Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology

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Operations on cohomology

Theorem
Let $A$ be an algebra, $M, N \in A\text{Mod}$ and $d \geq 0$. Let $O = (O^p)_{p \geq 0}$ be a sequence of natural transformations of functors of $A$-modules

$$O^p : \text{Ext}^p_A(N, -) \to \text{Ext}^{p+d}_A(M, -).$$

Assume that, for each short exact sequence $P' \rightarrowtail P \twoheadrightarrow P''$, the following diagram commutes:

$$
\begin{array}{ccc}
\text{Ext}^p_A(N, P'') & \xrightarrow{\partial} & \text{Ext}^{p+1}_A(N, P') \\
\downarrow O^p & & \downarrow O^{p+1} \\
\text{Ext}^{p+d}_A(M, P'') & \xrightarrow{\partial} & \text{Ext}^{p+d+1}_A(M, P')
\end{array}
$$

Then there exists exactly one $Y(O) \in \text{Ext}^d_A(M, N)$ such that

$$O^p(-) = (-) \circ Y(O).$$
Operations on cohomology

Corollary

There is an isomorphism of bifunctors of A-modules

\[ Y : s\text{Op}_A^\bullet(−, −) \cong \text{Ext}_A^\bullet(−, −). \]

Corollary

Let A be an algebra and \( d \geq 0 \). Let \( \mathcal{O} = (\mathcal{O}^p)_{p \geq 0} \) be a sequence of natural transformations of functors of A-bimodules

\[ \mathcal{O}^p : H^p(A, −) \to H^{p+d}(A, −) \]

which commutes with boundary maps. Then there exists exactly one \( Y(\mathcal{O}) \in \text{HH}^d(A) \) such that

\[ \mathcal{O}^p(−) = (−) \circ Y(\mathcal{O}). \]
Corollary

Let $\phi : A \to B$ be a map of rings, and let $M \in A\text{Mod}$. For each $q \geq 1$ there is a unique class

$$\zeta^q \in \text{Ext}^2_B(\text{Tor}^A_{q-1}(B, M), \text{Tor}^A_q(B, M))$$

such that the differential

$$d^{p,q}_2 : \text{Ext}^p_B(\text{Tor}^A_q(B, M), -) \to \text{Ext}^{p+2}_B(\text{Tor}^A_{q-1}(B, M), -)$$

of the spectral sequence

$$E^{p,q}_2 = \text{Ext}^p_B(\text{Tor}^A_q(B, M), -) \Rightarrow \text{Ext}^\bullet_A(M, -)$$

is given on $\alpha \in \text{Ext}^p_B(\text{Tor}^A_q(B, M), -)$ by

$$d^{p,q}_2(\alpha) = \alpha \circ \zeta^q.$$
Change of rings

Theorem
Let $\phi : A \to B$ be an epimorphism of algebras. There exists a spectral sequence, functorial on $B$-bimodules,

$$E_{2}^{p,q} \cong \text{Ext}_{B^{e}}^{p}(\text{Tor}_{q}^{A}(B, B), -) \Rightarrow H^{\bullet}(A, -)$$

which has $E_{2}^{\bullet,0} \cong H^{\bullet}(B, -)$.

For each $q \geq 1$ there exists a unique class

$$\zeta^{q} \in \text{Ext}_{B^{e}}^{2}(\text{Tor}_{q-1}^{A}(B, B), \text{Tor}_{q}^{A}(B, B))$$

such that $d_{2}^{p,q}(-) = (-) \circ \zeta^{q}$.

If $\phi$ is surjective and $I = \ker \phi$, $\text{Tor}_{1}^{A}(B, B) \cong I/I^{2}$ and

$$\zeta^{1} \in \text{Ext}_{B^{e}}^{2}(B, I/I^{2}) = H^{2}(B, I/I^{2})$$

is the class of the infinitesimal extension

$$0 \longrightarrow I/I^{2} \longrightarrow A/I^{2} \longrightarrow B \longrightarrow 0$$
Monogenic algebras

**Theorem**

Let $k$ be a field and fix a monic $f = \sum_{i=0}^{N} a_i X^i \in k[X]$. Let $d = (f, f')$, pick $q \in k[X]$ such that $f = qd$, and put

$$u = q^2 \sum_{i=0}^{N} a_i \frac{i(i-1)}{2} X^{i-2} = q^2 \Delta_2(f).$$

Let $A = k[X]/(f)$. There is an isomorphism of graded commutative algebras

$$HH^\bullet(A) \cong \frac{k[x_0, \tau_1, \zeta_2]}{(f(x), d(x)\tau, f'(x)\zeta, \tau^2 - u(x)\zeta)}.$$
Monogenic algebras

**Proposition**

Let

\[ w = \sum_{i=0}^{N} \sum_{s,t \geq 0 \atop s+t+1=i} a_i (s+1)X^s q X^t. \]

The Gerstenhaber Lie structure on \( HH^\bullet(A) \) is such that

\[ [\tau, x] = q(x), \]
\[ [\zeta, \tau] = w(x)\zeta, \]
\[ [x, x] = [\tau, \tau] = [\tau, \zeta] = [x, \zeta] = 0. \]
Nice morphisms

**Theorem**

Let $\phi : A \to B$ be an epimorphism of algebras. The following statements are equivalent:

a) $\phi : A \to B$ is a homological epimorphism;

b) $\text{Tor}^A_+(B, M) = 0$ for all $M \in \mathcal{A}\text{Mod}$;

c) $\text{Tor}^A_+(B, B) = 0$;

d) $\phi^e : A^e \to B^e$ is a homological epimorphism.

When they hold, there is an isomorphism of functors of $B$-bimodules

$$H^\bullet(B, -) \xrightarrow{\cong} H^\bullet(A, -).$$
Corollary

Let $\phi : A \to B$ be a surjective homological epimorphism and let $I = \ker \phi$. There is a long exact sequence

$$\cdots \to \Ext^p_{\mathcal{A}e}(A, I) \to \HH^p(A) \to \HH^p(B) \to \Ext^{p+1}_{\mathcal{A}e}(A, I) \to \cdots$$
Nice ideals

Proposition

Let $\phi : A \to B$ be a surjective homological epimorphism such that $I = \ker \phi$ is $A$-flat on one side. Then $H^0(B, -) \cong H^0(A, -)$ on $\text{BMod}_B$ and there is a natural long exact sequence of functors of $B$-bimodules

\[ \cdots \to H^p(B, -) \to H^p(A, -) \to \]

\[ \to \text{Ext}^{p-1}_{Ae}(I/I^2, -) \xrightarrow{\zeta} H^{p+1}(B, -) \to \cdots \]

with $\zeta \in H^2(B, I/I^2)$ the class of the infinitesimal extension

\[ 0 \to I/I^2 \to A/I^2 \to B \to 0 \]
Nice ideals

Lemma
Let $\phi : A \rightarrow B$ be a surjective morphism of algebras and put $I = \ker \phi$. Then

$$\text{Tor}_q^A(B, B) \cong \begin{cases} 
B, & \text{if } q = 0; \\
I/I^2, & \text{if } q = 1; \\
\ker \left( I \otimes_A I \xrightarrow{\mu} I \right), & \text{if } q = 2; \\
\text{Tor}_{q-2}^A(I, I), & \text{if } q > 2.
\end{cases}$$
Nice ideals: an example

Let \( A = kQ/J \) be an admissible quotient of the path algebra on a quiver \( Q \) and let \( e \in Q_0 \).
Assume

- Every minimal relation in \( J \) involving a path passing through \( e \) also involves a path not passing through \( e \); and
- \( e \) is on no oriented cycle of \( Q \).

Then \( I = AeA \triangleleft A \) is homological and, if \( B = A/I \), there is a long exact sequence

\[
\cdots \to \text{Ext}^p_A(D(eA), Ae) \to \text{HH}^p(A) \to \text{HH}^p(B) \to \text{Ext}^{p+1}_A(D(eA), Ae) \to \cdots
\]
Nice morphisms
Nice morphisms
Nice morphisms

\[
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\]
Nice morphisms
