Maxwell Theory from Matrix Model

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We present a scenario for deriving Maxwell theory from IIB matrix model. Four dimensional spacetime and theories on it relate different dimensional ones by applying appropriate limits of the backgrounds of matrix model. It is understood by looking at open strings bits as bi-local fields on the spacetime, which are decoupled from the bulk in the limits. The origin of electric-magnetic duality is also discussed in matrix model.

1. Introduction

It is necessary to make constructive definition of superstring theory that derives well known physics. IIB matrix model\cite{1} is a candidate and it seems to be suitable to the brane world scenario\cite{2}. It is defined in ten dimension and coordinates are generalized to be noncommutative. We present the way to derive commutative four dimensional theory from it.

In IIB matrix model, we can find 3-brane classical solution of the action. This 3-brane has a NNCYM4 on it, where gauge fields are from the quantum fluctuation while spacetime is from the classical background. Since the 3-brane solution forms Heisenberg algebra, 4 dimensional space is constructed as Hilbert space. Although spacetime coordinate has uncertainty relation, the coherent states make a intuitive 4 dimensional spacetime as von Neumann lattice\cite{3}. They showed open strings on the lattice and relates the noncommutativity to the string scale.

There is the correspondence between the electromagnetic field on the open strings and the spacetime noncommutativity in the matrix model\cite{4}. We are going to see the decouple limits in matrix model and get the intuitive explanation by using open strings bits on the von Neumann lattice. This is directly understood from relation between the noncommutativity $\theta_{01}^{01}$ from the matrix model and $\theta_{a}^{01}$ from string theory.

2. Noncommutative U(1) theory with direction dependent noncommutativities

In this section NCU(1) is derived from the matrix model. In order to see the duality of the matrix model, we first look into a theory around it, that is, NCU(1).

We start by following IIB matrix model action\cite{5}:

\begin{equation}
S = -\frac{1}{g^2} \text{Tr} (\frac{1}{4} [A_\mu, A_\nu] [A_\mu, A_\nu] + \frac{1}{2} \bar{\psi} \Gamma_\mu [A_\mu, \psi]),
\end{equation}

(2.1)

Now $A_\mu$ and $\psi$ are $n \times n$ Hermitian matrices and each component of $\psi$ is 10 dimensional Majorana-spinor. We expand $A_\alpha = \hat{p}_\alpha + \hat{A}_\alpha$, (for $\alpha = 0 \sim d$) around the following classical solution

\begin{equation}
[\hat{p}_\alpha, \hat{p}_\beta] = i \frac{\theta_{01}}{\theta_{01}},
\end{equation}

(2.2)

\begin{equation}
\begin{pmatrix}
0 & -1/\theta_{01} & 0 & 0 \\
1/\theta_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & -1/\theta_{23} \\
0 & 0 & 1/\theta_{23} & 0 \\
& & & \ldots
\end{pmatrix},
\end{equation}

(2.3)

where $\theta_{01}, \theta_{23}, \ldots$ are c-numbers. The noncommutative coordinates are introduced as:

\begin{equation}
\hat{x}^\alpha := \theta^{\alpha\beta} \hat{p}_\beta,
\end{equation}

(2.4)

and satisfy the relation:

\begin{equation}
[\hat{x}^\alpha, \hat{x}^\beta] = -i \theta^{\alpha\beta}.
\end{equation}

(2.5)

Since we are going to see different limits for different direction, the duality and the decoupling limit, not all $\theta$ are the same, namely, $\theta_{01} \neq \theta_{23}$ etc.

Followed Ref.\cite{5,6}, $\phi := \{ A_\alpha, \hat{\phi}_i := A_i, \hat{\psi} \}$, ($\alpha, \beta = 0 \sim d, i, j = d + 1 \sim 9$) are mapped

\begin{equation}
\text{Email: takata@imsc.ernet.in. Talk presented at LATTICE 2000.}
to usual functions on phase space formed by non-commutative coordinates explicitly:
\[
\hat{\phi} \to \phi(x) = \sum_k \hat{\phi}_k e^{ik_\alpha x^\alpha}.
\]  
(2.6)

The summation over \(k_\alpha\) is performed as follows:
\[
k_\alpha = l_\alpha \sqrt{\frac{2\pi}{\theta^{\alpha\alpha+1}}} \quad l_\alpha = \pm 1, \pm 2, \cdots, \pm \frac{n^{2}}{2}.
\]  
(2.7)

Then we get the action of NCU(1) from eq.(2.1):
\[
S_{NCU(1)} = \frac{1}{g_{YM}^2} \int d^4x \left(\frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} + \frac{i}{2} \bar{\psi} \gamma^\alpha [D_{\alpha}, \psi] - \frac{1}{4} [\psi_i, \psi_j] [\psi_i, \psi_j] \right).  
\]  
(2.8)

Inside \((\_\_\_\_\_\_\_\_\_\_)\), the products should be understood as the star product:
\[
\phi_1(x) \ast \phi_2(x) = e^{\frac{\theta^{\alpha\beta} k_{\alpha}}{2\theta^{\alpha\alpha+1}}} \phi_1(x + \xi) \phi_2(x + \eta)|_{\xi = \eta = 0}  
\]  
(2.9)

The covariant derivative and the field strength are defined as:
\[
D_\alpha := \partial_\alpha - i A_\alpha , \quad F_{\alpha\beta} := i [D_\alpha, D_\beta].
\]  
(2.10)

The Yang-Mills coupling is related to the non-commutativity as:
\[
g_{YM}^2 = g^2(2\pi)^{\frac{2}{2}} \theta^{01} \theta^{23} \cdots \theta^{d-2} d^{-1}.
\]  
(2.11)

3. Limits for four dimensional commutative Maxwell theory

In this section we are going to define a decoupling limit and a commutative limit in the matrix model. We assume the spacetime dimension is almost equal to 4 and coordinates are almost commutative. It dose not have to be exact 4 dimensional commutative spacetime. The stand point of this paper is in 10 dimensional noncommutative one. Our strategy is getting the above brane world from the matrix model in 10 dimension, by fine tuning the parameters \(\theta^{\mu\nu}\), \((\mu, \nu = 0 \sim 9)\).

To explain this by string terminology, we need identify strings in the matrix model. It is possible.\[\text{We will summarize it in our case. The}\]

von Neumann lattice is the best representing the intuitive spacetime. It is constructed by using coherent state of operators of the noncommutative coordinates which forms Heisenberg algebra: eq(2.5). The lattice spacing is \(\sqrt{2\pi \theta^{00}}\) for 0, 1 directions and \(\sqrt{2\pi \theta^{23}}\) for 2, 3 etc., which are written as \(l_{NC}^{2}\). Because of the noncommutativity, states cannot be localized. So, fields are naturally represented as bi-local ones, which are functions of two points. Small momentum modes correspond ordinary (commutative) field. Large momentum modes correspond open strings. Define \(d_0 := \theta^{\alpha\beta} k_\beta\) and decompose \(d = d_0 + \delta d\), where \(d_0\) is a vector which connects two points on the lattice and \(|\delta d\alpha| < l_{NC}^{2}\). The length of open string is \(d_0\) and the momentum which can be associated with the center of mass motion of open string is \(k_{c\alpha} := (1/\theta)_{\alpha\beta} \delta^\beta d\). There is an inequality:
\[
|k_{c\alpha}| < \sqrt{\frac{2\pi}{\theta^{\alpha\alpha+1}}}. \quad (3.1)
\]

And the length and momenta are explicitly:
\[
d_0^\alpha = m^\alpha \sqrt{2\pi \theta^{\alpha\alpha}}, \quad m^\alpha = 0, \pm 1, \pm 2, \cdots, \pm \frac{n^{2}}{2} \quad (3.2)
\]

\[
k_{c\alpha} = \frac{m_\alpha}{n^{2}} \sqrt{\frac{2\pi}{\theta^{\alpha\alpha}}}, \quad m_\alpha = 0, \pm 1, \cdots, \pm (n^2 - 1) \quad (3.3)
\]

3.1. Getting four dimensional spacetime

We define the decoupling limit as:
\[
\theta^{45,67,\cdots-d-2} d^{-1} \to \infty. \quad (3.4)
\]

In this limit,
\[
[\hat{p}_\alpha, \hat{p}_\beta] \to 0 \quad \alpha, \beta = 1 \sim d - 1. \quad (3.5)
\]

This means the dimension of brane get down by two in \(\alpha\beta\) directions, so it is natural definition. The lengths and the momenta are
\[
d_0^\alpha \to 0 \quad \text{or} \quad \infty, \quad k_{c\alpha} \to 0. \quad (3.6)
\]

The momenta of the open strings \(k_c\) goes to zero. If one considers the higher order correction to propagator of bi-local field, then the oscillation of open string are seen by collecting open strings bits(see fig.). In the limit, however, the
momentum of the bits are goes to zero and the open string cannot make loop and is decoupled from closed strings in bulk.  

3.2. Commutative limit

Let us think about the opposite limit:

\[ \theta^{01,23} \to 0 \quad , \quad n^2 \theta^{01,23} : fix \]  

Since

\[ [x^\alpha, x^\beta] = -i \theta^{\alpha\beta} \quad , \]  

coordinates commute to each other in this limit. The momentum \( k_\alpha \) is eigenvalue of \( \hat{P}_\alpha := [\hat{p}_\alpha, \cdot] \) (not of \( \hat{p}_\alpha \)). Momenta commute to themselves without any limits because \( [\hat{P}_\alpha, \hat{P}_\beta] = 0 \). So this limit can be called commutative limits.

The lengths and momenta are

\[ d^{0-3} = 0 \sim \frac{1}{2} \sqrt{2\pi n^2 \theta^{01,23}} < \infty \]  
\[ (\#d = n^2 + 1) \]  
\[ k_{c:0-3} = m_{0-3} \sqrt{\frac{2\pi}{n^2 \theta^{01,23}}} \leq \infty \]  
\[ m_{0-3} = 0, \pm 1, \ldots, \pm (n^2 - 1) \]  

Now we can draw a scenario of getting an almost commutative near 4 dimensional spacetime. We start from \( d \) dimensional solution of the matrix model. There are \( d/2 \) noncommutativity parameters \( \theta^{01}, \theta^{23}, \ldots, \theta^{d-2,d-1} \). We think of regions near following limits:

\[ \theta^{01,23} \to 0 \quad , \quad \theta^{45-6d-2d-1} \to \infty \]  

\[ \theta^{01} \to \infty \quad , \quad \theta_+^{01} : finite \quad , \]  

and consistent with string theory approach.

Then we have a 4 dimensional commutative spacetime. This is not dynamical determination, but just a fine tuning.

![Diagram](https://via.placeholder.com/150)

open string consisting of string bits.

4. Electric-Magnetic Duality in Matrix Model (d=4 case)

In this section we consider the electric-magnetic duality of NCU(1) on a D3-brane in the matrix model.

The relation between them are[10]:

\[ g_{YM D} = \frac{1}{g_{YM}} \]  
\[ \theta^{\alpha\beta}_D = \frac{g^2_{YM}}{2} \epsilon^{\alpha\beta\gamma\delta} \theta^{\gamma\delta} \quad , \]  
\[ F_{\alpha\beta D} = \frac{1}{2 g_{YM}} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} + O(\theta) \quad . \]  

(4.1)

This is a spacetime-spacetime duality as well as electric-magnetic and strong-weak.

Next we will see this in our case. By using eq.\([2.11]\), the dual Yang-Mills coupling and noncommutativity of eq.\([4.3]\) are written as:

\[ g_{YM D}^2 := \frac{1}{(2\pi g)^2 \theta^{01} \theta^{23}} \]  
\[ \theta^{01}_D := (2\pi g)^2 \theta^{01} (\theta^{23})^2 \]  
\[ \theta^{23}_D := (2\pi g)^2 (\theta^{01})^2 \theta^{23} \quad . \]  

(4.2)

Since there is a relation eq.\([2.11]\) in original theory, we would like the dual theory also to have the same one: \( g_{YM D}^2 = (2\pi g_D)^2 \theta_D^{01} \theta_D^{23} \). This requirement determine how the coupling \( g \) changes to \( g_D \):

\[ g_D := \frac{1}{(2\pi)^2 g^2 (\theta^{01} \theta^{23})^2} \]  

(4.3)
We try to explain this by imaging a duality web. The partition function of the matrix model is not changed under suitable rescaling of matrices. That is, rescaling of coupling $g$ does not change the model. We have started with a $g$ and chosen an arbitrary background $\theta$. On the other hand, we can start by another $g_D$ and $\theta_D$, and if those satisfy the condition:

\[(2\pi g)^2 \theta_0^1 \theta_2^3 \cdot (2\pi g_D)^2 \theta_D^0 \theta_D^2 = 1 , \quad (4.4)\]

then two NCU(1)'s are the dual to each other. This duality transformation is just rescaling: $g \to g_D$, which is a symmetry. It is natural to understand this if we remind type IIB superstring is self-S-dual and its matrix model too.

Finally let us see, in particular, more simple and familiar case. We can find the dual pair of NCU(1) from the matrix model with the same $g$, and find the electric-magnetic duality for usual commutative Maxwell equations. The U(1) coupling $g_{YM}$ cannot to be identity by rescaling of the gauge field $A$ in noncommutative case. But, in the matrix model framework, it is possible. For given $g$, we choose the back ground solution with $\theta_0^1$ and $\theta_2^3$ which satisfy $(2\pi g)^2 \theta_0^1 \theta_2^3 = 1$, namely, $g_{YM} = 1$. Then the dual noncommutativities also satisfy the same condition. Now the dual transformation is:

\[
\begin{align*}
(\theta_0^1, \theta_2^3) &= (\theta_2^3, \theta_0^1) , \\
(\mathcal{E}_D, \mathcal{B}_D) &= (\mathcal{B}, \mathcal{E}) + O(\theta) .
\end{align*}
\]

Thus, Maxwell equations without the sources have a duality in the following commutative limit:

\[
\theta, \theta_D \to 0 , \quad (2\pi g)^2 \theta_0^1 \theta_2^3 = 1 . \quad (4.6)
\]

5. Conclusions

We have considered the decoupling-commutative limit and the electric-magnetic duality in the matrix model. d-dimensional spacetime can be constructed as D(d−1)-brane solution of the model. It has non-selfdual solutions which we have treated in this paper. Then there are $d/2$ non commutativity parameters: $\theta^0_1, \theta^2_3, \ldots, \theta^{d-2d-1}$. Electric-magnetic duality transformation changes those parameters as eq.(4.3), as well as Yang-Mills coupling and electromagnetic fields. In addition, it corresponds to the rescaling of matrices in the original matrix model, which has S-duality symmetry. In particular solution related to a $g$, its duality is just the duality of U(1) Maxwell theory: eq.(4.3).

Decoupling limits have been defined as eq.(3.4). Open strings are decoupled from closed strings by looking into their momenta, and the tension goes to zero. This has been also seen from the relation $\theta^0_1$ and $\theta^0_1$ clearly Commutative limits are defined as eq.(3.8). Noncommutativity parameters manage making our 4 dimensional commutative spacetime into 4 direction commutes as $\theta^0_1, \theta^0_2 \to 0$. Staring from higher dimensional brane, we can get an almost commutative and near 4 dimension spacetime, by fine tuning of those parameters.

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