TWISTING OF SIEGEL PARAMODULAR FORMS

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Abstract. Let $S_k(\Gamma^\text{para}(N))$ be the space of Siegel paramodular forms of level $N$ and weight $k$. Let $p \nmid N$ and let $\chi$ be a nontrivial quadratic Dirichlet character mod $p$. Based on [JR2], we define a linear twisting map $T_\chi : S_k(\Gamma^\text{para}(N)) \to S_k(\Gamma^\text{para}(Np^4))$. We calculate an explicit expression for this twist and give the commutation relations of this map with the Hecke operators and Atkin-Lehner involution for primes $\ell \neq p$.

1. Introduction

Let $k$ and $N$ be positive integers and let $p$ be a prime with $p \nmid N$. Let $S_k(\Gamma_0(N))$ denote the space of elliptic modular cusp forms of weight $k$ with respect to $\Gamma_0(N) \subset \text{SL}(2,\mathbb{Z})$ and let $\chi$ be a nontrivial quadratic Dirichlet character mod $p$. There is a natural twisting map $T_\chi : S_k(\Gamma_0(N)) \to S_k(\Gamma_0(Np^2))$ such that if $f \in S_k(\Gamma_0(N))$ then

$$T_\chi(f) = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(u) f \left| \frac{u/p}{1} \right|^k.$$ (1)

Moreover, the Fourier expansion of the twist $T_\chi(f)$ is given by

$$T_\chi(f)(z) = \sum_{n=1}^{\infty} W(\chi(n)) a_n e^{2\pi inz},$$ (2)

where $W(\chi) = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(u) e^{2\pi i u/p}$ is the Gauss sum of $\chi$ and $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi inz}$ is the Fourier expansion of $f$. See, for example, [Sh] Proposition 3.64. A calculation verifies that for a prime $\ell \neq p$,

$$T(\ell) T_\chi = \chi(\ell) T_\chi T(\ell) \quad \text{and} \quad (T_\chi f)|_k W_\ell = \chi(\ell)^{\text{val}_\ell(N)} T_\chi(f|_k W_\ell),$$ (3)

where $f \in S_k(\Gamma_0(N))$, $T(\ell)$ is the Hecke operator, and $W_\ell$ is the Atkin-Lehner involution at $\ell$ as defined in Section 2.4 of [Cr], for example. By identifying $S_k(\Gamma_0(N))$ with automorphic forms on the adeles of $\text{GL}(2)$, it is evident that this twisting only acts on the automorphic form at the prime $p$. Hence the global twist is induced by a local twisting map for representations of $\text{GL}(2,\mathbb{Q}_p)$. In our previous work, [JR2], we constructed an analogous local twisting map for paramodular representations of $\text{GSp}(4,\mathbb{Q}_p)$ with trivial central character.

In this paper, we investigate the twisting map, $T_\chi$, on the space of Siegel paramodular forms induced by the local map from [JR2]. Our main result, Theorem 3.1, proves a formula for $T_\chi$ similar to, but more involved than, the formula (1) and proves commutation relations for the paramodular Hecke operators and Atkin-Lehner involution analogous to (3). Moreover, it follows from Theorem 1.2 of [JR2] that the map $T_\chi$ is not zero in general. Hence, our theorem may provide a source of examples to study conjectures such as the paramodular conjecture [BK] and the paramodular Böcherer’s conjecture [RT]. Finally, the formula (5) in Theorem 3.1 enables the computation of...
the Fourier coefficients of the twisted paramodular form $T_\chi(F)$, resulting in a formula analogous to (2). This will appear in a subsequent paper.

The paper is organized as follows. Section 2 introduces some necessary definitions and notation. In Section 3, we present and prove the main theorem. However, the proof relies on lengthy local calculations which are presented in Section 4 and the appendix [JR-a]. Note that in Section 4, GSp(4) is defined with respect to the symplectic form used in [RS] rather than the one given in Section 2. Finally in Section 5, the local results from Section 4 are translated to the setting of Section 3.

2. Notation

Let $M$ be a positive integer and let $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. We let $A$ denote the adeles of $\mathbb{Q}$ and define an associated Hecke character $\mathbb{Q}^\times \backslash A^\times \to \mathbb{C}^\times$, denoted by $\chi$, as follows. Recall that $A^\times = \mathbb{Q}^\times \mathbb{R}_{>0}^{\prod_{\ell<\infty} \mathbb{Z}_\ell^\times}$, with the groups embedded in $A^\times$ in the usual ways. In fact, the map defined by $(q,r,n) \mapsto qrn$ defines an isomorphism of topological groups

$$\mathbb{Q}^\times \times \mathbb{R}_{>0}^{\prod_{\ell<\infty} \mathbb{Z}_\ell^\times} \xrightarrow{\sim} A^\times.$$

Let $M = \ell_1^{\text{val}_1(M)} \ldots \ell_t^{\text{val}_t(M)}$ be the prime factorization of $M$. We consider the composition

$$Z_{\ell_1}^\times \times \cdots \times Z_{\ell_t}^\times \to Z_{\ell_1}^\times/(1 + \ell_1^{\text{val}_1(M)}Z_{\ell_1}) \times \cdots \times Z_{\ell_t}^\times/(1 + \ell_t^{\text{val}_t(M)}Z_{\ell_t}) \xrightarrow{\sim} (Z/\ell_1^{\text{val}_1(M)}Z) \times \cdots \times (Z/\ell_t^{\text{val}_t(M)}Z) \xrightarrow{\sim} (Z/M\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

We denote the restriction of this composition to $Z_{\ell_i}^\times$ by $\chi_{\ell_i}$. If $p \nmid M$, then we define $\chi_{\ell_i} : Z_{\ell_i}^\times \to \mathbb{C}^\times$ to be the trivial character. For each finite prime $p$, $\chi_\ell$ is a continuous character of $Z_{\ell_i}^\times$, and $\chi_\ell(1 + \ell^{\text{val}(M)}Z) = 1$. We define the corresponding Hecke character $\chi : \mathbb{Q}^\times \backslash A^\times \to \mathbb{C}^\times$ as the composition

$$A^\times \xrightarrow{\sim} \mathbb{Q}^\times \times \mathbb{R}_{>0}^{\prod_{\ell<\infty} \mathbb{Z}_\ell^\times} \xrightarrow{\text{proj}} \prod_{\ell<\infty} \mathbb{Z}_\ell^\times \xrightarrow{\prod \chi_\ell} \mathbb{C}^\times. \quad (4)$$

We see that if $a \in \mathbb{Z}$ with $(a, M) = 1$, then

$$\chi(a) = \chi_{\ell_1}(a) \cdots \chi_{\ell_t}(a).$$

Let

$$J = \begin{bmatrix} -1_2 & 1_2 \end{bmatrix}.$$

We define the algebraic $\mathbb{Q}$-group $\text{GSp}(4)$ as the set of all $g \in \text{GL}(4)$ such that $^t g J g = \lambda(g) J$ for some $\lambda(g) \in \text{GL}(1)$ called the multiplier of $g$. Let $\text{GSp}(4, \mathbb{R})^+$ be the subgroup of $g \in \text{GSp}(4, \mathbb{R})$ such that $\lambda(g) > 0$. The kernel of $\lambda : \text{GSp}(4) \to \text{GL}(1)$ is the symplectic group $\text{Sp}(4)$. Let $N$ and $k$ be positive integers. We define the paramodular group of level $N$ to be

$$\Gamma_{\text{para}}(N) = \text{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} Z & Z & N^{-1}Z & Z \\ NZ & Z & Z & Z \\ NZ & NZ & Z & NZ \\ NZ & Z & Z & Z \end{bmatrix}.$$

We also define local paramodular groups. For $\ell$ a prime of $\mathbb{Q}$ and $r$ an non-negative integer, let $K_{\text{para}}(\ell^r)$ be the paramodular subgroup of $\text{GSp}(4, \mathbb{Q}_\ell)$ of level $\ell^r$, i.e., the subgroup of elements...
$k \in \operatorname{GSp}(4, \mathbb{Q}_\ell)$ such that $\lambda(k) \in \mathbb{Z}_\ell^\times$ and
\[
k \in \begin{bmatrix}
Z_\ell & Z_\ell & \ell^{-1}Z_\ell & Z_\ell \\
\ell'Z_\ell & Z_\ell & Z_\ell & Z_\ell \\
\ell'Z_\ell & \ell'Z_\ell & Z_\ell & \ell'Z_\ell \\
\ell'Z_\ell & Z_\ell & Z_\ell & Z_\ell
\end{bmatrix}.
\]

Note that
\[
\Gamma_{\text{para}}(N) = \operatorname{GSp}(4, \mathbb{Q}) \cap \operatorname{GSp}(4, \mathbb{R})^+ \prod_{\ell < \infty} K_{\text{para}}(\mathfrak{p}_{\ell}(N)),
\]
with intersection in $\operatorname{GSp}(4, \mathbb{A})$.

For $n$ a positive integer, let $\mathcal{H}_n$ denote the Siegel upper half plane of degree $n$ with $I = i1_{2n}$. The group $\operatorname{GSp}(4, \mathbb{R})^+$ acts on $\mathcal{H}_2$ via
\[
h \langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad h = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad Z \in \mathcal{H}_2.
\]

Denote the factor of automorphy by $j(h, Z) = \det(CZ + D)$. If $F : \mathcal{H}_2 \to \mathbb{C}$ is a function and $h \in \operatorname{GSp}(4, \mathbb{R})^+$, then we define $F|_{k}h : \mathcal{H}_2 \to \mathbb{C}$ by
\[
(F|_{k}h)(Z) = \lambda(h)^k j(h, Z)^{-k} F(h(Z)), \quad Z \in \mathcal{H}_2.
\]

Let $\Gamma \subset \operatorname{GSp}(4, \mathbb{Q})$ be a group commensurable with $\operatorname{Sp}(4, \mathbb{Z})$. We define $S_k(\Gamma)$ to be the complex vector space of functions $F : \mathcal{H}_2 \to \mathbb{C}$ such that
\begin{enumerate}
  \item $F$ is holomorphic;
  \item $F|_{k}\gamma = F$ for all $\gamma \in \Gamma$;
  \item $\lim_{t \to \infty} (F|_{k}\gamma)(\begin{bmatrix} it \\ \z \end{bmatrix}) = 0$ for all $\gamma \in \operatorname{Sp}(4, \mathbb{Z})$ and $z \in \mathcal{H}_1$.
\end{enumerate}

For further background see, for example, [PY].

3. A Twisting Operator on Paramodular Cusp Forms

3.1. Statement of the main theorem. Let $Q$ be a $2 \times 2$ symmetric matrix, and let $P$ be an invertible $2 \times 2$ matrix. Then the matrices
\[
U(Q) = \begin{bmatrix} 1 & Q \\ & 1 \end{bmatrix} \quad \text{and} \quad A(P) = \begin{bmatrix} P & tP^{-1} \end{bmatrix}.
\]

are in $\operatorname{GSp}(4)$. In the following theorem, we extend the slash operator $|_{k}$ to formal $\mathbb{C}$-linear combinations of elements of $\operatorname{GSp}(4, \mathbb{R})^+$.

Theorem 3.1. Let $N$ and $k$ be positive integers, $p$ a prime with $p \nmid N$, and $\chi$ a nontrivial quadratic Dirichlet character mod $p$. Then, the local twisting map from Theorem 1.2 of [JR2] induces a linear map
\[
T_{\chi} : S_k(\Gamma_{\text{para}}(N)) \to S_k(\Gamma_{\text{para}}(Np^4)).
\]

If $F \in S_k(\Gamma_{\text{para}}(N))$, then this map is given by the formula
\[
T_{\chi}(F) = \sum_{i=1}^{14} F|_{k}T^i_{\chi}, \quad (5)
\]
where

\[
\begin{align*}
T^1_x &= p^{-11} \sum_{a,b,x \in \mathbb{Z}/p^3\mathbb{Z} \times z \in \mathbb{Z}/p^4\mathbb{Z}} \chi(ab)U\left(\begin{bmatrix} zp^{-4} & -bp^{-2} \\ -bp^{-2} & -x^{-1}p^{-1} \end{bmatrix}\right)A\left(\begin{bmatrix} 1 \\ (a + xb)p^{-1} \end{bmatrix}\right), \\
T^2_x &= p^{-11} \sum_{a,b,c \in \mathbb{Z}/p^2\mathbb{Z} \times x,y \in \mathbb{Z}/p^3\mathbb{Z}} \chi(abxy)U\left(\begin{bmatrix} -ab(1 - (1 - y)^{-1}x)p^{-3} & -ap^{-2} \\ -ap^{-2} & -a^{-1}(1 - x)^{-1}p^{-1} \end{bmatrix}\right)A\left(\begin{bmatrix} p &bp^{-1} \end{bmatrix}\right), \\
T^3_x &= p^{-6} \sum_{a \in \mathbb{Z}/p^3\mathbb{Z} \times b \in \mathbb{Z}/p^3\mathbb{Z} \times x \in \mathbb{Z}/p^4\mathbb{Z} \times z \in \mathbb{Z}/p^3\mathbb{Z} \times z \neq 1(p) \mathbb{Z}} \chi(b(1 - z))U\left(\begin{bmatrix} -bp^{-3} & ap^{-2} \\ ap^{-2} & -a^{-2}bp^{-2} \end{bmatrix}\right)A\left(\begin{bmatrix} p \\ xp^{-2} \end{bmatrix}\right), \\
T^4_x &= p^{-10} \sum_{a \in \mathbb{Z}/p^4\mathbb{Z} \times b \in \mathbb{Z}/p^3\mathbb{Z} \times x \in \mathbb{Z}/p^3\mathbb{Z}} \chi(b)U\left(\begin{bmatrix} (ax - bp)p^{-4} & ap^{-2} \\ ap^{-2} & 1 \end{bmatrix}\right)A\left(\begin{bmatrix} p \\ xp^{-2} \end{bmatrix}\right), \\
T^5_x &= p^{-9} \sum_{a,b,c \in \mathbb{Z}/p^3\mathbb{Z} \times x \in \mathbb{Z}/p^4\mathbb{Z}} \chi(b)U\left(\begin{bmatrix} (ax - b)p^{-3} & ap^{-2} \\ ap^{-2} & 1 \end{bmatrix}\right)A\left(\begin{bmatrix} p \\ xp^{-1} \end{bmatrix}\right), \\
T^6_x &= p^{-6} \sum_{a,b \in \mathbb{Z}/p^3\mathbb{Z} \times x \in \mathbb{Z}/p^4\mathbb{Z}} \chi(bx)U\left(\begin{bmatrix} (b + xp)p^{-2} & ap^{-2} \\ ap^{-2} & 1 \end{bmatrix}\right)A\left(\begin{bmatrix} p^2 \\ 1 \end{bmatrix}\right), \\
T^7_x &= p^{-7} \sum_{a \in \mathbb{Z}/p^3\mathbb{Z} \times b \in \mathbb{Z}/p^3\mathbb{Z} \times z \in \mathbb{Z}/p^4\mathbb{Z}} \chi(ab)U\left(\begin{bmatrix} zp^{-4} & bp^{-1} \\ bp^{-1} & 1 \end{bmatrix}\right)A\left(\begin{bmatrix} 1 \\ -ap^{-1} \end{bmatrix}\right), \\
T^8_x &= p^{-9} \sum_{a,b,z \in \mathbb{Z}/p^3\mathbb{Z} \times z \neq 1(p) \mathbb{Z}} \chi(abz(1 - z))U\left(\begin{bmatrix} ab(1 - z)p^{-3} & ap^{-1} \\ ap^{-1} & 1 \end{bmatrix}\right)A\left(\begin{bmatrix} p \\ bp^{-1} \end{bmatrix}\right), \\
T^9_x &= p^{-6} \sum_{a \in \mathbb{Z}/p^3\mathbb{Z} \times b \in \mathbb{Z}/p^2\mathbb{Z} \times x \in \mathbb{Z}/p^2\mathbb{Z}} \chi(b)U\left(\begin{bmatrix} bp^{-1} \\ xp^{-1} \end{bmatrix}\right)A\left(\begin{bmatrix} p^2 \\ a \end{bmatrix}\right), \\
T^{10}_x &= p^{-6} \sum_{a \in \mathbb{Z}/p^3\mathbb{Z} \times b \in \mathbb{Z}/p^2\mathbb{Z} \times x \in \mathbb{Z}/p^2\mathbb{Z}} \chi(b)U\left(\begin{bmatrix} bp^{-1} \\ a \end{bmatrix}\right)A\left(\begin{bmatrix} p^2 \\ a \end{bmatrix}\right), \\
T^{11}_x &= p^{-10} \sum_{a \in \mathbb{Z}/p^3\mathbb{Z} \times b \in \mathbb{Z}/p^3\mathbb{Z} \times x \in \mathbb{Z}/p^2\mathbb{Z} \times z \in \mathbb{Z}/p^2\mathbb{Z}} \chi(ab)U\left(\begin{bmatrix} zp^{-4} & (ap + xb)p^{-3} \\ (ap + xb)p^{-3} & xp^{-2} \end{bmatrix}\right)A\left(\begin{bmatrix} 1 \\ bp^{-2} \end{bmatrix}\right),
\end{align*}
\]
Moreover, for every prime \( \ell \neq p \) we have the following commutation relations for the Hecke operators (9) and Atkin-Lehner operator (12):

\[
T(1,1,\ell,\ell)\mathcal{T}_\chi = \mathcal{T}_\chi T(1,1,\ell,\ell),
\]

\[
T(1,\ell,\ell,\ell^2)\mathcal{T}_\chi = \mathcal{T}_\chi T(1,\ell,\ell,\ell^2),
\]

\[
\mathcal{T}_\chi(F)_{|k} U_\ell = \chi(\ell)\text{val}_\ell(N) \mathcal{T}_\chi(F_{|k} U_\ell),
\]

for \( F \in S_k(\Gamma^{para}(N)) \).

### 3.2. Twisted automorphic forms.

In order to present the proof of Theorem 3.1 we must explain the connection between Siegel modular forms and automorphic forms on \( \text{GSp}(4, \mathbb{A}) \). Let \( k \) be a positive integer, and let \( \mathcal{F} = \{ K_\ell \}_\ell \), where \( \ell \) runs over the finite primes of \( \mathbb{Q} \), be a family of compact, open subgroups of \( \text{GSp}(4, \mathbb{Q}_\ell) \) such that \( K_\ell = \text{GSp}(4, \mathbb{Z}_\ell) \) for almost all \( \ell \) and \( \lambda(K_\ell) = \mathbb{Z}_\ell^\times \) for all \( \ell \). To \( \mathcal{F} \) and \( k \) we will associate a space of automorphic forms on \( \text{GSp}(4, \mathbb{A}) \) and a space of Siegel modular forms of degree two. Set

\[
K_\mathcal{F} = \prod_{\ell < \infty} K_\ell \subset \text{GSp}(4, \mathbb{A}_f).
\]

Since \( \lambda(K_\ell) = \mathbb{Z}_\ell^\times \) for all finite \( \ell \), strong approximation for \( \text{Sp}(4) \) implies that

\[
\text{GSp}(4, \mathbb{A}) = \text{GSp}(4, \mathbb{Q})\text{GSp}(4, \mathbb{R})^+ K_\mathcal{F}.
\]

Let \( \chi : \mathbb{A}^\times \to \mathbb{C}^\times \) be a quadratic Hecke character, and let \( \chi = \prod_{\ell \leq \infty} \chi_\ell \) be the decomposition of \( \chi \) as a product of local characters. Then \( \chi_\infty(\mathbb{R}^\times_{>0}) = 1 \). Also let

\[
K_\infty = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in \text{GL}(4, \mathbb{R}) : {}^tAA + {}^tBB = 1, {}^tAB = {}^tBA \right\}.
\]

We define \( S_k(K_\mathcal{F}, \chi) \) to be the space of continuous functions \( \Phi : \text{GSp}(4, \mathbb{A}) \to \mathbb{C} \) such that

1. \( \Phi(\rho g) = \Phi(g) \) for all \( \rho \in \text{GSp}(4, \mathbb{Q}) \) and \( g \in \text{GSp}(4, \mathbb{A}) \);
2. \( \Phi(gz) = \Phi(g) \) for all \( z \in \mathbb{A}^\times \) and \( g \in \text{GSp}(4, \mathbb{A}) \);
3. \( \Phi(gk_\ell) = \chi_\ell(\lambda(\kappa_\ell))\Phi(g) \) for all \( \kappa_\ell \in K_\ell, g \in \text{GSp}(4, \mathbb{A}) \) and finite primes \( \ell \) of \( \mathbb{Q} \);
4. \( \Phi(gk_\infty) = j(k_\infty, I)^{-k}\Phi(g) \) for all \( k_\infty \in K_\infty \) and \( g \in \text{GSp}(4, \mathbb{A}) \);
5. For any proper parabolic subgroup \( P \) of \( \text{GSp}(4) \)

\[
\int_{N_P(\mathbb{Q}) \backslash N_P(\mathbb{A})} \Phi(n g) dn = 0
\]
for all \( g \in \text{GSp}(4, \mathbb{A}) \); here \( N_F \) is the unipotent radical of \( P \);

(6) For any \( g_f \in \text{GSp}(4, \mathbb{A}) \), the function \( \text{GSp}(4, \mathbb{R})^+ \to \mathbb{C} \) defined by \( g_\infty \mapsto \Phi(g_f g_\infty) \) is smooth and is annihilated by \( p_{\mathbb{C}} \), where we refer to Section 3.5 of [AS] for the definition of \( p_{\mathbb{C}} \).

On the other hand, to \( \mathcal{F} \) we associate a subgroup of \( \text{Sp}(4, \mathbb{Q}) \) that is commensurable with \( \text{GSp}(4, \mathbb{Z}) \),

\[
\Gamma_\mathcal{F} = \text{GSp}(4, \mathbb{Q}) \cap \text{GSp}(4, \mathbb{R})^+ \prod_{\ell < \infty} K_\ell.
\]

We define \( S_k(\Gamma_\mathcal{F}) \) to be the complex vector space of functions \( F : \mathfrak{H}_2 \to \mathbb{C} \) such that

1. \( F \) is holomorphic;
2. \( F|_{k}\gamma = F \) for all \( \gamma \in \Gamma_\mathcal{F} \);
3. \( \lim_{t \to \infty}(F|_{k}\gamma)\left(\begin{array}{c}it \\ z \end{array}\right) = 0 \) for all \( \gamma \in \text{Sp}(4, \mathbb{Z}) \) and \( z \in \mathfrak{H}_1 \).

**Lemma 3.2.** Let \( \chi, \mathcal{F} \) and \( k \) be as above. For \( F \in S_k(\Gamma_\mathcal{F}) \), define \( \Phi_F : \text{GSp}(4, \mathbb{A}) \to \mathbb{C} \) by

\[
\Phi_F(\rho h \kappa) = (F|_{k} h)(I) = \lambda(h)^{k} j(h, I)^{-k} \cdot \prod_{\ell < \infty} \chi_\ell(\lambda(\kappa_\ell)) \cdot F(h(I))
\]

for \( \rho \in \text{GSp}(4, \mathbb{Q}), h \in \text{GSp}(4, \mathbb{R})^+, \kappa \in K_\mathcal{F} = \prod_{\ell < \infty} K_\ell \). Then \( \Phi_F \) is a well-defined element of \( S_k(K_\mathcal{F}, \chi) \), so that there is a complex linear map

\[
S_k(\Gamma_\mathcal{F}) \to S_k(K_\mathcal{F}, \chi).
\]

Conversely, let \( \Phi \in S_k(K_\mathcal{F}, \chi) \). Define \( F_\Phi : \mathfrak{H}_2 \to \mathbb{C} \) by

\[
F_\Phi(Z) = \lambda(h)^{-k} j(h, I)^k \Phi(h_\infty)
\]

for \( Z \in \mathfrak{H}_2 \) with \( h \in \text{GSp}(4, \mathbb{R})^+ \) with \( h(I) = Z \). Then \( F_\Phi \) is well-defined and contained in \( S_k(\Gamma_\mathcal{F}) \), so that there is complex linear map

\[
S_k(K_\mathcal{F}, \chi) \to S_k(\Gamma_\mathcal{F}).
\]

Moreover, the maps (6) and (7) are inverses of each other.

**Proof.** To prove that \( \Phi_F \) is well defined, suppose that \( \rho \kappa = \rho' \kappa' \) for \( \rho, \rho' \in \text{GSp}(4, \mathbb{Q}), h, h' \in \text{GSp}(4, \mathbb{R})^+ \), and \( \kappa, \kappa' \in \prod_{\ell < \infty} K_\ell \). Comparing components, we have that \( \rho h = \rho' h' \in \text{GSp}(4, \mathbb{R})^+ \) and \( \rho \kappa_\ell = \rho' \kappa'_\ell \in \text{GSp}(4, \mathbb{Q}_\ell) \) for \( \ell < \infty \). Therefore \( \rho_0 \rho'^{-1} \rho \in \Gamma_\mathcal{F} \) and we have that

\[
\begin{align*}
\lambda(h')^k j(h', I)^{-k} \prod_{\ell < \infty} \chi_\ell(\lambda(\kappa'_\ell)) F(h'(I)) \\
\quad = \lambda(\rho_0 h)^k j(\rho_0 h, I)^{-k} \prod_{\ell < \infty} \chi_\ell(\lambda(\rho_0 \kappa_\ell)) F((\rho_0 h)(I)) \\
\quad = \lambda(\rho_0)^k \lambda(h)^k j(\rho_0, h(I))^{-k} j(h, I)^{-k} \prod_{\ell < \infty} \chi_\ell(\lambda(\kappa_\ell)) F(\rho_0(h(I))) \\
\quad = \lambda(h)^k j(h, I)^{-k} \prod_{\ell < \infty} \chi_\ell(\lambda(\kappa_\ell)) F(h(I)) \\
\quad = \lambda(h)^k j(h, I)^{-k} \prod_{\ell < \infty} \chi_\ell(\lambda(\kappa_\ell)) F(h(I)).
\end{align*}
\]

This shows that \( \Phi_F \) is well-defined. Straightforward calculations show that the function also satisfies first four conditions in the definition of \( S_k(K_\mathcal{F}, \chi) \). The proofs of the fifth and sixth conditions are
similar to the proofs of Lemma 5 and Lemma 7, respectively, in [AS]. Similar calculations show that \( F_\Phi \in S_k(\Gamma_F) \), for \( \Phi \in S_k(K_F, \chi) \). Finally, it is clear that these two maps are inverses of each other.

**Lemma 3.3.** Let \( k \) be a positive integer and let \( \mathcal{F}_1 = \{ K^1_\ell \} \) and \( \mathcal{F}_2 = \{ K^2_\ell \} \), where \( \ell \) runs over the finite primes of \( \mathbb{Q} \), be families of compact, open subgroups of \( \text{GSp}(4, \mathbb{Q}_\ell) \) such that \( K^1_\ell = K^2_\ell = \text{GSp}(4, \mathbb{Z}_\ell) \) for almost all \( \ell \) and \( \lambda(K^1_\ell) = \lambda(K^2_\ell) = \mathbb{Z}_\ell^x \) for all \( \ell \). Let \( \chi_1 \) and \( \chi_2 \) be quadratic Hecke characters. Suppose that there is a linear map

\[
T : S_k(K_{\mathcal{F}_1}, \chi_1) \rightarrow S_k(K_{\mathcal{F}_2}, \chi_2)
\]

given by a right translation formula at the \( p \)-th place,

\[
T(\Phi_1) = \sum_{i=1}^t c_i R(B_{i,p}) \Phi_1,
\]

for \( \Phi_1 \in S_k(K_{\mathcal{F}_1}, \chi_1) \). Here \( c_i \in \mathbb{C}^x \) and \( B_i \in \text{Sp}(4, \mathbb{Q})^+ \) are such that \( B_i \in K^1_\ell \) for \( \ell \neq p \), \( i \in \{1, ..., t\} \). Then the composition, \( T \),

\[
\begin{array}{ccc}
S_k(\Gamma_{\mathcal{F}_1}) & \xrightarrow{T} & S_k(\Gamma_{\mathcal{F}_2}) \\
\downarrow & & \uparrow \downarrow \\
S_k(K_{\mathcal{F}_1}, \chi_1) & \xrightarrow{T} & S_k(K_{\mathcal{F}_2}, \chi_2)
\end{array}
\]

is given by the formula

\[
T(F) = \sum_{i=1}^t c_i \cdot F|_{k}(B_{i})^{-1}
\]

for \( F \in S_k(\Gamma_{\mathcal{F}_1}) \).

**Proof.** Let \( F \in S_k(\Gamma_{\mathcal{F}_1}) \). By the isomorphism (6) for the family \( \mathcal{F}_1 \) with character \( \chi_1 \) we have \( \Phi_1 = \Phi_F \in S_k(K_{\mathcal{F}_1}, \chi_1) \). Using the isomorphism (7) for the family \( \mathcal{F}_2 \) with character \( \chi_2 \), we calculate the composition \( F_{T(\Phi_1)} \). Let \( Z \in \mathcal{H}_2 \) and let \( h \in \text{GSp}(4, \mathbb{R})^+ \) be such that \( h(I) = Z \). Then, using that \( \lambda(B_i) = 1 \), we have that

\[
F_{T(\Phi_1)}(Z) = \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \Phi_1(h_{\infty} B_{i,p})
\]

\[
= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \Phi_1(B_{i}^{-1} h_{\infty} B_{i,p})
\]

\[
= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \Phi_1(B_{i}^{-1} h_{\infty})
\]

\[
= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \lambda(B_{i}^{-1} h)^k j(B_{i}^{-1} h, I)^{-k} F((B_{i}^{-1} h)\langle I \rangle)
\]

\[
= j(h, I)^k \sum_{i=1}^t c_i j(B_{i}^{-1}, Z)^{-k} j(h, I)^{-k} F(B_{i}^{-1} \langle Z \rangle)
\]

\[
= \sum_{i=1}^t c_i j(B_{i}^{-1}, Z)^{-k} F(B_{i}^{-1} \langle Z \rangle)
\]
\[ = \sum_{i=1}^{t} c_i (F|_{kB_i^{-1}})(Z). \]

Hence,
\[ T(F) = F_{T(\Phi_1)} = \sum_{i=1}^{t} c_i \cdot F|_{kB_i^{-1}}. \]

This completes the proof. \[\square\]

3.3. **Paramodular Hecke operators.** Throughout this section, let \( M \) and \( k \) be positive integers, \( \chi \) a quadratic Hecke character, and \( \ell \) a fixed rational prime. We turn now to the paramodular family \( \mathcal{F} = \{K_{\text{para}}(\ell^{|\mathcal{F}|})\}_\ell \). We will determine the explicit relationship between the Hecke operators for Siegel paramodular forms and those for twisted automorphic forms. Let \( \ell \) be a rational prime and define the Hecke operators \( T(1, 1, \ell, \ell) \) and \( T(1, \ell, \ell, \ell^2) \) on \( S_k(\Gamma_{\text{para}}(M)) \) by
\[ T(1, 1, \ell, \ell) F = p^{k-3} \sum_i F|_{ka_i}, \quad T(1, \ell, \ell, \ell^2) F = p^{2(k-3)} \sum_j F|_{kb_j} \tag{9} \]

where
\[ \Gamma_{\text{para}}(M) \begin{bmatrix} 1 & 1 \\ \ell & \ell \end{bmatrix} \Gamma_{\text{para}}(M) = \bigsqcup \Gamma_{\text{para}}(M)a_i \tag{10} \]

and
\[ \Gamma_{\text{para}}(M) \begin{bmatrix} 1 & \ell \\ \ell^2 & \ell \end{bmatrix} \Gamma_{\text{para}}(M) = \bigsqcup \Gamma_{\text{para}}(M)b_j, \tag{11} \]

are disjoint decompositions. We define the Atkin-Lehner involution \( U_\ell \) on \( S_k(\Gamma_{\text{para}}(M)) \) as follows. Choose a matrix \( \gamma_\ell \in \text{Sp}(4, \mathbb{Z}) \) such that
\[ \gamma_\ell \equiv \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix} \mod \ell^{|\mathcal{F}|} \quad \text{and} \quad \gamma_\ell \equiv \begin{bmatrix} 1 & 1 \\ \ell^{|\mathcal{F}|} & \ell^{|\mathcal{F}|} \\ 1 & 1 \end{bmatrix} \mod M\ell^{-\text{val}(M)}. \]

Then,
\[ U_\ell = \gamma_\ell \begin{bmatrix} \ell^{|\mathcal{F}|} & \ell^{|\mathcal{F}|} \\ \ell^{|\mathcal{F}|} & \ell^{|\mathcal{F}|} \\ 1 & 1 \end{bmatrix} \tag{12} \]

normalizes \( \Gamma_{\text{para}}(M) \) and \( U_\ell^2 \) is contained in \( \ell^{|\mathcal{F}|}\Gamma_{\text{para}}(M) \), implying that \( F \mapsto F|_{kU_\ell} \) is indeed an involution of \( S_k(\Gamma_{\text{para}}(M)) \). The proof of the following lemma provides explicit forms for the representatives \( a_i \) and \( b_j \) from (10) and (11).
Lemma 3.4. Let $M$ be a positive integer, let $\ell$ be a prime and set $r = \text{val}_\ell(M)$. There exist finite disjoint decompositions

$$K_{\text{para}}(\ell^r) \begin{bmatrix} \ell & 1 \\ 1 & \ell \end{bmatrix} K_{\text{para}}(\ell^r) = \bigsqcup g_i K_{\text{para}}(\ell^r)$$

and

$$K_{\text{para}}(\ell^r) \begin{bmatrix} \ell^2 & 1 \\ 1 & \ell \end{bmatrix} K_{\text{para}}(\ell^r) = \bigsqcup h_j K_{\text{para}}(\ell^r)$$

such that

$$\Gamma_{\text{para}}(M) \begin{bmatrix} 1 & 1 \\ \ell & \ell \end{bmatrix} \Gamma_{\text{para}}(M) = \bigsqcup \Gamma_{\text{para}}(M) \ell g_i^{-1}$$

and

$$\Gamma_{\text{para}}(M) \begin{bmatrix} 1 & \ell^2 \\ \ell & \ell \end{bmatrix} \Gamma_{\text{para}}(M) = \bigsqcup \Gamma_{\text{para}}(M) \ell^2 h_j^{-1}.$$
and

\[
\begin{align*}
\text{GSp}(4, \mathbb{Z}_\ell) \begin{bmatrix} \ell^2 & \ell & 1 \\ \ell & 1 & \ell \end{bmatrix} \text{GSp}(4, \mathbb{Z}_\ell) &= \\
&\bigg\{ \bigg\lfloor \begin{array}{ccc} 1 & x & 1 \\ 1 & 1 & -x \\ 1 & 1 & 1 \end{array} \bigg\rfloor \bigg\lfloor \begin{array}{ccc} z & y \ell & 1 \\ 1 & 1 & \ell \\ 1 & 1 & 1 \end{array} \bigg\rfloor G\text{Sp}(4, \mathbb{Z}_\ell) \\
&\bigg\lfloor \begin{array}{ccc} 1 & c & 1 \\ 1 & d & 1 \\ c & d \ell & 1 \end{array} \bigg\rfloor G\text{Sp}(4, \mathbb{Z}_\ell) \\
&\bigg\lfloor \begin{array}{ccc} 1 & x \ell & 1 \\ 1 & 1 & -x \ell \\ x \ell & 1 & 1 \end{array} \bigg\rfloor G\text{Sp}(4, \mathbb{Z}_\ell) \\
&\bigg\lfloor \begin{array}{ccc} 1 & \ell \ell^2 \ell \ell^2 \end{array} \bigg\rfloor G\text{Sp}(4, \mathbb{Z}_\ell) \\
&\bigg\lfloor \begin{array}{ccc} 1 & d \ell & 1 \\ u \ell & 1 & 1 \\ 1 & 1 & 1 \end{array} \bigg\rfloor G\text{Sp}(4, \mathbb{Z}_\ell) \\
&\bigg\lfloor \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & -\lambda \ell \\ 1 & 1 & 1 \end{array} \bigg\rfloor G\text{Sp}(4, \mathbb{Z}_\ell). 
\end{align*}
\]

For \( r \geq 1 \), we again refer to [RS] Section 6.1 to deduce the decompositions

\[
K^{\text{para}}(\ell^r) \begin{bmatrix} \ell & \ell \\ \ell & 1 \end{bmatrix} K^{\text{para}}(\ell^r) = \\
\bigg\{ \bigg\lfloor \begin{array}{ccc} 1 & x & y \ell \\ 1 & y & \ell \\ 1 & 1 & \ell \end{array} \bigg\rfloor \bigg\lfloor \begin{array}{ccc} z \ell^{-r} & y \ell & 1 \\ 1 & 1 & \ell \\ 1 & 1 & 1 \end{array} \bigg\rfloor K^{\text{para}}(\ell^r) \\
\bigg\lfloor \begin{array}{ccc} 1 & x \ell^{-r} \ell & 1 \\ 1 & 1 & -x \ell \\ x \ell^{-r} \ell & 1 & 1 \end{array} \bigg\rfloor K^{\text{para}}(\ell^r)
\]

\(|x, y, z|_{\mathbb{Z}/\ell \mathbb{Z}} \in [1, \ell] \mathbb{Z} \)

\[ \begin{pmatrix} -\ell^{-r} & 1 \\ \ell^r & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ \ell & 1 \end{pmatrix} \begin{pmatrix} \ell & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -x \end{pmatrix} \begin{pmatrix} \ell & 1 \\ 1 & 1 \end{pmatrix} K_{\text{para}}(\ell^r) \]

\[ \bigcup_{x, y, z \in \mathbb{Z}/\ell \mathbb{Z}} \begin{pmatrix} -\ell^{-r} & 1 \\ \ell^r & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ \ell & 1 \end{pmatrix} \begin{pmatrix} \ell & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -x \end{pmatrix} \begin{pmatrix} \ell & 1 \\ 1 & 1 \end{pmatrix} K_{\text{para}}(\ell^r) \]

and

\[ \begin{pmatrix} \ell^2 & 1 \\ \ell & 1 \end{pmatrix} K_{\text{para}}(\ell^r) = \bigcup_{x, y, z \in \mathbb{Z}/\ell \mathbb{Z}} \begin{pmatrix} 1 & y \\ \ell & 1 \end{pmatrix} \begin{pmatrix} \ell & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -x \end{pmatrix} \begin{pmatrix} \ell & 1 \\ 1 & 1 \end{pmatrix} K_{\text{para}}(\ell^r) \]

\[ \bigcup_{x, y, z \in \mathbb{Z}/\ell \mathbb{Z}} \begin{pmatrix} -\ell^{-r} & 1 \\ \ell^r & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ \ell & 1 \end{pmatrix} \begin{pmatrix} \ell & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -x \end{pmatrix} \begin{pmatrix} \ell & 1 \\ 1 & 1 \end{pmatrix} K_{\text{para}}(\ell^r). \]

The explicit form of these decompositions implies that we can choose representatives \( g_i \) and \( h_j \) such that

\[ K_{\text{para}}(\ell^r) \begin{pmatrix} \ell^2 & \ell \\ \ell & 1 \end{pmatrix} K_{\text{para}}(\ell^r) = \bigcup_{i=1}^{D} g_i K_{\text{para}}(\ell^r), \]

\[ K_{\text{para}}(\ell^r) \begin{pmatrix} \ell^2 & \ell \\ \ell & 1 \end{pmatrix} K_{\text{para}}(\ell^r) = \bigcup_{j=1}^{D'} h_j K_{\text{para}}(\ell^r), \]

and

\[ \ell g_i^{-1} \in \Gamma_{\text{para}}(M) \begin{pmatrix} 1 & \ell \\ \ell & \ell \end{pmatrix} \Gamma(M), \quad \ell^2 h_j^{-1} \in \Gamma_{\text{para}}(M) \begin{pmatrix} 1 & \ell \ell^2 \\ \ell \ell^2 & \ell \end{pmatrix} \Gamma(M), \]

for \( i \in \{1, \ldots, D\} \) and \( j \in \{1, \ldots, D'\} \). For this, it is useful to note that \( \Gamma_{\text{para}}(M) \) contains several symmetry elements:

\[ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -M^{-1} \\ M & 1 \end{pmatrix}, \]
and in the case that \( r = 0 \), the element \([A \ell^{-1}A]^{A}\) where \( A \in \Gamma_0(M) \subset \text{SL}(2, \mathbb{Z}) \) and \( A \begin{bmatrix} 1 \\ \ell \end{bmatrix} A^{-1} = \begin{bmatrix} \ell & 1 \\ 1 \end{bmatrix}\). It follows that the cosets \( \Gamma_{\text{para}}(M)g_i^{-1} \) are mutually disjoint and contained in the first double coset, and the cosets \( \Gamma_{\text{para}}(M)\ell^2h_i^{-1} \) are mutually disjoint and contained in the second double coset. It remains to prove that the number of disjoint cosets in the first and second double cosets are \( D \) and \( D' \), respectively. Suppose that

\[
\Gamma_{\text{para}}(M) \begin{bmatrix} 1 \\ \ell \end{bmatrix} \Gamma_{\text{para}}(M) = \bigcup_{i=1}^{d} \Gamma_{\text{para}}(M)g_i^t
\]

is a disjoint decomposition. We have already shown that \( d \geq D \); we need to prove that \( D \geq d \). We have

\[
K_{\text{para}}(\ell^r) \begin{bmatrix} 1 \\ \ell \end{bmatrix} K_{\text{para}}(\ell^r) \supset K_{\text{para}}(\ell^r) \begin{bmatrix} 1 \\ \ell \end{bmatrix} \Gamma_{\text{para}}(M) = \bigcup_{i=1}^{d} K_{\text{para}}(\ell^r)g_i^t.
\]

We claim that these cosets are disjoint. For suppose \( K_{\text{para}}(\ell^r)g_i^t = K_{\text{para}}(\ell^r)g_j^t \). This implies that \( g_i^tg_j^{-1} \in K_{\text{para}}(\ell^r) \). Since for any prime \( q \) with \( q \neq \ell \) we have \( g_i^tg_j^{-1} \in K_{\text{para}}(q^{\text{val}_q(M)}) \) as all the elements of

\[
\Gamma_{\text{para}}(M) \begin{bmatrix} 1 \\ \ell \end{bmatrix} \Gamma_{\text{para}}(M)
\]

are contained in \( K_{\text{para}}(q^{\text{val}_q(M)}) \), we have \( g_i^tg_j^{-1} \in \Gamma_{\text{para}}(M) \). This implies that \( i = j \). It follows that \( D \geq d \). The proof that \( d' = D' \) is similar. \qed

Let \( V \) be the \( \mathbb{C} \) vector space of functions \( \Phi : \text{GSp}(4, \mathbb{A}) \to \mathbb{C} \) such that \( \Phi \in V \) if and only if there exists a compact, open subgroup \( \Gamma_1 \) of \( \text{GSp}(4, \mathbb{Q}) \) such that \( \Phi(gk) = \Phi(g) \) for \( g \in \text{GSp}(4, \mathbb{A}) \) and \( k \in \Gamma_1 \), and \( \Phi(gz) = \Phi(g) \) for \( g \in \text{GSp}(4, \mathbb{A}) \) and \( z \in \mathbb{A}^\times \). The group \( \text{GSp}(4, \mathbb{Q}) \) acts smoothly on \( V \) by right translation, and for this action, denoted by \( \pi \), the center of \( \text{GSp}(4, \mathbb{Q}) \) acts trivially. Assume that \( \chi_\ell \) is unramified. Then \( S_k(K_F, \chi) \subset V(\text{val}_\ell(M)) \), where the last space consists of the vectors in \( V \) that are fixed by \( K_{\text{para}}(\ell^{\text{val}_\ell(M)}) \) as in [RS]. Let \( T_{1,0} \) and \( T_{0,1} \) be the local Hecke operators and \( u_{\text{val}_\ell(M)} \) be the local Atkin-Lehner operator acting on \( V(\text{val}_\ell(M)) \) as in [RS]. These operators preserve the subspace \( S_k(K_F, \chi) \subset V(\text{val}_\ell(M)) \).

**Lemma 3.5.** Assume that \( \chi_\ell \) is unramified. Let \( \Phi \in S_k(K_F, \chi) \) and let \( F_\Phi \in S_k(\Gamma_{\text{para}}(M)) \) be the Siegel modular form corresponding to \( \Phi \) under the isomorphism (7) of Lemma 3.2. Then,

1. \( T(1,1,\ell)F_\Phi = \ell^{k-3} \chi_\ell(\ell)F_{T_{1,0}} \Phi \),
2. \( T(1,\ell,\ell^2)F_\Phi = \ell^{2(k-3)}F_{T_{0,1}} \Phi \),
3. \( F_{k[U_{\ell^r}} = \chi_\ell(\ell)\text{val}_\ell(F_{u_{\text{val}_\ell(M)}}) \Phi \).

**Proof.** Let \( Z \in S_2 \) and let \( g_\infty \in \text{Sp}(4, \mathbb{R}) \) such that \( g_\infty[I] = Z \). For the first assertion, we will use the coset representatives from Lemma 3.4. Note that from the explicit forms given in the proof, we have that \( g_i \in K_{\text{para}}(q^{\text{val}_q(M)}) \) for primes \( q \neq \ell \) and \( \lambda(g_i) = \ell \) for each \( i \). We calculate

\[
(T(1,1,\ell)F_\Phi)(Z) = \ell^{k-3} \sum_i (F_{\Phi|_{\ell^i}} g_i^{i-1})(Z)
\]
We let $T_{c,3.5}$ we have $S_{c,3.4}$. for this action $g$.

Turning to the Hecke operators and the Atkin-Lehner operator, let $K$ be a rational prime with $\ell \neq p$ and $B_i \in \text{Sp}(4, \mathbb{Q})$ for $i \in \{1, \ldots, t\}$. The space $S_k(K_{F_1}, \chi_1)$ is contained in $V$, and moreover one verifies that the image of the restriction $T$ of $T_{\chi}$ to $S_k(K_{F_2}, \chi_1)$ is contained in $S_k(K_{F_2}, \chi_2)$. By Lemma 3.3, the map $T : S_k(\Gamma_{\text{para}}(N)) \to S_k(\Gamma_{\text{para}}(NP^4))$ given by (8) has the formula

$$T(F) = \sum_{i=1}^{t} c_i \cdot F|_{k} B_i^{-1}, \quad F \in S_k(\Gamma_{\text{para}}(N)).$$

We let $T_{\chi} = T$; the formula in the statement of the theorem is a consequence of the expressions for $c_i$ and $B_i$ in Corollary 5.1.

Turning to the Hecke operators and the Atkin-Lehner operator, let $F \in S_k(\Gamma_{\text{para}}(N))$ and let $\Phi \in S_k(K_{F_1}, \chi_1)$ with $F = F_{\Phi}$. Let $\ell$ be a rational prime with $\ell \neq p$ so that $\chi_{1,\ell}$ and $\chi_{2,\ell}$ are unramified. Moreover, we have that $\chi_{1,\ell}$ is trivial and $\chi_{2,\ell}(\ell) = \chi(\ell)$. Then, using (8) and Lemma 3.5 we have

$$T(1,1,\ell)\mathcal{T}_{\chi}(F_{\Phi}) = T(1,1,\ell,\ell)F_{T(\Phi)} = \ell^{k-3} \chi(\ell)F_{T_{1,0}(\ell)T(\Phi)}$$
\[ \ell \kappa - 3 \chi(\ell) F_{\mathcal{T}}(T_{1,0}(\ell) \Phi) \]
\[ = \chi(\ell) \mathcal{T}_\chi(\ell) F_{\mathcal{T}}(T_{1,0}(\ell) \Phi) \]
\[ = \chi(\ell) \mathcal{T}_\chi(T(1, 1, \ell, \ell) F_{\mathcal{T}}). \]

The proofs for the other two operators are similar. \qed

4. LOCAL CALCULATIONS

Throughout this section, we will use the following notation. Let \( F \) be a nonarchimedean local field of characteristic zero, with ring of integers \( \mathfrak{o} \) and generator \( \mathfrak{w} \) of the maximal ideal \( \mathfrak{p} \) of \( \mathfrak{o} \). We let \( q \) be the number of elements of \( \mathfrak{o} / \mathfrak{p} \) and use the absolute value on \( F \) such that \( |\mathfrak{w}| = q^{-1} \). We use the Haar measure on the additive group \( F \) that assigns \( \mathfrak{o} \) measure 1 and the Haar measure on the multiplicative group \( F^\times \) that assigns \( \mathfrak{o}^\times \) measure \( 1 - q^{-1} \). We \( \chi \) be a quadratic character of \( F^\times \) of conductor \( \mathfrak{p} \).

Let

\[ J' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}. \]

For this section only, we define \( \text{GSp}(4,F) \) as the subgroup of all \( g \in \text{GL}(4,F) \) such that \( t_g J' g = \lambda(g) J' \) for some \( \lambda(g) \in F^\times \) called the multiplier of \( g \). For \( n \) a non-negative integer, we let \( K(\mathfrak{p}^n) \) be the subgroup of \( k \in \text{GSp}(4,F) \) such that \( \lambda(k) \in \mathfrak{o}^\times \) and

\[ k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \] (13)

Throughout this section, \((\pi, V)\) is a smooth representation of \( \text{GSp}(4,F) \) for which the center acts trivially. If \( n \) is a non-negative integer, then \( V(n) \) is the subspace of vectors fixed by the paramodular subgroup \( K(\mathfrak{p}^n) \); also, we let \( V(n,\chi) \) be the subspace of vectors \( v \in V \) such that \( \pi(k)v = \chi(\lambda(k)) v \) for \( k \in K(\mathfrak{p}^n) \). Finally, let

\[ \eta = \begin{pmatrix} \mathfrak{w}^{-1} & 1 \\ 1 & \mathfrak{w}^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & \mathfrak{w}^{-1} \\ \mathfrak{w}^{-1} & \mathfrak{w} \end{pmatrix}, \quad t_4 = \begin{pmatrix} 1 & -\mathfrak{w}^{-4} \\ \mathfrak{w}^4 & 1 \end{pmatrix}. \]

Usually, we will write \( \eta \) and \( \tau \) for \( \pi(\eta) \) and \( \pi(\tau) \), respectively.

In [JR2] we constructed a twisting map,

\[ T_\chi : V(0) \rightarrow V(4,\chi), \] (14)

given by

\[ T_\chi(v) = q^3 \int_{\mathfrak{o} \times \mathfrak{o}^\times} \chi(ab) \pi( \begin{pmatrix} 1 & a\mathfrak{w}^{-1} & b\mathfrak{w}^{-2} & z\mathfrak{w}^{-4} \\ x & 1 & b\mathfrak{w}^{-2} & a\mathfrak{w}^{-1} \\ 1 & 1 & 1 & 1 \end{pmatrix}) v \, da \, db \, dx \, dz \] (P1)
Remark 4.1. The Iwasawa decomposition asserts that $GSp(4, F) = B \cdot GSp(4, o)$ where $B$ is the Borel subgroup of upper-triangular matrices in $GSp(4, F)$. Hence, if $v \in V(0)$ so that $v$ is invariant under $GSp(4, o)$, then it is possible to obtain a formula for $T_\chi(v)$ involving only upper-triangular matrices. The remainder of this section will be devoted to calculating formulas for the terms (P1), (P2), (P3), (P4) involving only upper-triangular matrices. The resulting formula for $T_\chi(v)$ is given in the following theorem. The proof of this theorem follows from several technical lemmas. The full proofs of these lemmas are provided in an appendix to this paper [JR-a]. In some cases, we directly provide an Iwasawa identity $g = bk$ where $g \in GSp(4, F)$, $b \in B$, and $k \in GSp(4, o)$. In many cases, we are able to obtain an appropriate Iwasawa identity by using the following formal matrix identity
\[
\begin{bmatrix}
1 & x & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 
\end{bmatrix} =
\begin{bmatrix}
1 & x^{-1} & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
1 & -x^{-1} & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
1 & x^{-1} & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 
\end{bmatrix}.
\]\(15\)
Both methods will require that we decompose the domains of integration in an advantageous manner. The assumptions on the character, $\chi \neq 1$, $\chi^2 = 1$ and $\chi(1 + p) = 1$, also play a significant role in the computations.

Theorem 4.2. Let $v \in V(0)$. Then the twisting operator (14) is given by the formula
\[
T_\chi(v) = \sum_{k=1}^{14} T^k_\chi(v)
\]
where
\[
T^1_\chi(v) = q^2 \int \int \int \chi(ab) \pi(t_4) \begin{bmatrix}
1 \\
1 \\
1 \\
1 
\end{bmatrix}
\begin{bmatrix}
1 & -(a + xb)z^{-1} & 1 \\
1 & 1 & (a + xb)z^{-1} \\
1 & bz^{-2} & 1 \\
1 & 1 & 1 
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1 
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1 
\end{bmatrix}
\int v da db dx dz
\]
\( T_\chi^2(v) = q \eta \int \int \int \int \int \chi(abxy) \pi \left( \begin{bmatrix} 1 & b\varpi^{-1} \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) dx \, da \, db \, dy, \)

\( T_\chi^3(v) = \eta \int \int \int \int \chi(b(1-z)) \pi \left( \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) v \, da \, db \, dz, \)

\( T_\chi^4(v) = q \eta \int \int \int \chi(b) \pi \left( \begin{bmatrix} 1 & x\varpi^{-2} \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & (b\varpi - ax)\varpi^{-4} \\ & a\varpi^{-2} \end{bmatrix} \right) v \, da \, db \, dx, \)

\( T_\chi^5(v) = \eta \int \int \int \chi(b) \pi \left( \begin{bmatrix} 1 & x\varpi^{-1} \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & (b - ax)\varpi^{-3} \\ & a\varpi^{-2} \end{bmatrix} \right) v \, da \, db \, dx, \)

\( T_\chi^6(v) = q^{-1} \eta^2 \int \int \int \chi(bx) \pi \left( \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & b(1-x)\varpi^{-2} \\ & a\varpi^{-2} \end{bmatrix} \right) v \, da \, db \, dx, \)

\( T_\chi^7(v) = q \tau \int \int \int \chi(ab) \pi \left( \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & b\varpi^{-1} \\ & b\varpi^{-1} \end{bmatrix} \right) v \, da \, db \, dz, \)

\( T_\chi^8(v) = \eta \tau \int \int \int \chi(abz(1-z)) \pi \left( \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & -b\varpi^{-2} \\ & -b\varpi^{-2} \end{bmatrix} \right) v \, da \, db \, dz, \)

\( T_\chi^9(v) = q^{-2} \eta^2 \tau \int \int \int \chi(b) \pi \left( \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \right) \left( \begin{bmatrix} 1 & x\varpi^{-1} \\ & -b\varpi^{-1} \end{bmatrix} \right) v \, da \, db \, dx, \)
$$T_{\chi}^{10}(v) = q^{-3} \eta_2 \tau^2 \int_{\phi} \int \chi(b) \pi \left( \begin{array}{cc} 1 & -b \omega^{-1} \\ 1 & 1-a \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db,$$

$$T_{\chi}^{11}(v) = q^3 \tau^{-1} \int_{\phi} \int \int \chi(ab) \pi \left( \begin{array}{cc} 1 & b \omega^{-1} \\ 1 & 1-b \omega^{-1} \\ 1 \end{array} \right)$$

$$\left[ \begin{array}{ccc} 1 & (xb+a \omega) \omega^{-3} & z \omega^{-4} \\ 1 & -x \omega^{-2} & (xb+a \omega) \omega^{-3} \\ 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz,$$

$$T_{\chi}^{12}(v) = q^2 \eta_1 \tau^{-1} \int_{\phi} \int \int \int \chi(abz(1-z)) \pi \left( \begin{array}{cc} 1 & a \omega^{-1} \\ 1 & 1 -a \omega^{-1} \end{array} \right)$$

$$\left[ \begin{array}{ccc} 1 & y \omega^{-3} & a(b(1-z) \omega - y) \omega^{-4} \\ 1 & -a^{-1}(y+b \omega) \omega^{-2} & y \omega^{-3} \\ 1 & 1 \end{array} \right] v \, da \, db \, dy \, dz,$$

$$T_{\chi}^{13}(v) = \eta^2 \tau^{-1} \int_{\phi} \int \int \chi(bx) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & b(1-x) \omega^{-1} \\ 1 & a^2b^{-1} \omega^{-3} & a \omega^{-2} \\ 1 \end{array} \right)$$

$$\left[ \begin{array}{ccc} 1 & x \omega^{-4} & -b \omega^{-1} \\ 1 & 1 & a \omega^{-2} \end{array} \right] v \, da \, db \, dx,$$

$$T_{\chi}^{14}(v) = q \eta^2 \tau^{-2} \int_{\phi} \int \int \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & -b \omega^{-1} \\ 1 & x \omega^{-4} & a \omega^{-2} \\ 1 \end{array} \right)$$

$$\left[ \begin{array}{ccc} 1 & b \omega^{-1} \\ 1 \end{array} \right] v \, da \, db \, dx.$$

Proof. Substituting the formulas from Lemmas 4.3, 4.4, 4.5 and 4.6 we have that the twisting operator is given by the formula

$$T_{\chi}(v) =$$

$$q^2 \int_{\phi} \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & -(a+xb) \omega^{-1} \\ 1 & 1 & (a+xb) \omega^{-1} \end{array} \right)$$

$$\left[ \begin{array}{ccc} 1 & -b \omega^{-1} \\ 1 & 1 & -b \omega^{-1} \end{array} \right] v \, da \, db \, dx \, dz$$

$$+ q\chi(-1) \eta \int_{\phi} \int \int \int \chi(abx) \pi \left( \begin{array}{ccc} 1 & b \omega^{-1} \\ 1 \end{array} \right)$$

$$\left[ \begin{array}{ccc} 1 & 1 & -b \omega^{-1} \end{array} \right] v \, da \, db \, dx.$$
\[
\begin{align*}
&\left[\begin{array}{ccc}
1 & a\omega^{-2} & -ab^{-1}(1 + x - z)\omega^{-3} \\
1 & -ab^{-1}.xz(1 - z + x) & 1 \\
1 & -ab^{-1} & 1 \\
\end{array}\right] v \, dx \, da \, db \, dz \\
+ \chi(-1)\eta \int_{\omega\times(1+p)} \int_{\omega\times\omega} \int_{\omega\times\omega} \chi(b(1 - z)) \pi(\begin{array}{ccc}
1 & a\omega^{-2} & -b\omega^{-3} \\
1 & -a^{2}b^{-1}z\omega^{-1} & 1 \\
1 & -a^{2}b^{-1} & 1 \\
\end{array}) v \, da \, db \, dz \\
+ q\eta \int_{\omega\times(1+p)} \int_{\omega\times\omega} \int_{\omega\times\omega} \chi(b) \pi(\begin{array}{ccc}
1 & x\omega^{-2} & (b\omega - ax)\omega^{-4} \\
1 & -x\omega^{-2} & 1 \\
1 & -a\omega^{-2} & 1 \\
\end{array}) v \, da \, db \, dx \\
+ \eta \int_{\omega\times(1+p)} \int_{\omega\times\omega} \int_{\omega\times\omega} \chi(ab) \pi(\begin{array}{ccc}
1 & -a\omega^{-2} & b(1 + x)\omega^{-2} \\
1 & a\omega^{-2}b^{-1}\omega^{-2} & 1 \\
1 & a\omega^{-2} & 1 \\
\end{array}) v \, da \, db \, dy \\
+ q^{-1}\chi(-1)\eta^{2} \int_{\omega\times(1+p)} \int_{\omega\times\omega} \int_{\omega\times\omega} \chi(bx) \pi(\begin{array}{ccc}
1 & a\omega^{-2} & b(1 + x)\omega^{-2} \\
1 & 1 & 1 \\
1 & a\omega^{-2} & 1 \\
\end{array}) v \, da \, db \, dx \\
+ q\tau \int_{\omega\times\omega\times(1+p)} \int_{\omega\times\omega\times\omega} \int_{\omega\times\omega\times\omega} \chi(abz(1 - z)) \pi(\begin{array}{ccc}
1 & b\omega^{-1} & z\omega^{-4} \\
1 & -a\omega^{-1} & 1 \\
1 & b\omega^{-1} & 1 \\
\end{array}) v \, da \, db \, dz \\
+ \eta\tau \int_{\omega\times(1+p)} \int_{\omega\times\omega} \int_{\omega\times\omega} \chi(abz(1 - z)) \pi(\begin{array}{ccc}
1 & a\omega^{-1} & -ab(1 - z)\omega^{-3} \\
1 & 1 & 1 \\
1 & a\omega^{-1} & 1 \\
\end{array}) v \, da \, db \, dz \\
+ q^{-2}\chi(-1)\eta^{2}\tau \int_{\omega\times\omega\times(1+p)} \int_{\omega\times\omega\times\omega} \int_{\omega\times\omega\times\omega} \chi(b) \pi(\begin{array}{ccc}
1 & a\omega^{-1} & b\omega^{-1} \\
1 & 1 & 1 \\
1 & -a\omega^{-1} & 1 \\
\end{array}) v \, da \, db \, dx \\
+ q^{-3}\chi(-1)\eta^{2}\tau^{2} \int_{\omega\times\omega\times\omega} \int_{\omega\times\omega\times\omega} \int_{\omega\times\omega\times\omega} \chi(b) \pi(\begin{array}{ccc}
1 & a\omega^{-2} & b\omega^{-1} \\
1 & 1 & 1 \\
1 & -a\omega^{-2} & 1 \\
\end{array}) v \, da \, db
\end{align*}
\]
+ \eta^{-1} \int_{a \omega < (1+p)} \int_{a \omega < a \omega} \int_{a \omega < a \omega} \chi(ab) \pi \left( \begin{array}{cc} 1 & -a \omega^{-1} \\ 1 & a \omega^{-1} \end{array} \right)
\left( \begin{array}{c} x \omega^{-2} \\ 1 \\ x \omega^{-4} \end{array} \right) v \, da \, db \, dy \, dz

+ q^2 \chi(-1) \eta^{-1} \int_{a \omega < (1+p)} \int_{a \omega < a \omega} \int_{a \omega < a \omega} \chi(ab) \pi \left( \begin{array}{cc} 1 & -a \omega^{-1} \\ 1 & a \omega^{-1} \end{array} \right)
\left( \begin{array}{c} x \omega^{-2} \\ 1 \\ x \omega^{-4} \end{array} \right) v \, da \, db \, dz

+ \chi(-1) \eta^2 \tau^{-1} \int_{a \omega < a \omega} \int_{a \omega < a \omega} \int_{a \omega < a \omega} \chi(abx) \pi \left( \begin{array}{cc} 1 & -a^2 b^{-1} \omega^{-3} \\ 1 & a \omega^{-2} \end{array} \right)
\left( \begin{array}{c} a \omega^{-2} \\ 1 \\ x \omega^{-4} \end{array} \right) v \, da \, db \, dx

+ q \chi(-1) \eta^2 \tau^{-2} \int_{a \omega < a \omega} \int_{a \omega < a \omega} \int_{a \omega < a \omega} \chi(ab) \pi \left( \begin{array}{cc} 1 & -a^2 b^{-1} \omega^{-3} \\ 1 & a \omega^{-2} \end{array} \right)
\left( \begin{array}{c} a \omega^{-2} \\ 1 \\ x \omega^{-4} \end{array} \right) v \, da \, db \, dx
\[ + \eta^2 \tau^{-2} \int \int \int_{o \times o \times o} \chi(a) \pi \left( \begin{array}{ccc} 1 & b \omega^{-2} & -a \omega^{-1} \\ 1 & y \omega^{-3} & b \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db \, dy. \]

For the remainder of the proof, we will simplify by combining pairs of terms and rewriting certain domains. First we combine the terms involving \( \eta^2 \tau^{-2} \).

\[ q \chi(-1) \eta^2 \tau^{-2} \int \int \int_{o \times o \times o} \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & b \omega^{-1} \\ 1 & x \omega^{-4} & a \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db \, dx \]

\[ + \eta^2 \tau^{-2} \int \int \int_{o \times o \times o} \chi(a) \pi \left( \begin{array}{ccc} 1 & b \omega^{-2} & -a \omega^{-1} \\ 1 & y \omega^{-3} & b \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db \, dy \]

\[ = q \chi(-1) \eta^2 \tau^{-2} \int \int \int_{o \times o \times o} \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & b \omega^{-1} \\ 1 & x \omega^{-4} & a \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db \, dx \]

\[ + q \chi(-1) \eta^2 \tau^{-2} \int \int \int_{o \times o \times o} \chi(a) \pi \left( \begin{array}{ccc} 1 & b \omega^{-2} & a \omega^{-1} \\ 1 & y \omega^{-4} & b \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db \, dy \]

\[ = q \chi(-1) \eta^2 \tau^{-2} \int \int \int_{o \times o \times o} \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & b \omega^{-1} \\ 1 & x \omega^{-4} & a \omega^{-2} \\ 1 & 1 \end{array} \right) v \, da \, db \, dx. \]

Similarly, we combine the terms involving \( \eta \tau^{-1} \),

\[ q \eta \tau^{-1} \int \int \int_{o \times (1+p) o \times o} \chi(ab) \pi \left( \begin{array}{ccc} 1 & -a \omega^{-1} \\ 1 & 1 \end{array} \right) v \, da \, db \, dy \, dz \]

\[ + q^2 \chi(-1) \eta \tau^{-1} \int \int \int_{o \times (1+p) o \times o \times o} \chi(ba(1-z) \pi) \left( \begin{array}{ccc} 1 & b \omega^{-1} \\ 1 & 1 \end{array} \right) v \, dx \, db \, dz \]

\[ \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right) v \, dx \, da \, db \, dz \]
Then

\[ q^2 \chi(-1) \eta \tau^{-1} \int \int \int \chi(abz(1-z)) \pi \left( \begin{array}{cc} 1 & a \omega^{-1} \\ 1 & 1 - a \omega^{-1} \end{array} \right) \left( \begin{array}{cc} y \omega^{-3} & -a(y + b(1-z) \omega) \omega^{-4} \\ 1 & 1 \end{array} \right) \right) v \, da \, db \, dy \, dz, \]

the two terms involving \( \tau^{-1} \),

\[ q^3 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{cc} 1 & b \omega^{-1} \\ 1 & 1 - b \omega^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & -(b - x a \omega) x \omega^{-3} \\ 1 & x \omega^{-2} \end{array} \right) \right) v \, da \, db \, dx \, dz + q^2 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{cc} 1 & b \omega^{-1} \\ 1 & 1 - b \omega^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & (a + b y) \omega^{-2} \\ 1 & -y \omega^{-1} \end{array} \right) \right) v \, da \, db \, dy \, dz \]

\[ = q^3 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{cc} 1 & b \omega^{-1} \\ 1 & 1 - b \omega^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & (x b + a \omega) \omega^{-3} \\ 1 & -x \omega^{-2} \end{array} \right) \right) v \, da \, db \, dx \, dz, \]

and two of the terms that involve the \( \eta \) operator,

\[ q \eta \int \int \int \chi(b) \pi \left( \begin{array}{cc} 1 & x \omega^{-2} \\ 1 & 1 - x \omega^{-2} \end{array} \right) \left( \begin{array}{cc} 1 & a \omega^{-2} - (b \omega - ax) \omega^{-4} \\ 1 & 1 \end{array} \right) \right) v \, da \, db \, dx \]

\[ + \eta \int \int \int \chi(ab) \pi \left( \begin{array}{cc} 1 & a \omega^{-2} \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & y \omega^{-1} - a(y + b) \omega^{-3} \\ 1 & 1 \end{array} \right) \right) v \, da \, db \, dy \]
We rewrite one of the terms involving $\eta$ after making the observation that if $z \in \mathfrak{o}^\times - (1 + p)$ and $f$ is a locally constant function on $\mathfrak{o}^\times$, then

$$
\int_{\mathfrak{o}^\times - A(z)} f(x) dx = \int_{\mathfrak{o}^\times - (1 + p)} f((z^{-1} - 1)(w^{-1} - 1)^{-1}) dw.
$$

Hence

$$
q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - A(z)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} q(\chi(bx)\pi(\mathfrak{o}^\times) a^\omega - 2 \left( b^\omega - ax \right) a^\omega - 4)
\begin{bmatrix}
1 \\
1 - x^\omega - 2 \\
1 \\
1 - x^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - x^\omega - 2 \\
1 - x^\omega - 1 \\
1 - x^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - x^\omega - 2 \\
1 - x^\omega - 1 \\
1 - x^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
$$

= $q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} q(\chi(abx(1 - z)w(1 - w)\pi(\mathfrak{o}^\times)) a^\omega - 2 \left( ab(1 + zw^{-1}(1 - w)) a^\omega - 3 \right))
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
$$

= $q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} q(\chi(az^{-1}(1 - z)bw^{-1}(1 - w))\pi(\mathfrak{o}^\times)) a^\omega - 2 \left( ab(1 + zw^{-1}(1 - w)) a^\omega - 3 \right))
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
$$

= $q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} \int_{\mathfrak{o}^\times - \mathfrak{o}^\times - \mathfrak{o}^\times - (1 + p)} q(\chi(a(z^{-1} - 1)bw^{-1} - 1))\pi(\mathfrak{o}^\times)) a^\omega - 2 \left( ab(1 + zw^{-1}(1 - w)) a^\omega - 3 \right))
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 - ab^{-1} x^\omega - 2 \\
1 - ab^{-1} x^\omega - 2 \\
1
\end{bmatrix}
\begin{bmatrix}
1 - b^\omega - 1 \\
b^\omega - 1 \\
1 - b^\omega - 1 \\
1
\end{bmatrix}$
Finally, we are able to eliminate the factor \( \chi(-1) \) from all terms using an appropriate change of variables. Substituting the simplified terms into the formula for \( T_\chi(v) \), we obtain the result. \( \square \)

The proofs of the following four lemmas are provided in an appendix to this paper, [JR-a].

**Lemma 4.3.** If \( v \in V(0) \), then we have that (P1) is given by

\[
q^3 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \chi(ab)y \pi \left( \begin{array}{cccc}
1 & -a & -b & 1 \\
1 & x & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{array} \right) \tau v da db dx dz = q\tau \int_0^\infty \int_0^\infty \chi(ab) y \pi \left( \begin{array}{cccc}
1 & -a & -b & 1 \\
1 & a & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{array} \right) v da db dz
\]
\[ q^3 \tau^{-1} \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc} 1 & b \tau^{-1} & 1 \\ 1 & 1 & -b \tau^{-1} \\ 1 & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz + q^2 \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc} 1 \end{array} \right] v \, da \, db \, dxdz \]

\[ q^3 \tau^{-1} \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc} 1 & -b \tau^{-1} \end{array} \right] v \, da \, db \, dx \, dz \]

Proof. See the appendix [JR-a].

Lemma 4.4. If \( v \in V(0) \), then we have that (P2) is given by

\[ q^2 \int \int \int \chi(ab) \pi(t_4) \left[ \begin{array}{ccc} 1 & -a \tau^{-1} & b \tau^{-2} \\ 1 & 1 & b \tau^{-2} \\ 1 & 1 & a \tau^{-1} \end{array} \right] v \, da \, db \, dxdz = q^2 \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc} 1 & -a \tau^{-1} & b \tau^{-2} \\ 1 & 1 & b \tau^{-2} \\ 1 & 1 & a \tau^{-1} \end{array} \right] v \, da \, db \, dxdz \]

Proof. See the appendix [JR-a].

Lemma 4.5. If \( v \in V(0) \), then we have that (P3) is given by

\[ q^2 \int \int \int \chi(ab) \pi(t_4) \left[ \begin{array}{ccc} 1 & -a \tau^{-1} & b \tau^{-2} \\ 1 & 1 & b \tau^{-2} \\ 1 & 1 & a \tau^{-1} \end{array} \right] v \, da \, db \, dxdz \]

\[ = q^2 \tau^{-1} \int \int \int \chi(abx(1-z)) \eta \tau^{-1} \pi \left[ \begin{array}{ccc} 1 & b \tau^{-1} & 1 \\ 1 & 1 & -b \tau^{-1} \\ 1 & 1 & 1 \end{array} \right] v \, dx \, da \, db \, dz \]

Proof. See the appendix [JR-a].
\[ + \int_{\phi^-(1+p) \phi \phi \phi} \int \int \chi(abz(1-z)) \eta \tau \pi \begin{bmatrix} 1 & b^{-2} \\ a^{-1} & 1 \\ 1 & -b^{-2} \\ 1 & 1 \end{bmatrix} v \, dx \, da \, db \, dz \]

\[ + q \chi(-1) \int_{\phi^-(1+p) \phi \phi \phi - A(z)} \int \int \int \chi(abx) \eta \pi \begin{bmatrix} 1 & b^{-1} \\ a^{-2} & 1 \\ 1 & -b^{-1} \\ 1 & 1 \end{bmatrix} v \, dx \, da \, db \, dz \]

\[ + \chi(-1) \int_{\phi^-(1+p) \phi \phi \phi} \int \int \int \chi(b(1-z)) \eta \pi \begin{bmatrix} 1 & a^{-2} \\ b^{-1} & 1 \\ 1 & -a^{-2} \\ 1 & 1 \end{bmatrix} v \, dx \, da \, db \, dz \]

\[ + q \chi(-1) \int_{\phi \phi \phi \phi} \int \int \chi(b) \eta^2 \tau^2 \pi \begin{bmatrix} 1 & a^{-2} & b^{-1} \\ x^{-4} & 1 & a^{-2} \\ 1 & 1 & 1 \end{bmatrix} v \, da \, db \, dx \]

\[ + q^{-1} \chi(-1) \int_{\phi \phi \phi \phi} \int \int \chi(bx) \eta^2 \pi \begin{bmatrix} 1 & a^{-2} \\ b^{-1} & 1 \\ 1 & -a^{-2} \\ 1 & 1 \end{bmatrix} v \, da \, db \, dx \]

\[ + q^{-2} \chi(-1) \int_{\phi \phi \phi \phi} \int \int \chi(b) \eta^2 \tau \pi \begin{bmatrix} 1 & a^{-1} \\ b^{-1} & 1 \\ 1 & -a^{-1} \\ 1 & 1 \end{bmatrix} v \, da \, db \, dx \]

\[ + q^{-3} \chi(-1) \int_{\phi \phi \phi \phi} \int \chi(b) \eta^2 \tau^2 \pi \begin{bmatrix} 1 & a^{-2} \\ b^{-1} & 1 \\ 1 & -a^{-2} \\ 1 & 1 \end{bmatrix} v \, da \, db \]

\[ + q \int_{\phi \phi \phi \phi} \int \chi(b) \eta \pi \begin{bmatrix} 1 & x^{-2} \\ 1 & 1 \\ 1 & -x^{-2} \\ 1 & 1 \end{bmatrix} v \, da \, db \, dx \]

**Proof.** See the appendix [JR-a]. \[\square\]
Lemma 4.6. If \( v \in V(0) \), then we have that (P4) is given by

\[
q^2 \int \int \int \int _{o \circ o^\times \circ o^\times} \chi(ab) \pi(t_4) \left[ \begin{array}{ccc} 1 & -1 & 1 \\ 1 & y & 1 \\ 1 & \tau \end{array} \right] \left[ \begin{array}{ccc} 1 & -a_\omega^{-1} & b_\omega^{-2} \\ 1 & 1 & b_\omega^{-2} \\ 1 & 1 & a_\omega^{-1} \end{array} \right] \tau v \ da \ db \ dy \ dz
\]

\[
= \int \int \int \int _{o \circ o^\times \circ o^\times} \chi(ab) \eta \pi \left[ \begin{array}{ccc} 1 & -a_\omega^{-2} & 1 \\ 1 & a_\omega^{-2} & 1 \\ 1 & \tau \end{array} \right] \left[ \begin{array}{ccc} 1 & y_\omega^{-1} & a(y + b)_\omega^{-3} \\ 1 & 1 & y_\omega^{-1} \\ 1 & 1 & 1 \end{array} \right] \eta \tau^{-1} \pi \left( \begin{array}{ccc} 1 & y_\omega^{-2} & a(y + b)_\omega^{-3} \\ 1 & a^{-1}(y + bz(z - 1)^{-1})_\omega^{-1} & y_\omega^{-2} \\ 1 & 1 & 1 \end{array} \right) \)v da db dy dz
\]

\[
+ q \int \int \int \int _{o^\times -(1+p) \circ o^\times o^\times} \chi(ab) \eta \pi \left[ \begin{array}{ccc} 1 & -a_\omega^{-1} & 1 \\ 1 & 1 & 1 \\ 1 & \tau \end{array} \right] \left[ \begin{array}{ccc} 1 & b_\omega^{-2} & -a_\omega^{-1} \\ 1 & y_\omega^{-3} & b_\omega^{-2} \\ 1 & 1 & 1 \end{array} \right] \eta \tau^{-1} \pi \left( \begin{array}{ccc} 1 & b_\omega^{-2} & -a_\omega^{-1} \\ 1 & y_\omega^{-3} & b_\omega^{-2} \\ 1 & 1 & 1 \end{array} \right) \)v da db dy.
\]

Proof. See the appendix [JR-a].

5. Formulas for \( \mathbb{Q}_p \)

In this section we will write the formula for the twisting operator (14) in the case where \( F = \mathbb{Q}_p \) and \( \text{GSp}(4) \) is written with respect to \( J \), as given in the introduction. We use the isomorphism between \( \text{GSp}(4) \) as defined with respect to \( J \) and \( \text{GSp}(4) \) as defined with respect to \( J' \), as in Section 4, given in both directions by conjugation by the matrix

\[
C = \left[ \begin{array}{ccc} 1 & \circ & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right].
\]

Under this isomorphism, the paramodular group \( K(p^n) \) of level \( p^n \) as defined in (13) with respect to the \( J' \) realization, as in Section 4, is mapped to \( K(p^n) \). Let

\[
\eta' = \left[ \begin{array}{ccc} 1 & p^{-1} & 1 \\ p & 1 & 1 \end{array} \right] \quad \text{and} \quad \tau' = \left[ \begin{array}{ccc} 1 & p^{-1} & 1 \\ 1 & 1 & p \end{array} \right].
\]

Corollary 5.1. Let \( p \) be a prime of \( \mathbb{Q} \). Let \( (\pi, V) \) be a smooth representation of \( \text{GSp}(4, \mathbb{Q}_p) \) for which the center acts trivially, and let \( \chi \) be a quadratic character of \( \mathbb{Q}_p^\times \) of conductor \( p \). Let \( v \in V \) be such that \( \pi(k)v = v \) for \( k \in \text{GSp}(4, \mathbb{Z}_p) \). Then, the twisting map \( T_\chi : V(0) \to V(4, \chi) \) is given
by the formula $T_{\chi}(v) = \sum_{i=1}^{14} T^i_{\chi}(v)$ for $v \in V(0)$, where

$$T^1_{\chi}(v) = p^{-11} \sum_{a,b,x \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^{\times}} \chi(ab) \pi\left(1 - (a + xb)p^{-1} \begin{pmatrix} 1 & 1 \\ (a + xb)p^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & zp^{-4} & bp^{-2} \\ 1 & bp^{-2} & x^{-1}p^{-1} \end{pmatrix} \right)v,$$

$$T^2_{\chi}(v) = p^{-11} \eta' \sum_{a,b \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^{\times}} \chi(abxy) \pi\left(1 \begin{pmatrix} bp^{-1} \\ -bp^{-1} \end{pmatrix} \begin{pmatrix} 1 & -ap^{-2} \\ -ap^{-2} & ab^{-1}(1-x)^{-1}p^{-1} \end{pmatrix} \right)v,$$

$$T^3_{\chi}(v) = p^{-6} \eta' \sum_{a \in (\mathbb{Z}_p/p^4\mathbb{Z}_p)^{\times}} \chi(b(1-z)) \pi\left(1 \begin{pmatrix} bp^{-3} \\ ap^{-2} \\ a^2b^{-1}zp^{-1} \end{pmatrix} \right)v,$$

$$T^4_{\chi}(v) = p^{-10} \eta' \sum_{a \in (\mathbb{Z}_p/p^4\mathbb{Z}_p)^{\times}} \chi(b) \pi\left(1 \begin{pmatrix} xp^{-2} \\ -xp^{-2} \end{pmatrix} \begin{pmatrix} 1 & (bp - ax)p^{-4} & ap^{-2} \\ 1 & ap^{-2} & 1 \end{pmatrix} \right)v,$$

$$T^5_{\chi}(v) = p^{-9} \eta' \sum_{a,b \in (\mathbb{Z}_p/p^5\mathbb{Z}_p)^{\times}} \chi(b) \pi\left(1 \begin{pmatrix} xp^{-1} \\ -xp^{-1} \end{pmatrix} \begin{pmatrix} 1 & (b - ax)p^{-3} & ap^{-2} \\ 1 & ap^{-2} & 1 \end{pmatrix} \right)v,$$

$$T^6_{\chi}(v) = p^{-6} \eta'^2 \sum_{a,b \in (\mathbb{Z}_p/p^5\mathbb{Z}_p)^{\times}} \chi(bx) \pi\left(1 \begin{pmatrix} b(1-x)p^{-2} \\ ap^{-2} \\ a^2b^{-1}p^{-2} \end{pmatrix} \right)v,$$

$$T^7_{\chi}(v) = p^{-7} \tau' \sum_{a \in (\mathbb{Z}_p/p^5\mathbb{Z}_p)^{\times}} \chi(ab) \pi\left(1 \begin{pmatrix} -ap^{-2} \\ ap^{-2} \end{pmatrix} \begin{pmatrix} 1 & zp^{-4} & bp^{-1} \\ 1 & bp^{-1} & 1 \end{pmatrix} \right)v,$$

$$T^8_{\chi}(v) = p^{-9} \eta' \tau' \sum_{a,b,z \in (\mathbb{Z}_p/p^7\mathbb{Z}_p)^{\times}} \chi(abz(1-z)) \pi\left(1 \begin{pmatrix} bp^{-2} \\ -bp^{-2} \end{pmatrix} \right)v.$$
\[
\begin{pmatrix}
1 & -ab(1 - z)p^{-3} & ap^{-1} \\
1 & ap^{-1} & 1 \\
1 & 1 & 1
\end{pmatrix}
v, \quad T_{\chi}^0(v) = p^{-6} \eta'^2 \tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
b, x \in (\mathbb{Z}_p/p \mathbb{Z}_p) \times}} \chi(b)\pi(1)
\begin{pmatrix}
ap^{-1} & 1 & 1 \\
1 & -ap^{-1} & 1 \\
1 & 1 & 1
\end{pmatrix}v, \quad T_{\chi}^1(v) = p^{-6} \eta'^2 \tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
b \in (\mathbb{Z}_p/p^3 \mathbb{Z}_p) \times \\
x \in \mathbb{Z}_p/p^3 \mathbb{Z}_p \\
z \in \mathbb{Z}_p/p^4 \mathbb{Z}_p}} \chi(b)\pi(1)
\begin{pmatrix}
ap^{-2} & -bp^{-1} & 1 \\
1 & 1 & -ap^{-2} \\
1 & -bp^{-1} & 1
\end{pmatrix}v, \quad T_{\chi}^2(v) = p^{-10} \tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
b \in (\mathbb{Z}_p/p^3 \mathbb{Z}_p) \times \\
x \in \mathbb{Z}_p/p^3 \mathbb{Z}_p \\
z \in \mathbb{Z}_p/p^4 \mathbb{Z}_p}} \chi(ab\pi(1)) \begin{pmatrix}1 & ap^{-1} \\
1 & 1 \\
1 & -ap^{-1} \end{pmatrix}v, \quad T_{\chi}^3(v) = p^{-12} \eta'^2 \tau' \sum_{\substack{y \in \mathbb{Z}_p/p^4 \mathbb{Z}_p \\
a \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
b, z \in (\mathbb{Z}_p/p^3 \mathbb{Z}_p) \times \\
x \neq 1(p)}} \chi(ab \pi(1 - z)) \begin{pmatrix}1 & ap^{-1} \\
1 & 1 \\
1 & -ap^{-1} \end{pmatrix}v, \quad T_{\chi}^4(v) = p^{-6} \eta'^2 \tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
b \in (\mathbb{Z}_p/p^3 \mathbb{Z}_p) \times \\
x \in \mathbb{Z}_p/p^3 \mathbb{Z}_p}} \chi(bx)\pi(1)
\begin{pmatrix}1 & b(1 - x)p^{-1} & ap^{-2} \\
1 & ap^{-2} & a^2 b^{-1} p^{-3} \\
1 & 1 & 1
\end{pmatrix}v, \quad T_{\chi}^5(v) = p^{-6} \eta'^2 \tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
b \in (\mathbb{Z}_p/p^2 \mathbb{Z}_p) \times \\
x \in \mathbb{Z}_p/p^2 \mathbb{Z}_p}} \chi(b)\pi(1)
\begin{pmatrix}1 & -bp^{-1} & ap^{-2} \\
1 & ap^{-2} & xp^{-1} \\
1 & 1 & 1
\end{pmatrix}v.
Proof. This result follows directly from Theorem 4.2. Converting to $\mathbb{Q}_p$ and conjugating by $C$ gives:

$$T_1^1(v) = p^2 \int \int \int \int \chi(ab) \pi(C) \left[ \begin{array}{ccc} 1 & -(a+xb)p^{-1} & 1 \\ 1 & (a+xb)p^{-1} & 1 \\ 1 & 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc} 1 & bp^{-2} & zp^{-4} \\ 1 & x^{-1}p^{-1} & bp^{-2} \\ 1 & 0 & 1 \end{array} \right] C \, da \, db \, dx \, dz$$

$$= p^2 \int \int \int \int \chi(ab) \pi(\left[ \begin{array}{ccc} 1 & -(a+xb)p^{-1} & 1 \\ 1 & (a+xb)p^{-1} & 1 \\ 1 & 0 & 1 \end{array} \right])$$

$$\left[ \begin{array}{ccc} 1 & bp^{-2} & zp^{-4} \\ 1 & x^{-1}p^{-1} & bp^{-2} \\ 1 & 0 & 1 \end{array} \right] v \, da \, db \, dx \, dz$$

$$= p^{-11} \sum_{z \in \mathbb{Z}_p/p^2 \mathbb{Z}_p} \sum_{a,b,x \in (\mathbb{Z}_p/p^3 \mathbb{Z}_p)^\times} \chi(ab) \pi(\left[ \begin{array}{ccc} 1 & -(a+xb)p^{-1} & 1 \\ 1 & (a+xb)p^{-1} & 1 \\ 1 & 0 & 1 \end{array} \right])$$

$$\left[ \begin{array}{ccc} 1 & bp^{-2} & zp^{-4} \\ 1 & x^{-1}p^{-1} & bp^{-2} \\ 1 & 0 & 1 \end{array} \right] v.$$