COUNTING VISIBLE CIRCLES ON THE SPHERE AND KLEINIAN GROUPS

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ABSTRACT. For a circle packing \( \mathcal{P} \) on the sphere invariant under a geometrically finite Kleinian group, we compute the asymptotic of the number of circles in \( \mathcal{P} \) of spherical curvature at most \( T \) which are contained in any given region.

1. INTRODUCTION

In the unit sphere \( S^2 = \{ x^2 + y^2 + z^2 = 1 \} \) with the Riemannian metric induced from \( \mathbb{R}^3 \), the distance between two points is simply the angle between the rays connecting them to the origin \( o \). Let \( \mathcal{P} \) be a circle packing on the sphere \( S^2 \), i.e., a union of circles which may intersect with each other.

In the beautiful book Indra’s pearls, Mumford, Series and Wright ask the question (see [13, 5.4 in P. 155])

How many visible circles are there?

The visual size of a circle \( C \) in \( S^2 \) can be measured by its spherical radius \( 0 < \theta(C) \leq \pi/2 \), that is, the half of the visual angle of \( C \) from the origin \( o = (0,0,0) \). We label the circles by their spherical curvatures given by

\[
\text{Curv}_S(C) := \cot \theta(C).
\]

We suppose that \( \mathcal{P} \) is infinite and locally finite in the sense that for any \( T > 1 \), there are only finitely many circles in \( \mathcal{P} \) of spherical curvature at most \( T \). We then set for any subset \( E \subset S^2 \),

\[
N_T(\mathcal{P}, E) = \{ C \in \mathcal{P} : C \cap E \neq \emptyset, \text{ Curv}_S(C) < T \} < \infty.
\]

In order to present our main result on the asymptotic for \( N_T(\mathcal{P}, E) \), we consider the Poincare ball model \( \mathbb{B} = \{ x_1^2 + x_2^2 + x_3^2 < 1 \} \) of the hyperbolic 3-space with the metric given by \( \frac{2\sqrt{dx_1^2+dx_2^2+dx_3^2}}{1-(x_1^2+x_2^2+x_3^2)^2} \). The geometric boundary of \( \mathbb{B} \) naturally identifies with \( S^2 \).

Let \( G \) denote the group of orientation preserving isometries of \( \mathbb{B} \). A torsion-free discrete subgroup \( \Gamma \) of \( G \) is called a Kleinian group. A Kleinian group \( \Gamma \) is called geometrically finite if \( \Gamma \) admits a finite sided fundamental domain in \( \mathbb{B} \), and non-elementary if \( \Gamma \) has no abelian subgroup of finite index. We denote by \( \Lambda(\Gamma) \subset S^2 \) the limit set of \( \Gamma \), that is, the set of accumulation points of an orbit of \( \Gamma \) in \( \mathbb{B} \cup S^2 \). The critical exponent \( 0 \leq \delta_\Gamma \leq 2 \) of \( \Gamma \)

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is known to be positive for $\Gamma$ non-elementary and equal to the Hausdorff
dimension of the limit set $\Lambda(\Gamma)$ for $\Gamma$ geometrically finite [20].

For a vector $u$ in the unit tangent bundle $T^1(\mathbb{B})$, denote by $u^+ \in S^2$ the
forward end point of the geodesic determined by $u$, and by $\pi(u) \in \mathbb{B}$ the
basepoint of $u$. For $x_1, x_2 \in \mathbb{B}$ and $\xi \in S^2$, $\beta_\xi(x_1, x_2)$ denotes the signed
distance between horospheres based at $\xi$ and passing through $x_1$ and $x_2$.

For a non-elementary geometrically finite Kleinian group $\Gamma$, denote by
$\{\nu_x : x \in \mathbb{B}\}$ the Patterson-Sullivan density for $\Gamma$ ([17], [20]), i.e., a $\Gamma$-
invariant conformal density of dimension $\delta_\Gamma$ on $\Lambda(\Gamma)$, which is unique up to
homothety.

**Definition 1.1** (The $\Gamma$-skinning size of $P$). For a circle packing $P$ on $S^2$
invariant under $\Gamma$, we define $0 \leq \text{sk}_\Gamma(P) \leq \infty$ as follows:

$$\text{sk}_\Gamma(P) := \sum_{i \in I} \int_{s \in \text{Stab}_\Gamma(C_i^\perp)} e^{\delta_\Gamma \beta_{\pi(x)}(x, \pi(s))} d\nu_s(s^+)$$

where $x \in \mathbb{B}$, $\{C_i : i \in I\}$ is a set of representatives of $\Gamma$-orbits in $P$ and
$C_i^\perp \subset T^1(\mathbb{B})$ is the set of unit normal vectors to the convex hull of $C_i$.

By the conformal property of $\{\nu_x\}$, the definition of $\text{sk}_\Gamma(P)$ is independent
of the choice of $x$ and the choice of representatives $\{C_i\}$.

**Theorem 1.2.** Let $\Gamma$ be a non-elementary geometrically finite Kleinian

$\text{group}$ and $P = \bigcup_{i \in I} \Gamma(C_i)$ be an infinite, locally finite, and $\Gamma$-invariant circle

$\text{packing}$ on the sphere $S^2$ with finitely many $\Gamma$-orbits.

Suppose one of the following conditions hold:

1. $\Gamma$ is convex co-compact;
2. all circles in $P$ are mutually disjoint;

$\text{a discrete subgroup } \Gamma \triangleleft G \text{ is called convex co-compact if the convex core } C_\Gamma\text{ (that}$

is, the minimal convex set in $\Gamma \backslash \mathbb{B}$ containing all geodesics connecting any two points in
$\Lambda(\Gamma)$) is compact.
\[(3) \bigcup_{i \in I} C_i^o \subset \Omega(\Gamma) \text{ where } C_i^o \text{ denotes the open disk enclosed by } C_i \text{ and } \Omega(\Gamma) = S^2 - \Lambda(\Gamma) \text{ is the domain of discontinuity for } \Gamma.\]

Then for any Borel subset \(E \subset S^2\) whose boundary has zero Patterson-Sullivan measure,
\[
N_T(\mathcal{P}, E) \sim \frac{\text{skr}(\mathcal{P})}{\delta_T \cdot |m_{\text{BMS}}^\Gamma|} \cdot \nu_o(E) \cdot (2T)^{\delta_1} \quad \text{as } T \to \infty
\]
where \(o = (0, 0, 0), 0 < \text{skr}(\mathcal{P}) < \infty\) and \(0 < |m_{\text{BMS}}^\Gamma| < \infty\) is the total mass of the Bowen-Margulis-Sullivan measure associated to \(\{\nu_x\}\) (Def. 2.2) on \(T^1(\Gamma \setminus \mathbb{B})\).

**Remark 1.3.** The Patterson-Sullivan density is known to be atom-free, and hence the above theorem works for any Borel subset \(E\) intersecting \(\Lambda(\Gamma)\) in finitely many points. If \(\Gamma\) is Zariski dense in \(G\), then any proper real subvariety of \(S^2\) has zero Patterson-Sullivan density [6, Cor. 1.4] and hence Theorem 1.2 holds for any Borel subset of \(S^2\) whose boundary is contained in a countable union of real algebraic curves.

**Example 1.4.**

1. If \(X\) is a finite volume hyperbolic 3 manifold with totally geodesic boundary, its fundamental group \(\Gamma := \pi_1(X)\) is geometrically finite and \(X\) is homeomorphic to \(\Gamma \setminus \mathbb{B} \cup \Omega(\Gamma)\) [8]. The universal cover \(\tilde{X}\) developed in \(\mathbb{B}\) has geodesic boundary components which are Euclidean hemispheres normal to \(S^2\). Then \(\Omega(\Gamma)\) is the union of a countably many disjoint open disks corresponding to the geodesic boundary components of \(\tilde{X}\). The Ahlfors finiteness theorem [1] implies that the circle packing \(\mathcal{P}\) on \(S^2\) consisting of the geodesic boundary components of \(\tilde{X}\) is locally finite and has finitely many \(\Gamma\)-orbits. Hence provided \(\mathcal{P}\) is infinite, our theorem 1.2 applies to counting circles in \(\mathcal{P}\).

2. Starting with four mutually tangent circles on the sphere \(S^2\), one can inscribe into each of the curvilinear triangle a unique circle by an old theorem of Apollonius of Perga (c. BC 200). Continuing to inscribe the circles this way, one obtains an Apollonian circle packing on \(S^2\) (see Fig. 1). Apollonian circle packings are examples of circle packing obtained in the way described in (1) (cf. [5] and [10]). In the case when \(\pi_1(X)\) is convex co-compact, then no disks in \(\Omega(\Gamma)\) are tangent to each other and \(\Lambda(\Gamma)\) is known to be homeomorphic to a Sierpinski curve [4] (see Fig. [1]).

3. Take \(k \geq 1\) pairs of mutually disjoint closed disks \(\{(D_i, D_i') : 1 \leq i \leq k\}\) in \(S^2\) and choose \(\gamma_i \in G\) which maps \(D_i\) and \(D_i'\) and sends the interior of \(D_i\) to the exterior of \(D_i'\). The group, say, \(\Gamma_i\) generated by \(\{\gamma_i\}\) is called a Schottky group of genus \(k\) (cf. [11, Sec. 2.7]). The \(\Gamma\)-orbit of the disks nest down onto the limit set \(\Lambda(\Gamma)\) which is totally disconnected. If we set \(\mathcal{P} := \bigcup_{1 \leq i \leq k} \Gamma(C_i) \cup \Gamma(C_i')\) where \(C_i\) and \(C_i'\) are the boundaries of \(D_i\) and \(D_i'\) respectively, \(\mathcal{P}\) is locally finite, as the nesting disks will become smaller and smaller (cf. [13, 4.5]). The
common exterior of hemispheres above the initial disks \( D_i \) and \( D'_i \) is a fundamental domain for \( \Gamma \) in \( \mathbb{H} \) and hence \( \Gamma \) is geometrically finite. Since \( P \) consists of disjoint circles, Theorem 1.2 applies to counting circles in \( P \), called Schottky dance (see [13, Fig. 4.11]).

Since for \( o = (0,0,0) \), \( \sin \theta(C) = \frac{1}{\cosh d(\hat{C}, o)} \) for the convex hull \( \hat{C} \) of \( C \) (cf. [22] P.24), we deduce

\[
\text{Curv}_S(C) = \sinh d(\hat{C}, o).
\]

Hence Theorem 1.2 follows from the following:

**Theorem 1.5.** Keeping the same assumption as in Theorem 1.2, we have, for any \( o \in \mathbb{H} \),

\[
\#\{C \in P : C \cap E \neq \emptyset, \ d(\hat{C}, o) < t\} \sim \frac{\text{sk}_\Gamma(P)}{\delta \Gamma \cdot |m_{BMS}^{\Gamma}|} \cdot \nu_o(E) \cdot e^{\delta \Gamma \cdot t} \quad \text{as} \ t \to \infty.
\]

The main result in this paper was announced in [14] and an analogous problem of counting circles in a circle packing of the plane was studied in [9] and [15].

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2. Preliminaries and expansion of a hyperbolic surface

In this section, we set up notations as well as recall a result from [16] on the weighted equidistribution of expansions of a hyperbolic surface by the geodesic flow.

Denote by \( G \) the group of orientation preserving isometries of \( \mathbb{H} \) and fix a circle \( C_0 \subset S^2 \). Denote by \( \hat{C}_0 \subset \mathbb{H} \) the convex hull of \( C_0 \). Fix \( p_0 \in \hat{C}_0 \) and \( o \in \mathbb{H} \). As \( G \) acts transitively on \( \mathbb{H} \), there exists \( g_0 \in G \) such that

\[
o = g_0(p_0).
\]

Denote by \( K \) the stabilizer subgroup of \( p_0 \) in \( G \) and by \( H \) the stabilizer subgroup of \( \hat{C}_0 \) in \( G \). We note that \( H \) is locally isomorphic to \( \text{SO}(2,1) \) and has two connected components one of which is the orientation preserving isometries of \( \hat{C}_0 \). There exist commuting involutions \( \sigma \) and \( \theta \) of \( G \) such that the Lie subalgebras \( \mathfrak{h} = \text{Lie}(H) \) and \( \mathfrak{t} = \text{Lie}(K) \) are the +1 eigenspaces of \( d\sigma \) and \( d\theta \) respectively. With respect to the symmetric bilinear form on \( \mathfrak{g} = \text{Lie}(G) \) given by

\[
B_\theta(X,Y) = \text{Tr}(\text{ad}(X) \circ \text{ad}(\theta(Y))),
\]
we have the orthogonal decomposition
\[ g = \mathfrak{k} \oplus p = h \oplus q \]
where \( p \) and \( q \) are the \(-1\) eigenspaces of \( d\sigma \) and \( d\theta \) respectively. Let \( a \) be a one dimensional subalgebra of \( p \cap q \), \( A := \exp (a) \), and \( M \) the centralizer of \( A \) in \( K \). The map \( K \times p \to G \) given by \( (k, X) \mapsto k \exp X \) is a diffeomorphism and for the canonical projection \( \pi : G \to G/K = \mathbb{B} \), the differential \( d\pi : p \to T_{p_0}(G/K) = T_{p_0}(\mathbb{B}) \) is an isomorphism.

Choosing an element \( X_0 \in a \) of norm one, we can identify the unit tangent bundle \( T^1(\mathbb{B}) \) with \( G.X_0 = G/M \) where \( g.X_0 \) is given by \( d\lambda(g)(X_0) \) where \( \lambda(g) : G \to G \) is the left translation \( \lambda(g)(g') = gg' \) and \( d\lambda = \) its derivative at \( p_0 \).

Setting \( A^+ = \{ \exp (tX_0) : t \geq 0 \} \) and \( A^- = \{ \exp (tX_0) : t \leq 0 \} \), we have the following generalized Cartan decompositions (cf. [19]):
\[ G = K A^- K = H A^+ K. \]
in the sense that every element of \( g \in G \) can be written as \( g = k_1 a_s k_2 = ha_t k \), \( s \leq 0, \ t \geq 0, \ h \in H, k_1, k_2, k \in K \). Moreover, \( k_1 a_s k_2 = k_1'h^a_s k_2' \) implies \( s = s', \ k_1 = k_1'm_1, \) and \( k_2 = m_1^{-1}k_2' \) for some \( m_1 \in M \), and \( ha_t k = h'a_t' k' \) implies that \( t = t', \ h = h'm_2, \) and \( k = m_2^{-1}k' \) for some \( m_2 \in H \cap K \).

The set \( K.X_0 = K/M \) represents the set of unit tangent vectors at \( p_0 \), and as \( X_0 \) is orthogonal to \( h \cap p = T_{p_0}(C_0) \), \( H.X_0 = H/M \) corresponds to the set of unit normal vectors to the convex hull \( C_0 = H/H \cap K \), which will be denoted by \( C_0^0 \). Moreover if \( a_t = \exp(tX_0) \), the set \( (H/M)a_t = (Ha_tM)/M \) represents the orthogonal translate of \( C_0 \) by distance \( |t| \). We refer to [16] for the above discussion.

Let \( \Gamma < G \) be a non-elementary geometrically finite discrete subgroup of \( G \) in the rest of this section. Recall that \( g \in G \) is parabolic if and only if \( g \) has a unique fixed point in \( S^2 \).

Proposition 2.1 ([15]).
1. If \( \Gamma(C_0) \) is infinite, then \( [\Gamma : H \cap \Gamma] = \infty \).
2. \( \Gamma(C_0) \) is locally finite if and only if the natural projection map \( \Gamma \cap H' \backslash \hat{C}_0 \to \Gamma \backslash \mathbb{B} \) is proper.

3. Suppose one of the following conditions hold:
   a. \( \Gamma \) is convex co-compact;
   b. \( \gamma(C_0)'s \) are disjoint for \( \gamma \in \Gamma \);
   c. \( C_0^0 \subset \Omega(\Gamma) \) for the open disk \( C_0^0 \) enclosed by \( C_0 \).

   Then any parabolic element of \( \Gamma \) fixing a point on \( C_0 \) stabilizes \( C_0 \).

We denote by \( \{ \nu_x : x \in \mathbb{B} \} \) the Patterson-Sullivan density for \( \Gamma \), which is unique up to homothety: for any \( x, y \in \mathbb{B}, \ \xi \in S^2 \) and \( \gamma \in \Gamma, \)
\[ \gamma_* \nu_x = \nu_{\gamma x}; \quad \text{and} \quad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-d_\gamma \beta_\xi(y, x)}, \]
where \( \gamma_* \nu_x(R) = \nu_x(\gamma^{-1}(R)) \) and the Busemann function \( \beta_\xi(y_1, y_2) \) is given by \( \lim_{t \to \infty} d(y_1, \xi_t) - d(y_2, \xi_t) \) for a geodesic ray \( \xi_t \) toward \( \xi \).
For $u \in T^1(\mathbb{B})$, we define $u^+ \in S^2$ (resp. $u^- \in S^2$) the forward (resp. backward) endpoint of the geodesic determined by $u$ and $\pi(u) \in \mathbb{B}$ the basepoint. The map
\[ u \mapsto (u^+, u^-, \beta_u(\pi(u), o)) \]

yields a homeomorphism between $T^1(\mathbb{B})$ with $(S^2 \times S^2 - \{(\xi, \xi) : \xi \in S^2\}) \times \mathbb{R}$.

**Definition 2.2** (The Bowen-Margulis-Sullivan measure). The Bowen-Margulis-Sullivan measure $m_\Gamma^{BMS}$ ([2], [12], [21]) associated to $\nu_x$ is the measure on $T^1(\Gamma \backslash \mathbb{B})$ induced by the following $\Gamma$-invariant measure on $T^1(\mathbb{B})$: for $x \in \mathbb{B}$,
\[ dm^{BMS}(u) = e^{\delta \gamma \beta_u(x, \pi(u))} e^{\delta \gamma \beta_{u^-}(x, \pi(u))} d\nu_x(u^+) d\nu_x(u^-) dt. \]

It follows from the conformality of $\{\nu_x\}$ that this definition is independent of the choice of $x$. Sullivan showed that $m_\Gamma^{BMS}$ is ergodic for the geodesic flow and that the total mass $|m_\Gamma^{BMS}|$ is finite [21].

We denote by $\{m_x : x \in \mathbb{B}\}$ a $G$-invariant conformal density of $S^2$ of dimension 2, which is unique up to homothety. Each $m_x$ defines a measure on $S^2$ which is invariant under the maximal compact subgroup $\text{Stab}_G(x)$.

**Definition 2.3** (The Burger-Roblin measure). The Burger-Roblin measure $m_\Gamma^{BR}$ ([3], [13]) associated to $\nu_x$ and $m_x$ is the measure on $T^1(\Gamma \backslash \mathbb{B})$ induced by the following $\Gamma$-invariant measure on $T^1(\mathbb{B})$:
\[ dm^{BR}(u) = e^{2 \beta_u(x, \pi(u))} e^{\delta \gamma \beta_{u^-}(x, \pi(u))} d\nu_x(u^+) d\nu_x(u^-) dt \]

for $x \in \mathbb{B}$. By the conformal properties of $\{\nu_x\}$ and $\{m_x\}$, this definition is independent of the choice of $x \in \mathbb{B}$.

On $H/M = C_{\mathbb{B}}^1$, we consider the following two measures:
\begin{equation}
(2.4) \quad d\mu_{C_{\mathbb{B}}^1}^{\text{Leb}}(s) = e^{2 \beta_u(\pi(s))} d\nu_x(s) \quad \text{and} \quad d\mu_{C_{\mathbb{B}}^1}^{\text{PS}}(s) := e^{\delta \gamma \beta_u(\pi(s))} d\nu_x(s^+) \end{equation}

for $x \in \mathbb{B}$. These definitions are independent of the choice of $x$ and $\mu_{C_{\mathbb{B}}^1}^{\text{Leb}}$ (resp. $\mu_{C_{\mathbb{B}}^1}^{\text{PS}}$) is left-invariant by $H$ (resp. $H \cap \Gamma$). Hence we may consider the measures $\mu_{C_{\mathbb{B}}^1}^{\text{Leb}}$ and $\mu_{C_{\mathbb{B}}^1}^{\text{PS}}$ on the quotient $(H \cap \Gamma) \backslash C_{\mathbb{B}}^1$.

For the following theorem 2.5 and the proposition 2.6 we assume that the natural projection map $\Gamma \cap H \backslash C_0 \to \Gamma \backslash \mathbb{B}$ is proper and that any parabolic element of $\Gamma$ fixing a point on $C_0$ stabilizes $C_0$.

**Theorem 2.5.** [16] For $\psi \in C_c(\Gamma \backslash G/M)$, as $t \to \infty$,
\[ e^{(2 - \delta \gamma)t} \int_{s \in (\Gamma \cap H) \backslash C_{\mathbb{B}}^1} \psi(sa_1) d\mu_{C_{\mathbb{B}}^1}^{\text{Leb}}(s) \sim \frac{|\mu_{C_{\mathbb{B}}^1}^{\text{PS}}|}{|m_\Gamma^{BMS}|} m_{\Gamma}^{BR}(\psi) \]

where $|\mu_{C_{\mathbb{B}}^1}^{\text{PS}}| < \infty$. Moreover $|\mu_{C_{\mathbb{B}}^1}^{\text{PS}}| > 0$ if $[\Gamma : H \cap \Gamma] = \infty$. 

Proposition 2.6. \textsuperscript{[10]} For $\psi \in C_c(\Gamma \backslash G/M)$, there exists a compact subset $H_\psi \subset \Gamma \cap H \backslash H$ such that

$$\psi(ha_t) = 0$$

for all $h \notin H_\psi$ and all $t \in \mathbb{R}$.

Letting $dm$ the probability invariant measure on $M$ and writing $h = sm \in C_0^+ \times M$, $dh = d\mu_{C_0^+}(s)dm$ is a Haar measure on $H$, and the following defines a Haar measure on $G$: for $g = ha_t k \in HA^+K$,

$$dg = 4 \sinh 2t \cdot \cosh 2t \, dh \, dm_{p_0}(k)$$

where $dm_{p_0}(k) := dm_{p_0}((kX_0)^+)$.

We denote by $d\lambda$ the unique measure on $H \backslash G$ which is compatible with the choice of $dg$ and $dh$: for $\psi \in C_c(G)$,

$$\int_G \psi \, dg = \int_{[g] \in H \backslash G} \int_{h \in H} \psi(h[g]) \, dh \, d\lambda[g].$$

Hence for $[e] = H$, $d\lambda([e]a_t k) = 4 \sinh 2t \cdot \cosh 2t \, dt \, dm_{p_0}(k)$.

3. Density of the Burger-Roblin measure on $T_{p_0}^1(\mathbb{B})$

Fixing $p_0, o \in \mathbb{B}$, let $g_0 \in G$ be such that $g_0(p_0) = o$. Let $\Gamma < G$ be a non-elementary geometrically finite discrete subgroup of $G$. We use the same notation for $K = \text{Stab}_G(p_0), A, A^+, X_0, M, \ldots$ as in section 2.

Let $N$ denote the expanding horospherical subgroup of $G$ for $A^+$:

$$N = \{g \in G : a_t a_i^{-1} a_t^{-1} \to e \, \text{ as } t \to \infty\}.$$

The product map $A \times N \times K \to G$ is a diffeomorphism.

We fix a Borel subset $E \subset \mathbb{S}^2$ for the rest of this section.

Definition 3.1. Define a function $R_E$ on $G$ as follows: for $g = a_t nk \in ANK$,

$$R_E(g) := e^{-\delta t} \cdot \chi(g_0^{-1} E)_{p_0}(k^{-1})$$

where $(g_0^{-1} E)_{p_0} := \{u \in K : uX_0^{-1} \in g_0^{-1}(E)\}$ and $\chi(g_0^{-1} E)_{p_0}$ is its characteristic function.

Lemma 3.2. For any Borel subset $E \subset \mathbb{S}^2$,

$$\int_{k \in K/M} R_E(k^{-1} g_0) \, d\nu_{p_0}(kX_0^{-1}) = \nu_o(E).$$
Proof. Write $k^{-1}g_0 = a_{i\ell}n^k \in ANK$. Since $X_0^- = \lim_{s \to \infty} a_{-s}(p_0)$ and $\lim_{s \to \infty} a_{s+t}na_{-s-t} = e$, we obtain

$$\beta_{kX_0^-}(o, p_0) = \beta_{X_0^-}(k^{-1}o, p_0)$$

$$= \lim_{s \to \infty} d(k^{-1}g_0 p_0, a_{-s}p_0) - d(p_0, a_{-s}p_0)$$

$$= \lim_{s \to \infty} d(a_{t}np_0, a_{-s}p_0) - d(p_0, a_{-s}p_0)$$

$$= \lim_{s \to \infty} d((a_{s+t}na_{-s-t})a_{s+t}p_0, p_0) - d(p_0, a_{-s}p_0)$$

$$= \lim_{s \to \infty} ((s + t) - s) = t.$$

On the other hand, since $NA$ fixes $X_0^-$, $k_0^{-1}(X_0^-) = g_0^{-1}k(X_0^-)$, and hence $\chi_{(g_0^{-1}E)p_0}(k_0^{-1}) = \chi_{g_0^{-1}E}(k_0^{-1}X_0^-) = \chi_E(k(X_0^-))$.

So

$$\mathcal{M}_E(k^{-1}g_0) = e^{-\delta_E \beta_{kX_0^-}(o, p_0)} \chi_E(k(X_0^-)).$$

Therefore by the conformal property of $\{\nu_x\}$,

$$\int_{k \in K/M} \mathcal{M}_E(k^{-1}g_0) d\nu_{p_0}(kX_0^-) = \int_{\xi \in E} e^{-\delta_E \beta_{E}(o, p_0)} d\nu_{p_0}(\xi) = \nu_{p_0}(E).$$

Fixing a left-invariant metric on $G$, we denote by $U_\epsilon$ an $\epsilon$-ball around $e$, and for $S \subset G$, we set $S_\epsilon = S \cap U_\epsilon$.

Lemma 3.3. (cf. [16] Lem. 6.1) There exists $\ell \geq 1$ such that for any $a_{i\ell}nk \in ANK$ and small $\epsilon > 0$,

$$a_{i\ell}nk(g_0^{-1}U_\epsilon g_0) \subset A_\ell a_{i\ell}NK_\epsilon k.$$  

For each small $\epsilon > 0$, we choose a non-negative function $\psi^\epsilon \in C_c(G)$ supported inside $U_\epsilon$ and of integral $\int_G \psi^\epsilon dg$ one, and define $\Psi^\epsilon \in C_c(\Gamma \setminus G)$ by

$$\Psi^\epsilon(g) = \sum_{\gamma \in \Gamma} \psi^\epsilon(\gamma g).$$

Definition 3.5. Define a function $\Psi^\epsilon_E$ on $\Gamma \setminus G$ by

$$\Psi^\epsilon_E(g) = \int_{k^{-1} \in (g_0^{-1}E)p_0} \Psi^\epsilon(gk g_0^{-1}) dm_{p_0}(k).$$

For each $\epsilon > 0$, define

$$E^+ = g_0 U_\epsilon g_0^{-1}(E) \quad \text{and} \quad E^- = \cap_{u \in U_\epsilon} g_0 u g_0^{-1}(E).$$

By Lemma 3.3 it follows that there exists $c > 0$ such that for all $g \in U_\epsilon$ and $g_1 \in G$,

$$1 - c\epsilon \mathcal{M}_{E^+}(g_1 g_0) \leq \mathcal{M}_E(g_1 g g_0) \leq (1 + c\epsilon)\mathcal{M}_{E^+}(g_1 g_0).$$
Proposition 3.8. There exists \( c > 0 \) such that for all small \( \epsilon > 0 \),
\[
(1 - ce)\nu_o(E^-_\epsilon) \leq m^{\text{BR}}_\Gamma(\Psi^\epsilon_E) \leq (1 + ce)\nu_o(E^+_\epsilon).
\]

Proof. Using the decomposition \( G = ANK \), we have for \( g = a_tnk \),
\[
dg = dt \, dn \, dm_{p_0}(k)
\]
where \( dn \) is the Lebesgue measure on \( N \).

We use the following formula for \( \tilde{m}^{\text{BR}} \) (cf. [16]): for any \( \psi \in C_c(G)^M \),
\[
\tilde{m}^{\text{BR}}(\psi) = \int_K \int_A \int_N \psi(ka_tn)e^{-\delta \tau} \, dn \, dt \, dm_{p_0}(k)(X^-_0).
\]

For \( \psi^\epsilon_E(g) := \int_{k^{-1}} \psi^\epsilon(\nu g_0^{-1}) \, dm_{p_0}(k) \), we have
\[
m^{\text{BR}}_\Gamma(\Psi^\epsilon_E) = \tilde{m}^{\text{BR}}(\psi^\epsilon_E)
\]
\[
= \int_{g \in G} \int_{k^{-1} \in (g_0^{-1})} \psi^\epsilon(\nu g_0^{-1}) \, dm_{p_0}(k) \, d\tilde{m}^{\text{BR}}(g)
\]
\[
= \int_{KAN} \int_{k \in K} \psi^\epsilon(k_0a_tnk_0^{-1}) \, dm_{p_0}(k) \, e^{-\delta \tau} \, dn \, dt \, d\nu_{p_0}(k_0 X^-_0)
\]
\[
= \int_{K_0 \in K} \int_{AN} \psi^\epsilon(k_0a_tnk_0^{-1}) \, dm_{p_0}(k) \, e^{-\delta \tau} \, dn \, dt \, d\nu_{p_0}(k_0 X^-_0)
\]
\[
= \int_{k \in K} \int_{G} \psi^\epsilon(gk_0^{-1}) \, d\nu_{p_0}(k_0 X^-_0).
\]

Hence applying (3.7), the identity \( \int \psi^\epsilon \, dg = 1 \) and Lemma 3.2, we deduce that
\[
m^{\text{BR}}_\Gamma(\Psi^\epsilon_E) \leq (1 + c \epsilon) \int_{k \in K} \left( \int_{G} \psi^\epsilon(g) \, dg \right) \nu_{E^+_\epsilon}(k^{-1}g_0) \, d\nu_{p_0}(kX^-_0)
\]
\[
= (1 + c \epsilon) \int_{k \in K} \nu_{E^+_\epsilon}(k^{-1}g_0) \, d\nu_{p_0}(kX^-_0)
\]
\[
= (1 + c \epsilon) \nu_o(E^+_\epsilon).
\]

The other inequality follows similarly. \( \square \)

4. Simpler proof of Theorem 1.5 for the special case of \( E = S^2 \).

The result in this section is covered by the proof of Theorem 1.5 (for general \( E \)) given in section 6. However we present a separate proof for this special case as it is considerably simpler and uses a different interpretation of the counting function.

We may assume without loss of generality that \( \mathcal{P} = \Gamma(C_0) \). We use the notations from section 2.

Set
\[
\mathcal{N}_T(\mathcal{P}) = \# \{ C \in \mathcal{P} : d(\hat{C}, o) < t \}.
\]
Lemma 4.1. For $T > 1$,

$$
\mathcal{N}_T(\mathcal{P}) = \#[e] \Gamma \cap [e] A_1^- T g_0^{-1}
$$

where $[e] = H \in H \backslash G$ and $A_1^+ = \{ a_t : 0 \leq t \leq T \}$.

Proof. Note that $\mathcal{N}_T(\mathcal{P})$ is equal to the number of hyperbolic planes $\gamma(\hat{C}_0)$ such that $d(o, \gamma(\hat{C}_0)) < T$, or equivalently, $d(\gamma^{-1}(o), \hat{C}_0) < T$. Since $\{ x \in \mathbb{B} : d(x, \hat{C}_0) < T \} = HA_1^+(p_0)$, $\mathcal{N}_T(\mathcal{P})$ is equal to the number of $[\gamma] \in \Gamma/\text{Stab}_T(\hat{C}_0)$ such that $\gamma^{-1} g_0 p_0 \in H A_1^+ T p_0$, or alternatively, the number of $[\gamma] \in H \cap \Gamma \backslash \Gamma$ such that $\gamma g_0 \in H A_1^+ T$, which is equal to $\# [e] \Gamma g_0 \cap [e] A_1^+ T$. \qed

Define the following counting function $F_T$ on $\Gamma \backslash G$ by

$$
F_T(g) := \sum_{\gamma \in \Gamma \cap H \backslash \Gamma} \chi_{B_T}([e] \gamma g)
$$

where $B_T = [e] A_1^+ T g_0^{-1} \subset H \backslash G$. Note that $F_T(e) = \mathcal{N}_T(\mathcal{P})$.

By the strong wave front lemma (see [7]), for all small $\epsilon > 0$, there exists $\ell > 0$ and $t_0 > 0$ such that for all $t > t_0$,

$$
Ka_\ell k g_0^{-1} U_\epsilon \subset Ka_\ell A_\epsilon k K g_0^{-1}.
$$

It follows that for all $T \gg 1$,

$$(B_T - B_0) U_\epsilon \subset B_{T+\ell \epsilon} \quad \text{and} \quad B_{T-\ell \epsilon} - B_{t_0} \subset \cap u \in U_\epsilon B_T u.
$$

Hence there exists $m_0 \geq 1$ such that for all $g \in U_\epsilon$ and $T \gg 1$,

$$
F_{T-\ell \epsilon}(g) - m_0 \leq F_T(e) \leq F_{T+\ell \epsilon}(g) + m_0.
$$

Integrating against $\Psi^\epsilon$ (see (3.4)), we obtain

$$
\langle F_{T+\ell \epsilon}, \Psi^\epsilon \rangle - m_0 \leq F_T(e) \leq \langle F_{T+\ell \epsilon}, \Psi^\epsilon \rangle + m_0,
$$

where the inner product is taken with respect $dg$.

Setting $\Xi_t = 4 \sinh 2t \cdot \cosh 2t$, we have

$$
\langle F_{T+\ell \epsilon}, \Psi^\epsilon \rangle = \int_{g \in \Gamma \cap H \backslash G} \chi_{B_T}([e] g) \Psi^\epsilon (g g_0^{-1}) \, dg
$$

$$
= \int_{k \in K} \int_0^{T+\ell \epsilon} \int_{s \in \Gamma \cap H \backslash C_0} \left( \int_{m \in M} \Psi^\epsilon(s a_t m k g_0^{-1}) \, dm \right) \Xi_t \, d\mu_{C_0^1}(s) \, dt \, dm_{p_0}(k)
$$

$$
= \int_{k \in K} \int_0^{T+\ell \epsilon} \int_{s \in \Gamma \cap H \backslash C_0} \Psi^\epsilon g_0^{-1}(s a_t) \Xi_t \, d\mu_{C_0^1}(s) \, dt \, dm_{p_0}(k)
$$

where $\Psi^\epsilon_{g_1} \in C_c(\Gamma \backslash G)^M$ is given by $\Psi^\epsilon_{g_1}(g) = \int_{m \in M} \Psi^\epsilon(g m g_1) \, dm$.

Hence by Proposition 2.1 and Theorem 2.5 and using $\Xi_t \sim e^{2t}$, we deduce that as $T \to \infty$,

$$
\langle F_{T+\ell \epsilon}, \Psi^\epsilon \rangle \sim \frac{\mu_{C_0^1}^{PS}}{\delta_T} \cdot \left| \frac{m_{\Gamma}^{BR}(\Psi^\epsilon S_2)}{m_{\Gamma}^{BS} \cdot \mu_{C_0^1}^{BR} \Psi^\epsilon} \right| \cdot e^{\delta_T(T+\ell \epsilon)}
$$
where

$$\Psi_{S^2}^\varepsilon(g) = \int_{k \in K} \Psi^\varepsilon(gkg_0^{-1})dm_{p_0}(k).$$

By Prop. 3.8

$$m^{BR}_\Gamma(\Psi_{S^2}^\varepsilon) = (1 + O(\varepsilon))|\nu_0|.$$ Therefore it follows, as $$\varepsilon > 0$$ is arbitrary,

$$\limsup_T \frac{F_T(\varepsilon)}{e^{\delta_T T}} \leq \frac{|\nu_0| \cdot |\mu_{C_0}^{PS}|}{\delta_T \cdot |m^{BMS}_\Gamma|}.$$ Similarly

$$\liminf_T \frac{F_T(\varepsilon)}{e^{\delta_T T}} \geq \frac{|\nu_0| \cdot |\mu_{C_0}^{PS}|}{\delta_T \cdot |m^{BMS}_\Gamma|}.$$ This finishes the proof, as $$|\mu_{C_0}^{PS}| = \text{sk}_\Gamma(P).$$

5. Uniform distribution along $$b_T(W)$$

In this section, fix a Borel subset $$W \subset K$$ with $$MW = W$$.

**Definition 5.1.** For $$T > 1$$, we set

$$b_T(W) = H \setminus HKA_T^+ W \subset H \setminus G$$

where $$A_T^+ = \{a_t \in A : 0 \leq t \leq T\}.$$ We assume that the natural projection map $$\Gamma \cap H \setminus \hat{C}_0 \to \Gamma \setminus \mathbb{B}$$ is proper and that any parabolic element of $$\Gamma$$ fixing a point on $$C_0$$ stabilizes $$C_0$$.

**Theorem 5.2.** For any $$\psi \in C_c(\Gamma \setminus G)$$, we have

$$\int_{g \in b_T(W)} \int_{h \in \Gamma \cap H \setminus \hat{G}} \psi(hg)dh \sim \frac{\mu_{C_0}^{PS}}{\delta_T \cdot |m^{BMS}_\Gamma|} \int_{k \in W} m^{BR}_\Gamma(\psi_k) dm_{p_0}(k) \cdot e^{\delta_T T}$$

as $$T \to \infty$$, where $$\psi_k \in C_c(\Gamma \setminus G)^M$$ is given by $$\psi_k(g) = \int_{m \in M} \psi(gmk)dm.$$ 

**Proof.** (cf. [15, Thm 4.3])

Set $$K'_\varepsilon = \bigcup_{k \in K} kK'_\varepsilon$$ and define $$\psi_\varepsilon^+ \in C_c(\Gamma \setminus G)$$ by

$$\psi_\varepsilon^+(g) := \sup_{u \in K'_\varepsilon} \psi(gu) \text{ and } \psi_\varepsilon^-(g) := \inf_{u \in K'_\varepsilon} \psi(gu).$$

Note that for a given $$\eta > 0$$, there exists $$\varepsilon = \varepsilon(\eta) > 0$$ such that for all $$g \in \Gamma \setminus G$$, $$|\psi_\varepsilon^+(g) - \psi_\varepsilon^-(g)| \leq \eta$$ by the uniform continuity of $$\psi$$.

We can deduce from Theorem 2.5 that for all $$t > T_1(\eta) \gg 1,$$

$$\int_{h \in \Gamma \cap H \setminus \hat{G}} \psi_\varepsilon^+(ha_k)dh = (1 + O(\eta)) \frac{\mu_{C_0}^{PS}}{|m^{BMS}_\Gamma|} m^{BR}_\Gamma(\psi_\varepsilon^+, k) e^{(\delta - 2)t}$$

where $$\psi_\varepsilon^+, k$$ is defined similarly as $$\psi_k$$ and the implied constant can be taken uniformly over all $$k \in K$$. Defining

$$K_T(t) := \{k \in K : a_t k \in HKA_T^+\},$$
by Prop. 4.8 and Corollary 4.11 in [15], we have $H K A_T^+ = \cup_{0 \leq t \leq T} H a_t K_T(t)$ and there exists a sufficiently large $T_0(\epsilon) > T_1(\eta)$ such that $\epsilon \in K_T(t) \subset K_\epsilon M$ for all $T_0(\epsilon) < t < T$.

For $[e] = H \in H \setminus G$ and $s > 0$, set

$$V_T(s) := (\cup_{s \leq t \leq T} [e] a_t K_T(t)) \mathcal{W}$$

so that

$$b_T(\mathcal{W}) = V_T(s) \cup (b_T(\mathcal{W}) - V_T(s)).$$

Let $[g] = [e] a_k k_1 \in V_T(T_0(\epsilon))$ where $k_1 \in K$ and $k \in \mathcal{W}$. For $t > T_0(\epsilon)$, there exist $h_0 \in H$ and $u \in K_\epsilon^*$ such that $a_k k_1 k = h_0 a_t k u$ and hence

$$\psi^H(g) := \int_{h \in \Gamma \cap H^T} \psi(hg)dh = \int_{h \in \Gamma \cap H^T} \psi(hh_0 a_t k u)dh \leq \int_{h \in \Gamma \cap H^T} \psi^+(h a_t k)dh.$$

Therefore

$$\int_{V_T(T_0(\epsilon))} \psi^H(g)d\lambda(g) \leq \int_{k \in \mathcal{W}} \int_{T_0(\epsilon) < t < T} \int_{h \in \Gamma \cap H^T} \psi^+(h a_t k) \Xi_T dh dt dm_{p_0}(k)$$

where $\Xi_T = 4 \sinh 2t \cosh 2t$.

Using $\Xi_T \sim e^{2t}$, we then deduce

$$\int_{k \in \mathcal{W}} \int_{T_0(\epsilon) < t < T} \int_{h \in \Gamma \cap H^T} \psi^+(h a_t k) \Xi_T dh dt dm_{p_0}(k)$$

$$= (1 + O(\eta)) \frac{|\mu_{C_0}|}{\delta_T \cdot |m_{BMS}|} \int_{k \in \mathcal{W}} m_{T}^\text{BR}(\psi_k) dm_{p_0}(k) \cdot (e^{\delta_T} - e^{\delta_T T_0(\epsilon)})$$

since $m_{T}^\text{BR}(\psi_{e,k}) = (1 + O(\eta)) m_{T}^\text{BR}(\psi_k)$.

Hence

$$\limsup_T \int_{V_T(T_0(\epsilon))} \psi^H(g)d\lambda(g) \leq (1 + O(\eta)) \frac{|\mu_{C_0}|}{\delta_T \cdot |m_{BMS}|} \int_{k \in \mathcal{W}} m_{T}^\text{BR}(\psi_k) dm_{p_0}(k).$$

On the other hand, it follows from Proposition 2.6 that

$$\int_{[g] \in b_T(\mathcal{W}) - V_T(T_0(\epsilon))} \psi(hg)dh d\lambda(g) = O(1).$$

As $\eta > 0$ is arbitrary and $\epsilon(\eta) \to 0$ as $\eta \to 0$, it follows that

$$\liminf_T \int_{[g] \in b_T(\mathcal{W})} \psi^H(g)d\lambda(g) \geq \frac{|\mu_{C_0}|}{\delta_T \cdot |m_{BMS}|} \int_{k \in \mathcal{W}} m_{T}^\text{BR}(\psi_k) dm_{p_0}(k).$$

By a similar argument, one can prove

$$\limsup_T \int_{[g] \in b_T(\mathcal{W})} \psi^H(g)d\lambda(g) \leq \frac{|\mu_{C_0}|}{\delta_T \cdot |m_{BMS}|} \int_{k \in \mathcal{W}} m_{T}^\text{BR}(\psi_k) dm_{p_0}(k).$$

□
COUNTING VISIBLE CIRCLES ON THE SPHERE

6. Proof of Theorem 1.3

Without loss of generality, we may assume that \( \mathcal{P} = \Gamma(C_0) \). We keep the notations from section 2.

**Definition 6.1.** A subset \( E \subset \mathbb{S}^2 \) is said to be \( \mathcal{P} \)-admissible if, for any \( C \in \mathcal{P}, \ C^\circ \cap E \neq \emptyset \) implies \( C^\circ \subset E \), possibly except for finitely many circles.

For a subset \( E \subset \mathbb{S}^2 \), we define \( E_{p_0} \subset K \) by

\[
E_{p_0} := \{ k \in K : k(X_0^-) \in E \}. 
\]

We also set

\[
\mathcal{N}_T(\mathcal{P}, E) := \{ C \in \mathcal{P} : C \cap E \neq \emptyset, \ d(\hat{C}, o) < T \}. 
\]

**Lemma 6.2.** Fix a \( \mathcal{P} \)-admissible subset \( E \subset \mathbb{S}^2 \). For all \( T > 1 \),

\[
\mathcal{N}_T(\mathcal{P}, E) = \# \{ \alpha \} \Gamma \cap \{ \alpha \} KA_T^+(g_0^{-1}E)_{p_0}^{-1}g_0^{-1}
\]

up to a uniform finite additive constant where \( \{ \alpha \} = H \in H \setminus G \).

**Proof.** Since \( g_0K/M \) represents the set of all unit vectors based at \( o \), and the set \( \{ u \in T_o^1(\mathbb{B}) : u^- \in E \} \) is identified with \( g_0(g_0^{-1}E)_{p_0} = \{ g_0k[M] : kX_0^- \in g_0^{-1}E \} \), the set \( g_0(g_0^{-1}E)_{p_0}A^- (p_0) \) represents the set of all points in \( \mathbb{B} \) lying in the cone consisting of geodesic rays connecting \( o \) with a point in \( E \). Therefore the condition \( C \subset E \) is equivalent to that \( \hat{C} \subset g_0(g_0^{-1}E)_{p_0}A^- (p_0) \). Hence by the \( \mathcal{P} \)-admissibility condition, we may assume without loss of generality that \( \mathcal{N}_T(\mathcal{P}, E) \) is equal to the number of hyperbolic planes \( \gamma(\hat{C}_0) \) such that \( d(o, \gamma(\hat{C}_0)) < T \) and \( \gamma(\hat{C}_0) \subset g_0(g_0^{-1}E)_{p_0}A^- (p_0) \). Since \( \{ x \in \mathbb{B} : d(o, x) < T \} = g_0KA_T^-(p_0) \) where \( A_T^- = \{ a_t : 0 \leq t \leq T \} \), the former condition is again same as \( \gamma(\hat{C}_0) \cap g_0KA_T^-(p_0) \neq \emptyset \). Hence

\[
\mathcal{N}_T(\mathcal{P}, E) = \{ \gamma(C_0) : \gamma(\hat{C}_0) \cap g_0KA_T^-(p_0) \neq \emptyset, \ \gamma(\hat{C}_0) \subset g_0(g_0^{-1}E)_{p_0}A^- (p_0) \} 
\]

\[
= \{ [\gamma] \in \Gamma/T \cap H : \gamma \in g_0KA_T^-KH \cap g_0(g_0^{-1}E)_{p_0}A^-KH \} 
\]

\[
= \{ [\gamma] \in \Gamma/T \cap H : \gamma \in g_0(g_0^{-1}E)_{p_0}A^-KH \}.
\]

In the last equality, we have used the fact that if \( a_{-t} \in KA_T^-KH \) for some \( t > 0 \), then \( t < T \) (see [15] Lem 4.10).

By taking the inverse, we obtain that

\[
\mathcal{N}_T(\mathcal{P}, E) = [\{ \alpha \} \Gamma \cap [\{ \alpha \} KA_T^+(g_0^{-1}E)_{p_0}^{-1}g_0^{-1}].
\]

Fixing a Borel subset \( E \subset \mathbb{S}^2 \), recall the definition of \( E^\pm \) from (3.6):

\[
E^+ := g_0U_e\circ g_0^{-1}(E) \quad \text{and} \quad E^- := \cap_{u \in U_e} g_0ug_0^{-1}(E).
\]

We can find a \( \mathcal{P} \)-admissible Borel subset \( \tilde{E}^+_\varepsilon \) such that \( E \subset \tilde{E}^+_\varepsilon \subset E^+ \) by adding all the open disks inside \( E^+ \) intersecting the boundary of \( E \). Similarly we can find a \( \mathcal{P} \)-admissible Borel subset \( \tilde{E}^-_\varepsilon \) such that \( E^- \subset \tilde{E}^-_\varepsilon \subset E \) by
adding all the open disks inside $E$ intersecting the boundary of $E^-_. By the local finiteness of $\mathcal{P}$, there are only finitely many circles intersecting $\partial(E)$ (resp. $\tilde{E}^+_e$) which are not contained in $\tilde{E}^+_e$ (resp. $E$). Therefore there exists $q_e \geq 1$ (independent of $T$) such that

\begin{equation}
N_T(\mathcal{P}, \tilde{E}^+_e) - q_e \leq N_T(\mathcal{P}, E) \leq N_T(\mathcal{P}, \tilde{E}^+_e) + q_e.
\end{equation}

Setting

\[ B_T(E) := [e]KA^+_T(g_0^{-1}E)_{p_0}^{-1}g_0^{-1} \subset H \setminus G, \]
we define functions $F^\pm_T$ on $\Gamma \setminus G:

\[ F^\pm_T(g) := \sum_{\gamma \in \Gamma \setminus G} \chi_{B_T(\pm(e)_{(\ell+1)\epsilon})}([e]\gamma g). \]

**Lemma 6.4.** There exists $m_\epsilon \geq 1$ such that for all $g \in U_\epsilon$ and $T \gg 1$,

\[ F^\epsilon_+(g) - m_\epsilon \leq N_T(\mathcal{P}, E) \leq F^\epsilon_+(g) + m_\epsilon. \]

**Proof.** It follows from (4.2) that \[ B_T(E^+_e)U_\epsilon \subset B_{T+\epsilon}(E^+_e) \text{ and } B_T(E^-_e) \subset \cap_{u \in U_\epsilon} B_T(E^-_e)u. \]
Hence for any $g \in U_\epsilon$, as $U_\epsilon$ is symmetric,

\[ \# [e] \Gamma \cap B_T(\tilde{E}^+_e) \leq \# [e] \Gamma \cap B_T(\tilde{E}^-_e)U_\epsilon g^{-1} \leq \# [e] \Gamma g \cap B_{T+\epsilon}(E^+_e). \]

By Lemma 6.2 and (6.3), it follows that for some fixed $m_\epsilon \geq 1$,

\[ N_T(\mathcal{P}, E) \leq F^\epsilon_+(g) + m_\epsilon. \]

The other inequality can be proved similarly. \hfill \Box

Hence by integrating against $\Psi^\epsilon$ (see (3.4)), we obtain

\begin{equation}
\langle F^\epsilon_-, \Psi^\epsilon \rangle - m_\epsilon \leq N_T(\mathcal{P}, E) \leq \langle F^\epsilon_+, \Psi^\epsilon \rangle + m_\epsilon.
\end{equation}

We note that

\[ B_T(E) = b_T((g_0^{-1}E)_{p_0}^{-1}) g_0^{-1} \]
where $b_T(W)$ is defined as in Def. 5.1.

Since

\[ \langle F^\epsilon_+, \Psi^\epsilon \rangle = \int_{\Gamma \setminus H \setminus G} \chi_{B_T(\pm(e)_{(\ell+1)\epsilon})}([e]g) \Psi^\epsilon(g) \, dg \]

\[ = \int_{[e] \in B_T(\pm(e)_{(\ell+1)\epsilon})} \int_{\lambda \in \Gamma \setminus H} \Psi^\epsilon(hg) \, dhd\lambda(g) \]

\[ = \int_{[e] \in b_T(\pm(e)_{(\ell+1)\epsilon})_{p_0}^{-1}} \int_{\lambda \in \Gamma \setminus H} \Psi^\epsilon(hg_0^{-1}) \, dhd\lambda(g) \]
we deduce from Proposition 2.1 and Theorem 5.2 that

\begin{equation}
\langle F^\epsilon_+, \Psi^\epsilon \rangle \sim \frac{\text{skr}(C_0)}{\delta_T \cdot |m^\text{BMS}_{\Gamma}|} \cdot m^\text{BR}_{\Gamma}(\Psi^\epsilon_{E^+_e}) \cdot e^{\delta_T(T+\epsilon)}
\end{equation}
where $\Psi_E(g) = \int_{k^{-1} \in (g^{-1}E)_{p_0}} \Psi'(gk^{-1})dm_{p_0}(k)$ (see Def. 3.5).

Therefore by (6.5) and Prop. 3.8 we have

$$\limsup_T \frac{N_T(P, E)}{e^{\delta_T T}} \leq (1 + O(\epsilon)) \frac{\text{sk}_T(C_0)}{\delta_T \cdot |m_{\text{BMS}}(\Gamma)|} \cdot \nu_o(E_{(\ell+1)\epsilon}).$$

Since $\nu_o(\partial(E)) = 0$ by the assumption, $\nu_o(E_{(\ell+1)\epsilon} - E) \to 0$ as $\epsilon \to 0$. As $\epsilon$ can be taken arbitrarily small, it follows that

$$\limsup_T \frac{N_T(P, E)}{e^{\delta_T T}} \leq \frac{\text{sk}_T(C_0)}{\delta_T \cdot |m_{\text{BMS}}(\Gamma)|} \cdot \nu_o(E).$$

Similarly, we can prove

$$\liminf_T \frac{N_T(P, E)}{e^{\delta_T T}} \geq \frac{\text{sk}_T(C_0)}{\delta_T \cdot |m_{\text{BMS}}(\Gamma)|} \cdot \nu_o(E).$$

This completes the proof.

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