LINEAR TOPOLOGICAL INVARIANTS FOR KERNELS OF
CONVOLUTION AND DIFFERENTIAL OPERATORS

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ABSTRACT. We establish the condition $(\Omega)$ for smooth kernels of various types of
convolution and differential operators. By the $(DN)$-$(\Omega)$ splitting theorem of Vogt and
Wagner, this implies that these operators are surjective on the corresponding spaces
of vector-valued smooth functions with values in a product of Montel (DF)-spaces
whose strong duals satisfy the condition $(DN)$, e.g., the space $\mathcal{D}'(Y)$ of distributions
over an open set $Y \subseteq \mathbb{R}^n$ or the space $\mathcal{D}'(\mathbb{R}^n)$ of tempered distributions. Most
notably, we show that:

(i) $\mathcal{E}_P(X) = \{ f \in \mathcal{E}(X) | P(D)f = 0 \}$ satisfies $(\Omega)$ for any differential operator
$P(D)$ and any open convex set $X \subseteq \mathbb{R}^d$.

(ii) Let $P \in \mathbb{C}[\xi_1, \xi_2]$ and $X \subseteq \mathbb{R}^2$ open be such that $P(D) : \mathcal{E}(X) \to \mathcal{E}(X)$ is
surjective. Then, $\mathcal{E}_P(X)$ satisfies $(\Omega)$.

(iii) Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ be such that $\mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$, $f \mapsto \mu * f$ is surjective. Then,
$\{ f \in \mathcal{E}(\mathbb{R}^d) | \mu * f = 0 \}$ satisfies $(\Omega)$.

The central result in this paper is that the space of smooth zero solutions of a general
convolution equation satisfies the condition $(\Omega)$ if and only if the space of distribu-
tional zero solutions of the equation satisfies the condition $(P\Omega)$. The above and
related statements then follow from known results concerning $(P\Omega)$ for distributional
kernels of convolution and differential operators [3, 15, 16].

Keywords: Convolution operators; Differential operators; Linear topological invari-
ants; Surjectivity of convolution and differential operators on spaces of vector-valued
smooth functions; Homological algebra methods in functional analysis.

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1. INTRODUCTION

The aim of this paper is to study certain linear topological invariants for kernels of
convolution and differential operators. This topic goes back to the seminal article of
Vogt [27] and has gained a renewed interest by the work of Bonet and Domański [3, 4].
As we shall explain later on, this problem is closely connected to and motivated by the
question of surjectivity of such operators on spaces of vector-valued smooth functions and distributions:

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Question 1.1. Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ and $X_1, X_2 \subseteq \mathbb{R}^d$ be open sets such that
\begin{equation}
X_2 - \text{supp} \mu \subseteq X_1.
\end{equation}

Let $E$ be a locally convex space.

(i) Suppose that $\mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$, $f \mapsto \mu \ast f$ is surjective. When is the associated map
\begin{equation}
\mathcal{E}(X_1; E) \rightarrow \mathcal{E}(X_2; E), \ f \mapsto \mu \ast f
\end{equation}
surjective?

(ii) Suppose that $\mathcal{D}'(X_1) \rightarrow \mathcal{D}'(X_2)$, $f \mapsto \mu \ast f$ is surjective. When is the associated map
\begin{equation}
\mathcal{D}'(X_1; E) \rightarrow \mathcal{D}'(X_2; E), \ f \mapsto \mu \ast f
\end{equation}
surjective?

If $E$ is a space of functions or distributions, Question 1.1 is equivalent to the problem of parameter dependence of solutions of convolution equations. For differential equations this problem has a long and rich tradition, see [3–6, 17–19, 24, 25].

We now explain the connection between Question 1.1 and linear topological invariants for kernels of convolution operators. We assume that the reader is familiar with this problem has a long and rich tradition, see [3–6, 17–19, 24, 25].

We refer to [3, 21] for more information on these conditions and examples of spaces satisfying them. Suppose that $\mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$, $f \mapsto \mu \ast f$ is surjective and denote by $\mathcal{E}_\mu(X_1, X_2)$ the kernel of this map. A result of Grothendieck [10] yields that the map (1.2) is always surjective if $E$ is a Fréchet space. This is no longer true in general if $E$ is a (DF)-space, as shown by Vogt [27]. The splitting theory for Fréchet spaces [28] implies that for $E = s'$, with $s$ the space of rapidly decreasing sequences, the map (1.2) is surjective if and only if $\mathcal{E}_\mu(X_1, X_2)$ satisfies $(\Omega)$ (cf. [27]). Furthermore, if $\mathcal{E}_\mu(X_1, X_2)$ satisfies $(\Omega)$, then the map (1.2) is surjective for any $E$ isomorphic to a product of Montel (DF)-spaces whose strong duals satisfy (DN). Counterparts of these results for distributions were obtained by Bonet and Domaniński [3]. Suppose that $\mathcal{D}'(X_1) \rightarrow \mathcal{D}'(X_2)$, $f \mapsto \mu \ast f$ is surjective and denote by $\mathcal{D}_\mu(X_1, X_2)$ the kernel of this map. For $E = \mathcal{D}'(\mathbb{R}) \cong \prod_{n \in \mathbb{N}} s'$ the map (1.3) is surjective if and only if $\mathcal{D}_\mu(X_1, X_2)$ satisfies (PΩ). Moreover, if $\mathcal{D}_\mu(X_1, X_2)$ satisfies (PΩ), then the map (1.3) is surjective for any $E$ isomorphic to a product of (DFN)-spaces whose strong duals satisfy (DN).

In view of the above results, it is natural to study when the spaces $\mathcal{E}_\mu(X_1, X_2)$ and $\mathcal{D}_\mu(X_1, X_2)$ satisfy $(\Omega)$ and (PΩ), respectively. This problem has been mainly considered for differential operators. Let $P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ be a polynomial and consider the corresponding differential operator $P(D) = \mathcal{P}(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_d})$. For an open set $X \subseteq \mathbb{R}^d$ we write $\mathcal{D}_p(X) = \{ f \in \mathcal{D}'(X) \mid P(D)f = 0 \}$ and $\mathcal{E}_p(X) = \mathcal{D}_p(X) \cap \mathcal{E}'(X)$. Suppose that $P(D) : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X)$ is surjective. As mentioned above, $\mathcal{D}_p(X)$ satisfies (PΩ) if and only if $P(D) : \mathcal{D}'(X; \mathcal{D}'(\mathbb{R})) \rightarrow \mathcal{D}'(X; \mathcal{D}'(\mathbb{R}))$ is surjective. Hence, the Schwartz kernel theorem yields that $\mathcal{D}_p(X)$ satisfies (PΩ) if and only if $P^+(D) : \mathcal{D}'(X \times \mathbb{R}) \rightarrow \mathcal{D}'(X \times \mathbb{R})$ is surjective, where $P^+(\xi_1, \ldots, \xi_{d+1}) = P(\xi_1, \ldots, \xi_d) \in \mathcal{D}'(\mathbb{R}^d)$.
surjectivity of differential operators on distribution spaces \[12\] (applied to \(P^+(D)\)), gives a powerful method to study \((PΩ)\) for \(D′_P(X)\). For instance, it gives directly that \(D′_P(X)\) always satisfies \((PΩ)\) if \(X\) is convex \[3\] and it may be used to show that \(D′_P(X)\) satisfies \((PΩ)\) if \(P(D)\) is elliptic and \(X\) is an arbitrary open set \[9, 27\]. In \[15, 16\] the second author used this method to show that \(D′_P(X)\) satisfies \((PΩ)\) whenever \(P(D)\) is surjective on \(D′(X)\) in the following cases: \(d = 2\); \(P(D)\) is semi-elliptic with a single characteristic direction; \(P(D)\) acts along a subspace and is elliptic on this subspace; \(P(D)\) is a product of first order operators. We remark that in all the above cases also a geometric characterization of the sets \(X\) for which \(P(D) : D′(X) → D′(X)\) is surjective is known \[12, 13, 16\]. On the negative side, in \[14\] this method was used to provide a concrete example of a (hypoelliptic) surjective operator \(P(D)\) on \(D′(X)\) such that \(D′_P(X)\) does not satisfy \((PΩ)\).

There does not seem to exist an analogue of the above approach to study \((Ω)\) for \(E_P(X)\). However, if \(P(D)\) is hypoelliptic, then \(E_P(X) = D′_P(X)\) as locally convex spaces. Since for a (FS)-space the conditions \((Ω)\) and \((PΩ)\) are equivalent, we obtain that \(E_P(X)\) satisfies \((Ω)\) if and only if \(D′_P(X)\) satisfies \((PΩ)\) in the hypoelliptic case. The above mentioned results concerning \((PΩ)\) for \(D′_P(X)\) may therefore be transferred to \((Ω)\) for \(E_P(X)\) if \(P(D)\) is hypoelliptic, e.g., \(E_P(X)\) satisfies \((Ω)\) if \(P(D)\) is hypoelliptic and \(X\) is convex or if \(P(D)\) is elliptic and \(X\) is arbitrary. These results were obtained previously by Petzsche \[22\] and Vogt \[27\], respectively, via different more direct methods. Note that the example from \[14\] also shows that for surjective operators \(P(D)\) on \(E(X)\) the smooth kernel \(E_P(X)\) in general does not satisfy \((Ω)\).

The principal goal of this paper is to establish \((Ω)\) for smooth kernels of (non-hypoelliptic) differential operators and, more generally, convolution operators. Our main tool is the following result, which extends the equivalence of \((Ω)\) for \(E_P(X)\) and \((PΩ)\) for \(D′_P(X)\) in the hypoelliptic case to general convolution operators:

**Theorem 1.2.** Let \(μ ∈ E′(R^d)\) and \(X_1, X_2 ⊆ R^d\) be open sets such that \(12\) holds.

(i) Suppose that \(E(X_1) → E(X_2), f ↦ μ ∗ f\) is surjective. If \(D′_μ(X_1, X_2)\) satisfies \((PΩ)\), then \(E_μ(X_1, X_2)\) satisfies \((Ω)\).

(ii) Suppose that \(D′(X_1) → D′(X_2), f ↦ μ ∗ f\) is surjective. If \(E_μ(X_1, X_2)\) satisfies \((Ω)\), then \(D′_μ(X_1, X_2)\) satisfies \((PΩ)\).

Theorem 1.2 should be compared with a classical result of Meise, Taylor and Vogt \[20\] stating that a differential operator \(P(D)\) admits a continuous linear right inverse on \(E(X), X ⊆ R^d\) open, if and only if it does so on \(D′(X)\).

The proof of Theorem 1.2 is given in Section 3 and is based on some abstract results about the derived projective limit functor \[30\] and a simple regularization procedure. The necessary notions and results from the theory of the derived projective limit functor are discussed in the preliminary Section 2.

By combining Theorem 1.2 with results concerning \((PΩ)\) for distributional kernels of various types of convolution and differential operators, we show \((Ω)\) for smooth kernels of these operators in Section 4. Most notably, we prove that:

(i) \(E_P(X)\) satisfies \((Ω)\) for any differential operator \(P(D)\) and any open convex set \(X ⊆ R^d\) (Theorem 1.2).
(ii) Let $P \in \mathbb{C}[\xi_1, \xi_2]$ and $X \subseteq \mathbb{R}^2$ open be such that $P(D) : \mathcal{E}(X) \to \mathcal{E}(X)$ is surjective. Then, $\mathcal{E}_P(X)$ satisfies $(\Omega)$ (Theorem 4.4 (iii)).

(iii) Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ be such that $\mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$, $f \mapsto \mu \ast f$ is surjective. Then, $\mathcal{E}_\mu(\mathbb{R}^d, \mathbb{R}^d)$ satisfies $(\Omega)$ (Theorem 4.6).

(iv) Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ belong to the class $\mathcal{R}$ of Berenstein and Dostal [2] (see also [17]). Let $X \subseteq \mathbb{R}^d$ be open and convex, and set $X_\mu = \{x \in \mathbb{R}^d | x - \text{supp} \mu \subseteq X\}$. Then, $\mathcal{P}_\mu(X, X_\mu)$ satisfies $(P\Omega)$ and $\mathcal{E}_\mu(X, X_\mu)$ satisfies $(\Omega)$ (Theorem 4.9).

We mention that every distribution with finite support is of class $\mathcal{R}$ [2]. Hence, (iv) particularly applies to difference-differential operators as they are precisely the convolution operators whose kernels have finite support (Corollary 4.10). To the best of our knowledge, (iv) is the first result about linear topological invariants for kernels of convolution operators on convex sets other than the whole space $\mathbb{R}^d$.

2. Preliminaries

In this preliminary section we recall various notions and results concerning the derived projective limit functor [30], the linear topological invariants $(\Omega)$ and $(P\Omega)$ [3][21], and convolution operators [12].

Throughout this article we use standard notation from functional analysis [21][30] and distribution theory [11][12][23]. In particular, given a locally convex space $X$, we denote by $\mathcal{U}_0(X)$ the filter basis of absolutely convex neighborhoods of 0 in $X$ and by $\mathcal{B}(X)$ the family of all absolutely convex bounded sets in $X$.

2.1. Projective spectra. A projective spectrum $\mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}}$ consists of vector spaces $X_n$ and linear maps $\varrho_{n+1}^n : X_{n+1} \to X_n$, called the spectral maps. We define $\varrho_n^0 = \text{id}_{X_n}$ and $\varrho_m^n = \varrho_{m+1}^m \circ \cdots \circ \varrho_{m+1}^n : X_m \to X_n$ for $n, m \in \mathbb{N}$ with $m > n$. The projective limit of $\mathcal{X}$ is defined as

$$\text{Proj } \mathcal{X} = \left\{ (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n \mid x_n = \varrho_{n+1}^n(x_{n+1}), \forall n \in \mathbb{N} \right\}.$$

For each $n \in \mathbb{N}$ we write $\varrho^n : \text{Proj } \mathcal{X} \to X_n$, $(x_j)_{j \in \mathbb{N}} \mapsto x_n$.

Given two projective spectra $\mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}}$ and $\mathcal{Y} = (Y_n, \sigma_{n+1}^n)_{n \in \mathbb{N}}$, a morphism $S = (S_n)_{n \in \mathbb{N}} : \mathcal{X} \to \mathcal{Y}$ consists of linear maps $S_n : X_n \to Y_n$ such that $S_n \circ \varrho_{n+1}^n = \sigma_{n+1}^n \circ S_{n+1}$ for all $n \in \mathbb{N}$. We define $\text{ker } S$ as the projective spectrum $(\text{ker } S_n, \varrho_{n+1}^n|_{\text{ker } S_{n+1}})_{n \in \mathbb{N}}$ and

$$\text{Proj } S : \text{Proj } \mathcal{X} \to \text{Proj } \mathcal{Y}, (x_n)_{n \in \mathbb{N}} \mapsto (S_n(x_n))_{n \in \mathbb{N}}.$$

The derived projective limit of a projective spectrum $\mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}}$ is defined as

$$\text{Proj}^1 \mathcal{X} = \prod_{n \in \mathbb{N}} X_n/B(\mathcal{X}),$$

where

$$B(\mathcal{X}) = \left\{ (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n \mid \exists (u_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : x_n = u_n - \varrho_{n+1}^n(u_{n+1}), \forall n \in \mathbb{N} \right\}.$$
We shall use the following fundamental property of the derived projective limit.

**Proposition 2.1.** [30, Theorem 3.1.8] Let \( \mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}} \) and \( \mathcal{Y} = (Y_n, \sigma_{n+1}^n)_{n \in \mathbb{N}} \) be projective spectra and let \( S = (S_n)_{n \in \mathbb{N}} : \mathcal{X} \to \mathcal{Y} \) be a morphism. Suppose that for all \( n \in \mathbb{N} \) there is \( m \geq n \) such that \( \sigma_m^n(Y_m) \subseteq S_n(X_n) \). Then, there is an exact sequence of linear maps

\[
0 \to \text{Proj} \ker S \to \text{Proj} \mathcal{X}^{\text{Proj} S} \to \text{Proj} \mathcal{Y} \to \text{Proj}^1 \ker S \to \text{Proj}^1 \mathcal{X}.
\]

Consequently, \( \text{Proj}^1 \ker S = 0 \) implies that \( \text{Proj} S : \text{Proj} \mathcal{X} \to \text{Proj} \mathcal{Y} \) is surjective. If \( \text{Proj}^1 \mathcal{X} = 0 \), the converse holds true as well.

By a projective spectrum of locally convex spaces we mean a projective spectrum \( \mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}} \) consisting of locally convex spaces \( X_n \) and continuous linear spectral maps \( \varrho_{n+1}^n \). In such a case, we endow \( \text{Proj} \mathcal{X} \) with its natural projective limit topology.

For projective spectra of Fréchet spaces the vanishing of the derived projective limit may be characterized as follows (this result is known as the Mittag-Leffler lemma).

**Theorem 2.2.** [30, Theorem 3.2.8] Let \( \mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}} \) be a projective spectrum of Fréchet spaces. Then, the following statements are equivalent:

(i) \( \text{Proj}^1 \mathcal{X} = 0 \).
(ii) \( \forall n \in \mathbb{N}, U \in \mathcal{U}_0(X_n) \exists m \geq n \forall k \geq m : \varrho_{m}^n(X_m) \subseteq \varrho_k^n(X_k) + U \).
(iii) \( \forall n \in \mathbb{N}, U \in \mathcal{U}_0(X_n) \exists m \geq n : \varrho_{m}^n(X_m) \subseteq \varrho^n(\text{Proj} \mathcal{X}) + U \).

For projective spectra of (DFS)-spaces the following characterization holds.

**Theorem 2.3.** [30, Theorem 3.2.18] Let \( \mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}} \) be a projective spectrum of (DFS)-spaces. Then, \( \text{Proj}^1 \mathcal{X} = 0 \) if and only if

\[
\forall n \in \mathbb{N} \exists m \geq n \forall k \geq m \exists B \in \mathcal{B}(X_n) \forall M \in \mathcal{B}(X_m) \exists K \in \mathcal{B}(X_k) : \varrho_{m}^n(M) \subseteq B + \varrho_k^n(K).
\]

2.2. The condition \((\Omega)\) for Fréchet spaces. A Fréchet space \( X \) is said to satisfy the condition \((\Omega)\) [21] if

\[
\forall U \in \mathcal{U}_0(X) \exists V \in \mathcal{U}_0(X) \forall W \in \mathcal{U}_0(X) \exists C, s > 0 \forall \varepsilon \in (0, 1) : V \subseteq \varepsilon U + \frac{C}{\varepsilon^s} W.
\]

The following result concerning \((\Omega)\) for projective limits of spectra of Fréchet spaces is sometimes useful for verifying this condition in concrete situations. For us it will play a vital role in the next section. We believe this result is essentially known, but we include a proof here as we could not find one in the literature.

**Lemma 2.4.** Let \( \mathcal{X} = (X_n, \varrho_{n+1}^n)_{n \in \mathbb{N}} \) be a projective spectrum of Fréchet spaces. Then, \( \text{Proj} \mathcal{X} \) satisfies \((\Omega)\) and \( \text{Proj}^1 \mathcal{X} = 0 \) if and only if

\[
\forall n \in \mathbb{N}, U \in \mathcal{U}_0(X_n) \exists m \geq n, V \in \mathcal{U}_0(X_m) \forall k \geq m, W \in \mathcal{U}_0(X_k) \exists C, s > 0 \forall \varepsilon \in (0, 1) : \varrho_{m}^n(V) \subseteq \varepsilon U + \frac{C}{\varepsilon^s} \varrho_k^n(W).
\]
Proof. By definition of the projective limit topology, \((\Omega)\) for \(X = \text{Proj} \mathcal{X}\) means that
\[
\forall n \in \mathbb{N}, U \in \mathcal{U}_0(X_n) \exists m \geq n, V \in \mathcal{U}_0(X_m) \forall k \geq m, W \in \mathcal{U}_0(X_k)
\]
\[
(2.2) \quad \exists C, s > 0 \forall \varepsilon \in (0, 1) : (\varphi^m)^{-1}(V) \subseteq \varepsilon(\varphi^n)^{-1}(U) + \frac{C}{\varepsilon^s}(\varphi^k)^{-1}(W).
\]
Now suppose that \((2.1)\) holds. This condition clearly implies condition (ii) from Theorem 2.2 and thus that \(\text{Proj} \mathcal{X}^* = 0\). We now show (2.2). Let \(n \in \mathbb{N}\) and \(U \in \mathcal{U}_0(X_n)\) be arbitrary and choose \(m \geq n\) and \(V \in \mathcal{U}_0(X_m)\) according to \((2.1)\). Let \(k \geq m\) and \(W \in \mathcal{U}_0(X_k)\) be arbitrary. By condition (iii) from Theorem 2.2 there is \(\tilde{k} \geq k\) such that
\[
(2.3) \quad \varphi^\tilde{k}(X_{\tilde{k}}) \subseteq \varphi^k(X) + (W \cap (\varphi^n)^{-1}(U)).
\]
Set \(\tilde{W} = (\varphi^\tilde{k})^{-1}(W)\). Condition \((2.1)\) (applied to \(\tilde{k}\) and \(\tilde{W}\) instead of \(k\) and \(W\)) yields that there are \(C, s > 0\) such that for all \(\varepsilon \in (0, 1)\)
\[
(2.4) \quad \varphi^\tilde{n}(V) \subseteq \varepsilon U + \frac{C}{\varepsilon^s} \varphi^\tilde{n}(\tilde{W}).
\]
Let \(x \in (\varphi^m)^{-1}(V)\) and \(\varepsilon \in (0, 1)\) be arbitrary. Note that \(\varphi^n(x) = \varphi^m(\varphi^m(x)) \in \varphi^\tilde{n}(V)\). Condition \((2.4)\) implies that there is \(\tilde{y}_\varepsilon \in \frac{2\varepsilon C}{\varepsilon^s} \tilde{W}\) such that \(\varphi^n(x) - \varphi^\tilde{n}((\tilde{y}_\varepsilon)) \in \varepsilon U\). By multiplying both sides of \((2.3)\) with \(\varepsilon/2\), we find that there is \(y_\varepsilon \in X\) such that \(\varphi^\tilde{k}(\tilde{y}_\varepsilon) - \varphi^k(y_\varepsilon) \in \frac{\varepsilon}{\varepsilon^s} \tilde{W} \cap (\varphi^n)^{-1}(U)\). We have that
\[
\varphi^n(x - y_\varepsilon) = (\varphi^n(x) - \varphi^\tilde{n}((\tilde{y}_\varepsilon))) + (\varphi^\tilde{n}(\tilde{y}_\varepsilon) - \varphi^n(y_\varepsilon))
\]
\[
= (\varphi^n(x) - \varphi^\tilde{n}((\tilde{y}_\varepsilon))) + \varphi^\tilde{n}(\varphi^\tilde{k}(\tilde{y}_\varepsilon) - \varphi^k(y_\varepsilon)) \in \frac{\varepsilon}{2} U + \frac{\varepsilon}{2} U = \varepsilon U
\]
and
\[
\varphi^k(y_\varepsilon) = (\varphi^k(y_\varepsilon) - \varphi^\tilde{k}(\tilde{y}_\varepsilon)) + \varphi^\tilde{k}(\tilde{y}_\varepsilon) \in \frac{1}{\varepsilon^s} W + \frac{2\varepsilon C}{\varepsilon^s} W = \frac{2\varepsilon C + 1}{\varepsilon^s} W.
\]
Thus,
\[
x = (x - y_\varepsilon) + y_\varepsilon \in (\varphi^n)^{-1}(U) + \frac{2\varepsilon C + 1}{\varepsilon^s}(\varphi^k)^{-1}(W).
\]
Next, assume that \(X\) satisfies \((\Omega)\) (and thus \((2.2)\)) and \(\text{Proj} \mathcal{X}^* = 0\). Let \(n \in \mathbb{N}\) and \(U \in \mathcal{U}_0(X_n)\) be arbitrary. Choose \(\tilde{m} \geq n\) and \(V \in \mathcal{U}_0(X_{\tilde{m}})\) according to \((2.2)\). Since \(\text{Proj} \mathcal{X}^* = 0\), Theorem 2.2 implies that there is \(m \geq \tilde{m}\) such that
\[
(2.5) \quad \varphi^\tilde{m}(X_{\tilde{m}}) \subseteq \varphi^\tilde{m}(X) + (\tilde{V} \cap (\varphi^n)^{-1}(U)).
\]
Set \(V = \frac{1}{2}(\varphi^\tilde{m})^{-1}(\tilde{V})\). Let \(k \geq m\) and \(W \in \mathcal{U}_0(X_k)\) be arbitrary. Condition \((2.2)\) implies that there are \(C, s > 0\) such that for all \(\varepsilon \in (0, 1)\)
\[
(2.6) \quad (\varphi^\tilde{m})^{-1}(\tilde{V}) \subseteq \varepsilon(\varphi^n)^{-1}(U) + \frac{C}{\varepsilon^s}(\varphi^k)^{-1}(W).
\]
Let \(x \in V\) and \(\varepsilon \in (0, 1)\) be arbitrary. By multiplying both sides of \((2.5)\) with \(\varepsilon/2\), we find that there are \(y_\varepsilon \in X\) and \(z_\varepsilon \in \frac{1}{2}(\tilde{V} \cap (\varphi^\tilde{m})^{-1}(U)) \subseteq \frac{1}{2}\tilde{V} \cap \frac{1}{4}(\varphi^\tilde{m})^{-1}(U)\) such that
The spectrum $X$ is given by $\hat{\rho}_{m}(y_{c}) + z_{\ell}$. Note that $y_{c} \in (\hat{\rho}^{n})^{-1}(V)$. Hence, by (2.7),

$$\hat{\rho}^{n}_{m}(x) = \hat{\rho}^{n}_{m}(y_{c}) + \hat{\rho}^{n}_{m}(z_{\ell}) \in \frac{\varepsilon}{2} U + \frac{2^{|C|}}{\varepsilon^{s}} \hat{\rho}^{n}((\hat{\rho})^{-1}(W)) + \frac{\varepsilon}{2} U \subseteq \varepsilon U + \frac{2^{|C|}}{\varepsilon^{s}} \hat{\rho}^{n}_{m}(W).$$

\[ \Box \]

2.3. The condition (PΩ) for (PLS)-spaces. A locally convex space $X$ is called a (PLS)-space if it can be written as the topological projective limit of a spectrum of (DFS)-spaces.

Let $X = (X_{n}, \hat{\rho}^{n}_{m+1})_{n \in \mathbb{N}}$ be a spectrum of (DFS)-spaces. We call $X$ strongly reduced if

$$\forall n \in \mathbb{N} \exists m \geq n : \hat{\rho}^{n}_{m}(X_{m}) \subseteq \hat{\rho}^{n}(\text{Proj } X)^{X_{n}}.$$ 

The spectrum $X$ is said to satisfy (PΩ) if

$$\forall n \in \mathbb{N} \exists m \geq n \forall k \geq m \exists B \in \mathcal{B}(X_{n}) \forall M \in \mathcal{B}(X_{m}) \exists K \in \mathcal{B}(X_{k}) \exists C, s > 0$$

$$\forall \varepsilon \in (0, 1) : \hat{\rho}^{n}_{m}(M) \subseteq \varepsilon B + \frac{C}{\varepsilon^{s}} \hat{\rho}^{n}_{k}(K).$$

A (PLS)-space $X$ is said to satisfy (PΩ) if $X = \text{Proj } X$ for some strongly reduced spectrum $X$ of (DFS)-spaces that satisfies (PΩ). This notion is well-defined as $\mathbb{Z}$ Proposition 3.3.8] yields that all strongly reduced projective spectra $X$ of (DFS)-spaces with $X = \text{Proj } X$ are equivalent (in the sense of $\mathbb{Z}$ Definition 3.1.6]). The bipolar theorem and $\mathbb{Z}$ Lemma 4.5] imply that the above definition of (PΩ) is equivalent to the original one from $\mathbb{Z}$.

2.4. Convolution operators. Let $\mu \in \mathcal{E}'(\mathbb{R}^{d})$. For all open sets $X_{1}, X_{2} \subseteq \mathbb{R}^{d}$ such that (1.1) holds the convolution operators

$$S_{\mu} : \mathcal{D}'(X_{1}) \to \mathcal{D}'(X_{2}), f \mapsto \mu * f$$

and

$$S_{\mu} : \mathcal{E}(X_{1}) \to \mathcal{E}(X_{2})$$

are well-defined continuous linear maps. Hörmander characterized when the maps (2.7) and (2.8) are surjective $\mathbb{Z}$ Section 16.5]. We need some preparation to state his results. As customary, we define the Fourier transform of an element $\mu \in \mathcal{E}'(\mathbb{R}^{d})$ as

$$\hat{\mu}(\zeta) = \langle \mu(x), e^{-ix\cdot \zeta} \rangle, \quad \zeta \in \mathbb{C}^{d}.$$

Then, $\hat{\mu}$ is an entire function such that

$$|\hat{\mu}(\zeta)| \leq C(1 + |\zeta|)^{N} e^{H_{u}(\text{Im } \zeta)}, \quad \zeta \in \mathbb{C}^{d},$$

for some $C, N > 0$, where $H_{u}$ denotes the supporting function of supp $\mu$. A distribution $\mu \in \mathcal{E}'(\mathbb{R}^{d})$ is called invertible $\mathbb{Z}$ Definition 16.3.12] (see also $\mathbb{Z}$) if

$$\exists c, R, M > 0 \forall \xi \in \mathbb{R}^{d} \exists \zeta \in \mathbb{C}^{d}, |\zeta - \xi| < R \log(1 + |\xi|) \quad |\hat{\mu}(\zeta)| \geq c(1 + |\xi|)^{-M}.$$ 

We refer to $\mathbb{Z}$ Theorem 16.3.10] for various characterizations of invertibility.

Let $\mu \in \mathcal{E}'(\mathbb{R}^{d})$ and $X_{1}, X_{2} \subseteq \mathbb{R}^{d}$ be open sets such that (1.1) holds. We set $\hat{\mu}(x) = \mu(-x)$. Note that $S_{\mu}^{s} = S_{\mu}^{s} : \mathcal{E}'(X_{2}) \to \mathcal{E}'(X_{1})$. Hörmander $\mathbb{Z}$ Theorem 16.5.7 and Corollary 16.5.19] showed that $S_{\mu} : \mathcal{E}(X_{1}) \to \mathcal{E}(X_{2})$ is surjective if and
only if $\mu$ is invertible and the pair $(X_1, X_2)$ is $\mu$-convex for supports, i.e., for every compact $K_1 \subseteq X_1$ there is a compact $K_2 \subseteq X_2$ such that for all $f \in \mathcal{E}'(X_2)$

$$\supp \tilde{\mu} \ast f \subseteq K_1 \implies \supp f \subseteq K_2,$$

while $S_\mu : \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)$ is surjective if and only if the pair $(X_1, X_2)$ is $\mu$-convex for supports and singular supports, where the latter means that for every compact $K_1 \subseteq X_1$ there is a compact $K_2 \subseteq X_2$ such that for all $f \in \mathcal{E}'(X_2)$

$$\text{sing supp } \tilde{\mu} \ast f \subseteq K_1 \implies \text{sing supp } f \subseteq K_2.$$

Furthermore, if $(X_1, X_2)$ is $\mu$-convex for singular supports, then $\mu$ is invertible [12 Proposition 16.5.12]. Hence, $S_\mu : \mathcal{E}(X_1) \to \mathcal{E}(X_2)$ is surjective if $S_\mu : \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)$ is so. By [11 Theorem 4.3.3] and [12 Corollary 16.3.15], $\mu$ is invertible if and only if $S_\mu : \mathcal{E}(\mathbb{R}^d) \to \mathcal{F}(\mathbb{R}^d)$ is surjective if and only if $S_\mu : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ is surjective. This result was first shown by Ehrenpreis [8 Theorem 1]. We will use the above facts without explicitly referring to them.

Of course, every differential operator $P(D)$, where $P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ and $D = (-i\partial_{\xi_1}, \ldots, -i\partial_{\xi_d})$, may be seen as a convolution operator with kernel $\mu = P(D)\delta$. 

3. Equivalence of $(\Omega)$ and (PΩ) for kernels of convolution operators

The goal of this section is to show Theorem [1.2]. Fix $\mu \in \mathcal{E}'(\mathbb{R}^d)$ and two open sets $X_1, X_2 \subseteq \mathbb{R}^d$ such that (1.1) holds. We endow the spaces

$$\mathcal{E}_\mu(X_1, X_2) = \{f \in \mathcal{E}(X_1) \mid \mu * f = 0 \text{ in } \mathcal{E}(X_2)\}$$

and

$$\mathcal{D}_\mu(X_1, X_2) = \{f \in \mathcal{D}(X_1) \mid \mu * f = 0 \text{ in } \mathcal{D}(X_2)\}$$

with the relative topology induced by $\mathcal{E}(X_1)$ and $\mathcal{D}(X_1)$, respectively. Since both these spaces are closed, $\mathcal{E}_\mu(X_1, X_2)$ is a Fréchet space and $\mathcal{D}_\mu(X_1, X_2)$ is a (PLS)-space.

Let $(X_1(n))_{n \in \mathbb{N}}$ and $(X_2(n))_{n \in \mathbb{N}}$ be exhausted by relatively compact open subsets of $X_1$ and $X_2$ such that $X_{2,n} - \text{supp } \mu \subseteq X_{1,n}$ for all $n \in \mathbb{N}$. Set $K_{j,n} = X_{j,n}$ for $j = 1, 2$ and note that $K_{2,n} - \text{supp } \mu \subseteq K_{1,n}$ for all $n \in \mathbb{N}$. We define

$$\mathcal{E}_\mu(K_{1,n}, K_{2,n}) = \{f \in \mathcal{E}(K_{1,n}) \mid \mu * f = 0 \text{ in } \mathcal{E}(K_{2,n})\}$$

and

$$\mathcal{D}_\mu(K_{1,n}, K_{2,n}) = \{f \in \mathcal{D}(K_{1,n}) \mid \mu * f = 0 \text{ in } \mathcal{D}(K_{2,n})\}.$$

Then, $(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}$ is a projective spectrum of Fréchet spaces such that

$$\mathcal{E}_\mu(X_1, X_2) = \text{Proj}(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}$$

and $(\mathcal{D}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}$ is a projective spectrum of (DFS)-spaces such that

$$\mathcal{D}_\mu(X_1, X_2) = \text{Proj}(\mathcal{D}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}.$$

In both cases we tacitly assumed that the spectral maps are the restriction maps. In the sequel we shall not write these maps.

---

1Let $Y \subseteq \mathbb{R}^d$ be relatively compact and open. $\mathcal{E}(Y)$ is the Fréchet space of smooth functions $f \in \mathcal{E}(Y)$ such that $f^{(\alpha)}$ can be continuously extended to $\overline{Y}$ for all $\alpha \in \mathbb{N}^d$. $\mathcal{D}(Y)$ is the Fréchet space of smooth functions with support in $Y$ and $\mathcal{D}'(Y)$ is the strong dual of $\mathcal{D}(Y)$. 

---
Lemma 3.1.

For $n, N \in \mathbb{N}$ we define
\[ \|f\|_{n,N} = \max_{x \in K_{1,n},|\alpha| \leq N} |f^{(\alpha)}(x)|, \quad f \in \mathcal{E}(K_{1,n}), \]
and
\[ \|f\|_{n,N}^* = \sup\{|(f, \varphi)| : \varphi \in \mathcal{D}(K_{1,n}), \|\varphi\|_{n,N} \leq 1\}, \quad f \in \mathcal{D}'(K_{1,n}). \]

We set
\[ U_{n,N} = \{f \in \mathcal{E}_\mu(K_{1,n}, K_{2,n}) : \|f\|_{n,N} \leq 1\}, \quad U_n = U_{n,n}, \]
and
\[ B_{n,N} = \{f \in \mathcal{D}'_\mu(K_{1,n}, K_{2,n}) : \|f\|_{n,N}^* \leq 1\}. \]

Then, $(\frac{1}{N+1}U_{n,N})_{N \in \mathbb{N}}$ is a decreasing fundamental sequence of absolutely convex neighborhoods of 0 in $\mathcal{E}_\mu(K_{1,n}, K_{2,n})$ and $(NB_{n,N})_{N \in \mathbb{N}}$ is an increasing fundamental sequence of absolutely convex bounded sets in $\mathcal{D}'_\mu(K_{1,n}, K_{2,n})$. We need the following lemma.

Lemma 3.1.

(i) The map $S_\mu : \mathcal{E}(X_1) \to \mathcal{E}(X_2)$ is surjective if and only if $\mu$ is invertible and
\[ \text{Proj}^1(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0. \]

(ii) The map $S_\mu : \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)$ is surjective if and only if $\mu$ is invertible and
\[ \text{Proj}^1(\mathcal{D}'_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0. \]

Proof. (i) We start by showing that for invertible $\mu$ it holds that
\[ (3.1) \quad \forall n \in \mathbb{N} \forall f \in \mathcal{E}(K_{2,n+1}) \exists g \in \mathcal{E}(\mathbb{R}^d) : \mu \ast g = f \text{ on } K_{2,n}. \]

Since $\mu$ is invertible, there is $E \in \mathcal{D}'(\mathbb{R}^d)$ such that $\mu \ast E = \delta$. Choose $\psi \in \mathcal{D}(K_{2,n+1})$ such that $\psi = 1$ on $K_{2,n}$. Set $g = E \ast f \psi \in \mathcal{E}(\mathbb{R}^d)$ and note that $\mu \ast g = f \psi = f$ on $K_{2,n}$. The sufficiency of the condition for surjectivity is therefore a direct consequence of Proposition 2.1. For its necessity we note that $\mu$ is invertible because $S_\mu : \mathcal{E}(X_1) \to \mathcal{E}(X_2)$ is surjective, and thus that (3.1) holds. Furthermore, a standard cut-off and regularization argument shows that $(\mathcal{E}(K_{1,n}))_{n \in \mathbb{N}}$ satisfies condition (iii) from Theorem 2.2, whence $\text{Proj}^1(\mathcal{E}(K_{1,n}))_{n \in \mathbb{N}} = 0$. Proposition 2.1 now implies that also $\text{Proj}^1(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0$.

(ii) The proof is similar to the one of (i) and is therefore omitted. We only remark that $\text{Proj}^1(\mathcal{D}'(K_{1,n}))_{n \in \mathbb{N}} = 0$ since the restriction maps $\mathcal{D}'(K_{1,n+1}) \to \mathcal{D}'(K_{1,n})$ are surjective.

Fix $\chi \in \mathcal{D}(\mathbb{R}^d)$ with $\chi \geq 0$, $\text{supp} \chi \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \chi(x)dx = 1$, and set $\chi_\varepsilon(x) = \varepsilon^{-d} \chi(x/\varepsilon)$ for $\varepsilon > 0$. We are ready to show Theorem 1.2.

Proof of Theorem 1.2. (i) By Lemma 2.4 and a rescaling argument, it suffices to show that
\[ (3.2) \quad U_m \subseteq \varepsilon C_1 U_n + \frac{C_2}{\varepsilon^s} U_k. \]
We start by showing that the spectrum \( (\mathcal{D}_\mu'(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} \) is strongly reduced. We need to show that for all \( n \in \mathbb{N} \) there is \( m \geq n \) such that for all \( U \in \mathcal{U}_0(\mathcal{D}_\mu'(K_{1,n}, K_{2,n})) \) it holds
\[
\mathcal{D}_\mu'(K_{1,m}, K_{2,m}) \subseteq \mathcal{D}_\mu'(X_1, X_2) + U.
\]
As \( S_\mu : \mathcal{E}(X_1) \to \mathcal{E}(X_2) \) is surjective, \( \text{Proj}^1(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0 \) by Lemma 3.1(i).

Let \( n \in \mathbb{N} \) be arbitrary. There is \( \delta > 0 \) such that for all \( \varepsilon \in (0, 1) \) \( \varepsilon \in \mathbb{N} \), \( \exists m \geq n \) such that
\[
\mathcal{D}_\mu'(K_{1,m}, K_{2,m}) \subseteq \mathcal{D}_\mu'(X_1, X_2) + U_{n,0}.
\]
Set \( m = \tilde{m} + 1 \) and let \( U \in \mathcal{U}_0(\mathcal{D}_\mu'(K_{1,n}, K_{2,n})) \) be arbitrary. Let \( f \in \mathcal{D}_\mu'(K_{1,m}, K_{2,m}) \) be arbitrary. There is \( \varepsilon > 0 \) such that for all \( \varepsilon \in (0, 1) \) \( \varepsilon \in \mathbb{N} \), \( \exists m \geq n \) such that for all \( \delta \in (0, 1) \) and \( \varepsilon \in (0, \varepsilon_0) \) we define \( f_{\delta, \varepsilon} = f \ast \chi_\varepsilon \in \mathcal{D}_\mu'(K_{1,k}, K_{2,k}) \). The mean value theorem implies that
\[
\exists N \in \mathbb{N}, K \geq N, C, r > 0: \forall \delta \in (0, 1) : |K_{1,m}|B_{m,N} \subseteq \delta B_{n+1,N} + \frac{C}{\delta^r}B_{k+1,K},
\]
where \( |K_{1,m}| \) denotes the Lebesgue measure of \( K_{1,m} \). Let \( f \in U_m \) be arbitrary. Since \( U_m \subseteq |K_{1,m}|B_{m,N} \), there is \( f_{\delta, \varepsilon} \in \delta^{-r}CB_{k+1,K} \) with \( f_{\delta, \varepsilon} \in \delta B_{n+1,N} \) for all \( \delta \in (0, 1) \).

Set \( \varepsilon_0 = \min\{1, d(K_{1,k}, K_{1,k+1}), d(K_{1,n}, K_{1,n+1})\} \). For \( \delta \in (0, 1) \) and \( \varepsilon \in (0, \varepsilon_0) \) we define \( f_{\delta, \varepsilon} = f \ast \chi_\varepsilon \in \mathcal{D}_\mu'(K_{1,k}, K_{2,k}) \). The mean value theorem implies that
\[
\exists N \in \mathbb{N}, K \geq N, C, r > 0: \forall \delta \in (0, 1) : |K_{1,m}|B_{m,N} \subseteq \delta B_{n+1,N} + \frac{C}{\delta^r}B_{k+1,K},
\]
where \( |K_{1,m}| \) denotes the Lebesgue measure of \( K_{1,m} \). Let \( f \in U_m \) be arbitrary. Since \( U_m \subseteq |K_{1,m}|B_{m,N} \), there is \( f_{\delta, \varepsilon} \in \delta^{-r}CB_{k+1,K} \) with \( f_{\delta, \varepsilon} \in \delta B_{n+1,N} \) for all \( \delta \in (0, 1) \).

Set \( \varepsilon_0 = \min\{1, d(K_{1,k}, K_{1,k+1}), d(K_{1,n}, K_{1,n+1})\} \). For \( \delta \in (0, 1) \) and \( \varepsilon \in (0, \varepsilon_0) \) we define \( f_{\delta, \varepsilon} = f \ast \chi_\varepsilon \in \mathcal{D}_\mu'(K_{1,k}, K_{2,k}) \). The mean value theorem implies that
\[
\exists N \in \mathbb{N}, K \geq N, C, r > 0: \forall \delta \in (0, 1) : |K_{1,m}|B_{m,N} \subseteq \delta B_{n+1,N} + \frac{C}{\delta^r}B_{k+1,K},
\]
where \( |K_{1,m}| \) denotes the Lebesgue measure of \( K_{1,m} \). Let \( f \in U_m \) be arbitrary. Since \( U_m \subseteq |K_{1,m}|B_{m,N} \), there is \( f_{\delta, \varepsilon} \in \delta^{-r}CB_{k+1,K} \) with \( f_{\delta, \varepsilon} \in \delta B_{n+1,N} \) for all \( \delta \in (0, 1) \).

Set \( \varepsilon_0 = \min\{1, d(K_{1,k}, K_{1,k+1}), d(K_{1,n}, K_{1,n+1})\} \). For \( \delta \in (0, 1) \) and \( \varepsilon \in (0, \varepsilon_0) \) we define \( f_{\delta, \varepsilon} = f \ast \chi_\varepsilon \in \mathcal{D}_\mu'(K_{1,k}, K_{2,k}) \). The mean value theorem implies that
\[
\exists N \in \mathbb{N}, K \geq N, C, r > 0: \forall \delta \in (0, 1) : |K_{1,m}|B_{m,N} \subseteq \delta B_{n+1,N} + \frac{C}{\delta^r}B_{k+1,K},
\]
(ii) Lemma $3.1(ii)$ yields that $\text{Proj}^1(\mathcal{D}'_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0$. By $[30]$ Theorem 3.2.9, we obtain that $(\mathcal{D}'_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}$ is strongly reduced. Hence, it suffices to prove that $(\mathcal{D}'_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}$ satisfies (PΩ). By a rescaling argument, it is enough to show that

$$\forall n \in \mathbb{N} \exists m \geq n \forall k \geq m \exists N \in \mathbb{N} \forall M \geq N \exists K \geq N, C_1, C_2, s, \varepsilon_0 > 0$$

(3.4)

$$\forall \varepsilon \in (0, \varepsilon_0) : B_{m,M} \subseteq \varepsilon C_1 B_{n,N} + \frac{C_2}{\varepsilon^s} B_{k,K}.$$  

$S_\mu : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ is surjective because $S_\mu : \mathcal{D}'(X_1) \rightarrow \mathcal{D}'(X_2)$ is so. Hence, Lemma $3.1(i)$ implies that $\text{Proj}^1(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0$. Therefore, by Lemma $2.4(\mathcal{E}_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}}$ satisfies (2.1). As we have that $\text{Proj}^1(\mathcal{D}'_\mu(K_{1,n}, K_{2,n}))_{n \in \mathbb{N}} = 0$, Theorem $2.3$ yields that

$$\forall n \in \mathbb{N} \exists m \geq n \forall k \geq m \exists N \in \mathbb{N} \forall M \geq N \exists K \geq N, C > 0 :$$

(3.5)

$$B_{m,M} \subseteq B_{n,N} + CB_{k,K}.$$  

We now show (3.4). Let $n \in \mathbb{N}$ be arbitrary. Choose $m_1 \geq n$ according to (2.1) and $m_2 \geq n$ according to (3.5). Set $m = \max\{m_1, m_2\} + 1$. Let $k \geq m$ be arbitrary. Condition (2.1) implies that there are $C, r > 0$ such that for all $\delta \in (0, 1)$

(3.6)

$$U_{m_1} \subseteq \delta U_{n,0} + \frac{C}{\delta^s} U_{k,0}.$$  

Choose $N \in \mathbb{N}$ according (3.5). Let $M \geq N$ be arbitrary. Condition (3.5) (applied to $M + 1$ instead of $M$) implies that there are $K \geq M + 1$ and $C' > 0$ such that

(3.7)

$$B_{m_2,M+1} \subseteq B_{n,N} + C'B_{k,K}.$$  

Set $\varepsilon_0 = \min\{1, d(K_{1,m-1}, K_{1,m})\}$. Let $f \in B_{m,M}$ be arbitrary. For $\varepsilon \in (0, \varepsilon_0)$ we define $f_\varepsilon = f \star \chi_\varepsilon \in \mathcal{E}_\mu(K_{1,m-1}, K_{2,m-1})$. The mean value theorem and (3.7) imply that

$$f - f_\varepsilon \in \varepsilon \sqrt{d} B_{m_2,M+1} \subseteq \varepsilon \sqrt{d} B_{n,N} + \sqrt{d} C'B_{k,K}.$$  

We have that

$$f_\varepsilon \in \|\chi\|_{\infty, M+m_1} \frac{1}{\varepsilon^{M+m_1+d}} U_{m_1},$$

Set $s = r(M + m_1 + d + 1) + M + m_1 + d$. By (3.6) (applied to $\delta = \varepsilon^{M+m_1+d+1}$), we obtain that

$$\frac{1}{\|\chi\|_{\infty, M+m_1}} f_\varepsilon \in \frac{1}{\varepsilon^{M+m_1+d}} U_{m_1} \subseteq \varepsilon U_{n,0} + \frac{C}{\varepsilon^s} U_{k,0} \subseteq \varepsilon |K_{1,n}| B_{n,N} + \frac{C}{\varepsilon^s} |K_{1,k}| B_{k,K}.$$  

Set $C_1 = \sqrt{d} + |K_{1,n}||\chi|_{\infty, M+m_1}$ and $C_2 = \sqrt{d} C' + |K_{1,k}||\chi|_{\infty, M+m_1} C$. The above inclusions yield that

$$f \in C_1 \varepsilon B_{n,N} + \frac{C_2}{\varepsilon^s} B_{k,K}.$$  

Recall from the introduction that for a differential operator $P(D)$ and an open set $X \subseteq \mathbb{R}^d$ we simply write

$$\mathcal{E}_P(X) = \{ f \in \mathcal{E}(X) \mid P(D)f = 0 \}$$
and
\[ \mathcal{D}'(X) = \{ f \in \mathcal{D}(X) \mid P(D)f = 0 \}. \]

**Remark 3.2.** Theorem 1.2 with \((\Omega)\) replaced by \((\overline{\Omega})\) \cite{26} and \((P\Omega)\) replaced by \((P\overline{\Omega})\) \cite{3} does not hold. In fact, Vogt \cite{29}, Theorem 14] showed that for any open convex set \(X \subseteq \mathbb{R}^d\), \(d > 1\), and any \(P \in \mathbb{C}[\xi_1, \ldots, \xi_d]\), the space \(\mathcal{E}'(X)\) does not satisfy \((\overline{\Omega})\). On the contrary, as \(\mathcal{D}'(X)\) satisfies \((P\overline{\Omega})\) \cite{3} Corollary 6.1] and this condition is inherited by complemented subspaces, \(\mathcal{D}'(X)\) satisfies \((P\overline{\Omega})\) for any open set \(X \subseteq \mathbb{R}^d\) and any \(P(D)\) such that \(P(D) : \mathcal{D}'(X) \to \mathcal{D}'(X)\) admits a continuous linear right inverse \cite{20}.

4. **The condition \((\Omega)\) for smooth kernels of convolution and differential operators**

4.1. **The augmented operator.** Let \(X_1, X_2 \subseteq \mathbb{R}^d\) be open and let \(T : \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)\) be a surjective continuous linear map. Bonet and Domański \cite{3} Proposition 8.3] showed that \(\ker T\) satisfies \((P\Omega)\) if and only if
\[ T \otimes \text{id}_{\mathcal{D}'(\mathbb{R})} : \mathcal{D}'(X_1 \times \mathbb{R}) \to \mathcal{D}'(X_2 \times \mathbb{R}) \]
is surjective.\(^2\)

Let \(\mu \in \mathcal{E}'(\mathbb{R}^d)\) and \(X_1, X_2 \subseteq \mathbb{R}^d\) be open sets such that \((\Omega)\) holds. Suppose that \(S_\mu : \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)\) is surjective. By the above result for \(T = S_\mu\), we find that \(\mathcal{D}'(X_1, X_2)\) satisfies \((P\Omega)\) if and only if
\[ S_\mu \otimes \text{id}_{\mathcal{D}'(\mathbb{R})} = S_{\mu \otimes \delta} : \mathcal{D}'(X_1 \times \mathbb{R}) \to \mathcal{D}'(X_2 \times \mathbb{R}) \]
is surjective. We call \(S_{\mu \otimes \delta}\) the **augmented operator** of \(S_\mu\). In particular, for a differential operator \(P(D)\) and an open set \(X \subseteq \mathbb{R}^d\), the augmented operator of \(P(D) : \mathcal{D}'(X) \to \mathcal{D}'(X)\) is given by the differential operator \(P^+(D) : \mathcal{D}'(X \times \mathbb{R}) \to \mathcal{D}'(X \times \mathbb{R})\), where \(P^+(\xi_1, \ldots, \xi_{d+1}) = P(\xi_1, \ldots, \xi_d)\) (cf. the introduction).

Theorem 1.2 enables us to also relate the condition \((\Omega)\) for the space of smooth zero solutions of a convolution operator to the surjectivity of its augmented operator – as explained in the introduction, before this was only known for hypoelliptic differential operators. More precisely, the following result holds.

**Theorem 4.1.** Let \(\mu \in \mathcal{E}'(\mathbb{R}^d)\) and \(X_1, X_2 \subseteq \mathbb{R}^d\) be open sets such that \((\Omega)\) holds. Suppose that \(S_\mu : \mathcal{D}'(X_1) \to \mathcal{D}'(X_2)\) is surjective. The following statements are equivalent:

(i) \(\mathcal{E}_\mu(X_1, X_2)\) satisfies \((\Omega)\).
(ii) \(\mathcal{D}'(X_1, X_2)\) satisfies \((P\Omega)\).
(iii) \(S_{\mu \otimes \delta} : \mathcal{D}'(X_1 \times \mathbb{R}) \to \mathcal{D}'(X_2 \times \mathbb{R})\) is surjective.
(iv) \((X_1 \times \mathbb{R}, X_2 \times \mathbb{R})\) is \(\mu \otimes \delta\)-convex for singular supports.

\(^2\)In fact, they only showed this statement for \(X_1 = X_2\). Since \(\mathcal{D}'(X_1) \cong (s')^N \cong \mathcal{D}'(X_2)\) for any two open sets \(X_1, X_2 \subseteq \mathbb{R}^d\), the above more general result is a consequence of the particular case \(X_1 = X_2\). Alternatively, the proof of \cite{3} Proposition 8.3] can be readily adapted.
Proof. \( S_\mu : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2) \) is surjective as \( S_\mu : \mathcal{D}'(X_1) \rightarrow \mathcal{D}'(X_2) \) is so.

(i) \( \Leftrightarrow \) (ii) This is shown in Theorem 4.2.

(ii) \( \Leftrightarrow \) (iii) As explained above, this follows from \[3\] Proposition 8.3.

(iii) \( \Leftrightarrow \) (iv) \( S_{\mu \otimes \delta} : \mathcal{D}'(X_1 \times \mathbb{R}) \rightarrow \mathcal{D}'(X_2 \times \mathbb{R}) \) is surjective if and only if \( (X_1 \times \mathbb{R}, X_2 \times \mathbb{R}) \) is \( \mu \otimes \delta \)-convex for supports and singular supports. Hence, (iii) \( \Rightarrow \) (iv) is trivial. For the converse direction, it suffices to note that \( (X_1 \times \mathbb{R}, X_2 \times \mathbb{R}) \) is \( \mu \otimes \delta \)-convex for supports since \( (X_1, X_2) \) is \( \mu \)-convex for supports (cf. \[9\] Proposition 1).

In the next two subsections, we will use Theorem 4.1 to show (\( \Omega \)) for the space of smooth zero solutions of several types of convolution and differential operators.

4.2. Differential operators. In \[3\] Corollary 8.4 it is shown that \( \mathcal{D}'(X) \) satisfies (P\( \Omega \)) for any differential operator \( P(D) \) and any open convex set \( X \subseteq \mathbb{R}^d \). In view of Theorem 4.1 this follows from the fact that \( P^+(D) : \mathcal{D}'(X \times \mathbb{R}) \rightarrow \mathcal{D}'(X \times \mathbb{R}) \) is surjective (as \( X \times \mathbb{R} \) is convex). Hence, Theorem 4.1 yields the following counterpart of this result in the smooth setting.

**Theorem 4.2.** Let \( P \in \mathbb{C}[\xi_1, \ldots, \xi_d] \) and let \( X \subseteq \mathbb{R}^d \) be open and convex. Then, \( \mathcal{E}_P(X) \) satisfies (\( \Omega \)).

**Remark 4.3.** (i) Theorem 4.2 has been shown by Petzsche in \[22\] Corollary 4.5] under the additional hypothesis that \( P \) is hypoelliptic. A careful inspection of his proof, which is based on the fundamental principle of Ehrenpreis, actually shows that his method may be used to prove Theorem 4.2 in its full generality.

(ii) We will extend Theorem 4.2 to difference-differential operators in Corollary 4.10 below.

Next, we combine Theorem 4.1 with several results of the second author \[15,16\] about the surjectivity of augmented operators to show (\( \Omega \)) for spaces of smooth zero solutions of certain (non-hypoelliptic) differential operators. We say that a polynomial \( P \in \mathbb{C}[\xi_1, \ldots, \xi_d] \) acts along a subspace \( W \subseteq \mathbb{R}^d \) if \( P(x) = P(\pi_W x) \) for all \( x \in \mathbb{R}^d \), where \( \pi_W \) denotes the orthogonal projection onto \( W \). A polynomial \( P \) which acts along a subspace \( W \) is said to be elliptic on \( W \) if for its principal part \( P_m \) it holds that \( P_m(x) \neq 0 \) for every \( x \in W \setminus \{0\} \).

**Theorem 4.4.** Let \( P \in \mathbb{C}[\xi_1, \ldots, \xi_d] \) and \( X \subseteq \mathbb{R}^d \) be open such that \( P(D) : \mathcal{E}_m(X) \rightarrow \mathcal{E}(X) \) is surjective. Then, \( \mathcal{E}_P(X) \) satisfies (\( \Omega \)) in the following cases:

(i) \( P \) acts along a subspace of \( \mathbb{R}^d \) and is elliptic there.

(ii) \( P(D) \) is a product of first order operators, i.e., with \( \alpha \in \mathbb{C} \setminus \{0\} \), \( c_j \in \mathbb{C} \) and \( N_j \in \mathbb{C}^{d_j} \setminus \{0\} \), \( j = 1, \ldots, l \) it holds that \( P(x) = \alpha \prod_{j=1}^l (N_j \cdot x - c_j) \).

(iii) \( d = 2 \).

Proof. In view of Theorem 4.2 this follows from the fact that in all three cases the surjectivity of \( P(D) : \mathcal{E}(X) \rightarrow \mathcal{E}(X) \) implies that \( P(D) : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X) \) is surjective and that \( \mathcal{D}'(X) \) satisfies (P\( \Omega \)) \[16\] Theorem 9 and Theorem 18(b)] for (i); \[16\] Corollary 10 and Theorem 18(c)] for (ii); \[15\] Theorem 21] for (iii). \( \square \)
Remark 4.5. In all three cases considered in Theorem 14, a geometric characterization of the sets $X$ for which $P(D):\mathcal{E}(X)\to\mathcal{E}(X)$ is surjective is known. A function $f:X\to[0,\infty]$ is said to satisfy the minimum principle in a closed set $F\subseteq\mathbb{R}^d$ if for every compact subset $K$ of $X\cap F$ it holds that 
\[
\inf_{x\in K} f(x) = \inf_{x\in\partial_F K} f(x),
\]
where $\partial_F K$ denotes the boundary of $K$ in $F$. Then, $P(D):\mathcal{E}(X)\to\mathcal{E}(X)$ is surjective if and only if
\begin{itemize}
  \item[(i)] ($P$ acts along a subspace $W$ of $\mathbb{R}^d$ and is elliptic there) The boundary distance $d_X:X\to\mathbb{R}$ satisfies the minimum principle in the affine subspace $x+W$ for every $x\in\mathbb{R}^d$.
  \item[(ii)] ($P(x) = \alpha \prod_{j=1}^l (N_j \cdot x - c_j)$ for some $\alpha \in \mathbb{C}\setminus\{0\}$, $c_j \in \mathbb{C}$ and $N_j \in \mathbb{C}^d\setminus\{0\}$, $j = 1,\ldots,l$) The boundary distance $d_X:X\to\mathbb{R}$ satisfies the minimum principle in the affine subspace $x+\text{span}\{\text{Re}N_j,\text{Im}N_j\}$ for every $x\in\mathbb{R}^d$ and $j = 1,\ldots,l$.
  \item[(iii)] ($d = 2$) The intersection of every characteristic line of $P$ with any connected component of $X$ is an interval.
\end{itemize}
(i) and (ii) are shown in [16] Theorem 9 and Corollary 10, while (iii) is a classical result of Hörmander [12] Theorem 10.8.3.

4.3. Convolution operators - the convex case. In [3] Corollary 8.5 it is shown that $\mathcal{E}_\mu'(\mathbb{R}^d,\mathbb{R}^d)$ satisfies (PΩ) for any invertible $\mu \in \mathcal{E}'(\mathbb{R}^d)$. This follows from Theorem 4.1, $\mu \otimes \delta$ is invertible because $\mu$ is so, whence $S_{\mu\otimes\delta} : \mathcal{E}'(\mathbb{R}^{d+1}) \to \mathcal{E}'(\mathbb{R}^{d+1})$ is surjective. Theorem 4.1 therefore yields the following counterpart of this result in the smooth setting.

Theorem 4.6. Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ be invertible. Then, $\mathcal{E}_\mu'(\mathbb{R}^d,\mathbb{R}^d)$ satisfies (Ω).

In the rest of this section we extend [3] Corollary 8.5 and Theorem 4.6 to arbitrary convex sets for a certain class of convolution operators. Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$. For a convex open set $X \subseteq \mathbb{R}^d$ we set 
\[
X_\mu = \{x \in \mathbb{R}^d | x - \text{supp } \mu \subseteq X\}.
\]
Let $Y \subseteq \mathbb{R}^d$ be an open convex set such that $Y - \text{supp } \mu \subseteq X$. By [12] Example 16.5.5 and Proposition 16.5.6], the pair $(X,Y)$ is $\mu$-convex for supports if and only if $Y = X_\mu$. Hence, if $\mu$ is invertible, $S_\mu : \mathcal{E}(X) \to \mathcal{E}(Y)$ is surjective if and only if $Y = X_\mu$. On the contrary, there exist invertible $\mu \in \mathcal{E}'(\mathbb{R}^d)$ and open convex sets $X \subseteq \mathbb{R}^d$ such that $(X,X_\mu)$ is not $\mu$-convex for singular supports, or equivalently, that $S_\mu : \mathcal{E}'(X) \to \mathcal{E}'(X_\mu)$ is not surjective. For example, let $\mu$ be the area element of the boundary of an ellipsoid in $\mathbb{R}^d$ and let $X$ be an open half-space. Then, [12] Theorem 16.3.20 and Proposition 16.5.14] imply that $(X,X_\mu)$ is not $\mu$-convex for singular supports. Consequently, the simple argument used to show Theorem 4.2 cannot be extended to general convolution operators. However, we now show that this can be done for convolution operators whose kernels are distributions of class $\mathcal{B}$. This class
of distribution kernels was introduced and studied by Berenstein and Dostal [12, 7].
A distribution $\mu \in \mathcal{E}'(\mathbb{R}^d)$ is said to be of class $\mathcal{R}$ [2, Definition 1] if
\[ \exists c, R, M > 0 \forall \zeta \in \mathbb{C}^d \exists z \in \mathbb{C}^d, |\zeta - z| < R : |\hat{\mu}(z)| \geq c(1 + |\text{Re}\, \zeta|)^{-M} e^{H_\mu(\text{Im}\, \zeta)}. \]
We recall that $H_\mu$ denotes the supporting function of $\text{supp}\, \mu$. The following examples
and properties of distributions of class $\mathcal{R}$ are taken from [2]:

**Example 4.7.**

(i) Every distribution with finite support is of class $\mathcal{R}$.

(ii) The characteristic function of a compact polyhedron and the area element of
its boundary are of class $\mathcal{R}$.

(iii) Let $\mu, \nu \in \mathcal{E}'(\mathbb{R}^d)$. Then, $\mu * \nu$ is of class $\mathcal{R}$ if and only if $\mu, \nu$ are of class $\mathcal{R}$.

(iv) The characteristic function of an ellipsoid and the area element of its boundary
are not of class $\mathcal{R}$.

Our interest in distributions of class $\mathcal{R}$ stems from the following result.

**Proposition 4.8.** [2, Proposition 1 and Proposition 2] Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ be of class $\mathcal{R}$. For every open convex set $X \subseteq \mathbb{R}^d$ it holds that $S_\mu : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X_\mu)$ is surjective.

Proposition 4.8 enables us to show the following result.

**Theorem 4.9.** Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ be of class $\mathcal{R}$. For every open convex set $X \subseteq \mathbb{R}^d$ it holds that $\mathcal{D}'(X, X_\mu)$ satisfies $(P_\Omega)$ and $\mathcal{E}(X, X_\mu)$ satisfies $(\Omega)$.

**Proof.** Note that $H_{\mu \otimes \delta}(\xi_1, \ldots, \xi_d) = H_\mu(\xi_1, \ldots, \xi_d)$. Hence, $\mu \otimes \delta$ is of class $\mathcal{R}$ because $\mu$ is so. The result therefore follows from Theorem 4.7 and Proposition 4.8. $\square$

As difference-differential operators are precisely the convolution operators whose
kernels have finite support, Example 4.7(i) and Theorem 4.9 yield the next result.

**Corollary 4.10.** Let $a_1, \ldots, a_l \in \mathbb{R}^d$ and $P_1, \ldots, P_l \in \mathbb{C}[\xi_1, \ldots, \xi_d]$. Let $X \subseteq \mathbb{R}^d$ be

convex and set $Y = \bigcap_{k=1}^l (a_k + X)$. Consider the difference-differential operator

\[ S : \mathcal{D}'(X) \rightarrow \mathcal{D}'(Y), \quad S(f)(x) = \sum_{k=1}^l P_k(D)f(x - a_k). \]

Then, $\ker(S : \mathcal{D}'(X) \rightarrow \mathcal{D}'(Y))$ satisfies $(P_\Omega)$ and $\ker(S : \mathcal{E}(X) \rightarrow \mathcal{E}(Y))$ satisfies $(\Omega)$.

We end this article by posing the following problem:

**Problem 4.11.** Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$ be invertible and $X \subseteq \mathbb{R}^d$ be open and convex.

(i) Suppose that $S_\mu : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X_\mu)$ is surjective. Does $\mathcal{D}'(X, X_\mu)$ satisfy $(P_\Omega)$?

(ii) Does $\mathcal{E}(X, X_\mu)$ satisfy $(\Omega)$? Recall that $S_\mu : \mathcal{E}(X) \rightarrow \mathcal{E}(X_\mu)$ is always surjective.

By using the theory of plurisubharmonic functions, Hörmander [12, Proposition 16.5.14]
gave a geometrical characterization of $\mu$-convexity for singular supports for pairs of
open convex sets in terms of the family $\mathcal{H}(\mu)$ [12, Definition 16.3.2]. It would be interesting to investigate if this characterization could be used to tackle Problem 4.11(i).

More precisely, one could try to show condition (iv) from Theorem 4.1 by verifying
the condition in Hörmander’s result for $\mu \otimes \delta$. To this end, one would need a good
description of $\mathcal{H}(\mu \otimes \delta)$ in terms of $\mathcal{H}(\mu)$. It is unclear to the authors how to obtain
this.

Incidentally, for distributions $\mu$ with finite support it holds that $\text{supp} \, \mu = \text{sing supp} \, \mu$
and $\mathcal{H}(\mu) = \{ H_\mu \}$ as well as $\mathcal{H}(\mu \otimes \delta) = \{ H_{\mu \otimes \delta} \}$ (see the remark after [12 Corollary
16.3.18]). Hence, $\mu$ is invertible [12, Theorem 16.3.10] and $(X,X_\mu)$ is $\mu$-convex for
singular supports and $(X \times \mathbb{R}, X_\mu \times \mathbb{R})$ is $\mu \otimes \delta$-convex for singular supports [12 Corollary
16.5.15]. In view of Theorem 4.1, this gives an alternative proof of Corollary 4.10 (recall
that $(X, X_\mu)$ is always $\mu$-convex for supports).

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