Unification types and union splittings in intermediate logics

WOJCIECH DZIK, Institute of Mathematics, Silesian University, Bankowa 14, Katowice 40-007, Poland; wdzik@wdzik.pl

SLAWOMIR KOST, Institute of Computer Science, University of Opole, Oleska 48, Opole 45-052, Poland; skost@uni.opole.pl

PIOTR WOJTYLAK, Institute of Computer Science, University of Opole, Oleska 48, Opole 45-052, Poland; pwojtylak@uni.opole.pl

Abstract
Following a characterization of locally tabular logics with finitary (or unitary) unification by their Kripke models we determine the unification types of some intermediate logics (extensions of INT). There are exactly four maximal logics with nullary unification \( L(\mathcal{R}_2^+) \), \( L(\mathcal{R}_2) \cap L(\mathcal{F}_2) \), \( L(\mathcal{F}_2) \) and \( L(\mathcal{G}_3^+) \) and they are tabular. There are only two minimal logics with hereditary finitary unification: \( L(\mathcal{F}_{un}) \), the least logic with hereditary unitary unification, and \( L(\mathcal{F}_{pr}) \) the least logic with hereditary projective approximation; they are locally tabular. Unitary and non-projective logics need additional variables for mgu’s of some unifiable formulas, and unitary logics with projective approximation are exactly projective. None of locally tabular intermediate logics has infinitary unification. Logics with finitary, but not hereditary finitary, unification are rare and scattered among the majority of those with nullary unification, see the example of \( \mathcal{H}_3 \mathcal{B}_2 \) and its extensions.

Keywords: unification types, intermediate logics, locally tabular logics, Kripke models.

1 Introduction.
Unification, in general, is concerned with finding a substitution that makes two terms equal. Unification in logic is the study of substitutions under which a formula becomes provable in a a given logic \( L \). In this case the substitutions are called the unifiers of the formula in \( L \) (\( L \)-unifiers). If an \( L \)-unifier for a formula \( A \) exists, \( A \) is called unifiable in \( L \). An \( L \)-unifier \( \sigma \) for \( A \) can be more general than the other \( L \)-unifier \( \tau \), in symbols \( \sigma \preceq \tau \); the pre-order \( \preceq \) of substitutions gives rise to four unification types: 1, \( \omega \), \( \infty \), and 0, from the "best" to the "worst", see [2, 1]. Unification is unitary, or it has the type 1, if there is a most general unifier (mgu) for every unifiable formula. Unification is finitary or infinitary if, for every unifiable formula, there is a (finite or infinite) basis of unifiers. Nullary unification means that no such basis of unifiers exists at all.

Silvio Ghilardi introduced unification in propositional (intuitionistic [14] and modal [15]) logic. In [14] he showed that unification in \( \text{INT} \) is finitary, but in \( \text{KC} \) it is unitary and any intermediate logic with unitary unification contains \( \text{KC} \). Dzik [5] uses the particular splitting of the lattice of intermediate logics by the pair \( (L(\mathcal{F}_2), \text{KC}) \), where \( L(\mathcal{F}_2) \) is the logic determined by the "2-fork frame" \( \mathcal{F}_2 \) depicted in Figure 3 to give location of logics with finitary but not unitary unification: they all are included in \( L(\mathcal{F}_2) \). In Wroński [29, 30], see also [7], it is shown that unification in
any intermediate logic $L$ is projective iff $L$ is an extension of $LC$ (that is it is one of Gödel-Dummett logics); projective implies unitary unification. In Ghilardi [17] first examples of intermediate logics with nullary unification are given. Iemhoff [19] contains a proof-theoretic account of unification in fragments of intuitionistic logics. Many papers concern unification in modal logics, see e.g. [15, 21, 3, 8, 22], and also in intuitionistic predicate logic, see [9]. No (modal or intermediate) logic with infinitary unification has been found so far and it is expected that no such logic exists.

Generally, similar results on unification types in transitive modal logics and corresponding intermediate logics are given in [10].

In [17] Ghilardi studied unification in intermediate logics of finite slices (or finite depths). He applied his method, based on Category Theory, of finitely presented projective objects (see [13]) and duality, and characterized injective objects in finite posets. He gave some positive and negative criteria for unification to be finitary. From these criteria it follows, for instance, that bounded depth axioms $H_n$ plus bounded width axioms $B_k$ keep unification finitary. It also follows that there are logics without finitary unification. He considered, among others, the following frames:

![Fig. 1. Ghilardi’s Frames](image)

Since $L(\mathcal{G}_1)$, the logic of $\mathcal{G}_1$, coincides with $H_3B_2$, it has finitary unification by [17]. Theorem 9, p.112 of [17] says that, if $\mathcal{G}_3$ is a frame of any intermediate logic with finitary unification, then $\mathcal{G}_2$ is a frame of this logic, as well. It means, in particular, that $L(\mathcal{G}_3)$ has not finitary unification. (the unification type of $L(\mathcal{G}_2)$ and $L(\mathcal{G}_3)$ was not determined). Ghilardi announced that ‘attaching a final point everywhere’ provide examples in which unification is nullary. Thus, $L(\mathcal{G}_3+)$ has nullary unification. He also showed that replacing one of maximal elements in $\mathcal{G}_3$, with any finite (rooted) po-set $\mathcal{P}$, gives a frame of logic without finitary unification, see Figure 2.

![Fig. 2. Frames of Logics with Nullary Unification](image)

Hence, there are infinitely many intermediate logics without finitary (by [11]: with

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1Ghilardi’s original notation of frames, as well as our notation of frames in [10], was quite different. All frames depicted in this paper represent finite po-sets.

2The frame received from $\mathcal{G}$, by adding a top (=final) element is denoted by $\mathcal{G}+$.
nullary) unification. In [10] we gave necessary and sufficient conditions for finitary (or unitary) unification in locally tabular logics solely in terms of mappings between (bounded) Kripke models. Our approach was entirely different from that in [17]. A simpler variant of the conditions characterizes logics with projective approximation. Then we applied the conditions to determine the unification types of logics (intermediate or modal) given by relatively simple frames. In particular, we studied tabular modal and intermediate logics determined by the frames in Figure 3.

We proved that unification in the modal (as well as intermediate) logics of the frames $L_1, L_2, L_3, R_2$ and $R_3$ is unitary, in (the logic of) $\mathcal{F}_2$ and $\mathcal{F}_3$ it is finitary and in $\mathcal{G}_3$ and $\mathcal{G}_3+$ it is nullary. We have also considered $n$-forks $\mathcal{F}_n$ and $n$-rhombuses $\mathcal{R}_n$, for any $n \geq 2$, see Figure 4. We showed that the logic of any fork (including the infinite ‘fork frame’ $\mathcal{F}_\infty$) has projective approximation, and hence it has finitary unification. The logic of any rhombus (including $\mathcal{R}_\infty$) has unitary unification.

Still many questions about unification of intermediate logics and location of particular types remained open. Here is a summary of the results in the present paper. 1) We give another proof that our conditions (see Theorem 4.10) are necessary and sufficient for finitary/unitary unification, as well as for projective approximation (Theorem 4.18) in locally tabular intermediate logics. Variants of the frames in Figure 3 are considered and we determine the unification types of their logics. In particular, we prove that unification in $L(\mathcal{G}_2)$ is finitary and though (we know that) it is also...
finitary in $L(\mathfrak{F}_3)$, it is nullary in their intersection $L(\mathfrak{G}_2) \cap L(\mathfrak{F}_3)$.

2) It turns out that intermediate logics with unitary unification are either projective (hence they are extensions of $L(C)$) or they need new variables for mgu’s of some unifiable formulas. It means that any (non-projective) logic with unitary unification has a unifiable formula $A(x_1, \ldots, x_n)$ which do not have any mgu in $n$-variables (but its mgu’s must introduce additional variables – like in filtering unification). The same result for transitive modal logics is proved in [10].

3) We prove that a locally tabular intermediate logic with infinitary unification does not exist and we think that no intermediate logic has infinitary unification.

4) We claim (and give some evidences) that 'most of' intermediate logics have nullary unification. For instance, logics of the following frames are nullary:

![Frames of Logics with Nullary Unification](image)

**FIG. 5. Frames of Logics with Nullary Unification**

Intermediate logics with nullary unification can be found 'almost everywhere'. Extensions of finitary/unitary logics may have nullary unification, intersections of finitary logics may be nullary. We cannot put apart logics with finitary/unitary unification from those with the nullary one.

5) In structurally complete logics the situation is somehow similar. A.Citkin (see Tzitkin [27]) characterized hereditary structurally complete logics (instead of structurally complete) and showed that a logic $L$ is hereditary structurally complete iff $L$ omits (i.e. $L$ is falsified in) the following frames:

![Citkin’s Frames](image)

**FIG. 6. Citkin’s Frames**

We consider logics with *hereditary finitary unification* that is logics all their extensions have either finitary or unitary unification. We prove that there are exactly four maximal logics with nullary unification: $L(\mathfrak{G}_1)$, $L(\mathfrak{R}_2)$ \(\cap\) $L(\mathfrak{F}_3)$, $L(\mathfrak{G}_3)$ and $L(\mathfrak{G}_3^+)$. Thus, an intermediate logic has hereditary finitary unification if it omits $\mathfrak{G}_1$, $\mathfrak{F}_3$, $\mathfrak{G}_3^+$.

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3We consider rules $r.A/B$, where $A, B$ play the role of formula schemata, i.e. $r$ enables us to derive $\varepsilon(B)$ from $\varepsilon(A)$, for any substitution $\varepsilon$. The rule is said to be *admissible* in an intermediate logic $L$ (or $L$-admissible), if $r \vdash \varepsilon(A)$ implies $r \vdash \varepsilon(B)$, for any substitution $\varepsilon$, that is any $L$-unifier for $A$ must be an $L$-unifier for $B$. The rule is *$L$-derivable* if $A \vdash_r B$. A logic $L$ is *structurally complete* if every its admissible rule is derivable (the reverse inclusion always holds). *Hereditary structural completeness* of $L$ means that any extension of $L$ is structurally complete.

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and one of the frames \( \mathcal{R}_2, \mathcal{F}_3 \). This characterization is not optimal as, for instance, omitting \( \mathcal{F}_2 \) the logic omits \( \mathcal{G}_3 \); omitting \( \mathcal{R}_2 \) it omits \( \mathcal{G}_3 + \) and \( \mathcal{Y}_1 \).

There is no correlation between structural completeness and finitary unification. In particular, since \( \mathcal{C}_1 = \mathcal{F}_3 \) the logic of \( \mathcal{C}_1 \) has projective approximation (and therefore it is finitary), since \( \mathcal{C}_2 = \mathcal{R}_3 \), \( \mathcal{C}_2 \) is unitary and we will show that the fifth \( \mathcal{L}(\mathcal{C}_5) \) is finitary but not hereditary finitary. The remaining frames \( \mathcal{C}_3 \) and \( \mathcal{C}_4 \) coincide with \( \mathcal{G}_3 \) and \( \mathcal{G}_3+ \) and their logics have nullary unification.

6) Two additional classes of logics emerge here: logics with hereditary unitary unification and logics with hereditary projective approximation. We show that an intermediate logic \( L \) has hereditary unitary unification iff \( L \) omits the frames \( \mathcal{Y}_1, \mathcal{F}_2 \) and \( \mathcal{G}_3+ \). A logic \( L \) has hereditary projective approximation iff \( L \) omits the frames \( \mathcal{R}_2 \) and \( \mathcal{G}_3 \). Thus, \( L \) has hereditary finitary unification iff either \( L \) has hereditary unitary unification or \( L \) has hereditary projective characterization. Logics with hereditary projective approximation can be characterized by frames \( \mathcal{L}_d + \mathcal{F}_n \), for any \( d, n \geq 0 \) (that is forks on chains), whereas logics with hereditary unitary unification by \( \mathcal{L}_d + \mathcal{R}_n \), for any \( d, n \geq 0 \) (that is rhombuses on chains); see Figure 7.

\[
\begin{align*}
L(L(H_{pa})) & \text{ is the least intermediate logic with hereditary projective approximation and } \\
L(L(H_{un})) & \text{ is the least logic with hereditary unitary unification. The logics } L(L(H_{pa})) \text{ and } \\
& L(L(H_{un})) \text{ are locally tabular and they are (the only) minimal logics with hereditary finitary unification. We have } \\
& L(L(H_{pa}) \cup L(H_{un})) = LC \text{ as, it is proved that, any } \\
& \text{unitary intermediate logic with projective approximation is projective.}
\end{align*}
\]

2 Basic Concepts.

2.1 Intermediate Logics.

We consider the standard language of intuitionistic propositional logic \( \{ \rightarrow, \vee, \land, \bot \} \) where \( \leftrightarrow, \neg, \top \) are defined in the usual way. Let \( \text{Var} = \{ x_1, x_2, \ldots \} \) be the set of propositional variables and \( \text{Fm} \) be the set of (intuitionistic) formulas, denoted by \( A, B, C, \ldots \). For any \( n \geq 0 \), let \( \text{Fm}^n \), be the set of formulas in the variables \( \{ x_1, \ldots, x_n \} \), that is \( A \in \text{Fm}^n \iff \text{Var}(A) \subseteq \{ x_1, \ldots, x_n \} \iff A = A(x_1, \ldots, x_n) \).

Substitutions \( \alpha, \beta, \ldots \) are finite mappings; for each \( \alpha \) there are \( k, n \geq 0 \) such that \( \alpha : \{ x_1, \ldots, x_n \} \rightarrow \text{Fm}^k \). The extension of \( \alpha \) to an endomorphism of \( \text{Fm} \) is also denoted by \( \alpha \). Thus, \( \alpha(A) \) means the substitution of a formula \( A \). Let \( \alpha \circ \tau \) be the composition of the substitutions, that is a substitution such that \( \alpha \circ \tau(A) = \alpha(\tau(A)) \), for any \( A \).
An intermediate logic $L$ is any set of formulas containing the intuitionistic logic $\text{INT}$, closed under the modus ponens rule MP and closed under substitutions\footnote{Intermediate logics may be regarded as fragments of transitive modal logics (or extensions of $\text{S4}$, or $\text{Grz}$); the intuitionistic variable $x_i$ is meant as $\Box^+ x_i$ and $A \rightarrow B = \Box^+ (\neg A \vee B)$.}. All intermediate logics form, under inclusion, a (complete distributive) lattice where $\text{inf}\{L_i\}_{i \in I} = \bigcap_{i \in I} L_i$. Let $L(X)$, for any set $X$ of formulas, mean the least intermediate logic containing $X$. Given two intermediate logics $L$ and $L'$, we say $L'$ is an extension of $L$ if $L \subseteq L'$. The least intermediate logic is $\text{INT}$. Consistent logics are proper subsets of $\text{Fm}$. We will refer to the following list of formulas/logics:

$$
\begin{align*}
\text{LC} &: (x_1 \rightarrow x_2) \lor (x_2 \rightarrow x_1); \\
\text{KC} &: \neg x \lor \neg \neg x; \\
\text{SL} &: ((\neg \neg x \rightarrow x) \rightarrow (\neg x \lor \neg \neg x)) \rightarrow (\neg x \lor \neg \neg x); \\
\text{PWL} &: (x_2 \rightarrow x_1) \lor ((x_1 \rightarrow x_2) \rightarrow x_1); \\
\text{H}_n &: H_1 = x_1 \lor \neg x_1, \quad H_{n+1} = x_{n+1} \lor (x_{n+1} \rightarrow H_n); \\
\text{B}_n &: \bigwedge_{i=1}^{n+1} \left( (x_i \rightarrow \bigvee_{j \neq i} x_j) \rightarrow \bigvee_{j \neq i} x_j \right) \rightarrow \bigvee_{i=1}^{n+1} x_i.
\end{align*}
$$

Fig. 8. Intermediate Logics.

$\text{KC}$ is called the logic of weak excluded middle or Jankov logic or de Morgan logic (see \cite{14}). $\text{SL}$ is Scott logic and PWL is the logic of weak law of Peirce, see \cite{11}.

We define the consequence relation $\vdash_L$, for any given intermediate logic $L$, admitting only the rule MP in derivations. Then we prove the deduction theorem

$$(DT) \quad X, A \vdash_L B \iff X \vdash_L A \rightarrow B.$$ 

The relation of $L$-equivalent formulas,

$$A =_L B \iff \vdash_L A \leftrightarrow B,$$

leads to the standard Lindenbaum-Tarski algebra. The relation $=_L$ extends to substitutions, $\varepsilon =_L \mu$ means that $\varepsilon(A) =_L \mu(A)$, for each formula $A$. We define a pre-order (that is a reflexive and transitive relation) on the set of substitutions:

$$\varepsilon \trianglelefteq_L \mu \iff (\alpha \circ \varepsilon =_L \mu, \text{ for some } \alpha).$$

Note that $\varepsilon \trianglelefteq_L \mu \land \mu \trianglelefteq_L \varepsilon$ does not yield $\varepsilon =_L \mu$. If $\varepsilon \trianglelefteq_L \mu$, we say that $\varepsilon$ is more general than $\mu$. If it is not misleading, we omit the subscript $L$ and write $=\!$ and $\trianglelefteq$, instead of $=_L$ and $\trianglelefteq_L$, correspondingly.

A frame $\mathfrak{F} = (W, R, w_0)$ consists of a non-empty set $W$, a pre-order $R$ on $W$ and a root $w_0 \in W$ such that $w_0 R w$, for any $w \in W$. For any set $U$, let $P(U) = \{V : V \subseteq U\}$. Let $n$ be a natural number. Any $n$-model $\mathfrak{M}^n = (W, R, w_0, V^n)$, over the frame $(W, R, w_0)$, contains a valuation $V^n : W \rightarrow P(\{x_1, \ldots, x_n\})$ which is monotone:

$$uRw \Rightarrow V^n(u) \subseteq V^n(w), \text{ for each } u, w \in W.$$

Thus, $n$-models, are (bounded) variants of usual Kripke models $\mathfrak{M} = (W, R, w_0, V)$ where all variables are valued: $V : W \rightarrow P(\text{Var})$. Given $\mathfrak{M}^n$ and $\mathfrak{M}^k$ (for $n \neq k$),

\footnote{Sometimes the reverse pre-order is used; in this case $\mu \not\trianglelefteq \varepsilon \iff (\alpha \circ \varepsilon =_L \mu, \text{ for some } \alpha).$}
we do not assume that $\mathfrak{M}^n$ and $\mathfrak{M}^k$ have anything in common. In particular, we do not assume that there is any model $\mathfrak{M}$ such that $\mathfrak{M}^n$ and $\mathfrak{M}^k$ are its fragments. If $\mathfrak{M}^k = (W, R, w_0, V^k)$ and $n \leq k$, then $\mathfrak{M}^k|_n$ is the restriction of $\mathfrak{M}^k$ to the $n$-model. Thus, $\mathfrak{M}^k|_n = (W, R, w_0, V^n)$ is the $n$-model over the same frame as $\mathfrak{M}^k$ in which $V^n(w) = V^k(w) \cap \{x_1, \ldots, x_n\}$, for each $w \in W$. We say $(W, R, w_0)$ is a po-frame, and $(W, R, w_0, V^n)$ is a po-model, if the relation $R$ is a partial order.

Let $\mathfrak{F} = (W, \leq, w_0)$ be a finite po-frame. We define the depth, $d_\mathfrak{F}(w)$, of any element $w \in W$ in $\mathfrak{F}$. We let $d_\mathfrak{F}(w) = 1$ if $w$ is a $\leq$-maximal element (i.e., maximal elements are also called end elements) and $d_\mathfrak{F}(w) = i + 1$ if all elements in $\{u \in W : w < u \}$ are of the depth at most $i$ and there is at least one element $u > w$ of the depth $i$. The depth of the root, $d_\mathfrak{F}(w_0)$, is the depth of the frame $\mathfrak{F}$ (or any $n$-model over $\mathfrak{F}$).

Let $\mathfrak{F} = (W, \leq, w_0)$ and $\mathfrak{G} = (U, \leq, u_0)$ be two disjoint (that is $W \cap U = \emptyset$) po-frames. The join $\mathfrak{F} + \mathfrak{G}$ of the frames is the frame $(W \cup U, \leq, w_0)$ where

$$x \leq y \iff x \leq w y \text{ or } x \leq v y \text{ or } (x \in W \land y \in U).$$

If $\mathfrak{F}$ and $\mathfrak{G}$ are not disjoint, we take their disjoint isomorphic copies and the join of the copies is called the join of $\mathfrak{F}$ and $\mathfrak{G}$ (it is also denoted by $\mathfrak{F} + \mathfrak{G}$). Thus, the join of frames is defined up to an isomorphism. The join is associative (up to an isomorphism) and it is not commutative. Instead of $\mathfrak{F} + \mathfrak{G}$ and $\mathfrak{G} + \mathfrak{F}$, we write $\mathfrak{F} + \mathfrak{G}$ and $\mathfrak{G} + \mathfrak{F}$, correspondingly.

Let $(W, R, u, V^n)$ be an $n$-model. The subsets $\{V^n(w)\}_{w \in W}$ of $\{x_1, \ldots, x_n\}$ are usually given by their characteristic functions $f^{n}_{w}: \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$ or binary strings $f^{n}_{w} = i_1 \ldots i_n$, where $i_k \in \{0, 1\}$. Thus, $n$-models may also appear in the form $(W, R, u, \{V^n(w)\}_{w \in W})$, or $(W, R, u_0, \{f^{n}_{w}\}_{w \in W})$. $n$-Models are usually depicted as graphs whose nodes are labeled with binary strings.

The forcing relation $\mathfrak{M}^n \forces w A$, for any $w \in W$ and $A \in \mathcal{Fm}^n$, is defined as usual

$$\mathfrak{M}^n \forces w x_i \iff x_i \in V^n(w), \text{ for any } i \leq n;$$

$$\mathfrak{M}^n \forces w \perp, \text{ for none } w \in W;$$

$$\mathfrak{M}^n \forces w (A \rightarrow B) \iff \forall u (uRw \text{ and } \mathfrak{M}^n \forces u A \Rightarrow \mathfrak{M}^n \forces u B);$$

$$\mathfrak{M}^n \forces w (A \lor B) \iff (\mathfrak{M}^n \forces w A \text{ or } \mathfrak{M}^n \forces w B);$$

$$\mathfrak{M}^n \forces w (A \land B) \iff (\mathfrak{M}^n \forces w A \text{ and } \mathfrak{M}^n \forces w B).$$

**Lemma 2.1**

If $uRw$ and $\mathfrak{M}^n \forces w A$, then $\mathfrak{M}^n \forces w A$, for any $u, w \in W$ and any $A \in \mathcal{Fm}^n$.

Let $(W)_w = \{u \in W : uRw\}$, for any $w \in W$. The subframe of $(W, R, w_0)$ generated by $w$ is $((W)_w, R|_{(W)_w}, w)$; the submodel of $\mathfrak{M}^n$ generated by $w$ is

$$(\mathfrak{M}^n)_w = ((W)_w, R|_{(W)_w}, w, V^n|_{(W)_w}).$$

We write $\mathfrak{M}^n \models A$ if $\mathfrak{M}^n \models w A$ and we obviously have $(\mathfrak{M}^n)_w \models A \Rightarrow \mathfrak{M}^n \models w A$. For any $n$-model, we put $\text{Th}(\mathfrak{M}^n) = \{A \in \mathcal{Fm}^n : \mathfrak{M}^n \models A\}$. Given two $n$-models $\mathfrak{M}^n$ and $\mathfrak{N}^n$, we say they are equivalent, in symbols $\mathfrak{M}^n \sim \mathfrak{N}^n$, if $\text{Th}(\mathfrak{M}^n) = \text{Th}(\mathfrak{N}^n)$.
Let \((W, \leq, w_0, V^n)\) and \((W, \leq, w_0, V'^n)\) be \(n\)-models over the same po-frame, we say they are (mutual) variants if \(V(w) = V'(w)\) for each \(w \neq w_0\).

Let \(F\) be a class of frames and \(M^n(F)\), for any \(n \geq 0\), be the class of \(n\)-models over the frames \(F\); we write \(M^n\), instead of \(M^n(F)\), if there is no danger of confusion. The intermediate logic determined by \(F\) is denoted by \(L(F)\). Thus, if \(A \in Fm^n\), then
\[
A \in L(F) \iff (M^n \vdash A, \text{ for every } M^n \in M^n).
\]
We say that \(F\) are frames of an intermediate logic \(L\) if \(L \subseteq L(F)\) and \(L\) omits a frame \(\mathfrak{F}\) if \(\mathfrak{F}\) is not a frame of \(L\). A logic \(L\) is Kripke complete if \(L = L(F)\) for some \(F\). The logic \(L(F)\) is said to be tabular if \(F\) is a finite family of finite frames. \(L\) is Halldén complete (H-complete) if for any formulas \(A, B\) with \(\text{Var}(A) \cap \text{Var}(B) = \emptyset\), we have
\[
L \vdash A \lor B \implies L \vdash A \quad \text{or} \quad L \vdash B.
\]

**Theorem 2.2**

Let \(F\) be finite. Then \(L(F)\) is H-complete iff \(L(F) = L(\mathfrak{F})\) for some \(\mathfrak{F} \in F\).

A logic \(L\) is locally tabular if \(Fm^n/\sim = L\) is finite, for each \(n \geq 0\). Tabular logics are locally tabular but not vice versa. For each locally tabular logic \(L\) there exists a family \(F\) of finite frames such that \(L = L(F)\). Thus, locally tabular logics have the finite model property but, again, the converse is false. A logic \(L\) is said to be in the \(n\)-slice if \(L = L(F)\) for a family \(F\) of finite po-frames such that \(d(\mathfrak{F}) \leq n\), for any \(\mathfrak{F} \in F\).

**Theorem 2.3**

Suppose that the family \(F\) consists of finite frames. Then \(L(F)\) is locally tabular iff \(M^n /\sim\) is finite, for each \(n\).

**Proof.** \((\Rightarrow)\) Using finitely many (up to equivalence) formulas we do not distinguish infinitely many models. \((\Leftarrow)\) is obvious.

**Corollary 2.4**

(i) If \(L\) and \(L'\) are locally tabular intermediate logics, then their intersection \(L \cap L'\) is also a locally tabular intermediate logic;

(ii) any extension of any locally tabular intermediate logic is locally tabular.

**Proof.** (i) Let \(L = L(F)\) and \(L' = L(G)\) for some classes \(F, G\) of finite frames. Then \(L \cap L' = L(F \cup G)\) and \(M^n(F \cup G) = M^n(F) \cup M^n(G)\). Thus, \(M^n(F \cup G)/\sim\) is finite if \(M^n(F)/\sim\) and \(M^n(G)/\sim\) are finite. (ii) is obvious.

Let us characterize po-frames of the logics in Figure 8. Thus, LC-frames are chains and we let \(\mathfrak{L}_d\), for any natural number \(d \geq 1\), be the chain on \(\{1, 2, \ldots, d\}\) with the reverse (natural) ordering \(\geq\), where \(d\) is the root and 1 is the top (=greatest) element. Finite KC-frames have top elements. \(H_n\)-Frames are of the depth \(\leq n\) and \(H_nB_m\)-frames have (additionally) \(m\)-bounded branching, that is each point has at most \(m\) immediate successors. To get PWL-frames we need unrooted frames; PWL-frames are
\[
\mathfrak{F}_n + \mathfrak{J}_{n_1} + \cdots + \mathfrak{J}_{n_k},
\]
where \(n \geq 0\) and \(n_1, \ldots, n_k \geq 1\); \(\mathfrak{F}_n + \mathfrak{J}_{n_1} + \cdots + \mathfrak{J}_{n_k}\) denotes the vertical union with \(\mathfrak{F}_n\) on the top and \(\mathfrak{J}_{n_k}\) on the bottom.
where \( \mathcal{F}_n \) is the frame with the identity relation on an \( n \)-element set (and we agree that \( \mathcal{F}_0 = \mathcal{L}_2 \) and \( \mathcal{F}_1 = \mathcal{L}_2 \)). Note that the frames in Figure 7 are PWL-frames and hence \( L(\mathcal{H}_{pa}) \) and \( L(\mathcal{H}_{an}) \) are extensions of PWL.

There are three pretabular intermediate logics, see [23]: \( \mathcal{L}_C \) of Gödel and Dummett, given by all chains \( \mathcal{L}_n \), \( \mathcal{L}_J \) of Jankov, given by all \( n \)-forks \( \mathcal{F}_n \), and \( \mathcal{L}_H \) of Hosoi, given by all rhombuses \( \mathcal{R}_n \); see Figure 4

A pair of logics \((\mathcal{L}_1, \mathcal{L}_2)\) is a splitting pair of the lattice of (intermediate) logics if \( \mathcal{L}_2 \subset \mathcal{L}_1 \) and, for any intermediate logic \( \mathcal{L} \), either \( \mathcal{L} \subset \mathcal{L}_1 \), or \( \mathcal{L}_2 \subset \mathcal{L} \). Then we say \( \mathcal{L}_1 \) splits the lattice and \( \mathcal{L}_2 \) is the splitting (logic) of the lattice, see [31]. Jankov [20] characteristic formula of a finite rooted frame \( \mathcal{F} \) is denoted by \( \chi(\mathcal{F}) \).

**Theorem 2.5**
The pair \((\mathcal{L}(\mathcal{F}), \mathcal{L}(\chi(\mathcal{F}))\) is a splitting pair, for any finite frame \( \mathcal{F} \). Thus, for any intermediate logic \( \mathcal{L} \) and any finite frame \( \mathcal{F} \), the logic \( \mathcal{L} \) omits \( \mathcal{F} \) iff \( \chi(\mathcal{F}) \in \mathcal{L} \).

For instance, \( \mathcal{K}_C = \mathcal{L}(\{\chi(\mathcal{F}_2)\}) \) is the splitting logic. If \( \{\mathcal{L}_i\}_{i \in I} \) is a family of splitting logics, then \( \mathcal{L}(\cup_{i \in I} \mathcal{L}_i) \) is called a union splitting. For instance, \( \mathcal{L}_C = \mathcal{L}(\{\chi(\mathcal{F}_2), \chi(\mathcal{R}_2)\}) \) is a union splitting but not a splitting.

**Corollary 2.6**
If \( \{(\mathcal{L}_i, \mathcal{L}_i)\}_{i \in I} \) is a family of splitting pairs and \( \mathcal{L} = \mathcal{L}(\cup_{i \in I} \mathcal{L}_i) \), then \( \mathcal{L} \) is a union splitting and, for any intermediate logic \( \mathcal{L}' \), either \( \mathcal{L}' \subset \mathcal{L}_i \) for some \( i \in I \), or \( \mathcal{L} \subset \mathcal{L}' \).

### 2.2 The Problem of Unification

A substitution \( \varepsilon \) is a unifier for a formula \( A \) in a logic \( \mathcal{L} \) (an \( \mathcal{L} \)-unifier for \( A \)) if \( \varepsilon(A) \in \mathcal{L} \). In any intermediate logic, the set of unifiable formulas coincides with the set of consistent formulas. A set \( \Sigma \) of \( \mathcal{L} \)-unifiers for \( A \) is said to be complete, if for each \( \mathcal{L} \)-unifier \( \mu \) of \( A \), there is a unifier \( \varepsilon \in \Sigma \) such that \( \varepsilon \leq \mu \).

The unification type of \( \mathcal{L} \) is 1 (in other words, unification in \( \mathcal{L} \) is unitary) if the set of unifiers of any unifiable formula \( A \) contains a least, with respect to \( \leq \), element called a most general unifier of \( A \), (an mgu of \( A \)). In other words, unification in \( \mathcal{L} \) is unitary if each unifiable formula has a one-element complete set of unifiers. The unification type of \( \mathcal{L} \) is \( \omega \) (unification in \( \mathcal{L} \) is finitary), if it is not 1 and each unifiable formula has a finite complete set of unifiers. The unification type of \( \mathcal{L} \) is \( \infty \) (unification in \( \mathcal{L} \) is infinitary) if it is not 1, nor \( \omega \), and each unifiable formula has a minimal (with respect to inclusion) complete set of unifiers. The unification type of \( \mathcal{L} \) is 0 (unification in \( \mathcal{L} \) is nullary) if there is a unifiable formula which has no minimal complete set of unifiers. In a similar way one defines the unification type of any \( \mathcal{L} \)-unifiable formula.

The unification type of the logic is the worst unification type of its unifiable formulas.

Ghilardi [13] introduced projective unifiers and formulas; an \( \mathcal{L} \)-unifier \( \varepsilon \) for \( A \) is called projective if \( A \vdash_{\mathcal{L}} \varepsilon(x) \leftrightarrow x \), for each variable \( x \) (and consequently \( A \vdash_{\mathcal{L}} \varepsilon(B) \leftrightarrow B \), for each \( B \)). A formula \( A \) is said to be projective in \( \mathcal{L} \) (or \( \mathcal{L} \)-projective) if it has a projective unifier in \( \mathcal{L} \). It is said that a logic \( \mathcal{L} \) enjoys projective unification.

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7In the same way, one can define a splitting pair in any complete lattice.

8Jankov originally defined \( \chi(\mathcal{F}) \) for any subdirectly irreducible finite Heyting algebra. By duality, finite rooted frames are tantamount to finite s.

### 9
if each $L$-unifiable formula is $L$-projective. An $L$-projective formula may have many non-equivalent in $L$-projective unifiers and each $L$-projective unifier is its mgu:

**Lemma 2.7**

If $\varepsilon$ is an $L$-projective unifier for $A$ and $\sigma$ is any $L$-unifier for $A$, then $\sigma \circ \varepsilon =_L \sigma$.

Thus, projective unification implies unitary unification. If $A \in \text{Fm}^n$ is $L$-projective, then $A$ has a projective unifier $\varepsilon: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n$ that is a mgu preserving the variables of $A$ (which is not always the case with unitary unification). In contrast to unitary unification, projective unification is also monotone:

**Lemma 2.8**

If $A$ is $L$-projective and $L \subseteq L'$, then $A$ is $L'$-projective.

Ghilardi [14] gives a semantical characterization of projective formulas. The condition (ii) is called the extension property.

**Theorem 2.9**

Let $F$ be a class of finite po-frames and $L = L(F)$. The followings are equivalent:

(i) $A$ is $L$-projective;

(ii) for every $n$-model $\mathfrak{M}^n = (W, \leq, w_0, V^n)$ over a po-frame $(W, \leq, w_0)$ of the logic $L$: if $(\mathfrak{M}^n)_w \models A$ for each $w \neq w_0$, then $\mathfrak{N}^n \models A$ for some variant $\mathfrak{N}^n$ of $\mathfrak{M}^n$.

Wroński [29, 30] proved that

**Theorem 2.10**

An intermediate logic $L$ has projective unification iff $L \subseteq L$.

There are unitary logics which are not projective. Following Ghilardi and Sachetti [18], unification in $L$ is said to be filtering if given two unifiers, for any formula $A$, one can find a unifier that is more general than both of them. Unitary unification is filtering. If unification is filtering, then every unifiable formula either has an mgu or no basis of unifiers exists (unification is nullary). It is known, see e.g. [3], that

**Theorem 2.11**

Unification in any intermediate logic $L$ is filtering iff $L \subseteq L$.

If $\varepsilon, \sigma: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k$ are unifiers of a formula $A(x_1, \ldots, x_n)$ in (any extension of) $KC$, then, as a more general unifier for $\varepsilon, \sigma$ the following substitution $\mu$ can be taken (where $y$ is a fresh variable, i.e., $y \notin \text{Fm}^k$):

$$
\mu(x_i) = (\varepsilon(x_i) \land \neg y) \lor (\sigma(x_i) \land \neg \neg y), \quad \text{for } i = 1, \ldots, n.
$$

Thus, unifiers in filtering unification introduce new variables. We have, see [3, 14],

**Theorem 2.12**

$KC$ is the least intermediate logic with unitary unification. All extensions of $KC$ have nullary or unitary unification. All intermediate logics with finitary unification are included in $L(\mathfrak{S}_2)$, the logic determined by the ‘fork frame’ $\mathfrak{S}_2$ see Figure 5 ($L(\mathfrak{S}_2), KC$) is a splitting pair of the lattice of intermediate logics.

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9 More specifically, the theorem says that the class of models of a projective formula enjoys extension property.
Logics with finitary and unitary unification are separated by the splitting $(L(\mathcal{F}_2), KC)$. Let us agree that having good unification means either unitary, or finitary one. Given a logic $L$ with good unification, it has unitary or finitary unification depending only on that if $L$ contains $KC$ or not. Our aim would be to distinguish logics with good unification from those with nullary one. We show in later that locally tabular intermediate logics with infinitary unification do not exist at all. Let us notice that the splitting generated by $(L(\mathcal{F}_2), KC)$ is irrelevant for logics with nullary unification; there are extensions of $KC$, as well as sublogics of $L(\mathcal{F}_2)$, that have nullary unification.

A logic $L$ is said to have projective approximation if, for each formula $A$ one can find a finite set $\Pi(A)$ of $L$-projective formulas such that:

(i) $\mathbf{Var}(B) \subseteq \mathbf{Var}(A)$ and $B \vdash_L A$, for each $B \in \Pi(A)$;
(ii) each $L$-unifier of $A$ is an $L$-unifier of some $B \in \Pi(A)$.

If a finite $\Pi(A)$ exists we can assume that all $B \in \Pi(A)$ are maximal (with respect to $\vdash_L$) $L$-projective formulas fulfilling (i). But, even if there is finitely many maximal $L$-projective formulas fulfilling (i), we cannot be sure (ii) is fulfilled.

**Theorem 2.13**

Each logic with projective approximation has finitary (or unitary) unification.

Logics with projective approximation play a similar role for finitary unification as projective logics do for unitary unification, even though projective approximation is not monotone. Ghilardi [13, 14] proved that

**Theorem 2.14**

Intuitionistic propositional logic $\text{INT}$ enjoys projective approximation and hence unification in $\text{INT}$ is finitary.

### 3 Intuitionistic Kripke $n$-Models.

#### 3.1 $p$-Morphisms.

Let $(W, R, w_0, V^n)$ and $(U, S, u_0, V'^n)$ be $n$-models. A mapping $p: W \rightarrow U$, from $W$ onto $U$, is said to be a $p$-morphisms of their frames, $p: (W, R, w_0) \rightarrow (U, S, u_0)$, if

(i) $wRv \Rightarrow p(w)Sp(v)$, for any $w, v \in W$;
(ii) $p(w)Sa \Rightarrow \exists v \in W (wRv \land p(v) = a)$, for any $w \in W$ and $a \in U$;
(iii) $p(w_0) = u_0$.

A $p$-morphism of $n$-models, $p: (W, R, w_0, V^n) \rightarrow (U, S, u_0, V'^n)$ fulfills (additionally)

(iv) $V^n(w) = V'^n(p(w))$, for any $w \in W$.

If $p: \mathcal{M}^n \rightarrow \mathcal{M}'^n$ is a $p$-morphism, then $\mathcal{M}'^n$ is called a $p$-morphic image (or reduct, see [11]) of $\mathcal{M}^n$ and we write $p(\mathcal{M}^n) = \mathcal{M}'^n$. Reducing $\mathcal{M}'^n$ (by a $p$-morphism), we preserve its logical properties. In particular, $p(\mathcal{M}^n) \sim \mathcal{M}'^n$ as

**Lemma 3.1**

If $p: \mathcal{M}^n \rightarrow \mathcal{M}'^n$ is a $p$-morphism of $n$-models, $w \in W$ and $A \in \text{Fm}^n$, then

$$
\mathcal{M}'^n \models_w A \iff p(\mathcal{M}^n) \models_{p(w)} A.
$$

Ghilardi [13, 14] instead of assuming $\Pi(A)$ is finite, postulates $\mathbf{deg}(B) \leq \mathbf{deg}(A)$, for each $B \in \Pi(A)$, from which it follows that $\Pi(A)$ is finite. The condition $\mathbf{deg}(B) \leq \mathbf{deg}(A)$ is relevant for logics with disjunction property, like $\text{INT}$, but is irrelevant for locally tabular logics where $\mathbf{Var}(B) \subseteq \mathbf{Var}(A)$ is sufficient. We decided, therefore, to modify slightly Ghilardi’s formulations preserving, we hope, his ideas.
p-Morphisms are also used in modal logic. The above property is generally valid which means it also holds for modal models and modal formulas and it can be shown without assuming that $R$ is a pre-order and $V^n$ is monotone.

**Example 3.2**
Let $M^n = (W, R, w_0, V^n)$ be an $n$-model in which the pre-order $R$ is not a partial order. Let $w \approx v \iff wRv \land vRw$, for any $w, v \in W$. Then $\approx$ is an equivalence relation on $W$ and one can easily show that the canonical mapping $p(w) = [w]_\approx$, for any $w \in W$, is a p-morphism from $M^n$ onto the quotient model $M^n/\approx = (W/\approx, R/\approx, [w_0]_\approx, V^n/\approx)$.

Reducing all $R$-clusters to single points, we receive an equivalent $n$-model over a po-set; and hence po-sets (not pre-orders) are often taken as intuitionistic frames.

If a p-morphism $p: M^n \rightarrow N^n$ is one-to-one, then $wRv \iff p(w)Sp(v)$, for any $w, v \in W$ which means $p$ is an isomorphism and, if there is an isomorphism between the $n$-models, we write $M^n \equiv N^n$. It is usual to identify isomorphic objects.

### 3.2 Bisimulations.
Bisimulations (between Kripke frames) were introduced by K.Fine [12], by imitating Ehrenfeucht games. They found many applications. In particular, S.Ghilardi [14] used bounded bisimulation to characterize projective formulas. We show that bisimulations are closely related to p-morphisms. In our approach we follow A.Patterson [24].

A binary relation $B$ on $W$ is a bisimulation of the frame $(W, R, w_0)$ if

$$wBv \Rightarrow \forall w'. \exists v' (wRw' \Rightarrow vRv' \land w'Bv') \land \forall v'. \exists w' (vRv' \Rightarrow wRw' \land w'Bv').$$

Fig. 9. Bisimulation

Note that $wBv \Rightarrow \forall w'. \exists v' (wRw' \Rightarrow vRv' \land w'Bv')$ suffices if $B$ is symmetric. A bisimulation of the $n$-model $(W, R, w_0, V^n)$ additionally fulfills $V^n(w) = V^n(v)$ if $wBv$.

**Lemma 3.3**
(i) If $B$ is a bisimulation of $M^n$, then $B|_{(W),w}$ is a bisimulation of $(M^n)|_w$;
(ii) if $B$ is a bisimulation of $(M^n)|_w$, then $B$ is a bisimulation of $M^n$; for any $w \in W$.

**Lemma 3.4**
If $B$ is a bisimulation (of a frame or an $n$-model), then the least equivalence relation $B^*$ containing $B$ is also a bisimulation.

**Proof.** A proof of this lemma can be found in [24]. Let us only specify properties of bisimulations which are useful here.
Figure 10 commutes.

that the conditions (i), (iii) and (iv) are quite obvious.

We should show that \( [w]_B R/B [v]_B \iff \exists w' v' (wBw' \land vBv' \land w'Rv') \).

**Theorem 3.5**

If \( B \) is an equivalence bisimulation of an \( n \)-model \( \mathfrak{M}^n \), then \( \mathfrak{M}^n/B \) is an \( n \)-model and the canonical mapping \([\ ]_B : W \to W/B\) is a \( p \)-morphism of the \( n \)-models.

**Proof.** We should show that \( R/B \) is a pre-order. If \( w = v \), one can take \( w' = v' = w \) (in the definition of \( R/B \)) to show \( [w]_B R/B [w]_B \). Thus, \( R/B \) is reflexive.

Suppose that \( [w]_B R/B [v]_B R/B [u]_B \), for some \( w, v, u \in W \). Then \( wBw' \land vBv' \land wBu \), for some \( w', v', u \in W \). But \( B \) is an equivalence, hence \( v'Bv' \) and, by \( v''Ru'' \), we get \( v'Ru' \land u'Bu \), for some \( u' \in W \), as \( B \) is a bisimulation. By transitivity of \( R \), we have \( w'Ru' \land uBu' \) as \( B \) is an equivalence relation. Thus, \( [w]_B R/B [u]_B \); the relation \( R/B \) is transitive.

There remains to show that the canonical mapping is a \( p \)-morphism.

(i) If \( wRv \), then \( [w]_B R/B [v]_B \), by the definition of \( R/B \).

(ii) Suppose that \( [w]_B R/B [v]_B \), for some \( w, v \in W \). Then \( wBw' \land vBv' \), for some \( w', v' \in W \). As \( B \) is a bisimulation, \( wRv'' \land v''Bu' \), for some \( v'' \in B \).

Thus, \( wRv'' \land [v'']_B = [v]_B \), as required.

The conditions (iii) and (iv) are obviously fulfilled.

**Theorem 3.6**

If \( B \) and \( B' \) are equivalence bisimulations of an \( n \)-model \( \mathfrak{M}^n = (W, R, w_0, V^n) \) and \( B' \subseteq B \), then there is a \( p \)-morphism \( q : \mathfrak{M}^n/B' \to \mathfrak{M}^n/B \) such that the diagram in Figure 10 commutes.

![Fig. 10. Comparison of Bisimulations.](image)

**Proof.** Let us define \( q([w]_B) = [w]_B \) and notice that the mapping is well-defined and maps \( W/B' \) onto \( W/B \). We should only check that \( q \) is a \( p \)-morphism. Note that the conditions (i), (iii) and (iv) are quite obvious.

(ii) Suppose that \( q([w]_{B'}) R/B [u]_B \). By the definition of \( R/B \), there are \( w', u' \) such that \( wBw' Ru' Bu \). Since \( B \) is a bisimulation and \( wBw' Ru' \) there is an \( u'' \) such that \( wRu'' Bu' \). Thus, \( [w]_{B'} R/B' [u'']_B \) and \( q([u'']_B) = [u'']_B = [u]_B \) as required.  


Theorem 3.7
If \( p : \mathfrak{M}^n \to \mathfrak{N}^n \) is a p-morphism of \( n \)-models, then

\[
wBv \iff p(w) = p(v)
\]

is an equivalence bisimulation of the \( n \)-model \( \mathfrak{M}^n \), and \( \mathfrak{M}^n/B \equiv \mathfrak{N}^n \).

Proof. Let \( wBv \) and \( wRw' \) for some \( w, w', v \in W \) (see Figure 11). Then \( p(w) = p(v) \) and \( p(w)Sp(w') \), where \( S \) is the accessibility relation in \( \mathfrak{N}^n \). Thus, \( p(v)Sp(w') \). Since \( p \) is a p-morphism, \( vRv' \) and \( p(v') = p(w') \), for some \( v' \in W \). Thus, \( vRv' \) and \( w'Bv' \).

In the same way one shows \( wBv \) and \( wRw' \) give us \( wRw' \) and \( w'Bv' \), for some \( w' \), and we obviously have \( V^w = V^w(v) \) if \( wBv \).

Bisimulations preserve such properties of frames as reflexivity, symmetry, transitivity; consequently, p-morphic images preserve these properties, as well. There are, however, some properties which are not preserved by p-morphisms.

Example 3.8
Let \( W = \{ u_i : i \geq 0 \} \cup \{ v_i : i \geq 0 \} \cup \{ w_0 \} \) and a partial order \( R \) on \( W \), and a bisimulation \( B \) on \( W \), are defined as in the following picture (see Figure 11)

\[
[u_i]_B = \{ u_0, u_1, u_2, \ldots \} \\
[v_i]_B = \{ v_0, v_1, v_2, \ldots \} \\
[w_0]_B = \{ w_0 \}
\]

Fig. 11. Weak Asymmetry is not Preserved.

Thus, a p-morphic image of a partial order is not a partial order (only pre-order).

Note that the set \( W \) in the above Example is infinite which is essential as

Corollary 3.9
Any p-morphic image of any finite po-frame is a po-frame.

3.3 \( p \)-Irreducible \( n \)-Models.

An \( n \)-model \( \mathfrak{M}^n \) is said to be \( p \)-irreducible if each p-morphism \( p : \mathfrak{M}^n \to \mathfrak{N}^n \), for any \( n \)-model \( \mathfrak{N}^n \), is an isomorphism. Thus, any p-morphic image of any irreducible \( n \)-model is its isomorphic copy\(^{11}\). Irreducible \( n \)-models are po-sets, see Example 3.2 and we show any \( n \)-model can be reduced to a \( p \)-irreducible one.

\(^{11}\)The concept of \( p \)-irreducibility, in contrast to other concepts in this Section, would make no sense for frames.
Theorem 3.10
For each n-model \( \mathfrak{M}^n \) there exists a p-irreducible n-model \( \mathfrak{M}^n \) which is a p-morphic image of \( \mathfrak{M}^n \) (and \( \mathfrak{M}^n \) is unique up to \( \equiv \)).

Proof. Let \( \mathfrak{M}^n = (W, R, w_0, \{f^n_w\}_{w \in W}) \) and \( B \) be the least equivalence on \( W \) containing \( \bigcup \{ B_i; B_i \) is a bisimulation on \( \mathfrak{M}^n \} \). By Lemma 3.3 \( B \) is the greatest bisimulation on \( \mathfrak{M}^n \). Take \( \mathfrak{M}^n = \mathfrak{M}^n/B \), see Theorem 3.5. Since the composition of any two p-morphisms is a p-morphism, any p-morphic image \( \mathfrak{M}^n \) of \( \mathfrak{M}^n \) would be a p-morphic image of \( \mathfrak{M}^n \). Thus, by maximality of \( B \), we would get, by Theorem 3.6 an isomorphism \( p': \mathfrak{M}^n \equiv \mathfrak{M}^n \) which means \( \mathfrak{M}^n \) is p-irreducible.

The uniqueness of \( \mathfrak{M}^n \) also follows; if \( \mathfrak{M}^n \) were another p-irreducible p-morphic image of \( \mathfrak{M}^n \), we would get by Theorems 3.10 and 3.7 a p-morphism \( p': \mathfrak{M}^n \rightarrow \mathfrak{M}^n \) which would mean that \( \mathfrak{M}^n \) and \( \mathfrak{M}^n \) are isomorphic. 

The following theorem could give another characterization of p-irreducible n-models.

Theorem 3.11
If an n-model \( \mathfrak{M}^n \) is p-irreducible, then for any n-model \( \mathfrak{M}^n \) there is at most one p-morphism \( p: \mathfrak{M}^n \rightarrow \mathfrak{M}^n \).

Proof. Let \( \mathfrak{M}^n = (W, R, w_0, V^n) \) be p-irreducible and \( p, q: \mathfrak{M}^n \rightarrow \mathfrak{M}^n \) be two different p-morphisms for some \( \mathfrak{M}^n = (U, S, u_0, V'^n) \). Take \( B = \{ (p(v), q(v)) \); \( v \in V \} \) and let us show \( B \) is the greatest relation on \( \mathfrak{M}^n \). This would be a contradiction as, if \( B' \) were the least equivalence relation containing \( B \) (see Lemma 3.4), \( B' \) would be a non-isomorphic p-morphism, see Theorem 3.5 and it would mean that \( \mathfrak{M}^n \) were not p-irreducible.

Let \( p(v)Rw \), for some \( v \in V \) and \( w \in W \). As \( p \) is a p-morphism, \( p(v') = w \) and \( vSv' \) for some \( v' \in V \). Then \( q(v)Rq(v') \), as \( q \) is a p-morphism, and \( wBq(v') \) as \( w = p(v') \).

Similarly, if \( q(v)Rw \), for some \( v \in V \) and \( w \in W \), then \( q(v') = w \) and \( vSv' \), for some \( v' \in V \), and hence \( p(v)Rp(v') \) and \( p(v')Rw \) (as \( w = q(v') \)).

Theorem 3.12
If \( \mathfrak{M}^n \) is p-irreducible, then \( (\mathfrak{M}^n)_{w} \) is p-irreducible for each \( w \in W \).

Proof. Let \( \mathfrak{M}^n = (W, R, w_0, V^n) \) and suppose \( (\mathfrak{M}^n)_{w} \) is not p-irreducible for some \( w \in W \). By Theorem 3.7 there is a (non-trivial) bisimulation \( B \) on \( (\mathfrak{M}^n)_{w} \). Since \( B \) is a bisimulation of \( \mathfrak{M}^n \), if we extend \( B \) to an equivalence bisimulation \( B^* \) of \( \mathfrak{M}^n \), we get a (non-isomorphic) p-morphism of \( \mathfrak{M}^n \), see Theorem 3.5. Thus, \( \mathfrak{M}^n \) is not p-irreducible.

3.4 Finite n-Models.

It follows from Example 3.12 that, without losing any generality, we can confine ourselves to frames/n-models defined over partial orders (not pre-orders). So, in what follows, we assume that all frames/n-model are (defined over) po-sets even though we (sometimes) keep the notation \( \mathfrak{M}^n = (W, R, w_0, V^n) \). We examine here specific properties of finite n-models such as Corollary 3.9.

Theorem 3.13
If \( \mathfrak{M}^n \) is a finite n-model, then one can define \( \Delta(\mathfrak{M}^n) \in \operatorname{Fm}^n \) (called the character of \( \mathfrak{M}^n \)) such that \( \mathfrak{M}^n \models \Delta(\mathfrak{M}^n) \iff \operatorname{Th}(\mathfrak{M}^n) \subseteq \operatorname{Th}(\mathfrak{M}^n) \), for any n-model \( \mathfrak{M}^n \).
The next theorem is due to Patterson [24]:

**Theorem 3.14**

If \( \{ \text{Th}(\mathcal{M}_w^n) \} \) is finite (which is the case when \( \mathcal{M}^n \) is finite), then

\[
\text{Th}(\mathcal{M}^n) \subseteq \text{Th}(\mathcal{N}^n) \iff \mathcal{M}^n \sim (\mathcal{N}^n)_w, \text{ for some } w \in W, \text{ for any } n\text{-model } \mathcal{N}^n.
\]

**Proof.** The implication \((\Leftarrow)\) is obvious by Lemma 2.1. Let us prove \((\Rightarrow)\). If not all of \( \text{Th}(\mathcal{M}^n) \) is true at \( (\mathcal{M}^n)_w \), we pick \( A_w \in \text{Th}(\mathcal{M}^n) \) such that \( A_w \not\subseteq \text{Th}(\mathcal{M}^n)_w \) or \( A_w = \top \) otherwise. As \( \{ \text{Th}(\mathcal{M}^n)_w : w \in W \} \) is finite, we take \( A = \bigwedge A_w \) and notice \( \mathcal{M}^n \vDash A \) means that \( \text{Th}(\mathcal{M}^n) \subseteq \text{Th}(\mathcal{M}^n)_w) \).

If a formula not in \( \text{Th}(\mathcal{M}^n) \) is true at \( (\mathcal{M}^n)_w \), we pick \( B_w \not\in \text{Th}(\mathcal{M}^n) \) such that \( B_w \in \text{Th}(\mathcal{M}^n)_w \) (or \( B_w = \bot \) if \( \text{Th}(\mathcal{M}^n) \supseteq \text{Th}(\mathcal{M}^n)_w) \)), for each \( w \in W \). Take \( B = \bigvee B_w \) and notice \( \mathcal{M}^n \not\vDash B \) yields \( \text{Th}(\mathcal{M}^n) \supseteq \text{Th}(\mathcal{M}^n)_w) \).

Clearly, \((A \Rightarrow B) \not\in \text{Th}(\mathcal{M}^n) \). Thus, \((A \Rightarrow B) \not\in \text{Th}(\mathcal{M}^n) \) and hence \( \mathcal{M}^n \not\vDash B \), for some \( w \in W \), and this means that \( \text{Th}(\mathcal{M}^n) = \text{Th}(\mathcal{M}^n)_w) \).

**Theorem 3.15**

If \( \{ \text{Th}(\mathcal{M}^n)_w \}_{w \in W} \) is finite, then the greatest bisimulation \( B \) of \( \mathcal{M}^n \) is:

\[
wBv \iff (\mathcal{M}^n)_w \sim (\mathcal{M}^n)_v.
\]

**Proof.** Let \( wBv \wedge wRw' \). Then \( \text{Th}(\mathcal{M}^n)_v = \text{Th}(\mathcal{M}^n)_w \subseteq \text{Th}(\mathcal{M}^n)_w) \) and, by Theorem 3.14, \( wBv' \wedge wRv' \) for some \( v' \). Thus, \( B \) is a bisimulation as \( B \) is symmetric.

Let \( wB'v \) and \( B' \) be a bisimulation of \( \mathcal{M}^n \). By Theorem 3.9, there is a p-morphism \( p: \mathcal{M}^n \rightarrow \mathcal{M}^n/B' \) such that \( p(w) = p(v) \). Hence, by Lemma 3.1, \( (\mathcal{M}^n)_w \sim (\mathcal{M}^n)_v \) which means \( wBv \). Thus, we have showed \( B' \subseteq B \).

**Corollary 3.16**

If \( \{ \text{Th}(\mathcal{M}^n)_w \}_{w \in W} \) is finite, then there is a p-morphism from \( \mathcal{M}^n \) onto the \( n \)-model:

\[
\left( \{ \text{Th}(\mathcal{M}^n)_w \}_{w \in W}, \subseteq, \text{Th}(\mathcal{M}^n), \{ x_1, \ldots, x_n \} \cap \text{Th}(\mathcal{M}^n)_w \} \right)_{w \in W}.
\]

**Proof.** By the above Theorem and by Theorem 3.9.

**Corollary 3.17**

\( \mathcal{M}^n \) is finitely reducible (which means there is a p-morphism \( p: \mathcal{M}^n \rightarrow \mathcal{M}^n \) for some finite \( n \)-model \( \mathcal{M}^n \)) if and only if \( \{ \text{Th}(\mathcal{M}^n)_w \}_{w \in W} \) is finite.

**Corollary 3.18**

Let \( \mathcal{M}^n \) and \( \mathcal{N}^n \) be finite (or finitely reducible) \( n \)-models. Then \( \mathcal{M}^n \sim \mathcal{N}^n \) if and only if \( \mathcal{M}^n \) and \( \mathcal{N}^n \) have a common p-morphic image.

**Proof.** Let \( \mathcal{M}^n = (W, R, w), (V^n) \) and \( \mathcal{N}^n = (U, S, u), (V^n) \). It suffices to notice that \( \mathcal{M}^n \sim \mathcal{N}^n \) yields, by Theorem 3.14, \( \{ \text{Th}(\mathcal{M}^n)_w \}_{w \in W} = \{ \text{Th}(\mathcal{N}^n)_w \}_{w \in U} \).

**Corollary 3.19**

If \( \mathcal{M}^n = (W, R, w), (V^n) \) and \( \mathcal{N}^n = (U, S, u), (V^n) \) are finite and \( \mathcal{M}^n \sim \mathcal{N}^n \), then

(i) for every \( w \in W \) there is an element \( u \in U \) such that \( (\mathcal{M}^n)_w \sim (\mathcal{N}^n)_u); 
(ii) for every \( u \in U \) there is an element \( w \in W \) such that \( (\mathcal{M}^n)_w \sim (\mathcal{N}^n)_u). 

**Proof.** Let \( p \) and \( q \) be p-morphisms from \( \mathcal{M}^n \) and \( \mathcal{N}^n \), correspondently, onto a common p-morphic image. By Lemma 3.1, \( (\mathcal{M}^n)_w \sim (\mathcal{N}^n)_u \) if \( p(w) = q(u) \).

Idea of characterizing finite structures by formulas is due to Jankov [23] but the character should not be missed with the characteristic formula of a frame. If we consider \( n \)-models of a given locally tabular logic \( L \), where there is only finitely many \( (up to \equiv) \) formulas in \( n \)-variables, one could define the character of any finite \( n \)-model as the conjunction of the formulas (out of the finitely many) which are true in the model.
3.5 $\sigma$-Models.

This is the key notion and it was defined by Ghilardi [14]. Let $\sigma : \{x_1, \ldots, x_n\} \to \text{Fm}^k$, for $k, n \geq 0$. For any $\mathfrak{M}^k = (W, R, w_0, V^k)$, let $\sigma(\mathfrak{M}^k) = (W, R, w_0, V^n)$ where

\[ x_i \in V^n(w) \iff \mathfrak{M}^k \models_w \sigma(x_i), \quad \text{for any } w \in W \text{ and } i = 1, \ldots, n. \]

**Lemma 3.20**

For every $w \in W$ and every $A \in \text{Fm}^n$, we have

\[ \sigma(\mathfrak{M}^k) \models_w A \iff \mathfrak{M}^k \models_w \sigma(A). \]

**Lemma 3.21**

(i) $\mathfrak{M}^k$ and $\sigma(\mathfrak{M}^k)$ are models over the same frame;
(ii) $\sigma((\mathfrak{M}^k)_w) = (\sigma(\mathfrak{M}^k))_w$, for every $w \in W$;
(iii) if $\text{Th}(\mathfrak{M}^k) \subseteq \text{Th}(\mathfrak{N}^k)$, then $\text{Th}(\sigma(\mathfrak{M}^k)) \subseteq \text{Th}(\sigma(\mathfrak{N}^k))$.

**Proof.** We get (i) and (ii) by the definition of $\sigma(\mathfrak{M}^k)$. As concerns (iii):
\[ \sigma(\mathfrak{M}^k) \models A \iff \mathfrak{M}^k \models \sigma(A) \implies \mathfrak{N}^k \models \sigma(A) \iff \sigma(\mathfrak{N}^k) \models A. \]

**Lemma 3.22**

If $p: \mathfrak{M}^k \to \mathfrak{N}^k$ is a p-morphism of $k$-models, then $p: \sigma(\mathfrak{M}^k) \to \sigma(\mathfrak{N}^k)$ is also a p-morphism of $n$-models and hence $p(\sigma(\mathfrak{M}^k)) = \sigma(p(\mathfrak{M}^k))$ (see Figure. 12).

![Fig. 12. p-Morphic images of $\sigma$-models.](image)

The above does not mean that $\sigma$-models are closed under p-morphic images. Two (counter)examples below show that they may be not.

**Example 3.23**

Let $\sigma(x_1) = x_2 \lor (x_2 \to (x_1 \lor \neg x_1))$. The 1-model over the two-element chain (in Figure 13) cannot be any $\sigma$-model as to falsify $\sigma(x_1)$ at the root one needs at least three elements in the chain.

![Fig. 13. The First Counterexample.](image)
Let \( \sigma(x) = \neg\neg x \lor \neg x \) (we write \( x \) instead of \( x_1 \)). Models and the p-morphism are defined in Figure 14. The 1-model over a two-element chain cannot be any \( \sigma \)-model as to falsify \( \sigma(x) \) at the root one needs at least two end elements above the root.

\[
\begin{array}{c}
1 \\
\sigma \\
0
\end{array}
\] \quad
\[
\begin{array}{c}
1 \\
p \\
0
\end{array}
\]

**Fig. 14. The Second Counterexample.**

Nowhere (but Theorem 3.13) we have used the fact that valuations of any \( n \)-model are restricted to the \( n \)-initial variables. It would make no change in our argument if we replaced (everywhere) valuations \( V^n \) with \( V \), valuations of all variables. Thus, all results (but Theorem 3.13) of this section remain valid for usual Kripke models.

4 Locally Tabular Logics.

For any class \( F \) of frames, let \( sm(F) \) be the least class (of frames) containing \( F \) and closed under generated subframes and p-morphic images.

**Lemma 4.1**

\[ L(sm(F)) = L(F). \]

**Proof.** By Lemma 3.1 and Lemma 2.1

Extending any class of frames with generated subframes and p-morphic images does not change the logic but it enables us to characterize extensions of \( L(F) \).

**Theorem 4.2**

Let \( F \) be a class of finite frames and \( L = L(F) \) be locally tabular. If \( L' \) is an intermediate logic such that \( L \subseteq L' \), then \( L' = L(G) \), for some \( G \subseteq sm(F) \).

**Proof.** Let \( G = \{ \mathfrak{F} \in sm(F) : L' \subseteq L(\mathfrak{F}) \} \). Clearly, \( L' \subseteq L(G) \). We need to show the reverse inclusion. So, assume \( A \not\in L' \) and show \( A \not\in L(\mathfrak{F}) \) for some \( \mathfrak{F} \in G \).

Suppose that \( A = A(x_1, \ldots, x_k) \), for some \( k \geq 0 \), and let \( A_0, \ldots, A_j \) be all (non-equivalent in \( L \)) formulas in \( \text{Fm}^k \cap L' \). Let

\[ B = \bigwedge_{i=0}^{j} A_i \rightarrow A. \]

If \( B \in L(F) \), then \( B \in L' \) and it would give \( A \in L' \), a contradiction. Thus, we have \( B \not\in L(F) \). There is a \( k \)-model \( \mathfrak{M}^k = (W, R, w_0, V^k) \) over a frame from \( F \) such that \( \mathfrak{M}^k \models_w A_i \), for all \( i \leq j \), and \( \mathfrak{M}^k \not\models_w A \), for some \( w \in W \). Let \( p: \mathfrak{M}^k \rightarrow \mathfrak{M}^k \) be a p-morphism from \( \mathfrak{M}^k \) onto a p-irreducible \( k \)-model \( \mathfrak{M}^k \), see Theorem 3.10. We take the frame of \( (\mathfrak{M}^k)_{p(w)} \) as our \( \mathfrak{F} \). Let \( \mathfrak{F} = (U, \leq, p(w)) \). Since \( (\mathfrak{M}^k)_{p(w)} \) is a \( k \)-model over \( \mathfrak{F} \), we have \( A \not\in L(\mathfrak{F}) \). There remains to show that \( L' \subseteq L(\mathfrak{F}) \).

\[ ^{13} \text{The following theorem resembles (not without reasons) characterizations, see [26, 29], of extensions of logics given by logical matrices.} \]
Suppose that $C \notin \mathbf{L}(F)$ for some $C \in \mathbf{L}$. Let $C = C(x_1, \ldots, x_n)$ and let $\mathcal{M}^n$ be an $n$-model over $\mathcal{F}$ such that $\mathcal{M}^n \models C$. We define a substitution $\varepsilon : \{x_1, \ldots, x_n\} \to \text{Fm}^k$ taking $\varepsilon(x_i) = \bigvee\{\Delta(\mathcal{M}^k)_{u} : \mathcal{M}^u \models x_i\}$, for any $i \leq n$. Then we have $\mathcal{M}^k \models_v \varepsilon(x_i) \iff \exists_u \in U \mathcal{M}(\mathcal{M}^k)_{u} \models \mathcal{M}^u \models x_i \iff \exists_v \in U \mathcal{M}(\mathcal{M}^k)_{u} \models \mathcal{M}^u \models x_i$, for any $i \leq n$ and $v \in U$. Note that the last but one equivalence needs Corollary 3.10. This shows $\mathcal{M}^k \models_v \varepsilon(C) \iff \mathcal{M}^u \models_v C$, for any $v \in U$ and hence we get $\mathcal{M}^k \not\models_{p(w)} \varepsilon(C)$, that is $\mathcal{M}^k \not\models \varepsilon(C)$, which cannot happen as $\varepsilon(C)$ is one of the $A_i$’s and must be true at $(\mathcal{M}^k)_{w}$.

### 4.1 Substitutions in Locally Tabular Logics

Let $\mathbf{F}$ be a class of finite frames, $\mathbf{L} = \mathbf{L}(\mathbf{F})$ be locally tabular and $\mathcal{M}^n = \mathcal{M}^n(\mathbf{F})$, for any $n \geq 0$. Assume, additionally, that $\mathbf{F}$ is closed under generated subframes and p-morphic images, that is $\text{sm}(\mathbf{F}) = \mathbf{F}$, see Lemma 3.11. For any $\sigma : \{x_1, \ldots, x_n\} \to \text{Fm}^k$, define $H_{\sigma} : \mathcal{M}^k \to \mathcal{M}^n$ putting $H_{\sigma}(\mathcal{M}^k) = \sigma(\mathcal{M}^k)$, for each $\mathcal{M}^k$.

**Lemma 4.3**

Suppose that $\varepsilon, \sigma : \{x_1, \ldots, x_n\} \to \text{Fm}^k$. Then $\varepsilon =_{L} \sigma$ iff $H_{\sigma} \sim H_{\varepsilon}$.

**Proof.** ($\Rightarrow$) is obvious. ($\Leftarrow$). Let $H_{\sigma}(\mathcal{M}^k) \sim H_{\varepsilon}(\mathcal{M}^k)$, for any $\mathcal{M}^k \in \mathcal{M}^k$. Then $\mathcal{M}^k \models \sigma(A) \iff \varepsilon(\mathcal{M}^k) \models A \iff \mathcal{M}^k \models \varepsilon(A)$. Thus, $\varepsilon =_{L} \sigma$.

The assumptions that the frames $\mathbf{F}$ are finite and $\mathbf{L}(\mathbf{F})$ is locally tabular do not play any role in the above Lemma but they are essential in the subsequent theorem, to prove that the conditions (i)-(iii) of Lemma 3.21 characterize substitutions:

**Theorem 4.4**

Let $H : \mathcal{M}^k \to \mathcal{M}^n$. Then $H \sim H_{\sigma}$, for some $\sigma : \{x_1, \ldots, x_n\} \to \text{Fm}^k$ if and only if $H$ fulfills the following conditions:

(i) the $n$-model $H(\mathcal{M}^k)$ has the same frame as the $k$-model $\mathcal{M}^k$, for any $\mathcal{M}^k \in \mathcal{M}^k$;
(ii) $H((\mathcal{M}^k)_{w}) \sim (H(\mathcal{M}^k))_{w}$, for any $\mathcal{M}^k = (W, R, u_0, V^k) \in \mathcal{M}^k$ and $w \in W$;
(iii) if $\mathcal{M}^k \sim \mathcal{M}^k$, then $H(\mathcal{M}^k) \sim H(\mathcal{M}^k)$, for any $\mathcal{M}^k, \mathcal{M}^k \in \mathcal{M}^k$.

**Proof.** ($\Leftarrow$) follows from Lemma 3.21. The conditions (i)-(iii) of Lemma 3.21 seem to be stronger than the above ones, but they are not (see Theorem 3.14).

To prove ($\Rightarrow$) we assume $H : \mathcal{M}^k \to \mathcal{M}^n$ fulfills the above (i)-(iii). Let

$$\sigma(x_i) = \bigvee\{\Delta(\mathcal{M}^k) : \mathcal{M}^k \models x_i\}, \quad \text{for } i = 1, \ldots, n.$$  

By Theorem 2.3 we can claim that we have defined $\sigma : \{x_1, \ldots, x_n\} \to \text{Fm}^k$. For any $k$-model $\mathcal{M}^k = (W, R, u_0, V^k) \in \mathcal{M}^k$, we have

$$\mathcal{M}^k \models_x x_i \iff \mathcal{M}^k \models_x (\mathcal{M}^k)_u \models x_i \iff \mathcal{M}^k \models_x \sigma(x_i) \iff \mathcal{M}^k \models_x \mathcal{M}^k \models_x x_i,$$

for any $i = 1, \ldots, n$ and any $w \in W$. Hence $\sigma(\mathcal{M}^k) \sim H(\mathcal{M}^k)$.

---

14 Ghiardi wrote $\sigma(u)$ for any Kripke model $u$ and hence we have $\sigma(\mathcal{M}^k)$. We should, perhaps, wrote $\sigma : \mathcal{M}^k \to \mathcal{M}^n$ but we think it could be misleading as we already have $\sigma : \{x_1, \ldots, x_n\} \to \text{Fm}^k$ and $\sigma : \text{Fm} \to \text{Fm}$. Talking about the mapping $\sigma$, it would be unclear if we had in mind a mapping between formulas or models. For this reason we decided to introduce $H_\sigma$, to replace $\sigma$, though it could be seen as an excessive reaction.

15 where $H_\sigma \sim H_\varepsilon$ obviously means $H_\sigma(\mathcal{M}^k) \sim H_\varepsilon(\mathcal{M}^k)$ for each $\mathcal{M}^k$. 

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The above theorem is useful to define substitutions. However, the condition (iii) is hard to check if there is too many p-morphisms between models. So, we would prefer a variant of Theorem 4.3 given below, concerning p-irreducible models. The closure of p-morphic images is not necessary for the above theorem (it suffices the closure under generated subframes) but it is necessary for the subsequent theorem.

Let $M^n_n$, for any $n \geq 0$, be the class of p-irreducible $n$-models over the frames $F$. According to Theorem 3.10, for any $A \in Fm^n$,

\[ A \in L \iff (\mathfrak{M}^n \models A, \text{ for every } \mathfrak{M}^n \in M^n_n). \]

**Theorem 4.5**

If $H: M^n_k \to M^n_k$ fulfills

(i) the $n$-model $H(\mathfrak{M}^k)$ has the same frame as the $k$-model $\mathfrak{M}^k$, for any $\mathfrak{M}^k \in M^n_k$;

(ii) $H((\mathfrak{M}^k)_U) \sim (H(\mathfrak{M}^k))$, for any $\mathfrak{M}^k = (W, R, w_0, V^k) \in M^n_k$ and any $w \in W$;

(iii) if $\mathfrak{M}^k \equiv \mathfrak{M}^n$, then $H(\mathfrak{M}^k) \sim H(\mathfrak{M}^n)$, for any $\mathfrak{M}^k, \mathfrak{M}^n \in M^n_k$;

then there is exactly one (up to $=_{L}$) substitution $\sigma: \{x_1, \ldots, x_n\} \to FM^n_k$ such that $H(\mathfrak{M}^k) \sim H_{\sigma}(\mathfrak{M}^n)$, for each $\mathfrak{M}^k \in M^n_k$.

**Proof.** We proceed in the same way as above. In particular, we define

\[ \sigma(x_i) = \bigvee \{\Delta(\mathfrak{M}^i_k): \mathfrak{M}^i_k \in M^n_k \land H(\mathfrak{M}^i_k) \models x_i\}, \quad \text{for } i = 1, \ldots, n \]

and prove $H(\mathfrak{M}^k) \sim H_{\sigma}(\mathfrak{M}^n)$, for any $\mathfrak{M}^k \in M^n_k$. The crucial step in our argument

\[ \exists \mathfrak{M}^k ((\mathfrak{M}^k)_U \models \Delta(\mathfrak{M}^i_k) \land H(\mathfrak{M}^i_k) \models x_i) \Rightarrow H((\mathfrak{M}^k)_U) \models x_i \]

follows from the fact that, if $(\mathfrak{M}^i_k)_U \sim (\mathfrak{M}^i_k)_U$, for some $u$, then $(\mathfrak{M}^i_k)_U$ and $(\mathfrak{M}^i_k)_U$ are p-irreducible by Theorem 3.12 and hence $(\mathfrak{M}^k)_U \equiv (\mathfrak{M}^k)_U$ by Corollary 3.18. Thus, by (iii), we have $H((\mathfrak{M}^k)_U) \models x_i$ if $H(\mathfrak{M}^i_k) \models x_i$.

The uniqueness of $\sigma$ follows from Lemma 4.3 (and Theorem 3.10). \[\square\]

Suppose we need $H: M^n_k \to M^n_k$ fulfilling the above (i)–(iii). Let $\mathfrak{M}^k = (W, \leq, w_0, V^k)$ be a p-irreducible $k$-model. We should have $H(\mathfrak{M}^k) = (W, \leq, u_0, V^n)$, for some $V^n$, which means only the valuations $V^n$ are to be defined. By Theorem 3.12 $(\mathfrak{M}^k)_U$ is p-irreducible, for any $u \in W$, and hence we could define $H((\mathfrak{M}^k)_U)$ inductively with respect to $d_3(u)$, where $\mathfrak{F} = (W, \leq, w_0)$. First, we define $H((\mathfrak{M}^k)_U)$ for $n$-models over 1-element reflexive frames, that is we define $V^n(w)$ for end elements $w \in W$; any subset of \{x_1, \ldots, x_k\} can be taken as $V^n(w)$. Then, assuming $H((\mathfrak{M}^k)_U)$ has been defined for any $u > w$, and hence we have $V^n(w)$ for any $u > w$, we define $V^n(w)$. The only restriction is monotonicity, that is we should have $V^n(u) \subseteq V^n(u)$ for any $u > w$. The condition (ii) would be satisfied as we would even have $H((\mathfrak{M}^k)_U) = (H(\mathfrak{M}^k))_U$.

The condition (iii) should not produce any harm if we define $H$ on the equivalence classes $[\mathfrak{M}^k]_\equiv$. We should get $H: M^n_k \equiv \to M^n_k \sim$. Note that it would be much easier to satisfy this condition than to satisfy the corresponding (iii) of Theorem 4.3 where we should get $H: M^n_k \sim \to M^n_k \sim$.

Since $L$ is locally tabular, any formula $A \in Fm^n$ is a disjunction of the characters of all $A$-models (that is $n$-models $\mathfrak{M}^n \in M^n$ such that $A$ is true at $\mathfrak{M}^n$):

\[ A =_L \bigvee \{\Delta(\mathfrak{M}^n): \Delta(\mathfrak{M}^n) \rightarrow A \in L\} =_L \bigvee \{\Delta(\mathfrak{M}^n): \mathfrak{M}^n \models A\}. \]

(\star)
It also shows that each L-consistent (that is $\not\models_L \bot$) formula is L-unifiable.

**Lemma 4.6**
A substitution $\sigma: \{x_1, \ldots, x_n\} \to \text{Fm}^k$ is an L-unifier for $A \in \text{Fm}^n$ iff $\sigma(\mathcal{M}^k) \models A$ for each $k$-model $\mathcal{M}^k$. In other words, $\sigma$ is a unifier for $A$ iff all $\sigma$-models are $A$-models.

**Proof.** $\sigma(\mathcal{M}^k) \models A$ iff $\mathcal{M}^k \models \sigma(A)$, for each $\mathcal{M}^k$. Thus, $\sigma(\mathcal{M}^k) \models A$ for each $\mathcal{M}^k$ iff $\mathcal{M}^k \models \sigma(A)$ for each $\mathcal{M}^k$. But $\sigma$ is an L-unifier for $A$ iff $\mathcal{M}^k \models \sigma(A)$ for each $\mathcal{M}^k$. ■

Accordingly, $\sigma$ is a unifier for $A$ iff the disjunction $\bigvee\{\Delta(\mathcal{M}^n) : \mathcal{M}^n \models A\}$ in (*) contains the characters of all $\sigma$-models. To put it in other words:

**Corollary 4.7**
A substitution $\sigma: \{x_1, \ldots, x_n\} \to \text{Fm}^k$ is an L-unifier for a formula $A \in \text{Fm}^n$ iff $A_\sigma \to A \in L$, where $A_\sigma = \bigvee\{\Delta(\sigma(\mathcal{M}^n)) : \mathcal{M}^n \in \mathcal{M}^k\}$.

Thus, each substitution $\sigma$ is an L-unifier of some formulas $A \in \text{Fm}^n$ and among these formulas we can find the strongest (or the smallest in the Lindenbaum algebra) formula $A_\sigma$. Using Lemma 4.6 and Lemma 4.14 we can now characterize all unifiers of $A_\sigma$.

**Corollary 4.8**
Let $\sigma: \{x_1, \ldots, x_n\} \to \text{Fm}^k$. A substitution $\tau: \{x_1, \ldots, x_n\} \to \text{Fm}^m$ is an L-unifier for $A_\sigma$ iff all $\tau$-models are (equivalent to) $\sigma$-models, that is iff (iv) for every $\mathcal{M}^m \in \mathcal{M}^m$ there is an $\mathcal{M}^k \in \mathcal{M}^k$ such that $\tau(\mathcal{M}^m) \sim \sigma(\mathcal{M}^k)$.

The above condition (iv) can be also written down as the inclusion

$$\tau(\mathcal{M}^m)/\sim \subseteq \sigma(\mathcal{M}^k)/\sim.$$ 

There remains to characterize the relation $\preceq$ (of being a more general substitution) in terms of $\sigma$-models.

**Lemma 4.9**
A substitution $\tau: \{x_1, \ldots, x_n\} \to \text{Fm}^m$ is more general than $\sigma: \{x_1, \ldots, x_n\} \to \text{Fm}^k$ (that is $\tau \preceq \sigma$) iff there is $F: \mathcal{M}^k \to \mathcal{M}^m$, fulfilling (i)–(iii) of Theorem 4.4, such that (v) $\tau(F(\mathcal{M}^k)) \sim \sigma(\mathcal{M}^k)$, for every $\mathcal{M}^k \in \mathcal{M}^k$.

Consequently, if $\tau \preceq \sigma$, then all $\sigma$-models are (equivalent to) $\tau$-models and hence

$$\tau(\mathcal{M}^m)/\sim \supseteq \sigma(\mathcal{M}^k)/\sim.$$ 

**Proof.** Suppose that $\nu \circ \tau =_L \sigma$ for some $\nu: \{x_1, \ldots, x_m\} \to \text{Fm}^k$ and $F = H_\nu$. Then, $\tau(F(\mathcal{M}^k)) = \tau(\nu(\mathcal{M}^k))$ and, for each $B \in \text{Fm}^n$, $\tau(\nu(\mathcal{M}^k)) \models B \iff \nu(\mathcal{M}^k) \models \tau(B) \iff \mathcal{M}^k \models \nu(\tau(B)) \iff \mathcal{M}^k \models \sigma(B) \iff \mathcal{M}^k \models \sigma(\mathcal{M}^k) \models B$. Hence we get (v).

The reverse implication is shown in the same way. We get $\nu: \{x_1, \ldots, x_m\} \to \text{Fm}^k$ such that $F(\mathcal{M}^k) = \nu(\mathcal{M}^k)$, for each $\mathcal{M}^k$, by Lemma 4.6. Then, using (v), we show $\mathcal{M}^k \models \nu(\tau(B)) \iff \mathcal{M}^k \models \sigma(B)$ for each $B \in \text{Fm}^n$ and this yields $\nu \circ \tau =_L \sigma$. ■

### 4.2 Locally Tabular Logics with Finitary Unification

Now, we give a short proof of the main result of [113] characterizing locally tabular intermediate logics with finitary (or unitary) unification.

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**Theorem 4.10**

A locally tabular intermediate logic $L$ has finitary (or unitary) unification if and only if for every $n \geq 0$ there exists a number $m \geq 0$ such that for every $k \geq 0$ and every $\sigma: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k$ there are $G: M^m \rightarrow M^n$ and $F: M^k \rightarrow M^m$ such that

(i) $G$ preserves the frame of any $m$-model $M^m = (W, R, w_0, V^m) \in M^m$;

(ii) $F$ preserves the frame of any $k$-model $M^k = (W', R', w_0', V^k) \in M^k$;

(iii) $G(\langle M^m \rangle_w) \sim (G(\langle M^m \rangle)_w)$, for any $M^m = (W, R, w_0, V^m) \in M^m$ and $w \in W$;

(iv) for every $M^m \in M^m$ there is $M^k \in M^k$ such that $G(M^m) \sim \sigma(M^k)$;

(v) $G(F(M^k)) \sim \sigma(M^k)$, for every $M^k \in M^k$; see Figure 15 below.

**Proof.** Since $L$ is locally tabular, $\text{Fm}^n$ is finite (up to $\equiv_L$), for each $n$, and each $\sigma: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k$ is a unifier of $A_\sigma \in \text{Fm}^n$, see Corollary 4.7. Thus, if $L$ has finitary (or unitary) unification, one can find, for every $n \geq 0$, a number $m$ such that each $L$-unifiable formula $A \in \text{Fm}^n$ (and $A_\sigma$ in particular) has a complete set of unifiers among $\sigma: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^m$. Suppose $\tau: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^m$ is a unifier for $A_\sigma$ and $\tau \equiv \sigma$. Take $G = H_\tau$ and let $F$ be the mapping determined by Lemma 4.9. Then the conditions (i)–(v) are obviously fulfilled.

On the other hand, using the conditions (i)–(v) one shows that the set of $L$-unifiers $\tau: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^m$, for any unifiable $A \in \text{Fm}^m$, is complete. Since $L$ is locally tabular, there is only finitely many (up to $\equiv_L$) such unifiers $\tau$ and hence unification is finitary (unless unitary).

The above theorem do not need the assumption that $F$ is closed under p-morphic images; it suffices only the closure under generated subframes. But $F = \text{sm}(F)$ is needed for the following version of 4.10 where we refer to Theorem 4.5 (instead of 4.4). For this version of 4.10 we must, however, modify, the condition (v) as we cannot assume that $F(M^k)$ is p-irreducible even if $M^k$ is p-irreducible.

**Theorem 4.11**

A locally tabular intermediate logic $L$ has finitary (or unitary) unification if and only if for every $n \geq 0$ there exists a number $m \geq 0$ such that for every $k \geq 0$ and every $\sigma: \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k$ there are $G: M^m_\tau \rightarrow M^n$ and $F: M^k_\tau \rightarrow M^m$ such that

(i) $G$ preserves the frame of any $m$-model $M^m = (W, R, w_0, V^m) \in M^m_\tau$;

(ii) $F$ preserves the frame of any $k$-model $M^k = (W', R', w_0', V^k) \in M^k_\tau$;

(iii) $G(\langle M^m \rangle_w) \sim (G(\langle M^m \rangle)_w)$, for any $M^m = (W, R, w_0, V^m) \in M^m_\tau$ and $w \in W$;

(iv) for every $M^m \in M^m_\tau$ there is $M^k \in M^k_\tau$ such that $G(M^m) \sim \sigma(M^k)$;

(v) $G(F(M^k)) \sim \sigma(M^k)$, for every $M^k \in M^k_\tau$; see Figure 15 below.
\( M^k \equiv \mathfrak{M}^k \Rightarrow F(\mathfrak{M}^k) \sim F(\mathfrak{M}^k) \), for any \( \mathfrak{M}^k, \mathfrak{M}^k \in M^k \).

(iv) for every \( \mathfrak{M}^m \in M^m \) there is \( \mathfrak{M}^k \in M^k \) such that \( G(\mathfrak{M}^m) \sim \sigma(\mathfrak{M}^k) \).

(v) if \( \mathfrak{M}^m \sim F(\mathfrak{M}^k) \), then \( G(\mathfrak{M}^m) \sim \sigma(\mathfrak{M}^k) \), for every \( \mathfrak{M}^k \in M^k \) and \( \mathfrak{M}^m \in M^m \).

In [10] the above theorems is applied to determine the unification types of locally tabular logics. We use them, as well, in the present paper.

Let us recall that \( H_{un} \) (see Figure 7) consists of all frames \( \mathcal{L}_d + \mathfrak{R} \), where \( s, d \geq 0 \).

The logic \( L(H_{un}) \) is locally tabular as it extends \( PWL \), see [11].

**Theorem 4.12**

For any \( F \subseteq H_{un} \), the logic \( L(F) \) has unitary unification.

**Proof.** Let \( n \geq 1 \), and \( m = n \cdot 2^n + 1 \) and \( \sigma: \{x_1, \ldots, x_n\} \rightarrow \mathbb{F}^m \), for some \( k \). We need \( G: M^m \rightarrow M^n \) and \( F: M^m \rightarrow M^n \) fulfilling (i)-(v) of Theorem 4.11. Let \( code(f^k_0) = f^n_0 g^n_2 g^n_3 \ldots g^n_{2n-1} \) (the concatenation of \( f^n_0, g^n_i \)'s and the suffix 1) includes all \( g^n_i \)'s (in any order where repetitions are allowed) such that

\[
\begin{align*}
\sigma(\mathfrak{M}^k) &= \big( f^n_0 g^n_i \big), \quad \text{for some } g^n_i. \\
\text{Then we define } F(\mathfrak{M}^k) &:=
\end{align*}
\]

One checks the conditions (i)-(iii) for the mapping \( F \). Note that \( F(\mathfrak{M}^k) \) could be not p-irreducible, for some p-irreducible \( \mathfrak{M}^k \), but one could only collapse elements of the depth 2 or elements below \( s + 1 \) that is elements in the ‘leg’ of the model.

A. We put \( G(\mathfrak{M}^m) = \mathfrak{M}^m \wedge n \) if \( \mathfrak{M}^m \sim F(\mathfrak{M}^k) \) for some \( \mathfrak{M}^k \). Let \( \mathfrak{M}^m = (W, R, w_0, V^m) \) and \( \mathfrak{M}^k = (U, S, u_0, V^k) \). The conditions (i)-(v) for \( G \) seem to be obvious; as concerns (ii), for every \( w \in W \) there is \( u \in U \) such that \( (\mathfrak{M}^m)_w \sim F(\mathfrak{M}^k)_u \). To show (iv) and (v), let us note that \( F(\mathfrak{M}^k)|_n = \sigma(\mathfrak{M}^k) \), see Figure 16. Thus, \( G(F(\mathfrak{M}^k)) = \sigma(\mathfrak{M}^k) \) and \( G(\mathfrak{M}^m) = \mathfrak{M}^m \wedge n \sim F(\mathfrak{M}^k) \wedge n = \sigma(\mathfrak{M}^k) \) if \( \mathfrak{M}^m \sim F(\mathfrak{M}^k) \).

B. There remains to define \( G(\mathfrak{M}^m) \) for \( \mathfrak{M}^m \in M^m \) non-equivalent to any \( F(\mathfrak{M}^k) \). Our definition is inductive (to secure (ii)) and we should take into account that for

---

\[ ^{16}\text{There is no } \mathcal{L}_0 \text{ (or it would be the empty frame), so we should have here: all logics determined by } \mathfrak{R}_s \text{ and } \mathcal{L}_d + \mathfrak{R}_s, \text{ where } s \geq 0 \text{ and } d \geq 1, \text{ if we agree that } \mathfrak{R}_0 = \emptyset = \mathfrak{L}_1. \]

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some \( w \in W \) and some \( \mathfrak{M}^k \) we may have \( (\mathfrak{M}^m)_w \sim F(\mathfrak{M}^k) \) and hence we have defined \( G((\mathfrak{M}^m)_w) = (\mathfrak{M}^m)_w \upharpoonright n \). Since (iii) is obvious and (v) is irrelevant in this case, we have to bother only about (i) and (iv). Thus, the frame of the \( n \)-model \( G(\mathfrak{M}^m) \) should be the same as the frame of \( \mathfrak{M}^m \) and \( G(\mathfrak{M}^m) \) should always be equivalent with a \( \sigma \)-model. Let \( \mathfrak{M}^m \) be one of the following \( m \)-models:

![Diagram](image)

where \( s, d \geq 1 \)

**FIG. 17.**

B1. If \( (\mathfrak{M}^m)_i \)\(^1\) is not equivalent with any \( F(\mathfrak{M}^k) \), we take any valuation \( f^k \); let \( \sigma(\Sigma_1, \{ f^k \}) = (\Sigma_1, \{ f^n \}) \), for some \( f^n \), and put \( G(\mathfrak{M}^m) \sim (\Sigma_1, \{ f^n \}) \) (which uniquely determines \( G(\mathfrak{M}^m) \)). Then (i)–(iv) are clearly fulfilled and (v) is irrelevant. We do the same if \( 0 \) is the only vertex \( i \) for which \( (\mathfrak{M}^m)_i \sim F(\mathfrak{M}^k) \), with some \( \mathfrak{M}^k \); we only take \( f^k = f^n_0 \).

B2. We can defined \( G((\mathfrak{M}^m)_i) \), if \( 1 \leq i \leq s \), using A. or B1. Thus, if \( (\mathfrak{M}^m)_i = (\Sigma_2, \{ f^n_0, f^n_1 \}) \) and \( f^n_0 = f^n_0 g^n_2 \cdots g^n_s \cdot 1 = \text{code}(f^n_0) \) and \( f^n_1 = f^n_1 g^n_2 \cdots g^n_s \cdot 0 \), for some \( p \)-irreducible \( \mathfrak{M}^k = (\Sigma_2, \{ f^n_0, f^n_1 \}) \) (and we have \( \sigma(\mathfrak{M}^k) = (\Sigma_2, \{ f^n_0, f^n_1 \}) \)), we put \( G((\mathfrak{M}^m)_i) = \sigma(\mathfrak{M}^k) = (\Sigma_2, \{ f^n_0, f^n_1 \}) \). Note that \( f^n_1 \) must be one of \( g^n_i \)'s in \( \text{code}(f^n_0) \). Otherwise, \( G((\mathfrak{M}^m)_i) \sim \sigma(\Sigma_1, \{ f^n_0 \}) \), where \( G((\mathfrak{M}^m)_0) = \sigma(\Sigma_1, \{ f^n_0 \}) \). Thus, in each case we have \( G((\mathfrak{M}^m)_i) = \sigma(\mathfrak{M}^k) \), for some \( k \)-model \( \mathfrak{M}^k \) over \( \Sigma_2 \).

B3. The worst case is \( G((\mathfrak{M}^m)_{s+1}) \). By B2. and the definition of \( \text{code}(f^n_0) \) (included in \( f^n_0 \)), we can assume that, if \( 1 \leq i \leq s \), then \( G((\mathfrak{M}^m)_i) = G(\Sigma_2, \{ f^n_0, f^n_1 \}) = \sigma(\mathfrak{M}^k) \) for some \( k \)-model \( \mathfrak{M}^k = (\Sigma_2, \{ f^n_0, f^n_1 \}) \); note that we have the same \( f^n_0 \) in each of the models \( \mathfrak{M}^k \). There are different \( f^n_0 \)'s with the same \( \text{code}(f^n_0) \); but we can always decide which \( f^n_0 \) is chosen - for instance, our choose could be given according to the lexicographical order in which all valuations are given. Then, assuming \( (\mathfrak{M}^m)_{s+1} \) is not equivalent with any \( F(\mathfrak{M}^k) \), we take the \( k \)-valuation \( 0 \cdots 0 \) and define \( G((\mathfrak{M}^m)_{s+1}) \) as it is shown in Figure \( 18 \) below.

\(^1\)Let us agree that, in Figure 17 the vertex \( i \) is labeled with \( f^n_0 \).

\(^2\)We have added the suffix 0 or 1 at the end of \( \text{code}(f^n_0) \), see Figure 18 to be sure that \( F(\mathfrak{M}^k) \) is \( p \)-irreducible if \( \mathfrak{M}^k \) is a \( p \)-irreducible model over \( \Sigma_2 \).
Thus, we have \[ G \] as it is shown in Figure 19. There remains to define \( G((\mathfrak{M}^m)_{s+i+1}) \) assuming that \( i \geq 1, \) and \((\mathfrak{M}^m)_{s+i+1}\) is not equivalent with any \( F(\mathfrak{M}^k) \) and \( G((\mathfrak{M}^m)_{s+i}) \) has already been defined. Then we approach as it is shown in Figure 19.

\[ G((\mathfrak{M}^m)_{s+i+1}) = G((\mathfrak{M}^m)_{s+i}) \]

Fig. 18.

B4. There remains to define \( G((\mathfrak{M}^m)_{s+i+1}) \) assuming that \( i \geq 1, \) and \((\mathfrak{M}^m)_{s+i+1}\) is not equivalent with any \( F(\mathfrak{M}^k) \) and \( G((\mathfrak{M}^m)_{s+i}) \) has already been defined. Then we approach as it is shown in Figure 19.

\[ G((\mathfrak{M}^m)_{s+i+1}) = G((\mathfrak{M}^m)_{s+i}) \]

Fig. 19.

Thus, we have \( G(\mathfrak{M}^m) \sim G((\mathfrak{M}^m)_{s+i}). \)

\[ \]
We could make code($f^k$) unique, for any $f^k$. For instance, let us begin with the sequence $g_1^n h_1^n \ldots g_2^n h_2^n$, containing all possible pairs $g_i^n h_i^n$ of $n$-valuations (regarded as binary strings) put in a lexicographical order. The sequence contains $4^n$ bites. Then, step by step, we would remain the pair $g_i^n h_i^n$ or replace it with $f^n f^m$ depending if it fulfills (or not) the equation in Figure 20, for some $g_i^n$, $h_i^n$. Eventually, we add $f^n$ at the beginning and 11 at the end of the sequence receiving an $m$-valuation.

Now, we are ready to define $F(M^k)$, for any p-irreducible $k$-model $M^k$ over any frame in $F$. So, let $M^k$ be one of the following $k$-models:

![Diagram](image1)

**Fig. 21.**

and let $\sigma(M^k)$ be, correspondingly:

![Diagram](image2)

**Fig. 22.**

We define $F(M^k)$ as one of the following $m$-models:
The conditions (i)-(iii) of Theorem 4.11, as concerns the mapping \( F \), are obviously fulfilled. Note that it is important that all binary strings have the size \( m \) and we get intuitionistic models, but it is of no importance what is included in \( \text{code}(k) \). There remains to define the mapping \( G \). Let

\[
G(\mathfrak{M}^n) = \begin{cases} 
\mathfrak{M}^n \upharpoonright n & \text{if } \mathfrak{M}^n \sim F(\mathfrak{M}^k) \text{ for some } \mathfrak{M}^k; \\
\text{?} & \text{otherwise;}
\end{cases}
\]

where the \( n \)-model \( \mathfrak{M}^n \upharpoonright n \) results from \( \mathfrak{M}^n \) by the restriction of all valuations \( f_i^m \) of \( \mathfrak{M}^n \) to \( \{x_1, \ldots, x_n\} \). Note that \( \mathfrak{M}^n \sim F(\mathfrak{M}^k) \) yields \( \mathfrak{M}^n \upharpoonright n \sim F(\mathfrak{M}^k) \upharpoonright n \) and hence \( G(\mathfrak{M}^n) \sim \sigma(\mathfrak{M}^k) \) by the definition of \( F \) and \( G \). Thus, (v) is fulfilled. If an \( m \)-model \( \mathfrak{M}^m \) is equivalent with some \( F(\mathfrak{M}^k) \), then any its generated submodel \( (\mathfrak{M}^m)_w \) is equivalent with \( F((\mathfrak{M}^k)_w) \), for some \( w \); see Corollary 3.19. Thus, (i)-(v) are fulfilled if the first line of the definition \( G(\mathfrak{M}^m) \) applies.

There remains to define \( G(\mathfrak{M}^m) \) assuming \( \mathfrak{M}^m \) is not equivalent with any \( F(\mathfrak{M}^k) \). Our definition is inductive with respect to the depth of \( \mathfrak{M}^m \) to secure (ii). Since (v) is irrelevant here and (iii) is quite obvious, we should require \( G(\mathfrak{M}^m) \) and \( \mathfrak{M}^m \) were defined over the same frame, and \( G(\mathfrak{M}^m) \) should be equivalent with a \( \sigma \)-model.

\( \Sigma_1 \)-Models. If \( \mathfrak{M}^m = (\Sigma_1, f^m) \) is equivalent with some \( F(\mathfrak{M}^k) \), then we easily get \( \mathfrak{M}^m = F(\Sigma_1, f^k) \), for some \( f^k \), and hence \( f^m = \text{code}(f^k) \). Thus, we have

\[
G(\mathfrak{M}^m) = (\Sigma_1, \text{code}(f^k) \upharpoonright n) = \sigma(\Sigma_1, f^k).
\]

If \( f^m \neq \text{code}(f^k) \), for any \( f^k \), we could take \( G(\mathfrak{M}^m) = \sigma(\mathfrak{M}^k) \), for any \( k \)-model \( \mathfrak{M}^k \) over \( \Sigma_1 \). We can even take the same \( \mathfrak{M}^k \), for each \( \mathfrak{M}^m \). So, we choose some \( s^k \) and agree that, if \( (\Sigma_1, f^m) \) is not equivalent with any \( F(\mathfrak{M}^k) \), we have

\[
G(\mathfrak{M}^m) = \sigma(\Sigma_1, s^k) = (\Sigma_1, \text{code}(s^k) \upharpoonright n).
\]

\( \Sigma_2 \) and \( \Sigma_2 \)-Models. Let \( \mathfrak{M}^m = (\Sigma_2, \{f_0^m, f_0^0, f_1^m\}) \) be a \( p \)-irreducible \( m \)-model over \( \Sigma_2 \). Assume that \( \mathfrak{M}^m \sim F(\mathfrak{M}^k) \), for some \( \mathfrak{M}^k \). Then \( \mathfrak{M}^m = F(\mathfrak{M}^k) \) as the suffices 11,01 and 00 prevent any reduction of \( F(\mathfrak{M}^k) \), for any \( p \)-irreducible \( \mathfrak{M}^k \) of the
we have possibilities is worse. We have 

\[ f''_0 = code(f'_0) = g''_1 b''_1 \ldots g''_m b''_m + 11, \quad \text{and} \quad f''_1 = code(f'_1) = g''_1 b''_1 \ldots g''_m b''_m + 11, \quad \text{and} \quad f''_1 = f''(01) \]

where \( \sigma(M_k) = (\bar{g}_2, (f''_0, f''_1)) = G(M_k) \). By the definition of \( code() \), see the Figure \ref{fig:24} the pair \( f''_0 f''_1 \) must occur in \( code(f'_0) \) and \( f''_0 f''_1 \) in \( code(f'_1) \). We would like to preserve this condition even if \( M_m \) is not equivalent with any \( F(M_k) \).

Let \( M_m \) be not equivalent with any \( F(M_k) \). We have \( G(\mathcal{L}_1, f''_0) = \sigma(\mathcal{L}_1, f'_0) = (\mathcal{L}_1, f'_0) \) and \( G(\mathcal{L}_1, f''_1) = \sigma(\mathcal{L}_1, f'_1) = (\mathcal{L}_1, f'_1) \), for some \( f'_0, f'_1 \) and some \( f''_0, f''_1 \).

Take any \( f''_1 \) such that \( (\bar{g}_2, \{ f''_0, f''_1 \}) \) is an intuitionistic model and define

\[ G(M_m) = \sigma((\bar{g}_2, \{ f''_0, f''_1 \}) = (\bar{g}_2, \{ f''_0, f''_1 \}), \quad \text{for some} \quad f''_1. \]

We obviously have \( f''_0 f''_1 \) in \( code(f''_0) \) and \( f''_0 f''_1 \) in \( code(f''_1) \). For the choice of \( f''_1 \), use the lexicographical order which would also be sufficient to satisfy the condition (iii) of Theorem \ref{thm:1.1} saying that isomorphic models have isomorphic images.

Let \( M'' = (\mathcal{L}_2, \{ f''_0, f''_1 \}) \) be a \( p \)-irreducible \( m \)-model over \( \mathcal{L}_2 \) and \( M'' \sim F(M_k) \), for some \( M_k \). We have two possibilities: either \( M'' \) is a model over \( \mathcal{L}_2 \) or it is a model over \( \bar{g}_2 \). In the first case, \( G(M''') = \sigma(M_k) = \sigma(\mathcal{L}_2, \{ f''_0, f''_1 \}) = (\mathcal{L}_2, \{ f''_0, f''_1 \}) \), for some \( f''_0, f''_1 \) and \( f''_1 = code(f'_1) \). It is clear that \( f''_0 f''_1 \) occurs in \( code(f'_1) \). The second possibilities is worse. We have \( M'' = (\bar{g}_2, \{ f''_0, f''_1 \}) \) and \( f''_0 = code(f'_0) = code(f'_1) \) (which does not yield \( f''_0 f''_1 \)). Since \( \sigma(M_k) = (\bar{g}_2, \{ f''_0, f''_1 \}) \) and \( f''_0 = f''_0 \), we take

\[ G(M''') = (\mathcal{L}_2, \{ f''_0, f''_1 \}) = (\mathcal{L}_2, \{ f''_0, f''_1 \} \sim \sigma(M_k) \text{ and have } f''_0 f''_1 \text{ in } code(f'_1). \]

Suppose that \( M'''' = (\mathcal{L}_2, \{ f''_0, f''_1 \}) \) is not equivalent with any \( F(M_k) \). Then \( G(\mathcal{L}_1, f''_0) = \sigma(\mathcal{L}_1, f'_0) = (\mathcal{L}_1, f'_0) \), for some \( f'_0, f'_1 \). Let us take any \( k \)-valuation \( f''_1 \) such that \( (\mathcal{L}_2, \{ f''_0, f''_1 \}) \) is an intuitionistic model and define \( G(M''') = \sigma((\mathcal{L}_2, \{ f''_0, f''_1 \}) = (\mathcal{L}_2, \{ f''_0, f''_1 \}) \), for some \( f''_1 \). We obviously have \( f''_0 f''_1 \) in \( code(f''_0) \).

\( \mathfrak{C}_5 \) - and \( \mathfrak{C}_3 \)-Models. Let \( M'' = (\mathcal{L}_2, \{ f''_0, f''_1 \}) \) be a \( p \)-irreducible model over \( \mathfrak{C}_5 \) (or over \( \mathfrak{C}_3 \)) and assume it is not equivalent with any \( F(M_k) \). We have (see Figure \ref{fig:24})

\[ G((\mathcal{L}''')_1) \sim \sigma(M_k) = (\bar{g}_2, \{ f''_0, f''_1 \}), \quad \text{and} \]

\[ G((\mathcal{L}''')_1) \sim \sigma(M_k) = (\bar{g}_2, \{ f''_0, f''_1 \}) \quad \text{or} \quad \sigma(M_k) = (\bar{g}_2, \{ f''_0, f''_1 \}) \]

for some \( M_k = (\bar{g}_2, \{ f''_0, f''_1 \}) \) and \( M_k = (\bar{g}_2, \{ f''_0, f''_1 \}) \) (or \( \bar{g}_2, \{ f''_0, f''_1 \}) \), which are \( k \)-model over \( \bar{g}_2 \). We have \( f''_0 f''_1 \) and \( f''_0 f''_1 \) (or \( f''_0 f''_1 \)) in \( code(f''_0) \) (see Figure \ref{fig:24}). Then, for some \( g''_1, h''_1, g''_2, h''_2 \), we can define \( \mathbf{C}_5 \)-models such that

\[ \sigma \begin{pmatrix} g''_1 & f''_0 & f''_1 \ h''_1 \ h''_2 \ 0 \ldots 0 \end{pmatrix} = f''_0, f''_1 \quad \text{for some} \quad f''_2. \]

\[ \text{FIG. 24.} \]

The last \( n \)-model over \( \mathfrak{C}_5 \) can be taken as \( G(M_m) \). In the case of \( \mathfrak{C}_3 \) we have \( f''_0 = f''_0 \), and we can reduce the \( \mathbf{C}_5 \)-model to a \( \mathfrak{C}_3 \)-model taking \( 0'' = 0'' \).

\( \mathfrak{R}_2 \) - and \( \mathfrak{R}_2 \)-Models. Let \( M'' = (\mathcal{L}_2, \{ f''_0, f''_1 \}) \) be a \( p \)-irreducible model over \( \mathfrak{R}_2 \) and assume it is not equivalent with any \( F(M_k) \). By our inductive hypothesis, we have (see Figure \ref{fig:24})

\[ G((\mathcal{L}''')_1) \sim \sigma(M_k) = (\bar{g}_2, \{ f''_0, f''_1 \}), \quad G((\mathcal{L}''')_1) \sim \sigma(M_k) = (\bar{g}_2, \{ f''_0, f''_1 \}); \]

\[ 28 \]
for some $\mathfrak{M}^k = (\mathfrak{f}_2, \{t^{k}_{0}, t^{k}_{0}, t^{k}_{1}\})$ and $\mathfrak{M}^k = (\mathfrak{f}_2, \{t^{k}_{0}, t^{k}_{0}, t^{k}_{1}\})$. We have $t^{k}_{0} = t^{k}_{0}$ and $t^{k}_{0}, t^{k}_{0}$ occur in $\text{code}(\mathfrak{f}_{0})$ (see Figure 20). Then for some $\mathfrak{g}_{1}, \mathfrak{g}_{2}, \mathfrak{f}_{2}, \mathfrak{f}_{2}$

\[ \sigma(\mathfrak{g}_{1}^{k}, \mathfrak{g}_{2}^{k}) = f^{k}_{0}, \sim f^{k}_{0}, f^{k}_{1}, f^{k}_{2} \]

\[ 0 \cdots 0 \]

\[ \mathfrak{g}_{1}^{k}, \mathfrak{g}_{2}^{k} \]

\[ f^{k}_{0}, f^{k}_{0}, f^{k}_{1}, f^{k}_{2} \]

\[ f^{k}_{0} \]

\[ f^{k}_{0}, f^{k}_{1}, f^{k}_{1}, f^{k}_{2} \]

\[ f^{k}_{0}, f^{k}_{1}, f^{k}_{1}, f^{k}_{2} \]

\[ f^{k}_{0}, f^{k}_{1}, f^{k}_{1}, f^{k}_{2} \]

\[ f^{k}_{0}, f^{k}_{1}, f^{k}_{1}, f^{k}_{2} \]

The received $n$-model over $\mathfrak{M}_2$ can be taken as $G(\mathfrak{M}^n)$. A similar argument (as we used above for $\mathfrak{M}_2$-models) applies for models over $\mathfrak{M}_2$.

+ $\mathfrak{f}_2$- and $\mathfrak{f}_2$-Models. Let $\mathfrak{M}^m$ be a $p$-irreducible model over $\mathfrak{L}_3$, or over $+\mathfrak{f}_2$, and assume that it is not equivalent to any $F(\mathfrak{M}^k)$. We can also assume $G((\mathfrak{M}^m))$ has been defined (see Figure 21). If $G((\mathfrak{M}^m)) = \sigma(\mathfrak{M}^k)$, for some $\mathfrak{M}^k$, then $\mathfrak{M}^k$ is a model over $\mathfrak{L}_2$, or over $\mathfrak{f}_2$, and we can take $G(\mathfrak{M}^m) = \sigma(\mathfrak{M}^k)$ where $\mathfrak{M}^k$ is an extension of $\mathfrak{M}^k$ with any $\mathfrak{k}$-valuation $f^k_k$ (the extended model must be an intuitionistic model). Then $\mathfrak{M}^k$ is a model over $\mathfrak{L}_3$, or over $+\mathfrak{f}_2$, as required. The only problem arises if $G((\mathfrak{M}^m))$ is equivalent with some $\sigma(\mathfrak{M}^k)$ but it is not any $\sigma$-model. This can happen if $G((\mathfrak{M}^m))$ is a model over $\mathfrak{L}_2$ and $\mathfrak{M}^k$ over $\mathfrak{f}_2$. Then extending $\mathfrak{M}^k$ with $f^k_k$ we get a model $\mathfrak{M}^k$ over $+\mathfrak{f}_2$ such that $\mathfrak{M}^k$ is equivalent with an extension of $G((\mathfrak{M}^m))$ to a model over $\mathfrak{L}_3$; and the model over $\mathfrak{L}_3$ is $G(\mathfrak{M}^m)$.

4.3 Projective Formulas in Locally Tabular Logics.

Projective formulas are useful in the area of unification. By Ghilardi [13], any consistent Rasiowa-Harrop formula (e.g. $\neg B$) is projective in $\text{INT}$ and, consequently, in any intermediate logic, see Lemma 2.8. One can easily produce other examples of projective formulas. Although conjunctions of projective formulas may be not projective, for instance $(x_1 \rightarrow x_2 \lor x_3) \land x_1$ is not, we have

Lemma 4.14

If $A = \bigwedge_{i=1}^{s} (B_i \leftrightarrow z_i)$ for some distinct variables $z_1, \ldots, z_s$ which do not occur in the formulas $B_1, \ldots, B_s$, where $s \geq 1$, then $A$ is projective in $\text{INT}$.

Proof. Note that $\varepsilon : z_1/B_1 \cdots z_s/B_s$ is a projective unifier for $A$.

For locally tabular logics, we can show

Theorem 4.15

Let $\sigma : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k$ and $\varepsilon : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n$, for some $k, n \geq 0$. Then $\varepsilon$ is a projective unifier for $A_{\sigma}$ in the logic $\text{L}$ if and only if

(iv) for every $\mathfrak{M}^n \in \mathfrak{M}^n$ there is a $\mathfrak{M}^k \in \mathfrak{M}^k$ such that $\varepsilon(\mathfrak{M}^n) \sim \sigma(\mathfrak{M}^k)$;

(v) $\varepsilon(\sigma(\mathfrak{M}^k)) \sim \sigma(\mathfrak{M}^k)$, for every $\mathfrak{M}^k \in \mathfrak{M}^k$, see Figure 20

Proof. Assume (iv) and (v). Then $\varepsilon$ is a unifier for $A_{\sigma}$, by Corollary 4.8. We need to show $A_{\sigma} \models \varepsilon(x_i) \leftrightarrow x_i$, for each $i$. Let $\mathfrak{M}^k$ be any $k$-model. Then $\varepsilon(\sigma(\mathfrak{M}^k)) \sim \sigma(\mathfrak{M}^k)$
Theorem 4.18
Theorem 4.10 characterizes logics with projective approximation (see Theorem 2.13).

By Lemma 4.17, a substitution \( \varepsilon : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n \) is a projective unifier for \( A_\varepsilon \) in \( L \) iff \( \varepsilon \circ \varepsilon =_L \varepsilon \).

Corollary 4.16
A substitution \( \varepsilon : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n \) is a projective unifier for \( A_\varepsilon \) in \( L \).

Corollary 4.17
If \( \varepsilon : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n \) is an \( L \)-projective unifier for a formula \( A \in \text{Fm}^n \), then \( A =_L \bigwedge_{i=1}^n (x_i \leftrightarrow \varepsilon(x_i)) \).

Suppose that \( m = n \) and \( F = H_\sigma \) in Theorem 4.10. Then, by (iv) and (v), \( G \) would generate a retraction from \( M^n/\sim \) onto \( \sigma(M^k)/\sim \). This simplified version of Theorem 4.10 characterizes logics with projective approximation (see Theorem 2.13).

Theorem 4.18
The logic \( L \) has projective approximation iff for every \( \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k \), where \( n, k \geq 0 \), there is a mapping \( G : M^n \rightarrow M^k \) such that:

(i) \( G \) preserves the frame of any \( n \)-model \( M^n = (W, R, w_0, V^n) \);

(ii) \( G(M^n)_w \sim (G(M^k))_w \) for every \( w \in W \);

(iii) \( M^n \sim M^n \Rightarrow G(M^n) \sim G(M^k) \), for every \( M^n, M^k \in M^n \);

(iv) for every \( n \)-model \( M^n \), there is a \( k \)-model \( M^k \) such that \( G(M^n) \sim \sigma(M^k) \);

(v) for every \( k \)-model \( M^k \), we have \( G(\sigma(M^n)) \sim \sigma(M^k) \).

Proof.
Suppose that \( L \) has projective approximation and let \( \sigma : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k \), for \( n, k \geq 0 \). By Corollary 4.17, \( \sigma \) is a unifier for \( A_\sigma \in \text{Fm}^n \). Let \( B \in _L(A_\sigma) \) be a projective formula in \( \text{Fm}^n \) such that \( B \vdash _L A_\sigma \) and \( \vdash _L (B) \). If \( \varepsilon : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n \) is a projective unifier for \( B \), then \( \sigma \circ \varepsilon =_L \sigma \) and \( \varepsilon(A_\sigma) \), by Lemma 4.17. Thus, if one takes \( G = H_\varepsilon \), one easily shows (i)-(v) using Lemma 4.16 and Corollary 4.19.

Assume (i)-(v) and let \( \sigma : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^k \) be a unifier for a formula \( A \in \text{Fm}^n \). By Lemma 4.14, we have \( G \sim H_\varepsilon \) for some \( \varepsilon : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^n \). Then, by Lemma 4.16, \( G \) is a projective unifier for \( A_\sigma \). Thus, we have \( A_\sigma \) projective, and \( A_\sigma \vdash A \), and \( A_\sigma \in \text{Fm}^n \), and \( \sigma \) is a unifier for \( A_\sigma \). For each \( A \) one can define \( \Pi(A) \) so that to show projective approximation in \( L \).

In [10], using the above Theorem, we have shown that any logic determined by frames of the depth \( \leq 2 \) has projective approximation. Thus, \( L(\mathfrak{F}_m) \) for any \( m \)-fork \( \mathfrak{F}_m \) (see Figure 3) has finitary unification. Now we extend this result to the logic determined by all frames from \( \mathfrak{H}_p \) that is frames \( \mathfrak{L}_d + \mathfrak{F}_m \), where \( m, d \geq 0 \), see Figure 7. The logic \( L(\mathfrak{H}_{pd}) \) extends PWL, see [11], hence it is locally tabular.
THEOREM 4.19
For any $F \subseteq H_{pr}$, the logic $L(F)$ has projective approximation.

PROOF. Let $F \subseteq H_{pr}$. Since $H_{pr}$ is closed on $p$-morph images and generated subframes, we can assume $sm(F) = F$; see Lemma 4.14. For any $\sigma: \{x_1, \ldots, x_n\} \rightarrow F^m$, where $k \geq 0$, we need $G: M^a \rightarrow M^{\sigma}$ fulfilling (i)-(v).

A. If $M^a = (W, R, w_0, V^n) \in M^a$ is equivalent with a $\sigma$-model (i.e. $M^a \sim \sigma(M^k)$) for some $M^k$, we take $G(M^a) = M^k$. Then (i), (iv) and (v) are obvious.

(ii) If $M^a$ is equivalent with a $\sigma$-model, then so is $(M^a)_w$ for any $w \in W$ (see Corollary 3.19 and Lemma 3.21(iii)), and hence $G((M^a)_w) = (M^k)_w = (G(M^a))_w$.

(iii) If $M^a$ is equivalent with a $\sigma$-model and $M^a \sim M^b$, then $M^a$ is equivalent with the same $\sigma$-model and hence $G(M^a) = M^a \sim M^b = G(M^b)$.

B. Suppose that $M^a$ is not equivalent with any $\sigma$-model. Assume $M^a$ is $p$-irreducible and define $G(M^a)$ inductively with respect to the depth of $M^a$. We could not bother about (v) as it is irrelevant. Our inductive approach secure (ii) and (iii). see Theorem 4.33. Our task is to preserve (i) and (iv), that is $G(M^a)$ should be defined over the frame of $M^a$ and $G(M^a)$ should be equivalent with some $\sigma$-model.

B1. Let $M^a$ be an $n$-model over 1-element chain $\Sigma_1$, that is $M^a = (\Sigma_1, f^n)$ where $f^n = i_1 \ldots i_n$ with $i_j \in \{0, 1\}$. All $n$-models over $\Sigma_1$ are $p$-irreducible and we have (see [10]): any $n$-model over $\Sigma_1$ equivalent with some $\sigma$-model is a $\sigma$-model. We can take any $k$-model $M^k = (\Sigma_1, f^k)$ over $\Sigma_1$ and define $G(M^a) = \sigma(M^k)$.

B2. Let $M^a$ be an $n$-model over $m$-fork $\Sigma_m$, see Figure 27 for $m \geq 1$;

![Fig. 27. m-Fork](image)

By our inductive approach, $G((M^a)_i) = (\Sigma_1, f^i) = \sigma(\Sigma_1, f^i)$, for some $g^n$ and $f^k$, if $i < m$. We define $M^k$, see Figure 28 and put $G(M^a) = \sigma(M^k)$.

![Fig. 28.](image)

We should be more careful about defining the mapping $G$ on isomorphic $n$-models. Our mapping should be factorized by $\equiv$ and hence we should get the factorized mapping $G: M^a_{ir/\equiv} \rightarrow M^{\sigma}_{ir/\equiv}$. Thus, from any $\equiv$-class, we take only one $M^a$, define as above $G(M^a)$ and then extend $G$, on other members of the equivalence class, taking as the values for $G$ the appropriate isomorphic copy of $G(M^a)$.

B3. Eventually, suppose that the frame of $M^a$ is an $m$-fork extended with a $d$-element chain ‘leg’, where $m, d \geq 0$. Our subsequent definition of $G$ is inductive with respect to $d$. So, we assume that $d \geq 1$ and $G((M^a)_{m+d-1})$ has already been defined, see Figure 29. We define $G(M^a)$ extending $G((M^a)_{m+d-1})$ with $g^n_{m+d} = g^n_{m+d-1}$. So, we get $G(M^a) \sim G((M^a)_{m+d-1})$ which guarantees the condition (iv).
Proof. Let us prove \( L(\mathfrak{G}_2) \cap L(\mathfrak{G}_3) \) has finitary unification. We extend the family \( F \) (in our proof of Theorem 4.13) with the frames \( \mathfrak{G}_2 \) and \( \mathfrak{G}_3 \). The definition of \( F \), see Figure 24.23 needs the following two (additional) clauses:

\[
\begin{align*}
\sigma(f_0^{\mathfrak{G}_2}) &= f_0^{\mathfrak{M}^k}, \\
\sigma(f_0^{\mathfrak{G}_3}) &= f_0^{\mathfrak{M}^k} \\
F(\mathfrak{M}^k) &= \{ f_0^{\mathfrak{G}_2}, f_0^{\mathfrak{G}_3}, f_1^{\mathfrak{G}_2}, f_1^{\mathfrak{G}_3}, f_2^{\mathfrak{G}_2}, f_2^{\mathfrak{G}_3} \}
\end{align*}
\]

Then we need to extend the definition of \( G(\mathfrak{M}^n) \) with:

\( (\mathfrak{G}_2, \{ f_0^n, f_0^m, f_0^m, f_0^m, f_0^m \}) \) is a p-irreducible model over \( \mathfrak{G}_3 \) (see Figure 30) non-equivalent with any \( F(\mathfrak{M}^k) \), but \( (\mathfrak{G}_2, \{ f_0^n, f_0^n, f_0^n, f_0^n \}) \sim F(\mathfrak{M}^k) \), where \( \mathfrak{M}^k \) is a model over \( \mathfrak{G}_2 \). Then \( (\mathfrak{M}^n)_{i=1} \sim \mathfrak{G}_2 \), for some \( f_0^n, f_0^n, f_0^n, f_0^n \) where \( \sim \) cannot be replaced with \( = \). Let \( \mathfrak{M}^k = (\mathfrak{G}_2, \{ f_0^k, f_0^k, f_0^k, f_0^k \}) \) which means \( \sigma(\mathfrak{G}_3, f_0^n) = \sigma(\mathfrak{G}_3, f_0^n) = (\mathfrak{L}_1, f_0^n) \) and let \( G((\mathfrak{M}^n)_{i=1}) = \sigma(\mathfrak{L}_1, f_0^n) = (\mathfrak{L}_1, f_0^n) \), for some \( f_0^n, f_0^n, f_0^n, f_0^n \). Then, for some \( f_0^n, f_0^n, f_0^n, f_0^n \), we have

\[
\begin{align*}
\mathfrak{M}^n = & \mathfrak{G}_2 \mathfrak{M}^n \mathfrak{G}_2 \mathfrak{G}_3 \\
G((\mathfrak{M}^n)_{i=1}) = & \sigma(\mathfrak{G}_3, f_0^n) = (\mathfrak{L}_1, f_0^n) \\
G(\mathfrak{M}^n) = & (\mathfrak{L}_1, f_0^n)
\end{align*}
\]

FIG. 29.

So, we have \( G : \mathfrak{M}_n^m \to \mathfrak{M}_m^n \). Using Theorem 4.15 we extend the mapping to \( G : \mathfrak{M}_n^m \to \mathfrak{M}_m^n \) preserving the conditions (i)-(iii), see Lemma 3.21. It is an easy task to check that the conditions (iv) and (v) are also preserved by the extension.

Theorems 4.10 and 4.18 can also be used to show that certain locally tabular logics neither have finitary/unitary unification, nor projective approximation. Many examples follow in the next section. Here we only give

THEOREM 4.20
The logics \( L(\mathfrak{G}_2) \) and \( L(\mathfrak{G}_2) \cap L(\mathfrak{G}_3) \) (see Figure 21 and 22) have finitary unification but they do not have projective approximation.

FIG. 30.
and the $n$-model over $\mathfrak{S}_3$ may be $G(2^n)$; it is equivalent with a $\sigma$-model over $\mathfrak{S}_2$.20

The rest of the proof remains the same.

In the same way one shows $L(\mathfrak{S}_2)$ has finitary unification; from the proof that $L(\mathfrak{S}_2) \cap L(\mathfrak{C}_5)$ has finitary unification, it suffices to remove $\mathfrak{C}_5, \mathfrak{G}_2, \mathfrak{G}_2$ and $\mathfrak{G}_3$. Then we can even simplify our reasoning and delete $g^1, h^1, b^1, b_2^n$. So, $\text{code}(f^k)$. Thus, we can take $m = n + 2$ and define $F(2^n)$ using $m$-valuations $f^0, f^0, f^0$. The suffices 11, 01 and 00 are necessary.

We need to show that $L(\mathfrak{S}_2)$ does not have projective approximation. Consider the substitution $\sigma: \{x_1\} \rightarrow Fm^2$ such that $\sigma(x_1) = \neg x_1 \land (x_2 \lor (x_2 \rightarrow x_1 \lor \neg x_1))$. Any $\sigma$-model is equivalent with one of the 1-models:

![Fig. 32.](image)

The above model over $\mathfrak{L}_2$ is not a $\sigma$-model but is equivalent with some $\sigma$-model over $\mathfrak{S}_2$. Suppose $L(\mathfrak{S}_2)$ has projective approximation and let $G: M^1 \rightarrow M^1$ be the retraction given by Theorem 4.18. We have $G(\circ 0) = \circ 0$ and $G(\circ 1) = \circ 1$. But if

$$G(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ then } G(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which cannot happen as the last model is not equivalent to any model in Figure 32. Thus, we get a contradiction.

In the same way one shows $L(\mathfrak{S}_2) \cap L(\mathfrak{C}_5)$ does not have projective approximation.

We can also show that $L(\mathfrak{S}_1)$ (see Figure 1) has finitary unification; it suffices only to extend the above reasoning admitting the frames $\mathfrak{S}_1, \mathfrak{S}_3, \mathfrak{S}_3$ and $\mathfrak{S}_3$ (see Figure 3). But a detailed proof would take much time and place (and, above all, would be boring). We think that original Ghilardi’s reasoning, see [17], is conscious and elegant and, therefore, we resign from any alternative proof.

---

The problem would rise if we tried in the same way to show $L(\mathfrak{S}_3)$ is finitary, we have no $\sigma$-model over $\mathfrak{S}_3$ equivalent with $G(2^n)$. So, $L(\mathfrak{S}_3)$ is nullary and the above may be seen as Ghilardi’s Theorem 9, p.112. 21

Using them we avoid problems that could rise p-morphisms mentioned in Example 4.18, the first counterexample. For the second type of p-morphisms, mentioned in the Example, we have no such remedia. The whole proof is to be shown that for these particular frames we can deal somehow with p-morphisms which collapse $\mathfrak{S}_2$ onto $\mathfrak{L}_2$.22

---

\[
\sigma(\begin{array}{c}
g^k_0 \\
g^k_1 \\
g^k_2 \\
\end{array}) = \begin{array}{c}
f^0_0 \\
f^0_1 \\
f^0_2 \\
\end{array} \sim \begin{array}{c}
f^n_0 \\
f^n_1 \\
f^n_2 \\
\end{array}
\]

**FIG. 31.**
5 Unification Types.

5.1 Unitary Unification.

By Theorem 4.12 the logic $L(\mathcal{R}_2)$ of rhombus (see figure 4) has unitary unification. We have shown this directly, using Theorem 4.10 in [10]. We cannot have $m = n$ in this case. In other words, unification in $L(\mathcal{R}_2)$ is unitary but it is not true that every unifiable formula in $n$-variables has a mgu in $n$-variables. A unifiable $A(x_1, \ldots, x_n)$ has a mgu in $m$-variables where $(m$ is defined in the proof and) $n < m$.

Indeed, let $A = x_1 \lor x_2 \lor (\neg x_1 \land \neg x_2)$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be the following unifiers for $A$:

$$\varepsilon_1(x_1) = \top, \quad \varepsilon_1(x_2) = x_2; \quad \varepsilon_2(x_1) = x_1, \quad \varepsilon_2(x_2) = \top; \quad \varepsilon_3(x_1) = \varepsilon_3(x_2) = \bot.$$ 

Let $\varepsilon : \{x_1, x_2\} \rightarrow \text{Fm}^2$ be a mgu for $A$ in any intermediate logic $L \subseteq L(\mathcal{R}_2)$. Then for some $\alpha_i : \{x_1, x_2\} \rightarrow \text{Fm}^2$, where $i = 1, 2, 3$, we have $\alpha_i \circ \varepsilon = \varepsilon_i$, and hence the diagram in Figure 33 commutes (up to $\sim$) where $\text{M}^2 = M^2(sm(\mathcal{R}_2)) = M^2(\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{R}_2\})$:

Consider all 2-models over one-element frame $\mathcal{L}_1$. They are $(\mathcal{L}_1, 00), (\mathcal{L}_1, 01), (\mathcal{L}_1, 10)$ and $(\mathcal{L}_1, 11)$. Each of them is a $\varepsilon_i$-model for some $i$ and hence they are $\varepsilon$-models. The mapping $\varepsilon : \text{M}^2 \rightarrow \text{M}^2$ must be one-to-one on $\mathcal{L}_1$-models.

Consider the following 2-models over $\mathcal{L}_2$ (a two-element chain):

11 and 11
10 and 01

Both are $\varepsilon_i$-model for some $i = 1, 2$ and hence they are also $\varepsilon$-models. Thus, there are such $a, b, c, d, e, f \in \{0, 1\}$ that

$$\varepsilon\left(\begin{array}{c} ab \\ cd \end{array}\right) = \begin{array}{c} 11 \\ 10 \end{array} \quad \text{and} \quad \varepsilon\left(\begin{array}{c} ab \\ ef \end{array}\right) = \begin{array}{c} 11 \\ 01 \end{array}$$

and $ab \neq cd \neq ef \neq ab$. If we consider the following $\mathcal{R}_2$-model and its $\varepsilon$-image:

$$\varepsilon\left(\begin{array}{c} cd \\ ef \end{array}\right) = \begin{array}{c} 10 \\ 01 \end{array}$$

we conclude there is a $\varepsilon$-model which is not a model for $A$ (whatever ? is). Thus, $\varepsilon$ is not a unifier for $A$, a contradiction.

No unifier $\varepsilon : \{x_1, x_2\} \rightarrow \text{Fm}^2$ for $A$ is more general any $\varepsilon_i$, for $i = 1, 2, 3$. Clearly, using filtering unifications, we can find a unifier for $A$ which is more general than any $\varepsilon_i$, we can even find in this way a mgu for $A$. But filtering unifications require additional variables and thus, we get a mgu $\varepsilon : \{x_1, x_2\} \rightarrow \text{Fm}^m$, where $m > 2$. 

![Commute Diagram](fig33)

Fig. 33. Commutative Diagram
Theorem 5.1
If an intermediate logic $L \subseteq L(R_2)$ has unitary unification, then there are $L$-unifiable formulas such that all their mgu’s introduce new variables.

We do not assume here that $L$ is locally tabular. Now, we can apply a particular splitting in the lattice of intermediate logics. It is known (see Rautenberg [26]) that $(L(R_2), LC)$ is a splitting pair for all extensions of $KC$. It follows that for each $L$ extending $KC$, either $L \supseteq LC$, or $L \subseteq L(R_2)$. Recall also that, if $LC \subseteq L$, then $L$ is one of Gödel logics that is $L = LC$ or $L = L(L_n)$, for some $n \geq 1$. Thus, we arrive to the following dichotomy.

Corollary 5.2
For any intermediate logic $L$ with unitary unification, either $L$ is one of the Gödel logics and enjoys projective unification (and all unifiable formulas have mgu’s preserving their variables) or there are $L$-unifiable formulas such that all their mgu’s introduce new variables (that is mgu’s do not preserve variables of unifiable formulas).

We do not deny the existence of unifiable (non-projective) formulas with mgu’s preserving their variables. We claim that, if a non-projective logic has unitary unification, then there must occur unifiable formulas all their mgu’s introduce new variables.

Corollary 5.3
An intermediate logic $L$ has projective unification if and only if $L$ has unitary unification and projective approximation.

Proof. Suppose that $L$ has unitary unification and projective approximation. Then one notices that each unifiable formula $A$ has a one-elements projective approximation $Π(A)$. But it means that any unifiable $A$ has a mgu preserving its variables. By Corollary 5.2 we conclude $L$ has projective unification. □

Corollary 5.3 does not hold for transitive modal logics; there are unitary transitive modal logics with projective approximation that are non-projective. But for reflexive and transitive modal logics one can prove the counterpart of Corollary 5.3.

Other consequences of Theorem 5.1 are given in Section 5.4.

5.2 Infinitary Unification

Until now, no intermediate (nor modal) logic with infinitary unification has been found, see [17], [6]. Using Theorem 4.10 we show that

Theorem 5.4
Any locally tabular intermediate logic does not have infinitary unification.

Proof. Let $L$ be a locally tabular intermediate logic and suppose that unification is not finitary (nor unitary) in $L$. Then

$$(*) \quad \exists n > 0 \forall m > 0 \exists k > 0 \exists \sigma : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^m \forall \tau : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^m (\tau(A_0) \in L \Rightarrow \tau \not\vDash L \sigma).$$

Thus, $n > 0$ is given. Let us define a sequence of integers $n = m_0 < m_1 < m_2 \cdots$ and substitutions $\sigma_1, \sigma_2, \ldots$ such that, for each $i > 0$,

1. $\sigma_i : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^{m_i}$ and $\tau \not\vDash L \sigma_i$ if $\tau : \{x_1, \ldots, x_n\} \rightarrow \text{Fm}^{m_i-1}$ and $\tau(A_{\sigma_i}) \in L$;
(2) $\sigma_i(x_{i_1}) \land \cdots \land \sigma_i(x_{i_s})$ is $L$-projective, for any $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\}$;

(3) $\sigma_i$-models are $\land$-closed: $\mathfrak{M}^n \land \mathfrak{M}^n = (W, R, w_0, \{f^n_{w} \land g^n_{w}\}_{w \in W})$ is a $\sigma_i$-model, where $(f^n_{w} \land g^n_{w})(x_j) = f^n_{w}(x_j) \land g^n_{w}(x_j)$ for each $w \in W$ and each $j = 1, \ldots, n$, if $\mathfrak{M}^n = (W, R, w_0, \{f^n_{w}\}_{w \in W})$ and $\mathfrak{M}^n = (W, R, w_0, \{g^n_{w}\}_{w \in W})$ are $\sigma_i$-models.

Let $\mathfrak{M}^n \leq \mathfrak{M}^n$ mean $f^n_{w} \leq g^n_{w}$ for each $w \in W$, where the order between the valuations is the product order. Then $\mathfrak{M}^n \land \mathfrak{M}^n \leq \mathfrak{M}^n$ and $\mathfrak{M}^n \land \mathfrak{M}^n \leq \mathfrak{M}^n$.

Our definition is inductive with respect to $i$. Suppose that $m_i$ and $\sigma_i$ (if $i > 0$) are given. Then we apply $(\star)$, where $m = m_i$. To get $k$ and $\sigma$ fulfilling (1). We do not take $m_{i+1} = k$ and $\sigma_{i+1} = \sigma$ as we need (2)–(3) to be fulfilled.

For (4), we define $\pi_1 : \{x_1, \ldots, x_n\} \to Fm^{k+n}$ taking

$$
\pi(x_j) = (x_{k+j} \leftrightarrow \sigma(x_j)) \quad \text{for each } j = 1, \ldots, n.
$$

Note that $x_{k+j}$ does not occur in $\sigma(x_j)$ and hence, to show (2) for $\sigma_{i+1} = \pi$, Lemma 4.14 applies. We still have (1) (for $\sigma_{i+1} = \pi$) as taking $x_{k+1}/\top \cdots x_{k+n}/\top$ we get $\pi \equiv \sigma$. But we do not take $m_{i+1} = k + n$ and $\sigma_{i+1} = \pi$ as we need (3).

For (3), let $\nu : \text{Var} \to \text{Var}$, for any $r > 0$, be given by $\nu_r(x_i) = x_{i+r}$ for each $i$. We define $m_{i+1} = s(k + n)$ and, for each $j = 1, \ldots, n$, let

$$(\star) \quad \sigma_{i+1}(x_j) = \pi(x_j) \land \nu_{n+k}(\pi(x_j)) \land \nu_{n+k}^2(\pi(x_j)) \land \ldots \land \nu_{n+k}^{s-1}(\pi(x_j)),
$$

where $s$ is (sufficiently big and is) specified below. We have $\sigma_{i+1} = \bigwedge_{j=0}^{s-1} (\nu_{n+k}^j \circ \pi)$. Using the inverse mapping $\nu^{-1}$ one shows that $\sigma_{i+1} \equiv \sigma$ and hence (1) holds. For (2), we can use Lemma 4.14. Let us prove (3).

Suppose there are given $(n + k)$-models $\mathfrak{M}^{k+n}_0, \ldots, \mathfrak{M}^{k+n}_{s-1}$ having the same frame (and root). Let $\mathfrak{M}^{k+n}_i = (W, R, w_0, \{f^{k+n}_{w}\}_{w \in W})$, for $i = 0, \ldots, s - 1$, and suppose that the valuations $f^{k+n}_{w}, \ldots, f^{k+n}_{w-1} w$, for each $w \in W$, are given by binary strings (of the same length $k + n$). Take the concatenation of the strings

- $f^{k+n}_{w} = f^{k+n}_{w_0} \cdots f^{k+n}_{w-1} w$, for each $w \in W$.

Then we get an $s(k + n)$-model $\mathfrak{M}^{s(k+n)}_i = (W, R, w_0, \{f^{s(k+n)}_{w}\}_{w \in W})$, called the concatenation of $\mathfrak{M}^{k+n}_{0}, \ldots, \mathfrak{M}^{k+n}_{s-1}$ for which

$$
\sigma_{i+1}(\mathfrak{M}^{s(k+n)}) = \pi(\mathfrak{M}^{k+n}_1) \land \pi(\mathfrak{M}^{k+n}_2) \land \ldots \land \pi(\mathfrak{M}^{k+n}_{s-1}).
$$

Obviously, each $s(k + n)$-model $\mathfrak{M}^{s(k+n)}$ can be received as a ‘concatenation’ of its $(k + n)$-fragments, and hence each $\sigma_{i+1}$-model is a conjunction of some $\pi$-models. Since there is finitely many $n$-models ($\sigma_{i+1}$- and $\pi$-models are $n$-models), we can get as $\sigma_{i+1}$-models all conjunction of $\pi$-models if $s$ is sufficiently big. It means that $\sigma_{i+1}$-models are closed under conjunction if $s$ is big enough, see $(\star)$. We get (3).

We have substitutions $\sigma_1, \sigma_2, \ldots$ (and integers $m_0, m_1, \ldots$) fulfilling (1)–(3). Let us note that these conditions are also fulfilled by any subsequence of $\sigma_1, \sigma_2, \ldots$. Since $\sigma_i$-models, for each $i$, are $n$-models and there is only finitely many $n$-models, we can find a subsequence $\sigma_{i_1}, \sigma_{i_2}, \ldots$ such that each $\sigma_{i_k}$ has the same set of models. Thus, we can assume that the sequence $\sigma_1, \sigma_2, \ldots$ fulfills

(4) $\sigma_i(M^{m_i}) = \sigma_j(M^{m_j})$ for each $i, j \geq 1$.

The next (and crucial) step in our argument is to show that
(5) for each $i > 0$, there is a number $r > 0$ such that $\bigwedge_{j=0}^{r-1}(\nu_{m_i}^j \circ \sigma_i) \preceq \sigma_{i+1}$.

Let us show there is a mapping $F : M^{m_{i+1}} \rightarrow M^{r^m_i}$ fulfilling (see Theorem [4.4]):

(i) $\mathfrak{M}^{m_{i+1}}$ and $F(\mathfrak{M}^{m_{i+1}})$ have the same frame, for each $\mathfrak{M}^{m_{i+1}}$;

(ii) $F((\mathfrak{M}^{m_{i+1}})_w) \sim (F(\mathfrak{M}^{m_{i+1}}))_w$, for each $w \in W$ ($W$ is the domain of $\mathfrak{M}^{m_{i+1}}$);

(iii) If $\mathfrak{M}^{m_{i+1}} \sim \mathfrak{M}^{m_{i+1}}$, then $F(\mathfrak{M}^{m_{i+1}}) \sim F(\mathfrak{M}^{m_{i+1}})$;

such that the following diagram commutes (up to $\sim$)

\[ \begin{array}{ccc}
M^n & \xrightarrow{\Lambda_{j=0}^{r-1}(\nu_{m_i}^j \circ \sigma_i)} & M^{r^m_i} \\
& \xrightarrow{\sigma_i} & F \\
& \xrightarrow{\sigma_{i+1}} & M^{m_{i+1}} \\
\end{array} \]

\textbf{Fig. 34.}

To specify the number $r$, let us assume that the sequence:

\[
(* \ast \ast \ast) \quad \mathfrak{M}_0^{m_{i+1}}, \mathfrak{M}_1^{m_{i+1}}, \ldots, \mathfrak{M}_{r-1}^{m_{i+1}}
\]

contains all $p$-irreducible (!) $m_{i+1}$-models.

Suppose that we have defined $F_j : M^{m_{i+1}} \rightarrow M^{m_i}$, for any $j = 0, 1, \ldots, r-1$ fulfilling

(i) $\mathfrak{M}^{m_{i+1}}$ and $F_j(\mathfrak{M}^{m_{i+1}})$ have the same frame, for each $\mathfrak{M}^{m_{i+1}}$;

(ii) $F_j((\mathfrak{M}^{m_{i+1}})_w) = (F_j(\mathfrak{M}^{m_{i+1}}))_w$, for each $w \in W$;

(iii) If $\mathfrak{M}^{m_{i+1}} \sim \mathfrak{M}^{m_{i+1}}$, then $F_j(\mathfrak{M}^{m_{i+1}}) \sim F_j(\mathfrak{M}^{m_{i+1}})$

and let $F(\mathfrak{M}^{m_{i+1}})$ be the concatenation of the models $F_0(\mathfrak{M}^{m_{i+1}}), \ldots, F_{r-1}(\mathfrak{M}^{m_{i+1}})$.

We could not claim the following diagram commutes, for any $j$,

\[ \begin{array}{ccc}
M^n & \xrightarrow{\sigma_i} & M^{m_i} \\
& \xrightarrow{\sigma_{i+1}} & F_j \\
& \xrightarrow{F_j} & M^{m_{i+1}} \\
\end{array} \]

\textbf{Fig. 35.}

as it would give us $\sigma_{i+1} \preceq \sigma_i$ contradicting (1). But we have

(iv) \quad $\sigma_i(F_k(\mathfrak{M}^{m_{i+1}}_k)) = \sigma_{i+1}(\mathfrak{M}^{m_{i+1}}_k)$ and $\sigma_i(F_j(\mathfrak{M}^{m_{i+1}}_j)) \preceq \sigma_{i+1}(\mathfrak{M}^{m_{i+1}}_j)$,

for each $k, j \in \{0, \ldots, r-1\}$. It means the diagram in the Figure 35 commutes as

\[
\bigwedge_{j=0}^{r-1}(\nu_{m_i}^j \circ \sigma_i) \circ F(\mathfrak{M}^{m_{i+1}}_k) = \bigwedge_{j=0}^{r-1}(\sigma_i(F_j(\mathfrak{M}^{m_{i+1}}_j))) = \sigma_i(F_k(\mathfrak{M}^{m_{i+1}}_k)) = \sigma_{i+1}(\mathfrak{M}^{m_{i+1}}_k);
\]

and we know each $\mathfrak{M}^{m_{i+1}} \in M^{m_{i+1}}$ is equivalent with some $\mathfrak{M}^{m_{i+1}}_k$, so by (iii) we get

\[
\bigwedge_{j=0}^{r-1}(\nu_{m_i}^j \circ \sigma_i) \circ F(\mathfrak{M}^{m_{i+1}}) \sim \sigma_{i+1}(\mathfrak{M}^{m_{i+1}}).
\]
We could show (i)-(iii) for $F$ using the fact these conditions are fulfilled by each $F_j$. But such an argument would be too complicated and we could make it easier by the exact definition of a substitution $\alpha: \{x_1, \ldots, x_{m_i}\} \rightarrow \text{FM}^{m_{i+1}}$ such that $H_\alpha \sim F$. Then (i)-(iii) follow from Lemma 4.3. For any $F_j$ (where $j = 0, \ldots, r-1$), we should have $F_j \sim H_{\alpha_j}$ for some $\alpha_j: \{x_1, \ldots, x_{m_i}\} \rightarrow \text{FM}^{m_{i+1}}$. Then we define $\alpha$ as a disjoint union of the $\alpha_j$'s. More specifically:

$$\alpha(x_{l+jm_i}) = \alpha_j(x_l), \quad \text{for each } l = 1, \ldots, m_i.$$  

Obviously, $\alpha(\text{FM}^{m_{i+1}})$ is the concatenation of $\alpha_0(\text{FM}^{m_{i+1}}), \ldots, \alpha_{r-1}(\text{FM}^{m_{i+1}})$ for each $m_{i+1}$-model $\text{FM}^{m_{i+1}}$. There remains to define $F_j$, for $j = 0, \ldots, r-1$.

Let us define any $F_j: \text{FM}^{m_{i+1}} \rightarrow \text{FM}^m$ as a partial mapping; its domain $D(F_j)$ is an up-ward closed subset of $\text{FM}^{m_{i+1}}$, which means

$$\text{Th}(\text{FM}^{m_{i+1}}) \subseteq \text{Th}(\text{FM}^{m_{i+1}}) \text{ and } \text{FM}^{m_{i+1}} \in D(F_j), \text{ then } \text{FM}^{m_{i+1}} \in D(F_j).$$

Obviously, $F_j$ should also fulfill (i)-(iv), for models in $D(F_j)$. Then, step by step, we extend the domain $D(F_j)$ to the whole set $\text{FM}^{m_{i+1}}$; our definition of $F_j(\text{FM}^{m_{i+1}})$ is inductive with respect to depth of $\text{FM}^{m_{i+1}}$.

(A). By (4), $\sigma_{i+1}(\text{FM}^{m_{i+1}}) = \sigma_i(\text{FM}^m)$, for some $\text{FM}^m \in \text{FM}^m$. Thus, we take

$$F_j(\text{FM}^{m_{i+1}}) = \text{FM}^m,$$

which gives $\sigma_i(F_j(\text{FM}^{m_{i+1}})) = \sigma_{i+1}(\text{FM}^{m_{i+1}})$ and this guarantee the (first part of the) condition (iv). The models $\sigma_{i+1}(\text{FM}^{m_{i+1}})$, $\sigma_i(\text{FM}^m)$, $\text{FM}^{m_{i+1}}$, $\text{FM}^m$ have the same frame $(W, R, w_0)$ and only valuations could be different. To get (ii) we should take

$$F_j(\text{FM}^{m_{i+1}})_w = (\text{FM}^m)_w, \quad \text{for each } w \in W.$$

According to Theorem 3.12, $(\text{FM}^{m_{i+1}})_w$, for each $w \in W$, is p-irreducible and hence for each $\text{FM}^{m_{i+1}}$ equivalent with $(\text{FM}^{m_{i+1}})_w$ there is exactly one p-morphism (see Theorem 3.18) $p: \text{FM}^{m_{i+1}} \rightarrow (\text{FM}^{m_{i+1}})_w$. To satisfy (iii), we should take

$$F_j(\text{FM}^{m_{i+1}}) = p^{-1}(\text{FM}^m)_w,$$

where $p^{-1}(\text{FM}^m)_w$ is the only n-model on $(W, R, w_0)$ such that the mapping $p$ is a p-morphism $p: p^{-1}(\text{FM}^m)_w \rightarrow (\text{FM}^m)_w$ of n-models. Since the valuations are preserved by p-morphisms, $\sigma_i(F_j(\text{FM}^{m_{i+1}})) = \sigma_{i+1}(\text{FM}^{m_{i+1}})$ which guarantees the second part of (iv) if $\text{FM}^{m_{i+1}} = \text{FM}^{m_{i+1}}$, for some $k$. Our definition of the (partial) mapping $F_j$ is completed; its domain is an upset.

(B). We have no problems to define $F_j(\text{FM}^{m_{i+1}})$ for any $m_{i+1}$-model $\text{FM}^{m_{i+1}}$ over one-element frame (assuming the model does not belong to $D(F_j)$ by (A)). By (4), it suffices to define $F_j(\text{FM}^{m_{i+1}})$ in a such way that $\sigma_i(F_k(\text{FM}^{m_{i+1}})) = \sigma_{i+1}(\text{FM}^{m_{i+1}})$ for each $m_{i+1}$-model over one-element frame. Each $\text{FM}^{m_{i+1}} = (W, R, w_0, \{f_w\}_{w \in W})$ equivalent with a model with one-element frame has $f_w = f_u = f_{w+1}$, for each $u, w \in W$, and hence we can define $F_j(\text{FM}^{m_{i+1}})$ preserving $\sigma_i(F_k(\text{FM}^{m_{i+1}})) = \sigma_{i+1}(\text{FM}^{m_{i+1}})$. The conditions (i)-(iv) are fulfilled and $D(F_j)$ is an upset by Theorem 3.14.

(C). If all $\text{FM}^{m_{i+1}}$'s belong to $D(F_j)$ we have done. Suppose that some $\text{FM}^{m_{i+1}}$ does not belong to $D(F_j)$. We can assume that $\text{FM}^{m_{i+1}} = (W, R, w_0, \{f_w\}_{w \in W})$
and $(\mathcal{M}^{m_i+1})_w \in D(F_j)$ for each $w \neq w_0$. Thus, we have $F_j((\mathcal{M}^{m_i+1})_w)$ for each $w \neq w_0$ and we need to define $F_j(\mathcal{M}^{m_i+1})$. In other words, an $m_i$-model $\mathcal{M}^{m_i} = (W, R, w_0, \{F^{m_i}_w\}_{w \in W})$ is given such that $(\mathcal{M}^{m_i})_w = F_j((\mathcal{M}^{m_i+1})_w)$ for each $w \neq w_0$, what is $\mathcal{M}^{m_i}$ does not matter, we need its variant $\mathcal{M}^{m_i}_0 = (W, R, w_0, \{F^{m_i}_w\}_{w \in W})$ such that $\sigma, (\mathcal{M}^{m_i}_0) \geq \sigma_{i+1}(\mathcal{M}^{m_i+1})$; then we can take $F_j(\mathcal{M}^{m_i+1}) = \mathcal{M}^{m_i}_0$ fulfilling all requirements (except for (iii)). Let $x \in \{x_1, \ldots, x_n\}$. We have

$$\mathcal{M}^{m_i+1}_k \models w \models \sigma_{i+1}(x) \Rightarrow \mathcal{M}^{m_i}_k \models w \models \sigma_i(x), \text{ for each } w \neq w_0$$

and want a variant $\mathcal{M}^{m_i}_0$ of $\mathcal{M}^{m_i}$ such that:

$$\mathcal{M}^{m_i+1}_k \models w \models \sigma_{i+1}(x) \Rightarrow \mathcal{M}^{m_i}_0 \models w \models \sigma_i(x).$$

If $\mathcal{M}^{m_i+1}_k \not\models w \models \sigma_{i+1}(x)$ for some $w \neq w_0$ the implication holds. Thus, it suffices to consider the set $\{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n\}$ containing all $x$'s such that

$$\mathcal{M}^{m_i}_k \models w \models \sigma_i(x), \text{ for each } w \neq w_0.$$ 

By (2), $\sigma_i(x_{j_1}) \wedge \cdots \wedge \sigma_i(x_{j_s})$ is $L$-projective and hence, by Theorem 4.4, there is a variant $\mathcal{M}^{m_i}_0$ of $\mathcal{M}^{m_i}$ such that

$$\mathcal{M}^{m_i}_0 \models w \models \sigma_i(x_{j_1}) \wedge \cdots \wedge \sigma_i(x_{j_s}).$$

The definition of $F_j(\mathcal{M}^{m_i+1}_k)$ is completed. There remains to add that all $m_{i+1}$-models $\mathcal{M}^{m_{i+1}}$ with $\Theta(\mathcal{M}^{m_{i+1}}) \subseteq \Theta(\mathcal{M}^{m_i+1})$ are included in $D(F_j)$ by Lemma 3.14. If $\mathcal{M}^{m_{i+1}} \sim \mathcal{M}^{m_i+1}_k$, then there is a $m$-morph $p: \mathcal{M}^{m_{i+1}} \rightarrow \mathcal{M}^{m_{i+1}}$ and hence we can take $F_j(\mathcal{M}^{m_{i+1}}) = p^{-1}(\mathcal{M}^{m_i}_0)$. So, we claim $D(F_j)$ remains an upset.

We have shown (5) and can get to the proof of our theorem. Suppose that $L$ has finitary unification and let $A = A_{\sigma_i}$. It follows from (4) that $A = A_{\sigma_i}$, for each $i$, and hence all $\sigma_i$'s are unifiers for $A$. By (1), $A$ cannot have finitary unification in $L$ as there is no unifier of $A$ which would be more general than all $\sigma_i$'s. Let $\Sigma$ be a minimal complete set of unifiers for $A$. Then $\tau \preceq \sigma_i$ for some $\{x_1, \ldots, x_n\} \rightarrow \text{Fm}^m$ in $\Sigma$. Thus, there is a number $i$ such that $\tau \preceq \sigma_i$ and $\tau \not\preceq \sigma_{i+1}$. By $\tau \preceq \sigma_i$, we also get $\nu^j_m \circ \tau \preceq \nu^j_{m+1} \circ \sigma_i$, for any $j, k \geq 0$; it does not matter if $m = m_i$ or not. Thus,

$$\bigwedge_{j=0}^{r-1} (\nu^j_m \circ \tau) \preceq \bigwedge_{j=0}^{r-1} (\nu^j_{m+1} \circ \sigma_i)$$

But $\bigwedge_{j=0}^{r-1} (\nu^j_m \circ \tau) \preceq \tau$ and hence $\tau \preceq \bigwedge_{j=0}^{r-1} (\nu^j_m \circ \tau)$ as $\bigwedge_{j=0}^{r-1} (\nu^j_m \circ \tau)$ is a unifier for $A$, by (3), and $\tau \in \Sigma$. Thus, we get $\tau \preceq \sigma_{i+1}$, by (5), which is a contradiction. \(\blacksquare\)

Let $L$ be a locally tabular intermediate logic and suppose we have shown, using Theorem 4.10 or 4.11 that unification in $L$ is not finitary (nor unitary). It would mean, by the above Theorem 4.2.1 that $L$ has nullary unification. In the following Section, we prove in this way that very many intermediate logics has nullary unification.

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5.3 Nullary Unification

It is known that $L(G_3)$ and $L(G_3^+)$ (see Figure 1) have nullary unification, see Introduction. In [10], we proved that unification in the modal version of these logics is nullary. Below we present the intuitionistic version of our argument.

**Theorem 5.5**
The logics $L(G_3)$ and $L(G_3^+)$ have nullary unification.

**Proof.** Let $F = \{L_1, L_2, L_3, L_4, G_3\}$. Then $F = sm(G_3)$. Assume $L(F)$ has finitary unification. By Theorem 4.10, for every $n \geq 1$ there is a number $m \geq 1$ such that for any $\sigma: \{x_1\} \rightarrow F^m_k$ there are mappings $G: M^m \rightarrow M^1$ and $F: M^k \rightarrow M^m$ fulfilling the conditions (i)-(v). Let $n = 1 \leq m < k$ and $\sigma: \{x_1\} \rightarrow F^k$ be as follows

$$\sigma(x_1) = \neg(\bigvee_{i=1}^{k} x_i) \land \bigwedge_{i=1}^{k} (\neg\neg x_i \lor \neg x_i).$$

Take any $M^k$ over $G_3$: and notice $\sigma(M^k)$ is:

|   |   |   |   |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 |

a. if $f_0^k = 0 \cdots 0 = f_0^k$; b. if $f_0^k \neq 0 \cdots 0 = f_0^k$; c. if $f_0^k = 0 \cdots 0 \neq 0 \cdots 0$; d. if $f_0^k = f_0^k \neq 0 \cdots 0$; e. if $0 \cdots 0 \neq f_0^k \neq 0 \cdots 0$.

Thus, each $\sigma$-model is equivalent with a model of the depth $\leq 2$ and there are four $p$-irreducible 1-models equivalent with some $\sigma$-models. By (v)

$$G(F(\circ 0 \cdots 0)) = G(\circ g^m) = \sigma(\circ 0 \cdots 0) = \circ 0,$$

for some $g^m$. Since $m < k$, one can find $f_k^k \neq g^k$ such that $F(\circ f_k^k) = \circ f^m = F(\circ g^k)$, and $G(\circ f^m) = \circ 1$, for some $f^m$. By the characterization of all $\sigma$-models

$$G\left(\begin{array}{c}
\circ f^m \\
\circ \circ \circ \circ \\
n\end{array}\right) = \begin{array}{c}
1 \\
0 \\
1 \\
\end{array} \quad \text{or} \quad G\left(\begin{array}{c}
\circ f^m \\
\circ \circ \circ \circ \\
n\end{array}\right) = \begin{array}{c}
1 \\
0 \\
1 \\
\end{array}$$

for any $m$-valuation $\circ$. But if the first had happened, we would have

$$G\left(\begin{array}{c}
\circ g^m \\
\circ \circ \circ \circ \\
n\end{array}\right) = \begin{array}{c}
0 \\
0 \\
1 \\
\end{array}$$

Thus, by (v), we have $G(F(\circ 0 \cdots 0)) = G(\circ g^m)$, which means $\sigma(\circ 0 \cdots 0) = \circ 0$.
which would contradict (iv) as no $\sigma$-model is equivalent to the 1-model on the right hand side of the above equation. We conclude that

$$G\left( \begin{array}{c} \text{?} \\ \circ \end{array} \right) = \begin{array}{c} 1 \\ 1 \end{array} \sim \circ 1$$

and hence

$$G\left( \begin{array}{c} \circ \end{array} \right) = \begin{array}{c} 1 \\ 1 \end{array} \sim \circ 1$$

But this is in contradiction with e.

$$\sigma\left( \begin{array}{c} \circ \end{array} \right) = \begin{array}{c} 1 \\ 1 \end{array}$$

Let $F = \{ L_2, L_3, L_4, R_2, G_3 \}$. Then $F = sm(\mathfrak{G}_3^+)$. Suppose that $L(F)$ has unitary unification. Take $n = 2$. By Theorem 4.10 there is a number $m \geq 1$ such that for any $\sigma: \{ x_1, x_2 \} \to Fm^{k+1}$ there are mappings $G: M^m \to M^2$ and $F: M^{k+1} \to M^m$ fulfilling the conditions (i)-(v). Let $k > m$ and

$$\sigma(x_1) = x_1$$
$$\sigma(x_2) = \left( \left( \bigvee_{i=2}^{k+1} x_i \rightarrow x_1 \right) \rightarrow x_1 \right) \land \bigwedge_{i=2}^{k+1} \left( \left( x_i \rightarrow x_1 \right) \rightarrow x_1 \right) \lor \left( x_i \rightarrow x_1 \right).$$

If we take $\alpha \circ \sigma$, where $\alpha : x_1 / \bot$, we get $\sigma$ as used for $\mathfrak{G}_3$ (there is only a shift of variables from $x_1 \ldots x_k$ to $x_2 \ldots x_{k+1}$). If $x_1$ is false at the top element of any 2-model $\mathfrak{N}^2$ over $\mathfrak{G}_3^+$, then $\sigma(x_2)$ is true at the model and hence $\sigma(\mathfrak{N}^2)$ reduces to a model over $\mathfrak{L}_1$. Decapitation of the top element (and other elements at which $x_1$ is true) give us models over $\mathfrak{G}_3$. Then we can argue as in the case of $L(\mathfrak{G}_3)$.

We have relatively many $\sigma$-models of the depth $\leq 2$ but there are only two $p$-irreducible equivalents of $\sigma$-models of the depth $\geq 3$. They are

Cutting off the top element and erasing the first variable we get the $\sigma$-models for $L(\mathfrak{G}_3)$:

Let $F(\circ 1 \cdots 1) = \circ h^m$. Since $\sigma(\circ 1 \cdots 1) = \circ 11$, we have $G(\circ 1 \cdots 1) = \circ 11$, by (v). We also have $g^m, f^m$ and $g^k \neq f^k$ such that
\[ F(\begin{array}{c} 1 \cdots 1 \\ 0 \cdots 0 \end{array}) = \begin{array}{c} h^m \\ g^m \end{array}, \quad F(\begin{array}{c} 1 \cdots 1 \\ 0f^k \end{array}) = F(\begin{array}{c} 1 \cdots 1 \\ 0g^k \end{array}) = \begin{array}{c} h^m \\ f^m \end{array} \]

Then by (v)
\[ G(\begin{array}{c} h^m \\ g^m \end{array}) = \begin{array}{c} 11 \\ 00 \end{array}, \quad G(\begin{array}{c} h^m \\ 0f^m \end{array}) = \begin{array}{c} 11 \\ 01 \end{array} \]

By (i)–(ii), for any \( m \)-valuation \( ? \), we have
\[ G(\begin{array}{c} h^m \\ f^m \\ ? \end{array}) = \begin{array}{c} 11 \\ 01 \end{array} \quad \text{or} \quad G(\begin{array}{c} h^m \\ f^m \\ ? \end{array}) = \begin{array}{c} 11 \\ 01 \end{array} \]

Thus, either
\[ G(\begin{array}{c} h^m \\ g^m \\ ? \end{array}) = \begin{array}{c} 00 \\ 00 \end{array} \]

which would contradict (iv) as no \( \sigma \)-model is equivalent to the 2-model on the right hand side of the above equation, or we would have a contradiction:
\[ 01 \sim G\left(F\left(\begin{array}{c} 1 \cdots 1 \\ 0 \cdots 0 \end{array}\right)\right) = G\left(\begin{array}{c} h^m \\ f^m \end{array}\right) \sim 01 \]

By [17], there are infinitely many logics in which unification is not finitary, see Figure 2, and hence there are infinitely many logics with nullary unification. We can produce other infinite families of logics with nullary unification, see Figure 36.

Fig. 36. \( \mathfrak{F}_2 + \mathfrak{F}_s \).
Theorem 5.6
The logic $L(\mathfrak{G}_2 + \mathfrak{G}_s)$ has nullary unification, for any $s \geq 1$:

Proof. Let $F = sm(\{\mathfrak{G}_2 + \mathfrak{G}_s\})$ and suppose that $L(F)$ has finitary unification. Note that $\{L_1, L_2, L_3, L_4, R_2, R_2+\} \subseteq F$. Take $n = 3$ and let $m \geq 3$ be given by Theorem 4.10. Let $k > m$ and $\sigma: \{x_1, x_2, x_3\} \to F_{m+k+2}$ be a substitution defined as follows

$\sigma(x_1) = x_1$

$\sigma(x_2) = x_2$

$\sigma(x_3) = \bigwedge_{i=3}^{k+2} (x_i \to x_1) \lor ((x_i \to x_1) \to x_1)$.

There are mappings $G: M_m \to M_3$ and $F: M_{k+2} \to M_m$ fulfilling the conditions (i)-(v). Since $k > m$, there are $f_k \neq g_k$ such that (for some $f_m, g_m$)

$F\left(\begin{array}{c} 1 \cdots 1 \\ 01f^k \end{array}\right) = F\left(\begin{array}{c} 1 \cdots 1 \\ 01g^k \end{array}\right) = f^m, \quad G\left(\begin{array}{c} 1 \cdots 1 \\ g^m \end{array}\right) = 111. \quad \text{But}

\[ G\left(\begin{array}{c} 1 \cdots 1 \\ g^m \end{array}\right) = 111. \quad \text{Then}

\sigma\left(\begin{array}{c} 1 \cdots 1 \\ 01f^k \\ 01g^k \end{array}\right) = 011 \sim 011.

F\left(\begin{array}{c} 01f^k \\ 01g^k \\ 010 \cdots 0 \\
end{array}\right) = f^m \sim g^m,

for some $h^m$, and hence

$G\left(\begin{array}{c} 1 \cdots 1 \\ f^m \end{array}\right) = 111.$

On the other hand, we have

$\sigma\left(\begin{array}{c} 01f^k \\ 010 \cdots 0 \\
end{array}\right) = 011. \quad \text{Thus,}

F\left(\begin{array}{c} 01f^k \\ 00h^k \\
end{array}\right) = f^m, \quad \text{for some } f^m, \quad G\left(\begin{array}{c} 01f^k \\ 00h^k \\
end{array}\right) = 011.
We conclude

\[
G(h^m, f^m) = 010 \quad 011 \quad 001 \quad 010
\]

is a generated submodel of the above \( \mathfrak{F}_2 + L_2 \)-model. But the whole model is p-irreducible and hence the submodel on \( L_3 \) must be equivalent, by (iv), to a \( \sigma \)-model on \( +\mathfrak{F}_r \), for some \( r \geq 1 \). Thus, we get a model on \( +\mathfrak{F}_r \) such that

\[
\sigma(x_3) \quad \not\vdash \sigma(x_3)
\]

which is impossible.

In the same way one shows that \( L(\mathfrak{F}_r, \mathfrak{F}_s) \) has nullary unification, for \( r \geq 2, s \geq 1 \).

More than in the number (of logics with nullary unification) we are interested in their location in the lattice of all intermediate logics. In particular, we would like to know if they can be put apart from logics with finitary/unitary unification; just as logics with finitary unification are distinguished from unitary logics, by Theorem 2.12.

**Theorem 5.7**
The logic \( L(\{\mathfrak{R}_2, \mathfrak{F}_2\}) = L(\mathfrak{R}_2) \cap L(\mathfrak{F}_2) \) has nullary unification.

**Proof.** Suppose that \( L(\{\mathfrak{R}_2, \mathfrak{F}_2\}) \) has finitary unification and note that

\[
sm(\{\mathfrak{R}_2, \mathfrak{F}_2\}) = \{L_1, L_2, L_3, F_2, R_2\}.
\]

Let \( n = 2 \). According to Theorem 4.10 there is a number \( m \geq 2 \) such that for every \( \sigma: \{x_1, x_2\} \to Fm^k \) there are mappings \( G: \mathcal{M}^m \to \mathcal{M}^2 \) and \( F: \mathcal{M}^k \to \mathcal{M}^m \) fulfilling the conditions (i)-(v). Take any \( k > m \) and let \( \sigma: \{x_1, x_2\} \to Fm^k \) be as follows:

\[
\sigma(x_1) = \bigwedge_{i=1}^{k} (\neg x_i \lor \neg \neg x_i) \quad \text{and} \quad \sigma(x_2) = \bigwedge_{i=1}^{k} (\neg \neg x_i \to x_i).
\]

We have the following p-irreducible \( \sigma \)-models, on \( \mathfrak{R}_2 \) and \( \mathfrak{F}_2 \), correspondingly:

\[
\begin{array}{c c c c}
11 & 10 & 11 & 11 \\
11 & 10 & 01 & 00
\end{array}
\]

As \( k > m \), there are \( f^k \neq g^k \) such that \( F_k(L_1, f^k) = F_k(L_1, g^k) = (L_1, g^m) \), for some \( g^m \). Let \( f^k \neq 0 \cdots 0 \). Then
Let $F(x) = \sigma(x^k) = \circ \bullet 0 \cdots 0$ and $F(y) = \sigma(y^m) \sim 11 \circ \bullet 0 \cdots 0$.

for some $m$-valuations $b_1^m$ or $b_2^m$. Then, by (v),

$G(b_1^m) = 11 \circ \bullet 0 \cdots 0$ and $G(b_2^m) = 11 \circ \bullet 0 \cdots 0$.

Thus,

$G(b_1^m \sigma b_2^m) = 10 \circ \bullet 0 \cdots 0$.

The above 2-model on $R_2$ is not equivalent to any $\sigma$-model which contradicts (iv).

Thus, there are logics with finitary unification the intersection of whose has nullary unification. We could give many examples of such logics.

**Theorem 5.8**

The logic $L(\mathcal{E}_2) \cap L(\mathcal{F}_3)$ (see Figure 4) and 3 has nullary unification.

**Proof.** Let $L = L(\mathcal{E}_2) \cap L(\mathcal{F}_3)$ and $F = \{L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}\}$. Then $F = sm(\{\mathcal{E}_2, \mathcal{F}_3\})$ and $L = L(F)$. Suppose that $L$ has finitary unification and $n = 1$. By Theorem 4.10, there is a number $m \geq 1$ such that for any $\sigma: \{x_1, \ldots, x_n\} \rightarrow F^k$ there are $G: M^m \rightarrow M^1$ and $F: M^k \rightarrow M^m$ fulfilling the conditions (i)-(v).

The formula is valid in $\mathcal{F}_2$; to falsify it one needs three end-elements above the root, labeled with distinct valuation. Thus, $A$ is false in the following models over $\mathcal{F}_3$.

Let $m < k = 2l$ and $\sigma: \{x_1\} \rightarrow F^k$ be defined by

$$\sigma(x_1) = \neg\left(\bigvee_{i=1}^{k} x_i\right) \wedge \bigwedge_{i=1}^{l} A(x_{2i-1}, x_{2i}).$$

We have the following p-irreducible $\sigma$-models (or equivalents of $\sigma$-models):

$$\circ 0 \circ 1 \circ 0 \circ 1 \circ 0 \circ 1 \circ 0$$

The last 1-model is not any $\sigma$-model but it is equivalent with some $\sigma$-model.

Note $\sigma(\circ i^k) = \circ 0$ if $i^k = 0 \cdots 0 \circ 1 \circ 0 \cdots 0 \circ 1 \circ 0 \cdots 0 \circ 1 \circ 0 \cdots 0$.
Let $F(0 \cdots 0) = \circ g^m$. Since $m < k$, there are $f_0^k \neq f_0^m$ such that $F(F(f_0^k)) = F(0 \cdots 0)$. Then $f_0^k(j) \neq f_0^m(j)$ for some $j \leq k$. We do not know if $j$ is odd or even but for pairs of bits we have $f_0^k(2i - 1) f_0^m(2i) \neq f_0^m(2i - 1) f_0^m(2i)$ for some $i \leq l$ (where $2i - 1 = j$ or $2i = j$). Moreover, we can find a $k$-valuation $f_0^m$ such that

$$f_0^k(2i - 1) f_0^m(2i) \neq f_0^m(2i - 1) f_0^m(2i)$$

and $f_0^m(2i - 1) f_0^m(2i) \neq 0$. Let $F(0 \cdots 0) = \circ f_0^m$ and $F(0 \cdots 0) = \circ f_0^m$. We have $f_0^m \neq g^m$ and $f_0^m \neq g^m$ as, by (v), $G(\circ f_0^m) = G(\circ g^m) = 1$ and $G(\circ g^m) = 1$.

By (ii), (iv), and the above characterization of all $\sigma$-models, we conclude that

$$G(\begin{array}{cc}
\circ f_0^m & \circ f_0^m \\
? & 1 \end{array}) \sim 1$$

or

$$G(\begin{array}{cc}
\circ g^m & \circ g^m \\
? & 0 \end{array}) \sim 1$$

(whatever $?$ is). If the first had happened, we would have

$$G(\begin{array}{cc}
\circ f_0^m & \circ f_0^m \\
? & 1 \end{array}) = 0$$

which would contradict (iv) as no $\sigma$-model is equivalent to the 1-model on the right hand side of the above equation. We conclude that

$$G(\begin{array}{cc}
\circ f_0^m & \circ f_0^m \\
? & 1 \end{array}) = 1$$

and hence

$$G(F(\begin{array}{cc}
\circ f_0^m & \circ f_0^m \\
\circ f_0^m & \circ f_0^m \\
\circ f_0^m & \circ f_0^m \\
? & 0 \end{array})) = G(\begin{array}{cc}
\circ f_0^m & \circ f_0^m \\
? & 1 \end{array}) = 1$$

But this is in contradiction with (v) as

$$H_\sigma(\begin{array}{cc}
\circ f_0^m & \circ f_0^m \\
\circ f_0^m & \circ f_0^m \\
\circ f_0^m & \circ f_0^m \\
? & 0 \end{array}) = 1$$

There are, as well, families of logics with finitary unification closed under intersections. For instance, $L(\mathfrak{S}_2) \cap L(\mathfrak{C}_5)$ has finitary unification, see Theorem 4.19. The frames $\mathfrak{C}_5$ and $\mathfrak{G}_3$ are quite the same; the logics $L(\mathfrak{C}_5)$ and $L(\mathfrak{G}_3)$ have finitary unification (the second one even has projective approximation, see Theorem 4.19) but it does not mean they generate the same unification types in combination with other frames.
Lemma 5.9

If \( d \geq 1 \), a class \( \mathbf{F} \) of finite frames and \( \breve{\mathfrak{f}} \in \mathbf{F} \) are such that

1. \( d(\mathfrak{f}) = d \) and \( d(\mathfrak{g}) \leq d + 1 \), for each \( \mathfrak{g} \in \mathbf{F} \);
2. \( \mathfrak{g} \equiv +\mathfrak{f} \) or \( \mathfrak{g} \in \mathbf{F} \), for each p-morphic image \( \mathfrak{g} \) of \( +\mathfrak{f} \);
3. \( L(\mathbf{F}) \) is locally tabular and has finitary/unitary unification, then \( L(\mathbf{F} \cup \{ +\mathfrak{f} \}) \) is locally tabular and has finitary/unitary unification, as well.

Proof. \( L(\mathbf{F} \cup \{ +\mathfrak{f} \}) \) is locally tabular, by Corollary 2.4. Assume \( \mathbf{F} = sm(\mathbf{F}) \) and \( +\mathfrak{f} \notin \mathbf{F} \). Take \( M^k = M^k(\mathbf{F}) \) and \( N^k = M^k(sm(\mathbf{F} \cup \{ +\mathfrak{f} \})) \), for each \( k \geq 0 \). By (2), \( sm(\mathbf{F} \cup \{ +\mathfrak{f} \}) \) is not empty but it contains only isomorphic copies of \( +\mathfrak{f} \).

Let \( n \geq 1 \) and \( \sigma: \{ x_1, \ldots, x_n \} \to Fm^k \). There is a number \( m \geq 1 \) such that there are mappings \( F: M^m \to M^n \) and \( G: M^m \to M^n \) fulfilling the conditions (i)-(v) of Theorem 4.11. We take \( m' = m + n + d + 1 \) and would like to find mappings \( F': N^m \to N^n \) and \( G': N^m \to N^n \) fulfilling the conditions (i)-(v) of Theorem 4.11.

Let \( M^k = (W, R, w_0, \{ f^k_w \}) \in M^k \) and \( F(M^k) = (W, R, w_0, \{ f^m_w \}) \). We take \( F'(M^k) = (W, R, w_0, \{ g^m_w \}) \), where \( g^m_w = f^m_w \). We need to define \( F'(M^k) \) for \( M^k \in M^k \). Let \( M^k \in M^k \). Then \( M^k \) is a k-model over \( +\mathfrak{f} \). Let \( m \) be the root of \( +\mathfrak{f} \) and \( u \) its immediate successor (the root of \( \mathfrak{f} \)). Then \( (M^k)_u \) is a p-irreducible model in \( M^k \), see Theorem 3.12 and hence we have \( (F(M^k))_u \). Thus, we only need the valuation \( g^m_w \) at \( w_0 \) (to get \( F'(M^k) \)).

The definition of \( F': N^m \to N^n \) involves three cases.

(A) Suppose that \( M^m \) is not equivalent with \( F'(M^k) \), for any \( M^k \in M^k \). Then we take \( G'(M^m) = (G(M^m), m) \). The mapping \( F' \) preserves the depth of the frame. Thus, \( d(F'(M^k)) = d + 1 \) for any \( M^k \in M^k \) and it means that any generated submodel of \( M^m \) is not equivalent with \( F'(M^k) \), for any \( M^k \in M^k \), either. So, we can show (ii). The conditions (i) and (iii) are obvious. We also have \( G'(M^m) \sim \sigma(M^k) \), for some \( M^k \), as \( G \) fulfills (iv).

To show the condition (v) (of Theorem 4.11), let us suppose \( M^m \sim F'(M^k) \), for some \( M^k \in M^k \). Then \( M^m \sim F'(M^k) \sim M^m = F'(M^k) \) and hence we have \( G'(M^m) = G(M^m) \sim \sigma(M^k) \).

(B) Suppose \( M^m \in M^k \) is not equivalent with \( F'(M^k) \), for any \( M^k \in M^k \). Then \( M^m \) is a model over \( +\mathfrak{f} \). Let \( w_0 \) be the root of \( +\mathfrak{f} \) and \( u \) its immediate successor; that is \( u \) is the root of \( +\mathfrak{f} \). By (A), we have \( G'(M^m)_u = (G(M^m), m)_u \). Let \( f^m_u \) be the valuation at the root \( u \) of \( G'(M^m)_u \). To extend \( G'(M^m)_u \) to a model over \( +\mathfrak{f} \), it suffices to define the valuation \( f^m_w \) at \( w_0 \). Regardless of the choice, the conditions (i)-(iii) of Theorem 4.11 are always fulfilled. We take \( f^m_u = f^m_w \). Then \( G'(M^m) \sim G'(M^m)_u \sim \sigma(M^k) \), for some \( M^k \), which means the condition (iv) of Theorem 4.11 is also fulfilled and (v) is irrelevant in this case.

(C) Suppose that \( M^m \) is p-irreducible and \( M^m \sim F'(M^k) \), for some \( M^k \in M^k \).
Then there is a p-morphism $p: F'(\mathcal{M}^k) \to \mathcal{M}'$. We can assume $\mathcal{M}^k$ is p-irreducible and $\mathcal{M}'$ is a model over $+\mathfrak{F}$. Let $w_0$ be the root of $\mathcal{M}^k$ and $u$ its immediate successor. Then $p(w_0)$ is the root of $\mathcal{M}'$ and $p(u)$ its only immediate successor (in $\mathcal{M}'$). By the definition of $F'$, we have $F'((\mathcal{M}^k)_u) = m = F((\mathcal{M}^k)_u)$ and hence $G'(((\mathcal{M}')_{p(u)}) = G((((\mathcal{M}')_{p(u)}))[ m]) = G(F((\mathcal{M}^k)_u)) = (\sigma(\mathcal{M}^k))_u$.

To get $G'(\mathcal{M}')$ we need the valuation $f^n_{p(w_0)}$ at $p(w_0)$ (in $G'(\mathcal{M}')$). Since p-morphisms preserve the valuations we must have (in $\mathcal{M}'$ at $p(w_0)$) the valuation $g^n_{w_0} = 0 \ldots 00\ldots 0$, where $h^n_{w_0}$ is the valuation at $w_0$ in $\sigma(\mathcal{M}^k)$. It means, in particular, that for each $\mathcal{M}^k$ such that $\mathcal{M}' \sim F'(\mathcal{M}^k)$ we have the same valuation $h^n_{w_0}$ at $w_0$ in $\sigma(\mathcal{M}^k)$. Then we take $f^n_{p(w_0)} = h^n_{w_0}$ to complete our definition of $G'(\mathcal{M}')$. Since the valuation $f^n_{p(u)}$ at $p(u)$ (in $G'(\mathcal{M}')$) must be $h^n_u$ (the valuation at $u$ in $\sigma(\mathcal{M}^k)$), the monotonicity condition is fulfilled and our definition is correct. One checks (i)-(iii) of Theorem 4.11 and $G'(\mathcal{M}') \sim \sigma(\mathcal{M}^k)$ which gives (iv)-(v).

Our lemma is not sufficient to show $L(\{+\mathfrak{F}\})$ has finitary/unitary unification if $L(\mathfrak{F})$ does; it applies only to some frames $\mathfrak{F}$. But using it, we can show that $L(\{\mathfrak{F}, \mathcal{L}_n\})$ has finitary/unitary unification, for any $n \geq 1$, if $L(\mathfrak{F})$ does. We have $L(\{\mathfrak{F}, \mathcal{L}_n\}) = L(\mathfrak{F}) \cap L(\mathcal{L}_n)$ and $L(\mathcal{L}_n)$ has projective unification. Extending our argument we get:

**Theorem 5.10**

If $L$ is a locally tabular intermediate logics with finitary/unitary unification and $L'$ is projective, then their intersection $L \cap L'$ has finitary/unitary unification.

It follows from Theorem 5.8 that we cannot replace the assumption ‘$L$ has projective’ with ‘$L'$ has projective approximation’. This does not exclude that the intersection of two logics with projective approximation is a logic with finitary unification.

**Theorem 5.11**

$L(\mathcal{F}_2)$, $L(\mathcal{F}_2+)$, $L(\mathcal{F}_3)$, and $L(\mathcal{F}_3+)$ (see Figure 37) have nullary unification.

![Fig. 37. Frames of Logics with Nullary Unification.](image-url)
There are only four p-irreducible 1-models equivalent with some \( \sigma \)-models. They are:

\[
\begin{array}{ccc}
\circ 0 & \circ 1 & 1 \\
0 & 0 & \circ 0 \\
\end{array}
\]

The last 1-model is equivalent with some \( \sigma \)-model over \( \mathcal{F}_2 \) \( k \)-times.

Let \( F(\circ 0 \cdots 0) = \circ g^m \), for some \( g^m \). Since \( k > m \), there are \( k \)-valuations \( f^k \neq g^k \) such that \( F(\circ f^k) = F(\circ g^k) = \circ f^m \), for some \( f^m \), and \( f^k \neq 0 \cdots 0 \neq g^k \). By (v), \( G(\circ g^m) = \circ 0 \) and \( G(\circ f^m) = \circ 1 \). Thus, by (ii) and (iv), for any \( m \)-valuation?

either \( G\left( \begin{array}{c}
f^m \\
\end{array} \right) = \begin{array}{c}
1 \\
0 \\
\end{array} \), or \( G\left( \begin{array}{c}
f^m \\
\end{array} \right) = \begin{array}{c}
1 \\
1 \\
\end{array} \)

However, if the first happened, we would have

\[
G\left( \begin{array}{c}
f^m \\
\end{array} \right) = \begin{array}{c}
g^m \\
\end{array} \]

which would contradict (iv) as the received 1-model is not equivalent to any \( \sigma \)-model.

We conclude the second must happen and then we get the contradiction:

\[
\begin{array}{c}
1 \\
0 \\
\end{array} \sim G\left( \begin{array}{c}
\circ f^k \\
\circ g^k \\
0 \cdots 0 \\
\end{array} \right) = G\left( \begin{array}{c}
f^m \\
\circ f^m \\
\circ f^m \\
\end{array} \right) \sim G\left( \begin{array}{c}
f^m \\
\circ f^m \\
\end{array} \right) = \begin{array}{c}
1 \\
1 \\
\end{array}
\]

If \( F = \{ \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{F}_2, \mathcal{R}_2, +, \mathcal{F}_2, \mathcal{F}_3 \} \), then \( F = sm(\mathcal{F}_3) \). Suppose that \( L(F) \) has finitary unification. Take \( n = 2 \). By Theorem 4.10, there is a number \( m \geq 1 \) such that for any \( \sigma : \{ x_1, x_2 \} \rightarrow Fm^k \) there are mappings \( G : M^m \rightarrow M^2 \) and \( F : M^k \rightarrow M^m \) fulfilling the conditions (i)-(v). Take any \( k > m \) and let

\[
\sigma(x_1) = \bigwedge_{i=1}^{k} (\neg x_1 \lor \neg x_i) \quad \text{and} \quad \sigma(x_2) = \bigwedge_{i=1}^{k} (\neg x_i \lor \neg x_1) \rightarrow \left( \bigvee_{i=1}^{k} x_i \lor \neg \bigvee_{i=1}^{k} x_i \right)
\]

Note that \( \sigma(x_1) \land \sigma(x_2) \) is valid in \( \mathcal{L}_1 \). Let us prove \( \sigma \) is a unifier for \( x_1 \lor x_2 \) (in \( L(\mathcal{F}_3) \)). Namely, let be given any \( k \)-model over \( \mathcal{F}_3 \):

There are two cases to consider. If \( f^k_0 = f^k_0 \), then \( \sigma(x_1) \) is true in the model. Suppose that \( f^k_0 \neq f^k_0 \). Then \( \sigma(x_1) \) is satisfied only at the end nodes 0 and 0'. But it means that \( \sigma(x_2) \) is true in the model. In any case, \( \sigma(x_1 \lor x_2) \) is satisfied at any node of the model.

As \( m < k \), there are \( k \)-valuations \( f^k \neq g^k \) such that \( F(\circ f^k) = F(\circ g^k) = \circ g^m \), for some \( g^m \). Let \( f^k \neq 0 \cdots 0 \) and consider the following \( k \)-models over \( \mathcal{F}_2 \):

\[
\begin{array}{c}
f^k \\
\circ f^k \\
0 \cdots 0 \\
\end{array}
\quad \text{and} \quad \begin{array}{c}
f^k \\
\circ f^k \\
0 \cdots 0 \\
\end{array}
\]
Note that $\sigma(x_1)$ is false in the first model and $\sigma(x_2)$ in the second. Let

$$F\left(\begin{array}{c}
g^n_k \\
0 \cdots 0
\end{array}\right) = \begin{array}{c}
g^n_m \\
f^n_m_1
\end{array}; \quad F\left(\begin{array}{c}
g^n_k \\
0 \cdots 0
\end{array}\right) = \begin{array}{c}
g^n_m \\
f^n_m_1
\end{array}$$

for some $m$-valuations $f^n_m, f^n_m_1$. Then, by (v), we get

$$G\left(\begin{array}{c}
g^n_m \\
f^n_m_1
\end{array}\right) = \begin{array}{c}
11 \\
01
\end{array}; \quad G\left(\begin{array}{c}
g^n_m \\
f^n_m_1
\end{array}\right) = \begin{array}{c}
11 \\
10
\end{array}$$

Thus, we have

$$G\left(\begin{array}{c}
g^n_m \\
f^n_m_1
\end{array}\right) = \begin{array}{c}
11 \\
00
\end{array}$$

which contradicts (iv). Indeed, according to (iv), $G(2M^m)$ should be equivalent to a $\sigma$-model, for any $m$-model $2M^m$. But $\sigma$ is a unifier for $x_1 \lor x_2$ and hence $x_1 \lor x_2$ should be true in the received 2-model which is not the case.

Let us consider $L(\mathcal{Q}_3^+)$. We have $sm(\mathcal{Q}_3^+) = \{ L_1, L_2, L_3, L_4, R_2, +R_2, R_2^+, \mathcal{Q}_3^+ \}$. Assume that $L(\mathcal{Q}_3^+)$ has finitary unification. Take $n = 3$ and use Theorem 5.10 to get a number $m \geq 1$, mappings $G : M^m \rightarrow M^2$ and $F : M^k \rightarrow M^m$ fulfilling the conditions (i)-(v), where $k > m$ and $\sigma : \{ x_1, x_2, x_3 \} \rightarrow Fm^k$ is as follows

$$\sigma(x_1) = x_1 \quad \text{and} \quad \sigma(x_2) = \bigwedge_{i=2}^{k} (((x_i \rightarrow x_1) \rightarrow x_1)) \lor (x_i \rightarrow x_1)$$

$$\sigma(x_3) = \bigwedge_{i=2}^{k} (((x_i \rightarrow x_1) \rightarrow x_1)) \lor (x_i \rightarrow x_1) \lor \left( \bigvee_{i=2}^{k} (x_i \lor (\bigvee_{i=2}^{k} x_i) \rightarrow x_1) \right).$$

Note that if we take $\alpha \circ \sigma$, where $\alpha : x_1 \perp$ we get a slightly modified substitution $\sigma$ as used above for $\mathcal{Q}_3$ (we would have $x_1 = \perp$ and $x_2, x_3$ substituted instead of $x_1, x_2$). If $\sigma(x_1)$ (that is $x_1$) is true at any node of any $k$-model $M_k$ over $\mathcal{Q}_3^+$, then so are $\sigma(x_2)$ and $\sigma(x_3)$. Thus, if we characterize all $p$-irreducible $k$-models, we get $\circ 111$ and $\circ 011$, as models over $L_1$, and all $\sigma$-models as we got for $\mathcal{Q}_3$ (where valuations take the form $0ij$ instead of $ij$) with additional top node where the valuation is 111. It means that $\sigma$ is a unifier for $x_2 \lor x_3$ (instead of $x_1 \lor x_2$ as we had in $\mathcal{Q}_3$). Moreover, repeating the above argument we get two $m$-models over $R_2$ such that

$$G\left(\begin{array}{c}
g^m_0 \\
f^m_0
\end{array}\right) = \begin{array}{c}
011 \\
001
\end{array}; \quad G\left(\begin{array}{c}
g^m_0 \\
f^m_0
\end{array}\right) = \begin{array}{c}
011 \\
010
\end{array}$$
Thus,

\[
G\left(\begin{array}{c}
0_{f_1}^m \\
0_{f_2}^m \\
0_{f_2}^m \\
0_{f_1}^m \\
\end{array}\right) = \begin{array}{c}
011 \\
010 \\
001 \\
000 \\
\end{array}
\]

which would mean that \(\sigma\) is not a unifier for \(x_2 \lor x_3\), a contradiction.

In the case of \(L(\mathcal{Y}_2^+)\), we have \(sm(\mathcal{Y}_2^+) = \{L_1, L_2, L_3, L_4, R_2, +R_2, R_2^+, \mathcal{G}_3\}\). We take \(n = 2\) and consider the substitution \(\sigma : \{x_1, x_2\} \rightarrow \text{Fm}^k\) defined as follows

\[
\sigma(x_1) = x_1 \\
\sigma(x_2) = \left(\left(\bigvee_{i=2}^k x_i \rightarrow x_1\right) \rightarrow x_1\right) \land \bigwedge_{i=2}^k \left(\left(x_i \rightarrow x_1\right) \lor \left(x_i \rightarrow x_1\right)\right).
\]

If we take \(\alpha \circ \sigma\), where \(\alpha : x_1/\bot\) we (almost) get the substitution \(\sigma\) as used for \(\mathcal{Y}_2\).

Nodes of the graph in Figure 38 represent p-morphic images of \(\mathcal{G}_1\), see Figure 1 (or 2
for \(\mathcal{G}_3\)) and \(\mathcal{G}_3\), and edges p-morphisms between the frames. We omit, as usual, p-morphisms that are compositions of other p-morphisms. The graph represents also all (consistent) \(H\)–complete extensions of \(H_2\); each node is the logic \(L(\mathcal{Y})\) of the frame and the edges mean inclusions. Logics with nullary unification are denoted by a black square. There are 14 logics and 5 of them have nullary unification.

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\[\text{Fig. 38. Reducts of } \mathcal{G}_1.\]

\[\text{21 We think that one can show in this way that adding top element to } \mathcal{Y}, \text{ for any finite frame } \mathcal{Y}, \text{ does not improve the unification type if } L(\mathcal{Y}) \text{ has nullary unification.}\]
Let us agree that there is a chaos in the picture. Extensions of finitary/unitary logics may have nullary unification; intersections of some finitary logics are nullary. The chaos increases if we add intersections of the logics, see Figure 41. We can only try to identify maximal logics with nullary unification.

5.4 Hereditary Finitary Unification

Let us say that unification in a logic \( L \) is hereditary finitary if any extension of \( L \), including \( L \) itself, has finitary (or unitary) unification. Unification in \( L \) is hereditary unitary if any extension of \( L \) has unitary unification and \( L \) has hereditary projective approximation if any its extension has projective approximation. By Citkin [27],

**Theorem 5.12**

\((L(C_4), L(\chi(C_4)))\) (where \( C_4 = \Theta_3^+ \), see Figure 1 and 13) is a splitting pair in the lattice of intermediate logics (see Theorem 2.3) and the logic \( L(\chi(C_4)) \) is locally tabular.

Thus, if an intermediate logic \( L \) omits \( \Theta_3^+ \), then \( L \) is locally tabular.

We know that \((L(C_3), SL)\) (where \( SL \) is Scott logic, see Figure 5 and \( C_3 = \Theta_3 \)) is a splitting pair in extensions of \( \text{INT} \). It means that logics omitting \( \Theta_3 \) may be not locally tabular, like \( SL \) (the unification type of \( SL \) is not known, we conjecture it is \( \omega \)).

By Theorem 4.19, all logics determined by frames \( H_{pa} \) in Figure 7 (which are \( L_d + \delta_m \), with \( m, d \geq 0 \), and we agree that \( L_0 + \delta_m = \delta_m \), and \( \delta_1 = \Sigma_2 \) and \( \delta_0 = \Sigma_1 \)) have projective approximation.

**Theorem 5.13**

For any intermediate logic \( L \), the following conditions are equivalent:

(i) \( L \) has hereditary projective approximation;
(ii) \( L(H_{pa}) \subseteq L \);
(iii) \( L \) omits \( \Theta_3 \) and \( R_2 \);
(iv) \( \chi(\Theta_3), \chi(R_2) \in L \).

**Proof.** (ii)\( \Rightarrow \) (i). Since \( sm(H_{pa}) \subseteq H_{pa} \), then all extensions of \( L(H_{pa}) \) have projective approximation, by Theorem 4.22. Thus, \( L(H_{pa}) \) and any its extension enjoy hereditary projective approximation.

(i)\( \Rightarrow \) (iii). We know that \( L(\Theta_3) \) has nullary unification (see Theorem 5.5). Thus, if \( \Theta_3 \) were a frame of \( L \), the logic \( L \) would have a nullary extension. Therefore \( L \) must omit \( \Theta_3 \). It must also omit \( R_2 \) as \( L(R_2) \) is unitary and non-projective, hence it cannot have projective approximation, by Corollary 5.3.

(iii)\( \Rightarrow \) (ii). Assume that \( L \) omits \( \Theta_3 \) and \( R_2 \). If \( L \) omits \( R_2 \), then it also omits \( \Theta_3^+ \) and hence \( L \) is locally tabular, by Theorem 5.12. Thus, we can assume \( L = L(F) \) for some family \( F \) of finite frames.

For each \( \delta \in F \), let \( \delta^* \) denote its \( p \)-morph image resulting by gluing all its end elements (the relation of gluing end elements is obviously a bisimulation of \( \delta \)). Thus, \( L(\{ \delta^* : \delta \in F \}) \) is an extension of \( L \) and it must be an extension of \( KC \). As \( L(F) \) omits \( R_2 \), we infer \( L(\{ \delta^* : \delta \in F \}) \) is a logic with projective unification. It means \( \delta^* \), for each \( \delta \in F \), must be a chain. Since \( L(F) \) omits \( \Theta_3 \), we conclude each element of \( F \) must be of the form \( L_d + \delta_m \) for some \( m, d \geq 0 \).

(iii)\( \Leftrightarrow \) (iv) by Theorem 2.3

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22 We use again the known fact (see [20]) that \((L(R_2), L(C))\) is a splitting pair for all extensions of \( KC \).
Projective approximation is not hereditary, intuitionistic logic INT enjoys this property, see [14]. We conjecture that projective approximation is hereditary in locally tabular logics. The above theorem does not settle the question if there are locally tabular logics with projective approximation which are not extensions of $L(H_{pa})$.

**Corollary 5.14**

$L(L_{pa})$ is the least intermediate logic with hereditary projective approximation and $L(H_{pa}) = L\{\chi(\mathcal{G}_3), \chi(\mathcal{R}_2)\}$.

By Theorem 4.12, we know that all logics determined by frames $H_{un}$ in Figure 7 (which are $L_d + \mathcal{R}_m$, with $d, m \geq 0$, and we agree that $L_0 + \mathcal{R}_m = \mathcal{R}_m$ and $\mathcal{R}_0 = L_1$) have hereditary unitary unification.

**Theorem 5.15**

For any intermediate logic $L$ the following conditions are equivalent:

(i) $L$ has hereditary unitary unification;

(ii) $L(H_{un}) \subseteq L$;

(iii) $L$ omits the frames $\mathcal{R}_2+$, $\mathcal{G}_3+$ and $\mathcal{F}_2$;

(iv) $\chi(\mathcal{G}_3+), \chi(\mathcal{R}_2+), \chi(\mathcal{F}_2) \in L$.

**Proof.** (i)$\Rightarrow$(ii) follows from Theorem 4.12, note that the class $H_{un}$ is closed under p-morphic images and generated subframes, and then apply Theorem 4.12.

(i)$\Rightarrow$(iii). Since $\mathcal{F}_2 + \mathcal{F}_1 = \mathcal{F}_2 + \mathcal{L}_2 = \mathcal{R}_2+$ and $L(\mathcal{F}_2 + \mathcal{F}_1)$ has nullary unification (by Theorem 4.12), then $L$ omits $\mathcal{R}_2+$. We know that $L(\mathcal{G}_3)$ has nullary unification (see 4.17) and $L(\mathcal{F}_2)$ is not unitary, hence $L$ must omit $\mathcal{G}_3+$ and $\mathcal{F}_2$.

(ii)$\Rightarrow$(iii). Assume that $L$ omits $\mathcal{R}_2+,$, $\mathcal{G}_3+$ and $\mathcal{F}_2$. Since $L$ omits $\mathcal{G}_3+$, then $L$ is locally tabular, by Theorem 5.12. Thus, $L = L(F)$ for some family $F$ of finite frames such that $sm(F) = F$. Since $L$ omits $\mathcal{F}_2$, all frames in $F$ are KC-frames.

Let $\mathcal{F} = (W, R, w_0) \in F$ and $d(\mathcal{F}) = d$. We define a p-morphism $p: \mathcal{F} \to \mathcal{L}_d$ by taking $p(w) = d(\mathcal{F})(w)$, for any $w \in W$. There is only one element in $W$ of the depth 1, let it be denoted by $w_1$, and there is only one element of the depth $d$ (which is $w_0$). Every $w \in W$ sees $w_1$, that is $wRw_1$.

Let $d(w_2) = i$ and $d(w_3) = i - 1$, for some $w_2, w_3 \in W$ and $i > 1$. We prove $w_2$ sees $w_3$. Suppose it does not. By the definition of the depth function, there is an element $w_4 \in W$ such that $d(w_4) = i - 1$ and $w_4$ sees $w_3$. One can take a bisimulation gluing all elements of the depth $< i - 1$ (with $w_1$). Since $F$ is closed under p-morphic images, we could assume that the received (after gluing all elements of the depth $< i - 1$) frame is our $\mathcal{F}$. Thus, we have $w_1, w_2, w_3 \in W$ which could give (as we shall see) $\mathcal{G}_3+$ (as a p-morphic image of $\mathcal{F}$) if we add the root $w_0$, see the first frame in Figure 39.

We can use $sm(F) = F$ again to assume no other element of $W$ (only $w_0$) sees both $w_2$ and $w_3$; if not we would replace $\mathcal{F}$ with its subframe generated by an $R$-maximal element that sees both $w_2$ and $w_3$. There could be other elements in $\mathcal{F}$ but they all could be identified (by a bisimulation) with one of $\{w_0, w_1, w_2, w_3, w_4\}$. For instance, all elements $w \in W$ that see only $w_1$ (and do no see $\{w_0, w_2, w_3, w_4\}$) could be identified with $w_1$, etc. This identification gives us $\mathcal{G}_3+$ as a p-morphic image of $\mathcal{F}$ (the first frame in Figure 39) provided that there is no element $w_5 \in W$ which sees both $w_2$ and $w_3$ but does not see $w_2$. If there is such an element we would get the second frame in Figure 39. But this is $\mathcal{G}_2+$ and $\mathcal{G}_3+$ is its p-morphic image. Thus, in each case, $\mathcal{G}_3+$ would belong to $F$ which could not be the case.
Thus, we have shown that any element of the depth \( i \) sees any element of the depth \( i - 1 \) in \( \mathcal{F} \). Suppose that we have two (or more) elements of the depth \( i \) (let it be \( w_2 \) and \( w_3 \)) and two (or more) elements of the depth \( i - 1 \) (denoted by \( w_3 \) and \( w_4 \)). Then we would get (as a generated subframe of a \( p \)-morphic image of \( \mathcal{F} \)) the third frame in Figure 39. Thus, \( \mathcal{F}_+ \) would belong to \( \mathcal{F} \) which is impossible as \( L \) omits \( \mathcal{F}_+ \).

We conclude that each frame \( \mathcal{F} \in \mathcal{F} \) must be of the form \( \mathcal{F}_{n_1} \cdots \cup \mathcal{F}_{n_s} \), for some \( n_1, \ldots, n_s \geq 0 \). But \( L \) omits \( \mathcal{F}_+ \) and hence \( n_1 = n_2 = \cdots = n_{s-1} = 1 \) and, consequently, each \( \mathcal{F} \in \mathcal{F} \) must be of the form \( \mathcal{F}_d + \mathcal{F}_s \), for some \( d, s \geq 0 \). Thus, we have \( \mathcal{F} \subseteq H_{un} \) and \( L(H_{un}) \subseteq L \) as required.

**Corollary 5.16**

\( L(H_{un}) \) is the least intermediate logic with hereditary unitary unification and we have \( L(H_{un}) = L(\{\chi(\mathcal{F}_3^+), \chi(\mathcal{F}_2^+), \chi(\mathcal{F}_2)\}) = L(KC \cup \{\chi(\mathcal{F}_3^+), \chi(\mathcal{F}_2^+)\}) \).

\( H_{pa} \cap H_{un} \) consists of chains only. This is in accordance with Corollary 5.18 saying that unitary logics with projective approximation are projective.

**Theorem 5.17**

For any intermediate logic \( L \), the following conditions are equivalent:

(i) \( L \) has hereditary finitary unification;

(ii) \( L \) has hereditary projective approximation or \( L \) has hereditary unitary unification;

(iii) \( L \) omits \( \mathcal{F}_3^+, \mathcal{F}_3+ \) and one of the frames \( \{\mathcal{F}_2, \mathcal{R}_2\} \);

(iv) \( \chi(\mathcal{F}_3), \chi(\mathcal{F}_3^+), \chi(\mathcal{F}_2^+) \in L \) and, either \( \chi(\mathcal{F}_2) \in L \), or \( \chi(\mathcal{F}_2) \in L \).

**Proof.** If \( L \) has hereditary finitary unification, then \( L \) omits \( \mathcal{F}_3^+ \) (by Theorem 5.7), \( L \) omits \( \mathcal{F}_3 \) and \( \mathcal{F}_3^+ \) (as \( L(\mathcal{F}_3) \) and \( L(\mathcal{F}_3^+) \) are nullary, by Theorem 5.5) and it omits one of the frames \( \{\mathcal{F}_2, \mathcal{R}_2\} \), by Theorem 5.17 Thus, \( L \) has hereditary projective approximation or hereditary unitary unification (depending on whether \( L \) omits \( \mathcal{F}_2 \) or \( \mathcal{R}_2 \)), by Theorem 5.13 and 5.19.

All pretabular intermediate logics have hereditary finitary unification. More specifically, \( LC \) enjoys projective unification, \( LJ \) has hereditary projective approximation and \( LH \) has hereditary unitary unification.

**Corollary 5.18**

An intermediate logic \( L \) has hereditary finitary unification iff either \( L(H_{un}) \subseteq L \) or \( L(H_{pa}) \subseteq L \). Thus, there are exactly two minimal intermediate logics in the family of logics enjoying hereditary finitary unification.
There are exactly four maximal intermediate logics with nullary unification. They are: $L(\mathcal{R}_2^+)$, $L(\mathcal{R}_2) \cap L(\mathcal{F}_2)$, $L(\mathcal{G}_3)$, $L(\mathcal{G}_3^+)$; see Figure 40.

**Fig. 40. Frames of Maximal Logics with Nullary Unification.**

Similarly as in Theorem 2.12, we can use a particular para-splitting of the lattice of extensions of $\text{INT}$, given by the pair

$$(\{L(\mathcal{R}_2^+), L(\mathcal{R}_2) \cap L(\mathcal{F}_2), L(\mathcal{G}_3), L(\mathcal{G}_3^+)\}, \{L(\mathcal{H}_{un}), L(\mathcal{H}_{pr})\})$$

Corollary 5.20

For each intermediate logic $L$, either $L$ includes $L(\mathcal{H}_{un})$, or $L(\mathcal{H}_{pa})$, or $L$ is included in one of the logics $\{L(\mathcal{R}_2^+), L(\mathcal{R}_2) \cap L(\mathcal{F}_2), L(\mathcal{G}_3), L(\mathcal{G}_3^+)\}$. If $L$ includes $L(\mathcal{H}_{un})$ its unification is unitary, if $L$ includes $L(\mathcal{H}_{pa})$ it has projective approximation. If $L$ is included in one of the logics $\{L(\mathcal{R}_2^+), L(\mathcal{R}_2) \cap L(\mathcal{F}_2), L(\mathcal{G}_3), L(\mathcal{G}_3^+)\}$ its unification type is not determined but $L$ has an extension with nullary unification.

Corollary 5.21

It is decidable whether a recursive intermediate logic enjoys hereditary projective approximation, hereditary unitary unification or hereditary finitary unification.

We claim that most of intermediate logics has nullary unification. To give some evidence supporting our claim let us consider the lattice of all extensions of $H_3B_2$ (= $L(\mathcal{G}_1)$). Figure 38 contains all $H$-complete extensions of $H_3B_2$. Adding all intersections of the logics (included in Figure 38) we get the lattice in Figure 41 below. There are 42 logics and 31 of them have nullary unification. Logics with hereditary finitary unification (there are 7 such logics) are located at the top of the picture. The logic $L(H_{un})$ is represented by $L(\mathcal{R}_2)$, and $L(H_{pa})$ by $L(\mathcal{F}_2)$. We have two (out of four) maximal logics with nullary unification in the picture: $L(\mathcal{G}_3)$ and $L(\mathcal{R}_2) \cap L(\mathcal{F}_2)$. The appearance of 4 logics with finitary unification, that is $L(\mathcal{G}_1)$, $L(\mathcal{G}_3)$, $L(\mathcal{C}_5)$ and $L(\mathcal{G}_3) \cap L(\mathcal{C}_5)$ (see Theorem 4.13, Theorem 4.20 and comments after the theorem), is quite mysterious. They cannot be put apart from nullary ones by their location in the lattice of all logics and they are, rather, isolated points among the majority of those with nullary unification. Unexpectedly, there are no logics with unitary unification, but the extensions of $L(\mathcal{R}_2)$, which is due (we think) to the fact that our approach is

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23 The second graph represents an unrooted frame.

24 A ‘join’ of two join-splittings given by Theorems 5.13 and 5.15 that is $\{L(\mathcal{R}_2^+), L(\mathcal{F}_2), L(\mathcal{G}_3^+)\}$ and $\{L(\mathcal{H}_{un}), L(\mathcal{H}_{pr})\}$. 

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not enough representative. It would be much better if we had considered extensions of $H_4B_2$ instead, but this was quite beyond our capacity.

Fig. 41. Extensions of $L(G_1) = H_3B_2$.

To show that some of the logics $L(F)$ (assuming that $F$ is a family of frames for $H_3B_2$ and $F = sm(F)$) have nullary unification, we need results of the present paper (and these of [17, 10]), as well as their proofs. Thus, if $G_3 \in F$ but neither $G_2$ nor $G_5$ are in $F$, we can argue as in the proof of Theorem 5.5 showing that $L(G_3)$ has nullary unification. If $Y_2 \in F$ but $C_5 \notin F$, we can use the proof of Theorem 5.5 concerning $L(Y_2)$. If $R_2$ and $G_2$ are in $F$ but $Y_2$ is not in $F$, we can use fragments of the proof of Theorem 5.7 concerning $L(\{G_2, R_2\})$.

There are, however, some cases for which it happens that no our proof fits. These are $L(\{G_2, C_5, G\})$ and $L(\{C_5, G_3, F_2\})$. Then we define $\sigma: \{x_1, x_2\} \to \text{Fm}^k$ as follows

$$\sigma(x_1) = \neg\neg x_1 \land \bigwedge_{i=1}^{k} (\neg x_{2i} \lor \neg\neg x_{2i}) \quad \text{and} \quad \sigma(x_2) = \neg\neg x_1 \land \bigwedge_{i=2}^{k+1} (\neg x_{2i-1} \lor \neg\neg x_{2i-1});$$
we characterize all $\sigma$-models and easily get to a contradiction assuming that (any of the logics) has finitary unification. We claim that similar results given for intermediate logics in the present paper, can be reproven for transitive modal logics.

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