Analytical Calculation of Magnetic Field in Fractional-Slot Windings Linear Phase-Shifting Transformer Based on Exact Subdomain Model

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ABSTRACT Fractional-slot windings linear phase-shifting transformers (FW-LPSTs) are a new type of transformer that can be used as an important component of multi-rectifier and multi-inverter systems. The FW-LPSTs are proposed for the first time, and few scholars have studied their magnetic field. However, accurate calculation of magnetic field distribution of the FW-LPSTs is the premise of the optimal design of the ontology, and the establishment of an accurate analytical model which remains a challenging task is the key to solving the magnetic field distribution. In view of the limitation of traditional analytical theory in solving the magnetic field, the FW-LPST model based on the exact subdomain method is proposed to obtain the exact magnetic field distribution. Considering the influences of the permeability and the structural parameters of the slot subdomain and the interaction of all slots on the distribution of the magnetic field, the governing equations of the air gap and slot subdomains are established using magnetic vector potential as the variable. According to the boundary conditions of each subdomain, all equations of the boundary condition are listed based on the Fourier series method. Finally, the vector magnetic potential and open-circuit magnetic flux density distribution of each subdomain are solved. The model accurately analyzes the influence of the cogging effect on the magnetic field distribution. The inductances of the primary side and the induced potentials of the secondary side are solved using the magnetic flux linkage method. The accuracy of the proposed analytical model is verified by comparison of the results obtained with those obtained by the finite element method.

INDEX TERMS Linear phase-shifting transformer, fractional-slot windings, exact subdomain method, analytical calculation of magnetic field, the cogging effect, finite element method.

I. INTRODUCTION
Multiple superposed technology can effectively reduce voltage harmonics, reduce pollution to the power grid, improve output voltage waves, and improve output performance [1]. Phase-shifting transformers (PSTs) are essential devices of multi-phasing equipment, so it is of great significance to design and research PSTs.

Most of the conventional PSTs whose structures are complex are core-column transformers. As the capacity and the number of phases of the PSTs increase, the volume and weight of the transformer increase greatly. And then, the utilization rate of the iron core decreases, and the structure of the windings also becomes very complicated. WANG T J et al. [2] put forward the circular PSTs which had two major advantages: good function of phase-shifting and good symmetry of output voltage. But the leakage reactance of the transverse end of the transformer was particularly large, and the processing of iron cores and windings were very complex. The circular PSTs were not easy to expand. ZHAO J H et al. [3] proposed a double-layer windings linear PST, which was based on the principle of linear motor. The linear PSTs offer numerous advantages over conventional transformers, such as the capability of realizing arbitrary phase angle shifts, easy adjustment of the magnetic field in the air gap, suitability for high-power settings, and relatively simple structure, which makes linear PSTs easy to modularize and expand [3]. However, the end leakage reactance of the linear PSTs was large, the winding processing was difficult,
and the number of turns of windings was limited by the transverse end windings.

In recent years, fractional-slot windings have attracted the attention of domestic and foreign scholars because of their unique advantages [4]. The fractional-slot windings have many advantages, such as high utilization rate of slot, short end windings, less consumable materials, low loss, simple windings process, high efficiency, good sinusoidal electromagnetic force, etc. [4], [5]. Based on the above advantages, a fractional-slot windings linear phase-shifting transformer (FW-LPST) is proposed for the first time in this paper. However, the accurate calculation of magnetic field distribution is the premise of the optimized design of the ontology, and the establishment of an accurate analytical model which remains a challenging task is the key to solving the magnetic field. The finite element method (FEM) is a numerical method that has many advantages, such as high calculation accuracy and the ability to accommodate nonlinear materials. However, the calculation process of the FEM is time-consuming. In addition, the FEM cannot directly reflect any clear physical relationships between electromagnetic performance and the design parameters. The FEM is mostly used to verify the final results, but is not suitable for the initial design and intermediate optimization process. In other words, the final results of the analytical method need to be verified by the FEM. In order to solve the shortcomings of the FEM, it is very meaningful to propose an accurate analytical model for calculating the magnetic field distribution in the FW-LPSTs.

The magnetic field analytical methods can be applied to the FW-LPSTs by reference include the equivalent magnetic circuit model [6], equivalent magnetic network model [7], [8], layered model method [9], conformal transformation model [10], and exact subdomain method (ESM) [11]–[15], etc. The equivalent magnetic circuit model is difficult to deal with complex electromagnetic fields because the expression of structural parameters is not detailed enough and the calculation error is large. This has been resolved to some extent by the equivalent magnetic network model. The equivalent magnetic network model in essence is an enhancement of the previous one. However, the model is still not particularly accurate, and the accuracy is affected by the degree of mesh generation. The layered model method that is not very accurate is not ideally suited for modeling FW-LPSTs. In addition, the influence of the cogging effect is calculated by introducing the Cater factor coefficient model, the linear model of the cogging area permeability, and the air gap relative permeability function. However, the model cannot reflect the influence of the spatial position on the air gap permeance, and ignores the impact of adjacent slots. The method cannot calculate the tangential magnetic flux density in a FW-LPST. Differently, the conformal transformation model can accurately calculate the normal flux density and tangential flux density in a FW-LPST. However, the method is similar to numerical method, and it is difficult to express the relationship between structural parameters and electromagnetic parameters, so it is not widely used in linear motors. Similarly, it is also inappropriate to apply the method for calculating the magnetic field of a FW-LPST. The ESM is an accurate method to deal with complex magnetic field problems [10]. At present, the ESM has been applied for all kinds of motors. For example, HU H et al. [11]–[13] applied the ESM to model a rotating permanent magnet synchronous motor. The permanent magnet was equivalent to the current density in the polar coordinate system, and the ESM was used to solve the magnetic field of the rotating motor. WANG M J et al. [14] applied the ESM to model a permanent magnet linear synchronous motor, the magnetic field of that was solved. Scalar magnetic potential was used to establish the model based on the ESM, and the equivalent model of the end was established. The influences of cogging effect and end effect on the spatial distribution of air gap magnetic field were discussed. HU et al. [15] proposed a model of linear motors in the polar coordinate system. The ESM was applied to calculate the magnetic field. Then, according to the developed model, the tangential thrust and normal forces were calculated based on the Maxwell stress theory. The numerical results obtained by FEM were employed to validate the analytical model. T. L et al. [16] dealt with an analytical method for magnetic field calculation in the air gap of cylindrical electrical machines including cogging effects. The current source of the primary windings was equivalent to the current layer. The analytical method was based on the resolution of the two-dimensional Laplace’s equation in polar coordinates. The originality of the proposed model was to consider the mutual influence of slots on the air-gap magnetic field. The lap winding was used in the motor, while the fractional-winding was used in the FW-LPST. The magnetic coupling modes of the two windings were not the same, so the two kinds of windings could not be directly equivalent. The current equivalent method could not be directly applied to the FW-LPST model. In addition, the flux density of slot subdomains on the primary side could not be calculated by the model. T. L et al. [17] proposed an analytical computation of the magnetic field distribution in a magnetic gear. The analytical method was based on the resolution of Laplace’s and Poisson’s equations for each subdomain, i.e., magnets, air gap, and slots. The excitation source was permanent magnet, which could not be directly applied to the FW-LPST.

Different from the structures in the above literatures, the research object of this paper is a FW-LPST based on the linear motors, and the excitation source is the current source connected to the primary side windings. The magnetic field of the FW-LPSTs is different from that of ordinary rotating motors, which is not only affected by the cogging effect, but also affected by the end effect. However, the air gap of the FW-LPSTs is very small, so the end effect can be ignored in the modeling, and the model can be periodically extended indefinitely. Based on the structural characteristics of the FW-LPST, the Cartesian coordinate system would be more appropriate than the polar coordinate system.

Considering the above issues, an analytical model based on the ESM for calculating the open-circuit magnetic field
distribution of a FW-LPST is proposed. The model is established in a 2D Cartesian coordinate system regarding the magnetic vector potential as variables and considering the influence of the cogging effect on the magnetic field distribution. The Poisson equations are established in the slot subdomains of the primary side, and the Laplace equations are established in the slot subdomains of the secondary side and the air gap subdomain. The harmonic coefficients are obtained using the Fourier series method in conjunction with the boundary conditions. Finally, the distribution of the open-circuit magnetic flux density is solved in each subdomain. The accuracy of the proposed ESM-based model is verified by comparison of the results obtained by the FEM.

II. ANALYTICAL MODEL OF THE MAGNETIC

A schematic diagram illustrating the structure of a FW-LPST is presented in Figure 1-2. The primary side and secondary side have \( Q \) slots and \( N \) sets of windings respectively, and the number of pole pairs is 1. The primary and secondary sides are perfectly symmetrical and fixed. Each slot is divided vertically into two slot subdomains, left and right. Here, the characters represent the winding numbers, where the secondary side includes twelve windings (\( a_1 \), \( a_2 \), \( a_3 \), \( \ldots \), \( b_1 \), \( b_2 \), \( b_3 \), \( \ldots \)) and the primary side includes twelve windings (\( A_1 \), \( A_2 \), \( A_3 \), \( \ldots \), \( B_1 \), \( B_2 \), \( B_3 \), \( \ldots \), \( C_4 \), \( Y_4 \)).

Both the primary and secondary sides adopt fractional-slot windings. When the transformer is used for rectification, the three-phase windings on the primary side are connected with a set of symmetrical three-phase alternating currents, and each set of relative windings on the primary and secondary sides is coupled to produce an induced potential on the secondary sides separately. Finally, four groups of three-phase stepped wave potentials are induced in the secondary side windings, and the phase difference of the induced potentials of each adjacent winding on the secondary side is 15°.

A. MODEL BUILDING AND BASIC ASSUMPTIONS

The analytical modeling is based on the following assumptions [14]:

1) the permeability of the primary and secondary iron cores is infinite without magnetic iron-core saturation;
2) the permeability in the slot subdomains is regarded as the permeability of vacuum;
3) the periodic infinite extension model ignores the influence of end effects.

B. GENERAL SOLUTION OF THE SLOT SUBDOMAIN ON THE PRIMARY SIDE

\( J_{i1} \) and \( J_{i2} \) are the current densities of the left and right windings of the \( i \)-th slot subdomain on the primary side respectively. After the mirror transformation of the current density, a periodic signal is obtained, as shown in Figure 4. Fourier decomposition of the periodic signal yields the following equation.

\[
J_i(x) = J_{i0} + \sum_{m=1}^{\infty} J_{im} \cos(\beta_m(x - x_i))
\]
Here, \( m \) is the harmonic number, and \( \beta_m = m\pi/b_0 \), \( J_{i0} \) is the DC component of current density. \( J_{im} \) is the \( m \)-th harmonic component of the current density.

\[
J_{i0} = \frac{1}{2b_0} \int_{x_i}^{x_i+2b_0} J(x)dx = \frac{J_{i1} + J_{i2}}{2} \quad (4)
\]

\[
J_{im} = \frac{2}{2b_0} \int_{x_i}^{x_i+2b_0} J(x)cos[\beta_m(x-x_i)]dx
= \frac{2(J_{i1} - J_{i2})}{m\pi} \sin\left(\frac{m\pi}{2}\right) \quad (5)
\]

The current density of the \( i \)-th slot subdomain on the primary side is \( J_i \). Accordingly, the 2D Poisson equation in the \( i \)-th slot subdomain is respectively given in terms of Cartesian coordinates as follows.

\[
\begin{align*}
\frac{\partial^2 A_{i1}}{\partial x^2} + \frac{\partial^2 A_{i1}}{\partial y^2} &= -\mu_0 J_i \\
-H_1 \leq y \leq -H_2, \; x_i \leq x \leq x_i + b_0
\end{align*} \quad (6)
\]

Here, \( A_{i1} \) is the magnetic vector potential of the \( i \)-th slot subdomain on the primary side, and \( \mu_0 \) is the permeability of vacuum.

As is shown in Figure 5, because the permeability of the iron cores is infinite, the boundary conditions of the \( i \)-th slot subdomain on the primary side are respectively given as follows.

\[
\begin{align*}
\left. \frac{\partial A_{i1}}{\partial x} \right|_{x=x_i} &= \left. \frac{\partial A_{i1}}{\partial x} \right|_{x=x_i+b_0} = 0 \\
\left. \frac{\partial A_{i1}}{\partial y} \right|_{y=-H_1} &= 0
\end{align*} \quad (7)
\]

Then, \( A_{i1} \) can be obtained by solving Poisson equation (6) using the separation of variables method in conjunction with the boundary conditions in (7) as follows.

\[
A_{i1} = \sum_{m=1}^{\infty} \left( c_m e^{\beta_m y} + d_m e^{-\beta_m y} \right) \cos[\beta_m(x-x_i)] + c_0 y + d_0 ^i \quad (8)
\]

Here, \( m \) is the harmonic number, and \( \beta_m = m\pi/b_0 \), \( c_m \), \( d_m \) are harmonic coefficients, which can be determined by the boundary conditions.

The solutions of Poisson equation (6) consist of two parts: the particular solution and the general solution. The particular solution must satisfy the following (9).

\[
\frac{\partial^2 A_{i1}}{\partial x^2} + \frac{\partial^2 A_{i1}}{\partial y^2} = -\mu_0 J_{i0} - \sum_{m=1}^{\infty} \mu_0 J_{im} \cos[\beta_m(x-x_i)]
\]

\[
(9)
\]

Finally, the general solution of \( A_{i1} \) can be obtained as follows.

\[
A_{i1} = \sum_{m=1}^{\infty} \left( c_m e^{\beta_m y} + d_m e^{-\beta_m y} \right) \cos[\beta_m(x-x_i)] - \mu_0 J_{i0} \frac{y^2}{2} + c_0 y + d_0 ^i \quad (16)
\]

Substituting the second boundary condition of (7) into (16) yields the following equations.

\[
\begin{align*}
c_0 &= -\mu_0 J_{i0} H_1 \\
c_m &= d_m 2\beta_m H_1
\end{align*} \quad (17)
\]

Substituting (17) into (16) yields the following equations.

\[
A_{i1} = \sum_{m=1}^{\infty} \left[ d_m e^{\beta_m y}(e^{2\beta_m H_1} + e^{-2\beta_m y}) + \mu_0 J_{im} \frac{y^2}{\beta_m^2} \right] \cos[\beta_m(x-x_i)] - \mu_0 J_{i0} H_1 y + d_0 ^i \quad (18)
\]
C. GENERAL SOLUTION OF THE AIR GAP SUBDOMAIN

In the air gap subdomain, the electrical conductivity $\sigma = 0$. Therefore, the Laplace equation in the air gap subdomain is given as follows.

$$\begin{align*}
\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} &= 0 \\
-H_2 \leq y \leq H_2, \quad 0 \leq x \leq 2\tau
\end{align*}$$

(19)

Here, $A_2$ is the magnetic vector potential in the air gap subdomain. In addition, the periodic boundary condition within a single period of the air gap subdomain is given as follows.

$$A_2(x, y)|_{x=0} = A_2(x, y)|_{x=2\tau}$$

(20)

Finally, the general solution of $A_2$ can be obtained as follows.

$$A_2 = \sum_{n=1}^{\infty} (a_ne^{a_n y} + b_ne^{-a_n y}) \cos(\alpha_n x)$$

(21)

Finally, the general solution of $A_2$ can be obtained as follows.

$$A_2 = \sum_{n=1}^{\infty} (c_ne^{a_n y} + d_ne^{-a_n y}) \sin(\alpha_n x)$$

(22)

D. GENERAL SOLUTION OF THE SECONDARY SIDE SLOT SUBDOMAIN

In the secondary side slot subdomain, a slot can be regarded as a subdomain under an open-circuit condition. Accordingly, the Laplace equation of the $j$-th slot subdomain on the secondary side is given as follows.

$$\begin{align*}
\frac{\partial^2 A_{3j}}{\partial x^2} + \frac{\partial^2 A_{3j}}{\partial y^2} &= 0 \\
H_2 \leq y \leq H_1, \quad x_j \leq x \leq x_j + b_0
\end{align*}$$

(23)

Here, $A_{3j}$ is the magnetic vector potential of the $j$-th slot subdomain on the secondary side. As is shown in Figure 6, the $j$-th slot subdomain on the secondary side yields the following boundary conditions.

$$\begin{align*}
\frac{\partial A_{3j}}{\partial x} |_{x=x_j} &= \frac{\partial A_{3j}}{\partial x} |_{x=x_j+b_0} = 0 \\
\frac{\partial A_{3j}}{\partial y} |_{y=H_1} &= 0
\end{align*}$$

(24)

Finally, the general solution of $A_{3j}$ can be obtained as follows.

$$A_{3j} = \sum_{k=1}^{\infty} c_k e^{\beta_k y} (e^{-\beta_k H_1} + e^{-\beta_k H_2}) \cos(\beta_k (x - x_j)) + d_0^i$$

(25)

Here, $\beta_k = k\pi/b_0$, $k$ is harmonic number, $d_0^i$ and $c_k^i$ are harmonic coefficients, which can be determined by the boundary conditions.

III. HARMONIC COEFFICIENT SOLUTION

The general solution of each subdomain is obtained based on harmonic coefficients derived using the Fourier series method according to the related boundary conditions. The following parameters $[A_{1n}, B_{1n}, C_{1n}, D_{1n}, A_{2n}, B_{2n}, C_{2n}, D_{2n}, \eta_m, \chi_k]$ are assumed for the sake of simplicity of computation and have no special meaning. The following functions $[f_i(n), f_i(n, k), g_i(n), g_i(n, k), \ldots]$ are shown in the appendix.

A. INTERFACE BETWEEN THE PRIMARY SIDE SLOT SUBDOMAIN AND THE AIR GAP SUBDOMAIN

The normal magnetic flux density of the $i$-th slot subdomain on the primary side and the air gap subdomain is equal at the interface between the two subdomains. This yields the following equations.

$$A_{1i}(x, -H_2) = A_{2i}(x, -H_2), \quad x_i \leq x \leq x_i + b_0$$

(26)

Substituting $y = H_2$ into (18) and (21) yields the following equations.

$$A_{1i}(x, -H_2) = D_0^i + \sum_{m=1}^{\infty} D_m^i \cos[\beta_m (x - x_i)]$$

(27)

Finally, the decomposition of (26) yields the following equations.

$$D_0^i = \frac{1}{b_0} \int_{x_i}^{x_i+b_0} A_{1i}(x, -H_2) dx$$

(28)

$$D_m^i = \frac{2}{b_0} \sum_{n=1}^{\infty} \int_{x_i}^{x_i+b_0} A_{1i}(x, -H_2) \cos[\beta_m (x - x_i)] dx$$

(29)
Substituting (25) and (29) into both (32) and (33) yields the following equations.

\[
D_0^i = \sum_{n=1}^{\infty} A_{1n} \frac{f_i(n)}{b_0} + \sum_{n=1}^{\infty} C_{1n} \frac{g_i(n)}{b_0} \tag{34}
\]

\[
D_m^i = \sum_{n=1}^{\infty} A_{1n} \frac{2f_i(n, m)}{b_0} + \sum_{n=1}^{\infty} C_{1n} \frac{2g_i(n, m)}{b_0} \tag{35}
\]

Here, the functions \(f_i(n), f_i(n, k), g_i(n)\) and \(g_i(n, k)\) are given in the appendix.

The tangential magnetic induction intensity of the \(i\)-th slot subdomain on the primary side and the air gap subdomain are equal at their interface. This yields the following equation.

\[
\frac{\partial A_i}{\partial y} \bigg|_{y=-H_2} = \begin{cases} 
\frac{\partial A_{1i}}{\partial y} \bigg|_{y=-H_2} , & (x_i \leq x \leq x_i + b_0) \\
0, & \text{(otherwise)}
\end{cases} \tag{36}
\]

In addition, the two magnetic fields of the \(i\)-th slot subdomain on the primary side and the air gap subdomain are connected piecewise at their interface, which yields a piecewise function. Therefore, Fourier decomposition yields the following equation.

\[
\frac{\partial A_2}{\partial y} \bigg|_{y=-H_2} = \sum_{n=1}^{\infty} A_{2n} \cos(\alpha_n x) + \sum_{n=1}^{\infty} C_{2n} \sin(\alpha_n x) \tag{37}
\]

\[
A_{2n} = \alpha_n \alpha_0 e^{-\alpha_0 H_2} - \alpha_n \beta_n e^{\alpha_0 H_2}
\]

\[
C_{2n} = \alpha_n \beta_n e^{-\alpha_0 H_2} - \alpha_n \beta_0 e^{\alpha_0 H_2}
\]

Then, \(A_{2n}\) and \(C_{2n}\) can be defined as follows.

\[
A_{2n} = \frac{1}{\tau} \int_0^{2\tau} \frac{\partial A_2}{\partial y} \bigg|_{y=-H_2} \cos(\alpha_n x) dx \tag{40}
\]

\[
C_{2n} = \frac{1}{\tau} \int_0^{2\tau} \frac{\partial A_2}{\partial y} \bigg|_{y=-H_2} \sin(\alpha_n x) dx \tag{41}
\]

This yields the following equation.

\[
\frac{\partial A_{1i}}{\partial y} \bigg|_{y=-H_2} = \sum_{m=1}^{\infty} d_m^i \eta_m \cos[\beta_m(x - x_i)] + \mu_0 J_I H_2 - \mu_0 J_I H_1
\]

\[
\eta_m = \beta_m(e^{-\beta_0 H_2} + 2\beta_0 H_1 - e^{\beta_0 H_2}) \tag{42}
\]

Substituting (36) and (42) into (40) and (41) yields the following equations.

\[
A_{2n} = \sum_{i=1}^{Q} \mu_0 J_I(H_2 - H_1) \frac{f_i(n)}{\tau} + \sum_{i=1}^{Q} \sum_{m=1}^{\infty} d_m^i \eta_m \frac{f_i(n, m)}{\tau} \tag{44}
\]

\[
C_{2n} = \sum_{i=1}^{Q} \mu_0 J_I(H_2 - H_1) \frac{g_i(n)}{\tau} + \sum_{i=1}^{Q} \sum_{m=1}^{\infty} d_m^i \eta_m \frac{g_i(n, m)}{\tau} \tag{45}
\]

\section*{B. INTERFACE BETWEEN THE SECONDARY SIDE SLOT SUBDOMAIN AND THE AIR GAP SUBDOMAIN}

The normal magnetic flux density of the \(j\)-th slot subdomain on the secondary side subdomain and the air gap subdomain is equal at the interface between the two subdomains. This yields the following equations.

\[
A_{3j}(x, H_2) = A_{2j}(x, H_2), \quad x_j \leq x \leq x_j + b_0 \tag{46}
\]

Substituting \(y = H_2\) into (21) and (24) yields the following equations.

\[
A_{3j}(x, H_2) = C_0^j + \sum_{k=1}^{\infty} C_k^j \cos[\beta_k(x - x_j)] \tag{47}
\]

\[
C_0^j = C_0 \tag{48}
\]

\[
C_k^j = C_k \cos[\beta_k H_2(e^{-2\beta_k H_1} + e^{-2\beta_k H_2})] \tag{49}
\]

\[
A_{2j}(x, H_2) = \sum_{n=1}^{\infty} B_{1n} \cos(\alpha_n x) + \sum_{n=1}^{\infty} D_{1n} \sin(\alpha_n x) \tag{50}
\]

\[
B_{1n} = a_n e^{\alpha_0 H_2} + b_n e^{-\alpha_0 H_2} \tag{51}
\]

\[
D_{1n} = c_n e^{\alpha_0 H_2} + d_n e^{-\alpha_0 H_2} \tag{52}
\]

Fourier decomposition of (47) yields the following equations.

\[
C_0^j = \frac{1}{b_0} \int_{x_j}^{x_j + b_0} A_{3j}(x, H_2) dx \tag{53}
\]

\[
C_k^j = \frac{2}{b_0} \int_{x_j}^{x_j + b_0} A_{3j}(x, H_2) \cos[\beta_k(x - x_j)] dx \tag{54}
\]

Substituting both (46) and (50) into (53) and (54) yields the following equations.

\[
C_0^j = \sum_{n=1}^{\infty} B_{1n} \frac{f_j(n)}{b_0} + \sum_{n=1}^{\infty} D_{1n} \frac{g_j(n)}{b_0} \tag{55}
\]

\[
C_k^j = \sum_{n=1}^{\infty} B_{1n} \frac{2f_j(n, k)}{b_0} + \sum_{n=1}^{\infty} D_{1n} \frac{2g_j(n, k)}{b_0} \tag{56}
\]

The tangential magnetic flux density of the \(j\)-th slot subdomain of the secondary side and the air gap subdomain is equal at their interface. This yields the following equation.

\[
\frac{\partial A_2}{\partial y} \bigg|_{y=H_2} = \begin{cases} 
\frac{\partial A_3}{\partial y} \bigg|_{y=H_2} , & (x_j \leq x \leq x_j + b_0) \\
0, & \text{(otherwise)}
\end{cases} \tag{57}
\]

In addition, the two magnetic fields of the \(j\)-th slot subdomain on the secondary side and the air gap subdomain are connected piecewise at their interface, which yields a piecewise function. Therefore, Fourier decomposition yields the following equation.

\[
\frac{\partial A_2}{\partial y} \bigg|_{y=-H_2} = \sum_{n=1}^{\infty} B_{2n} \cos(\alpha_n x) + \sum_{n=1}^{\infty} D_{2n} \sin(\alpha_n x) \tag{58}
\]

\[
B_{2n} = a_n \alpha_0 e^{\alpha_0 H_2} - \alpha_n b_n e^{-\alpha_0 H_2} \tag{59}
\]

\[
D_{2n} = \alpha_n c_n e^{\alpha_0 H_2} - \alpha_n d_n e^{-\alpha_0 H_2} \tag{60}
\]
This yields the following equation.
\[
\frac{\partial A_{y}}{\partial y} \bigg|_{y=H_2} = \sum_{k=1}^{\infty} c_k \chi_k \cos[\beta_k(x - x_j)]
\]
(61)

\[
\chi_k = \beta_k e^{\beta_k H_2} (e^{-2\beta_k H_1} - e^{-2\beta_k H_2})
\]
(62)

Then, \(B_{2n}\) and \(D_{2n}\) can be defined as follows.
\[
B_{2n} = \frac{1}{\tau} \int_{0}^{2\tau} \frac{\partial A_2}{\partial y} \bigg|_{y=H_2} \cdot \cos(\alpha_n x) dx
\]
(63)

\[
D_{2n} = \frac{1}{\tau} \int_{0}^{2\tau} \frac{\partial A_2}{\partial y} \bigg|_{y=H_2} \cdot \cos(\alpha_n x) dx
\]
(64)

Substituting both (57) and (61) into both (63) and (64) yields the following equations.
\[
B_{2n} = \sum_{j=1}^{Q} \sum_{k=1}^{\infty} c_k \chi_k \frac{\int_{0}^{2\tau} f_j(n, k) dx}{\tau}
\]
(65)

\[
D_{2n} = \sum_{j=1}^{Q} \sum_{k=1}^{\infty} c_k \chi_k \frac{\int_{0}^{2\tau} g_j(n, k) dx}{\tau}
\]
(66)

Equations (4)(5)(27)(28)(34)(39)(44)(45)(48)(49)(51)(52)(55)(56)(59)(60)(64)(66) can be simplified into 8 equations, and the solutions of these equations yields the full set of harmonic coefficients. Substituting the appropriate harmonic coefficients into (18) (21) and (24) then yields the magnetic vector potential in each of the subdomains. Finally, the magnetic flux density in the normal direction (\(B_x\)) and tangential direction (\(B_y\)) are expressed in the 2D Cartesian coordinate system as follows.
\[
\begin{align*}
B_x &= -\frac{\partial A}{\partial x} \\
B_y &= \frac{\partial A}{\partial y}
\end{align*}
\]
(67)

IV. INDUCTANCE OF THE PRIMARY SIDE AND POTENTIALS OF THE SECONDARY SIDE

Under the condition of the primary windings connected to the current sources, the magnetic flux linkage of a coil can be obtained as the difference between the average magnetic vector potential between the left and right sides of the coil. Therefore, the self-inductance and mutual inductance of the phase winding can be solved by the flux method when a phase coil is connected to the DC current source [16].

The \(i\)-th coil on the primary side refers to the coil wound around the \(i\)-th cog of the primary side. The magnetic flux linkage of \(i\)-th coil is given as follows:

\[
\psi_i = \psi_{i1} - \psi_{i2}
\]
(68)

Here, \(\psi_{i1}\) is the average magnetic vector potential between the left side of the coil. \(\psi_{i2}\) is the average magnetic vector potential on the right side of the coil. The following definitions are applied.

\[
\psi_{i1} = L_b \frac{N_1}{S} \int_{x_i}^{x_i+b_0/2} \int_{-H_2}^{-H_1} A_{1(i+1)}(x, y) dxdy
\]
(70)

Here, \(L_b\) is the core width, \(N_1\) is the number of secondary coil turns, and \(S\) is the area associated with the side of each coil, \(S = b_0 h_c\). \(A_{1(i+1)}\) is the magnetic vector potential of the \((i + 1)\)-th slot subdomain on the primary side.

This yields the following equations.
\[
\psi_a = \sum_{i=1}^{N_p} \psi_i
\]
(71)

\[
\psi_c = \sum_{i=N_p+1}^{2N_p} \psi_i
\]
(72)

\[
\psi_b = \sum_{i=2N_p+1}^{3N_p} \psi_i
\]
(73)

Here, \(N_p\) is the number of series coils per phase on the primary side. \(\psi_{\gamma}\), where \(\gamma = a, b, c\), is the magnetic flux linkage of phase \(\gamma\).

The three-phase currents, where \(I_a = \text{constant}\) and \(I_b = I_c = 0\), are fed into the primary side. The ESM-based model deduced above can be applied to obtain the self-inductance \(L_{aa}\) of the A-phase winding and the respective mutual inductances \(M_{ab}\) and \(M_{ac}\) of the A-phase winding with the B and C phase windings as follows.

\[
\begin{align*}
L_{aa} &= \frac{\psi_a}{I_a} \\
M_{ab} &= \frac{\psi_b}{I_a} \\
M_{ac} &= \frac{\psi_c}{I_a}
\end{align*}
\]
(74)

In addition, we define the currents of the three-phase alternating current source connected to the primary side as follows.

\[
\begin{align*}
I_a &= \sqrt{2} I_m \cos(\omega t) \\
I_b &= \sqrt{2} I_m \cos(\omega t - \frac{\pi}{3}) \\
I_c &= \sqrt{2} I_m \cos(\omega t + \frac{\pi}{3})
\end{align*}
\]
(75)

Here, \(I_m\) is the magnitude of the current, \(\omega\) is the angular frequency, and \(t\) is time. The secondary side provides a 12-phase output, and the magnetic flux distribution over a single period of the LPST model is obtained using (68)(69)(70)(73). After the magnetic flux of a given phase winding is obtained, the open-circuit potential can be obtained by taking the derivative of the magnetic flux:

\[
E_\phi = -\frac{d\psi_\phi}{dt}
\]
(76)

V. FINITE-ELEMENT METHOD VALIDATION

The main parameters of the FW-LPST which can be used for multi-rectification are listed in Table 1. There are three-phase
windings on the primary side, and there are twelve-phase windings on the secondary side.

The magnetic field distribution, the inductances of primary side, and the potentials of the secondary side of the FW-LPST are calculated from the FEM-based model, and the FEM results are compared with those obtained from the proposed ESM-based model.

A. MAGNETIC INDUCTION
When a three-phase current source \( I_b = I_c = -0.5I_a \) is connected to the primary side, the output current of the secondary side is 0. The maximum harmonic number of the ESM is \( K_m = N_m = M_m = 150 \). The distribution of magnetic flux of the LPST obtained by FEM under an open-circuit condition is presented in Figure 7. Figure 7 shows that most of the magnetic lines of force pass through the iron cores and rarely pass through the slots. In addition, only the magnetic circuits of two adjacent pairs of cogs are coupled. Because of the special magnetic circuit distribution, the output voltages of the FW-LPSTs have good sinusoidal and symmetric properties.

The waveforms of \( B_y \) and \( B_x \) obtained using the FEM and the proposed ESM-based model at the centerline of the airgap \( (y = 0\text{mm}) \) and the centerline of the 2nd slot in the secondary side \( (y = 6\text{mm}) \) are presented in Figures 8–11. As it can be seen, the predicted flux density obtained by the proposed ESM-based model almost completely matches the results obtained by the FEM. The overall error is less than 5%. Especially in the harmonic analysis diagram, it can be found that there is little difference in the amplitude of each harmonic obtained by the two methods.

![FIGURE 7. Magnetic field distribution of the FW-LPST obtained by the FEM under an open-circuit condition.](image)

Figure 8(a) indicates normal flux density waveforms in the airgap at \( y = 0\text{mm} \), the amplitude of the two waveforms is basically the same. Figure 8(b) indicates that there is little difference in the amplitude of each harmonic obtained by the two methods. Due to the influence of the cogging effect, the harmonic content of the permeability of the air gap increases. In the regions around the slots, the air gap magnetic density obviously decreases.

B. INDUCTANCE OF PRIMARY SIDE WINDINGS
The self-inductances and mutual-inductances of the A, B, and C phases obtained on the primary side of the FW-LPST by the ESM and the FEM simulations are listed in Tables 2 and 3, respectively, and the errors between the FEM and ESM results are presented in Table 4.

It can be found by observing Table 3 that the self-inductances of the A, B, and C phases are almost equal, and the mutual inductances of the A, B, and C phases are very small, because the magnetic circuits of the A, B, and C phases on the primary side are not coupled. The ESM-based model cannot reflect the asymmetry of the inductor owing to the use of periodic extension, which ignores the end effect. Because the air gap of this type of transformer is very small, and the end effect is related to the air gap, the influence degree of the end effect is small. Accordingly, we note from Table 4 that the errors associated with the ESM predictions are very small. This also reflects that the model based on the ESM is very accurate.

C. OPEN-CIRCUIT POTENTIALS OF THE SECONDARY SIDE
The potentials of the a1, b1 and c1 phases of the secondary side arising from a three-phase current source connected to

| Symbol | Definition | Value |
|--------|------------|-------|
| \( N_1 \) | Number of primary coil turns | 166 |
| \( N_2 \) | Number of secondary coil turns | 198 |
| \( L_m \) | FW-LPST length (mm) | 448.8 |
| \( L_b \) | FW-LPST width (mm) | 100 |
| \( P \) | Power (KVA) | 1000 |
| \( g \) | Air gap width (mm) | 0.3 |
| \( h_s \) | Winding thickness (mm) | 12 |
| \( b_1 \) | Slot width (mm) | 22.4 |
| \( b_s \) | Slot pitch (mm) | 37.4 |
| \( p \) | Number of pole pairs | 1 |
| \( \tau \) | Polar distance (mm) | 224.4 |
| \( Q \) | Number of slots | 12 |
| \( N_{sp} \) | Number of series coils per phase on primary side | 4 |
TABLE 2. Self-inductance and mutual-inductance of primary side based on ESM.

| ESM (mH) | A    | B    | C    |
|----------|------|------|------|
| A        | 367.025 | 0   | 0    |
| B        | 0    | 367.025 | 0    |
| C        | 0    | 0    | 367.025 |

TABLE 3. Self-inductance and mutual-inductance of primary side based on FEM.

| FEM (mH) | A    | B    | C    |
|----------|------|------|------|
| A        | 370.041 | 0.02388 | 0.42246 |
| B        | 0.02388 | 369.842 | 0.42163 |
| C        | 0.42246 | 0.42163 | 370.832 |

TABLE 4. Absolute errors between the results presented in tables 2 and 3.

| FEM (mH) | A    | B    | C    |
|----------|------|------|------|
| A        | -3.016 | -0.02388 | -0.42246 |
| B        | -0.02388 | -2.817 | -0.42163 |
| C        | -0.42246 | -0.42163 | -3.807 |

FIGURE 8. FEM and ESM predicted normal flux density waveforms in the airgap at $y = 0$mm.

The primary side are obtained using the FEM and the proposed ESM-based model. The results are presented in Figure 12. The phase difference of the three phases is $120^\circ$. Analysis of the results indicates that the error of the waveforms obtained by the ESM-based model is less than 5%. This further confirms that the overall accuracy of the proposed ESM-based model meets the requirements for optimized FW-LPST design. In addition, the FEM takes about 1.5 hours, but the ESM-based model takes about ten seconds. The ESM-based model reduces the computation time greatly.
VI. CONCLUSION

In this paper, a fractional-slot windings linear phase-shifting transformer was proposed for the first time, which solved the problem that the windings of linear PSTs were difficult to process. The following conclusions could be drawn.

1) The proposed ESM-based model of the FW-LPST could solve the complex magnetic field analysis problems caused by the cogging effects. The influences of the current density and the parameters of the ontology on the magnetic field distribution in different mediums were considered in this analytical model, and the governing equations of the air gap and the slot subdomains were established.

2) The analytical expressions of magnetic vector potential and magnetic flux density of the ESM-based model of the FW-LPST were derived, and the magnetic field distribution in each subdomain was given. The FEM results showed that the analytical results were of high accuracy.

3) The inductances of the primary side and the induced potentials of the secondary side were obtained using the magnetic flux linkage method. The accuracy of the proposed analytical model was further verified by comparison of the results obtained by the FEM.

4) The results presented in Subsections V.A, V.B, and V.C demonstrated that the benefits of the proposed ESM-based model for conducting optimized design of FW-LPSTs far outweighed any small disadvantages regarding accuracy because the approach enabled very rapid calculation of FW-LPST performance for any arbitrary changes in the design parameters, and therefore greatly facilitated the design process.

APPENDIX

In order to give the detailed solution process, the equations (27)(28)(34)(35)(38)(39)(44)(45)(48)(49)(51)(52)(55)(56) (59)(60)(64)(66) in this paper are written in matrix form. Then the matrix is coded and solved by mathematical software.

\[ f_i(n) = \int_{x_i}^{x_i+b_0} \cos(\alpha_n x)dx \]
\[ = \frac{1}{\alpha_n}[\sin(\alpha_n x_i + \alpha_n b_0) - \sin(\alpha_n x_i)] \quad (A1) \]
\[ g_i(n) = \int_{x_i}^{x_i+b_0} \sin(\alpha_n x)dx \]
\[ = \frac{1}{\alpha_n}[\cos(\alpha_n x_i) - \cos(\alpha_n x_i + \alpha_n b_0)] \quad (A2) \]

In addition, the condition \( \beta_k \neq \alpha_n \) yields the following definitions for the \( i \)-th slot of the primary side subdomain.

\[ f_i(n, m) = \int_{x_i}^{x_i+b_0} \cos(\alpha_n x) \cos(\beta_m(x-x_i))dx \]
\[ = \frac{\alpha_n}{\beta_m^2 - \alpha_n^2}[\sin(\alpha_n x_i) - (-1)^m \sin(\alpha_n x_i + \alpha_n b_0)] \quad (A3) \]
\[ g_i(n, m) = \int_{x_i}^{x_i+b_0} \sin(\alpha_n x) \cos(\beta_m(x-x_i))dx \]
\[ = \frac{-\alpha_n}{\beta_m^2 - \alpha_n^2}[\cos(\alpha_n x_i) - (-1)^m \cos(\alpha_n x_i + \alpha_n b_0)] \quad (A4) \]

The condition \( \beta_k \neq \alpha_n \) yields the following definitions for the \( i \)-th slot of the primary side subdomain.

\[ f_i(n, m) = \int_{x_i}^{x_i+b_0} \cos(\alpha_n x) \cos(\beta_m(x-x_i))dx \]
\[ = \frac{1}{4\beta_m}[\sin(\alpha_n x_i + 2\alpha_n b_0) - \sin(\alpha_n x_i)] + \frac{b_0}{2}\cos(\alpha_n x_i) \quad (A5) \]
\[ g_i(n, m) = \int_{x_i}^{x_i+b_0} \sin(\alpha_n x) \cos(\beta_m(x-x_i))dx \]
\[ = \frac{1}{4\beta_m}[\cos(\alpha_n x_i) - \cos(\alpha_n x_i + 2\alpha_n b_0)] + \frac{b_0}{2}\sin(\alpha_n x_i) \quad (A6) \]
\[ f_j(n) = \int_{x_j}^{x_j+b_0} \cos(\alpha_n x)dx \]
\[ = \frac{1}{\alpha_n}[\sin(\alpha_n x_j + \alpha_n b_0) - \sin(\alpha_n x_j)] \quad (A7) \]
\[ g_j(n) = \int_{0}^{x_j+b_0} \sin(\alpha_n x) \, dx \]
\[ = \frac{1}{\alpha_n} [\cos(\alpha_n x_j) - \cos(\alpha_n x_j + \alpha_n b_0)] \quad (A8) \]

The condition \( \beta_k \neq \alpha_n \) yields the following definitions for the \( j \)-th slot of the secondary side subdomain.

\[ f_j(n, k) = \int_{x_j}^{x_j+b_0} \cos(\alpha_n x) \cos(\beta_k (x - x_j)) \, dx \]
\[ = \frac{1}{\beta_k - \alpha_n}[\sin(\alpha_n x_j) - (-1)^k \sin(\alpha_n x_j + \alpha_n b_0)] \quad (A9) \]

\[ g_j(n, k) = \int_{x_j}^{x_j+b_0} \sin(\alpha_n x) \cos(\beta_k (x - x_j)) \, dx \]
\[ = \frac{-\alpha_n}{\beta_k - \alpha_n}[\cos(\alpha_n x_j) - (-1)^k \cos(\alpha_n x_j + \alpha_n b_0)] \quad (A10) \]

Finally, the condition \( \beta_k \neq \alpha_n \) yields the following definitions for the \( j \)-th slot of the secondary side subdomain.

\[ f_j(n, k) = \int_{x_j}^{x_j+b_0} \cos(\alpha_n x) \cos(\beta_k (x - x_j)) \, dx \]
\[ = \frac{1}{4\beta_k}[\sin(\alpha_n x_j + 2\alpha_n b_0) - \sin(\alpha_n x_j)] \]
\[ + \frac{b_0}{2} \cos(\alpha_n x_j) \quad (A11) \]

\[ g_j(n, k) = \int_{x_j}^{x_j+b_0} \sin(\alpha_n x) \cos(\beta_k (x - x_j)) \, dx \]
\[ = \frac{1}{4\beta_k}[\cos(\alpha_n x_j) - \cos(\alpha_n x_j + 2\alpha_n b_0)] \]
\[ + \frac{b_0}{2} \sin(\alpha_n x_j) \quad (A12) \]

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