Elliptic Genus of E-strings

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Abstract

We study a family of 2d $\mathcal{N} = (0, 4)$ gauge theories which describes at low energy the dynamics of E-strings, the M2-branes suspended between a pair of M5 and M9 branes. The gauge theory is engineered using a duality with type IIA theory, leading to the D2-branes suspended between an NS5-brane and 8 D8-branes on an O8-plane. We compute the elliptic genus of this family of theories, and find agreement with the known results for single and two E-strings. The partition function can in principle be computed for arbitrary number of E-strings, and we compute them explicitly for low numbers. We test our predictions against the partially known results from topological strings, as well as from the instanton calculus of 5d $Sp(1)$ gauge theory. Given the relation to topological strings, our computation provides the all genus partition function of the refined topological strings on the canonical bundle over $\frac{1}{2}K3$. 
1 Introduction

Six dimensional superconformal theories with (2,0) and (1,0) supersymmetry enjoy a special status among all superconformal theories: they are at the highest possible dimension. They play a key role in various aspects of string dualities as well as in obtaining lower dimensional supersymmetric systems upon compactification. They are rather enigmatic as they include tensionless self-dual strings as their building blocks.

The study of these theories has recently intensified, leading to computations of their superconformal indices [1, 2, 3, 4], the elliptic genera of the self-dual strings in the Coulomb branch [5, 6, 7] (see 8 for an earlier work), as well as a partial classification of 6d supercon-
formal theories [9, 10, 11]. The aim of this paper is to take a step forward in this direction, in particular focusing on one of the most basic (1, 0) superconformal theories. The theory is known to arise in heterotic strings for small $E_8$ instantons [12, 13, 14], and also when an M5 brane approaches the M9 brane boundary [13, 14]. It also has an F-theory dual description given by blowing up a point on $\mathbb{C}^2$ base of F-theory [15, 16, 17]. This superconformal theory has an $E_8$ global symmetry. It also has a one dimensional Coulomb branch, parameterized by a real scalar in the (1,0) tensor multiplet. In the M-theory setup, the scalar parameterizes the distance between M5 and M9 branes [18]. In F-theory setup, it parameterizes the size of the $\mathbb{P}^1$ obtained by blowing up a point. On the Coulomb branch this theory has light strings, known as E-strings [19]. In the M-theory setup they arise by M2 branes stretched between M5 brane and M9 brane. In F-theory setup they arise by wrapping D3 branes on the blown up $\mathbb{P}^1$. It is natural to ask whether one can find a nice description of E-strings. The main aim of this paper is to find such a description and use it to compute the twisted partition function of such strings on $T^2$. More precisely we would be computing the elliptic genus of E-strings on $T^2$. Knowing the elliptic genus of E-strings is useful in its own right, as well as for uncovering aspects of the superconformal theory. For example, a basic quantity one may wish to compute for a superconformal theory is its superconformal index, which involves the computation of its partition function on $S^1 \times S^5$ with suitable fugacities turned on along $S^1$. As was argued in [2, 3] (see also [20, 21]), the computation of the superconformal index reduces to an integral over the Coulomb branch where the integrand consists of three copies of elliptic genus of the corresponding strings.

If one is computing supersymmetry protected quantities, such as elliptic genus, we can change parameters to make the computation easy. In particular one can change parameters and use string dualities to find a suitable description of the resulting strings. This strategy was employed in particular for M-strings and their orbifolds [4, 6]. Two basic ways were used to compute the elliptic genus of the M-strings: one was to use string dualities to map the 2d theory to a super-Yang-Mills type gauge theory and use the technique developed recently [22, 23, 24] to compute their elliptic genera. The other way was to use the relation of the elliptic genus to BPS quantities upon circle compactification of these theories, that can in principle be computed using topological strings.

In the context of E-strings we employ the former method, and identify the gauge theory which captures their low energy physics. This is done by considering the duality of M-theory with type IIA, by introducing a circle transverse to M5 brane, leading to a system involving NS5-brane and where the M9 brane is replaced by O8 plane with 8 D8 branes on it. The M2 branes suspended between M5 and M9 branes map to D2 branes suspended between NS5-brane and O8-D8 pair. We find a simple (0, 4) supersymmetric quiver describing this system with $O(n)$ gauge symmetry, where $n$ denotes the number of suspended M2 branes. We use it to compute the elliptic genus of $n$ E-strings by employing the techniques developed in [23, 24].
The other method of computing the elliptic genus of E-string involves the F-theory picture. Namely, we compactify the theory on a circle leading to an M-theory description, and consider the BPS states of wrapped M2 branes, which correspond to E-strings wound around $S^1$ [25]. M-theory geometry involves the canonical bundle over $\frac{1}{2}K3$. As is well known, the BPS states of M2 branes wrapped on it, are captured by topological string amplitudes [26, 27]. In this context the (refined) topological string for $\frac{1}{2}K3$ has been computed to a high genus [28, 29], though an all genus answer is not available. So our method leads to a complete answer for refined topological string on $\frac{1}{2}K3$. Our answer can also be related to $\mathcal{N} = 4$ Yang-Mills in $d = 4$ in two different ways. In the F-theory setup, E-strings arise by wrapping D3 branes on a $\mathbb{P}^1$. From this perspective the elliptic genus of $n$ E-strings gets mapped to the study of $n$ D3 branes on $T^2 \times \mathbb{P}^1$, i.e. the partition function of $\mathcal{N} = 4 U(n)$ Yang-Mills on this geometry. Except that the coupling constant of Yang-Mills $\tau$ is not a constant and varies over $\mathbb{P}^1$ according to the complex structure of the elliptic curve given by

$$y^2 = x^3 + f_4(z)x + g_6(z)$$

where $z$ parameterizes the $\mathbb{P}^1$ and $f_4$ and $g_6$ are polynomials of degree 4 and 6 respectively. Note that this takes into account the S-duality of $U(n)$ Yang-Mills. Moreover lifting this to M-theory leads to $n$ M5 branes on $T^2 \times \frac{1}{2}K3$, which gets mapped to $U(n) \mathcal{N} = 4$ Yang-Mills on $\frac{1}{2}K3$ [30] (for the $SU(2)$ case see [31] and for computations in related cases see [32]).

Explicit computations for the elliptic genus are now straightforward, but somewhat cumbersome. Nevertheless we carry it out explicitly for the case of $n$ E-strings for $n = 1, 2, 3, 4$, and also explain the concrete procedures needed to compute the elliptic genus in the case with general $n$. The case with $n = 1$ was already known in [19], and the case with $n = 2$ was recently found in [7]. For the other two cases we check our results against partial results from topological strings on $\frac{1}{2}K3$ (where low genus answer is known). We also check them at $n = 4$ against a recent proposal of [33], where the elliptic genus was proposed at a special value of $E_8$ fugacities with reduced symmetry $SO(8) \times SO(8) \subset E_8$. In all these cases we find agreements with our computations.

Finally, we explain an alternative method to compute the E-string elliptic genus, from the instanton calculus of 5d SYM theories with $Sp(1)$ gauge group and 8 fundamental hypermultiplets. The index for $k$ instantons captures the $k'$th order coefficient of the elliptic genus expanded in the modular parameter, but keeps the information on all higher E-strings' spectrum at this order. It was recently shown in [34] how to compute this index. Making double expansions of the indices of our 2d gauge theory and the instanton quantum mechanics, we confirm that the indices computed from the two approaches agree with each other.

The organization of this paper is as follows: In section 2 we describe the basic type IIA brane setup. In section 3 we use this to compute the elliptic genera of E-strings. We give the explicit details for 1, 2, 3, 4 E-strings and indicate how the higher case works. We also compare
with (partial) known results. In section 4 we also formulate how the E-string partition function can be computed using 5 dimensional Yang-Mills instantons, and compare the results with those obtained in section 3. In section 5 we present some concluding remarks. Some technical details are relegated to the appendices.

2 The brane setup and the 2d \((0, 4)\) gauge theories

We construct a brane system in the type IIA string theory, which at low energy engineers the 6d \(E_8\) SCFT and the 2d CFT for E-strings. We first take an NS5-brane to wrap the \(013456\) directions, located at \(x^2 = L \ (> 0), \ x^7 = x^8 = x^9 = 0\). An O8-plane and 8 D8-branes (or 16 D8-branes in the covering space of orientifold) wrap \(013456789\) directions, located at \(x^2 = 0\). To describe E-strings, \(n\) D2-branes are stretched between the NS5 and 8-brane system \((0 < x^2 < L)\), occupying \(012\) directions. \(x^1\) direction is compactified to a circle. This brane system has \(SO(4)_1 \times SO(3)_2 = SU(2)_L \times SU(2)_R \times SU(2)_I\) symmetry which rotates \(3456\) and \(789\) directions. We denote by \(\alpha, \beta, \cdots = 1, 2, \ \dot{\alpha}, \dot{\beta}, \cdots = 1, 2\) and \(A, B, \cdots = 1, 2\) the doublet indices of these three \(SU(2)\) symmetries. See Table 1 and Fig. 1.

|       | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-------|----|----|----|----|----|----|----|----|----|----|
| NS5   | ●● | ●● | ●● | ●● | ●● | ●● |●● |●● |●● |●● |
| D8-O8 | ●● | ●● | ●● | ●● | ●● | ●● |●● |●● |●● |●● |
| D2    | ●● | ●● |●● |●● |●● |●● |●● |●● |●● |●● |

Table 1: Brane configuration for the E-strings

The M-theory uplift of this brane configuration, with extra circle direction labeled by \(x^{10}\),
is given as follows. The NS5-brane lifts to the M5-brane transverse to the $x^{10}$ direction. The D8-O8 system uplifts to an M9-plane, or the Horava-Witten wall [18], longitudinal in $x^{10}$ direction. In order to get a weakly-coupled type IIA string theory at low energy, one has to turn on suitable $E_8$ Wilson line along $x^{10}$ to break $E_8 \rightarrow SO(16)$ [13]. See our section 4 for more details. D2-branes uplift to M2-branes transverse in $x^{10}$. In the strong coupling limit of the type IIA theory, the radius of the M-theory circle becomes large. The geometry $\mathbb{R}^3 \times S^1$ transverse to the 5-brane is replaced by $\mathbb{R}^4$. So the brane configuration contains the M5-M9 system, in the Coulomb branch of the 6d $E_8$ CFT. M2-branes suspended between them are the E-strings.

At an energy scale much lower than $L^{-1}$, one obtains a 2d QFT living at the intersection of these branes. At $g_{YM} \ll E \ll L^{-1}$ with $g_{YM}^2 \sim \frac{g_s}{\ell_s}$, where $\ell_s, g_s$ are the string scale and the coupling constant, one obtains a weakly coupled 2d Yang-Mills description with coupling constant $g_{YM}$. (One can take $g_s$ to be sufficiently small, and $L$ to be sufficiently larger than $\ell_s$.) When $E \ll g_{YM}$, the 2d Yang-Mills theory is strongly coupled and is expected to flow to an interacting SCFT. In terms of the Planck scale $\ell_P \sim g_s^{1/3} \ell_s$ of M-theory and the radius $R \sim g_s \ell_s$ of the $x^{10}$ circle, the strong coupling regime of the 2d Yang-Mills theory is $E \ll \frac{R}{L^{1/2} \ell_P^{1/2}}$. $L$ is related to the VEV $v$ of the scalar in the 6d tensor multiplet by $L \sim v \ell_P^{3/2}$. So the low energy limit is $E \ll \frac{R}{v^{1/2} \ell_P^{1/2}}$. In the Coulomb branch with fixed $v$, this low energy limit of the 2d theory is obtained by taking the M-theory limit $R \rightarrow \infty$. Thus our 2d gauge theory describes E-strings at its strong coupling fixed point.

Let us comment on the enhanced IR symmetries. We first consider the $SO(3) \times U(1)$ acting on $\mathbb{R}^3 \times S^1$. In the M-theory limit, this enhances to $SO(4) \sim SU(2)_r \times SU(2)_l$ of $\mathbb{R}^4$. $SO(3)$ is identified as the diagonally locked combination of $SU(2)_r$ and $SU(2)_l$. On the other hand, from the viewpoint of 6d superconformal symmetry, $SU(2)_r$ is the R-symmetry of the 6d $(1,0)$ SCFT and $SU(2)_l$ is a flavor symmetry. So it might appear that our 2d gauge theory is probing only a combination of the R-symmetry and a flavor symmetry. However, in the rank 1 system with only one M5-brane, the extra flavor $SU(2)_l$ completely decouples with the 6d CFT. For instance, these can be seen by studying the instanton partition functions of circle reduced 5d SYM [34], which will also be the subject of our section 4. Thus we can identify $SO(3)$ visible in our UV theory as the superconformal R-symmetry of the 6d CFT. It will be interesting to see how to generalize the brane constructions for E-strings in the higher rank 6d CFT.

We also discuss the $E_8$ global symmetry. The UV theory exhibits $SO(16)$ symmetry only. This should enhance to $E_8$ in the IR, which is naturally expected from the brane perspective. Namely, the type IIA brane system is obtained by compactifying M-theory brane system with an $E_8$ Wilson line which breaks $E_8$ to $SO(16)$. The IR limit on the 2d gauge theory is the strong coupling limit, which is the decompactification limit of the M-theory circle. So in this limit, the information on the Wilson line will be invisible, making us to expect an IR $E_8$ enhancement.
In section 3, we shall compute the elliptic genera of these gauge theories at various values of \( n \), which will be invariant under the \( E_8 \) Weyl symmetry and support the \( E_8 \) enhancement.

Let us study the SUSY of this system. The D2, D8 SUSY are associated with the projectors \( \Gamma^{012}, \Gamma^{013456789} \sim \Gamma^2 \), while the NS5-brane projector is \( \Gamma^{01} \Gamma^{3456} \). Various combinations of branes share different SUSY. We list the following projectors which should assume definite eigenvalues, for various combinations of branes:

\[
\begin{align*}
\text{D2-D8-NS5} & : \Gamma^{01}, \Gamma^2, \Gamma^{3456} \\
\text{D2-NS5} & : \Gamma^{01} \Gamma^2, \Gamma^{01} \Gamma^{3456} \\
\text{D2-D8-O8} & : \Gamma^{01}, \Gamma^2.
\end{align*}
\]

The projectors (2.1) will yield the SUSY preserved by our system. The SUSY given by (2.2) and (2.3) will constrain the boundary conditions of the 3d D2-brane fields at the two ends of the segment along \( x^2 \). Let us investigate them in more detail. The type IIA supercharges with 32 components can be arranged to be eigenstates of \( \Gamma^{01}, \Gamma^{3456}, \Gamma^2 \). The eigenspinors of \( \Gamma^{01} \) are 2d chiral spinors, while those of \( \Gamma^{3456} \) belong to either \( (\mathbf{2}, \mathbf{1}) \) or \( (\mathbf{1}, \mathbf{2}) \) representations of \( SU(2)_L \times SU(2)_R \). The 32 supercharges decompose into the sum of the \( (\mathbf{2}, \mathbf{1}, \mathbf{2})_{\pm \pm} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{\pm \pm} \) representations of \( SU(2)_L \times SU(2)_R \times SU(2) \), with all four possible choices of \( \pm \), where the first/second \( \pm \) subscripts denote 2d chirality and \( \Gamma^2 \) eigenvalues, respectively. The SUSY preserved by various combinations of branes are given by

\[
\begin{align*}
\text{D2-D8-NS5} & : (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-+} \\
\text{D2-NS5} & : (\mathbf{2}, \mathbf{1}, \mathbf{2})_{+-} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-+} \\
\text{D2-D8-O8} & : (\mathbf{2}, \mathbf{1}, \mathbf{2})_{--} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{+-}.
\end{align*}
\]

(2.1) yields the 2d \((0, 4)\) SUSY, which we write as \( Q^{-A}_{-} \). (2.5) yields 2d \((4, 4)\) SUSY \( Q^{+A}_{+}, Q^{-A}_{-} \). (2.6) yields 2d \((0, 8)\) SUSY \( Q^{+A}, Q^{-A} \). \( \pm \) subscripts of \( Q \) denote 2d left/right spinors.

We study the field contents of the 2d \( \mathcal{N} = (0, 4) \) gauge theory. This is obtained by starting from the 3d field theory living on D2-branes, together with the boundary degrees at \( x^2 = 0, L \), and then taking a 2d limit when \( E \ll L^{-1} \). The 3d fields living in the region \( 0 < x^2 < L \) are

\[
\begin{align*}
\text{D2-D2} & : A_\mu \ (\mu = 0, 1, 2) ; \ X^I \sim \varphi^{\alpha \dot{\beta}} \ (I = 3, 4, 5, 6) ; \ X^{I'} \ (I' = 7, 8, 9) \\
\lambda & \ (16 \text{ component spinor satisfying } \Gamma^{012} \lambda = \lambda).
\end{align*}
\]

The D2-D2 fields are in adjoint representation of \( U(n) \). One also finds boundary degrees at the brane intersections. At the intersection of D2-D8, open strings provide 2d \((0, 8)\) Fermi multiplet fields which we write as \( \Psi_l \ (l = 1, \ldots, 16) \). They will be in the bi-fundamental representation of \( O(n) \times SO(16) \) (after introducing the orientifold boundary condition on D2-D8). \( \Psi_l \) are Majorana-Weyl spinors.

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\end{align*}
\]
Let us consider the boundary conditions of the 3d fields. At the two ends $x^2 = 0, L$, we shall find separate boundary conditions. As our goal is to obtain the 2d theory, we shall only keep the zero modes of the 3d fields along the $x^2$ direction. This means that we shall keep the bosonic fields satisfying the Neumann boundary conditions on both ends, and the fermionic fields which survive suitable projection conditions at both ends. The SUSY conditions for the D2-D2 fields at $x^2 = 0, L$ take the form of

$$ (x^2 \text{ component of supercurrent}) = \text{tr} (\bar{\epsilon} \Gamma^{MN} F_{MN} \Gamma_2 \lambda) = 0 \quad (2.8) $$

in the 10d notation with $M, N = 0, \cdots, 9$. $\epsilon$ is chosen to be $(4, 4)$ on D2-NS5 ($x^2 = L$), and $(0, 8)$ on D2-D8 ($x^2 = 0$). One can follow the strategy of [35] to obtain the SUSY boundary conditions. With given SUSY $\epsilon$, one first imposes suitable bosonic boundary condition, depending on which brane D2’s are ending on. Then the condition (2.8) would determine the boundary condition for the fermions $\lambda$.

We study the D2-NS5 boundary condition first. Choosing the supercharge $Q$ to be in $(2, 1, 2)_{+-} \oplus (1, 2, 2)_{++}$, $\bar{\epsilon} = \bar{\epsilon} \Gamma^0$ should be chosen to have nonzero overlap with it. The D2-D2 fermion $\lambda$ has a definite $\Gamma^{012}$ eigenvalue (same as that of the supercharges), so is in

$$ (2, 1, 2)_{+-} \oplus (1, 2, 2)_{++} \oplus (1, 2, 2)_{+ -}. \quad (2.9) $$

We start from the boundary conditions for the bosonic fields that we know for D2-NS5:

$$ F_{\mu 2} = 0, \quad D_2 X^I = 0, \quad X^{I'} = 0 \quad (2.10) $$

with $\mu = 0, 1, I = 3, 4, 5, 6, I' = 7, 8, 9$. This provides the following constraints on $\lambda$:

$$ 0 = \bar{\epsilon} \lambda = \bar{\epsilon} \Gamma^{02I} \lambda = \bar{\epsilon} \Gamma^{IJ} \Gamma^2 \lambda = \bar{\epsilon} \Gamma^{I'} \lambda. \quad (2.11) $$

This requires $\lambda$ to be in

$$ (2, 1, 2)_{- +} \oplus (1, 2, 2)_{+-} \quad (2.12) $$

namely, with a right mover $\lambda^{aA}$ and a left mover $\lambda^{\dot{a}A}$. (The former will belong to a 2d $(0, 4)$ hypermultiplet and the latter will belong to a 2d $(0, 4)$ vector multiplet.)

Now we consider the D2-D8-O8 boundary conditions. The effect of having 8 D8-branes is simply adding $(0, 8)$ Fermi multiplet fields as explained above. So we focus on the effect of the O8-plane. Following [35], we consider the covering space of $x^2 > 0$ and consider the 3d SYM on $\mathbb{R}^{2,1}$. The reflection $x^2 \rightarrow -x^2$ of space is accompanied by an outer automorphism $\tau$ acting on $G = \text{U}(n)$ gauge group. The algebra $g$ of $G$ decomposes into $g^{(+)} \oplus g^{(-)}$, where $\tau$ acts on $g^{(\pm)}$ as $\pm 1$. In our case, $g^{(+)}$ is the algebra of $O(n) \subset \text{U}(n)$, and $g^{(-)}$ forms a rank 2 symmetric representation of $O(n)$. So any adjoint-valued field $\Phi$ can be written as $\Phi = \Phi^{(+)} + \Phi^{(-)}$. The reflection is further accompanied by $X^I \rightarrow -X^I$ for $I = 3, \cdots, 9$. The fields are required to be invariant under the net reflection:

$$ A_\mu (x^2) = A^{+}_\mu (-x^2), \quad A_2 (x^2) = -A^+ 2 (-x^2), \quad X_I (x^2) = -X^+_I (-x^2) \quad (2.13) $$
where $\Phi^r = \tau \Phi^{r^{-1}}$, $\mu = 0, 1$ and $I = 3, \cdots , 9$. So at the fixed plane $x^2 = 0$, the boundary condition is given by

$$ F^{(+)}_{\mu 2} = 0 \ , \ F^{(-)}_{\mu \nu} = 0 \ , \ D_2 X^{(-)}_I = 0 \ , \ X^{(+)}_I = 0 \ \ (I = 3, \cdots , 9) . \quad (2.14) $$

We can again find the fermionic boundary conditions from (2.8). This requires

$$ 0 = \bar{\epsilon} \lambda^{(+)} = \bar{\epsilon} \Gamma^I \lambda^{(+)} = \bar{\epsilon} \Gamma^{IJ} \lambda^{(+)} \, , \quad 0 = \bar{\epsilon} \Gamma^I \lambda^{(-)} = \bar{\epsilon} \Gamma^{IJ} \lambda^{(-)} \quad (2.15) $$

with $\mu = 0, 1$ and $I, J = 3, \cdots , 9$. $\epsilon$ is chosen so that $\bar{\epsilon}$ has nonzero overlap with the $(0, 8)$ SUSY (2.6), given by $(2, 1, 2)_- \oplus (1, 2, 2)_-$. Solving these constraints, the $O(n)$ adjoint fermion $\lambda^{(+)}$ and the $O(n)$ symmetric fermion $\lambda^{(-)}$ are required to be in

$$ \lambda^{(+)} : (2, 1, 2)_- \oplus (1, 2, 2)_- $$

$$ \lambda^{(-)} : (2, 1, 2)_- \oplus (1, 2, 2)_- . \quad (2.16) $$

We combine the D2-NS5 and D2-O8 boundary conditions to read off the 2d field contents. For bosons, requiring (2.10) and (2.14) yields the following 2d degrees:

$$ A_{\mu}^{(+)} \ , \ X_I^{(-)} \sim \varphi_{\alpha \beta} \ \ (I = 3, 4, 5, 6) . \quad (2.17) $$

Although $F^{(+)}_{\mu 2}$ is not required to be zero in the above consideration, one can make an $x^2$-dependent gauge transformation to set $A_2^{(+)} = 0$. For fermions, requiring (2.12) and (2.16) together, one finds that $\lambda_-^A \sim (2, 1, 2)_-$ is in the symmetric representation of $O(n)$, while $\lambda_+^A \sim (1, 2, 2)_-$ is in the adjoint (i.e. antisymmetric) representation. So from the D2-D2 modes, we obtain the $(0, 4)$ vector multiplet $A_{\mu}$, $\lambda_+^A$ of $O(n)$, and also a $(0, 4)$ hypermultiplet $\varphi_{\alpha \beta}$, $\lambda_-^A$ in the symmetric representation of $O(n)$.

So to summarize, one obtains the following 2d $\mathcal{N} = (0, 4)$ field contents:

- vector : $O(n)$ antisymmetric $(A_{\mu}, \lambda_+^A)$
- hyper : $O(n)$ symmetric $(\varphi_{\alpha \beta}, \lambda_-^A)$
- Fermi : $O(n) \times SO(16)$ bifundamental $\Psi_l$.

Fig. 2 shows the quiver diagram of this gauge theory. One can check the $SO(n)$ gauge anomaly cancelation of this chiral matter content. Note that we have no twisted hypermultiplets, whose scalars form doublets of $SU(2)_I$ and fermions form doublets of $SU(2)_R$.

We also explain how to get the full Lagrangian of this system. Viewing this as a special case of $\mathcal{N} = (0, 2)$ supersymmetric system, it suffices to determine the two holomorphic functions $E_\Psi(\Phi_i)$, $J_\Psi(\Phi_i)$ for each Fermi multiplet $\Psi$, depending on the $(0, 2)$ chiral multiplet fields $\Phi_i$. We choose $Q \equiv Q_1^\dagger$ and $Q^\dagger$ as the $(0, 2)$ subset, for further explanations. To have $(0, 4)$ SUSY,
the $E, J$ functions for the $(0, 2)$ Fermi multiplet $\Theta \equiv (\lambda_1^{12}, \lambda_1^{21})$ in the $(0, 4)$ vector multiplet are constrained\(^{(36)}\) as

$$J_{\Theta} = \varphi \tilde{\varphi} - \tilde{\varphi} \varphi, \quad E_{\Theta} = 0,$$

(2.19)

where $\varphi \equiv \varphi_{11}, \tilde{\varphi} \equiv \varphi_{21}$ are $(0, 2)$ chiral multiplet scalars which transform under $Q \equiv Q_{\tilde{1}}^1$. Note that, if the $(0, 4)$ theory has both hypermultiplets and twisted hypermultiplets, the full interaction has to be more complicated\(^{(36)}\). Without twisted hypermultiplets in our system, (2.19) provides the full interactions associated with $\Theta$. This induces a bosonic potential of the form $|J_{\Theta}|^2$, as well as the Yukawa interaction. Extra Fermi multiplets in the $(0, 2)$ viewpoint are $\Psi_l$ from D2-D8-O8 modes, so we should also determine their $E, J$. $E_{\Psi_l}, J_{\Psi_l}$ are simply zero, from $SO(16)$ symmetry. With all the $E, J$ functions determined, the supersymmetric action can be written down if $E^a J_a = 0$, where the index $a$ runs over all $(0, 2)$ Fermi multiplets. This condition is clearly met. With these data, the full action can be written down in a standard manner: see, for instance,\(^{(37, 36)}\). In our case, the bosonic potential consists of $|J_{\Theta}|^2$ and the usual D-term potential, making the D-term potential from the ‘$SU(2)_R$ triplet’ of D-terms. The classical Higgs branch moduli space, given by nonzero $\varphi, \tilde{\varphi}$, is real $4n$ dimensional. Semi-classically, these are the positions of $n$ E-strings.

One can also compute the central charges of the IR CFT from our UV gauge theory. Our gauge theory in principle could flow to more than one decoupled CFTs in IR, which will be explained shortly. Once we know the correct superconformal R-symmetry of the IR SCFT, the (right-moving) central charge of the IR CFT can be computed in UV by the anomaly of the superconformal R-symmetry. We closely follow\(^{(38, 37, 36)}\), which use the $(0, 2)$ superconformal R-symmetry to determine the central charges.

In our $(0, 4)$ system, at least there will be one CFT in IR, which admits a semi-classical description when $\varphi^{\alpha\beta}$ scalars are large. This is the CFT associated with the classical Higgs branch\(^{(39)}\). In this CFT, the superconformal R-symmetry can only come from $SU(2)_I$ in

\[\text{Figure 2: The quiver diagram of the 2d } \mathcal{N} = (0, 4) \text{ gauge theory for E-strings: solid/dotted lines denote hyper/Fermi multiplets, respectively.}\]
the UV theory. This is because the right sector contains the $O(n)$ symmetric scalar $\varphi_{\alpha\dot{\beta}}$, and the superconformal R-symmetry should not act on it \cite{39}. Following \cite{36}, let us choose the supercharge $Q \equiv Q^{12}$ and use the $(0,2)$ superconformal symmetry to determine the central charge. The right-moving central charge $c_R$ is given by

$$c_R = 3 \text{Tr}(\gamma^3 R^2) ,$$

with $\gamma^3 = \pm 1$ for the right/left moving fermions, respectively, and the trace acquires an extra $\frac{1}{2}$ factor for real fermions. The $(0,2)$ R-charge $R$ is normalized so that $R[Q] = -1$. In the Higgs branch CFT, this should be proportional to the Cartan of $SU(2)_I$, so we set $R = 2J_I$. Collecting the contribution from $O(n)$ symmetric $\lambda_{\alpha A}$ in the right sector and adjoint $\lambda^{\alpha A}$ in the left sector, one obtains

$$c_R = 3 \times \frac{1}{2} \times \frac{n^2 + n}{2} \times (4 \times 1^2) - 3 \times \frac{1}{2} \times \frac{n(n - 1)}{2} \times (4 \times 1^2) = 6n .$$

The left moving central charge $c_L$ is determined from $c_R$ by the gravitational anomaly \cite{37}:

$$c_R - c_L = \text{Tr}(\gamma^3) = \frac{1}{2} \times 4 \frac{n^2 + n}{2} - \frac{1}{2} \times 4 \frac{n^2 - n}{2} - \frac{1}{2} \times 16n = -6n \rightarrow c_L = 12n .$$

$c_L = 12n$ is consistent with the result obtained in \cite{30} (where $c_L = 12n - 4$ was found after eliminating 4 from the decoupled center-of-mass degrees.) One can semiclassically understand some of these results, by studying the region with large value of the Higgs scalar $\varphi_{\alpha\dot{\beta}}$. $c_R = 6n$ comes from the $n$ pairs of 4 scalars and 4 fermions for $n$ E-strings. As for $c_L = 12n$, the 4n scalars in the left moving sector accounts for 4n, and the $16n$ real fermions $\Psi_l$ accounts for 8n. For $n = 1$, we know that the last 8 is given by the $G = E_8$ current algebra at level $k = 1$ (with dual Coxeter number $c_2 = 30$) \cite{13,19}, whose central charge is indeed $\frac{k[G]}{k+c_2} = \frac{248}{1+30} = 8$.

One may also try to explore if the UV theory could flow to more than one decoupled conformal field theories in the IR. For instance, it happens in the $\mathcal{N} = (4,4)$ SCFT with both Higgs and Coulomb branches \cite{39}. Another type of example is a recently analyzed $(0,4)$ gauge theory for the D1-D5-D5’ system \cite{36}. This theory was proposed to have a ‘localized CFT’ whose ground state wavefunction is localized at the intersection of the two Higgs branches, which was suggested to be the holographic dual of type IIB strings on $AdS_3 \times S^3 \times S^3 \times S^1$. Morally, the last localized CFT should be coming from the D1-branes forming threshold bounds with D5 and D5’ branes. As our system will also exhibit threshold bounds of E-strings, it would be interesting to know if similar decoupled ‘localized CFTs’ exist like those of \cite{36}, other than the ‘Higgs branch CFT’ that we explained in the previous paragraph. If there exist localized CFTs with all E-strings fully bound, they will not have a regime which allows a semi-classical description (large $\varphi^{\alpha\dot{\beta}}$). So the argument of \cite{39} does not apply, and both $SU(2)_R$ and $SU(2)_I$ can participate in the superconformal R-symmetry \cite{36}. Following \cite{36}, we first determine the correct superconformal R-symmetry in this case (if it exists), again within the context of $(0,2)$ superconformal symmetry as in the previous paragraph. We take the two Cartans $J_R, J_I$ of
$SU(2)_R$, $SU(2)_I$, and consider their linear combination $R = -aJ_R + bJ_I$ which we take as a trial $U(1)$ R-symmetry. $R[Q] = -1$ demands $a + b = 2$. The superconformal R-symmetry should have no mixed anomaly with flavor charges, i.e. all global symmetries commuting with $Q, Q^\dag$. In our case, one only needs to consider the mixing with $V \equiv J_R + J_I$, chosen in $SU(2)_R \times SU(2)_I$ with $V[Q] = 0$. By demanding $\text{Tr}(\gamma^3 VR) = 0$, one finds $na + 2b = 0$. While computing this, one should exclude the decoupled center-of-mass modes for the $n$ E-strings, provided by $\text{tr}(\varphi_{\alpha\dot{\beta}})$. These decoupled modes always live in a ‘Higgs branch’ in which the R-symmetry is $SU(2)_I$. So if there is no accidental IR symmetry, the R-symmetry is given by

$$a + b = 2, \; na + 2b = 0.$$  

(2.23)

Note that these equations do not have solutions if $n = 2$, for two E-strings. This could be implying the absence of the localized CFT, decoupled to the Higgs branch CFT. In other cases, one finds $a = -\frac{4}{n-2}$, $b = \frac{2n}{n-2}$. The right central charge is again given by (2.20) with the new $R$, again without including the contributions from the center-of-mass modes. It is given by $c_R = \frac{6n(n-1)}{n-2}$. The left central charge is given by $c_L = c_R + 6n$. As emphasized, this result could be meaningful only at $n \neq 2$. When $n = 1$, one finds $c_R = 0$, which is consistent with the absence of the extra localized CFT for single E-string. For $n \geq 3$, it will be interesting to know whether such CFTs actually exist (when consistent with the c-theorem).

For $n = 2$, unless there are accidental IR symmetries, our study implies that there are no more decoupled CFTs. If this is true, one should be able to understand the elliptic genus of the 2 E-strings solely from the Higgs branch CFT. In the regime with large $\varphi^{\alpha\dot{\beta}}$, one can employ a semi-classical approximation to study the Higgs branch CFT. This requires us to study a free QFT, with $4n = 8$ bosonic fields given by eigenvalues $\varphi_{i}^{\alpha\dot{\beta}}$ and $16n = 32$ fermions $\Psi_{il}$ $(i = 1, 2)$. The spectrum of this QFT is subject to a gauge singlet condition for a discrete $D_4$ subgroup of $O(2)$ gauge symmetry, surviving in the Higgs branch. The two generators of $D_4$ are given by

$$x : (\varphi_1, \varphi_2) \to (\varphi_2, \varphi_1), \; (\Psi_{1l}, \Psi_{2l}) \to (-\Psi_{2l}, \Psi_{1l})$$

$$y : (\varphi_1, \varphi_2) \to (\varphi_1, \varphi_2), \; (\Psi_{1l}, \Psi_{2l}) \to (\Psi_{1l}, -\Psi_{2l}),$$  

(2.24)

which satisfy $x^4 = 1$, $y^2 = 1$, $xy^{-1} = x^3$ and define $D_4$. One can show that the index for the gauge invariant states, after adding twisted sectors, is simply given by the Hecke transformation of the single E-string index. This does not agree with the correct two E-string index [7], which we shall compute in section 3.2. This implies that the Higgs branch CFT for two E-strings should be more nontrivial than what we see in the semi-classical regime. It will be interesting to understand this Higgs branch CFT better.
3 E-string elliptic genera from 2d gauge theories

We consider the elliptic genus of the 2d $(0, 4)$ $O(n)$ gauge theory, constructed in the previous section. We pick a $(0, 2)$ SUSY and define the elliptic genus as follows:

\[ Z_n(q, \epsilon_{1,2}, m_l) = \text{Tr}_{RR} \left[ (-1)^F q^{H_L + \bar{H}_R} e^{2\pi i \epsilon_1 (J_1 + J_1)} e^{2\pi i \epsilon_2 (J_2 + J_1)} \prod_{l=1}^{8} e^{2\pi i m_l F_l} \right]. \tag{3.1} \]

$J_1, J_2$ are the Cartans of $SO(4)_{3456} = SU(2)_L \times SU(2)_R$ which rotate the 34 and 56 orthogonal 2-planes, and $J_I$ is the Cartan of $SU(2)_{789}$. $F_l$ are the Cartans of $SO(16)$, which we expect to be the Cartans of enhanced $E_8$ in IR. Note that $H_R \sim \{ Q, Q^\dagger \}$ with $Q = Q^1_1$ and $Q^\dagger = -Q^2_2$, and the remaining factors inside the trace commute with $Q, Q^\dagger$. Note also that, the 2d gauge theory itself has a noncompact Higgs branch spanned by $\varphi^{\alpha \beta}$. They are given nonzero masses by turning on $\epsilon_1, \epsilon_2$, so that the path integral for this index does not have any noncompact zero modes. The interpretation of the zero modes from $\varphi^{\alpha \beta}$ at $\epsilon_1, \epsilon_2 = 0$ is clearly the multi-particle positions, so by keeping nonzero $\epsilon_{1,2}$ we are computing the multi-particle index, as usual. The single particle spectrum can be extracted from the multi-particle index.

The general form of the index (3.1) for $\mathcal{N} = (0, 2)$ gauge theories was studied in [23, 24], by computing the path integral of the gauge theory on $T^2$. There appear compact zero modes from the path integral, coming from the flat connections on $T^2$. [23, 24] first fix the flat connections, integrate over the nonzero modes, and then integrate over the flat connections to obtain their final expression for the index.

Let us first explain the possible flat connections of our $O(n)$ gauge theories on $T^2$. These are given by two commuting $O(n)$ group elements $U_1, U_2$, the Wilson lines along the temporal and spatial circles of $T^2$. Note that $O(n)$ is a disconnected group so that $U_1$ and $U_2$ can each have two disconnected sectors, depending on whether their determinants are 1 or $-1$. The general $O(n)$ holonomies on $T^2$, up to conjugation, can be derived using a D-brane picture [30]. The $O(n)$ flat connections are the zero energy configurations of the $n$ D2-branes and an O2-plane wrapping $T^2$. By T-dualizing twice along the torus, one obtains $n$ D0-branes moving along the covering space $T^2$ of $T^2/\mathbb{Z}_2$ orientifold. The flat connections T-dualize to the positions of D0-branes on $T^2/\mathbb{Z}_2$. There are four O0-plane fixed points on the covering space $T^2$. It suffices for us to classify all possible positions of D0-branes. When two D0-branes on the covering space are paired as $\mathbb{Z}_2$ images of each other, they have one complex parameter $u$ as their position. Some D0-branes can also be stuck at the $\mathbb{Z}_2$ fixed points without a pair: they are fractional branes on $T^2/\mathbb{Z}_2$, whose positions are freezed at the fixed points. So the classification of $O(n)$ flat connections reduces to classifying the possible fractional brane configurations.

When $n = 2p$ is even, one can first have all $2p$ D0-branes to make $p$ pairs. In this branch, one finds $p$ complex moduli $u_i$ ($i = 1, \cdots, p$). Another possibility is to form $p - 1$ pairs to freely move, while having 2 fractional D-branes stuck at two of the 4 fixed points. Note that the two
fractional branes have to be stuck at different fixed points: otherwise they can pair and leave the fixed point, being a special case of the first branch. There are 6 ways of choosing 2 fixed points among 4, so we obtain 6 more sectors. Finally, one finds a sector in which $p - 2$ pairs freely move, while 4 fractional D-branes are stuck at 4 different fixed points (when $p \geq 2$). After T-dualizing, $U_1, U_2$ are exponentials of the D0-brane positions. The above 8 sectors are summarized by the following pairs of Wilson lines $U_1, U_2$, for $O(2p)$ with $p \geq 2$:

\[(ee) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}), \quad U_2 = \text{diag}(e^{iu_2\sigma_2})\]

\[U_1 = \text{diag}(e^{iu_1\sigma_2}, 1, -1, -1)_{p-2}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1, 1, -1)_{p-2};\]

\[(eo) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, 1, 1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1, -1)_{p-1};\]

\[(oe) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, -1, -1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1, -1)_{p-1};\]

\[(oo) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, 1, -1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, -1, -1)_{p-1}.\]  \hspace{1cm} (3.2)

\[(ee), (eo), (oe), (oo)\] are for $U_1, U_2$ in the even or odd elements of $O(n)$. The symbol ‘diag’ denotes a block-diagonalized matrix. The subscripts are the number of independent complex parameters. The parameters live on $u_i = u_{1i} + \tau u_{2i} \in \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, where $\tau$ is related to our fugacity $q$ by $q = e^{2\pi i \tau}$. For odd $n = 2p + 1$ with $n \geq 3$, one can make a similar analysis. There are 4 cases in which one has 1 fractional brane stuck at one of the 4 fixed points, and 4 more cases (when $p \geq 1$) in which 3 fractional branes are stuck at three of the 4 fixed points. So one obtains the following 8 sectors, for $p \geq 1$:

\[(ee) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, 1), \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1);\]

\[U_1 = \text{diag}(e^{iu_1\sigma_2}, -1, -1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1, -1)_{p-1};\]

\[(eo) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, 1), \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, -1);\]

\[U_1 = \text{diag}(e^{iu_1\sigma_2}, -1, -1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1, -1)_{p-1};\]

\[(oe) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, -1), \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1);\]

\[U_1 = \text{diag}(e^{iu_1\sigma_2}, 1, -1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, -1, -1)_{p-1};\]

\[(oo) : \quad U_1 = \text{diag}(e^{iu_1\sigma_2}, -1), \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, -1);\]

\[U_1 = \text{diag}(e^{iu_1\sigma_2}, 1, 1)_{p-1}, \quad U_2 = \text{diag}(e^{iu_2\sigma_2}, 1, -1)_{p-1}.\]  \hspace{1cm} (3.3)

There are two exceptional cases. For $O(1)$, the four sectors in (3.3) with rank $p - 1$ are absent. So we only have four rank 0 sectors

\[(U_1, U_2) = (1, 1), (1, -1), (-1, 1), (-1, -1).\]  \hspace{1cm} (3.4)

For $O(2)$, the second sector in (3.2) with rank $p - 2$ is absent. So we have seven sectors

\[(U_1, U_2) = (e^{iu_1\sigma_2}, e^{iu_2\sigma_2}), (1, \sigma_3), (-1, \sigma_3), (\sigma_3, 1), (\sigma_3, -1), (\sigma_3, \sigma_3), (\sigma_3, -\sigma_3).\]  \hspace{1cm} (3.5)
The Wilson lines can be more conveniently labeled by their exponents, which we call \( u = (u_1, \ldots, u_n) \) for \( O(n) \). In the \( 2 \times 2 \) blocks \( e^{iu_1}\sigma_z, e^{iu_2}\sigma_z \) with continuous elements, the associated two \( u \) parameters are given by the two eigenvalues \( \pm(u_1 + \tau u_2) \). In the blocks with discrete numbers, we assign \( u_i = 0 \) for each eigenvalue \((1, 1) \) of the Wilson line \( U_1, U_2 \), \( u_i = \frac{\tau}{2} \) for each eigenvalue \((-1, 1) \), \( u_i = \frac{1+\tau}{2} \) for \((-1, -1) \). For the above 8 sectors, one thus obtains

\[
\begin{align*}
\text{(ee)} & : \quad u = (\pm u_1, \ldots, \pm u_p) ; \quad u = (\pm u_1, \ldots, \pm u_{p-2}, 0, \frac{1}{2}, 1+\tau, \frac{\tau}{2}) \\
\text{(eo)} & : \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 0, \frac{\tau}{2}) ; \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 1, 1+\tau) \\
\text{(oe)} & : \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 0, \frac{1}{2}) ; \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 0, \frac{\tau}{2}, 1+\tau) \\
\text{(oo)} & : \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 0) \\
\end{align*}
\]

for \( O(2p) \), and

\[
\begin{align*}
\text{(ee)} & : \quad u = (\pm u_1, \ldots, \pm u_p, 0) ; \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 1, 1+\tau, \frac{\tau}{2}) \\
\text{(eo)} & : \quad u = (\pm u_1, \ldots, \pm u_p, \frac{\tau}{2}) ; \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 1, 1+\tau, 0) \\
\text{(oe)} & : \quad u = (\pm u_1, \ldots, \pm u_p, \frac{1}{2}) ; \quad u = (\pm u_1, \ldots, \pm u_{p-1}, 0, \frac{\tau}{2}, 1+\tau) \\
\text{(oo)} & : \quad u = (\pm u_1, \ldots, \pm u_p) \\
\end{align*}
\]

for \( O(2p+1) \). These \( u \) couple minimally to the matters in the fundamental representation. The parameters coupling to fields in a different representation of \( SO(n) \) are given by \( \rho(u) \), where \( \rho \) runs over the weights of the representation.

With the Wilson line backgrounds identified, we study the subgroup of \( O(n) \) gauge symmetry which acts within the \( U_1, U_2 \) specified above. This is the ‘Weyl group,’ defined in each disconnected sector of \( (U_1, U_2) \). When \( U_1, U_2 \) are given by \( 2 \times 2 \) blocks and an \( s \times s \) diagonal matrix with \( \pm 1 \) eigenvalues (with \( 2r + s = n \) and \( s \leq 4 \)), the Weyl group is given by

\[
\text{[Weyl group of } O(2r)] \times [O(s) \text{ elements commuting with the } s \times s \text{ block}] .
\]

The former part has order \( 2r! \), and the latter has order \( 2^s \) coming from \( \text{diag}_{s \times s}(\pm 1, \pm 1, \ldots, \pm 1) \). So the order of the Weyl group \( W(O(n))_s \), with given \( U_1, U_2 \), is given by

\[
\begin{align*}
|W(O(2p))_s| &= 2^p p! , \quad |W(O(2p))_2| = 2^{p+1}(p-1)! , \quad |W(O(2p))_4| = 2^{p+2}(p-2)! \\
|W(O(2p+1))_1| &= 2^{p+1} p!, \quad |W(O(2p+1))_3| = 2^{p+2}(p-1)!,
\end{align*}
\]

where the subscript denotes the value of \( s \) for \( U_1, U_2 \).
In the above background, the Gaussian path integral of non-zero modes yields $Z_{\text{1-loop}}$, which is the product of the following 1-loop determinants for various supermultiplets \[24\]:

\[
Z_{\text{sym. hyper}} = \prod_{\rho \in \text{sym}} \frac{i\eta(q)}{\theta_1(q, \epsilon_1 + \rho(u))} \cdot \frac{i\eta(q)}{\theta_1(q, \epsilon_2 + \rho(u))} \\
Z_{\text{SO}(16) \text{ Fermi}} = \prod_{\rho \in \text{fund}} \left( \frac{8}{\eta(q)} \right) \left( \frac{\theta_1(q, m_1 + \rho(u))}{\theta_1(q, m_2 + \rho(u))} \right) \\
Z_{\text{vector}} = \prod_{i=1}^r \left( \frac{2\pi \eta^2}{i} \cdot \left( \frac{\theta_1(\epsilon_1 + \epsilon_2)}{\eta(q)} \right) \right) \cdot \prod_{\alpha \in \text{root}} \frac{\theta_1(\alpha(u))\theta_1(\epsilon_1 + \epsilon_2 + \alpha(u))}{i^2 \eta^2}.
\]

Whenever we omit the modular parameters of the theta functions, it is understood as $\tau$. The ‘rank’ $r$ is the number of continuous complex parameters in $U_1, U_2$. $\alpha$ runs over the roots of $SO(n)$. Multiplying all these factors, one finally has to integrate over the continuous parameters in $u$ and then sum over distinct sectors of flat connections. The result is

\[
\sum_a \frac{1}{|W_a|} \cdot \frac{1}{(2\pi i)} \int Z_{\text{1-loop}}^{(a)} = Z_{\text{1-loop}}^{(a)} \equiv Z_{\text{vector}}^{(a)} Z_{\text{sym. hyper}}^{(a)} Z_{\text{SO}(16) \text{ Fermi}}^{(a)}.
\]

$a$ labels the disconnected sectors of the flat connection $U_1, U_2$. The integral is a suitable ‘contour integral’ over the continuous parameters $u$, to be explained shortly. $W_a$ is the Weyl group with given $U_1, U_2$ explained above.

Before proceeding, let us comment on the periodicity of (3.10) in $u$. Each $u_i$ (for $i = 1, \cdots, p$) lives on $T^2/\mathbb{Z}_2$, due to large gauge transformations on $T^2$, so is a periodic variable $u_i \sim u_i + 1 \sim u_i + \tau$. However, since $\theta_1(u, \tau)$ is only a quasi-periodic function,

\[
\theta_1(z + 1) = -\theta_1(z), \quad \theta_1(z + \tau) = -q^{-1/2} y^{-1} \theta_1(z), \quad \theta_1(z + 1 + \tau) = q^{-1/2} y^{-1} \theta_1(z)
\]

each measure in (3.10) is not invariant under these shifts. The failure of periodicity is related to the gauge anomaly of the chiral theory. The extra factors spoiling the periodicity cancel in the combination (3.11), due to the anomaly cancelation of our gauge theory.

Another subtlety is the determinant of the real scalars and Majorana fermions. Each real scalar or fermion contributes to a square-root of $\theta_1$ factor. Equivalently, each charge conjugate pair of fermion modes contributes a factor of $\frac{\theta_1(z)}{\theta_1(z)}$, while such a pair of bosons contributes $\frac{i\eta}{\theta_1(z)}$ in (3.10). In particular, on these modes, the discrete shifts on the holonomy (3.6), (3.7) given by $u_i = \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$ has to be understood with some care. When such a shift is made in the determinant, one obtains $(-1)^n$.

---

\footnote{One difference from [24] is that we put a factor $i$ in the denominator of the contribution $\frac{\theta_1(q, 2z)}{\theta_1(q, z)}$ from each Fermi multiplet. Of course this only affects the overall sign of the index, which is ambiguous in 2d without knowing the spin-statistics relation inherited from higher dimensional physics. We shall see that our choice is compatible with the physics of circle compactified 6d CFT, by comparing with some known results. Collecting all the factors of $i$ in $Z_{\text{1-loop}}$, one obtains $(-1)^n$.}
argument of $\theta_1$ coming from a pair of real fields, one should understand it as $\"theta_1(z + u_i)\" \sim \sqrt{\theta_1(z + u_i)} \theta_1(z - u_i)$. Having this in mind, and applying
\[
\theta_1(z + \frac{1}{2}) = \theta_2(z) , \quad \theta_1(z + \frac{3}{2}) = i q^{-1/8} y^{-1/2} \theta_4(z) , \quad \theta_1(z + \frac{1+i}{2}) = q^{-1/8} y^{-1/2} \theta_3(z) ,
\]
(3.13)
one can replace $\theta_1(z + \frac{1}{2}), \theta_1(z + \frac{r+1}{2}), \theta_1(z + \frac{r+3}{2})$ by $\theta_2(z), \theta_3(z), \theta_4(z)$, respectively, apart from the extra factors appearing in (3.13). These extra factors in (3.11) again cancel to 1. So the theta function $\theta_1$ with half-period shifts can be replaced by $\theta_2, \theta_3, \theta_4$.

Now we finally explain the meaning of the ‘contour integral’ in (3.11), following [23, 24]. The ‘contour integral’ is defined by providing a prescription for the residue sum which replaces the integral, whenever one encounters a pole on the parameter space of $(U_1, U_2)$. The prescription is derived in [24], using the so-called Jeffrey-Kirwan residues. At each pole $u = u_*$ on the $r$ complex dimensional $u$ space, there are $r$ or more hyperplanes of the form $\rho_i(u) + z_i = 0 \pmod{Z + \tau Z}$ which passes through it, where $i = 1, \ldots, d \geq r$. $z_i$ are linear combinations of the chemical potentials so that $\theta_1(\rho_i(u) + z_i)$ appear in the denominator of $\cal{Z}'_1\text{loop}$. In our problem, $z_i$ are either $\epsilon_1$ or $\epsilon_2$. When exactly $r$ hyperplanes intersect at a point $u = u_*$ (mod $Z + \tau Z$), this pole is called non-degenerate. When $d > r$, the pole is called degenerate.

Before explaining the Jeffrey-Kirwan residues (or JK-Res) of our integrand at $u = u_*$, let us first note that the results of [24] apply when the pole at $u_*$ is ‘projective.’ The pole is called projective when all the weight vectors $\rho_i$ associated with the hyperplanes meeting at $u = u_*$ are contained in a half space. Namely, the projective condition requires that there is a vector $v$ in the Cartan $\mathfrak{h}$ so that $\rho_i(v) > 0$. Note that all non-degenerate poles are projective. In our problem, even for degenerate poles, one can generally show that all poles should be projective, thus allowing us to use the results of [24]. To see this, first note that
\[
\rho_i(u_*) = -z_i + m_i + n_i \tau ,
\]
(3.14)
for suitable integers $m_i, n_i$. In our problem, since $\rho_i$ is chosen among the weight system of the $O(n)$ symmetric representation, it is either $\pm 2 \epsilon_I$ or $\pm \epsilon_I \pm \epsilon_J$ with $I, J = 1, \cdots, [\frac{n}{2}]$. Thus, we can take all $m_i, n_i$ to be either 0 or 1 to find all possible solutions for $u_*$, mod $Z + \tau Z$. Also, $z_i$ is either $\epsilon_1$ or $\epsilon_2$ for all $i$’s. Then, taking a solution $u_*(\epsilon_1, \epsilon_2)$ which depends on $\epsilon_{1,2}$, one deforms the solution to the regime in which $\epsilon_1, \epsilon_2$ are real and negative, taken to be $-\epsilon_{1,2} \gg 1$ and $-\epsilon_{1,2} \gg \text{Re}(\tau)$. Then one finds that $\rho_i \cdot \text{Re}(u_*) > 0$, fulfilling the projective condition. In fact, one can always provide this kind of argument on the projective nature of poles when the system has independent flavor symmetry for each matter supermultiplet. The $\cal{N} = (2, 2)$ or $(0, 2)$ models may exhibit non-projective poles if there are nonzero superpotentials so that flavor symmetries are restricted. In $\cal{N} = (0, 4)$ models, independent flavor symmetry can be found for each hypermultiplet. This is why it is easier to apply the results of [24] to $(0, 4)$ theories. For instance, the quantum mechanical version of this index formula is well applicable to the ADHM instanton quantum mechanics [34], as these systems always have $(0, 4)$ SUSY. (The results of [34] will be used in our section 4.)
[24] finds that the integral in (3.11) is given by

$$\frac{1}{(2\pi i)^r} \oint Z^{(a)}_{1\text{-loop}} = \sum_{u_*} \text{JK-Res}_{u_*}(Q_*, \eta) Z^{(a)}_{1\text{-loop}},$$

(3.15)

where $u_*$ runs over all the poles in the integrand. The JK-Res appearing in this expression is defined as follows. JK-Res is a linear functional which refers to an auxiliary vector $\eta$ in the charge space, and also to the set of charge vectors $Q_* = (Q_1, \cdots, Q_d)$ for the hyperplanes crossing $u_*$. The defining property of $\text{JK-Res}_{u_*}(Q_*, \eta)$ is

$$\text{JK-Res}_{u_*}(Q_*, \eta) = \left\{ \begin{array}{ll}
\text{sign } \det(Q_{j_1}, \cdots, Q_{j_r}) & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r}) \\
0 & \text{otherwise}
\end{array} \right.$$

(3.16)

or equivalently

$$\text{JK-Res}_{u_*}(Q_*, \eta) = \left\{ \begin{array}{ll}
|\det(Q_{j_1}, \cdots, Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r}) \\
0 & \text{otherwise}
\end{array} \right.$$

(3.17)

To make the condition $\eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r})$ unambiguous, one has to put $\eta$ at a sufficiently generic point, as explained in [24]. These rules are giving a definite residue when the integrand takes the form of a ‘simple pole.’ Although this definition apparently overdetermines JK-Res due to many relations among the forms $\bigwedge_{i=1}^r \frac{dQ_{j_i}(u)}{Q_{j_i}(u-u_*)}$, it turns out to be consistent (see [24] and references therein). As one expands the integrand $Z^{(a)}_{1\text{-loop}}$ around $u = u_*$, one will encounter not just simple poles, but also multiple poles and less singular homogeneous expressions in $u - u_*$, multiplied by $du_1 \wedge \cdots \wedge du_r$. The JK-Res of the last two classes of monomials are all (naturally) zero: this is also consistent with the alternative ‘constructive definition,’ which expresses JK-Res as an iterated integral over a cycle. Using this definition to compute the integral is especially simple for non-degenerate poles, in which case one can directly read off a unique integral of the form (3.17) at a given $u = u_*$. The case with degenerate poles require some more work, but of course coming with a clear rule. The final result (3.15) is independent of the choice of $\eta$ [24].

In the remaining part of this section, we first analyze the elliptic genera for $n = 1, 2, 3, 4$ E-strings in great detail. In section 3.5, we then illustrate the structure of the higher E-string indices. In particular, degenerate poles start to appear from $n \geq 6$. The residue evaluations are almost as simple as the non-degenerate poles for $n = 6, 7$, all coming from simple poles. Their residues are simply given by combinations of theta functions. For $n \geq 8$, we explain that there start to appear degenerate poles which are also multiple poles. Their residues are given by theta functions and their derivatives in the elliptic parameters.
3.1 One E-string

We consider the elliptic genus for the $O(1)$ theory. Since $O(1) = \mathbb{Z}_2$, there are four different flat connections $(1, 1), (1, -1), (-1, 1), (-1, -1)$. The indices in the four sectors are given by

$$Z_{1(i)} = - [1]_{\text{vec}} \cdot \left[ \frac{\eta^2}{\theta_1(e_1)\theta_1(e_2)} \right]_{\text{sym hyper}} \cdot \left[ \prod_{l=1}^{8} \frac{\theta_l(m_l)}{\eta} \right]_{\text{Fermi}},$$

(3.18)

where $i = 1, 2, 3, 4$ for the Wilson line $(1, 1), (1, -1), (-1, -1), (-1, 1)$, respectively. Combining all four contributions, and dividing by the Weyl group order $|W| = 2$ in each sector, the full index is given by

$$Z_1 = \sum_{i=1}^{4} \frac{Z_{1(i)}}{2} = - \frac{\Theta(q, m_l)}{\eta^6 \theta_1(e_1)\theta_1(e_2)},$$

(3.19)

where the $E_8$ theta function $\Theta$ is given by

$$\Theta(\tau, m_i) = \frac{1}{2} \sum_{n=1}^{4} \prod_{l=1}^{8} \theta_n(\tau, m_l).$$

(3.20)

Physically, $\frac{Z_{1(1)}+Z_{1(2)}}{2}$ simply imposes the $O(1) = \mathbb{Z}_2$ singlet condition, while the remainder $\frac{Z_{1(3)}+Z_{1(4)}}{2}$ is the contribution from the twisted sector.

In [19], the above result was derived using topological strings and was explained using an effective free string theory calculus, in which the left moving sector consists of the $E_8$ current algebra at level 1 and the right moving sector consists of a $(0, 4)$ supersymmetric string with target space $\mathbb{R}^4$. The four terms of $\Theta(\tau, m_i)$ can be understood as coming from the Ramond and Neveu-Schwarz sectors of the left-moving fermions, and then truncating the Hilbert space by a GSO projection. In our UV gauge theory calculus, the twisting and GSO projection is a consequence of the $O(1)$ gauge symmetry.

Since $\Theta(q, m_l)$ is given by the summation over the $E_8$ root lattice, $Z_1$ has a manifest $E_8$ symmetry, and is expanded as the sum of $E_8$ characters. This supports the IR enhancement $SO(16) \rightarrow E_8$ of global symmetry in our gauge theory.

3.2 Two E-strings

Now we consider the $O(2)$ theory. There are 7 sectors of $O(2)$ Wilson lines given by \[8\]. One in the $(ee)$ sector has a complex modulus, while the other six are all discrete. We name the sectors as follows, where $(a_+, a_-)$ are the two eigenvalues of $u$ in the discrete sectors which act
on the fundamental representation \([23]\):

\[
(0) \equiv (ee) : \quad (U_1, U_2) = (e^{iu_1\pi^2}, e^{iu_2\pi^2})
\]

\[(1), (2) \equiv (oe)_{\pm} : \quad (\sigma_3, \pm 1) \rightarrow (a_v, a_+, a_-) = (\tfrac{1}{2}, 0, \tfrac{1}{2}), \quad (\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1+i\tau}{2})
\]

\[(3), (4) \equiv (eo)_{\pm} : \quad (\pm 1, \sigma_3) \rightarrow (a_v, a_+, a_-) = (\tfrac{3}{2}, 0, \tfrac{1}{2}), \quad (\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1+i\tau}{2})
\]

\[(5), (6) \equiv (oo)_{\pm} : \quad (\pm \sigma_3, \sigma_3) \rightarrow (a_v, a_+, a_-) = (\tfrac{1+i\tau}{2}, 0, \tfrac{1+i\tau}{2}), \quad (\tfrac{1+i\tau}{2}, \tfrac{1}{2}, \tfrac{1+i\tau}{2}).
\]

All eigenvalues \(a_+, a_-\) are defined mod \(\mathbb{Z} + \tau \mathbb{Z}\). \(a_v = a_+ + a_-\) is the eigenvalue acting on the \(O(2)\) adjoint (antisymmetric) representation. The discrete holonomy eigenvalues acting on the \(O(2)\) symmetric representation are \(a_v = a_+ + a_-\), \(2a_+\), \(2a_-\). The contributions \(Z_{2(a)}\) (with \(a = 0, \cdots, 6\)) are given by

\[
Z_{2(0)} = \oint \left[ \eta^2 du \cdot \frac{\theta_1(2\epsilon_+) - c}{\eta} \right]_{\text{vec}} \cdot \left[ \frac{\eta^6}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_1 \pm 2u)\theta_1(\epsilon_2 \pm 2u)} \right]_{\text{sym}} \cdot \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)}{\eta^2} \right]_{\text{Fermi}} \quad (a = 1, \cdots, 6), \tag{3.21}
\]

where we defined \(\epsilon_+ = \frac{\epsilon_1 + \epsilon_2}{2}\). As explained after (3.11), \(\theta_1(z + a_v)\) factors should be understood as \(\theta_i\), with \(i = 1, 2, 3, 4\) for \(a_v = 0, \frac{1}{2}, \frac{1+i\tau}{2}, \frac{3}{2}\), respectively.

The contour integral in \(Z_{2(0)}\) can be done by taking residues from poles with positive \(SO(2)\) electric charge only: this is the simple rule for the rank 1 theory obtained by taking \(\eta = 1\) \([23]\).

The relevant poles are at \(\theta_1(\epsilon_1 + 2u) = 0\) and \(\theta_1(\epsilon_2 + 2u) = 0\). Using

\[
\frac{1}{2\pi i} \oint_{u=a+br} \frac{du}{\theta_1(\tau|u)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{\theta_1(\tau|0)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{2\pi \eta^3},
\]

one should pick the residues at \(u = -\frac{\epsilon_1 + \epsilon_2}{2}, -\frac{\epsilon_1 + \epsilon_2}{2} + \frac{1}{2}, -\frac{\epsilon_1 + \epsilon_2}{2} + \frac{3}{2}, -\frac{\epsilon_1 + \epsilon_2}{2} + \frac{1+i\tau}{2}\). The residue sum

\[
Z_{2(0)} = \frac{1}{2\eta^2} \theta_1(\epsilon_1)\theta_1(\epsilon_2) \sum_{i=1}^{4} \left[ \frac{\prod_{l=1}^{8} \theta_1(m_l \pm \eta^2)}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)} + \frac{\prod_{l=1}^{8} \theta_1(m_l \pm \eta^2)}{\theta_1(2\epsilon_2)\theta_1(\epsilon_1 - \epsilon_2)} \right]. \tag{3.23}
\]

Expressions with \(\pm\) signs mean \(\theta_i(x \pm y) \equiv \theta_i(x + y)\theta_i(x - y)\). The contributions from the other six sectors are

\[
Z_{2(1)} = \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{8} \theta_1(m_l)\theta_2(m_l)}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_2)}; \quad \quad Z_{2(2)} = \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{8} \theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_2)};
\]

\[
Z_{2(3)} = \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{8} \theta_1(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_2)}; \quad \quad Z_{2(4)} = \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{8} \theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_2)};
\]

\[
Z_{2(5)} = \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{8} \theta_1(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_2)}; \quad \quad Z_{2(6)} = \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{8} \theta_2(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_2)}. \tag{3.24}
\]
The two E-string index is given by

$$Z_2(\tau, \epsilon_{1,2}, m_i) = \frac{1}{2} Z_{2(0)} + \frac{1}{4} \sum_{a=1}^{6} Z_{2(a)} ,$$

(3.25)

by dividing the order of the ‘Weyl group,’ as defined around (3.9).

Recently, [7] obtained the 2 E-string elliptic genus. This was done by constraining its form with its modularity, the ‘domain wall’ ansatz of [5], and a few low orders in the genus expansion known from the topological string calculus. The result of [7] is given by

$$Z_2 = \frac{1}{576 \eta^{12} \theta_1(\epsilon_1) \theta_1(\epsilon_2) \theta_1(2\epsilon_1)} \left[ 4A_1^2(\phi_{0,1}(\epsilon_1)^2 - E_4\theta_{-2,1}(\epsilon_1)^2) + 3A_2(E_4^2\phi_{-2,1}(\epsilon_1)^2 - E_6\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) + 5B_2(E_6\phi_{-2,1}(\epsilon_1)^2 - E_4\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) \right] (\epsilon_1 \leftrightarrow \epsilon_2)$$

where $E_4(\tau)$, $E_6(\tau)$ are the Eisenstein series, summarized in appendix A,

$$\phi_{-2,1}(\epsilon, \tau) = -\frac{\theta_1(\epsilon, \tau)^2}{\eta(\tau)^6} \, , \, \phi_{0,1}(\epsilon, \tau) = 4 \left[ \frac{\theta_2(\epsilon, \tau)^2}{\theta_2(0, \tau)^2} + \frac{\theta_3(\epsilon, \tau)^2}{\theta_3(0, \tau)^2} + \frac{\theta_4(\epsilon, \tau)^2}{\theta_4(0, \tau)^2} \right],$$

(3.27)

and $A_1(m_i)$, $A_2(m_i)$, $B_2(m_i)$ are three of the nine Jacobi forms which are invariant under the Weyl group of $E_8$. See, for instance, the appendix of [29] for the full list. $A_1$ is simply the $E_8$ theta function $A_1 = \Theta(m_i, \tau)$, and

$$A_2 = \frac{8}{9} \left[ \frac{\Theta(2m_i, 2\tau)}{16} + \frac{\Theta(m_i, \frac{\tau+1}{2})}{16} \right],$$

(3.28)

$$B_2 = \frac{8}{15} \left[ (\theta_3^4 + \theta_4^4)\Theta(2m_i, 2\tau) - \frac{1}{16}(\theta_2^4 + \theta_3^4)\Theta(m_i, \frac{\tau}{2}) + \frac{1}{16}(\theta_2^4 - \theta_3^4)\Theta(m_i, \frac{\tau+1}{2}) \right].$$

We made a full analytic proof, at $\epsilon_1 = -\epsilon_2$ for simplicity (but keeping all $E_8$ masses and $\epsilon_- = \frac{\epsilon_+ - \epsilon_2}{2}$), that (3.25) and (3.26) agree with each other. See appendix C for our proof. On one side, this agreement shows that the ‘domain wall ansatz’ of [7] is at work. On the other hand, it also shows that our gauge theory index exhibits the Weyl symmetry of $E_8$, which is manifest in (3.26). So this supports the IR $E_8$ symmetry enhancement of our gauge theory.

### 3.3 Three E-strings

There are eight sectors of $O(3)$ holonomies on $T^2$, which we label as follows:

- (ee) : $\text{diag}(e^{iu_1\sigma_2}, 1), \text{diag}(e^{iu_2\sigma_2}, 1) \rightarrow (1)$ ; $\text{diag}(-1, -1, 1), \text{diag}(1, -1, -1) \rightarrow (1)'$ ;
- (eo) : $\text{diag}(e^{iu_1\sigma_2}, 1), \text{diag}(e^{iu_2\sigma_2}, -1) \rightarrow (4)$ ; $\text{diag}(-1, -1, 1), \text{diag}(1, -1, 1) \rightarrow (4)'$ ;
- (oe) : $\text{diag}(e^{iu_1\sigma_2}, -1), \text{diag}(e^{iu_2\sigma_2}, 1) \rightarrow (2)$ ; $\text{diag}(1, -1, 1), \text{diag}(-1, -1, 1) \rightarrow (2)'$ ;
- (oo) : $\text{diag}(e^{iu_1\sigma_2}, -1), \text{diag}(e^{iu_2\sigma_2}, -1) \rightarrow (3)$ ; $\text{diag}(1, 1, -1), \text{diag}(1, 1, 1) \rightarrow (3)'$. 


The indices in various sectors are given as follows. Firstly,

\[ Z_{3(1)} = - \int \frac{\eta^2 du}{\eta^3} \cdot \frac{\theta_1(2\epsilon_+)\theta_1(2\epsilon_+ \pm u)\theta_1(\pm u)}{i\eta^3} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)_{\text{sym}}} \]

\[ \prod_{t=1}^{8} \frac{\theta_1(m_t)\theta_1(m_t + u)\theta_1(m_t - u)}{\eta^3} \] \quad \text{vec} \quad \text{Fermi} \quad (3.29)

\[ Z_{3(1)'} = - \int \frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})_{\text{sym}}} \]

\[ \prod_{t=1}^{8} \frac{\theta_2(m_t)\theta_3(m_t)\theta_4(m_t)}{\eta^3} \] \quad \text{vec} \quad \text{vec} \quad \text{Fermi} \quad (3.30)

\[ Z_{3(1)} \] is obtained with discrete holonomy \((a_1, a_2, a_3) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)\) acting on the fundamental, \((a_1 + a_2, a_2 + a_3, a_3 + a_1) = \left( \frac{3}{2}, \frac{1}{2}, \frac{1}{2} \right)\) on adjoint, and \((2a_1, 2a_2, 2a_3, a_1 + a_2, a_2 + a_3, a_3 + a_1)\) on symmetric representations. Similarly, one obtains

\[ Z_{3(4)} = - \int \frac{\eta^2 du}{\eta^3} \cdot \frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+ \pm u)\theta_4(\pm u)}{i\eta^3} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_4(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)_{\text{sym}}} \]

\[ \prod_{t=1}^{8} \frac{\theta_4(m_t)\theta_1(m_t + u)\theta_1(m_t - u)}{\eta^3} \] \quad \text{vec} \quad \text{Fermi} \quad (3.31)

\[ Z_{3(4)'} = - \int \frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})_{\text{sym}}} \]

\[ \prod_{t=1}^{8} \frac{\theta_1(m_t)\theta_2(m_t)\theta_3(m_t)}{\eta^3} \] \quad \text{vec} \quad \text{vec} \quad \text{Fermi} \quad (3.32)

from the (eo) sectors with \((a_1, a_2, a_3) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), 0)\),

\[ Z_{3(2)} = - \int \frac{\eta^2 du}{\eta^3} \cdot \frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+ \pm u)\theta_2(\pm u)}{i\eta^3} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_2(\epsilon_{1,2} \pm u)\theta_1(\epsilon_{1,2} \pm 2u)_{\text{sym}}} \]

\[ \prod_{t=1}^{8} \frac{\theta_2(m_t)\theta_1(m_t + u)\theta_1(m_t - u)}{\eta^3} \] \quad \text{vec} \quad \text{Fermi} \quad (3.33)

\[ Z_{3(2)'} = - \int \frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6} \cdot \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})_{\text{sym}}} \]

\[ \prod_{t=1}^{8} \frac{\theta_1(m_t)\theta_3(m_t)\theta_4(m_t)}{\eta^3} \] \quad \text{vec} \quad \text{vec} \quad \text{Fermi} \quad (3.34)
from the (oe) sectors with \((a_1, a_2, a_3) = (\frac{1}{2}, \frac{1+\epsilon}{2}, 0)\), and

\[
Z_{3(3)} = -\oint \left[ \eta^2 du \frac{\theta_1(2\epsilon_+ + u)\theta_3(\epsilon_+ \pm u)}{i\eta^3} \right] \cdot \left[ \prod_{l=1}^{3} \frac{\theta_3(m_l)\theta_3(\epsilon_+)\theta_1(m_l - u)}{\eta^3} \right]_{\text{Fermi}} \cdot \left[ \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2 \theta_3(\epsilon_{1,2} \pm u) \theta_1(\epsilon_{1,2} \pm 2u)} \right]_{\text{sym}}
\]

\[
Z_{3(3)'} = -\left[ \frac{\theta_2(0)\theta_3(0)\theta_2(0)\theta_3(2\epsilon_+)\theta_3(2\epsilon_+)}{\eta^6} \right] \cdot \left[ \prod_{l=1}^{3} \frac{\theta_3(m_l)\theta_3(m_l + u)\theta_3(\epsilon_+)}{\eta^3} \right]_{\text{Fermi}} \cdot \left[ \frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3 \theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})} \right]_{\text{sym}}
\]  

(3.35)

from the (oo) sectors with \((a_1, a_2, a_3) = (0, \frac{\epsilon}{2}, \frac{1}{2})\). The contour integrals in \(Z_{3(i)}\) acquire residue contributions from poles \(u_* = -\frac{\epsilon_1}{2}, -\frac{\epsilon_1}{2} + \frac{1}{2}, -\frac{\epsilon_1}{2} + \frac{1+\epsilon}{2}, -\frac{\epsilon_1}{2} + \frac{1+\epsilon}{2}\) and \(u_* = -\epsilon_{1,2} + \cdots\), where \(\cdots\) part is decided by \(\theta_i(u + \epsilon_{1,2}) = 0\). The residue sums are given by

\[
Z_{3(i)} = -\frac{\eta^4}{\theta_1(\epsilon_{1,2})^2 \theta_1(\epsilon_2)^2} \left[ \frac{\eta^2\theta_1(\epsilon_1)\theta_1(\epsilon_2)}{\theta_2(2\epsilon_1)\theta_3(\epsilon_2 - \epsilon_1)\theta_1(3\epsilon_1)\theta_1(2\epsilon_2 - 2\epsilon_1)} \prod_{l=1}^{8} \frac{\theta_3(m_l)\theta_3(m_l + \epsilon_1)}{\eta^3} \right]
\]

\[
\quad+ \frac{1}{2} \sum_{a=1}^{4} \frac{\eta^2\theta_{\sigma(a)}(\frac{3\epsilon_1}{2} + \epsilon_2)\theta_{\sigma(a)}(-\frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(2\epsilon_2 - \epsilon_1)\theta_{\sigma(a)}(\frac{3\epsilon_1}{2} + \epsilon_2)\theta_{\sigma(a)}(\frac{3\epsilon_1}{2} + \epsilon_2)} \prod_{l=1}^{8} \frac{\theta_3(m_l)\theta_3(m_l + \epsilon_1)}{\eta^3} + (\epsilon_1 \leftrightarrow \epsilon_2)
\]

(3.37)

where the permutations are defined by

\[
\sigma_1(1, 2, 3, 4) = (1, 2, 3, 4), \quad \sigma_2(1, 2, 3, 4) = (2, 1, 3, 4), \\
\sigma_3(1, 2, 3, 4) = (3, 4, 1, 2), \quad \sigma_4(1, 2, 3, 4) = (4, 3, 2, 1).
\]

(3.38)

The full index is given by

\[
Z_3 = \sum_{i=1}^{4} \left( \frac{1}{4} Z_{3(i)} + \frac{1}{8} Z_{3(i)'} \right)
\]

(3.39)

after dividing by the Weyl factors \((3.9)\).

For simplicity, we study the indices at \(m_l = 0, \epsilon_1 = -\epsilon_2 \equiv \epsilon\) in more detail, which are

\[
Z_{3(i)} = \frac{\eta^4}{\theta_1(\epsilon)^4} \left[ \frac{2\theta_1(\epsilon)^2\theta_3(0)^8\theta_1(\epsilon)^{16}}{\eta^{22}\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2} + \sum_{a=1}^{4} \frac{\theta_{\sigma(a)}(\frac{\epsilon}{2})^2 \theta_{\sigma(a)}(0)^8 \theta_{\sigma(a)}(\frac{\epsilon}{2})^{16}}{\eta^{22}\theta_1(2\epsilon)^2\theta_{\sigma(a)}(\frac{3\epsilon}{2})^2} \right]
\]

(3.40)

and

\[
Z_{3(1)'} = \frac{\theta_2(0)^{10}\theta_3(0)^{10}\theta_4(0)^{10}}{\eta^{18}\theta_1(\epsilon)^6\theta_2(\epsilon)^2\theta_3(\epsilon)^2\theta_4(\epsilon)^2} = \frac{4\theta_2(0)^8\theta_3(0)^8\theta_4(0)^8}{\eta^{18}\theta_1(\epsilon)^6\theta_2(\epsilon)^2},
\]

(3.41)

with \(Z_{3(2)'} = Z_{3(3)'} = Z_{3(4)'} = 0\). We consider the genus expansion of \(Z_3\), where genus is defined for the topological string amplitudes on the CY3 which engineers our 6d CFT in the F-theory context. Namely, we expand

\[
F_3 \equiv Z_3 - Z_1 Z_2 + \frac{1}{3} Z_1^3 = \sum_{n \geq 0, g \geq 0} (\epsilon_1 + \epsilon_2)^n (\epsilon_1 \epsilon_2)^g \eta^{n-1} F^{(n, g, 3)}(\tau).
\]

(3.42)
Taking $\epsilon_+ = 0$, some known results on $F^{(0, g, 3)}$ are summarized in [31], which were computed in [41] up to genus 5. This can be compared with $F^{(0, g, 3)}$ obtained from our gauge theory index. Numerically, we checked the agreements for $g \leq 5$ up to first 10 terms in the $q$ expansions, starting at $q^{-3/2}$, with the last term that we checked at $q^{15/2}$. (The two terms at $q^{-1/2}$ and $q^{1/2}$ are all zero due to a vanishing theorem.)

We also analytically checked the agreements for $F^{(0, 0, 3)}$, $F^{(0, 1, 3)}$, and a refined amplitude $F^{(1, 0, 3)}$, against the results known from the topological string calculus. See appendix C for the details.

### 3.4 Four E-strings

The indices from the two sectors in the (ee) part of $O(4)$ holonomy are

$$Z_{4(1)} = - \oint \left[ \eta^4 d\eta_1 d\eta_2 \cdot \frac{\theta_1(2\epsilon_+)^2 \theta_1(2\epsilon_+ \pm u_1 \pm u_2) \theta_1(\pm u_1 \pm u_2)}{\eta^0} \right]_{\text{vec}} \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^2 \theta_1(\epsilon_{1,2} \pm u_1 \pm u_2) \theta_1(\epsilon_{1,2} \pm 2u_1) \theta_1(\epsilon_{1,2} \pm 2u_2)} \right]_{\text{sym}} \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u_1) \theta_1(m_l \pm u_2)}{\eta^4} \right]_{\text{Fermi}}$$

$$Z'_{4(1)} = \oint \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^4 \theta_2(\epsilon_{1,2})^2 \theta_3(\epsilon_{1,2})^2 \theta_4(\epsilon_{1,2})^2} \right]_{\text{vec}} \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l) \theta_2(m_l) \theta_3(m_l) \theta_4(m_l)}{\eta^4} \right]_{\text{Fermi}}$$

where $Z'_{4(1)}$ is obtained with discrete holonomy $(a_1, a_2, a_3, a_4) = (0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2})$ for the fundamental representation. We used a shorthand notation $\theta_i(\epsilon_{1,2}) \equiv \theta_i(\epsilon_1) \theta_i(\epsilon_2)$. The indices from the two sectors in the (oe) part are

$$Z_{4(2)} = \oint \left[ \eta^2 d\eta \cdot \frac{\theta_1(2\epsilon_+ \pm u) \theta_2(2\epsilon_+ \pm u) \theta_2(0) \theta_1(\pm u_1) \theta_2(\pm u)}{i\eta^{11}} \right]_{\text{vec}} \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u) \theta_1(\epsilon_{1,2})^3 \theta_2(\epsilon_{1,2}) \theta_1(\epsilon_{1,2} \pm u) \theta_1(\epsilon_{1,2} \pm u_1) \theta_1(\epsilon_{1,2} \pm u_2)} \right]_{\text{sym}} \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u) \theta_1(m_l) \theta_2(m_l)}{\eta^4} \right]_{\text{Fermi}}$$

$$Z'_{4(2)} = \oint \left[ \eta^2 d\eta \cdot \frac{\theta_1(2\epsilon_+ \pm u) \theta_2(2\epsilon_+ \pm u) \theta_2(0) \theta_3(\pm u) \theta_4(\pm u)}{i\eta^{11}} \right]_{\text{vec}} \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u) \theta_1(\epsilon_{1,2})^3 \theta_2(\epsilon_{1,2}) \theta_3(\epsilon_{1,2} \pm u) \theta_1(\epsilon_{1,2} \pm u_1) \theta_1(\epsilon_{1,2} \pm u_2)} \right]_{\text{sym}} \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u) \theta_3(m_l) \theta_4(m_l)}{\eta^4} \right]_{\text{Fermi}}$$

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where the holonomy \((a_1, a_2, a_3, a_4) = (u, -u, 0, \frac{1}{2})\) and \((u, -u, \frac{5}{2}, \frac{1+i}{2})\) are used for \(Z_{4(2)}\) and \(Z_{4(2)'}\), respectively. The indices from the two sectors in the (oo) part are

\[
\begin{align*}
Z_{4(3)} &= \int \left[ \eta^2 du \cdot \frac{\theta_1(2\epsilon_+u)\theta_3(2\epsilon_+u)\theta_1(2\epsilon_+u)\theta_3(2\epsilon_+u)\theta_3(0)\theta_1(\pm u)\theta_3(\pm u)}{i\eta^{11}} \right] \text{vec} \\
&\cdot \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u)\theta_1(\epsilon_{1,2})^3\theta_3(\epsilon_{1,2})\theta_1(\epsilon_{1,2} \pm u)\theta_3(\epsilon_{1,2} \pm u)} \right] \text{sym} \cdot \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)\theta_1(m_l)\theta_3(m_l)}{\eta^4} \right] \text{Fermi}
\end{align*}
\]

\[
\begin{align*}
Z_{4(3)'} &= \int \left[ \eta^2 du \cdot \frac{\theta_1(2\epsilon_+u)\theta_3(2\epsilon_+u)\theta_2(2\epsilon_+u)\theta_3(2\epsilon_+u)\theta_3(0)\theta_2(\pm u)\theta_4(\pm u)}{i\eta^{11}} \right] \text{vec} \\
&\cdot \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u)\theta_1(\epsilon_{1,2})^3\theta_3(\epsilon_{1,2})\theta_2(\epsilon_{1,2} \pm u)\theta_3(\epsilon_{1,2} \pm u)} \right] \text{sym} \cdot \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)\theta_2(m_l)\theta_3(m_l)}{\eta^4} \right] \text{Fermi}
\end{align*}
\]

where the holonomy \((a_1, a_2, a_3, a_4) = (u, -u, 0, \frac{1}{2})\) and \((u, -u, \frac{5}{2}, \frac{1}{2})\) are used for \(Z_{4(3)}\) and \(Z_{4(3)'}\), respectively. Finally, the indices from the two sectors in the (eo) part are

\[
\begin{align*}
Z_{4(4)} &= \int \left[ \eta^2 du \cdot \frac{\theta_1(2\epsilon_+u)\theta_4(2\epsilon_+u)\theta_1(2\epsilon_+u)\theta_4(2\epsilon_+u)\theta_4(0)\theta_1(\pm u)\theta_4(\pm u)}{i\eta^{11}} \right] \text{vec} \\
&\cdot \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u)\theta_1(\epsilon_{1,2})^3\theta_4(\epsilon_{1,2})\theta_1(\epsilon_{1,2} \pm u)\theta_4(\epsilon_{1,2} \pm u)} \right] \text{sym} \cdot \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)\theta_1(m_l)\theta_4(m_l)}{\eta^4} \right] \text{Fermi}
\end{align*}
\]

\[
\begin{align*}
Z_{4(4)'} &= \int \left[ \eta^2 du \cdot \frac{\theta_1(2\epsilon_+u)\theta_4(2\epsilon_+u)\theta_2(2\epsilon_+u)\theta_4(2\epsilon_+u)\theta_4(0)\theta_2(\pm u)\theta_3(\pm u)}{i\eta^{11}} \right] \text{vec} \\
&\cdot \left[ \frac{\eta^{20}}{\theta_1(\epsilon_{1,2} \pm 2u)\theta_1(\epsilon_{1,2})^3\theta_4(\epsilon_{1,2})\theta_2(\epsilon_{1,2} \pm u)\theta_3(\epsilon_{1,2} \pm u)} \right] \text{sym} \cdot \left[ \prod_{l=1}^{8} \frac{\theta_1(m_l \pm u)\theta_2(m_l)\theta_3(m_l)}{\eta^4} \right] \text{Fermi}
\end{align*}
\]

where the holonomy \((a_1, a_2, a_3, a_4) = (u, -u, 0, \frac{1}{2})\) and \((u, -u, \frac{5}{2}, \frac{1}{2})\) are used for \(Z_{4(4)}\) and \(Z_{4(4)'}\), respectively.

We also need to specify the residues which contribute to the above contour integrals. For the rank 1 cases, one just keeps all poles and residues associated with positively charged chiral fields. So for \(Z_{4(i)}\) with \(i = 2, 3, 4\), the relevant poles are at \(u_* = -\frac{\epsilon_1^2}{2} + \frac{\epsilon_2}{2}\), where \(p\) runs over \((p_1, p_2, p_3, p_4) = (0, 1, 1 + \tau, \tau)\), and \(u_* = -\epsilon_{1,2}, -\epsilon_{1,2} + \frac{m_2}{2}\). For \(Z_{4(i)'}\) with \(i = 2, 3, 4\), the poles are at \(u_* = -\frac{\epsilon_1^2}{2} + \frac{\epsilon_2}{2}\), again with \(p\) running over \((p_1, p_2, p_3, p_4) = (0, 1, 1 + \tau, \tau)\), and at \(u_* = -\epsilon_{1,2} + p_j\) with two possible values of \(j \neq 1, i\). The resulting residue sums are given by

\[
\begin{align*}
Z_{4(2)} &= \frac{1}{2} \sum_{i=1}^{4} \frac{\theta_2(\epsilon_1 + \epsilon_2)\theta_2\theta_2(\epsilon_2)\theta_2(\epsilon_2)\theta_2(\epsilon_2)\theta_2(\epsilon_1)\theta_2(\epsilon_1)\theta_2(\epsilon_1)\theta_2(\epsilon_1)}{\eta^{24}\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)} \theta_2(\epsilon_2) \theta_2(\epsilon_2) \theta_2(\epsilon_2) \\
&\quad \times \left( \prod_{j=1}^{4} \theta_2(m_j \pm \epsilon_1) \right) \frac{\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)}{\eta^{24}\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)} (\epsilon_1 \leftrightarrow \epsilon_2)
\end{align*}
\]

\[
\begin{align*}
Z_{4(2)'} &= \frac{1}{2} \sum_{i=1}^{4} \frac{\theta_2(\epsilon_1 + \epsilon_2)\theta_2\theta_2(\epsilon_1)\theta_2(\epsilon_1)\theta_2(\epsilon_1)\theta_2(\epsilon_1)\theta_2(\epsilon_1)\theta_2(\epsilon_1)}{\eta^{24}\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)\theta_1(2\epsilon_1)} \theta_2(\epsilon_2) \theta_2(\epsilon_2) \theta_2(\epsilon_2) \\
&\quad \times \left( \prod_{j=1}^{4} \theta_2(m_j \pm \epsilon_1) \right) \frac{\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)\theta_2(\epsilon_1 \pm \epsilon_1)}{\eta^{24}\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)\theta_1(3\epsilon_1)} (\epsilon_1 \leftrightarrow \epsilon_2)
\end{align*}
\]
where $\sigma_i$ are defined as (3.38). The expressions for $Z_{4(i)}$ and $Z_{4(i')}$ with $i = 3, 4$ are obtained by permuting the roles of the subscripts 2, 3, 4 of the theta functions and $\sigma_i$.

The rank 2 contour integral in $Z_{4(1)}$ can be done as follows. The charges of the $(0, 2)$ chiral multiplets, responsible for the poles in the integrand, are $\pm 2e_1$, $\pm e_1 \pm e_J$ ($I \neq J$) with $I, J = 1, 2$. We choose the vector $\eta$ to be in the cone between $e_1 + e_2$ and $2e_2$. Then, the poles with nonzero Jeffrey-Kirwan residues (after eliminating the fake poles due to vanishing numerators from Fermi multiplets) are at the following 104 positions:

\begin{align}
(1) : & \quad 2u_2 + \epsilon = 0, \quad u_1 + u_2 + \epsilon' = 0 \rightarrow u_2 = -\frac{\epsilon}{2} + \frac{p_i}{2}, \quad u_1 = -\epsilon' + \frac{\epsilon}{2} + \frac{p_i}{2} \quad (3.53) \\
(2) : & \quad 2u_2 + \epsilon = 0, \quad 2u_1 + \epsilon = 0 \rightarrow u_2 = -\frac{\epsilon}{2} + \frac{p_i}{2}, \quad u_1 = -\frac{\epsilon}{2} + \frac{p_i}{2} \quad (p_i \neq p_j) \\
(3) : & \quad 2u_2 + \epsilon = 0, \quad 2u_1 + \epsilon' = 0 \rightarrow u_2 = -\frac{\epsilon}{2} + \frac{p_i}{2}, \quad u_1 = -\frac{\epsilon'}{2} + \frac{p_i}{2} \\
(4) : & \quad 2u_2 + \epsilon = 0, \quad u_1 - u_2 + \epsilon = 0 \rightarrow u_2 = -\frac{\epsilon}{2} + \frac{p_i}{2}, \quad u_1 = -\frac{3\epsilon}{2} + \frac{p_i}{2} \\
(5) : & \quad u_2 - u_1 + \epsilon = 0, \quad u_1 + u_2 + \epsilon = 0 \rightarrow u_2 = -\epsilon + \frac{p_i}{2}, \quad u_1 = 0 + \frac{p_i}{2} \\
(6) : & \quad u_2 - u_1 + \epsilon = 0, \quad u_1 + u_2 + \epsilon' = 0 \rightarrow u_2 = -\epsilon + \frac{\epsilon'}{2} + \frac{p_i}{2}, \quad u_1 = -\epsilon + \frac{\epsilon'}{2} + \frac{p_i}{2} \\
(7) : & \quad u_2 - u_1 + \epsilon = 0, \quad 2u_1 + \epsilon = 0 \rightarrow u_2 = -\frac{3\epsilon}{2} + \frac{p_i}{2}, \quad u_1 = -\frac{\epsilon}{2} + \frac{p_i}{2} \\
(8) : & \quad -2u_1 + \epsilon = 0, \quad u_1 + u_2 + \epsilon = 0 \rightarrow u_1 = -\frac{\epsilon}{2} + \frac{p_i}{2}, \quad u_2 = -\frac{3\epsilon}{2} + \frac{p_i}{2}.
\end{align}

We defined $(p_1, p_2, p_3, p_4) = (0, 1, 1 + \tau, \tau)$. $\epsilon$ can be either $\epsilon_1$ or $\epsilon_2$, and $\epsilon' \neq \epsilon$ is the remaining parameter. In the second case, the four cases with $p_i = p_j$ do not provide poles since there are vanishing factors in the numerator. One can check that these poles are all non-degenerate.

The residue sums from these 8 cases are given by (the sectors labeled by (4), (7), (8) yield same result, shown on the second line)

\begin{align}
(1) : & \quad \sum_{i=1}^{4} \theta_1(2\epsilon_1 + \epsilon_2)\theta_1(-\epsilon_1) \prod_i \theta_i(m_i \pm (\epsilon_1 - \frac{\epsilon}{2}))\theta_i(m_i \pm \frac{\epsilon}{2}) \\
& \quad \quad \quad \quad \frac{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_2 - \epsilon_1)\theta_1(3\epsilon_1 - \epsilon_2)\theta_1(2\epsilon_2 - 2\epsilon_1)}{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_2 - \epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2) \\
(4) : & \quad \sum_{i=1}^{4} \theta_1(2\epsilon_1 + \epsilon_2)\theta_1(-\epsilon_1) \prod_i \theta_i(m_i \pm \frac{\epsilon}{2})\theta_i(m_i \pm \frac{3\epsilon}{2}) \\
& \quad \quad \quad \quad \frac{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(3\epsilon_1)\theta_1(4\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_2 - 3\epsilon_1)\theta_1(3\epsilon_1)}{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(3\epsilon_1)\theta_1(4\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_2 - \epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2) = (7) = (8) \\
(5) : & \quad \sum_{i=1}^{4} \theta_1(2\epsilon_1 + \epsilon_2)\theta_1(-\epsilon_1) \prod_i \theta_i(m_i)\theta_i(m_i \pm \epsilon_1) \\
& \quad \quad \quad \quad \frac{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(3\epsilon_1)\theta_1(\epsilon_2 - 2\epsilon_1)\theta_1(2\epsilon_2 - 3\epsilon_1)}{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(3\epsilon_1)\theta_1(\epsilon_2 - 2\epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2) \\
(6) : & \quad \sum_{i=1}^{4} \theta_1(2\epsilon_1 + \epsilon_2)\theta_1(-\epsilon_1) \prod_i \theta_i(m_i \pm \frac{\epsilon_1 + \epsilon_2}{2})\theta_i(m_i \pm \frac{\epsilon_1 - \epsilon_2}{2}) \\
& \quad \quad \quad \quad \frac{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_1 - \epsilon_2)\theta_1(2\epsilon_2)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_1 - \epsilon_2)\theta_1(2\epsilon_2 - \epsilon_1)}{2\eta^{24}\theta_1(\epsilon_1, 2\delta_1)\theta_1(2\epsilon_1)\theta_1(\epsilon_1 - \epsilon_2)\theta_1(2\epsilon_2)\theta_1(\epsilon_2 - \epsilon_1)\theta_1(2\epsilon_1 - \epsilon_2)\theta_1(2\epsilon_2 - \epsilon_1)}
\end{align}
and
\[
(2) : \left[ \theta_2(0) \theta_2(-\epsilon_1) \theta_2(\epsilon_1 + \epsilon_2) \theta_2(2\epsilon_1 + \epsilon_2) \left( \prod \theta_1(m_l \pm \frac{\tau}{2}) \theta_2(m_l \pm \frac{\tau}{2}) + \prod \theta_3(m_l \pm \frac{\tau}{2}) \theta_4(m_l \pm \frac{\tau}{2}) \right) \\
+ (2, 3, 4 \to 3, 4, 2) + (2, 3, 4 \to 4, 2, 3) \right] + (\epsilon_1 \leftrightarrow \epsilon_2)
\]
(3.55)

(3) : \[
\prod_{i,j=1}^{4} \frac{\theta_j(m_l \pm \frac{i}{2}) \theta_i(m_l \pm \frac{j}{2})}{2\eta^{24} \theta_1(\epsilon_1) \theta_1(2\epsilon_1) \theta_1(\epsilon_2 - \epsilon_1) \theta_1(2\epsilon_2) \theta_1(\epsilon_1 - \epsilon_2)} \theta_{\sigma_j(i)}(-\epsilon_1+\epsilon_2) \theta_{\sigma_j(i)}(\frac{3\epsilon_1+\epsilon_2}{2}) \theta_{\sigma_j(i)}(\frac{3\epsilon_1-\epsilon_2}{2})
\]
(3.56)

The full index is given by
\[
Z_4 = \frac{1}{8} \sum_{i=1}^{4} Z_{4(i)} + \frac{1}{8} \sum_{i=2}^{4} Z_{4(i)'} + \frac{1}{16} Z_{4(1)''},
\]
(3.57)

with the Weyl factors given by (3.9).

We test our results against various known ones. We first consider the case in which one sets
\[
\epsilon_1 = -\epsilon_2 \equiv \epsilon, \quad m_1 = m_2 = 0, m_3 = m_4 = \frac{1}{2}, \quad m_5 = m_6 = -\frac{1+\tau}{2}, \quad m_7 = m_8 = \frac{\tau}{2}.
\]
(3.58)

This case was considered recently in [33]. In particular, [33] wrote down the concrete forms of the elliptic genera in this limit for 2 and 4 E-strings. The case with 2 E-strings is a special case of [7], so also agrees with our results. The index of [33] at (3.58) is always zero for odd number of E-strings. By plugging in (3.58) to our 3 E-string indices in the previous subsection, all \(Z_{3(i)}, Z_{3(i)'}\) are identically zero, agreeing with the results of [33]. Now let us study our 4 E-string index. Plugging in (3.58), one finds that the contributions from the seven sectors are zero, and the only nonzero contribution is \(Z_{4(1)}\). The surviving contributions are
\[
(1) = (4) = (7) = (8) = \frac{4 \prod_{i=1}^{4} \theta_1(3\epsilon/2)^4 \theta_1(\epsilon/2)^4}{\eta^{24} \theta_1(\epsilon)^2 \theta_1(2\epsilon)^2 \theta_1(3\epsilon)^2 \theta_1(4\epsilon)^2}
\]
(3.59)

while (5), (6) become zero. So one obtains
\[
Z_{4(1)} = \frac{16 \prod_{i=1}^{4} \theta_i(\frac{3\epsilon}{2})^4 \theta_i(\frac{\epsilon}{2})^4}{\eta^{24} \theta_1(\epsilon)^2 \theta_1(2\epsilon)^2 \theta_1(3\epsilon)^2 \theta_1(4\epsilon)^2} + \frac{4 \prod_{i=1}^{4} \theta_i(\frac{3\epsilon}{2})^8}{\eta^{24} \theta_1(\epsilon)^4 \theta_1(2\epsilon)^4} \left[ \frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2} + \frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2} + \frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2} \right]
\]
\[
= \frac{16 \theta_1(\epsilon)^2 \theta_1(3\epsilon)^2}{\theta_1(2\epsilon)^2 \theta_1(4\epsilon)^2} + \frac{4 \theta_1(\epsilon)^4}{\theta_1(2\epsilon)^4} \left[ \frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2} + \frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2} + \frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2} \right].
\]
(3.60)

The four E-string index at (3.58) is given in [33] by
\[
\frac{\theta_1(\epsilon)^{20}}{2\eta^{48} \theta_1(2\epsilon)^2 \theta_1(4\epsilon)^2} \left[ 72(\varphi')^4 \varphi^2 - 18(\varphi'')^2(\varphi')^2 \varphi + 2(\varphi')^4 + (\varphi'')^4 \right],
\]
(3.61)

where \(\varphi(\tau, \epsilon)\) is the Weierstrass’s elliptic function. We checked that this agrees with our index \(\frac{1}{8} Z_{4(1)}\) in a serious expansion in \(q\) for the first 11 terms, up to and including \(O(q^{10})\).
We also compare our result with the genus expansion, at \( m_l = 0 \) and \( \epsilon_1 = -\epsilon_2 = \epsilon \). Our indices become

\[
Z_{4(1)} = \sum_{i=1}^{4} \left[ \frac{4\theta_{2}(\frac{\pi}{\sqrt{2}})\theta_{4}(\frac{\pi}{\sqrt{2}})^{16}}{\eta^{24}\theta_{1}(\epsilon)^{2}\theta_{3}(2\epsilon)^{2}\theta_{1}(3\epsilon)^{2}\theta_{1}(4\epsilon)^{2}} + \frac{2\theta_{2}(0)^{16}\theta_{4}(\epsilon)^{16}}{\eta^{24}\theta_{1}(\epsilon)^{2}\theta_{3}(2\epsilon)^{2}\theta_{1}(3\epsilon)^{2}} \right] + \frac{2}{\eta^{24}\theta_{1}(\epsilon)^{4}\theta_{1}(2\epsilon)^{4}} \left[ \frac{\theta_{2}(0)^{2}\theta_{3}(\frac{\pi}{\sqrt{2}})^{16}\theta_{2}(\frac{\pi}{\sqrt{2}})^{16} + \theta_{3}(\frac{\pi}{\sqrt{2}})^{16}\theta_{4}(\frac{\pi}{\sqrt{2}})^{16}}{\theta_{2}(2\epsilon)^{2}} + (3, 4, 2) + (4, 2, 3) \right]
\]

\[
Z_{4(2)\prime} = \sum_{i=1}^{4} \frac{\theta_{2}(0)^{2}\theta_{3}(\frac{\pi}{\sqrt{2}})^{2}\theta_{3}(i)^{2}\theta_{4}(0)^{8}\theta_{4}(1)^{8}\theta_{4}(\frac{\pi}{\sqrt{2}})^{16}}{\eta^{24}\theta_{1}(2\epsilon)^{2}\theta_{2}(\epsilon)^{2}\theta_{3}(\frac{\pi}{\sqrt{2}})^{4}\theta_{3}(\frac{\pi}{\sqrt{2}})^{2}} + \frac{2\theta_{2}(\epsilon)^{2}\theta_{3}(0)^{8}\theta_{4}(0)^{8}\theta_{3}(\epsilon)^{16} + \theta_{4}(\epsilon)^{16}}{\eta^{24}\theta_{1}(3\epsilon)^{2}\theta_{1}(2\epsilon)^{2}\theta_{1}(\epsilon)^{4}\theta_{2}(2\epsilon)^{2}} ,
\]

\( Z_{4(1)\prime} = 0, Z_{4(2)} = Z_{4(3)} = Z_{4(4)} = 0 \), and \( Z_{4(3)\prime}, Z_{4(4)\prime} \) are obtained from \( Z_{4(2)\prime} \) by changing the roles of 2, 3, 4 appearing in the subscripts of the theta functions and \( \sigma_{2}(i), \sigma_{3}(i), \sigma_{4}(i) \). We first confirmed numerically the agreement with \( F^{(0, g, 4)} \) computed from topological strings for \( g \leq 5 \) till \( q^{3} \), by checking the first 10 terms in the serious expansion in \( q \). We also exactly checked the agreements of \( F^{(0, 0, 4)}, F^{(0, 1, 4)}, F^{(0, 2, 4)} \). See appendix C for the details.

### 3.5 Higher E-strings

The computation of the elliptic genus using the methods of [24] quickly becomes complicated for higher rank gauge groups. In general, there could be a fundamental complication due to some poles failing to be projective. But we showed at the beginning of this section that this does not happen in our cases. With higher rank, the computational problem is that there is a large number of poles and residues to be considered. For \( U(n) \) indices, the possible poles are often completely classified by the so-called ‘colored Young diagrams,’ with a \( U(n) \) adjoint and several fundamental \((0, 4)\) hypermultiplets. This classification first appeared in the context of instanton counting [42,43], which was reproduced recently in the context of Jeffrey-Kirwan residues [34]. The resulting residues are often nicely arranged into a reasonably compact form [44,45]. However, for other gauge groups, we are not aware of systematic classifications of poles.\footnote{The pole structure of our \( O(n) \) index is similar to that of the \( Sp(N) \) instanton partition function, whose ADHM quantum mechanics comes with \( O(n) \) group for \( n \) instantons. The poles in our E-string index could be slightly simpler, because we only have \( O(n) \) symmetric hypermultiplets while the ADHM mechanics also has extra \( N \) fundamental hypermultiplets. In either case, we do not know the pole classification, apart from the basic rule given by the Jeffrey-Kirwan residues.}

In this subsection, we shall illustrate the pole structures for some higher E-strings, with \( O(5), O(6), O(7), O(8) \) gauge groups, and also make some qualitative classifications of these poles. Since the purpose is to illustrate the computations for higher ranks, we only consider the branch of \( O(n) \) holonomy with maximal number of continuous parameters, in the \((ee)\) sector.

We start by studying the \( O(5) \) index, for five E-strings. Taking \( \eta = \epsilon_{1} + \varepsilon\epsilon_{2} \) with \( 0 < \varepsilon \ll 1 \),
the following pair of weights \( \{\rho_1, \rho_2\} \) can potentially give nonzero JK-Res:

\[
\{2e_1, 2e_2\}, \{2e_1, e_2\}, \{2e_1, e_2 \pm e_1\}, \{e_1, 2e_2\}, \{e_1, e_2\}, \{e_1, e_2 \pm e_1\} \quad (3.63)
\]

\[
\{e_1 - e_2, 2e_2\}, \{e_1 - e_2, e_1 + e_2\}, \{e_1 - e_2, e_2\}, \{e_1 + e_2, -2e_2\}, \{e_1 + e_2, -e_2\}.
\]

These poles define the pole \( u_* \) by hyperplanes \( \rho_i(u_*) + z_i = 0 \) for suitable \( z_i \), chosen between \( e_1, e_2 \). Considering all possible values of \( u_* \), we find 142 poles, which are all non-degenerate. The evaluation of residue sum should be marginally more laborious than the \( O(4) \) case.

Next, we consider the \( O(6) \) contour integral. The poles come from the scalar fields with charges \( \pm 2e_I, \pm e_I \pm e_J \). We choose \( \eta \) to be \( \eta = e_1 + \varepsilon e_2 + \varepsilon^2 e_3 \) with \( 0 < \varepsilon \ll 1 \). The groups of 3 vectors which contain \( \eta \) in their cones are

\[
\{2e_1, 2e_2, 2e_3\}, \{2e_1, 2e_2, e_3 \pm e_1, 2\}, \{2e_1, 2e_3, e_2 \pm e_3\}, \{2e_1, e_2, e_2 \pm e_3\}, \{2e_1, -2e_3, e_2 + e_3\},
\]

\[
\{2e_1, e_2 \pm e_1, e_3 \pm e_2\}, \{2e_1, e_2 \pm e_1, e_3 \pm e_2\}, \{2e_1, e_2 \pm e_1, e_3 \pm e_2\}, \{2e_1, -e_3 \pm e_1, e_2 + e_3\},
\]

\[
\{2e_2, 2e_3, e_1 - e_3\}, \{2e_2, 2e_3, e_1 - e_3\}, \{2e_2, 2e_3, e_1 - e_3\}, \{2e_2, 2e_3, e_1 - e_3\}, \{2e_2, 2e_3, e_1 - e_3\},
\]

\[
\{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\},
\]

\[
\{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\},
\]

\[
\{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\},
\]

\[
\{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}, \{2e_3, e_1 \pm e_2, e_2 \pm e_3\}.
\]

(3.64)

With these chosen \( \{\rho_1, \rho_2, \rho_3\} \), the hyperplanes \( \rho_i(u_*) + z_i = 0 \) with \( i = 1, 2, 3 \) meet at a point \( u_* \) with suitable choices of \( z_i \), which are either \( e_1 \) or \( e_2 \). There may exist more than the chosen three hyperplanes which meet at the same point \( u_* \), in which case we have degenerate poles. Also, at some \( u_* \) there could be some vanishing theta functions in the numerator. Let us call the number of vanishing theta functions from the numerator and denominator as \( N_n(u_*) \) and \( N_d(u_*) \), respectively. When \( N_d - N_n < r = 3 \), then the corresponding \( u_* \) is not a pole due to too many vanishing terms in the numerator. The list below covers all the poles which have nonzero JK-Res, also provided with some illustrations on how to evaluate the residues:

1. When \( N_d = 3, N_n = 0 \), this is a non-degenerate and simple pole. We find 1680 poles in this class. Near \( u = u_* \), the integrand relevant for evaluating the residue approximately takes the form of

\[
\frac{1}{\prod_{i=1}^3(\rho_i(u) - \rho(u_*))} \cdot F(u_*) \quad , \quad (3.65)
\]

28
where $F(u)$ denotes the rest of the integrand, with $F(u_s) \neq 0$. The integral of the first factor of (3.65) can be immediately obtained from the basic definition (3.17).

2. There could be degenerate poles with $N_d = N_n + r$, $N_n \neq 0$. The leading divergences of the integrands are simple poles in this case, since $N_d - N_n = r$. Near the pole, the integrand relevant for computing the residue approximately takes the form of

$$\prod_{i=1}^{N_n} (\rho_i(u) - \rho_i(u_*)) \prod_{i=N_n+1}^{N_d+2N_n} (\rho_i(u) - \rho_i(u_*)) \cdot F(u_*), \quad (3.66)$$

where $F(u)$ is the rest of the integrand. The basic rule (3.17) has to be applied to the first factor of (3.66) after decomposing it into a linear combination of the expressions appearing in (3.17). In the $O(6)$ case with $r = 3$, we find two subclasses. Firstly, we find 104 poles with $N_d = 4$, $N_n = 1$. For all the poles in this class, we find

$$\text{JK-Res} \frac{\rho_1(u) - \rho_1(u_*)}{\prod_{i=2}^{2N_n}(\rho_i(u) - \rho_i(u_*))} = \frac{1}{2}, \quad (3.67)$$

thus all with nonzero residues. We illustrate how this is evaluated with an example among the 104 poles, defined with \{\rho_1, \rho_2, \rho_3, \rho_4\} = \{e_1 - e_2, e_1 + e_2, e_1 + e_3, -e_2 - e_3, -2e_2\}:

$$\text{JK-Res} \frac{\bigwedge_{a=1}^{3} du_a \cdot (\epsilon_1 + \epsilon_2 + u_1 - u_2)}{(\epsilon_1 - 2u_2)(\epsilon_2 + u_1 + u_2)(\epsilon_2 - u_2 - u_3)(\epsilon_1 + u_2 + u_3)} = \text{JK-Res} \frac{\bigwedge_{a=1}^{3} d\tilde{u}_a}{(\tilde{u}_1 + \tilde{u}_2)(-\tilde{u}_2 - \tilde{u}_3)} \left( \frac{1}{\tilde{u}_1 + \tilde{u}_2} + \frac{1}{-2\tilde{u}_2} \right) = \frac{1}{2} + 0 = \frac{1}{2}, \quad (3.68)$$

where $\tilde{u} = u - u_*$. Moreover, we find 72 poles with $N_d = 5$, $N_n = 2$, in which case we find either

$$\text{JK-Res} \frac{(\rho_1(u) - \rho_1(u_*))(\rho_2(u) - \rho_2(u_*))}{\prod_{i=3}^{7}(\rho_i(u) - \rho_i(u_*))} = \quad (3.69)$$

0 (32 cases), $-\frac{1}{4}$ (16 cases), $\frac{1}{4}$, (16 cases) $\frac{1}{2}$ (8 cases).

Thus we find 40 more poles. There are no more poles in this class with larger $N_d, N_n$.

3. In general, there could be degenerate poles with $N_d > N_n + r$. The integrand contains ‘multiple poles’ in this case. The integrand takes the form of

$$\prod_{i=1}^{N_n} \theta_1(\rho_i(u) - \rho_i(u_*)) \prod_{i=N_n+1}^{N_d+N_n} \theta_1(\rho_i(u) - \rho_i(u_*)) \cdot F(u), \quad (3.70)$$

where $F(u)$ is a combination of $\theta_1$ functions which are nonzero at $u_*$. Since the first factor contains multiple poles, one would have to expand both first and second factors to certain orders near $u = u_*$, until one obtains a linear combination of the functions appearing in (3.17). The residue will thus be expressed by $\theta_1$ functions and their suitable derivatives at $u_*$. This class of poles do not show up in the $O(6)$ case. (They will first appear in the $O(8)$ index, explained below.)
With the above $1680 + 104 + 40 = 1824$ poles and the computational rules stated in the list, clearly the $O(6)$ elliptic genus can be computed straightforwardly, although the resulting expression will be very long.

Let us explain the pole/residue structures of $O(7)$ index, with rank $r = 3$. The poles are again classified into the above three classes. To be definite, we chose $\eta = e_1 + \varepsilon e_2 + \varepsilon^2 e_3$. We simply list the number poles in each class.

1. non-degenerate poles ($N_d = 3, N_n = 0$): 2468 cases
2. degenerate (but simple) poles: With $N_d = 4, N_n = 1$, we find 106 degenerate and simple poles. The relevant integrals of the form of (3.67) are either $\frac{1}{2}$ or 1, depending on $u_*$. With $N_d = 5, N_n = 2$, we find 72 cases. The integral analogous to (3.69) are either $0, -\frac{1}{4}, \frac{1}{2}, \frac{1}{2}$. There are 32 cases with zero residues. So we find 40 poles in this class. Finally, there are 4 cases with $N_d = 6, N_n = 3$, and the JK-Res of the rational functions are either

$$\text{JK-Res} \frac{\Lambda^r_{n=1} d\tilde{u}_a \cdot \prod_{i=1}^3 \rho_i(\tilde{u})}{\prod_{i=4}^{r+6} \rho_i(\tilde{u})} = \frac{1}{2} \ (2 \text{ cases}), \text{ or } 0 \ (2 \text{ cases}). \quad (3.71)$$

So we have 2 poles in the last class. We do not find further degenerate simple poles with larger $N_n$.

3. degenerate multiple poles ($N_d > N_n + 3$): We do not find any poles in this case.

So we find $2468 + 106 + 40 + 2 = 2616$ poles with nonzero JK-Res.

As a final illustration, let us consider the $O(8)$ contour integral with rank $r = 4$. The number of poles quickly increases, as follows:

1. non-degenerate poles ($N_d = 4, N_n = 0$): 32304 poles
2. degenerate (but simple) poles: With $N_d = 5, N_n = 1$, we find 4424 poles. With $N_d = 6, N_n = 2$, we find 1696 poles. With $N_d = 7, N_n = 3$, we find 88 poles. Finally, with $N_d = 8, N_n = 4$, we finds 200 poles.
3. degenerate multiple poles ($N_d > N_n + 3$): We find 72 such poles.

So we find $32304 + 4424 + 1696 + 88 + 200 + 72 = 38784$ poles for the $O(8)$ contour integral.

## 4 E-strings from Yang-Mills instantons

In this section, we explain how one can alternatively compute the E-string elliptic genus from the instanton partition function of a suitable 5 dimensional super-Yang-Mills theory with $Sp(1)$
gauge group. The basic idea is that suitable circle reductions of 6d SCFTs sometimes admit 5d SYM descriptions at low energy. The latter SYM, despite being non-renormalizable, remembers the 6d KK degrees in its solitonic sector as the instanton solitons \[46, 47\]. The self-dual strings wrapping the circle become the W-bosons, quarks or their superpartner particles in 5d. So the Witten index for the threshold bounds of these particles with instantons in the Coulomb branch \[42, 43\] will carry information on the elliptic genera of wrapped self-dual strings. This idea has been used to study the elliptic genus of M-strings in the 6d \((2, 0)\) SCFT in \[8, 5\]. In this section, we make a similar study for the E-strings. Since the circle reduction of the \(E_8\) \((1, 0)\) SCFT is much subtler than that of the \((2, 0)\) theory, let us set up the problem first.

We start by considering the type IIA system consisting of 8 D8-branes and an O8-plane (or 16 D8-branes in the covering space), making a type I’ string background. The D8-branes are at the tip of the half-line \(\mathbb{R}^+\), formed by an O8. The worldvolume of the 8-branes hosts \(SO(16)\) gauge symmetry. Since the net 8-brane charges cancel, the asymptotic value of the dilaton on \(\mathbb{R}^+\) is a nonzero constant. So this system admits an M-theory uplift at strong coupling, on \(\mathbb{R}^{8+1} \times \mathbb{R}^+ \times S^1\). The D0-branes in the type I’ theory are identified as the Kaluza-Klein modes along the M-theory circle. In the uplifted background, an M9-plane (or the Horava-Witten wall) is located at the tip of \(\mathbb{R}^+\) and wraps \(\mathbb{R}^{8+1} \times S^1\). The M9-plane hosts an \(E_8\) gauge symmetry. When the M9 wraps a circle, one can turn on nonzero \(E_8\) Wilson line which reduces gauge symmetry. To get a background which admits a weakly coupled type I’ description with unbroken \(SO(16)\) gauge symmetry, one should turn on the Wilson line as follows. Let \(R\) be the radius of the M-theory circle, and \(A\) be the \(E_8\) gauge field on the circle. \(E_8\) has an \(SO(16)\) subgroup, in which the adjoint representation \(248\) of \(E_8\) decomposes into \(120 \oplus 128\). The Wilson line \(RA\) that we turn on in \(SO(16) \subset E_8\) is given by \[13\]

\[
RA = (0, 0, 0, 0, 0, 0, 0, 1) .
\]

This is in the convention that one picks the Cartans of \(SO(16)\) as rotations on the 8 orthogonal 2-planes. The circle holonomy generated by this Wilson line is \(\exp(2\pi i RA \cdot F)\), with \(F = (F_1, F_2, \ldots, F_8)\) being the Cartans of \(SO(16) \subset E_8\) in the same basis. The normalization is \(F_i = \pm \frac{1}{2}\) for \(SO(16)\) spinors. The holonomy with \(R,A\) acts on \(128\) as \(-1\), and on \(120\) as \(+1\). So \(E_8\) symmetry breaks down to \(SO(16)\). This is the background which admits the type I’ theory description for small \(R\).

Now let us consider the D4-D8-O8 system, by adding \(N\) D4-branes. This uplifts in M-theory to the M5-M9-branes wrapping the circle, in the above \(E_8\) Wilson line background. On the worldvolume of D4-branes, one obtains an \(Sp(N)\) gauge theory with 1 antisymmetric and 8 fundamental hypermultiplets.\(^3\) This 5d gauge theory is a low-energy description of the 6d \((1, 0)\) superconformal field theory compactified on a circle with \(E_8\) Wilson line. Note that, from the

\(^3\)Had one been reducing the M5-M9 system with zero Wilson line, one would have obtained the strongly interacting 5d SCFT with \(E_8\) symmetry \[19, 48\], discovered in \[49\].
worldvolume theory on D4 or M5-branes, \( SO(16) \) or \( E_8 \) act as global symmetries. So from the 5d/6d field theories, the Wilson line we explained above are nondynamical background fields.

Consider the system consisting of single M5-brane and an M9-plane, compactified on a circle with the above Wilson line. We have an \( Sp(1) \) gauge theory description in 5d. Taking into account the effect of the background Wilson line (4.1), we can identify various charges of the 5d SYM theory and the 6d \((1,0)\) theory on circle as follows:

\[
k = 2P + n(RA \cdot RA) - 2 \left( RA \cdot \tilde{F} \right) = 2P + n - 2\tilde{F}_8 \tag{4.2}
\]

\[
F_i = \tilde{F}_i - n(RA_i) \quad \rightarrow \quad F_8 = \tilde{F}_8 - n . \tag{4.3}
\]

Here, \( k \) is the Yang-Mills instanton charge on D4's (i.e. D0-brane number in the type I' theory), \( P \) is the momentum on E-strings along the circle, \( \tilde{F} \) is the \( E_8 \) Cartan charge in the 6d theory, and \( F \) is the \( SO(16) \) Cartan charges in the 5d SYM. \( n \) is the \( U(1) \subset Sp(1) \) electric charge in the Coulomb phase, which is identified as the winding number of the E-strings. This formula can be naturally inferred by starting from the charge relations of the fundamental type I' strings on \( \mathbb{R}^{8+1} \times I \) and the heterotic strings on \( \mathbb{R}^{8+1} \times S^1 \) [50, 51], where \( I \) is a segment, and then putting an M5-brane on \( I \) to decompose a heterotic string into two E-strings [7].

Later in this section, we shall consider an index for the E-strings, with the weight given by

\[
q^k e^{2\pi im_8 F_8} w^n \prod_{i=1}^{7} e^{2\pi im_i F_i} = q^{2P} (y'_8)^{\tilde{F}_8} (w')^n \prod_{i=1}^{7} e^{2\pi im_i \tilde{F}_i} \tag{4.4}
\]

with \( y_i \equiv e^{2\pi im_i} \), where

\[
y'_8 = y_8 q^{-2} , \quad w' = w q y_8^{-1} . \tag{4.5}
\]

The right hand side is the natural expression for the E-strings, while the instanton calculus will naturally use the expression on the left hand side. After doing the instanton calculus with the above weight, we shall use the fugacities \( y'_8, w' \) given by (4.3). This redefinition of fugacities plays the role of canceling the background \( E_8 \) Wilson line (4.1), which obscures the \( E_8 \) symmetry in the type I' instanton calculus.

Since the ADHM quantum mechanics is a UV completion of the 5d instanton quantum mechanics, it contains extra string theory degrees of freedom apart from the QFT degrees. So the partition function of the ADHM quantum mechanics may acquire contributions from the extra string theory states in the D4-D8-O8 background. Since the 5d/6d quantum field theories are obtained from the string theory background by taking low energy decoupling limit, the Hilbert space of this system factorizes at low energy. In particular, in the context of the

\[\text{Only in this section, the definition of } q \text{ is given by } q = e^{\pi i \tau}, \text{ instead of } q = e^{2\pi i \tau} \text{ used in all other sections of this paper. This is because the single instanton carries } q^{1/2} \text{ factor in the other convention, due to the fractional Wilson line, which we want to change to } q^1. \text{ This is the reason for the factor } q^{2P} \text{ in (4.4).}\]
Witten index of the ADHM quantum mechanics, one expects

\[ Z_{\text{ADHM}} = Z_{\text{QFT}} \cdot Z_{\text{other}}. \] (4.6)

The quantity of our interest is \( Z_{\text{QFT}} \). The unwanted factor \( Z_{\text{other}} \) was identified in [34]. For the purpose of studying the QFT spectrum, we simply divide the ADHM quantum mechanics partition function by \( Z_{\text{other}} \) identified in [34], to obtain \( Z_{\text{QFT}} \).

We will consider \( Z_{\text{QFT}} \) of the 5d \( Sp(1) \) gauge theory, i.e., the rank 1 6d (1, 0) SCFT compactified on circle with \( E_8 \) Wilson line. To see the E-string physics, for instance the \( E_8 \) symmetry, one should make a replacement (4.5) in the instanton partition function. The instanton partition function takes the form of series expansion in \( q \), \( Z(q, w, y) = \sum_{k=0}^{\infty} Z_k(w, y) q^k \). So at a given order in the modular parameter \( q \), one captures the spectrum of arbitrary number of E-strings by computing \( Z_k(w, y) \) exactly in \( w \). This is in contrast to our previous study of the E-string elliptic genus, keeping definite order \( Z_n(q, y) \) in \( w \) which is exact in \( q \). So to confirm that the two approaches yield the same result, we shall make a double expansion of \( Z(q, w, y) \) in \( q, w \) and compare, taking into account the shifts (4.5). We first take the E-string indices \( Z_n(q, y_k) \) and define \( \tilde{Z}_n(q, y_k') \sim Z_n(q, y_k q^{-2}) \) using (4.5). While making the study of instanton partition function of our \( Sp(1) \) gauge theory in [34], \( Z_k(w, y) \) was computed up to \( k = 5 \). So expanding \( \tilde{Z}_n \) up to \( O(q^5) \), and expanding \( Z_{\text{QFT}} \) computed from the instanton side to \( O(w^n) \) for some low \( n \), we shall find perfect agreement of the two results.

### 4.1 Instanton partition function

To take into account the effect of the Wilson line which breaks \( E_8 \) down to \( SO(16) \), we have to make a shift of the fugacities by (4.5). After inserting \( y_k' = y_k q^{-2} \) (or \( e^{2\pi i m_8} \rightarrow e^{2\pi i m_8 - 2\pi i r} \)) to the E-string indices of section 3, various E-string indices can be written as

\[ Z_1 = \left( \frac{y_8}{q} \right) \tilde{Z}_1, \quad Z_2 = \left( \frac{y_8}{q} \right)^2 \tilde{Z}_2, \quad Z_3 = \left( \frac{y_8}{q} \right)^3 \tilde{Z}_3, \quad Z_4 = \left( \frac{y_8}{q} \right)^4 \tilde{Z}_4 \] (4.7)

with

\[
\begin{align*}
\tilde{Z}_1 &= \frac{1}{2} (-Z_{1(1)} + Z_{1(2)} + Z_{1(3)} - Z_{1(4)}) \\
\tilde{Z}_2 &= \frac{1}{2} Z_{2(0)} + \frac{1}{4} (-Z_{2(1)} - Z_{2(2)} + Z_{2(3)} + Z_{2(4)} - Z_{2(5)} - Z_{2(6)}) \\
\tilde{Z}_3 &= \frac{1}{4} (-Z_{3(1)} - Z_{3(2)} + Z_{3(3)} + Z_{3(4)}) + \frac{1}{8} (-Z_{3(1)'} - Z_{3(2)}' + Z_{3(3)'} + Z_{3(4)'} \\
\tilde{Z}_4 &= \frac{1}{8} (Z_{4(1)} - Z_{4(2)} - Z_{4(2)'} - Z_{4(3)} - Z_{4(3)'} + Z_{4(4)} + Z_{4(4)'}) + \frac{1}{16} Z_{4(1)'},
\end{align*}
\] (4.8)

and so on, where \( Z_{n(i)} \)'s are all defined and computed in section 3. In all \( Z_{n(i)} \) on the right hand side, the arguments are \( y_8 \), not \( y_k' \). The overall factors \( (y_8 q^{-1})^n \) in (4.7) cancels with the
shift $w' = wqy_8^{-1}$ in $Z = \sum_{n=0}^{\infty} (w')^n Z_n$. Namely, the $E_8$ mass shift is inducing a different value of 2d theta angle, by changing various signs in [4.8]. We compute $\tilde{f}(w, q, \epsilon_{1,2}, m_i)$ defined by
\[
\tilde{Z} \equiv \sum_{n=0}^{\infty} w^n \tilde{Z}_n(q, \epsilon_{1,2}, m_i) = PE \left[ \hat{f} \right] \equiv \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \tilde{f}(w^n, q^n, n\epsilon_1, n\epsilon_2, nm_i) \right],
\]
and expand $\tilde{f} = \sum_{n=1}^{\infty} w^n \tilde{f}_n(q, \epsilon_{1,2}, m_i)$. The results up to $O(q^5)$ are as follows.

Defining $t \equiv e^{i\epsilon_1+i\epsilon_2}$, $u \equiv e^{i\epsilon_1-i\epsilon_2}$, $\tilde{f}_1$ is given by
\[
\begin{align*}
+q^0 \cdot & \chi_{16}^{SO(16)} + q^1 \cdot \chi_{128}^{SO(16)} \\
+q^2 \left[ (t + t^{-1})(u + u^{-1}) \chi_{16}^{SO(16)} + \chi_{560}^{SO(16)} + \chi_{16}^{SO(16)} \right] + q^3 \left[ (t + t^{-1})(u + u^{-1}) \chi_{128}^{SO(16)} + \chi_{1920}^{SO(16)} + \chi_{128}^{SO(16)} \right] \\
+q^4 \left[ (t + t^{-1})(u + u^{-1}) \chi_{560}^{SO(16)} + 2\chi_{16}^{SO(16)} \right] + \left( t^2 + 1 + t^{-2} \right)(u^2 + 1 + u^{-2} - 1) \chi_{16}^{SO(16)} \\
+ \chi_{13312}^{SO(16)} + 2\chi_{1920}^{SO(16)} + 4\chi_{128}^{SO(16)} \right] + O(q^6)
\end{align*}
\]

The boldfaced subscripts are the irreps of $SO(16) \subset E_8$ visible by the 5d $Sp(1)$ gauge theory with 8 fundamental flavors. $\chi_{R}^{SO(16)}$ is the $SO(16)$ character of the representation $R$. We computed the $Z_{QFT}$ of the 5d SYM, following the procedures outlined above (explained in [3.4]), up to five instantons. We further expanded it in the Coulomb VEV parameter to extract the $O(w^4)$ order. This completely agrees with (4.10).

\[
\begin{align*}
\tilde{f}_2 \text{ is given by } & \frac{t}{(1-tu)(1-t/u)} \text{ times} \\
- q^0 \cdot & (t + t^{-1}) - q^1 \left[ (t + t^{-1}) \chi_{128}^{SO(16)} \right] \\
- q^2 \left[ (t^3 + t + t^{-1} + t^{-3})(u^2 + 1 + u^{-2}) + (u + u^{-1}) + (t^2 + 1 + t^{-2})(u + u^{-1}) \chi_{120}^{SO(16)} + 1 \right] \\
+ (t + t^{-1})(\chi_{1820}^{SO(16)} + \chi_{120}^{SO(16)} + 2) \\
- q^3 \left[ (t + t^{-1})(t^2 + t^{-2})(u^2 + 1 + u^{-2}) - 1) \chi_{128}^{SO(16)} + (u + u^{-1}) \chi_{128}^{SO(16)} \right] \\
+ (t^2 + 1 + t^{-2})(u + u^{-1}) \chi_{1920}^{SO(16)} + 2\chi_{128}^{SO(16)} \right] + (t + t^{-1})(\chi_{13312}^{SO(16)} + \chi_{1920}^{SO(16)} + 4\chi_{128}^{SO(16)}) \right] \\
- q^4 \left[ (t^4 + t^{-4})(u + u^{-1}) + (t^3 + t + t^{-1} + t^{-3})(u^3 + u^{-3}) \\
+ (t^2 + 1 + t^{-2})(u^3 + u^{-3}) + (t + t^{-1})(u^2 + u^{-2}) \right] \\
+ (u + u^{-1})(\chi_{1820}^{SO(16)} + 2\chi_{120}^{SO(16)} + 3) + \left( t^4 + t^2 + 1 + t^{-2} + t^{-4}(u^3 + u^{-3}) \right) \\
+ (t^4 + t^{-4})(u + u^{-1}) + (t^3 + t^{-3}) + (t + t^{-1})(u^2 + u^{-2}) \chi_{120}^{SO(16)} + 1 \right] \\
+ (t^3 + t^{-3})(u^2 + 1 + u^{-2}) + (t + t^{-1})(u^2 + u^{-2}) \chi_{1820}^{SO(16)} + \chi_{135}^{SO(16)} + 2\chi_{120}^{SO(16)} + 5 \right] \\
+ (t^2 + 1 + t^{-2})(u + u^{-1})(\chi_{8008}^{SO(16)} + \chi_{7020}^{SO(16)} + 2\chi_{1820}^{SO(16)} + \chi_{135}^{SO(16)} + 6\chi_{120}^{SO(16)} + 8) \right] \\
+ (t + t^{-1})(\chi_{60060}^{SO(16)} + \chi_{8008}^{SO(16)} + \chi_{7020}^{SO(16)} + \chi_{5635}^{SO(16)} + \chi_{5304}^{SO(16)} + 4\chi_{1820}^{SO(16)} + 3\chi_{135}^{SO(16)} + 9\chi_{120}^{SO(16)} + 14) \right]
\end{align*}
\]
\[-q^5 \left[ (t^5 + t^{-5})(u^4 + u^2 + 1 + u^{-2} + u^{-4}) + (t^3 + t + t^{-1} + t^{-3})(u^4 + u^{-4}) + (t^2 + 1 + t^{-2})(u^3 + u^{-3}) + (t^4 + t^{-4})(u + u^{-1}) + (t + t^{-1})(u^2 + u^{-2}) \right] \chi_{SO(16)}^{\mathbb{128}} \]
\[+ \left[ (t^3 + t^{-3})(u^2 + 1 + u^{-2}) + (t + t^{-1})(u^2 + u^{-2}) \right] \left( 3\chi_{1128}^{SO(16)} + 7\chi_{128}^{SO(16)} + \chi_{13321}^{SO(16)} \right) \]
\[+ \left[ (t^2 + t^{-2})(u + u^{-1}) + (t + t^{-1})(u + u^{-1}) \right] \chi_{SO(16)}^{15360} + \chi_{SO(16)}^{56320} + 3\chi_{13312}^{SO(16)} + 7\chi_{1920}^{SO(16)} + 14\chi_{128}^{SO(16)} \]
\[+ (t^4 + t^{-4})(u^3 + u + u^{-1} + u^{-3}) + (t + t^{-1})(u^2 + u^{-2}) + (t^3 + t^{-3}) \left( 2\chi_{1920}^{SO(16)} + 2\chi_{128}^{SO(16)} \right) \]
\[+ (t + t^{-1})(\chi_{SO(16)}^{161280} + \chi_{SO(16)}^{141440} + 3\chi_{13312}^{SO(16)} + 5\chi_{1920}^{SO(16)} + 9\chi_{128}^{SO(16)}) \right] \right] + O(q^6)\]

This again agrees with the result obtained from the instanton calculus of [31].

We also computed \( \tilde{f_3} \) with all \( SO(16) \subset E_8 \) masses turned off. It again completely agrees with \( \tilde{f_3} \) computed from 5d instanton calculus, up to \( q^5 \) order that we checked. Also, for 3 and 4 E-strings, we have kept all \( E_8 \) masses and compared our 2d elliptic genus with the instanton partition function up to 1 instanton order, which all show agreements.

So we saw that the instanton calculus provides the correct index for the \( E_8 \) 6d SCFT. One virtue of this approach would be that, at a given order in \( q \), the index is computed exactly in \( w \). In particular, the chemical potential for the E-string number (the Coulomb VEV of 5d SYM) is an integration variable in the curved space partition functions, which can be used to study the conformal field theory physics. So knowing the exact form of the partition function in \( w \) will be desirable to understand the curved space partition functions.

### 5 Concluding remarks

In this paper we have found a description of E-strings which can be used to describe the IR degrees of freedom on it. This in particular includes the information about bound states of E-strings. The theory for \( n \) E-strings involves a \((0,4)\) supersymmetric quiver theory in 2 dimensions with \( O(n) \) gauge symmetry and some matter content. We in particular computed the elliptic genus of E-strings (including turning on fugacities for the \( E_8 \) flavor symmetry as well as \( SO(4) \) rotation transverse to the string in 6d) for small number of E-strings. We gave the explicit answer for \( n = 1, 2, 3, 4 \) and indicated how one can use these methods to obtain arbitrary \( n \) answers. Our results successfully pass the comparison checks with the partial results already known. Our results provide an all genus answer for the topological string on the canonical bundle over \( \frac{1}{2}K3 \). In addition, we explained how to compute the same elliptic genus using the instanton partition function of the 5d \( Sp(1) \) SYM theory coupled to 8 fundamental hypermultiplets.

There are a few natural extensions of the present work. First of all it would be nice to see if we can streamline the computation of the elliptic genus for arbitrary \( n \). Even though
our methods provide an answer, writing it explicitly is cumbersome. Secondly, it would be interesting to see if we can find an explicit description of the (0, 4) conformal theory they flow to. We have made some preliminary comments about it in this paper. Finally it would be interesting to see if we can use our results to come up with a domain wall description of the E-string amplitude as in [7]. Moreover one would like to use this to show that the partition function of a pair of $n$ E-strings can lead to the partition function of $n$ heterotic strings as is predicted by the Horava-Witten description of heterotic string.

Finally it would be interesting to generalize this to other (1, 0) superconformal field theories in 6d, and characterize all the 2d (0, 4) systems that one gets on the worldsheet of the associated strings.

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**A  Modular forms and Jacobi forms**

A modular form $f_n(\tau)$ of weight $n$ transforms under $SL(2, \mathbb{Z})$ as

$$f_n \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^n f_n(\tau), \quad ad - bc = 1.$$  \hfill (A.1)

An important class of modular forms is given by the Eisenstein series,

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$  \hfill (A.2)

where $q = e^{2\pi i \tau}$. The Bernoulli numbers $B_{2k}$ and the divisor functions $\sigma_k(n)$ are defined by

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}, \quad \sigma_k(n) = \sum_{d|n} d^k.$$  \hfill (A.3)

$E_{2k}(\tau)$ are modular forms of weight $2k$, except for $E_2(\tau)$ which involves an anomalous term,

$$E_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 E_2(\tau) + \frac{6}{i\pi} c(c\tau + d).$$  \hfill (A.4)
Another example of modular form is the Dedekind eta function \( \eta(\tau) \), defined by

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]  
(A.5)

Under the modular transformation, \( \eta(\tau) \) behaves as a weight \( \frac{1}{2} \) form up to a phase \( \epsilon(a, b, c, d) \),

\[
\eta \left( \frac{a\tau + b}{c\tau + d} \right) = \epsilon(a, b, c, d) \cdot (c\tau + d)^{1/2}\eta(\tau).
\]  
(A.6)

Jacobi forms have a modular parameter \( \tau \) and an elliptic parameter \( z \). Modular transformation for Jacobi forms \( \phi_{k,m}(\tau, z) \) of weight \( k \) and index \( m \) is given by

\[
\phi_{k,m} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k e^{2\pi imz^2} \phi_{k,m}(\tau, z),
\]  
(A.7)

Under the translation of the elliptic parameter \( z \), they behave as

\[
\phi_{k,m}(\tau, z + a\tau + b) = e^{-2\pi im(a^2 + 2az)} \phi_{k,m}(\tau, z).
\]  
(A.8)

where \( a, b \) are integers.

The Jacobi theta function \( \vartheta(\tau, z) \) is a Jacobi form of weight \( \frac{1}{2} \) and index \( \frac{1}{2} \), defined as

\[
\vartheta(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}}y)(1 + q^{n-\frac{1}{2}}y^{-1}) = \sum_{n \in \mathbb{Z}} q^{n^2/2}y^n
\]  
(A.9)

where \( q \equiv e^{2\pi i \tau} \) and \( y \equiv e^{2\pi iz} \). We define three other functions which are closely related to the Jacobi theta function, and define

\[
\theta_1(\tau, z) = -iq^{1/8}y^{1/2}\vartheta(\tau, z + \frac{1+i\tau}{2}) = -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty} (1 - q^n)(1 - q^n y)(1 - q^n y^{-1})
\]

\[
\theta_2(\tau, z) = q^{1/8}y^{1/2}\vartheta(\tau, z + \frac{\tau}{2}) = q^{1/8}y^{1/2}\prod_{n=1}^{\infty} (1 - q^n)(1 + q^n y)(1 + q^n y^{-1})
\]

\[
\theta_3(\tau, z) = \vartheta(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}}y)(1 + q^{n-\frac{1}{2}}y^{-1})
\]

\[
\theta_4(\tau, z) = \vartheta(\tau, z + \frac{i}{2}) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}}y)(1 - q^{n-\frac{1}{2}}y^{-1}).
\]  
(A.10)

From here, when we omit the parameter in various functions, it should be understood as \( \tau \). \( \theta_n(z) \)'s are related to others by the half-period shifts:

\[
\theta_1(z + \frac{1}{2}) = \theta_2(z), \quad \theta_1(z + \frac{1+i\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z), \quad \theta_1(z + \frac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_4(z)
\]

\[
\theta_2(z + \frac{1}{2}) = -\theta_1(z), \quad \theta_2(z + \frac{1+i\tau}{2}) = -iq^{-1/8}y^{-1/2}\theta_4(z), \quad \theta_2(z + \frac{\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z)
\]

\[
\theta_3(z + \frac{1}{2}) = \theta_4(z), \quad \theta_3(z + \frac{1+i\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_1(z), \quad \theta_3(z + \frac{\tau}{2}) = q^{-1/8}y^{-1/2}\theta_2(z)
\]

\[
\theta_4(z + \frac{1}{2}) = \theta_3(z), \quad \theta_4(z + \frac{1+i\tau}{2}) = q^{-1/8}y^{-1/2}\theta_2(z), \quad \theta_4(z + \frac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_1(z)
\]  
(A.11)
Various identities: The modular forms $E_4$, $E_6$, and $\eta$ can be expressed in terms of Jacobi theta functions with their elliptic parameters $z$ set to zero:

$$
E_4 = \frac{1}{2}(\theta_2(0)^8 + \theta_3(0)^8 + \theta_4(0)^8)
$$

$$
E_6 = \frac{1}{2}(\theta_2(0)^4 + \theta_3(0)^4)(\theta_3(0)^4 + \theta_4(0)^4)(\theta_4(0)^4 - \theta_2(0)^4)
$$

$$
\eta^3 = \theta_2(0)\theta_3(0)\theta_4(0).
$$

(A.12)

$\theta_n(z)$’s also satisfy

$$
\theta_2(z)^4 - \theta_1(z)^4 = \theta_3(z)^4 - \theta_4(z)^4, \quad \theta_2(0)^4 = \theta_3(0)^4 - \theta_4(0)^4.
$$

(A.13)

Further identities of $\theta_n(z)$’s with different elliptic parameters are

$$
\theta_1(a + b)\theta_1(a - b)\theta_4(0)^2 = \theta_3(a)^2\theta_2(b)^2 - \theta_3(a)^2\theta_2(b)^2 = \theta_1(a)^2\theta_4(b)^2 - \theta_4(a)^2\theta_1(b)^2
$$

(A.14)

$$
\theta_3(a + b)\theta_3(a - b)\theta_2(0)^2 = \theta_3(a)^2\theta_2(b)^2 + \theta_3(a)^2\theta_2(b)^2 = \theta_2(a)^2\theta_3(b)^2 + \theta_2(a)^2\theta_3(b)^2
$$

$$
\theta_3(a + b)\theta_3(a - b)\theta_3(0)^2 = \theta_1(a)^2\theta_1(b)^2 + \theta_3(a)^2\theta_3(b)^2 = \theta_2(a)^2\theta_2(b)^2 + \theta_4(a)^2\theta_4(b)^2
$$

(A.15)

$$
\theta_3(a + b)\theta_3(a - b)\theta_4(0)^2 = \theta_4(a)^2\theta_4(b)^2 - \theta_1(a)^2\theta_2(b)^2 = \theta_3(a)^2\theta_4(b)^2 - \theta_2(a)^2\theta_1(b)^2
$$

Remaining identities of this kind can be obtained through half-period shifts on $a$.

Under the shift of modular parameter $\tau \rightarrow \tau' = \tau + 1$, the corresponding changes are

$$
\theta_1(\tau + 1, z) = e^{i\pi}\theta_1(\tau, z), \quad \theta_2(\tau + 1, z) = e^{i\pi}\theta_2(\tau, z), \quad \theta_3(\tau + 1, z) = \theta_4(\tau, z), \quad \theta_4(\tau + 1, z) = \theta_3(\tau, z).
$$

(A.16)

Watson’s identities and Landen’s formulas involve doubling of modular parameter $\tau$,

$$
\theta_1(\tau, z)\theta_1(\tau, w) = \theta_3(2\tau, z + w)\theta_2(2\tau, z - w) - \theta_2(2\tau, z + w)\theta_3(2\tau, z - w)
$$

(A.17)

$$
\theta_3(\tau, z)\theta_3(\tau, w) = \theta_3(2\tau, z + w)\theta_3(2\tau, z - w) + \theta_2(2\tau, z + w)\theta_2(2\tau, z - w)
$$

$$
\theta_4(2\tau, 2z) = \theta_4(\tau, z)\theta_2(\tau, z)/\theta_4(2\tau, 0)
$$

(A.18)

$$
\theta_4(2\tau, 2z) = \theta_3(\tau, z)\theta_4(\tau, z)/\theta_4(2\tau, 0)
$$

Considering the case with $z = 0$, one obtains

$$
\theta_2(2\tau, 0) = \sqrt{\frac{\theta_3(\tau, 0)^2 - \theta_4(\tau, 0)^2}{2}}, \quad \theta_3(2\tau, 0) = \sqrt{\frac{\theta_3(\tau, 0)^2 + \theta_4(\tau, 0)^2}{2}}, \quad \theta_4(2\tau, 0) = \sqrt{\theta_3(\tau, 0)\theta_4(\tau, 0)}.
$$

(A.19)

Differentiations by $\tau, z$: The $\tau$ derivatives of $E_2, E_4, E_6$ can be obtained from the Ramanujan identities

$$
q \frac{d}{dq} E_2 = \frac{1}{12}(E_2^2 - E_4), \quad q \frac{d}{dq} E_4 = \frac{1}{3}(E_2 E_4 - E_6), \quad q \frac{d}{dq} E_6 = \frac{1}{2}(E_2 E_6 - E_4^2).
$$

(A.20)
The $\tau$ derivative of the eta function is given by

$$q \frac{d}{dq} \eta^3 = \frac{\eta^3}{8} E_2.$$  \hspace{1cm} (A.21)

As for the theta functions, first note that $\theta_n(z)$’s are solutions of

$$\left[ \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial \tau^2} - \frac{1}{i\pi} \frac{\partial}{\partial \tau} \right] \theta_n(\tau, z) = \left[ \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial z^2} - 2\frac{\partial}{\partial \tau} \right] \theta_n(\tau, z) = 0.$$ \hspace{1cm} (A.22)

$\theta_1$ is an odd function of $z$, while $\theta_2, \theta_3, \theta_4$ are even functions of $z$. The lowest non-vanishing derivatives of $\theta_n$’s at $z = 0$ are given by

$$\begin{align*}
\theta_1^{(1)}(0) &= 2\pi \eta^3 \\
\theta_2^{(2)}(0) &= -\frac{x^2}{3} \theta_2(0) (E_2 + \theta_3(0)^4 + \theta_4(0)^4) \\
\theta_3^{(2)}(0) &= -\frac{x^2}{3} \theta_3(0) (E_2 + \theta_2(0)^4 - \theta_4(0)^4) \\
\theta_4^{(2)}(0) &= -\frac{x^2}{3} \theta_4(0) (E_2 - \theta_2(0)^4 - \theta_3(0)^4),
\end{align*}$$ \hspace{1cm} (A.23)

where $(n)$ denotes $n$’th derivative with respect to the elliptic parameter. Using (A.22), (A.23), (A.20) and (A.21), one can also express the higher $z$ derivatives $\theta_1^{(2n+1)}(0), \theta_2^{(2n)}(0), \theta_3^{(2n)}(0), \theta_4^{(2n)}(0)$ at $z = 0$ in terms of $\theta_2(0), \theta_3(0), \theta_4(0), E_2$. See appendix C for more details, where this procedure will be illustrated and used to prove exact properties of the E-string indices.

**B  Genus expansions of topological string amplitudes**

In this appendix, we summarize some low genus results that we used in section 3. The low genus amplitudes have been studied in [19, 52, 30, 41, 29]. We list the unrefined results till $g \leq 5$ (as written in [11]), and some refined results that we used to compare with our results.

For three E-strings, the unrefined genus expansion coefficients $F^{(0, g, 3)}$ are given by

$$\begin{align*}
F^{(0,0,3)} &= \frac{54E_2^3E_4^3 + 216E_2^3E_4^2E_6 + 109E_4^4 + 197E_4E_6^2}{15552\eta^{36}} \\
F^{(0,1,3)} &= \frac{78E_2^3E_4^3 + 299E_2E_4^4 + 360E_2^2E_4^2E_6 + 472E_4^3E_6 + 439E_2E_4E_6^2 + 80E_6^3}{62208\eta^{36}} \\
F^{(0,2,3)} &= \frac{1}{2488320\eta^{36}} (575E_4^3E_6^2 + 3040E_2E_4^2E_6 + 4690E_2E_4E_6^2 + 3548E_2^2E_4^4 \\
&\quad \quad + 1600E_6^3E_2 + 10176E_6E_4E_2 + 2231E_4^5 + 5244E_2^2E_6^2) \\
F^{(0,3,3)} &= \frac{1}{20901880\eta^{36}} (138104E_4^4E_6 + 224024E_4^3E_2^2 + 36400E_2^3E_4E_6^2 + 622456E_2^3E_6^2E_4 \\
&\quad \quad + 49584E_4^3E_6^3 + 68460E_2E_4^2E_6^2 + 55006E_2E_4E_6^4 + 6055E_2^2E_4^3E_6^2 + 97431E_4^2E_2 + 33600E_6^3E_2^2) \\
F^{(0,4,3)} &= \frac{1}{7524679600\eta^{36}} (3164700E_4^2E_6^2 + 8993259E_4^4E_2^2 + 14111840E_2^7E_4E_6^4 + 806400E_6^4 \\
&\quad \quad + 25171632E_2E_6^4E_4 + 13855280E_3E_6^5E_4^3 + 8963520E_2E_4E_6^4 + 20453520E_2^2E_6^3E_4^2 \\
&\quad \quad + 4014627E_6^4 + 208985E_2^4E_4^3 + 201600E_6E_4E_2^2 + 1417920E_5E_4^2E_6 + 2638125E_4^4E_4^4)
\end{align*}$$

39
\[ F^{(0,5,3)} = \frac{1}{9932577177600\eta^{36}} (935093824E_6^2E_4^3E_2 + 233170300E_2^2E_6E_3^3 + 296640960E_2^2E_6^3E_4 + 83755072E_2^2E_6E_4 + 45368040E_3^2E_6^2E_2^2 + 16385600E_5^2E_6E_6 + 42513240E_2^2E_4E_6^2 + 20115192E_5^2E_2^2 + 36275085E_5^2E_4 + 53222400E_6E_2 + 266767491E_6E_2 + 40526828E_5E_6 + 268326944E_3^2E_6^3 + 33264000E_3^2E_4 + 2155615E_3^2E_4^3) . \]

A refined coefficient \( F^{(1,0,3)} \) that we studied in section 3.3 is given by

\[ F^{(1,0,3)} = -\frac{54E_2^3E_4^3 + 235E_4E_4^3 + 216E_2^2E_2E_6 + 776E_4^3E_6 + 287E_2E_4E_6^2 + 160E_6^3}{124416\eta^{36}} . \] (B.2)

For the four E-strings, \( F^{(0,g,4)} \) are given as follows (after correcting some typos in [11]):

\[ F^{(0,0,4)} = \frac{1}{62208\eta^{48}} E_4 \left( 272E_4^3E_6 + 154E_6^3 + 109E_2E_4 + 60E_2E_4E_4^2 + 144E_2^2E_4E_6^2 + 24E_2E_4^3 \right) \]

\[ F^{(0,1,4)} = \frac{1}{11943936\eta^{48}} \left( 37448E_2^2E_4^3E_6 + 68768E_2E_4^3E_6 + 29920E_2E_4E_6^3 + 13809E_4^6 + 57750E_2^3E_6^2 + 17416E_2E_6^4 + 4545E_6^6 + 16704E_3^4E_6 + 2472E_2E_4^4 \right) \]

\[ F^{(0,2,4)} = \frac{1}{17915904\eta^{48}} \left( 77280E_2E_6E_4^4 + 209200E_2^2E_2E_4^2E_6 + 547760E_2E_6E_4^4 + 214811E_2E_6E_2 
+ 203900E_2^3E_2^2E_4^2 + 103252E_5E_2^3E_4 + 827230E_6E_4^4E_2 + 10200E_5E_4^4 + 57375E_6E_2 + 426016E_4E_6 + 314360E_2E_4^3 \right) \]

\[ F^{(0,3,4)} = \frac{1}{9029615616\eta^{48}} \left( 28134630E_4^7 + 151049909E_2^3E_6E_4^2 + 25488295E_4E_4^3 + 966630E_2^2E_4^4 + 189296376E_2^3E_4^2 + 8172360E_5E_4^2E_6 + 3138800E_2E_4^4 + 88718416E_2E_4E_6^4 + 24977155E_2E_4E_6^4 + 1336678E_5E_4^2 + 12119625E_6E_6E_4^2 + 137926976E_2E_4^3E_6 + 51557313E_2E_6^4 + 192353224E_5E_6E_2 \right) \]

\[ F^{(0,4,4)} = \frac{1}{541776936960\eta^{48}} \left( 3336940980E_2^2E_4^3E_6^2 + 7817234620E_2^2E_2E_6^4 + 3248768730E_3^3E_4^3E_6^3 + 5085796952E_2^3E_5E_6 + 101280375E_6^6 + 355052500E_2^3E_5E_6^3 + 1290318725E_2E_4E_6^4 + 936363912E_4^3E_6^3 + 1481276055E_4^2E_6^2 + 2912603799E_4^3E_6^3 + 1216807640E_2E_6^4 + 152620090E_5^2E_4^5 + 78670608E_2E_6E_4^3 + 410158000E_5^2E_6E_4^2 + 274844990E_5^2E_6E_4^2 + 8381520E_7E_4^3 + 202702500E_6E_4^2 \right) \]

\[ F^{(0,5,4)} = \frac{1}{28605822714880\eta^{48}} \left( 12207942670E_2^6E_6^5 + 523849095E_3^3E_4^4 + 156150752805E_8^8 + 11201930320E_3^2E_6^4E_6 + 1311485716360E_5^2E_6E_6 + 1760563778482E_2^2E_6^5E_4 + 286289201000E_2E_4E_6^4 + 381058740370E_2E_4E_6^3E_6 + 1449394307792E_3^3E_4^3E_2 + 1106487740990E_2^2E_6^4 + 44575839000E_5E_6 + 1090255878484E_4^2E_2^2 + 774483173328E_2^2E_5E_6^5 + 531170439360E_2^2E_4^3E_6^3 + 5431290480E_2^2E_6^4E_4^3 + 37160939200E_5^3E_6E_4 + 337421738130E_4^2E_2 + 2143957730E_6E_6E_4^2 + 22344052500E_6E_6E_4^2 + 34499853724E_6E_4^2 \right) . \] (B.3)
C  Exact properties of the E-string elliptic genus

We explain the details on how we checked various exact properties of our E-string elliptic genera, using various identities of appendix A. We made lots of symbolic computations using computer. Below, we explain how one can simplify various expressions which can be put on a computer for further simplifications.

2 E-strings  We compare the two expressions for the elliptic genus of 2 E-strings, \( (3.25) \) and \( (3.20) \). Let us denote them by \( Z_2 \) and \( Z_2^{E8} \) respectively, in the sense that the latter expression shows manifest \( E_8 \) symmetry. After setting \( \epsilon_1 = -\epsilon_2 = \epsilon \) for simplicity, \( Z_2 \) is given by

\[
Z_2 = \sum_{n=1}^{4} \prod_{l=1}^{8} \theta_n(m_l \pm \frac{\tau}{2}) + \frac{1}{4\eta^{12}\theta_1(\epsilon)^2} \left[ \frac{\theta_2(0)^2}{\theta_2(\epsilon)^2} \left( \prod_{l=1}^{8} \theta_1(m_l)\theta_2(m_l) + \prod_{l=1}^{8} \theta_3(m_l)\theta_4(m_l) \right) \right] \ .
\]

(C.1)

Using the identity (A.13) with \( a = b \), one can write \( Z_2 = \frac{N^{E8}}{\eta^{12}\theta_1(\epsilon)^2\theta_2(\epsilon)^2} \) with

\[
N = \sum_{n=1}^{4} \prod_{l=1}^{8} \theta_n(m_l \pm \frac{\epsilon}{2}) + \frac{\theta_3(\epsilon)^2\theta_4(\epsilon)^2}{\theta_3(0)^2\theta_4(0)^2} \left( \prod_{l=1}^{8} \theta_1(m_l)\theta_2(m_l) + \prod_{l=1}^{8} \theta_3(m_l)\theta_4(m_l) \right) + \frac{\theta_2(0)^2\theta_3(0)^2}{\theta_2(\epsilon)^2\theta_3(\epsilon)^2} \left( \prod_{l=1}^{8} \theta_1(m_l)\theta_4(m_l) + \prod_{l=1}^{8} \theta_2(m_l)\theta_3(m_l) \right) + \frac{\theta_2(0)^2\theta_4(0)^2}{\theta_2(\epsilon)^2\theta_4(0)^2} \left( \prod_{l=1}^{8} \theta_1(m_l)\theta_3(m_l) + \prod_{l=1}^{8} \theta_2(m_l)\theta_4(m_l) \right) .
\]

(C.2)

We apply (A.14) to the first term of \( N \), where we take \( a = m_l, b = \epsilon/2 \). Then \( N \) can be expressed as a polynomial of \( \theta_n(m_l), \theta_n(\epsilon) \) and \( \theta_n(\epsilon/2) \), with coefficients given by \( \theta_n(0) \).

On the other side, expressing \( (3.26) \) as \( Z_2^{E8} = N^{E8}/(\eta^{12}\theta_1(\epsilon)^2\theta_2(\epsilon)^2) \), we consider

\[
N^{E8} = \frac{1}{72} A_1^2(\phi_{0,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)^2) + \frac{1}{96} A_2(E_4^2\phi_{-2,1}(\epsilon)^2 - E_6\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon))
\]

\[+ \frac{5}{288} B_2(E_6\phi_{-2,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon)) .
\]

(C.3)

We first insert (A.12) to replace \( E_4, E_6, \eta \) by expressions containing \( \theta_2(0), \theta_3(0), \theta_4(0) \) only. Looking at the definition of \( A_2 \) and \( B_2 \) in (3.23), there appear \( \theta_n(\frac{\tau}{2}, m_l) \) and \( \theta_n(\frac{\tau+1}{2}, m_l) \). To simplify them, we first consider the identities,

\[
\theta_1(\frac{\tau}{2}, m_1)\theta_1(\tau, m_2) = \theta_3(\tau, m_1 + m_2)\theta_2(\tau, m_1 - m_2) - \theta_2(\tau, m_1 + m_2)\theta_3(\tau, m_1 - m_2)
\]

\[
\theta_1(\frac{\tau+1}{2}, m_1)\theta_1(\tau+1, m_2) = e^{i\pi/4}\theta_4(\tau, m_1 + m_2)\theta_2(\tau, m_1 - m_2) - e^{i\pi/4}\theta_2(\tau, m_1 + m_2)\theta_4(\tau, m_1 - m_2) .
\]

(C.4)

The first identity can be obtained by replacing \( \tau, z, w \) in (A.17) by \( \frac{\tau}{2}, m_1, m_2 \), respectively, and the second one is obtained from the first identity by using (A.16). One can also obtain three
more copies of similar identities, replacing $\theta_1$ on the left hand sides by $\theta_2, \theta_3, \theta_4$, by using (A.11). The expressions appearing on the right hand sides of (C.4) can be written as polynomials of $\theta_n(\tau, m_l)$ by using (A.15). We apply these identities, and also those with $(m_1, m_2)$ replaced by $(m_3, m_4), (m_5, m_6), (m_7, m_8)$, to (C.3). Then one can express all theta functions with modular parameters $\frac{\tau}{2}$ or $\frac{\tau+1}{2}$ in terms of $\theta_n(\tau, m_l)$. Other terms including $\theta_n(2\tau, 2m_l)$ can be reorganized using (A.18) and (A.19), in terms of $\theta_n(\tau, m_l)$ and $\theta_n(\tau, 0)$. So finally, $N^{Es}$ is written as a polynomial of $\theta_n(\tau, m_l), \theta_n(\tau, \epsilon)$, with coefficients given by $\theta_n(\tau, 0)$.

Finally, to straightforwardly compare $N$ and $N^{Es}$, we want to express $\theta_n(\epsilon)$'s in terms of $\theta_n(\epsilon/2)$'s. Plugging $b = \frac{\tau}{2}$ and $a = \frac{\tau}{2} + \frac{\tau}{2}$ (with $p = 0, 1, \tau, \tau + 1$) into (A.14) and (A.15), one obtains the desired formulae. Then inserting them into $N, N^{Es}$, we obtain polynomials of $\theta_n(\tau, m_l), \theta_n(\tau, \frac{\tau}{2})$ with coefficients given by $\theta_n(\tau, 0)$. Now we can evaluate $N^{Es} - N$ on computer, by eliminating $\theta_1(m_l), \theta_1(\epsilon/2), \theta_2(0)$ by using (A.13). This yields zero, proving the equivalence of (3.25) and (3.26).

3 and 4 E-strings We compare our elliptic genera (3.39) and (3.57) against the known results summarized in Appendix B. The free energy is expanded as

$$F = \log Z = \sum_{n_b=1}^{\infty} w^{n_b} F_{n_b} = \sum_{n,g,n_b} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} w^{n_b} F^{(n,g,n_b)}, \quad (C.5)$$

where $F_1 = Z_1, F_2 = Z_2 - \frac{1}{2} Z_1^2, F_3 = Z_3 - Z_1 Z_2 + \frac{1}{2} Z_1^3$ and $F_4 = Z_4 - Z_1 Z_3 - \frac{1}{2} Z_2^2 + Z_1^2 Z_2 - \frac{1}{2} Z_1^4$. The coefficients $F^{(n,g,n_b)}$ computed from topological strings, summarized in appendix B, depend on $\eta, E_2, E_4, E_6$. Using (A.12), these can be arranged into expressions involving $E_2$ and $\theta_n(0)$ only.

On the other hand, if we set $m_l = 0$ and compute $F^{(n,g,n_b)}$ from our gauge theory indices, they will be rational functions of $\theta_n(0), \eta, \theta_n^{(k)}(0)$. The derivatives $\theta_n^{(k)}(0)$ appear because we are expanding the index with $\epsilon_1, \epsilon_2$. We want to express our gauge theory expressions for $F^{(n,g,n_b)}$ in terms of $\theta_n(0)$'s and $E_2$ only, to compare with the results summarized in appendix B. Firstly, (A.12) can be used to eliminate $\eta$. The remaining task is to write $\theta_1^{(k)}(0)$ in terms of $\theta_n(0)$'s and $E_2$, which can be done in the following way.

Starting from the lowest non-vanishing derivatives (A.23) at $z = 0$, we can iteratively obtain $\theta_n^{(k)}(0)$ for higher $k$'s. For example,

$$(\partial_z)^3 \theta_1(\tau, z)|_{z=0} = -8\pi^2 (\partial_z)(q\partial_q)\theta_1(\tau, z)|_{z=0} = -8\pi^2 (q\partial_q)(\partial_z \theta_1(\tau, z))|_{z=0}$$

$$= -16\pi^3 (q\partial_q)^3 \eta^3 = -2\pi^3 \eta^3 E_2 \quad (C.6)$$
where (A.22) and (A.21) are applied in the last step. If we look at another example,

\[(\partial_z)^4 \theta_2(\tau, z)|_{z=0} = -8\pi^2(\partial_z)^2(q\partial_q)\theta_2(\tau, z)|_{z=0} = -8\pi^2(q\partial_q)(\partial_z^2\theta_2(\tau, z))|_{z=0} \]
\[= \frac{8}{3}\pi^4 q\partial_q[\theta_2(0) \cdot (E_2 + \theta_3(0)^4 + \theta_4(0)^4)] \]
\[= \frac{1}{9}\pi^4 \theta_2(0)[\alpha_2^2 + 4\theta_3(0)^4\alpha_3 + 4\theta_4(0)^4\alpha_4 + \frac{1}{12}(E_2^2 - E_4)]. \quad (C.7) \]

for \(\alpha_2 \equiv E_2 + \theta_3(0)^4 + \theta_4(0)^4\), \(\alpha_3 \equiv E_2 + \theta_2(0)^4 - \theta_4(0)^4\), and \(\alpha_4 \equiv E_2 - \theta_2(0)^4 - \theta_3(0)^4\). In the last step, we applied (A.22) and (A.20). Going for higher derivatives involves no more difficulty, and this way we can always express \(F^{(n,g,n_b)}\) in terms of \(\theta_n(0)'s\) and \(E_2\) only.

So we find two expressions for \(F^{(n,g,n_b)}\), depending on \(\theta_n(0)'s\) and \(E_2\) only, one from the topological string calculus and another from our gauge theories. In particular, we focus on the 3 and 4 E-strings, obtained by expanding (3.39), (3.57). We computed the differences of the two expressions for \(F^{(0,0,3)}, F^{(0,1,3)}, F^{(1,0,3)}, F^{(0,0,4)}, F^{(0,1,4)}, F^{(0,2,4)}\) on computer, substituting \(\theta_2(0)^4 = \theta_3(0)^4 - \theta_4(0)^4\), and found zero in all cases. Of course, further analytic tests can also be easily done on computer for higher genus results.

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