Applications of methods of random differential geometry to quantum statistical systems

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1 Abstract

We apply concepts of random differential geometry connected to the random matrix ensembles of the random linear operators acting on finite dimensional Hilbert spaces. The values taken by random linear operators belong to the Liouville space. This Liouville space is endowed with topological and geometrical random structure. The considered random eigenproblems for the operators are applied to the quantum statistical systems. In the case of random quantum Hamiltonians we study both hermitean (self-adjoint) and non-hermitean (non-self-adjoint) operators leading to Gaussian and Ginibre ensembles Refs. [1], [2], [3].

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2 Introduction

We study generic quantum statistical systems with energy dissipation. Let us consider Hilbert’s space $\mathcal{V}$ with some basis $\{\ket{\Psi_i}\}$. The space of the linear bounded operators $\hat{X}$ acting on Hilbert space $\mathcal{V}$ is called Liouville space and is denoted $\mathcal{L}(\mathcal{V})$. The Liouville space is again Hilbert’s space with the scalar product: $\langle \hat{X}\mid\hat{Y}\rangle = \text{Tr}(\hat{X}^\dagger\hat{Y})$. Hence, it is a Banach space with norm: $||\hat{X}|| = (\langle \hat{X}\mid\hat{X}\rangle)^{1/2}$, and it is metric space with distance: $\rho(\hat{X},\hat{Y}) = ||\hat{X} - \hat{Y}||$. Finally, it is topological space with the balls $B(\hat{X}, r) = \{\hat{Y} | \rho(\hat{Y}, \hat{X}) < r\}$. The Liouville space is also a differentiable manifold. The exist a tangent space $T_{\hat{X}}\mathcal{L}(\mathcal{V})$, tangent bundle $T\mathcal{L}(\mathcal{V})$, cotangent space $T_{\hat{X}}^*\mathcal{L}(\mathcal{V})$, and cotangent bundle $T^*\mathcal{L}(\mathcal{V})$. The quantum operator $\hat{X} \in \mathcal{L}(\mathcal{V})$ is in the given basis a matrix with elements $X_{ij}$. We are allowed to define random operator variable $\hat{X} : \Omega \ni \omega \to \hat{X}(\omega) \in \mathcal{L}(\mathcal{V})$, where $\Omega$ is sample space, and $\omega$ is sample point. In
the basis it is reduced to random matrix variable \( X : \Omega \ni \omega \to X(\omega) \in \text{MATRIX}(N, N, F) \), where \( \text{MATRIX}(N, N, F) \) is set of all \( N \times N \) matrices with elements from the field \( F \) [1, 2, 3]. Thus, we define random Hamiltonian operator variable \( \mathcal{H} : \Omega \ni \omega \to \mathcal{H}(\omega) \in \mathcal{L}(V) \), and we define random Hamiltonian matrix variable \( \mathcal{H} : \Omega \ni \omega \to \mathcal{H}(\omega) \in \text{MATRIX}(N, N, F) \). Let us assume that the Hamiltonian \( \mathcal{H} \) is not hermitean operator, thus its eigenenergies \( Z_i \) are complex-valued random variables. We assume that distribution of matrix elements \( H_{ij} \) is governed by Ginibre ensemble [1, 2, 4, 5]. \( \mathcal{H}(\omega) \) belongs to general linear Lie group \( \text{GL}(N, C) \), where \( C \) is complex numbers field. Since \( \mathcal{H}(\omega) \) is not hermitean, therefore quantum system is dissipative system. Ginibre ensemble of random matrices is one of many Gaussian Random Matrix ensembles GRME. The above approach is an example of Random Matrix theory RMT [1, 2, 3]. The other RMT ensembles are for example Gaussian orthogonal ensemble GOE, unitary GUE, symplectic GSE, as well as circular ensembles: orthogonal COE, unitary CUE, and symplectic CSE. The distributions of the eigenenergies \( Z_1, ..., Z_N \) for \( N \times N \) Hamiltonian matrices is given by Jean Ginibre’s formula [1, 2, 4, 5]:

\[
P(z_1, ..., z_N) = \prod_{j=1}^{N} \frac{1}{\pi \cdot j!} \prod_{i<j}^{N} |z_i - z_j|^{2} \exp(-\sum_{j=1}^{N} |z_j|^2), \tag{1}
\]

where \( z_i \) are complex-valued sample points (\( z_i \in C \)). For Ginibre ensemble we define complex-valued spacings \( \Delta^1 Z_i \) and second differences \( \Delta^2 Z_i \):

\[
\Delta^1 Z_i = Z_{i+1} - Z_i, \quad i = 1, ..., (N - 1), \tag{2}
\]

\[
\Delta^2 Z_i = Z_{i+2} - 2Z_{i+1} + Z_i, \quad i = 1, ..., (N - 2). \tag{3}
\]

The \( \Delta^2 Z_i \) are extensions of real-valued second differences

\[
\Delta^2 E_i = E_{i+2} - 2E_{i+1} + E_i, \quad i = 1, ..., (N - 2), \tag{4}
\]

of adjacent ordered increasingly real-valued energies \( E_i \) defined for GOE, GUE, GSE, and Poisson ensemble PE (where Poisson ensemble is composed of uncorrelated randomly distributed eigenenergies) [6, 7, 8, 9, 10, 11].

There is an analogy of Coulomb gas of unit electric charges pointed out by Eugene Wigner and Freeman Dyson. A Coulomb gas of \( N \) unit charges moving on complex plane (Gauss’s plane) \( C \) is considered. The vectors of positions of charges are \( z_i \) and potential energy of the system is:

\[
U(z_1, ..., z_N) = -\sum_{i<j} \ln |z_i - z_j| + \frac{1}{2} \sum_i |z_i|^2. \tag{5}
\]

If gas is in thermodynamical equilibrium at temperature \( T = \frac{1}{2k_B} (\beta = \frac{1}{k_B}T = 2, k_B \) is Boltzmann’s constant), then probability density function of vectors of positions is \( P(z_1, ..., z_N) \) Eq. (1). Complex eigenenergies \( Z_i \) of quantum system are analogous to vectors of positions of charges of Coulomb gas. Moreover, complex-valued spacings \( \Delta^1 Z_i \) are analogous to vectors
of relative positions of electric charges. Finally, complex-valued second differences $\Delta^2 Z_i$ are analogous to vectors of relative positions of vectors of relative positions of electric charges.

The $\Delta^2 Z_i$ have their real parts $\text{Re} \Delta^2 Z_i$, and imaginary parts $\text{Im} \Delta^2 Z_i$, as well as radii (moduli) $|\Delta^2 Z_i|$, and main arguments (angles) $\text{Arg} \Delta^2 Z_i$.

### 3 Second Difference Distributions

We define following random variables for $N=3$ dimensional Ginibre ensemble:

$$Y_1 = \Delta^2 Z_1, A_1 = \text{Re} Y_1, B_1 = \text{Im} Y_1,$$

$$R_1 = |Y_1|, \Phi_1 = \text{Arg} Y_1,$$

and for the generic $N$-dimensional Ginibre ensemble [12]: $W_1 = \Delta^2 Z_1$.

Their distributions are given by following formulae [12]:

$$f_{Y_1}(y_1) = f_{(A_1,B_1)}(a_1, b_1) =$$

$$= \frac{1}{576\pi} [(a_1^2 + b_1^2)^2 + 24] \cdot \exp(-\frac{1}{6}(a_1^2 + a_2^2)).$$

$$f_{A_1}(a_1) = \frac{\sqrt{6}}{576\sqrt{\pi}} (a_1^4 + 6a_1^2 + 51) \cdot \exp(-\frac{1}{6}a_1^2),$$

$$f_{B_1}(b_1) = \frac{\sqrt{6}}{576\sqrt{\pi}} (b_1^4 + 6b_1^2 + 51) \cdot \exp(-\frac{1}{6}b_1^2),$$

$$f_{R_1}(r_1) =$$

$$\Theta(r_1) \frac{1}{288} r_1 (r_1^4 + 24) \cdot \exp(-\frac{1}{6}r_1^2),$$

$$f_{\Phi_1}(\phi_1) = \frac{1}{2\pi}, \phi_1 \in [0, 2\pi].$$

$$P_3(w_1) =$$

$$= \pi^{-3} \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} \sum_{j_3=0}^{N-1} \frac{1}{j_1!j_2!j_3!} I_{j_1,j_2,j_3}(w_1),$$

$$I_{j_1,j_2,j_3}(w_1) =$$

$$= 2^{-2j_2} \frac{\partial^{j_1+j_2+j_3}}{\partial^{j_1}\lambda_1 \partial^{j_2}\lambda_2 \partial^{j_3}\lambda_3} F(w_1, \lambda_1, \lambda_2, \lambda_3) |_{\lambda_i=0},$$

$$F(w_1, \lambda_1, \lambda_2, \lambda_3) =$$

$$= A(\lambda_1, \lambda_2, \lambda_3) \exp[-B(\lambda_1, \lambda_2, \lambda_3)|w_1|^2],$$

$$= A(\lambda_1, \lambda_2, \lambda_3) \exp[-B(\lambda_1, \lambda_2, \lambda_3)|w_1|^2],$$
$A(\lambda_1, \lambda_2, \lambda_3) = (15)$

$$= \frac{(2\pi)^2}{(\lambda_1 + \lambda_2 - \frac{5}{4}) \cdot (\lambda_1 + \lambda_3 - \frac{5}{4}) - (\lambda_1 - 1)^2}.$$ 

$B(\lambda_1, \lambda_2, \lambda_3) = (16)$

$$= (\lambda_1 - 1) \cdot \frac{2\lambda_1 - \lambda_2 - \lambda_3 + \frac{1}{2}}{2\lambda_1 + \lambda_2 + \lambda_3 - \frac{9}{2}}.$$ 

4 Conclusions

We compare second difference distributions for different ensembles by defining following dimensionless second differences:

$$C_\beta = \frac{\Delta^2 E_1}{< S_\beta >}, \quad (17)$$

$$X_1 = \frac{A_1}{< R_1 >}, \quad (18)$$

where $< S_\beta >$ are the mean values of spacings for GOE(3) ($\beta = 1$), for GUE(3) ($\beta = 2$), for GSE(3) ($\beta = 4$), for PE ($\beta = 0$) [6, 7, 8, 9, 10, 11], and $< R_1 >$ is mean value of radius $R_1$ for $N=3$ dimensional Ginibre ensemble [12].

On the basis of comparison of results for Gaussian ensembles, Poisson ensemble, and Ginibre ensemble we formulate homogenization law [6, 7, 8, 9, 10, 11, 12]: *Eigenenergies for Gaussian ensembles, for Poisson ensemble, and for Ginibre ensemble tend to be homogeneously distributed.* The second differences’ distributions assume global maxima at origin for above ensembles. For Coulomb gas the vectors of relative positions of vectors of relative positions of charges statistically vanish. It can be called stabilisation of structure of system of electric charges.

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