RICCI-POSITIVE GEODESIC FLOWS AND POINT-COMPLETION OF STATIC MONOPOLE FIELDS

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Abstract. Let \((\hat{M}, g)\) be a compact, oriented Riemannian three-manifold corresponding to the metric point-completion \(M \cup \{P_0\}\) of a manifold \(M\), and let \(\xi\) denote a geodesible Killing unit vector field on \(\hat{M}\) such that the Ricci curvature function \(\text{Ric}_g(\xi) > 0\) everywhere, and is constant outside a compact subset \(K \subset M\). Suppose further that \((E, \nabla, \varphi)\) supply the essential data of a monopole field on \(M\), smooth outside isolated singularities all contained in \(K\). The main theorem of this article provides a sufficient condition for smooth extension of \((E, \nabla, \varphi)\) across \(P_0\), in terms of the Higgs potential \(\Phi\), defined in a punctured neighbourhood of \(P_0\) by

\[\nabla_\xi \Phi - 2i[\varphi, \Phi] = \varphi.\]

The sufficiency condition is expressed by a system of equations on the same neighbourhood, which can be effectively simplified in the case that \(\hat{M}\) is a regular Sasaki manifold, such as the round \(S^3\).

1. Introduction

In its formal conception, every monopole on a three-dimensional manifold \(M\) is comprised of a smooth complex vector bundle \(E\) over \(M\) (or a subset of \(M\), if singularities are present) with structure group \(\mathbb{SU}(n)\), a connection \(\nabla\), and an endomorphism \(\varphi\) (the Higgs field) with coefficients in the corresponding Lie Algebra. A Riemannian metric \(g\) and associated volume form are also needed in order to define a Hodge-star operator on differential forms. These structures are then related via the Bogomolny equation

\[2 \ast \nabla \varphi = F_\nabla,
\]

where \(F_\nabla\) denotes the curvature two-form. On one hand, if the metric completion \(\hat{M}\) of \(M\) is compact, it is known that a non-trivial, non-singular solution of this equation cannot exist on \(\hat{M}\). On the other, if \(\hat{M}\) is not compact, non-singular solutions may exist provided certain asymptotic conditions are met at infinity. When \(\hat{M}\) corresponds to Euclidean or Hyperbolic three-space, on which the bundle structure \(E\) is effectively trivial if singularities are absent, these conditions were precisely formulated as part of a theory of moduli of solutions of the Bogomolny equation initially developed by Atiyah [3], Hitchin [8], Ward [11] and their collaborators.

By contrast, the concern of the present note will be with monopoles over three-manifolds \(M\) having a single-point compact Riemannian metric completion \(\hat{M} = M \cup \{P_0\}\). It will be assumed that the monopole field has singularities at isolated
points \( p_i \) all contained within a compact subset \( K \subset M \). Moreover, following \([5]\) and much of the tradition concerning singular monopoles, these may be assumed to be of Dirac type, which entails that in a neighbourhood \( B_\varepsilon(p_i) \) of each singularity, the bundle \( E \mid_{B_\varepsilon(p_i)\setminus\{p_i\}} \) splits as a direct sum of non-trivial complex line bundles. Setting aside their precise nature, the specific problem addressed here will be to formulate a sufficient condition for smooth extension of \( \varphi \) and \( \nabla \) from \( M \setminus K \) to \( \hat{M} \setminus K \), thereby making a removable singularity of \( P_0 \). In particular, such a condition must first imply the existence of a smooth trivialization of the bundle \( E \) in a punctured neighbourhood of \( P_0 \). Several years ago this problem was tentatively addressed by one of the present authors \([7]\) in the context of Euclidean monopoles, but our renewed interest has been inspired by recent work of Biswas and Hurtubise \([5]\), who have extended the investigation of monopoles to Sasakian three-folds. It is well-known that an anti self-dual structure on the pullback \( \pi^*E \) to the product \( M \times (0, \infty) \) is naturally supported by a Sasakian structure on \( \hat{M} \). As a result \( \pi^*E \) inherits a fully integrable holomorphic structure, while \( \hat{M} \times (0, \infty) \) becomes a Kähler surface \( X \). A central theme of \([5]\) is to develop the machinery of the Kobayashi-Hitchin correspondence for monopoles in the regular Sasakian setting by first compactifying the fibres of \( X \) and working instead with the properties of an available Gauduchon metric. The goal of the present article, however, is simply to use local tools of complex analysis on the Kähler surface \( X \) in order to study removable three-dimensional point singularities. In fact, for present purposes, the strict Sasakian property is only required in a neighbourhood of \( P_0 \).

In \([9]\) it was shown that if \((\hat{M}, g)\) is a compact, oriented Riemannian three-manifold, with a geodesible unit vector field \( \xi \) along which the Ricci tensor is strictly positive, i.e., \( \text{Ric}(\xi) > 0 \), then the dual one-form \( \vartheta \) corresponding to the contraction \( \iota_\xi g \) defines a contact structure on \( \hat{M} \), with \( \xi \) as its Reeb vector field. If \( \xi^\perp \) denotes the distribution corresponding to \( \ker(\vartheta) \), then a natural endomorphism \( j \) of \( \xi^\perp \) is defined, such that for any \( v \in \xi^\perp_p \) the set \( \{ v, jv, \xi \} \) forms a positively oriented orthonormal frame of \( T_p\hat{M} \). As will be discussed further in the next section (cf. Proposition 1), \( X \) must of course take on the structure of a Riemannian cone with respect to the metric tensor

\[
\bar{g} = t^2 g + \frac{1}{4f} dt \otimes dt , \quad \text{for } t \in (0, \infty) , \quad f := \text{Ric}(\xi) ,
\]

such that the Sasakian property is contingent on the identification

\[
d\vartheta = 2\sqrt{f} \ i_j g .
\]

This entails two basic hypotheses: (i) that the Reeb vector field \( \xi \) is Killing, hence its flow preserves \( j \) as well as the volume form, as discussed in \([9]\), and (ii) that the coefficient determined by \( f \) is a constant \( c \), at least in a neighbourhood of \( P_0 \), where the Kähler structure is needed. It is worth mentioning at this point that if one were instead to consider a Lorentzian metric

\[
\bar{g} = t^2 g - \frac{1}{4f} dt \otimes dt ,
\]
as in recent work of Aazami [4], where the Newman-Penrose formalism has been applied with interesting effect to methods of symplectic geometry, then the Kähler structure may be recovered from a simple reversal of orientation on \( \tilde{M} \), effectively replacing \( j \) by \(-j\).

The presence of a Kähler structure on \( \pi^{-1}(\tilde{M}\setminus K) \) enables the standard presentation of the Bogomolny equation as a time-independent reduction of the anti self-dual property of the connection

\[
\nabla' := d + \pi^* A + \frac{1}{t \sqrt{c}} \pi^* \varphi \otimes dt
\]

defined on \( \pi^* E \), and consequently induces a holomorphic structure on \( \pi^* E \). The first requirement for a removable singularity at \( P_0 \) is to obtain a trivialization of \( E|_{B_r(P_0) \setminus \{P_0\}} \). In earlier work [7] this was shown to follow from the existence of a relative holomorphic connection on \( \pi^* E \), by means of a Hartogs technique for holomorphically extending sections across \( \pi^{-1}(P_0) \). Existence of a relative holomorphic connection is itself obstructed by complex-analytic cohomology [2], expressible here in terms of solvability of a Cauchy-Riemann equation of the form

\[
\bar{\partial} \Psi = \iota_Z F_{\nabla'},
\]

where \( Z \) denotes a holomorphic vector field transverse to \( \pi^{-1}(P_0) \) (cf. section four below). Given the three-dimensional setting of our problem, however, it will be appropriate to formulate a sufficient condition for solution of this equation in terms of a \( t \)-independent reduction. In particular, a solution \( \Psi \) should be of the form \( \pi^* \psi \) (cf. Proposition 2). Existence of \( \psi \) can be made contingent on \( \varphi \), by first defining the Higgs potential \( \Phi \) such that

\[
\nabla_{\xi} \Phi - 2i[\varphi, \Phi] = \varphi
\]

in a punctured neighbourhood of \( P_0 \), and then setting \( \psi := -2i \nabla_Z \Phi \) (cf. section three). A sufficient condition for solution of the Cauchy-Riemann equation above can then be formulated in terms of the equations

\[
\nabla_Z(\nabla_Z \Phi) + \nabla_{\xi} \varphi = 0,
\]

\[
\nabla_{[Z, \xi]} \Phi - i[\nabla_Z \varphi, \Phi] = 0,
\]

(Proposition 3). We note in particular that if \( (\tilde{M}, g) \) corresponds to a regular Sasaki manifold, in which the Reeb flow endows \( \tilde{M} \) with the structure of an \( S^1 \)-principal bundle over a compact Riemann surface, then the Lie bracket \([Z, \xi] = 0\) (cf. [5]), and the equations of Proposition 3 are correspondingly simplified. It remains in section four to carry out the Hartogs extension of \( \pi^* E \) and thereby extend \( E \) smoothly across \( P_0 \), before proceeding in section five to consider extension of \( \varphi \) and \( \nabla \).

**Theorem:** (cf. Section five) Let \( (M, g) \) be a compact, oriented Riemannian three-manifold corresponding to the point-completion of a manifold \( M \), and let \( \xi \) denote a geodesible Killing unit vector field on \( M \) such that the Ricci curvature function \( \text{Ric}_g(\xi) > 0 \) everywhere, and is constant outside a compact subset \( K \subset M \). Suppose further that \( (E, \nabla, \varphi) \) supply the essential data of a monopole field on \( M \), smooth
outside isolated singularities all contained in $K$, and that the following hold in the complement of $K$:

(i) the associated Higgs potential $\Phi$ satisfies the equations

$$\nabla_{\bar{Z}}(\nabla_{Z}\Phi) + \nabla_{\xi}\varphi = 0,$$

and

$$\lambda\nabla_{Z}\Phi + [\nabla_{Z}\varphi,\Phi] = 0$$

where $Z = \sigma - ij\sigma$ with respect to an orthonormal framing $\{\sigma, j\sigma, \xi\} \in C^\infty(\hat{M} \setminus K, T\hat{M})$, and a real-valued function $\lambda = \frac{1}{4}g([\sigma, \xi], j\sigma)$,

(ii) the Higgs field $\varphi$ and monopole connection $\nabla$ are both uniformly $C^{1,\alpha}$ with respect to admissible $C^{1,\alpha}$-framings of $E$.

Then the bundle $E$ extends smoothly across $P_0$, and admits $C^1$-extensions of both $\varphi$ and $\nabla$, together with a continuous extension of $F_{\nabla}$.

**Corollary:** When $\hat{M} = S^3$, equipped with the round metric, and the flow of $\xi$ induces the Hopf fibration, then the simplified condition

$$(i) \quad \nabla_{\bar{Z}}(\nabla_{Z}\Phi) + \nabla_{\xi}\varphi = [\nabla_{Z}\varphi,\Phi] = 0,$$

together with condition (ii), is sufficient for extension of $E, \varphi, \nabla$ and $F_{\nabla}$ as above.

2. **Monopole fields and three-dimensional contact geometry**

Let $\hat{M}$ be a smooth, orientable three-manifold corresponding to the point compactification of a manifold $M$, i.e., $\hat{M} = M \cup \{P_0\}$. We will assume that $\hat{M}$ is equipped with a Riemannian metric $g$ and a geodesible unit vector field $\xi \in C^\infty(\hat{M}, T\hat{M})$. Let $\vartheta \in C^\infty(\hat{M}, T^*\hat{M})$ be the metric-dual of $\xi$, and $D$ the Levi-Civita connection associated with $g$. We note that the geodesibility requirement $D_\xi\xi = 0$ is easily seen to be equivalent to the condition $\iota_\xi d\vartheta = 0$. $\vartheta$ will then correspond to a contact form on $\hat{M}$ if $d\vartheta|_{\xi^\perp}$ is non-degenerate. A criterion of non-degeneracy was shown in [9], Lemma 1, to follow from the assumption $\text{Ric}(\xi) > 0$ on $\hat{M}$, where the function $\text{Ric}: S\hat{M} \to \mathbb{R}$ is naturally induced on the unit-sphere bundle by the Ricci tensor.

Under this assumption we may then identify $\xi$ with the Reeb vector field of $\vartheta$. In addition, for any $(p, v) \in S\hat{M}$, there is a unique linear map $j_{(p,v)}: v^\perp \to v^\perp \subset T\hat{M}$ such that $j_{(p,v)}^2 = -\text{id}|_{v^\perp}$, and for any unit vector $u \in v^\perp$, the triple $\{u, ju, v\}$ is a positively-oriented orthonormal basis for $T_p\hat{M}$. Given an orientation of $\hat{M}$ this in turn specifies a canonical pseudohermitian structure $j : \xi^\perp \to \xi^\perp$, a feature which arises uniquely in dimension three.

Now consider the product $\pi : N := \hat{M} \times (0, \infty) \to \hat{M}$, endowed with the structure of a Riemannian cone via the extended metric

$$\bar{g} = t^2 g + \frac{1}{4f}dt \otimes dt \text{ for } t \in (0, \infty), \quad f := \text{Ric}(\xi).$$
In a neighbourhood $U$ of any point $p \in \hat{M}$ we may choose a unit vector field $\sigma \in C^\infty(U, \xi^\perp)$ and define an orthonormal framing \( \{ \frac{1}{t}\sigma, \frac{1}{t}j\sigma, \frac{1}{t}\xi, 2\sqrt{f} \frac{\partial}{\partial t} \} \) for $T^*N$ over $\pi^{-1}(U)$. At the same time, we may extend $\pi^*j$ to an endomorphism $J : TN \to TN$ such that

\[
J\left(\frac{1}{t}\xi\right) = 2\sqrt{f} \frac{\partial}{\partial t} \quad ; \quad J\left(\frac{\partial}{\partial t}\right) = \frac{-1}{2t\sqrt{f}}\xi,
\]

with $J|_{\pi^*\xi^\perp} = \pi^*j$. $T^{1,0}N$ and $T^{0,1}N$ then correspond to the $\pm i$-eigenspaces of the $\mathbb{C}$-linear extension of $J$, endowing $N$ with an almost-complex structure in the usual way. Integrability (i.e., Frobenius-involutivity) of this structure is then well-known to be equivalent to the assumption $L_\xi j = 0$ on $\hat{M}$, i.e., for any $\sigma \in C^\infty(U, \xi^\perp)$, the Lie bracket satisfies

\[
[C, j\sigma] = j[C, \sigma].
\]

In smoothly preserving $j$, we say that the Reeb flow is \textit{conformal} \([9]\). Let the complex surface so obtained from $N$ be denoted $X$, and let $\Omega$ denote the natural hermitian form on the holomorphic tangent bundle, $TX$, corresponding to $\mathbb{C}$-linear extension of the contraction of $\bar{g}$ with $J$. More specifically,

\[
\Omega = \iota_J \bar{g} = t^2\pi^*\omega + \frac{t}{\sqrt{f}} \delta \wedge dt \quad (1),
\]

where $\omega|_{\xi^\perp} := \iota_J g$, $\iota_\xi \omega := 0$. Using the basic properties of the Levi-Civita connection on $T\hat{M}$, we note that for $u, v \in \xi^\perp$

\[
d\delta(u, v) = u(\delta(v)) - v(\delta(u)) - \delta([u, v])
\]

\[
= u(g(\xi, v)) - v(g(\xi, u)) - g(\xi, [u, v])
\]

\[
= g(D_u\xi, v) - g(D_v\xi, u) + g(\xi, Tor(u, v))
\]

\[
= (\iota_{\beta - \beta^*} g)(u, v),
\]

where $\beta : \xi^\perp \to \xi^\perp$ is the linear operator defined by $\beta(v) = D_v\xi$. We now recall from \([9]\), Lemma 3, the identity

\[
-2Ric(\xi) = \text{trace}(\beta^2) + \xi(\text{trace}(\beta)).
\]

By \([9]\), Proposition 2, the integrability condition $L_\xi j = 0$ is equivalent to $j\beta = \beta j$. If in addition it is assumed that the (geodesible) flow of $\xi$ is volume-preserving, i.e., $\text{div}(\xi) = \text{trace}(\beta) = 0$, then we have

\[
\beta = \sqrt{f} \cdot j, \quad \text{hence} \quad \beta - \beta^* = 2\sqrt{f} \cdot j,
\]

and so

\[
d\delta|_{\xi^\perp} = 2\sqrt{f} \cdot (\iota_j g) = 2\sqrt{f} \cdot (\omega|_{\xi^\perp}).
\]

Since both of these forms vanish under contraction with $\xi$, we may make the identification

\[
d\delta = 2\sqrt{f} \cdot \omega \quad \text{on} \quad T\hat{M}.
\]

We remark that the assumption that the flow of $\xi$ is both conformal and volume-preserving is equivalent to $\xi$ being a Killing vector field, via the equation

\[
L_\xi g|_{\xi^\perp} = (\text{div}(\xi))g
\]
derived in [9], Proposition 2. Finally, suppose there is a compact subset \( K \subset M \) outside which the function \( f = \text{Ric}(\xi) \) is equal to a positive constant \( c \). Returning to equation (1) above, we note that a simple comparison of types in this case implies that \( d\Omega |_{\pi^{-1}(\hat{M} \setminus K)} = 0 \) if and only if \( 2\sqrt{c} \cdot \omega = d\vartheta \). It follows that \( \Omega \) corresponds to a Kähler form on \( \pi^{-1}(\hat{M} \setminus K) \subseteq X \), i.e., \( \hat{M} \setminus K \) is a Sasakian three-manifold (cf. e.g., [5]).

We summarise with the following

**Proposition 1.** Let \( \xi \) be a geodesible Killing unit vector field on the compact Riemannian three-manifold \( \hat{M} \), such that \( \text{Ric}(\xi) > 0 \) and is constant outside a compact subset \( K \) of \( M \). Then \( \hat{M} \setminus K \) is Sasakian.

We now introduce a smooth complex vector bundle \( E \rightarrow M \), equipped with a connection
\[
\nabla : C^\infty(E) \rightarrow C^\infty(E \otimes T^*M) ,
\]
and consider the pullback \( \pi^*(E) \rightarrow N \setminus \pi^{-1}(P_0) \). A volume form on \( N \), compatible with both the Riemannian cone-structure and the Kähler structure on \( \pi^{-1}(\hat{M} \setminus K) \subseteq X \), is naturally defined by
\[
d\text{Vol}_{\hat{g}} := \frac{1}{2} \Omega \wedge \Omega ,
\]
noting that the induced volume-form/orientation on \( \hat{M} \), corresponding to the slice \( \{ t = 1 \} \), will then be
\[
t_{2\sqrt{c} \cdot \vartheta} d\text{Vol}_{\hat{g}} = \frac{1}{\sqrt{c}} d\vartheta \wedge \vartheta .
\]
This entails a relative sign-change between the action of the Hodge star-operator \( \star' \) on \( \bigwedge^2 T^*N |_{\pi^{-1}(\hat{M} \setminus K)} \) and the corresponding operator \( \star \) on \( \bigwedge^2 T^*\hat{M} \). In particular, if \( F_{\nabla} \) denotes the curvature 2-form associated with \((E, \nabla)\), then we have
\[
\star'(\pi^*F_{\nabla}) = \frac{-1}{2t\sqrt{c}} (\star\pi^*F_{\nabla}) \wedge dt \quad (2).
\]

Choose a globally defined endomorphism \( \varphi \in C^\infty(M, E \otimes E^*) \), and with it a natural extension of \( \nabla \) acting on sections of the pullback \( \pi^*(E) \big|_{\pi^{-1}(\hat{M} \setminus K)} \), i.e.,
\[
\nabla' := d + \pi^*(A_\nabla) + \frac{1}{t\sqrt{c}} \pi^*(\varphi) \otimes dt ,
\]
where \( A_\nabla \) denotes the connection matrix of \( \nabla \) with respect to an arbitrary smooth framing of \( E \). Consequently, we have
\[
F_{\nabla'} = \pi^*F_{\nabla} + \frac{1}{t\sqrt{c}} \nabla \varphi \wedge dt ,
\]
so that the identification
\[
F_{\nabla'} = \pi^*F_{\nabla} + \frac{1}{2t\sqrt{c}} (\star F_{\nabla}) \wedge dt = \pi^*F_{\nabla} - \star'(\pi^*F_{\nabla})
\]
follows automatically from the condition
\[ \nabla \varphi = \frac{1}{2} \ast F_\nabla \quad (3). \]

Equation (3) is of course the monopole equation for the ensemble \((E, \nabla, \varphi)\) over \(M\) \([10]\), from which we recover the well-known \(*\text{-anti self-duality condition for the curvature} 2\text{-form} F_\nabla, \text{relative to} (\pi^* (E), \nabla') |_{\pi^{-1}(M \setminus K)}, \text{and hence a holomorphic structure on the bundle} \pi^* (E) |_{\pi^{-1}(M \setminus K)} \) \([1]\).

The central problem of this discussion is to provide a sufficient condition for smooth completion of the monopole data \((E, \nabla, \varphi)\), defined initially on \(M\), over the compact Riemannian manifold \(\hat{M}\) with unit Killing vector field \(\xi\) as described in Proposition 1. Our approach will be to formulate the sufficiency in terms of existence of a holomorphic connection
\[ \mathcal{D} : \mathcal{O}(\pi^* (E) |_{\pi^{-1}(M \setminus K)}) \rightarrow \Omega^1_X \otimes \mathcal{O}(\pi^* (E) |_{\pi^{-1}(M \setminus K)}) \].

A “Hartogs method” for extension of holomorphic vector bundles across gap-loci of sufficiently large codimension, via holomorphic parallel transport of frames, was introduced in \([6]\), and adapted to completion of monopole fields at point singularities in \(\mathbb{R}^3\) in \([7]\). Existence of a holomorphic connection is well-known to be obstructed by analytic cohomology with coefficients in the sheaf \(\Omega^1_X \otimes \mathcal{O}(\text{End} (\pi^* (E) |_{\pi^{-1}(M \setminus K)}))\) \([2]\). As we shall see, for the purposes of parallel transport a relative holomorphic connection is all that is required, for which the obstruction may be formulated in terms of a Cauchy-Riemann equation of the form
\[ \bar{\partial} \Psi = i_Z F_{\nabla'}, \quad \text{for} \ \Psi \in C^\infty (\pi^{-1}(M \setminus K), \text{End} (\pi^* (E) |_{\pi^{-1}(M \setminus K)})) \],
and where \(Z\) denotes a holomorphic vector field on \(X\). In order to formulate the sufficiency expressly in terms of the three-dimensional structure \((E, \nabla, \varphi)\), we will assume moreover that \(\Psi = \pi^* \psi\) for some \(\psi \in C^\infty (M \setminus K, E \otimes E^*)\), and similarly that \(Z\) corresponds to the trivial lifting to \(\pi^{-1}(M \setminus K)\) of a smooth section of \(\mathbb{C} \otimes \xi^\perp\), i.e., we will define \(Z := \sigma - i J \sigma\). \(Z\) consequently lies in \(T^{1,0} N\) at each point of its definition, but must also be annihilated by the Cauchy-Riemann operator of the holomorphic tangent bundle \(TX\). If \(D'\) denotes the Levi-Civita connection associated with \(\bar{g}\) on \(TN\), we may identify this Cauchy-Riemann operator as \(\bar{\partial} = D' + i J D'\), noting the important general fact that because the complex structure is also Kähler, we have \(D' J = 0\). From this it follows easily that \(\bar{\partial} Z = 0\). Moreover, under these assumptions the Cauchy-Riemann equation for the obstruction \(i_Z F_{\nabla'}\) may be reduced to three dimensions as follows. A \(\bar{\partial}\)-operator for the holomorphic vector bundle corresponding to \(\pi^* (E) |_{\pi^{-1}(M \setminus K)}\) may similarly be defined in relation to \(\nabla'\) by \(\bar{\partial}_E := \nabla' + i J \nabla'\), so that we may write
\[ \bar{\partial}_E \pi^* \psi = \bar{\partial}_E \pi^* \psi + (\nabla \xi - 2 i \pi^* \varphi) \pi^* \psi \otimes \vartheta + \frac{1}{\sqrt{c}} (\pi^* \varphi + \frac{i}{2} \nabla \xi) \pi^* \psi \otimes dt \quad (4), \]
where \(\bar{\partial}_E = \nabla^\perp + i J \nabla^\perp\) denotes the natural restriction of the operator to \(\mathbb{C} \otimes \pi^* \xi^\perp\).

Now, by a simple rearrangement of terms,
\[ \bar{\partial}_E \pi^* \psi = \nabla_{\sigma + i j \sigma} \psi \otimes (\sigma^* - i (j \sigma)^*) = \nabla_{Z} \psi \otimes (\sigma^* - i (j \sigma)^*), \]
where, for convenience, we will continue to use \( Z \) to denote \( \sigma - iJ\sigma \) when considered simply as a section of \( \mathbb{C} \otimes T\hat{M} \). Relative to the same local framing of \( T^*\hat{M} \) we now write
\[
F_{\nabla} = F_{1,2}\sigma^* \wedge (j\sigma)^* + F_{1,3}\sigma^* \wedge \vartheta + F_{2,3}(j\sigma)^* \wedge \vartheta ,
\]
so that
\[
(\iota_Z F_{\nabla})^\perp = (\iota_Z \pi^* F_{\nabla})^\perp = iF_{1,2}(\sigma^* - i(j\sigma)^*) \text{ i.e.,}
\]
\[
\nabla Z \psi = iF_{1,2} .
\]
From the monopole equation we obtain, on the other hand,
\[
\nabla \varphi = \frac{1}{2}(F_{1,2}\vartheta - F_{1,3}(j\sigma)^* + F_{2,3}\sigma^*) ,
\]
hence \( \nabla \xi \varphi = \frac{1}{2}F_{1,2} \). In conclusion,
\[
\bar{\partial}^\perp E \pi^* \psi = (\iota_Z F_{\nabla})^\perp \text{ if and only if } \nabla Z \psi = 2i\nabla \xi \varphi .
\]
Turning now to the transversal terms of equation (4), we recall that
\[
F_{\nabla'} = \pi^* F_{\nabla} + \frac{1}{t\sqrt{c}}(\nabla \varphi) \wedge dt ,
\]
hence
\[
(\iota_Z F_{\nabla'})_\vartheta = (\iota_Z F_{\nabla})_\vartheta = (F_{1,3} - iF_{2,3}) = -2i\nabla Z \varphi .
\]
We consequently derive a pair of identifications
\[
(\nabla \xi - 2i\varphi)\psi = -2i\nabla Z \varphi \quad \text{and}
\]
\[
\frac{1}{t\sqrt{c}}(\varphi + \frac{i}{2}\nabla \xi)\psi = \frac{1}{t\sqrt{c}}\nabla Z \varphi ,
\]
which are clearly the same equation. In conclusion,

**Proposition 2.** The equation
\[
\bar{\partial}^\perp E \pi^* \psi = \iota_Z F_{\nabla'}
\]
is equivalent to the three-dimensional coupled system
\[
\nabla Z \psi = 2i\nabla \xi \varphi \quad ; \quad (\nabla \xi - 2i\varphi)\psi = -2i\nabla Z \varphi .
\]

Letting \( \phi := -2i\varphi \) and \( \hat{\nabla} := \nabla + \phi \otimes \vartheta \), we may write this more concisely in the form
\[
\hat{\nabla} Z \psi = -\hat{\nabla} \xi \phi
\]
\[
\hat{\nabla} \xi \psi = \hat{\nabla} Z \phi .
\]

We will take a closer look at this system in the next section.
3. Higgs Potential of the coupled system

The tensor $\varphi$ which comes as part the monopole data $(E, \nabla, \varphi)$ on the three-manifold $M$, commonly referred to as the Higgs Field, will be said to admit a Higgs Potential relative to the flow of a vector field $\eta$ if the equation

$$\hat{\nabla}_\eta \Phi = \varphi$$

is solvable on $M$. If all data are smooth, and $\eta$ is non-vanishing, then local existence of a solution $\Phi$ reduces to an elementary ODE problem. More specifically, given $p \in M$ and a sufficiently small neighbourhood $U_p$ in which the flow of $\eta$ is everywhere transversal to a disc $\Delta(p)$ centred at $p$, we may first take a smooth framing of $E|_{\Delta(p)}$, which supplies the initial data for parallel transport along the flow, and hence for a smooth framing of $E|_{U_p}$ in which our equation is uniquely solvable, after specifying, e.g., the condition

$$\Phi|_{\Delta(p)} \equiv 0.$$ 

Diffeomorphic transport of the disc $\Delta(p)$ along the flow of $\eta$ moreover allows smooth continuation of $\Phi$ as far as possible throughout a chain of such neighbourhoods. Now consider the unit Killing field $\xi$ of the previous section, and let us examine the problem when the data $(E, \nabla, \varphi)$ are restricted to $M \setminus K$, which we may effectively identify with a punctured neighbourhood of $P_0 \in \hat{M}$, for some compact subset $K$ of $M$. Again, we take $\Delta(P_0)$ to be a disc transversal to the flow of $\xi$ inside the complete neighbourhood $U_{P_0}$ in $\hat{M}$, which we may assume to be diffeomorphic to a product $\Delta(P_0) \times (-\varepsilon, \varepsilon)$. It is then straightforward to derive the existence of a solution to the equation

$$\hat{\nabla}_\xi \Phi = \varphi$$

inside $U_{P_0} \setminus \overline{B_\delta(P_0)}$, such that

$$\Phi|_{\Delta(P_0)} \equiv 0$$

for $B_\delta(P_0)$ a geodesic ball of arbitrarily small radius $\delta > 0$ centred at $P_0$. If we now define

$$\psi := -2i\hat{\nabla}_Z \Phi,$$

we have a solution of the coupled system of Proposition 2, in $U_{P_0} \setminus \{P_0\} = M \setminus K$, provided

$$\hat{\nabla}_Z(\hat{\nabla}_Z \Phi) + \hat{\nabla}_\xi(\hat{\nabla}_\xi \Phi) = 0,$$

and

$$\hat{\nabla}_Z(\hat{\nabla}_\xi \Phi) = \hat{\nabla}_\xi(\hat{\nabla}_Z \Phi).$$

Via a standard expansion of second-order covariant differentiation of endomorphisms, we note that

$$\hat{\nabla}_Z(\hat{\nabla}_\xi \Phi) - \hat{\nabla}_\xi(\hat{\nabla}_Z \Phi) = \hat{\nabla}_{[Z,\xi]} \Phi + [\iota_Z \iota_\xi F_{\hat{\nabla}} \Phi, \Phi]$$

(5),

where the bracket $[Z, \xi]$ denotes the $\mathbb{C}$-linear extension of the Lie bracket of vector fields, while $[\iota_Z \iota_\xi F_{\hat{\nabla}} \Phi, \Phi]$ corresponds to the formal commutator of matrices. If we now recall that

$$\hat{\nabla} = \nabla + \phi \otimes \partial$$
it is a direct calculation to show that
\[ \iota_Z t_\xi F_\nabla = \iota_Z t_\xi F_\nabla - \nabla_Z \phi. \]

Now \( F_\nabla = 2 \ast \nabla \varphi \) implies
\[
\iota_Z t_\xi F_\nabla = 2 \iota_Z t_\xi ((\nabla_\sigma \varphi) \otimes (j_\sigma)^* \wedge \vartheta - (\nabla j_\sigma \varphi) \otimes \sigma^* \wedge (j_\sigma)^*) = -2(-i \nabla_\sigma \varphi - \nabla j_\sigma \varphi) = 2i \nabla_Z \varphi = -\nabla_Z \phi,
\]
and consequently \( \iota_Z t_\xi F_\nabla = -2 \nabla_Z \phi. \)

Recalling now the natural properties of the Levi-Civita connection \( D \) on \( TM \),
together with the geodesibility of the Reeb vector field \( \xi \), and the assumption \( D_\sigma \xi = \beta(\sigma) = \sqrt{c} \cdot j_\sigma \) on \( M \setminus K \), we see that
\[
g(\sigma, [\xi, \xi], \xi) = -\varphi(\sigma, \xi, \xi) = -\sqrt{c} \cdot g(j_\sigma, \xi) = 0,
\]
and
\[
g(\sigma, [\xi, \xi], \sigma) = g(D_\sigma \xi, \xi) - g(D_\xi \sigma, \sigma),
\]
hence \( g(\sigma, [\xi, \xi], \sigma) = 0. \) This leaves us to conclude that \( [\sigma, \xi] = \lambda j_\sigma \) for some \( \mathbb{R} \)-valued function \( \lambda = g([\sigma, \xi], j_\sigma), \) and, with the additional integrability condition \( [\xi, j_\sigma] = j_\xi [\xi, \sigma], \) that \( [Z, \xi] = i \lambda Z. \) The commutator of covariant derivatives (5) may thus be rewritten
\[
\nabla_Z (\nabla_\xi \Phi) - \nabla_\xi (\nabla_Z \Phi) = i \lambda \nabla_Z \Phi + 4i [\nabla_Z \varphi, \Phi].
\]

A sufficient condition for solvability of the coupled equation of Proposition 2 may now be stated as follows.

Proposition 3. If the Higgs Potential \( \Phi \) relative to the Reeb vector field \( \xi \) above satisfies the equations
\[
\nabla_Z (\nabla_Z \Phi) + \nabla_\xi \varphi = 0,
\]
and
\[
\lambda \nabla_Z \Phi + [\nabla_Z \varphi, \Phi] = 0
\]
for a unit vector field \( \sigma \in C^\infty(M \setminus K, \xi^\perp) \) and a real-valued function \( \lambda = \frac{1}{2} g([\sigma, \xi], j_\sigma), \) then \( \psi := -2i \nabla_Z \Phi \) satisfies the coupled system
\[
\nabla_Z \psi = -\nabla_\xi \phi
\]
\[
\nabla_\xi \psi = \nabla_Z \phi.
\]

Remark: Following [5], we note that for the special case of \( (\hat{M}, g) \) a regular Sasaki manifold, in particular, a compact three-manifold on which the flow of the Reeb vector field \( \xi \) induces the structure of an \( S^1 \)-principal bundle over a compact Riemann surface, it may be assumed that the Lie bracket \( [Z, \xi] = 0, \) and hence \( \lambda = 0 \) in the second equation above.

At this point we recall the motivation for Propositions 2 and 3, namely to solve the Cauchy-Riemann equation \( \bar{\partial}_E \pi^* \psi = \iota_Z F_\nabla \) in \( \pi^{-1}(M \setminus K) \) thereby removing the obstruction to existence of a relative holomorphic connection on \( \pi^* E \big|_{\pi^{-1}(M \setminus K)}. \)
The role of holomorphic connections in defining unique extension of the holomorphic bundle \( \pi^*E \) to \( \pi^{-1}(\hat{M} \setminus K) \), and hence extension of \( E \) to \( \hat{M} \), will be recalled in the next section.

4. A Hartogs technique for point-completion of \( E \)

Let us begin by once again identifying \( M \setminus K \) with a punctured neighbourhood \( U \setminus \{P_0\} \) in \( \hat{M} \). Note that in a neighbourhood \( U' \) of \( P_0 \in \pi^{-1}(U) \) we are in a position to choose holomorphic coordinates \( (z,w) \) such that the \( t \)-axis, corresponding to the real line \( \pi^{-1}(P_0) \), lies wholly in the complex line defined in \( U' \) by \( z = 0 \), and of course contains the point \( P_0 = (0,0) \). Consider a sufficiently small bi-disc \( \Delta_\varepsilon \times \Delta_\varepsilon \) centred at the origin of this local holomorphic coordinate neighbourhood, where \( \varepsilon \) denotes the radius in each coordinate, and the domains

\[
V := (\Delta_\varepsilon \setminus \Delta_{\varepsilon^2}) \times \Delta_\varepsilon, \quad \text{and} \quad W := \Delta_\varepsilon \times D,
\]

where \( D \) is an open disc contained in \( \Delta_\varepsilon \setminus \pi^{-1}(P_0) \). From these we construct a Hartogs figure \( H := V \cup W \). Note that on each of the components of \( H \) we may appeal to the Oka-Grauert Principle for existence of a local holomorphic framing of \( \pi^*E \) - let us refer to these as \( f_V \) and \( f_W \) respectively. Moreover, on \( V \cap W \) suppose we have the transition relation \( f_W = \tau \cdot f_V \) with respect to a holomorphic gauge-transformation \( \tau \). Relative to \( f_V \), we denote the connection matrix \( A_{1,0}^{1,0} := \pi^*(A_\sigma - iA_{ij}\sigma) \), while the corresponding component \( A_{0,1}^{1,0} = \pi^*(A_\sigma + iA_{ij}\sigma) \), associated with the Cauchy-Riemann operator on \( \pi^*E \), can be taken to be zero. Consequently,

\[
\iota_Z F_V = \partial_E A_{1,0}^{1,0},
\]

so that the matrix \( A_{1,0}^{1,0} - \pi^*\psi \) is holomorphic. If we now choose \( z_0 \in \Delta_\varepsilon \setminus \Delta_{\varepsilon^2} \), the restriction \( f_V |_{\{z_0\} \times \Delta_\varepsilon} \) will provide the “initial data” for a first-order system of holomorphic differential equations

\[
\frac{\partial f}{\partial z} + (A_{1,0}^{1,0} - \pi^*\psi)f = 0 \quad (6)
\]

in one complex variable. The solution may be extended via uniqueness of analytic continuation around each annular fibre of the product corresponding to \( V \), with the sole obstruction to global continuation governed by a holonomy matrix at each point of \( \{z_0\} \times \Delta_\varepsilon \). A similar holomorphic system may be defined on \( W \), relative to the initial data provided by \( f_W |_{\{z_0\} \times D} \) - an important difference being that a covariantly constant solution \( \tilde{f} \) is unobstructed, given that the fibres of the product corresponding to \( W \) are simply connected.

Let \( \gamma : \Delta_\varepsilon \times [0,1] \to V \) denote a family of loops, holomorphically parametrized by \( w \) and traversing each annular fibre of \( V \), such that \( \gamma(w,0) = \gamma(w,1) = (w,z_0) \). If \( \alpha(w) \) denotes the holonomy matrix associated with analytic continuation of solutions to (6), then we may write \( \tilde{f}(\gamma(w,1)) = \alpha \cdot f(\gamma(w,0)) \) and hence, for \( w \in D \),

\[
\tilde{f}(\gamma(w,1)) = \tau \cdot \alpha \cdot \tau^{-1} \tilde{f}(\gamma(w,0)).
\]
Because the fibres of $W$ are simply connected, $\alpha |\Omega \equiv 1$, but $\alpha$ is holomorphic in $w$, hence we conclude that the holonomy everywhere in $V$ is trivial. The covariantly constant holomorphic framing of $\pi^*E|_V$ so obtained may easily be extended to $U' \setminus \pi^{-1}(P_0)$ using analytic continuation along homotopy equivalent paths, and hence we arrive at a unique holomorphic extension of $\pi^*E$ to $U'$ via the classical Hartogs extension of functions. It follows at once that we have

**Proposition 4.** If the equations of Proposition 3 are satisfied by $\varphi$ and $\Phi$ in $M \setminus K$, then the vector bundle $E$ admits a smooth extension across $P_0$.

5. **Completion of the Higgs field and monopole field**

Once a sufficient condition has been provided for smooth extension of the vector bundle $E$ over $\hat{M}$, a completion of the monopole field tensor corresponding to $F_\varphi$, as well as that of the Higgs field $\varphi$, can be addressed in a relatively straightforward way. Recall that a bounded smooth function $f$ defined on a punctured domain $U \setminus \{P_0\}$ in $\mathbb{R}^n$, for $n \geq 2$, need not admit a continuous extension at $P_0$ (let alone a smooth one). The problem of **boundary regularity** is removed of course if we are prepared to assume that $f$ is uniformly Hölder continuous (or uniformly $C^{k,\alpha}$, if continuous extension of derivatives up to order $k$ is required) on $U \setminus \{P_0\}$. If simple functions are replaced by sections of a vector bundle $E$ defined over a Riemannian manifold $M$, some care needs to be taken in defining analogous notions of fractional differentiability for local frames of the bundle. Let $\Phi$ be such a smooth frame, defined over a domain $U \subset M$. For an arbitrarily chosen base point $P \in U$ we note that there is a smooth map $\rho: U \to \text{GL}_{rk(E)}(\mathbb{C})$ such that $\Phi(p) = \rho(p) \cdot \Phi(P)$, where $\rho(P) = \text{id}$. This map will serve as a local *gauge* for $\Phi$ relative to $P$, and we will say that $\Phi$ is

(a) **admissible** if $\inf_{C} |\det(\rho)| > 0$, and

(b) **uniformly Hölder $\alpha$-continuous on $U$** if the associated gauge $\rho$ has this property, i.e., there exists a constant $C > 0$ such that for all $p, q \in \Omega$

$$
\|\rho(p) - \rho(q)\| \leq C \cdot d(p, q)^{\alpha}
$$

for some $0 < \alpha \leq 1$, where $d(p, q)$ denotes the Riemannian metric distance between any two points.

Similarly, we will say that $\Phi$ is uniformly $C^{1,\alpha}$ on $U \subset U'$ if, for every smooth unit vector field $\zeta \in C^\infty(U', S\hat{M})$, the matrix-valued function $d\rho(\zeta)$ is uniformly Hölder $\alpha$-continuous. Now let $\Phi'$ be another admissible smooth frame of $E$ on $\Omega$, hence there exists a gauge transformation $\tau : U \to \text{GL}_{rk(E)}(\mathbb{C})$ such that the gauge representing $\Phi'$, $\rho' = \tau^{-1} \cdot \rho \cdot \tau$. It is not difficult to see that a product of admissible $C^{1,\alpha}$-gauge transformations is again admissible and $C^{1,\alpha}$. Moreover, for $U$ of finite metric diameter, a $C^{1,\alpha}$-image $\tau(U)$ will be a relatively compact domain contained in $\text{GL}_{rk(E)}(\mathbb{C})$, hence the smooth involution $\iota$ acting on the general linear group by inversion will be uniformly Lipschitz when restricted to this image, and $\iota(\tau) = \tau^{-1}$ will again be $C^{1,\alpha}$ on $U$. Such $\tau$ consequently form a subgroup of $C^{1}$-gauge transformations.
on $\mathcal{U}$, and hence the orbit of an admissible $C^{1,\alpha}$-gauge $\rho$ under the action of this subgroup via conjugation preserves the fractional differentiability. Conversely, if $f'$ is assumed to be represented by a $C^{1,\alpha}$-gauge $\rho'$, such that $\inf_{\mathcal{U}} |\det(\rho')| > 0$, it is a similarly elementary consequence that the gauge transformation between these frames is also $C^{1,\alpha}$. As a result, we see that the orbit of a given $C^{1,\alpha}$-frame $f$ is independent of the choice of base-point $P \in \mathcal{U}$, and we may then simply refer to the orbit of admissible $C^{1,\alpha}$-frames of $E$ on this domain.

If $\varphi \in C^{1,\alpha}(\mathcal{U}, E \otimes E^*) = C^{1,\alpha}(\mathcal{U}, \mathfrak{gl}_{rk(E)}(\mathbb{C}))$ denotes the Higgs field of a monopole on $M$ with respect to a given admissible $C^{1,\alpha}$-frame, we may conclude using the same arguments that the action corresponding to the adjoint representation

$$Ad : G \to \mathfrak{gl}_{rk(E)}(\mathfrak{g})$$

(where $G := \mathfrak{gl}_{rk(E)}(\mathbb{C})$ and $\mathfrak{g}$ is the Lie Algebra), when restricted to admissible $C^{1,\alpha}$-gauge transformations on $\mathcal{U}$, induces an orbit of $C^{1,\alpha}$-representatives of $\varphi$. In a similar manner, if $A$ denotes the connection matrix locally representing $\nabla$ in the same local framing, we will say that $\nabla$ is uniformly $C^{1,\alpha}$ on $\mathcal{U}$, and write $A \in C^{1,\alpha}(\mathcal{U}, T^* M \otimes \mathfrak{g})$, if the contractions $dA(\zeta, \eta)$ are uniformly Hölder $\alpha$-continuous for every pair of smooth unit vector fields on $\mathcal{U}$. This is moreover a property that is preserved under $C^{1,\alpha}$-gauge transformations. With respect to the derived adjoint representation

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) ; \quad \psi \mapsto [\psi, \ast],$$

we note in particular that for every pair of smooth unit vector fields $\zeta, \eta$, the curvature

$$F_\nabla(\zeta, \eta) = dA(\zeta, \eta) + [A_\zeta, A_\eta] = dA(\zeta, \eta) + ad(A_\zeta)(A_\eta).$$

Given $\nabla$ uniformly $C^{1,\alpha}$ on $\mathcal{U} \subset \subset M$, $A_\zeta$ and $A_\eta$ are uniformly Lipschitz, hence it is a simple consequence of the triangle inequality, together with the skew-symmetry

$$ad(A_\zeta)(A_\eta) = -ad(A_\eta)(A_\zeta),$$

that

$$\|ad(A_\zeta(p))(A_\eta(p)) - ad(A_\eta(q))(A_\zeta(q))\| \le \|ad(A_\zeta(p))\|\|A_\eta(p) - A_\eta(q)\| + \|ad(A_\eta(q))\|\|A_\zeta(p) - A_\zeta(q)\|. $$

Since the linear operator norms of $ad(A_\zeta)$ and $ad(A_\eta)$ are themselves uniformly bounded, we conclude that the commutators $[A_\zeta, A_\eta]$ are uniformly Lipschitz on $\mathcal{U}$.

When $\mathcal{U}$ corresponds to a punctured neighbourhood $U \setminus \{P_0\} \subseteq \hat{M}$, supporting a frame of $E$ which extends smoothly to all of $U$, it follows directly from the assumption of uniform $C^{1,\alpha}$-regularity for $\varphi$ and $\nabla$ that both admit $C^1$-extensions across $P_0$ (and hence to $\hat{M}$). Moreover, the combination of uniform Hölder $\alpha$-continuity and Lipschitz continuity which these assumptions imply for the curvature, $F_\nabla$, allows continuous extension of this form across $P_0$ as well.

We may now summarize the results of this and the preceding sections in our main conclusion.
Theorem 1. Let $(\hat{M}, g)$ be a compact, oriented Riemannian three-manifold corresponding to the point-completion of a manifold $M$, and let $\xi$ denote a geodesible Killing unit vector field on $\hat{M}$ such that the Ricci curvature function $\text{Ric}_g(\xi) > 0$ everywhere, and is constant outside a compact subset $K \subset M$. Suppose further that $(E, \nabla, \varphi)$ supply the essential data of a monopole field on $M$, smooth outside isolated singularities all contained in $K$, and that the following hold in the complement of $K$:

(i) the associated Higgs potential $\Phi$ satisfies the equations
\[
\nabla_Z(\nabla_Z\Phi) + \nabla_\xi \varphi = 0,
\]
and
\[
\lambda \nabla_Z\Phi + [\nabla_Z\varphi, \Phi] = 0
\]
where $Z = \sigma - i\sigma$ with respect to an orthonormal framing $\{\sigma, j\sigma, \xi\} \in C^\infty(\hat{M} \setminus K, T\hat{M})$, and a real-valued function $\lambda = \frac{1}{4}g([\sigma, \xi], j\sigma),$

(ii) the Higgs field $\varphi$ and monopole connection $\nabla$ are both uniformly $C^{1,\alpha}$ with respect to admissible $C^{1,\alpha}$-framings of $E$.

Then the bundle $E$ extends smoothly across $P_0$, and admits $C^1$-extensions of both $\varphi$ and $\nabla$, together with a continuous extension of $F\nabla$.

Corollary 1. When $\hat{M} = S^3$, equipped with the round metric, and the flow of $\xi$ induces the Hopf fibration, then the simplified condition

(i) $\nabla_Z(\nabla_Z\Phi) + \nabla_\xi \varphi = [\nabla_Z\varphi, \Phi] = 0$, 

together with condition (ii), is sufficient for extension of $E, \varphi, \nabla$ and $F\nabla$ as above.

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