Spectra of Lindbladians on the infinite line: From non-Hermitian to full evolution via tridiagonal Laurent matrices

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Abstract
We determine the spectra of single-particle translation-invariant Lindblad operators on the infinite line. In the case where the Hamiltonian is given by the discrete Laplacian and the Lindblad operators are rank r, finite range and translates of each other, we obtain a representation of the Lindbladian as a direct integral of finite range bi-infinite Laurent matrices with rank-r-perturbations. By analyzing the direct integral we rigorously determine the complete spectrum in the general case and calculate it explicitly for several types of dissipation e.g. dephasing, coherent hopping. We further use the detailed information about the spectrum to prove gaplessness, absence of residual spectrum and a condition for convergence of finite volume spectra to their infinite volume counterparts. We finally extend the discussion to the case of the Anderson Hamiltonian, which enables us to study a Lindbladian recently associated to localization in open quantum systems.

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1 Introduction

Schrödinger operators and their spectra are one of the central objects studied in mathematical physics. Indeed, spectral properties provide signatures of and encode many important physical properties [1]. This includes dynamical properties such as the speed of propagation, localization versus delocalization as described by the RAGE theorem or the scattering behavior. Furthermore, symmetry constraints on the spectrum can lead to classification results in terms of symmetry protected topological phases.

Going beyond the closed system paradigm described by Schrödinger operators and unitary dynamics, a natural setting is that of Markovian time-evolution, described by a completely positive dynamical semi-group. These systems have been extensively studied from a quantum information perspective in particular by considering the generator of such evolutions, i.e. the Lindblad generator [2]. Here, gaps in the spectrum around the origin in the complex plane provide information about relaxation times towards the non-equilibrium steady state of the system (as we will discuss in Section 2.3). Understanding the interplay between disorder (e.g. in the form of a random potential) and dissipation (e.g. thermal noise) is emerging as an important problem [3, 4, 5, 6].

In addition, over the past decades, the theory of non-Hermitian Hamiltonians (i.e Non-self-adjoint Schrödinger operators) has developed rapidly [7]. Many applications of these operators both inside and outside of physics have been found [8, 9, 10]. One motivation for these investigations has been that non-Hermitian Hamiltonians model open quantum systems if quantum jumps are neglected [11]. The theory of non-Hermitian Hamiltonians is also closely connected to the study of tridiagonal Laurent matrices (see [12, 13, 14] and references therein). In particular, the stability of the spectra under perturbations has been investigated [15].

In the following, we extend this connection to the case of Markovian evolution in the single particle regime (see Figure 1). For Lindbladians with translation-invariant Hamiltonians and Lindblad operators $L_k$ that are rank-one, finite range and translates of each other we prove in Theorem 3.1 that the entire Lindbladian can be rewritten as a direct integral of tridiagonal Laurent matrices $T(q)$ corresponding to the non-Hermitian evolution subject to a rank-one perturbation $F(q)$ which corresponds to the quantum jump terms. The construction of $T(q)$ and $F(q)$ is explicit. This decomposition allows us to explore the spectral effects of the quantum jumps rigorously.

Since the Lindbladian $L$ is non-normal, to analyze its spectrum the notion of pseudo-spectrum can yield many insights (see for example [16, 7]). To determine the spectrum of the Lindbladian from the direct integral decomposition we extent the use of pseudo-spectra by providing a result of independent interest concerning the spectrum of a direct integral in terms of the pseudo-spectra of its fibers and thereby generalizing the corresponding result for the direct sum [17]. This is the content of Theorem 3.6.

The combination of the direct integral decomposition and Theorem 3.6 enables us to obtain information about the spectrum of the Lindbladian $L$.

In Section 4, we first use the methods to prove that this class of Lindbladians always have approximate point spectrum and are gapless. Then, in Section 5, we completely determine the spectrum of some Lindbladians which have received attention in the physics literature as the one-particle sector of open spin chains [18, 19] and complementing the exact results on the spectrum from [20]. We are particularly
interested in an example where the dissipators are non-normal and where the system shows signs of localization in an open quantum system [21]. Our rigorous analytic results also complement the large body of very recent work on random Lindblad operators studied from a random matrix theory point of view [22, 23, 24, 25, 26].

Finally, in order to connect the examples to the open quantum system in a random potential where it has received attention for showing signs of localization, in Theorem 6.1 we prove a Lindbladian analogue of the Kunz-Souliard theorem from the theory of random operators.

In contrast to many previous investigations, we work directly on the entire lattice \( \mathbb{Z} \). That allows us to utilise translation-invariance, Laurent matrices and some tools from random operator theory that do not work for finite systems. Furthermore, working in infinite volume directly allows us to determine closed formulas for the spectra explicitly. From one point of view one can look at these formulas as approximations to (some) large finite volume systems. In Theorem 4.5 we give conditions that ensure this convergence.

With some exceptions [26, 27, 28, 4, 6], the study of Lindbladian evolutions has, in recent decades, focused on finite spin systems. We therefore first present some results with conditions for the Lindbladian \( \mathcal{L} \) to be bounded as an operator on bounded, Hilbert-Schmidt and trace-class operators. Furthermore, we make some remarks on spectral independence and the relationship between spectra and dynamics.

2 Lindblad systems on the infinite lattice

We consider the Markovian, open-system dynamics of a single quantum particle on the one-dimensional lattice described by the Hilbert space \( \mathcal{H} = \ell^2(\mathbb{Z}) \). The underlying completely positive dynamical semigroup is generated by a Lindbladian \( \mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) of the form [29, 2]

\[
\mathcal{L}(\rho) = -i[H,\rho] + G \sum_k L_k \rho L_k^* - \frac{1}{2}(L_k^* L_k \rho + \rho L_k^* L_k),
\]

where \( H \) is the Hamiltonian of the system and \( G > 0 \) the coupling constant of the dissipation and we have used \( A^* \) to denote the adjoint of an operator \( A \). The operators \( L_k \in \mathcal{B}(\mathcal{H}) \) implementing the dissipation are referred to as Lindblad operators. Later, we will often choose our \( L_k \) to be local as for example \( L_k = |k\rangle \langle k| \). In the following we will refer to \( \mathcal{L} \) as the Lindblad form. We will similarly say that an operator \( \tilde{L} \) is in the adjoint Lindblad form if

\[
\tilde{L}(X) = i[H,X] + G \sum_k L_k^* X L_k - \frac{1}{2}(L_k^* L_k X + X L_k^* L_k).
\]

The case of a Markovian evolution on the infinite line is slightly under-represented in the literature as many works, in particular from the quantum information side, where open-system dynamics has been studied extensively, restrict themselves to finite dimensions. The infinite case adds some additional complications that we clarify without using the lattice structure of \( \ell^2(\mathbb{Z}) \).

The Lindblad form ensures dissipativity, which means that the spectrum is always contained within the half-plane of the complex plane with non-positive imaginary part (see also [30]). To see that, it is
enough to prove that \( \text{Re}(\mathcal{L}) \leq 0 \). Since the Hamiltonian part is skew we can omit it. Then we look at the Lindblad operators individually we can restrict ourselves to the Lindblad operator given by

\[
\mathcal{D}(\rho) = L_\rho L^* - \frac{1}{2} \{ L^* L, \rho \}.
\]

Correspondingly, the adjoint of the dissipator (with respect to the Hilbert-Schmidt inner product) becomes \( \mathcal{D}^*(\rho) = L^* \rho L - \frac{1}{2} \{ L^* L, \rho \} \) and thus

\[
2 \text{Re}(\mathcal{D})(\rho) = \mathcal{D}(\rho) + \mathcal{D}^*(\rho) = L_\rho L^* + L^* \rho L - \{ L^* L, \rho \} = L_\rho L^* - L^* \rho L + L^* L_\rho + \rho L^* L.
\]

Proving dissipativity of \( \mathcal{D} \) amounts to proving \( \text{Re}(\langle \rho, \mathcal{D}(\rho) \rangle) = \langle \rho, \text{Re}(\mathcal{D})\rho \rangle \leq 0 \). Thus,

\[
\langle \rho, 2 \text{Re}(\mathcal{D})(\rho) \rangle = \text{Tr}(\rho^* (L_\rho L^* + L^* \rho L - L^* L_\rho - \rho L^* L)) = \text{Tr}(-\langle \rho L - L_\rho \rangle (\rho L - L_\rho)^* ) \leq 0,
\]

where we used that any operator of the form \( AA^* \) is positive. Furthermore, for finite-dimensional systems \( \mathcal{L}(X^*) = \mathcal{L}(X)^* \) which implies that the spectrum is invariant under complex conjugation \( \sigma(\mathcal{L}) = \overline{\sigma(\mathcal{L})} \). This we prove also in the infinite dimensional case in Section 2.

In finite dimensions, it is always the case that there is a steady state \( \rho_\infty \) of the dynamics that satisfies \( \mathcal{L}(\rho_\infty) = 0 \) (see for example [31, Proposition 5]). It was discussed in [31, 32, 33] how the symmetries of \( L_k \) are inherited by \( \mathcal{L} \) and the steady state. However, Lindbladians with translation-invariance in the infinite system have not been studied extensively although some results exist [31, 4].

### 2.1 Boundedness of \( \mathcal{L} \) as an operator on \( \text{TC}(\mathcal{H}), \text{HS}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{H}) \)

In finite dimensions, the set of density matrices is defined to be the set of positive matrices with unit trace. In infinite dimensions, this notion is generalised to positive trace-class operators with unit trace. In the following, we will denote the space of trace-class operators by \( \text{TC}(\mathcal{H}) \), the Hilbert-Schmidt operators by \( \text{HS}(\mathcal{H}) \) and the space of bounded operators by \( \mathcal{B}(\mathcal{H}) \). See for example [2, 35] for more details on these spaces. All three spaces are Banach spaces with regards to their respective norms and \( \text{HS}(\mathcal{H}) \) is also a Hilbert space with the inner product

\[
\langle X, Y \rangle = \text{Tr}(X^* Y).
\]

In the following, we will use interpolation methods. These methods rely on the Schatten classes that interpolate between \( \text{TC}(\mathcal{H}), \text{HS}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{H}) \). To define them we consider operators \( A \in \mathcal{B}(\mathcal{H}) \) with \( \mathcal{H} = l^2(\mathbb{Z}) \), if \( p \in (1, \infty) \) we can define the \( p \)-norm by

\[
\|A\|_p = \text{Tr}(|A|^p)^{\frac{1}{p}},
\]

where \( |A| = (AA^*)^{\frac{1}{2}} \). The Schatten-\( p \)-class \( \mathcal{S}_p \) then consists of all bounded operators \( \mathcal{B}(\mathcal{H}) \) with finite \( p \)-norm. For \( p = \infty \) we set \( \mathcal{S}_\infty = \mathcal{K}(\mathcal{H}) \), the compact operators on \( \mathcal{H} \). Furthermore, the Schatten classes interpolate between these spaces in the sense that \( \mathcal{S}_1 = \text{TC}(\mathcal{H}), \mathcal{S}_2 = \text{HS}(\mathcal{H}) \). More information see [35] where the Schatten classes are treated extensively.

We first show that under a boundedness assumption on the \( L_k \)'s then any Lindblad generator of the form \( \mathcal{L}_1 \) will be a bounded operator on \( \text{TC}(\mathcal{H}), \text{HS}(\mathcal{H}) \) as well as \( \mathcal{B}(\mathcal{H}) \). To this end, assume that \( H \in \mathcal{B}(\mathcal{H}) \) and that

\[
(A_k) : \text{ Both } \sum_k L_k L_k^* \text{ and } \sum_k L_k^* L_k \text{ converge weakly (in } \mathcal{B}(\mathcal{H})) \text{ to a bounded operator.}
\]

A similar level of generality was used in [37] and [38]. We will use the Riesz-Thorin interpolation theorem in a non-commutative version, where the operators are defined on the Schatten classes \( \mathcal{S}_p \). We state it here for convenience.

**Theorem 2.1** ([39], Section IX.4). Let \( p, q \geq 1 \) and \( A \in \mathcal{B}(\mathcal{S}_p), \mathcal{B}(\mathcal{S}_q) \), then \( A \in \mathcal{B}(\mathcal{S}_t) \) where \( \frac{1}{r_t} = \frac{1}{p} + \frac{1-t}{q} \) for each \( t \in [0, 1] \) with

\[
\|A\|_{r_t \to r_t} \leq \|A\|_{p \to p}^{\frac{1}{r_t}} \|A\|_{q \to q}^{1-t}.
\]
The theorem enables us to prove that Assumption (A₁) is enough to ensure boundedness on all Schatten spaces, but we state the more relevant ones here for clarity.

**Lemma 2.2.** Suppose that \( \mathcal{L} \) is of the Lindblad form \([1]\) and that (A₁) holds. Then

\[
\mathcal{L} \in B(\mathcal{B}(\mathcal{H})), B(HS(\mathcal{H})), B(\text{TC}(\mathcal{H})).
\]

This is also true if \( \mathcal{L} \) is of the adjoint Lindblad form \([2]\).

**Proof.** The case \( \mathcal{L} \in B(\text{TC}(\mathcal{H})) \) follows from \([33]\) prop 6.4. In Appendix A.1 we prove that \( \mathcal{L} \in B(\mathcal{B}(\mathcal{H})) \). Then, to see that \( \mathcal{L} \in B(HS(\mathcal{H})) \) we use the non-commutative Riesz-Thorin theorem. Note that since the operator \( \mathcal{L} \) is bounded on \( \mathcal{B}(\mathcal{H}) \) it also bounded as an operator on the compact operators \( \mathcal{K}(\mathcal{H}) = S_\infty \) (see below). Thus, we obtain that

\[
\|\mathcal{L}\|_{2 \to 2} \leq \|\mathcal{L}\|_{1 \to 1} \|\mathcal{L}\|_{\infty \to \infty}.
\]

This shows that \( \mathcal{L} \in B(HS(\mathcal{H})) \). For further discussions and similar results see also \([10]\) and \([41]\). The result for the adjoint Lindblad form follows in the same way because of the assumption (A₁). \( \square \)

We have now proven that the Lindbladian \( \mathcal{L} \) is an element of the Banach algebras \( B(\text{TC}(\mathcal{H})), B(HS(\mathcal{H})) \) and \( B(\mathcal{B}(\mathcal{H})) \), where \( B(HS(\mathcal{H})) \) is also a \( C^* \)-algebra.

In the following, we will be concerned with the spectrum of \( A \) in each of these algebras. For any Banach algebra \( A \) the spectrum of an operator \( A \) with respect to the algebra \( A \) is defined as follows

\[
\sigma_A(A) = \{ \lambda \in \mathbb{C} \mid A - \lambda \text{ is not invertible in } A \}.
\]

Furthermore, we define the approximate point spectrum of an operator in a Banach algebra \( A = B(X) \) for some Banach space \( X \), is given by

\[
\sigma_{A,\text{appt}}(A) = \{ \lambda \in \mathbb{C} \mid \exists (\psi_n)_{n \in \mathbb{N}} \subset X, \|\psi_n\| = 1, \lim_{n \to \infty} \| (A - \lambda)\psi_n \| = 0 \}.
\]

A sequence \( (\psi_n)_{n \in \mathbb{N}} \) corresponding to a point \( \lambda \in \mathbb{C} \) as above we will call a Weyl sequence corresponding to \( \lambda \). It is always the case that \( \sigma_{A,\text{appt}}(A) \subset \sigma_A(A) \) and for normal operators equality holds. We will prove in Theorem 4.1 that the equality also holds in many of our cases of interest. The set

\[
\sigma_A(A) \setminus \sigma_{A,\text{appt}}(A) = \sigma_{A,\text{res}}(A)
\]

is called the residual spectrum of \( A \).

In the following, we will be particularly interested in the case where the Banach algebra \( A = B(S_\rho) \). It is a classical result that \( S_\rho \) (and in turn \( A \)) is Banach algebra in itself, and that \( S_\rho \) an ideal in \( B(H) \) for all \( \rho \in [1, \infty] \). The Calkin algebra \( Q(\mathcal{H}) \) is defined by \( Q(\mathcal{H}) = B(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H}) \) and the spectrum of an operator \( A \) in the Calkin algebra we call the essential spectrum \( \sigma_{\text{ess}}(A) = \sigma_{Q(\mathcal{H})}(A) \).

In the following, we are mainly concerned with translation-invariant operators with finite range. We formalize this by the following two assumptions.

(A₂a) : *Finite range* \( r < \infty : \langle y, L_k x \rangle = 0 \) whenever \( \max{|x - k|, |y - k|} > r \)

(A₂b) : *Translation-invariance:* \( S_1 L_k S_{-1} = L_{k+1} \) for each \( k \in \mathbb{Z} \)

where the operator \( S_n \) on \( \mathcal{H} \) is defined by \( (S_n \psi)(x) = \psi(x - n) \) with the convention that \( S = S_1 \). Notice that since \( \mathcal{H} = l^2(\mathbb{Z}) \) the operator \( S_n \) is unitary and \( S_n^{-1} = S_{-n} \).

If we further assume that the Hamiltonian \( H \) is translation-invariant meaning that \( H = S_1 HS_1^* \), then Assumption (A₂b) implies that \( \mathcal{L} \) is translation-covariant, i.e. it satisfies that

\[
\mathcal{L}(S_1 \rho S_1^*) = S_1 \mathcal{L}(\rho) S_1^*, \quad \text{for all } \rho \in \mathbb{C}.
\]

It is easy to see that (A₂) implies that \( \mathcal{L}(S_n \rho S_n^*) = S_n \mathcal{L}(\rho) S_n^* \) for all \( n \in \mathbb{Z} \). Furthermore, it is also true that (A₂a) and (A₂b) imply (A₁). To see this, notice that (A₂b) implies that \( \sum_{k} L_k L_k^* \) is translation-invariant and therefore constant on diagonals. Now, (A₂a) implies that there are only finitely many non-zero diagonals and the diagonal entries are finite. Thus, \( \sum_{k} L_k L_k^* = \sum_{n=-r}^{r} \alpha_n S_n \) for \( \alpha_n \in \mathbb{C} \) for \( -r \leq n \leq r \), which is clearly a bounded operator.
2.2 Remarks on spectral independence of Lindblad operators

In the following, we make some remarks on the spectrum of Lindblad operators when defined as an operator on the different Schatten spaces $\mathcal{S}_p$. To show how involved the situation is we start with the following remark.

Remark 2.3. Consider the Lindbladian with $H = V$ for some onsite operator $V$ and the dissipative part is 0. In that case $\mathcal{L}(\rho) = -i[V, \rho]$. It holds that $\mathcal{L}(|i\rangle \langle j|) = -i(V_i - V_j)(|i\rangle \langle j|)$ for each $i, j \in \mathbb{N}$. Thus, from one point of view $\mathcal{L}$ is a Schur multiplier with the matrix with $(i, j)$’th index $-i(V_i - V_j)$. If we think about the $V_i$ having bounded support then this is a Schur multiplier with bounded coefficients. An example was given in [42] where an example of a Schur multiplier with bounded coefficients, although not of this form, that does not map trace class operators to trace class operators is given.

However, the operator discussed in [42] is very far from having Lindblad form which is a much stronger condition. Therefore, we can ask under which assumptions, for example locality of the Lindblad operators, that spectral independence holds.

Question 2.4. Suppose that $H = \sum_i h_i$ and that $L_k$ are all local. Is $\sigma_{\mathcal{S}_p}(\mathcal{L})$ independent of $p$?

In particular, in the case $\mathcal{L}(\rho) = -i[V, \rho]$ where $V[x] = V(x)|x\rangle$ such that $|V(x)| \leq 1$ is it the case that $\sigma_{\mathcal{S}_n}(\mathcal{L})$ is independent of $p$?

In particular, we are concerned with the case $\mathcal{L}(\rho) = -i[H, \rho]$ where $H = -\Delta + V$ for some diagonal non-translation-invariant potential $V$ as an operator from $\mathcal{S}_p$ to $\mathcal{S}_p$ (i.e. $V$ could be a random potential, that we will study later). It is natural to conjecture that $\sigma_{\mathcal{S}_n}(\mathcal{L})$ is independent of $p \in [1, \infty]$, but this is beyond our current methods. However, we do present some partial results.

Proposition 2.5. Suppose that $1 \leq p \leq p' \leq \infty$ are Hölder conjugates, $\frac{1}{p} + \frac{1}{p'} = 1$ and that $\mathcal{L} \in \mathcal{B}(\mathcal{S}_p)$ and $\mathcal{L} \in \mathcal{B}(\mathcal{S}_{p'})$. Then $\mathcal{L} \in \mathcal{B}(\mathcal{S}_q)$ for all $p \leq q \leq p'$ and it holds that

$$\sigma_{\mathcal{S}_q}(\mathcal{L}) \subset \sigma_{\mathcal{S}_p}(\mathcal{L}) \cup \sigma_{\mathcal{S}_p}(\mathcal{L}).$$

Proof. The first claim follows directly from the non-commutative Riesz-Thorin theorem (Theorem 2.1). For the second claim notice that if $z \notin \sigma_{\mathcal{S}_n}(\mathcal{L}) \cap \sigma_{\mathcal{S}_n}(\mathcal{L})$ then $(\mathcal{L} - z)^{-1}$ is bounded both from $\mathcal{S}_p$ to $\mathcal{S}_p$ and from $\mathcal{S}_p$ to $\mathcal{S}_p$. Again, by the non-commutative Riesz-Thorin theorem this means that $(\mathcal{L} - z)^{-1}$ is bounded from $\mathcal{S}_q$ to $\mathcal{S}_q$ for all $q \in [p, p']$. \qed

In fact, we can say a bit more using the Lindblad form. Let us start with the following lemma proving that the spectrum is invariant under complex conjugation in our infinite volume setting.

Lemma 2.6. Suppose that $\mathcal{L}$ is of the Lindblad form $[\mathcal{L}]$ and that $(A_1)$ holds. Then for all $p \in [1, \infty]$ it holds that $\sigma(\mathcal{L})$ is closed under complex conjugation

$$\sigma_{\mathcal{S}_p}(\mathcal{L}) = \overline{\sigma_{\mathcal{S}_p}((\mathcal{L})^*)}.$$

Proof. Assume first that $1 \leq p \leq 2$ and let $\lambda \in \sigma_{\mathcal{S}_n}(\mathcal{L})$. If $\lambda \in \sigma_{\text{appt},\mathcal{S}_p}(\mathcal{L})$ there exists a sequence $\rho_n \subset \mathcal{S}_p(H)$ where

$$\|((\mathcal{L} - \lambda)\rho_n)\|_p \to 0.$$

Then consider the modified Weyl sequence $(\rho_n^*)_{n \in \mathbb{N}}$ which by the relation (1.2) in [36] satisfies that $\|\rho_n^*\|_p = \|\rho_n\|_p = 1$ is a Weyl sequence for $\lambda$ since

$$\|((\mathcal{L} - \bar{\lambda})\rho_n^*)\|_p = \|((\mathcal{L} - \lambda)(\rho_n^*))\|_p = \|((\mathcal{L} - \bar{\lambda})(\rho_n))\|_p \to 0.$$
We conclude that $\tilde{\lambda} \in \sigma_{\text{app},S\rho}(\mathcal{L}) \subset \sigma_{S\rho}(\mathcal{L})$. If there is no Weyl sequence corresponding to $\lambda$, then if $p'$ is the Hölder conjugate of $p$ then let $\hat{\mathcal{L}}^\dagger : S\rho' \rightarrow S\rho'$ be the Banach space adjoint of $\mathcal{L} : S\rho \rightarrow S\rho$. Now, it holds that $\lambda \in \sigma_{\rho}((\mathcal{L})^\dagger) = \sigma_{\rho'}(\hat{\mathcal{L}})$. Thus, there exists an operator $X \in S\rho'$ such that $\hat{\mathcal{L}}_{\rho'}(X) = \lambda X$ which implies that $\hat{\mathcal{L}}_{\rho'}(X^*) = \hat{\mathcal{L}}_{\rho'}(X)^* = \lambda X^*$ and thus $\lambda \in \sigma_{\rho}(\hat{\mathcal{L}}_{\rho'})$ which means that $\lambda \in \sigma_{S\rho}(\mathcal{L})$.

If on the other hand $2 \leq p \leq \infty$ then $\sigma_{S\rho}(\mathcal{L}) = \sigma_{S\rho}(\hat{\mathcal{L}})$ where $1 \leq p' \leq 2$. Since the same argument as above holds for operators of the adjoint Lindblad form for then $\sigma_{S\rho}(\mathcal{L}) = \sigma_{S\rho}(\hat{\mathcal{L}})$ which proves the Lemma also in this case.

Now, we collect the results in the following theorem using that $\mathcal{L}$ and $\hat{\mathcal{L}}$ on $S\rho$ and $S\rho'$ are (Banach)-adjoints.

**Theorem 2.7.** Suppose that $1 \leq p \leq p' \leq \infty$ are Hölder conjugates, $\frac{1}{p} + \frac{1}{p'} = 1$ and that $\mathcal{L} \in \mathcal{B}(S\rho)$ and $\mathcal{L} \in \mathcal{B}(S\rho')$. Then $\mathcal{L} \in \mathcal{B}(S\rho)$ for all $p \leq q \leq p'$ and it holds that

$$\sigma_{S\rho}(\mathcal{L}) \subset \sigma_{S\rho}(\mathcal{L}) \cup \sigma_{S\rho'}(\mathcal{L}) = \sigma_{S\rho}(\mathcal{L}) \cup \sigma_{S\rho}(\hat{\mathcal{L}}).$$

Furthermore, we have not quite used the Lindblad property $\mathcal{L}^*(\rho) = \mathcal{L}(\rho)^*$ which may help shed light on the following question.

**Question 2.8.** Suppose that $\mathcal{L}$ is of the Lindblad form [1] and that $(A_1)$ holds. Is it then the case that $\sigma_{S\rho}(\mathcal{L}) = \sigma_{S\rho}(\hat{\mathcal{L}})$?

In that case, which we consider fairly natural we would then obtain that for any $1 \leq p \leq q \leq 2$ $\sigma_{S\rho}(\mathcal{L}) \subset \sigma_{S\rho}(\mathcal{L})$ as well as the reverse inclusion in the case $2 \leq q \leq p \leq \infty$. In particular, it would hold that $\sigma_{S\rho}(\mathcal{L}) \subset \sigma_{S\rho}(\mathcal{L})$ meaning that our lower bounds to $\sigma_{S\rho}(\mathcal{L})$ would be lower bounds to $\sigma_{S\rho}(\mathcal{L})$.

### 2.3 Relation between spectra and dynamics for Lindblad systems

In the following, we are concerned with determining the spectra of certain infinite volume open quantum systems. In the Hamiltonian case there is a clear dynamical interpretation of the spectra and the different types of spectra. However, due to non-normality of Lindblad operators the dynamical implications of the spectra are more subtle and the details of the topic are still under discussion in the physics literature [13]. We discuss our knowledge in both the finite and infinite dimensional cases.

**Finite dimensions:** In the finite dimensional case the relationship between eigenvalues of $\mathcal{L}$ and the time evolution is given through the Jordan normal form. I.e. $\mathcal{L} = SAS^{-1}$ where $S$ is invertible and $A$ is of a certain almost-diagonal form. However, even in the cases where $A$ is diagonal, $\mathcal{L}$ is not necessarily normal.

The analysis is counter-intuitive to the person trained in Hamiltonian formalism due to the peculiarities of non-normality. Another peculiarity is the fact that all eigenvectors of $\mathcal{L}$ are traceless, which we describe in the following remark.

**Remark 2.9.** All eigenvectors of $\mathcal{L}$ with eigenvalues not equal to 0 are traceless. To see that, note that $e^{t\mathcal{L}}$ is trace preserving for all $t \in [0, \infty)$, so it holds that $\text{Tr}(\rho) = \text{Tr} \left( e^{t\mathcal{L}}(\rho) \right) = e^{t\lambda} \text{Tr}(\rho)$. Thus, if $\text{Tr}(\rho) \neq 0$ then for all $t \in [0, \infty)$ we get $1 = e^{t\lambda}$ which implies that $\lambda = 0$.

We can give guarantees about the dynamics in terms of the spectral gap $g$ of $\mathcal{L}$ which we define as follows

$$g = \sup \{ \text{Re}(\lambda) \mid \lambda \in \sigma(\mathcal{L}) \setminus \{0\} \}.$$ 

In the case where we have a unique steady state $\rho_{\infty}$, we can get a dynamical guarantee for the speed of decay towards the steady state in terms of the gap $g$. Namely that $\| e^{t\mathcal{L}}(\rho) - \rho_{\infty} \| \leq C e^{tg}$ where $C > 0$ is a constant that depends heavily on the size of the Jordan blocks of the systems.
In the literature, the cases where $\mathcal{L}$ is not diagonalizable are called exceptional points, there is evidence that these points can also lead to faster decay towards the steady state [44], although the mathematical guarantee gets worse.

**Infinite dimensions:** In infinite dimensions the relationship between spectra and dynamics can break down due to Jordan blocks of unbounded size (and more generally the breakdown of the Jordan normal form), due to the lack of a trace class steady state (a phenomenon that we will encounter in most examples in the following) and due to the lack of a spectral gap (which we prove for our models in Theorem 4.6).

However, we will encounter situations where $\sigma(\mathcal{L})$ has two or more disconnected parts. Suppose for simplicity that we just have two parts $\sigma(\mathcal{L}) = \Sigma_A \cup \Sigma_B$, with $\sup\{\text{Re}(z) \mid z \in \Sigma_A\} \leq g$ for some gap $g \in (-\infty, 0)$ and such that there exists a closed continuous curve encircling only $\Sigma_A$. Then we can define the Riesz projections by contour integration to get a decomposition of $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B$ for two orthogonal subspaces $\mathcal{H}_A, \mathcal{H}_B$ such that $\mathcal{L}$ leave each of the two subspaces invariant. Thus, we can decompose $\mathcal{L} = \mathcal{L}_A \oplus \mathcal{L}_B$. Suppose that $\rho \in \mathcal{H}_A$ then, since $\mathcal{L}_A$ is a semigroup by [45], it holds that

$$
\|e^{t\mathcal{L}_A}(\rho)\| = \|e^{t\mathcal{L}_A}(\rho)\| \leq C e^{tg} \|\rho\|,
$$

for some constant $C > 0$. Thus, if $\rho = \rho_A + \rho_B$ with $\rho_A \in \mathcal{H}_A$ and $\rho_B \in \mathcal{H}_B$ we see that the part $\rho_A$ decays quickly. Notice again that $\rho$ must be trace-free.

We leave it to future work to establish a stronger relationship between spectra and dynamics in the infinite dimensional case. In particular, one cannot use our work to gain many rigorous guarantees about the evolution of infinite open quantum systems, but we consider the results presented as steps towards such rigorous guarantees. Furthermore, due to the apparent convergence of the spectra of some finite dimensional Lindbladians (see Theorem 4.5 and the discussion in Section 7) one can also view the method presented as a way to compute large volume approximations to finite systems (which have discrete spectra and where the relation between spectra and dynamics is clearer).

In fact, given the question on spectral independence raised in the previous section one might even ask which Banach algebra (potentially $B(S_p)$ for some $p \in [1, \infty]$) enables us to transfer knowledge from spectra to dynamics of states $\rho \in S_1, \rho \geq 0$ and $\text{Tr}(\rho) = 1$.

## 3 Direct integral decompositions and their spectra

In this section, we specialize to the case where the Hamiltonian $H \in B(L^2(\mathbb{Z}))$ is the discrete Laplacian

$$
H = -\Delta = -\sum_{k \in \mathbb{Z}} |k\rangle \langle k + 1| + |k + 1\rangle \langle k| - 2|k\rangle \langle k|.
$$

(4)

Here $|k\rangle$ is the position eigenstate at site $k \in \mathbb{Z}$. In fact, since $H$ only enters through into the Lindbladian given by $[\mathcal{L}, \cdot]$ the Lindbladian does not change when disregarding the term $-2|k\rangle \langle k|$. Thus, we will often work with

$$
H = -\tilde{\Delta} = -\sum_{k \in \mathbb{Z}} |k\rangle \langle k + 1| + |k + 1\rangle \langle k| = -(S + S^*),
$$

(5)

where the operator $S_n$ on $\mathcal{H}$ was defined by $(S_n \psi)(x) = \psi(x - n)$ and we used the convention that $S = S_1$. For the Lindblad operators $L_k$ we made the following assumptions in the previous section.

$$(A_{2a}) : \text{Finite range } r < \infty : \langle y|, L_k|x\rangle = 0 \text{ whenever } \max\{|x - k|, |y - k|\} > r.$$

$$(A_{2b}) : \text{Translation-invariance: } S_n L_k S_{-n} = L_{k+n}.$$

Sometimes, we will also need the assumption

$$(A_{2c}) : \text{rank-one: } \text{Rank}(L_k) = 1.$$

Notice that in particular, we do not assume that each of the $L_k$ is normal or self-adjoint and in fact one of our motivating examples has non-normal $L_k$. Notice that $H$ is translation-invariant in the sense that $H = S_n H S_n^*$. The assumption $(A_{2b})$ implies that $\mathcal{L}$ is translation-covariant in the sense of $[3]$. 

8
3.1 From translation-invariance to a direct integral decomposition

We now show how we can use translation-invariance to get a direct integral composition in the case where we consider \( \mathcal{L} \in \mathcal{B}(\text{HS}(l^2(\mathbb{Z}))) \).

Let us first illuminate the \( \text{HS}(l^2(\mathbb{Z})) \) a bit. If \( A \in \text{HS}(l^2(\mathbb{Z})) = \mathcal{S}_2(l^2(\mathbb{Z})) \) is an operator then the 2-norm of \( A \) is given by \( \|A\|_2^2 = \sum_{i,j \in \mathbb{Z}} |A_{ij}|^2 \) where \( A_{ij} = \langle i, A(j) \rangle \) is the matrix elements of \( A \). Thus, \( \|A\|_2 = \left\| \{A_{ij}\}_{i,j \in \mathbb{Z}} \right\|_2 \) where \( \|\cdot\|_2 \) is the norm on \( l^2(\mathbb{Z})^2 \). Therefore, \( \text{HS}(l^2(\mathbb{Z})) \cong l^2(\mathbb{Z})^2 \) as Hilbert spaces.

Notice that the shift \( S_1 \rho S_1^* \) from [34] corresponds to the shift \( (x, y) \mapsto (x + 1, y + 1) \) on \( l^2(\mathbb{Z})^2 \). That \( \mathcal{L} \) is translation-covariant means that it is covariant under these joint translations. Thus, since we only have translation-invariance in one coordinate direction we can only hope to utilize it with a Fourier transform in one of the two variables. One may also think of this as relative and absolute position with respect to the diagonal. This change of coordinates was suggested in [46] and it was also used in [6].

Thereby we should be able to decompose the superoperator \( \mathcal{L} \) (which if we consider \( \mathcal{L} \) as an operator on Hilbert–Schmidt operators can be viewed as an operator on \( l^2(\mathbb{Z})^2 \)) into \( q \)-dependent operators on \( l^2(\mathbb{Z}) \), where \( q \) is the Fourier variable. This is formalized using the direct integral, that we introduce briefly since are important to us both for the main theorem and its applications. More information, for example on the details of measurability, can be obtained in [34] XII.16.

Operators on direct integral of Hilbert spaces: In the following, suppose that \( \{\mathcal{H}_q\}_{q \in I} \) is a family of Hilbert spaces indexed by some index set \( I \) (in the following \( I = [0, 2\pi] \)). The direct integral of Hilbert spaces over an index set \( I \) is the set of families of vectors in each of the Hilbert spaces and it is written by \( \int_I \mathcal{H}_q dq \) and it consists of equivalence classes up to sets of measure 0 of vectors \( v \) such that \( v_q \in \mathcal{H}_q \) for each \( q \in I \). The space \( \int_I \mathcal{H}_q dq \) is again a Hilbert space with inner product given by

\[
\langle v, w \rangle_{\int_I \mathcal{H}_q dq} = \int_I \langle v_q, w_q \rangle_{\mathcal{H}_q} dq.
\]

The particular decomposition of a Hilbert space as a direct integral that we are going to use is the following

\[
\text{HS}(l^2(\mathbb{Z})) \cong l^2(\mathbb{Z})^2 \cong l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z}) \cong L^2([0, 2\pi]) \otimes l^2(\mathbb{Z}) \cong L^2([0, 2\pi], l^2(\mathbb{Z})) \cong \int_{[0,2\pi]} l^2(\mathbb{Z})_q dq,
\]

where the third isomorphism is obtained by Fourier transform in the first component, \( L^2([0, 2\pi], l^2(\mathbb{Z})) \) is the space of (equivalence classes of) functions \( f : [0, 2\pi] \to l^2(\mathbb{Z}) \) such that \( \int_{[0,2\pi]} \|f(q)\|^2 dq < \infty \) and the third isomorphism is given by \( f \otimes \psi \mapsto \psi_f \) where \( \psi_f(x) = f(x)\psi \).

Now, if \( A(q) \) is a bounded operator on \( \mathcal{H}_q \) for each \( q \) and the family \( q \mapsto A(q) \) is measurable. Then we can define the integral operator \( A = \int_I A(q) dq \). Naturally, it acts as \( \int_I A(q) dv = \int_I A(q)v(q) dq \).

If for an operator \( A \) on \( \int_I \mathcal{H}_q dq \) the converse is true, i.e. there exists a measurable family of operators \( \{A(q)\}_{q \in I} \) such that \( A = \int_I A(q) dq \) then we say that \( A \) is decomposable. The norm of a decomposable operator is given as

\[
\|A\| = \text{esssup}_{q \in I} \|A(q)\|.
\]

The goal of this section is to exploit the translation-invariance of the Lindblad operators \( L_k \) under conjugation by shift operators in order to partially diagonalize the Lindbladian in vectorized form with the help of a Fourier transform. To this end, it will be useful to define \( \mathcal{F}_1 = \mathcal{F} \otimes \mathbb{I} \) to be the Fourier transform in the first component. The result is summarized in the following theorem. We remark that in [34] there is a result of similar flavour, but it is less informative since it has slightly weaker assumption than \((A_2a), A_2b)\) and it Fourier transforms both variables instead of just one.

**Theorem 3.1.** A Lindbladian \( \mathcal{L} \in \mathcal{B}(\text{HS}(\mathcal{H})) \cong \mathcal{B} \left( \int_{[0,2\pi]} l^2(\mathbb{Z})_q dq \right) \) of the form \([1]\) with Lindblad operators \( L_k \) satisfying assumption \((A_2a), A_2b)\) is unitarily equivalent to a decomposable operator of the form

\[
\int_{[0,2\pi]} T(q) + F(q) dq,
\]

as if we were reading it naturally.
with \(T(q)\) a bi-infinite \(r\)-diagonal Laurent matrix and \(F(q)\) a finite rank operator with finite range. Moreover, \(F(q) = [\Gamma_L(q)]/[\Gamma_R(q)]\) has rank-one if \(L_k\) has rank-one.

In the proof we will see that \(T = \int_0^{2\pi} \text{tr} \, T(q) \text{d}q\) corresponds to the non-Hermitian evolution, that is all terms in \(L\) in (1) except \(\sum_{k \in \mathbb{Z}} \, L_k \cdot L_k^\ast\). Conversely, \(F = \int_0^{2\pi} F(q) \text{d}q\) stems the quantum jumps of the Lindblad equation, i.e. the terms \(\sum_{k \in \mathbb{Z}} \, L_k \cdot L_k^\ast\).

To illuminate the form of \(T(q)\) and \(F(q)\) they are both bi-infinite matrices with the following form (in an example that we will return to in the next section):

\[
T(q) + F(q) = \begin{pmatrix}
\ldots \ldots \ldots \ldots \ldots \\
\ldots \beta \gamma 0 \ldots \\
\ldots \alpha \beta \gamma \ldots \\
\ldots 0 \alpha \beta \ldots \\
\ldots \ldots \ldots \ldots \ldots
\end{pmatrix} + \begin{pmatrix}
\ldots \ldots \ldots \ldots \ldots \\
\ldots 0 0 0 \ldots \\
\ldots 0 1 0 \ldots \\
\ldots 0 0 0 \ldots \\
\ldots \ldots \ldots \ldots \ldots
\end{pmatrix},
\]

where \(\alpha, \beta, \gamma : [0, 2\pi] \to \mathbb{C}\) are functions of \(q\).

The proof of Theorem 3.3 uses a unitary transformation based on conditional shifts that transforms a joint translation-invariance in two tensor factors into translation-invariance in the first tensor factor. The required properties of this transformation are collected in the following lemma.

**Lemma 3.2.** Consider the unitary conditional shift \(C = \sum_{l} |l\rangle|l\rangle \otimes S_l\) on \(l^2\mathbb{Z} \otimes l^2\mathbb{Z}\) and \(A \in \mathcal{B}(l^2\mathbb{Z})\) translation-invariant, then

1. \(C^\ast (\mathbb{1} \otimes A) C = \mathbb{1} \otimes A\)
2. \(C^\ast (A \otimes \mathbb{1}) C = F_1^\ast \left( \sum_{z} e^{iqz} \langle z|A|0\rangle S_z^\ast \right) F_1\)
3. \(C^\ast (S_k \otimes \mathbb{1}) C = S_k \otimes S_k^\ast\)
4. \(C^\ast (S_k \otimes S_k) C = S_k \otimes \mathbb{1}\)
5. \(C^\ast \left( \tilde{\Delta} \otimes \mathbb{1} \right) C = S \otimes S^\ast + S^\ast \otimes S\).

In particular, if \(A\) is finite range then \(C^\ast (\mathbb{1} \otimes A) C\) is also finite range and translation-invariant.

**Proof.** All identities in the enumeration follow from direct computation. For example, using translation-invariance of \(A\) in the last step we find the first one as

\[
C^\ast (\mathbb{1} \otimes A) C = \sum_{l,l'} |l\rangle \langle l'| |l\rangle \langle l' | S_{l'}^\ast AS_{l'} = \sum_{l} |l\rangle \langle l | S_{l}^\ast AS_{l} = \mathbb{1} \otimes A.
\]

Let us now consider the second claim. Evaluating the expression explicitly, we find

\[
C^\ast (A \otimes \mathbb{1}) C = \sum_{l,l'} |l\rangle \langle l'| |l\rangle \langle l' | S_{l'-l}^\ast
= \sum_{l,z} |l-z\rangle \langle l-z | A|l\rangle \otimes S_z
= F_1^\ast \left( \sum_{z} e^{iqz} \langle z|A|0\rangle S_z^\ast \right) F_1.
\]

As claimed, this operator is finite range if \(A\) is, i.e. \(\langle z|A|0\rangle = 0\) for \(|l|\) large enough. The last identity follows by realizing that \(\tilde{\Delta} = S + S^\ast\).

**Proof of Theorem 3.3.** Looking at \(L\) from (1) in vectorized form, as discussed in for example ([48, 49, (4.88)]), the Hamiltonian part \(-i[H, .]\) reads

\[
-i \tilde{\Delta} \otimes \mathbb{1} + i \mathbb{1} \otimes \tilde{\Delta},
\]

(7)
whereas the dissipation part yields the expression

$$
\sum_k \frac{G}{2} (L_k^* L_k \otimes \mathbb{I} - L_k^T \otimes (L_k L_k)^T) + G \sum_k L_k \otimes (L_k^* L_k)^T.
$$

(8)

Since $L_k = S_k L_0 S_k^*$ is assumed to have finite range and rank, so will $L_k^* L_k$ albeit with possibly squared rank and doubled range. Accordingly, $Q = \sum_k L_k^* L_k$ will be an at most $2r$-banded matrix with constant entries along the diagonals. Hence, combining the first sum of (8) with the Hamiltonian part from (7) yields a translation-invariant operator of the form

$$
\tilde{T} = \left(-i\tilde{\Delta} - \frac{G}{2} Q\right) \otimes \mathbb{I} + \mathbb{I} \otimes \left(i\tilde{\Delta} - \frac{G}{2} Q\right).
$$

(9)

This is also known as the *non-Hermitian evolution* which is the starting point for non-Hermitian Anderson models, their connection to Laurent matrices etc.

The remaining term is referred to as the quantum jump term \[11\]. We are again using the fact that the $L_k$ arise as translations of some common $L_0$ to deal with this term. Looking at its vectorized form in (8), we first notice that

$$
\tilde{F} = \sum_k L_k \otimes L_k = \sum_k S_k L_0 S_k^* \otimes S_k \overline{T}_0 S_k^*
$$

(10)

is invariant under joint translations of both tensor factors. Using the conditional shift $C$ introduced in Lemma 3.2, we can collect this joined shift invariance in the first tensor factor. Since $C^* S_a \otimes S_a C = S_a \otimes \mathbb{I}$, we get

$$(S_a \otimes \mathbb{I}) C^* \tilde{F} C (S_a^* \otimes \mathbb{I}) = C^* (S_a \otimes S_a) \tilde{F} (S_a^* \otimes S_a^*) C = C^* \tilde{F} C.$$

Evaluating $C^* \tilde{F} C$ explicitly, we find, after renaming indices, that

$$
C^* \tilde{F} C = G \sum_{l,l'} \left( \sum_k S_k \langle l | \langle l' | S_k^* \right) \otimes \left( \langle 0 | S_l^* L_0 S_{l'} | 0 \rangle S_l^* \overline{T}_0 S_{l'} \right)
$$

$$
= GF_1^* \left( \sum_{l,l'} e^{iql(l-l')} \langle 0 | S_l^* L_0 S_{l'} | 0 \rangle S_l^* \overline{T}_0 S_{l'} \right) F_1 = GF_1^* F(q) F_1,
$$

where we introduced the Fourier transform in the first tensor factor $F_1$. Hence, the quantum jump part is indeed unitarily equivalent to a direct integral as claimed. If $L_0$ is finite range, the same will be true for $F(q)$, since $(l | L_0 | l')$ will vanish for $l, l'$ large enough. If $L_0 = |\phi\rangle \langle \psi|$ is rank-one then

$$
F(q) = G \sum_{l,l'} e^{iql(l-l')} \langle 0 | S_l^* | \phi\rangle \langle \psi | S_{l'} | 0 \rangle S_l^* \overline{T}_0 S_{l'}
$$

$$
= G \left( \sum_l e^{iql} \langle 0 | S_l^* | \phi\rangle \overline{T}_0 \langle \psi | S_l | 0 \rangle \right) \left( \sum_{l'} e^{-iql'} \langle \psi | S_{l'} | 0 \rangle \langle 0 | S_l \overline{T}_0 \right)
$$

is still rank-one. Furthermore, if $L_0$ is rank $r$ then $F(q)$ has rank at most $r^2$.

Since $\tilde{T}$ in (9) is translation-invariant in both tensor factors independently, it is also invariant under joint translations and accordingly $C^* \tilde{T} C$ can be jointly diagonalized with translations in the first tensor factor. Hence, we only have to argue that the resulting operators $T(q)$ after the transformations will be Laurent matrices of width at most twice the range of $L_k$. However, each term of $\tilde{T}$ has either the form $A \otimes \mathbb{I}$ or $\mathbb{I} \otimes A$ for some translation-invariant operator $A$. Hence, Lemma 3.2 is applicable and $T$ will indeed translation-invariant and have range at most $2r$.

For the examples discussed later in Section 3, it will be helpful to derive more a more explicit expression of $T$, which we collect in the following Lemma where we also state the form of $F(q)$ for reference.
Lemma 3.3. Let $H$ be given by (1), then $T(q)$ in Theorem 3.1 has the form

$$T(q) = i\left(1 - e^{-iq}\right)S + i\left(1 - e^{iq}\right)S^* - \frac{G}{2} \left(\sum_k L_k L_k^* + \sum_z e^{iqz} \langle z| \sum_k L_k |0\rangle S_z^*\right).$$

In the case $L_0 = |\phi\rangle\langle\psi|$ is rank-one then $F(q) = |\Gamma_L\rangle\langle\Gamma_R|$ and are given by

$$F(q) = G \left(\sum_l e^{iq} \langle 0| S_l^* |\phi\rangle S_l \overline{\phi}\right) \left(\sum_l e^{-iq} \langle \psi| S_l^* |\overline{\psi}\rangle S_l\right).$$

In particular, if $|\phi\rangle = \sum_r \alpha_r |r\rangle$ and $|\psi\rangle = \sum_r \beta_r |r\rangle$ then

$$F(q) = G \left(\sum_{r_1, r_2} \alpha_{r_2}^* |r_2\rangle \langle r_2 - r_1|\right) \left(\sum_{r_1, r_2} \beta_{r_1} |r_1\rangle \langle r_1 - r_2|\right).$$

Proof. Looking at the proof of Theorem 3.1, we see that $T(q) = F_1 \left(C^*\tilde{T}C\right) F_1$ with $C$ the conditional shift introduced in Lemma 3.2. According to (1), $\tilde{T}$ is a sum of terms of the form $A \otimes I$ or $I \otimes A$, respectively, for $A$ translation-invariant and we can therefore directly apply Lemma 3.2 to each of the terms individually. All terms of the form $I \otimes A$ stay invariant by virtue of relation 1. and relations 2. and 5. allow us to evaluate the remaining terms of the form $A \otimes I$, which leads to the expression

$$C^*\tilde{T}C = -i(S \otimes S^* + S^* \otimes S) - \frac{G}{2} F_1 \left(\sum_z e^{iqz} \langle z|Q|0\rangle S_z^*\right) F_1 + I \otimes \left(i\hat{\Delta} - \frac{G}{2} Q\right).$$

Taking the Fourier transform in the first tensor factor, we arrive at the expression in the Lemma. The last expression follows from direct computation. \qed

Before we continue, let us remark that the decomposition also works in finite volume with periodic boundary conditions. Let $M_N$ be the set of $N \times N$ matrices with the Frobenius norm. Define $L^2_{per}$ to be the Lindblad operator $L$ defined on the space $M_N$ with periodic boundary conditions. For simplicity we let $\{1, \ldots, N\} = [N]$ and let $\mathcal{T}_N = \{\frac{2\pi k}{N} | k = 1, \ldots, N\}$. Then, similarly to the decomposition above we get that

$$M_N \cong l^2([N] \times [N]) \cong l^2([N]) \otimes l^2([N]) \cong L^2(T_N, \mathcal{T}, l^2([N])) \cong \bigoplus_{q \in \mathcal{T}_N} l^2_q([N]).$$

Mimicking the proof of Theorem 3.1, we obtain.

Theorem 3.4. A Lindbladian $L^2_{per} \in \mathcal{B}(M_N) \cong \mathcal{B}(\bigoplus_{q \in \mathcal{T}_N} l^2_q([N]))$ of the form (1) with Lindblad operators $L_k$ satisfying assumption $A_2(a), A_2(b)$ with periodic boundary conditions is unitarily equivalent to an operator of the form

$$\bigoplus_{q \in \mathcal{T}_N} \left(T^2_{per}(q) + F_N(q)\right)$$

with $T^2_{per}(q)$ an $r$-diagonal circulant $N \times N$ matrix and $F_N(q)$ a finite rank operator with finite range (uniformly in $N$). Moreover, $F_N(q) = |\Gamma^N_L(q)\rangle\langle\Gamma^N_R(q)|$ has rank-one if $L_k$ has rank-one.

### 3.2 Spectrum of direct integral of operators

In case of a self-adjoint, translation-invariant operator $N$ the spectrum of $N$ coincides with the union of the spectra of the operators contained in the direct integral representation of $N$ after the Fourier-transform (IX.8.5). However, in our case, due to the non-normality of $L$, the information about the pointwise spectrum of $T(q) + F(q)$ may not be sufficient to determine the spectrum of $L$. In fact, already
for the case of the direct sum (direct integral with respect to the counting measure) the spectrum is not the union of the spectra of the fibers as may be seen from the following example [50] Problem 98] where \( \mathcal{H} = \bigoplus_{n \geq 2} \mathbb{C}^n \) and where \( A = \bigoplus_{n \geq 2} A_n \) with

\[
A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ldots .
\]

here \( 1 \in \sigma(A) \), but \( \sigma(A_n) = 0 \) for all \( n \geq 2 \).

Instead, the correct concept to recover such a connection between the operator \( N \) and the operators forming its direct integral decomposition turns out to be the pseudospectrum, which where the resolvent has large norm. More precisely, for a bounded operator \( B \in \mathcal{B}(X) \) for a Banach space \( X \), we define the \( \varepsilon \)-pseudospectrum of \( B \) as the set \( \sigma_\varepsilon(B) \subset \mathbb{C} \) for which \( \| (B - \lambda)^{-1} \| \geq \frac{1}{\varepsilon} \). It is easy to see that \( \sigma(B) \subset \sigma_\varepsilon(B) \) as well as \( \sigma(B) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(B) \).

It is instructive to see the desired connection in the case of a direct-sum operator before turning to the direct integral. The following is a slight reformulation of [17 Theorem 5].

**Lemma 3.5** ([17], Theorem 5). Suppose that \( \mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n \) where each \( \mathcal{H}_n \) is a separable Hilbert space. Let \( A_n \in \mathcal{B}(\mathcal{H}_n) \) for each \( n \in \mathbb{N} \) and \( A = \bigoplus_{n \in \mathbb{N}} A_n \) be a bounded operator on \( \mathcal{H} \). Then for all \( \varepsilon > 0 \) it holds that

\[
\sigma_\varepsilon(A) = \bigcup_{n \in \mathbb{N}} \sigma_\varepsilon(A_n) \text{ and } \sigma(A) = \bigcap_{\varepsilon > 0} \bigcup_{n \in \mathbb{N}} \sigma_\varepsilon(A_n).
\]

**Proof.** Let first \( \lambda \in \bigcup_{n \in \mathbb{N}} \sigma_\varepsilon(A_n) \), then there exists an \( n_0 \in \mathbb{N} \) such that \( \| (A_{n_0} - \lambda)^{-1} \| \geq \frac{1}{\varepsilon} \). Thus, \( \sup_{n \in \mathbb{N}} \| (A_n - \lambda)^{-1} \| \geq \frac{1}{\varepsilon} \) and hence \( \lambda \in \sigma(A) \).

For the converse inclusion, we use contraposition. So suppose that \( \lambda \notin \bigcup_{n \in \mathbb{N}} \sigma_\varepsilon(A_n) \). Then for all \( n \in \mathbb{N} \) it holds that \( \| (A_n - \lambda)^{-1} \| \leq \frac{1}{\varepsilon} \). Thus, \( \sup_{n \in \mathbb{N}} \| (A_n - \lambda)^{-1} \| \leq \frac{1}{\varepsilon} \). It follows that \( \bigoplus_{n \in \mathbb{N}} (A_n - \lambda)^{-1} \) is a well-defined bounded operator. Since

\[
(\bigoplus_{n \in \mathbb{N}} (A_n - \lambda)^{-1}) (A - \lambda) = \bigoplus_{n \in \mathbb{N}} (A_n - \lambda)^{-1} (A_n - \lambda) = 1,
\]

we see that \( (A - \lambda) \) is invertible. Thus \( \lambda \notin \sigma(A) \). The second relation follows immediately. \( \Box \)

To state the analogue of Lemma 3.5 for the direct integral, we first need to define the essential union with respect to the Lebesgue measure on some interval (the following also holds for other measures, but we consider the Lebesgue measure for clarity). If \( M_q \) is a family of measurable sets we say that \( x \in \bigcup_{q \in I}^\text{ess} M_q \) if and only if there exists a set \( M \) of positive measure such that \( M \subset \{ q \mid x \in M_q \} \). The following proof is an extension of the proof techniques just employed and reduces to the case of Lemma 3.5 in the case of the counting measure. For more information on direct integrals and their spectral theory see [51, 52]. A related theorem is proven in the self-adjoint case in [17 XIII.85].

**Theorem 3.6.** Let \( I \subset \mathbb{R} \) be an interval and \( \mathcal{H} = \int_I^\oplus \mathcal{H}_q dq \) for some family of separable Hilbert spaces \( \{ \mathcal{H}_q \}_{q \in I} \). Suppose that \( \{ A(q) \}_{q \in I} \) is a measurable family of bounded operators \( A(q) \) on \( \mathcal{H}_q \) and that \( A = \int_I^\oplus A(q) dq \in \mathcal{B}(\mathcal{H}) \). Then for all \( \varepsilon > 0 \) it holds that

\[
\sigma(A) \subset \bigcup_{q \in I}^\text{ess} \sigma_\varepsilon(A(q)).
\]

Moreover,

\[
\sigma(A) = \bigcap_{\varepsilon > 0} \bigg( \bigcup_{q \in I}^\text{ess} \sigma_\varepsilon(A(q)) \bigg).
\]

**Proof.** We do the proof again by contraposition. So suppose that \( \lambda \notin \bigcup_{q \in I}^\text{ess} \sigma_\varepsilon(A(q)) \). Then

\[
\| (A(q) - \lambda)^{-1} \| \leq \frac{1}{\varepsilon}
\]

for all \( q \in I \).
for almost all \( q \in I \). Thus, \( \text{esssup}_{q \in I} \| (A(q) - \lambda)^{-1} \| \leq \frac{1}{\varepsilon} \). It follows that \( \int_I (A(q) - \lambda)^{-1} dq \) is a well defined bounded operator. Accordingly, considering

\[
\left( \int_I (A(q) - \lambda)^{-1} \right) dq(A - \lambda) = \int_I (A(q) - \lambda)^{-1} (A(q) - \lambda) dq = \mathbb{1},
\]

we see that \((A - \lambda)\) is invertible. Thus \( \lambda \notin \sigma(A) \).

To see the converse inclusion, suppose that \( \lambda \in \bigcap_{\varepsilon > 0} \bigcup_{q \in I} \mathbb{e}ss \sigma_{\varepsilon}(A(q)) \). Thus, for each \( n \in \mathbb{N} \) there exists a set \( I_n \) such that \( |I_n| > 0 \) with \( \| (A(q) - \lambda)^{-1} \| \geq n \) for all \( q \in I_n \). Now, our goal is to construct a vector in the direct integral of the Hilbert spaces out of this family of vectors which has large norm after application of the resolvent. For each \( q \in I_n \) and \( n \in \mathbb{N} \) there exists a \( v_{p,n} \in \mathcal{H}_p \) with \( \|v_{q,n}\| = 1 \) and such that \( \| (A(q) - \lambda)^{-1} v_{q,n} \| \geq \frac{2}{n} \). Now, defining \( w_n = \int_{[0,2\pi]} v_{q,n} \mathbb{1}_{q \in I_n} dq \) then

\[
\|w_n\|^2 = \int_{[0,2\pi]} \|v_{q,n} \mathbb{1}_{q \in I_n}\|^2 dq = \int_{I_n} 1 dq = |I_n|.
\]

Furthermore, it holds that

\[
\left\| \int_{[0,2\pi]} (A(q) - \lambda)^{-1} dq w_n \right\|^2 = \left\| \int_{[0,2\pi]} (A(q) - \lambda)^{-1} v_{q,n} dq \right\|^2 \]

\[
= \int_{I_n} \| (A(q) - \lambda)^{-1} v_{q,n} \|^2 dq \geq \int_{I_n} \left( \frac{n}{2} \right)^2 dq = |I_n| \left( \frac{n}{2} \right)^2.
\]

So we conclude that \( \left\| \int_{[0,2\pi]} (A(q) - \lambda)^{-1} \right\| \geq \frac{2}{n} \). This means that \( \int_{[0,2\pi]} (A(q) - \lambda)^{-1} dq \) is not bounded. By [52 Lemma 1.3] we have that if \( A - \lambda = \left( \int_I (A(q) - \lambda) \right) dq \) is invertible then the inverse is given by \( (A - \lambda)^{-1} = \left( \int_I E(q) \right) dq \) where \( E(q) = (A(q) - \lambda)^{-1} \) for almost all \( q \). Thus, we can conclude that then this inverse would also not be bounded and hence \( A - \lambda \) is not invertible and it holds that \( \lambda \in \sigma(A) \).

\[ \square \]

### 3.3 Spectrum of non-Hermitian Evolution and of the full Lindbladian

We now gradually move from the abstract operator-theoretic picture to the concrete cases of non-Hermitian and Markovian Evolution. With Theorem 3.6 in mind, we need to determine the essential union of the pseudo-spectra. One way to do that is using some continuity in \( q \) in the pseudospectra of \( T(q) \). The continuity is reminiscent of a theorem for self-adjoint operators which also finds the spectrum of the direct integral in terms of its fibers [53]. We prove it using resolvent estimates in the Appendix [A.3] A very recent related result that, in some sense, is in between the generality of Theorem 3.6 and Theorem 3.7 appeared in [54].

In the following we say that a family of operators \( \{B(q)\}_{q \in I} \) is norm continuous if the function \( q \mapsto \|B(q)\| \) is continuous.

From Lemma 3.3 it is clear to see that our assumptions imply this continuity since the operators \( T(q) \) and \( F(q) \) are 2\( r \)-diagonal and finite range with coefficients that polynomials in \( e^{i\theta}, e^{-i\theta} \) and hence continuous functions in \( q \).

**Theorem 3.7.** Let \( I \subset \mathbb{R} \) be a compact and suppose that \( \{A(q)\}_{q \in I} \) is norm continuous. Then

\[
\bigcup_{q \in I} \sigma(A(q)) = \sigma \left( \int_I A(q) dq \right).
\]

We give the proof in Appendix [A.3] We emphasize that the statement of Theorem 3.7 may look innocent, but it in some sense a statement of continuity of the pseudospectrum. Indeed, the spectrum

\[ ^1 \text{One may worry whether there is a measurable choice of } q \mapsto v_{q,n} \text{. This concern we address in Appendix A.2} \]
$q \mapsto \sigma(A(q))$ may be very discontinuous even when $q \mapsto \|A(q)\|$ is continuous (see e.g. Example 4.1), but $q \mapsto \sigma_e(A(q))$ will be continuous for every $\varepsilon > 0$. So when the norm of the $(A(q) - \lambda)^{-1}$ blows up for one $q$ the pseudospectrum of $qs$ in the neighborhood can feel it and $\lambda$ ends up in the essential union. The compactness of $I$ ensures that if $\|(A(q_n) - \lambda)^{-1}\| \to \infty$ for some $q_n \in I$ then that sequence has a subsequentially limit $q_0$, which we can prove is part of the spectrum.

As mentioned, we can use the Theorem 3.7 directly on our direct integral decomposition from Theorem 3.1 to obtain the following corollary.

**Corollary 3.8.** Let $\mathcal{L} \in \mathcal{B}(\text{HS}(\mathcal{H}))$ be a Lindbladian of the form (1) satisfying assumption $\mathcal{A}_2a$ and $\mathcal{A}_2b$ and let $T(q)$ and $F(q)$ be as in Theorem 3.1. Then

$$\sigma(\mathcal{L}) = \bigcup_{q \in [0,2\pi]} \sigma(T(q) + F(q)).$$

Furthermore, for the non-Hermitian evolution $T = \int_{0}^{\sigma} T(q)dq$ it holds that

$$\sigma(T) = \bigcup_{q \in [0,2\pi]} \sigma(T(q)), \text{ and } \sigma(T) \subset \sigma(\mathcal{L}).$$

**Proof.** The first two identities follows directly from Theorem 3.1, Theorem 3.7 and Lemma 3.3. So we only prove the last inclusion. Now, since by Theorem 3.1 we know that $F(q)$ is finite rank for each $q$ and as the essential spectrum of an operator is invariant under finite rank perturbations

$$\sigma(T(q)) = \sigma_{ess}(T(q)) = \sigma_{ess}(T(q) + F(q)) \subset \sigma(T(q) + F(q)) \subset \sigma(\mathcal{L}),$$

where we also used that $T(q)$ is translation-invariant, which means that it only has essential spectrum. Since this is true for each $q \in [0,2\pi]$ we obtain the inclusion. \qed 

Now, for our applications in the next section we know that each $L_k$ is rank-one, so it follows from Theorem 3.1 that $F(q) = |\Gamma_L)(\Gamma_R|$ is also rank-one. In that case, we can strengthen Theorem 3.7 even further.

**Corollary 3.9.** Let $\mathcal{L} \in \mathcal{B}(\text{HS}(\mathcal{H}))$ be a Lindbladian of the form (1) satisfying assumption $\mathcal{A}_2a$ and $\mathcal{A}_2b$ and let $T(q)$ and $F(q)$ be as in Theorem 3.7. Assume further that $F(q) = |\Gamma_L)(\Gamma_R|$ is rank-one. Then

$$\sigma(\mathcal{L}) = \bigcup_{q \in [0,2\pi]} \sigma(T(q)) \cup \{ \lambda \in \mathbb{C} \mid |\Gamma_R|\langle (T(q) - \lambda)^{-1}|\Gamma_L| = -1 \}. $$

**Proof.** By the decomposition in Theorem 3.7 it suffices to prove that

$$\sigma(T(q) + F(q)) = \sigma(T(q)) \cup \{ \lambda \in \mathbb{C} \mid |\Gamma_R|\langle (T(q) - \lambda)^{-1}|\Gamma_L| = -1 \}$$

for each $q \in [0,2\pi]$. This is a standard fact. For completeness, we give a proof in Appendix A.4. \qed

### 4 General applications of the direct integral decomposition

Before continuing with the concrete applications in the next section, we discuss some more abstract consequences of the results presented in previous section.
4.1 Approximate point spectrum of Lindblad operators

For normal operators the residual spectrum is always empty, whereas it might not be the case for non-normal operators. In the following, we use the results of the previous sections to prove that under our standing assumptions the spectrum of \( L \) is purely approximate point, i.e. the residual spectrum is empty. A result that we believe is of independent interest due to the normality aspect that it indicates. Furthermore, could hope that the theorem could take part in resolving Question 2.4 and Question 2.8.

For completeness, we note that covariance in the sense of [3] also implies that the entire spectrum is essential.

**Theorem 4.1.** Consider \( L \in \mathcal{B}(HS(H)) \). Under Assumptions (A2a) - (A2c) it holds that

\[
\sigma_{\text{appt}}(L) = \sigma(L).
\]

**Proof.** We do the split up as in Theorem 3.1 as \( L = T + F \), with \( T = \int_{[0,2\pi]} T(q) dq \) and \( F = \int_{[0,2\pi]} F(q) dq \). Then we show approximate point spectrum in several steps starting with Laurent operators.

**Step 1:** Laurent operators only have approximate point spectrum. Consider a Laurent operator \( T(q) \). The spectrum of \( T(q) \) is given by the symbol curve, which we define in Theorem 5.1. Since the symbol curve is a curve given by a polynomial of finite degree (see Appendix A.5) and the boundary of the spectrum is approximate point spectrum [7, Problem 1.18] it holds that

\[
\sigma(T(q)) = \partial \sigma(T(q)) \subseteq \sigma_{\text{appt}}(T(q)) \subseteq \sigma(T(q)).
\]

Thus, for fixed \( q \) and \( \lambda \in \sigma(T(q)) \) there exists a Weyl sequence \( \{v_{q,n}\}_{n \in \mathbb{N}} \) with \( \|v_{p,n}\| = 1 \) such that \( \|(T(q) - \lambda)v_{q,n}\| \to 0 \) as \( n \to \infty \).

**Step 2:** Direct integrals of Laurent operators only have approximate point spectrum. Let us prove that \( \lambda \in \sigma(T) \) is part of the approximate point spectrum, i.e. we construct a Weyl sequence for \( T \). From Corollary 3.8 it holds that \( \sigma(T) = \bigcup_{q \in [0,2\pi]} \sigma(T(q)) \), so let \( \lambda \in \sigma(T(q_0)) \). From Step 1 we have a Weyl sequence \( v_n \) corresponding to \( \lambda \) for the operator \( T(q_0) \). Now, define \( w_n = \int_{[0,2\pi]} v_n \sqrt{n} \mathbb{I}_{[q_0 - \frac{1}{n}, q_0 + \frac{1}{n}]} dq \).

Then

\[
\|w_n\|^2 = \int_{[0,2\pi]} \|v_n\|^2 n \mathbb{I}_{[q_0 - \frac{1}{n}, q_0 + \frac{1}{n}]} dq = 1.
\]

Furthermore,

\[
\left\| \int_{[0,2\pi]} (T(q) - \lambda) dq w_n \right\|^2 = \int_{q_0 - \frac{1}{n}}^{q_0 + \frac{1}{n}} n \|(T(q) - \lambda)v_n\|^2 dq \leq \int_{q_0 - \frac{1}{n}}^{q_0 + \frac{1}{n}} n \left(\|(T(q) - T(q_0))v_n\| + \|(T(q_0) - \lambda)v_n\|\right)^2 dq \to 0,
\]

where we used the continuity bound that \( \|(T(q) - T(q_0))v_n\| \to 0 \) as \( q \to q_0 \) as well as \( \|(T(q_0) - \lambda)v_n\| \to 0 \) as \( n \to \infty \).

**Step 3:** Direct integrals of Laurent operators with finite range perturbations only have approximate point spectrum. Assume that \( \lambda \in \sigma(L) \).

3a) Spectrum of the non-Hermitian evolution is approximate point: Suppose first that \( \lambda \in \sigma(T) \).

Then translation-invariance of \( T(q) \) means that \( S_1 T(q) S_{-1} = T(q) \). Consider a Weyl sequence \( v_n \) for \( T(q) \). Then \( \{S_a v_n\}_{n \in \mathbb{N}} \) is also a Weyl sequence for \( T(q) \) corresponding to \( \lambda \) for any \( a \in \mathbb{Z} \) since

\[
\|T(q) S_a v_n\| = \|S_a T(q) S_{-a} S_a v_n\| = \|S_a T(q) v_n\| = \|T(q) v_n\| \to 0.
\]

Since \( F(q) \) is finite range and \( v_n \) is finite norm for every \( \varepsilon > 0 \) there exist an \( a \in \mathbb{Z} \) such that \( \|F(q) S_a v_n\| \leq \varepsilon \). Thus, there exists a sequence \( a_n \) such that \( \|F(q) S_{a_n} v_n\| \leq \frac{\varepsilon}{n} \). Now, \( w_n = S_{a_n} v_n \) is a Weyl sequence for \( T(q) + F \) since

\[
\|(T(q) + F) S_{a_n} v_n\| \leq \|T(q) v_n\| + \|FS_{a_n} v_n\| \to 0.
\]
By an argument as in Step 2 we can use continuity to conclude that this eigenvector in the fiber gives rise to an approximate eigenvector in the direct integral.

3b) **Additional spectrum from quantum jump terms is approximate point:** Suppose that $\lambda \notin \sigma(T)$ then by Corollary 3.9 there is a $q$ such that $(\Gamma_R[(T(q) - \lambda)^{-1}] | \Gamma_L) = -1$. Now, consider the matrix acting on the vector $v = (T(q) - \lambda)^{-1} | \Gamma_L$

$$(T(q) - \lambda + |\Gamma_L|\Gamma_R)(T(q) - \lambda)^{-1} | \Gamma_L) = | \Gamma_L) + \langle \Gamma_R | (T(q) - \lambda)^{-1} | \Gamma_L \rangle | \Gamma_L) = 0.
$$

So we conclude that $(T(q) + |\Gamma_L|\Gamma_R)|v = \lambda v$ which means that $(T(q) - \lambda)^{-1} | \Gamma_L)$ is an eigenvector of $(T(q) + |\Gamma_L|\Gamma_R)$ with eigenvalue $\lambda$. Again, by an argument as in Step 2 we can use continuity to conclude that this eigenvector in the fiber gives rise to an approximate eigenvector in the direct integral.

Thus, we see that every point in the spectrum has a corresponding Weyl sequence. Although this Weyl sequence is not an eigenfunction it hint towards what an eigenfunction look like. In the case of the additional spectrum coming from quantum jumps terms we can say those Weyl sequences are classical in the case where $T(q)$ is tridiagonal.

**Proposition 4.2.** Suppose that $\lambda \in \sigma(L)/\sigma(T)$ and that $T = \int_{0,2\pi} T(q) dq$ where $T(q)$ is tridiagonal for each $q \in [0,2\pi]$. Then in the position basis there exists a Weyl sequence for $\lambda$ which exhibits exponential decay away from the diagonal.

**Proof.** By Corollary 3.9 there is a $q_0$ such that $(T(q_0) - \lambda)$ is invertible and it holds that

$$(\Gamma_R[(T(q_0) - \lambda)^{-1}] | \Gamma_L) = -1.
$$

This implies

$$(T(q_0) - \lambda) + |\Gamma_L|\Gamma_R)((T(q_0) - \lambda)^{-1} | \Gamma_L) = | \Gamma_L) + \langle \Gamma_R | (T(q_0) - \lambda)^{-1} | \Gamma_L \rangle | \Gamma_L) = 0.
$$

So $(T(q_0) - \lambda)^{-1} | \Gamma_L)$ is in the kernel of $(T(q_0) - \lambda) + |\Gamma_L|\Gamma_R)$ for fixed $q_0$. Let $c = \| (T(q_0) - \lambda)^{-1} | \Gamma_L \|$. Then, as before, define for each $n \in \mathbb{N}$ the vector $v_n$ by

$$v_n = \frac{\sqrt{n}}{c} \int_{q_0 - \frac{1}{2\pi}, q_0 + \frac{1}{2\pi}} (T(q_0) - \lambda)^{-1} | \Gamma_L) dq.
$$

Then $\| v_n \| = 1$ and $v_n$ is a Weyl sequence for $\int_{0,2\pi} (T(q) + |\Gamma_L|\Gamma_R)) dq$ since by continuity

$$\left\| \int_{0,2\pi} (T(q) + |\Gamma_L|\Gamma_R) dq \int_{q_0 - \frac{1}{2\pi}, q_0 + \frac{1}{2\pi}} \frac{\sqrt{n}}{c} (T(q_0) - \lambda)^{-1} | \Gamma_L) dq \right\|^2
$$

$$= \frac{n}{c} \int_{q_0 - \frac{1}{2\pi}}^{q_0 + \frac{1}{2\pi}} \| (T(q) + |\Gamma_L|\Gamma_R)(T(q_0) - \lambda)^{-1} | \Gamma_L) \|^2 dq \rightarrow 0.
$$

In the case where $(T(q_0) - \lambda)$ is tridiagonal, we can explicitly determine the inverse. We now want to unwind the Fourier transform. Back in $L^2(\mathbb{Z}) \otimes L^2(\mathbb{Z})$ we get that

$$v_n = \sum_{x,y \in \mathbb{Z}} \int_{q_0 - \frac{1}{2\pi}}^{q_0 + \frac{1}{2\pi}} e^{ipx} \langle y | (T(q_0) - \lambda)^{-1} | \Gamma_L) dq | x \rangle | y \rangle.
$$

For example in case $| \Gamma_L) = | 0 \rangle$ we get that $\langle y | (T(q_0) - \lambda)^{-1} | 0 \rangle = \frac{\lambda_2(q_0) y}{\sqrt{(|y|^2 - 2^2 - 4\lambda_2(q_0) y)}}$ where $| \lambda_2(q_0) | < 1$. Thus, we see that all coefficients are exponentially decaying away from the diagonal. From $| \lambda_2(q_0) |$ we can even calculate the coherence length. Note that the argument for this exponential decay remains valid even if $| \Gamma_L)$ is just local around 0 instead of on-site.
In Section 6.2 we prove a weak extension, i.e. there cannot be additional spectrum with Weyl sequences which are not concentrated along the diagonal. In that sense we can really say that the states are classical.

Notice that there is no eigenvector in the direct integral picture and this is intimately related to the absence of steady states and which space the operator is defined on. In particular, notice that for the case of dephasing noise if we instead defined \( \mathcal{L} \) on \( B(\mathcal{H}) \) then \( \mathcal{L}(\mathbb{I}) = 0 \) and in that case the identity would be a steady state which normalizable. Since \( \mathbb{I} \) is neither Hilbert-Schmidt nor trace-class \( \mathcal{L} \) does not have a steady state although \( 0 \in \sigma(\mathcal{L}) \).

### 4.2 Sufficient conditions for convergence of finite volume spectra to infinite volume spectra

The convergence behaviour of spectra of finite size approximations of Laurent and Toeplitz matrices to their infinite counterparts is a central topic in numerical analysis \[56\]. For Lindbladians this subject is, to our knowledge, still untouched, but in the view of Corollary 3.8 under our assumptions we can in some sense reduce the question of whether \( \sigma(\mathcal{L}_{N}^{\text{per}}) \to \sigma(\mathcal{L}) \) to the better studied questions of convergence in the case of Laurent and Toeplitz matrices. Since all the Schatten classes agree on finite volume our results also supports our agenda of studying the spectrum of \( \mathcal{L} \) as an operator on HS(\( \mathcal{H} \)).

To study convergence of subsets of \( \mathbb{C} \) we use the Hausdorff metric. Following \[55\] for a sequence of subsets of the complex plane \( \{S_n\}_{n \in \mathbb{N}} \) which are all non-empty define \( \liminf_{n \to \infty} S_n \) as the set of all \( \lambda \in \mathbb{C} \) that are limits of a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) which satisfies \( \lambda_n \in S_n \). Conversely, \( \limsup_{n \to \infty} S_n \) is defined as all subsequential limits of such sequences \( \{\lambda_n\}_{n \in \mathbb{N}} \) with \( \lambda_n \in S_n \). A central characterization of the Hausdorff metric is then the following.

**Theorem 4.3.** Let \( S \) and the members of the sequence \( \{S_n\}_{n \in \mathbb{N}} \) be nonempty compact subsets of \( \mathbb{C} \) then \( S_n \to S \) in the Hausdorff metric if and only if

\[
\limsup_{n \to \infty} S_n = S = \liminf_{n \to \infty} S_n.
\]

See \[57\] Sections 3.1.1 and 3.1.2 or \[58\] Section 2.8 for a proof. In \[55\] convergence is proven for periodic boundary conditions for tridiagonal matrices and diagonal perturbations except that it has not been proven that the symbol curve is fully captured by the finite size approximations. We will not use periodic boundary conditions for tridiagonal matrices and diagonal perturbations except that it has not been proven that the symbol curve is fully captured by the finite size approximations. We will not use this result, but we state it for comparison, since it is of a similar flavour as the condition that we have in Theorem 4.5.

**Theorem 4.4** (\[55\],Corollary 1.3). Suppose that \( T \) is tridiagonal with \( \alpha, \beta, \gamma \in \mathbb{C} \) on the diagonal and that \( K = \text{diag}(K_{11}, \ldots, K_{mm}) \) for some \( m \in \mathbb{N} \) then

\[
\lim_{n \to \infty} (\sigma(T_n^{\text{per}} + P_n K P_n) \cup \sigma(T)) = \sigma(T + K)
\]

where \( P_n \) is the projection onto the sites \( \{0, \ldots, n\} \).

It is further conjectured that \( \lim_{n \to \infty} \sigma(T_n^{\text{per}} + K) = \sigma(T + K) \) under the same assumptions (see \[55\] Conjecture 7.3).

Now, these results are not quite strong enough for our purposes, but we believe that one can show the assumption in the following theorem are satisfied at least for the model of local dephasing that we consider in Section 5.1. As we will see in Figures 6 and 7 in the next section it is important that we use periodic boundary conditions for \( \mathcal{L} \).

**Theorem 4.5.** Suppose that \( q \mapsto \sigma(T(q) + F(q)) \) is continuous and \( \{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N} \) is a sequence such that \( a_n \to \infty \). Assume that \( \tilde{f} \in \mathbb{T}_{a_n} \) for all \( n \in \mathbb{N} \) then

\[
\sigma(T_{a_n}^{\text{per}}(q) - F_{a_n}(q)) \to \sigma(T(q) - F(q))
\]

for each \( q \in [0, 2\pi] \) where the convergence is uniform. Then

\[
\sigma(L_{N}^{\text{per}}) \to \sigma(\mathcal{L}).
\]
Proof. Let us first prove that $\sigma(\mathcal{L}) \subset \liminf_{N \to \infty} \sigma(\mathcal{L}^\text{per}_N)$. So let $\lambda \in \sigma(\mathcal{L})$ by Corollary 3.8 it holds that $\lambda \in \sigma(T(q_0) + F(q_0))$ for some $q_0 \in [0, 2\pi]$. Then find a sequence $q_m \in \bigcup_{N \in \mathbb{N}} \mathbb{T}_N$ such that $q_m \to q_0$. By continuity of $q \mapsto \sigma(T(q) + F(q))$, we can find a sequence $\lambda_n \in \sigma(T(q_m) + F(q_m))$ such that $\lambda_n \to \lambda$. For each $q_m$, there is a sequence $\{a_m\}_{m \in \mathbb{N}} \subset \mathbb{T}_{a_m}$ such that $a_m \to \infty$ and $q_m \in \mathbb{T}_{a_m}$ for each $m \in \mathbb{N}$. Now, since $\sigma(T^\text{per}_{a_m}(q_m) - F_{a_m}(q_m)) \to \sigma(T(q_m) - F(q_m))$, that is, there exists a sequence $\lambda^m_n \in \sigma(T^\text{per}_{a_m}(q_m) - F_{a_m}(q_m))$ such that $\lambda^m_n \to \lambda_n$ as $m \to \infty$. Then we can finish with a diagonal argument. I.e. for each $n$ find $k(n) \in \mathbb{N}$ such that $|\lambda^m_n - \lambda_n| \leq \frac{1}{n}$ and define the sequence $\gamma_n = \lambda^m_{k(n)} \in \sigma(T^\text{per}_{a_{k(n)}}(q_m) - F_{a_{k(n)}}(q_m))$ satisfies that

$$|\gamma_n - \lambda| \leq |\lambda^m_{k(n)} - \lambda_n| + |\lambda_n - \lambda| \to 0$$

as $n \to \infty$. Thus, we conclude that $\sigma(\mathcal{L}) \subset \liminf_{N \to \infty} \sigma(\mathcal{L}^\text{per}_N)$. Let us then prove that $\limsup_{N \to \infty} \sigma(\mathcal{L}^\text{per}_N) \subset \sigma(\mathcal{L})$. So let $\lambda_n \in \sigma(\mathcal{L}^\text{per}_{a_n})$ and suppose that $\lambda_n \to \lambda$. By Theorem 3.4 it holds that

$$\sigma(\mathcal{L}^\text{per}_N) = \bigcup_{q \in \mathbb{T}_{a_n}} \sigma(T^\text{per}_{a_n}(q) + F_{a_n}(q)).$$

I.e. there is a sequence $q_n \in \mathbb{T}_{a_n}$ such that $\lambda_n \in \sigma(T^\text{per}_{a_n}(q_n) + F_{a_n}(q_n))$. Since $\sigma(T^\text{per}_{a_n}(q_n) + F_{a_n}(q_n)) \to \sigma(T(q_n) - F(q_n))$ and the convergence is uniform in $q_n$, there must be a sequence $\beta_n \in \sigma(T(q_n) - F(q_n)) \subset \sigma(\mathcal{L})$ such that $|\lambda_n - \beta_n| \to 0$. Now, since $\lambda_n \to \lambda$ this means that $\beta_n \to \lambda$, but as $\sigma(\mathcal{L})$ is closed it holds that $\lambda \in \sigma(\mathcal{L})$. \hfill \Box

4.3 Gaplessness of translation-covariant Lindblad generators

As a last application of the theory developed in this section we prove gaplessness of translation-covariant Lindblad generators. We saw in Theorem 4.6 how the Lindbladian that we study are gapless as long as we know that 0 is in the spectrum of $\mathcal{L}$. One might notice that the paper [4], which has a setup which is fairly similar to ours, assume that there is a spectral gap, in a similar, but different way this is the case in [6]. Here, we say that $\mathcal{L}$ is has a gap if 0 is an isolated point in the spectrum.

**Theorem 4.6.** Suppose that $\mathcal{L}$ has $H = \Delta$ and $\{L_k\}_{k \in \mathbb{Z}}$ satisfies $\mathcal{A}_{2a} - \mathcal{A}_{2c}$. Suppose $0 \in \sigma(\mathcal{L})$. Then $\mathcal{L}$ is either gapless or has an infinite dimensional kernel.

Notice, that if we add a random potential to $H$ and $\mathcal{L}$ is gapless then it stays gapless by Theorem 6.1.

**Proof.** If (as is the case in Section 5.2) the non-Hermitian evolution is gapless we are done by Corollary 3.8. If on the other hand the spectrum of the non-Hermitian evolution is gapped then by Corollary 3.9 the point 0 \in \sigma(\mathcal{L}) must be a solution of the equation

$$\langle F_R | (T(q) - z)^{-1} | F_L \rangle = -1$$

for $z$ for each $q \in [0, 2\pi]$. This equation is a polynomial equation in $z$ and its front factors are analytic functions and hence they have analytic solutions $z(q)$ (or in other words there exists an analytic choice of branches).

Since there is a $q_0$ such that $z(q_0) = 0$ then it is either the case that $z(q) = 0$ for all $q$ in which case the kernel is infinite dimensional or the branches extend away from 0, but since they are analytic by continuity the spectrum must be gapless in that case. \hfill \Box
5 Examples of spectra of translation-covariant Lindbladians

In this section, we are going to apply the techniques developed so far to several Lindblad models studied in the literature such as local dephasing and decoherent hopping. In all cases, we determine an expression for the spectrum of the Lindblad generator. In the following, we study Lindbladians \( L \) defined on \( \text{HS}(\mathbb{F}(\mathbb{Z})) \) which as in \([3]\) are of the form

\[
L(\rho) = -i[H, \rho] + G \sum_k L_k \rho L_k^* - \frac{1}{2}(L_k^* L_k + \rho L_k^* L_k),
\]

where \( G > 0 \) is the strength of the dissipation and \( L_k \) are the Lindblad operators.

We will use Theorem \([3, \text{Theorem } 1.2]\) to rewrite \( L \) as an direct of the operators \( T(q) + F(q) \) where \( T(q) \) is a banded Laurent operator and \( F(q) \) is finite range, finite rank. Further by Corollary \([3, \text{Corollary } 3.9]\) the spectrum of \( L \) is the union of the spectra of \( T(q) + F(q) \) which we again, in the rank-one case, by Corollary \([3, \text{Corollary } 3.9]\) can find as the spectrum of \( T(q) \) as well as all solutions \( z \in \mathbb{C} \) to the equation

\[
(\Gamma_R)(T(q) - z)^{-1} | \Gamma_L) = -1.
\]

Since \( T(q) \) is a banded Laurent operator its spectrum is easy to calculate Fourier transformation. Indeed, as \( T(q) \) has constant entries \( a_i \) on the \( i \)-th diagonal and \( a_i = 0 \) for \( |i| \geq r \) its Fourier transform is given as the multiplication \( a(z) = \sum z^i \) for \( z \in \mathbb{T} \), the unit circle. We will denote the curve that \( a \) traces out in the complex the symbol curve. It is a well known result that the spectrum of the Laurent operator is the range of the symbol curve.

**Theorem 5.1 \([3, \text{Theorem } 1.2]\).** Let \( L \) be a bi-infinite banded Laurent operator, i.e. \( L \) has constant entries \( a_i \) on the \( i \)-th diagonal and \( a_i = 0 \) for \( |i| \geq r \) then spectrum is the range of the symbol curve \( a(z) = \sum_{i=-r}^r a_i z^i \) for \( z \in \mathbb{T} \). I.e.

\[
\sigma(L) = \left\{ \sum_{i=-r}^r a_i z^i \mid z \in \mathbb{T} \right\}.
\]

Moreover, if \( 0 \notin \sigma(L) \) then the inverse of \( L \) has the inverse symbol \( a^{-1} \).

In the concrete applications, we consider we often end up considering tridiagonal matrices \( T(q) \) and therefore, we review some results about tridiagonal Laurent operators and their invertibility in Appendix \([A.5]\). If \( T(q) \) is tridiagonal we let \( \alpha, \beta, \gamma : [0, 2\pi] \rightarrow \mathbb{C} \) be the entries of \( T(q) \). Thus, the symbol curve is given by

\[
a(z) = \alpha z^{-1} + \beta + \gamma z, z \in \mathbb{T}
\]

which is a (possibly degenerate) ellipse. In particular, for the dephasing example considered in the next section we see from \([13]\) that \( \alpha(q) = i(1 - e^{-iq}), \beta(q) = -G, \gamma(q) = i(1 - e^{iq}) \).

5.1 Local dephasing

One prominent example, which has been broadly discussed in the physics literature is local dephasing. The spectrum was investigated numerically with free boundary conditions in \([13]\) and analytically many of the same considerations where made in finite volume with periodic boundary conditions \([13, 59]\). We plot some numerical results in finite volume in Figure \([2]\).

The model is given by choosing the Hamiltonian as nearest neighbor hopping

\[
H = -\Delta = -\sum_{k \in \mathbb{Z}} |k\rangle \langle k + 1| - |k + 1\rangle \langle k|\tag{13}
\]

and the local Lindblad operators as local dephasing, i.e. projectors onto single lattice sites \( k \in \mathbb{Z} \)

\[
L_k = |k\rangle \langle k|.	ag{14}
\]

We notice that each Lindblad generator satisfies \( L_k = L_k^* = L_k^* L_k = |k\rangle \langle k| \). Now, define \( Q \) by \( Q = \sum_k L_k^* L_k = 1 \) and therefore the model satisfies the assumptions of Theorem \([3, \text{Theorem } 3.1]\). Using Lemma \([3, \text{Lemma } 3.3]\) we can first determine the spectrum of the non-Hermitian part of the evolution \( \mathcal{T} = \int_{[0, 2\pi]} T(q) dq \) in Theorem \([3, \text{Theorem } 3.1]\) without the quantum jumps \( \mathcal{F} = \int_{[0, 2\pi]} F(q) dq \)
Proposition 5.2. Let $H = -\Delta$ as in (13) and let the Lindblad operators $L_k$ be given as $|k\rangle\langle k|$, then the generator of the non-Hermitian evolution $T \in \mathcal{B}(\mathcal{H}(l^2(\mathbb{Z})))$ according to (9) satisfies

$$\sigma(T) = -G + i[-4,4].$$

Proof. Using Lemma 3.3 and $Q = \sum_k L_k^* L_k = 1$, we see that $T(q)$ is given by

$$T(q) = i(1 - e^{-iq})S + i(1 - e^{iq})S^* - G. \quad (15)$$

For fixed $q$ this corresponds to a tridiagonal Laurent matrix with symbol curve $T(q,\theta) = i(1 - e^{-iq})e^{i\theta} + i(1 - e^{iq})e^{-i\theta} - G$. By Theorem 3.7 and Theorem 5.1 it follows that

$$\sigma(T) = \bigcup_q \sigma(T(q)) = \bigcup_q \{T(q,\theta) \mid \theta \in [0,2\pi]\}.$$

Putting in the explicit form of the parameters from (15), we find

$$T(q,\theta) = -G + 2i(\cos(\theta) - \cos(\theta - q)).$$

Since $\theta$ and $q$ can be varied independently over the interval $[-\pi,\pi]$ we can realize any value in $-G+i[-4,4]$ as claimed.

From (15) we see that $\alpha(q) = i(1 - e^{-iq})$, $\beta(q) = -G$, $\gamma(q) = i(1 - e^{iq})$ are the entries of the tridiagonal matrix $T(q)$.

Remark 5.3. Notice that the proof of Proposition 5.2 is in some sense equivalent to finding out that the polynomial $\alpha x^{-1} + \beta + \gamma x$ has a root of absolute value 1 if and only if $z \in -G + [-4i,4i]$, where $\alpha,\beta,\gamma : [0,2\pi] \to \mathbb{C}$ are the diagonals of the tridiagonal matrix $T(q)$. This has an interpretation in terms of transfer matrices: If one considers the Laurent operator with $\alpha,\beta$ and $\gamma$ on the diagonal and solves
it iteratively then a root of \( \alpha x^{-1} + \beta + \gamma x \) of absolute value less than 1 corresponds to an exponentially decaying and therefore normalizable solution. Conversely, a root of absolute value strictly larger than 1 corresponds to an exponentially increasing and therefore not normalizable solution. In this picture, one can interpret the rank-one perturbation \( |\Gamma_R \rangle \langle \Gamma_L| \) as boundary condition at 0 for an iterative solution for positive and negative entries.

Thus, we have computed the spectrum of the non-Hermitian Hamiltonian. Turning now to the full Lindblad generator \( \mathcal{L} \), we can characterize its spectrum depending on the dissipation strength by including the quantum jump part as a perturbation to the non-Hermitian part.

**Proposition 5.4.** Let \( H = -\Delta \) be the Laplacian defined in \( (15) \) and let the Lindblad operators \( L_k \) be given as \( |k\rangle\langle k| \), then the Lindblad generator \( \mathcal{L} \) satisfies

\[
\sigma(\mathcal{L}) = (-G + i[-4, 4]) \cup \begin{cases} [-G, 0], & \text{if } G \leq 4 \\ [-G + \sqrt{G^2 - 16}, 0], & \text{otherwise.} \end{cases}
\]

**Proof.** Since \( T(q) \) is translation invariant its spectrum is essential. Thus, when \( T(q) \) is compactly perturbed the spectrum can only be extended. So, as in the proof of Corollary 3.8 we directly get that

\[-G + [-4i, 4i] = \sigma(T) \subset \sigma(\mathcal{L}).\]

The coefficients of the perturbation also follow from our consideration in Section 3. Based on Lemma 3.3, we find that \( F(q) = G \langle 0|\langle 0| \), independent of \( q \), which means that we may choose \( |\Gamma_L \rangle = |\Gamma_R \rangle = \sqrt{G} |0\rangle \). Hence, both \( |\Gamma_L \rangle \) and \( |\Gamma_R \rangle \) are bounded vectors and \( q \mapsto \langle 0|(T - z)^{-1}|0\rangle \) is continuous for \( z \notin -G + [-4i, 4i] \). Therefore, we may determine \( \sigma(\mathcal{L}) \) from Corollary 3.9 which leads to the condition

\[G\langle 0|(T - z)^{-1}|0\rangle = -1.\]

Thus, we now have to compute \( \langle 0|(T - z)^{-1}|0\rangle \) which we elaborate on in Appendix A.5. Define \( \lambda_+ \) and \( \lambda_- \) by (with a convention on taking square roots of complex numbers elaborated on in Appendix A.5)

\[\lambda_{\pm} = -\frac{\beta}{2\gamma} \pm \sqrt{\left(\frac{\beta}{2\gamma}\right)^2 - \frac{\alpha}{\gamma}}.\]  

(16)

Let further, \( |\lambda_2| \leq |\lambda_1| \) such that \( \{\lambda_1, \lambda_2\} = \{\lambda_+, \lambda_-\} \). Then notice that the conditions of Lemma A.3 are satisfied and we may therefore \( \lambda_2 < 1 < |\lambda_1| \). Thus, we can use Lemma A.4 to find the inverse of the Laurent matrix to obtain

\[-1 = (-1)^{1 + |\lambda_+| < 1 < |\lambda_-|} \frac{G}{\sqrt{(\beta - z)^2 - 4\alpha \gamma}}.\]  

(17)

Our strategy is to square the equation, solve to find a set a possible \( z \) and then reinsert into (17) to see which sign is correct. The potential solutions satisfy

\[z = \beta \pm \sqrt{G^2 + 4\alpha \gamma}.\]

In the specific example with dephasing noise we saw that \( \beta = -G, \alpha = -ie^{-iq} + i \) and \( \gamma = -ie^{iq} + i \). As \( \alpha \gamma = 2(cos(q) - 1) \), we consider

\[z = -G \pm \sqrt{G^2 + 8(cos(q) - 1)}.\]

As \( G^2 + 8(cos(q) - 1) \in [-16 + G^2, G^2] \) it is natural to consider the cases \( G < 4 \) and \( G \geq 4 \). In the case \( G < 4 \) we have \(-16 + G^2 < 0 \). Thus,

\[\bigcup_{q \in [0, 2\pi]} \{ \sqrt{G^2 + 8(cos(q) - 1)} \} = i[0, \sqrt{16 - G^2}] \cup [0, G].\]
Therefore, the potential values of $z$ are
\[ z \in [-G, 0] \cup [-2G, 0] \cup \left( -G + i \left( -\sqrt{16 - G^2} \right), \sqrt{16 - G^2} \right). \]
Notice that since $[-\sqrt{16 - G^2}, \sqrt{16 - G^2}] \subset [-4i, 4i]$ this does not give additional spectrum.

It now remains to check the sign of the remaining possible solutions. From (17) we see that the square root must yield a (positive) real number and we at the same time need that $|\lambda_+| < 1 < |\lambda_-|$, so we only need to deal with the case where $0 \leq G^2 + 8(\cos(q) - 1) \leq G^2$. Then
\[ \beta - z = \mp \sqrt{G^2 + 8(\cos(q) - 1)}. \]
Using $\mp z$ and $\mp \lambda$ to denote two, on the outset, independent signs we obtain from (16) that
\[ 2\gamma \lambda \mp = - (\beta - z) \pm \lambda \sqrt{(\beta - z)^2 - 4\alpha \gamma} = \pm z \sqrt{G^2 + 8(\cos(q) - 1)} \pm \lambda \sqrt{G^2}. \]
If $\pm z = +$ then $|\lambda_-| < |\lambda_+|$ and $\pm z = -$ then $|\lambda_+| < |\lambda_-|$. Thus, from (17) we see that $\pm z = +$ is the only valid solution. Thus, only $z \in [-G, 0]$ are valid solutions.

In the case $G \geq 4$. It holds that $-16 + G^2 \geq 0$ and therefore there is only one segment $[\sqrt{G^2 - 16}, G]$. This observation translates into
\[ z \in [-G + \sqrt{G^2 - 16}, 0] \cup [-2G, -G - \sqrt{G^2 - 16}]. \]
Using a similar argument as above one finds that only the part $[-G + \sqrt{G^2 - 16}, 0]$ has the correct sign.

**Emergence of two timescales:** From Proposition 5.4 we see that if we change $G$ from a value below 4 to a value above 4 it becomes possible to change the connected component of the spectrum containing $\{0\}$ shrinks. This indicates the emergence of two timescales in the dynamics. The first one corresponding to the fast decay at rate $e^{-tG}$ and a second one with a much slower decay. On the infinite lattice it is difficult to discuss the density of states, but it is noticed numerically in (19) that there are of the order of $L$ eigenvalues on the real axis close to 0 and $L^2 - L$ eigenvalues with real part close to $-G$. In general, such a phenomenon can be explained from symmetry as in (22) Appendix B.9.

**Constructing states with exact decay properties in finite dimensions:** Along the lines of the discussion in Section 2.3 we want to indicate how the spectrum has direct dynamical consequences. In finite volume with $N$ lattice sites the steady state is $\frac{1}{N} \rho_\infty$. We further note that the eigenvectors are never states due to the trace preserving nature of $L$, see Remark 2.9 therefore the effect may not be directly observable. However, if $\lambda$ is an eigenvalue and $A$ is an eigenvector such that $L(A) = \lambda A$. Then it holds that $L(A^*) = L(A)^* = \lambda A^*$. Thus, if $\lambda$ is real then $A + A^*$ is a self-adjoint eigenvector of $L$. Still if $\lambda \neq 0$ we must have that $\text{Tr}(A + A^*) = 0$. Therefore $\rho = \rho_\infty + c(A + A^*)$ has trace 1 and is self-adjoint for any $c \in \mathbb{R}$ (here $\rho_\infty = \frac{1}{N}$). Furthermore, since $(A + A^*)$ is self-adjoint it is unitarily equivalent to some diagonal matrix with real eigenvalues. If we pick $c$ smaller than $N$ times the most negative of these eigenvalues is sufficient to ensure that $\rho$ is also positive and therefore a state which has the property that $\rho - \rho_\infty$ is in the eigenspace of $L$ with eigenvalue $\lambda$.

### 5.2 Non-normal dissipators

Next, we turn to a model with non-normal Lindblad operators. The following family of dissipative models was studied in (61). For the Hamiltonian $H$ we shall take the discrete Laplacian given in (13). The Lindblad operators are of the form
\[ L_k = \left( \begin{array}{cc} |k\rangle \langle k| + e^{i\delta} |k + l\rangle \langle k + l| & \end{array} \right) \left( \begin{array}{c} |k\rangle \langle k| - e^{-i\delta} |k + l\rangle \langle k + l| \end{array} \right) \]
for some $\delta \in [0, 2\pi]$ and $l \in \mathbb{N}$. Notice that $L_k$ is not normal. We have that
\[ L_k^* L_k = 2|k\rangle \langle k| + 2|k + l\rangle \langle k + l| - 2e^{i\delta} |k\rangle \langle k + l| - 2e^{-i\delta} |k + l\rangle \langle k|. \]
Define $Q$ by
\[ Q = \sum_k L_k^* L_k = 4\mathbb{I} - 2D_l, \]
where $D_l$ is the operator which has $e^{i\delta}$ and $e^{-i\delta}$ on the $l$'th sub- and superdiagonal. To compute the form of the rank-one perturbation notice that $\alpha_0 = 1, \alpha_l = e^{i\delta}$ and all other entries are 0. Similarly, $\beta_0 = 1, \beta_l = -e^{-i\delta}$ and all other entries are 0.

Thus, from Lemma [3.3] we obtain
\[ \frac{\langle \Gamma_l \rangle}{\sqrt{G}} = \sum_{r_1, r_2} \alpha_{r_1} \alpha_{r_2} e^{i\delta} |r_2 - r_1| = (1 + e^{i\delta})|0\rangle + e^{i\delta} e^{i\delta} |\ell\rangle + e^{-i\delta} |l\rangle \]
as well as
\[ \frac{\langle \Gamma_l \rangle}{\sqrt{G}} = \sum_{r_1', r_2'} \beta_{r_1'} e^{-i\delta} \beta_{r_2'} (r_2' - r_1') = (1 + e^{-i\delta})|0\rangle - e^{i\delta} e^{-i\delta} (-l\rangle - e^{i\delta} |l\rangle. \]

We see that the non-Hermitian evolution $T$ is given by
\[ (i\Delta - \frac{G}{2} Q) \otimes \mathbb{I} + \mathbb{I} \otimes (i\Delta - \frac{G}{2} Q) = (i\Delta + GD_l) \otimes \mathbb{I} + \mathbb{I} \otimes (i\Delta + GD_l) - 4G \otimes \mathbb{I} \]
which is tridiagonal. Let us examine how this $Q$ transforms under the shift and Fourier transformation in the first variable that we do in Theorem [5.5]. The term $-4G \otimes \mathbb{I}$ is left invariant. The term $(iGD_l) \otimes \mathbb{I}$ becomes:
\[ Ge^{i\delta} e^{-i\delta} \sum_j |j\rangle \langle j - l| + Ge^{-i\delta} e^{i\delta} \sum_j |j\rangle \langle j + l|. \]

For the term $\mathbb{I} \otimes (GD_l)$ we get
\[ Ge^{-i\delta} \sum_j |j\rangle \langle j + l| + Ge^{i\delta} \sum_j |j\rangle \langle j - l|. \]

In total,
\[ T(q) = G \sum_j e^{-i\delta} \left( 1 + e^{i\delta} \right) |j\rangle \langle j + l| + Ge^{i\delta} \sum_j |j\rangle \langle j - l| - 4G \otimes \mathbb{I} + i(1 - e^{-i\delta})S + i(1 - e^{i\delta})S^*. \]

We can see that the full operator has non-zero entries in 5 diagonals if $l > 1$ so from now on we will assume that $l = 1$ which is also the case mainly studied in [21] and [60]. In that case we find the infinite band matrix with diagonals:
\[ \alpha = i(1 - e^{i\delta}) + Ge^{-i\delta} \left( 1 + e^{i\delta} \right) \quad \beta = -4G \quad \gamma = i(1 - e^{-i\delta}) + Ge^{i\delta} \left( 1 + e^{-i\delta} \right). \]

Let us first determine the union of the $q$-wise spectra without the perturbation $F(q)$. See also Figure [4] for an illustration.

Proposition 5.5. The spectrum of the non-Hermitian evolution $A$ is for the Lindbladian from [18] in the case $l = 1$ is given by
\[ \sigma(T) = \bigcup_{q, \theta \in [0, 2\pi]} \{-4G + 2i \cos(\theta) - 2i \cos(q - \theta) + 2G \cos(\delta + \theta) + 2G \cos(q - \theta - \delta)\}. \]
For $\delta = 0$ and $\delta = \pi$ this set is the convex envelope of the points $-8G, -4i, 0, 4i$. 

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Figure 3: Spectrum $\mathcal{L}$ with non-normal Lindblad operators given by (18) with a random potential in orange and without in blue in the complex plane. On the right picture blue points are plotted on the top and in the left it is the orange points. In this case $G = 1$, the lattice size $n = 70$ and the support of the distribution of the strength of the external potential $V = 2$. Notice how the blue points are ordered very regularly except that there is a vertical hole in the middle and those eigenvalue tend to collapse to the real axis. It seems that the main effect of the external field is to push the eigenvalues vertically.
Figure 4: The ellipses corresponding to the spectrum of the Laurent matrices $T(q)$ in the model with non-normal Lindblad operators see for example [19]. For the plot we chose $\delta = 0$ corresponding to Proposition 5.5. Notice how the union of the ellipses make up the quadrilateral described in Proposition 19, compare also to the shape in Figure 3.

Proof. As before, it holds by Corollary 3.8 and the fact that the spectrum of a Laurent matrix is the image of the symbol curve that
\[
\sigma(T) = \bigcup_{q \in [0,2\pi]} \sigma(T(q)) = \bigcup_{q \in [0,2\pi]} \{ \delta(q)e^{-i\theta} + \beta + \gamma(q)e^{i\theta} \mid \theta \in [0,2\pi] \}.
\]

The result now follows from direct computation. For $\delta = 0$ and $\delta = \pi$ it reduces to
\[
\sigma(T) = \bigcup_{q,\theta \in [0,2\pi]} \{-4G + \cos(\theta) (2i \pm 2G) + \cos(q - \theta) (-2i \pm 2G)\}
= \bigcup_{x, y \in [-2,2]} \{-4G + x (i \pm G) + y (-i \pm G)\} = \text{conv}(-8G, -4i, 0, 4i).
\]

Thus, we have proven that $\mathcal{T}$ is gapless even before adding the quantum jump term and therefore also afterwards when considering the full $\mathcal{L}$. We now turn to study the spectral effects of the perturbation.

Notice now that for $\delta = 0$ then $\alpha \gamma = 2G^2 \cos(q) + 2G^2 + 2 \cos(q) - 2 \in \mathbb{R}$. We now ask whether the set
\[
\bigcup_{q \in [0,2\pi]} \{ z \in \mathbb{C} \mid \langle \Gamma_R | (T(q) - z)^{-1} | \Gamma_L \rangle = -1 \}
\]
increases the spectrum. Absorbing $z$ into $\beta$ yields that $\beta = -4G - z$ and from Lemma 3.3 we find that
\[
|\Gamma_L\rangle = \sqrt{G} \begin{pmatrix} e^{iq} \\ 1 + e^{iq} \end{pmatrix}
\text{ as well as } \langle \Gamma_R \rangle = \sqrt{G} \begin{pmatrix} -1 & 1 + e^{-iq} & -e^{-iq} \end{pmatrix},
\]
highlighting that we do allow for $q$-dependent $|\Gamma_L\rangle$ and $\langle \Gamma_R \rangle$. 

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Now, let $\omega = \sqrt{\beta^2 - 16r\gamma}$ and then from Lemma A.4 the equation reduces to
\[
\frac{-\omega}{\mathcal{G}} = \frac{(\Gamma_{\mathcal{R}}|T^{-1} - \Gamma_{\mathcal{L}})}{\mathcal{G}} = \begin{pmatrix} -1 + e^{-i\lambda} & -e^{-i\lambda} \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} & \frac{1}{\lambda_2} & \frac{1}{\lambda_2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\lambda} \\ 1 + e^{i\lambda} \\ 1 + e^{i\lambda} \end{pmatrix}
\]
\[
= -e^{-i\lambda} + 2 + 2 \cos(q) - e^{i\lambda} + \lambda_1^2 (-1 - e^{-i\lambda} + e^{i\lambda} + 1)
- \lambda_1^2 + \lambda_2(1 + e^{-i\lambda} - 1 - e^{i\lambda}) - \lambda_2^2
= 2 + 2 \left( \frac{1}{\lambda_1 - \lambda_2} \sin(q) - \left( \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right) \right),
\]
where $\lambda_1, \lambda_2$ are defined right after (16). This equation in $z$ can be numerically solved for fixed $q$ (and $\mathcal{G}$). Numerical evidence suggest that the curve stays inside the quadrilateral $\text{conv}(-8G, -4i, 0, 4i)$ and therefore leads us the following conjecture.

**Conjecture 5.6.** For the Lindbladian $\mathcal{L}$ with Lindblad operators given by (18) and non-Hermitian evolution $A$ it holds that

$$\sigma(\mathcal{L}) = \sigma(\mathcal{T}) = \text{conv}(-8G, -4i, 0, 4i).$$

The reader can further compare with the plot of the spectrum in finite volume in Figure 3. Notice further, how Corollary 3.8 gives us the inclusion $\sigma(\mathcal{L}) \supset \sigma(\mathcal{T})$. In particular, we obtain that $\mathcal{L}$ is gapless (independently of the conjecture). Interestingly, this gaplessness could be related to the dynamical behavior in a random potential observed in [21]. A topic that we return to in Section 6.

### 5.3 Incoherent hopping

We now turn to a model of incoherent hopping that was studied numerically in [19]. Here, the Lindblad dissipators are hopping terms, they are given by $L_k = |k\rangle \langle k + 1|$. The numerical finding for finite sections with free boundary conditions of the lattice [19] is that the gap is uniformly positive as the length of the lattice increases. We find the spectrum for periodic boundary conditions.

**Theorem 5.7.** Let $l \in \mathbb{Z}$. For the Lindbladian with $H = -\Delta$ and $L_k = |k\rangle \langle k + l|$ it holds that

$$\sigma(\mathcal{L}) = -G + i[-4, 4] \cup \bigcup_{q \in [0, 2\pi]} \left\{ -G \pm q \sqrt{e^{-2i\tilde{\Delta}G^2} + 8(\cos(q) - 1)} \right\},$$

where $\pm q$ is either $+$ or $-$ for each $q \in [0, 2\pi]$.

**Proof.** Notice that $L_k^*L_k = |k + l\rangle \langle k + l|$ so still $Q = \sum_k L_k^*L_k = \mathbb{I}$. So $\left( -i\tilde{\Delta} - \frac{G^2}{2} \right) \mathbb{I} = \mathbb{I} \otimes \left( -i\tilde{\Delta} - \frac{G^2}{2} \right)$. The diagonalization matrix $\alpha = i(1 - e^{i\phi})$, $\beta = -G$, $\gamma = i(1 - e^{-i\phi})$ on the diagonals. Furthermore,

$$F(q) = \mathcal{G} \left( \sum_{r_1, r_2} \alpha_{r_1} e^{ir_1q} |r_2 - r_1\rangle \langle r_2 - r_1| \right) \left( \sum_{r_1, r_2} \beta_{r_1} e^{-ir_1q} \frac{\mathcal{G}}{|r_2\rangle \langle r_2|} |r_2 - r_1\rangle \langle r_2 - r_1| \right)$$

with $\alpha_r = \delta_{r,0}$ and $\beta_r = \delta_{r,1}$. Thus,

$$|\Gamma_L\rangle = \sqrt{\mathcal{G}} \sum_{r_1, r_2} \alpha_{r_1} e^{ir_1q} |r_2 - r_1\rangle = \sqrt{\mathcal{G}} |0\rangle \text{ and }$$

$$|\Gamma_R\rangle = \sqrt{\mathcal{G}} \sum_{r_1, r_2} \beta_{r_1} e^{-ir_1q} \frac{\mathcal{G}}{|r_2\rangle \langle r_2|} |r_2 - r_1\rangle = \sqrt{\mathcal{G}} e^{-iq} |0\rangle.$$
That means,
\[ (\Gamma_R | \frac{1}{T-z} | \Gamma_L) = e^{-iqt} G \langle 0 | \frac{1}{T-z} | 0 \rangle = \frac{(-1)^{\|\lambda_-\| < \lambda_+ \|} e^{-iqt} G}{\sqrt{(\beta - z)^2 - 4\alpha\gamma}}. \]

So by Corollary 3.9 we need to solve
\[ -1 = (-1)^{\|\lambda_-\| < \lambda_+ \|} \frac{e^{-iqt} G}{\sqrt{(\beta - z)^2 - 4\alpha\gamma}}. \]

Squaring yields
\[ 1 = \frac{e^{-2iqt} G^2}{(\beta - z)^2 - 4\alpha\gamma}, \]
so solving for \( z \) gives
\[ z = \beta \pm \sqrt{e^{-2iqt} G^2 + 4\alpha\gamma} = -G \pm \sqrt{e^{-2iqt} G^2 + 8(\cos(q) - 1)}, \]
where we again, as in the proof of Proposition 5.4 have to throw some of the solutions away according to get the correct sign in (20).

In Figure 5 and 6 we explicitly plot the solutions as a function of \( q \) as well as the spectra obtained by of exact diagonalization of \( L \) in finite for volume. Notice how the predicted spectrum fits well with the numerical spectra for finite size system only for the system with periodic boundary condition, as is consistent with Theorem 4.5. This dramatic dependence on boundary conditions is sometimes called the non-Hermitian Skin effect [62, 63, 64]. It states that the spectra of non-Hermitian operators may exhibit dramatic dependence on boundary conditions. One could also view the difference between Toeplitz and Laurent operators this way. Furthermore, it is a feature of the non-Hermitian skin effect that the spectrum is pushed inwards and real eigenvalues start to appear [65]. This effect we also see in Figure 5 and 6.

An interesting question for further mathematical investigations could be to determine the convergence of the finite volume free boundary condition spectrum in the infinite volume limit. This is not just a Toeplitz matrix question, but it is related and one possible approach could be to extend the theory.

5.4 Single particle sector of a quantum exclusion process

In a many-body setting a model with \( L_k = |k\rangle\langle k+1| \) and \( L_k' = |k+1\rangle\langle k| \) was studied analytically in [66]. We briefly discuss how to adopt our methods to that case and rederive the spectrum of the Lindbladian. Going through the proof of Theorem 3.1, we see that we just get two independent contributions that diagonalize in the same way. Since we still use discrete Laplacian as the Hamiltonian, it holds that \( \alpha = i(1 - e^{iq}) \) and \( \gamma = i(1 - e^{-iq}) \) are unchanged. On the other hand, we get the sum \( \sum_k L_k^* L_k = \mathbb{I} \) twice, which means that \( \beta = -2G \). Therefore we have from Proposition 5.2 the spectrum of the non-Hermitian evolution \( \sigma(T) = -2G + [-4i, 4i] \).

We now turn to the spectrum of the full Lindbladian.

**Theorem 5.8.** If the system has Lindblad operators \( L_k = |k\rangle\langle k+1| \) and \( L_k' = |k+1\rangle\langle k| \) then the spectrum is given by
\[ \sigma(L) = [-2G, 0] \cup \{-2G + i[-4, 4]\}. \]

**Proof.** We saw that the spectrum of the non-Hermitian evolution was given by \( \sigma(T) = -2G + i[-4, 4] \). Now, we calculate the spectrum of the quantum jump terms as follows:
\[ F_1(q) = G|0\rangle\langle 0| e^{-iq}, \]
\[ F_2(q) = G|0\rangle\langle 0| e^{iq}. \]
Figure 5: Exact diagonalization of the Lindbladian $\mathcal{L}$ in Section 5.3 for $l = 1, G = 1, 2, 5$ and $N = 70$ comparison of the predicted curve with numerics with free boundary conditions. The red circle is the unit circle. Notice how the two curves due not match due to the non-Hermitian Skin effect.

Figure 6: Exact diagonalization of the Lindbladian in Section 5.3 for $l = 1, G = 1, 2, 5$ and $N = 50$ comparison of the predicted curve with numerics with periodic boundary conditions. The red circle is the unit circle. Notice how the numerics and the analytical spectrum in the infinite volume fit.
Thus, in total the quantum jump contribution is
\[ F(q) = F_1(q) + F_2(q) = 2G \cos(q)|0\rangle\langle 0|, \]
which is still rank-one and in constrast to the incoherent hopping discussed in Theorem 5.7 this expression is real. We use the same method to compute the spectrum as before
\[ -1 = \langle \Gamma_R \rangle \frac{1}{T - z} |\Gamma_L \rangle = \pm \frac{2 \cos(q)G}{\sqrt{(\beta - z)^2 - 4\alpha \gamma}}, \]
squaring and solving for \( z \) and inserting \( \alpha, \beta, \gamma \) yields that
\[ z = -2G \pm \sqrt{4 \cos^2(q)G^2 + 8(\cos(q) - 1)}. \]
Let us analyze the function \( f(q) = 4 \cos^2(q)G^2 + 8(\cos(q) - 1) \). It has extremal points \( q \) satisfying
\[ 0 = 8 \cos(q) \sin(q)G^2 + 8 \sin(q). \]
If \( q \not\in \{0, \pi\} \) then
\[ \cos(q) = -\frac{1}{G^2} \]
which has a solution if and only if \( G \geq 1 \).

This means that if \( G < 1 \) then the only extremal points are at \( q \in \{0, \pi\} \). The values are \( f(0) = 4G^2 \) and \( f(\pi) = 4G^2 - 16 \). Thus, by continuity and the intermediate value theorem it holds that the range of \( f \) is \([4G^2 - 16, 4G^2] = [4G^2 - 16, 0] \cup [0, 4G^2] \). The first interval corresponds to the segments \( \pm i[0, 2\sqrt{4 - G^2}] \subset i[-4, 4] \). The second one to \([0, 2G] \) analogously to the dephasing case from Section 5.1.

For \( G \geq 1 \) there is in addition the solution \( \cos(q) = -\frac{1}{G^2} \) which has values \(-\frac{1}{G^2} - 8 \). Since \( G \geq 1 \), the potential values of \( f \) are extended to the interval \([-\frac{1}{G^2} - 8, 0] \cup [0, 4G^2] \) which corresponds to solutions \( i \left[-\sqrt{\frac{8}{G^2} + \frac{1}{G^2}}, + \sqrt{\frac{8}{G^2} + \frac{1}{G^2}}\right] \) and \([0, 2G]\) respectively. Still, since \( G \geq 1 \), it holds that \( \sqrt{8 + \frac{1}{G^2}} \leq 4 \) and thus \( i \left[-\sqrt{\frac{8}{G^2} + \frac{1}{G^2}}, + \sqrt{\frac{8}{G^2} + \frac{1}{G^2}}\right] \subset i[-4, 4] \). Thus, the spectrum is not extended in that case. \( \square \)

6 Lindbladians with random potentials

In the following, we study models as in the previous section where we have added a random potential \( V \) to the Hamiltonian \( H \). The potential \( V \) is a random such that \((V(n))_{n \in \mathbb{Z}} \) is i.i.d. uniformly distributed potential in some range \([-\lambda, \lambda]\) for some \( \lambda > 0 \). The the closed system case, the study of operators of that type was initiated in the celebrated work by Anderson [67] and has led to the field random operator theory and the topic of Anderson localization.

Although, some results exist (e.g. [21][6]), it is not clear what the effects of a random potential in an open quantum system are. In addition, there has recently been interest in random Lindblad systems from the point of view of random matrix theory [22][23]. Our methods are however more along the lines of random operator theory [1]. Different methods that were also more random operator theoretic were pioneered in [6].

To study the spectrum of our random Lindbladians we use the following Lindbladian version of the Kunz-Soulliard Theorem from [69] generalised in [70]. We first need a bit of notation. Recall, from Section 2 that we say that a (super)-operator \( \mathcal{L} \in \mathcal{B}(\mathcal{H}) \) is translationally covariant if for all \( \rho \in \mathcal{H} \)
\[ \mathcal{L}(S \rho S^{-1}) = S^{-1} \mathcal{L}(\rho) S, \]
where \( S \) is the translation with respect to the computational basis in \( \mathcal{H} \). In the examples without a random field the Lindbladian is translation-invariant. We can separate the action of the potential by \( \mathcal{E}_V \), where \( \mathcal{E}_V(\rho) = -i[V, \rho] \).
We also need the numerical range of an operator \( A \) which we define as follows:

\[
W(A) = \{\langle v, Av \rangle \mid \|v\| = 1\}.
\]

For example \( W(\mathcal{E}_V) = -i[\lambda, \lambda] \). The numerical range will be useful because its closure is an upper bound to the spectrum by the Toeplitz–Hausdorff Theorem \cite{71, 72}.

**Theorem 6.1.** Let \( \mathcal{L}_0 \) be a translation-covariant operator on \( \text{HS}(\mathcal{H}) \) and \( V \) be an i.i.d. random potential. Define \( \mathcal{L} = \mathcal{L}_0 + \mathcal{E}_V \) which is also an operator on \( \text{HS}(\mathcal{H}) \). Then

\[
\sigma(\mathcal{L}_0) \subset \sigma(\mathcal{L}) \subset W(\mathcal{L}_0) + W(\mathcal{E}_V).
\]

**Proof.** Notice first that by the Toeplitz–Hausdorff Theorem \cite{71, 72} it holds that

\[
\sigma(\mathcal{L}_0) \subset \sigma(\mathcal{L}) \subset W(\mathcal{L}_0) + W(\mathcal{E}_V).
\]

For the first inclusion, let us first consider \( \lambda \in \sigma_{\text{app}}(\mathcal{L}_0) \) and let \( \{\varrho_n\}_{n \in \mathbb{N}} \) Weyl sequence corresponding to \( \lambda \) for \( \mathcal{L}_0 \). Without loss of generality assume that each \( \varrho_n \) is compactly supported in the sense that in the computational basis we have that \( \varrho_n(x, y) = 0 \) for \((x, y) \notin \Lambda_{R_n} \subset \mathbb{Z}^2 \) for some large but finite box \( \Lambda_{R_n} \subset \mathbb{Z}^2 \). Then define

\[
\Omega_n = \left\{ \text{for some } j \in \mathbb{Z} : \sup_{(x, y) \in \text{supp}(\varrho_n)} \left| (-i)(V_{x+j_n} - V_{y+j_n}) \right| \leq \frac{1}{n} \right\},
\]

and notice that each \( \Omega_n \) is a set of full measure. Hence \( \Omega_0 = \cap_{n \in \mathbb{N}} \Omega_n \) must have probability 1 as well.

Now, the sequence \( \gamma_n = \rho_n(-j_n) \) is a sequence approximating the eigenvalue \( \mu + \lambda \) since by translation covariance of \( \mathcal{L}_0 \) it holds that

\[
\left\| (\mathcal{L} - \lambda) \gamma_n \right\| \leq \left\| (\mathcal{L}_0 - \lambda) \rho_n \right\| + \|\mathcal{E}_V(\gamma_n)\|.
\]

The first term is small since \( \rho_n \) is a Weyl sequence. For \( \mathcal{L}_0 \) for the second term we estimate:

\[
\left\| \mathcal{E}_V(\varrho_n(-j_n)) \right\| = \left\| \mathcal{E}_V \left( \sum_{x, y \in \text{supp}(\varrho_n)} \varrho_n(x - j_n, y - j_n) |x \rangle \langle y| \right) \right\|
\]

\[
= \left\| \sum_{x, y \in \text{supp}(\varrho_n)} ((-i)(V_x - V_y)) \varrho_n(x - j_n, y - j_n) |x \rangle \langle y| \right\|
\]

\[
= \left\| \sum_{x, y \in \text{supp}(\varrho_n)} ((-i)(V_{x+j_n} - V_{y+j_n})) \varrho_n(x, y) |x + j_n \rangle \langle y + j_n| \right\|. \]

By the lemma below it suffices to have \( \left| ((-i)(V_{x+j_n} - V_{y+j_n})) \right| \leq \frac{1}{n} \) for each \((x, y) \in \text{supp}(\varrho_n)\).

**Lemma 6.2.** Suppose that \( |b_{x, y}| \leq \varepsilon \) and \( a = \sum a_{x, y} |x \rangle \langle y| \). Then

\[
\left\| \sum_{x, y} a_{x, y} b_{x, y} |x \rangle \langle y| \right\| \leq \varepsilon \|a\|.
\]

**Proof.** We use the definition of the Hilbert-Schmidt norm to conclude that

\[
\left\| \sum_{x, y} a_{x, y} b_{x, y} |x \rangle \langle y| \right\|^2 = \text{Tr} \left( \sum_{x, y} a_{x, y} b_{x, y} |x \rangle \langle y| \sum_{x', y'} a_{x', y'} b_{x', y'} |y' \rangle \langle x'| \right)
\]

\[
= \sum_{x, y} |a_{x, y}|^2 |b_{x, y}|^2 \leq \varepsilon^2 \sum_{x, y} |a_{x, y}|^2 = \varepsilon^2 \|a\|^2.
\]
Now, for $\lambda \in \sigma_{\text{res}}(\mathcal{L}_0)$ we know that $\tilde{\lambda}$ is an eigenvalue of $\mathcal{L}_0^\ast = \tilde{\mathcal{L}}_0$ where $\tilde{\mathcal{L}}_0$ is of the adjoint Lindblad form given by (4). Let $\rho_0$ be an eigenvector and consider an approximation $\rho_n$ with compact support. Then translation-invariance of $(\mathcal{L}_0 - \lambda)^\ast = \tilde{\mathcal{L}}_0 - \lambda$ enables us to translate $\rho_n$ as before Since it further holds that $\mathcal{E}_\mathcal{V}^\ast (\rho) = +i[V, \rho] = -\mathcal{E}_\mathcal{V} (\rho)$ we can use a similar argument as above to finish the proof.

In the following sections, we will see how in general both of the two inclusions in Theorem 6.1 are in general strict.

Furthermore, one can see in Figure 3 how it seems as if the random field generally pushes eigenvalues vertically as would be the straightforward generalization of the theorem by Kunz and Soulliard. However, the effect is much stronger in the bulk of the spectrum, whereas close to $\{0\}$ it does not seem as if the spectrum changes. A model that describes this phenomenon which is exactly solvable (even without the theory we have developed) is the following.

### 6.1 Exactly solvable model with random potential

Define on the Hilbert space $l^2(\mathbb{Z})$ the Lindbladian by $H = 0$ and Lindblad operators $L_k = |k\rangle\langle k|$. Then for any $i, j \in \mathbb{Z}$ we have that

$$\mathcal{L}_0(|i\rangle\langle j|) = G \sum_k |k\rangle\langle k||i\rangle\langle j||k\rangle - \frac{1}{2} \{ |k\rangle\langle k|, |i\rangle\langle j| \} = G (|i\rangle\langle i| + |j\rangle\langle j| - |i\rangle\langle j|).$$

This means that all states of the form $|i\rangle\langle j|$ are eigenvalues with eigenvalue $-G$ if $i \neq j$ and 0 if $i = j$. In particular, all states of the form $|i\rangle\langle i|$ for $i \in \mathbb{Z}$ are steady states. We conclude that $\sigma(\mathcal{L}_0) = \{0, -G\}$. Notice that in this case we do have trace-class steady states in infinite volume.

To use the Lindblad analogue of Kunz–Soulliard we first need to consider the numerical range of $\mathcal{L}_0$. However, $\mathcal{L}_0$ is normal (indeed self-adjoint). That means that the numerical range is the convex hull of $\lambda \mathcal{L}$. We can find the exact spectrum of the model in a random potential by noticing that

$$\mathcal{L}(|i\rangle\langle i|) = 0 + \mathcal{E}_\mathcal{V}(|i\rangle\langle i|) = (V_i - V_j)|i\rangle\langle i| = 0$$

as well as

$$\mathcal{L}(|i\rangle\langle j|) = -G|i\rangle\langle j| + \mathcal{E}_\mathcal{V}(|i\rangle\langle j|) = -G|j\rangle\langle i| + i(V_i - V_j)|i\rangle\langle j|.$$ 

Hence the spectrum $\sigma(\mathcal{L}) = \{-G + \lambda i[-1,1]\} \cup \{0\}$. Notice, how this is an example where both inclusions in Theorem 6.1 are strict. In the case at hand we can see that this happens because $\mathcal{L}$ leaves both the diagonal and off-diagonal subspaces of $\text{HS}(l^2(\mathbb{Z})) \cong l^2(\mathbb{Z}^2)$ invariant. The potential $\mathcal{E}_\mathcal{V}$ is zero on the diagonal part and therefore only the spectrum of the off-diagonal can be extended by $V$. In the case of $H = -\Delta$ the two subspaces get mixed, the effects are more complicated and rigorous guarantees are harder to obtain.

### 6.2 Improving the upper bound on the spectrum

In the following we prove an upper bound for the Anderson model with local dephasing. The model recently attracted attention in [73, 74]. In the following we use the model to exemplify how we can use a new method to prove a non-trivial upper bound to the spectrum using the numerical range.

**Theorem 6.3.** Consider the Anderson model $H = -\Delta + V$ with local dephasing defined in (14). Then for any $\rho \in \text{HS}(\mathcal{H})$ with $\|\rho\|_2 = 1$ such that

$$\sum_{x \in \mathbb{Z}} |\rho(x, x)|^2 = a \in [0, 1]$$

it holds that

$$\langle \rho, \mathcal{L}_\rho \rangle \in Ga + \langle \rho, \mathcal{E}_\mathcal{V}_\rho \rangle + \langle \rho, \mathcal{T}_\rho \rangle \in G(a - 1) + i[f(a, \lambda), f(a, \lambda)]$$

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with $f$ defined by $f(a, L) = 4(1 - a + 2\sqrt{a} \sqrt{1 - a}) + (1 - a)\lambda$. It follows that

$$\sigma(\mathcal{L}) \subset \bigcup_{a \in [0, 1]} (G(a - 1) + i[f(a, \lambda), f(a, \lambda)]).$$

We have plotted the upper bound to the spectrum in Figure 7.

**Proof.** We first rewrite $\mathcal{L}$ using Theorem 3.1.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{E}_V = \int_{[0, 2\pi]} T(q) dq + \int_{[0, 2\pi]} F(q) dq + \mathcal{E}_V = \mathcal{T} + \mathcal{F} + \mathcal{E}_V.$$  

Again, we want to use the upper bound to the spectrum given the numerical range $\sigma(\mathcal{L}) \subset \mathcal{W}(\mathcal{L})$, so we try to bound the numerical range of each term. So let $\rho \in \text{HS}(\mathcal{H})$ with $\|\rho\|_2 = 1$. Notice first how that identity looks in our two pictures:

$$1 = \|\rho\|_2^2 = \sum_{x,y \in \mathbb{Z}} |\rho(x, y)|^2 = \sum_{n \in \mathbb{Z}} \int_0^{2\pi} |\langle \rho(q)\rangle| n|^2 dq.$$  

Furthermore, the value

$$\int_0^{2\pi} |\langle \rho(q)\rangle| 0|^2 dq = \sum_{x \in \mathbb{Z}} |\rho(x, x)|^2 = a,$$

which is always between 0 and 1 gives an indication about how classical a state is. Notice that then $\sum_{x,y|x \neq y} |\rho(x, y)|^2 = 1 - a$. Now, we go term by term. For the first term we estimate

$$\langle \rho, \mathcal{E}_V \rho \rangle = \text{Tr} \left( \sum_{x,x',y,y' \in \mathbb{Z}} \bar{\rho(x', y')} \langle x'| y' \rangle \langle i(V_x - V_y) \rho(x, y) | x \rangle | y \rangle \right) = i \sum_{x,y|x \neq y} |\rho(x, y)|^2 (V_x - V_y) \in i[0, \lambda]$$

since $V$ is supported in $[0, \lambda]$. Notice that the more classical a state is the less it is affected by an external field.

For the $T(q)$ term suppose that $T(q)$ is tridiagonal with $\alpha, \beta, \gamma$ on the diagonals. Let us assume that $\alpha(q) = -\gamma(q)$ and that $\beta(q) = \beta$ is constant. Then

$$\langle \rho, T dq \rho \rangle = \langle \rho, \int_{[0, 2\pi]} T(q) dq \rho \rangle = \int_0^{2\pi} \langle \rho(q), T(q) \rho(q) \rangle dq = \sum_{n \in \mathbb{Z}} \int_0^{2\pi} \langle \rho(q)\rangle n |\alpha(q)\rangle n |\rho(q)\rangle + |\beta(q)\rangle n |\rho(q)\rangle + |\gamma(q)\rangle n |\rho(q)\rangle dq$$

$$= \beta(q) \sum_{n \in \mathbb{Z}} \int_0^{2\pi} |\langle \rho(q)\rangle n|^2 dq + \sum_{n \in \mathbb{Z}} 2i \int_0^{2\pi} \text{Im} (\gamma(q) \langle \rho(q)\rangle n |\rho(q)\rangle n dq$$

$$= \beta(q) + \sum_{n \in \mathbb{Z}} 2i \int_0^{2\pi} \text{Im} (\gamma(q) \langle \rho(q)\rangle n |\rho(q)\rangle n dq.$$  

We want to bound the second term. Define the sequence $l_{\eta, n} = \langle \rho(q)\rangle n$. And let $l_{\eta, n}^{\geq j} = l_{\eta, n} \mathbb{1}_{n \geq j}$ and similarly with $l_{\eta, n}^{\leq i}$. Suppose further that $\int_0^{2\pi} |\langle \rho(q)\rangle | 0|^2 dq = a \in [0, 1]$. Using Cauchy–Schwarz in a few
different spaces we obtain

\[ \left| \sum_{n \in \mathbb{Z}} \int_{0}^{2\pi} \text{Im} \left( \gamma(q)\ell_{q,n-1} \right) dq \right| \leq \| \gamma \|_{\infty} \int_{0}^{2\pi} \left| \sum_{n \leq -1} \ell_{q,n} \right| dq + \| \gamma \|_{\infty} \int_{0}^{2\pi} \left| \sum_{n \geq 2} \ell_{q,n} \right| dq + \| \gamma \|_{\infty} \int_{0}^{2\pi} \left| \sum_{n \leq -1} \ell_{q,n} \right| dq + \| \gamma \|_{\infty} \int_{0}^{2\pi} \left| \sum_{n \geq 2} \ell_{q,n} \right| dq \]

\[ \leq \| \gamma \|_{\infty} \int_{0}^{2\pi} \left( \| \ell_{q,-1} \|_{2} + \| \ell_{q,2} \|_{2} \right) dq \]

\[ \leq \| \gamma \|_{\infty} \left( 1 - a + \sqrt{\int_{0}^{2\pi} \| \rho(q) \|_{2}^2 dq} \right) \]

Notice that in particular this vanishes as \( a \to 1 \), i.e. for classical \( \rho \). Notice also that in our case of interest \( \beta(q) = -G \) and \( \| \gamma \|_{\infty} = 2 \) yielding the second part of the bound.

For the \( F(q) = |\Gamma_L\rangle\langle \Gamma_R| \) we obtain

\[ \langle \rho, \int_{[0,2\pi]} |\Gamma_L\rangle\langle \Gamma_R| dq \rho \rangle = \int_{0}^{2\pi} \langle \rho(q) |\Gamma_L\rangle\langle \Gamma_R| \rho(q) \rangle dq \]

Thus, in the case where \( |\Gamma_L\rangle\langle \Gamma_R| = |0\rangle \langle 0| \) in which case the term becomes

\[ G \int_{0}^{2\pi} \| \rho(q) \|_{2}^2 dq = G \sum_{x \in \mathbb{Z}} \langle \rho(x, x) \rangle^2 = Ga. \]

Notice that \( \text{Re}(\langle \rho, \mathcal{L} \rho \rangle) = G(a - 1) = \text{Re}(\langle \rho, \mathcal{L}_0 \rho \rangle) \). This means that the states as seen from the numerical range get more classical closer to 0. This also implies the absence of a non-classical eigenvalue since one would then be able to find it in the numerical range. Thus, for the dephasing model we see that the long-lived states are the classical even in the presence of an external (random) potential. In other words the states that survive longest are very diagonal so we see very explicitly how the dephasing noise suppresses coherences something which was also noted for a simpler model in Chapter 8 of [75].

6.3 Further discussion of spectral effects of random potentials

Spectra of random Lindbladians have been studied in random matrix theory approaches in for example [68]. There a lemon-like shape of the spectrum was found. This shape is reminiscent of the spectrum in both Figure 3 and Figure 7 where there seems to be a tendency that the spectrum close to 0 extends less in the direction of the imaginary axis. This is also mimicked in the exactly solvable example in Section 6.1 and the discussion in the previous section. Furthermore, inspecting the non-random spectrum with and without the perturbation one can see how there is a similar phenomenon with eigenvalues jumping from the bulk of the non-Hermitian spectrum and down to the real line [19]. An analytical explanation stemming from the symmetries of the Lindbladian is given in [22]. The one also sees in a corresponding RMT model [24] and the previous example gives an indication of a mechanism for this behaviour.

7 Discussion and further questions

We believe that the methods developed here can be applied to many one-particle open quantum systems of Lindblad form. In particular, it is easy to see that our results generalize to \( \mathbb{Z}^d \) for \( d \geq 1 \). Furthermore, directions for study could be many-body systems – in fact the model with non-normal
Figure 7: The numerical spectrum of the dephasing Lindbladian $\mathcal{L}$ in a random potential studied in Section 6.2 for $N \in \{10, 30, 50, 70\}, G = 2, V = 5$ along with the analytically calculated upper bound for the numerical range (and therefore the spectrum) sketched. Notice the line with real part close to $-1$ which is consistent with, but not predicted by our analysis. Notice further how it seems that the spectrum does not extend much when the real part is less than 1.
Lindblad operators we studied in Section 5.2 is already a candidate for many-body localization in the many-body setting [76]. A simple example where the number of particles is not fixed could be to have a Lindbladian where the particle number can only decrease. If the system start with one excitation, such models are used to describe the transport of an excitation during photosynthesis [3]. This would be one way to study dynamics of similar systems which are not as intimately related to the spectra of $\mathcal{L}$ as discussed in Section 2.3. Another approach could be through the transport in the steady state as is for example done for a non-random system in [77]. A system that is almost within our framework, where transport in the steady state is studied is discussed in [78].

We have also seen how the quantum-jump terms increases the number of real eigenvalues, this effect was also observed in [19]. We can now explain the phenomenon in that particular case corresponding to our first example. For finite dimensional systems an explanation was given using symmetries of the Lindbladian in [22, Appendix B.9]. The proof there relies on some rather old results, so it would be of value to obtain a self-contained proof. Along this line of investigation one could try to obtain results on the density of states. Since relatively many eigenvalues are confined to the the real line the density of states will potentially be challenging to define.

Furthermore, we have seen how the spectrum for the finite size numerics in some cases matches the analytic results for all of $\mathbb{Z}$ fairly well. This we have given an explanation of in Theorem 4.5 but it still remains to give a proof of ([55, Conjecture 7.3]), both in general and in our cases of interest. As shown by the examples in Figures 5 and 6 it will be important in that regard to use periodic boundary conditions, as it is the case in the corresponding question for Laurent operators compared to Toeplitz operators in [79].

Another direction of investigation is to expand on the spectral theory of Lindblad operators in a random potential and possibly prove a stronger lower bound to Theorem 6.1. In the self-adjoint case the spectrum and dynamics of the Hamiltonian in infinite volume are related through the RAGE theorem [80, 81, 82] much closer related than what we can prove for $\mathcal{L}$. Such a relation is probably difficult to recover since the notion of continuous spectrum breaks down in the non-normal case. However, we can still define the point spectrum of $\mathcal{L}$ to be spectrum that corresponds to a normalizable eigenvectors. In that case we can ask, in analogy with the case of the Anderson model, whether for large enough strength of the random potential $\lambda$ it is the case that $\sigma(\mathcal{L})$ has only point spectrum, in the sense of exponentially decaying (normalizable) eigenfunctions.

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A Appendix

In the appendix we provide some of the technicalities that we have postponed from the main part for clarity.

A.1 Proof of Lemma 2.2

We prove that $\mathcal{L} \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$. It is easy to see that by $(A_1)$ the commutator and anti-commutator term in the Lindbladian are bounded. Thus, we are left with the operator $\mathcal{J}$ defined by $\mathcal{J}(X) = \sum_k L_k X L_k^*$.

Let $N$ be the weak limit of $\sum_k L_k L_k^*$. By assumption $\|N\|_\infty < \infty$. Since $\sum_k L_k L_k^*$ is also self-adjoint it follows by the spectral radius theorem for every $|x\rangle \in \mathcal{H}$ that

$$\sum_k \|L_k|x\|^2 = \sum_k \langle x, L_k L_k^* |x\rangle = \sum_k \langle x, L_k L_k^* |x\rangle \leq \|N\|_\infty \|x\|^2.$$ 

Then consider $X \in \mathcal{B}(\mathcal{H})$ with $X \geq 0$. Then $X = AA^*$ for some $A \in \mathcal{B}(\mathcal{H})$ and it holds that

$$\sum_k L_k X L_k^* = \sum_k L_k AA^* L_k^*$$
is a sum of positive operators and therefore positive. We bound the norm
\[
\langle x, \sum_k L_k AA^* L_k^* x \rangle = \sum_k \langle x, L_k AA^* L_k^* x \rangle = \sum_k \|L_k AA^* L_k^* x\|^2 \leq \sum_k \|AA^*\|^2 \|L_k^* x\|^2
\]
\[
\leq \|AA^*\|^2 \|N\|_\infty\|x\|^2.
\]

Again, by self-adjointness of \(\sum_k L_k AA^* L_k^*\) the spectral radius theorem holds and therefore if the numerical range is bounded then so is the norm. We conclude that
\[
\left\| \sum_k L_k AA^* L_k^* \right\| \leq \|A\|^2 \|N\|_\infty.
\]

Now, for any element \(X \in \mathcal{B}(\mathcal{H})\) we write \(X = P_1 - P_2 + iP_3 - iP_4\) where \(P_1, P_2, P_3, P_4\) are all positive and satisfy \(\|P_i\| \leq \|X\|\) for each \(i = 1, \ldots, 4\) (see \[84\] Theorem 11.2 and 9.4). Then
\[
\|\mathcal{J}(X)\| \leq 4\|X\|\|N\|_\infty,
\]
where we also used the \(C^\star\)-identity since we could write \(P_i = A_i A_i^\star\) and that it holds that
\[
\|A_i\|^2 = \|A_i A_i^\star\| = \|P_i\| \leq \|X\|.
\]

We conclude that \(\mathcal{J}\) as an operator on \(\mathcal{B}(\mathcal{H})\) has a norm bounded by \(4\|N\|_\infty\).

### A.2 Measurability in the proof of Theorem 3.6

In this appendix we prove that for each fixed \(n \in \mathbb{N}\) there exists a measurable choice of the vectors \(q \mapsto v_{q,n}\) in the proof of Theorem 3.6. We do that with inspiration from \[84\] and let \(\{a_m\}_{m \in \mathbb{N}}\) be a countable dense subset of \(\mathcal{H}\) which does not contain 0. Then define \(b_m = \frac{a_m}{\|a_m\|}\). Recall that \(I_n\) is a set such that \(|I_n| > 0\) with \(\|(A(q) - \lambda)^{-1}\| \geq n\). Now, consider the function \(N : I_n \rightarrow \mathbb{N}\) defined by
\[
N(q) = \min \left\{ m \in \mathbb{N} \mid \|(A(q) - \lambda)^{-1}b_m\| \geq \frac{n}{2} \right\}.
\]

Notice that \(N\) is well-defined since \((A(q) - \lambda)^{-1}\) is bounded and hence continuous and by density of \(\{a_m\}_{m \in \mathbb{N}}\). We claim that \(N\) is also \(\mathcal{B}(I_n) - \mathcal{P}(\mathbb{N})\) measurable, where \(\mathcal{B}(I_n)\) is the Borel sigma-algebra on \(I_n\). To see that, notice first that since \(q \mapsto (A(q) - \lambda)^{-1}\) is measurable then also \(q \mapsto \|(A(q) - \lambda)^{-1}b_m\|\) will be measurable for each \(m\). Thus, the set
\[
\left\{ q \in I_n \mid \|(A(q) - \lambda)^{-1}b_m\| \geq \frac{n}{2} \right\}
\]
is measurable. Now, it holds that
\[
N^{-1}(\{1, \ldots, k\}) = \bigcup_{m=1}^k \left\{ q \in I \mid \|(A(q) - \lambda)^{-1}b_m\| \geq \frac{n}{2} \right\}
\]
and this is sufficient to prove measurability of \(N\). Now, since any function from \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\) to any measure space is measurable and compositions of measurable functions are measurable it holds that \(q \mapsto b_{N(q)}\) is measurable. We then pick \(v_{q,n} = b_{N(q)}\).

### A.3 Proof of Theorem 3.7

**Proof.** We use the characterisation from Theorem 3.6. Whenever \(\lambda \not\in \sigma(A(q))\) define the resolvent \(R(q) = (A(q) - \lambda)^{-1}\). Then if both \(R(p), R(q)\) exist it follows from the resolvent equation that
\[
\|R(p)\| - \|R(q)\| \leq \|R(p)\| \|A(p) - A(q)\| \|R(q)\|.
\]

"C": We prove the following stronger assertion.
Claim A.1. For any $p_0 \in [0, 2\pi]$ and $\varepsilon > 0$ it holds that

$$\sigma_\varepsilon(A(p_0)) \subset \bigcup_{p \in [0, 2\pi]} \sigma_{2\varepsilon}(A(p)). \quad (22)$$

Let us first prove the claim.

Proof of Claim. To prove the claim, let $\lambda \in \sigma_\varepsilon(A(p_0))$.

Case 1: Suppose first that $\lambda \not\in \sigma(A(p_0))$. This means that $\frac{1}{\varepsilon} \leq \|R(p_0)\| < \infty$. Suppose that $\|R(p_0)\| \leq K < \infty$. Assume for contradiction that there is a sequence $p_n \to p_0$ such that $\|R(p_n)\| \to \infty$. By continuity there exists a $\delta > 0$ such that $\|A(p) - A(q)\| \leq \frac{1}{2K}$ as long as $|p - q| < \delta$. If we pick $N$ such that $|p_n - p_0| < \delta$ for $n \geq N$ this means that

$$\|R(p_n) - R(p_0)\| \leq \|R(p_0)\| \|A(p_0) - A(p_n)\| \|R(p_n)\| \leq \frac{\|R(p_0)\|}{2K}.$$  

Thus, $\frac{\|R(p_n)\|}{\varepsilon} \leq \|R(p_0)\|$ and hence $\|R(p_0)\| = \infty$ which is a contradiction. We conclude that there exists a $\gamma > 0$ such that $\sup_{|p - p_0| \leq \gamma} \|R(p)\| \leq K_2 < \infty$. Thus, the resolvent bound $[\text{21}]$ ensures that there is an $\theta > 0$ such that if $|p - p_0| \leq \theta$ then $\|R(p)\| \geq \frac{1}{\varepsilon}$. This means that $\lambda \in \sigma_{2\varepsilon}(A(p))$ for all $p$ with $|p - p_0| \leq \theta$. Again this implies that $\lambda \in \bigcup_{p \in [0, 2\pi]} \sigma_{2\varepsilon}(A(p))$.

Case 2: Now, let $\lambda \in \sigma(A(p_0)) \subset \sigma_\varepsilon(A(p_0))$. Absorb $\lambda$ into $A$ for ease of notation. Now, suppose that $(p_n)_{n \in \mathbb{N}}$ is a sequence converging to $p_0$ and assume that $\|R(p_n)\| \leq C$ uniformly in $n$. Then

$$\|R(p_n)A(p_0) - \mathbb{I}\| \leq \|R(p_n)(A(p_0) - A(p_n))\| \leq \|R(p_n)\| \|A(p_0) - A(p_n)\| \leq C \|A(p_0) - A(p_n)\| \to 0.$$  

Similarly, $\|A(p_0)R(p_n) - \mathbb{I}\| \to 0$. Since further

$$\|R_0A(p_0) - \mathbb{I}\| \leq \|R(p_0)A(p_0) - \mathbb{I}\| + \|R_0 - R(p_n)\| \|A(p_0)\| \to 0,$$

we conclude that $R_0A(p_0) = \mathbb{I}$ and similarly $A(p_0)R_0 = \mathbb{I}$. This contradicts that $A(p_0)$ is invertible. So we conclude that no such sequence $p_n$ exists. Hence for our given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|R(p)\| \geq \frac{1}{\varepsilon}$ for every $p \in [p_0 - \delta, p_0 + \delta]$. This means that $\lambda \in \bigcup_{p \in [0, 2\pi]} \sigma_{2\varepsilon}(A(p))$.

So we conclude that for all $\varepsilon > 0$ it holds that

$$\sigma_\varepsilon(A(p_0)) \subset \bigcup_{p \in [0, 2\pi]} \sigma_{2\varepsilon}(A(p))$$

and therefore by Theorem 3.6

$$\sigma(A(p_0)) \subset \bigcap_{\varepsilon > 0} \sigma_\varepsilon(A(p_0)) \subset \bigcap_{\varepsilon > 0} \bigcup_{p \in [0, 2\pi]} \sigma_{2\varepsilon}(A(p)) = \sigma \left( \int_{[0, 2\pi]} A(p) \, dp \right)$$

Since this holds for any $p_0 \in [0, 2\pi]$ we have proven one inclusion.

"$\supset$": For the converse let $\lambda \in \bigcap_{\varepsilon > 0} \left( \bigcup_{\varepsilon \in \mathbb{Q}, \varepsilon > 0} \sigma_{\varepsilon}(A(q)) \right)$. Suppose for contradiction that $\lambda \not\in \sigma(A(q))$ for any $q \in I$. It then follows from the resolvent bound $[\text{21}]$ that the function $N : I \to \mathbb{R}$ defined by $N(q) = \|A(q) - \lambda\|^{-1}$ is continuous. Now,

$$S_n = \left\{ q \in I \mid \lambda \in \sigma_{\varepsilon_n}(A(q)) \right\} = \left\{ q \in I \mid \|A(q) - \lambda\|^{-1} \geq n \right\} = N^{-1}\left( [n, \infty) \right).$$

Since $N$ is continuous and $[n, \infty)$ is closed we must have that $S_n$ is closed. Further, we must have that $S_n \subset S_{n-1}$ and that $S_n$ is non-empty for each $n \in \mathbb{N}$. Thus, by the finite intersection property (since $I$ is
compact) it holds that $\bigcap_{n\in\mathbb{N}} S_n$ is nonempty. So let $q_0 \in S_n$ for each $n \in \mathbb{N}$. Then $\| (A(q_0) - \lambda)^{-1} \| \geq n$ for all $n \in \mathbb{N}$ and thus $\lambda \in \sigma(A(q_0))$ which is a contradiction. So we conclude that $\lambda \in \sigma(A(q))$ for at least one $q \in I$ and then $\lambda \in \bigcup_{q \in I} \sigma(A(q))$.

\[ \square \]

### A.4 Spectrum of rank-one perturbation of Laurent operators

In this appendix we prove the following relation that was used in the proof of Corollary 3.9:

$$\sigma(T(q) + F(q)) = \sigma(T(q)) \cup \{ \lambda \in \mathbb{C} \mid (\Gamma_R|(T(q) - \lambda)^{-1}|\Gamma_L) = -1 \}. $$

"⊃" From the proof of Corollary 3.8, we saw that $\sigma(T(q)) \subset \sigma(T(q) + F(q))$. So let $\lambda \notin \sigma(T(q))$ and $(\Gamma_R|(T(q) - \lambda)^{-1}|\Gamma_L) = -1$. Assume for contradiction that $T(q) + F(q) - \lambda$ is invertible. Let for ease of notation $T = T(q) - \lambda$. Then the resolvent equation states that

$$\frac{1}{T + |\Gamma_L|}\Gamma_R = \frac{1}{T} - \frac{1}{T + |\Gamma_L|}\Gamma_R|\Gamma_L|^{1/2} \frac{1}{|\Gamma_L|}.$$  (23)

Suppose that $\langle \Gamma_R|\frac{1}{T}|\Gamma_L \rangle = -1$ then multiplying with $|\Gamma_L|$ from the right yields that $T^{-1}|\Gamma_L| = 0$, which is a contradiction.

"⊂" Assume that $T$ is invertible and that $(\Gamma_R|\frac{1}{T}|\Gamma_L) \neq -1$. It is then straightforward to check that the following operator is well defined and an inverse to $T + |\Gamma_L|\langle \Gamma_R|$,

$$\frac{1}{T} - \frac{1}{(\Gamma_R|\frac{1}{T}|\Gamma_L) + 1} \frac{1}{T}|\Gamma_L|\langle \Gamma_R| \frac{1}{T},$$

so we conclude that $T + |\Gamma_L|\langle \Gamma_R|$ is invertible.

### A.5 Invertibility of bi-infinite tridiagonal Laurent matrices

In this appendix we discuss the inversion of bi-infinite tridiagonal Laurent matrices.

In Corollary 3.9, we saw that we could find the spectrum of $\mathcal{L}$ which was not just the spectrum of the non-Hermitian evolution by finding solutions of the equation

$$\langle \Gamma_R|(T(q) - \lambda)^{-1}|\Gamma_L \rangle = -1.$$  

In our applications $T(q) - \lambda$ is a tridiagonal Laurent matrix and thus to find explicit expressions we need to be evaluate matrix elements of the inverse of such a matrix. For a tridiagonal operator with $\alpha, \beta$ and $\gamma$ on the diagonal the symbol curve is given by

$$a(z) = \alpha z^{-1} + \beta + \gamma z, z \in T,$$

which is a (possibly degenerate) ellipse. Thus, the spectrum always forms a (possibly degenerate) ellipse in that case. In the cases where our Lindblad operators are supported on at most two lattice sites, by Theorem 3.1, the corresponding Laurent matrix will be tridiagonal.

If the matrix has $\alpha(q), \beta(q), \gamma(q)$ on the diagonals it will be useful to study the following equation

$$\alpha + \beta x + \gamma x^2 = 0.$$  (24)

In particular, we would like to know whether the two solutions $\lambda_1, \lambda_2$ to the equation satisfy that

$$|\lambda_1| < 1 < |\lambda_2|.$$  

To define square roots we will use the following convention.

**Convention A.2.** For a $z \in \mathbb{C}$ the two branches $\pm \sqrt{z}$ are defined such that $\text{Re}(+\sqrt{z}) \geq 0$ and $\text{Re}(-\sqrt{z}) \geq 0$. If $\text{Re}(\sqrt{z}) = 0$ then we use the convention that $\text{Im}(\sqrt{z}) \geq 0$.  

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The following lemma will be useful. We also sketch the situation in Figure 8.

**Lemma A.3.** Let $\alpha, \beta, \gamma : S^1 \to \mathbb{C}$ be continuous functions. Suppose that $T(q)$ is a family of tridiagonal Laurent matrices with $\alpha(q), \beta(q), \gamma(q)$ on the diagonals, which is invertible for all $q$. Assume further that $\gamma(q) = 0$ for at most one $q$ and that there exists $q_0, q_1$ such that $|\alpha(q_0)| \geq |\gamma(q_0)|$ and $|\alpha(q_1)| \leq |\gamma(q_1)|$.

Let $\lambda_1(q), \lambda_2(q)$ be the two solutions of $\alpha(q) + \beta(q)x + \gamma(q)x^2 = 0$ such that $|\lambda_1| \leq |\lambda_2|$. Then for all $q$

$$|\lambda_2(q)| < 1 < |\lambda_1(q)|.$$ 

**Proof.** For all $q$ such that $\gamma(q) \neq 0$ we can find the roots as

$$\lambda_\pm = -\frac{\beta}{2\gamma} \pm \sqrt{\left(\frac{\beta}{2\gamma}\right)^2 - \frac{\alpha}{\gamma}},$$

where we used the convention. From Vieta’s formula, we also know that $\lambda_+ \lambda_- = \frac{\alpha}{\gamma}$. Notice that since we have assumed that $T(q)$ is invertible we know that $|\lambda_+|$ and $|\lambda_-|$ are never equal to 1.

Thus, for at least one $q_0$ it holds that $|\lambda_+ \lambda_-| \geq 1$. That means either $|\lambda_1(q_0)| \geq |\lambda_2(q_0)| \geq 1$ or $|\lambda_1(q_0)| < 1 < |\lambda_2(q_0)|$. Similarly since for at least one $q_1$ it holds that $|\lambda_+ \lambda_-| \leq 1$ this means that either $|\lambda_1(q_1)| < 1 < |\lambda_2(q_1)|$ or $|\lambda_1(q_1)| \leq |\lambda_2(q_1)| < 1$.

Now, since the solutions are given by equation (25) it means that $\lambda_-(q), \lambda_+(q)$ are continuous functions of $p$ as long as $\gamma(q) \neq 0$ which happens in at most one point $q_2$. Since $S^1 \setminus \{q_2\}$ is connected the image of the set under the map $p \mapsto (\lambda_-(q), \lambda_+(q))$ is still connected. Since we have the properties for the points $q_0$ and $q_1$ and we at the same time never have eigenvalues with absolute value 0 it must mean that we are in the case

$$|\lambda_1(q)| < 1 < |\lambda_2(q)|$$

for all $q \neq q_2$. For $q = q_2$ we have the equation $\alpha(q_2) + \beta(q_2)x = 0$.

Knowledge about the modulus of the solutions can be transferred into explicitly knowing the inverse of the operator. The two solutions $\lambda_\pm$ are defined through (25) using Convention A.2. However, to be able to
deal with the solution with largest and smallest absolute value define \( \lambda_1, \lambda_2 \) such that \( \{ \lambda_1, \lambda_2 \} = \{ \lambda_+, \lambda_- \} \) and \( |\lambda_2| \leq |\lambda_1| \). Whether \( \lambda_+ \) is equal to \( \lambda_1 \) or \( \lambda_2 \) will introduce a sign in the following lemma. This we write as \((-1)^{\mathbb{I}[|\lambda_1| < 1 < |\lambda_+|]}\) where \( \mathbb{I} \) is the indicator function. Finally, we note that for finite size matrices the problem of inverting Laurent operators is a lot more intricate and has been studied in [53]. For related results see [59, Chap 3].

Lemma A.4. Suppose that \( T \) is an invertible tridiagonal Laurent matrix with \( \alpha, \beta, \gamma \) on the diagonals and such that \( \gamma \neq 0 \). Let \( \lambda_{\pm} \) be given by (29) and \( \lambda_1, \lambda_2 \) as above. Assume that \( |\lambda_2| < 1 < |\lambda_1| \) as is for example ensured by Lemma A.3. Then for \( k \geq 0 \) it holds that

\[
\langle n | T^{-1} | n + k \rangle = \frac{(-1)^{\mathbb{I}[|\lambda_1| < 1 < |\lambda_+|]} |}{\lambda_1^2 \sqrt{\beta^2 - 4\alpha \gamma}} \quad \text{as well as} \quad \langle n | T^{-1} | n - k \rangle = \frac{\lambda_2^2 (-1)^{\mathbb{I}[|\lambda_-| < 1 < |\lambda_1|]} |}{\sqrt{\beta^2 - 4\alpha \gamma}}.
\]

In particular, for \( k = 0 \) we have

\[
\langle n | T^{-1} | n \rangle = \frac{(-1)^{\mathbb{I}[|\lambda_1| < 1 < |\lambda_+|]} |}{\sqrt{\beta^2 - 4\alpha \gamma}}.
\]

The proof is a rather standard computation that we repeat for completeness.

Proof. First, \( \sigma(T) \) is the image of the symbol curve \( a(z) = \alpha z^{-1} + \beta + \gamma z \) for \( z \in \mathbb{T} \). Since \( T \) is invertible it holds that \( a(z) \neq 0 \) for all \( z \in \mathbb{T} \) and therefore, using Theorem 5.1 the symbol curve of the inverse is given by

\[
\frac{1}{a(z)} = \frac{\alpha}{z} + \beta + \gamma z = \frac{z}{\alpha + \beta z + \gamma z^2} = \frac{z}{\gamma\left(\frac{\alpha}{\gamma} + \frac{\beta}{\gamma} z + z^2\right)}.
\]

We can rewrite the denominator \( \gamma(z - \lambda_+)(z - \lambda_-) \) with

\[
\lambda_{\pm} = -\frac{\beta}{2\gamma} \pm \sqrt{\left(\frac{\beta}{2\gamma}\right)^2 - \frac{\alpha}{\gamma}}.
\]

Notice that

\( \lambda_+ \lambda_- = \frac{\alpha}{\gamma}, \lambda_+ + \lambda_- = -\frac{\beta}{\gamma} \) and \( \lambda_+ - \lambda_- = 2\sqrt{\left(\frac{\beta}{2\gamma}\right)^2 - \frac{\alpha}{\gamma}} \).

Now, the assumption \( |\lambda_2| < 1 < |\lambda_1| \) has implications on how we write this up as a geometric series:

\[
\frac{1}{a(z)} = \frac{z}{\gamma(z - \lambda_+)(z - \lambda_-)} = \frac{z}{\gamma(\lambda_1 - \lambda_2)} \left( \frac{1}{z - \lambda_1} - \frac{1}{z - \lambda_2} \right)
\]

\begin{align*}
&= \frac{z}{\gamma(\lambda_1 - \lambda_2)} \left( \frac{1}{\lambda_1} \frac{1}{1 - \frac{\lambda_2}{z}} - \frac{1}{z} \frac{1}{1 - \frac{\lambda_1}{z}} \right) \\
&= \frac{z}{\gamma(\lambda_1 - \lambda_2)} \left( -\frac{1}{\lambda_1} \sum_{n=0}^{\infty} \left(\frac{z}{\lambda_1}\right)^n - \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{\lambda_2}{z}\right)^n \right) \\
&= \frac{1}{\gamma(\lambda_2 - \lambda_1)} \left( \sum_{n=1}^{\infty} \frac{z^n}{\lambda_1^n} + \sum_{n=0}^{\infty} \left(\frac{\lambda_2}{z}\right)^n \right)
\end{align*}

From this formula we can read off the coefficients. The computation

\[
\gamma(\lambda_2 - \lambda_1) = (-1)^{\mathbb{I}[|\lambda_1| < 1 < |\lambda_+|]} \gamma(\lambda_+ - \lambda_-) = (-1)^{\mathbb{I}[|\lambda_-| < 1 < |\lambda_1|]} \sqrt{\beta^2 - 4\alpha \gamma}
\]

proves the last formula. The formulas for \( k \geq 1 \) then follow from reading off the coefficients. \( \square \)