A Lower Bound of the First Eigenvalue of a Closed Manifold with Negative Lower Bound of the Ricci Curvature

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Abstract
Along the line of the Yang Conjecture, we give a new estimate on the lower bound of the first non-zero eigenvalue of a closed Riemannian manifold with negative lower bound of Ricci curvature in terms of the in-diameter and the lower bound of Ricci curvature.

1 Introduction
It has been proved by Li and Yau [7] that if $M$ is an $n$-dimensional closed Riemannian manifold with Ricci curvature $\text{Ric}(M)$ bounded below by $(n-1)\kappa$, with constant $\kappa < 0$, then the first non-zero eigenvalue $\lambda$ of the Laplacian of $M$ has the lower bound
\[ \lambda \geq \frac{1}{2(n-1)d^2} \exp\{-1 - \sqrt{1 + 4(n-1)^2d^2|\kappa|}\}, \]
where $d$ is the diameter of $M$.

H. C. Yang [11] obtained later the following estimate
\[ \lambda \geq \frac{\pi^2}{d^2} \exp\{-C_n \sqrt{(n-1)|\kappa|d^2}\}, \]
where $C_n = \max\{\sqrt{n-1}, \sqrt{2}\}$. Yang further conjectured that
\[ \lambda \geq \frac{1}{2}(n-1)\kappa + \frac{\pi^2}{d^2}, \]

Along the line of the Yang Conjecture, we give a new estimate on the lower bound of the first non-zero eigenvalue of a closed Riemannian manifold
with negative lower bound of Ricci curvature in terms of the in-diameter and the lower bound of Ricci curvature. Instead of using the Zhong-Yang’s canonical function or the ”midrange” of the normalized eigenfunction of the first eigenvalue in the proof, we use a function $\xi$ that the author constructed in [8] for the construction of the suitable test function and use the structure of the nodal domains of the eigenfunction. That provides a new way to sharpen the bound. We have the following result.

**Theorem 1.** If $M$ is an $n$-dimensional closed Riemannian manifold and if the Ricci curvature of $M$ has a lower bound

$$\text{Ric}(M) \geq (n-1)\kappa$$

for some constant $\kappa < 0$, then the first non-zero eigenvalue $\lambda$ of the Laplacian $\Delta$ of $M$ satisfies the inequality

$$\lambda \geq \frac{1}{1 - (n-1)\kappa/(2\lambda)} \frac{\pi^2}{d^2} > 0$$

and $\lambda$ has the following lower bound,

$$\lambda \geq \frac{1}{2} (n-1)\kappa + \frac{\pi^2}{d^2},$$

where $\tilde{d}$ is the diameter of the largest interior ball in the nodal domains of the first eigenfunction.

Note that from the proof of the Theorem 1 the in-diameter $\tilde{d}$ can be replaced by the larger of the diameters of the two nodal domains of an eigenfunction of the first non-zero eigenvalue.

If $\text{Ric}(M) \geq (n-1)\kappa$ with constant $\kappa > 0$, it is known that the first non-zero eigenvalue $\lambda$ has a lower bound as the above. Therefore the lower bound in (2) is universal for all three cases, no matter constant $\kappa > 0$, $= 0$ or $< 0$.

We derive some preliminary estimates and conditions for test functions in the next section and construct the needed test function and prove the main result in the last section.

## 2 Preliminary Estimates

Let $v$ be a normalized eigenfunction of the first non-zero eigenvalue $\lambda$ of the Laplacian $\Delta$ such that

$$\sup_M v = 1, \quad \inf_M v = -k$$
with $0 < k \leq 1$. The function $v$ satisfies the following equation

$$\Delta v = -\lambda v \quad \text{in } M,$$

where $\Delta$ is the Laplacian of $M$.

We first use gradient estimate in [4]-[7] and [10] to derive following estimate.

**Lemma 1.** The function $v$ satisfies the following

$$\frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda(1 + \beta),$$

where $\beta = -(n - 1)\kappa/\lambda > 0$ and $b > 1$ is an arbitrary constant.

**Proof.** Consider the function

$$P(x) = |\nabla v|^2 + Av^2,$$

where $A = \lambda - (n - 1)\kappa + \epsilon$ for small $\epsilon > 0$. Function $P$ must achieve its maximum at some point $x_0 \in M$. We claim that $\nabla v(x_0) = 0$.

If on the contrary, $\nabla v(x_0) \neq 0$, then we can rotate the local orthonormal frame about $x_0$ such that

$$|v_1(x_0)| = |\nabla v(x_0)| \neq 0 \quad \text{and} \quad v_i(x_0) = 0, \quad i \geq 2.$$

Since $P$ achieves its maximum at $x_0$, we have

$$\nabla P(x_0) = 0 \quad \text{and} \quad \Delta P(x_0) \leq 0.$$

That is, at $x_0$

$$0 = \frac{1}{2} \nabla_i P = \sum_{j=1}^{n} v_j v_{ji} + Av_i,$$

or

$$v_{11} = -Av \quad \text{and} \quad v_{1i} = 0 \quad i \geq 2,$$

(7)
and

\[ 0 \geq \Delta P(x_0) = \sum_{i,j=1}^{n} (v_{ji}v_{ji} + v_{j}v_{jii} + A v_{i}v_{i} + Av_{ii}) \]

\[ = \sum_{i,j=1}^{n} (v_{ji}^2 + v_{j}(v_{ii})_j + R_{ji}v_{j}v_{i} + Av_{ii}^2 + Av_{ii}) \]

\[ = \sum_{i,j=1}^{n} v_{ji}^2 + \nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v) + A |\nabla v|^2 + Av \Delta v \]

\[ \geq v_{11}^2 + \nabla v \nabla (\Delta v) + (n - 1)\kappa |\nabla v|^2 + A |\nabla v|^2 + Av \Delta v \]

\[ = (-Av)^2 - \lambda |\nabla v|^2 + (n - 1)\kappa |\nabla v|^2 + A |\nabla v|^2 - \lambda Av^2 \]

\[ = [A - \lambda + (n - 1)\kappa] |\nabla v|^2 + A(A - \lambda)v^2, \]

where we have used (7) and (1). Therefore at \(x_0\),

\[ 0 \geq [A - \lambda + (n - 1)\kappa] |\nabla v|^2 + A(A - \lambda)v^2. \]  

That is,

\[ \epsilon |\nabla v(x_0)|^2 + [- (n - 1)\kappa + \epsilon] |\lambda - (n - 1)\kappa + \epsilon] v(x_0)^2 \leq 0. \]

Thus \(\nabla v(x_0) = 0\). This contradicts the assumption \(\nabla v(x_0) \neq 0\). So the above claim is right.

Therefore we have \(\nabla v(x_0) = 0\), and

\[ P(x_0) = |\nabla v(x_0)|^2 + Av(x_0)^2 = Av(x_0)^2 \leq A. \]

Now for all \(x \in M\) we have

\[ |\nabla v(x)|^2 + Av(x)^2 = P(x) \leq P(x_0) \leq A \]

and

\[ |\nabla v(x)|^2 \leq A(1 - v(x)^2). \]

Letting \(\epsilon \to 0\) in the above inequality, the estimate (5) follows. \(\square\)

We want to improve the above upper bound in (5) further and proceed in the following way.

Define a function \(Z\) on \([- \sin^{-1}(k/b), \sin^{-1}(1/b)]\) by

\[ Z(t) = \max_{x \in M, t = \sin^{-1}(v(x)/b)} \frac{|\nabla v|^2}{b^2 - v^2}/\lambda. \]
The estimate in (5) becomes

\[(9) \quad Z(t) \leq 1 + \beta \quad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)].\]

Throughout this paper let

\[(10) \quad \alpha = \frac{1}{2}(n - 1)\kappa < 0 \quad \text{and} \quad \delta = \alpha/\lambda < 0.\]

We have the following theorem on conditions for the test function.

**Theorem 2.** If the function \( z : [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \rightarrow \mathbb{R}^1 \) satisfies the following

1. \( z(t) \geq Z(t) \quad t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \),
2. there exists some \( x_0 \in M \) such that at point \( t_0 = \sin^{-1}(v(x_0)/b) \)
   \( z(t_0) = Z(t_0) \),
3. \( z(t_0) \geq 1 \),
4. \( z'(t_0) \sin t_0 \leq 0 \),

then we have the following

\[(11) \quad 0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta \cos^2 t_0.\]

**Proof.** Define

\[ J(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda z \right\} \cos^2 t, \]

where \( t = \sin^{-1}(v(x)/b) \). Then

\[ J(x) \leq 0 \quad \text{for } x \in M \quad \text{and} \quad J(x_0) = 0. \]

If \( \nabla v(x_0) = 0 \), then

\[ 0 = J(x_0) = -\lambda z \cos^2 t. \]

This contradicts Condition 3 in the theorem. Therefore

\[ \nabla v(x_0) \neq 0. \]

The Maximum Principle implies that

\[(12) \quad \nabla J(x_0) = 0 \quad \text{and} \quad \Delta J(x_0) \leq 0.\]
\( J(x) \) can be rewritten as
\[
J(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda z \cos^2 t.
\]
Take normal coordinates about \( x_0 \). \((12)\) is equivalent to
\[
\left. \frac{2}{b^2} \sum_i v_i v_{ij} \right|_{x_0} = \lambda \cos t [z' \cos t - 2z \sin t] t_j \right|_{x_0}
\]
and
\[
0 \geq \left. \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \sum_{i,j} v_i v_{ijj} - \lambda (z''|\nabla t|^2 + z' \Delta t) \cos^2 t \right|_{x_0}.
\]
Rotate the frame so that \( v_1(x_0) \neq 0 \) and \( v_i(x_0) = 0 \) for \( i \geq 2 \). Then \((13)\) implies
\[
\left. v_{11} \right|_{x_0} = \frac{\lambda b}{2} \left( z' \cos t - 2z \sin t \right) \right|_{x_0} \text{ and } \left. v_{1i} \right|_{x_0} = 0 \text{ for } i \geq 2.
\]
Now we have
\[
\left. |\nabla v|^2 \right|_{x_0} = \lambda b^2 z \cos^2 t \right|_{x_0},
\]
\[
\left. |\nabla t|^2 \right|_{x_0} = \frac{\left. |\nabla v|^2 \right|_{x_0}}{b^2 - v^2} = \lambda z \right|_{x_0},
\]
\[
\left. \frac{\Delta v}{b} \right|_{x_0} = \Delta \sin t = \cos t \Delta t - \sin t \left. |\nabla t|^2 \right|_{x_0},
\]
\[
\left. \Delta t \right|_{x_0} = \frac{1}{\cos t} (\sin t |\nabla t|^2 + \frac{\Delta v}{b})
\]
\[
\left. = \frac{1}{\cos t} \left[ \lambda z \sin t - \frac{\lambda}{b} v \right] \right|_{x_0}, \text{ and}
\]
\[
\left. \Delta \cos^2 t \right|_{x_0} = \Delta \left( 1 - \frac{v^2}{b^2} \right) = -\frac{2}{b^2} |\nabla v|^2 - \frac{2}{b^2} v \Delta v
\]
\[
\left. = -2\lambda z \cos^2 t + \frac{2}{b^2} \lambda v^2 \right|_{x_0},
\]
Therefore,
\[
\frac{2}{b^2} \sum_{i,j} v_{ij}^2 \right|_{x_0} \geq \frac{2}{b^2} v_{11}^2
\]
\[
= \frac{\lambda^2}{2} (z')^2 \cos^2 t - 2\lambda^2 zz' \cos t \sin t + 2\lambda^2 z^2 \sin^2 t \right|_{x_0}.
\]
\[
\frac{2}{b^2} \sum_{i,j} v_i v_{ij} \bigg|_{x_0} = \frac{2}{b^2} \left( \nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v) \right) \\
\geq \frac{2}{b^2} \left( \nabla v \nabla (\Delta v) + (n - 1)\kappa |\nabla v|^2 \right) \\
= -2\lambda^2 z \cos^2 t + 4\alpha \lambda z \cos^2 t \bigg|_{x_0},
\]
\[
-\lambda (z''|\nabla t|^2 + z'\Delta t) \cos^2 t \bigg|_{x_0} \\
= -\lambda^2 z z'' \cos^2 t - \lambda^2 z' z t \cos t \\
+ \frac{1}{b} \lambda^2 z' \cos t \bigg|_{x_0},
\]
and
\[
4\lambda z' \cos t |\nabla t|^2 - \lambda z \Delta \cos^2 t \bigg|_{x_0} \\
= 4\lambda^2 z' \cos t + 2\lambda^2 z^2 \cos^2 t - \frac{2}{b} \lambda^2 z v \sin t \bigg|_{x_0}.
\]
Putting these results into (14) we get
\[
0 \geq -\lambda^2 z z'' \cos^2 t + \frac{\lambda^2}{2} (z')^2 \cos^2 t + \lambda^2 z' \cos t \left( z \sin t + \sin t \right) \\
+ 2\lambda^2 z^2 - 2\lambda^2 z' + 4\alpha \lambda z \cos^2 t \bigg|_{x_0},
\]
(16)
where we used (15). Now
(17)
\[
z(t_0) > 0,
\]
by Condition 3 in the theorem. Dividing two sides of (16) by \(2\lambda^2 z \bigg|_{x_0},\) we have
\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + \frac{1}{2} z'(t_0) \cos t_0 \left( \sin t_0 + \frac{\sin t_0}{z(t_0)} \right) + z(t_0) \\
- 1 + 2\delta \cos^2 t_0 + \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0.
\]
Therefore,
\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta \cos^2 t_0 \\
+ \frac{1}{2} z'(t_0) \sin t_0 \cos t_0 \left[ \frac{1}{z(t_0)} - 1 \right] + \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0.
\]
(18)
By the conditions 3 and 4 in the theorem, the last two terms are nonnegative. Therefore (11) follows.

\section{Proof of the Main Result}

We now prove our main result.

\textit{Proof of Theorem 1.} Let

\begin{equation}
    z(t) = 1 + \delta \xi(t),
\end{equation}

where $\xi$ is the functions defined by (27) in Lemma 2 below and $\delta$ is the negative constant in (10). We claim that

\begin{equation}
    Z(t) \leq z(t) \quad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)].
\end{equation}

Lemma 2 implies that for $t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]$, we have the following

\begin{equation}
    \frac{1}{2} z'' \cos^2 t - z' \cos t \sin t - z = -1 + 2\delta \cos^2 t,
\end{equation}

\begin{equation}
    z'(t) \sin t \leq 0, \quad \text{ (since } \delta < 0 \text{) and}
\end{equation}

\begin{equation}
    z(t) \geq z(\pi/2) = 1.
\end{equation}

\text{Let } P \in \mathbb{R}^1 \text{ and } t_0 \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \text{ such that}

\begin{equation}
    P = \max_{t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]} (Z(t) - z(t)) = Z(t_0) - z(t_0).
\end{equation}

Thus

\begin{equation}
    Z(t) \leq z(t) + P \quad \text{on } [-\sin^{-1} \frac{k}{b}, \sin^{-1} \frac{1}{b}] \quad \text{and} \quad Z(t_0) = z(t_0) + P.
\end{equation}

Suppose that $P > 0$. Then $z + P$ satisfies the conditions in Theorem 2. (11) implies that

\begin{align*}
    z(t_0) + P &= Z(t_0) \\
    &\leq \frac{1}{2} (z + P)''(t_0) \cos^2 t_0 - (z + P)'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\
    &= \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \\
    &= z(t_0).
\end{align*}
This contradicts the assumption $P > 0$. Thus $P \leq 0$ and (20) must hold. That means

(25) \[ \sqrt{\lambda} \geq \frac{|\nabla t|}{\sqrt{z(t)}}. \]

Note that the eigenfunction $v$ of the first nonzero eigenvalue has exactly two nodal domains $D^+ = \{x : v(x) > 0\}$ and $D^- = \{x : v(x) < 0\}$ and the nodal set $v^{-1}(0)$ is compact. Take $q_1$ on $M$ such that $v(q_1) = 1 = \sup_M v$ and $q_2 \in v^{-1}(0)$ such that distance $d(q_1, q_2) = \text{distance} d(q_1, v^{-1}(0))$. Let $L$ be the minimum geodesic segment between $q_1$ and $q_2$. We integrate both sides of (25) along $L$ and change variable and let $b \to 1$. Let $d_+, d_-$ be the diameter of the largest interior ball in $D^+$, $D^-$ respectively, then

\[ d_+ = 2r_+ \quad \text{and} \quad r_+ = \max_{x \in D^+} \text{dist}(x, v^{-1}(0)) \]

and

\[ d_- = 2r_- \quad \text{and} \quad r_- = \max_{x \in D^-} \text{dist}(x, v^{-1}(0)). \]

Then $\tilde{d} = \max\{d_+, d_-\}$

(26) \[ \sqrt{\lambda} \frac{d_+}{2} \geq \int_L \frac{|\nabla t|}{\sqrt{z(t)}} \, dt = \int_0^{\pi/2} \frac{1}{\sqrt{z(t)}} \, dt \geq \frac{\left( \int_0^{\pi/2} \frac{dt}{z(t)} \right)^{1/2}}{\left( \int_0^{\pi/2} (z(t))^{1/2} \, dt \right)^{1/2}}. \]

Square the two sides, we get

\[ \lambda \geq \pi \frac{3}{2(d_+)^2} \int_0^{\pi/2} z(t) \, dt. \]

Now

\[ \int_0^{\pi/2} z(t) \, dt = \int_0^{\pi/2} [1 + \delta \xi(t)] \, dt = \frac{\pi}{2} (1 - \delta), \]

by (30) in Lemma 2. Therefore

\[ \lambda \geq \frac{\pi^2}{(1 - \delta)(d_+)^2} \quad \text{and} \quad \lambda \geq \frac{1}{2} (n - 1) \kappa + \frac{\pi^2}{(d_+)^2}. \]

Since $\tilde{d} \geq d_+$ and $\tilde{d} \geq d_-$, we complete the proof. \qed

We now state and prove Lemma 2 used in the proof of Theorem 1.
Lemma 2. Let
\[ \xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}]. \]

Then the function \( \xi \) satisfies the following
\[ \frac{1}{2} \xi'' \cos^2 t - \xi' \sin t \cos t - \xi = 2 \cos^2 t \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}), \]
\[ \xi' \cos t - 2 \xi \sin t = 4t \cos t \quad \text{on} \quad [-\pi, \pi], \]
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt = -\frac{\pi}{2} \]
\[ 1 - \frac{\pi^2}{4} = \xi(0) \leq \xi(t) \leq \xi(\pm \frac{\pi}{2}) = 0 \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}], \]
\[ \xi' \text{ is increasing on } [-\frac{\pi}{2}, \frac{\pi}{2}] \quad \text{and} \quad \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}. \]
\[ \xi'(t) < 0 \quad \text{on} \quad (-\frac{\pi}{2}, 0) \quad \text{and} \quad \xi'(t) > 0 \quad \text{on} \quad (0, \frac{\pi}{2}), \]
\[ \xi''(\pm \frac{\pi}{2}) = 2, \quad \xi''(0) = 2(3 - \frac{\pi^2}{4}) \quad \text{and} \quad \xi''(t) > 0 \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}], \]
\[ \frac{\xi'(t)}{t} > 0 \quad \text{on} \quad (0, \pi/2) \quad \text{and} \quad 2(3 - \frac{\pi^2}{4}) \leq \frac{\xi'(t)}{t} \leq \frac{4}{3} \quad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}], \]
\[ \xi''(\frac{\pi}{2}) = \frac{8\pi}{15}, \quad \xi''(0) < 0 \quad \text{on} \quad (-\frac{\pi}{2}, 0) \quad \text{and} \quad \xi''(t) > 0 \quad \text{on} \quad (0, \frac{\pi}{2}). \]

Proof of Lemma 2. For convenience, let \( q(t) = \xi'(t) \), i.e.,
\[ q(t) = \xi'(t) = \frac{2(2t \cos t + t^2 \sin t + \cos^2 t \sin t - \frac{\pi^2}{4} \sin t)}{\cos^3 t}. \]

Equation (28), and the values \( \xi(\pm \frac{\pi}{2}) = 0 \), \( \xi(0) = 1 - \frac{\pi^2}{4} \), and \( \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3} \) can be verified directly from (27) and (31). The values of \( \xi'' \) at 0 and \( \pm \frac{\pi}{2} \) can be computed via (28), (29), (30), and (31). By (28), \( (\xi(t) \cos^2 t)' = 4t \cos^2 t \). Therefore \( \xi(t) \cos^2 t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4s \cos^2 s \, ds \), and
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt = 2 \int_{0}^{\frac{\pi}{2}} \xi(t) \, dt = -8 \int_{0}^{\frac{\pi}{2}} \left( \frac{1}{\cos^2(t)} \int_{t}^{\frac{\pi}{2}} s \cos^2 s \, ds \right) \, dt \]
\[ = -8 \int_{0}^{\frac{\pi}{2}} \left( \int_{0}^{s} \frac{1}{\cos^2(t)} \, dt \right) s \cos^2 s \, ds = -8 \int_{0}^{\frac{\pi}{2}} s \cos s \sin s \, ds = -\pi. \]
It is easy to see that $q$ and $q'$ satisfy the following equations

\[(32) \quad \frac{1}{2} q'' \cos t - 2q' \sin t - 2q \cos t = -4 \sin t, \]

and

\[(33) \quad \frac{\cos^2 t}{2(1 + \cos^2 t)} (q'')' - \frac{2 \cos t \sin t}{1 + \cos^2 t} (q')' - 2(q') = \frac{-4}{1 + \cos^2 t}. \]

The last equation implies $q' = \xi''$ cannot achieve its non-positive local minimum at a point in $(-\frac{\pi}{2}, \frac{\pi}{2})$. On the other hand, $\xi''(\pm \frac{\pi}{2}) = 2$, by equation (28), $\xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}$. Therefore $\xi''(t) > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\xi'$ is increasing. Since $\xi'(t) = 0$, we have $\xi'(t) < 0$ on $(-\frac{\pi}{2}, 0)$ and $\xi'(t) > 0$ on $(0, \frac{\pi}{2})$. Similarly, from the equation

\[(34) \quad \frac{\cos t}{2(1 + \cos^2 t)} (q'')' - \frac{\cos t \sin t}{1 + \cos^2 t} (q')' - 2\xi''(t) = \frac{-8 \cos t \sin t}{(1 + \cos^2 t)^2}, \]

we get the results in the last line of the lemma.

Set $h(t) = \xi''(t) t - \xi'(t)$. Then $h(0) = 0$ and $h'(t) = \xi'''(t) t > 0$ in $(0, \frac{\pi}{2})$. Therefore $(\frac{\xi'(t)}{t})' = \frac{h(t)}{t^2} > 0$ in $(0, \frac{\pi}{2})$. Note that $\frac{\xi'(t)}{t} = \frac{\xi'(t)}{t}|_{t=0} = \xi'(0) = 2(3 - \frac{\pi^2}{4})$ and $\frac{\xi'(t)}{t}|_{t=\pi/2} = \frac{4}{3}$. This completes the proof of the lemma. \(\square\)

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