HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE OF AN ANNULUS

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Abstract. A bounded operator $S$ on a Hilbert space is hyponormal if $S^*S - SS^*$ is positive. In this work we find necessary conditions for the hyponormality of the Toeplitz operator $T_{f+g}$ on the Bergman space of the annulus \( \{1/2 < |z| < 1\} \), where $f$ and $g$ are analytic and $f$ satisfies a smoothness condition.

1. Introduction

A bounded operator $S$ on a Hilbert space is hyponormal if $S^*S - SS^*$ is positive. Hyponormality of Toeplitz operators has been studied by many authors. Hyponormality of these operators on the Hardy space was considered in [3, 4]. Hyponormality of these operators with a symbol of the form $g_1 + g_2$ on the Bergman space of the unit disk was first considered in [8]. Therein a necessary condition was proved, which was later improved in [1]. Some special cases are treated in [7]. A sufficient condition when $g_1$ is a monomial and $g_2$ is a polynomial is proved in [9]. An improvement of the necessary condition in the case when $g_1$ and $g_2$ are binomials is given in [5]. Basic material on Toeplitz operators on the Bergman space of the unit disk can be found in [2]. In this work we consider hyponormality of Toeplitz operators on the Bergman space of an annulus.

We start with definitions and notations. Denote by $A^2_{1/2}$ the space of holomorphic functions on the annulus $C_{1/2} = \{z \in \mathbb{C} : 1/2 < |z| < 1\}$ such that $\int |h|^2 \, dm(z) < \infty$, where $dm(z) = (4/3\pi) \, d\lambda(z)$ and $\lambda$ is the Lebesgue measure on the annulus. If $h \in A^2_{1/2}$ we write $h = a_0 + \sum_{n=1}^{\infty} a_n z^n + a_{-n} z^{-n}$ and we have $\|h\|^2 = \sum_{n=0}^{\infty} \frac{4(1-(1/2)^{2n+2})}{3(n+1)} |a_n|^2 + \frac{8}{3} \ln 2 |a_{-1}|^2 + \sum_{n=2}^{\infty} \frac{4(2^{2n-2}-1)}{3(n-1)} |a_{-n}|^2$. We denote by $L^2(C_{1/2})$ the space of measurable and square integrable functions with respect to $dm$ on $C_{1/2}$. Toeplitz operators on $A^2_{1/2}$ are defined by $T_f(h) = P(hf)$, where $f$ is bounded and measurable on $C_{1/2}$, $P$ is the orthogonal projection on

2010 Mathematics Subject Classification. Primary 47B35, 47B20; Secondary 15B48.

Key words and phrases. Toeplitz operator; Bergman space of an annulus; hyponormal; positive matrix.

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this work through the Research Group project No. RG-1435-069.

303
$A^2_{1/2}$, and $h$ is in $A^2_{1/2}$. The Hankel operators on the space $A^2_{1/2}$ are defined by $H_f(h) = (I - P)(hf)$. The space $A^2_{1/2}$ has an orthonormal basis given by the union of the sets

$$
\{ e_n = \frac{\sqrt{3}(n + 1)}{2\sqrt{(1 - (1/2)^{2n+2})^2}} z^n, \ n \geq 0 \},
$$

$$
\{ e_{-1} = \frac{\sqrt{3}}{\sqrt{8 \ln 2}} \}, \ \text{and}
$$

$$
\{ e_{-n} = \frac{\sqrt{3}(n - 1)}{2\sqrt{(2^{2n-2} - 1)}} \frac{1}{z^n}, \ n \geq 2 \}.
$$

We consider hyponormality of Toeplitz operators with a symbol of the form $f = g_1 + \frac{g_2}{z}$, where $g_1$ and $g_2$ are bounded analytic functions on $C_{1/2}$. We begin by recalling some known properties of Toeplitz operators.

2. SOME BASIC PROPERTIES

**Lemma 2.1.** Let $f$ and $g$ be bounded and measurable on $C_{1/2}$. The following properties hold:

a) $T_{f+g} = T_f + T_g$.

b) $T_f^* = T_{f^*}$.

c) $T_f T_g = T_{fg}$ if $g$ is analytic on $C_{1/2}$ or $f$ is conjugate analytic.

d) $T_f T_f - T_{f^*} T_f = H_f^* H_f$ if $f$ is analytic.

The next proposition is easy to prove and its proof is omitted.

**Proposition 2.2.** Let $g_1$ and $g_2$ be polynomials. The following are equivalent:

a) $T_{g_1 + \frac{g_2}{z}}$ is hyponormal.

b) $T_{g_1}^* T_{g_2} - T_{g_2} T_{g_1}^* \leq T_{g_1} T_{g_1} - T_{g_1}^* T_{g_1}$.

c) $H_{g_2}^* H_{g_2} \leq H_{g_1}^* H_{g_1}$.

d) $H_{g_2} = KH_{g_1}$, where $K$ is an operator of norm less than one.

The following lemma provides computations that will be needed.

**Lemma 2.3.** The projection $P$ on $A^2_{1/2}$ satisfies the following relations:

1) $P(z^m \bar{z}^n) = \frac{(m - n + 1)(1 - (1/2)^{2m+2})}{(m + 1)(1 - (1/2)^{2m-2n+2})} z^{m-n}$, if $m \geq n$.

2) $P(z^m \bar{z}^n) = \frac{(n - m - 1)(1 - (1/2)^{2m+2})}{(m + 1)(2^{2n-2m-2} - 1)} \frac{1}{z^{n-m}}$, if $n \geq m + 2$.

3) $P(z^m \bar{z}^{m+1}) = \frac{(1 - (1/2)^{2m+2})}{2 \ln 2 (m + 1)} \frac{1}{z}$, if $n = m + 1$.

4) $P \left( \frac{1}{z^m \bar{z}^n} \right) = \frac{(m + n - 1)(2^{2m-2} - 1)}{(2^{2(m+n)-2} - 1)(m - 1)} \frac{1}{z^{m+n}}$, if $m \geq 2$. 

Rev. Un. Mat. Argentina, Vol. 61, No. 2 (2020)
5) \( P \left( \frac{1}{z^n} \right) = \frac{2n \ln 2}{(2^n - 1)^{\frac{1}{n+1}}} \), if \( n \geq 1 \).

6) \( P \left( \frac{1}{z^m z^n} \right) = \frac{(m + n + 1)((1 - (1/2)^{2n+2})}{(n + 1)(1 - (1/2)^{2(m+n)+2})} \), if \( m \geq n, n \neq 1 \).

7) \( P \left( \frac{1}{z^m z^n} \right) = \frac{((m - n) + 1)(2^{2n-2} - 1)}{(n - 1)(1 - (1/2)^{2(m-n)+2})} \), if \( m \geq n, n \neq 1 \).

8) \( P \left( \frac{1}{z^m z^n} \right) = \frac{2m \ln 2}{(1 - (1/2)^{2m})} \), if \( m \geq 1 \).

9) \( P \left( \frac{1}{z^m z^n} \right) = \frac{(n - m - 1)(2^{2n-2} - 1)}{(n - 1)(2^{(m-n)-2} - 1)} \), if \( m \geq 1, n - m > 1 \).

10) \( P \left( \frac{1}{z^m z^n} \right) = \frac{(2^{m-1} - 1)}{2m \ln 2} \), if \( m \geq 1 \).

3. First main result

We begin with a matrix computation.

**Lemma 3.1.** Let \( f = \sum_1^\infty a_k z^k \) be bounded on \( C_{1/2} \). Then for \( i, j \geq 1 \) we have

\[
\langle T_f T_f - T_f T_f(e_j), e_i \rangle = \sum_{1 \leq k+j-i} \frac{\bar{a}_{k+j-i} a_k}{\sqrt{1 + \frac{(i + 1)(1 - (1/2)^{2k+j+2})}{\sqrt{1 - (1/2)^{2i+2}}} \sqrt{1 - (1/2)^{2j+2}}} (k + j + 1)}
\]

\[
- \sum_{1 \leq k \leq j, k \neq i-j} \frac{\bar{a}_k a_{k+i-j}}{\sqrt{1 - (1/2)^{2i+2}}} \sqrt{1 - (1/2)^{2j+2}}} \frac{(j - k + 1)}{(1 - (1/2)^{2(j-k)+2}) \sqrt{i + 1} \sqrt{j + 1}}
\]

\[
- a_{j+1} a_{i+1} \frac{\sqrt{1 - (1/2)^{2i+2}}} \frac{\sqrt{1 - (1/2)^{2j+2}}} \frac{2 \ln 2}{\sqrt{i + 1} \sqrt{j + 1}}
\]

\[
- \sum_{j+2 \leq k, 1 \leq k+i-j} \frac{\bar{a}_k a_{k+i-j}}{\sqrt{1 - (1/2)^{2i+2}}} \frac{\sqrt{1 - (1/2)^{2j+2}}} \frac{\sqrt{1 - (1/2)^{2j+2}}} \frac{2 \ln 2}{\sqrt{i + 1} \sqrt{j + 1}}
\]

**Proof.** We have

\[
\langle T_f T_f(e_j), e_i \rangle = \sum_{i=1}^\infty \frac{\sqrt{3(i + 1)}}{2 \sqrt{1 - (1/2)^{2i+2}}} \frac{\sqrt{3(j + 1)}}{2 \sqrt{1 - (1/2)^{2j+2}}} \langle z^{k+j}, z^{i+1} \rangle
\]

\[
= \sum_{1 \leq k \leq j-i} \frac{\bar{a}_{k+j-i} a_k (1 - (1/2)^{2(k+j)+2}) \sqrt{(i + 1)(j + 1)}}{(k + j + 1) \sqrt{1 - (1/2)^{2i+2}} (1 - (1/2)^{2j+2})}
\]

Rev. Un. Mat. Argentina, Vol. 61, No. 2 (2020)
Similarly, we get
\[
\langle T_f T_f(e_j), e_i \rangle = \sum_{1 \leq k+i-j \leq i, j \leq k} \overline{a_k} a_{k+i-j} (j-k+1) \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2j+2}} \\
\times (1 - (1/2)^{2(j-k)+2}) \sqrt{i+1} \sqrt{j+1} \\
+ a_{j+1} a_{i+1} \sqrt{(1 - (1/2)^{2i+2})} \sqrt{(1 - (1/2)^{2j+2})} \\
+ \sum_{j+2 \leq k \leq i, k+j = k} \overline{a_k} a_{k+i-j} (k-j-1) \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2j+2}} \\
\times \sqrt{(i+1)(j+1)}.
\]
\[\square\]

Set \(\beta_{i,j} = \langle T_f T_f - T_f T_f(e_j), e_i \rangle\), \(i, j \geq 1\). By rewriting the expression for \(\beta_{i,j}\) we obtain
\[
\beta_{i+p,i} = \sum_{1 \leq k \leq i, 1 \leq k \leq p} \overline{a_k} a_{k+p} \sqrt{i+1} \sqrt{i+p+1} (1 - (1/2)^{2(k+p+i)+2}) \\
- \sum_{1 \leq k \leq i, 1 \leq k \leq p} \overline{a_k} a_{k+p} (i+k+1) \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2(i+p)+2}} \\
\times (1 - (1/2)^{2(i-k)+2}) \sqrt{i+1} \sqrt{i+p+1} \\
+ \overline{a_{i+1}} a_{i+p+1} \sqrt{i+1} \sqrt{i+p+1} (1 - (1/2)^{2(i+1+p)+2}) \\
\times \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2(i+p)+2}} (2(i+1)+p) \\
- \overline{a_{i+1}} a_{i+p+1} \sqrt{i+1} \sqrt{i+p+1} \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2(i+p)+2}} \\
\times 2 \sqrt{i+1} \sqrt{i+p+1} \\
+ \sum_{i+2 \leq k} \overline{a_k} a_{k+p} \sqrt{i+1} \sqrt{i+p+1} (1 - (1/2)^{2(k+p+i)+2}) \\
\times \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2(i+p)+2}} (k+p+i+1) \\
- \sum_{i+2 \leq k} \overline{a_k} a_{k+p} (k-i-1) \sqrt{1 - (1/2)^{2i+2}} \sqrt{1 - (1/2)^{2(i+p)+2}} \\
\times \sqrt{i+1} \sqrt{i+p+1} \\
= \sum_{1 \leq k \leq i, 1 \leq k \leq p} \overline{a_k} a_{k+p} Q_{i,k,p} + \overline{a_{i+1}} a_{i+p+1} R_{i,p} + \sum_{i+2 \leq k} \overline{a_k} a_{k+p} S_{i,k,p}.
\]

**Lemma 3.2.** We have \(\lim_{i \to \infty} i^2 \beta_{i+p,i} = \gamma_{i+p,i}\), where \((\gamma_{i,j})\) is the matrix of the Hardy space Toeplitz operator \(T_{f'}^2\).

**Proof.** An elementary computation shows that \(\lim_{i \to \infty} i^2 Q_{i,k,p} = k(k+p)\). Set \(h_i(k) = i^2 \chi_{\{1, \ldots, i\}}(k) \overline{a_k} a_{k+p} Q_{i,k,p}\). The first sum in the above expression of \(\beta_{i+p,i}\) can be written as \(\int h_i(k) \, d\mu(k)\), where \(d\mu\) is the counting measure. It is easy to see that for \(i\) sufficiently large, \(|h_i(k)| \leq 2|a_k a_{k+p}| \leq k^2|a_k|^2 + (k+p)^2|a_{k+p}|^2 = M(k)\). Since \(f' \in H^2\), the function \(M(k)\) is integrable with respect to the counting measure.

*Rev. Un. Mat. Argentina, Vol. 61, No. 2 (2020)*
By the dominated convergence theorem we obtain:
\[
\lim_{i \to \infty} i^2 \sum_{1 \leq k \leq i} \overline{a_k} a_{k+p} Q_{i,k,p} = \sum k(k+p)\overline{a_k} a_{k+p}.
\]

Also, for \( i \) large, there exists a constant \( C \) such that
\[
|i^2 \overline{a_{i+1}} a_{i+p+1} R_{i,p}| \leq C \left( (i+1)^2|a_{i+1}|^2 + (i+p+1)^2|a_{i+p+1}|^2 \right).
\]

Thus \( \lim_{i \to \infty} i^2 \overline{a_{i+1}} a_{i+p+1} R_{i,p} = 0 \). Finally, it is not difficult to see that
\[
|i^2 S_{i,k,p}| \leq k(k+p).
\]

Using the dominated convergence theorem we obtain
\[
\lim_{i \to \infty} i^2 \sum_{i+2 \leq k} \overline{a_k} a_{k+p} S_{i,k,p} = 0.
\]

We deduce that \( \lim_{i \to \infty} i^2 \beta_{i+p,i} = \sum k(k+p)\overline{a_k} a_{k+p} \) and recognize this last limit as being equal to \( \gamma_{i+p,i} \), where \( (\gamma_{i,j}) \) is the matrix of the Hardy space Toeplitz operator \( T_{|f'|^2} \). \( \square \)

We are led to the following necessary condition for hyponormality.

**Theorem 3.3.** Let \( f = \sum a_k z^k \) and \( g = \sum b_k z^k \) be bounded on \( C_{1/2} \). Assume that \( f' \in H^2 \). If \( T_{f+g} \) is hyponormal then \( g' \in H^2 \) and \( |g'| \leq |f'| \) a.e. on the unit circle.

**Proof.** If \( (\theta_{i,j}) \) denotes the matrix of \( T_f\overline{T_f} - T_f T_{\overline{g}} - T_g T_{\overline{f}} \) and \( (\sigma_{i,j}) \) denotes the matrix of \( T_f T_g - T_g T_f \), then the inequality \( \sigma_{i,i} \leq \beta_{i,i} \) leads to
\[
\sum_{1 \leq k \leq i} |b_k|^2 Q_{i,k,0} + |b_{i+1}|^2 R_{i,0} + \sum_{i+2 \leq k} |b_k|^2 S_{i,k,0} \leq \sum_{1 \leq k \leq i} |a_k|^2 Q_{i,k,0} + |a_{i+1}|^2 R_{i,0} + \sum_{i+2 \leq k} |a_k|^2 S_{i,k,0}.
\]

We deduce that \( \sum_{1 \leq k \leq i} i^2 |b_k|^2 Q_{i,k,0} \leq i^2 \beta_{i,i} \). Since \( \lim_{i \to \infty} i^2 Q_{i,k,0} = k^2 \), writing the left hand side of this last inequality as an integral with respect to the counting measure and using Fatou’s lemma we get \( \int k^2 |b_k|^2 \leq \sum k^2 |a_k| \) and \( g' \in H^2 \). From the previous lemma, \( \lim_{i \to \infty} i^2 \theta_{i+p,i} = \lambda_{i+p,i} \), where \( (\lambda_{i,j}) \) denotes the matrix of the Hardy space Toeplitz operator \( |f'|^2 - |g'|^2 \). Hyponormality and a property of Toeplitz matrices lead to \( |g'| \leq |f'| \) a.e. on the unit circle. \( \square \)

**Corollary 3.4.** Let \( f = \sum a_k z^k \) and \( g = \sum b_k z^k \) be analytic and univalent in an open set containing \( C_{1/2} \). Then \( T_{f+g} \) is normal if and only if \( g = cf \), where \( c \) is a constant with \( |c| = 1 \).

**Proof.** Only the necessary condition needs to be shown. Normality implies that \( |g'| = |f'| \) on the unit circle. Thus \( f' \) and \( g' \) have the same finite number of zeros (if any) with the same multiplicity. We thus have \( \frac{|f'|}{|g'|} = \frac{|g'|}{|f'|} = 1 \) on the unit circle. By the maximum principle, \( g' = cf' \) with \( |c| = 1 \). We get \( g = cf \). \( \square \)
Lemma 3.5. Let $f = \sum_1^\infty a_kz^k$ be bounded on $C_{1/2}$. Then for $i \geq 3$, $j \geq 3$ we have

$$\langle T f T \rangle - T f T \langle e_{-j}, e_{-i} \rangle = \sum_{1 \leq k < j-1}^{1 \leq k+i-j} \frac{a_k + a_k}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{j-i}}{\sqrt{2^{j-2} - 1}} \frac{(2^{(j-k)-2} - 1)}{(j-k-1)}$$

$$+ 2 \ln 2 a_{i-1}a_{j-1} \frac{\sqrt{i-1}}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{j-i}}{\sqrt{2^{j-2} - 1}}$$

$$+ \sum_{j \leq k} a_k + a_k \frac{\sqrt{i-1}}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{j-i}}{\sqrt{2^{j-2} - 1}} \frac{(1 - (1/2)^2(k-j+2))}{k-j+1}$$

$$- \sum_{1 \leq k} \frac{a_k + a_k}{\sqrt{2^{i+k-j-2} - 1}} \frac{(2^{i+k-j-2} - 1)}{(2^{j+k-2} - 1)\sqrt{i-1}\sqrt{j-1}}.$$

Proof. We have

$$\langle T f T \rangle \langle e_{-j}, e_{-i} \rangle = \sum_{1 \leq k < j-1}^{1 \leq k+i-j} \frac{a_k + a_k}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{j-i}}{\sqrt{2^{j-2} - 1}} \frac{(2^{(j-k)-2} - 1)}{(j-k-1)}$$

$$+ 2 \ln 2 a_{i-1}a_{j-1} \frac{\sqrt{i-1}}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{j-i}}{\sqrt{2^{j-2} - 1}}$$

$$+ \sum_{j \leq k} a_k + a_k \frac{\sqrt{i-1}}{\sqrt{2^{i-2} - 1}} \frac{\sqrt{j-i}}{\sqrt{2^{j-2} - 1}} \frac{(1 - (1/2)^2(k-j+2))}{k-j+1}.$$

Similarly,

$$\langle T f T \rangle \langle e_{-j}, e_{-i} \rangle = \sum_{k,l=1}^{\infty} \frac{a_k a_l}{2^{(2^{i-l-2} - 1)}} \frac{\sqrt{3(i-l)}}{\sqrt{2(2^{j-l-2} - 1)}} \frac{\sqrt{3(j-l)}}{\sqrt{2(2^{j-l-2} - 1)}} \left\langle P \left( \frac{z^k}{z^l} \right), P \left( \frac{z^k}{z^l} \right) \right\rangle$$

$$= \sum_{1 \leq k} \frac{a_k + a_k}{\sqrt{2^{i+k-j-2} - 1}} \frac{(k+j-1)\sqrt{2^{i+k-j-2} - 1}\sqrt{2^{j+k-2} - 1}}{(2^{j+k-2} - 1)\sqrt{i-1}\sqrt{j-1}}.$$

Let $\beta_{-i,-j} = \langle (T f T - T f T) \rangle \langle e_{-j}, e_{-i} \rangle$ and denote by $(\zeta_{i,j})$ the matrix of the Toeplitz operator $T_{f_{1/2}}$ on the Hardy space of the unit disk, where $f_{1/2}(z) = \sum \frac{a_k}{z^k}$. We can show the following lemma.
Lemma 3.6. We have $\lim_{i \to \infty} i^2 \beta_{-i-p,-i} = \zeta_{i+p,i}$.

Proof.

\[
\beta_{-i-p,-i} = \sum_{1 \leq k \leq i-1 \atop 1 \leq k+p} \frac{a_{k+p}a_k}{\sqrt{(2i-2-1)} \sqrt{(2i+p-2-1)}} \frac{\sqrt{(i-1)}}{(i-k-1)} \\
+ 2 \ln 2a_{i+p-1}a_{i-1} - \frac{\sqrt{i+p-1}}{\sqrt{2i+p-2-1}} \frac{\sqrt{i-1}}{\sqrt{2i-2-1}} \\
+ \sum_{i \leq k} \frac{a_{k+p}a_k}{\sqrt{(2i-2-1)} \sqrt{(2i+p-2-1)}} \frac{\sqrt{i-i}}{\sqrt{2i+p-2}} \frac{k-i+1}{1} \\
- \sum_{1 \leq k \leq i} \frac{a_{k+p}a_k}{\sqrt{(2i-2-1)} \sqrt{(2i+p-2-1)}} \frac{(k+i-1)}{(2(i+k+p)-2-1)} \\
= \sum_{1 \leq k \leq i-1 \atop 1 \leq k+p} \frac{a_{k+p}a_k}{\sqrt{(2i-2-1)} \sqrt{(2i+p-2-1)}} \frac{\sqrt{i+p-1}}{\sqrt{2i+p-2-1}} \\
+ 2 \ln 2a_{i+p-1}a_{i-1} - \frac{\sqrt{i+p-1}}{\sqrt{2i+p-2-1}} \frac{\sqrt{i-1}}{\sqrt{2i-2-1}} \\
+ \sum_{i \leq k} \frac{a_{k+p}a_k}{\sqrt{(2i-2-1)} \sqrt{(2i+p-2-1)}} \frac{\sqrt{i-i}}{\sqrt{2i+p-2}} \frac{k-i+1}{1} \\
- \sum_{1 \leq k \leq i} \frac{a_{k+p}a_k}{\sqrt{(2i-2-1)} \sqrt{(2i+p-2-1)}} \frac{(k+i-1)}{(2(i+k+p)-2-1)} \\
+ \sum_{1 \leq k \leq i-1 \atop 1 \leq k+p} a_{k+p}a_k Q'_{i,p,k} + a_{i+p-1}a_{i-1} R'_{i,p} + \sum_{1 \leq k \leq i} a_{k+p}a_k S'_{i,k,p}.
\]

A computation shows that $\lim_{i \to \infty} i^2 Q'_{i,p,k} = \frac{1}{2k+p}$. As in the proof of the previous theorem we can show that

\[
\lim_{i \to \infty} i^2 \sum_{1 \leq k \leq i-1 \atop 1 \leq k+p} a_{k+p}a_k Q'_{i,p,k} = \sum_{1 \leq k \leq i} k(k+p)2k^2 \frac{a_{k+p}a_k}{2k+p}.
\]

We see that this last limit is equal to $\zeta_{i,i+p}$. We also show that

\[
\lim_{i \to \infty} i^2 \sum_{1 \leq k \leq i-1 \atop 1 \leq k+p} a_{k+p}a_k R'_{i,p} = 0
\]
and

\[ \lim_{i \to \infty} i^2 \sum_{i \leq k} a_{k+i}a_k S_{i,k,p} = 0. \]

We deduce that

\[ \lim_{i \to \infty} i^2 \beta_{-i-p,-i} = \zeta_{i+p,i}. \]

If \( f = \sum_1^\infty a_k z^k \) is bounded analytic on \( C_{1/2} \), then clearly \( \sum \frac{k^2}{2\pi} |a_k|^2 < \infty \). We can also see that \(|g(1/2)| \leq |f(1/2)| \) a.e. on the unit circle is equivalent to \(|g'| \leq |f'| \) a.e. on \( \{z : |z| = 1/2\} \). \( \square \)

**Theorem 3.7.** Let \( f = \sum_1^\infty a_k z^k \) and \( g = \sum_1^\infty b_k z^k \) be bounded on \( C_{1/2} \). If \( T_{f+g} \) is hyponormal then \(|g'| \leq |f'| \) a.e. on \( \{z : |z| = 1/2\} \).

The proof is similar to the proof of the previous theorem and is omitted. Combining the previous two theorems we get our first main result.

**Theorem 3.8.** Let \( f = \sum_1^\infty a_k z^k \) and \( g = \sum_1^\infty b_k z^k \) be bounded on \( C_{1/2} \) and assume that \( f' \in H^2 \). If \( T_{f+g} \) is hyponormal then \( g' \in H^2 \) and \(|g'| \leq |f'| \) a.e. on \( \{z : |z| = 1\} \cup \{z : |z| = 1/2\} \).

**4. Second main result**

We now put \( f = \sum_1^\infty a_k \frac{1}{z^k} \) and \( g = \sum_1^\infty b_k \frac{1}{z^k} \) and assume that \( f \) and \( g \) are bounded on \( C_{1/2} \). We need the following computation.

**Lemma 4.1.** For \( i \geq 1, j \geq 1 \) we have

\[
\langle T_f T_f - T_f T_f(e_j), e_i \rangle
\]

\[
= \sum_{1 \leq k, k+i-j} a_{k+i-j}a_k \frac{\sqrt{(i+1)\sqrt{(j+1)(1-(1/2)^{2(j-k)+2})}}}{\sqrt{(1-(1/2)^2i+2}\sqrt{(1-(1/2)^2j+2)(j-k+1)}}
\]

\[
- \sum_{1 \leq k, k+j-i} a_{k}a_{k+j-i} \frac{\sqrt{(1-(1/2)^{2j+2})\sqrt{(1-(1/2)^{2j+2}(j+k+1)}}}{\sqrt{i+1\sqrt{j+1(1-(1/2)^{2(j+k)+2})}}}
\]

Proof. We have

\[
\langle T_f T_f(e_j), e_i \rangle = \sum_{k,l=1}^\infty a_l a_k \frac{\sqrt{3(i+1)}}{2\sqrt{(1-(1/2)^{2i+2})}} \frac{\sqrt{3(j+1)}}{2\sqrt{(1-(1/2)^{2j+2})}} (z^{j-k}, z^{i-l})
\]

\[
= \sum_{1 \leq k, k+i-j} a_{k+i-j}a_k \frac{\sqrt{(i+1)}}{\sqrt{(1-(1/2)^{2i+2})}} \frac{\sqrt{(j+1)}}{\sqrt{(1-(1/2)^{2j+2})}} \frac{(1-(1/2)^{2(j-k)+2}}{j-k+1}
\]
Lemma 4.3. which is omitted.

In this case, \(|f|\) a.e. on the unit circle. The condition \(\tilde{T}\) on the Hardy space of the unit disk with \(\tilde{f}\) obtained the following theorem.

If we set \(f_2(z) = \sum 2^k a_k z^k\), then \(f_2' \in H^2\) is equivalent to \(\sum k^2 2^{k} |a_k|^2 < \infty\). In this case, \(|g_2'| \leq |f_2'|\) a.e. on the unit circle is equivalent to \(|g'| \leq |f'|\) a.e. on \(\{z : |z| = 1/2\}\). Let \((\rho_{i,j})\) denote the matrix of the Hardy space Toeplitz operator \(T_{f_2'}\). Using the same notations we can show the following lemma, the proof of which is omitted.

Lemma 4.3. \(\lim_{i \to \infty} \beta_{i-p,i} = \rho_{i+p,i}\).

We obtain our second main result.
Theorem 4.4. Let $f = \sum_{k=1}^{\infty} a_k \frac{1}{z^k}$ and $g = \sum_{k=1}^{\infty} b_k \frac{1}{z^k}$ be bounded on $C_{1/2}$, with $\sum k^2 2^{2k} |a_k|^2 < \infty$. If $T_{f+\overline{g}}$ is hyponormal then $\sum k^2 2^{2k} |b_k|^2 < \infty$ and $|g'| \leq |f'|$ a.e. on $\{z : |z| = 1\} \cup \{z : |z| = 1/2\}$.

An application of the maximum modulus principle allows us to describe the normality of $T_{f+\overline{g}}$ under the condition of univalence.

Corollary 4.5. Let $f = \sum_{k=1}^{\infty} a_k \frac{1}{z^k}$ and $g = \sum_{k=1}^{\infty} b_k \frac{1}{z^k}$ be analytic and univalent in an open set containing $C_{1/2}$. Then $T_{f+\overline{g}}$ is normal if and only if $g = cf$, where $c$ is a constant with $|c| = 1$.

We list two more results which are shown using methods similar to the ones used for the previous theorems.

Theorem 4.6. Let $f = \sum_{k=1}^{\infty} a_k z^k$ and $g = \sum_{k=1}^{\infty} b_k \frac{1}{z^k}$ be bounded on $C_{1/2}$. Assume that $\sum k^2 |a_k|^2 < \infty$. If $T_{f+\overline{g}}$ is hyponormal then $\sum k^2 |b_k|^2 < \infty$ and $|g'(e^{i\theta})| \leq |f'(e^{i\theta})|$ a.e. on the unit circle.

Corollary 4.7. Let $f = \sum_{k=1}^{\infty} a_k z^k$ and $g = \sum_{k=1}^{\infty} b_k \frac{1}{z^k}$ be bounded on $C_{1/2}$. Assume that $f$ and $\overline{g}$ are univalent in an open set containing $C_{1/2}$. Then $T_{f+\overline{g}}$ is normal if and only if $\overline{g} = cf$ for some constant $c$ with $|c| = 1$.

Theorem 4.8. Let $f = \sum_{k=1}^{\infty} a_k z^k$ and $g = \sum_{k=1}^{\infty} b_k \frac{1}{z^k}$ be bounded on $C_{1/2}$. If $T_{f+\overline{g}}$ is hyponormal then $\sum k^2 2^{2k} |b_k|^2 < \infty$ and $|g'(\frac{1}{2} e^{i\theta})| \leq |f'(\frac{1}{2} e^{i\theta})|$ for almost all $\theta$.

Corollary 4.9. Let $f = \sum_{k=1}^{\infty} a_k z^k$ and $g = \sum_{k=1}^{\infty} b_k \frac{1}{z^k}$ be bounded on $C_{1/2}$ and assume that $T_{f+\overline{g}}$ is hyponormal. The following holds:

i) $\sum k^2 2^{2k} |b_k|^2 < \infty$ and $|g'(\frac{1}{2} e^{i\theta})| \leq |f'(\frac{1}{2} e^{i\theta})|$ for almost all $\theta$.

ii) If $f' \in H^2$ then $|g'(e^{i\theta})| \leq |f'(e^{i\theta})|$ a.e. on the unit circle.

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Received: March 13, 2019
Accepted: August 26, 2019