Many currently available universal approximation theorems affirm that deep feedforward networks defined using any suitable activation functions can approximate any integrable function locally in $L^1$-norm. Though different approximation rates are available for deep neural networks defined using other classes of activation functions, there is little explanation for the empirically confirmed advantage that ReLU networks exhibit over their classical (e.g. sigmoidal) counterparts. Our main result demonstrates that deep networks with piecewise linear activation (e.g. ReLU or PReLU) are fundamentally more expressive than deep feedforward networks with analytic (e.g. sigmoid, Swish, GeLU, or Softplus). More specifically, we construct a strict refinement of the topology on the space $L_{loc}^1(\mathbb{R}^d, \mathbb{R})$ of locally Lebesgue-integrable functions, in which the set of deep ReLU networks with (bilinear) pooling $\text{NN}_{\text{ReLU}}$ is dense (i.e. universal) but the set of deep feedforward networks defined using any combination of analytic activation functions with (or without) pooling layers $\text{NN}_{\omega}$ is not dense (i.e. not universal). The “separation phenomenon” persists when comparing deep ReLU networks with pooling to classical polynomial functions; which we show are also not dense in this space. Our main result is further explained by quantitatively demonstrating that this “separation phenomenon” between the networks in $\text{NN}_{\text{ReLU}}$ and those in $\text{NN}_{\omega}$ by showing that the networks in $\text{NN}_{\text{ReLU}}$ are capable of approximate any compactly supported Lipschitz function while simultaneously approximating its essential support; whereas, the networks in $\text{NN}_{\omega}$ cannot.

1 Introduction

The classical universal approximation theorems of Hornik et al. (1989) concern neural networks with sigmoidal activation functions, of which the sigmoid activation $\sigma(x) \equiv \frac{e^x}{1+e^x}$ is the prototype. In contrast, in most contemporary universal approximation theorems (Yarotsky, 2018; Gühring et al., 2020a; Lu et al., 2021; Shen et al., 2022; Opschoor et al., 2022) networks with ReLU activation function $\text{ReLU}(x) \equiv \max\{0, x\}$ are studied. The reason for this is largely the fact that ReLU networks are more popular in practice $\sigma$ networks, since ReLU networks tend not to encounter vanishing gradients during training and tend to learn sparser weights and biases after training, than sigmoid networks. Nevertheless, it still remains unclear if ReLU networks are genuinely more expressive than sigmoid networks. This paper takes a first step towards answering this open question.

Our main result can be informally states as follows. Suppose that Alice and Bob both want to approximate a target function $f$ with their deep learning models. Alice has access to a very large state-of-the-art supercomputer with that can train a very deep and wide feedforward network with sigmoid activation function and Bob has a small dated
The set $\mathbb{R}$ each neuron can have any analytic activation function $f$ wherein, given enough of a computational advantage, Alice could in principle approximate Theorem 2 (2018), or of Galindo & Sanchis (2004)). This is because, the set of multivariate polynomial functions from $\mathbb{R}$ Theorem 1 (Separation: Neural Networks with Piecewise-Linear vs. Analytic Activation Functions) qualitative Theorem 1 puts forth the following Let $\log \sigma$ approximation efficiency Gühring et al. (2020b); Suzuki (2019); Yarotsky & Zhevnerchuk (2020a), carry over to a polynomial regression methods and the quantitative edge exhibited over polynomial methods in terms of uniform dense in the Weierstrass approximation theorem and its numerous contemporary variants Prolla (1994), Timofte et al. Interestingly, the phenomenon identified in Theorem 1 does indeed persist when comparing deep ReLU networks with analytic activation function, or any combination of any such networks. At this point, we ask: bilinear pooling layer at their output is strictly more expressive than the set $\mathbb{R}$ $\sigma$ $\tau$ $\omega$ $\beta$ “Can Theorem 1 help explain the edge of deep ReLU networks over polynomial regressors seen in practice?” Interestingly, the phenomenon identified in Theorem 1 does indeed persist when comparing deep ReLU networks in $\mathbb{R}$ to classical polynomial-regressor methods (as are well-understood by classical results such as the Weierstrass approximation theorem and its numerous contemporary variants Proïla (1994), Timofte et al. (2018), or of Galindo & Sanchis (2004)). This is because, the set of multivariate polynomial functions from $\mathbb{R}$ to $\mathbb{R}$, which we denoted by $\mathbb{R}[x_1, \ldots, x_d] \overset{\text{def.}}{=} \left\{ \sum_{k=0}^{K} \prod_{i=1}^{d} \beta_{n,i}^{k} : n_1,0, \ldots, n_d,K \in \mathbb{N}, \beta_{n,i} \in \mathbb{R} \right\}$, also fails to be dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for $\tau$. Therefore, the empirically-observed advantage which deep ReLU networks have over polynomial regression methods and the quantitative edge exhibited over polynomial methods in terms of uniform approximation efficiency Gühring et al. (2020b); Suzuki (2019); Yarotsky & Zhevnerchuk (2020a), carry over to a qualitative gap between the two methods; in the sense of Theorem 1 and the following result.

**Theorem 2** (Separation: Deep Networks with Piecewise-Linear Activation Functions vs. Polynomial-Regressors). The set $\mathbb{R}[x_1, \ldots, x_d]$ is not dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for $\tau$. 

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laptop on which they train a deep ReLU network with much fewer neurons than Alice’s model. We assume that both have access to an idealized optimizer and an infinite noiseless dataset. Our main result shows that there are non-pathological functions whose “sharpe” features cannot be approximately encoded by Alice’s model, irrespective of how much more computing power they have access to; however, Bob’s modest setup can.

In this way, we demonstrate a **qualitative** (or fundamental) gap between the approximation capabilities of deep feedforward networks with sigmoidal vs. ReLU activation functions. This is in contrast to a **quantitative gap** wherein, given enough of a computational advantage, Alice could in principle approximate $f$ just as well as Bob could. A fortiori, the phenomenon which we study persists even if Alice instead implements deep feedforward networks with any choice of very smooth activation function at each neuron and even if Bob implements any single other piecewise linear activation function (non-affine).

Rigorously, let $\sigma_{\text{PW-Lin}}$ be a piecewise linear activation function with at-least 2 (distinct) pieces and let $\text{NN}_{\text{ReLU+Pool}}$ denote the set of deep feedforward networks mapping $\mathbb{R}^d$ to $\mathbb{R}^D$ with bilinear pooling layer, defined by $\text{Pool}(x_1, \ldots, x_{2n}) \overset{\text{def.}}{=} (x_1 x_2, \ldots, x_{2n-1} x_{2n})$, at their output. Let $\text{NN}_{\text{ReLU+Pool}}$ denote the set of deep feedforward networks where each neuron can have any analytic activation function (e.g. sigmoid, Swish, GeLU, Softplus, sin, etc...) and any number of bi-linear pooling layers (possibly 0). In this paper, we exhibit a topology $\tau$ (constructed in Section 3) on the set of locally integrable functions from $\mathbb{R}^d$ to $\mathbb{R}^D$, denoted by $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$, in which $\text{NN}_{\text{ReLU+Pool}}$ is dense (i.e. universal) but $\text{NN}_{\text{ReLU+Pool}}$ fails to be. Since the topology $\tau$ is stronger than the usual topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ (details below) then the gap is not vacuous, in the sense that, density with respect to $\tau$ implies density in the classical sense on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$. However, since $\tau$ is strictly stronger than the usual topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ then the converse is generally false. Thus, exhibiting $\tau$ reveals the qualitative approximation gap between both neural network models. Let $C^\omega(\mathbb{R})$ denote the set of analytic functions from $\mathbb{R}$ to itself; i.e. $\sigma \in C^\omega(\mathbb{R})$ if $\sigma$ is locally given by a convergent power series. We call a function $\sigma \in C(\mathbb{R})$ piecewise linear with at-least two pieces if $\mathbb{R}$ can be covered by a sequence of intervals on which $\sigma$ is affine and there is at-least one point at which $\sigma$ is not differentiable. We prove the following theorem.

**Theorem 1** (Separation: Neural Networks with Piecewise-Linear vs. Analytic Activation Functions). Let $\log_2(d) \in \mathbb{N}_+$. There exists a strict refinement $\tau$ of the topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ which refines the metric topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ and whose restriction to $L^1(\mathbb{R}^d, \mathbb{R}^D)$ is also a strict refinement of the $L^1$-norm topology satisfying:

(i) If $\sigma_{\text{PW-Lin}} \in C(\mathbb{R})$ is piecewise linear with at-least 2 pieces then $\text{NN}_{\sigma_{\text{PW-Lin}}+\text{Pool}}$ is dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ with respect to $\tau$,

(ii) $\text{NN}_{\sigma+\text{Pool}}$ is not dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ with respect to $\tau$.

Theorem 1 puts forth the following **qualitative** implication. Namely, that the set of deep ReLU networks with bilinear pooling layer at their output is strictly more expressive than the set $\text{NN}_{\sigma+\text{Pool}}$ any set of deep feedforward networks with analytic activation function, or any combination of any such networks. At this point, we ask:

“Can Theorem 1 help explain the edge of deep ReLU networks over polynomial regressors seen in practice?”
Our last result probes the underlying mechanism responsible for the separation between deep ReLU networks and polynomial regressors as well as the networks in $NN_{\text{ReLU+Pool}}^{d}$ exhibited above. Informally, our last quantitative result, shows that networks in $NN_{\text{ReLU+Pool}}^{d}$ can approximate “any” function supported on a “low-dimensional bounded submanifold of $\mathbb{R}^d$” both in $L^1$-norm while and simultaneously identifying its support. In contrast, both the above classical methods cannot perform such a simultaneous approximation of a function’s output and its support.

A rigorous statement of our result requires some some terminology. Denote the $d$-dimensional Lebesgue measure by $\mu$. The essential support of a function $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$, which generalizes the support of a continuous real-valued function to elements in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$. The essential support of such an $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ is defined by $\text{ess-sup}(f) \equiv \mathbb{R}^d - \bigcup \{ U \subseteq \mathbb{R}^d : U \text{ open and } \| f \| (x) = 0 \text{ } \mu\text{-a.e. } x \in U \}$. We say that an $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ is essentially compactly supported if $\text{ess-sup}(f)$ is contained in a closed and bounded subset of $\mathbb{R}^d$. The regularity of a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^D$ (i.e. a function with at-most linear growth) is quantified by its Lipschitz constant $\text{Lip}(f) \equiv \sup_{x_1, x_2 \in \mathbb{R}^d, x_1 \neq x_2} \| f(x_1) - f(x_2) \|$. The “complexity” of a subset $X \subseteq \mathbb{R}^d$ is quantified both in terms of its size $\text{diam}(X) \equiv \sup_{x_1, x_2 \in X} \| x_1 - x_2 \|$ and its “fractal dimension” as quantified by its metric capacity

$$\text{cap}(X) \equiv \sup \{ n \in \mathbb{N}_+ : (\exists x_1, \ldots, x_n \in X, (\exists r > 0) \cup_{i=1}^n B_2(x_i, r/5) \subset B_2(x_0, r) \},$$

where $\cup$ denotes the union of disjoint subsets of $\mathbb{R}^d$ and where $B_2(x, r) \equiv \{ u \in \mathbb{R}^d : \| u - x \| < r \}$. We mention that, for a compact Riemannian manifold, the log-$2$-metric capacity is always a multiple of the manifold’s topological dimension and the log-$2$-metric capacity of a $d$-dimensional cube in $\mathbb{R}^d$ is proportional to $d$; (see (Acciaio et al., 2022, 2.13) for further details).

One can show that $\tau$ is not a metric topology, therefore, a quantitative counterpart of Theorem 1 generally does not exist. However, in the case where the target function is Lipschitz and compactly supported, we obtain the following quantitative version of Theorem 1 (i) and variant of (Shen et al., 2022, Theorem 1.1) the approximation has controlled support made to match that of the target function. To the best of the authors’ knowledge, the result is also the only quantitative universal approximation which encodes the target function $f$’s complexity in terms of its Lipschitz regularity, as well as the size and dimension of its essential support.

**Theorem 3** (Support Detection and Uniform + $\tau$ Approximation of ReLU Networks with Pooling). Let $f : \mathbb{R}^d \to \mathbb{R}^D$ be Lipschitz and compactly-supported and $\text{log}_2(d) \in \mathbb{N}_+$. For every “width parameter” $N \in \mathbb{N}_+$ and every sequence $\{e_n\}_{n=1}^{\infty}$ in $(0, \infty)$ converging to $0$, there is a sequence $\{\hat{f}^{(n)}\}_{n=1}^{\infty}$ in $NN_{\text{ReLU+Pool}}^{d}$ satisfying:

(i) **Quantitative Worst-Case Approximation:** for each $n \in \mathbb{N}_+$ max$_{x \in [n f, n f]^d} \| f(x) - \hat{f}^{(n)}(x) \| \leq e_n$.

(ii) **Convergence in Separating Topology $\tau$:** $\{\text{Pool} \circ f^{(n)}\}_{n=1}^{\infty}$ converges to $f$ in the separating topology $\tau$.

(iii) **Support Identification:** $\text{ess-sup}(\hat{f}^{(n)}) \subseteq \left( -\sqrt{2^{-d} e_n + n f}, \sqrt{2^{-d} e_n + n f} \right)$, where $n_f$ is defined by $n_f \equiv \min \{ n \in \mathbb{N}_+ : \text{ess-sup}(f) \subseteq [-n, n]^d \}$.

Moreover, each $\hat{f}^{(n)}$ is specified by:

(iv) **Width:** $\hat{f}^{(n)}$ has width $C_3 + C_4 \max \{ d [N^{1/d}], N + 1 \}$.

(v) **Depth:** $\hat{f}^{(n)}$ has depth $\frac{e_n^{d/2}}{c \text{log}_3(N+2)^2} \left( \log_2(\text{cap}(\text{ess-sup}(f))) \text{diam}(\text{ess-sup}(f)) \text{Lip}(f) \right)^d C_1 + C_2$

(vi) **Number of bilinear pooling layers:** $\hat{f}^{(n)}$ uses $\text{log}_2(d) + 1$ bilinear pooling layers.

where the dimensional constants are $C_1 \equiv c 2^{d} D^3 / d^d + 3d$, $C_2 \equiv +2d + 2$, $C_3 \equiv \max \{ d(d-1) + 2, D \}$, $C_4 \equiv d(D+1) + 3d^3$, and where $c > 0$ is an absolute constant independent of $X, d, D,$ and $f$.

Furthermore, if $f$ is not identically 0 then there is some sequence $(e_n)_{n=1}^{\infty}$ in $(0, \infty)$ converging to 0 for which no $\hat{f} \in NN_{\text{ReLU+Pool}}^{d}(\mathbb{R}[x_1, \ldots, x_d])$ satisfies (i)-(iii) simultaneously.
Omitting the constants, the depth of the ReLU networks \( \hat{\mathcal{f}}^{(n)} \) with pooling in Theorem 3 encodes three factors. The first is the desired approximation quality, with more depth translating to better approximation capacity, and the second is the target function’s regularity; both these factors are present in most quantitative approximation theorems available in the literature (Yarotsky, 2017b; Gühring et al., 2020a; Jiao et al., 2021; Lu et al., 2021; Shen et al., 2022; Opschoor et al., 2022).

\[
\begin{align*}
\text{Depth}(\hat{\mathcal{f}}^{(n)}) & \approx \frac{\epsilon_{n}^{d/2}}{N \log_2(N + 2)^{1/2}} \left( \frac{\text{Lip}(f)}{d} \right)^{d} \left( \frac{\log_2(\text{cap}(\text{ess-sup}(f)))}{d} \right) \frac{\text{diam}(\text{ess-sup}(f))}{d} \\
\end{align*}
\]

Part of the novelty of Theorem 3 is that it identifies a third impacting the approximation quality of a ReLU network with pooling; namely, the complexity of the target function’s support. This third factor can be decomposed into two parts, the diameter of the target function’s essential support, which other approximation theorems have also considered Siegel & Xu (2020); Kratsios & Papon (2021), but what is most interesting here is the effect of the fractal dimension (via the metric capacity; see Brü et al. (2021) for details) of the target function’s essential support. In particular, the result shows that functions essentially supported on low-dimensional sets (e.g. low-dimensional latent manifolds) must be simpler to approximate than those with full support.

### 1.1 Connection to Other Deep Learning Literature

Our results are perhaps most closely related to Park et al. (2021) which demonstrates, to the best of our knowledge, the only other qualitative gap in the deep learning theory. Namely, therein, the authors identify a minimum width under which all networks become too narrow to approximate any integrable function; equivalently, the set of “very narrow” deep feedforward networks is qualitatively less expressive than the set of “arbitrary deep feedforward networks”. Just as our main results are qualitative, the results of Park et al. (2021) can be contrasted against the main result of Shen et al. (2022) which quantifies the exact impacts of depth and width on approximation error of deep feedforward networks.

Our results also add to the recent scrutiny given to deep feedforward networks deploying several activation functions (Jiao et al., 2021; Yarotsky & Zhevnerchuk, 2020b; Beknazaryan, 2021; Yarotsky, 2021; Acciaio et al., 2022). The connection to this branch of deep learning theory happens on two distinct fronts. First \( \text{NN}^{\text{ReLU+Pool}} \) is clearly a family of deep feedforward networks simultaneously utilizing several activation functions. However, more interesting, is the second connection between networks in \( \text{NN}^{\text{ReLU+Pool}} \) and the approximation theory of deep feedforward networks with generalized ReLU activation function \( \text{ReLU}_r(x) \defeq \max\{x, 0\}^r \), where \( r \in \mathbb{R} \) is a trainable parameter. This is because, Pool can be implemented by a feedforward network with \( \text{ReLU}_2 \) activation function, since \( x^2 = \text{ReLU}_2(x) + \text{ReLU}_2(-x) \) (where \( x \in \mathbb{R} \)) and (Kidger & Lyons, 2020, Lemma 4.3) shows that the multiplication map \( \mathbb{R}^2 \ni (x_1, x_2) \mapsto x_1 x_2 \) can be exactly implemented by a neural network with one hidden layer and with activation function \( x \mapsto x^2 \). Therefore, any \( f \in \text{NN}^{\text{ReLU+Pool}} \) there are \( f_1, \ldots, f_t \in \text{NN}^{\text{ReLU}_2} \cup \text{NN}^{\text{ReLU}} \) representing \( f \) via

\[
f = f_t \circ \cdots \circ f_1.
\]

We note that networks with activation function in \( \{\text{ReLU}_r\}_{r \in \mathbb{R}} \) have recently rigorous study in Gribonval Rémi et al. (2021) and are related to the the constructive approximation theory of splines where \( \text{ReLU}_r \) are known as truncated powers (see (DeVore & Lorentz, 1993, Chapter 5, Equation (1.1))). We also mention that Theorem 3 is related to recent deep learning research considering the approximation of a function or probability measure’s support. The former case is considered by Kratsios & Zamanlooy (2022), where the authors consider an exotic neural network architecture specialized in the approximation of piecewise continuous functions in a certain sense. In the latter case, Puthawala et al. (2022) use a GAN-like architecture to approximate probability distributions supported on a low-dimensional manifold by approximating their manifold and the density thereon using a specific neural network architecture. In contrast, our results compare the approximation capabilities of feedforward networks built using different activation functions.
Organization of Paper

This paper is organized as follows. Section 2 reviews the necessary deep learning terminology, measure theoretic, and topological background needed in the formulation of our main result. Section 3 is devoted to the construction of the “separating topology” \( \tau \), the examination of its properties so as to ablate the meaning of the qualitative gap in Theorem 1, and then our main result is formally stated. The proofs of all supporting and technical lemmas are relegated to the paper’s appendix.

2 Preliminaries

We use \( \mathbb{N}_+ \) to denote the set of positive integers, fix \( d, D \in \mathbb{N}_+ \), and let \( \| \cdot \| \) denote Euclidean distance on \( \mathbb{R}^D \).

To simplify the analysis, we emphasize that \( d \) will always be assumed to be a power of 2; i.e. \( d = 2^d' \) where \( d' \in \mathbb{N}_+ \).

2.1 Deep Feedforward Networks

Originally introduced by McCulloch & Pitts (1943) as a prototypical model for artificial neural computation, deep feedforward networks have since lead to computational breakthroughs across various areas from biomedical imaging Ronneberger et al. (2015) to quantitative finance Buehler et al. (2019); Jaimungal (2022). Though deep learning tools has become pedestrian in most contemporary scientific computational endeavors, the mathematical foundations of deep learning are still in their early stages.

Therefore, in this paper, we study the approximation-theoretic properties of what is arguably the most basic deep learning model; namely, the feedforward (neural) network. These are models which iteratively process inputs in \( \mathbb{R}^d \) by repeatedly applying affine transformations (as in linear regression) and simple component-wise non-linearity called activation functions, until an output in \( \mathbb{R}^D \) is eventually produced.

Our discussion naturally begins with the formal definition of the class of deep feedforward neural networks defined by a (non-empty) family of (continuous) activation functions \( \Sigma \subseteq C(\mathbb{R}) \). In the case where \( \Sigma = \{ \sigma \} \) is a singleton, one recovers the classical definition of a feedforward network studied in Cybenko (1989); Hornik et al. (1989); Leshno et al. (1993); Yarotsky (2017b); Kidger & Lyons (2020) and when \( \Sigma = \{ \sigma \}_{r \in \mathbb{R}} \) and the map \( (r, x) \mapsto \sigma_r(x) \) is Lebesgue a.e. differentiable then one obtains so-called trainable activation functions as considered in Cheridito et al. (2021a); Kratsios et al. (2022); Acciaio et al. (2022) of which the \( \text{PReLU} \) activation function of He et al. (2015) is prototypical. More broadly, neural networks build using families of activation functions \( \Sigma \) exhibiting sub-exponential approximation rates have also recently become increasingly well-studied; e.g. Yarotsky & Zhevnerchuk (2020b); Jiao et al. (2021); Yarotsky (2021); Beknazaryan (2021).

Consider the bilinear pooling layer, from computer vision (Lin et al., 2015; Kim et al., 2016; Fang et al., 2019), given for any even \( n \in \mathbb{N}_+ \) and \( x \in \mathbb{R}^n \) as

\[
\text{Pool}(x) \equiv (x_i x_{n/2+i})_{i=1}^n.
\]

Alternatively, Pool can be thought of as a masking layer with non-binary values, similar to the bi-linear masking layers or bi-linear attention layers used in the computer-vision literature Fang et al. (2019); Lin et al. (2015) or in the low-rank learning literature Kim et al. (2016), or as the Hadamard product of the first \( n/2 \) components of a vector in \( \mathbb{R}^n \) with the last \( n \) components.

Fix a depth \( J, d, D \in \mathbb{N}_+ \). A function \( \hat{f} : \mathbb{R}^d \to \mathbb{R}^D \) is said to be a deep feedforward network with (bilinear) pooling if for every \( j = 0, \ldots, J - 1 \) there are Boolean pooling parameters \( \alpha^{(j)} \in \{0, 1\} \), \( d_{j,2} \times d_{j+1,1} \)-dimensional matrices \( A^{(j)} \) with \( d_{j+1,1}/2 = d_{j,2} \) if \( d_{j,2} \) is even and if \( \alpha = 1 \) and \( d_{j+1,1} = d_{j,2} \) otherwise which are called weights, \( b^{(j)} \in \mathbb{R}^{d_{j,2}} \).
and a $c \in \mathbb{R}^d$ called biases, and activation functions $\sigma^{(j,i)} \in \Sigma$ such that $\hat{f}$ admits the iterative representation

$$
\hat{f}(x) \overset{\text{def.}}{=} x^{(0)} + c
$$

$$
x^{(j+1)} \overset{\text{def.}}{=} \begin{cases} 
\text{Pool}(\hat{\phi}^{(j+1)}) & : \alpha^{(j)} = 1 \text{ and } d_{j+1} \text{ is even} \\
\hat{\phi}^{(j+1)} & : \text{else}
\end{cases}
$$

for $j = 0, \ldots, J-1$

(1)

We denote by $\text{NN}^{\Sigma+\text{Pool}}$ the set of all deep feedforward networks with pooling and activation functions belonging to $\Sigma$. If, in the above notation, $\hat{f}$ is such that $x^{(j+1)} = \hat{\phi}^{(j+1)}$ then, we say that $\hat{f}$ is a deep feedforward network (without pooling). The collection of all deep feedforward networks (without pooling) is denoted by $\text{NN}^{\Sigma}$ and activation functions belonging to $\Sigma$.

In either case, if $\Sigma$ consists only of a single activation function $\sigma$ then, we use $\text{NN}^{\Sigma+\text{Pool}}$ to denote $\text{NN}^{\sigma+\text{Pool}}$. Similarly, if $\Sigma = \{ \sigma \}$ then we set $\text{NN}^{\sigma} \overset{\text{def.}}{=} \text{NN}^{\Sigma}$. Let us consider some examples of activation functions.

**Example 1** (Piecewise Linear Networks with at-least two distinct pieces). An activation function $\sigma \in \mathcal{C}(\mathbb{R})$ is called piecewise linear with at-least 2 distinct pieces if: there exist $-\infty = t_0 < t_1 < \cdots < t_p < t_{p+1} = \infty$ and some $m_1, \ldots, m_p, b_1, \ldots, b_p \in \mathbb{R}$ for which

(i) $\sigma(x) = m_i x + b_i$ for every $t \in (t_i, t_{i+1})$ for each $i = 0, \ldots, p$.

(ii) There exist some $i \in \{1, \ldots, p\}$ for which $\sigma'(t_i)$ is undefined.

The prototypical example of such an activation function is $\text{ReLU}(x) \overset{\text{def.}}{=} \max\{0, x\}$.

**Example 2** (Deep Feedforward Networks with “Adaptive” Analytic Activation Functions ($\text{NN}^{\omega}$)). Let $\mathcal{C}_0^\omega(\mathbb{R})$ denote the set of a analytic maps from $\mathbb{R}$ to itself. We set $\text{NN}^{\omega} \overset{\text{def.}}{=} \text{NN}^{\omega}(\mathbb{R})$ and we use $\text{NN}^{\omega+\text{Pool}} \overset{\text{def.}}{=} \text{NN}^{\omega}(\mathbb{R})+\text{Pool}$

**Remark 1** (Scope of $\text{NN}^{\omega+\text{Pool}}$). The class $\text{NN}^{\omega}$ is rather broad and it contains all “classical” feedforward networks with the following common activation functions: the classical tanh, logistic, and sigmoid $\sigma_{\text{sigmoid}}(x) \overset{\text{def.}}{=} \frac{e^x}{1+e^x}$ activation functions, the GeLU activation function $\sigma_{\text{GeLU}}(x) \overset{\text{def.}}{=} \frac{1}{2}x(1 + \text{erf}\left(\frac{x}{2}\right))$ of Hendrycks & Gimpel (2016), $\sigma_{\text{Softplus}}(x) \overset{\text{def.}}{=} \ln(1 + e^x)$ of Glorot et al. (2011), $\sigma_{\text{Swish}}(x) \overset{\text{def.}}{=} \frac{x}{1 + e^x}$ of Ramachandran et al. (2018), any polynomial activation function (as used in neural ODEs Cuchiero et al. (2020) literature).

2.2 Measure Theory

Following (Schwartz, 1966, Chapter 1), we call Borel measurable function $f : \mathbb{R}^d \to \mathbb{R}^D$ is called locally integrable if, on each compact subset $K \subset \mathbb{R}^d$ the Lebesgue integral $\int_{x \in K} \|f(x)\| \, dx$ is finite. Let $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ denote the set of locally integrable functions from $\mathbb{R}^d$ to $\mathbb{R}^D$; with equivalence relation $f \sim g$ if and only if $f$ and $g$ differ only on a set of Lebesgue measure 0. The set $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ is made into a complete metric space by equipping it with the distance function $d_{L^1_{\text{loc}}}$ defined on any two $f, g \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ by

$$
d_{L^1_{\text{loc}}}(f, g) \overset{\text{def.}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{\|x\| \leq n} \|f(x) - g(x)\| \, dx}{1 + \int_{\|x\| \leq n} \|f(x) - g(x)\| \, dx}.
$$

The subset of $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ consisting of all integrable “functions”, i.e. all $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for which the integral $\int_{x \in \mathbb{R}^d} \|f(x)\| \, dx$ is finite, is denoted by $L^1(\mathbb{R}^d, \mathbb{R}^D)$. The set $L^1(\mathbb{R}^d, \mathbb{R}^D)$ is made into a Banach space, called the Bochner-Lebesgue space, by equipping it with the norm $\|f\|_{L^1} \overset{\text{def.}}{=} \int_{x \in \mathbb{R}^d} \|f(x)\| \, dx$. 

2.3 Point-Set Topology

In most of analysis one uses the language of metric spaces, i.e.: an (abstract) set of points $X$ together with a distance function $d : X^2 \to [0, \infty)$ satisfying certain axioms (see (Heinonen, 2001)), to the similarity of dissimilarity between different mathematical objects. However, not all notions of similarity can be described by a metric structure and this is in particular true for several very finer notions of similarity playing central roles in functional analysis (see Narayanaswami & Saxon (1986)).

In such situations, one instead turns to the notion of a topology to qualify closeness of two objects without relying on the quantitative notion of distance defined though by a metric. Briefly, a topology $\tau_X$ on a set $X$ is a collection of subsets of $X$ declared as being “open”; we require only that $\tau_X$ satisfy certain axioms reminiscent of the familiar open neighborhoods build using balls in metric space theory. Namely, $\tau_X$ contains the empty set and the “total” set $X$, the union of elements in $\tau_X$ are again a member of $\tau_X$, and the countable intersection of sets in $\tau_X$ are again a set in $\tau$. A topological space is a pair $(X, \tau_X)$ of a set $X$ and a topology $\tau_X$ on $X$. If clear from the context, we denote $(X, \tau_X)$ by $X$.

**Example 3** (Metric Topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$). The metric topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$, which exists, is the smallest topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ containing all the open balls

$$B_{L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)}(f, \varepsilon) \defeq \{ g \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D) : d_{L^1_{\text{loc}}}(f, g) < \varepsilon \},$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ and $\varepsilon > 0$. We denote this topology by $\tau_{\text{loc}}$.

A topology on the subset $L^1(\mathbb{R}^d, \mathbb{R}^D)$ of $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ can always be defined by restricting $\tau_{\text{loc}}$ as follows.

**Example 4** (Subspace Topology on $L^1(\mathbb{R}^d, \mathbb{R}^D)$). The subspace topology on $L^1(\mathbb{R}^d, \mathbb{R}^D)$, relative to the metric topology on $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$, is the collection $\{ U \cap L^1(\mathbb{R}^d, \mathbb{R}^D) : U \in \tau_{\text{loc}} \}$.

A topology $\tau_X$ on $X$ is said to be strictly stronger than another topology $\tau_X$ on $X$ if $\tau_X \subset \tau_X$. The key relation between $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ and $L^1(\mathbb{R}^d, \mathbb{R}^D)$ is that even if former is strictly larger as a set, the topology on the latter induced by the norm $\| \cdot \|_{L^1}$ is strictly stronger than $\tau_{\text{loc}}$. The norm topology on $L^1(\mathbb{R}^d, \mathbb{R}^D)$ is defined as follows.

**Example 5** (Norm Topology on $L^1(\mathbb{R}^d, \mathbb{R}^D)$). The norm topology on $L^1(\mathbb{R}^d, \mathbb{R}^D)$, which exists, is the smallest topology on $L^1(\mathbb{R}^d, \mathbb{R}^D)$ which contains all the open balls

$$B_{L^1(\mathbb{R}^d, \mathbb{R}^D)}(f, \varepsilon) \defeq \{ g \in L^1(\mathbb{R}^d, \mathbb{R}^D) : \| f - g \|_{L^1} < \varepsilon \},$$

where $f \in L^1(\mathbb{R}^d, \mathbb{R}^D)$ and $\varepsilon > 0$. We denote this topology by $\tau_{\text{norm}}$.

The qualitative statement being put forth by a universal approximation theorem (e.g. Leshno et al. (1993); Petrushev (1999); Yarotsky (2017a); Suzuki (2019); Grigoryeva & Ortega (2019); Heinecke et al. (2020); Kidger & Lyons (2020); Zhou (2020); Kratsios & Bilokopytov (2020); Siegel & Xu (2020); Kratsios & Hyndman (2021); Kratsios et al. (2022); Yarotsky (2022)) is a statement about the topological genericness of a machine learning model, such as a neural network model, in specific sets topological “function” spaces. Topological genericness is called denseness, and we say that a subset $F \subseteq X$ is dense with respect to a topology $\tau_X$ on $X$ if: for every open subset $U \in \tau_X$ there exists an element $f \in F$ which also belongs to $U$.

Related is the notion of convergence of a sequence in a general topological space. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$. The sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $x \in X$ with respect to the topology $\tau_X$ if for every open subset $U \in \tau_X$ containing $x$ there is some $n \in \mathbb{N}_+$ such that $\{x_k\}_{k \geq n}$ belongs to $U$. Two key observations in our analysis are the following. A subset $A \subseteq X$ is dense with respect to the topology $\tau_X$ if for every $x \in X$ one can form a sequence of elements $\{x_n\}_{n=1}^{\infty}$ of $A$ which “approximate a”; meaning that $\{x_n\}_{n=1}^{\infty}$ converges to $x$ with respect to $\tau_X$. Conversely, if no such sequence can be formed for some $x \in X$, then $A$ is not dense in $X$ with respect to $\tau_X$. 
2.4 Limit-Banach Spaces (LB-Spaces)

Our construction will exploit a special class of topological vector spaces, i.e. vector spaces wherein addition and scalar multiplication are continuous operators, formed by inductively gluing together ascending sequences of Banach spaces. Specifically, a topological vector space $X$ is a limit-Banach space, nearly always referred to as an LB-space in the literature, if first, one can exhibit sequence of strictly nested Banach spaces $\{X_n\}_{n=1}^{\infty}$ (i.e. each $X_n$ is a proper subspace of $X_{n+1}$) such that $X = \bigcup_{n=1}^{\infty} X_n$.

Then, the topology on $X$ must be smallest topology containing every convex subset $B \subseteq X$ for which $kb \in B$ whenever $k \in [-1, 1]$ and $b \in B$, and $B \cap X_n$ contained in some ball about the origin in $X_n$ for each $n \in \mathbb{N}$.

Conversely, given a sequence of strictly nested Banach spaces $\{X_n\}_{n=1}^{\infty}$ one can always form an “optimal” LB-space as follows. Define $X \overset{\text{def.}}{=} \bigcup_{n=1}^{\infty} X_n$ and equip $X$ with the finest topology making $X$ into an LB-space and such that, for every $n \in \mathbb{N}_+$, the inclusion $X_n \subseteq X$ is continuous. Indeed, as discussed in (Osborne, 2014, Section 3.8), such a topology always exists\(^1\). We will henceforth refer to $X$ as the LB-space glued together from $\{X_n\}_{n=1}^{\infty}$.

3 The Separating Topology $\tau$

We now construct the separating topology $\tau$ of Theorem 1 the set $L^1_{\mu, \text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$, in three steps. However, before beginning our construction, we fix an arbitrary “good a.e. partition” of $\mathbb{R}^d$. As we will see shortly, the construction of the separating topology $\tau$ is independent of the choice of “good a.e. partition” of $\mathbb{R}^d$; and thus, the construction is natural (in the precise algebraic sense describe in Proposition 1, below). However, to establish this surprising algebraic property of the separating topology $\tau$, it is more convenient to describe the construction (for any arbitrary choice of $\{K_n\}_{n=1}^{\infty}$) once and for all.

**Definition 1** (Good a.e. partition of $\mathbb{R}^d$). A collection $\{K_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}^d$ is called a good a.e. partition if it satisfies the following conditions:

(i) The set $\mathbb{R}^d - \bigcup_{n=1}^{\infty} K_n$ has Lebesgue measure 0,

(ii) For every $n \in \mathbb{N}_+$, $K_n$ has positive Lebesgue measure,

(iii) For each $n, m \in \mathbb{N}_+$, if $n \neq m$ then $K_n \cap K_m$ has Lebesgue measure 0.

For instance, since our construction will be shown to be independent of our choice of a good a.e. partition of $\mathbb{R}^d$ made when constructing $\tau$. Once we show this, we may, without loss of generality, henceforth only consider the following partition of $\mathbb{R}^d$. This partition is illustrated in Figure 1.

---

\(^1\)In the language of category theory, $X$ is the colimit of the inductive system $(\{X_n\}_{n=1}^{\infty}, \subseteq)$ in the category of locally-convex topological vector spaces with bounded linear maps as morphisms.
The next section outlines the main steps in the theorem’s derivation, with the details being relegated to our paper’s appendix.
4 Outline of the Proof of Theorem 1

To better understand Theorem 1, we overview the principal steps undertaken in its derivation. We begin by establishing the universality of $\text{NN}^{\text{ReLU}+\text{Pool}}$ for the separating topology, as guaranteed by Theorem 1 (i). Then, we show the non-universality of $\text{NN}^{\omega+\text{Pool}}$ for the separating topology, given in Theorem 1 (ii). Other curious approximation-theoretic properties of the separating topology are discussed along the way, in order to gain a fuller picture of our main result; such as the failure of the set of polynomial functions to be dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for $\tau$.

4.1 Establishing Theorem 1 (i): The universality of $\text{NN}^{\text{ReLU}+\text{Pool}}$ in the separating topology

In order to establish Theorem 1 (i), we must first understand how density in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$, for the metric topology interacts with density in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology. The next lemma accomplishes precisely this, by showing how dense subsets of $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the metric topology can be used to construct dense subsets of $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology. This construction happens in two phases. First, each “function” in the original dense subset is localized so that it is essentially supported on a part $K_n$ in (any) good a.e. partition $\{K_n\}_{n=1}^N$ of $\mathbb{R}^d$. Then, each of these localized “functions” are then pieced back together to form a new “function” which is essentially supported on the compact subset $\bigcup_{i=1}^N K_n$.

Let $\text{Lip}_\omega(\mathbb{R}^d, \mathbb{R}^D)$ denote the set of “compact support” Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}^D$; i.e. $f$ is Lipschitz and $\text{ess-sup} \ f$ is a compact subset of $\mathbb{R}^d$. The first key observation in the proof of Theorem 1 (i) is that, $\text{Lip}_\omega(\mathbb{R}^d, \mathbb{R}^D)$ is dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology $\tau$.

**Lemma 1** (Density of compactly-supported Lipschitz functions in the separating topology $\tau$). The set $\text{Lip}_\omega(\mathbb{R}^d, \mathbb{R}^D)$ is dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology $\tau$.

The second key observation, also contained in the next lemma, is a sufficient condition for approximating a “compact support” Lipschitz function with respect to the separating topology $\tau$. Briefly, the approximation of such a function in $\tau$ involves the simultaneous approximation of its outputs as well as its essential support.

**Lemma 2** (Approximation of compactly-supported Lipschitz functions in the separating topology $\tau$). Let $\{K_n\}_{n=1}^N$ be the cubic-annuli of Example 6. If $\{f_n\}_{n=1}^N$ is a sequence in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for which there is an $n_f \in \mathbb{N}_+$ with

$$
\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} = 0 \text{ and } \text{ess-sup} \ f \cup \bigcup_{n=1}^N \text{ess-sup} \ f_n \subseteq [-n_f - 1, n_f + 1]^d,
$$

then $\{f_n\}_{n=1}^N$ converges to $f$ in the separating topology $\tau$.

Together, Lemmata 2 and 1 provide a sufficient condition for universality with respect to the separating topology. Furthermore the condition is in a sense quantitative. We say in a sense, since the topology $\tau$ is non-metrizable (see Narayanaswami & Saxon, 1986, Corollary 3) and consequently $\tau$ is non-metrizable); thus there is no metric describing the approximation of a function in $\tau$. I.e. no genuine quantitative statement is possible$^4$.

**Lemma 3** (Approximation of a compactly essentially-supported functions in the separating topology $\tau$). Let $\mathcal{F} \subseteq L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$. If for every $f \in \text{Lip}_\omega(\mathbb{R}^d, \mathbb{R}^D)$ there exists a sequence $\{f_n\}_{n=1}^\infty$ in $\mathcal{F}$ satisfying the condition equation 2 then, $\mathcal{F}$ is dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology $\tau$.

By Lemma 3, it therefore remains to construct a subset of networks in $\text{NN}^{\text{ReLU}+\text{Pool}}$ which can approximate any compactly supported Lipschitz function in the $L^1$-norm and simultaneously correctly identify its essential support via the cubic annuli partition of $\mathbb{R}^d$.

---

$^4$Another example of a non-metric universal approximation theorem in the deep learning literature is the universal classification result of Kratsios & Bilokopytov, 2020, Corollary 3.12).
Accordingly, our next lemma is an extension of the main theorem of Shen et al. (2022), which gives an estimate on the 2-dimensional case is illustrated by Figure 2, which shows the target function \( f \) and an approximation of it by a ReLU network with bi-linear pooling (illustrated in blue). The value output of by function and network is represented by the vividness (alpha) of the each respective color. The Figure illustrates the main points of the next lemma; namely, if the target function is compactly supported then its output can be closely approximated by a ReLU network which also simultaneously correctly identifies which number of parts the target function is supported in (possibly up to one extra part if \( f \) is supported near any part’s boundary).

Accordingly, our next lemma is an extension of the main theorem of Shen et al. (2022), which gives an estimate on the width and depth of the smallest deep ReLU network approximating a Lipschitz map from a compact subset \( X \) of \( \mathbb{R}^d \) to \( \mathbb{R} \). For every “depth parameter” \( L \in \mathbb{N}_+ \) and “width parameter” \( N \in \mathbb{N}_+ \) there exists a \( \hat{f} \in \mathbb{N}^{\text{ReLU}} \) satisfying the uniform estimate

\[
\max_{x \in X} \| f(x) - \hat{f}(x) \| \lesssim \log_2(\text{cap}(X)) \text{diam}(X) \text{ Lip}(f) \frac{D^{3/2}d^{1/2}}{N^{2/d}L^{2/d} \log_2(N + 2)^{1/d}},
\]

where \( \lesssim \) hides an absolute positive constant independent of \( X, d, D, \) and \( f \). Furthermore, \( \hat{f} \) satisfies

1. **Width**: \( \hat{f}' \)’s width is at-most \( d(D + 1) + 3d + 3 \max\{d[N^{1/d}], N + 2\} \)
2. **Depth**: \( \hat{f}' \)’s depth is at-most \( D(11L + 2d + 19) \).

In order to apply Lemma 4, we need our approximating model to have support which “matches” the support of the target function \( f \in L^1_1(\mathbb{R}^d, \mathbb{R}^D) \approx \bigcup_{n \in \mathbb{N}_+} L^1_1(\mathbb{R}^d, \mathbb{R}^D) \) being approximated. The next lemma describes how, given a ReLU network how one can build a new ReLU network with one pooling layer at its output, which coincides with the original network on an arbitrarily cubic-annuli (as in Example 6) and vanishes straightaway outsides the correct number of cubic-annuli (+1).

**Lemma 5** (Adjusting a ReLU network to have support on the union of the first \( n + 1 \) cubic annuli).

Let \( \log_2(d) \in \mathbb{N}_+ \) and \( \hat{f} \in \mathbb{N}^{\text{ReLU}} \) have depth \( d_f \) and width \( w_f \). For every \( n \in \mathbb{N}_+ \) and each \( 0 < \delta < 1 \), there exists a \( \hat{f}_{\text{pool}} \in \mathbb{N}^{\text{ReLU+Pool}} \) with width \( \max\{d(d - 1) + 2, D\} + w_f \) and depth \( 2 + 3d + d_f \) satisfying:

(i) **Implementation on the cube**: For each \( x \in [-n, n]^d \) it holds that \( \hat{f}(x) = \hat{f}_{\text{pool}}(x) \).

(ii) **Controlled Support**: \( \text{ess-supp}(\hat{f}) \subseteq \left[ -\sqrt{2d} \epsilon + n^d, \sqrt{2d} \epsilon + n^d \right]^d \).

(i) **Control of Error near the Boundary**: \( \| \hat{f} - \hat{f}_{\text{pool}} \|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} < \epsilon \).

Figure 2: Approximation of a compactly supported Lipschitz function by a ReLU network with bi-linear pooling.
Lemma 1, 2, and 3 imply that $\mathbb{N}^{\text{ReLU+Pool}}$ is dense in $l_1^1(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology $\tau$ only if $\mathbb{N}^{\text{ReLU+Pool}}$ has a subset which can approximate any essentially compactly-supported Lipschitz function while having almost correct support (as detected by the cubic-annuli partition) as formalized by condition 2. Since Lemma 5 implies that such a subset of networks in $\mathbb{N}^{\text{ReLU+Pool}}$ exists then, Theorem 1 (i) follows.

Proof of Theorem 1 (i). The result for PW-Lin = ReLU is a direct consequence of Lemma 4 and 5 applied to Lemma 3. The result for general piecewise linear activation functions with at-least 2 parts follows from the ReLU case by (Yarotsky, 2017b, Proposition 1). This is because (Yarotsky, 2017b, Proposition 1) states that any network in $\mathbb{N}^{\text{ReLU-Lin}}$ can be implemented by a network in $\mathbb{N}^{\text{ReLU+Pool}}$.

We are now equally in a position to prove the first claim in theorem Theorem 3.

Proof of Theorem 3. Since $f$ is compactly essentially-supported, by Lemma 4 there is an $\hat{f}^{\varepsilon_n/2} \in \mathbb{N}^{\text{ReLU}}$ satisfying

$$
\max_{x \in \text{ess-sup}(f)} \|f(x) - \hat{f}^{\varepsilon_n/2}(x)\| < \frac{\varepsilon_n}{2},
$$

(3)

with width $w_{\hat{f}^{\varepsilon_n/2}}$ at-most $d(D + 1) + 3^d + 3 \max\{d[N^{1/d}], N + 1\}$ and depth $d_{\hat{f}^{\varepsilon_n/2}}$ equal to

$$
d_{\hat{f}^{\varepsilon_n/2}} \overset{\text{def.}}{=} \frac{\varepsilon_n^{-d/2}}{N \log_3 (N + 2)^{1/2}} \left(2 \log_2(\text{cap}(\text{ess-sup}(f))) \text{diam}(\text{ess-sup}(f)) \text{Lip}(f)\right)^d(cD^{3/d}d^d),
$$

(4)

where $c > 0$ is an absolute constant independent of $X, d, D,$ and $f$. Set $n_f \overset{\text{def.}}{=} \min \{n \in \mathbb{N}_+ : \text{ess-sup}(f) \subseteq [-n, n]^d\}$ and apply Lemma 5 to $\hat{f}^{\varepsilon_n/2}$ there exists an $\hat{f}(n) \in \mathbb{N}^{\text{ReLU+Pool}}$ with

$$
\text{ess-sup}(\hat{f}(n)) \subseteq \left[-\frac{d}{2} - d^{-1} \varepsilon_n + n_f^d, \frac{d}{2} - d^{-1} \varepsilon_n + n_f^d\right]^d,
$$

equal to $\hat{f}^{\varepsilon_n/2}$ on $[-n_f, n_f]^d$ and such that

$$
\|\hat{f}(n) - f^\text{pool}\|_{L^1([0, \varepsilon_n])} < \frac{\varepsilon_n}{2}.\text{ Therefore, the estimate in equation 3 and implies that}
$$

$$
\max_{x \in \text{ess-sup}(f)} \|f(x) - \hat{f}(n)(x)\| \leq \max_{x \in \text{ess-sup}(f)} \|f(x) - \hat{f}^{\varepsilon_n/2}(x)\| + \max_{x \in \text{ess-sup}(f)} \|\hat{f}(n)(x) - \hat{f}^{\varepsilon_n/2}(x)\| \leq 2^{-1} \varepsilon_n + 2^{-1} \varepsilon_n = \varepsilon_n.
$$

Similarly, equation 3 implies that

$$
\|f - \hat{f}(n)\|_{L^1} \leq \|f - \hat{f}^{\varepsilon_n/2}\|_{L^1} + \|\hat{f}(n) - \hat{f}^{\varepsilon_n/2}\|_{L^1}
$$

and that both $\hat{f}(n)$ and $f$ are essentially-supported in $[-n_f - 1, n_f + 1]^d$; whence, for each $n \in \mathbb{N}_+$ the condition equation 2 is met. Therefore, Lemma 2 implies that the sequence $\{\hat{f}(n)\}_{n=1}^\infty$ in $\mathbb{N}^{\text{ReLU+Pool}}$ converges to $f$ in the separating topology $\tau$.

It remains to count each of $\hat{f}(n)$’s parameters. By construction, Lemma 5 and the estimate on $w_{\hat{f}^{\varepsilon_n/2}}$ (below equation 3) imply that $\hat{f}(n)$ has width at-most $\max\{d(d - 1) + 2, D\} + d(D + 1) + 3^d + 3 \max\{d[N^{1/d}], N + 1\}$. Similarly, Lemma 5 and equation 4 imply that each $\hat{f}(n)$ has depth equal to

$$
\frac{\varepsilon_n^{-d/2}}{N \log_3 (N + 2)^{1/2}} \left(\log_2(\text{cap}(\text{ess-sup}(f))) \text{diam}(\text{ess-sup}(f)) \text{Lip}(f)\right)^d(c2^dD^{3/d}d^d + 3d + 2d + 2).
$$

Relabeling $C_1 \overset{\text{def.}}{=} c2^dD^{3/d}d^d + 3d$, $C_2 \overset{\text{def.}}{=} 2d + 2$, $C_3 \overset{\text{def.}}{=} \max\{d(d - 1) + 2, D\}$, $C_4 \overset{\text{def.}}{=} d(D + 1) + 3^d + 3$, yields the first conclusion. \qed
4.2 Establishing Theorem 1 (ii): The lack of university of $\text{NN}^{\omega+\text{Pool}}$ with respect to the separating topology

The main step in showing that $\text{NN}^\sigma$ fails to be dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology is the following necessary condition for a sequence $\{f_n\}_{n=1}^\infty$ in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ to convergence to some essentially compactly supported $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ therein with respect to $\tau$.

**Proposition 4** (Necessary condition for convergence in the separating topology $\tau$). A sequence $\{f_k\}_{k \in \mathbb{N}^+}$ in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ converges to some $f \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ with respect to the separating topology $\tau$, only if all but a finite number of $f_k$ are in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$.

Together, Proposition 4 and the fact that if any analytic function is 0 on a non-empty open subset of $\mathbb{R}^d$ then it must be identically 0 everywhere on $\mathbb{R}^d$ (see (Griffiths & Harris, 1994, page 1)) imply that no analytic function can converge to an essentially compactly supported “function” in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ with respect to the separating topology.

**Lemma 6** (Families of analytic functions cannot be dense with respect to the separating topology $\tau$). If $\mathcal{F}$ is a set of analytic functions from $\mathbb{R}^d$ to $\mathbb{R}^D$ then

1. $\mathcal{F}$ is not dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology $\tau$.
2. If $f : \mathbb{R}^d \to \mathbb{R}^D$ is Lipschitz, compact essential-supported, and not identically 0 then, is a sequence $\{\varepsilon_n\}_{n=1}^\infty$ in $(0, \infty)$ converging to 0 such that no $\hat{f} \in \mathcal{F}$ satisfies both Theorem 3 (i) and (iii).

The proof of Theorem 1 (i) is a consequence of Lemma 5 and the observation that any network in $\text{NN}^{\omega+\text{Pool}}$ is an analytic function.

**Proof of Theorem 1 (ii)**. By Lemma 6, the class of analytic functions from $\mathbb{R}^d$ to $\mathbb{R}^D$, denoted by $C^\omega(\mathbb{R}^d, \mathbb{R}^D)$, is not dense in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology. Now, the composition and the addition of analytic functions is again analytic. Since every affine function is analytic and since every activation function $\sigma \in C^\omega(\mathbb{R})$ is by definition analytic then, every $f \in \text{NN}^\omega$ must be analytic. I.e, $\text{NN}^\omega \subseteq C^\omega(\mathbb{R}^d, \mathbb{R}^D)$. Therefore, $\text{NN}^\omega$ cannot be in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology.

The proof of Theorem 2 now also follows from Lemma 6.

**Proof of Theorem 2**. Since every polynomial function is analytic then, the result follows from Lemma 6.

**Proof of Theorem 3 (Continued)**. If $f : \mathbb{R}^d \to \mathbb{R}^D$ is Lipschitz, compactly-supported, and not identically 0 then Lemma 6 and the fact that every $\hat{f} \in \text{NN}^{\omega+\text{Pool}} \cup \mathbb{R}[x_1, \ldots, x_d]$ is an analytic function implies that Theorem 3 (i)-(iii) cannot all hold simultaneously. This completes the proof of Theorem 3.

**Discussion**

Our main result finds that there is a qualitative gap between the approximation capacity of networks deploying piecewise linear activation functions and those utilizing analytic activation functions. This begs the question, “Are ReLU networks always better than networks with analytic activation functions?”

As one may expect, the answer is a mixed “yes and no”. The reason is that is the task is to learn a solution to a PDE (e.g. Han et al. (2018); Beck et al. (2021a; b) physics-informed neural networks Raissi et al. (2019); Shin et al. (2020); Mishra & Molinaro (2021)). Then, the networks should exhibit non-trivial (higher-order) partial derivatives, and the approximation should be in the $C^k$-norm (for some $k > 0$). In such cases, it is known that ReLU networks are less effective than sigmoid or sine networks; see Markidis (2021) or Hornik et al. (1990); Siegel & Xu (2020). This is the “no” part of the answer to the above question.

The “yes” part of the answer to the above question is more delicate. Notice that the $k$-fold anti-derivative of ReLU is the function $\max\{0,t^{k+1}\} \equiv \text{ReLU}_k$ (called a truncated power in (DeVore & Lorentz, 1993, Chapter 5, Equation
(1.1)). Therefore, ReLU_k has non-vanishing k first derivatives on the positive half-line. Thus, networks build with it as an activation function have enough flexibility to approximate all the first k partial derivatives of a k-dimension continuously differentiable functions. However, and more subtle, is that by construction, \( \partial^k \text{ReLU}_k = \text{ReLU} \) therefore, if the target function is k-times continuous differentiable and its \( k^{\text{th}} \)-partial derivatives vanishes outside some compact subset of the input space, then neural networks with ReLU_k activation function should also be able to identify the compact set on which \( f \) has non-zero \( k^{\text{th}} \)-partial derivatives; in the same way as Theorems 1 and 3 showed that neural networks ReLU activation function could identify the support of the “\( \psi^k \) order derivative” of a Lipschitz function.

In other words, the authors expect that a construction similar to \( \tau \) should be possible by replacing the sets \( L_n(\mathbb{R}^d, \mathbb{R}^D) \) with some suitable subset of a Sobolev space \( W^{1,1}(\mathbb{R}^d) \) whose elements are functions with \( \partial^k f = 0 \) outside of \([-n,n]^d\) for each \( i = 1, \ldots, d \). The authors plan to explore this extension in a subsequent work.

5 Appendix: Proof Details

This appendix contains proofs of the lemmas leading up to the derivation of Theorem 1 (i) and (ii), in the paper’s main body as well as a proof of Theorem 3. Thus, this proof contains the bulk of the derivations in our paper.

5.1 Proofs of Propositions Relating to the Separating Topology \( \tau \)'s Properties

Remark 3 (Comment of Background for the Proof of Proposition 1). The following proof is the only proof in this paper which makes used of category-theoretic tools, namely colimits of inductive diagrams/systems. Since these tools are not used anywhere else in the paper and since a proper overview of these tools is beyond the scope of this paper, we refer the interested reader which to (MacLane, 1971, Chapters I: 1-4, II 4, and III 1-3).

In the proof of Proposition 1, we denote the colimit of a direct system \( \{ (X_i)_{i \in I}, \leq, (\mathcal{L}_i, \leq_i) \} \) in the category with locally convex spaces as objects and continuous linear maps as morphisms by \( \text{colimit}_{\text{LCS}} \).

Proof of Proposition 1. For every compact subset \( K \subseteq \mathbb{R}^d \), let \( L^1(K, \mathbb{R}^D) \equiv \{ f \in L^1(\mathbb{R}^d, \mathbb{R}^D) : \text{ess-sup}(f) \subseteq K \} \), and define \( \mathcal{L'} \equiv \{ L^1(K, \mathbb{R}^D) : K \subseteq \mathbb{R}^d \text{ compact} \} \). Define the partial order \( \prec \) on the collection \( \mathcal{L} \) by \( L^1(K, \mathbb{R}^D) \prec L^1(K', \mathbb{R}^D) \) if and only if \( \mu(K' - K) = 0 \), where \( K, K' \subseteq \mathbb{R}^d \) are compact. In particular, \((\mathcal{L'}, \prec, \leq)\) defines an inductive diagram/system (see (MacLane, 1971, page 67) for a definition).

Fix a good a.e. partition of \( \mathbb{R}^d \). By the Heine-Borel Theorem and the compactness of \( K \subseteq \mathbb{R}^d \), the set \( K \) is closed and bounded. Since \( \{ K_n \}_{n=1}^\infty \) is a good a.e. partition of \( \mathbb{R}^d \) then \( \mu(\mathbb{R}^d - \bigcup_{n=1}^\infty K_n) = 0 \) and therefore, there must exist some \( N \in \mathbb{N} \) for which \( \mu(\mathbb{R}^d - \bigcup_{n=1}^N K_n) = 0 \); i.e. \( L^1(K, \mathbb{R}^D) \prec L^1(\bigcup_{n=1}^N K_n, \mathbb{R}^D) \). Thus, \( \{ L^1(K_n, \mathbb{R}^D) \}_{n=1}^\infty \) is cofinal (see (MacLane, 1971, Definition 217)); whence (MacLane, 1971, Theorem 1 on page 217) implies that

\[
L^1_1(\mathbb{R}^d, \mathbb{R}^D) \equiv \text{colimit}_{\text{LCS}} \{ L^1_1(\bigcup_{n=1}^N K_n, \mathbb{R}^D), \leq \} = \text{colimit}_{\mathcal{L}} \{ \mathcal{L'}, \prec, \leq \}. \tag{5}
\]

Since the right-hand inequality holds, independently of and for every good a.e. partition of \( \mathbb{R}^d \) then, the topology \( \tau_c \) is independent of the chosen good a.e. partition of \( \mathbb{R}^d \) used to construct it. Since \( \tau \) is defined to be the topology with subbase \( \tau \cup \tau_{\text{norm}} \cup \tau_{\text{loc}} \) and since the definition of \( \tau_{\text{norm}} \) and of \( \tau_{\text{loc}} \) do not depend on any good a.e. partition of \( \mathbb{R}^d \) then, \( \tau \) is independent of the good a.e. partition of \( \mathbb{R}^d \) used to construct it.

Proof of Proposition 2. By construction \( \tau \supseteq \tau_{\text{loc}} \); we will demonstrate that the inclusion is strict. We argue by contradiction. Suppose that \( \tau = \tau_{\text{loc}} \) then they’re subspace topologies must agree; in particular, \( \tau \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D) = \tau_{\text{loc}} \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D) \). By equation 7 in the proof of Lemma 7, we have that \( \tau_c = \tau \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D) \) and thus

\[
\tau_c = \tau \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D) = \tau_{\text{loc}} \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D). \tag{6}
\]

Since \( \tau_{\text{loc}} \) is a metrizable by \( d_{\text{loc}} \) (where \( d_{\text{loc}} \) defined in Section 2.2) then so are its subspace topology; in particular the subspace topology \( \tau_{\text{loc}} \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D) \) is metrizable. However, (Narayanaswami & Saxon, 1986, Corollary 3) states that \( \tau_c \) is not metrizable since \( (L^1_1(\mathbb{R}^d, \mathbb{R}^D), \tau_c) \) is an LB-space; whence \( \tau_c \neq \tau \cap L^1_1(\mathbb{R}^d, \mathbb{R}^D) \) and therefore we have a contradiction of equation 6. Hence, the inclusion \( \tau \supseteq \tau_{\text{loc}} \) must be strict.
Proof of Proposition 3. The proof of Proposition 3 is the same as the proof of Proposition 2 (mutatis mutandis) except we argue by contradiction on $\tau_{\text{norm}} = \tau \cap L^1(\mathbb{R}^d, \mathbb{R}^D)$ (instead of $\tau_{\text{loc}} = \tau$).

5.2 Proof of Lemmas Supporting the Derivation of Theorem 1 (i)

The following technical lemma will be of usea. Briefly, the result gives easily verifiable conditions under which one can extend the density of a subset $F$ in a "generic subspace" $X$ of a larger topological space $Y$, to all of $Y$, where the smaller space $X$ is equipped with a stronger topology than $Y$ is. A fortiori, the density of $F$ can even be guaranteed in $Y$ for the smallest topology containing all the open sets in $Y$ and all the open sets in $X$ (for its stronger topology not its subspace topology).

The lemma’s relevance comes from the fact that we have built the separating topology $\tau$ by iteratively gluing larger spaces with weaker topologies to smaller spaces with stronger topologies. Therefore, the lemma is rather useful, even if it is simple, since it reduces the problem of establishing the NNRelU+Pool’s density in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^D)$ for the separating topology to establishing its dense in $L^1(\mathbb{R}^d, \mathbb{R}^D)$ for the LB-space topology $\tau_1$ (which is built from more well-studied tools in the topological vector space literature of the 50s.)

Lemma 7 (Extension results for glued spaces). Let $\tau_X$ and $\tau_Y$ be topologies on $X$ and on $Y$, respectively, and let $\tau_{Y|X}$ denote the subspace topology on $X$ induced by restriction of $\tau_Y$. Denote the smallest topology on $Y$ containing $\tau_X \cup \tau_Y$ by $\tau_Y \vee \tau_Y$. Suppose that:

(i) $\tau_{Y|X} \subseteq \tau_X$,

(ii) $X$ is dense in $(Y, \tau_Y)$.

Then the following hold:

1. Extension of convergence: If $\{x_n\}_{n=1}^\infty$ is a sequence in $X$ converging to some $x \in X$ with respect to $\tau_X$ then, $\{x_n\}_{n=1}^\infty$ converges to $x$ with respect to $\tau_X \vee \tau_Y$,

2. Extension of density: If $\mathcal{D}$ is dense in $(X, \tau_X)$ then, $\mathcal{D}$ is dense in $Y$ for $\tau_X \vee \tau_Y$,

3. Extension of topological partial order: If $\tau_X$ is strictly finer than $\tau_Y$ then $\tau_X \vee \tau_Y$ is strictly finer than $\tau_Y$.

Proof. Since, $\tau_{Y|X} \subseteq \tau_X$, then the intersection of any $U \in \tau_Y$ and $W \in \tau_X$ satisfies $U \cap W \in \tau_X$. Therefore, the set $\tau_X \cup \tau_Y$ is closed under finite intersection. Hence, every $U \in \tau_X \cup \tau_Y$ must be of the form

$$U = \bigcup_{i \in I_X} U_{X,i} \cup \bigcup_{j \in I_Y} U_{Y,j},$$

for some indexing sets $I_X$ and $I_Y$, and some subsets $\{U_{X,i}\}_{i \in I_X} \subseteq \tau_X$ and $\{U_{Y,j}\}_{j \in I_Y} \subseteq \tau_Y$. In particular, equation 7 implies 3.

Let us now show 1. Suppose that $\{x_n\}_{n=1}^\infty$ is a sequence in $X$ converging to some $x \in X$ in $\tau_X$. Then, for every $U \in \tau_X$ containing $x$, there exists some $n_x \in \mathbb{N}_+$ such that $\{x_n\}_{n \geq n_x} \subseteq U$. Now since any $U \in \tau_X \cup \tau_Y$ is of the form equation 7 then either $x \in U_{X,i}$ for some $U_{X,i} \in \tau_X$; in which case there must exist some $n_x \in \mathbb{N}_+$ for which $\{x_n\}_{n \geq n_x} \subseteq U_{X,i} \subseteq U$.

Otherwise, there exists some $U_{Y,j}$ containing $x$; but since $\tau_{Y|X} \subseteq \tau_X$ then, there exists $x \in U_{Y,j} \cap X \in \tau_X$. Thus, there exists some $n_x \in \mathbb{N}_+$ such that $\{x_n\}_{n \geq n_x} \subseteq U_{Y,j} \cap X \subseteq U_{Y,j} \subseteq U$. In either case, $\{x_n\}_{n=1}^\infty$ converges to $x$ in $\tau_X \cup \tau_Y$.

To see 2, suppose that $\mathcal{D}$ is dense in $X$ with respect to $\tau_X$, and $\tau_{Y|X} \subseteq \tau_X$, then $\mathcal{D}$ is dense in $(X, \tau_{Y|X})$. Since density is transitive, and $X$ is dense in $(Y, \tau_Y)$ then $\mathcal{D}$ is dense in $(Y, \tau_Y)$. Assume that $I_X$ and $I_Y$ are non-empty or else there is nothing to show. Since $\mathcal{D}$ is dense in $(Y, \tau_Y)$ and $(X, \tau_X)$ then, there exist $x_1, x_2 \in \mathcal{D}$ such that

$$x_1 \in \bigcup_{i \in I_X} U_{X,i} \text{ and } x_2 \in \bigcup_{j \in I_Y} U_{Y,j}.$$ 

Therefore, $\mathcal{D} \cap \bigcup_{i \in I_X} U_{X,i} \cup \bigcup_{j \in I_Y} U_{Y,j}$ is non-empty. Whence, $\mathcal{D}$ is dense in $(Y, \tau_X \cup \tau_Y)$. 

\qed
We may now return to the proof of our main lemmata.

**Proof of Lemma 2.** By (Dieudonné & Schwartz, 1949, Proposition 2), the topology on $L_{n}(\mathbb{R}^{d}, \mathbb{R}^{D})$ coincides with the subspace topology inherited from restriction of the LB-space topology $\tau_{c}$ on $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$. Therefore, conditions equation 2 imply that $\{f_{n}\}_{n=1}^{\infty}$ converges to $f$ in the LB-space topology $\tau_{c}$ on $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$.

The result now follows upon applying Lemma 7 twice. This is because $\tau_{c}$ is finer than the subspace topology obtained by restricting $\tau_{n}$ norm to $\bigcup_{n=1}^{\infty} L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ and $\bigcup_{n=1}^{\infty} L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ is dense in $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ with respect to the topology $\tau_{n}$. Similarly, the topology $\tau_{n}$ norm on $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ is finer than the subspace topology $\tau_{\text{loc}}$ restricted to the subset $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ of $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$. \(\square\)

**Proof of Lemma 1.** Let $f \in L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ then $f = \sum_{i=1}^{D} f^{(i)} e_{i}$ where $\{e_{i}\}_{i=1}^{D}$ is the standard orthonormal basis of $\mathbb{R}^{d}$ and $f^{(1)}, \ldots, f^{(D)} \in L_{1}(\mathbb{R}^{d}, \mathbb{R})$ (see (Ryan, 2002, Section 2.3)). Since the set of smooth compactly supported “bump” functions $C_{c}(\mathbb{R}^{d})$ is dense in $L_{1}(\mathbb{R}^{d}, \mathbb{R})$ then, for each $i = 1, \ldots, D$ and every $\varepsilon > 0$ there exist $f^{(i, \varepsilon)} \in C_{c}(\mathbb{R}^{d})$ each satisfying $\|f^{(i)} - f^{(i, \varepsilon)}\|_{L_{1}(\mathbb{R}^{d}, \mathbb{R})} < \varepsilon$. Set $\tilde{f}_{\varepsilon} \equiv \sum_{i=1}^{D} f^{(i, \varepsilon)} e_{i}$ and observe that

$$\|f - \sum_{i=1}^{D} f^{(i, \varepsilon)} e_{i}\|_{L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} \leq D \max_{i=1, \ldots, D} \|f^{(i)} - f^{(i, \varepsilon)}\|_{L_{1}(\mathbb{R}^{d}, \mathbb{R})} < \varepsilon;$$

whence, the set $C_{c}(\mathbb{R}^{d}, \mathbb{R}^{D}) \equiv \{f : (\exists f_{1}, \ldots, f_{D} \in C_{c}(\mathbb{R}^{d})) f = \sum_{i=1}^{D} f_{i} e_{i}\}$ is dense in $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ for the norm topology. Consequently, $\text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$ is dense $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ for the norm topology since $C_{c}(\mathbb{R}^{d}, \mathbb{R}^{D}) \subset \text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$.

By Proposition 1, we may without loss of generality assume that $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ is defined using the cubic-annuli $\{K_{n}\}_{n=1}^{\infty}$ of Example 6. Therefore, for any $f \in L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$, there must exist some $n_{f} \in \mathbb{N}_{+}$ with ess-sup f $\subset [-n_{f}, n_{f}]^{d}$. Set $\delta \equiv \sqrt{2}^{-d-1} \left(\max_{x \in [-n_{f}, n_{f}]^{d}} \left\|f^{(i)}(x)\right\|\right)^{-1} \varepsilon + n_{f}^{d} - n_{f}$ and define the piece-wise affine map

$$f^{\text{mask}} \equiv \begin{cases} 1 & : \|x\| \leq n_{f} \\ 0 & : \|x\| \geq n_{f} + \delta \\ \frac{-\|x\|-n_{f}}{\delta} & : n_{f} < \|x\| \leq n_{f} + \delta. \end{cases} \quad (8)$$

Since $f \in L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ then, for every $\varepsilon > 0$, the density of $\text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$ in $L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})$ there exists some $\tilde{f}_{\varepsilon} \in \text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$ for which $\|f - \tilde{f}_{\varepsilon}\|_{L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} < 2^{-1} \varepsilon$. Define $f^{\varepsilon} \equiv f^{\text{mask}, \tilde{f}_{\varepsilon}}$ and note that $f^{\varepsilon} \in \text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$ (since $f^{\varepsilon}$ is supported in $[-n_{f} - 1, n_{f} + 1]^{d}$, $f^{\varepsilon}(x) = f^{\varepsilon}(x)$ for every $x \in [-n_{f}, n_{f}]^{d}$, and therefore the following estimate holds

$$\|f - f^{\varepsilon}\|_{L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} = \|f - f^{\varepsilon}\|_{[-n_{f}, n_{f}]^{d} \cap L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} + \|f - f^{\varepsilon}\|_{[-n_{f}, n_{f} + \delta]^{d} \cap L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} + \|f - f^{\varepsilon}\|_{[-n_{f} - \delta, n_{f}]^{d} \cap L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} + \|f - f^{\varepsilon}\|_{\delta \cap L_{1}(\mathbb{R}^{d}, \mathbb{R}^{D})} \leq 2^{-1} \varepsilon + \max_{x \in [-n_{f}, n_{f}]^{d}} \|f^{\varepsilon}(x)\| \mu([-n_{f}, n_{f} + \delta]^{d} - [-n_{f}, n_{f}]^{d}) + 0 + \varepsilon \leq \varepsilon.$$

For every $n \in \mathbb{N}_{+}$, we may choose a sequence $\{f^{(n)}\}_{n=1}^{\infty}$ in $\text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$ satisfying equation 2. Thus, by Lemma 2, $\{f^{(n)}\}_{n=1}^{\infty}$ converges to $f$ in $\tau$. Therefore, $\text{Lip}(\mathbb{R}^{d}, \mathbb{R}^{D})$ is dense in $L_{\text{loc}}(\mathbb{R}^{d}, \mathbb{R}^{D})$ for the separating topology $\tau$. \(\square\)

**Proof of Lemma 3.** Direct consequence of Lemma 2, Lemma 1, and the transitivity of density. \(\square\)

**Proof of Lemma 4.** By (Brüe et al., 2021, Theorem 4.1), there exists a Lipschitz map $F : \mathbb{R}^{d} \rightarrow \mathbb{R}^{D}$ with $\text{Lip}(F) \leq \tilde{c}(\log \lambda) \text{Lip}(f)$ and where $\tilde{c} > 0$ is an absolute constant and $\lambda$ is the doubling constant of $X$ (note that such a constant exists by (Heinonen, 2001, Chapter 10)). By (Brüe et al., 2021, Proposition 1.7 (i)) we have

$$\text{Lip}(F) \leq \tilde{c} \log_{2} (\text{cap}(X)) \text{Lip}(f). \quad (9)$$
By Jung’s Theorem, there exists an \( \bar{x} \in \mathbb{R}^d \) such that \( X \subseteq B_2 \left( \bar{x}, \frac{\text{diam}(X) \sqrt{d}}{\sqrt{d+2}} \right) \). Since \( \| \cdot \| \leq \| \cdot \|_{\infty} \) (i.e. the component-wise max-norm on \( \mathbb{R}^d \)) then, \( X \subseteq \left[ \bar{x} - \frac{\text{diam}(X)}{\sqrt{d+2}}, \bar{x} + \frac{\text{diam}(X)}{\sqrt{d+2}} \right]^d \). Let \( \bar{1} \equiv (1, \ldots, 1) \in \mathbb{R}^d \) and define the affine map \( A : \mathbb{R}^d \ni x \mapsto \frac{\bar{1}^T}{\text{diam}(X) \sqrt{d+2}} (x - \bar{x} + \frac{\text{diam}(X)}{\sqrt{d+2}})^T \). Note that \( A(X) \subseteq [0, 1]^d \), that \( A \) is a linear isomorphism of \( \mathbb{R}^d \) onto itself, and \( \text{Lip}(A) = \sqrt{d+1} \text{diam}(X) / \sqrt{2} \). Define \( \tilde{f} \equiv F \circ A^{-1} \) and note that, for all \( x \in X \), we have
\[
\tilde{f}(x) = (F \circ A^{-1})(x) \overset{\text{def.}}{=} \tilde{f} \circ A(x). \tag{10}
\]
In particular, \( \text{Lip}(\tilde{f}) \) has Lipschitz constant bounded by
\[
\text{Lip}(\tilde{f}) \leq \bar{c} \log_2(\text{cap}(X)) \text{Lip}(f) \text{diam}(X) d / \sqrt{d} (d+1)^{-1/2}. \tag{11}
\]
Since \( \tilde{f} \) is defined on all of \([0, 1]^d\), then for each \( i = 1, \ldots, d \), we may therefore apply (Shen et al., 2022, Theorem 1.1) to conclude that there exists a \( \tilde{f}^{(i)} \in \text{NN}_{\text{ReLU}} \) satisfying
\[
\max_{x \in [0,1]^d} \left\| f(x) - \tilde{f}^{(i)}(x) \right\| \leq \text{Lip}(F) 131 \sqrt{d} (N^2 L^2 \log_3(N+1))^{-1/d}, \tag{12}
\]
where \( \tilde{f}^{(i)} \) denotes the \( i \)-th component for the vector \( \tilde{f}(x) \). Furthermore, each \( \tilde{f}^{(i)} \) has
\[
\text{width: } 3d^3 \max \{ d | N^{1/d} \}, N+2 \} \text{ and depth: } 11L + 2d + 18. \tag{13}
\]
Let \( \{ e_i \}_{i=1}^d \) denote the standard orthonormal basis of \( \mathbb{R}^d \). Since \( \text{ReLU} \) has the 2-identity property (as defined in (Cheridito et al., 2021b, Definition 4)) then, applying (Cheridito et al., 2021a, Proposition 5), we find that there exists a \( \tilde{f} \in \text{NN}_{\text{ReLU}} \) satisfying
\[
\tilde{f} = \sum_{i=1}^D \tilde{f}^{(i)}, \tag{14}
\]
and which has
\[
\text{width: } d(D+1) + 3d^3 \max \{ d | N^{1/d} \}, N+2 \} \text{ and depth: } D(1+11L+2d + 18). \tag{15}
\]
Incorporating equation 14 into the “component-wise estimates” of equation 12 yields
\[
\max_{x \in [0,1]^d} \left\| f(x) - \tilde{f}(x) \right\| \leq \max_{x \in [0,1]^d} \left\| \sum_{i=1}^D f(x)_i e_i - \sum_{i=1}^D \tilde{f}^{(i)}(x)_i e_i \right\|
\leq \max_{x \in [0,1]^d} \sum_{i=1}^D \left\| f(x)_i - \tilde{f}^{(i)}(x)_i \right\| \| e_i \|
\leq \max_{x \in [0,1]^d} D \max_{i=1, \ldots, d} \left\| f(x)_i - \tilde{f}^{(i)}(x)_i \right\|
\leq \text{Lip}(\tilde{f}) 131D^{3/2} (N^2 L^2 \log_3(N+1))^{-1/d}. \tag{16}
\]
Since the pre-composition of any element of \( \text{NN}_{\text{ReLU}} \) be a linear isomorphism on \( \mathbb{R}^d \) is again an element thereof with the same depth and with then defined \( \hat{f} \overset{\text{def.}}{=} \tilde{f} \circ A \) and note that equation 9, equation 10 and equation 16 imply our final estimate
\[
\max_{x \in X} \left\| f(x) - \hat{f}(x) \right\| \leq \max_{x \in \left[ \bar{x} - \frac{\text{diam}(X)}{\sqrt{d+2}}, \bar{x} + \frac{\text{diam}(X)}{\sqrt{d+2}} \right]^d} \left\| F(x) - \tilde{f}(x) \right\|
\leq \max_{x \in [0,1]^d} \left\| \tilde{f}(x) - \hat{f}(x) \right\|
\leq \bar{c} \log_2(\text{cap}(X)) \text{Lip}(f) \text{diam}(X) d / \sqrt{d} (d+1)^{-1/2} 131D^{3/2} (N^2 L^2 \log_3(N+1))^{-1/d}. \tag{17}
\]
Relabeling the absolute constant yields the conclusion. \( \square \)
Proof of Lemma 5. Set $\delta \overset{\text{def.}}{=} \sqrt[3]{2^{-d} \varepsilon + n^d} - n$. The proof of the result is undertaken in three steps.

Step 1 - Implementing a piecewise linear mask matching $f$’s essential support: By (Kidger & Lyons, 2020, Lemma B.1) there exists a $\hat{f}^\text{mask}: \delta \overset{\text{def.}}{=} NN_{\text{ReLU}}$ with width 2 and depth 2 implementing the following real-valued piecewise linear function defined on $\mathbb{R}$

$$
\hat{f}^\text{mask}: \delta (x) \overset{\text{def.}}{=} \begin{cases} 
1 & : |x| \leq n \\
0 & : |x| \geq n + \delta \\
\frac{|x|}{\delta} + (1 + \frac{n}{\delta}) & : n < |x| < n + \delta.
\end{cases}
$$

(18)

For each $i = 1, \ldots, d$ let $P_i : \mathbb{R}^d \ni x \mapsto x_i \in \mathbb{R}$ be canonical (linear) map projecting vectors in $\mathbb{R}^d$ onto their $i^{th}$ coordinate. Using the projections $P_1, \ldots, P_d$, we may extend $\hat{f}^\text{mask}: \delta$ to the map $\hat{f}^\text{mask}: \delta, i \overset{\text{def.}}{=} \hat{f}^\text{mask}: \delta \odot P_i$. Since the pre-composition of feedforward networks by affine maps (such as the $P_i$) is again a feedforward network with the same depth, then each $\hat{f}^\text{mask}: \delta, i \overset{\text{def.}}{=} NN_{\text{ReLU}}$ and has width $d = \max\{d, 2\}$ and depth 2. By equation 18 and (Cheridito et al., 2021b, Proposition 5), there exists a $\hat{f}^\text{mask}: \delta \overset{\text{def.}}{=} NN_{\text{ReLU}}$ having

width: $d(d - 1) + 2$ and depth: $3d$,  

(19)

and implementing the following piecewise linear map from $\mathbb{R}^d$ to $\mathbb{R}^D$

$$
\hat{f}^\text{mask}: \delta \overset{\text{def.}}{=} (\hat{f}^\text{mask}: \delta, 1, \ldots, \hat{f}^\text{mask}: \delta, d).
$$

(20)

Let $\vec{1} \overset{\text{def.}}{=}(1, \ldots, 1) \in \mathbb{R}^D$ and $\vec{0} \overset{\text{def.}}{=}(0, \ldots, 0) \in \mathbb{R}^D$. The map $NN_{\text{ReLU}} + \text{Pool} \ni \hat{f}^\text{mask}: \delta \overset{\text{def.}}{=} \vec{1} \cdot \text{Pool} \odot \cdots \odot \text{Pool} \odot \hat{f}^\text{mask}: \delta$

takes values in $[0, 1]^D$ and satisfies $\hat{f}^\text{mask}: \delta (x) = \vec{1}$ whenever $\|x\|_\infty \leq n$, $\hat{f}^\text{mask}: \delta (x) = \vec{0}$ whenever $\|x\|_\infty \geq n + 2\delta$. By construction, $\hat{f}^\text{mask}: \delta$ has depth $3d + 1$ and width $\max\{d(d - 1) + 2\}$. By equation 18 and (Cheridito et al., 2021b, Proposition 5), there exists a $\hat{f} \overset{\text{def.}}{=} NN_{\text{ReLU}}$ satisfying

$$
\text{width: } \max\{d(d - 1) + 2, D\} + w_j \text{ and depth: } 2 + 3d + d_j.
$$

(21)

Define $\hat{f}^\text{pool} \overset{\text{def.}}{=} \text{Pool} \odot \hat{f}$. Note that (i) and (ii) hold by construction. It therefore remains to verify (iii).

Step 2 - Assembling the “mask network” with the original network: Since the ReLU activation function satisfies the 2-identity property (as defined in (Cheridito et al., 2021b, Definition 4)) then, we may apply (Cheridito et al., 2021b, Proposition 5) to conclude that there exists a $\tilde{f} \overset{\text{def.}}{=} NN_{\text{ReLU}}$ satisfying

$$
\text{width: } \max\{d(d - 1) + 2\} + w_j \text{ and depth: } 2 + 3d + d_j.
$$

(21)

Define $\tilde{f}^\text{pool} \overset{\text{def.}}{=} \text{Pool} \odot \tilde{f}$. Note that (i) and (ii) hold by construction. It therefore remains to verify (iii).

Step 3 - Approximating the target function while simultaneously controlling the network’s support: We have the following estimate

$$
\left\| \hat{f}^\text{pool} - f \right\|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \leq \left( \left\| \hat{f}^\text{pool} - f \right\|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \right) \left|_{n, \bar{n}} \right| + \left( \left\| \hat{f}^\text{pool} - f \right\|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \right) \left|_{n - \delta, \bar{n} + \delta} \right| + \left( \left\| \hat{f}^\text{pool} - f \right\|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \right) \left|_{n - \delta, \bar{n} + \delta} \right| + \left( \left\| \hat{f}^\text{pool} - f \right\|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \right) \left|_{n - \delta, \bar{n} + \delta} \right| + 0 \\
\leq 0 + \mu \left( \left| n - \delta, \bar{n} + \delta \right| - \left| n, \bar{n} \right| \right) \\
= 2d \left( n + \delta \right) - n^d \\
= \varepsilon.
$$

This completes the proof. \qed
5.3 Proof of Lemmas Supporting the Derivation of Theorem 1 (ii)

Proof of Proposition 4. By Proposition 1, we can without loss of generality assume that \( \{K_n\}_{n=1}^\infty \) is the cubic-annuli of Example 6. By Lemma 7, \( \tau = \tau \cap L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \). Therefore, (Dieudonné & Schwartz, 1949, Proposition 2) implies that the topology \( \tau \) coincides with the Banach space topology on \( L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \) (i.e. defined by restricting the norm \( \| \cdot \|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \) on \( L^1(\mathbb{R}^d, \mathbb{R}^D) \) to the linear subspace \( L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \)). Since Banach spaces are complete then any sequence converging to some \( f \in L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \) must lie entirely in \( L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \).

Proof of Lemma 6. By Proposition 1 we without loss of generality assume that \( \{K_n\}_{n=1}^\infty \) is the cubic-annuli partition of Example 6. Set \( \bar{I} \equiv (1, \ldots, 1) \in \mathbb{R}^D \) and consider the simple function
\[
f(\bar{I}) = I_{[-1,1]^d}(\bar{I}) \cdot \bar{I}.
\] (22)

NB, by the definition of the Lebesgue integral \( f \in L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \). A fortiori, \( f \in L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \) for every \( n \in \mathbb{N}_+ \).

Let \( \{f_n\}_{n=1}^\infty \) be a sequence of analytic functions mapping \( \mathbb{R}^d \) to \( \mathbb{R}^D \). We argue by contradiction. Suppose that \( f_n \) converges to \( f \) in the separating topology \( \tau \) then, by Lemma 6 there must exist \( N_1, N_2 \in \mathbb{N}_+ \) for which: for every \( n \geq N_1 \) the following hold
\[
f_n \in L^1_{N_2}(\mathbb{R}^d, \mathbb{R}^D) \quad \text{and} \quad \| f - f_n \|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} < 1.
\] (23)

Note that equation 23 implies that each \( f_n \) is non-zero; whenever \( n \geq N_2 \). Since each \( f_n \) is analytic, since \( \| \cdot \|_2 : \mathbb{R}^d \rightarrow \mathbb{R} \) is a polynomial function, and since the composition of analytic functions is again analytic then, the map \( F_n : \mathbb{R}^d \ni x \mapsto \| f_n(x) \|_2 \in \mathbb{R} \) is an analytic function. By equation 23, if \( n \geq N_1 \) then \( f_n \in L^1_{N_2}(\mathbb{R}^d, \mathbb{R}^D) \) and therefore \( F_n \) is identically 0 on the non-empty open subset \( \mathbb{R}^d \setminus [N_2, N_2]^2 \). Thus, by (Griffiths & Harris, 1994, page 1) \( F_n \) must be identically on all of \( \mathbb{R}^d \) since \( F_n \) coincides with the 0 function \( \| \mathbb{R}^d \ni x \mapsto 0 = (0, \ldots, 0) \in \mathbb{R}^{DD}, \text{which is itself an analytic function} \) on a non-empty open subset of \( \mathbb{R}^d \). Whence, the definition of \( f \) in equation 22 implies that: for every \( n \geq N_1 \) the following holds
\[
\| f - f_n \|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} \leq \| (f-f_n) I_{[-1,1]^d} \|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} = \| (\bar{I} - \bar{0}) \cdot I_{[-1,1]^d} \|_{L^1(\mathbb{R}^d, \mathbb{R}^D)} = 1\mu([-1,1]^d) = 2^d.
\] (24)

Since \( d > 0 \) then equation 24 and equation 23 cannot holds simultaneously. We have thus arrived at a contradiction; whence \( \{f_n\}_{n=1}^\infty \) cannot converge to \( f \).

The second claim follows similarly. Let \( f : \mathbb{R}^d \rightarrow \mathbb{R}^D \) be Lipschitz and not identically equal to 0. Then, there exists some \( x_0 \in \mathbb{R}^d \) for which \( \| f(x) \| > 0 \). Set \( \varepsilon_n = (2n)^{-1} \| f(x) \| \). Arguing as before, if \( f \in \mathcal{F} \) and satisfies Theorem 3 (iii) then, its analyticity implies that \( f \) is identically 0 since it is zero on the non-empty open set \( \mathbb{R}^d \setminus [-\sqrt{2d} \varepsilon_n + nD, \sqrt{2d} \varepsilon_n + nD], \text{Whence} \)
\[
\| f(x) - f(x) \| > \varepsilon_1;
\]
thus, we again have a contradiction. Therefore, if \( f \) is Lipschitz, essentially compactly supported, and not identically equal to 0 then no analytic function (and in particular those in \( \mathcal{F} \)) from \( \mathbb{R}^d \) to \( \mathbb{R}^D \) can simultaneously satisfy (i) and (iii) for any sequence \( \{\varepsilon_n\}_{n=1}^\infty \) in \((0, \infty)\) converging to 0.

Proof of Theorem 2. Since every polynomial is analytic then the conditions of Lemma 6 are met; whence, the result follows.

Proof of Theorem 3. The first statement follows from Lemmata 4 and 5. Now, the last statement, namely the non-existence of an \( f \in NN^{m+Pool} \cup \mathbb{R}[x_1, \ldots, x_d] \) to simultaneously satisfy (i), (ii), and (iii), follows from the same argument (mutatis mutandis) as in the proof of Lemma 6 but with the map \( f \) of equation 23 replaced by any \( f \in L^1_\text{loc}(\mathbb{R}^d, \mathbb{R}^D) \).

\[\Box\]
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