Syzygies of Veronese embeddings

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Abstract - We prove that the Veronese embedding $\varphi_{O_{\mathbb{P}^n}(d)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ with $n \geq 2$, $d \geq 3$ does not satisfy property $N_p$ (according to Green and Lazarsfeld) if $p \geq 3d - 2$. We make the conjecture that also the converse holds. This is true for $n = 2$ and for $n = d = 3$.

Introduction

Let $\mathbb{P}^n$ be the projective $n$-space over an algebraically closed field of characteristic zero and let $\varphi_{O_{\mathbb{P}^n}(d)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be the Veronese embedding associated to the complete linear system $| O_{\mathbb{P}^n}(d) |$. In order to understand the homogeneous ideal $\mathcal{I}$ of $\mathbb{P}^n$ in $\mathbb{P}^N$ as well as its syzygies, it is useful to study some properties about a minimal free resolution of $\mathcal{I}$.

M. Green and R. Lazarsfeld (see [G2], [GL]) introduced the property $N_p$ (Definition 1.3) for a complete projective non singular variety $X \hookrightarrow \mathbb{P}^N$ embedded in $\mathbb{P}^N$ with an ample line bundle $L$. When property $N_p$ holds for every integer $p$ the resolution of $\mathcal{I}$ is ”as nice as possible”. M. Green proves in [G2], Theorem 2.2, that $\varphi_{O_{\mathbb{P}^n}(d)}$ satisfies $N_p$ if $p \leq d$. L. Manivel has generalized this result to flag manifolds [M]. The rational normal curves (which are the Veronese embeddings of $\mathbb{P}^1$) satisfy $N_p \ \forall p$. C. Ciliberto showed us that the results of [G1] imply that $\varphi_{O_{\mathbb{P}^2}(d)}$ with $d \geq 3$ satisfies $N_p$ if $p \leq 3d - 3$. This sufficient condition has been found also by C. Birkenhake in [B1] as a corollary of a more general result. Here we prove that this condition is also necessary (Theorem 3.1) and we formulate (for $n \geq 2$) the following.

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Conjecture  \( \varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} \) satisfies \( N_p \) \( \iff \) \[
\begin{array}{l}
\{ \begin{array}{l}
n = 2, d = 2, \forall p \\
n \geq 3, d = 2, p \leq 5 \\
n \geq 2, d \geq 3, p \leq 3d - 3
\end{array} \mid
\end{array}
\]

Our precise result is the following:

**Theorem** The implication ‘\( \Rightarrow \)’ of the previous conjecture is true.

Moreover we remark that the implication ‘\( \Leftarrow \)’ of the previous conjecture is true in the cases \( n = 2 \) ([G1]), \( n = d = 3 \) ([G1]), \( d = 2 \) ([JPW]). This solves the Problem 4.5 of [EL] (raised by Fulton) in the first cases given by the projective plane and by the cubic embedding of the projective 3-dimensional space.

We remark also that our conjecture would be overcomed by the knowledge of the minimal resolution of the Veronese variety. This is stated as an open problem in [G2] (remark of section 2). Our results can be seen as a step towards this problem.

The paper is organized as follows: in section 1 we recall some definitions we will need later and we improve a known cohomological criterion for the property \( N_p \); in section 2 we prove our main results and in section 3 we fit our results into the literature.

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**KEYWORDS:** Veronese embedding, free resolution, syzygy.

**SUBJECT CLASSIFICATION:** 14M20, 14F17.

**1 - Notations and preliminaries**

Let \( V \) be a vector space of dimension \( n + 1 \) over an algebraically closed field \( \mathbb{K} \) of characteristic 0 and let \( \mathbb{P}^n = \mathbb{P}(V^*) \) the projective space associated to the dual space of \( V \).

Note that \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong S^d V \) \( \forall d \geq 0 \).

For any vector bundle \( E \) over \( \mathbb{P}^n \), we will denote by \( H^i(E) \) the \( i \)-th cohomology group of \( E \) over \( \mathbb{P}^n \) and by \( E(t) \) the tensor product \( E \otimes \mathcal{O}_{\mathbb{P}^n}(t) \).

The following bundles will play a fundamental role in this paper:
Definition 1.1  For any positive integer \( d \), the line bundle \( \mathcal{O}_{\mathbb{P}^n}(d) \) is generated by global sections \( H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \cong S^dV \) so that the evaluation map \( ev : S^dV \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(d) \) is surjective. Call \( E_d \) the kernel. Thus the vector bundle \( E_d \) is defined by the exact sequence:

\[
0 \rightarrow E_d \rightarrow S^dV \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^n}(d) \rightarrow 0. 
\]

It follows immediately from the definition that the bundle \( E_d \) has rank \( N := \text{rk} E_d = (n+d)n - 1 \) and first Chern class \( c_1(E_d) = -d \).

Note that, if \( d = 1 \), (1.2) is the dualized Euler sequence so that:

\[
E_1 \cong \Omega^1_{\mathbb{P}^n}(1) \quad \text{and} \quad \bigwedge^q E_1 \cong \Omega^q_{\mathbb{P}^n}(q). 
\]

For any integer \( d \geq 0 \), we will denote by \( \varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N \) the Veronese embedding \( \varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N \) associated to the complete linear system \( |\mathcal{O}_{\mathbb{P}^n}(d)| \) of dimension \( N + 1 := (n+d) \). Recall that if \([x_0 : \ldots : x_n]\) is a system of homogeneous coordinates on \( \mathbb{P}^n \) and \([y_0 : \ldots : y_N]\) on \( \mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d))^*) \), then \( \varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} \) is the embedding:

\[
[x_0 : \ldots : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_n^d].
\]

With the above notation, let \( S := \bigoplus_{k \geq 0} S^k (H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))) \) be the homogeneous coordinate ring of \( \mathbb{P}^N \) and define the graded \( S \)-module \( R := \bigoplus_{k \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kd)) \). Let

\[
0 \rightarrow \bigoplus_j S(-j)\tilde{b}_{ij} \rightarrow \cdots \rightarrow \bigoplus_j S(-j)\tilde{b}_{ij} \rightarrow R \rightarrow 0
\]

be a minimal free resolution of \( R \) with graded Betti numbers \( \tilde{b}_{ij} \).

Definition 1.3  For any integer \( p \geq 0 \) the embedding \( \varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N \) is said to satisfy property \( N_p \) if

\[
\tilde{b}_{0j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{b}_{ij} = 0 \quad \text{for } j \neq i + 1 \text{ when } 1 \leq i \leq p.
\]
Thus:

- $N_0$ means that $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}(\mathbb{P}^n)$ is projectively normal in $\mathbb{P}^N$;
- $N_1$ means that $N_0$ holds and the ideal $I$ of $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is generated by quadrics;
- $N_2$ means that, moreover, the module of syzygies among quadratic generators $Q_i \in I$ is spanned by the relations of the form $\sum L_i Q_i = 0$ where the $L_i$ are linear polynomials;
- and so on.

**Remark 1.4** Let $C \hookrightarrow \mathbb{P}^d$ be the rational normal curve (of degree $d$) in $\mathbb{P}^d$. If $V$ is a vector space of dimension 2, then $C \cong \mathbb{P}(V^*) \hookrightarrow \mathbb{P}^d = \mathbb{P}(S^dV^*)$ is the image of the Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^1}(d)} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$.

It is well known (e.g. by using the Eagon-Northcott complex) that the sheaf ideal $\mathcal{I}$ of $C$ in $\mathcal{O}_{\mathbb{P}^d}$ has the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^d}(-d)^{\oplus b_d} \to \mathcal{O}_{\mathbb{P}^d}(-d+1)^{\oplus b_{d-1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^d}(-2)^{\oplus b_2} \to \mathcal{I} \to 0$$

where $b_k := (k-1)(d^2)$. So the Veronese embeddings of $\mathbb{P}^1$ satisfy $N_p \forall p$.

From [B2], Remark 2.7, and [G1] we have the following cohomological criterion:

**Proposition 1.5** The Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies property $N_p$ if and only if

$$H^1\left(\bigwedge^q E_d(jd)\right) = 0 \quad \text{for} \quad 1 \leq q \leq p + 1 \quad \text{and} \quad \forall j \geq 1.$$

$\diamond$

We have the following cohomological criterion, which improves slightly the previous one (in fact $H^2(\bigwedge^q E_d) \simeq H^1\left(\bigwedge^{q-1} E_d(d)\right)$).

**Theorem 1.6** The Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies property $N_p$ if and only if $H^2\left(\bigwedge^q E_d\right) = 0 \quad \text{for} \quad 1 \leq q \leq p + 2$.

The proof of Theorem 1.6 relies on the following
Proposition 1.7 If $H^2(\bigwedge^q E_d) = 0$ for $1 \leq q \leq k$, then $H^2(\bigwedge^q E_d(t)) = 0$ for $1 \leq q \leq k$ and $\forall t \geq 0$.

Proof
Consider the two exact sequences:

(*) $0 \to \bigwedge^q E_d(t-1) \to \bigwedge^q E_d(t) \to \bigwedge^q E_d(t)|_{\mathbb{P}^{n-1}} \to 0$,

(**) $0 \to \bigwedge^q E_d(t-1) \to \bigwedge^q (S^d V) \otimes \mathcal{O}_{\mathbb{P}^n}(t-1) \to \bigwedge^{q-1} E_d(t+d-1) \to 0$.

The proof is by double induction on $n$ and $k$. The statement is true for $n = 2$ (Serre duality) and for $k = 1$ (it follows immediately from (1.2)). From the cohomology sequence associated to (**) with $t = 0$ and the inductive hypothesis on $k$ we get $H^3(\bigwedge^q E_d(-1)) = 0$ for $1 \leq q \leq k$. Since $E_d|_{\mathbb{P}^{n-1}} \cong \tilde{E}_d \oplus \mathcal{O}_{\mathbb{P}^{n-1}}^\oplus(n+d-1)$, where $\tilde{E}_d$ is the vector bundle $E_d$ over $\mathbb{P}^{n-1}$, the previous vanishing implies in the cohomology sequence associated to (*) with $t = 0$ that the hypothesis of the proposition are true on $\mathbb{P}^{n-1}$. Hence by induction on $n$, $H^2(\mathbb{P}^{n-1}, \bigwedge^q E_d(t)|_{\mathbb{P}^{n-1}}) = 0$ for $1 \leq q \leq k$ and $\forall t \geq 0$. From the cohomology sequence associated to (*) with $q = k$ we get that the map $H^2(\mathbb{P}^n, \bigwedge^k E_d(t-1)) \to H^2(\mathbb{P}^n, \bigwedge^k E_d(t))$ is surjective $\forall t \geq 0$ and the thesis follows easily. $\diamond$

Proof of Theorem 1.6
The implication '$\Rightarrow$' is a consequence of Proposition 1.5. To prove the converse, we may apply Proposition 1.7 and then Proposition 1.5 again.

Proposition 1.8 If $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies $N_p$ then $\varphi_{\mathcal{O}_{\mathbb{P}^m}(d)}$ satisfies $N_p$ $\forall m \leq n$.

Proof
It follows by the remark of section 2 of [G2] (which is an insight into representation theory). $\diamond$
2 - Necessary conditions on property $N_p$ for the Veronese embedding $\varphi O_{\mathbb{P}^n}(d)$

In this section we will prove the following:

**Theorem 2.1** The Veronese embedding $\varphi O_{\mathbb{P}^n}(d)$ does not satisfy $N_{3d-2}$ for $n \geq 2$, $d \geq 3$.

**Proof**
By Proposition 1.8, we can let $n=2$. By Theorem 1.6 and Serre duality it is enough to show that $H^0\left(\mathbb{P}^2, \Lambda^K E_d(d-3)\right) \neq 0$ with $K := \frac{d(d-3)}{2}$. So the theorem will follow from the following:

**Lemma 2.2** The bundle $\Lambda^q E_d(t)$ has a non-zero global section for $1 \leq q \leq N$, $q+1 \leq \left(\frac{n+t}{n}\right)$ and $t \geq 1$.

**Proof**
The exact sequence $0 \rightarrow \Lambda^q E_d \rightarrow \Lambda^q S^d V \otimes O_{\mathbb{P}^n} \rightarrow \Lambda^{q-1} E_d(d) \rightarrow 0$ implies that

$$H^0\left(\Lambda^q E_d(t)\right) = \text{Ker}\left(\Lambda^q S^d V \otimes S^t V \xrightarrow{\alpha_t} \Lambda^{q-1} S^d V \otimes S^{t+d} V\right).$$

Now, there is a Koszul complex

$$\rightarrow \Lambda^{q+1} S^d V \otimes O(t-d) \rightarrow \Lambda^q S^d V \otimes O(t) \xrightarrow{\alpha_t} \Lambda^{q-1} S^d V \otimes O(t+d) \rightarrow$$

with $\alpha_t = H^0(a_t)$. For $t \geq d$, global sections of $\Lambda^{q+1} S^d V \otimes O(t-d)$ will therefore give sections of $\Lambda^q E_d(t)$. In particular, for $d = t$, we get that for each family $s_0, \ldots, s_q$ of degree $d$ polynomials,

$$\sum_{i=0}^{q} (-1)^i s_0 \wedge \cdots \wedge \hat{s_i} \wedge \cdots \wedge s_q \otimes s_i$$

is in the kernel of $\alpha_d$. Let now be $1 \leq t < d$. If we can factor $s_i = uw_i$ with $u$ of degree $d-t$, then

$$\sum_{i=0}^{q} (-1)^i s_0 \wedge \cdots \wedge \hat{s_i} \wedge \cdots \wedge s_q \otimes w_i$$

is in the kernel of $\alpha_d$. Let now be $1 \leq t < d$. If we can factor $s_i = uw_i$ with $u$ of degree $d-t$, then
must be in the kernel of $\alpha_t$, and therefore defines a global section of $\bigwedge^q E_d(t)$. Thus to get a non-zero section of $\bigwedge^q E_d(t)$, it suffices to find $q + 1$ linearly independent polynomials of degree $t$, which is possible as soon as $q + 1 \leq \binom{n + t}{n}$.

Remark 2.3 The bundles $\bigwedge^q E_d$ are semistable (see [P], Proposition 5.6), so $H^0(\bigwedge^q E_d(t)) = 0$ if $\mu(\bigwedge^q E_d(t)) = t - \frac{qd}{N} < 0$. In particular,

$$H^0(\bigwedge^q E_d(t)) = 0 \, \forall t \leq 0.$$ 

3 - Conclusions

In this section we will fit our results into the literature. In particular, we will prove that:

Theorem 3.1 Let $d$ be an integer s.t. $d \geq 3$. Then the Veronese embedding $\varphi_{O_{\mathbb{P}^2}(d)} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^N$ satisfies property $N_p$ if and only if $0 \leq p \leq 3d - 3$. Moreover, if $d = 2$, the embedding $\varphi_{O_{\mathbb{P}^2}(2)} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ satisfies $N_p \, \forall p$.

We have the following:

Proposition 3.2 (M. Green, C. Birkenhake) Let $d \geq 2$ and $p = \begin{cases} 3d - 3 & \text{if } d \geq 3 \\ 2 & \text{if } d = 2 \end{cases}$. Then the complete Veronese embedding $\varphi_{O_{\mathbb{P}^2}(d)} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^N$ satisfies property $N_p$.

Proof
See [B1], Corollary 3.2.

The result follows also applying Theorem 3.b.7 of [G1] (which says that the minimal resolution of a Veronese variety restricts to the minimal resolution of its curve hyperplane section) and Theorem 4.a.1 of [G1] (which says that a line bundle of degree $2g + 1 + p$ on a curve of genus $g$ satisfies $N_p$). 

In the same way we get:

Lemma 3.3 The Veronese embedding $\varphi_{O_{\mathbb{P}^3}(3)} : \mathbb{P}^3 \hookrightarrow \mathbb{P}^{19}$ satisfies $N_6$. 

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Proof
The curve hyperplane section of the image of the cubic Veronese embedding of \( \mathbb{P}^3 \) is the space curve complete intersection of two cubics embedded by \(|\mathcal{O}_{\mathbb{P}^3}(3)|\) and it has genus 10. The result follows again applying Theorem 3.b.7 and Theorem 4.a.1 of [G1].

Lemma 3.4 The ideal \( \mathcal{I} \) of \( \varphi_{\mathbb{P}^2(2)}(\mathbb{P}^2) \) in \( \mathbb{P}^5 \) has the following resolution:

\[
\mathcal{O} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 3} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 8} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 6} \rightarrow \mathcal{I} \rightarrow 0.
\]

In particular the Veronese embedding \( \varphi_{\mathbb{P}^2(2)} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \) satisfies \( N_p \) for all \( p \).

Proof
Easy computation.

Proof of Theorem 3.1
By Proposition 3.2 and Lemma 3.4 we just need to show that if \( d \geq 3 \) then property \( N_p \) does not hold for \( p \geq 3d - 2 \). But this is exactly the bound coming from Theorem 2.1.

When \( d = 2 \) the minimal free resolution of the quadratic Veronese variety is known from the work of Joze{\d{e}}fiak, Pragacz and Weyman [JPW], in which they prove a conjecture made by Lascoux. As a corollary of the above paper we have the following result (which agrees with our Conjecture formulated in the Introduction):

Theorem 3.5 The quadratic Veronese embedding \( \varphi_{\mathbb{P}^n(2)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N \) satisfies \( N_p \) if and only if \( p \leq 5 \) when \( n \geq 3 \) and \( \forall p \) when \( n = 2 \).

The following nice characterization, probably well known, was found during discussions with E. Arrondo:

Theorem 3.6 The only (smooth) varieties in \( \mathbb{P}^n \) such that \( N_p \) holds for every \( p \geq 0 \) are the quadrics, the rational normal scrolls and the Veronese surface in \( \mathbb{P}^5 \).

Proof
Suppose \( X \) is a variety satisfying \( N_p \) for every \( p \geq 0 \). Then \( H^i(\mathcal{O}_X(t)) = 0 \) for \( t \geq 0 \) and \( 1 \leq i \leq \dim X - 1 \). Hence from Theorem 3.b.7 in [G1] it follows that the minimal free resolution of \( X \) restricts to the minimal resolution of its generic curve section \( C \). This implies that \( H^1(\mathcal{O}_C) = 0 \) and \( C \) is linearly normal, hence
$C$ is a rational normal curve. In particular, $X$ has minimal degree and we get the result.

We remark that the only Veronese varieties appearing in Theorem 3.6 are the rational normal curves and the Veronese surface in $\mathbb{P}^5$.

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