ON COMPLEMENTED COPIES OF THE SPACE $c_0$ IN SPACES $C_p(X \times Y)$

J. KĄKOL, W. MARCISZEWSKI, D. SOBOTA, AND L. ZDOMSKYY

ABSTRACT. Cembranos and Freniche proved that for every two infinite compact Hausdorff spaces $X$ and $Y$ the Banach space $C(X \times Y)$ of continuous real-valued functions on $X \times Y$ endowed with the supremum norm contains a complemented copy of the Banach space $c_0$. We extend this theorem to the class of $C_p$-spaces, that is, we prove that for all infinite Tychonoff spaces $X$ and $Y$ the space $C_p(X \times Y)$ of continuous functions on $X \times Y$ endowed with the pointwise topology contains either a complemented copy of $\mathbb{R}^\omega$ or a complemented copy of the space $(c_0)_p = \{(x_n)_{n \in \omega} \in \mathbb{R}^\omega : x_n \to 0\}$, both endowed with the product topology. We show that the latter case holds always when $X \times Y$ is pseudocompact.

On the other hand, assuming the Continuum Hypothesis (or even a weaker set-theoretic assumption), we provide an example of a pseudocompact space $X$ such that $C_p(X \times X)$ does not contain a complemented copy of $(c_0)_p$.

As a corollary to the first result, we show that for all infinite Tychonoff spaces $X$ and $Y$ the space $C_p(X \times Y)$ is linearly homeomorphic to the space $C_p(X \times Y) \times \mathbb{R}$, although, as proved earlier by Marciszewski, there exists an infinite compact space $X$ such that $C_p(X)$ cannot be mapped onto $C_p(X) \times \mathbb{R}$ by a continuous linear surjection. This provides a positive answer to a problem of Arkhangel’ski for spaces of the form $C_p(X \times Y)$.

Our another corollary—analogous to the classical Rosenthal–Lacey theorem for Banach spaces $C(X)$ with $X$ compact and Hausdorff—asserts that for every infinite Tychonoff spaces $X$ and $Y$ the space $C_k(X \times Y)$ of continuous functions on $X \times Y$ endowed with the compact-open topology admits a quotient map onto a space isomorphic to one of the following three spaces: $\mathbb{R}^\omega$, $(c_0)_p$ or $c_0$.

1. INTRODUCTION

Recall that a Banach space $E$ is a Grothendieck space or has the Grothendieck property if every weakly* convergent sequence in the dual $E^*$ of $E$ converges weakly, i.e. every sequence $(\varphi_n)_{n \in \omega}$ of continuous functionals on $E$ satisfying the condition that $\lim_{n \to \infty} \varphi_n(x) = 0$ for every $x \in E$ satisfies also the condition that $\lim_{n \to \infty} \psi(\varphi_n) = 0$ for every $\psi$ in $E^{**}$, the bidual space of $E$. Grothendieck [21] proved that spaces of the form $\ell_\infty(\Gamma)$ are Grothendieck spaces. Later, many other Banach spaces were recognized to be Grothendieck, e.g. von Neumann algebras (Pfitzner [38]), the space $H^\infty$ of bounded analytic functions on the unit disc (Bourgain [11]), spaces of the form $C(K)$ for $K$ an F-space (Seever [41]), etc. On the other hand, the space $c_0$ of all sequences convergent to 0 is not Grothendieck, since a separable Banach space is Grothendieck if and only if it is reflexive. It follows that closed...
linear subspaces of Grothendieck spaces need not be Grothendieck, although this property is preserved by complemented subspaces. Cembranos [12] proved that a space $C(K)$ is Grothendieck if and only if it does not contain any complemented copy of the space $c_0$.

The following results due to Cembranos [12] and Freniche [19] are the main motivation for our paper.

**Theorem 1.1 ([12], [19]).** Let $K$ and $L$ be infinite compact spaces and let $E$ be an infinite dimensional Banach space.

(1) $C(K, E)$ contains a complemented copy of (the Banach space) $c_0$ and hence it is not a Grothendieck space.

(2) The Banach space $C(K \times L)$ contains a complemented copy of $c_0$.

The second statement of Theorem 1.1 follows from the first one combined with the fact that $C(K \times L)$ is isomorphic to $C(K, C(L))$.

A strengthening of Theorem 1.1.(2) was obtained by Kąkol, Sobota and Zdomskyy [27] for spaces of the form $C_p(K \times L)$ (note that it follows from the Closed Graph Theorem that a complemented copy of $(c_0)_p$ in $C_p(K \times L)$ is actually a complemented copy of $c_0$ in $C(K \times L)$). Here, by $C_p(X)$ and $C_b(X)$ we mean the space $C(X)$ of continuous real-valued functions over a Tychonoff space $X$ endowed with the pointwise and compact-open topology, respectively. $C_p^*(X)$ denotes the vector subspace of $C_p(X)$ consisting of bounded functions. It is well known that for every infinite compact space $K$ the space $C(K)$ contains a copy of $c_0$, while it is also easy to see that $C_p(K)$ over an infinite compact $K$ contains a (closed) subspace isomorphic to $(c_0)_p = \{(x_n)_{n \in \omega} \in \mathbb{R}^\omega : x_n \to 0\}$ endowed with the product topology of $\mathbb{R}^\omega$.

**Theorem 1.2 ([27]).** For every infinite compact spaces $K$ and $L$ the space $C_p(K \times L)$ contains a complemented copy of the space $(c_0)_p$.

In [27] many other spaces $C_p(X)$ were recognized to contain a complemented copy of $(c_0)_p$, i.a. those $C_p(X)$ where $X$ is a compact space such that the Banach space $C(X)$ is not Grothendieck. In fact, for compact spaces $K$, the existence of a complemented copy of $(c_0)_p$ in the space $C_p(K)$ appeared to be equivalent to the property that the Banach space $C(K)$ does not have the so-called $\ell_1$-Grothendieck property, a variant of the Grothendieck property defined as follows: for a given compact space $K$ we say that a Banach space $C(K)$ has the $\ell_1$-Grothendieck property if every weakly* convergent sequence of Radon measures on $K$ with countable supports (equivalently, with finite supports) is weakly convergent; see [27] for details. Trivially, if $C(K)$ is Grothendieck, then it has the $\ell_1$-Grothendieck property; a counterexample for the reverse implication was first constructed by Plebanek ([39], see also [6]), in [27] Section 7] another example and a more detailed discussion on this topic were provided.

The research in [27] was motivated, i.a., by the following theorem of Banakh, Kąkol and Śliwa [5], especially by the equivalence $(1) \Leftrightarrow (2)$. We refer the reader to the paper [27] for a detailed discussion concerning properties of sequences of measures from (2).

**Theorem 1.3 ([5]).** For a Tychonoff space $X$ the following conditions are equivalent:
Theorem 1.4. Let $X$ and $Y$ be infinite Tychonoff spaces. Then:

1. If the space $X \times Y$ is pseudocompact, then $C_p(X \times Y)$ contains a complemented copy of $(c_0)_p$.
2. If the space $X \times Y$ is not pseudocompact, then $C_p(X \times Y)$ contains a complemented copy of $\mathbb{R}^\omega$.

Consequently, if for infinite Tychonoff spaces $X$ and $Y$ the product $X \times Y$ is pseudocompact, then $C_p(X \times Y)$ contains a complemented copy of $(c_0)_p$, so by the Closed Graph Theorem the Banach space $(C(X \times Y), \| \cdot \|_\infty)$ contains a complemented copy of the Banach space $c_0$. This provides a stronger version of Cembranos–Freniche theorem. However, if the product $X \times Y$ is not pseudocompact, then consistently $C_p(X \times Y)$ may fail to have the complemented copy of $(c_0)_p$. As our next theorem shows, this may happen even for the squares. We refer the reader to the paragraph before Theorem 1.5 for the exact formulation of the set-theoretic assumption we use in the proof of the next theorem (and which is called $(\dagger)$ by us). Let us only mention here that this assumption $(\dagger)$ is satisfied if either the Continuum Hypothesis or Martin’s axiom holds true.

Theorem 1.5. It is consistent that there exists an infinite pseudocompact space $X$ such that the spaces $C_p(X \times X)$ and $C_p^*(X \times X)$ do not contain a complemented copy of $(c_0)_p$.

The proofs of Theorems 1.4 and 1.5 are provided in Sections 2 and 5 respectively. Unfortunately, we do not know whether the negation of the conclusion of Theorem 1.5 may consistently hold. We are also not aware of any model of ZFC where $(\dagger)$, our set-theoretic assumption used to prove Theorem 1.5, fails.
Problem 1.6. Is it consistent that for any infinite pseudocompact space $X$ the space $C_p(X \times X)$ (respectively, $C_p^*(X \times X)$) contains a complemented copy of $(c_0)_p$?

Recall that every countably compact space is pseudocompact, thus the following question seems natural in the context of Theorems 1.2 and 1.5.

Problem 1.7. Is it consistent that there exists an infinite countably compact space $X$ such that the space $C_p(X \times X)$ (respectively, $C_p^*(X \times X)$) does not contain a complemented copy of $(c_0)_p$?

The methods applied to prove Theorem 1.5 cannot be used to answer Problem 1.7 in affirmative since they produce spaces which are very far from being countably compact. However, recall that there exist countably compact spaces whose squares are not pseudocompact, see e.g. [17, Example 3.10.9].

Theorem 1.4 has two important applications concerning the following remarkable problem in $C_p$-theory posed by Arkhangel’ski, see [1, 2], and the famous Rosenthal–Lacey theorem.

Problem 1.8 (Arkhangel’ski). Is it true that for every infinite (compact) space $K$ the space $C_p(K)$ is linearly homeomorphic to $C_p(K) \times \mathbb{R}$?

For a wide class of spaces the answer to Problem 1.8 is affirmative, e.g. if the space $X$ contains a non-trivial convergent sequence, or $X$ is not pseudocompact (see [2, Section 4]), yet, in general, the answer is negative even for compact spaces $X$, see Marciszewski [33]. It appears however that for finite products of Tychonoff spaces the answer is still positive—the following corollary follows immediately from Theorem 1.4.

Corollary 1.9. Let $X$ and $Y$ be infinite Tychonoff spaces. Then $C_p(X \times Y)$ is linearly homeomorphic to the product $C_p(X \times Y) \times \mathbb{R}$.

The famous Rosenthal–Lacey theorem [40], [32], see also [24, Corollary 1], asserts that for each infinite compact space $K$ the Banach space $C(K)$ admits a quotient map onto $c_0$ or $\ell_2$; we refer the reader to a survey paper [18, Theorem 18] for a detailed discussion on the theorem. The case of $C_p$-spaces remains however open, namely, it is still unknown whether for every infinite compact space $K$ the space $C_p(K)$ admits a quotient map onto an infinite-dimensional metrizable space, see [29]. Nevertheless, Theorem 1.4 yields the following corollary.

Corollary 1.10. Let $X$ and $Y$ be infinite Tychonoff spaces. Then

1. $C_p(X \times Y)$ admits a quotient map onto $\mathbb{R}^\omega$ or $(c_0)_p$.
2. $C_k(X \times Y)$ admits a quotient map onto a space isomorphic to one of the following spaces: $\mathbb{R}^\omega$, $(c_0)_p$ or $c_0$.

The proofs of Theorem 1.4 and its consequences, e.g. Corollary 1.10, are provided in Section 2.

Notation and terminology. Our notation and terminology are standard, i.e., we follow monographs of Tkachuk [42] (function spaces), Engelking [17] (general topology), Halmos
In particular, we assume that all topological spaces we consider are Tychonoff, that is, completely regular and Hausdorff.

The cardinality of a set $X$ is denoted by $|X|$. By $\omega$ we denote the first infinite cardinal number, i.e., the cardinality of the space of natural numbers $\mathbb{N}$. Usually we identify $\omega$ with $\mathbb{N}$, so $\omega$ is an infinite countable discrete topological space, thus, e.g., such notions like the Čech–Stone compactifications $\beta\omega$ of $\omega$ have sense. We denote the remainder $\beta\omega \setminus \omega$ of this compactification by $\omega^\ast$. The continuum, i.e. the cardinality of the real line $\mathbb{R}$, is denoted both by $c$ and $2^\omega$. If $X$ is a set and $\kappa$ is a (finite or infinite) cardinal number, then by $[X]^{\kappa}$ we denote the family of all subsets of $X$ of cardinality $\kappa$; in particular, $[X]^{\omega}$ denotes the family of all finite subsets of $X$. Finally, $\omega^\omega$ denotes the family of all functions from $\omega$ into $\omega$.

All other necessary and possibly non-standard notions will be defined in relevant places of the text.

2. Proof of Theorem 1.4 and its consequences

For a space $X$ and a point $x \in X$ let $\delta_x : C_p(X) \to \mathbb{R}$, $\delta_x : f \mapsto f(x)$, be the Dirac measure concentrated at $x$. The linear hull $L_p(X)$ of the set $\{\delta_x : x \in X\}$ in $\mathbb{R}C_p(X)$ can be identified with the dual space of $C_p(X)$. Elements of the space $L_p(X)$ will be called finitely supported signed measures (or simply signed measures) on $X$.

Each $\mu \in L_p(X)$ can be uniquely written as a linear combination of Dirac measures

$$\mu = \sum_{x \in F} \alpha_x \delta_x$$

for some finite set $F \subset X$ and some non-zero real numbers $\alpha_x$. The set $F$ is called the support of the signed measure $\mu$ and is denoted by $\text{supp}(\mu)$. The measure $\sum_{x \in F} |\alpha_x| \delta_x$ will be denoted by $|\mu|$ and the real number

$$\|\mu\| = \sum_{x \in F} |\alpha_x|$$

coincides with the norm of $\mu$ (in the dual Banach space $C(\beta X)^\ast$).

A sequence $(\mu_n)_n$ of finitely supported signed measures on $X$ such that $\|\mu_n\| = 1$ for all $n \in \omega$, and $\lim_{n} \mu_n(f) = 0$ for each $f \in C_p(X)$ is called a Josefson–Nissenzweig sequence or, in short, a JN-sequence on $X$. A JN-sequence $(\mu_n)_n$ is supported on a subset $A$ of a space $X$ if the supports of all measures $\mu_n$ are contained in $A$.

We say that $C_p(X)$ has the Josefson–Nissenzweig property or, in short, JNP if $X$ admits a JN-sequence.

The following proposition was proved in [27]. We provide a brief sketch of the proof for the sake of completeness.

**Proposition 2.1.** The product $\beta\omega \times \beta\omega$ has a JN-sequence $(\mu_n)_n$ supported on $\omega \times \omega$. Moreover, the supports of $\mu_n$ have pairwise disjoint projections onto each axis.
it follows that \( \| \) converges to defined as follows:

**Theorem 2.2.** Clearly it admits a following strengthening:

\[
\Theta \quad \Omega = \bigcup_{n} \Omega_n \quad \Sigma = \bigcup_{n} \Sigma_n, \quad \text{and endow these two sets with the discrete topology. This enables us to look at the the product space } \Omega \times \Sigma \text{ as a countable union of pairwise disjoint discrete rectangles } \Omega_k \times \Sigma_m \text{ of size } m2^k \text{—the rectangles } \Omega_n \times \Sigma_n, \text{ lying along the diagonal, will bear a special meaning, namely, they will be the supports of measures from a } JN\text{-sequence } (\mu_n) \text{ on the space } \beta \Omega \times \beta \Sigma \text{ defined as follows:}
\]

\[
\mu_n = \sum_{s \in \Omega_n, i \in \Sigma_n} \frac{s(i)}{n2^n} \delta_{(s,i)}, \quad n \in \omega.
\]

It turns out that the sequence \((\mu_n)\) defined above is a \( JN \)-sequence, see [27, Section 7] for details. Note that \( \omega \) is homeomorphic to both \( \Omega \) and \( \Sigma \), so \( \beta \omega, \beta \Omega \) and \( \beta \Sigma \) are mutually homeomorphic. Consequently, \( \beta \omega \times \beta \omega \) has the \( JN \)-sequence with the required properties.

In [27, Section 7] this result was used to prove that, for any infinite compact spaces \( K, L \), the space \( C_p(K \times L) \) has the \( JNP \). By a modification of this argument we can obtain the following strengthening:

**Theorem 2.2.** For any infinite spaces \( X, Y \), if the product \( X \times Y \) is pseudocompact, then it admits a \( JN \)-sequence \((\mu_n)\). Moreover, we can require that \((\mu_n)\) is supported on the product \( D \times E \), where \( D \subset X \) and \( E \subset Y \) are countable discrete, and the supports of \( \mu_n \) have pairwise disjoint projections onto each axis.

**Proof.** Let \( D \) and \( E \) be countable discrete subsets of \( X \) and \( Y \), respectively. Let \( \varphi : \omega \rightarrow D \) and \( \psi : \omega \rightarrow E \) be bijections. By the Stone Extension Property of \( \beta \omega \), there are continuous maps \( \Phi : \beta \omega \rightarrow \beta X \) and \( \Psi : \beta \omega \rightarrow \beta Y \) such that \( \Phi \upharpoonright \omega = \varphi \) and \( \Psi \upharpoonright \omega = \psi \).

We denote by \( \Theta \) the product map

\[
\Phi \times \Psi : \beta \omega \times \beta \omega \rightarrow \beta X \times \beta Y.
\]

Clearly \( \Theta \) maps \( \omega \times \omega \) injectively into \( X \times Y \).

Let \((\mu_n)_{n \in \omega}\) be a \( JN \)-sequence of measures on \((\beta \omega)^2\) supported on \( \omega^2 \), given by Proposition 2.1. For each \( n \in \omega \), we consider the image of \( \mu_n \) under \( \Theta \), i.e., the measure \( \nu_n \) on \( X \times Y \) defined as follows:

\[
\nu_n = \sum_{z \in \text{supp}(\mu_n)} \mu_n(\{z\}) \cdot \delta_{\Theta(z)},
\]

it follows that \( \| \nu_n \| = 1 \) and \( \text{supp}(\nu_n) \) is finite. We will show that the sequence \((\nu_n)\) converges to 0 on every \( f \in C(X \times Y) \) which will demonstrate that \( X \times Y \) admits a \( JN \)-sequence.

Recall that by Glicksberg’s theorem [17, 3.12.20(c)], pseudocompactness of \( X \times Y \) implies that \( \beta X \times \beta Y \) is the Čech–Stone compactification of \( X \times Y \), i.e., every continuous function on \( X \times Y \) is continuously extendable over \( \beta X \times \beta Y \).
Fix \( f \in C(X \times Y) \) and let \( F \) be its continuous extension over \( \beta X \times \beta Y \). Then the composition \( F \circ \Theta \) is a continuous function on \( \beta \omega \times \beta \omega \) and \( \nu_n(f) = \mu_n(F \circ \Theta) \) for each \( n \in \omega \). Therefore \( \lim_n \nu_n(f) = 0 \).

The additional properties of the supports of the measures \( \nu_n \) follow easily from the definition of \( \nu_n \) and the corresponding properties of supports of the measures \( \mu_n \).

We are in the position to prove the main result of this paper, Theorem 1.4.

**Proof of Theorem 1.4** If the space \( X \times Y \) is pseudocompact, then by the above theorem and Theorem 1.3 the space \( C_p(X \times Y) \) has a complemented copy of \( (c_0)_p \).

On the other hand, it is well known that if a space \( X \) is not pseudocompact, then \( C_p(X) \) has a complemented copy of \( \mathbb{R}^\omega \), cf. [2, Section 4] or [28, Theorem 14]. This completes the proof of part (2) of Theorem 1.4. \( \square \)

Observe that if a subspace \( Y \) of a space \( X \) has a \( JN \)-sequence, then this sequence is also a \( JN \)-sequence on \( X \). By Theorem 1.3 it follows that if \( C_p(Y) \) contains a complemented copy of \( (c_0)_p \), then \( C_p(X) \) also contains such a copy. Therefore Theorem 1.4.(1) immediately gives the following

**Corollary 2.3.** If a space \( Z \) contains a topological copy of a pseudocompact product \( S \times T \) of infinite spaces \( S \) and \( T \), then \( C_p(Z) \) contains a complemented copy of \( (c_0)_p \). In particular, if spaces \( X \) and \( Y \) contain infinite compact subsets, then \( C_p(X \times Y) \) contains a complemented copy of \( (c_0)_p \).

Note that there exist infinite spaces \( X \) without infinite compact subsets such that the square \( X \times X \) is pseudocompact, see Example 3.4.

Recall that a subset \( B \) of a locally convex space \( E \) is **bounded** if for every neighbourhood of zero \( U \) in \( E \) there exists a scalar \( \lambda > 0 \) such that \( \lambda B \subset U \).

Note the following fact connected with the next Corollary 2.6.

**Lemma 2.4.** For a space \( X \) the following assertions are equivalent:

1. The space \( C_k(X) \) is covered by a sequence of bounded sets.
2. The space \( C_p(X) \) is covered by a sequence of bounded sets.
3. \( X \) is pseudocompact.

**Proof.** (1) \( \Rightarrow \) (2) is clear, since the compact-open topology is stronger than the pointwise one of \( C(X) \).

(2) \( \Rightarrow \) (3): Assume \( C_p(X) \) is covered by a sequence of bounded sets but \( X \) is not pseudocompact. Then \( C_p(X) \) contains a complemented copy of \( \mathbb{R}^\omega \). On the other hand, \( \mathbb{R}^\omega \) cannot be covered by a sequence \( (S_n)_n \) of bounded sets. Indeed, we may assume that each \( S_n \) is absolutely convex and closed and \( S_n \supseteq S_{n+1} \) for all \( n \in \omega \). By the Baire theorem some \( S_n \) is a neighbourhood of zero in \( \mathbb{R}^\omega \). Consequently \( \mathbb{R}^\omega \) must be a normed space by a theorem of Day, see [22, Proposition 6.9.4], a contradiction (since \( \mathbb{R}^\omega \) is not normable).

(3) \( \Rightarrow \) (1): If \( X \) is pseudocompact and

\[
S = \{ f \in C(X) : |f(x)| \leq 1, x \in X \},
\]

then...
then the sequence of bounded sets \((nS)_n\) in the compact-open topology covers \(C_k(X)\). □

Remark 2.5. Since the image of a bounded subset of a topological vector space under a continuous linear operator is bounded, from the above lemma we can easily deduce that if \(T: C_k(X) \to C_k(Y)\) \((T: C_p(X) \to C_p(Y))\) is a linear continuous surjection and the space \(X\) is pseudocompact, then \(Y\) is also pseudocompact. This fact for the pointwise topology is well known, cf. [36, Proposition 6.8.6].

Since the pseudocompactness is not transferred even to finite products (see [17]), there exist spaces \(X\) and \(Y\) such that both \(C_k(X)\) and \(C_k(Y)\) are covered by a sequence of bounded sets but \(C_k(X \times Y)\) lacks this property, by Lemma 2.4.

A sequence \((S_n)_n\) of bounded sets in a locally convex space \(E\) is fundamental if every bounded set in \(E\) is contained in some set \(S_n\). Every \((DF)\) (in particular, every normed) space \(C_k(X)\) admits a fundamental sequence of bounded sets. By Warner [44] the space \(C_k(X)\) admits a fundamental sequence of bounded sets if and only if the following condition holds:

\[(*) \text{ Given any sequence } (G_n)_n \text{ of pairwise disjoint non-empty open subsets of } X \text{ there is a compact set } K \subset X \text{ such that } \{n \in \omega : K \cap G_n \neq \emptyset\} \text{ is infinite.}\]

This characterization easily implies that if the space \(C_k(X)\) has a fundamental sequence of bounded sets, and the space \(X\) is infinite, then \(X\) contains an infinite compact subspace. Therefore, by Corollary 2.3 we obtain

Corollary 2.6. Let \(X\) and \(Y\) be two infinite spaces. If \(C_k(X)\) and \(C_k(Y)\) admit fundamental sequences of bounded sets, then \(C_p(X \times Y)\) contains a complemented copy of \((c_0)_p\).

Remark 2.7. Note that if \(X\) is a pseudocompact space without infinite compact subsets (cf. Section 3), then by Warner’s characterization, \(C_k(X)\) does not have a fundamental sequence of bounded sets, although it is covered by a sequence of bounded sets, by Lemma 2.4.

Theorem 1.4 applies also to get the following

Corollary 2.8. Let \(X\) be an infinite pseudocompact space such that \(C(\beta X)\) is a Grothendieck space. Then for no infinite spaces \(Y\) and \(Z\) does exist a continuous linear surjection from \(C_p(X)\) onto the space \(C_p(Y \times Z)\).

Proof. Assume that for some infinite spaces \(Y\) and \(Z\) there exists a continuous linear surjection \(T: C_p(X) \to C_p(Y \times Z)\). Then the space \(Y \times Z\) is pseudocompact, see Remark 2.5. By Theorem 1.4 \(C_p(Y \times Z)\) contains a complemented copy of \((c_0)_p\). Hence \(C_p(X)\) maps onto \((c_0)_p\) by a continuous linear map, and then by Theorem 1.3 the space \(C_p(X)\) has a complemented copy of \((c_0)_p\). This implies that \(C(\beta X)\) contains a complemented copy of \(c_0\), a contradiction with the Grothendieck property of \(C(\beta X)\). □

For the proof of Corollary 1.10 will need some auxiliary facts.

The next proposition is known, the case of the pointwise topology can be found in [12], the general result covering the cases of both topologies is stated in [34] Exercise 2,
p. 36], but without a proof, hence, for the sake of completeness, we include the proof for the compact-open topology. Recall that a subset \( Y \) of a topological space \( X \) is \( C \)-embedded (\( C^* \)-embedded) in \( X \) if every (bounded) continuous real-valued function on \( Y \) can be continuously extended over \( X \). Given a compact subspace \( K \) of a space \( X \) and \( \varepsilon > 0 \), we denote the set \( \{ f \in C_k(X) : f(K) \subset (-\varepsilon, \varepsilon) \} \), a basic neighborhood of zero in \( C_k(X) \), by \( U_X(K, \varepsilon) \).

**Proposition 2.9.** Let \( Y \) be a subspace of a topological space \( X \), and \( R : C_k(X) \to C_k(Y) \) \((R : C_p(X) \to C_p(Y))\) be the restriction operator defined by \( R(f) = f \restriction Y \). \( R \) is open if and only if \( Y \) is closed and \( C \)-embedded in \( X \).

**Proof.** As we mentioned, the proof for the case of the pointwise topology can be found in [12 S.152], therefore we will only give the proof for the case of the compact-open topology.

Assume first that \( R \) is open. Then obviously it is a surjection, which means that \( Y \) is \( C \)-embedded in \( X \). Suppose, towards a contradiction, that \( Y \) is not closed in \( X \) and pick a point \( x \in \overline{Y} \setminus Y \). We will show that the image \( R(U_X(\{x\}, 1)) \) is not open in \( C_k(Y) \). Consider arbitrary basic neighborhood of zero \( U_Y(K, \varepsilon) \) in \( C_k(Y) \), given by a compact set \( K \subset Y \) and \( \varepsilon > 0 \). The set \( L = \{x\} \cup K \) is compact in \( X \), hence it is \( C \)-embedded in \( X \), see [20 3.11].

Let \( f : L \to \mathbb{R} \) be the (continuous) function which takes value 1 at \( x \) and value 0 on \( K \), and let \( F \) be its continuous extension over \( X \). Put \( g = F \restriction Y \). Obviously, for any \( h \in C_k(X) \) such that \( h \restriction Y = g \restriction Y \), the functions \( h \) and \( F \) must agree on the closure of \( Y \), so \( h(x) = F(x) = 1 \). This shows that

\[
g \in U_Y(K, \varepsilon) \setminus R(U_X(\{x\}, 1)).
\]

Now, assume that \( Y \) is closed and \( C \)-embedded in \( X \). To prove that \( R \) is open its enough to verify that, for any compact \( K \subset X \) and \( \varepsilon > 0 \), we have

\[
R(U_X(K, \varepsilon)) = U_Y(K \cap Y, \varepsilon).
\]

In the last equality, one inclusion is obvious. Take any

\[
g \in U_Y(K \cap Y, \varepsilon).
\]

We will show that \( g \) is an image under \( R \) of some \( f \in U_X(K, \varepsilon) \). Let \( G \) be a continuous extension of \( g \) over \( X \). Put

\[
L = \{x \in K : |G(x)| \geq \varepsilon\}.
\]

Clearly, \( L \) is a compact subset of \( X \) disjoint with \( Y \). Hence we can use the well known fact (see [20 3.11]), that a compact set can be separated from a closed set by a continuous function, i.e., we can find a continuous \( h : X \to [0, 1] \) which sends \( L \) to 0 and \( Y \) to 1. One can easily verify that \( f = hG \) has the required properties.

Recall that if \( X \) is a pseudocompact space and \( K \) is a compact space, then \( X \times K \) is pseudocompact (see [17 Corollary 3.10.27]), so \( C_p(X \times K) \) contains a complemented copy of \( (c_0)_p \), provided that both \( X \) and \( K \) are infinite. For the compact-open topology in this function space we have the following special case of Corollary [1.10].
Proposition 2.10. Assume that $X$ is an infinite pseudocompact space and $K$ is an infinite compact space.

(1) If $X$ contains an infinite compact subset, then $C_k(X \times K)$ contains a complemented copy of the Banach space $c_0$.

(2) If $X$ has no infinite compact subsets, then $C_k(X \times K)$ contains a complemented copy of $(c_0)_p$.

Proof. Part (1): By Domański’s and Drewnowski’s theorem [16], the space $C_k(X, C_k(K))$ contains a complemented copy of $c_0$. On the other hand, by [34, Corollary 2.5.7] the spaces $C_k(X \times K)$ and $C_k(X, C_k(K))$ are linearly homeomorphic.

Part (2): Since the product $X \times K$ is pseudocompact, we can apply Theorem 2.2 to obtain a JN-sequence $(\mu_n)_n$ on $X \times K$ such that $(\mu_n)_n$ is supported on the product $D \times E$, where $D \subset X$ and $E \subset Y$ are countable discrete, and the supports of $\mu_n$ have pairwise disjoint projections onto each axis. Since the sets $D$ and $E$ are discrete and countable, we can find families $\{U_d : d \in D\}$ and $\{V_e : e \in E\}$ of pairwise disjoint sets, such that, for each $d \in D$, $U_d$ is an open neighborhood of $d$ in $X$ and for each $e \in E$, $V_e$ is an open neighborhood of $e$ in $Y$. Let $A_n = \text{supp}(\mu_n)$ and $A = \bigcup_{n \in \omega} A_n$. Given $a = (d_a, e_a) \in A$, put $W_a = U_{d_a} \times V_{e_a}$. Clearly, the family $\{W_a : a \in A\}$ of neighborhoods of points of $A$ consists of pairwise disjoint sets. Moreover, if we define, for each $n \in \omega$,

$$W_n = \bigcup \{W_a : a \in A_n\},$$

then the properties of supports $A_n$ imply that the sets $W_n$ have pairwise disjoint projections onto each axis. For each $a \in A$, take a continuous function

$$g_a : X \times K \to [0, 1]$$

such that $g_a(a) = 1$ and $g$ takes value 0 on $(X \times K) \setminus W_a$. For every $n \in \omega$ define

$$f_n : X \times K \to [-1, 1]$$

by

$$f_n = \sum_{a \in A_n} \frac{\mu_n(a)}{|\mu_n(a)|} g_a.$$

The functions $f_n$ and the sets $W_n$ have the following properties for all $n \in \omega$:

(a) $\mu_n(f_n) = 1$;
(b) the support of $f_n$ is contained in $W_n$;
(c) the support of $\mu_n$ is contained in $W_n$;
(d) the projections of the sets $W_k$, $k \in \omega$, onto $X$ are pairwise disjoint.

Consider the linear operator

$$S : (c_0)_p \to C_k(X \times K)$$

defined by

$$S((t_n)) = \sum_{n \in \omega} t_n f_n.$$
We will show that $Z \in S \in X$ is finite, hence, by properties (b) and (d), $(c_0)_p$ is an isomorphic copy of $(c_0)_p$ which is complemented in $C_k(X \times K)$. Let $T : C_k(X \times K) \rightarrow (c_0)_p$ be defined by $T(f)(n) = \mu_n(f)$ for $f \in C_k(X \times K)$ and $n \in \omega$. Obviously, the operator $T$ is continuous. Using properties (a)–(d) one can easily verify that

$$(T \upharpoonright Z) \circ S = \text{Id}_{(c_0)_p}$$

and $S \circ (T \upharpoonright Z) = \text{Id}_Z$, hence the spaces $Z$ and $(c_0)_p$ are isomorphic. Let

$$P = S \circ T : C_k(X \times K) \rightarrow C_k(X \times K).$$

The identities $Z = S((c_0)_p)$ and $S \circ (T \upharpoonright Z) = \text{Id}_Z$ imply that $P$ is a continuous projection of $C_k(X \times K)$ onto $Z$. \qed

We are ready to provide a proof of Corollary 1.10.

Proof of Corollary 1.10 Part (1) is a direct consequence of Theorem 1.4.

For the proof of part (2) we shall consider several cases:

Case 1. If both spaces $X$ and $Y$ contain infinite compact subsets, say $K$ and $L$, respectively, then $K \times L$ is $C$-embedded in $X \times Y$ (see [20, 3.11]), and the restriction map $C_k(X \times Y) \rightarrow C_k(K \times L)$ is a continuous and open surjection by Proposition 2.9. Now we can use Theorem 1.11 (2).

Case 2. If neither $X$ nor $Y$ admits an infinite compact subset, then for $C_p(X \times Y) = C_k(X \times Y)$ applies part (1).

Case 3. If one of the spaces $X$ or $Y$ is not pseudocompact, the product $X \times Y$ is also not pseudocompact, and the space $C_k(X \times Y)$ contains a complemented copy of $\mathbb{R}^\omega$, see [28, Theorem 14].

Case 4. The only remaining case to be checked is that when both $X$ and $Y$ are pseudocompact, and one of these spaces contains an infinite compact subset but the other one lacks infinite compact subsets. Without loss of generality we can assume that $Y$ contains an infinite compact subset $K$. We will verify that $X \times K$ is $C$-embedded in $X \times Y$. Let $f$ be a continuous function on $X \times K$. Since $X \times K$ is pseudocompact, by Glicksberg’s theorem [17, 3.12.20(c)], $f$ can be extended to a continuous function $g$ on $\beta X \times K$. By compactness of $\beta X \times K$, we can extend $g$ continuously over $\beta X \times Y$ obtaining a function $h$. Now, the restriction of $h$ to $X \times K$ is the desired extension of $f$. Proposition 2.9 implies that the restriction operator $R : C_k(X \times Y) \rightarrow C_k(X \times K)$ is open, hence also surjective. From Proposition 2.10 we obtain a projection $P$ of $C_k(X \times K)$ onto a subspace $Z$ isomorphic to $(c_0)_p$. Since $P$ is open (as a map onto $Z$), the composition $P \circ R$ is an open continuous surjection of $C_k(X \times Y)$ onto a copy of $(c_0)_p$. \qed
3. Haydon's construction of a pseudocompact space with no infinite compact subsets

Theorem \[14\] may suggest a question whether \(C_p(X \times X)\) contains a complemented copy of \((c_0)_p\) for any infinite pseudocompact space \(X\). In Section \[5\] assuming the Continuum Hypothesis (or even a weaker set-theoretic assumption), we will answer this question negatively. Our examples use the following general scheme of constructing pseudocompact spaces with no infinite compact subsets given by Haydon in \[26\].

For each \(A \in [\omega]^\omega\), choose an ultrafilter \(u_A \in \omega^*\) in the closure of \(A\) in \(\beta\omega\). Let \(X = \omega \cup \{u_A : A \in [\omega]^\omega\}\) be topologized as a subspace of \(\beta\omega\). To simplify the notation we will denote the family of all such spaces \(X\) by HS.

We start with the following fact collecting some properties of every space \(X \in \text{HS}\).

**Proposition 3.1.** Every space \(X \in \text{HS}\) has the following properties:

1. \(X\) is pseudocompact of cardinality continuum;
2. all compact subspaces of \(X\) are finite;
3. \(C_p(X)\) does not have the JNP;
4. \(C_p(X)\) admits an infinite dimensional metrizable quotient isomorphic to the subspace \((\ell_\infty)_p = \{(x_n) \in \mathbb{R}^\omega : \sup_n |x_n| < \infty\}\) of \(\mathbb{R}^\omega\) endowed with the product topology and the Banach space \((C(X), \|\cdot\|_\infty)\) (i.e. endowed with the sup-norm topology) is isometric to the Banach space \(\ell_\infty\);
5. for every infinite compact space \(K\) both spaces \(C_p(K \times X)\) and \(C_k(K \times X)\) contain a complemented copy of \((c_0)_p\).

**Proof.** Obviously \(|X| \leq 2^\omega\). To see that \(X\) must contain continuum many ultrafilters, it is enough to take an almost family \(A \subset [\omega]^\omega\) (i.e., distinct members of \(A\) have finite intersection) of cardinality continuum. Then the closures of elements of \(A\) in \(\beta\omega\) have pairwise disjoint intersections with \(\omega^*\). This proves the second item of (1).

The first item of (1) has been shown by Haydon in the third example of \[26\]. Indeed, if \(C_p(X)\) is not pseudocompact, then there exists \(f \in C(X)\) unbounded. As \(\omega\) is dense in \(X\) one gets a sequence \((n_k)\) in \(\omega\) such that \(|f(n_k)| > k\). For the set \(A = \{n_k : k \in \omega\}\) the corresponding \(u_A\) in \(\beta\omega\) belongs to the closure in \(\beta\omega\) of \(\{n_k : k \geq m\}\) for each \(m \in \omega\). Hence \(|f(u_A)| > m\) for each \(m \in \omega\), a contradiction.

Proof of (2): Since infinite compact subsets of \(\beta\omega\) have cardinality \(2^\omega\), every compact subset of \(X\) is finite by (1).

Proof of (3): Note that \(\beta X = \beta\omega\) and the Banach space \((C(X), \|\cdot\|_\infty)\) is isometric to \(\ell_\infty = C(\beta\omega)\). On the other hand, as \(C(\beta\omega)\) is a Grothendieck space, \(C(X)\) does not have a complemented copy of the Banach space \(c_0\). Consequently, by applying the closed graph theorem \[37\] Theorem 4.1.10], the space \(C_p(X)\) does not contain a complemented copy of \((c_0)_p\). Now by Theorem \[13\] we know that \(C_p(X)\) does not have JNP.

Proof of (4): Since the space \(X\) is pseudocompact and contains discrete \(\omega\) which is \(C^*\)-embedded into \(X\), we apply Theorem 1 from \[1\] to deduce that \(C_p(X)\) has a quotient
\[ C_p(X)/W \text{ isomorphic to the subspace } (\ell_\infty)_p \text{ of } \mathbb{R}^\omega \text{ endowed with the product topology, where} \]
\[ W = \bigcap_n \{ f \in C_p(X) : \sum_{x \in F_n} f(x) = 0 \} \]
and \((F_n)_n\) is any sequence of non-empty, finite and pairwise disjoint subsets of \(\omega\) with \(\lim_n |F_n| = \infty\). The second part of the item (4) is clear since \(C(\beta \omega)\) and \((C(X), \| \cdot \|_\infty)\) are isometric.

Proof of (5): Since \(K \times X\) is pseudocompact, the first claim of (5) follows from Theorem 1.4. The other claim follows from Proposition 2.10. \(\square\)

The following proposition characterizes those subspaces \(X\) of \(\beta \omega\) containing \(\omega\) which are in the class HS.

**Proposition 3.2.** For a subspace \(X\) of \(\beta \omega\) containing \(\omega\), the following conditions are equivalent:

1. \(X \in \text{HS}\);
2. \(X\) is pseudocompact and of cardinality continuum;
3. \(X\) is of cardinality continuum, and every infinite subset of \(\omega\) has an accumulation point in \(X\).

**Proof.** The implication (1) \(\Rightarrow\) (2) was explained in the previous proposition.

The implication (2) \(\Rightarrow\) (3) is obvious, a pseudocompact \(X\) cannot contain an infinite clopen discrete subset.

To verify (3) \(\Rightarrow\) (1), first observe that each ultrafilter in \(X \cap \omega^*\) is in the closure of continuum many sets \(A \in [\omega]^\omega\). Also, given \(A \in [\omega]^\omega\), the closure of \(A\) in \(X\) has cardinality continuum, which is witnessed by an almost disjoint family \(A\) of subsets of \(A\) of cardinality \(|A| = 2^\omega\). Therefore, by a standard back-and-forth inductive argument, we can construct a bijection \(f\) between \([\omega]^\omega\) and \(X \cap \omega^*\) such that \(u_A = f(A)\) is in the closure of \(A\). To this end, enumerate \([\omega]^\omega\) as \(\{A_\alpha : \alpha < 2^\omega\}\), and \(X \cap \omega^*\) as \(\{u_\alpha : \alpha < 2^\omega\}\). Inductively, for each \(\alpha < 2^\omega\), define \(f\) on \(\{A_\beta : \beta < \alpha\}\) and \(f^{-1}\) on \(\{u_\beta : \beta < \alpha\}\). \(\square\)

In [33], Marciszewski constructed an example of a space \(X \in \text{HS}\) such that there is no continuous linear surjection from the space \(C_p(X)\) onto \(C_p(X) \times \mathbb{R}\). In particular, \(C_p(X)\) is not linearly homeomorphic to \(C_p(X) \times \mathbb{R}\), hence \(C_p(X)\) does not contain any complemented copy of \(c_0\) or \(\mathbb{R}^\omega\). To achieve this it is enough to require that all ultrafilters \(u_A\) in \(X\) are weak P-points and they are pairwise non-isomorphic.

Recall that a point \(p \in \omega^*\) is a weak P-point if \(p\) is not in the closure of any countable set \(D \subset \omega^* \setminus \{p\}\), see [35]. Two ultrafilters \(u, v \in \beta \omega\) are called isomorphic if \(h(u) = v\) for some homeomorphism \(h\) of \(\beta \omega\); equivalently, there is a bijection \(f : \omega \to \omega\) such that \(u = \{f^{-1}(A) : A \in v\}\).

The Rudin–Keisler preorder on \(\beta \omega\) is the binary relation \(\leq_{\text{RK}}\) on \(\beta \omega\) given by \(v \leq_{\text{RK}} u\) if there is \(f : \omega \to \omega\) such that \(u \supset \{f^{-1}(A) : A \in v\}\), i.e., \(v = f(u)\), where \(f(u) = \{A \subset \omega : f^{-1}(A) \in u\}\). This preorder becomes an order if we identify isomorphic ultrafilters, see [13] Corollary 9.3. We say that ultrafilters \(u, v \in \omega^*\) are incompatible with respect to the Rudin–Keisler preorder if there is no \(w \in \omega^*\) such that \(w \leq_{\text{RK}} u\) and \(w \leq_{\text{RK}} v\). An
Rudin–Keisler preorder. The Continuum Hypothesis or even a weaker set-theoretic assumptions, like Martin’s Axiom, imply that there exists an antichain of size continuum, consisting of weak P-points in \( \omega^* \), cf. [13, Theorem 9.13, Lemma 9.14] or [7, Theorem 2]. Blass and Shelah [9] proved the consistency, relative to \( \text{ZFC} \), of the statement that any two ultrafilters in \( \omega^* \) are compatible with respect to the Rudin–Keisler preorder.

**Proposition 3.3.** Let \( X = \omega \cup \{u_A : A \in [\omega]^{\omega}\} \) be a space in \( \text{HS} \) such that for distinct \( A, B \in [\omega]^{\omega} \) the ultrafilters \( u_A, u_B \) are not isomorphic. Then the square \( X \times X \) is not pseudocompact.

**Proof.** Take two disjoint infinite sets \( A, B \subset \omega \) and any bijection \( f : A \to B \). Observe that the graph \( \Gamma = \{(x, f(x)) : x \in A\} \) of \( f \) is an open discrete subspace of \( X \times X \). We will show that \( \Gamma \) is closed in \( X \times X \).

Suppose, towards a contradiction that, there exists \( (u_1, u_2) \in \text{Cl}_{X \times X}(\Gamma) \setminus \Gamma \). For any \( n \in \omega \), the intersections \( \Gamma \cap \{(n) \times X\} \) and \( \Gamma \cap (X \times \{n\}) \) contain at most one point. Therefore, they are closed in \( \{n\} \times X \) and \( X \times \{n\} \), respectively. Since the latter sets are open in \( X \times X \), it follows that \( u_1, u_2 \in \omega^* \). Since \( \Gamma \subset A \times B \), we have

\[
(u_1, u_2) \in \text{Cl}_{X \times X}(A \times B) = \text{Cl}_X(A) \times \text{Cl}_X(B).
\]

The disjoint sets \( A \) and \( B \) have disjoint closures in \( X \), so \( u_1 \neq u_2 \).

Let \( g : \omega \to \omega \) be any bijection extending \( f \). We claim that the ultrafilter \( \{g^{-1}(C) : C \in u_2\} \) is equal to \( u_1 \). Suppose that these two ultrafilters are not equal. Then we can find \( D \subset \omega \) such that \( D \in u_1 \) and

\[
(\omega \setminus D) \in \{g^{-1}(C) : C \in u_2\},
\]

i.e., \( E = g(\omega \setminus D) \in u_2 \). Observe that the product \( D \times E \) is disjoint with the graph of \( g \), hence is also disjoint with \( \Gamma \). Therefore the product \( \text{Cl}_X(D) \times \text{Cl}_X(E) \) is a neighborhood of \( (u_1, u_2) \) disjoint with \( \Gamma \) (because \( \Gamma \subset \omega \times \omega \) and

\[
(\text{Cl}_X(D) \times \text{Cl}_X(E)) \cap \omega \times \omega = D \times E,
\]

and the latter is disjoint with \( \Gamma \), a contradiction. We obtained that the ultrafilters \( u_1 \) and \( u_2 \) are isomorphic, contradicting our assumption on \( X \).

Finally, we verified that the infinite discrete subspace \( \Gamma \) is clopen in \( X \times X \), hence the space \( X \times X \) is not pseudocompact. \( \square \)

On the other hand, we have the following example.

**Example 3.4.** There exists a space \( Y \in \text{HS} \) such that the square \( Y \times Y \) is pseudocompact. Consequently, \( C_p(Y \times Y) \) contains a complemented copy of \( (c_0)_p \).

**Proof.** For any space \( X \in \text{HS} \), the square \( \omega \times \omega \) is dense in \( X \times X \), therefore it is enough to construct a space \( Y \in \text{HS} \) such that any infinite subset of \( \omega \times \omega \) has an accumulation point in \( Y \times Y \), see the proof of Proposition 3.1 (1).
Let \( \{ A_\alpha : \alpha < 2^\omega \} \) be an enumeration of \([\omega]^\omega\), and \( \{ C_\alpha : \alpha < 2^\omega \} \) be an enumeration of \([\omega \times \omega]^\omega\). For each \( \alpha < 2^\omega \), choose an ultrafilter \( u_\alpha \) from \( \omega^* \) in the closure of \( A_\alpha \), and an accumulation point \( (p_\alpha, q_\alpha) \) of \( C_\alpha \) in \( \beta \omega \times \beta \omega \). Let

\[
Y = \omega \cup \{ u_\alpha, p_\alpha, q_\alpha : \alpha < 2^\omega \}.
\]

One can easily verify that \( Y \) satisfies condition (c) from Proposition 3.2 so it is in HS, and any infinite subset of \( \omega \times \omega \) has an accumulation point in \( Y \times Y \). The last claim follows from Theorem 1.4

\[
\qed
\]

4. The Bounded Josefson–Nissenzweig Property

In this section we define a “bounded” version of the Josefson–Nissenzweig property. A sequence \( (\mu_n) \) of finitely supported signed measures on \( X \) such that \( |\mu_n| = 1 \) for all \( n \in \mathbb{N} \), and \( \lim \mu_n(f) = 0 \) for each bounded \( f \in C_p(X) \) is called a bounded Josefson–Nissenzweig sequences or, in short, a BJN-sequence. We say that \( C_p(X) \) has the Bounded Josefson–Nissenzweig Property or, in short, the BJNP, if \( X \) admits a BJN-sequence.

Obviously, for pseudocompact spaces \( X \), these bounded versions coincide with the standard ones. It is also trivial that the JNP implies the BJNP and each JN-sequence is a BJN-sequence.

One can easily construct examples of BJN-sequences, which are not JN-sequences. Actually, if \( X \) is not pseudocompact and \( C_p(X) \) has the JNP, then \( X \) has a BJN-sequence which is not a JN-sequence. Indeed, if \( X \) is such a space, take a JN-sequence \( (\mu_n) \) on \( X \) and a continuous unbounded function \( f \) on \( X \). For each \( n \), pick \( x_n \in X \) such that \( |f(x_n)| > n \). Define \( \nu_n = \mu_n + (1/n)\delta_{x_n} \). One can easily verify that the sequence \( (\nu_n) \) (after normalizing) is a BJN-sequence but not a JN-sequence.

In Example 12 we construct a space with the BJNP but without the JNP. For a verification of the properties of the example we will need an auxiliary fact concerning JN-sequences. Recall that a subset \( A \) of a topological space \( X \) is bounded if for every \( f \in C_p(X) \) the set \( f(A) \) is bounded in \( \mathbb{R} \).

**Proposition 4.1.** Let \( (\mu_n)_{n \in \omega} \) be a JN-sequence of measures on a space \( X \). Then the union \( \bigcup_{n \in \omega} \text{supp}(\mu_n) \) of supports of \( \mu_n \) is bounded in \( X \).

**Proof.** We sketch the proof of the proposition using operators between linear spaces. In Section 4 we provide a more direct and elementary proof.

Let \( S = \{ 0 \} \cup \{ 1/n : n = 1, 2, \ldots \} \) be equipped with the Euclidean topology. We can define a continuous linear operator \( T : C_p(X) \to C_p(S) \) in the following way: for \( f \in C_p(X) \), put \( T(f)(0) = 0 \) and

\[
T(f)(1/(n + 1)) = \mu_n(f)
\]

for \( n \in \omega \). From the definition of a JN-sequence immediately follows that the operator \( T \) is well-defined. Observe that, for any \( n \geq 1 \), the support \( \text{supp}_T(1/n) \) of \( 1/n \) in \( X \) with respect to \( T \) (see Chapter 6.8) is equal to \( \text{supp}(\mu_n) \). Obviously the set \( A = \{ 1/n : n \geq 1 \} \) is
bounded in $S$, $S$ being compact. By Theorem 6.8.3 in [36], the support

$$\text{supp}_T(A) = \bigcup_{n \geq 1} \text{supp}_T(1/n) = \bigcup_{n \in \omega} \text{supp}(\mu_n)$$

is bounded in $X$. \hfill \square

Example 4.2. There exists a countable space $X$ such that $C_p(X)$ has the BJNP, but does not have the JNP.

Proof. Recall the well known method of obtaining a countable space $N_F$ associated with a filter $F$ on a countable set $T$ (we consider only free filters on $T$, i.e., filters containing all cofinite subsets of $T$).

$N_F$ is the space $T \cup \{\infty\}$, where $\infty \notin T$, equipped with the following topology: All points of $T$ are isolated and the family $\{A \cup \{\infty\} : A \in F\}$ is a neighborhood base at $\infty$.

Let $F$ be the filter on $\omega$ consisting of sets of density 1, i.e.,

$$F = \left\{ A \subset \omega : \lim_{n} |A \cap \{1, 2, \ldots, n\}| \frac{n}{n} = 1 \right\}.$$

We will show that the space $X = N_F$ has the required properties (this space in the context of function spaces was investigated in [15]).

Let $S_n = \{2^n + 1, 2^n + 2, \ldots, 2^{n+1}\}$, $n \in \omega$, and

$$\lambda_n = \sum_k \left\{ \frac{1}{2^{n+1}} \delta_k : k \in S_n \right\} - \frac{1}{2} \delta_\infty.$$

Let us check that $(\lambda_n)_{n \in \omega}$ is a BJN-sequence. Take a bounded continuous function $f$ on $N_F$; for simplicity, we can assume that $|f| \leq 1$. Fix an $\varepsilon > 0$, and find a set $A \in F$ such that the image under $f$ of the neighborhood $A \cup \{\infty\}$ of $\infty$ has the diameter less than $\varepsilon$.

Choose $k \in \omega$ such that, for $m > k$,

$$\frac{|A \cap \{1, 2, \ldots, m\}|}{m} > 1 - \varepsilon,$$

Observe that, for $n > k$, we have $2^{n+1} > k$, $|S_n| = 2^n$ and $S_n \subset \{1, 2, \ldots, 2^{n+1}\}$, hence the above inequality applied for $m = 2^{n+1}$, gives us the estimate $|S_n \setminus A| < 2^{n+1} \varepsilon$. Hence, for
n > k, we have
\[ |\lambda_n(f)| = \left| \sum \left\{ \frac{1}{2^{n+1}} f(k) : k \in S_n \right\} - \frac{1}{2} f(\infty) \right| \leq \sum \left\{ \frac{1}{2^{n+1}} |f(k) - f(\infty)| : k \in S_n \right\} = \sum \left\{ \frac{1}{2^{n+1}} |f(k) - f(\infty)| : k \in S_n \cap A \right\} + \sum \left\{ \frac{1}{2^{n+1}} |f(k) - f(\infty)| : k \in S_n \setminus A \right\} < |S_n \cap A| \cdot \frac{\varepsilon}{2^{n+1}} + |S_n \setminus A| \cdot \frac{2}{2^{n+1}} < 2^n \cdot \frac{\varepsilon}{2^{n+1}} + 2^{n+1} \varepsilon \cdot \frac{2}{2^{n+1}} = \frac{5}{2} \varepsilon, \]
which shows that the sequence \((\lambda_n(f))\) converges to 0.

It is clear that for any JN-sequence \((\mu_n)\) in a space \(X\), the union \(S\) of supports of \(\mu_n\) is infinite, and by Proposition 4.1, \(S\) is also bounded. Since all bounded subsets of \(N_F\) are finite (cf. [15] Example 7.1), \(C_p(N_F)\) does not have the JNP. \(\square\)

Recall that \(C^*_p(X)\) is the subspace of \(C_p(X)\) consisting of bounded functions.

**Theorem 4.3.** For a space \(X\) the following conditions are equivalent:

1. \(C_p(X)\) has the BJNP;
2. \(C^*_p(X)\) has a complemented copy of \((c_0)_p\);
3. \(C^*_p(X)\) has a quotient isomorphic to \((c_0)_p\);
4. \(C^*_p(X)\) admits a continuous linear surjection onto \((c_0)_p\).

**Proof.** The implication \((1) \Rightarrow (2)\) follows from Lemma 1 in [4] and its proof. It is enough to apply this lemma for \(K = \beta X\), and observe that the subspace \(L\) of \(C_p(X)\) given by this lemma contains \(C^*_p(X)\) and the range of the projection \(P\) from \(L\) onto the subspace isomorphic to \((c_0)_p\) is contained in \(C^*_p(X)\).

Implications \((2) \Rightarrow (3) \Rightarrow (4)\) are obvious.

\((4) \Rightarrow (1)\). Let \(T : C^*_p(X) \to (c_0)_p\) be a continuous linear surjection. For each \(n \in \omega\), let \(p_n : (c_0)_p \to \mathbb{R}\) be the projection onto \(n\)th axis, and \(\lambda_n\) be the finitely supported signed measure on \(X\) corresponding to the functional \(p_n \circ T\). Let \(C^*(X)\) be the Banach space of all bounded continuous functions on \(X\), equipped with the standard supremum norm. By the Closed Graph Theorem, \(T\) can be treated as continuous linear surjection between the Banach spaces \(C^*(X)\) and \(c_0\), which gives the estimate \(\|\lambda_n\| \leq \|T\|\) for all \(n\). By the Open Mapping Theorem we can also get a constant \(c > 0\) such that \(\|\lambda_n\| \geq c\) for all \(n\). Now it is clear that \((\lambda_n/\|\lambda_n\|)_{n \in \omega}\) is a BJN-sequence on \(X\). \(\square\)

**Corollary 4.4.** If \(C_p(X)\) has the BJNP, then \(C(\beta X)\) contains a complemented copy of the Banach space \(c_0\).

**Corollary 4.5.** If \(C^*_p(X)\) does not have any complemented copy of \((c_0)_p\), then \(C_p(X)\) does not have it either.
Proof. If $C_p(X)$ does have a complemented copy of $(c_0)_p$, then $C_p(X)$ has the JNP and so $C_p(X)$ has the BJNP, which implies that $C_p^*(X)$ has a complemented copy of $(c_0)_p$. □

Example 4.2 combined with Theorems 1.3 and 4.3 gives us the following

Corollary 4.6. There exists a countable space $X$ such that $C_p^*(X)$ has a complemented copy of $(c_0)_p$, but $C_p(X)$ does not have such a copy. Consequently, the Banach space $C(\beta X)$ contains a complemented copy of the Banach space $c_0$.

5. A pseudocompact space $X$ such that its square $X \times X$ does not have a BJN-sequence

First we will prove several auxiliary results. The proof of the following standard fact is similar to that of [33, Lemma 3.1].

Lemma 5.1. Let $\{S_n : n \in \omega\}$ be a family of pairwise disjoint subsets of $\omega$. Then, for every subset $C$ of $\beta \omega$ of the cardinality less than $2^\omega$, there exists an infinite subset $A \subset \omega$ such that

$$C \cap (\text{Cl}_{\beta \omega} \bigcup_{n \in A} S_n) \subset \bigcup_{n \in A} \text{Cl}_{\beta \omega} S_n.$$

Proof. Let $A \subset [\omega]^\omega$ be an almost disjoint family of size $|A| = \mathfrak{c}$. It is clear that

$$(\text{Cl}_{\beta \omega} \bigcup_{n \in A} S_n) \cap (\text{Cl}_{\beta \omega} \bigcup_{n \in A'} S_n) = \bigcup_{n \in A \cap A'} \text{Cl}_{\beta \omega} S_n$$

for any distinct $A, A' \in A$. Thus the family

$$\{ (\text{Cl}_{\beta \omega} \bigcup_{n \in A} S_n) \setminus \bigcup_{n \in A} \text{Cl}_{\beta \omega} S_n : A \in A \}$$

is disjoint, and hence one of its elements is disjoint from $C$ as the latter has size $< \mathfrak{c} = |A|$. This completes our proof. □

Lemma 5.2. Let $X, Y$ be spaces in HS, and $(A_1^n)_{n \in \omega}$, $(A_2^n)_{n \in \omega}$ be two sequences of non-empty subsets of $\omega$ such that, for each $k, n \in \omega$, $k \neq n$, $i=1,2$, $A_k^n \cap A_i^n = \emptyset$. Put

$$U = \bigcup_{n \in \omega} \text{Cl}_{X \times Y}(A_1^n \times A_2^n)$$

and let $f_1, f_2 : \omega \rightarrow \omega$ be functions such that, for $i = 1, 2$, $f_i(A_1^n) = \{2n\}$, $f_i$ takes odd values on $\omega \setminus \bigcup_{n \in \omega} A_i^n$, and is injective on this complement.

If $(u_1, u_2) \in \text{Cl}_{X \times Y}(U) \setminus U$, then $u_1, u_2 \in \omega^*$ and $f_1(u_1) = f_2(u_2) \in \omega^*$, hence the ultrafilters $u_1$ and $u_2$ are compatible with respect to the Rudin–Keisler preorder. Moreover,

$$u_1 \in \text{Cl}_X \left( \bigcup_{n \in \omega} A_1^n \right) \setminus \bigcup_{n \in \omega} \text{Cl}_X (A_1^n)$$

and

$$u_2 \in \text{Cl}_Y \left( \bigcup_{n \in \omega} A_2^n \right) \setminus \bigcup_{n \in \omega} \text{Cl}_Y (A_2^n).$$
Proof. Let \((u_1, u_2) \in \text{Cl}_{X \times Y}(U) \setminus U\). For any \(n \in \omega\), the intersections \(U \cap \{n\} \times Y\) and \(U \cap (X \times \{n\})\) are either empty or, for some \(k \in \omega\), are equal to \(\{n\} \times \text{Cl}_Y(A^2_k)\) or \(\text{Cl}_X(A^1_k) \times \{n\}\), respectively. Therefore they are closed in \(\{n\} \times Y\) and \(X \times \{n\}\), respectively. Since the sets \(\{n\} \times Y\) and \(X \times \{n\}\) are open in \(X \times Y\), it follows that \(u_1, u_2 \in \omega^*\).

Let \(V = \bigcup_{n \in \omega} A_n^1 \times A_n^2\). Obviously, \(V\) is dense in \(U\), hence \((u_1, u_2)\) belongs to \(\text{Cl}_{X \times Y}(V) \setminus U\). Set \(p_i = f_i(u_i), i = 1, 2\). If \(p_1 \neq p_2\), then we can find \(C \subset \omega\) such that \(C \in p_1\) and \((\omega \setminus C) \in p_2\). We claim that the neighborhood
\[
O = \text{Cl}_X(f_1^{-1}(C)) \times \text{Cl}_Y(f_2^{-1}(\omega \setminus C))
\]
of \((u_1, u_2)\) is disjoint with \(V\). Indeed, suppose that there are \(n \in \omega\) and
\[
(m_1, m_2) \in (A_n^1 \times A_n^2) \cap (\text{Cl}_X(f_1^{-1}(C)) \times \text{Cl}_Y(f_2^{-1}(\omega \setminus C))),
\]
i.e., \(m_1 \in \text{Cl}_X(f_1^{-1}(C))\) and \(m_2 \in \text{Cl}_Y(f_2^{-1}(\omega \setminus C))\). Since \(\omega\) is a discrete subspace of \(X\) and \(Y\), we conclude that \(m_1 \in f_1^{-1}(C)\) and \(m_2 \in f_2^{-1}(\omega \setminus C)\), which means \(2n = f_1(m_1) \in C\) and \(2n = f_2(m_2) \in \omega \setminus C\), a contradiction. Therefore \(O \cap V = \emptyset\), which together with \((u_1, u_2) \in O\) and \((u_1, u_2) \in \text{Cl}_{X \times Y}(V)\) again leads to a contradiction.

Hence \(p_1 = p_2 = p\), and it remains to observe that \(p \in \omega^*\). Indeed, if \(\{n\} \in p\), then we have two cases:

If \(n\) is odd, then \(|f_i^{-1}(n)| \leq 1\), so \(f_i^{-1}(n)\) cannot belong to \(u_i \in \omega^*,\ i = 1, 2\).

If \(n = 2k\), then \(A^1_k \in u_i, \ i = 1, 2,\) hence
\[
(u_1, u_2) \in \text{Cl}_{X \times Y}(A^1_k \times A^2_k) \subset U,
\]
which again gives a contradiction.

Finally, it is clear that \(u_1 \in \text{Cl}_X(\bigcup_{n \in \omega} A_n^1)\). If \(u_1 \in \text{Cl}_X(A_n^1)\) for some \(n\), we would get that \(\{2n\} = f_1(A_n^1) \in f_1(u_1) = p\), thus contradicting \(p \in \omega^*\). Analogously for \(u_2\).

\[\square\]

Lemma 5.3. Let \((\mu_n)_{n \in \omega}\) be a BJN-sequence on a space \(X\). If \(Y\) is a clopen subset of \(X\) such that the sequence \((|\mu_n|(Y))_{n \in \omega}\) does not converge to 0, then there exist an increasing sequence \((n_k)\) and a BJN-sequence \((\nu_k)_{k \in \omega}\) in \(X\) such that \(\text{supp} \nu_k \subset Y \cap \text{supp} \mu_{n_k}\) for every \(k \in \omega\), in particular \((\nu_k)_{k \in \omega}\) is a BJN-sequence on \(Y\).

Proof. Pick an \(a > 0\) and an increasing sequence \((n_k)\) such that \(|\mu_{n_k}(Y)| > a\) for all \(k \in \omega\). Let
\[
\nu_k = (\mu_{n_k} \upharpoonright Y)/|\mu_{n_k}(Y)|.
\]
Since any continuous bounded function on \(Y\) can be extended to a continuous bounded function on \(X\) by declaring value 0 outside \(Y\), it easily follows that \((\nu_k)_{k \in \omega}\) is a BJN-sequence on \(X\) and in \(Y\).

\[\square\]

Lemma 5.4. Let \(X\) and \(Y\) be spaces in HS such that all ultrafilters in \(X \setminus \omega\) and \(Y \setminus \omega\) are weak P-points. If \(C_p(X \times Y)\) has the BJNP (JNP), then there exist a BJN-sequence (a JN-sequence) \((\mu_n)_{n \in \omega}\) in \(X \times Y\), and sequences \((A_n^1)_{n \in \omega}, (A_n^2)_{n \in \omega}\) of finite subsets of \(X, Y\), respectively, such that
\[
(1) \ A_k^1 \cap A_n^2 = \emptyset \text{ for } k, n \in \omega, \ k \neq n, \ i = 1, 2;
\]
Given $n \in \omega$, the subspaces $\{n\} \times Y$ and $X \times \{n\}$ are homeomorphic to $X$ and $Y$, respectively, and $C_p(\{n\} \times Y)$ and $C_p(X \times \{n\})$ do not have the BJNP by Proposition 3.1. Since these subspaces are clopen, Lemma 5.3 implies that

$$\lim_\kappa |\nu_k|(\{n\} \times Y) = 0, \quad \lim_\kappa |\nu_k|(X \times \{n\}) = 0.$$ 

Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be projections. For a subset $B$ of $\omega$, we denote $\pi_X^{-1}(B) \cup \pi_Y^{-1}(B)$ by $\mathcal{C}(B)$. Clearly, for any finite $B \subset \omega$, we have $\lim_\kappa |\nu_k|(\mathcal{C}(B)) = 0$.

By induction we choose an increasing sequence $(k_i)_{i \in \omega}$ and a sequence $(B_i)_{i \in \omega}$ of pairwise disjoint finite subsets of $\omega$ such that

1. $(\omega \times \omega) \cap \text{supp} \nu_{k_i} \subset \mathcal{C}(\bigcup_{j=0}^{i-1} B_j) \cup (B_i \times B_i);$
2. $|\nu_{k_i}|(\mathcal{C}(\bigcup_{j=0}^{i-1} B_j)) < 1/(i+1).$

For $i \in \omega$, let

$$Z_i = (X \times Y) \setminus \mathcal{C}(\bigcup_{j=0}^{i-1} B_j).$$

Define

$$\lambda_i = (\nu_{k_i} \upharpoonright Z_i)/|\nu_{k_i} \upharpoonright Z_i|.$$

Using condition $(ii)$ above, one can easily verify that $\lim_i |\lambda_i - \nu_{k_i}| = 0$, hence $(\lambda_i)_{i \in \omega}$ is a BJN-sequence on $X \times Y$. The first condition implies that

$$(\omega \times \omega) \cap \text{supp} \lambda_i \subset B_i \times B_i.$$

Let

$$C = (\pi_X(\bigcup_{i \in \omega} \text{supp} \lambda_i) \cup \pi_Y(\bigcup_{i \in \omega} \text{supp} \lambda_i)) \setminus \omega \subset \omega^*.$$

By Lemma 5.3 we can find an infinite subset $T \subset \omega$ such that

$$C \cap (\text{Cl}_{\beta \omega} \bigcup_{i \in T} B_i) = \emptyset.$$

Then the subsequence $(\lambda_i)_{i \in T}$ has the property that the set

$$D := \pi_X(\bigcup_{i \in T} \text{supp} \lambda_i) \cup \pi_Y(\bigcup_{i \in T} \text{supp} \lambda_i)$$

is a discrete subspace of $\beta \omega$. Indeed, the set $D' = D \cap \omega^*$ is a discrete subspace of $\beta \omega$ because it is countable and consists of weak $P$-points. In addition, $\text{Cl}_{\beta \omega}(D \cap \omega)$ is disjoint with $D'$ by our choice of $T$ because $D \cap \omega \subset \bigcup_{i \in T} B_i$. Thus $D$ is a union of two discrete subsets of $\beta \omega$ so that the closure of any of them does not intersect another one, and hence it is also discrete.
Set \( D_X = \pi_X(\bigcup_{i \in T} \text{supp } \lambda_i) \) and \( D_Y = \pi_Y(\bigcup_{i \in T} \text{supp } \lambda_i) \), so that \( D = D_X \cup D_Y \). Fix a point \( p \in D_X \), and find a clopen set \( V \) in \( X \) such that \( V \cap D_X = \{p\} \). Let \( U = V \times Y \). \( U \) is clopen in \( X \times Y \) and
\[
U \cap \bigcup_{i \in T} \text{supp } \lambda_i \subset \{p\} \times Y.
\]
Observe that each continuous function on \( \{p\} \times Y \) can be extended to a bounded continuous function on \( U \), therefore, by Lemma 5.3, we conclude that
\[
\lim_{i \in T} |\lambda_i|(\{p\} \times Y) = 0.
\]
In the same way we can show that, for any \( q \in D_Y \), we have \( \lim_{i \in T} |\lambda_i|(X \times \{q\}) = 0 \).

Now, we can repeat our inductive construction from the first part of the proof. We choose an increasing sequence \((i_n)_{n \in \omega} \); \( i_n \in T \), and sequences \((A^n_1)_{n \in \omega}, (A^n_2)_{n \in \omega} \) of finite subsets of \( D_X, D_Y \), respectively, such that the following conditions are satisfied (where \( E_n = \bigcup_{i=1,2} \bigcup_{j=0}^{n-1} A^n_j \)):

(a) \( (A^n_1 \cup A^n_2) \cap E_n = \emptyset \);

(b) \( \text{supp } \lambda_{i_n} \subset (\pi_X^{-1}(X \cap E_n) \cup \pi_Y^{-1}(Y \cap E_n)) \cup (A^n_1 \times A^n_2) \);

(c) \( |\lambda_{i_n}|(\pi_X^{-1}(X \cap E_n) \cup \pi_Y^{-1}(Y \cap E_n)) < 1/(n+1) \).

For \( n \in \omega \), let
\[
S_n = (X \times Y) \setminus (\pi_X^{-1}(X \cap E_n) \cup \pi_Y^{-1}(Y \cap E_n)).
\]

Define \( \mu_n = (\lambda_{i_n} \restriction S_n)/||\lambda_{i_n} \restriction S_n|| \). Conditions (1) and (2) follow from (a). Condition (c) implies that
\[
\lim_n ||\lambda_{i_n} - \mu_n|| = 0,
\]
hence \((\mu_n)_{n \in \omega}\) is a BJN-sequence on \( X \times Y \). From condition (b) we deduce that
\[
\text{supp } \mu_n \subset A^n_1 \times A^n_2.
\]
Condition (4) follows from inclusions \( A^n_1 \subset D_X, A^n_2 \subset D_Y \).

Observe, that the union of supports of measures \( \mu_n \) that we constructed above is contained in the union of supports of measures \( \nu_k \) from our initial BJN-sequence. If we start with a JN-sequence \((\nu_k)_{k \in \omega}\), then, by Proposition 5.11 \( \bigcup_{k \in \omega} \text{supp } (\nu_k) \) is bounded in \( X \times Y \), therefore, if we repeat the above argument, the resulting BJN-sequence \((\mu_n)_{n \in \omega}\) will be also a JN-sequence.

Before we prove Theorem 5.5, which is the main result of this section, we will show its weaker version, since it has an essentially simpler proof and a less technical set-theoretic assumption.

**Theorem 5.5.** Assume that there exist two incompatible with respect to the Rudin–Keisler preorder weak P-points \( u, v \) in \( \omega^* \). Then there exist infinite pseudocompact spaces \( X, Y \) in HS such that \( C_p(X \times Y) \) does not have the BJNP, and hence it does not have the JNP.
Proof. It is an easy and well known observation that, given an ultrafilter \( w \in \omega^* \) and a set \( A \in [\omega]^\omega \), there is an ultrafilter \( w' \) in the closure of \( A \) in \( \beta\omega \), which is isomorphic to \( w \). Therefore, we may take spaces
\[
X = \omega \cup \{ u_A : A \in [\omega]^\omega \}, \quad Y = \omega \cup \{ v_A : A \in [\omega]^\omega \}
\]
in HS such that all ultrafilters \( u_A \) are isomorphic to \( u \), and all ultrafilters \( v_A \) are isomorphic to \( v \). We will show that the product \( X \times Y \) does not have a BJN-sequence.

Suppose the contrary, and let
\[
(\mu_n)_{n \in \omega}, \quad (A^1_n)_{n \in \omega}, \quad (A^2_n)_{n \in \omega}
\]
be sequences of measures and sets given by Lemma 5.4. Let
\[
A^i = \bigcup_{n \in \omega} A^i_n, \quad i = 1, 2.
\]
Recall that basic clopen sets in \( X \) and \( Y \) are closures of subsets of \( \omega \). For \( i = 1, 2 \), since \( A^i \) is discrete, using a simple induction, we can find a family \( \{ U^i_p : p \in A^i \} \) of pairwise disjoint subsets of \( \omega \), such that the closure of \( U^i_p \) is a neighborhood of \( p \). For \( i = 1, 2 \) and \( n \in \omega \), let
\[
V^i_n = \bigcup \{ U^i_p : p \in A^i_n \}.
\]
For fixed \( i \), the family \( \{ V^i_n : n \in \omega \} \) is disjoint, and we have
\[
\text{supp} \mu_n \subset \text{Cl}_{X \times Y}(V^1_n \times V^2_n).
\]
From our construction of spaces \( X \) and \( Y \) and Lemma 5.2 we infer that the set
\[
U = \bigcup_{n \in \omega} \text{Cl}_{X \times Y}(V^1_n \times V^2_n)
\]
is clopen in \( X \times Y \). Let \( f_n : \text{supp} \mu_n \to [-1, 1] \) be defined by
\[
f_n(z) = \mu_n(\{ z \}) / |\mu_n(\{ z \})|
\]
for \( z \in \text{supp} \mu_n \). Since \( \text{supp} \mu_n \) is finite, we can extend each \( f_n \) to a continuous function
\[
F_n : \text{Cl}_{X \times Y}(V^1_n \times V^2_n) \to [-1, 1].
\]
Since sets \( \text{Cl}_{X \times Y}(V^1_n \times V^2_n) \) are clopen, the union of all \( F_n \) is continuous on \( U \), and we can extend this union to a bounded continuous \( F \) on \( X \times Y \), declaring value 0 outside \( U \). Clearly, for all \( n \in \omega \), we have \( \mu_n(F) = 1 \), a contradiction. \( \square \)

For a set \( A \) by \( \triangle A \) we denote the diagonal \( \{(a, a) : a \in A\} \) in \( A \times A \).

**Lemma 5.6.** Let \( E, F \) be non-empty finite subsets of a set \( X \), and \( \mu \) be a signed measure on \( E \times F \). Then there exist disjoint \( G \subset E \) and \( H \subset F \) such that
\[
|\mu|(G \times H) \geq |\mu|((E \times F) \setminus \triangle X)/6.
\]
Proof. Let $A = E \cap F$. First, we will show that there exist disjoint subsets $B, C$ of $A$ such that

$$|\mu|(B \times C) \geq |\mu|((A \times A) \setminus \Delta_A)/4.$$  

The case when $|A| \leq 1$ is easy, so we also assume that $|A| > 1$. Suppose the contrary, then, for any partition $(B, C)$ of $A$, we have

$$|\mu|(B \times C) < |\mu|((A \times A) \setminus \Delta_A)/4.$$  

**Case 1.** $|A| = 2n + 1$. Let $\mathcal{P}$ be the family of all partitions $(B, C)$ of $A$ such that $|B| = n$, $|C| = n + 1$. We have $|\mathcal{P}| = \binom{2n+1}{n}$ and each point from $(A \times A) \setminus \Delta_A$ appears in $\binom{2n-1}{n-1}$ sets $B \times C$, where $(B, C) \in \mathcal{P}$. Hence we have

$$\binom{2n-1}{n-1}|\mu|((A \times A) \setminus \Delta_A) = \sum \{|\mu|(B \times C) : (B, C) \in \mathcal{P}\} < \binom{2n+1}{n} |\mu|((A \times A) \setminus \Delta_A)/4.$$  

Since

$$\binom{2n+1}{n}/\binom{2n-1}{n-1} = 2(2n+1)/(n+1) = 4 - 2/(n+1) \leq 4,$$  

we arrive at a contradiction.

**Case 2.** $|A| = 2n$. Repeat the argument from the first case using partitions into sets of size $n$ and the estimate $\binom{2n}{n}/\binom{2n-2}{n-1} = 2(2n-1)/n \leq 4$.

Next, observe that

$$(E \times F) \setminus \Delta_X = ((E \setminus F) \times F) \cup (A \times (F \setminus E)) \cup ((A \times A) \setminus \Delta_A),$$  

and the sets $A, E \setminus F, F \setminus E$ are all disjoint. If the variation $|\mu|$ of one of the rectangles $(E \setminus F) \times F$ and $A \times (F \setminus E)$ is at least equal to $|\mu|((E \times F) \setminus \Delta_X)/6$, then we can take this rectangle as $G \times H$. If this is not the case, then we have

$$|\mu|((A \times A) \setminus \Delta_A) > (4/6) \cdot |\mu|((E \times F) \setminus \Delta_X),$$  

which implies that

$$|\mu|(B \times C) > (1/4) \cdot (4/6) \cdot |\mu|((E \times F) \setminus \Delta_X),$$  

and we can take $G = B$ and $H = C$. \qed

**Lemma 5.7.** Let $(\mu_n)_{n \in \omega}$ be a BJN-sequence on a space $X$. If $Z$ is a subset of $X$ such that

$$\lim_{n} \|\mu_n \upharpoonright Z\| = 0,$$  

then there exist an increasing sequence $(n_k)$ and a BJN-sequence $(\nu_k)_{k \in \omega}$ in $X$ such that $\text{supp} \nu_k \subset \text{supp} \mu_{n_k} \setminus Z$ for every $k \in \omega$. 

Theorem 5.8. Let $A \mapsto u_A$ assigning to each $A \in [\omega]^{<\omega}$ a weak $P$-point $u_A \ni A$ on $\omega$ such that for every pair $(f_1, f_2)$ of functions from $\omega$ to $\omega$, there exists $A \subset [\omega]^{<\omega}$ of size $|A| < c$ such that for all $A_1 \notin A$ and $A_2 \in [\omega]^{<\omega} \setminus \{A_1\}$ we have $f_1(u_{A_1}) \neq f_2(u_{A_2})$, provided that $f_1(u_{A_1}), f_2(u_{A_2}) \in \omega^*$. We do not know whether $(\dagger)$ can be proved outright in ZFC. However, at the end of this section we present a constellation of cardinal characteristics of the continuum which implies $(\dagger)$ and holds in many standard models of set theory as well as is implied by the Continuum Hypothesis or Martin’s axiom.

Theorem 5.8. Let $[\omega]^{<\omega} \ni A \mapsto u_A \in \mathcal{C}_{\beta\omega}(A) \cap \omega^*$ be a witness for $(\dagger)$ and $X = \omega \cup \{u_A : A \in [\omega]^{<\omega}\}$. Then $C_p(X \times X)$ does not have the BJNP, and hence it does not have the JNP.

Proof. Suppose the contrary, and let $(\mu_n)_{n \in \omega}$, $(A^1_n)_{n \in \omega}$, and $(A^2_n)_{n \in \omega}$ be sequences of measures and sets given by Lemma 5.4. Let

$$A = \bigcup_{i=1,2} \bigcup_{n \in \omega} A^i_n.$$ 

By condition (4) of Lemma 5.4, the set $A$ is discrete.

First we will show that

$$(\ast) \quad \lim_{n} |\mu_n|((A^1_n \times A^2_n) \setminus \triangle X) = 0.$$ 

Assume towards the contradiction that there exist an $a > 0$ and an increasing sequence $(n_k)$ such that

$$|\mu_{n_k}|((A^1_{n_k} \times A^2_{n_k}) \setminus \triangle X)) > a$$

for all $k \in \omega$. Using Lemma 5.6, for each $k \in \omega$, we can find disjoint sets $B^i_k \subset A^i_{n_k}$, $i = 1, 2$, such that

$$|\mu_{n_k}|(B^1_k \times B^2_k) > a/6.$$ 

Let $B^i = \bigcup_{k \in \omega} B^i_k$ for $i = 1, 2$. From conditions (1), and (2) of Lemma 5.4 it follows that the sets $B^1, B^2 \subset A$ are disjoint.

Since the set $A$ is discrete, we can find a disjoint family $\{U_p : p \in A\}$ of subsets of $\omega$, such that the closure of $U_p$ is a neighborhood of $p$.

For $i = 1, 2$ and $k \in \omega$, let

$$V^i_k = \bigcup\{U_p : p \in B^i_k\} \text{ and } W^i_k = \bigcup\{U_p : p \in A^i_{n_k}\}.$$ 

These sets have the following properties:
In particular, let \( \{W_i^k : k \in \omega\} \) be disjoint;

(2) \( B_1^1 \times B_2^1 \subset \text{Cl}_{X \times X}(V_1^1 \times V_2^1) \subset \text{Cl}_{X \times X}(W_1^1 \times W_2^1) \);

(3) \( \text{supp} \mu_{nk} \subset \text{Cl}_{X \times X}(W_1^k \times W_2^k) \).

Moreover, the sets \( V_i = \bigcup_{k \in \omega} V_i^k \), \( i = 1, 2 \), are disjoint, hence have disjoint closures in \( X \).

Consider maps \( f_i : \omega \to \omega \) such that \( f_i^{-1}(\{2k\}) = V_i^k \) and \( f_i \upharpoonright (\omega \setminus V_i) \) is injective and takes odd values, where \( i = 1, 2 \). Set

\[
U' = \bigcup_{k \in \omega} \text{Cl}_{X \times X}(V_1^k \times V_2^k)
\]

and note that if \( (u_1, u_2) \in \text{Cl}_{X \times X}(U') \setminus U' \), then \( u_1 \neq u_2 \) and, by Lemma 5.2, \( f_1(u_1) = f_2(u_2) \in \omega^* \). Applying (†) to \( (f_1, f_2) \) we get \( A \subset [\omega]^\omega \) of size \( |A| < \mathfrak{c} \) such that for all \( A_1 \not\subset A \) and \( A_2 \in [\omega]^\omega \setminus \{A_1\} \) we have \( f_1(u_{A_1}) \neq f_2(u_{A_2}) \), provided that \( f_1(u_{A_1}), f_2(u_{A_2}) \in \omega^* \). It follows from the above that

\[
\text{Cl}_{X \times X}(U') \setminus U' \subset \{u_A : A \in \mathcal{A}\}^2.
\]

Applying Lemma 5.1 we can find \( I \in [\omega]^\omega \) such that the set

\[
\text{Cl}_{X}(\bigcup_{k \in I} V_1^k) \setminus \bigcup_{k \in I} \text{Cl}_{X}(V_1^k)
\]

is disjoint from \( \{u_A : A \in \mathcal{A}\} \), where \( i = 1, 2 \). Thus, the set

\[
U = \bigcup_{k \in I} \text{Cl}_{X \times X}(V_1^k \times V_2^k)
\]

is clopen in \( X \times X \). Indeed, since \( U \) is clearly open, it remains to prove that it is also closed. Otherwise there exists \( (u_1, u_2) \in \text{Cl}_{X \times X}(U) \setminus U \). Since for every \( n \not\in I \) we have that \( \text{Cl}_{X \times X}(V_n^1 \times V_n^2) \) is a clopen subset of \( X \times X \) disjoint from \( U \), it is also disjoint from \( \text{Cl}_{X \times X}(U) \), and hence

\[
\text{Cl}_{X \times X}(U) \setminus U \subset \text{Cl}_{X \times X}(U') \setminus U',
\]

which yields

\[
(u_1, u_2) \in \{u_A : A \in \mathcal{A}\}^2.
\]

In particular \( u_1 \in \{u_A : A \in \mathcal{A}\} \). However,

\[
u_1 \in \text{Cl}_{X}(\bigcup_{k \in I} V_1^k) \setminus \bigcup_{k \in I} \text{Cl}_{X}(V_1^k)
\]

by the last clause of Lemma 5.2 which is impossible by our choice of \( I \).

For every \( k \in I \) set \( C_k = \text{supp} \mu_{nk} \cap U \). Properties (1)–(3) imply the inclusions

\[
\text{supp} \mu_{nk} \cap (B_1^k \times B_2^k) \subset C_k \subset \text{Cl}_{X \times X}(V_1^k \times V_2^k).
\]

Let \( f_k : C_k \to [-1, 1] \) be defined by

\[
f_k(z) = \mu_{nk}([z])/\mu_{nk}([z])
\]

for \( z \in C_k \). Since \( C_k \) is finite, we can extend each \( f_k \) to a continuous function

\[
F_k : \text{Cl}_{X \times X}(V_1^k \times V_2^k) \to [-1, 1].
\]
Since sets $\text{Cl}_{X \times Y}(V_k^1 \times V_k^2)$ are clopen, the union of all $F_k$, for $k \in I$, is continuous on $U$, and we can extend this union to a bounded continuous $F$ on $X \times X$, declaring value 0 outside $U$. Clearly, for all $k \in I$, we have $\mu_{\kappa}(F) > \alpha/6$, a contradiction.

Now, condition (*) and Lemma 5.7 imply that $X \times X$ admits a BJN-sequence $(\nu_k)_{k \in \omega}$ supported on $\triangle_X$. Clearly, $\triangle_X$ is homeomorphic to $X$, and each continuous bounded function on $\triangle_X$ extends to a continuous bounded function on $X \times X$. We obtain a contradiction with the lack of BJNP for $C_p(X)$, see Proposition 5.11.

Now, we present a sufficient condition for (†). Recall that $u$ denotes the minimal cardinality of a base of neighborhoods for an ultrafilter in $\omega^*$, and $\mathfrak{d}$ is the minimal cardinality of a cover of $\omega^*$ by compact sets. These cardinal characteristics can consistently attain arbitrary uncountable regular values $\leq \mathfrak{c}$, independently one from another, see [11]. If $u < \mathfrak{d}$, then $\mathfrak{d}$ is regular, see [13] and [8] for more information on these as well as other cardinal characteristics of the continuum.

Recall that $u \in \omega^*$ is a $\mathcal{P}$-point, if for any countable family $\mathcal{B}$ of open neighbourhoods of $u$ in $\omega^*$ the intersection $\cap \mathcal{B}$ contains $u$ in its interior. Clearly, every $\mathcal{P}$-point is also a weak $\mathcal{P}$-point. $\mathcal{P}$-points exist under various set-theoretic assumptions, e.g., under $\mathfrak{d} = \mathfrak{c}$ every filter on $\omega$ generated by $< \mathfrak{c}$ many sets can be enlarged to a $\mathcal{P}$-point, see [30]. On the other hand, there are models of ZFC without $\mathcal{P}$-points (see [15]). In contrast with $\mathcal{P}$-points the weak $\mathcal{P}$-points exist in ZFC by [31].

A family $\mathcal{A} \subset [\omega]^\omega$ is called strongly centered, if $\cap \mathcal{A}' \subset [\omega]^\omega$ for any finite $\mathcal{A}' \subset \mathcal{A}$. A sequence of elements of $[\omega]^\omega$ is strongly centered if the set of its elements is strongly centered.

Lemma 5.9. If $\mathfrak{d} = \mathfrak{c} \leq u^+$, then (†) holds. Moreover, there is a map $A \mapsto u_A$ witnessing for (†) such that $u_A$ is a $\mathcal{P}$-point for every $A \in [\omega]^\omega$.

Proof. We need the following auxiliary lemma.

Sublemma 5.10. Suppose that $\kappa$ is a cardinal such that $\kappa \leq u$ and $\kappa < \mathfrak{c} = \mathfrak{d}$. Suppose that $\{u_\xi : \xi < \kappa\} \subset \omega^*$ and $\{(f_1^\xi, f_2^\xi) : \xi < \kappa\}$ is a family of pairs of maps from $\omega$ to $\omega$. Then for every $A \in [\omega]^\omega$ there exists a $\mathcal{P}$-point $u \ni A$ such that $f_1^\xi(u) \neq f_2^\xi(u_\xi)$ for all $\xi < \kappa$, provided that $f_1^\xi(u), f_2^\xi(u_\xi), \in \omega^*$.

Proof. Without loss of generality we may assume that $f_2^\xi(u_\xi) \in \omega^*$ for all $\xi < \kappa$. Fix $A \in [\omega]^\omega$. Set $A_{-1} = A$ and suppose that for some $\eta < \kappa$ we have already constructed a strongly centered sequence $A_\eta = \langle A_\eta : -1 \leq \xi < \eta \rangle$ of infinite subsets of $\omega$ with the following property (we associate $\xi$ with the set $\{\alpha : \alpha < \xi\}$):

(*) For every $\xi < \eta$, if $f_1^\xi(\bigcap_{\gamma \in A_\eta} A_\gamma) \in [\omega]^\omega$ for all $\alpha \cap [\xi \cup \{-1\}]^\omega$, then $A_\xi = (f_1^\xi)^{-1}(B_\xi)$ for some $B_\xi \in [\omega]^\omega \setminus f_2^\xi(u_\xi)$ (hence $f_1^\xi(A_\xi) \notin f_2^\xi(u_\xi)$).

If $f_1^\eta(\bigcap_{\gamma \in A_\eta} A_\gamma)$ is finite for some $\alpha \subset [\eta \cup \{-1\}]^\omega$, then we simply set $A_\eta = \omega$. If $f_1^\eta(\bigcap_{\gamma \in A_\eta} A_\gamma)$ is infinite for all $\alpha \subset [\eta \cup \{-1\}]^\omega$ and $f_1^\eta(\bigcap_{\gamma \in A_\eta} A_\gamma) \notin f_2^\eta(u_\eta)$ for some finite
\[ a_\eta \in [\eta \cup \{-1\}]^{<\omega}, \text{ then we set} \]
\[ A_\eta = (f^n_1)^{-1}(f^n_1\left( \bigcap_{\gamma \in a_\eta} A_\gamma \right)). \]

Otherwise, notice that for the family
\[ y = \left\{ f^n_1\left( \bigcap_{\gamma \in a} A_\gamma \right) : a \in [\eta \cup \{-1\}]^{<\omega} \right\} \subset f^n_2(u_\eta) \]
the family
\[ \{ Cl_\beta \omega (Y) \setminus Y : Y \in y \} \]
cannot be a base of neighborhoods of \( f^n_2(u_\eta) \) in \( \omega^* \), because it has size \( < u \), and thus there exists \( X \in f^n_2(u_\eta) \) with \( |Y \setminus X| = \omega \) for all \( Y \in y \). Set \( B_\eta = \omega \setminus X \) and \( A_\eta = (f^n_1)^{-1}(B_\eta) \).

It follows from the above that \( A_{\eta + 1} = \langle A_\xi : -1 \leq \xi \leq \eta \rangle \) is strongly centered and satisfies \( (*)_{\eta + 1} \). This completes our recursive construction of a sequence \( A_\eta = \langle A_\xi : -1 \leq \xi < \kappa \rangle \).

Since \( \kappa < c = 0 \), there exists a \( P \)-point \( u \supset \{ A_\xi : -1 \leq \xi < \kappa \} \); see [30, 1.2 Theorem].

For every \( \xi \in \kappa \) such that \( f^1_2(u) \in \omega^* \) we have that \( f^1_2(A_\xi) \in f^2_2(u) \setminus f^2_2(u_\xi) \), which completes our proof. \( \Box \)

Now we proceed with the proof of Lemma 5.9. Let \( \{ A_\alpha : \alpha < c \} \) and \( \{ \langle f^1_1, f^2_2 \rangle : \alpha < c \} \) be enumerations of \( [\omega]^{<\omega} \) and the family of all pairs of functions from \( \omega \) to \( \omega \), respectively. Recursively over \( \alpha \in c \) we shall construct a \( P \)-point \( u_\alpha \supset A_\alpha \) as follows: Assuming that \( u_\beta \) is already constructed for all \( \beta \in \alpha \), let us fix an enumeration
\[ \{ \langle u_\xi, (f^1_2, f^2_2) \rangle : \xi < \kappa \} \]
of
\[ \{ \langle u_\beta, (f^1_2, f^2_2) \rangle : \beta, \gamma \in \alpha \}, \]
where \( \kappa = |\alpha \times \alpha| \). By Sublemma 5.10 there exists a \( P \)-point \( u_\alpha \supset A_\alpha \) such that \( f^1_2(u_\alpha) \neq f^2_2(u_\xi) \) for all \( \xi < \kappa \), provided that \( f^1_2(u_\alpha), f^2_2(u_\xi) \in \omega^* \). In other words,
\[ \forall \beta, \gamma \in \alpha \left( f^1_2(u_\alpha), f^2_2(u_\beta) \in \omega^* \Rightarrow f^1_2(u_\alpha) \neq f^2_2(u_\beta) \right). \]
The map \( A_\alpha \mapsto u_\alpha \) constructed this way clearly witnesses for \( (\dagger) \).

**Remark 5.11.** If there exists an \( RK \)-antichain of size \( c \) consisting of weak \( P \)-points, then \( (\dagger) \) holds. Indeed, let \( A \) be such an antichain. Using a standard transfinite inductive construction, we can easily obtain a family \( \{ u_A : A \in [\omega]^{<\omega} \} \subset \omega^* \) such that, for any \( A \in [\omega]^{<\omega} \), \( A \in u_A \), and the ultrafilters \( u_A \) and \( u_B \) are isomorphic to distinct elements of \( A \) for \( A \neq B \). Clearly, the assignment \( A \mapsto u_A \) is as required in \( (\dagger) \). However, by Lemma 5.9 \( (\dagger) \) holds in all models of \( c = 0 \leq \omega_2 \), in particular it holds in the Blass–Shelah model constructed in [9] in which the \( RK \)-preorder is downwards directed. Thus the existence of such \( RK \)-antichains as mentioned above is sufficient but not necessary for \( (\dagger) \) and thus also for Theorem 5.8.

Theorem 5.10 follows now easily.
Proof of Theorem 1.5 The theorem follows from Theorems 5.8 and 4.3, Corollary 4.5, and the consistency of \( \ddagger \).

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Faculty of Mathematics and Computer Science, A. Mickiewicz University, Poznań, Poland, and Institute of Mathematics, Czech Academy of Sciences, Prague, Czech Republic.

E-mail address: kakol@amu.edu.pl

Institute of Mathematics and Computer Science, University of Warsaw, Poland Warszawa, Poland.

E-mail address: wmarcisz@mimuw.edu.pl

Universität Wien, Institut für Mathematik, Kurt Gödel Research Center, Augasse 2-6, UZA 1 — Building 2, 1090 Wien, Austria.

E-mail address: damian.sobota@univie.ac.at

Universität Wien, Institut für Mathematik, Kurt Gödel Research Center, Augasse 2-6, UZA 1 — Building 2, 1090 Wien, Austria.

E-mail address: lzdomsay@gmail.com