1. INTRODUCTION

Modified nodal analysis (MNA) is a widely used technique for modelling RLC circuits. It has been first introduced in Ho et al. (1975). It is based on regarding a circuit as a graph, and results in a differential-algebraic model. This model provides a structure which allows a mathematically elegant analysis of essential properties and their physical interpretation. Among these properties is the index, i.e., the order of smoothness of perturbations entering the solution of the differential-algebraic equation, see Lamour et al. (2013); Kunkel and Mehrmann (2006); it is shown in Estévez Schwarz and Tischendorf (2000); Günther and Feldmann (1999a,b), see also Bächle (2007); März et al. (2003); Estévez Schwarz and Lamour (2001); Freund (2005); Reis (2014); Riaza (2006) that asymptotical stability requires some further conditions. It is shown in Riaza and Tischendorf (2000); Günther and Feldmann (1999a,b), see also Bächle (2007); März et al. (2003); Estévez Schwarz and Lamour (2001); Freund (2005); Reis (2014); Riaza (2006) that the index is not dependent on system parameters (such as values of resistances, capacitances and inductances), but rather on the interconnection structure, i.e., the topology, of the circuit. Further important possible properties of the circuit system are stability and asymptotic stability. Whereas MNA models of RLC circuits are always stable as long as the parameter values of resistances, capacitances and inductances are positive, asymptotic stability requires some further conditions. It is shown in Riaza and Tischendorf (2010, 2007); Riaza (2006) that asymptotical stability is guaranteed, if certain parameter-independent criteria on the circuit interconnection structure are fulfilled. The general idea of these articles is used in Berger and Reis (2014), where topological criteria for asymptotic stability and autonomy of the zero dynamics are presented for the purpose of adaptive tracking control of circuits.

In this article, we analyse further systems theoretic properties of the MNA equations. Besides presenting sufficient topological criteria for behavioral stabilizability, we derive expressions for the system space and the space of consistent initial values, and conclude topological conditions for controllability at infinity and impulse controllability.

In Sec. 2 we present the required tools from graph theory and Sec. 3 collects the basics on RLC circuit models. Sec. 4 and Sec. 5 contain the results on stability and stabilizability of the circuit model and their topological interpretation. Sec. 6 is devoted to the system space of the MNA equations, whereas we specify the space of consistent initial values and give topological conditions for controllability at infinity and impulse controllability in Sec. 7 and Sec. 8.

1.1 Nomenclature

\( \mathbb{N}_0 \) is the set of nonnegative integers, \( \mathbb{R}(s) \) is the field of real rational functions, and \( \mathbb{C}_+ \), \( \mathbb{C}_\infty \) are, resp., the open and closed complex right half planes. For a field \( K \), \( K^{n \times m} \) is the set of \( n \times m \) matrices with entries in \( K \). We use \( \text{rk}_K M \), \( \ker_K M \), \( \text{im}_K M \) for the rank, kernel and image of a matrix \( M \) over \( K \). If \( K = \mathbb{R} \), we omit the subindex indicating the underlying field. Further, \( M^T \) and \( M^* \) resp. stand for the transpose and conjugate transpose of a matrix \( M \), and by writing \( M > 0 \) \((M \geq 0) \), we mean that the square matrix \( M \) is symmetric positive (semi-)definite.

The identity matrix of size \( n \times n \) is denoted by \( I_n \) and the zero matrix of size \( m \times n \) by \( 0_{m,n} \). We omit the subindices, if they are clear from context. \( V^\perp \) denotes the orthogonal space of a subspace \( V \subset \mathbb{R}^n \), and we call the matrix \( Z \) a basis matrix of \( V \), if \( \ker Z = \{0\} \) and \( \text{im} Z = V \).

2. GRAPH THEORETIC PRELIMINARIES

For the purpose of this article, we consider finite and loop-free directed graphs, see Diestel (2017). We present some basics of graphs and incidence matrices along with some results about the correspondence between the topological structure of a graph and properties of its incidence matrix.
Definition 1. (Graph theoretic concepts). A directed graph is a quadruple $G = (V, E, \text{init}, \text{ter})$ consisting of a vertex set $V$, a edge set $E$ and two maps $\text{init} : E \to V$ assigning to each edge $e$ an initial vertex $\text{init}(e)$ and a terminal vertex $\text{ter}(e)$. The edge $e$ is said be directed from $\text{init}(e)$ to $\text{ter}(e)$. $G$ is said to be loop-free, if $\text{init}(e) \neq \text{ter}(e)$ for all $e \in E$. Let $V' \subseteq V$ and $E' \subseteq E$ with
\[ E' \subseteq E'_{|V'} \equiv \{e \in E : \text{init}(e) \in V' \land \text{ter}(e) \in V'\}. \]
Then the triple $(V', E', \text{init}|_{E'}, \text{ter}|_{E'})$ is called a subgraph of $G$. If $E' = E'_{|V'}$, then the subgraph is called the induced subgraph on $V'$. If $V' = V$, then the subgraph is called spanning. Additionally a proper subgraph is one where $E' \neq E$, $G$ is called finite, if $V$ and $E$ are finite.

For each $e \in E$ define $-e \not\in E$ as an edge with $\text{init}(-e) = \text{ter}(e)$ and $\text{ter}(-e) = \text{init}(e)$. Define $\overline{E}$ to be the set which contains all $e \in E$ and all corresponding $-e$. An $r$-tuple $e = (e_1, \ldots, e_r) \in \overline{E}^r$ is called a path from $v$ to $w$, if
\[ \text{init}(e_1), \ldots, \text{init}(e_r) \text{ are distinct, } \text{ter}(e_1) = \text{init}(e_1+1) \forall i \in \{1, \ldots, r-1\}, \text{ter}(e_r) = w. \]

A cycle is a path from $v$ to $v$. Two vertices $v, w$ are connected, if there is a path from $v$ to $w$. This gives an equivalence relation on the vertex set. A graph is called connected, if there is only one equivalence class. The induced subgraph on an equivalence class of connected vertices gives a connected component of the graph.

A spanning subgraph is a set of edges with state being composed of vertex potentials, inductive currents, and currents through voltage sources, i.e., $x = (v^T, i^T)^T$ and input consisting of voltages at voltage sources and currents at current sources, i.e., $u = (v^T, i^T)^T$.

The matrices $A$ and $B$ in (3) are given by
\[ sE - A = \begin{bmatrix} sA_C C_A^T + A_G G_A^T A_L & A_P \\ -A^T & sL \\ -A^T & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_j \ 0 \ 0 \ 0 \ 0 \ -I_{m_j} \end{bmatrix}, \]
where $s$ has to be regarded as a formal variable. The expression $sE - A$ is called a matrix pencil. Here, $G \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_x \times n_x}$, $C \in \mathbb{R}^{n_c \times n_c}$ are the conductance, inductance and capacitance matrix, and
\[ A \in \mathbb{R}^{n_x \times n_x}, \quad A_L \in \mathbb{R}^{n_x \times n_x}, \quad A_C \in \mathbb{R}^{n_c \times n_c}, \quad A_P \in \mathbb{R}^{n_p \times n_p}, \quad A_I \in \mathbb{R}^{n_I \times n_I}, \]
are the element-specific incidence matrices with sizes $n = n_x + n_c + n_v$, $m = n_x + n_v$. The matrices $G, L, C$ contain the parameters of capacitances, resistances, and inductions. Further, $A_R$ is an incidence matrix of the spanning subgraph consisting of all edges that contain resistances. Similarly, the incidence matrices $A_L, A_C, A_P, A_I$ then resp. correspond to the spanning subgraphs with the edges to inductances, capacitances, voltage and current source. An incidence matrix of the finite and loop-free directed graph modeling the circuit is consequently given by $A = [A_R \ A_L \ A_C \ A_P \ A_I]$. It is also reasonable to assume

which slightly differs from ours, as, for instance, cycles are called loops therein. Our notation is oriented by the standard reference Diestel (2017) for graph theory.

Proposition 1. (Estévez Schwarz and Tischendorf, 2000, Lem. 2.1 & 2.3) Let $G$ be a finite and loop-free connected graph with incidence matrix $A$. Furthermore let $K$ be a spanning subgraph, and assume that the incidence matrix is partitioned as in (1). Then the following holds:

(i) $G$ does not contain any $K$-cuts if, and only if, $\text{ker}(A_{G-K}) = \{0\}$.
(ii) $G$ does not contain any $K$-cycles if, and only if, $\text{ker}(A_K) = \{0\}$.

Let $G$ be a connected graph with incidence matrix $A$. Let $K$ be a spanning subgraph of $G$, and $L$ a spanning subgraph of $K$. Then, as in (1), we can, after possibly rearranging the columns, assume that the incidence matrix of $G$ reads
\[ A = [A_L A_K-L A_G-K], \quad A_K = [A_L A_K-L]. \]

Proposition 2. ([Riaza and Tischendorf, 2007, Prop. 4.4 & 4.5]) Let $G$ be a finite and loop-free connected graph with incidence matrix $A$. Let $K$ be a spanning subgraph of $G$, and $L$ a spanning subgraph of $K$. Further assume that the incidence matrix $A$ of $G$ is partitioned as in (2). Then the following holds:

(i) $G$ does not contain $K$-cycles except for $L$-cycles if, and only if, $\text{ker}(A_{G-L}) = \{0\}$.
(ii) $G$ does not contain $K$-cuts except for $L$-cuts if, and only if, $\text{ker}(A_{G-L}) = \{0\}$.

3. CIRCUIT EQUATIONS
that the circuit graph is connected, as any connected com-
ponent corresponds to a subcircuit which does not phys-
ically interact with the remaining components, so one may
simply consider the connected components separately. We
consider circuits with passive devices. This leads to the
assumption that the conductance matrix is dissipative,
whereas the inductance and capacitance matrices are pos-
itive definite. Altogether, this means
\[ \text{rk}[A_L A_C A_P A^T] = n_c, \quad (5a) \]
\[ g + g^T > 0, \quad L = L^T > 0, \quad C = C^T > 0. \quad (5b) \]

4. REGULARITY AND STABILITY

This section will take a closer look at the properties of the
properties of the pencil \( sE - A \) with matrices as in (4).
First we recall some results from Berger and Reis (2014).

Proposition 3. Let \( E, A \in \mathbb{R}^{n \times n} \) as in (4) and assume that
(5) holds. Then there exist invertible \( W, T \in \mathbb{R}^{n \times n} \) with
\[ W(sE - A)T = \text{diag}(sI - \tilde{A}, sN - I, 0_{n_0, n_0}), \quad (6) \]
where \( n_0 \in \mathbb{N} \), \( N \) is nilpotent with \( N^2 = 0 \), and \( \tilde{A} \) is a
square matrix with the property that all its eigenvalues
have nonpositive real part. Further, all eigenvalues of \( \tilde{A} \)
on the imaginary axis are semi-simple (i.e., their respective
geometric and algebraic multiplicities coincide). The pencil
\( sE - A \) further fulfills
\[ \text{ker}_{\mathbb{R}(s)}(sE - A) = \text{ker}_{\mathbb{R}(s)}[A_L A_C A_P A^T] \times \{0\} \times \text{ker}_{\mathbb{R}(s)} A^v, \quad (7) \]
\[ \text{im}_{\mathbb{R}(s)}(sE - A) = \text{im}_{\mathbb{R}(s)}[A_L A_C A_P A^v] \times \mathbb{R}(s)^{n/v} \times \text{im}_{\mathbb{R}(s)} A^v. \]

Proof. Since (5) implies \( E = E^T \geq 0 \) and \( A + A^T \leq 0 \),
the existence of invertible \( W, T \in \mathbb{R}^{n \times n} \) with (6) follows
from (Berger and Reis, 2014, Lem. 2.6), whereas (7) is
a consequence of (Berger and Reis, 2014, Thm. 4.3). \( \square \)

A direct consequence of Prop. 3 is that
\[ \forall \lambda \in \mathbb{C}_+ : \quad \text{ker}_{\mathbb{C}(\lambda E - A)} = \text{ker}_{\mathbb{C}(\lambda E - A)}[A_L A_C A_P A^T] \times \{0\} \times \text{ker}_{\mathbb{C}} A^v, \quad (8) \]
\[ \forall \lambda \in \mathbb{C}_+ : \quad \text{im}_{\mathbb{C}(\lambda E - A)} = \text{im}_{\mathbb{C}(\lambda E - A)}[A_L A_C A_P A^v] \times \mathbb{C}(s)^{n/v} \times \text{im}_{\mathbb{C}} A^v. \]

We further characterize regularity, i.e., the invertibility
of \( sE - A \) in \( \mathbb{R}(s)^{n \times n} \). Note that regularity translates
to the property of a differential-algebraic equation having
a solution for all smooth right hand sides, which is more
over unique by specification of the initial condition, see
Kunkel and Mehrmann (2006). Prop. 1 and Prop. 3 allow
to characterize regularity in terms of the circuit topology.

Corollary 4. Let \( E, A \in \mathbb{R}^{n \times n} \) as in (4) and assume that
(5) holds. Then the pencil \( sE - A \) is regular, if and only if,
the underlying circuit neither contains \( T \)-cycles nor \( I \)-cuts;
equivalently (by Prop. 1)
\[ \text{ker}[A_L A_C A_P A^T] = \{0\} \quad \text{and} \quad \text{ker} A^v = \{0\}. \]

Next we consider generalized eigenvalues of \( sE - A \). This is a
complex number \( \lambda \) with \( \text{rk}_{\mathbb{C}} \lambda E - A < \text{rk}_{\mathbb{R}(s)} sE - A \).
We see from Prop. 3 that all generalized eigenvalues of
\( sE - A \) have nonpositive real part. In the following we
discuss the possible absence of purely imaginary gener-
alized eigenvalues. The absence of generalized eigenvalues
on \( \mathbb{C}_+ \) corresponds to stabilizability of the circuit equation
\[ \frac{d}{dt} Ex(t) = Ax(t) \]. The latter refers to the properties that
for all \( x_0 \in \mathbb{R}^n \) such that there exists a solution \( x \) of
\[ \frac{d}{dt} Ex(t) = Ax(t) \text{ with } Ex(0) = Ex_0 \], there also exists
a solution \( x \) of \( \frac{d}{dt} Ex(t) = Ax(t) \text{ with } Ex(0) = Ex_0 \) which
vanishes at infinity, see (Berger and Reis, 2013, Sec. 5).

Proposition 5. ([Berger and Reis, 2014, Thm. 4.6]) Let \( E, A \in \mathbb{R}^{n \times n} \) as in (4) and assume that (5) holds. Then all
generalized eigenvalues of \( sE - A \) have negative real part,
if at least one of the following two assertions holds:

(i) The circuit neither contains \( T \)-cycles except for \( T \)-cycles,
not \( LCI \)-cuts except for \( LCI \)-cuts; equivalently (by Prop. 2)
\[ \text{ker}[A_L A_C A_P A^T] = \{0\} \times \text{ker} A^v, \quad \wedge \text{ker}[A_L A_C A_P A^T] = \text{ker}[A_L A_C A_P A^T]. \]

(ii) The circuit neither contains \( CI \)-cuts except for \( I \)-cuts,
not \( LC \)-cycles except for \( CV \)-cycles; equivalently (by Prop. 2)
\[ \text{ker}[A_L A_C A_P A^T] = \text{ker}[A_L A_C A_P A^T], \quad \wedge \text{ker}[A_L A_C A_P A^T] = \{0\} \times \text{ker}[A_L A_C A_P A^T]. \]

Prop. 5 slightly generalizes (Riaza and Tischendorf, 2007,
Thm. 5.2), where regularity (i.e., the absence of \( T \)-cycles and
\( I \)-cuts) is presumed. Now we combine Prop. 3 with
Prop. 5 to show a condition for \( \text{ker}_{\mathbb{C}} \lambda E - A = \{0\} \) for all
\( \lambda \in \mathbb{C}_+ \). The latter refers to asymptotic stability, i.e.,
after all solutions of \( \frac{d}{dt} Ex(t) = Ax(t) \) vanish at infinity.

Proposition 6. Let \( E, A \in \mathbb{R}^{n \times n} \) as in (4) and assume that
(5) holds. Then \( \text{ker}_{\mathbb{R}} \lambda E - A = \{0\} \) for all \( \lambda \in \mathbb{C}_+ \), if
at least one of the following two assertions holds:

(i) The circuit neither contains \( T \)-cycles, nor \( LCI \)-cuts except for \( LCI \)-cuts which are no \( I \)-cuts; equivalently (by Prop. 1 & Prop. 2)
\[ \text{ker}[A_L A_C A_P A^T] = \{0\}, \quad \wedge \text{ker}[A_L A_C A_P A^T] = \text{ker}[A_L A_C A_P A^T], \quad \wedge \text{ker}[A_L A_C A_P A^T] = \{0\}, \]

(ii) The circuit neither contains \( CI \)-cuts, nor \( LC \)-cycles except for \( CV \)-cycles which are no \( CV \)-cycles; equivalently (by Prop. 1 & Prop. 2)
\[ \text{ker}[A_L A_C A_P A^T] = \{0\}, \quad \wedge \text{ker}[A_L A_C A_P A^T] = \{0\} \times \text{ker}[A_L A_C A_P A^T], \quad \wedge \text{ker}[A_L A_C A_P A^T] = \{0\}. \]

5. BEHAVIORAL STABILIZABILITY

Loosely speaking, behavioral stabilizability of a differen-
tial-algebraic control system (3) means that \( x \) can always
be asymptotically steered to zero by a suitable choice of
the input \( u \). More precisely, for any \( x_0 \in \mathbb{R}^n \) for which
there exists a control \( u \) such that a solution \( x \) of (3) with
initial conditions \( Ex(0) = Ex_0 \) exists, there especially
exists some control \( u \) such that a solution \( x \) of (3) with
initial condition \( Ex(0) = Ex_0 \) exists which vanishes at
infinity. It is proven in (Berger and Reis, 2013, Sec. 5) that
this is equivalent to
\[ \forall \lambda \in \mathbb{C}_+: \quad \text{rk}_{\mathbb{C}} \lambda E - A B = \text{rk}_{\mathbb{C}} \lambda E - A B. \]
Now consider the circuit model $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ as in (4) and assume that (5) holds. Then

$$\text{Prop. 3} \quad \begin{aligned}
\text{im}_{\mathbb{R}(s)}[sE - A B] &= \text{im}_{\mathbb{R}(s)}(sE - A) + \text{im}_{\mathbb{R}(s)} B \\
&= \text{im}_{\mathbb{R}(s)}[A_x A_C A_r^T \times \mathbb{R}^n \times \text{im}_{\mathbb{R}(s)}] \\
&\quad + \text{im}_{\mathbb{R}(s)} A_r^T \times \{0\} \times \mathbb{R}^n \\
&= \text{im}_{\mathbb{R}(s)}[A_x A_C A_r A_r^T] \times \mathbb{R}^n \times \mathbb{R}^n.
\end{aligned}$$

Likewise, by using (8), the circuit model (4) with assumption (5) fulfills

$$\forall \lambda \in \mathbb{R}_+ : \text{im}_{\mathbb{C}}[AE - A B] = \mathbb{C}^n. \quad (10)$$

As a consequence, the circuit model is behaviorally stabilizable if, and only if, $\text{rk}_C[sE - A B] = n$ for all $s \in \mathbb{R}$. This is used in the following result, where we present sufficient conditions for behavioral stabilizability in terms of the circuit topology.

**Proposition 7.** Let $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ as in (4) and assume that (5) holds. Then (3) is behaviorally stabilizable, if at least one of the following two statements holds:

(i) The circuit neither contains $L$-cycles, nor $\mathcal{L}$-cuts except for $L$-cuts; equivalently (by Prop. 1 & Prop. 2)

$$\ker A_L = \{0\},$$

$$\land \ker [A_x A_r A_v A_l^T] = \ker [A_x A_v A_l^T].$$

(ii) The circuit neither contains $C$-cuts, nor $\mathcal{L}$-cycles except for $C$-cycles; equivalently (by Prop. 1 & Prop. 2)

$$\ker [A_x A_r A_v A_l^T] = \{0\},$$

$$\land \ker [A_x A_r^T] = \{0\} \times \ker A_C.$$

**Proof.** By the findings prior to this proposition, it suffices to show that the aforementioned topological conditions imply that for all $\omega \in \mathbb{R}$

$$\ker_C \left[ \omega E^T - A^T \right] = \{0\}. \quad (11)$$

Let $\omega \in \mathbb{R}$ and $x = (x_1, x_2, x_3)^T \in \ker_C \left[ \omega E^T - A^T \right]$ be partitioned according to the blocks in $E$ and $A$, i.e.,

$$\begin{bmatrix}
\omega A_r A_r^T + A_x A_r A_l^T & A_x A_v \\
- A_l^T & \omega L + 0 \\
A_l^T & 0
\end{bmatrix} (x_1, x_2, x_3)^T = 0.
$$

This gives $x_3 = 0$, $x_1 \in \ker [A_x A_r A_l^T]$ and

$$\begin{bmatrix}
\omega A_r A_r^T + A_x A_r A_l^T & A_x A_v \\
- A_l^T & \omega L + 0 \\
A_l^T & 0
\end{bmatrix} (x_1, x_2) = 0.$$

A multiplication of the latter equation with $(x_1^T, x_2^T)$ and taking the real part, one arrives at $0 = x_1 A_x (G + g^T) A_r^T x_1$. Then $G + g^T > 0$ gives $x_1 \in \ker A_x$. Altogether, $x \in \ker_C \left[ \omega E^T - A^T \right]$ leads to

$$x_1 \in \begin{bmatrix} A_x^T \\ A_v^T \\ A_l^T \end{bmatrix} \quad \land \ x_3 = 0 \quad \land \begin{bmatrix}
\omega A_r A_r^T + A_x A_r A_l^T & A_x A_v \\
- A_l^T & \omega L + 0 \\
A_l^T & 0
\end{bmatrix} (x_1, x_2) = 0. \quad (12)$$

First assume that (i) holds. Since we obtain from (12) that $x_1 \in \ker [A_x A_r A_l^T]$, (i) leads to $x_1 \in \ker A_r^T$, whence (12) gives rise to $A_r x_2 = 0$. Again making use of (i), we obtain $x_2 = 0$, and thus $A_r^T x_1 = 0$. We altogether have $x_2 = 0$, $x_3 = 0$ and $x_1 \in \ker [A_x A_r A_v A_l^T]$, and we again obtain $x_1 = 0$ by (5a), whence $x = 0$.

Now assume that (ii) holds: We use (12) to see that

$$\omega A_r A_r^T x_1 + A_r x_2 = 0,$$ i.e.,

$$\begin{bmatrix} \omega A_r^T x_1 \\ x_2 \end{bmatrix} \in \ker [A_r A_r^T] = \ker A_r \times \{0\},$$

and thus $x_2 = 0$. Then (12) gives $A_r^T x_1 = 0$, and we obtain $x_1 \in \ker [A_r A_v A_l^T]$. The latter space is trivial by (ii). Consequently, $x_1 = 0$, and thus $x = 0$.

6. SYSTEM SPACE

A useful space to understand differential-algebraic systems is the system space, which is the minimal subspace $V \subset \mathbb{R}^{n+m}$ in which all solutions $(x(t)^T, u(t)^T)^T$ of (3) evolve pointwisely. This space plays a crucial role, for instance in optimal control and dissipativity analysis of differential-algebraic systems, see Reis and Voigt (2015, 2019).

The main result in this section is an expression for the system space of the MNA equations (4).

**Theorem 8.** Let $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ as in (4) and assume that (5) holds. Let $Z_C$ and $Z_{\mathbb{R}^m}$ be basis matrices of $\ker A_l^T$ and, resp., $\ker [A_x A_r A_v A_l^T]$. Then the system space of (3) is given by

$$\ker \begin{bmatrix}
Z_C^T A_x G A_x^T & Z_C A_r A_v A_l^T & Z_v^T A_v^T & 0 \\
0 & A_v^T & -I_{v_2} & .
\end{bmatrix} \begin{bmatrix}
Z_C^T A_r G A_x^T & Z_C A_r A_v A_l^T & Z_v^T A_v^T & 0 \\
0 & A_v^T & -I_{v_2} & .
\end{bmatrix}.$$ 

Thm. 8 means that a vector $(x_1^T, x_2^T, u_1^T, u_2^T)^T$ partitioned according to the blocks in $[A \ B]$ as in (4) in the system space of (3) if, and only if, it satisfies

$$\begin{bmatrix}
Z_C^T A_x G A_x^T x_1 + A_r x_2 + A_v x_3 + A_r u_1 \\
A_v^T x_1 - u_2 \\
Z_{\mathbb{R}^m}^T A_r L^{-1} A_l^T x_1
\end{bmatrix} = 0.$$

The remaining part is devoted to the proof of Thm. 8 along with some preparatory results. We first recall a geometric characterization of the system space.

**Lemma 9.** (Reis et al., 2015, Prop. 3.3) Let $E, A \in \mathbb{R}^{k \times n}$ and $B \in \mathbb{R}^{k \times m}$. Consider the sequence $(V_i)_{i \in \mathbb{N}_0}$ of subspaces of $\mathbb{R}^{n+m}$ with $V_0 = \mathbb{R}^{n+m}$ and

$$V_{i+1} = \left\{ \begin{bmatrix} x \end{bmatrix} \in \mathbb{R}^{n+m} : A x + B u \in [E \ 0] \ V_i \right\} \forall i \in \mathbb{N}_0.$$

Then $V_i \supset V_{i+1}$ for all $i \in \mathbb{N}_0$. Further, there exists some $i_0 \in \mathbb{N}_0$ $V_{i_0} = V_{i_0+1}$ for some $i_0 \in \mathbb{N}_0$. Then the system space of (3) is $V_{i_0}$.

**Remark 1.** Consider the matrices $A = [A \ B] \in \mathbb{R}^{k \times (n+m)}$, $E = [E \ 0] \in \mathbb{R}^{n \times (n+m)}$. Then $V_{i+1}$ is the preimage of $E V_i$ under $A$, i.e., $V_{i+1} = A^{-1}(E V_i)$.

To determine the system space, we advance some helpful results.

**Lemma 10.** (Basile and Marro, 1992, Property 3.1.3))

Let $M \in \mathbb{R}^{k \times l}$ and $V \subset \mathbb{R}^k$ a subspace. Then

$$(M^T V)^\perp = M^{-1} (V^\perp).$$

By taking $V = \mathbb{R}^k$, Lem. 10 implies

$$\text{im} M^T = (\ker M)^\perp \ \forall M \in \mathbb{R}^{k \times l}. \quad (13)$$

**Lemma 11.** Let $E, A \in \mathbb{R}^{k \times n}$, $B \in \mathbb{R}^{k \times m}$ and consider the sequence $(V_i)$ as in Lem. 9. Then $(V_i) := (V_i^\perp)$ fulfills $W_0 = \{0\}$ and

$$W_{i+1} = \begin{bmatrix} A_l^T \\ B \end{bmatrix} \begin{bmatrix} E^T \\ 0 \end{bmatrix}^{-1} W_i \ \forall i \in \mathbb{N}_0. \quad (14)$$
Prove: We prove the statement via induction on \( i \). The
induction start \( i = 0 \) is fulfilled by \( W_0 = \{0\} \). For the
induction step, assume that \( i \in \mathbb{N}_0 \) with \( W_i = W_i^+ \). Then

\[
W_{i+1} = \left[ A \right]^{-1} \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] \right) W_i = \left[ A \right]^{-1} \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] W_i^+ \right)
\]

\[
\text{Lem. 10: } \left[ A \right]^{-1} \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] W_i^+ \right) = \left[ E^{\top} \right]^{-1} W_i^+ = W_i^{+\perp}.
\]

\[
\text{Lem. 10: } \left( \left[ A \right] \right)^* \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] \right) W_i = W_i^{+\perp}.
\]

Lemma 12. Consider an electrical circuit with incidence
matrices as in (4). Let \( Z \) be a basis matrix of \( \ker A \) and, resp., \( \ker \left[ A \right] \). Then there exists a basis matrix \( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \). Then

\[
W_{i+1} = \left[ A \right]^{-1} \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] \right) W_i = \left[ A \right]^{-1} \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] W_i^+ \right)
\]

\[
\text{Lem. 10: } \left[ A \right]^{-1} \left( \left[ \begin{array}{c} \left[ E \right] \mid 0 \end{array} \right] W_i^+ \right) = \left[ E^{\top} \right]^{-1} W_i^+ = W_i^{+\perp}.
\]

Now we present a proof of Thm. 8. In doing so, we use the
subspace iteration in Lem. 9. Instead of a direct calculation,
we determine the orthogonal space via Lem. 11.

Proof of Thm. 8. Let \( \left( W_i \right) \) be a sequence of subspaces
as in Lem. 11. For \( i \in \mathbb{N}_0 \) define

\[
Z_{i+1} := \left( \left[ E \right] \right)^{-1} W_i.
\]

Then \( W_i = \left[ A \right]^{+} \cdot Z_i \) for all \( i \in \mathbb{N}_0 \) with \( i \geq 1 \). Further, let
\( Z_{\mathbb{R}_y} \) be a basis matrix of \( \ker \left[ A \right] \), such that
\( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \). Then there exists a matrix \( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \), and
\( \ker Z_{\mathbb{R}_y} = \{0\} \) implies \( \ker Z_{\mathbb{R}_y} = \{0\} \). Then, with
\( k := \dim \ker A \), the result follows from

\[
Z_{\mathbb{R}_y} = \{z \in \mathbb{R}^k : z = \sum z \}\.
\]

\[
= \{z \in \mathbb{R}^k : z \in \ker \left[ A \right] \}
\]

\[
= \ker \left[ A \right] \in \mathbb{R}^k \text{ where } Z_{\mathbb{R}_y}.
\]

Now present a proof of Thm. 8. In doing so, we use the
subspace iteration in Lem. 9. Instead of a direct calculation,
we determine the orthogonal space via Lem. 11.

Proof of Thm. 8. Let \( \left( W_i \right) \) be a sequence of subspaces
as in Lem. 11. For \( i \in \mathbb{N}_0 \) define

\[
Z_{i+1} := \left( \left[ E \right] \right)^{-1} W_i.
\]

Then \( W_i = \left[ A \right]^{+} \cdot Z_i \) for all \( i \in \mathbb{N}_0 \) with \( i \geq 1 \). Further, let
\( Z_{\mathbb{R}_y} \) be a basis matrix of \( \ker \left[ A \right] \), such that
\( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \). Then there exists a matrix \( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \), and
\( \ker Z_{\mathbb{R}_y} = \{0\} \) implies \( \ker Z_{\mathbb{R}_y} = \{0\} \). Then, with
\( k := \dim \ker A \), the result follows from

\[
Z_{\mathbb{R}_y} = \{z \in \mathbb{R}^k : z = \sum z \}\.
\]

\[
= \{z \in \mathbb{R}^k : z \in \ker \left[ A \right] \}
\]

\[
= \ker \left[ A \right] \in \mathbb{R}^k \text{ where } Z_{\mathbb{R}_y}.
\]

Now present a proof of Thm. 8. In doing so, we use the
subspace iteration in Lem. 9. Instead of a direct calculation,
we determine the orthogonal space via Lem. 11.

Proof of Thm. 8. Let \( \left( W_i \right) \) be a sequence of subspaces
as in Lem. 11. For \( i \in \mathbb{N}_0 \) define

\[
Z_{i+1} := \left( \left[ E \right] \right)^{-1} W_i.
\]

Then \( W_i = \left[ A \right]^{+} \cdot Z_i \) for all \( i \in \mathbb{N}_0 \) with \( i \geq 1 \). Further, let
\( Z_{\mathbb{R}_y} \) be a basis matrix of \( \ker \left[ A \right] \), such that
\( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \). Then there exists a matrix \( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \), and
\( \ker Z_{\mathbb{R}_y} = \{0\} \) implies \( \ker Z_{\mathbb{R}_y} = \{0\} \). Then, with
\( k := \dim \ker A \), the result follows from

\[
Z_{\mathbb{R}_y} = \{z \in \mathbb{R}^k : z = \sum z \}\.
\]

\[
= \{z \in \mathbb{R}^k : z \in \ker \left[ A \right] \}
\]

\[
= \ker \left[ A \right] \in \mathbb{R}^k \text{ where } Z_{\mathbb{R}_y}.
\]

Now present a proof of Thm. 8. In doing so, we use the
subspace iteration in Lem. 9. Instead of a direct calculation,
we determine the orthogonal space via Lem. 11.

Proof of Thm. 8. Let \( \left( W_i \right) \) be a sequence of subspaces
as in Lem. 11. For \( i \in \mathbb{N}_0 \) define

\[
Z_{i+1} := \left( \left[ E \right] \right)^{-1} W_i.
\]

Then \( W_i = \left[ A \right]^{+} \cdot Z_i \) for all \( i \in \mathbb{N}_0 \) with \( i \geq 1 \). Further, let
\( Z_{\mathbb{R}_y} \) be a basis matrix of \( \ker \left[ A \right] \), such that
\( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \). Then there exists a matrix \( Z_{\mathbb{R}_y} = Z_{\mathbb{R}_y} \), and
\( \ker Z_{\mathbb{R}_y} = \{0\} \) implies \( \ker Z_{\mathbb{R}_y} = \{0\} \). Then, with
\( k := \dim \ker A \), the result follows from

\[
Z_{\mathbb{R}_y} = \{z \in \mathbb{R}^k : z = \sum z \}\.
\]

\[
= \{z \in \mathbb{R}^k : z \in \ker \left[ A \right] \}
\]

\[
= \ker \left[ A \right] \in \mathbb{R}^k \text{ where } Z_{\mathbb{R}_y}.
\]
\[ Y_2 = W_2 = \left( \begin{array}{c} \text{im} \left[ A_G A^T \begin{bmatrix} Z_C & A_V \end{bmatrix} A_L L^{-1} A^T L \end{bmatrix} \right] \right) \perp \]

\[ = \ker \left[ \begin{bmatrix} Z_C^T A_G A^T \begin{bmatrix} Z_A & A_L & A^T_L & Z_{Q_L} \end{bmatrix} & 0 & 0 & 0 & 0 \end{bmatrix} \\ A_G^T Z_A & 0 & 0 & 0 & 0 \end{bmatrix} \right] \]

which completes the proof.

7. CONSISTENT INITIAL VALUES AND CONTROLLABILITY AT INFINITY

Here we analyze the space of consistent initial values, which is the space of all \( x_0 \in \mathbb{R}^n \) for which there exists some control \( u \) for which there is a weakly differentiable solution \( x(t) \) of (3) with initial condition \( x(0) = x_0 \). If this space is the entire \( \mathbb{R}^n \), then the system (3) is called controllable at infinity. It is proven in (Berger and Reis, 2013, Sec. 5) that controllability at infinity is equivalent to \( \mathbb{R}^n \) and \( \mathbb{R}^n \) are trivial, i.e., these matrices have zero columns. Consequently, we also obtain from Thm. 14 that the absence of \( \mathbb{R}^n \)-cuts causes that any vector in \( \mathbb{R}^n \) is a consistent initial value for the MNA system (cf. Prop. 13).

8. CONSISTENT INITIAL DIFFERENTIAL VALUES AND IMPULSE CONTROLLABILITY

We now consider another type of initialization, namely (3) with initial condition \( E x(0) = E x_0 \). \( x_0 \in \mathbb{R}^n \) is called a consistent initial differential value, if there exists a control \( u \) for which a solution \( x(t) \) of (3) with initial condition \( E x(0) = E x_0 \) exists. If this space equals \( \mathbb{R}^n \), then the system (3) is called impulse controllable. It is proven in (Berger and Reis, 2013, Sec. 5) that impulse controllability is equivalent to \( \mathbb{R}^n \) for some (and hence any) basis matrix \( Z \) of \( \mathbb{R}^n \) as in (4) and assume that (5) holds. Then the system (3) is controllable at infinity if, and only if, the underlying differential equation does not contain any \( \mathbb{R}^n \)-cuts; equivalently (by Prop. 1)

\[ \ker [ A_C A_I ]^T = \{ 0 \} \]

It can be concluded from (Reis and Voigt, 2019, Lem. 3.7) that the space system \( V_{\text{sys}} \) and the space \( V_{\text{init}} \) of consistent initial values of the system (3) fulfill the identity

\[ V_{\text{sys}} = \{ x \in \mathbb{R}^n : 3u \in \mathbb{R}^m \text{ s.t. } (u) \in V_{\text{init}} \}. \]

This identity is the essential ingredient in the proof of the following result which contains an expression of the space of consistent initial differential values for the MNA system.

**Theorem 14.** Let \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) as in (4) and assume that (5) holds. Let \( Z_{\mathbb{R}^n \cap \mathbb{R}^m} \) and \( Z_{\mathbb{R}^n} \) be basis matrices of \( \ker [ A_C A_R A_L A_I ]^T \) and, resp., \( \ker [ A_C A_I ]^T \). Then the space of consistent initial values of (3) is given by

\[ \ker \left[ \begin{bmatrix} Z_{\mathbb{R}^n \cap \mathbb{R}^m} & A_L L^{-1} A^T \end{bmatrix} \right] \]

**Proof.** Analogous to Lem. 12, there exists a basis matrix \( Z_{\mathbb{R}^n \cap \mathbb{R}^m} \) of \( \ker [ A_C A_R A_L A_I ]^T \) such that \( Z_{\mathbb{R}^n \cap \mathbb{R}^m} = Z_{\mathbb{R}^n \cap \mathbb{R}^m} \). \( \square \)

\[ Z_{\mathbb{R}^n \cap \mathbb{R}^m}^T (A_R G A^T x_1 + A_L x_2 + A_V x_3 + A_I u_1) = 0, \]

\[ Z_{\mathbb{R}^n \cap \mathbb{R}^m}^T A_L L^{-1} A^T x_1 = 0. \]

Then a multiplication of the first equation with \( Z_{\mathbb{R}^n \cap \mathbb{R}^m}^T \) gives

\[ Z_{\mathbb{R}^n \cap \mathbb{R}^m}^T (A_R G A^T x_1 + A_L x_2 + A_V x_3 + A_I u_1) = 0, \]

\[ Z_{\mathbb{R}^n \cap \mathbb{R}^m}^T A_L L^{-1} A^T x_1 = 0. \]

\[ \therefore \]

\[ \square \]
This identity is the essential ingredient in the proof of the following result on the space of consistent initial differential values for the MNA system. We will use make of the following preparatory result.

**Lemma 16.** For any subspace \( V \subset \mathbb{R}^l \) and \( M \in \mathbb{R}^{k \times l} \) holds
\[
M^{-1}(MV) = V + \ker M.
\]

**Proof.** “\( \supset \)”: Let \( x \in M^{-1}(MV) \). Then \( Mx = My \) for some \( y \in \mathbb{R}^l \), whence \( x = (x, y) + y \in \ker M + V \).

“\( \subset \)”: Let \( x \in V + \ker M \), i.e., \( x = v + e \) for some \( v \in V \) and \( e \in \ker M \). Thus \( Mx = Mv \), whence \( x \in M^{-1}(MV) \). \( \square \)

**Theorem 17.** Let \( E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) as in (4) and assume that (5) holds. Let \( Z_{\text{RCVI}} \) be a basis matrix of \( \ker[A_G A_R A_{G'} A_{R'}]^\top \). Then the space of consistent initial differential values of (3) is given by
\[
\ker \{ 0 Z_{\text{RCVI}} A_L \}.
\]

**Proof.** Let \( Z_{\text{CI}} \) be a basis matrix of \( \ker[A_G A_R]^\top \). Analogous to Lem. 12, there exists a basis matrix \( Z_{\text{RCVI}} - Z_{\text{CI}} \) of \( \ker[A_G A_R A_{G'} A_{R'}]^\top Z_{\text{CI}} \), such that \( Z_{\text{RCVI}} = Z_{\text{CI}} Z_{\text{RCVI}} - Z_{\text{CI}} \).

**Step 1:** We show that the space \( \mathcal{V}_\text{diff} \) of consistent initial differential values fulfills
\[
E^{-1} \mathcal{V}_\text{diff} = \ker A_L^\top \times \mathcal{L}^{-1} A_L^\top Z_{\text{RCVI}} \times \mathbb{R}^{n_v}.
\]

“\( \subset \)”: Let \( x = (x_1^T x_2^T x_3^T)^\top \in E^{-1} \mathcal{L}^\top \) with \( x_1 \in \mathbb{R}^{n_0}, x_2 \in \mathbb{R}^{n_x}, x_3 \in \mathbb{R}^{n_v} \). Now using Thm. 14 together with (13), we obtain that there exist vectors \( z_1, z_2 \) with
\[
\begin{pmatrix}
A_G A_R A_{G'} A_{R'} Z_{\text{RCVI}} z_1 + A_R A_{G'} Z_{\text{CI}} z_2 \\
A_G A_R A_{G'} z_1 & A_R A_{G'} z_2 & A_G A_R A_{G'} z_1 & A_R A_{G'} z_2
\end{pmatrix} = 0.
\]

A multiplication of the first equation in (25) with \( Z_{\text{RCVI}}^\top \) yields \( Z_{\text{RCVI}}^\top E^{-1} \mathcal{V}_\text{diff} = 0 \), and the positive definiteness of \( \mathcal{L} \) now gives rise to \( A_L^\top Z_{\text{RCVI}} z_1 = 0 \). It follows that \( Z_{\text{RCVI}} Z_1 \in \ker A_L^\top \).

**Step 2:** We conclude from Step 1 that
\[
E \cdot (E^{-1} \mathcal{V}_\text{diff})^\top \supset \mathcal{V}_\text{diff}.
\]

Step 3: We conclude the statement of Thm. 17: By using the symmetry of \( E \), we obtain
\[
\ker \{ 0 Z_{\text{RCVI}} A_L \} \supset \mathcal{V}_\text{diff} \supset \ker \{ 0 Z_{\text{RCVI}} A_L \}.
\]

In the case where the circuit does not contain any \( \mathcal{L} \)-cuts, we can conclude from Prop. 1 that \( Z_{\text{RCVI}} \) are trivial, i.e., it has zero columns. As a consequence, we also obtain from Thm. 17 that in the case of absence of \( \mathcal{L} \)-cuts, any vector in \( \mathbb{R}^n \) is a consistent initial differential value for the MNA system (cf. Prop. 15).

**REFERENCES**

Bächle, S. (2007). Numerical solution of differential-algebraic systems arising in circuit simulation. Doctoral dissertation.

Basile, G. and Marro, G. (1992). *Controlled and Conditioned Invariants in Linear System Theory*. Prentice-Hall, Englewood Cliffs, NJ.

Berger, T. and Reis, T. (2014). Zero dynamics and funnel control for linear electrical circuits. *J. Franklin Inst.*, 351(11), 5099–5132.

Berger, T. and Reis, T. (2013). Controllability of linear differential-algebraic systems - a survey. In A. Ilchmann and T. Reis (eds.), *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, 1–61. Springer, Berlin-Heidelberg.

Diestel, R. (2017). *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, 5th edition.

Estévez Schwarz, D. (2002). A step-by-step approach to compute a consistent initialization for the mna. *Int. J. Circuit Theory Appl.*, 30(1), 1–16.

Estévez Schwarz, D. and Lamour, R. (2001). The computation of consistent initial values for nonlinear index-2 differential-algebraic equations. *Numer. Algorithms*, 26(1), 49–75.

Estévez Schwarz, D. and Tischendorf, C. (2000). Structural analysis for electric circuits and consequences for MNA. *Int. J. Circuit Theory Appl.*, 28(2), 131–162.

Freund, R. (2005). RCL circuit equations. In P. Benner, V. Mehrmann, and D. Sorensen (eds.), *Dimension Reduction of Large-Scale Systems*, volume 45 of *Lecture Notes in Computational Science and Engineering*, 367–371. Springer-Verlag, Berlin/Heidelberg.

Günther, M. and Feldmann, U. (1999a). CAD-based electric-circuit modeling in industry I. Mathematical structure and index of network equations. *Surv. Math. Ind.*, 8, 97–129.

Günther, M. and Feldmann, U. (1999b). CAD-based electric-circuit modeling in industry II. Impact of circuit configurations and parameters. *Surv. Math. Ind.*, 8, 131–157.

Ho, C.W., Ruehli, A., and Brennan, P. (1975). The modified nodal approach to network analysis. *IEEE Trans. Circuits Syst.*, CAS-22(6), 504–509.

Kunkel, P. and Mehrmann, V. (2006). *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland.

Lamour, R., März, R., and Tischendorf, C. (2013). *Differential Algebraic Equations: A Projector Based Analysis,*
volume 1 of Differential-Algebraic Equations Forum.
Springer-Verlag, Heidelberg-Berlin.

März, R., Estévez Schwarz, D., Feldmans, U., Sturtzel, S., and Tischendorf, C. (2003). Finding beneficial dae structures in circuit simulation. In H.K. W. Jäger (ed.), Mathematics – key technology for the future – Joint Projects between Universities and Industry, 413–428. Springer-Verlag, Berlin.

Reis, T. (2014). Mathematical modeling and analysis of nonlinear time-invariant RLC circuits. In P. Benner, R. Findeisen, D. Flockerzi, U. Reichl, and K. Sundmacher (eds.), Large-Scale Networks in Engineering and Life Sciences, Modeling and Simulation in Science, Engineering and Technology, 125–198. Birkhäuser, Basel.

Reis, T., Rendel, O., and Voigt, M. (2015). The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. Linear Algebra Appl., 86, 153–193.

Reis, T. and Voigt, M. (2015). The Kalman-Yakubovich-Popov inequality for differential-algebraic systems: Existence of nonpositive solutions. Systems Control Lett., 86, 1–8.

Reis, T. and Voigt, M. (2019). Linear-quadratic optimal control of differential-algebraic systems: the infinite time horizon problem with zero terminal state. SIAM J. Control Optim., 57(3), 1567–1596.

Riaza, R. (2006). Time-domain properties of reactive dual circuits. Int. J. Circ. Theor. Appl., 34(3), 317–340.

Riaza, R. (2013). Surveys in Differential-Algebraic Equations I, volume 2 of Differential-Algebraic Equations Forum, chapter DAEs in Circuit Modelling: A Survey, 97–136. Springer.

Riaza, R. and Tischendorf, C. (2007). Qualitative features of matrix pencils and DAEs arising in circuit dynamics. Dynamical Systems, 22(2), 107–131.

Riaza, R. and Tischendorf, C. (2010). The hyperbolicity problem in electrical circuit theory. Math. Meth. Appl. Sci., 33(17), 2037–2049.

Takamatsu, M. and Iwata, S. (2010). Index characterization of differential-algebraic equations in hybrid analysis for circuit simulation. Int. J. Circuit Theory Appl., 38, 419–440.