ON THE SCATTERING PROBLEM FOR INFINITELY MANY FERMIONS IN DIMENSIONS $d \geq 3$ AT POSITIVE TEMPERATURE

THOMAS CHEN, YOUNGHUN HONG, AND NATAŠA PAVLOVIĆ

Abstract. In this paper, we study the dynamics of a system of infinitely many fermions in dimensions $d \geq 3$ near thermal equilibrium and prove scattering in the case of small perturbation around equilibrium in a certain generalized Sobolev space of density operators. This work is a continuation of our previous paper [18], and extends the important recent result of M. Lewin and J. Sabin in [35] of a similar type for dimension $d = 2$. In the work at hand, we establish new, improved Strichartz estimates that allow us to control the case $d \geq 3$.

1. Introduction

In this paper, we study the dynamics of a system of infinitely many fermions in dimensions $d \geq 3$ near thermal equilibrium. In particular, we prove scattering in the case when the perturbation around equilibrium is small in a certain generalized Sobolev space of density operators. This work is a continuation of our previous paper [18], and extends some important recent result of M. Lewin and J. Sabin in [35] of a similar type for two dimensions ($d = 2$). In the work at hand, we are employing new, improved Strichartz estimates that allow us to access higher dimensions.

To set up the problem, we start with a finite system of $N$ fermions interacting via a pair potential $w$ in mean-field description. The dynamics is described by $N$ coupled Hartree equations

$$\begin{aligned}
i \partial_t u_1 &= (-\Delta + w \ast \rho)u_1 , & u_1(t = 0) &= u_{1,0} \\
&\cdots & \cdots \\
i \partial_t u_N &= (-\Delta + w \ast \rho)u_N , & u_N(t = 0) &= u_{N,0}
\end{aligned}$$

(1.1)

where $\rho$ is the total density of particles

$$\rho(t, x) = \sum_{j=1}^{N} |u_j(t, x)|^2.$$  

(1.2)

In order to be in agreement with the Pauli principle, we require that the initial data $\{u_{j,0}\}_{j=1}^{N}$ is an orthonormal family. Given that the Cauchy problem is well-posed in a suitable solution space, the solution $\{u_{j,t}\}_{j=1}^{N}$ continues to be an orthonormal family for $t > 0$. 

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We introduce the one-particle density matrix corresponding to (1.1),

$$\gamma_N(t) = \sum_{j=1}^{N} |u_j(t)\rangle\langle u_j(t)|.$$  

(1.3)

It corresponds to the rank-$N$ orthogonal projection onto the span of the orthonormal family \(\{u_j(t)\}_{j=1}^{N}\). The system (1.1) is then equivalent to a single operator-valued equation

$$i\partial_t \gamma_N = [-\Delta + w \ast \rho_{\gamma_N}, \gamma_N]$$

(1.4)

with initial data

$$\gamma_N(t = 0) = \sum_{j=1}^{N} |u_{j,0}\rangle\langle u_{j,0}|,$$

(1.5)

where the density function is given by

$$\rho_{\gamma_N}(t, x) = \gamma_N(t, x, x).$$

(1.6)

Orthornormality of the family \(\{u_j\}_{j=1}^{N}\) implies that \(0 \leq \gamma \leq 1\).

The expected particle number \(\int \rho_N dx\) diverges as \(N \to \infty\) for the system (1.1) - (1.2), respectively (1.4) - (1.6). Therefore, the one-particle density matrix \(\gamma = \sum_{j=1}^{\infty} |u_j\rangle\langle u_j|\) is not of trace class; on the other hand, it has a bounded operator norm \(L^2 \to L^2\).

For a dilute gas with a finite density (for instance, with \(\rho(t,x) = \frac{1}{N} \sum_{j=1}^{N} |u_j(t, x)|^2\) as \(N \to \infty\), or \(\rho(t, x) = \sum_{j=1}^{\infty} \lambda_j |u_j(t, x)|^2\) with \(\lambda_j > 0\) and \(\sum \lambda_j = 1\)), the system (1.1) has been extensively analyzed in the literature, see for instance [1, 8, 9, 10, 12, 44]. In this setting, \(\gamma = \lim_{N \to \infty} \gamma_N\) is trace class. See also for instance [4, 3, 19, 26, 5, 39] and the references therein for its derivation from a quantum system of interacting fermions; we remark that the fermionic exchange term is negligible in this limit.

The Cauchy problem, obtained from (1.6) as \(N \to \infty\) but with \(\rho_\gamma \notin L^1\), is much more difficult than in the earlier works noted above. The main problem is to understand in which framework the Cauchy problem

$$i\partial_t \gamma = [-\Delta + w \ast \rho_\gamma, \gamma]$$

(1.7)

with initial data

$$\gamma(0) = \gamma_0,$$

(1.8)

and density

$$\rho_\gamma(t, x) = \gamma(t, x, x),$$

(1.9)

can be meaningfully posed. Lewin and Sabin were the first authors who introduced a framework for this problem [34, 35], which can be described as follows. First, we observe that given a non-negative function \(f : \mathbb{R} \to \mathbb{R}_{\geq 0}\), the operator \(\gamma_f = f(-\Delta)\) is a stationary solution to (1.7) having infinite particle number, i.e., \(\rho_{\gamma_f} \notin L^1\), since the density function \(\rho_{\gamma_f}\) is a constant function. Examples of \(\gamma_f\) include the Fermi sea of the non-interacting

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\(^1\)Again, it is required that \(0 \leq \gamma_0 \leq 1\), to be in agreement with the Pauli principle; hence, \(\gamma\) has a bounded operator norm.
system. For inverse temperature $\beta > 0$ and chemical potential $\mu > 0$, the Fermi sea $\gamma_f$ is given by the Fermi-Dirac distribution
\[
\gamma_f(x, y) = \int_{\mathbb{R}^d} \frac{e^{ip(x-y)}}{e^{\beta(p^2 - \mu)} + 1} dp = \frac{1}{e^{\beta(-\Delta - \mu)} + 1}(x, y).
\] (1.10)
while in the zero temperature limit,
\[
\gamma_f = \Pi_\mu = 1(-\Delta \leq \mu).
\] (1.11)
Then, the main idea is to consider a perturbation
\[
Q := \gamma - \gamma_f
\] (1.12)
from the reference state $\gamma_f$, which evolves according to the following Cauchy problem:
\[
\begin{cases}
i \mathcal{L} Q = [-\Delta + w \ast \rho Q, Q + \gamma_f], \\
Q(0) = Q_0.
\end{cases}
\] (1.13)
In [34], Lewin and Sabin proved that the Cauchy problem (1.13) for $Q$ is globally well-posed for $d \geq 2$ in a suitable subspace of the space of compact operators, provided that the pair interaction $w$ is sufficiently regular. An important tool used in [34] was a Strichartz estimate for density functions originally established in [23], which is extended to the optimal range [24]. The case of a more singular interaction potential, with $w = \delta$ given by the Dirac delta, was analyzed by authors of the paper at hand; in [18], we proved global well-posedness of the perturbative system (1.13), at zero temperature $\gamma_f = \Pi_\mu$, by employing new Strichartz estimates for regular density functions and those for operator kernels, which were established in the same paper [18].
In the case of a sufficiently regular potential $w$, Lewin-Sabin in [35] proved scattering for $Q$ in $d = 2$ via Strichartz estimates from [23]. The case of higher dimensions was left open, and the purpose of the paper at hand is to address it.
Before we state the main result of this paper in Theorem 1.1, we present a brief review of the notation. For $p \geq 1$, the Schatten class $\mathcal{S}^p$ is defined via
\[
\|A\|_{\mathcal{S}^p} = (\text{Tr}(|A|^p))^{1/p},
\]
while for $\alpha \geq 0$ a Hilbert-Schmidt Sobolev space $\mathcal{H}^\alpha$ is equipped with the norm
\[
\|Q\|_{\mathcal{H}^\alpha} = \|\langle \nabla \rangle^\alpha Q \langle \nabla \rangle^\alpha\|_{\mathcal{S}^2}.
\]
Also, we use the standard notation
\[
\hat{g}(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} g(\xi) \, d\xi
\]
to denote the inverse Fourier transform of a function $g$.\footnote{For details, see [32].}
Theorem 1.1. Let \( d \geq 3 \), \( \alpha > \frac{d-2}{2} \) and \( \alpha_0 \) be given by
\[
\begin{cases}
\alpha_0 = 2\alpha - \frac{d-1}{2} & \text{if } \alpha < \frac{d}{2}, \\
\alpha_0 < \alpha & \text{if } \alpha = \frac{d-1}{2}, \\
\alpha_0 = \alpha & \text{if } \alpha > \frac{d}{2},
\end{cases}
\]
and \( \beta > \frac{d+2}{2} \).

We assume that
\[(i) \text{ (assumptions on } f \text{) } f \text{ is real-valued, } \langle \gamma \rangle^\beta f \in L^\infty_{r \geq 0}, f'(r) < 0 \text{ for } r > 0, \]
\[
\int_0^\infty (r^{d/2-1} |f(r)| + |f'(r)|) \, dr < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{\hat{g}(x)}{|x|^{d-2}} \, dx < \infty,
\]
where \( g(\xi) = f(|\xi|^2) \).

(ii) (assumption on } w \text{) The interaction potential } w = w_1 * w_2 \in L^1 \text{ is even,}
\[
\|\hat{w}_1\|_{L^{\frac{2d}{d-2}}}, \|\hat{w}_2\|_{L^{\frac{2d}{d-2}}}, \|\langle \gamma \rangle^{\alpha_0 + \frac{\beta}{2}} \hat{w}_1\|_{L^\infty}, \|\hat{w}_2\|_{L^\infty}, \|\cdot^{1/2} \langle \gamma \rangle^{-\alpha_0} \hat{w}_2\|_{L^\infty} < \infty,
\]
and
\[
\|\hat{w}_-\|_{L^\infty} < 2|\mathbb{S}^{d-1}| \left( \int_{\mathbb{R}^d} \frac{|\hat{g}(x)|}{|x|^{d-2}} \, dx \right)^{-1} \quad \text{and} \quad \hat{w}_+(0) < \frac{2}{\epsilon_g}|\mathbb{S}^{d-1}|,
\]
where \( A_\pm = \max\{\pm A, 0\} \) and \( \epsilon_g \) is given by (4.4).

Then, there exists small \( \epsilon > 0 \) such that \( \|Q_0\|_{H^\alpha} \leq \epsilon \), there exists a unique global solution \( Q(t) \in C_t(\mathbb{R}; \mathbb{S}^{2d}) \) to the equation (1.13) with initial data \( Q_0 \). Moreover, the associated density function \( \rho_Q \) obeys the global space-time bound,
\[
\|w \ast \rho_Q\|_{L^1_t(\mathbb{R}^d)} < \infty, \tag{1.18}
\]
and \( Q(t) \) scatters in \( \mathbb{S}^{2d} \) as \( t \to \pm \infty \); in other words, there exist \( Q_\pm \in \mathbb{S}^{2d} \) such that \( e^{-it \Delta} Q(t) e^{it \Delta} \to Q_\pm \) converges strongly in \( \mathbb{S}^{2d} \) as \( t \to \pm \infty \).

Remark 1.2. (i) In Theorem 1.1 various conditions are imposed on the reference state \( \gamma_f \), the interaction potential \( w \), and the initial data \( Q_0 \). Our main goal is to prove scattering in high dimensions. We do not pursue any optimality on the hypotheses. Some physically important examples, such as the Fermi-Dirac distribution (1.10), satisfy these assumptions. The assumptions on \( f \) and the assumptions in (1.17) are used for the linear response theory (see Proposition 4.1). The assumptions in (1.16) are used for the proof of the global space-time bound (1.18) (see Section 7).

(ii) The method in our paper might be applied to the two-dimensional case with different conditions on the interaction potential \( w \) and initial data \( Q_0 \) from Lewin and Sabin [35]. However, we omit the case \( d = 2 \), as it was already proved in [35]; moreover, some exponents would have to be modified in the proof. For instance, we are using the endpoint Strichartz estimate for convenience, but the endpoint estimate is known to be false in \( \mathbb{R}^2 \) [43].

(iii) As a crucial new ingredient that allow us to extend the work of Lewin-Sabin [35] to dimensions higher than 2, we establish new Strichartz estimates for density functions and
density matrices in Section 3. Compared to the Strichartz estimates derived in [23], and used in [35], our Strichartz estimates exhibit an improved summability gain by imposing more regularity on the initial data.

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2. Outline of the Proof of Theorem 1.1

In this part of our analysis, we explain the strategy to prove the main result of this article, Theorem 1.1. First, in Section 2.1, we show that if the density function \( \rho_Q \) of the solution to (1.13) satisfies the global space-time bound (see (2.10)), then the solution \( Q(t) \) scatters. Next, in §2.2, we set up a suitable contraction map \( \Gamma \) (see (2.19)) to construct a solution obeying the desired global space-time bound.

2.1. A global space-time bound for a density function implies scattering. We follow the strategy in Lewin and Sabin [35]. For simplicity, we present the argument only for the forward-in-time direction, as it can be easily modified to prove scattering backward in time.

Given a time-dependent potential \( V = V(t, x) \), we denote by \( \mathcal{U}_V(t) \) the linear propagator for the linear Schrödinger equation

\[
    i\partial_t u + \Delta u - Vu = 0,
\]

i.e., \( \mathcal{U}_V(t)\phi \) is the solution to (2.1) with initial data \( \phi \). We define the "finite-time" wave operator \( \mathcal{W}_V(t) \) by

\[
    \mathcal{W}_V(t) := e^{-it\Delta} \mathcal{U}_V(t).
\]

Iterating the Duhamel formula

\[
    \mathcal{U}_V(t) = e^{it\Delta} - i \int_0^t e^{i(t-t_1)\Delta} V(t_1) \mathcal{U}_V(t_1) dt_1
\]

infinitely many times, the wave operator can be written as an infinite sum,

\[
    \mathcal{W}_V(t) := \sum_{n=0}^{\infty} \mathcal{W}_V^{(n)}(t),
\]

where \( \mathcal{W}_V^{(0)}(t) := \text{Id} \), and for \( n \geq 1 \),

\[
    \mathcal{W}_V^{(n)}(t) := (-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 e^{-it_n \Delta} V(t_n) e^{it_n \Delta} \ldots e^{-it_1 \Delta} V(t_1) e^{it_1 \Delta}
\]

\[
    = (-i) \int_0^t dt_n e^{-it_n \Delta} V(t_n) e^{it_n \Delta} \mathcal{W}_V^{(n-1)}(t_n).
\]
By the definition of the finite-time wave operator, the equation (1.13) is equivalent to
\[
Q(t) = e^{it\Delta} W_{w*\rho_Q}(t) (\gamma_f + Q_0) W_{w*\rho_Q}(t)^* e^{-it\Delta} - \gamma_f,
\]  
(2.6)
because \( Q(t) = \gamma(t) - \gamma_f \) and
\[
\gamma(t) = U_V(t) \gamma_0 U_V(t)^*.
\]  
(2.7)
Inserting the sum (2.4) into the equation (2.6), it becomes
\[
Q(t) = e^{it\Delta} \left( \sum_{m=0}^{\infty} W_{w*\rho_Q}^{(m)}(t) \right) \gamma_f \left( \sum_{n=0}^{\infty} W_{w*\rho_Q}^{(n)}(t) \right)^* e^{-it\Delta} - \gamma_f
\]
\[
+ e^{it\Delta} \left( \sum_{m=0}^{\infty} W_{w*\rho_Q}^{(m)}(t) \right) Q_0 \left( \sum_{n=0}^{\infty} W_{w*\rho_Q}^{(n)}(t) \right)^* e^{-it\Delta}
\]
\[
= e^{it\Delta} Q_0 e^{-it\Delta} + \sum_{(m,n)\neq (0,0)} e^{it\Delta} W_{w*\rho_Q}^{(m)}(t) \gamma_f W_{w*\rho_Q}^{(n)}(t)^* e^{-it\Delta}
\]
\[
+ \sum_{(m,n)\neq (0,0)} e^{it\Delta} W_{w*\rho_Q}^{(m)}(t) Q_0 W_{w*\rho_Q}^{(n)}(t) e^{-it\Delta}.
\]  
(2.8)
In [23], Frank, Lewin, Lieb and Seiringer prove that if \( d \geq 2 \), then
\[
\| W_{V}^{(n)}(t_0) \|_{\mathcal{S}^{2d}} \leq \frac{1}{(n!)^{\frac{3}{2}} \pi^{\frac{d}{2}}} \left( C \| V \|_{L^2_w([0,\infty];L^d_w)} \right)^n, \quad \forall n \geq 1
\]  
(2.9)
for any small \( \epsilon > 0 \) (see Theorem 2 for \( n = 1 \) and Theorem 3 for \( n \geq 2 \) in [23]). Therefore, if the density function obeys the space-time norm bound
\[
\| w * \rho_Q \|_{L^2([0,\infty];L^d_w)} < \infty,
\]  
(2.10)
by (2.9) with \( V = w * \rho_Q \), the series is absolutely convergent in \( C_t([0,\infty);\mathcal{S}^{2d}) \), so it is well-defined.

Using this series expansion, we prove that the global space-time bound (2.10) implies scattering.

**Lemma 2.1** (A global space-time bound for a density function implies scattering). Let \( d \geq 3 \). Suppose that \( Q(t) \in C_t([0,\infty);\mathcal{S}^{2d}) \) is a solution to the equation (2.6) and its density satisfies (2.10). Then, \( Q(t) \) scatters in \( \mathcal{S}^{2d} \) as \( t \to +\infty \).

**Proof.** As in the proof of the absolute convergence of the series, applying the inequality (2.9) to the series expansion of the difference between \( e^{-it_1\Delta} Q(t) e^{it_1\Delta} \) and \( e^{-it_2\Delta} Q(t) e^{it_2\Delta} \), one can show that \( e^{-it_1\Delta} Q(t) e^{it_1\Delta} - e^{-it_2\Delta} Q(t) e^{it_2\Delta} \to 0 \) in \( \mathcal{S}^{2d} \) as \( t_1, t_2 \to +\infty \). Therefore, \( e^{-it\Delta} Q(t) e^{it\Delta} \) has a strong limit \( Q_+ \) in \( \mathcal{S}^{2d} \) as \( t \to +\infty \). That is, \( Q(t) \) scatters in \( \mathcal{S}^{2d} \) as \( t \to +\infty \). \( \square \)
2.2. Set-up for the contraction mapping argument. By Lemma 2.3, the goal is now to prove that the equation (2.10) has a unique solution \( Q(t) \) in a suitable space obeying the space-time bound (2.10). To this end, as in Lewin-Sabin [35], we write the equation (2.6) as an equation for density functions,

\[
\rho_Q(t) = \rho \left[ e^{it\Delta} W_{w \ast \rho_Q}(t)(\gamma_f + Q_0) W_{w \ast \rho_Q}(t)^* e^{-it\Delta} \right] - \rho_{\gamma_f}.
\] 

One of the advantages of this wave operator formulation in density is that the unknown is given only by the density function, and there is no unknown operator.

We further simplify the equation by splitting the interaction potential \( w \) into \( w = w_1 \ast w_2 \), and subsequently convolving the density function \( \rho_Q \) with \( w_2 \),

\[
w_2 \ast \rho_Q(t) = w_2 \ast \rho \left[ e^{it\Delta} W_{w_1 \ast (w_2 \ast \rho_Q)}(t)(\gamma_f + Q_0) W_{w_1 \ast (w_2 \ast \rho_Q)}(t)^* e^{-it\Delta} \right] - w_2 \ast \rho_{\gamma_f}.
\] 

Now we consider the equation for \( w_2 \ast \rho_Q \). The motivation for this formulation is that the solution \( w_2 \ast \rho_Q \) is expected to be contained in a larger function space (or bounded in a weaker norm) than the one for \( \rho_Q \), provided that \( w_2 \) is sufficiently nice; our constructions will exploit this fact.

Next, inserting the sum (2.14) for the finite time wave operators acting on \( \gamma_f \), we write

\[
w_2 \ast \rho_Q(t) = w_2 \ast \rho \left[ e^{it\Delta} \left( \sum_{m=0}^{\infty} W_{w_1 \ast (w_2 \ast \rho_Q)}^{(m)}(t) \right) \gamma_f \left( \sum_{n=0}^{\infty} W_{w_1 \ast (w_2 \ast \rho_Q)}^{(n)}(t) \right)^* e^{-it\Delta} \right] + w_2 \ast \rho \left[ e^{it\Delta} \left( W_{w_1 \ast (w_2 \ast \rho_Q)}^{(1)}(t) \gamma_f W_{w_1 \ast (w_2 \ast \rho_Q)}^{(1)}(t)^* e^{-it\Delta} \right) \right] - w_2 \ast \rho_{\gamma_f}.
\] 

Then, introducing the operators,

\[
L(\phi)(t) := -w_2 \ast \rho \left[ e^{it\Delta} \left( W_{w_1 \ast \phi}^{(1)}(t) \gamma_f W_{w_1 \ast \phi}^{(1)}(t)^* e^{-it\Delta} \right) \right],
\] 

\[
A_{m,n}(\phi)(t) := w_2 \ast \rho \left[ e^{it\Delta} W_{w_1 \ast \phi}^{(m)}(t) \gamma_f W_{w_1 \ast \phi}^{(n)}(t)^* e^{-it\Delta} \right],
\] 

\[
B(\phi)(t) := w_2 \ast \rho \left[ e^{it\Delta} W_{w_1 \ast \phi}(t) Q_0 W_{w_1 \ast \phi}(t)^* e^{-it\Delta} \right],
\] 

we write

\[
w_2 \ast \rho_Q = -L(w_2 \ast \rho_Q) + \left\{ \sum_{m,n=1}^{\infty} A_{m,n}(w_2 \ast \rho_Q) + B(w_2 \ast \rho_Q) \right\}.
\] 

We note that compared to the formulation in [35], the equation (2.17) is slightly simpler in that \( B(w_2 \ast \rho_Q) \) is not expanded as an infinite sum. However, due to the linear nature of the operator \( L \), which is not perturbative even for small functions, the series expansion \( \sum_{m,n=1}^{\infty} A_{m,n}(w_2 \ast \rho_Q) \) does not seem to be avoidable.
Later in Section 4, it will be shown that \((1 + \mathcal{L})\) is invertible on \(L^2_{t \geq 0}L^2_x\). As a result, the equation can be reformulated as

\[
w_2 * \rho_Q = (1 + \mathcal{L})^{-1} \left\{ \sum_{m,n=1}^{\infty} \mathcal{A}_{m,n}(w_2 * \rho_Q) + \mathcal{B}(w_2 * \rho_Q) \right\}.
\] (2.18)

Our goal is now to show that the map \(\Gamma\), defined by

\[
\Gamma(\phi) = (1 + \mathcal{L})^{-1} \left\{ \sum_{m,n=1}^{\infty} \mathcal{A}_{m,n}(\phi) + \mathcal{B}(\phi) \right\},
\] (2.19)

is contractive in a suitable function space, and its solution satisfies the space-time bound

\[
\|\phi\|_{L^2_{t \geq 0}L^2_x} < \infty.
\] (2.20)

Then, the main theorem follows (see Section 7).

3. Strichartz estimates for density functions

In this section we present the Strichartz estimates that will be used in our analysis. First, we give an overview of the notation.

3.1. Notation. As already mentioned in Section 1, we denote by \(\mathfrak{S}^p\) the Schatten spaces, equipped with the norms

\[
\|Q\|_{\mathfrak{S}^p} := \left( \mathrm{Tr}|Q|^p \right)^{1/p},
\] (3.1)

for \(p \geq 1\).

For \(\alpha \geq 0\), we define the Hilbert-Schmidt Sobolev space \(\mathcal{H}^\alpha\) as the collection of Hilbert-Schmidt operators (which are not necessarily self-adjoint) with a finite norm

\[
\|\gamma_0\|_{\mathcal{H}^\alpha} := \|\langle \nabla^\alpha \gamma_0 \nabla^\alpha \rangle\|_{\mathfrak{S}^2} = \|\langle \nabla_y^\alpha \langle \nabla_x^\alpha \rangle^\alpha \gamma_0(x, x') \|_{L^2_t L^2_x}.\] (3.2)

Here, \(\gamma_0(x, x')\) is the integral kernel of \(\gamma_0\), i.e.,

\[
(\gamma_0 g)(x) = \int_{\mathbb{R}^d} \gamma_0(x, x') g(x') dx'.
\] (3.3)

In order to review Strichartz estimates for operator kernels in Subsection 3.3, we need to recall notation from [13] related to Strichartz norms. An exponent pair \((q, r)\) is (Strichartz) admissible if \(2 \leq q, r \leq \infty\), \((q, r, d) \neq (2, \infty, 2)\) and

\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.
\] (3.4)

Assume that \(\gamma(t)\) is a time-dependent operator on an interval \(I \subset \mathbb{R}\). Then, its Strichartz norm is defined by

\[
\|\gamma(t)\|_{\mathcal{S}^\alpha(I)} := \sup_{(q, r): \text{admissible}} \left\{ \|\langle \nabla^\alpha_x \langle \nabla^\alpha_x \rangle^\alpha \gamma(t, x, x') \|_{L^q_t(L^r_x L^2_x)} + \|\langle \nabla^\alpha_y \langle \nabla^\alpha_y \rangle^\alpha \gamma(t, x, x') \|_{L^q_t(L^r_y L^2_y)} \right\}.
\] (3.5)

It is clear that \(\mathcal{S}^\alpha(I) \hookrightarrow L^\infty_t(I; \mathcal{H}^\alpha)\).
We identify the operator $e^{it\Delta} \gamma_0 e^{-it\Delta}$ with its integral kernel
\[
(e^{it\Delta} \gamma_0 e^{-it\Delta})(x, x') = (e^{it(\Delta - \Delta_x')} \gamma_0)(x, x').
\] (3.6)

3.2. Strichartz estimates for density functions. In this section, we prove new Strichartz estimates for density functions, which extend Strichartz estimates proved in the authors’ previous work \[18\] by allowing asymmetric derivatives ($\alpha_1$ not necessarily equal to $\alpha_2$). Those are presented in Theorem 3.1, and as a main application, we obtain Corollary 3.2, which we use to control the operators $A_{m,n}$.

**Theorem 3.1** (Strichartz estimates for density functions). Suppose that $\alpha_0, \alpha_1, \alpha_2 \geq 0$. When $d = 1$, we assume that $\alpha = \min\{\alpha_1, \alpha_2\}$. When $d \geq 2$, we assume that
\[
\alpha_1 + \alpha_2 > \frac{d-1}{2}
\] and
\[
\begin{cases}
\alpha_0 = \alpha_1 + \alpha_2 - \frac{d-1}{2} & \text{if } \max\{\alpha_1, \alpha_2\} < \frac{d-1}{2}, \\
\alpha_0 < \min\{\alpha_1, \alpha_2\} & \text{if } \max\{\alpha_1, \alpha_2\} = \frac{d-1}{2}, \\
\alpha_0 = \min\{\alpha_1, \alpha_2\} & \text{if } \max\{\alpha_1, \alpha_2\} > \frac{d-1}{2}.
\end{cases}
\] (3.8)

Then,
\[
\|\nabla\|^\frac{1}{2} \rho_{e^{it\Delta}} \gamma_0 \|_{L^2_t H^\alpha_x} \lesssim \|\langle \nabla \rangle^{\alpha_1} \gamma_0 \langle \nabla \rangle^{\alpha_2} \|_{\mathfrak{S}^2}.
\] (3.9)

**Corollary 3.2.** Suppose that $\alpha_0, \alpha_1$ and $\alpha_2$ satisfy the assumptions in Theorem 3.1. Then,
\[
\left\| \langle \nabla \rangle^{-\alpha_1} \int_\mathbb{R} e^{-it\Delta} V(t)e^{it\Delta} dt \langle \nabla \rangle^{-\alpha_2} \right\|_{\mathfrak{S}^2} \leq c \| V(t) \|_{L^2_t L^\frac{2d}{d+1}},
\] (3.10)

**Proof of Corollary 3.2, assuming Theorem 3.1.** For a compactly supported smooth function $V(t, x)$ and a finite rank smooth operator $\gamma_0$, we write
\[
\text{Tr} \left( \langle \nabla \rangle^{-\alpha_1} \int_\mathbb{R} e^{-it\Delta} V(t)e^{it\Delta} dt \langle \nabla \rangle^{-\alpha_2} \right) \gamma_0
\] = \int_\mathbb{R} \text{Tr} \left( e^{it\Delta} \langle \nabla \rangle^{-\alpha_2} \gamma_0 \langle \nabla \rangle^{-\alpha_1} e^{-it\Delta} V(t) \right) dt
\] (3.11)
= \int_\mathbb{R} \int_\mathbb{R}^{d} \rho_{e^{it\Delta}} \langle \nabla \rangle^{-\alpha_2} \gamma_0 \langle \nabla \rangle^{-\alpha_1} e^{-it\Delta} (x) V(t, x) dxdt,
where the first identity is from cyclicity of trace. Therefore, (3.11) is dual to
\[
\left\| \rho_{e^{it\Delta}} \gamma_0 e^{-it\Delta} \right\|_{L^2_t L^{\frac{2d}{d+1}}} \lesssim c \left\| \langle \nabla \rangle^{\alpha_2} \gamma_0 \langle \nabla \rangle^{\alpha_1} \right\|_{\mathfrak{S}^2},
\] (3.12)
which follows from (3.9) and the Sobolev inequality. \(\square\)

The main strategy to prove the Strichartz estimate for density functions is to reformulate it as an integral estimate through the space-time Fourier transformation. This approach, via bilinear estimates based on the space-time $L^2$-norm, has been introduced by Klainerman and Machedon \[31, 32\], and subsequently developed by many authors.
Lemma 3.3 (Reduction to an integral estimate). Let $\alpha$ be any real number. Then if the integral

$$I_{\tau, \xi} := \int_{|\eta| \leq |\xi - \eta|} \frac{|\xi|^{2\alpha}|\xi|^{2\alpha}}{|\eta|^{2\alpha}(\alpha_1, \alpha_2)} \delta(|\tau + |\eta|^2 - |\xi - \eta|^2|) d\eta$$

(3.13)

is bounded uniformly in $\tau$ and $\xi$, the Strichartz estimate

$$\|\nabla |^5 \rho_{e^{it\Delta} \gamma_0 e^{-it\Delta}} \|_{L^2_t L^2_x H^s_x} \lesssim \|\langle \nabla \rangle^{\alpha_1} \gamma_0 \langle \nabla \rangle^{\alpha_2} \|_{L^2_x}$$

(3.14)

holds.

Proof. The Fourier transform of the density function of $\gamma$ is given by

$$\hat{\gamma}(\xi) = \mathcal{F}_\xi \left\{ \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \hat{\gamma}(\eta, \zeta) e^{ix(\eta + \zeta)} d\eta d\zeta \right\} (\xi)$$

$$= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \hat{\gamma}(\eta, \zeta) \mathcal{F}_\xi \left\{ e^{ix(\eta + \zeta)} \right\} (\xi) d\eta d\zeta$$

$$= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \hat{\gamma}(\eta, \zeta) \cdot (2\pi)^d \delta(\xi - \eta - \zeta) d\eta d\zeta$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\gamma}(\eta, \xi - \eta) d\eta.$$ 

(3.15)

Hence, the space-time Fourier transform of the density function $\rho_{e^{it\Delta} \gamma_0 e^{-it\Delta}}$ is

$$(\rho_{e^{it\Delta} \gamma_0 e^{-it\Delta}}) (\tau, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}_\xi \left\{ e^{-it(|\eta|^2 - |\xi - \eta|^2)} \right\} \hat{\gamma}_0(\eta, \xi - \eta) d\eta$$

$$= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} \delta(|\tau + |\eta|^2 - |\xi - \eta|^2|) \hat{\gamma}_0(\eta, \xi - \eta) d\eta.$$ 

(3.16)

Thus, by the Plancherel theorem and Cauchy-Schwarz, we get

$$\|\nabla |^5 \rho_{e^{it\Delta} \gamma_0 e^{-it\Delta}} \|^2_{L^2_t L^2_x H^s_x}$$

$$= \frac{1}{(2\pi)^{2(d+1)}} \int_{\mathbb{R}^d} \langle |\xi|^2 \rangle^{\alpha_0} \langle \rho_{e^{it\Delta} \gamma_0 e^{-it\Delta}} \rangle^2 (\tau, \xi) d\xi$$

$$= \frac{1}{(2\pi)^{2(d+1)}} \int_{\mathbb{R}^d} \langle |\xi|^2 \rangle^{\alpha_0} d\xi \int_{\mathbb{R}^d} \delta(|\tau + |\eta|^2 - |\xi - \eta|^2|) \hat{\gamma}_0(\eta, \xi - \eta) d\eta$$

$$\leq \frac{1}{(2\pi)^{4d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle |\xi|^2 \rangle^{\alpha_0} \left\{ \int_{\mathbb{R}^d} \delta(|\tau + |\eta|^2 - |\xi - \eta|^2|) \langle \eta \rangle^{2\alpha_1} \langle \xi - \eta \rangle^{2\alpha_2} d\eta \right\} d\xi d\tau$$

$$\leq \sup_{\tau, \xi} \frac{1}{(2\pi)^{4d}} \left\{ \int_{\mathbb{R}^d} \langle |\xi|^2 \rangle^{\alpha_0} \delta(|\tau + |\eta|^2 - |\xi - \eta|^2|) \langle \eta \rangle^{2\alpha_1} \langle \xi - \eta \rangle^{2\alpha_2} d\eta \right\}$$

$$\cdot \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta(|\tau + |\eta|^2 - |\xi - \eta|^2|) \langle \eta \rangle^{2\alpha_1} \langle \xi - \eta \rangle^{2\alpha_2} d\eta d\xi d\tau.$$
Thus, integrating out the delta function with respect to \( \tau \) and using the Plancherel theorem again,

\[
\| \nabla [\rho_{\text{ext}}(s_0 e^{-itA})] \|_{L^2_{x} H^0}\leq \sup_{\tau, \xi} \left\{ \int_{\mathbb{R}^d} \frac{\langle \xi \rangle^{2\alpha} \langle \xi - \eta \rangle^{2\alpha} \delta(\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\alpha}\langle \xi - \eta \rangle^{2\alpha}} d\eta \right\} \frac{1}{(2\pi)^{2d}} \| \langle \nabla \rangle^\alpha \gamma_0 \langle \nabla \rangle^\alpha \|_{L^2}\quad (3.18)
\]

Therefore, it suffices to show that \( \sup_{\tau, \xi} \{ \cdots \} \) is bounded.

We decompose

\[
\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha} \langle \xi \rangle^{2\alpha} \delta(\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\alpha}\langle \xi - \eta \rangle^{2\alpha}} d\eta = \int_{|\eta| \leq |\xi - \eta|} + \int_{|\eta| \geq |\xi - \eta|} \frac{|\xi|^{2\alpha} \langle \xi \rangle^{2\alpha} \delta(\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\alpha}\langle \xi - \eta \rangle^{2\alpha}} d\eta. (3.19)
\]

By change of the variable \( (\xi - \eta) \rightarrow \eta \), the second integral becomes

\[
\int_{|\eta| \geq |\xi - \eta|} \frac{|\xi|^{2\alpha} \langle \xi \rangle^{2\alpha} \delta(\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\alpha}\langle \xi - \eta \rangle^{2\alpha}} d\eta = \int_{|\eta| \leq |\xi - \eta|} \frac{|\xi|^{2\alpha} \langle \xi \rangle^{2\alpha} \delta(-\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\alpha}\langle \xi - \eta \rangle^{2\alpha}} d\eta. (3.20)
\]

Thus, by the assumption (3.23), we prove the desired uniform bound,

\[
\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha} \langle \xi \rangle^{2\alpha} \delta(\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\alpha}\langle \xi - \eta \rangle^{2\alpha}} d\eta \leq I_{\tau, \xi} + I_{-\tau, \xi} \leq 2 \sup_{\tau, \xi} I_{\tau, \xi} < \infty. (3.21)
\]

\( \square \)

**Proof of Theorem 3.1.** By Lemma 3.3, the proof of Theorem 3.1 can be reduced to the proof of a uniform bound on the integral

\[
I_{\tau, \xi} = \int_{|\eta| \leq |\xi - \eta|} \frac{|\xi| \langle \xi \rangle^{2\alpha} \delta(\tau + |\eta|^2 - |\xi - \eta|^2)}{\langle \eta \rangle^{2\max\{\alpha_1, \alpha_2\}} \langle \xi - \eta \rangle^{2\min\{\alpha_1, \alpha_2\}}} d\eta (3.22)
\]

Here, we may assume that \( \tau > 0 \), since if \( \tau < 0 \), then \( \tau + |\eta|^2 - |\xi - \eta|^2 < 0 \) in the integral domain, so the delta function in (3.22) is zero.

When \( d = 1 \), using the trivial inequality

\[
|\xi| \leq |\eta| + |\xi - \eta| \leq 2|\xi - \eta| (3.23)
\]

in the integral domain, we obtain

\[
I_{\tau, \xi} \leq \frac{\langle \xi \rangle^{2\alpha}}{\langle \xi \rangle^{\min\{\alpha_1, \alpha_2\}}} \int_{|\eta| \leq |\xi - \eta|} |\xi| \delta(\tau - \xi^2 + 2\xi \eta) d\eta \sim 1.
\]
Suppose that $d \geq 2$. Given $\xi \in \mathbb{R}^d$, changing the variable $\eta$ by a rotation making $(1, 0, \cdots, 0) \in \mathbb{R}^d_\eta$ parallel to $\xi$ and then integrating out the delta function, we write the integral as

$$I_{\tau, \xi} = \int_{\mathbb{R}^{d-1}} \int_{|\eta_1| \leq |\eta| - |\xi|} \frac{|\xi\langle \xi \rangle^{2\alpha} \delta(\tau - |\xi|^2 + 2|\xi|\eta_1)}{(\langle \eta_1, \eta' \rangle)^{2\max\{\alpha_1, \alpha_2\}} (\langle (\eta_1 - |\xi|, \eta') \rangle)^{2\min\{\alpha_1, \alpha_2\}}} d\eta_1 d\eta'$$

$$= \frac{1}{2} \int_{\mathbb{R}^{d-1}} \frac{\langle \xi \rangle^{2\alpha} d\eta'}{(\langle \eta_1, \eta' \rangle)^{2\max\{\alpha_1, \alpha_2\}} (\langle (\eta_1 - |\xi|, \eta') \rangle)^{2\min\{\alpha_1, \alpha_2\}}} ,$$

where $\eta = (\eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $\eta_1^* = \frac{|\xi|^2 - \tau}{2|\xi|}$ with $|\eta_1^*| \leq |\eta_1^* - |\xi||$. Note that by the trivial inequality as in (3.23), we have $|\eta_1^* - |\xi|| \geq \frac{|\xi|}{2}$. Thus, Theorem 3.1 follows from the uniform bound on

$$\tilde{I}_{\tau, \xi} := \int_{\mathbb{R}^{d-1}} \frac{\langle \xi \rangle^{2\alpha} d\eta'}{(\langle \eta_1, \eta' \rangle)^{2\max\{\alpha_1, \alpha_2\}} (\langle \frac{|\xi|}{2}, \eta' \rangle)^{2\min\{\alpha_1, \alpha_2\}}} .$$

We decompose

$$\tilde{I}_{\tau, \xi} = \int_{|\eta'| \leq |\xi|} + \int_{|\eta'| > |\xi|} \frac{\langle \xi \rangle^{2\alpha} d\eta'}{(\langle \eta_1, \eta' \rangle)^{2\max\{\alpha_1, \alpha_2\}} (\langle \frac{|\xi|}{2}, \eta' \rangle)^{2\min\{\alpha_1, \alpha_2\}}} =: \tilde{I}_{\tau, \xi}^{(1)} + \tilde{I}_{\tau, \xi}^{(2)}. (3.25)$$

For the first integral, using that $\frac{|\xi|}{2} \leq |(\frac{|\xi|}{2}, \eta')| \leq \frac{\sqrt{2}}{2} |\xi|$ in the integral domain, we get

$$\tilde{I}_{\tau, \xi}^{(1)} \sim \int_{|\eta'| \leq |\xi|} \frac{\langle \xi \rangle^{2\alpha} d\eta'}{(\langle \eta_1, \eta' \rangle)^{2\max\{\alpha_1, \alpha_2\}}} \sim \left\{ \begin{array}{ll} 0 & \text{if } 0 \leq \max\{\alpha_1, \alpha_2\} < \frac{d-1}{2}, \\
\langle \xi \rangle^{2\alpha - 2\min\{\alpha_1, \alpha_2\}} \ln\langle \xi \rangle & \text{if } \max\{\alpha_1, \alpha_2\} = \frac{d-1}{2}, \\
\langle \xi \rangle^{2\alpha - 2\min\{\alpha_1, \alpha_2\}} & \text{if } \max\{\alpha_1, \alpha_2\} > \frac{d-1}{2}. \\
\end{array} \right. (3.26)$$

The second integral $\tilde{I}_{\tau, \xi}^{(2)}$ is bounded by

$$\int_{|\eta'| > |\xi|} \frac{\langle \xi \rangle^{2\alpha} d\eta'}{(\langle \eta_1, \eta' \rangle)^{2\max\{\alpha_1, \alpha_2\} + 2\min\{\alpha_1, \alpha_2\}}} = \int_{|\eta'| > |\xi|} \frac{\langle \xi \rangle^{2\alpha} d\eta'}{(\langle \eta_1, \eta' \rangle)^{2(\alpha_1 + \alpha_2) + (d-1)}} \lesssim \langle \xi \rangle^{2\alpha - 2(\alpha_1 + \alpha_2) + (d-1)} , (3.27)$$

since $2(\alpha_1 + \alpha_2) > d - 1$. Both $\tilde{I}_{\tau, \xi}^{(1)}$ and $\tilde{I}_{\tau, \xi}^{(2)}$ are uniformly bounded due to the assumption (3.8). \hfill \Box

Next, we prove optimality of the Strichartz estimate (3.9).

**Theorem 3.4** (Optimality of Theorem 3.1). The assumptions in Theorem 3.1 are necessary.

The following dual formulation is useful to find the necessary conditions on the Strichartz estimate (3.9).

**Lemma 3.5** (Dual inequality). The Strichartz estimate (3.14) holds if and only if

$$\| |\xi|^{\hat{\alpha}} \langle \xi \rangle^{\alpha_0} \hat{V}(-|\eta|^2 + \xi - \eta)^2, \xi \rangle \|_{L_x^2 L_t^2} \lesssim \| \hat{V} (\tau, \xi) \|_{L_x^2 L_t^{2\hat{\alpha}_0}} , (3.28)$$
Proof. Using the Plancherel theorem and (3.16) and then integrating out the delta function, we write

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( |\nabla|^a \langle \xi \rangle^{\alpha_0} \rho_0 e^{i|\xi| \theta} \right)(x) \overline{V(t, x)} dx dt = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{ \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d} \delta(\tau + |\eta|^2 - |\xi - \eta|^2) \hat{\gamma}_0(\eta, \xi - \eta) d\eta \right\} \\
\cdot |\xi|^a \langle \xi \rangle^{\alpha_0} \overline{V(\tau, \xi)} d\xi d\tau
\]

(3.29)

By Hölder inequality and the Plancherel theorem, it is bounded by

\[
\frac{1}{(2\pi)^{2d}} \| \langle \nabla \rangle^{\alpha_1} \gamma_0 \langle \nabla \rangle^{\alpha_2} \|^2 \| \xi^{\frac{1}{2}} \langle \xi \rangle^{\alpha_0} \overline{V(-|\eta|^2 + |\xi - \eta|^2, \xi)} \|_{L^2(\mathbb{R}^d)} \|_{L^2(\mathbb{R}^d)} \]

(3.30)

Therefore, by duality, (3.14) is equivalent to (3.28).

\[\Box\]

Proof of Theorem 3.4. By the duality lemma (Lemma 3.5), the inequality (3.9) holds if and only if

\[
\| \xi^{\frac{1}{2}} \langle \xi \rangle^{\alpha_0} \overline{V(-|\eta|^2 + |\xi - \eta|^2, \xi)} \|_{L^2(\mathbb{R}^d)} \|_{L^2(\mathbb{R}^d)} \leq \| \overline{V(\tau, \xi)} \|_{L^2(\mathbb{R}^d)} \|
\]

(3.31)

The square of the left hand side is

\[
\left\| \xi^{\frac{1}{2}} \langle \xi \rangle^{\alpha_0} \overline{V(-|\eta|^2 + |\xi - \eta|^2, \xi)} \right\|_{L^2(\mathbb{R}^d)}^2 \]

(3.32)

Changing the variable \(\eta\) by a rotation making \((1, 0, \cdots, 0) \in \mathbb{R}^d\) parallel to \(\xi\) and then changing the variable \(\tau = |\xi|^2 - 2|\xi| \eta_1\) as in the proof of Theorem 3.1, we write

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left| \xi^{2\alpha_1} \langle \xi \rangle^{2\alpha_0} \overline{V(|\xi|^2 - 2|\xi| \eta_1, \xi)} \right|^2 d\eta_1 d\eta' d\xi
\]

(3.33)

\[
= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left| \xi^{2\alpha_1 - 1} \langle \xi \rangle^{2\alpha_0} \overline{V(\tau, \xi)} \right|^2 d\tau d\eta' d\xi
\]

where \(\eta = (\eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^d\).

1. Necessity of the condition (3.7): From the inner integral \(\cdots\) over \(\mathbb{R}^{d-1}\) in (3.33), we see that it is necessary to assume that \(\alpha_1 + \alpha_2 > \frac{d-1}{2}\) in (3.7), because if \(\alpha_1 + \alpha_2 \leq \frac{d-1}{2}\), the inequality (3.9) fails.
2. Necessity of the homogeneous half derivative on the left hand side of (3.9): Suppose that \( \alpha_1 + \alpha_2 > \frac{d-1}{2} \). Let
\[
\tilde{V}_n(\tau, \xi) = n^{\frac{d+2}{2}} \delta_{\left[ -\frac{1}{2}, \frac{1}{2} \right]}(\tau) 1_{B_{0, r}\{ \xi \}}(\xi),
\]
where \( B_{0, r} \) is the ball of radius \( r \) centered at 0 in \( \mathbb{R}^d \). Note that for large \( n \), \( \tilde{V}_n \) is localized in low frequencies. We observe that by (3.33), if \( \tilde{\alpha} < \frac{1}{2} \), then
\[
\left\| \frac{\xi^\alpha \xi^{\alpha_2}}{\langle \xi \rangle^{\alpha_2}} \tilde{V}(\xi) \right\|_{L_x^2 L_t^\infty} \sim n \int_\mathbb{R} d\tau \int_\mathbb{R} \frac{n^{1-2\tilde{\alpha}} d\eta}{\langle \eta \rangle^{2(\alpha_1+\alpha_2)}} \sim n^{1-2\tilde{\alpha}} \to \infty, \tag{3.34}
\]
while \( \|\tilde{V}_n\|_{L_{x,t}^2 L_t^\infty} \sim 1 \). Thus, the inequality (3.9) fails when \( \tilde{\alpha} < \frac{1}{2} \).

3. Necessity of the condition (3.8): Suppose that \( \alpha_1 + \alpha_2 > \frac{d-1}{2} \) and \( \tilde{\alpha} = \frac{1}{2} \). We further assume that \( \alpha_\xi \geq \alpha_2 \). Now we define the sequence \( \{V_n\}_{n=1}^\infty \) by
\[
\tilde{V}_n(\tau, \xi) = 1_{\left[ n^2 - \frac{1}{2}, n^2 + \frac{1}{2} \right]}(\tau) 1_{\left[ \frac{1}{2}, \frac{1}{2} \right]}(\xi), \tag{3.35}
\]
where \( \xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1} \), so that \( \|\tilde{V}_n\|_{L_x^2 L_t^\infty} = 1 \). Then, \( |\tilde{\xi}|^{2-\tau} \leq 1 + o_n(1) \) and
\[
|\tilde{\xi}|^{2-\tau} = -n + o_n(1) \text{ in the support of } \tilde{V}(\tau, \xi), \text{ where } o_n(1) \to 0 \text{ as } n \to \infty. \]
Hence, by (3.33),
\[
\left\| \xi^{\alpha_1} \xi^{\alpha_2} \tilde{V}_n(\xi) \right\|_{L_x^2 L_t^\infty} \sim n \int_\mathbb{R} d\tau \int_\mathbb{R} \frac{n^{1-2\tilde{\alpha}} d\eta}{\langle \eta \rangle^{2(\alpha_1+\alpha_2)}} \sim n^{1-2\tilde{\alpha}} \to \infty. \tag{3.36}
\]
for sufficiently large \( n \). Thus, (3.31) fails unless (3.8) is not satisfied.

When \( \alpha_1 \leq \alpha_2 \), we use the sequence \( \{V_n\}_{n=1}^\infty \) given by
\[
\tilde{V}(\tau, \xi) = 1_{\left[ n^2 - \frac{1}{2}, n^2 + \frac{1}{2} \right]}(\tau) 1_{\left[ \frac{1}{2}, \frac{1}{2} \right]}(\xi) \tag{3.37}
\]
to prove that the condition (3.8) is necessary.

\[\square\]

3.3. Strichartz estimates for operator kernels. We finish this section by recalling the statement of the Strichartz estimates for operator kernels, that we established in [18].

Theorem 3.6 (Strichartz estimates for operator kernels). Let \( I \subset \mathbb{R} \). Then, we have
\[
\left\| e^{it\Delta} \gamma_0 e^{-it\Delta} \right\|_{S^0(\mathbb{R})} \lesssim \|\gamma_0\|_{H^s}, \tag{3.38}
\]
\[
\left\| \int_0^t e^{i(t-s)\Delta} R(s) e^{-i(t-s)\Delta} ds \right\|_{S^0(\mathbb{R})} \lesssim \|R(t)\|_{L^1(\mathbb{R}; H^s)}. \tag{3.39}
\]

4. Linear response theory: invertibility of \((1 + \mathcal{L})\)

We review the linear response theory from Section 3 of Lewin and Sabin [35], which addresses the invertibility of the operator \((1 + \mathcal{L})\), with \(\mathcal{L}\) defined by

\[
\mathcal{L}(\phi) = -w_2 * \rho\left[ e^{i t \Delta} \left( W_{w_1 \phi}^{(1)}(t) \gamma_f + \gamma_f W_{w_1 \phi}^{(1)}(t) \right) e^{-i t \Delta} \right] = iw_2 * \rho\left[ \int_0^t e^{i(t-t_1) \Delta} \left( (w_1 \phi)(t_1), \gamma_f \right) e^{-i(t-t_1) \Delta} dt_1 \right],
\]

where \(w = w_1 \ast w_2\). Roughly speaking, it asserts that \((1 + \mathcal{L})\) is invertible on \(L^2_{t \geq 0} L^2_x\), provided that \(f\) is strictly decreasing, and that \(\hat{w}_+(0)\) and \(\hat{w}_-\) are not too large, where \(A_\pm = \max\{\pm A, 0\}\) so that \(A = A_+ - A_-\).

**Proposition 4.1** (Invertibility of \((1 + \mathcal{L})\)). Let \(d \geq 3\). We assume that \(f \in L^\infty_{r \geq 0}\) is real-valued, \(f'(r) < 0\) for \(r > 0\),

\[
\int_0^\infty \left( d/2 - 1 \right) |f(r)| + |f'(r)| dr < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{\tilde{g}(x)}{|x|^{d-2}} dx < \infty,
\]

where \(g(\xi) = f(|\xi|^2)\). Moreover, we assume that the interaction potential \(w \in L^1\) is even,

\[
\|\hat{w}_-\|_{L^\infty} < 2|S^{d-1}| \left( \int_{\mathbb{R}^d} \frac{\tilde{g}(x)}{|x|^{d-2}} dx \right)^{-1} \quad \text{and} \quad \hat{w}_+(0) < \frac{2}{\epsilon_g} |S^{d-1}|,
\]

where

\[
\epsilon_g := - \liminf_{(\tau, \xi) \to (0, 0)} \frac{\text{Re}(m_f(\tau, \xi))}{2|S^{d-1}|}\]

and

\[
(F^{-1}_r m_f(t, \xi))(t, \xi) = 21_{t \geq 0} \sqrt{2\pi} \sin(t|\xi|^2) \tilde{g}(2t\xi).
\]

Then, \(1 + \mathcal{L}\) is invertible on \(L^2_{t \geq 0} L^2_x\).

**Sketch of the proof.** We sketch the proof for the sake of completeness of the article and for the convenience of the reader. For details, we refer the reader to [35] Proposition 1, Proposition 2 and Corollary 1. We assume \(d \geq 3\) for brevity, however, the invertibility of \((1 + \mathcal{L})\) was proved in [35] for any dimension \(d \geq 1\).

The space-time Fourier transformation of \(\mathcal{L}(\phi)\) is directly computed as

\[
(\mathcal{L}(\phi))^\sim(\tau, \xi) = \hat{w}^\sim(\xi)m_f(\tau, \xi)\hat{\phi}(\tau, \xi), \quad \forall \phi \in \mathcal{D}([0, +\infty) \times \mathbb{R}^d),
\]

with (4.5), in other words,

\[
\hat{\mathcal{L}}\hat{\phi}(t, \xi) = 2\sqrt{2\pi} \hat{w}(\xi) \int_0^\infty \sin(s|\xi|^2) \tilde{g}(2s\xi) \hat{\phi}(t-s, \xi) ds.
\]

Note that the operator \(\mathcal{L}\) maps \(L^2_{t \geq 0} L^2_x\) to itself, because \(\hat{\mathcal{L}}\hat{\phi}(t, \xi) = 0\) for \(t < 0\). Moreover, we have

\[
\|m_f\|_{L^\infty_{r, \xi}} \leq \frac{1}{2|S^{d-1}|} \left( \int_{\mathbb{R}^d} \frac{\tilde{g}(x)}{|x|^{d-2}} dx \right)
\]
and
\[
\|L\|_{L^2_{t>0}L^2_x \to L^2_{t>0}L^2_x} \leq \frac{\|\hat{w}\|_{L^\infty}}{2|\mathbb{S}^{d-1}|} \left( \int_{\mathbb{R}^d} \frac{|\hat{g}(x)|}{|x|^{d-2}} \, dx \right)
\] (4.9)
(see \cite{35} Proposition 1). We remark that the operator \(L\) looks different from the corresponding linear operator \(L_1\) in Lewin-Sabin \cite{35} at first glance, however they are indeed the same, since
\[
(L\phi)^\wedge(t,\xi) = \hat{w}_2(\xi) \left( \rho \int_0^t e^{i(t-t_1)\Delta} \left[ (w_1 \ast \phi)(t_1), \gamma_f e^{-i(t-t_1)\Delta} \, dt_1 \right] \right)^\wedge(t,\xi)
\]
= \(\hat{w}_2(\xi) \hat{w}_1(\xi) m_f(\tau,\xi) \hat{\phi}(\tau,\xi)\) (by \cite{35} Proposition 1)
\[
= \hat{w}(\xi) m_f(\tau,\xi) \hat{\phi}(\tau,\xi).
\] (4.10)

When \(\gamma_f = 1_{(-\Delta \leq \mu)}\), one can compute the multiplier \(m^F_d(\mu, \tau, \xi) := m_f(\tau, \xi)\) as
\[
m^F_d(\mu, \tau, \xi) = \frac{|\mathbb{S}^{d-2}|^2 \mu^{d-1}}{(2\pi)^{d-2}} \int_0^1 m^F_1(\mu(1-r^2), \tau, \xi) r^{d-2} \, dr,
\] (4.11)
where
\[
m^F_1(\mu, \tau, \xi) = \frac{1}{2\sqrt{2\pi} |\xi|} \log \left| \frac{|\xi|^2 + 2|\xi| \sqrt{\mu} - \tau^2}{|\xi|^2 - 2|\xi| \sqrt{\mu} - \tau^2} \right| + i \frac{\sqrt{\pi}}{2\sqrt{2}|\xi|} \left( 1_{|\tau+|\xi|^2| \geq 2\sqrt{\mu} |\xi|} - 1_{|\tau-|\xi|^2| \geq 2\sqrt{\mu} |\xi|} \right)
\] (4.12)
(see \cite{35} Proposition 2). By the relation \(\gamma_f = f(-\Delta) = -\int_0^\infty 1_{(-\Delta \leq \mu)} f'(s) \, ds\), \(m_f\) can be written in terms of \(m^F_d\) as
\[
m_f(\tau, \xi) = -\int_0^\infty m^F_d(s, \tau, \xi) f'(s) \, ds.
\] (4.13)

For \(\phi \in L^2_{t \geq 0} L^2_x\), the space-time Fourier transformation of \((1 + L)\phi\) is given by \((1 + \hat{w}(\xi)m_f(\tau, \xi)) \hat{\phi}(\tau, \xi)\). Thus, the invertibility of \((1 + L)\) follows from a uniform lower bound on \(|1 + \hat{w}m_f|\). Let
\[
A := \left\{ \xi \in \mathbb{R}^d : |\hat{w}(\xi)| \geq \frac{1}{4|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} \frac{|\hat{g}(x)|}{|x|^{d-2}} \, dx \right\}.
\] (4.14)

Then, by the bound \((4.8)\), \(|(1 + \hat{w}m_f)| \geq \frac{1}{2}\) on \(A\). Note that \(A^c\) is a compact subset in \(\mathbb{R}^d\), because \(\hat{w}(\xi) \to 0\) as \(\xi \to \infty\). Moreover, by \((4.13)\), \(m_f\) is continuous on \(\mathbb{R} \times (\mathbb{R}^d \setminus \{0\})\) (so as \((1 + \hat{w}m_f)\) by the Riemann-Lebesgue lemma), since \(m^F_d\) is continuous on \(\mathbb{R} \times (\mathbb{R}^d \setminus \{0\})\). Therefore, it suffices to show that \((1 + \hat{w}m_f)\) is non-zero for all \(\xi\).

We consider the four cases separately.

Case 1 \(((\tau, \xi) = (0, \xi) \text{ with } \xi \neq 0)\) We observe that \(m_f(0, \xi) \geq 0\) for \(\xi \neq 0\), since \(f'(s) < 0\)
and \(m_f^2(s, 0, \xi) \geq 0\) in the integral (4.13) (see (4.12)). Hence, it follows that
\[
m_f(0, \xi) \hat{w}(\xi) + 1 \geq 1 - \hat{w}_-(\xi)m_f(0, \xi) \\
\geq 1 - \|\hat{w}_-\|_{L^2} \frac{1}{2|\mathcal{S}^{d-1}|} \left( \int_{\mathbb{R}^d} \frac{|\hat{g}(x)|}{|x|^{d-2}} dx \right) \quad \text{(by (4.8))} \tag{4.15}
\]
\[
> 0 \quad \text{(by the assumption (4.3) on } \hat{w}_-). \]

**Case 2** \((\tau, \xi) = (\tau, 0) \text{ with } \tau \neq 0\) In this case, \(m_f^2(\tau, 0) = 0\), so \((m_f(\tau, 0)\hat{w}(0) + 1) = 1. \)

**Case 3** \((\tau, \xi) \text{ with } \tau \neq 0 \text{ and } \xi \neq 0\) It suffices to show that \(\text{Im}(m_f(\tau, \xi)) \neq 0\). By the relation \(\text{Im}(m_f(-\tau, \xi)) = -\text{Im}(m_f(\tau, \xi))\), we may assume that \(\tau > 0\). By (4.13) and (4.11), one can write the imaginary part of \(m_f(\tau, \xi)\) explicitly as
\[
\text{Im}(m_f(\tau, \xi)) = \frac{|\mathcal{S}^{d-2}|}{4(2\pi)^{d-2}} \int_0^1 r^{d-2} \left\{ \int_{\mathcal{S}^{d-2}} \frac{(\tau + r|\xi|^2)^2}{4(\tau + r|\xi|^2)^2} s^\frac{d-3}{2} f'(s) ds \right\} dr, \tag{4.16}
\]
Since by the assumption \(f'(s) < 0\), we conclude from (4.16) that \(\text{Im}(m_f(\tau, \xi)) \neq 0\).

**Case 4** \((\tau, \xi) \text{ in the neighborhood of } (0, 0)\) By the definition of \(m_f\) and (4.13), one can show that
\[
- \epsilon \tilde{g}^2|\mathcal{S}^{d-1}| \leq \text{Re}(m_f(\tau, \xi)) \leq \frac{1}{2|\mathcal{S}^{d-1}|} \left( \int_{\mathbb{R}^d} \frac{|\hat{g}(x)|}{|x|^{d-2}} dx \right) \tag{4.17}
\]
near \((0, 0)\) (see [35] for details). Thus, by the assumptions on \(\hat{w}_+\), \(\text{Re}(\hat{w}(\xi)m_f(\tau, \xi) + 1) > 0). \)

## 5. Bound on \(A_{m,n}(\phi)\)

In this section, we estimate the operator \(A_{m,n}\).

**Proposition 5.1** (Bounds on \(A_{m,n}\)). Let \(d \geq 3\), \(\beta > \frac{d+2}{2}\) and \(\beta_0 > \frac{1}{4}\). Then, there exists \(C_A > 0\) such that for any \((m, n)\) with \(m, n \geq 1, \)
\[
\|A_{m,n}(\phi)\|_{L^2_{t,x}} \leq C_A^{m+n+1}\|\langle \cdot \rangle^\beta f\|_{L^\infty}\|\phi\|_{L^2_{t,x}}^{m+n} \tag{5.1}
\]
and
\[
\|A_{m,n}(\phi) - A_{m,n}(\psi)\|_{L^2_{t,x}} \leq (m + n)C_A^{m+n+1}\|\langle \cdot \rangle^\beta f\|_{L^\infty}\left\{\|\phi\|_{L^2_{t,x}} + \|\psi\|_{L^2_{t,x}}\right\}^{m+n-1}\|\phi - \psi\|_{L^2_{t,x}}, \tag{5.2}
\]
where the constant \(C_A\) depends only on \(d\), \(\|\langle \cdot \rangle^\beta_0 \hat{w}_1\|_{L^\infty}, \|\hat{w}_2\|_{L^\infty}, \|\hat{w}_1\|_{L^{2d}2}, \|\hat{w}_2\|_{L^{2d}2}\) and \(\|\hat{w}_2\|_{L^{2d}2}.\)
Proof. We will prove the proposition by the standard duality argument. For notational convenience, we denote \( W = w_1 \ast \phi \). By the definition of \( A_{m,n} \), we write

\[
\int_0^\infty \int_{\mathbb{R}^d} A_{m,n}(\phi)(t)U(t,x)dxdt \\
= \int_0^\infty \int_{\mathbb{R}^d} w_2 \ast \rho \left[ e^{it\Delta} W^{(m)}_W(t) \gamma_f W^{(n)}_W(t) \ast e^{-it\Delta} \right] U(t,x)dxdt \\
= \int_0^\infty \int_{\mathbb{R}^d} \rho \left[ e^{it\Delta} W^{(m)}_W(t) \gamma_f W^{(n)}_W(t) \ast e^{-it\Delta} \right] (w_2 \ast U)(t,x)dxdt. 
\]

Then, by the formal identity

\[
\int_{\mathbb{R}^d} \rho \gamma_0 V dx = \text{Tr}(\gamma_0 V) 
\]

and the cyclicity of the trace, it becomes

\[
\int_0^\infty \int_{\mathbb{R}^d} A_{m,n}(\phi)(t)U(t,x)dxdt \\
= \text{Tr} \left( \int_0^\infty e^{it\Delta} W^{(m)}_W(t) \gamma_f W^{(n)}_W(t) \ast e^{-it\Delta} (w_2 \ast U)(t)dt \right) \\
= \text{Tr} \left( \int_0^\infty W^{(m)}_W(t) \gamma_f W^{(n)}_W(t) \ast e^{-it\Delta} (w_2 \ast U)(t)e^{it\Delta}dt \right). 
\]

Note that the application of the formal identity (5.5) in (5.9) will be justified by the estimates below.

First, we consider the higher order terms with \( m + n \geq 3 \). In this case, we employ the following two inequalities,

\[
\left\| \int_0^\infty e^{-it\Delta} V(t)e^{it\Delta} dt \right\|_{\mathfrak{g}^{2\beta}} \leq c\|V\|_{L^2_t L^4_x}, \tag{5.6}
\]

\[
\left\| \int_0^\infty e^{-it\Delta} V(t)e^{it\Delta} dt \langle \nabla \rangle^{-\tilde{\beta}} \right\|_{\mathfrak{g}^{2\beta}} \leq c\|V\|_{L^2_t L^4_x}, \tag{5.7}
\]

where \( \tilde{\beta} = \beta - 2 > \frac{d+2}{2} \). Here, (5.6) is from Theorem 2 in [23] and (5.7) can be obtained from the complex interpolation between (5.6) and (3.10) with \( \alpha_1 = 0 \) and \( \alpha_2 > \frac{d-1}{2} \). Expanding \( W^{(m)}_W(t) \) and \( W^{(n)}_W(t) \) in the expression (5.5) (see (2.5)) and applying the inequality
\[ |\text{Tr}(AB)| \leq \text{Tr}(|A||B|) \], we write
\[ \left| \int_0^\infty \int_{\mathbb{R}^d} A_{m,n}(\phi)(t)U(t,x) \, dx \, dt \right| \]
\[ \leq \text{Tr}\left\{ \int_0^\infty \left( \int_0^\infty \cdots \int_0^\infty e^{-it_1 \Delta} |W(t_1)|e^{it_1 \Delta} \cdots e^{-it_1 \Delta} |W(t_1)|e^{it_1 \Delta} dt_1 \cdots dt_n \right) \gamma_f \right. \]
\[ \left. \left( \int_0^\infty \cdots \int_0^\infty e^{-it_n \Delta} |W(t_n)|e^{it_n \Delta} \cdots e^{-it_n \Delta} |W(t_n)|e^{it_n \Delta} dt_1 \cdots dt_n \right) \right\} \]
\[ = \text{Tr}\left\{ \left( \int_0^\infty e^{-it_1 \Delta} |W(t_1)|e^{it_1 \Delta} dt_1 \right) \cdots \left( \int_0^\infty e^{-it_n \Delta} |W(t_n)|e^{it_n \Delta} dt_n \right) \gamma_f \right. \]
\[ \cdot \left( \int_0^\infty e^{-it_n \Delta} |W(t_n)|e^{it_n \Delta} dt_n \right) \cdots \left( \int_0^\infty e^{-it_n \Delta} |W(t_n)|e^{it_n \Delta} dt_n \right) \}
\[ \cdot \left( \int_0^\infty e^{-it_1 \Delta} w_1 * U(t) \right) e^{it_1 \Delta} \}
\[ \tag{5.8} \]

When \( m, n \geq 1 \), by the Hölder inequality in the Schatten spaces, (5.6) and (5.7), we obtain
\[ \int_0^\infty \int_{\mathbb{R}^d} A_{m,n}(\phi)(t)U(t,x) \, dx \, dt \]
\[ \leq \left\| \int_0^\infty e^{-it \Delta} |W(t)|e^{it \Delta} dt \right\|_{\mathcal{S}^{m-1}} \left\| \int_0^\infty e^{-it \Delta} |W(t)|e^{it \Delta} \langle \nabla \rangle^{-\frac{\beta}{2}} dt \right\|_{\mathcal{S}^{2d}} \]
\[ \cdot \left( 1 - \Delta \right)^{\frac{\beta}{2}} \gamma_f \left\| B(L^2) \right\| \left\| \langle \nabla \rangle^{-\frac{\beta}{2}} \int_0^\infty e^{-it \Delta} |W(t)|e^{it \Delta} dt \right\|_{\mathcal{S}^{2d}} \]
\[ \cdot \left( 1 - \Delta \right)^{\frac{\beta}{2}} \gamma_f \left\| B(L^2) \right\| \left\| \langle \nabla \rangle^{-\frac{\beta}{2}} \int_0^\infty e^{-it \Delta} |W(t)|e^{it \Delta} dt \right\|_{\mathcal{S}^{2d}} \]
\[ \leq (c\| W \|_{L_t^1 L_x^2})^{m+n-2} (c\| W \|_{L_t^1 L_x^2})^{2} \cdot (1 + | \cdot |)^{\beta} f_{L^\infty} \cdot c\| W \|_{L_t^1 L_x^2} \quad \text{for } W = w_1 \ast \phi \]
\[ \leq c^{m+n+1} \| w_1 \|_{L_t^2 L_x^2}^{m+n-2} \| w_1 \|_{L_t^\infty} \| w_2 \|_{L_t^2 L_x^2} \cdot (1 + | \cdot |)^{\beta} f_{L^\infty} \| \phi \|_{L_t^1 L_x^2}^{m+n} \| U \|_{L_t^1 L_x^2}^2, \]

where \( \mathcal{B}(L^2) \) is the operator norm and in the last step, we used that if \( r \geq 2 \),
\[ \| w \ast \phi \|_{L^r} \leq \| \hat{w} \hat{\phi} \|_{L^{r'}} \quad \text{(by Hausdorff-Young)} \]
\[ \leq \| \hat{w} \|_{L^{2r}} \| \hat{\phi} \|_{L^2} = \| w \|_{L^{2r}} \| \phi \|_{L^2} \quad \text{(by Plancherel)}. \]
\[ \tag{5.10} \]

When either \( m = 0 \) or \( n = 0 \), we give the negative derivative \( \langle \nabla \rangle^{-\frac{\beta}{2}} \) to the integral having \( U \) and use (5.7) for that term. Then, estimating as above, we can show that
\[ \int_0^\infty \int_{\mathbb{R}^d} A_{m,n}(\phi)(t)U(t,x) \, dx \, dt \]
\[ \leq c^{m+n} \| w_1 \|_{L_t^2 L_x^2}^{m+n-1} \| w_1 \|_{L_t^\infty} \| w_2 \|_{L_t^\infty} \cdot (1 + | \cdot |)^{\beta} f_{L^\infty} \| \phi \|_{L_t^1 L_x^2}^{m+n} \| U \|_{L_t^1 L_x^2}. \]
\[ \tag{5.11} \]

Therefore, by duality, we complete the proof of (5.1) for higher order terms.
Thus, repeating (5.9) but using (5.14) instead of (5.7), indeed, by complex interpolation between (5.6) and (3.10) with

\[ \| \langle \nabla \rangle^{-\beta_0} e^{-it\Delta} (w_2 \ast U)(t) e^{it\Delta} \langle \nabla \rangle^{-\beta_0} dt' dt_1 dt \|. \]  

(5.13)

We now claim that

\[ \text{Tr} \left( \int_0^\infty \int_0^t \int_0^\infty e^{-it_1 \Delta} V_1(t_1) e^{it_1 \Delta} \gamma_f e^{-it'_1 \Delta} V_2(t'_1) e^{it'_1 \Delta} \langle \nabla \rangle^{-\beta_0} e^{-it\Delta} (w_2 \ast U)(t) e^{it\Delta} \langle \nabla \rangle^{-\beta_0} dt' dt_1 dt \right) \]

Indeed, by complex interpolation between (5.6) and (3.10) with \( \alpha_1 = \alpha_2 > \frac{d-1}{4} \), we have

\[ \| \langle \nabla \rangle^{-\beta_0} \int_0^\infty e^{-it\Delta} V(t) e^{it\Delta} dt \langle \nabla \rangle^{-\beta_0} dt \|_{\mathcal{S}^d} \lesssim \| V \|_{L_t^2 L_x^\infty}. \]  

(5.14)

Thus, repeating (5.9) but using (5.14) instead of (5.7),

\[ \text{Tr} \left( \int_0^\infty \int_0^t \int_0^\infty e^{-it_1 \Delta} V_1(t_1) e^{it_1 \Delta} \gamma_f e^{-it'_1 \Delta} V_2(t'_1) e^{it'_1 \Delta} \langle \nabla \rangle^{-\beta_0} e^{-it\Delta} (w_2 \ast U)(t) e^{it\Delta} \langle \nabla \rangle^{-\beta_0} dt' dt_1 dt \right) \]

\[ \lesssim \| \int_0^\infty e^{-it\Delta} |V_1(t)| e^{it\Delta} dt \langle \nabla \rangle^{-\beta} \|_{\mathcal{S}^d} \| (1 - \Delta)^{\frac{\beta}{2}} \gamma_f \|_{B(L^2)} \]

\[ \cdot \left( \| \langle \nabla \rangle^{-\beta} \int_0^\infty e^{-it\Delta} |V_2(t)| e^{it\Delta} dt \|_{\mathcal{S}^d} \right) \]

\[ \cdot \left( \| \langle \nabla \rangle^{-\beta_0} \int_0^\infty e^{-it\Delta} |w_2 \ast U(t)| e^{it\Delta} dt \|_{\mathcal{S}^d} \right) \]

\[ \lesssim \| V_1 \|_{L_t^2 L_2^\infty} \| V_2 \|_{L_t^2 L_2^\infty} \| (1 + | \cdot |^{\frac{\beta}{2}}) f \|_{L^\infty} \| w_2 \ast U \|_{L_t^2 L_x^\infty} \]

\[ \lesssim \| \hat{w}_2 \|_{L_x^{\frac{d+1}{d}}} \| (1 + | \cdot |)^{\frac{\beta}{2}} f \|_{L^\infty} \| V_1 \|_{L_t^2 L_2^\infty} \| V_2 \|_{L_t^2 L_2^\infty} \| U \|_{L_t^2 L_2^\infty}, \]

where in the last step, we used (5.10).
Hence, interpolating it with (5.13), we get

\[
\text{Tr} \left( \int_0^\infty \int_0^t \left( (1 - \Delta) e^{-it_1 \Delta} V_1(t_1) e^{it_1 \Delta} \gamma f e^{-it_1 \Delta} V_2(t_1') e^{it_1 \Delta} (1 - \Delta) \right) \langle \nabla \rangle^{-\beta_0} e^{-it \Delta} (w_2 \ast U)(t) e^{it \Delta} \langle \nabla \rangle^{-\beta_0} dt_1 dt_1 dt \right) 
\]

\[
\lesssim \| (1 - \Delta) V_1 \|_{L_t^2 L_x^2} \| (1 - \Delta) V_2 \|_{L_t^2 L_x^2} \left( 1 + | \cdot | \right)^{\beta + 1} f \|_{L_x^\infty} \| w_2 \ast U \|_{L_t^2 L_x^{2d}} 
\]

\[
\lesssim \| \hat{w}_2 \|_{L_t^\infty L_x^{2d}} \left( 1 + | \cdot | \right)^{\beta} f \|_{L_x^\infty} \| V_1 \|_{L_t^2 H_x^\infty} \| V_2 \|_{L_t^2 H_x^\infty} \| U \|_{L_t^2 L_x^2}. 
\]

Finally, coming back to (5.12), applying this inequality, we prove that

\[
\int_0^\infty \int_{\mathbb{R}^d} A_{1,1}(\phi)(t) U(t, x) dx dt 
\]

\[
\lesssim \| \hat{w}_2 \|_{L_t^\infty L_x^{2d}} \left( 1 + | \cdot | \right)^{\beta} f \|_{L_x^\infty} \| w_1 \ast \phi \|_{L_t^2 H_x^\infty} \| w_1 \ast \phi \|_{L_t^2 H_x^\infty} \| U \|_{L_t^2 L_x^2} 
\]

\[
\lesssim \left\langle \langle \nabla \rangle^{\beta_0} \hat{w}_1 \|_{L_x^\infty} \| \hat{w}_2 \|_{L_t^\infty L_x^{2d}} \left( 1 + | \cdot | \right)^{\beta} f \|_{L_x^\infty} \| \phi \|_{L_t^2 L_x^2} \| U \|_{L_t^2 L_x^2}. 
\]

For (5.2), we decompose \( \mathcal{W}_{w_1 \ast \phi}(t) - \mathcal{W}_{w_1 \ast \psi}(t) \) into the sum of \( n \) integrals,

\[
(-i)^n \int_0^t dt_n \int_0^{t_n} dt_n-1 \cdots \int_0^{t_2} dt_1 e^{-it_n \Delta} (w_1 \ast (\phi - \psi))(t_n) e^{it_n \Delta} 
\]

\[
\times e^{-it_{n-1} \Delta} (w_1 \ast \phi)(t_{n-1}) e^{it_{n-1} \Delta} \cdots e^{-it_1 \Delta} (w_1 \ast \phi)(t_1) e^{it_1 \Delta} 
\]

\[
+ (-i)^n \int_0^t dt_n \int_0^{t_n} dt_n-1 \cdots \int_0^{t_2} dt_1 e^{-it_n \Delta} (w_1 \ast \psi)(t_n) e^{it_n \Delta} 
\]

\[
\times e^{-it_{n-1} \Delta} (w_1 \ast (\phi - \psi))(t_{n-1}) e^{it_{n-1} \Delta} \cdots e^{-it_1 \Delta} (w_1 \ast \psi)(t_1) e^{it_1 \Delta} 
\]

\[
+ \cdots 
\]

\[
+ (-i)^n \int_0^t dt_n \int_0^{t_n} dt_n-1 \cdots \int_0^{t_2} dt_1 e^{-it_n \Delta} (w_1 \ast \psi)(t_n) e^{it_n \Delta} 
\]

\[
\times e^{-it_{n-1} \Delta} (w_1 \ast \psi)(t_{n-1}) e^{it_{n-1} \Delta} \cdots e^{-it_1 \Delta} (w_1 \ast (\phi - \psi))(t_1) e^{it_1 \Delta}. 
\]
Using this sum, we decompose the difference
\[
A_{m,n}(\phi)(t) - A_{m,n}(\psi)(t) = w_2 \ast \rho \left[ e^{it\Delta} \left( W_{w_1 * \phi}^{(m)}(t) - W_{w_1 * \psi}^{(m)}(t) \right) \gamma_f W_{w_1 * \phi}^{(n)}(t) \ast e^{-it\Delta} \right] + w_2 \ast \rho \left[ e^{it\Delta} W_{w_1 * \phi}^{(m)}(t) \gamma_f \left( W_{w_1 * \phi}^{(n)}(t) - W_{w_1 * \psi}^{(n)}(t) \right) e^{-it\Delta} \right],
\]
(5.19)
into \((m + n)\) terms. For each term, we estimate as in the proof of (5.1). Collecting all, we obtain (5.2).

\[ \square \]

6. Bounds on \(B(\phi)\)

We prove the bounds on the operator
\[
B(\phi)(t) = w_2 \ast \rho \left[ e^{it\Delta} W_{w_1 * \phi}^{(m)}(t) Q_0 W_{w_1 * \phi}^{(n)}(t) \ast e^{-it\Delta} \right]
\]
introduced in (2.10).

**Proposition 6.1** (Bounds on \(B(\phi)\)). Let \(d \geq 3\), \(\alpha > \frac{d-2}{2}\) and \(\alpha_0\) be given by (3.8) with \(\alpha_1 = \alpha_2 = \alpha\). Suppose that \(u = w_1 * w_2\), and \(\| \cdot \|_{L^1}^{1/2} \langle \cdot \rangle^{\alpha_0} \hat{w}_1, \| \cdot \|_{L^\infty}^{1/2} \langle \cdot \rangle^{-\alpha_0} \hat{w}_2 \in L^\infty\). Then, there exist small \(\epsilon_B > 0\) and large \(C_B, C_B' > 0\) such that if \(\| \phi \|_{L^2_{t,x}}, \| \psi \|_{L^2_{t,x}} \leq \epsilon_B\), then
\[
\| B(\phi) \|_{L^2_{t,x}} \leq C_B \| Q_0 \|_{H^\alpha},
\]
\[
\| B(\phi) - B(\psi) \|_{L^2_{t,x}} \leq C_B' \| Q_0 \|_{H^\alpha} \| \phi - \psi \|_{L^2_{t,x}}.
\]
(6.1)

The constants \(\epsilon_B, C_B\) and \(C_B'\) depend only on \(d\), \(\| \cdot \|_{L^1}^{1/2} \langle \cdot \rangle^{\alpha_0} \hat{w}_1 \|_{L^\infty}\) and \(\| \cdot \|_{L^\infty}^{1/2} \langle \cdot \rangle^{-\alpha_0} \hat{w}_2 \|_{L^\infty}\).

**Proof.** For notational convenience, we denote
\[
Q_\phi(t) := e^{it\Delta} W_{w_1 * \phi}^{(m)}(t) Q_0 W_{w_1 * \phi}^{(n)}(t) \ast e^{-it\Delta} = U_{w_1 * \phi}(t) Q_0 U_{w_1 * \phi}(t)^*.
\]
(6.2)

Note that by definition, \(B(\phi) = w_2 \ast \rho Q_\phi(t)\).

Recalling (2.1) and (2.2), we see by differentiating (6.2) in \(t\) that \(Q_\phi\) solves the following equation:
\[
i \partial_t Q_\phi = [-\Delta + w_1 * \phi, Q_\phi]
\]
with initial data \(Q_\phi(0) = Q_0\), equivalently,
\[
Q_\phi(t) = e^{it\Delta} Q_0 e^{-it\Delta} - i \int_0^t e^{i(t-s)\Delta} \left[ w_1 * \phi, Q_\phi \right](s) e^{-i(t-s)\Delta} ds.
\]
(6.3)

Hence, by the Strichartz estimates in Theorem 3.6 we get
\[
\| Q_\phi \|_{S^\alpha} \leq c \| Q_0 \|_{H^\alpha} + c \langle \nabla_x \rangle^{\alpha} \langle \nabla_{x'} \rangle^{\alpha} \left[ w_1 * \phi, Q_\phi \right](t, x, x') \|_{L^1_x L^2_{x'}} \leq c \| Q_0 \|_{H^\alpha} + 2c \langle \nabla_x \rangle^{\alpha} \langle \nabla_{x'} \rangle^{\alpha} \left[ (w_1 * \phi)(t, x) Q_\phi(t, x, x') \right] \|_{L^1_x L^2_{x', x'}} + 2c \langle \nabla_x \rangle^{\alpha} \langle \nabla_{x'} \rangle^{\alpha} \left[ (w_1 * \phi)(t, x') Q_\phi(t, x, x') \right] \|_{L^1_x L^2_{x, x'}},
\]
(6.4)
where the time interval \([0, +\infty)\) is omitted in the norms for notational convenience. Moreover, applying the triangle inequality, the Strichartz estimate in Theorem 3.11 with \(\alpha_1 = \alpha_2 = \alpha\) to the density of \([6.3]\), we get

\[
\left\| \nabla^{1/2} \rho Q_\phi \right\|_{L^2_t H^\alpha_x} \leq \left\| \nabla^{1/2} \rho e^{it\Delta} Q_\phi e^{-i\Delta} \right\|_{L^2_t H^\alpha_x} + \int_{\mathbb{R}} \left\| \nabla^{1/2} \rho e^{i(t-s)\Delta} [w_1 * \phi, Q_\phi](s)e^{-i(t-s)\Delta} \right\|_{L^2_t H^\alpha_x} ds \\
\leq c \|Q_0\|_{H^\alpha} + c \int_{\mathbb{R}} \|e^{-is\Delta} [w_1 * \phi, Q_\phi](s) e^{is\Delta}\|_{H^\alpha} ds \\
\leq c \|Q_0\|_{H^\alpha} + 2c \left\| \left( (w_1 * \phi)(t,x) Q_\phi(t, x, x') \right) \right\|_{L^1_t L^2_{x,x'}} + 2c \left\| \left( (w_1 * \phi)(t, x') Q_\phi(t, x, x') \right) \right\|_{L^1_t L^2_{x,x'}}.
\]

By the fractional Leibniz rule and Sobolev inequalities with the choices of \(\alpha_0\) and \(\alpha\) (both are applied only for the \(x\)-variable),

\[
\left\| \left( (w_1 * \phi)(t, x') Q_\phi(t, x, x') \right) \right\|_{L^1_t L^2_{x,x'}} \\
\leq \left\| (w_1 * \phi) \right\|_{L^1_t L^2_x} \left\| \left( \nabla_x \left( \nabla_x \cdot \phi \right) Q_\phi(t, x) \right) \right\|_{L^2_t L^\infty_{x,x'}} \\
+ \left\| \left( \nabla_x \cdot (w_1 * \phi) \right) \right\|_{L^2_t L^\infty_{x,x'}} \left\| \left( \nabla_x \cdot (\nabla_x \phi) Q_\phi(t, x, x') \right) \right\|_{L^2_t L^\infty_{x,x'}} \\
\leq \left\| \nabla^{1/2} (w_1 * \phi) \right\|_{L^1_t H^\alpha_x} \left\| \left( \nabla_x \phi \right)^{2d} (w_1 * \phi) \right\|_{L^2_t L^\infty_{x,x'}} \left\| \left( \nabla_x \phi \right)^{2d} Q_\phi(t, x) \right\|_{L^2_t L^\infty_{x,x'}} \\
\leq \left\| \left( \nabla_x \cdot (w_1 * \phi) \right) \right\|_{L^1_t H^\alpha_x} \left\| \left( \nabla_x \phi \right)^{2d} \right\|_{L^1_t L^\infty_{x,x'}} \left\| \left( \nabla_x \phi \right)^{2d} Q_\phi(t, x) \right\|_{L^2_t L^\infty_{x,x'}}.
\]

We estimate \((w_1 * \phi)(t, x') Q_\phi(t, x, x')\) in a similar way, interchanging \(x\) and \(x'\). Thus, we prove that if \(\|\phi\|_{L^2_{t,x}} \leq \epsilon_B\), then

\[
\|Q_\phi\|_{S^0} + \|\nabla^{1/2} \rho Q_\phi\|_{L^2_t H^\alpha_x} \leq 2c \|Q_0\|_{H^\alpha} + 2c' \left\| \cdot \nabla^{1/2} \left( \phi \right) \right\|_{L^2_t L^\infty_{x,x'}} \|Q_\phi\|_{S^0} \\
\leq 2c \|Q_0\|_{H^\alpha} + 2c' \epsilon_B \left\| \cdot \nabla^{1/2} \left( \phi \right) \right\|_{L^2_t L^\infty_{x,x'}} \|Q_\phi\|_{S^0}.
\]

We take \(\epsilon_B := \frac{1}{4c' \left\| \cdot \nabla^{1/2} \left( \phi \right) \right\|_{L^2_t L^\infty_{x,x'}} \|Q_0\|_{H^\alpha}}\). Then, we get

\[
\|Q_\phi\|_{S^0} + \|\nabla^{1/2} \rho Q_\phi\|_{L^2_t H^\alpha_x} \leq 4c \|Q_0\|_{H^\alpha}. \tag{6.7}
\]

As a result, by \([6.10]\), we conclude that

\[
\|B(\phi)\|_{L^2_{t,x}} = \|w_2 \ast \rho Q_\phi\|_{L^2_{t,x}} \leq \left\| \cdot \nabla^{1/2} \left( \phi \right) \right\|_{L^\infty_{x,x'}} \|\nabla^{1/2} \rho Q_\phi\|_{L^2_t H^\alpha_x} \leq C_B \|Q_0\|_{H^\alpha}, \tag{6.9}
\]

where \(C_B = 4c \left\| \cdot \nabla^{1/2} \left( \phi \right) \right\|_{L^\infty_{x,x'}}\).

For the difference

\[
Q_\phi(t) - Q_\psi(t) = -i \int_0^t e^{i(t-s)\Delta} \left[ w_1 * (\phi - \psi), Q_\phi \right](s) e^{-i(t-s)\Delta} ds \\
- i \int_0^t e^{i(t-s)\Delta} \left[ w_1 * \psi, Q_\phi - Q_\psi \right](s) e^{-i(t-s)\Delta} ds, \tag{6.10}
\]
repeating the estimates in the proof of (6.7), we prove that if \( \|\phi\|_{L^2_{t,x}}, \|\psi\|_{L^2_{t,x}} \leq \epsilon_B \), then

\[
\|Q_\phi - Q_\psi\|_{S^0} + \|\nabla\|^{1/2}_x \rho Q_\phi - Q_\psi\|_{L^2_{t}H^{\alpha_0}} \\
\leq c' \|\nabla\|^{1/2}_x (\phi - \psi)\|_{L^2_{t}H^{\alpha_0}} \|Q_\phi\|_{S^0} + c' \|\nabla\|^{1/2}_x (\phi - \psi)\|_{L^2_{t}H^{\alpha_0}} \|Q_\phi - Q_\psi\|_{S^0} \\
\leq c' \|\nabla\|^{1/2}_x \hat{\omega}_1 \|\nabla\|^{1/2}_x \phi - \psi\|_{L^2_{t,x}} \|Q_\phi - Q_\psi\|_{S^0} \\
+ c' \|\nabla\|^{1/2}_x \hat{\omega}_1 \|\nabla\|^{1/2}_x \phi - \psi\|_{L^2_{t,x}} \cdot 4c \|Q_0\|_{\mathcal{H}^\alpha} \quad \text{(by (6.8))} \\
+ c' \|\nabla\|^{1/2}_x \hat{\omega}_1 \|\nabla\|^{1/2}_x \phi - \psi\|_{L^2_{t,x}} \cdot \epsilon_B \cdot \|Q_\phi - Q_\psi\|_{S^0}. \tag{6.11}
\]

By the choice of \( \epsilon_B \),

\[
\|\nabla\|^{1/2}_x \rho \|Q_\phi - Q_\psi\|_{L^2_{t}H^{\alpha_0}} \leq 4cc' \|\nabla\|^{1/2}_x \hat{\omega}_1 \|\nabla\|^{1/2}_x \phi - \psi\|_{L^2_{t,x}}. \tag{6.12}
\]

Thus, by (5.10), we conclude that

\[
\|\mathcal{B}(\phi) - \mathcal{B}(\psi)\|_{L^2_{t,x}} = \|w_2 \ast (\rho Q_\phi - \rho Q_\psi)\|_{L^2_{t,x}} \\
\leq \|\cdot \|\nabla\|^{1/2}_x \hat{\omega}_2 \|\|\nabla\|^{1/2}_x \rho \|Q_\phi - Q_\psi\|_{L^2_{t}H^{\alpha_0}} \tag{6.13}
\]

where \( C_\mathcal{B}' = 4cc' \|\nabla\|^{1/2}_x \hat{\omega}_2 \|\phi - \psi\|_{L^2_{t,x}} \).

\[\square\]

7. Proof of the main theorem

First, we prove that

\[
\Gamma(\phi) = (1 + \mathcal{L})^{-1} \left\{ \sum_{m,n=1}^\infty A_{m,n}(\phi) + \mathcal{B}(\phi) \right\} \tag{7.1}
\]

is contractive in a small ball in \( L^2_{t,x} \). Let \( \epsilon > 0 \) be a sufficiently small number. Suppose that \( \|Q_0\|_{\mathcal{H}^\alpha} \leq \epsilon \) and

\[
\|\phi\|_{L^2_{t,x}}, \|\psi\|_{L^2_{t,x}} \leq 2C_B \|1 + \mathcal{L}\|^{-1}_{L^2_{t,x} \to L^2_{t,x}} \|Q_0\|_{\mathcal{H}^\alpha} =: R. \tag{7.2}
\]

Note that \( R \) is also a sufficiently small number, since \( \|Q_0\|_{\mathcal{H}^\alpha} \) is assumed to be small. Then, by Proposition 5.1 and 6.1

\[
\|\Gamma(\phi)\|_{L^2_{t,x}} \leq 1 + \mathcal{L}^{-1}_{L^2_{t,x} \to L^2_{t,x}} \left\{ \sum_{m,n=1}^\infty C_{m,n} A_{m,n} \|\cdot\|_{L^2_{t,x}} R^{m+n} + C_B \|Q_0\|_{\mathcal{H}^\alpha} \right\} \leq R. \tag{7.3}
\]
where in the second inequality, we used that the sum $\sum_{m,n=1}^{\infty} C_A^{m+n+1} |\langle \gamma f \rangle| L^\infty R^{m+n}$ is $O(R^2)$, so it is bounded by $C_B \|Q_0\|_{H^\alpha} = O(R)$. Similarly, we prove that
\[
\|\Gamma(\phi) - \Gamma(\psi)\|_{L^2_{t,x}} \\
\leq 1 + \|\mathcal{L}\|^{-1}_{L^2_{t,x} \to L^2_{t,x}} \left\{ \sum_{m,n=1}^{\infty} (m+n) C_A^{m+n+1} |\langle \gamma f \rangle| L^\infty (2R)^{m+n-1} + C_B \epsilon \right\} \|\phi - \psi\|_{L^2_{t,x}} \quad (7.4)
\]
\[
\leq \frac{1}{2} \|\phi - \psi\|_{L^2_{t,x}}.
\]
Thus, by the contraction mapping theorem, there exists a unique $\phi \in L^2_{t,x}$ such that $\phi = \Gamma(\phi)$.

Next, we derive the equation (2.6) from $\phi = \Gamma(\phi)$. Precisely, we claim that $Q(t)$, defined by
\[
Q(t) := e^{it\Delta} \mathcal{W}_{w_1 \ast \phi}(t)(\gamma f + Q_0) \mathcal{W}_{w_1 \ast \phi}(t)^* e^{-it\Delta} - \gamma f,
\]
is a solution to (2.6). Indeed, it follows from the series expansion for the wave operator (see (2.4)) and its boundedness (see (2.9)) that $Q(t)$ is well-defined in $\mathcal{S}^{2d}$. Moreover, we have
\[
\|w_2 \ast \rho_Q - \phi\|_{L^2_{t,x}} = \left\| - \mathcal{L}(\phi) + \sum_{m,n=1}^{\infty} A_{m,n}(\phi) + B(\phi) - \phi \right\|_{L^2_{t,x}}
\]
\[
= \left\| - \mathcal{L}(\phi) + (1 + \mathcal{L})(1 + \mathcal{L})^{-1} \left\{ \sum_{m,n=1}^{\infty} A_{m,n}(\phi) + B(\phi) \right\} - \phi \right\|_{L^2_{t,x}} \quad (7.6)
\]
\[
= \| - \mathcal{L}(\phi) + (1 + \mathcal{L})\Gamma(\phi) - \phi\|_{L^2_{t,x}}
\]
\[
= \| - \mathcal{L}(\phi) + (1 + \mathcal{L})\phi - \phi\|_{L^2_{t,x}} = 0 \quad \text{(by $\Gamma(\phi) = \phi$)},
\]
where the first identity follows from straightforward calculations using the infinite series expansion of the wave operator and the definitions of $\mathcal{L}$, $A_{m,n}$ and $B$. Now, inserting $\phi = w_2 \ast \rho_Q$ into (7.6), we conclude that $Q$ satisfies the equation (2.6),
\[
Q(t) = U_{w_1 \ast \rho_Q}(t)(\gamma f + Q_0) U_{w_1 \ast \rho_Q}(t)^* - \gamma f
\]
\[
= U_{\rho_Q}(t)(\gamma f + Q_0) U_{\rho_Q}(t)^* - \gamma f \quad (7.7)
\]
in $C_t([0, +\infty); \mathcal{S}^{2d})$.

Finally, by (5.10), we prove the desired global-in-time bound,
\[
\|w \ast \rho_Q\|_{L^2_{t,x} L^d_{x}} \leq \|w_1 \ast w_2 \ast \rho_Q\|_{L^2_{t,x} L^d_{x}} \leq \|w_1\|_{L^{6d-2}_{t,x}} \|w_2 \ast \rho_Q\|_{L^2_{t,x}}
\]
\[
\leq \|\tilde{w}_1\|_{L^{6d-2}_{t,x}} \|\tilde{w}_2\|_{L^\infty} \|\phi\|_{L^2_{t,x}} < \infty,
\]
which implies scattering in $\mathcal{S}^{2d}$ by Lemma 2.1.

References

[1] W. Abou Salem, T. Chen and V. Vougalter, On the generalized semi-relativistic Schrödinger-Poisson system in $\mathbb{R}^n$. Doc. Math. 18 (2013), 343357.
[2] M. Aizenman, E.H. Lieb, R. Seiringer, J.P. Solovej and J. Yngvason, Bose-Einstein quantum phase transition in an optical lattice model, Phys. Rev. A 70 (2004), 023612.

[3] C. Bardos, L. Erdös, F. Golse, N. Mauser and H.-T. Yau, Derivation of the Schrödinger-Poisson equation from the quantum N-body problem, C. R. Math. Acad. Sci. Paris 334 (2002), 515–520.

[4] C. Bardos, F. Golse, A. D. Gottlieb and N. J. Mauser, Mean field dynamics of fermions and the time-dependent Hartree-Fock equation, J. Math. Pures Appl. (9), 82 (2003), 665–683.

[5] N. Benedikter, M. Porta and B. Schlein, Mean-field evolution of fermionic systems. Comm. Math. Phys. 331 (2014), no. 3, 1087–1131.

[6] J. Bourgain, Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity, Internat. Math. Res. Notices 1998, no. 5, 253–283.

[7] J. Bourgain, New Global Well-Posedness Results for Nonlinear Schrödinger Equations, AMS Publications, Providence, RI, 1999.

[8] A. Bove, G. Da Prato and G. Fano, An existence proof for the Hartree-Fock time-dependent problem with bounded two-body interaction. Comm. Math. Phys. 37 (1974), 183–191.

[9] A. Bove, G. Da Prato and G. Fano, On the Hartree-Fock time-dependent problem. Comm. Math. Phys. 49 (1976), no. 1, 25–33.

[10] F. Brezzi and P. Markowich. The three-dimensional Wigner-Poisson problem: existence, uniqueness and approximation. Math. Methods Appl. Sci. 14 (1991), no. 1, 3561.

[11] T. Cazenave, Semilinear Schrödinger equations. Courant Lecture Notes in Mathematics, 10. AMS, 2003.

[12] J. M. Chadam, The time-dependent Hartree-Fock equations with Coulomb two-body interaction. Comm. Math. Phys., 46 (1976), 99–104.

[13] X. Chen, J. Holmer, Correlation structures, Many-body Scattering Processes and the Derivation of the Gross-Pitaevskii Hierarchy, International Mathematics Research Notices (2015) online first.

[14] T. Chen and N. Pavlović, Higher order energy conservation and global well-posedness of solutions for Gross-Pitaevskii hierarchies. Comm. Partial Differential Equations 39 (2014), no. 9, 1597–1634.

[15] T. Chen and N. Pavlović, The quintic NLS as the mean field limit of a boson gas with three-body interactions. J. Funct. Anal. 260 (2011), no. 4, 959–997.

[16] T. Chen and N. Pavlović, Derivation of the cubic NLS and Gross-Pitaevskii hierarchy from many-body dynamics in $d = 3$ based on spacetime norms. Ann. Henri Poincaré 15 (2014), no. 3, 543–588.

[17] T. Chen, C. Hainzl, N. Pavlović and R. Seiringer, Unconditional uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de Finetti. Comm. Pure Appl. Math. 68 (2015), no. 10, 1845–1884

[18] T. Chen, Y. Hong, N. Pavlović, Global Well-posedness of the NLS System for infinitely many fermions, preprint available at [http://arxiv.org/abs/1512.04674](http://arxiv.org/abs/1512.04674).

[19] A. Elgart, L. Erdös, B. Schlein, and H.-T. Yau, Nonlinear Hartree equation as the mean field limit of weakly coupled fermions, J. Math. Pures Appl., 83 (2004), pp. 1241–1273.

[20] L. Erdös, B. Schlein, H.-T. Yau, Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. Invent. Math. 167 (2007), no. 3, 515614.

[21] L. Erdös, B. Schlein, H.-T. Yau, Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. Ann. of Math. (2) 172 (2010), no. 1, 291370.

[22] R. Frank, M. Lewin, E. Lieb and R. Seiringer, A positive density analogue of the Lieb-Thirring inequality. Duke Math. J. 162 (2013), no. 3, 435–495.

[23] R. Frank, M. Lewin, E. Lieb and R. Seiringer, Strichartz inequality for orthonormal functions. J. Eur. Math. Soc. (JEMS) 16 (2014), no. 7, 1507–1526.

[24] R. Frank and J. Sabin, Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates (2014), preprint available at [http://arxiv.org/abs/1404.2817](http://arxiv.org/abs/1404.2817).
[25] J. Fröhlich, S. Graffi, S. Schwarz, Simon Mean-field- and classical limit of many-body Schrödinger dynamics for bosons. Comm. Math. Phys. 271 (2007), no. 3, 681-697.

[26] J. Fröhlich and A. Knowles, A microscopic derivation of the time-dependent Hartree-Fock equation with Coulomb two-body interaction. J. Stat. Phys. 145 (2011), 23–50.

[27] K. Hepp, The classical limit for quantum mechanical correlation functions. Comm. Math. Phys. 35 (1974), 265–277.

[28] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), no. 5, 955-980.

[29] C. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Comm. Pure Appl. Math. 46 (1993), no. 4, 527-620.

[30] K. Kirkpatrick, B. Schlein and G. Staffilani, Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics. Amer. J. Math. 133 (2011), no. 1, 91–130.

[31] S. Klainerman and M. Machedon, Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math. 46 (1993), no. 9, 1221–1268.

[32] S. Klainerman and M. Machedon, On the uniqueness of solutions to the Gross-Pitaevskii hierarchy. Comm. Math. Phys. 279 (2008), no. 1, 169–185.

[33] M. Lewin, P. T. Nam, N. Rougerie, Derivation of Hartree’s theory for generic mean-field Bose systems. Adv. Math. 254 (2014), 570–621.

[34] M. Lewin and J. Sabin, The Hartree equation for infinitely many particles I. Well-posedness theory. Comm. Math. Phys. 334 (2015), no. 1, 117–170.

[35] M. Lewin and J. Sabin, The Hartree equation for infinitely many particles, II: Dispersion and scattering in 2D. Anal. PDE 7 (2014), no. 6, 1339–1363.

[36] E. Lieb, W. Thirring, Bound on kinetic energy of fermions which proves stability of matter. Phys. Rev. Lett. 35, 687–689 (1975)

[37] E. Lieb, W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities. Studies in Mathematical Physics, Princeton Univ. Press, 269–303 (1976) Zbl 0342.35044

[38] E.H. Lieb, R. Seiringer, J.P. Solovej and J. Yngvason, The mathematics of the Bose gas and its condensation. Oberwolfach Seminars, 34. Birkhäuser Verlag, Basel, 2005. viii+203 pp.

[39] H. Narnhofer and G. Sewell, Vlasov hydrodynamics of a quantum mechanical model. Comm. Math. Phys. 79 (1981), no. 1, 9–24.

[40] B. Simon, Trace ideals and their applications. Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, 2005. viii+150 pp.

[41] H. Spohn, On the Vlasov hierarchy. Math. Methods Appl. Sci. 3 (1981), no. 4, 445-455.

[42] B. Schlein, Derivation of effective evolution equations from microscopic quantum dynamics. Evolution equations, 511572, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[43] T. Tao, A counterexample to an endpoint bilinear Strichartz inequality. Electron. J. Differential Equations 2006, No. 151, 6 pp.

[44] S. Zagatti, The Cauchy problem for Hartree-Fock time-dependent equations, Ann. Inst. H. Poincaré Phys. Théor., 56 (1992), 357–374.
T. Chen, Department of Mathematics, University of Texas at Austin.  
E-mail address: tc@math.utexas.edu

Y. Hong, Department of Mathematics, Yonsei University, Seoul, 120-749, Republic of Korea.  
E-mail address: younghun.hong@yonsei.ac.kr

N. Pavlović, Department of Mathematics, University of Texas at Austin.  
E-mail address: natasa@math.utexas.edu