INVISCID LIMIT FOR 2D STOCHASTIC NAVIER-STOKES EQUATIONS

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Abstract. We consider stochastic Navier-Stokes equations in a 2D-bounded domain with the Navier with friction boundary condition. We establish the existence and the uniqueness of the solutions and study the vanishing viscosity limit. More precisely, we prove that solutions of stochastic Navier-Stokes equations converge, as the viscosity goes to zero, to solutions of the corresponding stochastic Euler equations.

Key words: Stochastic Navier-Stokes equations, Stochastic Euler equations, Navier slip boundary conditions, vanishing viscosity, boundary layer, turbulence.

Mathematics Subject Classification (2000): 60H15, 60H30, 76B99, 76D05.

1. Introduction

The study of the inviscid limit of the solutions of the Navier-Stokes equations is a classical issue in fluid mechanics. The knowledge of the behavior of the solutions for small viscosities (very high Reynolds numbers) is crucial to understand the turbulence phenomena. The mathematical resolution of the inviscid limit problem should have strong consequences in many branches of engineering (technology involving heat and mass transfer), as aircraft production, turbine blades, nanotechnology, etc..

The investigation of this problem for domains without boundary was performed, for instance, in [4], [14], [15], [19]. When the Navier-Stokes equations are considered with a stochastic random force, the inviscid limit of its solutions is studied in [9].

In the case of bounded domains, the Navier-Stokes equations should be supplemented with boundary conditions. The most studied and widely accepted is the Dirichlet boundary condition which prescribes the value of the velocity field on the surface boundary. In the presence of the impermeable boundary, the normal and the tangential components of the velocity are assumed to be zero on the surface. For the Euler equations it is just required that the velocity field be tangent to the boundary. In the vanishing viscosity strong boundary layers arise, which are very difficult to treat and the inviscid limit remains an open problem. Just partial results have been obtained (see [28], [32]). Other physical meaningful boundary condition is the so called Navier slip boundary condition. This boundary condition was initially introduced by Navier [26] in 1827 and due to recent experimental results (see [11], [18], [27]), has been renewed interest in this boundary condition.

To be more precise, we suppose that \( \mathcal{O} \) is a bounded simply connected domain in \( \mathbb{R}^2 \) with boundary \( \Gamma \) sufficiently regular. The Navier slip with friction boundary condition, for the Navier-Stokes equations, is written by

\[
2D(u)\mathbf{n} \cdot \mathbf{t} + \alpha u \cdot \mathbf{t} = 0 \quad \text{on} \quad ]0, T[ \times \Gamma
\]

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where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the rate-of-strain tensor; $\mathbf{n}$ and $\mathbf{t}$ are the unit exterior normal and the unit tangent vector, respectively, to $\Gamma$, $\{\mathbf{n}, \mathbf{t}\}$ being a direct basis; and $\cdot$ defines the scalar product on $\mathbb{R}^2$. Here the tangent component of the fluid velocity at the boundary, rather than being fixed, is proportional to the tangential stress. The normal component of the fluid velocity at the boundary is zero and corresponds to the impermeability of the boundary:

$$u \cdot \mathbf{n} = 0 \quad \text{on } ]0, T[ \times \Gamma.$$  

(2)

The key feature of this boundary condition (1)-(2) is that it can be expressed in terms of the vorticity $\xi$ of the vector field $u$ as

$$\xi(u) = (2\kappa - \alpha)u \cdot \mathbf{t} \quad \text{and} \quad u \cdot \mathbf{n} = 0 \quad \text{on } ]0, T[ \times \Gamma,$$

(3)

which permits to handle the vorticity formulation of the Navier-Stokes equations. The coefficient $\alpha$ is a known function describing physical properties and $\kappa$ is the curvature of the boundary.

A particular case of this boundary condition, with $\alpha = 2k$,

$$\xi = 0 \quad \text{on } ]0, T[ \times \Gamma$$

(4)

was considered in [23]; where an energy type estimate for $\xi$ was established, allowing to prove the convergence of the solutions of the Navier-Stokes equations to solutions of the Euler equations. This boundary condition is also known as the Lions boundary condition or free boundary condition. Besides its mathematical importance this particular boundary condition do not permits the creation of the vorticity on the boundary. The deterministic methods were extended in [5] and [9], to obtain some well posedness results for 2D stochastic Euler equations. In both articles, the stochastic Euler equations are regularized by the corresponding viscous stochastic Navier-Stokes equations supplemented with the Lions boundary condition; the zero-viscosity limit provides the solution for the stochastic Euler equations. In [5] is considered an additive noise and the inviscid limit is a strong solution (in the probability sense) of stochastic Euler equations. Moreover, a uniqueness result is established if the initial vorticity belongs to $L^\infty$. In [9], a less regular multiplicative noise is considered and the inviscid limit gives a martingale solution to the stochastic Euler equations. More recently, [6] handled this particular case of the Navier slip boundary condition for stochastic Navier-Stokes equations with a multiplicative noise and studied the viscous limit using the large deviations techniques, taking the square root of the viscosity in front of the noise.

In the deterministic framework, the study of the inviscid limit for the solutions of the Navier-Stokes equations with the physical Navier slip boundary conditions (1) has been greatly developed. In [13], the solvability of the Navier Stokes equations with the boundary condition (1) was established in the class of $L^\infty$-bounded vorticity. It was also proved that the vanishing viscosity limit is well described by Euler equations. Later on, in [24], this result was generalized for the class of $L^p$-bounded vorticity with $p > 2$. A rate of the vanishing viscous convergence of solutions of the Navier-Stokes equations to solutions of the Euler equations, in the class of almost $L^\infty$-bounded vorticity, was obtained in [21].

In the present work we consider stochastic Navier-Stokes equations, with an additive noise, on a bounded domain of $\mathbb{R}^2$, subjected to the Navier slip with friction boundary condition (1)-(2), which provides creation of the vorticity on the boundary proportional
to the tangential velocity, and tackle the problem of the inviscid limit. In some sense, our result is the probability counterpart of the deterministic result obtained in [24].

The article is organized as follows: in the section 2, we introduce the functional spaces, construct the appropriate Wiener process and state the main result Theorem 2.1. In the Section 3, we deduce the $L^2$ a priori estimates for the viscous solutions independent of $\nu$. The section 4 contains the relevant $L^p$ a priori estimates for the viscous vorticity independent of $\nu$. These estimates permit to establish the well posedness for the Navier-Stokes equations with the Navier boundary condition. In the last section we obtain crucial path-wise estimates independent of the viscosity, that allow to establish the inviscid limit.

2. VELOCITY EQUATIONS WITH ADDITIVE NOISE

We consider the following stochastic Navier-Stokes equations in dimension 2:

$$\begin{align*}
\frac{\partial u'^{r}(t)}{\partial t} - \nu \Delta u'^{r}(t) + (u'^{r}(t) \cdot \nabla)u'^{r}(t) + \nabla p(t) &= f(t) + \sqrt{Q} \dot{W} \quad \text{in } [0, T[\times \mathcal{O}, \\
\text{div } u'^{r} &= 0 \quad \text{in } [0, T[\times \mathcal{O}, \\
w'^{r}(0) &= u_0 \quad \text{in } \mathcal{O}, \\
u'^{r} \cdot n &= 0 \quad \text{on } [0, T[\times \Gamma, \\
2D(u'^{r})n \cdot t + \alpha u'^{r} \cdot t &= 0 \quad \text{on } [0, T[\times \Gamma}
\end{align*}$$

where $\nu > 0$ is the coefficient of kinematic viscosity, $\Delta$ denotes the Laplacian, $\nabla$ denotes the gradient, $\text{div } u'^{r} = \nabla \cdot u'^{r} = \sum_{i=1}^{2} \partial_{i} u'^{r,i}$, $\alpha(x)$ is a given positive twice continuously differentiable function defined on $\Gamma = \partial \mathcal{O}$, $u'^{r}$ is the velocity and $p$ is the pressure. $f(t, x)$ is a given deterministic force and $\sqrt{Q} \dot{W}$ is the formal derivative of a Gaussian random field in time and correlated in space that will be set below.

We introduce the following Hilbert spaces

$$H = \left\{ v \in \left[ L^2(\mathcal{O}) \right]^2 : \nabla \cdot v = 0 \text{ in } \mathcal{O} \text{ and } v \cdot n = 0 \text{ on } \Gamma \right\},$$

$$V = \left\{ v \in \left[ H^1(\mathcal{O}) \right]^2 : \nabla \cdot v = 0 \text{ in } \mathcal{O} \text{ and } v \cdot n = 0 \text{ on } \Gamma \right\},$$

$$\mathcal{W} = \left\{ v \in V \cap \left[ H^2(\mathcal{O}) \right]^2 : 2D(u'^{r})n \cdot t + \alpha u'^{r} \cdot t = 0 \text{ on } \Gamma \right\}.$$

It can be verified (see Lemma 2.1 of [13]) that

$$\mathcal{W} = \left\{ v \in V \cap \left[ H^2(\mathcal{O}) \right]^2 : \text{curl } v = (2\kappa - \alpha) v \cdot t \text{ on } \Gamma \right\}$$

where $\kappa$ denotes the curvature of $\Gamma$. We recall that $\text{curl } v = \partial_{1} v^{2} - \partial_{2} v^{1}$.

We consider on $H$ the $L^2$- inner product and norm that we denote by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|_{L^2}$. $V$ is endowed with the inner product

$$\langle u, v \rangle_{V} = \langle \nabla u, \nabla v \rangle$$

and the associated norm $\| \cdot \|_{V}$. We recall that, from the Poincaré's inequality, this norm is equivalent to the $H^1$-norm.
Let us denote by $V'$ the topological dual of $V$ and by $(\cdot, \cdot)_{V', V}$ the corresponding duality. We define the operator $\mathcal{A} : V \to V'$ by
\[
(\mathcal{A}u, v)_{V', V} = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Gamma} (\kappa - \alpha) u \cdot v,
\]
for all $u, v \in V$. Since
\[
| (\mathcal{A}u, v)_{V', V} | \leq C \| u \|_{V} \| v \|_{V}
\]
$\mathcal{A}$ is a continuous operator form $V$ to $V'$. Moreover $\mathcal{A} : W \to H$ coincides with the stokes operator $-P_{H} \Delta$, where $P_{H}$ denotes the Leray projector. More precisely we have
\[
(\mathcal{A}u, v)_{V', V} = \langle -\Delta u, v \rangle, \quad u \in W, v \in V.
\]

We also define $\mathcal{B} : V \to V'$ as $\mathcal{B}(u) = (u \cdot \nabla)u$, that is,
\[
(\mathcal{B}(u), v) = \int_{\Omega} (u \cdot \nabla) u \cdot v,
\]
for all $u, v \in V$.

From Lemma 2.2 of [13], there exists a basis $\{v_{k}\} \subset W$ for $V$, of eigenfunctions of the operator $\mathcal{A}$, being simultaneously an orthonormal basis for $H$. The corresponding sequence $\{\lambda_{k}\}$ of eigenvalues verifies $\lambda_{k} > 0, \forall k \in \mathbb{N}$ and $\lambda_{k} \to \infty$ as $k \to \infty$. Henceforth we shall consider this basis.

To be more specific, we shall take in the following $Q = A^{-2m}$, where $m \in \mathbb{N}$ will be fixed later and $W(t) = \sum_{k=1}^{\infty} \beta_{k}(t)v_{k}, \ t \geq 0$. Here $\{\beta_{k}\}$ denotes a sequence of standard Brownian motion mutually independent defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_{t}\}_{t \geq 0})$. In fact,
\[
\sqrt{Q}W(t) = \sum_{k=1}^{\infty} \beta_{k}(t)\sqrt{Q}v_{k} = \sum_{k=1}^{\infty} \lambda_{k}^{-m}v_{k}\beta_{k}(t)
\]
is a $H$-valued centered Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, with covariance $Q$ in $H$. We take $m \in \mathbb{N}$ such that
\[
\mathcal{M} := \sum_{k=1}^{\infty} \lambda_{k}^{-2m+3} < \infty.
\]
Then, with this choice of $m$ we have that $Q$ is an operator of trace class. We denote the trace of $Q$ by $\text{tr}(Q) = \sum_{k=1}^{\infty} \langle Qv_{k}, v_{k} \rangle = \sum_{k=1}^{\infty} \lambda_{k}^{-2m}$. Let us mention that a similar noise was considered in [3].

In terms of $\mathcal{A}, \mathcal{B}$ and $f$ we can write Equation (5) as the following stochastic evolution equation in $V'$:
\[
\begin{cases}
    d\nu^\nu = F(t, \nu^\nu(t)) \ dt + \sqrt{Q}dW(t) & \text{in } [0, T] \times \Omega, \\
    \nu^\nu(0) = u_{0} & \text{in } \Omega,
\end{cases}
\tag{9}
\]
where $F(t, \nu^\nu) = f - \nu \mathcal{A}\nu^\nu - \mathcal{B}(\nu^\nu)$.

**Definition 2.1.** Given $u_{0} \in L^{2}(\Omega; H)$, an adapted stochastic process $\nu^\nu$ with sample paths in $C([0, T]; H) \cap L^{2}(0, T; V)$ is said a weak solution of the stochastic Navier-Stokes equation (9) if
\[
\langle \nu^\nu(t), v \rangle = \langle u_{0}, v \rangle + \int_{0}^{t} \langle F(s, \nu^\nu(s)), v \rangle \ ds + \int_{0}^{t} \langle \sqrt{Q}dW(s), v \rangle, \tag{10}
\]
in $[0, T]$, for all $v \in V$ and a.e. $\omega \in \Omega$. 
For the viscosity equal to zero we consider the stochastic two-dimensional Euler equations

\[
\begin{aligned}
\frac{\partial u(t)}{\partial t} + (u(t) \cdot \nabla)u(t) + \nabla p(t) &= f(t) + \sqrt{q} \dot{W} & \text{in } ]0, T[ \times \Omega,
\text{div } u &= 0 & \text{in } ]0, T[ \times \Omega,
\end{aligned}
\]

\[u(0) = u_0 \quad \text{in } \Omega,
\]

\[u \cdot n = 0 \quad \text{on } ]0, T[ \times \Gamma
\]

which can be written in terms of the operator \(B\) and \(f\) by the following stochastic evolution equation in \(V'\):

\[
\begin{aligned}
du(t) &= \{f(t) - B(t, u(t))\} \ dt + \sqrt{q} \ dW(t) & \text{in } ]0, T[ \times \Omega,
u^0(0) = u_0 \quad \text{in } \Omega.
\end{aligned}
\]

The main result of this article is the following:

**Theorem 2.1.** Let \(T > 0\), \(\nu_0 > 0\) and \(p > 2\). Suppose that \(f \in L^2(0, T; H)\), \(\text{curl } f \in L^1(0, T; L^p(\Omega))\), \(u_0 \in L^p(\Omega; H)\) and \(u_0 \in L^p(\Omega; L^p(\Omega))\). Then we have:

(i) For any \(\nu \in [0, \nu_0]\), there exists a unique weak solution \(u^\nu\) of the stochastic Navier-Stokes equation (9) such that

\[u^\nu \in L^p(\Omega; C([0, T]; H)) \cap L^2(\Omega; L^2(0, T; V)) \cap [L^4(0, T; [\Omega \times \Omega])],\]

\(\text{curl } u^\nu \in L^2(\Omega; L^\infty(0, T; L^p(\Omega))).\)

(ii) In addition, if \(\text{curl } f \in L^1(0, T; L^\infty(\Omega))\), there exists a measurable stochastic process \(u\) that is a solution of the incompressible 2D stochastic Euler equation (12), in the sense that

\[
\langle u(t), v \rangle = \langle u_0, v \rangle - \int_0^t \langle B(u(s)), v \rangle \ ds + \int_0^t \langle f(s), v \rangle \ ds
\]

\[
+ \int_0^t \langle \sqrt{q} dW(s), v \rangle
\]

for all \(v \in V\) and \(\mathbb{P}\text{-a.e. } \omega \in \Omega\). Furthermore, taking \(\text{curl } u_0 \in L^p(\Omega; L^\infty(\Omega))\), for \(\mathbb{P}\text{-a.e. } \omega \in \Omega\)

\[u^\nu(\omega) \rightarrow u(\omega) \quad \text{strongly in } C([0, T]; H), \quad \text{as } \nu \rightarrow 0.
\]

3. **\(L^2\) A PRIORI ESTIMATES FOR THE VELOCITY AND SOLVABILITY OF THE NAVIER-STOKES EQUATIONS**

We consider the following Faedo-Galerkin approximations of Equation (9). Let \(H_n = \text{span} \{v_1, \ldots, v_n\}\) and define \(u^\nu_n\) as the solution of the following stochastic differential equation:

For each \(v \in H_n\),

\[
d\langle u^\nu_n(t), v \rangle = \langle F(t, u^\nu_n(t)), v \rangle \ dt + \langle \sqrt{q} dW(t), v \rangle,
\]

with \(u^\nu_n(0) = \sum_{k=1}^n \langle u_0, v_k \rangle v_k\).
Notice that Equation (14) defines a system of stochastic ordinary differential equations in \( \mathbb{R}^n \) with locally Lipschitz coefficients. Therefore, we need some a priori estimate to prove the global existence of a solution \( u_n^\nu(t) \) as an adapted process in the space \( C([0, T]; H_n) \).

**Proposition 3.1.** Let \( T > 0 \) and \( \nu_0 > 0 \). Suppose that \( f \in L^1(0, T; H) \) and \( u_0 \in L^2(\Omega; H) \). Let \( u_n^\nu(t) \) be an adapted process in the space \( C([0, T]; H_n) \) solution of Equation (14). Then

\[
\sup_{0 < \nu \leq \nu_0} \sup_n \left\{ \mathbb{E} \left( \sup_{0 \leq r \leq T} \| u_n^\nu(r) \|_{L^2}^2 \right) + \nu \int_0^T \mathbb{E} \left( \| u_n^\nu(s) \|_{L^2}^2 \right) \, ds \right\} \leq C(f, \Theta, \nu_0) \left( \mathbb{E} \left( \| u_0 \|_{L^2}^2 \right) + 1 \right).
\]  

(15)

Furthermore we have

\[
\| u_n^\nu(t \wedge \tau_N) \|_{L^2}^2 = \| u_n^\nu(0) \|_{L^2}^2 + 2 \int_0^{t \wedge \tau_N} \langle \nabla u_n^\nu(s), u_n^\nu(s) \rangle \, ds + 2 \int_0^{t \wedge \tau_N} \langle F(s, u_n^\nu(s)), u_n^\nu(s) \rangle \, ds
\]

\[
+ 2 \int_0^{t \wedge \tau_N} \langle \sqrt{Q} \, dW(s), u_n^\nu(s) \rangle + \int_0^{t \wedge \tau_N} \text{tr}(Q) \, ds.
\]  

(16)

**Proof.** For each \( N \in \mathbb{N} \), let us consider the stopping time \( \tau_N = \inf \{ t \geq 0 : \| u_n^\nu(t) \|_{L^2} \geq N \} \wedge T \). From Itô’s formula

\[
\| u_n^\nu(t \wedge \tau_N) \|_{L^2}^2 = \| u_n^\nu(0) \|_{L^2}^2 + 2 \int_0^{t \wedge \tau_N} \langle \nabla u_n^\nu(s), u_n^\nu(s) \rangle \, ds
\]

\[
+ 2 \int_0^{t \wedge \tau_N} \langle \sqrt{Q} \, dW(s), u_n^\nu(s) \rangle + \int_0^{t \wedge \tau_N} \text{tr}(Q) \, ds.
\]  

(17)

Applying to (17) the definition of operator \( \mathcal{A} \) (6) and of operator \( \mathcal{B} \) (7), respectively, and integration by parts formula, we obtain expression (18).

\[
\| u_n^\nu(t \wedge \tau_N) \|_{L^2}^2 = \| u_n^\nu(0) \|_{L^2}^2 + 2 \int_0^{t \wedge \tau_N} \langle \nabla u_n^\nu(s), u_n^\nu(s) \rangle \, ds
\]

\[
+ 2 \int_0^{t \wedge \tau_N} \langle \sqrt{Q} \, dW(s), u_n^\nu(s) \rangle + \int_0^{t \wedge \tau_N} \text{tr}(Q) \, ds.
\]  

(18)

Moreover,

\[
\int_\Gamma (\kappa - \alpha) u_n^\nu(s) \cdot u_n^\nu(s) \, d\mathcal{S} \leq \sup_\Gamma |\kappa - \alpha| \| u_n^\nu(s) \|_{L^2(\Gamma)}^2
\]

\[
\leq \sup_\Gamma |\kappa - \alpha| C(\Theta) \frac{1}{\sqrt{2\varepsilon}} \| u_n^\nu(s) \|_{L^2(\mathcal{S})} \sqrt{2\varepsilon} \| \nabla u_n^\nu(s) \|_{L^2}
\]

\[
\leq \varepsilon \| \nabla u_n^\nu(s) \|_{L^2}^2 + C(\varepsilon) \| u_n^\nu(s) \|_{L^2}^2.
\]  

(19)

where \( C(\varepsilon) = \sup_\Gamma |\kappa - \alpha| C(\Theta)^2 \frac{1}{4\varepsilon} \).
The application of Cauchy-Schwarz’s inequality gives
\[ |\langle f(s), u_n^\nu(s) \rangle| \leq \|f(s)\|_{L^2} \|u_n^\nu(s)\|_{L^2} \leq \|f(s)\|_{L^2} (1 + \|u_n^\nu(s)\|_{L^2}^2). \] (20)

Applying Burkholder-Davis-Gundy’s inequality
\[
E \left( \sup_{0 \leq r \leq t} \left\{ 2 \int_0^{r \wedge T_N} \langle \sqrt{Q} dW(s), u_n^\nu(s) \rangle \right\} \right) \leq 2C_1 \text{tr}(Q)^{1/2} E \left( \int_0^{T_N} \|u_n^\nu(s)\|_{L^2}^2 ds \right)^{1/2}
\leq C_1 \text{tr}(Q) + C_1 E \left( \int_0^{T_N} \|u_n^\nu(s)\|_{L^2}^2 ds \right). \] (21)

Using expression (18), estimates (19), (20) and (21), and \( \nu \leq \nu_0 \), we obtain
\[
E \left( \sup_{0 \leq r \leq t} \|u_n^\nu(r \wedge T_N)\|_{L^2}^2 \right) + 2\nu(1 - \varepsilon) \int_0^{T_N} \text{E} \left( \|u_n^\nu(s)\|_V^2 \right) ds
\leq E \left( \|u_0\|_{L^2}^2 \right) + 2\nu_0(C(\varepsilon) + 1) \int_0^{T_N} \text{E} \left( \sup_{0 \leq r \leq s} \|u_n^\nu(r \wedge T_N)\|_{L^2}^2 \right) ds
+ \int_0^{T_N} \left[ \text{tr}(Q) + 2\|f(s)\|_{L^2} \right] ds + 2\nu_0 \int_0^{T_N} \text{E} \left( \sup_{0 \leq r \leq s} \|u_n^\nu(r \wedge T_N)\|_{L^2}^2 \right) ds
+ C_1 \text{tr}(Q) + C_1 \int_0^{T_N} \text{E} \left( \sup_{0 \leq r \leq s} \|u_n^\nu(r \wedge T_N)\|_{L^2}^2 \right) ds. \] (22)

Finally, in (22) set \( \varepsilon = 1/2 \) and apply Gronwall-Bellman inequality (see pp. 651-652 in [22]) to
\[
X(t) \doteq E \left( \sup_{0 \leq r \leq t} \|u_n^\nu(r \wedge T_N)\|_{L^2}^2 \right), \quad Y(t) \doteq \nu \int_0^{T_N} \text{E} \left( \|u_n^\nu(s)\|_V^2 \right) ds,
\]
\[
Z(t) \doteq E \left( \|u_0\|_{L^2}^2 \right) + C_1 \text{tr}(Q) + \int_0^{T_N} \left[ \text{tr}(Q) + 2\|f(s)\|_{L^2} \right] ds,
\]
\[
\lambda(t) \doteq Z(t) - Y(t),
\]
\[
\varphi(t) \doteq 2\nu_0(C(1/2) + 1) + C_1 + 2\|f(s)\|_{L^2} \geq 0 \quad \text{with} \quad \mathcal{K} \doteq \int_0^{T} \varphi(s) ds.
\]
Then,
\[
X(t) + Y(t) \leq Z(T)(1 + \mathcal{K}e^{\mathcal{K}}) \leq C(f, Q, \nu_0) \left( E \left( \|u_0\|_{L^2}^2 \right) + 1 \right), \] (23)
uniformly in \( N, n \), and \( \nu \leq \nu_0 \).

In particular we take \( t = T \). The estimate (23) gives that \( \tau_N \) increases to \( T \) a.s. as \( N \to \infty \). Passing to the limit, as \( N \to \infty \), (15) holds.

Taking the limit, as \( N \to \infty \), in equality (18) we deduce (16).

This ends the proof.

\[ \blacksquare \]

**Corollary 3.2.** Assume hypotheses of Proposition 3.1 and \( u_0 \in L^p(\Omega; H) \). Then for any \( p \geq 4 \)
\[
\sup_{0 < \nu \leq \nu_0} \sup_n \left\{ E \left( \sup_{0 \leq r \leq T} \|u_n^\nu(r)\|_{L^2}^p \right) + \nu \int_0^{T} E \left( \|u_n^\nu(s)\|_{L^2}^{p-2} \|u_n^\nu(s)\|_{L^2}^2 \right) ds \right\}
\leq C(p, f, Q, \nu_0) \left( E \left( \|u_0\|_{L^2}^p \right) + 1 \right). \] (24)
Proof. For each $N \in \mathbb{N}$, let us consider the stopping time $\tau_N = \inf\{t \geq 0 : \|u_n^\nu(t)\|_{L^2} \geq N\} \wedge T$. Applying Itô’s formula to expression (18) and function $g(z) = z^{p/2}$,

$$\|u_n^\nu(t \wedge \tau_N)\|^2_{L^2} + p\nu \int_{0}^{t \wedge \tau_N} \|u_n^\nu(s)\|^2_{L^2} \|\nabla u_n^\nu(s)\|^2_{L^2} ds$$

$$= \|u_n^\nu(0)\|^2_{L^2} + p\nu \int_{0}^{t \wedge \tau_N} \left(\|u_n^\nu(s)\|^2_{L^2} \int_{\Gamma}(\kappa - \alpha)u_n^\nu(s) \cdot u_n^\nu(s) d\mathcal{S}\right) ds$$

$$+ \int_{0}^{t \wedge \tau_N} \left\{\frac{p}{2} \text{tr}(\mathcal{Q})\|u_n^\nu(s)\|^2_{L^2} + \frac{p}{2}(p-2)\|u_n^\nu(s)\|^4_{L^2} \langle \mathcal{Q}u_n^\nu(s), u_n^\nu(s) \rangle\right\} ds. \quad (25)$$

Using (19),

$$\|u_n^\nu(s)\|^2_{L^2} \int_{\Gamma}|\kappa - \alpha|u_n^\nu(s) \cdot u_n^\nu(s) d\mathcal{S}$$

$$\leq \epsilon_1 \|u_n^\nu(s)\|^2_{L^2} \|\nabla u_n^\nu(s)\|^2_{L^2} + C(\epsilon_1) \|u_n^\nu(s)\|^p_{L^2}. \quad (26)$$

By Cauchy-Schwarz’s inequality, we also get

$$\|u_n^\nu(s)\|^2_{L^2} \langle f(s), u_n^\nu(s) \rangle \leq \|u_n^\nu(s)\|^2_{L^2} \|f(s)\|_{L^2} \|u_n^\nu(s)\|_{L^2}$$

$$\leq \|u_n^\nu(s)\|^p_{L^2} \|f(s)\|_{L^2} \leq \|f(s)\|_{L^2} \left(1 + \|u_n^\nu(s)\|^2_{L^2}\right). \quad (27)$$

Applying first Burkholder-Davis-Gundy’s inequality and next Young’s inequality with $p' = q' = 1/2$,

$$\mathbb{E}\left| \sup_{0 \leq s \leq t} \left\{p \int_{0}^{t \wedge \tau_N} \|u_n^\nu(s)\|^2_{L^2} \langle \sqrt{\mathcal{Q}} dW(s), u_n^\nu(s) \rangle\right\}\right|$$

$$\leq pC_1 \mathbb{E}\left(\int_{0}^{t \wedge \tau_N} \text{tr}(\mathcal{Q})\|u_n^\nu(s)\|^2_{L^2} ds\right)^{1/2}$$

$$= \mathbb{E}\left(2\epsilon_2 \sup_{0 \leq r \leq t} \left\{\|u_n^\nu(r \wedge \tau_N)\|^p_{L^2}\right\}\frac{1}{2\epsilon_2} \int_{0}^{t \wedge \tau_N} p^2C_1^2 \text{tr}(\mathcal{Q})\|u_n^\nu(s)\|^p_{L^2} ds\right)^{1/2}$$

$$\leq \epsilon_2 \mathbb{E}\left(\sup_{0 \leq r \leq t} \left\{\|u_n^\nu(r \wedge \tau_N)\|^p_{L^2}\right\}\right) + C(\epsilon_2, p)\mathbb{E}\left(\int_{0}^{t \wedge \tau_N} \text{tr}(\mathcal{Q})\|u_n^\nu(r)\|^2_{L^2} ds\right), \quad (28)$$

where $C(\epsilon_2, p) \equiv \frac{p^2C_1^2}{4\epsilon_2}$.

The last term of the right hand side of (25) can be estimated by

$$\frac{1}{2}p(p-1) \int_{0}^{t \wedge \tau_N} \text{tr}(\mathcal{Q})\|u_n^\nu(s)\|^2_{L^2} ds.$$ 

Finally, notice that we can estimate

$$\int_{0}^{t \wedge \tau_N} \text{tr}(\mathcal{Q})\|u_n^\nu(s)\|^2_{L^2} ds \leq \int_{0}^{t \wedge \tau_N} \text{tr}(\mathcal{Q}) \left(1 + \|u_n^\nu(s)\|^2_{L^2}\right) ds. \quad (29)$$

Thus, using expression (25) and estimates (26), (27), (28) and (29), following the arguments used in the proof of Proposition 3.1, and taking $\epsilon_1 = 1 - (2p)^{-1}$ and $\epsilon_2 = 1/2$, one can complete the proof of this Corollary. \hfill \blacksquare
The next lemma gives an important monotonicity property of operator $F$ in order to prove the existence and uniqueness for the weak solution, according to the Definition 2.1, to Equation (9). As we shall see, from the stochastic point of view it will be a strong solution. Concerning weak solutions for stochastic Navier-Stokes equations, in the stochastic sense, we refer [2] and the more recent paper [33] (see also the references therein).

Lemma 3.3. For a given $r > 0$ we consider the following (closed) $L^4$-ball $B_r$ in the space $V$:

$$B_r = \{ v \in V : \|v\|_{L^4(O)}^2 \leq r \}.$$

Then the nonlinear operator $u \mapsto F(t,u)$, $t \in [0,T]$, is monotone in the convex ball $B_r$, that is, for any $u \in V$, $v \in B_r$, there exists a positive constant $C \equiv C(\nu_0, O, \alpha)$, depending on $\nu_0$, the domain $O$ and $\alpha$ such that

$$\langle F(t,u) - F(t,v), u - v \rangle \leq C \left( 1 + \frac{r^4}{\nu^3} \right) \|u - v\|_{L^2}^2. \tag{30}$$

Proof. Taking into account the definition of the operator $A$ (6), we have

$$\langle F(u) - F(v), u - v \rangle + \nu \int_O |\nabla(u - v)|^2 \, dx$$

$$= - \langle B(u) - B(v), u - v \rangle + \nu \int_{\Gamma} (k - \alpha)|u - v|^2 \, dS.$$

As in (19), we derive the inequality

$$\nu \int_{\Gamma} (k - \alpha)|u - v|^2 \, dS \leq \frac{\nu}{2} \|\nabla(u - v)\|_{L^2}^2 + C \nu \|u - v\|_{L^2}^2$$

where $C$ is a constant which depends on $O$ and $\alpha$.

For more details see the proof of Lemma 2.4 in [25] and Proposition 2.2 in [30]. ■

Now we shall prove the path-wise uniqueness of Equation (9).

Proposition 3.4. Assume the hypotheses of Proposition 3.1. Let $u^\nu$ be a solution of Equation (9), that is, an adapted stochastic process $u^\nu(t,x,\omega)$ satisfying (9) and such that

$$u^\nu \in L^2(O; C(0,T; H) \cap L^2(0,T; V)) \cap [L^4([0,T[\times O \times \Omega)]^2).$$

If $v^\nu$ is another solution of Equation (9) as an adapted stochastic process in the space $C(0,T; H) \cap L^2(0,T; V)$, then

$$\|u^\nu(t) - v^\nu(t)\|_{L^2}^2 \exp \left\{ -2C \int_0^t \left( 1 + \frac{1}{\nu^3} \|u^\nu(s)\|_{L^4(O)}^4 \right) \, ds \right\} \leq \|u^\nu(0) - v^\nu(0)\|_{L^2}^2,$$

with probability 1, for any $0 \leq t \leq T$, where $C$ is the positive constant that appears in Lemma 3.3. In particular $u^\nu = v^\nu$, if $v^\nu$ satisfies the same initial condition as $u^\nu$.

Proof. Using Lemma 3.3, it follows the same arguments as those in the proof of Proposition 3.2 in [25]. We should mention that this idea to prove the path-wise uniqueness for the two dimensional stochastic Navier-Stokes equation already appear in [29]. ■

The existence of solution to Equation (9) is given in the following proposition.
Proposition 3.5. Suppose the hypotheses of Corollary 3.2. Then there exists an adapted process $u^\nu(t, x, \omega)$ such that

$$u^\nu \in L^p(\Omega; C(0, T; H)) \cap L^2(\Omega; L^2(0, T; V)) \cap [L^4([0, T[ \times \mathcal{O} \times \Omega])^2,$$

and verifying Equation (9). Furthermore,

$$\sup_{0 < \nu \leq \nu_0} \mathbb{E} \left\{ \sup_{0 \leq r \leq T} \| u^\nu(r) \|_{L^2}^p + \nu \int_0^T \| u^\nu(s) \|_V^2 \, ds + \nu \int_0^T \| u^\nu(s) \|_{L^2}^{p-2} \| u^\nu(s) \|_V^2 \, ds \right\} \leq C(p, f, \mathcal{Q}, \nu_0) \left( \mathbb{E} \left( \| u_0 \|_{L^2}^p \right) + 1 \right).$$

(31)

Proof. Borrowing the arguments of the proof of Proposition 3.3 in [25] and using the a priori estimates (15) and (24) and Lemma 3.3, the proof of this Proposition can be completed.

In the following section we shall consider the vorticity equation associated with Equation (5) in order to improve the estimates (31). More precisely, we shall estimate the $L^p$-norms of the vorticity process $\xi^\nu$ by the initial data, independently of the viscosity.

4. $L^p$ A PRIORI ESTIMATES FOR THE VORTICITY INDEPENDENT OF $\nu$

Set $\xi^\nu = \text{curl} u^\nu$. We apply the operator curl to Equation (5), obtaining the following vorticity equation:

$$\begin{align*}
\frac{\partial \xi^\nu}{\partial t} - \nu \Delta \xi^\nu(t) + (u^\nu(t) \cdot \nabla) \xi^\nu(t) &= \text{curl}(\sqrt{Q} \dot{W}(t)) \quad \text{in } [0, T[ \times \mathcal{O}, \\
\xi^\nu(0) &= \text{curl} u_0 \quad \text{in } \mathcal{O}, \\
\xi^\nu &= (2\kappa - \alpha) u^\nu \cdot t \quad \text{on } [0, T[ \times \Gamma}
\end{align*}$$

(32)

Notice that

$$\text{curl}(\sqrt{Q} \, dW) = \sum_{k=1}^{\infty} \lambda_k^{-m} \text{curl} v_k \, d\beta_k.$$ 

In the following we shall denote by $\hat{H}$ the space $L^2(\mathcal{O})$ endowed with the $L^2$-norm. We use the same notation for the $L^2$-norm of vector functions and scalar functions.

In the space $\hat{H}$ consider the operator $\hat{\mathcal{A}} : D(\hat{\mathcal{A}}) \subset \hat{H} \rightarrow \hat{H}$ with domain $D(\hat{\mathcal{A}}) = \{ \zeta \in L^2(\mathcal{O}) : \Delta \zeta \in L^2(\mathcal{O}) \}$, defined by $\hat{\mathcal{A}} \zeta = -\Delta \zeta$ for all $\zeta \in D(\hat{\mathcal{A}})$.

Set

$\zeta_k = \frac{\text{curl} v_k}{\|\text{curl} v_k\|_{L^2}}.$

We recall that the basis $\{v_k\}$ fixed previously was constructed in [13] verifying the properties that $\{\text{curl} v_k\}$ is orthogonal in $L^2(\mathcal{O})$ and for each $k$, $\text{curl} v_k \in \mathcal{W}$ is an eigenfunction of the operator $\hat{\mathcal{A}}$ with eigenvalue $\lambda_k$. Then the sequence $\{\zeta_k\}$ is an orthonormal basis for the space $\hat{H}$, that verifies $\hat{\mathcal{A}} \zeta_k = \lambda_k \zeta_k$. Thus,

$$\text{curl}(\sqrt{Q} \, dW) = \sum_{k=1}^{\infty} \lambda_k^{-m} \text{curl} v_k \, d\beta_k = \sum_{k=1}^{\infty} \lambda_k^{-m} \|\text{curl} v_k\|_{L^2} \zeta_k \, d\beta_k.$$ 

We define $\hat{Q} \in L(\hat{H}, \hat{H})$ by

$$\hat{Q} \zeta_k = \lambda_k^{-2m} \mu_k^2 \zeta_k,$$
where $\mu_k = \|\text{curl} v_k\|_{L^2}$, and $\tilde{W} = \sum_{k=1}^{\infty} \zeta_k \beta_k$ is a new cylindrical Wiener process in $\tilde{H}$.

Notice that

$$\|\text{curl} v_k\|_{L^2}^2 \leq C(1 + \lambda_k)\|v_k\|_{L^2}^2.$$  \hfill (33)

Indeed, (33) is a consequence of the following fact. We consider the following spectral problem that appears in the proof of Lemma 2.2 in [13]:

$$\begin{cases}
\Delta^2 \psi = -\lambda \psi & \text{in } \mathcal{O}, \\
-\Delta \psi = -(2\kappa - \alpha) \nabla \psi \cdot n & \text{on } \Gamma, \\
\psi = 0 & \text{on } \Gamma.
\end{cases}$$  \hfill (34)

Its variational form reads: find $\psi \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$ and $\lambda \neq 0$ such that

$$\int_{\mathcal{O}} \Delta \psi \Delta \varphi \, dx - \int_{\Gamma} (2\kappa - \alpha) \nabla \psi \cdot n \nabla \varphi \cdot n \, dS = \lambda \int_{\mathcal{O}} \nabla \psi \nabla \varphi \, dx, \quad \forall \varphi \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}).$$

Finally, notice that $v_k = -\nabla^\perp \psi_k := (\partial_2 \psi_k, -\partial_1 \psi_k)$ and $\text{curl} v_k = -\Delta \psi_k$, for some $\psi_k$ solution of the spectral problem (34).

Since $\sum_{k=1}^{\infty} \lambda_k^{-2m+1} < \infty$ we obtain that $\tilde{Q}$ is a trace class operator.

Hence $\tilde{Q}^{1/2}\tilde{W}$ is an $\tilde{H}$-valued centered Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, with covariance $\tilde{Q}$ in $\tilde{H}$.

In terms of $\tilde{A}$ and $\tilde{Q}^{1/2}\tilde{W}$ we can write Equation (32) as

$$\begin{cases}
d\xi^\nu(t) + \left\{ \nu \tilde{A} \xi^\nu(t) + (u^\nu(t) \cdot \nabla) \xi^\nu(t) \right\} \, dt \\
= \text{curl} f(t) \, dt + \tilde{Q}^{1/2} \, d\tilde{W}(t) \quad \text{in } ]0, T[ \times \mathcal{O}, \\
\xi^\nu(0) = \text{curl} u_0 \quad \text{in } \mathcal{O}, \\
\xi^\nu = (2\kappa - \alpha) u^\nu \cdot t \quad \text{on } ]0, T[ \times \Gamma.
\end{cases}$$  \hfill (35)

The following Lemma establishes a useful estimate for the elements of the basis $\{v_j\}$.

**Lemma 4.1.** Let $\{v_j\}$ be the previous fixed basis for $V$. Set $\xi_j = \text{curl} v_j$. Then

$$\|\xi_j\|_{H^1(\mathcal{O})} \leq C(\lambda_j + 1)\|\xi_j\|_{L^2(\mathcal{O})}.$$  \hfill (36)

**Proof.** We know that $\xi_j$ is solution of the Dirichlet problem

$$\begin{cases}
-\Delta \xi_j = \lambda_j \xi_j & \text{in } \Omega, \\
\xi_j = (2\kappa - \alpha) v_j \cdot t & \text{on } \Gamma
\end{cases}$$

The functions $\xi_j$ can be written in the form $\xi_j = h_j + g_j$, where $h_j$ and $g_j$ verify

$$\begin{cases}
-\Delta h_j = \lambda_j \xi_j & \text{in } \Omega, \\
h_j = 0 & \text{on } \Gamma
\end{cases} \quad \text{and} \quad \begin{cases}
-\Delta g_j = 0 & \text{in } \Omega, \\
g_j = (2\kappa - \alpha) v_j \cdot t & \text{on } \Gamma
\end{cases}$$
The functions \( h_j \) and \( g_j \) satisfy the Calderon-Zygmund’s estimates (see for example Theorems 1.8, 1.10 on pages 12, 15 and Proposition 1.2, p. 14 in Girault and Raviart [17])

\[
\|h_j\|_{H^2(\mathcal{O})} \leq C\|\lambda_j \xi_j\|_{L^2(\mathcal{O})}, \quad \|g_j\|_{H^1(\mathcal{O})} \leq C\|(2\kappa - \alpha)v_j \cdot t\|_{H^{1/2}(\Gamma)}. 
\]

From trace’s theory \( \|v_j\|_{H^{1/2}(\Gamma)} \leq \|v_j\|_{H^1(\mathcal{O})} \). On the other hand, we also know that \( \psi_j \) verify

\[
\begin{cases}
-\Delta \psi_j = \xi_j & \text{ in } \mathcal{O}, \\
\psi_j = 0 & \text{ on } \Gamma
\end{cases}
\]

and \( v_j = -\nabla \cdot \psi_j \). Therefore

\[
\|v_j\|_{H^1(\mathcal{O})} \leq \|\psi_j\|_{H^2(\mathcal{O})} \leq C\|\xi_j\|_{L^2(\mathcal{O})}.
\]

Then we have

\[
\|\xi_j\|_{H^1(\mathcal{O})} \leq C(\lambda_j + 1)\|\xi_j\|_{L^2(\mathcal{O})}.
\]

The improvement on the a priori estimates obtained in Proposition 3.1 and Corollary 3.2 is given in the following result:

**Proposition 4.2.** Suppose hypotheses of Proposition 3.1. Assume also that \( p > 2 \), \( \text{curl} \, f \in L^2(0, T; L^p(\mathcal{O})) \) and \( \text{curl} \, u_0 \in L^p(\Omega; L^p(\mathcal{O})) \). Let \( \xi^\nu \) be the vorticity of \( u^\nu \), then we have

\[
\sup_{\nu} \mathbb{E} \left( \sup_{0 \leq t \leq T} \|\xi^\nu(t)\|_{L^p}^p \right) 
\leq C \left( \text{curl} \, f, \tilde{Q}, T, p, \mathcal{O}, \alpha \right) \left\{ \mathbb{E} \left( \|u_0\|_{L^2}^p \right) + \mathbb{E} \left( \|\text{curl} \, u_0\|_{L^p}^p \right) + 1 \right\}.
\]

(36)

**Proof.** Let \( u^\nu \) be a stochastic process which is solution of the stochastic Navier-Stokes equation (9) with vorticity process \( \xi^\nu \) solution of (35).

Let us denote by \( w \) the solution of the following linear equation

\[
\begin{cases}
dw(t) + \left\{ \nu \tilde{A}w(t) + (u^\nu(t) \cdot \nabla)w(t) \right\} \, dt = 0 & \text{ in } [0, T[ \times \mathcal{O}, \\
w(0) = 0 & \text{ in } \mathcal{O}, \\
w = (2\kappa - \alpha)u^\nu \cdot t & \text{ on } [0, T[ \times \Gamma.
\end{cases}
\]

(37)

We introduce the process \( \rho = \xi^\nu - w \). We can verify that \( \rho \) is solution of the following stochastic differential equation:

\[
\begin{cases}
d\rho(t) + \left\{ \nu \tilde{A}\rho(t) + (u^\nu(t) \cdot \nabla)\rho(t) \right\} \, dt \\
= \text{curl} \, f(t) \, dt + \tilde{Q}^{1/2} \, d\tilde{W}(t) & \text{ in } [0, T[ \times \mathcal{O}, \\
\rho(0) = \text{curl} \, u_0 & \text{ in } \mathcal{O}, \\
\rho = 0 & \text{ on } [0, T[ \times \Gamma.
\end{cases}
\]

(38)

Using minor adaptation of the proof of Lemma 3 in [24], for \( p > 2 \) we obtain that the solution to Equation (37) satisfies
where $\epsilon$ is an arbitrary small parameter. Using Proposition 3.1, we have
\begin{equation}
\|w\|_{L^\infty(0,T);L^p(\Omega)} \leq C(p, \mathcal{O}, \alpha, \epsilon) \|u''\|_{L^\infty(0,T);L^2(\Omega)} + \epsilon\|\xi''\|_{L^\infty(0,T);L^p(\Omega)}, \quad \mathbb{P} \text{-a.s.} - \omega \tag{39}
\end{equation}

As regards Equation (38), let us denote by $\tilde{\mathcal{H}}$ the Cameron-Martin space of the $\tilde{H}$-valued Wiener process $\tilde{\mathcal{W}}$. We introduce the class $R(\tilde{\mathcal{H}}, L^p)$ of the so-called radonifying operators (see Definition 4.2 in [9], [8] and [10]). Let \( \{b_k\} \) be a sequence of mutually independent $N(0,1)$-distributed random variables and \( \{h_k\} \) an orthonormal basis for $\tilde{\mathcal{H}}$. The norm of an operator $K$ in this class of operators is defined by $\|K\|_{R(\tilde{\mathcal{H}}, L^p)} = \mathbb{E} \left( \|\sum_{k=1}^{\infty} b_k K h_k\|^2_{L^p} \right)$. We remark that the notion of a radonifying operator is a generalization of the notion of a Hilbert-Schmidt operator to the case where $L^p$ is not a Hilbert space. In the particular case $p = 2$, $R(\tilde{\mathcal{H}}, L^p)$ is the space of Hilbert-Schmidt operators.

We can verify that the inclusion $I : \mathcal{H} \to L^p(\mathcal{O})$ belongs to the class $R(\mathcal{H}, L^p)$. In fact it is enough to verify that $I : \mathcal{H} \to H^1 \cap L^p(\mathcal{O})$ is a Hilbert Schmidt operator (see Remark 6.1 in [7] and Theorem 2.3 in [10]). Considering the orthonormal basis $h_k = \tilde{Q}^{1/2} \zeta_k$, the Sobolev Imbedding Theorem (see Theorem 4.1.2 page 85 in [1]), Lemma 4.1 and estimates (33) and (8), we obtain
\begin{equation}
\|I\|^2_{R(\mathcal{H}, L^p)} = \mathbb{E} \left( \left\| \sum_{k=1}^{\infty} \beta_k \tilde{Q}^{1/2} \zeta_k \right\|_{L^p}^2 \right) \leq \mathbb{E} \left( \left\| \sum_{k=1}^{\infty} \beta_k \tilde{Q}^{1/2} \zeta_k \right\|_{H^1}^2 \right)
\leq \mathbb{E} \left( \sum_{k=1}^{\infty} \beta_k^2 \left\langle \tilde{Q}^{1/2} \zeta_k, \tilde{Q}^{1/2} \zeta_k \right\rangle_{H^1} \right) + \mathbb{E} \left( 2 \sum_{j<k} \beta_j \beta_k \left\langle \tilde{Q}^{1/2} \zeta_j, \tilde{Q}^{1/2} \zeta_k \right\rangle_{H^1} \right)
= \sum_{k=1}^{\infty} \mathbb{E} (\beta_k^2) \left\langle \zeta_k, \zeta_k \right\rangle_{H^1} + 2 \sum_{j<k} \mathbb{E} (\beta_j \beta_k) \left\langle \tilde{Q}^{1/2} \zeta_j, \tilde{Q}^{1/2} \zeta_k \right\rangle_{H^1}
= \sum_{k=1}^{\infty} \lambda_k^{-2m} \mu_k^2 \|\zeta_k\|_{H^1}^2 = \sum_{k=1}^{\infty} \lambda_k^{-2m} \|\text{curl } v_k\|_{H^1}^2 \leq \mathcal{M}.
\end{equation}

Henceforth, $\langle \cdot, \cdot \rangle$ denotes the duality between $L^p$ and $L^{p/(p-1)}$ for some $1 < p < \infty$. For each $N \in \mathbb{N}$, we set $\tau_N = \inf \{t \geq 0 : \|\rho\|_{L^2} \geq N \} \wedge T$. Taking the function $\Phi : L^p \to \mathbb{R}$, $\Phi(x) = \|x\|_{L^p}^p$ and applying the Itô’s formula to the processes $\Phi(\rho(t))$ (see Theorem 4.3 of [9]), we have
\begin{align}
\|\rho(t \wedge \tau_N)\|_{L^p}^p &= \|\rho(0)\|_{L^p}^p - p \int_0^{t \wedge \tau_N} \left\langle \nu \tilde{\mathcal{A}} \rho(s), |\rho(s)|^{p-2} \rho(s) \right\rangle ds \\
&\quad - p \int_0^{t \wedge \tau_N} \left\langle u_u^v(s) \cdot \nabla \rho(s), |\rho(s)|^{p-2} \rho(s) \right\rangle ds + p \int_0^{t \wedge \tau_N} \left\langle \text{curl } f(s), |\rho(s)|^{p-2} \rho(s) \right\rangle ds \\
&\quad + p \int_0^{t \wedge \tau_N} \left\langle \tilde{Q}^{1/2} d\tilde{W}(t), |\rho(s)|^{p-2} \rho(s) \right\rangle + \frac{1}{2} \int_0^{t \wedge \tau_N} \text{tr}_I \Phi''(\rho(s)) ds, \tag{41}
\end{align}
where
\[ \text{tr}_I \Phi''(v) \leq p(p-1)\|v\|_{L^p}^{p-2} \|I\|_{R(\mathcal{H}, L^p)}^2 \leq p(p-1)\|v\|_{L^p}^{p-2} \mathcal{M}. \]
Hence
\[
\frac{1}{2} \int_0^{t \land \tau_N} \text{tr}_I \Phi''(\rho(s)) \, ds \leq \frac{1}{2} p(p-1) M \int_0^{t \land \tau_N} \|\rho(s)\|_{L^p}^{p-2} \, ds \\
\leq \frac{1}{2} p(p-1) M \int_0^{t \land \tau_N} (1 + \|\rho(s)\|_{L^p}^p) \, ds. \tag{42}
\]

Applying that \(\rho = 0\) on \(\Gamma\), integration by parts formula and the fact that
\[
\nabla \left[ |\rho(s)|^{p-2} \rho(s) \right] = (p - 1) |\rho(s)|^{p-2} \nabla \rho(s), \tag{43}
\]
we have
\[
\langle u^\nu(s) \cdot \nabla \rho(s), |\rho(s)|^{p-2} \rho(s) \rangle = 0. \tag{44}
\]

On the other hand, we consider the following identities and estimates for the remainder of the terms in (41).

\[
- p \int_0^{t \land \tau_N} \langle \nu A \rho(s), |\rho(s)|^{p-2} \rho(s) \rangle \, ds \\
= - \nu p(p-1) \int_0^{t \land \tau_N} \left( \int_\Omega |\nabla \rho(s,x)|^2 |\rho(s,x)|^{p-2} \, dx \right) \, ds. \tag{45}
\]

Indeed, using integration by parts formula and (43),
\[
\langle A \rho(s), |\rho(s)|^{p-2} \rho(s) \rangle = \langle -\Delta \rho(s), |\rho(s)|^{p-2} \rho(s) \rangle \\
= \int_\Omega \nabla \rho(s,x) \cdot \nabla \left[ |\rho(s,x)|^{p-2} \rho(s,x) \right] \, dx \\
= (p - 1) \int_\Omega |\rho(s,x)|^{p-2} \nabla \rho(s,x) \cdot \nabla \rho(s,x) \, dx \\
= (p - 1) \int_\Omega |\nabla \rho(s,x)|^2 |\rho(s,x)|^{p-2} \, dx.
\]

For the stochastic term, using Burkholder-Davies-Gundy inequality (see (6.10) page 1890 in [6] or Theorem 4.2 in [9], for instance) and Young’s inequality with \(p' = q' = 1/2\), we obtain
\[
\mathbb{E} \left| \sup_{0 \leq r \leq t} \left\{ p \int_0^{r \land \tau_N} \left\langle \mathcal{Q}^{1/2} d\tilde{W}(t), |\rho(s)|^{p-2} \rho(s) \right\rangle \right\} \right| \\
\leq \varepsilon \mathbb{E} \left( \sup_{0 \leq r \leq t} \|\rho(r \land \tau_N)\|_{L^p}^p \right) + \frac{C_\varepsilon p^2 \mathcal{M}}{4\varepsilon} \int_0^{t \land \tau_N} \left\{ 1 + \mathbb{E} \left( \sup_{0 \leq r \leq s} \|\rho(r)\|_{L^p}^p \right) \right\} \, ds. \tag{46}
\]
In fact,

$$
\begin{align*}
\mathbb{E} \left| \sup_{0 \leq r \leq t} \left\{ p \int_0^{t \wedge \tau_N} \left\langle \dot{\mathcal{Q}}^{1/2} d\bar{W}(t), |\rho(s)|^{p-2}\rho(s) \right\rangle \right| \right| \\
\leq C_1 p \mathbb{E} \left( \int_0^{t \wedge \tau_N} \|I\|_{L^p}^2 \|\rho(s)\|_{L^p}^{(p-1)} \, ds \right)^{1/2} \\
\leq C_1 p \mathbb{E} \left( \int_0^{t \wedge \tau_N} \mathcal{M} \|\rho(s)\|_{L^p}^{(p-1)} \, ds \right)^{1/2} \\
\leq \mathbb{E} \left( \left\{ 2\varepsilon \sup_{0 \leq r \leq t} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right\}^{1/2} \left\{ \frac{1}{2\varepsilon} \int_0^{t \wedge \tau_N} C_1^p \mathcal{M} \|\rho(s)\|_{L^p}^{p-2} \, ds \right\}^{1/2} \right) \\
\leq \varepsilon \mathbb{E} \left( \sup_{0 \leq r \leq t} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) + \frac{C_1^p \mathcal{M}}{4\varepsilon} \mathbb{E} \left( \int_0^{t \wedge \tau_N} \|\rho(s)\|_{L^p}^{p-2} \, ds \right) \\
\leq \varepsilon \mathbb{E} \left( \sup_{0 \leq r \leq t} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) + \frac{C_1^p \mathcal{M}}{4\varepsilon} \int_0^{t \wedge \tau_N} \left\{ 1 + \mathbb{E} \left( \sup_{0 \leq r \leq s} \|\rho(r)\|_{L^p}^p \right) \right\} ds.
\end{align*}
$$

Finally for the term with \( \text{curl} f \), applying Hölder inequality for \( p > 2 \) and \( q = p/(p-1) \)

$$
\langle \text{curl} f(s), |\rho(s)|^{p-2}\rho(s) \rangle \leq \|\text{curl} f(s)\|_{L^p} \|\rho(s)\|_{L^p}^{p-1} \|
= \|\text{curl} f(s)\|_{L^p} \|\rho(s)\|_{L^p}^{p-1} \\
\leq \|\text{curl} f(s)\|_{L^p} (1 + \|\rho(s)\|_{L^p}^p). \tag{47}
$$

To sum up, applying to (41) the estimates (42), (44), (45), (46) and (47), we obtain

$$
\begin{align*}
\mathbb{E} \left( \sup_{0 \leq r \leq t} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) + \nu p(p-1) \mathbb{E} \left( \int_0^{t \wedge \tau_N} \left( \int_0^{t \wedge \tau_N} |\nabla \rho(s,x)|^2 |\rho(s,x)|^{p-2} \, dx \right) \, ds \right) \\
\leq \mathbb{E} \left( \|\text{curl} u_0\|_{L^p}^p \right) + \int_0^{t \wedge \tau_N} p \|\text{curl} f(s)\|_{L^p} \, ds \\
+ \int_0^{t \wedge \tau_N} p \|\text{curl} f(s)\|_{L^p} \mathbb{E} \left( \sup_{0 \leq r \leq s} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) \, ds \\
+ \varepsilon \mathbb{E} \left( \sup_{0 \leq r \leq t} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) + \frac{C_1^p \mathcal{M}}{4\varepsilon} \int_0^{t \wedge \tau_N} \left\{ 1 + \mathbb{E} \left( \sup_{0 \leq r \leq s} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) \right\} ds \\
+ \frac{1}{2} p(p-1) \mathcal{M} \int_0^{t \wedge \tau_N} \left\{ 1 + \mathbb{E} \left( \sup_{0 \leq r \leq s} \|\rho(r \wedge \tau_N)\|_{L^p}^p \right) \right\} ds. \tag{48}
\end{align*}
$$
Finally, using (48) with \( \varepsilon = 1/2 \) and applying Gronwall-Bellman inequality for

\[
X(t) = \mathbb{E}\left( \sup_{0 \leq r \leq t} \| \rho(r) \|_{L^p}^p \right),
\]

\[
Y(t) = \nu p (p - 1) \mathbb{E}\left( \int_0^{T \wedge \tau_N} \left( \int_0^1 |\nabla \rho(s, x)|^2 |\rho(s, x)|^{p-2} \, dx \right) \, ds \right),
\]

\[
Z(t) = \mathbb{E}\left( \| \text{curl } u_0 \|_{L^p}^p \right) + \int_0^{T \wedge \tau_N} p \| \text{curl } f(s) \|_{L^p} \, ds + t \left\{ C_1^2 p^2 + p(p - 1) \right\} \mathcal{M}/2,
\]

\[
\lambda(t) = Z(t) - Z(t),
\]

\[
\varphi(t) = p \| \text{curl } f(t) \|_{L^p} + \left\{ C_1^2 p^2 + p(p - 1) \right\} \mathcal{M}/2 \geq 0
\]

we obtain

\[
X(t) + Y(t) \leq Z(T)(1 + K \varepsilon^2) \leq C(T, p, \text{curl } f, \mathcal{M}) \left( \mathbb{E}\left( \| \text{curl } u_0 \|_{L^p}^p \right) + 1 \right),
\]

uniformly in \( n \) and \( \nu \).

In particular, for \( t = T \),

\[
\mathbb{E}\left( \sup_{0 \leq r \leq T} \| \rho(r) \|_{L^p}^p \right) + 2 \nu p (p - 1) \mathbb{E}\left( \int_0^T \left( \int_0^1 |\nabla \rho(s, x)|^2 |\rho(s, x)|^{p-2} \, dx \right) \, ds \right) \leq C(T, p, \text{curl } f, \mathcal{M}) \left( \mathbb{E}\left( \| \text{curl } u_0 \|_{L^p}^p \right) + 1 \right),
\]

which gives that \( \tau_N \) increases to \( T \) a.s. as \( N \to \infty \). Taking the limit as \( N \to \infty \), we deduce

\[
\sup_{\nu} \mathbb{E}\left( \sup_{0 \leq r \leq T} \| \rho(r) \|_{L^p}^p \right) \leq C \left( T, p, \text{curl } f, \dot{\mathcal{Q}} \right) \left( \mathbb{E}\left( \| \text{curl } u_0 \|_{L^p}^p \right) + 1 \right).
\]  

Hence, estimates (40) and (49) yield the following estimate for the vorticity:

\[
\mathbb{E}\left( \| \xi'' \|_{L^\infty(0, T; L^p)}^p \right) \leq C(p) \mathbb{E}\left( \| \rho \|_{L^\infty(0, T; L^p)}^p + \| u \|_{L^\infty(0, T; L^p)}^p \right) \leq C \left( T, p, \text{curl } f, \dot{\mathcal{Q}}, \varepsilon \right) \mathbb{E}\left( \| \text{curl } u_0 \|_{L^p}^p + \| u_0 \|_{L^2}^p + 1 \right) + C(p) \varepsilon \mathbb{E}\left( \| \xi'' \|_{L^\infty(0, T; L^p)}^p \right).
\]

Taking \( \varepsilon \) small enough we obtain (36).

Using Propositions 3.1 and 4.2, we can deduce the following result:

**Proposition 4.3.** Assume the hypotheses of Proposition 4.2. Then

\[
\mathbb{E}\left( \| u'' \|_{L^\infty(0, T; W^{1,p}(\Omega))]^{2} \right) \leq C,
\]

with a constant \( C > 0 \) independent of viscosity.

**Proof.** Owing to Poincaré’s inequality, Lemma 3.1 in [20] and a priori estimates (36) for the vorticity of \( u'' \), (51) holds.
5. Vanishing viscosity limit

In this section we shall prove our main result (Theorem 2.1), that is, the sequence of solutions \( \{u^\nu\}_{0<\nu\leq v_0} \) to Equation (5) converges to a solution of the stochastic Euler equations with the same initial velocity as viscosity vanishes.

To establish the existence of solution for the stochastic Euler equations, we follow a path-wise approach similar to [5]. In our problem, the vorticity of the involved processes do not vanish at the boundary, so, we need to estimate the boundary terms which increases the difficulty. To overcome such difficulties, we proceed analogously to deterministic methods in the articles [13], [24]. Since the estimates, independent of the viscosity, are based on the maximum principle, we need to consider a regular Wiener process. To be precise, in the next two Lemmas the Wiener process \( \sqrt{Q}W(t) \) has covariance \( Q = A^{-2m} \), \( m > 4 \).

**Lemma 5.1.** Assume that for a.e. \( \omega \in \Omega, u_0 \in L^p(\Omega) \) and curl \( f \in L^1(0,T;L^\infty(\Omega)) \). Let \( u^\nu \) be the weak solution of (9), then we have

\[
\|u^\nu(\omega)\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq C(\omega),
\]

(52)

where \( C(\omega) \) does not depend on the viscosity \( \nu \), for a.e. \( \omega \in \Omega \) but depends on \( \omega \).

Moreover, if we assume for a.e. \( \omega \in \Omega, u_0 \in L^\infty(\Omega) \), the estimate (52) holds for \( p = \infty \).

**Proof.** Let us consider the stochastic process \( v^\nu(t,\omega) = u^\nu(t,\omega) - \sqrt{Q}W(t,\omega) \) which satisfies a deterministic equation similar to (16) in [5]. By handling such equation we deduce

\[
\sup_{0 \leq t \leq T} \|v^\nu(\omega)\|_{L^2(\Omega)} \leq C.
\]

To simplify, we represent by \( C \) a constant independent of the viscosity.

Taking into account the regularity of the process \( \sqrt{Q}W(t,\omega) \), we obtain

\[
\sup_{0 \leq t \leq T} \|u^\nu(\omega)\|_{L^2(\Omega)} \leq C.
\]

Next set \( z^\nu(t,\omega) = \text{curl} u^\nu(t,\omega) - \text{curl} (\sqrt{Q}W(t,\omega)) \). We consider just the case \( 2 < p < \infty \), since for \( p = \infty \) is easier, the result follow directly by the maximum principle. For a.e. \( \omega \in \Omega \), the sample paths of the process \( z^\nu(t) \) verify, in the sense of the distributions, the following equation:

\[
\begin{cases}
\frac{\partial z^\nu(t)}{\partial t} - \nu \Delta z^\nu(t) + (u^\nu(t) \cdot \nabla)z^\nu(t) = g(t) & \text{on } ]0,T[ \times \Omega, \\
z^\nu(0) = z_0 & \text{in } \Omega, \\
z^\nu = (2\kappa - \alpha)v^\nu \cdot t & \text{on } ]0,T[ \times \Gamma
\end{cases}
\]

(53)

where \( g(t) = \text{curl} f(t) - (u^\nu(t) \cdot \nabla)Z(t) + \nu \Delta Z(t) \) and \( Z(t) = \text{curl} (\sqrt{Q}W(t)) \). Let us denote by \( \lambda = \|(2\kappa - \alpha)v^\nu \cdot t\|_{L^\infty([0,T] \times \Gamma)} \) and \( L(t) = \|g(t)\|_{L^\infty(\Omega)} \). Given \( v^\nu(t) \), the linear problem

\[
\begin{cases}
\frac{\partial z(t)}{\partial t} - \nu \Delta z(t) + (u^\nu(t) \cdot \nabla)z(t) = L(t) & \text{on } ]0,T[ \times \Omega, \\
z(0) = |z_0| & \text{in } \Omega, \\
z = \lambda & \text{on } ]0,T[ \times \Gamma
\end{cases}
\]

(54)
is well posed with solution \( z(t) \in L^2(0, T; H^1(\mathcal{O})) \). Since the function \( z(t) = z'(t) - \bar{z}(t) \) verifies the inequality

\[
\frac{\partial z(t)}{\partial t} - \nu \Delta z(t) + (u'(t) \cdot \nabla)z(t) \leq 0 \quad \text{on } [0, T],
\]

and is non positive on \([0, T] \times \Gamma\) and at \( t = 0\), the maximum principle implies that \( z \) is a non positive function, i.e. \( z' \leq \bar{z} \). Analogously, we show that \( z = -z' - \bar{z} \leq 0 \). So, we conclude that

\[
|z'(t)| \leq \bar{z}(t), \quad \text{a.e. in } [0, t) \times \mathcal{O}.
\]  

(55)

The difference process \( \hat{z}(t) = \bar{z}(t) - \lambda \) verifies the following equation:

\[
\begin{cases}
\frac{\partial \hat{z}(t)}{\partial t} - \nu \Delta \hat{z}(t) + (u'(t) \cdot \nabla)\hat{z}(t) = L(t) & \text{on } [0, T], \times \mathcal{O}, \\
\hat{z}(0) = |z_0| - \lambda & \text{in } \mathcal{O}, \\
\hat{z} = 0 & \text{on } [0, T] \times \Gamma.
\end{cases}
\]  

(56)

Multiplying the first equation of (56) by \( G = p|\hat{z}|^{p-2}\hat{z} \) and integrating over \( \mathcal{O} \), we obtain

\[
\frac{d}{dt}||\hat{z}||^p_{L^p(\mathcal{O})} + \nu p(p-1) \int_\mathcal{O} |\hat{z}|^{p-2} |\nabla \hat{z}|^2 \, dx \leq | \int_\mathcal{O} L(t) \, G \, dx |. 
\]  

(57)

Having

\[
| L(t) \int_{\Omega} G \, dx | \leq C \left[ ||\text{curl} f||_{L^\infty(\mathcal{O})} + ||u'||_{C(\mathcal{O})} ||Z||_{L\infty(\mathcal{O})} + \nu ||\Delta Z||_{L\infty(\mathcal{O})} ||\hat{z}||^p_{L^p(\mathcal{O})} \right],
\]

we verify that \( ||\hat{z}||^p_{L^p(\Omega)} \) satisfies a Bihari’s type inequality, that gives the following estimate

\[
||\hat{z}(t)||_{L^p(\mathcal{O})} \leq C ||z(0)||_{L^p(\mathcal{O})} + \int_0^t ||v'(r)||_{C(\mathcal{O})} \, dr + 1.
\]

Considering Nirenberg-Gagliardo’s interpolation inequality

\[
||v'(t)||_{L^\infty(\mathcal{O})} \leq C \left( ||v'(t)||_{W^{1,\theta}(\mathcal{O})} ||v'(t)||_{W^{1,\frac{p}{2}(\mathcal{O})}} + ||v'(t)||_{L^2(\mathcal{O})} \right), \quad \theta = \frac{p}{2(p-1)},
\]

the embedding theorem

\[
W^{1,p}(\mathcal{O}) \hookrightarrow C^\alpha(\overline{\mathcal{O}}) \quad \text{with} \quad \alpha = 1 - 2/p
\]

we can write

\[
||v'(t)||_{C(\overline{\mathcal{O}})} = ||v'(t)||_{L^\infty(\mathcal{O})} \leq C (||v'(t)||_{L^2(\mathcal{O})} + ||z'||_{L^p(\mathcal{O})}).
\]

Combining with (55) we derive the following Gronwall’s inequality for \( z'(t) \):

\[
||z'(t)||_{L^p(\mathcal{O})} \leq C (||z(0)||_{L^p(\mathcal{O})} + \int_0^t ||z'(r, \cdot)||_{L^p(\mathcal{O})} \, dr + 1),
\]

which implies \( ||z'||_{L^\infty(0,T;L^p(\mathcal{O}))} \leq C \), where \( C \) is a constant independent of the viscosity. Therefore, we have

\[
||\text{curl} \, u'||_{L^\infty(0,T;L^p(\mathcal{O}))} \leq C
\]

and consequently (52) holds.  

\[\blacksquare\]
Lemma 5.2. Under the assumptions of Lemma 5.1. Then exists a stochastic process \( u \) with sample paths in \( C([0, T]; H) \cap L^\infty(0, T; W^{1,p}(\Omega)) \), \( p > 2 \) that is solution of the Euler equation (12), in the sense of (13). Moreover, in the case \( p = \infty \), such solution is unique.

Proof. Using estimates (52) and borrowing the arguments of Theorem 1 in [24] and Theorem 1.1 of [5], it can be proven the existence of \( u \), which is a solution of (12) in the sense of (13). To obtain a measurable solution \( u \), we can use a measurable selection theorem (see Chapter 5 in [31] and also Lemma 3.1 in [5] for a more specific result). The proof of the uniqueness is standard.

Finally, we can already prove our main result:

Proof of Theorem 2.1. Notice that (i) is a consequence of Propositions 3.4 and 3.5. Regarding (ii), observe that a stochastic process being solution of the Euler equation already exists, from Lemma 5.2. Let us suppose \( u^\nu(\omega) \in L^\infty(\Omega) \) for a.e. \( \omega \in \Omega \) and consider \( u^\nu \) and \( u \) the unique solutions to Navier-Stokes equations and Euler equations, respectively. It remains to prove that \( u^\nu \) converges to \( u \) as the viscosity goes to zero.

Let us consider the difference process \( u^\nu - u \).

\[
\langle B(u^\nu) - B(u), u^\nu - u \rangle = \int_\Omega [(u^\nu \cdot \nabla)(u^\nu - u) + ((u^\nu - u) \cdot \nabla)u] \cdot (u^\nu - u) \, dx
\]

\[
= \int_\Gamma (u \cdot n) \frac{|u^\nu - u|^2}{2} \, dS + \int_\Omega ((u^\nu - u) \cdot \nabla) u \cdot (u^\nu - u) \, dx.
\]

Since \( u \cdot n = 0 \), the first term in the right hand side is zero. For the second term we have

\[
\left| \int_\Omega ((u^\nu - u) \cdot \nabla) u \cdot (u^\nu - v) \, dx \right| \leq \|
abla u\|_{L^\infty} \|u^\nu - u\|_{L^2}^2.
\]

On the other hand, taking into account the definition of the operator \( \mathcal{A} \) in (6), We have

\[
\langle \mathcal{A}(u^\nu), u^\nu - u \rangle = \nu \int_\Omega \nabla u^\nu \cdot \nabla (u^\nu - u) \, dx - \nu \int_\Gamma (k - \alpha) u^\nu \cdot (u^\nu - u) \, dS.
\]

Therefore, the difference process verifies the following Gronwall inequality:

\[
\frac{\partial}{\partial \nu} \|u^\nu(t) - u(t)\|_{L^2}^2 \leq C \nu + \|
abla u(t)\|_{L^\infty} \|u^\nu(t) - u(t)\|_{L^2}^2
\]

for a.e. \( \omega \in \Omega \), which implies

\[
\|u^\nu(t) - u(t)\|_{L^2}^2 \leq C \nu \exp \left( \int_0^\nu \|
abla u(s)\|_{L^\infty} \, ds \right)
\]

where \( C \) is a constant independent of \( \nu \). Then \( \sup_{t \in [0, T]} \|u^\nu(t) - u(t)\|_{L^2}^2 \to 0 \), as \( \nu \to 0 \).

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