Modeling and study of a novel electrothermal oscillator based on shape memory alloys

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Abstract. The paper deals with mathematical modelling and experimental investigation of a novel electrothermal oscillator based on Shape Memory Alloys (SMA). The operation of the oscillator is based on a heated by DC electric current SMA wire, mounted with a pre-tension between the free ends of two symmetrically arranged cantilever beams. Crystallographic changes in the SMA wire, consisting in the transition between austenitic and martensitic phases due to the periodic heating and cooling, lead to its intermittent shortening and stretching and thus mechanical and electrical vibrations are generated. The mathematical model of the system is derived using Lagrange-Maxwell formalism, Joule-Lenz law and Newton’s law of cooling. Simulation results show two types of periodic motion – the first one with low frequency is due to the temperature variation in the wire and the second one is with much more higher frequency and is due to the mass and elastic properties of the system. Experimental validation of the mathematical model shows its suitability for description of the considered system dynamic behaviour.

1. Introduction
The modern smart materials allow a precise control of their physical properties by applying external influences. This unique feature makes it possible, by utilizing simple physical principles, to build a variety of technical devices with relatively complex behavior. Particularly widespread in the past two decades have become the Shape Memory Alloys (SMA). They are used for a considerable variety of practical purposes, some of which are vibration damping in mechanical devices [1], driving various devices by actuators and artificial muscles with reduced weight [2,3,4], natural frequency tuning of civil structures [5,6] and intensive application in the modern medicine and dentistry [7,8]. The nonlinear
behavior of the SMA is determined primarily by the presence of a hysteresis and by the reversible crystallographic changes in the material due to the heating or cooling. SMA are particularly suitable for designing mechanical devices with cyclic behavior - so called oscillators. The authors [9,10,11] adequately demonstrate the possibility of using SMA to create oscillators that produce periodic, quasi-periodic or chaotic vibrations. Combining systems from different physical domains into one system is especially promising, mostly from the point of view of building technical devices with simple structure and reach dynamic behaviour. For example, combining an electrical, thermal and mechanical system in [12] makes it possible to develop a DC-driven electrothermal oscillator with a simple design and stable vibrations. The same approach is used in [13] to develop a mechanical oscillator with frequency range from 38 Hz to 1200 Hz with electrothermal actuation and without using any electrical components.

The principal aim of the present paper is to develop a conceptual design and mathematical model of a novel nonlinear electrothermal oscillator based on Shape Memory Alloys.

2. Conceptual design of the studied electrothermal oscillator

Figure 1 shows the developed design of the oscillator. It is composed of two symmetrically arranged elastic cantilever beams (pos.1a), fixed on an unmovable base 2, and a SMA wire (pos.3a), mounted with pre-tension between the free ends of the beams. The electric current, generated by the external power supply 5, is submitted to the SMA wire through the electrode 4, which contacts the wire in its initial position (pos.3a). The generated due to the flow of the electric current heat in the wire leads to crystallographic changes in the SMA material – a transition from martensitic to austenitic phase, resulting in shortening of the wire. This causes a symmetric deformation of the beams free ends (pos.1b), accompanied by an upward motion of the wire (pos.3b). Opening of the contact between the electrode and the wire terminates the flow of electric current through the wire whereby it begins to cool by convection. This provokes a reverse crystallographic change in the wire - the transition from austenitic to martensitic phase, resulting in a stretching of the wire, reverse deformation of the beams (pos.1a) accompanied by a downward motion of the wire (pos.3a) resulting in a restoration of the contact between the electrode and the wire whereby the heating process is repeated. The resulting motion of the free ends of the cantilever beams has a periodic behavior with a certain frequency. Deformation of the beams is used to produce electrical energy by two piezoelectric transducers 6, glued on the beams deformed surfaces near to their fixed ends. It should be specified that the electrode 4 is mounted in height so as to create a certain pre-tension of the wire in its initial position 3a.

![Figure 1. Conceptual design of the studied electrothermal oscillator.](image-url)

3. Mathematical model of the oscillator

The electrothermal oscillator consist of three subsystems, belonging to different physical domains – mechanical, electrical and thermal. Every subsystem has its own dynamics, which affect the dynamics of the overall system.
3.1 Mathematical model of the mechanical subsystem

Figure 2 depicts the mechanism layout for an arbitrary wire length \( s \), obtained for arbitrary wire temperature \( T \). The following notations are used: \( \delta_y \) is the beam deflection; \( \delta_{y_1} \) and \( \delta_{x_1} \) are the projections of the beam deflection on the \( y \) and \( x \)-axes correspondingly; \( F_s \) is the force in the SMA wire and \( F_{s1}^t \) and \( F_{s1}^n \) are its tangential and normal projections correspondingly.

For the determination of the forces and deflections, it was assumed that before the SMA wire mounting, the undeformed cantilevers are in the initial positions \( OA_0 \) and \( CB_0 \). The deformed cantilever beams positions \( OA_1 \) and \( CB_1 \) correspond to the mounted in the cold martensitic condition wire. The following assumptions are adopted: 1) Deflections of the cantilevers beams are small; 2) The free end of the beam during the deformation is moving along the perpendicular to its undeformed position; 3) The Hook law is valid for the wire in low temperature. If the initial length of the wire in the undeformed condition is \( s_0 \) then the mounting requirement has the form:

\[
s_0 < A_0 B_0
\]

Mounting of the wire in the cold condition leads to its extension and its length is denoted by \( s_1 = A_1 B_1 \).

![Figure 2. Layout of the mechanical structure of the oscillator.](image)

Considering some geometric and force conditions, for the positions \( OA \) and \( CB \) corresponding to the coordinate \( y \) of the wire, the tensile force in the wire is determined as

\[
F_s = E_m A_s \varepsilon_s = E_m A_s \frac{3EI(2\cos \alpha + d)}{2E_m A_s l^3 \sin^2 \alpha + 3EI}
\]

In equation (2), the following notations are used: \( E_m \) is Young modulus of the SMA wire in martensitic phase; \( A_s \) is the area of the wire cross-section; \( \varepsilon_s \) is the wire strain; \( l \) is the length of the cantilever beam; \( \alpha \) is the angle of inclination of the beams; \( d \) is the distance between the beams fixed ends; \( E_s = E_s(T) \) is the Young modulus of the SMA wire for an arbitrary temperature \( T \), computed according to the SMA model, presented in [14]; \( E \) is the Young modulus of the beam; \( I \) is the area moment of inertia of the beam cross-section.

The following expressions for the \( y \)-coordinate of the wire and for its length \( s \) are obtained:

\[
y = l \sin \alpha + \frac{3EI(l^2 \sin 2\alpha + dl \sin \alpha) \cos \alpha}{2E_m A_s l^3 \sin^2 \alpha + 3EI}
\]

\[
s = \varepsilon_s s_0 = \frac{3EI(2\cos \alpha + d)s}{2l^3 E_s A_s \sin^2 \alpha + 3EI}
\]

For the arbitrary temperature \( T \), the vertical translation \( \Delta y \) of the wire according to the initial mounting position is given by the following equation:

\[
\Delta y = y - y_1 = 3EI(l^2 \sin 2\alpha + dl \sin \alpha) \cos \alpha \left( \frac{1}{2l^3 E_s A_s \sin^2 \alpha + 3EI} - \frac{1}{2l^3 E_m A_s \sin^2 \alpha + 3EI_0} \right)
\]
3.2 Mathematical model of the piezoelectric subsystem

The electromechanical piezoelectric subsystem is considered as a system with two degrees of freedom (DOF). The first DOF is mechanical, representing the deflection of the beam along the $x$-axis – figure 3(a). For the electrical generalized coordinate is chosen the electric charge $q$ of the piezoelectric transducer circuit – figure 3(b) and figure 3(c). The mass of the cantilever beam and the half of the SMA wire mass are represented as a concentrated mass $m$, positioned at the free end of the beam at point $A$. The electric circuit consists of a piezoelectric transducer with capacitance $C$ and load resistance $R_L$.

![Figure 3. Dynamical model of the cantilever piezoelectric beam: (a) mechanical model; (b) electric circuit; (c) electromechanical model](image)

Close to the beam fixed end three layers are applied – for the bottom electrode, for the piezoelectric film and for the top electrode and both electrodes are being connected by the load resistance. Also, it is assumed that the polarization of the piezoelectric layer is along the longitudinal axis of the beam. Since along the same axis is the strain of the piezoelectric layer, the mode of operation of the piezoelectric transducer is $d_{33}$.

Interaction of the mechanical and electrical variables is represented by the Lagrangian function:

$$W_L = W_k - W_p - W_{pr}$$

where $W_k$ is the kinetic energy of the mass, and $W_p$ is its potential energy. The third member $W_{pr}$ is the full electromechanical energy of the piezoelectric transducer.

The cantilever beam and the SMA wire are considered as springs connected in parallel with stiffness depending on the deformations and the wire temperature. Stiffness of the cantilever beam along the $x$ – axis is

$$k_x = \frac{3EI}{l^3 \sin \alpha}$$

Using the Lagrange equations of the second kind and the Lagrangian (6) one obtains the following system of differential equations:

$$\begin{align*}
\dddot{x} + k_x \left( x + \delta_n \right) - \frac{C'}{2C^2 \left(1 - k_p^2\right)} \dddot{q} + \frac{d_{33} K_x p_x C'}{C^2 \left(1 - k_p^2\right)} q \left( x + \delta_n \right) - \frac{d_{33} K_x p_x}{C \left(1 - k_p^2\right)} q \left( x + \delta_n \right) &= -\beta \ddot{x} + F_s \\
\frac{1}{C \left(1 - k_p^2\right)} \dddot{q} + \frac{d_{33} K_x p_x}{C \left(1 - k_p^2\right)} \left( x + \delta_n \right) &= -R_i \dot{q}
\end{align*}$$

where:

$$C' = \frac{dC}{dx} = \frac{C_{\varepsilon} p_x}{g_p}$$
where

\[ p_s = \frac{3h_p I_p}{l^3 \sin \alpha} \left( \frac{1 - I_p}{2} \right) \]

(10)

\[ K_p \] is the stiffness of the piezoelectric transducer with short-circuited electrodes; \( k_p \) is the electromechanical coupling factor; \( d_{33} \) is the piezoelectric constant; \( C \) is the capacitance of the transducer; \( \beta \) is the damping factor; \( \delta_{nt} \) is the deformation of the beam in the initial mounting position; \( \varepsilon \) is the dielectric constant of the piezoelectric material; \( g_p \) is the distance between the electrodes of the piezoelectric layer and \( w_p \) is the width of the piezoelectric layer; \( l_p \) is the length of the piezoelectric layer; \( h_p \) is the distance between the beam neutral axis and the middle layer of the piezoelectric transducer. The force in the SMA wire \( F_s \) is presented in the form

\[
F_s = \begin{cases} 
\left[ E_u - (E_u - E_m)R_m(T) \right] e_s A_s, & 0 \leq \varepsilon_i \leq \varepsilon^m \varepsilon^d \\
\left[ E_u - (E_u - E_m)R_m(T) \right] e_s A_s - (E_r - E_m) e^m_s R_m(T) A_s, & \varepsilon^m \varepsilon \leq \varepsilon_i \leq \varepsilon^d \varepsilon \\\n\left[ E_u - (E_u - E_m)R_m(T) \right] e_s A_s - (E_r - E_m) R_m(T) e^m_s A_s - (E_d - E_r) R_m(T) e^d_s A_s, & \varepsilon^d \varepsilon \leq \varepsilon_i
\end{cases}
\]

(11)

where \( E_u, E_r, \) and \( E_d \) are the Young’s modules at austenite phase, partly twinned martensite, and detwinned martensite respectively, \( \varepsilon_i \) is the strain of the SMA wire, \( \varepsilon^m \varepsilon^d \) is the yield strain of twinned martensite, \( \varepsilon^d \varepsilon \) is the minimum strain of detwinned martensite. The hysteretic behavior of the SMA wire is presented by the martensitic fraction \( R_m(T) \), calculated according to the model, presented in [14].

It is assumed that the position \( x \) of the mass is equal to 0 when the SMA wire has length \( s_i \) (see figure 2), which corresponds to the preliminary deformation of the beam \( \delta_{nt} \):

\[
\delta_{nt} = \frac{(l^2 \sin 2\alpha \sin \alpha + dl \sin^2 \alpha) E_m A_s l^2}{2E_m A_s l^2 \sin^2 \alpha + 3\gamma_0 EI}
\]

(12)

### 3.3 Mathematical model of the thermal subsystem

In the considered system for the heat generation is utilized the flow of an electric current through a resistance, i.e. the Joule heating is present. Under the assumption of convective cooling and taking into account the Joule–Lenz law and Newton’s law of cooling, one obtains the differential equation, describing the change of the temperature in the SMA wire:

\[
\dot{T} = \frac{1}{\rho c_s V c_p} \begin{cases} 
R i^2 + A_i h_i (T_c - T), & \text{if } x \leq x_T \\
A_i h_i (T_c - T), & \text{if } x > x_T
\end{cases}
\]

(13)

where \( x_T \) is the horizontal position of the beam free end, corresponding to the wire height \( y_T \), for which the contact between the electrode and the wire is lost. In (13) the following notations are used: \( R \) is the electric resistance of the wire; \( i \) is the electric current in the wire; \( T_c \) is the temperature of the environment; \( \rho \) is the density of the wire material; \( V \) is the volume of wire; \( c_s \) is the specific heat of the wire; \( A_i \) is the surface area of the wire; \( h_i \) is the convection heat transfer coefficient.

The following additional simplifying assumptions are made: 1) The inclination of the SMA wire relative to the \( x \)-axis when the electrode and the wire are in a contact is neglected; 2) The change of the resistance in the contact point between the wire and the electrode due to the change of the contact pressure is neglected; 3) The change of the resistance of the SMA wire due to the increase of the temperature is neglected.

### 4. Numerical solution of the equations and discussion of the results

The final system of differential equations is composed by equations (8) and (13), represented as a system of first-order equations. The backward Euler method with a suitable step is used for the solution of the system for a time period of 2 s. Initial conditions for the variables are: \( x(0) = 0 \), \( \dot{x}(0) = 0 \), \( T(0) = T_c \). At time \( t=0 \) the charge \( q \) has a value, computed from the second equation of (8) for \( \dot{q} = 0 \):

\[
\frac{d}{dt} (\rho V c_p) \begin{cases} 
\frac{R i^2 + A_i h_i (T_c - T)}{\rho c_s V c_p} = 0, & \text{if } x \leq x_T \\
\frac{A_i h_i (T_c - T)}{\rho c_s V c_p}, & \text{if } x > x_T
\end{cases}
\]
The obtained graphs for the output characteristics are shown in figure 4.

Figure 4(a) shows the graph of the horizontal translation $x$. As one can see, after a short transient period of 0.05 s the system switches to a steady-state mode with two types of periodic motion. The first periodic motion has a period $\tau_1 = 0.591$ s and frequency $f_1 = 1.692$ Hz. A comparison with the figure 4(b) graph shows that this frequency is equal to the frequency of the temperature variation. As one can see, all output characteristics of the system are modulated by this variation. The translations with this frequency are nonlinear and are caused by the temperature variation due to the consecutive heating and cooling of the SMA wire. They depend on the temperature convection time constant $\tau_r = \rho \nu c_p/(A \phi l_i)$ and the gradient of the temperature $k_r = R i^2/(\rho \nu c_p)$. For that reason, $f_r$ is denoted as a temperature-dependent frequency or macro frequency. The analysis of the graphs shows that except the macro frequency exists another much more higher frequency (micro frequency), which depends mainly on the mass and elastic parameters of the mechanical subsystem. The exponential decrease of the amplitude of the micro vibrations is caused by the assumed vicious damping. This vibration is modulated by the macro frequency.

![Figure 4](image-url)

**Figure 4.** Graphs of the system output variables: (a) horizontal translation $x$ of the beam free end; (b) horizontal acceleration; (c) temperature $T$ variation; (d) voltage, generated by the piezoelectric transducer
5. Experimental setup and validation of the mathematical model

Validation of the mathematical model is conducted by a comparison between the simulation and the experimental data. Figure 5(a) depicts the experimental setup, whose main elements are the fixed base 1, the cantilever beams 2 with piezoelectric transducers 3, the SMA wire 5, the electrode 6 and the power source 9. It is equipped with sensors for measuring the acceleration 7 (ADXL 335) and the SMA wire temperature 4. Analog outputs from the sensors and from the piezoelectric transducers are converted to digital by an analog-to-digital converter (National Instruments 8 inputs, 14 bit) and sent to a computer measurement system 8, where a computer program LabView is used for the initial processing and visualization of data [15].

The experimental data for the acceleration $a_x$ of the beam end in $x$-direction, for the generated voltage from the piezoelectric transducers and for the temperature of the SMA wire are shown in figure 5(b), figure 5(c) and figure 5(d). Although some differences between the experimental and simulated data shown in figure 4(b), figure 4(c) and figure 4(d) are present, the degree of coincidence is satisfactory and it can be considered that the developed mathematical model with good accuracy represents the real system. It should be noted that the difference between the experimental (figure 5(d)) and theoretical (figure 4(c)) temperatures is larger, but this is due to the high inertia of the thermocouple used.

![Experimental setup](image)

**Figure 5.** Experimental and simulation results: (a) layout of the experimental setup; (b) acceleration $a_x$ of the beam end in $x$-direction; (c) generated voltage from the piezoelectric transducers; d) temperature of the SMA wire

6. Conclusions

This paper studied issues related to the development and investigation of a novel electrothermal oscillator. The design of such a system, containing subsystems from different physical domains, namely mechanical, electromechanical and thermal, is difficult and the modelling method must take into account
their interaction and its influence on the overall system behaviour. The developed and validated mathematical model considerably facilitates the oscillator design process. The conducted investigation clearly demonstrates the possibility to generate a periodic motion of the system using Shape Memory Alloy wire by supplying the system input with DC electric current. Such type of system is extremely sensitive to changes in system parameters, that’s why by tuning the system parameters a considerable variety of alternative operating dynamic modes could be achieved. The primary direction for improvement of the system performance is its miniaturization which will cause an increase in its frequency, also its stability.

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