Mathematical pendulum and its variants

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Abstract: In this paper we show that there are applications that transform the movement of a pendulum into movements in \( \mathbb{R}^3 \). This can be done using Euler top system of differential equations. On the constant level surfaces, Euler top system reduces to the equation of a pendulum. Those properties are also considered in the case of system of differential equations with delay argument and in the fractional case. Another aspect presented here is stochastic Euler top system of differential equations and stochastic pendulum.

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1 Introduction

The dynamics of some mechanical systems is described using the rigid body dynamics with a fixed point, mathematical pendulum or oscillators. These systems belong to a class of differential equations from \( \mathbb{R}^3 \) with the right side polynomial functions of degree greater or equal to two. From this category we will consider Euler top system of differential equations. We begin our study from mathematical pendulum (and its variants: with delay, fractional and stochastic) approach.

The Euler top system of fractional differential equation belongs to a class of
differential equations that are described using polynomial functions. It has the form
\[
\begin{align*}
    \dot{x}_1(t) &= x_2(t)x_3(t), \\
    \dot{x}_2(t) &= -x_1(t)x_3(t), \\
    \dot{x}_3(t) &= x_1(t)x_2(t).
\end{align*}
\] (1)

Because system (1) has three Hamilton-Poisson realizations, three conservation laws are given by the Hamiltonians \(H_1, H_2\) and \(H_3\) [2]:
\[
\begin{align*}
    1. & \quad H_1(x_1(t), x_2(t), x_3(t)) := \frac{1}{2}(x_1^2(t) + x_2^2(t)); \\
    2. & \quad H_2(x_1(t), x_2(t), x_3(t)) := -\frac{1}{2}(x_2^2(t) + x_3^2(t)); \\
    3. & \quad H_3(x_1(t), x_2(t), x_3(t)) := x_1^2(t) - x_2^2(t),
\end{align*}
\]
and the other three conservation laws are given by the corresponding Casimir functions of the above realizations [2]:
\[
\begin{align*}
    1. & \quad C_1(x_1(t), x_2(t), x_3(t)) := \frac{1}{2}(x_2^2(t) + x_3^2(t)); \\
    2. & \quad C_2(x_1(t), x_2(t), x_3(t)) := \frac{1}{2}(x_1^2(t) - x_2^2(t)); \\
    3. & \quad C_3(x_1(t), x_2(t), x_3(t)) := x_1^2(t) + x_2^2(t).
\end{align*}
\]

A simple mathematical pendulum is the mathematical model of a ball, having the mass \(m\), which hangs in a point \(O\) by a bar of length \(l\), and the point \(O\) performs movement in a plane [9].

The Euler-Lagrange equation that describes the movement of a pendulum is given by
\[
l\ddot{\theta}(t) + g \sin \theta(t) + \ddot{x}_0 \cos \theta(t) - \ddot{y}_0 \sin \theta(t) = 0.
\] (2)
The dumping pendulum equations with periodic force is
\[
\ddot{\theta}(t) + 2h \sin \theta(t) + f_1(t) \cos \theta(t) + f_2(t) \sin \theta(t) + \sum_{p=0}^{N} \alpha_p \dot{\theta}(t)\dot{\theta}(t)|^{p-1} = 0, \quad (3)
\]
and for \(f_1 := 0, \quad f_2 := 0\) and \(\alpha_p := 0, \quad p = 1...N\), then (3) reduces to \(\ddot{\theta}(t) + 2h \sin \theta(t) = 0\).

In the first section we will determine the analytical solutions for Euler top system taking into consideration the conservation laws that it owns, and point out the analytical solution for pendulum. In the second section we have presented the connection between Euler top system and pendulum: the restriction of the system to a constant level surface represents the pendulum equations. The third section presents
the Euler top system of differential equations with delay argument, along the OZ and OX axes. These new systems have also conservation laws and the restriction of the orbits at these surfaces of constant level determined by the conservation laws are mathematical pendulums with delay argument. In the forth section we presented the Euler top system of fractional differential equations. We have used Caputo fractional derivative in OZ and OX directions. As in the previous case, this system of fractional differential equations have conservation laws and the restriction of the system to the constant level surfaces is a fractional pendulum. In Section 5 we presented stochastic Euler top system and stochastic pendulum. We considered Itô and Stratonovich integrals for describing the stochastic process, using a Wiener process. For all these cases numerical simulations are done. In the last section some conclusions are presented and ideas for future work.

2 Euler top system and simple pendulum - analytical solutions

Let us consider the Euler top system of differential equations (1) and the integrals of motion given by

\[ x_1(t) + x_2(t) = 2H^2, \quad x_2(t) + x_3(t) = 2K^2. \] (4)

From (4), results that

\[ x_1(t) = 2H^2 - x_2(t), \quad x_2(t) = 2K^2 - x_3(t). \] (5)

Replacing (5) in the first equation in (1) we get:

\[ (\dot{x}_2)^2(t) = (x_1)^2(t) + (x_3)^2(t) = (2H^2 - x_2(t))(2K^2 - x_3(t)) \] (6)

and so,

\[ t = \int_{x_2(0)}^{x_2(t)} \frac{1}{\sqrt{(2H^2 - u^2)(2K^2 - u^2)}} du, \] (7)

that shows that \( x_2(t) \), respectively \( x_1(t) \) and \( x_2(t) \) are elliptic functions of time [8].

In the case when \( H = K \), the quartic under the square root has double roots and (7) can be explicitly integrated by means of elementary functions in the following manner. The equation

\[ \dot{x}_2(t) = \pm(2H^2 - x_2(t)), \]
with \( x_2(0) = 0 \), has the solution

\[ x_2(t) = \pm H \sqrt{2} \tanh(H \sqrt{2}t). \] (8)
Substituting (8) in (7), we get
\[ x_1(t) = \pm H \sqrt{2} \text{sech}(H \sqrt{2}t), \quad x_3(t) = \pm H \sqrt{2} \text{sech}(H \sqrt{2}t). \] (9)

So, the equations (9) and (3) represent the two heteroclinic orbits for the Euler top system and are given by
\[ \left( \pm H \sqrt{2} \text{sech}(H \sqrt{2}t), \pm H \sqrt{2} \tanh(H \sqrt{2}t), \pm H \sqrt{2} \text{sech}(H \sqrt{2}t) \right). \]

In the case when \( H \neq K \), the integral (7) can be computed using Jacobi’s elliptic functions [7]. We use relations
\[ \frac{d}{dt} \text{sn} u = \text{cn} u \text{dn} u, \quad \text{cn}^2 u = 1 - \text{sn}^2 u, \quad \text{dn}^2 u = 1 - m^2 \text{sn}^2 u \]
and
\[ x_2(t) = H \sqrt{2} \text{sn} \left( H \sqrt{2}t; \frac{\sqrt{H}}{\sqrt{K}} \right), \] (10)
with the initial condition \( x_2(0) = 0 \). Choosing the time deviation, appropriately, we can assume that \( \dot{x}_2(0) > 0 \). From (5) results that
\[ x_1(t) = H \sqrt{2} \text{cn} \left( H \sqrt{2}t; \frac{\sqrt{H}}{\sqrt{K}} \right), \quad x_3(t) = K \sqrt{2} \text{sn} \left( H \sqrt{2}t; \frac{\sqrt{H}}{\sqrt{K}} \right). \] (11)

If \( \phi \) denotes the period invariant of Jacobi’s elliptic functions, then \( x_1(t) \) and \( x_2(t) \) have the period \( 4\phi/H \sqrt{2} \), whereas \( x_3(t) \) has the period \( 2\phi/H \sqrt{2} \).

**Proposition 1**

a) If \( H = K \), then Euler top system (1) has an analytical solution given by (8) and (9);

b) If \( H \neq K \), then the Euler top system has the analytical solution given by (10) and (11).

\[ \square \]

**Proposition 2** [1] The analytical solution for simple pendulum \( \ddot{\theta}(t) + \sin \theta(t) = 0 \), with initial conditions \( \theta(0) = \theta_0 \) and \( \dot{\theta}(0) = 0 \) is given by
\[ \theta(t) = 2 \arcsin \left\{ \sin \theta_0 \text{sn} \left( \sin^2 \frac{\theta_0}{2} - \sqrt{2ht}; \sin^2 \frac{\theta_0}{2} \right) \right\}. \]

\[ \square \]
3 Euler top system and simple pendulum

In this section we will show the way the Euler top system and the simple pendulum are linked. We will show that the movement of the Euler top system is reduced to pendulum movement on the constant level surfaces $H$ and $K$, described by the conservation laws:

\[
\frac{1}{2}(x_1(t))^2 + \frac{1}{2}(x_2(t))^2 = H, \quad (12)
\]

\[
\frac{1}{2}(x_2(t))^2 + \frac{1}{2}(x_3(t))^2 = K. \quad (13)
\]

Since $H$ and $K$ are conserved, the Euler top motion takes place along the intersections of the level surfaces of the energy and the angular momentum in $\mathbb{R}^3$.

**Proposition 3** Let us consider the Euler top system of differential equations (1).

1. The function $H$, given by (12) is a conservation law for system (1);

2. The solution of (1) on the constant level surface defined by (12), given by

\[
(x_1(t))^2 + (x_2(t))^2 = 2H = \text{const}, \quad H > 0 \quad (14)
\]

is

\[
x_1(t) = \sqrt{2H} \cos \frac{\theta(t)}{2}, \quad x_2(t) = \sqrt{2H} \sin \frac{\theta(t)}{2}, \quad x_3(t) = -\frac{1}{2} \dot{\theta}(t), \quad (15)
\]

where $\theta(t)$ is the solution of pendulum equation $\ddot{\theta}(t) + 2H \sin \theta(t) = 0$.

**Proof:**

1. By deriving (12) and by replacing it in (1), we have

\[
\dot{H}(t) = x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t) = 0.
\]

And so $H$ is a conservation law.

2. Using a direct calculus, it can easily checked that (15) is a solution for (1) and reciprocal. $\square$

**Proposition 4** 1. The function $K$, given by (13) is a conservation law for the Euler top system (1);
2. The solution of (1), on the constant level surface defined by (13), given by

\[(x_2(t))^2 + (x_3(t))^2 = 2K = \text{const}, \ K > 0\]  \quad (16)

is

\[x_1(t) = -\frac{1}{2}\dot{\theta}(t), \ x_2(t) = \sqrt{2K}\cos\frac{\theta(t)}{2}, \ x_3(t) = \sqrt{2K}\sin\frac{\theta(t)}{2},\]  \quad (17)

where \(\theta(t)\) is the solution of pendulum equation \(\ddot{\theta}(t) + 2K\sin\theta(t) = 0\).

**Proof:**

1. By deriving (13) and by replacing it in (1), we have that \(K\) is a conservation law because

\[
\dot{K}(t) = x_2(t)\dot{x}_2(t) + x_3(t)\dot{x}_3(t) = 0.
\]

2. By direct calculations, it can be easily checked that (17) is a solution for (1) and reciprocal. \(\square\)

**Remark 5** The dynamics of Euler top system of differential equations in \(\mathbb{R}^3\) is a union of two-dimensional simple pendula.

\(\square\)

For the initial conditions \(x_1(0) = 0.1, x_2(0) = 0.1\) and \(x_3(0) = 0.2\), the Euler top system is represented in the first figure and the pendulum is represented for the initial condition \(\theta(0) = -3.8\).
4 Euler top system and simple pendulum - with delay argument and fractional derivative

A differential equation with delay argument is defined in [4]. A second order differential equation with delay argument is given by

\[ \ddot{\theta}(t) = c \sin(\theta(t - \tau)), \tag{18} \]

where \( c \in \mathbb{R} \) is a solution of a differential equation on the circle \( S^1 = \{ y \in \mathbb{R}^2 | y_1^2 + y_2^2 = 1 \} \), \( \theta \) an angle variable determined up to a multiple of \( 2\pi \), and \( \tau > 0 \).

From Proposition 3 and Proposition 4 we can deduce the following results.

**Proposition 6** The Euler top system of differential equations with delay argument is given by

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t)x_3(t), \\
\dot{x}_2(t) &= -x_1(t)x_3(t), \\
\dot{x}_3(t) &= x_1(t - \tau)x_2(t - \tau). \tag{19}
\end{align*}
\]

The system (19) has the following properties

a) The function \( H \) given by (12);

b) The solution of system (19) on the constant level surface (12) is given by (15), where \( \theta(t) \) is the solution of

\[ \ddot{\theta}(t) + 2H \sin \theta(t - \tau) = 0 \tag{20} \]

and reciprocal. \( \square \)

**Proposition 7** The Euler top system of differential equations with delay argument given by

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t - \tau)x_3(t - \tau), \\
\dot{x}_2(t) &= -x_1(t)x_3(t), \\
\dot{x}_3(t) &= x_1(t)x_2(t). \tag{21}
\end{align*}
\]

has the following properties

a) The function \( K \) given by (13) is a conservation law for the system (21);

b) The solution of system (21) on the constant level surface (13) is given by (17), where \( \theta(t) \) is solution of the equation

\[ \ddot{\theta}(t) + 2K \sin \theta(t - \tau) = 0. \tag{22} \]
This system is considered to be a starting point in studying differential equations with delay argument for differential manifold.

For $H = 0.5$, and $\tau = 1$, the pendulum equation with delay argument and with initial condition $\theta(0) = 2$, is represented in the following figure. The Euler top system with delay argument (19) is represented in the second figure, for the initial conditions $x_1(0) = 0.1$, $x_2(0) = 0.05$, $x_3(0) = 0.2$.

Pendulum with delay $\tau = 1$, $H = 0.5$  
Euler top system with delay $\tau = 1$ $H = 0.5$

For $K = 0.3$, and $\tau = 1$, the pendulum equation with delay argument and with initial condition $\theta(0) = 2$, is represented in the following figure. The Euler top system with delay argument (19) is represented in the second figure, for the initial conditions $x_1(0) = 0.1$, $x_2(0) = 0.05$, $x_3(0) = 0.2$.

Pendulum with delay $\tau = 1$, $K = 0.3$  
Euler top system with delay $\tau = 1$ $K = 0.3$

Using Caputo fractional derivative [3], the following propositions take place.
**Proposition 8** The Euler top system of fractional differential equations, given by

\[
\begin{aligned}
\dot{x}_1(t) &= x_2(t)x_3(t), \\
\dot{x}_2(t) &= -x_1(t)x_3(t), \\
D^\alpha x_3(t) &= x_1(t)x_2(t),
\end{aligned}
\]  

(23)

with \( \alpha \in (0, 1) \), has the following properties

a) The function \( H \) is a conservation law for (23);

b) The solution of the system (23) on the constant level surface (12), with \( \theta(t) \) is the solution of the fractional equation

\[
D_t^{\alpha+1}\theta(t) + 2H \sin \theta(t) = 0,
\]  

(24)

and reciprocal.

□

**Proposition 9** The Euler top system of fractional differential equations, given by

\[
\begin{aligned}
D^\alpha x_1(t) &= x_2(t)x_3(t), \\
\dot{x}_2(t) &= -x_1(t)x_3(t), \\
\dot{x}_3(t) &= x_1(t)x_2(t),
\end{aligned}
\]  

(25)

with \( \alpha \in (0, 1) \), has the following properties

a) The function \( H \) is a conservation law for (25);

b) The solution of the system (25) on the constant level surface (13), with \( \theta(t) \) is the solution of the fractional equation

\[
D_t^{\alpha+1}\theta(t) + 2K \sin \theta(t) = 0,
\]  

(26)

and reciprocal.

□

By using the Adams-Moulton method for integration, for the initial condition \( \theta(0) = -3.1 \), the solution of the fractional differential equation (24) is represented in the following graphics for \( \alpha = 0.8 \), respectively for \( \alpha = 1 \).
It can be observed that the pendulum solution is asymptotically stable for $0 < \alpha < 1$ and it is oscillatory for $\alpha = 1$.

The solution for the system of fractional differential equations (23), respectively for (25), is represented in the above graphics, for the initial conditions $x_1(0) = 0.1$, $x_2(0) = 0.1$ and $x_3(0) = 0.3$. The cases of $\alpha = 0.8$ and $\alpha = 1$ are illustrated.
Stochastic Euler top system and stochastic pendulum

A Wiener process describes rapidly fluctuating random phenomena. Stochastic differential equations (SDE) are stochastic integral equations and are written symbolically in a differential form. We will consider such a Wiener process of the form

\[ dx(t) = f(x(t))dt + g(x(t))dW(t), \]  

(27)

where \( f \) is the slowly varying continuous component called drift coefficient and \( g \) is the rapidly varying continuous component called diffusion coefficient. The integral representation is of the form

\[ x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))ds + \int_{t_0}^{t} g(x(s))dW(s), \]  

(28)

where \( W(t) \) is a Wiener process, a Gaussian process with \( W(0) = 0 \) and \( N(0, t) \)-distributed \( W(t) \) for each \( t \geq 0 \), so

\[ \mathbb{E}(W(t)) = 0, \mathbb{E}((W(t))^2) = t. \]

The first integral is a Riemann-Stieltjes integral and the second one is a stochastic integral. The most studied interpretation of the stochastic integral are those of Itô and Stratonovich. The choice of interpretation depends on the type of analysis required for solution [5]. Itô stochastic calculus is closely related to diffusion processes and martingale theory [5]. The solution of (27) is a diffusion process with transition probability \( p = u(x(t)) \), satisfying the Fokker-Planck equation

\[ \frac{\partial}{\partial t}u(x(t)) = -\frac{\partial}{\partial x(t)}f(x(t))u(x(t)) + \frac{1}{2} \frac{\partial^2}{\partial (x(t))^2}[(g(x(t))g^T(x(t))u(x(t))]. \]  

(29)

Equations (27) and (29) contain the same statistical information from a one-particle process point of view (but not if we think the Itô equation as describing a random dynamical system) [6].

An Itô SDE is written in the form (27) and a Stratonovich SDE is written symbolically in the form

\[ dx(t) = f(x(t))dt + g(x(t)) \circ dW(t), \]  

(30)

and in the integral form as

\[ x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))ds + \int_{t_0}^{t} g(x(s)) \circ dW(s). \]  

(31)
It is possible to switch between these two approaches, in the sense that the Itô SDE (27) has the same solution as Stratonovich SDE

\[ dx(t) = f(x(t))dt + g(x(t)) \circ dW(t), \]

with modified drift coefficient

\[ f(x(t)) = f(x(t)) - \frac{1}{2}g(x(t)) \frac{\partial g}{\partial x(t)}(x(t)). \]

If \( W(1), ..., W(d) \) are \( d \) independent Wiener processes, and \( x(t) = (x_1(t), ..., x_n(t)) \) then the multi-Wiener process case can be written in the form

\[ dx_i(t) = f^i(x(t))dt + \sum_{j=1}^d g_{ij}(x(t))dW(j), \]

with \( g(x(t)) \) a \( n \times d \) matrix and \( dW \) a \( d \times 1 \) matrix.

In Stratonovich case, the stochastic system of differential equations with a multi-Wiener process, can be written in the following manner

\[ dx(t) = f(x(t))dt + \sum_{j=1}^d g_j(x(t)) \circ dW(j), \]

where

\[ f(x(t)) = f(x(t)) - \frac{1}{2} \sum_{k=1}^n \sum_{j=1}^d g_{k,j}(x(t)) \frac{\partial g_j}{\partial x_k(t)}. \]

The Euler top system of stochastic differential equations can be represented in the following form,

\[
\begin{align*}
    dx_1(t) &= x_2(t)x_3(t)dt + x_1(t)dW^1(t) \\
    dx_2(t) &= -x_1(t)x_3(t)dt, \\
    dx_3(t) &= x_1(t)x_2(t)dt + dW^3(t),
\end{align*}
\]

with the Wiener process \( W(t) = (W^1(t), 0, W^2(t)) \), the drift coefficients \( f^1(x(t)) = x_2(t)x_3(t), f^2(x(t)) = -x_1(t)x_3(t), f^3(x(t)) = x_1(t)x_2(t), x(t) = (x_1(t), x_2(t), x_3(t))^T \), \( f(x(t)) = (f^1(x(t)), f^2(x(t)), f^3(x(t)))^T \) and the diffusion coefficient vectors

\[
g^1(x(t)) = \begin{pmatrix} x_1(t) \\ 0 \\ 0 \end{pmatrix}, \quad g^2(x(t)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad g^3(x(t)) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]
The corresponding (Itô) Fokker-Planck equation for the probability density \( p = u(x(t)) \) reads

\[
\frac{\partial}{\partial t} u(x(t)) = -\frac{\partial}{\partial x_1(t)}[x_2(t)x_3(t)u(x(t))] + \frac{\partial}{\partial x_2(t)}[x_1(t)x_3(t)u(x(t))] - \frac{\partial}{\partial x_3(t)}[x_1(t)x_2(t)u(x(t))]
+ \frac{1}{2}\frac{\partial^2}{\partial(x_1(t))^2}[(x_1(t))^2u(x(t))] + \frac{1}{2}\frac{\partial^2}{\partial(x_3(t))^2}u(x(t)).
\]

In the Stratonovich case, stochastic system (35) can be written using relation (34) in the following manner

\[
\begin{align*}
    dx_1(t) &= (x_2(t)x_3(t) - \frac{1}{2}x_2(t))dt + x_1(t)dW^1(t) \\
    dx_2(t) &= -x_1(t)x_3(t)dt, \\
    dx_3(t) &= x_1(t)x_2(t)dt + dW^3(t).
\end{align*}
\]

(36)

The stochastic system (35), respectively (36), is implemented in Matlab, using Milstein scheme, for initial conditions \( x_1(1) = 0.1, x_2(1) = 0.1, x_3(1) = 0.1 \) and orbits are represented in the following figures.
If the SDE of Euler top system has the form

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= x_2(t)x_3(t)dt + \sqrt{x_1(t)}dW^1(t) \\
\frac{dx_2(t)}{dt} &= -x_1(t)x_3(t)dt + \sqrt{x_2(t)}dW^2(t), \\
\frac{dx_3(t)}{dt} &= x_1(t)x_2(t)dt + \sqrt{x_3(t)}dW^3(t),
\end{align*}
\]

(37)

then drift coefficients are

\[
\begin{align*}
f^1(t) &= x_2(t)x_3(t), & f^2(x(t)) &= -x_1(t)x_3(t), & f^3(x(t)) &= x_1(t)x_2(t),
\end{align*}
\]

with \(x(t) = (x_1(t), x_2(t), x_3(t))^T\), \(f(x(t)) = (f^1(x(t)), f^2(x(t)), f^3(x(t)))^T\), and the diffusion coefficient vectors

\[
\begin{align*}
g^1(x(t)) &= \begin{pmatrix} \sqrt{x_1(t)} \\ 0 \\ 0 \end{pmatrix},
& g^2(x(t)) &= \begin{pmatrix} 0 \\ \sqrt{x_2(t)} \\ 0 \end{pmatrix},
& g^3(x(t)) &= \begin{pmatrix} 0 \\ 0 \\ \sqrt{x_3(t)} \end{pmatrix},
\end{align*}
\]

then the associated (Itô) Fokker-Planck equation for the probability density \(p = u(x(t))\) is

\[
\begin{align*}
\frac{\partial}{\partial t} u(x(t)) &= -\frac{\partial}{\partial x_1(t)}[x_2(t)x_3(t)u(x(t))] + \frac{\partial}{\partial x_2(t)}[x_1(t)x_3(t)u(x(t))] - \frac{\partial}{\partial x_3(t)}[x_1(t)x_2(t)u(x(t))] \\
&+ \frac{1}{2}\frac{\partial^2}{\partial(x_1(t))^2}[x_1(t)u(x(t))] + \frac{1}{2}\frac{\partial^2}{\partial(x_2(t))^2}[x_2(t)u(x(t))] + \frac{1}{2}\frac{\partial^2}{\partial(x_3(t))^2}[x_3(t)u(x(t))].
\end{align*}
\]

The Stratonovich stochastic Euler top system is written in the following way

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= (x_2(t)x_3(t) - \frac{1}{4})dt + \sqrt{x_1(t)}dW^1(t) \\
\frac{dx_2(t)}{dt} &= -(x_1(t)x_3(t) + \frac{1}{4})dt + \sqrt{x_2(t)}dW^2(t), \\
\frac{dx_3(t)}{dt} &= (x_1(t)x_2(t))dt + \sqrt{x_3(t)}dW^3(t),
\end{align*}
\]

(38)

Stochastic system (37), respectively (38), can be implemented using stochastic Euler method which represents a square-root model. For initial values \(x_1(0) = 1, x_2(0) = 0.8, x_3(0) = 0.2\), orbits are represented in the following figures.
The stochastic pendulum equation is considered in the following manner. The dynamics of a non-dissipative classical pendulum of the form $\ddot{\theta}(t) + 2H \sin \theta(t) = 0$, can be expressed as a system of stochastic differential equations expressed like

$$\begin{align*}
  dx_1(t) &= x_2(t)dt + \sqrt{x_1(t)}dW^1(t), \\
  dx_2(t) &= -2H \sin(x_1(t))dt + \sqrt{x_2(t)}dW^2(t),
\end{align*}$$

(39)

and the Stratonovich stochastic pendulum equations are

$$\begin{align*}
  dx_1(t) &= (x_2(t) - \frac{1}{4})dt + \sqrt{x_1(t)}dW^1(t), \\
  dx_2(t) &= -(2H \sin(x_1(t)) + \frac{1}{4})dt + \sqrt{x_2(t)}dW^2(t),
\end{align*}$$

(40)

For the probability density $p = u(x(t))$, the corresponding (Itô) Fokker-Planck equation is given by

$$\begin{align*}
  \frac{\partial}{\partial t}u(x(t)) &= -\frac{\partial}{\partial x_1(t)}[x_2(t)u(x(t))] + \frac{\partial}{\partial x_2(t)}[2H \sin(x_1(t))u(x(t))] \\
  &\quad + \frac{1}{2} \frac{\partial^2}{\partial (x_1(t))^2}[x_1(t)u(x(t))] + \frac{1}{2} \frac{\partial^2}{\partial (x_2(t))^2}[x_2(t)u(x(t))].
\end{align*}$$

Using stochastic Euler method on square root process, for initial conditions $x_1(1) = 1$, $x_2(1) = 0.8$ we get the following graphics for stochastic systems (39) and (40).
6 Conclusions

In this paper we presented the Euler top system in $\mathbb{R}^3$ and the mathematical pendulum, but also the connections between them: the existence of some applications that transform the movement of a pendulum into a movement in $\mathbb{R}^3$. That means that the restriction of the Euler top system on a constant level surface is the pendulum equation. This property is also true in the case of Euler top system of differential equations with delay argument, respectively mathematical pendulum with delay argument, and in the case of fractional system of differential equations, respectively fractional pendulum. We have also studied the Euler top system and mathematical pendulum from the stochastic point of view, using Itô and Stratonovich integrals for a Wiener process. Numerical simulations were done using Maple 12 and Matlab. In the case of fractional Euler top system and fractional pendulum we used the Adams-Moulton integration method for their representation, and in the stochastic case we used the Milstein scheme, that is a convergent numerical algorithm. In the future we will study other aspects of these problems, such as stochastic Lyapunov functions, stochastic Lyapunov exponents for determining the stochastic stability in the equilibrium points of a considered system, classical, with delay of fractional.

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