Boundary conditions in the Unruh problem

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We have analyzed the Unruh problem in the frame of quantum field theory and have shown that the Unruh quantization scheme is valid in the double Rindler wedge rather than in Minkowski spacetime. The double Rindler wedge is composed of two disjoint regions (R- and L-wedges of Minkowski spacetime) which are causally separated from each other. Moreover the Unruh construction implies existence of boundary condition at the common edge of R- and L-wedges in Minkowski spacetime. Such boundary condition may be interpreted as a topological obstacle which gives rise to a superselection rule prohibiting any correlations between r- and l- Unruh particles. Thus the part of the field from the L-wedge in no way can influence a Rindler observer living in the R-wedge and therefore elimination of the invisible "left" degrees of freedom will take no effect for him. Hence averaging over states of the field in one wedge can not lead to thermalization of the state in the other. This result is proved both in the standard and algebraic formulations of quantum field theory and we conclude that principles of quantum field theory does not give any grounds for existence of the "Unruh effect".

I. INTRODUCTION

It was proposed more than 20 years ago that a detector moving with constant proper acceleration in empty Minkowski spacetime (MS) responses as if it had been immersed into thermal bath of Fulling particles at Davies-Unruh temperature \[ T_{DU} = \frac{\hbar g}{2\pi c k_B}, \] (1.1)
where \( g \) is proper acceleration of the observer and \( k_B \) is Boltzmann constant. Moreover it is claimed that such response is universal in the sense that it is the same for any kind of the detector. This statement is now referred as the "Unruh effect", see e.g. Refs. [3–13] and citation therein. More precisely the Unruh effect means that from the point of view of a uniformly accelerated observer the usual vacuum in MS occurs to be a mixed state described by thermal density matrix with effective temperature (1.1). In this paper we will give a critical analysis of this statement and will show that fundamental principles of quantum field theory do not give any physical grounds to assert that the Unruh effect exists.

In fact there are two aspects of the Unruh problem: (i) behaviour of a particular accelerated detector and (ii) interpretation of properties of quantum field restricted to a subregion of MS. The second aspect seems to be more fundamental since one doesn’t need to consider the structure of the detector and details of its interaction with the quantum field. Indeed the original derivation of the Unruh effect [1] (see also the later publications [2-6]) is based only on quantum field theory principles and use some special models of detectors only as illustration. Moreover exactly this approach gives grounds for the assertion about the universality of the detector response.

In this paper we will deal basically with the quantum field theory aspect (ii) of the Unruh problem. It should be emphasized that this aspect of the problem is of general interest for quantum field theory. There are serious
arguments (see e.g. [13]) to think that the Unruh effect is closely related to the effect of quantum evaporation of black holes predicted by Hawking [14]. It is claimed that both effects arise due to the presence of event horizons and that Schwarzschild observer in Kruskal spacetime may be considered by analogy with Rindler observer in MS. Furthermore very recently there were proposed some arguments that evaporation of an eternal Schwarzschild black hole may be considered as Unruh effect in the six dimensional embedding MS [15,16].

The standard explanation of the Unruh effect is based on existence of the aforementioned event horizons $h_{\pm}$ bordering the part of MS which is accessible for a Rindler observer, the so-called $R$- wedge, see Fig. 1. In the context of (ii) aspect of the Unruh problem we refer the term "Rindler observer" to a uniformly accelerated point object whose trajectory is entirely located in the $R$-wedge so that the totality of world lines of all Rindler observers completely cover the interior of the $R$-wedge.

Since such observer due to the presence of horizons have access only to a part of information possessed by inertial observers it is commonly accepted that he sees the usual vacuum state in MS as a mixed state. To put this idea on precise grounds Unruh suggested [1] a new quantization scheme for a free field in MS alternative to the standard one. There are two sorts of particles in this scheme, namely $r$- particles living everywhere but in $L$- wedge and $l$-particles living everywhere but in $R$- wedge. $r$-particles as seen by a Rindler observer turn out to be nothing but the Fulling particles [17]. The corresponding modes carry only positive frequencies with respect to $(t, z)$- plain Lorentz boost generator. The parameter of Lorentz boost may be chosen as time variable in the interior of the $R$-wedge and is called Rindler time. $r$- and $l$-particle content of Minkowski vacuum in the Unruh construction can be found by some formal manipulations [1] and after elimination of non-visible for a Rindler observer degrees of freedom corresponding to $l$-particles the "thermal" density matrix with temperature (1.1) can be obtained, see e.g. Refs. [8,13].

Such construction is however inconsistent because the Unruh quantization is unitary inequivalent to the standard one associated with Minkowski vacuum. Therefore the aforementioned expression for "Minkowski vacuum" in terms of $r$- and $l$- particles content doesn’t make direct mathematical sense and the "thermal" density matrix which arises after elimination of $l$- particles degrees of freedom actually vanishes. This difficulty was pointed out in literature by many authors (see e.g. Refs. [13]).

Therefore mathematically more rigorous methods based on algebraic approach to quantum field theory were applied to the problem. In the frame of this approach the notion of Kubo-Martin-Schwinger (KMS) state [19,20] is usually used instead of thermal equilibrium state (which can not be rigorously defined in this problem). Reformulation of the Unruh construction on the language of algebraic approach was presented in Ref. [21]. It is worth to note that mathematical physicists commonly identify [19,22,20] the Unruh effect and the so-called Bisogniano- Wichmann theorem [23,24]. This theorem is equivalent to the statement that the Minkowski vacuum state (understood in algebraic sense) when restricted to the wedge $R$ of MS satisfies the KMS condition with respect to Rindler time and the corresponding "temperature" parameter after evaluation in terms of the observer proper time is exactly the same as given by Eq. (1.1).

But apart from the difficulties related to unitary inequivalence between the Unruh and standard quantization schemes there are also inconsistencies in physical interpretation of the Unruh construction which (unlike mathematical difficulties) can not be resolved. Indeed it is well known that before any measurement could be carried out one should have prepared the initial state of the quantum system, the Minkowski vacuum state in our case. However only a part of MS consisting of the interior of the $R$-wedge is accessible for a Rindler observer. We will refer to the interior of $R$- or $L$- wedges as Rindler spacetimes (RS). These spacetimes are separated from the rest of MS by the event horizons. As a consequence any well defined scheme of quantization in RS should imply that the quantum field satisfies boundary condition at the edge $h_{0}$ of RS. In fact this condition is nothing but the usual requirement of vanishing of the field at spatial infinity.

Certainly the horizons $h_{\pm}$ arise due to excessive idealization of the problem. It is evident that in any physical situation the only thing one can achieve is to accelerate the detector during arbitrary long but finite period of time. In the latter case no horizons arise at all. But in this case one deals with the (i) aspect of the Unruh problem which has nothing to do with quantization of field in RS and hence with the notion of Fulling particles.

In virtue of boundary conditions the Unruh quantization actually can be performed only in the so-called double Rindler wedge rather than in MS. The former is a disjoint union of the interiors of the wedges $R$ and $L$ separated causally and in addition by a sort of "topological obstacle" . The role of the topological obstacle is played by the

\footnote{For definiteness we assume that Rindler observers are moving in $z$- direction with respect to some inertial observer.}
boundary condition necessary for the notion of Fulling particles and absent in the real Minkowski spacetime. It follows then that Rindler observers have nothing in common with the field in MS and that Minkowski vacuum can not be prepared by any manipulations in double RS. Moreover since the left and right RS are separated state vectors of the field in double Rindler wedge are represented by tensor products of state vectors describing the fields from R- and L- Rindler spacetimes. Since the fields in these spacetimes eternally don’t have any influence on each other only such states are physically realizable which do not have any correlations between r and l particles. In other words there is a ”superselection rule” acting in the Hilbert space of states representing quantum fields in the double Rindler wedge. Let us stress that this sort of ”superselection rule” arises due to eternal absence of interaction between the R and L parts of the field. Therefore Minkowski vacuum state which is formally represented in the Unruh construction [1] as a ”superposition” of states with different number of r - particles and the same number of l - particles is physically unrealizable for the quantum field in the double RS and discussion of it’s thermal or other properties becomes meaningless.

Since the difficulties with interpretation of the Unruh construction are of physical nature they of course arise also when one interprets the results obtained in algebraic approach. In Ref. [21] the notion of KMS state was related to the usual notion of the bath of heated up r- and l- particles. But it occurs that these states may be mathematically well-defined only for the observables which vanish at some neighborhood of the common edge of the right and left RS. It is nothing but the mentioned above boundary condition which leads to loss of any connection between a Rindler observer and MS.

Elucidation of the role of boundary conditions in the Unruh problem is the central point of this paper. Therefore we begin in Section II with brief consideration of boundary conditions at spatial infinity for a free field quantized in a plain-wave basis in MS. The existence of boundary conditions is always implied but their discussion is usually omitted in text-books. Nevertheless disregarding of this point may lead to mistakes in treating some delicate problems as it happened in our opinion with the Unruh problem. In Section III we consider boundary conditions for quantum field in RS. This case slightly differs in some technical details from the case of a free field in MS due to the absence of mass gap for Fulling particles. Section IV is devoted to consideration of quantization of a free field in MS in the basis of ”boost modes”. This scheme of quantization is unitary equivalent to the usual plain wave quantization and is exploited in the Unruh construction. We consider the Unruh construction in Section V. We show that the Unruh quantization scheme is valid only for the double Rindler wedge and can not be used for derivation of ”thermal properties” of Minkowski vacuum with respect to a Rindler observer. Algebraic approach to the Unruh problem is discussed in Section VI. Our results are summarized in Section VII.

In Appendix A we present some technical details of derivation of the expression for boost mode annihilation operator in terms of the field values on a Cauchy surface. The derived formula allows one to understand difference between the Uruh and Fulling operators. Appendix B includes discussion of analogy between the Unruh construction and the construction of squeezed states for a two dimensional harmonic oscillator. In this Appendix we have also included a proof of unitary inequivalence of the standard plain wave and Unruh schemes of quantization. Although this issue seems to be a well-known fact we actually could not find a detailed discussion of it in physical literature.

In the paper we restrict our discussion to the case of massive neutral scalar field in 1+1 dimensional MS. Generalization to higher dimensions may be obtained straightforwardly by introducing components of momentum $\vec{q}$ orthogonal to the direction of motion of a Rindler observer just changing definition of the mass of the field as $m \rightarrow \sqrt{m^2 + q^2}$ and inserting additional integration over $\vec{q}$ in appropriate places (see e.g. Ref. [12] for 1 + 3 dimensional case and Ref. [10] for the general case of $1 + n$ dimensions).

Short presentation of our results can be found in Refs. [25,26].

II. QUANTIZATION OF A NEUTRAL SCALAR FIELD IN D=1+1 MINKOWSKI SPACETIME (PLANE WAVE MODES)

In this section we will discuss the boundary conditions for a quantized free scalar field in two-dimensional Minkowski space-time (MS). Let

$$x = \{t, z\}, \quad ds^2 = dt^2 - dz^2,$$

(2.1)
be global coordinates and metric of pseudoeuclidean plain.

Operator of a free neutral scalar field \( \phi_M(x) \) of mass \( m \) satisfies the Klein-Fock-Gordon (KFG) equation

\[
\left\{ \frac{\partial^2}{\partial t^2} + \mathcal{K}_M(z) \right\} \phi_M(x) = 0, \quad \mathcal{K}_M(z) = -\frac{\partial^2}{\partial z^2} + m^2.
\]  

(2.2)

The plain waves

\[
\Theta_p(x) = (2\epsilon_p)^{-1/2} e^{-i\epsilon_p t} \varphi_p(z), \quad \varphi_p(z) = (2\pi)^{-1/2} e^{ipz}, \quad \epsilon_p = \sqrt{p^2 + m^2}, \quad -\infty < p < \infty,
\]  

(2.3)

form a complete set of solutions of the equation (2.2) orthonormalized relative to scalar product in MS

\[
(f,g)_M = i \int_{-\infty}^{\infty} f^*(x) \frac{\partial}{\partial t} g(x) dz.
\]  

(2.4)

The completeness of the set (2.3) allows one to perform quantization by setting

\[
\phi_M(x) = \int_{-\infty}^{\infty} dp \left[ a_p \Theta_p(x) + a_p^\dagger \Theta_p^*(x) \right],
\]  

(2.5)

where \( a_p \) and \( a_p^\dagger \) are annihilation and creation operators obeying canonical commutation relations. The vacuum state \( |0\rangle_M \) in MS is defined by the relations

\[
a_p |0\rangle_M = 0, \quad -\infty < p < \infty,
\]  

(2.6)

and operators \( a_p \) can be expressed in terms of field operator \( \phi_M(x) \) values on an arbitrary spacelike surface by

\[
a_p = i \int_{-\infty}^{\infty} \Theta_p^*(x) \frac{\partial}{\partial t} \phi_M(x) dz.
\]  

(2.7)

It is commonly assumed that the field \( \phi_M(x) \) vanishes at spatial infinity. Nevertheless it is worth emphasizing that the operators \( a_p, a_p^\dagger \), \( \phi_M(x) \) are unbounded ones and therefore the requirement \( \phi_M(x) \to 0, \ z \to \pm \infty \) as well as relations (2.3), (2.7) should be understood in weak sense [27–29]. The latter means that these statements relate to arbitrary matrix elements of operators under discussion.

Note that the requirement of vanishing of the field \( \phi_M(x) \) in the weak sense at spatial infinity is a necessary condition for finiteness of the energy of the field. For illustration of this statement we will consider one particle amplitude

\[
\phi_f(x) = \langle 0_M | \phi_M(x) | f \rangle,
\]  

(2.8)

which determines all matrix elements of the free field operator. One particle state \( |f\rangle \) in Eq.(2.8) is defined by

\[
|f\rangle = a_p^\dagger |0\rangle_M, \quad a_p^\dagger = \int_{-\infty}^{\infty} dp \ f(p) a_p^\dagger, \quad \langle f|f\rangle = \int_{-\infty}^{\infty} dp |f(p)|^2 = 1.
\]  

(2.9)

The field Hamiltonian expectation value in the state \( |f\rangle \) is given by the following expression

\[
From this place we use natural units \( \hbar = c = k_B = 1 \) throughout the paper.
\[ \langle f|H|f \rangle = \langle f \rangle \int_{-\infty}^{\infty} dz : T^{00} : |f \rangle = \frac{1}{2} \int_{-\infty}^{\infty} dz \left\{ \left| \frac{\partial \phi_f(x)}{\partial t} \right|^2 + \left| \frac{\partial \phi_f(x)}{\partial z} \right|^2 + m^2 |\phi_f(x)|^2 \right\}. \] (2.10)

One can easily see that the finiteness of the field energy \( \langle f|H|f \rangle \) implies

\[ \int_{-\infty}^{\infty} |\phi_f(x)|^2 dz < \infty, \quad \int_{-\infty}^{\infty} \left| \frac{\partial \phi_f(x)}{\partial z} \right|^2 dz < \infty, \] (2.11)

and hence leads to continuity of \( \phi_f(x) \) and its vanishing at spatial infinity

\[ \phi_f(t, z) \to 0, \quad z \to \pm \infty. \] (2.12)

Indeed from the inequality

\[ |\phi_f^2(t, z_2) - \phi_f^2(t, z_1)| \leq 2 \int_{z_1}^{z_2} dz |\phi_f(x)\frac{\partial \phi_f(x)}{\partial z}| \leq 2 \left( \int_{z_1}^{z_2} dz |\phi_f(x)|^2 \cdot \int_{z_1}^{z_2} dz \left| \frac{\partial \phi_f(x)}{\partial z} \right|^2 \right)^{1/2} \]

and square integrability of \( \phi_f(x) \) and \( \partial \phi_f(x) / \partial z \) it is evident that the function \( \phi_f(t, z) \) is continuous and there exists the limit \( \lim_{z \to \pm \infty} \phi_f(t, z) \). But if \( \phi_f(x) \) is square integrable then this limit should be zero (for details of the proof in more general case see Sec. 5.6 in Ref. [30]). Hence if we refuse from the condition (2.12) then the energy becomes infinite.

This boundary condition in terms of Wickramtunga functions is equivalent to vanishing of the two-point function for infinite spacelike separations. From Eqs. (2.3), (2.6) for positive-frequency function \( \Delta^{(+)}(x, m) = i \langle 0_M|\phi_M(x)\phi_M(0)|0_M \rangle \) we obtain

\[ \Delta^{(+)}(x, m) = \frac{1}{4} \times \begin{cases} H_0^{(2)}(m\sqrt{t^2 - z^2}), & t > |z|, \\ \frac{2m}{\pi} K_0(m\sqrt{z^2 - t^2}), & |t| < |z|, \\ -H_0^{(1)}(m\sqrt{t^2 - z^2}), & t < -|z|, \end{cases} \] (2.13)

where \( H_0^{(1,2)} \) are Hankel and \( K_0 \) - Macdonald (modified Bessel) functions. Therefore using asymptotic expansions for \( H_0^{(1,2)} \) and \( K_0 \) we get for \( |z| \to \infty, t = 0 \)

\[ \Delta^{(+)}(x, m) \propto (m|z|)^{-1/2}e^{-m|z|}. \] (2.14)

The two-point commutator function \( \Delta(x - x', m) = i \langle \phi_M(x), \phi_M(x') \rangle \) reads

\[ \Delta(x, m) = \frac{1}{4} \{ \text{sgn}(t - z) + \text{sgn}(t + z) \} J_0(m\sqrt{t^2 - z^2}), \] (2.15)

where \( J_0 \) denotes Bessel function, \( \theta(\tau) \) is the Heavicide step function and \( \text{sgn}(\tau) = \theta(\tau) - \theta(-\tau) \). Note that the Cauchy data of the function \( \Delta(x, m) \) on the surface \( t = 0 \) is

\[ \Delta(x, m)|_{t=0} = 0, \quad \frac{\partial \Delta(x, m)}{\partial t} \bigg|_{t=0} = \delta(z), \] (2.16)

in full analogy with the Cauchy data for the Pauli-Jordan function in four dimensional case.

### III. QUANTIZATION OF A NEUTRAL SCALAR FIELD IN D=1+1 RINDLER SPACE

In this section we will consider quantization of a neutral scalar field in D=1+1 Rindler space the geometry of which is described by the metric

\[ ds^2 = \rho^2 dt^2 - d\rho^2, \quad -\infty < \eta < +\infty, \quad 0 < \rho < \infty. \] (3.1)

This issue plays an important role for the Unruh problem because it defines the notion of Fulling particles. In the Sec. III A we will define the Fulling modes which form a basis for quantization and introduce the notion of Fulling particles. In the Sec. III B we will discuss boundary conditions arising in the procedure of Fulling quantization.
the inner product in RS, and one can define vacuum state |0\rangle.

Note that the functions \(\psi\) quantizing the field \(\phi\) with measure \(d\sigma(\rho) = d\rho/\rho\) and inner product \((\chi, \psi)_{L^2} = \int_0^\infty \chi(\rho)\psi(\rho) d\rho/\rho\). Functions from the domain of it’s definition \(\mathcal{D}(\mathcal{K}_R)\) obey the conditions

\[
(\psi, \psi)_{L^2} = \int_0^\infty \frac{d\rho}{\rho} |\psi(\rho)|^2 < \infty, \quad \int_0^\infty \frac{d\rho}{\rho} \left| \rho \frac{\partial \psi(\rho)}{\partial \rho} \right|^2 < \infty, \quad \psi(0) = 0, \quad (3.3)
\]

where the last restriction is a consequence of the two previous (compare to Eqs. (2.11), (2.12)). The condition \(\psi(0) = 0\) is an automatic or built-in boundary condition, see Ref. [30]. This means that at the point \(\rho = 0\) we encounter the case of the Weyl limit-point. From mathematical point of view it results from the fact that deficiency indices of the operator \(\mathcal{K}_R\) are equal to \((0, 0)\). The physical meaning of this condition was discussed by Fulling [17] using the language of wave packets.

The Eigenfunctions of the operator \(\mathcal{K}_R\),

\[
\psi_\mu(\rho) = \pi^{-1}(2\mu \sinh \pi \mu)^{1/2} K_{\mu}(\rho), \quad \mathcal{K}_R \psi_\mu(\rho) = \mu^2 \psi_\mu(\rho), \quad (3.4)
\]

satisfy the orthogonality and completeness conditions,

\[
\int_0^\infty \frac{d\rho}{\rho} \psi_\mu(\rho)\psi_{\mu'}(\rho) = \delta(\mu - \mu'), \quad \int_0^\infty d\mu \psi_\mu(\rho)\psi_\mu^*(\rho') = \rho \delta(\rho - \rho'). \quad (3.5)
\]

Note that the functions \(\psi_\mu(\rho)\) satisfy the third condition in Eq. (3.3) in the sense of distributions.

For the solution of the Cauchy problem for Eq. (3.2) we have

\[
\phi_R(\eta, \rho) = e^{-i\eta K_{-1/2}^R} \psi(\rho) + e^{i\eta K_{-1/2}^R} \psi^*(\rho), \quad \psi(\rho) = \frac{1}{2} \phi_R(0, \rho) + \frac{i}{2} \mathcal{K}_R^{-1/2} \frac{\partial}{\partial \eta} \phi_R(0, \rho). \quad (3.6)
\]

Therefore positive-frequency with respect to timelike variable \(\eta\) modes, Fulling modes [17], read

\[
\Phi_\mu(\xi) = (2\mu)^{-1/2} \psi_\mu(\rho) e^{-i\mu \eta}, \quad \mu > 0. \quad (3.7)
\]

These modes are orthonormal relative to the inner product in RS,

\[
(F, G)_R = i \int_0^\infty \frac{d\rho}{\rho} F^*(\xi) \frac{\partial}{\partial \eta} G(\xi), \quad (3.8)
\]

and together with \(\Phi_\mu\) form a complete set of solutions of KFG equation (3.2). Therefore they may be used for quantizing the field \(\phi_R\),

\[
\phi_R(\xi) = \int_0^\infty d\mu \left\{ c_\mu \Phi_\mu(\xi) + c^\dagger_\mu \Phi^*_\mu(\xi) \right\}, \quad [c_\mu, c^\dagger_{\mu'}] = \delta(\mu - \mu'), \quad [c_\mu, c_{\mu'}] = [c^\dagger_\mu, c^\dagger_{\mu'}] = 0, \quad (3.9)
\]

and one can define vacuum state \(|0_R\rangle\) for Rindler space by the condition

\[
c_\mu|0_R\rangle = 0, \quad \mu > 0. \quad (3.10)
\]
The states which are created from this vacuum by operators $c^\dagger_\mu$ correspond to Fulling (or sometimes also called Rindler) particles.

The annihilation operator $c_\mu$ may be expressed in terms of the field $\phi_R$ by

$$c_\mu = (\Phi_\mu, \phi_R)_R = \left. \frac{i}{\sqrt{2\mu}} \int_0^\infty \frac{d\rho}{\rho} \psi_\mu(\rho) \left( \frac{\partial \phi_R(\xi)}{\partial \eta} - i \mu \phi_R(\xi) \right) \right|_{\eta=0}. \quad (3.11)$$

The secondly quantized operator corresponding to the Killing vector $i\partial/\partial \eta$,

$$K = \int_0^\infty \mu c^\dagger_\mu c_\mu d\mu, \quad (3.12)$$

plays a role of Hamiltonian.

With the help of Eqs. (3.9), (3.10) one can calculate the two-point commutator $D(\eta - \eta', \rho, \rho') = i[\phi_R(\xi), \phi_R(\xi')]$ and the positive-frequency Whightman function $D^{(+)}(\eta - \eta', \rho, \rho') = i(0_R|\phi_R(\xi)|\phi_R(\xi')|0_R)$ for RS,

$$D(\eta - \eta', \rho, \rho') = \frac{2}{\pi^2} \int_0^\infty d\mu \sin \pi \mu \sin \mu(\eta - \eta') \, K_{i\mu}(m\rho)K_{i\mu}(m\rho'), \quad (3.13a)$$

$$D^{(+)}(\eta - \eta', \rho, \rho') = \frac{i}{\pi^2} \int_0^\infty d\mu \sin \pi \mu \cos \mu(\eta - \eta') \, K_{i\mu}(m\rho)K_{i\mu}(m\rho') + \frac{1}{2}D(\eta - \eta', \rho, \rho'). \quad (3.13b)$$

Using the relation between Minkowski coordinates $(t, z)$ and Rindler coordinates $(\eta, \rho)$ in the R-wedge of MS,

$$\eta = \text{artanh} \left(\frac{z}{t}\right), \quad \rho = (z^2 - t^2)^{1/2}, \quad (3.14)$$

one can easily see that the two point commutation functions $D$ and $\Delta$ coincide. In particular for $\Delta s^2 = 2\rho\rho' \cosh(\eta - \eta') - \rho^2 - \rho'^2 > 0$ we have $D(\eta - \eta', \rho, \rho') = \frac{1}{2}J_0 \left( m(\Delta s^2)^{1/2} \right)$. This coincidence of two-point commutators means that the local properties of the quantum fields $\phi_M$ and $\phi_R$ are the same. Nevertheless global properties of these fields are different due to different definitions of vacuums $|0_M\rangle$ and $|0_R\rangle$, $\Delta^{(+)} \neq D^{(+)}$. Note that singularities of these functions for coinciding points are the same and cancel when one takes their difference,

$$\langle 0_M|\phi_M^2(\xi)|0_M\rangle - \langle 0_R|\phi_R^2(\xi)|0_R\rangle = \frac{1}{\pi^2} \int_0^\infty d\mu e^{-\pi \mu}K_{i\mu}^2(m\rho) > 0. \quad (3.15)$$

**B. Boundary condition**

Let us now discuss the boundary conditions for the field $\phi_R(\xi)$. As in section [11] we will consider one-particle amplitude

$$\phi_g(\xi) = \langle 0_R|\phi_R(\xi)|g\rangle = \exp(-i\eta K^{1/2}_R)\psi_\eta(\rho). \quad (3.16)$$

(compare to Eq. (3.6)), where

$$|g\rangle = c^\dagger(g)|0_R\rangle, \quad c^\dagger(g) = \int_0^\infty \frac{d\mu}{\mu^{1/2}} g(\mu)c^\dagger_\mu. \quad (3.17)$$
The spatial part $\psi_g$ of the one particle amplitude (3.16) is expressed in terms of the weight function $g$ as follows,

$$
\psi_g(\rho) = \int_0^\infty d\mu \, G(\mu, \rho), \quad G(\mu, \rho) = \frac{g(\mu)}{\pi} \left( \frac{\sinh \pi \mu}{\mu} \right)^{1/2} K_{i\mu}(m\rho).
$$

(3.18)

For inner product in RS we have

$$
(\phi_g, \phi_h)_R = 2(K_{1/4}^R \psi_g, K_{1/4}^R \psi_h)_{L^2},
$$

(3.19)

and the state $|g\rangle$ is normalized by the condition

$$
\langle g|g\rangle = \int_0^\infty d\mu \, |g(\mu)|^2 = 2 \int_0^\infty \rho \, |K_{1/4}^R \psi_g|^2 = 1.
$$

(3.20)

We will discuss boundary conditions at both boundaries of RS, namely at the points $\rho = \infty$ and $\rho = 0$. It’s worth to note that the point $\rho = 0$ may also be considered as spatial infinity. It becomes evident after the Langer transformation $u = \ln(m\rho)$ ($-\infty < u < \infty$) mapping the singular point $\rho = 0$ into $-\infty$. After this transformation operator $K_R$ takes the form

$$
K_R = -\frac{\partial^2}{\partial u^2} + V(u), \quad V(u) = m^2 e^{2u}.
$$

(3.21)

Since $V(u)$ is a confining potential at $u \to +\infty$ the boundary condition

$$
\phi_g(\eta, +\infty) = 0,
$$

(3.22)

is obvious and is satisfied not only for the amplitude (3.16) but even for Eigenfunctions $\psi_\mu(\rho)$ of operator $K_R$.

Therefore we concentrate below on the less evident case $\rho = 0$ or $u = -\infty$.

The requirement of finiteness of the average energy (3.12) in the state $|g\rangle$,

$$
\langle g|K|g\rangle = \frac{1}{2} \int_0^\infty \rho \, \left\{ \frac{\partial \phi_g}{\partial \eta} \right\}^2 + \rho^2 \left\{ \frac{\partial \phi_g}{\partial \rho} \right\}^2 + m^2 \rho^2 |\phi_g|^2 \right\} = \int_0^\infty \rho |g(\mu)|^2 < \infty,
$$

(3.23)

leads to the restrictions

$$
\int_0^\infty \frac{d\rho}{\rho} \left| \frac{\partial \phi_g}{\partial \rho} \right|^2 < \infty, \quad \int_0^\infty \frac{d\rho}{\rho} |\rho \phi_g|^2 < \infty.
$$

(3.24)

But from these restrictions (unlike the case we had in MS) it does not immediately follow that

$$
\phi_g(\eta, 0) = 0.
$$

(3.25)

Therefore we need use more delicate procedure to prove the condition (3.25).

Let us split the integral in Eq.(3.18) into three parts,

$$
\psi_g(\rho) = I_1(\rho) + I_2(\rho) + I_3(\rho),
$$

$$
I_1 = \int_{\mu_1} \rho \, G \, d\mu, \quad I_2 = \int_{\mu_2} \rho \, G \, d\mu, \quad I_3 = \int_{\mu_2}^\infty G \, d\mu,
$$

(3.26)

where $\mu_1, \mu_2$ are arbitrary numbers such that $0 < \mu_1 \ll 1 \ll \mu_2 < \infty$ and the function $G(\mu, \rho)$ is defined in Eq.(3.18). After applying the known asymptotic behaviour of the Macdonald function \[31\] we obtain for $u = \ln(m\rho) < 0, |u| \gg 1$
Then from Eqs. (3.26), (3.27) we obtain the estimation

$$G(\mu, \rho) = \frac{g(\mu)}{\sqrt{\pi \mu}} \begin{cases} -\sin(\mu u - \mu \ln 2), & \mu \ll 1, \\ \cos(\mu u - \mu \ln 2 - \arg \Gamma(i\mu)), & \mu \sim 1, \\ \sin(\mu \ln \mu - \mu u + \mu(\ln 2 - 1) + \pi/4), & \mu \gg 1. \end{cases}$$

(3.27)

Let us first proceed with the last two terms in Eq. (3.26). It follows from the normalization condition Eq. (3.20) and the evident inequality $|g| \leq \frac{1}{2}(1 + |g|^2)$ that the integral $\int_{\mu_2}^{\infty} |g(\mu)| \, d\mu/\mu$ should converge. Therefore we may apply the Riemann-Lebesgue lemma to conclude that $I_2(\rho)$ vanishes when $\rho \to 0$. To estimate $I_3$ we use the Schwartz inequality and get

$$|I_3(\rho)|^2 \leq \frac{1}{\pi \mu_2} \int_{\mu_2}^{\infty} |g(\mu)|^2 \, d\mu.$$

(3.28)

Now using the condition of finiteness of the energy Eq. (3.23) we conclude that $I_3(\rho)$ may be done arbitrary small by the appropriate (independent of the value of $\rho$) choice of $\mu_2$. Therefore the sum $I_2(\rho) + I_3(\rho)$ tends to zero when $\rho \to 0$. Note also that for differentiable functions $g(\mu)$ one has the estimation $I_2 + I_3 \propto \ln^{-1}(1/\rho m)$ for $\rho \ll m^{-1}$.

Consider now the first part $I_1(\rho)$ of the integral in Eq. (3.20). We will discuss first the case of weight functions $g(\mu)$ which are continuous at the point $\mu = 0$. From Eq. (3.20) it immediately follows that such weight functions should vanish for $\mu \to 0$, $g(0) = 0$.

Let $g(\mu)$ vanish for $\mu \to 0$ as a power of $\mu$,

$$g(\mu) = a \mu^\alpha, \quad \alpha > 0, \quad \mu \to 0.$$  

(3.29)

Then from Eqs. (3.26), (3.27) we obtain the estimation

$$I_1(\rho) = \frac{a \Gamma(\alpha) \sin(\pi \alpha/2)}{\sqrt{\pi} (\ln \frac{2}{m\rho})^\alpha}, \quad \rho \ll m^{-1}, \quad \alpha \neq 2, 4, 6, ... .$$

(3.30)

Thus for this case $I_1(\rho)$ decreases logarithmically when $\rho \to 0$. We see that for $0 < \alpha < 1$ the term $I_1(\rho)$ dominates in the sum in Eq. (3.20) for small $\rho$ and thus the whole integral over $\mu$ is defined by the behaviour of $g(\mu)$ at small $\mu$ while for $\alpha > 1$ generally $\psi_\rho(\rho)$ does not depend on behaviour of $g(\mu)$ at small $\mu$. For even values of $\alpha$ the leading term $I_1(\rho)$ decreases for $\rho \to 0$ even faster. For example, for the particular weight function

$$g_0(\mu) = \frac{a_0 \mu (\sinh \mu)^{1/2}}{\sqrt{\pi \Gamma^2(\epsilon)}} |\Gamma(\epsilon + i\mu)|^2, \quad (\epsilon > 0), \quad g_0(\mu) \approx a_0 \mu^2, \quad \mu \to 0,$$

(3.31)

($a_0$ - normalization constant) by performing Kontorovich - Lebedev transform [35, 36] we obtain $\psi_\rho(\rho) \propto (m\rho)^\epsilon \exp(-m\rho) \propto \rho^\epsilon$ for $\rho \ll m^{-1}$.

If the weight function vanishes for small $\mu$ logarithmically,

$$g(\mu) = b \left( \ln \frac{1}{\mu} \right)^{-\beta}, \quad \beta > 0, \quad \mu \to 0,$$

(3.32)

then for small $\rho$ we have

$$I_1(\rho) = \frac{\sqrt{\pi b}}{2 \left( \ln \frac{2}{m\rho} \right)^\beta} \left( 1 - \frac{\beta C}{\left( \ln \frac{2}{m\rho} \right)^\beta} + ... \right),$$

(3.33)

where $C = 0.577...$ is Euler constant. This asymptotic representation is valid in particular for $0 < \beta < 1/2$ when the normalization integral [3.20] diverges. Since the faster $g(\mu)$ decreases for $\mu \to 0$ the faster $I_1(\rho)$ tends to zero, we conclude that the one particle amplitude $\Phi_\rho(\xi)$ tends to zero when $\rho \to 0$ for all weight functions which are continuous at $\mu = 0$, satisfy the normalization condition and correspond to the states with finite energy.
Note that continuity of weight functions $g(\mu)$ is not a necessary condition for the validity of the condition Eq.(3.25). As an example we will consider the weight function which for small $\mu$ behaves as

$$g(\mu) = c\sigma^\gamma \exp(-d\sigma^2\sin^2\sigma), \quad \gamma \geq 0, \quad \sigma = \ln(1/\mu).$$

(3.34)

For big values of $\sigma$ this function is localized almost near the points $\sigma_n = \pi n$ with characteristic width $\Delta \sigma_n = d^{-1/2}(\pi n)^{-\delta/2}$. Therefore for the contribution of small $\mu$ to the normalization integral Eq.(3.20) we have

$$\int_0^{\mu_1} \frac{d\mu}{\mu} |g(\mu)|^2 = \int_{\sigma_1}^{\infty} d\sigma |g(e^{-\sigma})|^2 \propto \sum_n d^{-1/2}(\pi n)^{2\gamma-\delta/2},$$

where $\sigma_1 = \ln(\frac{1}{\mu_1})$. Therefore the function $g(\mu)$ with behaviour at small $\mu$ as given by Eq.(3.34) may be normalizable if

$$\delta > 4\gamma + 2$$

(3.36)

(the case $\gamma = 1/2$, $\delta = 6$ is known as Dirichlet-Reymond example [37,30]). But let us note that for the function Eq.(3.34) with parameters satisfying Eq.(3.36) the integral $\int_0^{\mu_1} |g(\mu)|^2 d\mu/\mu = \int_{\sigma_1}^{\infty} |g(e^{-\sigma})| d\sigma$ also converges and thus in virtue of the Riemann-Lebesgue lemma we conclude that even in this case we have $I_1(\rho) \to 0$ as $\rho \to 0$. It looks as if the condition Eq.(3.25) is valid for discontinues at $\rho = 0$ weight functions as well.

Let us also note that for arbitrary small but finite accuracy of measuring the energy of Fulling particles the weight functions of the type Eq.(3.34) with parameters satisfying Eq.(3.36) the integral in Eq.(3.13b) becomes principal. Thus we have

$$\langle g | K^{-1} | g \rangle = \int_0^{\mu_1} \frac{d\mu}{\mu^2} |g(\mu)|^2 = 2 \int_0^\infty \frac{d\rho}{\rho} |\psi\rangle |\psi\rangle < \infty,$$

(3.37)

then the boundary condition Eq.(3.25) may be proved straightforwardly in the same way as it was done in section $\underline{1}$. The condition Eq.(3.37) corresponds to the regularity condition which was proposed by Kay in Ref. [21] to guarantee the existence of thermal states for Fulling quantization system.

At the end of the section let us discuss the asymptotic behaviour of Whightman function $D^{(+)}(\eta - \eta', \rho, \rho')$ for fixed value of $\rho'$ and $\rho \to 0$. In this case the two-point commutator vanishes and the contribution of small $\mu$ to the integral in Eq.(3.13b) becomes principal. Thus we have

$$D^{(+)}(\eta - \eta', \rho, \rho') = \frac{i K_0(m\rho')}{\pi \ln(2/m\rho)} + O(\ln^{-2}(1/m\rho)), \quad (\Delta s^2 < 0).$$

(3.38)

We see that physically significant quantities die out at point $\rho = 0$ in RS as they do at spatial infinity in MS.

IV. QUANTIZATION OF A NEUTRAL SCALAR FIELD IN D=1+1 MINKOWSKI SPACETIME (BOOST MODES)

The Rindler observer world line coincides with one of the orbits of Lorentz group and $R$-wedge is one of domains of MS invariant under Lorentz rotation. Therefore it is convenient dealing with the Unruh problem to quantize the field in the basis of Eigenfunctions of Lorentz boost operator rather than in the plane-wave basis.

Since the secondary quantized boost operator $L = M_{tx}$ does not commute with the energy $H$ and momentum $P$ operators,
\[ [L, H] = iP, \quad [L, P] = iH, \quad (4.1) \]

it is not diagonal in terms of annihilation, creation operators of particles with given momentum \( p \). In order to
diagonalize the operator \( L \) we replace first the variable \( p \) by the rapidity \( q = \text{artanh}(p/\epsilon_p) \) and introduce new
operators \( \alpha_q \) as

\[
\alpha_q = (m \cosh q)^{1/2} a_p, \quad [\alpha_q, \alpha_q^\dagger] = \delta(q - q'). \quad (4.2)
\]

The energy, momentum and boost operators are expressed in terms of new operators as follows

\[
H = m \int_{-\infty}^{\infty} dq \cosh q \alpha_q^\dagger \alpha_q, \quad P = m \int_{-\infty}^{\infty} dq \sinh q \alpha_q^\dagger \alpha_q, \quad L = \frac{i}{2} \int_{-\infty}^{\infty} dq \alpha_q^\dagger \frac{\partial}{\partial q} \alpha_q, \quad (4.3)
\]

It is easy to see that in terms of operators \( b_\kappa \) which are Fourier transforms of operators \( \alpha_q \)

\[
b_\kappa = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\kappa q} \alpha_q dq, \quad \alpha_q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\kappa q} b_\kappa d\kappa. \quad (4.4)
\]

the boost operator \( L \) is diagonal

\[
L = \int_{-\infty}^{\infty} \kappa b_\kappa^\dagger b_\kappa d\kappa. \quad (4.5)
\]

Using the definition \((4.2)\) and the relations \((4.4)\) one can easily transform the Eq.(2.5) to the form

\[
\phi_M(x) = \int_{-\infty}^{\infty} d\kappa \{ b_\kappa \Psi_\kappa(x) + b_\kappa^\dagger \Psi_\kappa^\ast(x) \}, \quad (4.6)
\]

where functions \( \Psi_\kappa \) are defined by the integral representation \([38,39,25,26]\)

\[
\Psi_\kappa(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dq \exp\{im(z \sinh q - t \cosh q) - ik\kappa\}. \quad (4.7)
\]

It is assumed that an infinitely small negative imaginary part added to \( t \), see Appendix A. It may be shown that
functions \((4.7)\) are Eigenfunctions of the boost generator \( B \),

\[
B \Psi_\kappa(x) = \kappa \Psi_\kappa(x), \quad -\infty < \kappa < +\infty, \quad B = i(z\partial/\partial t + t\partial/\partial z). \quad (4.8)
\]

They are orthonormal relative to inner product in MS,

\[
\langle \Psi_\kappa, \Psi_{\kappa'} \rangle_M = \delta(\kappa - \kappa'), \quad (\Psi_\kappa^\ast, \Psi_{\kappa'})_M = 0, \quad (4.9)
\]

and together with their adjoints \( \Psi_\kappa^\ast \) form a complete set of solutions for KFG equation in MS. We will call this set of
functions boost modes.

The boost modes can serve as a basis for a new quantization scheme for the field \( \phi_M(x) \). Indeed according to
Eqs.(4.2), \((4.4)\) the commutation relations for \( b_\kappa \) read

\[
[b_\kappa, b_{\kappa'}^\dagger] = \delta(\kappa - \kappa'), \quad [b_\kappa, b_{\kappa'}] = [b_\kappa^\dagger, b_{\kappa'}^\dagger] = 0. \quad (4.10)
\]

The vacuum state with respect to operators \( b_\kappa \) which obeys the condition

\[
b_\kappa|0_M\rangle = 0. \quad (4.11)
\]
is exactly the usual Minkowski vacuum. This is because the transition from the operators \( \alpha_q \) to \( b_\kappa \) (4.4) is a unitary transformation,

\[
b_\kappa = F \alpha_{-\kappa} F^\dagger, \quad FF^\dagger = F^\dagger F = 1, \quad F = \exp \left( i \frac{\pi}{4} \int_{-\infty}^{\infty} dq \{ \partial_q \alpha_q^\dagger \partial_0 \alpha_q + (q^2 - 1) \alpha_q^\dagger \alpha_q \} \right),
\]

(4.12)

and hence the solutions \( \Psi_\kappa \) correspond to positive frequencies relative to global time \( t \). Note that quantization of scalar field defined by Eqs.(4.10), (4.11), (4.6) is equivalent to the one performed in Ref. [10] by analytical extension of Green functions.

There exists another representation of boost modes [25,26] (see also Ref. [8] for the fermion case) corresponding to splitting of MS into the right (R), future (F), left (L) and past (P) wedges, see Fig.1.

\[
\Psi_\kappa = \theta(x_+)\theta(-x_-)\Psi_\kappa^{(R)} + \theta(x_+)\theta(x_-)\Psi_\kappa^{(F)} + \theta(-x_+)\theta(x_-)\Psi_\kappa^{(L)} + \theta(-x_+)\theta(-x_-)\Psi_\kappa^{(P)},
\]

where \( x_\pm = t \pm x \) are null coordinates in MS. By performing integration in Eq.(4.7) under assumption \( x_+ > 0, x_- < 0 \) and using integral representation for Macdonald functions [31] we obtain for \( \Psi_\kappa^{(R)} \)

\[
\Psi_\kappa^{(R)} = \frac{1}{\pi \sqrt{2}} \exp \left( \frac{\pi \kappa}{2} - i \frac{\kappa}{2} \ln \left( \frac{x_+}{-x_-} \right) \right) K_{i\kappa} \left( m \sqrt{-x_-x_+} \right).
\]

(4.14a)

The explicit form for the boost modes on other wedges may be obtained from (4.14a) by analytical extension. The branch points of the function Eq.(4.14) lie on the light cone and for transition from one wedge to another one should use the substitutions \((-x_-) \rightarrow x_- e^{i\pi} \) for the transition \( R \rightarrow F \), \( x_+ \rightarrow (-x_-) e^{-i\pi} \) for \( F \rightarrow L \), \( x_- \rightarrow (-x_-) e^{-i\pi} \) for \( L \rightarrow P \) and \((-x_+) \rightarrow x_+ e^{i\pi} \) for \( P \rightarrow R \). Thus we obtain

\[
\Psi_\kappa^{(R)} = \frac{-i}{2 \sqrt{2}} \exp \left( - \frac{\pi \kappa}{2} - i \frac{\kappa}{2} \ln \left( \frac{-x_-}{x_+} \right) \right) H_{i\kappa}^{(2)} \left( m \sqrt{-x_-x_+} \right).
\]

(4.14b)

\[
\Psi_\kappa^{(L)} = \frac{1}{\pi \sqrt{2}} \exp \left( - \frac{\pi \kappa}{2} - i \frac{\kappa}{2} \ln \left( \frac{x_+}{x_-} \right) \right) K_{i\kappa} \left( m \sqrt{-x_-x_+} \right).
\]

(4.14c)

\[
\Psi_\kappa^{(P)} = \frac{i}{2 \sqrt{2}} \exp \left( - \frac{\pi \kappa}{2} - i \frac{\kappa}{2} \ln \left( \frac{-x_-}{x_+} \right) \right) H_{i\kappa}^{(1)} \left( m \sqrt{-x_-x_+} \right).
\]

(4.14d)

After transition \( P \rightarrow R \) we return to Eq.(4.14a). The second linearly independent set of solutions for KFG equation \( \Psi_\kappa \) may be obtained from Eqs.(4.14a) - (4.14d) by substitutions \(-x_- \rightarrow x_- e^{-i\pi} \). The possibility of unique recovery of the values of \( \Psi_\kappa(x) \) (and hence the values of the field \( \phi_M(x) \)) in the full MS using its values only in \( R \)-wedge and the requirement of positivity of the energy is an illustration of the content of the Reeh-Schlieder theorem (see e.g. Refs. [41,42]).

The splitting (4.13) corresponds to the four families of orbits of the two-dimensional Lorentz group (compare to §6 of Chapter V in Ref. [43]). As it was already mentioned the functions Eqs.(4.14a) - (4.14d) have branch points at the light cone which corresponds to the four degenerate orbits \( x_\pm = 0 \), \( \text{sngt} = \pm 1 \). For example if \( t \rightarrow z > 0 \) then using the expansion of Macdonald function \( K_{\nu}(\zeta) \) for \( \zeta \rightarrow 0 \) [31] we obtain

\[
\Psi_\kappa^{(R)} = \frac{1}{2 \sqrt{2} \pi} \exp \left( \frac{\pi \kappa}{2} / 2 \right) \left\{ \Gamma(i\kappa) \left( \frac{mx_+}{2} \right)^{-i\kappa} + \Gamma(-i\kappa) \left( -\frac{mx_-}{2} \right)^{i\kappa} + \ldots \right\}.
\]

(4.15a)

The light cone asymptotic behaviour of the other functions Eqs.(4.14b) - (4.14d) may be derived from Eq.(4.15a) by the described above procedure of analytical extension. For example,

\[
\Psi_\kappa^{(F)} = \frac{1}{2 \sqrt{2} \pi} \exp \left( \frac{\pi \kappa}{2} / 2 \right) \left\{ \Gamma(i\kappa) \left( \frac{mx_+}{2} \right)^{-i\kappa} + \exp(-\pi \kappa \Gamma(-i\kappa) \left( \frac{mx_-}{2} \right)^{i\kappa} + \ldots \right\}.
\]

(4.15b)
After substituting these expressions in Eq. (4.13) we find the light-cone behaviour of the boost mode which reads
\[ \Psi_\kappa(x) = \frac{1}{2^{3/2} \pi} e^{\pi \kappa/2} \left\{ \Gamma \left( i \kappa \right) \left( \frac{m x^+ - i 0}{2} \right)^{-i \kappa} + \Gamma \left( -i \kappa \right) \left( -\frac{m x^- + i 0}{2} \right)^{i \kappa} \right\}. \] (4.16)

The distributions \( (\zeta \pm i 0)^\lambda = \zeta^\lambda \theta(\zeta) + e^{\pm i \lambda \pi} (\zeta^-)^\lambda \) in Eq. (4.16) were defined and studied in Ref. [44]. It is clear from Eq. (4.16) that in spite of the presence of \( \theta \)-functions in Eq. (4.13) the modes \( \Psi_\kappa(x) \) obey the KFG equation without sources. In the vertex of the light cone \( t = z = 0 \) (which is the fixed point for the Lorentz group) from Eq. (4.7) we get
\[ \Psi_\kappa(0,0) = \frac{1}{\sqrt{2}} \Gamma(\kappa). \] (4.17)

The same result may be derived either from Eqs. (4.14a) - (4.14d) by taking into account that
\[ \frac{i}{2} H_\kappa^{(1)}(0) = -\frac{i}{2} H_\kappa^{(2)}(0) = \frac{1}{\pi} K_\kappa(0) = \delta(\kappa), \] (4.18)
or from Eq. (4.16). This result means that all modes \( \Psi_\kappa(x) \) except for the singular zero mode vanish at the vertex of the light cone.

The expression for the annihilation operator \( b_\kappa \) in terms of field operator on an arbitrary Cauchy surface (compare Eqs. (2.7), (3.11)) read
\[ b_\kappa = (\Psi_\kappa, \phi_M)_\mathcal{M} = i \int_{-\infty}^{\infty} \Psi^*_\kappa(t,z) \frac{\partial}{\partial t} \phi_M(t,z) \, dz, \quad t \neq 0. \] (4.19)

For the surface \( t = 0 \) we have
\[ b_\kappa = \frac{i}{\pi \sqrt{2}} \left( e^{\pi \kappa/2} \int_{-\infty}^{0} F_R(z, \kappa) \, dz + e^{-\pi \kappa/2} \int_{0}^{\infty} F_L(z, \kappa) \, dz \right), \quad F_{R,L} = K_{\kappa \mp 1}(\pm m z) \left( \frac{\partial \phi_M}{\partial t} \mp \frac{\partial \phi_M}{\partial z} \right)_{t=0} \pm \right)^{\mp i \kappa} \left( \frac{\partial \phi_M}{\partial z} \right)_{t=0} \pm m \left( K_{\kappa \mp 1}(\pm m z) - \frac{1}{2} \Gamma(1 \mp i \kappa) \left( \pm \frac{m z}{2} \right)^{i \kappa - 1} \right) \phi_M(0,z), \] (4.20)

with upper (lower) signs corresponding to the indices \( R \) (L) consequently. (The derivation of this very important formula is given in Appendix A).

Note that calculation of the Whightman function \( \Delta^{(+)}(x,m) \) by use of Eqs. (4.6), (4.11) of course leads to the result (2.13). It gives the independent proof that the plain waves quantization is unitary equivalent to the boost modes one.

V. THE UNRUH CONSTRUCTION

In this Section we will consider the quantum field theory aspect of the Unruh problem. The study of this problem was inspired by Fulling who suggested [17] in 1973 a valid scheme for quantization of a massive scalar field in RS (see Sec. III). Fulling treated RS as a part of MS and hence considered Fulling-Rindler vacuum as a state of quantum field in MS. Therefore he tried to express the annihilation and creation operators of Fulling-Rindler particles (3.11) in terms of plain-wave operators (2.7) and argued that the Minkowski vacuum state could be interpreted as a many-particle Fulling-Rindler state. In virtue of the boundary conditions which the field \( \phi_R \) must obey in RS and which we have considered in Sec. III Fulling procedure in MS is physically meaningless. But even if one disregards the existence of boundary conditions the quantization scheme suggested by Fulling is incorrect for MS since it implied the assumption that the field modes which he used for quantization were equal to zero outside \( R \)-wedge of MS. In other words only the first term from Eq. (4.13) for boost modes was involved in the procedure of quantization. But due to the presence
of $\theta$-functions this term obeys not the KFG equation for the free field in MS but the equation with sources of infinite power localized on the light cone.

To avoid this difficulty Unruh with the help of a rather elegant trick made an attempt to construct a new scheme of quantization which should be valid in MS and in some sense repeat the Fulling scheme in $R$-wedge. The central point of Unruh suggestion was to use such superpositions $R_\mu, L_\mu$ of boost modes $\Psi_\kappa$ with positive and $\Psi^*_\kappa$ with negative frequencies that they vanish either in the left or right wedges of MS and coincide with Fulling modes respectively in the right or left wedges. We will present the explicit form of the Unruh modes and discuss the Unruh quantization scheme in Sec.V A. Then in the Sec.V B we will discuss the so-called Unruh effect.

A. The Unruh quantization

The Unruh modes can be expressed in terms of boost modes as follows

$$R_\mu(x) = \frac{1}{\sqrt{2} \sinh \pi \mu} \left\{ e^{\pi \mu/2} \Psi_\mu(x) - e^{-\pi \mu/2} \Psi^*_\mu(x) \right\}, \quad L_\mu(x) = \frac{1}{\sqrt{2} \sinh \pi \mu} \left\{ e^{\pi \mu/2} \Psi^*_\mu(x) - e^{-\pi \mu/2} \Psi_\mu(x) \right\},$$

(5.1)

where $\mu > 0$. These functions obey the normalization conditions

$$(R_\mu, R_{\mu'})_M = -(L_\mu, L_{\mu'})_M = \delta(\mu - \mu'), \quad (R_\mu, R'_{\mu'})_M = (L_\mu, L'_{\mu'})_M = (R_\mu, L'_{\mu'})_M = 0.$$  

(5.2)

With the help of Eqs.(4.14a) - (4.14d) one can easily check that for $x$ belonging to the right wedge $R$ the ”left” Unruh modes $L_\mu(x)$ vanish while the ”right” modes $R_\mu(x)$ coincide with Fulling modes $\Phi_\mu(\xi)$, Eq.(3.7). For the light cone behaviour of Unruh modes in the right wedge $R$ we have

$$R_\mu(x) = \Phi_\mu(\xi) \sim \frac{\sqrt{\sinh \pi \mu}}{2\pi} \left\{ \Gamma(i\mu) \left( \frac{m x^+}{2} \right)^{-i\mu} + \Gamma(-i\mu) \left( -\frac{m x^-}{2} \right)^{i\mu} \right\}, \quad x \in R, \quad x_+ \to 0.$$  

(5.3)

In the future wedge $F$ they can be represented as

$$R_\mu(x) = -\frac{i}{2\sqrt{\sinh \pi \mu}} \left( \frac{x_+}{x_-} \right)^{-i\mu/2} J_{-i\mu} \left( m \sqrt{x_+ x_-} \right) \sim \frac{\sqrt{\sinh \pi \mu}}{2\pi} \Gamma(i\mu) \left( \frac{m x^+}{2} \right)^{-i\mu}, \quad x \in F, \quad x_+ \to 0,$$  

(5.4a)

$$L_\mu(x) = \frac{i}{2\sqrt{\sinh \pi \mu}} \left( \frac{x_+}{x_-} \right)^{-i\mu/2} J_{i\mu} \left( m \sqrt{x_+ x_-} \right) \sim -\frac{\sqrt{\sinh \pi \mu}}{2\pi} \Gamma(-i\mu) \left( -\frac{m x^-}{2} \right)^{i\mu}, \quad x \in F, \quad x_- \to 0.$$  

(5.4b)

Note that in spite of Unruh statement made in Ref. [4] functions (5.1) cannot be obtained by analytical extension of Fulling modes (3.7) through the future and past horizons into the $F$- and $P$-wedges. As a matter of fact the Unruh modes are combinations of functions defined on different sheets of Riemannian surface, see the rules for enclosing of brunch points of the boost modes in Sec.V.

By inverting the relations Eq.(5.1) and substituting the result in Eq.(5.1) one obtains

$$\phi_M(x) = \int_0^\infty d\mu \left\{ r_\mu R_\mu(x) + r^*_\mu R^*_\mu(x) + l_\mu L^*_\mu(x) + l^*_\mu L_\mu(x) \right\}.$$  

(5.5)

The Eq.(5.5) holds everywhere in MS except the origin of Minkowski coordinate frame because the boost mode $\Psi_\kappa(x)$ is singular at the point $x = (0,0)$ (see Eq.(4.17)) and hence it is impossible to perform integration in Eq.(5.5) at this point over positive and negative values of $\kappa$ independently. This means that expansion (5.5) is valid for the field in MS with cut out point $x = (0,0)$.

\footnote{This fact was earlier proposed as the reason for "thermal properties" of "Minkowski vacuum" in Refs. [44, 45].}
Then considering for the sake of simplicity the case of equal times $t = t' = 0$ we have

$$
0_M|\tilde{\phi}_M(0, z)\tilde{\phi}_M(0, z')|0_M\rangle = \frac{1}{\pi^2} \lim_{\epsilon \to 0} \int \pi \mu K_{i\mu}(mz)K_{i\mu}(mz'), \quad z, z' > 0.
$$

Since $(1/2\pi)K_{i\mu}(0) = 0$ at $\mu > 0$ (see Eq. (4.13)) we obtain for $z' = 0$

$$
\langle 0_M|\tilde{\phi}_M(0, z)\tilde{\phi}_M(0, 0)|0_M\rangle = 0.
$$

At the same time for the Whightman function in MS we have

$$
\langle 0_M|\phi_M(0, z)|0_M\rangle = \frac{1}{2\pi^2} \int dk \cosh \pi \kappa K_{i\kappa}(mz)K_{i\kappa}(0) = \frac{1}{2\pi} K_0(mz), \quad z > 0,
$$
in full agreement with Eq. $(2.13)$. It is clear after comparison of Eqs. $(5.12), (5.13)$ that $\Delta^+(x, m)$ is not equal to zero in the $R$-wedge only due to existence of singular zero boost mode $(4.17)$. Since this mode is absent in the Unruh set $(2.11)$ the latter is not a complete set of solutions for the KFG equation. Therefore from this moment we will denote the l.h.s. of Eq. $(3.4)$ as $\phi_{DW}(x)$ instead of $\phi_M(x)$.

Consider quantization of the field $\phi_{DW}(x)$ in the double Rindler wedge now. Since there exists a timelike variable, namely Rindler time, with respect to which the Unruh modes $(5.1)$ are positive frequency solutions of KFG equation we can attach to the Unruh operators the meaning of annihilation, creation operators of particles living in the double Rindler wedge. But the fields in $R$- and $L$-wedges are absolutely independent of each other since any two points belonging to different wedges are separated by spacelike interval. Therefore double Rindler wedge is a disjoint union of $R$- and $L$-wedges and quantization in these wedges should be carried out separately. We have discussed already quantization procedure in the $R$-wedge and found that it implies existence of boundary condition ensuring finiteness of the field energy. It is clear that the field in double Rindler wedge $\phi_{DW}(x)$ should satisfy the same boundary condition. Taking into account these considerations we should rewrite Eq. $(5.5)$ in the form

$$\phi_{DW}(x) = \int_0^\infty d\mu \{r_\mu R_\mu(x) + r_\mu^\dagger R_\mu^\dagger(x)\} + \int_0^\infty d\mu \{l_\mu L_\mu^\dagger(x) + l_\mu^\dagger L_\mu(x)\}, \quad x \in R \cup L,$$

(5.14)

where Unruh operators should coincide with the corresponding Fulling operators $c_\mu, c_\mu^\dagger$ and $c'_\mu, c''_\mu$ for particles living in $R$- and $L$-wedges respectively if the field $\phi_{DW}(x)$ satisfies the boundary condition

$$\phi_{DW}(0, 0) = 0.$$

(5.15)

We will prove the latter statement for the operator $r_\mu$ as an example.

Substituting the expression $(4.19)$ for operators $b_\kappa, b'_\kappa$ into the first formula in Eq. $(5.6)$ we obtain after transition to Rindler coordinates $(3.14)$

$$r_\mu = \lim_{\rho \to 0} (\hat{c}_\mu(\rho) - c_\mu(\rho)), \quad \hat{c}_\mu(\rho) = \frac{i}{\sqrt{2\mu}} \int d\rho \psi_\mu(\rho) \left( \frac{\partial \phi_M}{\partial \eta} - i\mu \phi_M \right)_{\eta=0},$$

(5.16a)

$$c_\mu(\rho) = \frac{i \sqrt{\sinh \pi \mu}}{2\pi} \phi_M(0, \rho) \left\{ \Gamma(-i\mu) \left( \frac{m\rho}{2} \right)^{i\mu} - \Gamma(i\mu) \left( \frac{m\rho}{2} \right)^{-i\mu} \right\},$$

(5.16b)

(5.16c)

Note that both $\hat{c}_\mu(\rho)$ and $c_\mu(\rho)$ become singular for $\rho \to 0$ if $\phi_M(0, 0) \neq 0$ but the singularities cancel in their difference in Eq. $(5.16a)$. It is clear from Eqs. $(5.16a), (5.16c)$ that although Eqs. $(5.16b), (3.11)$ look similar, one can not identify the right Unruh operator $r_\mu$ and the Fulling annihilation operator $c_\mu$ unless

$$\phi_M(0, 0) = 0.$$

(5.17)

This condition should be understood in weak sense of course.

From Eqs. $(4.6), (4.17)$ we however formally have

$$\phi_M(0, 0) = \frac{b_0 + b_0^\dagger}{\sqrt{2}} = \int_{-\infty}^{\infty} \frac{dp}{4\pi\epsilon_p} (a_p + a_p^\dagger),$$

(5.18)

and therefore for the value of one-particle amplitude $(2.8)$ at the vertex of the light cone we obtain

$$\phi_f(0, 0) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{4\pi\epsilon_p}} f(p).$$

(5.19)
Of course there are no any physical reasons to require vanishing of this quantity in MS. Thus if one understands Eq. (5.17) as a condition for the field in MS it means nothing but cutting out the point $t = z = 0$. Cutting out even a single point is not however a "painless operation" for MS because it dramatically changes its properties. In particular MS loses its property to be a globally hyperbolic spacetime. As a consequence the Cauchy data for the two-point commutator corresponds now to zero solution of KFG equation unlike what we have in real MS. Since in four-dimensional MS the Pauli-Jordan function has Cauchy data similar to (2.16) this result is not a specific property of two dimensional case.

This inconsistency disappears if we apply the Unruh construction to the double Rindler wedge rather than to MS. That means that we should use the expansion instead of $\phi_M$ in Eqs. (5.16b), (5.16c) and read Eq. (5.17) as Eq. (5.15). As a result we conclude that the Unruh construction is a valid quantization scheme only in the double Rindler wedge.

### B. The Unruh "effect"

There are several ways to "give proof" of existence of the Unruh effect in the frames of conventional quantum field theory [1,5,6,8,10–13].

One of them is based on the equation

$$\langle 0_M | r_\mu^\dagger r_{\mu'} | 0_M \rangle = (e^{2\pi\mu} - 1)^{-1} \delta(\mu - \mu'),$$

which can be easily obtained using Eqs. (5.6), (4.11). The l.h.s. of the Eq. (5.20) at $\mu = \mu'$ is interpreted after integration over $\mu$ as Minkowski vacuum expectation value of the operator of Fulling particles, while the r.h.s. of this equation after the standard trick is written as

$$\delta(\mu - \mu')|_{\mu = \mu'} = \int_{-\infty}^{\infty} e^{i(\mu - \mu')\eta}|_{\mu = \mu'} \frac{d\eta}{2\pi} = \frac{g\Delta \tau}{2\pi},$$

where $\tau = \eta/g$ is proper time and $g$ is proper acceleration of Rindler observer. Then Eq. (5.20) is transformed to the form

$$\frac{\Delta \tilde{N}}{\Delta \tau} = \int_{0}^{\infty} \frac{d\omega}{2\pi} (e^{2\pi\omega/g} - 1)^{-1},$$

where $\omega = g\mu$ is commonly understood as the energy of Fulling quanta. Finally the integrand in Eq. (5.21) is identified with the thermal spectrum corresponding to Davies-Unruh temperature [1,1].

However, as it was shown in the previous section, one can not identify Unruh operators $r_\mu$, $r_\mu^\dagger$ with Fulling annihilation and creation operators $c_\mu$, $c_\mu^\dagger$ in MS. Moreover operators $r_\mu$, $r_\mu^\dagger$ can not serve as annihilation and creation operators of any particles in MS. Therefore the l.h.s. of Eq. (5.20) can not be interpreted as Minkowski vacuum expectation value of number of particles.

The Unruh operators coincide with the corresponding Fulling operators for the field obeying the boundary condition (5.15) in the double Rindler wedge. But an observer living in RS can not define Minkowski vacuum state. To conclude that the state of the field is Minkowski vacuum one must have possibilities to perform measurements in every point of Cauchy surface in the whole MS. This is impossible for an observer living in $R$- (or $L$-) wedge because he can not perform measurements at the points belonging to $L$- (or $R$-) wedge. From mathematical point of view this statement is a direct consequence of Reeh-Schlieder theorem, see Refs. [41,42]. On the other hand observers living in the double Rindler wedge are not able to perform such measurements due to existence of the boundary condition (5.15).

Let us also note that the Bose factor in the r.h.s. of Eq. (5.20) should not be necessarily interpreted as thermal distribution. This factor appears entirely due to specific properties of Bogolubov transformation (5.6) and is encountered in many physical problems where in no way does the notion of temperature arise. Two-mode squeezed photon states in quantum optics [49,50] is a well known example of such situation, see also Ref. [71]. Two-dimensional harmonic oscillator can serve as another example, see Appendix B.
Another "derivation" of the Unruh effect is based on the relation \[ |0_M\rangle = Z^{-\frac{1}{2}} \sum_{n=0}^{\infty} \int_0^{\infty} d\mu_1 \cdots \int_0^{\infty} d\mu_n \ e^{-\pi \sum_{i=1}^{n} \mu_i} |1_{\mu_1}, \ldots, 1_{\mu_n}\rangle_L \otimes |1_{\mu_1}, \ldots, 1_{\mu_n}\rangle_R. \quad (5.22) \]

This formula "determines r- and l-particle content of Minkowski vacuum" and allows one to introduce the density matrix describing states of the field in R-wedge, see e.g. [15]. The latter is achieved by taking the tensor product of the r.h.s. of Eq. (5.22) with its dual and then taking trace over the states of the field in the L-wedge. So for an arbitrary observable \( \omega \) depending on the "values of the field" \( \phi_M(x) \) for \( x \) belonging only to the right Rindler wedge \( R \) we have

\[ \langle 0_M | \mathcal{R} | 0_M \rangle = \text{Sp}(\mathcal{R} \rho_R), \quad \rho_R = Z^{-1} \exp(-H_R/T_{DU}). \quad (5.23) \]

In this equation \( \rho_R \) is the density matrix and \( H_R = gK = g \int_0^\infty \mu \frac{d\mu}{e^{\mu} - 1} \) is a secondly quantized Hamiltonian with respect to proper time \( \tau \) of the accelerating observer.

However the l.h.s. of Eq. (5.22) may not be considered as Minkowski vacuum state because as we have shown in the previous section the notion of Rindler particles which is essentially used in derivation of Eq. (5.22) makes sense only in double Rindler wedge rather than in global MS. Therefore Eq. (5.22) could describe vacuum state only in double Rindler wedge. But it loses any physical meaning if one takes into account the existence of boundary condition (5.15). Indeed the derivation of Eq. (5.22) assumes (see e.g. [13]) that one-particle Hilbert space in the double Rindler wedge is the direct sum of one-particle Hilbert spaces in \( R \) and \( L \)-wedges \( \mathcal{H}_{DW} = \mathcal{H}_R \oplus \mathcal{H}_L \) and Fock space of states of the field in double Rindler wedge \( \mathcal{F} \mathcal{H}_{DW} \) is a tensor product of Fock spaces on \( \mathcal{H}_R \) and \( \mathcal{H}_L \), \( \mathcal{F} \mathcal{H}_{DW} \cong \mathcal{F} \mathcal{H}_R \otimes \mathcal{F} \mathcal{H}_L \). But in virtue of boundary condition, which is equivalent to cutting out the vertex of light cone, \( R \)- and \( L \)-wedges have no common points and therefore never interact. Therefore only such superpositions of state vectors from \( \mathcal{F} \mathcal{H}_{DW} \) can have physical sense which do not contain correlations between r- and l-particles. This "superselection rule" prohibits states of the type Eq. (5.22).

Besides the normalization constant \( Z \) in Eq. (5.22) which also has the meaning of partition function in Eq. (5.23) is infinite, namely

\[ Z = \exp(\delta(0) \pi^2/12), \quad (5.24) \]

see Appendix B. The divergence of constant \( Z \) means that representations of the canonical commutation relations in terms of Unruh and boost modes operators are unitary inequivalent.

VI. ALGEBRAIC APPROACH

Algebraic approach to quantum theory [18, 20, 21] allows one to compare states which cannot be represented by vectors or density matrices in the same Hilbert space representation of algebra of observables of the system. It is because the states in this approach are primarily considered as positive normalized linear functionals over the algebra of observables rather than vectors in Hilbert space. The physical meaning of the state \( \omega \) in algebraic approach is that the value \( \omega(A) \) is the expectation value of the observable \( A \) in the state \( \omega \). The algebraic counterpart of usual thermal equilibrium state is called the KMS state [19, 22]. Unlike usual thermal equilibrium state the KMS state exists even if partition function of the system diverges. On the language of algebraic approach the Unruh effect means that the algebraic state corresponding to Minkowski vacuum state coincides with the KMS state for double Fulling quantization. In this section we will show that such conclusion implies existence of boundary condition at the origin of Minkowski reference frame. Our consideration will make clear that in algebraic derivation of the Unruh effect the same inconsistencies are present as in traditional approach.
A. One mode model

In order to use simple and suitable notation, let us first present the construction of KMS state over the one mode quantum system (i.e. one-dimensional harmonic oscillator). We will see that the case of free Bose field in $D = 1 + 1$ MS requires just a trivial generalization.

Let $R$ be one-mode quantum system. Its algebra of observables (which we denote by $U_R$) may be characterized either in terms of unbounded generators, annihilation and creation operators $r, r^\dagger$ satisfying commutation relation
\begin{equation}
[r, r^\dagger] = 1,
\end{equation}
or in terms of Weyl generators
\begin{equation}
W(f) = \exp(f r - f^* r^\dagger),
\end{equation}
labeled by complex number $f$ and which satisfy the following requirements
\begin{equation}
W(f_1)W(f_2) = \exp(\frac{i}{2}(f_1^* f_2 - f_2^* f_1))W(f_1 + f_2), \quad W(f)^* = W(-f).
\end{equation}
An arbitrary observable $R$ may be written in the form $R = R(r, r^\dagger)$. Time evolution of observables is determined by the equation
\begin{equation}
R(t) = U(t) R U(t), \quad U(t) = \exp(-iH_Rt),
\end{equation}
with $H_R = e r^\dagger r$ being the one-mode Hamiltonian. In particular,
\begin{equation}
R(t) = R(re^{-i\epsilon t}, r^\dagger e^{i\epsilon t}), \quad W(f, t) = W(f(t)), \quad f(t) = fe^{-i\epsilon t}.
\end{equation}

Vacuum state $\omega_R$ is defined by the relation $r|0_R\rangle = 0$ and the Hilbert space $H_R$ where the considered operators act is generated by the basis $|n_R\rangle = (r^\dagger)^n/\sqrt{n!}|0_R\rangle$. The algebraic vacuum state $\omega_R$ is a prescription for calculating expectation values in the vacuum state
\begin{equation}
\omega_R(R) = \langle 0_R| R(r, r^\dagger) |0_R\rangle.
\end{equation}
The vacuum expectation value of Weyl generator $W(f)$ may be easily shown to be equal to
\begin{equation}
\omega_R(W(f)) = \exp(-\frac{1}{2}|f|^2).
\end{equation}

Thermal equilibrium state with inverse temperature $\beta$ is defined as follows
\begin{equation}
\omega^{(\beta)}_R(R) = \text{Sp}(\rho_\beta), \quad \rho_\beta = Z^{-1}_\beta \exp(-\beta H_R) = Z^{-1}_\beta \sum_n \exp(-\beta \epsilon n)|n_R\rangle\langle n_R|,
\end{equation}
where $Z_\beta = \sum_n \exp(-\beta \epsilon n)$. Of course $Z_\beta$ is finite for this simple one-mode model, $Z_\beta = (1 - e^{-\beta \epsilon})^{-1}$. But since this may be not the case for quantum systems with infinite number of degrees of freedom it is important to reformulate Eq.\((6.8)\) in the form not containing the value of $Z_\beta$ explicitly.

In order to do it one should introduce another copy of system $R$, say $L$ which does not interact with $R$, and consider the combined quantum system $R \otimes L^*$, the "double system".\(^4\) The asterisk here indicates that we choose Hamiltonian of the combined system to be

\[^4\]The unbounded operators $r, r^\dagger$ may be expressed in terms of Weyl generators, for example $r = \frac{1}{i}(W'(f) - iW'(if))|f=0$ where derivatives are taken with respect to $f$.

\[^5\]Normally in applications of KMS states to statistical mechanics this additional copy of initial system is considered as a mathematical trick used with the purpose to describe thermal state by a vector in Hilbert space and there are no attempts to interpret it as a really existent.
The property served as a reason for the choice of Hamiltonian \( H \) rather than \( H \otimes 1 + 1 \otimes H_L \) (compare to Eq. (B1)). One can interpret it saying that time direction at \( L \) is inverted. The vacuum state of the system \( R \otimes L^* \) is defined by the relations
\[
 r|0_{R\otimes L^*}\rangle = 0, \quad l|0_{R\otimes L^*}\rangle = 0,
\]
where \( l \) is annihilation operator for \( L \) and the vectors
\[
 |n_R\rangle \otimes |m_L\rangle = \frac{(r\dagger)^n (l\dagger)^m}{\sqrt{n!} \sqrt{m!}} |0_{R\otimes L^*}\rangle,
\]
constitute the basis of the Hilbert space \( \mathcal{H}_{R\otimes L^*} \). Let us introduce the state
\[
|\Omega_\beta\rangle = Z_\beta^{-1/2} \sum_n \exp(-\beta cn/2) |n_R\rangle \otimes |n_L\rangle. \tag{6.10}
\]
It can be immediately verified that the thermal expectation value (6.8) may be rewritten in the form
\[
\omega^{(\beta)}_R(\mathcal{R}) = \langle \Omega_\beta| \mathcal{R}(r \otimes 1, r\dagger \otimes 1)|\Omega_\beta\rangle, \tag{6.11}
\]
where calculation is performed in the space \( \mathcal{H}_{R\otimes L^*} \). Note that the expectation value (6.11) does not depend on time. This property served as a reason for the choice of Hamiltonian \( H_{R\otimes L^*} \) in the form (6.9).

Now let us consider operators
\[
b_+ = \cosh \theta r \otimes 1 - \sinh \theta 1 \otimes l\dagger, \quad b_- = - \sinh \theta r\dagger \otimes 1 + \cosh \theta 1 \otimes l,
\]
where \( \theta \) is defined by the equation \( \tanh \theta = e^{-\beta t/2} \). Note that these operators depend on time as \( b_\pm(t) \propto e^{\mp i\theta t} \).

The key observation is that these operators annihilate the state \( |\Omega_\beta\rangle \) and together with \( b_+^\dagger, b_-^\dagger \) satisfy the usual commutation relations
\[
[b_\pm, |\Omega_\beta\rangle] = 0, \quad [b_+, b_-] = 0, \quad [b_+, b_-^\dagger] = 0, \quad [b_\pm, b_-^\dagger] = 1. \tag{6.13}
\]

The span of the vectors of the form
\[
\frac{(b_+^\dagger)^n (b_-^\dagger)^m}{\sqrt{n!} \sqrt{m!}} |\Omega_\beta\rangle
\]
constitutes the Hilbert space \( \mathcal{H} \) which in our one-mode case coincides with the space \( \mathcal{H}_{R\otimes L^*} \). Expressing operators \( r, r\dagger \) in terms of operators \( b_\pm \) we can rewrite Eq. (6.11) in the form
\[
\omega^{(\beta)}_R(\mathcal{R}) = \langle \Omega_\beta| \mathcal{R}(b_+ \cosh \theta + b_- \sinh \theta, b_- \cosh \theta + b_+ \sinh \theta)|\Omega_\beta\rangle. \tag{6.14}
\]
The r.h.s. of this equation does not contain \( Z_\beta \) and in virtue of Eqs. (5.13) is just the vacuum expectation value of the observable \( \mathcal{R} \) but calculated in the space \( \mathcal{H} \) and with respect to vacuum \( |\Omega_\beta\rangle \). In general the algebraic state defined by Eq. (6.14) is called "the KMS state."\footnote{Another equivalent definition of KMS state is given by the requirement \( \omega^{(\beta)}_R(\mathcal{R}_1(t)\mathcal{R}_2(t')) = \omega^{(\beta)}_R(\mathcal{R}_2(t')\mathcal{R}_1(t + i\beta)) \) where \( \mathcal{R}_1, \mathcal{R}_2 \) are arbitrary observables of system \( R \), (5.16).}

In our case the KMS state (6.14) is just the usual thermal equilibrium state. One can generalize the definition of KMS state to the observables of the double system \( R \otimes L^* \) which have the form \( A(r, r\dagger, l, l\dagger) \) by setting
\[
\omega^{(\beta)}(A) = \langle \Omega_\beta| A(b_+ \cosh \theta + b_-^\dagger \sinh \theta, b_- \sinh \theta + b_+^\dagger \cosh \theta, b_- \cosh \theta + b_-^\dagger \sinh \theta, b_-^\dagger \cosh \theta + b_+ \sinh \theta)|\Omega_\beta\rangle. \tag{6.15}
\]
The state defined by Eq. (6.15) is called the double KMS state \[21\]. Given definitions of KMS and double KMS states in the evident way may be generalized to the case of any finite or infinite number of degrees of freedom. For the latter case the usual definition of thermal equilibrium state is in general not valid.

Let us give the formulas for expectation values of Weyl generators in KMS and double KMS states. By simple computation one gets

\[
\omega^{(\beta)}_R(W(f)) = \exp \left( -\frac{1}{2} \coth \left( \frac{\beta \varepsilon}{2} \right) |f|^2 \right). 
\]

We define the Weyl generator \( W(f_R, f_L) \) for the double system by the equation

\[
W(f_R, f_L) = \exp(f_{RR} - f_R^* f_L^1). 
\]

The advantage of this definition is that time dependence of Weyl generator takes the form (6.7):

\[
\omega^{(\beta)}(W(f_R, f_L)) = \exp \left\{ -\frac{1}{2} \coth \left( \frac{\beta \varepsilon}{2} \right) (|f_R|^2 + |f_L|^2) - \frac{1}{\sinh(\beta \varepsilon/2)} \Re f^*_R f_L \right\}. 
\]

B. The Unruh problem in algebraic approach

Now let us turn back to the Unruh problem. At first sight Eqs. (6.12) and the definition of the state \( |\Omega_A\rangle \) (6.13) look very similar to inverted Eqs. (5.9) expressing boost operators \( b_\kappa \) in terms of Unruh operators \( r_\mu, l_\mu \) and the definition of the state \( |\Omega_M\rangle \) in Eq. (4.11). But we will show that it is not correct to apply the notion of double KMS state to the Unruh problem. The physical reason is that free field in Minkowski spacetime cannot be decomposed into two non-interacting fields living in the interior of right and left Rindler wedges.

To reformulate the Eq. (5.23) in terms of algebraic approach let us introduce the required definitions. The algebra \( \mathcal{U} \) of observables of the free field in MS is a \( C^* \) algebra with Weyl generators \( W(\Phi) = \exp\left\{ -\frac{1}{2} \coth \left( \frac{\beta \varepsilon}{2} \right) (|f_R|^2 + |f_L|^2) - \frac{1}{\sinh(\beta \varepsilon/2)} \Re f^*_R f_L \right\} \) (compare to Eqs. (6.2), (6.3)). Note that solutions \( \Phi(x) \) are required to decrease sufficiently fast at spatial infinity (say, have compact support on any Cauchy surface).

The expectation value of Weyl generator \( W(\Phi) \) in Minkowski vacuum state \( \omega_M \) may be obtained by generalization of the formula (5.7):

\[
\omega_M(W(\Phi)) = \exp \left( -\frac{1}{2} \int_{-\infty}^{\infty} dx |f_\kappa|^2 \right), 
\]

where coefficients \( f_\kappa = (\Psi_\kappa, \Phi)_M \) are defined with respect to a complete set of boost modes \( \Psi_\kappa \) (4.7). By inverting relations (5.1) one can rewrite eq. (6.21) in terms of Unruh modes. The result is
\[ \omega_M(W(\Phi)) = \exp \left\{ -\frac{1}{2} \int_0^{\infty} d\mu \left( \coth \pi \mu \left[ |f_{\mu}^{(L)}|^2 + |f_{\mu}^{(R)}|^2 \right] + \frac{2}{\sinh \pi \mu} \Re (f_{\mu}^{(L)\ast} f_{\mu}^{(R)}) \right) \right\}, \quad (6.22) \]

where \( f_{\mu}^{(R)} = (R_{\mu}, \Phi)_M, f_{\mu}^{(L)} = (L_{\mu}, \Phi)_M \).

Finite linear combinations of elements from \( \mathcal{U} \) of the form \( W(\Phi) \) with \( \Phi \) vanishing in the closed wedge \( \tilde{L} \) and the limits of sequences of such linear combinations in uniform sense (i.e. limits in the sense of convergence in \( C^* \) norm) constitute a \( C^* \) subalgebra \( \mathcal{U}_R \) of \( \mathcal{U} \) which is called the right wedge algebra. The left wedge algebra \( \mathcal{U}_L \) and the double wedge algebra \( \mathcal{U} \) are defined similarly by restricting to solutions which vanish in closed wedge \( \mathcal{R} \) and in a neighborhood of \( h_0 \) respectively, see Fig.1.

Now let us evaluate expectation value of Weyl generator in a double KMS state with temperature \( \beta^{-1} \) with respect to Fulling quantization prescription. By generalizing Eq.(6.18) one obtains \( \zeta_{\mu}^{(R)}(W(\Phi)) = \exp \left\{ -\frac{1}{2} \int_0^{\infty} d\mu \left( \coth \left( \frac{\beta \mu}{2} \right) \left[ |\zeta_{\mu}^{(R)}|^2 + |\zeta_{\mu}^{(L)}|^2 \right] + \frac{2}{\sinh (\beta \mu/2)} \Re (\zeta_{\mu}^{(L)\ast} \zeta_{\mu}^{(R)}) \right) \right\}, \quad (6.23) \]

where \( \Phi = \Phi_R \oplus \Phi_L, \zeta_{\mu}^{(R)} = (\Phi_{\mu}^{(R)}, \Phi)_R, \zeta_{\mu}^{(L)} = (\Phi_{\mu}^{(L)}, \Phi)_L, \Phi_{\mu}^{(R)} \) is complete set of Fulling modes (3.7) and \( \Phi_{\mu}^{(L)} \) is their analog in the wedge \( L \). Note that the expression in the r.h.s. of Eq.(6.23) is well-defined only if test functions \( \Phi \) in double RS obey the requirement

\[ \int_0^{\infty} d\mu \left| \zeta_{\mu}^{(R,L)} \right|^2 < \infty, \]

which is referred as regularity condition in Ref. [21] (compare to Eq.(3.37)).

Let us obtain the relation between the coefficients \( f_{\mu}^{(R)} \) and \( \zeta_{\mu}^{(R)} \) in Eqs.(6.22), (6.23). For this purpose we first evaluate \( f_{\mu}^{(R)} = (R_{\mu}, \Phi)_M \) supposing that the surface of integration is a surface of constant positive small \( t \) and then take limit \( t \to 0 \). For Unruh mode we use the expression

\[ R_{\mu}(x) = R_{\mu}^{(R)}(x) \theta(x_+) \theta(-x_-) + R_{\mu}^{(P)}(x) \theta(x_+) \theta(x_-) + R_{\mu}^{(F)}(x) \theta(-x_+) \theta(-x_-), \quad (6.24) \]

which can be easily obtained from Eqs.(4.13, 5.1). To calculate the inner product

\[ f_{\mu}^{(R)} = i \int_{-\infty}^{\infty} dx \ R_{\mu}^{*}(x) \frac{\partial}{\partial t} \Phi(x), \quad (6.25) \]

we need a time derivative of (6.24). Taking into account that \( t > 0 \) we write it in the following way:

\[ \frac{\partial}{\partial t} R_{\mu}(x) = \left( \frac{\partial}{\partial t} R_{\mu}^{(R)}(x) \right) \theta(x_+) \theta(-x_-) + \left( \frac{\partial}{\partial t} R_{\mu}^{(F)}(x) \right) \theta(x_+) \theta(x_-) + \left( \frac{\partial}{\partial t} R_{\mu}^{(P)}(x) \right) \delta(x_+) + \left( R_{\mu}^{(F)}(x) - R_{\mu}^{(R)}(x) \right) \delta(-x_-). \quad (6.26) \]

It is not very hard to verify using Eqs.(3.1), (3.7), (4.14a) and (3.14) that

\[ \lim_{t \to 0} \int_{-\infty}^{\infty} dx \ R_{\mu}^{(R)}(x) \frac{\partial}{\partial t} \Phi(x) = \zeta_{\mu}^{(R)} . \quad (6.27) \]

\footnote{Compare to section 1.4 in Ref. [21].}
The second term in the r.h.s. of Eq. (6.26) vanishes when \( t \to 0 \). Therefore we consider only the last two terms. Substituting the expansions Eq. (5.3) and Eq. (5.4a) into Eq. (6.25) and taking limit \( t \to 0 \) we obtain

\[
f^{(R)}_{\mu} = \zeta^{(R)}_{\mu} + \frac{i}{2\pi} \sqrt{\sinh \pi \mu} \lim_{z \to 0} \Phi(0, z) \left\{ \Gamma(i \mu) \left( \frac{mz}{2} \right)^{-i \mu} - \Gamma(-i \mu) \left( \frac{mz}{2} \right)^{i \mu} \right\},
\]

(6.28)

and the similar relation between \( f^{(L)}_{\mu} \) and \( \zeta^{(L)}_{\mu} \).

One concludes after comparing Eqs. (6.22), (6.23) that equation

\[
\omega_{M}(W(\Phi)) = \tilde{\omega}_{F}^{(2\pi)}(W(\Phi)),
\]

(6.29)

holds if and only if

\[
\Phi(0, 0) = 0,
\]

(6.30)

(compare to Eq. (5.17) and hence by linearity

\[
\omega_{M} = \tilde{\omega}_{F}^{(2\pi)} \quad \text{on } \hat{U}.
\]

(6.31)

This equation is an analog of Eq. (5.23) in algebraic approach (see Ref. [21]). The restriction of Eq. (6.31) to the right wedge algebra \( U_{R} \) is usually referred as Bisogniano - Wichmann theorem [23,24].

We see that Eq. (6.31) holds only on the double wedge subalgebra \( \hat{U} \subset U \), which corresponds to the space of those solutions for the field equation which satisfy the boundary condition (6.30). The r.h.s. of Eq. (6.31) doesn’t admit continuation to the whole algebra \( U \) while the l.h.s. admits such continuation. Therefore functionals \( \omega_{M} \) and \( \tilde{\omega}_{F}^{(2\pi)} \) describe different algebraic states over the algebra of observables of the free field in MS.

Let us consider two opportunities to interpret Eq. (6.31). The first one is to treat \( U \) as the true algebra of observables for the accelerated observer. In this case Eq. (6.31) does not hold for all observables and therefore Minkowski vacuum does not coincide with the thermal state \( \tilde{\omega}_{F}^{(2\pi)} \).

The second opportunity is to propose that \( \hat{U} \) should be the true algebra of observables for accelerated observer. In this case the true Minkowski vacuum state \( \omega_{M} \) (which is the state over the algebra \( U \)) is unrealizable state for such observer. Then Eq. (6.31) is satisfied for all physical observables and hence the restriction \( \omega_{M}|_{\hat{U}} \) of the state \( \omega_{M} \) to \( \hat{U} \) coincides with the state \( \tilde{\omega}_{F}^{(2\pi)} \) and admits interpretation in terms of Fulling – Unruh quanta. But let us stress that the Minkowski vacuum state is physically distinguished among the other possible states of the theory not so much due to it’s explicit expression (which of course is inherited by it’s restrictions to the subalgebras of \( U \)) as by it’s key physical properties such as Poincaré invariance, spectral conditions, local commutativity and cluster property (see the Whightman reconstruction theorem, [42]). Although some of these properties are inherited by the restrictions to subalgebras, the other properties such as Poincaré invariance generally are not.

But exactly these properties are mentioned when one assumes that some quantum system is prepared initially in the state of Minkowski vacuum. Since subalgebra \( U \) is not Poincaré invariant there are no any physical reasons to consider the state \( \omega_{M}|_{\hat{U}} \) as the initial state of the field. Moreover, since the automorphisms of \( \hat{U} \) corresponding to boost time evolution don’t mix up observables from \( U_{R} \) and \( U_{L} \) there is still no way for Rindler observer to prepare the state \( \omega_{M}|_{\hat{U}} \). We see that consideration of Unruh problem in algebraic approach leads to the same results as in usual field-theoretical approach.

VII. CONCLUSIONS

We have analyzed the Unruh problem in the frame of quantum field theory and have shown that the Unruh quantization scheme is valid in the double Rindler wedge rather than in MS. The double Rindler wedge is composed

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8 Compare to Eqs. (5.16a)-(5.16d), see also Appendix A.

9 Lacking of Poincaré invariance for field theory with boost time evolution is a consequence of the fact that 1-parameter boost group does not constitute normal subgroup in Poincaré group.
of two disjoint regions which causally do not communicate with each other. Moreover the Unruh construction implies existence of boundary condition at the point $\eta_0$ (or 2-dimensional plain in the case of $1+3$-dimensional spacetime) of MS. Such boundary condition may be interpreted as a topological obstacle which gives rise to a superselection rule prohibiting any correlations between $r$- and $l$-particles. Thus a Rindler observer living in the $R$-wedge in no way can influence the part of the field from the $L$-wedge and therefore elimination of the invisible "left" degrees of freedom will take no effect for him. Hence averaging over states of the field in one wedge can not lead to thermalization of the state in the other.

In algebraic approach the Unruh effect is commonly identified with the Bisogniano - Wichmann theorem. According to the Bisogniano - Wichmann theorem the Minkowski vacuum expectation value of only those observables which are entirely localized in the interior of the $R$-wedge constitutes the algebraic state which satisfies the KMS condition with respect to Rindler timelike variable $\eta$. This statement implies two points essential for its physical interpretation. First, it is assumed that the observer which carries out measurements lives in MS. Only then he could prepare the Minkowski vacuum state as the initial state of the field. Second, the variable $\eta$ must coincide with proper time of the observer. Only then he can interpret the KMS state as a thermal bath with Davies - Unruh temperature. But the Rindler observer can carry out measurements only inside the $R$-wedge and hence can not prepare the Minkowski vacuum state. From the other hand the variable $\eta$ can not coincide with proper time of an observer which is an inertial one at least asymptotically in far past and far future. Nevertheless only such observer for whom inertial in- and out- regions exist is able to prepare a state with finite number of particles in MS. These are the reasons why the Bisogniano - Wichmann theorem is irrelevant for consideration of the Unruh problem.

Hence considerations of the Unruh problem both in the standard and algebraic formulations of quantum field theory lead us to conclusion that principles of quantum field theory does not give any grounds for existence of the "Unruh effect".

Nevertheless there exists another aspect of the Unruh problem dealing with behaviour of a particular detector uniformly accelerated in MS. The direct consideration of the behavior of a constantly accelerating physical detector is a very difficult problem and its treatment in literature is very contradictory and often is simply erroneous. The major difficulty is that an object moving with a constant proper acceleration must be considered as a point object. This is because different points of a finite size body rigid with respect to Rindler coordinate frame move actually with different accelerations. Thus one should use an elementary particle or a microscopic bound system as a detector. In both cases the detector is a quantum object moving along a definite classical trajectory. Such assumption is in contradiction with the uncertainty principle, its range of applicability is very limited and therefore it must be used with proper care. Moreover, it was shown by Nikishov and Ritus \[58\] that elementary particles placed in a constant electric field do not demonstrate the universal thermal response.

It is clear that a heavy atom for which WKB approach is valid satisfies physical claims for the detector much better than an elementary particle. Unfortunately a systematic relativistic theory of bound states is still absent. Utilization of non relativistic bound systems as detectors was discussed in Ref. \[54\]. The ionization rate of a heavy ion moving with a constant acceleration was considered. It was shown that the ionization rate differs from the one obtained by virtue of the detailed balance principle applied to an atom immersed in a thermal bath with the Davies-Unruh temperature. It was also shown that the time of "thermal ionization" (if it was at all possible) is parametrically much greater than the time of destruction of the atom due to the tunneling ionization process in electric field.

It is worth to add that in literature the Unruh effect is usually explained by existence of event horizons for a constantly accelerated observer. But we understand that the notion of a constantly accelerated observer is an inadmissible idealization. It is clear that for any physical object the horizons are absent.

We certainly understand that behavior of accelerated detectors will differ from those at rest. We admit that under some circumstances detectors of some special configuration will follow Unruh behavior. But no conclusive proof exists that this behavior is universal and does not depend on the nature of the detector and the accelerating field.

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APPENDIX A: DERIVATION OF EQ.(4.20) FOR $b_\kappa$

To derive Eq.(4.20) let us introduce small parameter $z_0$ and split the integral in Eq.(4.19) as follows,

$$b_\kappa = b_\kappa^{(R)}(z_0) + b_\kappa^{(0)}(z_0) + b_\kappa^{(L)}(z_0), \quad b_\kappa^{(R)}(z_0) = \int_{z_0}^{\infty} dz \Psi_\kappa^{(R)*}(0, z) \left\{ \frac{\partial \phi_M}{\partial t} - i \frac{\kappa}{z} \phi_M \right\}_{t=0},$$

$$b_\kappa^{(0)} = i \int_{-z_0}^{z_0} dz \left( \Psi_\kappa^{(0)}(t, z) \frac{\partial}{\partial t} \phi_M(t, z) \right)_{t=0}, \quad b_\kappa^{(L)}(z_0) = i \int_{-\infty}^{-z_0} dz \Psi_\kappa^{(L)*}(0, z) \left\{ \frac{\partial \phi_M}{\partial t} - i \frac{\kappa}{z} \phi_M \right\}_{t=0}.$$

(A1)

In the first and last integrals of Eq.(A1) we used Eq.(4.8) for calculation of the time derivative of boost mode at $t = 0$.

Let us first calculate the first term in Eq.(A1). Using Eq.(4.14a) for $\Psi_\kappa^{(R)}$ we find \[^{10}\]

$$-i \frac{\kappa}{z} K_{1\kappa}(mz) = m \{ K_{1\kappa}'(mz) + K_{1\kappa-1}(mz) \},$$

we may rewrite Eq.(A2) in the form

$$b_\kappa^{(R)} = \frac{ie^{\pi \kappa/2}}{2\sqrt{\pi}} \int_{z_0}^{\infty} dz \left( K_{1\kappa}(mz) \frac{\partial \phi_M}{\partial t} + mK_{1\kappa}'(mz)\phi_M + mK_{1\kappa-1}(mz)\phi_M \right)_{t=0}. \quad (A3)$$

Finally after substitution $mK_{1\kappa}'(mz)\phi_M = (K_{1\kappa}(mz)\phi_M)'_z - K_{1\kappa}(mz)(\phi_M)'_z$ and adding and subtracting to the integrand the term $\Gamma(-ik)\{ (mz/2)^{ik} \phi_M \}'_z$ we obtain

$$b_\kappa^{(R)} = \frac{ie^{\pi \kappa/2}}{2\sqrt{\pi}} \left\{ \int_{z_0}^{\infty} dz F_R(z, \kappa) + \frac{1}{2} \left( \Gamma(-ik) \left( \frac{mz_0}{2} \right)^{ik} - \Gamma(ik) \left( \frac{mz_0}{2} \right)^{-ik} \right) \phi_M(0, z_0) \right\}, \quad (A4)$$

where $F_R$ was defined in Eq.(4.20). Note that we have chosen the regularization term by the requirement that the integral in Eq.(A4) converges when $z_0$ tends to zero. Note also that we assumed that the field vanishes at spatial infinity.

Substitution of Eq.(4.14d) into the third integral in Eq.(A1) yields

$$b_\kappa^{(L)(L)} = \frac{ie^{-\pi \kappa/2}}{2\sqrt{\pi}} \int_{-\infty}^{-z_0} dz K_{1\kappa}(-mz) \left\{ \frac{\partial \phi_M}{\partial t} - i \frac{\kappa}{z} \phi_M \right\}_{t=0}. \quad (A5)$$

It is easy to see that the r.h.s. of Eq.(A3) may be obtained from the r.h.s. of Eq.(A2) by changing the variable of integration $z \to -z$ and substitutions $\kappa \to -\kappa, \phi_M(t, z) \to \phi_M(t, -z)$. Thus we obtain

\[^{10}\]One should add a small negative imaginary part to $t$ in order to choose the right branches of the functions contained in the expression for $\Psi_\kappa^{(R)}$. 

25
Since \( b^{(R)}_\kappa(z_0) + b^{(L)}_\kappa(z_0) \) becomes singular when \( z_0 \) tends to zero one should also consider the contribution of the second integral in Eq. (A3). Using the integral representation of boost modes (4.7) this integral may be written in the form

\[
\begin{equation}
 b^{(0)}_\kappa = \frac{i}{2^{3/2} \pi} \int_{-\infty}^{z_0} \int_{-\infty}^{\infty} dq \exp(-imz \sinh(q) + i\kappa q) \left\{ \frac{\partial \phi_M}{\partial t} - im \cosh(q) \phi_M \right\},
\end{equation}
\]

(A7)

Let \( z_0 \) be sufficiently small we could change \( \phi_M(0, z) \) to \( \phi_M(0, 0) \). Then after performing integration over \( z \) and changing variable of integration \( q \rightarrow u = mz_0 \sinh q \) this expression may be reduced to

\[
 b^{(0)}_\kappa = \frac{\sqrt{2}}{\pi} \phi_M(0, 0) \int_0^\infty du \sin u \cos(\kappa q)u,
\]

(A8)

(the term with time derivative of the field vanishes when \( z_0 \) tends to zero). Since the effective values of the variable \( u \) in the integral (A8) are of order 1 we may use approximation

\[
\cos \kappa q \approx \frac{1}{2} \left\{ \left( \frac{mz_0}{2u} \right)^{-i\kappa} + \left( \frac{mz_0}{2u} \right)^{i\kappa} \right\},
\]

(A9)

which is valid if \( u \sim 1 \) and \( z_0 \) is small enough. Substitution of Eq. (A7) into Eq. (A8) and evaluation of the integral yields

\[
 b^{(0)}_\kappa = \frac{i \sinh(\pi \kappa/2)}{\pi \sqrt{2}} \phi_M(0, 0) \left\{ \Gamma(i\kappa) \left( \frac{mz_0}{2} \right)^{-i\kappa} - \Gamma(-i\kappa) \left( \frac{mz_0}{2} \right)^{i\kappa} \right\}.
\]

(A10)

Finally substituting Eqs. (A4), (A6) and (A10) into Eq. (A1) and taking limit \( z_0 \rightarrow 0 \) we obtain Eq. (4.20).

**APPENDIX B: ANALOGY BETWEEN THE UNRUH STATES AND SQUEEZED STATES OF A HARMONIC OSCILLATOR**

1. Consider a two dimensional harmonic oscillator in \{\( x, y \)\}-plain. The Hamiltonian of such oscillator reads

\[
 H_{osc} = b_+^\dagger b_+ + b_-^\dagger b_- + 1,
\]

(B1)

where

\[
 b_+ = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right), \quad b_- = \frac{1}{\sqrt{2}} \left( y + \frac{\partial}{\partial y} \right),
\]

and the commutation relations are of the usual form,

\[
 [b_\pm, b_\mp^\dagger] = 1, \quad [b_\pm, b_\mp] = 0.
\]

(B2)

The ground state \( |0\rangle \) satisfies the condition

\[
 b_\pm |0\rangle = 0.
\]

(B3)

Let us introduce the new operators \( r_\nu, l_\nu \) as
where the unitary operator \( S(\nu) \) reads
\[
S(\nu) = e^{i\theta\mathcal{G}} = \exp\{\theta(b_+b_- - b_+^\dagger b_-^\dagger)\} = \exp(-e^{-\nu}b_+b_-^\dagger)\exp\left(\frac{1}{4}\ln(1 - e^{-2\nu})H_{osc}\right)\exp(e^{-\nu}b_+b_-),
\]
(B5)

The operators (B4) obey the commutation relations in the form
\[
[r_\nu, r^\dagger_\mu] = [l_\nu, l^\dagger_\mu] = 1, \quad [r_\nu, l^\dagger_\nu] = 0.
\]
(B6)

One can write the relation between the Unruh and boost operators in the form
\[
\text{r}_\nu |s_\nu\rangle = 0, \quad l_\nu |s_\nu\rangle = 0,
\]
(B8)

For the amplitude of transition between the ground and squeezed vacuum states according to Eq.(B9) we have
\[
\langle 0 | s_\nu \rangle = Z^{-1/2}_\nu, \quad Z_\nu = (1 - e^{-2\nu})^{-1},
\]
(B11)

The analogy between Eqs.(B11, B12) and Eq.(5.23) becomes evident after substitution \( \nu = \pi \mu \). It is clear that the appearance of the Bose factor in Eq.(5.23) results completely from the properties of Bogohubov transformation and does not relate to any sort of thermal behaviour.

2. We will show now that due to infinite number of degrees of freedom in the Unruh problem the considered analogy is not complete and representation of canonical commutation relations in terms of the Unruh operators is unitary inequivalent to the standard one (in terms of the plain waves or boost operators).

One can write the relation between the Unruh and boost operators in the form
\[
r_\mu = \cosh \theta_\mu b_\mu + \sinh \theta_\mu b^\dagger_\mu, \quad l_\mu = \sinh \theta_\mu b^\dagger_\mu + \cosh \theta_\mu b_\mu,
\]
(B14)

with \( \theta_\mu \) defined by \( \tanh \theta_\mu = \exp(-\pi \mu) \). The vacuum state of the field in the double Rindler wedge \(|0_{DW}\rangle \) (called sometimes the Fulling vacuum \([2]\) ) is defined by the requirements
\[
r_\mu |0_{DW}\rangle = 0, \quad l_\mu |0_{DW}\rangle = 0, \quad \mu > 0.
\]
(B15)
If the Fulling vacuum could be represented by the vector \( |0_{DW} \rangle \) in the same Hilbert space where Minkowski vacuum state \( |0_M \rangle \) is defined then there should exist a unitary operator \( S \) such that

\[
|0_{DW} \rangle = S|0_M \rangle, \quad Sb_\mu S^\dagger = r_\mu, \quad Sb_{-\mu} S^\dagger = l_\mu.
\] (B16)

A simple calculation shows that such operator \( S \) has the following formal representation (compare to Eq.(B17)):

\[
S = \exp \left( \int_0^\infty d\mu \theta_\mu (b_\mu b_{-\mu} - b_{-\mu}^\dagger b_\mu^\dagger) \right).
\] (B17)

It is obvious from this representation that \( |0_{DW} \rangle \) has the form

\[
|0_{DW} \rangle = \exp \left( -\frac{1}{2} \int_0^\infty d\mu \ln \cosh \theta_\mu \right) |0_M \rangle.
\] (B18)

Consider the matrix element

\[ f[\theta_\mu] = \langle 0_{DW} | 0_M \rangle = \langle 0_M | S^\dagger [\theta_\mu] | 0_M \rangle, \] (B19)

for arbitrary function \( \theta_\mu \). The derivative of the functional \( f[\theta_\mu] \) can be expressed as follows (see e.g. Sec. 2.4 of Ref. [56]),

\[
\frac{\delta f}{\delta \theta_\mu} = -\langle 0_M | b_\mu b_{-\mu} S^\dagger | 0_M \rangle = \langle 0_M | S^\dagger (Sb_\mu S^\dagger) (Sb_{-\mu} S^\dagger) | 0_M \rangle = \langle 0_M | S^\dagger r_\mu l_\mu | 0_M \rangle = -f \cosh \theta_\mu \sinh \theta_\mu \delta (0) - \sinh^2 \theta_\mu \frac{\delta f}{\delta \theta_\mu}.
\] (B20)

For the last transformation in Eq.(B20) we have used Eqs.(B14) and the obvious formula

\[
\frac{\delta f}{\delta \theta_\mu} = \langle 0_M | S^\dagger b_{-\mu} b_\mu | 0_M \rangle.
\] (B21)

After simplification of Eq.(B20) one obtains for the functional \( f[\theta_\mu] \) the differential equation ,

\[
\frac{\delta f[\theta_\mu]}{\delta \theta_\mu} = -\delta (0) \tanh \theta_\mu f[\theta_\mu],
\] (B22)

the formal solution of which we can write as

\[
f = \exp \left( -\delta (0) \int_0^\infty d\mu \ln \cosh \theta_\mu \right),
\] (B23)

(the constant of integration is fixed by the requirement that \( f \) should be equal to 1 for \( \theta_\mu = 0 \)). Evaluation of the integral yields

\[
\int_0^\infty d\mu \ln \cosh \theta_\mu = -\frac{1}{2} \int_0^\infty d\mu \ln (1 - e^{-2\pi \mu}) = \frac{\pi^2}{24}.
\]

Thus we obtain

\[
f = \langle 0_{DW} | 0_M \rangle = \exp (-\delta (0) \pi^2/24) = 0,
\] (B24)

i.e. \( K^{(0)} \) in Eq.(B18) vanishes. Further with the help of Eqs.(B21),(B23),(B24) we get
\[ \langle 0_{DW} | b_\mu^\dagger b_{-\mu}^\dagger | 0_M \rangle = \frac{\delta f[\theta_\mu]}{\delta \theta_\mu} = -\delta(0) \exp(-\pi \mu - \delta(0) \pi^2/24) = 0, \quad (B25) \]

i.e. \( K^{(2)} = 0 \). Processing further in such a way we conclude that \( |0_{DW}\rangle = 0 \). It means that there is no Unruh vacuum state in the same Hilbert space where Minkowski vacuum exists and that the Unruh operators (B14) form unitary inequivalent representation of commutation relations.

It is clear from Eq. (6.10) (where for current consideration one should change \( |\Omega_\beta\rangle \) to \( |0_M\rangle \), \( \beta \) to \( 2\pi \) and \( |0_R\rangle \otimes |0_L\rangle \) to \( |0_{DW}\rangle \)) that \( \langle 0_{DW} | 0_M \rangle = Z^{-1/2} \). Therefore we can also express Eq. (B24) in the form \( Z^{2\pi} = \exp(\delta(0) \pi^2/12) = \infty \) (compare to Eq. (5.24)).

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FIG. 1. Splitting of MS into wedges $R$, $F$, $L$, $P$. 
