The geometric sieve and the density of squarefree values of invariant polynomials

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February 28, 2022

Abstract

We develop a method for determining the density of squarefree values taken by certain multivariate integer polynomials that are invariants for the action of an algebraic group on a vector space. The method is shown to apply to the discriminant polynomials of various prehomogeneous and coregular representations where generic stabilizers are finite. This has applications to a number of arithmetic distribution questions, e.g., to the density of small degree number fields having squarefree discriminant, and the density of certain unramified nonabelian extensions of quadratic fields. In separate works, the method forms an important ingredient in establishing lower bounds on the average orders of Selmer groups of elliptic curves.

1 Introduction

The purpose of this article is to develop a method for determining the density of squarefree values taken by certain multivariate integer polynomials that are invariants for an algebraic group acting on a vector space. In the case of general polynomials in one or two variables having degree at most three or six, respectively, methods of Hooley [28] or Greaves [26], respectively, may be applied; in other cases, if the degree of the polynomial is quite small relative to the number of variables, then the circle method may be used to extract squarefree values of the polynomial in question. In contrast, our method may be applied to polynomials of high degree—even when the degree and the number of variables are comparable—so long as the polynomial has some extra structure, such as symmetry under the action of a “suitably large” algebraic group defined over $\mathbb{Z}$ (this condition will be made more precise in Section 2).

1.1 The density of number fields having squarefree discriminant

The most classical specific cases of arithmetic interest that our method addresses is that of determining the density of small degree number fields having squarefree discriminant. Building on the works of Levi [32], Wright–Yukie [42], and Gan–Gross–Savin [24], it was shown in [21], [2], and [3] that the integers that occur as the discriminants of orders in cubic, quartic, and quintic number fields, respectively, correspond to suitable integer values taken by certain fixed multivariate integral polynomials $f_3$, $f_4$, and $f_5$, having degrees 4, 12, and 40 in 4, 12, and 40 variables, respectively. This correspondence between number field discriminants and integers represented by these special polynomials was indeed what was used in [20], [4], and [5], in conjunction with geometry-of-numbers arguments, to determine the density of discriminants of cubic, quartic, and quintic number fields, respectively.

To determine the density of such number fields having squarefree discriminant, we must thus determine the density of squarefree integer values taken by these special polynomials $f_3$, $f_4$, 


and $f_5$. As we have noted, for general polynomials of large degree $d$ in about $d$ variables, this is an unsolved problem. However, using the structure of these special polynomials—namely, that they are invariants for the action of a “suitably large” algebraic group—we determine in §4 the density of squarefree values taken by these polynomials.

As a consequence, we prove that a positive density of all $S_n$-number fields of degrees $n = 3, 4,$ and $5$ have squarefree discriminant, and we determine this density precisely. We similarly determine the density of such number fields that have fundamental discriminant. Specifically, we prove:

**Theorem 1.1** Let $n = 3, 4,$ or $5$, and let $N_{n}^{\text{sqf}}(X)$ (resp. $N_{n}^{\text{fund}}(X)$) denote the number of isomorphism classes of number fields of degree $n$ having squarefree (resp. fundamental) discriminant of absolute value less than $X$. Then

\[
\begin{align*}
(a) \quad N_{n}^{\text{sqf}}(X) &= \frac{r_2(S_n)}{3n!} \zeta(2)^{-1} \cdot X + o(X); \\
(b) \quad N_{n}^{\text{fund}}(X) &= \frac{r_2(S_n)}{2n!} \zeta(2)^{-1} \cdot X + o(X),
\end{align*}
\]

where $r_2(S_n)$ denotes the number of $2$-torsion elements in the symmetric group $S_n$.

Note that Theorem 1.1 is true also for $n = 2$, provided that we count each quadratic field $K$ with weight $\frac{1}{2}$ (i.e., with weight $\frac{1}{\#\text{Aut}(K)}$). We conjecture that Theorem 1.1 holds for general $n$.

In conjunction with the main results of [20], [4], and [5], which give the total density of $\Sigma$-number fields of degree $n$, we conclude:

**Corollary 1.2** When ordered by absolute discriminant, the proportion of $S_n$-number fields of degree $n$ ($n \in \{2, \ldots, 5\}$) having fundamental discriminant is given by

\[
\begin{cases}
1 & \text{if } n = 2; \\
\zeta(2)^{-1} \zeta(3) & \text{if } n = 3; \\
\zeta(2)^{-1} \prod_p (1 + p^{-2} - p^{-3} - p^{-4})^{-1} & \text{if } n = 4; \\
\zeta(2)^{-1} \prod_p (1 + p^{-2} - p^{-4} - p^{-5})^{-1} & \text{if } n = 5.
\end{cases}
\]

Furthermore, the proportion of $S_n$-number fields of degree $n$ ($n \in \{2, \ldots, 5\}$) having squarefree discriminant is exactly $2/3$ of the proportion having fundamental discriminant.

Both Theorem 1.1 and Corollary 1.2 follow from a general theorem that our methods allow us to prove, concerning the asymptotic count of $S_n$-number fields of degree $n \leq 5$ satisfying any desired finite or suitable infinite set of local conditions:

**Theorem 1.3** Let $n = 2, 3, 4,$ or $5$. Let $\Sigma = (\Sigma_\infty, \Sigma_2, \Sigma_3, \ldots)$ denote an acceptable set of local specifications for degree $n$ extensions of $\mathbb{Q}$, i.e., $\Sigma_\nu$ is any subset of (isomorphism classes of) étale degree $n$ extensions of $\mathbb{Q}_\nu$ for each place $\nu$ of $\mathbb{Q}$, such that for sufficiently large primes $p$, the set $\Sigma_p$ contains all étale extensions $K_p$ of $\mathbb{Q}_p$ of degree $n$ such that $p^2 \nmid \text{Disc}(K_p/\mathbb{Q}_p)$. Let $N_{n,\Sigma}(X)$ denote the number of $S_n$-number fields $K$ of degree $n$ having absolute discriminant at most $X$ such that $K \otimes \mathbb{Q}_\nu \in \Sigma_\nu$ for all places $\nu$ of $\mathbb{Q}$. Then

\[
\lim_{X \to \infty} \frac{N_{n,\Sigma}(X)}{X} = \left( \sum_{K \in \Sigma_\infty} \frac{1}{\#\text{Aut}(K)} \right) \prod_p \left( \sum_{K \in \Sigma_p} \frac{p-1}{p} \cdot \frac{1}{\text{Disc}_p(K)} \cdot \frac{1}{\#\text{Aut}(K)} \right).
\]
The above theorem thus allows one to count number fields of degree at most five satisfying very general sets of local conditions. In particular, it proves a more general version (namely, where we allow infinitely many local conditions) of the heuristics given in [8] (4.2).

Since having squarefree or fundamental discriminant is a local condition of the type occurring in Theorem 1.3, Theorem 1.1 will follow from Theorem 1.3 once the sums in the Euler factors in (1), i.e., the local masses, are computed (see §4 for details).

1.2 Unramified nonabelian \((A_n-\text{ and } S_n \times C_2-)\) extensions of quadratic fields

The density of degree \(n\) number fields having squarefree discriminant is directly related to the distribution of certain unramified nonabelian extensions of quadratic fields. More precisely, given a finite group \(G\) and a quadratic field \(K\), we may consider the set \(U(K; G)\) of all isomorphism classes of unramified \(G\)-extensions of \(K\), i.e., Galois extensions of \(K\) with Galois group \(G\). An extension \(L \in U(K; G)\) is not necessarily normal over \(\mathbb{Q}\), and its normal closure over \(\mathbb{Q}\) has Galois group \(G' \subset G \wr C_2 = (G \times G) \rtimes C_2\). It is thus natural to partition \(U(K; G)\) into the sets \(U(K; G, G')\), where \(U(K; G, G')\) denotes the set of all isomorphism classes of unramified \(G\)-extensions \(L\) of \(K\) such that the Galois closure of \(L\) over \(\mathbb{Q}\) has Galois group \(G'\). If \(L \in U(K; G, G')\), then we say that \(L\) is an unramified extension of \(K\) of type \((G, G')\), or simply an unramified \((G, G')\)-extension.

**Theorem 1.4** Let \(n = 3, 4, \text{ or } 5\), and let \(E^+(G, G')\) (resp. \(E^-(G, G')\)) denote the average number of unramified \((G, G')\)-extensions that real (resp. imaginary) quadratic fields possess, where quadratic fields are ordered by their absolute discriminants. Then

\[
\begin{align*}
(a) \quad E^+(A_n, S_n) &= \frac{1}{n!}; \\
(b) \quad E^-(A_n, S_n) &= \frac{1}{2(n-2)!}; \\
(c) \quad E^+(S_n, S_n \times C_2) &= \infty; \\
(d) \quad E^-(S_n, S_n \times C_2) &= \infty.
\end{align*}
\]

In other words, the average number of unramified \(A_n\)-extensions \((n = 3, 4, \text{ or } 5)\) possessed by real or imaginary quadratic fields is positive, and the average number of unramified \(S_n \times C_2\)-extensions is also positive, and in fact infinite! For \(n = 2\), note that Theorem 1 is still true, except that the constants in (a) and (b) must each be multiplied by 2, again reflecting the fact that a quadratic extension has two automorphisms.

The case \(n = 3\) in Theorems 1.4(a)–(b) corresponds to abelian \((A_3-)\) extensions, and is due to Davenport–Heilbronn [20], who obtained these results via the use, in particular, of methods that amount essentially to class field theory (see [18] for this nice interpretation). The cases \(n = 4\) and \(n = 5\) of Theorems 1.4(a)–(b) are both new, and to our knowledge are independent of and cannot be treated by class field theory. Indeed, they yield information on the distribution of certain nonabelian unramified extensions of quadratic fields, namely, those corresponding to the groups \(A_4\) and \(A_5\); in particular, the case \(n = 5\) yields information about the distribution of unramified extensions of a quadratic field of a nonsolvable type, namely \(A_5\). Theorems 1.4(c)–(d) are also new.

Returning to the statement of Theorem 1.4, it is an interesting question as to which groups \(G, G'\) lead to quantities \(E^+(G, G')\) and \(E^-(G, G')\) that exist and are finite, and what their values are when they are finite. Both possibilities of finite and infinite already occur in Theorem 1.4.
the case of abelian $G$, we must have that $G' = G \times C_2$ (where the nontrivial element of $C_2$ acts on $G$ by inversion). The Cohen–Lenstra heuristics [17] can then be shown to imply that, for $G$ abelian,

$$E^+(G, G \times C_2) = \frac{1}{|\text{Aut}(G)| \cdot |G|},$$

$$E^-(G, G \times C_2) = \frac{1}{|\text{Aut}(G)|}$$

whenever $|G|$ is odd.

Note that the cases in Theorems 1.4(a)–(b) in which $G$ is abelian occur when $n = 3$, and in these cases the values agree with those predicted by (2) and (3). It would be interesting to have more general heuristics for $E^\pm(G, G')$ that include both the abelian results and conjectures above as well as the nonabelian results of Theorem 1.4.

In parts (c) and (d) of Theorem 1.4, it is actually possible to say something more precise; namely, the methods of Section 4 show that

$$\sum_{0 < \text{Disc}(K) < X} |U(K; S_n, S_n \times C_2)| \sim c^+_n X \log X;$$

$$\sum_{-X < \text{Disc}(K) < 0} |U(K; S_n, S_n \times C_2)| \sim c^-_n X \log X,$$

for $n = 3, 4,$ and $5$, where $c^+_n$ and $c^-_n$ are certain positive constants which depend on $n$.

### 1.3 Squarefree values taken by polynomials such as $f_3$, $f_4$, and $f_5$

As we have mentioned, to prove Theorem 1.1, Corollary 1.2, Theorem 1.3, and Theorem 1.4, one must determine the densities of lattice points in $\mathbb{R}^m$ where the values of certain polynomials—namely, the discriminant polynomials $f_3$, $f_4$, or $f_5$—are squarefree. In general, counting the number of lattice points of bounded height where a polynomial takes squarefree values is an unsolved problem, although conjecturally it is easy to guess what should happen. Namely, if $f(x_1, \ldots, x_m)$ is any squarefree polynomial over $\mathbb{Z}$ then, barring congruence obstructions, one expects that $f$ takes infinitely many squarefree values on $\mathbb{Z}^m$. More precisely, one expects

$$\lim_{N \to \infty} \frac{\#\{x \in \mathbb{Z}^m \cap [-N, N]^m : f(x) \text{ squarefree}\}}{(2N + 1)^m} = \prod_p (1 - c_p/p^{2m}),$$

where, for each prime $p$, the quantity $c_p$ is the number of elements $x \in (\mathbb{Z}/p^2\mathbb{Z})^m$ satisfying $f(x) = 0$ in $\mathbb{Z}/p^2\mathbb{Z}$.

When $m = 1$, this assertion is relatively easy to prove in degrees $\leq 2$, while for cubic polynomials it was proven by Hooley [28]. For degrees $\geq 4$, it appears that no single example is known of a univariate irreducible polynomial $f$ satisfying (6)! As for polynomials in more than one variable, Greaves has shown that (6) holds for all binary forms of degree at most 6.

Conditionally, Granville [25] showed that (6) follows, for all univariate polynomials of any degree, from the ABC Conjecture. More recently, Poonen [34] proved that the ABC Conjecture implies that (a slightly weaker version of) equation (6) is true also for all multivariate polynomials.

In this article, we give three special examples of polynomials $f$ for which we can prove unconditionally that (6) holds; namely, these are the three polynomials that we use to prove Theorem 1.1 Corollary 1.2 and Theorems 1.3–1.4. More precisely, let $f (= f_3, f_4,$ or $f_5)$ denote the primitive integral polynomial that generates the ring of invariants for:
(i) the action of SL₂(ℂ) on Sym₃(ℂ²), the space of binary cubic forms over ℂ;

(ii) the action of SL₂ × SL₃(ℂ) on ℂ² ⊗ Sym₂(ℂ³), the space of pairs of ternary quadratic forms over ℂ; or

(iii) the action of SL₄ × SL₅(ℂ) on ℂ⁴ ⊗ ∧²(ℂ⁵), the space of quadruples of 5 × 5 skew-symmetric matrices over ℂ,

respectively. Then for (i), (ii), or (iii), \( f \) is a polynomial of degree \( m \) in \( m \) variables, where \( m = 4, 12, \) or 40, respectively (see [38], or see [2]–[3] for explicit constructions of these invariant polynomials).

We prove:

**Theorem 1.5** The polynomials \( f \) in (i)–(iii) above are each irreducible over \( \overline{\mathbb{Q}} \) and are of degree \( m \) in \( m \) variables, where \( m = 4, 12, \) and 40 respectively. Moreover, for each of these polynomials \( f \),

\[
\lim_{N \to \infty} \frac{\# \{ x \in \mathbb{Z}^m \cap [-N, N]^m : f(x) \text{ squarefree} \}}{(2N + 1)^m} = \prod_p (1 - c_p/p^{2m}) = \frac{2}{3} \zeta(2)^{-1},
\]

where \( c_p \) is the number of elements \( x \in (\mathbb{Z}/p^2\mathbb{Z})^m \) satisfying \( f(x) = 0 \) in \( \mathbb{Z}/p^2\mathbb{Z} \).

For these three discriminant polynomials \( f \), particularly in the cases (ii) and (iii) where the degrees are large (≥ 4) in each individual variable (and the number of variables is equal to the degree), we do not believe that any of the previously known unconditional results and methods as described above would apply. Thus these \( f \) give new examples of polynomials satisfying (6). It is interesting to note that the density of squarefree values taken by each of these three discriminant polynomials \( f \) is exactly \( \frac{2}{3} \zeta(2)^{-1} \), independent of \( f \).

### 1.4 Squarefree values of discriminants of genus one models

The method, which we will describe more axiomatically in the next subsection and in Section 2, may also be applied to various other polynomials that are invariant under the action of a suitably large algebraic group defined over \( \mathbb{Z} \). Another family of classical examples on which the method applies are the discriminant polynomials of models of genus one curves.

There are many such models of genus one curves of interest. Genus one curves with maps to \( \mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \) or \( \mathbb{P}^4 \), via complete linear systems of degrees 2, 3, 4, or 5, are called *genus one normal curves* of degree 2, 3, 4, or 5, respectively. They can be realized as: a double cover of \( \mathbb{P}^1 \) ramified at four points; a cubic curve in \( \mathbb{P}^2 \); the intersection of a pair of quadrics in \( \mathbb{P}^3 \); or the intersection of five quadrics in \( \mathbb{P}^4 \) arising as the \( 4 \times 4 \) sub-Pfaffians of a \( 5 \times 5 \) skew-symmetric matrix of linear forms on \( \mathbb{P}^4 \). A genus one model of degree one may be viewed simply as an elliptic curve in Weierstrass form. (See, e.g., [23] for a beautiful exposition.)

We note that we may also consider genus one models in products of projective spaces. For example, a genus one curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is cut out by a bidegree \((2, 2)\)-form on \( \mathbb{P}^1 \times \mathbb{P}^1 \); and a genus one curve in \( \mathbb{P}^2 \times \mathbb{P}^2 \) is similarly cut out by three bidegree \((3, 3)\)-forms on \( \mathbb{P}^2 \times \mathbb{P}^2 \). These cases will be carried out in more detail in [10]. (See [1] also for other examples of such spaces of genus one models.)

For all these genus one models over \( \mathbb{Z} \), we show that the discriminant polynomials of these genus one curves all take the expected (positive) densities of squarefree values. (Recall that the discriminant of a genus one model is the polynomial whose nonvanishing is equivalent to the smoothness of the corresponding genus one curve.) For genus one models of degree one, i.e., Weierstrass
elliptic curves \(y^2 = x^3 + Ax + B\), the result is easy, as the discriminant polynomial \(-4A^3 - 27B^2\) is only of degree 2 as a polynomial in \(B\). For higher degree genus one models, the result is much more difficult to obtain.

More precisely, let \(g\) (which we will denote by \(g_2, g_3, g_4, g_5\), respectively) denote the primitive integral discriminant polynomial of any of the following representations:

(i) the action of \(\text{SL}_2(\mathbb{C})\) on \(\text{Sym}^4(\mathbb{C}^2)\), the space of binary quartic forms over \(\mathbb{C}\);

(ii) the action of \(\text{SL}_3(\mathbb{C})\) on \(\text{Sym}^3(\mathbb{C}^3)\), the space of ternary cubic forms over \(\mathbb{C}\);

(iii) the action of \(\text{SL}_2 \times \text{SL}_4(\mathbb{C})\) on \(\mathbb{C}^2 \otimes \text{Sym}^2(\mathbb{C}^4)\), the space of pairs of quaternary quadratic forms over \(\mathbb{C}\);

(iv) the action of \(\text{SL}_5 \times \text{SL}_5(\mathbb{C})\) on \(\mathbb{C}^4 \otimes \Lambda^2(\mathbb{C}^5)\), the space of quintuples of \(5 \times 5\) skew-symmetric matrices over \(\mathbb{C}\),

respectively. Then the discriminant polynomial \(g\) on each of these representations detects stable orbits, i.e., \(g\) does not vanish precisely when the orbit is closed and has finite stabilizer. The discriminant \(g\) of an element in any of these representations also corresponds to the discriminant of the associated genus one model, i.e., \(g\) does not vanish precisely when this associated genus one curve is smooth.

The dimensions of the representations in (i)–(iv) above are given by 5, 10, 20, and 50, respectively, while the degrees of the corresponding discriminant polynomials are given by 6, 12, 24, and 60, respectively (see [23] for explicit constructions of these invariant polynomials). Then we prove:

**Theorem 1.6** The polynomials \(g\) in (i)–(iv) above are each irreducible. Moreover, for each of these polynomials \(g\), we have

\[
\lim_{N \to \infty} \frac{\# \{ x \in \mathbb{Z}^m \cap [-N, N]^m : g(x) \text{ squarefree} \}}{(2N + 1)^m} = \prod_p (1 - c_p/p^{2m})
\]

where \(c_p\) denotes the number of elements \(x \in \mathbb{Z}/p^2\mathbb{Z}\) satisfying \(g(x) = 0\) in \(\mathbb{Z}/p^2\mathbb{Z}\).

Thus a positive density of genus one models over \(\mathbb{Z}\) mapping into \(\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \text{ or } \mathbb{P}^4\), have squarefree discriminant. In particular, a positive density of binary quartic forms over \(\mathbb{Z}\), and a positive density of ternary cubic forms over \(\mathbb{Z}\), have squarefree discriminant.

These results, and the methods behind them, play an important role in establishing lower bounds on the average sizes of Selmer groups of families of elliptic curves in [11, 12, 13, 14] and in [10]. They also play a key role in proving that the local–global principle fails for a positive proportion of plane cubic curves over \(\mathbb{Q}\) (see [8]).

### 1.5 Method of proof

Let \(f\) be an integral polynomial on \(V(\mathbb{Z}) \cong \mathbb{Z}^m\). As is standard in squarefree sieves (see, e.g., §3.4 for more details), the equality (6) can be proven for \(f\) whenever sufficiently good upper bounds on sums involving \(w_p(f, H)\) are obtained, where \(w_p(f, H)\) denotes the number of points \(v \in V(\mathbb{Z})\) having height at most \(H\) (= the maximum of the absolute values of the coordinates) satisfying \(p^2 \mid f(v)\). It is natural to partition the set \(W_p = W_p(V) \subset V(\mathbb{Z})\) of elements \(v \in V(\mathbb{Z})\) such that \(p^2 \mid f(v)\) into two sets: \(W_p^{(1)}\), consisting of elements \(v \in V(\mathbb{Z})\) on which \(f\) vanishes modulo \(p^2\) for
“mod \(p\) reasons”, i.e., \(f(v') \equiv 0 \pmod{p^2}\) for any \(v' \equiv v \pmod{p}\); and \(W_p^{(2)}\), consisting of the elements \(v \in V(\mathbb{Z})\) on which \(f\) vanishes modulo \(p^2\) for “mod \(p^2\) reasons”, i.e., there exist \(v' \equiv v \pmod{p}\) such that \(f(v') \not\equiv 0 \pmod{p^2}\).

As a consequence, we may write \(w_p(f, H)\) as a sum \(w_p^{(1)} + w_p^{(2)}\), where \(w_p^{(i)}\) denotes the portion of the count of elements in \(W_p\) coming from \(W_p^{(i)}\). It is well-known that good estimates on the relevant sums involving \(w_p^{(1)}\) can be obtained by “geometric sieve” or “closed-point sieve” methods (the latter terminology is due to Poonen), as introduced in the work of Ekedahl [22]; see also Poonen [33, 34] for a very clear treatment. We will prove a precise and quantitative version of Ekedahl’s sieve estimates in §3.2, which will be useful in the applications.

The difficulty in squarefree sieves for values taken by integral polynomials thus arises in the estimation of sums involving \(w_p^{(2)}\). It is essentially here that Granville [25] and Poonen [34] use the ABC Conjecture to obtain the desired estimates. For the polynomials arising in Theorems 1.5 and 1.6 we sidestep the use of the ABC Conjecture by using instead the invariance of these polynomials under the action of an algebraic group \(G\) defined over \(\mathbb{Z}\). Specifically, for the polynomials \(f\) arising in Theorem 1.5, we show that for any element \(v \in W_p^{(2)}\), there always exists an element \(\gamma \in G(\mathbb{Q})\) such that \(\gamma v \in V(\mathbb{Z})\) and \(f(\gamma v) = f(v)/p^2\). Together with estimates from the geometry-of-numbers in [19, 11, 15] giving uniform upper bounds on the number of “irreducible” \(G(\mathbb{Z})\)-classes on \(V(\mathbb{Z})\) having bounded absolute discriminant, this is sufficient to obtain the desired upper bounds on \(w_p^{(2)}\).

With a related construction, for all but one of the polynomials \(g\) arising in Theorem 1.6 we show that for any element \(v \in W_p^{(2)}\), there always exists an element \(\gamma \in G(\mathbb{Q})\) such that \(\gamma v \in W_p^{(1)}\) and \(f(\gamma v) = f(v)\); i.e., via the action of \(G(\mathbb{Q})\), we turn \(v \in V(\mathbb{Z})\) on which \(f\) vanishes modulo \(p^2\) for mod \(p^2\) reasons into \(v'\) for which \(f\) vanishes modulo \(p^2\) for mod \(p\) reasons! As before, we combine this construction with estimates from the geometry-of-numbers as in [11, 12, 13, 14], which give uniform upper bounds on the number of “irreducible” \(G(\mathbb{Z})\)-classes on \(V(\mathbb{Z})\) having bounded absolute discriminant, to deduce the desired upper bounds on \(w_p^{(2)}\).

In Case (i) of Theorem 1.6, however, this argument does not work; we find that the group \(G(\mathbb{Q})\) in this case is just too small to do the job. We get around this problem via a further argument that we call the “embedding sieve”. Namely, we find a representation \(G'\) on \(V'\), defined over \(\mathbb{Z}\), and an invariant polynomial \(f'\) for this action, such that: there is a map of orbits \(\phi : G(\mathbb{Z})\backslash V(\mathbb{Z}) \to G'(\mathbb{Z})\backslash V'(\mathbb{Z})\), having preimages of absolutely bounded cardinality, for which \(f'(\phi(v)) = f(v)\). Furthermore, we choose \((G', V')\) such that \(G'(\mathbb{Q})\) is sufficiently larger than \(G(\mathbb{Q})\), while the set of irreducible orbits of \(G'(\mathbb{Z})\) on \(V'(\mathbb{Z})\) is not too large; this allows one to obtain an estimate \(w_{p'}^{(2)}\) on \(V(\mathbb{Z})'\), which then leads to a good estimate also for \(w_p^{(2)}\). Amusingly, in the case of \(g_2\) in Theorem 1.6 we embed \((G, V)\) into the representation \((G', V')\) corresponding to the polynomial \(f_4\) in Theorem 1.5.

Indeed, the latter argument (which will be described in more detail in §5) shows that the method of this paper may in fact be applied to some polynomials \(f\) that do not have a very large group of symmetries; in such cases, we simply attempt to arrange a suitable embedding where the method does apply to give the desired estimates. Although we only apply this embedding sieve in one case in this paper, it will serve as a starting point in a sequel to this paper where we study squarefree values of more general polynomials that may have fewer symmetries.

This paper is organized as follows. In Section 2, we enumerate a natural set of axioms on an integral multivariate polynomial \(f\) which is sufficient to deduce that \(f\) takes the expected density of squarefree values (i.e., \(f\) satisfies (6)). In Section 3, we then prove the latter assertion, by developing the geometric sieve method that we use to extract squarefree values of such polynomials satisfying...
these axioms. Finally, in Sections 4 and 5, we then prove Theorems 1.5 and 1.6 by proving that all but one of the polynomials occurring in these theorems satisfy the axioms of Section 2. For the remaining polynomial \( g_2 \), we describe an extension of these axioms (the “embedding sieve”) that allows us to prove (6) also for this polynomial.

2 Some general criteria for extracting squarefree values of invariant polynomials

Let \( V \) be a representation of an algebraic group \( G \) defined over \( \mathbb{Z} \), and let \( f \) be an integer polynomial of degree \( d \) that is a relative invariant for the action of \( G \) on \( \mathbb{Z} \) and whose squarefree values we wish to extract. Let \( m := \dim(V) \). We use \( G^1 \) to denote the kernel of the determinant map \( G \to \text{GL}(V) \to \mathbb{G}_m \).

Suppose \( f, G, \) and \( V \) have the following properties:

1. There is a notion of a generic element of \( V(\mathbb{Z}) \); the subset \( V(\mathbb{Z})^{\text{gen}} \) of generic elements in \( V(\mathbb{Z}) \) is \( G(\mathbb{Z}) \)-invariant, and satisfies

\[
\mu(V(\mathbb{Z})^{\text{gen}}) := \lim_{N \to \infty} \frac{\#\{x \in V(\mathbb{Z})^{\text{gen}} \cap [-N, N]^m \}}{(2N + 1)^m} = 1.
\]

2. The order of the stabilizer in \( G(\bar{\mathbb{Q}}) \) of any element in \( V(\mathbb{Z})^{\text{gen}} \) is finite and absolutely bounded.

3. There is a continuous (but not necessarily polynomial) invariant \( I \) for the action of \( G^1(\mathbb{Z}) \) on \( V(\mathbb{Z})^{\text{gen}} \) that is homogeneous of degree \( d \), i.e., \( I(\lambda v) = \lambda^d I(v) \).

4. There is a fundamental domain \( \mathcal{F} \) for the action of \( G^1(\mathbb{Z}) \) on \( V(\mathbb{R}) \) such that the region \( \mathcal{F}_X := \{ v \in \mathcal{F} : |I(v)| < X \} \) is measurable and homogeneously expanding, i.e., \( \mathcal{F}_X = X^{1/d} \mathcal{F}_1 \), and the volume \( \text{Vol}(\mathcal{F}_X) \) of \( \mathcal{F}_X \) is finite.

5. For any subset \( S \) of \( V(\mathbb{Z}) \) defined by congruence conditions modulo finitely many prime powers, we have

\[
N(S; X) := \#\{v \in S \cap \mathcal{F}_X \text{ generic} \} = \text{Vol}(\mathcal{F}_X) \cdot \prod_p \mu_p(S) + o(X^{m/d}),
\]

where \( \mu_p(S) \) denotes the density of the \( p \)-adic closure of \( S \) in \( V(\mathbb{Z}_p) \).

6. Fix a prime \( p \). If \( v \in V(\mathbb{Z})^{\text{gen}} \) is an element such that \( f(v) \) is a multiple of \( p^2 \), then there is a nonnegative real number \( a = a_v \), an absolutely bounded integer \( k = k_v \geq 0 \), and an element \( g = g_v \in G(\bar{\mathbb{Q}}) \) such that

(i) \( |I(gv)| = p^{-a} |I(v)| \);

(ii) the element \( gv \) lies in \( V(\mathbb{Z})^{\text{gen}} \) in the reduction \( \text{mod } p \) of a closed \( G \)-invariant subscheme \( Y_k \) of \( V \) (viewed as affine \( n \)-space) defined over \( \mathbb{Z} \), depending only on \( k \), that has codimension \( \geq k \);

(iii) for each fixed \( k \), every point of \( Y_k(\mathbb{Z}) \) arises as \( g_v v \) for some \( v \in V(\mathbb{Z})^{\text{gen}} \) at most \( c \) times up to \( G(\mathbb{Z}) \)-equivalence, where \( c \) is an absolute constant;

(iv) \( \frac{m}{d} \cdot a + k - 1 \) is bounded below by an absolute positive constant \( \eta \).
Theorem 2.1 If \( f, G, V \) satisfy Conditions 1–6, then \( f \) takes the expected density of squarefree values, i.e., \( f \) satisfies (6).

While Conditions 1–6 may seem very restrictive, we will see in Sections 4 and 5 that they are satisfied by all but one of the polynomials in Theorems 1.5 and 1.6 (and, indeed, by many other polynomials, e.g., by a number of the discriminant polynomials occurring in [2]). In general, the notion of generic in Condition 1 is chosen so that the cusps of the fundamental domain \( F \) in Condition 4 contain mostly non-generic points. Indeed, the integral points in the cusps of such fundamental domains \( F \) tend to lie primarily on certain subvarieties; the lattice points in \( V(\mathbb{Z}) \) that lie outside the union of these subvarieties are then called generic. Condition 2 is of course common, and will generally be satisfied in any representation that has stable orbits in the language of geometric invariant theory. With Conditions 1–4 satisfied, Condition 5 can then be proven using geometry-of-numbers methods (as developed, e.g., in the works [19, 4, 5, 11]).

Finally, Condition 6 is also true for all the representations and polynomials in Theorem 1.5 and all but one of the representations and polynomials in Theorem 1.6. However, it is not true for the very first polynomial \( g_2 \) in Theorem 1.6. In general, Condition 6 can be quite restrictive (as opposed to Conditions 1–5), because there may not be enough symmetries in \( G \) to satisfy the condition. In such cases, we may attempt to embed \( V \) into a larger representation that has more symmetries for which (a suitable version of) Condition 6 is satisfied! This argument indeed works for the remaining representation, and will be important in future applications.

Remark 2.2 In the course of proving Theorem 2.1, we will also show that the polynomials \( f \) in this theorem—in addition to satisfying (6)—also satisfy

\[
\lim_{X \to \infty} \frac{\# \{ x \in F_X \cap V(\mathbb{Z})^{\text{gen}} : f(x) \text{ squarefree} \}}{\# \{ x \in F_X \cap V(\mathbb{Z})^{\text{gen}} \}} = \prod_p (1 - c_p/p^{2m})
\]

where again \( c_p \) denotes the number of elements \( x \in (\mathbb{Z}/p^2\mathbb{Z})^m \) satisfying \( f(x) = 0 \) in \( \mathbb{Z}/p^2\mathbb{Z} \).

3 A geometric squarefree sieve

In this section, we describe the geometric sieve method that we use to extract squarefree values of polynomials.

In §3.1–3.2 (see in particular Theorem 3.3), we lead up to a quantitative version of a certain uniformity estimate due to Ekedahl [22] (see also Poonen [33, 34] and Poonen–Stoll [35]). Ekedahl shows that, in appropriate situations, the usual inclusion–exclusion tail becomes “negligible” as the cut off defining the tail gets larger and larger. For our applications here as well as in future applications, we require precise quantitative versions of these tail estimates (i.e., how negligible is “negligible”?), when counting lattice points in homogeneously expanding regions. Given any variety of codimension at least 2 defined over \( \mathbb{Z} \), these estimates will, in particular, yield a method for sifting out those lattice points that, for some sufficiently large \( p \), reduce (mod \( p \)) to a point on the reduction of that variety (mod \( p \)). The quantitative versions of the relevant tail estimates that we prove in §3.2 enable one also to obtain second order terms or power-saving error terms in the applications.

In this article, we are particularly concerned with sifting out those lattice points on which a given polynomial takes non-squarefree values. The application to this scenario is described in Subsection 3.3. In the final Subsection 3.4, we then prove Theorem 2.1 of Section 2, namely, that any integral polynomial \( f \) satisfying the axioms of Section 2 takes the expected number of squarefree values.
3.1 The number of lattice points in a homogeneously expanding region lying on a subvariety

We start with the following simple and oft-used lemma that states that the number of lattice points on a given variety in a homogeneously expanding region in $\mathbb{R}^n$ grows at most polynomially in the linear scaling factor, where the degree of the polynomial is the dimension of the variety. Though this result is well-known, we include a proof here for completeness, and as a preparatory ingredient for the sieve estimates in §3.2.

**Lemma 3.1** Let $B$ be a compact region in $\mathbb{R}^n$ having finite measure. Let $Y$ be a variety in $\mathbb{R}^n$ of codimension $k \geq 1$. Then we have

$$\#\{a \in rB \cap Y \cap \mathbb{Z}^n\} = O(r^{n-k}),$$

(9)

where the implied constant depends only on $B$ and on $Y$.

**Proof:** We may clearly assume that $Y$ is irreducible, for otherwise we could simply sum over the irreducible components of $Y$. Since $Y$ has codimension $k$ in $\mathbb{R}^n$, there exist polynomials $f_1, \ldots, f_k$ for which

$$Y \subseteq Y'' := \{a \in \mathbb{R}^n \mid f_1(a) = f_2(a) = \cdots = f_k(a) = 0\}$$

(10)

such that the irreducible component $Y'$ of $Y''$ containing $Y$ also has codimension $k$.

We prove the estimate of Lemma 3.1 for $Y'$ in place of $Y$, by induction on $n$, using the polynomials $f_1, \ldots, f_k$. We always write the $f_i$ as polynomials in the arguments $x_1, \ldots, x_n$. For the proof, we may clearly assume that each $f_i$ is irreducible, for otherwise we can simply replace each $f_i$ by the irreducible factor of $f_i$ that vanishes on $Y'$. If all the $f_i$ do not involve some variable, say $x_n$, then the result follows by the induction hypothesis. So we may assume that every variable $x_1, \ldots, x_n$ occurs in at least one $f_i$.

We now show, via elimination theory, that we may reduce to the case where $k-1$ of the $f_i$, say $f_1, \ldots, f_{k-1}$, all do not involve some fixed variable, say $x_n$. Indeed, by reordering the $f_i$ if necessary, let us assume that $f_k$ is nonconstant as a polynomial in $x_n$. Let $R_n(f_i, f_k)$ denote the resultant of $f_i$ and $f_k$ with respect to $x_n$. Since $R_n(f_i, f_k) = A_i f_i + B_i f_k$ for some polynomials $A_i$ and $B_i$ with $A_i$ nonzero, the irreducible component containing $Y$ in the variety cut out by $R_n(f_1, f_k), \ldots, R_n(f_1, f_{k-1}), f_k$ (the variety cut out by $A_1 f_1, \ldots, A_{k-1} f_{k-1}, f_k$) is still $Y'$. Thus we may simply replace each $f_i$ involving $x_n$ (for $i \in \{1, \ldots, k-1\}$) by $R_n(f_i, f_k)$, and we see that the irreducible component containing $Y$ of the new $Y''$ cut out by the new $f_i$ is still the variety $Y'$ of codimension $k$, where now $f_1, \ldots, f_{k-1}$ do not involve $x_n$.

Thus it suffices to prove the lemma when $f_1, \ldots, f_{k-1}$ are polynomials only in $x_1, \ldots, x_{n-1}$. Let $h_k$ denote the leading coefficient of $f_k$ as a polynomial in $x_n$, so $h_k$ is a polynomial in $x_1, \ldots, x_{n-1}$. We may assume that $h_k$ does not vanish on $Y'$, for otherwise we might as well eliminate the leading term of $f_k$, and $f_1, \ldots, f_k$ would still cut out a variety $Y''$ whose irreducible component containing $Y$ is $Y'$. Let $Z$ be the union of the irreducible components intersecting $Y' \cap \{h_k = 0\}$ of the variety cut out by $f_1, \ldots, f_{k-1}, h_k$ in $\mathbb{R}^{n-1}$. Then $Z$ is of codimension $k$ in $\mathbb{R}^{n-1}$.

We now partition $\mathbb{Z}^n$ into two sets of points: those on which $h_k$ vanishes and those on which it does not. For the set of points where $h_k$ vanishes, we have

$$\#\{a \in rB \cap Y \cap \mathbb{Z}^n \mid h_k(a) = 0\} = O(r^{n-1-k}) \cdot O(r) = O(r^{n-k}),$$

(11)
since there are at most \( O(r^{n-1-k}) \) eligible values for the first \( n - 1 \) arguments by the induction hypothesis applied to \( Z \), and then there are at most \( O(r) \) possible values for the last coordinate of a point \( a \in rB \).

To handle the points where \( h_k \) does not vanish, if \( d \) denotes the degree of \( f_k \) as a polynomial in \( x_n \), then once values of \( x_1, \ldots, x_{n-1} \) are fixed satisfying \( h_k(x_1, \ldots, x_{n-1}) \neq 0 \), then there are at most \( d \) values for \( x_n \) satisfying \( f_k(x_1, \ldots, x_n) = 0 \). Therefore, using again the induction hypothesis on the irreducible component containing \( Y' \) of \( \{ f_1 = \cdots = f_{k-1} = 0 \} \) of codimension \( k - 1 \), we see that

\[
\#\{ a \in rB \cap Y \cap \mathbb{Z}^n \mid h_k(a) \neq 0 \} = O(r^{(n-1)-(k-1)}) \cdot d = O(r^{n-k}),
\]

as desired. \( \square \)

**Remark 3.2** Note that the \( O(r^{n-k}) \) estimate of Lemma 3.1 is optimal and is achieved for varieties of degree 1. In the case of varieties of degree \( > 1 \), the result can be improved, in some cases significantly, depending on the variety; see e.g., the work of Heath-Brown [27] and more recently of Salberger and Wooley [37]. We do not include their results here because, for our particular application, we require a result that includes also the case of degree 1.

### 3.2 The number of lattice points in a homogeneously expanding region that reduce modulo some sufficiently large prime \( p \) to a point on the reduction of a given variety modulo \( p \)

We are now in a position to prove an upper asymptotic estimate for the number of lattice points in a homogeneously expanding region in \( \mathbb{R}^n \) that reduce modulo \( p \), for some sufficiently large \( p > M \), to an \( \mathbb{F}_p \)-point of a given variety \( Y \subset \mathbb{A}^n \) defined over \( \mathbb{Z} \). In [22], Ekedahl proved that the asymptotic proportion of such points in a box of sidelength \( r \), as \( r \to \infty \), approaches 0 as \( M \to \infty \), assuming that \( Y \) has codimension at least two.

We prove here the following precise quantitative version of Ekedahl’s result:

**Theorem 3.3** Let \( B \) be a compact region in \( \mathbb{R}^n \) having finite measure, and let \( Y \) be any closed subscheme of \( \mathbb{A}_Z^n \) of codimension \( k \geq 1 \). Let \( r \) and \( M \) be positive real numbers. Then we have

\[
\# \{ a \in rB \cap \mathbb{Z}^n \mid a \ (\text{mod} \ p) \in Y(\mathbb{F}_p) \text{ for some prime } p > M \} = O \left( \frac{r^n}{M^{k-1} \log M} + r^{n-k+1} \right)
\]

where the implied constant depends only on \( B \) and on \( Y \).

**Proof:** We may again assume that \( Y \) is irreducible, for otherwise we could simply sum over the irreducible components of \( Y \). Since \( Y \) is a closed subscheme of \( \mathbb{A}_Z^n \) of codimension \( k \), there exist integral polynomials \( f_1, \ldots, f_k \) such that for any ring \( T \), we have

\[
Y(T) \subseteq Y''(T) := \{ a \in T^n \mid f_1(a) = f_2(a) = \cdots = f_k(a) = 0 \}
\]

whose irreducible component \( Y'' \) containing \( Y \) has codimension \( k \). Furthermore, there exists a constant \( M_0 > 0 \) such that \( Y'' \ (\text{mod} \ p) \) has codimension \( k \) in \( \mathbb{A}_{\mathbb{F}_p}^n \) for all primes \( p > M_0 \). Since \( M_0 \) depends only on \( Y \), for the purposes of proving Theorem 3.3, we may assume that \( M > M_0 \).

Now, by Lemma 3.1, we know that the number of points \( a \in rB \cap \mathbb{Z}^n \) such that \( a \in Y''(\mathbb{R}) \) is \( O(r^{n-k}) \). Thus it suffices to restrict ourselves to considering those points \( a \in rB \cap \mathbb{Z}^n \) for which \( a \notin Y''(\mathbb{Z}) \).
Since the result is trivial for \( k = 1 \), we may assume that \( k \geq 2 \). In this case, we will prove a slight strengthening of the theorem by showing that

\[
\# \{ (a, p) \mid a \in rB \cap \mathbb{Z}^n, p > M, a \notin Y'(\mathbb{Z}), a \equiv (\text{mod } p) \in Y'(\mathbb{F}_p) \} = O\left( \frac{r^n}{M^{k-1} \log M} + r^{n-k+1} \right). \tag{15}
\]

We first count those pairs \((a, p)\) on the left side of (15) for each prime \( p \) satisfying \( p \leq r \); such primes arise only when \( r > M \). In this case, since \( \#Y'(\mathbb{F}_p) = O(p^{n-k}) \) and \( rB \) can be covered by \( O((r/p)^n) \) boxes each of whose sides have length \( p \), we conclude that the number of \( a \in rB \cap \mathbb{Z}^n \) such that \( a \equiv (\text{mod } p) \) is in \( Y'(\mathbb{F}_p) \) is \( O(p^{n-k}) \cdot O(r^n/p^n) = O(r^n/p^k) \). Thus the total number of desired pairs \((a, p)\) with \( p \leq r \) is at most

\[
\# \{ (a, p) \mid a \in rB \cap \mathbb{Z}^n, M < p \leq r, a \equiv (\text{mod } p) \in Y'(\mathbb{F}_p) \} = \sum_{M < p \leq r} O\left( \frac{r^n}{p^n} \right) = O\left( \frac{r^n}{M^{k-1} \log M} \right). \tag{16}
\]

We next turn to the case of counting pairs \((a, p)\) where \( p > r \), and show that

\[
\# \{ (a, p) \mid a \in rB \cap \mathbb{Z}^n, p > r, a \notin Y'(\mathbb{Z}), a \equiv (\text{mod } p) \in Y'(\mathbb{F}_p) \} = O\left( r^{n-k+1} \right). \tag{17}
\]

Note first that (17) is true also for \( k = 1 \): if \( a \notin Y'(\mathbb{Z}) \), then some \( f_i \) does not vanish on \( a \), and since \( f_i(a) = O(r^{\deg(f_i)}) \), we see that \( f_i(a) \) can have at most \( O(1) \) prime factors \( p > r \).

For \( k \geq 2 \), we prove (17) by induction on \( n \). As before, we write the \( f_i \) as polynomials in the arguments \( x_1, \ldots, x_n \), and for the same reasons as in the proof of Lemma 3.1, we may assume that: 1) each \( f_i \) is irreducible; 2) \( f_1, \ldots, f_{k-1} \) are polynomials only in \( x_1, \ldots, x_{n-1} \); and, 3) the leading coefficient \( h_k \) of \( f_k \), as a polynomial in \( x_n \), does not vanish on \( Y' \). Let \( Y_{k-1} \) denote the irreducible component containing \( Y' \) of the closed subscheme of \( \mathbb{A}^n_{\mathbb{Z}} \) cut out by \( f_1, \ldots, f_{k-1} \), and let \( Z \) denote the union of irreducible components intersecting \( Y' \cap \{ h_k = 0 \} \) of the closed subscheme of \( \mathbb{A}^n_{\mathbb{Z}} \), via ignoring the last free coordinate) of codimensions \( k - 1 \) and \( k \), respectively. We may write

\[
\{(a, p) \mid a \in rB \cap \mathbb{Z}^n, p > r, a \notin Y'(\mathbb{Z}), a \equiv (\text{mod } p) \in Y'(\mathbb{F}_p) \} \tag{18}
\]

\[
\subseteq \{(a, p) \mid a \in rB \cap \mathbb{Z}^n, p > r, a \in Y_{k-1}(\mathbb{Z}), f_k(a) = 0 \equiv 0 \pmod{p}, a \equiv (\text{mod } p) \in Z(\mathbb{F}_p) \} \cup \tag{19}
\]

\[
\{(a, p) \mid a \in rB \cap \mathbb{Z}^n, p > r, a \notin Y_{k-1}(\mathbb{Z}), a \equiv (\text{mod } p) \in Y'(\mathbb{F}_p), h_k(a) = 0 \equiv 0 \pmod{p} \} \cup \tag{20}
\]

\[
\{(a, p) \mid a \in rB \cap \mathbb{Z}^n, p > r, a \notin Y_{k-1}(\mathbb{Z}), a \equiv (\text{mod } p) \in Y'(\mathbb{F}_p), h_k(a) = 0 \equiv 0 \pmod{p} \}. \tag{21}
\]

We estimate the size of the set (18) by giving estimates for each of the sets in (19)-(21).

Let \( P(rB) \) denote the projection of \( rB \) onto the first \( n - 1 \) coordinates. To give an upper estimate for the set in (19), we note that the number of points \( b \in P(rB) \cap \mathbb{Z}^{n-1} \) such that \( (b, \cdot) \in Y_{k-1}(\mathbb{Z}) \) is \( O(r^{(n-1)-(k-1)}) \) by Lemma 3.1. If furthermore \( (b, a_n) \in rB \cap \mathbb{Z}^n \) satisfies \( f_k(b, a_n) \neq 0 \), then again \( f_k(b, a_n) \) has at most \( O(1) \) prime factors \( p > r \). Thus the total number of pairs \((a, p)\), where \( a = (b, a_n) \in rB \cap \mathbb{Z}^n \) and \( p > r \), such that \( a \in Y_{k-1}(\mathbb{Z}) \), \( f_k(a) \neq 0 \), and \( f_k(a) \equiv 0 \pmod{p} \), is at most

\[
O\left( r^{(n-1)-(k-1)} \right) \cdot O(r) \cdot O(1) = O\left( r^{n-k+1} \right),
\]

giving the desired estimate for set (19).

Next, we see that the total number of pairs \((a, p)\) in the set (20) is also at most \( O\left( r^{n-k+1} \right) \), since there are at most \( O\left( r^{(n-1)-(k-1)} \right) \) values for the first \( n - 1 \) arguments by the
induction hypothesis applied to $Z$, and then at most $O(r)$ possible values for the last coordinate for a point in $rB$.

Finally, we give an upper estimate for the size of the set \((21)\). By the induction hypothesis applied to $Y_{k-1}$, the total number of pairs $(b, p)$, where $b \in P(rB) \cap \mathbb{Z}^{n-1}$ and $p > M$, such that $(b, \cdot) \notin Y_{k-1}(\mathbb{Z})$, $(b, \cdot) \pmod{p} \in Y_{k-1}(\mathbb{F}_p)$, and $h_k(b, \cdot) \equiv 0 \pmod{p}$ is $O(r^{(n-1)-(k-1)+1})$. Given such a pair $(b, p)$, the number of values of $a_n$ such that $f_k(b, a_n) \equiv 0 \pmod{p}$ and $a = (b, a_n) \in rB \cap \mathbb{Z}^n$ is at most $d \cdot O(1)$, where $d$ denotes the degree of $f_k$ as a polynomial in $x_n$. Indeed, $a_n \pmod{p}$ must be one of the $\leq d$ roots of $f_k \pmod{p}$ in that case, and the number of such integers $a_n$ in the union of a bounded number of intervals in $\mathbb{R}$ having total measure at most $O(r) = O(p)$, that are congruent (mod $p$) to one of these $\leq d$ values, is at most $d \cdot O(1) = O(1)$ (since $d$ is a constant depending only on $Y$). We conclude that the total number of pairs $(a, p)$, where $a = (b, a_n)$, that lie in the set \((21)\) is at most $O(r^{(n-1)-(k-1)+1}) \cdot O(1) = O(r^{n-k+1})$, as desired. \(\Box\)

Remark 3.4 Tracing through the proof, it is clear that the bound in Theorem 3.3 can be achieved for suitable choices of $Y$, and so the bound is essentially optimal without further assumptions on $Y$.

Theorem 3.3 also has a number of variations, which can be useful in various sieves depending on context. One natural variation is when the region of interest in $\mathbb{R}^n$ is not just homogeneously expanding, but is also being applied with more general linear transformations in $GL_n$, such as diagonal matrices and shears.

Theorem 3.5 Let $B$ be a compact region in $\mathbb{R}^n$ having finite measure, and let $Y$ be any closed subscheme of $\mathbb{A}^n_\mathbb{Z}$ of codimension $k \geq 2$ such that the Zariski closure of the projection of $Y$ onto the first $n - k + j$ coordinates has codimension $j$ in $\mathbb{A}^{n-k+j}_\mathbb{Z}$ for $j = 0, \ldots, k$. Let $r$ and $M$ be positive real numbers, and let $t = \text{diag}(t_1, \ldots, t_n)$ be a diagonal element of $\text{SL}_n(\mathbb{R})$. Suppose that $\kappa > 0$ is a constant such that $rt_i \geq \kappa$ for all $i$ and $t_i \geq \kappa$ for all $i > n - k$. Then we have

$$\#\{a \in rtB \cap \mathbb{Z}^n \mid a \pmod{p} \in Y(\mathbb{F}_p) \text{ for some prime } p > M\} = O\left(\frac{r^n}{M^{k-1} \log M} + r^{n-k+1}\right),$$

\[\text{(22)}\]

where the implied constant depends only on $B$, $Y$, and $\kappa$.

To prove Theorem 3.5 we note first that the analogue of Lemma 3.1 holds equally well when $rB$ is replaced with $rtB$, even without the condition that $t_i \geq \kappa$ for all $i > n - k$. The proof is then identical to that of Theorem 3.3

3.3 Polynomials taking values that are multiples of squares of primes

Let $f$ be a polynomial with integer coefficients in the variables $x_1, \ldots, x_n$. To count squarefree values taken by $f$, we wish to sieve out those points in $\mathbb{Z}^n$ where $f$ is a multiple of $p^2$ for some prime $p$. Now if $f(a) \equiv 0 \pmod{p^k}$ for some $k > 1$ and $a \in \mathbb{Z}^n$, then this can happen in two distinct ways, namely, we have either

$$f(a') \equiv 0 \pmod{p^k} \quad \forall a' \equiv a \pmod{p} \quad \text{(23)}$$

or

$$\exists a' \equiv a \pmod{p} \text{ such that } f(a') \not\equiv 0 \pmod{p^k}. \quad \text{(24)}$$

In the first case, we say that $f$ is strongly a multiple of $p^k$ at $a$, and otherwise we say that $f$ is weakly a multiple of $p^k$ at $a$. In other words, \((23)\) says that $f(a)$ is a multiple of $p^k$ for “mod $p$ reasons”;

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Theorem 3.3 can thus be applied in order to estimate the asymptotic number of points in homogeneously expanding region, on which $f$ then we have for all primes $p$

§ of strong multiples via more linear algebraic methods (see prove the necessary estimates by ring-theoretic methods, or by reducing weak multiples to the case of strong multiples via geometric techniques. For example, if $f$ is strongly a multiple of $p$ at $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, and not all the coefficients of $f$ vanish at $a$ (as a polynomial in $x_n$) modulo $p$, then $f(a_1, \ldots, a_{n-1}, x_n) \pmod{p}$ must have a root of multiplicity $k$ at $x_n \equiv a_n \pmod{p}$.

It follows that if $Y_k$ denotes the closed subscheme of $\mathbb{A}^n_\mathbb{Z}$ defined by

$$f = \frac{\partial f}{\partial x_n} = \cdots = \frac{\partial^{k-1}f}{\partial x_n^{k-1}} = 0,$$

then we have for all primes $p$ that

$$\{a \in \mathbb{Z}^n \mid f \text{ is strongly a multiple of } p^k \text{ at } a\} \subseteq \{a \in \mathbb{Z}^n \mid a \pmod{p} \in Y_k(\mathbb{F}_p)\}.$$ 

Theorem 3.3 can thus be applied in order to estimate the asymptotic number of points in $\mathbb{Z}^n$, in a homogeneously expanding region, on which $f$ is strongly a multiple of $p^k$.

Generically, if the degree of $f$ is large enough, then the subscheme $Y_k$ will have codimension $k$. In practice, this can be checked in any given example; for our purposes, the following lemma will suffice:

**Lemma 3.6** Let $f$ be an irreducible integral polynomial in $n \geq 2$ variables. Then there exists a subscheme $Y$ of $\mathbb{A}^n_\mathbb{Z}$ of codimension two such that, for all primes $p$, we have

$$\{a \in \mathbb{Z}^n \mid f \text{ is strongly a multiple of } p^2 \text{ at } a\} \subseteq \{a \in \mathbb{Z}^n \mid a \pmod{p} \in Y(\mathbb{F}_p)\}.$$ 

**Proof:** Without loss of generality, we may assume that $f$ is nonconstant as a polynomial in $x_n$. We let $Y = Y_k$ as defined in (25), with $k = 2$. Then since $f$ is irreducible, $f$ and $\partial f / \partial x_n$ do not share a common factor, and so they cut out a subscheme in $\mathbb{A}^n_\mathbb{Z}$ of codimension two. It follows that $Y = Y_k$ as defined in (25), with $k = 2$, has codimension two in $\mathbb{A}^2_\mathbb{Z}$, as desired. \qed

Thus, to sieve out lattice points in homogeneously expanding regions in $\mathbb{R}^n$ where a multivariate irreducible polynomial is a multiple of $p^2$, one may first use the estimates of §3.2 to handle the strong multiples of $p^2$. It remains to handle the weak multiples; the key idea then is to utilize extra structure on the polynomials to reduce weak multiples to strong multiples via appropriate rational changes of variable!

### 3.4 Proof of Theorem 2.1

In this subsection, we prove Theorem 2.1 i.e., if $f$ is a polynomial that satisfies the axioms of Section 2, then $f$ takes the expected density of squarefree values.

To this end, suppose that $f$ (with given $G$ and $V$) satisfies the set of axioms of Section 2. We begin by proving Equation (8) of Remark 2.2 for $f$. Let $\mathcal{F}_1$ and $\mathcal{F}_X = X^{1/d} \mathcal{F}_1$ be as in Condition 4. Then Condition 5, in the special case $S = V(\mathbb{Z})$, states that

$$|\mathcal{F}_X \cap V(\mathbb{Z})^\text{gen}| = \text{Vol}(\mathcal{F}_X) + o(X^{m/d}).$$

(26)
For any small \( \epsilon > 0 \), let \( \mathcal{F}_1^{1-\epsilon} \) denote a compact measurable subset of \( \mathcal{F}_1 \) such that

\[
\text{Vol}(\mathcal{F}_1^{1-\epsilon}) = (1 - \epsilon)\text{Vol}(\mathcal{F}_1).
\]

(That is, \( \mathcal{F}_1^{1-\epsilon} \) is obtained from \( \mathcal{F}_1 \) by cutting off the cusps of \( \mathcal{F}_1 \) sufficiently far out.) Let \( \mathcal{F}_X^{1-\epsilon} = X^{1/d} \cdot \mathcal{F}_1^{1-\epsilon} \), so that

\[
\text{Vol}(\mathcal{F}_X^{1-\epsilon}) = (1 - \epsilon)\text{Vol}(\mathcal{F}_X).
\]

Then

\[
|\mathcal{F}_1^{1-\epsilon} \cap V(\mathbb{Z})^\text{gen}| = \text{Vol}(\mathcal{F}_1^{1-\epsilon}) + o(X^{m/d}) = (1 - \epsilon)\text{Vol}(\mathcal{F}_1) \cdot X^{m/d} + o(X^{m/d}),
\]

since we are simply counting lattice points in a bounded homogeneously expanding region, and then subtracting away the count of non-generic points which have density zero by Condition 1. Similarly, for a set \( S \subset V(\mathbb{Z}) \) defined by finitely many congruence conditions, we have

\[
|\mathcal{F}_X^{1-\epsilon} \cap S^\text{gen}| = (1 - \epsilon)\text{Vol}(\mathcal{F}_1) \prod_p \mu_p(S) \cdot X^{m/d} + o(X^{m/d}),
\]

where \( S^\text{gen} \) denotes the subset of generic points in \( S \). Note that, by (26) and (27), we have

\[
|\mathcal{F}_X \cap \mathcal{F}_1^{1-\epsilon} \cap S^\text{gen}| \leq \epsilon \cdot \text{Vol}(\mathcal{F}_1) \cdot X^{m/d} + o(X^{m/d}).
\]

Now, for each prime \( p \), let \( S_p \) be a subset of \( V(\mathbb{Z}) \) defined by finitely many congruence conditions such that for sufficiently large \( p \), the set \( S_p \) contains all elements \( v \in V(\mathbb{Z}) \) such that \( p^2 \nmid f(v) \). Let \( S = \bigcap_p S_p \). Then, to prove (8) for \( f \), it suffices to determine, asymptotically, the cardinality of \( \mathcal{F}_X \cap S^\text{gen} \); indeed, the special case where \( S_p \) is exactly the set of elements \( v \in V(\mathbb{Z}) \) such that \( p^2 \nmid f(v) \) will correspond to (8).

Let \( M \) be any positive integer. It follows from Condition 5 that

\[
\lim_{X \to \infty} \frac{|\mathcal{F}_X \cap (\bigcap_{p \leq M} S_p^\text{gen})|}{X^{m/d}} = \text{Vol}(\mathcal{F}_1) \prod_{p \leq M} \mu_p(S).
\]

Letting \( M \) tend to \( \infty \), we conclude that

\[
\limsup_{X \to \infty} \frac{|\mathcal{F}_X \cap S^\text{gen}|}{X^{m/d}} \leq \text{Vol}(\mathcal{F}_1) \prod_p \mu_p(S).
\]

To obtain a lower bound for \( |\mathcal{F}_X \cap S^\text{gen}| \), we note that

\[
\bigcap_{p \leq M} S_p \subset (S \cup \bigcup_{p > M} W_p),
\]

where \( W_p \) denotes the set of points in \( V(\mathbb{Z}) \) having discriminant a multiple of \( p^2 \). We use the geometric sieve estimates of the previous section to estimate the size of \( \mathcal{F}_X \cap (\bigcup_{p > M} W_p^\text{gen}) \). More precisely, we prove:

**Lemma 3.7** We have

\[
|\mathcal{F}_X \cap (\bigcup_{p > M} W_p^\text{gen})| = O_\epsilon(X^{m/d} / (M^{\min\{1, \eta\}} \log M) + X^{m-1/d}) + O(\epsilon X^{m/d}),
\]

where the implied constants are independent of \( M \).
\textbf{Proof:} We write \( W_p^{\text{gen}} = W_p^{(1)} \cup W_p^{(2)} \), where \( W_p^{(1)} \) denotes the set of points where the discriminant is strongly a multiple of \( p^2 \) and \( W_p^{(2)} \) denotes the set of points where the discriminant is weakly a multiple of \( p^2 \).

By Lemma 3.6 there exists an arithmetic subscheme \( Y \) of \( \mathbb{A}^m \) of codimension \( \geq 2 \) such that \( x \in W_p^{(1)} \) implies that \( x \pmod{p} \) is a point on \( Y(\mathbb{F}_p) \). Since \( F_X^{1-\epsilon} \) is a bounded and homogeneously expanding region, by (29) and Theorem 3.3 we conclude that

\[
\left| F_X \cap (\bigcup_{p > M} W_p^{(1)}) \right| = \left| F_X^{1-\epsilon} \cap (\bigcup_{p > M} W_p^{(1)}) \right| + O(\epsilon X^{m/d}) = O(\epsilon X^{m/d}/(M \log M) + X^{(m-1)/d}) + O(\epsilon X^{m/d}).
\]

In particular, any \( v \in W_p^{(1)} \) satisfies Condition 6 with \( g = 1, a = 0, \) and \( k = 2 \).

To handle \( W_p^{(2)} \) for primes \( p \) with \( M < p \leq X^{1/2d} \), we may use the same argument used to prove (16) to obtain

\[
\# \{ (v, p) | v \in F_X^{1-\epsilon} \cap W_p^{(2)}, M < p \leq X^{1/2d} \} = \sum_{M < p \leq X^{1/2d}} O\left( \frac{X^{m/d}}{p^2} \right) = O\left( \frac{X^{m/d}}{M \log M} \right). \tag{34}
\]

For primes \( p > X^{1/2d} \), we use Condition 6. Let us write \( W_p^{(2)} = \bigcup_{k \geq 0} W_p^{(2)}(k) \), where \( W_p^{(2)}(k) \) is the portion of \( W_p^{(2)} \) having given value of \( k \) in Condition 6(ii). For this fixed \( k \), let \( \alpha \) be the infimum of \( \alpha \) over all \( v \in W_p^{(2)}(k) \). Then we have

\[
N(W_p^{(2)}(k); X) = O(N(V(\mathbb{Z}); X/p^\alpha)) = O((X/p^\alpha)^{m/d}), \tag{35}
\]

where the first equality follows from Conditions 6(i) and 6(iii), and the fact that \( f(v) \) (for \( v \in F_X \cap V(\mathbb{Z}) \)) has at most \( d \) prime factors \( p \) greater than \( X^{1/2d} \) such that \( p^2 \pmod{f(v)} \); and the second equality follows from Condition 5. By summing over \( p > M' = \max\{M, X^{1/2d}\} \), this is sufficient to obtain the estimate of Lemma 3.7 in cases where \( k = 0 \). If \( k \geq 1 \), then we may strengthen \( (35) \), when counting in the union of the \( W_p^{(2)}(k) \) over all \( p > M' \), using Condition 6(ii), Estimate (29), and Theorem 3.3

\[
N(\bigcup_{p > M'} W_p^{(2)}(k); X) = O\left( \left| \{ v \in V(\mathbb{Z}) : v \in F_X/p^\alpha \pmod{p} \in Y_k(\mathbb{F}_p) \text{ for some } p > M' \} \right| \right) = O\left( \left| \{ v \in F_X^{1-\epsilon}/M^\alpha \cap V(\mathbb{Z}) : v \pmod{p} \in Y_k(\mathbb{F}_p) \text{ for some } p > M' \} \right| + \epsilon (X/M'\alpha)^{m} \right) = O(\epsilon (X/M'\alpha)^{m}/(M'k^{-1}\log M') + (X/M'\alpha)^{m/k+1} + O(\epsilon (X/M'\alpha)^{m/d}). \tag{36}
\]

Combining (33), (34), and (36), we obtain Lemma 3.7 \( \Box \)

By (30), (31), and Lemma 3.7, we see that

\[
\liminf_{X \to \infty} \frac{|F_X \cap S^{\text{gen}}|}{X^{m/d}} \geq \text{Vol}(F_1) \prod_{p \leq M} \mu_p(S) - O(1/M\min\{a, 1\}) = O(\epsilon). \tag{37}
\]

Letting \( M \) tend to infinity, and combining with Condition 6(iv) that \( \eta > 0 \), gives

\[
|F_X \cap S^{\text{gen}}| \geq \text{Vol}(F_1) \prod_{p \leq M} \mu_p(S) \cdot X^{m/d} - O(\epsilon \cdot X^{m/d}). \tag{38}
\]
Finally, letting $\epsilon$ tend to 0, and combining with (31), yields
\[
|\mathcal{F}_X \cap S^{\text{gen}}| = \text{Vol}(\mathcal{F}_1) \prod_p \mu_p(S) \cdot X^{m/d} + o(X^{m/d}). \tag{39}
\]

This proves (8) of Remark 2.2 under the assumption that $f$ satisfies the axioms of Section 2.

To prove also (6) for $f$ (i.e., Theorem 2.1), let $B_N = [-N,N]^m \subset V(\mathbb{R})$, and for each prime $p$, let $c_p$ denote the number of elements $x \in (\mathbb{Z}/p^2\mathbb{Z})^m$ satisfying $f(x) = 0$ in $\mathbb{Z}/p^2\mathbb{Z}$. Given any positive integer $M$, let $S_M \subset \mathbb{Z}$ denote the set of all integers that are not multiples of $p^2$ for any prime $p \leq M$. Then it is clear that
\[
\lim_{N \to \infty} \frac{\# \{ x \in \mathbb{Z}^m \cap B_N : f(x) \in S_M \}}{(2N + 1)^m} = \prod_{p \leq M} (1 - c_p/p^2), \tag{40}
\]

since the set of points being counted is a union of finitely many translates of lattices, all defined by congruence conditions modulo a single fixed modulus (namely, $\prod_{p \leq M} p^2$). Letting $M$ tend to infinity, we see that
\[
\limsup_{N \to \infty} \frac{\# \{ x \in \mathbb{Z}^m \cap B_N : f(x) \text{ squarefree} \}}{(2N + 1)^m} \leq \prod_p (1 - c_p/p^2). \tag{41}
\]

(This upper bound indeed holds for any polynomial, and we have not yet used any special property of $f(x)$.)

To obtain a lower bound, we note that by Condition 1, the density of non-generic points in $B_N$ approaches 0 as $N \to \infty$, and hence such non-generic points may be ignored for the purposes of proving Theorem 2.1. We now treat separately the generic points on which $f$ is strongly a multiple of $p^2$ and on which $f$ is weakly a multiple of $p^2$. In the case of the set $B_N \cap W^{(1)}_p$ of points in $B_N$ where $f$ is strongly a multiple of $p^2$, we immediately have by Lemma 3.6 and Theorem 3.3 that
\[
\left| B_N \cap (\bigcup_{p \geq M} W^{(1)}_p) \right| = O(N^m/(M \log M) + N^{m-1}) \tag{42}
\]
for any $M > 0$.

In order to obtain an analogous estimate for the set $B_N \cap W^{(2)}_p$ of points in $B_N$ where $f$ is weakly a multiple of $p^2$, our strategy is to cover a certain large portion of $B_N$ by fundamental domains for the action of $G(\mathbb{Z})$ on $V(\mathbb{R})$, and then apply (36) to each such fundamental domain. To carry out this plan, we note that $B_N$ may be covered by a countable union $\bigcup_{i=1}^\infty \gamma_i \mathcal{F}_X$ ($\gamma_i \in G(\mathbb{Z})$) of translates of $\mathcal{F}_X$, where $X$ is sufficiently large so that $|I(v)| < X$ for all $v \in B_N$; since $I$ has degree $d$, we may take $X = c N^d$ for some fixed constant $c > 0$.

Let $B_{N,s} = B_N \cap (\bigcup_{i=1}^s \gamma_i \mathcal{F}_X)$. Since we have the estimate of Lemma 3.7 for any $G(\mathbb{Z})$-translate of $\mathcal{F}_X$, and $B_{N,s}$ is a union of $s$ translates of $\mathcal{F}_X$, we conclude that
\[
\left| B_{N,s} \cap (\bigcup_{p > M} W^{(2)}_p) \right| = s O \left( \frac{X^m}{(M^{\min\{\eta,1\}} \log M) + X^{m-1} d} \right) + o(\epsilon X^m) \tag{43}
\]

where the implied constant is independent of $s$, $M$, and $N$.

It follows from (42) and (43) that
\[
\liminf_{N \to \infty} \frac{\# \{ x \in \mathbb{Z}^m \cap B_{N,s} : f(x) \text{ squarefree} \}}{\text{Vol}(B_{N,s})} \geq \prod_{p \leq M} (1 - c_p/p^2) - s O(1/M^{\min\{\eta,1\}}) - s O(\epsilon). \tag{44}
\]
Evidently, \( \text{Vol}(B_{N,s}) \) approaches \( \text{Vol}(B_N) = (2N)^m \) from below as \( s \to \infty \). Letting \( M \) tend to infinity, and then \( \epsilon \) to 0, and finally \( s \) to \( \infty \) in (44) now yields the desired result

\[
\lim_{N \to \infty} \frac{\# \{ x \in \mathbb{Z}^m \cap B_N : f(x) \text{ squarefree} \}}{(2N + 1)^m} = \prod_p (1 - c_p/p^2),
\]

since \( \text{Vol}(B_N)/(2N + 1)^m \to 1 \) as \( N \to \infty \).

4 The density of squarefree values taken by \( f_3, f_4, \) and \( f_5 \) (Proofs of Theorems 1.1–1.4)

In this section, we show that the polynomials \( f_3, f_4, \) and \( f_5 \) as in the introduction satisfy all the axioms of Section 2. It will therefore follow by Theorem 2.1 that these polynomials take the expected density of squarefree values. We will also then deduce Theorem 1.1, Corollary 1.2, Theorem 1.3, Theorem 1.4, and Theorem 1.5.

We begin by describing the polynomials \( f_3, f_4, \) and \( f_5 \) in more detail, and their interpretations in terms of cubic, quartic, and quintic rings.

4.1 The parametrization of rings of small rank by prehomogeneous vector spaces

Let \( n = 3, 4, \) or \( 5 \). For any ring \( T \) (commutative, with unit), let \( V(T) \) be:

(a) the space \( \text{Sym}_3 T^2 \) of binary cubic forms with coefficients in \( T \), if \( n = 3 \);

(b) the space \( T^2 \otimes \text{Sym}_2 T^3 \) of pairs of ternary quadratic forms with coefficients in \( T \), if \( n = 4 \); or

(c) the space \( T^4 \otimes \wedge^2 T^5 \) of quadruples of \( 5 \times 5 \) skew-symmetric matrices with entries in \( T \), if \( n = 5 \).

Then the group \( G(T) \) naturally acts on \( V(T) \), where we set \( G(T) = \text{GL}_2(T), \ \text{GL}_2(T) \times \text{SL}_3(T), \) or \( \text{GL}_4(T) \times \text{SL}_5(T) \) in accordance with whether \( n = 3, 4, \) or \( 5 \), respectively. In the case of \( n = 3 \), we use the “twisted action”, i.e., an element \( \gamma \in \text{GL}_2 \) acts on a binary cubic form \( x(s,t) \in V \) by

\[
\gamma \cdot x(s,t) = (\det \gamma)^{-1} x((s,t) \cdot \gamma).
\]

For each \( n = 3, 4, \) or \( 5 \), there is a natural invariant polynomial \( f_n \) for the action of \( G(\mathbb{Z}) \) on \( V(\mathbb{Z}) \), called the discriminant, which in fact generates the ring of polynomial invariants. This discriminant polynomial has degree 4, 12, or 40 on \( V(\mathbb{Z}) \), in accordance with whether \( n = 3, 4, \) or \( 5 \). We say that an orbit of \( G(T) \) on \( V(T) \) is nondegenerate if the discriminant of any element in that orbit is nonzero.

The nondegenerate orbits of \( G(T) \) on \( V(T) \) in the case of fields \( T \) were classified by Wright and Yukie [42], and were shown to be in natural correspondence with étale degree \( n \) extensions of \( T \). The orbits of \( G(\mathbb{Z}) \) on \( V(\mathbb{Z}) \) were classified in [21], [2], and [3], where the following theorem was proved:

**Theorem 4.1** The nondegenerate \( G(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \) are in canonical bijection with isomorphism classes of pairs \( (R, R') \), where \( R \) is a ring of rank \( n \) and \( R' \) is a resolvent ring of \( R \). In this bijection, the discriminant of an element \( v \in V(\mathbb{Z}) \) equals the discriminant of the corresponding ring \( R \) of rank \( n \). Furthermore, every isomorphism class of ring \( R \) of rank \( n \) occurs in this bijection, and every isomorphism class of maximal ring occurs exactly once.
Recall that a ring of rank $n$ is a ring $R$ (commutative, with unit) such that $R$ is free of rank $n$ as a $\mathbb{Z}$-module. A ring of rank 2, 3, 4, 5, or 6 is called a quadratic, cubic, quartic, quintic, or sextic ring, respectively. A resolvent ring $R'$ of a cubic, quartic, or quintic ring $R$ is a quadratic, cubic, or sextic ring, respectively, that satisfies certain properties, whose precise definition will not be needed here (see [2] and [3] for details). A ring $R$ of rank $n$ is said to be maximal if it is not a proper subring of any other ring of rank $n$; equivalently, $R$ is maximal if it is the maximal order in a product of number fields.

Now a ring $R$ of rank $n$ that has squarefree (or fundamental) discriminant is automatically maximal. In particular, such a ring will arise exactly once in the bijection of Theorem 4.1. We are interested, however, only in those maximal rings of rank $n$ that are actually orders in $S_n$-number fields (rather than, say, in a nontrivial product of number fields). We thus say that a point $v \in V(\mathbb{Z})$ is generic if the ring $R$ of rank $n$ associated to it under the bijection of Theorem 4.1 is an order in a $S_n$-number field of degree $n$. With this definition of generic in hand, we may now use the results of [19, §2.4], or [6, §2.1], while Condition 5 is [20, §5], [4, Eqn. (32)], or [2, §2.1], respectively.

It remains to check the crucial Condition 6. We will prove Condition 6 for $f_3$, $f_4$, and $f_5$ with $a = 2$, $c = \binom{n}{2}$, and $k = 0$. Condition 6(ii) is then automatically satisfied (noting that genericity is a $G(\mathbb{Q})$-invariant condition and that $k = 0$). Also, since $m = d$ in these cases, Condition 6(iv) is also then automatically satisfied.

We now verify Conditions 6(i) and (iii). The proofs of these two important subconditions are where the “largeness” of the symmetry group $G$ of the polynomial $f_n$ is used.

We begin by noting that a general (i.e., nondegenerate) element of $V(\mathbb{F}_p)$ determines $n$ distinct points in $\mathbb{P}^{n-2}(\mathbb{F}_p)$. Indeed, when $n = 3$, we have a binary cubic form, which generally has 3 zeros in $\mathbb{P}^1$. Similarly, when $n = 4$, we have a pair of conics in $\mathbb{P}^2$, which generally intersect in 4 points in $\mathbb{P}^2$. And when $n = 5$, we have four $5 \times 5$ skew-symmetric matrices, the $4 \times 4$ sub-Pfaffians of which give quadrics in $\mathbb{P}^3$ that generally intersect in 5 points in $\mathbb{P}^3$. The discriminant of an element $\bar{x} \in V(\mathbb{F}_p)$ vanishes precisely when two or more of these $n$ points come together, or when $\bar{x}$ is so degenerate that the variety that is cut out by $\bar{x}$ in $\mathbb{P}^{n-2}$ is of dimension greater than zero.

The latter case, where a variety of dimension greater than zero is cut out by $x \in V(\mathbb{F}_p)$, happens on an algebraic set (defined over $\mathbb{Z}$) that is of codimension greater than one in $V(\mathbb{F}_p)$. Similarly, the case where strictly fewer than $n - 1$ points cut are out by $x$ (not including multiplicity) also occurs on a set of codimension greater than one in $V(\mathbb{F}_p)$. Indeed, these two sets in $V(\mathbb{F}_p)$ together comprise the image of the set $W_p^{(1)}$ in $V(\mathbb{F}_p)$.

The image of the set $W_p^{(2)}$ in $V(\mathbb{F}_p)$ consists of elements $x \in V(\mathbb{F}_p)$ that cut out $n$ points (counting multiplicity) in $\mathbb{P}^{n-2}(\mathbb{F}_p)$, such that two of those $n$ points are the same and the rest are distinct and different. Thus we have a description of those points in $V(\mathbb{Z}/p\mathbb{Z})$ on which the discriminant polynomial $f_n$ vanishes (mod $p$) that potentially lift to points in $V(\mathbb{Z}/p^2\mathbb{Z})$ where $f_n$ is weakly a multiple of $p^2$.

To determine what precisely the set is in $V(\mathbb{Z}/p^2\mathbb{Z})$ where the discriminant is weakly a multiple of $p^2$, we consider each $n = 3, 4, 5$ separately. If $n = 3$, then we see from the above
discussion that the image of $W_p^{(2)}$ in $V(\mathbb{F}_p)$ consists only of binary cubic forms over $\mathbb{F}_p$ having a double (but not triple) root in $\mathbb{P}^1$. A binary cubic form in $V(\mathbb{F}_p)$ having a double root in $\mathbb{P}^1$ is always $G(\mathbb{F}_p)$-equivalent to one of the form $\bar{x}(s,t) = \bar{a}s^3 + \bar{b}s^2t$, where $\bar{a}, \bar{b} \in \mathbb{F}_p$ and $\bar{b} \neq 0$. If $x(s,t) = as^3 + bs^2t + cst^2 + dt^3 \in V(\mathbb{Z})$ is a lift of $\bar{x}$ to $V(\mathbb{Z})$, then $c$ and $d$ are multiples of $p$, $b$ is prime to $p$, and we compute that the discriminant $f_3(x)$ of $x$ is given by

$$f_3(x) \equiv -4b^3d \pmod{p^2}$$

implying (for $p > 2$) that $d$ must then be a multiple of $p^2$ for $x$ to have discriminant that is (weakly) a multiple of $p^2$.

Given such a form $x(s,t) = as^3 + bs^2t + cst^2 + dt^3 \in V(\mathbb{Z})$, with $p \nmid b$, $p \nmid c$, and $p^2 \mid d$, we may multiply $x$ by $p$ and then apply $g = [1\ 1/p] \in \text{GL}_2(\mathbb{Q})$ to obtain a form $x'(s,t) = pas^3 + bs^2t + (c/p)st^2 + (d/p^2)t^3 \in V(\mathbb{Z})$, and we see that $f_3(x') = f_3(x)/p^2$. This proves Condition 6(i) for $f_3$ with $a = 2$.

To prove Condition 6(iii), note that an element $x'(s,t) = a's^3 + b's^2t + c's^2t^2 + d't^3 \in V(\mathbb{Z})$ may be sent to an element $x \in V(\mathbb{Z})$ of the above type via the inverse transformation $g^{-1}$ precisely when $p \mid a'$ but $p \nmid b'$. It follows that the $\text{GL}_2(\mathbb{Z})$-class of $x'$ can lead to at most 3 $\text{GL}_2(\mathbb{Z})$-classes of elements $x \in V(\mathbb{Z})$ in this way (since $x'$ can have at most 3 simple roots in $\mathbb{P}^1(\mathbb{F}_p)$), yielding Condition 6(iii) with $c = 3$.

The case $n = 4$ can be treated similarly. The image of $W_p^{(2)}$ in $V(\mathbb{F}_p)$ consists only of pairs $(\bar{A}, \bar{B})$ of ternary quadratic forms over $\mathbb{F}_p$ that have three common zeroes in $\mathbb{P}^2(\mathbb{F}_p)$ (i.e., two simple common zeroes, and one common zero of multiplicity two). By a transformation in $\text{SL}_3(\mathbb{F}_p)$, we may assume that that the common zero of multiplicity two in $\mathbb{P}^2$ of such an element $(\bar{A}, \bar{B}) \in V(\mathbb{F}_p)$ is at $[1:0:0] \in \mathbb{P}^2(\mathbb{F}_p)$. Furthermore, via a transformation in $\text{GL}_2(\mathbb{F}_p)$ we may assume that $\bar{B}$ cuts out a union of two distinct lines in $\mathbb{P}^2(\mathbb{F}_p)$ that intersect at $[1:0:0] \in \mathbb{P}^2$, and $\bar{A}$ cuts out a nonsingular conic passing through the intersection point $[1:0:0]$ of these two lines, but not tangent to either of these two lines.

It follows that an element in the image of $W_p^{(2)}$ in $V(\mathbb{F}_p)$ will always be $G(\mathbb{F}_p)$-equivalent to an element $(\bar{A}, \bar{B}) \in V(\mathbb{F}_p)$ of the matrix form

$$\begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{23} \\ \bar{a}_{13} & \bar{a}_{23} & \bar{a}_{33} \end{pmatrix}, \begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} & \bar{b}_{13} \\ \bar{b}_{12} & \bar{b}_{22} & \bar{b}_{23} \\ \bar{b}_{13} & \bar{b}_{23} & \bar{b}_{33} \end{pmatrix}$$

where $\bar{a}_{11} = \bar{b}_{11} = \bar{b}_{12} = \bar{b}_{13} = 0$ and $\bar{b}_{22}\bar{b}_{33} - \bar{b}_{23}^2 \neq 0$.

Now if $(\bar{A}, \bar{B}) \in V(\mathbb{Z})$ is a lift of $(\bar{A}, \bar{B})$ to $V(\mathbb{Z})$, where $A$ and $B$ have entries $a_{ij}$ and $b_{ij}$ respectively, then $a_{11}, b_{11}, b_{12},$ and $b_{13}$ are multiples of $p$ and $b_{22}b_{33} - b_{23}^2$ is prime to $p$. In that case, we compute the discriminant $f_4((A,B))$ of $(A,B)$ to be

$$f_4((A,B)) = \text{Disc}(\det(As + Bt)) \equiv b_{11}(b_{22}b_{33} - b_{23}^2)C^3 \pmod{p^2},$$

where $C$ is the coefficient of $s^2t$ in $\det(As + Bt)$; this implies that $b_{11}$ must be a multiple of $p^2$ (and $C$ not a multiple of $p$) for $f$ to be (weakly) a multiple of $p^2$ at $(A,B)$.

Given such an element $(A,B) \in V(\mathbb{Z})$, with $a_{11} \equiv b_{11} \equiv b_{13} \equiv 0 \pmod{p}$, $b_{11} \equiv 0 \pmod{p^2}$, $b_{22}b_{33} - b_{23}^2 \neq 0 \pmod{p}$, and $C \neq 0 \pmod{p}$, we may multiply $A$ by $p$ and then divide the first row and column of both $A$ and $B$ by $p$—this corresponds to the application of the transformation

$$g = \begin{pmatrix} 1 & 1/p \\ 1/p & 1 \end{pmatrix} \in G(\mathbb{Q}).$$

(45)
Hence we obtain an element \((A', B') \in V(\mathbb{Z})\) such that \(f_4((A', B')) = f_4((A, B))/p^2\), yielding Condition 6(i) with \(a = 2\).

To check Condition 6(ii), we note that an element \((A', B') \in V(\mathbb{Z})\), where \(A'\) and \(B'\) have entries \(a'_{ij}\) and \(b'_{ij}\) respectively, may be sent to an element \((A, B) \in V(\mathbb{Z})\) as above via the inverse transformation \(g^{-1}\) precisely when \(a'_{22} \equiv a'_{23} \equiv a'_{33} \equiv 0 \pmod{p}\), i.e., the ternary quadratic form \(A'(\mod p)\) factors into two rational linear factors with a distinguished linear factor, namely, \(x\) (where \(x\) denotes the first variable of the quadratic form), which vanishes at two of the four common points of intersection of \((A', B')\) in \(\mathbb{P}^2(\mathbb{F}_p)\). It follows that the \(G(\mathbb{Z})\)-class of \((A', B')\) can lead to at most six \(G(\mathbb{Z})\)-classes of elements \(x \in V(\mathbb{Z})\) in this way (since there are at most \(6 = (\frac{4}{2})\) lines in \(\mathbb{P}^2(\mathbb{F}_p)\) passing through two of four given points), yielding Condition 6(ii) with \(c = 6\).

The case \(n = 5\) is more difficult to treat due to the complexity of the discriminant polynomial, but in the end the same idea still applies. The image of \(W_p^{(1)}\) in \(V(\mathbb{F}_p)\) consists of quadruples \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) of \(5 \times 5\) skew-symmetric matrices over \(\mathbb{F}_p\) whose \(4 \times 4\) sub-Pfaffians have four common zeroes in \(\mathbb{P}^3\) (i.e., three simple common zeroes, and one double common zero). Recall that these common zeroes in \(\mathbb{P}^3(\mathbb{F}_p)\) correspond to the linear combinations of \(\bar{A}, \bar{B}, \bar{C}, \bar{D}\), up to scaling, that yield rank 2 skew-symmetric matrices.

By a change of basis in \(\text{GL}_4(\mathbb{F}_p)\), we may assume that \(\bar{A}\) is the rank 2 matrix that corresponds to the common double zero in \(\mathbb{P}^3(\mathbb{F}_p)\) of \((\bar{A}, \bar{B}, \bar{C}, \bar{D}) \in V(\mathbb{F}_p)\). By a change of basis in \(\text{SL}_5(\mathbb{F}_p)\), we may assume that \(\bar{A}\) is the \(5 \times 5\) matrix having \(\pm 1\) in the \((1,2)\)- and \((2,1)\)-entries, and zeroes elsewhere. We may then use a further \(\text{GL}_4(\mathbb{F}_p)\)-transformation to clear out the \((1,2)\) and \((2,1)\) entries of \(\bar{B}, \bar{C}\), and \(\bar{D}\).

We now claim that, after a suitable \(G(\mathbb{F}_p)\)-transformation, \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) can be expressed in the form

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & * & * & * \\
0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\end{array}
\right)
\]

where the *’s denote elements of \(\mathbb{F}_p\).

Indeed, if \(\bar{A}\) corresponds to a double and only multiple common zero of \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\), then the coordinate ring of the scheme cut out by the \(4 \times 4\) sub-Pfaffians of \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) in \(\mathbb{P}^3\), as given by (16)–(22) in [3], is isomorphic to \(\mathbb{F}_p[\alpha_1]/(\alpha_1^3) \oplus K\), where \(K\) is an étale cubic \(\mathbb{F}_p\)-algebra. In the notation of [3], this means that we must have the equalities

\[
Q(\bar{A}, M_1) \cdot M_2 \cdot Q(M_3, \bar{A}) = 0
\]

for any matrices \(M_1, M_2,\) and \(M_3\) that are \(\mathbb{F}_p\)-linear combinations of \(\bar{B}, \bar{C},\) and \(\bar{D}\). Now if the space spanned by the three bottom right \(3 \times 3\) matrices of \(\bar{B}, \bar{C},\) and \(\bar{D}\) were three-dimensional, then by assuming that the bottom right \(3 \times 3\) submatrices of \(\bar{B}, \bar{C},\) and \(\bar{D}\) are \(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\), and \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\), respectively, we see that the conditions (17) would not hold since, e.g., \(Q(\bar{A}, \bar{B}) \cdot \bar{C} \cdot Q(\bar{D}, \bar{A}) \neq 0\). If this space were one-dimensional, then we see that the discriminant of (any lift to \(V(\mathbb{Z})\) of) \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) would be strongly a multiple of \(p^2\). We conclude that this space must be two-dimensional, and a suitable \(\text{GL}_4(\mathbb{F}_p)\) transformation then transforms \((\bar{A}, \bar{B}, \bar{C}, \bar{D})\) into the form (46).

We now proceed in a manner similar to the case of \(n = 4\). Let \((A, B, C, D) \in V(\mathbb{Z})\) be an element that reduces modulo \(p\) to (46). Then evaluating the discriminant function \(f_5\) on this
element, we see that, for those values of the *’s in (46) where the discriminant is not strongly a multiple of \( p^2 \), the discriminant of \((A, B, C, D)\) is a multiple of \( p^2 \) precisely when the \((4,5)\)-entry of \( A \) is a multiple of \( p^2 \). In that case, we can multiply \( B, C, \) and \( D \) by \( p \), and then divide the fourth and fifth rows and columns of each of \( A, B, C, D \) by \( p \); this corresponds to the transformation

\[
g = (\text{diag}(1,p,p,p), \text{diag}(1,1,1,1/p, 1/p)) \in G(\mathbb{Q}).
\]  

(48)

After applying this transformation, we obtain an element \((A', B', C', D') \in V(\mathbb{Z})\) such that \( f_5((A', B', C', D')) = f_5((A, B, C, D))/p^2 \), again giving Condition 6(i) with \( a = 2 \).

Finally, to check Condition 6(iii), note that an element \((A', B', C', D') \in V(\mathbb{Z})\) may be sent to such an element \((A, B, C, D) \in V(\mathbb{Z})\) via the inverse transformation \( g^{-1} \) precisely when the top left \( 3 \times 3 \) sub-matrices of \( B', C', D' \) are all zero. This means that the fourth and fifth \( 4 \times 4 \) sub-Pfaffians of \( wA' + xB' + yC' + zD' \) then factor and are both multiples of \( w \); it follows that \( w \) cuts out a plane in \( \mathbb{P}^2 \) that passes through at least 3 of the 5 points of intersection (counting multiplicity) in \( \mathbb{P}^2 \) of the \( 4 \times 4 \) sub-Pfaffians of \( wA' + xB' + yC' + zD' \). Hence each \( G(\mathbb{Z}) \)-class of \((A', B', C', D')\) can lead to at most \( \binom{5}{3} = 10 \) \( G(\mathbb{Z}) \)-classes of elements \((A, B, C, D) \in V(\mathbb{Z})\) in this way, proving that Condition 6(iii) holds with \( c = 10 \). This completes the proofs of all the axioms for \( f_3, f_4, \) and \( f_5 \).

In summary, given a polynomial \( f \) on a vector space \( V \) that has symmetry under the action of an algebraic group \( G \) on \( V \) (all defined over \( \mathbb{Z} \)), our general strategy to extract squarefree values taken by \( f \) is: a) describe geometrically or algebraically what it means for a point in \( V(\mathbb{F}_p) \) to have vanishing \( f \) \( (\text{mod } p) \); b) ascertain which lifts \( (\text{mod } p^2) \) of such points have strongly vanishing and which have weakly vanishing \( f \) \( (\text{mod } p^2) \); c) treat the points where \( f \) strongly vanishes \( (\text{mod } p^2) \) via the geometric sieve estimates in \( \S3.2 \); and, finally, d) for points \( x \in V(\mathbb{Z}) \) where \( f \) weakly vanishes \( (\text{mod } p^2) \), effect transformations in \( g \in G(\mathbb{Q}) \) so that the relevant \( (\text{mod } p^2) \) conditions on \( x \in V(\mathbb{Z}) \) are transformed into \( (\text{mod } p) \) conditions on \( gx \in V(\mathbb{Z}) \)! We will see that this strategy also works for discriminants of genus one models in \( \S5 \).

**Remark 4.2** If the arguments of \( \S3.4 \) (with Theorem 3.5 used in place of Theorem 3.3) are applied to each set \( H(u, s, \lambda, X) \), in the averaging method employed in \([5, \S2.2]\) (and its analogues in \([14, \S5.3]\) and \([1, \S2.2]\)), then one obtains Lemma 3.7 for \( f_3, f_4, \) and \( f_5 \) without the dependence on \( \epsilon \) and without the \( O(\epsilon X^{m/d}) \) term. This may be used, e.g., to obtain power saving error terms in \( \mathbb{S} \), via the methods of \([1, 15], \) and \([39]\).

### 4.3 Proof of Theorems 1.1 and 1.3

We now consider the consequences for number fields of small degree having squarefree discriminant. Let \( n \in \{3, 4, 5\} \). Let \( \Sigma = (\Sigma_\nu)_\nu \) denote a set of local specifications for degree \( n \) number fields, i.e., \( \Sigma_p \) is a set of (isomorphism classes of) étale degree \( n \) extensions of \( \mathbb{Q}_p \), such that for all \( p \) larger than some constant \( C \), we have that \( \Sigma_p \) contains all unramified and simply ramified étale degree \( n \) extensions of \( \mathbb{Q}_p \).

For each prime \( p \), let \( S_p \) denote the subset of points of \( V(\mathbb{Z}) \) corresponding to rings \( R \) such that \( R \otimes \mathbb{Z}_p \) gives the ring of integers of some étale extension in \( \Sigma_p \). If we set \( S = \cap_p S_p \), then the generic points of \( S \) are exactly the points of \( V(\mathbb{Z}) \) corresponding to rings \( R \) that are the rings of integers in \( S_n \)-number fields of degree \( n \) that agree with the local specifications of \( \Sigma \). Note also that \( S \) then satisfies the hypotheses of \( \S3.4 \). Hence Equation (39) holds for \( S \).

It thus remains only to compute \( \text{Vol}(\mathcal{F}_1) \prod_p \mu_p(S) \) for the set \( S \subset V(\mathbb{Z}) \) corresponding to the local specifications \( \Sigma = (\Sigma_\infty, \Sigma_2, \Sigma_3, \ldots) \). For \( n = 5 \), this has been carried out in \([3]\) Pf. of
Lemma 20], where it is shown that

\[
\text{Vol}(\mathcal{F}_1) \prod_p \mu_p(S) = \left( \sum_{K \in \Sigma_\infty} \frac{1}{2} \cdot \frac{1}{\#\text{Aut}(K)} \right) \prod_p \left( \sum_{K \in \Sigma_p} \frac{p-1}{p} \cdot \frac{1}{\text{Disc}_p(K)} \cdot \frac{1}{\#\text{Aut}(K)} \right),
\]

and the identical arguments there show that the same formula holds also in the cases \(n = 3\) and \(n = 4\). We have proven Theorem 1.3.

**Remark 4.3** Although we do not carry this out here, using the methods of [1, 15, 39], it is possible to also estimate the \(o(X^{m/d})\) in these various deductions and thus obtain a power-saving error term in (39).

Theorem 1.1 is of course the particular case of Theorem 1.3, where \(\Sigma\) corresponds to the local specifications for fields of squarefree (resp. fundamental) discriminant. In the squarefree case, \(\Sigma_p\) is the set of all isomorphism classes of étale algebras of dimension \(n\) over \(\mathbb{Q}_p\) such that \(p^2 \nmid \text{Disc}(K)\).

To complete the proof of Theorem 1.1, we must evaluate

\[
\sum_{K \in \Sigma_p} \frac{1}{\#\text{Aut}(K)} \cdot \frac{1}{\#\text{Disc}_p(K)}.
\]

Since, for \(p \neq 2\), \(p^2 \mid \text{Disc}(K)\) is equivalent to \(p\) at most simply ramifying in \(K\), while for \(p = 2\) it means that \(p\) does not ramify, we obtain by [6, Prop. 2.2] that

\[
\sum_{K \in \Sigma_p} \frac{1}{\#\text{Aut}(K)} \cdot \frac{1}{\#\text{Disc}_p(K)} = \begin{cases} 1 + 1/p & \text{if } p \neq 2 \\ 1 & \text{if } p = 2 \end{cases}.
\] (49)

Let \(\Sigma_\infty\) denote the set of all étale extensions of \(\mathbb{R}\) of degree \(n\). Then [6, Prop. 2.4] gives

\[
\sum_{K \in \Sigma_\infty} \frac{1}{\#\text{Aut}(K)} = \frac{r_2(S_n)}{n!}.
\] (50)

Combining (49) and (50), we obtain

\[
\left( \sum_{K \in \Sigma_\infty} \frac{1}{\#\text{Aut}(K)} \right) \prod_p \left( \sum_{K \in \Sigma_p} \frac{p-1}{p} \cdot \frac{1}{\text{Disc}_p(K)} \cdot \frac{1}{\#\text{Aut}(K)} \right) = \frac{r_2(S_n)}{2n!} \cdot \frac{1}{2} \cdot \prod_{p \neq 2} \left( 1 - \frac{1}{p^2} \right),
\]

yielding Theorem 1.1(a).

To obtain Theorem 1.1(b), we simply must change the local conditions at \(p = 2\) to include also simply ramified extensions in \(\Sigma_2\). This replaces the value 1 for \(p = 2\) in (49) by \(1 + 1/2\) (again, by [6, Prop. 2.2]), thus multiplying the final constant in Theorem 1.1(a) by \(3/2\). This proves Theorem 1.1(b).

### 4.4 Unramified extensions of quadratic fields, and proof of Theorem 1.4

It is known (see, e.g., [31]) that, for \(n \leq 5\), an \(S_n\)-extension of \(\mathbb{Q}\) is unramified over its quadratic subfield precisely when its associated degree \(n\) subfield is at most simply ramified at all places. Moreover, in this scenario, the quadratic field ramifies exactly at the places where the degree \(n\) field is simply ramified. In particular, the quadratic field is real precisely when the degree \(n\) field \(K\) is totally real, i.e., \(K \otimes \mathbb{R} \cong \mathbb{R}^n\), and it is imaginary precisely when \(K\) satisfies \(K \otimes \mathbb{R} \cong \mathbb{R}^{n-2} \times \mathbb{C}\).
As in the proof of the fundamental discriminant case of Theorem 1.1, let \( \Sigma_p \) denote the set of all unramified or simply ramified étale extensions of \( \mathbb{Q}_p \); furthermore, let \( \Sigma_\infty = \{ \mathbb{R}^n \} \). Then by the arguments of Theorem 1.1 we have that the number of degree \( n \) fields that are simply ramified at all places and have absolute discriminant less than \( X \) is

\[
\left( \sum_{K \in \Sigma_\infty} \frac{1}{2} \frac{1}{\# \text{Aut}(K)} \right) \prod_p \left( \sum_{K \in \Sigma_p} \frac{p-1}{p} \frac{1}{\text{Disc}_p(K)} \frac{1}{\# \text{Aut}(K)} \right) \cdot X + o(X),
\]

which evaluates to

\[
\frac{1}{2n!} \prod_p \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p} \right) \cdot X + o(X) = \frac{1}{2n!} \cdot \zeta(2)^{-1}. \tag{51}
\]

On the other hand, the number of real quadratic fields having discriminant less than \( X \) is \( \frac{1}{2} \zeta(2)^{-1} \cdot X + o(X) \). We conclude that the average number of unramified \((A_n, S_n)\)-extensions of real quadratic fields, over all real quadratic fields of discriminant less than \( X \), as \( X \to \infty \), is

\[
\frac{1}{2n!} \zeta(2)^{-1} = \frac{1}{n!},
\]

yielding Theorem 1.4(a).

The proof of Theorem 1.4(b) is similar. We put instead \( K_\infty = \{ \mathbb{R}^{n-2} \times \mathbb{C} \} \); this changes the factor of \( \frac{1}{2n!} \) in (51) to \( \frac{1}{2 \cdot 2(n-2)!} \). Since the number of imaginary quadratic fields of absolute discriminant less than \( X \) is again \( \frac{1}{2} \zeta(2)^{-1} \cdot X + o(X) \), we conclude that the average number of unramified \((A_n, S_n)\)-extensions of imaginary quadratic fields over all imaginary quadratic fields having absolute discriminant less than \( X \), as \( X \to \infty \), is

\[
\frac{1}{2 \cdot 2(n-2)!} \zeta(2)^{-1} = \frac{1}{2(n-2)!},
\]

yielding Theorem 1.4(b).

**Remark 4.4** The same argument can also be used to show that the constants occurring in Theorem 1.4 remain the same even when one averages only over quadratic fields satisfying any desired local conditions at finitely many primes.

**Remark 4.5** We note that analogous results can be proved also for \( A_n \)-extensions of quadratic fields that are unramified away from some finite set of primes. For example, if we are interested only in “weakly unramified extensions”, that is, extensions unramified at all finite places, then the analogous methods would apply; in Theorem 1.4(a)–(b), the constants \( 1/n! \) and \( 1/(2(n-2)!) \) would then be replaced by \( r_2(S_n)/n! \) and \( r_2(S_n)/n! \), respectively, where \( r_2(S_n) \) and \( r_2(S_n) \) denote the number of 2-torsion elements in \( S_n \) having signature +1 and −1, respectively.

Next, we may prove by the identical arguments that an unramified \( S_n \times C_2 \)-extension \( M \) of a quadratic field \( F \) necessarily arises as the compositum of the quadratic field \( F \) and the Galois closure \( L \) of a number field \( K \) of degree \( n \) having fundamental discriminant. In that case, \( \text{Gal}(L/F) = S_n \), and for \( M = LF \) to be unramified over \( F \) at all finite places, it is necessary and sufficient that \( \text{Disc}(K) \) divides \( \text{Disc}(F) \). Furthermore, for \( M = LF \) to be unramified over \( F \) also at the infinite
places, it is necessary and sufficient that $K$ be totally real if $F$ is real, and that $K \otimes \mathbb{R} \cong \mathbb{R}^{n-2} \times \mathbb{C}$ if $F$ is imaginary.

Let $N$ be any positive squarefree integer. Then for $X > N$ sufficiently large, we see that the total number of unramified extensions of real quadratic fields, where we range over all quadratic fields—is identical. For such $(K, F)$ we have $\big| \sum_{\text{unr}} \text{Gal}(K/F) \big| \approx X \log X$, proving Theorem 1.4(c). The argument for Theorem 1.4(d)—the case of imaginary quadratic fields—is identical.

## 5 The density of squarefree values taken by the polynomials $g_2$, $g_3$, $g_4$, $g_5$

Again, most of the axioms of Section 2 follow for the polynomials $g_2$, $g_3$, $g_4$, $g_5$ of §1.4, using the geometry-of-numbers works [11, 12, 13, 14]. In this section, we outline how to deduce Conditions 1–6 for these polynomials from the results of these works, with a emphasis again on Condition 6.

First, we define a genus one model of degree 2 over $\mathbb{Q}$ (or a corresponding element of $V(\mathbb{Q})$) to be generic if none of the four ramification points (viewed as a double cover of $\mathbb{P}^1$) is rational (i.e., the corresponding binary quartic form has no linear factor over $\mathbb{Q}$). Similarly, we define a genus one model of degree 3, 4, or 5 over $\mathbb{Q}$ (or a corresponding element of $V(\mathbb{Q})$) to be generic if it does not have a rational hyperosulation point (e.g., in the case of degree 3: does not have a rational flex point). Alternatively, a genus one model of degree $n = 2, 3, 4, 5$ over $\mathbb{Q}$ is generic if it does not correspond to the trivial element of the $n$-Selmer group of its Jacobian.

Next, we use the following groups $G$ of symmetries of $g_n$ (as a polynomial on $V$) for $n \in \{2, 3, 4, 5\}$:

- $n = 2$: $G = \text{PGL}_2$. Note that $\gamma \in \text{GL}_2$ naturally acts on a binary quartic form $x(s, t) \in V$ by
  \[ \gamma \cdot x(s, t) = (\det \gamma)^{-2}x((s, t) \cdot \gamma) \]
  yielding an action of $\text{PGL}_2$ on $V$.

- $n = 3$: $G = \text{PGL}_3$. In this case, $\gamma \in \text{GL}_3$ naturally acts on a ternary cubic form $x(r, s, t) \in V$ by
  \[ \gamma \cdot x(r, s, t) = (\det \gamma)^{-1}x((r, s, t) \cdot \gamma) \]
  inducing an action of $\text{PGL}_3$ on $V$.

- $n = 4$: $G = \{(\gamma_2, \gamma_4) \in \text{GL}_2 \times \text{GL}_4 : \det(\gamma_2) \cdot \det(\gamma_4) = 1\}/\{(\lambda^{-2}I_2, \lambda I_4)\},$ where $I_2$ and $I_4$ denote the identity elements of $\text{GL}_2$ and $\text{GL}_4$, and $\lambda \in \mathbb{G}_m$.

- $n = 5$: $G = \{(\gamma_1, \gamma_2) \in \text{GL}_5 \times \text{GL}_5 : \det(\gamma_1)^2 \cdot \det(\gamma_2) = 1\}/\{(\lambda I_5, \lambda^{-2}I_5)\},$ where $I_5$ denotes the identity element of $\text{GL}_5$ and $\lambda \in \mathbb{G}_m$. 

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For $n \in \{2, 3, 4, 5\}$, one easily checks that $g_n$ is an invariant polynomial for the action of $G$ on $V$.

With these definitions of the groups $G$ and the notion of generic in hand, Condition 1 then follows again from Hilbert irreducibility or [11] §2.2, [12] §2.5, [13] §3.5, and [14] 3.6, and Condition 2 from [11] Thm. 3.2, [12] Prop. 28, [13] Prop. 7, and [14] Thm. 7. For Condition 3, we take $f$ to be the height $H = \text{max}\{|c_4|^3, |c_6|^2\}$ on $V(V)$, where $c_4$ and $c_6$ denote the two generating invariants for the action of $G(V)$ on $V(V)$ (see [23] for constructions of these invariants). Condition 4 is then obtained in [11] §2.1, [12] §2.1, [13] §3.1, and [14] §3.1, while Condition 5 is [11] Thm. 2.11, [12] Thm. 17, [13] Thm. 19, and [14] Thm. 25.

Finally, Conditions 6(i)–(iii) for $n = 3, 4$, and 5 is contained in [12] Lem. 26, [13] Lem. 24, and [14] Lem. 28, respectively, with $a = 0$, $c = 3$, and $k = 2$. Thus Subcondition 6(iv) is then automatically satisfied. The proofs of these three important subconditions 6(i)–(iii) are again where the “largeness” of the symmetry group $G$ of the polynomial $g_n$ is used. We describe now the special case of plane cubics ($n = 3$) in detail to illustrate.

First, recall that for any $n \in \{3, 4, 5\}$, a general element of $V(F_p)$ (i.e., an element for which the discriminant is nonzero) determines a smooth genus one curve in $\mathbb{P}^{n-1}$ over $F_p$. The discriminant of an element $\bar{x} \in V(F_p)$ vanishes precisely when the associated curve in $\mathbb{P}^{n-1}$ is not smooth, or when $\bar{x}$ is so degenerate that the variety that is cut out by $\bar{x}$ in $\mathbb{P}^{n-1}$ is of dimension greater than one.

The latter case, where a variety of dimension greater than one is cut out by $x \in V(F_p)$, happens on an algebraic set (defined over $\mathbb{Z}$) that is of codimension greater than one in $V(F_p)$. Similarly, the case where the associated curve in $\mathbb{P}^{n-1}$ has a cuspidal singularity also occurs on a set of codimension greater than one in $V(F_p)$. Indeed, these two sets in $V(F_p)$ together comprise the image of the set $W_p^{(1)}$ in $V(F_p)$.

The image of the set $W_p^{(2)}$ in $V(F_p)$ then consists of elements $x \in V(F_p)$ that cut out a genus one curve in $\mathbb{P}^{n-1}$ with a nodal singularity. Thus we have a description of those points in $V(\mathbb{Z}/p\mathbb{Z})$ on which the discriminant polynomial $g_n$ vanishes (mod $p$) that potentially lift to points in $V(\mathbb{Z}/p^2\mathbb{Z})$ on which $g_n$ is weakly a multiple of $p^2$.

We may now determine precisely the set in $V(\mathbb{Z}/p^2\mathbb{Z})$ where the discriminant is weakly a multiple of $p^2$. When $n = 3$, then via a transformation in $G(\mathbb{Z})$, we may assume that the plane cubic curve over $F_p$ corresponding to a given element $x \in W_p^{(2)}$ has a node at $(0 : 0 : 1) \in \mathbb{P}^{2}(F_p)$. Thus the corresponding ternary cubic form $\bar{x}(r, s, t)$ has $t^3$, $rt^2$, and $st^2$-coefficients equal to zero. If $x(r, s, t)$ is a lift of $\bar{x}$ to $V(\mathbb{Z})$, then the $t^3$, $rt^2$, and $st^2$-coefficients of $x(r, s, t)$ are multiples of $p$. Evaluating the discriminant of such an element $x$, we see that $g_3(x) \equiv a_{333}h(x) \pmod{p^2}$, where $a_{333}$ is the coefficient of $x^3$ and $h(x)$ is an irreducible polynomial in the coefficients of $x$. As $x \in W_p^{(2)}$, we see that $h(x) \not\equiv 0 \pmod{p}$. Therefore, since $p^2 \mid g_3(x)$, we obtain that $p^2 \mid a_{333}$. Now the element $\gamma$ defined by

$$\begin{pmatrix} 1 \\ p^{-1} \end{pmatrix} \cdot x$$

(52)

has the same discriminant as $x$ and moreover is in $W_p^{(1)}$, since its $r^3$, $r^2s$, $rs^2$, and $s^3$-coefficients are zero (mod $p$). We therefore obtain a discriminant-preserving map $\phi$ from $G(\mathbb{Z})$-orbits on $W_p^{(2)}$ to $G(\mathbb{Z})$-orbits on $W_p^{(1)}$. This proves Conditions 6(i) and (ii) for $g_3$ with $a = 0$ and $k = 2$.

To prove Condition 6(iii), we note that an element $x' \in V(\mathbb{Z})$ may be sent to an element $x \in V(\mathbb{Z})$ of the above type via the inverse transformation $\gamma^{-1}$ only when $x'(r, s, t)$ is a multiple of $t$. It follows that the $G(\mathbb{Z})$-class of $x'$ can lead to at most 3 $G(\mathbb{Z})$-classes of elements $x \in V(\mathbb{Z})$ in this way (since $x'$ can have at most 3 linear factors over $F_p$), yielding Condition 6(iii) with $c = 3$. 

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The cases \( n = 4 \) and \( n = 5 \) can be treated in an analogous fashion; we refer the reader to [13, Lem. 24] and [14, Lem. 28], respectively.

**Remark 5.1** As in Remark 4.2 if the arguments of §3.4 (with Theorem 3.5 used in place of Theorem 3.3) are applied to each set \( B(n, t, \lambda, X) \), in the averaging method employed in [11, §2.3] (and its analogues in [12, §2.2], [13, §3.2], and [14, 3.2]), then one again obtains Lemma 3.7 for \( P_3 \), \( P_4 \), and \( P_5 \) without the dependence on \( \epsilon \) and without the \( O(\epsilon X^{m/d}) \) term. As before, this may be used, for example, to obtain power saving error terms in \( \mathcal{E} \), via the methods of [1], [15], and [39].

Condition 6, however, does not hold for the discriminant polynomial \( g_n \) when \( n = 2 \). Indeed, in the case \( n = 2 \), the image of \( W_p^{(2)} \) in \( V(F_p) \) consists of binary quartic forms over \( F_p \) having exactly one double (but not triple) root in \( \mathbb{P}^1 \). A binary quartic form in \( V(F_p) \) having exactly one double root in \( \mathbb{P}^1 \) is always \( G(F_p) \)-equivalent to one of the form \( x(s, t) = \tilde{a}s^4 + bs^3t + \tilde{c}s^2t^2 \) where \( \tilde{a}, b, \tilde{c} \in F_p \) and \( \tilde{c} \neq 0 \) (to prevent a triple root) and \( \tilde{b}^2 - 4\tilde{a}\tilde{c} 
eq 0 \) (to prevent a second double root). If \( x(s, t) = as^4 + bs^3t + cs^2t^2 + dst^3 + et^4 \in V(Z) \) is a lift of \( \tilde{x} \) to \( V(Z) \), then \( d \) and \( e \) are multiples of \( p \), \( c \) and \( b^2 - 4ac \) are prime to \( p \), and we compute that the discriminant \( g_2(x) \) of \( x \) is given by

\[
f(x) \equiv -4c^3(b^2 - 4ac)e \pmod{p^2}
\]

implying (for \( p > 2 \)) that \( e \) must then be a multiple of \( p^2 \) for \( x \) to have discriminant that is weakly a multiple of \( p^2 \).

It is now easy to see that there is no transformation in \( G(Q) \) that removes the mod \( p^2 \) condition; in particular, unlike the cases \( n = 3, 4, \) and \( 5 \), there is in general no transformation in \( G(Q) \) that maps such an element \( x \in W_p^{(2)} \) to \( W_p^{(1)} \). (The best potential candidate is the transformation \( \left( \begin{array}{c} 1 \\ p^{-1} \end{array} \right) \), but this sends \( x(s, t) = as^4 + bs^3t + cs^2t^2 + dst^3 + et^4 \) to \( x'(s, t) = ap^2s^4 + bps^3t + cs^2t^2 + (d/p)st^3 + (e/p^2)t^4 \), which in general is again in \( W_p^{(2)} \).) The group \( G(Q) \) of symmetries of \( g_2 \) is too small to directly apply the methods of Section 3.

To remedy this problem, we use what we call the “embedding sieve”, where we attempt to embed our orbit space \( G(Z) \setminus V(Z) \), via some map \( \phi \), into another orbit space \( G'(Z) \setminus V'(Z) \) for which the group \( G'(Z) \) is sufficiently large. Moreover, we assume that the invariant polynomial \( g \) of interest on \( Z \) is mapped to a corresponding invariant polynomial \( g' \) for the action of \( G' \) on \( V' \), i.e., for \( x \in V(Z) \), we have \( g(x) = g'(\phi(x)) \). If an analogue of Condition 6 then holds for \( g', G' \), and \( V' \), then the resulting estimate for \( W_p^{(2)} \subset V'(Z) \) can be pulled back to give an estimate for \( W_p^{(2)} \subset V(Z) \), and this may be sufficient to deduce (6) for the polynomial \( g \).

For example, for the space \( V(Z) \) of binary quartic forms, there are a number of possibilities for the space \( V' \) that yield the desired estimates for \( W_p^{(2)} \). We give here, for simplicity, an example of \( V' \) that we have already treated in the previous section, namely, the representation \( V' \) on which \( f_4 \) is an invariant polynomial! Indeed, the group \( \text{PGL}_2 \) may be viewed as the special orthogonal group of the three-dimensional quadratic space \( W \) of \( 2 \times 2 \) matrices of trace zero, with quadratic form \( A_1 \) given by the determinant; the representation of \( \text{PGL}_2 \) on the space \( \text{Sym}^4(Z) \) of binary quartic forms can then be viewed as the action of \( \text{O}(W) \) by conjugation on the space of self-adjoint operators \( T : W \rightarrow W \) with trace zero [7]. Alternatively, we can view the latter representation as the action of \( \text{SO}(W) \) on pairs \( (A, B) \) of quadratic forms \( (A, B) \), where \( A = A_1 \) and \( B \) is the quadratic form given by \( B(w, w) = \langle w, Tw \rangle \). The space \( V'(Z) \) of pairs \( (A, B) \) of ternary quadratic forms can then be viewed as a representation of the larger group \( G'(Z) = \text{GL}_2(Z) \times \text{SL}_3(Z) \), which has polynomial invariant \( f_4 \). We thus obtain a natural map

\[
\phi : \text{PGL}_2(Z) \setminus \text{Sym}^4(Z^2) = G(Z) \setminus V(Z) \rightarrow G'(Z) \setminus V'(Z) = \text{GL}_2(Z) \times \text{SL}_3(Z) \setminus \mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^3).
\]
Explicitly in terms of coordinates, the map $\phi$ is given by

$$
\phi : ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \mapsto \begin{bmatrix}
1/2 & 0 \\
-1 & d/2 \\
1/2 & c \\
0 & b/2 \\
0 & a
\end{bmatrix} ; \quad (54)
$$

see [41] §2.3 and [11] (30). One easily checks then that for any $v \in V$, we have $g_2(v) = f_4(\phi(v))$. Furthermore, it was shown in [11, Prop. 2.16] that the map $\phi$ defined by (53) and (54) is at most 12-to-1.

Note that if $x(s, t) = as^4 + bs^3t + cs^2t^2 + dst^3 + et^4$ satisfies $d \equiv 0 \pmod{p}$ and $e \equiv 0 \pmod{p^2}$, then even though there is no transformation in $G'(Q)$ that can be applied to $x$ to remove the mod $p^2$ condition, there is a transformation in $G'(Q)$ that removes the mod $p^2$ condition on $\phi(x)$, namely, the transformation given by (45).

We may now proceed as in Section 3. Let $W_p^{(1)}$ and $W_p^{(2)}$ be the subsets of $V(Z)$ as defined in §3.4, and let $W_p^{(1)'}$ and $W_p^{(2)'}$ be their analogues for $V'(Z)$. The estimate (53) for $W_p^{(1)}$ is obtained in the identical manner. Meanwhile, for small primes $p \leq X^{1/6}$, by the argument used to prove the individual estimates (10) for $W_p^{(1)}$ for $p \leq r$ (see also Remark 4.2) gives a useful estimate also for $W_p^{(2)}$; we have

$$
|\mathcal{F}_X \cap W_p^{(2)}| = O(\max\{X^{5/6}/p^2, X^{4/6}\}) \text{ for all } p. \quad (55)
$$

We use this estimate for $p \leq X^{1/6}$.

To handle $p > X^{1/6}$, we use the map $\phi$. Let $\mathcal{F}_X'$ be the analogue of $\mathcal{F}_X$ for $G'$, $V'$, and $f_4$. Then in the previous section, we have shown that

$$
|\mathcal{F}_X' \cap W_p^{(2)'}| = O(X/p^2) \text{ for all } p. \quad (56)
$$

Since we have a map $\phi : \mathcal{F}_X \cap V(Z) \to \mathcal{F}_X' \cap V'(Z)$ that is at most 12-to-1 and satisfies $g_2(x) = f_4(\phi(x))$, we conclude using (55) and (56) that

$$
|\mathcal{F}_X \cap (\cup_{p > M} W_p^{(2)})| = \sum_{M < p \leq X^{1/6}} O(\max\{X^{5/6}/p^2, X^{4/6}\}) + \sum_{p > \max\{M, X^{1/6}\}} O(X/p^2) \quad (57)
$$

$$
= O(X^{5/6}/\log M).
$$

The analogue of Lemma 3.7 for the space $V(Z)$ of binary quartic forms then becomes

**Lemma 5.2** We have

$$
|\mathcal{F}_X \cap (\cup_{p > M} W_p^{\text{gen}})| = O(X^{5/6}/\log M + X^{4/6}) + O(\epsilon X^{5/6}),
$$

where the implied constants are independent of $M$.

As in Remark 5.1, the dependence on $\epsilon$ and the $O(\epsilon X^{5/6})$ term may again be removed if desired by combining with the averaging method of [11].

The remainder of the argument in §3.4 now gives (8) for these polynomials $g = g_2, g_3, g_4,$ and $g_5$, which was already used in [11] [12] [13] [14] to determine the average orders of 2-, 3-, 4-, and 5-Selmer groups of elliptic curves over $\mathbb{Q}$. Finally, it also then yields Theorem 1.6 giving the density of squarefree values taken by $g_2, g_3, g_4,$ and $g_5$. 
Acknowledgments

I am extremely grateful to Jordan Ellenberg, Benedict Gross, Jonathan Hanke, Wei Ho, Kiran Kedlaya, Juergen Klueners, Hendrik Lenstra, Henryk Iwaniec, Barry Mazur, Carl Pomerance, Bjorn Poonen, Peter Sarnak, Arul Shankar, Frank Thorne, and Jerry Wang for many helpful conversations. This work was done in part while the author was at MSRI during the special semester on Arithmetic Statistics. The author was also partially supported by NSF grant DMS-1001828 and a Simons Investigator Grant.

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