Identifying the Spectral Representation of Hilbertian Time Series

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Abstract

We provide $\sqrt{n}$-consistency results regarding estimation of the spectral representation of covariance operators of Hilbertian time series, in a setting with imperfect measurements. This is a generalization of the method developed in Bathia et al. (2010). The generalization relies on an important property of centered random elements in a separable Hilbert space, namely, that they lie almost surely in the closed linear span of the associated covariance operator. We provide a straightforward proof to this fact. This result is, to our knowledge, overlooked in the literature. It incidentally gives a rigorous formulation of PCA in Hilbert spaces.

Keywords: covariance operator, dimension reduction, Hilbertian time series, $\sqrt{n}$-consistency, functional PCA

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1. Introduction

In this paper, we provide theoretical results regarding estimation of the spectral representation of the covariance operator of stationary Hilbertian time series. This is a generalization of the method developed in Bathia et al. (2010) to a setting of random elements in a separable Hilbert space. The approach taken in Bathia et al. (2010) relates to functional PCA and, similarly to the

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latter, relies strongly on the Karhunen-Loève (K-L) Theorem. The authors
develop the theory in the context of curve time series, with each random curve
in the sequence satisfying the conditions of the K-L Theorem which, together
with a stationarity assumption, ensures that the curves can all be expanded in
the same basis – namely, the basis induced by their zero-lag covariance function.
The idea is to identify the dimension of the space $M$ spanned by this basis (finite
by assumption), and to estimate $M$, when the curves are observed with some
degree of error. Specifically, it is assumed that the statistician can only observe
the curve time series $(Y_t)$, where

$$Y_t = X_t + \epsilon_t,$$

whereas the curve time series of interest is actually $(X_t)$. Here $Y_t$, $X_t$ and
$\epsilon_t$ are random functions (curves) defined on $[0, 1]$. Estimation of $M$ in this
framework was previously addressed in Hall and Vial (2006) assuming the curves
are iid (in $t$), a setting in which the problem is indeed unsolvable in the sense
that one cannot separate $X_t$ from $\epsilon_t$. Hall and Vial (2006) propose a Deus ex
machina solution which consists in assuming that $\epsilon_t$ goes to 0 as the sample size
grows. Bathia et al. (2010) in turn resolve this issue by imposing a dependence
structure in the evolution of $(X_t)$. Their key assumption is that, at some lag
$k$, the $k$-th lag autocovariance matrix of the random vector composed by the
Fourier coefficients of $X_t$ in $M$, is full rank. In our setting this corresponds to
Assumption (A1) (see below).

In Bathia et al. (2010) it is assumed that each of the stochastic processes
$(X_t(u) : u \in [0, 1])$ satisfy the conditions of the K-L Theorem (and similarly for
$\epsilon_t$), and as a consequence the curves are in fact random elements with values
in the Hilbert space $L^2[0, 1]$. Therefore, since every separable Hilbert space is
isomorphic to $L^2[0, 1]$, the idea of a generalization to separable Hilbert spaces of
the aforementioned methodology might seem, at first, rather dull. The issue is
that in applications transforming the data (that is, applying the isomorphism)
may not be feasible nor desirable. For instance, the isomorphism may involve
calculating the Fourier coefficients in some ‘rule-of-thumb’ basis which might
yield infinite series even when the curves are actually finite dimensional.

The approach that we take here relies instead on the key feature that a centered Hilbertian random element of strong second order, lies almost surely in the closed linear span of its corresponding covariance operator. This result allows one to dispense with considerations of ‘sample path properties’ of a random curve by addressing the spectral representation of a Hilbertian random element directly. In other words, the Karhunen-Loève Theorem is just a special case\(^1\) of a more general phenomena. The result below (which motivates – and for that matter, justifies – our approach) is not a new one: it appears, for example, in a slightly different guise as an exercise in Vakhania et al. (1987). However, it is in our opinion rather overlooked in the literature. The proof that we give is straightforward and, to our knowledge, a new one. In this paper \(H\) is always assumed to be a real Hilbert space, but with minor adaptations all stated results hold for complex \(H\).

**Theorem 1.** Let \(H\) be a separable Hilbert space, and assume \(\xi\) is a centered random element in \(H\) of strong second order, with covariance operator \(R\). Then \(\xi \perp \ker(R)\) almost surely.

**Corollary 1.** In the conditions of Theorem 1, let \((\lambda_j : j \in J)\) be the (possibly finite) non-increasing sequence of nonzero eigenvalues of \(R\), repeated according to multiplicity, and let \(\{\varphi_j : j \in J\}\) denote the orthonormal set of associated eigenvectors. Then

\[
(i) \quad \xi(\omega) = \sum_{j \in J} \langle \xi(\omega), \varphi_j \rangle \varphi_j \text{ in } H, \text{ almost surely;}
\]

\[
(ii) \quad \xi = \sum_{j \in J} \langle \xi, \varphi_j \rangle \varphi_j \text{ in } L_2^2(\Omega).
\]

Moreover, the scalar random variables \(\langle \xi, \varphi_i \rangle\) and \(\langle \xi, \varphi_j \rangle\) are uncorrelated if \(i \neq j\), with \(\mathbb{E}(\langle \xi, \varphi_j \rangle^2) = \lambda_j\).

\(^1\)This is not entirely true since the Karhunen-Loève Theorem states uniform (in \([0,1]\)) \(L^2(\Omega)\) convergence.
Remark. (a) Although it is beyond the scope of this work, we call attention to the fact that Theorem 1 and Corollary 1 provide a rigorous justification of PCA for Hilbertian random elements. (b) In Corollary 1 either $J = \mathbb{N}$ or, whenever $R$ is of rank $d < \infty$, $J = \{1, \ldots, d\}$.

Proofs to the above and subsequent statements are given in Appendix B. We can now adapt the methodology of Bathia et al. (2010) to a more general setting.

2. The model

In what follows $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed complete probability space. Consider a stationary process $(\xi_t : t \in T)$ of random elements with values in a separable Hilbert space $H$. Here $T$ is either $\mathbb{N} \cup \{0\}$ or $\mathbb{Z}$. We assume throughout that $\xi_0$ is a centered random element in $H$ of strong second order. Of course, the stationarity assumption ensures that these properties are shared by all the $\xi_t$. Now let

$$R_k(h) := \mathbb{E} \langle \xi_0, h \rangle \xi_k, \quad h \in H,$$

denote the $k$-th lag autocovariance operator of $(\xi_t)$, and let $(\lambda_j : j \in J)$ be the (possibly finite) non-increasing sequence of nonzero eigenvalues of $R_0$, repeated according to multiplicity. Here either $J = \mathbb{N}$ or, whenever $R_0$ is of rank $d < \infty$, $J = \{1, \ldots, d\}$. Now for $j \in J$, let $\varphi_j \in H$ be defined by

$$R_0(\varphi_j) = \lambda_j \varphi_j,$$

and assume the set $\{\varphi_j : j \in J\}$ is orthonormal in $H$. Corollary 1 and the stationarity assumption ensure that the spectral representation

$$\xi_t = \sum_{j \in J} Z_{tj} \varphi_j$$

holds almost surely in $H$, for all $t$, where the $Z_{tj} := \langle \xi_t, \varphi_j \rangle$ are centered scalar random variables satisfying $\mathbb{E} Z_{tj}^2 = \lambda_j$ for all $t$, and $\mathbb{E} Z_{ti} Z_{tj} = 0$ if $i \neq j$. In applications, an important case is that in which the above sum has only finitely
many terms: that is, the case in which $R_0$ is a finite rank operator. In this setting, the stochastic evolution of $(\xi_t)$ is driven by a vector process $(Z_t : t \in T)$, where $Z_t = (Z_{t1}, \ldots, Z_{td})$, in $\mathbb{R}^d$ (here $d$ is the rank of $R_0$). The condition that $R_0$ is of finite rank models the situation where the data lie (in principle) in an infinite dimensional space, but it is reasonable to assume that they in fact lie in a finite dimensional subspace which must be identified inferentially.

We are interested in modeling the situation where the statistician observes a process $(\zeta_t : t \in T)$ of $H$-valued random elements, and we shall consider two settings; the simplest one occurs when

$$\zeta_t = \xi_t.$$  

(1)

This is to be interpreted as meaning that perfect measurements of a ‘quantity of interest’ $\xi_t$ are attainable. A more realistic scenario would admit that associated to every measurement there is an intrinsic error – due to rounding, imprecise instruments, etc. In that case observations would be of the form

$$\zeta_t = \xi_t + \epsilon_t.$$  

(2)

In fact, the latter model nests the ‘no noise’ one if we allow the $\epsilon_t$ to be degenerate. Equation (2) is analogous to the model considered in Hall and Vial (2006) and in Bathia et al. (2010). Here $(\epsilon_t : t \in T)$ is assumed to be noise, in the following sense: \(i\) for all $t$, $\epsilon_t \in L_2^2(H)$, with $E\epsilon_t = 0$; \(ii\) for each $t \neq s$, $\epsilon_t$ and $\epsilon_s$ are strongly orthogonal.

In the above setting, for $h, f \in H$ one has $E\langle h, \zeta_t \rangle \langle f, \zeta_t \rangle = \langle R_0(h), f \rangle + E\langle h, \epsilon_t \rangle \langle f, \epsilon_t \rangle$ and thus estimation of $R_0$ via a sample $(\zeta_1, \ldots, \zeta_n)$ is spoiled (unless the $\epsilon_t$ are degenerate). This undesirable property has been addressed by Hall and Vial (2006) and Bathia et al. (2010) respectively in the iid scenario and in the time series (with dependence) setting. The clever approach by Bathia et al. (2010) relies on the fact that $E\langle h, \zeta_t \rangle \langle f, \zeta_{t+1} \rangle = \langle R_1(h), f \rangle$ (lagging filters the noise) and therefore $R_1$ can be estimated using the data $(\zeta_1, \ldots, \zeta_n)$. Now an easy check shows that $\text{ran}(R_1) \subset \text{ran}(R_0)$. The key assumption in Bathia et al. (2010) is asking that this relation hold with equality:
Consider the operator $S := R_1 R_1^*$, where $*$ denotes adjoining. It is certainly positive, and compact (indeed nuclear) since $\text{ran}(R_1 R_1^*) = \text{ran}(R_1)$. Let $(\theta_j : j \in J')$ be the (possibly finite) non-increasing sequence of nonzero eigenvalues of $S$, repeated according to multiplicity, and denote by $\{\psi_j : j \in J'\}$ the orthonormal set of associated eigenvectors. Under Assumption (A1) we have $J' = J$, and the representation

$$\xi_t = \sum_{j \in J} W_{tj} \psi_j$$

is seen to hold, for all $t$, almost surely in $H$ for centered scalar random variables $W_{tj} = \langle \xi_t, \psi_j \rangle$. Again, when $R_0$ is finite rank, say rank($R_0$) = $d$, then the stochastic evolution of $\xi_t$ is driven by the finite-dimensional vector process $(W_t : t \in T)$, where $W_t = (W_{t1}, \ldots, W_{td})$.

3. Main results

Before stating our result, let us establish some notation. Define the estimator $\hat{S} := \hat{R}_1 \hat{R}_1^*$, where $\hat{R}_1$ is given by

$$\hat{R}_1(h) := \frac{1}{n-1} \sum_{t=1}^{n-1} \langle \zeta_t, h \rangle \zeta_{t+1}, \quad h \in H.$$ 

Notice that $\hat{R}_1$ is almost surely a finite rank operator, say of rank $q$, with $q \leq n-1$ almost surely, and thus $\hat{S}$ is also of finite rank $q$. Let $(\hat{\theta}_1, \hat{\theta}_2, \ldots)$ denote the non-increasing sequence of eigenvalues of $\hat{S}$, repeated according to multiplicity. Clearly $\hat{\theta}_j = 0$ if $j > n - 1$. Denote by $\{\hat{\psi}_1, \hat{\psi}_2, \ldots\}$ the orthonormal basis of associated eigenfunctions. Also, for a closed subspace $V \subset H$, let $\Pi_V$ denote the orthogonal projector onto $V$. Let $M := \text{ran}(R_0)$, and for conformable $k$ put $\hat{M}_k := \bigvee_{j=1}^k \hat{\psi}_j$.

Theorem 2. Let (A1) and the following conditions hold.

(A2) $(\zeta_t : t \in T)$ is strictly stationary and $\psi$-mixing, with the mixing coefficient satisfying the condition $\sum_{k=1}^{\infty} k \psi^{1/2}(k) < \infty$.


(A3) \( \zeta_t \in L^2(H) \), for all \( t \);

(A4) \( \epsilon_t \) and \( \xi_s \) are strongly orthogonal, for all \( t \) and \( s \).

Then,

(i) \( \| \hat{S} - S \|_2 = O_p(n^{-1/2}) \);

(ii) \( \sup_{j \in J} |\hat{\theta}_j - \theta_j| = O_p(n^{-1/2}) \).

Moreover, if

(A5) \( \ker(S - \theta_j) \) is one-dimensional, for each nonzero eigenvalue \( \theta_j \) of \( S \),

holds, then

(iii) \( \sup_{j \in J} \| \hat{\psi}_j - \psi_j \| = O_p(n^{-1/2}) \).

If additionally \( S \) is of rank \( d < \infty \), then

(iv) \( \hat{\theta}_j = O_p(n^{-1}), \) for all \( j > d \);

(v) \( \| \Pi_M(\hat{\psi}_j) \| = O_p(n^{-1/2}), \) for all \( j > d \).

Remark. (a) Assumption (A5) ensures that \( \psi_j \) is an identifiable statistical parameter. It is assumed that the ‘correct’ version (among \( \psi_j \) and \( -\psi_j \)) is being picked. See Lemma 4.3 in Bosq (2000); (b) Since the operator \( \hat{S} \) is almost surely of finite rank, items (ii) and (iv) imply the following. If \( \text{rank}(S) = d < \infty \), then for \( j = 1, \ldots, d, \) \( \hat{\theta}_j \) is eventually non-zero and arbitrarily close to \( \theta_j \), and the remaining nonzero \( \hat{\theta}_j \) for \( j > d \) (if any) are eventually arbitrarily close to zero. Otherwise, eventually \( \hat{\theta}_j > 0 \) for all \( j \) (but notice that this cannot occur uniformly in \( j \): it is always the case that \( \hat{\theta}_j = 0 \) for \( j > n - 1 \)). This property can be used to propose consistent estimators of \( d \).

**Corollary 2.** Let Assumptions (A1)–(A4) hold. Let \( N_j := \ker(S - \theta_j) \) and \( \hat{N}_j := \ker(\hat{S} - \hat{\theta}_j) \). Then,

(i) \( \| \Pi_{\hat{N}_j} - \Pi_{N_j} \|_2 = O_p(n^{-1/2}) \), for all \( j \) such that \( N_j \) is one-dimensional;
(ii) if \( S \) is of rank \( d < \infty \), then
\[
\left\| \Pi \widehat{M}_d - \Pi_M \right\|_2 = O_P(n^{-1/2});
\]

(iii) if \( S \) is of rank \( d < \infty \), there exists a metric \( \rho \) on the collection of finite-dimensional subspaces of \( H \) such that
\[
\rho(\widehat{M}_d, M) = O_P(n^{-1/2}).
\]

Remark. (a) Observe that, when the process \( (\xi_t) \) is not centered, evidently all the above results would still hold by replacing \( \xi_t \) by \( \xi_t - \mathbb{E} \xi_0 \) and \( \xi_t \) by \( \xi_t - \mathbb{E} \xi_0 \), but this is not practical since in general \( \mathbb{E} \xi_0 \) is not known to the statistician. However, this does not pose a problem, since under mild conditions we have
\[
1/n \sum_{t=1}^n \xi_t \xrightarrow{a.s.} \mathbb{E} \xi_0,
\]
and thus all the results still hold with \( \xi_t \) and \( \xi_t \) replaced respectively by \( \xi_t - 1/n \sum_{t=1}^n \xi_t \) and \( \xi_t - 1/n \sum_{t=1}^n \xi_t \). (b) The key assumption in Bathia et al. (2010) would be translated in our setting to the condition that, for some \( k \geq 1 \), the identity \( \text{ran}(R_k) = \text{ran}(R_0) \) holds. For simplicity we have assumed that \( k = 1 \), but of course the stated results remain true if we take \( k \) to be any integer \( \geq 1 \) and redefine \( S \) and \( \widehat{S} \) appropriately. Indeed the stated results remain true if we define \( S = (n-p)^{-1} \sum_{k=1}^p R_k R_k^* \), where \( p \) is an integer such that \( \text{ran}(R_k) = \text{ran}(R_0) \) holds for some \( k \leq p \). In statistical applications, a recommended approach would be to estimate \( S \) defined in this manner. In any case, computation of the eigenvalues and eigenvectors of \( \widehat{S} \) can be carried out directly through the spectral decomposition of a convenient \( n-p \times n-p \) matrix. The method is discussed in Bathia et al. (2010). Notice that if \( R_0 \) is of rank one, then asking that \( \text{ran}(R_k) = \text{ran}(R_0) \) holds for some \( k \) corresponds to the requirement that the times series \( (Z_{t1} : t \in T) \) is correlated at some lag \( k \). Otherwise we would find ourselves in the not very interesting scenario (for our purposes) of an uncorrelated time series.

4. Concluding remarks

In this paper we have provided consistency results regarding estimation of the spectral representation of Hilbertian time series, in a setting with imperfect measurements. This generalizes a result from Bathia et al. (2010). The generalization relies on an important property of centered random elements in a
A separable Hilbert space – see Theorem 1. Further work should be directed at obtaining a Central Limit Theorem for the operator $\hat{S}$, which would have the important consequence of providing Central Limit Theorems for its eigenvalues (via Theorem 1.2 in Mas and Menneteau (2003)), potentially allowing one to propose statistical tests for these parameters. The term ‘spectral’ in the title of this work refers, of course, to the spectral representation of the operator $S$ and not to the spectral representation of the time series $(\xi_t)$ in the usual sense.

A. Notation and mathematical background

As in the main text we let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space, i.e. a probability space with the additional requirement that subsets $N \subset \Omega$ with outer probability zero are elements of $\mathcal{F}$. Let $H$ be a separable Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A Borel measurable\footnote{There are notions of strong and weak measurability but for separable spaces they coincide.} map $\xi : \Omega \to H$ is called a random element with values in $H$ (also: Hilbertian random element). For $q \geq 1$, if $\mathbb{E}\|\xi\|^q < \infty$ we say that $\xi$ is of strong order $q$ and write $\xi \in L^q(\mathbb{P})$. In this case, there is a unique element $h_\xi \in H$ satisfying the identity $\mathbb{E}\langle \xi, f \rangle = \langle h_\xi, f \rangle$ for all $f \in H$. The element $h_\xi$ is called the expectation of $\xi$ and is denoted be $\mathbb{E}\xi$. If $\mathbb{E}\xi = 0$ we say that $\xi$ is centered. If $\xi$ and $\eta$ are centered random elements in $H$ of strong order 2, they are said to be (mutually) strongly orthogonal if, for each $h, f \in H$, it holds that $\mathbb{E}\langle h, \xi \rangle \langle f, \eta \rangle = 0$.

Denote by $\mathcal{L}(H)$ the Banach space of bounded linear operators acting on $H$. Let $A \in \mathcal{L}(H)$. If for some (and hence, all) orthonormal basis $(e_j)$ of $H$ one has $\|A\|_2 := \sum_{j=1}^{\infty} \|A(e_j)\|^2 < \infty$, we say that $A$ is a Hilbert-Schmidt operator. The set $L_2(H)$ of Hilbert-Schmidt operators is itself a separable Hilbert space with inner-product $\langle A, B \rangle_2 = \sum_{j=1}^{\infty} \langle A(e_j), B(e_j) \rangle$, with $\|\cdot\|_2$ being the induced norm. An operator $T \in \mathcal{L}(H)$ is said to be nuclear, or trace-class, if $T = AB$ for some Hilbert-Schmidt operators $A$ and $B$. If $\xi \in L_2^\mathbb{P}(H)$, its covariance operator is the nuclear operator $R_\xi(h) := \mathbb{E}\langle \xi, h \rangle \xi$, $h \in H$. More generally, if $\xi, \eta \in L_2^\mathbb{P}(H)$,
their cross-covariance operator is defined, for \( h \in H \), by \( R_{\xi,\eta}(h) := \mathbb{E}(\langle \xi, h \rangle \eta) \). In the main text we denote by \( R_k \) the cross-covariance operator of \( \xi_0 \) and \( \xi_k \).

For a survey on strong mixing processes, including the definition of \( \psi \)-mixing in Assumption (A2), we refer the reader to Bradley (2005).

B. Proofs

Proof of Theorem 1. Let \((e_j)\) be a basis of \(\ker(R)\). It suffices to show that \(\mathbb{E}|\langle \xi, e_j \rangle|^2 = 0\) for each \(j\). Indeed, this implies that there exist sets \(E_j, \mathbb{P}(E_j) = 0\) and \(\langle \xi(\omega), e_j \rangle = 0\) for \(\omega \notin E_j\). Thus \(\langle \xi(\omega), e_j \rangle = 0\) for all \(j\) as long as \(\omega \notin \bigcap E_j\) with \(\mathbb{P}(\bigcap E_j) = 0\). But \(\mathbb{E}|\langle \xi, e_j \rangle|^2 = \mathbb{E}\langle \xi, e_j \rangle \mathbb{E}\langle \xi, e_j \rangle = \langle \mathbb{E}\langle \xi, e_j \rangle, e_j \rangle = \langle R(e_j), e_j \rangle = 0\).

Proof of Corollary 1. Item (i) is just another way of stating the Lemma. For item (ii), first notice that the functions \(\omega \mapsto \langle \xi(\omega), \varphi_j \rangle \varphi_j, j \in J\), form an orthogonal set in \(L_2^2(\mathbb{P})\) (although not orthonormal). We must show that \(\int \|\xi(\omega) - \sum_{j=1}^n \langle \xi(\omega), \varphi_j \rangle \varphi_j \|^2 d\mathbb{P}(\omega) \to 0\). Let \(g_n(\omega) := \|\xi(\omega) - \sum_{j=1}^n \langle \xi(\omega), \varphi_j \rangle \varphi_j \|\). By item (i) \(g_n(\omega) \to 0\) almost surely. Also, \(0 \leq g_n(\omega) \leq 2\|\xi(\omega)\|\). So \(g_n^2(\omega) \to 0\) and \(g_n^2(\omega) \leq 4\|\xi(\omega)\|^2\). Now apply Lebesgue’s Dominated Convergence Theorem.

Proof of Theorem 2. One only has to consider an isomorphism \(U : H \to L^2[0, 1]\). The proof is the same as in Bathia et al. (2010).

Proof of Corollary 2. See the proof of Theorem 2 in Bathia et al. (2010).

Remark. The hypothesis that \(\xi\) is centered in Theorem 1 cannot be weakened, as the following simple example shows. Let \(H = \mathbb{R}^2\) and let \(\xi = (\xi_1, \xi_2)\) where \(\xi_1\) is a (real valued) standard normal and \(\xi_2 = 1\) almost surely. Then \(R \equiv (R_{ij})\) is the matrix with all entries equal to zero except for \(R_{11}\) which is equal to 1, and obviously one has \(\mathbb{P}(\xi \perp \ker(R)) = 0\).
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