New exponential, logarithm and q-probability in the non-extensive statistical physics

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In this paper, a new exponential and logarithm related to the non-extensive statistical physics is proposed by using the q-sum and q-product which satisfy the distributivity. And we discuss the q-mapping from an ordinary probability to q-probability. The q-entropy defined by the idea of q-probability is shown to be q-additive.

I. INTRODUCTION

Boltzman-Gibbs statistical mechanics shows how fast microscopic physics with short-range interaction has as effect on much larger space-time scale. The Boltzman-Gibbs entropy is given by

\[ S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i = k \sum_{i=1}^{W} p_i \ln \left( \frac{1}{p_i} \right) \]  

(1)

where \( k \) is a Boltzman constant, \( W \) is a total number of microscopic possibilities of the system and \( p_i \) is a probability of a given microstate among \( W \) different ones satisfying \( \sum_{i=1}^{W} p_i = 1 \). When \( p_1 = \frac{1}{W} \), we have \( S_{BG} = k \ln W \).

Boltzman-Gibbs theory is not adequate for various complex, natural, artificial and social system. For instance, this theory does not explain the case that a zero maximal Lyapunov exponent appears. Typically, such situations are governed by power-laws instead of exponential distributions. In order to deal with such systems, the non-extensive statistical mechanics is proposed by C.Tsallis [1,2]. The non-extensive entropy is defined by

\[ S_q = k(\sum_i^W p_i^q - 1)/(1 - q) \]  

(2)

The non-extensive entropy has attracted much interest among the physicist, chemist and mathematicians who study the thermodynamics of complex system [3]. When the deformation parameter \( q \) goes to 1, Tsallis entropy (2) reduces to the ordinary one (1). The non-extensive statistical mechanics has been treated along three lines:

1. Mathematical development [4, 5, 6]
2. Observation of experimental behavior [7]
3. Theoretical physics (or chemistry) development [8]

The basis of the non-extensive statistical mechanics is q -deformed exponential and logarithmic function which is different from those of Jackson’s [9]. The q-deformed exponential and q-logarithm of non-extensive statistical mechanics is defined by [10]

\[ \ln_q x = \frac{t^{1-q} - 1}{1-q}, \quad (t > 0) \]  

(3)

\[ e_q(t) = (1 + (1-q)t)^{-\frac{1}{1-q}}, \quad (x, q \in R) \]  

(4)

where \( 1 + (1-q)t > 0 \).

From the definition of q-exponential and q-logarithm, q-sum, q-difference, q-product and q-ratio are defined by [5, 6]

\[ x \oplus y = x + y + (1-q)xy \]
\[
x \otimes y = \frac{x - y}{1 + (1 - q)y} \\
x \oslash y = \frac{x^{1-q} + y^{1-q} - 1}{1 - q} \\
x \ominus y = \frac{x^{1-q} - y^{1-q} + 1}{1 - q}
\]  

(5)

It can be easily checked that the operation \( \oplus \) and \( \otimes \) satisfy commutativity and associativity. For the operator \( \oplus \), the identity additive is 0, while for the operator \( \oslash \) the identity multiplicative is 1. Indeed, there exist an analogy between this algebraic system and the role of hyperbolic space in metric topology [11]. Two distinct mathematical tools appears in the study of physical phenomena in the complex media which is characterized by singularities in a compact space [12].

For the new algebraic operation, q-exponential and q-logarithm have the following properties:

\[
\ln_q(xy) = \ln_q x \oplus \ln_q y \\
\ln_q(x \otimes y) = \ln_q x \oplus \ln_q y \\
\ln_q(x/y) = \ln_q x \otimes \ln_q y \\
\ln_q(x \ominus y) = \ln_q x \ominus \ln_q y
\]

(6)

From the associativity of \( \oplus \) and \( \otimes \), we have the following formula:

\[
t \oplus t \oplus \cdots \oplus t \underbrace{\oplus \cdots \oplus}_{n \text{ times}} t = \frac{1}{1 - q} \left\{ [1 + (1 - q)t]^n - 1 \right\}
\]

(7)

\[
t \otimes^n = \underbrace{t \otimes t \otimes \cdots \otimes}_{n \text{ times}} t = [nt^{1-q} - (n - 1)]^{1-q}
\]

(8)

II. NEW Q-CALCULUS

In this section, we discuss the new algebraic operation related to the non-extensive statistical physics. The q-sum, q-difference, q-product and q-ratio defined in the eq.(5) does not obey distributivity. To resolve this problem, the new multiplication is introduced [13].

Inserting \( t = 1 \) in eq.(7), we have

\[
1 \oplus 1 \oplus 1 \oplus \cdots \oplus 1 \underbrace{\oplus \cdots \oplus}_{n \text{ times}} 1 = \frac{1}{1 - q} \left\{ (2 - q)^n - 1 \right\}
\]

(9)

We will denote \( 1 \oplus 1 \oplus 1 \oplus \cdots \oplus 1 \) by \( n_q \). Here we call \( n_q \) a q-number of \( n \), where \( n_q \) reduces to \( n \) when \( q \) goes to 1.

For real number \( x \), we can define the q-number \( x_q \) as follows:

\[
x_q = \frac{1}{1 - q} \left\{ (2 - q)^x - 1 \right\}
\]

(10)

Here we have the following:

\[
0_q = 0, \quad 1_q = 1
\]

(11)

Then q-number satisfies the following:

\[
x_q \oplus y_q = (x + y)_q
\]

(12)

For this addition, we have the identity \( 0_q \) obeying

\[
x_q \oplus 0_q = x_q
\]

(13)

Letting the inverse of \( x_q \) by \( (-x)_q \), we have

\[
x_q \oplus (-x)_q = 0_q
\]

(14)
The q-sum satisfies the following property.

\[
(x_1)_q \oplus (x_2)_q \oplus \cdots \oplus (x_n)_q = \frac{(2 - q) \sum_{i=1}^{n} x_i - 1}{1 - q} = \sum_{i=1}^{n} (x_i)_q
\] (15)

The q-difference is defined in a similar way:

\[
x_q \ominus y_q = x_q \oplus (-y)_q = (x - y)_q
\] (16)

The new q-product \( \hat{\otimes} \) is defined in [13] as follows:

\[
x_1 \hat{\otimes} x_2 \hat{\otimes} \cdots \hat{\otimes} x_n = \frac{(2 - q) \prod_{i=1}^{n} \ln(1 + (1 - q)x_i)}{(1 - q)^n} - 1
\] (17)

Indeed, \( x \hat{\otimes} y \) reduces to \( xy \) when \( q \) goes to 1. For this q-product, we have the following:

\[
x_q \hat{\otimes} y_q = (xy)_q
\] (18)

For q-sum and q-product, the distributive law holds:

\[
x_q \hat{\otimes} (y_q \oplus z_q) = (x_q \hat{\otimes} y_q) \oplus (x_q \hat{\otimes} z_q) = (x(y + z))_q
\] (19)

The q-product of \( n \) variables is defined by

\[
x_1 \hat{\otimes} x_2 \hat{\otimes} \cdots \hat{\otimes} x_n = \frac{1}{1 - q} \left( (2 - q) \prod_{i=1}^{n} \frac{\ln(1 + (1 - q)x_i)}{(1 - q)^n} - 1 \right)
\] (20)

The formula is applied to \( n \) q-numbers as follows:

\[
(x_1)_q \hat{\otimes} (x_2)_q \hat{\otimes} \cdots \hat{\otimes} (x_N)_q = \frac{1}{1 - q} \left( (2 - q) \prod_{i=1}^{n} x_i - 1 \right) = \left( \prod_{i=1}^{n} x_i \right)_q
\] (21)

When \( n \) variables are same, the eq.(21) becomes

\[
x \hat{\otimes}^n = \frac{(2 - q)^n \prod_{i=1}^{n} \ln(1 + (1 - q)x)}{(1 - q)^n} - 1
\] (22)

The eq.(22) is easily proved by mathematical induction. Assume that the eq.(22) holds for \( n \). For \( n + 1 \), we have

\[
x \hat{\otimes}^{n+1} = x \hat{\otimes}^n \hat{\otimes} x = \frac{(2 - q)^n \prod_{i=1}^{n} \ln(1 + (1 - q)x_i)}{(1 - q)^n} - 1
\] (23)

Thus, for all integers \( n \), the eq.(22) holds.

Replacing \( x \) with \( x_q \) in the eq.(22), we have

\[
x_q \hat{\otimes}^n = (x^n)_q
\] (24)

Using the eq.(24), we have the following q-binomial theorem.

\[
(x + y)_q \hat{\otimes}^n = (x_q \oplus y_q) \hat{\otimes}^n = \frac{(2 - q) (x + y)^n - 1}{1 - q}
\] (25)

From the new q-product, we obtain the inverse of \( x_q \), denoted by \( (x^{-1})_q \), as follows:

\[
(x^{-1})_q = \frac{1}{1 - q} \left( (2 - q)x^{-1} - 1 \right)
\] (26)
Indeed, \((x^{-1})_q\) is an inverse of \(x_q\) because
\[
x_q \hat{\otimes} (x^{-1})_q = 1
\] (27)

The q-factorial is defined by
\[
n_q! = 1_q \hat{\otimes} 2_q \hat{\otimes} \cdots \hat{\otimes} n_q = \frac{(2 - q)^{n!} - 1}{1 - q}
\] (28)

The recurrence relation of the q-factorial is then given by
\[
(n + 1)_q! = \frac{1}{1 - q}[(1 + (1 - q)n_q!)^{n+1} - 1]
\] (29)

With a help of the inverse, we can define the q-ratio as follows:
\[
x_q \otimes y_q = x_q \hat{\otimes} (y^{-1})_q = (2 - q)^{\frac{\ln(1 + (1 - q)x) \ln(1 + (1 - q)(y^{-1})_q)}{\ln(2 - q)^2}} - 1
\] (30)

For the q-ratio, the following holds:
\[
x_q \otimes y_q = (xy^{-1})_q
\] (31)

Form the definition of q-number, we have the following property:
\[
(x + ky)_q = (1 + (1 - q)(ky)_q)x_q + (ky)_q
\] (32)

or
\[
(x + ky)_q = (1 + (1 - q)x_q)(ky)_q + x_q,
\] (33)

where \(k\) is an arbitrary real number.

III. NEW Q-EXPONENTIAL AND Q-LOGARITHM

In this section we will investigate the q-exponential and q-logarithm using the q-sum and q-product given in section II. The q-exponential can be expressed in two ways.
\[
e_q(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)_q^n
\] (34)

or
\[
e_q(x) = \lim_{n \to \infty} \left(1 \hat{\otimes} \frac{x}{n}\right)_{\hat{\otimes}n}
\] (35)

Let us discuss the first definition (34). Then, the q-exponential is given by
\[
e_q(x) = (e^x)_q = \frac{(2 - q)^{e^x} - 1}{1 - q}
\] (36)

Indeed, we have
\[
\lim_{q \to 1} e_q(x) = e^x
\] (37)

For the q-product of q-exponentials, we have
\[
e_q(x) \hat{\otimes} e_q(y) = e_q(x + y)
\] (38)

As an inverse function of the q-exponential, we can define the q-logarithm as follows:
\[
\ln_q x = \ln \left(\frac{\ln(1 + (1 - q)x)}{\ln(2 - q)}\right)
\] (39)
Indeed, we have

\[ \lim_{q \to 1} \ln_q x = \ln x \]  

(40)

The q-logarithm has the following property:

\[ \ln_q (x \otimes y) = \ln_q x + \ln_q y \]  

(41)

For the second definition (35), the q-exponential has the following form:

\[ e_q(x) = \frac{(2 - q)^{\ln_q x} - 1}{1 - q} \]  

(42)

The proof is easy. From the eq. (35), we have

\[
\begin{align*}
\lim_{n \to \infty} e_q(x) &= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{\otimes n} \\
&= \lim_{n \to \infty} \left( 1 + (2 - q) \frac{x}{n} \right)^{\otimes n} \\
&= \lim_{n \to \infty} \frac{1}{1 - q} \left[ (2 - q)^{\ln(1 + (1 - q) x)} n - 1 \right] \\
&= \frac{(2 - q)^{\ln_q x} - 1}{1 - q},
\end{align*}
\]

(43)

where we used

\[
\lim_{n \to \infty} \left( 1 + \frac{1}{\ln(2 - q)} \ln(1 + (1 - q) \frac{x}{n}) \right)^n = e^{\ln_q x}
\]

Indeed, we have

\[ \lim_{q \to 1} e_q(x) = e^x \]

(44)

For the q-product of q-exponentials, we have

\[ e_q(x) \otimes e_q(y) = e_q(x + y) \]

(45)

As an inverse function of the q-exponential, we can define the q-logarithm as follows:

\[ \ln_q x = \frac{\ln(2 - q)}{1 - q} \frac{\ln(1 + (1 - q) x)}{\ln(2 - q)} \]

(46)

Indeed, we have

\[ \lim_{q \to 1} \ln_q x = \ln x \]

(47)

The q-logarithm has the following property:

\[ \ln_q (x \otimes y) = \ln_q x + \ln_q y \]

(48)

IV. Q-PROBABILITY AND Q-ADDITIVITY OF THE Q-ENTROPY

In this section we discuss the q-additivity of the q-entropy which is expressed by using the idea of q-number. Let W be the number of microstates. As the probability of a given microstate among W different ones, we adopt \((p_i)_q\) instead of \(p_i\), which is defined by

\[ (p_i)_q = \frac{(2 - q)^{p_i} - 1}{1 - q} \]

(49)
It is easy to check that \((p_i)_q\) becomes \(p_i\) when \(q\) goes to 1. The eq.(49) is a kind of mapping from \(p_i\) into \((p_i)_q\). When the condition \(\sum_{i=1}^W p_i = 1\), we have

\[
\bigoplus_{i=1}^W (p_i)_q = 1
\]  

where \(\bigoplus_{i=1}^N a_i\) is defined by

\[
\bigoplus_{i=1}^N a_i = a_1 \oplus a_2 \oplus \cdots \oplus a_N
\]  

Thus, the sum of the probability of each microstate is still unity under replacing an ordinary addition with \(q\)-sum and replacing \(p_i\) with \((p_i)_q\).

Now let us consider the binomial distribution. At each trial, we may set

\[
p_1 + p_2 = 1,
\]

where \(p_1\) is a probability that a certain event occurs at each trial. If we use a mapping given in the eq.(49), we have

\[
(p_1)_q \oplus (p_2)_q = 1
\]

and

\[
[(p_1)_q \oplus (p_2)_q] \tilde{\ominus}^n = 1
\]

The \(q\)-binomial expansion is then as follows:

\[
[(p_1)_q \oplus (p_2)_q] \tilde{\ominus}^n = \bigoplus_{r=1}^n \left\{ \frac{1}{1-q} \right\} \left\{ (1 + (1-q)p_1 \tilde{\ominus}^r p_2 \tilde{\ominus}^{n-r})^n C_r - 1 \right\}
\]  

Let us denote the \(q\)-binomial distribution by \(B_q(n, p)\) corresponding to the ordinary binomial distribution \(B(n, p)\). The probability the the event occurs \(r\) times among \(n\) trials is given by

\[
(P_r(n))_q = \frac{1}{1-q} \{(1 + (1-q)p_1 \tilde{\ominus}^r p_2 \tilde{\ominus}^{n-r})^n C_r - 1\}
\]  

Then we have

\[
\bigoplus_{r=1}^n (P_r(n))_q = 1
\]

If we define the \(q\)-expectation value of \(A\) by

\[
\langle A \rangle_q = \bigoplus_{r=1}^n (AP_r(n))_q
\]

Using this, the \(q\)-mean is given by

\[
m_q = \langle r \rangle_q = \frac{(2-q)^n p_1 - 1}{1-q}
\]

and \(q\)-variance is given by

\[
V_q = \frac{(2-q)^n p_1 p_2 - 1}{1-q}
\]

Following the above discussion, we can define the \(q\)-entropy as follows:

\[
S_q = -k \bigoplus_{i=1}^W ((p_i)_q \tilde{\ominus} (\ln p_i)_q) = -k(\sum_{i=1}^W p_i \ln p_i)_q
\]  

Indeed the \(q\)-entropy given in the eq.(61) is not additive but \(q\)-additive. It obeys the following:

\[
S_q(A + B) = S_q(A) \oplus S_q(B)
\]
V. CONCLUSION

In this paper, we constructed two types of new exponential and logarithm related to the non-extensive statistical physics by using the q-sum and q-product which satisfy the distributivity. And we discussed the q-mapping from an ordinary probability to q-probability and investigated the q-binomial distribution. The q-entropy defined through the idea of q-probability was shown to be q-additive.

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