1 Introduction

Residential segregation is defined as the physical separation of two or more groups into different neighborhoods [Massey and Denton, 1988]. It is pervasive in metropolitan areas, where large homogeneous regions inhabited by residents belonging to the same ethnic group emerged over time. For more than five decades, residential segregation has been intensively studied by sociologists, as a high degree of segregation has severe consequences for the inhabitants of homogeneous neighborhoods. It negatively impacts their health [Acevedo-Garcia and Lochner, 2003], their mortality [Jackson et al., 2000], and in general their socioeconomic conditions [Massey and Denton, 1993]. While in the early days of research on segregation the emergence of homogeneous neighborhoods was attributed to the individual intolerance of the citizens, it was shown by [Schelling, 1971] that residential segregation also emerges in a tolerant population. In his landmark model, he considers two types of agents that live on a one-dimensional grid as residential area. Every agent has a tolerance level $\tau$ and is content with her position, if at least a $\tau$-fraction of her direct neighbors are of her type. Discontent agents randomly jump to other empty positions or swap positions with another discontent agent. Schelling found that even for $\tau < \frac{1}{2}$, i.e., even if everyone is content with being in the minority within her neighborhood, random initial placements are over time transformed to placements having large homogeneous regions, i.e., many agents are surrounded by same-type neighbors, by the individual random movements of the agents. It is important to note that the agent behavior is driven by a slight bias towards preferring a certain number of same-type neighbors and that this bias on the microlevel is enough to tip the macrolevel state towards segregation. Schelling coined the term "micromotives versus macrobehavior" for such phenomena [Schelling, 2006].

Since its inception, Schelling’s influential model was thoroughly studied by sociologists, mathematicians and physicists via computer simulations. But only in the last decade progress has been made to understand the involved random process from a theoretical point of view. Even more recently, the Algorithmic Game Theory and the AI communities became interested in residential segregation and game-theoretic variants of Schelling’s model were studied [Chauhan et al., 2018; Echzell et al., 2019; Bilò et al., 2020; Elkind et al., 2021; Kanellopoulos et al., 2021; Bullinger et al., 2021]. In these strategic games the agents do not perform random moves but rather jump or swap to positions that maximize their utility. These models incorporate utility functions that are monotone in the fraction of same-type neighbors, i.e., the utility of an agent is proportional to the fraction of same-type neighbors in her neighborhood. See Figure 1 (left). However,
representative sociological polls, in particular data from the
General Social Survey\textsuperscript{2} (GSS) [Smith et al., 2019], indicate
that this assumption of monotone utility functions should be
challenged. For example, in 1982 all black respondents where
asked “If you could find the housing that you would want and
like, would you rather live in a neighborhood that is all black;
mostly black; half black, half white; or mostly white?” and
54\% responded with “half black, half white” while only 14\% chose “all black”. Later, starting from 1988 until 2018 all
respondents (of whom on average 78\% were white) where
asked what they think of “Living in a neighborhood where
half of your neighbors were blacks?” a clear majority\textsuperscript{3}
responded “strongly favor”, “favor” or “neither favor nor
oppose”. This shows that the maximum utility should not be
attained in a homogeneous neighborhood.

Based on these real-world empirical observations, this paper
sets out to explore a game-theoretic variant of Schelling’s
model with non-monotone utility functions. In particular, we
will focus on single-peaked utility functions with maximum
utility at a $\Lambda$-fraction of same-type neighbors (see Figure 1
(middle and right)), with $\Lambda \in (0, 1)$, satisfying mild assump-
tions. More precisely, we only require a function $p(x)$ to be
zero-valued at $x = 0, 1$, to be strictly increasing in the inter-
val $[0, \Lambda]$ and to be such that $p(x) = p(\frac{\Lambda(1-x)}{1-\Lambda})$ for each
$x \in [\Lambda, 1]$, that is, both sides of $p$ approach the peak, one from
the left and the other from the right, in the same way, up to a
rescaling due to the width of their domains $([0, \Lambda], [\Lambda, 1])$.
Our main findings shed light on the existence of equilibrium
states and their quality in terms of the recently defined degree
of integration [Elkind et al., 2021] that measures the number
of agents that live in a heterogeneous neighborhood.

1.1 Model
We consider a strategic game played on a given underlying
connected graph $G = (V, E)$, with $|V| = n$ and $|E| = m$.
For any node $v \in V$, let the closed neighborhood of $v$ in $G$
be $V[v] = \{v\} \cup \{u \in V : \{v, u\} \in E\}$ where $d(v) = |
V[v]| - 1$ denotes the degree of $v$, and $\delta(G)$ and $\Delta(G)$
denote the minimum and the maximum degree over all nodes in $G$,
respectively. We call a graph $G$ $\delta$-regular, if $\delta(G) = \Delta(G)$
and $\delta(G) \geq 1$, and almost regular, if $\Delta(G) - \delta(G) \leq 1$.
We denote with $\alpha(G)$ the independence number of $G$, i.e.,
the cardinality of the maximum independent set in $G$.

A Single-Peaked Swap Schelling Game $(G, \Lambda)$, called
the game, is defined by a graph $G = (V, E)$, a positive in-
teger $b \leq n/2$ and a peak position $\Lambda$. There are $n$
strategic agents who choose nodes in $V$ such that every node is
occupied by exactly one agent. Every agent belongs to one of two
types that are associated with the colors blue and red. There
are $b$ blue agents and $r = n - b$ red agents, with blue being
the color of the minority type. Let $c(i)$ be the color of agent $i$.

A strategy profile $\sigma$ is an $n$-dimensional vector where all
strategies are pairwise disjoint, i.e., $\sigma$ is a permutation of $V$.

The $i$-th entry of $\sigma$ corresponds to the strategy of the $i$-th
agent. We treat $\sigma$ as a bijective function mapping agents to
nodes, with $\sigma^{-1}$ being its inverse function. Thus, any strat-
egy profile $\sigma$ corresponds to a bi-coloring of $G$ in which exac-
tly $b$ nodes of $G$ are colored blue and $n - b$ nodes are colored
red. We say that agent $i$ occupies node $v$ in $\sigma$ if the $i$-th entry
of $\sigma$, denoted as $\sigma(i)$, is $v$ and, equivalently, if $\sigma^{-1}(v) = i$.

We use the notation $1_{ij}(\sigma)$, with $1_{ij}(\sigma) = 1$ if agents $i$
d and $j$ occupy adjacent nodes in $\sigma$ and $1_{ij}(\sigma) = 0$ otherwise.

For an agent $i$ and a feasible strategy profile $\sigma$, we de-
note the set of nodes of $G$ which are occupied by agents
having the same color as agent $i$ by $C_i(\sigma) = \{v \in V : c(\sigma^{-1}(v)) = c(i)\}$. Observe that $C_i(\sigma)$ includes node $\sigma(i)$. Let
$f_i(\sigma) := \frac{|C_i(\sigma)|}{n}$ be the fraction of agents of
her own color in $i$'s neighborhood including herself. Thus,
agents are aware of their own contribution to the diversity of
their neighborhood. The utility of an agent $i$ in $\sigma$ is defined as
$U_i(\sigma) = p(f_i(\sigma))$, where $p$ is a single-peaked function with
domain $[0, 1]$ and peak at $\Lambda \in (0, 1)$ that satisfies the follow-
ing two properties: (i) $p$ is a strictly monotonically increasing
function in the interval $[0, \Lambda]$ with $p(0) = 0$; (ii) for each
$x \in [\Lambda, 1]$, $p(x) = p(\frac{\Lambda(1-x)}{1-\Lambda})$ and $p(\Lambda) = 1$. Each agent
aims at maximizing her utility. We say an agent $i$ is below the
peak when $f_i(\sigma) < \Lambda$, above the peak when $f_i(\sigma) > \Lambda$, at
the peak when $f_i(\sigma) = \Lambda$, and segregated when $f_i(\sigma) = 1$.

A game $(G, b, \Lambda)$ depends also on the choice of $p$. However,
as all our results are independent of $p$, we remove it from the
notation for the sake of simplicity.

An agent can change her strategy only via a swap, i.e.,
exchanging node occupation with another agent. Consider two
agents $i$ and $j$, on nodes $\sigma(i)$ and $\sigma(j)$, respectively, per-
foming a swap. This yields the new strategy profile $\sigma_{ij}$. As
agents are strategic, we only consider profitable swaps,
i.e., swaps which strictly increase the utility of both agents.

Hence, profitable swaps can only occur between agents of
different colors. A strategy profile $\sigma$ is a swap equilibrium
(SE), if $\sigma$ does not admit profitable swaps, i.e., if for each pair
of agents $i, j$, we have $U_i(\sigma) \geq U_i(\sigma_{ij})$ or $U_j(\sigma) \geq U_j(\sigma_{ij})$.

We measure the quality of a strategy profile $\sigma$ via the
degree of integration (DoI), defined by the number of non-
segregated agents. The DoI is a simple segregation mea-
sure that captures how many agents have contact with other-
type agents. We prefer it to the standard utilitarian wel-
fare since it measures segregation independently of the value
of $\Lambda$. For any fixed game $(G, b, \Lambda)$ let $\sigma^*$ denote a feasible
strategy profile maximizing the DoI and let $SE(G, b, \Lambda)$
denote the set of swap equilibria for $(G, b, \Lambda)$. We study
the impact of the agents’ selfishness by analyzing the Price of
Anarchy (PoA), which is defined as $PoA(G, b, \Lambda) = \frac{\min_{\sigma \in SE(G, b, \Lambda)} DoI(\sigma^*)}{DoI(\sigma^*)}$ and the Price of Stability (PoS), which
is defined as $PoS(G, b, \Lambda) = \frac{\max_{\sigma \in SE(G, b, \Lambda)} DoI(\sigma^*)}{DoI(\sigma^*)}$.

We investigate the finite improvement property (FIP) [Monden-
ner and Shapley, 1996], i.e., if every sequence of profitable
swaps is finite, which is equivalent to the existence of an ordinal
potential function. For this, let $\Phi(\sigma) = \{\{u, v\} : \in E : c(\sigma^{-1}(u)) = c(\sigma^{-1}(v))\}$,
counting the number of monochromatic edges of $G$ under $\sigma$,
i.e., the edges whose endpoints are occupied by agents of the same color, the potential function of $\sigma$.

1.2 Related Work

In the last decade progress has been made to thoroughly understand the involved random process in Schelling’s influential model, e.g., [Brandt et al., 2012; Barmpalias et al., 2014; Immorlica et al., 2017].

[Zhang, 2004a; 2004b] investigated the random Schelling process via evolutionary game theory. In particular, [Zhang, 2004b] proposes a model that is similar to our model. There, agents on a toroidal grid graph with degree 4 also have a non-monotone single-peaked utility function. However, in contrast to our model, random noise is added to the utilities and transferable utilities are assumed. Zhang analyzes the Markov process of random swaps and shows that this process converges with high probability to segregated states.

The investigation of game-theoretic models for residential segregation was initiated by [Chauhan et al., 2018]. There, agents are equipped with a utility function as shown in Figure 1 (left) and the finite improvement property and the PoA in terms of the number of content agents is studied. Later, [Echzell et al., 2019] significantly extended these results and generalized them to games with more than two agent types.

[Elkind et al., 2021] introduce a simplified model with $\tau = 1$. They prove results on the existence of equilibria, in particular that equilibria are not guaranteed to exist on trees, and on the complexity of deciding equilibrium existence. Moreover, they study the PoA in terms of the utilitarian social welfare and in terms of the newly introduced degree of integration, that counts the number of non-segregated agents. For the latter, they give a tight bound of $\frac{11}{6}$ on the PoA and the PoS that is achieved on a tree. In contrast, they derive a constant PoS on paths. [Bilò et al., 2020] strengthened the PoA results for the simplified swap version w.r.t. the utilitarian social welfare function and investigated the model on almost regular graphs, grids and paths. Additionally, they introduce a variant with locality.

Recently, a model was introduced where the agent itself is included in the computation of the fraction of same-type neighbors [Kanellopoulos et al., 2021]. We adopt this modified version also in our model. [Bullinger et al., 2021] consider the number of agents with non-zero utility as social welfare function. They prove hardness results for computing the social optimal state and they discuss other stability notions such as Pareto optimality.

Also related are hedonic games [Drèze and Greenberg, 1980; Bogomolnaia and Jackson, 2002] where selfish agents form coalitions and the utility of an agent only depends on her coalition. Especially close are hedonic diversity games [Brederick et al., 2019; Boehmer and Elkind, 2020], where agents of different types form coalitions and the utility depends also on the type distribution in a coalition.

Our main focus is on single-peaked utility functions. This can be understood as single-peaked preferences, which date back to [Black, 1948] and are a common theme in the Economics and Game Theory literature. In particular, such preferences yield favorable behavior in the above mentioned hedonic diversity games and in the realm of voting and social choice [Walsh, 2007; Yu et al., 2013; Betzler et al., 2013; Elkind et al., 2014; Brandt et al., 2015].

1.3 Our Contribution

In this work we initiate the study of game-theoretic models for residential segregation with non-monotone utility functions. This departs from the recent line of work focusing on monotone utility functions and it opens up a promising research direction. Non-monotone utility functions are well-justified by real-world data and hence might be more suitable for modeling real-world segregation.

We focus on a broad class of non-monotone utility functions well-known in Economics and Algorithmic Game Theory: single-peaked utilities. We emphasize that our results hold for all such functions that satisfy our mild assumptions. See Table 1 for a detailed result overview.

For games with integration-oriented agents, i.e., $\Lambda \leq 1/2$, we show that swap equilibria exist on almost regular graphs and that improving response dynamics are guaranteed to converge to such stable states. Moreover, for $\Lambda = \frac{1}{2}$ swap equilibria exist on the broad class of graphs that admit an independent set that is large enough to accommodate the minority type agents. In particular, this implies equilibrium existence and efficient computability on bipartite graphs, including trees, which is in contrast to the non-existence results by [Elkind et al., 2021].

Another contrast are our bounds on the PoA. On general graphs we prove a tight bound on the PoA that depends on $b$, the number of agents of the minority color, and we give a bound of $\Delta(G)$ on all graphs $G$, that is asymptotically tight on $\delta$-regular graphs. Also for the PoS we get stronger positive results compared to [Elkind et al., 2021]. For $\Lambda = \frac{1}{2}$ we give a tight PoS bound of $2$ on bipartite graphs and show that the PoS is $1$ on almost regular graphs with maximum degree $3$, or if the size of the maximum independent set of the graph is at most $b$. The latter implies a PoS of $1$ on regular graphs for balanced games, i.e., if there are equally many agents of both colors. Even more general, for constant $\Lambda \leq \frac{1}{2}$ we prove a constant PoS on almost regular graphs via a sophisticated proof technique that relies on the greedy algorithm for the K-MAX-CUT problem.

Additional complexity results and all omitted details can be found in [Bilò et al., 2022].

2 Preliminaries

In this section, we provide some facts and lemmas that will be widely exploited throughout the paper. We start by observing the following fundamental relationship occurring between $f_i(\sigma)$ and $f_j(\sigma_{ij})$ for two swapping agents $i$ and $j$:

$$f_i(\sigma) = \frac{x}{y}, \text{ then } f_j(\sigma_{ij}) = \frac{y + 1 - x - 1_{ij}(\sigma)}{y}.$$  \hspace{1cm} (1)

Using property (1), we claim the following observation.

\footnote{For the sake of conciseness, from now on, whenever we write $f_i(\sigma) = \frac{x}{y}$ for some agent $i$, we implicitly mean that $x := |N[\sigma(i)] \cap C_i(\sigma)|$ and $y := |N[\sigma(i)]|$. Observe that, under this assumption, $f_i(\sigma) = 3/6$ is different than $f_i(\sigma) = 1/2$.}
Existence of Equilibria

For any network, if 

\( f_i(\sigma) = x/y < 1/2 \), then 
\( f_j(\sigma_{ij}) > 1/2 \).

If 
\( f_i(\sigma) = x/y > 1/2 \), then 
\( f_j(\sigma_{ij}) \leq 1/2 \) unless 
\( y = 2x-1 \) and 
\( 1_{ij}(\sigma) = 0 \), for which 
\( f_j(\sigma_{ij}) = f_i(\sigma) = x/y > 1/2 \).

The following series of lemmas characterizes the conditions under which a profitable swap can take place.

**Lemma 1.** For any \( \Lambda \leq 1/2 \), no profitable swaps can occur between agents below the peak.

**Lemma 2.** For any \( \Lambda \leq 1/2 \), no profitable swaps can occur between adjacent agents at different sides of the peak.

**Proof.** Assume towards a contradiction, that \( i \) and \( j \) can perform a profitable swap in \( \sigma \), and, w.l.o.g., that 
\( f_i(\sigma) = x/y < \Lambda \) and 
\( f_j(\sigma_{ij}) = x'/y' > \Lambda \). By Observation 1, \( j \) ends up above the peak in \( \sigma_{ij} \). As \( j \) improves after the swap, we have 
\( U_j(\sigma_{ij}) = p(1-x/y) > U_j(\sigma) = p(x'/y') \) which, given that 
\( 1-x/y > \Lambda \) and 
\( x'/y' > \Lambda \), yields 
\( 1-x/y < x'/y' \). This implies that 
\( f_i(\sigma_{ij}) = 1-x'/y' < 1-1/x/y = x/y = f_i(\sigma) \) which, given that 
\( f_i(\sigma) < \Lambda \), contradicts the fact that \( i \) improves after the swap.

**Lemma 3.** For any \( \Lambda \leq 1/2 \), no profitable swaps can occur between agents at different sides of the peak in games on almost regular graphs.

**3 Existence of Equilibria**

In this section, we provide existential results for games played on some specific graph topologies. We start by showing that games on almost regular graphs enjoy the FIP property and converge to a SE in at most \( m \) steps in any game in which the peak does not exceed \( 1/2 \). This result does not hold when the peak exceeds \( 1/2 \), as we prove the existence of a game played on a 2-regular graph (i.e., a ring) admitting no SE.

**Theorem 1.** For any \( \Lambda \leq 1/2 \), fix a game \((G, b, \Lambda)\) on an almost regular graph \( G \) and a strategy profile \( \sigma \). Any sequence of profitable swaps starting from \( \sigma \) ends in a SE after at most \( m \) steps.

**Proof.** We show that, after a profitable swap, \( \Phi \) decreases by at least 1. Consider a profitable swap performed by agents \( i \) and \( j \) such that 
\( f_i(\sigma) = x/y \) and 
\( f_j(\sigma_{ij}) = x'/y' + t \), with 
\( t \in \{0, 1\} \) since \( G \) is almost regular. By Lemmas 1 and 3, we have that both \( i \) and \( j \) are above the peak, i.e., 
\( x/y > \Lambda \) and 
\( x'/y' + t > \Lambda \). Thus, a necessary condition for the swap to be profitable is that 
\( f_i(\sigma_{ij}) < f_i(\sigma) \) and 
\( f_j(\sigma_{ij}) < f_j(\sigma) \).

By Observation 1, the latter yields 
\( x'/y' + t > 1-x/y + (1-1_{ij}(\sigma))/y \), which gives 
\( x' > y-x+1-1_{ij}(\sigma)+t(1-x/y+1-1_{ij}(\sigma))/y \geq y-x+1-1_{ij}(\sigma) \). Since \( x, x', y \), and \( 1_{ij}(\sigma) \) are integers, we derive 
\( x' \geq y-x+2-1_{ij}(\sigma) \).

As it holds that 
\( \Phi(\sigma) - \Phi(\sigma_{ij}) \) equals 
\( -x-1+x'-1-(y-x-1_{ij}(\sigma)+y+t-x'-1_{ij}(\sigma)) = 2(x+x'-1+1_{ij}(\sigma)) - 2y - t \), we get 
\( \Phi(\sigma) - \Phi(\sigma_{ij}) \geq 1 \).

**Theorem 2.** For any \( \Lambda > 1/2 \), there exists a game played on a 2-regular graph admitting no SE.
Proof. Consider an instance of a game played on a ring with 6 nodes, where $b = r = 3$. Only the following two complementary cases may occur: Either, the blue agents occupy nodes that induce a path of length 2. In this case, there are two segregated agents of different colors, both with utility 0. As $p(0) = 0$ and $p(x) > 0$ for $x \in (0, 1)$, the two agents swap their positions. Or, there are two neighboring agents $i$ and $j$ of different colors being below the peak. In this case, as $p(1/3) < p(2/3)$, both $i$ and $j$ prefer to swap their positions. □

A fundamental question is whether a SE always exists in games with tolerant agents, i.e., for $\Lambda \leq 1/2$. Next result shows that Theorem 1 cannot be generalized to all graphs.

Theorem 3. There cannot exist an ordinal potential function in games on arbitrary graphs for $\Lambda = 1/2$.

For the special case of $\Lambda = 1/2$, however, existence of a SE is guaranteed in any graph whose independence number is at least the number of blue agents.

Theorem 4. Fix a game $(G, b, \Lambda)$ with
$$\frac{1}{\Delta(G)+1} \leq \Lambda \leq 1/2.$$
Any strategy profile in which all agents of a same color are located on an independent set of $G$ is a SE.

Proof. Let $\sigma$ be a strategy profile in which all agents of a same color are located on an independent set of $G$. Assume, w.l.o.g., that all blue agents are assigned to the nodes of an independent set of $G$ and consider a profitable swap performed by a blue agent $i$ and a red agent $j$. If $1_{ij}(\sigma) = 0$, since $i$ is only adjacent to red agents other than $j$, it holds that $f_j(\sigma_{ij}) = 1$, which gives $U_j(\sigma_{ij}) = 0$, thus contradicting the fact that $j$ performs a profitable swap. If $1_{ij}(\sigma) = 1$, instead, we obtain $f_i(\sigma) = \frac{1_{ij}(\sigma)}{\Delta(G)+1} \leq \frac{1}{\Delta(G)+1} \leq \Lambda$. The numerator comes from the fact that $i$ is only adjacent to red agents. Knowing that $i$ cannot be at the peak, we conclude that she is below the peak. If $j$ is also below the peak, Lemma 1 contradicts the fact that the swap is profitable, while, if $j$ is above the peak, the contradiction comes from Lemma 2. □

Corollary 1. For $\Lambda = 1/2$, games played on bipartite graphs always admit a SE which can be efficiently computed.

4 Price of Anarchy

In this section, we give bounds on the PoA for games played on different topologies, even in those cases for which existence of a SE is not guaranteed.

4.1 General Graphs

Next lemmas provide a necessary condition that needs to be satisfied by any SE and an upper bound of the value on the social optimum, respectively.

Lemma 4. In a SE for any game $(G, b, \Lambda)$, no agents of different colors can be segregated.

Proof. Fix a strategy profile $\sigma$. If there exist two agents $i$ and $j$ such that $f_j(\sigma) = f_j(\sigma) = 1$, they can perform a profitable swap, as $f_j(\tau) = f_j(\sigma) = 1$ and $f_j(\sigma_{ij}) = f_j(\sigma_{ij}) \notin \{0, 1\}$. So, $\sigma$ cannot be a SE for $(G, b, \Lambda)$. □

Figure 2: Lower bounds for PoA$(G, b, \Lambda)$ when (a) $b = 1$, and (b) $b > 1$. Left: the socially optimal placement $\sigma^*$. Right: the SE $\sigma$ with minimum social welfare.

Lemma 5. For any game $(G, b, \Lambda)$, we have
$$\text{DoI}(\sigma^*) \leq \min\{\Delta(G)+1, b\}.$$"
Theorem 7. For every $\delta \geq 2$ and $\Lambda \leq 1/2$, there exists a game $(G, b, \Lambda)$ on a $\delta$-regular graph such that
\[
\text{PoA}(G, b, \Lambda) \geq \frac{\delta(\delta+1)}{2\delta+1} = \frac{\delta+1}{2} - \frac{\delta+1}{2\delta+1}.
\]
The lower bound given in Theorem 7 holds for all values of $\delta$. It may be the case then that, for fixed values of $\delta$, better bounds are possible. For $\delta = 2$ indeed, lower bounds matching the upper bounds given in Theorems 5 and 6 can be derived.

Theorem 8. For any $\epsilon > 0$, there exists a game $(G, b, \Lambda)$ on a ring such that $\text{PoA}(G, b, 1/2) > 2 - \epsilon$ and $\text{PoA}(G, b, \Lambda) > 3/2 - \epsilon$ for $\Lambda < 1/2$.

5 Price of Stability

In this section, we give bounds on the PoS for games played on different topologies.

5.1 General Graphs

We give a lower bound on the PoS on general graphs which asymptotically matches the upper bound on the PoA when $b = \Theta(\sqrt{n})$ and $\Lambda$ is a constant w.r.t. $n$.

Theorem 9. For every $\Lambda$, there is a game $(G, b, \Lambda)$ such that $\text{PoS}(G, b) = \Omega(\sqrt{n\Lambda})$.

5.2 Bipartite Graphs

For bipartite graphs, we provide a tight bound of 2 for the PoS of games for which the peak is at 1/2. We start with the upper bound.

Theorem 10. For any game $(G, b, 1/2)$ on a bipartite graph $G$, we have $\text{PoS}(G, b, 1/2) \leq 2$.

We now give the matching lower bound.

Theorem 11. There exists a game $(G, b, 1/2)$ on a bipartite graph such that $\text{PoS}(G, b, 1/2) \geq 2$.

5.3 Almost Regular Graphs

We provide upper bounds to the PoS for games played on almost regular graphs. We start by considering the case of graphs with small degree.

Theorem 12. For any game $(G, b, \Lambda)$ on an almost regular graph with $\Delta(G) \leq 3$ and $\Lambda \leq 1/2$, $\text{PoS}(G, b, \Lambda) = 1$.

An analogous result holds for the case in which $b \geq \alpha(G)$.

Theorem 13. For any game $(G, b, \Lambda)$ on an almost regular graph with $b \geq \alpha(G)$ and $\frac{1}{\Delta(G)+1} \leq \Lambda \leq 1/2$, we have $\text{PoS}(G, b, \Lambda) = 1$.

A game $(G, b, \Lambda)$ is balanced if $b = \lfloor n/2 \rfloor$. Using Theorem 13, we show that the PoS is 1 in balanced games on regular graphs.

Corollary 2. For any balanced game $(G, b, \Lambda)$ on a $\delta$-regular graph $G$ and $\frac{1}{\delta+1} \leq \Lambda \leq 1/2$, we have $\text{PoS}(G, b, \Lambda) = 1$.

Proof. We have that $b = \lfloor n/2 \rfloor$. We show that $\alpha(G) \leq \lfloor n/2 \rfloor$ using a simple counting argument. This allows us to use Theorem 13 to prove the claim.

To show the upper bound on $\alpha(G)$, we count all the edges that are incident to the nodes of a fixed maximum independent set of $G$ and bound this value from above by the number of edges of the graph, thus obtaining the following inequality $\delta \alpha(G) \leq \frac{\delta}{2} n$, i.e., $\alpha(G) \leq n/2$. Using the fact that $\alpha(G)$ is an integer value, we derive $\alpha(G) \leq \lfloor n/2 \rfloor$.

We now give the upper bound to the PoS for games played on almost regular graphs when $b < \alpha(G)$.

Theorem 14. For any game $(G, b, \Lambda)$ on an almost regular graph $G$ with $b < \alpha(G)$ and $\Lambda \leq 1/2$, we have $\text{PoS}(G, b, \Lambda) = \min\{\Delta(G) + 1, O(1/\Lambda)\}$.

We can derive the following upper bound to the PoS.

Corollary 3. For any game $(G, b, \Lambda)$ on an almost regular graph with a constant value of $\Lambda \leq 1/2$, we have $\text{PoS}(G, b, \Lambda) = O(1)$.

Proof. By Theorem 5, the PoS is constant if $\Delta(G)$ is constant. The result when $\Delta(G)$ is not constant is divided into two cases. For the case $b \geq \alpha(G)$ the claim immediately follows from Theorem 13. For the case $b < \alpha(G)$ the claim follows from Theorem 14 and the fact that $\Lambda$ is constant by assumption.

6 Conclusion and Future Work

We study game-theoretic residential segregation with integration-oriented agents and thereby open up the novel research direction of considering non-monotone utility functions. Our results clearly show that moving from monotone to non-monotone utilities yields novel structural properties and different results in terms of equilibrium existence and quality. We have equilibrium existence for a larger class of graphs, compared to [Elkind et al., 2021], and it is an important open problem to prove or disprove if swap equilibria for our model with $\Lambda \leq \frac{1}{2}$ are guaranteed to exist on any graph.

So far we considered single-peaked utilities that are supported by data from real-world sociological polls. However, also other natural types of non-monotone utilities could be studied. Also ties in the utility function could be resolved by breaking them consistently towards favoring being in the minority or being in the majority. The non-existence example of swap equilibria used in the proof of Theorem 2 also applies to the case with $\Lambda = \frac{1}{2}$ and breaking ties towards being in the majority. Interestingly, by breaking ties the other way we get the same existence results as without tie-breaking and also our other results hold in this case. This is another indication that tolerance helps with stability.

Moreover, all our existence results also hold for utility functions having a symmetric plateau shape around $\Lambda$. Investigating the PoA for these utility functions is open.

Regarding the quality of the equilibria, we analyze the degree of integration as social welfare function, as this is in line with considering integration-oriented agents. Of course, studying the quality of the equilibria in terms of the standard utilitarian social welfare, i.e., $\text{SUM}(\sigma) = \sum_{i=1}^{n} U_i(\sigma)$, would also be interesting. We note in passing that on ring topologies the PoA and the PoS with respect to both social welfare functions coincide.
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