New spectral representation and evaluation of $f_\pi$ and the quark condensate $\langle \bar{q}q \rangle$ in the terms of string tension

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Abstract

New spectral representations for $f_\pi$ and chiral condensate are derived in QCD and used for calculations in the large $N_c$ limit. Both quantities are expressed in this limit through string tension $\sigma$ and gluon correlation length $T_g$ without fitting parameters. As a result one obtains $\langle \bar{q}q \rangle = -N_c \sigma^2 T_g a_1$, $f_\pi = \sqrt{N_c \sigma T_g a_2}$, with $a_1 = 0.0823, a_2 = 0.30$. Taking $\sigma = 0.18$GeV$^2$ and $T_g = 1$ GeV$^{-1}$, as known from analytic and lattice calculations, this yields $\langle \bar{q}q \rangle (\mu = 2$GeV$) = -(0.225$GeV$)^3$, $f_\pi = 0.094$ GeV, which is close to the standard values.

1 Introduction

The Chiral Symmetry Breaking (CSB) is known to occur in QCD at large $N_c$, if confinement is preserved in this limit [1]. Lattice calculations for $N_c = 2, 3$ indicate that confinement and CSB coexist in the confinement phase at $T \leq T_c$ and disappear simultaneously above $T_c$ [2]. At larger $N_c$ it was found on the lattice that the $1/N_c$ corrections to all observables studied are not large [3], suggesting that a smooth limit at large $N_c$ is possible.

In the framework of the Field Correlator Method FCM [4] the dynamics of confinement and deconfinement is associated with the set of field correlators $D_{\mu_1 \nu_1, ... \mu_n \nu_n}^{(n)}(x_1, ... x_n) = \langle F_{\mu_1 \nu_1}(x_1) ... F_{\mu_n \nu_n}(x_n) \rangle$ of which the lowest one

\footnote{parallel transporters are here omitted for simplicity}
$D^{(2)}(x_1, x_2) \equiv D^{(2)}(x_1 - x_2)$ plays the dominant role \cite{4}. Moreover, $D^{(2)}(x)$ was calculated on the lattice \cite{3} and its confining part, $D(x)$, was shown to disappear exactly above $T_c$ \cite{7}.

In \cite{8,9} also CSB was found as a consequence of confinement and in \cite{9,10} the Effective Chiral Lagrangian (ECL) was derived from the 4$q$ interaction term using $D(x)$ as a kernel.

The resulting ECL in \cite{9,10} has a general structure which can be reduced to the expressions derived in the framework of the instanton model \cite{11} or the NJL model \cite{12}, when the corresponding kernels are introduced there.

In the case of confinement, the effective quark mass operator $M(x)$ in QCD obtained in \cite{9,10} contains the effect of the scalar confining string connecting the quark to the nearest antiquark. Moreover all invariant quark Green’s functions can be expressed at large $N_c$ through the string spectrum as it was done in \cite{10} in the PS channel.

The phenomenon of CSB was shown in \cite{9,10} as occurring due to the spontaneous creation of the scalar string (similar to the creation of the scalar condensate in nonconfining models \cite{12,13}) which generates CSB and chiral Nambu-Goldstone (NG) fields (see eqs. (50-54) in \cite{9} and eqs. (21-24) in \cite{10}).

Since confinement is present in our formalism (in the form of $M(x)$) one can ask the question how confinement fits in the chiral picture of NG spectrum, and in particular how CSB modifies the lowest PS states computed in FCM (or in any quark model) taking into account confinement and disregarding CSB. Two such lowest states, $\pi(0)$ and its first radial excitation $\pi(1)$ with masses $m(\pi(0)) \equiv m_0 \simeq 0.4$ GeV and $m(\pi(1)) \equiv m_1 \simeq 1.35$ GeV have been computed in FCM, see Appendix 2 below in this paper. It was shown in \cite{10} that the ECL obtained there with account of confinement, has a remarkable property: the PS spectrum of confinement transforms due to CSB in such a way that $\pi(0)$ becomes a NG pion with the mass satisfying Gell-Mann-Oakes-Renner (GOR) relation \cite{13} while the first radial excitation shifts only slightly.

In deriving that property it was essential that all basic quantities in the ECL and in particular the pion self-energy operator can be expressed as a spectral decomposition in the confinement (string-like) spectrum states, which is possible in the large $N_c$ limit.

In this paper we follow this line to obtain a more fundamental relation, namely, to calculate the quark condensate $\langle \bar{q}q \rangle$ and the pion decay constant $f_\pi$ using new spectral representations for these quantities. Since in the latter
all masses and coupling constants are expressed via \( D(x) \), i.e. via the string tension \( \sigma \) and the gluon correlation length \( T_g \), we have an expression for \( \langle \bar{q}q \rangle \) and \( f_\pi \) in terms of \( \sigma \) and \( T_g \). The most important role in the spectral representations is played by the lowest PS meson \( \pi^{(0)} \) – the ”to be pion” – which is the quark model analog of the pion with mass \( m_0 \) shifted by the hyperfine interaction from the \( \rho \)-meson mass. In Appendix 2 we derive the mass \( m_0 \) and the corresponding wave-function in the framework of FCM in terms of \( \sigma \) and \( \alpha_s \).

Having established the connection of \( \langle \bar{q}q \rangle \) and \( f_\pi \) with \( \sigma, T_g \) stated in abstract above, and explained in the text below, it is easy to understand that at the deconfinement transition when \( \sigma \) vanishes at \( T = T_c \), also \( \langle \bar{q}q \rangle \) and \( f_\pi \) vanish in agreement with lattice data [2].

Some specification with respect to the notion of ”magnetic confinement” [14] is needed at this point since magnetic counterpart of \( D(x) \) and the corresponding spacial string tension stay nonzero above \( T_c \). This topic will be studied elsewhere.

The paper is organized as follows. In the next section the ECL is written down together with the appropriate expressions for \( \langle \bar{q}q \rangle \) and \( f_\pi \). In section 3 the spectral representations for these quantities are derived, with coefficients depending on eigenfunctions of the \( \bar{q}q \) system in the pseudoscalar channel. Section 4 is devoted to the discussion of results in comparison to lattice data and to the concluding remarks. Four appendices are included in the paper, containing respectively the evaluation of \( M(0) \), derivation of spectral representation, explicit calculation of eigenvalues and eigenfunctions in the pseudoscalar spectrum, and the contribution of the small-distance region.

## 2 The Effective Chiral Lagrangian

The quadratic part of the ECL for pions was derived in [10] and has the form

\[
W^{(2)}(\phi) = \frac{N_c}{2} \int \phi_a(k)\phi_a(-k)\tilde{N}(k)\frac{d^4k}{(2\pi)^4}
\]

where notations of [10] have been used, \( \phi_a = \frac{2\pi}{f_\pi} \) and

\[
\tilde{N}(k) = \frac{1}{2}[G^{(MM)}(k) + tr(\Lambda M_S)] = (m_\pi^2 + k^2)\frac{f_\pi^2}{4N_c} + O(k^4),
\]
\[ G^{(MM)}(k) \equiv -\int tr(\Lambda(y,x)\gamma_5 M_S(x)\Lambda(x,y)\gamma_5 M_S(y))e^{ik(x-y)}d^4(x-y), \]  
\[ \Lambda(x,y) = (\hat{\partial} + m + M_S)^{-1}. \]  

As it was shown in [10], two terms in the square brackets in (2) cancel for \( k^2 = m = 0 \) and one obtains the GOR relation for the pion mass [13]

\[ m_N tr \Lambda \equiv m|\langle \bar{\psi}\psi \rangle_M| = \frac{1}{2}(m_u + m_d)|\langle \bar{u}u + \bar{d}d \rangle| = m^2 f^2_\pi. \]  

To calculate the quark condensate, defined in the Minkowskian space time, one can write \( \langle \bar{\psi}\psi \rangle_M = -N_c tr \Lambda \) and use identical transformation

\[ tr \Lambda_{xx} = tr(\frac{1}{M_S + m + \hat{\partial}}) = \int \langle \gamma_5 \Lambda(x,y)\gamma_5(M_S+m)\Lambda(y,x) \rangle d^4y \equiv -\int G^{(M)}(x,y)d^4y \equiv -G^{(M)}(k = 0). \]  

Hence \( tr \Lambda_{xx} \) reduces to the zero-momentum component of the \( q\bar{q} \) Green’s function in the PS channel, which differs from (3) only by vertex operators.

To define \( f_\pi \), one needs the first term in the \( k^2 \) expansion of \( G^{(MM)}(k) \) Eq.(3), (c.f. Eq.(2)) so that one has

\[ G^{(MM)}(k) - G^{(MM)}(0) = \frac{k^2 f^2_\pi}{2N_c} + O(k^4) \]  

As it was argued in [10], both \( G^{(MM)}(k) \) and \( G^{(M)}(k) \) have spectral representations in the large \( N_c \) limit, with the same set of poles \( m_n, n = 0, 1, 2, ..., m_0 \equiv m(\pi(0)) \),

\[ G^{(MM)}(k) = -\sum_{n=0}^{\infty} \frac{(c_n^{(M)})^2}{k^2 + m_n^2}, \quad G^{(M)}(k) = -\sum_{n=0}^{\infty} \frac{c_n c_n^{(M)}}{k^2 + m_n^2}. \]  

In the next section we shall determine the coefficients \( c_n, c_n^{(M)} \) and \( m_n \) for the lowest states, and now we define \( \langle \bar{q}q \rangle \) and \( f_\pi \) in terms of spectral sums [8]. From (8) and (7) one has

\[ \langle \bar{\psi}\psi \rangle_M = -2N_c \sum_{n=0}^{\infty} \frac{c_n c_n^{(M)}}{m_n^2}, \quad f^2_\pi = 2N_c \sum_{n=0}^{\infty} \frac{(c_n^{(M)})^2}{m_n^2}. \]
The coefficients $c_n$ and $c_{n}^{(M)}$ differ by the presence of the vertex operator $M_S \equiv M(0)$ in the latter which is a constant computed in Appendix 1, therefore one has $c_{n}^{(M)} = M(0)c_n$, and limiting oneself to the first term in the sum (9) one obtains
\[
|\langle \bar{\psi}\psi \rangle_M| \geq m_0^2 f_\pi^2 \frac{c_n}{c_{n}^{(M)}} = \frac{m_0^2 f_\pi^2}{M(0)}.
\] (10)
Inserting $M(0) = 148$ MeV from Appendix 1 and $|\bar{\psi}\psi| = (225$ MeV$)^3$, $f_\pi = 94$ MeV, one obtains $m_0 \cong 437$ MeV which is close to the value $m_0 = 400$ MeV calculated in Appendix 3. On the other hand, the sum (9) for $\langle \bar{\psi}\psi \rangle$ is converging more slowly than that for $f_\pi$, and therefore one has inequality in (10) due to the presence of higher terms in $\langle \bar{\psi}\psi \rangle$.

3 Calculation of $\langle \bar{q}q \rangle$ and $f_\pi$

The integration region in the space-time integrals in (3), (10) can be split in two parts: $|x - y| > T_g$ and $|x - y| \leq T_g$. In the first (long distance) region the relativistic local potential-type dynamics sets in at space-time distances exceeding $T_g$ [15], [16] and the result can be expressed in terms of the spectrum, as will be done below in this section. The second region can be treated in the OPE formalism [17] and is considered in the Appendix 4. It is shown there that the contribution of this region is parametrically small in the parameter $\sigma T_g^2 \ll 1$. Only the long-distance contribution is calculated in this section below.

We start with the calculation of $f_\pi$ and to this end we write the $q\bar{q}$ Green’s function $G^{(MM)}(k)$ in terms of c.m. and relative coordinates as follows (another derivation is given in Appendix 2):
\[
-G^{(MM)}(k) = \int d^4 X G^{(MM)}(r_{12} = 0; R = 0, r'_{12} = 0, R' = X, T)e^{i kX} =
\]
\[
= M^2(0) \sum_n |\varphi_n(0)|^2 \frac{d^3P}{(2\pi)^3}dT d^3X e^{-E(P)T - iP\cdot X + kX} =
\]
\[
= M^2(0) \sum_n |\varphi_n(0)|^2 \int_0^\infty e^{-E(k)T}dT = M^2(0) \sum_n \frac{|\varphi_n(0)|^2}{\sqrt{m_n^2 + k^2}}.
\] (11)
Expanding \( (11) \) in \( k^2 \) and comparing to \( (7) \) one finds \( f_\pi^2 \),
\[
f_\pi^2 = N_c M^2(0) \sum_{n=0}^{\infty} \frac{|\varphi_n(0)|^2}{m_n^3}.
\] (12)

Comparing \( (9) \) and \( (12) \) one finds
\[
e_n^{(M)} = \sqrt{\frac{m_n}{2}} M(0) \varphi_n(0), \quad c_n = \sqrt{\frac{m_n}{2}} \varphi_n(0).
\] (13)

In a similar way one computes \( \langle \bar{q}q \rangle \) from \( (6) \) and finds
\[
- \langle \bar{q}q \rangle = N_c M(0) \sum_{n=0}^{\infty} \frac{|\varphi_n(0)|^2}{m_n}.
\] (14)

Here \( \varphi_n(r) \) is the 3d spin-singlet wave-function of \( q\bar{q} \) system, as obtained e.g. in the relativistic Hamiltonian method of FCM \[15\], or else in the Bethe-Salpeter equation with the kernel not depending on relative time, as it is discussed in Appendix 2.

The accuracy of the method with respect to calculation of \( |\varphi_n(0)|^2 \) can be checked by comparison of predicted leptonic width with experiment, as it is done in Appendix 3. Taking into account both color Coulomb and confining interaction one has
\[
|\varphi_n(0)|^2 = \frac{\mu_n(\sigma + \frac{4}{3} \alpha_s \frac{1}{\sqrt{\sigma/\mu_n}})}{4\pi}
\] (15)

where \( \mu_n \) is the constituent energy (mass) computed through \( \sigma \) \[15\]; we refer the reader to the Appendix 3 for the details of calculation of \( m_n \) and \( |\varphi_n(0)|^2 \).

As it is shown in \[16\] and discussed in Appendix 2, the masses \( m_n^2, \mu_n^2 \) grow linearly with \( n \) in the large \( N_c \) limit, hence the sum \( (14) \) for \( \langle \bar{q}q \rangle \) is formally diverging if the spectrum of radially excited mesons extends to infinitely large masses. In fact the experimental spectrum can be followed up to the mass values around \( m_{cont} \approx 2.5 \) GeV, where resonances become very wide and strongly mix between themselves and with hybrids, forming the continuum of states. Following the ideology of the QCD sum rules \[17\] one could replace this continuum by the perturbative diagrams, which do not contribute to \( \langle \bar{q}q \rangle \). Therefore we shall keep the first 3 terms in the sum \( (14) \) over \( n \) (the term with \( n = 3 \) gives negligible contribution). As was mentioned in the beginning of this section the relativistic potential description of \( G^{(MM)}(r_{12}, R, 0, r_{12}, R', T) \) is possible only for the time \( T \gtrsim T_0 \), while for
$T < T_0, T_0 \sim T_g$ one should use the properties of the $\bar{q}q$ Green’s function $\tilde{G}$, as given by the OPE \[17\]. As it is discussed in Appendix 4, the region of small times and relative distances covered by the OPE treatment, gives a contribution to $\langle \bar{q}q \rangle$ proportional to $O(\sigma^{5/2}T_0^4/T_g^2, \sigma m)$ and therefore can be disregarded for light quarks and small $T_0 < T_g$. As a result one should exclude from the integration over $dT$ in (11) the region $(0, T_0)$ which brings about the following factor in (14) instead of $1/m_n$

$$1/m_n \to e^{-m_nT_0}/m_n$$  \hspace{1cm} (16)

and in (12)

$$1/m_n^3 \to e^{-m_nT_0}/m_n^3(1 + m_nT_0).$$  \hspace{1cm} (17)

Keeping for $\langle \bar{q}q \rangle$ the first 3 terms in the sum (14) and 2 terms in (12) one has

$$- \langle \bar{q}q \rangle = N_cM(0)\left\{ \frac{\varphi_0^2(0)e^{-m_0T_0}}{m_0} + \frac{\varphi_1^2(0)e^{-m_1T_0}}{m_1} + \frac{\varphi_2^2(0)e^{-m_2T_0}}{m_2} \right\}$$  \hspace{1cm} (18)

$$f_\pi^2 = N_cM(0)\left\{ \frac{\varphi_0^2(0)e^{-m_0T_0}}{m_0^3}(1 + m_0T_0) + \frac{\varphi_1^2(0)e^{-m_1T_0}}{m_1^3}(1 + m_1T_0) \right\}.$$  \hspace{1cm} (19)

Using (A.25), (A.30) one has

$$\varphi_0^2(0) = \frac{0.109\text{GeV}^3}{4\pi}, \quad \varphi_1^2(0) = \frac{0.097\text{GeV}^3}{4\pi}, \quad \varphi_2^2(0) = \frac{0.115\text{GeV}^3}{4\pi}.$$  \hspace{1cm} (20)

$m_0 = 0.4\text{GeV}, \quad m_1 = 1.35\text{GeV}, \quad m_2 = 1.85\text{GeV}.$

For a reasonable estimate we put $T_0 = T_g = 1\text{GeV}^{-1}$ and the value $M(0) = 0.148\text{ GeV}$ from Appendix 1, and obtain.

$$- \langle \bar{q}q \rangle = (0.195\text{GeV})^3, \quad f_\pi = 0.094\text{GeV}.$$  \hspace{1cm} (21)

One can check that the behaviour of (18) for $\langle \bar{q}q \rangle$ at small $T_0$ is smooth, e.g. when changing $T_0$ from $T_g = 1\text{ GeV}^{-1}$ to 0.5 $T_g$, the result changes by roughly 10%.

To check the sensitivity to the change of $T_g$, we have taken $T_g = 1/1.5\text{ GeV}^{-1}$ and recalculated all quantities, e.g. from (A.7) one has $M(0) = 0.12\text{ GeV}$. The resulting values are not much changed from (21),

$$- \langle \bar{q}q \rangle(T_g = \frac{1}{1.5}\text{GeV}^{-1}) = (0.189\text{GeV})^3, \quad f_\pi = 0.076\text{GeV}.$$  \hspace{1cm} (22)
It is remarkable that \( f_\pi \) in (21) is very close to the value obtained from the pion decay and used in the chiral perturbation theory \[ f_\pi = 93 \text{ MeV} \] and we discuss in the concluding section the scale dependence and comparison to existing lattice measurements.

4 Discussion and conclusions

The quark condensate and \( f_\pi \) are given by Eqs. (18), (19) and (22), where all quantities can be expressed through \( m, \sigma \) and \( T_g \), since \( \varphi_n^2(0), m_n \) and \( M(0) \) are expressed through these quantities, while \( T_0 \) can be taken in the region of plateau and e.g. equal to \( T_g \). In this way one obtains (\( m = 0, \sigma = 0.18 \text{ GeV}^2, \ T_0 = T_g = 1 \text{ GeV}^{-1} \), and Eq. (22) for \( \langle \bar{q}q \rangle \))

\[
\begin{align*}
 f_\pi &\approx 0.094 \text{GeV}, \quad -\langle \bar{q}q \rangle \approx (0.20 \text{GeV})^3. \\
\end{align*}
\]

Several corrections should be added to this result. First of all, the short distance contribution to \( \langle \bar{q}q \rangle \) is of relative order \( \sqrt{\sigma T_g} \sim 0.45 \) and can substantially increase the result. Another essential point is the value of \( T_g \), which increases in the presence of dynamical quarks, and can be smaller if gluelump data \[19\] are instead taken into account, \( T_g = 0.7 \text{ GeV}^{-1} \). This influences significantly the value of \( M(0) \), however an independent check can be made since \( M(0) \) also enters the strong decay matrix element, and the value \( M(0) = 0.148 \text{ GeV} \) is reasonably close to the phenomenological value known from the \( ^3P_0 \) model \[20\].

We are now in position to compare (23) with the lattice data. There the computation was done in the quenched case for Wilson fermions \[21\] and also for the overlap action \[22\]. Before using the evaluation coefficient for \( \langle \bar{q}q \rangle \), one can compare the result (23) which does not contain any scale \( \mu \), and any evolution corrections, with the so-called Renormalization Group Invariant (RGI) lattice measurements, which yield \[21\]

\[ -\langle \bar{q}q \rangle_{\text{lat}}^{\text{RGI}} = [(206 \pm 44 \pm 8 \pm 5) \text{ MeV}]^3. \]

This value is in reasonable agreement with (23). As the next step we take the evolution coefficient for \( \langle \bar{q}q \rangle \) computed in \[23\] \( (n_f = 0, N_c = 3) \)

\[
C_s^{\overline{MS}}(\mu) = [\alpha_s(\mu)]^{-4/11} \{1 - 0.219\alpha_s - 0.1054\alpha_s^2 \}. \]

\[25\]
For $\mu = 2$ GeV taking $\alpha_s \approx 0.3$, and identifying $\langle \bar{q}q \rangle$ in (23) with $\langle \bar{q}q \rangle^\text{RGI}$ one obtains for the long-distance contribution to the condensate

$$\langle \bar{q}q \rangle(\mu = 2\text{GeV}) \cong \langle \bar{q}q \rangle^\text{RGI} C_s^\text{MS} \cong - (225 \text{MeV})^3.$$  

(26)

This value, given in the abstract of the paper, is obtained without inclusion of the coefficient used on the lattice [21] to calculate the transition from the lattice RGI result to the $\overline{\text{MS}}$ scheme, this coefficient is anyhow close to unity.

The lattice value at $\mu = 2$ GeV for Wilson quarks in [20]

$$\langle \bar{q}q \rangle^\text{MS}(\mu = 2\text{GeV}) = -[(242 \pm 9)\text{MeV}]^3$$  

(27)

and differs from the result [22]: $-(282 (6) \text{MeV})^3 \left( \frac{a^{-1}}{1766\text{MeV}} \right)^3$. An independent estimate from the QCD sum rules yields [24]

$$\langle \bar{q}q \rangle(\mu = M_N) = -[(225 \pm 9)\text{MeV}]^3.$$  

(28)

As a result one can see that our long-distance contribution to $\langle \bar{q}q \rangle$, Eq. (26), is somewhat smaller than the lattice data (27), but is certainly in the same ballpark, and the evaluation of the short-distance contribution is important to improve the accuracy of calculation.

At the same time the resulting value $f_\pi$ [23] is in good agreement with the standard value, obtained from the pion decay and used in the chiral perturbation theory [18].

The method used above can be easily applied to the case of nonzero quark mass $m$ and the $\text{SU}(3)$ flavour group to calculate $\langle \bar{s}s \rangle$, $f_K$ etc., which will be published elsewhere [25].

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Appendix 1

Calculation of the vertex mass $M(0)$

One starts with the definition of the nonlocal mass operator $M_S(u,v)$, given in [9, 10] (see e.g. Eq.(24) in [10])

$$M_S(u,v) = (\gamma_\mu \Lambda(u,v) \gamma_\mu)_{sc} J(u,v). \quad (A.1)$$

The mass operator enters in the gauge-invariant Green’s function $s$, see e.g. Eq.(3), via the quark propagator $\Lambda(x,y)$, Eq. (4), where $M_S(z,u)$ enters at all intermediate points, and also at initial and final points $x_4$ and $y_4$, where the nonlocal pion $\phi(x, \bar{x})$ is emitted. According to the prescription given in [10], we choose the set of contours $C(z)$ for all intermediate points $z$ in the Green’s function $G(x,y)$, which minimizes the mass eigenvalues. One simple choice is to take the contours $C(z)$ from $z$ along the shortest way to the $x_4$ axis passing through $x_4$ and $y_4$, and along $x_4$ axis to the origin at the point $x_4^{\pm_2}$.

When $M_S$ is situated at the initial or final point of the $q\bar{q}$ Green’s function, i.e. at the points $M_S(x, \bar{x})$ or $M_S(y, \bar{y})$, where the $q\bar{q}$ or pion is created or annihilated, then it is convenient to choose points $x, \bar{x}$ on the axis 1 with the origin at $x_4^{\pm_2}$. In this way one obtains for $x_1 > 0, \bar{x}_1 \equiv y_1 < 0, y_4 \equiv x_4$.

$$J(x,y) = \int_0^{x_1} du \int_{y_1}^0 dv D(u-v, x_4-y_4). \quad (A.2)$$

It is convenient to use for $D$ the Gaussian form,

$$D(x,x_4) = D(0)e^{-\frac{x^2+y^2}{\sigma^2}} = \frac{\sigma}{2\pi T^2 g} e^{-\frac{x^2+y^2}{\sigma^2}}, \quad (A.3)$$

which yields

$$J(x,y) = \frac{\sigma}{\pi} e^{-\frac{(x_4-y_4)^2}{4T^2 g}} (1 - e^{-\frac{(x-y)^2}{4T^2 g}}). \quad (A.4)$$

Now one has to estimate the scalar part of the quark Green’s function $\Lambda(x,y)$ in [11], for which in [12] it was found that it behaves as a smeared $\delta$-function with the smearing radius equal to $1/\sqrt{\sigma}$ (see Eq.(24) in the second ref. of [12]). We simplify this form taking

$$\Lambda(x,y) = \left(\frac{\sigma}{\pi}\right)^{3/2} e^{-(x-y)^2/\sigma}, \int \Lambda(x,y) d^3(x-y) = 1. \quad (A.5)$$
To obtain the localized form of the vertex function

\[ M_S(x) \equiv M(0) = \int M_S^{(0)}(x, y) d^4(x - y) \]  \hspace{1cm} (A.6)

we substitute in (A.6) \( J(x, y) \) from (A.4) and \( \Lambda(x, y) \) from (A.5) to get finally.

\[ M(0) = \left[ 1 - \left( \frac{\sigma T_g^2}{\sigma T_g^2 + 1} \right)^{3/2} \right] \frac{2\sigma T_g}{\sqrt{\pi}} \equiv \eta \frac{2\sigma T_g}{\sqrt{\pi}}. \]  \hspace{1cm} (A.7)

In the limit \( \sigma T_g^2 \to 0 \) one obtains \( M(0) \approx \frac{2\sigma T_g}{\sqrt{\pi}} \), i.e. exactly the value which appears in the strong decay vertex of the string in [20]. This is not surprising since in both cases \( M(0) \) is a mass corresponding to a piece of the string with the length of the order of \( T_g \), hence \( M(0) \sim \sigma T_g \). The factor \( \eta \) in (A.7) describes the attenuation due to the nonlocality of \( \Lambda(x, y) \) at small \( |x - y| \) for light quarks. For heavy quarks this factor tends to zero since the localization of \( \Lambda(x, y) \) becomes more strong, indeed the quark Green’s function \( \Lambda \) for \( m \to \infty \) is proportional to \( \delta(3)(x - y) \) see [8]. Effectively for nonzero \( m \) this can be described replacing \( \eta \) in (A.7) by the factor

\[ \eta \to \eta(m) = \left[ 1 - \left( \frac{\sigma T_g + m}{\sigma T_g + m + 1} \right)^{3/2} \right]. \]  \hspace{1cm} (A.8)

For light quarks and \( \sigma = 0.18 \text{ GeV}^2, T_g = 1 \text{ GeV}^{-1} \), the factor \( \eta \equiv \eta(0) \) is \( 1 - 0.27 \cong 0.73 \), and from (A.7) one gets \( M(0) \approx 0.148 \text{ GeV} \).

Appendix 2

Derivation of the spectral representations, Eqs. (12) and (13)

Consider the \( q\bar{q} \) Green’s function of the type given in Eq. (6)

\[ G_\Gamma(x, y) = \langle tr \Gamma \Lambda(x, y) \Gamma \Lambda(y, x) \rangle \]  \hspace{1cm} (A.9)

where \( \Gamma = \gamma_5, \gamma_\mu, ... \), and the \( q\bar{q} \) Green’s function in the 4 \( \times \) 4 spinor representation,

\[ G^{(q\bar{q})}_{\alpha\beta, \gamma\delta}(x, \bar{x}; y, \bar{y}) = \langle \Lambda_{\alpha\beta}(x, y) \Lambda_{\gamma\delta}(\bar{y}, \bar{x}) \rangle. \]  \hspace{1cm} (A.10)

Following the standard procedure from [20] one can introduce the c.m. and relative coordinates, e.g.

\[ X = \frac{x + \bar{x}}{2}, Y = \frac{y + \bar{y}}{2}, r = x - \bar{x}, r' = y - \bar{y} \]  \hspace{1cm} (A.11)
and define
\[
G^{(q\bar{q})}(x, \bar{x}; y, \bar{y}) = \int d^4P e^{iP(x-y)} \frac{d\varepsilon}{2\pi} e^{-i\varepsilon r_0} G_P(r, r', \varepsilon, r'_0)
\]
(A.12)

\(G_P\) satisfies an equation
\[
(E - E_2 - H_1)(E_2 - H_2)G_P = \beta_1 \beta_2 \hat{1}
\]
(A.13)

where \(E = E_1 + E_2 = P_0, E_1 - E_2 = 2\varepsilon\), and
\[
H_i = m_i \beta_i + p\alpha_i + \beta_i M_S.
\]
(A.14)

At this point one can exploit the property of \(H_i\) that it does not depend on relative time \(r_0\), and therefore one can integrate in (A.12) over \(d\varepsilon\) with the result \[26, 27\]
\[
G_P \bigg|_{r_0=0, r'_0=0} = \beta_1 \beta_2 \int_{-\infty}^{\infty} \frac{d\varepsilon/2\pi}{(E_1 - H_1)(E_2 - H_2)} = i \beta_1 \beta_2 \frac{1}{E - \hat{H}} \langle r | \hat{H} | r' \rangle.
\]
(A.15)

As the result one obtains
\[
\int d^4(X - Y)G^{(q\bar{q})}(x, \bar{x}; y, \bar{y}) \bigg|_{r_0=r'_0=0} = \langle r | \frac{i \beta_1 \beta_2}{\hat{H}} | r' \rangle = \sum_n \langle | r n \rangle i \beta_1 \beta_2 \frac{1}{E_n} \langle n | r' \rangle.
\]
(A.16)

One can now express \(G_\Gamma\), with \(\Gamma = \gamma_5\),
\[
\int d^4(X - Y)G_\Gamma(x, y, \bar{x}, \bar{y}) = \sum_{n=0}^{\infty} \frac{\psi_n(0)\psi_n^+(0)}{E_n}
\]
(A.17)

where we have defined the relativistic wave-function \(\psi_n(r) \sim \gamma_5 (r | n\rangle, \psi_n^+(r) \sim \beta_1 \beta_2 (n | r\rangle\), satisfying the Hamiltonian equation
\[
\hat{H} \psi_n(r) = E_n \psi_n(r).
\]
(A.18)

As it is known from dynamical calculations with the Bethe-Salpeter equation \[28\] with the scalar confining kernel the dominant role in \(\psi_n(r)\) is played by the \(^1S_0\) component \(\varphi_n(r)\) of the wave-function, which satisfies the relativistic Schroedinger equation with the hyperfine interaction, discussed in the Appendix 3. Therefore one can identify \(\psi_n(r) \to \varphi_n(r), E_n \to m_n\) and the Eq. (6) with the help of (A.17) goes over into Eq. (14).
Appendix 3

Calculation of the masses $m_n$ and $\varphi_n(0)$ through $\sigma$ in FCM

The mass eigenvalue $\bar{m}_n$ of the spin-averaged state $\frac{3m_n^0 + m_n^s}{4}$ for $L = 0$ can be written as \[15, 16\]

\[
\bar{m}_n = M_0(n, 0) + \Delta_{SE} + \Delta_C, \quad (A.19)
\]

where $M_0(n, 0)$ is the eigenvalue of the spinless Salpeter equation, which can be written as

\[
M_0(n, 0) = 4\mu_0(n) = 4\sqrt{p^2 + m^2} n_0. \quad (A.20)
\]

For $m = 0$, $\mu_0(n, 0)$ is expressed through $\sqrt{\sigma}$ and dimensionless coefficients $a(n)$ – zeros of Airy functions \[15\]

\[
\mu_0(n) = \sqrt{\sigma} \left( \frac{a(n)}{3} \right)^{3/4}, \quad a(0) = 2.338, \quad a(1) = 4.088. \quad (A.21)
\]

Taking into account nonzero $m$ one finds $\mu_0(n)$ from the equation

\[
1 = \frac{m^2}{\mu_0^2} + \frac{\sigma^{2/3}}{3\mu_0^{4/3}} a(n). \quad (A.22)
\]

For large $m \gg \sqrt{\sigma}$ the solution of (A.22) is

\[
\mu_0^2(n) \approx m^2 \left[ \frac{2\sigma \mu}{m^2(m + \mu)} \right]^{2/3}. \quad (A.23)
\]

The term $\Delta_{SE}$ is the self-energy correction \[19\] which can be written as

\[
\Delta_{SE}(n) = -\frac{4\sigma \eta(m)}{\pi \mu_0(n)} \quad (A.24)
\]

and $\eta(m)$ is computed through $m$, for $m = 0$ it is $\eta(m = 0) = 0.9 \div 1$.

Taking all contributions into account one obtains for the light quarks ($m = 0$)

\[
\bar{m}_0 = 0.652 \text{GeV}, \quad \bar{m}_n^2 = \bar{m}_0^2 + \Omega_0 n, n = 0, 1, 2,.. \quad (A.25)
\]

where $\Omega_0$ is computed solely through $\sigma$ and is equal $\Omega_0 \approx 1.6 \text{ GeV}^2$, which is close to the experimental slope $\Omega_{\text{exp}}(L = 0) = 1.64 \pm 0.11 \text{ GeV}^2$, see \[16\] for refs. and discussion.
Now we take into account the hyperfine interaction which produces the HF splitting between $\rho$ and $\pi$ states.

$$\Delta_{HF} = \Delta_{HF}^{Pert} + \Delta_{HF}^{NP}, \quad \Delta_{HF}^{Pert} = \frac{8\alpha_s(\mu_{HF})|R_n(0)|^2}{9\mu_0^2(n)}. \quad (A.26)$$

Here $R_n(0) = \sqrt{4\pi} \varphi_n(0)$ is the radial meson w.f., which can be found also from the leptonic width of $\rho$ meson. One has

$$|R_0(0)|^2 = \mu_0(0)(\sigma + \frac{4}{3}\alpha_s(r^{-2})) = \left\{ \begin{array}{ll} 0.091\text{GeV}^3, & \alpha_s = 0 \\ 0.109\text{GeV}^3, & \alpha_s = 0.3. \end{array} \right. \quad (A.27)$$

These values can be checked vs the leptonic width of $\rho$, $\Gamma_{e^+e^-} = \left\{ \begin{array}{ll} 6.36\text{KeV}, & \alpha_s = 0 \\ 7.62\text{KeV}, & \alpha_s = 0.3 \end{array} \right.$,

while $\Gamma_{e^+e^-}^{exp} = (6.85 \pm 0.11)\text{KeV}$.

Thus one obtains $\Delta_{HF}^{Pert}$ from (A.26), $\Delta_{HF}^{Pert} = \left\{ \begin{array}{ll} 0.26\text{GeV}, & \alpha_s = 0 \\ 0.24\text{GeV}, & \alpha_s = 0.3. \end{array} \right.$

The nonperturbative part $\Delta_{HF}^{NP}$ is expressed through the correlator $D(x)$ and depends on the accepted value of $G_2 \equiv \frac{\alpha_s}{\pi}\langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle$,

$$\Delta_{HF}^{NP} \approx 50\text{MeV} \left( \frac{G_2}{0.012\text{GeV}^4} \right). \quad (A.28)$$

We take two values of $G_2 = G_2^{st} = 0.012 \text{ GeV}^4$ and $G_2 = 2G_2^{st}$. Thus one obtains for the lowest mass of PS state in the $q\bar{q}$ approach (no chiral effects)

$$m_0 = \bar{m}_0 - \frac{3}{4}\Delta_{HF} = [0.652 - \frac{3}{4}(0.3 \div 0.35)]\text{GeV} = (0.39 - 0.43)\text{GeV}. \quad (A.29)$$

As a result we accept the following values for $m_0$ and $m_1$ (the latter is calculated in the same way using (A.25) and $\Delta_{HF}(n = 1)$)

$$m_0 = 0.4\text{GeV}, \quad m_1 = 1.35\text{GeV}. \quad (A.30)$$

**Appendix 4**

**Small distance contribution to $\langle \bar{q}q \rangle$**

To separate the small-distance contribution we start from Eq. (4) where we take into account the nonlocal structure of $M_S(u, v)$ and put $m = 0$,

$$tr\Lambda_{xx} = \int d^4u d^4y tr(\gamma_5\Lambda(x, y)\gamma_5M_S(y, u)\Lambda(u, x)). \quad (A.31)$$
In the limit when one keeps the most singular part – the free part of $\Lambda(x, y)$, one has

$$\Lambda(x, y) \approx \Lambda_{\text{free}}(x, y) = \frac{1}{2\pi^2} \left( \frac{\hat{x} - \hat{y}}{(x - y)^4} + \frac{m}{2(x - y)^2} \right) + ... \quad (A.32)$$

From (A.1) one can derive the behaviour of $M_S(y, u)$ at small $|y - u| \leq T_0, T_g$,

$$M_S(y, u) \sim \frac{\sigma}{T_g^2} c |y - u|^2 \Lambda(x, y), \quad (A.33)$$

where the coefficient $c$ is of the order of unity.

The nonperturbative part of $\Lambda(x, y)$ is not singular (modulo logarithms) and proportional to $\sigma^{3/2}$, c.f. Eq.(A.5) (apart from the OPE part of $\Lambda$ which has $m\langle \bar{q}q \rangle$ and $\langle F^2 \rangle$ terms and even less singular at small $x$). Finally inserting (A.33), (A.32) into (A.31) and integrating in the region $|x - y|, |y - u|, |u - x| \leq T_0$ one can write the short-distance contribution to (A.31) as

$$tr \Lambda_{xx}(\text{small distance} \leq T_g) = O(\sigma m, (\sigma^{5/2} T_0^4 / T_g)). \quad (A.34)$$

As it is seen from (18) the long distance part is $O(\sigma^2 T_g)$ and is dominant at $\sigma T_g^2 \to 0, T_0 \leq T_g$.

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