Nambu-Jona-Lasinio Model in Curved Space-Time

T. Inagaki, T. Muta, and S. D. Odintsov

Department of Physics, Hiroshima University
Higashi-Hiroshima, Hiroshima 724, Japan

ABSTRACT

The phase structure of Nambu-Jona-Lasinio model with N-component fermions in curved space-time is studied in the leading order of the 1/N expansion. The effective potential for composite operator $\bar{\psi}\psi$ is calculated by using the normal coordinate expansion in the Schwinger proper-time method. The existence of the first-order phase transition caused by the change of the space-time curvature is confirmed and the dynamical mass of the fermion is calculated as a simultaneous function of the curvature and the four-fermion coupling constant. The phase diagram in the curvature and the coupling constant is obtained.

*Work supported in part by the Monbusho Grant-in-Aid for Scientific Research (C) No. 04640301 and the Monbusho Grant-in-Aid for Encouragement of Young Scientists No. 92011.
†On leave of absence from Tomsk Pedagogical Institute, 634041 Tomsk, Russia.
It is quite interesting to make investigations of how the phase transition in quantum field theory takes place under the circumstance of the early universe. In particular much interest has been taken in clarifying the mechanism of the spontaneous symmetry breaking in the grand unified theories under the influence of the temperature, density and external gravity\[1\].

In the standard grand unified theories the Higgs field appears as an elementary field and has a Yukawa coupling to fermion fields giving masses to the fermions. As in the technicolor model for electroweak theory it is possible to assume that the Higgs field may be a composite system of some fundamental fermion fields. We shall call the model based on such an assumption generically the composite Higgs model.

As far as we know a very little number of works have been reported in the study of the composite Higgs models under the circumstance of the early universe\[2\]. Therefore we launched our plan to make a systematic study of the composite Higgs models under the circumstance of the early universe. Before going into a direct investigation of the composite Higgs models it may be better to start with a fundamental study of prototype models of the composite Higgs particle with finite temperature, finite density and external gravity. In the present communication we take Nambu-Jona-Lasinio model as one of such prototype models and make a brief report of our work in the study of the phase structure of Nambu-Jona-Lasinio model in external gravity.

The Nambu-Jona-Lasinio model\[3\] in curved space-time is defined by the action,

\[
S = \int d^4x \sqrt{-g} \left[ \bar{\psi} i \gamma^\mu(x) \nabla_\mu \psi + \frac{\lambda}{2N} \left\{ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right\} \right], \tag{1}
\]

where \( g \) is the determinant of the space-time metric \( g_{\mu\nu} \), \( \gamma_\mu(x) \) the Dirac matrix in curved space-time, \( \nabla_\mu \psi \) the covariant derivative of the fermion field \( \psi \), \( N \) the number of the fermion species. We work in the scheme of the \( 1/N \) expansion and perform our calculation in the leading order of the expansion. Our notation is basically in conformity with the (---) convention in the book by Misner, Thorne and Wheeler\[4\]. In practical calculations it is more convenient to introduce auxiliary fields \( \sigma(x) \) and
\[ S = \int d^4x \sqrt{-g} \left[ \bar{\psi} i\gamma^\mu(x) \nabla_\mu \psi - \frac{N}{2\lambda} (\sigma^2 + \pi^2) - \bar{\psi}(\sigma + i\gamma_5 \pi)\psi \right] . \] (2)

In the flat space-time it is well-known that the global abelian chiral symmetry posessed by the Lagrangian (1) and (2) is broken spontaneously if the coupling constant \( \lambda \) exceeds a critical value. Our purpose in the present paper is to see whether this phenomenon is modified under the influence of external gravity. We start with the generating functional given by

\[ Z[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} D\sigma D\pi \exp \left\{ iS + i\bar{\eta}\psi + i\bar{\psi}\eta \right\} , \] (3)

where \( \eta \) and \( \bar{\eta} \) are source functions. Integrating over the fermion fields with \( \eta = \bar{\eta} = 0 \) we find

\[ Z[0, 0] = \int D\sigma D\pi \exp\left\{-iNS_{\text{eff}}\right\} , \] (4)

where we pulled out an obvious factor \( N \) in defining the semiclassical effective action \( S_{\text{eff}} \):

\[ S_{\text{eff}} = \int d^4x \sqrt{-g} \left\{ -\frac{1}{2\lambda} (\sigma^2 + \pi^2) \right\} - i \ln \text{Det} \left\{ i\gamma^\mu(x) \nabla_\mu - (\sigma + i\gamma_5 \pi) \right\} . \] (5)

In the leading order of the \( 1/N \) expansion the effective action \( \Gamma \) (with \( N \) factored out) is just equal to \( S_{\text{eff}} \). Thus we obtain

\[ \Gamma[\sigma, \pi] = S_{\text{eff}}[\sigma, \pi] + O(1/N) . \] (6)

The effective potential (with \( N \) factored out) in the leading order of the \( 1/N \) expansion is then given by

\[ V(\sigma, \pi) = \frac{1}{2\lambda} (\sigma^2 + \pi^2) + i \text{Tr} \ln \langle x | \{ i\gamma^\mu(x) \nabla_\mu - (\sigma + i\gamma_5 \pi) \} | x \rangle . \] (7)

In Eq. (7) the variables \( \sigma \) and \( \pi \) are regarded as constant.

To estimate the second term in the right-hand side of Eq. (7) we adopt the Schwinger proper time method\(^5\). For this purpose we rewrite Eq. (7) in the following form,

\[ V(\sigma, \pi) = \frac{1}{2\lambda} (\sigma^2 + \pi^2) - i \text{Tr} \ln S(x, x; s) \Bigg|_{s=\sigma+i\gamma_5 \pi} , \] (8)
where the Green function defined by

\[ S(x, y; s) = \langle x | (i\gamma^\mu \nabla_\mu - s)^{-1} | y \rangle \]  

(9)

is the solution of the equation

\[ \{ i\gamma^\mu \nabla_\mu - s \} S(x, y; s) = \frac{1}{\sqrt{-g(x)}} \delta^4(x - y) . \]  

(10)

We would like to calculate the Green function \( S(x, y, s) \) in the approximation of weakly varying gravity where we neglect any terms involving derivatives of the metric tensor higher than the third derivative. We use the Riemann normal coordinate expansion[6]. Here we keep only terms independent of curvature \( R \) and terms linear in \( R \). The calculation proceeds just in parallel with the one given in the paper by Parker and Toms[7]. The result is

\[
S(x, x; s) = \int \frac{d^4k}{(2\pi)^4} \left[ (\gamma^{\hat{\alpha}} k_{\hat{\alpha}} + s) \frac{1}{k^2 - s^2} - \frac{1}{12} R(\gamma^{\hat{\alpha}} k_{\hat{\alpha}} + s) \frac{1}{(k^2 - s^2)^2} \right. \\
\left. + \frac{2}{3} R_{\mu\nu} k^\mu k^\nu (\gamma^{\hat{\alpha}} k_{\hat{\alpha}} + s) \frac{1}{(k^2 - s^2)^3} - \frac{1}{2} \gamma^{\hat{\beta}} \mathcal{J}_{\hat{\alpha}\hat{\beta}} R_{\hat{\alpha}\hat{\beta}\mu\nu} k^\mu \frac{1}{(k^2 - s^2)^2} \right] ,
\]

(11)

where

\[
\mathcal{J}_{\hat{\alpha}\hat{\beta}} = \frac{1}{4} [\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}] ,
\]

(12)

and Latin indices with a caret symbol are vierbein indices. It may be seen from Eqs. (8) and (11) that \( V(\sigma, \pi) \) is symmetric in \( \sigma \) and \( \pi \) and so it is enough to discuss \( V(\sigma, 0) \) instead of the full expression in the following arguments. Inserting Eq. (11) in Eq. (8) and making the Fourier transformation we obtain

\[
V(\sigma, 0) = \frac{1}{2\lambda} \sigma^2 \\
- i \text{Tr} \int_0^\sigma ds \int \frac{d^4k}{(2\pi)^4} \left[ (\gamma^{\hat{\alpha}} k_{\hat{\alpha}} + s) \frac{1}{k^2 - s^2} - \frac{1}{12} R(\gamma^{\hat{\alpha}} k_{\hat{\alpha}} + s) \frac{1}{(k^2 - s^2)^2} \right. \\
\left. + \frac{2}{3} R_{\mu\nu} k^\mu k^\nu (\gamma^{\hat{\alpha}} k_{\hat{\alpha}} + s) \frac{1}{(k^2 - s^2)^3} - \frac{1}{2} \gamma^{\hat{\beta}} \mathcal{J}_{\hat{\alpha}\hat{\beta}} R_{\hat{\alpha}\hat{\beta}\mu\nu} k^\mu \frac{1}{(k^2 - s^2)^2} \right] .
\]

(13)
To perform the momentum integration we make the Wick rotation and regularize the divergent integral by the cut-off method. We derive the following expression for the effective potential,
In Fig. 1 a typical behavior of the effective potential $V(\sigma, 0)$ as a function of curvature $R$ is given for fixed four-fermion coupling constant $\lambda$. As is well-known in the flat space-time, the chiral symmetry is broken if $\lambda > \lambda_0$ with

$$\lambda_0 = \frac{4\pi^2}{\Lambda^2}.$$  

In drawing Fig. 1 the coupling constant $\lambda$ is kept in the region $\lambda > \lambda_0$. It is clearly seen in Fig. 1 that the first-order phase transition takes place as $R$ changes. Note, however, that the transition is found to be of second order for $\lambda \leq \lambda_0$. The dynamical mass of the fermion is calculated by analyzing the gap equation $\partial V(\sigma, 0)/\partial \sigma = 0$

$$\partial V(\sigma, 0) = V(0, 0) + \frac{1}{2\lambda} \sigma^2 - \frac{1}{4\pi^2} \left[ \sigma^2 \Lambda^2 + \Lambda^4 \ln \left( 1 + \frac{\sigma^2}{\Lambda^2} \right) - \sigma^4 \ln \left( 1 + \frac{\Lambda^2}{\sigma^2} \right) \right] - \frac{R}{(4\pi)^2} 6 \left[ - \sigma^2 \ln \left( 1 + \frac{\Lambda^2}{\sigma^2} \right) + \frac{\Lambda^2 \sigma^2}{\Lambda^2 + \sigma^2} \right].$$  

The solution $\sigma_0$ of the gap equation (16) corresponds to the vacuum expectation value of the composite field $\bar{\psi} \psi$ in the true vacuum and is equal to the dynamical mass of the fermion, $m = \sigma_0$. In Fig. 2 the dynamical mass of the fermion is plotted as a function of the curvature $R$ for fixed $\lambda$. The coupling constant $\lambda$ is kept again in the range $\lambda > \lambda_0$. The behavior of the dynamical mass as shown in Fig. 2 is characteristic for the relatively small coupling constant, $\lambda_0 < \lambda < 2\lambda_0$. For larger coupling, $\lambda \geq 2\lambda_0$, the behavior near the critical point $R = R_{cr}$ is quite different from the one in Fig. 2: the curve representing the dynamical mass is bent upward near $R = R_{cr}$. At any rate there is observed the gap in the dynamical mass at the critical point $R = R_{cr}$ reflecting the nature of the first-order phase transition. By the
direct numerical analysis we find that there is no gap at \( R = R_{cr} = 0 \) if \( \lambda \leq \lambda_0 \) and so the phase transition is of second order.

It is possible to obtain the critical value of the curvature \( R \) and the coupling constant \( \lambda \) by observing the behavior of the effective potential. The critical values \( R_{cr} \) and \( \lambda_{cr} \) constitute a critical curve in the \( R - \lambda \) plane as shown in Fig. 3. It is found from Fig. 3 that for large positive \( R \) the chiral symmetry is restored even if \( \lambda \) is kept in the region of the broken phase for \( R = 0 \). On the other hand the chiral symmetry is always broken for negative \( R \) irrespective of the value of \( \lambda \). This latter conclusion comes from the particular behavior of the effective potential \((14)\) with \( \pi = 0 \), i.e., the term proportional to \( R \) in Eq. \((14)\) with \( \pi = 0 \) dominates over other terms if \( R < 0 \) and makes the point \( \sigma = 0 \) an unstable stationary point. The above peculiar behavior of our effective potential may be due to our approximation to keep only the first-order term in \( R \) in the normal coordinate expansion and should be left for the future detailed study.

In the above discussions concerning Figs. 1, 2 and 3 we dealt with the relatively large value of the curvature, e.g., in Fig. 1 \( R_{cr}/\Lambda^2 = 0.656 \) which is large compared with \( \sigma_0/\Lambda = 0.2 \sim 0.3 \). Since we work in the normal coordinate expansion and keep only terms up to the one linear in \( R \), large \( R_{cr}/\Lambda^2 \) compared with \( \sigma/\Lambda \) may spoil the validity of the expansion. One should, however, note that the terms quadratic or higher in \( R \) are not divergent by dimensional reason and the coefficient of \( R^2 \) does not carry logarithmic functions of \( \sigma^2/\Lambda^2 \). Hence the \( R^2 \) term is relatively suppressed compared with the term linear in \( R \). Accordingly relatively large \( R_{cr}/\Lambda^2 \) may be within the scope of our approximation.

In the present model we found within our approximation that the first-order phase transition takes place as the curvature varies and the fermion mass is generated when the curvature is smaller than the critical value. Our conclusion has some similarity with the previous results in the massless scalar theory with \( \phi^4 \) coupling\[8\]. The four-fermion theory in 2 dimensions (Gross-Neveu model\[9\]) has been investigated in curved space-time\[10\] and there the second-order phase transition is observed in contrast with our result.
REFERENCES

1. For references see, e. g., I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, *Effective Action in Quantum Gravity* (IOP Publishing, Bristol and Philadelphia, 1992).

2. C. Hill and D. S. Salopek, *Ann. of Phys.* 213 (1992) 21;
   T. Muta and S. D. Odintsov, *Mod. Phys. Lett.* A6 (1991) 3641.

3. Y. Nambu and G. Jona-Lasinio, *Phys. Rev.* 122 (1961) 345.

4. C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., San Francisco, 1973).

5. J. Schwinger, *Phys. Rev.* 82 (1951) 664.

6. See, e. g. [2].

7. L. Parker and D. J. Toms, *Phys. Rev.* D29 (1984) 1584.

8. G. M. Shore, *Ann. of Phys.* 128 (1980) 376;
   B. Allen, *Nucl. Phys.* B226 (1983) 228;
   K. Ishikawa, *Phys. Rev.* D28 (1983) 2445;
   B. L. Hu and D. J. O’Connor, *Phys. Rev.* D30 (1984) 743.

9. D. J. Gross and A. Neveu, *Phys. Rev.* D10 (1974) 3235.

10. H. Itoyama, *Prog. Theor. Phys.* 64 (1980) 1886;
    I. L. Buchbinder and E. N. Kirillova, *Int. J. of Mod. Phys.* A4 (1989) 143.
FIGURE CAPTIONS

Fig. 1. The typical behavior of the effective potential $V$ is shown for fixed $\lambda$ ($= 1.25 \lambda_0, \lambda_0 = 4\pi^2/\Lambda^2$) as a function of the curvature $R$ where $R_{cr} = 0.656\Lambda^2$.

Fig. 2. The solution $\sigma_0$ of the gap equation (17) is shown as a function of the curvature $R$ where $\lambda = 1.25\lambda_0$ and $R_{cr} = 0.656\Lambda^2$.

Fig. 3. The phase diagram in $\lambda$ and $R$ where $\lambda_{cr}$ and $R_{cr}$ represent the critical values of $\lambda$ and $R$ with which the region preserving the chiral symmetry is divided from the one of the broken chiral symmetry.