String solutions with non-constant scalar fields

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Abstract

We discuss charged string solutions of the effective equations of the $D = 4$ heterotic string theory with non-constant dilaton $\phi$ and modulus $\varphi$ fields. The effective action contains the generic moduli-dependent coupling function in the gauge field kinetic term and non-perturbative scalar potential.

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1. Introduction

Scalar fields play a very important role in string theory. They are related to the two basic infra-red problems: why there is a correspondence with the Einstein theory (why there is no extra scalar mode of gravity in spite of the existence of the massless universally coupled dilaton in the free string spectrum) and why the cosmological constant should be zero. Both the dilaton and moduli (string modes associated with compact extra dimensions) are natural partners of the metric and thus should play an important role in string gravitational physics.

Scalar fields are known to couple to the kinetic terms of the gauge fields. In particular, in string theory, the dilaton field determines the strength of the gauge couplings at the tree level of the effective action, while string one-loop (genus-one) contributions give moduli dependent corrections to such couplings. Thus, in general, a scalar function $f$ that couples to the gauge field kinetic energy is a function of both the dilaton as well as the moduli.

The dilaton and the moduli have no potential in the effective action to all orders in string loops. To avoid a contradiction with observations they should acquire masses. Currently proposed scenarios rely on non-perturbatively induced potential $V$ due to gaugino condensation in the hidden gauge group sector. Such a potential would generate masses for the dilaton and the moduli and at the same time provide a mechanism of supersymmetry breaking. While these scalar fields eventually get masses, they may change with distance (or time, in the cosmological context) at small scales (or times), i.e. may participate in the dynamics at small scales (or times), or in the region where non-perturbative effects presumably can be neglected. String solutions are usually discussed in perturbation theory in $\alpha'$. For example, new charged black hole string solutions [1][2] have been recently obtained by taking into account the tree level coupling of the dilaton to the gauge fields (for a review see [3]). Below we shall consider (following [4]) charged string solutions taking into account perturbative (genus-one, moduli dependent threshold corrections to the gauge couplings), and non-perturbative string effects (non-perturbatively induced potential for the dilaton and moduli). Such corrections may substantially modify the tree level solutions and it is thus important to include them in order to understand predictions of string theory. We

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1 Aspects of charged dilatonic black hole solutions with non-perturbatively induced dilaton mass included were addressed in [3][4]. However, the potentials for the dilaton were not always taken to be ‘realistic’ or well motivated from the point of view of non-perturbative dynamics like gaugino condensation in the hidden gauge sector of the gauge group.
shall consider abelian electric and magnetic solutions in flat space concentrating on the role of the non-trivial functions $f$, a coupling function of scalars to the gauge field kinetic term, and $V$, a non-perturbative potential for the dilaton and the moduli.

The solutions we shall describe should have generalisations to the curved space. We shall assume that they approximate the exact solutions of the whole set of equations (including the gravitational one) in the region where the curvature is small. As in the case of solitonic solutions in field theory in flat space one can ignore gravitational effects if the scale of the solutions is large compared to the gravitational scale $\sim E/M^2_P$ (i.e. if the energy of the solutions is small enough). It is true that in the absense of a non-perturbative potential the dilaton and the metric are on an equal footing. Once the potential is generated, it introduces a new scale (different from the Planck one). This makes it possible in principle to ‘disentangle’ the metric from the dilaton and to consider dilatonic solutions in flat space with a characteristic scale being larger than the gravitational one.

We shall include the dilaton $\phi$ and only one modulus field $\varphi$ (which is associated with an overall compactification scale) and look for stable spherically symmetric finite-energy solutions with a regular gauge field strength in flat $D = 4$ space-time. We shall consider a general class of functions $f$ and $V$. We shall treat examples with $f$ modified by the string loop corrections, as they appear in a class of orbifold-type compactifications, and with the non-perturbative potential $V$ due to the gaugino condensation in the hidden sector of the theory, as special cases. We shall ignore higher derivative terms, assuming that the fields change slowly in space.

It turns out that the abelian electric solutions are regular, have finite energy, and are stable when the abelian subgroup is embedded in a non-abelian gauge group. They have the effective string coupling $e^{\phi}$ increasing from zero at the origin ($r = 0$) to a finite value $e^{\phi_0}$ at $r = \infty$. The asymptotic value $\phi_0$ of the dilaton corresponds to the minimum of the potential $V$. Thus the small distance region is a weak coupling region and can be studied ignoring non-perturbative corrections. The large distance region corresponds to the ‘observed’ world where the dilaton is trapped in the minimum of $V$. The modulus field $\varphi$ is slowly varying with $r$; at large scales it is fixed at the minimum of the potential $V$, while at small scales its value decreases slightly. Generic existence of such ‘non-topological’ solitonic charged configurations may have potentially interesting applications. As for abelian magnetic solutions, here at small distances ($r \to 0$) the dilaton approaches the strong coupling region, $e^{\phi} \to \infty$, while the modulus goes to zero, $\varphi \to 0$, i.e., the small scale region is the compactification region. The role of the non-perturbative potential is again to fix the asymptotic values of $\phi_0$ and $\varphi_0$ to be at its minimum.
2. Low-energy string effective equations

The low-energy $D = 4$ effective action of the heterotic string theory has the following structure

$$ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - 2\partial_\mu \phi \partial^\mu \phi - 2\partial_\mu \varphi \partial^\mu \varphi - f(\phi, \varphi)F_{\mu\nu}F^{\mu\nu} - 4V(\phi, \varphi) + \ldots \right]. $$

(2.1)

For simplicity we consider, along with the dilaton field $\phi$, only one modulus field $\varphi$, associated with an overall scale of compactification and ignore the axionic partners of $\phi$ and $\varphi$ as well as other matter fields. At the tree level

$$ f_{\text{tree}} = e^{-2\phi} , \quad V_{\text{tree}} = e^{2\phi} = 0. $$

(2.2)

In the case of the supersymmetric $D = 4$ heterotic string $V$ remains zero to all orders in the string perturbation theory. On the other hand, the gauge coupling function $f$ receives a non-trivial, $\varphi$-dependent, string one-loop (genus-one) correction [7]

$$ f_{\text{perturb}} = e^{-2\phi} + f_2(\varphi) , \quad V_{\text{perturb}} = 0. $$

(2.3)

The modulus dependent function $f_2$ depends on a type of superstring vacuum one is considering. In particular, for toroidal compactifications and a class of orbifolds, it is invariant under the duality symmetry ($\varphi \to -\varphi$) and can be schematically written in the form:

$$ f_2(\varphi) = b_0 \ln \left[ (T + T^*) |\eta(T)|^4 \right] + b_1 , \quad T = e^{2\varphi/\sqrt{3}}. $$

(2.4)

It turns out that $\ln \left[ (T + T^*) |\eta(T)|^4 \right]$ is always negative, has a maximum at $\varphi = 0 (T = 1)$ and approaches $-\frac{\pi}{3} T = -\frac{\pi}{3} e^{2\varphi/\sqrt{3}}$ as $\varphi \to \infty$. The constant $b_0$ is related to the one-loop $\beta$-function coefficients associated with the $N = 2$ subsector of the massless spectrum in a symmetric orbifold compactification. Generically $b_0 = O(1/100)$ and is negative (positive) in the case of the abelian (non-abelian) gauge group factors.

Since the string coupling is related to the dilaton, both $f$ and $V$ may contain non-perturbative contributions which are non-trivial functions of $\phi$. Such terms in $f$ do not actually appear in the proposed gaugino condensation scenario for supersymmetry breaking. In general, however, one should allow for a possibility that $f$ may contain additional, non-perturbative terms which depend on $\phi$, e.g., $\exp[-k \exp(-2\phi)]$ (implying $1/g^2 \to 1/g^2 + a e^{-k/g^2}$). A detailed structure of a non-perturbative potential $V(\phi, \varphi)$ depends on a particular mechanism of supersymmetry breaking. We shall describe the form of $V$ due to the gaugino condensation in the hidden sector of the gauge group [8][9][10][11][12].
In the case of symmetric orbifolds with the compactification moduli of all three two-tori equal to $T$ the one finds \cite{10} \cite{11} \cite{12}

$$V_{\text{non-perturb.}} \equiv V(S, T) = \frac{|H|^2}{|\eta(T)|^{12} S_R T_R^3} \left[ S_R \frac{\partial \ln H}{\partial S} - 1 \right]^2 + \frac{3}{4\pi^2} T_R^2 \hat{G}_2(T)^2 - 3 \right],$$

where

$$H(S) = \sum_{i=1}^J d_i e^{-a_i S}, \quad a_i = \frac{3}{2b_{0i}}, \quad (2.6)$$

$J$ is a number of gaugino condensates and $b_{0i}$ are the (one-loop $N = 1$) $\beta$-functions of the gauge group factors of the hidden gauge group sector. Also, $S_R = 2\text{Re} S$, $T_R = 2\text{Re} T$, and

$$\hat{G}_2(T) = G_2(T) - 2\pi T_R^{-1} = -4\pi \frac{1}{\eta(T)} \frac{\partial \eta(T)}{\partial T} - 2\pi T_R^{-1}.$$

This potential vanishes in the weak coupling limit $S = e^{-2\phi} \to \infty$ and has an extremum in $S$ if $S_R \frac{\partial V}{\partial S} - W = 0$. A minimum exists if $J > 1$, i.e., in cases with more than one gaugino condensate. As for the extrema in $T$, $\partial V/\partial T = 0$, they are achieved at the self-dual points $T = 1$ and $T = e^{i\pi/6}$, which are saddle points of $V$, and at $T \sim 1.2$, which is the minimum of $V$. At a fixed, extremal value of $T$ and a fixed $\text{Im} S$ the potential $V$ can be represented in the following form

$$V = S_R^{-1} \sum_{i=1}^J e^{-a_i S_R} (c_i + d_i S_R + e_i S_R^2), \quad (2.7)$$

where $a_i$, $c_i$, $d_i$, and $e_i$ are constants and $S_R = 2e^{-2\phi}$. For example, in the case of two gaugino condensates, $J = 2$, this potential starts from zero in the weak coupling region $\phi \to -\infty$, grows and reaches a local maximum, then decreases to a local minimum (with negative value of $V$), then has the second local maximum and finally goes to $-\infty$ at $\phi \to +\infty$. Since the potential has a local minimum in $\phi$, it may fix the value of the dilaton\footnote{This would give a mass to the fluctuating part of the dilaton. A generic property of this potential is that starting from a weak coupling region it first increases and has a local maximum and only then decreases to a minimum, namely, the potential is not convex everywhere. Another problem is that the value of the potential at the minimum, i.e. the effective cosmological constant, is negative in general. A local minimum with a zero cosmological constant can be achieved in the case with more than two gaugino condensates. Note, however, that in this case there are usually also other minima with negative cosmological constants (see e.g., \cite{13}).}.
We shall discuss charged solutions with non-trivial dilaton $\phi$ and the modulus field $\varphi$ in flat four dimensional ($D = 4$) space-time, i.e., $ds^2 = -dt^2 + dr^2 + r^2d\Omega^2$. The relevant part of the action (2.1) can be put in the form

$$S = \frac{1}{4\pi} \int d^4x \left\{ -\frac{1}{2} \left( \partial \Phi_i \right)^2 - \frac{1}{4} f(\Phi_i) F_{\mu\nu} F^{\mu\nu} - V(\Phi_i) \right\} ,$$

so that the corresponding field equations are

$$D_\mu (f F^{\mu\nu}) = 0 , \quad D_\mu F^{*\mu\nu} = 0 ,$$

$$D^2 \Phi_i - \frac{1}{4} \partial_i f F_{\mu\nu} F^{\mu\nu} - \partial_i V = 0 , \quad i = 1, 2 ,$$

where $\partial_i = \partial / \partial \Phi_i$ and $\Phi_i = (\phi, \varphi)$. It is easy to see that this system transforms into the same one under $f \to f^{-1}$, $F_{\mu\nu} \to f F_{\mu\nu}^*$, $\Phi_i \to \Phi_i$, so that electric solutions for the action (2.8) with the gauge coupling function $f(\Phi_i)$ are related to the magnetic solutions of the theory with the coupling $f^{-1}(\Phi_i)$.

Let us consider first the electric solution:

$$F_{01} = E(r) , \quad F_{0k} = F_{lk} = F_{kl} = 0 \ (k, l = 2, 3) , \quad \Phi_i = \Phi_i(r) .$$

Then

$$\frac{d}{dr} (r^2 f E) = 0 , \quad E = \frac{q}{r^2 f(\Phi_i(r))} ,$$

so that (2.10) becomes ($i = 1, 2$)

$$\Phi''_i + \partial_i U - \frac{1}{x^4} \partial_i V = 0 , \quad U \equiv -\frac{q^2}{2f} .$$

We have introduced the new coordinate $x = 1/r$ with the range $0 \leq x \leq \infty$, and $\Phi'_i \equiv d\Phi_i/dx$. Equation (2.13) can be interpreted as corresponding to a mechanical system with the action which, at the same time, gives the energy of the field configuration as derived from (2.8):

$$E = \mathcal{S} = \int_0^\infty dr \ r^2 \left[ \frac{1}{2} \left( \frac{d}{dr} \Phi_i \right)^2 + \frac{q^2}{2r^4 f} + V \right] = \int_0^\infty dx \left( \frac{1}{2} \Phi_i'^2 - U + \frac{1}{4x^4} V \right) .$$

The case with $V = 0$ corresponds to the mechanical system with the conservative potential $U$, while the case with $V \neq 0$ corresponds to the mechanical system with non-conservative (time-dependent) potential

$$U(\Phi_i, x) \equiv U - \frac{1}{x^4} V .$$
One can show that a sufficient condition of ‘linearized’ stability of the abelian electric solutions is

\[ V_{ij} \geq 0 , \quad V_{ij} \equiv -\partial_i \partial_j U + r^4 \partial_i \partial_j V = -\partial_i \partial_j U . \]  

(2.16)

This condition has an obvious interpretation that perturbations should not decrease the energy (2.14) of the system.

The set of equations, the energy and the condition of stability for the magnetic solution

\[ F_{23} = h \sin \theta , \quad h = \text{const.} , \quad F_{01} = F_{0k} = F_{1k} = 0 , \quad k = 2, 3 , \quad \Phi = \Phi_i (r) , \]  

(2.17)

are found by replacing \( f \) by \( f^{-1} \) and \( q \) by \( h \) in the above equations or by using

\[ U \equiv -\frac{1}{2} h^2 f . \]  

(2.18)

3. Solutions in the case of zero scalar potential

The properties of the solutions are different depending on whether \( V \) is zero or not: in the former case the system is conservative, in the latter it is not. Here we shall consider the case when the effect of the potential can be ignored, i.e., when non-perturbative corrections are small. In addition, we shall start with the case when only one scalar field \( \Phi \) is changing with \( r \). Such a reduction to only one field could be possible if \( f \) had extrema with respect to the other field and thus the other field could be ‘frozen’, i.e., put to a fixed \( r \)-independent value. \(^3\)

If \( V = 0 \) eq. (3.6) has one integral of motion, i.e., the ‘energy’ of the corresponding mechanical system:

\[ \frac{1}{2} \Phi^2 + U(\Phi) = C = \text{const} , \]  

(3.1)

where \( U = -\frac{1}{2} q^2 f^{-1} \) and \( U = -\frac{1}{2} h^2 f \) for the electric and the magnetic solutions, respectively. We shall assume that \( f \) is always positive (since it is the coupling function in the gauge field kinetic term) and thus, \( U \) is always negative. In looking for minima of \( \mathcal{E} \) it is useful to represent (2.14) in the form

\[ \mathcal{E} = \int_0^\infty dx \frac{1}{2} \left( \Phi' \pm \sqrt{2C - 2U(\Phi)} \right)^2 \mp \int_{\Phi_0}^{\Phi_\infty} d\Phi \sqrt{2C - 2U(\Phi)} - \int_0^\infty dx C , \]  

(3.2)

\(^3\) In principle, \( \Phi \) may be either the dilaton or the modulus, but it does not seem to be possible to ‘freeze out’ the \( r \) dependence of the dilaton in general.
where $\Phi_\infty \equiv \Phi(x = \infty)$ and $\Phi_0 \equiv \Phi(x = 0), \ x \equiv 1/r$. We shall consider solutions with a regular field value, i.e., $\Phi_0 \neq \infty$ at $x = 0$. However, in the core of the solution, i.e., at $x \to \infty$ the field may or may not blow up.

The energy is minimized if (cf. Bogomol’nyi conditions)

$$\Phi' = \mp \sqrt{2C - 2U(\Phi)} .$$

(3.3)

The constant $C$ is chosen such that the second and the third term in (3.2) give a finite value of the energy. Eq. (3.3) implies the explicit form of solution

$$\int_{\Phi_0}^{\Phi(x)} \frac{1}{\sqrt{2C - 2U(\Phi)}} \, d\Phi = \mp x .$$

(3.4)

The upper (lower) sign solutions correspond to $\Phi$ decreasing (increasing) with increasing $x = 1/r$. Then the energy $\mathcal{E}$, the charge $Q$ and the scalar charge $D$ of the electric solution are

$$\mathcal{E} = \mp \int_{\Phi_0}^{\Phi(x)} d\Phi \sqrt{2C - 2U(\Phi)} - \int_{0}^{\infty} dx C ,$$

$$Q = [r^2 E(r)]_{r \to \infty} = \frac{q}{f(\Phi_0)} ; \quad D = -[r^2 \frac{d\Phi}{dr}]_{r \to \infty} = \mp \sqrt{2C + \frac{q^2}{f(\Phi_0)}} .$$

(3.5)

The magnetic solution is found from the electric solution by the replacements $f \to f^{-1}$ and $q \to h$, or in other words, by taking $U = -\frac{1}{2} h^2 f$.

In order to have a finite energy, regular solution the potential $U(\Phi) = -\frac{1}{2} q^2 f^{-1}(\Phi)$ (for the electric solution) and $U(\Phi) = -\frac{1}{2} h^2 f(\Phi)$ (for the magnetic solution) should satisfy certain conditions. The nature of the solution is different in the cases with $C = 0$ or $C \neq 0$. One can show that the regular, finite energy solutions with the upper sign, exist only for the choice of $C = 0$. Such solutions have the property $\Phi_\infty \neq \infty$ and as $\Phi \to \Phi_\infty$, $U(\Phi)$ approaches zero faster than $(\Phi - \Phi_\infty)^2$. Solutions of this type correspond to the case of ‘dilatonic’-type electric solutions and a class of ‘moduli’-type magnetic solutions. Solutions with the lower sign exist for $C = 0$ or $C \neq 0$, depending on the nature of $U$. If $C = 0$ one obtains $\Phi_\infty = \infty$ and as $\Phi \to \infty$, $U \to 0$ faster than $\Phi^{-2}$ . We shall see that such properties are found for a class of ‘moduli’-type electric solutions and a special case of the ‘dilatonic’-type magnetic solution. On the other hand, there also exist regular, finite energy solutions for a specific value of $C = C_0 > 0$; this is the case if $\Phi_\infty = \infty$ and as $\Phi \to \infty , U(\Phi) - \frac{1}{2} C_0$ approaches zero faster than $\Phi^{-1}$. 
The simplest example of the dilatonic solution corresponds to the case of the tree level dilaton coupling
\[ f = e^{-2\Phi}, \quad \Phi = \phi. \] (3.6)
The solution is given by eq. (3.5) with the upper sign and \( C = 0 \) (in order to avoid a singularity at finite \( x \)). The explicit form of the solution, its mass \( M \), charge \( Q \) and the scalar charge \( D \) are
\[ \phi = \phi_0 - \ln \left(1 + \frac{M}{r}\right), \quad M \equiv E = |q|e^{\phi_0}, \] (3.7)
\[ E(r) = \frac{Q}{(r + M)^2}, \quad Q = Me^{\phi_0}, \quad D = -M. \] (3.8)
The string coupling grows from zero at \( r = 0 \) to a finite value \( e^{\phi_0} \) at large distances. The small distance region is thus a weak coupling region. Therefore it is consistent to ignore the non-perturbative potential \( V \) in this region. We shall see that once the potential \( V \) is included, \( \phi \) will be evolving to its minimum at large \( r \). The solution is stable since the condition (2.16) is satisfied for (3.6).

The regular magnetic solution with the finite energy, a counterpart of the electric solution (3.7), is obtained from the lower sign solution of eq. (3.4) where \( f \) in (3.6) is replaced with \( f^{-1} \), and \( q \) with \( h \), and \( C = 0 \). Then
\[ \phi = \phi_0 + \ln \left(1 + \frac{M}{r}\right), \quad \mathcal{E} \equiv M = |h|e^{-\phi_0}, \quad D = M. \] (3.9)
The small distance region of the magnetic solution is a strong coupling region and hence there non-perturbative corrections (inducing \( V \neq 0 \)) can be significant. Like the corresponding regular electric solution, this magnetic solution is also stable.

The above expressions for the dilaton in the electric and the magnetic solutions coincide with the expressions for the dilaton in the electric and magnetic black hole solutions of [1], [2] if \( r \) is identified with the coordinate \( \hat{r} = r - M \), where \( r \) is defined outside the horizon. In terms of \( r \) we get asymptotic large \( r \) expressions for the dilaton of [1], [2]. For example, in the case of the electric black hole solution the metric (in the Einstein frame) takes the form (see, e.g., [3])
\[ ds^2 = -(1 - \frac{m}{r})(1 + \frac{M}{r})^{-2}dt^2 + (1 - \frac{m}{r})^{-1}d\hat{r}^2 + \hat{r}^2d\Omega^2, \]
while the expressions for the dilaton and the electric field coincide with (3.7) and (3.8) where \( r \) is replaced by \( \hat{r} \). The physical mass is \( \mu = M + m \) and \( Q = (M\mu)^{1/2} \). As was
discussed in [3] the small \( \hat{r} \) region is a weak coupling region for the electric solution but a strong coupling region for the magnetic one (which is obtained from the electric solution by the duality transformation \( \phi \to -\phi, \ F_{\mu\nu} \to e^{-2\phi}F_{\mu\nu}^* \)).

An important generalization of (3.6) corresponds to

\[
f = e^{-2\Phi} + b , \quad \Phi = \phi , \quad (3.10)
\]

where the constant \( b \) can be interpreted, e.g., as a contribution of threshold corrections to \( f \) in the case when the space dependence of the modulus field can be ignored. Assuming that \( b > 0 \), we get the electric solution from eqs.(3.4),(3.5) with the choice of the upper sign and \( C = 0 \):

\[
[-\sqrt{ e^{-2\phi} + b } + \sqrt{b} \text{Arccosh}(\sqrt{be^\phi})]_{\phi_0}^{(r)} = -|q| r , \quad E = |q| \sqrt{b} \text{Arccosh}(\sqrt{be^{\phi_0}}) . \quad (3.11)
\]

Here \( \phi_\infty = -\infty \), i.e., the string coupling is increasing with \( r \) from zero to a finite value \( e^{\phi_0} \) and the energy is finite. The stability condition (2.16) is satisfied if \( \phi_0 \) is such that \( e^{-2\phi_0} - b \geq 0 \).

It turns out that the only finite energy, regular magnetic solution exists for \( b < 0 \) and the choice of \( C = -h^2b > 0 \). In this case:

\[
\frac{1}{\sqrt{-b}}[\text{Arccosh}(\sqrt{-be^\phi})]_{\phi_0}^{(r)} = |h| r , \quad E = |h|(\sqrt{ e^{-2\phi_0} - b } - \sqrt{-b}) . \quad (3.12)
\]

For \( b > 0 \) one gets a regular solution with \( \phi_\infty = \infty \) but infinite energy.

Next, let us consider ‘modulus-type’ solutions corresponding to

\[
f(\Phi) = p^2(\cosh a\Phi + s)^2 . \quad (3.13)
\]

For large negative \( \Phi \) and \( a = 1 \) this function is the same as in (3.6). Being symmetric under the ‘duality’ symmetry \( \Phi \to -\Phi \) this \( f \) (with \( \Phi = \phi, \ a = 1/\sqrt{3} \)) is a good approximation for the modular invariant coupling function \( f_2(\varphi) \) in (2.4). A non-zero constant \( s \) may be considered as accounting for a modulus independent contribution to \( f \). Without loss of generality one can fix the boundary condition \( \varphi_0 \equiv \varphi(1/r = 0) > 0 \) (due to the duality symmetry \( \varphi \to -\varphi \) solutions with the boundary condition \( \varphi_0 < 0 \) are related to the ones with \( \varphi_0 > 0 \)). Then for the regular, positive energy electric solutions one finds from eqs.(3.4),(3.5) (with the lower sign and \( C = 0 \))

\[
[\sinh a\varphi + sa\varphi]_{\varphi_0}^{(r)} = \frac{|q|a}{|p|r} , \quad (3.14)
\]
\[ E = \frac{2|q|}{a|p|\sqrt{1 - s^2}} \left[ \arctan \left( \frac{\sqrt{1 - s^2}}{1 + s} \tanh \frac{a\varphi}{2} \right) \right]_{\varphi_0}^{\infty}, \quad s^2 < 1, \quad (3.15) \]

\[ E = \frac{q}{r^2 p^2 |\cosh a\varphi + s|^2}, \quad Q = \frac{q}{p^2 (\cosh a\varphi_0 + s)^2}, \quad D = \frac{q}{|p|(\cosh a\varphi_0 + s)}. \quad (3.16) \]

For \( s > -1 \) the solution is regular for any \( \varphi_0 > 0 \). On the other hand, for \( s < -1 \), the regular solution exists when \( \varphi_0 > 0 \) satisfies \( \cosh a\varphi_0 + s > 0 \). The condition of stability (3.12), i.e., \( \frac{d^2}{d\varphi^2} f^{-1} \geq 0 \), is satisfied if \( \varphi_0 \) is such that \( 2\cosh^2 a\varphi - s \cosh a\varphi - 3 \geq 0 \) for all values of \( \varphi(r) \). Since there always exists a choice of \( \varphi_0 \) for which both of the above constraints are satisfied we get a class of stable, regular finite energy electric solutions.

In general, the electric solution (3.14)–(3.16) has the property that it increases from a positive finite value \( \varphi_0 \equiv \varphi(1/r = 0) > 0 \) to \( \varphi_\infty \equiv \varphi(1/r = \infty) = +\infty \). Namely, in the core of the solution, i.e., as \( r \to 0 \), a decompactification (\( \varphi \to \infty \)) takes place. There is an obvious analogy with the corresponding ‘dilatonic’ electric solutions with \( f \) in (3.10): the role of the weak coupling at the core of the ‘dilatonic’ electric solution is now played by the decompactification at the core of the ‘modulus’ electric solution.

We thus see that in the case of the duality invariant function \( f(\varphi) \) (3.13) there are stable, regular, finite energy electric solutions. It turns out that there are no regular, finite energy magnetic solutions corresponding to such \( f(\varphi) \), unless \( s = -1 \) \[4\]. In the latter case

\[ \coth \frac{a\varphi(r)}{2} - \coth \frac{a\varphi_0}{2} = \frac{|p||h|a}{r}, \quad E = \frac{|h||p|}{a} (\sinh a\varphi_0 - a\varphi_0). \quad (3.17) \]

Here \( \varphi_\infty = 0 \), i.e., the magnetic solution corresponds to the compactification at the self-dual point \( \varphi = 0 \). For the duality invariant \( f \) with \( f(0) = 0 \) the electric and the magnetic solutions have complementary features, similar to the ones of the dilatonic solution with \( f \) in (3.6). Now, however, the role of the strong-weak coupling regions is played by the compactification - decompactification regions. One can show \[4\] that the finite energy electric and magnetic solutions with qualitatively the same behaviour exist for a general positive definite, duality invariant function \( f \) with the following properties: \( f(\varphi) \) has the minimum at \( \varphi = 0 \), \( f(0) = 0 \), and as \( \varphi \to \infty \), \( f \) grows faster than \( \varphi^2 \).

Let us now turn to the solutions in a more ‘realistic’ case when both the dilaton and the modulus field can change in space. We shall choose the coupling function in the following form

\[ f(\phi, \varphi) = f_1(\phi) + f_2(\varphi), \quad (3.18) \]
which is a generalization of the perturbative expressions (2.3),(2.4): $f_1 = e^{-2\phi}$ and $f_2 = b_0 \ln [(T + T^*)|\eta(T)|^4] + b_1$, $T = e^{2\varphi/\sqrt{3}}$. The system of equations for the two scalars $\Phi_i = (\phi, \varphi)$ in the case of the electric solution becomes

$$\Phi_i'' + \frac{q^2}{2(f_1 + f_2)^2} \frac{df_i}{d\Phi_i} = 0, \quad i = 1, 2.$$  

(3.19)

It reduces to the one-scalar case considered in the previous section if one of the scalars $\Phi_i$ is fixed to be at the extremum of the corresponding function $f_i$. While the tree-level dilatonic coupling $f_1$ does not have a local extremum (and thus the dilaton cannot be ‘frozen’ at a constant value) the modulus coupling $f_2$ does have an extremum at $\varphi = 0$.

It is possible to find electric solutions of eq. (3.19) using a perturbative approach, i.e. assuming that $f_1 \gg f_2$ [4]. This assumption is satisfied, in fact, in the case of the threshold correction in (2.4). In such a case one can reduce the system of the two second-order coupled differential equations (3.19) to a set of two first-order coupled differential equations. For example, one can approximate the function $f_2$ in (2.4) by a simple duality invariant function $f_2 = p^2 \sinh^2 a\varphi$. Then

$$\ln \left[ \frac{\tanh a\varphi(r)}{\tanh a\varphi_0} \right] = \frac{1}{6} a^2 p^2 e^{2\phi_0} \left[ 1 - \left( 1 + \frac{|q|e^{\phi_0}}{r} \right)^{-2} \right].$$  

(3.20)

As $r \to 0$, $\varphi$ increases (decreases) for $p^2 > 0$ ($p^2 < 0$). One can show that the solutions with $f_2 > 0$ (abelian case) are unstable (since the tree-level dilaton solution is stable, the instability is due to the modulus sector), while those with $f_2 < 0$ (embedding in a non-abelian gauge group) are stable.

4. Case of non-vanishing scalar potential

When $V \neq 0$ the system of equations (2.13)

$$\phi'' + \frac{\partial U}{\partial \phi} - \frac{1}{x^4} \frac{\partial V}{\partial \phi} = 0, \quad \varphi'' + \frac{\partial U}{\partial \varphi} - \frac{1}{x^4} \frac{\partial V}{\partial \varphi} = 0,$$  

(4.1)

(where $U = -\frac{1}{2} q^2 f^{-1}$ and $U = -\frac{1}{2} h^2 f$ for the electric and magnetic cases) does not have integrals of motion. The non-perturbative potential due to gaugino condensation in the hidden sector of the gauge group (2.7) depends on both the dilaton, $S = e^{-2\phi}$ and the modulus, $T = e^{2\varphi/\sqrt{3}}$ and vanishes in the limit of small string coupling $e^{\phi} \to 0$. The potential is not convex everywhere; in addition to the minimum, it also has saddle points and local maxima. We shall consider a class of functions $f = f_1(\phi) + f_2(\varphi)$ (eqs.(2.3),(2.4)).
Let us consider first the electric solutions. At small radius $r \to 0$ the electric solutions correspond to the weak coupling region, $\phi \to -\infty$, and thus, the potential $V$ term can be neglected in both equations in (4.1) and we are back to the case discussed in the previous section. To determine the large $r$ asymptotic behaviour of $\phi$ and $\varphi$ let us again assume that in the large distance region $\phi$ and $\varphi$ approach constant values $\phi_0$ and $\varphi_0$. Using the expansion

$$\Phi_i = \Phi_{0i} + k_i x + l_i x^2 + m_i x^3 + n_i x^4 + \ldots, \quad \Phi_i = (\phi, \varphi),$$

one finds that (4.1) is satisfied to the leading order in $x$ if

$$k_i = l_i = m_i = 0, \quad n_i \neq 0, \quad (\partial V / \partial \Phi_i)_{\Phi_0} = 0,$$

$$q^2 (1 \over f^2 \partial \Phi_i)_{\Phi_0} - \sum_{j=1}^2 n_j (\partial^2 V / \partial \Phi_i \partial \Phi_j)_{\Phi_0} = 0.$$ 

As a result,

$$\Phi_i (r \to \infty) = \Phi_{0i} + {n_i \over r^4} + O(1 \over r^5),$$

where the asymptotic value $\Phi_{0i} = (\phi_0, \varphi_0)$ corresponds to an extremum of $V(\Phi)$. As follows from (2.4), $f$ is positive, $\partial f / \partial \phi$ is negative and $\partial f / \partial \varphi$ is positive (negative) for the abelian (non-abelian embedding) case. For consistency with the behaviour of the solutions at small $r$ (Section 3) we are to assume that as $r$ increases $\phi$ should be growing and $\varphi$ should be falling (growing) in the abelian (non-abelian embedding) case, namely, $n_1 < 0$, and $n_2 > 0$ ($n_2 < 0$). These constraints in turn imply that in the case when the mixed second derivative term $(\partial^2 V / \partial \phi \partial \varphi)_{\Phi_0}$ is small, the matrix $(\partial^2 V / \partial \Phi_i \partial \Phi_j)_{\Phi_0}$ is positive definite, i.e., both $\phi_0$ and $\varphi_0$ correspond to the minimum of $V$. One can show that the potential $V$ in (2.5) has actually the mixed second derivative equal to zero, $(\partial^2 V / \partial \phi \partial \varphi)_{\Phi_0} = 0$.

The magnetic solutions have the strong coupling region $\phi \to +\infty$ at small radius, $r \to 0$. The potential $V$ term, however, does not qualitatively modify the particular dilaton solution in this region. In addition, the potential term can be neglected in the second equation in (4.1) since it is proportional to $e^{2\phi} x^{-4} \propto r^{-2}$. Therefore, as $r \to 0$, solutions for the modulus can be discussed as in Section 3. Thus, in both the electric and the magnetic, cases the presence of the potential $V$ term in the equations of motion fixes the asymptotic values of $\phi_0$ and $\varphi_0$ to be at the minimum of $V$.

In the above discussion we have assumed the metric to be flat. This is consistent provided the energy of the solutions is small enough. It is important of course to extend
the analysis to the gravitational case, i.e. to find the corresponding black hole - type solutions (generalizing those of [1][2][3][4]) of the string effective action with the threshold corrected coupling $f$ as well as the non-perturbative potential $V$ included. Choosing the metric \([1]\)

\[
ds^2 = -e^{2\nu} dt^2 + e^{4\zeta - 2\nu} dy^2 + e^{2\zeta - 2\nu} d\Omega^2 ,
\]

where $\nu$ and $\zeta$ are functions of $y$ one can represent the system of equations that follow from (2.1) and generalise (2.10) as corresponding to the following mechanical system (together with the zero energy constraint $T + U = 0$)

\[
L = T - U , \quad T = \frac{1}{2} \nu'^2 + \frac{1}{2} \phi'^2 + \frac{1}{2} \varphi'^2 - \frac{1}{2} \zeta'^2 ,
\]

\[
U = \frac{1}{2} e^{2\zeta} - \frac{1}{2} q^2 e^{2\nu} [f_1(\phi) + f_2(\varphi)]^{-1} - e^{4\zeta - 2\nu} V(\phi, \varphi) .
\]

One may find the large distance asymptotics of the fields which will generalise (4.2)–(4.5) and the results of [3].

In conclusion, let us emphasize again the generic features of the stable, finite energy electric solution. It is a particle-like configuration with the weak coupling region inside the core. It seems that the proper dilaton boundary condition for analogous solutions in string theory should be $\phi \to -\infty$ at $r \to 0$, i.e., small string coupling at small scales. This corresponds to an appealing ‘asymptotic freedom’ scenario in which both perturbative and non-perturbative string corrections are negligible in the small distance region. Thus, in this region the tree-level string theory applies, supersymmetry and other symmetries are unbroken (in the cosmological context the small distance region corresponds to an early time era). On the other hand, the growth of the dilaton with $r$ implies that at large distances, or in ‘our world’, the string coupling becomes relatively strong, so that non-perturbative corrections can no longer be ignored. In this region supersymmetry is spontaneously broken and both the dilaton and the modulus fields are stabilized at the minimum of the non-perturbative potential.

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References

[1] G.W. Gibbons and K. Maeda, Nucl. Phys. B298 (1988) 741.
[2] D. Garfinkle, G.T. Horowitz and A. Strominger, Phys. Rev. D43 (1991) 3140.
[3] G.T. Horowitz, “The dark side of string theory: black holes and black strings”, in Proceedings of the 1992 Trieste Spring School on String Theory and Quantum Gravity, preprint UCSBTH-92-32.
[4] M. Cvetiˇc and A.A. Tseytlin, preprint CERN-TH.6911/93; hep-th/9307123.
[5] R. Gregory and J.A. Harvey, Phys. Rev. D47 (1993) 2411.
[6] J.H. Horne and G.T. Horowitz, Nucl. Phys. B399 (1993) 169.
[7] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649.
[8] J.P. Derendinger, L.E. Ibáñez and H.P. Nilles, Phys. Lett. B155 (1985) 65.
[9] M. Dine, R. Rohm, N. Seiberg and E. Witten, Phys. Lett. B156 (1985) 55.
[10] A. Font, L. Ibáñez, D. Lüst and F. Quevedo, Phys. Lett. B245 (1990) 401; Nucl. Phys. B361 (1991) 194.
[11] S. Ferrara, N. Magnoli, T. Taylor and G. Veneziano, Phys. Lett. B245 (1990) 409.
[12] P. Nilles and M. Olechowski, Phys. Lett. B248 (1990) 268.
[13] R. Brustein and P. Steinhhardt, Phys. Lett. B302 (1993) 196.