General Matrix-Valued Inhomogeneous Linear
Stochastic Differential Equations and Applications

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Abstract: The expressions of solutions for general $n \times m$ matrix-valued inhomogeneous linear stochastic differential equations are derived. This generalizes a result of Jaschke (2003) for scalar inhomogeneous linear stochastic differential equations. As an application, some $\mathbb{R}^n$ vector-valued inhomogeneous nonlinear stochastic differential equations are reduced to random differential equations, facilitating pathwise study of the solutions.

1 A Review of Stochastic Exponential Formulas

We first review some existing results about solution formulas for linear stochastic differential equations (SDEs) or for their integral formulations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a standard stochastic basis. For the following stochastic integral equation

$$X_t = 1 + \int_0^t X_s \, dZ_s,$$  \hfill (1.1)

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where $Z$ is a semimartingale with $Z_0 = 0$, Doléans-Dade (1970) proved that the unique solution of (1.1) is given by

$$X_t = \exp \left\{ Z_t - \frac{1}{2} \langle Z^c, Z^c \rangle_t \right\} \prod_{0 < s \leq t} (1 + \Delta Z_s) e^{-\Delta Z_s}. \quad (1.2)$$

In the literature $X$ is called *Doléans exponential (or stochastic exponential)* of $Z$, and is denoted by $\mathcal{E}(Z)$. The formula (1.2) is called the *Doléans (or stochastic) exponential formula*.

In an unpublished paper, Yoeurp and Yor (1977) proved the following result for the solution formula of scalar SDEs (see also Revuz-Yor (1999) and Protter (2005) for the case where $Z$ is a continuous semimartingale, and Melnikov-Shiryaev (1996)) for the general case).

**Theorem 1.1 (Yoeurp and Yor 1977)** Let $Z$ and $H$ be semimartingales, and $\Delta Z_s \neq -1$ for all $s \in [0, \infty]$. Then the unique solution of the inhomogeneous scalar linear SDE

$$X_t = H_t + \int_0^t X_s dZ_s, \quad t \geq 0, \quad (1.3)$$

is given by

$$X_t = \mathcal{E}(Z)t \left\{ H_0 + \int_0^t \mathcal{E}(Z)^{-1}_s dG_s \right\}, \quad (1.4)$$

where

$$G_t = H_t - \langle H^c, Z^c \rangle_t - \sum_{0 < s \leq t} \frac{\Delta H_s \Delta Z_s}{1 + \Delta Z_s}. \quad (1.5)$$

Jaschke (2003) extended equation (1.3) to the case where $H$ is an adapted cadlag process, not necessarily a semimartingale. He proved that in this case the solution of (1.3) is given by:

$$X_t = H_t - \mathcal{E}(Z)t \int_0^t H_s d(\mathcal{E}(Z)^{-1}_s). \quad (1.6)$$

By using (1.6) Jaschke (2003) obtained a new proof of (1.4).

On the other hand, Emery (1978) considered the following $n \times n$ matrix-valued stochastic equation

$$U(t) = I + \int_0^t (dL(s)) U(s-), \quad (1.7),$$

where $I$ is an $n \times n$ identity, $L$ is a given $n \times n$ matrix-valued semimartingale with $L(0) = 0$, such that $I + \Delta L(s)$ is invertible for all $s \in [0, \infty]$, where $\Delta L(s) = L(s) - L(s-)$. Emery proved that the equation (1.7) admits a unique solution $U$, which is $n \times n$ matrix-valued semimartingale. We call it the stochastic exponential of $L$ and denote it by $\mathcal{E}(L)$. However, there is no explicit expression for such a stochastic exponential in general.
Jacod (1982) has studied the following inhomogeneous matrix-valued stochastic integral equation

\[ X(t) = H(t) + \int_0^t (dL(s))X(s^-) \]

where \( L \) is a given \( n \times n \) matrix-valued semimartingale with \( L_0 = 0 \), such that \( I + \Delta L(s) \) is invertible for all \( s \in [0, \infty) \), and \( H \) is an \( n \times m \) matrix valued semimartingale. For an \( n \times n \) matrix valued semimartingale \( A \) and an \( n \times m \) matrix valued semimartingale \( B \), we let

\[ [A, B](t) = \langle A^c, B^c \rangle(t) + \sum_{0<s\leq t} \Delta A(s) \Delta B(s). \]

Here \( A^c \) denotes its continuous martingale part defined componentwise by \( (L^c)^i_j = (L^i_j)^c \), and

\[ \langle A^c, B^c \rangle_j^i = \sum_k \langle (A^c)^i_k, (B^c)^k_j \rangle. \]

Using these notations the result of Jacod (1982) implies the following

**Theorem 1.2 (Jacod 1982)** The unique solution of (1.8) is given by

\[ X(t) = E(L)(t) \left\{ H(0) + \int_0^t E(L)(s^-)^{-1}dG(s) \right\}, \]

where

\[ G(t) = H(t) - \langle L^c, H^c \rangle(t) - \sum_{0<s\leq t} (1 + \Delta L(s))^{-1} \Delta L(s) \Delta H(s). \]

In particular, if \( L \) and \( H \) are continuous semimartingales, an expression for the solution of (1.8) is given in Revuz-Yor (1999) as follows:

\[ X(t) = E(L)(t) \left\{ H(0) + \int_0^t E(L)(s^-)^{-1}(dH(s) - d[L, H](s)) \right\}. \]

The objective of the present note is to generalize equation (1.8) to the case where \( H(t) \) is a given \( n \times m \) matrix-valued adapted cadlag process, not necessarily a semimartingale. We give an expression of the solution of (1.8) for this case and also give a simpler proof for Theorem 1.1. Our result extends (1.6) of Jaschke (2003) to matrix-valued case. As an application, we reduce some \( \mathbb{R}^n \)-valued inhomogeneous nonlinear SDEs to random differential equations (RDEs) — differential equations with random coefficients. This facilitates pathwise study of solutions and is an important step in the context of random dynamical systems approaches; see Arnold (1998).
2 Matrix-valued Inhomogeneous Linear SDEs

Léandre (1985) obtained the following result about stochastic equation (1.7). If we denote by $V$ the inverse of $\mathcal{E}(L)$, then $V$ is the solution of the following equation:

$$V(t) = I + \int_0^t V(s-)dW(s),$$

where

$$W(t) = -L(t) + \langle L^c, L^c \rangle(t) + \sum_{0<s\leq t} (1 + \Delta L(s))^{-1}(\Delta L(s))^2.$$  

That means $\mathcal{E}(L)^{-1} = \mathcal{E}(W^\tau)^\tau$. We refer the reader to Karandikar (1991) for a detailed proof of this result.

Now we will use this result of Léandre (1985) to solve the following inhomogeneous stochastic integral equation

$$X(t) = H(t) + \int_0^t (dL(s))X(s-),$$

(2.1),

where $L$ is a given $n \times n$ matrix-valued semimartingale with $L0 = 0$, such that $I + \Delta L(s)$ is invertible for all $s \in [0, \infty]$, $H(t)$ is a given $n \times m$ matrix-valued adapted cadlag process.

Our main result is the following.

**Theorem 2.1** The unique solution of (2.1) is given by

$$X(t) = H(t) - \mathcal{E}(L)(t) \int_0^t (d\mathcal{E}(L)(s)^{-1})H(s-).$$

(2.2)

If $H$ is $n \times m$ matrix-valued semimartingale, then $X(t)$ has the same expression as given by (1.10) and (1.11).

**Proof.** We denote $\mathcal{E}(L)$ and $\mathcal{E}(L)^{-1}$ by $U$ and $V$, respectively. We are going to show that the process

$$X(t) = H(t) - U(t) \int_0^t (dV(s))H(s-),$$

defined by (2.2), satisfies equation (2.1). Since $UV = I$, by the integration by parts formula (see Karandikar (1991)) we get

$$0 = d(U(t)V(t)) = U(t-)dV(t) + (dU(t))V(t-) + d[U, V](t)$$

$$= d(U(t)V(t)) = U(t-)dV(t) + dL(t) + d[U, V](t).$$

Once again by the integration by parts formula, using the above result and the fact that $dU(t) = (dL(t))U(t-)$, we have

$$d(X(t) - H(t)) = -U(t-)dV(t)H(t-) - (dU(t))(\int_0^{t-} dV(s)H(s-)) - (d[U, V](t))H(t-)$$

$$= (dL(t))[H(t-) - U(t-)(\int_0^{t-} dV(s)H(s-))] = (dL(t))X(t-).$$
This shows that the process \((X(t))\) defined by (2.2) satisfies (2.1).

Now we assume that \((H(t))\) is an \(n \times m\) matrix-valued semimartingale. Using the notations in Section 1 we can verify that

\[
G(t) = H(t) + [W, H](t). \tag{2.3}
\]

By the integration by parts formula, using (2.3) and the fact that \(dV(t) = V(t-)dW(t)\), we obtain

\[
0 = d(V(t)H(t)) = V(t-)dH(t) + (dV(t))H(t-) + V(t-)d[W, H](t)
= V(t-)dG(t) + dL(t) + (dV(t))H(t-),
\]

from which we get

\[
H(t) = U(t)\left\{H(0) + \int_0^t V(s-)dG(s) + \int_0^t dV(s)H(s-)\right\}.
\]

Thus, if we let \((G(t))\) be defined by (2.3) then \((X(t))\) has the expression of (2.2), and consequently it satisfies equation (2.1). The proof of the theorem is complete. \(\square\)

3 An Application to Nonlinear SDEs

In this section we will apply our results in Theorem 2.1 to reduce an inhomogeneous nonlinear SDE to a RDE (random differential equation). Now we consider the following \(n\)-dimensional nonlinear SDE (but with a linear multiplicative noise term):

\[
dX^i(t) = f^i(t, X(t))dt + \sum_{j=1}^n C^i_j(t)X^j(t)dB^j(t), \quad X^i(0) = x^i, \tag{3.1}
\]

where \(C(t)\) is an \(n \times n\) matrix-valued measurable function, \(f(t, x)\) is a \(\mathbb{R}^n\)-valued measurable function on \([0, \infty) \times \mathbb{R}^n\), and \(B(t) = (B^1(t), \ldots, B^n(t))^\tau\) is a \(n\)-dimensional Brownian motion. Put

\[
L^i_j(t) = \int_0^t C^i_j(s)dB^j(s), \quad i, j = 1, \ldots, n; \quad H(t) = X(0) + \int_0^t f(s, X(s))ds. \tag{3.2}
\]

Then (3.1) can be rewritten in the form of (2.1). For such \(L\), the equation (1.9) is reduced to the following linear equation:

\[
dU^i_j(t) = \sum_{k=1}^n C^i_k(t)U^k_j(t)dB^k(t), \quad U^i_j(0) = \delta^i_j; \tag{3.3}
\]

In the present case we have \(G(t) = H(t)\). So according to (2.2), the solution of (3.1) can be expressed as

\[
X(t) = U(t)\left\{x + \int_0^t U(t)^{-1}f(s, X(s))ds\right\}, \tag{3.4}
\]
where $U$ is the solution of (3.3). Unfortunately, even in this case we are not able to give an explicit expression for $U(t)$. Let $Y(t) = U(t)^{-1}X(t)$, then

$$Y(t) = \left\{ x + \int_0^t U(s)^{-1} f(s, U(s)Y(s)) \, ds \right\}.$$

(3.5)

This is the integral formulation of a RDE (random differential equation). Once we have sample path solution $Y(t)$ of this transformed RDE, we obtain the solution of the original SDE via $X(t) = U(t)Y(t)$.

**References**

[1] Arnold, L. *Random Dynamical Systems*. Springer-Verlag, New York, 1998.

[2] Doléans-Dade, C. (1970): Quelques applications de la formule de changement de variables pour les semimartingales, Z. Wahrsch. verw. Gebiete 16, 181-194.

[3] Emery, M. (1978): Stabilité des solution des equations differentielles stochastiques: application aux integrales multiplicatives stochastique. Z. Wahrsch. verw. Gebiete 41, 241-262.

[4] Jacod, J. (1982): Equations différentielles lineaires: la methode de variation des constantes, Séminaire de Probabilités XVI, LN in Math. 920, Springer-Verlag, 442-446.

[5] Jascke, S. (2003): A note on the inhomogeneous linear stochastic differential equation, *Insurance: Mathematics and Finance* 32, 461-464.

[6] Karandikar, R.L. (1991): Multiplicative decomposition of nonsingular matrix valued semimartingales, Séminaire de Probabilités XXV, LN in Math. 1485, Springer-Verlag, 262-269.

[7] Léandre, R. (1985) Flot d’une équation différentielle stochastique, Séminaire de Probabilités XIX, LN in Math., Springer-Verlag, 271-274.

[8] Melnikov, A.V. and Shiryaev, A.N. (1996): Criteria for the absence of arbitrage in the financial market, *Frontiers in Pure and Appl. Probab. II*, Shiryaev, A.N. (Eds.), TVP Science Publishers, Moscow, 121-134.

[9] Oksendal B. (1998): *Stochastic differential equations*, 5th Edition, Springer-Verlag.

[10] Protter, P. (2005): Stochastic integration and differential equations, 2nd Edition, Springer-Verlag, New York.

[11] Revuz, D. and Yor, M. (1999): *Continuous martingales and Brownian motion*, 3rd edition, Springer-Verlag, Berlin.

[12] Yoeurp, C., Yor, M. (1977): Espace orthogonal à une semimartingale, Unpublished.