MODULI OF BRIDGELAND SEMISTABLE OBJECTS ON 3-FOLDS AND DONALDSON-THOMAS INVARIANTS

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Abstract. We show that the moduli stacks of Bridgeland semistable objects on smooth projective 3-folds are proper algebraic stacks of finite type, if they satisfy the Bogomolov-Gieseker (BG for short) inequality conjecture proposed by Bayer, Macrì and the second author. The key ingredients are the equivalent form of the BG inequality conjecture and its generalization to arbitrary very weak stability conditions. This result is applied to define Donaldson-Thomas invariants counting Bridgeland semistable objects on smooth projective Calabi-Yau 3-folds satisfying the BG inequality conjecture, for example on étale quotients of abelian 3-folds.

1. Introduction

1.1. Motivation and background. Let $X$ be a smooth projective variety over $\mathbb{C}$. Following Douglas’s work \cite{Dou02} on II-stability in physics, Bridgeland introduced the complex manifold

$$\text{Stab}(X)$$

called the space of stability conditions on the bounded derived category $D^b\text{Coh}(X)$ of coherent sheaves on $X$ (see \cite{Bri07}). Each point of Stab$(X)$ determines certain semistable objects in $D^b\text{Coh}(X)$, and the whole space is expected to contain the stringy Kähler moduli space of $X$. Also the space Stab$(X)$ has potential applications to Donaldson-Thomas (DT for short) invariants \cite{Tod10}, \cite{Tod13b}, \cite{Tod13a}, \cite{Tod14a}, and birational geometry \cite{BM14b}, \cite{BM14a}, \cite{ABCH13}, \cite{Tod13c}, \cite{Tod14b}. Here one needs to construct the moduli spaces of Bridgeland semistable objects, and then to study the variations of these moduli spaces under the changes of stability conditions.

However, in general the space of stability conditions is a difficult object to study, and there exist several foundational issues. At least, the following conjecture needs to be be settled at firsthand:

**Conjecture 1.1.** We have the following for $X$:

(i) $\text{Stab}(X) \neq \emptyset$, and

(ii) for each known stability condition $\sigma \in \text{Stab}(X)$, the moduli stack of $\sigma$-semistable objects with a fixed Chern character is a proper algebraic stack of finite type.
The above conjecture is known to be true when \( \dim X \leq 2 \). The construction problem (i) for surfaces was solved by Bridgeland [Bri08] and Arcara-Bertram [AB13], by tilting coherent sheaves. However, this is an open problem in \( \dim X \geq 3 \). For 3-dimensional case, Bayer, Macri and the second author [BMT14] reduced the problem (i) to a conjectural Bogomolov-Gieseker (BG for short) type inequality involving the Chern characters of certain two term complexes, called \textit{tilt semistable objects} (see Conjecture 1.3 below). On the other hand, the moduli problem (ii) was solved by the second author [Tod08] for K3 surfaces, and the same argument also applies to any surface. The main purpose of this paper is to solve problem (ii) for 3-folds satisfying the above mentioned BG inequality conjecture in [BMT14]. A rough statement is as follows:

\textbf{Theorem 1.2.} Let \( X \) be a smooth projective 3-fold satisfying the BG inequality conjecture in [BMT14]. Then it satisfies Conjecture 1.1 (ii).

So far the BG inequality conjecture in [BMT14] is known to hold in the following cases:

- \( X = \mathbb{P}^3 \) by Macri [Mac14].
- \( X \subset \mathbb{P}^4 \) is a smooth quadric threefold by Schmidt [Sch14].
- \( X \) is an abelian 3-fold by Maciocia and the first author [MPa], [MPb], [Piy14], and by Bayer, Macri and Stellari [BMS].
- \( X \) is an \( \acute{e} \)tale quotient of an abelian 3-fold by Bayer, Macri and Stellari [BMS].

The result of Theorem 1.2 is applied to the above 3-folds, and it gives new non-trivial Bridgeland moduli spaces of 3-folds. Furthermore, we use Theorem 1.2 to construct Donaldson-Thomas invariants counting Bridgeland semistable objects on Calabi-Yau 3-folds satisfying the BG inequality conjecture, for example on A-type Calabi-Yau 3-folds [BMS], fulfilling the expected properties.

1.2. \textbf{BG type inequality conjecture.} Let \( X \) be a smooth projective 3-fold. Let \( B \in \text{NS}(X)_{\mathbb{Q}} \) and \( \omega \in \text{NS}(X)_{\mathbb{R}} \) be an ample class with \( \omega^2 \) rational; that is \( \omega = mH \) for some ample divisor class \( H \in \text{NS}(X) \) with \( m^2 \in \mathbb{Q}_{>0} \). In [BMT14], Bayer, Macri and the second author constructed data

\[ \sigma_{\omega,B} = (Z_{\omega,B}, A_{\omega,B}) \]

for a conjectural Bridgeland stability condition on \( X \). Here \( A_{\omega,B} \) is the heart of a bounded t-structure on \( D^b \text{Coh}(X) \) given as a double tilt of \( \text{Coh}(X) \), and \( Z_{\omega,B} : K(X) \to \mathbb{C} \) is the group homomorphism defined by

\[ Z_{\omega,B}(E) := -\int_X e^{-i\omega} \text{ch}^B(E), \]

where \( \text{ch}^B(E) := e^{-B} \text{ch}(E) \). The pair \( (1) \) is shown to give a Bridgeland stability condition on \( D^b \text{Coh}(X) \), if the following conjectural BG inequality holds for certain two term complexes, called \textit{tilt semistable objects} (see Conjecture 3.8):
Conjecture 1.3. (BG Inequality Conjecture, [BMT14]) For a tilt semistable object $E \in D^b\text{Coh}(X)$ with $\text{Im} Z_{\omega,B}(E) = 0$, we have the inequality

$$\text{ch}^3_B(E) \leq \frac{1}{18} \omega^2 \text{ch}^1_B(E).$$

One of the important statements in [BMS] is that, when $B$ and $\omega$ are proportional, Conjecture 1.3 is equivalent to another conjectural inequality for tilt semistable objects $E$ without the condition $\text{Im} Z_{\omega,B}(E) = 0$. Our proof of Conjecture 1.1 (ii) for 3-folds rely on this equivalent inequality, which we generalize to the case when $B$ and $\omega$ are not proportional:

Theorem 1.4. ([BMS], $B$ and $\omega$ are proportional; Theorem 3.15, in general) Conjecture 1.3 is equivalent to the following:

any tilt semistable object $E$ satisfies the inequality

$$(\omega^2 \text{ch}^1_B(E))^2 - 2\omega^3 \text{ch}^B_0(E)\omega \text{ch}^2_B(E) + 12(\omega \text{ch}^B_2(E))^2 - 18\omega^2 \text{ch}^1_B(E) \text{ch}^3_B(E) \geq 0.$$

(2)

The advantage of the inequality (2) is that it can be used to imply the support property of $\sigma_{\omega,B}$ in some sense. More precisely, Conjecture 1.3 implies

$$\sigma_{\omega,B} \in \text{Stab}^0_{\omega,B}(X),$$

where $\text{Stab}^0_{\omega,B}(X)$ is a connected component of the space of stability conditions whose central charges are written as linear combinations of $\omega^{3-j} \text{ch}^B_j(E)$ for $0 \leq j \leq 3$. The inequality (2) also plays a key role to generalize our moduli problem for an arbitrary very weak stability condition and associated tilting, which we discuss in this paper.

1.3. Moduli stacks of Bridgeland semistable objects on 3-folds.

Now we give the precise statement of Theorem 1.2 as follows:

Theorem 1.5. (Theorem 4.1) Suppose that $X$ is a smooth projective 3-fold satisfying Conjecture 1.3. Then for any $\sigma = (Z,A) \in \text{Stab}^0_{\omega,B}(X)$, the moduli stack of $\sigma$-semistable objects $E \in A$ with a fixed vector

$$(\omega^3 \text{ch}^B_0(E), \omega^2 \text{ch}^1_B(E), \omega \text{ch}^2_B(E), \text{ch}^3_B(E))$$

is a proper algebraic stack of finite type.

The strategy of the proof of Theorem 1.5 is as follows. In [Tod08], the second author reduced the moduli problem of Bridgeland semistable objects to the following two problems:

(i) generic flatness of the corresponding heart.

(ii) boundedness of semistable objects.

We prove the properties (i), (ii) for the stability condition $\sigma_{\omega,B}$ defined in terms of data (1), and show that they are preserved under the deformations of stability conditions. In order to show (i), (ii) for $\sigma_{\omega,B}$, we generalize the tilting construction which appeared in the construction of Bridgeland stability on surfaces, tilt stability and its further tilting. Following [BMS], we introduce the notion of very weak stability conditions (see Definition 2.1) on $D^b\text{Coh}(X)$, which generalizes classical slope stability on sheaves and tilt stability on two term complexes of sheaves. Roughly speaking, a very weak
stability condition consists of a pair \((Z, A)\) as similar to Bridgeland stability, but we allow some objects in \(A\) are mapped to zero by \(Z\). For a very weak stability condition \((Z, A)\), we introduce the notion of its BG type inequality in the form

\[ \text{Re} Z(E) \Delta_R(E) + \text{Im} Z(E) \Delta_I(E) \geq 0 \]  

for any semistable object \(E \in A\) such that \(\Delta_R, \Delta_I\) satisfying certain conditions. The inequality (3) turns out to be a generalization of the classical BG inequality for surfaces, and the conjectural inequality (2) for tilt semistable objects. We construct the tilting of \((Z, A)\) as a one parameter family of very weak stability conditions of the form

\[ (Z, A) \rightsquigarrow (Z^t, A^t), \quad t > 0, \]  

where \(A^t\) is a tilt of \(A\), and \(Z^t\) is defined by \(-iZ + t\Delta_I\). Applying the tilting process (4) twice starting from the classical slope stability, we get the construction (6) in [BMT14]. We show that, in some sense, the above properties (i), (ii) are preserved under the operation (4). This yields the desired properties for the stability condition (1), and so proves Theorem 1.5. Finally, the properness of the moduli stack follows from the valuative criterion due to Abramovich-Polishchuk [AP06].

1.4. Donaldson-Thomas invariants. Suppose that \(X\) is a Calabi-Yau 3-fold satisfying Conjecture 1.3 for example an A-type Calabi-Yau 3-fold (see [BMS]). We use Theorem 1.5 to define Donaldson-Thomas invariants counting Bridgeland semistable objects on such Calabi-Yau 3-folds. Following the construction of generalized DT invariants [JST12], [KS] counting semistable sheaves, we construct a map

\[ \text{DT}_\sigma(v): \text{Stab}^\sigma_{\omega, B}(X) \to \mathbb{Q}, \]

for each \(v \in H^*(X, \mathbb{Q})\). The invariant \(\text{DT}_\sigma(v)\) counts \(\sigma\)-semistable objects in \(D^b \text{Coh}(X)\) with Chern character \(v\). In particular, the invariant

\[ \text{DT}_{\omega, B}(v) := \text{DT}_{\omega, B}(v) \]

counts \(Z_{\omega, B}\)-semistable objects in \(A_{\omega, B}\) that are certain three term complexes in the derived category. In a forthcoming paper, we will pursue the wall-crossing formula relating the invariants (5) with the original DT invariants counting semistable sheaves, and show that they are invariant under the deformations of the complex structure on \(X\).

1.5. Relation to the existing works. As we mentioned before, Conjecture 1.1 (ii) for surfaces essentially follows from [Tod08], and Theorem 1.5 is a 3-fold generalization. Moreover, the arguments in this paper contain several improvements toward Conjecture 1.1 (ii). One of the key points of our approach is the formulation of the BG type inequality in the form (3), and to show that several properties are inherited from the tilting process (4). This generalized tilting argument would be convenient for the future study of Conjecture 1.1 (ii) for higher dimensional varieties, which may require further tilting.

Regarding the moduli problems involving objects in the derived category of 3-folds, the second author constructed moduli spaces of limit stable
objects [Tod09] on Calabi-Yau 3-folds. These are special cases of Bayer’s polynomial stability conditions [Bay09]. Also they can be interpreted as limiting degenerations of Bridgeland stability conditions, and the moduli spaces constructed in [Tod09] may appear as moduli stacks in Theorem 1.5 at points which are sufficiently close to the so called large volume limit. Also there exist works by Jason Lo [Lo11], [Lo13] constructing moduli spaces of certain polynomial semistable objects called PT semistable objects, which may appear near the large volume limit as well. Similarly, the DT type invariants constructed in [PT09], [Tod09] also may coincide with the invariants (5) near the large volume limit.

1.6. Plan of the paper. In Section 2 we introduce the notion of very weak stability conditions on triangulated categories, their BG type inequality and the associated tilting. The results of this section contain several generalizations of known results for tilt stability. In Section 3 we interpret tilt stability and the associated double tilting in the framework of Section 2, and proves an equivalent form of Conjecture 1.3 in Theorem 1.4. In Section 4 we prove Theorem 1.5 by showing general results on the generic flatness and the boundedness under the tilting (1). In Section 5 using the result of Section 4, we define Donaldson-Thomas invariants counting Bridgeland semistable objects on Calabi-Yau 3-folds satisfying Conjecture 1.3.

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1.8. Notations and conventions. Throughout this paper, all the varieties are defined over \( \mathbb{C} \). For a variety \( X \), by \( \text{Coh}(X) \) we denote the category of coherent sheaves on \( X \), and \( \text{Coh}_{\leq d}(X) \) denote its subcategory of coherent sheaves whose supports have dimension less than or equal to \( d \). For simplicity, we write \( \text{Coh}_{\leq 0}(X) \) as \( \text{Coh}_0(X) \). When \( \mathcal{A} \) is the heart of a bounded t-structure on a triangulated category \( \mathcal{D} \), by \( \mathcal{H}^i_{\mathcal{A}}(\ast) \) we denote the corresponding \( i \)-th cohomology functor. When \( \mathcal{A} = \text{Coh}(X) \) and \( \mathcal{D} = D^b \text{Coh}(X) \), we simply write \( \mathcal{H}^i(\ast) \) for \( \mathcal{H}^i_{\text{Coh}(X)}(\ast) \). For a set of objects \( \mathcal{S} \subset \mathcal{D} \), by \( \langle \mathcal{S} \rangle \subset \mathcal{D} \) we denote its extension closure, that is the smallest extension closed subcategory of \( \mathcal{D} \) which contains \( \mathcal{S} \). We denote the upper half plane \( \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) by \( \mathbb{H} \).

2. Tilting via very weak stability conditions

In this section, we develop general arguments of very weak stability conditions, which are the variants of weak stability of [Tod10] introduced in [BMS, Definition B.1].
2.1. **Very weak stability conditions.** Let $\mathcal{D}$ be a triangulated category, and $K(\mathcal{D})$ its Grothendieck group. We fix a finitely generated free abelian group $\Gamma$ and a group homomorphism

$$\text{cl} : K(\mathcal{D}) \to \Gamma.$$  

We first give the following definition:

**Definition 2.1.** A very weak pre-stability condition on $\mathcal{D}$ is a pair $(Z, A)$, where $A$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z : \Gamma \to \mathbb{C}$ is a group homomorphism satisfying the following conditions:

(i) For any $E \in A$, we have

$$Z(E) \in \mathbb{H} \cup \mathbb{R}_{\leq 0}.$$  

Here $Z(E) := Z(\text{cl}(E))$ and $\mathbb{H}$ is the upper half plane.

(ii) Let

$$\mu := \frac{\text{Re } Z}{\text{Im } Z} : A \to \mathbb{R} \cup \{\infty\}$$

be the associated slope function. Here we set $\mu(E) = \infty$ if $\text{Im } Z(E) = 0$. Then $\mu$ satisfies the Harder-Narasimhan (HN for short) property.

We say that $E \in A$ is $\mu$-(semi)stable if for any non-zero subobject $F \subset E$ in $A$, we have the inequality

$$\mu(F) < (\leq) \mu(E/F).$$

The HN filtration of an object $E \in A$ is a chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

in $A$ such that each $F_i = E_i / E_{i-1}$ is $\mu$-semistable with $\mu(F_i) > \mu(F_{i+1})$. If such HN filtrations exists for all objects in $A$, we say that $\mu$ satisfies the HN property.

**Remark 2.2.** The definition of the $\mu$-semistable objects is equivalent to the usual definition of slope semistability: $E \in A$ is $\mu$-semistable if and only if for any non-zero $F \subset E$ in $A$, we have $\mu(F) \leq \mu(E)$. However, the $\mu$-stable objects are different from those defined by the inequality $\mu(F) < \mu(E)$.

For a given a very weak pre-stability condition $(Z, A)$, we define its slicing on $\mathcal{D}$ (see [Bri07, Definition 3.3])

$$\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}, \quad \mathcal{P}(\phi) \subset \mathcal{D}$$

as in the case of Bridgeland stability conditions (see [Bri07, Proposition 5.3]). Namely, for $0 < \phi \leq 1$, the category $\mathcal{P}(\phi)$ is defined to be

$$\mathcal{P}(\phi) = \{E \in A : E \ is \ \mu\text{-semistable with } \mu(E) = -1/\tan(\pi\phi)\} \cup \{0\}.$$  

Here we set $-1/\tan \pi = \infty$. The other subcategories are defined by setting

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1].$$  

For an interval $I \subset \mathbb{R}$, we define $\mathcal{P}(I)$ to be the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{P}(\phi)$ for each $\phi \in I$.

Note that the category $\mathcal{P}(1)$ contains the following category

$$C := \{E \in A : Z(E) = 0\}.$$  

(6)
It is easy to check that $C$ is closed under subobjects and quotients in $\mathcal{A}$. In particular, $C$ is an abelian subcategory of $\mathcal{A}$. We say that $(Z, \mathcal{A})$ is a pre-stability condition if $C = \{0\}$. We define the subgroup $\Gamma_0 \subset \Gamma$ by

$$\Gamma_0 := [\text{cl}(C)] \subset \Gamma.$$  

Here for a subset $S \subset \Gamma$, by $[S]$ we denote the saturation of the subgroup of $\Gamma$ generated by $S$. Note that $E \in \mathcal{A}$ satisfies $\text{cl}(E) \in \Gamma_0$ if and only if $E \in C$. Also $Z$ descends to the group homomorphism

$$\overline{Z} : \Gamma/\Gamma_0 \to \mathbb{C}.$$  

We define the following analogue of support property introduced by Kontsevich-Soibelman [KS] for Bridgeland stability conditions:

**Definition 2.3.** A very weak pre-stability condition $(Z, \mathcal{A})$ is a very weak stability condition if it satisfies the support property: there is a quadratic form $Q$ on $\Gamma/\Gamma_0$ satisfying

(i) $Q(E) \geq 0$ for any $\mu$-semistable object $E \in \mathcal{A}$, and

(ii) $Q|_{\ker Z}$ is negative definite on $\Gamma/\Gamma_0$.

For $v \in \Gamma$, we denote by $\overline{v}$ its image in $\Gamma/\Gamma_0$, and $\|\cdot\|$ is a fixed norm on $(\Gamma/\Gamma_0) \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly to the Bridgeland stability, the support property is also interpreted in the following way:

**Lemma 2.4.** Let $(Z, \mathcal{A})$ be a very weak pre-stability condition. Then it satisfies the support property if and only the following holds:

(7) \[ C := \sup \left\{ \left\| \text{cl}(E) \right\| : E \text{ is } \mu \text{-semistable with } E \notin C \right\} < \infty. \]

**Proof.** If $(Z, \mathcal{A})$ is a pre-stability condition, then the lemma is stated in [KS] and the precise proof is available in [BMS, Lemma A.4]. The same proof also applies for very weak pre-stability conditions. Indeed if (7) holds, then the quadratic form $Q$ in Definition 2.3 is given by

$$Q(v) = C \overline{Z(v)}^2 - \|v\|^2$$

for $v \in \Gamma/\Gamma_0$. The proof of the converse is also the same as in [BMS, Lemma A.4]. \qed

A very weak stability condition $\sigma = (Z, \mathcal{A})$ is called a stability condition if it is also a pre-stability condition, i.e. $C = \{0\}$. This notion coincides with the notion of Bridgeland stability conditions [Bri07] satisfying the support property. In this case, we also call $\mu$-semistable objects as $Z$-semistable objects, or as $\sigma$-semistable objects.

**Example 2.5.** When $\mathcal{A}$ is the heart of a bounded t-structure on an arbitrary triangulated category $\mathcal{D}$, by definition, trivial pair $(Z = 0, \mathcal{A})$ defines a very weak stability condition.

**Example 2.6.** Let $\mathcal{A}$ be a finite dimensional $\mathbb{C}$-algebra and $\mathcal{A} = \text{mod} \mathcal{A}$ the category of finitely generated right $\mathcal{A}$-modules. Then there is a finite number of simple objects

$$S_1, S_2, \cdots, S_k \in \mathcal{A}$$
such that $K(\mathcal{A})$ is freely generated by $[S_i]$ for $1 \leq i \leq k$. We set $\Gamma = K(\mathcal{A})$ and $\text{cl} = \text{id}$. Then for any group homomorphism $Z : \Gamma \to \mathbb{C}$ with $Z([S_i]) \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$, it is easy to show that the pair $(Z, \mathcal{A})$ is a weak stability condition on $D^b(\mathcal{A})$. The subcategory $\mathcal{C} \subset \mathcal{A}$ is generated by $S_i$ with $Z(S_i) = 0$.

Let $\text{Stab}^w_\Gamma(\mathcal{D})$ be the set of very weak stability conditions on $\mathcal{D}$ with respect to the data $(\Gamma, \text{cl})$, and $\text{Slice}(\mathcal{D})$ the set of slicing on $\mathcal{D}$ with its topology introduced in [Bri07, Section 6]. The set $\text{Stab}^w_\Gamma(\mathcal{D})$ has a topology induced by the inclusion

$$\text{Stab}^w_\Gamma(\mathcal{D}) \subset \text{Hom}(\Gamma, \mathbb{C}) \times \text{Slice}(\mathcal{D}).$$

Let $\text{Stab}_\Gamma(\mathcal{D}) \subset \text{Stab}^w_\Gamma(\mathcal{D})$ be the subset consists of stability conditions. By [Bri07, Theorem 1.2], the map $\text{Stab}_\Gamma(\mathcal{D}) \to \text{Hom}(\Gamma, \mathbb{C})$ sending $(Z, \mathcal{A})$ to $Z$ is a local homeomorphism.

### 2.2. Tilting of a very weak stability condition

Following the idea to construct Bridgeland stability conditions on surfaces [Bri08], [AB13], and tilt stabilities on 3-folds [BMT14], we construct a ‘tilting’ of a very weak stability condition $(Z, \mathcal{A})$. The additional required data is the following generalization of Bogomolov-Gieseker (BG) inequality:

**Definition 2.7.** We say that $(Z, \mathcal{A})$ satisfies the BG inequality if there exist linear maps

$$\Delta_R, \Delta_I : \Gamma \to \mathbb{Q}$$

satisfying the following conditions:

(i) For any $\mu$-semistable $E \in \mathcal{A}$, we have the inequality

$$(8) \quad \Delta(E) := \text{Re} Z(E) \Delta_R(E) + \text{Im} Z(E) \Delta_I(E) \geq 0.$$

(ii) $\Delta_R|_{\mathcal{C}} = 0$, $\Delta_I|_{\mathcal{C}} \leq 0$ and $\Delta_I|_{\mathcal{C}} \neq 0$.

As we will see in Subsection 3.3, the inequality $\Delta(E) \geq 0$ is a generalization of the classical Bogomolov-Gieseker inequality for torsion free semistable sheaves on smooth projective varieties. The inequality $\Delta(E) \geq 0$ also appears in an algebraic situation as follows:

**Example 2.8.** Let $Q_l$ be the quiver with two vertex $\{1, 2\}$, and $l$-arrows from 1 to 2 (see the following picture for $l = 2$)

$$Q_2 : \begin{tikzpicture}[baseline=-.5ex]
\node (A) at (0,0) {$1$};
\node (B) at (1,0) {$2$};
\draw[->] (A) -- (B);
\end{tikzpicture}$$

Let $\mathcal{A}$ be the category of finite dimensional $Q_l$-representations, and set $\mathcal{D} = D^b(\mathcal{A})$. The group $\Gamma = K(\mathcal{D})$ is generated by $S_1$ and $S_2$, where $S_i \in \mathcal{A}$ is the simple object corresponding to the vertex $i$. We set the group homomorphism $Z : \Gamma \to \mathbb{C}$ by $Z(S_1) = 0$ and $Z(S_2) = i$. Then $(Z, \mathcal{A})$ is a very weak stability condition with $\mathcal{C} = \langle S_1 \rangle$, and $E \in \mathcal{A} \setminus \mathcal{C}$ is $\mu$-semistable if and only if $\text{Hom}(S_1, E) = 0$. If we write $[E] = v_1[S_1] + v_2[S_2]$ in $\Gamma$, the condition $\text{Hom}(S_1, E) = 0$ implies $v_1 \leq lv_2$. By setting

$$(\Delta_R, \Delta_I) = (0, lv_2 - v_1)$$

the data $(Z, \mathcal{A})$ satisfies the BG inequality.
Lemma 2.11. We have the following lemma:
\[ \Delta = \operatorname{Re} Z_s \Delta_R + \operatorname{Im} Z_s (\Delta_I - s \Delta_R) \geq 0 \]

is the BG inequality for \((Z_s, \mathcal{A})\). It requires \((\Delta_I - s \Delta_R)|_C \leq 0\) for any \(s \in \mathbb{Q}\), which implies \(\Delta_R|_C = 0\).

Suppose that \((Z, \mathcal{A})\) satisfies the BG inequality, and fix the quadratic form \(\Delta\) as above. For \(t \in \mathbb{R}_{\geq 0}\), we define the group homomorphism \(Z^t_\mu : \Gamma \to \mathbb{C}\) to be
\[ Z^t_\mu := -iZ + t \Delta_I = (\operatorname{Im} Z + t \Delta_I) - \operatorname{Re} Z \cdot i. \]

We also define the heart \(\mathcal{A}^t \subset \mathcal{D}\) to be
\[ \mathcal{A}^t := \mathcal{P}((1/2, 3/2)). \]

Remark 2.10. The heart \(\mathcal{A}^1\) is also described in the following way. Let \((T, \mathcal{F})\) be the pair of subcategories of \(\mathcal{A}\) defined by
\[ T := \langle \mu\text{-semistable} \in \mathcal{A} \text{ with } \mu(E) > 0 \rangle, \]
\[ \mathcal{F} := \langle \mu\text{-semistable} \in \mathcal{A} \text{ with } \mu(E) \leq 0 \rangle. \]

Then \((T, \mathcal{F})\) is a torsion pair as in [HRS96], and \(\mathcal{A}^1\) coincides with the associated tilt:
\[ \mathcal{A}^1 = \langle \mathcal{F}[1], T \rangle. \]

In particular, any object \(E \in \mathcal{A}^1\) fits into the exact sequence in \(\mathcal{A}^1\)
\[ 0 \to \mathcal{H}^{-1}_\mathcal{A}(E)[1] \to E \to \mathcal{H}^0_\mathcal{A}(E) \to 0. \]

We also set
\[ (9) \quad C^t := \{ F \in \mathcal{C} : \Delta_I(F) = 0 \}. \]

We have the following lemma:

Lemma 2.11. For any \(E \in \mathcal{A}^t\), we have \(Z^t_\mu(E) \in \mathbb{H} \cup \mathbb{R}_{\leq 0}\). Moreover, we have
\[ (10) \quad \{ E \in \mathcal{A}^t : Z^t_\mu(E) = 0 \} = \begin{cases} C, & t = 0, \\ C^t, & t > 0. \end{cases} \]

Proof. By the construction of \(\mathcal{A}^t\), we have \(\operatorname{Im} Z^t_\mu(E) = -\operatorname{Re} Z(E) \geq 0\) for any \(E \in \mathcal{A}^t\). Suppose that \(\operatorname{Im} Z^t_\mu(E) = 0\). We have the short exact sequence
\[ 0 \to U[1] \to E \to F \to 0 \]
in \(\mathcal{A}^t\), where \(U = \mathcal{H}^{-1}_\mathcal{A}(E)\) and \(F = \mathcal{H}^0_\mathcal{A}(E)\). We have \(\operatorname{Im} Z^t_\mu(U) = \operatorname{Im} Z^t_\mu(F) = 0\), and so \(U \in \mathcal{P}(1/2)\) and \(F \in \mathcal{C}\). We have
\[ \operatorname{Re} Z^t_\mu(E) = -\operatorname{Im} Z(U) - t \Delta_I(U) + t \Delta_I(F). \]

If \(U \neq 0\) then \(\operatorname{Im} Z(U) > 0\), and \((8)\) implies \(\Delta_I(U) \geq 0\). Combined with \(\Delta_I(F) \leq 0\) as \(F \in \mathcal{C}\), we obtain \(\operatorname{Re} Z^t_\mu(E) < 0\). If \(U = 0\) then \(E \in \mathcal{C}\), and \(\operatorname{Re} Z^t_\mu(E) = 0\) if and only if \(t \Delta_I(E) = 0\). Therefore, \((10)\) also holds. \(\square\)
Similarly to $\mu$, we define the slope function $\mu^\dagger_t : \mathcal{A}^\dagger \to \mathbb{R} \cup \{\infty\}$ to be

$$
\mu^\dagger_t := \frac{-\text{Re} Z^\dagger_t}{\text{Im} Z^\dagger_t} = \frac{\text{Im} Z + t\Delta_I}{\text{Re} Z}.
$$

Here we set $\mu^\dagger_t(E) = \infty$ if $\text{Re} Z(E) = 0$.

**Aside 2.12.** The following discussion highlights some potential applications of our general framework. However, it is not directly relevant to our main purposes in this paper.

An object of an abelian category is called a **minimal object** when it has no proper subobjects or equivalently no nontrivial quotients in the category. For example skyscraper sheaves of closed points are the only minimal objects of the abelian category of coherent sheaves on a scheme. This is a very useful algebraic notion and such minimal objects are preserved under an equivalence of abelian categories. In our setting one can prove that (see Lemma 2.19 (iv) for the notation and a similar argument)

$$
\langle \mathcal{P}(1/2)[1], \mathcal{C} \rangle = \mathcal{P}^\dagger_0(1) \subset \mathcal{A}^\dagger.
$$

Therefore, it is an easy exercise to check that if $E \in \mathcal{A}$ is

- $\mu$-stable with $\mu(E) = 0$, and
- $\text{Ext}^1_{\mathcal{A}}(\mathcal{C}, E) = 0$,

then $E[1] \in \mathcal{A}^\dagger$ has no proper subobjects in $\mathcal{A}^\dagger$. That is, $E[1] \in \mathcal{A}^\dagger$ is a minimal object. The classes of minimal objects in [Huy08 Proposition 2.2] and [MPa Lemma 2.3] can be realized as examples of this result in the setup that we will introduce in Section 3 for varieties. Those classes of minimal objects played a crucial role in Bridgeland stability for surfaces and 3-folds [Huy08, MPa, MPb, Piy14].

### 2.3. Harder-Narasimhan property after tilting.

In order to proceed further, we need to show the HN property and the support property of $(Z^\dagger_t, \mathcal{A}^\dagger)$. For this purpose, we introduce the following technical condition.

**Definition 2.13.** Let $(Z, \mathcal{A})$ be a very weak stability condition satisfying a BG inequality. We say that $(Z, \mathcal{A})$ is good if the following conditions are satisfied:

(i) There exist constants $\zeta_i \in \mathbb{R}_{>0}$, $i = 1, 2$ such that $Z(\Gamma) \subset (\zeta_1 \mathbb{Q}) + (\zeta_2 \mathbb{Q})$.

(ii) $\mathcal{A}$ is a noetherian abelian category.

(iii) We have $|\text{cl}(\mathcal{C}^\dagger)| = \Gamma^\dagger_0 := \Gamma_0 \cap \ker(\Delta_I)$.

(iv) For any $U \in \mathcal{P}(\phi)$ with $\phi \in (0, 1)$, there is an injection $U \hookrightarrow \hat{U}$ in $\mathcal{A}$ such that $\text{Hom}(\mathcal{C}^\dagger, \hat{U}[1]) = 0$.

**Remark 2.14.** When $(Z, \mathcal{A})$ is a stability condition, it is good if and only if the condition (i) holds. (The condition (ii) follows from (i) by [AP06 Proposition 5.0.1].) If $\zeta_i = 1$, such a stability condition was called algebraic in [Tod08].
Lemma 2.15. Suppose that a very weak stability condition $(Z, \mathcal{A})$ is good. Then for any $U \in \mathcal{P}((0, 1))$, there is no infinite sequence in $\mathcal{A}$

$$U = U_1 \subset U_2 \subset \cdots \subset U_{i-1} \subset U_i \subset \cdots$$

such that $U_i \in \mathcal{P}((0, 1))$ and $U_i/U_{i-1} \in \mathcal{C}$ for all $i$.

Proof. Suppose that such a sequence exists. Let $U \rightarrow Q$ be the minimal destabilizing quotient with respect to the $\mu$-stability, and $P$ the kernel of $U \rightarrow Q$. Let $Q'_i$ be the cokernel of the composition $P \hookrightarrow U \rightarrow U_i$, and define $Q_i$ to be the quotient of $Q'_i$ by its maximal subobject $T_i \subset Q'_i$ in $\mathcal{A}$ with $T_i \in \mathcal{C}$. Then we obtain the sequence in $\mathcal{A}$

$$Q_1 := Q \subset Q_2 \subset Q_3 \subset \cdots \subset Q_{i-1} \subset Q_i \subset \cdots$$

such that $F_i := Q_i/Q_{i-1} \in \mathcal{C}$ for all $i$. By the equality

$$\Delta_I(Q_i) = \Delta_I(Q_{i-1}) + \Delta_I(F_i)$$

and $\Delta_I(F_i) \leq 0$, we have $\Delta_I(Q_i) \leq \Delta_I(Q_{i-1})$. On the other hand, $Q_i$ is $\mu$-semistable by Sublemma 2.16 below, and so the BG inequality gives $\Delta(Q_i) \geq 0$. Combined with $Z(Q_i) = Z(Q)$ and $\Delta_R(Q_i) = \Delta_R(Q)$, we have

$$\text{Re} \ Z(Q)\Delta_R(Q) + \text{Im} \ Z(Q)\Delta_I(Q_i) \geq 0.$$ 

Since $\text{Im} \ Z(U) > 0$, we have $\text{Im} \ Z(Q) > 0$, and so $\Delta_I(Q_i)$ is bounded below. Therefore, we may assume that $\Delta_I(Q_i)$ is constant, which implies that $F_i \in \mathcal{C}$. Now we take $Q \subset \hat{Q}$ as in Definition 2.13 (iv). The condition $\text{Hom}(\hat{C}^\dagger, \hat{Q}[1]) = 0$ and $F_i \in \mathcal{C}$ implies that

$$\cdots \subset Q_{i-1} \subset Q_i \subset \cdots \subset \hat{Q}.$$ 

Since $\mathcal{A}$ is noetherian, the above sequence must terminate. The result now follows by the induction on the number of HN factors of $U$. \hfill \Box

Sublemma 2.16. For $E \in \mathcal{P}(\phi)$ with $\phi \in (0, 1]$, let $0 \rightarrow E \rightarrow E' \rightarrow F \rightarrow 0$ be an exact sequence in $\mathcal{A}$ with $F \in \mathcal{C}$ and $\text{Hom}(C, E') = 0$. Then $E' \in \mathcal{P}(\phi)$.

Proof. Let $0 \rightarrow P' \rightarrow E' \rightarrow Q' \rightarrow 0$ be an exact sequence in $\mathcal{A}$ with $P', Q' \neq 0$, and let $F' \subset F$ be the image of the composition $P' \rightarrow E' \rightarrow F$. Since $C$ is closed under subobjects in $\mathcal{A}$, we have $F' \in \mathcal{C}$. Let $P$ be the kernel of $P' \rightarrow F'$. Then $P \subset E$, and the assumption $\text{Hom}(C, E') = 0$ implies that $P \neq 0$. As $\mu(P') = \mu(P)$ and $\mu(E') = \mu(E)$, the $\mu$-semistability of $E$ implies the $\mu$-semistability of $E'$. \hfill \Box

Lemma 2.17. Suppose that a very weak stability condition $(Z, \mathcal{A})$ is good. Then $\mathcal{A}^\dagger$ is noetherian.

Proof. Suppose that there is an infinite sequence of surjections

$$E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_i \rightarrow E_{i+1} \rightarrow \cdots$$

in $\mathcal{A}^\dagger$. Since $\text{Im} \ Z^\dagger_i = - \text{Re} \ Z$ is discrete and non-negative on $\mathcal{A}^\dagger$, we may assume that $\text{Im} \ Z^\dagger_i(E_i)$ is independent of $i$. Let us consider the sequence of short exact sequences in $\mathcal{A}^\dagger$

$$0 \rightarrow F_i \rightarrow E := E_1 \rightarrow E_i \rightarrow 0.$$
Then we have the chain of surjections
\[ H_\mathcal{A}^0(E) \to H_\mathcal{A}^0(E_2) \to H_\mathcal{A}^0(E_3) \to \cdots \]
in \mathcal{A}. Hence, we may assume that \( H_\mathcal{A}^0(E) \cong H_\mathcal{A}^0(E_i) \) for \( \mathcal{A} \) is noetherian. Also we have the chain of inclusions
\[ H_\mathcal{A}^{-1}(F_i) \subset H_\mathcal{A}^{-1}(F_{i+1}) \subset \cdots \subset H_\mathcal{A}^{-1}(E) \]
in \mathcal{A}. So we may assume that \( H_\mathcal{A}^{-1}(F_i) \) is independent of \( i \). By setting \( V = H_\mathcal{A}^{-1}(E)/H_\mathcal{A}^{-1}(F_i) \), we have the exact sequence in \( \mathcal{A} \)
\[ 0 \to V \to H_\mathcal{A}^{-1}(E_i) \to H_\mathcal{A}^0(F_i) \to 0. \]
Note that \( H_\mathcal{A}^0(F_i) \in \mathcal{C} \) as \( \text{Re} \, Z(F_i) = 0 \). We write \( U_i = H_\mathcal{A}^{-1}(E_i) \in \mathcal{P}((0,1/2)) \). Since \( \mathcal{C} \) is closed under subobjects and quotients in \( \mathcal{A} \) and \( \text{Hom}(C,U_i) = 0 \), we have the chain of inclusions in \( \mathcal{A} \)
\[ U := U_1 \subset U_2 \subset \cdots \subset U_i \subset U_{i+1} \subset \cdots \]
whose subquotients \( U_i/U_{i-1} \) are contained in \( \mathcal{C} \). By Lemma 2.15, the above sequence terminates, and so the sequence [1] also terminates as required.

\begin{lemma}
If \((Z, \mathcal{A})\) is good, then the data \((Z_1^\dagger, \mathcal{A}^\dagger)\) is a very weak pre-stability condition.
\end{lemma}

\begin{proof}
Since \( \mathcal{A}^\dagger \) is noetherian by Lemma 2.17, it is enough to show that there is no infinite sequence of subobjects in \( \mathcal{A}^\dagger \)
\[ \cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_2 \subset E_1 \]
with \( \mu_1^\dagger(E_i) > \mu_1^\dagger(E_i/E_{i-1}) \) (see [Tod10, Proposition 2.12]). Suppose that such a sequence exists. Since \( \text{Im} \, Z_1^\dagger \) values are non-negative and discrete on \( \mathcal{A}^\dagger \), we may assume that \( \text{Im} \, Z_1^\dagger(E_{i+1}) = \text{Im} \, Z_1^\dagger(E_i) \) for all \( i \). Then \( \mu_1^\dagger(E_i/E_{i-1}) = \infty \); this is the required contradiction.
\end{proof}

\section{Support property after tilting}

Let \((Z, \mathcal{A})\) be a very weak stability condition which satisfies a BG inequality and is good. In this subsection, we show the support property of \((Z_1^\dagger, \mathcal{A}^\dagger)\). Let \( \mathcal{P}_1^\dagger(\phi) \) be the slicing associated with \((Z_1^\dagger, \mathcal{A}^\dagger)\). We first investigate the category \( \mathcal{P}_1^\dagger(\phi) \) when \( t = 0 \).

\begin{lemma}
The category \( \mathcal{P}_0^\dagger(\phi) \) is described in the following way:
\begin{enumerate}
\item[(i)] If \( 0 < \phi < 1/2 \), then \( \mathcal{P}_0^\dagger(\phi) = \mathcal{P}(\phi + 1/2) \).
\item[(ii)] \( \mathcal{P}_0^\dagger(1/2) \) consists of \( E \in \mathcal{P}(1) \) with \( \text{Hom}(\mathcal{C}, E) = 0 \).
\item[(iii)] If \( 1/2 < \phi < 1 \), then \( \mathcal{P}_0^\dagger(\phi) \) consists of objects \( E \in \langle \mathcal{P}(\phi + 1/2), \mathcal{C} \rangle \) with \( \text{Hom}(\mathcal{C}, E) = 0 \).
\item[(iv)] \( \mathcal{P}_0^\dagger(1) \) coincides with \( \langle \mathcal{P}(3/2), \mathcal{C} \rangle \).
\end{enumerate}
\end{lemma}

\begin{proof}
We only prove the case when \( 1/2 < \phi < 1 \). One can prove the other cases similarly. Let us take \( E \in \mathcal{P}_0^\dagger(\phi) \) with \( 1/2 < \phi < 1 \), i.e. \( E \in \mathcal{A}^\dagger \) is \( \mu_0 \)-semistable with \( \mu_0^\dagger(E) = -1/\tan(\pi \phi) > 0 \). We have the exact sequence
\[ 0 \to H_\mathcal{A}^{-1}(E)[1] \to E \to H_\mathcal{A}^0(E) \to 0 \]
in $A^\dagger$. Let $F \subset \mathcal{H}_A^0(E)$ be the maximum subobject in $A$ with $F \in C$, and let $T = \mathcal{H}_A^0(E)/F$. If $T \neq 0$ then $\mu_0^\dagger(T) < 0$, and this is not possible as $E$ is $\mu_0^\dagger$-semistable. Hence, $\mathcal{H}_A^0(E) \in C$. As $Z_0^\dagger(\mathcal{H}_A^0(E)) = 0$, by considering the HN filtration of $\mathcal{H}_A^{-1}(E)$, one can easily check that $\mathcal{H}_A^{-1}(E)$ is $\mu$-semistable. Hence, $E \in \langle \mathcal{P}(\phi + 1/2), C \rangle$ holds. Also the $\mu_0^\dagger$-semistability of $E$ implies that $\text{Hom}(C, E) = 0$.

Conversely, let us take an object $E \in \langle \mathcal{P}(\phi + 1/2), C \rangle$ for $1/2 < \phi < 1$ satisfying $\text{Hom}(C, E) = 0$, and an exact sequence

$$0 \to P \to E \to Q \to 0$$

in $A^\dagger$ with $P, Q \neq 0$. Note that $P \notin C$ as $\text{Hom}(C, E) = 0$. By taking the long exact sequence of the cohomology functor $H^\ast_A(*)$, we get $H_A^{-1}(P) \in \mathcal{P}((0, \phi - 1/2))$. Since $\mathcal{H}_A^0(P) \in \mathcal{P}(1/2, 1])$ and $P \notin C$, we have the inequalities

$$\mu_0^\dagger(P) \leq \mu_0^\dagger(\mathcal{H}_A^{-1}(P)[1]) \leq \mu_0^\dagger(E).$$

Therefore, $E$ is $\mu_0^\dagger$-semistable. \hfill $\square$

Let $Q$ be a quadratic form on $\Gamma/\Gamma_0$ which gives the support property of $(Z, A)$, and $\Gamma_0^\dagger = \Gamma_0 \cap \ker(\Delta_I)$ the subgroup of $\Gamma_0$ considered in Definition \ref{def:4}. Note that $Z_I^\dagger$ descends to the group homomorphism

$$Z_I^\dagger: \Gamma/\Gamma_0^\dagger \to \mathbb{C}.$$

**Lemma 2.20.** There exists $k_0 > 0$ such that the quadratic form

$$\Delta_{k,t} := kQ + t\Delta$$

on $\Gamma/\Gamma_0^\dagger$ is negative definite on $\ker(Z_I^\dagger)$ for any $k \in (0, k_0)$ and $t > 0$.

**Proof.** We first check that $\Delta_{k,t}$ is a quadratic form on $\Gamma/\Gamma_0^\dagger$. It is enough to check that $\Delta$ descends to $\Gamma/\Gamma_0^\dagger$, or equivalently for $\gamma_1 \in \Gamma_0^\dagger$ and $\gamma_2 \in \Gamma$, we have

$$\Delta(\gamma_1 + \gamma_2) - \Delta(\gamma_1) - \Delta(\gamma_2) = 0. \tag{12}$$

The LHS of \ref{eq:12} is calculated as

$$\text{Re} Z(\gamma_1)\Delta_R(\gamma_2) + \text{Re} Z(\gamma_2)\Delta_R(\gamma_1) + \text{Im} Z(\gamma_1)\Delta_I(\gamma_2) + \text{Im} Z(\gamma_2)\Delta_I(\gamma_1).$$

Since $\gamma_1 \in \Gamma_0^\dagger$, we have $\text{Re} Z(\gamma_1) = \text{Im} Z(\gamma_1) = \Delta_I(\gamma_1) = 0$. Also $\Delta_R|C = 0$ in Definition \ref{def:4} implies $\Delta_R(\gamma_1) = 0$. Therefore, \ref{eq:12} holds.

We take a non-zero $\gamma \in \Gamma/\Gamma_0^\dagger$ such that $Z_I^\dagger(\gamma) = 0$, i.e. $\text{Re} Z(\gamma) = 0$ and $\text{Im} Z(\gamma) + t\Delta_I(\gamma) = 0$. So we have

$$\Delta_{k,t}(\gamma) = kQ(\gamma) - (\text{Im} Z(\gamma))^2. \tag{13}$$

Since $Q$ is negative definite on $\ker(Z)$ in $\Gamma/\Gamma_0$, one can find $k_0 > 0$, which is independent of $t$ and $\gamma$, such that the RHS of \ref{eq:13} is always negative for any $k \in (0, k_0)$.

**Proposition 2.21.** For any $k \in (0, k_0)$, $t > 0$ and a $\mu_0^\dagger$-semistable object $E \in A^\dagger$, we have $\Delta_{k,t}(E) \geq 0$. In particular, we have $\Delta(E) \geq 0$. \hfill $\square$
Proof. Note that for any \( E \in \mathcal{A}_1 \), we have \( \text{Im} Z_t^\dagger(E) = -\text{Re} Z(E) \geq 0 \). We show the claim by the induction of \( -\text{Re} Z(E) \). We define \( R > 0 \) to be

\[
R := \min\{-\text{Re} Z(E) > 0 : E \in \mathcal{A}_1\}.
\]

If \( -\text{Re} Z(E) \) is either zero or \( R \), then any \( \mu_t^\dagger \)-semistable object \( E \in \mathcal{A}_1 \) is also \( \mu_t^\dagger \)-semistable for \( 0 < t \ll 1 \), and so it is \( \mu_t^\dagger \)-semistable. By Lemma 2.19, the object \( E \) satisfies \( Q(E) \geq 0 \) and \( \Delta(E) \geq 0 \), and so the inequality \( \Delta_{k,t}(E) \geq 0 \) holds. Suppose that \( -\text{Re} Z(E) > R \). If \( E \) is \( \mu_t^\dagger \)-semistable for \( 0 < t \ll 1 \), then again Lemma 2.19 shows that \( \Delta_{k,t}(E) \geq 0 \). Otherwise, there is \( 0 < t' < t \) such that \( E \) is \( \mu_t^\dagger \)-semistable, and an exact sequence

\[ 0 \to E_1 \to E \to E_2 \to 0 \]

with \( \mu_t^\dagger(E_1) = \mu_t^\dagger(E_2) \). By the induction hypothesis, we have \( \Delta_{k',t'}(E_i) \geq 0 \) for any \( k' \in (0, k_0) \) and \( i = 1, 2 \). Let \( C_{E,k',t} \subseteq \Gamma_{\mathbb{R}} \) be the cone defined by

\[
C_{E,k',t} := \{ v \in \Gamma_{\mathbb{R}} : Z_t^\dagger(v) \in \mathbb{R}_{<0}Z_t^\dagger(E), \Delta_{k,t}(v) \geq 0 \}.
\]

By [BMS, Lemma A.7] the cone \( C_{E,k',t} \) is convex, and so \( \Delta_{k',t'}(E) \geq 0 \). By setting \( k' = k't/t \), we obtain the inequality \( \Delta_{k,t}(E) = t\Delta_{k',t'}(E)/t' \geq 0 \). The last statement follows by taking \( k \to +0 \). \( \square \)

We have the following corollary:

**Corollary 2.22.** Suppose that \((Z, A)\) is good. Then \((Z_t^\dagger, \mathcal{A}_1)\) is a very weak stability condition such that the map defined by

\[
\mathbb{R}_{>0} \to \text{Stab}_{\text{wt}}(\mathcal{D}), \ t \mapsto (Z_t^\dagger, \mathcal{A}_1)
\]

is continuous.

**Proof.** By Lemma 2.20 and Proposition 2.21 \((Z_t^\dagger, \mathcal{A}_1)\) satisfies the support property. Hence, it is a very weak stability condition. In order to show that the map \((15)\) is continuous, it is enough to show that the map

\[
\mathbb{R}_{>0} \to \text{Slice}(\mathcal{D}), \ t \mapsto \{\mathcal{P}_t^\dagger(\phi)\}_{\phi \in \mathbb{R}}
\]

is continuous. Let \( K \subseteq \mathbb{R}_{>0} \) be a closed interval. Since the quadratic form \( \Delta_{k,t} \) is continuous with respect to \( t \), we can choose a quadratic form \( \Delta_K \) which is independent of \( t \) giving the support property of \((Z_t^\dagger, \mathcal{A}_1)\) for any \( t \in K \). In particular, for any \( t, t' \in K \) and \( E \in \mathcal{P}_t^\dagger(\phi) \) with \( \phi \in (0, 1) \), we have the inequality (see Remark 2.3)

\[
1 - \frac{Z_t^\dagger(E)}{Z_t^\dagger(\phi)} < C|t - t'|,
\]

where \( C > 0 \) is a constant which is independent of \( t, t' \) and \( \phi \). If necessary, we may replace \( K \) by a smaller interval such that \( C|t - t'| \) is small enough, say less than \( 1/8 \). Let \( E \to F \) be a destabilizing quotient of \( E \) in \( \mathcal{A}_1 \) with respect to the \( \mu_t^\dagger \)-stability. Applying \((17)\) to \( E \) and \( F \), we have

\[
|\arg Z_t^\dagger(E) - \arg Z_t^\dagger(\phi)| < \pi \theta,
\]

\[
|\arg Z_t^\dagger(F) - \arg Z_t^\dagger(\phi)| < \pi \theta.
\]
Here $\theta \in [0, 1)$ is determined by $\sin(\pi \theta / 2) = C |t - t'|$. On the other hand, we have $\arg Z^t_0(E) \leq \arg Z^t_0(F)$ and $\arg Z^t_0(E) > \arg Z^t_0(F)$. Therefore, 
\[ |\arg Z^t_0(E) - \arg Z^t_0(F)| < \pi \theta. \]

Similar arguments show the same inequality holds for destabilizing subobjects of $E$. As a result, we obtain 
\[ \mathcal{P}_t^\dagger(\phi) \subset \mathcal{P}_t^\dagger((\phi - \theta, \phi + \theta)). \]

As $\theta \to 0$ when $t' \to t$, and $\mathcal{P}_t^\dagger(1)$ is independent of $t$, the map (10) is continuous.

The following corollary follows from Lemma 2.11 and Corollary 2.22.

**Corollary 2.23.** Suppose that $(Z, A)$ is good. Then the following conditions are equivalent for $t > 0$:

(i) $(Z^t, A^t)$ is a stability condition.

(ii) $Z^t|_{\mathcal{P}(1)}$ is a stability condition on $\mathcal{P}(1)$.

(iii) $\Delta_I$ is strictly negative on $C \setminus \{0\}$.

**Remark 2.24.** Even if $C^\dagger \neq 0$, as $C^\dagger \subseteq C$, the very weak stability condition $(Z^t, A^t)$ for $t > 0$ is closer to a stability condition.

Finally in this subsection, we give a speculative argument for the BG inequality of $(Z^t, A^t)$. Let $E \in A^t$ be a $\mu^t_1$-semistable object. By Proposition 2.21 $\Delta(E) \geq 0$. Hence, we have 
\[ \text{Re} Z^t_1(E)\Delta_I(E) + \text{Im} Z^t_1(E)(-\Delta_R(E)) \geq t\Delta_I(E)^2 \geq 0. \]

Moreover, $\Delta_I|_{C^\dagger} = 0$ and $\Delta_R|_{C^\dagger} = 0$ holds. This implies that, although the inequality (18) is not a BG inequality for $(Z^t, A^t)$, it is very close to it. We expect that a BG inequality for $(Z^t, A^t)$ is obtained by adding some natural correction term $\nabla_I$ on $-\Delta_R$ satisfying $\nabla_I|_{C} \leq (\neq) 0$. Namely, by setting 
\[ (\Delta^t_R, \Delta^t_I) = (\Delta_I, \nabla_I - \Delta_R), \]

we may obtain a BG inequality for $(Z^t, A^t)$, i.e. for any $\mu^t_1$-semistable object $E \in A^t$, we may have 
\[ \text{Re} Z^t_1(E)\Delta^t_R(E) + \text{Im} Z^t_1(E)\Delta^t_I(E) \geq 0. \]

In Subsection 3.3 we will see that the above form of the BG inequality (20) exactly matches with the conjectural BG inequality for tilt semistable objects on 3-folds in [BMT14].

### 2.5. The limit $t \to +0$

We will carry the notation introduced in the previous subsection. The purpose of this subsection is to investigate the $\mu^t_1$-semistable objects for $0 < t \ll 1$. We first prepare the following lemma:

**Lemma 2.25.** Let $E \in A^t$ be a $\mu^t_1$-semistable object with $\mu^t_1(E) < \infty$. Let $0 \to E_1 \to E \to E_2 \to 0$ be a short exact sequence in $A^t$ with $\mu^t_1(E_1) = \mu^t_1(E_2)$. Let $R_i := \text{Re} Z(E_i)$ and $I_i := \text{Im} Z(E_i)$. Then 
\[ \Delta(E) \geq \Delta(E_1) + \Delta(E_2) + \frac{R_1R_2}{t} \left( \frac{I_1}{R_1} - \frac{I_2}{R_2} \right)^2. \]
Proof. The condition $\mu^1_i(E_1) = \mu^1_i(E_2)$ is equivalent to
\[
\frac{I_1 + t \Delta I(E_1)}{R_1} = \frac{I_2 + t \Delta I(E_2)}{R_2}.
\]
Also since $\Im Z_i(E_i) = -R_i > 0$, the condition $\Delta(E_i) \geq 0$ in Proposition 2.21 is equivalent to
\[
\Delta_R(E_i) \leq -\frac{I_i \Delta I(E_i)}{R_i}.
\]
We have
\[
\Delta(E) - \Delta(E_1) - \Delta(E_2) = R_1 \Delta R(E_2) + R_2 \Delta R(E_1) + I_1 \Delta I(E_2) + I_2 \Delta I(E_1)
\geq -R_1 \frac{I_2 \Delta I(E_2)}{R_2} - R_2 \frac{I_1 \Delta I(E_1)}{R_1} + I_1 \Delta I(E_2) + I_2 \Delta I(E_1)
= R_1 R_2 \left(\frac{I_1}{R_1} - \frac{I_2}{R_2}\right) \left(\frac{\Delta I(E_2)}{R_2} - \frac{\Delta I(E_1)}{R_1}\right)
= \frac{R_1 R_2}{t} \left(\frac{I_1}{R_1} - \frac{I_2}{R_2}\right)^2.
\]
Here we have used (22) for the first inequality, and (21) for the last equality. Therefore, we obtain the desired result.\]

For $v \in \Gamma$, we define $M_i^1(\varpi)$ to be the set of isomorphism classes of $\mu_i^1$-semistable objects $E \in \mathcal{A}^I$ with $c_1(E) = \varpi$ in $\Gamma/\Gamma_0^I$. Note that, similarly to the existence of wall and chamber structures in the space of Bridgeland stability conditions (see [Bri08, Proposition 9.3]), the result of Corollary 2.22 implies the existence of locally finite set of points $W \subset \mathbb{R}_{>0}$ called walls, such that $M_i^1(\varpi)$ is constant on each connected component of $\mathbb{R}_{>0} \setminus W$. In the following lemma, we show that there is no point $t \in W$ for $0 < t < 1$.

**Lemma 2.26.** There is $t_0 > 0$ such that $M_i^1(\varpi)$ is constant for $0 < t < t_0$.

Proof. We may assume that $\Im Z_i(v) = -\Re Z(v)$ is positive, and let $R := \Re Z(v)$. For $E \in M_i^1(\varpi)$, suppose that there is an exact sequence
\[
0 \to E_1 \to E \to E_2 \to 0
\]
in $\mathcal{A}^I$ such that $\mu_i^1(E_1) = \mu_i^1(E_2)$ and $\mu_i^1(E_1) > \mu_i^1(E_2)$ for $0 < \varepsilon \ll 1$. Using the same notation in the proof of Lemma 2.25, these conditions imply that
\[
\frac{I_1}{R_1} - \frac{I_2}{R_2} > 0.
\]
We take $r \in \mathbb{R}_{>0}$ so that $r \Im Z$ is an integer valued function on $\Gamma$. Then the LHS of (23) is bigger than or equal to $1/r R_1 R_2$. Since $0 < -R_i < -R$ with $R_1 + R_2 = R$, and $\Delta(E_i) \geq 0$, from Lemma 2.25 we have
\[
\Delta(E) \geq \frac{1}{tr^2 R_1 R_2} \geq \frac{4}{tr^2 R^2}.
\]
By the proof of Lemma 2.20, the quadratic form \( \Delta \) descends to \( \Gamma/\Gamma_0^1 \), and so \( \Delta(E) = \Delta(v) \). Therefore, by setting \( t_0 := 4A(e)^{-2}\Re Z \), the set of objects \( M_1^1(\tau) \) is constant for \( 0 < t < t_0 \).

We consider the restriction of \( Z_t^1 \) to \( P_0^1(\phi) \). For \( 0 < \phi < 1 \), the image of \( Z_t^1|_{P_0^1(\phi)} \) is contained in the upper half plane, and \( (Z_t^1|_{P_0^1(\phi)}, P_0^1(\phi)) \) is a pre-stability condition. The restriction of \( \mu_t^1 \) to \( P_0^1(\phi) \) is given by

\[
\mu_t^1|_{P_0^1(\phi)} = \tan(\pi \phi) + \frac{\Delta_I}{\Re Z}.
\]

Hence, \( \mu_t^1|_{P_0^1(\phi)} \)-semistable objects on \( P_0^1(\phi) \) do not depend on \( t \), and coincide with the \( \lambda|_{P_0^1(\phi)} \)-semistable objects, where \( \lambda \) is the slope function

\[
(24) \quad \lambda := \frac{\Delta_I}{\Re Z}.
\]

**Proposition 2.27.** Let \( v \in \Gamma \) and let \( t_0 \) be the corresponding positive real number as in Lemma 2.20. Then for \( 0 < t < t_0 \), the set \( M_1^1(\tau) \) consists of isomorphism classes of \( \lambda|_{P_0^1(\phi)} \)-semistable objects \( E \in P_0^1(\phi) \) with \( \overline{\text{cl}(E)} = \tau \). Here \( \phi \) is determined by \( \mu_0^1(v) = -1/\tan(\pi \phi) \).

**Proof.** Let us take an object \( [E] \in M_1^1(\tau) \) for \( 0 < t < t_0 \). Then \( E \) must be \( \mu_0^1 \)-semistable, and so it is an object in \( P_0^1(\phi) \) for some \( 0 < \phi \leq 1 \) satisfying \( \mu_0^1(v) = -1/\tan(\pi \phi) \). Since \( E \) is \( \mu_t^1 \)-semistable, it must be \( \lambda|_{P_0^1(\phi)} \)-semistable.

Conversely, take a \( \lambda|_{P_0^1(\phi)} \)-semistable object \( E \in P_0^1(\phi) \) with \( \overline{\text{cl}(E)} = \tau \). Let us take a short exact sequence

\[
0 \to P \to E \to Q \to 0
\]

in \( \mathcal{A}^1 \). Since \( E \) is \( \mu_0^1 \)-semistable, we have \( \mu_0^1(P) \leq \mu_0^1(Q) \). If \( \mu_0^1(P) < \mu_0^1(Q) \) then \( \mu_t^1(P) < \mu_t^1(Q) \) for \( 0 < t < t_0 \) as there are no walls on \((0, t_0)\). If \( \mu_0^1(P) = \mu_0^1(Q) \) then \( P, Q \in P_0^1(\phi) \) and the \( \lambda|_{P_0^1(\phi)} \)-semistability of \( E \) implies \( \mu_t^1(P) \leq \mu_t^1(Q) \) for \( 0 < t < t_0 \). Therefore, we have \( [E] \in M_1^1(\tau) \). \( \square \)

### 3. Bogomolov-Gieseker inequality conjecture

In this section, we interpret the tilt stability and the double tilting construction in [BMT14] in terms of tilting in the general framework of very weak stability conditions in the previous section. We also recall the BG inequality conjecture in [BMT14], and give two equivalent forms of it generalizing [BMS] Theorem 4.2, Theorem 5.4 due to Bayer, Macrì and Starr.

#### 3.1. The space of Bridgeland stability conditions

Let \( X \) be an \( n \)-dimensional smooth projective variety. Let

\[
(25) \quad B \in \text{NS}(X)_\mathbb{Q}, \quad \text{and} \quad \omega \in \text{NS}(X)_\mathbb{R} \text{ an ample class with } \omega^2 \text{ rational}.
\]

That is \( \omega = mH \) for some ample divisor class \( H \in \text{NS}(X) \) with \( m^2 \in \mathbb{Q}_{>0} \).
Let $\text{ch}^B(E) := e^{-B} \text{ch}(E)$ be the twisted Chern character of $E$ with respect to $B$. We set $v^B_j(E) := \omega^{n-j} \text{ch}^B_j(E)$, and

$$v^B(E) := (v^B_0(E), v^B_1(E), \cdots, v^B_n(E)).$$

Let

$$\Gamma_{\omega,B} \subset m^n \oplus \cdots \oplus m \oplus Q \approx Q^{n+1}$$

be the free abelian group of rank $(n+1)$ given by the image of the map

$$v^B : K(X) \rightarrow \Gamma_{\omega,B}.$$ (26)

In what follows, we write an element of $\Gamma_{\omega,B}$ as $(v^B_0, v^B_1, \cdots, v^B_n)$. Applying the definitions in Subsection 2.1 to the following setting

$$D = D^b \text{Coh}(X), \quad \Gamma = \Gamma_{\omega,B}, \quad \text{cl} = v^B$$

we have the space of very weak stability conditions

$$\text{Stab}^{vw}_{\omega,B}(X) := \text{Stab}^{vw}_{\Gamma_{\omega,B}}(D^b \text{Coh}(X)).$$

The subset $\text{Stab}_{\omega,B}(X) \subset \text{Stab}^{vw}_{\omega,B}(X)$ of Bridgeland stability conditions admit the local homeomorphism by the forgetful map

$$\text{Stab}_{\omega,B}(X) \rightarrow \text{Hom}(\Gamma_{\omega,B}, \mathbb{C}), \quad (Z, A) \mapsto Z$$

as long as $\text{Stab}_{\omega,B}(X)$ is non-empty.

3.2. Tilting via slope stability. We interpret the classical slope stability on $\text{Coh}(X)$ as a very weak stability condition. We set $Z : \Gamma \rightarrow \mathbb{C}$ to be

$$Z(v) = -v^B_1 + v^B_0 i.$$ (27)

Here we have denoted $\Gamma = \Gamma_{\omega,B}$ for simplicity. It is easy to check that

$$\text{Stab}^{vw}_{\omega,B}(X) \subset \text{Stab}^{vw}_{\omega,B}(X).$$

Indeed, the associated slope function on $\text{Coh}(X)$ is given by the twisted slope

$$\mu_{\omega,B}(E) := \frac{v^B_1(E)}{v^B_0(E)}.$$ (28)

Here we set $\mu_{\omega,B}(E) = \infty$ when $E$ is a torsion sheaf. Since $\mu_{\omega,B} = \mu_{\omega} - (B\omega^2)/\omega^3$ for $\mu_{\omega} := \mu_{0,\omega}$, $\mu_{\omega,B}$-stability is independent of $B$. Also the existence of HN filtrations with respect to the $\mu_{\omega}$-stability is well-known (see [HL10, Section 1] for further details). Moreover, the trivial quadratic form $Q = 0$ gives the support property of (27).

Remark 3.1. If we take $\Gamma$ to be the image of the Chern character map in $H^*(X, \mathbb{Q})$ rather than $\Gamma_{\omega,B}$, the pair $(Z, \text{Coh}(X))$ does not satisfy the support property when the Picard number is bigger than one.

Suppose that $n \geq 2$. The category $\mathcal{C}$ in (28) is given by

$$\mathcal{C} = \text{Coh}_{\leq n-2}(X)$$

and the saturated subgroup $\Gamma_0 \subset \Gamma$ generated by $v^B(\mathcal{C})$ is given by

$$\Gamma_0 = \{(v^B_0, v^B_1, \cdots, v^B_n) \in \Gamma : v^B_0 = v^B_1 = 0\}.$$
We define the quadratic form $\overline{\omega}_{\omega,B}$ on $\Gamma$ by
\[
\overline{\omega}_{\omega,B}(v) := (v_1^B)^2 - 2v_0^B v_2^B.
\]

**Lemma 3.2.** The very weak stability condition $(Z, \text{Coh}(X))$ satisfies the BG inequality $\overline{\omega}_{\omega,B}(E) \geq 0$ in the sense of Definition 2.7.

**Proof.** For $E \in D^b \text{Coh}(X)$, we have
\[
\overline{\omega}_{\omega,B}(E) = (\omega^{n-1} \text{ch}_1^B(E))^2 - \omega^n (\omega^{n-2} \text{ch}_1^B(E)^2) + \omega^n \Delta_\omega(E),
\]
where $\Delta_\omega(E)$ is defined by
\[
\Delta_\omega(E) = \omega^{n-2} (\text{ch}_1^B(E)^2 - 2 \text{ch}_0^B(E) \text{ch}_2^B(E))
\]
which is independent of $B$. The classical Bogomolov-Gieseker inequality is $\Delta_\omega(E) \geq 0$ for any torsion free $\mu_{\omega,B}$-semistable sheaf $E$. Together with the Hodge index theorem, the inequality $\overline{\omega}_{\omega,B}(E) \geq 0$ follows for any $\mu_{\omega,B}$-semistable sheaf $E \in \text{Coh}(X)$. Note that $\overline{\omega}_{\omega,B}$ can be written as
\[
\overline{\omega}_{\omega,B}(v) = \text{Re} Z(v) \cdot (-v_1^B) + \text{Im} Z(v) \cdot (-2v_2^B).
\]
We set
\[
(\Delta_R, \Delta_I) = (-v_1^B, -2v_2^B).
\]
Since $v_1^B = 0$ on $\text{Coh}_{\leq n-2}(X)$, and $-2v_2^B(F) \leq 0$ for $F \in \text{Coh}_{\leq n-2}(X)$ with the equality if and only if $F \in \text{Coh}_{\leq n-3}(X)$, the pair $(Z, \text{Coh}(X))$ satisfies the BG inequality.

Note that the category $\mathcal{C}^1$ defined in (9) is given by
\[
\mathcal{C}^1 = \text{Coh}_{\leq n-3}(X).
\]

**Lemma 3.3.** The very weak stability condition $(Z, \text{Coh}(X))$ is good in the sense of Definition 2.13.

**Proof.** The conditions (i), (ii) in Definition 2.13 are obvious. As for (iii), by (30), the saturation of the subgroup of $\Gamma_0$ generated by $v_1^B(\mathcal{C}^1) \subset \Gamma_0$ coincide with $v_1^B = 0$ in $\Gamma_0$. As for (iv), any torsion free sheaf $U \in \text{Coh}(X)$ fits into the short exact sequence
\[
0 \to U \to U^{\vee \vee} \to T \to 0
\]
with $T \in \text{Coh}_{\leq n-2}(X)$. Since $U^{\vee \vee}$ is reflexive, we have $\text{Hom}(\mathcal{C}, U^{\vee \vee}[1]) = 0$ as required for condition (iv).

3.3. Tilt stability via very weak stability conditions. Let $(Z, \text{Coh}(X))$ be the very weak stability condition given by (27). By Corollary 2.22 we have the associated very weak stability conditions
\[
(Z^t, \text{Coh}^t(X)) \in \text{Stab}_{\omega,B}^w(\mathcal{X}), \ t \in \mathbb{R}_{>0}.
\]
Using (27), the group homomorphism $Z^t_1 = -iZ + t\Delta_I$ is given by
\[
Z^t_1(v) = (-2tv_2^B + v_0^B) + v_1^B i.
\]
The associated slope function $\mu^t_1$ is given by
\[
\mu^t_1 = \frac{2tv_2^B - v_0^B}{v_1^B}.
\]
Let us describe Coh$^\dagger(X)$ in terms of tilting of Coh$(X)$.

**Definition 3.4.** For a given subset $I \subset \mathbb{R} \cup \{\infty\}$, the subcategory $\text{HN}_{\omega,B}^\mu(I) \subset \text{Coh}(X)$ is defined by

$$\text{HN}_{\omega,B}^\mu(I) = \{E \in \text{Coh}(X) : E \text{ is $\mu_{\omega,B}$-semistable with } \mu_{\omega,B}(E) \in I\}.$$  

Here $(*)$ means the extension closure. We also define the pair of subcategories $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ of Coh$(X)$ to be

$$\mathcal{T}_{\omega,B} = \text{HN}_{\omega,B}^\mu((0, \infty]), \quad \mathcal{F}_{\omega,B} = \text{HN}_{\omega,B}^\mu((\infty, 0]).$$

If $I = \{\vartheta\}$ for some $\vartheta \in \mathbb{R} \cup \{\infty\}$, we just write $\text{HN}_{\omega,B}^\mu(I)$ as $\text{HN}_{\omega,B}^\mu(\vartheta)$. Therefore, in terms of the slicing $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ for the $\mu_{\omega,B}$-stability

$$\mathcal{P}(\phi) = \text{HN}_{\omega,B}^\mu(-1/\tan(\pi\phi)).$$

**Remark 3.5.** Since $\text{ch}_1^B = \text{ch}_1 - B \cdot \text{ch}_0$, we have $\mu_{\omega,B}(E) - \vartheta = \mu_{\omega,B+\vartheta}(E)$ for any $\vartheta \in \mathbb{R}$. Therefore, for any $r \in \mathbb{R}_{>0}$ we have

$$\text{HN}_{\omega,B}^\mu(\vartheta) = \text{HN}_{\omega,B+\vartheta}^\mu(0).$$

The existence of Harder-Narasimhan filtrations with respect to $\mu_{\omega,B}$-stability implies that the pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ is a torsion pair (see [HRS96]) on Coh$(X)$. Let the abelian category

$$\mathcal{B}_{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle \subset D^b\text{Coh}(X)$$

be the corresponding tilt of Coh$(X)$. Obviously, we have Coh$^\dagger(X) = \mathcal{B}_{\omega,B}$.

**Remark 3.6.** If $\dim X = 2$, then $\mathcal{C}^\dagger = \{0\}$ and $(Z^\dagger_1, \mathcal{B}_{\omega,B})$ is a Bridgeland stability condition constructed by [Bri08], [AB13].

From here onwards we assume $\dim X = 3$. We relate $(Z^\dagger_1, \mathcal{B}_{\omega,B})$ with the tilt stability introduced in [BMT14]. Let $Z_{\omega,B} : K(X) \to \mathbb{C}$ be the group homomorphism defined by

$$Z_{\omega,B}(E) = -\int_X e^{-i\omega} \text{ch}^B(E) = \left(-v_3^B(E) + \frac{1}{2} v_1^B(E)\right) + i \left(v_2^B(E) - \frac{1}{6} v_3^B(E)\right).$$

Note that $Z_{\omega,B}$ factors through $v^B : K(X) \to \Gamma_{\omega,B}$. Following [BMT14], the tilt-slope $\nu_{\omega,B}(E)$ of $E \in \mathcal{B}_{\omega,B}$ is defined by

$$\nu_{\omega,B}(E) = \frac{\text{Im} Z_{\omega,B}(E)}{v_1^B(E)}.$$  

Here we set $\nu_{\omega,B}(E) = \infty$ when $v_1^B(E) = 0$. In the above notation, we have

$$6\nu_{\omega,B}(E) = \mu_{i=3}(E)$$

from the formula [32]. The associated $\nu_{\omega,B}$-stability on $\mathcal{B}_{\omega,B}$ was called tilt-stability in [BMT14].

Let us discuss the BG inequality for $(Z^\dagger_1, \mathcal{B}_{\omega,B})$. By [30], the category $\mathcal{C}^\dagger$ is given by Coh$_0(X)$. Therefore, a natural choice of the correction term $\nabla_l$
in (20) may be of the form \( av^3_B \) for some \( a < 0 \). By (29), the functions (19) may be of the form 
\[
\langle \Delta^1_R, \Delta^1_I \rangle = (-2v^2_B, v^1_B + av^3_B)
\]
and the inequality (20) may be of the following form 
\[
(-2tv^2_B + v^0_B) \cdot (-2v^2_B) + v^1_B \cdot (v^1_B + av^3_B) \geq 0.
\]
We require that the above inequality becomes an equality when \( v^B_i = 1/i! \); i.e. when \( B \) is proportional to \( \omega \), it is the Chern character of a line bundle \( L \) with \( c_1(L) \) proportional to \( \omega \). This requirement uniquely determines \( a = -6t \), and the above inequality becomes 
\[
(v^1_B)^2 - 2v^0_B v^2_B + 2t(2(v^2_B)^2 - 3v^1_B v^3_B) \geq 0.
\]
Now we define the quadratic form \( \nabla_{\omega,B} \) on \( \Gamma \) to be 
\[
\nabla_{\omega,B}(v) := 2(v^2_B)^2 - 3v^1_B v^3_B.
\]
By setting \( t = 3 \), we arrive at the following conjecture:

Conjecture 3.7. Let \( X \) be a smooth projective 3-fold. Then for any \( \nu_{\omega,B} \)-semistable object \( E \in B_{\omega,B} \), we have the inequality
\[
\Delta_{\omega,B}(E) + 6\nabla_{\omega,B}(E) \geq 0.
\]

Note that by setting \( \nu_{\omega,B}(E) = 0 \), we obtain the following conjecture stated in [BMT14]:

Conjecture 3.8 ([BMT14]). Let \( X \) be a smooth projective 3-fold. Then for any \( \nu_{\omega,B} \)-semistable object \( E \in B_{\omega,B} \) with \( \nu_{\omega,B}(E) = 0 \), i.e. \( v^2_B(E) = v^0_B(E)/6 \), we have the inequality
\[
v^3_B(E) \leq \frac{1}{18} v^1_B(E).
\]

Obviously Conjecture 3.8 is implied by Conjecture 3.7 Conversely from [BMS] Theorem 4.2, if Conjecture 3.8 holds for all \( (B, \omega) \) with \( B \) proportional to \( \omega \), then Conjecture 3.7 also holds for all \( (B, \omega) \) with \( B \) proportional to \( \omega \). In Subsection 3.7 we will generalize the result of [BMS] Theorem 4.2, and show that Conjecture 3.7 and Conjecture 3.8 are indeed equivalent.

3.4. Double tilting construction. In this subsection, we change the notation and set \( W = Z_{t=3}^1 \) where \( Z^1_t \) is given by (31), i.e.
\[
W(v) = (-6v^2_B + v^0_B) + v^1_B i.
\]
We consider the very weak stability condition \( (W, B_{\omega,B}) \). Note that the associated semistable objects coincide with \( \nu_{\omega,B} \)-semistable objects in \( B_{\omega,B} \).

Proposition 3.9. Suppose that Conjecture 3.7 holds. Then \( (W, B_{\omega,B}) \) satisfies the BG inequality and is good.

Proof. The inequality in Conjecture 3.7 implies that for any \( \nu_{\omega,B} \)-semistable object \( E \in B_{\omega,B} \), we have
\[
\text{Re} W(E) \Delta^1_R(E) + \text{Im} W(E) \Delta^1_I(E) \geq 0,
\]
where
\[(\Delta^R_\nu, \Delta^I_\nu) = (-2v^B_2, v^B_1 - 18v^B_3). \tag{36}\]
Note that the category $C^\dagger \subset B_{\omega,B}$ of objects $E \in B_{\omega,B}$ with $W(E) = 0$ coincides with $\text{Coh}_0(X)$. Since $\Delta^R|_{\text{Coh}_0(X)} \equiv 0$ and $\Delta^I|_{\text{Coh}_0(X)}\{0\} < 0$, the very weak stability condition $(W, B_{\omega,B})$ satisfies the BG inequality.

We check that $(W, B_{\omega,B})$ is good. The condition (i) in Definition 2.13 is obvious. The condition (ii) follows from Lemma 2.17 since $B_{\omega,B}$ is a tilt of a good very weak stability condition on $\text{Coh}(X)$. Also the category $C^{\dagger\dagger}$ of objects $F \in C^\dagger = \text{Coh}_0(X)$ with $\Delta^I_\nu(F) = 0$ is $\{0\}$. Therefore, we also have the conditions (iii) and (iv) as required.

As we observed in the proof of Proposition 3.9, the category $C^{\dagger\dagger}$ is zero. Hence, by Corollary 2.22 we have the associated one parameter family of stability conditions
\[(W^t, B^\dagger_{\omega,B}) \in \text{Stab}_{\omega,B}(X), \quad t \in \mathbb{R}_{>0}. \tag{37}\]
The group homomorphism $W^t = -iW + t\Delta^I_\nu$ is given by
\[W^t_\nu(v) = ((1 + t)v^B_1 - 18tv^B_3) + (6v^B_2 - v^B_0)i. \]
The associated slope function $\nu^t_\nu$ is given by
\[\nu^t_\nu = \frac{18tv^B_3 - (1 + t)v^B_1}{6v^B_2 - v^B_0}. \]
By comparing it with $Z_{\omega,B}$ given by (33), we have
\[\frac{9}{4} \nu^t_{1/8} = -\frac{\text{Re} Z_{\omega,B}}{\text{Im} Z_{\omega,B}}. \tag{38}\]
We describe $B^{\dagger\dagger}_{\omega,B}$ in terms of tilt stability. Similar to Definition 3.4 for tilt-slope $\nu_{\omega,B}$ stability on $B_{\omega,B}$ we define the following:

**Definition 3.10.** For a given subset $I \subset \mathbb{R} \cup \{\infty\}$, the subcategory $\text{HN}^\nu_{\omega,B}(I) \subset B_{\omega,B}$ is defined by
\[\text{HN}^\nu_{\omega,B}(I) = \langle E \in B_{\omega,B} : E \text{ is } \nu_{\omega,B}-\text{semistable with } \nu_{\omega,B}(E) \in I \rangle. \]
We also define the pair of subcategories $(T'_{\omega,B}, F'_{\omega,B})$ of $B_{\omega,B}$ to be
\[T'_{\omega,B} = \text{HN}^\nu_{\omega,B}(0, \infty), \quad F'_{\omega,B} = \text{HN}^\nu_{\omega,B}((-\infty, 0]). \]
We also set $\text{HN}^\nu_{\omega,B}(I) = \text{HN}^\nu_{\omega,B}(\emptyset)$ when $I = \{\emptyset\}$. The HN property of tilt stability implies that the pair of subcategories $(T'_{\omega,B}, F'_{\omega,B})$ forms a torsion pair on $B_{\omega,B}$. By tilting, we have another heart
\[A_{\omega,B} = \langle F'_{\omega,B}[1], T'_{\omega,B} \rangle \subset D^b\text{Coh}(X). \]
By the construction, we have $B^{\dagger\dagger}_{\omega,B} = A_{\omega,B}$. By (38), the condition (37) for $t = 1/8$ is equivalent to the statement
\[\sigma_{\omega,B} := (Z_{\omega,B}, A_{\omega,B}) \in \text{Stab}_{\omega,B}(X), \tag{39}\]
which was conjectured in [BMT14].
3.5. Slope bounds for cohomology sheaves of objects in \( B_{\omega,B} \). In this and next subsections, we prepare some results on tilt stability to show the equivalence of Conjectures 3.7 and 3.8. We will carry the notations introduced in Subsections 3.3 and 3.4.

It is straightforward to check that, for any \( \vartheta \in \mathbb{R} \)

\[
\sum_{j=0}^{i} \frac{(-1)^j}{j!} \vartheta^j v_{i-j},
\]

and for any \( \beta \in \mathbb{R}_{>0} \), we have \( \Delta_{\vartheta, B_{\omega}} \) for \( E \in D^b \text{Coh}(X) \), we have the following identities:

\[
\text{Im} Z_{\omega,B}(E) - \vartheta v_1(E) = \frac{1}{\eta} \text{Im} Z_{\eta \vartheta, B_{\omega}}(E)
\]

\[
v_0(E) \text{Im} Z_{\omega,B}(E) = -\frac{1}{2} \Delta_{\vartheta, B_{\omega}}(E) + \frac{1}{2} v_1(E) v_1^{B+\omega/\sqrt{3}}(E).
\]

Proof. The identity (41) follows easily by simplifying \( \text{Im} Z_{\eta \vartheta, B_{\omega}}(E) \) with the relations (40).

Let us show the identity (42). Let \( B' = B - \omega/\sqrt{3} \). By the definition of \( \text{Im} Z_{\omega,B}(E) \), and \( \text{ch} B(E) = e^{-\omega/\sqrt{3}} \text{ch} B'(E) \), we have

\[
\text{Im} Z_{\omega,B}(E) = v_2(B') - \frac{1}{\sqrt{3}} v_1(E).
\]

Therefore,

\[
v_0(E) \text{Im} Z_{\omega,B}(E)
\]

\[
= v_0(E) v_2(B') - \frac{1}{2} (v_1(B'))^2 + \frac{1}{2} v_1(B') \left( v_1(B') - \frac{2}{\sqrt{3}} v_0(E) \right)
\]

\[
= -\frac{1}{2} \Delta_{\omega,B}(E) + \frac{1}{2} v_1(B') v_1^{B+\omega/\sqrt{3}}(E)
\]

as required. \( \square \)

Consequently, we have

\[
\nu_{\omega,B}(E) - \vartheta = \frac{\text{Im} Z_{\omega,B}(E) - \vartheta v_1(B')}{v_1(B')}
\]

\[
= \frac{\text{Im} Z_{\eta \vartheta, B_{\omega}}(E)}{\eta v_1(B')}(E).
\]

Assume \( v_0^B(E) \neq 0 \). From multiplying both the denominator and the numerator by \( v_0^B(E) \), we get

\[
\nu_{\omega,B}(E) - \vartheta = \frac{-\Delta_{\omega,B}(E) + v_1(B+(\vartheta-\eta/\sqrt{3})\omega)(E) v_1(B+(\vartheta+\eta/\sqrt{3})\omega)(E)}{2 \eta v_0^B(E) v_1^B(E)}.
\]
We have the following proposition, which generalizes the claims in [MPa Prop. 3.1, 3.2] and [MPb Prop. 4.1, 4.2]. Our proofs are also somewhat similar to them.

**Proposition 3.12.** Let $\vartheta$ be any real number and let $\eta = \sqrt{3}\vartheta^2 + 1$. Let $E \in \mathcal{B}_{\omega,B}$ and $E_i = \mathcal{H}^i(E)$. Then we have the following:

(i) if $E \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta))$, then $E_{-1} \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta - \eta/3))$;

(ii) if $E \in \mathcal{H}^\nu_{\omega,B}((\vartheta, \infty))$, then $E_0 \in \mathcal{H}^\mu_{\omega,B}((\vartheta + \eta/3, \infty))$; and

(iii) if $E$ is tilt semistable with $\nu_{\omega,B}(E) = \vartheta$, then

(a) $E_{-1} \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta - \eta/3))$ with equality $\mu_{\omega,B}(E_{-1}) = \vartheta - \eta/3$

holds if and only if $v_2^{B+\vartheta}(1 - \eta/3)(E_{-1}) = 0$, and

(b) when $E_0$ is torsion free $E_0 \in \mathcal{H}^\nu_{\omega,B}((\vartheta + \eta/3, \infty))$ with equality

$\mu_{\omega,B}(E_0) = \vartheta + \eta/3$ holds if and only if $v_2^{B+\vartheta}(1 - \eta/3)(E_0) = 0$.

**Proof.** Note that any object $E \in \mathcal{B}_{\omega,B}$ fits into the short exact sequence

$$0 \to E_{-1}[1] \to E \to E_0 \to 0$$

in $\mathcal{B}_{\omega,B}$.

(i) If $E \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta))$ then we have $E_{-1}[1] \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta))$. Let $E_{-1}$ be the HN $\mu_{\omega,B}$-semistable factor of $E_{-1}$ with the highest $\mu_{\omega,B}$ slope. From the HN filtration of $E_{-1}$ with respect to the $\mu_{\omega,B}$-stability, $E_{-1}[1]$ fits into the short exact sequence

$$0 \to E_{-1}^+[1] \to E_{-1}[1] \to Q[1] \to 0$$

in $\mathcal{B}_{\omega,B}$, where $Q$ is the cokernel of $E_{-1}^+ \to E_{-1}$ in $\text{Coh}(X)$. Hence, $E_{-1}^+[1] \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta))$.

Assume the opposite for a contradiction; so that

$$v_1^{B+\vartheta}(1 - \eta/3)(E_{-1}^+) \geq 0.$$ 

We have

$$\nu_{\omega,B}(E_{-1}^+[1]) - \vartheta = \nu_{\omega,B}(E_{-1}^+) - \vartheta.$$ 

Since $E_{-1}^+$ is $\mu_{\omega,B}$-semistable, by the classical Bogomolov-Gieseker inequality

$$\mathfrak{G}_{\mu_{\omega,B}}(E_{-1}^+) \geq 0$$

and since $E_{-1}^+ \in \mathcal{F}_{\omega,B} = \mathcal{H}^\nu_{\omega,B}((-\infty, 0])$, we have $\nu_{\omega,B}(E_{-1}^+[1]) \neq \infty$. Also for any $\vartheta \in \mathbb{R}$ we have $\vartheta + \eta/3 > 0$, and so

$$v_1^B(E_{-1}^+) < 0$$

and $v_1^{B+\vartheta}(1 - \eta/3)(E_{-1}^+) < 0$.

Hence, as $v_0(E_{-1}^+) > 0$, by (i3), we have $\nu_{\omega,B}(E_{-1}^+) - \vartheta > 0$. But this is not possible as $E_{-1}^+[1] \in \mathcal{H}^\nu_{\omega,B}((-\infty, \vartheta))$. This is the required contradiction to complete the proof.

(ii) If $v_0(E_0) = 0$ then clearly we have the required result. So we may assume $v_0(E_0) > 0$. Since $E \in \mathcal{H}^\nu_{\omega,B}((\vartheta, \infty))$, we have $E_0 \in \mathcal{H}^\nu_{\omega,B}((\vartheta, \infty))$.

Let $E_0^-$ be the HN $\mu_{\omega,B}$-semistable factor of $E_0$ with the lowest $\mu_{\omega,B}$ slope. By the HN filtration of $E_0$ with respect to $\mu_{\omega,B}$-stability, since $v_0(E_0) > 0$ we have $E_0^-$ is a non-trivial torsion free quotient of $E_0$. So

$$0 \to K \to E_0 \to E_0^- \to 0$$
is a short exact sequence in \( \mathcal{B}_{\omega,B} \), where \( K \) is the kernel of \( E_0 \to E_0^- \) in \( \text{Coh}(X) \). Since \( E_0 \in \text{HN}^\nu_{\omega,B}((\vartheta, \infty]) \) we have \( E_0^- \in \text{HN}^\nu_{\omega,B}((\vartheta, \infty]) \).

Assume the opposite for a contradiction; so that
\[
v_1^B(\vartheta + \eta/\sqrt{3}\omega)(E_0^-) \leq 0.
\]
Since \( E_0^- \) is \( \mu_{\omega,B} \)-semistable, by the classical Bogomolov-Gieseker inequality,
\[
\Delta_{\mu_{\omega,B}}(E_0^-) \geq 0.
\]
Also since \( E_0^- \in T_{\omega,B} = \text{HN}^\mu_{\omega,B}((0, \infty]) \) is torsion free, for any \( \vartheta \in \mathbb{R} \) we have \( \vartheta - \eta/\sqrt{3} < 0 \), and so
\[
v_2^B(\vartheta, \omega, B)(E_0^-) > 0 \quad \text{and} \quad v_1^B(\vartheta + \eta/\sqrt{3}\omega)(E_0^-) > 0.
\]
Hence, as \( v_0(E_0^-) > 0 \), by [43], we have \( \nu_{\omega,B}(E_0^-) - \vartheta \leq 0 \). But this is not possible as \( E_0^- \in \text{HN}^\nu_{\omega,B}((\vartheta, \infty]) \). This is the required contradiction to complete the proof.

(iii) Similarly to (i) one can show that if \( E \in \text{HN}^\nu_{\omega,B}((\infty, \vartheta]) \) then \( E_{-1} \in \text{HN}^\nu_{\omega,B}((-\infty, \vartheta - \eta/\sqrt{3})) \). Hence, for tilt semistable \( E \in \mathcal{B}_{\omega,B} \) with \( \nu_{\omega,B}(E) = \vartheta \), we have \( E_{-1} \in \text{HN}^\nu_{\omega,B}((-\infty, \vartheta - \eta/\sqrt{3})) \). The equality \( v_{-1}^B(\vartheta - \eta/\sqrt{3}\omega)(E_{-1}^-) = 0 \) holds when \( E_{-1} \) is slope semistable, and so it satisfies the classical BG inequality. Since \( \nu_{\omega,B}(E_{-1}) - \vartheta \leq 0 \), from [43] we have \( v_1^B(\vartheta - \eta/\sqrt{3}\omega)(E_{-1}) = 0 \) if and only if \( \Delta_{\nu_{\omega,B}}(E_{-1}) = 0 \) equivalently \( v_2^B(\vartheta - \eta/\sqrt{3}\omega)(E_{-1}) = 0 \). This completes the proof of (a). The proof of (b) is similar to that of (a). \( \square \)

3.6. Some properties of tilt stable objects. When \( E \in \text{Coh}(X) \) is a \( \mu_{\omega,B} \)-(semi)stable sheaf, it is straightforward to check that it is also \( \mu_{\nu_{\omega,B} + \vartheta_\omega} \)-(semi)stable for any \( r \in \mathbb{R}_{>0} \) and \( \vartheta \in \mathbb{R} \). In this subsection we obtain a somewhat similar result for tilt stable objects.

**Lemma 3.13.** Let \( E \in \mathcal{B}_{\omega,B} \) and for given \( \vartheta \in \mathbb{R} \) let \( \eta = \sqrt{3}\vartheta^2 + 1 \). Then \( E \) is \( \nu_{\omega,B} \)-stable with \( \nu_{\omega,B}(E) = \vartheta \) if and only if \( E \in \mathcal{B}_{\nu_{\omega,B} + \vartheta_\omega} \) is \( \nu_{\nu_{\omega,B} + \vartheta_\omega} \)-stable with \( \nu_{\nu_{\omega,B} + \vartheta_\omega}(E) = 0 \).

**Proof.** Suppose that \( E \in \mathcal{B}_{\omega,B} \) is \( \nu_{\omega,B} \)-stable with \( \nu_{\omega,B}(E) = \vartheta \).

Let \( E_i := H^i(E) \). By (iii) of Proposition 3.12 we have
\[
E_{-1} \in \text{HN}^\nu_{\omega,B}((-\infty, \vartheta - \eta/\sqrt{3})) \subset \text{HN}^\nu_{\nu_{\omega,B} + \vartheta_\omega}'((-\infty, 0)),
\]
and
\[
E_0 \in \text{HN}^\nu_{\omega,B}([\vartheta + \eta/\sqrt{3}, \infty]) \subset \text{HN}^\nu_{\nu_{\omega,B} + \vartheta_\omega}'(0, \infty]).
\]
Therefore, \( E \in \mathcal{B}_{\nu_{\omega,B} + \vartheta_\omega} \). Recall that
\[
\nu_{\nu_{\omega,B} + \vartheta_\omega}(E) = \vartheta \Longleftrightarrow \frac{\text{Im } Z_{\nu_{\omega,B} + \vartheta_\omega}(E)}{\eta^B_1(E)}.
\]
Therefore, \( \text{Im } Z_{\nu_{\omega,B} + \vartheta_\omega}(E) = 0 \) and so \( \nu_{\nu_{\omega,B} + \vartheta_\omega}(E) = 0 \).

Assume \( E \) is a \( \nu_{\nu_{\omega,B} + \vartheta_\omega} \) tilt unstable object for a contradiction. Therefore, from the Harder-Narasimhan property, there exist \( F \in \text{HN}^\nu_{\nu_{\omega,B} + \vartheta_\omega}'((0, \infty]), \) \( G \in \text{HN}^\nu_{\nu_{\omega,B} + \vartheta_\omega}'((-\infty, 0]), \) and a short exact sequence
\[
(44) \quad 0 \to F \to E \to G \to 0
\]
in \( \mathcal{B}_{\nu \omega, B + \theta \omega} \). By considering the long exact sequence of Coh(X)-cohomologies, we have \( F_{-1} \to E_{-1} \) and \( E_0 \to G_0 \) in Coh(X). So

\[
F_{-1} \in \text{HN}^\mu_{\omega, B}((-\infty, \vartheta - \eta/\sqrt{3})) \subset \text{HN}^\mu_{\omega, B}((-\infty, 0)),
\]

and

\[
G_0 \in \text{HN}^\mu_{\omega, B}([\vartheta + \eta/\sqrt{3}, \infty]) \subset \text{HN}^\mu_{\omega, B}((0, \infty]).
\]

By (ii) of Proposition 3.12,

\[
F_0 \in \text{HN}^\mu_{\omega, B + (\vartheta + \eta/\sqrt{3})}((0, \infty]) \subset \text{HN}^\mu_{\omega, B}((0, \infty]),
\]

and by (i) of Proposition 3.12,

\[
G_{-1} \in \text{HN}^\mu_{\omega, B + (\vartheta - \eta/\sqrt{3})}((-\infty, 0)) \subset \text{HN}^\mu_{\omega, B}((-\infty, 0)).
\]

Therefore, \( \text{HN}^\mu_{\omega, B + (\theta - \eta/\sqrt{3})}((-\infty, 0)) \subset \text{HN}^\mu_{\omega, B}((-\infty, 0)) \).

Let \( \alpha \in \mathbb{R}_{>0} \) and \( \beta \in \mathbb{R} \). Then from the above result we have

\[
\text{HN}^\nu_{\omega, B + \beta \omega} (\tau) = \text{HN}^\nu_{\alpha \sqrt{3} \tau^2 + \omega, B + (\beta + \tau \alpha) \omega} (0).
\]

Moreover, we have

\[
\text{HN}^\nu_{\omega, B} (\vartheta) = \text{HN}^\nu_{\alpha \sqrt{3} \tau^2 + \omega, B + \theta \omega} (0).
\]

Therefore, \( \text{HN}^\nu_{\omega, B} (\vartheta) = \text{HN}^\nu_{\alpha \omega, B + \beta \omega} (\tau) \) when \( \sqrt{3} \vartheta^2 + 1 = \alpha \sqrt{3} \tau^2 + 1 \) and \( \vartheta = \beta + \tau \alpha \) hold. So we get the following, which is a generalization of [BMS] Lemma 4.3.

**Lemma 3.14.** Let the object \( E \in \mathcal{B}_{\omega, B} \) be \( \nu_{\omega, B} \)-stable. Then \( E \in \mathcal{B}_{\omega, B + \beta \omega} \) is \( \nu_{\alpha \omega, B + \beta \omega} \)-stable for all \( \alpha \in \mathbb{R}_{>0} \) and \( \beta \in \mathbb{R} \) such that

\[
\alpha^2 + 3 (\beta - \nu_{\omega, B} (E))^2 = 3 \nu_{\omega, B} (E)^2 + 1.
\]

**3.7. Generalized conjectural BG inequality.** Let \( E \in \mathcal{B}_{\omega, B} \) be a \( \nu_{\omega, B} \)-stable object with \( \nu_{\omega, B} (E) = \vartheta \) for some \( \vartheta \in \mathbb{R} \). So

\[
(45) \quad v_2^B (E) - \frac{1}{6} v_0^B (E) = \vartheta v_1^B (E).
\]

From Proposition 3.13 \( E \in \mathcal{B}_{\nu \omega, B + \theta \omega} \) is \( \nu_{\nu \omega, B + \theta \omega} \)-stable with \( \nu_{\nu \omega, B + \theta \omega} (E) = 0 \). So the inequality for \( E \) in Conjecture 3.8 reads as:

\[
v_3^{B + \vartheta \omega} (E) \leq \frac{(3 \vartheta^2 + 1)}{18} v_1^{B + \vartheta \omega} (E).
\]

That is,

\[
v_3^B (E) - \vartheta v_2^B (E) + \frac{(6 \vartheta^2 - 1)}{18} v_1^B (E) + \frac{\vartheta}{18} v_0^B (E) \leq 0.
\]

As \( \vartheta \neq \infty \), \( v_3^B (E) > 0 \). Therefore, from multiplying both sides of the above inequality by \( v_1^B (E) \) and then using (45), we get the inequality

\[
18 v_1^B (E) v_3^B (E) - 12 (v_2^B (E))^2 + 2 v_0^B (E) v_2^B (E) - (v_1^B (E))^2 \leq 0.
\]
Since $\Delta_{\omega,B}(E) = (v^B_1(E))^2 - 2v^B_0(E)v^B_1(E)$ and $\nabla_{\omega,B}(E) = 2(v^B_2(E))^2 - 3v^B_1(E)v^B_3(E)$, we obtain the following result:

**Theorem 3.15.** For a given smooth projective 3-fold $X$, Conjecture 3.7 holds if and only if Conjecture 3.8 holds for all the complexified ample classes $B + i\omega$.

### 3.8. Another equivalent form of BG inequality conjecture

In this subsection we formulate an equivalent form of Conjecture 3.7 which only considers BG type inequalities for a small class of tilt stable objects. This generalizes [BMS, Conjecture 5.3] and we show that it is equivalent to Conjecture 3.7. The following discussion is not needed in the rest of the paper, and so the reader is safe to skip this subsection.

Most of our arguments in this subsection are closely related to Section 5 of [BMS], and also we try to follow somewhat similar notations.

Let us consider the complexified classes parametrized by $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$:

$$(B + \beta \omega) + i(\alpha \omega).$$

By definition

$$\nu_{\alpha\omega,B+\beta\omega} = \frac{\omega \chi^2_{B+\beta\omega} - \frac{1}{6} \alpha^2 \omega^3 \chi_0}{\alpha \omega^3 \chi_1^{B+\beta\omega}}.$$  

Therefore, we can consider

$$Z^\nu_{\alpha,\beta}(E) = -\left( \omega \chi^2_{B+\beta\omega} - \frac{1}{6} \alpha^2 \omega^3 \chi_0 \right) + i\omega^2 \chi_1^{B+\beta\omega}$$

as the associated group homomorphism in $\nu_{\alpha\omega,B+\beta\omega}$ tilt stability. For a given object $E$, if we have

$$\lim_{\alpha \to +0} - \text{Re} Z^\nu_{\alpha,\beta}(E) = 0,$$

when $\beta \to \bar{\beta}$, then $\bar{\beta}$ satisfies $\omega \chi^2_{B+\beta\omega}(E) = 0$. That is

$$v^B_0(E)\bar{\beta}^2 - 2v^B_1(E)\bar{\beta} + 2v^B_2(E) = 0.$$

We consider one of its root defined by

$$(46) \quad \bar{\beta}(E) := \frac{2v^B_0(E)}{v^B_1(E) + \sqrt{\Delta_{\omega,B}(E)}}.$$  

We conjecture the following, which generalizes [BMS, Conjecture 5.3]. Also our claim is directly adapted from their formulation.

**Conjecture 3.16.** Let $E$ be an object of $D^b \text{Coh}(X)$. Suppose there is an open neighborhood $U \subset \mathbb{R}^2$ containing $(0, \bar{\beta}(E))$ such that for any $(\alpha, \beta) \in U$ with $\alpha > 0$, $E \in \mathcal{B}_{\alpha\omega,B+\beta\omega}$ is $\nu_{\alpha\omega,B+\beta\omega}$-stable. Then

$$\chi^3_{B+\beta\omega}(E) \leq 0.$$  

Let $E \in \mathcal{B}_{\omega,B}$ be $\nu_{\omega,B}$-stable. Recall Lemma 3.14. $E \in \mathcal{B}_{\alpha\omega,B+\beta\omega}$ is $\nu_{\alpha\omega,B+\beta\omega}$-stable along the wall in $\mathbb{R}^2$ defined by

$$W_{\omega,B}(E) := \left\{ (\alpha, \beta) : \alpha^2 + 3(\beta - \nu_{\omega,B}(E))^2 = 3\nu_{\omega,B}(E)^2 + 1, \alpha > 0 \right\}.$$
Let $D_{\omega,B}(E)$ be the interior of $W_{\omega,B}(E)$ in $\alpha \geq 0$. That is for any $E \in B_{\omega,B}$, we define

$$D_{\omega,B}(E) := \left\{ (\alpha, \beta) : \alpha^2 + 3 (\beta - \nu_{\omega,B}(E))^2 < 3\nu_{\omega,B}(E)^2 + 1, \alpha \geq 0 \right\}.$$ 

The $\Delta_{\omega,B}$ values of objects in $B_{\omega,B}$ are very important for us. In particular, [BMT14 Corollary 7.3.2] says if $E \in B_{\omega,B}$ is $\nu_{\omega,B}$-stable, then

$$\Delta_{\omega,B}(E) \geq 0.$$ 

Moreover, we have the following:

**Proposition 3.17.** The inequalities in Conjectures 3.7 and 3.16 hold for corresponding tilt stable objects $E$ with $\Delta_{\omega,B}(E) = 0$.

**Proof.** Similar to the proof of [BMS Lemma 5.6]. □

The following result generalizes the claim in [BMS Lemma 5.5].

**Proposition 3.18.** Let $E \in B_{\omega,B}$ be $\nu_{\omega,B}$-stable with $\nu_{\omega,B}(E) > 0$. Then

$$(0, \beta(E)) \in D_{\omega,B}(E).$$

**Proof.** Let $E \in B_{\omega,B}$ be $\nu_{\omega,B}$-stable with $\nu_{\omega,B}(E) = \vartheta$ for some $\vartheta \in \mathbb{R}$. So $v_2^B(E) = \frac{1}{6} v_0^B(E) + \vartheta v_1^B(E)$, and by (46)

$$\beta(E) = \frac{3v_1^B(E)}{\nu_{\omega,B}(E)} + \vartheta.$$  

By Proposition 3.12 we have

$$\omega^2 \text{ch}_1^{B+(\vartheta+\sqrt{\vartheta^2+1/3})\omega}(H^0(E)) \geq 0,$$

$$\omega^2 \text{ch}_1^{B+(\vartheta-\sqrt{\vartheta^2+1/3})\omega}(H^{-1}(E)) \leq 0.$$

Since $\vartheta + \sqrt{\vartheta^2 + 1/3} > 0$ and $\vartheta - \sqrt{\vartheta^2 + 1/3} < 0$, we have

$$v_1^B(E) - \left( \vartheta + \sqrt{\vartheta^2 + 1/3} \right) v_0^B(E) \geq 0,$$

$$v_1^B(E) - \left( \vartheta - \sqrt{\vartheta^2 + 1/3} \right) v_0^B(E) \geq 0.$$ 

Therefore, by dividing $v_1^B(E) > 0$,

$$-\vartheta - \sqrt{\vartheta^2 + 1/3} \leq \frac{v_0^B(E)}{v_1^B(E)} \leq -\vartheta + \sqrt{\vartheta^2 + 1/3}.$$ 

Since $v_1^B(E) > 0$ and $\Delta_{\omega,B}(E) > 0$, from (17) together with the above inequalities we get

$$\vartheta - \sqrt{\vartheta^2 + 1/3} < \beta(E) < \vartheta + \sqrt{\vartheta^2 + 1/3}.$$ 

That is, we have $(0, \beta(E)) \in D_{\omega,B}(E)$ as required. □

Now we have the main result in this subsection.

**Theorem 3.19.** Conjectures 3.7 and 3.16 are equivalent.

**Proof.** The proof is identical to that of [BMS Theorem 5.4] as we have Propositions 3.17 and 3.18 in general setup. □
Remark 3.20. Consequently, we have two equivalent forms Conjectures 3.7 and 3.16 of Conjecture 3.8. In [BMS], the authors showed that Conjecture 3.8 holds for abelian 3-folds. They firstly reduced Conjecture 3.8 for $B$ and $\omega$ are parallel cases by using the multiplication map in abelian varieties, and then proved Conjecture 3.16 for those cases. Since we have the equivalent formulations of the conjectures in general, following similar arguments in [BMS, Section 7], one can directly prove Conjecture 3.16 for abelian 3-folds.

4. Moduli stacks of semistable objects

Our main aim of this section is to complete the proof of Theorem 1.5.

4.1. Notation. Let $X$ be a smooth projective variety. Due to Lieblich [Lie06], there is an algebraic stack $M$ locally of finite type parameterizing objects $E \in D^bCoh(X)$ with $\text{Ext}^< 0(E, E) = 0$.

Let $\sigma = (Z, A)$ be a stability condition on $D^bCoh(X)$ with respect to some data $(\Gamma, cl)$. For given $v \in \Gamma$, one can consider the substack $M_\sigma(v) \subset M$ which parametrizes $\sigma$-semistable objects $E \in A$ with $cl(E) = v$. A priori, we do not know whether $M_\sigma(v)$ is an algebraic stack nor is of finite type.

Suppose that $\text{dim } X = 3$, and let $B \in \text{NS}(X)_\mathbb{Q}$ and $\omega \in \text{NS}(X)_\mathbb{R}$ an ample class with $\omega^2$ rational as in (25). If we assume Conjecture 3.8, then we have the associated Bridgeland stability condition $\sigma_{\omega, B}$ given by (39). Note that $\sigma_{\omega, B}$ is good in the sense of Definition 2.13 by Remark 2.14. Let $\text{Stab}^\circ_{\omega, B}(X) \subset \text{Stab}_{\omega, B}(X)$ be the connected component which contains $\sigma_{\omega, B}$. The purpose of this section is to prove the following result:

Theorem 4.1. Let $X$ be a smooth projective 3-fold satisfying Conjecture 3.8. Then for any $v \in \Gamma_{\omega, B}$ and $\sigma \in \text{Stab}^\circ_{\omega, B}(X)$, the stack $M_\sigma(v)$ is a proper algebraic stack of finite type over $\mathbb{C}$, such that the embedding (48) is an open immersion.

The proof of the above result will be given in Subsection 4.7. The key ingredients are the boundedness and the generic flatness statements. First we recall the boundedness.

Definition 4.2. A set of isomorphism classes of objects $S$ in $D^bCoh(X)$ is called bounded if there is a $\mathbb{C}$-scheme $S$ of finite type, and an object $E \in D^bCoh(X \times S)$ such that any object in $S$ is isomorphic to $E_s := \text{Li}_s^*E$ for some $s \in S$. Here $i_s: X \times \{s\} \hookrightarrow X \times S$ is the inclusion.

Next we recall the generic flatness. Let $S$ be a smooth projective variety, and $\mathcal{L}$ an ample line bundle on $S$. Let $A \subset D^bCoh(X)$ be the heart
of a bounded t-structure on $D^b \text{Coh}(X)$ which is noetherian. By [AP06, Theorem 2.6.1], the category

$$(49) \quad A_S := \{ E \in D^b \text{Coh}(X \times S) : R p_{X^*}(E \otimes p_S^* L_{\otimes m}) \in A, m \gg 0 \}$$

is the heart of a bounded t-structure on $D^b \text{Coh}(X \times S)$. Here $p_X$, $p_S$ are the projections from $X \times S$ onto the corresponding factors.

**Definition 4.3.** We say that $A$ satisfies the generic flatness if for any smooth projective variety $S$ and an object $E \in A_S$, there is a non-empty open subset $U \subset S$ such that $E_s \in A$ for any $s \in U$.

Note that if $\sigma = (Z, A)$ is a good stability condition, then the heart $A$ is noetherian (see Remark 2.14). We say that a good stability condition $\sigma = (Z, A)$ satisfies the generic flatness.

Our strategy for Theorem 4.1 is to show that the boundedness and the generic flatness are preserved (in some sense) under the tilting of very weak stability conditions $(Z, A) \Rightarrow (\tilde{Z}, \tilde{A})$ for $t > 0$. This argument shows the required results for the stability condition $\sigma_{\omega, B}$ (see Corollary 2.22). Then we show that these properties are preserved under the deformations of a stability condition (see Proposition 4.11), and together with Abramovich-Polishchuk’s valuative criterion (see Theorem 4.21) we conclude the result.

4.2. Induction argument for the boundedness. Let $X$ be a smooth projective variety, and take $D = D^b \text{Coh}(X)$. We fix the data $(\Gamma, \text{cl})$ to consider very weak stability conditions on $D$, and use notation as in Section 2. Let $(Z, A)$ be a very weak stability condition on $D$ satisfying a BG inequality and also good (see Definition 2.13). For $t > 0$, let $(Z^t, A^t)$ be the associated tilting given in Corollary 2.22. The purpose of this subsection is to prove the boundedness of $\mu^t$-semistable objects in $A^t$, assuming some kind of boundedness for $\mu$-semistable objects. We first prepare some notation. Let us fix an isomorphism

$$(50) \quad \Gamma_0 \otimes_Z \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}^r.$$ 

**Definition 4.4.** We say a subset $S \subset \mathbb{Q}^r$ is bounded below (resp. above) if there exist functions $f_i(x_1, \ldots, x_{i-1})$ for each $1 \leq i \leq r$ such that every element $(s_1, \ldots, s_r) \in S$ satisfies

$s_i \geq f_i(s_1, \ldots, s_{i-1}), \quad (\text{resp. } s_i \leq f_i(s_1, \ldots, s_{i-1})).$

**Remark 4.5.** It is easy to see that a subset $S \subset \mathbb{Q}^r$ is bounded below (above) if and only if $S \cap S'$ is a finite set for any bounded above (below) subset $S' \subset \mathbb{Q}^r$.

If we choose an isomorphism (50), it defines the notion of bounded below (above) subsets in $\Gamma_0$. For $v \in \Gamma$ and a subset $S \subset \Gamma_0$, let $M(v, S)$ be the set of isomorphism classes of $\mu$-semistable objects $E \in A$ with $\text{cl}(E) \in v + S$.

**Definition 4.6.** A very weak stability condition $(Z, A)$ satisfies the boundedness if either (i) or (ii) holds:
(i) $\Gamma_0 \neq 0$, and under a suitable isomorphism $\tilde{\mathcal{M}}$, the set $M_\mu(v, S)$ of isomorphism classes of objects is bounded for any $v \in \Gamma$ with $\mu(v) < \infty$ and any bounded below subset $S \subset \Gamma_0$.

(ii) $\Gamma_0 = 0$ and the same condition of (i) holds for any $v \in \Gamma$.

We put the following assumption (recall the category $\mathcal{P}_0^1(\phi)$ in Lemma 2.19 and the slope function $\lambda$ in (24)):

**Assumption 4.7.**

(i) The very weak stability condition $(Z, \mathcal{A})$ satisfies the boundedness.

(ii) For any $v \in \Gamma$ with $\mu(v) < \infty$ and any bounded below subset $S \subset \Gamma_0^1$, the isomorphism classes of objects

$$E \in \langle U[1], C : U \in \mathcal{A} \text{ is } \mu\text{-semistable} \rangle$$

satisfying $\text{Hom}(C, E) = 0$ and $\text{cl}(E) \in v + S \uparrow$ is bounded.

(iii) For any $v \in \Gamma \setminus \Gamma_0$, and any bounded below subset $S \subset \Gamma_0^1$, the isomorphism classes of $\lambda|_{\mathcal{P}_0^1(1/2)}$-semistable objects $E \in \mathcal{P}_0^1(1/2)$ with $\text{cl}(E) \in v + S \uparrow$ is bounded.

We have the following proposition:

**Proposition 4.8.** Suppose that Assumption 4.7 holds. If $\Gamma^1_0 \neq 0$, then the very weak stability condition $(Z_1^1, \mathcal{A}^1)$ satisfies the boundedness for any $t > 0$.

**Proof.** Let us take $v \in \Gamma$ with $\mu^1_1(v) < \infty$, i.e. $\text{Re} Z(v) \neq 0$, and a bounded below subset $S^\uparrow \subset \Gamma_0^1$. We denote by $M^1_1(v, S^\uparrow)$ the set of isomorphism classes of $\mu^1_1$-semistable objects $E \in \mathcal{A}^1$ with $\text{cl}(E) \in v + S^\uparrow$. We show the boundedness of $M^1_1(v, S^\uparrow)$ by induction on $\text{Im} Z^1_1(v) = -\text{Re} Z(v) > 0$. We set $R > 0$ as in (14), and suppose that $-\text{Re} Z(v) = R$. Then any $E \in M^1_1(v, S^\uparrow)$ is $\mu^1_1$-semistable for $0 < t' \ll 1$. If $0 < \mu(v) < \infty$, then $E \in \mathcal{A}$ and it is $\mu$-semistable by Proposition 2.27. Hence, the boundedness of $M^1_1(v, S^\uparrow)$ follows from Assumption 4.7 (i). If $\mu(v) \leq 0$, by Proposition 2.27 any object $E \in M^1_1(v, S^\uparrow)$ fits into the exact sequence in $\mathcal{A}^1$

$$0 \to U[1] \to E \to F \to 0$$

where $F \in \mathcal{C}$ and $U \in \mathcal{A}$ is a non-zero $\mu$-semistable object with $\mu(U) = \mu(v)$. Note that $\text{Hom}(\mathcal{C}, E) = 0$ as $E$ is $\mu^1_1$-semistable with $\mu^1_1(E) < \infty$ and $\mu^1_1(T) = \infty$ for any $T \in \mathcal{C}$. Hence, the boundedness of $M^1_1(v, S^\uparrow)$ follows from Assumption 4.7 (ii). If $\mu(v) = \infty$, then $E \in \mathcal{P}_0^1(1/2)$, and it is $\lambda|_{\mathcal{P}_0^1(1/2)}$-semistable by Proposition 2.27. Hence, the boundedness of $M^1_1(v, S^\uparrow)$ follows from Assumption 4.7 (iii).

Suppose that $-\text{Re} Z(v) > R$. Let $U \subset \mathbb{R}_{>0}$ be the set of $t \in \mathbb{R}_{>0}$ such that $M^1_1(v, S^\uparrow)$ is bounded. It is enough to show that $U$ is non-empty, open and closed. The same argument of the case of $-\text{Re} Z(v) = R$ together with Lemma 2.26 and Proposition 2.27 show that $(0, t_0) \subset U$ for some $t_0 > 0$. In particular, $U$ is non-empty. Also as in the proof of Lemma 2.26, there is a locally finite set of walls $W \subset \mathbb{R}_{>0}$ such that $M^1_1(v, S^\uparrow)$ is constant on each connected component of $\mathbb{R}_{>0} \setminus W$, which implies that $U$ is open. In order to
show that $U$ is closed, we take a chamber $V \subset \mathbb{R}_{>0}\setminus W$ satisfying $V \subset U$ and take $t \in \overline{V \cap W}$. It is enough to show that $t \in U$. Let $M_t^{s\ddagger}(v, S')$ be the subset of $M_t^{\ddagger}(v, S')$ consisting of $\mu_t^\ddagger$-stable objects. Then $M_t^{s\ddagger}(v, S') \subset M_t^{\ddagger}(v, S')$ for $t' \in V$, and so $M_t^{s\ddagger}(v, S')$ is bounded. If $E \in M_t^{\ddagger}(v, S')$ is not $\mu_t^\ddagger$-stable, then there is an exact sequence

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

in $\mathcal{A}^\ddagger$ with $\mu_t^\ddagger(E_1) = \mu_t^\ddagger(E_2)$. By the support property of $(Z_t^\ddagger, \mathcal{A}^\ddagger)$, the set of possible vectors $\text{cl}(E_i) \in \Gamma/\Gamma_0^\ddagger$ is bounded. Also $0 < -\text{Re} Z(E_i) < -\text{Re} Z(v)$, and so the induction hypothesis is applied. Noting Remark 4.5, it follows that there is a finite subset $S' \subset \Gamma/\Gamma_0^\ddagger$ and a bounded above subset $S'' \subset \Gamma_0^\ddagger$ such that all the possible $E_i$ are objects in $M_t^\ddagger(v', S'')$ for some $v' \in S'$. As $S'$ is bounded below, one can take $S''$ to be a finite set, and so $M_t^\ddagger(v', S'')$ is bounded by the induction hypothesis. Therefore, $M_t^\ddagger(v, S')$ is also bounded, and $t \in U$ holds.

When $(Z_t^\ddagger, \mathcal{A}^\ddagger)$ becomes a stability condition, we have the following result on the boundedness:

**Lemma 4.9.** Suppose that Assumption 4.7 holds, $\Gamma_0^\ddagger = 0$ and the set of $F \in \mathcal{C}$ with fixed $\text{cl}(F)$ is bounded. Then the stability condition $(Z_t^\ddagger, \mathcal{A}^\ddagger)$ satisfies the boundedness for any $t > 0$.

**Proof.** By Proposition 4.8, we only need to show the boundedness of $\mu_t^\ddagger$-semistable objects $E \in \mathcal{A}^\ddagger$ with a fixed $\text{cl}(E) = v \in \Gamma$ satisfying $\mu_t^\ddagger(v) = \infty$. Any such object $E$ fits into a short exact sequence in $\mathcal{A}^\ddagger$

$$0 \rightarrow U[1] \rightarrow E \rightarrow F \rightarrow 0$$

for some $\mu$-semistable $U \in \mathcal{A}$ with $\mu(U) = 0$, and $F \in \mathcal{C}$. From the support property of $(Z_t^\ddagger, \mathcal{A}^\ddagger)$, $\text{cl}(U)$ and $\text{cl}(F)$ are bounded. By Assumption 4.7 (i) and the boundedness of $F$, we have the boundedness of possible $E$. \qed

### 4.3. Generic flatness

We carry on the setting in the previous subsection. Here we discuss the generic flatness under tilting. The proof of the following proposition is almost the same as [Tod08, Proposition 3.18], but we include the proof for the reader’s convenience.

**Proposition 4.10.** Let $(Z, \mathcal{A})$ be a very weak stability condition on $\mathcal{D}$ which satisfies a BG inequality and is good. If $(Z, \mathcal{A})$ satisfies Assumption 4.7 and $\mathcal{A}$ satisfies the generic flatness, then $\mathcal{A}^\ddagger$ satisfies the generic flatness.

**Proof.** Let $S$ be a smooth projective variety and $\mathcal{L}$ an ample line bundle on it. Since the category $\mathcal{A}^\ddagger$ is noetherian (see Lemma 2.17), a similar construction of [19] defines the heart

$$\mathcal{A}_S^\ddagger \subset D^b \text{Coh}(X \times S).$$

Let us take $\mathcal{E} \in \mathcal{A}_S^\ddagger$. By the definition of $\mathcal{A}_S$ and $\mathcal{A}_S^\ddagger$, we have

$$\mathcal{R}p_{X*}(\mathcal{E} \otimes p_S^* \mathcal{L}^{\otimes m}) \in \mathcal{A}^\ddagger \subset (\mathcal{A}[1], \mathcal{A})$$

$$\mathcal{R}p_{X*}(\mathcal{H}^i_{\mathcal{A}_S}(\mathcal{E}) \otimes p_S^* \mathcal{L}^{\otimes m}) \in \mathcal{A}$$
for any $i \in \mathbb{Z}$ and $m \gg 0$. Therefore, we have

$$R^i \mathcal{P}_{X, s}(\mathcal{H}^i_{As}(\mathcal{E}) \otimes p_S^* \mathcal{L}^m) = 0$$

for any $i \neq 0, -1$ and $m \gg 0$. This implies that $\mathcal{H}^i_{As}(\mathcal{E}) = 0$ for $i \neq 0, -1$. By the assumption of the generic flatness of $\mathcal{A}_S$, there is a non-empty open subset $U \subset S$ such that $\mathcal{H}^i_{As}(\mathcal{E})_s \in \mathcal{A}$ for any $s \in U$ and $i \in \{-1, 0\}$. It remains to show that there is a non-empty open subset $U' \subset U$ such that $\mathcal{H}^0_{As}(\mathcal{E})_s \in \mathcal{P}((1/2, 1)), \mathcal{H}^{-1}_{As}(\mathcal{E})_s \in \mathcal{P}((0, 1/2))$.

for any $s \in U'$. The condition $\mathcal{H}^0_{As}(\mathcal{E})_s \notin \mathcal{P}((1/2, 1])$ is equivalent to the existence of a surjection $\mathcal{H}^0_{As}(\mathcal{E})_s \rightarrow Q$ in $\mathcal{A}$ with $\mu(Q) \leq 0$. By Assumption 4.7, for any $v \in \Gamma$ with $\mu(v) < \infty$, the set of $\mu$-semistable objects $E \in \mathcal{A}$ with $\text{cl}(\mathcal{E}) = v$ is bounded. Hence, by [Tod08 Proposition 3.17] there is a $\mathbb{C}$-scheme of finite type

$$\pi_Q : \text{Quot}(\mathcal{H}^0_{As}(\mathcal{E})) \rightarrow U$$

whose fiber at a point $s \in U$ parametrizes the quotients $\mathcal{H}^0_{As}(\mathcal{E})_s \rightarrow Q$ with $\mu(Q) \leq 0$. On the other hand, by [AP06 Proposition 3.5.3] the set of points $s \in U$ with $\mathcal{E}_s \in \mathcal{A}^1$ is dense in $U$. This implies that the complement $U \setminus \text{image} \pi_Q$ is dense, and because it is constructible, there is a non-empty open subset $U_1 \subset U \setminus \text{image} \pi_Q$. By the construction of $\pi_Q$, we have $\mathcal{H}^0_{As}(\mathcal{E})_s \in \mathcal{P}((1/2, 1])$ for any $s \in U_1$. One can similarly find a non-empty open subset $U_2 \subset U$ such that $\mathcal{H}^{-1}_{As}(\mathcal{E})_s \in \mathcal{P}((0, 1/2])$ holds for any $s \in U_2$. We obtain a desired $U'$ by setting $U' = U_1 \cap U_2$.

4.4. Boundedness and generic flatness under deformations. In the setting of the previous subsection, we show that the boundedness and the generic flatness are preserved under the deformations of stability conditions. The following statement is stronger than [Tod08 Theorem 3.20], but the essential arguments were already there.

**Proposition 4.11.** Let $\text{Stab}^\sigma_0(\mathcal{D}) \subset \text{Stab}_1(\mathcal{D})$ be a connected component. Suppose that there is a good stability condition $\sigma \in \text{Stab}^\sigma_0(\mathcal{D})$ satisfying the boundedness and the generic flatness. Then any good stability condition $\tau \in \text{Stab}_1(\mathcal{D})$ satisfies the boundedness and the generic flatness.

**Proof.** Let $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ and $\{\mathcal{Q}(\phi)\}_{\phi \in \mathbb{R}}$ be the slicing determined by $\sigma$ and $\tau$ respectively. By connecting $\sigma$ and $\tau$ via a path and cutting it into several pieces, we may assume that

$$\mathcal{Q}(\phi) \subset \mathcal{P}(\langle \phi - \varepsilon, \phi + \varepsilon \rangle)$$

holds for any $\phi \in \mathbb{R}$ and some $0 < \varepsilon < 1/8$. Then the argument of [Tod08 Theorem 3.20, Step 1] shows that, for a fixed $v \in \Gamma$, the stack $\mathcal{M}_\tau(v)$ is an algebraic substack of finite type such that $\mathcal{M}_\tau(v) \subset \mathcal{M}$ is an open immersion. In particular, $\tau$ satisfies the boundedness. Indeed, one can replace $(\sigma, \sigma')$ in the proof of [Tod08 Theorem 3.20, Step 1] by $(\tau, \sigma)$, and the rest of the arguments are the same.

It remains to show the generic flatness of $\tau$. Let $\mathcal{B}$ be the heart $\mathcal{Q}((0, 1])$, and take an object $\mathcal{E} \in \mathcal{B}_S$ for a smooth projective variety $S$. We set

$$S' := \{s \in S : \mathcal{E}_s \in \mathcal{B}\}.$$
We need to show that $S'$ contains an open subset of $S$. By the result of [AP06, Lemma 2.6.2], there is an object $G \in B$, an ample line bundle $L$ on $S$ and a surjection $G \boxtimes L^{-1} \to E$ in $B_S$. By [AP06, Lemma 2.5.7], the functor

$$\text{Li}^s_* : D^b \text{Coh}(X \times S) \to D^b \text{Coh}(X)$$

for any $s \in S$ is right exact with respect to the t-structures with hearts $B_S$ and $B$ respectively. Therefore, we have the surjection $G \to E_s$ in $B$ for $s \in S'$. Since there is $\psi \in (0, 1]$ such that $G \in \mathcal{Q}((\psi, 1])$, we have $E_s \in \mathcal{Q}((\psi, 1])$ for any $s \in S'$. By the support property of $\tau$, there is only a finite number of ways to write $\text{cl}(F)$ for an object $F \in \mathcal{Q}((\psi, 1])$ as $v_1 + \cdots + v_l$ for $v_i \in \Gamma$ of the form $\text{cl}(F_i)$ for some $F_i \in \mathcal{Q}(\phi_i)$ with $\phi_i \in (\psi, 1]$. As we proved above, the stack of objects $F_i \in \mathcal{Q}(\phi_i)$ with fixed $v^{B_1}(F_i) = v_i$ is an algebraic stack of finite type. Hence, the set of closed points of the stack

$$\text{Obj} \left(\mathcal{Q}((\psi, 1])\right) \subset \mathcal{M}$$

of objects in $\mathcal{Q}((\psi, 1])$ is locally constructible. Therefore, the set of points $S'$ is constructable. On the other hand, $S'$ is dense in $S$ by [AP06, Proposition 3.5.3]. Therefore, $S'$ contains an open subset. 

4.5. Boundedness of tilt semistable objects. Below we assume that $X$ is a smooth projective 3-fold, and take $v^B, \Gamma = \Gamma_{\omega, B}$ and $Z$ as in Subsection 4.2. We have the very weak stability condition $(Z, \text{Coh}(X))$ on $D^b \text{Coh}(X)$, and let $\mu$ be the associated slope function [28]. In this setting, the category $\mathcal{C}$ is given by $\text{Coh}_{\leq 1}(X)$ and $\mathcal{C}_1$ is given by $\text{Coh}_0(X)$. The subgroup $\Gamma_0 \subset \Gamma$ is given by $v^{B_0} = v^{B_1} = 0$, and $\Gamma^1 \subset \Gamma$ is given by $v^{B_0} = v^{B_1} = v^{B_2} = 0$. We have the natural isomorphism

$$(v^{B_2}_2, v^{B_3}_2) : \Gamma_0 \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}^2.$$ 

We repeatedly use the following result by Langer:

Theorem 4.12. ([Lan04, Theorem 4.4]) (i) The set of isomorphism classes of $\mu_{\omega, B}$-semistable $E \in \text{Coh}(X)$ with fixed $(v^{B_0}_1, v^{B_1}_1, v^{B}_2)$ and bounded below $v^{B}_3$, is bounded.

(ii) The set of isomorphism classes of $\hat{\mu}_{\omega, B}$-semistable two dimensional $E \in \text{Coh}_{\leq 2}(X)$ with fixed $(v^{B}_1, v^{B}_2)$ and bounded below $v^{B}_3$, is bounded.

Here the $\hat{\mu}_{\omega, B}$-semistability on $\text{Coh}_{\leq 2}(X)$ is given by the slope function $v^{B}_3(s)/v^{B}_2(s)$. Using the above result, we prove the following proposition:

Proposition 4.13. The very weak stability condition $(Z, \text{Coh}(X))$ satisfies Assumption 4.7.

Proof. The condition (i) in Assumption 4.7 follows from Theorem 4.12 (i) together with the classical BG inequality $(v^{B}_1)^2 \geq 2v^{B_0}_0 v^{B}_2$ for torsion free semistable sheaves. The condition (iii) follows from Theorem 4.12 (ii). The condition (ii) follows from Lemma 4.16 below.

In order to show (ii) in the proof of Proposition 4.13 we prepare some notation and a lemma. For $\vartheta \in \mathbb{R}$, we set

$$\mathcal{C}_\vartheta := (U[1], \text{Coh}_{\leq 1}(X) : U \text{ is } \mu_{\omega}-\text{semistable with } \mu_{\omega}(U) = \vartheta).$$
Remark 4.14. The category $\mathcal{C}_\theta$ consists of objects $E \in \mathcal{B}_{\omega, \theta}$ with $v_1^\theta(E) = 0$. In particular, $\mathcal{C}_\theta$ is an abelian subcategory of $\mathcal{B}_{\omega, \theta}$.

We define the following subcategories of $\mathcal{C}_\theta$

$$\mathcal{C}_\theta^\circ := \{ E \in \mathcal{C}_\theta : \text{Hom}(\text{Coh}_{\leq 1}(X), E) = 0 \}$$

(52) $\mathcal{C}_\theta^\circ := \{ E \in \mathcal{C}_\theta : \mathcal{H}^0(E) \in \text{Coh}_0(X), \text{Hom}(\text{Coh}_0(X), E) = 0 \}.$

Let $\mathbb{D}$ be the dualizing functor defined by $\mathbb{R}\text{Hom}(-, \mathcal{O}_X)[2]$. We have the following lemma:

Lemma 4.15. We have the anti-equivalence of categories

$$\mathbb{D} : \mathcal{C}_\theta^\circ \xrightarrow{\sim} \mathcal{C}_{\circ \theta}^\circ.$$  

Proof. For an object $E \in \mathcal{C}_\theta^\circ$, the proof of [Tod13a, Lemma 3.8] shows that $\mathbb{D}(E) \in \mathcal{C}_{\circ \theta}^\circ$. We only show that $\mathbb{D}(E) \in \mathcal{C}_\theta^\circ$ for $E \in \mathcal{C}_{\circ \theta}^\circ$. We have an exact sequence in $\mathcal{C}_{\circ \theta}^\circ$

$$0 \to U[1] \to E \to Q \to 0$$

(54)

for a $\mu_\omega$-semistable sheaf $U$ with $\mu_\omega(U) = -\theta$ and $Q \in \text{Coh}_0(X)$. Applying $\mathbb{D}$ and taking the long exact sequence of cohomologies with respect to $\text{Coh}(X)$, we obtain the isomorphisms $\mathcal{H}^i \mathbb{D}(E) \cong 0$ for $i \neq -1, 0, 1$, $\mathcal{H}^{-1} \mathbb{D}(E) \cong U^\vee$ and the exact sequence of sheaves

$$0 \to \mathcal{H}^0 \mathbb{D}(E) \to \mathcal{E}xt^1_X(U, \mathcal{O}_X) \to Q^\vee \to \mathcal{H}^1 \mathbb{D}(E) \to \mathcal{E}xt^2(U, \mathcal{O}_X) \to 0.$$  

In particular, $\mathcal{H}^0 \mathbb{D}(E)$ is at most one dimensional, and $\mathcal{H}^1 \mathbb{D}(E)$ is zero dimensional. By dualizing $\text{Hom}(\text{Coh}_0(X), E) = 0$, we have $\mathcal{H}^1 \mathbb{D}(E) = 0$, and so $\mathbb{D}(E) \in \mathcal{C}_\theta$. It remains to show $\text{Hom}(\text{Coh}_{\leq 1}(X), \mathbb{D}(E)) = 0$. By dualizing, this is equivalent to $\text{Hom}(E, F) = 0$ for any pure one dimensional sheaf $F$ and $\text{Hom}(E, \text{Coh}_0(X)[-1]) = 0$. These conditions obviously follow from the condition $E \in \mathcal{C}_\theta^\circ$. \hfill \qed

Lemma 4.16. For any bounded below subset $S_1 \subset \Gamma_0^1$ and $v \in \Gamma$, the set of $E \in \mathcal{C}_\theta^\circ$ with $v^B(E) \in v + S_1$ is bounded.

Proof. By Lemma 4.15, it is enough to show the boundedness of $E \in \mathcal{C}_\theta^\circ$ with $v^B(E) \in v + S'$ for a bounded above subset $S' \subset \Gamma_0^1$. By Theorem 4.12 (i), the set of possible $v_3^B(\mathcal{H}^{-1}(E)[1])$ is bounded, and we have $v_3^B(\mathcal{H}^0(E)) \geq 0$ since $\mathcal{H}^0(E)$ is zero dimensional. Hence, one can take $S'$ to be a finite set, and the result follows using Theorem 4.12 (i) again. \hfill \qed

Applying Proposition 4.13 to $(Z, \text{Coh}(X))$, we have the following corollary:

Corollary 4.17. The set of $\nu_{\omega, B}$-semistable objects $E \in \mathcal{B}_{\omega, B}$ with fixed $(v_0^B, v_1^B, v_2^B)$ satisfying $v_1^B > 0$, and bounded below $v_3^B$, is bounded.

4.6. Boundedness of Bridgeland semistable objects. We carry on the notation in the previous subsection. Let $W$ be the group homomorphism given by (55). We have the very weak stability condition $(W, \mathcal{B}_{\omega, B})$ studied in Subsection 3.4 and denote by $\{ \mathcal{Q}(\phi) \}_{\phi \in \mathbb{R}}$ the slicing determined by
(W, B_ω, B). Let (W^t, B^{1}_{ω, B}) be its tilting, and \( v^t_1 \), \( \{Q^1_t(\phi)\}_{\phi \in \mathbb{R}} \) the slope function, slicing determined by \((W^t, B^{1}_{ω, B})\) respectively. By the argument in Subsection 2.3, we have the slope function on \( Q^1_t(\phi) \)

\[
\xi := \frac{\Delta^t_0}{\text{Re}W} = \frac{18v^B_0 - v^B_1}{6v^B_2 - v^B_0}
\]

where \( \Delta^t_0 \) is given by (56). In this notation, we have the following lemma:

**Lemma 4.18.** For a fixed \( v \in \Gamma \), the set of \( \xi|_{Q^1_{0}(1/2)} \)-semistable objects \( E \in Q^1_{0}(1/2) \) with \( v^B(E) = v \) is bounded.

**Proof.** Note that the category \( Q(1) \) is given by \( Q(1) = \langle U[1], \text{Coh}_{≤1}(X) : U \text{ is } \mu_{ω, B}\text{-semistable with } \mu_{ω, B}(U) = 0 \rangle \) i.e. it coincides with \( C_{B=(ω, B)/(ω)} \). By Lemma 2.19, the category \( Q^1_{0}(1/2) \) consists of \( ξ \)-semistable objects \( E \in C_{(ω, B)/(ω)} \) satisfying Hom(\( \text{Coh}_0(X), E \)) = 0. The restriction of \( ξ \) to \( Q^1_{0}(1/2) \) is given by

\[
ξ|_{Q^1_{0}(1/2)} = \frac{18v^B_0}{6v^B_2 - v^B_0}
\]

We fix \( v \in \Gamma \) which is written as \( v^B(E) = v \) for some \( ξ|_{Q^1_{0}(1/2)} \)-semistable object \( E \in Q^1_{0}(1/2) \). Then we have \( v^B_1 = 0, v^B_2 ≥ 0 \) and \( v^B_0 ≤ 0 \). There is an exact sequence in \( Q^1_{0}(1/2) \)

\[
0 \rightarrow F \rightarrow E \rightarrow E' \rightarrow 0
\]

where \( F \in \text{Coh}_{≤1}(X) \) and \( E' \in C_{(ω, B)/(ω)} \). As \( v^B_2(\ast) \) is non-negative on \( Q^1_{0}(1/2) \), the possible \( v^B_2(E') \) are bounded. Combined with the \( ξ|_{Q^1_{0}(1/2)} \)-semistability of \( E \), we obtain the lower bound of \( v^B_3(E') \), and so the possible \( E' \) is bounded by Lemma 4.16. It remains to show the boundedness of possible \( F \). Note that the \( ξ|_{\text{Coh}_{≤1}(X)} \)-stability on \( \text{Coh}_{≤1}(X) \) coincides with the \( B \)-twisted \( ω \)-Gieseker stability on it. Let \( F_1, \cdots, F_k \) be the BN factors of \( F \) with respect to the \( ξ|_{\text{Coh}_{≤1}(X)} \)-stability such that \( F_1 \) has the maximal \( ξ \)-slope. Since \( F_1 \subset E \) in \( Q^1_{0}(1/2) \), the \( ξ|_{Q^1_{0}(1/2)} \)-semistability of \( E \) implies the upper bound of the \( ξ \)-slope of \( F_1 \). Because the \( ξ \)-slope of \( F_1 \) is also bounded below \( ξ(F_1) ≥ ξ(F) \), and \( v^B_3(F_1) \) is bounded, the set of possible \( v^B_3(F_1) \) is also bounded. The same argument shows that the possible \( k \geq 1 \) together with \( v^B(F_i) \) are bounded for all \( 1 ≤ i ≤ k \). By the boundedness of \( ξ|_{\text{Coh}_{≤1}(X)} \)-semistable objects in \( \text{Coh}_{≤1}(X) \) with fixed \( v^B_2 \) and \( v^B_3 \), we have the boundedness of possible \( F_i \), and so the boundedness of possible \( F \). □

**Proposition 4.19.** Suppose that \( X \) satisfies Conjecture 3.3 Then the very weak stability condition \((W, B_ω, B)\) satisfies Assumption 4.7.

**Proof.** In the notation of the proof of Proposition 3.9, we have \( C^\dagger = \text{Coh}_0(X), \ C^\dagger_0 = \{0\} \), and so \( Π^\dagger_0 = \{0\} \). Therefore, the conditions (i), (ii) in Assumption 4.7 follow from Corollary 4.17. The condition (iii) in Assumption 4.7 follows from Lemma 4.18 □
By combining the results so far, we obtain the following result which is required to prove Theorem 4.1.

**Corollary 4.20.** Suppose that $X$ satisfies Conjecture 3.8. Then any good stability condition $\sigma \in \text{Stab}_{\omega,B}^\circ(X)$ satisfies the boundedness and the generic flatness.

**Proof.** Note that the set of $F \in \text{Coh}_0(X)$ with fixed $v^B_3(F)$ is bounded. Applying Lemma 4.9 to $(W, B^\omega_B)$, we conclude that $(Z, A^\omega_B)$ satisfies the boundedness. Also by using Proposition 4.10 twice, we see that $A^\omega_B$ satisfies the generic flatness. The result now follows from Proposition 4.11. □

### 4.7. Proof of Theorem 4.1

**Proof.** In [Tod08, Theorem 3.20], the second author gave general arguments for the moduli stacks of Bridgeland semistable objects to be algebraic stacks of finite type. First, because the good stability conditions form a dense subset in $\text{Stab}_{\omega,B}^\circ(X)$, and the walls are defined over $\mathbb{Q}$, one can perturb $\sigma \in \text{Stab}_{\omega,B}^\circ(X)$ and may assume that $\sigma$ is algebraic (see [Tod08, Theorem 3.20, Step 3]), i.e. $\sigma$ is good with $\zeta = 1$ in Definition 2.13. Next for an algebraic stability condition $\sigma$, the problem is reduced to showing the boundedness and the generic flatness of $\sigma$, which are proved in Corollary 4.20. Therefore, by [Tod08, Theorem 3.20] the stack $M_\sigma(v)$ is an algebraic stack of finite type such that (48) is an open immersion.

It remains to show that $M_\sigma(v)$ is proper. We write $\sigma = (Z, A)$, and let $\{P(\phi)\}_{\phi \in \mathbb{R}}$ be the associated slicing. Let us take $\phi_0 \in (0, 1]$ such that $Z(v) \in \mathbb{R}_{>0}\mathbb{e}^{\pi \phi_0}$. Then the rotated stability condition

$$(57) \quad (-e^{-\pi \phi_0}Z, P((\phi_0 - 1, \phi_0)))$$

is also algebraic. Then the properness follows from Theorem 4.21 (i) applied to the stability condition (57). □

Here we have used the following result by Abramovich-Polishchuk [AP06].

**Theorem 4.21.** ([AP06 Theorem 4.1.1]) Let $\sigma \in \text{Stab}_{\omega,B}^\circ(X)$ be a good stability condition, and $\{P(\phi)\}_{\phi \in \mathbb{R}}$ the associated slicing. Let $C$ be a smooth curve, $p \in C$ a closed point and set $U = C \setminus \{p\}$.

(i) Every family $F_U$ of objects in $\text{P}(1)$ over $U$ extends to a family $F$ of objects in $\text{P}(1)$.

(ii) Let $F_1$ and $F_2$ be families of objects in $\text{P}(1)$ over $S$, and $\phi_U : (F_1)_U \to (F_2)_U$ an isomorphism. Then $(F_1)_p$ and $(F_2)_p$ are $S$-equivalent.

When all the objects $[E] \in M_\sigma(v)$ are $\sigma$-stable, we also have the following result:

**Corollary 4.22.** Suppose that any object $[E] \in M_\sigma(v)$ is $\sigma$-stable. Then $M_\sigma(v)$ is a $\mathbb{C}^*$-gerbe over a proper and separated algebraic space $M_\sigma(v)$ of finite type.

**Proof.** Let $M$ be the algebraic space of simple objects in $D^b \text{Coh}(X)$ constructed by Inaba [Ina02], and write $\sigma = (Z, A)$. Similarly to the proof of Theorem 4.1 we may assume that $\sigma$ is good. The proof of Theorem 4.1...
shows that the subset \( M_\sigma(v) \subset M \) of \( \sigma \)-stable \( E \in A \) with \( v^B(E) = v \) is an open sub algebraic space. The valuative criterion in Theorem 4.21 applied to (57) shows that \( M_\sigma(v) \) is proper and separated. By the assumption, any \( E \in M_\sigma(v) \) is \( \sigma \)-stable, and so \( \text{Aut}(E) = \mathbb{C}^*. \) The natural morphism \( M_\sigma(v) \to M_\sigma(v) \) gives the desired \( \mathbb{C}^* \)-gerbe structure of \( M_\sigma(v). \)

5. DONALDSON-THOMAS INVARIANTS FOR BRIDGELAND SEMISTABLE OBJECTS

In this section, we mainly use Theorem 4.1 to define Donaldson-Thomas invariants counting Bridgeland semistable objects on Calabi-Yau 3-folds which satisfy the BG inequality conjecture.

5.1. Donaldson-Thomas invariants. Let \( X \) be a smooth projective Calabi-Yau 3-fold, i.e.

\[
K_X = 0, \ H^1(X, \mathcal{O}_X) = 0.
\]

Throughout this section, we assume that \( X \) satisfies Conjecture 3.8. So far, the only known examples of Calabi-Yau 3-folds satisfying Conjecture 3.8 are A-type Calabi-Yau 3-folds, that are étale quotients of abelian 3-folds [BMS]. See [OS01] for a classification of such Calabi-Yau 3-folds. We describe one of such examples.

Example 5.1. Let \( E_1, E_2, E_3 \) be three elliptic curves, and set \( A = E_1 \times E_2 \times E_3. \) Let \( \tau_i \in E_i \) be 2-torsion elements. We define the automorphisms \( g_1, g_2 \) on \( A \) to be

\[
\begin{align*}
g_1(z_1, z_2, z_3) &= (z_1 + \tau_1, -z_2, -z_3), \\
g_2(z_1, z_2, z_3) &= (-z_1, z_2 + \tau_2, -z_3 + \tau_3).
\end{align*}
\]

Then \((g_1, g_2)\) defines the free action of \( G = (\mathbb{Z}/2\mathbb{Z})^\oplus 2 \) on \( A \), and \( X = A/G \) is an A-type Calabi-Yau 3-fold.

Let us take the classes \( B \) and \( \omega \) as in (25). Assuming Conjecture 3.8 we have the Bridgeland stability condition \( \sigma_{\omega,B} \) given by (39). We take the connected component

\[
\text{Stab}^\circ_{\omega,B}(X) \subset \text{Stab}_{\omega,B}(X)
\]

which contains \( \sigma_{\omega,B}. \) Our goal is to construct a map

\[
(58) \quad \text{DT}_\sigma(v) : \text{Stab}^\circ_{\omega,B}(X) \to \mathbb{Q}
\]

for each \( v \in H^*(X, \mathbb{Q}) \), such that \( \text{DT}_\sigma(v) \) virtually counts \( \sigma \)-semistable objects \( E \in D^b \text{Coh}(X) \) with \( \text{ch}(E) = v. \)

5.2. DT invariants via virtual classes. In some cases, the DT invariants may be defined along with the original idea by Thomas [Tho00]. Let \( \Lambda \) be the image of the Chern character map

\[
\Lambda := \text{im}(\text{ch} : K(X) \to H^*(X, \mathbb{Q})).
\]

Note that the group homomorphism \( v^B \) in (26) factors through the Chern character map

\[
v^B : K(X) \xrightarrow{\text{ch}} \Lambda \xrightarrow{\alpha} \Gamma_{\omega,B}.
\]
For $v \in \Lambda$ and $\sigma \in \text{Stab}_{\omega,B}^0(X)$, Theorem 1.1 shows that the stack $\mathcal{M}_\sigma(\alpha(v))$ is an algebraic stack of finite type. The substack

$$\mathcal{M}_\sigma(v) \subset \mathcal{M}_\sigma(\alpha(v))$$

of $\sigma$-semistable objects with Chern character $v$ is an open and closed substack. Hence, it is also an algebraic stack of finite type.

Suppose that any object $[E] \in \mathcal{M}_\sigma(v)$ is $\sigma$-stable. Then by Corollary 4.22, the stack $\mathcal{M}_\sigma(v)$ is a $\mathbb{C}^*$-gerbe over a proper and separated algebraic space $M_\sigma(v)$ of finite type. The argument of [HT10] shows that there is a symmetric perfect obstruction theory on $\mathcal{M}_\sigma(v)$, and so the zero dimensional virtual class $[\mathcal{M}_\sigma(v)]^{\text{vir}}$. Since $M_\sigma(v)$ is proper and separated, we can integrate the virtual class and define the DT invariant:

**Definition 5.2.** Suppose that any object $[E] \in \mathcal{M}_\sigma(v)$ is $\sigma$-stable. Then we define

$$\text{DT}_\sigma(v) := \int_{[\mathcal{M}_\sigma(v)]^{\text{vir}}} 1 \in \mathbb{Z}. \quad (60)$$

Note that, by [Beh09], the invariant (60) is also described as the weighted Euler characteristic

$$\text{DT}_\sigma(v) = \int_{\mathcal{M}_\sigma(v)} \chi_B \, de := \sum_{m \in \mathbb{Z}} m \cdot e(\chi_B^{-1}(m)).$$

Here $\chi_B$ is Behrend’s constructible function on $M_\sigma(v)$, and $e$ is the topological Euler characteristic.

**Remark 5.3.** The work of [BBB05] shows that $M_\sigma(v)$ is locally written as a critical locus of some algebraic function $f$. The Behrend function $\chi_B$ is locally described by the Euler number of the Milnor fiber of $f$.

Suppose that $v$ and $\sigma = \sigma_{\omega,B}$ satisfies the assumption in Definition 5.2. Then we have the associated invariant

$$\text{DT}_{\omega,B}(v) := \text{DT}_{\sigma_{\omega,B}}(v).$$

The advantage of defining the DT invariant via the virtual cycle is that its deformation invariance automatically follows. Indeed, the argument of [BF97] shows the following:

**Theorem 5.4.** Suppose that $v$ and $\sigma_{\omega,B}$ satisfy the condition in Definition 5.2. Then the invariant $\text{DT}_{\omega,B}(v)$ is invariant under the deformations of complex structures of $X$.

We explain the precise statement of Theorem 5.4. Let $0 \in \Delta \subset \mathbb{C}$ be a small disc, and

$$\pi : \mathcal{X} \to \Delta$$

a smooth one parameter family of smooth projective Calabi-Yau 3-folds. Let $B$ be a $\mathbb{Q}$-divisor on $\mathcal{X}$, and $\omega$ be an $\pi$-ample $\mathbb{R}$-divisor on $\mathcal{X}$ with $\omega^2$ rational. For $v \in \Gamma(\Delta, \mathcal{R}_\pi \mathcal{Q})$, the claim in Theorem 5.4 means that the invariant

$$\text{DT}_{\omega,B_t}(v_t) \in \mathbb{Z}, \quad t \in \Delta$$
is independent of $t \in \Delta$. Here $B_t, \omega_t, v_t$ are the restrictions of $B, \omega, v$ to $\pi^{-1}(t)$.

In general, there may be strictly semistable objects $[E] \in \mathcal{M}_\sigma(v)$. In that case, we are not able to define the DT invariant by the virtual cycle at the moment. Instead, we will use the Hall algebras and the Behrend function following the idea of [KS, JS12].

5.3. Hall algebras. Here we use the notation in Subsection 4.1. Let $\mathcal{A} \subset \mathbb{D}^b \text{Coh}(X)$ be the heart of a bounded t-structure which is noetherian, and satisfies the generic flatness. Then the stack of objects in $\mathcal{A}$

$$\text{Obj}(\mathcal{A}) \subset \mathcal{M}$$

is realized as an open substack of $\mathcal{M}$. In particular, it is an algebraic stack locally of finite type. The Hall algebra $H(\mathcal{A})$ is $\mathbb{Q}$-spanned by the isomorphism classes of the symbols

$$[\rho: \mathcal{X} \to \text{Obj}(\mathcal{A})]$$

where $\mathcal{X}$ is an Artin stack of finite type over $\mathbb{C}$ with affine geometric stabilizers and $\rho$ is a 1-morphism. The relation is generated by

$$[\rho: \mathcal{X} \to \text{Obj}(\mathcal{A})] \sim [\rho|_\mathcal{Y}: \mathcal{Y} \to \text{Obj}(\mathcal{A})] + [\rho|_\mathcal{U}: \mathcal{U} \to \text{Obj}(\mathcal{A})]$$

where $\mathcal{Y} \subset \mathcal{X}$ is a closed substack and $\mathcal{U} := \mathcal{X} \setminus \mathcal{Y}$. There is an associative $*$-product on $H(\mathcal{A})$ based on the Ringel-Hall algebras (see [Joy07a, Section 5.1]). The unit is given by $1 = [\text{Spec} \mathbb{C} \to \text{Obj}(\mathcal{A})]$, which corresponds to $0 \in \mathcal{A}$. Also there is a Lie subalgebra

$$H^{\text{Lie}}(\mathcal{A}) \subset H(\mathcal{A})$$

consisting of elements supported on virtual indecomposable objects. See [Joy07a, Section 5.2] for further details on the definition of $H^{\text{Lie}}(\mathcal{A})$.

The algebra $H(\mathcal{A})$ is graded by $\Lambda$

$$H(\mathcal{A}) = \bigoplus_{v \in \Lambda} H_v(\mathcal{A})$$

where $H_v(\mathcal{A})$ is generated by symbols [61] which factors through $\text{Obj}_{v}(\mathcal{A}) \subset \text{Obj}(\mathcal{A})$. Here $\text{Obj}_{v}(\mathcal{A})$ is the stack of objects $E \in \mathcal{A}$ with $\text{ch}(E) = v$. By the Riemann-Roch theorem, the Euler pairing on $K(X)$ descends to the anti-symmetric pairing

$$\chi: \Lambda \times \Lambda \to \mathbb{Z}.$$ 

Let $C(\Lambda)$ be the Lie algebra

$$C(\Lambda) := \bigoplus_{v \in \Lambda} \mathbb{Q} \cdot c_v$$

with the bracket given by

$$[c_{v_1}, c_{v_2}] := (-1)^{\chi(v_1, v_2)} \chi(v_1, v_2) c_{v_1 + v_2}.$$ 

By [JS12 Theorem 5.12], there is a $\Lambda$-graded linear map

$$\Pi: H^{\text{Lie}}(\mathcal{A}) \to C(\Lambda)$$
such that if $\mathcal{X}$ is a $\mathbb{C}^*$-gerbe over an algebraic space $\mathcal{X}'$, we have

$$\Pi([\rho: \mathcal{X} \to \text{Obj}(\mathcal{A})]) = - \left( \sum_{k \in \mathbb{Z}} k \cdot e(\chi^{-1}_B(k)) \right) c_v.$$ 

Here $\rho$ is an open immersion and $\chi_B$ is Behrend’s constructible function on $\mathcal{X}'$. The map (64) is shown to be a Lie algebra homomorphism if $\mathcal{A} = \text{Coh}(X)$ by [JS12] Theorem 5.12.

5.4. DT invariants for Bridgeland semistable objects. Let us take the classes $B, \omega$ as in the previous section (also as in (25)). Consider a good stability condition

$$\sigma = (Z, \mathcal{A}) \in \text{Stab}^g_{\omega, B}(X).$$

For each element $v \in \Lambda$, the stack $\mathcal{M}_\sigma(v)$ in (59) determines an element

$$\delta_\sigma(v) := \{ \mathcal{M}_\sigma(v) \subset \text{Obj}(\mathcal{A}) \} \in H(\mathcal{A}).$$

Let $C(\mathcal{A}) \subset \Lambda$ be the image of $\text{ch}|_A$. We also define $\epsilon_\sigma(v) \in H(\mathcal{A})$ in the following way:

$$(65) \quad \epsilon_\sigma(v) := \sum_{l \geq 1} \sum_{v_1, \ldots, v_l \in C(\mathcal{A}), \arg Z(v_1) = \ldots = \arg Z(v_l)} \frac{(-1)^{l-1}}{l} \delta_\sigma(v_1) \ast \cdots \ast \delta_\sigma(v_l).$$

By the support property and the boundedness of $\sigma$, the sum (65) is a finite sum, and so it is well-defined. Then the argument of [Joy07b] Theorem 8.7 shows that $\epsilon_\sigma(v) \in H^{\text{Lie}}(\mathcal{A})$. Following the construction of generalized DT invariants in [JS12], we give the following definition:

**Definition 5.5.** For $v \in \Lambda$, we define the invariant $\text{DT}_\sigma(v) \in \mathbb{Q}$ in the following way: if $v \in C(\mathcal{A})$, we define it by the formula

$$\Pi \epsilon_\sigma(v) = - \text{DT}_\sigma(v) \cdot c_v.$$ 

Otherwise, we set

$$\text{DT}_\sigma(v) := \begin{cases} \text{DT}_\sigma(-v), & \text{if } -v \in C(\mathcal{A}), \\ 0, & \text{if } \pm v \notin C(\mathcal{A}). \end{cases}$$

As a summary, we have the following result:

**Theorem 5.6.** Let $X$ be a smooth projective Calabi-Yau 3-fold satisfying Conjecture 5.8. Then for each $v \in \Lambda$, there is a map

$$\text{DT}_\sigma(v): \text{Stab}^g_{\omega, B}(X) \to \mathbb{Q}$$

such that $\text{DT}_\sigma(v)$ virtually counts $\sigma$-semistable objects $E \in D^b\text{Coh}(X)$ with $\text{ch}(E) = v$. If any $\sigma$-semistable object $E$ with $\text{ch}(E) = v$ is $\sigma$-stable, then $\text{DT}_\sigma(v)$ coincides with (67).

**Proof.** If $\sigma$ is good, then the invariant $\text{DT}_\sigma(v)$ is defined in Definition 5.5. Suppose $\sigma$ is not good. Since the set of good points in $\text{Stab}^g_{\omega, B}(X)$ is dense, and the walls are defined over rational numbers, one can perturb $\sigma$ to a good stability condition $\sigma'$ so that stable factors of objects in $\mathcal{M}_\sigma(v)$ and those in $\mathcal{M}_{\sigma'}(v)$ are the same. We define $\text{DT}_\sigma(v)$ to be $\text{DT}_{\sigma'}(v)$, which is obviously independent of $\sigma'$. \qed
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