Ward Identities and Renormalization of General Gauge Theories

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Abstract

We introduce the concept of general gauge theory which includes Yang-Mills models. In the framework of the causal approach and show that the anomalies can appear only in the vacuum sector of the identities obtained from the gauge invariance condition by applying derivatives with respect to the basic fields. Then we provide a general result about the absence of anomalies in higher orders of perturbation theory. This result reduces the renormalizability proof to the study of lower orders of perturbation theory. For the Yang-Mills model one can perform this computation explicitly and obtains its renormalizability in all orders.

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1 Introduction

The causal approach to perturbative renormalization theory of by Epstein and Glaser [21], [22] gives significant simplification of the conceptual and computational aspects for quantum electrodynamics [39], [11], [26], Yang-Mills theories [14], [15], [17], [18], [1], [3], [7], [8], [33], [35], [19], [23], [24], [25], [40], [41], [28], gravity [30], [31], [44], the analysis of scale invariance can be done [27], [37], Wess-Zumino model [29], etc. In this approach one uses exclusively the Bogoliubov axioms of renormalization theory [6] imposed on the scattering matrix: this is an operator acting in the Hilbert space of the model, which is a Fock space generated from the vacuum by the quantum fields corresponding to the particles of the model. If one considers the S-matrix as a perturbative expansion in the coupling constant of the theory, one can translate these axioms on the chronological products. Epstein-Glaser approach is an inductive procedure to construct the chronological products in higher orders starting from the first-order of the perturbation theory - the interaction Lagrangian - which is a Wick polynomial. For gauge theories one can construct a non-trivial interaction only if one considers a larger Hilbert space generated by the fields associated with the particles of the model and the ghost fields. The condition of gauge invariance becomes in this framework the condition of factorization of the S-matrix to the physical Hilbert space in the adiabatic limit. To avoid infra-red problems one works with a formulation of this factorization condition which corresponds to a formal adiabatic limit and it is perfectly rigorously defined [15]. The obstructions to the implementation of the condition of gauge invariance are called anomalies. The most famous is the Adler-Bell-Bardeen-Jackiw anomaly (see [36] for a review). The most convenient way to organize the combinatorial argument seems to be the following one [12], [11]. One constructs the chronological products $T(W_1(x_1), \ldots, W_n(x_n))$ associated to arbitrary Wick monomials $W_1(x_1), \ldots, W_n(x_n)$ according to Epstein-Glaser prescription [21] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [38] (which reduces the renormalization procedure to the process of extension of distributions).

If $T(x)$ is the interaction Lagrangian (i.e. the first order chronological product) and $d_Q$ the BRST operator, we suppose the validity of some “descent” equations of the type:

$$d_Q T(x) = i \partial_x T(x), \quad d_Q T^\mu(x) = i \partial^\mu T(x), \ldots,$$

$$d_Q T^{\mu_1 \ldots \mu_{p-1}}(x) = i \partial_{\nu_p} T^{\mu_1 \ldots \mu_p}(x), \quad d_Q T^{\mu_1 \ldots \mu_p}(x) = 0$$

(1.0.1)

for some finite $p$. One denotes by $A^k(x)$, $k = 1, 2, \ldots$, the expressions $T(x)$, $T^\mu(x)$, $T^{\mu_1 \mu_2 \ldots \mu_p}(x)$, and we suppose that these expressions have a well defines ghost number, i.e all terms of the Wick polynomial $A^k(x)$ have the same ghost degree. Then we can write the preceding equation in the compact form

$$d_Q A^k(x) = i \sum_m c^k_m \partial_x A^m(x), \quad k = 1, 2, \ldots$$

(1.0.2)

for some constants $c^k_m$. The gauge invariance condition has the generic form

$$d_Q T(A^{k_1}(x_1), \ldots, A^{k_n}(x_n)) = i \sum_{l=1}^n (-1)^n \sum_m c^k_m \partial_{x_i} T(A^{k_1}(x_1), \ldots, A^m(x_l), \ldots, A^{k_n}(x_n))$$

(1.0.3)
for all $n \in \mathbb{N}$ and all $k_1, \ldots, k_p = 1, 2, \ldots$. Here the expression

$$s_l \equiv \sum_{i=1}^{l-1} gh(A^{k_i}). \quad (1.0.4)$$

One can define [11] the notion of derivative of a Wick polynomial with respect to the basic fields of the model. In particular one can apply these derivatives operators to the polynomials $A^k(x), \ k = 1, 2, \ldots$.

Then we can prove that the gauge invariance condition can be reduced to some identities verified by the vacuum expectation values of the chronological products of the following type: $<\Omega, T(DA^{k_1}(x_1), \ldots, DA^{k_n}(x_n))\Omega>$; here $DA^k(x), \ k = 1, 2, \ldots$ are polynomials in the derivatives of the basic expressions $A^k(x), \ k = 1, 2, \ldots$. These are the so-called C-g identities in the language of [14] - [17]; it is plausible to expect that they are equivalent to the Ward (Slavnov - Taylor) identities from the usual formulation of gauge theories so we prefer to call them Ward identities.

This idea was used in [11] to study the conservation of the electromagnetic current in quantum electrodynamics. The generalization of these idea to non-Abelian gauge theories is under current investigation [12].

We can show that the anomalies of these Ward identities are absent in higher orders of the perturbation theory (more precisely in orders greater than the dimension of the space-time +1). In this way, one can check the gauge invariance of the Yang-Mills model if one establishes that there are no anomalies in the orders 2, 3, 4 and 5.

We start in the next Section with a systematic study of the Wick monomials. In particular, we circumvent the complications associated with the signs coming from the fields with Fermi-Dirac statistics using Grassmann variables (following a suggestion from [39]). In Section 3 we sketch the framework of the perturbative renormalization theory of Bogoliubov. In the next Section we formulate the notion of general gauge theory and we derive the Ward identities. In the last Section we check the absence of anomalies in lower orders of the perturbation theory for the Yang-Mills model.
2 The General Framework

2.1 Free Fields

We define here the general framework of a free field theory in the Fock space following closely the point of view of [14]. Some standard notions from quantum relativistic mechanics are used [15], [21]. We define a system of free fields to be the ensemble $(\phi^A(x), \mathcal{F}, \Omega, U_{a,L})$ where:

(i) $\phi^A(x)$, $A = 1, \ldots, N$ are distribution-valued operators acting in the Fock space $\mathcal{F}$ with a common dense domain $D_0$. Here $x \in M$ where $M$ is the Minkowski space.

(ii) $\Omega \in D_0$ is called the vacuum state. The vectors $\phi^{A_1}(x_1) \cdots \phi^{A_n}(x_n)\Omega$ generate the Fock space $\mathcal{F}$.

(iii) $a, L \mapsto U_{a,L}$ is a unitary representation of the group $SL(2, \mathbb{C})$ acting in $\mathcal{F}$ such that

$$U_{a,L}\phi^A(x)U_{a,L}^{-1} = S(L^{-1})^A_B\phi^B(\delta(L) \cdot x + a); \quad (2.1.1)$$

here $SL(2, \mathbb{C}) \ni L \mapsto \delta(L) \in L_+^\mathbb{R}$ is the covering map and $SL(2, \mathbb{C}) \ni L \mapsto S(L)$ is a $N \times N$ representation of $SL(2, \mathbb{C})$.

(iv) $\text{supp}(\tilde{\psi}^A) \subset V^+_M \cup V^-_M$ where $M_A \geq 0$ is called the mass of the field $\psi^A$, $A = 1, \ldots, N$.

(v) Let us denote by $\phi^A_\pm$ the positive (negative) frequency components of $\psi^A$ such that $\text{supp}(\tilde{\phi}^A_\pm) \subset V^\pm_\mathbb{M}$; then there exist a system of numbers $z_A \in \mathbb{Z}$, $A = 1, \ldots, N$ such that the following canonical (anti)commutation relations are true:

$$\phi^A_\pm(x)\phi^B_\pm(y) = (-1)^{z_Az_B}\phi^B_\pm(y)\phi^A_\pm(x),$$

$$\phi^A_\pm(x)\phi^B_\mp(y) - (-1)^{z_Az_B}\phi^B_\mp(y)\phi^A_\pm(x) = D^{AB}_\pm(x-y) \times 1_\mathcal{F} \quad (2.1.2)$$

where $D^{AB}_\pm(x)$ are distributions verifying:

$$D^{AB}_\pm(x) = 0, \quad \text{iff} \quad z_A + z_B \neq 0, \quad (2.1.3)$$

and if we define

$$D^{AB}(x) \equiv D^{AB}_+(x) + D^{AB}_-(x) \quad (2.1.4)$$

then these distributions have causal support: $\text{supp}(D^{AB}(x)) \subset V^+ \cup V^-$. (vi) One has

$$\phi^A(x)\Omega = 0, \quad \forall A = 1, \ldots, N. \quad (2.1.5)$$

(vii) Equations of motion of the type

$$\sum_\alpha u^\alpha_\beta \partial_\alpha \phi^A(x) = 0, \quad A = 1, \ldots, N \quad (2.1.6)$$

for some constants $u^\alpha_\beta$ are verified; here we use Schwartz multi-indices $\alpha, \beta, \ldots$ but one can also use alternative notation $u^{\mu
u\ldots}_A$ from jet-bundle extension theory. One cannot avoid the existence of the equations of motion: indeed, because of the requirement (iv) the fields will verify Klein-Gordon equation:

$$\partial^2\phi^A + M^2_A\phi^A = 0, \quad A = 1, \ldots, N \quad (2.1.7)$$
i.e. the preceding equation is true for

\[ u_A^{\mu\nu} = M_A^2 g^{\mu\nu} u_A \tag{2.1.8} \]

for arbitrary numbers \( u_A \). For Dirac fields one has a first order system of equations of motion: the Dirac equation.

(viii) For some of the fields some reality condition might be imposed, connecting the Hermitian conjugates \((\phi^A)^*\) and the original fields \(\phi^B\).

Let us remind the fact that from (ii) one can derive that if the (graded) commutators of some operator \(X\) with all fields \(\phi^A\) are zero, then this operator is proportional to the unit operator \(\mathbf{1}\) from the Fock space.

One can easily see that all known models in Fock spaces can be accommodated in this scheme.

We avoid the complications due to the signs from (2.1.2) if we consider a \(\mathbb{Z}\)-graded Grassmann algebra \(\mathcal{G} = \sum_{n \in \mathbb{Z}} \mathcal{G}_n\) over \(\mathbb{C}\) and some Grassmann numbers \(g_A \in \mathcal{G}\) which are invertible and of parity \(z_A\), \(\forall A = 1, \ldots, N\). Then we consider distributions with values in \(\mathcal{G} \otimes \mathcal{L}(\mathcal{F})\) (here \(\mathcal{L}(\mathcal{F})\) are the linear operators from \(\mathcal{F}\)) given by:

\[ \varphi^A(x) \equiv g_A \otimes \phi^A(x), \quad \forall A = 1, \ldots, N. \tag{2.1.9} \]

We call these operators the *supersymmetric associated fields*. We consider \(J^r(\mathbb{R}^N, M)\) the \(r\)-th order jet bundle extension of the trivial fibre bundle \(\mathbb{R}^N \times M \to M\) which describes classical fields with \(N\) components defined over the Minkowski space \(M \sim \mathbb{R}^4\) and we consider the jet bundle coordinates \(u^a_A, A = 1, \ldots, N, |\alpha| \leq r\). The natural number \(r\) should be chosen large enough. Then we define the operators:

\[ \varphi^\pm_u(x) \equiv \sum_{\alpha, A} u^a_A \partial_\alpha \varphi^A_{\pm}(x) \tag{2.1.10} \]

and

\[ \varphi_u(x) \equiv \varphi^+_u(x) + \varphi^-_u(x). \tag{2.1.11} \]

We call *mass-shell* the linear subspace:

\[ \mathcal{M} \equiv \{ u \in J^r(\mathbb{R}^N, M) | \varphi_u = 0 \} \tag{2.1.12} \]

and we denote \([u] \equiv u \mod \mathcal{M}\). We see that in fact the operators \(\varphi_u(x)\) depend only on the equivalence class \([u]\) i.e. we can consistently use the notation:

\[ \varphi_{[u]}(x) \equiv \varphi_u(x). \tag{2.1.13} \]

One can verify elementary that we have the following form of the canonical commutator relations:

\[ [\varphi^+_u(x), \varphi^+_w(y)] = 0, \]

\[ [\varphi^+_u(x), \varphi^-_w(y)] = \Delta^+_uw(x - y), \]

\[ [\varphi_u(x), \varphi_w(y)] = \Delta_{uw}(x - y) \tag{2.1.14} \]
where in the left hand side we have the usual commutator and we have defined:

\[
\Delta_{\pm}^{uw}(x - y) \equiv \sum_{\alpha, A} \sum_{\beta, B} g^{A} g^{B} u_{\alpha}^{A} w_{\beta}^{B} \partial_{\alpha}^{x} \partial_{\beta}^{y} D^{AB}(x - y),
\]

\[
\Delta_{uw}(x - y) \equiv \Delta_{+}^{uw}(x - y) + \Delta_{-}^{uw}(x - y).
\] (2.1.15)

One can easily see that the distribution \( \Delta_{uw}(x - y) \) has causal support and we have the symmetry property

\[
\Delta_{uw}(x - y) = \Delta_{wu}(y - x).
\] (2.1.16)

It is convenient to choose the Grassmann algebra \( G \) such that \( G_{0} = \mathbb{C} \); then we have \( \Delta_{\pm}^{uw}(x - y) \in \mathbb{C} \).

The fact that in (2.1.14) we have the ordinary commutator will make all the following computations more convenient because will we not worry about the Jordan signs appearing for fields with Fermi-Dirac statistics.

We will need in the following a derivative operation defined on the classical fields jet bundle: if \( \alpha \) is multi-index, then we define \( \partial_{\alpha} : J^{r}(\mathbb{R}^{N}, M) \rightarrow J^{r}(\mathbb{R}^{N}, M) \) according to:

\[
(\partial_{\alpha} u)^{\beta}_{A} \equiv \sum_{\alpha + \beta = \gamma} u_{\gamma}^{\gamma}.
\] (2.1.17)

Then we have three elementary facts:

**Proposition 2.1**  
(i) If \( u \in M \) then \( \partial_{\alpha} u \in M \) for any multi-index \( \alpha \).

**Proof:** One applies to the equations of motion (2.1.6) the partial derivative operator \( \partial_{\alpha} \). \( \blacksquare \)

**Proposition 2.2**  
The following relations are true:

\[
\partial_{\alpha} \varphi_{u}(x) = \varphi_{\partial_{\alpha} u}(x),
\] (2.1.18)

\[
\partial_{\alpha} \Delta_{uw} = \Delta_{\partial_{\alpha} u, w}.
\] (2.1.19)

**Proof:** The first relation is a result of a elementary computation. For the second relation, we apply the partial derivative operator \( \partial_{\alpha} \) to the last canonical commutation relation (2.1.14) and use the first relation. \( \blacksquare \)

We also note that there is a natural group action \( l, u \mapsto l \cdot u \) of the group \( SL(2, \mathbb{C}) \) on the elements of \( J^{r}(\mathbb{R}^{N}, M) \). The Hermitian conjugation operation postulated at item (viii) of the preceding Subsection induces a natural conjugation operation \( u \mapsto u^{*} \) on the classical fields from \( J^{r}(\mathbb{R}^{N}, M) \). We will need these operations later.

We end this Subsection pointing the fact that in the literature one usually uses Grassmann valued classical fields instead of the classical fields from \( J^{r}(\mathbb{R}^{N}, M) \). The connection is \( J^{r}(\mathbb{R}^{N}, M) \ni u_{\alpha}^{A} \rightarrow g^{A} u_{\alpha}^{A} \in J^{r}(G \otimes \mathbb{R}^{N}, M) \).

The classical structure \( J^{r}(\mathbb{R}^{N}, M) \) associated to the Wick monomials algebra was used somewhat differently in \[4\] and \[11\].
2.2 Supersymmetric Wick Monomials

A complete and rigorous investigation of the Wick combinatorial arguments can be found in [13]. Here we give an approach which does not use Feynman graphs. We use consistently Bourbaki conventions \( \sum_{\emptyset} \equiv 0, \prod_{\emptyset} \equiv 1 \). We will define Wick monomials through the following proposition:

**Proposition 2.3** The operator-valued distributions \( N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) \) are uniquely determined through the following properties:

\[
N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n))\Omega = \varphi_{u_1}^+(x_1), \ldots, \varphi_{u_n}^+(x_n)\Omega; \quad (2.2.1)
\]

\[
[N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)), \varphi_w(y)] = \sum_{l=1}^n N(\varphi_{u_1}(x_1), \ldots, \varphi^{\sim}_{u_l}, \ldots, \varphi_{u_n}(x_n))\Delta_{u_lw}(x_l - y); \quad (2.2.2)
\]

\[
N(\emptyset) \equiv 1. \quad (2.2.3)
\]

In the first two relations \( n \) is arbitrary.

**Proof:** Is is elementary. For \( n = 1 \) we find from the second property that \( N(\varphi_u(x)) - \varphi_u(x) \) commutes with every operator \( \varphi_w(y) \) so it must be a of the form \( \text{const} \times 1 \). But the first relation fixes this constant to 0. Next, we suppose that we have defined the expressions \( N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_{n-1}}(x_{n-1})) \) and we use the second and the first relation to define the action of \( N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) \) on vectors of the type \( \varphi_{w_1}(y_1), \ldots, \varphi_{w_k}(y_k)\Omega \); from (ii) of the previous Subsection we know that they generate the whole Fock space. ■

We call the operators \( N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) \) supersymmetric Wick (or normal) monomials in \( n \) variables.

Let us note that in fact, the Wick monomial \( N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) \) depends only on the equivalence classes \( [u_1], \ldots, [u_n] \). Using induction, one can easily prove that it is completely symmetric in the arguments.

We can easily establish the connection with the usual definition of the Wick monomials:

**Proposition 2.4** The following relation is true:

\[
N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) = \sum_{I,J} \prod_{i \in I} \varphi_{u_i}^+(x_i) \prod_{j \in J} \varphi_{u_j}^-(x_j). \quad (2.2.4)
\]

**Proof:** We first note that the order of the factors in the two products is irrelevant because of the commutativity property \( (2.1.14) \). The proof consists in denoting the right hand side of the relation by \( N'(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) \) and proving they three relations appearing in the preceding proposition are true. Then we use the uniqueness assertion. ■

As a immediate corollary we obtain:
Corollary 2.5  The following relations are true:

\[
N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n), \varphi_w(y)) = N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n))\varphi_w(y)
- \sum_{l=1}^{n} <\Omega, \varphi_{u_l}(x_l)\varphi_w(y)\Omega > N(\varphi_{u_1}(x_1), \ldots, \hat{\varphi}_{u_l}, \ldots, \varphi_{u_n}(x_n));
\]  

(2.2.5)

\[
N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n), \varphi_w(y)) = \varphi_w(y)N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n))
+ \sum_{l=1}^{n} \Delta_{u_lw}(x_l - y)N(\varphi_{u_1}(x_1), \ldots, \hat{\varphi}_{u_l}, \ldots, \varphi_{u_n}(x_n))
\]  

(2.2.6)

Proof: The first relation follows immediately from the preceding proposition. If we combine it with the second property from Prop. 2.3, then we obtain the second relation. ■

If we apply the preceding results we can also obtain using induction

Corollary 2.6  The normal products can be expressed as

\[
N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)) = \sum_k \sum_{i_1 < \cdots < i_k} d^+(x_1, \ldots, x_n)\varphi_{u_{i_1}}(x_{i_1}), \ldots, \varphi_{u_{i_k}}(x_{i_k})
\]  

(2.2.7)

where \(d^+(x_1, \ldots, x_n)\) are distributions.

In fact, one can express the distributions from the statement as sum of distributions \(d^+_U\) labelled by Feynman graphs \([13]\). However, we do not need this result here. Now, a non-trivial observation is that if we formally “colapse” all arguments \(x_1, \ldots, x_n \mapsto x\) in the expression \(N(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n))\) we obtain well-defined operators:

Proposition 2.7  The expressions

\[
W_{u_1, \ldots, u_n}(x) \equiv N(\varphi_{u_1}(x), \ldots, \varphi_{u_n}(x))
\]  

(2.2.8)

are well defined and they are completely symmetric in the indices \(u_1, \ldots, u_n\).

Proof: We collapse the arguments in the relations of the preceding proposition and formally obtain:

\[
W_{u_1, \ldots, u_n}(x)\Omega = \varphi_{u_1}^+(x), \ldots, \varphi_{u_n}^+(x)\Omega;
\]  

(2.2.9)

\[
[W_{u_1, \ldots, u_n}(x), \varphi_w(y)] = \sum_{l=1}^{n} W_{u_1, \ldots, \hat{u_l}, \ldots, u_n}(x)\Delta_{u_lw}(x_l - y);
\]  

(2.2.10)

\[
W_\emptyset \equiv 1.
\]  

(2.2.11)

The only non-trivial step is to prove that the right hand side of the first relation is well defined; this is done in [16]. The rest of the proof is identical. ■

If \(U \equiv \{u_1, \ldots, u_n\}\) then we can use consistently the notation \(W_U(x)\). We call expressions of this type a supersymmetric Wick monomials of rank \(n\) in one variable. Again we note that the dependence on the classical fields \(u_1, \ldots, u_n\) is only through the equivalence classes. The action \(l, u \mapsto l \cdot u\) of the group \(SL(2, \mathbb{C})\) on the jet bundle coordinates extends naturally to the action \(l, U \mapsto l \cdot U\) componentwise. The same assertion is valid for the hermitian conjugation \(u \mapsto u^*\) which extends componentwise to \(U \mapsto U^*\). We now give an elementary result:
Proposition 2.8  The following formula is true:

\[
\partial_\alpha W_{u_1,\ldots,u_n}(x) = \sum_{l=1}^n W_{u_1,\ldots,\partial_\alpha u_l,\ldots,u_n}(x). \tag{2.2.12}
\]

Proof: It follows by induction commuting both sides with an arbitrary field \( \varphi_w(y) \).

By definition, a (supersymmetric) Wick polynomial is any linear combination (with coefficients from \( G \)) of Wick monomials. The set of all (supersymmetric) Wick polynomials in the Fock space \( \mathcal{F} \) is denoted by \( \text{sWick}(\mathcal{F}) \). The action of \( SL(2,\mathbb{C}) \) and the Hermitian conjugation extend naturally to the set of (supersymmetric) Wick polynomials. When no ambiguity is possible we abandon the attribute supersymmetric.

A generalization of the collapsing procedure used above is available and essential to the perturbation theory. Namely, we consider the expression \( N(\varphi_{u_1}(x_1),\ldots,\varphi_{u_k}(x_k)) \) for \( k > n \) and group the variables \( x_1,\ldots,x_k \) in \( n \) subsets; then we collapse the arguments to distinct points inside every subset.

Proposition 2.9  The expressions

\[
N(W_{U_1}(x_1),\ldots,W_{U_n}(x_n)) \equiv N(\prod_{u \in U_1} \varphi_u(x_1),\ldots,\prod_{u \in U_n} \varphi_u(x_n)). \tag{2.2.13}
\]

are well-defined and completely symmetric in the arguments.

Proof: As before, we obtain from the first proposition the following relations:

\[
N(W_{U_1}(x_1),\ldots,W_{U_n}(x_n))\Omega = \prod_{i=1}^n \prod_{u \in U_i} \varphi_u^+(x_i)\Omega; \tag{2.2.14}
\]

\[
[N(W_{U_1}(x_1),\ldots,W_{U_n}(x_n)),\varphi_w(y)] = \sum_{l=1}^n \sum_{u \in U_l} N(W_{U_1}(x_1),\ldots,W_{U_l-\{u\}},\ldots,W_{U_n}(x_n))\Delta_{uw}(x_l - y); \tag{2.2.15}
\]

\[
N(W(x)) \equiv W(x) \tag{2.2.16}
\]

and we can use recursion.

We have results similar to Corollary 2.3:

Corollary 2.10  The following relations are true:

\[
N(W_{U_1}(x_1),\ldots,W_{U_k}(x_k),\varphi_w(y)) = N(W_{U_1}(x_1),\ldots,W_{U_k}(x_k))\varphi_w(y)
- \sum_{l=1}^k \sum_{u \in U_l} <\Omega,\varphi_u(x_l)\varphi_w(y)\Omega > N(W_{U_1}(x_1),\ldots,W_{U_l-\{u\}},\ldots,W_{U_k}(x_k)); \tag{2.2.17}
\]
Proof: We take in Corollary (2.5) \( \{ u_1, \ldots, u_n \} = \cup_{i=1}^k U_i \) and we “colapse” the variables \( x_j \) pertaining to the same set \( U_i \). In this way the first two relations follow. The last relation follows from the first one. ■

The relation (2.2.13) from the proof of Proposition (2.9) is remarkable and deserves a special name. We call an ensemble of operator-valued distributions \( E(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) \) of (supersymmetric) Wick type if and only if the following two conditions are verified:

\[
E(\emptyset, \ldots, \emptyset) = \text{const.},
\]

\[
E(\emptyset, \ldots, \varphi_u(x_i), \ldots, \emptyset) = \Omega E(\emptyset, \ldots, \varphi_u(x_i), \ldots, \emptyset) \Omega > 1 + E(\emptyset, \ldots, \emptyset) \varphi_u(x_i), \quad (2.2.20)
\]

\[
\left[ E(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)), \varphi_w(y) \right] = \sum_{i=1}^n \sum_{u \in U_i} E(W_{U_1}(x_1), \ldots, W_{U_i-u}(x_i), \ldots, W_{U_n}(x_n)) \Delta_{uw}(x_i - y); \quad (2.2.21)
\]

It is easy to note that if \( E(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) \) and \( F(W_{U_{n+1}}(x_{n+1}), \ldots, W_{U_{n+m}}(x_{n+m})) \) are expression of Wick type, then \( E(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) F(W_{U_{n+1}}(x_{n+1}), \ldots, W_{U_{n+m}}(x_{n+m})) \) is also an expression of Wick type. The assertion stays true for more than two factors.

We can extend by linearity a expression of Wick type to Wick polynomials: if \( W_j(x_j) \) are Wick monomials, and \( c_{i_1 j_1}, \ldots, c_{i_n j_n} \in \mathcal{G} \) then we define:

\[
E(\sum_{i_1 j_1} c_{i_1 j_1} W_{j_1}(x_1), \ldots, \sum_{i_n j_n} c_{i_n j_n} W_{j_n}(x_n)) \equiv \sum_{i_1 j_1} \ldots c_{i_n j_n} E(W_{j_1}(x_1), \ldots, W_{j_n}(x_n)) \quad (2.2.22)
\]

where the convention about the order of factors is important because of the non-commutativity of the elements of the Grassmann algebra.

The well-known 0-theorem of Epstein-Glaser asserts that expressions of the type

\[
E(W_1(x_1), \ldots, W_n(x_n)) \equiv d(x_1, \ldots, x_n) N(W_1(x_1), \ldots, W_n(x_n)) \quad (2.2.23)
\]

where \( d(x_1, \ldots, x_n) \) is a translation-invariant distribution, are well-defined. They are obviously expressions of Wick type. If we take the distribution \( d \) of the form

\[
d(x_1, \ldots, x_n) = p(\partial)\delta^{n-1}(X) \quad (2.2.24)
\]
where $p(\partial)$ is a polynomial in the partial derivatives and
\[
\delta^{n-1}(X) \equiv \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n)
\] (2.2.25)
then we obtain some special expressions of Wick type called quasi-local operators \[^{[3]}\]. Such type of operator have a distinguished rôle in perturbative renormalization theory.

This analysis culminates with an extremely neat form of Wick theorem.

**Theorem 2.11** Let $E(W_{U_1}(x_1), \ldots, W_{U_n}(x_n))$ be an expression of Wick type. Then the following relation is valid:
\[
E(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) = \sum_{U_i' \subset U_i} <\Omega, E(W_{CU_i'}(x_1), \ldots, W_{CU_i'}(x_n))\Omega > (N(W_{U_i'}(x_1), \ldots, W_{U_i'}(x_n))); \quad (2.2.26)
\]
here $CU_i' \equiv U_i - U_i'$ are the set-theoretically complements.

**Proof:**
It is done by induction over the rank $r \equiv |U_1| + \cdots |U_n|$. For $r = 1$ the formula from the statement is trivial. We suppose that the formula is true for $|U_1| + \cdots |U_n| = r - 1$ and we prove it for $|U_1| + \cdots |U_n| = r$. One commutes both sides of the identity to be proven with any arbitrary $\varphi_w(y)$ and, using the induction hypothesis, obtains equality. It follows that the relation to be proven is valid up to a constant operator. If we average on the vacuum we obtain that the constant is, in fact, zero. \(\blacksquare\)

### 2.3 Derivatives of Wick Polynomials

We can give alternative expressions for this theorem if we introduce the notion of derivative of a Wick monomial \[^{[11]}\]. We give here a more compact treatment. Let us denote the coordinates on the dual of the classical fields bundle $(J^r(\mathbb{R}^N, M))^\ast$ by $v^A_{\alpha}$; the duality form is:
\[
<v, u> \equiv \sum_{A,\alpha} v^A_{\alpha} u^A_{\alpha}. \quad (2.3.1)
\]
We consider the polar of the mass-shell:
\[
\mathcal{M}^0 \equiv \{ v \in (J^r(\mathbb{R}^N, M))^\ast | <v, u> = 0, \quad \forall u \in J^r(\mathbb{R}^N, M) \}. \quad (2.3.2)
\]
Then we have the following elementary result:

**Proposition 2.12** Let $v \in \mathcal{M}^0$; the the operator $\rho(v) : s\text{Wick}(\mathcal{F}) \rightarrow s\text{Wick}(\mathcal{F})$ is well defined by
\[
\rho(v)W_U(x) \equiv \sum_{u \in U} <v, u> W_{U - \{u\}}(x) \quad (2.3.3)
\]
and linearity. Moreover, these operators commute among themselves:
\[
[\rho(v_1), \rho(v_2)] = 0, \quad \forall v_1, v_2 \in \mathcal{M}^0. \quad (2.3.4)
\]
We call $\rho(v)$ derivative operators of Wick polynomials. Because of the commutativity it makes sense to define for any set $V = \{v_1, \ldots, v_n\}$ of elements from $M^0$ derivative operators of higher order through:

$$\rho(V) \equiv \prod_{i=1}^{n} \rho(v_i).$$  \hfill (2.3.5)

We can provide an alternative expression for the normal products and Wick monomials.

**Proposition 2.13** Let us consider $\{v_j\}_{j \in J}$ a basis in $M^0$ and $\{v^*_j\}_{j \in J}$ a dual basis in a supplement $M'$ of $M \subset J'(\mathbb{R}^N, M)$ such that the completeness relation is valid:

$$\sum_{j \in J} (v_j^*)_A^\alpha(v_j)_B^\beta = \delta^\alpha_\beta \delta^B_A. \hfill (2.3.6)$$

Then the following formulæ are valid:

$$N(\varphi_u(x_1), \ldots, \varphi_u(x_n)) = \sum_{j_1, \ldots, j_n \in J} \prod_{k=1}^{n} <v_{j_k}, u_k> \Delta v^*_{j_k} w(x - y) = \Delta w(x - y)$$  \hfill (2.3.7)

$$W_{u_1, \ldots, u_n}(x) = \sum_{j_1, \ldots, j_n \in J} \prod_{k=1}^{n} <v_{j_k}, u_k> W_{v^*_{j_1}, \ldots, v^*_{j_n}}(x). \hfill (2.3.8)$$

**Proof:** We use a technique familiar by now. Let us denote the right hand side of the first relation by $N'(\varphi_u(x_1), \ldots, \varphi_u(x_n))$ and check that the properties from Proposition 2.3 are true. One must use the relation

$$\sum_{j \in J} <v_j, u> \Delta_{v^*_j w}(x - y) = \Delta_{uw}(x - y) \hfill (2.3.9)$$

which is a consequence of the completeness relation. The second relation from the statement follows if we “collapse” the arguments into the first one. $\blacksquare$

Now we can give two alternative formulation of Wick theorem. First we have:

**Theorem 2.14** Every Wick expression $E(W_1(x_1), \ldots, W_n(x_n))$ (here $W_1(x_1), \ldots, W_n(x_n)$ are Wick polynomials) verify the following relation:

$$[E(W_1(x_1), \ldots, W_n(x_n)), \varphi_w(y)] = \sum_{l=1}^{n} \sum_{j \in J} \Delta_{v^*_j w}(x_l - y) E(W_1(x_1), \ldots, \rho(v_j) W_l(x_l), \ldots, W_n(x_n)). \hfill (2.3.10)$$

In particular we have for every Wick polynomial:

$$[W(x), \varphi_w(y)] = \sum_{j \in J} \Delta_{v^*_j w}(x - y) \rho(v_j) W(x). \hfill (2.3.11)$$
Proof: It is sufficient to consider that $W_1(x_1), \ldots, W_n(x_n)$ are Wick monomials. Then we use the defining relation for a expression of Wick type and the relation (2.3.9). ■

Now we can give another compact form of Wick theorem.

**Theorem 2.15** The following formula is valid:

$$E(W_1(x_1), \ldots, W_n(x_n)) = \sum_{V_i} <\Omega, E(\rho(V_1)W_1(x_1), \ldots, \rho(V_n)W_n(x_n))\Omega > N(W_{V_1^*}(x_1), \ldots, W_{V_n^*}(x_n)); \quad (2.3.12)$$

where the sum runs over all sets $V_i$ of elements of the type $v_j$ ($j \in J$) from $\mathcal{M}^0$.

**Proof:** As before, it is sufficient to consider that the expressions $W_i$ are Wick monomials. If we use Proposition 2.13 we obtain that the right hand side of the relation from the statement coincides with the right hand side of the relation from Wick theorem (2.2.26). ■

We can extend the operation of derivation $\partial_\alpha$ to elements of the polar $\mathcal{M}^0$ through duality: we have

**Proposition 2.16** Let us define $\partial_\alpha : (J^r(\mathbb{R}^N, M))^* \to (J^{r-|\alpha|}(\mathbb{R}^N, M))^*$ according to:

$$(\partial_\alpha v)^A_{\beta} \equiv v^{A+\beta}_A. \quad (2.3.13)$$

Then: (i) If $v \in \mathcal{M}^0$ we also have $\partial_\alpha v \in \mathcal{M}^0$ for any multi-index $\alpha$.

(ii) The following commutation relation is valid:

$$[\rho(v), \partial_\alpha] = \rho(\partial_\alpha v). \quad (2.3.14)$$

**Proof:** The first assertion follows from the corresponding property for the derivatives of the elements of $J^r(\mathbb{R}^N, M)$ and the duality relation:

$$<\partial_\alpha v, u > = <v, \partial_\alpha u>. \quad (2.3.15)$$

This identity, as well as the last relation from the statement can be proved directly from the definitions by elementary computations. ■

It it convenient to introduce some particular derivatives of the type $\rho(v)$. We consider some field $\varphi^A$ (the index $A$ is fixed) constrained only by the Klein-Gordon equation (2.1.7). Then we define the elements $v_A, \; v^\mu_A \in \mathcal{M}^0$ by giving only the non-zero entries:

$$(v_A)^B = \delta_A^B, \; (v^\mu_A)^B_{\nu} = -\frac{1}{4}M^2\delta^B_A g^\nu_\rho \delta^\mu_{\rho\sigma}; \quad (2.3.16)$$

and respectively

$$(v^\mu_A)^B_{\nu} = \delta^B_A \delta^\mu_{\nu}, \; (v^\mu_A)^B_{\nu\sigma} = -\frac{1}{4}M^2\delta^B_A \delta^\mu_{\nu\sigma}; \quad (2.3.17)$$

indeed we immediately have

$$<v_A, u > = 0, \; <v^\mu_A, u > = 0, \; \forall u \in \mathcal{M}. \quad (2.3.18)$$
Then we denote:
\[
\frac{\partial}{\partial \phi^A} \equiv \rho(v_A), \quad \frac{\partial}{\partial \phi^A_\mu} \equiv \rho(v^\mu_A);
\] (2.3.18)
these are in fact derivatives with respect to the basic fields and their first order jet extension.

Next we have the following result \[11\] following directly from the second formula of theorem 2.14. We say that \(W_U\) is a Wick monomial of first order if all elements \(u \in U\) verify \(u_A = 0, \forall |\alpha| > 1\). A first order Wick polynomial is a sum of first order Wick monomials.

**Proposition 2.17** Suppose that \(W\) is a Wick polynomial of first order. Then we have:
\[
[W(x), \varphi^A(y)] = \frac{\partial}{\partial \varphi^B} W(x) \Delta^{AB}(x-y) \rho(v_j) + \frac{\partial}{\partial \varphi^A_\mu} W(x) \varphi^A \Delta^{AB}(x-y).
\] (2.3.19)
The proof is elementary. We also define:
\[
\partial_{\sigma} \cdot \frac{\partial}{\partial \varphi^A} \equiv \rho(\partial_{\sigma} \cdot v_A), \quad \partial_{\sigma} \cdot \frac{\partial}{\partial \varphi^A_\mu} \equiv \rho(\partial_{\sigma} \cdot v^\mu_A).
\] (2.3.20)

Then we have for any first order Wick polynomial \(W\) the following formulæ:
\[
\partial_{\sigma} \cdot \frac{\partial}{\partial \varphi^A} W(x) = -\frac{1}{4} M^2_A \frac{\partial}{\partial \varphi^A} W(x), \quad \partial_{\sigma} \cdot \frac{\partial}{\partial \varphi^A_\mu} W(x) = \delta^\mu_{\sigma} \frac{\partial}{\partial \varphi^A} W(x).
\] (2.3.21)

We finally notice that we can define in a natural way the ghost number of the derivatives \(\frac{\partial}{\partial \varphi^A}, \frac{\partial}{\partial \varphi^A_\mu}\) to be \(z_A\). If the elements of the set \(V = \{v_1, \ldots, v_k\}\) are of defined ghost number, then
\[
gh(V) \equiv \sum gh(v_i).
\] (2.3.22)

### 2.4 Wick Monomials

In this Subsection, we make the connection with the ordinary Wick monomials defined in the original Fock space \(\mathcal{F}\). Loosely speaking, if we strip a supersymmetric Wick monomial of their Grassmann factors in a consistent way, we obtain the usual Wick monomials. First we have:

**Proposition 2.18** Let \(\sigma\) be section of the fibre bundle: \(J^r(\mathbb{R}^N, M) \rightarrow J^r(\mathbb{R}^N, M)/\mathcal{M}\). Then every Wick monomial can be uniquely written in the form
\[
W_{u_1, \ldots, u_n}(x) = \prod_{i=1}^n (\tilde{u}_i)^{\alpha_i}_{A_i} \ W_{\alpha_1, \ldots, \alpha_n}^{A_1, \ldots, A_n}(x) = \prod_{i=1}^n (\tilde{u}_i)^{\alpha_i}_{A_i} g_{A_i} \ \mathcal{W}_{\alpha_1, \ldots, \alpha_n}^{A_1, \ldots, A_n}(x)
\] (2.4.1)
where \(\tilde{u}_i \equiv \sigma([u_i])\), \(W_{\alpha_1, \ldots, \alpha_n}^{A_1, \ldots, A_n}(x)\) are operator-valued distributions with values in \(\mathcal{G} \otimes \mathcal{L}(\mathcal{F})\) and \(\mathcal{W}_{\alpha_1, \ldots, \alpha_n}^{A_1, \ldots, A_n}(x)\) are operator-valued distributions with values in \(\mathcal{L}(\mathcal{F})\). A similar writing can be established for supersymmetric expressions of Wick type.
Proof: We define recurrently the expressions $W_{\alpha_1,\ldots,\alpha_n}^A(x)$ through the following properties:

\[ W_{\alpha_1,\ldots,\alpha_n}^A(x)\Omega = \prod_{i=1}^n \partial_{\alpha_i} \varphi^A_i(x)\Omega; \]  

\[ [W_{\alpha_1,\ldots,\alpha_n}^A(x), \partial_\alpha \varphi^B(x)] = \sum_{l=1}^n W_{\alpha_1,\ldots,\alpha_n}^{A_l} (x) \partial_{\alpha_l} \partial_\beta \Delta_{AB}(x-y); \]  

\[ W_{\emptyset}^0(x) = 1. \]  

One can prove that these relations define uniquely the expressions $W_{\alpha_1,\ldots,\alpha_n}^A(x)$ using a familiar argument. Then we obtain the first equality from the statement using the uniqueness argument from Proposition 2.7.

The expressions $W_{\alpha_1,\ldots,\alpha_n}^A(x)$ can be defined quite similarly and we obtain the second equality from the statement. □

One can see that the expressions $W_{\alpha_1,\ldots,\alpha_n}^A(x)$ are in fact supersymmetric Wick monomials: they can be obtained for some special choice of the classical fields $u_i$. Moreover, they are completely symmetric in the couples $(A, \alpha)$ and are not linearly independent. In fact, we have “equation of motion” of the type:

\[ \sum_{\alpha_l} u_{A_l}^{\alpha_l} W_{\alpha_1,\ldots,\alpha_n}^A(x) = 0, \quad \forall u \in \mathcal{M}. \]  

(2.4.5)

Similar assertion are valid for $W_{\alpha_1,\ldots,\alpha_n}^A(x)$; more precisely, we have skew-symmetry in the couples $(A, \alpha)$ and appropriate equations of motion. The expression $W_{\alpha_1,\ldots,\alpha_n}^A(x)$ are called Wick monomials and $W_{\alpha_1,\ldots,\alpha_n}(x)$ is the associated supersymmetric Wick monomial. A Wick polynomial is an operator acting in $\mathcal{F}$ of the following form:

\[ \mathcal{L}(x) = \sum C_{\alpha_1,\ldots,\alpha_n} W_{\alpha_1,\ldots,\alpha_n}^A(x) \]  

(2.4.6)

where $C_{\alpha_1,\ldots,\alpha_n}$ are complex constants with convenient (anti)-symmetry properties. We denote by Wick($\mathcal{F}$) the set of Wick polynomials in $\mathcal{F}$. If we express the operators $W_{\alpha_1,\ldots,\alpha_n}(x)$ in term of $W_{\alpha_1,\ldots,\alpha_n}^A(x)$ then we canonically associate to the Wick polynomial $\mathcal{L}(x)$ a super-symmetric Wick polynomial $L(x)$ acting in $\mathcal{G} \otimes \mathcal{L}(\mathcal{F})$.

We can define now some derivative operators. First, we note that the derivation $\rho(v)$ induces a derivation, also denoted $\rho(v)$ on the space of Wick polynomials. Next, we have the following result:

**Proposition 2.19** Let us define the operators: $\partial_\beta^B : \text{sWick} \rightarrow \text{sWick}$ according to:

\[ \partial_\beta^B W_{\alpha_1,\ldots,\alpha_n}^{A_1,\ldots,A_n}(x) \equiv \sum_{l=1}^n \delta_{B_l}^\beta \delta_{A_l}^{A_i} W_{\alpha_1,\ldots,\alpha_n}^{A_1,\ldots,A_n}(x). \]  

(2.4.7)

Then the following relations are true:

\[ \partial_\beta^B W_U(x) = \sum_{u \in U} \tilde{u}_B^\beta W_{U-\{u\}}(x), \quad \rho(v) = \sum_{\alpha, A} v_{\alpha}^{A} \partial_\alpha A, \quad \forall v \in \mathcal{M}^0. \]  

(2.4.8)
3 Perturbation Theory in the Causal Approach

We give here the essential ingredients of perturbation theory using the supersymmetric formalism described in the preceding Section.

3.1 Bogoliubov Axioms

We use, essentially, the point of view of of Stora and Fredenhagen [12], [11], [4] using chronological product. An equivalent point of view uses retarded products [43]. By a perturbation theory in the sense of Bogoliubov we mean an ensemble of operator-valued distributions \( T(W_1(x_1), \ldots, W_n(x_n)) \in \mathcal{G} \otimes \mathcal{L}(\mathcal{F}) \), \( n = 1, 2, \ldots \) called (super-symmetric) chronological products (here \( W_1(x_1), \ldots, W_n(x_n) \) are supersymmetric Wick polynomials) verifying the following set of axioms:

- Symmetry in all arguments \( W_1(x_1), \ldots, W_n(x_n) \);
- Poincaré invariance: for all \((a, L) \in inSL(2, \mathbb{C})\) we have:
  \[ U_{a,L}T(W_1(x_1), \ldots, W_n(x_n))U_{a,L}^{-1} = T(L \cdot W_1(\delta(L) \cdot x_1 + a), \ldots, L \cdot W_n(\delta(L) \cdot x_n + a)); \quad (3.1.1) \]

Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available. Also some other global symmetry with respect to some internal symmetry group might be imposed.

- Causality: if \( x_i \geq x_j, \quad \forall i \leq k, \quad j \geq k + 1 \) then we have:
  \[ T(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_k(x_k))T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n)); \quad (3.1.2) \]

- Unitarity: We define the (super-symmetric) anti-chronological products according to

\[
(-1)^n \hat{T}(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum_{r=1}^{n}(-1)^r \sum_{I_1, \ldots, I_r \in Part\{1, \ldots, n\}} T_{I_1}(X_1) \cdots T_{I_r}(X_r) \quad (3.1.3)
\]

where the we have used the notation:

\[
T_{\{i_1, \ldots, i_k\}}(x_{i_1}, \ldots, x_{i_k}) \equiv T(W_{i_1}(x_{i_1}), \ldots, W_{i_k}(x_{i_k})). \quad (3.1.4)
\]

Then the unitarity axiom is:

\[
\hat{T}(W_1(x_1), \ldots, W_n(x_n)) = T(W_1^*(x_1), \ldots, W_n^*(x_n)) \quad (3.1.5)
\]

- The “initial condition”

\[
T(W(x)) = W(x). \quad (3.1.6)
\]

Remark 3.1 From (3.1.2) one can derive easily that if we have \( x_i \sim x_j, \quad \forall i \leq k, \quad j \geq k + 1 \) then:

\[
[T(W_1(x_1), \ldots, W_k(x_k)), T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n))] = 0. \quad (3.1.7)
\]
3.2 Epstein-Glaser Construction

Epstein-Glaser construction provides an explicit solution for Bogoliubov axioms. We sketch briefly the proof.

**Theorem 3.2** There exists a solution of Bogoliubov axioms.

**Proof:** Goes by induction. One suppose that the chronological products are constructed up to the order \( n - 1 \) such that all Bogoliubov axioms are verified. We supplement the induction hypothesis with the requirement that the chronological products (up to the order \( n - 1 \)) are expressions of Wick type; this means that we have for all \( p = 1, \ldots, n - 1 \):

\[
[T(W_{U_1}(x_1), \ldots, W_{U_p}(x_p)), \varphi_w(y)] = \sum_{l=1}^{p} \sum_{u \in U_l} T(W_{U_1}(x_1), \ldots, W_{U_l-\{u\}}, \ldots, W_{U_p}(x_p))\Delta_{uw}(x_l - y) \tag{3.2.1}
\]

so, according to Wick theorem 2.11 we have the expansion:

\[
T(W_{U_1}(x_1), \ldots, W_{U_p}(x_p)) = \sum_{U_i' \subset U_i} <\Omega, T(W_{CU_{U_i'}}(x_1), \ldots, W_{CU_{U_i'}}(x_p))\Omega > N(W_{U_1}(x_1), \ldots, W_{U_p}(x_p)). \tag{3.2.2}
\]

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials \( W_1, \ldots, W_p \); explicitly:

\[
\omega(<\Omega, T(W_1(x_1), \ldots, W_p(x_p))\Omega >) \leq \sum_{l=1}^{p} \omega(W_l) - 4(p - 1), \quad p = 1, \ldots, n - 1 \tag{3.2.3}
\]

where by \( \omega(d) \) we mean the order of singularity of the (numerical) distribution \( d \) and by \( \omega(W) \) we mean the canonical dimension of the Wick monomial \( W \). It is easy to check that the induction hypothesis is true for \( n = 1 \).

The construction of Epstein-Glaser is based on the commutator \( D(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) \) with causal support. The explicit expression of this commutator is known in terms of the chronological products up to the order \( n - 1 \). Moreover, from the explicit formula it is clear that this expression is also of Wick type (it is a sum of products of expressions of Wick type -according to the induction hypothesis). So, Wick theorem can be applied and gives an expression of the type:

\[
D(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) = \sum_{U_i' \subset U_i} <\Omega, D(W_{CU_{U_i'}}(x_1), \ldots, W_{CU_{U_i'}}(x_n))\Omega > N(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)). \tag{3.2.4}
\]

One can show that the order of singularity of the numerical distributions

\[
d(x_1, \ldots, x_n) \equiv <\Omega, D(W_{U_1}(x_1), \ldots, W_{U_n}(x_n))\Omega > \tag{3.2.5}
\]
verifies a restriction of the type (3.2.3); this is the content of the so-called power-counting theorem. Next, one can provide in a standard way a causal splitting of the distribution $d(x_1, \ldots, x_n)$ such that Poincaré covariance and the order of singularity are preserved. This induces a causal splitting for the operator-valued distribution

$$D(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) = A(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) - R(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)) \quad (3.2.6)$$

which can be used to construct the $n$-order chronological product $T(W_{U_1}(x_1), \ldots, W_{U_n}(x_n))$ The unitarity can be also fixed quite elementary [21]. The induction is finished. ■

From the construction it follows that one can define the chronological products such that we have (3.2.1), (3.2.2) and (3.2.3) for all $p = 1, 2, \ldots$. The first relation is the normalisation condition (N3) of [11], [4]. According to the previous Section, we also have alternative formulations for the first two of them, namely: we have for all $n \in \mathbb{N}$:

$$[T(W_1(x_1), \ldots, W_n(x_n)), \varphi_w(y)] = \sum_{l=1}^{n} \sum_{j \in J} \Delta_{u^j w}(x_l - y) \ T(W_1(x_1), \ldots, \rho(v_j)W_l(x_l), \ldots, W_n(x_n)) \quad (3.2.7)$$

and

$$= \sum_{V_i \subset M^{u^0}} < \Omega, T(\rho(V_1)W_1(x_1), \ldots, \rho(V_n)W_n(x_n)) \Omega > N(W_{V_1^+}(x_1), \ldots, W_{V_n^+}(x_n)). \quad (3.2.8)$$

We still have some freedom on the chronological products which can be used to impose another condition. Let

$$\Delta_{uw} = \Delta_{uw}^{adv} - \Delta_{uw}^{ret} \quad (3.2.9)$$

be a causal splitting of the distribution with causal support $\Delta_{uw}$. By definition the Feynman propagator and the Feynman antipropagator are:

$$\Delta_{uw}^F \equiv \Delta_{uw}^{adv} - \Delta_{uw}^- = \Delta_{uw}^{ret} + \Delta_{uw}^+, \quad \Delta_{AF} \equiv \Delta_{uw}^+ - \Delta_{uw}^{adv} = -\Delta_{uw}^{ret} - \Delta_{uw}^- \quad (3.2.10)$$

Then we have the following result [12]:

**Theorem 3.3** Suppose that the chronological products have been chosen such that they are expressions of Wick type. Then they can be chosen such that one also has for all $n \in \mathbb{N}$:

$$T(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y)) = \sum_{U_i \subset U_i} < \Omega, T(W_{CU_1}(x_1), \ldots, W_{CU_n}(x_n)) \Omega > N(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y))$$

$$+ \sum_{l=1}^{n} \sum_{u \in U_l} \Delta_{uw}^F(x_l - y) \ T(W_1(x_1), \ldots, W_{U_l-\{u\}}, \ldots, W_n(x_n)) \quad (3.2.11)$$

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**Proof:** It is also based on induction on the rank \(|U_1| + \cdots + |U_n|\) (see also [1]). One can easily see that the relation from the statement is trivial for \(r = 1\). We suppose that it is true for \(|U_1| + \cdots |U_n| = r - 1\) and prove that it can be fixed for \(|U_1| + \cdots |U_n| = r\) also. We use a familiar technique, namely we consider for this case the commutator of both sides of (3.2.8) with an arbitrary \(\varphi_w(z)\) and, using the induction hypothesis, we get zero. So, the relation from the statement for \(|U_1| + \cdots |U_n| = r\) can be affected by a constant “anomaly”:

\[
c(x_1, \ldots, x_n, y) \equiv <\Omega, T(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y))\Omega> \\
- \sum_{l=1}^{n} \sum_{u \in U_l} \Delta_{uw}^F(x_l - y) <\Omega, T(W_{U_1}(x_1), \ldots, W_{U_l-\{u\}}, \ldots, W_{U_n}(x_n))\Omega> . \tag{3.2.12}
\]

Using the causal factorisation property of the chronological products, the induction hypothesis and property (3.2.2) one can prove that the support of the distribution \(c(x_1, \ldots, x_n, y)\) is contained in the diagonal set \(x_1 = \ldots = x_n = y\). This means that we have the generic form

\[
c(x_1, \ldots, x_n, y) = p(\partial)\delta(x_1 - y) \cdots \delta(x_n - y) \tag{3.2.13}
\]

with \(p(\partial)\) some polynomials in the partial derivatives. Moreover, this numerical distribution has convenient covariance properties and a limitation on the degree of \(p\) comes from the power counting limitations in the right hand side of (3.2.12). In the end, it follows that we can absorb the anomaly \(c(x_1, \ldots, x_n, y)\) into the vacuum sector of the chronological product \(T(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y)), \ |U_1| + \cdots |U_n| = r\) without affecting the Epstein-Glaser induction construction. ■

The relation appearing in this proposition is called in [1], [4] the normalization condition (N4).

As in the preceding Section, one can define the chronological products acting in the Hilbert space \(\mathcal{F}\) by stripping the Grassmann variables.
4 General Gauge Theories

4.1 The Supersymmetric BRST Operator

In the general setting of Subsection 2.1 we define a BRST operator $d_Q$ on the set of polynomials in the fields $\phi^A_{\pm}$ through the following properties:

- It gives zero on the constant operator:
  \begin{equation}
  d_Q 1 = 0. \tag{4.1.1}
  \end{equation}

- It is linear over $\mathbb{C}$.

- It acts on the basic fields as follows:
  \begin{equation}
  d_Q \phi^A_{\pm}(x) = -i \sum_{|\alpha| \leq s} \sum_B (q^\alpha)^A_B \partial_{\alpha} \phi^B_{\pm}(x) \tag{4.1.2}
  \end{equation}
  where $q^\alpha$ are real $N \times N$ matrices constrained by
  \begin{equation}
  (q^\alpha)^A_B = 0, \quad \text{if} \quad z_B - z_A \neq 1; \tag{4.1.3}
  \end{equation}
  here $s \in \mathbb{N}^*$ is called the rank of the BRST operator. The usual case is $s = 1$.

- It is a (graded) derivative operator in the sense that for all $\epsilon_1, \ldots, \epsilon_n = \pm$ we have:
  \begin{equation}
  d_Q[\phi^{A_1}_{\epsilon_1}(x_1) \ldots \phi^{A_n}_{\epsilon_n}(x_n)] = \sum_{l=1}^{n} \prod_{i<l} (-1)^{z_{A_i}} \phi^{A_1}_{\epsilon_1}(x_1) \ldots d_Q \phi^{A_l}_{\epsilon_l}(x_l) \ldots \phi^{A_n}_{\epsilon_n}(x_n) \tag{4.1.4}
  \end{equation}

- It commutes with the derivative operators:
  \begin{equation}
  [d_Q, \partial_\beta] = 0. \tag{4.1.5}
  \end{equation}

It is clear that the usual BRST operator appearing in Yang-Mills models is a particular case of this general framework. One can naturally extend the operator $d_Q$ to the set of polynomials in the fields $\varphi^A_{\pm}$, then the following properties are true:

- It gives zero on the constant operator:
  \begin{equation}
  d_Q 1 = 0. \tag{4.1.6}
  \end{equation}

- It is linear over $\mathcal{G}$.

- It acts on the basic fields as follows:
  \begin{equation}
  d_Q \varphi^A_{\pm}(x) = -i \sum_{|\alpha| \leq s} \sum_B (Q^\alpha)^A_B \partial_{\alpha} \varphi^B_{\pm}(x) \tag{4.1.7}
  \end{equation}
  where
  \begin{equation}
  (Q^\alpha)^A_B = g_A g_B^{-1} (q^\alpha)^A_B \tag{4.1.8}
  \end{equation}
  and $\text{deg}((Q^\alpha)^A_B) = -1$. 

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• It is a derivative operator:
\[ d_Q [\varphi_{\xi_1} A_1 (x_1) \ldots \varphi_{\xi_n} A_n (x_n)] = -i \sum_{l=1}^{n} (Q^\alpha)_{B_l} A_l \varphi_{\xi_1} A_1 (x_1) \ldots \partial_\alpha \varphi_{\xi_l} A_l (x_l) \ldots \varphi_{\xi_n} A_n (x_n) \]  
\hfill (4.1.9)

• It commutes with the derivative operators:
\[ [d_Q, \partial_\beta] = 0. \]  
\hfill (4.1.10)

We compute the action of the operator on the supersymmetric fields. We have:

**Proposition 4.1** The following formula is true:

\[ d_Q \varphi_u (x) = \zeta^{-1} \varphi_{Q \cdot u} (x) \]  
\hfill (4.1.11)

where \( \zeta \in \mathcal{G}_1 \) is a fixed invertible element and we have defined:

\[ (Q \cdot u)_A^\alpha \equiv - \sum_{\beta+\gamma=\alpha} \sum_{B} (Q^\beta)_B^A u_B^\gamma. \]  
\hfill (4.1.12)

The proof is elementary. We have introduced the factor \( \zeta \) because \( (Q \cdot u)_A^\alpha \in \mathcal{G}_{-1} \) and in this way \( \zeta Q \cdot u \) has real values and it can be considered as an element of the classical manifold \( J^r (\mathbb{R}^N, M) \).

If we apply the operator \( d_Q \) to the last commutation relation (2.1.14) we obtain:

\[ \Delta \zeta Q \cdot u, w = -\Delta u, \zeta Q \cdot w. \]  
\hfill (4.1.13)

Another consequence of the preceding proposition is:

**Corollary 4.2** The following formulæ are true:

\[ d_Q [\varphi_{u_1}^{\xi_1} (x_1) \ldots \varphi_{u_n}^{\xi_n} (x_n)] = i\zeta^{-1} \sum_{l=1}^{n} \varphi_{u_l}^{\xi_l} (x_1) \ldots \varphi_{Q \cdot u_l}^{\xi_l} (x_l) \ldots \varphi_{u_n}^{\xi_n} (x_n), \]  
\hfill (4.1.14)

\[ d_Q N (\varphi_{u_1}^{\xi_1} (x_1) \ldots \varphi_{u_n}^{\xi_n} (x_n)) = i\zeta^{-1} \sum_{l=1}^{n} N (\varphi_{u_l}^{\xi_l} (x_1) \ldots \varphi_{Q \cdot u_l}^{\xi_l} (x_l) \ldots \varphi_{u_n}^{\xi_n} (x_n)), \]  
\hfill (4.1.15)

\[ d_Q W_{u_1, \ldots, u_n} (x) = i\zeta^{-1} \sum_{l=1}^{n} W_{u_1, \ldots, Q \cdot u_l, \ldots, u_n} (x). \]  
\hfill (4.1.16)

\[ d_Q W_{\alpha_1, \ldots, \alpha_n}^{A_1, \ldots, A_n} (x) = -i \sum_{l=1}^{n} \sum_{B, \beta} (Q^\beta)_B^A W_{\alpha_1, \ldots, \alpha_l-1, B, \alpha_l+1, \ldots, \alpha_n}^{A_1, \ldots, A_l, B, \alpha_l+1, \ldots, A_n} (x) \]  
\hfill (4.1.17)

\[ d_Q W_{\alpha_1, \ldots, \alpha_n}^{A_1, \ldots, A_n} (x) = -i \sum_{l=1}^{n} \sum_{B, \beta} (Q^\beta)_B^A W_{\alpha_1, \ldots, \alpha_l-1, B, \alpha_l+1, \ldots, \alpha_n}^{A_1, \ldots, A_l, B, \alpha_l+1, \ldots, A_n} (x). \]  
\hfill (4.1.18)
Proof: The first relation goes by direct computations from the derivative property of the operator $d_Q$, the second relation follows from the first if we use Proposition 2.4 and the third one follows if we “colapse” the variables into the preceding one. The last two relations are direct consequences of the definitions. ■

We will have to extend the action of the BRST operator to the dual space of classical fields $(J^r(\mathbb{R}^N, M))^*$. For this we need the following result:

**Proposition 4.3** (i) If $u \in \mathcal{M}$ then $\zeta Q \cdot u \in \mathcal{M}$. 
(ii) If $v \in (J^r(\mathbb{R}^N, M))^*$ let us define $Q \cdot v \in (J^{r-s}(\mathbb{R}^N, M))^*$ according to

$$(Q \cdot v)^A_\alpha \equiv \sum_{B,\beta} (Q^\beta)^A_B v^B_{\alpha+\beta}.$$ (4.1.19)

Suppose now that $v \in \mathcal{M}^0$; then $\zeta Q \cdot v \in \mathcal{M}^0$. 
(iii) The following relation is valid:

$$[d_Q, \rho(v)] = i\zeta^{-1} \rho(\zeta Q \cdot v)$$ (4.1.20)

Proof: The proof of the first two assertions are based on elementary computations. For the last relation, it is sufficient to prove it to be true when applied on a Wick monomial $W_{U}(x)$. ■

### 4.2 Gauge Invariant Models

We generalize the framework outlined in the Introduction i.e. we suppose that we have a set of Wick polynomials $A^i(x), \quad i = 1, \ldots, p$, which we organize as a Wick multiplet (a column matrix) $\mathcal{A}$ and some $p \times p$ matrices $c^\alpha$ such that the following relation is true:

$$d_Q A(x) = i \sum_{\alpha} c^\alpha \partial_\alpha A(x).$$ (4.2.1)

The we say that we have a general gauge theory. If $A^i(x)$ are the supersymmetric Wick polynomials associated to $A^i(x), \quad i = 1, \ldots, p$ then a similar relation is verified by them:

$$d_Q A(x) = i \sum_{\alpha} c^\alpha \partial_\alpha A(x).$$ (4.2.2)

We have the following consequence:

**Proposition 4.4** Let $A(x)$ be a general gauge theory. Then we also have:

$$d_Q [\rho(v_1) \cdots \rho(v_k) A(x)] = i \sum_{\alpha} c^\alpha \partial_\alpha [\rho(v_1) \cdots \rho(v_k) A(x)]$$

$$+ \sum_{l=1}^{k} \left[ \zeta^{-1} \rho(v_1) \cdots \rho(\zeta Q \cdot v_l) \cdots \rho(v_k) + \sum_{\alpha} c^\alpha \rho(v_1) \cdots \rho(\partial_\alpha v) \cdots \rho(v_k) \right] A(x)$$ (4.2.3)
The proof is done elementary using the commutation relations (2.3.14) and (1.1.20). One can write the preceding relation more compactly introducing some notations. We denote:

\[ Q \cdot \rho(v_1, \ldots, v_k) \equiv \zeta^{-1} \sum_{l=1}^{k} \rho(v_1, \ldots, \zeta Q \cdot v_l, \ldots, v_k), \]  

(4.2.4)

and

\[ \delta \cdot \rho(v_1, \ldots, v_k) \equiv \sum_{l=1}^{k} \sum_{\alpha} c^\alpha \rho(v_1, \ldots, \partial_\alpha v_l, \ldots, v_k). \]  

(4.2.5)

We also define:

\[ D \equiv \sum_\alpha c^\alpha \partial_\alpha. \]  

(4.2.6)

Then the relation from the preceding proposition can be rewritten as follows:

\[ d_Q \rho(V) A(x) = i(D + Q + \delta) \rho(V) A(x) \]  

(4.2.7)

where \( V = \{v_1, \ldots, v_k\} \).

For further use, we give the following commutation relations:

\[ Q \cdot \rho(v, V) - \rho(v) Q \cdot \rho(V) = \zeta^{-1} \rho(\zeta Q \cdot v) \rho(V) \]  

(4.2.8)

and

\[ \delta \cdot \rho(v, V) - \rho(v) \delta \cdot \rho(V) = \sum_\alpha c^\alpha \rho(\partial_\alpha v) \rho(V). \]  

(4.2.9)

We define a perturbation theory of the general gauge theory \( A(x) \). We say that the chronological products verifying, beside Bogoliubov axioms (and other normalization conditions imposed in the analysis from the preceding Section) verify gauge invariance of rank \( k \) if the following identity is true for any \( |V| = k \):

\[
\sum_{V_1, \ldots, V_n \in \text{Part}(V)} \left\{ d_Q T(\rho(V_1) A(x_1), \ldots, \rho(V_k) A(x_k)) \right. \\
- i \sum_{l=1}^{n} [D_l \cdot T(\rho(V_1) A(x_1), \ldots, \rho(V_k) A(x_k)) \\
+ T(\rho(V_1) A(x_1), \ldots, (Q + \delta) \cdot \rho(V_l) A(x_l), \ldots, \rho(V_k) A(x_k))] \} = 0;
\]  

(4.2.10)

here

\[ D_l \equiv \sum_\alpha (1 \otimes \cdots \otimes c^\alpha \otimes \cdots 1) \partial_\alpha^l. \]  

(4.2.11)

It is not very hard to see that for a Yang-Mills model, the preceding relation for \( k = 0 \) goes into the usual gauge invariance condition for the chronological product of the Wick monomials \( A^i(x) \) so we call this case simply gauge invariance. The cases \( k > 0 \) give the behaviour with respect to the BRST operator of the chronological products of derivatives of the Wick monomials \( A^i(x) \). There is a connection between gauge invariance of rank \( k \) and rank invariance of rank \( k + 1 \) described in the following theorem which is the analogue of the result from Appendix B of [11].
Theorem 4.5 Suppose that \( A(x) \) verifies the gauge invariance condition of rank \( k + 1 \). Then the anomalies of the gauge invariance condition of rank \( k \) can only appear into the vacuum sector.

Proof: As in [11] we consider the gauge invariance condition of rank \( k \) and commute both sides with an arbitrary \( \varphi_w(y) \). Using the induction hypothesis, Wick theorem and some commutation relations derived before we get (after some tedious but straightforward computations) the same expression. This means that the anomaly if proportional to 1 so it can show up only in the vacuum sector. 

We say that a BRST transformation is normal if it can be expressed as the (graded) commutator with some operator \( Q \) verifying \( Q\Omega = 0 \), \( Q^*\Omega = 0 \). The operator \( Q \) is called supercharge. In usual gauge models the BRST transformation is always normal. We have now:

Corollary 4.6 If \( dQ \) is a normal BRST transformation, then the gauge invariance condition is equivalent to the following set of identities:

\[
\sum_{V_1, \ldots, V_n \in \text{Part}(V)} \left\{ \sum_{l=1}^{n} D_l < \Omega, T(\rho(V_1)A(x_1), \ldots, \rho(V_k)A(x_k))\Omega > + < \Omega, T(\rho(V_1)A(x_1), \ldots, (Q + \delta) \cdot \rho(V_l)A(x_l), \ldots, \rho(V_k)A(x_k))\Omega > \right\} = 0 \tag{4.2.12}
\]

for any set of derivatives \( V \).

Proof: First we note the fact that gauge invariance of rank \( k \) is always true for \( k \) large enough. Indeed, the anomaly of the gauge invariance of rank 0 is a quasi-local operator where there is a limitation on the degree of the polynomial in the partial derivatives - see the relation (2.2.24); details of the argument can be found, for instance, in [24]. If one considers instead of the Wick polynomials \( A'(x) \) their derivatives \( \rho(V_i)A'(x) \) one can easily see that every derivative lowers the restriction on the degree of the anomaly with at least one unit. This proves the preceding assertion. Now the Ward identities are the vacuum averages of the gauge invariance relations of arbitrary rank. We apply the preceding theorem iteratively and we obtain the conclusion. 

It is not so simple to eliminate the anomalies from the vacuum sector. In the case of quantum electrodynamics [11] this can be done using charge conjugation invariance. In the case of a Yang-Mills model, we have some restrictions coming from ghost number counting and PCT invariance, but they do not eliminate all anomalies. However, we have another trick which eliminates the anomalies in higher orders if they are absent in lower orders of perturbation theory. We say that a gauge model \( A(x) \) is of degree \( r \) if \( c^\alpha = 0 \), \( \forall \alpha \neq r \). The case considered in the Introduction corresponds to \( r = 1 \) so we refer from now on to the notations introduced in the formulæ (1.0.1) and (1.0.3).

Theorem 4.7 Let \( dQ \) be a normal BRST transformation and \( A(x) \) a gauge model of degree 1 such that \( \omega(A(x)) = d \equiv \dim(M) \) and only first order derivatives of the basic fields do appear. Suppose that the gauge invariance of rank \( k \) is valid up to the order \( k \) of the perturbation theory \( \forall k = 0, 1, \ldots, d + 1 \). Then the chronological products can be chosen in such a way that the the gauge invariance of arbitrary order \( k \) is valid in every order of the perturbation theory.
Proof: (i) First we prove that one can choose the chronological products in such a way that
\[ \omega \left( \frac{\partial}{\partial x_1^\mu} T(T^\nu...(x_1), A_2(x_2), \ldots, A_n(x_n)) \right) = \omega \left( T(T^\nu...(x_1), A_2(x_2), \ldots, A_n(x_n)) \right) + 1 \quad (4.2.13) \]

Indeed, for \( n = 1 \) the assertion is true because the only way to break this equality is through the use of the equations of motion. But only Wick monomials of the type
\[ \partial^\mu \varphi, \quad \overline{\psi} \gamma^\mu \psi \] (4.2.14)
(where \( \psi \) is a Dirac field) can break the preceding equality and they are of canonical dimension \( \omega \leq 3 \). If the assertion is true for \( 1, 2, \ldots, n-1 \) then we have a formula of this type for the corresponding commutator function \[ \text{(3.2.4)}. \] A similar formula is valid for the commutator function with derivatives if the first factor is not differentiated. It follows that if we consider the Wick expansion of these commutator function we will have terms for which a formula of type \( (4.2.13) \) is valid for the orders of singularity of the numerical distributions. The order of singularity is not modified by a suitable causal distribution splitting so the formula \( (4.2.13) \) is pushed to the order \( n \).

(ii) Now, gauge invariance of rank \( d + 2 \) is valid because of the restrictions imposed on the degree anomalies: the argument was also used in the preceding Corollary. We consider now the gauge invariance of rank \( d + 1 \). We use induction on the order of perturbation theory. According to the hypothesis of the theorem the gauge condition is true up to the order \( d + 1 \) of the perturbation theory. We suppose that it is true up to the order \( n - 1 \). The obstructions in order \( n \) are given by the relations \[ (4.2.12) \]. More precisely one has to investigate if it possible to split causally the relations:
\[ \sum_{V_1,...,V_n \in \text{Part}(V)} \left\{ \sum_{l=1}^n D_l < \Omega, D(\rho(V_1)A(x_1), \ldots, \rho(V_n)A(x_n))\Omega \right. \\
+ < \Omega, D(\rho(V_1)A(x_1), \ldots, (Q + \delta) \cdot \rho(V_l)A(x_l), \ldots, \rho(V_n)A(x_n))\Omega \right\} = 0 \quad (4.2.15) \]
where \( D(A^i(x_1), \ldots, A^i(x_n)) \) are the commutator distributions with causal support introduced in Subsection 3.2.

Because the gauge model is of degree 1 these identities have the generic form:
\[ \sum_{p \in \text{Part}(V)} \sum_{l=1}^n \frac{\partial}{\partial x_l^\mu} d_{l,p}^{\mu\nu...(x_1, \ldots, x_n)} = \sum_{p \in \text{Part}(V)} d_p^{\nu...(x_1, \ldots, x_n)} \quad (4.2.16) \]
where the distributions \( d_{l,p}^{\mu\nu...(x_1, \ldots, x_n)} \) and \( d_p^{\nu...(x_1, \ldots, x_n)} \) have causal support. For certain partitions \( p \) these identities might be purely algebraic i.e. the terms with derivatives are missing and the equations are of the type
\[ \sum_{p \in \text{Part}(V)} d_p^{\nu...(x_1, \ldots, x_n)} = 0. \quad (4.2.17) \]

We call them \textit{algebraic} Ward identities; the other Ward identities are called \textit{non-algebraic}. The corresponding partitions \( a \) are also called \textit{algebraic} and \textit{non-algebraic} as well. In this
case, we consider only a independent set of distributions, eliminate the others algebraically and substitute the result into the non-algebraical Ward identities (4.2.16). The result will be a system of equations of the same type where in all equations the derivative terms do appear explicitly.

The resulting system equation has the following generic form: we define

\[ d_{\nu \ldots}^{\mu \ldots} \equiv \frac{\partial}{\partial x^\mu} d_{\nu \ldots}^{\mu \ldots}, \]

and we must have

\[ d_p^{\nu \ldots} = \sum_{l=1}^n d_{\nu \ldots}^{\mu \ldots}; \]

here the index \( p \) runs over the the set of non-algebraic partitions \( P' \subset Part(V) \).

The anomalies can appear because for some of these indices \( p \in P' \) we can have

\[ \omega(\frac{\partial}{\partial x^\mu} d_{\nu \ldots}^{\mu \ldots}) < \omega(d_{\nu \ldots}^{\mu \ldots}) + 1; \]

we denote by \( A_l \) the set of these (non-algebraic) anomalous partitions and by \( A'_l \) the complement in \( P' \); Let us observe that \( A'_l \neq \emptyset \). We consider some causal splitting of the distributions from the relations (4.2.18)

\[ d_{\nu \ldots}^{\mu \ldots} = a_{\nu \ldots}^{\mu \ldots} - r_{\nu \ldots}^{\mu \ldots}, \quad d_{\nu \ldots}^{\mu \ldots} = a_{\nu \ldots}^{\mu \ldots} - r_{\nu \ldots}^{\mu \ldots} \]

such that Lorentz covariance and the order of singularity is preserved. Some anomalies might appear because of the existence of partitions of type \( A_l \); let the sum of all anomalies be denote by \( P \). There is a limitation on the degree of the anomaly, namely it cannot exceed \( \text{supp}(\omega(d_{\nu \ldots}^{\mu \ldots})) \).

The maximum is reached for some non-anomalous partition \( a \in A'_l \); then we can add the anomaly to the expression \( a_{\nu \ldots}^{\mu \ldots} \) (resp. \( r_{\nu \ldots}^{\mu \ldots} \)) i.e. we can make the finite renormalization of the advanced part \( a_{\nu \ldots}^{\mu \ldots} \rightarrow a_{\nu \ldots}^{\mu \ldots} + P \) and similarly for the retarded part. In this way we do not spoil Lorentz covariance and we preserve the order of singularity. It can be seen that this renormalization procedure can be done in a non-contradictory way in all Ward identities: this follows from the fact that distinct Ward identities correspond to distinct choices of the Wick polynomials \( A^i_p, \quad p = 1, \ldots, n \) and/or distinct sets of derivatives \( V \) and we have eliminated all algebraic constraints on the chronological products. ■
5 The Gauge Invariance of the Yang-Mills Model

So, the gauge invariance of a gauge model of degree 1 can be reduced to the investigation of a finite number of Ward identities. By tedious computation one can prove that this is true for the generalization of the standard model considered in [24] and [25]. Let us give some details.

5.1 Yang-Mills Fields

In [23] - [25] we have considered the following scheme for the standard model (SM): we construct the auxiliary Hilbert space $H_{Y M}^{gh,r}$ from the vacuum $\Omega$ by applying the free fields $A_{\alpha\mu}$, $u_\alpha$, $\bar{u}_\alpha$, $\Phi_\alpha$, $\psi_A$, $A = 1, \ldots, N$. The fields $\psi_A$ are, in general Dirac fields describing the matter and have the masses $M_A$, $A = 1, \ldots, N$. We give the spin structure and the statistics for the other fields: first we postulate that $A_{\alpha\mu}$ (resp. $u_\alpha$, $\bar{u}_\alpha$, $\Phi_\alpha$, $\psi_A$, $A = 1, \ldots, r$) has vector (resp. scalar) transformation properties with respect to the Poincaré group. In other words, every vector field has three scalar partners. Also $A_{\alpha\mu}$, $\Phi_\alpha$ are Boson and $u_\alpha$, $\bar{u}_\alpha$ $a = 1, \ldots, r$ are Fermion fields.

Moreover, if for some index $a$ the vector field $A_a^\mu$ has non-zero mass $m_a$ then we suppose that all the other scalar partners fields $u_\alpha$, $\bar{u}_\alpha$, $\Phi_\alpha$ have the same mass $m_a$.

If for some index $a$ the vector field $A_a^\mu$ has zero mass then the scalar partners fields $u_\alpha$, $\bar{u}_\alpha$ also have the zero mass but the corresponding scalar field $\Phi_\alpha$ can have a arbitrary mass $m_a^*$ or might be absent.

Finally, we admit that for some indices $a$ all the fields $A_a^\mu$, $u_\alpha$, $\bar{u}_\alpha$ might be absent and the corresponding scalar field $\Phi_\alpha$ can have a arbitrary mass $m_a^*$.

The canonical (anti)commutation relations are:

$$[A_{\alpha\mu}(x), A_{\beta\nu}(y)] = -\delta_{\alpha\beta}g_{\mu\nu}D_{ma}(x - y) \times 1,$$
$$\{u_\alpha(x), \bar{u}_\beta(y)\} = \delta_{\alpha\beta}D_{ma}(x - y) \times 1, \quad \{\Phi_\alpha(x), \Phi_\beta(y)\} = \delta_{\alpha\beta}D_{ma}^*(x - y) \times 1;$$
$$\{\psi_A(x), \bar{\psi}_B(y)\} = \delta_{AB}S_{Ma}(x - y) \quad (5.1.1)$$

all other (anti)commutators are null.

In the Hilbert space $H_{Y M}^{gh,r}$ we suppose given a sesquilinear form $<\cdot, \cdot>$ such that:

$$A_{\alpha\mu}(x)^\dagger = A_{\alpha\mu}(x), \quad u_\alpha(x)^\dagger = u_\alpha(x), \quad \bar{u}_\alpha(x)^\dagger = -\bar{u}_\alpha(x), \quad \Phi_\alpha(x)^\dagger = \Phi_\alpha(x). \quad (5.1.2)$$

The ghost degree is $\pm 1$ for the fields $u_\alpha$ (resp. $\bar{u}_\alpha$), $a = 1, \ldots, r$ and 0 for the other fields. One can define the BRST supercharge $Q$ by:

$$\{Q, u_\alpha\} = 0 \quad \{Q, \bar{u}_\alpha\} = -i(\partial_\mu A_\alpha^\mu + m_\alpha \Phi_\alpha)$$
$$[Q, A_\alpha^\mu] = i\partial_\mu u_\alpha \quad [Q, \Phi_\alpha] = im_\alpha u_\alpha, \quad \forall a = 1, \ldots, r \quad (5.1.3)$$

and

$$Q\Omega = 0. \quad (5.1.4)$$

Then one can justify that the physical Hilbert space of the Yang-Mills system is a factor space

$$H_{YM}^r \equiv \mathcal{H} \equiv Ker(Q)/Ran(Q). \quad (5.1.5)$$
The sesquilinear form \(<\cdot,\cdot>\) induces a \textit{bona fide} scalar product on the Hilbert factor space. Let us consider the set of Wick monomials \(\mathcal{W}\) constructed from the free fields \(A^\mu_a, u_a, \tilde{u}_a\) and \(\Phi_a\) for all indices \(a = 1, \ldots, r\); we define the BRST operator \(d_Q : \mathcal{W} \rightarrow \mathcal{W}\) as the (graded) commutator with the supercharge operator \(Q\). Then one can prove easily that:

\[
d^2_Q = 0. \tag{5.1.6}
\]

Let us consider the first order Lagrangian:

\[
T(x) \equiv f_{abc} \left[ \frac{1}{2} : A_{a\mu}(x) A_{b\nu}(x) F^\mu_\nu(x) : - : A^\mu_a(x) u_b(x) \partial_\mu \tilde{u}_c(x) : \right] + f'_{abc} [ : \Phi_a(x) \partial_\mu \Phi_b(x) A^\mu_c(x) : - m_b : \Phi_a(x) A_{b\mu}(x) A^\mu_c(x) : - m_b : \Phi_a(x) \tilde{u}_b(x) u_c(x) : ] + f''_{abc} : \Phi_a(x) \Phi_b(x) \Phi_c(x) : + j_a^\mu(x) A_{a\mu}(x) + j_a(x) \Phi_a(x) \tag{5.1.7}
\]

where:

\[
F^\mu_\nu(x) \equiv \partial^\mu A^\nu_a(x) - \partial^\nu A^\mu_a(x) \tag{5.1.8}
\]

is the Yang-Mills field tensor and the so-called \textit{currents} are:

\[
j_a^\mu(x) = : \bar{\psi}_A(x)(t_a)_{AB} \gamma^\mu \psi_B(x) : + : \bar{\psi}_A(x)(t'_a)_{AB} \gamma^\mu \gamma_5 \psi_B(x) : \tag{5.1.9}
\]

and

\[
j_a(x) = : \bar{\psi}_A(x)(s_a)_{AB} \gamma_5 \psi_B(x) : + : \bar{\psi}_A(x)(s'_a)_{AB} \gamma_5 \gamma_5 \psi_B(x) : \tag{5.1.10}
\]

where a number of restrictions must be imposed on the various constants (see [23]-[25]).

Moreover, if we define

\[
T^\mu(x) = f_{abc} \left[ : u_a(x) A_{b\nu}(x) F^\mu_c(x) : - \frac{1}{2} : u_a(x) u_b(x) \partial^\mu(x) \tilde{u}_c(x) : \right] + f'_{abc} [ m_a : A^\mu_a(x) \Phi_b(x) u_c(x) : + : \Phi_a(x) \partial^\mu \Phi_b(x) u_c(x) : ] + u_a(x) j^\mu_a(x) \tag{5.1.11}
\]

and

\[
T^{\mu\nu}(x) = \frac{1}{2} f_{abc} : u_a(x) u_b(x) F^{\mu\nu}_c(x) : \tag{5.1.12}
\]

then we have the relation (1.0.1) from the Introduction for \(p = 2\).

All these Wick polynomials are \(SL(2, \mathbb{C})\)-covariant, are causally commuting and are Hermitian. Moreover we have the following ghost content:

\[
gh(T(x)) = 0, \quad gh(T^\mu(x)) = 1, \quad gh(T^{\mu\nu}(x)) = 2. \tag{5.1.13}
\]

We will construct a perturbation theory verifying Bogoliubov axioms using this set of free fields and imposing the usual axioms of causality, unitarity and relativistic invariance on the chronological products \(T(A^i(x_1), \ldots, A^i(x_n))\) (where the Wick polynomials \(A^i(x)\) must be \(T(x), \ T^\mu(x)\) or \(T^{\mu\nu}(x)\)) such that we have the relation (1.0.3) from the Introduction which
amounts to some factorizes property of the chronological products to the physical Hilbert space in the formal adiabatic limit. This generalizes the gauge invariance condition from [1], [19]:

\[ d_Q T(T(x_1), \ldots, T(x_n)) = i \sum_{l=1}^n \frac{\partial}{\partial x_i^l} T(T(x_1), \ldots, T_i^\mu(x_l), \ldots, T(x_n)). \] (5.1.14)

We work from now on with the usual chronological products. The various signs from some of the relations below are obtained by conveniently eliminating the Grassmann variables.

Let us consider now some elements \( v_1, \ldots, v_k \in \mathcal{M}^0 \) of fixed ghost number and let us define:

\[ g_l = \sum_{i=1}^{l-1} gh(v_i), \quad g_l' = \sum_{i=l}^{k} gh(v_i). \] (5.1.15)

Then after some computation one obtains from (4.2.3):

\[ d_Q [\rho(v_1) \ldots \rho(v_k) T(x)] = i \partial_\mu [\rho(v_1) \ldots, \rho(v_k) T^\mu(x)] + i \sum_{l=1}^{k} \left[ (-1)^{g_l} \rho(v_1) \ldots, \rho(q \cdot v_l) \rho(v_k) T(x) + \rho(v_1) \ldots, \rho(\partial_\mu \cdot v_l) \rho(v_k) T^\mu(x) \right] \] (5.1.16)

\[ d_Q [\rho(v_1) \ldots \rho(v_k) T^{\mu\nu}(x)] = i \partial_\nu [\rho(v_1) \ldots, \rho(v_k) T^{\mu\nu}(x)] + i \sum_{l=1}^{k} \left[ (-1)^{g_l'} \rho(v_1) \ldots, \rho(q \cdot v_l) \rho(v_k) T^{\mu\nu}(x) + \rho(v_1) \ldots, \rho(\partial_\nu \cdot v_l) \rho(v_k) T^{\mu\nu}(x) \right] \] (5.1.17)

\[ d_Q [\rho(v_1) \ldots \rho(v_k) T^{\mu\nu}(x)] = i \sum_{l=1}^{k} (-1)^{g_l} \rho(v_1) \ldots, \rho(q \cdot v_l) \rho(v_k) T^{\mu\nu}(x). \] (5.1.18)

Here the expressions \( \partial_\mu \cdot v \) are defined according to (2.3.20) and

\[ q \cdot \frac{\partial}{\partial A_{a\mu}} = -\frac{1}{4} m^2 \frac{\partial}{\partial \tilde{u}_{a,\mu}}, \quad q \cdot \frac{\partial}{\partial u_a} = \frac{1}{4} m^2 g_{\mu\nu} \frac{\partial}{\partial A_{a\mu;\nu}}, \quad q \cdot \frac{\partial}{\partial A_{a\mu;\nu}} = 0, \quad q \cdot \frac{\partial}{\partial \Phi_a} = m_a \frac{\partial}{\partial u_{a}}; \]

\[ q \cdot \frac{\partial}{\partial A_{a\mu;\nu}} = g^{\mu\rho} \frac{\partial}{\partial \tilde{u}_{a,\nu}}, \quad q \cdot \frac{\partial}{\partial u_{a,\mu}} = -\frac{\partial}{\partial A_{a\mu}}, \quad q \cdot \frac{\partial}{\partial \tilde{u}_{a,\mu}} = 0, \quad q \cdot \frac{\partial}{\partial \Phi_{a;\mu}} = m_a \frac{\partial}{\partial \tilde{u}_{a,\mu}}; \] (5.1.19)

where the derivatives with respect to the fields are defined according to the general formulae (2.3.18).

Using these relations one can easily write now explicitly all Ward identities. We will not list them here.
5.2 Lower Order Ward Identities

We consider the identities (4.2.12) for $2 \leq n \leq 5$. It follows from the preceding Section that we have something non-trivial only if

$$|V| \leq 5, \quad gh(V) = \sum_{l=1}^{n} gh(A^{il}) - 1. \quad (5.2.1)$$

Also, in the sum over the partitions of $V$ it is sufficient to consider only those terms for which all subsets $V_1, \ldots, V_n$ are non-void. The other partitions are not dangerous: they can be treated as in the last theorem of the preceding Section. Finally, at least one term from (4.2.12) which is differentiated should be of the form

$$\frac{\partial}{\partial x_{l}^\mu} T(\ldots, \partial^\mu \psi(x_l), \ldots), \quad \frac{\partial}{\partial x_{l}^\mu} T(\ldots, \bar{\psi}(x_l)\gamma^\mu \psi(x_l), \ldots) \quad (5.2.2)$$

for $\psi$ a Dirac field. Only in this cases we will have a anomalous partition and some anomaly might appear.

The list of these Ward identities is too long to give in detail. We will only mention the choices for the set $V$ and insist on those identities which are producing anomalies. Afterwards we will specify the finite renormalizations which do eliminate the anomalies. In all these computations we heavily rely on the various relations verified by the constants appearing in the first order Lagrangian $T(x)$; all these constraints can be found in [24] and [25].

(i) $n = 2$

In the second order perturbation theory we have three possibilities

(ii) $A^{i1}(x) = A^{i2}(x) = T(x)$

In this case we must have $gh(V) = 1$ so we have the following cases:

$$V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_d} \right\},$$
$$V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\},$$

$$V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\} \quad (5.2.3)$$

and the cases obtained from the first six ones by appending a derivatives to one of the fields. In all, there are 24 such relations. We give below only the anomalous Ward identities:

1) $V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\}$

Let us give the Ward identity in detail in the case:

$$\frac{\partial}{\partial x_{1}^\mu} < \Omega, T \left( \frac{\partial^2}{\partial u_a \partial A_{b\nu}} T^\mu(x_1), \frac{\partial^2}{\partial A_{c\rho} \partial A_{d\sigma}} T(x_2) \right) \Omega >$$
\[ + \langle \Omega, T \left( \frac{\partial^2}{\partial u_a \partial A_{b\nu}} T^\lambda(x_1), \frac{\partial^2}{\partial A_{c\rho} \partial A_{d\sigma}} T(x_2) \right) \rangle \Omega \]
\[ + (b\nu \leftrightarrow c\rho) + (x_1 \leftrightarrow x_2) + \cdots = 0 \]  
\quad \tag{5.2.4}

where by \ldots we mean terms which do not produce anomalies.

We will illustrate the procedure of getting the anomaly on this case. The first chronological product comes from the causal commutator
\[ \left[ \frac{\partial^2}{\partial u_a \partial A_{b\nu}}, \frac{\partial^2}{\partial A_{c\rho} \partial A_{d\sigma}} \right] T(x_1), \frac{\partial^2}{\partial A_{c\rho} \partial A_{d\rho}} T(x_2) \right) = f_{abe} f_{cde} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \partial^\mu D_m(x_1 - x_2) \]  
\quad \tag{5.2.5}

and it produces the anomaly
\[ f_{abe} f_{cde} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \delta(x_1 - x_2). \]  
\quad \tag{5.2.6}

The total anomaly produced by the preceding Ward identity is
\[ A_{1,abcd}^{\nu\rho\sigma\lambda} = 2 i f_{abe} f_{cde} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \delta(x_1 - x_2). \]  
\quad \tag{5.2.7}

2) \( V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\} \) with the anomaly
\[ A_{2,abcd}^{\nu\rho\sigma} = i f_{abe} f_{cde} (g^{\rho\sigma} \partial^\nu - g^{\nu\rho} \partial^\sigma) \delta(x_1 - x_2) + (b\nu \leftrightarrow c\rho) + (b\nu \leftrightarrow d\sigma) \]  
\quad \tag{5.2.8}

3) \( V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial A_{c\sigma}}, \frac{\partial}{\partial A_{d\lambda}} \right\} \)

In this case we get a algebraic Ward identity:
\[ - \langle \Omega, T \left( \frac{\partial^2}{\partial A_{a\nu} \partial A_{b\rho}} T^\nu(x_1), \frac{\partial^2}{\partial A_{c\sigma} \partial A_{d\lambda}} T(x_2) \right) \rangle \Omega \]
\[ + (b\rho \leftrightarrow c\sigma) + (b\rho \leftrightarrow d\lambda) + (x_1 \leftrightarrow x_2) + \cdots = 0 \]  
\quad \tag{5.2.9}

4) \( V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_{d\sigma}} \right\} \)

with the anomaly
\[ A_{3,abcd}^{\rho\sigma} = -2 i g^{\rho\sigma} f'_{dea} f'_{ceb} \delta(x_1 - x_2) \]  
\quad \tag{5.2.10}

5) \( V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\} \)
with the anomaly
\[ A^\rho_{abcd} = -2i(f'_{dea}f'_{ceb} + f'_{debf'_{cea}})\partial^\rho \delta(x_1 - x_2) \] (5.2.11)

6) \[ V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial A_{b\sigma}}, \frac{\partial}{\partial \Phi_c^\rho}, \frac{\partial}{\partial \Phi_d^\rho} \right\} \]

In this case we get an algebraic Ward identity.

7) \[ V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial A_{b\sigma}}, \frac{\partial}{\partial \Phi_c^\rho}, \frac{\partial}{\partial \Phi_d^\rho} \right\} \]

with the anomaly
\[ A_{5;abcd} = -12iS_{bcd}(f'_{bea}f''_{cdde})\delta(x_1 - x_2) \] (5.2.12)

(i2) \( A^{i1}(x) = T(x), \quad A^{i2}(x) = T^\nu(x) \)

In this case we must have \( gh(V) = 2 \) so we have the following possibilities:

\[ V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial u_b^\rho}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial A_{b\sigma}}, \frac{\partial}{\partial \Phi_c^\rho}, \frac{\partial}{\partial \Phi_d^\rho} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial u_b^\rho} \right\} \]

(5.2.13)

and the cases obtained from the first four ones by appending a derivatives to one of the fields.

In all, there are 14 such relations. We give below only the anomalous Ward identities:

8) \[ V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial u_b^\rho}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma};\lambda} \right\} \]

with the anomaly
\[ A^\rho_{6;abcd} = -if_{abe}f_{cd}(g^{\rho\nu}g^{\nu\lambda} - g^{\rho\lambda}g^{\nu\sigma})\delta(x_1 - x_2) \] (5.2.14)

9) \[ V = \left\{ \frac{\partial}{\partial u_a^\rho}, \frac{\partial}{\partial u_b^\rho}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\} \]

with the anomaly
\[ A^{\nu\rho\sigma\lambda}_{7;abcd} = 2if_{abe}f_{cde}(g^{\nu\rho}\partial^\sigma - g^{\nu\sigma}\partial^\rho)\delta(x_1 - x_2) + (b\nu \leftrightarrow cp) + (bv \leftrightarrow d\sigma) \] (5.2.15)

10) \[ V = \left\{ \frac{\partial}{\partial u_a^\nu}, \frac{\partial}{\partial u_b^\nu}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\lambda}} \right\} \]

In this case we get an algebraic Ward identity;

11) \[ V = \left\{ \frac{\partial}{\partial u_a^\nu}, \frac{\partial}{\partial u_b^\nu}, \frac{\partial}{\partial \Phi_c^\rho}, \frac{\partial}{\partial \Phi_d^\rho} \right\} \]

with the anomaly
\[ A^\rho_{8;abcd} = -ig^{\rho\sigma}[m_a(f'_{ead}f'_{ceb} + f'_{debf'_{cea}}) - (a \leftrightarrow b)]\delta(x_1 - x_2) \] (5.2.16)
12) \[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_{d\rho}} \right\} \]

with the anomaly

\[ A^\rho_{abcd} = -ig^{\rho\sigma} f_{abc} f'_{dec} \delta(x_1 - x_2) \] (5.2.17)

13) \[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_{d\rho}} \right\} \]

In this case we get a algebraic Ward identity.

(i3) \( A^1(x) = T(x), \ A^2(x) = T^\rho(x) \)

In this case we must have \( gh(V) = 3 \) so we have the following possibilities:

\[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial A_{d\rho}} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_{d\rho}} \right\} \] (5.2.18)

and the cases obtained by appending a derivatives to one of the fields. There are 6 such relations and the corresponding Ward identities do not give anomalies.

(i4) \( A^1(x) = T(x), \ A^2(x) = T^\rho(x) \)

In this case we also have \( gh(V) = 3 \) so we have the same possibilities as in case (i3):

\[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial A_{d\rho}} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_{d\rho}} \right\} \] (5.2.19)

and the cases obtained by appending a derivatives to one of the fields. There are 6 such relations. The anomalous Ward identities correspond to:

14) \[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial A_{d\sigma}} \right\}; \]

In this case we get a algebraic Ward identity;

15) \[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_{d\rho}} \right\} \]

with the anomaly

\[ A^{\rho\sigma}_{10abcd} = -ig^{\rho\sigma} A_{abc} (f_{fde} f'_{dec}) m_{e} \delta(x_1 - x_2) \] (5.2.20)

(i5) \( A^1(x) = T^\nu(x), \ A^2(x) = T^{\rho\sigma}(x) \)

In this case we take \( gh(V) = 4 \) so we have only:

\[ V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial u_d} \right\} \] (5.2.21)

and the case obtained by appending a derivatives to one of the fields. There are 2 such relations and the corresponding Ward identities do not produce anomalies.
All anomalies can be removed if we perform the following finite renormalization of the chronological products:

\[ T \left( \frac{\partial^2}{\partial u_a \partial A_{ds}} T^\lambda(x_1), \frac{\partial^2}{\partial A_{bs} \partial A_{c\lambda}} T(x_2) \right) \rightarrow \cdots + f_{ade} f_{bce} (g^{\mu\alpha} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\alpha}) \delta(x_1 - x_2) \]

\[ T \left( \frac{\partial^2}{\partial A_{as} \partial A_{bp}} T^\lambda(x_1), \frac{\partial^2}{\partial A_{cs} \partial A_{d\lambda}} T(x_2) \right) \rightarrow \cdots - f_{abe} f_{cde} (g^{\mu\alpha} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\alpha}) \delta(x_1 - x_2) \]

\[ T \left( \frac{\partial^2}{\partial u_a \partial \Phi_d} T^\sigma(x_1), \frac{\partial^2}{\partial A_{bp} \partial \Phi_c} T(x_2) \right) \rightarrow \cdots + f'_{ceb} f'_{dea} (g^{\mu\alpha} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\alpha}) \delta(x_1 - x_2) \]

\[ T \left( \frac{\partial^2}{\partial u_a \partial \Phi_f} T^\sigma(x_1), \frac{\partial^2}{\partial \Phi_c \partial \Phi_d} T(x_2) \right) \rightarrow \cdots - f'_{ceb} f'_{dea} g^{\mu\sigma} \delta(x_1 - x_2) \]

\[ T \left( \frac{\partial^2}{\partial u_a \partial \Phi_f} T^\lambda(x_1), \frac{\partial^2}{\partial A_{bp} \partial \Phi_c} T^\nu(x_2) \right) \rightarrow \cdots + f'_{ceb} f'_{dea} g^{\mu\nu} \delta(x_1 - x_2) \]

\[ T \left( \frac{\partial^2}{\partial u_a \partial A_{ds}} T^\lambda(x_1), \frac{\partial^2}{\partial u_b \partial A_{c\mu}} T^\nu(x_2) \right) \rightarrow \cdots + f_{ace} f_{bde} g^{\mu\nu} \delta(x_1 - x_2) \]

(5.2.23)

All these renormalizations are made in the vacuum sector so there is no need to take the vacuum average. Let us note that all these finite renormalization are consistent with the symmetry properties of the chronological products. If we use the formula (3.2.2) we can obtain the finite renormalizations for the original chronological products:

\[ T(T(x_1), T(x_2)) \rightarrow \cdots + N(x_1) \delta(x_1 - x_2) \]

\[ T(T^\mu(x_1), T(x_2)) \rightarrow \cdots + N^\mu(x_1) \delta(x_1 - x_2) \]

\[ T(T^\mu(x_1), T^\nu(x_2)) \rightarrow \cdots + N^{\mu\nu}(x_1) \delta(x_1 - x_2) \]

\[ T(T(x_1), T^\mu(x_2)) \rightarrow \cdots + N^{\mu}(x_1) \delta(x_1 - x_2) \]

(5.2.24)

where:

\[ N \equiv \frac{1}{4} f_{abc} f_{cde} : A_{\mu A_c A_{sp}} : A_{\nu A_d} + \frac{1}{2} f'_{ceb} f_{dea} : A_{\mu A_c} \Phi_a \Phi_d : + \frac{1}{2 m_a} f'_{bea} f'_{cde} : \Phi_a \Phi_b \Phi_c \Phi_d : \]

\[ N^\mu \equiv - f_{ade} f_{bce} : u_{\mu A_c A_{sp}} : - f'_{ceb} f_{dea} : u_{\mu A_c} \Phi_a \Phi_d : \]

\[ N^{\mu\nu} \equiv f_{abc} f_{cde} : u_{\nu A_c} A_{sp} A_{\mu} : \]

(ii) \( n = 3 \)

The situation in the third order of the perturbation theory can be analysed as in [25]. One can see that only in two situations anomalies can appear:
(ii2) When the chronological products involve at least one Fermionic loop. The relevant choices for the set $V$ are

$$V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{bw}}, \frac{\partial}{\partial A_{cp}} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{bw}}, \frac{\partial}{\partial \Phi_b} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}, \frac{\partial}{\partial \Phi_c} \right\} \quad (5.2.25)$$

and other relations with one of the fields differentiated. There are 10 relations of this type. The Ward identities which can produce anomalies correspond only to the choices without derivatives. They are respectively:

- for $A^{i_1}(x) = A^{i_2}(x) = A^{i_3}(x) = T(x)$

$$\frac{\partial}{\partial x^\mu_1} < \Omega, T(j_\mu(x_1), j_\nu(x_2), j_\rho(x_3)) \Omega > -m_a < \Omega, T(j_\mu(x_1), j_\nu(x_2), j_\rho(x_3)) \Omega >$$

$$+(b \nu \leftrightarrow c \rho) + (x_1 \leftrightarrow x_2) + (x_1 \leftrightarrow x_3) + \cdots \quad (5.2.26)$$

- for $A^{i_1}(x) = T^\nu(x), A^{i_2}(x) = A^{i_3}(x) = T(x)$

$$\frac{\partial}{\partial x^\mu_1} < \Omega, T(j_\mu(x_1), j_\nu(x_2), j_\rho(x_3)) \Omega > -m_a < \Omega, T(j_\mu(x_1), j_\nu(x_2), j_\rho(x_3)) \Omega >$$

$$+(b \leftrightarrow c) + (x_1 \leftrightarrow x_2) + (x_1 \leftrightarrow x_3) + \cdots \quad (5.2.27)$$

- for $A^{i_1}(x) = T^\nu(x), A^{i_2}(x)T^\rho(x), A^{i_3}(x) = T(x)$

$$A_{abc} \frac{\partial}{\partial x^\mu_1} < \Omega, T(j_\mu(x_1), j_\nu(x_2), j_\rho(x_3)) \Omega > -m_c < \Omega, T(j_\mu(x_1), j_\nu(x_2), j_\rho(x_3)) \Omega >$$

$$+(x_1 \leftrightarrow x_2) + (x_2 \leftrightarrow x_3) + \cdots \quad (5.2.28)$$

One can show as in [25] that these Ward identities are not anomalous if the axial vertex anomaly vanishes. Indeed, one ca show that the preceding Ward identities can be fulfilled if the eqs. (5.1.49) - (5.1.60) from [25] can be causally split; this in turns happens iff the axial anomaly vanishes.

(iii2) We also have some Ward identities where anomalies can appear because of the finite renormalizations (5.2.23). One can easily see that these cases correspond to the choice

$$A^{i_1}(x) = A^{i_2}(x) = A^{i_3}(x) = T(x)$$
and the following assignments for the derivatives $V$:

\[
V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial A_{c\sigma}}, \frac{\partial}{\partial A_{d\lambda}}, \frac{\partial}{\partial A_{f\nu}} \right\}, \quad V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\sigma}}, \frac{\partial}{\partial A_{f\lambda}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\}, \\
V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d}, \frac{\partial}{\partial \Phi_e} \right\}
\] (5.2.31)

The first choice gives the Ward identity:

\[
\frac{\partial}{\partial x^\mu} \Omega, T \left( \frac{\partial^2}{\partial u_a \partial A_{f\nu}} T^\mu(x_1), \frac{\partial}{\partial A_{b\rho}} T(x_2), \frac{\partial^2}{\partial A_{c\sigma} \partial A_{d\lambda}} T(x_3) \right) \Omega > + \text{perm}(b\nu, c\sigma, d\lambda, f\nu) + (x_1, \leftrightarrow x_2) + (x_1, \leftrightarrow x_3) + \cdots = 0
\] (5.2.32)

The chronological product involves the causal splitting of the following commutator

\[
\left[ \frac{\partial^2}{\partial u_a \partial A_{f\nu}} T^\mu(x_1), T \left( \frac{\partial}{\partial A_{b\rho}} T(x_2), \frac{\partial^2}{\partial A_{c\sigma} \partial A_{d\lambda}} T(x_3) \right) \right] = f_{afg} f_{bde} f_{ced} (g^{\rho\lambda} g^{\nu\sigma} - g^{\rho\sigma} g^{\nu\lambda}) \partial^\mu D_m (x_1 - x_2) \delta(x_2 - x_3) + \cdots
\] (5.2.33)

which produces the anomaly

\[
A_{abcd}^{\rho\lambda\nu\sigma} = 2 f_{afg} f_{bde} f_{ced} (g^{\rho\lambda} g^{\nu\sigma} - g^{\rho\sigma} g^{\nu\lambda}) \partial^\mu D_m (x_1 - x_2) + \text{perm}(b\nu, c\sigma, d\lambda, f\nu) = 0.
\] (5.2.34)

The other two cases can be treated similarly and do not produce anomalies. Let us note that no finite renormalizations of the third order chronological products are necessary to implement gauge invariance.

(iii) $n = 4, 5$

In these cases, one can argue like in [25] that only when the chronological products involve at least one Fermionic loop one can have anomalies. The relevant choices for the derivative set $V$ are similar to the case (ii1) studied above. One obtains that the corresponding ward identities might be broken by the box and the pentagon anomalies [36]. If this anomalies are also zero, then we have gauge invariance up to the fifth order of the perturbation theory, so according to the general theorem from the preceding Section, we have gauge anomaly in all orders.
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