Packing parameters in graphs: new bounds and a solution to an open problem

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Abstract
In this paper, we investigate the packing parameters in graphs. By applying the Mantel’s theorem, we give upper bounds on packing and open packing numbers of triangle-free graphs along with characterizing the graphs for which the equalities hold and exhibit sharp Nordhaus–Gaddum type inequalities for packing numbers. We also solve the open problem of characterizing all connected graphs with $\rho_o(G) = n - \omega(G)$ posed in Hamid and Saravanakumar (Discuss Math Graph Theory 35:5–16, 2015).

Keywords Packing number · Open packing number · Nordhaus–Gaddum inequality · Open problem · Triangle-free graph

Mathematics Subject Classification 05C69

1 Introduction
Throughout this paper, let $G$ be a finite graph with vertex set $V(G)$ and edge set $E(G)$. We use (West 2001) as a reference for terminology and notation which are not defined here. The open neighborhood of a vertex $v$ is denoted by $N(v)$, and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The subset $S \subseteq V(G)$ is said to be 2-independent if the maximum degree of the subgraph induced by it is less than two.

A set $S \subseteq V(G)$ is a dominating set if each vertex in $V(G) \setminus S$ has at least one neighbor in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set (Haynes et al. 1998).

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A subset $B \subseteq V(G)$ is a 2-packing in $G$ if for every pair of vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$. The 2-packing number (or packing number) $\rho(G)$ is the maximum cardinality of a 2-packing in $G$. The open packing, as it is defined in Henning and Slater (1999), is a subset $B \subseteq V(G)$ for which the open neighborhoods of the vertices of $B$ are pairwise disjoint in $G$ (clearly, $B$ is an open packing if and only if $|N(v) \cap B| \leq 1$, for all $v \in V(G)$. The open packing number, denoted $\rho_o(G)$, is the maximum cardinality among all open packings in $G$.

Gallant et al. (2010) introduced the concept of limited packing in graphs. They exhibited some real-world applications of it to network security, NIMBY, market saturation and codes. In fact as it is defined in Gallant et al. (2010), a set of vertices $B \subseteq V(G)$ is called a $k$-limited packing in $G$ provided that for all $v \in V(G)$, we have $|N[v] \cap B| \leq k$. The limited packing number $L_k(G)$ is the largest number of vertices in a $k$-limited packing set. It is easy to see that $L_1(G) = \rho(G)$. More results on this topic can be found in Mojdeh et al. (2017).

In this paper, as an application of the classic theorem of Mantel (1907) we give upper bounds on packing and open packing numbers of triangle-free graphs and characterize the graphs obtaining equality in these bounds. In Sect. 3, we give lower bounds on $L_k(G)$, for $k = 1, 2$, in terms of $k$ and the diameter of $G$. Also, we prove sharp Nordhaus–Gaddum inequalities for packing numbers.

In Hamid and Saravanakumar (2015), the problem of finding all connected graphs with $\rho_o(G) = n - \omega(G)$ was posed as an open problem. In Sect. 4, we exhibit a solution to this problem.

### 2 Applications of Mantel’s theorem

Our aim in this section is to establish upper bounds on $\rho_o(G)$ and $\rho(G)$ for a triangle-free graph $G$ in terms of its order and size. Furthermore, we characterize all triangle-free graphs attaining these bounds. We need the following well-known theorem of Mantel from extremal graph theory.

**Lemma 2.1** (Mantel 1907)(Mantel’s Theorem) *If $G$ is a triangle-free graph of order $n$, then*

$$|E(G)| \leq \lfloor n^2/4 \rfloor$$

*with equality if and only $G$ is isomorphic to $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$.*

In order to characterize all triangle-free graphs attaining the upper bounds in the following theorem, we define the family $\Omega$ to consist of all triangle-free graphs $G$ that can be obtained from the disjoint union of a complete bipartite graph $K_{t,1}$ and $pK_2$ ($p \geq 1$) by adding exactly one edge $uv$ such that $u \in V(K_{t,1})$ and $v \in V(pK_2)$, for every $u \in V(K_{t,1})$. Also, we define the collection $\Omega'$ by replacing $pK_2$ with $pK_1$ in the definition of $\Omega$. 
Theorem 2.2 Let \( G \) be a triangle-free graph of order \( n \) and size \( m \). If \( G \) has no isolated vertex, then

\[
\rho_o(G) \leq n + 1 - \sqrt{4m - 2n + 1}.
\]

Moreover,

\[
\rho(G) \leq n + 2 - 2\sqrt{1 + m}.
\]

The first inequality holds with equality if and only if \( G \in \Omega \) and the second holds with equality if and only if \( G \in \Omega' \).

Proof Let \( B \) be a maximum open packing set in \( G \). Then, \( |E(G[B])| \leq |B|/2 \) and \( |[B, V \setminus B]| \leq n - |B| \). Since \( G \) is triangle-free, \( |E(G[V \setminus B])| \leq (n - |B|)^2/4 \), by Lemma 2.1. Clearly,

\[
m = |E(G[B])| + |[B, V \setminus B]| + |E(G[V \setminus B])|.
\]

Therefore, \( m \leq |B|/2 + n - |B| + (n - |B|)^2/4 \) and hence

\[
|B|/2 + n - |B| + (n - |B|)^2/4 - m \geq 0.
\]

Solving the above inequality for \( |B| \) we obtain

\[
\rho_o(G) = |B| \leq n + 1 - \sqrt{4m - 2n + 1}.
\]

The upper bound on \( \rho(G) \) can be proved by a similar fashion. It suffices to note that \( |E(G[B])| = 0 \) when \( B \) is a packing set in \( G \).

Now, we characterize all triangle-free graphs with no isolated vertices for which the equality in the first upper bound holds. By (1) we deduce that the inequality (2) holds with equality if and only if \( |E(G[B])| = |B|/2 \), \( |[B, V \setminus B]| = n - |B| \) and \( |E(G[V \setminus B])| = (n - |B|)^2/4 \).

Let \( G \in \Omega \) and \( B \) be the set of vertices of \( pK_2 \). It is easy to see that \( B \) is an open packing set satisfying the three above equality. So, \( B \) is a maximum open packing in \( G \) and \( \rho_o(G) = |B| = n + 1 - \sqrt{4m - 2n + 1} \).

Conversely, suppose that \( G \) satisfies the equality in (2) and \( B \) is a maximum open packing set in \( G \). Since \( |E(G[B])| = |B|/2 \), \( |E(G[V \setminus B])| = (n - |B|)^2/4 \) and \( G \) is triangle-free, \( G[V \setminus B] \) is the complete bipartite graph \( K_{n - |B|/2, n - |B|} \), by Lemma 2.1. Taking into account the facts that \( B \) is an open packing and \( |E(G[B])| = |B|/2 \), we have \( G[B] = (|B|/2)K_2 \). On the other hand, the equality \( |[B, V \setminus B]| = n - |B| \) implies that every vertex in \( G[V \setminus B] \) has exactly one neighbor in \( V(G[B]) = V((|B|/2)K_2) \).

This shows that \( G \in \Omega \).

It is easy to see that the second upper bound holds with equality for \( G \in \Omega' \).

Conversely, suppose that \( G \) satisfies the equality in the second upper bound and \( B \) is a maximum packing set in \( G \). Then \( B \) is independent. Moreover, \( |[B, V \setminus B]| = n - |B| \) and \( |E(G[V \setminus B])| = (n - |B|)^2/4 \). Thus, every vertex in \( V \setminus B \) has exactly one neighbor.
in $B$. Also, $|E(G[V \setminus B])| = (n - |B|)^2/4$ shows that $G[V \setminus B] = K_{n - |B|, n - |B|}$, by Lemma 2.1. Therefore, $G \in \Omega'$. This completes the proof. \hfill \Box

3 Diameter and Nordhaus–Gaddum inequalities for packing number

Many results in domination theory have relationship with the diameter of graphs (see Haynes et al. 1998). In this section we exhibit tight bounds on $L_k(G)$ ($k = 1, 2$) and the sum and product of the packing number $G$ and $\overline{G}$ involving the diameter. The following well-known lower bound on the domination number for a connected graph $G$ was given in Haynes et al. (1998):

$$\gamma(G) \geq \lceil \frac{diam(G) + 1}{3} \rceil.$$ (3)

In the next result we bound the $k$-limited packing numbers, $k \in \{1, 2\}$, of a general connected graph $G$ from below just in terms of $k$ and its diameter. Since $\rho(G) \leq \gamma(G)$ (see Gallant et al. 2010), it improves the lower bound (3) for the case $k = 1$.

**Proposition 3.1** For any connected graph $G$ and integer $k \in \{1, 2\}$,

$$\left\lfloor \frac{k \ \text{diam}(G) + k}{3} \right\rfloor \leq L_k(G).$$

**Proof** Let $P$ be a diametral path in $G$ with the set of vertices $V(P) = \{v_1, \ldots, v_{\text{diam}(G)+1}\}$. For $k = 1$, $V_1(P) = \{v_1, \ldots, v_{3i+1}, \ldots, v_{3i+\text{diam}(G)+1}\}$ is a packing in $G$. For otherwise, there exists a vertex $v$ adjacent to at least two vertices in $V_1(G)$. This yields to a path between $v_1$ and $v_{\text{diam}(G)+1}$ by $v$ with length less than $\text{diam}(G)$, a contradiction. So, $\rho(G) \geq |V_1(P)| = \lceil \frac{\text{diam}(G)+1}{3} \rceil$.

For $k = 2$, $V_2(P) = V(P) \setminus \{v_3, \ldots, v_{3\lfloor \text{diam}(G)+1/3 \rfloor}\}$ is a 2-limited packing in $G$, by a similar fashion. Therefore, $L_2(G) \geq |V_2(P)| = \lceil \frac{2 \ \text{diam}(G)+2}{3} \rceil$. \hfill \Box

Nordhaus and Gaddum in 1956, gave lower and upper bounds on the sum and product of the chromatic number of a graph and its complement, in terms of the order. Since then, bounds on $\psi(G) + \psi(\overline{G})$ or $\psi(G)\psi(\overline{G})$ are called Nordhaus–Gaddum inequalities, where $\psi$ is a graph parameter. For more information about this subject the reader can consult (Aouchiche and Hansen 2013).

The Nordhaus–Gaddum inequalities for limited packing parameters was initiated by exhibiting the sharp upper bound $L_2(G) + L_2(\overline{G}) \leq n + 2$, for $k = 2$, in Samadi (2016). We conclude this section by establishing upper bounds on the sum and product of the packing number ($k = 1$). We first need the following useful observation.

**Observation 3.2** For any graph $G$, $\rho(G) = 1$ if and only if $\text{diam}(G) \leq 2$.

Clearly $\rho(G) + \rho(\overline{G}) = 2$ ($\rho(G)\rho(\overline{G}) = 1$) if and only if $\text{diam}(G)$ and $\text{diam}(\overline{G}) \leq 2$, by Observation 3.2. Thus, we restrict our attention to the case $\max\{\text{diam}(G), \text{diam}(\overline{G})\} \geq 3$.\hfill \square
Theorem 3.3 Let \( G \) and \( \overline{G} \) be both connected with \( \Delta' = \min\{\Delta(G), \Delta(\overline{G})\} \) and \( M = \max\{\text{diam}(G), \text{diam}(\overline{G})\} \geq 3 \). Then,

\[
\rho(G) + \rho(\overline{G}) = \rho(G)\rho(\overline{G}) = 4 \text{ if } \text{diam}(G) = \text{diam}(\overline{G}) = 3,
\]

If \( \text{diam}(G) \neq \text{diam}(\overline{G}) \), then

\[
\rho(G) + \rho(\overline{G}) \leq n - \left\lfloor \frac{2M + 3\Delta' - 11}{3} \right\rfloor \text{ and } \rho(G)\rho(\overline{G}) \leq n - \left\lfloor \frac{2M + 3\Delta' - 8}{3} \right\rfloor.
\]

Furthermore, these bounds are sharp.

Proof Let \( \text{diam}(G) = \text{diam}(\overline{G}) = 3 \) and \( u \) and \( v \) be the end vertices of a diametrical path of length 3. It is easy to see that \( \{u, v\} \) is a dominating set in \( \overline{G} \). Therefore, \( \gamma(\overline{G}) \leq 2 \). On the other hand, \( \rho(\overline{G}) \leq \gamma(\overline{G}) \). Now Observation 3.2 implies that \( \rho(\overline{G}) = 2 \). A similar argument shows that \( \rho(G) = 2 \).

Now let \( \text{diam}(G) \neq \text{diam}(\overline{G}) \). Without loss of generality we may assume that \( \text{diam}(G) \geq \text{diam}(\overline{G}) \). Since \( \text{diam}(G) \geq 3 \) implies \( \text{diam}(\overline{G}) \leq 3 \) (see West 2001), we have \( \text{diam}(G) \geq 3 \) and \( \text{diam}(\overline{G}) \leq 2 \). Thus, \( \rho(\overline{G}) = 1 \). Now let \( B \) be a maximum packing in \( G \) and \( u \) be a vertex of the maximum degree. Then, at most one of the vertices in \( N[u] \) belongs to \( B \). Let \( x \) and \( y \) be the end vertices of a diametrical path \( P \) of the length \( \ell(P) = \text{diam}(G) \geq 3 \) in \( G \). Since \( \text{diam}(G[N[u]]) \leq 2 \), at least one of the end vertices, say \( x \), is in \( G \setminus N[u] \) and at most three vertices of \( P \) are in \( N[u] \). Then \( H = P \setminus N[u] \) is a disjoint union of two subpaths \( P_x \) and \( P_y \) of \( P \) beginning at \( x \) and \( y \), respectively, or a disjoint union of \( P_x \) and \( P_y \) and a singleton \( \{z\} \), for some \( z \in V(P) \) (if \( y \in N[u] \), then \( P_y = \emptyset \)). Therefore, \(|V(P_x)| + |V(P_y)| \geq \text{diam}(G) - 2 \). Since \( \rho(P_m) = \lceil \frac{\ell(P)}{3} \rceil \) (see Gallant et al. 2010), at most \( \lceil |V(P_x)|/3 \rceil + \lceil |V(P_y)|/3 \rceil \) vertices of \( P_x \cup P_y \) belong to \( B \) and therefore at least \( \lfloor 2|V(P_x)|/3 \rfloor + \lfloor 2|V(P_y)|/3 \rfloor \) vertices of \( P_x \cup P_y \) belong to \( V(G) \setminus B \). Thus,

\[
|V(G) \setminus B| \geq \Delta(G) + \lfloor 2|V(P_x)|/3 \rfloor + \lfloor 2|V(P_y)|/3 \rfloor
\]

\[
= \Delta(G) + \lfloor (2|V(P_x)| - 2)/3 \rfloor + \lfloor (2|V(P_y)| - 2)/3 \rfloor
\]

\[
\geq \Delta(G) + \lfloor (2|V(P_x)| + |V(P_y)| - 2)/3 \rfloor
\]

\[
\geq \Delta(G) + \lfloor (2\text{diam}(G) - 8)/3 \rfloor.
\]

So, \( \rho(G) = |B| \leq n - \lfloor \frac{2\text{diam}(G) + 3\Delta(G) - 8}{3} \rfloor \). This implies the upper bounds.

That these bounds are sharp, may be seen as follows. Let \( G \) be a graph obtained from the star \( K_{1,t} \), \( t \geq 3 \), with the central vertex \( u \) by adding new edges among the pendant vertices of \( K_{1,t} \) provided that there exist two nonadjacent vertices \( u_1 \) and \( u_2 \) in \( N(u) \) and a vertex \( w \in N(u) \) which is neither adjacent to \( u_1 \) nor \( u_2 \). We add two vertices \( x \) and \( y \) (\( x \notin N[u] \) and \( y \neq u \)) and consider two paths \( P_x \) and \( P_y \) as above, with \( \ell(P_x) \geq \ell(P_y) \) and \( \ell(P_x) \equiv 0 \) (mod 3), for which the other end vertices of them are adjacent to \( u_1 \) and \( u_2 \) (if \( \ell(P_y) \geq 1 \)), respectively. Show this graph by \( H \). Then \( \Delta(G) = \Delta(H) \), \( d(x, y) = \text{diam}(H) \) and the three vertices \( u, u_1 \) and \( u_2 \) of the diametrical path belong to \( N[u] \). It is easy to see that the maximum packing \( B \) of
\( H \) contains one vertex of \( N[u] \), say \( w \), and \([|V(P_x)|/3] + [|V(P_y)|/3]\) vertices of \( V(P_x) \cup V(P_y) \). So,

\[
|V(H) \setminus B| = \Delta(H) + [(2|V(P_x)| - 2)/3] + [(2|V(P_y)| - 2)/3].
\]

(4)

Moreover, since \( \ell(P_x) \equiv 0 \pmod{3} \) and three vertices of the \( x, y \)-path belong to \( N[u] \), we have

\[
|V(H) \setminus B| = \Delta(H) + [2(|V(P_x)| + |V(P_y)| - 2)/3]
\]

\[
= \Delta(H) + [2diam(H) - 8)/3],
\]

by (4). Hence, \(|B| = n - \lceil 2diam(H) + 3\Delta(H) - 8 \rceil/3 \). Taking into account this, the sharpness of the upper bounds follows from \( \rho(H) = 1 \).

\( \square \)

4 Characterization of graphs with \( \rho_o(G) = n - \omega(G) \)

Hamid and Saravanakumar (2015) posed the following open problem:

Characterize the connected graphs of order \( n \geq 3 \) for which \( \rho_o(G) = n - \omega(G) \), where \( \omega(G) \) denotes the clique number of \( G \).

We conclude the paper by exhibiting a solution to this problem. We note that a characterization was independently given by Hamid and Saravanakumar (2017). In spite of this, the proof presented here is different and shorter. For this purpose, we let \( \Pi_1 \) be \( \{P_4, P_5, P_6, C_4, K_{1,3}\} \) for \( \omega(G) = 2 \), and for \( \omega(G) \geq 3 \) we define \( \Pi_2 \) to be the union of all families of connected graphs described as follows (the figures (a)–(j) depict examples of graphs in the families (a)–(j)). In each case, we let \( S \) be a maximum clique.

- (a) All graphs \( G \) with \( \delta(G) = 1 \) and \( \Delta(G) = n - 1 = \omega(G) + 1 \);
- (b) all graphs \( G \) for which the subgraph induced by \( V(G) \setminus S \) is 2-independent and each vertex in \( S \) has at most one neighbor in \( V(G) \setminus S \);
  In remaining cases each vertex in \( S \) has at most one neighbor in \( V(G) \setminus S \).
- (c) All graphs \( G \) formed from adding a new vertex \( y \) and joining it to at least two vertices in \( S \);

(d) all graphs \( G \) formed from adding two new vertices \( y \) and \( z \) with \( N(y) \subseteq S \setminus \{x\} \) and \( N(z) = \{x\} \);
(e) all graphs \( G \) obtained by adding two new vertices \( y \) and \( z \) for which \( xy \notin E(G) \), \( yz \in E(G) \) and \( N(z) \cap S = \emptyset \);
(f) the family of graphs $G \cup \{xz\}$ in which $G$ is a graph described in (e);

![Diagram](image)

(g) the family of graphs formed from adding a new vertex $t$ to a graph described in (e) and joining it to $x$;

(h) all graphs $G$ obtained by adding new vertices $y, z$ and $w$ for which $xy \notin E(G)$, $yz \in E(G)$ and $N(\{z, w\}) \cap S = \emptyset$;

(i) the family of graphs formed from adding a new vertex $t$ to a graph described in (h) and joining it to $x$;

(j) all graphs obtained by adding two new vertices $y$ and $z$ for which $yz, xz \in E(G)$ and $N(y) \cap S = \emptyset$.

![Diagram](image)

We first need the following useful lemma.

**Lemma 4.1** Let $G$ be a connected graph of order $n$. Then, $\rho_o(G) \leq n - \Delta(G) + 1$. Moreover, the equality holds if and only if $\Delta(G) = n - 1$ and $\delta(G) = 1$.

**Proof** Let $B$ be an open packing in the connected graph $G$ of maximum size and $u$ be a vertex of the maximum degree $\Delta(G)$. Then at most two vertices in $N[u] \cap B$, otherwise $\rho_o(G) \leq n - \Delta(G)$ and this is a contradiction. On the other hand, by the definition of the open packing one of these two vertices is $u$ and the other must be a pendant vertex adjacent to $u$, necessarily. Thus,

$$\rho_o(G) = |B| \leq n - \Delta(G) + 1. \quad (5)$$

We now show that the equality in (5) holds if and only if $\Delta(G) = n - 1$ and $\delta(G) = 1$. Let the equality holds for the graph $G$. If $u$ is a vertex of the maximum degree $\Delta(G)$, then there exists two vertices in $N[u] \cap B$, otherwise $\rho_o(G) \leq n - \Delta(G)$ and this is a contradiction. On the other hand, by the definition of the open packing one of these two vertices is $u$ and the other must be a pendant vertex adjacent to $u$, necessarily. Moreover, $V(G) \setminus N[u] \subseteq B$. Now the connectedness of $G$ shows that $\Delta(G) = n - 1$ and $\delta(G) = 1$. Conversely, let $\Delta(G) = n - 1$ and $\delta(G) = 1$. Then every maximum open packing in $G$ contains the vertex of the maximum degree and a pendant vertex. So, $\rho_o(G) = n - \Delta(G) + 1.$

We are now in a position to exhibit the solution to the problem.
Theorem 4.2  Let $G$ be a connected graph of order $n \geq 3$. Then, $\rho_o(G) = n - \omega(G)$ if and only if $G \in \Pi_1$ for $\omega(G) = 2$, and $G \in \Pi_2$ for $\omega(G) \geq 3$.

Proof  We distinguish two cases depending on the value of $\omega(G)$.

Case 1 Let $\omega(G) = 2$. It is a routine matter to see that $\rho_o(G) = n - 2$ if and only if $G \in \Pi_1 = \{P_4, P_5, P_6, C_4, K_{1,3}\}$.

Case 2 Let $\omega(G) \geq 3$. Suppose that the equality holds. Clearly, $\omega(G) - 1 \leq \Delta(G)$. On the other hand, if $\Delta(G) \geq \omega(G) + 2$ then $\rho_o(G) \leq n - \omega(G) - 1$, by Lemma 4.1, and this is a contradiction. Therefore, $\Delta(G) \in \{\omega(G) - 1, \omega(G), \omega(G) + 1\}$.

If $\Delta(G) = \omega(G) - 1$, then $G$ is a complete graph and hence $\rho_o(G) \neq n - \omega(G)$. Therefore, $\Delta(G) = \omega(G)$ or $\omega(G) + 1$.

Let $\Delta(G) = \omega(G) + 1$. Then, $\rho_o(G) \leq n - \omega(G)$ with equality if and only if $\Delta(G) = n - 1$ and $\delta(G) = 1$, by Lemma 4.1. In this case, $G$ belongs to the family (a).

Let $\Delta(G) = \omega(G)$. Let $S$ be a maximum clique in $G$. Since $S$ is a clique and $\rho_o(G) = n - |S|$, we deal with two possible subcases:

Subcase 2.1 Let $V(G) \setminus B = S$. Then of the vertices in $V(G) \setminus S$ belong to $B$. Therefore the set $V(G) \setminus S$ is 2-independent. Moreover, each vertex in $S$ is adjacent to at most one vertex in $V(G) \setminus S$. In this subcase, $G$ belongs to the family (b).

Subcase 2.2 Let $V(G) \setminus B = (S \setminus \{x\}) \cup \{y\}$, for some vertices $x \in S$ and $y \in V(G) \setminus S$. Since $V(G) \setminus (S \cup \{y\}) \subseteq B$, then the vertices of the subgraph $H$ induced by this set is 2-independent. Therefore, the components of $H$ are isolated vertices or copies of $P_2$. Moreover, $H$ has at most two components. For otherwise, if there exist $k \geq 3$ components of $H$, then $x$ has a neighbor in at least $k - 1$ components. Thus, $|N(x) \cap B| \geq k - 1 \geq 2$, a contradiction. If $y$ has no neighbor in $V(H)$, then $y$ has all of its neighbors in $S$ and $H$ is an isolated vertex adjacent to $x$ or empty, necessarily, and we deal with all graphs in the families (c) or (d). If $y$ is adjacent to a vertex in a component $F$ of $H$, then either $y$ has a neighbor in $S \setminus \{x\}$ or $F$ is an isolated vertex adjacent to $x$ (which in this case $H$ has only one component). Considering the possible cases we can see that $G$ belongs to one of the families (e)–(j).

The above argument implies that $G \in \Pi_2$. On the other hand, it is easy to see that the equality holds for all graphs in $\Pi_2$. This completes the proof. □

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