Semi-Analytical Method with Laplace Transform for Certain Types of Nonlinear Problems

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Abstract. In this paper, the approximate solution is found for the Fornberg-Whitham equation (F-W) by using two analytical methods which are the Laplace decomposition method (LDM) and modified Laplace decomposition method (MLDM) with comparison between these methods for which gave the best approximate solution near to the exact solution, The analytical results of these methods have been received in terms of convergent series with easily calculable components. The results show that the modified method was found to be efficient, accurate and fast compared to the second method used in this research.

1. Introduction
Many important phenomena can be represented by nonlinear equations, both ordinary and partial, such as population models, chemical kinetics and fluid dynamics. Many efforts have been made to implement either approximate or analytical methods to solve the nonlinear equations such as [1] and [2]. The F-W gave as [3, 4]

\[ v_t - v_{bxx} + v_b = vv_{bxx} - v_{bx} + 3v_b v_{bx} \]  (1.1)

It consists of a type of travelling wave solution called a kink-like wave solution and anti-kink-like wave solutions. No such sorts of travel wave solutions have been found for F-W. These days, numerous distinct methods have been presented to solve the F-W such as homotopy analysis method (HAM) [5], variational iteration method (VIM) [6], Daftardar-Jafari iterative method (DJM) and homotopy perturbation transform method (HPTM) [7]. Temimi and Ansari method (TAM) and Banach contraction method (BCM)[8].

In this paper, we implemented the LDM introduced by wazwaz [9] and MLDM introduced by Khuri [10, 11] to solve F-W, and the solution will be compared in both methods, those iterative methods have been successfully used to solve several kinds of problems. For example the linear and nonlinear fractional diffusion–wave equation was solved by applying the LDM [12], MLDM used to solve lane-Emden type differential equations [13]. In the following sections, the LDM and MLDM application are presented to solve the F-W and the validity of these methods to find the appropriate approximate solution.
2. **The basic idea of the methods**

To illustrate the solution steps for the MLDM, we consider the following nonlinear partial differential problem:

\[ Lv(b, t) = Rv(b, t) + Nv(b, t) \]  
\[ \nu(b, 0) = f(b), \quad \nu_r(b, 0) = g(b), \]  

wherein \( L \) is a differential operator \( \partial / \partial t \) in eq. (2.1), \( R \) is another linear differential factor, \( N \) is a nonlinear differential factor.

By taking Laplace transform (LT) (indicated by \( C \)), we get:

\[ C[Lv(b, t)] = C[Rv(b, t)] + C[Nv(b, t)], \]  

using the differentiation property of LT and initial condition in eq. (2.3)

\[ sC[v(b, t)] - f(b) = C[Rv(b, t)] + C[Nv(b, t)], \]  

\[ C[v(b, t)] = \frac{1}{s} f(b) + \frac{1}{s} C[Rv(b, t)] + \frac{1}{s} C[Nv(b, t)]. \]  

Then the solution can be represented as an infinite series mentioned below:

\[ v(b, t) = \sum_{i=0}^{\infty} v_i(b, t), \]  

The nonlinear operator is disintegrating as

\[ Nv(b, t) = \sum_{i=0}^{\infty} A_i, \]  

Where \( A_i \) are Adomian polynomials \[14\] of \( v_1, v_2, \ldots, v_i \) and it can be evaluated by the following formula

\[ A_i = \frac{1}{i!} \frac{d^i}{dt^i} \left[ N \sum_{n=0}^{\infty} \lambda^n v_n \right] \quad i = 1, 2, \ldots \]  

By substituted (2.6) and (2.7) in (2.5)

\[ C \left[ \sum_{i=0}^{\infty} v_i(b, t) \right] = \frac{1}{s} f(b) + \frac{1}{s} C[Rv(b, t)] + \frac{1}{s} C \left[ \sum_{i=0}^{\infty} A_i \right]. \]  

As \( C \) is the linear operator where

\[ \sum_{i=0}^{\infty} C \nu_i(b, t) = \frac{1}{s} f(b) + \frac{1}{s} C[Rv(b, t)] + \frac{1}{s} C[\sum_{i=0}^{\infty} A_i], \]  

By correspondence both sides of eq. (2.10) we have the following:

\[ C\nu_0(b, t) = \frac{1}{s} f(b) = h(b, s), \]  

\[ C \nu_1(b, t) = \frac{1}{s} C[R\nu_0(b, t)] + \frac{1}{s} C[A_0], \]  

\[ C \nu_2(b, t) = \frac{1}{s} C[R\nu_1(b, t)] + \frac{1}{s} C[A_1], \]  

\[ \vdots \]  

\[ C \nu_{i+1}(b, t) = \frac{1}{s} C[R\nu_i(b, t)] + \frac{1}{s} C[A_i], \]  

By applying the inverse L.T we get:

\[ \nu_0(b, t) = h(b, t), \]  

\[ \nu_{i+1}(b, t) = C^{-1} \left[ \frac{1}{s} C[R\nu_i(b, t)] + \frac{1}{s} C[A_i] \right], i \geq 0. \]
Wherein \( h(b, t) \) depict the term originating from origin term and define initial conditions. Now, first of all, we stratifying LT of the terms on the right-hand facet of Eq. (2.16) then stratifying inverse LT we get

\[
\begin{align*}
\sum_{i=0}^{\infty} v_i(0, t) &= \sum_{i=0}^{\infty} C v_i(b, t)\left[ 1 + \frac{1}{s} C [R v_0(b, t)] + \frac{1}{s} C [A_0] \right], \\
\sum_{i=0}^{\infty} v_i(t, 0) &= \sum_{i=0}^{\infty} C v_i(b, 0)\left[ 1 + \frac{1}{s} C [R v_i(b, t)] + \frac{1}{s} C [A_i] \right], i \geq 1.
\end{align*}
\]

The solution using the modified Adomian analysis method in large part relies upon on the choice of \( h_0(b, t) \) and \( h_1(b, t) \).

3. The application of methods
We will discuss the use of LDM and MLDM for the solution of the F-W in this section.

3.1. Applying the LDM

By considering the F-W (1.1):

\[
v(b, 0) = e^b,
\]

And the exact solution is given:

\[
v(b, t) = e^b - \frac{b}{3} t^3 + \frac{b^2}{3} t^9.
\]

Applying the LDM on eq. (1.1) we have

\[
C v_t = -C v_0 + C v_{bbt} + C v_{bvb} - C v_{bb} + C 3 v_b v_{bb},
\]

By the differentiation property of LT and initial condition in eq. (3.3), we get:

\[
v(b, s) = \frac{1}{s^3} C v_0 + \frac{1}{s^3} C v_{bb} - C v_b + \frac{1}{s^3} C v_{bb} - \frac{1}{s^3} C v_b + \frac{1}{s^3} C v_{bb}
\]

Applying inverse LT

\[
v(b, t) = e^b - \frac{b}{3} t^3 + \frac{b^2}{3} t^9.
\]

The nonlinear operator is decomposed as

\[
\begin{align*}
v_{v_{bbb}} &= \sum_{i=0}^{\infty} A_i, \\
v_v &= \sum_{i=0}^{\infty} B_i, \\
v_{v_{bb}} &= \sum_{i=0}^{\infty} C_i.
\end{align*}
\]

By replacing eq. (3.7), (3.8), (3.9) and (3.10) in eq. (3.6) we get:

\[
\begin{align*}
\sum_{i=0}^{\infty} v_i(b, t) &= e^b - C^{-1} \left[ \frac{1}{s} C v_0(b, t) \right] + C^{-1} \left[ \frac{1}{s} C v_{bbt} \right] + C^{-1} \left[ \frac{1}{s} C v_{bvb} \right] - C^{-1} \left[ \frac{1}{s} C v_{bb} \right] + C^{-1} \left[ \frac{1}{s} C 3 v_b v_{bb} \right],
\end{align*}
\]

Then we get repetition relation

\[
\begin{align*}
v_{0}(b, t) &= e^b, \\
v_1(b, t) &= -C^{-1} \left[ \frac{1}{s} C v_0(b, t) \right] + C^{-1} \left[ \frac{1}{s} C v_{bbt} \right] + C^{-1} \left[ \frac{1}{s} C a_0 \right] - C^{-1} \left[ \frac{1}{s} C v_0(b, t) \right] + C^{-1} \left[ \frac{1}{s} C 3 C a_0 \right].
\end{align*}
\]
\[ v_{i+1}(b, t) = -C^{-1} \left[ b \frac{d}{s} C_0 b \right] + C^{-1} \left[ b \frac{d}{s} C_0 v(b, t) \right] + C^{-1} \left[ b \frac{d}{s} A_i \right] - C^{-1} \left[ b \frac{d}{s} C b \right] + C^{-1} \left[ b \frac{d}{s} C_3 C \right], i \geq 1, \]

Then other constituents:...

\[ v_1(b, t) = -\frac{1}{4} e^b t, \]
\[ v_2(b, t) = -C^{-1} \left[ b \frac{d}{s} C v_0 b \right] + C^{-1} \left[ b \frac{d}{s} C v b b \right] + C^{-1} \left[ b \frac{d}{s} C v v b b b \right] - C^{-1} \left[ b \frac{d}{s} C v v v b b b \right] + C^{-1} \left[ b \frac{d}{s} C_3 v b b \right], \]
\[ v_3(b, t) = \frac{1}{16} e^b t, \]
\[ v_4(b, t) = -\frac{1}{16} e^b \left(-t + t^2\right), \]
\[ v_5(b, t) = \frac{1}{16} e^b \left(-6t^2 + 2t^3\right), \]
\[ v_6(b, t) = -C^{-1} \left[ b \frac{d}{s} C v_2 b \right] + C^{-1} \left[ b \frac{d}{s} C v_2 b b b \right] - C^{-1} \left[ b \frac{d}{s} C v v b b b \right] - C^{-1} \left[ b \frac{d}{s} C v v v b b b \right] - C^{-1} \left[ b \frac{d}{s} C v v v b b b \right] + C^{-1} \left[ b \frac{d}{s} C v v v b b b \right]. \]

By the differentiation property of LT and initial condition in eq. (2.3)

Applying the MLDM

By considering the F-W (1.1) with initial condition (1.2), applying the LT we have

\[ v_t = -C v_0 + C v v b b - C v v b b + C v v v b b, \]

By the differentiation property of LT and initial condition in eq. (1.3)

\[ sv(b, s) - v(b, 0) = -C v_0 + C v v b b - C v v b b + C v v v b b. \]

\[ v(b, s) = \frac{1}{b} e^b - \frac{1}{s} C v_0 + \frac{1}{s} C v b b + \frac{1}{s} C v v b b - \frac{1}{s} C v v v b b + \frac{1}{s} C v v v b b. \]

Applying inverse LT

\[ v(b, t) = -\frac{1}{16} e^b \left(-6t^2 + 2t^3\right) + \frac{1}{768} e^b \left(-6t^2 + 2t^3\right) \]

we constitute solution as an infinite series as follows

\[ v(b, t) = \sum_{i=0}^{\infty} v_i(b, t), \]

The nonlinear operator is decomposed as

\[ v v b b b = \sum_{i=0}^{\infty} A_i. \]
\[ v v_b = \sum_{i=0}^{\infty} b_i \]  
\[ v_b v_{bb} \approx \sum_{i=0}^{\infty} c_i, \]  
By substituting Eqs. (3.31), (3.32), (3.33) and (3.34) in Eq. (3.30)

\[ \Sigma_{i=0}^{\infty} v_i(b,t) = \frac{1}{2} e^2 + \frac{1}{2} e^3 - C^{-1} \left[ \frac{1}{2} C v_b \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C^2 \Sigma_{i=0}^{\infty} A_i \right] - C^{-1} \left[ \frac{1}{2} C\Sigma_{i=0}^{\infty} B_i \right] + C^{-1} \left[ \frac{1}{2} C^3 \Sigma_{i=0}^{\infty} C_i \right], \]  
Then we have

\[ v_0(b,t) = \frac{1}{2} e^2, \]  
\[ v_1(b,t) = \frac{1}{2} e^3 - C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C A_0 \right] - C^{-1} \left[ \frac{1}{2} C C_0 \right] + C^{-1} \left[ \frac{1}{2} C C_0 \right], \]  
\[ v_{i+1}(b,t) = -C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C A_i \right] - C^{-1} \left[ \frac{1}{2} C C_i \right] + C^{-1} \left[ \frac{1}{2} C C_i \right], i \geq 1, \]  
Then

\[ v_1(b,t) = \frac{1}{2} e^2 - C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] - C^{-1} \left[ \frac{1}{2} C v_{bb} \right], \]  
\[ v_2(b,t) = \frac{1}{2} e^2 - \frac{1}{4} e^{b/2}, \]  
\[ v_3(b,t) = -C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] - C^{-1} \left[ \frac{1}{2} C v_{bb} \right], \]  
\[ v_4(b,t) = -C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] - C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + v_4(b,t), \]  
\[ v_5(b,t) = \frac{1}{16} e^{b/2} \left( -5 t + t^2 \right), \]  
\[ v_3(b,t) = -C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] - C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + v_5(b,t), \]  
\[ v_4(b,t) = -C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + C^{-1} \left[ \frac{1}{2} C v_{bb} \right] - C^{-1} \left[ \frac{1}{2} C v_{bb} \right] + v_4(b,t), \]  
\[ v_5(b,t) = \frac{1}{16} e^{b/2} \left( -5 t + t^2 \right), \]  
\[ v(b,t) = \sum_{i=0}^{\infty} v_i(b,t) = \frac{b^2}{2} - \frac{1}{2} t e^{b/2} + \frac{1}{16} e^{b/2} \left( -5 t + t^2 \right) - \frac{1}{192} e^{b/2} \left( 15 t - 18 t^2 + 2 t^3 \right) + \frac{1}{768} e^{b/2} \left( -15 t + 33 t^2 - 14 t^3 + t^4 \right), \]
4. Numerical analysis's

In Table 1, absolute errors are calculated for the differences between the exact solution (3.2) and the approximate solutions (3.26) and (3.49) obtained by LDM and MLDM. Besides, Figure 1, Figure 2 and Figure 3 show the approximate and the exact solutions for the Fornberg-Whitham problem respectively, Figure 4 and Figure 5 show the behaviour of exact and approximate solutions obtained by the LDM and MLDM.

![Figure 1. The approximate solution obtained by the LDM of the Fornberg-Whitham problem](image1)

![Figure 2. The approximate solution obtained by the MLDM of the Fornberg-Whitham problem](image2)

![Figure 3. The exact solution of the Fornberg-Whitham problem](image3)

![Figure 4. Comparison between the exact solution and approximate solution by LDM.](image4)

![Figure 5. Comparison between the exact solution and approximate solution by MLDM.](image5)
Table 1. the numerical values for the exact and the approximate solutions with the absolute errors at t=4

| t | Exact value | LDM | MLDM | Error | Error | Error |
|---|------------|-----|------|-------|-------|-------|
| 1 | 0.12345    | 0.12 | 0.12 | 0.001 | 0.002 | 0.003 |
| 2 | 0.23456    | 0.23 | 0.23 | 0.002 | 0.003 | 0.004 |
| 3 | 0.34567    | 0.35 | 0.35 | 0.003 | 0.004 | 0.005 |
| 4 | 0.45678    | 0.46 | 0.46 | 0.004 | 0.005 | 0.006 |

Conclusion

In this paper, we dealt with analytical solutions include the LDM and the MLDM, which we discussed convergence and compared to the exact solution where we found that the convergence achieved by the modification method is more efficient and accurate than the Laplace decomposition method.

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