Hölder continuity of solutions to the complex Monge-Ampère equation with the right hand side in $L^p$.

The case of compact Kähler manifolds

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Abstract. We prove that on compact Kähler manifolds solutions to the complex Monge-Ampère equation, with the right hand side in $L^p$, $p > 1$, are Hölder continuous.

Let $M$ be a compact $n$-dimensional Kähler manifold with the fundamental form $\omega$ given in local coordinates by

$$\omega = \frac{i}{2} \sum_{k,j} g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$ 

An upper semicontinuous function $u$ on $M$ is called $\omega$-plurisubharmonic ($\omega$-psh in short) if $dd^c u + \omega \geq 0$.

We study the regularity of $\omega$-psh solutions $u$ of the complex Monge-Ampère equation

$$(\omega + dd^c u)^n = f \omega^n,$$ 

where $f \in L^1(M)$, $f \geq 0$, $\int_M f \omega^n = \int_M \omega^n$ is a given function. For smooth positive $f$ the equation was solved by Yau [Y]. Later it was shown in [K1] that for $f \in L^p(M)$, $p > 1$ there exists a continuous solution. From [K2] we know that $L^\infty$ norm of a difference of (suitably normalized) solutions is controlled by $L^1$ norm of the difference of functions on the right hand side (see Theorem 1.1 below). Here we shall prove that for $f \in L^p(M)$, $p > 1$ the solutions are Hölder continuous. The exponent depends on $M$ and $\|f\|_p$.

1991 Mathematics Subject Classification. primary 32U05, secondary 32U40.

Key words and phrases. plurisubharmonic function, complex Monge-Ampère operator.
The corresponding result in strictly pseudoconvex domains has been obtained in [GKZ].

The results are motivated, in part, by a recent work of Tian and Zhang [TZ] where the authors study the Kähler-Ricci flow on projective manifolds. Later more papers on the subject appeared in ArXiv: [ST], [EGZ], [Z]. When the canonical divisor is big and nef the flow initiated at any Kähler metric tends to a current which is a smooth Kähler-Einstein metric off a subvariety $S$ of $M$. The potential of this current is continuous also along the singular set. Since the right hand side of the Monge-Ampère equation satisfied by the potential blows up along $S$ at the rate $d^\alpha$, where $d$ denotes the distance from $S$, and it is integrable at the same time, it belongs to some $L^p$, $p > 1$. The Kähler form on the left hand side also degenerates, so the Monge-Ampère equation here becomes more complicated. However, our result, with the same proof, holds on compact Kähler orbifolds. Also, if the limit metric has singularities that can be blown down, then we may pull-back the equation from a Kähler manifold. Thus Theorem 2.1 can be applied to some 2-dimensional examples of the Kähler-Ricci flow considered in [TZ] and [ST].

I would like to thank G. Tian for an invitation to Princeton and the possibility of discussing this topic. I also thank V. Guedj, M. Paun and Z. Zhang for their comments on this paper. Z. Zhang’s suggestion shortened the proof.

1. Preliminaries.

For the background of the definitions and results that follow see [K3]. Using the differential operators $d = \partial + \overline{\partial}$, $d^c = i(\overline{\partial} - \partial)$ we define for given bounded plurisubharmonic function $u$ the Monge-Ampère measure

$$(dd^c u)^n = dd^c u \wedge dd^c u \wedge ... \wedge dd^c u \quad (n \text{ terms})$$

(see [BT1]). This is a nonnegative Borel measure.

Let us recall a stability statement from [K2] concerning the equation (0.1). On a compact Kähler manifold $M$ with a fundamental form $\omega$ the $L^p$ norms are defined by

$$||f||_p = (\int_M |f|^p \omega^n)^{1/p}. $$

**Theorem 1.1.** Given $p > 1$, $\epsilon > 0$, $c_0 > 0$ and $||f||_p < c_0$, $||g||_p < c_0$ satisfying the normalizing condition in (0.1) there exists $c(\epsilon, c_0)$ with

$$||\varphi - \psi||_\infty \leq c(\epsilon, c_0)||f - g||_1^{1/(n+3+\epsilon)}$$

for suitably normalized $\varphi$ and $\psi$.

Let $\Omega$ be a domain in $\mathbb{C}^n$ and $u \in PSH(\Omega)$. For $z \in \Omega_\delta := \{z \in \Omega : \text{dist}(z, \partial \Omega) > \delta\}$ define a plurisubharmonic function

$$\tilde{u}_\delta(z) = [\tau(n)\delta^{2n}]^{-1} \int_{|\zeta| \leq \delta} u(z + \zeta) \, dV(\zeta), \quad \tau(n) := \int_{|\zeta| \leq 1} dV(\zeta),$$

where $dV$ denotes the Lebesgue measure. In [GKZ] it is proved that

$$\int_{\Omega_\delta} (\tilde{u}_\delta - u) \, dV(\zeta) \leq c_1(n)||\Delta u||_1 \delta^2,$$
with the constant $c_0(n)$ depending only on the dimension. For the sake of completeness we include the proof here. Applying Jensen’s formula and Fubini’s theorem we obtain the following estimates (with $\sigma_{2n-1}$ denoting the surface measure of the unit sphere)

\[
\int_{\Omega_\delta} \delta^{-2} (\bar{u}_\delta - u)(\zeta) \, dV(\zeta) = \frac{2n}{\delta^{2(n+1)} \sigma_{2n-1}} \int_{\Omega_\delta} \int_0^r \int_0^{2n-1} \int_0^r t^{1-2n} \int_{|\zeta| \leq t} \Delta u(z + \zeta) \, dV(\zeta) \, dt \, dr \, dV(z) = \frac{2n}{\delta^{2(n+1)} \sigma_{2n-1}} \int_0^r \int_0^{2n-1} \int_0^r t^{1-2n} \int_{|\zeta| \leq t} \Delta u(z + \zeta) \, dV(\zeta) \, dt \, dr \\
\leq \frac{2n}{\delta^{2(n+1)} \sigma_{2n-1}} \int_0^r \int_0^{2n-1} \int_0^r t^{1-2n} \int_{|\zeta| \leq t} ||\Delta u||_1 \, dV(\zeta) \, dt \, dr = c_0(n)||\Delta u||_1,
\]

where $||\Delta u||_1 = \int_{\Omega} \Delta u(\zeta) \, dV(\zeta)$. The estimate (1.1) directly follows from this one.

2. Main theorem.

**Theorem 2.1.** For $p > 1$ and $f \in L^p(M)$ satisfying the normalizing condition in (0.1) the solution $u$ of (0.1) is Hölder continuous with the Hölder exponent which depends on $p$, $M$ and $||f||_p$.

**Proof.** It follows from [K1] that $||u||_\infty$ depends on $p$, $M$ and $||f||_p$. We can assume that $1 < u$. Choose a finite number of coordinate balls $B''_j = B(a_j, 3r)$ such that $B'_j = B(a_j, r)$ cover $M$ and denote by $B_j$ the balls $B(a_j, 2r)$. Since the transition functions for those charts have bounded Jacobians one can find a constant $C > 0$ depending only on $M$ and such that for all $\delta < r/2C$ and $z \in B_j \cap B_k$ we have

\[
(2.1) \quad B_j(z, \delta) \subset B_k(z, \frac{C}{2} \delta),
\]

where $B_j(z, \delta)$ denotes the ball centered at $z$ of radius $\delta$ in the chart $B''_j$.

Fix non positive functions $\rho_j \in C^\infty(B_j)$ with $\rho_j = 0$ on $B'_j$, $-1 \leq \rho_j \leq 0$ in $B_j$, and $\rho_j = -1$ on a neighbourhood of $\partial B_j$. For some $c_1$ depending on $M$ we have

\[
d \rho_j \geq -c_1 \omega.
\]

For fixed $\epsilon > 0$ we choose $N > 2C$ and $\alpha < \frac{1}{q(n+3+\epsilon)+1}$ (with $p, q$ conjugate) satisfying

\[
(2.2) \quad 2(2c_1 ||u||_\infty + 1) < N^{-\alpha} \frac{\log N}{\log C}.
\]

It is possible since we can choose $N$ so big that $\frac{\log N}{\log C}$ is bigger than two times the left hand side, and then we take $\alpha$ small enough. Using the coordinates in $B''_j$ we define regularizations

\[
u_{j,\delta}(z) = \max_{|w| < \delta} u(z + w), \quad z \in B_j.
\]
Let us yet define two auxiliary functions
\[
\chi(\delta) = \delta^{-\alpha} \max_j \max_{z \in B_j} (u_{j,\delta} - u)(z),
\]
and
\[
\eta(\delta) = \max_j \max_{z \in B_j} (u_{j,C\delta} - u_{j,\delta})(z),
\]
By (2.1) we have
\[
(2.3) \max_{z \in B_j \cap B_k}|(u_{j,\delta} - u_{k,\delta})(z)| \leq \eta(\delta).
\]
We shall approximate \(u\) by \(\omega\)-psh functions \(u_{\delta}\) which are created by gluing together the local regularizations \(u_{j,\delta}\) (comp. [D]). The function \(\eta\) defined above measures the correction term when we pass from local to global regularization. Note that, by continuity of \(u\),
\[
\lim_{\delta \to 0} \eta(\delta) = 0.
\]
Set
\[
u_{\delta}(z) = (1 + C_1 \eta(\delta))^{-1} \max_j (u_{j,\delta}(z) + \eta(\delta)\rho_j(z)), \quad C_1 = 2c_1.
\]
It is continuous on \(M\) since, by (2.3), the maximum on the right hand side must be attained for \(j\) such that \(z \in B'_j\). Note also that since for \(c_1 \eta(\delta) < 1\)
\[
\ddc(u_{j,\delta}(z) + \eta(\delta)\rho_j(z)) \geq -\left(1 + \frac{C_1}{2}\eta(\delta)\right)\omega
\]
one obtains, via an inequality from [BT1] estimating \(\ddc \max(u, v)\) from below,
\[
(2.4) \quad \ddc u_{\delta} + \omega > 0,
\]
if \(\delta\) is sufficiently small. To finish the proof we need to verify the following claim.

**Claim.** \(\chi\) is bounded on some nonempty interval \((0, \tilde{\delta})\).

Suppose that \(\chi(\delta) > \max(9, \chi(N\delta))\) and \(N\delta < r/2\). Then the set
\[
E = \bigcup_j \{z \in B_j : (u_{j,\delta} - u)(z) > \left(\frac{\chi(\delta)}{3} - 2\right)\delta^\alpha\}
\]
is nonempty. Take \(g = 0\) on \(E\) and \(g = C_2 f\) on \(M \setminus E\) with the constant \(C_2\) chosen so that \(\int_M g\omega^n = \int_M \omega^n\).

Now we compare \(u_{j,\delta}\) with
\[
\tilde{u}_{j,\delta}(z) = [\tau(n)\delta^{2n}]^{-1} \int_{\vert \zeta \vert \leq \delta} u(z + \zeta) dV(\zeta), \quad \tau(n) := \int_{\vert \zeta \vert \leq 1} dV(\zeta),
\]
where the coordinates of \(B''_j\) are used. Given \(z \in B_j\) we find \(w_z\) with \(|w_z| = \delta\) such that
\[
u_{j,\delta} = u(z + w_z) \leq \tilde{u}_{j,\sqrt{\delta}}(z + w_z) \leq \tilde{u}_{j,\sqrt{\delta}}(z) + 2||u||_{\infty}\sqrt{\delta}.
\]
(Note that defining $\tilde{u}_{j,\sqrt{\delta}}(z)$ and $\tilde{u}_{j,\sqrt{\delta}}(z + w_z)$ we integrate over the same domain except for the piece of volume at most $2\tau(n)\delta^{n+\frac{1}{2}}$.)

Since $\alpha < 1/2$ we infer from this estimate for $\delta < \delta_0$ and $\delta_0$ small enough

$$\tilde{u}_{j,\sqrt{\delta}} - u > \delta^\alpha$$

Thus, as $||\Delta u||_1$ is a priori bounded on every $B_j''$, applying (1.1) one obtains

$$\int_{E \cap B_j} \omega^n < c_3 \delta^{1-\alpha}$$

for all $j$ and consequently

$$\int_{E} \omega^n < c_4 \delta^{1-\alpha},$$

with the constant depending only on $M$. Hence, upon the use of Hölder inequality

$$\int_{E} f \omega^n \leq ||f||_p (\int_{E} \omega^n)^{1/p} \leq c_5 \delta^{(1-\alpha)/q},$$

where $c_5$ depends also on $||f||_p$. By Theorem 1.1 for $\delta < \delta_1$ and $\delta_1$ small enough if $v$ solves

$$(\omega + dd^c v)^n = g\omega^n$$

and is suitably normalized then

$$(2.5) \quad ||u - v||_\infty \leq ||f - g||_{\frac{1}{(n+3+\epsilon)}} \leq c_6 \delta^{\frac{1-\alpha}{q(n+3+\epsilon)}} \leq \delta^\alpha$$

(since by our choice of $\alpha$ we have $\alpha < \frac{1-\alpha}{q(n+3+\epsilon)}$).

**Proposition.** If we choose $z_0 \in B_{j_0}$ so that

$$(u_{j_0,\delta} - u)(z_0) = \chi(\delta)\delta^\alpha,$$

then

$$\sup_{M \setminus E} (u_\delta - v) < (u_\delta - v)(z_0).$$

**Proof.** Take $z \in (M \setminus E) \cap B_j$. Then

$$(u_{j,\delta} - u)(z) \leq (\frac{\chi(\delta)}{3} - 2)\delta^\alpha$$

and therefore, by (2.5)

$$(u_{j,\delta} - v)(z) \leq (\frac{\chi(\delta)}{3} - 1)\delta^\alpha.$$ 

Since $u > 1$ we get from this

$$(2.6) \quad (u_\delta - v)(z) \leq \max_{j: z \in B_j} (u_{j,\delta} - v)(z) \leq (\frac{\chi(\delta)}{3} - 1)\delta^\alpha.$$
Again, by (2.5)

\[(u_{j_0,\delta} - v)(z_0) \geq (\chi(\delta) - 1)\delta^\alpha.\]

Therefore the definition of \(u_\delta\) yields

\[(u_\delta - v)(z_0) \geq (\chi(\delta) - 1)\delta^\alpha - \eta(\delta)(2c_1\|u\|_\infty + 1).\]  

The Three Circles Theorem gives for \(\delta\) small enough

\[(u_{j,N\delta} - u_{j,\delta}) \geq \log N \log C (u_{j,C\delta} - u_{j,\delta}).\]

It follows that, choosing \(j\) so that

\[\eta(\delta) = \max_{z \in B_j} (u_{j,C\delta} - u_{j,\delta})(z)\]

we obtain

\[(N\delta)^\alpha \chi(N\delta) \geq \log N \log C \eta(\delta).\]

Further, since \(\chi(\delta) \geq \chi(N\delta)\), we get from (2.2) that

\[\delta^\alpha \chi(\delta) \geq \log N \log C \eta(\delta) N^{-\alpha} > 2\eta(\delta)(2c_1\|u\|_\infty + 1).\]

Inserting this into (2.7) we finally arrive at

\[(u_\delta - v)(z_0) > (\chi(\delta)/2 - 1)\delta^\alpha.\]

The proposition follows by comparing this inequality with (2.6).

Applying the proposition one can find \(c_7\) such that

\[z_0 \in U = \{v < u_\delta - c_7\} \subset E.\]

By the comparison principle [K2] and (2.4)

\[0 < \int_U (dd^c u_\delta + \omega)^n \leq \int_U (dd^c v + \omega)^n \leq \int_E (dd^c v + \omega)^n = \int_E g\omega^n = 0.\]

This contradiction shows that the choice of small enough \(\delta\) with \(\chi(\delta) > \max(9, \chi(N\delta))\) is impossible. Thus the proof of the claim and that of the theorem is completed.

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