The non-supersymmetric $AdS_4$ vacua
from Sasaki-Einstein manifolds are brane-jet stable

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Abstract

On a given arbitrary Sasaki-Einstein manifold, there is the Freund-Rubin solution which is supersymmetric $AdS_4$ solution of eleven-dimensional supergravity. There are also non-supersymmetric solutions: the skew-whiffed Freund-Rubin, the Pope-Warner, and the Englert solutions. From particular Sasaki-Einstein manifolds of $Q^{1,1,1}$ and $M^{1,1,1}$ there are the non-supersymmetric $AdS_4$ solutions discovered by Cassani, Koerber and Varela. Among those, the skew-whiffed solutions and the solutions on $Q^{1,1,1}$ and $M^{1,1,1}$ are known to be perturbatively stable, i.e., masses are above the Breitenlohner-Freedman (BF) bound. In this paper, we examine a new decay channel known as the brane-jet instability. It turns out that all $AdS_4$ solutions we consider are brane-jet stable. Hence, the skew-whiffed solutions and the solutions on $Q^{1,1,1}$ and $M^{1,1,1}$ are both BF and brane-jet stable.
1 Introduction and conclusions

Recently, there has been a number of approaches expanding our understanding of the structure of vacua from string and M-theory. At the level of principles, conjectures are proposed which restrict the possible landscape of quantum gravity. In particular, a strong version of the weak gravity conjecture, \[1\], implies the non-existence of stable non-supersymmetric \(AdS\) vacua from quantum gravity, \[2, 3\]. From gauged supergravity, due to the complexity of the scalar potential, search of critical points for \(AdS\) vacua was a daunting task. However, new machine learning technique enables us to discover larger landscape of critical points, \[4, 5, 6, 7\]. Furthermore, as an application of exceptional field theory, a powerful tool of Kaluza-Klein spectroscopy for mass spectrum was developed in \[8, 9, 10, 11, 12\]. It is useful in checking the perturbative stability of \(AdS\) vacua by the Breitenlohner-Freedman (BF) bound, \[13, 14, 15\].

In addition, a new decay channel for \(AdS\) vacua, called the brane-jet instability, was proposed in \[16\]. Along the line of previous studies, \[17, 18, 19\], it employs probe branes to test the instability. When the force acting on the probe brane is repulsive, the vacuum is determined to be unstable. In close relation to the approaches we mentioned, the brane-jet instabilities of numerous \(AdS\) vacua from gauged supergravity in diverse dimensions have been tested, \[16, 20, 21\]. See also \[22\]. In the end, among the non-supersymmetric \(AdS\) vacua which have been tested in the literature, only seven \(AdS_4\) vacua of massive type IIA supergravity, \[24, 25, 26\], are proven to be both BF, \[23\], and brane-jet stable, \[21\]. In this paper, we search for BF and brane-jet stable non-supersymmetric \(AdS_4\) vacua from M-theory.\[1\]

For a Sasaki-Einstein manifold, \(SE_7\), there is the Freund-Rubin solution, \[28\], which is \(N = 2\) supersymmetric \(AdS_4 \times SE_7\) solution of eleven-dimensional supergravity, \[29\]. There are

\[1\]Recently, in \[27\] so-called dilaton bubble solution was explicitly constructed for the \(G_2\)-symmetric vacuum which is one of the seven \(AdS_4\) vacua of massive type IIA supergravity known to be BF and brane-jet stable.
also non-supersymmetric solutions: the skew-whiffed Freund-Rubin, the Pope-Warner, and the Englert solutions.

The skew-whiffed Freund-Rubin solutions, \[28\], are obtained by flipping the orientation of \(SE_7\) of supersymmetric Freund-Rubin solutions. This solution breaks all the supersymmetry. However, if \(SE_7\) is \(S^7\), it preserves the full supersymmetry. As this solution inherits its mass spectrum from the superymmetric one, it is automatically perturbatively stable by the Breitenlohner-Freedman bound, \[30\].

The Pope-Warner solutions, \[31, 32\], break all the supersymmetry. However, they were shown to be stable within the massive truncations of \[33, 34, 35\]. On the other hand, when \(SE_7\) is \(S^7\), unstable modes were found in the \(SU(4)^{-}\)-invariant sector of gauged \(N = 8\) supergravity, \[36\], which is the consistent truncation on the solutions, \[37\]. They were also proved to be unstable within the massive truncation on tri-Sasakian manifolds, \[38\]. The final perturbative instability of these solutions on arbitrary Sasaki-Einstein manifolds was proved in \[39\].

The Englert solutions were first found on \(S^7\) in \[40\], and were generalized to \(SE_7\) in \[41\]. The Englert solutions break all the supersymmetry, \[42, 43\]. The perturbatively unstable modes of the solutions were found in \[44\] and also within the massive truncations of \[33, 34\]. This solution corresponds to the \(SO(7)^{-}\) fixed point of gauged \(N = 8\) supergravity and the instability of the fixed point was shown in \[44, 45\].

To recapitulate, among the non-supersymmetric \(AdS_4\) solutions from Sasaki-Einstein manifolds, the skew-whiffed Freund-Rubin solutions are known to be perturbatively stable and the Pope-Warner and the Englert solutions are proved to be unstable. For a more detailed account on the stability analysis of the solutions, see the introduction of \[39\] and section 3 of \[34\].

Additionally, and compared to the other solutions, fairly recently, non-supersymmetric \(AdS_4\) solutions from particular Sasaki-Einstein manifolds of \(Q^{1,1,1}\) and \(M^{1,1,1}\) were found by Cassani, Koerber and Varela in \[35\]. These solutions are proved to be perturbatively stable within the truncation performed there.

In this paper, we examine the brane-jet instability of the \(AdS_4 \times SE_7\) solutions. It turns out that all solutions we consider are brane-jet stable. Therefore, the skew-whiffed Freund-Rubin solutions and the solutions on \(Q^{1,1,1}\) and \(M^{1,1,1}\) are BF and also brane-jet stable non-supersymmetric \(AdS_4\) solutions. It would be most interesting to construct the precise AdS/CFT correspondence, \[46\], in these non-supersymmetric settings. For the skew-whiffed solutions there are some studies on the dual field theories, \[47, 48\]. However, there is always a possibility of unknown instabilities we have not discovered. Possible instabilities from global singlet marginal operators and tunneling into bubble of nothing are discussed in \[47, 49\].

Unlike the usual cases, the solutions we consider are not warped but direct products of \(AdS_4\) and internal manifolds. Therefore, the probe brane potentials are independent of the
internal coordinates and are constant. There are similar examples of direct product solutions studied previously for brane-jet stability. The $SO(7)^-$ solution of eleven-dimensional supergravity was proved to be brane-jet stable in [16] and the $G_2$ and $SO(7)$ solutions of massive type IIA supergravity were proved to be brane-jet stable and unstable, respectively, in [21].

In section 2, we consider the supersymmetric Freund-Rubin, the skew-whiffed Freund-Rubin, the Pope-Warner, and the Englert solutions and calculate the M2-brane probe potentials to examine the brane-jet instability. In section 3, we consider the solutions from $Q^{1,1,1}$ and $M^{1,1,1}$ manifolds and calculate the M2-brane probe potentials to examine the brane-jet instability.

2 AdS$_4$ vacua from arbitrary Sasaki-Einstein manifolds

2.1 Freund-Rubin, skew-whiffed, Pope-Warner, and Englert solutions

We consider AdS$_4$ solutions of eleven-dimensional supergravity on arbitrary seven-dimensional Sasaki-Einstein manifolds. In particular, we review the supersymmetric Freund-Rubin, the skew-whiffed Freund-Rubin, the Pope-Warner, and the Englert solutions. To present the solutions in a uniform manner, we employ the ansatz used for the consistent truncation of eleven-dimensional supergravity on arbitrary seven-dimensional Sasaki-Einstein manifolds, [33, 34].

Locally Sasaki-Einstein manifold is a fibration over a Kähler-Einstein manifold,

$$ds^2_{SE_7} = ds^2_{KE_6} + \eta \otimes \eta,$$

where $\eta$ is the one-form dual to the Reeb Killing vector from $d\eta = 2J$ and $J$ is the Kähler form of $KE_6$. The $(3,0)$-form on $KE_6$ is denoted by $\omega$ and satisfies $d\Omega = 4i\eta \wedge \Omega$. Then the volume form is $vol_{SE_7} = \eta \wedge J^3/3! = (i/8)\eta \wedge \Omega \wedge \Omega^*$. The metric employed for the consistent truncation on general Sasaki-Einstein manifolds is given by, [34],

$$
\frac{1}{(2L)^2} ds^2 = e^{-6U-V} ds^2_4 + e^{2U} ds^2_{KE_6} + e^{2V} (\eta + A_1) \otimes (\eta + A_1),
$$

and the four-form flux is

$$
\frac{1}{(2L)^3} G_4 = 6e^{-18U-3V} (\epsilon + h^2 + |\chi|^2) vol_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J
+ dh \wedge J \wedge (\eta + A_1) + 2hJ \wedge J
+ \sqrt{3} \left[ \chi (\eta + A_1) \wedge \Omega - \frac{i}{4} D\chi \wedge \Omega + c.c. \right],
$$
where $\epsilon = \pm 1$, $D\chi = d\chi - 4iA_1\chi$ and $L$ is an overall scale parameter. $U$, $V$, $h$ are real scalar fields and $\chi$ is a complex scalar field in four dimensions. In four dimensions there are also one- and two-form fields, $A_1$, $B_1$ and $B_2$, with field strengths,

\begin{align*}
F_2 &= dA_1, \\
H_3 &= dB_2, \\
H_2 &= dB_1 + 2B_2 + hF_2.
\end{align*}

There are previously known $AdS_4 \times SE_7$ solutions. The supersymmetric Freund-Rubin solution, \cite{28}, is

\begin{align*}
\epsilon &= +1, \quad U = 0, \quad V = 0, \quad \chi = 0, \quad h = 0, \quad R^2_{AdS_4} = \frac{1}{4},
\end{align*}

and is explicitly given by

\begin{align*}
\frac{1}{(2L)^2} ds^2 &= \frac{1}{4} ds^2_{AdS_4} + ds^2_{SE_7}, \\
\frac{1}{(2L)^3} G_4 &= \epsilon^3 \frac{3}{8} vol_{AdS_4}.
\end{align*}

Flipping the sign of the four-form flux by choosing $\epsilon = -1$, we obtain the skew-whiffed Freund-Rubin solution which breaks all the supersymmetry.

The Pope-Warner solution, \cite{31,32}, is

\begin{align*}
\epsilon &= -1, \quad e^U = 2^{-1/6}, \quad e^V = 2^{1/3}, \quad \chi^2 = 2/3, \quad h = 0, \quad R^2_{AdS_4} = \frac{3}{16},
\end{align*}

and is explicitly given by

\begin{align*}
\frac{1}{(2L)^2} ds^2 &= 2^{2/3} \left[ \frac{3}{16} ds^2_{AdS_4} + \frac{1}{2} ds^2_{KE_6} + \eta \otimes \eta \right], \\
\frac{1}{(2L)^3} G_4 &= 2 \left[ -\frac{9}{64} vol_{AdS_4} + \frac{1}{\sqrt{2}} (\eta \wedge \Omega + c.c.) \right].
\end{align*}

This solution breaks all the supersymmetry.

The Englert solution, \cite{40}, is

\begin{align*}
\epsilon &= -1, \quad e^U = (4/5)^{1/6}, \quad e^V = (4/5)^{1/6}, \quad \chi^2 = 4/15, \quad h^2 = 1/5, \quad R^2_{AdS_4} = \frac{12}{25\sqrt{5}},
\end{align*}

In the consistent truncation to four-dimensional gauged supergravity, \cite{33,34}, the solutions we consider are fixed points of the scalar potential,

\begin{align*}
\mathcal{P} &= 48e^{-8U-V} - 6e^{-10U+V} - 24h^2 e^{-14U-V} - 18 (\epsilon + h^2 + |\chi|^2)^2 e^{-18U-3V} - 24e^{-12U-3V} |\chi|^2.
\end{align*}
and is explicitly given by
\[
\frac{1}{(2L)^2} ds^2 = \left(\frac{4}{5}\right)^{1/3} \left[ \frac{3}{10} ds^2_{AdS_4} + ds^2_{KE_6} + \eta \otimes \eta \right],
\]
\[
\frac{1}{(2L)^3} G_4 = \left(\frac{4}{5}\right)^{1/2} \left[ -\frac{9}{25} vol_{AdS_4} + J \wedge J + (\eta \wedge \Omega + c.c.) \right].
\]
(2.11)

This solution also breaks all the supersymmetry.

### 2.2 M2-brane probes

At the $AdS_4$ fixed points, we have
\[
ds_4^2 = e^{2A} \left( -dx_0^2 + dx_1^2 + dx_2^2 \right) + dr^2,
\]
(2.12)
where
\[
A = \frac{r}{l},
\]
(2.13)
and $l$ is the radius of $AdS_4$. We obtain that the three-form potential is
\[
\frac{1}{(2L)^3} A_3 = \frac{l}{3} e^{3A} 6 e^{-18U - 3V} \left( \epsilon + h^2 + |\chi|^2 \right) dx_0 \wedge dx_1 \wedge dx_2 + \ldots,
\]
(2.14)

We partition the spacetime coordinates,
\[
x^a = \{x_0, x_1, x_2\}, \quad y^m = \{r, \ldots\},
\]
(2.15)
and choose the static gauge,
\[
x_0 = t = \xi^0, \quad x^a = \xi^a, \quad y^m = y^m(t),
\]
(2.16)
where $\xi^a$ are the worldvolume coordinates. The pull-back of the metric is
\[
\tilde{G}_{ab} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b}.
\]
(2.17)

Now we study the worldvolume action of the M2-branes which is given by a sum of DBI and WZ terms. If the probe branes move slowly, the worldvolume action is
\[
S = - \int d^3 \xi \sqrt{-\det(\tilde{G})} + \int \tilde{A}_3
\]
\[
= -(2L)^3 \int d^3 \xi \left( e^{-9U - 3V/2 + 3A} - \frac{1}{2} e^{-3U - V/2 + A} G_{mn} \dot{y}^m \dot{y}^n + \ldots \right)
\]
\[
+ (2L)^3 \int \frac{l}{3} e^{3A} 6 e^{-18U - 3V} \left( \epsilon + h^2 + |\chi|^2 \right) dx_0 \wedge dx_1 \wedge dx_2,
\]
(2.18)
where $\bar{A}_3$ is the pull-back of the three-form potential. Then the worldvolume action reduces to

$$S = (2L)^3 \int d^3 \eta \left( K - V \right), \quad (2.19)$$

where the kinetic and the potential terms are

$$K = \frac{1}{2} e^{-3U-V/2} + A G_{mn} \dot{y}^m \dot{y}^n + \ldots,$$
$$V = e^{3A} \left( e^{-9U-3V/2} - \frac{1}{3} 6 e^{-18U-3V} \left( \epsilon + h^2 + |\chi|^2 \right) \right). \quad (2.20)$$

For the $AdS_4$ solutions of the supersymmetric Freund-Rubin, the skew-whiffed Freund Rubin, the Pope-Warner, and the Englert solutions in (2.6), (2.8), and (2.10), respectively, we obtain

$$e^{-3A} V|_{SUSY} = 0,$$
$$e^{-3A} V|_{skew-whiffed} = 2,$$
$$e^{-3A} V|_{Pope-Warner} = 2 + \frac{2}{\sqrt{3}},$$
$$e^{-3A} V|_{Englert} = \frac{5^{5/4}}{4\sqrt{3}} + \frac{5^{7/4}}{8\sqrt{2}}. \quad (2.21)$$

All M2-brane probe potentials obtained here for non-supersymmetric solutions are positive. We conclude that all solutions we consider are brane-jet stable. However, the Pope-Warner and the Englert solutions are known to be BF unstable. On the other hand, the skew-whiffed solutions are BF and also brane-jet stable.

### 3 $AdS_4$ vacua from particular Sasaki-Einstein manifolds

#### 3.1 Non-supersymmetric $AdS_4$ solutions on $Q^{1,1,1}$ and $M^{1,1,1}$

In this section, we consider the non-supersymmetric $AdS_4$ solutions found from the consistent truncation of eleven-dimensional supergravity on seven-dimensional homogeneous Sasaki-Einstein manifolds, specifically on $Q^{1,1,1}$ and $M^{1,1,1}$, [35].

We review gauged $\mathcal{N} = 2$ supergravity in four dimensions from the consistent truncation of eleven-dimensional supergravity on $Q^{1,1,1}$ manifolds, [35]. The truncation on $M^{1,1,1}$ manifolds are obtained from the truncation on $Q^{1,1,1}$ by identifying $t^3 = t^1$. The field content consists of 1 gravity multiplet, $\{g_{\mu\nu}, A^0_\mu\}$, 3 vector multiplets, $\{A^i_\mu, t^i\}$, and 1 hypermultiplet, $\{\phi, a, \xi^0, \tilde{\xi}_0\}$, where $i = 1, 2, 3$. There are 4 real scalar fields, $\phi, a, \xi^0, \tilde{\xi}_0$, where $\phi$ and $a$ are dilaton and axion fields in four dimensions. There are 3 complex scalar fields, $t^i$, for which we also employ the parametrizations,

$$t^i = b^i + i v^i, \quad (3.1)$$

...
The scalar fields from the vector multiplets and the hypermultiplet parametrize the coset manifolds,
\[ M_v \times M_h = SU(1,1) \times SU(2,1) / U(1) U(1), \]
which is a product of special Kähler and quaternionic manifolds, respectively. The metric of the special Kähler and quaternionic manifolds are, respectively, given by
\[ ds^2 = \sum_{i=1}^{3} \left[ (du_i)^2 + \frac{1}{4} e^{-2u_i} (db_i)^2 \right], \]
and
\[ h_{uv} dq^u dq^v = (d\phi)^2 + \frac{1}{4} e^{4\phi} \left[ da + \frac{1}{2} \left( \xi^0 d\tilde{\xi}_0 - \tilde{\xi}_0 d\xi^0 \right) \right]^2 + \frac{1}{4} e^{2\phi} (d\xi^0)^2 + \frac{1}{4} e^{2\phi} (d\tilde{\xi}_0)^2. \]
The scalar potential is given by
\[ \mathcal{P} = -8 e^{2\phi} (e^{-2u_1} + e^{-2u_2} + e^{-2u_3}) + e^{4\phi} \left[ e^{-2u_1+2u_2+2u_3} + e^{2u_1-2u_2+2u_3} + e^{2u_1+2u_2-2u_3} \right] \\
+ e^{4\phi-2u_1-2u_2-2u_3} \left[ e^{4u_1} (b^2 + b^3) + e^{4u_2} (b^1 + b^3)^2 + e^{4u_3} (b^1 + b^2)^2 \right] \\
+ \frac{1}{4} e^{4\phi-2u_1-2u_2-2u_3} \left[ e_0 + 2b^1 b^2 + 2b^1 b^3 + 2b^2 b^3 + 2 (\xi^0)^2 + 2 (\tilde{\xi}_0)^2 \right]^2 \\
+ 4 e^{4\phi-2u_1-2u_2-2u_3} \left( (\xi^0)^2 + (\tilde{\xi}_0)^2 \right). \]
The supersymmetric and non-supersymmetric fixed points of $AdS_4 \times M^{1,1,1}$ are obtained from \((3.7)\) and \((3.8)\) as particular cases of $t^3 = t^1$ and $U_2 = U_1$.

Now we present the uplift formula to eleven-dimensional supergravity. The metric is given by

$$ds^2 = e^{2V}K^{-1}ds_4^2 + e^{-V}ds^2(B_6) + e^{2V}(\theta + A^0)^2,$$  \hspace{1cm} (3.9)

with the six-dimensional base space of

$$e^{-V}ds^2(B_6) = \frac{1}{8}e^{2U_1}ds_{CP^1}^2 + \frac{1}{8}e^{2U_2}ds_{CP^1}^2 + \frac{1}{8}e^{2U_3}ds_{S^2}^2.$$ \hspace{1cm} (3.10)

The warp factors are

$$u_1 = U_1 + \frac{1}{2}V, \quad u_2 = U_2 + \frac{1}{2}V, \quad u_3 = U_3 + \frac{1}{2}V, \quad \phi = -U_1 - U_2 - U_3.$$ \hspace{1cm} (3.11)

The Kähler potential is

$$K = \frac{1}{6}K_{ijk}v_i^iv_j^iv_k^k,$$ \hspace{1cm} (3.12)

and for $Q^{1,1,1}$ solutions,

$$K_{123} = 1.$$ \hspace{1cm} (3.13)

There is a relation which we employ later,

$$e^{2\phi} = e^{3V}K^{-1}.$$ \hspace{1cm} (3.14)

The four-form flux is given by

$$G_4 = dA_3 + G_4^{\text{flux}} = H_4 + dB \wedge (\theta + A^0) + H_i^j \wedge \omega_i + Db^i \wedge \omega_i \wedge (\theta + A^0)$$

$$+ D\xi^A \wedge \alpha_A - D\xi_A \wedge \beta^A + \chi_i \tilde{\omega}^i$$

$$+ \left[(b^I Q^I + \Xi)^A \alpha_A - (b^I Q^I + \Xi)^A \beta^A\right] \wedge (\theta + A^0),$$ \hspace{1cm} (3.15)

where

$$H_4 = K^{-1}e^{4\phi}\left(b^I \xi_I + \frac{1}{2}K_{ijk}m^imb^k\right) \ast 1,$$ \hspace{1cm} (3.16)

and

$$\xi_I = e_I + Q_I^T \tilde{C} \xi - \frac{1}{2}\delta_0^I \xi^T \tilde{C} \tilde{U} \xi.$$ \hspace{1cm} (3.17)

On $SE_7$ there exists a set of real differential forms, an one-form, $\theta$, $n_V$ two-forms, $\omega_i$, $2n_H$ three-forms, $\alpha_A$, $\beta^A$, $n_V$ four-forms, $\tilde{\omega}^i$, and a six form, $\tilde{\omega}^0$ where $n_V$ and $n_H$ are the numbers of vector and hypermultiplets. Four-form fluxes, $(p^A, q_A)$, and geometric fluxes, $(m_i^A, e_iA)$, and $(v^A_B, t^{AB}, s_{AB}, u_{A}^B)$, define matrices of

$$Q_I = \begin{pmatrix} p^A & m^A_i \\ q_A & e_iA \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} v^A_B & t^{AB} \\ s_{AB} & u_{A}^B \end{pmatrix},$$ \hspace{1cm} (3.18)
where we have
\[ v^A_B = -u_B^A. \]  
(3.19)

The \( Sp(2n_H, \mathbb{R}) \) matrix is defined to be
\[ C = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}, \]  
(3.20)
and the scalar fields from hypermultiplets parametrize
\[ \xi = \begin{pmatrix} \xi_A \\ \tilde{\xi}_A \end{pmatrix}. \]  
(3.21)

Some parameters are introduced to be
\[ e_I = (e_0, e_i), \quad m^I = (0, m^i), \quad b^I = (1, b^j), \]  
(3.22)
with a choice of
\[ e_i = 0. \]  
(3.23)

The constant, \( e_0 \), is a dualization of three-form potential on four-dimensional external spacetime.

When the Betti number, \( b_3 = 0 \), which is the case we consider, we have the four-form and geometric fluxes to be
\[ e_i = 0, \quad p^A = 0, \quad q_A = 0. \]  
(3.24)

For \( Q^{1,1,1} \) and \( M^{1,1,1} \) solutions, we have the geometric fluxes of
\[ e_{iA} = 0, \quad m_i^A = 0, \]  
(3.25)
and
\[ u_A^B = 0, \quad s_{00} = -4, \quad t^{00} = 4. \]  
(3.26)

Also for \( Q^{1,1,1} \) and \( M^{1,1,1} \) solutions, we have
\[ m^1 = m^2 = m^3 = 2. \]  
(3.27)

### 3.2 M2-brane probes

At the \( AdS_4 \) fixed points, we have
\[ ds_4^2 = e^{2A} \left( -dx_0^2 + dx_1^2 + dx_2^2 \right) + dr^2, \]  
(3.28)
where
\[ A = \frac{r}{l}, \]  
(3.29)
and \( l \) is the radius of \( AdS_4 \). The relevant part of four-form flux is

\[
G_4 = \mathcal{K}^{-1} e^{4\phi} \left( e_0 + 2 \left( (\xi^0)^2 + (\tilde{\xi}_0)^2 \right) + 2b^1b^2 + 2b^2b^3 + 2b^3b^1 \right) \text{vol}_4. \tag{3.30}
\]

Thus we obtain that the three-form potential is

\[
A_3 = \frac{l}{3} e^{3A} \mathcal{K}^{-1} e^{4\phi} \left( e_0 + 2 \left( (\xi^0)^2 + (\tilde{\xi}_0)^2 \right) + 2b^1b^2 + 2b^2b^3 + 2b^3b^1 \right) dx_0 \wedge dx_1 \wedge dx_2. \tag{3.31}
\]

We partition the spacetime coordinates,

\[
x^a = \{ x_0, x_1, x_2 \}, \quad y^m = \{ r, \ldots \}, \tag{3.32}
\]

and choose the static gauge,

\[
x_0 = t = \eta^0, \quad x^a = \eta^a, \quad y^m = y^m(t), \tag{3.33}
\]

where \( \eta^a \) are the worldvolume coordinates. The pull-back of the metric is

\[
\tilde{G}_{ab} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \eta^a} \frac{\partial x^\nu}{\partial \eta^b}. \tag{3.34}
\]

Now we study the worldvolume action of the M2-branes which is given by a sum of DBI and WZ terms. If the probe branes move slowly, the worldvolume action is

\[
S = -\int d^3\eta \sqrt{-\det(\tilde{G})} + \int \tilde{A}_3
\]

\[
= -\int d^3\eta \left( e^{3V+3A} \mathcal{K}^{-3/2} - \frac{1}{2} e^{V+A} \mathcal{K}^{-1/2} G_{mn} \dot{y}^m \dot{y}^n + \ldots \right)
\]

\[
+ \int \frac{l}{3} e^{3A} \mathcal{K}^{-1} e^{4\phi} \left( e_0 + 2 \left( (\xi^0)^2 + (\tilde{\xi}_0)^2 \right) + 2b^1b^2 + 2b^2b^3 + 2b^3b^1 \right) dx_0 \wedge dx_1 \wedge dx_2, \tag{3.35}
\]

where \( \tilde{A}_3 \) is the pull-back of the three-form potential. Then the worldvolume action reduces to

\[
S = \int d^3\eta \left( K - V \right), \tag{3.36}
\]

where the kinetic and the potential terms are

\[
K = \frac{1}{2} e^{V+A} \mathcal{K}^{-1/2} G_{mn} \dot{y}^m \dot{y}^n + \ldots,
\]

\[
V = e^{3A} \left( e^{3V} \mathcal{K}^{-3/2} - \frac{l}{3} e^{6V} \mathcal{K}^{-3} \left( e_0 + 2 \left( (\xi^0)^2 + (\tilde{\xi}_0)^2 \right) + 2b^1b^2 + 2b^2b^3 + 2b^3b^1 \right) \right), \tag{3.37}
\]

where we employed the relation in (3.14). For the supersymmetric fixed point in (3.7), we obtain

\[
e^{-3A} V|_{\text{SUSY}} = 0. \tag{3.38}
\]
For the non-supersymmetric fixed point in (3.8), we obtain
\[ e^{-3A} |_{\text{non-SUSY}} = \frac{1}{\sqrt{15}} a^{21/4} \left( \frac{7}{2} \right)^{3/4} + \frac{7\sqrt{14}}{45} l^{9/4} , \]  
(3.39)

where the free parameter, \( a > 0 \), and \( l \) is the radius of AdS\(_4\). We conclude that both supersymmetric and non-supersymmetric vacua are brane-jet stable. They are also known to be BF stable within the truncation of [35].

So far we have considered the solutions of AdS\(_4 \times Q^{1,1,1}\). The analysis for the AdS\(_4 \times M^{1,1,1}\) solutions goes parallel by identifying \( t^3 = t^1 \) and \( U_2 = U_1 \). Therefore, the vacua from \( M^{1,1,1}\) are also BF and brane-jet stable.

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