Non-Abelian Chern-Simons Theory from a Hubbard-like Model

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Here, we provide a simple Hubbard-like model of spin-1/2 fermions that gives rise to the SU(2) symmetric Thirring model that is equivalent, in the low-energy limit, to Yang-Mills-Chern-Simons model. First, we identify the regime that simulates the SU(2) Yang-Mills theory. Then, we suitably extend this model so that it gives rise to the SU(2) level k Chern-Simons theory with k ≥ 2 that can support non-Abelian anyons. This is achieved by introducing multiple fermionic species and modifying the Thirring interactions, while preserving the SU(2) symmetry. Our proposal provides the means to theoretically and experimentally probe non-Abelian SU(2) level k topological phases.

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Interacting systems are in general too hard to track analytically. An interesting approach is to employ low-dimensional interacting relativistic quantum field theories at zero temperature for which bosonisation can be applied. Some of these theories can be simultaneously analytically tractable and amendable to experimental verification, e.g. with cold atoms. The (2 + 1)-dimensional Thirring model [1], that describes interacting Dirac fermions, provides such example. If the interaction term possesses U(1) symmetry then the model is equivalent through bosonisation to the Maxwell-Chern-Simons theory [2]. If the interaction term is SU(2) symmetric then the model can be described by Yang-Mills-Chern-Simons theory [3]. Unfortunately, the anyons supported by this model are Abelian, namely SU(2) level k = 1 anyons.

The goal of this report is twofold. First, to present a Hubbard-like model of spin-1/2 fermions that gives rise in the continuum limit to the SU(2) symmetric Thirring model. In particular, we identify the coupling regime where the Yang-Mills theory is predominant in the bosonised version of the model. Hence, the model could serve as a quantum simulator for demonstrating confinement in 2 + 1 dimensions, e.g. with current cold atom technology. Although quantum simulators for lattice Yang-Mills theory in cold atomic systems have been recently proposed in [4–6], our model simulates a continuum non-Abelian gauge theory. Second, we employ multiple species of fermions so that the low energy of the model is described by the SU(2) level k ≥ 2 Chern-Simons theory. This theory can support non-Abelian anyons such as Ising or Fibonacci anyons. Hence, its physical realisation can serve for the implementation of topological quantum computation [7].

A finite temperature implementation of our work is also possible. Indeed, the non-Abelian Chern-Simons theory can be induced by fermions also at finite temperature [8,9]. This analysis goes beyond the scope of the paper and it will be left to future work.

Our starting point is a tight-binding model with low energy behaviour described by the SU(2) symmetric Thirring model in 2 + 1 dimensions. The Thirring model comprises of interacting relativistic Dirac fermions. To simulate it we employ tight-binding fermions in a honeycomb lattice configuration, as shown in Fig. 1 (Left). We introduce the Hubbard-like Hamiltonian

\[ H = -t \sum_{\langle i,j \rangle,s} (b_{i,s}^\dagger w_{j,s} + b_{j,s} w_{i,s}^\dagger) - \mu \sum_{i,s} (n_{s,i}^b - n_{s,i}^w) - \sum_{\langle i,j \rangle,s} \chi_{i,j} t' (b_{i,s}^\dagger b_{j,s} - w_{i,s}^\dagger w_{j,s}) + U \left( \sum_{i,s,s'} n_{s,i}^b n_{s',i}^w - \sum_{i,\alpha} n_{s,i}^\alpha n_{s,i}^{\alpha'} \right), \]

where \( n_{s}^\alpha = n_{s,\alpha} + n_{s,\alpha'} \) is the population of particle \( \alpha = b, w \), distinguished by their position in the unit cell, with spin \( s = \uparrow, \downarrow \). The phase factor \( \chi_{i,j} = \pm i \) is defined in Fig. 1 (Left). The t-term of the Hamiltonian corresponds to tunnelling along the honeycomb lattice. In the continuum limit it gives rise to two massless Dirac fermions corresponding to the Fermi points \( P_{\pm} = (0, \pm 4\pi/(3\sqrt{3})) \) in Cartesian coordinates. The chemical potential \( \mu \)-term and the next-to-nearest tunnelling t'-term give rise to energy gaps at the two Fermi points of the form

\[ \Delta E_{\pm} = 2| - \mu \pm 3t'|. \]

For \( \Delta E_{+} \ll \Delta E_{-} \), as shown in Fig. 1 (Right), we can adiabatically eliminate the \( P_{-} \) Fermi point from the low energy dynamics of the system [10]. Hence, we can isolate the dynamics of the single Fermi point \( P_{+} \). An alternative approach to the adiabatic elimination is to consider three-dimensional topological insulator with an isolated Dirac cone at its boundary [11]. Introducing suitable boundary fields generates an energy gap, so the surface state can be effectively described by a massive Dirac fermion.

By introducing the spinor \( \psi = (\psi_{\uparrow}, \psi_{\downarrow})^T = (b_{\uparrow}, w_{\uparrow}, b_{\downarrow}, w_{\downarrow})^T \) with \( \psi_s = (b_s, w_s)^T \) and \( s = \uparrow, \downarrow \), we can write the interaction U-term of Hamiltonian in the form

\[ \frac{1}{2} U (\bar{b} T^a \gamma^\mu \psi)(\bar{w} T^a \gamma^\mu \psi) \]

that acts locally within the unit cell.
cell. Here $\bar{\psi} = \psi^+ \gamma_2$, $\gamma_\mu = \sigma_\mu \otimes I_2$ for $\mu = x, y, z$ are $4 \times 4$ Euclidean Dirac matrices written in terms of the Pauli matrices, where $I_2$ acts on the spin subspace, and $T^a = \sigma^a / 2$, for $a = x, y, z$, are the generators of SU(2).

The arrangement of the tight-binding interactions that give rise to the self-interaction of the Dirac fermion is shown in Fig. 2 (Left).

In the low energy limit the behaviour of the model around $P_4$ is given by the Hamiltonian

$$ H = \int d^3x \left[ \psi^+ \left( \gamma_\mu \gamma_5 \mathbf{p} + \gamma_\mu M v^2 \right) \psi + g^2 / 2 j^{a\mu} j^a_\mu \right], $$

where $j^{a\mu} = \bar{\psi} T^a \gamma^\mu \psi$, $v = \tfrac{3}{2} t$, $M v^2 = -\mu + \sqrt{3} v'$, $g^2 = 4 / 3 U$. For simplicity we take from now on $v = 1$.

Hamiltonian (3) corresponds to the $(2 + 1)$-dimensional Thirring model with SU(2) symmetry. This non-Abelian symmetry is manifested by the invariance of the Hamiltonian under transformations of the spinor $\psi^a_\nu = V_{\alpha\nu}^a \psi^a_\nu$, for $V \in$ SU(2). Note that this symmetry of the interacting term is also exact in the discrete model.

It is known that in $2 + 1$ or higher dimensions even the Abelian Thirring model is perturbatively non-renormalisable. Nevertheless, it has been shown to become renormalisable in the non-perturbative large-N limit [12, 13]. In our case we are only interested in the low energy sector of the tight-binding model and, consequently, in the infrared limit of the corresponding SU(2) Thirring model. In the following we show how this model maps to a renormalisable gauge theory to leading order in $1 / M$ [3]. This mass fixes the validity energy range of our effective theory.

We show now the connection between the SU(2) symmetric Thirring model and the Yang-Mills-Chern-Simons theory [3]. To proceed we employ the path integral formalism with Euclidean signature. The Non-Abelian Thirring action that corresponds to Hamiltonian (3) is given by

$$ S_{\text{Th}} = \int d^3x \left[ \bar{\psi} \left( \partial - M \right) \psi - g^2 / 2 j^{a\mu} j^a_\mu \right], $$

and the corresponding partition function is defined as

$$ Z_{\text{Th}} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \ e^{-S_{\text{Th}}}. $$

To treat the interaction term that is quartic in the fermionic operators we employ the Hubbard-Stratonovich transformation

$$ \exp \left[ \int d^3x \frac{g^2}{2} j^{a\mu} j^a_\mu \right] = \int \mathcal{D}a_\mu \exp \left[ - \int d^3x \text{tr} \left( \frac{1}{2} a^{a\mu} a_\mu + g j^{a\mu} a_\mu \right) \right], $$

which introduces the vector field $a_\mu = a^{a\mu} T^a$. At this point we can integrate out the fermions that now appear quadratically. The resulting effective action is given by

$$ S_{\text{eff}}[a] = - \ln \det \left( \partial - M + g \gamma_5 \right) = \frac{i}{8\pi} \frac{M}{|M|} \int d^3x L_{\text{CS}}[a] + \mathcal{O} \left( \frac{\partial}{M} \right), $$

where

$$ L_{\text{CS}}[a] = g^2 e^{\lambda_{\mu\nu}} \text{tr} \left( a_\lambda \partial_\mu a_\nu + \frac{2}{3} g a_\lambda a_\mu a_\nu \right). $$

The term $\mathcal{O} \left( \frac{\partial}{M} \right)$ has a negligible contribution to the low energy behaviour that we are interested in. Note that the action $S_{\text{eff}}[a]$ is not gauge invariant for large gauge transformations [13]. It is possible to cure this global gauge anomaly by introducing a gauge-invariant regularisation such as the Pauli-Villars one [11, 13]. In this scheme the

**FIG. 2:** The fermionic interactions, given by grey lines, within a single unit cell that includes one black and one white site (see Fig. 1). (Left) The interactions for the single fermionic species model between populations $n^\alpha_n = \alpha \beta \delta \varepsilon$ with $\alpha = b, w$, $s = \uparrow, \downarrow$ and strength $\pm U$. (Right) The interactions for the two fermionic species model. We can consider this as a bilayered system with the interactions between populations $n^\alpha_n = \alpha \beta \delta \varepsilon$ with $\alpha = b, w$, $s = \uparrow, \downarrow$ and $\beta = 1, 2$ given explicitly by Hamiltonian (14).
regularised action $S^R_{\text{eff}}[a] = S_{\text{eff}}[a] - \lim_{M^2 \to \infty} S_{\text{eff}}[a](M_0)$ is given by

$$S^R_{\text{eff}}[a] = \lim_{M^2 \to \infty} \frac{1}{2} \left( \frac{M}{|M|} - \frac{M_0}{|M_0|} \right) \frac{i}{4\pi} \int d^3x \, L_{\text{CS}}[a].$$  \hspace{1cm} (8)

When $\text{sign}(M_0) = -\text{sign}(M)$, we obtain the standard non-Abelian Chern-Simons action with level $k = 1$ [16, 17]. It is worth noticing that changing the value of the coefficient $g$ in (7) does not change the value of the level of the non-Abelian theory [16, 17]. To simplify the next calculations we rescale $a_\mu \to a_\mu / g$ and take $M$ positive.

Still the total action is not gauge invariant due to the $\int d^3x \, tr (a^\mu a_\mu)$ term in (5). It is possible to recast the total action in terms of a gauge invariant and renormalisable theory by introducing the interpolating action [18, 19]

$$S_1[a, A] = \int d^3x \left\{ \frac{1}{2g^2} \text{tr} \, a^\mu a_\mu + \frac{i}{2\pi} \epsilon^{\mu
u\lambda} \text{tr} \, a_\mu \left( F_{\nu\lambda}(A) + A_\nu a_\lambda \right) + \frac{i}{4\pi} \epsilon^{\mu
u\lambda} \text{tr} \left( A_\lambda \partial_\mu A_\nu + \frac{2}{3} A_\lambda A_\mu A_\nu \right) \right\}.$$  \hspace{1cm} (9)

If we shift the vector potential $A_\mu = A^\mu_0 T^a$ as $A_\mu = \tilde{A}_\mu - a_\mu$ and then integrate over $\tilde{A}_\mu$, we find that the corresponding partition function becomes $Z_1 \approx Z_{\text{TH}}$, where the approximation is due to neglecting the $\mathcal{O}(\frac{g}{M})$ term. If, on the other hand, we directly perform the $a_\mu$ integration in $Z_1$ we obtain the following partition function

$$Z_{\text{FCS}} = \int \mathcal{D}A_\mu \exp \left\{ -\int d^3x \, \frac{g^2}{2\pi^2} \text{tr} \left( F_\mu S^{\mu\nu} F_\nu \right) - \frac{i}{4\pi} \int d^3x \, \epsilon^{\mu
u\lambda} \text{tr} \left( A_\lambda \partial_\mu A_\nu + \frac{2}{3} A_\lambda A_\mu A_\nu \right) \right\},$$  \hspace{1cm} (10)

where $S^{\mu\nu} = (\delta^{\mu\nu} + \frac{i g^2}{2\pi} \epsilon^{\mu
u\lambda} A_\lambda)^{-1}$ and $F_\mu = \frac{1}{2} \epsilon^{\mu
u\lambda} F_{\nu\lambda}$. The first term of this action is a non-Abelian gauge theory that does not admit direct interpretation. The second term is the SU(2) Chern-Simons theory at level $k = 1$ that gives mass to the gauge field and a finite correlation length $\xi$. As a result the large distance behaviour compared to $\xi$ is dominated by the Chern-Simons term with the contribution of the first term decaying exponentially fast away from the sources.

The partition function (10) describes our model for any value of $g$. Consider now the limit $g^2 \ll 1$, where $S^{\mu\nu} \sim \delta^{\mu\nu}$. In this limit the short distance behaviour compared to $\xi$ of the FCS theory is described by the SU(2) Yang-Mills action

$$S_{\text{YM}}[A] = \frac{g^2}{8\pi^2} \int d^3x \, \text{tr} \, F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (11)

and (10) defines a topologically massive gauge theory [16, 20]. Thus the original field theory, after the interpolating procedure, becomes the gauge invariant Yang-Mills-Chern-Simons theory in the limit $g^2 \ll 1$ and large mass $M$. In particular the $(2+1)$-dimensional Yang-Mills theory supports confinement, one of the most intriguing challenges in high energy physics. Confinement can explain why free quarks cannot be experimentally detected. Nevertheless, this behaviour is analytically intractable to prove in $3 + 1$ dimensions [21]. To probe this property of the SU(2) Yang-Mills theory, given by $\sigma \sim g^{-4}$ [22], loops $K$ that can probe this short-distance regime are shown in Fig. 3 (Left). It is important to remark that there exists just a single quantum phase with different long- and short-range behaviours. Indeed the behaviour of the Wilson loop changes drastically when we consider distances where the Chern-Simons term becomes relevant. This is achieved when we consider loops $K$ with geometric characteristics that are large compared to the correlation length $\xi$ of the system, as shown in Fig. 3 (Right). In this large-distance/low-energies regime we can ignore the Yang-Mills term and consider exclusively the Chern-Simons term that gives rise to a topological behaviour. This is the regime that we consider next.

We now extend Hamiltonian (1) in order to obtain theories with general integer $k$ that can support non-Abelian anyons. This is possible by introducing more than one species of spin-1/2 fermions. To illustrate that we start by considering $N$ copies of the model and parameterising the fermionic species by the index $\beta = 1, \ldots, N$. Then we modify the interaction term to obtain the SU(2) Thirring model with $N$ fermion species, namely

$$H = \int d^3x \left[ \sum_{\beta=1}^N \psi_\beta^\dagger (\gamma_\mu \gamma_\nu p + \gamma_\nu M) \psi_\beta + \frac{g^2}{2} J^{a\mu} J^a_\mu \right],$$  \hspace{1cm} (14)

where $\psi_\beta = (\psi_{\beta \uparrow}, \psi_{\beta \downarrow})^T = (b_{\beta \uparrow}, w_{\beta \uparrow}, b_{\beta \downarrow}, w_{\beta \downarrow})^T$, $J^{a\mu} = \sum_{\beta=1}^N J^{a\mu}_\beta$ and $J^a_\mu = \psi_\beta^\dagger T^a \gamma_\mu \psi_\beta$. This new interaction can be directly given in terms of the tight-binding fermions, $b_{\beta \uparrow}$ and $w_{\beta \downarrow}$, for $\beta = 1, \ldots, N$ and $s = \uparrow, \downarrow$. It represents a spin non-preserving interaction because it mixes the different fermionic species, as shown in Fig. 2 (Right). By performing the same bosonisation procedure as in the case of single species we obtain the effective ac-
The expectation value of the corresponding Wilson loop is a topological invariant of the link \( K \) which does not depend on its geometrical characteristics, but only on its topology. For a simple loop considered here it is \( V_K(q) = 1 \). This is in stark contrast to the conforming regime where \( \langle W(K) \rangle \) tends to zero as \( A_K \) increases.

The Jones polynomial \( V_K \) is given in terms of the Jones polynomial \( [25] \) for anyons supported, which are universal for Fibonacci anyons. The string tension \( \sigma \) is a topological invariant of the SU(2) level \( k \) Chern-Simons bulk theory. It is a topological invariant of the link \( K \). This relation holds for any link \( K \) of any size and position is a witness of the model’s topological order provided its ground state is not a trivial product state \([10]\).

It is worth noting that we do not have a direct way to measure the Wilson loop in terms of fermionic observables as we did in the Abelian case \([10]\). Nevertheless, it is possible to probe the topological order of the model through its behaviour at the boundary. For large characteristic geometries of the boundary so that short range correlations do not get involved the topological properties of the model can be isolated. It was shown in \([25, 26]\) that the SU(2) level \( k \) Chern-Simons bulk theory induces at its edge the SU(2) Wess-Zumino-Witten model, which is a conformal field theory \([27]\). To probe the Wess-Zumino-Witten model it is possible to measure the thermal currents at the boundary. For this model the thermal conductance \( K_Q \) of the edge modes is given by \([28, 29]\)

\[
K_Q = \frac{\partial J_Q}{\partial T} = \frac{\pi}{6} e k_B T, \tag{18}
\]

which holds in the low temperature limit \( T \to 0 \). Here \( J_Q \) is the thermal current carried by the edge modes, \( k_B \) is the Boltzmann constant and \( e \) is the corresponding central charge. By employing \([18]\) we can evaluate the level \( k \) of the theory and thus determine the particular species of anyons present in our model \([7]\). The physical realisation of our model could be performed with cold atom methods proposed in \([30–37]\), while a possible method to detect the chiral edge states is given in \([38]\).

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