Quantum Domain Theory - Definitions and Applications

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Abstract

Classically domain theory is a rigorous mathematical structure to describe denotational semantics for programming languages and to study the computability of partial functions. Recently, the application of domain theory has also been extended to the quantum setting. In this note we review these results and we present some new thoughts in this field.

1 Introduction

In 1930s the two main models of computation were introduced: 1) Turing machines by Alan Turing and 2) Lambda calculus by Alonzo Church. The former is the foundation of all existing von-Neumann computers, computational complexity analysis, and imperative programming languages. The latter is more suitable for the study of formal methods and functional programming languages. Both directions have been extensively studied in classical computer science and many equivalent models of computation have been introduced where each new model addresses a different aspect of information processing.

Quantum computation is traditionally studied via quantum circuit models or in terms of quantum Turing machines, which fit into the first model of computation \[10,11,45,5]. In this approach, one specifies a concrete recipe for how to build more complicated quantum processes from a few basic building blocks, and it lays the foundations for computational complexity and the design of new quantum algorithms. In order to analyse other aspects of quantum computation different models are required. For example, the one-way quantum computer is a new model for quantum information processing where measurements play a central role \[34\]; it is a universal model and proven to be polynomially equivalent to the quantum network model. The one-way quantum computer presents new aspects of quantum processing, such as temporal complexity, that can not be analysed properly in other models \[35\]. Recently there have been also some developments in the area of programming languages, which require models of computation with a higher level of abstraction \[31,30,36,8,39\].

In this paper we will work within yet another model of computation, namely domain theory. Classically domain theory is a suitable mathematical structure for descriptions of denotational semantics for programming languages and to study the computability of partial functions \[1,12\]. Recently, the application of domain theory has also been extended
to the quantum setting. Here we review these results and present some new thoughts in this field. In sections 2 and 3 we review the basic concepts of quantum computation and classical domain theory. In Section 4 we introduce quantum domain theory and summarise its applications in Quantum Computability, Quantum Semantics and Quantum Information Theory. Finally Section 5 contains discussion and indicates further research directions.

2 Quantum Computation

Quantum computation is a rapidly growing cross-disciplinary field which explores the relation between quantum physics and computation, and is of great importance from both a fundamental as well as technological perspective. The exciting discovery was that quantum computer is in fact provably more efficient than any classical computer [10, 40, 19, 4]. One of the key effect leading to this efficiency is the quantum superposition phenomenon which allows a quantum computer to perform a multitude of different tasks simultaneously (in parallel). For a general introduction to quantum computing we refer to [29]. Here we review some standard physical background which is required for the discussion of this paper.

The state of a closed quantum physical system which is not interacting with an environment (pure state) is described by a unit vector in a Hilbert space, which in Dirac notation is denoted by \( |\psi\rangle \in \mathcal{H} \). The simplest quantum mechanical system is a qubit, which has a two dimensional Hilbert space, \( \mathcal{H}_2 = \mathbb{C}^2 \), as the state space. The state of a composite quantum system (made of \( n \) qubits) is described by the tensor product of state of the component physical systems:

\[
|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle \in \mathcal{H}_{2^n},
\]

where

\[
\mathcal{H}_{2^n} = \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2 = \mathbb{C}^{2^n}.
\]

In Dirac notation, \( \langle \psi | \) is used for the dual vector to the vector \( |\psi\rangle \) which is a linear operator over the corresponding Hilbert space defined as following:

\[
\langle \psi | : \mathcal{H} \to \mathbb{C}, \quad |\phi\rangle \mapsto \langle \psi |\phi\rangle,
\]

where \( \langle .| . \rangle \) denotes the inner product of the Hilbert space.

The evolution of a closed quantum system is described by a unitary transformation 1 on the corresponding Hilbert space. The matrix representations of the quantum operations used in this paper are:

Hadamard\[
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},
\]

Pauli-X\[
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

Phase\[
P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix},
\]

Rotation-\( \pi/8 \)\[
T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix},
\]

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1A transformation \( U \) on a Hilbert space is unitary iff \( UU^\dagger = U^\dagger U = I \).
controlled-Not \( \text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \).

A general unitary transformation \( U \) on a finite dimensional Hilbert space can be decomposed into a product of (one and two)-level unitary matrices where act on 1 and 2 qubits. It is known that the set of single qubits operators e.g. \( H, X, P, T \) and controlled-NOT operator (CNOT) are universal. The set of all bounded operators on a Hilbert space is represented with \( B(\mathcal{H}) \).

Mixed states arise when we do not have complete information about the state of the physical system. This is always the case in experiments, since the system that we are trying to prepare in a pure state interacts with an uncontrolled environment. A mixed state is a probabilistic mixture of pure states, denoted by \( \{p_i, |\psi_i\rangle\} \) or alternatively by a density matrix

\[ \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \, . \]

A density matrix \( \rho \in B(\mathcal{H}_{2^n}) \) is a hermitian (i.e. \( \rho = \rho^\dagger \)) positive semi-definite matrix of dimension \( 2^n \otimes 2^n \) with trace \( \text{trace}(\rho) = 1 \). Note that a given pure state \( |\psi\rangle \) can also be represented with the density matrix \( |\psi\rangle \langle \psi| \).

The most general operation on quantum states are the transformations of density matrices i.e. linear operators on operators (super-operator). The physically allowed superoperators are linear completely positive and trace-preserving operators, called \( \text{CP maps} \) in short. A super-operator \( T \), is positive if it sends positive semi-definite Hermitian matrices to positive semi-definite Hermitian matrices; it is completely positive if \( T \otimes I_d \) is positive, where \( I_d \) is the identity operator on \( d \)-dimensional Hilbert space.

In order to observe a quantum system, a measurement should be applied. Quantum measurements are described by a collection \( M_m \) of measurements operators. These are operators acting on the state space of the system being measured. The index \( m \) refers to the measurements outcome that may occur in the experiment. If the state of the quantum system is \( \rho \) immediately before the measurement then the probability that the result \( m \) occurs is given by

\[ p(m) = \text{Tr}(M_m^\dagger M_m \rho) \, , \]

and the state of the system after the measurement is

\[ \frac{M_m \rho M_m^\dagger}{\text{Tr}(M_m^\dagger M_m \rho)} \, . \]

The measurements operators satisfy the completeness equation,

\[ \sum_m M_m^\dagger M_m = I \, . \]

3 Classical Domain Theory

Domain theory was introduced independently by Scott \[38\] for the study of denotational semantics and by Ershov \[16\] as a tool for the study of partial computable functions. A complete survey of domain and its applications can be found in \[1\] \[12\]. Domain Theory has been developed towards the following key applications:
• A mathematical theory of computation for the semantics of programming languages;
• A mathematical theory of computation over partial information;
• An algebraic approach to computability;

Informally speaking, a domain is a partially ordered set with added structures to model information processing. In this pictures of computation, an specific input (output) is represented with a sequence of elements approximating it. An algorithm is a function from domain of the input to the domain of the output. In order to describe this model precisely first we introduce the standard basic language of domain theory and all the notations that we will use later in this paper.

**Definition 1** A partially ordered set (poset) is a pair \((P, \sqsubseteq)\), where \(\sqsubseteq\) is a binary relation on \(P\) such that the following conditions are satisfied:

- **Reflexivity**. \(\forall x \in P : x \sqsubseteq x\).
- **Transitivity**. \(\forall x, y, z \in P : x \sqsubseteq y \& y \sqsubseteq z \Rightarrow x \sqsubseteq z\).
- **Anti-symmetry**. \(\forall x, y \in P : x \sqsubseteq y \& y \sqsubseteq x \Rightarrow x = y\).

An element \(\bot \in P\) is called a least element iff \(\forall x \in P : \bot \sqsubseteq x\).

It is easy to see that if a poset has a least element, then it is unique.

The poset structure appears in many different fields of computer science and physics and in each context the ordering, \(\sqsubseteq\), is interpreted differently. In this paper, \(\sqsubseteq\), refers to a notion of information which will be described more precisely later. The notion of a sequence of data is captured via the following structures.

**Definition 2** A subset \(A\) in a poset \(P\) is called a chain iff

\[\forall x, y \in A : x \sqsubseteq y \lor y \sqsubseteq x\.

Assume \(A\) is a chain of the poset \(P\). An upper bound of \(A\) is an element \(u \in P\) such that

\[\forall x \in A : x \sqsubseteq u;\]

The least upper bound of \(A\) is denoted by \(\sqcup A\).

Not every chain in a poset has a least upper bound. Adding this property to a poset (chain completeness) will result in a structure rich enough to model denotational semantics, as we describe later.

**Definition 3** The partially ordered set \(P\) is chain-complete (CCPO) iff all chains \(A\) in \(P\) have a least upper bound \(\sqcup A\) in \(P\).

We shall be interested in continuous functions:

**Definition 4** Assume \((P_1, \sqsubseteq_1)\) and \((P_2, \sqsubseteq_2)\) are given posets. A function \(f : P_1 \rightarrow P_2\) is called continuous iff it is :

- **Monotone**: \(\forall x, y \in P_1 : x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y)\).
- It preserves the least upper bounds of the chains, i.e. for all chains \(A\) in \(P_1\):

\[\sqcup_2\{f(x) : x \in A\} = f(\sqcup_1 A)\]
For a given function \( f \), define \( f^0 \) to be the identity function and \( f^{(n+1)} = f \circ f^n \). Now, we can state the fixed-point theorem which is a canonical tool to construct the mathematical object corresponding to a recursive definitions.

**Theorem 1** Knaster-Tarski Fixed-Point Theorem Assume \( f : P \rightarrow P \) is a continuous function on the chain complete poset \( P \) with a least element \( \bot \). Then

\[
\text{Fix}_f = \sqcup \{ f^n(\bot) \mid n \geq 0 \},
\]

defines an element of \( P \) which is the least fixed-point of \( f \).

A similar structure to a chain in a poset is a directed set:

**Definition 5** A non-empty subset \( A \subset P \) of a poset \( (P, \sqsubseteq) \) is directed iff:

\[
\forall x, y \in A \exists z \in A : x, y \sqsubseteq z.
\]

A directed set corresponds to a consistent set of data. We denote by \( \sqcup A \) the least upper bound of a directed set, if it exists.

**Definition 6** A partially ordered set in which every directed subset has a least upper bound, is called a domain.

The notion of approximation in domain theory is described via the following relation:

**Definition 7** Assume that \( x \) and \( y \) belongs to a domain \( D \). We say that \( x \) is way-below \( y \) or equivalently \( x \) approximates \( y \), denoted by \( x \ll y \), iff for every directed subset \( A \subset D \):

\[
y \sqsubseteq \sqcup A \Rightarrow \exists a \in A : x \sqsubseteq a.
\]

A constructive structure for a domain can be introduced via basis elements:

**Definition 8** A subset \( B \) of the domain \( D \) is called a basis iff for each \( d \in D \):

\[
A = \{ b \in B \mid b \ll d \} \text{ is directed and } d = \sqcup A.
\]

A domain with a basis is called a continuous domain and if the basis is also countable the domain is called an \( \omega \)-continuous domain.

The following definitions provide a topological structure for a domain.

**Definition 9** An open set \( O \subset D \) of the Scott topology of \( D \) is a set which satisfies the following conditions:

i) \( x \in O \text{ and } x \sqsubseteq y \Rightarrow y \in O \).

ii) For any directed subset \( A \) of \( D \) we have \( \sqcup A \in O \Rightarrow \exists x \in A : x \in O \).

Dually a closed set \( C \subset D \) is defined with the following conditions:

i) \( x \in C \text{ and } y \sqsubseteq x \Rightarrow y \in C \).

ii) For any directed subset \( A \subset C \) we have \( \sqcup A \in C \).

In any continuous domain, subsets \( \uparrow b = \{ x \mid b \ll x \} \) where \( b \) belongs to a given basis of the domain, forms a basis for the Scott topology.

We denote by \( [D \rightarrow D'] \) the set of all continuous functions (with respect to the Scott topology) between two domains \( D \) and \( D' \), which also forms a domain with pointwise ordering:

\[
f \sqsubseteq g \text{ iff } \forall x \in D : f(x) \sqsubseteq g(x).
\]
In summary, in the domain picture of information processing, data are elements of an \( \omega \)-continuous domain \( D \), and represented as least upper bound of the basis elements. A program is an element of domain of continuous functions, \([D \to D]\) and can be represented as the least upper bound of the corresponding basis elements in \([D \to D]\). In what follows we review the main applications of domain theory in computability analysis and denotational semantics. As we show in each scenario a suitable domain will be constructed.

### 3.1 Computability Analysis

There exist two main approaches to computability analysis in the literature. One is the machine-oriented framework and the other one is the analysis-oriented approach \[43\]. In the former scenario, the computation is performed on a certain kind of abstract machine. Whereas in the latter, concepts from classical analysis are extended to develop a computability theory for real numbers or any other mathematical spaces.

Recently, a new approach to computability has also been developed which is based on domain theory and fits into the second main framework for computability \[14, 17, 7, 14\]. In his famous article \[38\], Scott pointed out the relationship between continuity versus computability. For most purposes to detect whether some construction is computationally feasible it is sufficient to check that it is continuous, although continuity is an algebraic condition, which is much easier to handle than computability. We describe briefly how to define computability via domain theory. In the next section we extend this concept to the quantum setting. We define the notion of **effectively given \( \omega \)-continuous domain** by putting a proper recursive structure on the elements of a basis of the domain \[41, 12\].

**Definition 10** Assume domain \( D \) is \( \omega \)-continuous with a countable basis 
\( B = \{b_0, b_1, b_2, \ldots \} \). We say \( D \) is effectively given with respect to \( B \), if the relation \( b_n \ll b_m \) is recursively enumerable \(^2\) in \( n \) and \( m \).

The definition of computable elements is:

**Definition 11** Assume that \( D \) is effectively given. An element \( x \in D \) is called computable, if the set \( \{ n \in \mathbb{N} | b_n \ll x \} \) is recursively enumerable.

We state the following important theorem (without proof) which present a constructive definition of computability.

**Theorem 2** \[14\] Assume domain \( D \) is effectively given, \( x \in D \) is computable iff it is the least upper bound of an effective given chain in the basis \( B \) i.e., iff there exists a total recursive function \( f : \mathbb{N} \to \mathbb{N} \) such that
\[
b_{f(0)} \sqsubseteq b_{f(1)} \sqsubseteq b_{f(2)} \sqsubseteq \cdots \quad \text{and} \quad x = \bigsqcup_{n \in \mathbb{N}} b_{f(n)}.
\]

Moreover, the chain can be chosen to be a \( \ll \)-chain, i.e. such that
\[
b_{f(0)} \ll b_{f(1)} \ll b_{f(2)} \ll \cdots.
\]

Finally the computability of a function is defined as follows.

**Definition 12** Assume that domains \( D \) and \( D' \) are effectively given with respect to the basis sets \( B \) and \( B' \). A continuous function \( f : D \to D' \) is called computable, if the relation \( b_m' \ll f(b_n) \) is recursively enumerable in \( n \) and \( m \).

\(^2\) A set is called recursively enumerable if it is accepted by some Turing machine \[43\].
3.2 Denotational Semantics

The main problem which gave rise to domain theory was that of describing the meaning of recursive definitions of objects or data-types [38]. An important result in this direction is the fixed-point theorem. Traditionally, semantics studies the meaning of programs, mainly in order to be able to state some correctness properties. The meaning of each phrase in a program is the computation that it describes. There are two main directions in the area of semantics of programming languages that differ in the areas they are based on:

- **Operational Semantics**, basically uses infinite automata, and programs are studied in terms of the steps or operations by which each program is executed.

- **Denotational Semantics**, where programs are interpreted as mathematical functions.

Denotational semantics was developed in the early 1970s by Strachey and Scott [37]. They aimed to place the semantics of programming languages on a purely mathematical basis. Denotational semantics assigns a mathematical function not only to a complete program but also to every phrase in the language. This approach has important benefits such as the ability of predicting the behaviour of each program without actually executing it on a computer or reasoning mathematically about programs, for example to prove that one program is equivalent to another.

In this subsection we review a denotational semantics introduced by Kozen for probabilistic computation [26]. This framework will be the basis of our approach to quantum semantics in the most general case. We will show that quantum computation over density matrices with completely positive maps, has a similar semantical structure as probabilistic computation over random variables. First we present some standard basic definitions for vector spaces [6, 26].

**Definition 13** A subset $P$ in a vector space $V$ is called positive cone iff it satisfies the following conditions:

\[
\forall x, y \in P \text{ and positive scalars } a, b : ax + by \in P \\
\forall x \in P : x, -x \in P \Rightarrow x = 0.
\]

$P$ induces a partial order on $V$ with the following relation:

\[
x \sqsubseteq_P y \text{ iff } y - x \in P.
\]

A similar structure to a domain where every directed set has a least upper bound is a lattice where every pair of element has a least upper bound. **Vector lattices** are the main mathematical structure of the Kozen’s denotational semantics for probabilistic computation.

**Definition 14** Let $V$ be a normed vector space and $P \subset V$ a positive cone, $(V, P)$ is called a vector lattice iff every pair $x, y \in V$ has a $\sqsubseteq_P$-least upper bound in $V$. A vector lattice is called conditionally complete if every set of elements of $V$ with an $\sqsubseteq_P$-upper bound has a least upper bound.

To partially order a measurable space we will consider Banach lattices.
Definition 15 Assume that \( B \) is a normed vector space with norm \( \| \cdot \| \), if 
\((B, P, \| \cdot \|)\) is both a Banach space and vector lattice such that:
\[
\| x \| = \| x \| \quad \text{and} \quad \forall x, y \in P : x \leq_P y \Rightarrow \| x \| \leq \| y \| ,
\]
then \( B \) is called a Banach lattice.

In the semantics introduced by Kozen for probabilistic computation, programs are interpreted as continuous linear operators on Banach space of distributions \([26]\). In this framework one could work only with the joint distribution of the program variables instead of dealing directly with variables. Any simple program \( P \) maps the input distributions \( \mu \) to the output distribution \( P(\mu) \). Kozen has considered a probabilistic WHILE program over the variables \( x_1, \ldots, x_n \). Syntactically, there are five types of statements in the language described by Kozen \([26]\).

Core syntax of probabilistic WHILE:

- simple assignment: \( x_i := f(x_1, \cdots, x_n) \), where \( f : X^n \to X \) is a measurable function.
- random assignment: \( x_i := \text{random} \).
- composition: \( S; T \).
- conditional: \( \text{if } B \text{ then } S \text{ else } T \).
- while loop: \( \text{while } B \text{ do } S \).

Let \((X, M)\) be a measurable space and let \( B = B(X^n, M^n) \) be the set of measures on the cartesian product \((X^n, M^n)\). Then \( B \) consists of all possible joint distributions of the program variables \( x_1, x_2, \cdots, x_n \), plus all their linear combinations. Denote by \( P \) the set of all positive measures and by \( \| \cdot \| \) the total variation norm, then \((B, P, \| \|)\) is a conditionally complete Banach lattice \([26]\).

Every program \( P \) will map a probability distribution into a subprobability measure. This can be extended uniquely to a linear transformation in \( B \to B \). Moreover, this extension will be \( \| \cdot \| \)-bounded and therefore continuous. Thus, each program will define a continuous linear operator in \( B \to B \) \([26]\).

The space \( B' \) of operators in \( B \to B \) forms a Banach space which is conditionally complete. The partial order on \( B' \) is defined as follows:
\[
S \leq T \quad \text{iff} \quad S(\mu) \leq T(\mu) \quad \text{for all} \quad \mu \in P .
\]

Programs will be interpreted as elements of this space. The semantics of the probabilistic WHILE language, introduced above, is:

- **simple assignment**: If \( P \) is the program \( \text{“} x_i := f(x_1, \cdots, x_n) \text{”} \) where \( f : X^n \to X \) is a measurable function, then the meaning of \( P \), \( \| [P] \| \), is the linear operator \( P : B \to B \) such that:
\[
P(\mu) = \mu \circ F^{-1} ,
\]
where \( F : X^n \to X^n \) is the measurable function
\[
F(a_1, \cdots, a_n) = (a_1, \cdots, a_{i-1}, f(a_1, \cdots, a_n), a_{i+1}, \cdots, a_n) .
\]

Since \( f \) is measurable, so is \( F \), thus \( \mu \circ F^{-1} \) is indeed a measure.
• random assignment: If $P$ is the program "$x_i := \text{random}$" then the meaning of $P$, $[P]$, is the linear operator $P : B \rightarrow B$ such that:

$$P(\mu)(B_1 \times \cdots \times B_n) = \mu(B_1 \times \cdots, B_i, X, B_{i+1}, \cdots B_n)\rho(B_i),$$

where $\rho$ is a fixed distribution.

• composition: The meaning of the program "$S; T$" is the composition of operators $[S] \circ [T]$.

• conditional: Let $\mu_B$ denote the measure $\mu_B(A) = \mu(A \cap B)$. The conditional test checks the membership of $x_1, \cdots, x_n$ in $B$, which will occur with probability $\mu(B)$ and hence $S$ will be executed on the conditional probability distribution $\mu_B/\mu(B)$. Similarly, with probability $\mu(\neg B)$ the program $T$ will be executed on $\mu_{\neg B}/\mu(\neg B)$.

Formally, the semantics of the program "if $B$ then $S$ else $T$" is the linear operator $P : B \rightarrow B$ such that:

$$A \mapsto \mu(B)S(\mu_B/\mu(B))(A) + \mu(\neg B)T(\mu_{\neg B}/\mu(\neg B))(A)$$

which can be written as $S \circ e_B + T \circ e_{\neg B}$ where $e_B$ is the operator $e_B(\mu) = \mu_B$ and $+\,$ is addition in $B'$.

• while loop: The meaning of the program "while $B$ do $S$" is equivalent to the program

if $\neg B$ then $I$ else $S$; while $B$ do $S$,

therefore the meaning of a "while statement" must be a solution of

$$W = e_{\neg B} + W \circ P \circ e_B.$$  

Using well established techniques one can solve the above equation to drive the following solution. The meaning of a "while statement" is the fixed-point of the affine transformation $\tau : B' \rightarrow B'$ defined by

$$\tau(W) = e_{\neg B} + W \circ S \circ e_B,$$

which is equal to

$$\tau^n(0) = \sum_{0 \leq k \leq n-1} e_{\neg B} \circ (S \circ e_B)^k.$$

4 Quantum Setting

In this section we present some of the applications of domain theory in the framework of quantum computation. In the first subsection we study the domain computability for quantum computation. Subsequently a denotational semantics for quantum computation is presented. Finally we review a recent work on information aspects of quantum domain theory by Coecke and Martin [9]. By introducing a domain framework for quantum computation we aim to address different aspects of information processing which has not yet been studied in other existing models of quantum computation.
4.1 Computability Analysis

The Church-Turing thesis is about classical computability, (i.e. the computability which is defined based on a computing machine which obeys classical mechanics). Hence, it might be thought that quantum mechanical computing can violate the Church-Turing thesis. However, Deutsch [10] and Jozsa [25] discussed this problem and showed that the class of functions computable by a deterministic quantum Turing machine, is equal to the class of recursive functions (computable by a classical Turing machine). Ozawa extended this argument to the probabilistic quantum Turing machine [32]. He also distinguished the notation of measurability from computability to answer the problem that has been alleged by Nielsen in [28].

Apart from these few discussions, there have been no further attempts in this direction. We believe, by introducing a rigorous framework for quantum computability, we can address more interesting questions. Furthermore, quantum domain theory provides a topological structure for quantum computation that can be useful for the study in other fields of quantum computation.

To develop a computational model to analyse quantum computability, it would be enough to consider a model for a Hilbert space. Different effective structures for metric spaces can be found in the literature. We use the domain of the closed balls [44, 13] to introduce a model for quantum pure states and the power domain of the former domain [24, 12, 27] will capture the quantum mixed states. Most of the definitions of this subsection have been already appeared in [13, 12] under the theory of computability for Metric spaces. We rephrase these results in order to suit our purposes of defining a mathematical foundation for quantum computability.

Pure quantum states

An standard way to construct a partially ordered set for a given metric space \((X, d)\) is based on ordering of the set of closed balls [22]. Define a closed ball \(C(x, r)\) of given metric space \((X, d)\) with \(x \in X\) and \(r \in \mathbb{R}\) to be the following set:

\[
C(x, r) = \{ y \in X \mid d(x, y) \leq r \}.
\]

The Hilbert space \(\mathcal{H}\) of the quantum pure state is a metric space by virtue of the metric induced by the standard scalar product. Denote the poset of all closed balls of \(\mathcal{H}\) by \(\mathcal{C}_H\) with the following partial order:

\[
C(|\phi\rangle, r) \subseteq C(|\psi\rangle, s) \quad \text{iff} \quad C(|\phi\rangle, r) \supseteq C(|\psi\rangle, s).
\]

This relation reflects a natural notion of information: \(C(|\phi\rangle, r) \subseteq C(|\psi\rangle, s)\) can be read as the statement that \((|\phi\rangle, r)\) has less information than \((|\psi\rangle, s)\). The quantum pure state \(|\phi\rangle \in \mathcal{H}\) can be identified with the maximal closed ball \(C(|\phi\rangle, 0) \in \mathcal{C}_H\), i.e. the maximal element of the poset \(\mathcal{C}_H\) is in one-to-one correspondence with \(\mathcal{H}\). The following results from [13] prove that the poset \(\mathcal{C}_H\) has the required structure for the foundation of a computational model.

**Theorem 3** [13] Let \(B\) be a dense subset of a separable Hilbert space \(\mathcal{H}\). Then \(B \times \mathbb{Q}^+\) is a basis of \(\mathcal{C}_H\) where \(\mathbb{Q}^+\) is the set of all non-negative rational numbers.

There are many different choices for a dense subset of \(\mathcal{H}\). Any universal set of quantum gates defines a different dense subset of quantum states of a Hilbert space \(\mathcal{H}\). To see
this fact consider a discrete set of universal quantum gates, \( S \), (e.g. \( H, X, P, T, CNOT \)), therefore any unitary operator on \( H \) can be approximated by combination of elements in \( S \). In other words a universal set of gates is a dense subset of the set of all unitary operators on \( H \). Denote by \(< S >\) the set of all finite combination of elements of \( S \). The following lemma gives a dense subset of \( H \).

**Lemma 1** The image of \(< S >\) on state \( |0\rangle \in H \) is a dense subset of \( H \).

**Theorem 4** [13] The poset of the closed balls of a separable Hilbert space, ordered by reversed inclusion, is an \( \omega \)-continuous domain.

It is easy to see that the way-below relation is nothing but

\[
C(|\phi\rangle, r) \ll (|\psi\rangle, s) \iff C(|\phi\rangle, r) \supset C(|\psi\rangle, s).
\]

The embedding of \( H \) into \( CH \) is defined with the following function:

\[
e_P : H \rightarrow CH
\]

\[
| \phi \rangle \mapsto (| \phi \rangle, 0).
\]

Clearly, the elements of \( CH^+ = \{(|\phi\rangle, 0) | | \phi \rangle \in H \} \) are the maximal elements of \( CH \). Following the definitions of Subsection 3.1 we can introduce a topological structure for \( CH \). It is easy to check that for any given element \((|\phi\rangle, r) \in CH \) we have:

\[
e_P^{-1}(\uparrow(|\phi\rangle, r)) = O(|\phi\rangle, r),
\]

where \( O(|\phi\rangle, r) \) is the open ball with center \(|\psi\rangle\) and radius \( r \). The subsets \( \uparrow(|\phi\rangle, r) \) form a basis of the Scott topology on \( CH \), while the open balls \( O(|\phi\rangle, r) \) are a basis for metric topology on \( H \). Hence, \( e_P \) is a topological embedding, which makes \( H \) homomorphic to the subspace of maximal elements of \( CH \).

The \( \omega \)-continuity of \( CH \) introduces an effective structure along the lines of Subsection 3.1. The homomorphism between \( H \) and maximal elements of \( CH \), derives an effective structure for \( H \) and hence it provides a computational framework for quantum pure states. In a similar way to the Subsection 3.1 we can define a computable pure state as follows.

**Definition 16** An quantum pure state \( |\psi\rangle \) is called computable, if its domain image \( e_P(|\psi\rangle) = (|\psi\rangle, 0) \) is computable in \( CH \), i.e. iff the set \( \{n \in \mathbb{N} | b_n \ll (|\psi\rangle, 0) \} \) is recursively enumerable (where \( \{b_n\} \) are elements of the basis \( B_{CH} \)).

**Mixed quantum states**

The Gleason theorem provides a correspondence between density matrices and probability measures on \( H \) [18].

**Theorem 5** [13] Let \( \mu \) be a probability measure on the closed subspaces of a separable Hilbert space \( H \) of dimension at least three. There exists a positive semi-definite self-adjoint operator \( T \) of the trace class (density matrix) such that for all closed subspaces \( A \) of \( H \)

\[
\mu(A) = \text{Tr}(TP_A),
\]

where \( P_A \) is the orthogonal projection of \( H \) onto \( A \).
Therefore, to present a computational framework for mixed states, it is enough to construct such a framework for probability measure on $\mathcal{H}$. To this end we need the following notations and results [20, 12, 3, 27].

The domain of probability measures will be defined in terms of continuous valuation functions, a finite measure which is defined on open subsets of a topological space [6, 21, 12].

**Definition 17** Assume that $X$ is a topological space. A function $\nu$ from open sets of $X$ to non-negative real number, $\mathbb{R}^+$, is called a continuous valuation function iff the following conditions are satisfied:

- **Strictness.** $\nu(\emptyset) = 0$;
- **Monotonicity.** $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$;
- **Modularity.** $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$;
- **Continuity.** whenever $I$ is a directed subset of open sets (with respect to $\subseteq$), $\nu(\bigcup I) = \sup_{A \in I} \nu(A)$.

A continuous valuation on an $\omega$-continuous domain is a continuous valuation on its Scott topology.

**Definition 18** [24] Assume that $X$ is a topological space. The probabilistic power domain $PX$ of $X$ consists of all continuous valuations $\nu$ on $X$ with $\nu(X) \leq 1$, ordered pointwise, i.e.

$$\mu \sqsubseteq \nu \iff \mu(O) \leq \nu(O) \text{ for all open sets in } X.$$  

Simple valuation functions provide a basis for probabilistic power domain.

**Definition 19** [24] For any point $x \in X$ the point valuation, $\delta_x$, is defined as follows:

$$\delta_x(O) = \begin{cases} 
1 & \text{if } x \in O \\
0 & \text{if } x \notin O 
\end{cases}$$

A finite linear combination of point valuations i.e. $\sum_{i=1}^{n} r_i \delta_{x_i}$, with $x_i \in X$ and positive rational numbers $r_i$, satisfying $\sum_{i=1}^{n} r_i \leq 1$, is called a simple valuation.

**Theorem 6** [24] The probabilistic power domain of an $\omega$-continuous domain is also $\omega$-continuous with a basis of simple valuation.

Now, we can introduce the domain of quantum mixed states. The set of all closed subspaces of $\mathcal{H}$ is the $\sigma$-algebra, $M$, of the measurable sets. Let $M(\mathcal{H})$ denote the set of all probability measures on $\mathcal{H}$. Based on Gleason theorem (Theorem 5), a mixed state can be considered to be an element of $M(\mathcal{H})$. We embed $M(\mathcal{H})$ into the probabilistic power domain $PCH$ of the closed ball domain $CH$, which forms an $\omega$-continuous domain.

The set of maximal elements of $PCH$ contains all valuations $\nu$ such that:

$$\nu(O) = 1 \text{ for all open subsets } O \in CH^+.$$  

The embedding of $M(\mathcal{H})$ into $PCH$ is defined with the following function:

$$e_M : M(\mathcal{H}) \rightarrow PCH \quad \mu \mapsto \mu \circ e_p^{-1}.$$  

The following result from [12] provides the correspondence between $M(\mathcal{H})$ and $PCH^+$:
Theorem 7 \[12] The space $M_H$ is homomorphic with the space of maximal elements of the $\omega$-continuous domain $PC_H$. These maximal elements are characterised by $\nu(CH^+) = 1$.

Every mixed state on $H$ can be obtained via this homomorphism as the least upper bound of an increasing chain of simple valuations on $CH$.

Similar to the case of pure states, we define the computability of a mixed state via the computational framework of $PC_H$.

Definition 20 A quantum mixed state $\rho$ is called computable, if its corresponding measure $\mu_\rho \in M(H)$ is computable i.e. if the domain element $e_M(\mu)$ is computable in $PC_H$, i.e. iff the set $\{n \in \mathbb{N} \mid b_n \ll e_M(\mu)\}$ is recursively enumerable (where $\{b_n\}$ are elements of the basis $B_{PC_H}$).

The process of quantum computation over a pure state is described with a unitary operator and over a mixed state is described with a CP map. As we explained before, in a domain-picture of computation, programs are functions from the domain of the input to the domain of the output. The set of all continuous function forms the domain of operators. This is exactly the same in the case of quantum computation.

### Unitary Operators

We denote by $[CH \rightarrow CH]$, the domain of the all continuous functions on $CH$ with pointwise ordering. Every unitary operator $U : H \rightarrow H$ has a Scott-continuous extension to the domain of $[CH \rightarrow CH]$, i.e. there exists a Scott-continuous function $\tilde{U}$ in $[CH \rightarrow CH]$ such that

$$\tilde{U}(C(|\phi\rangle,0)) = C(U|\phi\rangle,0) \quad \text{for all} \quad |\phi\rangle \in H,$$

and it is explicitly given by

$$\tilde{U}(C(|\phi\rangle,r)) = C(U|\phi\rangle,r).$$

The following lemma shows that the $\tilde{U}$ is well-defined.

Lemma 2 Let $U$ to be a unitary operator on $H$. The extension function $\tilde{U}$ (defined above), maps closed balls in $CH$ to another closed ball.

We define the computability of a unitary function via domain theory with:

Definition 21 A unitary function $U : H \rightarrow H$ is computable iff its extension, $\tilde{U}$, is computable in $[CH \rightarrow CH]$ (in the terms of Subsection 3.1).

### CP Maps

For simplicity, we denote by $T_\mu$ the corresponding operator for a given measure $\mu$ which is derived from Theorem 3

$$\forall \mu \exists T : \mu(A) = \text{Tr}(TP_A) \quad \text{for all closed subspace} \quad A.$$

A CP map $A$ is an operator over $B(H)$ and can also be considered as a function in $M(H) \rightarrow M(H)$ (from Gleason theorem). Denote by $[PC_H \rightarrow PC_H]$ the domain of all continuous functions on $PC_H$ (with pointwise ordering).
Every CP map \( A : \mathbf{M}(\mathcal{H}) \to \mathbf{M}(\mathcal{H}) \) has a Scott-continuous extension to the domain of \([\mathbf{PCH} \to \mathbf{PCH}]\), i.e. there exists a Scott-continuous function \( \tilde{A} \in [\mathbf{PCH} \to \mathbf{PCH}] \) such that for every probability measure \( \mu \in \mathbf{M}(\mathcal{H}) \) we have:

\[
\tilde{A}(\mu) = \Lambda(\mu).
\]

The extension function \( \tilde{A} \) for a continuous valuation function \( \nu \in \mathbf{PCH} \) is explicitly given by:

\[
\tilde{A}(\nu)(B) = \operatorname{Tr}(T_{A(\nu)} \rho_B) \quad \text{where} \ B \ \text{is a closed subspace of} \ \mathcal{H}.
\]

The following lemma shows that the above definition is well-defined:

**Lemma 3** Any CP map \( A \) on \( \mathbf{M}(\mathcal{H}) \), maps continuous valuation functions to another continuous valuation function.

We define the computability of a CP map function via domain theory with:

**Definition 22** A CP map \( A : \mathbf{M}(\mathcal{H}) \to \mathbf{M}(\mathcal{H}) \) is computable iff its extension, \( \tilde{A} \), is computable in \([\mathbf{PCH} \to \mathbf{PCH}]\).

**Quantum Measurements**

At the end of computation a measurement operator will be applied. A measurement can be viewed as a CP map which takes a density matrix (the final state) to another density matrix (the probabilistic mixture of the outcomes).

Assume \( M_m \) is a collection of measurement operators. The corresponding measurement of this collection can be considered as a CP map over \( \mathbf{B}(\mathcal{H}) : \mathbf{M} : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H}) \rho \mapsto M_m \rho M_m^\dagger \operatorname{Tr}(M_m^\dagger M_m \rho) \).

Hence, the extension function and computability can be also defined exactly in the same way that we defined before for a given CP map.

**4.2 General Setting**

In this subsection we present a denotational semantics for quantum computation using domain theory, which could be considered as a foundation for designing a functional programming language for quantum computation. The recent literature contains several proposals for quantum programming languages. The first contribution in this direction is the Knill’s paper on QRAM model [15]. The other attempts to define a true quantum programming language are two imperative languages. The first approach by Ömer [31, 30] has a C-like syntax, while a second proposal by Sanders and Zuliani [36] is based on Dijkstra’s guarded-command language. A similar approach to the work of this subsection has been developed independently by Selinger [39]. He has presented the first functional programming language and discussed the denotational semantics of his proposed language. Our work is based on the Kozen’s semantics for probabilistic computation [26].

We aim to develop a denotational semantics for a basic programming language, called Quantum WHILE. In this approach, we show how to define the mathematical object corresponding to the language constructors. We will consider a simple quantum computational
machine with quantum memory registers. To develop the proper foundation for quantum semantics, in the most general setting, we consider density matrices and CP maps. Aharonov, Kitaev and Nisan in [2] introduced the first computational model based on mixed state where possible operators are represented by CP maps. We show in this subsection that the same structure of the classical probabilistic semantics which has been introduced by Kozen [26] can also capture the semantics of quantum computation.

To follow the procedure introduced by Kozen [26], we define a measurable space $(X^n, M^n)$ with the set of all measures $\mathcal{B} = \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n, M^n)$ such that the set of all probability measures in this space is in correspondence with the set of all density matrices over $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. In this way, input to a quantum program $P$ is represented by a probability measure $\rho \in \mathcal{B}$ which is the same as the corresponding density matrix of all input pure states $|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$.

Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ denote the Hilbert space spanned by all the quantum variables which are involved in the computation. Define $X_i$ to be the set of all unit vectors in $\mathcal{H}_i$ and $X$ to be the set of all unit vectors in $\mathcal{H}$. The set of all closed subspaces of $\mathcal{H}$ is the $\sigma$-algebra, $\mathcal{M}$, of the measurable sets. Gleason theorem determines all measures on $\mathcal{M}$ and shows a correspondences between operators in $\mathcal{B}(\mathcal{H})$ and measures on $\mathcal{M}^\mathcal{H}$, we use interchangeably any of the two notions of measure and operator. In the same way as the classical case, the set of positive measures (positive self-adjoint operators) $\mathcal{P} \subset \mathcal{B}$ is the positive cone of the measure space $\mathcal{B}$. The definition of ordering of measures is defined as follows

$$\mu \sqsubset \nu \text{ iff } \nu - \mu \in \mathcal{P}.$$ 

In semantics of the general setting for quantum computation, each program will define a CP map. For simplicity, in what follows the symbol $P$ refers to both a program and the corresponding CP map. A quantum program $P$ maps distributions $\mu$ on $(X, \mathcal{M})$ to distribution $P(\mu)$ on $(X, \mathcal{M})$, or equivalently, maps a density matrix $\mu$ on $\mathcal{H}$ to the density matrix $P(\mu)$.

For the completeness of the discussion we will give the full semantics of the quantum WHILE language in the general setting. The syntax of this Language is the same as the syntax of the classical probabilistic WHILE language (Subsection 3.2). The only difference is that instead of “random assignment” we have “quantum measurement”.

- **simple assignment**: If $P$ is the program “$x_i := f(x_1, \cdots, x_n)$” where $f : X \rightarrow X_i$ is a measurable function, then the meaning of $P$ is the following CP map:

$$\begin{align*}
\mu & \mapsto P(\mu) \\
P(\mu) &= \mu \circ F^{-1},
\end{align*}$$

where $F : X \rightarrow X_i$ is the measurable function

$$F(a_1, \cdots, a_n) = (a_1, \cdots, a_i, f(a_1, \cdots, a_n), a_{i+1}, \cdots a_n).$$

- **measurement assignment**: If $P$ is the program “$x_i := \text{measure}$” then the meaning of $P$ is the following CP map:

$$\begin{align*}
\mu & \mapsto P(\mu) \\
P(\mu)(B_1 \otimes \cdots \otimes B_n) &= \mu(B_1 \otimes \cdots \otimes B_{i-1} \otimes X_i \otimes B_{i+1} \otimes \cdots \otimes B_n) \\
T(\rho)(B_i),
\end{align*}$$

15
where $T_\rho$ is a fixed distribution corresponding to the measurement process. To be more precise, assume that the collection $\{M_m\}$ describes the quantum measurement that has been applied, then

$$\rho = \frac{M_m \mu M_m^\dagger}{\text{tr}(M_m^\dagger M_m \mu)}$$

and $T_\rho$ is the corresponding measure obtained from Gleason theorem.

- **composition**: The meaning of the program “$S; T$” is the functional composition of the CP maps $T$ and $S$, $T \circ S$.
- **conditional**: The semantics of the program “if $B$ then $S$ else $T$” is the CP map

$$S \circ e_B + T \circ e_{\sim B},$$

where $e_B$ is the CP map $e_B(\mu) = \mu_B$.
- **while loop**: The meaning of the program “while $B$ do $S$” is the fixed-point of the affine transformation $\tau : B' \rightarrow B'$ defined by

$$\tau(W) = e_{\sim B} + W \circ S \circ e_B,$$

which is equal to

$$\tau^n(0) = \sum_{0 \leq k \leq n-1} e_{\sim B} \circ (S \circ e_B)^k.$$

### 4.3 Information Theory

Recently a new application of domain theory has been introduced by Coecke and Martin [9]. One of the main results of their work is to show a domain formulation of the existing results from information theory. They have shown that Shannon entropy and Von Neumann entropy can be captured as Scott continuous functions over the corresponding domain. Here we briefly review their work in order to give a complete picture of quantum domain theory. All the definitions and results in this subsection is taken from [9].

Coecke and Martin have constructed a domain structure over mixed states such that pure states are the maximal elements. They first order classical states recursively in terms of Bayesian order.

**Definition 23** Let $n \geq 2$. The classical states are

$$\Delta^n = \left\{ x \in [0, 1]^n | \sum_{i=1}^n x_i = 1 \right\}$$

A classical state $x \in \Delta^n$ is pure, when $x_i = 1$ for some $i \in 1, \ldots, n$. The set of all pure states is denoted by $\{e_i | i = 1, \ldots, n\}$.

A classical state in $x \in \Delta^n$ can be interpreted as the information that an observer has about the results of an event in which $n$ different outcomes is possible, i.e. $x_i$ indicates the probability of obtaining the outcome $i$. If we know $x$ and after measuring we determine that outcome $i$ is not possible, our knowledge improves to

$$p_i(x) = \frac{1}{1 - x_i} (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \Delta^{n-1},$$
where \( p_i(x) \) is obtained first by removing \( x_i \) from \( x \) and then re-normalising. The partial functions \( p_i \):

\[
p_i : \Delta^n \to \Delta^{n-1},
\]

with \( \text{Dom}(p_i) = \Delta^n \setminus e_i \), are called the Bayesian projections. The classical states are partially ordered with the following recursive relation.

**Definition 24** Assume that \( x \) and \( y \) are in \( \Delta^n \we write \( x \sqsubseteq_B y \) iff:

\[
\forall i : \ x, y \in \text{Dom}(p_i) \Rightarrow p_i(x) \sqsubseteq_B p_i(y).
\]

For \( x, y \in \Delta^2 \) we have:

\[
x \sqsubseteq_B y \text{ iff } (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1).
\]

The above relation is called Bayesian order.

The Bayesian order leads to a domain of classical states where the pure states are the maximal elements.

**Theorem 8** \([9]\) \((\Delta^n, \sqsubseteq_B)\) is a domain with the following set of maximal elements:

\[
\{e_i \mid 1 \leq i \leq n\},
\]

and least element \( \perp = (1/n, \cdots, 1/n) \).

Coecke and Martin have generalised the idea of Bayesian order to the quantum setting using the spectral order. Informally speaking, to compare the amount of information of two given mixed states it is enough to consider an observable and measure both mixed states. The result of measurements are two classical states and can be ordered via Bayesian order.

Following the notation of \([9]\), we denote by \( \Omega^n \) the set of all density matrices on \( \mathcal{H}^n \). For simplicity we also consider the following definition.

**Definition 25** Assume that \( O \) is a non-degenerate observable on \( \mathcal{H}^n \) i.e. it has \( n \) different eigenvalues with orthogonal eigenvector spaces \( \{P_i\}_{i=1}^n \). For a density matrix \( \rho \) on \( \mathcal{H}^n \) we define:

\[
\text{Spec}(\rho|O) = (\text{Tr}(P_1 \cdot \rho), \cdots, \text{Tr}(P_n \cdot \rho)) \in \Delta^n.
\]

**Definition 26** Let \( n \geq 2 \), for quantum states \( \rho, \sigma \in \Omega^n \), we have \( \rho \sqsubseteq_S \sigma \) iff there exists a non-degenerate observable \( O : \mathcal{H}^n \to \mathcal{H}^n \) such that \( [\rho, O] = [\sigma, O] = 0 \) and

\[
\text{Spec}(\rho|O) \sqsubseteq_B \text{Spec}(\sigma|O).
\]

This is called the spectral order.

Finally the domain of the quantum states can be defined with:

**Theorem 9** \([9]\) \((\Omega^n, \sqsubseteq_S)\) is a domain with the following set of pure states as the maximal elements and least element \( \perp = I/n \), where \( I \) is the identity matrix.

The final part of the Coecke and Martin’s work that we review here is concerned with measuring of information content \([9]\).
Definition 27 A Scott continuous map $\mu$ from a given domain $D$ to the domain of $^3 \mathbb{R}^*$ measure the content of $x \in D$ if

$$x \in U \Rightarrow (\exists \epsilon > 0) \ x \in \mu_\epsilon (x) \subseteq U,$$

whenever $U$ is Scott open in $D$ and

$$\mu_\epsilon (x) = \{ y \in D | y \sqsubseteq x \land |\mu(x) - \mu(y)| < \epsilon \}.$$

The map $\mu$ measures $X$ if it measures the content of each $x \in X$.

A map $\mu$ is a measure of content if it distinguishes the maximal (in content) elements.

Definition 28 Assume $D$ is a domain, a measurement is a Scott continuous map $\mu : D \rightarrow [0, \infty)^*$ that measures the set $\{ x \in D | \mu(x) = 0 \}$.

The following results from [9] present the domain picture of the well known functions, the Shannon entropy and the von Neumann entropy.

Theorem 10 [9] Shannon entropy

$$\mu(x) = - \sum_{i=1}^n x_i \log(x_i),$$

is a measurement of type $\Delta^n \rightarrow [0, \infty)^*$.

von Neumann entropy

$$\sigma(\rho) = -\text{Tr}(\rho \log\rho),$$

is a measurement of type $\Omega^n \rightarrow [0, \infty)^*$.

5 Discussions

In this paper, we have discussed a new framework for quantum computation via quantum domain theory. Using domain theory a rigours framework for quantum computability has been also introduced. Although it is known that the class of computable functions with a quantum model is equal to the class of classical computable functions (Church-Turing Principle [10, 25, 32]), however we believe that by considering different frameworks for quantum computability, interesting questions can be addressed. Quantum domain theory also provides a topological structure for quantum computation that could be useful for the study in other fields of quantum computation.

Furthermore we introduced a denotational semantics for quantum computation and we showed that quantum computation over density matrices with completely positive maps, has a similar semantical structure as probabilistic computation over random variables. This could be considered as a foundation for designing a functional programming language for quantum computation.

Finally we briefly reviewed a domain structure for quantum information theory, introduced by Coecke and Martin [9]. The partial order introduced in their work has interesting connections with theory of entanglement as they have discussed in their paper. Therefore, a domain theoretical approach to the theory of entanglement manipulation may provide us with a uniform framework for measuring the entanglement in the same line as [42].

$^3$The set $[0, \infty)^*$ is the domain of nonnegative real numbers in their opposite order.
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References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3. Clarendon Press, 1994.

[2] D. Aharonov, A. Kitaev, and N. Nisan. Quantum circuits with mixed states. In *Proceedings STOCK’98 – Symposium on Theory of Computing*, Dallas, TX USA, 1998. ACM.

[3] M. Alvarez-Manilla. *Measure Theoretic Results for Continuous Valuations on Partially Ordered Spaces*. PhD thesis, University of London, Imperial College, 2000.

[4] C. Bennett, E. Bernstein, G. Brassard, and U. Vazirani. Strengths and weaknesses of quantum computing. *SIAM Journal on Computing*, 26:1510, 1997.

[5] E. Bernstein and U. Vazirani. Quantum complexity theory. *SIAM Journal of Computing*, 5(26):1411, 1997.

[6] G. Birkhoff. *Lattice Theory*, volume 25 of R. I. Amer. Math. Soc. Colloquium, 1967.

[7] J. Blanck. Domain representability of metric spaces. *Annuals of Pure and Applied Logic*, 43, 1997.

[8] S. Bettelliand T. Calarco and L. Serafini. Toward an architecture for quantum programming. arXive:cs.PL/0103009, 2001.

[9] B. Coecke and K. Martin. A partial order on classical and quantum states. Programming Research Group Research Report RR-02-07, 2002.

[10] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. In *Proceedings of the Royal Society of London*, volume A400, page 97, 1985.

[11] D. Deutsch. Quantum computational networks. In *Proc. Roy. Soc. Lond A*, volume 425, page 467, 1989.

[12] A. Edalat. Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3(4), 1997.

[13] A. Edalat and R. Heckmann. A computationa model for metric spaces. *Theoretical Computer Science*, 1996.

[14] A. Edalat and P. Sündenhauf. A domain theoretic approach to computability on the real line. *Theoretical Computer Science*, 1997.

[15] E.H.Knill. Conventions for quantum pseudocede. LANL report LAUR-96-2724, 1996.

[16] Y.L. Ershov. Computable functions of finite types. *Algebra and Logic*, 11(4), 1972.

[17] P. Di Gianantonio. Real number computability and domain theory. *Information and Computation*, 127(1), 1996.

[18] A.M. Gleason. Measures of closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885, 1957. reprint in 23.
[19] L.K. Grover. A fast quantum mechanical algorithm for database search. In Proceedings of STOC’96 – Symposium on the Theory of Computing, page 212, Philadelphia, Pennsylvania, 1996. ACM.
[20] R. Heckmann. Probabilistic domains. CAAP ’94, 1994.
[21] R. Heckmann. Space of valuations. In Papers on General Topology and its Applications, volume 806. Annals of the New York Academy of Science, 1996.
[22] R. Heckmann. Approximation of metric spaces by partial metric spaces. Applied Categorical Structures, 7, 1999.
[23] C.A. Hooker, editor. The Logico-Algebraic Approach to Quantum Mechanics, volume I – Historical Evolution. Reidel, Dordrecht – Boston, 1975.
[24] C. Jones and G. Plotkin. A probabilistic powerdomain of evaluations. Logic in Computer Science, 1989.
[25] R. Jozsa. Characterising classes of functions computable by quantum parallelism. In Proceedings of the Royal Society of London, volume A435, 1991.
[26] D. Kozen. Semantics of probabilistic programs. Computer and System Sciences, 22(3), 1981.
[27] K. Martin. Powerdomains and zero finding. QAPL, 59(3), 2001.
[28] M. A. Nielsen. Computable functions, quantum measurements, and quantum dynamics. Phys.Rev.Lett., 79, 1997.
[29] M.A. Nielsen and I.L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, Cambridge, 2000.
[30] B. Ömer. Quantum Programming in QCL. PhD thesis, Institute of Information Systems – Technical University Vienna, Vienna, 2000.
[31] B. ÓOmer. A procedural formalism for quantum computing, 1998.
[32] M. Ozawa. Measurability and computability. quant-ph/9809048, 1998.
[33] C.H. Papadimitriou. Computational Complexity. Addision-Wesley, 1994.
[34] R. Raussendorf and H. Briegel. Computational model underlying the one-way quantum computer. QIC, 2(6), 2002.
[35] R. Raussendorf, D.E. Browne, and H.-J. Briegel. The one-way quantum computer - a non-network model of quantum computation. Journal of Modern Optics, 49, 2002.
[36] J.W. Sanders and P. Zuliani. Quantum programming. In R. Backhouse and J.S. Nuno Oliveira, editors, Proceedings of MPC’00 – Mathematics of Program Construction, volume 1837 of Lecture Notes in Computer Science. Springer-Verlag, 2000.
[37] D. Scott and C. Strachey. Towards a mathematical semantics for computer languages. Tech. mono. PRC6, 1971.
[38] D. S. Scott. Outline of a mathematical theory of computation. In 4th Annual Princeton Conference on Information Science and Systems, 1970.
[39] P. Selinger. Towards a quantum programming language. submitted, 2003.
[40] P.W. Shor. Algorithms for quantum computation: Discrete logarithms and factoring. In Proceedings of FOCS’94 – Symposium on Foundations of Computer Science, page 124, Santa Fe, New Mexico, 1994. IEEE Press.
[41] M. B. Smyth. Effectively given domains. Theoretical Computer Science, 5, 1977.
[42] V. Vedral and E. Kashefi. Uniqueness of entanglement measure and thermodynamics. *Phys. Rev. Lett.*, 89, 2002.

[43] K. Weihrauch. *Computable Analysis*. Springer, 2000.

[44] K. Weihrauch and U. Schreiber. Embedding metric spaces into cpo’s. *Theoretical Computer Science*, 16, 1981.

[45] A. C. C. Yao. Quantum circuit complexity. In *Proceedings of FOCS’93 – Symposium on Foundations of Computer Science*. IEEE Press, 1993.