (spin 0), $n_{1/2}$ is the number of Weyl or Majorana spinor fields (spin 1/2), $n_1$ is the number of real vector fields (spin 1), and $n_2$ is the number of graviton fields (spin 2). Importantly, of the standard model particles, the fermions account for $n_{1/2} = 48$ particles. These include six leptons: $(e, \nu_e)$, $(\mu, \nu_\mu)$, $(\tau, \nu_\tau)$, six quark flavours $(u, d)$, $(c, s)$, $(t, b)$, each of which can have three colour quantum numbers thus yielding 18 quarks. Overall this makes 24 fermions. Together with the corresponding antifermions the total comes to 48 spin 1/2 fermions. The bosons that are the force carriers include 1 photon ($\gamma$) mediating EM interactions, the $W^+$, $W^-$ and $Z$ bosons mediating weak interactions, and eight gluons ($g$) that mediate strong interactions, which add up to twelve spin 1 bosons. The standard electroweak theory also includes four spin 0 scalar fields that are Higgs bosons, and one spin 2 graviton yet to be discovered. This configuration is summarized in in the upper chart in Table 1.

This configuration of particles gives rise to a non-zero Weyl anomaly, as easily checked by substituting the appropriate quantities of particle species in (3.1). However, with precisely 36 spin 0 scalar fields that have mass dimension zero, the Weyl anomaly vanishes. Specifically, the new Weyl anomaly has the form (3.1) with (3.2) and (3.3), but with $a$ and $c$ given instead of (3.4) and (3.5) by

$$a = n_0 + \frac{11}{2} n_{1/2} + 62n_1 + 1142n_2 - 28n'_0 \tag{3.6}$$

$$c = n_0 + 3n_{1/2} + 12n_1 + 522n_2 - 8n'_0 \tag{3.7}$$

where $n'_0$ is the number of spin 0 dimension 0 scalar fields in the theory. Note that if $n'_0 = 36$, then as a result $a = 0$, and independently $c = 0$ as well. Consequently the Weyl anomaly cancels with this configuration of particles. This modified configuration of particles is summarized in the lower chart in Table 1.
The observation that 36 dimension 0 scalar fields leads to a vanishing Weyl anomaly is one of the motives for the composite Higgs model proposed in this article. Not only does the cancellation of the Weyl anomaly require precisely 36 dimension 0 scalars, but the action with non-minimal coupling of fermions to gravity (given in (3.8) below) admits precisely 36 different coupling coefficients, as explained below.

The non-minimal coupling action was given in (2.20). The piece containing the non-minimal coupling of fermions to gravity concerning this discussion is

\[ S_f = \frac{1}{6} \epsilon_{abcd} \int dx \, \theta^a \wedge e^b \wedge e^c \wedge e^d, \]  
(3.8)
where \( \theta^a \) contributions from all of the fermions in the standard model:

\[
\theta^a = \theta^a_l + \theta^a_q ,
\]

\[
\theta^a_l = \frac{i}{2} \sum_{A,B=e,\mu,\tau} \bar{\psi}_A (1 + (\xi_l)_{AB}) \gamma^a D\psi_B - \frac{i}{2} \sum_{A,B=e,\mu,\tau} \overline{D\psi_A} (1 + (\xi^\dagger_l)_{AB}) \gamma^a \psi_B \tag{3.10}
\]

\[
\theta^a_q = \frac{i}{2} \sum_{A,B=u,s,t} \bar{\psi}_A (1 + (\xi_q)_{AB}) \gamma^a D\psi_B - \frac{i}{2} \sum_{A,B=u,s,t} \overline{D\psi_A} (1 + (\xi^\dagger_q)_{AB}) \gamma^a \psi_B \tag{3.11}
\]

If it assumed that each fermion in the standard model couples to the background vielbein field differently, then there are 36 different coefficients, denoted as \( \xi \), required to couple each one of the fermions to the background tetrad.

In our model these coupling coefficients become dimension 0 scalar fields in their own right. Accordingly, the action of our model is comprises the kinetic action of the scalar fields themselves and the coupling piece with non-minimal coupling of fermions to gravity through 36 coupling coefficients, where the very coupling coefficients are scalar fields.

A \textit{dimension-zero} conformally coupled scalar field \( \phi \) has the action

\[
S_{\text{scalar}}[\phi] = \frac{1}{2} \int dx \sqrt{g} \phi \Delta_4 \phi \tag{3.12}
\]

where \( \Delta_4 \) is the unique conformally-invariant fourth order differential operator

\[
\Delta_4 = \Box^2 + 2R^{\alpha\beta} \nabla_\alpha \nabla_\beta - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^\alpha R) \nabla_\alpha .
\]

The action in Eq. \( \text{[3.12]} \) is invariant under the Weyl transformation \( g_{\alpha\beta}(x) \rightarrow \Omega^2(x)g_{\alpha\beta}(x) \), \( \phi(x) \rightarrow \Omega^0(x)\phi(x) \).

The total action of our model is

\[
S = S_{\text{scalar}}[e] + S_f[e] \tag{3.14}
\]

where \( S_{\text{scalar}}[e] \) is the conformally invariant action of dimension 0 scalar fields (see \( \text{[3.12]} \))

\[
S_{\text{scalar}} = \sum_{A,B,C,D=1,2,3} \sum_{\text{generations}} \alpha_{ABCD}^{uv} \int dx \left( \xi_{AB}^{CD} \right)^* \Box^2 \xi_{uv}^{CD} ,
\]

where \( \alpha_{ABCD}^{uv} \) are coupling constants. \( S_f[e] \) is the covariant Dirac action with 36 constant coupling coefficients replaced by dimension 0 scalar fields:

\[
S_f[e] = \frac{1}{6} \epsilon_{abcd} \int dx \theta^a \wedge e^b \wedge e^c \wedge e^d
\]

\[
\theta^a = \theta^a_l + \theta^a_q \tag{3.17}
\]

\[
\theta^a_l = \frac{i}{2} \sum_{A,B=e,\mu,\tau} \bar{\psi}_A [1 + (\xi_l)_{AB}] \gamma^a D\psi_B - \frac{i}{2} \sum_{A,B=e,\mu,\tau} \overline{D\psi_A} [1 + (\xi^\dagger_l)_{AB}] \gamma^a \psi_B \tag{3.18}
\]

\[
\theta^a_q = \frac{i}{2} \sum_{A,B=u,s,t} \bar{\psi}_A [1 + (\xi_q)_{AB}] \gamma^a D\psi_B - \frac{i}{2} \sum_{A,B=u,s,t} \overline{D\psi_A} [1 + (\xi^\dagger_q)_{AB}] \gamma^a \psi_B \tag{3.19}
\]
where $\xi_l$, $\xi_q$ are $3 \times 3$ complex-valued fields.

In [4] a simplified toy-model version of this action was formed, comprising the dominant contribution from the 3rd generation of quarks and 1 scalar field $\xi$, viz

$$S = S_{\text{scalar}} + S_q$$

(3.20)

where

$$S_{\text{scalar}} = \frac{1}{2} \int dx \, \sqrt{g} \Delta_4 \xi$$

(3.21)

with $\Delta_4$ given by (3.13) and

$$S_q = \int dx \left[ \alpha \xi^* \Box^2 \xi + \frac{i}{2} Q (1 + \xi) \psi Q - \frac{i}{2} \overline{\psi} Q (1 + \xi^*) Q \right]$$

(3.22)

where $Q = \begin{pmatrix} t \\ b \end{pmatrix}$. It was assumed that the $t$ and $b$ quarks couple in the same way to the background tetrad and that their masses are equal. In this simplified model, dynamical symmetry breaking occurs in the action with the consequent generation of a mass term ascribing mass to the quarks. In the following section this model is generalized to the case where dynamical symmetry breaking leads to different mass terms for the $t$ and $b$ quarks.

4. FUNDAMENTAL SCALARS IN MODELS WITH THREE GENERATIONS OF SM FERMIONS

4.1. Non-minimal coupling of fermions with gravity

The starting point of the discussion in this section is the action for Dirac fermions coupled to the vielbein field as given in Eq. (3.8). In turn, the action for the right - handed Weyl fermions may be written as Eq. (3.8) with $\theta^a \rightarrow \theta^a_R$, where

$$\theta^a_R = \frac{i}{2} \left[ \bar{\psi}_R \sigma^a D_\mu \psi_R - \overline{D_\mu \psi_R} \sigma^a \psi_R \right] dx^\mu ,$$

(4.1)

while the action for the left - handed Weyl fermions may be written as Eq. (3.8) with $\theta^a \rightarrow \theta^a_L$ and

$$\theta^a_L = \frac{i}{2} \left[ \bar{\psi}_L \bar{\sigma}^a D_\mu \psi_L - \overline{D_\mu \psi_L} \bar{\sigma}^a \psi_L \right] dx^\mu ,$$

(4.2)

A non - minimal coupling of fermions to gravity is achieved via the substitution [2]

$$\theta^a \rightarrow \theta^a_R + \frac{i}{2} \left[ \bar{\psi}_R \xi_R \sigma^a D_\mu \psi_R - \xi^*_R \overline{D_\mu \psi_R} \sigma^a \psi_R \right] dx^\mu$$

(4.3)
and

\[
\theta_a^a \to \theta_a^a + \frac{i}{2} \left[ \bar{\psi}_L \xi_L \sigma^a D_\mu \psi_L - \xi_L^* \bar{\psi}_L \sigma^a \psi_L \right] dx^\mu
\]  \hspace{1cm} (4.4)

with complex-valued constants \( \xi_R \) and \( \xi_L \). Notice that in the absence of the spin connection, \( D_\mu = \partial_\mu \) and the imaginary parts of \( \xi_R \) and \( \xi_L \) decouple and do not interact with fermions. At the same time the real parts of these constants may be absorbed by a rescaling of the fermion fields.

That said, suppose that this construction is extended to the case where \( \xi_R \) and \( \xi_L \) become coordinate-dependent fields. In this case both the real and imaginary parts of \( \xi_R \) and \( \xi_L \) interact with the fermions of the theory, and cannot be removed by any rescaling of the fields. Further more, let \( \xi_R \) and \( \xi_L \) be assigned flavour indices. In the presence of \( SU(3) \otimes SU(2) \otimes U(1) \) gauge fields, there are \( n_1 = 8 + 3 + 1 = 12 \) vector fields. In accordance the number of Weyl fermions must equal \( n_{1/2} = 4n_1 = 48 \). This number coincides with the number of Weyl fermions in the SM with three generations: \( n_{1/2} = (1 \text{ leptons} + 3 \text{quarks}) \times 2_{\text{up \& down}} \times 2_{\text{left \& right}} \times 3_{\text{generations}} = 48 \).

### 4.2. Parity breaking interactions with zero dimension scalar fields

The theory can be extended one stage further to the case where \( \xi_L \) and \( \xi_R \) carry not only flavor indices but also generation indices. Even more they can be further distinguished by assigning different forms for the matrices \( \xi \) for the left-handed and the right-handed particles, while \( \xi \) the matrices for the quarks and leptons remain identical. In this approach the fermion term \( \theta_a \) in (3.8) now reads

\[
\theta_a = \theta_a^a + \theta_a^R,
\]  \hspace{1cm} (4.5)

where

\[
\theta_a^R = \frac{i}{2} \left[ \bar{\psi}_R \left( 1 + \xi_R \right) \sigma^a D_\mu \psi_R - \left( D_\mu \bar{\psi}_R \right) \left( 1 + \xi_R^\dagger \right) \sigma^a \psi_R \right] dx^\mu,
\]  \hspace{1cm} (4.6)

and

\[
\theta_a^L = \frac{i}{2} \left[ \bar{\psi}_L \left( 1 + \xi_L \right) \sigma^a D_\mu \psi_L - \left( D_\mu \bar{\psi}_L \right) \left( 1 + \xi_L^\dagger \right) \sigma^a \psi_L \right] dx^\mu.
\]  \hspace{1cm} (4.7)

This gives rise to \( n_0' = 2_{\text{chiralities}} \times 2_{\text{real \& imaginary parts}} \times 3_{\text{generations}} \times 3_{\text{generations}} = 36 \) components of the scalar fields. This is precisely the number of scalars \( n_0' = 3n_1 \) needed for the cancellation of the Weyl anomaly [3].
4.3. Interactions that conserve parity

There would be the same number of scalar fields as above if \( \xi_L \) were identical to \( \xi_R \), but with the matrices \( \xi \) being different for quarks and leptons. In this case the interaction with \( \xi \) does not break parity. The fermion action is still defined as in Eq. (3.8), but instead of Eq. (4.5),

\[
\theta^a = \theta^a_l + \theta^a_q ,
\]

where

\[
\theta^a_q = \frac{i}{2} \left[ \bar{\psi}_q (1 + \xi_q) \gamma^a D_\mu \psi_q - (D_\mu \bar{\psi}_q)(1 + \xi_q^\dagger) \gamma^a \psi_q \right] dx^\mu , \quad (4.9)
\]

and

\[
\theta^a_l = \frac{i}{2} \left[ \bar{\psi}_q (1 + \xi_l) \gamma^a D_\mu \psi_l - (D_\mu \bar{\psi}_q)(1 + \xi_l^\dagger) \gamma^a \psi_l \right] dx^\mu . \quad (4.10)
\]

Here both \( \xi_l \) and \( \xi_q \) are the 3 \times 3 complex - valued matrices in flavor space. As before, \( n'_0 = 2_{\text{lgrs} \& \text{qrks}} \times 2_{\text{real} \& \text{imag parts}} \times 3_{\text{generations}} \times 3_{\text{generations}} = 36 \) zero dimension scalar fields.

4.4. Additional coupling constants

The above mentioned scheme may be extended to allow different couplings of the same field \( \xi_q \) with different quarks/leptons. Namely, we consider the action of the form

\[
S = \int dx \left[ \alpha^{uv}_{ABCD} (\xi^A_u)^\ast \Box^2 \xi^C_v + \frac{i}{2} \bar{Q}_a L (1 + \beta_{cdL} \xi_q^c) \gamma^e Q^e_{bL} - \frac{i}{2} \Box \bar{Q}_a L (1 + \beta_{cdL} \xi_q^c) Q_{bL} \\
+ \frac{i}{2} \bar{U}_a R (1 + \beta_{cdU} \xi_q^c) \gamma^e U^e_{bR} - \frac{i}{2} \Box \bar{U}_a R (1 + \beta_{cdU} \xi_q^c) U_{bR} \\
+ \frac{i}{2} \bar{D}_a R (1 + \beta_{cdD} \xi_q^c) \gamma^e D^e_{bR} - \frac{i}{2} \Box \bar{D}_a R (1 + \beta_{cdD} \xi_q^c) D_{bR} \\
+ \frac{i}{2} \bar{L}_a L (1 + \gamma_{cdL} \xi_q^c) \gamma^e L^e_{bL} - \frac{i}{2} \Box \bar{L}_a L (1 + \gamma_{cdL} \xi_q^c) L_{bL} \\
+ \frac{i}{2} \bar{N}_a R (1 + \gamma_{cdN} \xi_q^c) \gamma^e N^e_{bR} - \frac{i}{2} \Box \bar{N}_a R (1 + \gamma_{cdN} \xi_q^c) N_{bR} \\
+ \frac{i}{2} \bar{E}_a R (1 + \gamma_{cdE} \xi_q^c) \gamma^e E^e_{bR} - \frac{i}{2} \Box \bar{E}_a R (1 + \gamma_{cdE} \xi_q^c) E_{bR} \right] \quad (4.11)
\]

Here \( Q_{aL} \) is the \( SU(2) \) doublet of left - handed quarks \((u, d), (c, s), (t, b)\) (index \( a \) takes values 1, 2, 3). \( U_{aR} \) is the \( SU(2) \) singlet of right - handed quarks \( u, c, t \). \( D_{aR} \) is the \( SU(2) \) singlet of right - handed
quarks $d, s, b$. In a similar way $L_{aL}$ is the $SU(2)$ doublet of left-handed leptons $(\nu, e)$, $(\nu_\mu, \mu)$, $(\nu_\tau, \tau)$. $N_{aR}$ is the $SU(2)$ singlet of right-handed neutrinos $\nu, \nu_\mu, \nu_\tau$. $E_{aR}$ is the $SU(2)$ singlet of right-handed leptons $e, \mu, \tau$. Tensors $\beta$ and $\gamma$ contain coupling constants of the field $\xi_q$ and $\xi_l$ to fermions.

5. ELECTROWEAK SYMMETRY BREAKING AND MASS GENERATION

5.1. Introduction of the toy model with top and bottom quarks

Consider the SM with three generations and the fundamental scalar $\xi$ fields constructed in §4.3. With variations of the vielbein and spin connection restrained, define an action for the zero dimension scalar fields as

$$S_B = \alpha_{ABCD}^{uv} \int dx (\xi_u^{AB})^* \Box^2 \xi_v^{CD},$$

(5.1)

where $A, B, C, D = 1, 2, 3$ are generation indices, $u, v = q, l$ are flavor indices, and $\alpha_{ABCD}^{uv}$ are a set of coupling constants. The effective four-fermion interactions arise from the exchange by quanta of the fields $\xi_q$ and $\xi_l$. The effective four-fermion interaction is non-local.

Consider now a certain sector of the theory that describes the dominant contributions from the sector corresponding to the third generation of quarks, and the field $\xi^{33}_q$. Let the kinetic piece of the action given in (5.1) be added to the action in (3.8), with $\theta^a$ given by (4.8), with only the sector describing the fields $\xi^{33}_q$ included. Note that the left- and right-handed fermion fields $\psi_L, \psi_R$ appearing in (3.8) in this sector correspond to the doublets $Q_{3L} = \begin{pmatrix} t_L \\ b_L \end{pmatrix}$ and $Q_{3R} = \begin{pmatrix} t_R \\ b_R \end{pmatrix}$, where $t_L, t_R, b_L, b_R$ stand for left- and right-handed top and bottom quark fields. The action obtained has the form

$$S = \int dx \left[ \alpha^{qq}_{3333} \xi^{33}_q \Box^2 \xi^{33}_q + \frac{i}{2} \bar{Q}_{3L} \left( 1 + \beta_L \xi^{33}_q \right) \not{D} Q_{3L} - \frac{i}{2} \bar{D} Q_{3L} \left( 1 + \beta_L^{*} \xi^{33*}_q \right) Q_{3L} \\
+ \frac{i}{2} \bar{Q}_{3R} \left( 1 + \beta_R \xi^{33}_q \right) \not{D} Q_{3R} - \frac{i}{2} \bar{D} Q_{3R} \left( 1 + \beta_R^{*} \xi^{33*}_q \right) Q_{3R} \right].$$

(5.2)

Here, $\beta_L$ and $\beta_R$ are non-identical constant coupling coefficients. Note that the covariant derivative $D$ contains gauge fields. This action admits global $SU(2)$ symmetry and, given the necessary form for the gauge fields, local $SU(2)$ symmetry as well. Next this action is modified to a different expression. The symmetry of the left-handed fields remains, but that of the right-handed fields is broken by allowing different couplings of the top and bottom quarks to the scalar field. The modified
expression is

\[ S = \int dx \left[ \alpha^{qq}_{3333} \xi^{33*} \Box^2 \xi + \frac{i}{2} Q_L (1 + \beta_L \xi^q ) \bar{\Psi} Q_L - \frac{i}{2} \overline{\bar{\Psi} Q_L} (1 + \beta_{L}^{*} \xi^{33*} ) Q_L \right. \]
\[ + \frac{i}{2} i_R (1 + \beta_i \xi^q ) \bar{\Psi} t_R - \frac{i}{2} \overline{\bar{\Psi} t_R} (1 + \beta_{i}^{*} \xi^{33*} ) t_R \\
\left. + \frac{i}{2} b_R (1 + \beta_b \xi^q ) \bar{\Psi} b_R - \frac{i}{2} \overline{\bar{\Psi} b_R} (1 + \beta_{b}^{*} \xi^{33*} ) b_R \right] \]  

(5.3)

where \( \beta_i \) and \( \beta_b \) are non-identical constant coupling coefficients corresponding to the couplings of the top and bottom quarks to the scalar field, respectively. To save on notation, use the shortened symbols \( \xi_q \equiv \xi^{33}_q, \alpha \equiv \alpha^{qq}_{3333}, Q \equiv Q_3, Q_L \equiv Q_{3L} = \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \) and \( Q_R \equiv Q_{3R} = \begin{pmatrix} t_R \\ b_R \end{pmatrix} \). Like this Eq. (5.3) reads

\[ S = \int dx \left[ \alpha \xi^{*} \Box^2 \xi + \frac{i}{2} Q_L (1 + \beta_L \xi^q ) \bar{\Psi} Q_L - \frac{i}{2} \overline{\bar{\Psi} Q_L} (1 + \beta_{L}^{*} \xi^{33*} ) Q_L \right. \]
\[ + \frac{i}{2} i_R (1 + \beta_i \xi^q ) \bar{\Psi} t_R - \frac{i}{2} \overline{\bar{\Psi} t_R} (1 + \beta_{i}^{*} \xi^{33*} ) t_R \\
\left. + \frac{i}{2} b_R (1 + \beta_b \xi^q ) \bar{\Psi} b_R - \frac{i}{2} \overline{\bar{\Psi} b_R} (1 + \beta_{b}^{*} \xi^{33*} ) b_R \right] \]  

(5.4)

Let the last two terms on the right be re-written in such a way that

\[ S = \int dx \left[ \alpha \xi^{*} \Box^2 \xi + \frac{i}{2} Q_L (1 + \beta_L \xi^q ) \bar{\Psi} Q_L - \frac{i}{2} \overline{\bar{\Psi} Q_L} (1 + \beta_{L}^{*} \xi^{33*} ) Q_L \right. \]
\[ + \frac{i}{2} i_R (1 + \beta_i \xi^q ) \bar{\Psi} t_R - \frac{i}{2} \overline{\bar{\Psi} t_R} (1 + \beta_{i}^{*} \xi^{33*} ) t_R \\
\left. + \frac{i}{2} b_R (1 + \beta_b \xi^q ) \bar{\Psi} b_R - \frac{i}{2} \overline{\bar{\Psi} b_R} (1 + \beta_{b}^{*} \xi^{33*} ) b_R \right] \]  

(5.5)

Define the top and bottom projection operators \( \Pi_t \) and \( \Pi_b \)

\[ \Pi_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]  

(5.6)
such that given the doublet $Q = \begin{pmatrix} t \\ b \end{pmatrix}$,

$$\Pi_t Q = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \Pi_b Q = \begin{pmatrix} 0 \\ b \end{pmatrix},$$

$$\bar{Q} \Pi_t = \begin{pmatrix} \bar{t} \\ 0 \end{pmatrix}, \quad \bar{Q} \Pi_b = \begin{pmatrix} 0 \\ \bar{b} \end{pmatrix},$$

(5.7)

Note the properties

$$\Pi_t^2 = \Pi_t, \quad \Pi_b^2 = \Pi_b, \quad \Pi_t \Pi_b = \Pi_b \Pi_t = 0.$$  

(5.8)

Accordingly (5.5) may be written in terms of $\Pi_t$ and $\Pi_b$ as

$$S = \int dx \left[ \alpha \xi^* \Box^2 \xi \\
+ \frac{i}{2} \bar{Q}_L (1 + \beta_L \xi) \slash{D} Q_L - \frac{i}{2} \bar{Q}_L (1 + \beta^*_L \xi^*) Q_L \\
+ \frac{i}{2} \bar{Q}_R \Pi_t (1 + \beta_t \xi) \slash{D} Q_R - \frac{i}{2} \bar{Q}_R (1 + \beta^*_t \xi^*) \Pi_t Q_R \\
+ \frac{i}{2} \bar{Q}_R \Pi_b (1 + \beta_b \xi) \slash{D} Q_R - \frac{i}{2} \bar{Q}_R \Pi_b (1 + \beta^*_b \xi^*) \Pi_b Q_R \right]$$

(5.9)

Note that the projection matrices $\Pi_t, \Pi_b$ and $\gamma$ matrices act on different spaces ($\Pi_t, \Pi_b$ act on $SU(2)$ doublets in flavour space while the $4 \times 4$ $\gamma$ matrices act on four-component Dirac spinors), so $\Pi_t, \Pi_b$ matrices in can be commuted past the $\gamma$ matrices in the operators $\slash{D}$ to yield

$$S = \int dx \left[ \alpha \xi^* \Box^2 \xi \\
+ \frac{i}{2} \bar{Q}_L (1 + \beta_L \xi) \slash{D} Q_L - \frac{i}{2} \bar{Q}_L (1 + \beta^*_L \xi^*) Q_L \\
+ \frac{i}{2} \bar{Q}_R (\Pi_t^2 + \beta_t \Pi_t) \slash{D} Q_R - \frac{i}{2} \bar{Q}_R (\Pi^*_t \Pi_t) Q_R \\
+ \frac{i}{2} \bar{Q}_R (\Pi_b^2 + \beta_b \Pi_b) \slash{D} Q_R - \frac{i}{2} \bar{Q}_R (\Pi^*_b \Pi_b) Q_R \right]$$

(5.10)
where in the 2nd step \((5.8)\) was used. This can be further simplified to

\[
S = \int dx \left[ \alpha \xi^* \Box^2 \xi \\
+ \frac{i}{2} \dot{P}_R (1 + \beta_L \xi) \dot{\Psi} Q_L - \frac{i}{2} \dot{\overline{\Psi}} Q_L (1 + \beta_L^* \xi^*) Q_L \\
+ \frac{i}{2} \dot{P}_L (\Pi_t + \Pi_b + \beta_t \xi \Pi_t + \beta_b \xi \Pi_b) \dot{\Psi} Q_R - \frac{i}{2} \dot{\overline{\Psi}} Q_R (1 + \beta_t^* \xi^* \Pi_t + \beta_b^* \xi^* \Pi_b) Q_R \right]
\]

\[
= \int dx \left[ \alpha \xi^* \Box^2 \xi \\
+ \frac{i}{2} \dot{P}_L (1 + \beta_L \xi) \dot{\Psi} Q_L - \frac{i}{2} \dot{\overline{\Psi}} Q_L (1 + \beta_L^* \xi^*) Q_L \\
+ \frac{i}{2} \dot{P}_R (1 + \beta_t \xi \Pi_t + \beta_b \xi \Pi_b) \dot{\Psi} Q_R - \frac{i}{2} \dot{\overline{\Psi}} Q_R (1 + \beta_t^* \xi^* \Pi_t + \beta_b^* \xi^* \Pi_b) Q_R \right], \tag{5.11}
\]

where in the last step the fact that \(\Pi_t + \Pi_b = 1\) was used. According to the definitions in Eqs. (A.19) and (A.21) and the relations in (A.25),

\[
\dot{\Psi} Q_L = D_\mu \gamma^\mu P_L Q = D_\mu P_R \gamma^\mu Q = P_R \dot{\Psi} Q, \tag{5.12a}
\]

\[
\dot{\Psi} Q_R = D_\mu \gamma^\mu P_R Q = D_\mu P_L \gamma^\mu Q = P_L \dot{\Psi} Q, \tag{5.12b}
\]

such that

\[
\begin{align}
\dot{\overline{\Psi}} Q_L &= \overline{P_R \dot{\Psi} Q} = (P_R \dot{\Psi} Q)^\dagger \gamma^0 = (\Psi Q)^\dagger P_R^\dagger \gamma^0 = (\Psi Q)^\dagger P_R^\dagger = \overline{\Psi} P_L Q, \tag{5.12c} \\
\dot{\overline{\Psi}} Q_R &= \overline{P_L \dot{\Psi} Q} = (P_L \dot{\Psi} Q)^\dagger \gamma^0 = (\Psi Q)^\dagger P_L^\dagger \gamma^0 = (\Psi Q)^\dagger P_L^\dagger = \overline{\Psi} P_R Q. \tag{5.12d}
\end{align}
\]

With the help of these relations Eq. (5.11) can be cast in the form

\[
S = \int dx \left[ \alpha \xi^* \Box^2 \xi \\
+ \frac{i}{2} \dot{P}_R^2 (1 + \beta_L \xi) \dot{\Psi} Q - \frac{i}{2} \dot{\overline{\Psi}} Q P_R^2 (1 + \beta_L^* \xi^*) Q \\
+ \frac{i}{2} \dot{P}_L^2 (1 + \beta_t \xi \Pi_t + \beta_b \xi \Pi_b) \dot{\Psi} Q - \frac{i}{2} \dot{\overline{\Psi}} Q P_L^2 (1 + \beta_t^* \xi^* \Pi_t + \beta_b^* \xi^* \Pi_b) Q \right]. \tag{5.13}
\]
But $P^2_R = P_R$ and $P^2_L = P_L$ (see Eqs. (A.17)), hence

$$ S = \int dx \left[ \alpha \xi \Box^2 \xi 
+ \frac{i}{2} \overline{Q} P_R (1 + \beta_L \xi) \Phi Q - \frac{i}{2} \overline{\Phi} Q P_L (1 + \beta_L^* \xi^*) Q 
+ \frac{i}{2} \overline{Q} P_L (1 + \beta_t \Pi t + \beta_b \Pi b) \Phi Q - \frac{i}{2} \overline{\Phi} Q P_R (1 + \beta_t^* \xi^* \Pi t + \beta_b^* \xi^* \Pi b) Q \right] $$

$$ = \int dx \left[ \alpha \xi \Box^2 \xi 
+ \frac{i}{2} \overline{Q} (P_R + P_L + \beta_L \xi P_R + [\beta_t \Pi t + \beta_b \Pi b] \xi P_L) \Phi Q 
- \frac{i}{2} \overline{\Phi} Q (P_L + P_R + \beta_t^* \xi^* P_L + [\beta_t^* \Pi t + \beta_b^* \Pi b] \xi^* P_R) Q \right] $$

$$ = \int dx \left[ \alpha \xi \Box^2 \xi 
+ \frac{i}{2} \overline{Q} (1 + \beta_L \xi P_R + [\beta_t \Pi t + \beta_b \Pi b] \xi P_L) \Phi Q 
- \frac{i}{2} \overline{\Phi} Q (1 + \beta_t^* \xi^* P_L + [\beta_t^* \Pi t + \beta_b^* \Pi b] \xi^* P_R) Q \right] $$

$$ = \int dx \left[ \alpha \xi \Box^2 \xi + \frac{i}{2} \overline{Q} \Phi Q + \frac{i}{2} \overline{Q} \Gamma \xi \Phi Q - \frac{i}{2} \overline{\Phi} Q Q - \frac{i}{2} \overline{\Phi} Q \chi \xi^* Q \right] , \quad (5.14) $$

where

$$ \Gamma = \beta_L P_R + (\beta_t \Pi t + \beta_b \Pi b) P_L , \quad \chi = \beta_t^* P_L + (\beta_t^* \Pi t + \beta_b^* \Pi b) P_R . \quad (5.15) $$

Note that in accordance with the identities in (A.27),

$$ \gamma^0 \chi = \beta_L^* \gamma^0 P_L + (\beta_t^* \Pi t + \beta_b^* \Pi b) \gamma^0 P_R = \left( \beta_L^* P_R^\dagger + (\beta_t^* \Pi t + \beta_b^* \Pi b) P_L^\dagger \right) \gamma^0 = \Gamma^\dagger \gamma^0 \quad (5.16) $$

such that

$$ \overline{\Phi} Q \chi Q = (\Phi Q)^\dagger \gamma^0 \chi Q = (\Phi Q)^\dagger \Gamma^\dagger \gamma^0 Q = (\Gamma \Phi Q)^\dagger \gamma^0 Q = \left( \overline{\Gamma \Phi Q} \right) Q \quad (5.17) $$

and consequently (5.14) takes the succinct form

$$ S = \int dx \left[ \alpha \xi \Box^2 \xi + \frac{i}{2} \overline{Q} \Phi Q + \frac{i}{2} \overline{Q} \Gamma \xi \Phi Q - \frac{i}{2} \overline{\Phi} Q Q - \frac{i}{2} \overline{\Phi} Q \chi \xi^* Q \right] . \quad (5.18) $$

### 5.2. Effective action resulting from the integration over scalar fields

Define an effective action, $S_{\text{eff}}$ as

$$ e^{i S_{\text{eff}}} = \frac{1}{Z_0} \int D\xi \ D\xi^* e^{i S} \quad (5.19) $$
where $S$ appearing in the integrand is the action in (5.18). To integrate out the scalar fields, let

\[ \xi' = \xi - \frac{i}{2\alpha} \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y) \]  
\[ (5.20) \]

\[ \xi'' = \xi* + \frac{i}{2\alpha} \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y). \]  
\[ (5.21) \]

where $\square^{-2}(x, y)$ is the square of the inverse propagator of the boson field $\xi = \xi_3^3$. In terms of the new variables $S$ has the form

\[
S = \int dx \left[ \alpha \left( \xi'' - \frac{i}{2\alpha} \int dz \Gamma \bar{\psi} Q \square^{-2}(x, z) \right) \square_x^2 \left( \xi' + \frac{i}{2\alpha} \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y) \right) 
+ \frac{i}{2} \bar{\psi} \Gamma \xi' \psi Q + \frac{i}{2} \bar{\psi} \Gamma \left( \frac{i}{2\alpha} \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y) \right) \psi Q 
- \frac{i}{2} \Gamma \bar{\psi} Q \xi'' Q + \frac{i}{2} \Gamma \bar{\psi} Q \left( \frac{i}{2\alpha} \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y) \right) Q 
+ \frac{i}{2} \bar{\psi} \Gamma \psi Q - \frac{i}{2} \bar{\psi} \Gamma Q \right] 
= \int dx \left[ \alpha \xi'' \square_x^2 \xi' - \frac{i}{2} \int dz \Gamma \bar{\psi} Q \square^{-2}(x, z) \square_x^2 \xi' + \xi'' \square_x^2 \frac{i}{2} \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y) 
+ \frac{1}{4\alpha} \int dz \Gamma \bar{\psi} Q \square^{-2}(x, z) \square_x^2 \int d\gamma \Gamma \bar{\psi} Q \square^{-2}(x, y) 
+ \frac{i}{2} \bar{\psi} \Gamma \xi' \psi Q - \frac{1}{4\alpha} \bar{\psi} \Gamma \int d\gamma \left( \Gamma \bar{\psi} Q \square^{-2}(x, y) \right) \psi Q 
- \frac{i}{2} \Gamma \bar{\psi} Q \xi'' Q - \frac{1}{4\alpha} \Gamma \bar{\psi} Q \int d\gamma \left( \bar{\psi} \Gamma \psi Q \square^{-2}(x, y) \right) Q 
+ \frac{i}{2} \bar{\psi} \Gamma \psi Q - \frac{i}{2} \bar{\psi} \Gamma Q \right] 
= \int dx \left[ \alpha \xi'' \square_x^2 \xi' - \frac{i}{2} \bar{\psi} \Gamma \psi Q \xi' + \xi'' \frac{i}{2} \bar{\psi} \Gamma Q \right.
+ \frac{1}{4\alpha} \int d\gamma \bar{\psi} Q \Psi(x, y) Q \square^{-2}(x, y) \bar{\psi} Q \Psi Q(x) 
+ \frac{i}{2} \bar{\psi} \Gamma \xi' \psi Q - \frac{1}{4\alpha} \bar{\psi} \Gamma \int d\gamma \left( \Gamma \bar{\psi} Q \square^{-2}(x, y) \right) \psi Q 
- \frac{i}{2} \Gamma \bar{\psi} Q \xi'' Q - \frac{1}{4\alpha} \Gamma \bar{\psi} Q \int d\gamma \left( \bar{\psi} \Gamma \psi Q \square^{-2}(x, y) \right) Q 
+ \frac{i}{2} \bar{\psi} \Gamma \psi Q - \frac{i}{2} \bar{\psi} \Gamma Q \right]. 
\]  
\[ (5.22) \]

The 2nd and 5th terms cancel, the 3rd and 7th cancel, and the 4th and 8th terms cancel, leaving

\[
S = \int dx \left[ \alpha \xi'' \square_x^2 \xi' - \frac{1}{4\alpha} \bar{\psi} \Gamma \int d\gamma \left( \Gamma \bar{\psi} Q \square^{-2}(x, y) \right) \psi Q + \frac{i}{2} \bar{\psi} \Gamma \psi Q - \frac{i}{2} \bar{\psi} \Gamma Q \right]. 
\]  
\[ (5.23) \]
or equivalently

\[ S = \int dx \left[ \alpha \xi' \Box^2 \xi' + \frac{i}{2} \bar{Q} \gamma \bar{Q} - \frac{i}{2} \bar{Q} i \gamma_5 Q \right] \\
- \frac{1}{4\alpha} \int dx \, dy \left( \bar{Q} \Gamma Q \right) (x) \Box^{-2}(x,y) \left( \Gamma \bar{Q} Q \right) (y). \]  

(5.24)

By substituting (5.24) in (5.19) and integrating out the scalar fields it is obtained that

\[ e^{i S_{\text{eff}}} = \frac{1}{Z_0} \left( \text{Det} \left( \frac{i \Box}{\pi} \right) \right)^{-1} \exp \left[ \alpha \int dx \left( \frac{i}{2} \bar{Q} \gamma_5 Q - \frac{i}{2} \bar{Q} i \gamma_5 Q \right) \right] \\
- \frac{1}{4\alpha} \int dx \, dy \left( \bar{Q} \Gamma Q \right) (x) \Box^{-2}(x,y) \left( \Gamma \bar{Q} Q \right) (y). \]  

(5.25)

such that the effective action is found to be

\[ S_{\text{eff}} = - \frac{1}{4\alpha} \int dx \, dy \left( \bar{Q} \Gamma Q \right) (x) \Box^{-2}(x,y) \left( \Gamma \bar{Q} Q \right) (y). \]  

(5.26)

Here \( \Box^{-2}(x,y) \) is the square of the inverse propagator of the boson field \( \xi^{33} \). The one-loop contribution to the two-point Green function is straightforwardly read off Eq. (5.26). The form of the self-energy of the third generation quarks then follows. The corresponding Feynman diagram is shown in Fig. 2.

![Feynman diagram of the fermion self-energy corresponding to Eq. (6.1). The incoming and outgoing lines are initial and final fermion states with momentum \( p \). The dashed line in the loop is a scalar boson propagator carrying virtual momentum \( p - k \), and the thick horizontal line corresponds to the full fermion propagator with renormalized mass \( \Sigma(k) \) inclusive of all self-energy corrections, as related in Eq. (6.1).](image)

6. SCHWINGER-DYSON APPROACH FOR CALCULATING FERMION MASS

6.1. Schwinger-Dyson equation in rainbow approximation

Analogous to the method used for the toy model in [4], at this point a Schwinger-Dyson equation is assembled to express the inverse of the full propagator, \( D^{-1}(p) \) in terms of the self-energy function \( \Sigma(p) \), through which \( \Sigma(p) \) can be determined. The self-energy function, \( \Sigma(p) \) of the third generation quarks through leading order, has the form

\[ \Sigma(p) = \frac{1}{\alpha} \int dk \, \Gamma \gamma_5 k \, \frac{i}{\gamma_5 k - \Sigma(k)} \, \Gamma \gamma_5 k \, \frac{1}{(p - k)^4}, \]  

(6.1)
where the standard notation \( \int dk (\ldots) = (2\pi)^{-4} \int d^4k (\ldots) \) has been used. Note that this is equivalent to expression (19) in [4] but with \( \Gamma \) set to 1. Now apply a Wick rotation (see Appendix [B] for details) to obtain

\[
i \Sigma(p) = \frac{1}{\alpha} \int dk_E \Gamma \gamma_E k_E \frac{1}{(\gamma_E k_E - i \Sigma(k))} \Gamma \gamma_E k_E \frac{1}{(p_E - k_E)^4}.
\] (6.2)

Since all expressions from now on are in terms of Euclidean-space variables the subscript ‘\( E \)’ can be dropped.

The self-energy function contains combinations of \( \{ \gamma_\mu \} \) matrices and \( \gamma_5 \), and in accordance is a \( 4 \times 4 \) matrix on the space of four-spinors. It also contains combinations of the \( 2 \times 2 \) projection matrices \( \Pi_1, \Pi_2 \) making it a function on the space of \( SU(2) \) doublets. (In all \( \Sigma(p) \) is an \( 8 \times 8 \) matrix.) So, \( \Sigma(p) \) can be expressed in terms of a linear combination of a basis of \( 4 \times 4 \) matrices on the space of Dirac spinors and a basis of \( 2 \times 2 \) matrices on the space of \( SU(2) \) doublets. A basis of \( 2 \times 2 \) matrices is

\[
\Pi_1 := \Pi_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Pi_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Pi_4 := \Pi_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\] (6.3)

A basis of \( 4 \times 4 \) matrices may be constructed from the \( \gamma \) matrices as the set \( \{ 1, \gamma_\mu, \gamma_5, \gamma_\mu \gamma_5, \gamma_\mu \gamma_\nu \} \), where their precise definitions can be found in Appendix [A].

\[
\Sigma(p) = a_1(p)\Pi_1 + a_2(p)\Pi_2 + a_3(p)\Pi_3 + a_4(p)\Pi_4
\]
\[
+ b_1(p)\gamma_5\Pi_1 + b_2(p)\gamma_5\Pi_2 + b_3(p)\gamma_5\Pi_3 + b_4(p)\gamma_5\Pi_4
\]
\[
+ c_1^\mu(p)\gamma_\mu\Pi_1 + c_2^\mu(p)\gamma_\mu\Pi_2 + c_3^\mu(p)\gamma_\mu\Pi_3 + c_4^\mu(p)\gamma_\mu\Pi_4
\]
\[
+ d_1^\mu(p)\gamma_\mu\gamma_5\Pi_1 + d_2^\mu(p)\gamma_\mu\gamma_5\Pi_2 + d_3^\mu(p)\gamma_\mu\gamma_5\Pi_3 + d_4^\mu(p)\gamma_\mu\gamma_5\Pi_4
\]
\[
+ f_1^{\mu\nu}(p)\gamma_\mu\gamma_\nu\Pi_1 + f_2^{\mu\nu}(p)\gamma_\mu\gamma_\nu\Pi_2 + f_3^{\mu\nu}(p)\gamma_\mu\gamma_\nu\Pi_3 + f_4^{\mu\nu}(p)\gamma_\mu\gamma_\nu\Pi_4
\] (6.4)

Certain symmetry properties of \( \Sigma(p) \) means that a handful of these terms can be dropped. \( \Sigma(p) \) depends on the absolute value of the momentum \( p_\mu \), so the \( d^{\mu\nu}(p) \) could only have symmetric combinations of \( p^\mu p^\nu \) or \( p^\delta p^{\mu\nu} \), which contract on \( \Sigma_{\mu\nu} \) to give zero. Since (6.2) only contains \( \Pi_1 \) and \( \Pi_2 \) terms containing \( \Pi_3 \) and \( \Pi_4 \) shall be disregarded. Hence \( \Sigma(p) \) has the form

\[
\Sigma(p) = a_1(p)\Pi_1 + a_2(p)\Pi_2 + a_3(p)\Pi_3 + a_4(p)\Pi_4
\]
\[
+ b_1(p)\gamma_5\Pi_1 + b_2(p)\gamma_5\Pi_2 + b_3(p)\gamma_5\Pi_3 + b_4(p)\gamma_5\Pi_4
\]
\[
+ c_1^\mu(p)\gamma_\mu\Pi_1 + c_2^\mu(p)\gamma_\mu\Pi_2 + c_3^\mu(p)\gamma_\mu\Pi_3 + c_4^\mu(p)\gamma_\mu\Pi_4
\]
\[
+ d_1^\mu(p)\gamma_\mu\gamma_5\Pi_1 + d_2^\mu(p)\gamma_\mu\gamma_5\Pi_2 + d_3^\mu(p)\gamma_\mu\gamma_5\Pi_3 + d_4^\mu(p)\gamma_\mu\gamma_5\Pi_4
\] (6.5)

In light of this structure for \( \Sigma(p) \) a suitable ansatz for the Schwinger-Dyson equation is

\[
D^{-1}(p) = \gamma_p - i \Sigma(p)
\] (6.6)
where

\[ D^{-1}(p) = \left[ A_1(p^2)\Pi_1 + A_2(p^2)\Pi_2 + A_3(p^2)\Pi_3 + A_4(p^2)\Pi_4 \right] \gamma_p \]

\[ - i \left[ B_1(p^2)\Pi_1 + B_2(p^2)\Pi_2 + B_3(p^2)\Pi_3 + B_4(p^2)\Pi_4 \right] \]

\[ + \left[ C_1(p^2)\Pi_1 + C_2(p^2)\Pi_2 + C_3(p^2)\Pi_3 + C_4(p^2)\Pi_4 \right] \gamma_p \gamma_5 \]

\[ - i \left[ D_1(p^2)\Pi_1 + D_2(p^2)\Pi_2 + D_3(p^2)\Pi_3 + D_4(p^2)\Pi_4 \right] \gamma_5 \]

\[ = \hat{A}\gamma_p - i\hat{B} \otimes 1_{4 \times 4} + \hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5 . \tag{6.7} \]

where \( \hat{A} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \) and similarly for \( \hat{B}, \hat{C} \) and \( \hat{D} \). Note the identity

\[ (\hat{A}\gamma_p - i\hat{B} + \hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5) \left( \hat{A}\gamma_p + i\hat{B} + \hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5 \right) \]

\[ = (\hat{A}\gamma_p - i\hat{B}) (\hat{A}\gamma_p + i\hat{B}) + (\hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5) (\hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5) \]

\[ + (\hat{A}\gamma_p - i\hat{B}) (\hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5) + (\hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5) (\hat{A}\gamma_p + i\hat{B}) \]

\[ = \hat{A}^2(\gamma_p)^2 + \hat{B}^2 + i[\hat{A},\hat{B}]\gamma_p + \hat{C}^2\gamma_p \gamma_5 \gamma_5 - \hat{D}^2\gamma_5^2 - i[\hat{C},\hat{D}] \gamma_p \gamma_5 \]

\[ + [\hat{A},\hat{C}](\gamma_p)^2 \gamma_5 - i[\hat{B},\hat{C}]\gamma_p \gamma_5 - i[\hat{A},\hat{D}]\gamma_p \gamma_5 - [\hat{B},\hat{D}] \gamma_5 \]

\[ = \hat{A}^2(\gamma_p)^2 + \hat{B}^2 + i[\hat{A},\hat{B}]\gamma_p - \hat{C}^2(\gamma_p)^2 \gamma_5 - \hat{D}^2\gamma_5^2 - i[\hat{C},\hat{D}] \gamma_p \gamma_5 \]

\[ + [\hat{A},\hat{C}](\gamma_p)^2 \gamma_5 - i[\hat{B},\hat{C}]\gamma_p \gamma_5 - i[\hat{A},\hat{D}]\gamma_p \gamma_5 - [\hat{B},\hat{D}] \gamma_5 \]

\[ = \left( \hat{A}^2 - \hat{C}^2 \right) p^2 + \hat{B}^2 - \hat{D}^2 \]

\[ + i \left( [\hat{A},\hat{B}] - [\hat{C},\hat{D}] \right) \gamma_p + [\hat{A},\hat{C}] p^2 \gamma_5 - i \left( [\hat{B},\hat{C}] + [\hat{A},\hat{D}] \right) \gamma_p \gamma_5 - [\hat{B},\hat{D}] \gamma_5 . \tag{6.8} \]

At this stage we seek solutions for which the second line of (6.8) vanishes. This is the case provided the commutator of any pair of non-identical matrices from \( \{\hat{A}, \hat{B}, \hat{C}, \hat{D}\} \) vanish, namely

\[ [\hat{A},\hat{B}] = 0 , \quad [\hat{A},\hat{C}] = 0 , \quad [\hat{A},\hat{D}] = 0 \quad [\hat{B},\hat{C}] = 0 \quad [\hat{B},\hat{D}] = 0 \quad [\hat{C},\hat{D}] = 0 . \tag{6.9} \]

After imposing (6.9) then

\[ \left( \hat{A}\gamma_p - i\hat{B} + \hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5 \right) \left( \hat{A}\gamma_p + i\hat{B} + \hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5 \right) \]

\[ = (\hat{A}^2 - \hat{C}^2) p^2 + \hat{B}^2 - \hat{D}^2 \tag{6.10} \]

and the inverse of (6.6) is found to be

\[ \frac{1}{\gamma_p - i\Sigma(p)} = \left( \hat{A}\gamma_p + i\hat{B} + \hat{C}\gamma_p \gamma_5 - i\hat{D}\gamma_5 \right) \cdot \frac{1}{(\hat{A}^2 - \hat{C}^2) p^2 + \hat{B}^2 - \hat{D}^2} . \tag{6.11} \]
where \( \frac{1}{(A^2 - C^2)p^2 + B^2 - D^2} \) on the right stands for the inverse of the matrix \((A^2 - C^2)p^2 + B^2 - D^2\).

Now (6.11) is substituted inside (6.2) to obtain

\[
i \Sigma(p) = \frac{1}{\alpha} \int dk \frac{1}{\gamma k} \left( \dot{A}\gamma k + i\dot{B} + \dot{C}\gamma k\gamma_5 - i\dot{D}\gamma_5 \right) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \Gamma \frac{k^2\gamma k}{(p - k)^4} \cdot (6.12)
\]

Note that the matrix \( \Gamma \) defined in (5.15) \( \Gamma = \beta_L P_R + (\beta_t \Pi_1 + \beta_b \Pi_4) P_L \) where \( \Pi_t, \Pi_b \) have been replaced with \( \Pi_1, \Pi_4 \) in accordance with (6.3)) can be anti-commuted past \( \gamma \) matrices in accordance with (A.25). Let

\[
\tilde{\Gamma} = \beta_L P_L + (\beta_t \Pi_1 + \beta_b \Pi_4) P_R .
\]

Then the following relations are derived from (A.25) and (A.26):

\[
\Gamma \gamma^\mu = \gamma^\mu \tilde{\Gamma} , \quad \gamma^\mu \Gamma = \tilde{\Gamma} \gamma^\mu , \quad \Gamma \gamma^\mu = \gamma^\mu \tilde{\Gamma} , \quad \gamma^\mu \tilde{\Gamma} = \Gamma \gamma^\mu , \quad \Gamma \gamma_5 = \gamma_5 \Gamma , \quad \tilde{\Gamma} \gamma_5 = \gamma_5 \tilde{\Gamma} .
\]

By using (6.14) and the anticommutation relation in (A.13), then Eq. (6.12) becomes

\[
i \Sigma(p) = \frac{1}{\alpha} \int dk \Gamma \gamma k \left( \dot{A}\gamma k + i\dot{B} + \dot{C}\gamma k\gamma_5 - i\dot{D}\gamma_5 \right) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \frac{1}{\gamma k} \frac{1}{(p - k)^4} \\
= \frac{1}{\alpha} \int dk \Gamma \gamma k \left( \dot{A}\gamma k + i\dot{B} + \dot{C}\gamma k\gamma_5 - i\dot{D}\gamma_5 \right) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \tilde{\Gamma} \frac{1}{(p - k)^4} \\
= \frac{1}{\alpha} \int dk \Gamma \gamma k \left( \dot{A}k^2 + i\dot{B}k - \dot{C}k^2\gamma_5 + i\dot{D}k\gamma_5 \right) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \tilde{\Gamma} \frac{1}{(p - k)^4} \\
= \frac{1}{\alpha} \int dk \Gamma \left( \dot{A}\gamma^k + i\dot{B}k + \dot{C}\gamma k\gamma_5 - i\dot{D}\gamma_5 \right) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \tilde{\Gamma} \frac{1}{(p - k)^4} \\
= \frac{1}{\alpha} \Gamma \int dk (\dot{A} + \dot{C}\gamma_5) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \Gamma \frac{k^2\gamma k}{(p - k)^4} \\
+ \frac{i}{\alpha} \Gamma \int dk (\dot{B} + \dot{D}\gamma_5) \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \tilde{\Gamma} \frac{k^2}{(p - k)^4} .
\]

Let (6.15) be written as

\[
i \Sigma(p) = \frac{1}{\alpha} \Gamma \Sigma_1(p) \Gamma + \frac{i}{\alpha} \Gamma \Sigma_2(p) \tilde{\Gamma} ,
\]

where

\[
\Sigma_1(p) = \int dk (\dot{A} + \dot{C}\gamma_5) \cdot \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \cdot \frac{k^2\gamma k}{(p - k)^4},
\]

\[
\Sigma_2(p) = \int dk (\dot{B} + \dot{D}\gamma_5) \cdot \frac{1}{(A^2 - C^2)k^2 + B^2 - D^2} \cdot \frac{k^2}{(p - k)^4} .
\]

\( \Sigma_1(p) \) needs to be in a form in which \( \gamma p \) appears explicitly such that \( \dot{A}(p^2) \) and \( \dot{C}(p^2) \) can be easily read off (6.6) by comparing coefficients. Multiply \( \Sigma_1 \) by \( \gamma p \) to obtain

\[
\gamma p \Sigma_1(p) = -\frac{1}{2} \int dk (\dot{A} + \dot{C}\gamma_5) \cdot \frac{k^2}{(A^2 - C^2)k^2 + B^2 - D^2} \left( \frac{1}{(p - k)^2} - \frac{p^2 + k^2}{(p - k)^4} \right) .
\]
Now multiply (6.19) by $\gamma p$, bearing in mind that by (B.3) $(\gamma p)^2 = p^2$, to obtain

$$p^2 \Sigma_1(p) = -\frac{1}{2} \gamma p \int dk \ (\hat{A} + \hat{C} \gamma_5) \cdot \frac{k^2}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left(\frac{1}{(p-k)^2} - \frac{p^2 + k^2}{(p-k)^4}\right). \tag{6.20}$$

In this form the $\gamma p$ term is explicit and (6.16) reads

$$i \Sigma(p) = -\frac{1}{2\alpha p^2} \Gamma \gamma p \int dk \ (\hat{A} + \hat{C} \gamma_5) \cdot \frac{k^2}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left(\frac{1}{(p-k)^2} - \frac{p^2 + k^2}{(p-k)^4}\right) \Gamma
+ \frac{i}{\alpha} \int dk \ (\hat{B} + \hat{D} \gamma_5) \cdot \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{k^2}{(p-k)^4} \tilde{\Gamma}. \tag{6.21}$$

Now substitute (6.21) back into (6.6) to obtain

$$\hat{A} \gamma p - i \hat{B} + \hat{C} \gamma p \gamma_5 - i \hat{D} \gamma_5
= \gamma p + \frac{1}{2\alpha p^2} \Gamma \gamma p \int dk \ (\hat{A} + \hat{C} \gamma_5) \cdot \frac{k^2}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left(\frac{1}{(p-k)^2} - \frac{p^2 + k^2}{(p-k)^4}\right) \Gamma
- \frac{i}{\alpha} \int dk \ (\hat{B} - \hat{D} \gamma_5) \cdot \frac{1}{(A^2 + \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{k^2}{(p-k)^4} \tilde{\Gamma}
= \gamma p + \frac{1}{2\alpha p^2} \Gamma \int dk \ (\hat{A} - \hat{C} \gamma_5) \cdot \frac{k^2}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left(\frac{1}{(p-k)^2} - \frac{p^2 + k^2}{(p-k)^4}\right) \tilde{\Gamma} \gamma p
- \frac{i}{\alpha} \int dk \ (\hat{B} - \hat{D} \gamma_5) \cdot \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{k^2}{(p-k)^4} \tilde{\Gamma}. \tag{6.22}$$

The coefficients of the (linearly-independent) terms 1, $\gamma p$, $\gamma_5$, $\gamma_5 \gamma p$ may be equated to end up with the relations

$$\hat{A}(p^2) = 1 + \frac{1}{2\alpha p^2} \Gamma \int dk \ \hat{A} \cdot \frac{k^2}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left(\frac{1}{(p-k)^2} - \frac{p^2 + k^2}{(p-k)^4}\right) \tilde{\Gamma}, \tag{6.23a}$$

$$\hat{B}(p^2) = \frac{1}{\alpha} \Gamma \int dk \ \hat{B} \cdot \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{k^2}{(p-k)^4} \tilde{\Gamma}, \tag{6.23b}$$

$$\hat{C}(p^2) = \frac{1}{2\alpha p^2} \Gamma \int dk \ \hat{C} \cdot \frac{k^2}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left(\frac{1}{(p-k)^2} - \frac{p^2 + k^2}{(p-k)^4}\right) \tilde{\Gamma}, \tag{6.23c}$$

$$\hat{D}(p^2) = -\frac{1}{\alpha} \Gamma \int dk \ \hat{D} \cdot \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{k^2}{(p-k)^4} \tilde{\Gamma}. \tag{6.23d}$$
6.2. Calculation of integrals over angular degrees of freedom

Write Eqs. (6.23) in terms of four-dimensional polar coordinates:

\begin{align*}
\hat{A}(p^2) &= 1 + \frac{1}{2\alpha p^2 (2\pi)^4} \Gamma \int_0^\infty dk \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\theta_3 k^3 \sin^2 \theta_1 \sin \theta_2 \\
& \quad \times \left( 1 + \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \left( \frac{1}{p^2 + k^2 + 2pk \cos \theta_1} - \frac{p^2 + k^2}{(p^2 + k^2 - 2pk \cos \theta_1)^2} \right) \tilde{\Gamma} \right)
\end{align*}

\begin{align*}
\hat{B}(p^2) &= \frac{1}{\alpha (2\pi)^4} \Gamma \int_0^\infty dk \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\theta_3 k^3 \sin^2 \theta_1 \sin \theta_2 \\
& \quad \times \left( 1 + \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{1}{(p^2 + k^2 - 2pk \cos \theta_1)^2} \tilde{\Gamma} \right)
\end{align*}

\begin{align*}
\hat{C}(p^2) &= \frac{1}{2\alpha p^2 (2\pi)^4} \Gamma \int_0^\infty dk \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\theta_3 k^3 \sin^2 \theta_1 \sin \theta_2 \\
& \quad \times \left( 1 + \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{1}{p^2 + k^2 - 2pk \cos \theta_1} - \frac{p^2 + k^2}{(p^2 + k^2 - 2pk \cos \theta_1)^2} \right) \tilde{\Gamma}
\end{align*}

\begin{align*}
\hat{D}(p^2) &= -\frac{1}{\alpha (2\pi)^4} \Gamma \int_0^\infty dk \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\theta_3 k^3 \sin^2 \theta_1 \sin \theta_2 \\
& \quad \times \left( 1 + \frac{1}{(A^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \frac{1}{p^2 + k^2 - 2pk \cos \theta_1} \right) \tilde{\Gamma}
\end{align*}

The \( \theta_1 \) integrals have the forms

\begin{align*}
I_1 &= \int_0^\pi d\theta_1 \frac{\sin^2 \theta_1}{a - b \cos \theta_1}, \\
I_2 &= \int_0^\pi d\theta_1 \frac{\sin^2 \theta_1}{(a - b \cos \theta_1)^2}
\end{align*}
where \( a = p^2 + k^2 > 0 \) and \( b = 2pk \), such that Eqs. (6.24) read

\[
\hat{A}(p^2) = 1 + \frac{\pi}{\alpha} \frac{1}{(2\pi)^4} \Gamma \int_0^\infty dk^2 k^4 \hat{A} \cdot \frac{1}{(\hat{A}^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left( I_1 - (p^2 + k^2)I_2 \right) \tilde{\Gamma} , \tag{6.27a}
\]

\[
\hat{B}(p^2) = \frac{2\pi}{\alpha} \frac{1}{(2\pi)^4} \Gamma \int_0^\infty dk^2 k^4 \hat{B} \cdot \frac{1}{(\hat{A}^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left( I_1 - (p^2 + k^2)I_2 \right) \tilde{\Gamma} , \tag{6.27b}
\]

\[
\hat{C}(p^2) = \frac{\pi}{\alpha} \frac{1}{(2\pi)^4} \Gamma \int_0^\infty dk^2 k^4 \hat{C} \cdot \frac{1}{(\hat{A}^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} \cdot \left( I_1 - (p^2 + k^2)I_2 \right) \tilde{\Gamma} , \tag{6.27c}
\]

\[
\hat{D}(p^2) = - \frac{2\pi}{\alpha} \frac{1}{(2\pi)^4} \Gamma \int_0^\infty dk^2 k^4 \hat{D} \cdot \frac{1}{(\hat{A}^2 - \hat{C}^2)k^2 + \hat{B}^2 - \hat{D}^2} I_2 \tilde{\Gamma} . \tag{6.27d}
\]

The values of the integrals \( I_1 \) and \( I_2 \) are calculated in Appendix §C. Their final forms are given in Eqs. (C.16) and (C.17). After substituting them in Eqs. (6.27) the resulting expressions are

\[
\hat{A}(p^2) = 1 + \frac{1}{16\pi^2 \alpha^2} \Gamma \int_0^\infty dk^2 k^4 \hat{A} \cdot \left[ \frac{p^2}{k^2(p^2 - k^2)} \theta(k - p) + \frac{k^2}{p^2(k^2 - p^2)} \theta(p - k) \right] \tilde{\Gamma} , \tag{6.28a}
\]

\[
\hat{B}(p^2) = \frac{1}{16\pi^2 \alpha} \Gamma \int_0^\infty dk^2 k^4 \hat{B} \cdot \left[ \frac{1}{k^2(p^2 - k^2)} \theta(k - p) + \frac{1}{p^2(k^2 - p^2)} \theta(p - k) \right] \tilde{\Gamma} , \tag{6.28b}
\]

\[
\hat{C}(p^2) = \frac{1}{16\pi^2 \alpha^2} \Gamma \int_0^\infty dk^2 k^4 \hat{C} \cdot \left[ \frac{p^2}{k^2(p^2 - k^2)} \theta(k - p) + \frac{k^2}{p^2(k^2 - p^2)} \theta(p - k) \right] \tilde{\Gamma} , \tag{6.28c}
\]

\[
\hat{D}(p^2) = - \frac{1}{16\pi^2 \alpha} \Gamma \int_0^\infty dk^2 k^4 \hat{D} \cdot \left[ \frac{1}{k^2(p^2 - k^2)} \theta(k - p) + \frac{1}{p^2(k^2 - p^2)} \theta(p - k) \right] \tilde{\Gamma} . \tag{6.28d}
\]

Clearly the integrands in both expressions diverge at \( k = p \). However, the integration limits will be adjusted to take values that agree with current experimental data, such that this singular point will lie outside of the integration range. This is elucidated in the next paragraph. The next step is to drop the contributions to the loop integrals in Eqs. (6.28) from the region \( k < p \) while retaining just the pieces from the region \( k > p \). This too is justified in the following paragraph. To aid the
discussion below it is useful to expand the remaining terms in $A$ and $B$ in powers of $p^2$ to yield

$$
\hat{A}(p^2) = 1 + \frac{1}{16\pi^2\alpha} \Gamma \int_0^\infty dk^2 \frac{\hat{A}}{(A^2 - \hat{C}^2_k)k^2 + \hat{B}^2 - \hat{D}^2} \left[ 1 - \frac{p^2}{k^2} - \frac{p^4k^4}{k^4} \right] \tilde{\Gamma} + O(p^6) , \quad (6.29a)
$$

$$
\hat{B}(p^2) = \frac{1}{16\pi^2\alpha} \Gamma \int_0^\infty dk^2 \hat{B} \frac{1}{(A^2 - \hat{C}^2_k)k^2 + \hat{B}^2 - \hat{D}^2} \left[ 1 + \frac{p^2}{k^2} + \frac{p^4}{k^4} \right] \tilde{\Gamma} + O(p^6) , \quad (6.29b)
$$

$$
\hat{C}(p^2) = \frac{1}{16\pi^2\alpha} \Gamma \int_0^\infty dk^2 \hat{C} \frac{1}{(A^2 + \hat{C}^2_k)k^2 + \hat{B}^2 - \hat{D}^2} \left[ 1 - \frac{p^2}{k^2} - \frac{p^4k^4}{k^4} \right] \tilde{\Gamma} + O(p^6) , \quad (6.29c)
$$

$$
\hat{D}(p^2) = - \frac{1}{16\pi^2\alpha} \Gamma \int_0^\infty dk^2 \hat{D} \frac{1}{(A^2 - \hat{C}^2_k)k^2 + \hat{B}^2 - \hat{D}^2} \left[ 1 + \frac{p^2}{k^2} + \frac{p^4}{k^4} \right] \tilde{\Gamma} + O(p^6) . \quad (6.29d)
$$

### 6.3. The leading order approximation at $\Lambda \gg \lambda$.

The leading order in $p^2$ contributions to $A$ and $B$ are straightforwardly read off Eqs. (6.29). By inserting a cut-off at the ultra-violet and at the infra-red end of the spectrum, these LO contributions have the forms

$$
\hat{A}_0 = 1 - \frac{1}{16\pi^2\alpha} \Gamma \int_{\lambda^2}^{A^2} dk^2 \hat{A}_0 \frac{1}{(A^2_0 - \hat{C}^2_0)k^2 + \hat{B}^2_0 - \hat{D}^2_0} \tilde{\Gamma} , \quad (6.30a)
$$

$$
\hat{B}_0 = \frac{1}{16\pi^2\alpha} \Gamma \int_{\lambda^2}^{A^2} dk^2 \hat{B}_0 \frac{1}{(A^2_0 - \hat{C}^2_0)k^2 + \hat{B}^2_0 - \hat{D}^2_0} \tilde{\Gamma} , \quad (6.30b)
$$

$$
\hat{C}_0 = - \frac{1}{16\pi^2\alpha} \Gamma \int_{\lambda^2}^{A^2} dk^2 \hat{C}_0 \frac{1}{(A^2_0 - \hat{C}^2_0)k^2 + \hat{B}^2_0 - \hat{D}^2_0} \tilde{\Gamma} , \quad (6.30c)
$$

$$
\hat{D}_0 = - \frac{1}{16\pi^2\alpha} \Gamma \int_{\lambda^2}^{A^2} dk^2 \hat{D}_0 \frac{1}{(A^2_0 - \hat{C}^2_0)k^2 + \hat{B}^2_0 - \hat{D}^2_0} \tilde{\Gamma} . \quad (6.30d)
$$

Let

$$
\hat{B}_0^2 - \hat{D}_0^2 = M^2(\hat{A}_0^2 - \hat{C}_0^2) . \quad (6.31)
$$

Then Eqs. (6.30) reduce to

$$
\hat{A}_0 = 1 - \frac{1}{16\pi^2\alpha} \Gamma \hat{A}_0 \int_{\lambda^2}^{A^2} dk^2 \left[ (k^2 + M^2)(\hat{A}_0^2 - \hat{C}_0^2) \right]^{-1} \tilde{\Gamma} , \quad (6.32a)
$$

$$
\hat{B}_0 = \frac{1}{16\pi^2\alpha} \Gamma \hat{B}_0 \int_{\lambda^2}^{A^2} dk^2 \left[ (k^2 + M^2)(\hat{A}_0^2 - \hat{C}_0^2) \right]^{-1} \tilde{\Gamma} , \quad (6.32b)
$$

$$
\hat{C}_0 = - \frac{1}{16\pi^2\alpha} \Gamma \hat{C}_0 \int_{\lambda^2}^{A^2} dk^2 \left[ (k^2 + M^2)(\hat{A}_0^2 - \hat{C}_0^2) \right]^{-1} \tilde{\Gamma} , \quad (6.32c)
$$

$$
\hat{D}_0 = - \frac{1}{16\pi^2\alpha} \Gamma \hat{D}_0 \int_{\lambda^2}^{A^2} dk^2 \left[ (k^2 + M^2)(\hat{A}_0^2 - \hat{C}_0^2) \right]^{-1} \tilde{\Gamma} . \quad (6.32d)
$$
Note that \[ (k^2 + M^2)(\hat{A}_0^2 - \hat{C}_0^2) \]^{-1} = (\hat{A}_0^2 - \hat{C}_0^2)^{-1}(k^2 + M^2)^{-1}. Hence Eqs. (6.32) can be written as

\[ \hat{A}_0 = 1 - \frac{1}{16\pi^2\alpha} \Gamma \hat{A}_0 \frac{1}{\hat{A}_0^2 - \hat{C}_0^2} \int_{\Lambda^2} dk^2 \frac{1}{k^2 + M^2} \tilde{\Gamma}, \]  

(6.33a)

\[ \hat{B}_0 = \frac{1}{16\pi^2\alpha} \Gamma \hat{B}_0 \frac{1}{\hat{A}_0^2 - \hat{C}_0^2} \int_{\Lambda^2} dk^2 \frac{1}{k^2 + M^2} \tilde{\Gamma}, \]  

(6.33b)

\[ \hat{C}_0 = - \frac{1}{16\pi^2\alpha} \Gamma \hat{C}_0 \frac{1}{\hat{A}_0^2 - \hat{C}_0^2} \int_{\Lambda^2} dk^2 \frac{1}{k^2 + M^2} \tilde{\Gamma}, \]  

(6.33c)

\[ \hat{D}_0 = - \frac{1}{16\pi^2\alpha} \Gamma \hat{D}_0 \frac{1}{\hat{A}_0^2 - \hat{C}_0^2} \int_{\Lambda^2} dk^2 \frac{1}{k^2 + M^2} \tilde{\Gamma}. \]  

(6.33d)

From (6.33b),

\[ \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0 = \frac{1}{16\pi^2\alpha} \frac{1}{\hat{A}_0^2 - \hat{C}_0^2} \int_{\Lambda^2} dk^2 k^4 \frac{1}{k^2 + M^2} \tilde{\Gamma}. \]  

(6.34)

Therefore, (6.34) may be substituted in Eqs. (6.33) to yield

\[ \hat{A}_0 = 1 - \Gamma \hat{A}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0, \]  

(6.35a)

\[ \hat{B}_0 = \Gamma \hat{B}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0, \]  

(6.35b)

\[ \hat{C}_0 = - \Gamma \hat{C}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0, \]  

(6.35c)

\[ \hat{D}_0 = - \Gamma \hat{D}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0. \]  

(6.35d)

The commutation relations (6.9) are assumed to hold at each order of \( p^2 \), hence they apply also for \( \{\hat{A}_0, \hat{B}_0, \hat{C}_0, \hat{D}_0\} \). And if \([M, N] = 0\) for matrices \( M, N \in \mathbb{C}^{m \times n} \), then it can be shown that \([M, N^{-1}] = [M^{-1}, N] = [M^{-1}, N^{-1}] = 0\). From this \( \hat{A}_0, \hat{B}_0^{-1} \) can be swapped in Eqs. (6.35), as well as \( \hat{C}_0, \hat{B}_0^{-1} \) and \( \hat{D}_0, \hat{B}_0^{-1} \). Hence Eqs. (6.35) can be reduced to

\[ \hat{A}_0 = 1 - \Gamma \hat{A}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0, \]  

(6.36a)

\[ \hat{B}_0 = \Gamma \hat{B}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0, \]  

(6.36b)

\[ \hat{C}_0 = - \Gamma \hat{C}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0, \]  

(6.36c)

\[ \hat{D}_0 = - \Gamma \hat{D}_0 \hat{B}_0^{-1} \Gamma^{-1} \hat{B}_0. \]  

(6.36d)

The simplest solution is when all of the \( \{\hat{A}_0, \hat{B}_0, \hat{C}_0, \hat{D}_0\} \) are diagonal. Then, given that \( \Gamma, \tilde{\Gamma} \) are
diagonal as well, Eqs. (6.36) reduce to

\[ \hat{A}_0 = 1 - \frac{1}{16\pi^2\alpha} \Gamma \hat{A}_0 \int_\lambda^2 d\lambda 1 \frac{1}{k^2 + M^2} \Gamma, \quad (6.37a) \]

\[ \mathbf{1}_{2\times2} = \frac{1}{16\pi^2\alpha} \Gamma \frac{1}{A_0^2} \int_\lambda^2 d\lambda 1 \frac{1}{k^2 + M^2} \Gamma, \quad (6.37b) \]

\[ \hat{C}_0 = 0, \quad (6.37c) \]

\[ \hat{D}_0 = 0. \quad (6.37d) \]

In relation to this solution Eq. (6.31) implies

\[ \hat{B}_0 = M \hat{A}_0. \quad (6.38) \]

Eqs. (6.37a) and (6.37b) are 2 × 2 matrix equations, with completely diagonal matrices on both sides. Importantly, by constraining \( \hat{A}_0 \) and \( \hat{B}_0 \) to be diagonal, then consistent with (6.31) the matrix \( M^2 \) is forced to be diagonal. In light of this let it be written as

\[ M = \begin{pmatrix} m_t & 0 \\ 0 & m_b \end{pmatrix}. \quad (6.39) \]

\( M \) will be referred to as the mass matrix. Then

\[ (k^2 + M^2)^{-1} = \begin{pmatrix} \frac{1}{k^2 + m_t^2} & 0 \\ 0 & \frac{1}{k^2 + m_b^2} \end{pmatrix}. \quad (6.40) \]

According to the definitions in (5.15) and (6.13),

\[ \Gamma = \begin{pmatrix} \beta_L P_R + \beta_t P_L & 0 \\ 0 & \beta_L P_R + \beta_b P_L \end{pmatrix}, \quad (6.41) \]

\[ \tilde{\Gamma} = \begin{pmatrix} \beta_L P_L + \beta_t P_R & 0 \\ 0 & \beta_L P_L + \beta_b P_R \end{pmatrix}. \quad (6.42) \]

Let \( \hat{A}_0, \hat{B}_0 \) be written in a notation more fitting with these relations:

\[ \hat{A}_0 = \begin{pmatrix} A_t & 0 \\ 0 & A_b \end{pmatrix}, \quad (6.43) \]

and

\[ \hat{B}_0 = \begin{pmatrix} B_t & 0 \\ 0 & B_b \end{pmatrix} = \begin{pmatrix} m_t A_t & 0 \\ 0 & m_b A_b \end{pmatrix}, \quad (6.44) \]
where the last equality follows directly from Eqs. (6.38), (6.39) and (6.43). Altogether the two diagonal elements of the matrix equation in Eq. (6.37a) are

\[
A_t = 1 - \frac{1}{16\pi^2\alpha} \beta_L \beta_t A_t \int_{\lambda^2}^{A^2} \frac{d\lambda^2}{k^2 + m_t^2},
\]

(6.45a)

\[
A_b = 1 - \frac{1}{16\pi^2\alpha} \beta_L \beta_b A_b \int_{\lambda^2}^{A^2} \frac{d\lambda^2}{k^2 + m_b^2},
\]

(6.45b)

while the two diagonal elements of the matrix equation in Eq. (6.37b) are

\[
1 = \frac{1}{16\pi^2\alpha} \beta_L \beta_t A_t \int_{\lambda^2}^{A^2} \frac{d\lambda^2}{k^2 + m_t^2},
\]

(6.46a)

\[
1 = \frac{1}{16\pi^2\alpha} \beta_L \beta_b A_b \int_{\lambda^2}^{A^2} \frac{d\lambda^2}{k^2 + m_b^2}.
\]

(6.46b)

Eq. (6.46a) may be substituted into (6.45a), to obtain

\[
A_t = 1 - A_t \Rightarrow A_t = \frac{1}{2},
\]

(6.47)

Now substitute (6.47) in (6.46a) to find

\[
1 = \frac{1}{4\pi^2\alpha_t} \ln \left( \frac{A^2 + m_t^2}{\lambda^2 + m_t^2} \right),
\]

(6.48)

where

\[
\alpha_t = \frac{\alpha}{\beta_L \beta_t}.
\]

(6.49)

Equally Eq. (6.46b) may be substituted into (6.45b) to obtain

\[
A_b = 1 - A_b \Rightarrow A_b = \frac{1}{2},
\]

(6.50)

which may be substituted in (6.46b) to yield

\[
1 = \frac{1}{4\pi^2\alpha_b} \ln \left( \frac{A^2 + m_b^2}{\lambda^2 + m_b^2} \right),
\]

(6.51)

where

\[
\alpha_b = \frac{\alpha}{\beta_L \beta_b}.
\]

(6.52)

We assume that the ultraviolet cutoff is at the scale of Plank mass while the infrared scale is 1 TeV. Then the critical value of the coupling constant \( \alpha_t \) is given by

\[
\alpha_t^{(c)} = \frac{1}{4\pi^2} \ln \left( \frac{A^2}{\lambda^2} \right) \sim 1.866405176474873.
\]

(6.53)

At \( \alpha_t < \alpha_t^{(c)} \) the top quark mass is generated dynamically, with the value

\[
m_t = \Lambda e^{-2\pi^2\alpha_t} \sqrt{1 - e^{4\pi^2(\alpha_t - \alpha_t^{(c)})}}.
\]

(6.54)
while for \( \alpha_b < \alpha^{(c)} \) the \( b \) quark mass is generated dynamically, with the value

\[
m_b = \Lambda e^{-2\pi^2\alpha_b} \sqrt{1 - e^{4\pi^2(\alpha_b - \alpha^{(c)})}}.
\]  

(6.55)

It follows from (6.54) that the generated top mass varies from 0 at \( \alpha_t = \alpha^{(c)} \) to a value that approaches \( \Lambda \) at strong coupling \( 1/\alpha_t \gg 1 \). The top quark mass, which is approximately 175 GeV, is generated for

\[
\alpha_t = \frac{1}{4\pi^2} \log \frac{A^2 + m_t^2}{\lambda^2 + m_t^2} = 1.865641077603585.
\]  

(6.56)

In order to generate the observable bottom quark mass \( m_b = 4.18 \) GeV we need

\[
\alpha_b = \frac{1}{4\pi^2} \log \frac{A^2 + m_b^2}{\lambda^2 + m_b^2} = 1.86640733897677.
\]  

(6.57)

Without loss of generality we can set \( \beta_L = 1/2 \). Then the original coupling constants to be set up are \( \alpha, \beta_b, \beta_t \). The following choice of their values leads to the observable values of top quark and bottom quark masses:

\[
\beta_L = \alpha = 1/2; \quad \beta_b = 0.53578946829590691109, \quad \beta_t = 0.53600878111265624799.
\]

At \( \beta_L = \alpha = 1/2 \) the critical values of \( \beta_b, \beta_t \) corresponding to nearly vanishing values of masses are

\[
\beta^{(c)}_L = \alpha^{(c)} = 1/2; \quad \beta^{(c)}_b = \beta^{(c)}_t = 0.53578934124514456055.
\]

It is immediately clear that the above adjustment of the couplings is a certain kind of fine tuning, given that the values of the couplings that provide physical values of the quark masses are very close to their critical values. In fact, the value of \( \beta_b \) differs from the value of \( \beta^{(c)}_b \) by an order of \( 10^{-7} \).

As a consistency check, take the integrals in (6.28a) and (6.28b) but with \( C = D = 0 \), and with the cutoffs \( k^2 = \lambda^2 \) at the lower end and \( k^2 = A^2 \) at the upper end of the spectrum, where recall that \( \lambda \gg p \) is assumed such that only the \( k > p \) contribution survives. Then substitute for \( A \) and \( B \) inside the integrands, the leading-order values found in (6.47) and (6.54). Select the values \( \Lambda = 10^{19} \) Gev (the Planck mass), \( \lambda = 1 \) TeV, \( m_t = 175 \) GeV and \( \alpha_t = 1.865641077603585 \) found in (6.56), then evaluate the integrals over \( k^2 \). In this regime, the integrals as functions of \( p \) labeled as \( \tilde{A}_t(p) \) and \( \tilde{B}_t(p) \), have the forms

\[
\tilde{A}_t(p) = 1 + \frac{1}{16\pi^2\alpha_t} \int_{\lambda^2}^{A^2} dk^2 \frac{A_{t0}k^4}{A_{t0}^2k^2 + B_{t0}^2(p^2 - k^2)},
\]  

(6.58)

\[
\tilde{B}_t(p) = \frac{1}{16\pi^2\alpha_t} \int_{\lambda^2}^{A^2} dk^2 \frac{B_{t0}k^2}{A_{t0}^2k^2 + B_{t0}^2(k^2 - p^2)},
\]  

(6.59)

where in Eqs. (6.58) and (6.59), \( A_{t0} = 1/2 \) and \( B_{t0} = m_t/2 = 87.5 \) GeV should be inserted. The values for \( \tilde{A}_t(p) \) and \( \tilde{B}_t(p)/\tilde{A}_t(p) \) as functions of \( p \) are shown in the plots in Figs 3 and Fig. 4. The values of
FIG. 3: The function $\tilde{A}_t(p)$ found by evaluating the integral in Eq. (6.27a) for the values $\Lambda = 10^{19}$ Gev (the Planck mass), $\lambda = 1$ TeV, $m_t = 175$ GeV and $\alpha_t = 1.86564$. The values of $\tilde{A}_t(p)$ are very close to $A_0 = \frac{1}{2}$ as expected.

FIG. 4: The function $\tilde{B}_t(p)/\tilde{A}_t(p)$ found by evaluating the integral in (6.27b) with the same numerical parameters as Fig. 3. The values of $\tilde{B}_t(p)/\tilde{A}_t(p)$ are very close to $m_t = 175$ GeV, as expected.

The function $\tilde{A}_t(p)$ are very close to $A_{t0} = \frac{1}{2}$ as expected, while the values of $\tilde{B}_t(p)/\tilde{A}_t(p)$ are very close to $m_t = 175$ GeV, also as expected. Now we generate analogous plots for $A_b$ and $B_b$. We take the integrals in (6.28a) and (6.28b) with $C = D = 0$, and cutoffs $k^2 = \lambda^2$ at the lower end and $k^2 = \Lambda^2$ at the upper end of the spectrum, but this time we substitute for $A$ and $B$, the leading-order values found in (6.50) and (6.55), with the same values for $\Lambda$ and $\lambda$, $m_b = 4.18$ GeV and $\alpha_b = 1.866404733897677$ as found in (6.57), then evaluate the integrals over $k^2$. In this regime, the integrals as functions of $p$ labeled as $\tilde{A}_b(p)$ and $\tilde{B}_b(p)$, have the forms

$$\tilde{A}_b(p) = 1 + \frac{1}{16\pi^2\alpha_b} \int_{\lambda^2}^{\Lambda^2} dk^2 \frac{A_{b0}k^4}{A_{b0}^2k^2 + B_{b0}^2} \frac{1}{(p^2 - k^2)},$$

$$\tilde{B}_b(p) = \frac{1}{16\pi^2\alpha_b} \int_{\lambda^2}^{\Lambda^2} dk^2 \frac{B_{b0}k^2}{A_{b0}^2k^2 + B_{b0}^2} \frac{1}{(k^2 - p^2)},$$

where in Eqs. (6.58) and (6.59), $A_{b0} = \frac{1}{2}$ and $B_{b0} = \frac{m_b}{2} = 2.09$ GeV should be inserted. The values for $\tilde{A}_b(p)$ and $\tilde{B}_b(p)/\tilde{A}_b(p)$ as functions of $p$ are shown in the plots in Figs 5 and Fig. 6. The values of $\tilde{A}_b(p)$ are very close to $A_{b0} = \frac{1}{2}$ as expected, while the values of $\tilde{B}_b(p)/\tilde{A}_b(p)$ are very close to $m_b = 4.18$ GeV, also as expected.
FIG. 5: The function $\tilde{A}_t(p)$ found by evaluating the integral in Eq. (6.27a) for the values $A = 10^{19}$ Gev (the Planck mass), $\lambda = 1$ TeV, $m_b = 4.18$ GeV and $\alpha_b = 1.86640473897677$. The values of $\tilde{A}_b(p)$ are very close to $A_{b0} = \frac{1}{2}$ as expected.

FIG. 6: The function $\tilde{B}_b(p)/\tilde{A}_b(p)$ found by evaluating the integral in (6.27b) with the same numerical parameters as Fig. 5. The values of $\tilde{B}_b(p)/\tilde{A}_b(p)$ are very close to $m_b = 4.18$ GeV, as expected.

7. CONCLUSIONS AND DISCUSSION

In this paper we have extended our previous model [4], which contained a single fermion, namely the top quark, to that comprising both top and bottom quarks with distinct masses. In particular we have inserted couplings into the action in a way that allows for distinguishable masses between the top and bottom quarks. This form of the action leads to mass-gap equations bearing solutions corresponding to non-equal masses of top and bottom quarks. By suitably adjusting the couplings, our model yields predicted masses for the top and bottom quarks that match experimental observations. In the particular case where the ultraviolet cutoff is set to the Plank mass, while the infrared cutoff of the theory is around 1 TeV, the gap equations can be brought to the simple form of (6.48) and (6.51). Then the above adjustment of the couplings is a form of fine tuning, because the critical values of the coupling constants are very close to those specific values that give rise to physical masses. This indicates that the two cutoffs should in reality be close to each other, i.e. the UV completion of the proposed theory should enter the game at an energy much lower than the Plank mass. In this situation the approximation giving rise to Eqs. (6.48) and (6.51) does not work, and we should deal
directly with the more complicated form of the gap equations, namely Eqs. \[6.28\]. Nonetheless, our solution to these equations at extreme values of $\Lambda$, give a reasonably good description of the phenomena.

Concerning future prospects for the development of our proposed theory, there are several questions yet to be answered, as detailed below.

1. The first question concerns the accuracy of our calculations. We use the truncated Schwinger - Dyson equations to calculate the quark masses. We have seen that the values of the coupling constants needed to generate the observable values of quark masses (at least, at $\Lambda \sim m_P$) are of the order of 0.5. Therefore, our model cannot be used in the weak coupling regime. In this case, any application of the truncated Schwinger - Dyson equations will be limited. Nevertheless it is generally accepted that even at non - small values of couplings, these equations in the rainbow approximation give a reasonable approximation to certain physical quantities. Still this should be checked using a more refined scheme of calculations. Feasibly lattice simulations could be used to check this hypothesis.

2. In this paper we concentrate on the calculation of the top and bottom quark masses in the toy model constructed here. In actual fact this model predicts appearance of not just the standard model composite Higgs but also several scalar excitations in addition. The former Higgs boson mass should equal 125 GeV as observed. At the same time the masses of the remaining composite scalar bosons should be larger than present experimental bounds. Those experimental bounds also include values of the decay constants of the scalar bosons. Thus, it is necessary to calculate these decay constants as well as the extra Higgs boson masses. For this purpose, the Bethe - Salpeter equations for the corresponding excitations should be solved. This is beyond the scope of this paper but will be addressed in a follow-up paper.

3. The theory put forward here inclusive of fundamental scalar fields admits two types of divergences: ultraviolet and infrared. Correspondingly, we insert two cutoffs. The physical interpretation of these cut-offs is that the theory is an approximation to a more complicated one, in which the presence of zero dimension scalar bosons is suppressed at energies below $\lambda$, while at the energies above $\Lambda$ extra excitations are present. Since our model originates from Riemann - Cartan gravity coupled to SM fermions non - minimally, we assume that the UV completion of the proposed model has to be an extension of Riemann - Cartan gravity. In the same extended model, the vielbein and spin connection have extra indices, and as such become matrices in flavor space. The construction of such a theory is an important task, that while beyond the scope of this work, will be addressed in our next paper.
The authors are grateful to G.E. Volovik for useful discussions. M.A.Z. is indebted to V.A. Miransky for numerous discussions in the past on the Schwinger - Dyson equations technique.

Appendix A: Conventions for Dirac matrices and chiral operators

1. Dirac matrices

Throughout these notes the representation of [126] (see p.41) is used for the Dirac matrices, often called the Chiral representation. The Dirac matrices matrices have the form

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (A.1) \]

\( i = 1, 2, 3 \) where \( \sigma^i \) are the Pauli spin matrices,

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (A.2) \]

which have the property of all of them being Hermitian: \( \sigma^i \dagger = \sigma^i \). Note the identities

\[ (\sigma^i)^2 = 1, \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}, \quad \sigma^i \sigma^j = i\epsilon^{ijk} \sigma_k + \delta^{ij} \sigma_i \sigma_j, \quad [\sigma^i, \sigma^j] = 2i\epsilon^{ijk} \sigma_k, \quad (A.3) \]

from which the Dirac matrices are straightforwardly shown to satisfy the Clifford algebra

\[ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (A.4) \]

where \( \eta^{\mu\nu} \) is the Minkowski metric, which in the convention used here is assumed to be “mostly positive”:

\[ \eta^{\mu\nu} = \text{diag} (1, -1, -1, -1) \quad (A.5) \]

It follows from the representation in \((A.1)\) and the Hermitian property of the \( \sigma^i \) that

\[ \gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i, \quad (A.6) \]

which equally can be expressed as

\[ \gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0. \quad (A.7) \]

The matrix usually referred to as \( \gamma_5 \) is defined as

\[ \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad (A.8) \]
or equivalently, taking into account the effect of the anticommutation relation in (A.4) on re-ordering of $\gamma$ matrices,

$$\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma,$$

(A.9)

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally-antisymmetric Levi-Civita symbol with four indices. Note that indices are raised and lowered by contracting with the Minkowski metric, with the effect that $\epsilon_{0123} = -\epsilon^{0123}$. In the convention used here,

$$\gamma_5 = \begin{pmatrix} -\mathbb{1}_{2\times2} & 0 \\ 0 & \mathbb{1}_{2\times2} \end{pmatrix}.$$

(A.10)

It is easily verified by using (A.4), that

$$(\gamma_5)^2 = 1.$$  

(A.11)

From the relation in (A.6) or (A.7) for obtaining the Hermitian conjugate of the $\gamma^\mu$,

$$\gamma_5^\dagger = \gamma_5,$$

(A.12)

and from the Clifford algebra it is easily verified that

$$\{\gamma_5, \gamma^\mu\} = 0.$$  

(A.13)

Note that these identities for $\gamma_5$ are derived independent of the explicit form of $\gamma_5$, it is nonetheless useful to write down it’s explicit form in the representation used here:

2. Chiral projection operators

The chiral projection operators are $4 \times 4$ matrices defined as

$$P_R = \frac{1}{2}(1 + \gamma_5) = P_R^\dagger, \quad P_L = \frac{1}{2}(1 - \gamma_5) = P_L^\dagger,$$

(A.14)

where the indices $R$ and $L$ refer to right-handed and left-handed. In the convention of these notes, according to (A.10),

$$P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{2\times2} \end{pmatrix}, \quad P_L = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & 0 \end{pmatrix}.$$  

(A.15)

$P_R$ and $P_L$ satisfy a completeness relation,

$$P_R + P_L = 1,$$

(A.16)

are idempotent, i.e.,

$$P_R^2 = P_R, \quad P_L^2 = P_L,$$

(A.17)
and respect the orthogonality relations

\[ P_R P_L = P_L P_R = 0. \tag{A.18} \]

The combined properties of Eqs. (A.16) – (A.18) guarantee that \( P_R \) and \( P_L \) are indeed projection operators which project from the Dirac field variable \( \psi \) to its chiral components \( \psi_R \) and \( \psi_L \), defined as

\[ \psi_R \equiv P_R \psi, \quad \psi_L \equiv P_L \psi. \tag{A.19} \]

For the goal of analyzing the symmetry of an action with respect to independent global transformations of the left- and right-handed fields, the identities below are required.

\[ \bar{\psi} \Gamma_i \psi = \begin{cases} 
\bar{\psi}_R \Gamma_1 \psi_R + \bar{\psi}_L \Gamma_1 \psi_L & \text{for } \Gamma_1 \in \{\gamma^\mu, \gamma^\mu \gamma_5\} \\
\bar{\psi}_R \Gamma_2 \psi_L + \bar{\psi}_L \Gamma_2 \psi_R & \text{for } \Gamma_2 \in \{1, \gamma_5, \gamma[^{\mu,\nu}]\} 
\end{cases}, \tag{A.20} \]

where

\[ \bar{\psi}_R = \bar{\psi} P_L, \quad \bar{\psi}_L = \bar{\psi} P_R. \tag{A.21} \]

Equation (A.20) is easily proven by inserting the completeness relation of Eq. (A.16) both to the left and the right of \( \Gamma_i \):

\[ \bar{\psi} \Gamma_i q = \bar{\psi}(P_R + P_L) \Gamma_i (P_R + P_L) \psi, \tag{A.22} \]

and by noting \( \{\Gamma_1, \gamma_5\} = 0 \) and \([\Gamma_2, \gamma_5] = 0\). Together with the orthogonality relations of Eq. (A.18) it is then obtained

\[ P_R \Gamma_1 P_R = \Gamma_1 P_L P_R = 0, \quad P_R \Gamma_1 P_L = P_R^2 \Gamma_1 = P_R \Gamma_1, \quad P_L \Gamma_1 P_R = P_L^2 \Gamma_1 = P_L \Gamma_1, \tag{A.23} \]

and similarly

\[ P_L \Gamma_1 P_L = 0, \quad P_R \Gamma_2 P_L = 0, \quad P_L \Gamma_2 P_R = 0. \tag{A.24} \]

Due to (A.13),

\[ \gamma^\mu P_L = P_R \gamma^\mu, \quad \gamma^\mu P_R = P_L \gamma^\mu. \tag{A.25} \]

By (A.11) and (A.14),

\[ \gamma_5 P_L = P_L \gamma_5, \quad \gamma_5 P_R = P_R \gamma_5. \tag{A.26} \]

And it follows from (A.14) and (A.13) that

\[ P_L \equiv P_L^\dagger \gamma^0 = P_L \gamma^0 = \gamma^0 P_R \tag{A.27a} \]
\[ P_R \equiv P_R^\dagger \gamma^0 = P_R \gamma^0 = \gamma^0 P_L \tag{A.27b} \]
Appendix B: Wick rotations

In Minkowski space the $\gamma$ matrices satisfy the Clifford algebra given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu \nu}, \quad (\mu, \nu = 0, 1, 2, 3), \quad (B.1)$$

where $\eta^{\mu \nu} = \text{diag}(1, -1, -1, -1)$. A wick rotation comprises a transformation of the $\gamma$ matrices to new matrices, $\gamma_E$ defined by

$$\gamma_E^4 = \gamma^0, \quad \gamma^i_E = -i\gamma^i, \quad (i = 1, 2, 3) \quad (B.2)$$

such that

$$\{\gamma_E^\mu, \gamma_E^\nu\} = 2\delta^{\mu \nu}, \quad (\mu, \nu = 1, 2, 3, 4). \quad (B.3)$$

Four-vector components get transformed under a Wick rotation to new components. For example given components $k^\mu$ of the four vector $k$ in Minkowski space,

$$k_E^4 = ik^0, \quad k^i_E = k^i, \quad (i = 1, 2, 3). \quad (B.4)$$

The implication is that

$$k^2 = (k^0)^2 - (k^i)^2 = -(k_E^4)^2 - (k_E^1)^2 - (k_E^2)^2 - (k_E^3)^2 \equiv -k_E^2. \quad (B.5)$$

$$\gamma k = \gamma^0 k^0 - \gamma^i k_i = -i\gamma^4_E k^4_E - i\gamma^1_E k^1_E - i\gamma^2_E k^2_E - i\gamma^3_E k^3_E \equiv -i\gamma_E k_E. \quad (B.6)$$

Appendix C: Angular integral

The integrals in (C.1) and (C.4) are solved in this appendix, whose forms are

$$I_1 = \int_0^\pi d\theta \frac{\sin^2 \theta}{(a - b \cos \theta)}, \quad (C.1)$$

$$I_2 = \int_0^\pi d\theta \frac{\sin^2 \theta}{(a - b \cos \theta)^2}, \quad (C.2)$$

where

$$a = p^2 + k^2, \quad b = 2pk. \quad (C.2)$$

They both have the same structure, namely

$$I_n = \int_0^\pi d\theta \frac{\sin^2 \theta}{(a - b \cos \theta)^n}, \quad (C.3)$$

and are evaluated in the same way.
First write it as an integral from 0 to $2\pi$ so that it can be converted into a contour integral along a closed circular path in the complex plane. Since $\cos \theta = \cos(2\pi - \theta)$ and $\sin^2 \theta = \sin^2(2\pi - \theta)$ then $I_n$ can equally be written as

$$I_n = \int_0^{2\pi} d\theta \frac{\sin^2(2\pi - \theta)}{(a - b \cos(2\pi - \theta))^n}$$

$$= \int_\pi^{2\pi} d\xi \frac{\sin^2 \xi}{(a - b \cos \xi)^n} ,$$

such that

$$I_n = \frac{1}{2} \int_0^{2\pi} d\theta \frac{\sin^2 \theta}{(a - b \cos \theta)^n} .$$

Now write this as a closed integral over the variable $z = e^{i\theta}$ in the complex plane, with $\sin \theta = (-i/2)(z - z^{-1})$ and $\cos \theta = (1/2)(z + z^{-1})$:

$$I_n = \frac{i}{8} \oint dz \frac{2^n z^{n-3} (z^2 - 1)^2}{(2az - bz^2 - b)^n} .$$

Write the denominator of the integrand as

$$(2az - bz^2 - b)^n = (-b)^n (z - z_1)^n (z - z_2)^n ,$$

where

$$z_1 = \frac{a + \sqrt{a^2 - b^2}}{b} , \quad z_2 = \frac{a - \sqrt{a^2 - b^2}}{b},$$

to bring the integral into the form

$$I_n = 2^{n-3} i \int dz \frac{z^{n-3} (z^2 - 1)^2}{(-b)^n (z - z_1)^n (z - z_2)^n} .$$

Now the integral has the familiar form $\int dz \frac{f(z)}{(z - z_1)^n(z - z_2)^m}$ where the contour is the unit circle with $f(z)$ analytic and continuous at $z_1$ and $z_2$, and it can be solved by the standard method of summing over the residues of $f(z)$.

Note carefully the location of the poles. From (C.2) it follows that

$$a - b = p^2 + k^2 - 2pk = (p - k)^2 > 0 ,$$

and there it is always true that

$$a > b .$$

Accordingly

$$z_1 = \frac{k}{p} \theta(k - p) + \frac{p}{k} \theta(p - k) ,$$

$$z_2 = \frac{p}{k} \theta(k - p) + \frac{k}{p} \theta(p - k) .$$
In conclusion \(|z_2| < 1\) while \(|z_1| > 1\), so the pole at \(z_1\) is discounted because it is not inside the unit circle. This leaves two poles in the expression (C.9): one at \(z = 0\) and one at \(z = z_2\).

For \(n = 2\) the integral in (C.9), by the Cauchy formula, has the form

\[
I_2 = \frac{i}{2} \left( \frac{(2\pi i)}{b^2} \right) \left[ \left( \frac{(z^2 - 1)^2}{(z - z_1)^2 (z - z_2)^2} \right) \right]_{z=0} + \frac{d}{dz} \left( \frac{(z^2 - 1)^2}{z (z - z_1)^2} \right) \right]_{z=z_2}
\]

\[
= -\frac{\pi}{(2pk)^2} \left[ \frac{1}{(z_1 z_2)^2} - \left( \frac{z_2 - 1}{z_2^2 (z_2 - z_1)^2} \right) - \frac{2 (z_2 - 1)^2}{z_2 (z_2 - z_1)^3} + \frac{4 (z_2 - 1)}{(z_2 - z_1)^2} \right], \quad (C.14)
\]

while for \(n = 1\) it is

\[
I_1 = \frac{i}{4} \left( \frac{(2\pi i)}{(-b)} \right) \left[ \frac{d}{dz} \left( \frac{(z^2 - 1)^2}{(z - z_1) (z - z_2)} \right) \right]_{z=0} + \left( \frac{(z^2 - 1)^2}{z^2 (z - z_1)^2} \right) \right]_{z=z_2}
\]

\[
= \frac{\pi}{4pk} \left[ \frac{z_1 + z_2}{z_1^2 z_2^2} + \left( \frac{z_2^2 - 1}{z_2^2 (z_2 - z_1)} \right) \right]. \quad (C.15)
\]

Finally substitute the explicit forms of \(z_1\) and \(z_2\) to obtain

\[
I_1 = \frac{\pi}{2k^2} \theta(k - p) + \frac{\pi}{2p^2} \theta(p - k), \quad (C.16)
\]

and

\[
I_2 = -\frac{\pi}{2k^2 (p^2 - k^2)} \theta(k - p) - \frac{\pi}{2p^2 (k^2 - p^2)} \theta(p - k). \quad (C.17)
\]

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