A new method for driven-dissipative problems: Keldysh-Heisenberg equations

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Driven-dissipative systems have recently attracted great attention due to the existence of novel physical phenomena with no analog in the equilibrium case. The Keldysh path-integral theory is a powerful tool to investigate these systems. However, it has still been challenge to study strong nonlinear effects implemented by recent experiments, since in this case the photon number is few and quantum fluctuations play a crucial role in dynamics of system. Here we develop a new approach for deriving exact steady states of driven-dissipative systems by introducing the Keldysh partition function in the Fock-state basis and then mapping the standard saddle-point equations into Keldysh-Heisenberg equations. We take the strong Kerr nonlinear resonators with/without the nonlinear driving as two examples to illustrate our method. It is found that in the absence of the nonlinear driving, the exact steady state obtained does not exhibit bistability and agree well with the complex P-representation solution. While in the presence of the nonlinear driving, the multiphoton resonance effects are revealed and are consistent with the qualitative analysis. Our method provides an intuitive way to explore a variety of driven-dissipative systems especially with strong correlations.

I. INTRODUCTION

In recent years, the driven-dissipative systems have got a lot of attentions both theoretically and experimentally. In these systems, the nonlinear interactions can be significantly enhanced by controlling both the driving and dissipation processes. For example, strong optical nonlinearities at the single-photon level have already been observed in cavity quantum electrodynamics (QED) [1, 2], Rydberg atomic systems [3–5], optomechanical systems [6], and superconducting circuit QED systems [7–12]. These advances in experimental methods have greatly promoted the development of quantum metrology, quantum information and quantum optical devices [13, 14]. On the other hand, they also provide good platforms for studying novel nonequilibrium physical phenomena, such as the dynamical critical phenomena [15–17], time crystals [18], driven-dissipative strong correlations [19, 20]. In this context, how to understand the nonlinear effects in nonequilibrium phenomena has become an important topic.

The Keldysh functional integral formalism in the coherent-state basis is a general approach to study nonequilibrium physics [21]. This technique provides a well-developed toolbox of perturbation techniques to study the nonlinear effects [22–24]. For example, in some systems such as the polariton condensates [15, 16] and atomic ensembles in cavity [25–30], the single-particle actions are quadratic and the diagrammatic perturbation theory, based on the Wick’s theorem, can be performed. However, for coherently driven systems such as optomechanical systems [31–34], the single-particle actions are no longer quadratic and the Wick’s theorem cannot be applied directly. Fortunately, when the coherent driving is strong and the nonlinear interaction is weak, the mean photon number is large and the standard saddle-point approximation can be well introduced. In this approach, the mean value of operators are mainly determined by the classical path, which satisfies the saddle-point equations, and quantum fluctuations are treated as perturbation [22–24]. However, recent researches have focused on the strongly nonlinear effects at the level of individual photons, which are benefit for processing quantum information [14]. Experimentally, these require the systems are weakly driven and the nonlinear interactions are strong. As a result, the mean photon number is few and quantum fluctuations play a crucial role in dynamics of system. This indicates that the standard saddle-point approximation is not reasonable.

To solve this crucial problem, we develop the Keldysh path-integral theory in the Fock-state basis, from which the standard saddle-point equations are mapped into quantum Hamiltonian equations named as Keldysh-Heisenberg equations. As a result, the exact steady states induced by the quantum fluctuation effect can be well derived. We take the strong Kerr nonlinear resonators with/without the nonlinear driving as two examples to illustrate our method. It is found that in the absence of the nonlinear driving, the exact steady state obtained does not exhibit bistability and agree well with the complex P-representation solutions. While in the presence of the nonlinear driving, the multiphoton resonance effects are revealed and are consistent with

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corresponds to coupling the system to a zero-temperature \( \gamma \) operator, especially with strongly-correlated photons, based on the qualitative analysis. Our method offers an effective way to explore a variety of driven-dissipative systems with the powerful toolbox of quantum field theory [22–24], in which the evolution takes place along the closed time contour.

We begin to consider a dissipative Kerr nonlinear resonator with a coherent driving term, in which the Hamiltonian is written as \( \hbar = 1 \) hereafter

\[
\hat{H} = \Delta_c \hat{a}^\dagger \hat{a} + \chi \hat{a}^\dagger \hat{a}^2 + i\Omega (\hat{a}^\dagger - \hat{a}),
\]

(1)

where \( \hat{a} (\hat{a}^\dagger) \) is the annihilation (creation) operator of the resonator, \( \Delta_c = \omega_c - \omega_p \) is the detuning between the resonator and driving fields, \( \chi \) is the Kerr nonlinearity, and \( \Omega \) is the driving amplitude. The dynamics of such system is described by the Lindblad master equation [35, 36]

\[
\frac{d}{dt} \hat{\rho} (t) = \mathcal{L} \hat{\rho} (t) = -i \left[ \hat{H}, \hat{\rho} (t) \right] + \gamma \mathcal{D}[\hat{a}] \hat{\rho} (t),
\]

(2)

where \( \hat{\rho} (t) \) is the density matrix, \( \mathcal{L} \) is the Liouville superoperator, \( \gamma \) is the one-photon decay rate, and \( \mathcal{D}[\hat{a}] \hat{\rho} (t) = \hat{\rho} (t) \hat{a}^\dagger - [\hat{a} \hat{\rho} (t) + \hat{\rho} (t) \hat{a}] / 2 \) is the standard dissipator in the Lindblad form. Note that the dissipative evolution corresponds to coupling the system to a zero-temperature bath [35, 36]. This Lindblad master equation can be investigated by the Keldysh nonequilibrium quantum field theory [22–24], in which the evolution takes place along the closed time contour.

We suppose \( |\alpha\rangle \) as a coherent state, which is the eigenstate of the annihilation operator \( \hat{a} \) with the complex eigenvalue \( a \) (i.e., \( \hat{a} |\alpha\rangle = a |\alpha\rangle \)). Note that the Keldysh closed time contour can be divided into a sequence of infinitesimal time steps, as shown in Fig. 1(a). Then the completeness relation in terms of the coherent state, \( \hat{1}_{coh} = \int \int (da^* da / \pi) e^{-|a|^2} |\alpha\rangle \langle \alpha| \), is inserted in between consecutive time steps [24]. In this coherent-state basis, the partition function, which is corresponding to the Lindblad master equation (2), is given by

\[
Z = \int \mathcal{D} [a_+, a_-] \exp (iS),
\]

(3)

where + and − denote the forward and backward branches and the action

\[
S = \int_{-\infty}^{+\infty} dt \left\{ a^*_{+} \left( i\partial_t - \Delta_c \right) a_{+} - \chi \alpha^2 a^2_{+} - i\Omega (a^*_{+} - a_{+}) \right. \\
- a^*_+ \left( i\partial_t - \Delta_c \right) a_{-} - \chi \alpha^2 a^2_{-} + i\Omega (a^*_{-} - a_{-}) \\
\left. - i\gamma a_{+} a_{-}^* + i\gamma c (a^*_{+} a_{+} + a^*_{-} a_{-}) \right\}.
\]

(4)

It is more convenient to discuss Eq. (4) in the Keldysh basis,

\[
a_{cl} = \frac{1}{\sqrt{2}} (a_{+} + a_{-}), \quad a_q = \frac{1}{\sqrt{2}} (a_{+} - a_{-}),
\]

(5)

where \( a_{cl} \) and \( a_q \) are the classical and quantum fields [22–24]. After a straightforward calculation, the action is rewritten as

\[
S = \int_{-\infty}^{+\infty} dt \left\{ a^*_{cl} \left( i\partial_t - \Delta_c \right) a_{q} + a^*_q \left( i\partial_t - \Delta_c \right) a_{cl} \right. \\
\left. - i\gamma \alpha (a^*_q a_{cl} - a_{cl} a^*_q) + i\gamma a^*_q a_q - i\sqrt{2} \Omega (a^*_q - a_{q}) \right. \\
\left. - \chi \left( a^2_{cl} a_{cl} a_q + a^2_{cl} a_q - a_{cl} a^2_{cl} a_{cl} a_q + \alpha^2 a^2_{cl} a_{cl} a_q \right) \right\}.
\]

(6)

Note that in the presence of the coherent driving \( (\Omega \neq 0) \), the first two lines of Eq. (6) are not quadratic. Therefore we cannot directly apply the diagrammatic perturbation theory, which is based on the Wick’s theorem, to calculate the nonlinear term. Fortunately, when the coherent driving is strong and the nonlinear interaction is weak, the mean photon number circulating inside the resonator is large and the light field behaves as a semi-classical field [35]. In such a case, the saddle-point approximation can be well used to investigate the dynamics of system [22–24]. As show in Fig. 1(b), the mean value of operators are mainly determined by the classical path and the quantum fluctuations are treated as perturbation. The classical path is determined by the principle of least action:

\[
\frac{\delta S}{\delta a_{cl}} = 0, \quad \frac{\delta S}{\delta a_q} = 0,
\]

(7)
which lead to two saddle-point equations

\[ i\partial_t a_q = \frac{1}{2} (2\Delta_c + i\gamma) a_q + \chi(2a^*_c a_c a_q + a^*_q a^*_q a^2_q + a^*_q a^*_q a_q^2) , \]  

(8)

\[ i\partial_t a_{c,l} = i\sqrt{2}\Omega - i\gamma a_q + \frac{1}{2} (2\Delta_c - i\gamma) a_{c,l} + \chi(2a_{c,l} a^*_q a_q + a^*_c a^*_c a_{c,l} + a^*_c a^*_q a^2_q). \]  

(9)

Equation (8) is always solved by

\[ a_q = a^*_q = 0. \]  

(10)

By substituting \( a_q = a^*_q = 0 \) into Eq. (6), we find that the action \( S = 0 \) in the steady-state case. Indeed, in this case, \( a_+ = a_- \) and the action on the forward part of the contour is canceled by that on the backward part [23]. In addition, we also obtain \( a_{c,l} = i\sqrt{2}\Omega_0 \), where \( \Omega_0 = a_+ = a_- \) is the steady-state mean value of \( \hat{a} \), i.e., \( \langle \hat{a} \rangle = \Omega_0 \) [25]. By substituting \( a_q = 0 \) and \( a_{c,l} = i\sqrt{2}\Omega_0 \) into Eq. (9) and making \( i\partial_t a_{c,l} = 0 \), we obtain \( \Omega_0 = -2i\Omega / (2\Delta_c - i\gamma + 4\chi |a_0|^2) \), from which the mean photon number

\[ \langle \hat{a}^\dagger \hat{a} \rangle = |a_0|^2 = \frac{4\Omega^2}{4(\Delta_c + 2\chi |a_0|^2)^2 + \gamma^2}. \]  

(11)

This solution is identical to the mean-field solution of the steady state [35, 36]. In fact, the saddle-point approximation is equivalent to the mean-field approach, named the linearization approximation, in quantum optics [35]. In the spirit of the linearization approximation, the operator \( \hat{a} \) can be split into an average amplitude and a fluctuation term, i.e., \( \hat{a} \rightarrow \langle \hat{a} \rangle + \delta \hat{a} \), where \( \langle \hat{a} \rangle \) is determined by the mean-field equation. The correspondence between these two methods is shown in Fig. 1(b). As pointed out in Ref. [36], when the sign of \( \Delta_c \) is opposite to that of \( \chi \), Eq. (11) may have two stable solutions; see the red lines of Fig. 2. In other words, the action of system has two classical paths; see Fig. 1(c). As a result, the perturbation calculation around the classical path maybe not reasonable. This phenomenon, named as optical bistability, signals the failure of both the linearization approximation and the saddle-point approximation. On the contrary, Drummond and Walls derived a complex \( P \)-representation solution for the steady state [36]. In that method, they considered the quantum fluctuation effect and found that the exact steady-state solution does not exhibit bistability.

### III. KELDYSH-HEISENBERG EQUATIONS

We note that the standard saddle-point equations are based on the coherent-state basis. The coherent state is the closest quantum mechanical state to a classical description of the field. It is a suitable representation for optical fields when the total photon number is large and the quantum fluctuations are weak [35]. Obviously, this condition is not satisfied in the bistable region. As shown in Fig. 2, in that region the coherent driving is weak and the Kerr nonlinearity is the same order as the other parameters. Therefore the mean photon number is not so large and quantum fluctuations induced by the Kerr nonlinearity can not be ignored. To overcome this shortcoming, we introduce the Fock state, which is the eigenstate of the photon number operator. In the Fock-state basis, we develop a new method called Keldysh-Heisenberg equations that governs the quantum fluctuation effect.

In the Fock-state basis, the completeness relation inserted in between consecutive time steps of the Keldysh close time contour becomes \( \hat{1}_F = \sum_n |n\rangle \langle n| \); see Fig. 3. In this case, the Keldysh partition function for stationary states reads (see Appendix A for details)

\[ Z = \text{Tr} \left[ \exp(i\hat{S}) \right], \]  

(12)

where \( \text{Tr} \) denotes the trace operation which connects the two time branches, giving rise to the closed Keldysh contour [24]. \( \hat{S} \) is the quantum action (like a time evolution
operator) that
\[
\dot{S} = - \int_{-\infty}^{+\infty} \mathcal{H} \, dt.
\]  
(13)

In Eq. (13), \( \mathcal{H} \) is a generalized Hamiltonian operator. As shown in Appendix A, \( \mathcal{H} \) consists of operators acting on different branches of the Keldysh closed time contour. For the driven-dissipative Kerr nonlinear resonator described in Eq. (2), the generalized Hamiltonian operator has the form
\[
\mathcal{H} = \Delta_+ a_+^\dagger a_+ + \chi a_+^\dagger a_-^\dagger a_- + i \Omega \left( a_+^\dagger - a_- \right)
- \Delta_- a_-^\dagger a_- - \chi a_-^\dagger a_- - i \Omega \left( a_-^\dagger - a_- \right)
+ i \gamma a_+ a_-^\dagger - i \frac{\gamma}{2} \left( a_+^\dagger a_+ + a_-^\dagger a_- \right),
\]  
(14)

where \( \hat{a}_\pm \) (\( \hat{a}_\pm^\dagger \)) are the annihilation (creation) operators and the subscript + (−) means that the operator only acts on the forward (backward) time branch. Note that these operators obey the commutation relations:
\[
\left[ \hat{a}_+, \hat{a}_-^\dagger \right] = \left[ \hat{a}_-, \hat{a}_-^\dagger \right] = 1 \quad \text{and} \quad \left[ \hat{a}_+, \hat{a}_- \right] = 0.
\]

Comparing Eq. (13) with Eq. (4), it can be found that we only need to replace the complex variable \( a_+ \) (\( a_- \)) by the corresponding operator \( \hat{a}_+ \) (\( \hat{a}_- \)) and omit the derivative with respect to time. We further define \( \hat{a}_{cl} \) (\( \hat{a}_{cl}^\dagger \)) as the mean photon number basis and \( \beta_m \) is the expansion coefficient. Interestingly, using Eqs. (15), (16) and (19), we verify \( \mathcal{H} \vert \Psi_0 \rangle = 0 \). In other words, the steady state is the ground state of \( \mathcal{H} \). At the same time, it is straightforward to see \( \dot{S} \langle \Psi_0 \vert = - \int_{-\infty}^{+\infty} \mathcal{H} \, dt \langle \Psi_0 \vert = 0 \), which is corresponding to the above discussion in section II that \( S \) = 0 in steady-state case.

Finally, using \( \mathcal{H} \vert \Psi_0 \rangle = 0 \), we get a recursion relation for the expansion coefficient as
\[
\beta_m = \sqrt{\frac{2}{m \, x + m - 1}} \beta_{m-1},
\]  
(20)

with \( \epsilon = \Omega / (i \chi) \) and \( x = (2 \Delta_+ - i \gamma) / (2 \chi) \). Based on this recursion relation, the steady-state wave function
\[
\vert \Psi_0 \rangle = \frac{1}{\sqrt{N}} \langle 0 \vert \sum_{m=0}^{+\infty} \frac{(\sqrt{\pi} \gamma)^m}{\sqrt{m!}} \frac{\Gamma(x)}{\Gamma(x + m)} \vert m \rangle_{cl},
\]  
(21)

where \( \Gamma(x) \) is the gamma special function and \( N = \frac{\Gamma(x + 2 \vert \epsilon \vert^2)}{\Gamma(x + 2 \vert \epsilon \vert^2)} \) is the normalization constant with 
\[
\frac{\Gamma(x + 2 \vert \epsilon \vert^2)}{\Gamma(x + 2 \vert \epsilon \vert^2)} = \sum_{m=0}^{+\infty} \frac{\Gamma(x + 2 \vert \epsilon \vert^2)}{\Gamma(x + 2 \vert \epsilon \vert^2)} \frac{\Gamma(x + m)}{\Gamma(x + m)} \frac{2 \vert \epsilon \vert^2}{m!}
\]  
being the generalized hypergeometric function. According to the relation \( \hat{a}_{cl} = (\hat{a}_+ + \hat{a}_-) / \sqrt{2} \), we obtain the steady-state correlation function
\[
\langle \hat{a}_+^\dagger \hat{a}_+ \rangle = \frac{1}{\sqrt{2}} \langle \Psi_0 \vert a_{cl}^\dagger a_{cl} \vert \Psi_0 \rangle
\]  
(22)

which is equivalent to the complex \( P \)-representation solution in Ref. [36]. In Fig. 2, we plot the steady-state mean photon number \( \langle \hat{a}_+^\dagger \hat{a}_+ \rangle \) as a function of the coherent driving amplitude \( \Omega \). Obviously, the exact steady state does not exhibit bistability; see the blue dashed line.

IV. NONLINEAR DRIVING CASE

In this section, we extend our method to the two-photon nonlinear driving case implemented recently in
superconducting quantum circuits [37]. The effective Hamiltonian reads
\[
\hat{H} = \Delta_c \hat{a}_c^\dagger \hat{a}_c + \chi \hat{a}_c^\dagger \hat{a}_c^2 + i \Omega \left( \hat{a}_c^\dagger - \hat{a}_c \right) + \frac{1}{2} \left( \Lambda \hat{a}_c^\dagger \Lambda^* \hat{a}_c^2 \right),
\]  
(23)
where \( \Lambda \) is the complex amplitude of the two-photon driving term. The Lindblad master equation becomes
\[
\frac{d}{dt} \hat{\rho}(t) = -i \left[ \hat{H}, \hat{\rho}(t) \right] + \gamma D[\hat{a}] \hat{\rho}(t) + \kappa D[\hat{a}^2] \hat{\rho}(t),
\]  
(24)
where \( \kappa \) is the two-photon loss rate.

In the presence of the two-photon driving and loss terms, we rewrite \( \hat{H} = \hat{H}_1 + \hat{H}_2 \) as
\[
\hat{H}_1 = \frac{1}{2} (2 \Delta_c - i \gamma) \hat{a}_c^\dagger \hat{a}_c + \chi \left( \hat{a}_c^\dagger \hat{a}_c + \hat{a}_c^\dagger \hat{a}_c^\dagger \hat{a}_c \right) + \frac{1}{2} \left( \Lambda \hat{a}_c^\dagger \Lambda^* \right) + \frac{1}{2} \left( \Lambda^* \hat{a}_c^\dagger \Lambda \right),
\]  
(25)
\[
\hat{H}_2 = \frac{1}{2} (2 \Delta_c + i \gamma) \hat{a}_c^\dagger \hat{a}_c + \chi \left( \hat{a}_c^\dagger \hat{a}_c \hat{a}_c^\dagger \hat{a}_c + \hat{a}_c^\dagger \hat{a}_c^\dagger \hat{a}_c \right) - \frac{i}{2} \left( \Lambda \hat{a}_c^\dagger \Lambda^* \right) - \frac{i}{2} \left( \Lambda^* \hat{a}_c^\dagger \Lambda \right).
\]  
(26)
As shown in section II, we define the steady-state wave function \( |\Psi_0\rangle = |0\rangle \sum_{m=0}^\infty \beta_m |m\rangle \). The condition \( \hat{H}[\Psi_0] = 0 \) induces a recursion relation for the expansion coefficient as
\[
[(2 \Delta_c - i \gamma) + (2 \chi - i \kappa) (m - 1)] \beta_{m-1} = -i \kappa \sqrt{2 \Omega} \beta_m - 2 \Delta \sqrt{m} - 1 \beta_{m-2}.
\]  
(27)
The last term in Eq. (27), which corresponds to the term \( \Lambda \hat{a}_c^\dagger \hat{a}_c^\dagger \), in \( \hat{H}_1 \), makes the recursion relation difficult to solve. To simplify the calculation, we use a displacement transformation \( \hat{H}' = e^{\Lambda \hat{a}^\dagger} (\hat{H} + \hat{H}_1) e^{-\Lambda \hat{a}} = \hat{H}_1 + \hat{H}_2 \). Under this transformation, \( \hat{a} \) does not change \( (e^{\Lambda \hat{a}} e^{-\Lambda \hat{a}} \rightarrow \hat{a} \) ), but \( \hat{a}_c \) has a displacement \( (e^{\Lambda \hat{a}} e^{-\Lambda \hat{a}} \rightarrow \hat{a}_c - \lambda) \). When choosing \( \lambda = \sqrt{2 \Delta / (2 \chi - i \kappa)} \), the term \( \Lambda \hat{a}_c^\dagger \hat{a}_c^\dagger \) can be eliminated and the condition \( \hat{H}[\Psi_0] = 0 \) is thus equivalent to \( \hat{H}'[\Psi_0] = 0 \) with \( |\Psi_0\rangle = e^{\Lambda \hat{a}} |\Psi_0\rangle = |0\rangle \sum_{m=0}^\infty \phi_m |m\rangle \). Since \( \hat{H}_2 \) is proportional to \( \hat{a}_q \), the equation \( \hat{H}'[\Psi_0] = 0 \) reduces to \( \hat{H}_1'[\Psi_0] = 0 \), with
\[
\hat{H}_1' = \chi \left[ \hat{a}_c^\dagger \hat{a}_c \hat{a}_c + \Lambda \hat{a}_c^\dagger \hat{a}_c + (\hat{a}_c - \lambda) \right] \hat{a}_c^\dagger
\]  
(28)
And the recursion relation for the expansion coefficient is given by
\[
\phi_m = \frac{2 \lambda \sqrt{m + 1}}{\sqrt{m + 1} - \phi_{m-1}},
\]  
(29)
where \( y = -(i \kappa \sqrt{2 \Omega} + \lambda (2 \Delta_c - i \gamma))/(2 \lambda (2 \chi - i \kappa)) \) and
\[
z = (2 \Delta_c - i \gamma)/(2 \chi - i \kappa) \]. This recursion relation is solved by \( \phi_m = (2 \lambda \sqrt{m})^{\frac{(y+z)}{m}} \), from which the steady-state wave function \( |\Psi_0\rangle = e^{-\Lambda \hat{a}} |\Phi_0\rangle \) is obtained by (see Appendix B for details)
\[
|\Psi_0\rangle = \frac{1}{\sqrt{N}} |0\rangle \sum_{n=0}^{\infty} (-\lambda)^{-n} F_1\left(-m, y; z; 2\right) \sqrt{m} |m\rangle.
\]  
(30)
where \( N = \sum_{n=0}^{\infty} \frac{|\lambda|^{-2n}}{n!} 2 F_1\left(-m, y; z; 2\right) \) is the normalization constant and
\[
F_1\left(-m, y; z; 2\right) = \sum_{n=0}^{\infty} \frac{(-m)_n (y)_n}{(z)_n} \frac{2^n}{n!}.
\]  
(31)
where \( F_{m+k} = (-\lambda)^{m+k} 2 F_1\left(-m+k, y; z; 2\right) \). It can be verified that Eq. (31) is equivalent to the solution in Ref. [38].

Using Eq. (31), we can study the influence of different driving processes on the nonlinear effects. For example, we consider the multiphoton resonances in the weak driving regime, which are easy to observe in experiments. In this situation, the mean photon number is small and the mean-field approach is not reasonable. We firstly make a qualitative prediction from the Hamiltonian (23). When the energy of \( n \) incident photons is equivalent to the energy of \( n \) photons inside the resonator, that is
\[ n\omega_p = n\omega_c + \chi n (n-1), \]  
the absorption of \( n \) pumping photons is favored. Expressed in terms of the detuning \( \Delta_c = \omega_c - \omega_p \), this relation reads \( \Delta_c/\chi = -(n-1) \). On the other hand, the parity of \( n \) depends on the driving processes. In the absence of the one-photon driving (\( \Omega = 0 \) and \( \Lambda \neq 0 \)), even number of pumping photons are favored \( (n \text{ is even}) \) and \( \Delta_c/\chi = -1, -3, -5, \ldots \), while in the presence of both the one- and two-photon driving (\( \Omega \neq 0 \) and \( \Lambda \neq 0 \)), \( n \) can be any integer greater than 0 and \( \Delta_c/\chi = 0, -1, -2, -3, -4, \ldots \). In Fig. 4, we plot the steady-state mean photon number \( \langle a^\dagger a \rangle \) as a function of the detuning \( \Delta_c/\chi \), based on Eq. (31). This figure shows clearly that in the absence of the one-photon pumping (see the blue dashed line), the photon resonances arise around \( \Delta_c/\chi = -1 \) and \(-3\), while in the presence of both the one- and two-photon drivings (see the red solid line), the photon resonances arise around \( \Delta_c/\chi = 0, -1, -2, \) and \(-3\). These results are consistent with the qualitative analysis.

V. CONCLUSIONS

In summary, we have established the Keldysh path-integral theory in the Fock-states basis, from which the Keldysh-Heisenberg equations are successfully introduced. In contrast to the standard saddle-point equations, these quantum operator equations can well describe the quantum fluctuation effect and thus present the exact steady-state solutions. We have also considered two examples about the driven-dissipative Kerr nonlinear resonators with/without the two-photon nonlinear driving. Our results agree well with the qualitative analysis and those obtained by the complex \( P \)-representation method [36, 38] and the coherent quantum-absorber method [39, 40].

Before ending up this paper, we compare our method with the complex \( P \)-representation method [36] and the coherent quantum-absorber method [39, 40], both of which have also considered the quantum fluctuation effect. For the complex \( P \)-representation method, an operator master equation has been transformed to a c-number Fokker-Planck equation, and many complicated integration operations have to be faced. While for the coherent quantum-absorber method, an auxiliary resonator, which has the same Hilbert space dimension as the original resonator, should be introduced. By constructing the Hamiltonian for the auxiliary resonator appropriately, this cascaded system has a “dark” state. Then, one can get the steady state of the original system by tracing out the auxiliary cavity. We developed Keldysh functional-integral method with the Keldysh-Heisenberg equations is more physical and intuitive. Moreover, it can be extended to deal with more complex problems such as strongly-correlated photons [20, 41], based on the powerful toolbox of quantum field theory.

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**Appendix A: Keldysh partition function in the Fock-states basis**

In this appendix, we drive the Keldysh partition function in the Fock-states basis in details. A general Lindblad master equation reads

\[
\frac{d}{dt} \rho(t) = \mathcal{L} \rho(t) = -i \left[ \hat{H}, \rho(t) \right] + \gamma \mathcal{D}[\hat{o}] \hat{o} \rho(t),
\]

(A1)

where \( \hat{H} \) is the any Hamiltonian of system. For one or two photon loss process, \( \hat{o} \) can be chosen as \( \hat{a} \) or \( \hat{a}^\dagger \). Without loss of generality, we set \( \hat{o} = \hat{a} \) hereafter. Using the master equation (A1), the time evolution of the density matrix from \( t_0 \) to \( t_f \) is formally solved by

\[
\rho(t_f) = e^{(t_f-t_0)\mathcal{L}} \rho(t_0) = \lim_{N \to \infty} (1 + \delta t \mathcal{L})^N \rho(t_0),
\]

(A2)

where we have decomposed the time evolution into a sequence of infinitesimal steps of duration \( \delta t = (t_f - t_0) / N \). We focus on a single time step, and denote the density matrix after the \( j \)-th step \( (t_j = t_0 + j \delta t) \) by \( \hat{\rho}_j = \rho(t_j) \). Then we have

\[
\hat{\rho}_{j+1} = e^{\delta t \mathcal{L}} \hat{\rho}_j = (1 + \delta t \mathcal{L}) \hat{\rho}_j + O(\delta t^2).
\]

(A3)

Since the Liouville superoperator \( \mathcal{L} \) acts on the density matrix “from both sides”. It is more convenient to represent the density matrix in the Keldysh closed time contour. As shown in Fig. 3 of the main text, this can be achieved by projecting the density matrix into the two time branches [24]:

\[
\hat{\rho}_j \equiv \hat{P}_{+,j} \hat{\rho}_j \hat{P}_{-,j},
\]

(A4)

where \( \hat{P}_{+,j} (\hat{P}_{-,j}) \) is the projection operator on the forward (backward) branch at the time \( t_j \). Obviously, if we choose \( \hat{P} \) as a unit operator of the coherent state, i.e., \( \hat{P} = \hat{1}_{coh} \int \int (d\alpha^+ da) e^{-|\alpha|^2} |\alpha \rangle \langle \alpha| \), we can get the partition function in Sec. II [24]. Instead, here we choose \( \hat{P} \) as the identity in the Fock space, i.e., \( \hat{P}_\pm = \sum_n |n_\pm \rangle \langle n_\pm| \). In this case, \( \hat{\rho}_j \) can be written as

\[
\hat{\rho}_j \equiv \sum_{m,n} |m_+ \rangle \langle m_+| \hat{\rho}_j |n_- \rangle \langle n_-| = \sum_{m,n} \langle m_+| \hat{\rho}_j |n_- \rangle |m_+ \rangle \langle n_-|.
\]

(A5)
We now consider $\hat{\rho}_{j+1} = \sum_{k,l} \langle k_+ | \hat{\rho}_{j+1} | l_- \rangle | k_+ \rangle \langle l_- |$ in terms of the corresponding matrix element at the previous time step $t_j$. Inserting Eq. (A5) into Eq. (A3), we obtain

\[
\langle k_+ | \hat{\rho}_{j+1} | l_- \rangle = \sum_{m,n} \langle k_+ | [(1 + \delta t \mathcal{L}) (\langle m_+ | \hat{\rho}_j | n_- \rangle | m_+ \rangle | n_- \rangle)] | l_- \rangle = \sum_{m,n} (\delta_{k,m} \delta_{l,n} - i \delta t \mathcal{H}_{k,l,m,n}) \langle m_+ | \hat{\rho}_j | n_- \rangle , \tag{A6}
\]

where

\[
\mathcal{H}_{k,l,m,n} = i \langle k_+ | \mathcal{L} | m_+ \rangle \langle n_- | l_- \rangle = \langle k_+ | \hat{H} | m_+ \rangle \langle n_- | l_- \rangle - \langle k_+ | m_+ \rangle \langle n_- | l_- \rangle + i \gamma \langle k_+ | \hat{a} | m_+ \rangle \langle n_- | \hat{a}^\dagger | l_- \rangle - \frac{i \gamma}{2} \langle k_+ | \hat{a}^\dagger \hat{a} | m_+ \rangle \langle n_- | \hat{a}^\dagger \hat{a} | l_- \rangle . \tag{A7}
\]

Equation (A7) shows that the operators act on the forward or backward time branch, respectively. Therefore, we can introduce a generalized Hamiltonian operator:

\[
\hat{\mathcal{H}} = \hat{H}_+ - \hat{H}_- + i \gamma \hat{a} \hat{a}^\dagger - \frac{i \gamma}{2} (\hat{a}^\dagger \hat{a}^\dagger \hat{a} + \hat{a}^\dagger \hat{a}^\dagger \hat{a}) , \tag{A8}
\]

where $\hat{H}_\pm$ are the Hamiltonians of the forward and backward time branches, respectively. Based on Eqs. (A7) and (A8), $\mathcal{H}_{k,l,m,n}$ can be seen as a matrix element of $\hat{\mathcal{H}}$, i.e.,

\[
\mathcal{H}_{k,l,m,n} = \langle n_- | k_+ | \hat{\mathcal{H}} | l_- \rangle | m_+ \rangle , \tag{A9}
\]

and the trace of $\hat{\rho}_{j+1}$ can thus be expressed as a simple form:

\[
\text{Tr} \hat{\rho}_{j+1} = \text{Tr} \sum_{k,l,m,n} (\delta_{k,m} \delta_{l,n} - i \delta t \mathcal{H}_{k,l,m,n}) \langle m_+ | \hat{\rho}_j | n_- \rangle | k_+ \rangle \langle l_- | m_+ \rangle \hat{\rho}_j = \text{Tr} \left( - i \delta t \hat{\mathcal{H}} \right) \hat{\rho}_j = \text{Tr} \left( e^{-i \delta t \hat{\mathcal{H}}} \right) \hat{\rho}_j + O(\delta t^2) . \tag{A10}
\]

By iteration of Eq. (A10), the density matrix can be evolved from $\hat{\rho}(t_0)$ at $t_0$ to $\hat{\rho}(t_f)$ at $t_f$. This implies that in the limit $N \to \infty$ (and hence $\delta t \to 0$),

\[
Z_{t_f,t_0} = \text{Tr} \hat{\rho}(t_f) = \text{Tr} \left[ \exp(i \hat{S}) \hat{\rho}(t_0) \right] , \tag{A11}
\]

with $\hat{S} = - \int_{t_0}^{t_f} \hat{\mathcal{H}} dt$.

Finally, we perform the limit, $t_0 \to -\infty$ and $t_f \to +\infty$, to get the Keldysh partition function for stationary states. Since in a Markov process, the initial state in the infinite past does not affect the stationary state [24], we can ignore the boundary term, i.e., $\hat{\rho}(t_0)$ in Eq. (A11), and obtain the final expression of the Keldysh partition function as

\[
Z = \text{Tr} \left[ \exp(i \hat{S}) \right] , \tag{A12}
\]

with the quantum action

\[
\hat{S} = - \int_{-\infty}^{+\infty} \hat{\mathcal{H}} dt . \tag{A13}
\]

Appendix B: Steady-state wave function for the nonlinear driving case

We present the detailed derivation of Eq. (30) of the main text.

\[
| \Psi_0 \rangle = e^{-\lambda \hat{a}^\dagger j m} | \Phi_0 \rangle = \frac{1}{\sqrt{N}} e^{-\lambda \hat{a}^\dagger j m} | \Phi \rangle \sum_{k=0}^{+\infty} \phi_k | k \rangle_{cl} = \frac{1}{\sqrt{N}} | 0 \rangle \sum_{j=0}^{+\infty} (-\lambda j) (2 \lambda)^{j} j! \phi_k | k \rangle_{cl}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{j,k=0}^{+\infty} (-\lambda j) (2 \lambda)^{j} j! \phi_k | k \rangle_{cl} \sum_{j,k=0}^{+\infty} \frac{2k (\lambda)^{j} j! \phi_k | k \rangle_{cl} \langle j+k |_{cl} \langle j+k |_{cl}}{j!k!}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{j,k=0}^{+\infty} \frac{2k (\lambda)^{j} j! \phi_k | k \rangle_{cl} \langle j+k |_{cl}}{j!k!}
\]

Let $j+k = m$, we obtain

\[
| \Psi_0 \rangle = \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]

\[
= \frac{1}{\sqrt{N}} | 0 \rangle \sum_{m=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{(2 \lambda)^{m} (\lambda)^{-m} \phi_k | k \rangle_{cl} \langle m \rangle_{cl}}{\sqrt{m!} (m-j)! k! (z)^{m}}
\]
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