BOUNDDED WEIGHT MODULES OF THE LIE ALGEBRA OF
VECTOR FIELDS ON $\mathbb{C}^2$

ANDREW CAVANESS AND DIMITAR GRANTCHAROV

Abstract. We study weight modules of the Lie algebra $W_2$ of vector fields on $\mathbb{C}^2$. A classification of all simple weight modules of $W_2$ with a uniformly bounded set of weight multiplicities is provided. To achieve this classification we introduce a new family of generalized tensor $W_n$-modules. Our classification result is an important step in the classification of all simple weight $W_n$-modules with finite weight multiplicities.

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Introduction

Lie algebras of vector fields have been studied since the appearance of infinite Lie groups in the works of S. Lie in the late 19th century. Based on the fundamental works of E. Cartan in the early 20th century, some of these infinite-dimensional Lie algebras are known as Lie algebras of Cartan type. A classical example of a Cartan type Lie algebra is the Lie algebra $W_n$ consisting of derivations of the polynomial algebra $\mathbb{C}[x_1,...,x_n]$, or, equivalently, the Lie algebra of polynomial vector fields on $\mathbb{C}^n$. The first classification results concerning representations of $W_n$ and other Cartan type Lie algebras were obtained by A. Rudakov in 1974-1975, [14], [15]. These results address the classification of a class of irreducible $W_n$-representations that satisfy some natural topological conditions. The so-called tensor modules, that is modules $T(\nu,S)$ whose underlying spaces are tensor products $x^\nu \mathbb{C}[x_1^\pm 1,...,x_n^\pm 1] \otimes S$ of a “shifted” Laurent polynomial ring and a finite-dimensional $\mathfrak{gl}_n$-module $S$, play an important role in the works of Rudakov. Tensor $W_1$-modules and extensions of tensor modules were studied extensively in the 1970’s and in the 1980’s by B. Feigin, D. Fuks, I. Gelfand, and others, see for example, [3], [4].

In this paper we focus on the category of weight representations of $W_n$, namely those that decompose as direct sums of weight spaces relative to the subalgebra $\mathfrak{b}_{W_n}$ of $W_n$ spanned by the derivations $x_1 \partial_1,...,x_n \partial_n$. Weight representations of Lie algebras of vector fields (in particular, of $W_n$) are subject of interest by both mathematicians and theoretical physicists in the last 30 years. Another important example of a Lie algebra of vector fields is the Witt algebra $\text{Witt}_n$ consisting of the derivations of the Laurent polynomial algebra $\mathbb{C}[x_1^\pm,...,x_n^\pm]$, or, equivalently, the Lie algebra of polynomial vector fields on the $n$-dimensional complex torus. In particular, Witt_1 is the centerless Virasoro algebra. The classification of all simple weight representations with finite weight multiplicities of $W_1$

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and Witt$_1$ (and hence of the Virasoro algebra) was obtained by O. Mathieu in 1992, [11], proving a conjecture of V. Kac, [10]. Following a sequence of works of S. Berman, Y. Billig, C. Conley, S. Eswara Rao, X. Guo, C. Martin, O. Mathieu, V. Mazorchuk, V. Kac, G. Liu, R. Lu, A. Piard, Y. Su, K. Zhao, very recently, Y. Billig and V. Futorny managed to extend Mathieu’s classification result to Witt$_n$ for arbitrary $n \geq 1$ (see [1] and the references therein). The classification theorem in [1] states roughly that every nontrivial simple weight Witt$_n$-module with finite weight multiplicities is either a submodule of a tensor module or a module of highest weight type.

In contrast with Witt$_n$, the classification of the simple weight $W_n$-modules $M$ with finite weight multiplicities is still an open problem for $n > 1$. The possible supports (sets of weights) of all such $M$ have been described by I. Penkov and V. Serganova in [13]. In addition, in [13], a parabolic induction theorem for such modules $M$ is proven. More precisely, it is shown that $M$ is a quotient of a parabolically induced module from a parabolic subalgebra $p$ of $W_n$. Unfortunately, the parabolic subalgebras $p = l \oplus n^+$ that appear in the parabolic induction theorem are quite complicated and have Levi components isomorphic to a semi-direct sum of Lie algebras of Cartan type and finite dimensional-reductive Lie algebras. Another obstacle in the study of weight $W_n$-modules is the fact that the $W_n$-modules $T(\nu, S)$ are highly reducible - they may contain $2^n$ simple subquotients.

The purpose of this paper is to make the first step towards the classification of the simple weight $W_n$-modules with finite weight multiplicities. Namely, we classify the simple bounded modules of $W_2$, that is, all simple weight $W_2$-modules whose sets of weight multiplicities is uniformly bounded. The classification is given in Theorem 4.20 (the tensor modules are introduced in Definition 2.4). The second step in the weight module classification is to classify all simple bounded $l$-modules, where $l$ is a Levi subalgebra of a parabolic subalgebra of $W_n$. The last step is, based on the parabolic induction theorem of Penkov-Serganova, to complete the classification in question. The second and the third steps will be addressed in a subsequent paper. We note that for $n = 2$, the classification of simple bounded $l$-modules, where $l$ is a Levi subalgebra of a parabolic subalgebra of $W_2$, is obtained in the present paper and, in fact, is used to classify the simple bounded $W_2$-modules. It turns out that in this case $l \simeq \text{Der } \mathbb{C}[x] \ltimes \mathbb{C}[x]$.

In addition to obtaining the classification of simple weight modules with finite weight multiplicities, the results in the present paper will be essential for the study of the category $\mathcal{B}$ of bounded representations of $W_n$. This category is intimately related to the corresponding category of bounded $\mathfrak{sl}_{n+1}$-modules. It is expected that, like in the case of $\mathfrak{sl}_{n+1}$, the indecomposable injectives of $\mathcal{B}$ will have a nice geometric realizations in terms of twisted functions and twisted differential forms on algebraic varieties, [7], [8].

An important tool in the present paper is the twisted localization functor, a functor used by O. Mathieu in the proof of another fundamental result: the classification of all simple weight modules with finite weight multiplicities of finite-dimensional reductive Lie algebras, [12]. Also, in order to deal with the reducibility of $T(\nu, S)$, we introduce a family of (generalized) tensor modules $T(\nu, S, J)$. The modules $T(\nu, \lambda, J) = T(\nu, S, J)$ are defined for a tuple $J = (a_1, ..., a_k)$ of signed integers $a_i = b_i^+$ or $a_i = b_i^-$, where the $b_i$’s are in the set of all indices $j$ such that $\lambda_j - \nu_j \in \mathbb{Z}$, and $\lambda$ is the highest weight of $S$. Our
main result is that all nonzero simple bounded $W_2$-modules are isomorphic to $T(\nu, \lambda, J)$ for some $\nu, \lambda, J$. A similar result will hold for the simple bounded weight $W_n$-modules ($n \geq 2$) as well, but additional conditions for the tensor modules $T(\nu, \lambda, J)$ corresponding to fundamental weights $\lambda$ have to be imposed.

The content of the paper is as follows. In Section 2 we collect important results on the twisted localization functor, parabolic subalgebras and tensor modules of $W_n$. In particular we provide an explicit list of the possible parabolic subalgebras $\mathfrak{p}$ of $W_2$. In Section 3, we classify all simple bounded modules over the Lie algebra $A = \text{Der} \mathbb{C}[x] \ltimes \mathbb{C}[x]$. In Section 4, based on the results of Section 3, we complete the classification of simple bounded $W_2$-modules.

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1. Notation and Conventions

Throughout the paper the ground field is $\mathbb{C}$. All vector spaces, algebras, and tensor products are assumed to be over $\mathbb{C}$ unless otherwise stated.

By $W_n$ we denote the Lie algebra $\text{Der} \mathbb{C}[x_1, \ldots, x_n]$ of derivations of $\mathbb{C}[x_1, \ldots, x_n]$. Also, $A_n$ will be the semi-direct product $A_n = \text{Der} \mathbb{C}[x_1, \ldots, x_n] \ltimes \mathbb{C}[x_1, \ldots, x_n]$. For simplicity we set $A := A_1$ and $\partial_i := \frac{\partial}{\partial x_i}$. Every element $w$ of $W_n$ can be written uniquely as $w = \sum_{i=1}^{n} f_i \partial_i$, for some $f_i \in \mathbb{C}[x_1, \ldots, x_n]$.

By $\mathbb{Z}_{\geq k}$ we denote the set of all integers $n$ such that $n \geq k$. We similarly define $\mathbb{Z}_{\leq k}$, $\mathbb{Z}_{> k}$, $\mathbb{R}_{\geq k}$, etc. If $M$ is a set of real numbers, and $S$ is a subset of a real vector space $V$, then by $MS$ we denote the set of all $M$-linear combinations of elements in $S$.

For a Lie algebra $\mathfrak{a}$ by $U(\mathfrak{a})$ we denote the universal enveloping algebra of $\mathfrak{a}$.

Throughout the paper we use the multi index-notation for monomials: $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ if $x = (x_1, \ldots, x_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$. If $n$ is fixed, we set $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$, $\mathbb{C}[x^{\pm 1}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $x^\nu \mathbb{C}[x^{\pm 1}] = x_1^{\nu_1} \cdots x_n^{\nu_n} \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where the latter is the span of all (formal) monomials $x_1^{k_1+\nu_1} \cdots x_n^{k_n+\nu_n}$, $k_i \in \mathbb{Z}$.

For an $n$-tuple $\nu = (\nu_1, \ldots, \nu_n)$ in $\mathbb{C}^n$, we set $\text{Int}(\nu) := \{i \mid \nu_i \in \mathbb{Z}\}$.

2. Preliminaries

2.1. The Lie algebra $A$. Recall that $A = \text{Der} \mathbb{C}[x] \ltimes \mathbb{C}[x]$ and $W_1 = \text{Der} \mathbb{C}[x]$. Note that by definition $[D, f] = Df$ for $D \in \text{Der} \mathbb{C}[x]$ and $f \in \mathbb{C}[x]$. In terms of generators and relations, $A$ can be defined as follows:

$$A = \text{Span}\{D_i, I_j \mid i \in \mathbb{Z}_{\geq -1}, j \in \mathbb{Z}_{\geq 0}\}$$

with

$$[D_i, D_j] = (j - i)D_{i+j},$$
$$[D_i, I_j] = jI_{i+j},$$
$$[I_i, I_j] = 0.$$
Here $D_i$ and $I_j$ correspond to $x^{i+1}\partial$ and $x^j$, respectively. Note that the center of $\mathcal{A}$ is generated by $I_0$. We say that an $\mathcal{A}$-module $M$ has central charge $c$ if $I_0m = cm$ for every $m \in M$. In particular, every irreducible $\mathcal{A}$-module $M$ has a central charge.

We say that $\mathfrak{h} \subset \mathcal{A}$ is a Cartan subalgebra of $\mathcal{A}$ if $\mathfrak{h}$ is both self-normalizing and nilpotent. In what follows, we fix the Cartan subalgebra of $\mathcal{A}$ to be $\mathfrak{h}_\mathcal{A} = \text{Span}\{D_0, I_0\}$. We also have the triangular decomposition $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{A}^0 \oplus \mathcal{A}^+$, where $\mathcal{A}^- = \text{Span}\{D_{-1}\}$, $\mathcal{A}^0 = \mathfrak{h}_\mathcal{A}$, and $\mathcal{A}^+ = \text{Span}\{D_i, I_j \mid i, j \geq 1\}$. Define $\varepsilon, \delta \in \mathfrak{h}_\mathcal{A}^*$ by the identities

$$\varepsilon(D_0) = 1, \quad \varepsilon(I_0) = 0; \quad \delta(D_0) = 0, \quad \delta(I_0) = 1.$$  

2.2. Injective and finite actions. Let $\mathfrak{g}$ be Lie algebra, and $M$ be a $\mathfrak{g}$-module. We say that an element $x$ of $\mathfrak{g}$ acts locally nilpotently (or, finitely) on a vector $m$ in $M$, if there is $N = N(x, m)$ such that $x^N(m) = 0$. If such $N$ does not exists we say that $x$ acts injectively on $m$. We say that $x$ acts injectively (respectively, finitely) on $M$ if $x$ acts injectively (respectively, finitely) on all $m \in M$.

We will often use the following setting. Let $x$ be an ad-nilpotent element in $\mathfrak{g}$ and let $M$ be a $\mathfrak{g}$-module. Then the set $M^{(x)}$ of all $m$ on which $x$ acts finitely is a submodule of $M$. In particular, if $M$ is simple, then every ad-nilpotent element $x$ of $\mathfrak{g}$ acts either finitely or injectively on $M$.

2.3. Weight modules. We first introduce weight modules in a general setting. Let $\mathcal{U}$ be an associative unital algebra and $H \subset \mathcal{U}$ be a commutative subalgebra. We assume in addition that $H$ is a polynomial algebra identified with the symmetric algebra of a vector space $\mathfrak{h}$, and that we have a decomposition

$$\mathcal{U} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathcal{U}^\mu,$$

where

$$\mathcal{U}^\mu = \{x \in \mathcal{U} \mid [h, x] = \mu(h)x, \forall h \in \mathfrak{h}\}.$$  

Let $Q_H = \mathbb{Z}\Delta_H$ be the $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ generated by $\Delta_H = \{\mu \in \mathfrak{h}^* \mid \mathcal{U}^\mu \neq 0\}$. We obviously have $\mathcal{U}^\mu \mathcal{U}^{\nu} \subset \mathcal{U}^{\mu + \nu}$.

We call a $\mathcal{U}$-module $M$ a generalized weight ($\mathcal{U}, H$)-module (or just generalized weight $\mathcal{U}$-module) if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^{(\lambda)}$, where

$$M^{(\lambda)} = \{m \in M \mid (h - \lambda(h)\text{Id})^N m = 0 \text{ for some } N > 0 \text{ and all } h \in \mathfrak{h}\}.$$  

We call $M^{(\lambda)}$ the generalized weight space of $M$ and $\dim M^{(\lambda)}$ the weight multiplicity of the weight $\lambda$. A vector $v$ in $M^{(\lambda)}$ is called a weight vector of weight $\lambda$ and we write $\text{wht}(v) = \lambda$. Note that

$$\mathcal{U}^\mu M^{(\lambda)} \subset M^{(\mu + \lambda)}.$$  

A generalized weight module $M$ is called a weight ($\mathcal{U}, H$)-module (or just weight $\mathcal{U}$-module) if $M^{(\lambda)} = M^{\lambda}$, where

$$M^{\lambda} = \{m \in M \mid (h - \lambda(h)\text{Id})m = 0 \text{ for all } h \in \mathfrak{h}\}.$$

In the case when $\mathcal{U} = U(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, a (generalized) weight ($\mathcal{U}, H$)-module will be called (generalized) weight ($\mathfrak{g}, \mathfrak{h}$)-module.
Definition 2.1. (i) An $A$-module $M$ is a weight $A$-module if $M$ is a weight $(A, H)$-module for $H = \mathbb{C}[a_A]$. If $M$ is a weight $A$-module we call the set of weights $\lambda \in h^*_A$ such that $M^\lambda \neq 0$ the $A$-support (or simply the support) of $M$ and denote it by $\text{Supp} M$.

(ii) We say that a weight $A$-module $M$ is bounded if there is $N > 0$ such that $\dim M^\lambda < N$ for all $\lambda \in h^*_A$. If $M$ is bounded we call $\sup\{\dim M^\lambda \mid \lambda \in h^*_A\}$ the $A$-degree (or simply the degree) of $M$.

The adjoint module $A$ is a weight module of central charge 0 such that $A^\lambda \neq 0$ if and only if $\lambda = n\varepsilon$ for $n \in \mathbb{Z}_{\geq -1}$. The set $\Delta_A = \{-\varepsilon, n\varepsilon \mid n \in \mathbb{Z}_{>0}\}$ is the root system of $A$, and

$$A^{-\varepsilon} = \text{Span}\{D_{-1}\}; \quad A^{n\varepsilon} = \text{Span}\{I_n, D_n\}, n \in \mathbb{Z}_{>0}$$

are the root spaces of $A$.

If $M$ has central charge $c$, then a weight of $M$ is of the form $\lambda \varepsilon + c\delta$, for some $\lambda \in \mathbb{C}$. If $c$ is fixed, with a slight abuse of notation we set $M^\lambda = M^{\lambda \varepsilon + c\delta}$, for all weight modules $M$ with central charge $c$. In particular, $M = \bigoplus_{\lambda \in \mathbb{C}} M^\lambda$ and $\text{Supp} M \subset \mathbb{C}$.

We similarly introduce the notions of weight and bounded $W_n$-modules. More precisely, let $\mathfrak{h}_{W_n}$ (or simply $\mathfrak{h}_W$ if $n$ is fixed) be the subalgebra $\text{Span}\{x_1\partial_1, \ldots, x_n\partial_n\}$. Then $\mathfrak{h}_W$ is a Cartan subalgebra of $W_n$.

Definition 2.2. A $W_n$-module $M$ is a weight $W_n$-module if $M$ is a weight $(W_n, H)$-module with $H = \mathbb{C}[b_W]$. We say that a weight $W_n$-module $M$ is bounded if there is $N > 0$ such that $\dim M^\lambda < N$ for all $\lambda \in h^*_W$. If $M$ is bounded we call $\sup\{\dim M^\lambda \mid \lambda \in h^*_W\}$ the $W_n$-degree (or simply the degree) of $M$.

Note that $W_n$ is a weight $W_n$-module whose support is $\Delta_{W_n} \cup \{0\}$, where $\Delta_{W_n}$ is the root system of $W_n$. We identify the root lattice of $W_n$ with $\mathbb{Z}^n$ and will often write every element $\alpha$ of $\mathbb{Z}\Delta_{W_n}$ as an $n$-tuple $(\alpha_1, \ldots, \alpha_n)$ of integers. In particular,

$$\Delta_{W_n} \cup \{0\} = \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \geq 0\} \cup \{(\alpha_1, \ldots, \alpha_n) \mid \exists i : \alpha_i = -1 \text{ and } \alpha_j \geq 0 \text{ for all } j \neq i\}.$$

For simplicity we will often write $\Delta_{W_1}$ for $\Delta_{W_n}$. In what follows we use the (root) basis of $W_n$ consisting of the elements $x^\alpha(x_i\partial_i)$, $\alpha \in \Delta_{W_n} \cup \{0\}$, $i = 1, \ldots, n$.

2.4. Tensor modules. We say that $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is a dominant integral $gl_n$-weight if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n - 1$. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a dominant integral weight, by $L_{gl}(\lambda) = L_{gl}(\lambda_1, \ldots, \lambda_n)$ we denote the simple finite-dimensional module with highest weight $\lambda$.

For a dominant integral $gl_n$-weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ and any $\nu = (\nu_1, \ldots, \nu_2)$ in $\mathbb{C}^n$, we define the $W_n$-modules $T(\nu, \lambda)$ as follows:

$$T(\nu, \lambda) = x^\nu \mathbb{C}[x^{\nu+1}] \otimes L_{gl}(\lambda)$$

with $W_n$-action defined by

$$(x^\alpha x_i \partial_i) \cdot (x^s \otimes v) = s_i x^{\alpha^s + s} \otimes v + \sum_{j=1}^n \alpha_j x^{\alpha^s + s} \otimes E_{ji} v,$$

where $\alpha \in \Delta_{W_n} \cup \{0\}$, $s \in \nu + \mathbb{Z}^n$, $v \in L_{gl}(\lambda)$, and $E_{ji}$ is the $(j, i)$th elementary matrix of $gl_n$. As indicated in the introduction, these modules play important role in the
classification of simple weight modules with finite weight multiplicities over various classes of Lie algebras. We easily extend the \( W_n \)-action on \( T(\nu, \lambda) \) to an \( A_n \)-action. Namely, for \( c \in \mathbb{C} \) we define \( T(\nu, \lambda, c) = T(\nu, \lambda) \) as vector space and set

\[
(x^j \cdot (x^s \otimes v) := cx^{j+s} \otimes v.
\]

The next theorem gives a necessary and sufficient condition when two tensor modules are isomorphic as \( W_n \)-modules and \( A_n \)-modules. The fact is well-known but for reader’s convenience a short proof suggested by M. Gorelik is provided.

**Proposition 2.3.** The following are equivalent.

(i) \( T(\nu, \lambda) \simeq T(\nu', \lambda') \) as \( W_n \)-modules.

(ii) \( T(\nu, \lambda) \simeq T(\nu', \lambda') \) as \( A_n \)-modules.

(iii) \( \nu - \nu' \in \mathbb{Z}^n \) and \( \lambda = \lambda' \).

**Proof.** The fact that (iii) implies (i) and (ii) is straightforward. Also, obviously (ii) implies (i). It remains to show that (i) implies (iii).

Let \( \psi : T(\nu, \lambda) \to T(\nu', \lambda') \) be an isomorphism. Since the \( \mu \)-weight space of \( T(\nu, \lambda) \) is \( x^\mu \otimes L(\lambda) \), we have that for every \( s \in \nu + \mathbb{Z}^n \) and \( u \in L(\lambda) \), \( \psi(x^s \otimes u) = x^s \otimes u' \) for some \( u' \in L(\lambda') \). Also, \( \dim L(\lambda) = \dim L(\lambda') \).

Let \( v_\lambda \) be a highest weight vector of \( L(\lambda) \), and let us fix \( s \in \nu + \mathbb{Z}^n \) such that \( s_i \neq 0 \) for every \( i \). Also, let \( \psi(x^s \otimes v_\lambda) = x^s \otimes v \) for every \( v \in L(\lambda') \). Denote by \( v_i \) the \( gl_n \)-weight components of \( v \), i.e. \( v = \sum_{i=1}^t v_i \), where \( v_i \in L(\lambda)^{n_i} \) are nonzero vectors and \( \eta_1, \ldots, \eta_t \) are distinct weights (of \( gl_n \)). Assume that \( \eta_i \) is a minimal element in \( \{ \eta_1, \ldots, \eta_t \} \) with respect to the standard partial order on \( \mathfrak{h}^{\ast} \). Then for \( 1 \leq j < i \leq n \) we have

\[
(x_i, x_j) \partial_j(x^s \otimes v) = x^s \otimes (s_i + E_{ji})(s_j - E_{jj})v;
\]

\[
(x_i, x_j) \partial_j(x^s \otimes v_\lambda) = s_i(s_j - \lambda_j \cdot x^s \otimes v_\lambda),
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \). Thus

\[
(s_i + E_{ji})(s_j - E_{jj})v = s_i(s_j - \lambda_j)v.
\]

Using the minimality of \( \eta_1 \), after taking the \( \eta_1 \)-components of the vector above, we obtain \( E_{jj}v_1 = \lambda_j v_1 \) for all \( j < n \). Thus \( \lambda - \eta_1 = c \varepsilon_n \) some \( c \in \mathbb{C} \). But since \( \eta_1 \) is in the support of \( L(\lambda') \) and \( \lambda' \) is a maximal weight in this support, we have \( \lambda' - \lambda \geq -c \varepsilon_n \). With similar reasoning we obtain \( \lambda - \lambda' \geq -c' \varepsilon_n \) for some \( c' \in \mathbb{C} \). Thus \( \lambda - \lambda' \in \mathbb{C} \varepsilon_n \). Now using this and the Weyl dimension formula for \( \dim L(\lambda) = \dim L(\lambda') \), we prove that \( \lambda = \lambda' \). \( \square \)

In what follows we introduce some important subquotients of the \( W_n \)-modules \( T(\nu, \lambda) \) defined above. First, for any \( z \in \mathbb{C}^n \), we set \( P_M(z) = \{ +, - \}^{\text{Int}(z)} \). Every element \( J : \text{Int}(z) \to \{ +, - \} \) of \( P_M(z) \) will be written as \( (i_1^{J(i_1)}, \ldots, i_k^{J(i_k)}) \), where \( \text{Int}(z) = \{ i_1, \ldots, i_k \} \) and \( i_1 < \cdots < i_k \). For example, if \( \text{Int}(z) = \{ 1, 2 \} \), then the elements of \( P_M(z) \) are: \( (1^+, 2^+), (1^+, 2^-), (1^-, 2^+), (1^-, 2^-) \).

For every element \( J \) in \( P_M(z) \) we write \( J^+ \) (respectively, \( J^- \)) for the subset of \( J \) consisting of all \( i_j^{J(i_j)} \) with \( J(i_j) = "^+" \) (respectively, \( J(i_j) = "^-" \)).

**Definition 2.4.** Let \( \lambda \) be a dominant integral \( gl_n \)-weight, \( \nu \in \mathbb{C}^n \), and \( J \) be in \( P_M(\lambda - \nu) \). We define \( T(\nu, \lambda, J) \) as follows.
(i) \( T(\nu, \lambda, \emptyset) := T(\nu, \lambda) \).
(ii) If \( J^+ \neq \emptyset \) and \( J^- = \emptyset \), then

\[
T(\nu, \lambda, J) := \text{Span}\{x^n \otimes v_\mu \mid v_\mu \in L_{gl}(\lambda)^\mu, \eta_i - \mu_i \in \mathbb{Z}_{\geq 0} \text{ for all } i \in J\}.
\]

(iii) If \( J^- \neq \emptyset \), then

\[
T(\nu, \lambda, J) := T(\nu, \lambda, J^+)/ \left( \sum_{j \in J^-} T(\nu, \lambda, J^+ \cup \{j^-\}) \right)
\]

It is easy to check that if \( J^- = \emptyset \), then \( T(\nu, \lambda, J) \) is a submodule of \( T(\nu, \lambda) \). Therefore all \( T(\nu, \lambda, J) \) are bounded weight \( W_n \)-modules of degree \( \dim L_{gl}(\lambda) \).

Using \([23]\), we easily endow \( T(\nu, \lambda, J) \) with an \( A_n \)-module structure and the resulting module \( T(\nu, \lambda, J, c) \) has central charge \( c \). In what follows, we will call both \( T(\nu, \lambda, J) \) and \( T(\nu, \lambda, J, c) \) generalized tensor modules, or simply tensor modules.

2.5. Tensor \( A_1 \)- and \( W_1 \)-modules and classification of simple weight \( W_1 \)-modules with finite weight multiplicities. In the case \( n = 1 \) we will use the notation \( T(\nu, \lambda) \) and \( T(\nu, \lambda, c) \) for the tensor modules corresponding to \( \nu, \lambda \in \mathbb{C} \). More explicitly, \( T(\nu, \lambda, c) = \text{Span}\{x^{\nu+\ell} \otimes v_\lambda \mid \ell \in \mathbb{Z}\} \), with action of \( A \) defined by:

\[
D_i(x^{\nu+\ell} \otimes v_\lambda) = (\nu + \ell + i\lambda)x^{\nu+\ell+i} \otimes v_\lambda,
\]

\[
I_j(x^{\nu+\ell} \otimes v_\lambda) = cx^{\nu+\ell+j} \otimes v_\lambda
\]

Remark 2.5. Another important class of modules that appears in the literature consists of the tensor densities modules. Namely these are the \( A \)-modules \( F(\nu, \lambda, c) = x^{\nu}[x^{\pm 1}](dx)^\lambda \) of central charge \( c \) and with the natural action of the generators \( D_i \) and \( I_j \). One easily can show that \( F(\nu, \lambda, c) \simeq T(\nu + \lambda, \lambda, c) \).

We will also write \( T(\lambda, \lambda, +) = T(\lambda, \lambda, 1^+) \) and \( T(\lambda, \lambda, -) = T(\lambda, \lambda, 1^-) \). In particular, \( T(\lambda, \lambda, +) = \text{Span}\{x^{\lambda+n} \otimes v_\lambda \mid n \in \mathbb{Z}_{\geq 0}\} \), and \( T(\lambda, \lambda, -) = T(\lambda, \lambda) / T(\lambda, \lambda, +) \). Furthermore, the corresponding \( A \)-modules to \( T(\lambda, \lambda, \pm) \) will be denoted by \( T(\lambda, \lambda, c, \pm) \). The Jordan-Hölder decomposition of the modules \( T(\nu, \lambda) \) and \( T(\nu, \lambda, c) \) is described in the following two propositions. The proof is standard and is omitted.

**Proposition 2.6.** Let \( \lambda, \nu \in \mathbb{C} \)

(i) The \( W_1 \)-modules \( T(\nu_1, \lambda_1) \) and \( T(\nu_2, \lambda_2) \) are isomorphic if and only if:

(a) \( \nu_1 - \nu_2 \in \mathbb{Z} \) and \( \lambda_1 = \lambda_2 \), or

(b) \( \nu_1 - \nu_2 \in \mathbb{Z} \), \( \nu_1 \notin \mathbb{Z} \), and \( \{\lambda_1, \lambda_2\} = \{0, 1\} \).

(ii) The \( W_1 \)-module \( T(\nu, \lambda) \) is simple if and only if \( \lambda - \nu \notin \mathbb{Z} \).

(iii) The \( W_1 \)-module \( T(\lambda, \lambda, +) \) is simple if and only if \( \lambda \neq 0 \). The \( W_1 \)-module \( T(0, 0, +) \) has length two with a simple submodule isomorphic to \( \mathbb{C} \) and a simple quotient isomorphic to \( T(1, 1, +) \).

(iv) The \( W_1 \)-module \( T(\lambda, \lambda, -) \) is simple if and only if \( \lambda \neq 1 \). The \( W_1 \)-module \( T(1, 1, -) \) has length two with a simple submodule isomorphic to \( T(0, 0, -) \), and a simple quotient isomorphic to \( \mathbb{C} \).
In view of the above proposition, for \( \lambda \neq 0 \) we set \( L(\lambda, +) := T(\lambda, \lambda, +) \) and \( L(\lambda, -) = T(\lambda + 1, \lambda + 1, -) \). We also set \( L(0) := \mathbb{C} \). Note that \( L(\lambda, +) \) and \( L(\lambda, -) \) are highest weight modules with highest weight \( \lambda \) with respect to the Borel subalgebras \( b(+) = \text{Span}\{D_i \mid i \in \mathbb{Z}_{\geq 0}\} \) and \( b(-) = \text{Span}\{D_i \mid i \in \{0, -1\}\} \), respectively.

We now look at the structure of the tensor \( \mathcal{A} \)-modules. In the case \( c = 0 \), we simply restate Proposition 2.6 replacing the statements for \( T(\nu, \lambda) \), \( T(\lambda, \lambda, \pm) \) by \( T(\nu, 0, 0) \), \( T(\lambda, 0, 0, \pm) \), respectively. We also write \( L(\lambda, 0, +) \) and \( L(\lambda, 0, -) \) for \( T(\lambda, \lambda, 0, +) \) and \( T(\lambda + 1, \lambda + 1, 0, -) \) if \( \lambda \neq 0 \), and \( L(0, 0) = \mathbb{C} \).

For tensor \( \mathcal{A} \)-modules with nonzero central charge we have the following.

**Proposition 2.7.** Let \( \lambda, \nu \in \mathbb{C} \) and let \( c \neq 0 \).

(i) The \( \mathcal{A} \)-modules \( T(\nu_1, \lambda_1, c) \) and \( T(\nu_2, \lambda_2, c) \) are isomorphic if and only if \( \nu_1 - \nu_2 \in \mathbb{Z} \) and \( \lambda_1 = \lambda_2 \).

(ii) The \( \mathcal{A} \)-module \( T(\nu, \lambda, c) \) is simple if and only if \( \lambda - \nu \notin \mathbb{Z} \).

(iii) The \( \mathcal{A} \)-modules \( T(\lambda, \nu, c, +) \) and \( T(\lambda, \lambda, c, -) \) are simple for all \( \lambda \in \mathbb{C} \).

Naturally, for \( c \neq 0 \) we set \( L(\lambda, c, +) := T(\lambda, \lambda, c, +) \) and \( L(\lambda, c, -) := T(\lambda + 1, \lambda + 1, c, +) \).

We finish this subsection with the classification theorem for all simple weight \( W_1 \)-modules with finite weight multiplicities due to O. Mathieu, see [11].

**Theorem 2.8.** Every simple weight \( W_1 \)-module with finite weight multiplicities is isomorphic to a module in the following list: \( T(\nu, \lambda) \), \( \lambda - \nu \notin \mathbb{Z} \), \( L(\eta, +) \), \( L(\eta, -) \), \( \eta \neq 0 \), \( L(0) \). The only isomorphisms among the modules in the list are: \( T(\nu, \lambda) \simeq T(\nu + n, \lambda) \) for \( n \in \mathbb{Z} \) and \( \lambda - \nu \notin \mathbb{Z} \); \( T(\nu, 0) \simeq T(\nu, 1) \), for \( \nu \notin \mathbb{Z} \).

2.6. **Twisted localization of weight modules.** We first introduce the twisted localization functor in a general setting. Let \( \mathcal{U} \), \( \mathcal{H} = \mathbb{C}[\mathfrak{h}] \) be as in [2,3]. Let \( a \) be an ad-nilpotent element of \( \mathcal{U} \). Then the set \( \langle a \rangle = \{a^n \mid n \geq 0\} \) is an Ore subset of \( \mathcal{U} \) (see for example §4 in [12]) which allows us to define the \( \langle a \rangle \)-localization \( D_{\langle a \rangle} \mathcal{U} \) of \( \mathcal{U} \). For a \( \mathcal{U} \)-module \( M \) by \( D_{\langle a \rangle} M = D_{\langle a \rangle} \mathcal{U} \otimes_{\mathcal{U}} M \) we denote the \( \langle a \rangle \)-localization of \( M \). Note that if \( a \) is injective on \( M \), then \( M \) is isomorphic to a submodule of \( D_{\langle a \rangle} M \). In the latter case we will identify \( M \) with that submodule, and will consider \( M \) as a submodule of \( D_{\langle a \rangle} M \).

We next recall the definition of the generalized conjugation of \( D_{\langle a \rangle} \mathcal{U} \) relative to \( x \in \mathbb{C} \). This is the automorphism \( \phi_x : D_{\langle a \rangle} \mathcal{U} \to D_{\langle a \rangle} \mathcal{U} \) given by

\[
\phi_x(u) = \sum_{i \geq 0} \binom{x}{i} \text{ad}(a)^i(u)a^{-i}.
\]

If \( x \in \mathbb{Z} \), then \( \phi_x(u) = a^xua^{-x} \). With the aid of \( \phi_x \) we define the twisted module \( \Phi_x(M) = M^{\phi_x} \) of any \( D_{\langle a \rangle} \mathcal{U} \)-module \( M \). Finally, we set \( D_{\langle a \rangle}^x M = \Phi_x D_{\langle a \rangle} M \) for any \( \mathcal{U} \)-module \( M \) and call it the twisted localization of \( M \) relative to \( a \) and \( x \). We will use the notation \( a^x \cdot m \) (or simply \( a^x m \)) for the element in \( D_{\langle a \rangle}^x M \) corresponding to \( m \in D_{\langle a \rangle} M \).

In particular, the following formula holds in \( D_{\langle a \rangle}^x M \):

\[
u(a^x m) = a^x \left( \sum_{i \geq 0} \binom{-x}{i} \text{ad}(a)^i(u)a^{-i}m \right)
\]
for \( u \in U, m \in D_{(a)} M \).

We easily check that if \( a \) is a weight element of \( U \) (i.e. \( a \in U^\mu \) for some \( \mu \in \mathfrak{h}^* \)) and \( M \) is a (generalized) \((U, \mathcal{H})\)-weight module, then \( D_{(a)}^\xi M \) is a (generalized) \((U, \mathcal{H})\)-weight module. We will apply the twisted localization functor for several pairs \((U, \mathcal{H})\), and in particular in the following two cases: \( U = U(A), \mathcal{H} = \mathbb{C}[\mathfrak{h}_A] \); and \( U = U(W_2), \mathcal{H} = \mathbb{C}[\mathfrak{h}_{W_2}] \).

**Lemma 2.9.** Let \( a \in U \) be an ad-nilpotent weight element in \( U \), \( M \) be a simple \( a \)-injective weight \( U \)-module, and let \( z \in \mathbb{C} \). If \( N \) is any simple submodule of \( D_{(a)}^z M \), then \( D_{(a)} M \simeq D_{(a)}^z N \). In particular, if \( a \) acts bijectively on \( M \), \( M \simeq D_{(a)}^z N \).

**Proof.** We use the fact that if \( M \) is a simple weight \( U \)-module, then \( D_{(a)} M \) and \( D_{(a)}^z M \) are simple \( D_{(a)} U \)-modules. So, if \( N \) is any simple submodule of \( D_{(a)}^z M \), then \( D_{(a)} N \) is a submodule of \( D_{(a)}^z M \). This forces \( D_{(a)} N \simeq D_{(a)}^z M \) and \( D_{(a)}^z N \simeq D_{(a)} M \). If \( a \) acts bijectively, then \( M \simeq D_{(a)} M \).

The above lemma will be applied both for \( U = U(A) \) and \( U = U(W_2) \). For reader’s convenience, we list some (but not all) ad-locally nilpotent weight elements \( a \) in \( U \) in these two cases:

1. \( a = D_{-1}, I_j, j \geq 0 \) for \( U = U(A) \);
2. \( a = \partial_1, \partial_2, x_1 \partial_2, x_2 \partial_1 \) for \( U = U(W_2) \).

**Lemma 2.10.** Let \( \nu \notin \mathbb{Z} \). Then the following \( A \)-module isomorphisms hold.

- (i) \( D_{(t_1)}^\nu T(\lambda, \lambda, c, +) \simeq T(\lambda + \nu, \lambda, c) \), whenever \( c \neq 0 \).
- (ii) \( D_{(\mathcal{D})}^\nu T(\lambda, \lambda, c, -) \simeq T(\lambda + \nu, \lambda, c) \).
- (iii) \( D_{(\mathcal{D})}^\nu T(\lambda + \nu, \lambda, c) \simeq T(\lambda + \eta, \lambda, c) \), whenever \( \eta \notin \mathbb{Z} \).

**Proof.** (i) Note that if \( c \neq 0 \), the set \( \{ I_1^{\nu+\ell}(x^\lambda \otimes v_\lambda) \mid \ell \in \mathbb{Z} \} \) forms a basis of \( D_{(t_1)}^\nu T(\lambda, \lambda, c, +) \). Then it is easy to check that the map

\[
I_1^{\nu+\ell}(x^\lambda \otimes v_\lambda) \mapsto c^{\nu+\ell}x^{\lambda+\nu+\ell} \otimes v_\lambda
\]

provides an isomorphism \( D_{(t_1)}^\nu T(\lambda, \lambda, c, +) \simeq T(\lambda + \nu, \lambda, c) \).

(ii) Note that the set \( \{ D_{(\mathcal{D})}^{-\nu-\ell}(x^{\lambda-1} \otimes v_\lambda + T(\lambda(\lambda, c, +)) \mid \ell \in \mathbb{Z} \} \) forms a basis of \( D_{(\mathcal{D})}^{-\nu} T(\lambda, \lambda, c, -) \). Then it is easy to check that the map

\[
D_{(\mathcal{D})}^{-\nu-\ell}(x^{\lambda-1} \otimes v_\lambda + T(\lambda(\lambda, c, +))) \mapsto D_{-1}(\nu, \ell)x^{\lambda+\nu+\ell} \otimes v_\lambda,
\]

where \( D_{-1}(\nu, \ell) = \nu(\nu-1)\ldots(\nu-(\ell-1)) \) if \( \ell < 0 \) and \( D_{-1}(\nu, \ell) = \frac{1}{(\nu+1)\ldots(\nu+\ell)} \) if \( \ell \geq 0 \), provides an isomorphism \( D_{(\mathcal{D})}^{-\nu} T(\lambda, \lambda, c, -) \simeq T(\lambda + \nu, \lambda, c) \).

Part (iii) easily follows from (ii) and the additive property of the twisted localization functors: \( D_{(\mathcal{D})}^{\nu_1+\nu_2} M \simeq D_{(\mathcal{D})}^{\nu_1} D_{(\mathcal{D})}^{\nu_2} M \). □

**Remark 2.11.** Note that the isomorphism in Lemma 2.10(ii) holds even in the case \( (\lambda, c) = (1, 0) \), namely when \( T(\lambda, \lambda, c, -) \) and \( T(\lambda + \nu, \lambda, c) \) are not simple. Also, the coefficients \( D_{-1}(\nu, \ell) \) in the proof of Lemma 2.10(ii) can also be defined in terms of Gamma-functions: \( D_{-1}(\nu, \ell) = \frac{\Gamma(\nu+\ell+1)}{\Gamma(\nu+\ell+1)} \).
2.7. **Parabolic subalgebras of $W_2$ and Penkov-Serganova Parabolic Induction Theorem.** To keep the content of the paper short we will avoid the general definition of a parabolic subalgebra of $W_n$. Such a definition in terms of flags of real subspaces can be found in §1 of [13]. Alternatively, one can use the general definition of a parabolic set of roots $P$ and then define a parabolic subalgebra $p_P$ associated with $P$ following [9]. The problem is that the root system of $W_n$ is neither symmetric nor finite.

In what follows we fix $\sigma$ to be the automorphism of $\Delta_{W_2}$ interchanging $\varepsilon_1$ and $\varepsilon_2$. This automorphism naturally defines an automorphism of $h_{W_2}, h_{W_2}$ and $W_2$. With a slight abuse of notation we will denote all resulting automorphisms by $\sigma$.

We have a natural embedding of $\mathfrak{s}l_{n+1}$ in $W_n$ arising from the infinitesimal action of the group $PSL(n+1)$ on $\mathbb{C}P^n$. In explicit terms, the embedding $\Phi$ is defined by $E_{ij} \mapsto x_i \partial_j$, $1 \leq i, j \leq n$, $E_{0k} \mapsto x_i \mathcal{E}$, $E_{k0} \mapsto -\partial_k$, $1 \leq k \leq n$, where $\mathcal{E} = \sum_{k=1}^{n} x_k \partial_k$. With the embedding of $\mathfrak{s}l(n+1)$ in $W_n$ in mind we fix the Cartan subalgebra $h$ of $\mathfrak{s}l(n+1)$ to be the one corresponding to $h_{W_2}$ under the embedding $\Phi$. Again with a slight abuse of notation, the root system of $\mathfrak{s}l_{n+1}$ relative to $h$ will be denoted by $\Delta_s = \{ \varepsilon_i - \varepsilon_j \mid 0 \leq i \neq j \leq n \}$. Since we will deal with parabolic subalgebras of $W_n$ that are induced from parabolic subalgebras of $\mathfrak{s}l_{n+1}$, we will limit out attention to this case only.

We now recall one of the few equivalent definitions of a parabolic subalgebra of $\mathfrak{s}l_{n+1}$. A *parabolic subset of roots* of $\mathfrak{s} = \mathfrak{s}l_{n+1}$ is a proper subset $P_s$ of $\Delta_s$ for which $
abla_s = P_s \cup (-P_s)$ and $\alpha, \beta \in P_s$ with $\alpha + \beta \in \Delta_s$ implies $\alpha + \beta \in P_s$.

For a parabolic subset of roots $P_s$ of $\nabla_s$, we call $L_s := P_s \cap (-P_s)$ the *Levi component* of $P_s$, $N^+_s := P_s \setminus (-P_s)$ the *nilradical* of $P_s$, and $P_s \nabla L_s \cup N^+_s$ the Levi decomposition of $P$. We call

$$p_{P_s} := \mathfrak{h} \oplus \bigoplus_{\alpha \in P_s} \mathfrak{sl}^{\alpha}$$

a parabolic subalgebra of $\mathfrak{s}$ associated with $P_s$.

If $P_s = L_s \cup N^+_s$ is a parabolic subset of roots of $\mathfrak{s}$, then we call $P = L \cup N^+$ a parabolic subset of roots of $W_n$ induced from $P_s$, where $L := \mathbb{Z}L_s \cap \Delta_W$ and $N^+ := (\mathbb{Z}_{\geq 0} P_s \cap \Delta_W) \setminus L$. If $P_s = L_s \cup N^+_s$ is a parabolic subset of roots of $\mathfrak{s}$, and $P = L \cup N^+$ is the parabolic subset of roots of $W_n$ induced from $P_s$, we set $N^-_s = \Delta_s \setminus P_s$ and $N^- = \Delta_W \setminus P$. The reader is referred to Lemma 3 in [13] for a proof of the fact that every parabolic subset of roots of $W_n$ induced from one of $\mathfrak{s}$ is indeed a parabolic subset of roots of $W_n$. We call $P^\perp = L \cup N^\perp$ the *opposite* to $P$ parabolic subset. Analogously to the case of $\mathfrak{s}l_{n+1}$ we define

$$p_P := \mathfrak{h} \oplus \bigoplus_{\alpha \in P} \mathfrak{sl}^{\alpha}$$

to be the parabolic subalgebra of $W_n$ associated with $P$ (or with $P_s$). By $p^\perp$ we denote the parabolic subalgebra associated with $P^\perp$ and call it the opposite to $p$ parabolic subalgebra.

If $P_s = L_s \cup N^+_s$ is a parabolic subset of roots of $\mathfrak{s}$ for which $L_s = \emptyset$, we will call the corresponding subalgebras $p_{P_s}$ and $p_P$ *Borel subalgebras* of $\mathfrak{s}l_{n+1}$ and $W_n$, respectively.

Below we list all parabolic subsets of roots of $W_2$ induced from parabolic subsets of roots of $\Delta_s$ of $\mathfrak{s}l(3)$ together with the corresponding parabolic subalgebras.
Example 2.12. For a subset $J$ of the real vector space $\mathbb{R} \Delta_W$, let $P(J) = \{ \alpha \in \Delta_W \mid (\alpha, s) \in \mathbb{R}_{\leq 0} \text{ for every } s \in J \}$. Then all parabolical subset of roots of $W_2$ induced by parabolic subsets of roots of $\mathfrak{sl}(3)$ are of the form $P(J)$ for a set $J$ consisting of one or two elements. All possible sets $J$ together with the notation that will be used for the corresponding parabolic subset of roots $P(J)$ and the parabolic subalgebra $p(J) = p_{p(J)}$ are listed below.

(i) $J = \{ \varepsilon_1 - \varepsilon_0 \}, P(1^+) = p(1^+)$.
(ii) $J = \{ \varepsilon_0 - \varepsilon_1 \}, P(1^-) = p(1^-)$.
(iii) $J = \{ \varepsilon_2 - \varepsilon_0 \}, P(2^+) = p(2^+)$.
(iv) $J = \{ \varepsilon_0 - \varepsilon_2 \}, P(2^-) = p(2^-)$.
(v) $J = \{ \varepsilon_1 + \varepsilon_2 \}, P(12^+) = p(12^+)$.
(vi) $J = \{ -\varepsilon_1 - \varepsilon_2 \}, P(12^-) = p(12^-)$.
(vii) $J = \{ \varepsilon_1 - \varepsilon_0, \varepsilon_2 - \varepsilon_0 \}, P(1^+, 2^-) = p(1^+, 2^-)$.
(viii) $J = \{ \varepsilon_0 - \varepsilon_1, \varepsilon_2 - \varepsilon_0 \}, P(1^-, 2^+) = p(1^-, 2^+)$.
(ix) $J = \{ \varepsilon_0 - \varepsilon_2, -\varepsilon_1 - \varepsilon_2 \}, P(2^-, 12^-) = p(2^-, 12^-)$.
(x) $J = \{ \varepsilon_1 - \varepsilon_0, \varepsilon_1 + \varepsilon_2 \}, P(1^+, 12^+) = p(1^+, 12^+)$.
(xi) $J = \{ \varepsilon_0 - \varepsilon_1, -\varepsilon_1 - \varepsilon_2 \}, P(1^-, 12^-) = p(1^-, 12^-)$.
(xii) $J = \{ \varepsilon_2 - \varepsilon_0, \varepsilon_1 + \varepsilon_2 \}, P(2^+, 12^+) = p(2^+, 12^+)$.

Remark 2.13. Note for example that $P(1^+) = \mathbb{Z}_{\leq 0} \varepsilon_1 + \mathbb{Z} \varepsilon_2$ which seems counterintuitive. This sign change is imposed in order to match the notation of the parabolic subalgebras with the notation of the corresponding induced modules, see for example Proposition 2.15.

The following parabolic induction theorem follows from Lemma 11 in [13].

Theorem 2.14. Let $M$ be a simple weight $W_n$-module with finite weight multiplicities. Then there is a parabolic subalgebra $p = l \oplus n^+$ of $W_n$ induced from a parabolic subalgebra of $s = \mathfrak{sl}_{n+1}$ and a simple $p$-module $S$ with a trivial action of $n^+$ such that $M$ is a quotient of the induced module $U(W_n) \otimes_{U(p)} S$. Moreover, there is $\lambda \in \text{Supp} L$ such that $\lambda + \alpha \notin \text{Supp} M$ for all $\alpha$ in $N^+ = \Delta^+$.

2.8. Supports of $W_2$-modules. In what follows we describe all possible supports of simple weight $W_2$-modules with finite weight multiplicities, see Example 2 in [13].

Proposition 2.15. All possible supports of simple weight $W_2$-modules with finite weight multiplicities are exactly in one of the following forms:

(i)-(xii) $\lambda + P(J)$ for $J$ being one in the list (i)-(xii) of Example 2.12.
(xiii) $(\lambda + P(2^-, 12^-)) \cap (\sigma(\lambda) + P(1^-, 12^-))$.
(xiv) $(\lambda + P(1^+, 12^+)) \cap (\sigma(\lambda) + P(2^+, 12^+))$.
(xv) $\lambda + \mathbb{Z}^2$.
(xvi) $\{0\}$.

with the conditions on $\lambda$ as follows: no restrictions on $\lambda$ in cases (i)-(vi), (xv); $\lambda \neq 0$ in cases (vi)-(viii); $\lambda_2 - \lambda_1 \notin \mathbb{Z}_{\geq 0}$ in cases (ix)-(x); $\lambda_1 - \lambda_2 \notin \mathbb{Z}_{\geq 0}$ in cases (xi)-(xii); $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0}$, $\lambda \neq 0$ in cases (xiii)-(xiv).

A simple weight module $M$ of type (xv), i.e. such that $\text{Supp} M = \lambda + \mathbb{Z}^2$, will be called a dense module.
3. Classification of simple bounded $\mathcal{A}$-Modules

We start with a general property of the bounded $\mathcal{A}$-modules.

**Proposition 3.1.** Every bounded $\mathcal{A}$-module (and, hence $W_n$-module) whose support is a subset of $\lambda + \mathbb{Z}^n$ for some $\lambda$ has finite length.

**Proof.** This follows from the fact that $\mathcal{A}$ has a subalgebra isomorphic to $\mathfrak{sl}_{n+1}$ and that the statement holds for bounded $\mathfrak{sl}_{n+1}$-modules with the same support property, see Lemma 3.3 in [12]. \qed

In the rest of this section we work with $n = 1$, i.e. with $\mathcal{A}$.

**Lemma 3.2.** Let $M$ be a simple weight $\mathcal{A}$-module of central charge $c$. If $D_{-1}$ acts finitely on $M$, then either $M \simeq L(\lambda, c, +)$ for some $\lambda, c$, $(\lambda, c) \neq (0, 0)$, or $M \simeq L(0, 0)$.

**Proof.** Let $v \in M^\lambda$ be such that $D_{-1}v = 0$. Then $M = U(\mathcal{A})v$ is an $\mathcal{A}^-$-highest weight module. Thus $U(\mathcal{A})v$ is the unique simple quotient of the induced module $U(\mathcal{A}) \otimes_{U(\mathcal{A}^- \oplus \mathfrak{a}^0)} C(\lambda, c)$, where $C(\lambda, c)$ is the 1-dimensional $\mathcal{A}^0$-module of weight $(\lambda, c)$ on which $\mathcal{A}^-$ acts trivially. However, we know that $L(\lambda, c, +)$ and $L(0, 0)$ are such simple highest weight modules. \qed

**Theorem 3.3.** Let $M$ be a simple bounded $\mathcal{A}$-module of central charge $c$. If $c = 0$, then $M$ is a simple bounded $W_1$-module, i.e. it is isomorphic to one of the modules listed in Theorem 2.2 with trivial action of $I_k$, $k \geq 0$. If $c \neq 0$, then $M$ is isomorphic to one of the following: $T(\nu, \lambda, c)$ $(\lambda - \nu \notin \mathbb{Z})$, $L(\lambda, c, +)$, $L(\lambda, c, -)$. The only isomorphisms of the listed $\mathcal{A}$-modules for $c \neq 0$ are: $T(\nu, \lambda, c) \simeq T(\nu + n, \lambda, c)$, $n \in \mathbb{Z}$, for $\nu \notin \mathbb{Z}$.

**Proof.** By Lemma 3.2 we know that the result holds if $D_{-1}$ acts finitely on $M$. So, for the rest of the proof, we can assume that $D_{-1}$ acts injectively on $M$. We split the proof in two parts depending on the central charge $c$. In all statements we assume that $M$ is a simple bounded $\mathcal{A}$-module of central charge $c$.

**Case 1: Nonzero central charge, i.e. $c \neq 0$.**

We split this case into two subclasses depending on the action of $I_1$ on $M$.

**Lemma 3.4.** If $c \neq 0$, $D_{-1}$ acts injectively on $M$, and $I_1$ acts finitely on $M$, then $M \simeq L(\lambda, c, -)$ for some $\lambda$.

**Proof of Lemma 3.4.** Let $\mathfrak{a} = \text{Span}\{D_{-1}, I_0, I_1\}$. Note that $\mathfrak{a}$ is a Lie subalgebra of $\mathcal{A}$ which is isomorphic to the three-dimensional Heisenberg Lie algebra. Furthermore, each weight space $M^\lambda$, $\lambda \in \mathfrak{h}_\mathfrak{a}^*$, is $I_1D_{-1}$-invariant, so $M$ considered as $\mathfrak{a}$-module is a generalized weight $(\mathfrak{a}, \mathfrak{h}_\mathfrak{a})$-module for $\mathfrak{h}_\mathfrak{a} = \text{Span}\{I_1D_{-1}\}$.

The classification of simple generalized weight $(\mathfrak{a}, \mathfrak{h}_\mathfrak{a})$-modules with nonzero central charge $c$ (equivalently, generalized weight modules of the Weyl algebra $U(\mathfrak{a})/(I_0 - c)$) on which $I_1$ acts finitely is well-known. All such modules are simple weight $(\mathfrak{a}, \mathfrak{h}_\mathfrak{a})$-modules and are isomorphic to the module $\mathbb{C}[D_{-1}]$, such that $D_{-1}(D_{-1}^k) = D_{-1}^{k+1}$, $I_0(D_{-1}^k) = cD_{-1}^k$, and $I_1(D_{-1}^k) = -ckD_{-1}^{k-1}$ (see for example §2 in [13]). Note that if $c = 0$ then we should add the trivial module $\mathbb{C}$ in that list, but this case is addressed in Case 2 below.
Let $d$ be the degree of $M$. Looking at the $\mathcal{A}$-support of $M$ we see that $M$ can not have more than $d$ simple $\alpha$-subquotients. Indeed, if the converse is true, all such subquotients will be isomorphic as $\alpha$-modules to $\mathbb{C}[D_{-1}]$, and then we can easily find an $\mathcal{A}$-weight space of $M$ of dimension bigger than $d$. In particular, $M$ has finitely many simple $\alpha$-subquotients $M_i$ and $M_i = \text{Span}\{D_{-1}^k m_i \mid k \geq 0\}$ for some $m_i \in M$. Hence the $\mathcal{A}$ support of $M$ is bounded from the right, i.e. there is $\lambda \in \text{Supp} M$ such that $\text{Supp} M \subset \lambda + \mathbb{Z}_{\leq 0}$. Therefore $M$ is a simple $\mathcal{A}^+$-highest weight module whose highest weight is $\lambda$, that is $M \simeq L(\lambda, c, -)$.

**Lemma 3.5.** Let $c \neq 0$ and let both $D_{-1}$ and $I_1$ act injectively on $M$.

(i) There are $\nu \in \mathbb{C}$ and a simple $\mathcal{A}$-module $N$ on which $D_{-1}$ acts finitely, such that $M \simeq D_{\langle I_1 \rangle}^\nu N$.

(ii) $M \simeq T(\nu, \lambda, c)$ for some $\lambda \in \mathbb{C}$.

**Proof of Lemma 3.5.** First note that since $D_{-1}$ and $I_1$ act injectively on $M$, then $I_1$ acts bijectively on $M$. In particular, $D_{\langle I_1 \rangle}^\nu M \simeq M$. Now consider $D_{\langle I_1 \rangle}^\nu M$, for any $\nu \in \mathbb{C}$. Let $\lambda \in \text{Supp} M$. Then $I_1D_{-1}|_{M^\lambda}$ is an endomorphism on the finite-dimensional vector space $M^\lambda$. Let $\alpha$ be an eigenvalue of this endomorphism and let $I_1D_{-1}v = \alpha v$ for $v \in M^\lambda$. Then

$$D_{-1}(I_1^{-\nu}v) = I_1^{-\nu} \left( \sum_{i \geq 0} \binom{\nu}{i} (\text{ad} I_1)^i(D_{-1})I_1^{-i}(v) \right)$$

$$= I_1^{-\nu} \left( (D_{-1} + \nu I_0 I_1^{-1})(v) \right)$$

$$= I_1^{-\nu-1} ((\alpha + \nu c)v)$$

We first note that since both $D_{-1}$ and $I_1$ act injectively on $M$ and the weight space of $M$ are finite dimensional, then $I_1$ (and $D_{-1}$) act bijectively on $M$, hence $M \simeq D_{\langle I_1 \rangle}^\nu M$. If $\nu = -\frac{2}{c}$, then $D_{-1}(I_1^{-\nu}m) = 0$. The elements of $D_{\langle I_1 \rangle}^\nu M$ on which $D_{-1}$ acts finitely form a submodule $N'$ of $D_{\langle I_1 \rangle}^\nu M$. Then by Proposition 3.1, $N'$ has finite length, so we can choose a simple $\mathcal{A}$-submodule $N$ of $N'$. Then by Lemma 2.9, $M \simeq D_{\langle I_1 \rangle}^\nu N$ which proves part (i).

To prove (ii), we apply Lemma 3.2 and obtain that $N \simeq T(\nu, \lambda, c)$. Therefore $M \simeq D_{\langle I_1 \rangle}^\nu M \simeq D_{\langle I_1 \rangle}^\nu L(\lambda, c, +) \simeq T(\lambda + \nu, \lambda, c)$. The last isomorphism follows from Lemma 2.10 (i).

**Case 2: Zero central charge, i.e. $c = 0$.**
In this case we have the following lemma.

**Lemma 3.6.** Suppose that $c = 0$ and $D_{-1}$ acts injectively on $M$. Then $I_k = 0$ on $M$ for all $k \geq 0$. In particular, $M$ is a simple $W_1$-module, and thus is isomorphic to one of the modules $T(\nu, \lambda, 0)$ ($\lambda - \nu \notin \mathbb{Z}$), $L(\eta, 0, -)$, $\eta \neq 0$.

**Proof of Lemma 3.6.** Let $d$ be the degree of $M$, let $\lambda \in \text{Supp} M$ and consider the endomorphisms $S = D_{-1}^2 I_2$ and $T = D_{-1} I_1|_{M^\lambda}$ on the vector space $M^\lambda$ of dimension at most $d$. Using that $D_{-1}$ and $I_1$ commute, we easily check that $[T, S] = 2T^2$ and $[T^N, S] = 2NT^{N+2}$. Therefore the trace of the endomorphism $T^N = \left[ -\frac{1}{2N-4} T^{N-2}, S \right]$ is zero for all $N > 2$. But then the sum of the $N$-th powers, $N > 2$, of the eigenvalues of $T$
is zero and hence \( T \) is nilpotent. Thus \( T^d = 0 \). But using again that \( I_1 \) and \( D_{-1} \) commute we find that \( I_1^d = 0 \) on \( M \). Fix \( N_0 > 0 \) such that \( I_1^{N_0} = 0 \) and \( I_1^{N_0-1} \neq 0 \) on \( M \). Let \( v_0 \in M \) be such that \( I_1^{N_0-1}(v_0) \neq 0 \). Then for \( k \geq 1 \), we have
\[
0 = D_{k-1}(I_1^{N_0}(v_0)) = I_1^{N_0}(D_{k-1}(v_0)) + N_0 I_1^{N_0-1}(v_0).
\]
Therefore \( I_k(v) = 0 \) for every \( k \geq 1 \) where \( v = I_1^{N_0-1}(v_0) \). This implies that the set \( M' \) of all \( w \) with the property \( I_k w = 0 \) for all \( k \geq 0 \) is nonzero. Since \( M' \) is an \( A \)-submodule of \( M \), we have \( M' = M \), which proves the first assertion of the lemma. The second part of the lemma follows from the classification of the simple weight \( W_1 \)-modules, i.e. from Theorem 2.8.

\[ \square \]

4. Classification of simple bounded \( W_2 \)-modules

In this section we classify all simple bounded \( W_2 \)-modules, i.e. weight \( W_2 \)-modules with uniformly bounded set of weight multiplicities.

In what follows we set \( a := \text{Span}\{x_i \partial_j \mid 1 \leq i, j \leq 2\} \). We know that the Cartan subalgebras of \( W_2 \), \( a \) and \( s \) coincide with \( \mathfrak{h}_{W_2} \). Using this and the isomorphisms \( s \simeq \mathfrak{sl}_3 \), \( a \simeq \mathfrak{gl}_2 \) we will often write the weights of \( \mathfrak{sl}_3 \) and \( \mathfrak{gl}_2 \) as pairs \((\lambda_1, \lambda_2)\). In some cases, when using representation theory results for \( \mathfrak{sl}_3 \) we will write an \( \mathfrak{sl}_3 \)-weight \( \lambda \) as \( \lambda = \lambda_0 \varepsilon_0 + \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 \) with \( \lambda_0 + \lambda_1 + \lambda_2 = 0 \) (in particular \( \lambda = (\lambda_1, \lambda_2) \) as an element of \( \mathfrak{h}_{W_2}^* \)).

We also set \( W(x_i) := \text{Der}(\mathbb{C}[x_i]), \ i = 1, 2, \mathcal{A}(x_1) := W(x_1) \ltimes (\mathbb{C}[x_1](x_2 \partial_2)), \) and \( \mathcal{A}(x_2) := W(x_2) \ltimes (\mathbb{C}[x_2](x_1 \partial_1)) \). In particular, \( \mathcal{A}(x_1) \simeq \mathcal{A}(x_2) \simeq \mathcal{A} \).

**Definition 4.1.** Let \( J \) be a set from the list (i)-(xii) of Example 2.13. We say that a simple weight \( W_2 \)-module \( M \) with finite weight multiplicities is of type \( J \) if \( M \) is the simple quotient of the module \( U(W_2) \otimes_{U(p(J))} L \) for some simple \( p(J) \)-module \( L \).

We proceed with the classification of simple bounded \( W_2 \)-modules \( M \) in three steps depending on the type of \( M \).

4.1. Classification of simple bounded highest weight \( W_2 \)-modules. In this subsection we classify all bounded highest weight \( W_2 \)-modules, namely all modules from cases (vii)-(xii) in the list of Example 2.12 and Proposition 2.15. For simplicity, in this section, we will not use bold symbols for the vectors and multi-indexes. For example, we write \( \lambda \) for \( \mathfrak{h} \), etc.

Recall that for every \( \lambda \in \mathbb{C}^2 \), the tensor modules \( T(\lambda, \lambda) \) has four subquotients \( T(\lambda, \lambda, (1^+, 2^+)), T(\lambda, \lambda, (1^+, 2^-)), T(\lambda, \lambda, (1^-, 2^+)), T(\lambda, \lambda, (1^-, 2^-)) \). Some important properties of these four modules are collected in the next proposition. A proof is provided in [2] and is based on the description of the highest weight bounded \( \mathfrak{sl}_3 \)-modules.

**Proposition 4.2.** Let \( \lambda \in \mathbb{C}^2 \).

(i) Let \( J = (1^+, 2^+) \). The \( W_2 \)-module \( T(\lambda, \lambda, J) \) is simple if and only if \( \lambda \neq (0, 0) \) and \( \lambda \neq (1, 0) \). The \( W_2 \)-module \( T((0, 0), (0, 0), J) \) has length 2 with simple submodule isomorphic to \( \mathbb{C} \). The \( W_2 \)-module \( T((1, 0), (1, 0), J) \) has length 2 with simple submodule isomorphic to \( T((0, 0), (0, 0), J)/\mathbb{C} \) and simple quotient isomorphic to \( T((1, 1), (1, 1), J) \).
(ii) Let \( J = (1^+, 2^-) \) or \( J = (1^-, 2^+) \). The \( W_2 \)-module \( T(\lambda, \lambda, J) \) is simple if and only if \( \lambda \neq (1, 0) \). The module \( T((1, 0), (1, 0), J) \) has length 3 with simple submodule \( T((0, 0), (0, 0), J) \), simple quotient \( T((1, 1), (1, 1), J) \), and simple subquotient \( \mathbb{C} \).

(iii) Let \( J = (1^-, 2^-) \). The \( W_2 \)-module \( T(\lambda, \lambda, J) \) is simple if and only if \( \lambda \neq (1, 0) \) and \( \lambda \neq (1, 1) \). The \( W_2 \)-module \( T((1, 0), (1, 0), J) \) has length 2 with simple submodule isomorphic to \( T((0, 0), (0, 0), J) \). The \( W_2 \)-module \( T((1, 1), (1, 1), J) \) has length 2 with simple quotient isomorphic to \( \mathbb{C} \) and simple submodule isomorphic to \( T((1, 0), (1, 0), J)/T((0, 0), (0, 0), J) \).

The character formulae of all tensor \( W_2 \)-modules \( T(\lambda, \lambda, J) \) follow directly from their definition. For a weight module \( M \) with finite weight multiplicities, we write \( \text{ch} M = \sum_{\lambda \in \text{Supp} M} \dim M^\lambda e^\lambda \).

**Proposition 4.3.** Let \( \lambda \in \mathbb{C}^2 \). Then the following identities hold.

(i) \( \text{ch} T(\lambda, \lambda, (1^+, 2^+)) = \frac{\text{ch} L_{\mathfrak{gl}}(\lambda)}{(1 - e^{\varepsilon_1})(1 - e^{\varepsilon_2})} \).

(ii) \( \text{ch} T(\lambda, \lambda, (1^+, 2^-)) = \frac{e^{-\varepsilon_2} \text{ch} L_{\mathfrak{gl}}(\lambda)}{(1 - e^{\varepsilon_1})(1 - e^{-\varepsilon_2})}, \quad \text{ch} T(\lambda, \lambda, (1^-, 2^+)) = \frac{e^{-\varepsilon_1} \text{ch} L_{\mathfrak{gl}}(\lambda)}{(1 - e^{-\varepsilon_1})(1 - e^{-\varepsilon_2})} \).

(iii) \( \text{ch} T(\lambda, \lambda, (1^-, 2^-)) = \frac{e^{\varepsilon_1 - \varepsilon_2} \text{ch} L_{\mathfrak{gl}}(\lambda)}{(1 - e^{\varepsilon_1})(1 - e^{-\varepsilon_2})} \).

In particular, the degrees of all four modules equal \( \dim L_{\mathfrak{gl}}(\lambda) = \lambda_1 - \lambda_2 + 1 \).

For any Borel subalgebra \( \mathfrak{b} \) of \( W_2 \) induced by a Borel subalgebra \( \mathfrak{b}_s \) of \( \mathfrak{s} \cong \mathfrak{sl}_3 \), by \( L_\mathfrak{b}(\lambda) \) (respectively, by \( L_{\mathfrak{gl}}^\mathfrak{s}(\lambda) \)) we denote the simple highest weight \( W_2 \)-module (respectively, \( \mathfrak{s} \)-module) relative to \( \mathfrak{b} \) (respectively, to \( \mathfrak{b}_s \)) with highest weight \( \lambda \). In the case when \( \mathfrak{b}_s \) is the standard Borel subalgebra \( \mathfrak{b}_s \) of \( \mathfrak{s} \cong \mathfrak{sl}_3 \), i.e. the one with base \( \Pi_{\mathfrak{a}} = \{ \varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2 \} \), we will write \( L(\lambda) \) and \( L_{\mathfrak{a}}^\mathfrak{s}(\lambda) \) for \( L_\mathfrak{b}(\lambda) \) and \( L_{\mathfrak{gl}}^\mathfrak{s}(\lambda) \), respectively. Note that the Borel subalgebra of \( W_2 \) induced by \( \mathfrak{b}_s \) is \( \mathfrak{g}(2^+, 12^+) \). For \( \mathfrak{g}_\mathfrak{a}^\mathfrak{s} \) (the opposite to the standard Borel subalgebra) and its induced Borel subalgebra \( \mathfrak{g}(2^-, 12^-) \) of \( W_2 \), the corresponding modules will be denoted by \( \tilde{L}(\lambda) \) and \( \tilde{L}_{\mathfrak{a}}^\mathfrak{s}(\lambda) \), respectively.

For a weight \( W_2 \)-module \( M = \bigoplus_{\lambda \in \mathfrak{b}} M^\lambda \) with finite weight multiplicities, by \( M^* \) we denote the restricted dual of \( M \), namely the module \( \bigoplus_{\lambda \in \mathfrak{b}} \text{Hom}_{\mathbb{C}}(M^\lambda, \mathbb{C}) \) with action defined by \( (uf)(m) = f(-um) \). It is clear that \( M^* \) is also a weight module with finite weight multiplicities. Moreover, \( (L_\mathfrak{b}(\lambda))^* \cong L_{\mathfrak{b}}^-(\lambda) \), where recall that \( \mathfrak{b}^-(\lambda) \) is the opposite to \( \mathfrak{b} \) Borel subalgebra. Certainly, the same isomorphism holds for the corresponding highest weight \( \mathfrak{s} \)-modules, and Borel subalgebras of \( \mathfrak{sl}_3 \).

A weight \( \lambda \) will be called \( (W_2, \mathfrak{b}) \)-bounded (respectively, \( (\mathfrak{sl}_3, \mathfrak{b}_s) \)-bounded) if \( L_\mathfrak{b}(\lambda) \) (respectively \( L_{\mathfrak{gl}}^\mathfrak{s}(\lambda) \)) is a bounded module. We will use the following classification of the \( (\mathfrak{sl}_3, \mathfrak{b}_s) \)-bounded weights, see Lemma 7.1 in [12].

**Lemma 4.4.** A weight \( \lambda \) of \( \mathfrak{sl}_3 \) is \( (\mathfrak{sl}_3, \mathfrak{b}_s) \)-bounded if and only if \( (\lambda + \rho_{\mathfrak{b}_s}, \alpha) \in \mathbb{Z}_{\geq 0} \) for some root \( \alpha \) of \( \mathfrak{b}_s \), where \( \rho_{\mathfrak{b}_s} \) is the half-sum of the \( \mathfrak{b}_s \)-positive roots of \( \Delta_{\mathfrak{sl}_3} \).

Note that in the lemma above we may have more than one root \( \alpha \) that satisfy the stated condition. In particular, if all three roots satisfy the condition, then \( L_{\mathfrak{gl}}^\mathfrak{s}(\lambda) \) is finite dimensional.
Lemma 4.5. Let \( \lambda \in \mathbb{C}^2 \). Then \( L(\lambda) \) is a bounded module if and only if \( \lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0} \).

Proof. If \( \lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0} \), by Proposition 4.2 we know that \( L(\lambda) \) is a subquotient of \( T(\lambda, \lambda, (1^+, 2^+)) \) (in fact, \( L(\lambda) \cong T(\lambda, \lambda, (1^+, 2^+)) \) if \( \lambda \neq (0, 0), (1, 0) \)). Hence, \( L(\lambda) \) is bounded.

For the “only if” direction, we will prove the following equivalent statement: If \( \tilde{L}(\mu) \) is bounded, then \( \mu_2 - \mu_1 \in \mathbb{Z}_{\geq 0} \). The two statements are equivalent because \( L(\lambda)^* = \tilde{L}(-\lambda) \).

Since \( \mu \) is an \((\mathfrak{sl}_3, \mathfrak{b}_\mu^-)\)-bounded weight and \( \rho_{\mathfrak{b}_\mu^+} = \varepsilon_2 - \varepsilon_0 \), by Lemma 4.4 \( \mu \) is one (or more than one) of the following three types:

Type 1: \( \mu_2 - \mu_1 \in \mathbb{Z}_{\geq 0} \);
Type 2: \( 2\mu_1 + \mu_2 \in \mathbb{Z}_{\geq 0} \);
Type 3: \( \mu_1 + 2\mu_2 + 1 \in \mathbb{Z}_{\geq 0} \).

Assume for the sake of contradiction that \( \mu_2 - \mu_1 \notin \mathbb{Z}_{\geq 0} \), in particular, \( \mu \) is of Type 2 or of Type 3. Then \( \tilde{L}(\mu) \) is \( \partial_1 \)-injective module. Indeed, if \( \tilde{L}(\mu) \) is \( \partial_1 \)-finite, then the \( \mathcal{A}(x_1) \)-module generated by a highest weight vector of \( \tilde{L}(\mu) \) must have finite support. But the only possible finite-dimensional \( \mathcal{A} \)-modules are the trivial modules, i.e. \( \mu_1 = \mu_2 = 0 \), contradicting \( \mu_2 - \mu_1 \notin \mathbb{Z}_{\geq 0} \).

Since \( \tilde{L}(\mu) \) is \( \partial_1 \)-injective, it can be considered as a submodule of \( D_{(\partial_1)} \tilde{L}(\mu) \). But then the quotient \( D_{(\partial_1)} \tilde{L}(\mu)/\tilde{L}(\mu) \) has a primitive vector relative to the Borel subalgebra \( \mathfrak{p}(1^+, 2^-) \). Namely, this is the vector \( \partial_1^{-1} v \) where \( v \) is a highest weight vector of \( \tilde{L}(\mu) \). As a result \((\mu_1 + 1, \mu_2)\) is a \((W_2, \mathfrak{p}(1^+, 2^-))\)-bounded weight. This implies that \(-\mu_1 - \mu_2 - 1)\varepsilon_0 + (\mu_1 + 1)\varepsilon_1 + \mu_2 \varepsilon_2 \) is \((\mathfrak{sl}_3, s_{\varepsilon_0 - \varepsilon_1} \mathfrak{b}_\mu^-)\)-bounded. Here \( s_\beta \) denotes the reflection of the Weyl group reflection corresponding to the \( \mathfrak{sl}_3 \)-root \( \beta \). We apply Lemma 4.4 again but this time for the weight \((\mu_1 + 1, \mu_2)\). Then one of the following conditions hold:

(a) \( \mu_1 + 2\mu_2 + 1 \in \mathbb{Z}_{\geq 0} \); (b) \( -2\mu_1 - \mu_2 - 2 \in \mathbb{Z}_{\geq 0} \); (c) \( \mu_2 - \mu_1 \in \mathbb{Z}_{\geq 0} \).

We already assumed that (c) does not hold. If (a) holds then \( \mu \) is of Type 3. If (b) holds then \( \mu \) can not be of Type 2. Hence, it remains to consider the case when \( \mu \) is of Type 3. Look again at the simple highest weight \( W_2 \)-module \( L = L_{\mathfrak{p}_1^{(1^+, 2^-)}}(\mu_1 + 1, \mu_2) \).

As mentioned above, this module has a simple \( \mathfrak{sl}_3 \)-subquotient \( L_0 \) with highest weight \((-\mu_1 - \mu_2 - 1)\varepsilon_0 + (\mu_1 + 1)\varepsilon_1 + \mu_2 \varepsilon_2 \) relative to \( \mathfrak{b}_\mu^- \). Since \( \mu_2 - (\mu_1 - \mu_2 - 1) \in \mathbb{Z}_{\geq 0} \), \( L_0 \) is \( \partial_2 \)-finite. Therefore \( L \) has a simple \( \mathfrak{sl}_3 \)-subquotient isomorphic to \( L_{\mathfrak{p}_1^{(1^+, 2^-)}}(\mu_2 + 1, \mu_1 - \mu_2 - 2) \). However one easily checks that since \( \mu_2 - \mu_1 \notin \mathbb{Z}_{\geq 0} \) and \( \mu_1 + 2\mu_2 + 1 \in \mathbb{Z}_{\geq 0} \), the weight \((\mu_2 + 1, \mu_1 - \mu_2 - 2)\) is not \((\mathfrak{sl}_3, s_{\varepsilon_0 - \varepsilon_1} \mathfrak{b}_\mu^-)\)-bounded. This contradicts with the fact that \( L_{\mathfrak{p}_1^{(1^+, 2^-)}}(\mu_2 + 1, \mu_1 - \mu_2 - 2) \) is a subquotient of the bounded module \( L \).

Theorem 4.6. Let \( \lambda \in \mathbb{C}^2 \). Then the highest weight \( W_2 \)-module \( L_0(\lambda) \) is bounded if and only if:

(i) \( \lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0} \) for \( b = \mathfrak{p}(2^+, 12^+) \) and \( b = \mathfrak{p}(1^-, 12^-) \),
(ii) \( \lambda_2 - \lambda_1 \in \mathbb{Z}_{\geq 0} \) for \( b = \mathfrak{p}(1^+, 12^+) \) and \( b = \mathfrak{p}(2^-, 12^-) \),
(iii) \( \lambda_1 - \lambda_2 + 1 \in \mathbb{Z}_{\geq 0} \) for \( b = \mathfrak{p}(1^-, 2^+) \),
(iv) \( \lambda_2 - \lambda_1 + 1 \in \mathbb{Z}_{\geq 0} \) for \( b = \mathfrak{p}(1^+, 2^-) \).

Proof. Using Lemma 4.5 and applying the duality functor \( M \mapsto M^* \) and the twist by the automorphism \( \sigma \), we easily prove (i) and (ii).
Again by duality and because \( p(1^+,2^-) = p(1^-,2^+) \), we see that it is enough to show (iii). For the “only if” direction we use that \( L_{p(1^-,2^+)}(\lambda_1, \lambda_2) \) is isomorphic to a subquotient of the bounded module \( T((\lambda_1 + 1, \lambda_2), (\lambda_1 + 1, \lambda_2), (1^-, 2^+)) \). Assume now that \( L = L_{p(1^-,2^+)}(\lambda_1, \lambda_2) \) is bounded. We reason as in the proof of Lemma 4.4. Namely, we first observe that \( L \) is \( \partial_1 \)-injective. Then the module \( D_{(\partial_1)} L/L \) has a \( p(2^+,12^+) \)-primitive vector of weight \((\lambda_1 + 1, \lambda_2)\) (namely the vector \( \partial_1 w \) where \( w \) is a highest weight vector of \( L \)). Then we use (i) for \((\lambda_1 + 1, \lambda_2)\) and \( p(2^+,12^+) \) and complete the proof. \( \square \)

4.2. Classification of simple bounded half-plane \( W_2 \)-modules. In this subsection we classify all simple bounded \( W_2 \)-modules \( M \) whose supports are half-planes. Namely we give a necessary and sufficient conditions for the modules listed in (i)–(vi) of Example 2.12 and Proposition 2.15 to be bounded. We call modules \( M \) in that list ((i)–(vi)) simple weight half-plane modules.

We first provide the decomposition of the half-plane tensor modules. It is not surprising that in this case the result is much more simple than the highest weight case described in Proposition 4.1. For example, the character formula for all tensor half-plane modules \( T(\nu,\lambda,J) \). Naturally, these formulas contain more terms than the ones for highest weight modules, see Proposition 4.1. For example, the character formula for \( T(\nu,\lambda,2^-) \) is:

\[
\text{ch} T(\nu,\lambda,2^-) = \frac{\left( \sum_{k \in \mathbb{Z}} e^{(\nu_1-\lambda_1+k)e_1-e_2} \right) \text{ch} L_{\mathfrak{gl}}(\lambda)}{1 - e^{-e_2}}.
\]

In this section we will use two parabolic induction functors. For a parabolic subalgebra \( p = l \oplus n^+ \) of \( W_2 \) induced from a parabolic subalgebra of \( \mathfrak{sl}_3 \), and a simple \( l \)-module \( S \) with trivial action of \( n^+ \), we define \( M_p(S) = U(W_2) \otimes_{U(p)} S \). Also, by \( L_p(S) \) we denote the simple quotient of \( M_p(S) \). Similarly we define the two parabolic induction functors for the algebras \( A_1, \mathfrak{sl}_3 \), and \( \mathfrak{gl}_2 \). We will use numerous times the facts that if \( S \) is dense \( \mathfrak{sl}_3 \)- or \( \mathfrak{gl}_2 \)-module, then \( S \) is a twisted localization of a bounded highest weight module, and that the twisted localization functors commute with the parabolic induction functors \( M_p \) and \( L_p \), see for example Proposition 6.2 and Lemma 13.2 in [12]. The proof that the twisted localization and the parabolic induction functors commute in [12] concerns the case of a finite-dimensional reductive Lie algebra \( g \), but one naturally extends Mathieu’s proof for \( W_2 \). For further properties and a more detailed exposition of the twisted localization functor, the reader is referred for example to [5].

We first deal with the last two cases in the list (i)–(vi) of Example 2.12.

Lemma 4.9. Let \( p = p(12^+) \) or \( p = p(12^-) \), and let \( S \) be a simple \( p \)-module with a trivial action of \( n^+ \). Assume that the support of \( M = L_p(S) \) is a half-plane. Then \( M \) is not bounded.
Proof. Assume that $M$ is bounded. In both cases for $\mathfrak{p}$, the Levi subalgebra of $\mathfrak{p}$ is $\mathfrak{a} \simeq \mathfrak{gl}_2$. Then since the support of $M$ is a half-plane, $S$ is a dense $x_2 \partial_1$-injective $\mathfrak{a}$-module. So, let us consider $\lambda \in \mathfrak{h}_a^* \cong \mathbb{C}^2$ and $\nu \in \mathbb{C}^2$ so that $S = D^\nu_{(x_2 \partial_1)} L_{\mathfrak{gl}}(\lambda)$, where recall that $L_{\mathfrak{gl}}(\lambda)$ is the simple highest weight $\mathfrak{a}$-module relative to the Borel subalgebra $\mathfrak{b}_a = \text{Span}\{x_1 \partial_2, x_1 \partial_1, x_2 \partial_2\}$ of $\mathfrak{a}$. But then

$$L_{\mathfrak{p}}(S) \simeq L_{\mathfrak{p}}\left(D^\nu_{(x_2 \partial_1)} L_{\mathfrak{gl}}(\lambda)\right) \simeq D^\nu_{(x_2 \partial_1)} (L_b(\lambda)),$$

where $\mathfrak{b} = \mathfrak{b}_a + \mathfrak{n}^+$. Since $L_b(\lambda)$ is bounded and $\mathfrak{b} = \mathfrak{p}(2^+, 12^+)$ or $\mathfrak{b} = \mathfrak{p}(1^-, 12^-)$, by Theorem $4.6$ we have that $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 0}$. But this implies that $L_{\mathfrak{gl}}(\lambda)$ is finite dimensional which contradicts to the fact that it is $x_2 \partial_1$-injective. \qed

For the four remaining cases (i)–(iv) of simple bounded half-plane modules $L_{\mathfrak{p}}(S)$, the parabolic subalgebra $\mathfrak{p}$ has Levi component isomorphic to $\mathcal{A}$. More precisely, we have the following straightforward result.

**Lemma 4.10.** The Levi component of $\mathfrak{p} = \mathfrak{p}(1^+)$ and $\mathfrak{p} = \mathfrak{p}(1^-)$ is $\mathcal{A}(x_2)$, while the Levi component of $\mathfrak{p} = \mathfrak{p}(2^+)$ and $\mathfrak{p} = \mathfrak{p}(2^-)$ is $\mathcal{A}(x_1)$.

Before we state our classification result for the bounded simple half-plane modules, recall that, by Theorem $3.3$ every simple dense bounded weight $\mathcal{A}$-module is isomorphic to $T(\nu, \lambda, c)$ for some $\lambda, \nu, c$, such that $\lambda - \nu \notin \mathbb{Z}$.

**Proposition 4.11.** Let $\nu, \lambda, c \in \mathbb{C}$ be such that $\lambda - \nu \notin \mathbb{Z}$. Then the following isomorphisms hold.

(i) If $\lambda - c \in \mathbb{Z}_{\geq 0}$ and $(\lambda, c) \neq (1, 0)$, then $L_{\mathfrak{p}(1^+)} T(\nu, \lambda, c) \simeq T((\lambda, \nu), (\lambda, c), 1^+)$ and $L_{\mathfrak{p}(2^+)} T(\nu, \lambda, c) \simeq T((\nu, \lambda), (\lambda, c), 2^+)$. Moreover, for any $\nu \notin \mathbb{Z}$, $L_{\mathfrak{p}(1^+)} T(\nu, 1, 0) \simeq T((0, \nu), (0, 0), 1^+)$ and $L_{\mathfrak{p}(2^+)} T(\nu, 1, 0) \simeq T((\nu, 0), (0, 0), 2^+)$.  

(ii) If $c + 1 - \lambda \in \mathbb{Z}_{\geq 0}$ and $(\lambda, c) \neq (0, 0)$, then $L_{\mathfrak{p}(1^-)} T(\nu, \lambda, c) \simeq T((\nu, \lambda), (c + 1, \lambda), 2^-)$. Moreover, for any $\nu \notin \mathbb{Z}$, $L_{\mathfrak{p}(1^-)} T(\nu, 0, 0) \simeq T((\nu, 1), (1, 1), 2^-)$.

Proof. We prove (i) for the parabolic subalgebra $\mathfrak{p}(1^+)$. The statements for the remaining three parabolic subalgebras are analogous. Let $\lambda - c \in \mathbb{Z}_{\geq 0}$ and $(\lambda, c) \neq (1, 0)$. To show that $L_{\mathfrak{p}(1^+)} T(\nu, \lambda, c) \simeq T((\lambda, \nu), (\lambda, c), 1^+)$, observe that the nilradical of $\mathfrak{p} = \mathfrak{p}(1^+)$ is $\mathfrak{n}^+ = \text{Span}\{x_1^k \partial_1 \mid k \geq 0\}$. We easily check that if $x^s \otimes v \in T((\lambda, \nu), (\lambda, c), 1^+)$ is such that $x_1^k \partial_1 (x^s \otimes v) = 0$, then $s_1 = c$ and the weight of $v$ must be $(c, \lambda)$. Therefore the $\mathfrak{n}^+$-invariants of $T((\lambda, \nu), (\lambda, c), 1^+)$ form an $\mathfrak{A}(x_2)$-module isomorphic to $T(\nu, \lambda, c)$, which proves the desired isomorphism for $\mathfrak{p}(1^+)$. The isomorphism for $\mathfrak{p}(2^+)$ follows with similar reasoning. It remains to consider the case $(\lambda, c) = (1, 0)$. In this case we use that $T(\nu, 1, 0) \simeq T(\nu, 0, 0)$ as $\mathfrak{A}(x_2)$-modules and apply the isomorphism we just proved for $(\lambda, c) = (0, 0)$ (possible because $(\lambda, c) \neq (1, 0)$). Part (ii) follows in a similar way. \qed

**Theorem 4.12.** Let $\nu, \lambda, c \in \mathbb{C}$ be such that $\lambda - \nu \notin \mathbb{Z}$. The simple weight half-plane module $M \simeq L_{\mathfrak{p}} T(\nu, \lambda, c)$ is bounded if and only if the following conditions hold.

(i) $\lambda - c \in \mathbb{Z}_{\geq 0}$ for $\mathfrak{p} = \mathfrak{p}(1^+)$ or $\mathfrak{p} = \mathfrak{p}(2^+)$.

(ii) $c + 1 - \lambda \in \mathbb{Z}_{\geq 0}$ for $\mathfrak{p} = \mathfrak{p}(1^-)$ or $\mathfrak{p} = \mathfrak{p}(2^-)$.
Proof. The “if” directions follow from Proposition 4.11. For the “only if” directions, we prove again the condition only for the parabolic subalgebra \( p(1^+) \) and then use similar reasoning for the remaining three parabolic subalgebras. We need to show that if \( L_{p(1^+)} T(\nu, \lambda, c) \) is bounded, then \( \lambda - c \in \mathbb{Z}_{\geq 0} \). If \( (\lambda, c) = (1, 0) \) the statement follows from the third isomorphism of Proposition 4.11(i). Assume now that \( (\lambda, c) \neq (1, 0) \). To prove the desired condition, we use that \( T(\nu, \lambda, c) \simeq D_{(\partial_2)}(\lambda, \lambda, c, -) \), see Lemma 2.10(ii). Then

\[
L_{p(1^+)} T(\nu, \lambda, c) \simeq L_{p(1^+)} D_{(\partial_2)}^{\lambda-c} T(\lambda, \lambda, c, -) \simeq D_{(\partial_2)}^{\lambda-c} L_{p(1^+)} L(\lambda-1, c, -) \simeq D_{(\partial_2)}^{\lambda-c} L_{p(1^+, 2^+)} (c, \lambda-1).
\]

The last isomorphism uses the fact that the Levi subalgebra of \( L_{p(1^+)} \) is \( \mathcal{A}(x_2) \), see Lemma 4.10. Hence \( L_{p(1^+, 2^+)} (c, \lambda-1) \) is bounded and the condition \( \lambda - c \in \mathbb{Z}_{\geq 0} \) follows from Theorem 4.6(iv). \( \square \)

4.3. Classification of simple bounded dense \( W_2 \)-modules. Recall that \( M \) is a dense module if \( \text{Supp} \, M = \lambda + \mathbb{Z}^2 \) for some \( \lambda \).

Lemma 4.13. Let \( M \) be a simple bounded dense \( W_2 \)-module on which \( x_1 \partial_2 \) or \( x_2 \partial_1 \) act finitely. Then the support of \( M \) is contained in a horizontal or vertical half-plane. In particular, if \( M \) is dense, then \( x_1 \partial_2 \) and \( x_2 \partial_1 \) act injectively (hence bijectively) on \( M \).

Proof. Assume that \( M \) is not isomorphic to \( \mathbb{C} \) and that \( x_2 \partial_1 \) acts finitely on \( M \). To identify the possible types of \( M \) we use representation theory of \( \mathfrak{gl}_2 \). Let \( \alpha = \varepsilon_1 - \varepsilon_2 \). Recall that \( \mathfrak{a} \simeq \mathfrak{gl}_2 \) is the subalgebra of \( W_2 \) generated by \( x_i \partial_j \), \( i, j = 1, 2 \). For a weight \( \mu \) in \( \text{Supp} \, M \), consider the \( \mathfrak{a} \)-module \( M[\mu] = \bigoplus_{k \in \mathbb{Z}} M^{\mu + k\alpha}. \) This is a bounded \( \mathfrak{a} \)-module on which \( x_2 \partial_1 \) acts finitely. Then the number of weights of \( (x_2 \partial_1) \)-primitive vectors in \( M[\mu] \) is bounded by \( 2 \deg M[\mu] \). Hence \( M[\mu] \) has an \( (x_2 \partial_1) \)-maximal weight, say \( \mu' = \mu + k\alpha \). Namely \( \text{Supp} \, M[\mu] \) is a subset of the \( \alpha \)-half-plane \( \mu' + \mathbb{Z}_{\leq 0} \alpha \). Thus the set \( (\mu + \mathbb{Z}_{\leq 0} \alpha) \cap \text{Supp} \, M \) is also on the \( \alpha \)-half-line \( \mu' + \mathbb{Z}_{\leq 0} \alpha \). By Proposition 2.15 the possible supports of \( M \) with empty \( \alpha \)-half-lines, are contained either in a horizontal, or a vertical, or a diagonal (i.e. \( M \) is of type \( 12^+ \) or \( 12^- \)) half-plane. Assume now that \( M \) is of type \( 12^+ \) (the case of type \( 12^- \) is analogous). Then \( M \) is a quotient of \( U(W_2) \otimes_{U(p(1^+))} S \) for some simple \( p(1^+) \)-module \( S \) whose support is a whole \( \alpha \)-line. But, on the other hand, \( M \), and therefore \( S \), is \( x_2 \partial_1 \)-finite. Thus \( S \) cannot be simple, which is a contradiction. \( \square \)

Lemma 4.14. Let \( M \) be a simple bounded dense \( W_2 \)-module.

Then there is \( \nu \notin \mathbb{Z} \) such that \( M \simeq D_{(x_1 \partial_2)} M_0 \), where

(i) \( M_0 = T(\nu', \lambda', 2^-) \) for some \( \nu', \lambda' \) with \( \lambda'_1 - \nu'_1 \not\in \mathbb{Z}, \lambda'_2 - \nu'_2 \in \mathbb{Z}, \) or

(ii) \( M_0 = T(\nu', \lambda', 1^+) \) for some \( \nu', \lambda' \) with \( \lambda'_1 - \nu'_1 \in \mathbb{Z}, \lambda'_2 - \nu'_2 \not\in \mathbb{Z}, \) or

(iii) \( M_0 = T(\nu', \lambda', (1^+, 2^-)) \) for some \( \nu', \lambda' \) with \( \lambda'_1 - \nu'_1 \in \mathbb{Z}, \lambda'_2 - \nu'_2 \in \mathbb{Z}. \)

Proof. By Lemma 4.13 \( x_1 \partial_2 \) and \( x_2 \partial_1 \) act injectively on \( M \). Let \( \lambda = (\lambda_1, \lambda_2) \) be in \( \text{Supp} \, M \) and consider the module \( D_{(x_1 \partial_2)} M \) for any \( \nu \in \mathbb{C} \). For any \( m \in M^\lambda \) we have

\[
(1.1) \quad x_2 \partial_1 (x_1 \partial_2)^{-\nu} m = (x_1 \partial_2)^{-\nu} (x_2 \partial_1 + \nu(\lambda_1 - \lambda_2 - \nu - 1)(x_1 \partial_2)^{-1}) m.
\]

Consider the endomorphism \( (x_1 \partial_2)(x_2 \partial_1)|_M \), and choose an eigenvector \( m \) with eigenvalue \( x \). If we choose now \( \nu \) to be a root of \( x + \nu(\lambda_1 - \lambda_2 - \nu - 1) = 0 \), we have that...
In (i)–(iii) of Lemma 4.14 are tensor modules as a submodule of a twisted localization. For this we will use that (recall the definition of a weight)

\[ \text{Hor}(\langle \nu \rangle) \]

be the set of weights of all \( \mathfrak{gl}_2 \) primitive vectors of localized tensor modules. To describe sets of weights of primitive vectors of localized tensor modules we introduce the following notation for \( \mathfrak{gl}_2 \) and its fixed weight.

**Lemma 4.15.** Let \( g = \mathfrak{gl}_2 \), \( \alpha = \varepsilon_1 - \varepsilon_2 \), \( e \in \mathfrak{g}^a \), and \( f \in \mathfrak{g}^{-\alpha} \). If \( M \) is a weight \( \mathfrak{gl}_2 \)-module for which \( \text{ch} M = \frac{e^\lambda}{1 - e^\alpha} \), then \( WP_{D(e)}(f) = \{ \lambda, s_\alpha \cdot \lambda \} \) if \( \lambda_1 - \lambda_2 \not\in \mathbb{Z} \) and \( WP_{D(e)}(f) = \{ \lambda \} \), otherwise. Here \( s_\alpha \cdot (\lambda_1, \lambda_2) = (\lambda_2 - 1, \lambda_1 + 1) \).

For convenience, in what follows we fix \( \alpha = \varepsilon_1 - \varepsilon_2 \) as a root of \( \mathfrak{a} = \text{Span}\{x_i \delta_j \mid i, j = 1, 2\} \). To describe sets of weights of primitive vectors of localized tensor modules we introduce some subsets of \( \mathbb{C}^2 \). For \( y, z_1, z_2 \in \mathbb{C} \) with \( z_1 - z_2 \in \mathbb{Z}_{\geq 0} \), set:

\[ \text{Hor}(y, [z_1, z_2]) = (y + Z) \times ([z_1, z_2] \cap (z_1 + Z)) \] (horizontal strip in \( (y, z_1) + \mathbb{Z}^2 \)), and

\[ \text{Ver}([z_1, z_2], y) = ([z_1, z_2] \cap (z_1 + Z)) \times (y + Z) \] (vertical strip in \( (z_1, y) + \mathbb{Z}^2 \)).

**Lemma 4.16.** Let \( M = D(x_1 \delta_2) M_0 \) and \( M_0 \) be one of the three modules listed in (i)–(iii) in Lemma 4.14. Then the following hold.

(i) If \( M_0 = T(\nu, \lambda, 2^-) \), then \( WP_M(x_2 \delta_1) = \text{Hor}(\nu_1, [\lambda_1 - 1, \lambda_2 - 1]) \).

(ii) If \( M_0 = T(\nu, \lambda, 1^+) \), then \( WP_M(x_2 \delta_1) = \text{Ver}([\lambda_1, \lambda_2], \nu_2) \).

(iii) If \( M_0 = T(\nu, \lambda, (1^+, 2^-)) \), then \( WP_M(x_2 \delta_1) \subset \text{Hor}(\nu_1, [\lambda_1 - 1, \lambda_2 - 1]) \cup \text{Ver}([\lambda_1, \lambda_2], \nu_2) \).

**Proof.** For part (i) we look at the character formula for \( M_0 \), see Remark 4.8. More precisely, if \( s \in \text{Supp} M_0 \), then the character of the \( \mathfrak{a} \)-module \( M_0[s] = \bigoplus_{k \in \mathbb{Z}} M^{s+k\alpha} \) equals the character of the \( \mathfrak{gl}_2 \)-module \( L_0 = \bigoplus_{i=0}^{\lambda_1 - \lambda_2} L_{\mathfrak{g}}(s_1 + s_2 - \lambda_1 - i + 1, \lambda_1 + i - 1) \). But since the central characters of the direct summands of \( L_0 \) are distinct, we have \( M_0[s] \simeq L_0 \). Hence \( WP_{M_0[s]}(x_2 \delta_1) = WP_{L_0}(E_{21}) \). We use the identity \( M_0 = \bigoplus_{\ell \in \mathbb{Z}} M_0[s + \ell \varepsilon_1] \) of non-integral \( \mathfrak{a} \)-modules. After applying the functor \( D(x_1 \delta_2) \) on that identity, and Lemma 4.15 on each \( D(x_1 \delta_2) M_0[s] \), we prove the desired identity in part (i). The proof of part (ii) is similar to the proof of (i). For part (iii) we use the same reasoning, and in particular apply Lemma 4.15 for the integral case. Unfortunately in this case, some direct summands of \( M_0[s] \) may have equal central characters, so we cannot claim that \( M_0[s] \) is a direct sum of simple...
highest modules. For that reason, we cannot claim that identity holds for $WP_M(x_2\partial_1)$, but we can prove that we have an inclusion.

In order to explicitly write the weights of the $x_2\partial_1$-primitive vectors in $D_{(x_1\partial_2)}^{-\nu}T(s,\lambda)$, we need to introduce additional notation. Recall that the elementary matrices of $\mathfrak{gl}_2$ are denoted by $E_{ij}$, $i,j = 1,2$. For the simple finite-dimensional $\mathfrak{gl}_2$-module $L_{\mathfrak{gl}}(\lambda)$ we use the following setting: $L_{\mathfrak{gl}}(\lambda) = \text{Span}\{v_0, \ldots , v_n\}$, $n = \lambda_1 - \lambda_2$, with $\mathfrak{gl}_2$-action defined by

\[
E_{12}v_i = iv_{i-1},
E_{21}v_i = (n-i)v_{i+1},
E_{11}v_i = (\lambda_1 - i)v_i,
E_{22}v_i = (\lambda_2 + i)v_i,
\]

for $i = 0,\ldots,n$. For any $x,y \in \mathbb{C}$ we introduce the following $(n+1) \times (n+1)$ matrices:

\[
(4.2) \quad A_n(x) = \begin{bmatrix}
x & 0 & \ldots & 0 & 0 \\
1 & x-1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & n & x-n \\
\end{bmatrix}, \quad B_n(y) = \begin{bmatrix}
y-n & 0 & \ldots & 0 & 0 \\
n & y-n+1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & y \\
\end{bmatrix}.
\]

In particular, $B_n(x)$ is the anti-diagonal transpose of $A_n(x)$.

**Lemma 4.17.** Let $s,\lambda$ be in $\mathbb{C}^2$, and let $T = T(s,\lambda)$.

(i) The matrix $M(s,\lambda)$ of the endomorphism $(x_1\partial_2)(x_2\partial_1)\vert_T$, relative to the basis $\{x^s \otimes v_0,\ldots , x^s \otimes v_n\}$ of $T^s = T(s,\lambda)^s$ is

\[
M(s,\lambda) := A_n(s_2 - \lambda_2 + 1)B_n(s_1 - \lambda_2).
\]

(ii) Let $v \in L_{\mathfrak{gl}}(\lambda)$ be such that $x_1\partial_2$ is injective on $x^s \otimes v$. Then for all $\nu \in \mathbb{C}$,

\[
(x_2\partial_1)(x_1\partial_2)^{-\nu}(x^s \otimes v) = (x_1\partial_2)^{-\nu-1}((x_1\partial_2)(x_2\partial_1) + \nu(s_1 - s_2 - \nu - 1)\text{Id}) (x^s \otimes v)
\]

in $D_{(x_1\partial_2)}^{-\nu}T(s,\lambda)$.

(iii) The characteristic polynomial of $A_n(x)B_n(y)$ is:

\[
(4.3) \quad \det(\mu I_{n+1} - A_n(x)B_n(y)) = (\mu - xy)(\mu - (x-1)(y-1))\ldots(\mu - (x-n)(y-n)).
\]

**Proof.** Part (i) follows by a direct verification using the formulas (2.2) and the explicit $\mathfrak{gl}_2$-action on $L_{\mathfrak{gl}}(\lambda)$. Part (ii) is also a subject of direct verification (see also (4.1)).

We can prove part (iii) with purely technical tools, but there is a more elegant proof using the structure of the modules $D_{(x_1\partial_2)}^{-\nu}T(s,\lambda)$. Let $P(\mu, x, y)$ be the characteristic polynomial of $A_n(x)B_n(y)$. Note that $P(\mu, x, y)$ is polynomial in $\mu, x, y$.

Consider $s,\lambda$ such that $s_1 - \lambda_1 \notin \mathbb{Z}$ and $s_2 - \lambda_2 \in \mathbb{Z}$, and let $\nu \in \mathbb{Z}$. In this case $x_1\partial_2$ is injective on $x^s \otimes v$ if and only if $x^s \otimes v \notin T(s,\lambda,2^+)$. If the latter holds, by part (ii) we have that $s - \nu \alpha$ is a weight of an $(x_2\partial_1)$-primitive vector in $D_{(x_1\partial_2)}^{-\nu}T(s,\lambda)$ if and only if $\nu(\nu + 1 - s_1 + s_2)$ is an eigenvalue of $M(s,\lambda)$. On the other hand, since $L \mapsto D_{(x_1\partial_2)}L$ is an exact functor, $\nu \in \mathbb{Z}$, and $D_{(x_1\partial_2)}T(s,\lambda,2^+) = 0$, we have

\[
D_{(x_1\partial_2)}^{-\nu}T(s,\lambda) \simeq D_{(x_1\partial_2)}T(s,\lambda) \simeq D_{(x_1\partial_2)}T(s,\lambda,2^-).
\]
By Lemma 4.16(i),

\[(s_1 + s_2 - \lambda_2 - i + 1, \lambda_2 + i - 1) = -s + (\lambda_2 - s_2 + i - 1)\alpha, \quad i = 0, \ldots, \lambda_1 - \lambda_2,\]

are the weights of the set of \((x_2\partial_1)\)-primitive vectors of \(D_{(x_1, \partial_2)} T(s, \lambda, 2^-)\) that are on the \(\alpha\)-line \(s + Z\alpha\). Therefore, \(x_i, (x - 1)(y - 1), \ldots, (x - n)(y - n)\) are all eigenvalues (with possible repetitions) of \((x_1\partial_2)(x_2\partial_1)|T^s\), where \(x = s_2 - \lambda_2 + 1\) and \(y = s_1 - \lambda_2\). Thus \(1.3\) holds for all \(\mu, x, y\). Since \(Z \times (\mathbb{C} \setminus \mathbb{Z})\) is a Zariski dense subset of \(\mathbb{C}^2\), we have that \(1.3\) holds for all \(\mu, x, y\). 

\[\square\]

**Lemma 4.18.** Let \(\lambda, s \in \mathbb{C}^2\) be such that \(\lambda_i - s_i \notin \mathbb{Z}, \quad i = 1, 2, \quad \text{and} \quad \lambda \neq (1, 0)\).

(i) If \(\lambda_1 + \lambda_2 - s_1 - s_2 \notin \mathbb{Z}\), then the following isomorphisms hold:

(a) \(D^{\nu_2}_{(x_1, \partial_2)} T(s - \nu_2\alpha, \lambda, 2^-) \simeq T(s, \lambda)\),

(b) \(D^{\nu_1}_{(x_1, \partial_2)} T(s - \nu_1\alpha, \lambda, 1^+) \simeq T(s, \lambda)\),

for \(\nu_2 = s_2 - \lambda_2 + 1\) and \(\nu_1 = s_1 - \lambda_1\).

(ii) If \(\lambda_1 + \lambda_2 - s_1 - s_2 \in \mathbb{Z}\), then

\[D^{\nu_2}_{(x_1, \partial_2)} T(s - \nu_2\alpha, \lambda, (1^+, 2^-)) \simeq T(s, \lambda)\]

for \(\nu_2 = s_2 - \lambda_2 + 1\) and for \(\nu_1 = s_1 - \lambda_1\).

**Proof.** For part (i)(a) consider first the module \(D^{-\nu_2}_{(x_2, \partial_2)} T(s, \lambda)\) (where \(\nu_2 = s_2 - \lambda_2\)), and let \(\nu \in \mathbb{C}\) be such that \(\nu - \nu_2 \in \mathbb{Z}\). By Lemma 4.17(ii) we know that \(s - \nu_2\alpha\) is a weight of an \((x_2\partial_1)\)-primitive vector in \(D^{-\nu_2}_{(x_2, \partial_2)} T(s, \lambda)\) if and only if \(\nu(\nu + 1 - s_1 + s_2)\) is an eigenvalue of \(M(s, \lambda)\). But by Lemma 4.17(ii), all such eigenvalues are \((s_2 - \lambda_2 - i + 1)(s_1 - \lambda_2 - i)\), \(i = 0, \ldots, \lambda_1 - \lambda_2\). Recall that \(\nu - s_2 + \lambda_2 \in \mathbb{Z}\) and \(\lambda_1 + \lambda_2 - s_2 \notin \mathbb{Z}\). Hence, \(s - \nu_2\alpha\) is a weight of an \((x_2\partial_1)\)-primitive vector in \(D^{-\nu_2}_{(x_2, \partial_2)} T(s, \lambda)\) if and only if \(\nu = s_2 - \lambda_2 - i + 1\) for some \(i \in \{0, 1, \ldots, \lambda_1 - \lambda_2\}\).

On the other hand, \(T(s, \lambda)\) is dense and \(D^{\nu_2}_{(x_2, \partial_2)} T(s, \lambda)\) has \((x_2\partial_1)\)-primitive vectors. Thus by Lemma 4.14, we have \(T(s, \lambda) = D^{\nu_2}_{(x_2, \partial_2)} T(s - \nu_2\alpha, \lambda', 2^-)\) for some \(\lambda' \in \mathbb{C}^2\). Hence, \(D^{-\nu_2}_{(x_2, \partial_2)} T(s, \lambda) \simeq D_{(x_1, \partial_2)} T(s - \nu_2\alpha, \lambda', 2^-)\). Using Lemma 4.16(i) and the description of the weights of \((x_2\partial_1)\)-primitive vectors in \(D^{-\nu_2}_{(x_2, \partial_2)} T(s, \lambda)\), we obtain

\[\text{Hor}(s_1 - \nu_2, [\lambda_1 - 1, \lambda_2 - 1]) = \text{Hor}(s_1 - \nu_2, [\lambda_1' - 1, \lambda_2' - 1]).\]

Thus \(\lambda = \lambda'\), which completes the proof of (i)(a).

For parts (i)(b) and (ii) we use the same reasoning, namely we apply again the corresponding parts of Lemma 4.17, Lemma 4.14, Lemma 4.16. Part (i)(b) will automatically follow, while for part (ii) we will have at the end

\[\text{Hor}(s_1 - \nu_2, [\lambda_1 - 1, \lambda_2 - 1]) \cup \text{Ver}([\lambda_1, \lambda_2], s_2 + \nu_2) \subset \text{Hor}(s_1 - \nu_2, [\lambda_1' - 1, \lambda_2' - 1]) \cup \text{Ver}([\lambda_1', \lambda_2'], s_2 + \nu_2).\]

However, the above condition is sufficient to conclude that \(\lambda = \lambda'\). 

\[\square\]

Using Lemma 4.14 and Lemma 4.18, we obtain the classification of simple bounded dense \(W_2\)-modules.

**Theorem 4.19.** If \(M\) is a simple bounded dense \(W_2\)-module, then \(M \simeq T(\nu, \lambda)\) for some \(\nu, \lambda\), such that \(\lambda_i - \nu_i \notin \mathbb{Z}, \quad i = 1, 2, \quad \lambda \neq (1, 0)\).
4.4. **Main Theorem.** Combining Theorems 4.6, 4.12, and 4.19 we obtain our main result in the paper.

**Theorem 4.20.** Let $M$ be a simple bounded $W_2$-module. Then either $M \cong \mathbb{C}$ or $M \cong T(\nu, \lambda, J)$ for some $\nu, \lambda \in \mathbb{C}^2$, and $J \in \mathcal{PM}(\lambda - \nu)$, such that:

$\lambda \neq (1, 0), (\nu, \lambda, J) \neq ((0, 0), (0, 0), (1^+, 2^+)), (\nu, \lambda, J) \neq ((1, 1), (1, 1), (1^-, 2^-))$.

Furthermore, two modules $T(\nu, \lambda, J)$ and $T(\nu', \lambda', J')$ in this list are isomorphic if and only if $\nu - \nu' \in \mathbb{Z}^2$, $\lambda = \lambda'$, and $J = J'$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT ARLINGTON, ARLINGTON, TX 76021, USA  
*E-mail address: cavaness@uta.edu*

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT ARLINGTON, ARLINGTON, TX 76021, USA  
*E-mail address: grandim@uta.edu*