Some critical point theorems and applications

Guangcun Lu *

School of Mathematical Sciences, Beijing Normal University,
Laboratory of Mathematics and Complex Systems, Ministry of Education,
Beijing 100875, The People’s Republic of China
(gclu@bnu.edu.cn)

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Abstract

This paper is a continuation of [36]. In Part I, applying the new splitting theorems developed therein we generalize previous some results on computations of critical groups and some critical point theorems to weaker versions. In Part II (in progress), they are used to study multiple solutions for nonlinear higher order elliptic equations described in the introduction of [36].

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Part I

Some critical point theorems

1 Introduction

In previous many critical point theorems involving computations of critical groups the functionals are often assumed to be at least $C^2$ smooth so that the usual splitting lemma can be used. Doubtlessly, it is possible to obtain some new critical point theorems or to generalize previous ones by combing our splitting lemmas for continuously directional differentiable functionals with and techniques and results in nonsmooth and continuous critical point theories. Firstly, we shall generalize the some results in [5] to weaker versions. Though they are not the weakest versions, our theorems are more convenient in applications because we do not need to compute such as subdifferentials and weak slopes, which are not easy actually. These are the main context in Section 5. Next, in Section 6 we shall present the corresponding version of the results on critical groups of sign-changing critical points in [4] and [34] in our framework and sketch how to prove them with our results in the previous sections.

2 Compactness conditions and deformation lemmas

Let $X$ be a normed vector space with dual space $X^*$, $U \subset X$ nonempty and open, and $f : U \to \mathbb{R}$ be a continuous functional. Recall in [22, 26, 27] that the weak slope of $f$ at $u \in U$ is the nonnegative extended real number $|df|(u)$, which is the supremum of the $\sigma$’s in $[0, +\infty]$ such that there exist $\delta > 0$ and $H : B_X(u, \delta) \times [0, \delta] \to X$ continuous with

$$\|H(v, t) - v\| \leq t \quad \text{and} \quad f(H(v, t)) \leq f(v) - \sigma t.$$ 

Clearly, $u \to |df|(u)$ is lower semicontinuous ([22 Prop.2.6]). A point $u \in X$ satisfying $|df|(u) = 0$ is called a lower critical point of $f$, and call $c = f(u)$ a lower critical value of $f$. By [20 Def.1.1], the strong slope of a continuous function $f : X \to \mathbb{R}$ at $u \in U$ is defined by

$$|\nabla f(u)| = \begin{cases} 0 & \text{if } u \text{ is a local minimum of } f, \\ \lim_{v \to u} \frac{f(v) - f(u)}{\|v - u\|} & \text{otherwise.} \end{cases}$$

Then $|df|(u) \leq |\nabla f(u)|$ for any $u \in U$ (see below Definition 2.8 in [22]), and

$$|\nabla f(u)| \leq \|f'(u)\|$$  \hspace{1cm} (2.1)
provided that \( f \) has F-derivative \( f'(u) \) at \( u \in U \).

For a locally Lipschitz continuous function \( f : U \to \mathbb{R} \), by [14] page 27 it has the (Clarke) generalized gradient at every \( u \in U \),

\[
\partial f(u) = \{g \in X^* | g(h) \leq f^o(u, h) \forall h \in X\},
\]

which is the subdifferential at \( \theta \) of the convex function \( X \to \mathbb{R} \), \( h \mapsto f^o(u, h) \), where

\[
f^o(u, h) = \lim_{s \to 0^+} \frac{f(u + sh) - f(u + w)}{s}
\]

is the generalized directional derivative of \( f \) at \( u \) in the direction \( h \). If \( X \) is a Banach space it was proved in [22] that

\[
|df|(u) \geq |\partial f|(u) := \min\{\|x^*\| : x^* \in \partial f(u)\} \quad \forall u \in U. \tag{2.2}
\]

This inequality may be strict by Example 1.1 in Kristály’s thesis [29].

By [14] Prop.2.2.1 or [45] Prop.3.2.4(iii), the function \( f : U \to \mathbb{R} \) is strictly H-differentiable at \( u_0 \in U \) if and only if \( f \) is locally Lipschitz continuous around \( x_0 \) and strictly G-differentiable at \( u_0 \in U \). In this case we have \( \partial f(u_0) = \{f'(u_0)\} \) by [14] Prop.2.2.4. Moreover, by [10] Prop.(6)] the set-valued mapping \( u \to \partial f(u) \) is weak* upper semi-continuous at \( u_0 \) in the sense that for any \( \epsilon > 0, v \in X \) there exists a \( \delta > 0 \) such that \( |\langle w - f'(u_0), v \rangle| < \epsilon \) for each \( w \in \partial f(u) \) with \( \|u - u_0\| < \delta \).

By [10] Prop.(7)] the function \( u \to \|\partial f|(u) \) is lower semi-continuous at \( u_0 \), i.e. \( \lim_{u \to u_0}\|\partial f|(u) \geq \|f'(u_0)\| \). These are summarized into the following proposition.

**Proposition 2.1** Let \( U \) be a nonempty open set of a normed vector space \( X \) with dual space \( X^* \), and let \( f : U \to \mathbb{R} \) be strictly H-differentiable at every \( u \in U \). Then

(i) \( \partial f(u) = \{f'(u)\} \) at any \( u \in U \), the map \( U \to X^* \), \( x \mapsto f'(x) \) is weak* continuous and the function \( \partial f(u) \) is lower semi-continuous.

(ii) \( \nabla f(u) = |df|(u) = \|f'(u)\| \forall u \in U \) provided that \( f \) is F-differentiable at \( u \in U \).

(ii) comes from (2.1) and (2.2). Clearly, Proposition 2.1 holds if \( f \) is \( C^1 \).

Let \( X \) be a Banach space and let \( f : X \to \mathbb{R} \) be a G-differential functional. Denote by \( K(f) = \{x \in X | f'(x) = 0\} \). For \( c \in \mathbb{R} \) let \( K(f)_c = \{x \in X | f(x) = c, f'(x) = 0\} \) and \( f^c = \{x \in X | f(x) \leq c\} \). If \( f \) is only continuous we write \( LK(f)_c = \{x \in X | f(x) = c, \ |df|(x) = 0\} \).

**Definition 2.2** Let \( X \) be a Banach space and let \( f : X \to \mathbb{R} \) be a strictly H-differentiable functional. For \( c \in \mathbb{R} \) the usual Palais-Smale compactness condition at the level \( c \), or \((PS)_c\) for short, means that every \((PS)_c\) sequence \( \{x_n\} \subset X \), i.e. satisfying \( f(x_n) \to c \) and \( f'(x_n) \to 0 \), has a convergent subsequence; moreover according to Cerami [8] we say that \( f \) satisfies the condition \((C)_c\) if every \((C)_c\) sequence \( \{x_n\} \subset X \), i.e. such that \( f(x_n) \to c \) and \((1 + \|x_n\|)\|f'(x_n)\| \to 0 \), has a convergent subsequence.
Clearly, the second condition is weaker than the first one. Note that the semi-continuity of the map \( u \mapsto \|f'(u)\| \) (by Proposition 2.1) implies the limit of a \((PS)_c\) or \((C)_c\) sequence \( \{x_n\} \) is in \( K(f)_c \). In particular, \( K(f)_c \) is compact if \( f \) satisfies \((PS)_c\) or \((C)_c\). The condition \((C)_c\) has the following equivalent form (11 Definition 1.1): (i) every bounded sequence \( \{x_n\} \) in \( X \) with \( f(x_n) \rightarrow c \) and \( f'(x_n) \rightarrow 0 \) has a convergent subsequence; (ii) there exist positive constants \( \sigma, R, \alpha \) such that \( \|f'(x)\| \cdot \|x\| \geq \alpha \) for any \( x \) with \( c - \sigma \leq f(x) \leq c + \sigma \) and \( \|x\| \geq R \).

Lemma 2.3 (First Deformation Lemma) For a strictly \( H \)-differentiable functional \( f : X \rightarrow \mathbb{R} \) on a (real) Banach space \( X \), and \( c \in \mathbb{R} \), suppose that \( f \) satisfies the condition \((C)_c\). Then for every \( \varepsilon_0 > 0 \), every neighborhood \( U \) of \( K(f)_c \) (if \( K(f)_c = \emptyset \) we take \( U = \emptyset \)), there exist an \( 0 < \varepsilon < \varepsilon_0 \) and a map \( \eta \in C([0,1] \times X, X) \) satisfying (i) \( \|\eta(t,u) - u\| \leq e(1 + \|u\|)t \), where \( e = \sum_{n=0}^{\infty} \frac{1}{n!} \); (ii) \( \eta(t,x) = x \) if \( x \notin f^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \); (iii) \( \eta([1] \times (f^{-\varepsilon} \setminus U)) \subset f^{\varepsilon - \varepsilon} \); (iv) \( f(\eta(s,x)) \leq f(\eta(t,x)) \) if \( s \geq t \); (v) \( \eta(t,x) \neq x \Rightarrow f(\eta(t,x)) < f(x) \).

In particular, (ii)-(iv) show that \( f \) satisfies the deformation condition \((D)_c\) in the sense of [3, Def. 3.1]. When \( f \) is even, \( \eta \) may be chosen so that \( \eta(t,\cdot) \) is odd for all \( t \in [0,1] \).

When \( f \) is \( C^1 \) and satisfies the \((PS)_c\) (resp. \((C)_c\)) this lemma was proved by Palais [10] (see also [13]), (resp. Cerami [8] and Bartolo-Benci-Fortunato [1]). For a \( C^{1-0}\)-functional on a reflexive Banach space \( X \), when \( f \) satisfies the \((PS)_c\) (resp. \((C)_c\)) Chang [10] (resp. Kourogenis and Papageorgiou [28]) proved this lemma.

Proof of Lemma 2.3 The ideas are following those of [28 Theorem 4]. In the present case the reflexivity of \( X \) is not required because we do not need to use the Eberlein separation theorem as in the proofs of [10 Lemma 3.3] and [28 Lemma 3]. Let us reprove Lemma 3 of [28] under our assumptions as follows. By [28 Lemma 2], for each \( \delta > 0 \) there exist \( \gamma > 0, \varepsilon > 0 \) such that for \( K_c = K(f)_c \),

\[
(1 + \|x\|)\|f'(x)\| \geq \gamma \quad \forall x \in f^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_{\delta}(K_c),
\]

where \( N_{\delta}(K_c) = \{x \in X \mid d(x, K_c) < \delta\} \). For each \( x \in f^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_{\delta}(K_c) \), we have \( \|f'(x)\| \geq \gamma/(1 + \|x\|) \). Note that \( \|f'(x)\| = \sup\{\langle f'(x), h \rangle \mid h \in X, \|h\| = 1\} \). We have \( h_x \in X \) such that \( \|h_x\| = 1 \) and

\[
\langle f'(x), h_x \rangle > \frac{3\gamma}{4(1 + \|x\|)} > \frac{\gamma}{2(1 + \|x\|)}.
\]

By Proposition 2.1(i) we have \( r_x > 0 \) such that

\[
\langle f'(y), h_x \rangle > \frac{\gamma}{2(1 + \|x\|)} \quad \forall y \in B_X(x, r_x).
\]
Now \( \{B_X(x, r_x)\} \) is an open cover of \( f^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_3(K_c) \). Repeating the remaining arguments in the proof of \cite[Lemma 2]{28} we get a locally Lipschitz vector field \( V : f^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_3(K_c) \to X \) such that

\[
\|V(x)\| \leq (1 + \|x\|) \quad \text{and} \quad \langle f'(x), V(x) \rangle \geq \frac{\gamma}{2}.
\]

Shrinking \( \varepsilon > 0, \gamma > 0, \delta > 0 \) so that \( N_{3\delta}(K_c) \subset U \) and that (8) of \cite{28} is satisfied, and almost repeating the proof of \cite[Theorem 4]{28} we may get the desired conclusions. The unique point which should be noted is the proof of (iii). By contradiction, suppose that \( f(\eta(1, x)) > c - \varepsilon \) for some \( x \in f^{c+\varepsilon} \setminus U \). Then \( c - \varepsilon < f(\eta(t, x)) \leq c + \varepsilon \) for all \( t \in [0, 1] \). As in the proof (c) of \cite[Theorem 4]{28} it must hold that \( \eta([0, 1] \times \{x\}) \cap (K(f)_c)_{2\delta} = 0 \). It follows that there exist 0 \( \leq t_1 < t_2 \leq 1 \) such that \( d(\eta(t_1, x), K(f)_c) = 2\delta, d(\eta(t_1, x), K(f)_c) = 3\delta \) and \( 2\delta < d(\eta(t, x), K(f)_c) < 3\delta \) for all \( t_1 \leq t \leq t_2 \). Repeating the remaining part of the proof (c) of \cite[Theorem 4]{28} yields (iii). \( \square \)

Corresponding to \cite[Corollary 3.3]{5} we have

**Corollary 2.4** For a strictly H-differentiable functional \( f : X \to \mathbb{R} \) on a (real) Banach space \( X \), we have

(i) If \( f \) satisfies the condition \((C)_c\) for all \( c \in [a, b] \) and \( K(f)_c = \emptyset \) for \( c \in [a, b] \) then there exist a deformation \( \eta : X \to X \) such that \( \eta_0 = \text{id}_X, \eta_t(x) = x \) if \( x \notin f^{-1}([a - 1, b + 1]) \), \( f(\eta_t(x)) \) is decreasing in \( t \) and \( \eta_1(f^b) \subset f^a \).

(ii) If \( f \) satisfies the condition \((C)_c\) for all \( c \geq a \) and \( K(f)_c = \emptyset \) for \( c \geq a \) then there exist a deformation \( \eta : X \to X \) such that \( \eta_0 = \text{id}_X, \eta_t(x) = x \) if \( f(x) \leq a - 1 \), \( f(\eta_t(x)) \) is decreasing in \( t \) and \( \eta(X) \subset f^a \).

**Proof.** We only outline the proof of (i). For each \( c \in [a, b] \) Lemma \cite{28} yields positive numbers \( \varepsilon_1^{(c)} < \varepsilon_2^{(c)} < 1 \) and a deformation \( \eta_t^{(c)} : X \to X \) such that \( \eta_0^{(c)} = \text{id}_X \) and

- \( \eta_t^{(c)}(t, x) = x \) if \( x \notin f^{-1}([c - \varepsilon_1^{(c)}, c + \varepsilon_2^{(c)}]) \);
- \( \eta_t^{(c)}(\{1\} \times (f^{c+\varepsilon_1^{(c)}}) \subset f^{c-\varepsilon_1^{(c)}}) \);
- \( f(\eta_t^{(c)}(s, x)) \leq f(\eta_t^{(c)}(t, x)) \) if \( s \geq t \).

Since \( [a, b] \) is compact there exist finite numbers \( a \leq c_1 < \cdots < c_k \leq b \) such that \( \{(c_i - \varepsilon_i^{(c_i)}, c_i + \varepsilon_i^{(c_i)})\}_{i=1}^k \) is an open cover of \([a, b] \). In particular we have

\[ c_1 - \varepsilon_1^{(c_1)} < a \leq c_1 < \cdots < c_k \leq b < c_k + \varepsilon_1^{(c_k)} \]

Then the composition \( \eta_t = \eta_t^{(c_1)} \circ \cdots \circ \eta_t^{(c_k)} \) satisfies the desired requirements. As in the proof of \cite[Corollary 3.3(b)]{5} we can derive (ii) from (i). \( \square \)

**Remark 2.5** Even if \( f \) is only continuous, if we replace the \((C)_c\) condition by the following \((PS)_c\) condition

\[ f(x_n) \to c \quad \text{and} \quad |df|(x_n) \to 0 \implies \exists \text{ a convergence subsequence of } \{x_n\} \]
then Lemma 2.3 holds provided that $K(f)_c$ is replaced by $LK(f)_c$. See [18, Theorem 2.14]. If $f$ is $F$-differentiable so that $|\nabla f(x_n)| = |df|(x_n) = \|f'(x_n)\|$ for all $n$, then under the $(PS)_c$ condition Lemma 2.3 is a corollary of [18, Theorem 2.14]. By the same reason, we may get the part I of the following the second deformation lemma from Theorem 2.3 of [16] or Theorem 4 and Remark 2 of [17].

Lemma 2.6 (Second Deformation Lemma) For a $F$-differentiable functional $f : X \to \mathbb{R}$ on a (real) Banach space $X$, and $-\infty < a < b \leq +\infty$ suppose that $f$ has only a finite number of critical points at the level $a$ and has no critical values in $(a, b)$. Then

I. If $f$ satisfies the condition $(PS)$ on $f^{-1}([a, c])$ for all $c \in [a, b] \cap \mathbb{R}$, then there exists a deformation $\eta : [0, 1] \times f^{bo} \to f^{bo} := \{f < b\}$ such that

(a) $f(\eta(t, u)) \leq f(u)$;
(b) $u \in K(f)_a \implies \eta(t, u) = u$;
(c) $\eta(\{1\} \times f^{bo}) \subset f^{ao} \cup K(f)_a$;
(d) if $b \in \mathbb{R}$ and $K(f)_b = \emptyset$, then $\eta$ can be extended to $[0, 1] \times X$, still denoted by $\eta$, such that $\eta(\{1\} \times f^b) \subset f^{ao} \cup K(f)_a$.

In particular, $f^{ao} \cup K(f)_a$ is a weak deformation retract of $f^{bo}$.

II. If $f$ is $C^1$ and satisfies the condition $(C)_c$ for all $c \in [a, b] \cap \mathbb{R}$, then $f^a$ is a strong deformation retract of $f^b \setminus K(f)_b$, i.e. there exists a map $\eta : [0, 1] \times (f^b \setminus K(f)_b) \to (f^b \setminus K(f)_b)$, called a strong deformation retraction of $f^b \setminus K(f)_b$ onto $f^b$, satisfying

(i) $\eta(0, u) = u$ for all $u \in f^b \setminus K(f)_b$;
(ii) $\eta(t, u) = x$ for all $(t, u) \in [0, 1] \times f^a$;
(iii) $\eta(\{1\} \times (f^b \setminus f^a)) = f^a$.

For the part II, under the condition $(PS)_c$, the proof is due to Rothe [44], Chang [11] and Wang [50]; and under the condition $(C)_c$ the proof can be found in Bartsch-Li [5], Perera-Schechter [42] and Perera-Agarwal-O’Regan [41].

Applying these two deformation lemmas and our splitting lemma, Theorem 2.1 in [36], the standard arguments as in [9, 38, 39, 40] may yield the following two theorems.

Theorem 2.7 Let $H$ be a Hilbert space and let $f : H \to \mathbb{R}$ be a $F$-differentiable and strictly $H$-differentiable functional. Suppose:

(i) for some small $\varepsilon > 0$ there exists a unique critical value $c$ in $[c - \varepsilon, c + \varepsilon]$;

(ii) $K_c$ is finite and $f$ satisfies the conditions of Theorem 2.1 in [36] near each of $K_c$;

(iii) Either $f$ satisfies the $(PS)$ condition on $f^{-1}([c - \varepsilon, c + \varepsilon])$ or $f$ is $C^1$ and satisfies the condition $(C)_d$ for every $d \in [c - \varepsilon, c + \varepsilon]$. 


Then for any abelian group $G$ one has

$$H_*(f_{c+\varepsilon}, f_{c-\varepsilon}; G) \cong \bigoplus_{z \in K_c} C_*(f, z),$$

which is finitely dimensional vector spaces over $G$ if $G$ is a field.

Let $B^m$ be the closed unit disk in $\mathbb{R}^m$. By a topological embedding $h : B^m \to H$ we mean that it is continuous bijection onto $h(B^m) \subset H$ and that $h$ is a homeomorphism between $B^m$ and $h(B^m)$ with respect to the induced topology on $h(B^m)$ from $H$.

**Theorem 2.8** (Handle Body Theorem). Under the assumptions of Theorem 2.7, if each of $K_c = \{z_j\}_{1}^{l}$ is also nondegenerate, then for some $0 < \varepsilon \leq \varepsilon_0$ there exist topological embeddings $h_i : B^{m_i} \to H$, $i = 1, \cdots, l$, such that

$$f_{c-\varepsilon} \cap h_j(B^{m_j}) = f^{-1}(c-\varepsilon) \cap h_j(B^{m_j}) = h_j(\partial B^{m_j})$$

for $j = 1, \cdots, l$, and $f_{c-\varepsilon} \cup \bigcup_{j=1}^{l} h_j(B^{m_j})$ is a deformation retract of $f_{c+\varepsilon}$, where $m_j$ is the Morse index of $z_j$.

Similarly, we can also give the versions on Hilbert manifolds.

The following is a slight variant of [24, Prop. 2.1].

**Proposition 2.9** Let $H$ be a Hilbert space and let $B(\infty) : H \to H$ be a bounded self-adjoint linear operator satisfying (C1$\infty$), i.e., 0 is at most an isolated point of the spectrum $\sigma(B(\infty))$, which implies $\pm(B(\infty)u, u)_H \geq 2\alpha_\infty \|u\|^2 \forall u \in H^\pm_\infty$. Assume:

(i) $g : H \to \mathbb{R}$ is strictly $H$-differentiable (i.e., locally Lipschitz continuous and strictly $G$-differentiable) and hence $\partial g(x) = \{g'(x)\}$ by Proposition 2.8.

(ii) $\|g'(x)\|$ is bounded, $g'$ is compact and $\nu_\infty = \dim H^0_\infty < \infty$.

(iii) For any $M > 0$, $g'(u^0 + u^\pm) \to 0$ uniformly in $u^\pm \in \bar{B}_H(\theta, M) \cap H_\infty^\pm$ as $\|u^0\| \to \infty$.

Then $\mathcal{L}(u) = \frac{1}{2}(B(\infty)u, u)_H + g(u)$ satisfies (PS) condition on $H \setminus C_{R,M}$, where

$$C_{R,M} = \{u = u^0 + u^\pm \ | \ \|u^0\| > R, \|u^\pm\| < M\}.$$

Consequently, for any $(PS)_c$ sequence $\{u_n\}$ of $\mathcal{L}$, either $\{u_n\}$ has a bounded subsequence (and hence a converging subsequence) or $c \in C_\infty(\mathcal{L})$ and there exists a subsequence $\{u_{n_k}\}$ such that $\|u^0_{n_k}\| \to \infty$, $\|u^\pm_{n_k}\| \to 0$ and $g(u_{n_k}) \to c$. Here $C_\infty(\mathcal{L})$ is a closed subset of $\mathbb{R}$ given by

$$C_\infty(\mathcal{L}) := \{c \in \mathbb{R} \ | \ \exists u^0_n \in H^0_\infty, u^\pm_n \in H^\pm_\infty \text{ with } \|u^0_n\| \to \infty, \|u^\pm_n\| \to 0 \text{ such that } g(u^0_n + u^\pm_n) \to c\}.$$

Consequently, $\mathcal{L}$ satisfies the $(PS)_c$ condition for $c \notin C_\infty$. 


Proof. Let \( \{u_n\} \subset H \setminus C_{R,M} \) be such that \( \mathcal{L}(u_n) \to c \) and \( B(\infty)u_n + g'(u_n) \to 0 \) as \( n \to \infty \). Since \( \|g'(x)\| \) is bounded, and \( \|u_n^\pm\| \leq \|B(\infty)\|_{H^\infty} \cdot \|B(\infty)u_n^\pm\| \) we infer that \( \{\|u_n^\pm\|\} \) is bounded. Let \( \|u_{n_k}^\pm\| \leq M_1 \forall n \). Suppose that a subsequence \( \|u_{n_k}^0\| \to \infty \). Then \( g'(u_{n_k}^0 + u_{n_k}^\pm) \to 0 \) and so
\[
\|u_{n_k}^\pm\| \leq \|B(\infty)\|_{H^\infty} \cdot \|B(\infty)u_{n_k}^\pm\| = \|B(\infty)\|_{H^\infty} \cdot \|\mathcal{L}'(u_{n_k}) - g'(u_{n_k}^0 + u_{n_k}^\pm)\| \to 0.
\]
Hence \( u_{n_k} = u_{n_k}^0 + u_{n_k}^\pm \in C_{R,M} \) for \( k \) large enough. This contradiction shows that \( \{\|u_n^0\|\} \) is bounded. Since \( g' \) is compact and \( \nu_\infty = \dim H^0_{\infty} < \infty \) we have a subsequence \( \{u_{n_k}\} \) such that \( u_{n_k}^0 \to u^0 \) and \( g'(u_{n_k}) \to v \). The latter implies that
\[
\|u_{n_k}^\pm - u_{n_k}^\pm\| \leq \|B(\infty)\|_{H^\infty} \cdot \|B(\infty)u_{n_k}^\pm - B(\infty)u_{n_k}^\pm\| = \|B(\infty)\|_{H^\infty} \cdot \|\mathcal{L}'(u_{n_k}) - \mathcal{L}'(u_{n_k}) - [g'(u_{n_k}) - g'(u_{n_k})]\| \to 0
\]
as \( k, l \to \infty \). Hence \( \{u_{n_k}\} \) converges to some \( v \). \( \square \)

3 Computations of critical groups

3.1 Critical groups at infinity and computations

In this subsection \( \mathbb{K} \) always denotes a commutative ring without special statements. For a strictly \( H \)-differentiable functional \( f : X \to \mathbb{R} \) on a Banach space \( X \), suppose that the set of critical values of \( f \) is strictly bounded from below by \( a \in \mathbb{R} \), and for all \( c \leq a \) that \( f \) satisfies the condition \((C)_c\). By Corollary 2.4(i), for every nonnegative integer \( m \),
\[
C_m(f, \infty; \mathbb{K}) := H_m(X, f^a; \mathbb{K}),
\]
\[
\tilde{C}_m(f, \infty; \mathbb{K}) := H_m(X, f^a; \mathbb{K})
\]
are independent of the choices of such \( a \), and are called the \( m \)th critical group of \( f \) at infinity and \( m \)th cohomological critical group of \( f \) at infinity, respectively (cf. Definition 3.4 of [23]). Here \( H_*(-; \mathbb{K}) \) and \( \tilde{H}^*(-; \mathbb{K}) \) denote the singular homology and cohomology with coefficients in \( \mathbb{K} \). It is well-known that
\[
C_m(f, \infty; \mathbb{K}) = H_m(X, f^a; \mathbb{K}) \cong \text{Hom}(H_m(X, f^a; \mathbb{K})) \]
\[
\cong H^m(X, f^a; \mathbb{K}) = \tilde{C}_m(f, \infty; \mathbb{K})
\]
if \( \mathbb{K} \) is a field. Let \( \tilde{H}^*(-; \mathbb{K}) \) denote Alexander-Spanier cohomology with coefficients in \( \mathbb{K} \), which has often some stronger excision and continuity properties. Now the Banach space \( X \) is an ANR (absolute neighborhood retract). By Section K on the page 30 of [25], every open subset of an ANR an ANR, and Hanner theorem claims that a metrizable space is an ANR if it has a countable open covers consisting of
ANR. Hence \( f^a = \cup_{n=1}^{\infty} \{ f < a - \frac{1}{n} \} \) is an ARN. From Section 9 of [18], Chapter 6 it follows that \( H^m(X, f^a; K) \approx H^m(X, f^a; K) \) for any field \( K \). In particular we have

\[
C_m(f, \infty; K) \approx C_m(f, \infty; K) \approx H^m(X, f^a; K)
\]

(3.3)

for any field \( K \) and nonnegative integer \( m \). These and Proposition 3.15 of [11] lead to

**Proposition 3.1** For a strictly \( H \)-differentiable functional \( f : X \to \mathbb{R} \) on a Banach space \( X \), suppose that the set of critical values of \( f \) is strictly bounded from below by \( a \in \mathbb{R} \), and that \( f \) satisfies the condition \((C)_c \) for all \( c \in \mathbb{R} \). Then for any field \( K \) and nonnegative integer \( m \) it holds

(i) \( C_m(f, \infty; K) \approx C_m(f, \infty; K) \approx \delta_{m0}K \) if \( f \) is bounded from below.

(ii) \( C_m(f, \infty; K) \approx C_m(f, \infty; K) \approx H^{m-1}(f^a; K) \) if \( f \) is unbounded from below. Here \( H^{m}(f^a; K) = H^m(f^a; K) / K \) and \( H^q(f^a; K) = H^q(f^a; K) \) for \( q \geq 1 \).

By Proposition B.1 of [36], the continuously directional differentiability is stronger than the strict \( H \)-differentiability. We have the following generalization of Theorem 3.9 in [5].

**Theorem 3.2** Suppose for \( V_\infty = H \):

(i) the assumptions of Theorem 4.1 of [36], \((S), (F1_\infty)-(F3_\infty)\) and \((C1_\infty)-(C2_\infty)\), \((D_\infty)\) and \((E'_\infty)\), are satisfied;

(ii) \( \mathcal{L}(u) = \frac{1}{2}(B(\infty)u, u)_H + o(\|u\|^2) \) as \( \|u\| \to \infty \);

(iii) \( \nabla \mathcal{L}(u) = B(\infty)u + o(\|u\|) \) as \( \|u\| \to \infty \), where \( \nabla \mathcal{L} \) is the gradient of \( \mathcal{L} \) defined by \( d\mathcal{L}(u)(v) = (\nabla \mathcal{L}(u), v)_H \) for all \( u, v \in H \); (Note: we do not assume \( \mathcal{L} \in C^1(H, \mathbb{R}) \).)

(iv) the critical values of \( \mathcal{L} \) are bounded below;

(v) \( \mathcal{L} \) satisfies the condition \((C)_c \) (or \((D)_c \)) for \( c \ll 0 \).

Then \( C_k(\mathcal{L}, \infty; K) = 0 \) for \( k \in [\mu_\infty, \mu_\infty + \nu_\infty] \) even if \( \mu_\infty = \infty \) or \( \nu_\infty = \infty \). Moreover, if \( \mu_\infty \ll \infty \) and \( \nu_\infty = 0 \) then \( C_{\mu_\infty}(\mathcal{L}, \infty; K) \not\cong K \) (even if \( H \) is not complete).

**Proof.** Step 1. Carefully checking the proof of Lemma 4.2 in [5] one easily sees that the conditions (i) and (ii) imply: for sufficiently large \( R > 0 \) and \( a \ll 0 \) the pair

\[
(B_{H^0_\infty}(\theta, R + 1) \oplus H^\pm_\infty, \mathcal{L}^0 \cap (B_{H^0_\infty}(\theta, R + 1) \oplus H^\pm_\infty))
\]

is homotopy to the pair

\[
(B_{H^0_\infty}(\theta, R + 1) \oplus \bar{B}_{H^0_\infty}(\theta, 1), B_{H^0_\infty}(\theta, R + 1) \oplus \partial \bar{B}_{H^0_\infty}(\theta, 1)).
\]

The homotopy equivalence leaves the \( H^0_\infty \)-component fixed.\footnote{For a possible method removing this condition, see below the end of this document.}
Step 2. Under the assumption (i), by Theorem 4.1 of \[36\] we can get Lemma 4.3 of \[5\]: There exist a sufficiently large $R > 0$, $a \ll 0$ and a continuous map $\gamma : B_{H_0}^{\infty}(\infty, R) \rightarrow [0, 1]$ with $\gamma(C) > 0$ for $C := B_{H_0}^{\infty}(\theta, R + 1) \cap B_{H_0}^{\infty}(\infty, R)$ such that the pair

$$(B_{H_0}^{\infty}(\infty, R) \times H_0^{\infty}, \mathcal{L}^a \cap (B_{H_0}^{\infty}(\infty, R) \times H_0^{\infty}))$$

is homotopy equivalent to the pair $(B_{H_0}^{\infty}(\infty, R) \times H_0^{-\infty}, \Gamma)$, where

$$\Gamma = \{(z, u) \in B_{H_0}^{\infty}(\infty, R) \times H_0^{-\infty} : \|u\| \geq \gamma(z)\}$$

and $\gamma(z) = \begin{cases} 
0 & \text{if } \mathcal{L}(z + h_0^\infty(z)) \leq a, \\
1 & \text{if } \mathcal{L}(z + h_0^\infty(z)) \geq a + 1,
\end{cases}$$

Moreover, the homotopy equivalence leaves the $H_0^{\infty}$-component fixed.

Step 3. By the assumptions (iv) and (v), $C_*(\mathcal{L}, \infty; K) = H_*(H, \mathcal{L}^a; K)$ for $a \ll 0$ is well-defined. Using Step 1 and Step 2 we may repeat the proof on the pages 428-429 of \[5\] to obtain at the desired conclusion. \(\Box\)

Using Corollary \[2.4\] we derive (i) and (ii) the following proposition, which are corresponding with Propositions 3.5 and 3.6 in \[5\].

**Proposition 3.3** For a strictly $H$-differentiable functional $f : X \rightarrow \mathbb{R}$ on a Banach space $X$, we have

(i) If $a < \inf f(K(f)) \leq \sup f(K(f)) < b$ and $f$ satisfies the condition $(C)_c$ (or $(D)_c$) for any $c \notin (a, b)$, then $C_*(f, \infty; K) \cong H_*(f^b, f^a; K)$.

(ii) If $f$ satisfies the condition $(C)_c$ (or $(D)_c$) for any $c \in \mathbb{R}$, then $C_*(f, \infty; K) \cong H_*(f^b, f^a; K)$.

(iii) If $f$ is $F$-differentiable, satisfies the condition $(PS)_c$ for every $c \in \mathbb{R}$ and has finite critical points, then for every field $K$ it holds that

$$\dim C_m(f, \infty; K) \leq \sum_{u \in K(f)} \dim C_m(f, u; K) \quad \forall m \in \mathbb{N} \cup \{0\}.$$
where \(a_{k+1} = b\). From the proof of Theorem 4 in \cite{17} we may see that \(f^{a_1^0}\) is a strong deformation retract of \(f^b\). This implies \(\bar{H}^m(f^{b_0}, f^{a_1^0}; \mathbf{K}) \cong \bar{H}^m(X, f^{a_1}; \mathbf{K})\), and so

\[
\dim \bar{H}^m(f^{b_0}, f^{a_1^0}; \mathbf{K}) = \dim C_m(f, \infty; \mathbf{K}) \quad \forall m.
\]

Now the part I of Lemma \ref{2.6} yields

\[
\bar{H}^m(f^{a_{i+1}^0}, f^{a_i^0}; \mathbf{K}) \cong \bar{H}^m(f^{a_{i+1}^0}, f^{c_i^0}; \mathbf{K}) \cong \bar{H}^m(f^{c_i^0} \cup K(f)_{c_i}, f^{c_i^0}; \mathbf{K})
\]

for any \(i = 1, \ldots, k\) and nonnegative integer \(m\). As showed in \cite{17} Remark 2 the subsets \(f^{c_i^0} \cup K(f)_{c_i}\) and \(f^{c_i^0}\) are ARN. Hence

\[
\bar{H}^m(f^{c_i^0} \cup K(f)_{c_i}, f^{c_i^0}; \mathbf{K}) \cong \bar{H}^m(f^{c_i^0} \cup K(f)_{c_i}, f^{c_i^0}; \mathbf{K}) \quad \forall m, i.
\]

Let \(K(f)_{c_i} = \{x_{1i}, \ldots, x_{di}\}\), and \(U_{1i}, \ldots, U_{di}\) be mutually disjoint open neighborhoods of \(x_{1i}, \ldots, x_{di}\), respectively. Then the excision property of singular cohomology groups lead to

\[
\bar{H}^m(f^{c_i^0} \cup K(f)_{c_i}, f^{c_i^0}; \mathbf{K}) \cong \bigoplus_{j=1}^{l_i} \bar{H}^m((f^{c_i^0} \cup \{x_{ij}\}) \cap U_{ij}, f^{c_i^0} \cap U_{ij}; \mathbf{K})
\]

for all \(m, i\). Moreover, it is easy to prove

\[
\bar{H}^m((f^{c_i^0} \cup \{x_{ij}\}) \cap U_{ij}, f^{c_i^0} \cap U_{ij}; \mathbf{K}) \cong \bar{H}^m(f^{c_i} \cap U_{ij}, (f^{c_i} \setminus \{x_{ij}\}) \cap U_{ij}; \mathbf{K}) \\
\cong \bar{H}^m(f^{c_i} \cap U_{ij}, (f^{c_i} \setminus \{x_{ij}\}) \cap U_{ij}; \mathbf{K}) \\
\cong C_m(f, x_{ij}; \mathbf{K}).
\]

The desirable conclusion follows from these immediately. \(\Box\)

We can also give generalizations of some computation results on critical groups at infinity such as Proposition 3.10 in \cite{5} and some parts of \cite{31}. We leave them intersecting readers.

### 3.2 Computations of critical groups at degenerate critical points

**Definition 3.4** Let \(X\) be a Banach space and let \(f : X \to \mathbb{R}\) be a strictly \(H\)-differentiable functional. For subsets \(S \subset X\) and \(A \subset X^*\) we call \(f\) to satisfy \((PS)\) condition with respect to \(A\) on \(S\) if every sequence \((x_n)\) in \(S\) with \((f(x_n))\) bounded and \(f'(x_n) \to y \in A\) has a convergent subsequence.

Clearly, the usual \((PS)\) condition is the \((PS)\) condition with respect to \(\{0\}\) on \(X\). We have the following generalization of Proposition 2.5 in \cite{5}.

**Proposition 3.5** (i) Let \(X \subset H\) be as in \((S)\) of \cite{36} §2.1, and let \(\mathcal{L}\) be a continuously directional differentiable functional defined in an open neighborhood \(V\) of \(x_0 \in X\) in \(H\); moreover we assume that the conditions \((F1)-(F3)\), \((C1)-(C2)\) and \((D)\) in \cite{36} §2.1 hold with \(\theta\) replaced by \(x_0\).
(ii) $x_0$ is an isolated critical point of $\mathcal{L}$, and $\mathcal{L}$ is $F$-differentiable near $x_0$.

(iii) $\nabla \mathcal{L}(u) = B(x_0)(u - x_0) + o(||u - x_0||)$ as $||u - x_0|| \to 0$, where $\nabla \mathcal{L}$ is the gradient of $\mathcal{L}$ defined by $d\mathcal{L}(u)(v) = (\nabla \mathcal{L}(u), v)_H$ for all $u, v \in H$. (Note: we do not assume $\mathcal{L} \in C^1(H, \mathbb{R})$.)

(iv) $\mathcal{L}$ satisfies the $(PS)$ condition with respect to $H^0$ on a closed ball $\bar{B}_H(x_0, \delta)$.

Let $\mu_0$ and $\nu_0$ be the Morse index and nullity of $\mathcal{L}$ at $x_0$. Then we have:

(a) $C_k(\mathcal{L}, x_0; K) = \delta_{k\mu_0} K$ provided that $\mathcal{L}$ also satisfies:
   
   (AC)* There exist $\varepsilon > 0$ and $\theta \in (0, \pi/2)$ such that $(\nabla \mathcal{L}(u + x_0), u^0)_H \geq 0$ for any $u = u^0 + u^\pm \in H^0 + H^\pm$ with $||u|| \leq \varepsilon$ and $||u^\pm|| \leq ||u|| \cdot \sin \theta$.

(b) $C_k(\mathcal{L}, x_0; K) = \delta_{k(\mu_0 + \nu_0)} K$ provided that $\mathcal{L}$ also satisfies:
   
   (AC) There exist $\varepsilon > 0$ and $\theta \in (0, \pi/2)$ such that $(\nabla \mathcal{L}(u + x_0), u^0)_H \leq 0$ for any $u = u^0 + u^\pm \in H^0 + H^\pm$ with $||u|| \leq \varepsilon$ and $||u^\pm|| \leq ||u|| \cdot \sin \theta$.

By Corollary 2.6 of [36] we have $C_q(\mathcal{L}, x_0; K) = 0$ if $q \notin [\mu_0, \mu_0 + \nu_0]$. So Proposition 3.5 may be viewed a refinement of this result. For the proof of it we also need the following stability property of critical groups for continuous functionals by Cingolani and Degiovanni [13], which is a very general generalization of the previous results due to Chang [9] page 53, Th.5.6, Chang and Ghoussoub [12] and in Mawhin and Willem [38, Th.8.8], and Corvellec and Hantoute [19].

**Theorem 3.6** ([13] Th.3.6) Let $\{f_t : t \in [0, 1]\}$ be a family of continuous functions from a metric space $X$ to $\mathbb{R}$, let $U$ be an open subset of $X$ and $[0, 1] \ni t \mapsto u_t \in U$ a continuous map. Assume:

(I) if $t_k \to t$ in $[0, 1]$, then $f_{t_k} \to f_t$ uniformly on $U$;

(II) $U$ is complete, and for every sequence $t_k \to t$ in $[0, 1]$ and $(v_k)$ in $U$ with $|df_{t_k}|(t_k) \to 0$ and $(f_{t_k}(v_k))$ bounded, there exists a subsequence $(v_{k_j})$ convergent to some $v$ with $|df|(v) = 0$;

(III) $|df_t|(v) > 0$ for every $t \in [0, 1]$ and $v \in U \setminus \{u_t\}$

Then $C_q(f_0, u_0; K) \cong C_q(f_1, u_1; K)$ for every $q \geq 0$.

**Proof of Proposition 3.5** Following the proof ideas of Proposition 2.5 in [5], we assume $x_0 = \theta$. For the case (a) (resp. (b)) we set $\mathcal{L}_t(u) = \mathcal{L}(u) + \frac{1}{2}t||u^0||^2$ (resp. $\mathcal{L}_t(u) = \mathcal{L}(u) - \frac{1}{2}t||u^0||^2$) for $t \in [0, 1]$. Using the assumptions (ii) and (iv), as in the proof of Proposition 2.5 in [5] we have a small $\varepsilon \in (0, 2/\delta)$ such that $\theta$ is the only critical point of each $\mathcal{L}_t$ in $B_H(\theta, 2\varepsilon)$. Clearly, $\mathcal{L}_1$ also satisfies the assumption of Theorem 2.1 of [36], and $\theta$ is a nondegenerate critical point of $\mathcal{L}_1$ with Morse index $\mu_0$ (resp. $\mu_0 + \nu_0$) in the case (a) (resp. (b)). It follows from Corollary 2.6 of [36] that

$$C_k(\mathcal{L}, \theta) = \delta_{k\mu_0} K \text{ in case (a) } \quad \text{resp. } C_k(\mathcal{L}, \theta) = \delta_{k(\mu_0 + \nu_0)} K \text{ in case (b)}. \quad (3.4)$$

The remaining is to prove $C_q(\mathcal{L}, \theta) \cong C_q(\mathcal{L}_1, \theta)$ for every $q \geq 0$ in both cases.
We only prove the case (a). Since $\mathcal{L}$ is continuously directional differentiable, and $F$-differentiable at $\theta$, so is each $\mathcal{L}_t$. By Proposition B.2(ii) of [36] and Proposition 2.1 every $\mathcal{L}_t$ is strictly $H$-differentiable (and thus locally Lipschitz continuous), and
\[
\partial \mathcal{L}_t(u) = \{ \mathcal{L}_t'(u) \} \quad \text{and} \quad |d\mathcal{L}_t|(u) = \| \mathcal{L}_t'(u) \| \quad \forall u \in B_H(\theta, \epsilon) \tag{3.5}
\]
and $|d\mathcal{L}_t|(u) > 0$ $\forall u \in B_H(\theta, \epsilon) \setminus \{ \theta \}$ because $\theta$ is only critical point of $\mathcal{L}_t$ in $B_H(\theta, 2\epsilon)$.

The second equality in (3.5) implies that $\theta$ is also a lower critical point of each $\mathcal{L}_t$. Because of (3.3), we hope to use Theorem 3.6 proving that $C_q(\mathcal{L}, \theta) \cong C_q(\mathcal{L}_1, \theta)$ $\forall q \geq 0$. It suffice to check the conditions of Theorem 3.6. Clearly, $\mathcal{L}_{t_k} \to \mathcal{L}_t$ uniformly on $B_H(\theta, \epsilon)$ as $t_k \to t$ in $[0, 1]$. Now we assume: $t_k \to t$ in $[0, 1]$, $(u_k) \subset B_H(\theta, \epsilon)$ is such that $(\mathcal{L}_{t_k}(u_k))$ is bounded and $|d\mathcal{L}_{t_k}|(u_k) \to 0$. These imply that $\mathcal{L}_{t_k}'(u_k) = \mathcal{L}'(u_k) + t_k u_k^0$ $\to \theta$. Since $\dim H^0 < \infty$ we may assume $u_k^0 \to u^0$ (passing a subsequence if necessary). Then $(\mathcal{L}(u_k))$ is bounded and $\mathcal{L}'(u_k) \to -tu^0 \in H^0$. By the assumption (iv) $(u_k)$ has a convergent subsequence $u_{k_i} \to u_0 \in B_H(\theta, \epsilon)$. Hence $u^0 = P^0 u_0$. Moreover, since $\mathcal{L}$ is continuously directional differentiable we get
\[
(\mathcal{L}'(u_k), v)_H \to (\mathcal{L}'(u_0), v)_H \quad \forall v \in H.
\]
Hence $(-tu^0, v)_H = (\mathcal{L}(u_0), v)_H \forall v \in H$, i.e. $\mathcal{L}'(u_0) + tP^0 u^0 = \mathcal{L}'(u_0) + tu^0 = \theta$. It follows from (3.5) that $|d\mathcal{L}_t|(u_0) = \| \mathcal{L}_t'(u_0) \| = 0$. Namely $\{ \mathcal{L}_t \mid t \in [0, 1] \}$ satisfies the conditions of Theorem 3.6 on $B_H(\theta, \epsilon)$. □

4 Morse inequalities and some critical point theorems

4.1 Morse inequalities

In this subsection, unless otherwise specified, let the functional $\mathcal{L} : H \to \mathbb{R}$ be as in Proposition 2.9. Then $\mathcal{L}$ satisfies the $(PS)_c$ condition for each $c \not\in C_\infty(\mathcal{L})$. We also assume that $\mathcal{L}$ is $C^1$ so that it has a pseudo-gradient vector field, $V : \tilde{H} \to H$. Note that $\tilde{H} \to H$, $u \mapsto V(u)/\|V(u)\|$ is also a locally Lipschitz continuous map. Consider the flow
\[
\dot{\eta}(t, u) = -\frac{V(\eta(t, u))}{\|V(\eta(t, u))\|} \quad \text{and} \quad \eta(t, 0) = u. \tag{4.1}
\]
Our goal is to present the Morse inequality established in [5] [24] under the above weaker setting. For $F \subset H$ let $\tilde{F} = \bigcup_{t \in \mathbb{R}} \eta(t, F)$.

For any isolated value $c$ in $C_\infty(\mathcal{L})$, let $\mathcal{L}^{c+\varepsilon}_{c-\varepsilon} := \mathcal{L}^{-1}([c-\varepsilon, c+\varepsilon])$ and $K^{c+\varepsilon}_{c-\varepsilon}(\mathcal{L}) := K(\mathcal{L}) \cap \mathcal{L}^{c+\varepsilon}_{c-\varepsilon}$. Define
\[
U_{R,M} = \{ u = u^0 + u^\pm \mid \| u \| \leq R \} \cup \{ u = u^0 + u^\pm \mid \| u \| > R, \| u^\pm \| \geq M \},
\]
\[
C_{R,M} = \{ u = u^0 + u^\pm \mid \| u \| > R, \| u^\pm \| < M \} = H \setminus U_{R,M},
\]
\[
U_{R,M}^{c+\varepsilon} = U_{R,M} \cap \mathcal{L}^{c+\varepsilon}, \quad C_{R,M}^{c+\varepsilon} = C_{R,M} \cap \mathcal{L}^{c+\varepsilon},
\]
\[
A_{R,M}^{c+\varepsilon} = U_{2R,M/2}^{c+\varepsilon} \cap C_{R,M}^{c+\varepsilon}.
\]
The following lemma corresponds to Lemma 2.1, Proposition 2.2 and Corollary 2.4 in [24] (see also Lemma 2.7 and Theorem 2.9 in [30]).

**Lemma 4.1** Assume that \( K(\mathcal{L})_{c+\varepsilon_0} = K(\mathcal{L})_c \) is compact for some \( \varepsilon_0 > 0 \). Then for \( R \) large and \( R > M > 0 \) with \( K(\mathcal{L})_c \subset B_H(\theta, R/2) \cup C_{3R,M/8} \) there exists \( \varepsilon_1 > 0 \) such that for any \( \varepsilon < \varepsilon_1 \) it holds that

(i) \( (\mathcal{L}_{c+\varepsilon} \cap U_{R,M/c}) \cap (\mathcal{L}_{c+\varepsilon} \cap C_{2R,M/4}) = \emptyset, \)

(ii) \( \mathcal{L}_{c+\varepsilon} \cap A_{R,M} = \mathcal{L}_{c+\varepsilon} \cap A_{R,M}. \)

Furthermore, if \( K(\mathcal{L})_c \) is compact then for any \( M > 0 \) there exist a large \( R > 0 \), and \( \varepsilon_1 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_1), \)

\[ H_q(\mathcal{L}_{c+\varepsilon}, \mathcal{L}_{c-\varepsilon}; K) \cong H_q(\mathcal{L}_{c+\varepsilon} \cap U_{2R,M/2}; \mathcal{L}_{c-\varepsilon} \cap U_{2R,M/2}; K) \]

\[ \oplus H_q(\mathcal{L}_{c+\varepsilon} \cap C_{R,M}; \mathcal{L}_{c-\varepsilon} \cap C_{R,M}; K) \quad \forall q = 0, 1, \ldots. \]

Hereafter \( K \) always denotes a commutative ring without special statements.

**Proof.** Since \( U_{3R,M/8} \setminus B_H(\theta, R/2) = H \setminus (B_H(\theta, R/2) \cup C_{3R,M/8}) \) is disjoint with \( K(\mathcal{L})_{c+\varepsilon_0} = K(\mathcal{L})_c \), and \( \mathcal{L} \) satisfies the (PS) condition in \( U_{3R,M/8} \) by Proposition 2.9, there exists an \( \varepsilon' > 0 \) such that

\[ \| \mathcal{L}'(u) \| \geq \| \mathcal{L}'(u) \| \geq \varepsilon' \quad \forall u \in \mathcal{L}_{c+\varepsilon_0} \cap (U_{3R,M/8} \setminus B_H(\theta, R/2)). \]

For \( 0 < \varepsilon < \min\{\varepsilon_0, \varepsilon'M/16\} \), suppose that \( \eta(s, x) \in \mathcal{L}_{c+\varepsilon} \cap C_{R+M/4,3M/4} \) for some \( x \in \mathcal{L}_{c-\varepsilon} \cap U_{R,M} \) and \( s > 0 \). Then there exists \( t_1 < t_2 \) such that

\( \eta(t_1, x) \in \mathcal{L}_{c+\varepsilon} \cap \partial U_{R,M}, \quad \eta(t_2, x) \in \mathcal{L}_{c+\varepsilon} \cap \partial U_{R+M/4,3M/4} \subset \mathcal{L}_{c+\varepsilon} \cap \partial U_{2R,M/2}, \)

\( \eta(t, x) \in \mathcal{C}_{R,M} \cup U_{R+M/4,3M/4} \forall t \in [t_1, t_2]. \)

Then \( M/4 \leq \| \eta(t_1, x) - \eta(t_2, x) \| \leq |t_2 - t_1|, \) and

\[ \mathcal{L}(\eta(t_2, x)) - \mathcal{L}(\eta(t_1, x)) = \int_{t_1}^{t_2} \frac{d}{dt} \mathcal{L}(\eta(t, x))dt \]

\[ = -\int_{t_1}^{t_2} \frac{\langle \mathcal{L}'(\eta(t, x)), V(\eta(t, x)) \rangle}{\| V(\eta(t, x)) \|} dt \]

\[ \leq -\frac{\varepsilon'}{2} |t_2 - t_1|. \]

Hence

\[ \varepsilon \frac{M}{4} \leq \varepsilon' |t_2 - t_1| \leq 2(\mathcal{L}(\eta(t_1, x)) - \mathcal{L}(\eta(t_2, x))) \leq 4\varepsilon. \]

This contradicts to the choice of \( \varepsilon \). So we get

\( (\mathcal{L}_{c+\varepsilon} \cap U_{R,M}) \cap (\mathcal{L}_{c+\varepsilon} \cap C_{R+M/4,3M/4}) = \emptyset. \)
Since \((L_{c+\varepsilon}^c \cap U_{2R,M}^c) \subset (L_{c-\varepsilon}^c \cap U_{R,M})\),

\[
(L_{c+\varepsilon}^c \cap U_{2R,M}^c) \cap (L_{c-\varepsilon}^c \cap C_{R+M/4,3M/4}) = \emptyset. \tag{4.2}
\]

Similarly, for \(0 < \varepsilon < \min\{\varepsilon_0, \varepsilon'M/16\}\) we have

\[
(L_{c+\varepsilon}^c \cap C_{R+M/4}^c) \cap (L_{c-\varepsilon}^c \cap U_{2R-M/4,2M/2}) = \emptyset.
\]

This and (4.2) together give (i).

As in the proof of [24, 30] we can get the remaining conclusions. □

Following [5 24], as in [9] and [38] Lemma 4.1 may lead to

**Theorem 4.2** Under the assumptions of Proposition 2.4, let \(L\) be \(C^1\), and let \(K(L)\) and \(C_\infty(L)\) be finite. Denote by \(\beta_k(f, x) := \dim C_k(f, x; K)\) for \(x \in K(L)\), and by

\[
\beta_k(L, c) = \dim H_k(L^{c+\varepsilon} \cap \tilde{C}_{R,M}, L^{c-\varepsilon} \cap \tilde{C}_{R,M}; K) \quad \forall c \in C_\infty(L),
\]

\[
P(L, \infty) := \sum_{k=0}^\infty \dim H_k(H, L^a; K)t^k
\]

for any \(a < \min\{x \mid x \in L(K(L)) \cup C_\infty(L) \cup \{0\}\}\). Then there exists a polynomial \(Q(t)\) with nonnegative integer coefficients such that

\[
P(L, \infty) + (1 + t)Q(t) = \sum_{x \in K(f)} P(L, x) + \sum_{c \in C_\infty(f)} P(L, c),
\]

where \(P(L, x) := \sum_{k=0}^\infty \beta_k(L, x)t^k\) and \(P(L, c) := \sum_{k=0}^\infty \beta_k(L, c)t^k\)

### 4.2 Some critical point theorems

Many critical point theorems, which were obtained by computations of critical groups, can be generalized with our methods. For example, the following is a generalization of Theorem 5.1 on the page 121 of [9].

**Theorem 4.3** (I) Let the Banach space \((X, \| \cdot \|)\) and the Hilbert space \((H, (\cdot, \cdot)_H)\) satisfy the condition (S) in [26 §2.1]. Let \(\tilde{H}\) (resp. \(\tilde{X}\)) be a \(C^1\) Hilbert (resp. \(C^2\) Banach) manifold modeled on \(H\) (resp. \(X\)). Suppose that \(\tilde{X} \subset \tilde{H}\) is dense in \(\tilde{H}\), and that for each point \(p \in \tilde{X}\) there exists a coordinate chart around \(p\) on \(\tilde{H}\), \(\Phi_p : U_p \to \Phi_p(U_p) \subset H\) with \(\Phi_p(p) = \theta\), such that it restricts to a coordinate chart around \(p\) on \(\tilde{H}\), \(\tilde{\Phi}_p : U_p \cap \tilde{X} \to \tilde{\Phi}_p(U_p \cap \tilde{X}) \subset X\).

(II) Let \(L : \tilde{H} \to \mathbb{R}\) be a continuously directional differentiable, and \(F\)-differentiable functional with the following properties.

(a) \(L\) satisfies the \((PS)\) condition, and restricts to a \(C^2\)-functional on \(\tilde{X}\);
(b) \(\text{rank}H_k(L^b, L^a; K) \neq 0\) for some \(k \in \mathbb{N}\) and regular values \(a < b\);
(c) \(\exists\) a finite set \(\{p_1, \ldots, p_m\} \subset \tilde{X}\) is contained in \(K(L) \cap L^{-1}[a, b]\).
(d) Around each \( p_i \) there exists a chart \( \Phi_{p_i} : U_{p_i} \to \Phi_{p_i}(U_{p_i}) \subset H \) as in (I) such that the functional \( \mathcal{L} \circ (\Phi_{p_i})^{-1} : \Phi_{p_i}(U_{p_i}) \to \mathbb{R} \) satisfies the conditions of Theorem 2.1 of [36]; so \( p_i \) has Morse index \( \mu_i \) and nullity \( \nu_i \);

(f) Either \( \mu_i > k \) or \( \mu_i + \nu_i < k \), \( i = 1, \cdots, m \).

Then \( \mathcal{L} \) has at least one more critical point \( p_0 \) with \( \text{rank} C_k(\mathcal{L}, p_0; K) \neq 0 \).

**Proof.** By the condition (II), the lower critical point set of \( \mathcal{L} \) coincides with \( K(\mathcal{L}) \), and \( \mathcal{L} \) satisfies the (PS) condition for continuous functionals. If the conclusion is not true then \( K(\mathcal{L}) = \{ p_1, \cdots, p_m \} \). By Corollary 2.6 of [36] and (f) we have \( C_k(\mathcal{L}, p_i) = 0, i = 1, \cdots, m \). As in the proof of Theorem 5.1 on [9, page 121] we may use Corvellec’s Morse theory for continuous functionals [16] to obtain a contradiction. \( \square \)

Similarly, a suitable weaker version of Theorem 5.4 on [9, page 121] may be given. In particular, by Theorem 2.12 of [36] we have the following generalization of Corollary 5.3 therein.

**Theorem 4.4** Under the assumptions of Theorem 2.1 of [36], suppose \( V = H \) and the following conditions hold:

(i) \( \mathcal{L} \) is bounded below, \( F \)-differentiable and satisfies the (PS) condition;

(ii) For a small \( \epsilon > 0 \), either \( \deg_{BS}(\nabla \mathcal{L}, B_H(\theta, \epsilon), \theta) = \pm 1 \), or \( \deg_{FPR}(A, B_X(\theta, \epsilon), \theta) = \pm 1 \) provided that the map \( A \) in the condition (F2) is \( C^1 \) near \( \theta \in X \), where the degrees \( \deg_{BS} \) and \( \deg_{FPR} \) are as in Theorem 2.12 of [36].

Then \( \mathcal{L} \) has at least three critical points.

Finally, we give a generalization of Theorem 3.12 in [5]. To this goal we also need the following results, which are Propositions 2.3 and 3.8 in [5].

**Proposition 4.5** Let a normed vector space \( X \) have a direct sum decomposition \( X = X_1 \oplus X_2 \), where \( k = \dim X_2 < \infty \). For \( f \in C(X, \mathbb{R}) \) we have:

(i) If there exist \( x_0 \in X \) and \( \epsilon > 0 \) such that \( f(x_0 + x) > f(x_0) \forall x \in H_1, 0 < \|x\| \leq \epsilon \), and that \( f(x_0 + x) \leq f(x_0) \forall x \in X_2, \|x\| \leq \epsilon \), then \( C_k(f, x_0) \neq 0 \). (32)

(ii) If \( f \) is bounded from below on \( X_1 \), and \( f(x) \to -\infty \) for \( x \in X_2 \) as \( \|x\| \to \infty \), then \( H_k(X, f^a) \neq 0 \) for \( a < \inf f|_{X_1} \). ([5, Proposition 3.8])

**Theorem 4.6** Under the assumptions (i)-(iv) of Theorem 3.2, let the condition (i) of Proposition 3.2 be satisfied. (Of course the corresponding densely imbedded Banach spaces in \( H \) are not necessarily same, we denote by \( A_\infty \) and \( B_\infty \) the corresponding maps in \( (i) \) of Theorem 3.2). Let \( H \) split as \( H = H^0 \oplus H^+ \oplus H^- \) (resp. \( H = H^0_\infty \oplus H^\infty_\infty \oplus H^-_\infty \)) according to the spectral decomposition of \( B(x_0) \) (resp. \( B_\infty(\infty) \)). Let \( \mu_0, \nu_0 \) (resp. \( \mu_\infty, \nu_\infty \)) be the Morse index and nullity at \( x_0 \) (resp. infinity).

(I) If (v) of Theorem 3.2 and the local linking condition as in Proposition 4.5(i) with \( X^- = H^- \) (resp. \( X^- = H^0 \oplus H^- \)) hold, then there exists a critical point different from \( x_0 \) provided \( \mu_0 \notin [\mu_\infty, \mu_\infty + \nu_\infty] \) (resp. \( \mu_0 + \nu_0 \notin [\mu_\infty, \mu_\infty + \nu_\infty] \)).
(II) If (ii)-(iv) and (AC$^+$) of Proposition 3.5 hold, and $\mathcal{L}$ is bounded from below on $H_0^0 \oplus H_\infty^\perp$, then there exists a nontrivial critical point provided $\mu_\infty \neq \mu_0$.

**Proof.** (I) Otherwise we have $K(\mathcal{L}) = \{x_0\}$. By Proposition 3.3(ii), $C_\ast(\mathcal{L}, \infty) \cong C_\ast(\mathcal{L}, x_0)$. Moreover, Proposition 1.5(i) implies $C_{\mu_0}(\mathcal{L}, x_0) \neq 0$. Hence $C_{\mu_0}(\mathcal{L}, \infty) \neq 0$. This gives a contradiction by Theorem 3.2.

(II) Similarly, assume $K(\mathcal{L}) = \{x_0\}$. By Proposition 3.3(ii), $C_\ast(\mathcal{L}, \infty) \cong C_\ast(\mathcal{L}, x_0)$. Proposition 3.5(a) yields $C_k(\mathcal{L}, x_0) = \delta_{k\mu_0} K$. By Proposition 4.5(ii), $C_{\mu_\infty}(\mathcal{L}, \infty) \neq 0$. This contradiction proves the desired conclusion. \(\square\)

5 Critical groups of sign-changing critical points

In this section we shall present the corresponding version of the results on critical groups of sign-changing critical points in [1] and [34] in our framework and sketch how to prove them with our results in the previous sections. It is also possible to generalize some of [2]. They are left to the interested reader.

Let the Hilbert space $(H, (\cdot, \cdot)_H)$ and the Banach space $(X, \| \cdot \|_X)$ satisfy the condition (S) as in [36, §2.1]. Let $P_H$ be a closed convex in $H$ so that $P \triangledown P_H \cap X$, a closed convex cone in $X$, satisfies:

(i) $\text{int}_X(P) \neq \emptyset$, (ii) $\exists \ e \in \text{int}_X(P)$ with $(u, e)_H > 0$ for all $u \in P \setminus \{\theta\}$. \hspace{1cm} (5.1)

Then $H$ (resp. $X$) is partially ordered by by $P_H$ (resp. $P$). For $u, v \in H$ (resp. $X$) we write: $u \geq v$ if $u - v \in P_H$ (resp. $u - v \in P$); $u > v$ if $u - v \in P_H \setminus \{\theta\}$ (resp. $u - v \in P \setminus \{\theta\}$). When $u, v \in X$ we also write $u \gg v$ if $u - v \in \text{int}_X(P)$. A map $f : H \to H$ (resp. $f : X \to X$) is called order preserving if $u \geq v \Rightarrow f(u) \geq f(v) \forall u, v \in H$ (resp. $X$). In particular, $f : X \to X$ is said to be strongly order preserving if $u > v \Rightarrow f(u) \gg f(v) \forall u, v \in X$.

Recall that above Proposition 2.1 we have showed that a continuously directional differentiable map is locally Lipschitz continuous and strictly G-differentiable.

Let the assumptions of Theorem 2.1 of [36] hold for $V = H$. We also assume:

(L$_0$) The assumptions of Proposition 2.26 of [36] hold for $V = H$, that is, the assumptions of Theorem 2.1 of [36] hold for $V = H$, the map $A : X \to X$ in the condition (F2) of [36] is Fréchet differentiable, and there exist positive constants $\eta'_0$ and $C'_2 > C'_1$ such that

$$C'_2 \|u\|^2 \geq (P(x)u, u) \geq C'_1 \|u\|^2 \quad \forall u \in H, \ \forall x \in B_H(\theta, \eta'_0) \cap X. $$

(L$_1$) $\mathcal{L} : H \to \mathbb{R}$ is $C^{2-0}$, and satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. Moreover, all critical points of $\mathcal{L}$ are contained in $X$, and $H$ and $X$ induce an equivalent topology on the critical set of $\mathcal{L}$.

\[\text{The final assumption is used in the proof of Claim 3, cf. the proof of Theorem 5.1. There is no such a assumption in [3].}\]
(L$_2$) The map $id_X - A : X \to X$ is strongly order preserving.

(L$_3$) The smallest eigenvalue of $B$ is equal to $\inf_{\|u\|=1} (B(\theta)u, u)_H$ by [6, Prop.6.9] and Proposition B.2 of [36], is simple and its eigenspace (contained in $X$ by (D1)) is spanned by a positive eigenvector (i.e. sitting in $\text{Int}_X(P)$).

(L$_4$) One of the following holds:

(i) $L$ is bounded below;

(ii) For every $u \in H \setminus \{\theta\}$ it holds that $L(tu) \to -\infty$ as $t \to \infty$. Moreover, there exists $b < 0$ such that $L(u) \leq b$ implies $DL(u)(u) < 0$;

(iii) There exist a compact self-adjoint linear operator $Q_\infty \in L(H)$ and a positive definite operator $P_\infty \in L(H)$ such that $\nabla L(u) = P_\infty u - Q_\infty u + o(\|u\|)$ as $\|u\| \to \infty$, where $\nabla L$ is the gradient of $L$. The smallest eigenvalue of $B_\infty := P_\infty - Q_\infty$ is simple and its eigenspace is spanned by a positive eigenvector $e_\infty \in \text{Int}_X(P)$ such that $(u, e_\infty)_H > 0$ for every $u \in P \setminus \{\theta\}$. Moreover, $B_\infty |_X \in L(X)$, and if a subset $S \subset X$ is bounded in $H$ then $A(S)$ is also bounded in $H$.

Let $\mu_0 := \text{dim} H^- \text{ and } \nu_0 := \text{dim} H^0$. Define $\mu_\infty = \nu_\infty = 0$ in the case (i) of (L$_4$), and $\mu_\infty = \nu_\infty = 0$ in the case (ii) of (L$_4$). For the case (iii) of (L$_4$), let $\mu_\infty$ be the number of negative eigenvalues of $B_\infty$ (counted with multiplicities) and $\nu_\infty = \text{dim Ker}(B_\infty)$. An element $x \in X$ is called a subcritical (resp. supercritical) critical point of $L$ if $\nabla L(x) \leq 0$ (resp. $\nabla L(x) \geq 0$).

By (F2) the map $A : X \to X$ is continuously directional differentiable. It follows from Proposition B.1(ii) of [36] that $A$ is a locally Lipschitz map from $X$ to $X$. Note that $\nabla L$ is $C^{1,0}$, and equal to $A$ on $X$ by (L$_2$), i.e. $\nabla L(x) = A(x)$ $\forall x \in X$.

Consider the negative gradient flow $\varphi(t, x)$ of $L$ on $H$ defined by

$$
\frac{d}{dt} \varphi(t, x) = -\nabla L(\varphi(t, x)) \quad \text{and} \quad \varphi(0, x) = x \in H.
$$

Let $\varphi(t, \cdot) = \varphi(t, \cdot)$. As in [4] it restricts to a continuous local flow on $X$, still denoted by $\varphi(t, \cdot)$, and $t \mapsto L(\varphi(t, x))$ is strictly decreasing for any $x \notin K(L)$. Let $D = P \cup (-P)$ and $D^* = D \setminus \{\theta\}$, and let $L^d = \{L(x) \leq d\}$ for $d \in \mathbb{R}$. The following result is a generalization of Theorem 3.4 in [4].

**Theorem 5.1** Under the above assumptions (L$_0$)-(L$_4$), if $\mu_0 \geq 2$ and $\mu_\infty + \nu_\infty \leq 1$ then $L$ has a sign-changing critical point $x_1$ with $L(x_1) < 0$. If all sign changing critical points (i.e. those in $X \setminus D$) with critical values contained in a bounded interval of $(-\infty, 0)$ are isolated, then there exists a sign changing critical point $x_1$ with $L(x_1) < 0$ and $C_1(L, x_1; K) \neq 0$. Furthermore, if near $x_1$ the conditions of Theorem 2.1 of [36] hold and $x_1$ has nullity 1 then $x_1$ is of mountain type and $C_k(L, x_1; K) \cong \delta_{k1} \mathbf{F}$. (Note: It is sufficient that $L$ satisfies the (PS)$_c$ condition for each $c < 0$).

**Proof.** Firstly, we prove:
Claim 1. Suppose that all sign changing critical points with critical values contained in a bounded interval of \((\mathbb{R}, 0, 0)\) are isolated. Then for any interval \([a, b] \subset (\mathbb{R}, 0, 0)\) there exist only finitely many sign changing critical points with critical values in \([a, b]\).

Otherwise, let \(\{u_n\}\) be infinite such points with \(\{L(u_n)\} \subset [a, b] \subset (\mathbb{R}, 0, 0)\). We may assume \(L(u_n) \to c\) (by passing to a subsequence if necessary). By the \((PS)\) condition we may assume \(u_n \to u_0\) in \(H\), and \(u_n \to u_0\) in \(X\) because of the final assumption in \((L_1)\). Clearly, \(u_0 \neq 0\) because of \(L(u_0) \leq b < 0\). Since each \(u_n\) sits in an open subset \((\mathbb{R}, 0, 0)\), it belongs to \(\mathbb{R} \cap (\mathbb{R}, 0, 0) = (\mathbb{R}, 0, 0)\) as well. Then either \(u_0 \in X \setminus \{(P \cup (-P))\} \) or \(u_0 \in \partial D \setminus \{\theta\}\). In the first case \(u_0\) also sits in \(H \setminus (\mathbb{R}, 0, 0)\). This contradicts the assumption that all sign changing critical points with critical values contained in \((\mathbb{R}, 0, 0)\).

Claim 2. 
\[
\L(u) = L(u) - L(\theta) = \int_0^1 D\L(tu)(tu)dt = \int_0^1 (A(tu), u)_H dt \leq MN.
\]
This also holds for all \( u \in B_H(\theta, N) \) because \( X \cap B_H(\theta, N) \) is dense in \( B_H(\theta, N) \). On the other hand

\[
\mathcal{L}(u) - \mathcal{L}(\bar{u}) = \int_0^1 (\nabla \mathcal{L}(tu + (1-t)\bar{u}), (u - \bar{u}))_H dt
\]

\[
\geq \int_0^1 (B_{\infty}(tu + (1-t)\bar{u}), u - \bar{u})_H dt - \frac{\lambda_2}{4} \|u - \bar{u}\| \cdot \|tu + (1-t)\bar{u}\|
\]

\[
\geq \frac{1}{2} (B_{\infty}(u - \bar{u}), u - \bar{u})_H + (B_{\infty}(\bar{u}), u - \bar{u}) - \frac{\lambda_2}{4} \|u - \bar{u}\| \cdot \|\bar{u}\|
\]

\[
\geq \frac{\lambda_2}{2} \|u - \bar{u}\|^2 + (B_{\infty}(\bar{u}), u - \bar{u})_H - \frac{\lambda_2}{4} \|u - \bar{u}\| \cdot \|\bar{u}\|.
\]

Claim 2 follows immediately.

This implies that any \( d < \inf \mathcal{L}|_{X_1} = \inf \mathcal{L}|_{H_1} \) belongs to \( \Gamma \). Hence \( c \) is finite. Since \( \mu_0 \geq 2 \) the two smallest eigenvalues of \( B(\theta) \) are negatives and the corresponding eigenspaces are contained in \( X \) by (D1). Let \( e_1 \) and \( e_2 \) two normalized eigenvectors belonging to the two smallest eigenvalues of \( B(\theta) \). By (L3), \( e_1 \in \text{Int}_X(P) \subset \text{Int}_X(D) \).

Let \( S_\rho \) be the sphere of radius \( \rho \) in \( \text{Span}\{e_1, e_2\} \). It easily follows from Proposition 2.26 of [30] that \( \max \mathcal{L}(S_\rho) < 0 \). As in the proof of [3] Lem.4.2 we can prove \( c < 0 \).

Claim 3. \( c \) is a critical value of \( \mathcal{L} \), and hence \( \mathcal{L} \) has a sign-changing critical point with negative critical value.

This may be proved by a standard deformation argument. In view of Claim 1 let us suppose that there exist only finitely many sign changing critical points \( x_1, \ldots, x_q \) at the level \( c \). Note that Claim 1 also implies that there exist a \( \eta > 0 \) such that no number in \( [c - \eta, c + \eta] \setminus \{c\} \) is a critical value of sign changing critical points. Repeating the remainder of proof of [3] Th.3.4 we get some \( i \in \{1, \ldots, q\} \) such that \( C_1(\mathcal{L}, x_i; K) \neq 0 \). The final conclusions follow from Corollary 2.9(ii) of [36]. \( \square \)

Corresponding with Theorem 3.5 in [3] we have:

**Theorem 5.2** Under the assumptions \((L_0)-(L_4)\) above, if \( \mu_0 \geq 2 \) and there exist a subcritical critical point \( x \) and a supercritical critical point \( \bar{x} \) of \( \mathcal{L} \) such that \( x \ll \theta \ll \bar{x} \), then the conclusions of Theorem 2.1 of [30] is still true.

Similarly, we can get the corresponding results with Theorems 3.6, 3.8 in [3] as follows.

**Theorem 5.3** Under the assumptions \((L_0)-(L_4)\) above, if \( \mu_{\infty} \geq 2 \) and \( \mu_0 + \nu_0 \leq 1 \) then \( \mathcal{L} \) has a sign-changing critical point \( x_1 \) with \( \mathcal{L}(x_1) > 0 \). If all sign changing critical points (i.e. those in \( X \setminus D \)) with critical values contained in a bounded interval of \([0, \infty)\) are isolated, then there exists a sign changing critical point \( x_1 \) with \( \mathcal{L}(x_1) > 0 \). Morse index \( \mu \in \{1, 2\} \) and \( C_0(\mathcal{L}, x_1; K) = 0 = C_1(\mathcal{L}, x_1; K), C_2(\mathcal{L}, x_1; K) \neq 0 \). Furthermore, if near \( x_1 \) the conditions of Theorem 2.1 of [36] hold then \( x_1 \) is neither a local minimum nor of mountain pass type, and \( C_k(\mathcal{L}, x_1; K) = \delta_{k2}K \) holds for all \( k \) provided that \( \mu = 2 \) or the nullity \( \nu \leq 1 \).

---

3This is only place where Proposition 2.26 of [36] is used. The assumptions of Theorem 2.1 of [36] and \((L_1)-(L_4)\) are sufficient to other arguments.
Theorem 5.4 Under the assumptions (L0)-(L4) above, let $\nu_0 = 0 = \nu_\infty$, $\mu_\infty \geq 1$ and $\mu_0 \neq \mu_\infty$. Suppose also that all sign changing critical points are isolated. Then

(i) If $\mu_0 \geq 1$ then $\mathcal{L}$ has a sign changing critical point $x_1$ which satisfies either $\mathcal{L}(x_1) > 0$ and $C_{\mu_0+1}(\mathcal{L}, x_1; K) \neq 0$ or $\mathcal{L}(x_1) < 0$ and $C_{\mu_0-1}(\mathcal{L}, x_1; K) \neq 0$.

(ii) If (L4) (iii) applies with $\mu_\infty \geq 2$ then $\mathcal{L}$ has a sign changing critical point $x_1$ with $C_{\mu_\infty}(\mathcal{L}, x_1; K) \neq 0$.

The last theorem can be obtained by completely repeating the proof of Theorem 3.8 in [4].

Proof of Theorem 5.3 For reader’s convenience we follow the proof ideas of [4, Theorem 3.6] to give necessary details. Since $\mu_\infty \geq 2$ either (L4)-(ii) or (L4)-(iii) occurs.

Claim 4. There exist two orthogonal unit vectors $v_\infty \in \text{Int}_X(P)$ and $u_\infty \in X$ such that $\mathcal{L}(u) < 0$ for $u \in \text{span}\{v_\infty, u_\infty\}$ with $\|u\| \geq R$.

In fact, In the latter case, the two smallest eigenvalue of $B_\infty$,

$$\lambda_1 := (B_\infty v_\infty, v_\infty)_H < \lambda_2 := \inf \{ (B_\infty u, u)_H \mid \|u\| = 1, u \in H_1 \}$$

are negative, where $v_\infty = e_\infty/\|e_\infty\|$ and $H_1 := H \cap \langle v_\infty \rangle^\perp$. Since $X$ is dense in $H$, $X \cap H_1 \neq \emptyset$. Let $u_\infty$ be a unit vector in $X \cap H_1$. Then $(v_\infty, u_\infty)_H = 0$.

As in the arguments below Claim 2 in the proof of Theorem 5.1 we have $N > 0$ and $M > 0$ such that

$$\|\nabla \mathcal{L}(u) - B_\infty u\| < \frac{|\lambda_2|}{4} \|u\| \quad \text{as} \quad \|u\| \geq N,$$

$$|\mathcal{L}(u)| \leq MN \quad \forall u \in B_H(\theta, N).$$

For any $u \in \text{span}\{v_\infty, u_\infty\}$ with $\|u\| > N$ let $\bar{u} = N \cdot u/\|u\|$. Then

$$\mathcal{L}(u) = \int_0^1 (\nabla \mathcal{L}(tu + (1-t)\bar{u}), (u - \bar{u}))_H dt + \mathcal{L}(\bar{u})$$

$$\leq \int_0^1 (B_\infty (tu + (1-t)\bar{u}), u - \bar{u})_H dt + \frac{|\lambda_2|}{4} \|u - \bar{u}\| \cdot \|tu + (1-t)\bar{u}\| + MN$$

$$\leq \frac{1}{2} (B_\infty (u - \bar{u}), u - \bar{u})_H + (B_\infty (\bar{u}), u - \bar{u})_H + \frac{|\lambda_2|}{4} \|u - \bar{u}\| \cdot \|ar{u}\| + MN$$

$$\leq \frac{\lambda_2}{2} \|u - \bar{u}\|^2 + (B_\infty (\bar{u}), u - \bar{u})_H + \frac{|\lambda_2|}{4} \|u - \bar{u}\| \cdot \|ar{u}\| + MN.$$

So Claim 4 follows from this in this case.

In the former case take any unit vector $v_\infty \in \text{Int}_X(P)$. As above we can choose another unit vector $u_\infty \in X$ which is orthogonal to $v_\infty$. Since for any $u \in H$ with $\mathcal{L}(u) \leq a$ it holds that $D\mathcal{L}(u)(u) < 0$, $\mathcal{L}^a$ is a manifold with $C^1$-boundary $\mathcal{L}^{-1}(a)$, and $\mathcal{L}^{-1}(a)$ is transversal to the radial vector field. Moreover, for any $u \in H \setminus \{\theta\}$, $\mathcal{L}(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. So for each $u \in \partial B_H(\theta, 1)$ there exists a $t_u \in (0, \infty)$ such that $\mathcal{L}(t_u \cdot u) = a$ and that $u \rightarrow t_u$ is continuous by the implicit function theorem. This shows that the map $\partial B_H(\theta, 1) \rightarrow \mathcal{L}^{-1}(a)$, $u \rightarrow t_u \cdot u$ is a homeomorphism. It follows
that \( \partial B_H(\theta, 1) \cap \text{span}\{v_\infty, u_\infty\} \) is compact and hence \( \partial B_H(\theta, 1) \cap \text{span}\{v_\infty, u_\infty\} \subset B_H(\theta, R) \) for some \( R > 0 \). We have still \( L(u) < 0 \) for \( u \in \text{span}\{v_\infty, u_\infty\} \) with \( \|u\| \geq R \).

That is, Claim 4 holds.

For \( R \) in Claim 4 let us set

\[
B_R := \{ sv_\infty + tu_\infty : |s| \leq R, \; 0 \leq t \leq R \},
\]

\[
\partial B_R := \{ sv_\infty + tu_\infty : |s| = R, \; \text{or} \; t \in \{0, 1\} \}.
\]

Then \( \partial B_R \subset L^0 \cup D \). Let \( \beta := \max L(B_R) \) and \( \xi_\beta \in H_2(L^\beta \cup D, L^0 \cup D; K) \) be the image of \( 1 \in K \cong H_2(B_R, \partial B_R; K) \) under the homomorphism

\[
K \cong H_2(B_R, \partial B_R; K) \rightarrow H_2(L^\beta \cup D, L^0 \cup D; K)
\]

induced by the inclusion \((B_R, \partial B_R) \hookrightarrow (B_R, \partial B_R)\). For \( \gamma \leq \beta \) let

\[
(j_\gamma)_2 : H_2(C^\gamma \cup D, L^0 \cup D; K) \rightarrow H_2(L^\beta \cup D, L^0 \cup D; K)
\]

be the homomorphism induced by the inclusion. Set

\[
\Gamma := \{ \gamma \leq \beta | \xi_\beta \in \text{Image}(j_\gamma)_2 \} \quad \text{and} \quad c := \inf \Gamma. \tag{5.3}
\]

Let \( e_1 \in \text{Int}_X(P) \) be the eigenvector of \( B(\theta) \) belonging to the first eigenvalue \( \lambda_1 = \inf_{\|u\|=1} (B(\theta)u, u)_H \), and let \( X_1 = \langle e_1 \rangle \) and \( X_2 := X_1^+ \cap X \). Since \( \mu_0 + \nu_0 \leq 1 \).

There are three cases: (a) \( \mu_0 = 0 = \nu_0 \), (b) \( \mu_0 = 1 \) and \( \nu_0 = 0 \), (c) \( \mu_0 = 0 \) and \( \nu_0 = 1 \). For the first two case we have \( X_2 \subset H^+ \). In the third case, \( \lambda_1 = 0 \) and \( X_1 = \text{Ker}(B(\theta)) \), and \( H^- = \{\theta\} \). We also get \( X_2 \subset H^+ \). It follows from \([36, (2.74)]\) that

\[
L(u) \geq \frac{a_1}{4}\|u\|^2
\]

for all \( u \in B_H(\theta, \rho_0) \cap H^+ \). Hence for some small \( \rho > 0 \) it always holds that \( \inf\{ L(u) | u \in X_2, \|u\| = \rho \} > 0 \). This is what is needed in the proof of \([4\; \text{Lem.4.3}]\).

It leads to \( \xi_\beta \neq 0 \) and hence \( \beta \in \Gamma \). Moreover, \((j_0)_2 = 0\) implies that \( 0 \notin \Gamma \).

Since we have assumed that \( \theta \) is an isolated critical point\footnote{In [4] it was claimed that since \( \mu_0 + \nu_0 \leq 1 \) the sign changing solutions cannot accumulate at 0.}, and that all sign changing critical points with critical values in a bounded interval of \([0, \infty)\), by the proof of Claim 3 in the proof of Theorem 5.1 we can derive that there exist only finitely many sign changing critical points with critical values in \([0, \beta]\). It follows that \( L^0 \cup D \) is a strong deformation retract of \( L^\gamma \cup D \) for \( \gamma > 0 \) small enough. So \( c > 0 \).

The remained arguments are the same as those of \([4\; \text{Th.3.6}]\) (as long as slightly modifications as in the proof of Theorem 5.1) \( \Box \)

\textbf{Remark 5.5} Let us outline a possible way to weaken the conditions of Theorem 5.1 that is, removing the assumption that \( L \) is \( C^{2-0} \), but adding the condition (6.5.1) “For any subset \( S \subset X \), which is bounded in \( H \), the image \( A(S) \) is bounded in \( X \)” in case (L4)-(i);
(6.5.2) “For any $c \in \mathbb{R}$ and small $\varepsilon > 0$, $A(X \cap L^{-1}([c-\varepsilon,c+\varepsilon]))$ is bounded in $X$” in case (L4)-(iii).

Since (F2) and Proposition B.1(ii) of [36] imply that the map $A : X \to X$ is locally Lipschitz, we get a (local) flow on $X \setminus K(L)$,

$$\begin{align*}
\frac{d}{dt}\sigma(t,x) &= -A(\sigma(t,x)) \\
\sigma(0,x) &= x \in X \setminus K(L),
\end{align*}$$

(5.4)

where $K(L)$ is the critical set of $L$. By (L2) we get that $\sigma(t,x) \in \text{Int}_X(D)$ for all $x \in D^*$ and $t > 0$. The key is how to prove Claim 3 in the present assumptions. Note that we have proved $c < 0$ above Claim 3. Then Claim 1 implies that $L^{-1}(c)$ contains at most finitely many sign changing critical points $x_1, \cdots , x_q$ and that there exist a $\eta > 0$ such that no number in $[c-\eta,c+\eta] \setminus \{c\}$ is a critical value of sign changing critical points.

By contradiction, suppose that $c$ is not a critical value of $L$. Then the PS condition implies that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\|\nabla L(u)\| = \|DL(u)\| \geq \delta \quad \forall u \in L^{-1}[c-\varepsilon,c+\varepsilon].$$

We may assume $\varepsilon < \eta$. By (L2) and $\|u\|_X \geq \|u\| \quad \forall u \in X$ we get

$$\|A(u)\|_X \geq \|A(u)\| = \|DL(u)\| \geq \delta \quad \forall u \in X \cap L^{-1}[c-\varepsilon,c+\varepsilon].$$

(5.5)

For $x \in X \cap L^{-1}[c-\varepsilon,c+\varepsilon]$ let $[0,T_x)$ be the maximal existence interval of the flow in (5.4) on $X \cap L^{-1}[c-\varepsilon,c+\varepsilon]$. Then

$$\eta + \varepsilon \geq L(x) - L(\sigma(t,x)) = \int_0^t \|A(\sigma(s,x))\|^2 ds \geq \delta^2 t$$

for any $t \in [0,T_x)$. So $T_x \leq (\eta + \varepsilon) / \delta^2$.

**In case (L4)-(i), $L$ is coercive by a result of [7].** It follows that $L^{c+\varepsilon}$ is bounded in $H$ and hence $A(X \cap L^{c+\varepsilon})$ is bounded in $X$ by the assumption (6.5.1). Namely, there exists a $N > 0$ such that $\|A(x)\|_X \leq N$ for all $x \in X \cap L^{c+\varepsilon}$. Then

$$\text{dist}_X(\sigma(t_2,x),\sigma(t_1,x)) \leq \int_{t_1}^{t_2} \left\| \frac{d}{dt}\sigma(t,x) \right\|_X dt = \int_{t_1}^{t_2} \|A(\sigma(t,x))\|_X dt \leq N(t_2 - t_1)$$

for any $0 \leq t_1 < t_2 < T_x$. It follows that the limit $\lim_{t \to T_x} \sigma(t,x)$ exists in $X$ and

$$L\left( \lim_{t \to T_x} \sigma(t,x) \right) = c - \varepsilon.$$ 

As usual we can use $\sigma$ to construct a deformation retract from $L^{c+\varepsilon}_X \cup D^*$ to $L^{c-\varepsilon}_X \cup D^*$, where $L^{d}_X := X \cap L^{d}$ for $d \in \mathbb{R}$. This is a contradiction. Hence $c$ is a critical value.

**In case (L4)-(iii), by the assumption (6.5.2), $A(X \cap L^{-1}([c-\varepsilon,c+\varepsilon]))$ is bounded in $X$ for some small $\varepsilon > 0$. Then the same method leads to a contradiction yet.**

Finally, we are going to generalize the following result, which is an abstract summary of the arguments in [34].
Theorem 5.6 Let $H$ be a Hilbert space, and let $P_H \neq H$ be a closed cone, i.e. $P_H = \bar{P}_H$ is convex and satisfies $\mathbb{R}^+ \cdot P_H \subset P_H$, $P_H \cap (-P_H) = \{\theta\}$. Suppose that a $C^2$-functional $\mathcal{L} : H \to \mathbb{R}$ has critical point $\theta$ and satisfies the following properties.

(i) $\mathcal{L}$ is bounded from below, and satisfies the (PS) condition.

(ii) $P_H$ is positively invariant under the negative gradient flow $\varphi^t$ of $\mathcal{L}$,

$$\frac{d}{dt} \varphi^t(u) = -\nabla \mathcal{L}(\varphi^t(u)) \quad \text{and} \quad \varphi^0(u) = u \in H.$$ 

(iii) There exists a positive element $e \in P_H \setminus \{\theta\}$ such that the cone

$$D := \{u \in H \mid u \geq e\} \subset P_H \ (\text{resp. } -D)$$

contains all positive (resp. negative) critical points of $\varphi$. (Note that $D \cap (-D) = \emptyset$). Let $D_\varepsilon := \{u \in H \mid \text{dist}(u, D) \leq \varepsilon\}$ for $\varepsilon > 0$.

(iv) There exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, $D_{\varepsilon_0} \cap (-D_{\varepsilon_0}) = \emptyset$, and $D_{\varepsilon}$ and $-D_{\varepsilon}$ are strictly positively invariant for the flow $\varphi^t$.

Let $W_\varepsilon := D_{\varepsilon} \cup (-D_{\varepsilon})$ and let $i_c : (\mathcal{L}^e \cup W_\varepsilon, W_\varepsilon) \to (H, W_\varepsilon)$ be the inclusion. Then

$$c_1 := \inf \{e \in \mathbb{R} \mid i_c^* : \bar{H}^1(H, W_\varepsilon; \mathbb{Z}_2) \to \bar{H}^1(\mathcal{L}^e \cup W_\varepsilon, W_\varepsilon; \mathbb{Z}_2) \text{ is a monomorphism}\} \quad (5.6)$$

is finite and $K_{c_1}^* := \{u \in H \setminus W_\varepsilon \mid \mathcal{L}(u) = c_1, \mathcal{L}'(u) = 0\} \neq \emptyset$ for $0 < \varepsilon \leq \varepsilon_0$, where $i_c^*$ is induced by the inclusion $i_c$. Moreover, $\mathcal{L}$ has at least two nontrivial critical points in $H \setminus W_\varepsilon$ provided that $c_1 < \mathcal{L}(\theta)$ and

each $u \in K_{c_1}^*$ with Morse index $\mu(u) = 0$ has nullity $\nu(u) \leq 1, \quad (5.7)$

$$C_q(\mathcal{L}, \theta; \mathbb{Z}_2) \cong \delta_{q\varepsilon_0} \mathbb{Z}_2 \text{ for some } n \geq 2 \text{ and any } q \in \mathbb{Z}. \quad (5.8)$$

Remark 5.7 (i) Actually, the assumptions (i)-(ii) in Theorem 5.6 imply that $\mathcal{L}$ has a critical point $u \in P_H$ with $\mathcal{L}(u) = \inf_{v \in P_H} \mathcal{L}(v)$ (33, Theorem 2.1). In the same way the assumption (iv) yields a critical point sitting in $-D \subset P_H$.

(ii) Note that (5.8) holds if $\theta$ is a nondegenerate critical point of $\mathcal{L}$ with Morse index $n \geq 2$.

(iii) If $K := id - \nabla \mathcal{L} : H \to H$ maps $D_{\varepsilon}$ (resp. $-D_{\varepsilon}$) into int($D_{\varepsilon}$) (resp. int($-D_{\varepsilon}$)), then (iv) holds by the proof of Proposition 3.2(ii) of [34].

We shall generalize Theorem 5.6 as follows.

Theorem 5.8 Let $H$ be a Hilbert space, and let $P_H \neq H$ be a closed cone, i.e. $P_H = \bar{P}_H$ is convex and satisfies $\mathbb{R}^+ \cdot P_H \subset P_H$, $P_H \cap (-P_H) = \{\theta\}$. Suppose that a $C^1$-functional $\mathcal{L} : H \to \mathbb{R}$ has critical point $\theta$ and satisfies the following properties.

\footnote{By a result of [34] these two conditions imply the coercivity of $\mathcal{L}$.}

\footnote{By considering the functional $f(x) = (x, e)_H$ we obtain $D \subset \{f > 0\}$ and $-D \subset \{f < 0\}$. This implies $\text{dist}(D, -D) \geq 2\|e\|$ and thus $\text{dist}(D_{\varepsilon_0}, -D_{\varepsilon_0}) \geq 2\|e\| - 2\varepsilon_0$ for $\varepsilon < \|e\|!$}
We claim: Moreover, (5.9) means that 

\[ D := \{ u \in H \mid u \geq e \} \subset P_H \text{ (resp. } -D) \]

contains all positive (resp. negative) critical points of \( \mathcal{L} \).

(iii) There exists a \( \varepsilon_0 \in (0, \|e\|) \) such that for each \( \varepsilon \in (0, \varepsilon_0] \),

\[ \mathcal{K} := \text{id} - \nabla \mathcal{L} : H \to H \text{ maps } D_\varepsilon \text{ (resp. } -D_\varepsilon) \]

into \( \text{int}(D_\varepsilon) \) (resp. \( \text{int}(-D_\varepsilon) \)).

Then there exists \( \varepsilon_1 \in (0, \varepsilon_0] \) such that for any \( 0 < \varepsilon \leq \varepsilon_1 \), \( c_1 \) defined by (5.6) is finite and \( K_{c_1}^* := \{ u \in H \setminus W_\varepsilon \mid \mathcal{L}(u) = c_1, \mathcal{L}'(u) = 0 \} \neq \emptyset \). Moreover, if \( c_1 < \mathcal{L}(\theta) \) and the conditions of Theorem 2.1 of [30] are satisfied near each \( u \in K_{c_1}^* \) and \( \theta \) then \( \mathcal{L} \) has at least two nontrivial critical points in \( H \setminus W_\varepsilon \) provided that (5.7) and (5.8) hold.

Since \( D_\varepsilon \) and \( -D_\varepsilon \) are positive invariant under the pseudo-gradient flow \( \tilde{\varphi}^t \) defined by (5.11) (see proof below (5.12)), as in Remark 5.7(i) we may show that \( \mathcal{L} \) has a critical point \( u \in D_\varepsilon \) (resp. \( u \in -D_\varepsilon \)) with \( \mathcal{L}(u) = \inf_{v \in D_\varepsilon} \mathcal{L}(v) \) (resp. \( \mathcal{L}(u) = \inf_{v \in -D_\varepsilon} \mathcal{L}(v) \)). Remark 5.7(iii) shows that Theorem 5.8(iii) is stronger than Theorem 5.6(iv).

**Proof of Theorem 5.8.** The basic ideas is almost the same as in [34]. However, since we only assume \( \mathcal{L} \) to be \( C^1 \), the negative gradient flow of it cannot be used. We shall overcome this difficulty by some methods in [3].

Note that \( V := \nabla \mathcal{L} \) is a \( C^0 \) pseudo-gradient vector field for \( \mathcal{L} \) in the sense of [3], i.e. \( V : H \to H \) is a continuous map satisfying

\[ \|V(x)\| < 2\|\nabla \mathcal{L}(x)\| \quad \text{and} \quad (\nabla \mathcal{L}(x), V(x))_H > \frac{1}{2}\|\nabla \mathcal{L}(x)\|^2 \quad \forall x \in H \setminus K(\mathcal{L}). \]

Moreover, (5.9) means that \( D \) and \( -D \) are \( \mathcal{K} \)-attractive in the sense of [3] Definition 3.3. By [3, Lemma 3.4] there exists \( \varepsilon_1 \in (0, \varepsilon_0] \) such that for every \( \sigma \in (0, \varepsilon_1] \) there is a pseudo-gradient vector field \( \tilde{V}_\sigma \) of \( \mathcal{L} \) such that for all \( \varepsilon \in [\sigma, \varepsilon_1] \) the sets \( D_\varepsilon \) and \( -D_\varepsilon \) are strongly positive invariant under \( \tilde{V}_\sigma \) in the following sense:

\[ \begin{align*}
\text{for any } u \in \partial D_\varepsilon, & \exists \varepsilon_0 > 0 \text{ such that } \forall \varepsilon \in (0, \varepsilon_0], \\
u + \varepsilon \tilde{V}_\sigma(u) \in \text{Int}(D_\varepsilon) \text{ and } -u + \varepsilon \tilde{V}_\sigma(-u) \in \text{Int}(-D_\varepsilon). \end{align*} \]

(5.10)

Let \( \tilde{K}_\sigma := \text{id} - \tilde{V}_\sigma \) and let \( \tilde{\varphi}^t \) be the flow of \( \tilde{V}_\sigma \), i.e.

\[ \frac{d}{dt} \tilde{\varphi}^t(u) = -\tilde{V}_\sigma(\tilde{\varphi}^t(u)) \quad \text{and} \quad \tilde{\varphi}^0(u) = u \in H. \]

(5.11)

We claim:

\[ \begin{align*}
\text{for any } \varepsilon \in [\sigma, \varepsilon_0] \text{ the sets } D_\varepsilon \text{ and } -D_\varepsilon \text{ are } \\
\text{strictly positive invariant for the flow } \tilde{\varphi}^t. \end{align*} \]

(5.12)
Indeed, from (5.10) and Theorem 5.2 in [23] we deduce that the sets $D_{\varepsilon}$ and $-D_{\varepsilon}$ are positive invariant for the flow $\varphi^t$ for any $\varepsilon \in [\sigma, \varepsilon_0]$.

As in the proof of [34] Proposition 3.2(ii)], suppose by contradiction that there exist $u \in D_{\varepsilon}$ and $t > 0$ such that $\varphi^t(u) \in \partial D_{\varepsilon}$. Then Mazur’s separation theorem yields $f \in H^*$ and $\beta > 0$ such that $f(\varphi^t(u)) = \beta$ and $f(u) > \beta$ for any $u \in \text{Int}(D_{\varepsilon})$. By (5.10) we have $\varepsilon_0 > 0$ such that $\varphi^t(u) + \varepsilon \tilde{V}(\varphi^t(u)) \in \text{Int}(D_{\varepsilon}) \forall \varepsilon \in (0, \varepsilon_0]$. Hence

$$\frac{d}{ds} f(\varphi^s(u)) \bigg|_{s=t} = f(-\tilde{V}(\varphi^t(u))) = \frac{1}{\varepsilon} f(-\varepsilon \tilde{V}(\varphi^t(u)))$$

$$= \frac{1}{\varepsilon} f(\varphi^t(u)) - \varepsilon \tilde{V}(\varphi^t(u)) - \frac{1}{\varepsilon} f(\varphi^t(u))$$

$$= \frac{1}{\varepsilon} f(\varphi^t(u)) - \varepsilon \tilde{V}(\varphi^t(u)) - \beta > 0.$$ 

This leads to a contradiction as in [34]. The assertion in (5.11) is proved.

Having (5.12) we may follow the lines of [34] to outline the remaining proof.

Let $\tilde{H}^*$ denote Alexander-Spanier cohomology with coefficients in the field $\mathbb{Z}_2$. For a critical point $u$ of $\mathcal{L}$ the cohomological critical groups of $\mathcal{L}$ at $u$ are defined by

$$C^q(\mathcal{L}, u) := \tilde{H}^q(\mathcal{L}^c, \mathcal{L}^c \setminus \{u\}) \quad \forall q \geq 0,$$

where $c = \mathcal{L}(u)$. We conclude

$$C^q(\mathcal{L}, u) \cong C_q(\mathcal{L}, u; \mathbb{Z}_2) \quad \forall q \in \mathbb{Z}. \quad (5.13)$$

This can be obtained from the proof of Proposition[5,3]. We can also prove its as follows. By excision property it holds that $C^q(\mathcal{L}, u) = \tilde{H}^q(\mathcal{L}^c \cap U, (\mathcal{L}^c \setminus \{u\}) \cap U) \forall q \geq 0$ for any open neighborhood $U$ of $u$. In particular, one can find a $U$ so that both $\mathcal{L}^c \cap U$ and $(\mathcal{L}^c \setminus \{u\}) \cap U$ are absolute neighborhood retracts ([21, Th.1.1] and [17, Remark 2]). It follows that

$$\tilde{H}^q(\mathcal{L}^c \cap U, (\mathcal{L}^c \setminus \{u\}) \cap U) \cong H^q(\mathcal{L}^c \cap U, (\mathcal{L}^c \setminus \{u\}) \cap U) \forall q \geq 0.$$

Moreover, $H^q(\mathcal{L}^c \cap U, (\mathcal{L}^c \setminus \{u\}) \cap U; G) \cong \text{Hom}(H_q(\mathcal{L}^c \cap U, (\mathcal{L}^c \setminus \{u\}) \cap U; G), G)$ for any divisible group $G$. If $C^1(\mathcal{L}, u) \neq 0$ we say $u$ to be of mountain-pass type.

**Lemma 5.9** ([34] Lemma 4.4). If $C^1(\mathcal{L}, u) \neq 0$, and $\nu(u) \leq 1$ in case $\mu(u) = 0$, then $C^q(\mathcal{L}, u) = \delta_{q1}\mathbb{Z}_2$ for $q \in \mathbb{Z}$.

**Proof.** Note that $\tilde{H}^q(B^n, S^{n-1}; G) = \delta_{q0}G$. If $\nu(u) = \text{Ker}(\mathcal{L}''(u)) = 0$, i.e. $u$ is nondegenerate, by Morse lemma we get that $C^q(\mathcal{L}, u) \cong H^q(\mathcal{L}''(u), S^{(n-1)}; \mathbb{Z}_2) = \delta_{q0}\mathbb{Z}_2$. The desired conclusion follows.

If $\nu(u) = \text{Ker}(\mathcal{L}''(u)) > 0$, from Corollary 2.6 of [30] it follows that $\mu(u) \leq 1 \leq \mu(u) + \nu(u)$ and $C^1(\mathcal{L}, u) \cong C^{1-\mu(u)}(\mathcal{L}^0, 0)$, where $\mathcal{L}^0$ is a function defined near the origin of a $\nu(u)$-dimensional space $\text{Ker}(\mathcal{L}''(u))$. If $\mu(u) = 1$ then Corollary 2.9(iii) of [30] gives the conclusion. If $\mu(u) = 0$ then $\nu(u) = 1$ by the fact that $1 \leq \mu(u) + \nu(u) = \nu(u)$. Corollary 2.9(ii) of [30] yields the conclusion. □

The following is a special case of the strong excision property (Theorem 5) on the page 318 of [48].
Lemma 5.10 ([34, Lemma 4.4]). Let $X$ be paracompact Hausdorff space and let $Y, Z \subset X$ be closed subsets such that $X = Y \cup Z$. Then the inclusion $(Y, Y \cap Z) \to (X, Z)$ induces an isomorphism $\tilde{H}^*(X, Z) \to \tilde{H}^*(Y, Y \cap Z)$.

By the assumption (i) in Theorem 5.8, $L$ is bounded from below, for each $u \in H$ the flow $t \mapsto \tilde{\varphi}^t(u)$ exists in an open interval containing $[0, \infty)$. For $-\infty \leq c \leq \infty$, $L^c$ is positively invariant for the flow $\tilde{\varphi}^t$, and strictly positively invariant if $c$ is a regular value of $L$.

Lemma 5.11 ([34, Lemma 4.3]). Let $u$ be a critical point of $L$ with $L(u) = c$ and such that $B_H(u, 2\varepsilon)$ contains no other critical point of $L$. Then for $\delta > 0$ sufficiently small there exists a closed neighborhood $N \subset B_H(u, \varepsilon) \cap L^{c+\delta}$ of of $u$ such that $N \cup L^{c-\delta}$ is positively invariant for the flow $\tilde{\varphi}^t$. Moreover,

$$C^q(L, u) \cong \tilde{H}^q(N \cup L^{c-\delta}, L^{c-\delta}) \cong \tilde{H}^q(N, L^{c-\delta} \cap N) \quad \forall q \in \mathbb{Z}.$$ 

Fix $\varepsilon \in [\sigma, \varepsilon_1]$, then $W_\varepsilon := D_\varepsilon \cup (-D_\varepsilon)$ is closed and strictly positively invariant under $\tilde{\varphi}^t$ by (5.11). In particular, $\partial W_\varepsilon$ contains no critical points of $L$. So the (PS) condition implies that $K^*_c := \{u \in H \setminus W_\varepsilon, |L(u) = c, L'(u) = 0\}$ is a compact subset of $H \setminus W_\varepsilon$ for every $c \in \mathbb{R}$. Clearly, every non-trivial critical point in $K^*_c$ changes sign.

Lemma 5.12 ([34, Lemma 4.5]). Suppose that $-\infty \leq a < b \leq c \leq d \leq \infty$ satisfy:

(i) $K^*_c = \emptyset$ for any $c' \in [b, d] \setminus \{c\}$

(ii) There is a neighborhood $N \subset L^d$ of $K^*_c$ such that $L^b \cup N$ is is positively invariant under $\tilde{\varphi}^t$.

Then the inclusion

$$(N \cup L^b \cup W_\varepsilon, L^a \cup W_\varepsilon) \to (L^d \cup W_\varepsilon, L^a \cup W_\varepsilon)$$

is a homotopy equivalence. In particular, if $K^*_c = \emptyset$ for any $c \in [b, d]$, then the inclusion $(L^b \cup W_\varepsilon, L^a \cup W_\varepsilon) \to (L^d \cup W_\varepsilon, L^a \cup W_\varepsilon)$ is a homotopy equivalence.

From Lemmas 5.10, 5.11 and 5.12 we have

Lemma 5.13 ([34, Lemma 4.7]). If the compact set $K^*_c$ consists of isolated critical points $u_1, \ldots, u_m$ for some $c \in \mathbb{R}$, then

$$\tilde{H}^q(L^{c+\delta} \cup W_\varepsilon, L^{c-\delta} \cup W_\varepsilon) \cong \bigoplus_{i=1}^m C^q(L, u_i)$$

for $q \in \mathbb{Z}$ and $\delta > 0$ sufficiently small.

Note that

$$\tilde{H}^q(H, W_\varepsilon; \mathbb{Z}_2) \cong \text{Hom}(H_q(H, W_\varepsilon; \mathbb{Z}_2); \mathbb{Z}_2) \cong \delta_q \mathbb{Z}_2.$$ (5.14)

By the definition of $c_1$ and the assumption (i) we get $c_1 > -\infty$. 

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Lemma 5.14 ([34] Lemma 4.8]).

(i) \( K_{c_1}^* \neq \emptyset \), and \( c_1 \leq \mathcal{L}(\theta) \) provided that \( \mathcal{L}(te) \leq \mathcal{L}(\theta) \) \( \forall t \in [-1,1] \).

(ii) If \( c_1 < \mathcal{L}(\theta) \) then \( K_{c_1}^* \) consists of non-trivial critical points.

(iii) If \( K_{c_1}^* \) consists of only one isolated critical point \( u \), then

\[
C^q(\mathcal{L}, u) \cong \bar{H}^q(\mathcal{L}^{c_1+\delta} \cup W_\varepsilon, \mathcal{L}^{c_1-\delta} \cup W_\varepsilon) \cong \delta q_1 \mathbb{Z}_2 \quad \text{for small} \quad \delta > 0.
\]

The claim that \( c_1 \leq \mathcal{L}(\theta) \) in (i) can be proved by (5.14). Lemma 5.12 leads to \( K_{c_1}^* \neq \emptyset \). Then the assumption and Lemma 5.13 imply \( C^q(\mathcal{L}, u) \cong \bar{H}^q(\mathcal{L}^{c_1+\delta} \cup W_\varepsilon, \mathcal{L}^{c_1-\delta} \cup W_\varepsilon) \) for small \( \delta > 0 \). From this and the definition of \( c_1 \) it follows that \( C^1(\mathcal{L}, u) \neq 0 \). Combing it with (5.7) together, we may use Lemma 5.9 to infer (iii).

Now (5.8), (5.14) and Lemmas 5.11, 5.12 and 5.14 lead to

Lemma 5.15 ([34] proposition 4.9]). Under the assumption (5.8), if \( c_1 < \mathcal{L}(\theta) \) and then there exists \( c \neq c_1 \) such that \( K_c^* \) contains a non-trivial critical point of \( \mathcal{L} \).

Summarizing these two lemmas we complete the proof for \( \varepsilon \in [\sigma, \varepsilon_1] \). Since \( \sigma \in (0, \varepsilon_0) \) is arbitrary Theorem 5.8 is proved. \( \square \)

Part II

Applications

In progress!

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