THE arc-TOPOLOGY

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ABSTRACT. We study a Grothendieck topology on schemes which we call the arc-topology. This topology is a refinement of the $v$-topology (the pro-version of Voevodsky’s $h$-topology) where covers are tested via rank $\leq 1$ valuation rings. Functors which are arc-sheaves are forced to satisfy a variety of gluing conditions such as excision in the sense of algebraic $K$-theory.

We show that étale cohomology is an arc-sheaf and deduce various pullback squares in étale cohomology. Using arc-descent, we reprove the Gabber-Huber affine analog of proper base change (in a large class of examples), as well as the Fujiwara-Gabber base change theorem on the étale cohomology of the complement of a henselian pair. As a final application we prove a rigid analytic version of the Artin-Grothendieck vanishing theorem from SGA4, extending results of Hansen.

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1. Introduction

The purpose of this paper is to study a Grothendieck topology on the category of (highly non-noetherian, but qcqs) schemes, which we call the arc-topology. This topology is a slight refinement of the \( v \)-topology of \([39, 4]\).

The benefit of such non-noetherian topologies arises when one studies invariants of schemes which satisfy descent with respect to them. In this case, one can try to work locally. For the \( v \)-topology, working locally essentially means that one can reduce many questions about these invariants (on any scheme) to potentially far simpler questions involving valuation rings, even ones which have algebraically closed fraction field. Our strengthening in this paper shows that, for arc-sheaves, one can restrict even further, to rank \( \leq 1 \)-valuation rings.

We will show that several natural invariants of schemes, such as étale cohomology with torsion coefficients and perfect complexes on perfect \( \mathbb{F}_p \)-schemes, satisfy descent for the arc-topology. In these cases \( v \)-descent was previously known.

The seemingly slight strengthening of topologies (from \( v \)- to arc-) turns out to have concrete consequences: namely, arc-descent additionally forces several excision-type squares. For instance, arc-sheaves (satisfying mild finiteness hypotheses) satisfy “excision” in the classical \( K \)-theoretic sense as well as an analog of the Beauville-Laszlo “formal glueing” theorem \([2]\). As applications, using the arc-topology, we will recover some classical results in étale cohomology, such as the Gabber-Huber affine analog of proper base change and the Fujiwara-Gabber theorem on the étale cohomology of punctured henselian pairs. We also prove new general results including a version of Artin-Grothendieck vanishing in rigid geometry (which improves on recent work of Hansen \([22]\)).

Conventions. All schemes appearing in this paper are assumed to be quasicompact and quasiseparated (qcqs). We use the term “rank” for a valuation ring synonymously with its Krull dimension.

1.1. The arc-topology. The starting point for us is the so-called \( v \)-topology or universally subtrusive topology, studied by \([39, 4]\):

Definition 1.1 (The \( v \)-topology). An extension of valuation rings is a faithfully flat map \( V \to W \) of valuation rings (equivalently, an injective local homomorphism). A map of qcqs schemes \( Y \to X \) is a \( v \)-cover if for any valuation ring \( V \) and any map \( \text{Spec}(V) \to X \), there is an extension of valuation rings \( V \to W \) and a map \( \text{Spec}(W) \to Y \) leading to a commutative square

\[
\begin{array}{ccc}
\text{Spec}(W) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(V) & \longrightarrow & X.
\end{array}
\]

For finite type maps of noetherian schemes, the \( v \)-topology coincides with Voevodsky’s \( h \)-topology \([48, \text{Sec. 3}]\), i.e., the one generated by étale covers and proper surjections. In general, every \( v \)-cover is a limit of \( h \)-covers. In this paper we study the following definition, which has also been explored by Rydh in the forthcoming work \([38]\).
**Definition 1.2** (The arc-topology). A map \( f : Y \to X \) of qcqs schemes is an arc-cover\(^1\) if for any rank \( \leq 1 \) valuation ring \( V \) and a map \( \text{Spec}(V) \to X \), there is an extension \( V \to W \) of rank \( \leq 1 \) valuation rings and a map \( \text{Spec}(W) \to Y \) lifting the composition \( \text{Spec}(W) \to \text{Spec}(V) \to X \) to a commutative square as in (1).

We shall show (Proposition 2.17) that the arc-topology coincides with a finer variant of the topology of universal submersions (or universal topological quotient maps) of [36, 39] that behaves better under limits. This result had been independently observed by Rydh in the forthcoming \([\text{38}]\).

For noetherian targets, there is no distinction between \( v \)-covers and arc-covers (Proposition 2.6). On the other hand, they do not coincide in general. The fundamental example (for the purposes of this paper) capturing this discrepancy is the following.

**Example 1.3.** Let \( V \) be a valuation ring of rank 2. If \( p \subset V \) denotes the unique height 1 prime, then both \( V_p \) and \( V/p \) are rank 1 valuation rings, and the map \( V \to V_p \times V/p \) is an arc-cover (Corollary 2.9) but not a \( v \)-cover. In fact, if \( f \in V - p \) is not invertible, then \( V \to V_f \times V/fV \) is a finitely presented arc-cover that is not a \( v \)-cover.

The existence of this example illustrates one of the subtleties in working with the arc-topology. Namely, even though every arc-cover is a limit of finitely presented arc-covers, a finitely presented arc-cover cannot be obtained as a base change of an arc-cover of noetherian schemes. In particular, noetherian approximation arguments do not work as well as they do in the \( v \)-topology.

1.2. **Instances of arc-descent.** Given a Grothendieck topology, we can ask when a functor satisfies descent with respect to it (or equivalently is a sheaf). To formalize this, fix a functor \( F : \text{Sch}^{op} \to \mathcal{C} \) on the category \( \text{Sch} \) of qcqs schemes with values in some target \( \infty \)-category\(^2\) \( \mathcal{C} \). We assume that \( F(Y_1 \sqcup Y_2) \simeq F(Y_1) \times F(Y_2) \) for qcqs schemes \( Y_1, Y_2 \).

**Definition 1.4.** We will say that \( F \) satisfies descent for a morphism \( Y \to X \) of qcqs schemes if it satisfies the \( \infty \)-categorical sheaf axiom with respect to \( Y \to X \), i.e., if the natural map

\[
F(X) \to \lim F(Y) \Rightarrow F(Y \times_X Y /\ldots).
\]

is an equivalence. If this property holds true for all maps \( f : Y \to X \) that are covers for a Grothendieck topology \( \tau \), we also say that \( F \) satisfies \( \tau \)-descent or is a \( \tau \)-sheaf.

A typical example is the following:

**Example 1.5.** Let \( \tau \) be a Grothendieck topology on the category of schemes. For any scheme \( X \) and a coefficient ring \( \Lambda \), write \( F(X) := R\Gamma(X, \Lambda) \) for the \( \tau \)-cohomology of \( X \), viewed as an object of the derived \( \infty \)-category \( D(\Lambda) \). The resulting functor \( F : \text{Sch}^{op} \to D(\Lambda) \) is a \( \tau \)-sheaf.

In this paper, we will be interested in invariants which satisfy arc-descent. As \( v \)-covers are arc-covers, it is clear that arc-sheaves are \( v \)-sheaves. Conversely, we prove the following general criterion for a \( v \)-sheaf to be an arc-sheaf; roughly speaking, it says that a \( v \)-sheaf (satisfying a mild finite presentation constraint) which also satisfies descent with respect to the covers from Example 1.3 (and slight variants) is automatically an arc-sheaf.

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\(^1\)The name “arc” was chosen in view of the natural analogy between rank 1 valuations and arcs. Thus, an arc cover is a map of schemes along which every arc lifts.

\(^2\)The theory of \( \infty \)-categories does not play a crucial role in this paper. However, it is convenient to use this language to formulate clean statements. Our main example of a target \( \infty \)-category \( \mathcal{C} \) shall be the derived \( \infty \)-category \( D(\Lambda) \) of a ring \( \Lambda \) (or variants), though we occasionally also use the \( \infty \)-category \( \text{Cat}_{\infty} \) of all \( \infty \)-categories when discussing “stacky” phenomenon.
Theorem 1.6 (Criteria for arc-descent, Theorem 4.1 below). Let $F : \text{Sch}^{op} \to D(\mathbb{Z})_{\geq 0}$ be a functor on qcqs schemes which is finitary\(^3\), i.e. $F$ takes filtered limits with affine transition maps into filtered colimits. Then the following are equivalent:

1. $F$ is an arc-sheaf.
2. $F$ is a $v$-sheaf and for every absolutely integrally closed valuation ring $V$ and $p \subset V$ a prime ideal, the square

$$
\begin{array}{ccc}
F(\text{Spec}(V)) & \longrightarrow & F(\text{Spec}(V/p)) \\
\downarrow & & \downarrow \\
F(\text{Spec}(V_p)) & \longrightarrow & F(\text{Spec}(\kappa(p)))
\end{array}
$$

is cartesian.

As an application, we give some examples of arc-sheaves. Our first example is étale cohomology with torsion coefficients; note that this invariant was known to be a $v$-sheaf, essentially because of the proper base change theorem, cf. Proposition 5.2. More generally, we show that constructible complexes on a scheme $X$ can be constructed arc-locally.

Theorem 1.7 (arc-descent for étale cohomology, Theorems 5.4 and 5.14 below). If $\Lambda$ is a finite ring, then the assignment $X \mapsto D^b_{\text{cons}}(X, \Lambda)$ satisfies arc-descent. In particular, the functor $X \mapsto R\Gamma(X_{\text{ét}}, \Lambda)$ is an arc-sheaf.

Our second example is that of perfect complexes on perfect schemes of characteristic $p$; again, this invariant was already known to be a $v$-sheaf [4, Sec. 11].

Theorem 1.8 (arc-descent for perfect complexes, Theorem 5.15 below). On qcqs $\mathbb{F}_p$-schemes, the functor $X \mapsto \text{Perf}(X_{\text{perf}})$ satisfies arc-descent.

As an application of the previous result, we obtain a completely categorical description of arc-covers on perfect schemes.

Theorem 1.9 (Characterization of arc-covers for perfect schemes, Theorem 5.16 below). On qcqs $\mathbb{F}_p$-schemes, a map $Y \to X$ is an arc-cover if and only if it is a universally effective epimorphism.

1.3. Consequences of arc-descent. It is sometimes easy to show (thanks largely to the relatively simple nature of rank $\leq 1$ valuation rings) that certain squares of schemes that “look like” they ought to be pushouts do in fact yield pushout squares on associated arc-sheaves. In particular, any arc-sheaf carries such a square of schemes to a pullback square. This leads to concrete consequences for arc-sheaves, such as excision squares or Mayer-Vietoris sequences, which we describe next.

Let us begin with the classical formulation for excision for functors on rings.

Definition 1.10. An excision datum is given by a map $f : (A, I) \to (B, J)$ where $A$ and $B$ are commutative rings, $I \subset A$ and $J \subset B$ are ideals, and $f : A \to B$ is a map that carries $I \subset A$.

\(^3\)For sheaves of sets, this property has been dubbed “local finite presentation” by Grothendieck and is pervasive in the classical literature [45, Tag 049J]; however, we avoid this terminology to avoid clashes with other similarly named notions, such as that of finitely presented objects in an $\infty$-category.
isomorphically onto $J \subset B$. In this situation, we obtain a commutative square of rings

\[
\begin{array}{ccc}
A & \rightarrow & A/I \\
\downarrow & & \downarrow \\
B & \rightarrow & B/J
\end{array}
\]

that is both cocartesian and cartesian. Such diagrams are also called Milnor squares (after [35, §2]). We say that a $D(\mathbb{Z})$-valued functor $F$ on commutative rings is excisive if for an excision datum as above, the square obtained by applying $F$ to \((2)\) is cartesian; equivalently, the natural map from the fibre of $F(A) \rightarrow F(A/I)$ to the fibre of $F(B) \rightarrow F(B/J)$ is an equivalence.

The question of excision has played a crucial role in algebraic $K$-theory, see for instance [46, 10, 19]. We prove the following result relating excisiveness to the arc-topology.

**Theorem 1.11** (arc-sheaves satisfy excision, Theorem 4.1 below). Let $F : \text{Sch}^{\text{op}} \rightarrow D(\mathbb{Z})$ be an arc-sheaf. Then (on rings) $F$ satisfies excision.

**Remark 1.12.** The hypothesis that $F$ is an arc-sheaf, and not just a $v$-sheaf, is essential to Theorem 1.11. Indeed, general $v$-sheaves can fail to be excisive. This distinction is explained by the following observation (which was also the basis of our discovery of the arc-topology): if $(A, I) \rightarrow (B, J)$ is an excision datum, then $\text{Spec}(B) \sqcup \text{Spec}(A/I) \rightarrow \text{Spec}(A)$ is always an arc-cover (Lemma 2.7) but not in general a $v$-cover; in fact, with notation as in Example 1.3, the map $(p, V) \rightarrow (pV_p, V_p)$ gives an example.

**Remark 1.13.** In [9, Th. 3.12], one finds a result deducing a descent property for certain presheaves (for the $cdh$-topology) from excision. On the other hand, in Theorem 4.1, we deduce excision as a consequence of arc-descent.

Next, we study “formal glueing” results for arc-sheaves. The formal glueing property of a functor captures whether its value on a variety can be reconstructed from its value on a formal neighbourhood of a subvariety and its value on the complement. We formulate this precisely as follows.

**Definition 1.14.** A formal glueing datum is given by a pair $(R \rightarrow S, I)$ where $R \rightarrow S$ is a map of commutative rings and $I \subset R$ is a finitely generated ideal such that $R/I^n \simeq S/I^nS$ for all $n \geq 0$. The corresponding square

\[
\begin{array}{ccc}
\text{Spec}(S) \setminus V(IS) & \rightarrow & \text{Spec}(S) \\
\downarrow & & \downarrow \\
\text{Spec}(R) \setminus V(I) & \rightarrow & \text{Spec}(R)
\end{array}
\]

of schemes is called the (weak) Mayer-Vietoris square attached to this datum (following [21]). We say that a contravariant functor $F$ on schemes satisfies formal glueing if for every formal glueing datum $(R \rightarrow S, I)$ as above, the functor $F$ carries the square \((3)\) to a cartesian square.

This property has been studied frequently in algebraic geometry. For example, the functor that assigns to a scheme its category of vector bundles satisfies formal glueing (at least when restricted to noetherian schemes, see [45, Tag 05E5] and the references therein). We prove the same holds for arc-sheaves without any noetherianness constraints:
Theorem 1.15 (Formal glueing squares for arc-sheaves, Theorem 6.5 below). Let $F : \text{Sch}^{op} \to D(\mathbb{Z})$ be an arc-sheaf. Then $F$ satisfies formal glueing.

Remark 1.16. Just like in Theorem 1.11, the hypothesis that $F$ be an arc-sheaf, and not merely a $v$-sheaf, is essential for Theorem 1.15. In fact, just like Remark 1.12, this can be explained by the following observation: if $(R \to S, I)$ is a formal glueing datum, then the corresponding map $f : \text{Spec}(S) \sqcup (\text{Spec}(R) \setminus V(I)) \to \text{Spec}(R)$ is an arc-cover (Proposition 6.3) but not in general a $v$-cover; see Example 6.7 to see what can go wrong.

Specializing the previous theorems to étale cohomology, we obtain:

Corollary 1.17. For any torsion abelian group $\Lambda$, the functor $X \mapsto R\Gamma(X_{\text{ét}}, \Lambda)$ is excisive (on rings, in the sense of Definition 1.10) and satisfies formal glueing (in the sense of Definition 1.14).

As applications, we give quick proofs of two foundational results in the étale cohomology of rings and schemes: the Gabber-Huber affine analog of proper base-change\textsuperscript{4}, at least when working over a henselian local ring, and the Fujisawa-Gabber theorem, generalized to the non-noetherian setting.

Corollary 1.18. Let $R$ be a commutative ring that is henselian with respect to an ideal $I \subset R$. Let $\mathcal{F}$ be a torsion étale sheaf on $\text{Spec}(R)$, viewed as a sheaf on all schemes over $\text{Spec}(R)$ via pullback.

1. (Gabber [16], Huber [23]) Assume that $R$ lives over a henselian local ring (ex: if $R$ lives over a field or the $p$-adics). Then $R\Gamma(\text{Spec}(R)_{\text{ét}}, \mathcal{F}) \simeq R\Gamma(\text{Spec}(R/I)_{\text{ét}}, \mathcal{F})$.

2. (Fujisawa-Gabber [14]) Assume $I$ is finitely generated. Then the natural map

$$R\Gamma(\text{Spec}(R \setminus V(I))_{\text{ét}}, \mathcal{F}) \to R\Gamma(\text{Spec}(R/I) \setminus V(I)_{\text{ét}}, \mathcal{F})$$

is an equivalence.

Unlike the proof of Corollary 1.18 (2) in [14], our proof is relatively soft and does not rely on Elkik’s approximation theorem or its variants. Similar techniques also allow us to prove a descent property for the étale cohomology (Corollary 6.17) that essentially amounts to the equality of the algebraic and analytic étale cohomology of affinoid spaces in rigid analytic geometry (which, under noetherian hypotheses, is Huber’s affinoid comparison theorem [24, Corollary 3.2.2] for constant coefficients and Hansen’s [22, Theorem 1.9] for general coefficients).

As a final application, we apply these descent-theoretic techniques to sharpen recent results of Hansen [22] and prove the following version of the classical Artin-Grothendieck vanishing theorem for rigid geometry. In loc. cit., this is proved when $K$ has characteristic zero and when $A$ arises as the base-change of an affinoid algebra over a discretely valued nonarchimedean field.

Theorem 1.19 (Artin-Grothendieck vanishing for the cohomology of affinoids, Theorem 7.9 below). Fix a complete and algebraically closed nonarchimedean field $K$. Let $A$ be a classical affinoid $K$-algebra of dimension $d$. Fix a prime $\ell$ and let $\mathcal{F}$ be an $\ell$-power torsion étale sheaf on $\text{Spec}(A)$. If $A$ is smooth or if $\ell$ is not the residue characteristic of $K$, then $H^i(\text{Spec}(A)_{\text{ét}}, \mathcal{F}) = 0$ for $i > d$. In general, we at least have $H^i(\text{Spec}(A)_{\text{ét}}, \mathcal{F}) = 0$ for $i > d + 1$.

Our proof of Theorem 1.19 is independent of Hansen’s work [22]. Moreover, with the exception of the affinoid comparison theorem mentioned above, our arguments can be formulated purely in terms of the étale cohomology of rigid spaces (though we do not do so in our exposition).

\textsuperscript{4}At its core, our proof has some similarity with Gabber’s proof: both proofs involve reduction to the absolutely integrally closed case. Once this reduction is made, Gabber proceeds to study the topology of closed sets inside absolutely integrally closed integral schemes directly. However, in our proof, we actually reduce to absolutely integrally closed valuation rings, where there is no nontrivial topology at all.
Conventions. For a commutative ring $R$, let $\text{Ring}_R$ (resp. $\text{Sch}_R$) denote the category of commutative $R$-algebras (resp. qcqs $R$-schemes). We simply write $\text{Sch} := \text{Sch}_\mathbb{Z}$ for the category of all qcqs schemes. Given a presheaf $F$ on $\text{Sch}_R$, we often write $F(A) = F(\text{Spec}(A))$ for an $R$-algebra $A$ if there is no confusion.

For a torsion étale sheaf $\mathcal{F}$ on a scheme $X$ and a morphism $f : Y \to X$, we often write $R\Gamma(Y, \mathcal{F})$ as shorthand for $R\Gamma(\text{ét}, f^* \mathcal{F})$ if there is no confusion. In particular, cohomology with constant coefficients is always computed with respect to the étale topology unless otherwise specified.

For a spectral space $S$ (such as the space underlying any qcqs scheme), we write $S^\text{cons}$ for the set $S$ equipped with the constructible topology inherited from the spectral topology on $S$ (so the open subsets of $S^\text{cons}$ are exactly the ind-constructible subsets of $S$); recall that $S^\text{cons}$ is a profinite set whose clopen subsets are exactly the constructible subsets of $S$.

Given a qcqs scheme $X$ and a specialization $x \leadsto y$ of points in $X$, we say that this specialization is witnessed by a map $f : \text{Spec}(V) \to X$ if $V$ is a valuation ring, and $f$ carries the generic point (resp. the closed point) to $x$ (resp. $y$); every specialization can be witnessed by a map from the spectrum of a valuation ring [45, Tag 00IA].

Following terminology from the theory of adic spaces, we shall refer to a nonzero nonunit in a rank $\leq 1$ valuation ring $V$ as pseudouniformizer.

Acknowledgments. The first author was supported by National Science Foundation under grant number 1501461 and a Packard fellowship. This work was done while the second author was a Clay Research Fellow. The second author would also like to thank the University of Michigan at Ann Arbor for its hospitality during a weeklong visit where this work was started. We thank Piotr Achinger, Benjamin Antieau, Dustin Clausen, David Hansen, Marc Hoyois, Johan de Jong, Matthew Morrow, David Rydh, and Peter Scholze for helpful conversations.
2. The arc-topology

The goal of this section is twofold: in §2.1, we collect some basic properties and examples of the arc-topology, while in §2.2, we explain why the arc-topology can be defined topologically via the notion of “universal spectral submersions”.

2.1. Properties of arc-covers. In this subsection, we collect various results on the arc-topology. We will work freely with the theory of valuation rings; see [45, Tag 0018, Tag 0ASF] or [7, Ch. 6] for an introduction. In particular, we use the basic properties that valuation rings are closed under localization and taking quotients by prime ideals, as well as the notion of a rank 1 valuation.

Proposition 2.1. Let \( f : Y \to X \) be a v-cover of qcqs schemes. Then \( f \) is an arc-cover.

Proof. Let \( V \) be a rank \( \leq 1 \) valuation ring with a map \( \Spec(V) \to X \); by assumption there exists a valuation ring \( W \) with a faithfully flat map \( q : V \to W \) and a map \( \Spec(W) \to Y \) making the diagram (1) commutative. The only issue is that \( W \) need not be rank \( \leq 1 \). To remedy this, we proceed as follows.

Let \( m \) be the maximal ideal of \( V \). The collection of prime ideals in \( W \) is totally ordered, so any intersection of prime ideals remains prime; thus, there exists a minimal prime \( q \subset W \) such that \( g^{-1}(q) = m \). Similarly, there exists a maximal prime ideal \( q_0 \subset W \) such that \( g^{-1}(q_0) = 0 \). The map \( V \to W \to W' \overset{q}{\to} (W/q_0)_q \) is faithfully flat as it is an injective local homomorphism, and \( W' \) is a rank one valuation ring. In view of the map \( \Spec(W') \to \Spec(W) \to Y \), we see that \( f \) is an arc-cover as desired. \[\square\]

Remark 2.2. The primes \( q \) and \( q_0 \) appearing in the preceding proof can be described more explicitly as follows. Recall that for any valuation ring \( W \) equipped with an element \( f \in W \) which is not a unit, the ideal \( q' := \sqrt{fW} \) is the minimal prime ideal containing \( f \), and the ideal \( q_0 := \cap_f f^nW \) is the maximal prime ideal contained in \( fW \); one proves this using the primality of radical ideals in valuation rings. In the situation of the proof above, one simply applies this to \( f \in W \) being the image of any element \( t \in V \) such that \( \sqrt{tW} \) is the maximal ideal.

In the preceding construction, if \( f \in W \) is nonzero and not invertible, then the resulting specialization \( q_0 \to q \) is an immediate one, i.e., that \( (W/q_0)_q \) has rank 1 (and \( f \) gives a pseudouniformizer in this valuation ring). Consequently, for any non-trivial specialization \( p \to p' \) in a valuation ring \( W \), there exist specializations \( p \to q_0 \to q \to p' \) where the middle one is an immediate specialization realized by applying the construction in the previous paragraph to some \( f \in p' - p \). In particular, the totally ordered sets arising as spectra of valuation rings cannot be arbitrary totally ordered sets admitting minimal and maximal elements. This observation (for all commutative rings more generally) already appears in [28, Sec. 1-1, Theorem 11].

Lemma 2.3. Consider a composite map \( Y' \overset{g}{\to} Y \overset{f}{\to} X \). If \( f \circ g \) is an arc-cover, then so is \( f \).

Proof. Clear. \[\square\]

Lemma 2.4. Let \((V, m)\) be a rank 1 valuation ring and let \( A \) be a \( V \)-algebra via the structure map \( f : V \to A \). Then the following are equivalent:

1. \( \Spec(A) \to \Spec(V) \) is a v-cover.
2. \( \Spec(A) \to \Spec(V) \) is an arc-cover.
3. There exists an inclusion \( p_1 \subset p_2 \subset A \) of prime ideals with \( 0 = f^{-1}(p_1) \) and \( f^{-1}(p_2) = m \).

That is, the specialization \( (0) \to m \) in \( \Spec(V) \) lifts to a specialization \( p_1 \to p_2 \) in \( \Spec(A) \).
Proof. (1) implies (2) in view of Proposition 2.1. If \( \text{Spec}(A) \to \text{Spec}(V) \) is a arc-cover, then there exists a rank 1 valuation ring \( W \), faithfully flat over \( V \), such that the map \( \text{Spec}(W) \to \text{Spec}(V) \) factors through \( \text{Spec}(A) \). The images of the generic and special points in \( \text{Spec}(A) \) then verify (3). Finally, for (3) implies (1), it is enough to observe that the valuation ring \( W \) of any valuation on \( \kappa(p_1) \) centered on the local ring \( (A/p_1)_{p_2} \subset \kappa(p_1) \) (which exists by [45, Tag 00IA]) is faithfully flat over \( V \).

\[ \Box \]

Corollary 2.5. Let \( f : Y \to X \) be a map of qcqs schemes. Then \( f \) is an arc-cover if and only if every base change of \( f \) along a map \( \text{Spec}(V) \to X \), for \( V \) a rank \( \leq 1 \) valuation ring, is a \( v \)-cover.

In the noetherian setting, there is no distinction between the \( v \)- and arc-topologies; we record this result here, though it is not essential to the sequel.

Proposition 2.6. Let \( f : Y \to X \) be a map of qcqs schemes with \( X \) noetherian. Then \( f \) is a \( v \)-cover if and only if it is an arc-cover.

Proof. This is [39, Theorem 2.8].

To deduce excision, we must necessarily work with very non-noetherian rings. In this setting, the arc-topology differs from the \( v \)-topology. To see this, let us first explain how to extract arc-covers from excision data; we shall later see that these covers are typically not \( v \)-covers.

Lemma 2.7. Consider an excision datum \( f : (A, I) \to (B, J) \) as in Definition 1.10. Then the map \( A \to A/I \times B \) is an arc-cover. More precisely, any map from \( A \) to a rank \( \leq 1 \) valuation ring factors through \( A/I \times B \).

Proof. Let \( W \) be a rank \( \leq 1 \) valuation ring and consider a map \( g : A \to W \). We claim that it factors either through the map \( A \to B \) or the map \( A \to A/I \). There are two cases:

1. Suppose \( g(I) = 0 \). Then \( g \) factors through the map \( A \to A/I \).
2. Suppose there exists \( x \in I \) such that \( g(x) \in W \) is nonzero; in this case, we claim that \( g \) factors over \( f : A \to B \).

Indeed, the map \( g : A \to W \) extends to a map \( A[1/x] \to W[1/g(x)] \to K := \text{Frac}(W) \); since \( g(x) \neq 0 \), \( W[1/g(x)] \) is either \( W \) or its fraction field \( K \). The map \( f[1/x] : A[1/x] \to B[1/x] \) is an isomorphism, so we get a commutative diagram

\[
\begin{array}{c}
A \\
\downarrow^{A[1/x]} \\
A[1/x] \\
\end{array} \quad \begin{array}{c}
g \quad \quad \quad W. \\
\quad \quad \quad \downarrow \\
\quad f[1/x] \\
B[1/x] \quad \quad \quad \quad \quad \quad \quad \quad K \\
\end{array}
\]

Restricting to \( B \to B[1/x] \) we obtain a map \( \tilde{g} : B \to K \) extending \( g \to W \subset K \). Note that \( \tilde{g}|_{J} = g|_{I} \) has image in \( W \subset K \). Therefore, for any \( y \in B \), we have \( f(x)y \in J \) and thus \( \tilde{g}(y) \in \frac{1}{g(x)}W \), so that the image of \( \tilde{g} \) has bounded denominators. But then \( \tilde{g} \) has image in \( W \) as desired: any subring between \( W \) and \( K \) with bounded denominators is necessarily equal to \( W \), as \( W \) has rank \( \leq 1 \).

\[ \Box \]

We now give the basic example (for this note) of an excision datum that will yield an arc-cover.

Proposition 2.8 (Excision data attached to valuation rings). Let \( V \) be a valuation ring. Fix \( p \in \text{Spec}(V) \). Then:
(1) The map $(V, p) \to (V_p, pV_p)$ is an excision datum.

(2) Fix an inclusion $p \subset q$ of prime ideals in $V$. Then $(V_q, pV_q) \to (V_p, pV_p)$ is an excision datum.

**Proof.** The second claim reduces to the first, as we may replace $V$ with $V_q$ to assume $q$ is the maximal ideal. Therefore, we need only prove (1). That is, we need to show that $p \simeq pV_p$. The map is clearly an injection (both sides are contained in the quotient field of $V$). Conversely, given a fraction $a/s$ with $a \in p$ and $s \in V \setminus p$, we have necessarily $s \mid a$ and $a = sb$ for $b \in p$. Thus, $a/s = b \in p$ as desired.

**Corollary 2.9** (arc-covers of valuation rings). Let $V$ be a valuation ring, and let $p \subset V$ be a prime ideal. Then $V \to V_p \times V/p$ is an arc-cover.

Thus, the arc-topology is strictly finer than the $v$-topology: if $V$ has rank $\geq 2$ and $p$ is a nonzero nonmaximal prime, then the map from Corollary 2.9 is clearly not a $v$-cover.

**Remark 2.10.** Using the easier "only if" part of Proposition 2.17 below, one may also deduce Corollary 2.9 from [36, Corollary 33] (see also [20, Ex. 4.5], and [39, Ex. 4.3]).

Finally, we observe that the condition of being an arc-cover is essentially module-theoretic.

**Proposition 2.11.** Let $f : \text{Spec}(B) \to \text{Spec}(A)$ be a map of affine schemes. Then the following are equivalent:

(1) $f$ is an arc-cover.

(2) The map $A \to B$ of modules has the property that after every base-change $A \to V$, for $V$ a rank $\leq 1$ valuation ring, it is pure.

**Proof.** We immediately reduce to the case where $A = V$ is a rank $\leq 1$ valuation ring. In this case, see [37, Lemme 1.3.1, Part 2] and [36, Prop. 16].

We reproduce the argument for the reader’s convenience. If $A \to B$ is pure, then $A \to B/\text{torsion}$ is also pure and hence faithfully flat, so a $v$-cover. It follows that $A \to B$ must be a $v$-cover. Conversely, if $A \to B$ is a $v$-cover, choose an inclusion of prime ideals $p_1 \subset p_2 \subset B$ which pull back to the zero and maximal ideal of $A$. Then $B/p_1$ is an integral domain which is faithfully flat over $A$, hence pure. Thus $A \to B$ is also pure. \qed

### 2.2. Relation to the submersions

In this subsection, we relate the arc-topology to the topology of universal submersions from [36, 39]. Recall that coverings in the latter are given by maps of schemes that are universally submersive, i.e., universally quotient maps on the underlying spaces. Since universally submersive maps are not stable under limits (Remark 2.21), we shall use instead the following variant where the quotient property is tested only using quasicompact open subsets:

**Definition 2.12.** A map $f : X \to Y$ of qcqs schemes is called **spectrally submersive** or a **spectral submersion** if it satisfies the following two conditions:

(1) $f$ is surjective.

(2) Given a subset $U \subset Y$, if the preimage $f^{-1}(U)$ is a quasicompact open, then $U$ is a quasicompact open.

We say that $f$ is universally spectrally submersive if for all $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is a spectral submersion.

**Remark 2.13.** Any map $f : X \to Y$ of qcqs schemes that is a topological quotient map is certainly a spectral submersion, so any $v$-cover is universally spectrally submersive; the converse fails without some finite dimensionality constraints, see Remark 2.21.
Remark 2.14. As universally spectrally submersive maps are closed under composition and base change, the property of being universally spectrally submersive can be checked Zariski locally (or even \( \mathfrak{v} \)-locally) on the source and the target.

Remark 2.15. Fix a surjective map \( f : X \to Y \) of qcqs schemes. Condition (2) in Definition 2.12 can be relaxed to the following:

(2') Given a constructible subset \( U \subset Y \), if the preimage \( f^{-1}(U) \) is open, then \( U \) is open.

To see why, fix \( U \subset Y \) with \( f^{-1}(U) \) quasicompact open in \( X \). We must show that \( U \) is a quasicompact open if \( f \) is surjective and satisfies (2') above. As \( f \) is surjective, the quasicompactness of \( U \) is clear. To show openness, thanks to (2'), it is enough to show that \( U \) is constructible. But this follows from the following general fact: if \( f : S \to T \) is a surjective spectral map of spectral spaces, then the induced surjection \( f_{\text{cons}} : S_{\text{cons}} \to T_{\text{cons}} \) for the constructible topology is a quotient map (as any continuous surjection of profinite sets is a quotient map). Now \( f^{-1}(U) \) is constructible, hence clopen for the constructible topology; thus \( U \) is also clopen for the constructible topology and hence constructible.

The following stability property of universal spectral submersions is crucial for our application.

Lemma 2.16. Let \( f : X \to Y \) be a map of qcqs schemes that can be written as a cofiltered inverse limit of morphisms \( f_i : X_i \to Y_i \) of qcqs schemes along affine transition maps. If each \( f_i \) is a (universal) spectral submersion, the same holds true for \( f \).

In Remark 2.21, we explain why this property fails if we drop the adjective “spectral”.

Proof. Let us first show that \( f \) is a spectral submersion when \( f_i \) is so. As each \( f_i \) is surjective, the map \( f \) is also surjective: if \( f_i \) is surjective, then \( f_i^{\text{cons}} \) is surjective, whence \( f^{\text{cons}} := \lim_{\bar{i}} f_i^{\text{cons}} \) is surjective by Tychonoff’s theorem, and thus \( f \) is surjective. For the rest, we use the criterion in Remark 2.15. Pick a constructible subset \( U \subset Y \) such that \( f^{-1}(U) \) is open. As \( Y \simeq \lim_i Y_i \), we also have \( Y^{\text{cons}} \simeq \lim_i Y_i^{\text{cons}} \), so the constructible set \( U \) arises as the pullback of some constructible set \( U_i \subset Y_i \) for some index \( i \). For \( j \geq i \), write \( U_j \) for the preimage of \( U_i \) in \( Y_j \). As each \( f_j \) is spectrally submersive, it is enough to show that the constructible set \( f_j^{-1}(U_j) \) is open for \( j \gg i \). As \( f^{-1}(U) \) is a quasicompact open in \( X \) and \( X \simeq \lim_i X_i \), we can realize \( f^{-1}(U) \) is the preimage of some quasicompact open \( V_k \subset X_k \) for some index \( k \); reindexing, we may assume \( k = i \). Write \( V_j \subset X_j \) for the preimage of \( V_i \). Then \( \lim_{j \geq i} V_j = f^{-1}(U) = \lim_{j \geq i} f_j^{-1}(U_j) \) as constructible subsets of \( X \). But then we must have \( V_j = f_j^{-1}(U_j) \) for \( j \) sufficiently large: by passage to the constructible topology, this reduces to the observation that in a cofiltered inverse limit \( S := \lim_i S_i \) of profinite sets \( S_i \), if we have clopen subsets \( W, W' \subset S_i \) for some \( i \) with the same preimage in \( S \), then the preimages of \( W \) and \( W' \) in \( S_j \) for \( j \gg i \) must coincide (as the collection of clopen subsets of \( S \) is the direct limit of the collection of clopen subsets of the \( S_i \)'s). In particular, we learn that \( f_j^{-1}(U_j) \) is open for \( j \gg i \), as wanted.

It remains to check that every base change of \( f \) is spectrally submersive if the \( f_i \)'s are universally spectrally submersive. But a base change of \( f \) can be realized as the cofiltered inverse limit of base changes of the \( f_i \)'s. As the \( f_i \)'s are universally spectrally submersive, the same holds true for their base changes, so we conclude using the previous paragraph. \( \square \)

We now arrive at our main topological characterization of arc-covers; this was independently observed by Rydh in the forthcoming [38].
Proposition 2.17 (arc-covers and universal spectral submersions). Let \( f : X \to Y \) be a map of qcqs schemes. Then \( f \) is universally spectrally submersive if and only if \( f \) is an arc-cover.

We note the following immediate corollary of Proposition 2.17 and Lemma 2.16.

Corollary 2.18. The collection of arc-covers of qcqs schemes is closed under filtered inverse limits via affine transition maps.

The proof of Proposition 2.17 will rely on a few (standard) facts about valuation rings that we recall next; these will also be useful later. For future reference, we also include assertions about absolutely integrally closed valuation rings (see Definition 3.22).

Lemma 2.19. Let \( W \) be a valuation ring with fraction field \( K \). Let \( K_0 \subset K \) be a subfield. Then the intersection \( W_0 = K_0 \cap W \) is a valuation ring too, and the map \( W_0 \to W \) is faithfully flat. If in addition \( K_0 \) is algebraically closed, the ring \( W_0 \) is absolutely integrally closed.

Proof. We see immediately that \( W_0 \) is a valuation ring in the field \( K_0 \). Moreover, the map \( W_0 \to W \) is a local injective homomorphism of valuation rings, so it is faithfully flat.

Suppose now that \( K_0 \) is algebraically closed. Then \( W_0 \) is absolutely integrally closed, since it is integrally closed and has algebraically closed fraction field. \( \square \)

Lemma 2.20. Let \( V \) be a valuation ring. Then \( V \) can be written as a filtered colimit of a family of valuation subrings \( V_i \subset V \) such that:

1. Each \( V_i \) has finite rank.
2. Each map \( V_i \to V_j \) is faithfully flat.

If in addition \( V \) is absolutely integrally closed, we may also take each \( V_i \) to be absolutely integrally closed.

Proof. Consider first the case of valuation rings (not assumed absolutely integrally closed). Let \( K \) be the fraction field of \( V \). For each finitely generated subfield \( K_i \subset K \), we let \( V_i = K_i \cap V \); this is a valuation subring of \( K_i \), and is finite rank since the transcendence degree of \( K_i \) is finite. Moreover, the map \( V_i \to V \) is a local homomorphism and is therefore faithfully flat. This gives the desired expression of \( V \) as a filtered colimit. When \( V \) is absolutely integrally closed, we run a similar argument but take the \( \{K_i\} \) to be the family of algebraically closed subfields of \( K \) of finite transcendence degree over the prime field. \( \square \)

Proof of Proposition 2.17. Assume first that \( f \) is universally spectrally submersive. We must show \( f \) is an arc-cover. By base change and Lemma 2.4, it is enough to show that the following holds: if \( Y := \text{Spec}(V) \) for a valuation ring \( V \) of rank 1, the nontrivial specialization \( (0) \to m \) in \( \text{Spec}(V) \) lifts to a specialization in \( X \). Now \( f^{-1}((0)) \subset X \) is a nonempty quasicompact open subset that is not closed: if it were closed, its complement \( f^{-1}(m) \subset X \) would be a quasicompact open, which, by spectral submersiveness of \( f \), would imply the absurd conclusion that the closed point of \( \text{Spec}(V) \) is open. As points in the closure of a quasicompact open of a spectral space are given by specializations [45, Tag 0903], there must exist a specialization \( x_0 \to x_m \) in \( X \) such that \( x_0 \in f^{-1}((0)) \) and \( x_m \in X - f^{-1}((0)) \). But then \( x_m \in f^{-1}(m) \), so we have found the desired specialization.

Conversely, assume that \( f \) is an arc-cover. We must show \( f \) is universally spectrally submersive. In fact, since arc-covers are stable under base change, it is enough to show that \( f \) is spectrally submersive. By Remark 2.15, we must show the following: a constructible subset \( U \subset Y \) is open if \( f^{-1}(U) \subset X \) is open. We may assume \( U \) is non-empty.
Let us first reduce to the case where \( Y \) is the spectrum of a valuation ring. Fix \( U \subset Y \) as above. As a constructible set in a spectral space is open exactly when it is stable under generalizations [45, Tag 0903], it is enough to show that \( U \) is stable under generalizations, i.e., if \( u \in U \) and \( y \in Y \) is a generalization of \( u \) in \( Y \), then \( y \in U \). Choose a valuation ring \( V \) and a map \( g : \text{Spec}(V) \to Y \) such that the specialization \( y \to u \) lifts to a specialization \( y' \to u' \) in \( \text{Spec}(V) \) along \( g \). Assume that the base change \( f'_V : X \times_Y \text{Spec}(V) \to \text{Spec}(V) \) of \( f \) along \( g \) is already known to be spectrally submersive. Now \( f'_V^{-1}(g^{-1}(U)) \) is a quasicompact open in \( X \times_Y \text{Spec}(V) \) by hypothesis, so \( g^{-1}(U) \) must be open by spectral submersiveness of \( f'_V \). In particular, since \( u' \in g^{-1}(U) \), we must have \( y' \in g^{-1}(U) \), whence \( y = g(y') \in U \), as wanted. In other words, we have made the promised reduction. Assume now from now on that \( Y := \text{Spec}(V) \) is the spectrum of a valuation ring.

Let us first assume that \( V \) has finite rank, say \( n \). In this case, \( \text{Spec}(V) \) is given by a finite totally ordered set \( \{ p_0 \rightsquigarrow p_1 \rightsquigarrow ... \rightsquigarrow p_n \} \) of prime ideals. Each immediate specialization \( p_i \rightsquigarrow p_{i+1} \) gives rise to a map \( V \to V_i := (V/p_i)_{p_{i+1}} \) where the target is a rank 1 valuation ring. By assumption on \( f \), these maps lift to \( X \) (after extending \( V_i \) if necessary). It is thus enough to check the claim when \( f \) equals the map \( \lim_{i=0}^n \text{Spec}(V_i) \to \text{Spec}(V) \). But this is an elementary fact about finite tosets. Our hypothesis on \( U \) implies that each \( U_i := U \cap \text{Spec}(V_i) \) is open in \( \text{Spec}(V_i) \) for all \( i \). As \( U = \bigcup_{i=1}^n U_i \) is non-empty, we may choose \( j \) maximal such that \( U_j \neq \emptyset \). If \( j = 0 \), the claim is clear. If not, then \( U_j \subset \text{Spec}(V_j) \) is either just the generic point \( \{ \mathfrak{p}_j \} \subset \text{Spec}(V_j) \) or the whole space \( \{ \mathfrak{p}_j, \mathfrak{p}_{j+1} \} = \text{Spec}(V_j) \). In either case, we have \( \mathfrak{p}_j \in U_j \), which then implies \( \mathfrak{p}_{j-1} \in U \) since \( U_{j-1} = U \cap \text{Spec}(V_{j-1}) \) is open in \( \text{Spec}(V_{j-1}) = \{ \mathfrak{p}_{j-1}, \mathfrak{p}_j \} \). Continuing this way, we find \( U = \{ p_0 \rightsquigarrow ... \rightsquigarrow p_k \} \) for \( k \in \{ j, j+1 \} \), so \( U \) is open.

In general, write \( V \) as a filtered colimit \( \lim_{i=0}^\infty V_i \) of finite rank valuation rings \( V_i \subset \subset V \) as in Lemma 2.20, and let \( g_i : \text{Spec}(V) \to \text{Spec}(V_i) \) be the projection. The induced map \( f_i : X \to \text{Spec}(V_i) \) is then an arc-cover by Proposition 2.1, and thus a spectral submersion by the previous paragraph. It follows from Lemma 2.16 that the inverse limit \( f : X \to \text{Spec}(V) \simeq \lim_{i=0}^\infty \text{Spec}(V_i) \) of the \( f_i \)'s is also a spectral submersion. \( \square \)

**Remark 2.21.** We give an example of a universal spectral submersion that is not a quotient map; thus, the converse to the first statement of Remark 2.13 need not be true.

Write \( \mathbb{N} \) for the set of positive natural numbers. Let \( V \) be a valuation ring with \( \text{Spec}(V) \) given by the totally ordered set \( T := \{ 0 \} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subset [0, 1] \), so \( 0 \in T \) corresponds to the generic point; the existence of such a valuation ring follows from [41, Theorem 3.9 (b), (e)]. Write \( \mathfrak{p}_n \in \text{Spec}(V) \) for the prime corresponding to \( \frac{1}{n} \in T \) for \( n \geq 1 \); set \( V_n := (V/\mathfrak{p}_n)_{\mathfrak{p}_{n+1}} \) for any \( n > 1 \) and set \( V_1 = \kappa(p_1) \) to be the residue field of \( V \). Then each \( V_n \) is a rank 1 valuation ring (with rank exactly 1 unless \( n = 1 \)) with residue field \( \kappa(V_n) \) identified with \( \kappa(p_{n-1}) \) if \( n > 1 \) and \( \kappa(p_1) \) when \( n = 1 \). Moreover, we have \( \cap_{n} \mathfrak{p}_n = \{ 0 \} \). In particular, the map \( V \to B := \prod_{n} V_n \) is injective, whence \( f : X := \text{Spec}(B) \to \text{Spec}(V) \) hits the generic point and is thus surjective (as non-generic points are obviously in the image). Moreover, the map \( f \) is also an arc-cover: the specializations \( \mathfrak{p}_n \rightsquigarrow \mathfrak{p}_{n-1} \) exhaust all the immediate specializations in \( \text{Spec}(V) \), so any map \( V \to W \) where \( W \) is a rank 1 valuation ring either factors over some \( V \to V_n \) or factors over a residue field of \( V \). It follows from Proposition 2.17 that \( f \) is a universal spectral submersion.

Let us show that \( f \) is not a quotient map. First, one checks \( \pi_0(X) = \beta(\mathbb{N}) \) is the Stone–Čech compactification of \( \mathbb{N} \). The connected component of \( X \) corresponding to an ultrafilter \( \mathcal{U} \in \beta(\mathbb{N}) \) (i.e., the fibre of \( X \to \pi_0(X) \simeq \beta(\mathbb{N}) \) over \( \mathcal{U} \in \beta(\mathbb{N}) \)) is given by the spectrum of the ultraproduct \( \prod_{\mathcal{U}} V_n \) (see also Remark 3.13 and Lemma 3.23 below). We claim that the preimage \( f^{-1}(\{ 0 \}) \) of the generic point of \( \text{Spec}(V) \) coincides with the preimage in \( X \) of the closed set \( \beta(\mathbb{N}) - \mathbb{N} \subset \beta(\mathbb{N}) \).
Granting this claim, it is clear that $f$ is not a quotient map: if it were, then the generic point of $\text{Spec}(V)$ is closed, which is absurd. To identify $f^{-1}((0))$ with $\beta(\mathbb{N}) - N$, note that the containment $f^{-1}((0)) \subset \beta(\mathbb{N}) - N$ is clear: the points in $\mathbb{N}$ map down to the $p_n$’s in $\text{Spec}(V)$. Conversely, given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we must show that the image on $\text{Spec}(−)$ of the map $V \to \prod_{\mathcal{U}} V_n$ is the generic point. Unwinding definitions, this amounts to checking that if $\mathcal{U}$ is not principal, then the map $V \to \prod_{\mathcal{U}} \kappa(V_n)$ is injective. If this map were not injective, then the kernel would be nonzero. But, by definition of the ultraproduct, this kernel is exactly those $a \in V$ such that $\{n \in \mathbb{N} \mid a = 0 \in \kappa(V_n)\} \in \mathcal{U}$; this set is finite if $a \neq 0$ since $a \notin p_n$ for $n \gg 0$, whence $a \in \kappa(V_n)^{\ast}$ for $n \gg 0$. Taking a nonzero $a$ in the kernel then shows that $\mathcal{U}$ contains a finite set, so it must be principal, as wanted.

In this example, if we write $V$ as a filtered direct limit $\lim_{\rightarrow} W_i$ of finite rank valuation rings $W_i$ along faithfully flat maps as in Lemma 2.20, then the induced maps $\text{Spec}(B) \to \text{Spec}(W_i)$ are all (universal) submersions, but the inverse limit $\text{Spec}(B) \to \text{Spec}(V) \simeq \lim_{\leftarrow} \text{Spec}(W_i)$ is not a quotient map. Thus, we also obtain an example of the failure of (universal) submersions to be stable under filtered inverse limits; this phenomenon cannot happen with spectral submersions by Lemma 2.16.
3. Studying $v$-sheaves via valuation rings

In this section, we shall describe how to control $v$-sheaves in terms of their behaviour on absolutely integrally closed valuation rings. We begin in §3.1 by isolating a class of $v$-sheaves that we call finitary; these are the ones that commute with filtered colimits of rings, and are thus determined by their values on finitely presented rings. In §3.2, we collect some tools for describing the behaviour of finitary $v$-sheaves on (infinite) products of rings in terms of ultraproducts. In §3.3, we prove some general results concerning morphisms of universal $F$-descent with respect to certain functors $F$. Finally, these ingredients are put together in §3.4 to explain why finitary $v$-sheaves can be controlled by their behaviour on absolutely integrally closed valuation rings.

3.1. Finitary $v$-sheaves. In the sequel, we will need to work with sheaves for the $v$- and arc-topology.

Definition 3.1. Let $R$ be a ring, and let $\mathcal{C}$ be an $\infty$-category with all limits. A functor $F : \text{Sch}^{\text{op}}_R \to \mathcal{C}$ is said to be a $v$-sheaf (resp. arc-sheaf) if $F$ carries finite coproducts of schemes to products and for every $v$-cover (resp. arc-cover) $Y \to X$ in $\text{Sch}_R$, the natural map

$$F(X) \to \lim \left( F(Y) \Rightarrow F(Y \times_X Y) \Rightarrow \cdots \right)$$

is an equivalence. Note that the limit is indexed over the simplex category $\Delta$, i.e., it is a totalization.

Suppose that $\mathcal{C}$ has filtered colimits. We will say that $F$ is finitary if whenever $\{Y_\alpha\}_{\alpha \in A}$ is a tower of qcqs $R$-schemes (indexed over some cofiltered partially ordered set) with affine transition maps, then $F(\lim \alpha Y_\alpha) \simeq \lim \alpha F(Y_\alpha)$. In this case, by “relative approximation” [45, Tag 09MU], $F$ is determined by its restriction to $R$-schemes of finite presentation.

We next review generalities on descent.

Definition 3.2 (Universal $F$-descent). Let $F : \text{Sch}^{\text{op}}_R \to \mathcal{C}$ be a functor. We say that a map $f : Y \to X$ in $\text{Sch}_R$ is of $F$-descent if the natural map

$$F(X) \to \lim (F(Y) \Rightarrow F(Y \times_X Y) \Rightarrow \cdots)$$

is an equivalence in $\mathcal{C}$. We say that $f$ is of universal $F$-descent if all base-changes $f' : Y' \to X'$ of $f$ (along maps $X' \to X$ in $\text{Sch}_R$) are of $F$-descent.

We recall the following basic result (cf. [30, Lemma 3.12]), which gives basic closure properties for the class of maps of universal $F$-descent:

Lemma 3.3. Fix a functor $F : \text{Sch}^{\text{op}}_R \to \mathcal{C}$ as before. Let $f : Y \to X$ and $g : Z \to Y$ be maps in $\text{Sch}_R$.

(1) If $f$ has a section, then $f$ is of universal $F$-descent.

(2) If $f, g$ are of universal $F$-descent, then $f \circ g$ is of universal $F$-descent.

(3) If $f \circ g$ is of universal $F$-descent, then $f$ is of universal $F$-descent.

(4) The collection of maps of universal $F$-descent is closed under base-change.

We now specialize to cases such as $C = D(\Lambda)^{\geq 0}$, where we have additional properties.

Lemma 3.4. Suppose $F : \text{Sch}^{\text{op}}_R \to D(\Lambda)^{\geq 0}$ is finitary. Then the collection of maps which are of universal $F$-descent is closed under filtered limits with affine transition maps.

Proof. This follows because totalizations commute with filtered colimits in $D(\Lambda)^{\geq 0}$. \qed
In addition, the $v$-topology has the following basic finiteness property. The analogous result is not true with the arc-topology.

**Proposition 3.5.** Let $R$ be a ring. A finitary functor $F : \text{Sch}_{R}^{op} \to D(\Lambda)^{\geq 0}$ is a $v$-sheaf if and only if it satisfies the following conditions:

1. $F$ carries disjoint unions of finitely presented $R$-schemes to products.
2. For every $v$-cover of finitely presented $R$-schemes $Y \to X$, the map
   $$F(X) \to \lim_{\leftarrow} (F(Y) \to F(Y \times X Y) \to \ldots)$$
   is an equivalence.

**Proof.** Let $Y' \to X'$ be a $v$-cover. We want to show that this map is of universal $F$-descent. Suppose that $Y', X'$ are finitely presented over $R$. Because any qcqs $X'$-scheme is a filtered limit of finitely presented $X'$-schemes, it follows that any base change of $Y' \to X'$ in this case is a filtered limit of $v$-covers between finitely presented $R$-schemes. Our hypotheses thus show that $Y' \to X'$ is of universal $F$-descent.

Now suppose $Y' \to X'$ is an arbitrary $v$-cover. Without loss of generality, we can assume that $X'$ is affine. Since $Y' \to X'$ is a filtered limit of finitely presented maps $Y'_\alpha \to X'$ (each of which is forced to be a $v$-cover) with affine transition maps, we can assume that $Y' \to X'$ is finitely presented. Then using [39, Th. 3.12], $Y' \to X'$ admits a refinement $Y'' \to X'$ which is a composite of a quasi-compact open cover and a proper finitely presented surjection. It suffices to show that this refinement is of universal $F$-descent. By general results (see, e.g., [45, Tag 01YT]) we can descend quasi-compact open covers and proper finitely presented surjections to a base finitely presented over $R$. Therefore, $Y'' \to X'$ is a filtered limit (with affine transition maps) of $v$-covers between finitely presented $R$-schemes. Since we saw earlier that these maps are of universal $F$-descent, it follows that $Y'' \to X'$ is of universal $F$-descent, as desired. The statement that $F$ preserves finite products is similar and easier.

**Corollary 3.6.** The $\infty$-category of finitary $D(\Lambda)^{\geq 0}$-valued $v$-sheaves on $\text{Sch}_{R}$ is equivalent to the $\infty$-category of $D(\Lambda)^{\geq 0}$-valued $v$-sheaves on finitely presented $R$-schemes.

In particular, it follows that finitary $v$-sheaves of sets form a topos, for instance. This has been studied by various authors, especially since it is equivalent to Voevodsky’s $h$-topology [48]. By contrast, it is less clear what happens for the arc-topology. Nonetheless in the next subsection we will prove some structural results for finitary arc-sheaves.

### 3.2. Ultraproducts and sheaves

To proceed further, we will need to show that equivalences of $v$-sheaves can be detected by their values on absolutely integrally closed valuation rings (at least in the noetherian case, this is a classical result). In order to do so, we first review some facts about ultraproducts and sheaves; our goal is to prove Corollary 3.15, explaining how certain functors defined on all commutative rings can be controlled when evaluated on an infinite product of rings. We also refer to [26] for a more detailed treatment. For the rest of this section, we fix a base ring $R$ and an $\infty$-category $\mathcal{C}$ admitting small limits and filtered colimits. Eventually we will assume that $\mathcal{C} = D(\Lambda)^{\geq 0}$.
Recall that under Stone duality (see [27] for a general reference), there is a duality between Boolean algebras and profinite sets given by sending such a space to its collection of clopen subsets. One can also describe sheaves on profinite sets in terms of the corresponding Boolean algebra:

**Proposition 3.7.** Let $X$ be a profinite set, and let $B$ be the poset of quasicompact open (or equivalently clopen) subsets of $X$. The $\infty$-category of $C$-valued sheaves on $X$ is identified via restriction with the $\infty$-category of functors $B^{op} \to C$ that carry finite disjoint unions to products.

**Proof.** Given a $\mathcal{C}$-valued sheaf $f$ on $X$, restriction certainly gives a functor $\overline{f} : B^{op} \to \mathcal{C}$ carrying finite disjoint unions to coproducts. Conversely, given such a functor $g : B^{op} \to \mathcal{C}$, we can define a presheaf $\overline{g}$ on $X$ by setting $\overline{g}(U) = \varprojlim g(V)$ where the limit runs over all quasicompact open subsets $V \subset U$. To see that $\overline{g}$ is a sheaf, it is enough to check that its restriction $g$ to quasicompact open subsets forms a sheaf; but this follows from the assumption on $g$ together with the observation any if $V \subset U$ is an inclusion of quasicompact open subsets of $X$, then both $U$ and $V$ are clopen, and $U = V \sqcup (U - V)$ is the disjoint union of $V$ with its complement in $U$. It is straightforward to check that these constructions give inverse equivalences (see [33, Corollary 1.1.4.5] for a more general statement). $\square$

The following definition is partially motivated by Proposition 3.7.

**Definition 3.8.** Let $T$ be a set. We let $P(T)$ denote the poset of all subsets of $T$. We say that a functor $f : P(T)^{op} \to \mathcal{C}$ is a sheaf if it carries finite disjoint unions to products.

**Example 3.9.** Suppose we have a functor $u : T \to \mathcal{C}$ (where $T$ is considered as a discrete category). Then we have a sheaf $f : P(T)^{op} \to \mathcal{C}$ defined via $f(T') = \prod_{t \in T'} u(t)$.

This notion of sheaf turns out to be equivalent to the classical notion of a sheaf on the profinite set corresponding to the Boolean algebra $P(T)$.

**Construction 3.10** (Stone-Čech compactification). Let $T$ be a set. An ultrafilter $\mathcal{U}$ on $T$ is a collection of subsets such that $\emptyset \notin \mathcal{U}$, $\mathcal{U}$ is closed under finite intersections, and for every $T' \subset T$, either $T'$ or $T \setminus T'$ belongs to $\mathcal{U}$. Each element $t \in T$ defines the principal ultrafilter $\mathcal{U}_t$ of all subsets of $T$ containing $t$.

The collection of all ultrafilters on $T$ is naturally a profinite set $\beta T$, called the Stone-Čech compactification on $T$. The assignment $t \mapsto \mathcal{U}_t$ gives an open embedding $T \hookrightarrow \beta T$ with dense image (where $T$ has the discrete topology). The Boolean algebra of all clopen subsets of $\beta T$ is identified with the power set $P(T)$ via $(V \subset \beta T) \mapsto (V \cap T \subset T)$. Under the equivalence of Proposition 3.7, the $\infty$-category of sheaves on $P(T)$ in the sense of Definition 3.8 is equivalent to the category of sheaves on $\beta T$ in the classical sense. In particular, given a sheaf $f : P(T)^{op} \to \mathcal{C}$ and an ultrafilter $\mathcal{U}$ on $T$, we may define the stalk of $f$ at $\mathcal{U}$ by the formula

$$f_\mathcal{U} = \lim_{T' \in \mathcal{U}} f(T'),$$

where the colimit is taken over the (filtered) partially ordered set of all $T' \in \mathcal{U}$.

**Lemma 3.11.** Suppose $\mathcal{C}$ is compactly generated. Suppose $f \to g$ is a map of sheaves in $\text{Fun}(P(T)^{op}, \mathcal{C})$. Suppose furthermore that for each ultrafilter $\mathcal{U}$ on $T$, the map $f_\mathcal{U} \to g_\mathcal{U}$ is an equivalence. Then $f \to g$ is an equivalence.

---

5A profinite set is a topological space that can be realized as a cofiltered inverse limit of finite sets. Any profinite set is a compact, totally disconnected Hausdorff space, and conversely any such space is a profinite set. In the literature, such spaces are often called Stone spaces as well.
Proof. For each compact object \( x \in \mathcal{C} \), it follows that the functors \( \pi_0 \text{Hom}_\mathcal{C}(x, f(\cdot)), \pi_0 \text{Hom}_\mathcal{C}(x, g(\cdot)) \) from \( \mathcal{P}(T)^{\text{op}} \rightarrow \text{Sets} \) are sheaves as well. They thus define sheaves on \( \beta T \) and the natural map induces an isomorphism on stalks in \( \beta T \); consequently \( \pi_0 \text{Hom}_\mathcal{C}(x, f(\cdot)) \simeq \pi_0 \text{Hom}_\mathcal{C}(x, g(\cdot)) \) for each compact \( x \in \mathcal{C} \), and therefore \( f \simeq g \) as desired. \( \square \)

To proceed further, recall the following classical definition.

**Definition 3.12** (Ultraproducts of rings). Given a set \( \{ A_t \}_{t \in T} \) of \( R \)-algebras and an ultrafilter \( \mathcal{U} \) on \( T \), we define the *ultraproduct* \( \prod_{\mathcal{U}} A_t \) via the formula

\[
\prod_{\mathcal{U}} A_t = \lim_{T' \in \mathcal{U} T' T} \prod_{t \in T'} A_t.
\]

Note that the colimit appearing above is filtered.

**Remark 3.13** (Ultraproducts via the spectrum of a product). Given a set \( \{ A_t \}_{t \in T} \) of commutative rings, the space \( \text{Spec}(\prod_{t \in T} A_t) \) comes equipped with a natural projection map \( \pi : \text{Spec}(\prod_{t \in T} A_t) \rightarrow \beta T \) determined by requiring that the preimage \( \pi^{-1}(U) \) of a quasicompact open \( U \subset \beta T \) corresponding to a subset \( T' \subset T \) is the clopen subscheme \( \text{Spec}(\prod_{t \in T'} A_t) \subset \text{Spec}(\prod_{t \in T} A_t) \). The ultraproduct \( \prod_{\mathcal{U}} A_t \) is then simply the co-ordinate ring of the closed (and pro-open) subscheme \( \pi^{-1}(\mathcal{U}) \subset \text{Spec}(\prod_{t \in T} A_t) \).

Next, given a family \( \{ A_t \}_{t \in T} \) of \( R \)-algebras, let us explain how certain functors defined on all rings give sheaves on \( \mathcal{P}(T) \).

**Construction 3.14.** Let \( \{ A_t \}_{t \in T} \) be a family of \( R \)-algebras. Let \( \mathcal{F} : \text{Ring}_R \rightarrow \mathcal{C} \) be a functor which preserves finite products and filtered colimits. Then we obtain a functor (using Example 3.9)

\[
f : \mathcal{P}(T)^{\text{op}} \rightarrow \mathcal{C}
\]

sending

\[
T' \subset T \mapsto \mathcal{F}(\prod_{t \in T'} A_t).
\]

The hypothesis that \( \mathcal{F} \) preserves finite products implies that \( f \) is a sheaf. Moreover, for any ultrafilter \( \mathcal{U} \) on \( T \), we have an identification \( f_\mathcal{U} \simeq \mathcal{F}(\prod_{\mathcal{U}} A_t) \) since \( \mathcal{F} \) commutes with filtered colimits. That is, the stalks of \( f \) can be identified with the values of \( \mathcal{F} \) on the ultraproducts.

For future reference, we record a slight variant of the preceding paragraph. Fix an \( A \)-algebra \( B \). Then we obtain a functor \( f_B : \mathcal{P}(T)^{\text{op}} \rightarrow \mathcal{C} \) defined by

\[
T' \subset T \mapsto \mathcal{F}(B \otimes_A \prod_{t \in T'} A_t).
\]

Then \( f_B \) is again a sheaf on \( \mathcal{P}(T) \), and \( (f_B)_\mathcal{U} \simeq \mathcal{F}(B \otimes_A \prod_{\mathcal{U}} A_t) \). That is, the stalk of \( f_B \) at an ultrafilter \( \mathcal{U} \) is \( \mathcal{F} \) applied to the base-change of \( B \) along the map from \( A \) to the \( \mathcal{U} \)-ultraproduct.

We can now prove the promised result, explaining how to control the behaviour on infinite products of certain functors defined on commutative rings.

**Corollary 3.15.** Suppose \( \mathcal{C} \) is compactly generated. Fix a base ring \( R \). Let \( \mathcal{F}_a, \mathcal{F}_b : \text{Ring}_R \rightarrow \mathcal{C} \) be two functors which commute with filtered colimits and finite products. Suppose given a map \( \mathcal{F}_a \rightarrow \mathcal{F}_b \) of functors. Fix a family \( \{ A_t \}_{t \in T} \) of \( R \)-algebras. Suppose for each ultrafilter \( \mathcal{U} \) on \( T \), the map \( \mathcal{F}_a(\prod_{\mathcal{U}} A_t) \rightarrow \mathcal{F}_b(\prod_{\mathcal{U}} A_t) \) is an equivalence. Then the map \( \mathcal{F}_a(\prod_T A_t) \rightarrow \mathcal{F}_b(\prod_T A_t) \) is an equivalence.

**Proof.** This follows from Lemma 3.11 in light of the above constructions. \( \square \)
3.3. Detection of universal $\mathcal{F}$-descent. Let $\mathcal{F} : \text{Sch}_R^{op} \to C$ be a functor. The goal of this section is to prove a result (Corollary 3.21) that explains how to detect $\mathcal{F}$-descent properties of morphisms of schemes by base changing to ultraproducts instead of products. To formulate this precisely, it is convenient to make the following definition.

Definition 3.16 (Detection of universal $\mathcal{F}$-descent). Let $X \in \text{Sch}_R$. Consider a family of maps $X_i \to X$, $i \in I$. We say that the family $\{X_i \to X\}_{i \in I}$ detects universal $\mathcal{F}$-descent if for every map $f : Y \to X$, $f$ is of universal $\mathcal{F}$-descent if and only if the base-change $f_i : Y \times_X X_i \to X_i$ is of universal $\mathcal{F}$-descent for each $i \in I$.

Example 3.17. Suppose a map $Y \to X$ is of universal $\mathcal{F}$-descent. Then the map $Y \to X$ detects universal $\mathcal{F}$-descent. This follows in view of Lemma 3.3.

Example 3.18. Suppose $\{\text{Spec}(A_i) \to \text{Spec}(A)\}_{i \in I}$ detects universal $\mathcal{F}$-descent. Then the map $f : \text{Spec}(\prod_{i \in I} A_i) \to \text{Spec}(A)$ is of universal $\mathcal{F}$-descent. In fact, the map admits a section after base-change to $\text{Spec}(A_j)$ for each $j$ and each such base-change is therefore of universal $\mathcal{F}$-descent. By assumption, this implies that $f$ is of universal $\mathcal{F}$-descent. As a partial converse, under mild hypotheses on $\mathcal{F}$, we shall show in Corollary 3.21 that if the singleton family $\{\text{Spec}(\prod_i A_i) \to \text{Spec}(A)\}$ detects universal $\mathcal{F}$-descent, then the family $\{\text{Spec}(\prod_{i \in I} A_i) \to \text{Spec}(A)\}_{i \in I}$ (obtained by localizing $\prod_i A_i$ at all possible ultrafilters on $I$) detects universal $\mathcal{F}$-descent.

Example 3.19. Suppose we have a family $\{X_i \to X\}_{i \in I}$ which detects universal $\mathcal{F}$-descent. Suppose for each $i \in I$, we have a set $J_i$ and a family of maps $\{Y_j \to X_i\}_{j \in J_i}$, which detects universal $\mathcal{F}$-descent. Then the family $\{Y_j \to X\}_{j \in J_i \text{ some } i}$ detects universal $\mathcal{F}$-descent.

Lemma 3.20 (Detection and filtered colimits). Let $\mathcal{F} : \text{Sch}_R^{op} \to D(\Lambda)^{\geq 0}$ be a finitary functor. Let $X$ be a scheme and let $\{Y_i\}_{i \in I}$ be a family of $X$-schemes. Suppose we can write $X$ as a filtered limit $X = \lim_{\to j} X_j$ in $\text{Sch}_R$ with affine transition maps. Suppose that for each $j$, the family of maps $\{Y_i \to X_j\}_{i \in I}$ detects universal $\mathcal{F}$-descent. Then the family of maps $\{Y_i \to X\}$ detects universal $\mathcal{F}$-descent.

Proof. Let $X' \to X$ be a map. Suppose that each base-change $Y_i \times_X X' \to Y_i$ is of universal $\mathcal{F}$-descent. For each $i, j$, we have a factorization

$$Y_i \times_X X' \to Y_i \times_{X_j} X' \to Y_i$$

and since the composition is of universal $\mathcal{F}$-descent, it follows that $Y_i \times_{X_j} X' \to Y_i$ is of universal $\mathcal{F}$-descent (Lemma 3.3). Since $\{Y_i \to X_j\}_{i \in I}$ detects universal $\mathcal{F}$-descent, we find that each map $X' \to X_j$ is of universal $\mathcal{F}$-descent. Taking the limit over $j$, we find that $X' \to X$ is of universal $\mathcal{F}$-descent (Lemma 3.4). $\square$

In the next result, for convenience, we switch to the notation of rings (rather than schemes).

Corollary 3.21. Let $\mathcal{F} : \text{Ring}_R^{op} \to D(\Lambda)^{\geq 0}$ be a finitary functor which preserves finite products. Let $\{A_i\}_{i \in I}$ be a family of $R$-algebras and let $A = \prod_{i \in I} A_i$. Then the maps $\{A \to \prod_{i \in I} A_i\}$ as $\mathcal{U}$ ranges over the ultrafilters in $T$, detect universal $\mathcal{F}$-descent.

Proof. Let $B \to C$ be a map of $A$-algebras. Suppose $B \otimes_A \prod_{i \in I} A_i \to C \otimes_A \prod_{i \in I} A_i$ is of $\mathcal{F}$-descent for each $\mathcal{U}$ on $T$. Then we claim that $B \to C$ is of $\mathcal{F}$-descent, which will imply the result. In fact, consider the augmented cosimplicial object of $D(\Lambda)^{\geq 0}$

$$\mathcal{F}(B) \to \mathcal{F}(C) \Rightarrow \mathcal{F}(C \otimes_B C) \Rightarrow \cdots$$

(4)
which we want to show is a limit diagram. By Construction 3.14, the augmented cosimplicial diagram upgrades to a diagram of sheaves (with values in $D(\Lambda)^{\geq 0}$) on $\mathcal{P}(T)$. Furthermore, by assumption, the diagram of $\mathfrak{U}$-stalks is a limit diagram for each ultrafilter $\mathfrak{U}$. Since totalizations and filtered colimits commute in $D(\Lambda)^{\geq 0}$, it follows by Lemma 3.11 that we have a limit diagram of sheaves, and thus $B \to C$ is of $\mathcal{F}$-descent as desired. □

3.4. Detection in the $v$-topology. The goal of this section is to prove that finitary $v$-sheaves are controlled by their behaviour on (certain) valuation rings (Propositions 3.26 and 3.28). To formulate these, we first recall the following basic definition:

**Definition 3.22.** A ring $R$ is called absolutely integrally closed if every monic polynomial in $R[x]$ admits a root in $R$.

We will only use this definition in the case where $R$ is an integral domain, in which case it is equivalent to the condition that $\text{Frac}(R)$ is algebraically closed and that $R$ is normal. Note that the class of absolutely integrally closed domains is preserved by localizations and quotients by prime ideals.

**Lemma 3.23.** An ultraproduct of absolutely integrally closed valuation rings is an absolutely integrally closed valuation ring.

*Proof.* The condition that a ring should be an absolutely integrally closed valuation ring is a first-order property in the language of commutative rings. Therefore, the result follows from Los’s theorem.

Alternately, one can argue directly as in [4, Lemma 6.2] to show that an ultraproduct $\prod_{\mathfrak{U}} V_i$ of a collection $\{V_i\}_{i \in T}$ of valuation rings is a valuation ring. If each $V_i$ is absolutely integrally closed, the same holds true for $\prod_{i \in T} V_i$, and thus also for any localization such as $\prod_{\mathfrak{U}} V_i$. □

Our next goal is to show that every ring admits a $v$-cover by a product of absolutely integrally closed valuation rings (Proposition 3.26); this is a variant of [4, Lemma 6.2], and is closely related to the fact that absolutely integrally closed valuation rings give a conservative system of points for the $h$-topology (at least for noetherian rings, see [20, Prop. 2.2, Cor. 3.8] and [17]). To get there, let us first study absolute integral closures of valuation rings.

**Lemma 3.24.** Let $V$ be a valuation ring. Then there exists an absolutely integrally closed valuation ring $V'$ and a map $V \to V'$ which is faithfully flat (hence a $v$-cover).

*Proof.* Consider an extension of the valuation to $\text{Frac}(V)$, and take $V'$ to be the valuation ring associated to that valuation. By Lemma 2.19, $V'$ satisfies the desired claims. □

To avoid set-theoretic inconsistencies, we bound the size of the absolutely integrally closed valuation rings required to probe a given ring.

**Lemma 3.25.** Let $A$ be an arbitrary commutative ring and let $\kappa = \max(\text{card}(A), \aleph_0)$. Then for every absolutely integrally closed valuation ring $W$ with a map $A \to W$, there exists a factorization $A \to W' \to W$ such that:

1. $W'$ is an absolutely integrally closed valuation ring with $\text{card}(W') \leq \kappa$.
2. $W' \to W$ is faithfully flat.

*Proof.* Fix a map $f : A \to W$ and consider the image $\text{im}(f) \subseteq W$. Let $K_0$ be the fraction field of $\text{im}(f)$ inside $K := \text{Frac}(W)$, and let $K_1 \subseteq K$ be the algebraic closure of $K_0$. Then $\text{card}(K_1) \leq \kappa$, so $W' := K_1 \cap W$ has cardinality at most $\kappa$. The rest of the result follows from Lemma 2.19. □
We can now prove the promised result.

**Proposition 3.26.** Let $F : \text{Sch}^{\text{op}}_R \to D(\Lambda)^{\geq 0}$ be a finitary $v$-sheaf. Let $A$ be an $R$-algebra. Then there exists a set of maps $\{A \to V_i\}_{i \in S}$ such that:

1. Each $V_i$ is an absolutely integrally closed valuation ring.
2. The maps $\{A \to V_i\}$ detect universal $F$-descent.
3. The map $A \to \prod_{i \in S} V_i$ is a $v$-cover.

**Proof.** Let $\kappa = \max(\text{card}(A), 8_0)$. Let $W_i, t \in T$ be a set of representatives of isomorphism classes of absolutely integrally closed valuation rings of cardinality $\leq \kappa$ receiving maps $A \to W_i$. We first claim that the map $A \to A' := \prod_{i \in T} W_i$ is a $v$-cover. This follows from Lemma 3.24 and Lemma 3.25. Explicitly, if $A \to V$ is a map to any valuation ring $V$, we can enlarge $V$ and assume $V$ absolutely integrally closed (by Lemma 3.24). Then the map $A \to V$ actually factors through $A \to A'$ (by Lemma 3.25); together, these imply that $A \to A'$ is a $v$-cover.

This produces a collection of aic valuation rings satisfying (3), but we have not yet shown detection of universal $F$-descent; for this we enlarge the family further. We construct the family $\{V_i\}$ of $A$-algebras as the collection of ultraproducts of the $W_i$. For each ultrafilter $\mathcal{U}$ on $T$, we consider the $A'$-algebra $A'_\mathcal{U} := \prod_{i \in T} W_i$. Then each $A'_\mathcal{U}$ is an absolutely integrally closed valuation ring (Lemma 3.23). Moreover the family of maps $\{A' \to A'_\mathcal{U}\}$ (as $\mathcal{U}$ ranges over all ultrafilters on $T$) detects universal $F$-descent thanks to Proposition 3.21. The assertion (3) now follows because $A \to A'$ is a $v$-cover. \qed

**Remark 3.27.** Proposition 3.25 and Lemma 3.23 imply that for every commutative ring $A$, there is a $v$-cover $A \to B$ such that each connected component of $B$ is a valuation ring. The analog for the arc-topology is false: already for $A = k[x, y]$ over a field $k$, there does not exist an arc-cover $A \to B$ such that that every connected component of $B$ is a rank $\leq 1$ valuation ring. Indeed, if such an arc-cover $A \to B$ existed, then this map would be a $v$-cover by Proposition 2.6, so every valuation on $A$ would extend to $B$, which is impossible: $A$ admits a rank 2 valuation, while every valuation on $B$ has rank $\leq 1$ by the assumption on connected components.

Using the preceding constructions, we can control finitary $v$-sheaves in terms of their behaviour on absolutely integrally closed valuation rings.

**Proposition 3.28.** Let $F, G : \text{Sch}^{\text{op}}_R \to D(\Lambda)^{\geq 0}$ be finitary $v$-sheaves and fix a map $f : F \to G$. Then the following are equivalent:

1. For every absolutely integrally closed valuation ring $V \in \text{Ring}_R$, $F(\text{Spec}(V)) \to G(\text{Spec}(V))$ is an equivalence.
2. $F \to G$ is an equivalence.

**Proof.** Without loss of generality, $G = 0$. Recall that ultraproducts of absolutely integrally closed valuation rings are absolutely integrally closed valuation rings. Therefore, by Corollary 3.15, we have $F(\text{Spec}(R)) = 0$ whenever $R$ is a product of absolutely integrally closed valuation rings. Now for any ring $A$, we have a $v$-cover $A \to R$ for $R$ an appropriate product of absolutely integrally closed valuation rings, by Proposition 3.26. It follows that the map of abelian groups $H^0(F(\text{Spec}(A))) \to H^0(F(\text{Spec}(R)))$ is injective, thanks to the $v$-sheaf condition, which forces $H^0(F(\text{Spec}(A))) = 0$ too. Therefore, $H^0(F) = 0$ and $F$ takes values in $D(\Lambda)^{\geq 1}$. Inducting, we obtain that $F = 0$. Note that this argument could also have been carried out by constructing a $v$-hypercover of $A$ where all of the terms are products of aic valuation rings. \qed
4. THE MAIN RESULT FOR ARC-DESCENT

In this section, we prove our main result, explaining the relationship of arc-descent to excision. Fix a base ring $R$ and a target ring $\Lambda$.

**Theorem 4.1 (Equivalence of arc-descent and excision).** Let $F : \text{Sch}^\op_R \to D(\Lambda)^{\geq 0}$ be a finitary functor (see Definition 3.1) which satisfies $v$-descent. Then the following are equivalent:

1. $F$ satisfies arc-descent.
2. $F$ satisfies excision.
3. $F$ satisfies aic-v-excision, i.e., for every absolutely integrally closed valuation ring $V$ and prime ideal $p \subset V$, the square in $D(\Lambda)^{\geq 0}$

$$
\begin{array}{ccc}
F(V) & \to & F(V/p) \\
\downarrow & & \downarrow \\
F(V_p) & \to & F(\kappa(p)) \\
\end{array}
$$

is cartesian. (Recall that the square of rings is a Milnor square, cf. Proposition 2.8.)

Clearly (2) implies (3). We will structure the proof of the theorem as follows. In subsection 4.1, we show that (1) and (3) are equivalent. In subsection 4.2, we show that (3) implies (2). Together these will complete the proof.

4.1. **Aic-v-excision and arc-descent.** In this subsection, we will prove that (1) and (3) in Theorem 4.1 are equivalent. First we show that (1) implies (3).

**Proposition 4.2.** Suppose $F$ satisfies arc-descent. Then $F$ satisfies aic-v-excision.

**Proof.** We consider the map $V \to \tilde{V} := V_p \times V/p$. By our assumptions and Corollary 2.9, $F$ satisfies descent for this morphism. Unwinding the definitions, and using the assumption that $F$ preserves finite products, we see that the cosimplicial object $F(\tilde{V}^{\bullet+1})$ computes precisely $F(V_p) \times_{F(\kappa(p))} F(V/p)$. In particular, the statement that $F$ satisfies descent for $V \to \tilde{V}$ is equivalent to the statement that $F$ satisfies excision for the excision datum $(V, p) \to (V_p, \kappa(p))$. □

We will next show that for a $v$-sheaf which satisfies aic-v-excision, there are enough maps to rank $\leq 1$ valuation rings which detect universal $F$-descent. To proceed, we will use the following lemma about maps between absolutely integrally closed valuation rings.

**Lemma 4.3.** Let $V \to W$ be a map between absolutely integrally closed valuation rings. Fix a prime $p \subset V$. Suppose that $pW \neq W$. Then:

1. The ideal $pW \subset W$ is prime and pulls back to $p$ in $V$.
2. There exists a largest prime $q \subset W$ whose pullback to $V$ equals $p$. For such $q$, we get $W \otimes_V V_p \simeq W_q$ and $W \otimes_V \kappa(p) \simeq W_q/pW_q$.

As the proof below shows, we do not need the full strength of absolute integral closedness above: it suffices to assume that the value groups of $V$ and $W$ are $n$-divisible for some integer $n > 1$.

**Proof.** By replacing $V$ by its image, we may assume that $V$ is a subring of $W$.

For (1), since radical ideals in a valuation ring are prime, it is enough to show that $pW$ is radical. Say $y \in W$ with $y^2 \in pW$. Then we can write $y^2 = \sum_{i=1}^n a_i x_i$ with $a_i \in p$ and $x_i \in W$. As $V$ is a valuation ring, we may assume after rearrangement that $a_1 \mid a_i$ for all $i$, so we can rewrite
$y^2 = a_1 \cdot z$ for some $z \in W$. As both $V$ and $W$ have 2-divisible value groups, we can choose $c_1 \in p$ such that $c_1^2 V = a_1 V$ and $w \in W$ such that $w^2 W = z W$. It follows that $y^2 W = c_1^2 w^2 W$, and hence $y W = c_1 w W$ as $W$ is a valuation ring. But this implies $y \in c_1 W \subset p W$, as wanted.

Next we show that $p W \subset W$ pulls back to $p \subset V$. Suppose there is $x \in V \setminus p$ such that $x \in p W$, so $x = \sum y_i z_i$ for some $y_i \in p, z_i \in W$. Using divisibility to collect terms, we may assume that there is only one term in the sum, i.e., $x = y z$ for $y \in p, z \in W$. Since $V$ is a valuation ring, it follows that we can write $y = x y'$ for some $y' \in p$. We get $x = x y' z$ in $W$, and canceling we find that $y' z = 1$ in $W$. This means that $p W = W$, a contradiction.

For the existence of $q$ in (2): note that the collection of primes $q' \subset W$ whose pullback to $V$ equals $p$ is directed: as $W$ is a valuation ring, any set of prime ideals of $W$ is even totally ordered. The existence of $q$ follows as a filtered colimit of prime ideals is prime (and because forming filtered colimits of primes in $W$ commutes with pulling back to $W$). Having constructed $q$, we immediately get a map $W \otimes_V V_p \to W_q$. This is an injective map of two valuation rings (as both sides are localizations of $W$). To prove bijectivity, it is thus enough to show that their spectra match up. The outermost square is a Milnor square as in (5) (i.e., belongs to the setting of aic-$v$-excision), and so is the bottom square (Proposition 2.8). Therefore, $F$ carries the outermost and bottom squares to pullback squares. It follows that $F$ carries the top square to a pullback, i.e., $F(W) \simeq F(W/p W) \times_{F(W \otimes_V V_p)} F(W \otimes_V k(p))$. □

We can now prove a crucial stability property of aic-$v$-excision: it passes up to algebras.

**Lemma 4.4.** Let $V$ be an absolutely integrally closed valuation ring, and fix a $V$-algebra $A$. Let $F : \text{Sch}_{V}^{op} \to D(\Lambda)_{\geq 0}$ be a finitary $v$-sheaf which satisfies aic-$v$-excision. Then for every prime ideal $p \subset V$, the map

$$F(A) \to F(A \otimes_V V_p) \times_{F(A \otimes_V k(p))} F(A/p)$$

is an equivalence.

**Proof.** Since both sides are $v$-sheaves in $A$, it suffices (by Proposition 3.28) to show that the map is an equivalence for $A = W$ an absolutely integrally closed valuation ring.

If $p W = W$, then $W \simeq W \otimes_V V_p$, $W/p W = 0$, and the result is clear. If $p W \neq W$, then Proposition 4.3 shows that $p W$ is a prime ideal in $W$. Since $p W$ pulls back to $p$, we obtain a natural map $V_p \to W_{p W}$. We consider the diagram

\[
\begin{array}{ccc}
W & \longrightarrow & W/p W \\
\downarrow & & \downarrow \\
W \otimes_V V_p & \longrightarrow & W \otimes_V k(p) \\
\downarrow & & \downarrow \\
W_{p W} & \longrightarrow & k(p W),
\end{array}
\]

where all rings are absolutely integrally closed valuation rings. There are three squares we can extract naturally from this diagram. The outermost square is a Milnor square as in (5) (i.e., belongs to the setting of aic-$v$-excision), and so is the bottom square (Proposition 2.8). Therefore, $F$ carries the outermost and bottom squares to pullback squares. It follows that $F$ carries the top square to a pullback, i.e., $F(W) \simeq F(W/p W) \times_{F(W \otimes_V V_p)} F(W \otimes_V k(p))$. □
We can now prove a variant of Proposition 3.26 for aic-v-excisive sheaves where we work exclusively with rank ≤ 1 valuation rings.

**Lemma 4.5.** Let \( \{A_i \to B_i\}_{i \in I} \) be a diagram of finitely presented rings maps indexed by a filtered category \( I \). Assume that for each map \( i \to j \) in \( I \), the map \( A_j \otimes_{A_i} B_i \to B_j \) is an isomorphism. If \( \lim_{\to} A_i \to \lim_{\to} B_i \) is a \( v \)-cover, then so is \( A_j \to B_j \) for some \( j \in I \) (and thus for all \( j \) large enough).

**Proof.** Let \( A_\infty = \lim_{\to} A_i, B_\infty = \lim_{\to} B_i \). The map \( \text{Spec}(B_\infty) \to \text{Spec}(A_\infty) \) is finitely presented and a \( v \)-cover and therefore [39, Theorem 3.12] admits a refinement which factors as a composite of a quasi-compact open covering and a proper finitely presented surjection. By general results of noetherian approximatin [45, Tag 09MV], some map \( \text{Spec}(B_j) \to \text{Spec}(A_i) \) admits a refinement which factors as a composite of a quasi-compact open covering and a proper finitely presented surjection, and is therefore a \( v \)-cover. \( \square \)

**Proposition 4.6.** Let \( \mathcal{F} : \text{Sch}^{op}_R \to D(\Lambda)^{\geq 0} \) be a finitary \( v \)-sheaf which is aic-v-excisive. Then \( \mathcal{F} \) is an arc-sheaf.

**Proof.** Let \( f : Y \to X \) be an arc-cover of \( R \)-schemes. We want to show that it is of universal \( \mathcal{F} \)-descent. Without loss of generality, we can assume \( X, Y \) are affine. By Proposition 3.26, we can further reduce to the case where \( X = \text{Spec}(V) \) is the spectrum of an absolutely integrally closed ring \( V \). Let \( Y = \text{Spec}(A) \), for \( A \) a \( V \)-algebra.

Now \( A \) is the filtered colimit of finitely presented \( V \)-algebras, each of which is also an arc-cover of \( V \). Since morphisms of universal \( \mathcal{F} \)-descent are closed under filtered colimits (of rings), it suffices to assume that \( A \) is finitely presented over \( V \).

We now consider the totally ordered set \( \text{Spec}(V) \) of prime ideals of \( V \). An interval \( I = [p, q] \) (for \( p \subset q \)) will denote the set of prime ideals of \( V \) which are contained between \( p \) and \( q \); note that this is also \( \text{Spec}((V/p)_{q}) \), so to each interval \( I \) we have an associated absolutely integrally closed valuation ring \( V_I = (V/p)_{q} \).

The collection of intervals of \( \text{Spec}(V) \) is totally ordered under inclusion: we have \( [p, q] \subseteq [p', q'] \) if and only if \( p \supseteq p' \) and \( q \subseteq q' \) as prime ideals. Given an inclusion of intervals \( I \subseteq J \), we have a map \( V_J \to V_I \), so we have a contravariant functor from the poset of intervals to absolutely integrally closed valuation rings. Finally, note that any chain \( C \) (i.e., totally ordered subset) in the poset of intervals admits a supremum and an infimum. The infimum is given by the intersection, and the functor \( I \mapsto V_I \) sends an intersection of intervals along a chain to the associated filtered colimit of the \( V_I, I \in C \).

We consider now those intervals \( I \) such that the map \( V_I \to A \otimes_V V_I \) (a finitely presented map of \( R \)-algebras) is of universal \( \mathcal{F} \)-descent. We call such intervals good; our goal is to show that the interval \( \text{Spec}(V) \) is good. The following:

1. If \( I \) is an interval of length \( \leq 1 \) (so \( I \) either consists of one point or \( I = [p, q] \) where \( p \not\subseteq q \) is an inclusion that cannot be refined further) then \( V_I \) is a rank \( \leq 1 \) valuation ring. Therefore, for such \( I \), the map \( V_I \to A \otimes_V V_I \) is actually a \( v \)-cover and therefore of universal \( \mathcal{F} \)-descent. Thus, \( I \) is good.

2. If \( I \) is good and \( J \subseteq I \) is a subinterval, then \( J \) is also good. This is evident since morphisms of universal \( \mathcal{F} \)-descent are stable under base-change.

3. Suppose \( p \in \text{Spec}(V) \) is not maximal. Then there exists \( q \supseteq p \) such that \( [p, q] \) is good.

Indeed, if \( p \) has an immediate successor, then we can take that successor as \( q \), in view of (1) above. Suppose \( p \) has no immediate successor. Then the interval \( \{p\} \) is the intersection...
of the intervals \([p, q']\) for \(q' \supseteq p\). It follows that
\[
\kappa(p) = \lim_{I = [p, q'], q' \supseteq p} V_I.
\]

Now \(\kappa(p) \to A \otimes_V \kappa(p)\) is a \(v\)-cover and is finitely presented. Therefore, by Lemma 4.5, there exists \(I = [p, q]\) for \(q \supseteq p\) such that \(V_I \to A \otimes_V V_J\) is a \(v\)-cover. We can take this \(q\), since \(\mathcal{F}\) is a \(v\)-sheaf so that \(I\) is good.

(4) Suppose \(q \in \text{Spec}(V)\) is nonzero. Then there exists \(p \subseteq q\) such that \([p, q]\) is good. This is proved similarly.

(5) Suppose \(I, J\) are overlapping good intervals, so that \(I \cup J\) is an interval. Then \(I \cup J\) is good.

To see this, we may first assume without loss of generality that \(I \cup J = \text{Spec}(V)\), by base-change. Suppose \(I = [0, p], J = [q, m]\) for \(p \supseteq q\) and \(m\) the maximal ideal. We claim that for any \(V\)-algebra \(B\), the diagram
\[
\begin{array}{ccc}
\mathcal{F}(B) & \xrightarrow{\cdot} & \mathcal{F}(B \otimes_V V_I) \\
\downarrow & & \downarrow \\
\mathcal{F}(B \otimes_V V_J) & \xrightarrow{\cdot} & \mathcal{F}(B \otimes_V V_{I \cap J})
\end{array}
\]

is cartesian.

If \(I \cap J\) is a single point, then this assertion follows from Lemma 4.4. To reduce to this case, let \(J' = [p, m]\) so that \(J' \subseteq J\) and \(I \cap J' = \{p\}\). We can extend the diagram
\[
\begin{array}{ccc}
\mathcal{F}(B \otimes_V V_J) & \xrightarrow{\cdot} & \mathcal{F}(B \otimes_V V_{I \cap J}) \\
\downarrow & & \downarrow \\
\mathcal{F}(B \otimes_V V_{J'}) & \xrightarrow{\cdot} & \mathcal{F}(B \otimes_V V_{I \cap J'})
\end{array}
\]

This square is cartesian, since we can consider the intervals \(J', I \cap J\) whose union is \(J\) and which intersect at a single point. Moreover, if we paste together the diagrams (6), (7) the outer square is cartesian (via the intervals \(J', I\)). Combining, we get that (6) is cartesian.

Since (6) is cartesian, it follows easily that the maps \(V \to V_I, V_J, V_{I \cap J}\) detect universal \(\mathcal{F}\)-descent. This proves the claim.

We claim that observations (1) through (5) above now essentially formally imply that \(A \to V\) is of universal \(\mathcal{F}\)-descent.

Indeed, consider the collection of good intervals. We will apply Zorn’s lemma. Suppose \(\mathcal{D}\) is a chain of good intervals \([p, q_n]\).

We let \(p = \bigcap p_i\) and \(q = \bigcup q_i\); the claim is that \([p, q]\) is also good, so \(\mathcal{D}\) has an upper bound. By assumption and (2) any interval properly contained in \([p, q]\) is good.

Now there exists a good interval \(I_0\) containing \(p\) and a strictly larger prime ideal, and a good interval \(I_1\) containing \(q\) and a strictly smaller prime ideal by (3, 4). By assumption, there exists a good interval \(I_2\) in \(\mathcal{D}\) which intersects \(I_0\) and \(I_1\). By (5), \(I_0 \cup I_1 \cup I_2\) is a good interval and it contains \([p, q]\), so \([p, q]\) is good.

Thus the collection of good intervals contains a maximal element by Zorn’s lemma. Let \([p_\infty, q_\infty]\) be such a maximal element. If \(p_\infty \neq 0\) or \(q_\infty \neq m\), then we can use (3) and (4) (and (5)) to construct a larger good interval. It follows that \(\text{Spec}(V)\) is good and that \(V \to A\) is of universal \(\mathcal{F}\)-descent, as desired.
Proof that (1) is equivalent to (3) in Theorem 4.1. We already saw that (1) implies (3) in Proposition 4.2. In Proposition 4.6, we saw (3) implies (1).

For future reference we record the following corollary: one can test equivalences of finitary arc-sheaves on aic valuation rings of rank \( \leq 1 \). This is a slight strengthening of Proposition 3.28 in this case.

**Corollary 4.7.** Let \( F_1, F_2 : \text{Sch}_{\mathbb{R}} \to D(\Lambda) \geq 0 \) be finitary arc-sheaves. Suppose given a map \( F_1 \to F_2 \) such that for every aic valuation ring \( W \) of rank \( \leq 1 \), the map \( F_1(W) \to F_2(W) \) is an equivalence. Then \( F_1 \to F_2 \) is an equivalence.

**Proof.** It suffices to prove the analogous assertion for finitary arc-sheaves of abelian groups and where \( F_2 \equiv 0 \). By Proposition 3.28, it suffices to show that for any aic valuation ring \( V \), the map \( F_1(V) = 0 \).

Indeed, fix \( x \in F_1(V) \). We use the notation of the proof of Proposition 4.6. For each interval \( I \subset \text{Spec}(V) \), we can consider the image of \( x \) in \( F_1(V_I) \). We call the interval \( I \) *good* if \( x \) maps to zero in \( F_1(V_I) \). It is easy to see that the good intervals satisfy the conditions necessary to run the proof, since \( F_1 \) is a finitary arc-sheaf. Therefore, the same Zorn’s lemma argument shows that \( \text{Spec}(V) \) is good, so that \( x = 0 \) as desired. \( \square \)

### 4.2. Aic-v-excision and excision

Finally, we show that a functor which satisfies arc-descent also satisfies excision. That is, we show that (1) implies (2) in Theorem 4.1. We follow a general argument going back to Voevodsky, though we find it more convenient to argue directly instead of quoting an axiomatization (such as [1, Th. 3.2.5]).

**Lemma 4.8.** Consider a Milnor square as in (2). Then for any map \( A \to V \) where \( V \) is a valuation ring, the base-change of the square (2) along \( A \to V \) is also a Milnor square.

**Proof.** In fact, the square

\[
\begin{array}{ccc}
V & \to & V/IV \\
\downarrow & & \downarrow \\
B \otimes_A V & \to & B/J \otimes_A V
\end{array}
\]

has at least the property that the map \( f : V \to V/IV \times_{B/J \otimes_A V} B \otimes_A V \) is surjective, by right-exactness of the tensor product. To see that the map is injective, it suffices to show that after base-change to the fraction field \( \text{Frac}(V) \), the target is nonzero. This holds because the map \( A \to \text{Frac}(V) \) factors through \( A/I \times B \) (Lemma 2.7), so \( (A/I \times B) \otimes_A \text{Frac}(V) \neq 0 \). \( \square \)

**Lemma 4.9.** Let \( F : \text{Sch}_{\mathbb{R}}^{op} \to D(\Lambda) \geq 0 \) be a finitary functor which is a v-sheaf. Given a Milnor square as in (2) such that \( A \to B \) is surjective, \( F \) carries it to a pullback square.

**Proof.** Since everything is local on \( A \) (and \( F \) is a v-sheaf), we may assume that \( A \) is an absolutely integrally closed valuation ring. By Lemma 4.8, base-change to an absolutely integrally closed valuation ring preserves Milnor squares. In this case, we must have that one of the maps \( A \to A/I \) or \( A \to B \) is an isomorphism; otherwise, we could not have a Milnor square as the ideals of \( A \) are totally ordered. In this case, it is clear that \( F \) carries the diagram to a pullback. \( \square \)

The following result shows that (1) implies (2), and completes the proof of Theorem 4.1.

**Proposition 4.10.** Let \( F : \text{Sch}_{\mathbb{R}}^{op} \to D(\Lambda) \geq 0 \) be a finitary functor which is an arc-sheaf. Then \( F \) is excisive.
Proof. Consider a Milnor square (2); we show that \( F \) carries it to a pullback square. Since everything is local on \( A \), and \( F \) is an arc-sheaf, it follows that we may assume that \( A \) is a rank \( \leq 1 \) valuation ring (Corollary 4.7, since we have already shown \( F \) is satisfies aic-v-excision).

In this case, either the map \( A \to A/I \) or the map \( A \to B \) admits a section thanks to Lemma 2.7. If the first map admits a section, then \( I = 0 \) and it is clear that \( F \) carries (2) to a pullback square. Suppose \( A \to B \) admits a section. Then we can form a new Milnor square

\[
\begin{array}{ccc}
B & \to & B/J \\
\downarrow & & \downarrow \\
A & \to & A/I
\end{array}
\]

where the section \( B \to A \) is surjective. Now it suffices to show that \( F \) carries (8) to a fiber square, by a two-out-of-three argument. However, \( F \) carries (8) to a fiber square thanks to Lemma 4.9. □

4.3. Excision via arc-sheafification. In this subsection, we give a slightly different formulation of the relation between excision and arc-descent. Namely, we shall prove that the square of schemes attached to an excision datum gives a pushout square of arc-sheaves of spaces on arc-sheafification; this implies Proposition 4.10 by the universal property of pushouts. We shall use the language of coherent objects in a topos and that of coherent topoi; we refer the reader to [13, Expose VI] as well as [31, Lectures 11-13] for the relevant generalities.

To have a sensible notion of arc-sheafification, we fix an uncountable strong limit cardinal \( \kappa \), and let \( \text{Sch} \) denote the category of qcqs schemes which can be written as a finite union of affine schemes whose coordinate rings have cardinality \( < \kappa \). The arc-topology on \( \text{Sch} \) is defined as the Grothendieck topology where covering families \( \{ U_i \to U \} \) are finite families of maps such that \( \sqcup_i U_i \to U \) has the arc-lifting property. In particular, this topology is finitary, so the resulting topos of sheaves of sheaves is coherent. If we write \( h^\#_X \) for the arc-sheaf associated to a qcqs scheme \( X \), then we can describe all coherent objects: all sheaves of the form \( h^\#_X \) are coherent (as we restrict to qcqs schemes), and the coherent objects are exactly those sheaves \( F \) that admit a surjection \( h^\#_X \to F \) for some \( X \) such that \( h^\#_X \times_F h^\#_Y \) is quasicompact (i.e., admits a surjection from some \( h^\#_Z \)). The following criterion for detecting surjections between coherent objects will be useful.

Lemma 4.11. Say \( F \to G \) is a map of arc-sheaves of sets.

1. Assume that \( G \) is coherent and \( F \) is quasicompact. Then \( F \to G \) is surjective if and only if it has the arc-lifting property, i.e., for every rank \( \leq 1 \) valuation ring \( V \) and every section \( g \in G(V) \), there exists an extension \( V \to W \) of rank \( \leq 1 \) valuation rings and a section \( f \in F(W) \) lifting the image of \( g \) in \( G(W) \).

2. Assume that both \( F \) and \( G \) are coherent. Then \( F \simeq G \) if and only if \( F(V) \simeq G(V) \) for a cofinal collection of rank \( \leq 1 \) valuation rings \( V \).

Note that (1) above is trivially false without the quasicompactness hypothesis: the canonical map \( \bigsqcup_p h^\#_{\text{Spec}(\mathbb{Z}(p))} \to h^\#_{\text{Spec}(\mathbb{Z})} \simeq \ast \) (where the coproduct is indexed by the prime numbers) has the arc-lifting property but is not a surjection of arc-sheaves.

Proof. In both cases, the “only if” direction is clear, so it suffices to prove the “if” direction.

1. When \( F \) and \( G \) are representable, i.e., have the form \( h^\#_X \) and \( h^\#_Y \), this is essentially a reformulation of the definition of an arc-covering combined with the observation that the map from a presheaf to the associated sheaf has the arc-lifting property. In general, one
first chooses a surjection $h^t_1 \to G$ (which is possible as $G$ is quasicompact); one may then then choose a surjection $h^t_1 \to h^t_1 \times_G F$ (which is possible as $G$ is quasiseparated and $F$ is quasicompact). By the stability of the arc-lifting property under fibre products (and the fact that it holds true for surjections), then map $h^t_1 \to h^t_1$ is surjective by the representable case. But then $h^t_1 \to h^t_1 \to G$ is also surjective, whence $F \to G$ is surjective as it factors a surjection.

(2) Note that $F \to G$ is an isomorphism if and only if both $F \to G$ and its diagonal $F \to F \times_G F$ are surjective. The claim now follows from (1) and the stability of coherent objects under fibre products since the hypothesis $F(V) \simeq G(V)$ implies that both $F \to G$ and its diagonal $F \to F \times_G F$ have the arc-lifting property.

We can now prove the promised statement,

**Proposition 4.12.** Let $(A, I) \to (B, J)$ be an excision datum. Consider the associated square

$$
\begin{array}{ccc}
\text{Spec}(B/J) & \longrightarrow & \text{Spec}(B) \\
\downarrow & & \downarrow \\
\text{Spec}(A/I) & \longrightarrow & \text{Spec}(A)
\end{array}
$$

of schemes. The associated square of arc-sheaves of spaces (or sets) is a pushout square.

**Proof.** Let $Q$ denote the pushout of $\text{Spec}(A) \leftarrow \text{Spec}(A/I) \to \text{Spec}(B/J)$ in the $\infty$-category of presheaves of spaces. As $\text{Spec}(A/I) \to \text{Spec}(A)$ is a closed immersion, it gives a monomorphism of presheaves, so $Q$ is discrete and hence we may regard it as a presheaf of sets. There is a natural map $\eta : Q \to \text{Spec}(B)$ of presheaves. To prove Proposition 4.10 for all arc-sheaves, it is enough to prove that $\eta$ gives an isomorphism on arc-sheafification. Note that the sheafification of $Q$ is a coherent object of the arc-topos: one readily checks that a pushout of coherent objects in a coherent topos will be coherent provided one of the maps is injective. By Lemma 4.11, it is thus enough to show that $\eta(V) : Q(V) \to \text{Spec}(B)(V)$ is bijective for any rank $\leq 1$ valuation ring $V$. The surjectivity is immediate from Lemma 2.7, so it suffices to prove injectivity. Say $x_1, x_2 \in Q(V)$ are two points that define the same point of $y \in \text{Spec}(B)(V)$. Since $\text{Spec}(A/I) \subset \text{Spec}(A)$ is the preimage of $\text{Spec}(B/J) \subset \text{Spec}(B)$, it is easy to see that $Q \to \text{Spec}(B)$ is an isomorphism after pullback to $\text{Spec}(B/J)$. We may thus assume that $y \in \text{Spec}(B)(V)$ corresponds to a map $y^* : B \to V$ with $y^*(J) \neq 0$. Then both $x_1$ and $x_2$ must lie in the image of $\text{Spec}(A)(V) \to Q(V)$: if not, then one of them would give a factorization of $y^*$ through $B \to B/J$, which we just ruled out. Thus, $x_1$ and $x_2$ correspond to two ring maps $a_1, a_2 : A \to V$ that factor $f$. Pick some $t \in J$ such that $y^*(t) \neq 0$. As $t \in J$, we have $A[t] = B[t]$, and so $a_1[t] = a_2[t]$ as maps $A[t] \to V[t]$. As $y^*(t) \neq 0$, the map $V \to V[\frac{1}{y^*(t)}]$ is injective, so we must have $a_1 = a_2$, whence $x_1 = x_2$ as wanted. □
5. Examples of arc-sheaves

In this section, we record several examples of functors which satisfy arc-descent. There are two classes of examples we consider: those arising from étale cohomology, and those arising from perfect $F$-schemes.

5.1. Étale cohomology. Let $R$ be a fixed base ring, and let $\mathcal{G}$ be a torsion sheaf on the small étale site of $\text{Spec}(R)$. In this section, we will consider the functor

$$\text{Sch}^{op}_R \to D(\mathbb{Z})$$

which sends a qcqs scheme $X$ with structure map $f : X \to \text{Spec}(R)$ to the étale cohomology complex $R\Gamma(X_{\text{ét}}, f^*\mathcal{G})$. Recall that this functor is finitary, cf. [45, Tag 03Q4]. Our main result is that it satisfies arc-descent (Theorem 5.4). To begin with, we review the (classical) result that it satisfies $\nu$-descent. This comes from the theory of cohomological descent [13, Exp. V-bis], and also appears explicitly in [8, Prop. 5.3.3].

**Lemma 5.1.** Let $\Lambda$ be a ring, and let $\mathcal{F} : \text{Sch}^{op}_R \to D(\Lambda)^{\geq 0}$ be a finitary functor. Suppose $\mathcal{F}$ satisfies étale descent. If $f : Y \to X$ is a map in $\text{Sch}_R$ such that the base-changes of $f$ to the strict henselizations of $X$ are of $\mathcal{F}$-descent, then $f$ is of $\mathcal{F}$-descent.

**Proof.** This follows from the fact that the étale topology has enough points, given by the strict henselizations. Suppose $f : Y \to X$ is such that after base-change to any strict henselization of $A$, $f$ admits $\mathcal{F}$-descent. We need to see that the natural map

$$(9) \quad \mathcal{F}(X) \to \varprojlim \mathcal{F}(Y) \Rightarrow \mathcal{F}(Y \times_X Y) \xrightarrow{\sim} \ldots$$

is an equivalence. To do this, we consider both sides as sheaves on the small étale site of $X$. For instance, for any étale $X' \to X$, we consider $X' \to \mathcal{F}(X')$, and similarly for the right-hand-side. Our assumption is that this map of étale sheaves becomes an equivalence on stalks for each strict henselization of $X$; therefore the map of sheaves is an equivalence, and so is (9). $\square$

**Proposition 5.2.** Let $\mathcal{G}$ be a torsion sheaf on $\text{Spec}(R)_{\text{ét}}$. Let $\mathcal{F} : \text{Sch}^{op}_R \to D(\Lambda)^{\geq 0}$ be the functor $(f : X \to \text{Spec}(R)) \mapsto R\Gamma(X_{\text{ét}}, f^*\mathcal{G})$. Then $\mathcal{F}$ satisfies $\nu$-descent.

**Proof.** Let $Y \to X$ be a $\nu$-cover of schemes, which we want to show is of universal $\mathcal{F}$-descent. Since $Y, X$ are qcqs, we can write $Y$ as a filtered limit of a tower of finitely presented $X$-schemes $\{Y_\alpha\}$ with affine transition maps. Since étale cohomology turns such filtered limits to filtered colimits, and since each $Y_\alpha \to X$ is a $\nu$-cover too, we may assume that $Y \to X$ is finitely presented. The map $Y \to X$ admits a refinement which factors as a composite of a quasi-compact open covering together with a proper finitely presented surjection [39, Th. 3.12].

Now quasi-compact open coverings are of universal descent for $\mathcal{F}$. We claim that proper surjections are too, thanks to proper base change. In fact, if $Y \to X$ is proper and surjective, with $X$ the spectrum of a strictly henselian ring, let $x \in X$ be the closed point with residue field $k(x)$. Let $Y_x$ be the fiber of $Y$ at $x$. Then $\mathcal{F}(Y) \simeq \mathcal{F}(Y_x)$ thanks to proper base-change, whereas the map $Y_x \to x$ admits a section after base-change along the universal homeomorphism $k(x) \to k(x)$ (which does not change the value of $\mathcal{F}$). The analogous result holds for fiber products of $Y$ over $X$. It follows that $Y \to X$ is of $\mathcal{F}$-descent by comparison with $Y_x \to x$. Thus any proper surjection $Y \to X$ with $X$ strictly henselian is of $\mathcal{F}$-descent; by Lemma 5.1, it follows that proper surjections are of $\mathcal{F}$-descent. Since proper surjections are closed under base-change, they are of universal $\mathcal{F}$-descent. $\square$
Next, we show that \( \mathcal{F} \) satisfies arc-descent.

**Lemma 5.3.** An absolutely integrally closed valuation ring \( V \) is strictly henselian.

**Proof.** Let \( \mathfrak{m} \subset V \) be the maximal ideal. Given a monic polynomial \( p(x) \in V[x] \), we know that \( p \) splits into linear factors; this forces that \( V \) is strictly henselian, and even that the residue field is algebraically closed. \( \square \)

**Theorem 5.4** (arc-descent for étale cohomology). Let \( \mathcal{G} \) be a torsion sheaf on \( \text{Spec}(R)_{\text{et}} \). Let \( \mathcal{F} : \text{Sch}^{\text{op}}_{R} \to D(\Lambda)^{\geq 0} \) be the functor \( (f : X \to \text{Spec}(R)) \mapsto R\Gamma(X_{\text{ét}}, f^{*}\mathcal{G}) \). Then \( \mathcal{F} \) satisfies arc-descent. In particular, it satisfies excision.

**Proof.** Proposition 5.2 already shows that \( \mathcal{F} \) satisfies \( \nu \)-descent. Also, \( \mathcal{F} \) is finitary as étale cohomology commutes with filtered colimits of rings. It is therefore enough to check the condition of aic-\( \nu \)-excision from Theorem 4.1. Fix an absolutely integrally closed valuation ring \( V \) and a prime ideal \( \mathfrak{p} \subset V \). Note that any reduced quotient of a localization of \( V \) is also an absolutely integrally closed valuation ring, and hence a strictly henselian local ring. In particular, as \( V \) and \( V/\mathfrak{p} \) are both strictly henselian with identical residue fields, we have \( \mathcal{F}(V) \simeq \mathcal{F}(V/\mathfrak{p}) \) by standard facts in étale cohomology. Similarly, we also have \( \mathcal{F}(V_{\mathfrak{p}}) \simeq \mathcal{F}(\kappa(\mathfrak{p})) \), so the cartesianness of the square from Theorem 4.1 (3) is clear. \( \square \)

We now prove part (1) of Corollary 1.18 from the introduction, recovering many special cases of the Gabber-Huber rigidity theorem for the étale cohomology of henselian pairs [16, 23].

**Proof of part (1) of Corollary 1.18: rigidity from excision.** Say \((A, I)\) is a henselian pair where \( A \) lives over a henselian local ring \( k \). Let \( \mathcal{F} \) be a torsion abelian sheaf on \( \text{Spec}(A) \). We must show that the map \( \eta_{\mathcal{F}} : R\Gamma(\text{Spec}(A), \mathcal{F}) \to R\Gamma(\text{Spec}(A/I), \mathcal{F}) \) is an isomorphism.

First, note that the collection of all sheaves \( \mathcal{F} \) for which \( \eta_{\mathcal{F}} \) is an isomorphism satisfies the “2-out-of-3” property and contains sheaves pushed forward from \( \text{Spec}(A/I) \). We may therefore assume \( \mathcal{F} \) has the form \( j_{\mathfrak{g}} \mathcal{G} \) for some torsion étale sheaf \( \mathcal{G} \) on \( \text{Spec}(A) \setminus V(I) \), where \( j : \text{Spec}(A) \setminus V(I) \to \text{Spec}(A) \) is the displayed open immersion.

Next, let us prove the claim when \( A/I \) is itself a henselian local ring with residue field \( E \). By profinite étale descent, this reduces to the case where \( A \) is a strictly henselian local ring. But the global sections functor is exact on such a ring, and \( \Gamma(\text{Spec}(A), \mathcal{H}) \simeq \Gamma(\text{Spec}(E), \mathcal{H}) \) for any étale sheaf \( \mathcal{H} \) on \( \text{Spec}(A) \). As \( A \) is a henselian local ring with residue field \( E \), the same holds true for \( A/I \). Applying the preceding reasoning to \( A/I \) then shows that \( \Gamma(\text{Spec}(A), \mathcal{H}) \simeq \Gamma(\text{Spec}(E), \mathcal{H}) \simeq \Gamma(\text{Spec}(A/I), \mathcal{H}) \) for any étale sheaf \( \mathcal{H} \) on \( \text{Spec}(A) \), so the claim follows.

We will now use excision to reduce to the special case treated above. Let \( k \to A \) be a map with \( k \) being a henselian local ring. Write \( B := k \times_{A/I} A \), so we can also view \( I \) as an ideal \( J \) of \( B \), and we get an excision datum \((B, J) \to (A, I)\) with \( B/I \simeq k \) being a henselian local ring. As \( B \to A \) is an isomorphism after inverting any element of \( J \), the sheaf \( \mathcal{G} \) can be viewed as a torsion étale sheaf on \( \text{Spec}(B) \setminus V(J) \). Write \( \mathcal{F}' := j'_{\mathfrak{g}} \mathcal{G} \), where \( j' : \text{Spec}(B) \setminus V(J) \to \text{Spec}(B) \) is the displayed open immersion. Consider the functor \( R\Gamma(-, \mathcal{F}') \) on the category of schemes over \( \text{Spec}(B) \). Theorem 5.4 implies that this functor carries the Milnor square associated to \((B, J) \to (A, I)\) to a pullback square. By proper base change, it is also clear that \( R\Gamma(\text{Spec}(A), \mathcal{F}') \simeq R\Gamma(\text{Spec}(A), \mathcal{F}) \), and similarly for \( \text{Spec}(A/I) \). We are thus reduced to the case where \( A = B \). But then \( A/I \) is a henselian local ring, so we are done by the special case shown earlier. \( \square \)

Next, we observe that excision for étale cohomology is also a quick consequence of Gabber-Huber’s affine analog of proper base change [16, 23].
Remark 5.5 (Excision from rigidity). Suppose we have an excision datum $f : (A, I) \to (B, J)$. To see that the induced square on $R\Gamma(\cdot, \Lambda)$ is cartesian, we may localize on $A$ and assume that $A$ is a henselian local ring with maximal ideal $m \subset A$. Note that the datum of an excision datum is preserved under flat base change in $A$. We need to see that the square

$$
\begin{array}{ccc}
R\Gamma(\text{Spec}(A)_{\text{ét}}, \Lambda) & \to & R\Gamma(\text{Spec}(A/I)_{\text{ét}}, \Lambda) \\
\downarrow & & \downarrow \\
R\Gamma(\text{Spec}(B)_{\text{ét}}, \Lambda) & \to & R\Gamma(\text{Spec}(B/J)_{\text{ét}}, \Lambda).
\end{array}
$$

Then, there are two cases:

1. $I = A$, so $1 \in I$. In this case, $J = B$ since $J$ is an ideal and therefore $A \simeq B$. Therefore, the excision assertion in $R\Gamma(\cdot, \Lambda)$ is evident.

2. Suppose $I \subset m$. Then $(A, I)$ is a henselian pair. It follows then by an observation of Gabber\[15\] that $(B, J)$ is a henselian pair. By the affine analog of proper base change, we have $R\Gamma(\text{Spec}(A)_{\text{ét}}, \Lambda) \simeq R\Gamma(\text{Spec}(A/I)_{\text{ét}}, \Lambda)$ and $R\Gamma(\text{Spec}(B)_{\text{ét}}, \Lambda) \simeq R\Gamma(\text{Spec}(B/J)_{\text{ét}}, \Lambda)$. Therefore, excision holds in this case as well: the horizontal maps in (10) are equivalences.

Of course, in our approach above, we do not need to invoke the affine analog of proper base change at any point.

5.2. Constructible étale sheaves. In this section we prove “categorified” versions of Theorem 4.4.

Remark 5.6. Theorem 4.1 is stated for finitary $v$-sheaves with values in $D(\Lambda) \geq 0$. The arguments show that it is valid slightly more generally. The main points necessary to run the argument are that:

1. The target $\infty$-category is compactly generated.
2. Filtered colimits commute with finite limits and totalizations.
3. We can run the argument of Proposition 3.28. For instance, this holds if sheaves (with values in the relevant $\infty$-category) are automatically hypercomplete.

For instance, the arguments would hold with $D(\Lambda) \geq 0$ replaced by the $\infty$-category of $n$-truncated spaces for some $n$. Similarly, they would hold for the $\infty$-category $\text{Cat}_n$ of $n$-categories (i.e., $(n,1)$-categories) for some $n < \infty$. We would not expect such a result to hold without imposing some such finiteness conditions.

We demonstrate the following result, which is due to Rydh in the forthcoming work [38]; we thank him for originally indicating the result to us. We give our own quick proof using Theorem 4.1.

Theorem 5.7 (Rydh [38]). Consider the following three functors:

1. The functor $\mathcal{F}_0 : \text{Sch}^{op} \to \text{Cat}_1$ sends a qcqs scheme $X$ to the category of étale qcqs $X$-schemes which are separated over $X$.
2. The functor $\mathcal{F}_1 : \text{Sch}^{op} \to \text{Cat}_1$ sends a qcqs scheme $X$ to the category of finite étale qcqs $X$-schemes.
3. The functor $\mathcal{F}_2 : \text{Sch}^{op} \to \text{Cat}_2$ sends a qcqs scheme $X$ to the category of constructible étale sheaves of sets on $X$, or equivalently the category of étale qcqs algebraic spaces over $X$.

Then $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ are finitary arc-sheaves.
Proof. First, the functors $F_0, F_1$ are finitary by general theory about limits of qcqs schemes along affine transition maps (cf. [45, Tag 081C]). Similarly $F_2$ is finitary, as a presentation of the algebraic space can be descended.

Now $F_0, F_1, F_2$ satisfy $v$-descent. Recall that since the functors are finitary, it suffices to check $v$-descent on schemes of finite type over Spec$(\mathbb{Z})$, where the $v$-topology reduces to the topology of universal submersions. Now Rydh’s result [39, Cor. 5.18] implies that $F_0, F_1, F_2$ satisfy $v$-descent. To complete the argument, it remains to show that $F_0, F_1, F_2$ satisfy aic-$v$-excision. Let $V$ be an aic valuation ring and let $p \subset V$ be a prime ideal. Then $V/p, V_p, \kappa(p)$ are aic valuation rings as well.

(1) For $F_1$, the categories of finite étale algebras over $V, V/p, V_p, \kappa(p)$ are all equivalent to the category of finite sets thanks to Lemma 5.3. Thus clearly $F_1$ satisfies aic-$v$-excision.

(2) For $F_0$, we see by Proposition 5.8 below that the category of separated étale quasi-compact $V$-schemes is equivalent to the category of disjoint unions of quasi-compact open sets of $V$. Similarly for $V/p, V_p, \kappa(p)$. Now any quasi-compact open subset of Spec$(V)$ which contains $\{p\}$ contains Spec$(V_p)$. Unwinding the definitions, we easily check that $F_0(\text{Spec}(V)) \simeq F_0(\text{Spec}(V_p)) \times F_0(\text{Spec}(\kappa(p)))$ as desired.

Very explicitly, we can reduce to the case where $V$ has finite rank by writing $V$ as a filtered colimit of aic valuation subrings (Lemma 2.20). Suppose $V$ has rank $n$, so Spec$(V)$ is the totally ordered set $\{1, 2, \ldots, n\}$ under specialization, and $p$ corresponds to the element $i \in [1, n]$. Then we observe that:

(a) The category of separated étale $V$-schemes is the category of functors $(1 \to 2 \to \cdots \to n) \to \text{FinSet}_{\text{inj}}$, the category of finite sets and injections.

(b) The category of separated étale $V_p$-schemes is the category of functors $(1 \to 2 \to \cdots \to i) \to \text{FinSet}_{\text{inj}}$.

(c) The category of separated étale $V/p$-schemes is the category of functors $(i \to i + 1 \to \cdots \to n) \to \text{FinSet}_{\text{inj}}$.

(d) The category of separated étale $\kappa(p)$-schemes is the category of functors $i \to \text{FinSet}_{\text{inj}}$.

In view of the above four identifications, it is evident that $F_0(\text{Spec}(V)) \simeq F_0(\text{Spec}(V_p)) \times F_0(\text{Spec}(\kappa(p)))$ as desired.

(3) For $F_2$, the category of constructible sheaves of sets on Spec$(V)$ is just the category functors $\{1, 2, \ldots, n\} \to \text{FinSet}$, again via Proposition 5.8 below which describes the étale site. One has a similar description $V_p, V/p, \kappa(p)$ and the result follows.

Therefore, $F_0, F_1, F_2$ satisfy aic-$v$-excision and are finitary $v$-sheaves. In light of Remark 5.6 we can conclude. □

Proposition 5.8. Let $V$ be an aic valuation ring. Let $f : X \to \text{Spec}(V)$ be a quasi-compact separated étale morphism. Then $X$ is a disjoint union of $V$-schemes of the form $\text{Spec}(V[1/t])$, $t \in V$. That is, the category of separated étale $V$-schemes is equivalent to the category of disjoint unions of quasi-compact opens of Spec$(V)$.

Proof. This is a general fact about integral normal schemes with separably closed function field, cf. [45, Tag 09Z8]. For the convenience of the reader, we also sketch a direct proof.

Using Lemma 2.20 and a standard noetherian approximation argument, we may assume $V$ has finite rank. We prove the claim by induction on the rank. When the rank is 0, then $V$ is an algebraically closed field, so the claim is clear. In general, since $V$ is strictly henselian, we can use Zariski’s main theorem to write $X := X' \cup U$ where $X' \to \text{Spec}(V)$ is finite étale, and $U \to \text{Spec}(V)$ does not hit the closed point. As $V$ is absolutely integrally closed, the map $X' \to \text{Spec}(V)$ is a disjoint union of sections: the ring of functions on each component of $X'$ is a normal domain finite
over $V$, but there are no such domains other than $V$ as the fraction field of $V$ is algebraically closed. Also, $f(U) \subset \text{Spec}(V)$ is a quasicompact open subset that misses the closed point. As $V$ is an aic valuation ring, each finitely generated ideal is principal, so $f(U) := \text{Spec}(V[t]/t^i)$ for some nonunit $t \in V$. In particular, $f(U)$ is the spectrum of aic valuation ring of rank strictly smaller than that of $V$. Applying the inductive hypothesis to the map $U \to f(U)$ now gives the claim. \hfill $\Box$

Let $\Lambda$ be a finite ring. Next, we prove that the functor which sends a qcqs scheme to the $\infty$-category $D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)$ of constructible sheaves satisfies arc-descent. This is a refinement of Theorem 5.4 (which we recover by taking derived endomorphisms of the constant sheaf).

**Notation 5.9 (Constructible sheaves).** Consider the functor $X \mapsto D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)$, $\text{Sch}^{op} \to \text{Cat}_\infty$ which assigns to a qcqs scheme $X$ the subcategory of the full derived category $D(X_{\acute{e}t}, \Lambda)$ of étale sheaves of $\Lambda$-modules spanned by those objects which are bounded with constructible cohomology.

For $n \geq 0$, we let $D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)^{[-n,n]} \subset D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)$ be the subcategory consisting of objects with amplitude in $[-n,n]$. For example, when $n = 0$ we recover the ordinary abelian category of constructible sheaves of $\Lambda$-modules on $X$.

**Remark 5.10.** When $\Lambda$ is a field or a product of fields, the preceding definition coincides with the standard one appearing in the definition of the $\ell$-adic derived category, c.f. [45, Tag 09C0]. When $\Lambda$ has a non-trivial Jacobson radical, one can also enforce finite $\Lambda$-Tor-dimension (which is necessary to obtain something functorial in $\Lambda$). The results below also hold for the variant where we force finite Tor-dimension.

**Proposition 5.11.** The functor $X \mapsto D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)$ is finitary. More generally, for each $n$, the functor $X \mapsto D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)^{[-n,n]}$ is finitary.

**Proof.** Clearly it suffices to prove that each functor $X \mapsto D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)^{[-n,n]}$ is finitary.

Suppose we can write $X = \varprojlim_{i \in I} X_i$ as a filtered inverse limit of a tower of qcqs schemes $X_i$ over a totally ordered set $I$, such that the transition maps $X_j \to X_i$ are affine. Let $p_i : X \to X_i$ and $p_{ji} : X_j \to X_i$ (for $j \geq i$) denote the transition maps.

Let $0 \in I$. First, we show that if $F,G \in D^b_{\text{cons}}((U_0)_{\acute{e}t}, \Lambda)$, then $\text{Hom}_{D^b_{\text{cons}}((U_0)_{\acute{e}t}, \Lambda)}(p_{0i}^*F, p_{0i}^*G) \simeq \varinjlim_{i \geq 0} \text{Hom}_{D^b_{\text{cons}}((X_i)_{\acute{e}t}, \Lambda)}(p_{0i}^*F, p_{0i}^*G)$. This easily implies that the map $\varinjlim_{i \geq 0} D^b_{\text{cons}}((X_i)_{\acute{e}t}, \Lambda) \to D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)$ is fully faithful (and similarly with truncations attached). In fact, by a straightforward d´evissage we reduce to the case where $F$ is obtained as the (discrete) sheaf $j_!(M)$ for some finite $\Lambda$-module $M$ and $j : U_0 \to X_0$ a separated étale map and where $G$ is also (up to shift) discrete. In this case, however, the result follows from the commutation of étale cohomology with filtered colimits.

Next, we need to show that the functor $\varinjlim D^b_{\text{cons}}((X_i)_{\acute{e}t}, \Lambda)^{[-n,n]} \to D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)^{[-n,n]}$ is essentially surjective. Since the functor is already known to be fully faithful (indeed on the whole bounded derived $\infty$-categories), it suffices by d´evissage to show the result in the case $n = 0$, where it follows from [45, Tag 095M]. \hfill $\Box$

**Proposition 5.12.** The functor $X \mapsto D^b_{\text{cons}}(X, \Lambda)$ on $\text{Sch}$ is a $v$-sheaf. More generally, for each $n$, the functor $X \mapsto D^b_{\text{cons}}(X, \Lambda)^{[-n,n]}$ is a $v$-sheaf (automatically hypercomplete).

Related results can be found in [29].

**Proof.** Note that the assertions for $D^b_{\text{cons}}(X, \Lambda)$ and $D^b_{\text{cons}}(X, \Lambda)^{[-n,n]}$ (for each $n$) are equivalent, since given a surjection of qcqs schemes $Y \to X$, amplitude in the derived category on $X$ can be tested by pullback to $Y$. Hence it suffices to show that $X \mapsto D^b_{\text{cons}}(X, \Lambda)^{[-n,n]}$ is a $v$-sheaf. Note
that this functor takes values in $\text{Cat}_{2n+2}$, where filtered colimits and totalizations commute. The functor is an étale sheaf by construction. Using the refinement result [39, Th. 3.12], it now suffices to prove that for a proper finitely presented surjection of noetherian schemes $Y \to X$, we have for each $n$,

$$D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)^{-n,n} \simeq \lim_{\longleftarrow} \left( D^b_{\text{cons}}(Y_{\text{ét}}, \Lambda)^{-n,n} \implies D^b_{\text{cons}}((Y \times_X Y)_{\text{ét}}, \Lambda)^{-n,n} \rightarrow \cdots \right).$$

Indeed, the map $Y \to X$ determines via the Čech nerve an augmented cosimplicial scheme (over $X$) and the diagram (11) is obtained by applying $D^b_{\text{cons}}((\cdot)_{\text{ét}}, \Lambda)^{-n,n}$ to it.

We verify the equivalence using the (dual of) the abstract result [32, Cor. 4.7.5.3]. Note that the pullback $D^b_{\text{cons}}(X_{\text{ét}}, \Lambda) \to D^b_{\text{cons}}(Y_{\text{ét}}, \Lambda)$ is clearly conservative since isomorphisms are checked on stalks. Since pullback is exact, $D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)^{-n,n} \to D^b_{\text{cons}}(Y_{\text{ét}}, \Lambda)^{-n,n}$ preserves totalizations (which are first computed in the whole derived $\infty$-category and then truncated). Moreover, for any cartesian diagram of qcqs schemes

$$\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & & \downarrow \\
F' & \longrightarrow & F
\end{array}$$

along proper maps, the square

$$\begin{array}{ccc}
D^b_{\text{cons}}(F_{\text{ét}}, \Lambda)^{-n,n} & \longrightarrow & D^b_{\text{cons}}(F'_{\text{ét}}, \Lambda)^{-n,n} \\
\downarrow & & \downarrow \\
D^b_{\text{cons}}(E_{\text{ét}}, \Lambda)^{-n,n} & \longrightarrow & D^b_{\text{cons}}(E'_{\text{ét}}, \Lambda)^{-n,n}
\end{array}$$

is right adjointable. The right adjoints are given by truncations of derived pushforward and the adjointability follows from proper base change. \qed

**Proposition 5.13.** Let $V$ be an aic valuation ring of finite rank $m$, so $X = \text{Spec}(V)$ (under specialization) is the partially ordered set $1 \to 2 \to \cdots \to m$. Then $D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)$ is the $\infty$-category of functors $(1 \to 2 \to \cdots \to m)^{\text{op}} \to D^b_f(\Lambda)$ for $D^b_f(\Lambda)$ the bounded derived category of finitely generated $\Lambda$-modules. Similarly $D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)^{-n,n}$ is the $\infty$-category of functors $(1 \to 2 \to \cdots \to m)^{\text{op}} \to D^b_f(\Lambda)^{-n,n}.$

**Proof.** Recall (Proposition 5.8) that the category of separated étale $V$-schemes is equivalent to the category of finite disjoint unions of quasi-compact open subsets of $\text{Spec}(V)$. Every quasi-compact open subset is of the form $\{1, 2, \ldots, i\}$ for some $i \leq m$, and the Grothendieck topology in this case is trivial. The result follows easily. \qed

**Theorem 5.14** (arc-descent for constructible sheaves). The functor $X \mapsto D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)$ is a hypercomplete arc-sheaf. More generally, for each $n$, the functor $X \mapsto D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)^{-n,n}$ is an arc-sheaf.

**Proof.** We first claim that it suffices to prove the second assertion. Indeed, the construction $X \mapsto D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)^{-n,n}$ is automatically hypercomplete if it is an arc-sheaf because it takes values in $\text{Cat}_{2n+2}$. Now given an arc-hypercover $Y_* \to X$, we obtain $D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)^{-n,n} \simeq \lim_{\longleftarrow} D^b_{\text{cons}}((Y_*)_{\text{ét}}, \Lambda)^{-n,n}$ for each $n$ and we can let $n \to \infty$ in the totalization (since all inclusions are fully faithful). Thus if we verify the second claim, we automatically prove that $X \mapsto D^b_{\text{cons}}(X_{\text{ét}}, \Lambda)$ is a hypercomplete arc-sheaf.
Since \( X \mapsto D^b_{\text{cons}}(X_{\acute{e}t}, \Lambda)^{[-n,n]} \) is already checked to be a \( \nu \)-sheaf (Proposition 5.11), it suffices to verify aic-\( \nu \)-excisiveness by Remark 5.6. Let \( X = \text{Spec}(V) \) for \( V \) an aic valuation ring and let \( p \subset V \) be a prime ideal. We need to show that \( D^b_{\text{cons}}(\cdot_{\acute{e}t}, \Lambda)^{[-n,n]} \) satisfies descent for the cover \( \text{Spec}(V/p) \sqcup \text{Spec}(V_p) \to \text{Spec}(V) \), i.e., that we have a homotopy cartesian square

\[
\begin{array}{ccc}
D^b_{\text{cons}}(\text{Spec}(V)_{\acute{e}t}, \Lambda)^{[-n,n]} & \to & D^b_{\text{cons}}(\text{Spec}(V/p)_{\acute{e}t}, \Lambda)^{[-n,n]} \\
\downarrow & & \downarrow \\
D^b_{\text{cons}}(\text{Spec}(V_p)_{\acute{e}t}, \Lambda)^{[-n,n]} & \to & D^b_{\text{cons}}(\text{Spec}(\kappa(p))_{\acute{e}t}, \Lambda)^{[-n,n]}
\end{array}
\]

Without loss of generality, we can assume that \( V \) has finite rank, so that \( \text{Spec}(V) = \{1,2,\ldots,m\} \) as a poset under specialization. Suppose \( p \) corresponds to the element \( i \); then \( \text{Spec}(V_p) = \{1,2,\ldots,i\} \), \( \text{Spec}(V/p) = \{i,i+1,\ldots,n\} \), and \( \text{Spec}(\kappa(p)) = \{i\} \). Using the description of \( \acute{e}t \) sheaves on spectra of aic valuation rings in Proposition 5.13, the result now follows easily.

5.3. **Perfect schemes of characteristic \( p > 0 \)***. Using exactly the same strategy used to prove arc-descent in \( \acute{e}t \)ale cohomology, one also obtains arc-hyperdescent for perfect complexes on perfect schemes, extending the analogous aic-hyperdescent result in [4, Theorem 11.2 (2)].

**Theorem 5.15** (arc-hyperdescent for perfect complexes on perfect schemes). Fix a prime number \( p \). Let \( \mathcal{F} \) be the functor on \( \text{Sch}_{\mathbf{F}_p} \) carrying a scheme \( X \) to the \( \infty \)-category \( \text{Perf}(X_{\text{perf}}) \) of perfect complexes on the perfection \( X_{\text{perf}} \) of \( X \). Then \( \mathcal{F} \) is a hypercomplete arc-sheaf.

**Proof.** By [4, Theorem 11.2 (2)], we already know that \( \mathcal{F} \) is a hypercomplete \( \nu \)-sheaf. Also, \( \mathcal{F} \) is clearly finitary. To show that \( \mathcal{F} \) is an arc-sheaf, it suffices to check aic-\( \nu \)-excision as in Theorem 4.1 (3), cf. Remark 5.6 above. Thus, fix an absolutely integrally closed valuation ring \( V \) of characteristic \( p \) with a prime ideal \( p \). We must check that applying \( \mathcal{F} \) to the Milnor square

\[
\begin{array}{ccc}
V & \to & V/p \\
\downarrow & & \downarrow \\
V_p & \to & \kappa(p)
\end{array}
\]

gives a cartesian square of \( \infty \)-categories. Note that all rings appearing above are perfect. Moreover, the map \( V \to B := V_p \times V/p \) is descendent in the sense of [4, Definition 11.14] as the above square is cartesian. By [4, Theorem 11.15] (which is [34, Proposition 3.21]), we have \( \mathcal{F}(V) \simeq \varprojlim \mathcal{F}(B^*) \), where \( B^* \) is the Čech nerve of \( V \to B \) in the \( \infty \)-category of \( E_\infty \)-rings. In fact, as both \( V \) and \( B \) are perfect, the terms of \( B^* \) coincide with the Čech nerve of \( V \to B \) in the category of ordinary commutative rings by [4, Lemma 3.16]. It is now easy to see, just as in Proposition 4.2, that the statement \( \mathcal{F}(V) \simeq \varprojlim \mathcal{F}(B^*) \) implies exactly that applying \( \mathcal{F} \) to the above Milnor square results in a cartesian square. Finally, to check that the arc-sheaf \( \mathcal{F} \) is hypercomplete, we argue exactly as in the proof of hypercompleteness of \( \mathcal{F} \) as a \( \nu \)-sheaf, as in [4, proof of Theorem 11.12 (2)].

Using the previous result, we show that arc-covers of perfect \( \mathbf{F}_p \)-schemes are precisely the universal effective epimorphisms. That is, a map of qcqs perfect \( \mathbf{F}_p \)-schemes is an arc-cover precisely if it is a cover in the canonical topology.

**Theorem 5.16** (Universally effective epimorphisms of perfect schemes). Fix a prime number \( p \). Let \( Y \) be a qcqs algebraic space over \( \mathbf{F}_p \). Consider the functor \( \mathcal{F} \) on \( \text{Sch}_{\mathbf{F}_p} \) carrying \( X \) to the set of maps \( X_{\text{perf}} \to Y \). Then \( \mathcal{F} \) is an arc-sheaf (of sets).
Moreover, a map of qcqs perfect $\mathbb{F}_p$-schemes $Y \to X$ is an arc-cover if and only if it is a universally effective epimorphism in the category of perfect $\mathbb{F}_p$-schemes.

Note that the functor $\mathcal{F}$ is finitary if $Y$ is in addition of finite type over $\mathbb{F}_p$, and the proof shows that it satisfies excision even if $Y$ is only assumed qcqs (one can reduce to the finite type case via noetherian approximation [40]).

Proof. For the sheafyness of $\mathcal{F}$, we use the Tannaka duality in the form of [3, Theorem 1.5]. Given $X$, we let $\text{Perf}(X_{\text{perf}})$ denote the symmetric monoidal $\infty$-category of perfect complexes on $X_{\text{perf}}$ as before. Then we have

$$\text{Hom}(X_{\text{perf}}, Y) \simeq \text{Fun}_{\text{ex}}(\text{Perf}(Y), \text{Perf}(X_{\text{perf}})),$$

where the target denotes symmetric monoidal exact functors. Since we have just seen that $X \mapsto \text{Perf}(X_{\text{perf}})$ satisfies arc-descent (Theorem 5.15), the result now follows.

The previous paragraph shows that arc-covers of perfect qcqs $\mathbb{F}_p$-schemes are universally effective epimorphisms. For the converse, it therefore suffices to show that if $X \to Y$ is a map of perfect $\mathbb{F}_p$-schemes is a universally effective epimorphism, then it is also a cover for the arc-topology. As the property of being a universally effective epimorphism is local, we may assume $Y := \text{Spec}(V)$ is a rank $\leq 1$ valuation ring. Moreover, as arc-covers are universally effective epimorphisms, we may refine $X$ by an arc-cover to assume $X$ has the form $\text{Spec}(R)$ where $R := \prod_{i \in I} W_i$ is a product of rank $\leq 1$ valuation rings $W_i$. There are two cases to consider:

Assume first that $V$ has rank 0, so $V$ is a field. We must show that $X \neq \emptyset$ or equivalently that $R \neq 0$. The sheaf property of $\text{Hom}(-, \mathbb{A}^1_{\text{perf}})$ with respect to the map $V \to R$ (assumed to be a universal effective epimorphism of qcqs perfect $\mathbb{F}_p$-schemes) shows that $V$ is the equalizer of the two maps $R \to R \otimes_V R$, and so $R \neq 0$ since $V \neq 0$.

Assume now that $V$ has rank 1. The previous paragraph shows that $I \neq \emptyset$. If one of the induced maps $V \to W_i$ is an injective local homomorphism, then it is also faithfully flat, so we are done. Assume towards contradiction then that each map $V \to W_i$ is either non-injective or non-local. Any such map must factor over either the residue field $k$ (if non-injective) or the fraction field $K$ (if non-local) of $V$. But then each map $V \to W_i$ factors over $V \to R' := K \times k$, and hence the same holds true for $V \to R = \prod W_i$. In particular, $\text{Spec}(R') \to \text{Spec}(V)$ is also a canonical cover. But one easily checks that $R' \otimes_V R' \simeq R'$ via the multiplication map in all cases. Applying the sheaf axiom for the sheaf $\text{Hom}(-, \mathbb{A}^1_{\text{perf}})$ then shows that $V \simeq R'$, which is absurd since $R'$ is a product of fields while $V$ is a rank 1 valuation ring.

Remark 5.17. The canonical topology on qcqs perfect $\mathbb{F}_p$-schemes is not quasi-compact. That is, a covering family in the canonical topology does not need to admit a finite refinement. Indeed, it is shown in [45, Tag 0EUE] that $\text{Spec}(\mathbb{Z})$ is not quasi-compact for the canonical topology on the category of all qcqs schemes. This example can be adapted to the setting of perfect $\mathbb{F}_p$-schemes by replacing $\text{Spec}(\mathbb{Z})$ with $\mathbb{A}^1$.
6. Consequences of arc-descent

6.1. Formal glueing squares. Next, we prove that any functor satisfying arc-descent also satisfies a “formal glueing square.” Recall the following assertion: if $A$ is a noetherian ring and $t \in A$, then we can form the square

$$
\begin{array}{ccc}
A & \rightarrow & \hat{A}_t \\
\downarrow & & \downarrow \\
A[1/t] & \rightarrow & \hat{A}_t[1/t]
\end{array}
$$

which is a pullback square. Given a functor on rings, one can ask whether it carries (12) to a homotopy pullback square.

Example 6.1. A basic example is that nonconnective $K$-theory $K$ does; that is, the square

$$
\begin{array}{ccc}
K(A) & \rightarrow & K(\hat{A}_t) \\
\downarrow & & \downarrow \\
K(A[1/t]) & \rightarrow & K(\hat{A}_t[1/t])
\end{array}
$$

is homotopy cartesian. This follows from the theory of localization sequences: the fibers of the horizontal maps are given by the $K$-theory of perfect $t$-power torsion $A$-modules (resp. $t$-power torsion $\hat{A}_t$-modules), cf. [47, Theorem 5.1], and these categories are clearly equivalent. More generally one can formulate a statement for finitely generated ideals.

Here we prove an analogous result for finitary arc-sheaves. First we need some preliminaries.

Proposition 6.2. Let $V$ be a rank 1 valuation ring, and let $t$ be a pseudouniformizer. Then the $t$-adic completion $\hat{V}_t$ is a rank 1 valuation ring, and the map $V \rightarrow \hat{V}_t$ is faithfully flat.

Proof. In fact, if $K$ is the fraction field of $V$, then the rank 1 valuation on $V$ defines a nonarchimedean absolute value $| \cdot | : K \rightarrow \mathbb{R}_{\geq 0}$. The completion $\hat{K}$ of $K$ with respect to the absolute value $| \cdot |$ also admits a canonical extension of the absolute value (denoted by $| \cdot |$ again). One checks that $\hat{V}_t$ is precisely the rank 1 valuation ring $\{ x \in \hat{K} : |x| \leq 1 \}$ and that $V \rightarrow \hat{V}_t$ is an extension of valuation rings, hence faithfully flat. \qed

Note in particular that if $V$ is a rank 1 valuation ring, then the condition that $V$ should be $t$-adically complete does not depend on the non-unit $t \neq 0$.

Proposition 6.3. Let $R$ be a ring and let $I \subset R$ be a finitely generated ideal. Then $\text{Spec}(\hat{R}_I) \sqcup \text{Spec}(R) \setminus V(I) \rightarrow \text{Spec}(R)$ is an arc-cover.

Proof. Let $V$ be a rank $\leq 1$ valuation ring. By Proposition 6.2, we may assume that $V$ is complete with respect to any element in the maximal ideal $m_V \subset V$. We then make a stronger claim: any map $\text{Spec}(V) \rightarrow \text{Spec}(R)$ factors through the above cover.

Consider a map $f : R \rightarrow V$. If $f$ carries $I$ into the maximal ideal of $V$, then $f(I) \subset tV$ for some pseudouniformizer $t \in m_V$ as $I$ is finitely generated. Clearly then $f$ factors over $\hat{R}_I$ since $V$ is $t$-adically complete. Conversely, if $f(I)$ generates the unit ideal in $V$, then $\text{Spec}(V) \rightarrow \text{Spec}(R)$ factors through the open locus $\text{Spec}(R) \setminus V(I)$ and we are done in this case too. \qed
Remark 6.4. In the following, we freely use the following fact: if $R$ is a ring (not necessarily noetherian) and $I \subset R$ is a finitely generated ideal, then the map $R \to \hat{R}_I$ induces an equivalence modulo $I^n$ for all $n \geq 0$ ([45, Tag 00M9]).

Theorem 6.5 (Formal gluing squares for arc-sheaves). Let $\mathcal{F} : \text{Sch}_{/R}^{op} \to D(\Lambda)^{\geq 0}$ be a finitary arc-sheaf. Then $\mathcal{F}$ satisfies formal gluing, i.e., if $(R \to S, I)$ is a formal gluing datum in the sense of Theorem 1.15, then the natural square

$$
\begin{array}{ccc}
\mathcal{F}((\text{Spec } R)) & \to & \mathcal{F}((\text{Spec } S)) \\
\downarrow & & \downarrow \\
\mathcal{F}((\text{Spec } R \setminus V(I))) & \to & \mathcal{F}((\text{Spec } S \setminus V(IS)))
\end{array}
$$

is cartesian.

Proof. The argument follows a familiar pattern, cf. the proof of Proposition 4.10. We first consider the case where $R \to S$ is surjective (this may happen in a non-noetherian case, e.g., $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{Z}_p$ for $I = (p)$). Then assertion is local on $R$, so we may assume that $R$ is a rank $\leq 1$ valuation ring. If $I = R$, then the assertion that (13) is homotopy cartesian is trivial. If $I$ is contained in the maximal ideal of $R$, then (since $I$ is finitely generated) the only way $R \to S$ can induce an isomorphism $R/I^n R \cong S/I^n S$ is for $R = S$. In this case, too, (13) is evidently homotopy cartesian.

Next, suppose $R \to S$ admits a section $s : S \to R$. In this case, $s : S \to R$ also induces an equivalence modulo each power of $IS \subset S$, so when we contemplate the diagram

$$
\begin{array}{ccc}
\mathcal{F}((\text{Spec } R)) & \to & \mathcal{F}((\text{Spec } S)) \\
\downarrow & & \downarrow \\
\mathcal{F}((\text{Spec } R \setminus V(I))) & \to & \mathcal{F}((\text{Spec } S \setminus V(IS)))
\end{array}
$$

we get that the rightmost square is homotopy cartesian by the previous paragraph. This also implies that the leftmost square is homotopy cartesian by an easy 2-out-of-3.

Now we consider the general case. The assertion is local on $R$, so we may assume that $R$ is a rank $\leq 1$ valuation ring which is complete. Then either $R$ is $I$-adically complete or $I$ is the unit ideal, since $I$ is finitely generated. If $I$ is the unit ideal, then the above square (13) is trivially homotopy cartesian. Suppose then that $R$ is $I$-adically complete, so then $R \to S$ admits a section. In this case we are also done by the previous paragraph. \hfill \Box

Let us also sketch an essentially equivalent proof that is formulated slightly differently and applies to all arc-sheaves (but uses arc-sheafification); this is analogous to the alternative proof of Proposition 4.10 given in §4.3.

Alternative proof of Theorem 6.5. This proof uses arc-sheafification. We adopt the set-theoretic conventions from §4.3 to make sense of this notion. Before proceeding further, let us remark that if a ring has size $< \kappa$, then so does its completion with respect to any ideal, so the category of rings under consideration is closed under this operation.

Let $\mathcal{F}$ be a $D(\Lambda)$-valued arc-sheaf on $\text{Sch}$. We shall check that (13) is homotopy cartesian. Let $Q$ denote the pushout of $\text{Spec}(R)/V(I) \leftarrow \text{Spec}(S)/V(IS) \to \text{Spec}(S)$ in the $\infty$-category of presheaves of spaces; note that $Q$ is discrete since the map $\text{Spec}(S)/V(IS) \to \text{Spec}(S)$ is an open immersion (and thus a monomorphism of presheaves). There is an evident map $\eta : Q \to \text{Spec}(R)$; to prove
that (13) is cartesian, it is enough to prove that \( \eta \) gives an isomorphism after arc-sheafification. Arguing as in Proposition 4.12, it is enough to prove that \( \eta(V) : Q(V) \to \text{Spec}(R)(V) \) is bijective for all complete rank \( \leq 1 \) valuation rings \( V \). The surjectivity of \( \eta(V) \) follows from Proposition 6.3. For injectivity, say we have two points \( x_1, x_2 \in Q(V) \) giving the same point of \( y \in \text{Spec}(R)(V) \). As \( \text{Spec}(S) \setminus V(IS) \subseteq \text{Spec}(S) \) is the preimage of \( \text{Spec}(R) \setminus V(I) \subseteq \text{Spec}(R) \), it formally follows that \( \eta \) is an isomorphism after pullback to \( \text{Spec}(R) \setminus V(I) \subseteq \text{Spec}(R) \). We may thus assume that the point \( y \in \text{Spec}(R)(V) \) does not lie in \( (\text{Spec}(R) \setminus V(I))(V) \), i.e., the corresponding map \( y^* : R \to V \) carries \( I \) into a nonunit ideal. But then both \( x_1, x_2 \in Q(V) \) must come from \( \text{Spec}(S)(V) \): if they came from \( (\text{Spec}(R) \setminus V(I))(V) \), then \( y^*(I) \) would generate the unit ideal, which is not possible. This means that the map \( y^* : R \to V \) has two factorizations \( x'_1, x'_2 : S \to V \) through \( S \). But \( V \) is \( I \)-adically complete (note that every ring is \( 0 \)-adically complete) while \( R \) and \( S \) have the same \( I \)-adic completions, so \( x'_1 = x'_2 \), whence \( x_1 = x_2 \). \( \square \)

**Example 6.6** (Formal glueing in \( \acute{e} \)tale cohomology). If \( I \) is generated by a single element, then Theorem 6.5 shows that \( \acute{e} \)tale cohomology (or any arc-sheaf) admits “formal glueing squares,” i.e., carries diagrams of shape (12) to homotopy pullbacks.

Let us give an example illustrating why working locally in the arc-topology is essential for Theorem 6.5, and the \( v \)-topology does not suffice.

**Example 6.7.** Let \( V \) be a rank 2 valuation ring. Write \( p \) for the height 1 prime. Choose \( f \in V - p \) a nonunit, and let \( I = (f) \), so \( \sqrt{I} \) is the maximal ideal. Write \( \hat{V}_I \) for the \( I \)-adic completion of \( V \). Consider the formal glueing datum \( (V \to \hat{V}_I, I) \). We claim that the corresponding map \( (\text{Spec}(V) \setminus V(I)) \sqcup \text{Spec}(\hat{V}_I) \to \text{Spec}(V) \) is not a \( v \)-cover. It suffices to show that there is no extension \( V \to W \) of \( V \) such that the \( \text{Spec}(W) \to \text{Spec}(V) \) factors over \( (\text{Spec}(V) \setminus V(I)) \sqcup \text{Spec}(\hat{V}_I) \to \text{Spec}(V) \). In fact, as \( \text{Spec}(W) \to \text{Spec}(V) \) is surjective with \( \text{Spec}(W) \) connected, the only possibility is that \( V \to W \) as factor as \( V \to \hat{V}_I \to W \). But this is impossible as \( V \to W \) is injective while \( V \to \hat{V}_I \) contains \( p \) in its kernel: we have \( p \subseteq I^n \) for all \( n \geq 0 \) since \( f^n \notin p \) for all \( n \geq 0 \). Note that this is not a problem if we are allowed to work arc-locally on \( V \) as the map \( V \to \hat{V}_I = V/p \times V_p \) is an arc-cover which factors over \( (\text{Spec}(V) \setminus V(I)) \sqcup \text{Spec}(\hat{V}_I) \to \text{Spec}(V) \) on spectra.

### 6.2. GAGA for rigid \( \acute{e} \)tale cohomology of affinoids

In this subsection, we collect various results related to the \( \acute{e} \)tale cohomology of “affinoids,” though we formulate them algebraically. Specifically, these results are related to the \( \acute{e} \)tale cohomology of rings obtained by inverting an element \( t \) on \( t \)-adically complete rings.

Namely, we reprove the Fujiwara-Gabber theorem (Theorem 6.10). Next, we show that \( \acute{e} \)tale cohomology of rings of the form \( A_t[1/t] \) satisfies descent with respect to a variant of the arc-topology where the element \( t \) is taken into account (Corollary 6.17). Finally, we record a Künneth theorem (Proposition 6.22).

Let us make the following definition.

**Definition 6.8.** We say that a functor \( \mathcal{F} \) on \( R \)-algebras is **rigid** if for every henselian pair \( (A, J) \) with \( A \) an \( R \)-algebra, we have \( \mathcal{F}(A) \simeq \mathcal{F}(A/J) \).

Rigid arc-sheaves exhibit some additional rigidity: they are insensitive to completion, even after “passage to the generic fibre”.

**Corollary 6.9.** Let \( R \) be a ring which is henselian along a finitely generated ideal \( I \subset R \). Let \( \mathcal{F} : \text{Sch}^{pp}_R \to D(A)_{20} \) be a finitary functor which satisfies arc-descent. Suppose \( \mathcal{F} \) is additionally rigid on \( R \)-algebras. Then the map \( \mathcal{F}(\text{Spec}(R) \setminus V(I)) \to \mathcal{F}(\text{Spec}(\hat{R}_I) \setminus V(I)) \) is an equivalence.
Proof. This follows from the fiber square (13) applied with \( S = \hat{R}_I \). By assumption, the top arrow is an equivalence via rigidity; therefore, the bottom arrow is too. \( \Box \)

By the affine analog of proper base change, it follows that étale cohomology with torsion coefficients is rigid. We thus recover the following result of Fujiwara-Gabber.

Theorem 6.10 (Fujiwara-Gabber, cf. [14, Cor. 6.6.4]). Let \((R, I)\) be a henselian pair with \( I \subset R\) finitely generated. Let \( f : R \to \hat{R}_I\) denote the \( I\)-adic completion of \( R \). Then for any torsion étale sheaf \( \mathcal{G}\) on \( \text{Spec}(R) \setminus V(I)\), the map

\[
R\Gamma(\text{Spec}(R) \setminus V(I), \mathcal{G}) \to R\Gamma(\text{Spec}(\hat{R}_I) \setminus V(I\hat{R}_I), f^*\mathcal{G})
\]

is an equivalence.

Proof. Note first that \( \mathcal{G}\) extends to a torsion étale sheaf \( \mathcal{G}_1\) on \( \text{Spec}(R)\), for example, by extension by 0. We consider the functor \( \mathcal{F} : \text{Sch}_{\text{ét}}^\text{op} \to D(\mathbb{Z})^{\geq 0}\) which sends \( p : Y \to \text{Spec}(R)\) to \( R\Gamma(Y_{\text{ét}}, p^*\mathcal{G}_1)\). By Theorem 5.4 and the affine analog of proper base change, it follows that the hypotheses of Corollary 6.9 apply to the functor \( \mathcal{F}\), and hence we can conclude. \( \Box \)

Remark 6.11. In [14], the above result is proved under the additional hypotheses that \( R\) is noetherian. The non-noetherian case of the Fujiwara-Gabber theorem is due to Gabber, and is outlined in [25, Exp. XX, Sec. 4.4].

Remark 6.12. Corollary 6.9 relies on the fact that equivalences in the derived category \( D(\Lambda)\) can be checked after pullback. In particular, the analogous result does not obviously imply to functors taking values in sets or categories. For instance, it does not obviously apply to the functor which sends a qcqs scheme to its category of finite étale covers (which we have seen satisfies arc-descent, and which is rigid). Nevertheless, the conclusion of Corollary 6.9 is valid for this functor.

In the noetherian case, Elkik [12] proved that if \((R, I)\) is a henselian pair with \( R\) noetherian, then finite étale covers of \( \text{Spec}(R) \setminus V(I)\) and \( \text{Spec}(\hat{R}_I) \setminus V(I\hat{R}_I)\) are identified. This statement has been generalized to the non-noetherian case in the important special case where \( I = (t)\) is principal, by Gabber-Ramero [18, Proposition 5.4.54] and in the general finitely generated case by Gabber [25, Exp. XX, Théorème 2.1.2]. The most general assertion is that one has a rigidity result for arbitrary sheaves of sets or sheaves of ind-finite groups on \( \text{Spec}(R) \setminus V(I)\), that is, \( H^0\) (resp. \( H^0, H^1\)) are identified over \( \text{Spec}(R) \setminus V(I)\) and \( \text{Spec}(\hat{R}_I) \setminus V(I\hat{R}_I)\). Compare also [21, Theorem 6.4] for another proof of the case of sets.

For our next application, we work (implicitly) in the category of \( t\)-adically complete \( \mathbb{Z}[t]\)-algebras \( R_0\) (endowed with the \( t\)-adic topology). Recall that each such \( R_0\) yields a Banach \( \mathbb{Z}((t))\)-algebra \( R := R_0[\frac{1}{t}]\) with unit ball \( R_0\) and thus corresponds to an affinoid rigid space \( \text{Spa}(R, R^\circ)\) over \( \mathbb{Z}((t))\); conversely, every affinoid rigid space has such a form as one may simply take \( R_0\) to be a ring of definition. Our next result is roughly that the assignment \( \text{Spa}(R, R^\circ) \to R\Gamma(\text{Spec}(R), \Lambda)\) on affinoids satisfies descent with respect to the étale topology on affinoid rigid spaces; in fact, we prove descent with respect to a much finer topology. In particular, it follows that the purely algebraically defined étale cohomology groups \( H^*(\text{Spec}(R), \Lambda)\) may serve as meaningful étale cohomology groups in rigid geometry. To avoid developing the language of rigid geometry, we formulate our statements purely algebraically.

Definition 6.13. We say that a map of \( \mathbb{Z}[t]\)-algebras \( R \to S\) is an arc-\( t\)-cover if for every rank 1 valuation ring \( V\) over \( \mathbb{Z}[t]\) where \( t\) is a pseudouniformizer and every map \( R \to V\), there is an extension of rank \( \leq 1\) valuation rings \( V \to W\) such that the map \( R \to V \to W\) extends over \( S\).
For the interested reader, we translate the above definition into the theory of adic spaces.

**Example 6.14.** Let \( f : (A, A^+) \to (B, B^+) \) be a map of Tate \((\mathbb{Z}(t)), \mathbb{Z}[[t]]\))-algebras, so \( t \) is a pseudouniformizer. Then \( f \) defines a map \( \text{Spa}(f) : \text{Spa}(B, B^+) \to \text{Spa}(A, A^+) \) on adic spectra. Then the map \( A^+ \to B^+ \) is an arc-cover if and only if the map \( \text{Spa}(f) \) is surjective on generic points (or equivalently that the map on associated Berkovich spaces is surjective). Indeed, this follows immediately from the definitions: the generic points of the adic spectrum \( \text{Spa}(A, A^+) \) are in bijective correspondence with maps \( A^+ \to V \) where \( V \) is a rank 1 valuation ring with \( t \) being a pseudouniformizer (up to refinements of \( V \)). In particular, the \( \text{arc}_t \)-topology on affinoid rigid spaces (as defined via pullback along \((A, A^+) \to A^+)\) is finer than the \( v \)-topology of [42, Definition 8.1].

**Construction 6.15** (Étale cohomology of the rigid generic fiber). Fix a torsion abelian group \( \Lambda \). Consider the functor \( F \) on \( \mathbb{Z}[t] \)-algebras with values in \( D(\mathbb{Z})^{\geq 0} \) that sends a \( \mathbb{Z}[t] \)-algebra \( R \) to \( R\Gamma(\text{Spec}(\hat{R}_t[1/t])_{\text{et}}, \Lambda) \). By Theorem 6.10, one can replace the \( t \)-adic completion with the \( t \)-henselization; therefore, \( F \) commutes with filtered colimits.

We show that \( F \) satisfies descent for the \( \text{arc}_t \)-topology. We will need the following basic fiber square for the functor \( F \). For convenience we formulate the result more generally with coefficients in a torsion sheaf.

**Proposition 6.16.** If \( R \) is any \( \mathbb{Z}[t] \)-algebra and \( \mathcal{G} \) is a torsion abelian sheaf on \( \text{Spec}(R)_{\text{et}} \), then we have a fiber square\(^6\)

\[
\begin{array}{c}
\text{R} \Gamma(\text{Spec}(R)_{\text{et}}, \mathcal{G}) \ar{r} \ar{d} & \text{R} \Gamma(\text{Spec}(R[1/t])_{\text{et}}, \mathcal{G}) \ar{d} \\
\text{R} \Gamma(\text{Spec}(R/t)_{\text{et}}, \mathcal{G}) \ar{r} & \text{R} \Gamma(\text{Spec}(\hat{R}_t[1/t])_{\text{et}}, \mathcal{G})
\end{array}
\]

**Proof.** We consider the fiber square (Example 6.6)

\[
\begin{array}{c}
\text{R} \Gamma(\text{Spec}(R)_{\text{et}}, \mathcal{G}) \ar{r} \ar{d} & \text{R} \Gamma(\text{Spec}(R[1/t])_{\text{et}}, \mathcal{G}) \ar{d} \\
\text{R} \Gamma(\text{Spec}(\hat{R}_t)_{\text{et}}, \mathcal{G}) \ar{r} & \text{R} \Gamma(\text{Spec}(\hat{R}_t[1/t])_{\text{et}}, \mathcal{G})
\end{array}
\]

The bottom left term is identified with \( \text{R} \Gamma(\text{Spec}(R/t)_{\text{et}}, \mathcal{G}) \) by the affine analog of proper base change. Thus, we conclude. \( \Box \)

**Corollary 6.17** (\( \text{arc}_t \)-descent for étale cohomology of affinoids). Let \( A \) be a ring with a distinguished element \( t \in A \). Let \( \mathcal{F} \) be a torsion sheaf on \( \text{Spec}(A)_{\text{et}} \). Consider the functor \( F \) which sends an \( A \)-algebra \( R \) to \( F(R) := \text{R} \Gamma(\text{Spec}(\hat{R}_t[1/t]), \mathcal{F}) \). Then \( F \) is a finitary \( \text{arc}_t \)-sheaf on \( A \)-algebras.

**Proof.** The statement that \( F \) is finitary follows from Proposition 6.16. If \( R \to S \) is an \( \text{arc}_t \)-cover, then \( R \to S \times R/t \times R[1/t] \) is an arc-cover: given a map \( R \to V \) with \( V \) a rank \( \leq 1 \) valuation ring, the image of \( t \) in \( V \) can either pseudouniformizer or zero or a unit, and these three possibilities correspond to factoring (up to extensions) through each of the three terms of the product. One sees easily that the desired statement for \( S \) is equivalent to the statement for \( S \times R/t \times R[1/t] \) (since \( t \)-adic completion followed by inverting \( t \) kills both any \( R/t \)-module and any \( \hat{R}[1/t] \)-module). Thus it suffices to show that \( F \) is an arc-sheaf. We use the fiber square (14). To see that \( F \) is an

\(^6\)For convenience we suppress the various pullbacks of \( \mathcal{G} \).
implies that the purely algebraically defined étale cohomology is an arc-sheaf and arc-covers are stable under base change.

In the language of adic spaces, Corollary \ref{cor:alg-arc} implies that the purely algebraically defined étale cohomology groups of affinoid adic spaces satisfy descent for the analytic étale topology; in this form, it is equivalent to Huber’s affinoid comparison theorem \cite[Corollary 3.2.2]{BH}. Let us sketch why this result, together with proper base change, also give a GAGA result for proper adic spaces. A more general statement can be found in \cite[Theorem 3.2.10]{BH} (at least under noetherian hypotheses).

**Corollary 6.18** (GAGA for rigid étale cohomology in the proper case). Let \( A \) be a ring with a distinguished element \( t \) such that \( (A, (t)) \) is a henselian pair. Let \( X \) be a proper \( \mathbb{A}[1/t] \)-scheme, and write \( X^\text{ad} \) for the associated adic space over \( \text{Spa}(\mathbb{A}[1/t], A) \). Then for any torsion étale sheaf \( F \) on \( X \) with pullback \( F^\text{ad} \) to \( X^\text{ad} \), the natural map gives an identification

\[ \check{R}\Gamma(X, F) \cong \check{R}\Gamma(X^\text{ad}, F^\text{ad}) \]

between algebraic and analytic étale cohomologies.

**Proof.** By Nagata compactification, we can find a proper \( A \)-scheme \( \hat{X} \) extending \( X \), i.e., with an identification \( \hat{X} \times_{\text{Spec}(A)} \text{Spec}(\mathbb{A}[1/t]) \cong X \). Let \( \mathfrak{F} \) be a torsion étale sheaf on \( \hat{X} \) extending \( F \). Write \( \hat{X} \) for the \( t \)-adic completion of \( \hat{X} \). Consider the commutative square

\[
\begin{array}{ccc}
X^\text{ad} & \longrightarrow & \hat{X} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \hat{X}
\end{array}
\]

of morphisms of locally ringed topoi (where each vertex is given the étale topology). The sheaf \( \mathfrak{F} \) defines via pullback an étale sheaf on each of the topoi above, and we abusively denote this pullback by \( \mathfrak{F} \) as well. We shall show that applying \( \check{R}\Gamma(-, \mathfrak{F}) \) to \( \check{R}\Gamma(X^\text{ad}, F^\text{ad}) \) gives a cartesian square; this implies the corollary as \( \check{R}\Gamma(\hat{X}, \mathfrak{F}) \cong \check{R}\Gamma(\hat{X}, \mathfrak{F}) \) by the proper base change theorem.

To prove that applying \( \check{R}\Gamma(-, \mathfrak{F}) \) to \( \check{R}\Gamma(X^\text{ad}, F^\text{ad}) \) gives a Cartesian square, fix an affine open cover \( \{U_i\} \) of the \( A \)-scheme \( \hat{X} \). Via pullback, this defines compatible open covers of \( X \) by affine schemes, of \( \hat{X} \) by affine formal schemes, and of \( X^\text{ad} \) by affinoid adic spaces. Using these covers to compute cohomology, and thanks to the affinoid comparison theorem \cite[Corollary 3.2.2]{BH}, it is therefore enough to prove that \( \check{R}\Gamma(-, \mathfrak{F}) \) to the analog of \( (\ref{eq:cart}) \) for each \( U_i \) gives a Cartesian square. But this follows from formal glueing for étale cohomology (Proposition \ref{prop:glee}).

For future reference we note the following consequences for the functor \( R \mapsto \check{R}\Gamma(\text{Spec}(\hat{R}_t[1/t]), F) \) of the above corollary.

**Definition 6.19** (arc\(_t\)-equivalences). Let \( A \) be a base ring containing an element \( t \). A map of \( A \)-algebras \( R \rightarrow R' \) is said to be an arc\(_t\)-equivalence if for every \( t \)-adically complete rank \( \leq 1 \) aic valuation ring \( V \) where \( t \) is a pseudouniformizer, we have that \( \text{Hom}_A(R, V) \cong \text{Hom}_A(R', V) \).

**Example 6.20.** Let us give some examples of arc\(_t\)-equivalences. Let \( R \) be an \( A \)-algebra. The following maps are arc\(_t\)-equivalences:

1. The map \( R \rightarrow R/(t^\infty\text{-torsion}) \).
2. The map \( R \rightarrow \hat{R}_t \).
3. If \( R^+ \) denotes the integral closure of \( R \) in \( R[\frac{1}{t}] \), then \( R \rightarrow R^+ \) is an arc\(_t\)-equivalence. Indeed, if \( V \) is any valuation ring over \( A \) with \( t \neq 0 \) in \( V \), then \( V \) is integrally closed in \( V[\frac{1}{t}] \).
(4) If $R^{\text{tir}}$ denotes the total integral closure of $R$ in $R[\frac{1}{l}]$ (see Theorem 6.25 for the definition), then $R \to R^{\text{tir}}$ is an arc$_t$-equivalence. This follows just as in (3) since rank $\leq 1$ valuation rings are totally integrally closed in their fraction fields.

**Proposition 6.21** (Invariance under arc$_t$-equivalences). Fix a pair $(A, t \in A)$ and let $F$ be a torsion sheaf on $\text{Spec}(A)_{\text{et}}$. Let $R \to R'$ be an arc$_t$-equivalence. Then $R\Gamma(\text{Spec}(\hat{R}_t[1/t], F)) \simeq R\Gamma(\text{Spec}(\hat{R}_t'[1/t], F))$.

**Proof.** Let $F$ be the functor on $A$-algebras given by $F(S) = R\Gamma(\text{Spec}(\hat{S}_t[1/t], S))$. We have just seen (Corollary 6.17) that $F$ satisfies arc$_t$-descent. Now $R \to R'$ is an arc$_t$-cover, so

$\text{(16)} \quad F(R) \simeq \lim_{\to} (F(R') \cong F(R' \otimes_R R') \leftarrow F(R' \otimes_R R')) \ldots$.

Moreover, $R' \otimes_R R' \to R'$ is arc$_t$-cover as well, and taking the Čech nerve of $R' \otimes_R R' \to R'$, we find that $F(R' \otimes_R R') \simeq F(R')$, and similarly $F(R' \otimes_R R' \otimes_R R') \simeq F(R')$, etc. Substituting this back into (16) gives the claim. \hfill $\Box$

Finally, as an application of Corollary 6.17, we obtain the following Künneth formula.

**Proposition 6.22** (Künneth formula). Let $V$ be an absolutely integrally closed valuation ring of rank 1 with pseudouniformizer $\pi$. Fix a prime number $\ell$ prime to the characteristic of the residue field of $V$. Consider the functor $\tilde{F} : \text{Ring}_{V} \to D(\mathbb{F}_\ell)^{\geq 0}$ given by $\tilde{F}(R) = R\Gamma(\text{Spec}(\hat{R}_\pi[1/\pi]), \mathbb{F}_\ell)$. Then $\tilde{F}$ is a symmetric monoidal functor. That is, for any $V$-algebras $R, R'$, the natural map

$R\Gamma(\text{Spec}(\hat{R}_\pi[1/\pi]), \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} R\Gamma(\text{Spec}(\hat{R}'_\pi[1/\pi]), \mathbb{F}_\ell) \to R\Gamma(\text{Spec}((R \otimes_V R')_\pi[1/\pi]), \mathbb{F}_\ell)$

is an equivalence.

**Proof.** Since the functor $\tilde{F}$ on $V$-algebras is a finitary arc$_\pi$-sheaf, we can work locally on $R, R'$ separately and reduce to the case where $R, R'$ are themselves absolutely integrally closed rank $\leq 1$ valuation rings (under $V$), thanks to Corollary 4.7. If $\pi = 0$ or $\pi$ is a unit in either of $R, R'$, then it is easy to see that everything vanishes, so we may assume that $\pi$ is a pseudouniformizer in each of $R, R'$. In this case, $\tilde{F}(R), \tilde{F}(R') = \mathbb{F}_\ell$ since the étale cohomology of $\hat{R}_\pi[1/\pi]$ is equal to that of the algebraically closed field $R[1/\pi]$, and similarly for $R'$. It remains to determine $\tilde{F}(R \otimes_V R')$.

That is, we need the étale cohomology of $(R \otimes_V R')_\pi[1/\pi]$.

For this, we use the formal glueing square. Let $K$ denote the fraction field of $V$ and let $R_K, R'_K$ denote the associated base-change. In view of the square (14), we get a homotopy pullback diagram

$$
\begin{array}{ccc}
R\Gamma(\text{Spec}(R \otimes_V R'), \mathbb{F}_\ell) & \to & R\Gamma(\text{Spec}R_K \otimes_K R'_K, \mathbb{F}_\ell) \\
\downarrow & & \downarrow \\
R\Gamma(\text{Spec}(R/\pi \otimes_{(V/\pi)} (R'/\pi)), \mathbb{F}_\ell) & \to & \tilde{F}(R \otimes_V R')
\end{array}
$$

Now the top right and bottom left squares are just $\mathbb{F}_\ell$, thanks to the Künneth formula in étale cohomology for qcqs schemes over a separably closed field [11, Cor. 1.11]. The top left square is $\mathbb{F}_\ell$ by a result of Huber, [24, Cor. 4.2.7]. Therefore, $\tilde{F}(R \otimes_V R') \simeq \mathbb{F}_\ell$ and the natural map $\tilde{F}(R) \otimes_{\mathbb{F}_\ell} \tilde{F}(R') \to \tilde{F}(R \otimes_V R')$ is an equivalence. This proves that $\tilde{F}$ is symmetric monoidal. \hfill $\Box$

**Remark 6.23.** One no longer has a Künneth formula for $p$-adic étale cohomology. We would expect that when one works with $\mathcal{O}_K$-algebras, for $K$ a nonarchimedean algebraically closed field of characteristic zero with residue field of characteristic $p$, then $p$-adic étale cohomology of the rigid
generic fiber can be obtained as the Frobenius fixed points of an arc-sheaf given by global sections of the tilted structure sheaf, as in [43], and that the latter (up to some completion) has a Künneth formula.

6.3. A variant of excision. We end with a slight generalization of excision; the statement is formulated in a fashion that could be potentially useful in relating the étale cohomology of different formal schemes giving the same Berkovich space. First, we record the following simple observation.

**Proposition 6.24.** Let $Y \to X$ be a universal homeomorphism of qcqs schemes. If $\mathcal{F}$ is a $v$-sheaf on $\text{Sch}$, then $\mathcal{F}(X) \to \mathcal{F}(Y)$ is an equivalence.

**Proof.** This is similar to Proposition 6.2. Since $Y \to X$ is a universal homeomorphism, it is clearly a $v$-cover (as we can lift specializations). Thus, we have

\[(17) \quad \mathcal{F}(X) \simeq \lim_{\to} (\mathcal{F}(Y) \subset \mathcal{F}(Y \times_X Y) \subset \ldots).\]

Moreover, the map $\Delta : Y \to Y \times_X Y$ is a universal homeomorphism too, and hence a $v$-cover, so we recover an expression for $\mathcal{F}(Y \times_X Y)$ in terms of $\mathcal{F}(Y), \mathcal{F}(Y \times_Y Y), \ldots$; since the map $\Delta$ is an immersion, this simplifies to $\mathcal{F}(Y) \simeq \mathcal{F}(Y \times_X Y)$. This also holds for the iterated fiber products of $Y$ over $X$. Returning to (17), we find also that $\mathcal{F}(X) \simeq \mathcal{F}(Y)$ as desired. \qed

The promised result is the following:

**Theorem 6.25.** Let $A \to B$ be a map of commutative rings, and let $t \in A$ be an element which is a nonzerodivisor on both $A$ and $B$ and such that $A^{\text{tic}} \simeq B^{\text{tic}}$, where $A^{\text{tic}}$ denotes the total integral closure$^7$ of $A$ in $A[1/t]$, and similarly for $B^{\text{tic}}$. Let $\mathcal{F}$ be any finitary $D(\mathbb{Z})^{\geq 0}$-valued arc-sheaf on $\text{Ring}_A$ (such as the functor from Proposition 5.2). Then the square

\[
\begin{array}{ccc}
\mathcal{F}(A) & \longrightarrow & \mathcal{F}(A/tA) \\
\downarrow & & \downarrow \\
\mathcal{F}(B) & \longrightarrow & \mathcal{F}(B/tB)
\end{array}
\]

is cartesian.

The conditions on $A \to B$ appearing above are satisfied, for example, if $A \to B$ is an integral map of $t$-torsionfree rings such that $A[1/t] \simeq B[1/t]$.

**Proof.** Set $A' := B \times_{B/tB} A/tA$, so we have a factorization $(A, tA) \to (A', tB) \to (B, tB)$ of maps of pairs with the second map being an excision datum. As $\mathcal{F}$ is excisive by Theorem 4.1, it is enough to show that $\mathcal{F}(A) \simeq \mathcal{F}(A')$. By Proposition 6.24, it suffices to show that $\text{Spec}(A') \to \text{Spec}(A)$ is a universal homeomorphism.

We first check that $A \to A'$ is integral. As $A'$ is an extension of $A/tA \simeq A'/tB$ by $tB$, we can write any $x \in A'$ as $a + tb$ where $a \in A$ and $tb \in tB \subset A'$. To show integrality of $x$ over $A$, it is enough to show the integrality of $tb$ over $A$. But $A^{\text{tic}} \simeq B^{\text{tic}}$, so the element $t^n b^n \in B$ actually lies in $A$ (in fact, in $t^{n-c} A$ for some constant $c$) for $n \gg 0$, which proves integrality.

As integral maps are universally closed, to see that $\text{Spec}(A') \to \text{Spec}(A)$ is a universal homeomorphism, it suffices to show that $f : A[1/t] \to A'[1/t]$ and $g : A/tA \to A'/tA'$ are universal

---

$^7$Concretely, $A^{\text{tic}}$ is the set of $x \in A[1/t]$ whose powers have bounded denominators, i.e., there exists some integer $c \geq 0$ with $x^{cl} \in \frac{1}{t^c} A \subset A[1/t]$. It is easy to see that $A^{\text{tic}}$ contains the integral closure of $A$ in $A[1/t]$, and equals it when $A$ is noetherian.
homeomorphisms on Spec(−). The claim for $f$ is clear since $f$ is an isomorphism. For $g$, note that $A/tA \to A'/tB$ is an isomorphism by construction, so it is enough to check that the surjection $A'/tA' \to A'/tB$ has nilpotent kernel; equivalently, we must show that for any $tb \in tB$, we have $(tb)^n \in tA'$ for $n \gg 0$, which was already shown in the previous paragraph. \qed
7. An Application: Artin-Grothendieck Vanishing in Rigid Geometry

The classical Artin-Grothendieck vanishing theorem in algebraic geometry gives a bound on the cohomological dimension of affine varieties.

**Theorem 7.1** (Artin-Grothendieck [13, Cor. 3.2, Exp. XIV]). Let $k$ be a separably closed field and let $\ell$ be a prime number invertible on $k$. Let $A$ be a finitely generated $k$-algebra of dimension $d$. Let $\mathcal{F}$ be an $\ell$-power torsion étale sheaf on $\text{Spec}(A)_{\text{ét}}$. Then $H^j(\text{Spec}(A), \mathcal{F}) = 0$ for $j > d$.

The assumption that $\ell$ differs from the characteristic of $k$ is not necessary: if they equal each other, we even have vanishing above degree 1 (and even degree 0 if $\dim(A) = 0$). For this reason, we focus on the interesting case discussed above in the sequel.

In this section, we prove an analog of the classical Artin-Grothendieck vanishing theorem in rigid analytic geometry, strengthening recent results of Hansen [22]. Before formulating the statement, let us introduce the basic objects of rigid analytic geometry.

**Notation 7.2.** Throughout this section, we let $K$ be a complete, algebraically closed nonarchimedean field with (nontrivial) absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$. We let $\mathcal{O}_K \subset K$ be the ring of integers, and $\pi \in \mathcal{O}_K$ a pseudouniformizer. We fix a prime number $\ell$ which is different from the characteristic of the residue field of $K$. Write $\mathcal{O}_K(X_1, \ldots, X_n)$ for the $\pi$-adic completion of the polynomial ring $\mathcal{O}_K[X_1, \ldots, X_n]$; this can be viewed as the ring of rigid analytic functions bounded by 1 on the closed unit ball $D^\pi_K$ of dimension $n$ over $K$. Write $T_n := K(X_1, \ldots, X_n) := \mathcal{O}_K(X_1, \ldots, X_n)[\frac{1}{\pi}]$, which can be viewed as the ring of all rigid analytic functions on $D^\pi_K$ and is sometimes called the Tate algebra of power series in $n$ variables; the ring $T_n$ is naturally a Banach $K$-algebra via the Gauss norm, and its subring $T^n_n \subset T_n$ of power bounded elements coincides with its unit ball which also equals $\mathcal{O}_K(X_1, \ldots, X_n)$. In this section, all ring theoretic completions are $\pi$-adic ones unless otherwise specified.

We start by reviewing facts about topologically finite type $K$-algebras. See [5, Ch. 6–7] for a general reference.

**Definition 7.3** (Topologically finite type $K$-algebras). Let $A$ be a $K$-algebra. We say that $A$ is topologically finite type (or tft) if there exists a surjection $T_n \twoheadrightarrow A$.

Recall that $T_n$ is a noetherian ring of Krull dimension $n$. It follows that any tft $K$-algebra $A$ is necessarily noetherian of finite Krull dimension. In particular, $A$ has finitely many minimal primes. For future use, let us also remark that finite $A$-algebras are automatically tft $K$-algebras. Moreover, $A$ has the natural structure of a Banach $K$-algebra, e.g., via any choice of quotient map $T_n \twoheadrightarrow A$ and the Gauss norm on $T_n$. The Banach norm is not uniquely determined, but the Banach topology is unique and homomorphisms of tft $K$-algebras are automatically continuous. In particular, each tft $K$-algebra $A$ has a well-defined $\mathcal{O}_K$-subalgebra $A^\circ \subset A$ consisting of the power bounded elements, i.e., those $f \in A$ such that the set $\{f^n \mid n \in \mathbb{N}\}$ lives inside a ball in $A$. To study these subrings more effectively, we shall use the following “integral” variant of the preceding notion:

**Definition 7.4** (Topologically finite type $\mathcal{O}_K$-algebras, cf. [6, Ch. 7]). Let $R$ be an $\mathcal{O}_K$-algebra. We say that $R$ is topologically of finite type (or tft) if $R$ is a quotient of $\mathcal{O}_K(T_1, \ldots, T_n)$ for some $n$. It is known that $R$ is automatically $\pi$-adically complete; conversely, if $R$ is $\pi$-adically complete and $R/\pi$ is finitely generated over $\mathcal{O}_K/\pi$ then $R$ is tft.

**Example 7.5.** Let $g_1, \ldots, g_n \in \mathcal{O}_K[T_1, \ldots, T_n]$. Then one constructs a tft $\mathcal{O}_K$-algebra in two equivalent ways: one as the $\pi$-adic completion of $\mathcal{O}_K[T_1, \ldots, T_n]/(g_1, \ldots, g_n)$, and another as
Let \( \pi \) provides the proper context for many of the statements that follow.

Construction 7.6. Let \( R \) be any \( \mathcal{O}_K \)-algebra. Suppose that \( R/\pi \) is a finitely generated \( \mathcal{O}_K/\pi \)-algebra. Then \( \hat{R}[1/\pi] \) is a \( \mathcal{O}_K \)-algebra.

In fact, we have that \( \hat{R} \) is \( \pi \)-adically complete, and \( \hat{R}/\pi \simeq R/\pi \) is a finitely generated \( \mathcal{O}_K/\pi \)-algebra by assumption. Therefore, \( \hat{R} \) is a \( \mathcal{O}_K \)-algebra, and inverting \( \pi \) in any presentation of \( \hat{R} \) as a \( \mathcal{O}_K \)-algebra shows that \( \hat{R}[1/\pi] \) is a \( \mathcal{O}_K \)-algebra.

Conversely, any \( \mathcal{O}_K \)-algebra arises in this way. Indeed, given a \( \mathcal{O}_K \)-algebra \( A \), we can choose an open bounded subring \( A_0 \subset A \) such that \( A_0 \) is a \( \mathcal{O}_K \)-algebra; explicitly, if \( A := T_n/I \) is a presentation for \( A \), then we can simply choose \( A_0 = T_n^0/(I \cap T_n^0) \subset A \). Then \( A_0 \) is \( \pi \)-adically complete and \( A = A_0[1/\pi] \).

Warning 7.7. If \( A \) is a \( \mathcal{O}_K \)-algebra, then power bounded subring \( \mathcal{A}^0 \subset A \) need not be a \( \mathcal{O}_K \)-algebra. For example, if \( A := K(X)/(X^2) \simeq K[X]/(X^2) \), then \( rX \) is power bounded for all \( r \in K \), so \( \mathcal{A}^0 \simeq (K \cdot 1) \oplus (K \cdot X) \). In particular, \( \mathcal{A}^0 \) is not \( \pi \)-adically complete, and thus not a \( \mathcal{O}_K \)-algebra. This phenomenon does not occur if \( A \) is reduced.

Let us recall also the following standard constructions with \( \mathcal{O}_K \)-algebras.

Definition 7.8. (Rational localizations) Given a \( \mathcal{O}_K \)-algebra \( A \) and elements \( f_1, \ldots, f_n, g \in A \) which generate the unit ideal, we can construct a map \( A \to \frac{A}{\langle f_i \rangle} \) with the following universal property: given a \( \mathcal{O}_K \)-algebra \( B \), to give a map \( A \to B \) such that \( g \) maps to an invertible element in \( B \) and such that \( \phi(f_i) \in B \) is powerbounded for \( i = 1, \ldots, n \).

For an explicit construction, fix an open bounded subring \( A_0 \subset A \). By rescaling the \( f_i \) and \( g \), we may assume they all lie in \( A_0 \). Then \( \hat{A}_0 := A_0(T_1, \ldots, T_m)/(gT_i - f_1, \ldots, gT_m - f_m) \) is a \( \mathcal{O}_K \)-algebra, and one may simply set \( A \langle \frac{1}{g} \rangle := \hat{A}_0[\frac{1}{g}] \).

(2) (Coproducts and pushouts) Given \( \mathcal{O}_K \)-algebras \( A, B \), we have the completed tensor product \( A \hat{\otimes} K B \). This is the coproduct in the category of \( \mathcal{O}_K \)-algebras. Explicitly, if \( A_0 \subset A \) and \( B_0 \subset B \) are open bounded subrings, then the \( \pi \)-adically completed tensor product \( A_0 \hat{\otimes} K B_0 \) is a \( \mathcal{O}_K \)-algebra, and we set \( A \hat{\otimes} B := (A_0 \hat{\otimes} B_0)[\frac{1}{\pi}] \). More generally, given maps \( B \leftarrow A \to C \) of \( \mathcal{O}_K \)-algebras, a similar construction defines a pushout \( B \hat{\otimes} C \) in the category of \( \mathcal{O}_K \)-algebras.

We will consider the étale cohomology of \( \mathcal{O}_K \)-algebras, considered as abstract rings. It is known that these agree with the étale cohomology of rigid analytic varieties (e.g., [24, Theorem 3.2.1]), but we will try to minimize the use of this language for simplicity; nevertheless, this comparison provides the proper context for many of the statements that follow.

Our main result about étale cohomology of \( \mathcal{O}_K \)-algebras is the following theorem.

Theorem 7.9. Let \( K \) be a complete, algebraically closed nonarchimedean field. Suppose \( A \) is a \( \mathcal{O}_K \)-algebra which is topologically finite type and let \( d = \dim(A) \). Let \( \mathcal{F} \) be a torsion abelian sheaf on \( \text{Spec}(A)_{\text{ét}} \). Then:

1. We have \( H^i(\text{Spec}(A), \mathcal{F}) = 0 \) for \( i > d + 1 \).
(2) If $F$ is an $\ell$-power torsion sheaf ($\ell$ prime to the characteristic of the residue field of $K$), then we have $H^i(\text{Spec}(A), F) = 0$ for $i > d$.

This extends recent results of Hansen [22]. In particular, in Theorem 1.3 of loc. cit. part (2) of the above result is proved in the case when $A$ descends to a discretely valued field and $K$ has characteristic zero. Our result confirms Hansen’s Conjecture 1.2 when $K$ has characteristic zero, in view of the comparison [22, Theorem 1.7].

Remark 7.10. We would expect that part (2) of Theorem 7.9 holds true for arbitrary torsion sheaves. By part (1), this amounts to showing the following: if $A$ is a tft $K$-algebra of dimension $d$ and $p$ is the residue characteristic of $K$, then $H^{d+1}(\text{Spec}(A), F) = 0$ for all $p$-torsion sheaves $F$ on $\text{Spec}(A)_{\text{ét}}$. Two special cases of this expectation are within reach:

(1) If $K$ itself has characteristic $p$, then this assertion is straightforward: the $\mathbb{F}_p$-étale cohomological dimension of any affine $\mathbb{F}_p$-scheme is $\leq 1$ (thanks essentially to the Artin-Schreier sequence).

(2) The algebraization method used to prove Lemma 7.14 below can be adapted to prove this statement when $A$ is smooth (in the sense of rigid spaces) and $F$ is constant, thanks to [12, Theorem 7 and Remark 2 on page 587].

Nevertheless, if $K$ has characteristic 0 with its residue field having characteristic $p$, the general case remains out of reach. One difficulty is that that over such $p$-adic fields, $p$-adic étale cohomology behaves quite differently on affinoids than its $\ell$-adic counterpart. For example, it is almost never finite dimensional (unlike the $\ell$-adic case), even though it does take on finite dimensional values for proper rigid analytic varieties [43], [44, Theorem 3.17]. More crucially, the tensor product trick used in Proposition 7.15 to improve the bound from $d+1$ to $d$ in the $\ell$-adic case is not available in the $p$-adic case.

We review some facts about the étale cohomology of affinoid varieties. First we need the Künneth formula.

**Proposition 7.11** (Künneth formula). Given tft $K$-algebras $A, B$, the natural map

$$R\Gamma(\text{Spec}(A), F_\ell) \otimes_{\mathbb{F}_\ell} R\Gamma(\text{Spec}(B), F_\ell) \rightarrow R\Gamma(\text{Spec}(A \otimes_K B), F_\ell)$$

is an equivalence in $D(\mathbb{F}_\ell)$.

*Proof.* Let $A_0 \subset A, B_0 \subset B$ be open bounded subrings. Then $A_0, B_0$ are $\pi$-adically complete, and $A \otimes_K B$ is obtained by inverting $\pi$ on $A_0 \otimes_{\mathcal{O}_K} B_0$. The result now follows from the Künneth formula of Proposition 6.22. \[\square\]

Next, we need to observe that (algebraic) étale cohomology satisfies descent in the analytic topology. We just treat a special case.

**Proposition 7.12** (Descent in the analytic topology). Let $A$ be a tft $K$-algebra and let $f, g \in A$ generate the unit ideal. Then for any torsion abelian sheaf $F$ on $\text{Spec}(A)$, we have a pullback square

$$
\begin{array}{c}
R\Gamma(\text{Spec}(A), F) \quad \rightarrow \quad R\Gamma(\text{Spec}(A \langle f \rangle), F) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
R\Gamma(\text{Spec}(A \langle g \rangle), F) \quad \rightarrow \quad R\Gamma(\text{Spec}(A \langle f, g \rangle), F)
\end{array}
$$
In the language of rigid spaces, this square captures the Mayer-Vietoris sequence for the affinoid space $X := \text{Spa}(A, A^\circ)$ for $A$ attached to the open cover $X\left(\frac{f}{g}\right) := \{x \in X | |f(x)| \leq |g(x)|\}$ and $X\left(\frac{g}{f}\right)$. In particular, the proposition follows immediately from the comparison between analytic and algebraic étale cohomology of affinoids. Alternately, we can argue directly as follows using the machinery of this paper.

\textbf{Proof.} We consider the functor $F$ on $\mathcal{O}_K$-algebras given by $F(S) = R\Gamma(\text{Spec}(\overline{S}[1/\pi]), \mathcal{F})$, which we have seen satisfies arc$\pi$-descent (Corollary 6.17).

Let $A_0 \subset A$ be an open bounded subring; rescaling $f, g$ we can assume that $f, g \in A_0$, and they generate an open ideal of $A_0$. We consider the $A_0$-algebras $S_1 = A_0[T]/(fT - g), S_2 = A_0[U]/(gU - f)$ and $S = S_1 \times S_2$. It is easy to see that $A_0 \to S$ is an arc$\pi$-cover and that $S_1 \otimes_{A_0} S_1 \to S_1, S_2 \otimes_{A_0} S_2 \to S_2$ are arc$\pi$-equivalences. Form the Čech nerve of $A_0 \to S$ and apply the functor $F$ to it, obtaining a limit diagram; in light of the above observation and Proposition 6.2, this translates to a pullback square

$$
\begin{array}{ccc}
F(A_0) & \longrightarrow & F(S_1) \\
\downarrow & & \downarrow \\
F(S_2) & \longrightarrow & F(S_1 \otimes_{A_0} S_2)
\end{array}
$$

This is precisely the claimed pullback square. \hfill \Box

Next, we review (what amounts to) a special case of Huber’s quasi-compact base change theorem. More precisely, we prove a continuity property for étale cohomology of affinoids that roughly says that the étale cohomology of a Zariski closed subset of an affinoid can be calculated as the filtered colimit of the étale cohomology of rational subsets that contain it.

\textbf{Proposition 7.13.} Let $R$ be a tft $K$-algebra. Let $f \in R$. This gives an evident inductive sequence of tft $K$-algebras $\{R \langle f/\pi^r \rangle\}$ with transition maps $R \langle f/\pi^r \rangle \to R \langle f/\pi^{r+1} \rangle \to \cdots$. For any torsion sheaf $\mathcal{F}$ on $\text{Spec}(R)_{\text{aff}}$, there is an equivalence

$$
\lim_{r} R\Gamma(\text{Spec}(R \langle f/\pi^r \rangle), \mathcal{F}) \simeq R\Gamma(\text{Spec}(R/f), \mathcal{F}).
$$

\textbf{Proof.} Let $R_0 \subset R$ be an open bounded $\mathcal{O}_K$-subalgebra which is topologically finite type over $\mathcal{O}_K$. Rescaling $f$, we may assume $f \in R_0$ as well.

We consider the functor $F : \text{Ring}_{R_0} \to D(\mathbb{Z})^{\geq 0}$ given by $F(T) = R\Gamma(\text{Spec}(\overline{T}[1/\pi]), \mathcal{F})$. As in Proposition 6.17 and the following discussion, we have that $F$ satisfies arc$\pi$-descent, and it commutes with filtered colimits.

For each $r$, we consider the algebra $S_r := R_0[T_r]/(\pi^r T_r - f)$. We have a sequence of maps $S_1 \to S_2 \to \cdots \to S_r \to S_{r+1} \to \cdots$, where the map $S_r \to S_{r+1}$ sends $T_r \mapsto \pi T_{r+1}$. We claim that the natural map

$$
\lim_{r} S_r \to R_0/f
$$

induces an equivalence upon applying $F$.

We have seen that $F$ is a finitary arc$\pi$-sheaf, so it suffices to show that $\lim_{r} S_r \to R_0/f$ is an arc$\pi$-equivalence (Proposition 6.2). In fact, given any rank 1 valuation ring $V$ (over $\mathcal{O}_K$) such that $\pi$ is a pseudouniformizer, maps from $\lim_{r} S_r \to V$ extend uniquely over maps $R_0/f \to V$. Indeed,
a map $R_0 \to V$ extends over $S_r$ if and only if the image of $f$ is divisible by $\pi^r$ in $R_0$, and the only element of $V$ divisible by all powers of $\pi$ is zero. It follows that
\[
\lim_{r \to 0} F(S_r) \simeq F(R_0/f),
\]
which implies the desired claim. Indeed, we have that $S_r[1/\pi] = R \langle f/\pi^r \rangle$. Also, $R_0/f$ is a tft $\mathcal{O}_K$-algebra and hence $\pi$-adically complete. Therefore, we recover (18). \hfill \Box

Next, we prove a special case of Artin-Grothendieck vanishing.

**Lemma 7.14.** Let $f_1, \ldots, f_m, g \in T_n$ generate the unit ideal. Let $A$ be an algebra which is finite étale over $T_n \left\langle \frac{f_1, \ldots, f_m}{g} \right\rangle$. Then for any torsion abelian group $\Lambda$, $R\Gamma(\text{Spec}(A), \Lambda) \in D(\mathbb{Z})^{\leq n}$ (i.e., Artin vanishing holds).

**Proof.** Note that $\min(|f_1|, \ldots, |f_m|, |g|)$ is uniformly bounded below on the unit ball of $K^n$ since $f_1, \ldots, f_m, g$ generate the unit ideal. Therefore, we can assume that $f_i, g \in K[X_1, \ldots, X_n] \subset T_n$ (for each $i$) without changing the rational subset $\{x : |f_i(x)| \leq |g(x)|, \forall i\}$ in the unit ball. Rescaling the $f_i$ and $g$ and possibly enlarging the set, we may assume furthermore that they belong to $\mathcal{O}_K[X_1, \ldots, X_n]$ and that $f_1$ is a power of $\pi$.

Now $R_0 = \mathcal{O}_K[X_1, \ldots, X_n, T_1, \ldots, T_m]/(gT_1 - f_1, \ldots, gT_m - f_m)$ is a ring of finite type over $\mathcal{O}_K$ such that $T_n \left\langle \frac{f_1, \ldots, f_m}{g} \right\rangle = \widehat{R_0}[1/\pi]$. Note that $R_0[1/\pi] \simeq K[X_1, \ldots, X_n, 1/g]$; imposing the equation $gT_1 - f_1$ forces $g$ to become invertible on inverting $\pi$ (as $f_1$ is a power of $\pi$), thus trivializing the other equations. In particular, this ring has Krull dimension $n$.

Let $R_0^n$ denote the henselization of $R_0$ along $\pi$. By [18, Proposition 5.4.54] (which is the non-noetherian version of a result of Elkik [12]) finite étale covers of $R_0^n[1/\pi]$ and $\widehat{R_0}[1/\pi]$ are identified. It follows that there exists a finite étale $R_0^n[1/\pi]$-algebra $A'$ such that

$$A \simeq A' \otimes_{R_0^n[1/\pi]} \widehat{R_0}[1/\pi].$$

By the Fujimura-Gabber theorem (Theorem 6.10), it follows that $A$ and $A'$ have the same étale cohomology. But $A'$ is finite étale over $R_0^n[1/\pi]$. Now $R_0^n[1/\pi]$ is a filtered colimit of étale $R_0$-algebras, so $R_0^n[1/\pi]$ is a filtered colimit of étale $R_0[1/\pi] \simeq K[X_1, \ldots, X_n, 1/g]$-algebras and therefore so is $A'$. In particular, it is a filtered colimit of finite type $K$-algebras of dimension $n$. We can apply classical Artin-Grothendieck vanishing (and the fact that étale cohomology of rings commutes with filtered colimits) to conclude that $R\Gamma(\text{Spec}(A), \Lambda) = R\Gamma(\text{Spec}(A'), \Lambda) \in D^{\leq n}(\mathbb{Z})$, as desired. \hfill \Box

Next, we prove Theorem 7.9 in the case of a constant sheaf.

**Proposition 7.15.** Let $R$ be an affinoid $K$-algebra of dimension $d$. Then:

1. For all torsion abelian groups $\Lambda$, we have $R\Gamma(\text{Spec}(R), \Lambda) \in D^{\leq d+1}(\mathbb{Z})$.
2. For all $\ell$-power torsion abelian groups $\Lambda$, we have $R\Gamma(\text{Spec}(R), \Lambda) \in D^{\leq d}(\mathbb{Z})$.

**Proof.** Consider the following assertion:

$(\text{Art}_d)$: For all tft $K$-algebras $S$ of dimension $d$ and all torsion abelian groups $\Lambda$, we have $R\Gamma(\text{Spec}(S), \Lambda) \in D(\mathbb{Z})^{\leq d+1}$.

We shall prove $(\text{Art}_d)$ by induction on $d$; this will imply part (1) of the Proposition. The case $d = 0$ is trivial. Suppose that we know $(\text{Art}_{d-1})$. Fix a $d$-dimensional tft $K$-algebra $R$. Without
loss of generality, we may assume that $R$ is reduced. By Noether normalization, there is a finite map

$$T_n \to R.$$  

Recall that $T_n$ is an integral domain. Hence, if char$(K) = 0$, there exists $f \in T_n$ such that $R[1/f]$ is finite étale over $T_n[1/f]$.

If char$(K) = p > 0$, we have to work slightly harder since the map $T_n \to R$ given by Noether normalization may have nonreduced geometric generic fiber. To fix this, for each $t$, consider the $t$th iterated Frobenius $\phi^t : T_n \to T_n$ and the new $T_n$-algebra $R_t = (R \otimes_{T_n,\phi^t} T_n)_\text{red}$. Each of these new $T_n$-algebras $R_t$ comes with a universal homeomorphism $R \to R_t$, so it suffices to prove the result for any $R_t$. But for $t > 0$, the map $T_n \to R_t$ has reduced geometric generic fiber. Therefore, up to replacing $R$ by some $R_t$, we find that there exists $f \in T_n$ such that $T_n[1/f] \to R[1/f]$ is finite étale.

For each $r > 0$, we consider the following rings:

1. $R_r^{(1)} = R \otimes_{T_n} T_n \langle \pi^r / f \rangle = R \langle \pi^r / f \rangle$.
2. $R_r^{(2)} = R \otimes_{T_n} T_n \langle f / \pi^r \rangle = R \langle f / \pi^r \rangle$.
3. $R_r^{(3)} = R \otimes_{T_n} T_n \langle \pi^r / f, f / \pi^r \rangle = R \langle \pi^r / f, f / \pi^r \rangle$.

We have a pullback square from Proposition 7.12,

$$
\begin{array}{ccc}
RT(\Spec(R), \Lambda) & \xrightarrow{\iota} & RT(\Spec(R_r^{(1)}), \Lambda) \\
\downarrow & & \downarrow \\
RT(\Spec(R_r^{(2)}), \Lambda) & \xrightarrow{\iota} & RT(\Spec(R_r^{(3)}), \Lambda)
\end{array}
$$

Since $R \langle \pi^r / f \rangle$ is finite étale over $T_n \langle \pi^r / f \rangle$ (note that the completed tensor products in the definitions of $R_r^{(i)}$, $i = 1, 2, 3$ can be replaced by tensor products), and similarly $R \langle \pi^r / f, f / \pi^r \rangle$ is finite étale over $T_n \langle \pi^r / f, f / \pi^r \rangle$, by Lemma 7.14 we find that the top right and bottom right corners belong to $D^{\leq d}(\mathbb{Z})$, for any $r$.

Suppose $x \in H^j(\Spec(R), \Lambda)$ for $j \geq d + 2$ is nonzero. Using the long exact sequence in cohomology from the above fiber square, we find that $x$ must map to a nonzero class in $H^j(\Spec(R \langle f / \pi^r \rangle), \Lambda)$ for each $r > 0$. This contradicts the fact that the colimit of these groups is zero by the inductive hypothesis and Proposition 7.13. This completes the proof of the claim (Art.$d$) and, by induction, the first half of the theorem.

For the second half of the theorem, we may assume $\Lambda = \mathbb{F}_\ell$ and use the “tensor power trick.” For any tft $K$-algebra $A$ which is $d$-dimensional, we have seen that $RT(\Spec(A), \mathbb{F}_\ell) \in D(\mathbb{F}_\ell)^{\leq d+1}$. Since $A \otimes_K A$ is $2d$-dimensional, we get that $RT(\Spec(A \otimes_K A), \mathbb{F}_\ell) \in D(\mathbb{F}_\ell)^{\leq 2d+1}$ again by the first half of the theorem. Using the Künneth formula (Proposition 7.11), we have $RT(\Spec(A \otimes_K A), \mathbb{F}_\ell) \simeq RT(\Spec(A), \mathbb{F}_\ell)^{\otimes 2}$, and this forces $RT(\Spec(A), \mathbb{F}_\ell) \in D(\mathbb{F}_\ell)^{\leq d}$.

We now explain the proof of the full result.

**Proof of Theorem 7.9.** For simplicity we treat the $\ell$-power torsion case (i.e., the second half of the theorem); the other case is analogous. By a filtering $\mathcal{F}$ in terms of its constructible subsheaves [45, Tag 03SA] and in terms of the $\ell$-adic filtration, it suffices to prove the following statement:

$(\ast_d)$: If $A$ is a tft $K$-algebra of dimension $d$ and $\mathcal{F}$ is a constructible étale $\mathbb{F}_\ell$-sheaf on $X := \Spec(A)$, then $H^i(X, \mathcal{F}) = 0$ for $i > d$. 

We already know (**d**) for all d when \( F \) is constant. We shall reduce the general case to this one by a standard devissage procedure. We remind the reader that if \( A \) is tft \( K \)-algebra of dimension \( d \), then any finite \( A \)-algebra (such as a quotient) is also a tft \( K \)-algebra and has dimension \( \leq d \).

Let us prove (**d**) by induction on \( d \). When \( d = 0 \), the statement is clear as \( A \) is a finite product of copies of \( K \). Assume that (**d** - 1) holds true. Pick \( X := \text{Spec}(A) \) and \( F \) as in (**d**). There exists a dense open \( j : U \hookrightarrow X \) such that \( L = F|_U \) is locally constant. As \( j \) is dense open, the cokernel of the canonical injective map \( j_! L \to F \) is supported on some closed subset of dimension \( < d \). By induction and the long exact sequence, we may thus assume \( F = j_! L \). Moreover, as \( F \) decomposes as a finite direct sum of its restrictions to each connected component of \( U \), we may replace \( F \) by a direct summand to assume \( U \) is connected (though it might no longer be dense in \( X \)). By [45, Tag 0A3R], there exists a finite étale morphism \( \pi : V \to U \) of degree prime to \( \ell \) such that \( \pi^*(L) \) is an iterated extension of constant sheaves. As \( \pi \) has degree prime to \( \ell \), the “métod de la trace” [45, Tag 03SH] coupled with the assumption on \( \pi^* L \) shows that we may assume \( L = \pi_* \mathbb{F}_\ell \). Zariski’s main theorem then gives a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{j'} & Y \\
\downarrow{\pi} & & \downarrow{\pi} \\
U & \xrightarrow{j} & X
\end{array}
\]

with \( j' \) an open immersion and \( \pi \) finite. In particular, \( Y \) is an affinoid of dimension \( \leq d \). By replacing \( Y \) with the closure of \( V \) if necessary, we may also assume \( j' \) has dense image. Now \( H^i(X, j_! \pi_* \mathbb{F}_\ell) \cong H^i(Y, j'_! \mathbb{F}_\ell) \) as \( \pi_! \cong \pi_* \) and similarly for \( \pi \) (and because these functors have no higher derived functors).

We are thus reduced to showing that \( H^i(Y, j'_! \mathbb{F}_\ell) = 0 \) for \( i > d \). If \( i : Z \to Y \) is the complementary embedding, then we have a short exact sequence

\[
0 \to j'_! \mathbb{F}_\ell \to \mathbb{F}_\ell \to i_* (\mathbb{F}_\ell) \to 0
\]

of sheaves on \( Y \). Taking the long exact sequence and using the result for constant sheaves (Proposition 7.15) gives the claim since \( \dim(Z) < d \). \( \square \)
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