CLASSIFICATION OF SIMPLE LIE SUPERALGEBRAS IN CHARACTERISTIC 2

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ABSTRACT. All results concern characteristic 2. Two procedures that to every simple Lie algebra assign simple Lie superalgebras, most of the latter new, are offered. We prove that every simple finite-dimensional Lie superalgebra is obtained as the result of one of these procedures, so we classified all simple finite-dimensional Lie superalgebras modulo non-existing at the moment classification of simple finite-dimensional Lie algebras.

This result concerns Lie superalgebras considered naively, as vector spaces. To obtain classification of simple Lie superalgebras in the category of supervarieties, one should list the non-isomorphic deformations (results of deformations) with odd parameter. This problem is open bar several examples described in arXiv 0807.3054.

For Lie algebras, in addition to the known — “classical” — restrictedness, we introduce a (2,4)-structure on the two non-alternating series: orthogonal and of Hamiltonian vector fields. For Lie superalgebras, the classical restrictedness of Lie algebras has two analogs: (2|4)- and (2|2)-structures; one more analog — a (2,4)|4-structure on Lie superalgebras is the analog of (2,4)-structure on Lie algebras.

1. INTRODUCTION

The Lie algebras and superalgebras we consider in this paper are finite-dimensional (except for §7) over an algebraically closed field \( \mathbb{K} \) of characteristic \( p > 0 \), mostly, for \( p = 2 \). (Observe that neither finite-dimensionality of algebras, nor algebraic closedness of the ground field is needed in most of the constructions below.) In examples, we need (generalized) Cartan prolongation (for more examples, see [BGLLS2]) and Lie (super)algebra with indecomposable Cartan matrix over \( \mathbb{K} \) classified in [BGL2].

1.1. What restricted Lie algebra in characteristic 2 is. In 2005, P. Deligne wrote several comments to a paper by the two of us, see his Appendix in [LL]. In particular, a part of his advice is (in our words): “Over \( \mathbb{K} \), to classify ALL simple Lie (super)algebras and their representations are, perhaps, not very reasonable problems, and definitely very tough; investigate first the restricted case: it is related to geometry, meaningful and of interest”.

But what is restrictedness if \( p = 2 \)? For possible answers, see [2] where a reason for difference in answers for \( p = 2 \) and \( p > 2 \) is conjecturally attributed to the fact (13).

Having read [LL], several algebraists argued that in certain problems of interest to experts in the field, non-restricted Lie algebras are important, see [Kos]. Non-restricted Lie algebras are also needed even to describe the simple restricted algebras, see [BW, SF, S, BGP, Sk]. So in our quest for simple Lie (super)algebras we do not restrict ourselves to restricted algebras (whatever they are, cf. [WZ]).

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1.2. **On classification of simple Lie algebras over \( \mathbb{K} \).** For \( p > 7 \), Kostrikin and Shafarevich suggested (ca 1966) a construction of simple Lie algebras and a conjectural list of all restricted simple Lie algebras thus obtained, see [KS]. This list lucidly describes \( \mathbb{Z} \)-graded Lie algebras, but their deforms were clearly described and classified only much later, e.g., see [Sk] and [W].

For \( p > 7 \), Block and Wilson proved the restricted version of the KSh-conjecture, [BW]. This classification is implicit when dealing with deforms: simple deforms of the divergence-free algebras ([W]) and of hamiltonian type algebras ([SK]) were classified only several years after [BW] was published; explicit formulas of the \( p \)-structure were obtained only recently in the divergence-free case and (despite huge computational difficulties) in some of Hamiltonian cases, see [BKLS].

For \( p > 3 \), the KSh-conjecture amended by adding Melikyan algebras (for \( p = 5 \)) was proved only recently, see [S, BGP]. Classification of filtered deforms of Hamiltonian Lie algebras due to Skryabin, see [Sk], is a very important ingredient difficult to prove.

After his teacher (A.I.Kostrikin) died, A. Dzhumadildaev wrote the paper [KD], where he suggested to improve the Kostrikin-Shafarevich construction by identifying certain Lie algebras as “standard” examples, every simple Lie algebra being either on the list of “standard” algebras, or a deform (i.e., the result of a deformation) of one of the standard objects.

For \( p \leq 5 \), the stock of “standard” examples should contain not only simple Lie algebras, but also their nontrivial central extensions (or even semi-simple Lie algebras, see [SkT1]): the (family of) Melikyan algebras are described in [KD] as deforms of certain Poisson Lie algebras (in [KD], the proof of the fact that the Melikyan algebra is a deform of the Poisson algebra is correct, although the Poisson algebra is called Hamiltonian; so the computer verification, see [McZu], is not needed). The KSh-construction thus improved embraces \( p = 5 \).

In [KD], Dzhumadildaev also claimed that the Érmolaev and (unspecified) Skryabin algebras are deforms of examples obtained by the KSh-construction. For Érmolaev algebras, this claim was announced almost a decade earlier in [KuJa]; for the full set of explicit cocycles corroborating the claim of [KuJa], see [BGL4]. The importance of investigating deforms, and isomorphism classes and central extensions thereof, is now manifest; for classification of deforms of simple Lie algebras with symmetric root system (containing together with every root its opposite of the same multiplicity) and of rank \( \leq 8 \) (the vital cases in the inductive proof of the future classification), see [BGL3].

True deforms. Further study revealed the following serious obstacle seldom discussed before. In [DK], one of the first papers on deformations of simple vectorial Lie algebras, it was observed that although \( H^2(g; g) \neq 0 \), and the cocycles representing the nontrivial classes of \( H^2(g; g) \) are integrable, the deforms corresponding to some or all of these cocycles can be isomorphic to \( g \). We call such cocycles, and deformations corresponding to them, semitrivial, see [BLW]; a wide class of them is characterized in [BGLLS, BGL3]; for examples for \( p = 0 \), see [RI]. Thus, we are only interested in **true deforms**, not trivial or semi-trivial deforms, of “standard” examples.

For \( p = 3 \), in addition to the simple Lie algebras obtained by the KSh-construction, and deforms thereof, there are known other examples: Frank algebra, Érmolaev algebra, and several rather mysterious Skryabin algebras; each of these algebras is, actually, a member of a family depending on the shearing vector \( N \). These Lie algebras were interpreted (demystified) in [GL3], where the number of parameters on which the vector \( N \) depends was corrected as compared with previous descriptions [S]. These examples from [GL3] conjecturally complete the stock of “standard” objects for \( p = 3 \).

For \( p = 2 \), conjecturally, the set of “standard” Lie algebras is the union of examples from [Ei, SKT1, BGL2, LeP, ILL, BGLLS, BGLLS1, BGLLS2], and those that might be obtained by methods of [LSh] from nontrivial central extensions and algebras of derivations listed in [BGLL1], and from still to be listed algebras with non-symmetric root system.
1.2.1. **On classification of simple Lie superalgebras.** Lie superalgebras appeared not in high energy physics in 1974 or in solid state physics in 1980s, as we sometimes hear and read, but in 1930s in topology. They appeared in the form of — in modern terms — super Lie rings, i.e., over \( \mathbb{Z} \), or over finite fields. For example, the homotopy groups constitute a super Lie ring with respect to the Whitehead multiplication. These examples are solvable, hence without nice structure like that of simple ones, and this fact delayed the study of modular Lie superalgebras.

The case of \( p = 0 \) illustrates how much more difficult classification of simple Lie superalgebras is as compared with that of simple Lie algebras of the same type.

The finite-dimensional simple Lie superalgebras were classified, thanks to several groups of researchers, inside two years or so, see [Kapp]. For a review not only of classification (with due references to results of Kaplansky, Djokovich–Hochschild, and Scheunert–Nahm–Rittenberg), but of impressive at that time results on the basics of representation theory, see [K0].

For classification of simple \( \mathbb{Z} \)-graded Lie superalgebras of polynomial growth a big chunk of the problem (in particular, of vectorial Lie superalgebras with polynomial and Laurent coefficients) is solved on hundreds of pages, see [LSH, Sh5, Sh14, GLS1, HS] and [K, K10, CCK, CKa] and references therein, but the conjecture formulated in [LSS] embracing all types of Lie superalgebras (\( \mathbb{Z} \)-graded of polynomial growth) is still open.

For \( p > 0 \), the classification of simple Lie superalgebras is even more difficult than classification of simple Lie algebras for \( p > 0 \), the smaller \( p \), the more difficult. And for any \( p > 0 \) it is more difficult than classification of simple Lie superalgebras for \( p = 0 \). For a conjectural classification of simple Lie superalgebras, rather complicated even for \( p \) even for restricted Lie superalgebras, see [BGLLS1]. (The absence of classification is no wonder: even the classification of restricted simple Lie algebras for \( p > 3 \) is implicit to this day: “The problem of restrictedness is approached. …[But] the family of Hamiltonian algebras … is not yet handable”, see [S] v.1, p.357.)

So the main result of this paper is unexpected: it reduces the list of “standard” examples needed to construct simple Lie superalgebras for \( p = 2 \) — presumably the most difficult case of all classifications spoken above — to the list of “standard” examples for simple Lie algebras.

1.3. **Summary of our results.** In this note we briefly overview the classification problem of simple Lie (super)algebras over \( K \) and formulate our results expounding [LeD], where A. Lebedev briefly described the following three phenomena existing for \( p = 2 \) only:

1) The two methods, see subsec. 3.2.1 (queerification) and 3.3 (“method 2”), producing simple Lie superalgebras from every simple Lie algebra. Here we prove that every simple Lie superalgebra is obtained by one of these procedures.

2) On Lie algebras: in addition to the known, “classical”, restrictedness there exists a \( (2,4) \)-structure, e.g., on the orthogonal Lie algebras \( o(2n+1) \) and the Lie algebras of non-alternating Hamiltonian vector fields \( \mathfrak{h}_I(2n+1;\mathbf{1}) \), where \( \mathbf{1} := (1,\ldots,1) \) is the shearing vector of heights of indeterminates.

3) On Lie superalgebras: there are three analogs of the classical restrictedness, namely, (a) the classical (direct) one, i.e., \( (2|4) \)-structure, (b) the \( (2|2) \)-structure, (c) the \( (2,4)|4 \)-structure. Hereafter by restricted Lie (super)algebra we mean a classically restricted one.

In §2 we consider various restrictednesses, with more details and examples than in [BGL2].

In §3 we describe the two methods (queerification and “method 2”) producing new simple Lie superalgebras from every simple Lie algebra if \( p = 2 \).

In §4 we prove our main Theorem: **if \( p = 2 \), every simple finite-dimensional Lie superalgebra is obtained from a simple Lie algebra by means of either queerification, or “method 2”**. So we have obtained classification of simple Lie superalgebras modulo classification of simple Lie algebras. Here “Lie superalgebra” is understood naively, as a \( \mathbb{Z}/2 \)-graded algebra satisfying certain
identities. To classify Lie superalgebras considered in the category of supervarieties we need to describe odd parameters of deformations of naively understood Lie superalgebras, see [BGL3].

Observe a “braking news” result: the list of $\mathbb{Z}/2$-gradings of simple Lie algebras — a vital ingredient for “method 2” — is much longer for $p = 2$ than for the namesakes of these algebras (“the same”, in a sense, algebras) considered for $p \neq 2$, see [KrLe].

In [5] and [6] examples of queerifications are given with details further clarifying the mechanism which does not exist for $p \neq 2$.

In [7] examples illustrating numerous new simple Lie superalgebras obtained by means of “method 2” from vectorial Lie algebras are explicitly described.

2. SEVERAL VERSIONS OF RESTRICTEDNESS IN CHARACTERISTIC 2

Here is an expounded with new results version of the respective sections from [LeD, BGL2].

2.1. Restrictedness on Lie algebras. Let the ground field $\mathbb{K}$ be of characteristic $p > 0$, and $\mathfrak{g}$ a Lie algebra. For every $x \in \mathfrak{g}$, the operator $(\text{ad}_x)^p$ is a derivation of $\mathfrak{g}$. If this derivation is an inner one, i.e., there is a map (called $p$-structure) $[p] : \mathfrak{g} \rightarrow \mathfrak{g}$, $x \mapsto x^{[p]}$ such that

\begin{align*}
1) & \quad [x^{[p]}, y] = (\text{ad}_x)^p(y) \quad \text{for any } x, y \in \mathfrak{g}, \\
2) & \quad (ax)^{[p]} = a^p x^{[p]} \quad \text{for any } a \in \mathbb{K}, x \in \mathfrak{g}, \\
3) & \quad (x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{1 \leq i \leq p-1} s_i(x,y) \quad \text{for any } x, y \in \mathfrak{g},
\end{align*}

where $is_i(x,y)$ is the coefficient of $\lambda^{i-1}$ in $(\text{ad}_{\lambda x+y})^{p-1}(x)$, then the Lie algebra $\mathfrak{g}$ is said to be restricted or having a $p$-structure.

2.1.1. Remarks. 1) If the Lie algebra $\mathfrak{g}$ is centerless, then the condition (1) implies (2) and (3).

A $p$-structure on a given Lie algebra $\mathfrak{g}$ does not have to be unique; all $p$-structures on $\mathfrak{g}$ agree modulo center. Hence, on any simple Lie algebra, there is not more than one $p$-structure.

2) According to [SF, Th. 2.3, p. 71], the following condition, due to Jacobson, is sufficient for a Lie algebra $\mathfrak{g}$ to have a $p$-structure: for a basis $\{g_i\}_{i \in I}$ of $\mathfrak{g}$, there exist elements $g_i^{[p]}$ such that

$$[g_i^{[p]}, y] = (\text{ad}_{g_i})^p(y) \quad \text{for any } y \in \mathfrak{g}.$$ 

2.1.2. Restricted modules. A $\mathfrak{g}$-module $M$ over a restricted Lie algebra $\mathfrak{g}$, and the representation $\rho$ defining $M$, are said to be restricted or having a $p$-structure if

$$\rho(x^{[p]}) = (\rho(x))^p \quad \text{for any } x \in \mathfrak{g}.$$ 

2.2. Lie superalgebras. Naively, the definition of Lie superalgebra is the same for any $p \neq 2$. Let us point at the subtleties for $p = 2$. For any $p$, a Lie superalgebra is a superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that the even part $\mathfrak{g}_0$ is a Lie algebra, the odd part $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module (made into the two-sided one by anti-symmetry, i.e., $[y, x] = -[x, y]$ for any $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_1$), and a squaring defined on $\mathfrak{g}_1$ as a map $S^2(\mathfrak{g}_1) \rightarrow \mathfrak{g}_0$:

$$x \mapsto x^2 \in \mathfrak{g}_0 \quad \text{such that } (ax)^2 = a^2 x^2 \text{ for any } x \in \mathfrak{g}_1 \text{ and } a \in \mathbb{K}, \text{ and } [x, y] := (x + y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_1 \text{ with values in } \mathfrak{g}_0.$$ 

(This extra requirement on squaring is needed, say, over $\mathbb{Z}/2$ where not any quadratic form that vanishes at the origin yields a bilinear form $[\cdot, \cdot]$.)
The Jacobi identity involving odd elements takes the following form:

\[ [x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_1, y \in \mathfrak{g}_0, \]
\[ [x^2, x] = 0 \text{ for any } x \in \mathfrak{g}_1. \]

For any Lie superalgebra \( \mathfrak{g} \), its derived algebras are defined to be (for \( i \geq 0 \))

\[ \mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = \begin{cases} [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] & \text{for } p \neq 2, \\ [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \operatorname{Span}\{g^2 \mid g \in \mathfrak{g}_1^{(i)}\} & \text{for } p = 2. \end{cases} \]

2.3. The \( p \mid 2p \)-structure or restricted Lie superalgebra. For a Lie superalgebra \( \mathfrak{g} \) of characteristic \( p > 0 \), let the Lie algebra \( \mathfrak{g}_0 \) be restricted and

\[ (x^p, y) = (\text{ad}_x)^p(y) \text{ for any } x \in \mathfrak{g}_0, y \in \mathfrak{g}. \]

This gives rise to the map (recall that the bracket of odd elements is the polarization of the squaring \( x \mapsto x^2 \))

\[ [2p] : \mathfrak{g}_1 \to \mathfrak{g}_0, \ x \mapsto (x^2)^p, \]

satisfying the condition

\[ [x^{2p}, y] = (\text{ad}_x)^{2p}(y) \text{ for any } x \in \mathfrak{g}_1, y \in \mathfrak{g}. \]

The pair of maps \([p]\) and \([2p]\) is called a \( p \)-structure (or, sometimes, a \( p \mid 2p \)-structure) on \( \mathfrak{g} \), and \( \mathfrak{g} \) is said to be restricted. It suffices to determine the \( p \mid 2p \)-structure on any basis of \( \mathfrak{g} \); on simple Lie superalgebras there are not more than one \( p \mid 2p \)-structure.

2.3.1. Remark. If (4) is not satisfied, the \( p \)-structure on \( \mathfrak{g}_0 \) does not have to generate a \( p \mid 2p \)-structure on \( \mathfrak{g} \); even if the actions of \((\text{ad}_x)^p\) and \(\text{ad}_{x^p}\) coincide on \( \mathfrak{g}_0 \), they do not have to coincide on the whole of \( \mathfrak{g} \). For example, consider \( p = 2 \) and \( \mathfrak{g} = \mathfrak{osp}(1|2) \) (for the definition of ortho-orthogonal Lie superalgebras, see [LeP, BGL2]; for clarity, we write \( \mathfrak{osp}(1|2) \) or \( \mathfrak{g}(1)(A) \) instead of \( \mathfrak{osp}(3; a) \) or \( \mathfrak{g}(A) \) and the like) with basis \( \{X^2, X_-, H, X_+, X^2_+\} \), where \( X^2, H, \) and \( X^2_+ \) are even elements while \( X_- \) and \( X_+ \) are odd ones, with the relations (other bracket being equal to 0)

\[ [H, X_\pm] = X_\pm; \quad [X_+, X_-] = [X^2_+, X^2_-] = H. \]

We can define a 2-structure on \( \mathfrak{g}_0 \simeq \mathfrak{sl}(2) \), which is nilpotent for \( p = 2 \), by setting

\[ (X^2)^2 = H; \quad H^{[2]} = H; \quad (X^2_+)^2 = 0, \]

and extending it to the whole \( \mathfrak{g}_0 \) by properties (2) and (3). This 2-structure on \( \mathfrak{g}_0 \) can not be extended to a 2|4-structure on \( \mathfrak{g} \), since, for example,

\[ [X^2, [X^2, X_-]] = 0 \neq ((X^2)^2, X_-) = X_-. \]

2.3.2. Restricted modules. A \( \mathfrak{g} \)-module \( M \) corresponding to a representation \( \rho \) of the restricted Lie superalgebra \( \mathfrak{g} \) is said to be restricted or having a \( p \mid 2p \)-structure if

\[ \rho(x^p) = (\rho(x))^p \text{ for any } x \in \mathfrak{g}_0, \]
\[ \rho(x^{2p}) = (\rho(x))^{2p} \text{ for any } x \in \mathfrak{g}_1. \]
2.4. **On 2|2-structures on Lie superalgebras.** Let \( p = 2 \), a Lie superalgebra \( \mathfrak{g} \) have a 2|4-structure, and \( F(\mathfrak{g}) \) be the Lie algebra one gets from \( \mathfrak{g} \) by forgetting the squaring and considering only brackets by setting \([x,x] := 2x^2 = 0 \) for \( x \) odd. Then \( F(\mathfrak{g}) \) has a 2|2-structure given by

the “2” part of 2|4-structure on \( \mathfrak{g}_0 \);

the squaring on \( \mathfrak{g}_1 \), i.e., \( x^{[2]} := x^2 \);

(5) the rule \((x + y)^{[2]} := \begin{cases} \frac{x^{[2]} + y^{[2]} + [x,y]}{2} & \text{if } x, y \in \mathfrak{g}_0, \\ \frac{x^{[2]} + y^{[2]} + [x,y]}{2} & \text{if } x \in \mathfrak{g}_1, y \in \mathfrak{g}_0, \\ \frac{x^{[2]} + y^{[2]} + [x,y]}{2} & \text{if } x, y \in \mathfrak{g}_1. \end{cases} \)

(Actually, the first and the third cases in (5) are redundant. If \( x \) and \( y \) are both in \( \mathfrak{g}_0 \) or both in \( \mathfrak{g}_1 \), then \( x + y \) is homogenous, and \((x + y)^{[2]} \) in \( F(\mathfrak{g}) \) is already given by \((x + y)^{[2]} \) or \((x + y)^2 \), correspondingly.) So one can say that if \( p = 2 \), then the restricted Lie superalgebra (i.e., the one with a 2|2-structure) also has a 2|2-structure which is defined even on inhomogeneous elements (unlike \( p|2p \)-structures). In future, for Lie superalgebras with 2|2-structure, we write \( x^{[2]} \) instead of \( x^2 \) for any odd or inhomogeneous \( x \in \mathfrak{g} \). The analog of sufficient condition 2) of Remarks 2.1.1 holds.

2.4.1. **Restricted modules.** A \( \mathfrak{g} \)-module \( M \) corresponding to a representation \( \rho \) of the Lie superalgebra \( \mathfrak{g} \) with 2|2-structure is said to be **restricted** or having a 2|2-structure if

\[ \rho(x^{[2]}) = (\rho(x))^2 \quad \text{for any } x \in \mathfrak{g}. \]

2.5. **Restrictedness of Lie (super)algebras with Cartan matrix, and of their relatives (the derived algebras, central extensions, and quotients thereof modulo center).**

2.5.1. **Lie (super)algebras with Cartan matrix.** Speaking about Lie (super)algebras \( \mathfrak{g} = \mathfrak{g}(A) \) with an \( n \times n \) Cartan matrix \( A = (A_{ij}) \), recall (see [LeD, BGL2]) that any nonzero element \( \alpha \in \mathbb{R}^n \) is called a root if the homogeneous subspace \( \mathfrak{g}_\alpha \) of \( \mathfrak{g} \) with grade (weight) \( \alpha \) is nonzero. Let \( R \) be the set of all roots of \( \mathfrak{g} \) and \( \mathfrak{h} \) the maximal torus.

2.5.1a. **Proposition.** 1) If \( p > 2 \) (or \( p = 2 \) and \( A_{ii} \neq 1 \) or 1 for any \( i \)) and \( \mathfrak{g}(A) \) is a Lie (super)algebra, then \( \mathfrak{g}(A) \) has a \( p \)-structure (resp. \( p|2p \)-structure) such that

\[ (x_\alpha)^{[p]} = 0 \quad \text{for any even } \alpha \in R \text{ and } x_\alpha \in \mathfrak{g}_\alpha, \]

(6) \[ (x_\alpha)^{[2p]} = 0 \quad \text{for any odd } \alpha \in R \text{ and } x_\alpha \in \mathfrak{g}_\alpha, \]

\[ h_i^{[p]} \subset \mathfrak{h}. \]

2) If \( A_{ij} \in \mathbb{Z}/p \) for all \( i, j \), then the derived Lie (super)algebra \( \mathfrak{g}^{(1)}(A) \) inherits the \( p \)-structure (resp. \( p|2p \)-structure) of \( \mathfrak{g}(A) \), and we can make the 3rd line of eq. (6) precise:

\[ h_i^{[p]} = h_i \quad \text{for any basis element } h_i \in \mathfrak{h}. \]

3) The quotient modulo center of a Lie (super)algebra \( \mathfrak{g} \) with a \( p \)-structure (resp. \( p|2p \)-structure) always inherits the \( p \)-structure (resp. \( p|2p \)-structure) of \( \mathfrak{g} \).

**Proof.** 1) The statement of line 1 of eq. (6) is contained in the proof of [S] v.1, Th.7.2.2, proof of the statement of line 2 is similar.

In our definition of roots, see [BGL2], instead of nonexisting pairing \((\gamma, h)\), where \( \gamma \in \mathbb{R}^n \) is a root and \( h \in \mathfrak{h} \), we introduce a function

\[ \text{ev} : R \times \mathfrak{h} \to \mathbb{K}, \quad \text{ev}(\gamma, h) = \text{the eigenvalue of } \text{ad}_h \text{ on } \mathfrak{g}_\gamma. \]
Clearly, $\text{ev}$ is linear in the second argument in the usual sense, and it is $\mathbb{Z}$-linear, i.e., additive, in the first argument in the sense that

$$
\text{ev} \left( \sum_{1 \leq i \leq k} c_i \gamma_i, h \right) = \sum_{1 \leq i \leq k} c_i \text{ev}(\gamma_i, h) \quad \text{for any } c_i \in \mathbb{Z} \text{ and any roots } \gamma_i \text{ such that } \sum c_i \gamma_i \in R,
$$

here the $c_i$ are considered as elements of $\mathbb{Z}$ or $\mathbb{R}$ in the left-hand side and as elements of $\mathbb{K}$ in the right-hand side. The other way round, if a function $f : R \to \mathbb{K}$ is such that

$$
f \left( \sum_{1 \leq i \leq k} c_i \gamma_i \right) = \sum_{1 \leq i \leq k} c_i f(\gamma_i) \quad \text{for any } c_i \in \mathbb{Z} \text{ and any roots } \gamma_i \text{ such that } \sum c_i \gamma_i \in R,
$$

then

$$
\text{ev}(\gamma, h) = f(\gamma)
$$

for any root $\gamma$.

To prove the claim (9), it suffices to choose $h$ so that the condition $\text{ev}(\alpha_j, h) = f(\gamma_j)$ is satisfied for any simple positive root $\alpha_j$. Such an $h$ exists because the roots $\alpha_j$ are linearly independent. Because any root can be represented as an integral linear combination of simple (positive) roots and due to the condition (8), this condition is satisfied for any root. This demonstrates validity of the statement of line 3 in (6).

Note that the derived Lie superalgebra $\mathfrak{g}^{(1)}(A)$ may fail to have property (9).

2) Note that for any integers $c_1, \ldots, c_k$ and any roots $\gamma_1, \ldots, \gamma_k$ such that $\gamma = \sum_{1 \leq i \leq k} c_i \gamma_i$ is a root, the eigenvalue of $(\text{ad}_h)^p$ on $\mathfrak{g}_\gamma$ is equal to

$$
\text{ev} \left( \sum_{1 \leq i \leq k} c_i \gamma_i, h \right)^p = \left( \sum_{1 \leq i \leq k} c_i \text{ev}(\gamma_i, h) \right)^p = \sum_{1 \leq i \leq k} (c_i \text{ev}(\gamma_i, h))^p = \sum_{1 \leq i \leq k} c_i (\text{ev}(\gamma_i, h))^p,
$$

since $c^p = c$ for any $c \in \mathbb{Z}/p$, and $\left( \begin{array}{c} p \\ m \end{array} \right) \equiv 0 \mod p$ for $m \neq 0 \mod p$. In other words, the function $\text{ev}$ on $\mathfrak{h}$ whose value at $h \in \mathfrak{h}$ is the eigenvalue of $(\text{ad}_h)^p$ on $\mathfrak{g}_\gamma$, satisfies the condition (8). Thus, there really exists an element $h^{[p]} \in \mathfrak{h}$ such that $\text{ad}_{h^{[p]}} = (\text{ad}_h)^p$.

If $A_{ij} \in \mathbb{Z}/p$ for all $i, j$, we consider $A_{ij}$ as integers from the set $\{0, \ldots, p - 1\}$, and apply Fermat’s little theorem.

If $A_{ij} \not\in \mathbb{Z}/p$ for some $i, j$, then $\mathfrak{g}^{(1)}(A)$ may have no $p$-structure (resp. $p\mid 2p$-structure) even if $\mathfrak{g}(A)$ has one.

3) Factorization modulo center fixes one $p$-structure (resp. $p\mid 2p$-structure) inherited from the nonfactorized algebra which may have several such structures. \(\square\)

2.5.2. **Examples.** 1) Observe that the center $c$ of the Lie algebra $\mathfrak{m}\mathfrak{t}(3; a)$ with Cartan matrix

$$A = \begin{pmatrix}
0 & a & 0 \\
a & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
$$

where $a \neq 0, 1$, is spanned by $ah_1 + h_3$ in the standard numeration of Chevalley generators $x_i^\pm$ and related elements $h_i := [x_i^+, x_i^-]$. The 2-structure on $\mathfrak{m}\mathfrak{t}(3; a)$ is given by the conditions $(x_\alpha)^2 = 0$ for any root vector $x_\alpha$, and the conditions (10).

For the matrix $B = (1, 0, 0)$ supplementing the Cartan matrix $A$ as explained in [BGL2] (where in a copy of eq. (10) and similar ones there are typos: $\text{ad}_h^a$ and $\text{ad}_d^a$ should be $h^a_i$ and $d^a_i$),
respectively) and the grading operator $d$ also defined in [BGL2], set:

\begin{align}
    h_1^{[2]} &= (1+at)h_1 + th_3 \quad \equiv h_1 \pmod{c}, \\
    h_2^{[2]} &= ah_1 + ah_2 + th_3 + (1+a)d \quad \equiv ah_2 + (1+a)d \pmod{c}, \\
    h_3^{[2]} &= (at+a^2)h_1 + th_3 \quad \equiv a^2h_1 \pmod{c}, \\
    d^{[2]} &= ah_1 + th_3 + d \quad \equiv d \pmod{c}.
\end{align}

(10)

Observe that the simple Lie algebra $\mathfrak{sl}(1;3)/c$ has no 2-structure.

2) The 2-structure on $\mathfrak{sl}(4;a)$ with Cartan matrix

\[
\begin{pmatrix}
0 & a & 0 & 0 \\
-a & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

where $a \neq 0,1$, is given by

the conditions $(x_\alpha)^{[2]} = 0$ for any root vectors $x_\alpha$, and

\begin{align}
    h_1^{[2]} &= ah_1 + (1+a)h_4, \\
    h_2^{[2]} &= ah_2, \\
    h_3^{[2]} &= h_3, \\
    h_4^{[2]} &= h_4.
\end{align}

(11)

2.5.3. **Proposition.** The superizations of simple Lie algebras of the form $\mathfrak{g}(A)$, and non-simple ones of the form $\mathfrak{g}(A)/c$, see [BGL2], have 2|4-structures.

**Proof.** The 2|4-structure is given by means of the first two lines of eq. (6) and — instead of the third line of eq. (6) — expressions (10), (11), or (7) if $A_{ij} \in \mathbb{Z}/p$ for all $i, j$. \qed

2.6. **(2, 4)- and (2, -)-structures on Lie algebras.** Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a $\mathbb{Z}/2$-grading of a Lie algebra (not superalgebra) $\mathfrak{g}$. We say that $\mathfrak{g}$ has a (2, -)-structure, if there is a map

\[ [2]: \mathfrak{g}_+ \to \mathfrak{g}_+, \quad x \mapsto [x]^{[2]} \]

such that (for simplicity, we consider the case of centerless $\mathfrak{g}$)

\[ [x^{[2]}, y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_+, \quad y \in \mathfrak{g}, \]

but there is no 2-structure on $\mathfrak{g}$.

A (2, 4)-structure (do not confuse with 2|4-structure!) on a $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is a pair: a 2-structure on $\mathfrak{g}_+$, and a map

\[ [4]: \mathfrak{g}_- \to \mathfrak{g}_+, \quad x \mapsto x^{[4]} \]

such that $[x^{[4]}, y] = [x, [x, [x, [x, y]]]]$ for any $x \in \mathfrak{g}_-,$ $y \in \mathfrak{g}$.

The analog of sufficient condition 2) of Remarks 2.1.1 holds.

In the examples we know, the (2, 4)-structure is related with the following fact:

Whereas the maximal dimension of the irreducible $\mathfrak{sl}(2)$-module with highest weight is equal to $p$ if $p > 2$, the maximal dimension of irreducible module with highest weight over the simple 3-dimensional Lie algebra for $p = 2$ is equal to 4, see [D].

2.6.1. **Examples.** 1) If indecomposable symmetrizable $n \times n$-matrix $A$ is such that

\[ A_{mn} = 1; \quad A_{ii} = 0 \text{ for } i < n, \]

then $\mathfrak{g}(A)$ has no 2-structure but has a (2, 4)-structure with the $\mathbb{Z}/2$-grading given on Chevalley generators by setting:

\[ \deg(h) = 0; \quad \deg(x_n^+) = 1; \quad \deg(x_i^+) = 0 \text{ for } i < n. \]
In particular, the Lie algebra $g = o(2n+1)$ with Cartan matrix

\[
\begin{pmatrix}
\vdots & \cdots & \cdots & \cdots \\
\ddots & 0 & 1 & 0 \\
& 1 & 0 & 1 \\
& \ddots & \ddots & \ddots \\
& & 0 & 1 & 1
\end{pmatrix}
\]

as the algebra of matrices of the form (recall that $ZD(n)$ is the space of symmetric matrices with zeros on the main diagonal)

\[
\begin{pmatrix}
A & X & B \\
Y^T & 0 & X^T \\
C & Y & A^T
\end{pmatrix}, \quad A \in gl(n); \ B, C \in ZD(n);
X, Y \text{ are column-vectors.}
\]

Then $g_+$ consists of matrices with $X = Y = 0$ and the map $[2]$ is the squaring of matrices, while $g_-$ consists of matrices with $A = B = C = 0$, and the map $[4]$ is rising matrices to the fourth power.

2) Let $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$, and one more indeterminate $x$ be indeterminates of the generating functions of the Lie algebra $\mathfrak{h}_I(2n+1; \mathbb{A})$. On it, there is a $(2,4)$-structure, see eq. (12), with respect to the grading

\[
\deg p_i = \deg q_i = \bar{0} \text{ for all } i, \deg x = \bar{1}.
\]

3) Example of a $\mathbb{Z}/2$-graded Lie algebra $g$ which has a $(2, -)$-structure, but not a $(2,4)$-structure: let $g$ be spanned by elements $a_n$, where $n \in \mathbb{Z}$, and $b$. Let the commutation relations be

\[
[a_m, a_n] = 0, \quad [b, a_n] = a_{n-1} \text{ for any } m, n \in \mathbb{Z}.
\]

Then $g$ has a $\mathbb{Z}/2$-grading such that $\deg a_n = n \mod 2$ and $\deg b = \bar{1}$. This $g$ has a $(2, -)$-structure which maps every even element to 0, but no element can be $b^{[4]}$.

2.6.2. $(2,4)$-structure on Lie superalgebras. The Lie superalgebra $o^{1(1)}_{11}(2k_0 + 1|2k_1)$ has, similarly to the above, a $(2,4)$-structure consisting of the rising to the fourth power defined on the odd part while on the even part a $(2,4)$-structure is defined, for any $y \in o^{1(1)}_{11}(2k_0 + 1|2k_1)$, satisfying the following conditions:

\[
\begin{align*}
\text{ad}_{x^{[2]}}(y) &= (\text{ad}_x)^2(y) \text{ for any } x \in (o^{1(1)}_{11}(2k_0 + 1|2k_1)_+)^0 \\
\text{ad}_{x^{[4]}}(y) &= (\text{ad}_x)^4(y) \text{ for any } x \in (o^{1(1)}_{11}(2k_0 + 1|2k_1)_- \text{ or } (o^{1(1)}_{11}(2k_0 + 1|2k_1)_+)^1
\end{align*}
\]

The analog of sufficient condition 2) of Remarks [2.1.1] holds.

2.6.3. $(2,4)$-restricted modules. A $g$-module $M$ corresponding to a representation $\rho$ of the Lie (super)algebra $g$ with $(2,4)$-structure (resp. $(2,4)$-structure) is said to be $(2,4)$-restricted (resp. $(2,4)$-structure) or having a $(2,4)$-structure (resp. $(2,4)$-structure) if

\[
\begin{align*}
\rho(x^{[2]}) &= (\rho(x))^2 \text{ for any } x \in (g_+)_0, \\
\rho(x^{[4]}) &= (\rho(x))^4 \text{ for any } x \in g_- \text{ or } (g_+)_{\bar{1}}.
\end{align*}
\]

2.7. Restricted vectorial Lie (super)algebras. In Proposition below, by "vectorial Lie (super)algebra" we only mean the one whose the Weisfeiler filtration and grading corresponding to the maximal subalgebra of least codimension (i.e., the degree of each indeterminate is nonzero). For the list of known vectorial Lie (super)algebras having simple derived, see [GLS, SK, BGL1, BGLS1, BGLLS2].

Let $g(sdim; N)$ or $g(a; N|b)$ denote the vectorial Lie (super)algebra with a “family name” $g$ realized on $sdim = (a|b)$ indeterminates, of which $a$ are even and $b$ are odd, and with shearing vector $N$; if $b = 0$ the notation usually shrinks to $g(dim; N)$.
2.7.1. **Fact.** Let vectorial Lie (super)algebra $\mathfrak{g}(\text{sdim}; \mathbb{1})$ be the prolong, i.e., the result of generalized (perhaps, partial) Cartan prolongation, see [Shch], of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a restricted Lie (super)algebra and $\mathfrak{g}_- = \bigoplus_{-d \leq i < 0} \mathfrak{g}_i$ is a restricted $\mathfrak{g}_0$-module. Then $\mathfrak{g}(\text{sdim}; \mathbb{1})$ is restricted. For the known simple derived of vectorial Lie (super)algebras the restricted structure is given by the following expressions, where $\mathfrak{h}$ is a maximal torus of $\mathfrak{g}_0$:

\begin{equation}
\mathfrak{h}^{[p]} \subset \mathfrak{h}, \text{ and if the structure constants lie in } \mathbb{Z}/p, \text{ then }
\end{equation}

\begin{align}
\mathfrak{h}^{[p]}_i &= h_i \text{ for the basis elements } h_i \text{ of } \mathfrak{h}; \\
\mathfrak{w}^{[p]} &= 0 \text{ (resp. } \mathfrak{w}^{[2p]} = 0) \text{ for the other even (resp. odd) weight elements } \mathfrak{w} \text{ of the basis of } \mathfrak{g} \text{ with weights relative a maximal torus of } \mathfrak{g}. \\
\end{align}

2.7.2. **Proposition.** 1) For $p > 0$, the vectorial Lie superalgebra $\mathfrak{g}(\text{sdim}; N)$ is not restricted if $N \neq \mathbb{1}$. 

2) If $\mathfrak{g} := \mathfrak{g}(\text{sdim}; \mathbb{1})$ is restricted, and the $i$-th derived $\mathfrak{g}^{(i)}$ of $\mathfrak{g}$ contains the maximal torus of $\mathfrak{g}$, then $\mathfrak{g}^{(i)}$ is restricted and formulas (14) are applicable.

**Proof.** Items 1) and 2): for Lie algebra case, see [S, v.1, Th. 7.2.2]; the super case is analogous. □

2.7.3. **Remarks.** 1) The restrictedness was established, so far, in the following cases of simple Lie algebras: for $p > 7$, by Block and Wilson ([BW]); for $p > 5$, see [S, v.1, Th. 7.2.2]; for the Lie algebras with indecomposable Cartan matrix for any $p$, see [BGL2, Prop. 2.5.1a].

2) Let $\tilde{x}_i$ be the highest possible (divided) power of $x_i$, and $\tilde{x} = \prod \tilde{x}_i$. Let

$$\mathfrak{svect}(\text{dim}; N) := \{(1 - \tilde{x})y \text{ for any } y \in \mathfrak{svect}(\text{dim}; N)\}.$$ 

The Lie algebra $\mathfrak{svect}(\text{dim}; \mathbb{1})$ is restricted, but expression (14) should be modified: the only difference between bases of $\mathfrak{svect}(\text{dim}; \mathbb{1})$ and $\mathfrak{svect}(\text{dim}; \mathbb{1})$ is that instead of $\partial_1$ we have $X_i := (1 - \tilde{x})\partial_1$ and while $\partial_i^{[p]} = 0$ we have $X_i^{[p]} = -\partial_i^{p-1}\tilde{x}$, see [BKLLS].

3. **The Two Methods of Superization for $p = 2$**

3.1. **Qu eerification for $p \neq 2$.** Let $\mathcal{A}$ be an associative algebra; let $\mathcal{A}_L$ be the Lie algebra with the same space as $\mathcal{A}$ and the multiplication being defined by the commutator instead of the dot product. The space of the Lie superalgebra $q(\mathcal{A})$, which we call the queerification of $\mathcal{A}$, is $\mathcal{A}_L \oplus \Pi(\mathcal{A})$, where $\Pi$ is the change of parity functor, so $q(\mathcal{A})_0 = \mathcal{A}_L$ and $q(\mathcal{A})_1 = \Pi(\mathcal{A})$, with the multiplication given by the following expressions

\begin{equation}
[x, y] = xy - yx; \quad [x, \Pi(y)] = \Pi(xy - yx); \quad [\Pi(x), \Pi(y)] = xy + yx \quad \text{for any } x, y \in \mathcal{A}.
\end{equation}

The term “queer”, now classical, is taken after the Lie superalgebra $q(n) := q(\text{Mat}(n))$, a “queer” analog (as explained in [LSos]) of $\mathfrak{gl}(n)$, where $\text{Mat}(n)$ is the associative algebra of $n \times n$ matrices. We express the elements of the Lie superalgebra $\mathfrak{g} = q(n)$ by means of a pair of matrices

\begin{equation}
(A, B) \longleftrightarrow \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in q(n), \text{ where } A, B \in \mathfrak{gl}(n).
\end{equation}

We will similarly denote by pairs $(A, B)$, where $A, B \in \mathcal{A}$ the elements of $q(\mathcal{A})$. The brackets between these basis elements are as follows:

\begin{equation}
[(A_1, 0), (A_2, 0)] = ([A_1, A_2], 0), \quad [(A, 0), (0, B)] = (0, [A, B]), \quad [(0, B_1), (0, B_2)] = (B_1B_2 + B_2B_1, 0).
\end{equation}
3.1.1. The only simple Lie superalgebras related with queerification for \( p \neq 2 \) are \( \text{psq}(n) \) for \( n > 2 \). Let \( \text{sq}(n) := q(n)^{(1)} \) denote the subsuperalgebra of queertraceless matrices, where the queer trace is \( \text{qtr} : (A, B) \mapsto \text{tr} B \). The Lie superalgebras \( q(n) \) and \( \text{sq}(n) \) are specifically “super” analogs of the general Lie algebra \( \mathfrak{gl}(n) \) and its special (traceless) subalgebra \( \mathfrak{s}(n) \); we define their projectivizations to be \( \text{pq}(n) := q(n)/\mathbb{K}1_{2n} \) and \( \text{psq}(n) := \text{sq}(n)/\mathbb{K}1_{2n} \).

3.2. Queerification for \( p = 2 \). For \( q = q(\mathcal{A}) \), where \( \mathcal{A} \) is an associative algebra, the multiplication is defined by the expressions (17), the bracket of odd elements being polarization of squaring of the odd elements:

\[
(0, B)^2 = (B^2, 0).
\]

If \( p = 2 \), then we can queerify any restricted Lie algebra \( \mathfrak{g} \), not only associative algebras, as follows. We set \( q(\mathfrak{g})_0 = \mathfrak{g} \) and \( q(\mathfrak{g})_1 = \Pi(\mathfrak{g}) \); define the multiplication involving the odd elements as follows:

\[
[\pi, \Pi(y)] = \Pi([\pi, y]); \quad (\Pi(x))^2 = x^{[2]} \quad \text{for any} \ x, y \in \mathfrak{g}.
\]

Clearly, if \( \mathfrak{g} \) is restricted and \( i \subset q(\mathfrak{g}) \) is an ideal, then \( i_0 \) and \( \Pi(i_1) \) are ideals in \( \mathfrak{g} \). So, if \( \mathfrak{g} \) is restricted and simple, then \( q(\mathfrak{g}) \) is a simple Lie superalgebra. (Note that \( \mathfrak{g} \) has to be simple as a Lie algebra, not just as a restricted Lie algebra, i.e., \( \mathfrak{g} \) is not allowed to have any ideals, not only restricted ones.) A generalization of the queerification is the following procedure producing as many simple Lie superalgebras as there are simple Lie algebras.

3.2.1. Method 1: generalized queerification. Let the 1-step restricted closure \( \mathfrak{g}^{<1>} \) of the simple Lie algebra \( \mathfrak{g} \) be the minimal subalgebra of the (classically) restricted closure \( \mathfrak{g} \) containing \( \mathfrak{g} \) and all the elements \( x^{[2]} \), where \( x \in \mathfrak{g} \). To any simple Lie algebra \( \mathfrak{g} \) the generalized queerification assigns the Lie superalgebra

\[
\tilde{q}(\mathfrak{g}) := \mathfrak{g}^{<1>} \oplus \Pi(\mathfrak{g})
\]

with squaring given by \( (\Pi(x))^2 = x^{[2]} \) for any \( x \in \mathfrak{g} \). Obviously, for \( \mathfrak{g} \) restricted, the generalized queerification coincides with the queerification: \( \tilde{q}(\mathfrak{g}) = q(\mathfrak{g}) \).

3.2.2. Theorem. For any simple Lie algebra \( \mathfrak{g} \), the Lie superalgebra \( \tilde{q}(\mathfrak{g}) \) is simple.

Proof. Assume that \( i = i_0 \oplus \Pi(i_1) \), where \( i_0 \subset \mathfrak{g}^{<1>} \) and \( i_1 \subset \mathfrak{g} \), is an ideal of \( \tilde{q}(\mathfrak{g}) \). Then \( i_1 \) is an ideal in \( \mathfrak{g} \) since \( [\mathfrak{g}, \Pi(i_1)] \subset \Pi(i_1) \), so either \( i_1 = 0 \) or \( i_1 = \mathfrak{g} \). If \( i_1 = \mathfrak{g} \), then, by construction, \( \Pi(i_1) \) generates the whole \( \mathfrak{g}^{<1>} \), and \( i = \tilde{q}(\mathfrak{g}) \), so in what follows we assume that \( i_1 = 0 \).

Similarly, \( i_0 \) is an ideal of \( \mathfrak{g}^{<1>} \). Since \( \mathfrak{g} \) is an ideal in \( \tilde{g} \), it follows that \( i_0 \cap \mathfrak{g} \) is an ideal of \( \mathfrak{g} \), so either \( i_0 \cap \mathfrak{g} = 0 \), or \( \mathfrak{g} \subset i_0 \). If \( \mathfrak{g} \subset i_0 \), then \( i_1 \neq 0 \) and this is a contradiction with our assumption.

If \( i_0 \cap \mathfrak{g} = 0 \), then \( i_0 \) commutes with \( \mathfrak{g} \). It follows from the minimality of the restricted closure that the centralizer of \( \mathfrak{g} \) in \( \tilde{g} \) is the center of \( \mathfrak{g} \), so in case \( i_0 = 0 \), and \( i = 0 \).

For a detailed description of several examples of generalized queerifications, see [5]. Additionally, let us also mention \( \tilde{q}(\mathfrak{m}\mathfrak{t}^{(1)}(3; a)/c) \), and numerous new simple Lie superalgebras that are generalized queerifications of simple vectorial Lie algebras (of serial type, such as \( \mathfrak{svect}, \mathfrak{svect}, \) and \( \mathfrak{h}_1, \mathfrak{h}_1, \mathfrak{t} \) with their divergence-free subalgebras, see [LeP, ILL], and their deform, see [SkKos]; and exceptional ones, see [BGLLS, BGLLS1, BGLLS2]).

3.3. Method 2: superization of the \( \mathbb{Z}/2 \)-graded simple Lie algebra \( \mathfrak{g} \). Observe that every simple Lie algebra has a \( \mathbb{Z}/2 \)-grading because every simple Lie algebra \( \mathfrak{g} \) has a nonzero toral subalgebra.

For \( p = 2 \), the space of every \( \mathbb{Z}/2 \)-graded simple Lie algebra \( \mathfrak{g} \) can be endowed with a Lie superalgebra structure (more precisely, unless \( \mathfrak{g} \) is restricted, its space has to be enlarged prior to superization, as described below). For restricted Lie algebras \( \mathfrak{g} \), these “hidden supersymmetries” were first discovered in [BGL2], but their mechanism remained unclear until now.
If $p = 2$, let $g = g_+ \oplus g_-$ be a simple Lie algebra with a $\mathbb{Z}/2$-grading $\text{gr}$. Let $(g, \text{gr})$ be the minimal Lie subalgebra of the restricted closure $\overline{g}$ containing $g$ and all the elements $x^2$, where $x \in g_-$. Clearly, there is a single way to extend the grading $\text{gr}$ from $g$ to $(g, \text{gr})$.

Let $s(g, \text{gr})$ be the Lie superalgebra structure on the space of $(g, \text{gr})$ given by

$$x^2 := x^2$$

for any $x \in g_-$.

### 3.3.1. Theorem

**For any simple Lie algebra $g$, the Lie superalgebra $s(g, \text{gr})$ is simple.**

**Proof.** Let $i$ be an ideal of $s(g, \text{gr})$. Let $F: s(g, \text{gr}) \rightarrow (g, \text{gr})$ be the desuperization functor. Then $F(i)$ is an ideal in $(g, \text{gr})$. Since $g$ is an ideal in $\overline{g}$, and therefore in $(g, \text{gr})$, we see that $F(i) \cap g$ is an ideal in $(g, \text{gr})$, and therefore in $g$. This means that either $g \subset F(i)$, or $F(i) \cap g = 0$. By construction, $F^{-1}(g)$ generates $s(g, \text{gr})$, so if $g \subset F(i)$, then $i = s(g, \text{gr})$. If $F(i) \cap g = 0$, then $F(i)$ commutes with $g$; it follows from the minimality of the 2-closure that the centralizer of $g$ in $\overline{g}$ is the center of $g$, so in this case $i = 0$. \qed

For a huge number of new simple Lie superalgebras, see §7 examples in [BGLLS1, BGLLS2] and references therein, and [KrLe].

### 3.4. A relation between queerification and “method 2”.

Let $g$ be an arbitrary Lie algebra over any field $K$, and $C$ an associative and commutative algebra over $K$. On the space $g_C := g \otimes_K C$, a Lie algebra structure is naturally defined (“extension of the base field”).

In particular, if $A = K[a]/(a^2 - 1)$, then $g_A = g \oplus ga$, as spaces. Even if $g$ is simple, the Lie algebra $g_A$ is never simple, since the subspaces $g \otimes (1 \pm a)$ are ideals in it. Assuming that $a$ is odd, we see that $g_A$ has a natural $\mathbb{Z}/2$-grading.

Now, let char $K = 2$ and $g$ be simple. Then we can apply “method 2” to $g_A$ considered with the natural grading. Then $s(g_A)$ is a simple Lie superalgebra. This $s(g_A)$ is, as is easy to see, a queerification of $g$.

### 4. Classification of simple Lie algebras yields the classification of simple Lie superalgebras if $p = 2$

Let $p = 2$, and $g = g_0 \oplus g_1$ a Lie superalgebra, $S = \text{Span}\{x^2 \mid x \in g_1\}$.

#### 4.1. Lemma

**The space $([g_1, g_1] + S) \oplus g_1$ is an ideal of $g$.**

**Proof.** The subspace $[g_1, g_1]$ is $g_0$-invariant due to Jacobi identity. We see that

$$[x^2, y] = [x, [x, y]] \in [g_1, g_1]$$

for any $x^2 \in S$, $y \in g_0$.

This means that $([g_1, g_1] + S) \oplus g_1$ is an ideal of $g$. \qed

Let now $g$ be a simple Lie superalgebra. By Lemma 4.1 we see that $g_0 = [g_1, g_1] + S$. Then

$$\mathfrak{h} := [g_1, g_1] \oplus \Pi(g_1)$$

is an ideal of the Lie algebra $F(g)$, whereas $g$ is obtained from $\mathfrak{h}$ by means of “method 2”. Notice that the $\mathbb{Z}/2$-grading of $g$ induces $\mathbb{Z}/2$-gradings on the Lie algebras $F(g)$ and $\mathfrak{h}$. In particular,

$$\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

where $\mathfrak{h}_+ = [g_1, g_1]$ and $\mathfrak{h}_- = \Pi(g_1)$.

A question arises: must the Lie algebra $\mathfrak{h}$ be simple?

#### 4.2. Lemma

**The $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{h}$ defined by eq. (19) for a simple $g$ has no nontrivial $\mathbb{Z}/2$-graded ideals.**
Classification of Simple Lie Superalgebras

Proof. Let \( i = i_+ \oplus i_- \) be a \( \mathbb{Z}/2 \)-graded ideal of \( \mathfrak{h} \). This means that \([\Pi(\mathfrak{g}_1), i] \subset i \), and hence \([\mathfrak{s}, i] \subset i \), implying that \( i \) is an ideal of the whole Lie algebra \( F(\mathfrak{g}) \). Then

\[
I = i + \text{Span}\{x^2 \mid x \in i_-\} \subset F(\mathfrak{g}),
\]

is an ideal that can be superized by means of “method 2”.

Indeed, if \( x \in i_- \), \( y \in F(\mathfrak{g}) \), then \([x^2, y] = [x, [x, y]] \in i \). If now \( s(I) \) is the superization of \( I \), then \( s(I) \) is an ideal of the Lie superalgebra \( \mathfrak{g} \), and since \( \mathfrak{g} \) is simple, it follows that either \( s(I) = 0 \), or \( s(I) = \mathfrak{g} \). In the first case, \( i = 0 \), while in the second case \( i \supset \Pi(\mathfrak{g}_1) \), and hence, \( i = \mathfrak{h} \). □

What nongraded ideal \( i \) might the Lie algebra \( \mathfrak{h} \) have?

Denote the projectors of \( \mathfrak{h} \) onto \( \mathfrak{h}_+ \) and \( \mathfrak{h}_- \) by \( \text{pr}_+ \) and \( \text{pr}_- \), respectively. Then \((i \cap \mathfrak{h}_+) \oplus (i \cap \mathfrak{h}_-) \) and \( \text{pr}_+(i) \oplus \text{pr}_-(i) \) are graded ideals of the Lie algebra \( \mathfrak{h} \). By Lemma 4.2 we see that

\[(i \cap \mathfrak{h}_+) = 0, \ (i \cap \mathfrak{h}_-) = 0, \ \text{pr}_+(i) = \mathfrak{h}_+, \ \text{pr}_-(i) = \mathfrak{h}_-.
\]

This means that there exists a bijection \( f : \mathfrak{h}_+ \to \mathfrak{h}_- \) such that \( i = \{x + f(x) \mid x \in \mathfrak{h}_+\} \). Now, since \( i \) is an ideal of \( \mathfrak{h} \), it immediately follows that

\[
[f(y), f(x)] = f^{-1}([f(y), x]) = [y, x] \text{ for any } x, y \in \mathfrak{h}_+.
\]

Eq. (20) means that \( \mathfrak{g} = \tilde{q}(\mathfrak{h}_+) \).

4.3. Theorem. If \( p = 2 \), then every simple finite-dimensional Lie superalgebra can be obtained from a simple Lie algebra by means of either (generalized) queerification, or “method 2”.

Proof. As it is written after Lemma 4.1 we can obtain \( \mathfrak{g} \) by “method 2” from the Lie algebra \( \mathfrak{h} \), see eq. (19). Just after Lemma 4.2 it is shown that if \( \mathfrak{h} \) is not simple, it may only have a nontrivial nongraded ideal, and then \( \mathfrak{g} = \tilde{q}(\mathfrak{h}_+) \). And the Lie algebra \( \mathfrak{h}_+ \) must be simple, because, if \( \mathfrak{h}_+ \) had a nontrivial ideal \( i \), then \( \mathfrak{g} \) would have had the nontrivial ideal \( \mathfrak{g}_0 \oplus \Pi(i) \).

So if the Lie algebra \( \mathfrak{h} \) is simple, then \( \mathfrak{g} \) can be obtained by “method 2” from \( \mathfrak{h} \). If \( \mathfrak{h} \) is not simple, then \( \mathfrak{g} \) can be obtained by queerification of the simple Lie algebra \( \mathfrak{h}_+ \). □

5. Examples of Queerifications for \( p = 2 \)

5.1. Queerification of Superalgebras. If \( \mathcal{A} \) is an associative superalgebra, we can construct \( q(\mathcal{A}) \) using eq. (15), where \( \mathcal{A} = \mathcal{A}_0 \oplus \Pi(\mathcal{A}_1) \). If \( p = 2 \), then we can also queerify any restricted Lie superalgebra \( \mathcal{G} \) using eq. (18) for \( \mathcal{G} = \mathcal{G}_0 \oplus \Pi(\mathcal{G}_1) \).

For the definition of the 2|4-structure on Lie superalgebras — an analog of the 2-structure on Lie algebras, see Subsection 2.3.

5.1.1. Lemma. Let \( p = 2 \).

1) Let \( \mathfrak{g} \) be a simple Lie algebra with a 2-structure. Every Lie superalgebra of the form \( q(\mathfrak{g}) \) has a natural 2|4-structure given by the 2-structure on \( \mathfrak{g} \).

2) If \( \mathcal{A} \) is an associative superalgebra, then \( q(\mathcal{A}) \simeq q(F(\mathcal{A})) \). If \( \mathfrak{g} \) is a Lie superalgebra with a 2|4-structure, then \( q(\mathfrak{g}) \simeq q(F(\mathfrak{g})) \).

Proof. Because \( \mathfrak{g} \) is restricted, it follows that for every \( x, y \in \mathfrak{g} \), we have

\[
[x^2, \Pi(y)] = \Pi(x^2, y) = \Pi(\text{ad}^2_x(y)) = \Pi([x, [x, y]]) = [x, [x, \Pi(y)]] = \text{ad}^2_x(\Pi(y)).
\]

Since condition (4) is met, the 2|4-structure can be defined as in subsec. 2.3.

Heading 1) follows from eq. (15); to prove heading 2) recall eq. (18) as well. □
5.2. **Queerifications of \( sl(n) \) and \( psl(n) \).** If \( p = 2 \), the supertrace on the Lie superalgebra \( q(n) \) does not vanish identically as it does for \( p \neq 2 \), so there are two traces on \( q(n) \) which induce two NISes (nondegenerate invariant symmetric bilinear forms) on the simple subquotient of \( q(n) \) which is a double extension of this simple subquotient, see \([BCLS]\). This simple subquotient of \( q(n) \) is a queerification of \( sl(n) \) if \( n \) is odd and \( psl(n) \) if \( n \) is even.

5.3. **Queerification of orthogonal Lie algebra \( o_B(n) \).** The Lie algebra \( o_B(n) \) preserving a non-degenerate symmetric bilinear form \( B \) on the \( n \)-dimensional space has a natural 2-structure: considered as Lie algebra of \( n \times n \) matrices (or linear operators) \( X \) such that \( XB + BX^T = 0 \), we see that

\[
X^2B + B(X^2)^T = X(XB + BX^T) + (XB + BX^T)X^T = 0.
\]

Thus, if \( X \in o_B(n) \), then \( X^2 \in o_B(n) \). So we can consider queerification \( q(o_B(n)) \) of this algebra. Recall (see \([LeP]\)) that there is one equivalence class of bilinear forms \( B \) if \( n \) is odd, and two classes if \( n \) is even. For the normal shape of the Gram matrix of the form of one of the two classes one can take \( B = I \), the unit matrix, for the normal shape of the Gram matrix of the form of the other class we take either \( B = \Pi \), the matrix with units on the side diagonal, the other matrix elements being 0, or an equivalent to it \( B = \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix} \). We will find simple (modulo center) derived of these queerifications for \( n \) large enough; more precisely, \( n \geq 3 \) for \( q(o_I(n)) \), and \( n \geq 6 \) for \( q(o_{\Pi}(n)) \).

First, consider the “infinite derived”

\[
q(o_B(n))^{(\infty)} := \bigcap_{i \geq 1} q(o_B(n))^{(i)}.
\]

Clearly, \( q(o_B(n))^{(\infty)}_0 \) and \( \Pi(q(o_B(n))^{(\infty)}_1) \) are subalgebras of \( o_B(n) \).

Let us give a general description of nontrivial ideals of \( q(o_B(n))^{(\infty)} \). Clearly, for any ideal \( i \subset q(o_B(n))^{(\infty)} \), we have

1) \( i_0 \) is an ideal of \( q(o_B(n))^{(\infty)}_0 \);
2) \( \Pi(i_1) \) is an ideal of \( \Pi(q(o_B(n))^{(\infty)}_1) \).

Let us specify our description. We’ll need a new notation \( \hat{q} \) for the quotient of \( q \) or its derived:

5.3.1. \( \hat{q}(o_I(n)) \). Recall that \( o_I(n) \) consists of all symmetric matrices. Let \( ZD(n) \) denote the space of symmetric \( n \times n \) matrices with zeros on the main diagonal. We have

\[
o_I(n)^{(\infty)} = o_I(n)^{(1)} = ZD(n).
\]

It follows that

\[
q(o_I(n))^{(1)} = o_I(n) \oplus \Pi(ZD(n));
q(o_I(n))^{(i)} = (o_I(n) \cap sl(n)) \oplus \Pi(ZD(n)) \text{ for } i > 1.
\]

Clearly, the Lie algebra defined on the space \( ZD(n) \) is simple, so any nontrivial ideal of \( q(o_I(n))^{(\infty)} \) has zero odd part. The restricted Lie algebra \( o_I(n) \cap sl(n) \) has a nontrivial ideal if and only if \( n \) is even, and this ideal is \( \mathbb{K}1_n \). Thus,

\[
\hat{q}(o_I(n)) := \begin{cases} (o_I(n) \cap sl(n)) \oplus \Pi(ZD(n)) & \text{if } n \text{ is odd}, \\
(o_I(n) \cap sl(n))/\mathbb{K}1_n \oplus \Pi(ZD(n)) & \text{if } n \text{ is even}. \end{cases}
\]
5.3.2. \( \hat{q}(\sigma_\Pi(2k)) \). The Lie algebra \( \sigma_\Pi(2k) \) consists of matrices of the following form:

\[
\begin{pmatrix}
A & C \\
D & A^T
\end{pmatrix}, \quad \text{where} \quad A \in \mathfrak{gl}(k), \quad C = C^T \text{ and } D = D^T.
\]

So the elements of \( \hat{q}(\sigma_\Pi(2k)) \) are of the form

\[
\begin{pmatrix}
A & C \\
D & A^T
\end{pmatrix} \oplus \Pi\left(\begin{pmatrix}
A' & C' \\
D' & A'^T
\end{pmatrix}\right), \quad \text{where} \quad A,A' \in \mathfrak{gl}(k), \quad C,C',D,D' \text{ are symmetric matrices.}
\]

Computations show that elements of \( \hat{q}(\sigma_\Pi(2k))^{(i)} \) satisfy the following conditions for different \( i \):

- \( i = 1 \): \( A,A' \in \mathfrak{gl}(k), \quad C,C',D,D' \in ZD(k) \);
- \( i = 2 \): \( A \in \mathfrak{gl}(k), \quad A' \in \mathfrak{sl}(k), \quad C,C',D,D' \in ZD(k) \);
- \( i \geq 3 \): \( A,A' \in \mathfrak{sl}(k), \quad C,C',D,D' \in ZD(k) \).

Thus,

\[
\hat{q}(\sigma_\Pi(2k))^{(\infty)} \simeq \Pi(\hat{q}(\sigma_\Pi(2k))^{(\infty)}) \neq \sigma_\Pi(2k)^{(2)} = \sigma_\Pi(2k)^{(\infty)}.
\]

Clearly, \( \sigma_\Pi(2k)^{(2)} \) has a nontrivial ideal if and only if \( k \) is even, and this ideal is \( \mathbb{K}1_{2k} \), so

\[
\hat{q}(\sigma_\Pi(2k)) := \begin{cases} 
\sigma_\Pi(2k)^{(2)} \oplus \Pi(\sigma_\Pi(2k)^{(2)}) & \text{if } k \text{ is odd}, \\
(\sigma_\Pi(2k)^{(2)} \oplus \Pi(\sigma_\Pi(2k)^{(2)}))/\langle \mathbb{K}1_{2k} \oplus \Pi(1_{2k}) \rangle & \text{if } k \text{ is even.}
\end{cases}
\]

6. A relation between Cartan prolongation and queerification

In what follows, we illustrate the queerification phenomenon in terms of Cartan prolongations for the simplest example. This illustration shows why the characteristic \( p = 2 \) is exceptional. For a detailed description of the generalized Cartan prolongation and recipes how to realize elements of Lie (super)algebras by vector fields, see [Shch]. In this section, the results do not depend on the shearing vector \( N \) (for related definitions, see [BGLLS2]), so we mostly do not indicate it.

6.1. Notation. Recall that we denote the elements of \( g := q(n + 1) \) by pairs of matrices \( (A,B) \), where \( A,B \in \mathfrak{gl}(n+1) \), see eq. (16). Let us shorthand the element \( (E_{ij},0) \in g_0 \), see eq. (16), by means of the matrix unit \( E_{ij} \), and \( (0,E_{ij}) \in g_1 \) by means of the same matrix unit but denoted \( X_{ij} \) and considered to be odd. Let 0 through \( n \) be labels of rows and columns of matrices \( A,B \in \mathfrak{gl}(n+1) \). Consider the following \( \mathbb{Z} \)-grading of \( g = g_{-1} \oplus g_0 \oplus g_1 \):

\[
\begin{align*}
g_{-1} & = \text{Span}(E_{0i}, X_{0i} \mid i = 1, \ldots, n), \\
g_0 & = \text{Span}(E_{00}, X_{00}, E_{ij}, X_{ij} \mid i,j = 1, \ldots, n) = q(n) \oplus q(1), \\
g_1 & = \text{Span}(E_{ij}, X_{ij} \mid i,j = 1, \ldots, n) \oplus q(1) = \text{Span}(E_{00}, X_{00}).
\end{align*}
\]

6.1.1. Homomorphism \( \varphi : q(n+1) \longrightarrow \text{vect}(n|n) \). We define \( \varphi \) by setting

\[
\begin{align*}
E_{0i} \mapsto \partial_{\xi_i}, \\
E_{ij} & \mapsto z_j \partial_{\xi_i} + \xi_j \partial_{\xi_i}, \\
X_{ij} & \mapsto z_i \partial_{\xi_j} + \xi_i \partial_{\xi_j} \\
E_{00} & \mapsto -\sum_i (z_i \partial_{\xi_i} + \xi_i \partial_{\xi_i}), \\
X_{00} & \mapsto \sum_i (z_i \partial_{\xi_i} + \xi_i \partial_{\xi_i}) + \\
E_{0i} & \mapsto -z_i \sum_j (z_j \partial_{\xi_j} + \xi_j \partial_{\xi_j}) - \xi_i \sum_j (z_j \partial_{\xi_j} + \xi_j \partial_{\xi_j}) + \\
X_{0i} & \mapsto z_i \sum_j (z_j \partial_{\xi_j} + \xi_j \partial_{\xi_j}) - \xi_i \sum_j (z_j \partial_{\xi_j} + \xi_j \partial_{\xi_j}).
\end{align*}
\]
The homomorphism \( \varphi : q(n + 1) \longrightarrow \text{vect}(n|n) \) has a kernel yielding an embedding

\[
\varphi : \begin{cases}
q(n + 1) \longrightarrow \text{vect}(n|n) & \text{if } p \neq 2, \\
psq(n + 1) \longrightarrow \text{vect}(n|n) & \text{if } p = 2.
\end{cases}
\]

Let \( J \) be the odd operator commuting with \( \varphi(q(n)) \), and \( \varphi(g_{-1}) \) the tautological \( \varphi(q(n)) \)-module.

We have \( \varphi(g_0) \cong \begin{cases} q(n) & \text{if } p = 2, \\
q(n) \oplus \mathbb{K} \cdot J & \text{if } p \neq 2.\end{cases} \)

### 6.1.2. Prolongs of the nonpositive part of \( pq(2) \) for \( p \neq 2 \)

In this particular case eqs. (22) take the form:

\[
\begin{align*}
E_{01} & \mapsto \partial_z, & X_{01} & \mapsto \partial_\xi, \\
E_{00} & \mapsto -z\partial_z - \xi \partial_\xi, & X_{00} & \mapsto -z\partial_\xi + \xi \partial_z, \\
E_{11} & \mapsto \partial_z + \xi \partial_\xi, & X_{11} & \mapsto \partial_\xi + \xi \partial_z, \\
E_{10} & \mapsto -\xi^2 \partial_z - 2z\xi \partial_\xi, & X_{10} & \mapsto -\xi^2 \partial_\xi.
\end{align*}
\]

Clearly, \( X_{11} - X_{00} \mapsto 2z\partial_\xi \) and \( X_{11} + X_{00} \mapsto 2\xi \partial_z \), so \( \varphi(g_0) \simeq \mathfrak{sl}(1|1) \), and hence

\[
(g_{-1}, \varphi(g_0))_{\ast\mathbb{N}} \simeq \text{svect}(1;\mathbb{N}[1]),
\]

the Lie superalgebra of divergence-free vector fields, see [BGLLS2].

Denote the 0th component \( \varphi(g_0) \) of the image of \( \text{psq}(2) \) by \( \mathfrak{h} \). Direct computations show:

\[
\begin{align*}
\mathfrak{h} & = \text{Span}\{z\partial_z + \xi \partial_\xi, z\partial_\xi\}; \\
(\varphi(g_{-1}), \mathfrak{h})_{\ast\mathbb{N}} & = \text{Span}\{f(z)\partial_z + f'(z)\xi \partial_\xi, \ g(z)\partial_\xi, \ f, g \in \mathcal{O}(1;\mathbb{N})\}.
\end{align*}
\]

Consider the generalized Cartan prolong \( \varphi(q(2))_{\ast\mathbb{N}} \), i.e., the maximal Lie subsuperalgebra of \( \text{vect}(1;\mathbb{N}[1]) \) whose \((-1)\)-st, 0-th and 1-st components coincide with the respective components of \( \varphi(q(2)) \). It is easy to see that this prolong coincides with the semi-direct sum (here \( i \ltimes \mathfrak{a} \) is a direct sum as spaces where \( i \) is an ideal)

\[
\varphi(q(2))_{\ast\mathbb{N}} = (\varphi(q_{-1}, \mathfrak{h}))_{\ast\mathbb{N}} \ltimes \mathbb{K} \xi \partial_z.
\]

An interpretation: the Cartan prolong \( (\varphi(g_{-1}), \mathfrak{h})_{\ast\mathbb{N}} \) consists of the divergence-free vector fields preserving the subspace of \( \varphi(g_{-1}) \) spanned by \( \partial_\xi \). Indeed, the subalgebra \( \mathfrak{h} \subset \mathfrak{sl}(1|1) \) reducibly acts on the component \( \varphi(g_{-1}) \) preserving \( \mathbb{K} \xi \partial_z \).

### 6.1.3. Prolong of the nonpositive part of \( pq(2) \) for \( p = 2 \)

For \( p = 2 \), all formulas of eq. (23) hold but the element \( X_{11} - X_{00} \) belongs to the kernel of the homomorphism, and hence

\[
\begin{align*}
[E_{10}, E_{01}] & = E_{11} - E_{00} \mapsto 0, & [E_{10}, X_{01}] & = X_{11} - X_{00} \mapsto 0, \\
[X_{10}, E_{01}] & = X_{11} - X_{00} \mapsto 0, & [X_{10}, X_{01}] & = X_{11} + X_{00} \mapsto 0.
\end{align*}
\]

In other words, the images of \( E_{10} \) and \( X_{10} \) in \( \text{svect}(1|1) \) must commute every element of \( \varphi(g_{-1}) \), hence these images should vanish. So the map \( \varphi \) has a big kernel and does not define embedding of either \( q(2) \), or \( pq(2) \), or \( \text{psq}(2) \) in \( \text{vect}(1;\mathbb{N}[1]) \) for any \( \mathbb{N} \).

### 6.2. Prolong of the nonpositive part of \( \varphi(pq(n + 1)) \) for \( n + 1 \geq 2 \)

Let \( \text{id} \) denote the tautological \( n|n \)-dimensional \( q(n) \)-module. Let us compute the Cartan prolongs of the nonpositive part of \( \varphi(pq(n + 1)) \), i.e., compute \( (\text{id}, q(n) \oplus \mathbb{K} \cdot J)_{\ast\mathbb{N}} \) for \( p \neq 2 \) and \( (\text{id}, q(n))_{\ast\mathbb{N}} \) for \( p = 2 \).

#### 6.2.1. Theorem.

If \( p = 2 \), then \( (\text{id}, q(n))_{\ast\mathbb{N}} = q(\text{vect}(n;\mathbb{L}|0)) \).

If \( p \neq 2 \), then \( (\text{id}, q(n))_{\ast i} = 0 \) for \( i \geq 1 \).
Proof. Let \( \mathfrak{h} := (\text{id}, q(n)) \ast \mathcal{N} \) and \( X = \sum(F_i(z, \xi) \partial_{z_i} + G_i(z, \xi) \partial_{\xi_i}) \in \mathfrak{h} \). Let us single out the image \( g \) of the 0-th component of \( \mathfrak{h} \) in \( \text{vect}(n; \mathcal{N}|n) \) by means of a system of linear differential equations using operators \( \partial_{z_i} \) and \( \partial_{\xi_i} = \partial_{\xi_i} \cdot \text{Py} \), where \( \text{Py}(x) := (-1)^{p(x)}x \) is the parity operator. The system of linear differential equations obtained defines the (generalized) Cartan prolong, see \([\text{Shch}]\).

The image of \( q(n) \) is described by the second line of eqs. (22) or, equivalently, by means of the following equations:

\[
\partial_{z_i} F_i = -(-1)^{p(G_i)} \partial_{\xi_i} G_i; \quad \partial_{z_j} G_i = -(-1)^{p(F_i)} \partial_{\xi_j} F_i \quad \text{for any } i, j \in 1, \ldots, n.
\]

Since the indices \( i \) and \( j \) in equations (25) are independent of each other, it suffices to find all pairs of functions \((F, G)\) satisfying the system of equations

\[
\partial_{z_i} F = -(-1)^{p(G)} \partial_{\xi_i} G; \quad \partial_{z_j} G = -(-1)^{p(F)} \partial_{\xi_j} F \quad \text{for any } j \in 1, \ldots, n.
\]

In these terms, the Lie superalgebra \( \mathfrak{h} \) is spanned by vector fields of the form \( F \partial_{z_i} + G \partial_{\xi_i} \) and \( G \partial_{z_i} + F \partial_{\xi_i} \) for any \( i = 1, \ldots, n \), minding conditions (26).

Let the function \( F \) be a sum of monomials, one of which is \( A z^k_i f \) (for \( p = 0 \)) or \( A z^k_i f \) (for \( p > 0 \)), where \( f \) does not depend on \( z_i \) and \( A \in \mathbb{K} \). Because \( \partial_{z_i} F = -(-1)^{p(G)} \partial_{\xi_i} G \), the monomial \( f \) should not depend on \( \xi_i \), either, and the function \( G \) should contain a monomial \( z^k_i - 1 \xi_i f \) (for \( p = 0 \)) or \( z^k_i \xi_i f \) (for \( p > 0 \)) with, perhaps, different coefficient. But since all equations are symmetric with respect to the transposition \( F \leftrightarrow G \), the degree \( k - 1 \) should be equal to 0. Thus, the degree of each of the functions \( F \) and \( G \) relative each even indeterminate should not exceed 1.

Let us explicitly describe the component \( \mathfrak{h}_1 \).

If the vector field \( X \in \mathfrak{h}_1 \) is homogeneous with respect to parity, then the functions \( F \) and \( G \) should be of opposite parity, and since the equations are symmetric with respect to the transposition \( F \leftrightarrow G \), let us assume, for definiteness sake, that

\[
F \text{ is even and } G \text{ is odd.}
\]

Therefore, these functions are of the form:

\[
F = \sum_{i < j} (a_{ij} z_i z_j + b_{ij} \xi_i \xi_j), \quad G = \sum_{i \neq j} c_{ij} z_i \xi_j.
\]

Hence

\[
\partial_{z_i} F = \sum_{j > i} a_{ij} z_j + \sum_{j < i} a_{ji} z_j, \quad \partial_{\xi_i} G = \sum_{j \neq i} c_{ij} \xi_j,
\]

\[
\partial_{\xi_i} F = \sum_{j > i} b_{ij} \xi_j - \sum_{j < i} b_{ji} \xi_j, \quad \partial_{z_i} G = \sum_{j \neq i} c_{ij} z_j.
\]

Now, eqs. (26) imply that

\[
c_{ij} = -b_{ij} = a_{ij} \text{ and } c_{ji} = b_{ij} = a_{ij} \text{ for } i < j.
\]

If \( p \neq 2 \), these conditions contradict each other, and hence

\[
(id, q(n))_i = 0 \quad \text{for } i \geq 1.
\]

If \( p = 2 \), eqs. (26) are free of contradictions, generate \( \binom{n}{2} \) pairs of functions \((F, G)\), and, accordingly, \( \binom{n}{2} (n|n) \) vector fields of the form:

\[
X_0 = (z_i z_j + \xi_i \xi_j) \partial_{z_k} + (z_i \xi_j + \xi_i z_j) \partial_{\xi_k},
\]

\[
X_1 = (z_i z_j + \xi_i \xi_j) \partial_{\xi_k} + (z_i \xi_j + \xi_i z_j) \partial_{z_k}.
\]
Let us describe now the general shape of the pairs of functions \((F, G)\) satisfying eq. (26) and defining the vector fields \(X \in \mathfrak{h}\). Let the field \(X\) be homogeneous with respect to the parity and \(\mathbb{Z}\)-grading. Then the functions \(F\) and \(G\) are also homogeneous with respect to the parity and \(\mathbb{Z}\)-grading and their parities are opposite; recall eq. (27).

Let \(F\) contain, as a summand, a monomial \(\alpha \xi_i \xi_j \cdot f\), where \(\alpha \in \mathbb{K}\) and \(f\) is a monomial independent of \(\xi_i\) and \(\xi_j\). By differentiating \(F\) with respect to \(\xi_i\) we see, by virtue of eqs. (26), that the function \(G\) contains the monomial \(\alpha z_i \xi_j \cdot f\) as a summand. Differentiating now the function \(G\) with respect to \(\xi_j\) we see that \(F\) contains also the term \(\alpha z_i z_j \cdot f\).

Consequently applying this procedure several times we conclude that \(F\) must contain monomials depending on \(z\) only, and the sum of all these monomials completely determines the whole pair \((F, G)\). Let us call this sum the main part of \(F\) and denote it \(F^m\). For \(F^m = z_1 z_2 \ldots z_k\), the pair \((F, G)\) is of the form:

\[
F = \sum_{1 \leq i_1 < i_2 < \ldots < i_k} \partial_{z_{i_1}} \partial_{z_{i_2}} \ldots \partial_{z_{i_k}} (z_1 z_2 \ldots z_k) \xi_i \xi_j + \ldots
\]

\[
G = \sum_i \xi_i (z_1 z_2 \ldots z_k) \xi_i + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq k} \partial_{z_{i_1}} \partial_{z_{i_2}} \partial_{z_{i_3}} \partial_{z_{i_4}} (z_1 z_2 \ldots z_k) \xi_i \xi_j \xi_k \xi_l + \ldots
\]

Let

\[
X = \sum_i (F_i(z, \xi) \partial_{z_i} + G_i(z, \xi) \partial_{\xi_i}) \quad \mapsto X^m = \sum_i F^m_i(z) \partial_{z_i}
\]

be the passage to the main part \(X^m\) of a given even vector \(X\). Clearly, \(X^m\) depends only on \(z\), and if \(X, Y\) are even fields, then

\[
[X^m, Y^m] = [X, Y]^m.
\]

This means that \(\mathfrak{h}_0 \simeq \text{vect}(n; \mathbb{Z}/2\mathbb{Z})\). This isomorphism is given by the map (28).

If the field \(X\) is odd, then the field \(X\), and its main part, are, due to our assumption (27), of the form

\[
X = \sum_i (G_i(z, \xi) \partial_{z_i} + F_i(z, \xi) \partial_{\xi_i}) \quad \mapsto X^m = \sum_i F^m_i(z) \partial_{z_i},
\]

If now \(Y\) is an even field, then

\[
[Y^m, X^m] = [Y, X]^m,
\]

i.e., as \(\mathfrak{h}_0\)-module, \(\mathfrak{h}_1\) is the adjoint one. It is easy to verify that \(\mathfrak{h} = q(\text{vect}(n; \mathbb{Z}/2\mathbb{Z}))\).

\section*{6.2.2. Theorem} If \(p \neq 2\) and \(n > 1\), then

\[
\mathfrak{h} := (\varphi(\mathfrak{g}_1), \varphi(\mathfrak{g}_0))_{\ast \ast} = (\text{id}, q(n) \oplus \mathbb{K} \cdot J)_{\ast \ast} = \varphi(q(n + 1)).
\]

\textbf{Proof.} As we repeatedly demonstrated describing the exceptional simple vectorial Lie superalgebras, see \cite{Sh5, Sh14}, having added one odd central element to the 0-th component (this is precisely what happens under the homomorphism \(\varphi\)) we can add to \(\mathfrak{h}_1\) either the \(\mathfrak{g}_0\)-module \(\Pi(\mathfrak{g}_1)^{\ast \ast}\), or nothing. Since \((\text{id}, q(n))_1 = 0\) by Theorem 6.2.1 we see that \(\mathfrak{h}_1 = \varphi(\mathfrak{g}_1)\), where \(\mathfrak{g}_1\) is defined in eq. (21).

Let us describe the component \(\mathfrak{h}_2\). Once again, let us write the system of differential equations on coordinates of the vector field \(X = \sum (F_i \partial_{z_i} + G_i \partial_{\xi_i}) \in \mathfrak{h}\). Because the difference from the case of Theorem 6.2.1 consists of one basis element in the 0-th component only, these equations do not differ much from eqs. (25). More precisely, the first group of eqs. (25) remains the same; the
second group is the same for all \(i = 1, \ldots, n\) and all \(j \neq i\), whereas \(n\) equations of the second groups for \(j = i\) turn into the following \(n - 1\) equations:

\[
\partial_i G_i + (-1)^{p(F_i)} \partial_{\xi_i} F_i = \partial_i G_1 + (-1)^{p(F_i)} \partial_{\xi_i} F_1, \text{ where } i = 2, \ldots, n.
\]

Let now \(X \in \mathfrak{h}_2\) be even. Then the degrees of all (even) functions \(F_i\) and all (odd) functions \(G_i\) are equal to 3, i.e.,

\[
F_i = f_i(z) + \sum_{j,k,l} c_{i}^{jkl} z_j \xi_k \xi_l, \text{ where } c_{i}^{jkl} \in \mathbb{K} \text{ and } \deg(f_i(z)) = 3,
\]

\[
G_i = \sum_j g_j^i(z) \xi_j + \sum_{j,k,l} d_{i}^{jkl} z_j \xi_k \xi_l, \text{ where } d_{i}^{jkl} \in \mathbb{K} \text{ and } \deg(g_j^i(z)) = 2.
\]

The following are all equations for the \(F_i\) and \(G_i\), their parities being taken into account:

\[
\begin{align*}
\partial_{z_i} F_i &= \partial_{\xi_i} G_i & \text{ for any } i, j = 1, \ldots, n \quad (30) \\
\partial_{z_i} G_i &= -\partial_{\xi_i} F_i & \text{ for any } i = 1, \ldots, n, j \neq i; \\
\partial_{z_i} G_i + \partial_{\xi_i} F_i &= \partial_{z_i} G_1 + \partial_{\xi_i} F_1 & \text{ for any } i = 2, \ldots, n.
\end{align*}
\]

The first two lines of eqs. (30) imply:

\[
\partial_{z_j} \partial_{\xi_k} (G_i) = \partial_{\xi_j} \partial_{z_k} (F_i) = \partial_{z_k} \partial_{\xi_j} (F_i) = -\partial_{z_k} \partial_{z_j} (G_i) \text{ for } j, k \neq i.
\]

In particular, for \(j = k \neq i\), we get

\[
\frac{\partial^2 G_i}{\partial z_j^2} = 0 \text{ for all } j \neq i.
\]

Analogously, \(\partial_{\xi_k} \partial_{\xi_j} (G_i) = -\partial_{\xi_j} \partial_{z_k} (G_i)\), and because partial derivatives with respect to the even variables commute, whereas those with respect to the odd ones anticommute, we get:

\[
\frac{\partial^2 G_i}{\partial z_j \partial z_k} = \frac{\partial^2 G_i}{\partial \xi_j \partial \xi_k} = 0 \text{ for all } j, k \neq i, \quad (31)
\]

implying that \(d_{ij}^{kkl} = 0\), and all functions \(g_j^i\) are linear combinations of monomials \(z_i^{(2)}\) and \(z_i z_k\), where \(k = 1, \ldots, n\) and \(k \neq i\).

We similarly deduce that for \(j, k \neq i\), we have

\[
\frac{\partial^2 F_i}{\partial z_j^2} = \frac{\partial^2 F_i}{\partial z_j \partial z_k} = \frac{\partial^2 F_i}{\partial \xi_j \partial \xi_k} = 0, \quad (32)
\]

implying that among the coefficients \(c_{ij}^{jkl}\) only those of the form \(c_{ij}^{jil}\) can be nonzero, and the function \(f_i\) is a linear combination of the monomials \(z_i^{(3)}\) and \(z_i (z_j)\).

Finally, eqs. (32) for \(j, k \neq i\) imply that

\[
\frac{\partial^2 G_i}{\partial z_j \partial \xi_k} = \frac{\partial^2 G_i}{\partial z_j \partial z_k} = 0.
\]

Therefore the function \(G_i\) is of the form

\[
G_i = z_i^{(2)} \sum_{\substack{j \leq k \leq n \atop j \neq i}} a_j^i \xi_j + z_i \xi_i \sum_{j \neq i} b_j^i z_j.
\]

Similarly, eq. (31) for \(j, k \neq i\) implies that

\[
\frac{\partial^2 F_i}{\partial \xi_j \partial \xi_k} = \frac{\partial^2 F_i}{\partial \xi_j \partial z_k} = 0 \text{ and } \frac{\partial^2 F_i}{\partial \xi_i \partial z_i} = \frac{\partial^2 G_i}{\partial \xi_i^2} = 0.
\]
These equalities imply that \( c_{ijkl} = 0 \) for any \( i, j, k \), and the function \( F_i \) is of the form

\[
F_i = \alpha_i z_i^{(3)} + \sum_{j \neq i} \beta_j^i \xi_j.
\]

Now, from eq. (30) for \( j \neq i \) we deduce that \( a_i^j = \beta_j^i \), whereas for \( j = i \) we have \( \beta_i^i = b_i^i \) and \( \alpha_i = a_i^i \). Because \( \partial_{\xi_i} F_i = 0 \) for all \( j \), the equality (30) implies \( b_j^i = 0 \) for all \( i, j \). As a result, \( F_i = \alpha_i z_i^{(3)} \), \( G_i = \alpha_i z_i^{(2)} \xi_i \) and

\[
\frac{\partial F_i}{\partial \xi_i} + \frac{\partial G_i}{\partial z_i} = \alpha_i z_i \xi_i.
\]

But since this sum should not, thanks to the third line of eqs. (30), depend on \( i \), we conclude that \( \alpha_i = 0 \) for any \( i \), and hence \( X = 0 \).

For any odd vector field \( X \) we similarly get \( h_i = 0 \) for any \( i \geq 2 \).

6.3. **Queerification and Cartan prolongation are commuting operations for** \( p = 2 \). Let \( \mathfrak{g}_0 \) be a Lie algebra with a 2-structure, \( \mathfrak{g}_{-1} \) a restricted \( \mathfrak{g}_0 \)-module, and \( \mathfrak{g} = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_{\mathbb{Z}} \) the Cartan prolong. Now, consider:

- \( \mathfrak{g}q = \mathfrak{g} \oplus \Pi(\mathfrak{g}) \), the queerification of the prolong,
- \( \mathfrak{g}q_0 = \mathfrak{g}_0 \oplus \Pi(\mathfrak{g}_0) \), the queerification of the 0-th component of the prolong,
- \( \mathfrak{g}q_{-1} = \mathfrak{g}_{-1} \oplus \Pi(\mathfrak{g}_{-1}) \), the queerification of the \(-1\)-st component of the prolong,
- \( \mathfrak{g}q := (\mathfrak{g}q_0, \mathfrak{g}q_{-1})_{\mathbb{Z}} \oplus \mathfrak{g}q_{k\mathbb{N}} \), the prolong of the queerifications.

6.3.1. **Theorem.** The Lie superalgebras \( \mathfrak{g}q \) and \( \mathfrak{g}q \) coincide: \( \mathfrak{g}q = \mathfrak{g}q \).

**Proof.** First of all, recall, see [BGLLS1], that the coordinates of the shearing vector that can not exceed a certain value are said to be *critical* and notice that if \( \dim \mathfrak{g}_{-1} = n \), then \( \mathfrak{g}q_0 \subset \mathfrak{q}(n) \).

Now, due to Theorem 6.2.1, \( \mathfrak{g}q \subset (\text{id}, \mathfrak{q}(n))_{\mathbb{Z}} = \mathfrak{q}(\mathfrak{vect}(n; \mathbb{Z})/0) \), and hence all coordinates of the shearing vector for \( \mathfrak{g}q \) are critical: \( N = \mathbb{Z} \).

Observe that the nonpositive parts of the \( \mathbb{Z} \)-graded Lie superalgebra \( \mathfrak{g}q \) and \( \mathfrak{g}q \) coincide by their definitions. Because the Cartan prolongation with a fixed \( N \), again by definition, is the maximal transitive Lie (super)algebra with given nonpositive part and \( N \), we get an embedding \( \mathfrak{g}q \subset \mathfrak{g}q \).

On the other hand, the map \( X \mapsto X^m \) constructed in eqs. (28), (29) determines the embedding \( \mathfrak{g}q \rightarrow \mathfrak{q}(\mathfrak{vect}(n; \mathbb{Z})/0) \). If \( X^m \) is even, then \( X^m \in \mathfrak{g} \) by definition of the Cartan prolong and equality of the shearing vectors for \( \mathfrak{g} \) and \( \mathfrak{g}q \). If \( X^m \) is odd, then \( X^m \in \Pi(\mathfrak{g}) \) due to the commutation relations in \( \mathfrak{q}(\mathfrak{vect}(n; \mathbb{Z})/0) \). Hence, \( \mathfrak{g}q \) is embedded into \( \mathfrak{g}q \) and \( \mathfrak{g}q = \mathfrak{g}q \).

7. **Examples of simple Lie superalgebras obtained by “Method 2”**

7.1. **How would “method 2” work if it were defined for** \( p \neq 2 \). Over any ground field, “method 2” applied to \( \mathfrak{sl}(n) \) boils down to the following: we declare several pairs of (positive and the corresponding to them negative) Chevalley generators odd and simultaneously change the corresponding diagonal elements of the Cartan matrix (replace 2 with a 0). Having changed the Cartan matrix and parities of generators we also change the defining relations. The Lie superalgebras \( \mathfrak{sl}(a|b) \), where \( a + b = n \), thus obtained are simple as long as \( a \neq b \), whereas \( \mathfrak{sl}(a|a) \) is simple modulo center.

Over \( \mathbb{C} \) and \( \mathbb{K} \) with \( \text{char} \mathbb{K} > 2 \), for any simple Lie algebra \( \mathfrak{g} \neq \mathfrak{sl}(n) \), having similarly declared any pair of Chevalley generators odd and factorized by the ideal of relations that replace Serre relations (for their description, see [BGLL] and references therein) we get a simple (perhaps, modulo
center) Lie superalgebra of infinite dimension; moreover, it is of rather fast growth as a \( \mathbb{Z} \)-graded algebra, cf. [CCLL, GL3]. Under such superization not only relations that replace Serre ones (the ones between root vectors of the same sign) change their form and affect the dimension of the quotient, but also Cartan matrix and ensuing weight relations become modified: compare \( \mathfrak{sl}(2) \) whose Cartan matrix is (2) and weight relations are \([H,X_\pm] = \pm 2X_\pm\) with \( \mathfrak{sl}(1|1) \) whose Cartan matrix is (0) and weight relations are \([H,X_\pm] = 0\).

7.2. **“Method 2” for vectorial Lie algebras:** \( p = 2 \) and \( \mathcal{N} = \mathbb{1} \). In [LeP], there are given examples of superization of simple vectorial Lie algebras by what we call here “method 2” for the cases where several of coordinates of \( \mathcal{N} \) are equal to 1. In this case, one can consider the corresponding indeterminates odd. In this way we get \( \mathfrak{h}_{11}(a;b_\mathcal{N}) \) from \( \mathfrak{h}_{11}(a+b;\mathcal{N}) \) (resp. \( \mathfrak{le}(a;\mathcal{N}) \) from \( \mathfrak{le}(2a;\mathcal{N}) \)), where \( \mathcal{N} \) is the part of \( \mathcal{N} \) corresponding to the even indeterminates. The same concerns other series and exceptional examples, see [BGLLS, Ei, SKTI]. These superizations are expected, to an extent. Numerous examples of simple Lie superalgebras given below are totally new (except for “occasional isomorphisms” that might occur for a small number of indeterminates): these Lie superalgebras are \( \mathbb{Z} \)-graded and this grading is compatible with \( \mathbb{Z}/2\text{-grading} \) whereas over \( \mathbb{C} \) only the following simple Lie superalgebras have italicized property: the series \( \mathfrak{t}(1|m) \), and exceptions \( \mathfrak{fas}, \mathfrak{vle}(4;3;K) = \mathfrak{vle}(3|6), \mathfrak{tse}(9;6;K) = \mathfrak{tse}(5|10) \) and \( \mathfrak{mb}(4;5;K) = \mathfrak{mb}(3|8) \), see [LS, Sh14].

In the cases below, we have to add squares of elements of degree \(-1\) (and \(-3\), if there are any; the only simple infinite-dimensional vectorial Lie algebras with Weisfeiler gradings known to us for \( p = 2 \) are of depth at most 3) but we do not have to add squares of elements positive degrees, see Proposition 7.4.

“Method 2”, first consciously used in [BGLLS], being applied to the deforms of “standard” vectorial Lie algebras, in particular, Kaplansky, Eick and Skryabin algebras, see [BGLLS, Ei, SKTI] yields many new simple Lie superalgebras.

7.3. **“Method 2” applied to simple \( \mathbb{Z}/2\text{-graded} \) vectorial Lie algebras for which all coordinates of \( \mathcal{N} \) corresponding to “odd” indeterminates are \( > 1 \).** We introduce \( \mathbb{Z}/2\text{-gradings} \) as the \( \mathbb{Z} \)-gradings modulo 2. Observe that the desuperized finite-dimensional simple vectorial Lie superalgebra \( \mathfrak{F}(\mathfrak{g})(\mathcal{N}) \) over \( \mathbb{K} \) may have more (types of) Weisfeiler \( \mathbb{Z} \)-gradings than its infinite-dimensional namesake \( \mathfrak{g} \) over \( \mathbb{C} \) has.

7.3.1. **The simple serial vectorial Lie algebras** \( \mathfrak{g} = \mathfrak{g}(\mathcal{N}) \) **in the standard \( \mathbb{Z} \)-grading**. Recall that in the standard \( \mathbb{Z} \)-grading the degree of each indeterminate is equal to 1, except for the contact case, where the degree of the “time” indeterminate is equal to 2.

In certain cases, we have to take not the Cartan prolong listed in the titles of subsections below but its derived since we superize the simple Lie algebras; but since the simple Lie superalgebra \( \mathfrak{s}(\mathfrak{g}, \text{gr}) \), where \( \mathfrak{g} = \mathfrak{g}(\mathcal{N}) \), see Statement 5.3.1, is completely determined by \( \mathcal{N} \), and the elements of degree \( \leq 0 \) which for \( \mathfrak{g}^{(i)} \) coincide with those of \( \mathfrak{g} \), see Proposition 7.4 we do not indicate such subtleties.

7.3.1a. \( \text{vect}^{(1)}(1;\mathcal{N}) \). We have to add the square of the basis element of degree \(-1\) (obviously, \( \mathcal{N} > 1 \)) and the square of \( x^{(2^{\mathcal{N}-1})} \); the Lie superalgebra \( \mathfrak{s}(\text{vect}^{(1)}(1;\mathcal{N}), \text{st}) \) is, clearly, isomorphic to \( \mathfrak{t}(1;\mathcal{N} - 1|1) \). (For far from obvious description of \( \mathfrak{t}(1;\mathcal{N}|1) \), see [BGLLS2].)

7.3.1b. \( \text{vect}(n;\mathcal{N}) \) **for** \( n > 1 \), **and** \( \text{s vect}(n;\mathcal{N}) \) **for** \( n > 2 \); **various Hamiltonian series (all in the standard grading st)**, **and filtered deforms thereof**. Set

\[
\mathfrak{s}(\mathfrak{g}, \text{st})_{-2} := \mathfrak{g}^{(2)}_{-1} := \text{Span}(\partial_i^2 \mid N_i > 1).
\]

In particular, if \( \mathcal{N} = \mathbb{1} \), then \( \mathfrak{s}(\mathfrak{g}, \text{st})_{-2} = 0 \).
7.3.1c. \( \mathfrak{t}(2n+1;N) \) for \( n > 0 \) in the standard grading \( st \), and filtered deformations of \( \mathfrak{t}(2n+1;N) \). Having added the trivial \( g_0 \)-module \( \mathfrak{g}^{[2]} := \text{Span}(\partial_i^2 \mid N_i > 1) \) we get \( \mathfrak{s}(\mathfrak{g}, st)_{-2} = \mathfrak{g}^{[2]}_{-1} \oplus \mathfrak{g}_{-2} \).

7.3.2. The simple vectorial Lie algebras in nonstandard \( \mathbb{Z} \)-gradings. In addition to the “standard” gradings there are finitely many (classes of) nonstandard \( \mathbb{Z} \)-gradings associated with maximal subalgebras (of finite codimension if the algebra is infinite-dimensional) and the corresponding Weisfeiler filtration. Generally, these gradings are given by setting \( \text{deg} x_i = 0 \) for \( i = 0, 1, \ldots, n-1 \) of the \( n \) indeterminates; for the classification of nonstandard gradings of simple vectorial Lie superalgebras over \( \mathbb{C} \), see \[LSH\].

7.3.2a. \( \mathfrak{t}(9;N) \) and \( \mathfrak{F}(\mathfrak{vle}(3;N[6])) \) with a grading \( gr \) described in \[BGLLS1\]. Having added the trivial \( g_0 \)-module \( \mathfrak{g}^{[2]} := \text{Span}(\partial_i^2 \mid N_i > 1) \) we get \( \mathfrak{s}(\mathfrak{g}, gr)_{-2} = \mathfrak{g}^{[2]}_{-1} \oplus \mathfrak{g}_{-2} \).

7.3.2b. \( \mathfrak{F}(\mathfrak{tsle}(5;N[10])) \) and \( \mathfrak{F}(\mathfrak{mb}(3;N[8])) \) described in \[BGLLS2\]. These Lie algebras \( g \) have \(-3\)rd components, so their superizations corresponding to such gradings, denote them \( gr 3 \), are of depth \( 6 \), the squares of the basis elements of degree \( -3 \) corresponding to the indeterminates whose heights are \( > 1 \) span the trivial \( g_0 \)-module we denote hereafter (for other algebras of depth \( 3 \) as well) by \( \mathfrak{g}^{[2]}_3 \). It is easy to see that if \( \bar{g} \) is the restricted closure of \( g \), then \( [\bar{g}, \bar{g}] \subseteq [g, g] \). In particular, \( [\bar{g}, \bar{g}] \subseteq g \). Therefore, \( \mathfrak{s}(\mathfrak{g}, gr 3)_{-4} = \mathfrak{s}(\mathfrak{g}, gr 3)_{-5} = 0 \) since \( g \) is of depth \( 3 \).

7.4. Proposition. Let \( g = g(\mathbb{N}) \) be the generalized Cartan prolong. To obtain the Lie superalgebra \( \mathfrak{s}(\mathfrak{g}, gr) \), we only have to add to \( g \) the squares of basis elements of negative degrees corresponding to the indeterminates whose heights are \( > 1 \).

Recall that in all known examples of simple vectorial Lie superalgebras except \( \mathfrak{vle} \), for which one of the coordinates of the shearing vector \( N \) can not exceed \( 2 \), if \( N_x \) can be \( > 1 \) it can be however big, see \[BGLLS2\]. The coordinates of the shearing vector with “in-built” bounds are called critical.

**Proof.** Let the depth of \( g \) be equal to \( 1 \). Let \( g = (g_{-1}, g_0)_s, N \) be the infinite-dimensional Cartan prolong corresponding to the shearing vector \( N \) without any constraints imposed on its coordinates, so all noncritical coordinates of \( N \) are equal to \( \infty \). Then \( g \) is the maximal transitive \( \mathbb{Z} \)-graded Lie algebra with the given nonpositive part. In particular, the maximality implies that the positive part is a restricted Lie algebra. So, to cook \( \mathfrak{s}(g, gr) \), it suffices to add squares of the basis elements of degree \( -1 \) to \( g \).

Take the new nonpositive part \( \mathfrak{G} := \bigoplus_{-2 \leq k \leq 0} \mathfrak{G}_k \) with \( \mathfrak{G}_{-1} := g_{-1} \) considered odd, \( \mathfrak{G}_0 := g_0 \), and \( \mathfrak{G}_{-2} := g^{[2]}_{-1} \), see eq. (33), added; let \( \mathfrak{G} := \bigoplus_{-2 \leq k} \mathfrak{G}_k \) be the Cartan prolong of the new nonpositive part. If we desuperize \( \mathfrak{G} \), then in the Lie algebra \( F(\mathfrak{G}) \) obtained, the \(-1\)st component is commutative, and \( F(\mathfrak{G})_{-2} \) is a trivial \( F(\mathfrak{G})_0 = g_0 \)-module. This means that \( i := \bigoplus_{-1 \leq k} \mathfrak{G}_k \) is an ideal in \( F(\mathfrak{G}) \) with the nonpositive part equal to \( g_{-1} \oplus g_0 \). But \( g \) is the maximal Lie algebra with such nonpositive part.

The arguments for depth \( > 1 \) are similar. \( \square \)

7.4.1. Example. Apply method 2 to \( g = \mathfrak{vect}(n;N) = \text{Span}(f_i(u) \partial_i)_{i=1}^n \) in its standard \( \mathbb{Z} \)-grading, where \( u = (u_1, \ldots, u_n) \). We get

\[
\mathfrak{G}_{-2} = \text{Span}(\partial_i x_i)_{i=1}^n, \quad \mathfrak{G}_{-1} = \text{Span}(D_i = \partial_i + x_i \partial_x)_{i=1}^n,
\]

and hence, as spaces, \( \mathfrak{G} = \text{Span}(f_i(x, \xi) D_i)_{i=1}^n \oplus \text{Span}(\partial_j)_{j=1}^n \). The one-to-one correspondence between \( g \) and \( i \), and the description of \( f(u) \) as \( f(x, \xi) \), are as follows:

\[
\partial u_i \leftrightarrow D_i, \quad u_i^{(2k)} \leftrightarrow x_i^{(k)}, \quad u_i^{(2k+1)} \leftrightarrow x_i^{(k)} \xi_i.
\]
8. Deform with an odd parameter and its desuperization

8.1. Simple Lie superalgebras. In the functorial approach, a superspace $g$ is a Lie superalgebra if $g(C) := (g \otimes C)_0$ is a Lie algebra for any supercommutative superalgebra $C$ and any superalgebra homomorphism $C_1 \to C_2$ induces a Lie algebra homomorphism $g(C_1) \to g(C_2)$ with the natural properties of the through and identity maps.

A given Lie superalgebra $g$ is said to be simple if $\dim g > 1$ and it has no proper ideals. The ideal is also defined functorially.

8.2. Deformations of Lie superalgebras. Infinitesimally, the deformation with parameter $t$ of the multiplication in a given Lie superalgebra $g$ is defined by the expression (34):

$$\begin{cases} [x,y] + tc(x,y) & \text{for } x \text{ not proportional to } y \text{ if both are odd}, \\ x^2 + tc(x,x) & \text{for } x = y \text{ odd}, \end{cases}$$

where $c$ is a 2-cocycle with adjoint coefficients and $p(t) = p(c)$, and $x, y \in g$.

Thus, eq. (34) means that the global deformation linearly depending on the parameter is a transition

$$g \to g \otimes \left\{\begin{array}{ll} O(1; 1) = \mathbb{K}[t; \frac{1}{t}] \simeq \mathbb{K}[t]/(t^2) & \text{if } p(t) = 0 \\ \Lambda(1) = \Lambda[t] & \text{if } p(t) = 1. \end{array}\right.$$ 

The desuperization functor $F$ sends $\Lambda[t]$ to $O(1; 1)$. (These spaces only differ by the parity of $t$; this difference can be seen at the level of (co)homology, more precisely (co)chains, of degree $> 1$.) Evaluating $O(1; 1) \to \mathbb{K}$ by sending $t \mapsto a \in \mathbb{K}$ for some $a$ we turn the Lie algebra over $O(1; 1)$ into a Lie algebra over $\mathbb{K}$, equivalently $t$ can be considered an even parameter. For examples, see [BGL3].

9. Related open problems

At the conference [http://reims.math.cnrs.fr/pevzner/records.html](http://reims.math.cnrs.fr/pevzner/records.html) in honor of A. Kirillov, P. Etingof and Yu. Neretin posed questions some of which we interpret as follows:

1) Describe (compare with [Prem]) the algebraic groups corresponding to the Brown algebra $br(3)$, the Weisfeiler-Kac algebras $\mathfrak{w}(1)(3; a)/c$ and $\mathfrak{w}(4; a)$, the orthogonal Lie algebras $\mathfrak{o}(2n)/c$; simple vectorial Lie algebras (especially, for $p = 2$), see [BGLLS1, BGLLS2, Ei], and to deforms thereof, see [WK, BGL2, BLW, BGL3].

2) Describe the algebraic supergroups corresponding to superizations of the Lie algebras of item 1) and those introduced in this paper (combine [Prem] with [FiGa]). Describe the automorphism groups for the simple Lie algebras, not considered in [FG] and [W], and establish isomorphisms between the deforms with the help of these groups à la [KCh, Ch]. Describe the automorphism supergroups of the simple Lie superalgebras obtained by the two methods given in §3.

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