A possible experimental check of the uncertainty relations by means of homodyne measuring photon quadrature

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Abstract

We suggest to use the photon homodyne detection experimental data for checking the Heisenberg and Schrödinger-Robertson uncertainty relations, by means of measuring optical tomograms of the photon quantum states.

Key words Quantum optical tomograms, homodyne detection, uncertainty relations.

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1 Uncertainty relations

In quantum mechanics and quantum optics a key role in distinguishing classical and quantum domains is played by the uncertainty relations of Heisenberg \(^1\) and Schrödinger-Robertson \(^2\) \(^3\). The aim of this work is to suggest a direct experimental check of the uncertainty relations and control accuracy of the experiments of photon homodyne detection \(^4\). The photon homodyne quadrature was measured in \(^5\) and in a series of later experiments, see, e.g., \(^6\) \(^7\) \(^8\). These experiments had the goal to measure the photon quantum state described by a Wigner function \(^9\).

As it was shown in \(^10\) \(^11\), the Wigner function can be reconstructed if one knows the optical tomogram of the quantum state. In the experiments \(^5\) \(^6\) \(^7\) \(^8\) with the measuring of the photon homodyne quadrature by a homodyne detector, the output of the experimental result was the photon state optical tomogram \(W(X, \theta)\). This is the non negative normalized distribution function of the homodyne quadrature \(X\) and of the local oscillator phase \(\theta\). By taking...
the Radon transform \[12\] of \( W(X, \theta) \) one gets the Wigner function of the photon quantum state.

Our aim is to show that the same experimental setup can be used to check the uncertainty relations using the measured optical tomograms of photon quantum states. Since quantum uncertainty relations play a key role in the foundation of quantum mechanics, it seems reasonable to have a direct method for an experimental check of both the uncertainty relations and the accuracy degree of the measurements. These can be controlled and, in principle, improved, in experiments with homodyne detection.

The experimental fulfilling of the uncertainty relations of Heisenberg, \textit{per se}, does not characterize a genuine quantum mechanical behaviour \[13\], see also \[14\]. In fact, there are examples of Hermitian trace class operators which fulfill Heisenberg uncertainty relations when used as quantum density states \[13, 15\]. Nevertheless these operators are non-positive, so they cannot represent any quantum state.

In view of this remark, to have direct independent experimental confirmation of quantum mechanics foundations, one should check experimentally not only the quantum uncertainty relations for the second moments of position and momentum (which are either the Heisenberg or the Schrödinger-Robertson uncertainty relations), but also the quantum inequalities, available in standard quantum mechanics (see, e.g., the review \[16\]).

We suggest in this paper a way to make such confirmation by using homodyne detection of photon quantum states. In the next section 2, we review the tomographic probability formulation of quantum mechanics \[17, 18, 19, 20, 21, 22\]. In section 3, we present the Heisenberg and Schrödinger-Robertson uncertainty relations in the tomographic probability representation of quantum states. In section 4, a suggestion of an experimental check of these uncertainty relations is discussed. Conclusions and perspective are given in section 5.

2 Tomograms of quantum states

According to \[17\] the quantum state of a photon with Wigner function \( W(q, p) \) is described by the Radon transform of the function

\[
W(X, \mu, \nu) = \int W(p, q) \delta(X - \mu q - \nu p) \frac{dp dq}{2\pi}, (\hbar = 1)
\]

which is called symplectic tomogram of the state. The tomogram is nonnegative and satisfies the normalization condition

\[
\int W(X, \mu, \nu) dX = 1.
\]

The real parameters \( \mu \) and \( \nu \), in the case when \( \mu = \cos \theta \) and \( \nu = \sin \theta \), provide the phase \( \theta \) of a local oscillator in the experiments with homodyne detecting photon states and \( W(X, \mu, \nu) \) becomes the optical tomogram \( W(X, \theta) \) of \[10, 11\].
which is directly measured in this experiments. For \( \mu = 1 \) and \( \nu = 0 \) the symplectic tomogram yields the position probability distribution in the quantum state (first quadrature probability distribution in the photon state). For \( \mu = 0 \) and \( \nu = 1 \) one has the momentum (second photon quadrature) probability distribution. The symplectic and optical tomograms are connected by an invertible relation

\[
W(X, \theta) = W(X, \cos \theta, \sin \theta) \quad (3)
\]

\[
W(X, \mu, \nu) = \frac{1}{\sqrt{\mu^2 + \nu^2}} W \left( \frac{X}{\sqrt{\mu^2 + \nu^2}}, \arctan \frac{\nu}{\mu} \right). \quad (4)
\]

The Wigner function of the photon quantum state is determined by the symplectic tomogram \cite{17}:

\[
W(p, q) = \int \frac{1}{2\pi} W(X, \mu, \nu) \exp \left[ i(X - \mu q - \nu p) \right] dX d\mu d\nu. \quad (5)
\]

Introducing polar coordinates \( \mu = \sqrt{\mu^2 + \nu^2} \cos \theta, \nu = \sqrt{\mu^2 + \nu^2} \sin \theta \) one can reduce Eq. (5) to a standard Radon integral \cite{12} used for reconstructing the Wigner function from the experimentally found optical tomogram \( W(X, \theta) \) in the aforementioned experiments. In view of the physical meaning of the optical tomogram one can calculate higher moments of the probability distribution

\[
\langle X^n \rangle (\mu, \nu) = \int X^n W(X, \mu, \nu) dX, \quad n = 1, 2, ...
\]

for any value of the parameters \( \mu \) and \( \nu \), in particular for any given phase of the local oscillator \( \theta \). This provides the possibility to check the inequalities for the quantum uncertainty relations.

### 3 Schrödinger-Robertson uncertainty relations

The Heisenberg uncertainty relation connects position and momentum variances \( \sigma_{QQ} \) and \( \sigma_{PP} \) by means of an inequality. In the tomographic probability representation the Heisenberg relation reads (see, e. g., \cite{23}):

\[
\sigma_{PP} \sigma_{QQ} = \left( \int X^2 W(X, 0, 1) dX - \left[ \int X W(X, 0, 1) dX \right]^2 \right) \times \quad (7)
\]

\[
\left( \int X^2 W(X, 1, 0) dX - \left[ \int X W(X, 1, 0) dX \right]^2 \right) \geq \frac{1}{4}.
\]

The Schrödinger-Robertson uncertainty relation contains the contribution of the position-momentum covariance \( \sigma_{QP} \) and reads

\[
\sigma_{QQ} \sigma_{PP} - \sigma_{QP}^2 \geq \frac{1}{4}. \quad (8)
\]
In view of Eq. (6), the variance $\sigma_{XX}$ of the homodyne quadrature $X$, in terms of the parameters $\mu, \nu$ and the quadratures variances and covariance, is

$$\sigma_{XX}(\mu, \nu) = \mu^2 \sigma_{QQ} + \nu^2 \sigma_{PP} + 2 \mu \nu \sigma_{QP}.$$  \hspace{1cm} (9)

Then one can get the expression of the covariance in terms of the tomographic characteristics of the state. Taking $\mu = \nu = \sqrt{2}/2$ corresponding to the local oscillator phase $\theta = \pi/4$ one has

$$\sigma_{QP} = \sigma_{XX} \left( \theta = \frac{\pi}{4} \right) - \frac{1}{2} \left( \sigma_{QQ} + \sigma_{PP} \right)$$  \hspace{1cm} (10)

where $\sigma_{PP}$ and $\sigma_{QQ}$ are the factors appearing in the left hand side of Eq. (7) respectively. The term $\sigma_{XX}(\theta = \pi/4)$ is given by Eq. (6) as

$$\sigma_{XX} \left( \theta = \frac{\pi}{4} \right) = \langle X^2 \rangle \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) - \left[ \langle X \rangle \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \right]^2.$$  \hspace{1cm} (11)

## 4 Checking uncertainty relations

On the base of the obtained formulae we suggest the following procedure to check the Heisenberg and Schrödinger-Robertson uncertainty relations. First one obtains the function $W(X, \theta)$, which is the optical tomogram, from the standard homodyne detection of a photon state. It means that one has also the symplectic tomogram $W(X, \mu, \nu)$ according to Eq. (4). Formula (7) can then be directly checked if one obtains from the experimental data the integrals in the left-hand side for $W(X, \theta = 0)$ and $W(X, \theta = \pi/2)$ and compares the product $\sigma_{XX}(\theta = 0)\sigma_{XX}(\theta = \pi/2)$ with $1/4$. The check of Schrödinger-Robertson uncertainty relations requires extra elaboration of the available experimentally obtained optical tomogram of photon quantum state. We express this procedure as the following inequality for optical tomogram. Let us calculate the function $F(\theta)$ which we call “tomographic uncertainty function”:

$$F(\theta) = \left( \int X^2 W(X, \theta) dX - \left[ \int X W(X, \theta) dX \right]^2 \right) \times$$

$$\left( \int X^2 W(X, \theta + \frac{\pi}{2}) dX - \left[ \int X W(X, \theta + \frac{\pi}{2}) dX \right]^2 \right)$$

$$- \left\{ \int X^2 W(X, \theta + \frac{\pi}{4}) dX - \left[ \int X W(X, \theta + \frac{\pi}{4}) dX \right]^2 \right\}$$

$$- \frac{1}{2} \left[ \int X^2 W(X, \theta) dX - \left[ \int X W(X, \theta) dX \right]^2 \right]$$

$$+ \int X^2 W(X, \theta + \frac{\pi}{2}) dX - \left[ \int X W(X, \theta + \frac{\pi}{2}) dX \right]^2 \right\} - \frac{1}{4}.$$  \hspace{1cm} (12)
The tomographic uncertainty function must be non-negative

\[ F(\theta) \geq 0 \]  

(13)

for all the values of the local oscillator phase angle \(0 \leq \theta \leq 2\pi\). The previous Eq. (12) for \(\theta = 0\) yields Eq. (8). Thus, choosing the values \(\theta = 0, \pi/4, \pi/2\) out from experimental optical tomogram \(\mathcal{W}(X, \theta)\) data, one can check both the Heisenberg uncertainty relation (Eq. 7) and the inequality 8. Moreover one can check also the above inequality 13 by using tomographic experimental data corresponding to all values of angles \(\theta, \theta + \pi/4, \theta + \pi/2\).

5 Conclusion

We point out the main results of this work. We suggest to use the known experimental data obtained by measuring quantum states by means of optical tomographic method, which in all the available experiments were used to find the Wigner function, as a tool to check the quantum uncertainty relations. Our suggestion consists of elaborating the experimental optical tomogram data for computing the tomographic uncertainty function \(F(\theta)\) defined in Eq. (12), instead of using the data, as usual, in a Radon integral transform leading to the Wigner function. The function of the local oscillator phase \(F(\theta)\) contains integrations for different fixed values of \(\theta\). In the case of the computation of the Wigner function the integration over the local oscillator phase is performed. However, even though the integrals to evaluate differ from the integrals in the Radon transform, they do not contain extra mathematical complications.

The suggested experimental checking of the quantum uncertainty relations can be used not only to test the degree of experimental accuracy with which the uncertainty relations are known today, but also to control the correctness of the experimental tools used in homodyne detection of photon states.

There exist inequalities in which the higher momenta of quadrature components are involved (see, e.g., the review [16]). One can reformulate these higher order inequalities in terms of the tomographic quadrature momenta given in Eq. (6) in order to obtain extra inequalities, again expressed in terms of the experimental values of the optical tomogram.

The tomographic probability approach can be applied also for two-mode and multi-mode photon states, in particular for Gaussian states, whose properties, like photon statistics, are sufficiently known (see, e.g., [24, 25]).

The tomographic entropic uncertainty relations which are associated with position and momentum probability distributions were discussed in [26].

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