RECURRENCE OF MULTI-DIMENSIONAL DIFFUSION PROCESSES IN BROWNIAN ENVIRONMENTS

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Abstract. We consider limiting behavior of multi-dimensional diffusion processes in two types of Brownian environments. One is given values at different $d$ points of a one-dimensional Brownian motion, which is supposed to be a multi-parameter environment, and other is given by $d$ independent one-dimensional Brownian motions. We show recurrence of multi-dimensional diffusion processes in both Brownian environments above for any dimension and almost all environments. Their limiting behavior is quite different from that of ordinary multi-dimensional Brownian motion. We also consider cases of reflected Brownian environments.

1. Introduction and results. It is well-known that an $\mathbb{R}^d$-dimensional Brownian motion, which is formed by $d$ independent one-dimensional Brownian motions, is recurrent if $d = 1$ or $2$, and transient otherwise. We consider such a problem for multi-dimensional diffusion processes in Brownian environments.

Let $W$ be the space of $\mathbb{R}$-valued continuous functions with vanishing at 0 and let $Q$ be the Wiener measure on $W$. We call $(W, Q)$ a Brownian environment. For a given $W(x)$, we consider a multi-dimensional diffusion process in Brownian environment $(W, Q)$

$$X_W = \{X^k_W(t), k = 1, 2, 3, ..., d\}$$

whose generator is

$$\sum_{k=1}^{d} \frac{1}{2} \exp\{W(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-W(x_k)\} \frac{\partial}{\partial x_k} \right\}. \quad (1)$$

We treat a one-dimensional Brownian motion and regard its values at different $d$ points as a multi-parameter environment. Such an $X_W$ is constructed from $d$ independent Brownian motions by a scale change and a time change (c.f. [5]). Each component of $X_W$ above is symbolically described by

$$dX^k_W(t) = dB^k(t) - \frac{1}{2} W'(X^k_W(t))dt, \quad X^k_W(0) = 0, \quad \text{for } k = 1, 2, 3, ..., d,$$

where $B^k(t)$ is a Brownian motion independent of a Brownian environment $W$.

In the case where $d = 1$, the diffusion $X_W$ is a continuous model of random walks in a random environment by Sinai([9]), and recurrent for almost all Brownian environments (see also [10]). Brox studied the case and showed that the distribution of $(\log t)^{-2} X_W(t)$ converges weakly as $t \to \infty$ in [1], that is, $X_W$ moves very slowly by the effect of environments.

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Because of the subdiffusive property of $X$, we expect to see exotic limiting behavior of a multi-dimensional diffusion process $X$. In this paper, we consider this problem for Brownian environments. Our main theorem is as follows:

**Theorem 1.1.** For almost all Brownian environments, $X$ corresponding to the generator (1) is recurrent for any dimension $d = 1, 2, 3, \ldots$.

Some investigations related to multi-dimensional diffusion processes in random environments have been conducted. In [2], Fukushima, Nakao and Takeda obtained recurrent property of the diffusion process with a generator

$$\frac{1}{2} \exp\{W(|x|)\} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\{-W(|x|)\} \frac{\partial}{\partial x_k} \right\},$$

where $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ and $W$ is a one-dimensional Brownian environment. Their environment is a function of the distance from the origin. In the case where an environment is a Lévy's Brownian motion with a multi-dimensional time, Tanaka showed that the diffusion process is recurrent for almost all environments in [13]. For a non-negative reflected Lévy's Brownian environment, Mathieu obtained some results of long time behavior in [8]. Our model is different from theirs, but it gives a certain multi-dimensional extension of the previous work by Brox. In the case of $d$ independent one-dimensional Brownian environments, we have the following:

**Theorem 1.2.** Let $W$ be a set of $d$ independent copies of a Brownian environment $(W, Q)$ and let $X_W$ be a multi-dimensional diffusion process in $d$ independent Brownian environments $W$ with a generator

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\{-W_j(x_k)\} \frac{\partial}{\partial x_k} \right\}.$$

Then for any dimension $d = 1, 2, 3, \ldots$, $X_W$ is recurrent for almost all environments.

2. **Proof of theorems.** We use Ichihara’s recurrent test studied in [3] in a similar manner to previous works in [2] and [13]. We can show that the generator (1) equals to

$$\frac{1}{2} \exp\left\{ \sum_{j=1}^{d} W_j(x_j) \right\} \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\left\{- \sum_{j=1}^{d} W_j(x_j) \right\} \frac{\partial}{\partial x_k} \right\}$$

through a simple calculation. We set

$$A_t := \frac{1}{2} \int_0^t \exp\left\{ \sum_{j=1}^{d} W_j(X_W^j(s)) \right\} ds \quad \text{and} \quad \tau_t := A_t^{-1}.$$

Using them, we construct a time changed process as

$$Y_W(t) := X_W(\tau_t),$$

whose generator is given by

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\left\{- \sum_{j=1}^{d} W_j(x_j) \right\} \frac{\partial}{\partial x_k} \right\}.$$

Since $\tau_t$ is strictly increasing, we have the following lemma:

**Lemma 2.1.** Recurrent or transient property of $X_W$ coincides with that of $Y_W$.

Hence, we study limiting behavior of $Y_W$. To show the assertion, we use the following lemma:
Lemma 2.2. We set 

\[ T_t W(x) := e^{-t/2} W(e^t x). \]

Then \( \{T_t, t \in \mathbb{R}\} \) is ergodic.

Proof. Since \( \{T_t, t \in \mathbb{R}\} \) is a family of measure preserving transformations, the covariance matrix of 

\[(W(x_1), W(x_2), \ldots, W(x_m), T_t W(x'_1), T_t W(x'_2), \ldots, T_t W(x'_n))\]

equals to

\[
\begin{pmatrix}
S & R^*(t) \\
R(t) & T
\end{pmatrix},
\]

where \( S \) and \( T \) are the correlation matrices of \((W(x_1), W(x_2), \ldots, W(x_m))\) and of \((W(x'_1), W(x'_2), \ldots, W(x'_n))\), respectively, the element \( r_{ij}(t), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \) of \( R(t) \) are equal to \( e^{-t/2} \min \{e^t x'_i, x_j\} \), and \( R^*(t) \) is the transposed matrix of \( R(t) \). As \( r_{ij}(t) \) converges to 0 as \( t \to 0 \), the ergodicity of \( \{T_t\} \) can be shown in the same way as the theorem in [4] or Theorem 9 in [7].

Lemma 2.3. For a fixed positive \( a_0 \) we have 

\[ Q \left\{ \min_{\sigma \in S^{d-1}} \left\{ \sum_{j=1}^{d} W(\sigma_j) \right\} > a_0 \right\} > 0. \]

Proof. We set \( \varphi(s) = 2a_0 s^2 \). Then \( \sum_{j=1}^{d} W(\sigma_j) = 2a_0 \). We denote by \( Q^\varphi \) the probability measure \( Q\{\cdot - \varphi\} \). Then Cameron-Martin formula and Itô formula imply that \( Q\{\cdot - \varphi > 0 \} \) if and only if \( Q\{\cdot > 0 \} \) and 

\[
\frac{dQ^\varphi}{dQ}_{|[0,1]} = \exp\left\{-\int_0^1 \varphi'(s) dW(s) - \frac{1}{2} \int_0^1 |\varphi'(s)|^2 ds\right\} = \exp\left\{-4a_0 W(1) + 4a_0 \int_0^1 W(s) ds - \frac{8}{3} a_0^2 \right\}. \tag{4}
\]

For some positive \( a \), we let \( \|W\|_\infty < a \) denote \( \{W : \sup_{u \in [0,1]} |W(u)| < a\} \). Then (4) implies that 

\[
Q \left\{ \min_{\sigma \in S^{d-1}} \left\{ \sum_{j=1}^{d} W(\sigma_j) \right\} > a_0 \right\} = Q \left\{ \min_{\sigma \in S^{d-1}} \left\{ \sum_{j=1}^{d} (W(\sigma_j) - \varphi(\sigma_j)) \right\} > -a_0 \right\} \\geq Q \left\{ \|W - \varphi\|_\infty < \frac{a_0}{d} \right\}^2 = Q^\varphi \left\{ \|W\|_\infty < \frac{a_0}{d} \right\}^2 \\geq \exp\{-16a_0^2(1/d^2 + 1/3)\} \left\{ Q \left\{ \|W\|_\infty < \frac{a_0}{d} \right\} \right\}^2 > 0.
\]

Proof of Theorem 1.1. According to Ichihara’s recurrent test for the generator (1), it is enough to show that for almost all environments 

\[
\int_1^\infty r^{1-d} \left\{ \int_{S^{d-1}} \exp\left\{ -\sum_{j=1}^{d} W(r\sigma_j) \right\} d\sigma \right\}^{-1} dr = \infty, \tag{5}
\]
where $\sigma$ is the normalized uniform measure on $S^{d-1}$. Setting $r = e^t$ for the left hand side of (9), we obtain that

$$\int_0^\infty e^{(2-d)t} \left\{ \int_{S^{d-1}} \exp \left\{ - \sum_{j=1}^d W(e^t \sigma_j) \right\} d\sigma \right\}^{-1} dt$$

$$= \int_0^\infty e^{(2-d)t} \left\{ \int_{S^{d-1}} \exp \left\{ -t^{d/2} \sum_{j=1}^d T_i W(\sigma_j) \right\} d\sigma \right\}^{-1} dt$$

$$\geq C_d \int_0^\infty \exp \left\{ (2-d)t + t^{d/2} \min_{\sigma \in S^{d-1}} \left\{ \sum_{k=1}^d T_i W(\sigma_k) \right\} \right\} dt,$$

where $C_d$ denotes the surface area of the unit sphere. We set

$$M(s) := \min_{\sigma \in S^{d-1}} \left\{ \sum_{j=1}^d T_i W(\sigma_j) \right\}.$$

Let $a$ be a positive number such that for any $t > 0$

$$(2-d)t + ae^{t/2} \geq 0.$$

We set $a_0 := \inf a$. Then Lemma 2.2 implies that

$$\lim_{t \to \infty} \left\{ \frac{1}{t} \int_0^t 1_{[a_0,\infty]}(M(s)) ds \right\} = E[1_{[a_0,\infty]}(M(0))]$$

$$= P \left\{ \min_{\sigma \in S^{d-1}} \left\{ \sum_{k=1}^d W(\sigma_k) \right\} > a_0 \right\}. \quad (6)$$

By this convergence and Lemma 2.3, we obtain that

$$C_d \int_0^t \exp \left\{ (2-d)s + e^{s/2} M(s) \right\} ds$$

$$\geq C_d \int_0^t 1_{[a_0,\infty]} (M(s)) ds$$

$$\geq C_d t \cdot \frac{1}{t} \int_0^t 1_{[a_0,\infty]} (M(s)) ds \to \infty \quad \text{as } t \to \infty;$$

which concludes Theorem 1.1. \qed

**Proof of Theorem 1.2.** In the same way as the case of $X$, we can show that recurrent or transient property of $X$ with the generator (2) coincides with that of $Y$ whose generator is given by

$$\sum_{k=1}^d \frac{\partial}{\partial x_k} \left\{ \exp \left\{ - \sum_{j=1}^d W_j(x_j) \right\} \frac{\partial}{\partial x_k} \right\}.$$

Ergodicity of $\{T_i\}$ under $Q^\otimes d$ is shown in a similar way to the case for the measure $Q$, namely we consider the covariance matrix of

$$(W_1(x_1), \ldots, W_d(x_1), W_1(x_2), \ldots, W_d(x_2), \ldots, W_1(x_m), \ldots, W_d(x_m),$$

$$T_iW_1(x'_1), \ldots, T_iW_1(x'_1), T_iW_2(x'_2), \ldots, T_iW_d(x'_2), \ldots, T_iW_1(x'_m), \ldots, T_iW_d(x'_m)),$$

and for any $x, y \in \mathbb{R}$ and $i, j \in \mathbb{N}$

$$\lim_{t \to \infty} E[W_i(x), T_iW_j(y)] = 0$$
implies the ergodicity.

We can show that
\[ Q \left\{ \sup_{u \in [0,1]} |W_j(u) - \varphi(u)| < \frac{a_0}{d} \right\} > 0 \tag{7} \]
for each component. Since \( W_j \)'s are independent, (7) implies
\[ Q^{\otimes d} \left\{ \min_{\sigma \in \tilde{S}^{d-1}} \left\{ \sum_{j=1}^{d} W_j(\sigma_j) \right\} > a_0 \right\} > 0. \tag{8} \]
This positivity and the ergodic property conclude Theorem 1.2.

3. The cases of reflected Brownian environments. Tanaka studied the diffusion process in a non-negative or non-positive reflected Brownian environment in [12]. He showed that each distribution of \( (\log t)^{-2} X(t) \) also converges weakly. Through a simple argument, they are recurrent for almost all environments when \( d = 1 \). Following the study, Kim obtained some limit theorems of multi-dimensional diffusion processes in [6]. He gave an example that the random environment is given by \( d \) independent one-dimensional reflected non-negative Brownian environments studied in [11]. In the case of reflected Brownian environments, we obtain the following results:

**Theorem 3.1.**

(i) Suppose non-negative reflected Brownian environment, namely we consider a multi-dimensional diffusion process in a non-negative reflected Brownian environment \( X_{|W|} \) with a generator
\[ \sum_{k=1}^{d} \frac{1}{2} \exp\{W(x_k)\} \frac{\partial}{\partial x_k} \left\{ \exp\{-|W(x_k)|\} \frac{\partial}{\partial x_k} \right\}. \]
Then, for almost all environments \( X_{|W|} \) is recurrent for any dimension \( d = 1, 2, 3, \ldots \). (ii) A diffusion process in non-positive reflected Brownian environment \( X_{-|W|} \), whose generator is
\[ \sum_{k=1}^{d} \frac{1}{2} \exp\{-|W(x_k)|\} \frac{\partial}{\partial x_k} \left\{ \exp\{|W(x_k)|\} \frac{\partial}{\partial x_k} \right\}, \tag{9} \]
is recurrent for \( d = 1 \) and transient for \( d = 2, 3, 4, \ldots \), for almost all environments.

**Proof.** The assertion (i) is shown in the same way as those for showing the cases \( X_W \), and we omit the proof.

On the assertion (ii), recurrence of the one-dimensional diffusion process is easily obtained from a general theory of one-dimensional diffusion process. Hence, we prove the case where \( d \geq 2 \). For the diffusion process \( X_{-|W|} \) with the generator (9), Lemma 2.1 implies that it is sufficient to consider the process \( Y_{-|W|} \) whose generator is
\[ \sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left\{ \exp\left\{ \sum_{j=1}^{d} |W(x_j)| \right\} \frac{\partial}{\partial x_k} \right\}. \tag{10} \]
According to Ichihara’s transient test for (10), it is enough to show that for almost all the environments
\[ \int_{1}^{\infty} r^{1-d} \exp \left\{ -\sum_{j=1}^{d} |W(r\sigma_j)| \right\} dr < \infty \]
for any \( \sigma \in \tilde{S} \), where \( \tilde{S} \subset S^{d-1} \) with \( |\tilde{S}| > 0 \). Since \( \exp\{|W(x)|\} > 1 \), it is sufficient to show only the case where \( d = 2 \).
Without loss of generality, we can assume that there exists a positive $a$ such that $0 < a < \sigma_1 \leq \sigma_2 < 1$. For such a $\sigma = (\sigma_1, \sigma_2) \in S^1$, we take the expectation with respect to $Q$ as follows:

$$
E \left[ r^{-1} \exp \left\{ - (|W(r\sigma_1)| + |W(r\sigma_2)|) \right\} \right] 
\leq E \left[ r^{-1} \exp \left\{ - \min_{a \leq \sigma_1 \leq 1} |W(r\sigma_1)| \right\} \right] 
= E \left[ r^{-1} \exp \left\{ -r^{1/2} \min_{a \leq \sigma_1 \leq 1} |W(\sigma_1)| \right\} \right].
$$

(11)

For any $x > 0$

$$
Q \left\{ \min_{0 \leq s \leq 1} |W(s)| \in dx \right\} / dx
= \frac{2}{\sqrt{2\pi}a} \int_{x}^{\infty} dy \cdot e^{-y^2/2a} Q \left\{ \min_{0 \leq s \leq 1} |W(s)| \in dx \mid W(a) = y \right\} / dx
= \frac{2}{\sqrt{2\pi}a} \int_{x}^{\infty} dy \cdot e^{-y^2/2a} \frac{2}{\sqrt{2\pi}b} e^{-(y-a)^2/2b} (b := 1 - a)
= \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \int_{x}^{\infty} \frac{2}{\sqrt{2\pi}ab} e^{-(y-a)^2/(2ab)} dy
\leq \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}ab} e^{-z^2/(2ab)} dz (z := y - ax)
= \frac{2}{\sqrt{2\pi}} e^{-x^2/2},
$$

which implies that for any $\sigma = \{(\sigma_1, \sigma_2) : a < \sigma_1 \leq \sigma_2 < 1\}$

$$
\int_{1}^{\infty} \frac{1}{r} \exp \left\{ - \sum_{j=1}^{2} |W(r\sigma_j)| \right\} dr
\leq \int_{1}^{\infty} \frac{1}{r} E \left[ \exp \left\{ -r^{1/2} \min_{0 \leq s \leq 1} |W(s)| \right\} \right] dr
\leq \int_{1}^{\infty} \frac{1}{r} \left\{ \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} e^{-r^{1/2}x} e^{-x^2/2} dx \right\} dr
= \int_{1}^{\infty} \frac{1}{r} \left\{ \frac{2}{\sqrt{2\pi}} e^{r/2} \int_{r/1}^{\infty} e^{-u^2/2} du \right\} dr (u := x + r^{1/2}).
$$

(12)

Using the inequality

$$
\int_{\xi}^{\infty} e^{-u^2/2} du \leq \frac{1}{\xi} e^{-\xi^2/2} \quad \text{for} \quad \xi > 0,
$$

we obtain that

$$
(\text{the right hand side of (12)})
\leq \int_{1}^{\infty} \frac{1}{r} \left\{ \frac{2}{\sqrt{2\pi}} e^{r/2} \frac{1}{r^{1/2}} e^{-r/2} \right\} dr
\leq \int_{1}^{\infty} r^{-3/2} dr = 2 < \infty,
$$

which implies assertion.
Models studied in [11] correspond to cases of \( d \) independent one-dimensional reflected Brownian environments. We also obtain the same results as a corollary of Theorem 3.1:

**Corollary 3.1.** (i) In the case of \( d \) independent non-negative reflected Brownian environment, namely the generator of a diffusion \( X_{|W|} \) is given by

\[
\sum_{k=1}^{d} \frac{1}{2} \exp\{|W_k(x_k)|\} \frac{\partial}{\partial x_k} \left\{ \exp\{-|W_k(x_k)|\} \frac{\partial}{\partial x_k} \right\},
\]

for any dimension \( d = 1, 2, 3, \ldots \), \( X_{|W|} \) is recurrent for almost all environments.

(ii) In the case of \( d \) independent non-positive reflected Brownian environment, namely the generator of a diffusion \( X_{-|W|} \) is given by

\[
\sum_{k=1}^{d} \frac{1}{2} \exp\{-|W_k(x_k)|\} \frac{\partial}{\partial x_k} \left\{ \exp\{|W_k(x_k)|\} \frac{\partial}{\partial x_k} \right\},
\]

\( X_{-|W|} \) is recurrent for \( d = 1 \) and transient for \( d = 2, 3, 4, \ldots \), for almost all environments.

**REFERENCES**

[1] T. Brox, A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.*, 14 (1986), 1206–1218.

[2] M. Fukushima, S. Nakao and M. Takeda, On Dirichlet form with random date - recurrence and homogenization, in “Stochastic Processes - Mathematics and Physics II (Bielefeld, 1985)” (eds. S. Albeverio, Ph. Blanchard and L. Streit), Lect. Notes in Math., 1250, Springer-Verlag, (1987), 87–97.

[3] K. Ichihara, Some global properties of symmetric diffusion processes. *Publ. RIMS, Kyoto Univ.*, 14 (1978), 441–486.

[4] K. Itô, On the ergodicity of a certain stationary process. *Proc. Imp. Acad.*, 20 (1944), 54–55.

[5] K. Itô and H.P. McKean,Jr., “Diffusion Processes and Their Sample Paths,” Springer-Verlag, New York, 1965.

[6] D. Kim, Some limit theorems related to multi-dimensional diffusions in random environments, *J. Korean Math. Soc.*, 48 (2011), 147–158.

[7] G. Maruyama, The harmonic analysis of stationary stochastic processes, *Mem. Fac. Sci. Kyushu Univ.*, A4 (1949), 45–106.

[8] P. Mathieu, Zero white noise limit through Dirichlet forms, with application to diffusions in a random environment, *Probab. Theory Related Fields*, 99 (1994), 549–580.

[9] Y. Sinai, The limit behavior of a one-dimensional random walk in a random environment, *Theory Probab. Appl.*, textbf27 (1982), 256–268.

[10] F. Solomon, Random walks in a random environment, *Ann. Probab.*, 3 (1975), 1–31.

[11] H. Takahashi, Recurrence and transience of multi-dimensional diffusion processes in reflected Brownian environments. *Statist. Prob. Lett.*, 69 (2004), 171–174.

[12] H. Tanaka, Limit distributions for one-dimensional diffusion process in self-similar random environments, in “Hydrodynamic Behavior and Interacting Particle Systems (Minneapolis, Minn., 1986)” (ed. G. Papanicolaou), IMA Vol. Math. Appl., 9, Springer, (1987), 189–210.

[13] H. Tanaka, Recurrence of a diffusion process in a multi-dimensional Brownian environment, *Proc. Japan Acad. Ser. A Math. Sci.*, 69 (1993), 377–381.

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