Accretion of perfect fluids onto a class of regular black holes

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We consider the stationary spherical accretion process of perfect fluids onto a class of spherically symmetric regular black holes corresponding to quantum-corrected Schwarzschild spacetimes. We show that the accretion rates can differ from the Schwarzschild case, suggesting that the de Sitter core inside these regular black holes, which indeed precludes the central singularity, can act for some cases as a sort of antigravitational source, decreasing the fluid’s radial infall velocity in the accretion process, and for others as a gravitational enhancer, increasing the fluid flow into the black hole horizon. Our analysis and results can be extended and applied also to the problem of black hole evaporation in cosmological scenarios with phantom fluids. In particular, we show that the mass of typical regular black holes could be used to constrain turnaround events in cyclic cosmologies.

I. INTRODUCTION

A Regular black hole (RBH) is a solution of the gravitational field equations without a singularity inside the event horizon. The first RBH ever created is due to Bardeen [1]. It possesses spherical symmetry and a mass function that depends on the radial coordinate. Inside the event horizon, the Bardeen RBH hides a de Sitter core which precludes either the point singularity of spherical RBHs [2-10] or the ring singularity in RBHs with axial symmetry [11-16]. As it is well known, the de Sitter core promotes energy conditions violations in order to avoid the conclusions of the singularities theorems (see [17] for a detailed study on the theorems). For example, the Bardeen RBH violates the strong energy condition, and RBHs with rotation ignores the weak energy condition [16].

The class of RBHs studied in this work adopts a generalized mass function that includes the Bardeen [1] and Hayward [8] mass functions. The mass function is

$$m(r) = M_0 \left(1 + \left(\frac{r_0}{r}\right)^q\right)^{-\frac{q}{2}},$$

(1)

and it provides RBHs solutions for any positive integer $q$, by assuming geometries with either spherical or axial symmetry. For $q = 2$, one has the Bardeen solution, and, on the other hand, when $q = 3$, the Hayward RBH is obtained. Notice that, as discussed in [16], it is possible to interpret the interior region of these RBHs, for any value of $q$, as a de Sitter solution with high matter-energy densities, in the same spirit of the pioneering works of Sakharov [18] and Gliner [19]. The parameter $M_0$ is the Arnowitt-Deser-Misner (ADM) mass for the spherical and asymptotically flat case. The length parameter $r_0$ has been recently investigated [20] and, by using a generalized uncertainty principle (GUP), a new interpretation of the Bardeen metric was proposed: the Bardeen RBH can be conceived of as a quantum-corrected Schwarzschild black hole instead of a BH that comes from a nonlinear electrodynamics as is well-known in Ayón-Beato and García’s work [21]. This new interpretation led to an upper bound on the length parameter, namely $r_0 < 10^{-25}$m, which is compatible with a microscopical origin. As we will see, by using this upper bound, we will calculate important quantities in the accretion process like the radial velocity and the critical point where the fluid’s velocity equals the critical speed.

The accretion process onto BHs is a remarkable issue in science today. The recent image of a BH constructed by the Event Horizon Telescope [22] confirms that. The accretion of perfect fluids onto BHs, particularly, is also an issue with a large literature. In a seminal paper in the 70’s, Michel [23] investigated the steady-state accretion of a perfect fluid onto the Schwarzschild BH. A steady-state accretion process does not depend on time, it is a stationary process. For an observer in the spherically symmetrical case, physical quantities in such a process are $r$-dependent only. Following Michel, Babichev et al. [24] studied the accretion of a phantom fluid onto the Schwarzschild metric. (For the analysis involving scalar fields instead of fluids, see [25] and references therein.) As a continuation, the same authors used the Reissner-Nordström geometry in order to obtain the accretion equations (in the same direction, see [27] for a study on accretion onto charged BHs). Cases with a cosmological constant were studied in [28, 29], and accretion onto general spherical BHs were discussed in [30, 31]. In the first work on accretion onto a RBH [32], the Bardeen metric was used. Studies on accretion onto other RBHs are shown in [33, 34]. In the present work, we investigate the accretion process of perfect fluids for RBHs generated by the mass function [1] and, then, compare to the Schwarzschild case.
As pointed out by Babichev *et al*, it is also worth considering the phantom fluid case because such a fluid decreases the BH mass in the accretion process. A perfect fluid is said to be phantom-like when its equation of state (EoS) is \( w = \frac{p}{\rho} < -1 \) or, equivalently, \( p + \rho < 0 \), where \( p \) and \( \rho \) are the fluid’s pressure and energy density, respectively. Recently, the Planck Collaboration [35], by using Planck + Pantheon supernovae + BAO data, constrained the dark energy EoS to \( w = -1.03 \pm 0.03 \). Some years before, the Supernova Cosmology Project obtained \( w = -0.997 \pm 0.077 \) for a flat universe [36]. With such an EoS, a phantom fluid that describes the dark energy would be pretty much ruled out. However, according to the authors of Ref. [37], who adopted the Union2 data from Supernova Project, the phantom energy could be a candidate for dark energy with \( \sim 1.9\sigma \). For that reason, disintegration of BHs by phantom-like fluids is an important issue in the cosmological scenario as well. In bouncing cosmologies or cyclic models [38, 39], which are options to the standard big bang model, elimination of BHs is essential in order to promote models beyond the standard cosmology [40, 41]. A given cycle cannot inherit large BHs from the previous cycle. The existence of cyclic models depends on eliminating BHs with the aid of a phantom energy or Hawking’s radiation [42–44].

With the turnaround critical mass, obtained by Ref. [44] within a cyclic scenario, we compare the Schwarzschild critical mass to the critical mass of the class of RBHs studied here. As we will see, there will be a slight difference over time.

The structure of this paper is as follows: in Section II we derived the equations that describe the steady-state accretion process for spherical RBHs with mass term [11]; in Section III we compared the accretion process regarding the Schwarzschild BH and RBHs; in Section IV some types of fluids and critical points are evaluated. In Section V, implications on cosmology are briefly discussed, and the final remarks are made in Section VI. In this work, we adopted geometric units such that \( G = c = 1 \), where \( G \) is the gravitational constant, and \( c \) is the speed of light in vacuum.

## II. THE STEADY-STATE ACCRETION PROCESS

Michel’s approach [23] assumes some important conditions in order to provide the accretion equations for static spacetimes:

1. General Relativity is the theory of gravitation and, hence, the matter energy-momentum tensor is conserved;
2. the accretion is a steady-state process with the accreted matter conceived of as a perfect fluid;
3. the fluid in the process does not modify the original geometry, i.e., back-reaction effects due to the fluid’s self-gravity are neglected.

Michel’s approach gives origin to both physical and non-physical solutions. We are interested in physical ones, *i.e.*, those ones corresponding to “real” accretion phenomenon, for which the perfect fluid departs from infinity at rest and has its infall velocity monotonically increased in the outside region of the black hole event horizon. For these solutions, the fluid experiences a subsonic flow in the regions far from the horizon, and then crosses a critical point \( (r = r_c) \), also outside the event horizon. At the critical point, the sonic point, the fluid has radial velocity equals to the speed of sound, and after that point in the inflow motion the fluid’s radial velocity increases to the supersonic levels. Therefore, regarding spherical spacetimes, we have a sonic sphere at \( r = r_c \), in which every fluid’s element has radial velocity equals to speed of sound.

Let us obtain the process equations for static metrics. First of all, it is assumed a metric with spherical symmetry in the \((t, r, \theta, \phi)\) coordinates in the form

\[
\text{ds}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( f(r) = 1 - 2m(r)/r \), and the mass function is given by Eq. (1). Due to the shape of \( m(r) \), the spacetime structure of (2) is quite different from the Schwarzschild geometry. That is, when \( r_0 \ll M_0 \), the geometry given by \( m(r) \) has two horizons, namely an inner \( (r_-) \) and an outer horizon, the event horizon \( (r_+) \). In the case when \( r_0 > M_0 \), there are no horizons (this is not the case studied here). In the latter case, neither horizons nor naked singularities are present because the mass function \( m(r) \) avoids a singularity, and \( f(r) \) does not provide any roots.

A perfect fluid has an energy-momentum tensor given by

\[
T_{\mu\nu} = (p + \rho)u_{\mu}u_{\nu} + pg_{\mu\nu},
\]

where \( p \) and \( \rho \) are the pressure and energy density, respectively, and obey an EoS \( p(\rho) \). The four-vector \( u^\mu \) is the four-velocity of the fluid and, in the radial process, is simply \( u^\mu = (u^t, u^r, u^\theta, u^\phi) \), with \( u^r = dx^\mu/\text{ds} \) and the following normalization: \( u^\mu u_{\mu} = -1 \). Following Michel, the first equation of accretion is obtained from the current conservation, \( J^\mu_{\nu} = 0 \), which indicates the conservation of mass-energy flux, in which a semicolon indicates a covariant derivative. Therefore, the first conservation, from \( J^\mu = -T^\mu_{\nu}u^\nu \), leads to

\[
pu^2 = C_1.
\]

The second equation for the accretion process comes from the energy-momentum conservation, \( T^\mu_{\nu,\mu} = 0 \). By using \( u^\mu u_{\mu} = -1 \) and \( u^t = \sqrt{\frac{1}{f(r)} + \frac{u^2}{f(r)^2}} \), the second equation can be written as

\[
(p + \rho) \left( f(r) + u^2 \right) \frac{3}{2} u^2 = C_2.
\]

As we pointed out, the component \( u \) is the radial velocity \( u = \frac{dx}{ds} \), and \( u < 0 \) for accretion, *i.e.*, for the fluid’s inflow.
motion. With the aid of Eq. (4), Eq. (3) is rewritten as

\[ (p + \rho) \frac{1}{\rho} (f(r) + u^2) = C_3 = \left(\frac{C_2}{C_1} \right)^2. \]  
(6)

It is worth mentioning some important results and limits that are obtained from Eq. (6). Assuming a dust fluid \((p = 0)\), alongside the condition of rest at infinity, namely \(u_\infty \rightarrow 0\), we have \(C_3 = 1\) and the Newtonian equation for freely falling bodies, that is to say, \(u_{Sch} = -\sqrt{2M_0/r}\) for each fluid’s particle, if the metric is the Schwarzschild one. Another interesting result is obtained for a cosmological constant, \(p = -\rho\), in the accretion process. As we can see, Eq. (6) indicates lack of accretion if the cosmological constant is conceived of as a perfect fluid with \(w = -1\). As pointed out in [13], a Schwarzschild-(a)-dS BH does not provide an accretion process of a cosmological constant as well.

Differentiating [4] and (6) with the elimination of \(d\rho\), we have

\[ \frac{du}{dr} = -\frac{u}{r} \left(2V^2 (f(r) + u^2) - \frac{1}{2} f'(r) \right), \]  
(7)

where

\[ V^2 = \frac{d\ln(p + \rho)}{d\ln \rho} - 1, \]  
(8)

with the symbol \((\cdot')\) denoting differentiation with respect to radial coordinate \(r\). As we said, acceptable or physical results have a critical point, \(r = r_c\). That is, the fluid’s velocity increases monotonically along its trajectory (at least until the event horizon is reached), and the flow is smooth in all points, which means that both the numerator and denominator in Eq. (7) vanish at the same point, namely, the critical point. Therefore, at the critical point one reads

\[ u_c^2 = \frac{r_c f'(r_c)}{4} \quad \text{and} \quad V_c^2 = \frac{r_c f'(r_c)}{4f(r_c) + r_c f'(r_c)}. \]  
(9)

For many physically relevant cases, at the critical point one has \(u_c^2 = c_s^2\), where

\[ c_s = \sqrt{\frac{dp}{d\rho}} \]  
(10)

stands for the speed of sound in the fluid. We will indeed use such condition here, but we need to keep in mind that this is not really obligatory, see [40] for further references on this issue.

It is worth mentioning that for \(r_0 = 0\) we can obtain the same results of Michel and the Schwarzschild case, i.e., \(u_c^2 = M_0/2r_c\) and \(V_c^2 = u_c^2/(1 - 3u_c^2)\). Acceptable solutions require \(u_c^2 > 0\) and \(V_c^2 > 0\) at the critical point, therefore that implies some restrictions or constraints on the mass function and its parameters:

\[ m'(r_c) - \frac{m(r_c)}{r_c} < 0, \]  
(11)

and

\[ m'(r_c) + \frac{3m(r_c)}{r_c} - 2 < 0. \]  
(12)

The second restriction, for the Schwarzschild case, \(^1\) is simply \(r_c > 3/2M_0\). The inequalities (11) and (12) should impose a critical point outside the event horizon. Then the critical velocity, \(u_c^2 = c_s^2\), is reached outside the BH.

The third equation for the accretion process is obtained from another conservation: \(u_\mu T^{\mu\nu}_\nu = 0\). This provides us a continuity equation, that is to say,

\[ u^\mu \rho_\mu + (p + \rho)u^\mu = 0, \]  
(13)

in which the comma indicates an ordinary differentiation. By using the metric (9), the above equation leads to

\[ w^2 \exp \left(\int_{\rho_\infty}^\rho \frac{dp}{p + \rho}\right) = -A. \]  
(14)

Since \(u < 0\) for accretion, then \(A\) must be positive. As we will see, the constant \(A\) is related to the accretion rate of the perfect fluid. From (13) and (14), one reads

\[ (p + \rho)(f(r) + u^2)^2 \exp \left(\int_{\rho_\infty}^\rho \frac{dp}{p + \rho}\right) = -\frac{C_2}{A} = C_4, \]  
(15)

and \(C_4 = p_\infty + \rho_\infty\) (values calculated at infinity with fluid at rest, \(u_\infty \rightarrow 0\)), assuming an asymptotically flat spacetime, like the Schwarzschild BH and the class of RBHs studied here.

The change in the BH mass due the accretion of matter-energy may be evaluated \([24]\) as

\[ \dot{M} = \int T_{0i}^i \sqrt{-g} d\theta d\phi, \]  
(16)

where dot indicates time derivative, and \(g\) is the metric determinant. For the metric given by Eq. (2), we have simply \(\dot{M} = -4\pi r^2 T_{01}\), and by using Eqs. (14) and (15), we conclude that the mass variation is \(^2\)

\[ \dot{M} = 4\pi A (p_\infty + \rho_\infty). \]  
(17)

Therefore, the BH mass decreases with a phantom fluid because, as we said, \(p + \rho < 0\) for this type of perfect fluid. This is a quite interesting conclusion pointed out by Babichev et al. \([24]\) with consequences in cosmology, as we will see.

The constants obtained in this section are important in order to calculate the radial velocity and the accretion rate, which are dependent on an EoS. In Sec. IV, we will use another EoS (instead of \(p = w\rho\)), which avoid both a nonsense speed of sound, in the case of negative values of \(w\), and hydrodynamic instabilities when \(\rho < 0\).

\(^1\) There is a misprint in Michel’s work \([23]\) concerning this point.

In the quoted article, we read \(r_c > 6M_0\) instead of \(r_c > 3/2M_0\).

\(^2\) Our constant \(A\) differs from \([24, 26]\) by a factor \(M_0^2\).
III. COMPARISON BETWEEN THE SCHWARZSCHILD BH AND RBHS

Regarding the Schwarzschild metric and the class of RBHs generated by the general mass function given by Eq. (14), it is possible to compare the radial velocity for the entire class of RBHs with the radial velocity of the accretion process onto the Schwarzschild BH, by using an approximation for the mass function (15). For that purpose, we write the mass function as

\[ m_l(r) \simeq M_0 \left( 1 - \frac{3}{q} \left( \frac{r_0}{r} \right)^q \right), \tag{18} \]

for large values of the radial coordinate. It is worth emphasizing that for small \( r \), it is problematic to speak of radial velocities due to the absence of a stationary observer inside the event horizon. On the other hand, for large values of \( r \), assuming the EoS \( w = \frac{p}{\rho} = 0 \), Eq. (15) presents the following result by adopting (18):

\[ u_l = - \sqrt{\frac{2M_0}{r}} \left( 1 - \frac{3}{q} \left( \frac{r_0}{r} \right)^q \right). \tag{19} \]

In Schwarzschild, as we said, \( u_{Sch} = -\sqrt{2M_0/r} \) for free fall. A straightforward interpretation indicates that the fluid’s radial velocity is smaller for the entire class of RBHs compared to the Schwarzschild accretion velocity. Of course, due to the very small size of \( r_0 \), the difference is very tiny in Eq. (19) for astrophysical objects, whether stellar or supermassive BHs. Nevertheless, we believe that the tiny difference between the class of RBHs and the Schwarzschild BH comes from the de Sitter core inside RBHs. The de Sitter core works as an antigravitational source, decreasing the radial velocity in the accretion process onto RBHs. This core’s capability was also indicated in geodesic studies of the Bardeen RBH in Ref. [17]. As we can see by adopting \( m(r) \simeq M_0 (r/r_0)^3 \) for the mass function when \( r \) is small, the metric term is \( f(r) \sim 1 - Cr^2 \) for those values of \( r \), with \( C \) playing the role of a positive cosmological constant and illustrating the de Sitter spacetime inside the RBHs of the class studied here.

Another interesting result comes from Eq. (7) and shows once again the de Sitter antigravitational capability in this case. By using the simplest EoS \( p = w\rho \), we have a simple differential equation that indicates the variation of the accretion radial velocity. With the mentioned EoS, Eq. (7) is easily written as

\[ \frac{du}{dr} = -\frac{f'(r)}{2u}. \tag{20} \]

As we can see, for the Schwarzschild BH \( du_{Sch}/dr = -M_0/u_{Sch}r^2 \) and, for the entire class of RBHs, one has

\[ \frac{du_l}{dr} = -\frac{M_0}{u_l r^2} + 3 \left( 1 + \frac{1}{q} \right) \frac{M_0 r_0^q}{u_l r^{q+2}}, \tag{21} \]

using the approximation (18). Thus, for the accretion process, in which \( u < 0 \) and \( dr < 0 \), \( du \) is negative for acceleration and is positive for deceleration. In the Schwarzschild BH, we have only a positive term for Eq. (20) during the accretion process. On the other hand, for the class of RBHs, the second term in Eq. (21) produces deceleration, that is to say, the second term is negative. Once again, the antigravitational feature of the de Sitter core for RBHs gets evident for this EoS.

IV. CRITICAL POINTS AND OTHER EXAMPLES OF FLUIDS

In this section, a different EoS is adopted. Following Babichev et al. [24, 26], we assume that

\[ p = \alpha (\rho - \rho_0), \tag{22} \]

where \( \rho_0 \) is a constant. According to the cited authors, the above EoS avoids the hydrodynamic instabilities related to negative values of \( \rho \). Another issue fixed by Eq. (22) concerns the positivity of speed of sound for any type of fluid. For fluids with \( w < 0 \), the EoS \( p = w\rho \) leads to a nonsense result because \( c_s = \sqrt{dp/d\rho} = w \). The EoS (22) avoids such a problem because \( \alpha \) may be adjusted in order to be positive. Then, from now, \( c_s^2 = u_l^2 = \alpha \). It is worth mentioning that \( \alpha \) has a maximum value in order to provide an accretion process in which the radial velocity is sonic outside the event horizon. As we can see in Fig. (1) and we will see from the equation that provides \( r_c \), if \( \alpha < \alpha_{max} \), we will have \( r_c > r_+ \). For the Schwarzschild BH, \( \alpha_{max} = 1/3 \). Expressions for \( \alpha_{max} \) in the class of RBHs are cumbersome, however for \( q = 3 \), or the Hayward RBH, its approximate form is quite simple, namely

\[ \alpha_{max} \approx \frac{27M_0^3 - 32r_+^3}{81M_0^3}, \tag{23} \]

and the corresponding Schwarzschild limit is straightforward. It is possible relate both EoS (inasmuch as \( w = \alpha (\rho - \rho_0) / \rho \) for some important fluids in cosmology, assuming \( \rho > 0 \) and \( 0 < \alpha < \alpha_{max} \):

1. \( w < -1 \), phantom fluid: \( -\frac{\rho}{\rho - \rho_0} < \alpha < \alpha_{max} \) with \( \rho_0 > 0 \) and \( 0 < \rho < \rho_0 \);
2. \( w = -1 \), cosmological constant: there is no accretion process;
3. \( w = 0 \), dust: \( \rho = \rho_0 > 0 \);
4. \( w = \frac{1}{3} \), radiation: \( \alpha = \frac{\rho}{3(\rho - \rho_0)} \) and \( \alpha_{max} > \frac{\rho}{3(\rho - \rho_0)} \) with \( \rho_0 \leq 0 \) and \( \rho > 0 \) or \( \rho_0 > 0 \) and \( \rho > \rho_0 \);
5. \( w = 1 \), stiff fluid: \( \alpha = \frac{\rho}{\rho - \rho_0} \) and \( \alpha_{max} > \frac{\rho}{\rho - \rho_0} \) with \( \rho_0 \leq 0 \) and \( \rho > 0 \) or \( \rho_0 > 0 \) and \( \rho > \rho_0 \).
With the choice of the EoS, physical quantities may be presented in terms of $\alpha$. From Eq. (14), by using the EoS (22), we have the following equation that relates thermodynamic and dynamic variables:

$$\left(\frac{p+\rho}{p_\infty+\rho_\infty}\right) = \left(-\frac{A}{ur^2}\right)^{1+\alpha}.$$  \hfill (24)

Moreover, with the aid of Eqs. (25) and (26), we obtain an easier expression in order to calculate the radial velocity:

$$f(r) + u^2 = \left(\frac{p_\infty + \rho_\infty}{p + \rho}\right) \frac{2\alpha}{\alpha} = \left(-\frac{ur^2}{A}\right)^{2\alpha}.$$  \hfill (25)

As we can see in Fig. 1, the radial velocity depends on $\alpha$. The larger $\alpha$, the closer to the event horizon the critical point is. That is, a sonic radial velocity for the fluid is reached just nearby $r_+$, then the perfect fluid (coming from infinity) takes more time in order to reach and surpass $c_s$ for large values of $\alpha$.

With the EoS (22) and the fact that $u^2 = c_s^2 = \alpha$ at the critical point, the constant $A$, given by Eq. (14), is rewritten as

$$A = r_c^2 \left(\frac{\alpha^2}{\alpha + f(r_c)}\right) \frac{1}{\alpha}.$$  \hfill (26)

By assuming the positivity of $\alpha$ for any value of $\alpha$ in order to obtain a general accretion process, the denominator of Eq. (26) must be positive, and it will be if $\alpha + f(r_c) > 0$, which is translated into Eq. (12). Moreover, for large values of $r$, we can see an easier expression for $A$ and compare to the Schwarzschild BH, that is to say,

$$A \approx r_c^2 \left(\frac{\alpha^2}{\alpha + 1 - \frac{2M_0}{r_c} + \frac{6M_0r_c^2}{q_0^2}}\right) \frac{1}{\alpha}.$$  \hfill (27)

For some values of $q$ and $r_0$ in Eq. (27), the term that contains those parameters may produce a smaller accretion process onto RBHs compared to the Schwarzschild BH (see Fig. 1 for comparison). With $r_0 \neq 0$, the denominator of (27) may be larger (for $q = 1$ and $q = 2$), then, according to Eq. (17), the accretion process takes more time for RBHs, that is to say, their accretion rates are smaller compared to the Schwarzschild BH. Differences in $A$ are due to different values assumed by $q$, $r_0$, and $r_c$, which depends on that parameters. For the entire class of RBHs, the critical point $r_c$ is smaller than the critical point of the Schwarzschild metric. And in the entire class in which $q = 1$ and $q = 2$, considering some values of $r_0$, the constant $A$ is smaller as well. On the other hand, the accretion rate is larger for different values of $q$ even with smaller radial velocities (compared to Schwarzschild BH) for the accreted fluid.

For $r = r_c$, Eq. (26) (left and right sides) provides the equation whose roots are the critical points. From Eqs. (25) and (26), by using both the metric and the mass

![Figure 1: On the top, graphic for speed of sound over values of the critical point $r_c$, according to Eq. (28) and the equality $c_s^2 = \alpha$. As we can see, there is a $s_{max}$ in order to have a critical point outside the event horizon. On the middle, graphic of the radial velocity for some values of $q$ in Eq. (1) and for the Schwarzschild black hole ($r_0 = 0$). It is interesting to note the fluid’s behavior according to values of $\alpha$: the larger $\alpha$, the larger the speed of sound and, consequently, the critical point is smaller. That is to say, for large values of $\alpha$, the fluid in the accretion process needs more distance in order to reach $c_s$, the speed of sound. On the bottom, graphic for $A$, which provides the accretion rate, according to Eq. (17). The dashed line indicates the value of $A$ for the Schwarzschild BH. In these graphics, we used $M_0 = 1$ and $r_0 = 0.1$, on the top, and $r_0 = 0.7$, on the middle.](image-url)
Table I: Values of the outer critical point, \( r_c = r_{c+} \), for the Schwarzschild BH and members of the class of RBHs by using perfect fluids with EoS \( \alpha = 0.1 \) and \( \alpha = 0.2 \) in order to describe the accretion processes for objects with one solar mass \( (M_\odot) \) and \( 10^6 \) solar masses like the Sagittarius A*. All critical point values for RBHs are obtained using the largest value for the quantum gravity parameter that produces \( r_0 \sim 10^{-23}\text{m} \), estimated in \([20]\). Such a value for \( r_0 \) turns indistinguishable BHs and RBHs critical points from that point of view. As we can see, the larger \( \alpha \) from EoS \((22)\), the closer the critical point is. Keep in mind that the event horizon for those objects is about 3 km for \( M_0 = M_\odot \) and \( 1.2 \times 10^7 \) km for \( M_0 = 4 \times 10^6 M_\odot \).

| Type of Black Hole | \( M_0 = M_\odot \) | \( M_0 = 4 \times 10^6 M_\odot \) |
|--------------------|-------------------|-------------------|
| Schwarzschild BH and RBHs | \( \alpha = 0.1 \) | \( \alpha = 0.2 \) |
| \( r_c \) | \( \approx 7.428 \) | \( \approx 2.986 \times 10^7 \) |
| \( r_c \) | \( \approx 3.714 \) | \( \approx 1.493 \times 10^7 \) |

*Values in kilometers.

function, the critical points are solutions of

\[
\frac{M_0}{2r_c} \left( 1 - 2 \left( \frac{r_0}{r_c} \right)^q \right) \left( 1 + \left( \frac{r_0}{r_c} \right)^q \right)^{-\frac{q+1}{q}} = \alpha. \tag{28}
\]

Another way to obtain the critical points and the above equation is simply solving \( u^2 = \alpha \), with the aid of Eq. \((23)\). In general, for \( r_0 \ll M_0 \) and any positive integer \( q \), Eq. \((23)\) presents two positive solutions or critical points: the inner \( r_- \) and the outer critical point \( r_{c+} \). These points, which are solutions of \((28)\), obey the following relation:

\[
0 < r_- < r_{c-} < r_+ < r_{c+}. \tag{29}
\]

For the Schwarzschild BH, there is just one critical point: \( r_{c+} = M_0/2\alpha \). As we said, \( r_c = r_{c+} > 3M_0/2 \), thus \( 0 < \alpha < 1/3 \). That is the origin for \( \alpha_{\text{max}} = 1/3 \) in the Schwarzschild spacetime as we already pointed out. And, according to Fig. 1, the class of RBHs also presents a maximum \( \alpha_{\text{max}} \), which is about 1/3 (as close as \( r_0 \approx 0 \)).

The role played by \( r_0 \) is important in order to obtain the critical points. Following Ref. \([48]\), where a GUP was applied to the RBHs studied here, the same GUP was used in the Bardeen metric in Ref. \([20]\), then the length parameter was obtained and written as

\[
r_0 = \frac{\lambda l_p}{3}, \tag{30}
\]

in which \( \lambda \) stands for the dimensionless quantum gravity parameter, and \( l_p \) is the Planck length. When \( \lambda = 0 \) in the generalized principle, we recover the Heisenberg uncertainty relation, and, according to several methods, \( \lambda \) (and consequently \( r_0 \)) may be estimated. In \([49, 50]\), some methods in order to obtain the quantum gravity parameter are commented. The best upper bound for \( \lambda \) is obtained from the scanning tunneling microscope, that is to say, \( \lambda < 10^{16} \) for that method. Thus, \( r_0 < 10^{-23}\text{m} \) is straightforwardly obtained from Eq. \((30)\), and Table I presents some values of \( r_{c+} \) estimated from some geometries and for perfect fluids with EoS \( \alpha = 0.1 \) and \( \alpha = 0.2 \) in the accretion processes of objects with one solar mass and \( \sim 10^6 \) solar masses, like Sagittarius A*. The super massive BH at the Milk Way center. Such as Fig. 1 indicates, large values of \( \alpha \) produce small values of the critical point, showing that the fluid in the accretion process needs more distances in order to acquire a radial velocity equals to speed of sound. Above all, the tiny value of \( r_0 \) imposes that the difference of critical points among members of the class of RBHs and the Schwarzschild BH is pretty much indistinguishable for astrophysical BHs. As we said, such a microscopical value becomes very important just at small scales, inside the event horizon. However, if clear differences between RBHs and singular BHs are not noticeable regarding large distances, it will be over a long time. In this sense, a cosmological scenario may provide us some differences.

V. IMPLICATIONS FOR COSMOLOGY

The accretion process onto black holes may be an important issue in cosmology. If, as we saw, a cosmological constant conceived of as a perfect fluid cannot be accreted by BHs and RBHs, other types of fluids will be. As we saw, according to \([24]\), a phantom fluid may decrease the BH mass during the accretion process. Then, somehow, destruction of BHs and RBHs by means of phantom energy turns into an issue in bouncing and cyclic cosmologies. According to cyclic cosmologies \([42, 43]\), the initial singularity and the big rip are avoided, and before the bounce and after the turnaround, a contraction phase is predicted. The scale factor reaches minimum and maximum values at the bounce and the turnaround, respectively. Whether before the bounce or the turnaround, BHs should be eliminated in order to promote a new cycle without such debris. According to the authors of \([43]\), BHs are torn apart in the expansion phase—dominated by phantom energy—before the turnaround. But in Ref. \([44]\), this very conclusion is criticized. Accordingly, BHs do not evaporate before the turnaround using a phantom fluid. Their masses decrease during the late expansion dominated by a phantom fluid and, then, the remaining masses are just eliminated during the contraction phase by the Hawking radiation. Following Ref. \([44]\), who adopts a modified Friedmann equation in order to
provide a turnaround, the BH mass at the turnaround for initial masses larger than the Planck mass is

$$M_c \approx \sqrt{\frac{3}{2\pi^2 \rho_c}} M_p^3.$$  \hspace{1cm} (31)

in which $\rho_c$, according to the author, is the phantom’s critical density at the turnaround, and $M_p \approx 10^{-8}$ kg is the Planck mass. Here, the initial mass is $M_0$, which is given at the beginning of the phantom dominated phase. Therefore, using Eq. (27) in order to compare accretion processes onto BHs and RBHs, we conclude that the RBH mass may be larger than the Schwarzschild mass at the turnaround in a cyclic cosmology, especially for $q = 1$ and $q = 2$. Such a result is in agreement with the previous conclusion of ours, according to which the accretion process is slower for RBHs with $q = 1$ or $q = 2$. Thus, a phantom dominated phase will steal more mass from the Schwarzschild BH than from that RBHs. Moreover, as we said, according to Ref. 44, the remaining mass, whether of BH or RBH, is evaporated by Hawking’s mechanism in the contraction phase, and a phantom cyclic cosmology is safe from that astrophysical objects in this perspective.

In order to promote another direct comparison between BHs and RBHs, we propose a critical mass ratio at the turnaround for those objects, that is to say, by using the approximate $A$, given by Eq. (27), and the critical mass defined above, we have

$$\frac{M_c(RBH)}{M_c(Sch)} \approx (1 + F(\alpha, r_0, M, q))^{\frac{1}{2\nu}}, \hspace{1cm} (32)$$

where $M_c(RBH)$ and $M_c(Sch)$ stand for the critical mass for RBHs and the Schwarzschild BH at the turnaround, respectively, and

$$F(\alpha, r_0, M, q) = \frac{6M_0}{(1 + \alpha) r_c - 2M_0} \left(\frac{r_0}{r_c}\right)^q$$  \hspace{1cm} (33)

is a positive function. Following [44], the critical mass $M_c$ is about the Planck mass at the turnaround. With a tiny quantum gravity parameter that provides $r_0$, the function $F(\alpha, r_0, M, q)$ renders an indistinguishable difference between astrophysical BHs and RBHs at the turnaround (astrophysical at the beginning of the phantom dominated era). That is to say, by assuming $M_0 \sim M_0$ or $M_0 \sim 10^6 \times M_0$ as the initial masses for the Schwarzschild BH and RBHs at the beginning of the phantom dominated phase, the difference between singular and RBHs is still very tiny at the turnaround. However, if the initial masses are similar to the primordial black hole (PBH) masses, it will be more interesting. PBHs are hypothetical objects generated during the radiation dominated era and are candidates for dark matter. Recently [51], the Subaru Telescope constrained the PBHs masses range to $10^{-11}M_\odot - 10^{-6}M_\odot$ by using the microlensing effect of stars in M31 (Andromeda Galaxy) caused by PBHs in the halo of the Milky Way. According to [51], with these masses, PBHs as candidates for the total amount of dark matter are pretty much ruled out. However, with $M_0 \approx 10^{-11}M_\odot$ as the initial mass during the phantom era, Eq. (32) with $q = 1$ and $\alpha = 0.2$ renders about $10^{-19}$ kg of difference between the Schwarzschild BH and RBHs. That is, both objects will have approximately Planck masses at the turnaround, but RBHs will have a tiny advantage in their masses ($\simeq 10^{-19}$ kg). Then, within a cosmological point of view, it is possible to conceived of a quantitative difference between the Schwarzschild BH and RBHs.

\section{VI. FINAL REMARKS}

We applied Michel’s [23] approach, which describes steady-state accretion onto BHs, to a class of RBHs developed in Ref. [16]. Compared to the Schwarzschild BH, some members of that class present a potentially slower accretion process. Regarding the accretion process of a phantom fluid, the RBHs masses may decrease with a slower rate compared to Schwarzschild’s. The origin for that is found in the accretion rate and the radial velocity of the accreted perfect fluid. For the entire class of RBHs, the radial velocity is smaller, and, for some members, the accretion rate is smaller than the rate of the Schwarzschild BH. A slower radial velocity for RBHs indicates the capability of the de Sitter core—that which generates regular solutions—of playing the role of an antigravity source.

By using the recent upper bound on the parameter that provides regular metrics, namely $r_0 < 10^{-20} m$ according to [21], we calculated the critical point, where the fluid’s radial velocity equals to speed of sound in the fluid. Due to the tiny value of $r_0$, members of the class of RBHs present indistinguishable values for the critical points compared to Schwarzschild BH.

However, from the cosmological point of view, differences between RBHs and the Schwarzschild BH are suggested. With a slower accretion process for some RBHs, the evaporation of RBHs due to a phantom fluid takes longer time, in agreement with other results. Therefore, the accretion process conceived over long time may produce different masses between BHs and RBHs. In particular, RBHs with $q = 1$ and $q = 2$ would have larger masses than the Schwarzschild BH at the turnaround in cyclic cosmologies. This result may be important for cyclic cosmologies, in which the existence of debris of previous cycles are avoided in order to build realistic models.
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