High-Dimensional Optimization in Adaptive Random Subspaces

Jonathan Lacotte∗ Mert Pilanci∗ Marco Pavone†

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Abstract

We propose a new randomized optimization method for high-dimensional problems which can be seen as a generalization of coordinate descent to random subspaces. We show that an adaptive sampling strategy for the random subspace significantly outperforms the oblivious sampling method, which is the common choice in the recent literature. The adaptive subspace can be efficiently generated by a correlated random matrix ensemble whose statistics mimic the input data. We prove that the improvement in the relative error of the solution can be tightly characterized in terms of the spectrum of the data matrix, and provide probabilistic upper-bounds. We then illustrate the consequences of our theory with data matrices of different spectral decay. Extensive experimental results show that the proposed approach offers significant speed ups in machine learning problems including logistic regression, kernel classification with random convolution layers and shallow neural networks with rectified linear units. Our analysis is based on convex analysis and Fenchel duality, and establishes connections to sketching and randomized matrix decomposition.

1 Introduction

Random Fourier features, Nystrom method and sketching techniques have been successful in large scale machine learning problems. The common practice is to employ oblivious sampling or sketching matrices, which are typically randomized and fixed ahead of the time. However, it is not clear whether one can do better by adapting the sketching matrices to data. In this paper, we show that adaptive sketching matrices can significantly improve the approximation quality. We characterize the approximation error on the optimal solution in terms of the smoothness of the function, and spectral properties of the data matrix.

Many machine learning problems end up being high dimensional optimization problems, which typically follow from forming the kernel matrix of a large dataset or mapping the data through a high dimensional feature map, such as random Fourier features [1] or convolutional neural networks [2]. Such high dimensional representations induce higher computational and memory complexities, and result in slower training of the models. Random projections are a classical way of performing dimensionality reduction, and are widely used in many algorithmic contexts [3]. Nevertheless, only recently these methods have captured great attention as an effective way of performing dimensionality reduction in convex optimization.

In the context of solving a linear system $Ax = b$ and least-squares optimization, the authors of [4] propose a randomized iterative method with linear convergence rate, which, at each iteration, performs a proximal update $x^{(k+1)} = \arg\min_{x \in T} \|x - x^{(k)}\|_2^2$, where the next iterate $x^{(k+1)}$ is restricted to lie within an affine subspace $T = x^{(k)} + \text{range}(A^TS)$, and $S$ is a $n \times m$ dimension-reduction matrix with $m \leq \min\{n, d\}$. In the context of kernel ridge regression, the authors of [5] propose to approximate the $n$-dimensional kernel matrix by sketching its columns to a lower $m$-dimensional subspace, chosen uniformly at random. From the low dimensional kernel ridge solution $\alpha^* \in \mathbb{R}^m$, they show how to reconstruct an approximation $\tilde{x} \in \mathbb{R}^n$ of the high dimensional solution $x^* \in \mathbb{R}^n$. Provided that the sketching dimension $m$ is large enough – as measured by the spectral properties of the kernel matrix $K$ –, the estimate $\tilde{x}$ retains some statistical properties of $x^*$, e.g., minimaxity. Similarly, in the

∗Department of Electrical Engineering, Stanford University
†Department of Aeronautics & Astronautics, Stanford University
broader context of classification through convex loss functions, the authors of \cite{5, 6} propose to project the \(d\)-dimensional features of a given data matrix \(A\) to a lower \(m\)-dimensional subspace, chosen independently of the data. After computing the optimal low-dimensional classifier \(\alpha^* \in \mathbb{R}^m\), their algorithm returns an estimate \(\tilde{x} \in \mathbb{R}^d\) of the optimal classifier \(x^* \in \mathbb{R}^d\). Even though they provide formal guarantees on the estimation error \(\|\tilde{x} - x^*\|_2\), their results rely on several restrictive assumptions, that is, the data matrix \(A\) must be low rank, or, the classifier \(x^*\) must lie in the span of the top few left singular vectors of \(A\). Further, random subspace optimization has also been explored for large-scale trust region problems \cite{7}, also using a subspace chosen uniformly at random. Our proposed approach draws connections with the Gaussian Kaczmarz method proposed in \cite{34} and the kernel sketching method in \cite{4}. Differently, we are interested in smooth, convex optimization problems with ridge regularization. In contrast to \cite{5, 6}, we do not make any assumption on the optimal solution \(x^*\).

Our work relates to the considerable amount of literature on randomized approximations of high dimensional kernel matrices \(K\). The typical approach consists of building a low-rank factorization of the matrix \(K\), using a random subset of its columns \cite{8, 9, 10, 11}. The so-called Nystrom method has proven to be effective empirically \cite{12}, and many research efforts have been devoted to improving and analyzing the performance of its many variants (e.g., uniform column sub-sampling, leverage-score based sampling), especially in the context of kernel ridge regression \cite{13, 14}. In a related vein, sketching methods have been proposed to reduce the space complexity of storing high-dimensional data matrices \cite{15, 4}, by projecting their rows to a randomly chosen lower dimensional subspace. Our theoretical findings build on known results for low-rank factorization of positive semi-definite (p.s.d.) matrices \cite{16, 17, 18, 19}, and show intimate connections with kernel matrices sketching \cite{4}. Lastly, our problem setting also draws connections with compressed sensing \cite{20} where the goal is to recover a high dimensional structured signal from a small number of randomized, and usually oblivious, measurements.

### 1.1 Contributions

In this work, we propose a novel randomized subspace optimization method with strong solution approximation guarantees which outperform oblivious sampling methods. We derive probabilistic bounds on the error of approximation for general convex functions. We show that our method provides a significant improvement over the oblivious version, and theoretically quantify this observation as function of the spectral properties of the data matrix. We also introduce an iterative version of our method, which converges to the optimal solution by iterative refinement.

### 1.2 An overview of our results

Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a convex and \(\mu\)-strongly smooth function, i.e., \(\nabla^2 f(w) \preceq \mu I_n\) for all \(w \in \mathbb{R}^n\), and \(A \in \mathbb{R}^{n \times d}\) a high-dimensional matrix. We are interested in solving the primal problem

\[
x^* = \underset{x \in \mathbb{R}^d}{\text{argmin}} \ f(Ax) + \frac{\lambda}{2} \|x\|_2^2,
\]

Given a random matrix \(S \in \mathbb{R}^{d \times m}\) with \(m \ll d\), we consider instead the sketched primal problem

\[
\alpha^* = \underset{\alpha \in \mathbb{R}^m}{\text{argmin}} \ f(AS\alpha) + \frac{\lambda}{2} \alpha^\top S^\top S\alpha,
\]

where we effectively restrict the optimization domain to a lower \(m\)-dimensional subspace. In this work, we explore the following questions: How can we estimate the original solution \(x^*\) given the sketched solution \(\alpha^*\)? Is a uniformly random subspace the optimal choice, e.g., \(S \sim \text{Gaussian i.i.d.}\)? Or, can we come up with an adaptive sampling distribution that is related to the matrix \(A\), which yields stronger guarantees?

By Fenchel duality analysis, we exhibit a natural candidate for an approximate solution to \(x^*\), given by \(\tilde{x} = -\lambda^{-1} A^\top \nabla f(AS\alpha^*)\). Our main result (Section 2) establishes that, for an adaptive sketching matrix of the form \(\hat{S} = A^\top \hat{S}\) where \(\hat{S}\) is typically Gaussian i.i.d., the relative error satisfies a high-probability guarantee of
there exist an unique primal solution where. We define the following deterministic functional We introduce the Fenchel conjugate of \( P \) where \( z \) is the orthogonal projector onto the range of \( A \). The relationship \( x^* = -\lambda^{-1} A^\top \nabla f(Ax^*) \) suggests the point \( \bar{x} = -\lambda^{-1} A^\top \nabla f(Ax^*) \) as a candidate for approximating \( x^* \). The Fenchel dual programs of \( 1 \) and \( 2 \) only differ in their quadratic regularization term, \( \|A^\top z\|^2_2 \) and \( \|P_S A^\top y\|^2_2 \), which difference is tied to the quantity \( \|P_{\perp S} A^\top (\bar{y} - y^*)\|^2_2 \). As it holds that \( \|\bar{x} - x^*\|^2_2 = \lambda^{-1} \|A^\top (z - y^*)\|^2_2 \), we show that the error \( \|\bar{x} - x^*\|^2_2 \) can be controlled in terms of the spectral norm \( \|P_{\perp S} A^\top\|_2 \), or more sharply, in terms of the quantity \( Z_f \), which satisfies \( Z_f \leq \|P_{\perp S} A^\top\|_2 \). We formalize this statement in our next result, which proof is deferred to Appendix B.1.

## 2 Convex optimization in adaptive random subspaces

We introduce the Fenchel conjugate of \( f \), defined as \( f^*(z) := \sup_{w \in \mathbb{R}^n} \{ w^\top z - f(w) \} \), which is convex and its domain \( \text{dom} f^* := \{ z \in \mathbb{R}^n \mid f^*(z) < +\infty \} \) is a closed, convex set. Our control of the relative error \( \|\bar{x} - x^*\|^2_2/\|x^*\|^2_2 \) is closely tied to controlling a distance between the respective solutions of the dual problems of \( 1 \) and \( 2 \). The proof of the next two Propositions follow from standard convex analysis arguments [21], and are deferred to Appendix C.

**Proposition 1 (Fenchel Duality).** Under the previous assumptions on \( f \), it holds that

\[
\min x \ f(Ax) + \frac{\lambda}{2} \|x\|^2_2 = \max z \ - f^*(z) - \frac{1}{2\lambda} \|A^\top z\|^2_2.
\]

There exist an unique primal solution \( x^* \) and an unique dual solution \( z^* \). Further, we have \( Ax^* \in \partial f^*(z^*) \), \( z^* = \nabla f(Ax^*) \) and \( x^* = -\frac{1}{\lambda} A^\top z^* \).

**Proposition 2 (Fenchel Duality on Sketched Program).** Strong duality holds for the sketched program

\[
\min \alpha \ f(AS\alpha) + \frac{\lambda}{2} \|S\alpha\|^2_2 = \max y \ - f^*(y) - \frac{1}{2\lambda} \|P_S A^\top y\|^2_2,
\]

where \( P_S = S(S^\top S)^{1/2} S^\top \) is the orthogonal projector onto the range of \( S \). There exist a sketched primal solution \( \alpha^* \) and an unique sketched dual solution \( y^* \). Further, for any solution \( \alpha^* \), it holds that \( AS\alpha^* \in \partial f^*(y^*) \) and \( y^* = \nabla f(AS\alpha^*) \).

We define the following deterministic functional \( Z_f \) which depends on \( f^* \), the data matrix \( A \) and the sketching matrix \( S \), and plays an important role in controlling the approximation error,

\[
Z_f \equiv Z_f(A, S) = \sup_{\Delta \in \text{dom} f^* - x^*} \left( \frac{\Delta^\top A P_{\perp S} A^\top \Delta}{\|\Delta\|^2_2} \right)^{1/2}, \tag{3}
\]

where \( P_{\perp S} = I - P_S \) is the orthogonal projector onto \( \text{range}(S)^\perp \). The relationship \( x^* = -\lambda^{-1} A^\top \nabla f(Ax^*) \) suggests the point \( \bar{x} = -\lambda^{-1} A^\top \nabla f(Ax^*) \) as a candidate for approximating \( x^* \). The Fenchel dual programs of \( 1 \) and \( 2 \) only differ in their quadratic regularization term, \( \|A^\top z\|^2_2 \) and \( \|P_S A^\top y\|^2_2 \), which difference is tied to the quantity \( \|P_{\perp S} A^\top (\bar{y} - y^*)\|^2_2 \). As it holds that \( \|\bar{x} - x^*\|^2_2 = \lambda^{-1} \|A^\top (z - y^*)\|^2_2 \), we show that the error \( \|\bar{x} - x^*\|^2_2 \) can be controlled in terms of the spectral norm \( \|P_{\perp S} A^\top\|_2 \), or more sharply, in terms of the quantity \( Z_f \), which satisfies \( Z_f \leq \|P_{\perp S} A^\top\|_2 \). We formalize this statement in our next result, which proof is deferred to Appendix B.1.
Theorem 1 (Deterministic bound). Let $\alpha^*$ be any minimizer of the sketched program (2). Then, under the condition $\lambda \geq 2\mu Z_f^2$, we have
\[
\|\bar{x} - x^*\|_2 \leq \frac{\mu}{2\lambda} Z_f \|x^*\|_2,
\]
which further implies
\[
\|\bar{x} - x^*\|_2 \leq \frac{\mu}{2\lambda} \|P_S^A \tilde{A}^\top\|_2 \|x^*\|_2.
\]

For an adaptive sketching matrix $S = A^\top \tilde{S}$, we rewrite $\|P_S^A \tilde{A}^\top\|_2 = \|K - \tilde{K}(S^\top \tilde{K} S)^\top \tilde{S}^\top K\|_2$, where $K = AA^\top$ is p.s.d. Combining our deterministic bound (5) with known results [16, 13, 17] for randomized low-rank matrix factorization in the form $K\tilde{S}(\tilde{K} S)^\top \tilde{S}^\top K$ of p.s.d. matrices $K$, we can give guarantees with high probability (w.h.p.) on the relative error for various types of matrices $\tilde{S}$. For conciseness, we specialize our next result to adaptive Gaussian sketching, i.e., $\tilde{S}$ Gaussian i.i.d. Given a target rank $k \geq 2$, we introduce a measure of the spectral tail of $A$ as $R_k(A) = (\sigma_2^2 + \frac{1}{k} \sum_{j=k+1}^p \sigma_j^2)^{\frac{1}{2}}$, where $p$ is the rank of the matrix $A$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p$ its singular values. The proof of the next result follows from a combination of Theorem 1 and Corollary 10.9 in [16], and is deferred to Appendix B.2.

Corollary 1 (High-probability bound). Given $k \leq \min(n, d)/2$ and a sketching dimension $m = 2k$, let $S = A^\top \tilde{S}$, with $\tilde{S} \in \mathbb{R}^{n \times m}$ Gaussian i.i.d. Then, for some universal constant $c_0 \leq 36$, provided $\lambda \geq 2\mu c_0^2 R_k^2(A)$, it holds with probability at least $1 - 12e^{-k}$ that
\[
\|\bar{x} - x^*\|_2 \leq c_0 \frac{\mu}{2\lambda} R_k(A) \|x^*\|_2.
\]

Remark 1. The quantity $Z_f := \sup_{\Delta \in (\text{dom} f^* - z^*)} \left( \frac{\Delta^\top A P_k^A \tilde{A}^\top \Delta}{\|\Delta\|_2^2} \right)^{\frac{1}{2}}$ is the eigenvalue of the matrix $P_k^A \tilde{A}^\top$, restricted to the spherical cap $\mathcal{K} := (\text{dom} f^* - z^*) \cap \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in dimension $n$. Thus, depending on the geometry of $\mathcal{K}$, the deterministic bound (4) might be much tighter than (5), and yield a probabilistic bound better than (6). The investigation of such a result is left for future work.

2.1 Theoretical predictions as a function of spectral decay

We study the theoretical predictions given by [5] on the relative error, for different spectral decays of $A$ and sketching methods, in particular, adaptive Gaussian sketching versus oblivious Gaussian sketching and leverage score column sub-sampling [15]. We denote $\nu_k = \sigma_k^2$ the eigenvalues of $AA^\top$. For conciseness, we absorb $\mu$ into the eigenvalues by setting $\nu_k = \mu \nu_k$ and $\mu \equiv 1$. This re-scaling leaves the right-hand side of the bound (5) unchanged, and does not affect the analysis below. Then, we assume that $\nu_1 = \mathcal{O}(1)$, $\lambda \in (\nu_\rho, \nu_1)$, and $\lambda \to 0$ as $n \to +\infty$. These assumptions are standard in empirical risk minimization and kernel regression methods [22], which we focus on in Sections 3 and 5. We consider three decaying schemes of practical interest. The matrix $A$ has either a finite-rank $\rho$, a $\kappa$-exponential decay where $\nu_j \sim e^{-\kappa j}$ and $\kappa > 0$, or, a $\beta$-polynomial decay where $\nu_k \sim j^{-2\beta}$ and $\beta > 1/2$. Among other examples, these decays are characteristic of various standard kernel functions, such as the polynomial, Gaussian and first-order Sobolev kernels [23]. Given a precision $\epsilon > 0$ and a confidence level $\eta \in (0, 1)$, we denote by $m_A$ (resp. $m_O$, $m_S$) a sufficient dimension for which adaptive (resp. oblivious, leverage score) sketching yields the following $(\epsilon, \eta)$-guarantee on the relative error. That is, with probability at least $1 - \eta$, it holds that $\|\bar{x} - x^*\|_2 / \|x^*\|_2 \leq \epsilon$.

We determine $m_A$ from our probabilistic regret bound (6). For $m_O$, using our deterministic regret bound (5), it then suffices to bound the spectral norm $\|P_k^A \tilde{A}^\top\|_2$ in terms of the eigenvalues $\nu_k$, when $S$ is a leverage score column sub-sampling matrix. To the best of our knowledge, the tightest bound has been given by [18] (see Lemma 5). For $m_S$, we leverage results from [5]. The authors provide an upper bound on the relative error $\|\bar{x} - x^*\|_2 / \|x^*\|_2$, when $S$ is Gaussian i.i.d. with variance $\frac{1}{2}$. It should be noted that their sketched solution $\alpha^*$ is slightly different from ours. They solve $\alpha^* = \arg\min f(AS\alpha) + (2\lambda)^{-1} \|\alpha\|_2^2$, whereas we do
include the matrix $S$ in the regularization term. One might wonder which regularizer works best when $S$ is Gaussian i.i.d. Through extensive numerical simulations, we observed a strongly similar performance. Further, standard Gaussian concentration results yields that $\|S\alpha\|_2^2 \approx \|\alpha\|_2^2$

Our theoretical findings are summarized in Table 1 and we give the mathematical details of our derivations in Appendix D. For the sake of clarity, we provide in Table 1 lower bounds on the predicted values $m_O$ and $m_S$, and, thus, lower bounds on the ratios $m_O/m_A$ and $m_S/m_A$. Overall, adaptive Gaussian sketching provides stronger guarantees on the relative error $\|\tilde{x} - x^*\|_2/\|x^*\|_2$.

Table 1: Sketching dimensions for a $(\varepsilon, \eta)$-guarantee on the relative error $\|\tilde{x} - x^*\|_2/\|x^*\|_2$.

| $\rho$-rank matrix | $\kappa$-exponential decay | $\beta$-polynomial decay |
|----------------------|-----------------------------|--------------------------|
| Adaptive Gaussian ($m_A$) | $(\rho + 1 + \log(4/\eta))$ | $\kappa^{-1} \log \left( \frac{\lambda_1}{\beta^2} \right) + \log \left( \frac{\lambda_2}{\eta} \right)$ |
| Oblivious Gaussian ($m_O$) | $(\rho + 1) \varepsilon^{-2} \log \left( \frac{2\rho}{\eta} \right)$ | $\kappa^{-1} \varepsilon^{-2} \log \left( \frac{2\rho}{\eta} \right)$ |
| Leverage score ($m_S$) | $(\rho + 1) \log \left( \frac{4\rho}{\eta} \right)$ | $\kappa^{-1} \log \left( \frac{1}{\varepsilon^2} \right) \log \left( \frac{1}{\eta} \right)$ |
| Lower bound on $m_O/m_A$ | $\varepsilon^{-2} \log \rho$ | $\varepsilon^{-2} + \log(2d/\eta)$, $\forall h > 0$ |
| Lower bound on $m_S/m_A$ | $\log \rho$ | $\min \left( \log \left( \frac{1}{\varepsilon^2} \right), \kappa^{-1} \log \left( \frac{1}{\varepsilon^2} \right) \right)$ |

We illustrate numerically our predictions for adaptive Gaussian sketching versus oblivious Gaussian sketching. With $n = 1000$ and $d = 2000$, we generate matrices $A^{\text{exp}}$ and $A^{\text{poly}}$, with spectral decay satisfying respectively $\nu_j \sim n e^{-0.1j}$ and $\nu_j \sim n j^{-2}$. First, we perform binary logistic regression, with $f(Ax) = n^{-1} \sum_{i=1}^n \ell_{y_i}(a_i^\top x)$ where $\ell_{y_i}(z) = y_i \log(1 + e^{-z}) + (1 - y_i) \log(1 + e^z)$, $y \in \{0,1\}^n$ and $a_i$ is the $i$-th row of $A$. For the polynomial (resp., exponential) decay, we observe the relative error $\|\tilde{x} - x^*\|_2/\|x^*\|_2$ to follow w.h.p. a decay proportional to $m^{-1}$ (resp., $e^{-0.05m}$). Figure 1 confirms those predictions. We repeat the same experiments with a second loss function, $f(Ax) = (2n)^{-1} \sum_{i=1}^n (a_i^\top x)^2 - 2(a_i^\top x) y_i$. The latter is a convex relaxation of the penalty $\frac{1}{2} \|Ax\|_2^2$ for fitting a shallow neural network with a ReLU non-linearity. Again, Figure 1 confirms our predictions, and we observe that the adaptive method performs much better than the oblivious sketch.

Figure 1: Relative error versus sketching dimension $m \in \{2^k \mid 3 \leq k \leq 10\}$ of adaptive Gaussian sketching (red) and oblivious Gaussian sketching (green), for the ReLU and logistic models, and the exponential and polynomial decays. We use $\lambda = 10^{-4}$ for all simulations. Results are averaged over 10 trials. Bar plots show (twice) the empirical standard deviations.

3 Algorithms for adaptive subspace sketching

3.1 Numerical conditioning and generic algorithm

A standard quantity to characterize the capacity of a convex program to be solved efficiently is its condition number \cite{24}, which, for the primal \cite{1} and (adaptive) sketched program \cite{2}, are given by

$$\kappa = \lambda + \sup_x \sigma_1 \left( A^\top \nabla^2 f(Ax)A \right) \lambda + \inf_x \sigma_d \left( A^\top \nabla^2 f(Ax)A \right), \quad \kappa_S = \frac{\sup_\alpha \sigma_1 \left( \tilde{S}^\top A(\lambda I + A^\top \nabla^2 f(AA^\top S\alpha)A)A^\top \tilde{S} \right)}{\inf_\alpha \sigma_d \left( \tilde{S}^\top A(\lambda I + A^\top \nabla^2 f(AA^\top S\alpha)A)A^\top \tilde{S} \right)}.$$
The latter can be significantly larger than $\kappa$, up to $\kappa_S \approx \kappa \frac{\sigma_1(S^\top A A^\top S)}{\sigma_m(S^\top A A^\top S)} \gg \kappa$. A simple change of variable overcomes this issue. With $A_{S,\dagger} = AS(S^\top S)^{-\frac{1}{2}}$, we solve instead the optimization problem

$$
\alpha_\dagger^t = \arg\min_{\alpha_t \in \mathbb{R}^n} f(A_{S,\dagger}\alpha_t) + \frac{\lambda}{2}\|\alpha_t\|_2^2.
$$

It holds that $\bar{x} = -\lambda^{-1} A^\top \nabla f(A_{S,\dagger}\alpha_\dagger^t)$. The additional complexity induced by this change of variables comes from computing the (square-root) pseudo-inverse of $S^\top S$, which requires $\mathcal{O}(m^3)$ flops via a singular value decomposition. When $m$ is small, this additional computation is negligible and numerically stable, and the re-scaled sketched program (7) is actually better conditioned that the original primal program (1), as stated in the next result that we prove in Appendix [C.3]

**Proposition 3.** Under adaptive sketching, the condition number $\kappa_1$ of the re-scaled sketched program (7) satisfies $\kappa_1 \leq \kappa$ with probability 1.

**Algorithm 1:** Generic algorithm for adaptive sketching.

**Input:** Data matrix $A \in \mathbb{R}^{n \times d}$, random matrix $S \in \mathbb{R}^{n \times m}$ and parameter $\lambda > 0$.

1. Compute the sketching matrix $S = A^\top S$, and, the sketched matrix $A_S = AS$.
2. Compute the re-scaling matrix $R = (S^\top S)^{-\frac{1}{2}}$, and the re-scaled sketched matrix $A_{S,\dagger} = A_SR$.
3. Solve the convex optimization problem (7), and return $\bar{x} = -\frac{1}{\lambda}A^\top \nabla f \left(A_{S,\dagger}\alpha_\dagger^t\right)$.

We observed a drastic practical performance improvement between solving the sketched program as formulated in (2) and its well-conditioned version (7).

If the chosen sketch dimension $m$ is itself prohibitively large for computing the matrix $Q = (S^\top S)^{-\frac{1}{2}}$, one might consider a pre-conditioning matrix $Q$, which is faster to compute, and such that the matrix $SQ$ is well-conditioned. Typically, one might compute a matrix $Q$ based on an approximate singular value decomposition of the matrix $S^\top S$. Then, one solves the optimization problem $\alpha_Q^t = \arg\min_{\alpha_t \in \mathbb{R}^m} f(ASQ\alpha) + \frac{\lambda}{2}\|SQ\alpha\|_2^2$.

Provided that $Q$ is invertible, it holds that $\bar{x}$ satisfies $\bar{x} = -\lambda^{-1} A^\top \nabla f(ASQ\alpha_Q^t)$.

### 3.2 Error contraction and almost exact recovery of the optimal solution

The estimate $\bar{x}$ satisfies a guarantee of the form $\|\bar{x} - x^*\|_2 \leq \varepsilon\|x^*\|_2$ w.h.p., and, with $\varepsilon < 1$ provided that $\lambda$ is large enough. Here, we extend Algorithm [1] to an iterative version which takes advantage of this error contraction, and which is relevant when a high-precision estimate $\bar{x}$ is needed.

**Algorithm 2:** Iterative adaptive sketching

**Input:** Data matrix $A \in \mathbb{R}^{n \times d}$, random matrix $S \in \mathbb{R}^{n \times m}$, iterations number $T$, parameter $\lambda > 0$.

1. Compute the sketched matrix $A_{S,\dagger}$ as in Algorithm 1. Set $\bar{x}^{(0)} = 0$.
2. for $t = 1, 2, \ldots, T$ do
3. Compute $a^{(t)} = \tilde{A}\bar{x}^{(t-1)}$, and, $b^{(t)} = (S^\top S)^{-\frac{1}{2}}S^\top\bar{x}^{(t-1)}$.
4. Solve the following convex optimization problem

$$
\alpha_\dagger^{(t)} = \arg\min_{\alpha_t \in \mathbb{R}^n} f(A_{S,\dagger}\alpha_t + a^{(t)}) + \frac{\lambda}{2}\|\alpha_t + b^{(t)}\|_2^2.
$$

Update the solution by $\bar{x}^{(t)} = -\frac{1}{\lambda}A^\top \nabla f(A_{S,\dagger}\alpha_\dagger^{(t)} + a^{(t)})$.
5. end
6. Return the last iterate $\bar{x}^{(T)}$.

A key advantage is that, at each iteration, the same sketching matrix $S$ is used. Thus, the sketched matrix $A_{S,\dagger}$ has to be computed only once, at the beginning of the procedure. The output $\bar{x}^{(T)}$ satisfies the following recovery property, which empirical benefits are illustrated in Figure 2.


Theorem 2. After $T$ iterations of Algorithm 2, provided that $\lambda \geq 2\mu Z_f^2$, it holds that

$$\|\tilde{x}^{(T)} - x^*\|_2 \leq \left( \frac{\mu Z_f^2}{2\lambda} \right)^{\frac{T}{2}} \|x^*\|_2.$$  \hspace{1cm} \text{(9)}$$

Further, if $S = A^\top \tilde{S}$ where $\tilde{S} \in \mathbb{R}^{n \times m}$ with i.i.d. Gaussian entries and $m = 2k$ for some target rank $k \geq 2$, then, for some universal constant $c_0 \leq 36$, after $T$ iterations of Algorithm 2, provided that $\lambda \geq 2c_0^2 \mu R_k^2(A)$, the approximate solution $\tilde{x}^{(T)}$ satisfies with probability at least $1 - 12e^{-k}$,

$$\|\tilde{x}^{(T)} - x^*\|_2 \leq \left( \frac{c_0^2 \mu R_k^2(A)}{2\lambda} \right)^{\frac{T}{2}} \|x^*\|_2.$$  \hspace{1cm} \text{(10)}$$

Figure 2: Relative error versus sketching dimension $m \in \{2^k \mid 3 \leq k \leq 10\}$ of adaptive Gaussian sketching for (a) the iterative method (Algorithm 2) and (b) the power method (see Remark 2). We use the MNIST dataset with images mapped through 10000-dimensional random Fourier features [1] for even-vs-odd classification using binary logistic loss, and, $\lambda = 10^{-5}$. Results are averaged over 20 trials. Bar plots show (twice) the empirical standard deviations.

Remark 2. An immediate extension of Algorithms 1 and 2 consists in using the power method [16]. Given $q \in \mathbb{N}$, one uses the sketching matrix $S = (A^\top A)^q A^\top \tilde{S}$. The larger $q$, the smaller the approximation error $\|AA^\top - AS(S^\top S)^{1/2} S^\top A^\top\|_2$ (see Corollary 10.10 in [16]). Of practical interest are data matrices $A$ with a spectral profile starting with a fast decay, and then becoming flat. This happens typically for $A$ of the form $A = \bar{A} + W$, where $\bar{A}$ has a fast decay and $W$ is a noise matrix with, for instance, independent subgaussian rows [33]. Our results easily extend to this setting and we illustrate its empirical benefits in Figure 2.

4 Application to empirical risk minimization and kernel methods

By the representer theorem, the primal program (1) can be re-formulated as

$$w^* \in \arg\min_{w \in \mathbb{R}^n} f(Kw) + \frac{\lambda}{2} w^\top Kw,$$  \hspace{1cm} \text{(11)}$$

where $K = AA^\top$. Clearly, it holds that $x^* = A^\top w^*$. Given a matrix $\tilde{S}$ with i.i.d. Gaussian entries, we consider the sketched version of the kernelized primal program (11),

$$\alpha^* \in \arg\min_{\alpha \in \mathbb{R}^m} f(K\tilde{S}\alpha) + \frac{\lambda}{2} \alpha^\top \tilde{S}^\top K\tilde{S}\alpha.$$  \hspace{1cm} \text{(12)}$$

The sketched program (12) is exactly our adaptive Gaussian sketched program (2). Thus, setting $\tilde{w} = -\lambda^{-1} \nabla f(K\tilde{S}\alpha^*)$, it holds that $\tilde{x} = A^\top \tilde{w}$. Since the relative error $\|\tilde{x} - x^*\|_2/\|x^*\|_2$ is controlled by the decay
of the eigenvalues of $K$, so does the relative error $\|A^T(\tilde{w} - w^*)\|_2/\|A^Tw^*\|_2$. More generally, the latter statements are still true if $K$ is any positive semi-definite matrix, and, if we replace $A$ by any square-root matrix of $K$. Here, we denote $Z_f \equiv Z_f \left( K^{\frac{1}{2}}, K^{\frac{1}{2}}S \right)$ (see Eq. (3)).

**Theorem 3.** Let $K \in \mathbb{R}^{n \times n}$ be any positive semi-definite matrix. Let $w^*$ be any minimizer of the kernel program (11) and $\alpha^*$ be any minimizer of its sketched version (12). Define the approximate solution $\tilde{w} = -\frac{1}{\lambda} \nabla f(K\tilde{S}n^*)$. If $\lambda \geq 2\mu Z_f^2$, then it holds that

$$\|K^{\frac{1}{2}}(\tilde{w} - w^*)\|_2 \leq \sqrt{\frac{\mu}{2\lambda}} Z_f \|K^{\frac{1}{2}}w^*\|_2. \quad (13)$$

For a positive definite kernel $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and a data matrix $A = [a_1, \ldots, a_n]^\top \in \mathbb{R}^{n \times d}$, let $K$ be the empirical kernel matrix, with $K_{ij} = k(a_i, a_j)$. Let $\varphi(\cdot) \in \mathbb{R}^D$ be a random feature map [11, 25], such as random Fourier features or a random convolutional neural net. We are interested in the computational complexities of forming the sketched versions of the primal (1), the kernel primal (11) and the primal (1) with $\varphi(A)$ instead of $A$. We compare the complexities of adaptive and oblivious sketching and uniform column sub-sampling. Table 2 shows that all three methods have similar complexities for computing $AS$ and $\varphi(A)S$. Adaptive sketching exhibits an additional factor 2 that comes from computing the correlated sketching matrices $S = A^\top \tilde{S}$ and $\tilde{S} = \varphi(A)^\top \tilde{S}$. In practice, the latter is negligible compared to the cost of forming $\varphi(A)$ which, for instance, corresponds to a forward pass over the whole dataset in the case of a convolutional neural network. On the other hand, uniform column sub-sampling is significantly faster in order to form the sketched kernel matrix $KS$, which relates to the well-known computational advantages of kernel Nyström methods [12].

Table 2: Complexity of forming the sketched programs, given $A \in \mathbb{R}^{n \times d}$. We denote $d_k$ the number of flops to evaluate the kernel product $k(a, a')$, and, $d_w$ the number of flops for a forward-pass $\varphi(a)$. Note that these complexities could be reduced through parallelization.

|                | $AS$       | $\varphi(A)S$                   | $KS$               |
|----------------|------------|---------------------------------|--------------------|
| Adaptive sketching | $O(2mdn)$ | $O(d_w n) + O(2mdn)$           | $O(d_k n^2) + O(mn^2)$ |
| Oblivous sketching  | $O(mdn)$   | $O(d_w n) + O(mdn)$            | -                  |
| Uniform column sub-sampling | $O(mdn)$   | $O(d_w n) + O(mdn)$            | $O(d_k nm)$        |

5 Numerical evaluation of adaptive Gaussian sketching

We evaluate Algorithm 1 on MNIST and CIFAR10. First, we aim to show that the sketching dimension can be considerably smaller than the original dimension while retaining (almost) the same test classification accuracy. Second, we aim to get significant speed-ups in achieving a high-accuracy classifier. To solve the primal program (1), we use two standard algorithms, stochastic gradient descent (SGD) with (best) fixed step size and stochastic variance reduction gradient (SVRG) [26] with (best) fixed step size and frequency update of the gradient correction. To solve the adaptive sketched program (2), we use SGD, SVRG and the sub-sampled Newton method [27, 28] – which we refer to as Sketch-SGD, Sketch-SVRG and Sketch-Newton. The latter is well-suited to the sketched program, as the low-dimensional Hessian matrix can be quickly inverted at each iteration. For both datasets, we use 50000 training and 10000 testing images. We transform each image using a random Fourier feature map $\varphi(\cdot) \in \mathbb{R}^D$, i.e., $\langle \varphi(a), \varphi(a') \rangle \approx \exp(-\gamma \|a - a'\|_2^2)$ [11, 29]. For MNIST and CIFAR10, we choose respectively $D = 10000$ and $\gamma = 0.02$, and, $D = 60000$ and $\gamma = 0.002$, so that the primal is respectively 10000-dimensional and 60000-dimensional. Then, we train a classifier via a sequence of binary logistic regressions – which allow for efficient computation of the Hessian and implementation of the Sketch-Newton algorithm –, using a one-vs-all procedure. First, we evaluate the test classification error of $\tilde{x}$. We solve to optimality the primal and sketched programs for values of $\lambda \in \{10^{-4}, 5 \cdot 10^{-5}, 10^{-5}, 5 \cdot 10^{-6}\}$ and sketching dimensions $m \in \{64, 128, 256, 512, 1024\}$. In
Table 3 are reported the results, which are averaged over 20 trials for MNIST and 10 trials for CIFAR10, and, empirical variances are reported in Appendix A. Overall, the adaptive sketched program yields a high-accuracy classifier for most couples $(\lambda, m)$. Further, we match the best primal classifier with values of $m$ as small as 256 for MNIST and 512 for CIFAR10, which respectively corresponds to a dimension reduction by a factor $\approx 40$ and $\approx 120$. These results additionally suggest that adaptive Gaussian sketching introduces an implicit regularization effect, which might be related to the benefits of spectral cutoff estimators. For instance, on CIFAR10, using $\lambda = 10^{-5}$ and $m = 512$, we obtain an improvement in test accuracy by more than 2% compared to $x^*$. Further, over some sketching dimension threshold under which the performance is bad, as the value of $m$ increases, the test classification error of $\tilde{x}$ increases to that of $x^*$, until matching it.

Further, we evaluate the test classification error of two sketching baselines, that is, oblivious Gaussian sketching for which the matrix $S$ has i.i.d. Gaussian entries, and, adaptive column sub-sampling (Nystrom method) for which $S = A^T \tilde{S}$ with $\tilde{S}$ a column sub-sampling matrix. As reported in Table 4, adaptive Gaussian sketching performs better for a wide range of values of sketching size $m$ and regularization parameter $\lambda$.

Table 4: Test classification error on MNIST and CIFAR10. "AG": Adaptive Gaussian sketch, "Ob": Oblivious Gaussian sketch, "N": Nystrom method.

| $\lambda$ | $x_{\text{MNIST}}$ | $\tilde{x}_{64}$ | $\tilde{x}_{128}$ | $\tilde{x}_{256}$ | $\tilde{x}_{512}$ | $\tilde{x}_{1024}$ | $x_{\text{CIFAR}}$ | $\tilde{x}_{64}$ | $\tilde{x}_{128}$ | $\tilde{x}_{256}$ | $\tilde{x}_{512}$ | $\tilde{x}_{1024}$ |
|-----------|-------------------|------------------|------------------|------------------|------------------|------------------|-----------------|------------------|------------------|------------------|------------------|------------------|
| $10^{-4}$  | 5.4               | 4.8              | 4.8              | 5.2              | 5.3              | 5.4              | -               | -                | -                | -                | -                | -                |
| $5 \cdot 10^{-5}$ | 4.6               | 4.1              | 3.8              | 4.0              | 4.3              | 4.5              | 51.6            | 52.1             | 50.5             | 50.6             | 50.8             | 51.0             |
| $10^{-5}$  | 2.8               | 8.1              | 3.4              | 2.4              | 2.5              | 2.8              | 48.2            | 60.1             | 54.5             | 47.7             | 45.9             | 46.2             |
| $5 \cdot 10^{-6}$ | 2.5               | 11.8             | 4.9              | 2.8              | 2.6              | 2.4              | 47.6            | 63.6             | 59.8             | 51.9             | 47.7             | 45.8             |

Then, we compare the test classification error versus wall-clock time of the optimization algorithms mentioned above. Figure 3 shows results for some values of $m$ and $\lambda$. We observe some speed-ups on the 10000-dimensional MNIST problem, in particular for Sketch-SGD and for Sketch-SVRG, for which computing the gradient correction is relatively fast. Such speed-ups are even more significant on the 60000-dimensional CIFAR10 problem, especially for Sketch-Newton. A few iterations of Sketch-Newton suffice to almost reach the minimum $\tilde{x}$, with a per-iteration time which is relatively small thanks to dimensionality reduction. Hence, it is more than 10 times faster to reach the best test accuracy using the sketched program. In addition to random Fourier features mapping, we carry out another set of experiments with the CIFAR10 dataset, in which we pre-process the images. That is, similarly to [31, 32], we map each image through a random convolutional layer. Then, we kernelize these processed images using a Gaussian kernel with $\gamma = 2 \cdot 10^{-5}$. Using our implementation, the best test accuracy of the kernel primal program [11] we obtained is 73.1%. Sketch-SGD, Sketch-SVRG and Sketch-Newton – applied to the sketched kernel program [12] – match this test accuracy, with significant speed-ups, as reported in Figure 3.

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9
Figure 3: Test classification error (percentage) versus wall-clock time (seconds).

References

[1] Ali Rahimi and Benjamin Recht. Random features for large-scale kernel machines. In Advances in neural information processing systems, pages 1177–1184, 2008.

[2] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. Deep learning. MIT press, 2016.

[3] Santosh S Vempala. The random projection method, volume 65. American Mathematical Soc., 2005.

[4] Yun Yang, Mert Pilanci, Martin J Wainwright, et al. Randomized sketches for kernels: Fast and optimal nonparametric regression. The Annals of Statistics, 45(3):991–1023, 2017.

[5] Lijun Zhang, Mehrdad Mahdavi, Rong Jin, Tianbao Yang, and Shenghuo Zhu. Recovering the optimal solution by dual random projection. In Conference on Learning Theory, pages 135–157, 2013.

[6] Lijun Zhang, Mehrdad Mahdavi, Rong Jin, Tianbao Yang, and Shenghuo Zhu. Random projections for classification: A recovery approach. IEEE Transactions on Information Theory, 60(11):7300–7316, 2014.

[7] Ky Vu, Pierre-Louis Poirion, Claudia D’Ambrosio, and Leo Liberti. Random projections for trust region subproblems. arXiv preprint arXiv:1706.02730, 2017.

[8] Christopher KI Williams and Matthias Seeger. Using the nyström method to speed up kernel machines. In Advances in neural information processing systems, pages 682–688, 2001.

[9] Alex J Smola and Bernhard Schölkopf. A tutorial on support vector regression. Statistics and computing, 14(3):199–222, 2004.

[10] Petros Drineas and Michael W Mahoney. On the nyström method for approximating a gram matrix for improved kernel-based learning. journal of machine learning research, 6(Dec):2153–2175, 2005.

[11] Sanjiv Kumar, Mehryar Mohri, and Ameet Talwalkar. Sampling methods for the nyström method. Journal of Machine Learning Research, 13(Apr):981–1006, 2012.

[12] Tianbao Yang, Yu-Feng Li, Mehrdad Mahdavi, Rong Jin, and Zhi-Hua Zhou. Nyström method vs random fourier features: A theoretical and empirical comparison. In Advances in neural information processing systems, pages 476–484, 2012.

[13] Ahmed Alaaoui and Michael W Mahoney. Fast randomized kernel ridge regression with statistical guarantees. In Advances in Neural Information Processing Systems, pages 775–783, 2015.

[14] Francis Bach. Sharp analysis of low-rank kernel matrix approximations. In Conference on Learning Theory, pages 185–209, 2013.

[15] Mert Pilanci and Martin J Wainwright. Randomized sketches of convex programs with sharp guarantees. IEEE Transactions on Information Theory, 61(9):5096–5115, 2015.
[16] Nathan Halko, Per-Gunnar Martinsson, and Joel A Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM review*, 53(2):217–288, 2011.

[17] Christos Boutsidis and Alex Gittens. Improved matrix algorithms via the subsampled randomized hadamard transform. *SIAM Journal on Matrix Analysis and Applications*, 34(3):1301–1340, 2013.

[18] Alex Gittens and Michael W Mahoney. Revisiting the nystrom method for improved large-scale machine learning. *The Journal of Machine Learning Research*, 17(1):3977–4041, 2016.

[19] Rafi Witten and Emmanuel Candes. Randomized algorithms for low-rank matrix factorizations: sharp performance bounds. *Algorithmica*, 72(1):264–281, 2015.

[20] Emmanuel J Candes, Justin K Romberg, and Terence Tao. Stable signal recovery from incomplete and inaccurate measurements. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 59(8):1207–1223, 2006.

[21] Ralph Tyrell Rockafellar. *Convex analysis*. Princeton university press, 2015.

[22] Jerome Friedman, Trevor Hastie, and Robert Tibshirani. *The elements of statistical learning*, volume 1. Springer, 2001.

[23] A. Berlinet and C. Thomas-Agnan. *Reproducing kernel Hilbert spaces in probability and statistics*. Kluwer Adademic, 2004.

[24] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[25] Adam Coates and Andrew Y Ng. Learning feature representations with k-means. In *Neural networks: Tricks of the trade*, pages 561–580. Springer, 2012.

[26] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in neural information processing systems*, pages 315–323, 2013.

[27] Raghu Bollapragada, Richard H Byrd, and Jorge Nocedal. Exact and inexact subsampled newton methods for optimization. *IMA Journal of Numerical Analysis*, 39(2):545–578, 2018.

[28] Murat A Erdogdu and Andrea Montanari. Convergence rates of sub-sampled newton methods. *arXiv preprint arXiv:1508.02810*, 2015.

[29] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay. Scikit-learn: Machine learning in Python. *Journal of Machine Learning Research*, 12:2825–2830, 2011.

[30] Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals. Understanding deep learning requires rethinking generalization. *arXiv preprint arXiv:1611.03530*, 2016.

[31] Stephen Tu, Rebecca Roelofs, Shivaram Venkataraman, and Benjamin Recht. Large scale kernel learning using block coordinate descent. *arXiv preprint arXiv:1602.05310*, 2016.

[32] Benjamin Recht, Rebecca Roelofs, Ludwig Schmidt, and Vaishaal Shankar. Do cifar-10 classifiers generalize to cifar-10? *arXiv preprint arXiv:1806.00451*, 2018.

[33] Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge University Press, 2018.

[34] Robert Gower and Peter Richtárik. Randomized iterative methods for linear systems. *SIAM Journal on Matrix Analysis and Applications*, 36(4):1660-1690, 2015.
A Additional experimental results and implementation details

A.1 Synthetic examples (Figure 1)

With $n = 1000$ and $d = 2000$, we sample two matrices with orthonormal columns $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{d \times n}$, uniformly at random from their respective spaces. We construct two diagonal matrices $\Sigma^{poly}$, $\Sigma^{exp} \in \mathbb{R}^{n \times n}$, such that their respective diagonal elements are $\Sigma^{poly}_{jj} = \sqrt{n}j^{-1}$ and $\Sigma^{exp}_{jj} = \sqrt{n}e^{-0.05j}$. We set $A^{exp} = U\Sigma^{exp}V^T$ and $A^{poly} = U\Sigma^{poly}V^T$, and we sample a planted vector $x_{gd} \in \mathbb{R}^d$ with iid entries $N(0, 1)$.

In the case of binary logistic regression, for each $A \in \{A^{exp}, A^{poly}\}$, we set $y_i = 0.5 \cdot \text{sign}(\langle a_i, x_{gd} \rangle) + 1$, for $i = 1, \ldots, n$.

For the ReLU model, we set $y_i = \langle a_i, x_{gd} \rangle_+$, where $(z)_+ = \max(0, z)$. Hence, each observation $y_i$ is the result of a linear operation $z_i = \langle x, a_i \rangle$ and a non-linear operation $y_i = (z_i)_+$. Additionally, it can be shown that the global minimum of the optimization problem

$$\min_x \frac{1}{2} \sum_{i=1}^{n} (a_i^T x)_+^2 - 2(a_i^T x)y_i,$$

is equal to $x_{gd}$, which motivates using such a convex relaxation.

A.2 Numerical illustration of the iterative and power methods (Figure 2)

We use the MNIST dataset with 50000 training images and 10000 testing images. We rescale the pixel values between $[0, 1]$. Each image is mapped through random cosines $\varphi(\cdot) \in \mathbb{R}^D$ which approximate the Gaussian kernel, i.e., $\langle \varphi(a), \varphi(a') \rangle \approx \exp(-\gamma \|a - a'\|_2^2)$. We choose $D = 10000$ and $\gamma = 0.02$.

We perform binary logistic regression for even-vs-odd classification of the digits.

For the iterative method, we use the sketching matrix $S = (A^T A)^q A^T \tilde{S}$, where $\tilde{S}$ is Gaussian iid. That is, we run the iterative method on top of the power method, with $q = 2$.

A.3 Adaptive Gaussian sketching on MNIST and CIFAR10 datasets (Table 3 and Figure 3)

Table 5: Empirical standard deviation of test classification error on MNIST and CIFAR10 datasets, mapped through Gaussian random Fourier features, respectively with $D = 10000$ and $\gamma = 0.02$, and, $D = 60000$ and $\gamma = 0.002$. The notation $\tilde{x}_m$ refers to the solution of (2), with sketching size $m$.

| $\lambda$ | $x^*_{\text{MNIST}}$ | $\tilde{x}_{64}$ | $\tilde{x}_{128}$ | $\tilde{x}_{256}$ | $\tilde{x}_{512}$ | $\tilde{x}_{1024}$ | $x^*_{\text{CIFAR}}$ | $\tilde{x}_{64}$ | $\tilde{x}_{128}$ | $\tilde{x}_{256}$ | $\tilde{x}_{512}$ | $\tilde{x}_{1024}$ |
|-----------|-----------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-----------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $10^{-4}$ | -                     | 0.2               | 0.2               | 0.2               | 0.2               | 0.2               | -                     | -                  | -                  | -                  | -                  | -                  |
| $5 \cdot 10^{-5}$ | -                     | 0.2               | 0.2               | 0.2               | 0.2               | 0.2               | -                     | 0.5                | 0.3                | 0.3                | 0.2                | 0.2                |
| $10^{-5}$ | -                     | 2.0               | 0.8               | 0.8               | 0.8               | 0.8               | -                     | 4.8                | 3.2                | 0.6                | 0.2                | 0.2                |
| $5 \cdot 10^{-6}$ | -                     | 3.2               | 2.1               | 1.0               | 1.0               | 1.0               | -                     | 4.1                | 3.5                | 2.1                | 0.6                | 0.6                |

Experiments were run in Python on a workstation with 20 cores and 256 GB of RAM. The MNIST and CIFAR10 datasets were downloaded through the PyTorch torchvision module and converted to NumPy arrays. We use the Sklearn.kernel_approximation.RBFSampler module to generate random cosines. We use our own implementation of each algorithm for a fair comparison.

For SGD, we use a batch size equal to 128. For SVRG, we use a batch size equal to 128 and update the gradient correction every 400 iterations. For Sketch-SGD, we use a batch size equal to 1024. For Sketch-SVRG, we use a batch size equal to 64 and update the gradient correction every 200 iterations. Each iteration of the sub-sampled Newton method (Sketch-Newton) computes a full-batch gradient, and, the Hessian with respect to a batch of size 1500.
For SGD and SVRG, we considered step sizes $\eta$ between $10^{-2}$ and $10^2$. We obtained best performance for $\eta = 10^1$. For the sub-sampled Newton method, we use a step size $\eta = 1$, except for the first 5 iterations, for which we use $\eta = 0.2$.

In Figure 3, we did not report results for SVRG for solving the primal (1) on CIFAR, as the computation time for reaching a satisfying performance was significantly larger than for the other algorithms.

In Table 3, we did not investigate results for CIFAR with $\lambda = 10^{-4}$, as the primal classifier had a test error significantly larger than smaller values of $\lambda$.

**B Proof of main results**

Here, we establish our main technical results, that is, the deterministic regret bounds (4) and (5) stated in Theorem 1 and its high-probability version stated in Corollary 1 along with its extension to the iterative Algorithm 2 as given in Theorem 2, and its variant for kernel methods, given in Theorem 3. Our analysis is based on convex analysis and Fenchel duality arguments.

**B.1 Proof of Theorem 1**

We introduce the Fenchel dual program of (1),

$$\min_z f^*(z) + \frac{1}{2\lambda} \|Az\|_2^2.$$  \hfill (14)

For a sketching matrix $S \in \mathbb{R}^{d \times m}$, the Fenchel dual program of (2) is

$$\min_y f^*(y) + \frac{1}{2\lambda} \|PSA^\top y\|_2^2.$$  \hfill (15)

Let $\alpha^*$ be any minimizer of the sketched program (2). Then, according to Proposition 2, the unique solution of the dual sketched program (15) is

$$y^* = \nabla f(AS\alpha^*)$$

and the subgradient set $\partial f^*(y^*)$ is non-empty. We fix $g_{y^*} \in \partial f^*(y^*)$.

According to Proposition 1, the dual program (14) admits a unique solution $z^*$, which satisfies

$$z^* = \nabla f(Ax^*)$$

and which subgradient set $\partial f^*(z^*)$ is non-empty. We fix $g_{z^*} \in \partial f^*(z^*)$.

We denote the error between the two dual solutions by $\Delta = y^* - z^*$. By optimality of $y^*$ with respect to the sketched dual (15) and by feasibility of $z^*$, first-order optimality conditions imply that

$$\langle \frac{1}{\lambda} PSA^\top y^* + g_{y^*}, \Delta \rangle \leq 0.$$  

Similarly, by optimality of $z^*$ with respect to the dual (14) and by feasibility of $y^*$, we get by first-order optimality conditions that

$$\langle \frac{1}{\lambda} AA^\top z^* + g_{z^*}, \Delta \rangle \geq 0.$$
It follows that
\[
\langle \frac{1}{\lambda}A_P S A^T \Delta, \Delta \rangle = \langle \frac{1}{\lambda}A_P S A^T y^*, \Delta \rangle - \langle \frac{1}{\lambda}A_P S A^T z^*, \Delta \rangle \\
\leq \langle g_{z^*} - g_{y^*}, \Delta \rangle - \langle \frac{1}{\lambda}A_P S A^T z^* + g_{z^*}, \Delta \rangle
\]
\[
\leq \langle g_{z^*} - g_{y^*}, \Delta \rangle - \langle \frac{1}{\lambda}A_P S A^T z^* + g_{z^*}, \Delta \rangle
\]
\[
\leq \langle g_{z^*} - g_{y^*}, \Delta \rangle + \langle \frac{1}{\lambda}A_P S A^T z^*, \Delta \rangle
\]
\[
= \langle g_{z^*} - g_{y^*}, \Delta \rangle + \langle \frac{1}{\lambda}A_P S A^T z^*, \Delta \rangle - \langle \frac{1}{\lambda}A A^T z^* + g_{z^*}, \Delta \rangle
\]
\[
\leq \langle g_{z^*} - g_{y^*}, \Delta \rangle + \langle \frac{1}{\lambda}A_P S A^T z^*, \Delta \rangle.
\]

Strong \( \mu \)-smoothness of \( f \) implies that the function \( f^* \) is \( \frac{1}{\mu} \)-strongly convex. Hence, it follows that
\[
\langle g_{z^*} - g_{y^*}, \Delta \rangle + \frac{1}{\mu} \| \Delta \|^2 \leq 0.
\]

Therefore, combining (17) with the previous set of inequalities (16), we get
\[
\langle \frac{1}{\lambda}A_P S A^T \Delta, \Delta \rangle + \frac{1}{\mu} \| \Delta \|^2 \leq \langle \frac{1}{\lambda}A_P S A^T z^*, \Delta \rangle,
\]
and, multiplying both sides by \( \lambda \),
\[
\langle A_P S A^T \Delta, \Delta \rangle + \frac{\lambda}{\mu} \| \Delta \|^2 \leq \langle A_P S A^T z^*, \Delta \rangle.
\]

By definition of \( Z_f \) and since \( \Delta \in \text{dom} f^* - z^* \), it holds that
\[
\frac{\Delta^T A_P S A^T \Delta}{\| \Delta \|^2} \leq Z_f^2,
\]
which we can rewrite as
\[
\langle A_P S A^T \Delta, \Delta \rangle \geq \langle A^T A^T \Delta, \Delta \rangle - Z_f^2 \| \Delta \|^2.
\]

Hence, combining (19) and (18), we obtain
\[
\left( \frac{\lambda}{\mu} - Z_f^2 \right) \| \Delta \|^2 + \| A^T \Delta \|^2 \leq \langle A_P S A^T z^*, \Delta \rangle,
\]
Under the assumption that \( \lambda \geq 2\mu Z_f^2 \), it holds that \( \lambda/\mu - Z_f^2 \geq \lambda/(2\mu) \). Thus,
\[
\left( \frac{\lambda}{\mu} - Z_f^2 \right) \| \Delta \|^2 + \| A^T \Delta \|^2 \geq \frac{\lambda}{2\mu} \| \Delta \|^2 + \| A^T \Delta \|^2
\]
\[
\geq \sqrt{\frac{2\lambda}{\mu} \| \Delta \|^2 \| A^T \Delta \|_2}.
\]

where we used the fact that for any \( a, b \geq 0 \), \( a + b \geq 2\sqrt{ab} \), with \( a = \frac{\lambda}{2\mu} \| \Delta \|^2 \) and \( b = \| A^T \Delta \|^2 \). Combining the former inequality with inequality (20), we obtain
\[
\sqrt{\frac{2\lambda}{\mu} \| \Delta \|^2 \| A^T \Delta \|_2} \leq \langle A_P S A^T z^*, \Delta \rangle.
\]
The right-hand side of the latter inequality can be bounded as
\[
\langle AP_S^\perp A^T z^*, \Delta \rangle = \langle A^T z^*, P_S^\perp A^T \Delta \rangle \\
\leq \|A^T z^*\|_2 \|P_S^\perp A^T \Delta\|_2^{(i)} \\
\leq \|A^T z^*\|_2 \|\Delta\|_2 \sup_{\Delta' \in \text{dom} f^* - z^*} \left( \|P_S^\perp A^T \Delta'\|_2 \right)^{(ii)} \\
= \|A^T z^*\|_2 \|\Delta\|_2 Z_f.
\]
where (i) follows from Cauchy-Schwarz inequality, and (ii) holds by definition of \(Z_f\). Thus, inequality (21) becomes
\[
\sqrt{\frac{2\lambda}{\mu}} \|\Delta\|_2 \|A^T \Delta\|_2 \leq \|A^T z^*\|_2 \|\Delta\|_2 Z_f.
\] (22)
From Propositions 1 and 2, we have that (23)
\[
P_S^\perp A^T \|_2 \leq c_0 \sqrt{\frac{r+1}{r}} \left( \frac{1}{rk} \sum_{j=k+1}^{\min(n,d)} \sigma_j^2 \right)^{\frac{1}{2}}.
\]
where \(\sigma_1 \geq \sigma_2 \geq \ldots\) are the singular values of \(A\), and the universal constant \(c_0\) satisfies \(c_0 \leq 36\). In particular, if \(m = 2k\), then it holds with probability at least \(1 - 6e^{-rk}\) that
\[
\|P_S^\perp A^T \|_2 \leq c_0 \left( \frac{1}{k} \sum_{j=k+1}^{\min(n,d)} \sigma_j^2 \right)^{\frac{1}{2}}.
\] (24)
From Theorem 1 if \(\lambda \geq 2\mu Z_f^2\), then
\[
\|\bar{x} - x^*\|_2 \leq \sqrt{\frac{\mu}{2\lambda}} \|P_S^\perp A^T \|_2 \|x^*\|_2.
\]
Hence, combining the latter inequality with Lemma 1 provided \(2 \leq k \leq \frac{1}{2} \min(n,d), m = 2k\) and \(\lambda \geq 2\mu Z_f^2\), it holds with probability at least \(1 - 6e^{-k}\) that
\[
\|\bar{x} - x^*\|_2 \leq c_0 \sqrt{\frac{\mu}{2\lambda}} R_k(A) \|x^*\|_2.
\]
We want to establish the latter inequality, but under the condition \( \lambda \geq 2c_0^2 \mu R_k^2(A) \). But, by Lemma 1, the condition \( \lambda \geq 2c_0^2 \mu R_k^2(A) \) implies that \( \lambda \geq 2\mu Z_f^2 \) with probability at least \( 1 - 6e^{-k} \). By union bound, it follows that if \( \lambda \geq 2c_0^2 \mu R_k^2(A) \), then
\[
\|\widetilde{x} - x^*\|_2 \leq c_0 \sqrt{\frac{\mu}{2\lambda} R_k(A)}\|x^*\|_2,
\]
with probability at least \( 1 - 12e^{-k} \).

### B.3 Proof of Theorem 2

First, we show that for any \( t \geq 0 \), provided \( \lambda \geq 2\mu Z_f^2 \),
\[
\|\bar{x}^{(t+1)} - x^*\|_2 \leq \sqrt{\frac{\mu}{2\lambda} Z_f \|\bar{x}^{(t)} - x^*\|_2}.
\]  
(25)

It should be noted that for \( t = 0 \), the latter inequality is exactly the regret bound \( 4 \). The proof for \( t > 0 \) follows similar steps.

Fix \( t \geq 0 \). Consider the optimization problem
\[
\min_{\delta \in \mathbb{R}^d} f(A\delta + A\bar{x}^{(t)}) + \frac{\lambda}{2} \|\delta + \bar{x}^{(t)}\|_2^2,
\]  
(26)

which is equivalent to the primal program \( 1 \), up to a translation of the optimization variable. Thus, the unique optimal solution of \( 26 \) – which exists by strong convexity of the objective – is given by \( \delta^* = x^* - \bar{x}^{(t)} \).

By Fenchel duality (Corollary 31.2.1, [21]), it holds that
\[
\min_{\delta} f(A\delta + A\bar{x}^{(t)}) + \frac{\lambda}{2} \|\delta + \bar{x}^{(t)}\|_2^2 = \max_{z} -f^*(z) - \frac{1}{2\lambda} z^T AA^T z,
\]
and the optimal dual solution \( z^* \) exists and is unique (by strong concavity of the dual objective). Further, by the Karush-Kuhn-Tucker conditions (Theorem 31.3, [21]), we have
\[
\begin{cases}
\delta^* = -\bar{x}^{(t)} - \frac{1}{\lambda} A^T z^* \\
z^* = \nabla f(A\delta^* + A\bar{x}^{(t)}).
\end{cases}
\]

Observe that, by using the change of variables \( \alpha = (S^T S)^{-\frac{1}{2}} \alpha_t \), the optimization problem \( 8 \) can be rewritten as
\[
\min_{\alpha \in \mathbb{R}^m} f(A\alpha + A\bar{x}^{(t)}) + \lambda \alpha^T S^T \bar{x}^{(t)} + \frac{\lambda}{2} \|S\alpha\|_2^2
\]  
(27)

Let \( \alpha^{(t+1)}_t \) be the unique solution of \( 8 \). Then, setting \( \alpha^{(t+1)} = (S^T S)^{-\frac{1}{2}} \alpha^{(t+1)}_t \), we have
\[
\alpha^{(t+1)} \in \underset{\alpha \in \mathbb{R}^m}{\text{argmin}} f(A\alpha + A\bar{x}^{(t)}) + \frac{\lambda}{2} \|S\alpha + \bar{x}^{(t)}\|_2^2.
\]  
(27)

By Fenchel duality, we get
\[
\min_{\alpha} f(A\alpha + A\bar{x}^{(t)}) + \frac{\lambda}{2} \|S\alpha + \bar{x}^{(t)}\|_2^2 = \max_{y} -f^*(y) + y^T A\bar{x}^{(t)} - \frac{1}{2\lambda} y^T AP_S A^T y + \frac{\lambda}{2} \bar{x}^{(t)} P_S^\perp \bar{x}^{(t)}.
\]  
16
By strong concavity of the dual objective, there exists a unique maximizer \( y^* \). Further, by the Karush-Kuhn-Tucker conditions (Theorem 31.3, [21]), we have

\[ AS\alpha^{(t+1)} + A\bar{x}^{(t)} \in \partial f^*(y^*) \]

and, thus, \( y^* = \nabla f (AS\alpha^{(t+1)} + A\bar{x}^{(t)}) \).

We define \( \Delta = y^* - z^* \). Following similar steps as in the proof of Theorem 1, we obtain

\[ \|A^\top\Delta\|_2^2 + \left( \frac{\lambda}{\mu} - Z_f^2 \right) \|\Delta\|_2^2 \leq \langle AP\frac{1}{2}S \delta^*, \Delta \rangle \]

\[ = -\lambda \langle AP\frac{1}{2}S \delta^*, \Delta \rangle. \]

Since \( \lambda \geq 2\mu Z_f^2 \), it follows that \( \frac{\lambda}{\mu} - Z_f^2 \geq \lambda/(2\mu) \). Using the fact that for any \( a, b \geq 0 \), we have

\[ 2\sqrt{ab} \leq a + b, \]

we obtain the inequality

\[ \sqrt{2\lambda/\mu} \|A^\top\Delta\|_2 \|\Delta\|_2 \leq -\lambda \langle AP\frac{1}{2}S \delta^*, \Delta \rangle \leq \lambda \|\delta^*\|_2 Z_f \|A^\top\Delta\|_2. \]

Dividing both sides by \( \lambda \|\Delta\|_2 \) and using the identities \( \delta^* = x^* - \bar{x}^{(t)} \) and \( A^\top\Delta/\lambda = x^* - \bar{x}^{(t+1)} \), we obtain the desired contraction inequality

\[ \|\bar{x}^{(t+1)} - x^*\|_2 \leq \sqrt{\frac{\mu}{2\lambda}} Z_f \|\bar{x}^{(t)} - x^*\|_2. \]

By induction, it immediately follows that for any number of iterations \( T \geq 1 \),

\[ \|\bar{x}^{(T)} - x^*\|_2 \leq \left( \frac{\mu}{2\lambda} Z_f^2 \right)^{\frac{T}{2}} \|x^*\|_2. \]

The high-probability version follows by immediate application of Lemma 1 to the previous inequality.

### B.4 Proof of Theorem 3

Define

\[ x^* = \arg\min_x f(K^{\frac{1}{2}}x) + \frac{\lambda}{2} \|x\|_2^2. \]

Set \( \bar{x} = -\frac{1}{\lambda} K^{\frac{1}{2}} \nabla f(K^{\frac{1}{2}}x^*) \), and

\[ Z_f = Z_f \left( K^{\frac{1}{2}}, K^{\frac{1}{2}} \bar{S} \right). \]

Then, by application of Theorem 1 with \( S = K^{\frac{1}{2}} \bar{S} \), it holds that \( \|\bar{x} - x^*\|_2 \leq \sqrt{\mu/(2\lambda)} Z_f \|x^*\|_2 \), provided that \( \lambda \geq 2\mu Z_f^2 \). We conclude by using the facts that \( \bar{x} = K^{\frac{1}{2}} \bar{w} \) and \( x^* = K^{\frac{1}{2}} w^* \).

### C Proofs of intermediate results

#### C.1 Proposition 1 – Strong duality and Karush-Kuhn-Tucker conditions of the primal objective

Denote \( g(x) = \frac{1}{2} \|x\|_2^2 \). The functions \( f \) and \( g \) are proper, closed, convex and their domains are respectively equal to \( \mathbb{R}^n \) and \( \mathbb{R}^d \). It is then trivial that for any \( x \in \mathbb{R}^d \), we have \( x \in \text{dom}(g) \) and \( Ax \in \text{dom}(f) \).
Hence, all conditions to apply strong Fenchel duality results hold (Theorem 31.2, [21]). Using the fact that 
\[ g^*(z) = \frac{1}{2\lambda} \|z\|^2_2, \]
we get
\[ \inf_x f(Ax) + \frac{\lambda}{2} \|x\|^2_2 = \sup_z -f^*(z) - \frac{1}{2\lambda} \|A^\top z\|^2_2, \]
and the supremum is attained for some \( z^* \in \text{dom}(f^*) \). The uniqueness of \( z^* \) follows from strong concavity of the dual objective, which comes from the \( \frac{1}{\mu} \)-strong convexity of \( f^* \).

Further, the primal objective is also strongly convex over \( \mathbb{R}^d \), which implies the existence and uniqueness of a minimizer \( x^* \).

The Karush-Kuhn-Tucker conditions (Theorem 31.3, [21]) imply that \( Ax^* \in \partial f^*(x^*) \). Since \( \partial f^* = (\nabla f)^{-1} \) (Theorem 23.5, [21]), it follows that \( z^* = \nabla f(Ax^*) \). Finally, by first-order optimality conditions of \( x^* \), we have that \( A^\top \nabla f(Ax^*) + \lambda x^* = 0 \), i.e., \( x^* = -\lambda^{-1}A^\top z^* \).

### C.2 Proposition 2 – Strong duality and Karush-Kuhn-Tucker conditions of the sketched primal objective \( [1] \)

Denote \( g(\alpha) = \frac{\lambda}{2} \|S\alpha\|^2_2 \). The functions \( f \) and \( g \) are proper, closed, convex and their domains are respectively equal to \( \mathbb{R}^n \) and \( \mathbb{R}^m \). It is then trivial that for any \( \alpha \in \mathbb{R}^m \), we have \( \alpha \in \text{dom}(g) \) and \( AS\alpha \in \text{dom}(f) \). Hence, all conditions to apply strong Fenchel duality results hold (Theorem 31.2, [21]). Using the fact that 
\[ g^*(y) = \frac{1}{2\lambda} y^\top (S^\top S) y, \]
we get
\[ \inf_{\alpha} f(AS\alpha) + \frac{\lambda}{2} \|S\alpha\|^2_2 = \sup_y -f^*(y) - \frac{1}{2\lambda} \|P_S A^\top y\|^2_2, \]
and the supremum is attained for some \( y^* \in \text{dom}(f^*) \). The uniqueness of \( y^* \) follows from strong concavity of the dual objective, which comes from the \( \frac{1}{\mu} \)-strong convexity of \( f^* \).

We establish the existence of a minimizer \( \bar{\alpha} \) of \( \alpha \mapsto f(AS\alpha) + \frac{\lambda}{2} \|S\alpha\|^2_2 \). The latter function is strongly convex over the subspace \( (\text{Ker}S)^\perp \). Thus, there exists a unique minimizer \( \bar{\alpha} \) over \( (\text{Ker}S)^\perp \). Then, for any \( \alpha \in \mathbb{R}^m \), writing \( \alpha = \alpha_\perp + \alpha_\parallel \) where \( \alpha_\perp \in \text{Ker}(S)^\perp \) and \( \alpha_\parallel \in \text{Ker}(S) \), we have
\[ f(AS\alpha) + \frac{\lambda}{2} \|S\alpha\|^2_2 = f(AS\alpha_\perp) + \frac{\lambda}{2} \|S\alpha_\perp\|^2_2 \]
\[ \geq f(AS\bar{\alpha}) + \frac{\lambda}{2} \|S\bar{\alpha}\|^2_2. \]
Thus, the point \( \bar{\alpha} \) is a minimizer.

Let \( \alpha^* \) be any minimizer. The Karush-Kuhn-Tucker conditions (Theorem 31.3, [21]) imply that \( AS\alpha^* \in \partial f^*(y^*) \). Since \( \partial f^* = (\nabla f)^{-1} \) (Theorem 23.5, [21]), it follows that \( y^* = \nabla f(AS\alpha^*) \).

### C.3 Proof of Proposition 3 – Numerical conditioning of the re-scaled sketched program

The condition number \( \kappa_1 \) of the re-scaled sketched program is equal to
\[ \kappa_1 = \frac{\sup_\alpha \lambda + \sigma_1 (I^\top S A^\top \nabla^2 f(AS\alpha) I_S)}{\inf_\alpha \lambda + \sigma_m (I^\top S A^\top \nabla^2 f(AS\alpha) I_S)}, \]
where \( I_S = S(S^\top S)^{-\frac{1}{2}} \).

In order to show that \( \kappa_1 \leq \kappa \), it suffices to upper bound the numerator in the definition of \( \kappa_1 \) by the numerator of \( \kappa \) and to lower bound the denominator of \( \kappa_1 \) by the denominator of \( \kappa \), i.e., it suffices to show that
\[ \sup_\alpha \sigma_1 (I^\top S A^\top \nabla^2 f(AS\alpha) I_S) \leq \sup_x \sigma_1 (A^\top \nabla^2 f(Ax) A), \]
\[ \inf_\alpha \sigma_m (I^\top S A^\top \nabla^2 f(AS\alpha) I_S) \geq \inf_x \sigma_d (A^\top \nabla^2 f(Ax) A). \]
By the trivial inclusion \( \{ S \alpha \mid \alpha \in \mathbb{R}^m \} \subseteq \mathbb{R}^d \), it holds that
\[
\sup_{\alpha} \sigma_1 \left( I_S^\top A^\top \nabla^2 f(AS\alpha)AIS \right) \leq \sup_{x} \sigma_1 \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right), \\
\inf_{\alpha} \sigma_m \left( I_S^\top A^\top \nabla^2 f(AS\alpha)AIS \right) \geq \inf_{x} \sigma_m \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right).
\]

Therefore, to establish that \( \kappa_i \leq \kappa \), it is sufficient to show that for any \( x \in \mathbb{R}^d \),
\[
\sigma_1 \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right) \leq \sigma_1 \left( A^\top \nabla^2 f(A)x \right), \\
\sigma_m \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right) \geq \sigma_d \left( A^\top \nabla^2 f(A)x \right).
\]
The first inequality follows from the fact that \( \| I_S \|_2 \leq 1 \). Hence, we have
\[
\sigma_1 \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right) = \sup_{w \neq 0} \frac{w^\top I_S^\top A^\top \nabla^2 f(Ax)AISw}{\| w \|_2} \\
= \sup_{w \neq 0} \frac{(I_Sw)\top A^\top \nabla^2 f(Ax)A(I_Sw)}{\| I_Sw \|_2} \| I_Sw \|_2 \leq 1 \\
\leq \sup_{z \neq 0} \frac{z^\top A^\top \nabla^2 f(Az)}{\| z \|_2} \\
= \sigma_1 \left( A^\top \nabla^2 f(A)x \right).
\]

For the second inequality, we distinguish two cases.
If the sketching matrix \( S \in \mathbb{R}^{d \times m} \) is full-column rank, then, the matrix \( I_S \) is actually an isometry, i.e., for any \( w \in \mathbb{R}^m \), we have \( \| S(S^\top S)^{-\frac{1}{2}} w \|_2 = \| w \|_2 \), which implies that
\[
\sigma_m \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right) = \inf_{w \neq 0} \frac{w^\top I_S^\top A^\top \nabla^2 f(Ax)AISw}{\| w \|_2} \\
= \inf_{w \neq 0} \frac{(I_Sw)\top A^\top \nabla^2 f(Ax)A(I_Sw)}{\| I_Sw \|_2} \| I_Sw \|_2 = 1 \\
\geq \inf_{z \neq 0} \frac{z^\top A^\top \nabla^2 f(Az)}{\| z \|_2} \\
= \sigma_d \left( A^\top \nabla^2 f(A)x \right).
\]

Suppose now that the sketching matrix \( S \in \mathbb{R}^{d \times m} \) is not full column-rank. By assumption, \( S = A^\top \tilde{S} \) where \( \tilde{S} \in \mathbb{R}^{n \times m} \) is Gaussian iid, hence, full-column rank almost surely. It implies that there exists a vector \( v \neq 0 \) such that \( Av = 0 \). Indeed, let \( v \neq 0 \) be a vector such that \( S^\top v = 0 \), which exists since \( m < d \). The equation \( S^\top v = 0 \) can be rewritten as \( \tilde{S}^\top Av = 0 \). Since \( \tilde{S}^\top \) is full row-rank, we get that \( Av = 0 \), i.e., \( \text{Ker} A = \{0\} \).
From \( \text{Ker} A = \{0\} \), we get \( \sigma_d \left( A^\top \nabla^2 f(A)x \right) = 0 \) and \( \sigma_m \left( I_S^\top A^\top \nabla^2 f(Ax)AIS \right) = 0 \), which concludes the proof.

D Proof of bounds in Table 1

D.1 Adaptive Gaussian sketching

From Corollary 11, we have that for a target rank \( k \) and a sketching dimension \( m_A = 2k \), with probability at least \( 1 - 12e^{-k} \),
\[
\frac{\| \tilde{x} - x^* \|_2}{\| x^* \|_2} \leq \lambda^{-\frac{1}{2}} \left( \nu_k + \frac{1}{k} \sum_{j=k+1}^{n} \nu_j \right)^{\frac{1}{2}}.
\]
For a matrix $A$ with rank $\rho \ll \min(n,d)$, with $k \geq \rho + 1$, the right hand side of the latter equation is equal to 0. In order to achieve this with probability at least $1 - \eta$, it is sufficient to oversample by an amount $\log(12/\eta)$, that is, $m_A = \rho + 1 + \log(12/\eta)$ is sufficient to achieve a $(\varepsilon = 0, \eta)$-guarantee.

For a $\kappa$-exponential decay with $\kappa > 0$, we have $\nu_j \sim e^{-\kappa j}$, and

$$\left(\nu_k + \frac{1}{k} \sum_{j=k+1}^{\rho} \nu_j\right)^{\frac{1}{2}} \sim e^{-\kappa(k+1)/2},$$

and it is sufficient for the sketching dimension $m_A$ to satisfy

$$m_A \gtrsim \kappa^{-1} \log \left(\frac{1}{\lambda \varepsilon}\right) + \log \left(\frac{12}{\eta}\right).$$

For a $\beta$-polynomial decay with $\beta > 1/2$, we have $\nu_j \sim j^{-2\beta}$ and

$$\left(\nu_k + \frac{1}{k} \sum_{j=k+1}^{\rho} \nu_j\right)^{\frac{1}{2}} \sim k^{-\beta},$$

and it is sufficient to have

$$m_A \gtrsim \lambda^{-\frac{1}{2\beta}} \varepsilon^{-\frac{1}{\beta}} + \log \left(\frac{12}{\eta}\right).$$

### D.2 Oblivious Gaussian sketching

For a $\rho$-rank matrix $A$, it has been shown in [5] that, provided the sketching dimension $m_O$ satisfies

$$m_O \gtrsim \frac{(\rho + 1) \log(2\rho/\eta)}{\varepsilon^2},$$

then,

$$\frac{\|\bar{x} - x^*\|_2}{\|x^*\|_2} \lesssim \varepsilon,$$

with probability at least $1 - \eta$, for any $\varepsilon \in (0, \frac{1}{2})$.

We now justify the bounds for the $\kappa$-exponential and $\beta$-polynomial decays. Let $\overline{\rho}$ be the effective rank of the matrix $AA^\top$, defined as

$$\overline{\rho} = \sum_{i=1}^{\rho} \frac{\nu_i}{\lambda + \nu_i},$$

In [5], the authors have shown that, provided

$$m_O \gtrsim \frac{\overline{\rho}}{\varepsilon^2(\rho + 1)} \log \left(\frac{2d}{\eta}\right),$$

then the relative error satisfies

$$\frac{\|\bar{x} - x^*\|_2}{\|x^*\|_2} \lesssim \varepsilon \left(1 + \sqrt{\frac{\lambda}{\nu_k}}\right),$$

with probability at least $1 - \eta$, under the additional condition that the minimizer $x^*$ lies in the subspace spanned by the top $k$-left singular vectors of $A$. For simplicity of comparison, we neglect the latter (restrictive) requirement on $x^*$, and the term $\sqrt{\lambda/\nu_k}$ in the latter upper bound, which yields a smaller lower bound on a
sufficient sketching size \( m_O \) to achieve a \((\varepsilon, \eta)\)-guarantee. Based on those simplifications, oblivious Gaussian sketching yields a relative error such that
\[
\frac{\|\tilde{x} - x^*\|_2}{\|x^*\|_2} \leq \varepsilon,
\]
with probability at least \( 1 - \eta \), provided that \( m_O \geq \rho \varepsilon^{-2} \log \left( \frac{2d}{\eta} \right) \).

For a \( \kappa \)-exponential decay, it holds that
\[
\rho = \rho \sum_{i=1}^{\rho} e^{-\kappa i} \geq \int_1^{\rho} \frac{1}{e^{-\kappa t} + \lambda} dt = \int_{e^{-\kappa \rho}}^{e^{-\kappa}} \frac{1}{\kappa u + \lambda} du = \frac{1}{\kappa} \log \left( \frac{e^{-\kappa} + \lambda}{e^{-\kappa \rho} + \lambda} \right).
\]
Since \( \lambda \in (\nu, 1) = (e^{-\kappa \rho}, 1) \), it follows that \( \rho \gtrsim \kappa^{-1} \log \frac{1}{\lambda} \).

Hence, their theoretical predictions state that the sketching dimension \( m_O \) must be greater than
\[
m_O \gtrsim \kappa^{-1} \varepsilon^{-2} \log \left( \frac{1}{\lambda} \right) \log \left( \frac{2d}{\eta} \right)
\]
in order to achieve a \((\varepsilon, \eta)\)-guarantee.

For a \( \beta \)-polynomial decay, it holds that
\[
\rho = \rho \sum_{i=0}^{\rho} \frac{1}{1 + \lambda^{2i\beta}} \geq -1 + \int_0^{\rho} \frac{1}{1 + \lambda^{2i\beta}} dt = -1 + \frac{\lambda^{-1/2\beta}}{2\beta} \int_0^{\rho^{1/2\beta}} \frac{u^{\beta-1}}{1 + u} du
\]
\[
\geq -1 + \frac{\lambda^{-1/2\beta}}{2\beta} \int_0^{1} \frac{u^{\beta-1}}{1 + u} du,
\]
where the last inequality is justified by the fact that the integrand is non-negative, and the fact that \( \lambda \geq \rho^{-2\beta} \).

Since the integral is finite and independent of \( \lambda \), it follows that
\[
\rho \gtrsim \lambda^{-\frac{1}{2\beta}},
\]
and the sketching dimension \( m_O \) must satisfy
\[
m_O \gtrsim \lambda^{-\frac{1}{2\beta}} \varepsilon^{-2} \log \left( \frac{2d}{\eta} \right)
\]
in order to achieve a \((\varepsilon, \eta)\)-guarantee, according to their theoretical predictions.

D.3 Leverage score column sampling.

Let \( A = U \Sigma V^T \) be a singular value decomposition of the matrix \( A \), where \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\rho) \), and \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\rho \). For a given target rank \( k \), let \( u_1, \ldots, u_k \) be the first \( k \) columns of the matrix \( U \), and denote \( U_1 = [u_1, \ldots, u_k] \in \mathbb{R}^{n \times k} \). For \( j = 1, \ldots, n \), define \( p_j = k^{-1} ||U_{1,j}||_2^2 \), where \( U_{1,j} \) is the \( j \)-th row of the matrix \( U_1 \). By orthonormality of the family \( (u_1, \ldots, u_k) \), it holds that \( \sum_{j=1}^n p_j = 1 \), and \( p_j \geq 0 \). The family \( \{p_j\}_{j=1}^n \) is called the leverage score probability distribution of the Gram matrix \( AA^T \).

Leverage based column sampling consists in, first, computing the exact or approximated leverage score distribution of the matrix \( AA^T \), and, second, sampling \( m \) columns of \( AA^T \) from the latter probability distribution, with replacement. Precisely, the sketching matrix \( S \in \mathbb{R}^{d \times m} \) is given as
\[
S = A^T RD,
\]
where $R \in \mathbb{R}^{n \times m}$ is a column selecting matrix drawn according to the leverage scores, and $D \in \mathbb{R}^{m \times m}$ is a diagonal rescaling matrix, with $D_{jj} = (mp_i)^{-\frac{1}{2}}$, if $R_{ij} = 1$.

In order to compare the theoretical guarantees of adaptive Gaussian sketching and leverage-based column sampling, we assume that the leverage scores are computed exactly. Note that if this is not the case, then the sketching size increases as the quality of approximation of the leverage score distribution decreases. As our primary goal is to lower bound the ratio $m_S/m_A$, our qualitative comparison is not affected (at least not in the favor of adaptive Gaussian sketching) by this assumption.

The authors of [18] showed that given $\delta \in (0, 1]$, provided $m_S$ satisfies

$$m_S \gtrsim \delta^{-2} k \log \left( \frac{k}{\eta} \right),$$

then, with probability at least $1 - \eta$,

$$\| P_S A^T \|_2 \leq \nu_k^2 + \delta^2 \sum_{j=k+1}^\rho \nu_j^2.$$

Using the combination of the latter concentration bound with our deterministic regret bound (4) on the relative error, it follows that, under the latter condition on $m_S$, with probability at least $1 - \eta$,

$$\frac{\| \tilde{x} - x^* \|_2}{\| x^* \|_2} \leq \lambda^{-\frac{1}{2}} \left( \nu_k^2 + \delta^2 \sum_{j=k+1}^\rho \nu_j^2 \right).$$

For a matrix $A$ with rank $\rho \ll \min(n, d)$, if the sketching size $m_S$ is greater than $(\rho + 1) \log \left( \frac{\rho + 1}{\eta} \right)$, then the relative error satisfies an $(\varepsilon = 0, \eta)$-guarantee.

For a $\kappa$-exponential decay, we have

$$\nu_k^2 + \delta^2 \sum_{j=k+1}^\rho \nu_j^2 \sim \left(1 + \frac{2\delta^2}{\kappa}\right) e^{-(k+1)/2},$$

Taking $\delta = 1/2$, it follows that the sketching size $m_S$ must be greater than $\kappa^{-1} \log \left( \frac{1}{\kappa \eta} \right)$ to satisfy

$$\frac{\| \tilde{x} - x^* \|_2}{\| x^* \|_2} \lesssim \varepsilon$$

with probability at least $1 - \eta$.

For a $\beta$-polynomial decay (with $\beta > 1$), we have

$$\nu_{k+1} + \delta^2 \sum_{j=k+1}^\rho \nu_j \sim k^{-\beta} + \delta^2 \beta^{-1} k^{1-\beta},$$

and, provided $m_S \gtrsim \delta^2 k \log \left( \frac{k}{\eta} \right)$,

$$\frac{\| \tilde{x} - x^* \|_2}{\| x^* \|_2} \leq \lambda^{-\frac{1}{2}} \left( k^{-\beta} + \delta^2 \beta^{-1} k^{1-\beta} \right).$$

To achieve a precision $\varepsilon$, it is sufficient to have

$$\varepsilon \gtrsim \lambda^{-\frac{1}{2}} \left( k^{-\beta} + \delta^2 \beta^{-1} k^{1-\beta} \right)$$
Suppose first that we choose $\delta \lesssim k^{-\frac{1}{2}}$. Then, the latter sufficient condition becomes $\varepsilon \gtrsim \lambda^{-\frac{1}{2}} k^{-\beta}$. Hence, we need $k$ to be at least $\lambda^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}$, which implies $m_S \gtrsim \delta^{-2} k \log \left(\frac{k}{\eta}\right)$. Since $\delta \lesssim k^{-\frac{1}{2}}$, we get that $m_S$ must be greater than $k^2 \log(1/\eta)$, which further implies

$$m_S \gtrsim \lambda^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}} \log \left(\frac{1}{\eta}\right).$$

Now, suppose that we choose $\delta \gtrsim k^{-\frac{1}{2}}$. Write $\delta^2 = k^{-1+\gamma}$, where $\gamma > 0$. Since $\delta < 1$, we must have $\gamma \in (0, 1)$. Further, we need $\beta \varepsilon \gtrsim \lambda^{-\frac{1}{2}} \delta^2 k^{1-\beta} = \lambda^{-\frac{1}{2}} k^{\gamma-\beta}$. By assumption, $\beta > 1$, hence, $\gamma - \beta < 0$. Hence, the smallest value of $k$ that satisfies the latter inequality is given by

$$k = \left(\varepsilon^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \delta\right)^{\frac{1}{1-\beta}}.$$

On the other hand, the smallest sketching size $m_S$ to achieve an $(\varepsilon, \eta)$ satisfies

$$m_S \gtrsim k^{2-\gamma} \log \left(\frac{1}{\eta}\right).$$

Plugging-in the value of $k$, we must have

$$m_S \gtrsim \left(\varepsilon^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \delta\right)^{\frac{2-\gamma}{1-\beta}} \log \left(\frac{1}{\eta}\right).$$

Optimizing over $\gamma \in (0, 1)$, we finally obtain that the best sufficient sketching size must satisfy

$$m_S \gtrsim \left(\varepsilon^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \delta\right)^{\min(2, \frac{\beta}{\beta-1})} \log \left(\frac{1}{\eta}\right).$$

**E. Extension to the non-smooth case**

Here, we present some results to the case where the function $f : \mathbb{R}^n \to \mathbb{R}$ is proper, convex, but not necessarily smooth. We make the assumption that the function is $L$-Lipschitz, that is, for any $x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\|_2 \leq L \|x - y\|_2.$$ 

In particular, this implies that the domain of the function $f^*$ is bounded, i.e., for any $z \in \text{dom} f^*$, it holds that $\|z\|_2 \leq L$.

Let $x^*$ be the solution of the primal program \([1]\), which exists and is unique by strong convexity of the primal objective.

For a sketching matrix $S \in \mathbb{R}^{d \times m}$, the sketched primal program \([2]\) admits a solution $\alpha^*$. Indeed, using arguments similar to the proof of Proposition \([2]\), the sketched program is strongly convex over $\text{Ker}(S)^\perp$, and admits a unique solution $\alpha^*$ over that subspace. Further, for any $\alpha \in \mathbb{R}^m$, we can decompose $\alpha = \alpha_\perp + \alpha_\parallel$, with $\alpha_\perp \in \text{Ker}(S)^\perp$ and $\alpha_\parallel \in \text{Ker}(S)$. Then,

$$f(AS\alpha) + \frac{\lambda}{2} \|S\alpha\|_2^2 = f(AS\alpha_\perp) + \frac{\lambda}{2} \|S\alpha_\perp\|_2^2 \geq f(AS\alpha^*) + \frac{\lambda}{2} \|S\alpha^*\|_2^2.$$

As for the smooth case, using convex analysis arguments, we obtain that the dual program \([14]\) has a solution $z^*$ which satisfies $z^* = \nabla f(Ax^*)$. Similarly, the sketched dual program \([15]\) has a solution $y^*$ which satisfies $y^* = \nabla f(AS\alpha^*)$. Further, by first-order optimality conditions of $x^*$, we have $x^* = -\lambda^{-1} A^\top \nabla f(Ax^*)$, i.e., $x^* = -\lambda^{-1} A^\top z^*$. 

23
As for the smooth case, we introduce the candidate approximate solution \( \tilde{x} \), defined as
\[
\tilde{x} = -\lambda^{-1} A^\top \nabla f(AS\alpha^*),
\]
where \( \alpha^* \) is any minimizer of (15).

**Theorem 4.** For any \( \lambda > 0 \), it holds that
\[
\|\tilde{x} - x^*\|_2 \leq \frac{6L}{\lambda} \sqrt{\sigma_1 Z_f}.
\]
where \( \sigma_1 \) is the top singular value of \( A \).

*Proof.* Let \( \alpha^* \) be any minimizer of the sketched primal program. Following similar lines as in the proof of Theorem 1 (see Appendix B.1), it holds that
\[
\|A^\top \Delta\|_2^2 \leq Z_f^2 \|\Delta\|_2^2 + z^*^\top \Lambda \Delta,
\]
where \( \Delta = y^* - z^* \). After applying Cauchy-Schwarz and using the definition of \( Z_f \), inequality (29) becomes
\[
\|A^\top \Delta\|_2 \leq \sqrt{Z_f^2 \|\Delta\|_2^2 + Z_f \|z^*\|_2 \|A^\top \Delta\|_2}.
\]
Using the fact that \( \sqrt{w + w'} \leq \sqrt{w} + \sqrt{w'} \) with \( w = Z_f^2 \|\Delta\|_2^2 \) and \( w' = Z_f \|z^*\|_2 \|A^\top \Delta\|_2 \), along with the inequality \( \|A^\top \Delta\|_2 \leq \sigma_1 \|\Delta\|_2 \), we obtain
\[
\|A^\top \Delta\|_2 \leq \sqrt{Z_f} \left( \sqrt{Z_f^2 \|\Delta\|_2^2} + \sqrt{\|z^*\|_2 \|\Delta\|_2 \sigma_1} \right).
\]
Using the inequality \( 2ww' \leq w^2 + w'^2 \) and the fact that \( Z_f \leq \sigma_1 \), it follows that
\[
\|A^\top \Delta\|_2 \leq 2 \sqrt{Z_f \sigma_1} (\|\Delta\|_2 + \|z^*\|_2)
\]
Dividing by \( \lambda \) and using the fact that \( \tilde{x} - x^* = -\lambda^{-1} A^\top \Delta \), we get
\[
\|\tilde{x} - x^*\|_2 \leq \frac{2}{\lambda} \sqrt{Z_f \sigma_1} (\|y^* - z^*\|_2 + \|z^*\|_2).
\]
Using the fact that \( \|y^*\|_2, \|z^*\|_2 \leq L \), we obtain the desired inequality (28). \( \square \)

As for the smooth-case, high-probability bounds follow from the previous deterministic bound on the relative error.