ON COHERENT STATES AND
q-DEFORMED ALGEBRAS

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Abstract

We review some aspects of the relation between ordinary coherent states and q-deformed generalized coherent states with some of the simplest cases of quantum Lie algebras. In particular, new properties of (q-)coherent states are utilized to provide a path integral formalism allowing to study a modified form of q-classical mechanics, to probe some geometrical consequences of the q-deformation and finally to construct Bargmann complex analytic realizations for some quantum algebras.

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1 Introduction

This work is devoted to the relation between q-continuous representations, or q-generalized coherent states, and some of the simplest deformed Lie algebras\(^1\), \(^2\), \(^3\), i.e the q-oscillator, \(su_q(2)\) and the \(su_q(1,1)\)\(^4\), \(^5\), \(^6\). We start by giving some basic notions of Hopf algebras, the underlying theory of quantum groups, such as co-product, non-co-commutativity, R-matrix, quantum group and algebra duality etc, by using the prototype of \(su_q(2)\) algebra (Chapt. 2). Then q-deformed coherent states\(^5\), \(^7\) are introduced and their completeness relations\(^8\), \(^9\), \(^10\), eigenvalue problem, minimum uncertainty and related properties are discussed\(^11\), \(^12\) (Chapt.3). Usual i.e non-deformed coherent states\(^13\), \(^14\) are also used to provide complex analytic realizations of these algebras and their co-products\(^15\). These realizations are given in terms of a series in powers of ordinary derivative operators which act on the Bargmann-Hilbert space of holomorphic functions endowed with the usual integration measure (Chapt. 4). Subsequently ordinary and q-deformed coherent states corresponding to the deformed oscillator and the quantum \(su(2)\) and \(su(1, 1)\) algebras are utilized to introduce a path integral formalism for these algebras. In the semiclassical limit, the resulting classical mechanics is studied by evaluating the symplectic forms and the metrics, for arbitrary values of the q-deformation parameters; both metrics are found to be modified by the deformation. Moreover, evaluation of the Riemann curvature scalar reveals a non-constant, deformation induced curvature, in the two dimensional subspace of the Hilbert space spanned by the coherent states vectors and attributing in this way a geometrical role to the q-deformation. When non-deformed coherent states are employed alternatively for the construction of the path integrals, the geometries remain unaffected by the deformation\(^12\). The latter modifies only the upper symbol of each quantum algebra generator, which is given as power series of the respective upper symbols of the non-deformed algebra generators. This result is further used in studying non-linear q-classical dynamics. Especially for this generalized q-mechanics the Liouville theorem of the incompressibility of the phase space is valid (Chapt. 5). Conclusions and future perspectives of the present work are finally concentrated in Chapter 6.

2 Quantum Lie algebras

Lef us start with the matrix

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

(1)

the entries of which are generating elements of the associative algebra of functions of the quantum group \(SU(2)\) denoted by \(Fun(SU_q(2))\). The six commutation relations, defining the algebra \(Fun(SU_q(2))\), \((\omega = q - q^{-1})\),
are written by using two auxiliary 4x4 matrices $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$ and the R-matrix

$$R = \begin{pmatrix} q & 0 \\ 1 & \omega \\ \omega^{-1} & 1 \\ q^{-1} & 0 \end{pmatrix},$$

in the compact form

$$RT_1T_2 = T_2T_1R.$$  \hfill (4)

The unitarity condition $T^+ = T^{-1}$

$$\left(\begin{array}{cc} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{array}\right) = \left(\begin{array}{cc} d & -q^{-1}b \\ -qc & a \end{array}\right)$$

(\text{where bar indicates an involution of the algebra of functions on the group which changes the order of factors in a product, leads to the restrictions } d = \bar{a}, b = -q\bar{c}, \text{ where } q = e^\gamma \in \mathcal{R}). \text{ The unimodularity condition uses the quantum determinant and implies}

$$\text{Det}_q T \equiv ad - qbc = a\bar{a} + q^2\bar{c}c = 1. \hfill (6)$$

Mathematically the $\text{Fun}(SU_q(2))$, is characterized as a Hopf algebra (see details in Ref.(1-3); this means that three more operations can be defined on the algebra. Indeed if $T = \{t^i_j\}$ and the t-elements are identified with the entries of the T-matrix in Eq.1, then there exists the operation of co-product $\Delta T = T \otimes T$, which in components reads $\Delta(t^i_j) = t^i_j \otimes t^k_j$, then the co-unit $\varepsilon(T) = 1$ which means that $\varepsilon(t^i_j) = \delta^i_j$ and finally, the co-inverse or antipode of the entries of T-matrix, $S(T) = T^{-1}$, which has the property $s(T)T = 1 = Ts(T)$, but in general $S^2 \neq 1$. Then the algebra $\text{Fun}(SU_q(2))$, is said to be a non-commutative and non-co-commutative Hopf algebra. Moreover the R-matrix satisfies the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where $R_{ij}$ is the R-matrix acting non-trivially in the ij-subspace of the triple tensor product.

The following property of the T-matrices can be qualified as the group property of the quantum group $\text{Fun}(SU_q(2))$ and reflects the group property of the underlying quantizing group: if the entries of $T$ and $T'$ are commuting and each of them satisfies Eq. (4) then their product $T'' = TT'$ also satisfies the same equation i.e,

$$RT''_1T''_2 = T''_2T''_1R.$$ \hfill (7)

Finally the representation theory of the $\text{Fun}(SU_q(2))$ algebra states that there two families of irreducible representations, both labelled my a real
index $0 < \phi < 2\pi$. A trivial one $a = e^{i\phi}$, $c = 0$ and a infinite dimensional one in the Fock representation space of the usual harmonic oscillator, $[A, \bar{A}] = 1$, $N = AA$, i.e

$$a = \sqrt{1 - q^{2(N+1)}} A, \quad c = e^{i\phi} q^N. \quad (8)$$

The introduction of the quantum algebra $U_q(su(2))$ i.e, the deformed enveloping algebra of $su(2)$ can be done via the construction of duality pairs between Hopf algebras. Defining, $R^+ \equiv R$ and $R^{-1} \equiv PR^{-1}R$ where $P$ is the permutation matrix of two factors, or in components $(R^-)_{ij} = (R^{-1})_{kj}^{i}$ and forming the matrices,

$$L^+ = \begin{pmatrix} q^{j_3} & -\omega j_3^q \\ 0 & -q^{-j_3^q} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-j_3^q} & 0 \\ -\omega j_3^q & q^{j_3^q} \end{pmatrix} \quad (9)$$

with entries in the $U_q(su(2))$ algebra, we set up the pairing between quantum group and quantum algebra elements as $(L^\pm = \{l_j^\pm\})$

$$< T \otimes L^\pm >= R^\pm , \quad < 1, L^\pm >= 1 , \quad < T, 1 >= 1 , \quad (10)$$

or in component form

$$< t^i_j, l^\pm_i >= (R^\pm)_{ij}^{k} t^k_i < t^i_j, 1 >= \delta^i_j = < 1, l^\pm_i >. \quad (11)$$

Using further the properties of the pairing bracket, such as, $< t^i_j t^k_l, l^{\pm r}_s > = < t^i_j \otimes t^k_l, \Delta(l^{\pm r}_s) >$ and $< t^i_j l^{\pm k}_l \otimes l^{\pm r}_s > = < \Delta(t^i_j), t^k_l \otimes l^{\pm r}_s >$, the commutation relations of the algebra are deduced ($\varepsilon = +++, --, +-+$):

$$L^\varepsilon_1 L^\varepsilon_2 R = R L^\varepsilon_2 L^\varepsilon_1, \quad (12)$$

where as before $L^\varepsilon_1 = L^\varepsilon \otimes 1, L^\varepsilon_2 = 1 \otimes L^\varepsilon$. The so constructed quantum algebra possesses the structure of a Hopf algebra as well, with co-product $\Delta L^\pm = L^\pm \otimes L^\pm$, co-unit $\varepsilon(L^\pm) = 1$ and antipode, $s(L^\pm) = (L^\pm)^{-1}$. From the unitarity of the quantum group matrix $T$ and the duality connecting group and algebra, follows that $(L^+)^+ = (L^{-1})^{-1}$, which implies for the elements of the quantum $su(2)$ algebra that $J^3_3 = J^3_3$ and $\bar{J}^3_\pm = \bar{J}^3_\pm$.

The explicit form of the commutation relations (12) is,

$$[J_3^3, J^\pm_3] = \pm J^\pm_3, \quad [J^\pm_3, J^3_3] = [2J^3_3] \quad (13)$$

where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh x}{\sinh 1}$, or if $J^3_1 = \frac{1}{2}(J^3_3 + J^3_3)$ and $J^3_2 = \frac{1}{2i}(J^3_3 - J^3_3)$ we obtain the commutation relations,

$$[J^3_1, J^3_2] = i[J^3_3], \quad [J^3_2 J^3_3] = iJ^3_1, \quad [J^3_3, J^3_1] = iJ^3_2. \quad (14)$$

Writing explicitly the co-product of the algebra yields

$$\Delta J^3_\pm = J^3_\pm \otimes q^{J^3_3} + q^{-J^3_3} \otimes J^3_\pm \quad (15)$$
This co-product permits to define representation of the algebra (Hopf algebra) in $V_1 \otimes V_2$ tensor product of two representations $V_1$, $V_2$. The irreducible representations of the $su_q(\frac{1}{2})$ algebra generators are similar to the usual ones,

$$J_3^3 |jm> = m |jm>$$

from them we deduce the deforming mapping connecting deformed, indexed by $q$ and non-deformed algebra generators,

$$J_3^q |jm> = \sqrt{[j \pm m][j \pm m + 1]} jm \pm 1 >, \quad (18)$$

The corresponding formulae for the $U_q(su(1, 1))$ algebra include the commutation relations,

$$[K_3^q, K_3^\pm] = \pm K_3^\pm, \quad [K_3^q, K_3^q] = -2K_3^3 \quad (20)$$

while the deforming mappings deduced from the Fock representation read,

$$a_q = a \sqrt{\frac{[N]}{N}} \equiv aF(N) \quad , \quad a_q^+ = \sqrt{\frac{[N]}{N}} a_q^+. \quad (24)$$

Let us remark finally that for the q-oscillator there is no satisfactory bialgebra and Hopf algebra structure.
3 Deformed coherent states and some of their properties

Deformed coherent states for the $q$-oscillator algebra are defined by ($\alpha \in \mathbb{C}$)

$$|\alpha\rangle_q = e^{\alpha a^+}|0\rangle = e^{\alpha T^+_a} |0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_q = e^{\alpha T^+_a} e^{-\alpha a^+} |\alpha\rangle$$ (25)

where

$$T^+_a = a^+ \frac{(N+1)}{[N+1]}$$ and $$T^-_a = \frac{(N+1)}{[N+1]} a_q,$$ (26)

and $[n]! = [1][2]...[n]$ , with the $q$-exponential function $e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let us remark that the introduction of the $q$-CS above may involve deformed and non-deformed exponential functions$^{19,7}$ and that the latter possibility allows us to connect the undeformed oscillator CS to the $q$-deformed one. The normalized states $|\alpha\rangle_q = 1/\sqrt{\alpha q^{|\alpha|} q^{|\alpha|}} |\alpha\rangle_q$ with $q^{|\alpha|} q^{|\alpha|} = e_q^{|\alpha|^2}$ are eigenstates of the annihilation operator $a_q |\alpha\rangle_q = e_q^{-|\alpha|^2} |\alpha\rangle_q$ and satisfy the completeness relation

$$1 = \int |\alpha\rangle_q d\mu_q(\alpha) q^{|\alpha|} q^{|\alpha|}$$ with $$d\mu_q(\alpha) = d^2 \alpha q e_q^{-|\alpha|^2}$$ (27)

where the integral is regarded as the Jackson $q$-integral. Indeed if $\alpha = \sqrt{r} e^{i\varphi}$, the completeness integral is factorized into an ordinary angular integral times a discrete Jackson integral, which leads to the moment problem

$$\int_0^\infty e_q^{-r} r^n d_q r = [n]!. $$ (28)

The derivation of this results in view of the fact that the $q$-exponential function has zeros on the real axis and can even become negative, requires some analysis which has been carried out in the literature. Here will be sufficient to remind that the completeness condition requires the antiderivative $D_x^{-1}$, of the discrete derivative,

$$D_x f(x) = x^{-1} \left[ x \partial_x \right] f(x) = \frac{f(q^{1/2} x) - f(q^{-1/2} x)}{(q^{1/2} - q^{-1/2}) x},$$ (29)

which by virtue of formulae $[x, \partial_x] = 1$, $g(x \partial_x) x = x g(x \partial_x + 1)$ and $q^{kx \partial_x} f(x) = f(q^k x)$, valid when acting on analytic functions of variable $x$ and by the expansion $(2sinh(u/2))^{-1} = \sum_{l=0}^{\infty} e^{u(l+1/2)}$, can be inverted as

$$D_x^{-1} f(x) = x \left[ x \partial_x + 1 \right]^{-1} f(x) = x(q^{-1/2} - q^{1/2})(2sinh - \gamma(x \partial_x + 1) \frac{1}{2})^{-1} f(x)$$

$$= x(q^{-1/2} - q^{1/2}) \sum_{l=0}^{\infty} q^{(l+1/2)} f(q^{(l+1/2)} x).$$ (30)
These $q$-CS are minimum-uncertainty states in the sense that they minimize the $[q, p]$ commutator

$$\Delta q \Delta p = \frac{1}{2} |q < \alpha |[q, p]| \alpha > | q \rangle \rangle (31)$$

where $a_q = \frac{1}{\sqrt{2}} (q + ip_q)$, $a_q^+ = \frac{1}{\sqrt{2}} (q - ip_q)$.

A final comment before closing the oscillator case concerns the kind of analytic functions as elements of the Bargmann-Hilbert space of representation are expandable in the $q$-CS basis. The square integrability condition of such elements

$$< \psi | \psi > = || \psi ||^2 = \frac{1}{\pi} \int d^2 \alpha (e^{\alpha^2})^2 |\psi(\alpha)|^2 < \infty \ (32)$$

requires in the $q = 1$ case, that their growth must be of order $0 < \rho \leq 2$ and type $\tau = \frac{1}{2}$. However due the fact that $e^{\alpha^2} \leq e^{1^2}$, in the $q \neq 1$ case, the possible expandable functions must be of slower growth in comparison to those of the $q = 1$ case. In light of some recent interest in the dynamics of wavefunction zeros, it would be desirable to find the factorization of these wavefunctions in terms of their roots.

The deformed CS for the $su_q(2)$ algebra, related to representations characterized by $j=1/2, 1, 3/2,...$ are defined by ($z \in C$)

$$|z\rangle = e^{zJ_3^+} | -j \rangle = e^{zT^+} | -j \rangle = \sum_{m=-j}^j \left[ \frac{2j}{j+m} \right]_{j}^{1/2} z^{j+m} |m\rangle = e^{zT^+} e^{-zJ^+} |z\rangle, \ (33)$$

where the $q$-binomial is defined as $[a]_q = \frac{[a]}{[a-b]}$.

$$T^+_j = J^+_q \frac{(J_3^q + j + 1)}{[j^3 + j + 1]} \quad \text{and} \quad T^-_j = \frac{(J_3^q + j + 1)}{[j^3 + j + 1]} J^-_q \ (34)$$

The factor $(1 + |z|^2)^{2j} q^2 \equiv q(z|z)_q$ normalizes the states, $|z\rangle = \frac{1}{\sqrt{q(z|z)_q}} |z\rangle$ and using the general formula

$$(x + y)^n_q \equiv \sum_{m=0}^n \left[ \begin{array}{c} n \\ m \end{array} \right]_q x^{n-m} y^m = \prod_{k=1}^n (x + q^{n-2k+1} y) \ (35)$$

derived with the help of $[2m + 1] = \sum_{\ell=-m}^m q^{2\ell}$, that factor is written as

$$q(z|z)_q = \sum_{m=0}^{2j} \left[ \begin{array}{c} 2j \\ m \end{array} \right]_q |z|^2 m = \prod_{k=1}^{2j} (1 + q^{2j-2k+1} |z|^2) \ (36)$$

The normalized $q$-CS are complete with resolution of unity.
\[ 1 = \int |z\rangle_q d\mu_q(z)q(z) \] with \[ d\mu_q(z) = \frac{[2j + 1]}{q(z)q(z)} d^2z , \] (37)

where again the Jackson \( q \)-integral is in use. This completeness relation leads to a moment problem which entails the integral form of the \( q \)-binomial,

\[ \int_0^\infty r^n(1 + r)^{(2j+2)}d\mu_q(r) = \frac{n!(2j + n)!}{[2j + 1]!} . \] (38)

It is also interesting that the \( q \)-CS satisfy the eigenvalue problem,

\[ (J_q^- + (q^j + q^{-j})z[K_q^3] - z^2J_q^+)|z > q = 0 \] (39)

which upon taking the zero deformation limit reduces to its \( q = 1 \) analogue equation which serves as a definition, up to a phase factor, for the \( SU(2) \) coherent state\(^{14}\). For future use we also record the formula

\[ J_q^\pm |z > q = z^\mp 1 [j \pm J_q^3]|z > q . \] (40)

The coherent states related to the quantum \( su(1, 1) \) algebra and associated with the discrete representations characterized by \( k=1, 3/2, 2, 5/2, \ldots \) are defined by the generators

\[ T_K^+ = K_q^+ \left( \frac{K_q^3 - k + 1}{K_q^3 - k + 1} \right) \quad \text{and} \quad T_K^- = \left( \frac{K_q^3 - k + 1}{K_q^3 - k + 1} \right) K_q^- , \] (41)

in a manner similar to the previous cases:

\[ |\xi\rangle_q = e^{\xi K_q^+} |k; 0 >= e^{\xi T_K^+} |k; 0 >= \sum_{n=0}^{\infty} \frac{[2k + n + 1]!}{n!![2k + 1]!!} \xi^n |k; n >= e^{\xi T_K^-} e^{-\xi K_q^+} |\xi\rangle , \] (42)

where \( \xi \in D^k = \{ |\xi|^2 < q^{k-1} \} \). With normalization factor obtained from the overlap of two states

\[ (1 - |\xi|^2)_{-2k} \equiv q(\xi|\xi)_q = \sum_{n=0}^{\infty} \frac{[2k + n + 1]!}{n!![2k + 1]!!} |\xi|^{2n} \] (43)

the normalized states are complete,

\[ 1 = \int |\xi\rangle_q d\mu_q(\xi)q(\xi) \] with \[ d\mu_q(\xi) = \frac{[2k - 1]}{q(\xi|\xi)_{-2}} d^2\xi , \] (44)

and obey the equations,

\[ (K_q^- + (q^k + q^{-k})\xi[K_q^3] + \xi^2K_q^+)|\xi > q = 0 \] (45)

and

\[ K_q^\pm |\xi > q = \xi^{\mp 1} [K_q^3 \mp k]|\xi > q . \] (46)
4 Bargmann realization for quantum algebras

The representation spaces of these deformed realizations will be the ordinary Hilbert spaces of square-integrable analytic functions $L^2(G/H, d\mu(\zeta))$, built on the corresponding cosets of the non-deformed Lie groups, i.e. $G/H = Weyl = SU(2)/U(1)$ and $SU(1,1)/U(1)$. The invariant measure of integration $d\mu(\zeta)$, is the so-called Bargmann measure. One feature of the obtained realizations of the quantum algebra generators is that they constitute a deformation of the ordinary Lie algebra generators in the sense that they involve a series in powers of ordinary derivatives (the coefficients of which depend on the deformation parameter $q$) that reproduces in the ‘classical’ $q \to 1$ limit the Lie algebra vector field generators. Using generic symbols for the deformed algebra generators, $G^q_0 \equiv N, J_3, K_3$ and $G^q_{\pm} \equiv a^\pm_q, J^q_{\pm}, K^q_{\pm}$, and for their co-products, which generically for our cases can be written in the form

$$\Delta G^q_{\pm} = G^q_{\pm} \otimes g(G_0) + g'(G_0) \otimes G^q_{\pm}$$

$$\Delta G^q_0 = G_0 \otimes 1 + 1 \otimes G_0 \ , \hspace{1cm} (47)$$

where $g(G_0)$ and $g'(G_0)$ are specific for each algebra separately.

As the co-product of each algebra generator acts in the tensor product of the representation space of the algebra, we shall look for its analytic functional realization carried by functions of two variables. Omitting the details of the construction, which can be found in Ref.15, we give only the final formulae, which for the realization of the step generators of the algebras read

$$\pi_\zeta(G^q_{\pm}) = \tau_{\pm} \sum_{n=0}^l \frac{b^\pm_n}{n!} \zeta^n \partial_\zeta^n \ , \hspace{1cm} (49)$$

while for the co-products it is obtained that

$$\pi_{\zeta_1,\zeta_2}(\Delta G^q_{\pm}) \Psi(\zeta_1, \zeta_2) = [\pi_{\zeta_1}(G^q_{\pm}) \pi_{\zeta_2}(g(G_0))] \Psi(\zeta_1, \zeta_2) + [\pi_{\zeta_1}(g'(G_0)) \pi_{\zeta_2}(G^q_{\pm})] \Psi(\zeta_1, \zeta_2) \ , \hspace{1cm} (50)$$

with

$$\pi_{\zeta_1}(g(G_0)) = \sum_{m=0}^l \frac{c_m}{m!} \zeta^m \partial_{\zeta_2}^m \ , \hspace{1cm} (51)$$

where the $b$’s and $c$’s are numerical factors, the limit $l$ has an appropriate value for each case and the $\tau$’s are specific for each algebra.
5 Path integrals and q-deformed classical mechanics

We proceed now with the CS propagator utilizing the coupleteness relations of the CS. Let \(\zeta = \alpha, z, \xi\), the transition amplitude between coherent states takes the form

\[
K = \langle \zeta'' | U(t'', t') | \zeta' \rangle = \int \mathcal{D} \mu (\zeta) \exp \left[ \sum_{\ell=1}^{L} \ell n < \zeta|\zeta_{\ell-1} > - \frac{i}{\hbar} \varepsilon < \zeta|H|\zeta_{\ell-1} > \right]
\]

where \(\zeta_0 \equiv \zeta', \zeta_L \equiv \zeta''\),

\[
\mathcal{D} \mu (\zeta) = \lim_{L \to \infty} \prod_{\ell=1}^{L-1} d \mu (\zeta_{\ell})
\]

and \(\varepsilon = \frac{t'' - t'}{L}\), while \(H\) stands for a Hamiltonian given in terms of generators of a quantum algebra. In the continuous limit \(\varepsilon \to 0, L \to \infty\), the lattice space path integral,

\[
K = \int \mathcal{D} \mu (\zeta) \exp \left[ \frac{i}{\hbar} (\int_{0}^{t'' - t'} \theta - H(\zeta, \bar{\zeta}) dt) \right]
\]

where \(H(\zeta, \bar{\zeta}) = < H >\), and \(< (\cdot) > = < \zeta| (\cdot) |\zeta >\), involves the usual one-form \(\theta = i\hbar < \zeta | d |\zeta >\). This one-form gives rise to a Kähler geometry i.e, the symplectic form and the metric of the generalized phase space, corresponding to the cosets, \(WH/U(1) \approx \mathbb{R}'\), \(SU(2)/U(1) \approx S^2\), \(SU(1,1)/U(1) \approx S^{1,1}\) of the three groups in question. The effect of deformation is only in the Hamiltonian which will involve the upper symbols \(< \zeta |\text{gen} |\zeta >\), of generators. These can be expressed in terms of upper symbols of non-deformed generators, resulting in expressions which manifest the nonlinear character of the deformation process. We take up the \(su_q(2)\) case and similar results hold for the two other cases too. Indeed for this case, employing the realizations of the previous section and the formulae

\[
z^m = < j - J_3 >^m, \quad < J^m > = \frac{2j}{(2j - m)!2j^m} < J >^m
\]

deduced by induction, we obtain

\[
< z | J^q | z > = \pi_z (J^q) (z|z) = z^m \sum_{m=0}^{2j} \frac{b^m_m}{m!} z^m \partial^m_z (z|z)
\]

\[
= \sum_{m=0}^{2j} \frac{b^m_m (2j)^m}{2j^m} < J >^m < j - J_3 >^m < J >^{-m+1}. \quad (55)
\]
Having used the ordinary CS in the path integral, the mechanics resulting from the semiclassical form of the propagator is not modified by the deformation. Indeed the $q$-deformation comes in only via the modification of the classical Hamiltonian, which by the last formulae will generate an involved dynamics due to its non-linearity.

Alternatively, we can use the $q$-CS to build up the path integral. The lattice-space form of the propagator is formally the same as it was previously but now the index $q$, should be added everywhere to emphasize the use of $q$-CS. In the continuous limit where $\varepsilon \to 0$, $L \to \infty$, while the deformation parameter $q$ is retained fixed, assuming that $\zeta_{\ell-1} \simeq \zeta_{\ell} - \Delta \zeta_{\ell}$, then from the definition of the CS and the short-time approximation follows that

$$\varepsilon \frac{1}{\varepsilon} \ln(q | \zeta_{\ell} | \zeta_{\ell-1} > q) \cong \frac{\varepsilon}{2} \frac{\Delta \zeta_{\ell}}{\varepsilon} q < \zeta_{\ell} | T_{i}^{-} | \zeta_{\ell} > q - \frac{\Delta \zeta_{\ell}}{\varepsilon} q < \zeta_{\ell} | T_{i}^{+} | \zeta_{\ell} > q \quad (56)$$

where $i = a, J, K$ and bar denotes complex conjugation. In the limit where $L \to \infty$ and $\varepsilon \to 0$ the r.h.s of the above expression is written formally as

$$\frac{1}{2} (\dot{\zeta} q < \zeta | T_{i}^{+} | \zeta > q - \dot{\zeta} q < \zeta | T_{i}^{-} | \zeta > q) dt,$$

where overdot denotes time derivative. In that limit

$$K = \int \mathcal{D} \mu_{q}(\zeta) \exp \left[ \frac{i}{\hbar} \int_{0}^{t'} \int dt \, L_{q}(\zeta, \bar{\zeta}, \dot{\zeta}, \bar{\dot{\zeta}}) \right] \quad (57)$$

and the Lagrangian is given by,

$$L_{q} = \frac{i}{2} \hbar (\dot{\zeta} q < \zeta | T_{i}^{+} | \zeta > q - \dot{\zeta} q < \zeta | T_{i}^{-} | \zeta > q) - \mathcal{H}_{q}(\zeta, \bar{\zeta}) \quad (58)$$

where $\mathcal{H}_{q} = q < \zeta | H | \zeta > q$, from which we extract the canonical 1-form

$$\theta_{q} = i \hbar \frac{1}{2} q < \zeta | d | \zeta > q = \frac{i \hbar}{2} (q < \zeta | T_{i}^{+} | \zeta > q d \zeta - q < \zeta | T_{i}^{-} | \zeta > q d \bar{\zeta}) \quad (59)$$

By using the properties of the $q$-CS as above we obtain for the three cases,

$$L = \frac{i \hbar}{2} q < \frac{N + 1}{[N + 1]} q (\dot{\alpha} \bar{\alpha} - \dot{\bar{\alpha}} \alpha) - \mathcal{H}(\alpha, \bar{\alpha}) \quad (60)$$

and

$$L = \frac{i \hbar}{2} q < J_{q}^{3} + j > q (\dot{\bar{z}} \bar{z}^{-1} - \dot{z} z^{-1}) - \mathcal{H}(z, \bar{z}) \quad (61)$$

and

$$L = \frac{i \hbar}{2} q < K_{q}^{3} - k > q (\dot{\xi} \xi^{-1} - \dot{\bar{\xi}} \bar{\xi}^{-1}) - \mathcal{H}(\xi, \bar{\xi}) \quad (62)$$

The closed generalized symplectic 2-form $\omega_{q} = d \theta_{q}$, is obtained by differentiation of the canonical 1-form above which we cast into the form,

$$\theta_{q} = \frac{i \hbar}{2} q \frac{1}{q(\zeta | \zeta) q} \frac{\partial_{q} (\zeta | \zeta)}{\partial | \zeta |^{2}} (\bar{\zeta} d \zeta - \zeta d \bar{\zeta}) \quad (63)$$
This yields the form,

$$\omega_q = i\hbar (q < T_i^- T_i^+ >_q - q < T_i^- >_q q < T_i^+ >_q) d\bar{\zeta} \wedge d\zeta$$

$$= i\hbar \frac{\partial}{\partial |\zeta|^2} \left[ \frac{|\zeta|^2}{q(\zeta|\zeta)_q} \frac{\partial q(\zeta|\zeta)_q}{\partial |\zeta|^2} \right] d\bar{\zeta} \wedge d\zeta \quad (64)$$

The modification of the resulting classical mechanics is obvious in the last formula from the appearance of the deformed overlaps of CS, which have been given earlier. Along with the symplectic metric the metrical distance gets modified. Proceeding by analogy with the case $q = 1$, writing the potential $\Phi_q = \ell n_q(\zeta|\zeta)_q$, the metric is written as $ds^2 = k\partial_{\zeta}\partial_{\bar{\zeta}}\Phi_q d\zeta d\bar{\zeta}$, where $k$ is a proportionality factor. Using the definition of $q$-CS we find that

$$ds^2 = (q < T_i^- T_i^+ >_q - q < T_i^- >_q q < T_i^+ >_q) d\bar{\zeta} d\zeta$$

$$= i\hbar \frac{\partial}{\partial |\zeta|^2} \left[ \frac{|\zeta|^2}{q(\zeta|\zeta)_q} \frac{\partial q(\zeta|\zeta)_q}{\partial |\zeta|^2} \right] d\bar{\zeta} d\zeta \quad (65)$$

The Riemann curvature scalar of that metric is not constant:

$$R = - (\partial_{\zeta}\partial_{\bar{\zeta}}\Phi_q)^{-1} \ell n(\partial_{\zeta}\partial_{\bar{\zeta}}\Phi_q)$$

for the $q$-oscillator for example, when the deformation parameter is small, $q = e^\gamma \approx 1 + \gamma$,

$$R = \gamma^2 12(1 + 2|\alpha|^2) \quad (67)$$

We conclude then that the $q$-deformation induces a non-constant curvature in the two-dimensional space spanned by each $q$-CS vector, which is a subspace of Hilbert space of representation for each algebra. The symplectic metric, distance and the curvature scalar all refer to that subspace labelling the $q$-CS, which in the non-deformed case coincides with the respective coset spaces of each of the groups. Here however the $q$-CS have been generated acting on a fiducial vector with a displacement operator not originated from a coset decomposition of some group element; with some abuse of language we call the labelling space of our $q$-CS a deformed coset space, which as shown above is qualified for generalized phase space. The equation of motion in this phase space is derived by standard variation of the Langrangian above in the limit when action $\ll \hbar$, and reads,

$$i\hbar \dot{\bar{\zeta}} = (q < T_i^- T_i^+ >_q - q < T_i^- >_q q < T_i^+ >_q)^{-1} \partial_{\bar{\zeta}} \mathcal{H} \quad (68)$$

Finally, since we deal with generalized symplectic mechanics the Liouville theorem holds true and we have verified explicitly that the phase space area element, given by the 2-form above, is an invariant of the canonical equation of motion.
6 Discussion

The classical notion of coherent states is extended in the case of quantum algebras in a rather fruitful way. These extensions offer generalizations to known existing CS for Lie groups, but also provide $q$-CS as a tool for probing the novelties of quantum algebras themselves. Based on the existing duality between quantum algebra and its respective quantum group as Hopf algebras\textsuperscript{21}, it would be possible to construct on the co-representations of quantum group, CS which will be in duality to CS defined, as in this work, on representations of the dual quantum algebra. Unlike of course the CS of the algebra, the CS on its dual quantum group will not have classical analogue.

Of particular importance is the use of CS in the study of geometrical manifestations of the algebraic operation of $q$-deformation, as it has been reviewed here. Based on the methodology of coherent states and their direct relation to path integrals, research can be carried out beyond the algebras studied here to larger quantum algebras and even to other generalized algebras associated with manifolds with richer geometry and potential applications; such work will be taken up elsewhere.
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