DEFROST: a new code for simulating preheating after inflation

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Abstract. At the end of inflation, dynamical instability can rapidly deposit the energy of homogeneous cold inflaton into excitations of other fields. This process, known as preheating, is rather violent, inhomogeneous and non-linear, and has to be studied numerically. This paper presents a new code for simulating scalar field dynamics in an expanding universe written for that purpose. Compared to available alternatives, it significantly improves both the speed and the accuracy of calculations, and is fully instrumented for 3D visualization. We reproduce previously published results on preheating in simple chaotic inflation models, and further investigate non-linear dynamics of the inflaton decay. Surprisingly, we find that the fields do not ‘want’ to thermalize in quite the way that one would think. Instead of directly reaching equilibrium, the evolution appears to be stuck in a rather simple but quite inhomogeneous state. In particular, a one-point distribution function of total energy density appears to be universal among various two-field preheating models, and is exceedingly well described by a log-normal distribution. It is tempting to attribute this state to scalar field turbulence.

Keywords: cosmological phase transitions, inflation, physics of the early universe

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1. Introduction

The idea of inflation is a cornerstone of the modern theory of the early universe. According to the inflationary paradigm, the universe at early times undergoes a period of rapid (quasi-exponential) expansion, which wipes the initial state of the universe clean while seeding the primordial inhomogeneities with quantum fluctuations generated during expansion [1, 2]. While the universe is inflating, all of its energy sits in the homogeneous scalar field or condensate (known as the inflaton), which is in a vacuum-like state with little entropy and few particle excitations. But eventually inflation ends, and this energy has to be deposited into excitations of other matter fields, starting the thermal history of the universe with a hot big bang.

The decay of the inflaton can be very efficient if the fields experience dynamical instability at the end of inflation; such a stage became known as preheating. In most chaotic inflation models, oscillations of the inflaton field can cause instability via parametric resonance [3]–[7]. Although linear development of this instability can be understood analytically [8, 9], it might be chaotic [10], and one needs to resort to numerical simulations to investigate the non-linear dynamics that soon takes over [11]–[14]. In hybrid inflation models [15], one has, in addition to parametric resonance [16], a tachyonic instability associated with the symmetry breaking [17], the dynamics of which has been explored in [18]–[20].

Non-equilibrium dynamics of preheating can lead to a multitude of interesting phenomena. Some of the topics discussed in the literature are the formation of topological defects [21, 22], the production of various particles (with applications to baryogenesis and leptogenesis) [23]–[26], the possibility of primordial black hole formation [27, 28], the generation of primordial magnetic fields [29, 30], and the production of a stochastic gravitational wave background [31]–[36]. Due to difficulties of dealing with non-linear evolution equations, most of these studies rely on numerical simulations.
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This paper describes a new code for simulating non-linear scalar field dynamics in an expanding universe developed to study preheating, called DEFROST, and the first results obtained with it. There are other codes available for this purpose, most notably LATTICEEASY developed by Gary Felder and Igor Tkachev [37], and its parallel version CLUSTEREASY [38]. Through the use of more advanced algorithms and careful optimization, the new code significantly improves both the accuracy and the performance achievable in simulations of preheating. An important design goal has been ease of visualization and analysis of the results, which is extremely important for understanding the dynamics of complex systems.

We present results on preheating in a simple two-field chaotic inflation model with a massive inflaton decaying into another scalar field via quartic coupling [14]. Through our simulations, a new and somewhat simpler picture of the late stages of preheating emerges. After an initial transient when instability develops, bubbles form and then break up, the matter distribution soon arranges itself in a clumpy state which persists with little change for a long time. The one-point probability distribution function of the total energy density in this state appears to be universally log-normal among various two-field preheating models. It is tempting to attribute this to relativistic turbulence [39]–[41]. We also see some evidence that the structure formed during preheating continues to grow in size on a much longer timescale. To some extent, this picture reminds one of large scale structure formation [43]–[45].

This paper is organized in the following way. In section 2, we introduce scalar field models of preheating, derive equations of motion, and discuss physical approximations that we use. Section 3 describes the detailed implementation of the numerical evolution algorithm. Initial conditions including quantum fluctuations of the fields produced during inflation are discussed in section 4, with particular attention paid to implementation of the Gaussian random field generator. We briefly recount the theory of preheating via a broad parametric resonance at the end of chaotic inflation in section 5, and present our simulations in section 6. We conclude by summarizing our main results in section 7.

2. Equations of motion

As our baseline model of reheating we will take a system of $N$ scalar fields $\{\phi^i\}$, minimally coupled to gravity and interacting through some (non-linear) potential $V(\phi^i)$ as described by the action

$$S = \frac{1}{16\pi G} \int \{ R - g^{\mu\nu} \delta_{ij} \phi^j_{,\mu} \phi^i_{,\nu} - 2V(\phi^i) \} \sqrt{-g} \, d^4x.$$  \hspace{1cm} (1)

The above action can be modified to describe more complicated geometrical quantities (like a vector field, for example) instead of a real scalar multiplet $\{\phi^i\}$. Ultimately what we care about is just field equations of motion and their gravitational effects, not actual gauge symmetries. Variation of the action (1) with respect to the metric gives the Einstein equation with a stress–energy tensor

$$T_{\mu\nu} = \delta_{ij} \phi^i_{,\mu} \phi^j_{,\nu} - \left[ \frac{1}{2} (g^{\alpha\beta} \delta_{ij} \phi^i_{,\alpha} \phi^j_{,\beta}) + V(\phi^i) \right] g_{\mu\nu},$$  \hspace{1cm} (2)

while variation with respect to the field $\phi^i$ gives the equation of motion for the field

$$\Box \phi^i \equiv g^{\mu\nu} \phi^i_{,\mu\nu} = \frac{\partial V}{\partial \phi^i}.$$  \hspace{1cm} (3)
Although in principle it is possible to solve the complete system of Einstein and scalar field equations numerically (see [46]–[49] for the formulation and some approaches), in practice, it is not such an easy thing to do. Einstein equation solvers in 3 + 1 dimensions are complex to implement, very expensive to run, and, despite marked improvement in recent years [49], might still have issues with numerical stability.

Fortunately for us, we do not have to solve the full Einstein equations. Although preheating is a rather violent process, and the stress–energy tensor becomes very inhomogeneous due to non-linear field dynamics, the smallness of the gravitational coupling constant ensures that the gravitational backreaction of these inhomogeneities is rather small (at the $10^{-3}$ level in the simulations presented in this paper). Thus we are going to treat the scalar field evolution as if it was happening in a homogeneous flat Friedmann–Robertson–Walker spacetime

$$ds^2 = -dt^2 + a^2(t) dx^2,$$

and calculate the inhomogeneous gravitational field due to matter distribution with stress–energy tensor (2) using linear perturbation theory [42]. We will ignore the backreaction of metric perturbations on the scalar field evolution.

With these simplifying assumptions, the problem becomes much more tractable: we just need to solve a system of coupled scalar field equations of motion (3), which in spacetime with the metric (4) become

$$\ddot{\phi}^i + 3H \dot{\phi}^i - \frac{\Delta}{a^2} \phi^i + \frac{\partial V}{\partial \phi^i} = 0. \quad (5)$$

Here and later, a dot denotes the derivative with respect to time $t$, and spatial differential operators (gradient $\nabla$ and Laplacian $\Delta$) are taken with respect to comoving coordinates and the three-dimensional flat metric. The expansion rate $H \equiv \dot{a}/a$ and acceleration $\ddot{a}$ are determined by the averaged Einstein equations

$$H^2 = \frac{\langle \rho \rangle}{3}, \quad \frac{\ddot{a}}{a} = -\frac{1}{6} \langle \rho + 3p \rangle, \quad (6)$$

where the energy density $\rho$ and isotropic pressure $p$ are components of the stress–energy tensor (2) given by

$$\rho \equiv -T^t_t = \sum_i \frac{(\dot{\phi}^i)^2}{2} + \sum_i \frac{(\nabla \phi^i)^2}{2a^2} + V(\phi^i), \quad (7)$$

$$p \equiv \frac{1}{3} T^a_a = \frac{1}{3} \sum_i \frac{(\dot{\phi}^i)^2}{2} - \sum_i \frac{(\nabla \phi^i)^2}{6a^2} - V(\phi^i), \quad (8)$$

and averages are taken over the whole simulation volume.

To solve equation (5) numerically, one needs to know the Hubble parameter value $H$. Rather than attempting to resolve the constraint equation (6) at every time step (which would result in an implicit evolution scheme), it is faster and more convenient to use the evolution equation

$$H = -H^2 - \frac{1}{6} \langle \rho + 3p \rangle = -\frac{1}{6} \langle \rho + p \rangle \quad (9)$$

to evaluate its value in the future. This is what is done in LATTICEEASY. However, we have one more trick up our sleeves: a disproportionately huge gain in numerical accuracy
can be realized by evolving the Hubble length \( L \equiv 1/H \) instead of the Hubble parameter \( H \) by using
\[
\dot{L} = -\frac{\dot{H}}{H^2} = 1 + \frac{L^2}{6} (\rho + 3p).
\] (10)

For constant equation of state \( p = w\rho \), the Hubble parameter evolves as \( H \propto 1/t \), while the Hubble length evolves as \( L \propto t \). The latter variable has vanishing second (and higher) derivatives and correspondingly smaller truncation error when discretized to second order. As an added bonus, the spatial gradients (which are the single most expensive thing to calculate) cancel out when taken in a \( \rho + 3p \) combination, and do not enter evolution equation in the form (10).

Once the field equations of motion (5) are solved, we can evaluate all the components of stress–energy tensor (2), in particular energy density (7) and pressure (8), as well as calculating the linear metric perturbations that they create in a homogeneous spacetime (4). In this paper, we will focus on scalar perturbations, the behaviors of which during preheating have not been widely studied yet. In the longitudinal gauge, the perturbed metric can be written as
\[
ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi) dx^2.
\] (11)

The two scalar gravitational potentials \( \Phi \) and \( \Psi \) are equal in the absence of anisotropic stress. This is in general not the case for the scalar field models, and can be expected to hold only approximately and in the average sense. Both gravitational potentials \( \Phi \) and \( \Psi \) are non-dynamical, being solutions of the Poisson-like equations. In particular, the equation for \( \Psi \) is
\[
\frac{\Delta}{a^2} \Psi = \frac{\rho}{2}.
\] (12)

Strictly speaking, what should stand in the right-hand side is not the density \( \rho \), but the gauge-invariant density variable \( \rho_m \), which has some extra terms in it [42]. At this stage of the code development, I will not make that distinction and simply ignore the missing terms (which usually works well on sub-horizon scales), along with anisotropic stress contribution to potential \( \Phi \). Full support of gauge-invariant perturbations including tensor and vector modes is planned for a future code release.

3. PDE solver implementation

Scalar field equations of motion (5) are coupled non-linear partial differential equations, and have to be solved numerically. Fortunately, all the non-linearity comes from the potential term only; the differential operator itself is simple, homogeneous and hyperbolic, which makes the numerical solution quite straightforward. For the solver, we adopt a second-order accurate finite difference scheme based on a leapfrog algorithm.

The scalar field values \( \phi^i \) are discretized on a three-dimensional cubic \( n \times n \times n \) grid in comoving coordinates with uniform spacing \( dx \) and periodic boundary conditions. Since the evolution equation (5) is second order, values of the fields on two consecutive time slices are required to advance to the next one. We will denote the previous, current, and next time slices by indices \( \text{dn}, \text{hr}, \) and \( \text{up} \) correspondingly. The time derivatives of a
quantity $X$ are discretized to second order as

$$
\dot{X} = \frac{X_{up} - X_{dn}}{2\, dt}, \quad \ddot{X} = \frac{X_{up} - 2X_{hr} + X_{dn}}{(dt)^2}.
$$

(13)

Discretization of the spatial differential operators

$$
\Delta X = \frac{D[X]}{(dx)^2}, \quad (\nabla X)^2 = \frac{G[X]}{(dx)^2}
$$

(14)

allows more freedom. Direct generalization of the second-derivative discretization in equation (13) to three spatial dimensions leads to the often-used second-order accurate expression for the Laplacian using six nearest neighbors of a point. However, this is not the only (or the best) choice. The truncation error for this scheme depends on the direction, introducing anisotropic artifacts in the field evolution at short length scales. One can mitigate this unpleasant feature by increasing resolution, but a smarter discretization scheme is a better solution. By using all 26 neighbors in a $3 \times 3 \times 3$ cube around a point, one can derive a family of discretizations of the Laplacian operator which are second-order accurate and fourth-order isotropic [50]. For discretization to be isotropic, the coefficients in a linear combination $D[X]$ approximating the Laplacian operator should only depend on the distance from the central point:

$$
D[X] \equiv \sum_{x=-1}^{x=1} \sum_{y=-1}^{y=1} \sum_{z=-1}^{z=1} c_{d(\alpha)} X_{\alpha},
$$

(15)

as illustrated in figure 1. The values of the coefficients for possible discretizations of the Laplacian operator are summarized in table 1, along with their computational cost and stability properties. Isotropic discretizations A and B offer reduced computational cost, while isotropic discretization C has the best accuracy and stability. As multiplications are cheap and additions are essentially free on modern CPUs, there is no reason not to use the best discretization scheme available. Thus, DEFROST is configured to use the
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Table 1. Summary of spatial discretization schemes.

| Coefficient | \( c_3 \) | \( c_2 \) | \( c_1 \) | \(-c_0\) | Cost | Stability |
|-------------|---------|---------|---------|--------|------|----------|
| Degeneracy  | 8       | 12      | 6       | 1      | (\( \times, + \)) | (\( \alpha > \cdot \cdot \cdot \)) |
| Standard    | 0       | 0       | 1       | 6      | 1, 6 | \( \sqrt{3} \) |
| Isotropic A | 0       | \( \frac{1}{6} \) | \( \frac{1}{3} \) | 4      | 3, 18 | \( \sqrt{2} \) |
| Isotropic B | \( \frac{1}{12} \) | 0       | \( \frac{1}{3} \) | \( \frac{1}{3} \) | 3, 14 | \( \sqrt{21}/3 \) |
| Isotropic C | \( \frac{1}{30} \) | \( \frac{1}{10} \) | \( \frac{1}{7} \) | \( \frac{1}{15} \) | 4, 26 | \( 8/\sqrt{30} \) |

isotropic discretization C by default. That can be changed by uncommenting the other coefficient definitions in the source code, although profiling shows little gain in doing so for large grids (for which the performance of the DEFROST solver is apparently memory bandwidth dominated).

To calculate the energy density (7), we need to discretize the square of spatial gradients as well. It is very important for discretized energy to be conserved by discretized equations of motion. Otherwise, it will leak off the grid in the course of a long simulation, affecting overall accuracy or even giving incorrect results (for the equation of state, for example). Simply squaring the first spatial derivative discretized like in equation (13), although second-order accurate, is not conservative. Using a discretized action approach [51], one can show that the conservative second-order accurate and fourth-order isotropic discretization of the gradient-square operator is

\[
G[X] \equiv \frac{1}{2} \sum_{x=1}^{x+1} \sum_{y=1}^{y+1} \sum_{z=1}^{z+1} c_d(\alpha) (X_\alpha - X_0)^2, \tag{16}
\]

where coefficients \( c_i \) are the same as in Laplacian (15). Evaluating this expression requires 30 multiplications per point for discretization C, which is significantly more expensive than computing the Laplacian. However, this does not place much of a burden as regards the total runtime. As we mentioned before, we eliminated gradient-square terms from evolution equations, so they only need to be calculated for output, which happens much less often.

Putting everything together, we end up with the following evolution scheme: the discretized field values are advanced to the next time slice using

\[
\phi_{i,\text{up}} = \frac{2 + D/(\alpha^2) - M_i^2 dt^2}{1 + (3/2) H dt} \phi_{i,\text{hr}} - \frac{1 - (3/2) H dt}{1 + (3/2) H dt} \phi_{i,\text{dn}}, \tag{17}
\]

where \( D \) is the discretized Laplace operator (15) and \( \alpha \equiv dx/dt \). The time step \( dt \) has to be small enough both for satisfying the Courant stability condition (\( \alpha > \text{const listed in table 1} \)) and for resolving the period of the fastest oscillating field. All the coefficients in expression (17) except for the effective mass of the \( i \)th field

\[
M_i^2 \equiv \frac{1}{\phi_i} \frac{\partial V}{\partial \phi_i}, \tag{18}
\]
are constant on the grid and the same for all fields, and thus can be pre-computed outside
the evolution loop. Inside the same loop, the kinetic and potential energy of the fields $\phi^i$
are accumulated to evaluate

$$\langle \rho + 3p \rangle_{\text{hr}} = \frac{1}{n^3} \sum_{x,y,z} \left[ \sum_i \left( \frac{(\phi^{i}_{\text{up}} - \phi^{i}_{\text{dn}})}{2 \Delta t^2} - 2V(\phi^i) \right)^2 \right],$$  \hspace{1cm} (19)$$

which is used to advance the Hubble length:

$$L_{\text{up}} = L_{\text{dn}} + \left[ 1 + \frac{L_{\text{hr}}^2}{6} \langle \rho + 3p \rangle_{\text{hr}} \right] 2 \Delta t.$$  \hspace{1cm} (20)$$

To avoid weak numerical instability associated with even–odd slice decoupling, we use
$L_{\text{hr}} = (L_{\text{up}} + L_{\text{dn}})/2$ instead of $L_{\text{hr}}$ in the above equation, which then can be solved for
$L_{\text{up}}$ either as an exact quadratic or iteratively (we use the latter in the code).

Equations (17) and (20) provide a complete recipe for advancing the field variables to
the next time step. Once in a while (at the user’s request) we would also like to calculate,
analyze, and output for visualization some auxiliary variables like the energy density $\rho$ and
gravitational potential $\Psi$. Energy density (7) and pressure (8) are easy to calculate once
we know the field values $\phi^i$ and their gradients (16). Finding the gravitational potential is
a little less trivial, as we need to solve the Laplace equation (12) with periodic boundary
conditions. The fastest way to do this is to use a fast Fourier transform (FFT). Applying
the FFT to the discretized Laplacian operator (15), we end up with an algebraic equation for $\Psi$ in the Fourier domain:

$$\Psi_k = \frac{\rho_k a^2 \Delta x^2}{2 P(\cos(2\pi/n)k_x, \cos(2\pi/n)k_y, \cos(2\pi/n)k_z)},$$  \hspace{1cm} (21)$$

where $k \in [0, n]$ is the integer Fourier mode wavenumber and polynomial $P(i, j, k)$ follows
discretization (15):

$$P(i, j, k) = \hat{c}_0 + \hat{c}_1(i + j + k) + \hat{c}_2(ij + ik + jk) + \hat{c}_3ijk,$$  \hspace{1cm} (22)$$

with $\hat{c}_k = 2^k c_k$. Here $c_k$ are once again the coefficients of discretization (15) with values
listed in table 1.

DEFROST implements the evolution scheme (17) in Fortran 90, fully taking
advantage of the capabilities of modern hardware and compilers by using both automatic
vectorization (over field variables) and automatic parallelization (of the evolution
loop). During code development, profiling showed that physical memory layout has an
unexpectedly large impact on performance, which became apparent as the solver loop
became optimized. Let us briefly discuss the storage model which was adopted after some
investigation. The fields $\phi^i$ are sampled and stored in a large multi-dimensional array
smp(N,0:p,0:p,0:p,3). The first index (minor in Fortran index ordering) enumerates
the $N$ fields. The next three indices enumerate the three dimensions of the spatial grid,
padded to $(p + 1)^3$ elements for reasons discussed below. The last index (major in Fortran
index ordering) enumerates the three time slices used in the evolution scheme. Rather
than copying large amounts of data around, the allocation of indices for dn, hr, and up
slices cycles every time step.
The layout of a single time slice is shown in figure 2 (with one spatial dimension suppressed for clarity). The $n \times n \times n$ simulation volume is surrounded by a single-cell-wide boundary layer, introduced to implement the boundary conditions without conditional logic inside the evolution loop. It also allows for an easy transition to parallel cluster implementation using MPI, as the required buffers are already in place. Somewhat counter-intuitively, padding the grid with a few extra empty cells can significantly improve the runtime for large grids. The root cause for this phenomenon probably lies in some interaction between the memory access pattern and hardware memory cache algorithm.

To get the best FFT performance, the grid size $n$ is usually taken to be a power of 2. While evaluating (17) on an unpadded array, memory would be accessed with a power of 2 stride, which could conceivably interfere with caching and prefetching done by the memory subsystem. Whatever the cause, padding the array so that its size $p + 1$ is prime (or a product of a few large primes) can improve the runtime, so the user is advised to experiment.

Finally, a few words should be said about the statistical estimators used to analyze the simulation data. The power spectral density (PSD) estimator is a fairly conventional one, implemented using the FFT. It employs anti-aliasing with a fourth-order polynomial kernel when folding the spectrum into wavenumber bins to reduce sampling noise. The implementation of the probability density function (PDF) estimator is less conventional, and does not use histogram binning at all. Instead, the PDF is derived from the cumulative density function (CDF) which is obtained by partially sorting the data cube into $n$ quantile
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brackets. Although more expensive and harder to parallelize, this approach is more robust and offers uniform sampling noise across the distribution.

4. Initial conditions

At the end of the inflation, most of the energy is still stored in the inflaton, and all the fields are homogeneous except for small quantum fluctuations. But the presence of these quantum fluctuations in the fields is essential for triggering the dynamical instability leading to preheating.

Initial conditions in the homogeneous field components depend on the model of inflation and are treated as an external input by DEFROST. They are straightforward to obtain by following the expanding Friedmann–Robertson–Walker solution during inflation, either analytically (if possible), or using any of the available numerical ordinary differential equation integrators. As the inflationary trajectory is an attractor [52], it is easy to find and no particular care is needed as regards where to start tracing it. The full three-dimensional simulation of preheating should take over at some time near the end of the inflation, but as there are other factors at play (such as limited spatial dynamic range available to simulation), the exact moment is best decided on a case by case basis.

To give a concrete example, figure 3 shows the expansion history of a typical chaotic inflation model with a massive inflaton \( V(\phi) = m^2 \phi^2 / 2 \) (discussed in more detail in section 5). As the universe is inflating, its horizon size \( L \equiv 1 / H \) stays relatively constant, while the physical wavelength of comoving modes grows with the scale factor \( a \). The modes which were originally inside the horizon expand and leave the horizon during inflation. Eventually, inflation ends and the horizon size starts growing faster than the scale factor (for instance, \( L \propto a^{3/2} \) during matter domination), at which point the modes begin to re-enter the horizon. The moment in time when comoving modes stop leaving the horizon and begin re-entry can be taken as the end of the inflation. This happens when

\[
\frac{d \ln L}{d \ln a} = 1, \tag{23}
\]

or in terms of a slow roll parameter

\[
\epsilon \equiv \frac{\ddot{H}}{H^2} = -1. \tag{24}
\]

For simulations presented in this paper, we start exactly at the moment when inflation ends (24), and select the comoving box size of \( \ell = 10/m \) to cover all the length scales of interest, as illustrated in figure 3.

As we already mentioned, it is crucial to include quantum field fluctuations in the mostly homogeneous initial conditions for the preheating instability to develop. The spectra of quantum field fluctuations are determined by the effective masses of the fields involved. Let us briefly recount the standard derivation [2] to establish notation. The canonically normalized massive field \( \varphi \) with Lagrangian

\[
\mathcal{L} = \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} m^2 \varphi^2 \tag{25}
\]

is quantized in flat Minkowski spacetime by promoting the field value \( \varphi \) and field momentum

\[
\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} \tag{26}
\]
to quantum operators $\hat{\phi}$ and $\hat{\pi}$ obeying the canonical commutation relation

$$[\hat{\phi}(x), \hat{\pi}(x')] = i \delta(x - x').$$  \hspace{1cm} (27)

In flat spacetime, the field operator can be represented using Fourier mode decomposition as

$$\hat{\phi}(x^\mu) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} [e^{ik_\mu x^\mu}\hat{a}^+_k + e^{-ik_\mu x^\mu}\hat{a}^-_k],$$  \hspace{1cm} (28)

where the mode creation–annihilation operators $\hat{a}^\pm$ obey

$$[\hat{a}^-_k, \hat{a}^+_q] = \delta(k - q),$$  \hspace{1cm} (29)

which directly follows from canonical commutation relation (27). The mode frequency $\omega$ of the massive field is related to its wavevector $k$ by a simple dispersion relation

$$\omega^2 = m^2 + k^2.$$  \hspace{1cm} (30)
Using mode decomposition (28) and commutation relation (29), it is easy to show that the two-point correlation functions of the field value and momentum are

\[ \langle \hat{\varphi}(\mathbf{x})\hat{\varphi}(\mathbf{x}') \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} , \]  

\[ \langle \hat{\pi}(\mathbf{x})\hat{\pi}(\mathbf{x}') \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} . \]  

The spectrum of the field $\varphi$ fluctuations is simply $1/(2\omega)$.

One can repeat this procedure in the background of an expanding homogeneous universe. In general, the time dependence of the modes will be different, but it turns out that quantum fluctuations of massive fields in de Sitter spacetime can be approximated quite well with the above expressions if one simply replaces the field mass $m$ with an effective mass

\[ m^2_{\text{eff}} = m^2 - \frac{9}{4} H^2 . \]  

We will use this approximation in DEFROST. In addition, we will follow the established practice of treating quantum operators as Gaussian random variables. This is an assumption, but not a totally unjustified one. Quantum modes essentially behave classically after they leave the horizon [53]. Even the sub-horizon modes (which are initially quantum) get large occupation numbers once the preheating instability kicks in, and can be treated classically [11]. Thus, we will initialize the field fluctuations as a Gaussian random field

\[ \hat{\varphi}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} e^{i\mathbf{k}\mathbf{x}} \left[ \hat{b}_k \cos \omega t + \hat{c}_k \sin \omega t \right] , \]  

where the complex random operators $\hat{b}_k$ and $\hat{c}_k$ obey

\[ \langle \hat{b}_k \hat{b}^*_k \rangle = \langle \hat{c}_k \hat{c}^*_k \rangle = \delta(k-k') , \]  

to reproduce the two-point correlation functions (31) and (32).

A lot of effort has gone into making the realization of random field initial conditions in DEFROST as statistically accurate as possible. The straightforward and often-used way to generate a Gaussian random field on a discrete grid is to directly discretize equation (34) in Fourier space, assigning the $k$th mode a random Gaussian number with amplitude $1/\sqrt{2\omega}$. Although simple, this procedure is ‘spoiled’ by the finite grid size effects, and does not reproduce correct two-point correlation functions in the real space [59]. One ends up with a substantial lack of power on scales comparable with the box size, which is not surprising if one considers that only a few long-wavelength modes ‘fit’ into the box, and naive discretization ignores all the power in the infrared part of the spectrum which should have been properly aliased into those few low-$k$ modes.

A wealth of literature is dedicated to the subject of generating Gaussian random fields of a given spectrum, particularly in the context of the $N$-body simulations of the large scale structure [54]–[61]. We draw on that experience, and in DEFROST for random field generator we adopt a method described in [58, 59]. Gaussian random field (34) with a $1/(2\omega)$ spectrum is realized by convolving white noise with a spherically symmetric
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The kernel function

\[
\xi(r) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} e^{i k x} \xi(k) = \frac{1}{\sqrt{\pi}} \int \frac{k^2 dk}{(k^2 + m_{\text{eff}}^2)^{1/4}} \sin kr.
\]

The kernel function \(\xi(r)\) can be evaluated analytically in terms of Bessel functions:

\[
\xi(r) = \frac{2^{3/4} m_{\text{eff}}^{1/4}}{4\pi r^{9/4}} \Gamma\left(\frac{3}{4}\right) \left[ K_{1/4}(mr) + 2mr K_{3/4}(mr) \right],
\]

and has a power-law ultraviolet divergence \(\xi(r) \propto r^{-5/2}\) in the limit \(r \to 0\). For the purposes of discretization on a finite grid, we have to regularize this divergence, which we do by introducing a Gaussian cut-off at some scale \(q\) below the Nyquist frequency:

\[
\xi(r) = \frac{1}{\sqrt{\pi}} \int \frac{k^2 dk}{(k^2 + m_{\text{eff}}^2)^{1/4}} \sin kr \exp\left[-\frac{k^2 q^2}{q^2}\right].
\]

The regularized kernel does not have a nice expression in terms of elementary functions, but is easy to evaluate numerically, of course. We use a one-dimensional discrete sine transform (DST) on a substantially larger grid to calculate it (simply because the DST is already provided by the FFTW libraries that we use), but other methods like using quadrature integrators could be implemented as well. Once the spherically symmetric kernel \(\xi(r)\) is evaluated, it is sampled on the three-dimensional grid in real space, and the random field is initialized as a convolution:

\[
\hat{\phi}(x, 0) = \frac{1}{(2\pi)^3} \int \int d^3k d^3x' \hat{b}_k(x') e^{i k (x - x')} \hat{\xi}(x').
\]

The convolution is implemented using a discrete FFT as

\[
\phi(x, 0) = \sum_k \sum_{x'} \frac{B_k \xi(x')}{2^{1/2} (d k)^{3/2} n^3},
\]

where \(n\) is the grid size, \(d k = 2\pi / \ell\) is the spacing of discrete wavemodes, and \(B_k\) is a complex Gaussian random number generated using the Box–Muller transformation

\[
B_k = \sqrt{-2 \ln U_1} e^{2\pi i U_2}
\]

from two real random numbers \(U_1\) and \(U_2\) uniformly distributed on a unit interval. The implementation of the initial velocity generator is entirely analogous (and handled by the same procedure), and is not worth repeating here.

Finally, I have to point out that as of current writing, there is a bug in the random number generator implementation in LATTICEEASY. The formula (41) is modified there to produce two complex numbers using only three uniformly distributed real numbers. This results in correlated random numbers with a quite non-Gaussian distribution. Fortunately, the results reported so far do not seem to be too much affected by this problem, but as non-Gaussianity studies are becoming more prominent in the modern cosmology literature, some care must be taken in proper implementation of random number generators.
5. Chaotic inflation and broad parametric resonance

So far, the discussion has been rather general, as DEFROST is designed to be easily adaptable for studying arbitrary models of preheating. This section describes the preheating model that we selected for the first simulations with DEFROST: chaotic inflation ending via broad parametric resonance [5]. The development of linear instability in this model can be largely understood analytically [8], and its non-linear dynamics has been widely studied numerically as well [12,14]. For all its simplicity, this model has very rich dynamics, and still holds surprises. Our very first simulations uncover new aspects of the evolution dynamics in this model, which are reported in section 6.

In a minimal form, the model consists of two scalar fields: the massive inflaton $\phi$ and the massless decay product $\psi$ interacting via the potential

$$V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2.$$  \hfill (42)

During inflation, the value of the inflaton $\phi$ is large, and the field is overdamped by the large Hubble friction, and slowly rolls down its potential. As it reaches the value of around 1 in Planck units, the damping dips below critical, and the homogeneous inflaton starts oscillating with decreasing amplitude:

$$\phi(t) \approx \Phi(t) \sin mt, \quad \Phi(t) = \frac{\Phi_0}{a^{3/2}} = \sqrt{2} \frac{2}{3} m t.$$  \hfill (43)

Decay field $\psi$ is coupled to the inflaton, and feels its oscillations through modulation of the effective mass; the equation of motion for the Fourier mode $\psi_k$ with wavevector $k$ is

$$\ddot{\psi}_k + 3H \dot{\psi}_k + \left( \frac{k^2}{a^2} + g^2 \Phi^2 \sin^2 mt \right) \psi_k = 0.$$  \hfill (44)

If the coupling $g$ is large enough, periodic modulation of the field mass leads to strong instability via parametric resonance. This can be understood analytically by applying the general theory of differential equations with periodic coefficients [8]. If one ignores the expansion and the Hubble drag term in equation (44), evolution for the Fourier mode $\psi_k$ is given by the Mathieu equation

$$\frac{d^2 \psi_k}{d\eta^2} + (A - 2q \cos 2\eta) \psi_k = 0,$$  \hfill (45)

where we have introduced dimensionless parameters

$$A = 2q + \frac{k^2}{m^2 a^2}, \quad q = \frac{g^2 \Phi^2}{4m^2},$$  \hfill (46)

and time variable $\eta \equiv mt$ to bring the equation into canonical form. According to Floquet’s theorem, a general solution of the Mathieu equation (45) is of the form $e^{i\mu} P(\eta)$, where $P(\eta)$ is a periodic function with period $\pi$. The Floquet exponent $\mu$ depends on parameters $A$ and $q$, and there is an elegant way to calculate its value [62], which we (somewhat reluctantly) will omit here, and just quote the final result. Figure 4 shows the dependence of Re $\mu$ on parameters $A$ and $q$ as a density plot. For certain parameter values the Floquet exponent $\mu$ has positive real part, leading to exponential instability of the solution; these unstable bands are marked in figure 4. The value of $A$ for our problem (46) is restricted to lie above the red line $A = 2q$ in figure 4, which corresponds...
to the homogeneous mode with $k = 0$. For sufficiently large coupling $q \gg 1$, a large portion of the available phase space volume is unstable, leading to fast development of instability. This regime is known as the broad parametric resonance. The instability grows on a timescale comparable to $1/m$ (as the Floquet exponent values are around $\text{Re } \mu \lesssim 1/3$), and manifests itself after a few dozen oscillations of the inflaton, which is very fast in cosmological terms. In section 6, we describe the non-linear field evolution after this instability develops.

6. Numerical results

Before we present our simulation results, a few words should be said about the units used throughout the code. As is clear from the action (1), we prefer to work with dimensionless scalar fields, rather than canonically normalized ones. In this convention, one can simply think of the values of the scalar fields as measured in units of the Planck mass $m_{\text{pl}}$. Note that we use the reduced Planck mass $m_{\text{pl}} = (8\pi G)^{-1/2}$ rather than $M_{\text{pl}} = G^{-1/2}$. The only other scale in the model (42) is the inflaton mass $m$. The coupling constant $g$ (which has the dimension of mass with our scalar field normalization), the frequency and wavevectors of the unstable modes, and everything else can be referred to it. It is convenient to scale all the quantities (except field values) by the appropriate power of $m$, making them dimensionless and suitable for numerical analysis. Thus we set $m = 1$ in the code.
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Figure 5. Expansion history (left) and the average equation of state (right) during preheating.

We simulate the preheating model (42) with inflaton mass \( m = 5 \times 10^{-6} m_{\text{pl}} \), and the value of the coupling constant \( g = 100 \) in a comoving box of size \( \ell = 10/m \) (the values chosen correspond to the ones used in [14]). The grid size is taken to be 256\(^3\), while the time step \( dt = 2^{-10}/m \) has to be reduced substantially below the Courant limit to resolve oscillations of the field \( \psi \) (which is initially a hundred times heavier than the inflaton \( \phi \)). The simulation is started at the end of the inflation (24), when the value of the inflaton is \( \phi \approx 1.009343 \) and the Hubble constant is \( H \approx 0.50467 m \). The simulation box is initially about five Hubble lengths across. We let the code run until \( t = 256/m \), which corresponds to \( 2^{18} \) time steps.

To get things going, we first reproduce the previously reported results [12] on the expansion history of the preheating model (42). The left panel of figure 5 shows the evolution of the horizon size during expansion, with \( a^{3/2} \) growth corresponding to matter-dominated expansion scaled out. The expansion history has a sharp break as instability develops, and energy gets deposited into relativistic inhomogeneous modes from a homogeneous oscillating inflaton (which behaves as a pressureless dust). This transition can be seen in terms of an effective equation of state parameter \( w \equiv \langle p \rangle / \langle \rho \rangle \), the value of which is plotted in the right panel of figure 5. It undergoes large amplitude oscillations (shown by thin pale line in figure 5), but when averaged over a few periods (with a Kaiser
The averaged equation of state (shown by the thick red line) switches over from the dust-like equation of state $w = 0$ to a value slightly less than a quarter [12], corresponding to a fairly relativistic fluid. (If evolved further, the residual homogeneous component in the inflaton will eventually come to dominate the evolution again, slowly lowering the equation of state toward $w = 0$ in the process.)

While we recover the results obtained by simulations with LATTICEEASY, the accuracy of the integrator used in DEFROST is significantly higher. The performance of integration scheme for expansion factor (20) is illustrated in figure 6, which shows the residual curvature

$$K = a^2 \left( \frac{\langle \rho \rangle}{3} - H^2 \right),$$

which should be zero in the flat model that we are evolving. As you can see, the constraint equation is satisfied to the $10^{-7}$ level (which is exceptionally good for a second-order scheme with a $10^{-3}$ time step), and the error does not accumulate with time. In fact, this error is mostly due to the fact that we neglected second-order corrections to the density from initial field fluctuations.

Having made sure that our code reproduces previously reported results, and after numerous checks of code integrity and accuracy, we move on to investigation of the field dynamics during preheating.

Evolutions of field distributions and spectra for inflaton $\phi$, decay field $\psi$, total density $\rho$ and gravitational potential $\Psi$ are presented in figure 7. The left panel shows the evolution of the median value (thick red line), along with 68% and 95% percentile brackets around it (which would correspond to $1\sigma$ and $2\sigma$ contours for a Gaussian distribution) shown by shaded outlines. The contour with the lightest shading spans the extremal values inside the simulation box, which serves to illustrate the extent of the tails of the distribution, although the exact percentile that it corresponds to depends on the spatial resolution of the simulation. Dilution due to expansion has been scaled out to highlight the relative change of the distributions as evolution proceeds.
Figure 7. Evolution of field value distributions (left) and spectra (right) of the inflaton $\phi$ (top row), decay field $\psi$ (second row), total density $\rho$ (third row), and gravitational potential $\Psi$ (bottom row). The onset of instability and the characteristic size of the structure formed are clearly visible in the plots.
The evolution of the distribution of the decay field values (second row of figure 7) clearly shows the onset of instability a little after $t = 100/m$, rapid spreading of the distribution due to exponential amplification of seed inhomogeneities, and self-limiting of the growth by non-linear interactions when the scaled value of the decay field becomes of order unity. As the decay field perturbation grows and becomes non-linear, it is drawing the energy from the zero mode of the inflaton (top row of figure 7), reducing the amplitude of its oscillations, and eventually forcing the inflaton to become strongly inhomogeneous as well, due to non-linear backreaction. We should note here that although the amplitude of the coherent inflaton oscillation decays, it does not go away altogether, and the leftover homogeneous mode will eventually come to dominate the universe expansion [13], as its equation of state is effectively that of pressureless dust, and it dilutes more slowly than inhomogeneous components, which have equation of state closer to the relativistic one.

Of particular interest to us is the distribution of the total energy density, shown in the third row of figure 7. It is clearly very inhomogeneous, with peak densities easily exceeding ten times the average. After a brief transient, it quickly settles to a nearly stationary distribution, which appears to be highly non-Gaussian (and is in fact plotted on a logarithmic scale). We will come back to this point after we inspect the spatial picture of the energy density distribution inside the simulation box.

The bottom row of figure 7 shows the evolution of the gravitational potential. Despite total energy density being highly inhomogeneous and having huge overdensities, the gravitational potential that it produces is rather small (at the $10^{-3}$ level), and is further diluted away by the expansion with a near-relativistic equation of state. The maximal potential well depth inside the simulation box is $2 \times 10^{-3}$, which is far too small for forming any primordial black holes. This result is in line with observations of [27, 28], although now we have a more transparent diagnostic of black hole formation as we calculate the gravitational potential directly.

The right panel of figure 7 shows evolution of the field spectra as the density plot in terms of time $t$ and wavenumber $k$, with dilution of the fields due to expansion and overall power-law dependence on $k$ scaled out. The spectra of total energy density $\rho_k$ and gravitational potential $\Psi_k$ show a very clear simultaneous peak soon after instability develops, which is sharply localized in both time and scale. The subsequent evolutions of the two differ, however. The peak power in energy density is evolving toward higher $k$ and smaller scales, while the peak power in potential is evolving to lower $k$, so the structure of potential wells is growing in spatial extent. What is going on will become clearer when we look at evolution in real space, which is what we are going to discuss next.

Figure 8 shows three-dimensional volume renderings of the contents of the simulation box. As the fields $\phi$ and $\psi$ oscillate rapidly in a standing wave pattern, quickly losing coherent phasing, the view of their values is messy and not very enlightening, and we will omit it here. Much more interesting is the picture of what is happening to the total energy density, which is an adiabatic invariant for the oscillating fields. The top row of figure 8 shows the density distribution inside the simulation box soon after the onset of instability (at $t = 124/m$, left) and at the end of the simulation (at $t = 256/m$, right). The density field is shaded using a logarithmic color map with a linear transparency ramp applied, so that only the peaks of the density distribution are visible.

Immediately after the onset of instability, the density distribution in figure 8 (top left) looks like smoke filling the box. What you are seeing is actually the overdense bubble...
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Figure 8. Volume rendering of the total density $\rho$ (top row), gravitational potential $\Psi$ (second raw), and PDF of density values $\rho$ inside the simulation box soon after the onset of instability (at $t = 124/m$, left) and during subsequent evolution (at $t = 256/m$, right). Animations for the density and potential evolutions are available at http://www.sfu.ca/physics/cosmology/defrost/.
walls forming a three-dimensional foam-like structure that fills the box. Its origin is easy
to understand if one thinks about how seed inhomogeneities are amplified by instability.
Broad parametric resonance amplifies wavemodes in a certain band, effectively serving
as a low-pass filter (with a kernel that can be approximated analytically) and sets the
characteristic size of the structure which grows out of the seed inhomogeneities. Original
fluctuations are a Gaussian random field, which already has the structure of peaks, ridges,
and valleys imprinted into it. The skeleton of this structure is essentially preserved
unchanged as the growth of inhomogeneities due to instability increases density contrast.
Once the density contrast becomes of order unity, non-linear evolution takes over. This
will happen to underdense regions first, with repulsive interaction term \( g^2 \phi^2 \psi^2 \)
helping to evacuate the bubble interiors, and pushing the matter density into the bubble walls, thus
forming the structure that you see in figure 8 (top left).

The evolution does not stop at forming bubbles, however. The repulsive interaction
soon breaks the extended bubble walls into smaller localized blobs, moving more or less
freely inside the simulation box (the animation of this process is available online). The
final state is depicted in figure 8 (top right), and persists with little change for a long, long
time. This state seems quite distinct from thermal equilibrium, yet it is still long-lived and
statistically simple in a certain way, which is quite surprising. Even more surprisingly, the
distribution of values of total energy density \( \rho \) in units of \( H^2 \) quickly becomes statistically
stationary and after a brief transient tends to a distribution with a probability density
function shown lower right in figure 8. It can be fitted with exceedingly high accuracy by
a log-normal distribution

\[
P(\rho) \, d\rho = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[ -\frac{(\ln \rho - \mu)^2}{2\sigma^2} \right] \frac{d\rho}{\rho},
\]

with one free parameter (\( \sigma = 0.6584 \) or \( \mu = -0.2197 \)), as the mean \( \bar{\rho} = \exp(\mu + \sigma^2/2) \)
is unit normalized by virtue of us scaling out the expansion. The corresponding median
is \( e^\mu = 0.8028 \). With statistical errors of the PDF estimator being what they are, the
apparent log-normality of a density distribution is undoubtedly not a mere coincidence,
but must have an explanation rooted in scalar field dynamics. Moreover, further
simulations of preheating models with different couplings and inflaton potentials seem to
suggest that log-normal distribution of density is a universal feature of two-field preheating
models. This observation presents a very interesting theoretical puzzle, which will be
explored in detail elsewhere [63].

Although the one-point distribution of the total energy density \( \rho \) quickly becomes
stationary as noted above, other quantities continue evolving on much longer timescales.
The blobs continue to fragment, and their characteristic size slowly decreases with time.
While this is obvious from visualizations, it can be further quantified by introducing the
(physical) correlation length of the total energy density configuration

\[
\ell^2_{\rho} \equiv \frac{\langle (a\rho)^2 \rangle}{\langle (\nabla \rho)^2 \rangle}.
\]

The evolution of the comoving correlation length \( \ell_\rho / a \) is plotted in the left panel of
figure 9. Initially it is very large (\( \gtrsim 100/m \)), as the density field is nearly homogeneous.
As instability develops and structure forms, it abruptly drops to about \( 10^{-1}/m \), and
then continues to decrease, but much more slowly. Although the graph clearly shows an
Figure 9. Evolution of characteristic structure size in the total energy density (left) and gravitational potential (right).

evolutionary trend in the density correlation length $\ell_\rho/a$, actual numbers should be taken with a grain of salt, as the density field does eventually become fragmented on a scale close to spatial grid resolution (which for the $256^3$ grid that we use would be reached at around $t \sim 10^3/m$).

While it is known that thermalization after preheating might take a long time [7], and there might be an intermediate scaling regime in the evolution of the fields [40, 41], the view of the process from the real space presented here is strikingly simple. One would think that the field distributions will thermalize eventually, presumably forming a homogeneous fluid-like state with Maxwellian distribution of particle velocities. Thermalization does not happen in our simulations, which lack the necessary quantum effects. Exactly how and when it does happen is an extremely interesting question, and one which requires further study.

Finally, let us discuss the gravitational potential $\Psi$, which is sourced from the evolving energy density distribution $\rho$, and is shown in the middle row of figure 8. To make the structure more visible, we have opted for a density plot on a 3-slice through the simulation box rather than using a volume rendering. The color map shows positive potential values (corresponding to underdense regions) as shades of red, and negative potential values (corresponding to overdense regions) as shades of blue, blending into white for zero potential value.

Immediately after the onset of instability, the gravitational potential in figure 8 (middle left) clearly traces the foam-like structure of matter distribution. The isolated potential peaks (red) in the interior of the bubbles are separated by extended potential valleys (blue) created by overdense bubble walls. The gravitational potential configuration is asymmetric between positive and negative values, and is clearly non-Gaussian. The subsequent evolution of the gravitational potential is rather interesting. As bubble walls break into smaller and smaller blobs, the structure of the gravitational potential does not follow suit. Instead, it begins to grow in spatial extent (the animation is available online). By the end of the simulation in figure 8 (middle right), the size of the structure in the gravitational potential spans almost the whole box. The growth of the structure can be quantified by introducing the correlation length $\ell_\Psi$ for the gravitational potential in the
way that we did for the energy density:

$$\ell_\Psi^2 \equiv \frac{\langle (a\Psi)^2 \rangle}{\langle (\nabla \Psi)^2 \rangle} = -\frac{\langle 2\Psi^2 \rangle}{\langle \rho \cdot \Psi \rangle}. \quad (50)$$

The evolution of the comoving correlation length $\ell_\Psi/a$ is plotted in the right panel of figure 9, with a pale line tracing its instantaneous value, and a thick red line giving a running average (with the Kaiser window) over a few oscillations. While the comoving correlation length grows overall, the evolution shown in figure 9 (right) still represents an initial transient, and the long-time asymptotic behavior is not reached in the simulation reported here. Further investigation shows that the correlation length $\ell_\Psi$ continues to grow faster than the comoving size, but not quite as fast as the horizon size. Eventually one might even have to worry about it outgrowing the finite simulation box size, but that would take a very long time, and is not reached by our simulations.

7. Conclusions

This paper presents a new numerical code that I developed for simulating preheating of the universe after the end of the inflation, which I call DEFROST. It is small (about 600 lines of Fortran code), fast, easy to modify, and is fully instrumented for 3D visualizations (using, for example, LLNL’s VisIt: http://wci.llnl.gov/codes/visit/) The source code is available for download online at http://www.sfu.ca/physics/cosmology/defrost/ and is distributed under the terms of the GNU Public License. While the main design goal of DEFROST has been accuracy of the simulations, the performance of the solver has also been significantly improved as compared to LATTICEEASY [37], which is the most mature and widely used reheating code publicly available today.

As a result of all the optimizations (and a bit of black magic), DEFROST outperforms LATTICEEASY by about a factor of 4 in raw PDE solver speed (for two fields on a 256$^3$ grid in double precision on a dual Xeon 5160 machine) while using more accurate (and more expensive) discretization. If one takes into account the time spent on analysis of the results, the difference is even larger, as FFTW libraries used by DEFROST are vastly faster than FFT routines shipped with LATTICEEASY (especially on multi-processor machines). The speed-up offered by DEFROST is so significant that the studies done a few years ago on a big parallel cluster [12,13] using the MPI version of LATTICEEASY [38] can now be carried out on a single fast workstation. The planned MPI version of DEFROST should be able to push the accessible simulation size over the 1024$^3$ barrier, provided that the code scales well.

The code was tested on a number of chaotic inflation models which end via parametric resonance. In this paper, we report the simulations of the simplest two-field preheating model with massive inflaton and quartic coupling to the decay field (42). We reproduce the previously published numerical results for this model [12,14] (and the ones for trilinear coupling [13], which we will not discuss here, although our simulation data and results for that model are available online as well). We further investigate the dynamics of the scalar field evolution in these preheating models, taking advantage of the advanced visualization and analysis capabilities that DEFROST offers. In particular, we study the behavior of the energy density distribution and scalar gravitational potential during preheating,
something which has not been looked at closely before. Our main scientific results are summarized as two observations, both novel and quite surprising.

First, the evolving scalar fields quickly end up in a simple state, which, although highly inhomogeneous, appears to have a certain universality to it. In this state, the one-point distribution function of the total energy density is nearly stationary (apart for the overall dilution due to expansion), and is described by a log-normal distribution for all two-field parametric resonance preheating models that we tried so far, namely the ones described by interaction potentials

- \( V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2, \)
- \( V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \sigma \phi \psi^2 + \frac{1}{4} \lambda \psi^4, \)
- \( V(\phi, \psi) = \frac{1}{4} \lambda \phi^4 + \frac{1}{2} g^2 \phi^2 \psi^2, \)
- \( V(\phi, \psi) = \frac{1}{4} \lambda (\phi^2 + \psi^2)^2. \)

This is true even if distributions of field values or other correlators might still be evolving, and appears to be a very general statement about the random scalar fields that one encounters in preheating. It is tempting to attribute this state to scalar field turbulence [40, 41], especially since log-normal density distributions are known to occur in supersonic isothermal turbulence in hydrodynamics [39]. We do not see obvious signs of thermalization, even if the simulations are run for a time much longer than the dynamical timescale of the problem (the longest done so far for a massive inflaton is \( 2^{12}/m \), corresponding to five e-folds since the end of inflation; this is limited mainly by my patience rather than the code stability).

Second, less general but still amusing, is the observation that the small scale structure in the gravitational potential can grow faster than the comoving box expands. It is not quite clear whether the reason for this happening is kinematic or dynamical in nature. As we neglected gravitational interactions in our simulations, the only thing that can cause the structure to grow is the interaction between scalar fields themselves. In our preheating model it is repulsive; yet the structure still grows! Although one might suspect that any inhomogeneity in a gravitational potential on sub-horizon scales would probably get washed away by subsequent evolution (and is too small to form primordial black holes), this effect still might have some interesting cosmological consequences.

All in all, we find that the picture of preheating dynamics is simpler in real space than it looks in a particle representation. The final stage of preheating, with growing structure and log-normal density distribution, eerily reminds one of large scale structure formation in later cosmology (although of course it occurs on vastly smaller scales and is driven by completely different physics). Perhaps the analytical methods developed for the latter [43]–[45] could be fruitfully applied to preheating as well. This is what we intend to explore next.

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