MULTIPLE OPERATOR INTEGRALS AND SPECTRAL SHIFT

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Abstract. Multiple scalar integral representations for traces of operator derivatives are obtained and applied in the proof of existence of the higher order spectral shift functions.

1. Introduction

For a large class of admissible functions \( f : \mathbb{R} \mapsto \mathbb{C} \), the operator derivatives \( \frac{d^j}{dx^j} f(H_0 + xV) \), where \( H_0 \) and \( V \) are self-adjoint operators on a separable Hilbert space \( \mathcal{H} \), exist and can be represented as multiple operator integrals [1, 14]. Let \( \mathcal{M} \) be a semi-finite von Neumann algebra acting on \( \mathcal{H} \) and \( \tau \) a semi-finite normal faithful trace on \( \mathcal{M} \). For \( H_0 = H_0^* \) affiliated with \( \mathcal{M} \) and \( V = V^* \) in the \( \tau \)-Hilbert-Schmidt class \( L_2(M, \tau) \) (that is, \( V \in \mathcal{M} \) and \( \tau(|V|^2) < \infty \)), we represent the traces of the derivatives \( \tau \left[ \frac{d^j}{dx^j} f(H_0 + xV) \right] \) as multiple scalar integrals, and, subsequently, as a distribution on \( f^{(j)} \), which is essentially a derivative of an \( L^\infty \)-function (see Theorem 3.11 and Corollary 3.12). Then these representations for \( \tau \left[ \frac{d^j}{dx^j} f(H_0 + xV) \right] \) are applied in the study of the higher order spectral shift functions (reviewed below) in Section 4.

Let \( R_p \equiv R_{p,H_0,V}(f) \) denote the remainder of the Taylor-type approximation

\[
f(H_0 + V) - \sum_{j=0}^{p-1} \frac{1}{j!} \left. \frac{d^j}{dx^j} \right|_{x=0} f(H_0 + xV)
\]

of the value of \( f(H_0 + V) \) at the perturbed operator \( H_0 + V \) by data determined by the initial operator \( H_0 \). Let \( \mathcal{W}_p(\mathbb{R}) \) denote the set of functions \( f \in C^p(\mathbb{R}) \) such that for each \( j = 0, \ldots, p \), the derivative \( f^{(j)} \) equals the Fourier transform \( \int_{\mathbb{R}} e^{it\lambda} d\mu_{f^{(j)}}(\lambda) \) of a finite Borel measure \( \mu_{f^{(j)}} \). There exist functions \( \xi \equiv \xi_{H_0+V,H_0} \) and \( \eta \equiv \eta_{H_0,H_0+V} \), called Krein’s and Koplienko’s spectral shift functions, respectively, such that when \( \tau(|V|) < \infty \),

\[
\tau[R_1(f)] = \int_{\mathbb{R}} f'(t)\xi(t) \, dt,
\]

(1.1)

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for $f \in W_1(\mathbb{R})$ [8] (see also [2, 4, 9, 13]), and when $\tau(|V|^2) < \infty$,

$$
\tau[R_2(f)] = \int_{\mathbb{R}} f''(t)\eta(t) \, dt,
$$

(1.2)

for $f \in W_2(\mathbb{R})$ [7] (see also [6, 10, 12, 16]).

It was conjectured in [7] that for $V$ in the Schatten $p$-class, $p \geq 3$, and $M = \mathcal{B}(\mathcal{H})$ (the algebra of bounded operators on $\mathcal{H}$), there exists a real Borel measure $\nu_p \equiv \nu_{p,H_0,V}$, with the total variation bounded by $\frac{\tau(|V|^p)}{p!}$, such that

$$
\tau[R_p(f)] = \int_{\mathbb{R}} f^{(p)}(t) \, d\nu_p(t),
$$

(1.3)

for bounded rational functions $f$. A proof of (1.3) was also suggested in [7], but, unfortunately, it contained a mistake (see [6] for details).

It was proved in [6, Theorem 5.1] that (1.3) holds for $f \in W_p(\mathbb{R})$ when $V$ is in the Hilbert-Schmidt class and $M = \mathcal{B}(\mathcal{H})$, with $\nu_p$ a real Borel measure whose total variation is bounded by $\frac{\tau(|V|^2)^{p/2}}{p!}$. It was shown in [6] and [15] that $\nu_p$ is absolutely continuous for a bounded and unbounded $H_0$, respectively. Moreover, an explicit formula for the density of $\nu_p$, called the spectral shift function of order $p$, was derived in [6, 15] (see, e.g., (4.3) of Theorem 4.1). The trace formula (1.3) was also obtained in the case of a general semi-finite von Neumann algebra and $p = 3$ [6, Theorem 5.2], with $\nu_3$ absolutely continuous when $H_0$ is bounded. When $H_0$ is unbounded, the trace formula (1.3) with an absolutely continuous measure $\nu_3$ was established in [15] for a set of functions $f$ disjoint from the one in [6, Theorem 5.2] for an unbounded $H_0$ (this discrepancy is explained in Remark 4.4).

The proof of existence of the measure $\nu_p$ in [6] relied on iterated operator integration techniques, while the proofs of the absolute continuity of $\nu_p$ in [6, 15] on the analysis of the Cauchy (also known as Borel) transform of $\nu_p$. By utilizing the results on operator derivatives and divided differences of Sections 3 and 2, respectively, we obtain a more direct, unified, proof of the established trace formula (1.3) and the absolute continuity of $\nu_p$ in Section 4. We also obtain a new representation for the density of $\nu_p$ (see (1.2) of Theorem 4.1) and, in the case of a general $\mathcal{M}$ and unbounded $H_0$, extend (1.3) with an absolutely continuous measure $\nu_3$ to a larger (as compared to [15]) set of functions $f$ (see Theorem 4.3).

2. DIVIDED DIFFERENCES AND SPLINES

In this section we collect facts on divided differences and splines to be used in the sequel.

**Definition 2.1.** The divided difference of order $p$ is an operation on functions $f$ of one (real) variable, which we will usually call $\lambda$, defined recursively as follows:

$$
\Delta^{(0)}_{\lambda_1}(f) := f(\lambda_1),
$$

$$
\Delta^{(p)}_{\lambda_1,\ldots,\lambda_{p+1}}(f) := \begin{cases} 
\frac{\Delta^{(p-1)}_{\lambda_1,\ldots,\lambda_{p-1},\lambda_{p+1}}(f) - \Delta^{(p-1)}_{\lambda_1,\ldots,\lambda_{p-1},\lambda_{p}}(f)}{\lambda_{p+1} - \lambda_p} & \text{if } \lambda_p \neq \lambda_{p+1} \\
\frac{p!}{\partial^p_t|_{t=\lambda_p} \Delta^{(p-1)}_{\lambda_1,\ldots,\lambda_{p-1},t}(f)} & \text{if } \lambda_p = \lambda_{p+1}.
\end{cases}
$$
The following facts are well-known.

**Proposition 2.2.**  
1. (See [5, Section 4.7, (a)].) \( \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) \) is symmetric in \( \lambda_1, \lambda_2, \ldots, \lambda_{p+1} \).
2. (See [5, Section 4.7].) For a sufficiently smooth function, 
   \[
   \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \sum_{i \in I} \sum_{j=0}^{m(\lambda_i) - 1} c_{ij}(\lambda_1, \ldots, \lambda_{p+1}) f^{(j)}(\lambda_i). 
   \]
   Here \( I \) is the set of indices \( i \) for which \( \lambda_i \) are distinct, \( m(\lambda_i) \) is the multiplicity of \( \lambda_i \), and \( c_{ij}(\lambda_1, \ldots, \lambda_{p+1}) \in \mathbb{C} \).
   In particular, if all points \( \lambda_1, \ldots, \lambda_{p+1} \) are distinct, then
   \[
   \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \sum_{j=1}^{p+1} f(\lambda_j) \prod_{k \in \{1, \ldots, p+1\} \setminus \{j\}} (\lambda_j - \lambda_k). 
   \]
3. (See [5, Section 4.7].) \( \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(a_p \lambda^p + a_{p-1} \lambda^{p-1} + \cdots + a_1 \lambda + a_0) = a_p \), where \( a_0, a_1, \ldots, a_p \in \mathbb{C} \).
4. (See [5, Theorem 6.2 and Theorem 6.3].) For \( f \in C^p[a, b] \), the function
   \[
   [a, b]^{p+1} \ni (\lambda_1, \ldots, \lambda_{p+1}) \mapsto \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) 
   \]
   is continuous.

We will need a more specific version of Proposition 2.2 (2).

**Lemma 2.3.** Let \( f \in C^1[a, b] \) and \( \lambda_1, \ldots, \lambda_p \) be distinct points in \([a, b]\). Then for any \( i \in \{1, \ldots, p\} \),

\[
\Delta_{\lambda_1, \ldots, \lambda_p, \lambda_i}^{(p)}(f) = \frac{f'(\lambda_i)}{\prod_{k \in \{1, \ldots, p\} \setminus \{i\}} (\lambda_i - \lambda_k)} + \sum_{j \in \{1, \ldots, p\} \setminus \{i\}} \frac{1}{(\lambda_i - \lambda_j)^2} \left( \frac{f(\lambda_j)}{\prod_{k \in \{1, \ldots, p\} \setminus \{i, j\}} (\lambda_j - \lambda_k)} - \frac{f(\lambda_i)}{\prod_{k \in \{1, \ldots, p\} \setminus \{i, j\}} (\lambda_i - \lambda_k)} \right) 
\]

**Proof.** Without loss of generality, we may assume that \( \lambda_i = \lambda_p \). By Proposition 2.2 (2),

\[
\Delta_{\lambda_1, \ldots, \lambda_{p-1}, s}^{(p-1)}(f) = \sum_{j=1}^{p-1} \frac{f(\lambda_j)}{\prod_{k \in \{1, \ldots, p-1\} \setminus \{j\}} (\lambda_j - \lambda_k)(\lambda_j - s)} + \frac{f(s)}{\prod_{k \in \{1, \ldots, p-1\}} (s - \lambda_k)}. 
\]
Next,

$$\Delta_{\lambda_1, \ldots, \lambda_p-1, \lambda_p, \lambda_p}^{(p)}(f) = \frac{\partial}{\partial s} \left( \Delta_{\lambda_1, \ldots, \lambda_p-1, s}^{(p-1)}(f) \right) \bigg|_{s=\lambda_p}$$

$$= \sum_{j=1}^{p-1} f(\lambda_j) \prod_{k \in \{1, \ldots, p-1\} \setminus \{j\}} (\lambda_j - \lambda_k)(\lambda_p - \lambda_j)^2$$

$$+ \frac{f'(\lambda_p)}{\prod_{k \in \{1, \ldots, p-1\} \setminus \{j\}} (\lambda_p - \lambda_k)} - f(\lambda_p) \sum_{j=1}^{p-1} \frac{1}{\prod_{k \in \{1, \ldots, p-1\} \setminus \{j\}} (\lambda_p - \lambda_k)(\lambda_p - \lambda_j)^2},$$

which coincides (upon regrouping the terms) with the expression in the statement of the lemma.

In the case of repeated knots, the order of the divided difference can be reduced, as it is done in the next lemma.

**Lemma 2.4.** Let \( f \in C^p[a, b] \) and \( \lambda_1, \ldots, \lambda_p \in [a, b] \). Then,

$$\sum_{i=1}^{p} \Delta_{\lambda_1, \ldots, \lambda_p, \lambda_i}^{(p)}(f) = \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)}(f').$$

*Proof.* In view of Proposition 2.2, it is enough to prove the lemma only in the case when all \( \lambda_1, \ldots, \lambda_p \) are distinct. Applying Lemma 2.3 ensures

$$\sum_{i=1}^{p} \Delta_{\lambda_1, \ldots, \lambda_p, \lambda_i}^{(p)}(f) = \sum_{i=1}^{p} \frac{f'(\lambda_i)}{\prod_{k \in \{1, \ldots, p\} \setminus \{i\}} (\lambda_i - \lambda_k)}$$

$$+ \sum_{i=1}^{p} \sum_{j \in \{1, \ldots, p\} \setminus \{i\}} \frac{1}{(\lambda_i - \lambda_j)^2} \left( \frac{f(\lambda_j)}{\prod_{k \in \{1, \ldots, p\} \setminus \{i, j\}} (\lambda_j - \lambda_k)} - \frac{f(\lambda_i)}{\prod_{k \in \{1, \ldots, p\} \setminus \{i, j\}} (\lambda_i - \lambda_k)} \right).$$

By Proposition 2.2, the first summand in (2.1) equals \( \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)}(f') \). The second summand in (2.1) with double summation sign equals zero; to see it, we group and cancel the terms with indices \((i, j) = (i_1, i_2)\) and \((i, j) = (i_2, i_1)\), where \(i_1 \neq i_2 \in \{1, \ldots, p\}\).

*Remark 2.5.* Depending on the number of repeated knots of the divided difference in Proposition 2.2 and, subsequently, in Lemma 2.4, the smoothness assumption on \( f \) can be relaxed; see for details [5, Theorem 6.2 and Theorem 6.3].

The divided difference of a function in \( W_p(\mathbb{R}) \) admits a useful representation as an integral of products of exponentials, each depending on only one knot of the divided difference.

**Proposition 2.6.** (See [1, Lemma 2.3].) For \( f \in W_p(\mathbb{R}) \),

$$\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \int_{\Pi^{(p)}} e^{i(s_0-s_1)\lambda_1} \cdots e^{i(s_{p-1}-s_p)\lambda_p} e^{is_{p+1}\lambda_{p+1}} d\sigma_f^{(p)}(s_0, \ldots, s_p).$$

Here

$$\Pi^{(p)} = \{(s_0, s_1, \ldots, s_p) \in \mathbb{R}^{p+1} : |s_p| \leq \cdots \leq |s_1| \leq |s_0|, \ sign(s_0) = \cdots = sign(s_p)\}$$
and \( d\sigma_{f}^{(p)}(s_0, s_1, \ldots, s_p) = ip\mu_f(ds_0)ds_1 \ldots ds_p \).

Below, we list properties of piecewise polynomials, which will appear in representations for the higher order spectral shift functions, and include a representation of the divided difference in terms of its Peano kernel.

**Proposition 2.7.** (1) (See [5, Section 5.2, (2.3) and (2.6)].)
The basic spline with the break points \( \lambda_1, \ldots, \lambda_{p+1} \), where at least two of the values are distinct, is defined by
\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p-1}) = \begin{cases} 
\frac{1}{|\lambda_2 - \lambda_1|}X(\min\{\lambda_1, \lambda_2\}, \max\{\lambda_1, \lambda_2\})(t) & \text{if } p = 1 \\
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p-1}) & \text{if } p > 1.
\end{cases}
\]

Here the truncated power is defined by \( x^k_+ = \begin{cases} 
x^k & \text{if } x \geq 0 \\
0 & \text{if } x < 0,
\end{cases} \) for \( k \in \mathbb{N} \).

The basic spline is non-negative, supported in
\[
[\min\{\lambda_1, \ldots, \lambda_{p+1}\}, \max\{\lambda_1, \ldots, \lambda_{p+1}\}]
\]
and integrable with the integral equal to 1/p. (Often the basic spline is normalized so that its integral equals 1.)

(2) (See [3, Section 5.2, (2.2) and Section 4.7, (c)].)
Let \( [a, b] = [\min\{\lambda_1, \ldots, \lambda_{p+1}\}, \max\{\lambda_1, \ldots, \lambda_{p+1}\}] \). For \( f \in C^p[a, b] \),
\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \begin{cases} 
\frac{1}{(p-1)!} \int_a^b f^{(p)}(t) \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p-1}) \, dt & \text{if } \exists i_1, i_2 \text{ such that } \lambda_{i_1} \neq \lambda_{i_2} \\
\frac{1}{p!} f^{(p)}(\lambda_1) & \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_{p+1}.
\end{cases}
\]

The first equality in (2.2) also holds for \( f \in C^{p-1}[a, b] \), with \( f^{(p-1)} \) absolutely continuous and \( f^{(p)} \) integrable on \([a, b] \).

Properties of an antiderivative of the basic spline are written below.

**Proposition 2.8.** (See [15, Lemma 3.1].) (i) If \( \lambda_1 = \cdots = \lambda_{p+1} \in \mathbb{R} \), with \( p \geq 0 \), then
\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p}) = \chi(-\infty, \lambda_1)(t). \tag{2.3}
\]

(ii) If not all \( \lambda_1, \ldots, \lambda_{p+1} \in \mathbb{R} \) coincide, let \( I \) be the set of indices \( i \) for which \( \lambda_i \) are distinct and let \( m(\lambda_i) \) be the multiplicity of \( \lambda_i \). Assume that \( p \geq 1 \) and \( M = \max_{i \in I} m(\lambda_i) \leq p \). Then, \( t \mapsto \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p}) \in C^{p-M}(\mathbb{R}) \) and
\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p}) = p \int_t^{\infty} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - s)_{+}^{p-1}) \, ds. \tag{2.4}
\]

**Proposition 2.9.** (See [15, Lemma 3.2].) Let \( (\lambda_1, \ldots, \lambda_{p+1}) \in \mathbb{R}^{p+1} \). Then the function \( \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p}) \) is decreasing; it is equal to 1 when \( t < \min_{1 \leq k \leq p+1} \lambda_k \) and equal to 0 when \( t \geq \max_{1 \leq k \leq p+1} \lambda_k \).
We will need a representation of the divided difference in terms of an antiderivative of the corresponding basic spline \((2.4)\).

Lemma 2.10. Let \([a, b] = [\min\{\lambda_1, \ldots, \lambda_{p+1}\}, \max\{\lambda_1, \ldots, \lambda_{p+1}\}]\). For \(f \in C^p[a, b]\),

\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \frac{1}{p!} f^{(p)}(a) + \frac{1}{p!} \int_a^b f^{(p+1)}(t) \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)^{p+1}) \, dt.
\]  

(2.5)

Proof. Assume first that not all \(\lambda_1, \ldots, \lambda_{p+1}\) coincide. Applying Proposition 2.8 and then integrating by parts and applying the representation \((2.4)\) of Proposition 2.7 provide

\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \frac{1}{(p-1)!} \int_a^b f^{(p)}(t) \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)^{p-1}) \, dt
\]

\[
= -\frac{1}{p!} \left( f^{(p)}(t) \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)^p) \right)_{a}^{b} + \frac{1}{p!} \int_a^b f^{(p+1)}(t) \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)^p) \, dt.
\]

By Proposition 2.9, the latter reduces to \((2.5)\). If \(\lambda_1 = \cdots = \lambda_{p+1}\), then by Proposition 2.7 \((2)\),

\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \frac{1}{p!} f^{(p)}(\lambda_1) = \frac{1}{p!} f^{(p)}(a) + \frac{1}{p!} \int_a^{\lambda_1} f^{(p+1)}(t) \, dt.
\]

With use of the representation \((2.3)\) of Proposition 2.8 the latter can be rewritten as \((2.5)\). \(\square\)

3. Traces of multiple operator integrals

In this section, we represent traces of certain multiple operator integrals as multiple scalar integrals. In particular, we obtain useful formulas for the traces of the Gâteaux derivatives \(\tau \left[ \frac{\partial}{\partial x} f(H_0 + xV) \right]\), where \(V = V^*\) is a Hilbert-Schmidt perturbation of a self-adjoint operator \(H_0\) and \(f \in W_p(\mathbb{R})\).

3.1. Multiple spectral measures. We will need the facts that certain multi-measures extend to finite countably additive measures.

Proposition 3.1. Let \(2 \leq p \in \mathbb{N}\) and let \(E_1, E_2, \ldots, E_p\) be projection-valued Borel measures from \(\mathbb{R}\) to \(\mathcal{M}\). Suppose that \(V_1, \ldots, V_p\) belong to \(L_2(\mathcal{M}, \tau)\). Assume that either \(\mathcal{M} = \mathcal{B}(\mathcal{H})\) or \(p = 2\). Then there is a unique (complex) Borel measure \(m\) on \(\mathbb{R}^p\) with total variation not exceeding the product \(\|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2\), whose value on rectangles is given by

\[
m(A_1 \times A_2 \times \cdots \times A_p) = \tau\left[ E_1(A_1) V_1 E_2(A_2)V_2 \cdots V_{p-1} E_p(A_p)V_p \right]
\]

for all Borel subsets \(A_1, A_2, \ldots, A_p\) of \(\mathbb{R}\).

Remark 3.2. In the case of \(\mathcal{M} = \mathcal{B}(\mathcal{H})\) and \(V\) a Hilbert-Schmidt operator, Proposition 3.1 was obtained in [3] [11]. For a general \(\mathcal{M}\) and \(V \in L_2(\mathcal{M}, \tau)\), the set function \(m\) is known to be of bounded variation only if \(p = 2\) (see [6] Section 4) for a positive result and a counterexample).
**Proposition 3.3.** (See [1] Corollary 4.3.) Under the assumptions of Proposition 3.1, there is a unique (complex) Borel measure \( m_1 \) on \( \mathbb{R}^p \) with total variation not exceeding the product \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2 \), whose value on rectangles is given by
\[
m_1(A_1 \times A_2 \times \cdots \times A_p \times A_{p+1}) = \tau \left[ E_1(A_1)V_1E_2(A_2)V_2 \cdots V_{p-1}E_p(A_p)V_pE_1(A_{p+1}) \right]
\]
for all Borel subsets \( A_1, A_2, \ldots, A_p, A_{p+1} \) of \( \mathbb{R} \).

In the sequel, we will work with the set functions
\[
m_{p,H_0,V}(A_1 \times A_2 \times \cdots \times A_p) = \tau \left[ E_{H_0}(A_1)V E_{H_0}(A_2)V \cdots V E_{H_0}(A_p)V \right],
\]
\[
m_{1}(A_1 \times A_2 \times \cdots \times A_p) = \tau \left[ E_1(A_1)V E_{H_0}(A_2)V \cdots V E_{H_0}(A_p)V \right],
\]
and their countably-additive extensions (when they exist), called multiple spectral measures. Here \( A_j, 1 \leq j \leq p, \) are measurable subsets of \( \mathbb{R} \), \( H_0 = H_0^\# \) is affiliated with \( \mathcal{M} \), and \( V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau) \). Clearly, the measures \( m_{p,H_0,V} \) and \( m_{1}(A_1 \times A_2 \times \cdots \times A_p) \) are particular representatives of the measures \( m \) and \( m_1 \), respectively.

**Proposition 3.4.** (See [1] Theorem 4.5.) Let \( \tau \) be a finite trace normalized by \( \tau(I) = 1 \) and let \( H_0 = H_0^\# \) be affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau) \). Assume that \((zI - H_0)^{-1} \) and \( V \) are free. Then the set functions \( m_{p,H_0,V} \) and \( m_{1}(A_1 \times A_2 \times \cdots \times A_p) \) extend to countably additive measures of bounded variation.

Upon evaluating a trace, some iterated operator integrals can be written as Lebesgue integrals with respect to “multiple spectral measures”.

**Proposition 3.5.** (See [1] Lemma 4.9.) Assume the hypothesis of Proposition 3.1. Assume that the spectral measures \( E_1, E_2, \ldots, E_p \) correspond to self-adjoint operators \( H_0, H_1, \ldots, H_p \) affiliated with \( \mathcal{M} \), respectively, and that \( V_1, V_2, \ldots, V_p \in \mathcal{L}_2(\mathcal{M}, \tau) \). Let \( f_1, f_2, \ldots, f_p \) be functions in \( C_0(\mathbb{R}) \) (vanishing at infinity). Then
\[
\tau[f_1(H_1)V_1f_2(H_2)V_2 \cdots f_p(H_p)V_p] = \int_{\mathbb{R}^p} f_1(\lambda_1)f_2(\lambda_2) \cdots f_p(\lambda_p) \, dm(\lambda_1, \lambda_2, \ldots, \lambda_p),
\]
with \( m \) as in Theorem 3.1.

**Remark 3.6.** A completely analogous result with \( m \) replaced by \( m_{p,H_0,V} \) or \( m_{1}(A_1 \times A_2 \times \cdots \times A_p) \) holds under the hypothesis of Proposition 3.1 or Proposition 3.3.

### 3.2. Reduction of traces of multiple operator integrals to scalar integrals.

**Definition 3.7.** ([1] Definition 4.1; see also [14]) Let \( H_k = H_k^\# \) and \( V_k = V_k^* \), with \( k = 1, \ldots, p + 1 \), be operators defined in \( \mathcal{H} \). Assume that \( V_k, k = 1, \ldots, p + 1 \), are bounded. Let \( \phi \) be a function representable in the form
\[
\phi(\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}) = \int_S \alpha_1(\lambda_1, s)\alpha_2(\lambda_2, s) \cdots \alpha_p(\lambda_p, s)\alpha_{p+1}(\lambda_{p+1}, s) \, d\sigma(s), \quad (3.1)
\]
where \((S, \sigma)\) is a finite measure space and \( \alpha_1, \ldots, \alpha_{p+1} \) are bounded Borel functions on \( \mathbb{R} \times S \). Then the multiple operator integral
\[
\int_{\mathbb{R}^{p+1}} \phi(\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}) \, dE_{H_1}(\lambda_1)V_1 \, dE_{H_2}(\lambda_2)V_2 \cdots dE_{H_p}(\lambda_p)V_p \, dE_{H_{p+1}}(\lambda_{p+1})
\]
is defined as the Bochner integral
\[ \int_S \alpha_1(H_1, s) V_1 \alpha_2(H_2, s) V_2 \ldots \alpha_p(H_p, s) V_p \alpha_{p+1}(H_{p+1}, s) \, d\sigma(s). \]

When the set functions \( m \) and \( m_1 \) admit extensions to finite countably additive measures, a trace of a multiple operator integral can be represented as a multiple scalar integral.

**Lemma 3.8.** Let \( H_1, \ldots, H_{p+1} \) be self-adjoint operators affiliated with \( \mathcal{M} \) and \( V_1, \ldots, V_p \) self-adjoint operators in \( L_2(\mathcal{M}, \tau) \). Let \( E_k = E_{H_k} \), for \( k = 1, \ldots, p + 1 \), and let \( \phi \) be a bounded Borel function admitting the representation (3.1). Then, the following representations hold.

1. For \( m_1 \) the measure provided by Proposition 3.3 or Proposition 3.4,
\[
\tau \left[ \int_{\mathbb{R}^{p+1}} \phi(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}) \, dE_{H_1}(\lambda_1) V_1 \ldots dE_{H_p}(\lambda_p) V_p \, dE_{H_{p+1}}(\lambda_{p+1}) \right] \\
= \int_{\mathbb{R}^{p+1}} \phi(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}) \, dm_1(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}).
\]

2. In the case of \( H_{p+1} = H_1 \), for \( m \) the measure provided by Proposition 3.1 or Proposition 3.4,
\[
\tau \left[ \int_{\mathbb{R}^{p+1}} \phi(\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}) \, dE_{H_1}(\lambda_1) V_1 dE_{H_2}(\lambda_2) V_2 \ldots dE_{H_p}(\lambda_p) V_p \, dE_{H_1}(\lambda_{p+1}) \right] \\
= \int_{\mathbb{R}^p} \phi(\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_1) \, dm(\lambda_1, \lambda_2, \ldots, \lambda_p).
\]

**Proof.** (1) By [1, Lemma 3.10 and Remark 4.2],
\[
\tau \left[ \int_S \alpha_1(H_1, s) V_1 \ldots \alpha_p(H_p, s) V_p \alpha_{p+1}(H_{p+1}, s) \, d\sigma(s) \right] \\
= \int_S \tau [\alpha_1(H_1, s) V_1 \ldots \alpha_p(H_p, s) V_p \alpha_{p+1}(H_{p+1}, s)] \, d\sigma(s).
\]
By Remark 3.6, the latter integral equals
\[
\int_S \int_{\mathbb{R}^{p+1}} \alpha_1(\lambda_1, s) \ldots \alpha_p(\lambda_p, s) \alpha_{p+1}(\lambda_{p+1}, s) \, d\sigma(s) \, dm_1(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}) \, d\sigma(s),
\]
which by Fubini’s theorem converts to
\[
\int_{\mathbb{R}^{p+1}} \int_S \alpha_1(\lambda_1, s) \ldots \alpha_p(\lambda_p, s) \alpha_{p+1}(\lambda_{p+1}, s) \, d\sigma(s) \, dm_1(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}) \\
= \int_{\mathbb{R}^{p+1}} \phi(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}) \, dm_1(\lambda_1, \ldots, \lambda_p, \lambda_{p+1}).
\]
(2) By [1, Lemma 3.10 and Remark 4.2] and cyclicity of the trace,
\[
\tau \left[ \int_S \alpha_1(H_1, s) V_1 \alpha_2(H_2, s) \ldots V_p \alpha_{p+1}(H_{p+1}, s) \, d\sigma(s) \right] \\
= \int_S \tau [\alpha_{p+1}(H_1, s) \alpha_1(H_1, s) V_1 \alpha_2(H_2, s) \ldots V_p] \, d\sigma(s).
\]
By Proposition 3.5, the latter integral equals
\[
\int_S \int_{\mathbb{R}^p} \alpha_{p+1}(\lambda_1, s)\alpha_1(\lambda_1, s)\alpha_2(\lambda_2, s) \ldots \alpha_p(\lambda_p, s) \, dm(\lambda_1, \lambda_2, \ldots, \lambda_p) \, d\sigma(s),
\]
which by Fubini’s theorem converts to
\[
\int_{\mathbb{R}^p} \int_S \alpha_1(\lambda_1, s)\alpha_2(\lambda_2, s) \ldots \alpha_p(\lambda_p, s)\alpha_{p+1}(\lambda_1, s) \, dm(\lambda_1, s) \, d\sigma(s) \, dm(\lambda_1, \lambda_2, \ldots, \lambda_p)
\]
\[
= \int_{\mathbb{R}^p} \phi(\lambda_1, \lambda_2, \ldots, \lambda_p, 1) \, dm(\lambda_1, \lambda_2, \ldots, \lambda_p).
\]
\[\square\]

We have the following representation for the derivative \(\frac{dp}{dx} f(H_0 + xV)\).

**Proposition 3.9.** ([14, Theorem 5.6]; see also [1, Theorem 5.7]) Let \(H_0 = H_0^*\) be an operator affiliated with \(\mathcal{M}\) and \(V = V^*\) an operator in \(\mathcal{M}\). Denote \(H_x = H_0 + xV\). Then for every \(f \in \mathcal{W}_p(\mathbb{R})\),
\[
\frac{dp}{dx} \bigg|_{x=0} f(H_0 + xV) = p! \int_{\mathbb{R}^{p+1}} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) \, dE_{H_0}(\lambda_1)V \ldots V \, dE_{H_0}(\lambda_{p+1}).
\]

The main assumptions of the following results are collected in the format of a hypothesis.

**Hypothesis 3.10.** Let \(H_0 = H_0^*\) be an operator affiliated with \(\mathcal{M}\) and \(V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau)\). Assume that one of the following three conditions is satisfied:

1. \(\mathcal{M} = \mathcal{B}(\mathcal{H}), p \geq 3,\)
2. \(p = 3,\)
3. \(\mathcal{M}\) is finite, \(p \geq 3,\) and \((zI - H_0)^{-1}\) and \(V\) are free in \((\mathcal{M}, \tau)\).

In the multiple operator integral representation for the derivative \(\frac{dp}{dx} f(H_x)\) provided by Proposition 3.9, the order of the divided difference can be reduced upon evaluating the trace.

**Theorem 3.11.** Assume Hypothesis 3.10. Then for \(f \in \mathcal{W}_p(\mathbb{R}),\)
\[
\tau \left[ \frac{dp}{dx} \bigg|_{x=0} f(H_0 + xV) \right] = p! \int_{\mathbb{R}^{p+1}} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) \, dm_{p,H_0,V}^{(1)}(\lambda_1, \ldots, \lambda_{p+1}) \quad (3.2)
\]
\[
= (p - 1)! \int_{\mathbb{R}^{p}} \Delta_{\lambda_1, \ldots, \lambda_{p}}^{(p-1)}(f') \, dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_{p}). \quad (3.3)
\]

**Proof.** By Proposition 2.6, the function \(\phi(\lambda_1, \ldots, \lambda_{p+1}) = \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f)\) admits the representation (3.1), where \(\alpha_1(\lambda_1, s) = e^{i(s_0 - s_1)\lambda_1}, \ldots, \alpha_p(\lambda_p, s) = e^{i(s_{p-1} - s_p)\lambda_p},\) and
\[ \alpha_{p+1}(\lambda_{p+1}, s) = e^{is\lambda_{p+1}}. \]

It follows from Proposition \ref{prop:3.9} and Lemma \ref{lem:3.8} that

\[
\tau \left[ \frac{d^p}{dx^p} \bigg|_{x=0} f(H_0 + xV) \right] = \tau \left[ \int_{\mathbb{R}^{p+1}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_{p+1}}^{(p)}(f) dE_{H_0}(\lambda_1)V dE_{H_0}(\lambda_2)V \ldots V dE_{H_0}(\lambda_{p+1}) \right]
\]

\[ = \int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_{p+1}}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p, \lambda_{p+1}) \quad (3.4) \]

\[ = \int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_1}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p). \quad (3.5) \]

Proposition \ref{prop:3.9} and the representation \ref{eq:3.4} imply \ref{eq:3.2}.

To prove that the expressions in \ref{eq:3.2} and \ref{eq:3.3} are equal, we note first that a trivial renumbering of the variables of integration gives

\[
\int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_1}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p)
\]

\[ = \int_{\mathbb{R}^{p}} \Delta_{\lambda_i,\lambda_{i+1},\ldots,\lambda_{p},\lambda_{i-1}}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_i, \lambda_{i+1}, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_p). \quad (3.6) \]

Cyclicity of the trace \( \tau \) ensures cyclicity of the measure \( m_{p,H_0,V} \), that is,

\[
dm_{p,H_0,V}(\lambda_i, \lambda_{i+1}, \ldots, \lambda_{p}, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_p) = dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{p}). \quad (3.7) \]

Symmetry of the divided difference (see Proposition \ref{prop:2.2} \( \star \)) along with \ref{eq:3.6} and \ref{eq:3.7} ensures the equality

\[
\int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_1}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p)
\]

\[ = \int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_1}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p). \quad (3.8) \]

It follows from \ref{eq:3.8} and Lemma \ref{lem:2.3} that

\[
p \int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_1}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p)
\]

\[ = \sum_{i=1}^{p} \int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\lambda_2,\ldots,\lambda_{p},\lambda_i}^{(p)}(f) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_p)
\]

\[ = \int_{\mathbb{R}^{p}} \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}(f') \, dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_p). \quad (3.9) \]

Combination of \ref{eq:3.5} and \ref{eq:3.9} completes the proof of the theorem. \( \square \)
Corollary 3.12. Assume Hypothesis 3.10. Assume, in addition, that $H_0$ is bounded. Let $[a, b]$ be a segment containing $\sigma(H_0) \cup \sigma(H_0 + V)$. Then for $f \in \mathcal{W}_p(\mathbb{R})$,

$$\tau \left[ \frac{d^p}{dx^p} \right] \left. f \right|_{x=0} (H_0 + xV) - \tau(V^p) f^{(p)}(a)$$

$$= \int_a^b f^{(p+1)}(t) \int_{[a,b]^p} \Delta^{(p)}_{\lambda_1, \ldots, \lambda_{p+1}} ((\lambda - t)^{p+1}_+) \, dm_{p,H_0,V}^{(1)}(\lambda_1, \ldots, \lambda_{p+1}) \, dt \quad (3.10)$$

$$= \int_a^b f^{(p+1)}(t) \int_{[a,b]^p} \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, dm_{p,H_0,V}^{(1)}(\lambda_1, \ldots, \lambda_p) \, dt. \quad (3.11)$$

Proof. First, notice that the measure $m_{p,H_0,V}$ is supported in $[a, b]$. Expanding the integrands in (3.2) and (3.3) according to (2.5) of Lemma 2.10 and then using Fubini’s theorem (the functions $f^{(p+1)}(\cdot)$, $\Delta^{(p)}_{\lambda_1, \ldots, \lambda_{p+1}} ((\lambda - \cdot)^{p+1}_+)$, and $\Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - \cdot)^{p-1}_+)$ are bounded and the measures $m_{p,H_0,V}^{(1)}$ and $m_{p,H_0,V}$ are finite) provide the representations (3.10) and (3.11). Here we used the fact that $m_{p,H_0,V}^{(1)}(\mathbb{R}^{p+1}) = m_{p,H_0,V}(\mathbb{R}^p) = \tau(V^p)$. \hfill \Box

4. Properties of the spectral shift measure

In this section, we prove existence of the higher order spectral shift functions and derive some of their properties by implementing a multiple operator integral approach.

Theorem 4.1. Assume Hypothesis 3.10. Assume, in addition, that $H_0$ is bounded.

1. There exists a unique finite real-valued absolutely continuous measure $\nu_p$ such that the trace formula

$$\tau[R_p(f)] = \int_\mathbb{R} f^{(p)}(t) d\nu_p(t) \quad (4.1)$$

holds for $f \in \mathcal{W}_p(\mathbb{R}) \cup \mathfrak{R}$, where $\mathfrak{R}$ denotes the set of rational functions on $\mathbb{R}$ with nonreal poles.

2. The density of $\nu_p$ is given by the formulas

$$\eta_p(t) = \frac{\tau(V^p)}{(p-1)!} - \nu_{p-1}((-\infty, t))$$

$$- \frac{1}{(p-1)!} \int_{\mathbb{R}^p} \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, dm_{p-1,H_0,V}^{(1)}(\lambda_1, \ldots, \lambda_p) \quad (4.2)$$

$$= \frac{\tau(V^p)}{(p-1)!} - \nu_{p-1}((-\infty, t))$$

$$- \frac{1}{(p-1)!} \int_{\mathbb{R}^{p-1}} \Delta^{(p-2)}_{\lambda_1, \ldots, \lambda_{p-1}} ((\lambda - t)^{p-2}_+) \, dm_{p-1,H_0,V}(\lambda_1, \ldots, \lambda_{p-1}). \quad (4.3)$$

3. The measure $\nu_p$ is supported in the convex hull of the set $\sigma(H_0) \cup \sigma(H_0 + V)$ and $\nu_p(\mathbb{R}) = \frac{\tau(V^p)}{p!}$.

Remark 4.2. Theorem 4.1, except for the representation (4.2), was originally proved in \cite{6} Theorem 5.1, Theorem 5.2, and Theorem 5.6. We provide a shorter proof.
Proof of Theorem 4.1. The proof can be accomplished by induction. The result is known to hold for \( p = 2 \) (see [6, 7]). Assume that the theorem holds when \( p \) is replaced with \( p - 1 \). Let \([a, b]\) be a segment containing \( \sigma(H_0) \cup \sigma(H_0 + V) \). Then \( \eta_{p-1} \) is supported in \([a, b]\). Clearly,

\[
\tau [R_p(f)] = \tau [R_{p-1}(f)] - \frac{1}{(p-1)!} \tau \left. \frac{d^{p-1}}{dx^{p-1}} \right|_{x=0} f(H_0 + xV) \right) . \tag{4.4}
\]

Let \( f \in \mathcal{W}_p(\mathbb{R}) \). By the induction hypothesis and the representation (3.10) of Corollary 3.12, the expression in (4.4) equals

\[
\int_a^b f^{(p-1)}(t) \eta_{p-1}(t) \, dt - \frac{\tau(V^{p-1})}{(p-1)!} f^{(p-1)}(a)
\]

\[
= f^{(p-1)}(b) \frac{\tau(V^{p-1})}{(p-1)!} - \int_a^b f^{(p-1)}(b) \left( \int_a^t \eta_{p-1}(s) \, ds \right) \, dt . \tag{4.5}
\]

Integrating by parts in the first integral in (4.5) gives

\[
\int_{[a,b]} f^{(p-1)}(t) \eta_{p-1}(t) \, dt = \left( f^{p-1}(t) \int_a^t \eta_{p-1}(s) \, ds \right) \bigg|_a^b - \int_a^b f^{(p)}(t) \left( \int_a^t \eta_{p-1}(s) \, ds \right) \, dt
\]

\[
= f^{(p-1)}(b) \frac{\tau(V^{p-1})}{(p-1)!} - \int_a^b f^{(p)}(t) \left( \int_a^t \eta_{p-1}(s) \, ds \right) \, dt . \tag{4.6}
\]

Combining (4.4) - (4.6) implies

\[
\tau [R_p(f)] = \left( f^{(p-1)}(b) - f^{(p-1)}(a) \right) \frac{\tau(V^{p-1})}{(p-1)!} - \int_a^b f^{(p)}(t) \int_a^t \eta_{p-1}(s) \, ds \, dt
\]

\[
- \int_a^b f^{(p)}(t) \frac{1}{(p-1)!} \int_{[a,b]} \Delta_{\lambda_1,\ldots,\lambda_p}^{p-1} ((\lambda - t)_+^{p-1}) \, dm^{(1)}_{p-1, H_0, V} (\lambda_1, \ldots, \lambda_p) \, dt
\]

\[
= \int_a^b f^{(p)}(t) \left( \frac{\tau(V^{p-1})}{(p-1)!} - \nu_p((a,t)) \right)
\]

\[
- \frac{1}{(p-1)!} \int_{\mathbb{R}^p} \Delta_{\lambda_1,\ldots,\lambda_p}^{p-1} ((\lambda - t)_+^{p-1}) \, dm^{(1)}_{p-1, H_0, V} (\lambda_1, \ldots, \lambda_p) \right) \, dt .
\]

from what the trace formula (4.1) follows for \( f \in \mathcal{W}_p(\mathbb{R}) \), with

\[
\eta_p(t) = \frac{\tau(V^{p-1})}{(p-1)!} - \nu_{p-1}((a,t))
\]

\[
- \frac{1}{(p-1)!} \int_{\mathbb{R}^p} \Delta_{\lambda_1,\ldots,\lambda_p}^{p-1} ((\lambda - t)_+^{p-1}) \, dm^{(1)}_{p-1, H_0, V} (\lambda_1, \ldots, \lambda_p) .
\]
Let \([c, d]\) denote the convex hull of \(\sigma(H_0) \cup \sigma(H_0 + V)\). By the induction hypothesis, 
\[ \nu_{p-1}((-\infty, c)) = \nu_{p-1}((d, \infty)) = 0 \] and 
\[ \nu_{p-1}([c, d]) = \frac{\tau(V^{p-1})}{(p-1)!}. \] 
By Proposition 2.9,
\[ \int_{\mathbb{R}^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)}((\lambda - t)^{(p-1)}_+) \, dm_{p-1,H_0,V}^{(1)}(\lambda_1, \ldots, \lambda_p) \]
\[ = \left\{ \begin{array}{ll}
\frac{\tau(V^{p-1})}{(p-1)!} & \text{if } t < c \\
0 & \text{if } t > d.
\end{array} \right. \]

Therefore, (4.2) holds and the measure \(\nu_p\) is supported in \([c, d]\). To extend the trace formula (4.1) to \(f\) a polynomial, we apply (4.1) to a function \(g \in \mathcal{W}_p(\mathbb{R})\), which coincides with \(f\) on a segment containing \(\cup_{x \in [-1, 1]} \sigma(H_0 + xV)\), and get
\[ \tau[R_p(f)] = \tau[R_p(g)] = \int_{\mathbb{R}} g^{(p)} \, d\nu_p(t) = \int_{\mathbb{R}} f^{(p)}(t) \, d\nu_p(t). \]

To obtain the equality \(\nu_p(\mathbb{R}) = \frac{\tau(V^{p})}{(p)!}\), we apply (4.1) to \(f = t^p\). The measure \(\nu_p\) is finite since it is compactly supported and its density is bounded.

The proof of (4.3) is completely analogous to the proof of (4.2), where the only difference consists in applying (3.11) (instead of (3.10)) to the second summand in (4.4).

The techniques used in the proof of Theorem 4.1 also work in the case of an unbounded operator \(H_0\), provided \(f^{(p)} \in L^1(\mathbb{R})\) and \(\nu_{p-1}\) is known to be finite.

**Theorem 4.3.** Let \(H_0 = H^*_0\) be an operator affiliated with \(\mathcal{M}\), \(V = V^*\) an operator in \(\mathcal{L}(\mathcal{M}, \tau)\) and \(p = 3\). Then for \(f \in C^p_+(\mathbb{R}) \cup \mathcal{R}_b\) where \(\mathcal{R}_b\) is the subset of bounded functions in \(\mathcal{R}\), the representations (4.1) - (4.3) hold.

**Proof.** The proof is very similar to the one of Theorem 4.1, so we provide only a brief sketch. Clearly,
\[ 0 = \lim_{a \to -\infty, b \to \infty} \left( f^{(p-1)}(b) - f^{(p-1)}(a) \right) \frac{\tau(V^{p-1})}{(p-1)!} = \int_{\mathbb{R}} f^{(p)}(t) \, \frac{\tau(V^{p-1})}{(p-1)!} \, dt. \] (4.7)

By letting \(a \to -\infty\) and \(b \to \infty\) in (4.6), we obtain
\[ \int_{\mathbb{R}} f^{(p)}(t) \, \eta_{p-1}(t) \, dt = -\int_{\mathbb{R}} f^{(p)}(t) \left( \int_{-\infty}^{t} \eta_{p-1}(s) \, ds \right) \, dt. \] (4.8)

By letting \(a \to -\infty\) and \(b \to \infty\) in (2.5), we obtain
\[ \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p}(f) = \frac{1}{(p-1)!} \int_{\mathbb{R}} f^{(p)}(t) \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p}( (\lambda - t)^{(p-1)}_+ ) \, dt \]
and, subsequently,
\[ \tau \left[ \frac{d^{(p-1)}}{dx^{(p-1)}} \bigg|_{x=0} f(H_0 + xV) \right] \]
\[ = \int_{\mathbb{R}} f^{(p)}(t) \int_{\mathbb{R}^p} \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p}( (\lambda - t)^{(p-1)}_+ ) \, dm_{p-1,H_0,V}^{(1)}(\lambda_1, \ldots, \lambda_p) \, dt \] (4.9)
\[ = \int_{\mathbb{R}} f^{(p)}(t) \int_{\mathbb{R}^{p-1}} \Delta^{(p-2)}_{\lambda_1, \ldots, \lambda_{p-1}}( (\lambda - t)^{(p-2)}_+ ) \, dm_{p-1,H_0,V}(\lambda_1, \ldots, \lambda_{p-1}) \, dt \] (4.10)
We note that the integral $\int_{-\infty}^{t} \eta_{p-1}(s) \, ds$ is well defined since $\eta_{p-1}$ is integrable (see, e.g., discussion in the introductory section of [15]). Combining (4.8) - (4.10), as it was done in the case of a bounded $H_0$, completes the proof. □

**Remark 4.4.** The trace formula (4.11) for $f \in C_c^\infty(\mathbb{R})$ (in fact, $f \in C^{p+1}_c(\mathbb{R})$ also works) was obtained in [6, Theorem 5.2], without establishing the absolute continuity of the measure $\nu_3$ when $H_0$ is unbounded. The trace formula

$$
\tau[R_3(f)] = \int_{\mathbb{R}} f''(t) \eta_3(t) \, dt,
$$

(4.11)

for an unbounded $H_0$, with $\eta_3$ given by (4.3), was proved in [15, Theorem 4.1] only for bounded rational functions $f$. The results of Theorem 3.11 have allowed to obtain (4.11) for both $f \in C^p_c(\mathbb{R})$ and $f \in \mathcal{R}_b$. The same approach proves existence of the spectral shift function of order $p \geq 3$ for an unbounded $H_0$, when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ (the original proof was obtained in [6 15]). A substantial obstacle in establishing (4.11) for $\tau(|V|^p) < \infty$, with $p > 2$ (unless $\tau(|V|^2) < \infty$), is non-extendibility of the set function $m_{p,H_0,V}$ to a finite measure on $\mathbb{R}^p$ (see a counterexample in [6, Section 4]). An analogous problem has caused a delay in establishing (4.11) in the von Neumann algebra setting; the set function $m_{3,H_0,V}$ can fail to extend to a finite measure even if $\tau$ is finite (and $\dim(\mathcal{H}) = \infty$) [6, Section 4]. That is why the approach of [6, 15] working for every Hilbert-Schmidt $V = V^* \in \mathcal{B}(\mathcal{H})$ was not so successful in the general von Neumann algebra setting.

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