Neutrino oscillations in the Kerr–Newman spacetime

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Received 2 December 2009, in final form 16 January 2010
Published 2 March 2010
Online at stacks.iop.org/CQG/27/065011

Abstract

The mass neutrino oscillation in the Kerr–Newman (K–N) spacetime is studied in the plane \( \theta = \theta_0 \), and general equations of the oscillation phases are given. The effect of the rotation and electric charge on the phase is presented. Then, we consider three special cases. (1) The neutrinos travel along the geodesics with angular momentum \( L = aE \) in the equatorial plane. (2) The neutrinos travel along the geodesics with \( L = 0 \) in the equatorial plane. (3) The neutrinos travel along the radial geodesics in the direction \( \theta = 0 \). Finally, we calculate the proper oscillation length in the K–N spacetime. The effect of the gravitational field on the oscillation length is embodied in the gravitational red shift factor. When the neutrino travels out of the gravitational field, a blue shift of the oscillation length takes place. We discuss the variation of the oscillation length influenced by the gravitational field strength, the rotation \( a^2 \) and charge \( Q \).

PACS numbers: 95.30.Sf, 14.60.Pq

1. Introduction

Mass neutrino mixing and oscillations were proposed by Pontecorvo [1], and Mikheyev, Smirnov and Wolfenstein (MSW for short) explored the effect of transformation of one neutrino flavor into another in a medium with varying density [2, 3]. Recently, the consideration of mass neutrino oscillations has been a hot topic. There have been many theoretical [4–11] and experimental [12–18] studies about neutrino oscillations. The neutrino oscillations in the flat spacetime were then extended to cases in the curved spacetime [19–26]. Neutrino oscillation experiments were also considered to test the equivalence principle [27]. Calculating the phase along the geodesic line will produce a factor of 2 in the high energy limit, compared with the value along the null line, which often exists in the flat and Schwarzschild spacetime.
This issue of the factor of 2 is due to the difference between the time-like and null geodesics. Furthermore, some alternative mechanisms have been proposed to account for the gravitational effect on the neutrino oscillation [32–34]. The inertial effects on neutrino oscillations and neutrino oscillations in non-inertial frames were also noted [35, 36]. As a further theoretical exploration, neutrino oscillations in spacetime with both curvature and torsion [37–39] have been studied.

In recent years, research on the neutrino oscillation has made new progress. A further mechanism to generate pulsar kicks, which was based on the spin flavor conversion of neutrinos propagating in a gravitational field, and the neutrino geometrical optics in a gravitational field (in particular in a Lense–Thirring background) were proposed by Lambiase [40, 41]. Some publications were centered on the theoretical study and experimental measurement of the mixing angle $\theta_{13}$ [42–44], and CP violation in neutrino oscillations was considered by some authors [45–48]. In addition, Cuesta and Lambiase studied the neutrino mass spectrum [49]. Akhmedov, Maltoni and Smirnov presented the neutrino oscillograms for different oscillation channels and discussed the effects of non-vanishing 1-2 mixing [50].

In this paper, we extend the mass neutrino oscillation work from the Schwarzschild spacetime to the Kerr–Newman spacetime, since the Kerr–Newman metric is rather important in black hole physics, where a most generally stationary solution with axial symmetry exists [51]. For the reason of simplicity, we confine our treatment to two generation neutrinos (electron and muon). We give general equations of the oscillation phases along the arbitrary null and the time-like geodesics, respectively, in the equal $\theta$ plane, $\theta = \theta_0$. The phase along the geodesic will also produce a factor of 2 in the K–N spacetime, $\Phi_{\text{null}} = 2 \Phi_{\text{null}}$, in the high energy limit. In our derivation we have not assumed a weak field approximation.

We discuss three special cases. Firstly, the oscillation phases along the geodesics with $L = aE$ are considered in the equatorial plane. $E$ is the energy per unit mass of the particle. $L$ and $a$ are the angular momentum per unit mass of the particle and the K–N spacetime, respectively. The geodesics with $L = aE$ in the K–N spacetime play the same roles as the radial geodesics in the Schwarzschild and in the Reissner–Nordstrom geometry. In this case, the phases along both the null geodesic and the time-like geodesic are similar in form to the phases along the radial geodesics (null and time-like) in the Schwarzschild spacetime. Secondly, we calculate the oscillation phases along the geodesics with $L = 0$ in the equatorial plane. This kind of geodesics is also important in the K–N spacetime. In the Schwarzschild spacetime with non-rotating spherical symmetry, particles with $L = 0$ can propagate along the radial geodesics. But in the K–N spacetime, because of the dragging effect, the coordinate $\varphi$ must change if a particle with $L = 0$ travels along the geodesics. Thirdly, the phases along the radial geodesics in the direction $\theta = 0$ are given. Only at the poles $\theta = 0$ and $\theta = \pi$ does the ergosphere coincide with the event horizon. In the direction $\theta = 0$, the effects of the rotation of the spacetime on the oscillation length are found to be more than those in the other directions.

Finally, we calculate the proper oscillation length in the K–N spacetime. The oscillation length is proportional to the local energy (local measurement), $E^{\text{loc}} = E/\sqrt{g_{00}}$, of the neutrino, where $E$ is a constant along the geodesic. The decrease in the local energy leads to the decrease in the oscillation length as the neutrino travels out of the gravitational field. So, blue shift of the oscillation length occurs, which is unlike the case of the gravitational red shift for the light signal. In the equatorial plane in the K–N spacetime, the rotation has no contribution to the oscillation length because $g_{00}$ has nothing to do with the rotating parameter $a$ in this plane. The rotation $a^2$ of the gravitational field shortens the oscillation length in the other equal $\theta$ plane, compared with the length in the R–N spacetime. We also find that the length varies...
according to $\theta$. The charge $Q$ shortens it too, compared with the Kerr spacetime case. But, the gravitational field lengthens it, compared with the case in the flat spacetime.

In this paper, we take the neutrino as a spin-less particle to go along the geodesic because the spin and the curvature coupling has a little contribution to the geodesic derivation [52]. Moreover, the neutrino is a high energy particle, so we do not think that the neutrino spin has more contribution to the geodesic.

The paper is organized as follows. In section 2, we briefly review the standard treatment of neutrino oscillation in the flat spacetime. In section 3, we give general expressions of the oscillation phases along the null and time-like geodesics in an arbitrary equal $\theta = \theta_0$ plane. In section 4, we discuss the neutrino phase in three special cases. In section 5, we discuss the proper oscillation length in the K–N spacetime. Finally, the conclusion and discussion are given. Throughout the paper, the units $G = c = \hbar = 1$ and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ are used.

2. The standard treatment of neutrino oscillation in the flat spacetime

In a standard treatment, the flavor eigenstate $|\nu_\alpha\rangle$ is a superposition of the mass eigenstates $|\nu_k\rangle$, i.e. [21, 22]

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k} \exp[-i\Phi_k]|\nu_k\rangle,$$

(1)

where

$$\Phi_k = E_k t - \vec{p}_k \cdot \vec{x}, \quad (k = 1, 2),$$

(2)

and the matrix elements $U_{\alpha k}$ comprise the transformation between the flavor and mass bases. $E_k$ and $\vec{p}_k$ are the energy and momentum of the mass eigenstates $|\nu_k\rangle$, respectively, and they are different for different mass eigenstates. If the neutrino is produced at a spacetime point $A(t_A, \vec{x}_A)$ and detected at $B(t_B, \vec{x}_B)$, the expression for the phase in equation (2), which is coordinate independent and suitable for application in a curved spacetime, is [21, 53]

$$\Phi_k = \int_A^B p^{(k)}_{\mu} \, dx^\mu,$$

(3)

where

$$p^{(k)}_{\mu} = m_k g_{\mu\nu} \frac{dx^\nu}{ds}$$

(4)

is the canonical conjugate momentum to the coordinate $x^\mu$ and $m_k$ is the rest mass corresponding to the mass eigenstate $|\nu_k\rangle$. $g_{\mu\nu}$ and $s$ are the metric tensor and an affine parameter, respectively.

The following assumptions are often applied in the standard treatment [4]: (1) the mass eigenstates are taken to be the energy eigenstates, with a common energy; (2) up to $O(m/E)$, there is the approximation $E \gg m$; (3) a massless trajectory is assumed, which means that the neutrino travels along the null trajectory. In the case of two neutrinos mixing $\nu_e - \nu_\mu$, we can write

$$\nu_e = \cos \theta \nu_1 + \sin \theta \nu_2, \quad \nu_\mu = -\sin \theta \nu_1 + \cos \theta \nu_2.$$

(5)

Here, $\theta$ is the vacuum mixing angle. The oscillation probability that the neutrino produced as $|\nu_e\rangle$ is detected as $|\nu_\mu\rangle$ is [54]

$$P(\nu_e \rightarrow \nu_\mu) = |\langle \nu_e | \nu_\mu(x, t) \rangle|^2 = \sin^2(2\theta) \sin^2 \left( \frac{\Phi_{ij}}{2} \right).$$

(6)
where $\Phi_{kj} = \Phi_k - \Phi_j$ is the phase shift. From the standard treatment of the neutrino oscillation [21–23], the standard result for the phase is

$$\Phi_k \simeq \frac{m_k^2}{2E_0} |\vec{x}_B - \vec{x}_A|.$$  

(7)

Here, $E_0$ is the energy for a massless neutrino. So, the phase shift responsible for the oscillation is given by

$$\Phi_{kj} \simeq \frac{\Delta m_{kj}^2}{2E_0} |\vec{x}_B - \vec{x}_A|,$$  

(8)

where $\Delta m_{kj}^2 = m_k^2 - m_j^2$.

3. Neutrino oscillation phase along the null and the time-like geodesic in the plane $\theta = \theta_0$

In this section, we study the neutrino oscillation in an equal $\theta = \theta_0$ surface. The line element of the K–N spacetime takes the form

$$ds^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\varphi^2 + 2g_{03} dt d\varphi.$$  

(9)

The relevant components of the canonical momentum of the $k$th massive neutrino in equation (4) are

$$p_t^{(k)} = p_0^{(k)} = m_k g_{00} \dot{t} + m_k g_{03} \dot{\varphi},$$

$$p_r^{(k)} = m_k g_{11} \dot{r},$$

$$p_\varphi^{(k)} = m_k g_{33} \dot{\varphi} + m_k g_{03} \dot{t},$$

(10)

where $i = \frac{\partial}{\partial t}, r = \frac{\partial}{\partial r}$ and $\varphi = \frac{\partial}{\partial \varphi}$. Because the metric tensor components do not depend on the coordinates $t$ and $\varphi$, their canonical momenta $p_t^{(k)}$ and $p_\varphi^{(k)}$ are constant along the trajectory. In fact, the momentum $p_0^{(k)}$ conjugate to $t$ is the asymptotic energy of the neutrino at $r = \infty$. It is stressed that the covariant energy $p_0$ (not $p^0$) is the constant of motion. Otherwise the ambiguous definition of the energy will lead to confusion in understanding the neutrino oscillation.

The phase along the null geodesic from point A to point B is given by [21, 23, 53]

$$\Phi_{null} = \int_A^B p_{\alpha}^{(k)} dx^\alpha = \int_A^B \left( p_0^{(k)} \frac{dt}{dr} + p_\varphi^{(k)} \frac{d\varphi}{dr} + p_r^{(k)} \right) dr.$$  

(11)

We can obtain the following relations which are useful in the calculation:

$$g_{00} = -\frac{g_{33}}{\Delta \sin^2 \theta}, \quad g_{33} = -\frac{g_{00}}{\Delta \sin^2 \theta},$$

$$g_{03} = \frac{g_{03}}{\Delta \sin^2 \theta}, \quad g_{30}^2 - g_{00}g_{33} = \Delta \sin^2 \theta,$$  

(12)

where $\Delta = r^2 - 2Mr + a^2 + Q^2$. Solving equation (10) for $t$ and $\varphi$, we obtain

$$t = -\frac{g_{33}E_k + g_{03}L_k}{\Delta \sin^2 \theta}, \quad \varphi = \frac{g_{03}E_k + g_{00}L_k}{\Delta \sin^2 \theta},$$

(13)

where $E_k = \frac{\dot{\varphi}^{(k)}}{m_k}$ and $L_k = -\frac{\dot{t}^{(k)}}{m_k}$ are the energy and angular momentum per unit mass, respectively.
In the standard treatment of the neutrino oscillation, the neutrino is usually supposed to travel along the null [4, 21, 22, 54–56]. Following the standard treatment, we will calculate the phase along the light-ray trajectory from $A$ to $B$.

The Lagrangian appropriate to motions in the plane (for which $\dot{\theta} = 0$ and $\theta = a$ constant) is [57]

$$L = \frac{1}{2}(g_{00}\dot{t}^2 + 2g_{03}\dot{t}\dot{\phi} + g_{11}\dot{r}^2 + g_{33}\dot{\phi}^2). \tag{14}$$

The Hamiltonian is given by

$$H = \mathcal{E}_k \dot{t} - \mathcal{L}_k \dot{\phi} + p(k)\dot{r} - \mathcal{L}. \tag{15}$$

Because of the independence of the Hamiltonian on $t$, we can deduce that

$$2H = \mathcal{E}_k \dot{t} - \mathcal{L}_k \dot{\phi} + p(k)\dot{r} = \delta_1 = \text{constant}. \tag{16}$$

Without loss of generality, we can set $\delta_1 = 1$ for time-like geodesics and $\delta_1 = 0$ for null geodesics. Substituting (13) into (16) and setting $\delta_1 = 0$ for null geodesics, we have the radial equation

$$g_{11}\dot{r}^2 = g_{33}\mathcal{E}_k^2 + 2g_{03}\mathcal{E}_k L_k + g_{00}L_k^2 \sin^2 \theta_0/\Delta_1. \tag{17}$$

We define a new function

$$V(r) = g_{33}\mathcal{E}_k^2 + 2g_{03}\mathcal{E}_k L_k + g_{00}L_k^2. \tag{18}$$

The different $V(r)$ determines the phase of the different trajectory. From (17), we get

$$\dot{r} = \frac{\sqrt{-V}}{\rho \sin \theta_0}, \tag{19}$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta_0$. So, the equations governing $t$ and $\phi$ are

$$\frac{dt}{dr} = -\frac{\rho (g_{33}\mathcal{E}_k + g_{03}L_k)}{\Delta \sin \theta_0 \sqrt{-V}}, \quad \frac{d\phi}{dr} = \frac{\rho (g_{03}\mathcal{E}_k + g_{00}L_k)}{\Delta \sin \theta_0 \sqrt{-V}}. \tag{20}$$

The mass-shell condition is [21]

$$m_k^2 = g_{\mu\nu}p^{(k)\mu}p^{(k)\nu} = p_\mu^{(k)\mu} = p_0^{(k)} + p_\phi^{(k)} + p_\psi^{(k)}, \tag{21}$$

Substituting $p^{(k)\mu} = g^{\mu\nu}p^{(k)}_{\nu}$, $p_\phi^{(k)} = g^{33}p_\phi^{(k)} + g^{30}p_0^{(k)}$ and (10) into the equation of the mass-shell condition (21), we obtain

$$p_\phi^{(k)} = \frac{m_k \sqrt{-V} - \sin^2 \theta_0 \Delta}{\rho \sin \theta_0}. \tag{22}$$

In the process of calculation, relations (12) are used. Applying the relativistic condition $p_\phi^{(k)} \gg m_k$, we have the relation

$$p_\phi^{(k)} \simeq \frac{m_k}{\rho \sin \theta_0} \left(\sqrt{-V} - \frac{\sin^2 \theta_0 \Delta}{2\sqrt{-V}}\right). \tag{23}$$

Adopting (20) and $p_\phi^{(k)}$, the phase along the null geodesics (11) is approximated by

$$\Phi_{k\text{null}} \simeq \int_A^B m_k \rho \sin \theta_0 \frac{dr}{2\sqrt{-V}}. \tag{24}$$

The phase (24) is a general result. The different function $V(r)$ corresponds to the different motion and consequently determines the different phase. If $a = 0$, we can obtain the oscillation
phase in the Reissner–Nordstrom spacetime; if \( Q = 0 \), the Kerr spacetime case is given. If \( a = 0, Q = 0 \), the function \( V(r) \) reduces to
\[
V(r) = -\frac{r^2 \sin^2 \theta_0}{E_k^2}.
\]
(25)
The phase (24) becomes
\[
\Phi_{\text{null}}^k = \int_{B}^{A} \frac{m_k^2}{2p_{0}^{(k)}} \frac{dr}{r} = \frac{m_k^2}{2p_{0}^{(k)}} (r_B - r_A).
\]
(26)
This is just the phase in the Schwarzschild spacetime [21, 22].

The velocity of an extremely relativistic neutrino is very close to the speed of light. In the standard treatment, the neutrino is supposed to travel along the null line [4, 21, 22, 54–56]. Despite this, the propagation difference between a massive neutrino and a photon can have important consequences and this tiny derivation becomes important for the understanding of the neutrino oscillation. Therefore, for more general situations, we start to calculate the phase along the time-like geodesic. A factor of 2 will be obtained, when we compare the time-like geodesic phase with the null geodesic phase in the high energy limit. The classical orbit is defined to a plane [21, 23], \( \theta = \theta_0, d\theta = 0 \). The phase along the time-like geodesic is [19, 23, 28, 53]
\[
\Phi_{k}^{\text{geod}} = \int_{A}^{B} p_{\mu}^{(k)} \frac{dx^{\mu}}{r} = \int_{A}^{B} \left( p_{0}^{(k)} \frac{dr}{dr} + p_{\phi}^{(k)} \frac{d\phi}{dr} + p_{r}^{(k)} \right) dr.
\]
(27)
For the time-like geodesic, \( \delta_1 = 1 \), equation (16) becomes
\[
E_k i - L_k \dot{\phi} + \frac{p_{r}^{(k)}}{m_k} \dot{r} = 1,
\]
(28)
while the equations for \( i \) and \( \phi \) (13) are the same for time-like geodesics [57]. Substituting \( i \) and \( \phi \), we have
\[
\frac{ds}{dr} = \frac{1}{\dot{r}} = \frac{\sqrt{-g_{11}}}{\sqrt{-1 - \frac{V}{\Delta \sin^2 \theta_0}}}.
\]
(29)
So, we obtain the equations for \( dt/dr \) and \( d\phi/dr \) for time-like geodesics
\[
\frac{dt}{dr} = \frac{\sqrt{-g_{11}} (g_{33}E_k + g_{03}L_k)}{\Delta \sin^2 \theta_0 \sqrt{-1 - \frac{V}{\Delta \sin^2 \theta_0}}}, \quad \frac{d\phi}{dr} = \frac{\sqrt{-g_{11}} (g_{00}L_k + g_{03}E_k)}{\Delta \sin^2 \theta_0 \sqrt{-1 - \frac{V}{\Delta \sin^2 \theta_0}}}.
\]
(30)
According to the mass-shell condition, \( p_{r}^{(k)} \) is given by
\[
p_{r}^{(k)} = -m_k \sqrt{-g_{11}} \sqrt{-1 - \frac{V}{\Delta \sin^2 \theta_0}}.
\]
(31)
Thus, the phase along the time-like geodesic is
\[
\Phi_{k}^{\text{geod}} = \int_{A}^{B} m_k \frac{\sqrt{-g_{11}}}{\sqrt{-1 - \frac{V}{\Delta \sin^2 \theta_0}}} dr.
\]
(32)
If the high energy limit is taken into account, equation (32) reduces to
\[
\Phi_{k}^{\text{geod}} \approx \int_{A}^{B} m_k \rho \sin \theta_0 \frac{dr}{\sqrt{-V}} = 2\Phi_{k}^{\text{null}}.
\]
(33)
It is often noted that the factor of 2 of the neutrino phase calculations exists in the flat spacetime [29, 30] and in the Schwarzschild spacetime [23, 24, 28], which is believed to be the difference.
between the null geodesic and the time-like geodesic. The neutrino phase induced by the null condition, as in the standard treatment, comes from the 4-momentum $p^\mu$ defined along the time-like geodesic, and equation (17) governing $\dot{r}$ to the null geodesic. If the 4-momentum defined along the null geodesic were instead used to compute the null phase, we would obtain zero because of the null condition. When we calculate the phase along the time-like geodesic, $\dot{r}$ is defined to the time-like geodesic. This is the difference producing the factor 2. It can be proved that the neutrino phase along the null is the half of the value along the time-like geodesic in the high energy limit in a general curved spacetime (see appendix A in [23]).

4. Three special cases

4.1. Oscillation phases along the geodesics with $L_k = aE_k$ in the equatorial plane

It is very important that the geodesic is described in the equatorial plane $\theta = \pi/2$ in the K–N spacetime. The geodesics with $L_k = aE_k$ play the same roles as the radial geodesics in the Schwarzschild and in the Reissner–Nordstrom geometry. In this case, for the null geodesic $\dot{t}$, $\dot{\phi}$ and $\dot{r}$ reduce to

$$\dot{t} = \frac{r^2 + a^2}{\Delta}E_k; \quad \dot{\phi} = \frac{a}{\Delta}E_k; \quad \dot{r} = \pm E_k.$$  

(34)

These equations in fact define the shear-free null-congruences which we use for constructing a null basis for a description of the K–N spacetime in a Newman–Penrose formalism [57]. The function $V(r)$ for the null geodesic becomes

$$V(r) = -\frac{r^2E_k^2}{96}.$$  

(35)

So, the phase along the null is

$$\Phi_{null} \sim \int_A^B \frac{m_k \rho \sin \theta_0}{2\sqrt{-V}} \, dr = \int_A^B \frac{m_k}{2E_k} \, dr = \frac{m_k^2}{2p_0^2} (r_B - r_A),$$  

(36)

which appears in the same form as that of the Schwarzschild spacetime radial oscillation case.

We now turn to a consideration of the time-like geodesic case. The equations for $\dot{t}$, $\dot{\phi}$ are the same as for the null geodesics, while $\dot{r}$ becomes

$$\dot{r} = \sqrt{E_k^2 + \frac{1}{g_{11}}}.$$  

(37)

Substituting $L_k = aE_k$ into (32), we obtain the phase along the time-like geodesic

$$\Phi_{k}^{good} = \int_A^B \frac{m_k \, dr}{\sqrt{\left(\frac{p_0^2}{m_k^2}\right)^2 + \frac{1}{g_{11}}}^{1/2}}.$$  

(38)

Compared with the phase along the radial time-like geodesic in the Schwarzschild spacetime [23],

$$\Phi_{k}^{good} (Sch) = \int_A^B \frac{m_k \, dr}{\sqrt{\left(\frac{p_0^2}{m_k^2}\right)^2 - g_{00}}} = \int_A^B \frac{m_k \, dr}{\sqrt{\left(\frac{p_0^2}{m_k^2}\right)^2 + \frac{1}{g_{11}}}},$$  

(39)

we find that the oscillation phase with $L_k = aE_k$ in the K–N spacetime has a similar form to the phase along the radial in the Schwarzschild spacetime. Substituting $g_{11} = -\frac{r^2}{\Delta}$ into equation (38), we have

$$\Phi_{k}^{good} = \int_A^B \frac{m_k \, dr}{\sqrt{b + \frac{2M}{r} - \frac{a^2 + Q^2}{r^2}}}.$$  

(40)
where \( b = \left( \frac{a}{m_0} \right)^2 - 1 \). Equation (40) can be integrated directly to give

\[
\Phi^\text{geod}_k = \frac{m_k}{b} \sqrt{b r_B^2 + 2 M r_B - a^2 - Q^2} - \frac{m_k}{b} \sqrt{b r_A^2 + 2 M r_A - a^2 - Q^2}
\]

\[
= \frac{M m_k}{b^{3/2}} \ln \frac{b r_B + M + \sqrt{b (b r_B^2 + 2 M r_B - a^2 - Q^2)}}{b r_A + M + \sqrt{b (b r_A^2 + 2 M r_A - a^2 - Q^2)}}.
\]

Equation (41) shows the effects of the rotation \( a^2 \) on the oscillation phase.

If \( a = 0 \), we can obtain \( \dot{t}, \dot{\phi} \) and \( \dot{r} \) along the radial null geodesics in the equatorial plane in the Reissner–Nordstrom spacetime:

\[
\dot{t} = r^2 / (r^2 - 2 M r + Q^2), \quad \dot{\phi} = 0, \quad \dot{r} = \pm E.
\]

Therefore, the phases along the radial null and time-like geodesic in the Reissner–Nordstrom spacetime are given by, respectively,

\[
\Phi^\text{null}_k (\text{RN}) = m_k^2 / 2 p_0^2 (r_B - r_A),
\]

\[
\Phi^\text{geod}_k (\text{RN}) = \int_A^B \frac{m_k \, dr}{\sqrt{b + 2 M / r - Q / r^2}}.
\]

Letting \( a = 0 \) in (41), the integral of equation (43) is given.

### 4.2. Oscillation phases along the geodesics with \( L = 0 \) in the equatorial plane

The geodesics with \( L_k = 0 \) is another important class of geodesics in the K–N spacetime. If the coordinates \( t \) and \( \phi \) have a relation \( d\phi / dt = - g_{03} / g_{33} \), then the canonical momentum \( p_\phi^{(i)} \) in (10) vanishes. The corresponding \( \dot{t}, \dot{\phi} \) and \( \dot{r} \) for the null geodesic are

\[
\dot{t} = - \frac{g_{33}}{\Delta} E_k; \quad \dot{\phi} = \frac{g_{03}}{\Delta} E_k; \quad \dot{r} = \sqrt{- g_{33}} \frac{E_k}{r}.
\]

\( \dot{r} \) for time-like geodesics is

\[
\dot{r} = \sqrt{\frac{1 + g_{33} E_k^2 / \Delta}{g_{11}}}.
\]

Substituting \( L_k = 0 \) into (24) and (32), the phases along the null and time-like geodesics are given by, respectively,

\[
\Phi^\text{null}_k = \int_A^B \frac{m_k^2}{2 p_0^2} \frac{r \, dr}{\sqrt{- g_{33}}} = \int_A^B \frac{m_k^2}{2 p_0^2} \sqrt{- g_{11} g_{00}} \, dr,
\]

\[
\Phi^\text{geod}_k = \int_A^B \frac{\sqrt{- g_{11} g_{00}} m_k \, dr}{\sqrt{\left( \frac{m_k}{m_0} \right)^2 - \tilde{g}_{00}}},
\]

where \( \tilde{g}_{00} = g_{00} - g_{33} / g_{33} \). It is difficult to integrate (46) and (47) directly. We can work them out by expanding as \( a^2 \), when \( a^2 \) is a small quantity.
4.3. Oscillation phase along the radial geodesic at $\theta = 0$

Unlike in the Schwarzschild and in the Reissner–Nordstrom spacetime, the event horizon does not coincide with the ergosphere where $g_{00}$ vanishes in the K–N spacetime. This is an important feature which distinguishes the K–N spacetime from the others. The ergosphere that is a stationary limit surface coincides with the event horizon only at the poles $\theta = 0$ and $\theta = \pi$. The phase along the null geodesic in the direction $\theta = 0$ can be written as

$$\Phi^\text{null}_k = \int_A^B \frac{m_k \rho \sin \theta_0 \, dr}{2 \sqrt{-V}}$$

(48)

Substituting $\theta_0 = 0$, equation (48) becomes

$$\Phi^\text{null}_k = \int_A^B \frac{m_k^2 \rho \, dr}{2 p_0^{(k)}} = \frac{m_k^2}{2 p_0^{(k)}} (r_B - r_A).$$

(49)

By a similar calculation, the phase along the time-like geodesics at $\theta = 0$ is given by

$$\Phi^\text{geod}_k = \int_A^B \frac{m_k \, dr}{(b + 2M r - Q^2) V}.$$  

(50)

where $b = \left( \frac{g^{(k)}_{\mu\nu}}{m_k} \right)^2 - 1$.

5. Proper oscillation length

The propagation of a neutrino is over its proper distance, but $dr$ in (24) is only a coordinate. The proper distance can be written as [58]

$$dl = \sqrt{\left( \frac{g_{\mu\nu} g_{\nu\nu}}{g_{00}} - g_{\mu\nu} \right) dx^\mu dx^\nu}. \quad (51)$$

In the K–N spacetime, we have

$$dl = \sqrt{-g_{11} dr^2 + \left( \frac{g_{13}}{g_{00}} - g_{33} \right) d\phi^2}. \quad (52)$$

Substituting $\frac{dw}{dr}$, we obtain

$$dr = \frac{\sqrt{-g_{00}}} {\sqrt{\Delta E_k} \sin \theta_0} \, dl. \quad (53)$$

For convenience of discussion, we adopt the differential form of (24)

$$d\Phi^\text{null}_k = \frac{m_k \rho \sin \theta_0 \, dr}{2 \sqrt{-V}}. \quad (54)$$

Substituting (53), we have

$$d\Phi^\text{null}_k = \frac{m_k^2}{2 p_0^{(k)}} \sqrt{g_{00}} \, dl. \quad (55)$$

It is assumed that the mass eigenstates are taken to be the energy eigenstates, with a common energy in the standard treatment. The equal energy assumption is considered to be correct by some authors [29, 31, 59] and is studied carefully in papers [24, 60, 61]. In addition, it
is adopted widely in a great deal of literature, for example [21–23, 62]. \( p_0 \) will represent the common energy of different mass eigenstates. In fact, the condition of equal momentum is also adopted to study the neutrino oscillation. In the flat spacetime, both conditions (the equal energy and the equal momentum) present practically the same neutrino oscillation results [24]. There are conditions of time translation invariance and space translation invariance in the flat spacetime. So, the energy conservation and momentum conservation of a free particle are right. In the curved (stationary) spacetime, the energy of a particle is conserved along the geodesic due to the existence of a time-like Killing vector field. However, the canonical conjugate momentum to \( r \), \( p_\rho \) is not conserved because \((\frac{\partial}{\partial r})^a\) is not Killing in the curved (stationary) spacetime. Consequently, it is very difficult to study neutrino oscillation if the condition of equal momentum is adopted in the curved spacetime. In this section, our discussion is on the basis of the results in the standard treatment in which the phase is calculated along the null.

Then, the phase shift which determines the oscillation is

\[
d\Phi_{null}^{kj} = d\Phi_{null}^k - d\Phi_{null}^j = \frac{\Delta m^2_{kj}}{2p_0}\sqrt{g_{00}}\,dl,
\]

where \( \Delta m^2_{kj} = m_{\beta}^2 - m_{\alpha}^2 \). Equation (56) can be rewritten as

\[
\frac{dl}{a(\Phi_{null}^j)} = \frac{4\pi p_0}{\Delta m^2_{kj}\sqrt{g_{00}}} = \frac{4\pi p^{h_{0c}}}{\Delta m^2_{kj} \sqrt{g_{00}}}.
\]

The term \( \frac{4\pi p_0}{\Delta m^2_{kj}\sqrt{g_{00}}} \) in (57) can be interpreted as the oscillation length \( L_{osc} \) (which is defined by the proper distance as the phase shift \( \Phi_{null}^{kj} \) changing \( 2\pi \)) measured by the observer at rest at a position \( r \), and \( p^{h_{0c}} = p_0/\sqrt{g_{00}} \) is the local energy. As \( r \to \infty \), \( p^{h_{0c}} \) approaches the energy \( p_0 \) measured by the observer at infinity. \( \frac{4\pi p_0}{\Delta m^2_{kj} \sqrt{g_{00}}} \) is the oscillation length in the flat spacetime. Equation (57) is of universal significance in the curved spacetime. In fact, \( \sqrt{g_{00}} \) is the gravitational red shift factor which shows the effect of the gravitational field on the oscillation length. Consider two static observers \( O \) (the radial coordinate \( r \)) and \( O' \) (the radial coordinate \( r' \)). The oscillation length measured by \( O \) and by \( O' \) is, respectively,

\[
\begin{align*}
L_{osc}(r) &= \frac{4\pi p_0}{\Delta m^2_{kj} \sqrt{g_{00}(r)}}, \\
L_{osc}(r') &= \frac{4\pi p_0}{\Delta m^2_{kj} \sqrt{g_{00}(r')}}.
\end{align*}
\]

We can obtain the relation

\[
\frac{L_{osc}(r')}{L_{osc}(r)} = \frac{\sqrt{g_{00}(r')}}{\sqrt{g_{00}(r)}}.
\]

If \( r' > r \), we have \( L_{osc}(r') < L_{osc}(r) \) and blue shift occurs. Physically, the oscillation length is proportional to the local energy of the neutrino. When the neutrino travels out of the gravitational field, the local energy decreases. Consequently, the neutrino oscillation length decreases and blue shift takes place. From equation (57), the oscillation length increases in the gravitation field because of \( 0 < g_{00} < 1 \) out of the the infinite red shift surface. The effect of the gravitational blue shift on the oscillation length may have a possible observable effect from experiments. In the Schwarzschild spacetime, \( g_{00} = 1 - 2M/r \), we have

\[
L_{osc}(Sch) = \frac{4\pi p_0}{\Delta m^2_{kj} \sqrt{1 - 2M/r}}.
\]

In order to study the influence of charge on the neutrino oscillation, we consider the oscillation length in the Reissner–Nordstrom spacetime:

\[
L_{osc}(RN) = \frac{4\pi p_0}{\Delta m^2_{kj} \sqrt{g_{00}}} = \frac{4\pi p_0}{\Delta m^2_{kj} \sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}}.
\]
Compared with the case in the Schwarzschild spacetime, the oscillation length decreases due to the influence of charge \( Q \).

The metric component \( g_{00} \) in the K–N spacetime is

\[
g_{00} = 1 - \frac{2Mr - Q^2}{\rho^2},
\]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \). In the equatorial plane, there is \( g_{00} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \), which is the same as \( g_{00} \) in the Reissner–Nordstrom spacetime. Thus, it is concluded that the neutrino oscillation length along the geodesics in the equatorial plane in the K–N spacetime is identical to that in the Reissner–Nordstrom spacetime, and the rotating parameter \( a^2 \) does not work in this plane. Therefore, we have to select the other plane \( \theta = \theta_0 \neq \pi/2 \) to highlight the effect of rotation on the oscillation length. In the plane \( \theta = \theta_0 \), the oscillation length can be written as

\[
L_{\text{OSC}}(K - N) = \frac{4\pi p_0}{\Delta m^2_{kj}} \frac{1}{\sqrt{1 - \frac{2Mr - Q^2}{r^2 + a^2 \cos^2 \theta_0}}},
\]

(63)

It is obvious that the oscillation length decreases too because of the rotation of the gravitational field compared with that in the R–N spacetime. Letting \( Q = 0 \) in (63), the oscillation length in the Kerr spacetime is given by

\[
L_{\text{OSC}}(\text{Kerr}) = \frac{4\pi p_0}{\Delta m^2_{kj}} \frac{1}{\sqrt{1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta_0}}},
\]

(64)

Compared with (63), the charge \( Q \) shortens the oscillation length. We can obtain that the oscillation length varies with \( \theta_0 \) by

\[
\frac{d}{d\theta_0} L_{\text{OSC}}(K - N) = \frac{4\pi p_0}{\Delta m^2_{kj}} \frac{1}{(g_{00})^{3/2}} \frac{2Mr - Q^2}{(r^2 + a^2 \cos^2 \theta_0)^2} a^2 \sin \theta_0 \cos \theta_0.
\]

(65)

In the K–N spacetime, we conclude that the oscillation length increases with \( \theta \) within \( 0 < \theta < \pi/2 \), and it reaches a maximum in the equatorial plane. Then, it decreases with \( \theta \) within \( \pi/2 < \theta < \pi \). In the direction \( \theta = 0 \) and \( \theta = \pi \), the oscillation length reaches a minimum,

\[
L_{\text{OSC}}(K - N) = \frac{4\pi p_0}{\Delta m^2_{kj}} \frac{1}{\sqrt{1 - \frac{2Mr - Q^2}{r^2 + a^2}}}.
\]

(66)

In summary, the gravitational field lengthens the oscillation length; both the rotation \( a^2 \) and the charge \( Q \) shorten the oscillation length.

6. Conclusion and discussion

In this paper, we have given the phase of the mass neutrino propagating along the null and the time-like geodesic in the gravitational field of a rotating symmetric and charged object, which is described by the Kerr–Newman metric. Most astrophysical bodies in the universe have rotation and charge generally. Thus, work on the neutrino oscillation in the K–N spacetime is important and meaningful for black hole astrophysics. We work out the general formula of the oscillation phase on the equal \( \theta = \theta_0 \) plane with the generality. The phase along the geodesic is the double of that along the null in the high energy limit, which is the same in the cases of the flat and the Schwarzschild spacetime. By setting \( \theta = \pi/2 \), the phases in the equatorial plane are given. As \( a = 0 \) or \( Q = 0 \), we obtain the phases in the R–N spacetime or in the
Kerr spacetime. Moreover, we study three special cases in the K–N spacetime: geodesics with \( L = aE \); geodesics with \( L = 0 \) and radial geodesics at \( \theta = 0 \). Among them, the geodesics with \( L = aE \) have the same importance as the radial geodesics in the Schwarzschild and in the R–N geometry. The phases obtained are very similar in form to the cases along the radial geodesics in the Schwarzschild and in the R–N spacetime.

In section 5, the proper oscillation length in the K–N spacetime is studied in detail. We find that oscillation length in the curved spacetime is proportional to the local energy, which is regraded as the neutrino ‘climbs out of the gravitational potential well’. So, a blue shift occurs. Then, the effects of rotation and charge of the spacetime on the oscillation length are given. Because of the correction of the gravitation field, the oscillation length increases, compared with the flat spacetime case. However, both the rotation \( a^2 \) and the charge \( Q \) shorten the oscillation length. It is noted that the rotation has a null contribution to the length in the equatorial plane in the K–N spacetime, because the red shift factor is independent of \( a^2 \) in this plane. Finally, we remark that our result can be exploited to study the neutrino oscillation near a rotating compact star, neutron star or black hole.

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