NON-HARMONIC CONES ARE SETS OF INJECTIVITY FOR THE TWISTED SPHERICAL MEANS ON C^n

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Abstract. In this article, we prove that a complex cone is a set of injectivity for the twisted spherical means for the class of all continuous functions on C^n as long as it does not completely lay on the level surface of any bi-graded homogeneous harmonic polynomial on C^n. Further, we produce examples of such level surfaces.

1. Introduction

Let \( \mu_r \) be the normalized surface measure on the sphere \( S_r(x) \). Suppose \( F \subseteq L^1_{\text{loc}}(\mathbb{R}^n) \). We say that \( S \subseteq \mathbb{R}^n \) is a set of injectivity for the spherical means for \( F \) if for \( f \in F \) with \( f \ast \mu_r(x) = 0 \), \( \forall r > 0 \) and \( \forall x \in S \), implies \( f = 0 \).

In an interesting article by Zalcman et al. [9], it has been shown that a real cone \( C \) in \( \mathbb{R}^d \) (\( d \geq 2 \)) is a set of injectivity for the spherical means for the class of all continuous functions on \( \mathbb{R}^d \) if and only if \( C \) does not lay on the level surface of any homogeneous harmonic polynomial on \( \mathbb{R}^d \). We look at this problem in the Heisenberg group set up. In particular, for the twisted spherical mean on C^n. We prove an analogous result for the twisted spherical means (TSM) for the class of all continuous functions on C^n.

Let \( \mu_r \) be the normalized surface measure on the sphere \( S_r(z) = \{ w \in \mathbb{C}^n : |z - w| = r \} \). For a locally integrable function \( f \) on \( \mathbb{C}^n \), we define its twisted spherical mean on the sphere \( S_r(z) \) by

\[
    f \times \mu_r(z) = \int_{|w|=r} f(z - w)e^{2\text{Im}(z.w)} d\mu_r(w).
\]

Let \( F \subseteq L^1_{\text{loc}}(\mathbb{C}^n) \). We say \( S \subseteq \mathbb{C}^n \) is a set of injectivity for twisted spherical means for \( F \) if for \( f \in F \) with \( f \times \mu_r(z) = 0 \), \( \forall r > 0 \) and \( \forall z \in S \), implies \( f = 0 \) a.e. The results on set of injectivity differ in the choice of sets and the class of functions considered. We would like to refer to [8, 14, 18], for some results on the sets of injectivity for the TSM.

A set \( C \) in \( \mathbb{C}^n \) (\( n \geq 2 \)) which satisfy \( \lambda C \subseteq C \), for all \( \lambda \in \mathbb{C} \) is called a complex cone. In this article we have proved the following result. Let \( f \) be a continuous function on \( \mathbb{C}^n \). Suppose \( f \times \mu_r(z) = 0 \), for all \( r > 0 \) and \( z \in C \).

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Then $f \equiv 0$ as as long as $C$ does not lay on the level surface of any bi-graded homogeneous harmonic polynomial on $\mathbb{C}^n$. We will call such cones as \textbf{non-harmonic} cones. A proof of this result for functions in $L^p(\mathbb{C}^n)$, $1 \leq p \leq 2$, has been worked out using real analytic expansion of the spectral projections, (see [20]). However, this proof does not stand for the class of all continuous functions on $\mathbb{C}^n$.

We would like to assert that our main result of this article is in sharp contrast (in terms of topological dimension) with the Euclidean results about the sets of injectivity for the spherical means. Since a non-trivial complex cone in $\mathbb{C}^n$ ($n \geq 2$) can have topological dimension at most $2n - 2$, therefore, it follows that a $(2n - 2)-$ dimensional entity is set of injectivity for the TSM on $\mathbb{C}^n$. All though, for the Euclidean set up in $\mathbb{R}^{2d}$, the least topological dimension required (in general) for a set to be set of injectivity for the spherical means is $2d - 1$. For instance, the boundary of a bounded domain in $\mathbb{R}^{2d}$ is a set of injectivity for the spherical means for $L^p(\mathbb{R}^{2d})$ with $1 \leq p \leq \frac{2d}{2d-1}$. (See [2]). The main result of this article is also distinct to the known results in terms of topological dimension of sets of injectivity for the TSM. The topological dimension of sets of injectivity for the TSM on $\mathbb{C}^n$ for the known results is $2n - 1$. For example, boundary of bounded domain and $(\mathbb{R} \cup i\mathbb{R}) \times \mathbb{C}^{n-1}$ are sets of injectivity for the TSM in $\mathbb{C}^n$ having topological dimension $2n - 1$. For more details see, [8, 14, 18].

On the basis of our result that any non-harmonic complex cone is a set of injectivity for the TSM, we can pose the following interesting question. A non-trivial complex submanifold $S$ of $\mathbb{C}^n$ ($n \geq 2$) is a set injectivity for the TSM if and only if $S$ does not contained in the zero set of any bi-graded homogeneous harmonic polynomial. We leave this question open for the time being.

In order to complete the argument of this result, we show that the zero set of polynomial $H(z) = az_1\bar{z}_2 + |z|^2$, where $a \neq 0$ and $z \in \mathbb{C}^n$ is a complex cone which does not contained in the zero set of any bi-graded homogeneous harmonic polynomial. The proof of the fact that $H^{-1}(0)$ is a non-harmonic cone is one of the most difficult pat of this article. We call a complex cone to be \textit{non-harmonic} if it does not contained in the zero set of any bi-graded homogeneous harmonic polynomial on $\mathbb{C}^n$ of the form

\begin{equation}
(1.1) \quad P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta,
\end{equation}

where $p, q \in \mathbb{Z}_+$, the set of non-negative integers. Let $P_{p,q}$ denotes the space of all bi-graded homogeneous polynomials defined by (1.1). For $p \geq 1$, we observe that each of the diagonal space $P_{p,p}$ has at least one member which corresponds to a non-harmonic complex cone. However, for $p, q \geq 1$, it is interesting to know that whether the non-diagonal space $P_{p,q}$ has at least one member which corresponds to a non-harmonic complex cone.
The question of sets of injectivity for the spherical means has been taken up by many authors in recent past. In an article by Agranovsky et al. [2], it has been proved that the boundary of any bounded domain in $\mathbb{R}^d$ ($d \geq 2$) is set of injectivity for the spherical means on $L^p(\mathbb{R}^d)$, with $1 \leq p \leq \frac{2d}{d-1}$.

In general, characterization of sets of injectivity for spherical mean is a long standing problem in integral geometry. In fact, an intuition about sets of injectivity is that these sets are essentially those sets which are seating outside zero set of a homogeneous harmonic polynomials union an algebraic variety of codimension at most $d - 2$. In 1996, Agranovsky and Quinto have given a characterization of sets of injectivity for spherical mean for compactly supported function on $\mathbb{R}^2$ in terms of zero set of a homogeneous harmonic polynomials union a finite set. They have conjectured this question in higher dimension in similar way, (see [7]). Following is their result.

**Theorem 1.1.** [7] A set $S \subset \mathbb{R}^2$ is a set of injectivity for the spherical means for $C_c(\mathbb{R}^2)$ if and only if $S \notin \omega(\Sigma_N) \cup F$, where $\omega$ is a rigid motion of $\mathbb{R}^2$, $\Sigma_N = \bigcup_{l=0}^{N-1} \{te^{\frac{i\pi l}{N}} : t \in \mathbb{R}\}$ is a Coxeter system of $N$ lines and $F$ is a finite set in $\mathbb{R}^2$.

Since, a homogeneous harmonic polynomial can be expressed as the product of homogeneous polynomials of degree 1. Therefore, $\Sigma_N$ in Theorem 1.1 is nothing but zero set of some homogeneous harmonic polynomial on $\mathbb{R}^2$. Further studies on sets of injectivity for spherical mean and related problems have been carried out in [3, 4, 7, 13, 15, 24]. Later, the question of sets of injectivity for the TSM has been considered by Agranovsky and Rawat, (see [8]). They have shown that the boundary of any bounded domain in $\mathbb{C}^n$ is set of injectivity for the TSM for a certain class of functions in $L^p(\mathbb{C}^n)$. For more histories and further work on this question, we refer [5, 6, 14, 18, 19, 20, 21, 25, 26, 27, 28, 29].

A continuous function $f$ on $\mathbb{R}^d$ can be decomposed in terms of spherical harmonics as

$$f(x) = \sum_{k=0}^{\infty} a_{k,j}(\rho)Y_{k,j}(\omega),$$

where $x = \rho \omega$, $\rho = |x|$, $\omega \in S^{n-1}$ and $\{Y_{kj} : 1, 2, \ldots, d_k\}$ is an orthonormal basis for the space $V_k$ of the homogeneous harmonic polynomials of degree $k$, restricted to the unit sphere $S^{n-1}$ with the series in the right-hand side converges locally uniformly to $f$. Let $H_k = \{ P_k : P_k(x) = \rho^k Y_k(\omega), Y_k \in V_k \}$. The space $H_k$ is called the space of solid spherical harmonics. For more details, see [22].

Let $\nu_r$ be the normalized surface measure on the sphere $S_r(x)$ in $\mathbb{R}^d$. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we define its mean over the sphere $S_r(x)$ by

$$f * \nu_r(x) = \int_{S_r(x)} f(y) d\mu_r(y).$$
A set $C$ in $\mathbb{R}^d$ ($d \geq 2$) which satisfies $\lambda C \subseteq C$, for all $\lambda \in \mathbb{R}$ is called a real cone. Zalcman et al. [9] have proved the following result.

**Theorem 1.2.** [9] Let $C$ be a real cone in $\mathbb{R}^d$ ($d \geq 2$). Let $f$ be a continuous function on $\mathbb{R}^d$. Suppose $f \times \nu_r(x) = 0$, for all $r > 0$ and $x \in C$. Then $f \equiv 0$ if and only if $C \notin P^{-1}(0)$, for any $P \in H_k$ and for all $k \in \mathbb{Z}_+$.

An example of such a cone had been produced by Armitage, (see [1]). Let $0 < a < 1$ and $C_k^a(x)$ denotes Gegenbauer polynomial of degree $k$ and order $\lambda$. Then $K_a = \{ x \in \mathbb{R}^d : |x_1|^2 = a^2|x|^2 \}$ is a non-harmonic cone if and only if $D^mC_k^{d-2}(a) \neq 0$, for all $0 \leq m \leq k - 2$, where $D^m$ denotes the $m$th derivative.

We would like to mention that the proof of Theorem 1.2 is being deduced by concentrating the problem to the unit sphere $S$. Zalcman et al. [9] have proved the following result.

**Lemma 1.3.** [9] Suppose $f \in C(S^{d-1})$ have spherical harmonic expansion $\sum_{k=0}^{\infty} Y_k$. Then $f(\omega, t) = 0$, $\forall t \in (-1, 1)$ if and only if $Y_k(\omega) = 0$, $\forall k \in \mathbb{Z}_+$. In particular, if $f(\omega, t) = 0$, $\forall t \in (-1, 1)$ then $f \equiv 0$ if and only if $\omega$ is not contained in the zero set of any homogeneous harmonic polynomial.

The results on the sphere $S^{d-1}$ for geodesic mean need not be same as results for spherical mean on $\mathbb{R}^d$. For instance, Theorem 1.4 says that sets of injectivity for spherical mean are essentially (up to a translation and rotation) sitting out side the zero set of a homogeneous harmonic polynomials union a finite set.

### 2. Preliminaries

We define the twisted convolution which arises in the study of group convolution on Heisenberg group. The group $\mathbb{H}^n$, as a manifold, is $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, t)(w, s) = \left( z + w, t + s + \frac{1}{2} \text{Im}(z \bar{w}) \right), \quad z, w \in \mathbb{C}^n \text{ and } t, s \in \mathbb{R}.$$ 

Let $\mu_s$ be the normalized surface measure on the sphere $\{(z, 0) : |z| = s\} \subset \mathbb{H}^n$.

The spherical means of a function $f$ in $L^1(\mathbb{H}^n)$ are defined by

$$f * \mu_s(z, t) = \int_{|w|=s} f((z, t)(-w, 0)) d\mu_s(w). \quad (2.1)$$
Thus the spherical means can be thought of as convolution operators. An important technique in many problem on $\mathbb{H}^n$ is to take partial Fourier transform in the $t$-variable to reduce matters to $\mathbb{C}^n$. Let

$$f^\lambda(z) = \int_{\mathbb{R}} f(z, t)e^{i\lambda t}dt$$

be the inverse Fourier transform of $f$ in the $t$-variable. Then a simple calculation shows that

$$(f * \mu_s)^\lambda(z) = \int_{-\infty}^{\infty} f * \mu_s(z, t)e^{i\lambda t}dt$$

$$= \int_{|w|=s} f^\lambda(z - w)e^{\frac{i\lambda}{2}1_{\mathbb{R}}(z, w)}d\mu_s(w)$$

$$= f^\lambda * \mu_s(z),$$

where $\mu_s$ is now being thought of as normalized surface measure on the sphere $S_s(o) = \{z \in \mathbb{C}^n : |z| = s\}$ in $\mathbb{C}^n$. Thus the spherical mean $f * \mu_s$ on the Heisenberg group can be studied using the $\lambda$-twisted spherical mean $f^\lambda * \mu_s$ on $\mathbb{C}^n$. For $\lambda \neq 0$, by scaling argument, it is enough to study the twisted convolution for the case $\lambda = 1$.

We need the following basic facts from the theory of bigraded spherical harmonics, (see [11, 12, 23] for details). Let $K = U(n)$ and $M = U(n - 1)$. Then, $S^{2n-1} \cong K/M$ under the map $kM \rightarrow k.e_n$, $k \in U(n)$ and $e_n = (0, \ldots, 1) \in \mathbb{C}^n$. Let $\hat{K}_M$ denote the set of all equivalence classes of irreducible unitary representations of $K$ which have a nonzero $M$-fixed vector.

For a $\delta \in \hat{K}_M$, which is realized on $V_\delta$, let $\{e_1, \ldots, e_{d(\delta)}\}$ be an orthonormal basis of $V_\delta$ with $e_1$ as the $M$-fixed vector. Let $t^\delta_j(k) = \langle e_j, \delta(k)e_j \rangle$, $k \in K$. By Peter-Weyl theorem, it follows that $\{\sqrt{d(\delta)}t^\delta_j : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M\}$ form an orthonormal basis for $L^2(K/M)$ (see [23], p.14 for details). Define $Y^\delta_j(\omega) = \sqrt{d(\delta)}t^\delta_j(k)$, where $\omega = k.e_n \in S^{2n-1}, k \in K$. Then $\{Y^\delta_j : 1 \leq j \leq d(\delta), \delta \in \hat{K}_M, \}$ becomes an orthonormal basis for $L^2(S^{2n-1})$.

For our purpose, we need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way. See [17], p.253, for details. For $p, q \in \mathbb{Z}_+$, let $H_{p,q} = \{P \in P_{p,q} : \Delta P = 0\}$. The group $K$ acts on $H_{p,q}$ in a natural way. It is easy to see that the space $H_{p,q}$ is $K$-invariant. Let $\pi_{p,q}$ denote the corresponding representation of $K$ on $H_{p,q}$. Then representations in $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_{p,q} : p, q \in \mathbb{Z}_+\}$.

Define the bi-graded spherical harmonic by $Y^{p,q}_j(\omega) = \sqrt{d(p,q)}t^{p,q}_j(k)$. Then $\{Y^{p,q}_j : 1 \leq j \leq d(p, q), p, q \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$. Therefore, a continuous function $f$ on $S^{2n-1}$ can be expressed in terms of
bi-graded spherical harmonics as

\[(2.2)\quad f(\omega) = \sum_{p,q \geq 0} \sum_{j=1}^{d(p,q)} Y_j^{p,q}(\omega).\]

For each \(k\), the space \(V_k\) is invariant under the action of \(SO(d)\). When \(d = 2n\), it is invariant under the the action of the unitary group \(U(n)\) as well, and under this action of \(U(n)\) the space \(V_k\) breaks up into an orthogonal direct sum of \(H_{p,q}\)'s where \(p + q = k\). (See [17], p. 255).

**Lemma 2.1.** [17]. Let \(\omega \in S^{2n-1}\) and \(Y_k \in V_k\). Then

\[Y_k(\omega) = \sum_{p+q=k} Y_{p,q}(\omega), \quad \text{where } Y_{p,q} \in H_{p,q}.\]

As a consequence to Lemma 2.1, we prove the following lemma which is required in the proof of our main result.

**Lemma 2.2.** Let \(\Omega = \{z : |z| \in C, z \neq 0\}\). Then \(Y_k = 0\) on \(\Omega\) if and only if \(Y_{p,q} = 0\) on \(\Omega\), \(\forall p, q \in \mathbb{Z}_+\) such that \(p + q = k\).

**Proof.** Let \(\omega \in \Omega\) and \(Y_k(\omega) = 0\). Then by Lemma 2.1, we have

\[\sum_{p+q=k} Y_{p,q}(\omega) = 0.\]

Since the cone \(C\) is closed under complex scaling, by replacing \(\omega\) for \(e^{i\theta} \omega\) in the above equation, we get

\[\sum_{p+q=k} e^{i(p-q)\theta} Y_{p,q}(\omega) = 0.\]

Using the fact that the set \(\{e^{i\beta} : \beta \in \mathbb{Z}\}\) form an orthogonal set and the sum vanishes on the diagonal \(p + q = k\), we conclude that \(Y_{p,q}(\omega) = 0, \forall p, q \in \mathbb{Z}_+\), such that \(p + q = k\). \(\square\)

We shall frequently need the following lemma which gives an unique decomposition of homogeneous polynomial in terms of homogeneous harmonic polynomials.

**Lemma 2.3.** Let \(P \in P_{p,q}\). Then \(P(z) = P_0(z) + |z|^2 P_1(z) + \cdots + |z|^{2l} P_l(z)\), where \(P_j \in H_{p-j,q-j}; j = 1, 2, \ldots, l\) and \(l \leq \min(p, q)\).

For a proof of this lemma, see [23], p. 66.

We shall also require the next two lemmas in the proof of existence of non-harmonic complex cones in \(\mathbb{C}^n\). Using the fact that Laplacian is rotation invariant, we prove the following lemma.

**Lemma 2.4.** Let \(0 \leq j \leq \min(p, q)\) and \(R_j \in H_{p-j,q-j}\). Then

\[\Delta (|z|^{2j} R_j) = c_j |z|^{2j-2} R_j, \quad \text{where } c_j = 4j(n+p+q-j-1).\]
Proof. Consider
\begin{equation}
\Delta \left( |z|^{2j} R_j \right) = 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} \left( |z|^{2j} R_j \right).
\end{equation}
We have
\[
\frac{\partial^2}{\partial z_k \partial \bar{z}_k} \left( |z|^{2j} R_j \right) = \frac{\partial}{\partial z_k} \left\{ j |z|^{2(j-1)} z_k R_j + |z|^{2j} \frac{\partial R_j}{\partial z_k} \right\} \\
= j \left\{ (j-1) |z|^{2(j-2)} |z_k|^2 R_j + |z|^{2(j-1)} R_j + |z|^{2j} \frac{\partial R_j}{\partial z_k} \right\} \\
+ j |z|^{2j-1} z_k \frac{\partial R_j}{\partial z_k} + |z|^{2j} \frac{\partial^2 R_j}{\partial z_k \partial \bar{z}_k}.
\]
By using Euler’s formula for homogeneous function, we get
\[
\sum z_k \frac{\partial R_j}{\partial z_k} = (p-j) R_j \quad \text{and} \quad \sum \bar{z}_k \frac{\partial R_j}{\partial \bar{z}_k} = (q-j) R_j.
\]
From Equation (2.3), it follows that
\[
\Delta \left( |z|^{2j} R_j \right) = 4j(n+p+q-j-1)|z|^{2j-2} R_j.
\]

For multi-indexes \( \alpha, \beta \in \mathbb{Z}_+^n \), write \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) and \( \bar{\partial}^\beta = \bar{\partial}_1^{\beta_1} \cdots \bar{\partial}_n^{\beta_n} \). Then for \( R \in P_{p,q} \), we can write
\[
R(\partial) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} \partial^\alpha \bar{\partial}^\beta.
\]
For \( R, S \in P_{p,q} \), define an inner product on \( P_{p,q} \) by \( \langle R, S \rangle = R(\partial) \bar{S} \). We shall prove the following lemma which is crucial for proof of the fact that complex cone \( H^{-1}(0) \) is a non-harmonic cone in \( \mathbb{C}^n \).

**Lemma 2.5.** For \( z_1, z_2 \in \mathbb{C} \), define \( A = \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} + z_1 \frac{\partial}{\partial z_2} \). Then \( A \) is a self-adjoint operator on the space \( P_{p,q} \).

**Proof.** Consider
\[
\langle z_1 \frac{\partial R}{\partial \bar{z}_2}, S \rangle = \left( \frac{\partial R}{\partial \bar{z}_2} \right) (\bar{\partial}) \frac{\partial}{\partial z_1} \bar{S} = \left( \frac{\partial R}{\partial \bar{z}_2} \right) (\bar{\partial}) \frac{\partial S}{\partial z_1}.
\]
That is,
\[
\langle z_1 \frac{\partial R}{\partial \bar{z}_2}, S \rangle = \frac{\partial R}{\partial \bar{z}_2}, \frac{\partial S}{\partial z_1} = \left( \frac{\partial S}{\partial z_1} \right) (\bar{\partial}) \frac{\partial R}{\partial \bar{z}_2}.
\]
This implies,
\[
\langle z_1 \frac{\partial R}{\partial \bar{z}_2}, S \rangle = \left( \frac{\partial (\bar{z}_2 S)}{\partial z_1} \right) (\bar{\partial}) \bar{R} = \left( \frac{\partial (\bar{z}_2 S)}{\partial z_1} \right) (\bar{\partial}) \bar{R} = \langle R, \frac{\partial (\bar{z}_2 S)}{\partial z_1} \rangle.
\]
Hence,
\[
\left\langle z_1 \frac{\partial R}{\partial z_2}, S \right\rangle = \left\langle R, \bar{z}_2 \frac{\partial S}{\partial \bar{z}_1} \right\rangle.
\]

Similarly, we can obtain the equation
\[
\left\langle \bar{z}_2 \frac{\partial R}{\partial \bar{z}_1}, S \right\rangle = \left\langle R, z_1 \frac{\partial S}{\partial z_2} \right\rangle.
\]

Thus by combining both these conditions, we get \( \langle AR, S \rangle = \langle R, AS \rangle \). That is, operator \( A \) is self-adjoint. □

We also need an expansion of functions on \( \mathbb{C}^n \) in terms of Laguerre functions \( \varphi_k^{n-1} \)'s, which is known as special Hermite expansion. The special Hermite expansion is a useful tool in the study of convolution operators and is related to the spectral theory of sub-Laplacian on the Heisenberg group \( H^n \). However, more details can be found in [23].

For \( \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \), let \( \pi_\lambda \) be the unitary representation of \( H^n \) on \( L^2(\mathbb{R}^n) \) given by
\[
\pi_\lambda(z, t) \varphi(\xi) = e^{i \lambda t} e^{i \lambda (x_\xi + \frac{1}{2} x_\xi y_\xi)} \varphi(\xi + y), \varphi \in L^2(\mathbb{R}^n).
\]
A celebrated theorem of Stone and von Neumann says that \( \pi_\lambda \) is irreducible and up to unitary equivalence \( \{ \pi_\lambda : \lambda \in \mathbb{R} \} \) are all the infinite dimensional unitary irreducible representations of \( H^n \). Let
\[
T = \frac{\partial}{\partial t} X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, j = 1, 2, \ldots, n.
\]
Then \( \{T, X_j, Y_j : j = 1, \ldots, n\} \) is a basis for the Lie Algebra \( \mathfrak{h}^n \) of all left invariant vector fields on \( H^n \). Define \( \mathcal{L} = - \sum_{j=1}^n (X_j^2 + Y_j^2) \), the second order differential operator which is known as the sub-Laplacian of \( H^n \). The representation \( \pi_\lambda \) induces a representation \( \pi_\lambda^* \) of \( \mathfrak{h}^n \), on the space of \( C^\infty \) vectors in \( L^2(\mathbb{R}^n) \) is defined by
\[
\pi_\lambda^*(X)f = \left. \frac{d}{dt} \right|_{t=0} \pi_\lambda(\exp tX)f.
\]
An easy calculation shows that \( \pi^*(X_j) = i \lambda x_j, \pi^*(Y_j) = \frac{\partial}{\partial x_j}, j = 1, 2, \ldots, n. \)
Therefore, \( \pi_\lambda^*(\mathcal{L}) = - \Delta_x + \lambda^2 |x|^2 = H(\lambda) \), the scaled Hermite operator. The eigenfunction of \( H(\lambda) \) are given by \( \phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{2}} \phi_\alpha(\sqrt{|\lambda|} |x|), \alpha \in \mathbb{Z}_+^n, \) where \( \phi_\alpha \) are the Hermite functions on \( \mathbb{R}^n \). Since \( H(\lambda) \phi_\alpha^\lambda = (2|\lambda| + n)|\lambda| \phi_\alpha^\lambda. \) Therefore,
\[
\mathcal{L} \left( \pi_\lambda(z, t) \phi_\alpha^\lambda, \phi_\beta^\lambda \right) = (2|\lambda| + n)|\lambda| \left( \pi_\lambda(z, t) \phi_\alpha^\lambda, \phi_\beta^\lambda \right).
\]
Thus the entry functions \( (\pi_\lambda(z, t) \phi_\alpha^\lambda, \phi_\beta^\lambda) \), \( \alpha, \beta \in \mathbb{Z}_+^n \) are eigenfunctions for \( \mathcal{L} \). As \( (\pi_\lambda(z, t) \phi_\alpha^\lambda, \phi_\beta^\lambda) = e^{i \lambda t} \left( \pi_\lambda(z) \phi_\alpha^\lambda, \phi_\beta^\lambda \right) \), these eigenfunctions are not in
$L^2(H^n)$. However for a fix $t$, they are in $L^2(C^n)$. Define $L$ by $L(e^{i\lambda t}f(z)) = e^{i\lambda t}L\lambda f(z)$. Then the functions $\phi_{\alpha\beta}^\lambda(z) = (2\pi)^{-n/2} (\pi_\lambda(z)\phi^\lambda_\alpha, \phi^\lambda_\beta)$, are eigenfunction of the operator $L\lambda$ with eigenvalue $2|\lambda| + n$. The functions $\phi_{\alpha\beta}^\lambda$'s are called the special Hermite functions and they form an orthonormal basis for $L^2(C^n)$ (see [23], Theorem 2.3.1, p.54). Thus, for $g \in L^2(C^n)$, we have the expansion $g = \sum_{\alpha,\beta} \langle g, \phi_{\alpha\beta}^\lambda \rangle \phi_{\alpha\beta}^\lambda$.

To further simplify this expansion, let $\varphi_{k,\lambda}^{n-1}(z) = \varphi_k^{n-1}(\sqrt{|\lambda|}z)$, the Laguerre function of degree $k$ and order $n - 1$. The special Hermite functions $\phi_{\alpha\alpha}^\lambda$ satisfy the relation

$$\sum_{|\alpha| = k} \phi_{\alpha\alpha}^\lambda(z) = (2\pi)^{-n/2} |\lambda|^{n/2} \varphi_{k,\lambda}^{n-1}(z).$$

Let $g$ be a function in $L^2(C^n)$. Then $g$ can be expressed as $g(z) = (2\pi)^{-n} |\lambda|^n \sum_{k=0}^{\infty} g \times \varphi_{k,\lambda}^{n-1}(z)$, whenever $\lambda \in \mathbb{R}^*$, (see [23], p.58). In particular, for $\lambda = 1$, we have

$$g(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} g \times \varphi_k^{n-1}(z),$$

which is called the special Hermite expansion for $g$. For radial functions, this expansion further simplifies as can be seen from the following lemma.

**Lemma 2.6.** [23] Let $f$ be a radial function in $L^2(C^n)$. Then

$$f = \sum_{k=0}^{\infty} B_k^n \langle f, \varphi_k^{n-1} \rangle \varphi_k^{n-1}, \quad \text{where} \quad B_k^n = \frac{k!(n-1)!}{(n+k-1)!}.$$

**3. Proofs of the main results**

In this section, we first prove our main result that a non-harmonic complex cone is a set of injectivity for the twisted spherical means for the class of all continuous functions on $C^n$. After that we prove the existence of examples of non-harmonic complex cone. In fact, we show that each of the diagonal space $P_{p,p}$ has at least one member which corresponds to a non-harmonic complex cone. At the end we mention some remarks and open problems related to the problem of sets of injectivity for the TSM.

**Theorem 3.1.** Let $C$ be a complex cone in $C^n (n \geq 2)$. Let $f$ be a continuous function on $C^n$. Suppose $f \times \mu_r(z) = 0$, for all $r > 0$ and $z \in C$. Then $f \equiv 0$ if and only if $C \not\subseteq H^{-1}(0)$, for any $P \in H_{p,q}$ and for all $p, q \in \mathbb{Z}_+$. 


Proof. Since $C$ is closed under complex scaling, by rotation we can assume that $z = (z_1, 0, \ldots, 0) \in C$, for all $z_1 \in \mathbb{C}$. By the hypothesis $f \times \mu_s(z) = 0$, $\forall s > 0$ and for all $z \in C$. Therefore, we can write

\begin{equation}
(3.1) \quad \int_{|w| \leq r} f(z + w)e^{-\frac{i}{2} \text{Im}(z, \bar{w})} dw = \int_0^r f \times \mu_s(z)s^{2n-1} dw = 0,
\end{equation}

for all $r > 0$ and $z \in C$. Let $z_1 = x_1 + iy_1$. Applying $2\partial z_1 = 2\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial y_1}$ to the above equation, we get

\[ \int_{|w| \leq r} \frac{\partial}{\partial w_1} \left( f(z + w)e^{-\frac{i}{2} \text{Im}(z, \bar{w})} \right) dw - \frac{1}{2} \int_{|w| \leq r} \bar{w}_1 f(z + w)e^{-\frac{i}{2} \text{Im}(z, \bar{w})} dw = 0, \]

for all $r > 0$ and $z \in C$. By Green’s theorem, we get

\[ \int_{|w| = r} \frac{\bar{w}_1}{r} \left( f(z + w)e^{-\frac{i}{2} \text{Im}(z, \bar{w})} \right) dw = \frac{1}{2} \int_{|w| \leq r} \bar{w}_1 f(z + w)e^{-\frac{i}{2} \text{Im}(z, \bar{w})} dw, \]

Let $g(z) = \bar{z}_1 f(z)$. Then we have

\[ r^{2n-2}g \times \mu_r(z) = \frac{1}{2} \int_0^r g \times \mu_s(z)s^{2n-1} ds. \]

Put $F(t) = t^{2n-1}g \times \mu_t(z)$. Then the above equation becomes

\[ \frac{F(r)}{r} = \frac{1}{2} \int_0^r F(s) ds. \]

By differentiating both sides, we get

\[ F'(r) = \left( \frac{r}{2} + \frac{1}{r} \right) F(r). \]

A general solution to this equation is

\[ F(r) = \frac{c(z)}{r} e^{\frac{r^2}{4}}. \]

That is,

\[ r^{2n-2}g \times \mu_r(z) = c(z)e^{\frac{r^2}{4}}. \]

By letting $r \to 0$, we get $c(z) = 0$. Hence $g \times \mu_r(z) = 0$, for all $r > 0$ and $z \in C$. Let $h(z) = z_1 f(z)$. Similarly, by applying $2\partial z_1 = 2\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial y_1}$ to the equation (3.1), we get $h \times \mu_r(z) = 0$, for all $r > 0$ and $z \in C$. Hence for any polynomial $P(z_1, \bar{z}_1)$, we deduce that $(P f) \times \mu_r(z) = 0$, $\forall r > 0$ and $z \in C$. Let $w_j = u_j + iv_j$, $j = 1, 2, \ldots, n$. Since $0 \in C$, by evaluating the means at $0$, we get

\[ \int_{S^{2n-1}} P(u_1) f(w) d\mu_r(w) = 0, \]

for all $r > 0$. Thus, we can approximate the above equation at $u_1 = t$. For $\epsilon > 0$, we can write

\[ \int_{S^{2n-1}} \frac{1}{\epsilon \sqrt{\pi}} e^{-\frac{(u_1 - t)^2}{\epsilon^2}} f(w) d\mu_r(w) = 0. \]
Letting $\epsilon \to 0$, we get
\[
\int_{rS^{2n-1} \cup \{u_1 = t\}} f \, d\tilde{\mu} = 0,
\]
for all $t \in (-r,r)$, where $\tilde{\mu}$ is the normalized surface measure on the geodesic sphere $rS^{2n-1} \cup \{u_1 = t\}$. Thus, the integral of $f$ vanishes over all $(2n-2)$-dimensional geodesic spheres on $rS^{2n-1}$ with poles lay on $rS^{2n-1}$. In view of Lemma 1.3, we infer that $f$ vanishes on $rS^{2n-1}$ if and only if poles are not contained in $Y_k^{-1}(0)$, $\forall \, k \in \mathbb{Z}_+$. Further by Lemma 2.1 we have
\[
Y_k = \sum_{p,q=k} Y_{p,q}.
\]
Hence, in view of Lemma 2.2 we get $f = 0$ on $rS^{2n-1}$ if and only if poles are not contained in $Y_{p,q}^{-1}(0)$, $\forall \, p,q \in \mathbb{Z}_+$. Since $r > 0$ is arbitrary and $f$ is continuous, we conclude that $f \equiv 0$ on $\mathbb{C}^n$. This completes the proof. \qed

**Remark 3.2.** In Theorem 3.1 we have shown that complex cone is set of injectivity for the twisted spherical means, however the question of the real cones to be sets of injectivity for the twisted spherical means is still unsolved. For instance, the author in the article [19] has shown that the real cone is set of injectivity for the TSM for the class $L^p(\mathbb{C}^n)$ ($n \geq 2$) with $1 \leq p \leq 2$. This real cone is not contained in the zero set of any bi-graded homogeneous harmonic polynomial. We would like to mention that the later result is a consequence of a result that $\mathbb{R} \cup i\mathbb{R}$ is set of injectivity for the TSM for $L^p(\mathbb{C})$, which has been proved by the author in the article [18].

In order to complete the argument of Theorem 3.1, we now prove that there exists a non-trivial complex cone which does not vanish on the zero set (or level surface) of any bi-graded homogeneous harmonic polynomial.

Let $0 \neq a \in \mathbb{C}$ and $z \in \mathbb{C}^n$, ($n \geq 2$). Write $H(z) = az_1\bar{z}_2 + |z|^2$. Then $H^{-1}(0)$ is a complex cone. In fact, we shall show that this is a non-harmonic complex cone. Since $H$ is a homogeneous polynomial, for $n = 2$, put $z_1 = wz_2$. Then $H(wz_2, z_2) = |z_2|^2(w\bar{w} + aw + 1)$. Since the polynomial $w\bar{w} + aw + 1$ can not be factorized into linear factors, therefore, $H$ is irreducible. It is clear from the context that $H$ is also irreducible for $n > 2$. Now, it only remains to show that $H^{-1}(0) \not\subset R^{-1}(0)$, for any $R \in H_{s,t}$ and for all $s, t \in \mathbb{Z}_+$. Otherwise, if $H^{-1}(0) \subset R^{-1}(0)$, then by using the fact that $H$ is irreducible, it implies that $H$ divides $R$ and hence $R = HQ$, for some $Q \in P_{p,q}$. Thus, it is enough to prove the following result in order to prove our claim.

**Theorem 3.3.** Suppose $\Delta(HQ) = 0$, for some $Q \in P_{p,q}$ with $p, q \in \mathbb{Z}_+$. Then $Q$ has to vanish identically.

In order to prove Theorem 3.3, we need the following lemma. Write
\[
A = \bar{z}_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2} \quad \text{and} \quad B = \frac{\partial^2}{\partial z_1 \partial z_2}.
\]
Lemma 3.4. Let $\Delta(HQ) = 0$, for some $Q \in P_{p,q}$. Then

$$HQ = a \left( z_1 \bar{z}_2 Q_0 + \gamma |z|^2 A Q_0 + \delta |z|^4 B Q_0 \right),$$

where $\gamma$ and $\delta$ are non-zero constants and independent of $a$.

Proof. Since $Q \in P_{p,q}$, therefore by Lemma 2.3, $Q$ can be uniquely decomposed in terms of spherical harmonics as

$$Q = Q_0 + |z|^2 Q_1 + |z|^4 Q_2 + \cdots + |z|^{2l} Q_l,$$

where $Q_j \in H_{p-j,q-j}$. We can write

$$HQ = a z_1 \bar{z}_2 (Q_0 + |z|^2 Q_1 + |z|^4 Q_2 + \cdots + |z|^{2l} Q_l) + |z|^2 Q_0$$

$$= a z_1 \bar{z}_2 Q_0 + |z|^2 (a z_1 \bar{z}_2 Q_1 + |z|^2 a z_1 \bar{z}_2 Q_2 + \cdots + |z|^{2l-2} a z_1 \bar{z}_2 Q_l + |z|^2 Q).$$

Using Lemma 2.3, once again to the above equation, we get

$$HQ = a z_1 \bar{z}_2 Q_0 + |z|^2 R_1 + |z|^4 R_2 + \cdots + |z|^{2m} R_m,$$

where $R_j \in H_{p-j,q-j}$. Now, using the condition that $HQ$ is harmonic, we show that $HQ$ is completely determined by $Q_0$, which is the key part of this proof. In fact, the later argument would enable us to assume that $Q$ is harmonic.

Since $\Delta(HQ) = 0$. Therefore, by Lemma 2.3, it follows that

$$\Delta(HQ) = a \Delta(z_1 \bar{z}_2 Q_0) + c_1 R_1 + c_2 |z|^2 R_2 + \cdots + c_m |z|^{2m-2} R_m = 0.$$

We have

$$\Delta(z_1 \bar{z}_2 Q_0) = 4 \left[ \frac{\partial^2(z_1 \bar{z}_2 Q_0)}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2(z_1 \bar{z}_2 Q_0)}{\partial \bar{z}_2 \partial z_2} + \sum_{k=3}^{n} \frac{\partial^2(z_1 \bar{z}_2 Q_0)}{\partial z_k \partial \bar{z}_k} \right]$$

$$= 4z_2 \frac{\partial Q_0}{\partial \bar{z}_1} + 4z_1 \frac{\partial Q_0}{\partial z_2} = 4AQ_0.$$

(3.5)

By equation (3.4), we have

$$aAQ_0 + c_1 R_1 + c_2 |z|^2 R_2 + \cdots + c_m |z|^{2m-2} R_m = 0.$$  

By applying $\Delta$ in the above equation, we get

$$a \Delta(AQ_0) + c_2 c_2' R_2 + c_3 c_3' |z|^2 R_3 \cdots + c_m c_m' |z|^{2m-4} R_m = 0.$$  

(3.6)

A straightforward calculation gives

$$\Delta(AQ_0) = 4A \Delta Q_0 + 8BQ_0 = 8BQ_0.$$  

(3.7)

Operator $B$ seems to be a degree reducing operator over the spaces $P_{p,q}$’s. By combining (3.6) and (3.7) we get

$$8aBQ_0 + c_2 c_2' R_2 + c_3 c_3' |z|^2 R_3 \cdots + c_m c_m' |z|^{2m-4} R_m = 0.$$  

(3.8)

Since the spaces $H_{p,q}$’s are orthogonal among themselves, it follows that

$$8aBQ_0 + c_2 c_2' R_2 = 0, \quad R_3 = 0, \ldots, R_m = 0.$$
By substituting these values in (3.3), we get

\[(3.9)\]

\[HQ = a z_1 \bar{z}_2 Q_0 + |z|^2 R_1 - \frac{8a}{c_2 c_2'} |z|^4 B Q_0.\]

Once again applying $\Delta$ to (3.9), we get

\[0 = 4a A Q_0 + 4(p + q) R_1 - 32a c_2 c_2' (p + q + 1) |z|^2 B Q_0.\]

Finally, after substituting the value of $R_1$ to (3.9), we can write

\[(3.10)\]

\[HQ = a (z_1 \bar{z}_2 Q_0 + \gamma |z|^2 A Q_0 + \delta |z|^4 B Q_0),\]

where $\gamma$ and $\delta$ are non-zero constants and independent of $a$. □

**Proof of Theorem 3.3.** From Lemma 3.4, we infer that $HQ$ is completely determined by $Q_0$. From (3.2) and in view of Lemma 2.3, it is clear that $HQ$ is harmonic part of $(a z_1 \bar{z}_2 Q_0)$, because the rest of the terms will not contribute to $HQ$, since $HQ$ is harmonic. This says that product $HQ$ depends only upon harmonic part $Q_0$ of $Q$ and hence, without loss of generality, we can assume $Q$ to be a homogeneous harmonic polynomial. That is, $\Delta Q = 0$.

By the given condition, we have $\Delta(HQ) = 0$. Therefore, by using the condition that $Q$ is harmonic, we get

\[a \Delta (z_1 \bar{z}_2) + \Delta \left( |z|^2 Q \right) = 0.\]

In view of equation (3.5), we get

\[(3.11)\]

\[a A Q + (n + p + q) Q = 0.\]

This says that $Q$ is an eigenfunction of the operator $A$. Since by Lemma 2.5 $A$ is a self-adjoint operator, therefore, its eigenvalues must be real. From (3.11), we have

\[A Q = -\frac{(n + p + q)}{a} Q.\]

If $a = \alpha + i \beta$ and $\beta \neq 0$. Then $Q$ has to be identically zero. If $\beta = 0$, then we can use the rotation

\[\sigma = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \in U(n),\]

where $\theta$ is any real number other than even multiple of $\pi$. This gives

\[\sigma \cdot (z_1, z_2, z_3, \ldots, z_n) = \left(e^{i\theta} z_1, e^{-i\theta} z_2, z_3, \ldots, z_n\right).\]

Since $\Delta(HQ) = 0$ and $\sigma \in U(n)$, therefore, by using the fact that $\Delta$ is rotation invariant, it implies that $\sigma \cdot \Delta(HQ) = 0$. That is, $\Delta(HQ)(\sigma \cdot z) = 0$, which in turn implies that

\[\Delta \left\{ (ae^{i\theta} z_1 \bar{z}_2 + |z|^2) Q^\sigma(z) \right\} = 0,\]
where $Q^\sigma(z) = Q(\sigma \cdot z)$. By a similar calculation as to the previous case, we can write

$$(n + p + q)Q^\sigma + ae^{\theta}AQ^\sigma = 0.$$ 

Thus $Q^\sigma$ is an eigenfunction of $A$ with complex eigenvalue having non-zero imaginary part. Therefore, $Q^\sigma = 0$ and hence $Q = 0$. Thus, we infer that Theorem 3.3 is valid for any complex number $a \neq 0$.

**Corollary 3.5.** Let $s \in \mathbb{Z}_+$ and $s \geq 1$. Then there exists $P_o \in P_{s,s}$ such that $P_o^{-1}(0)$ is a non-harmonic complex cone.

**Proof.** As in Theorem 3.3, let $H(z) = az_1\bar{z}_2 + |z|^2$. We show that $H^s$ is a required member which belongs to $P_{s,s}$ such that $(H^s)^{-1}(0)$ is a non-harmonic complex cone. We prove this using induction on $s$. For $s = 1$, by Theorem 3.3, $H \in P_{1,1}$ and $H^{-1}(0)$ is non-harmonic complex cone. When $s = 2$, we need to show that $(H^2)^{-1}(0) \not\subseteq R^{-1}(0)$, for any $R \in H_{s,t}$ and for all $s, t \in \mathbb{Z}_+$. On the contrary, suppose $(H^2)^{-1}(0) \subseteq R^{-1}(0)$, for some $R \in H_{s,t}$. Then either $H$ divide $R$ or $H^2$ divide $R$. Then as a consequence of Theorem 3.3 it implies that $R = 0$. That is, $H$ does not divide $R$. Therefore, $H^2$ divide $R$ and hence $R = H^2 Q$ for some $Q \in P_{p,q}$. This in turn implies that $\Delta \{H(HQ)\} = 0$. By applying Theorem 3.3 once again, we get $HQ = 0$. Hence $Q = 0$. This proves the result for $s = 2$. Similarly, the result for an arbitrary $s$ is being followed by induction on $s$. \qed

**Remark 3.6.** (a) In Corollary 3.5 we have shown that for $s \geq 1$, each of the diagonal space $P_{s,s}$ has at least one member which corresponds to a non-harmonic complex cone. Hence there are plenty of complex cones which do not completely lay in to the level surface of any bi-graded homogeneous harmonic polynomials. Roughly speaking, there are plenty of homogeneous surfaces other than that homogeneous harmonic surfaces. However, it is interesting to know that whether a given non-diagonal space $P_{s,t}$ has at least one member which corresponds to a non-harmonic complex cone. This question is open for the time being.

(b) In a recent article [19], we have observed some embedding property of set of injectivity in higher dimensions. For instance, the set $\mathbb{R} \cup i\mathbb{R}$ is a set of injectivity for the TSM for $L^p(\mathbb{C})$ with $1 \leq p \leq 2$. Then we deduced that the set $(\mathbb{R} \cup i\mathbb{R}) \times \mathbb{C}^n$ is a set of injectivity for the TSM for $L^p(\mathbb{C}^{n+1})$ with $1 \leq p \leq 2$. Therefore, for a given non-harmonic cone $C$ in $\mathbb{C}^n$, it is interesting to know that whether the set $C \times \mathbb{C}^m$ is a set of injectivity for the TSM for the certain class of continuous functions on $\mathbb{C}^{n+m}$. This question is still unanswered and for the time being we leave it open.

**4. Weighted twisted spherical mean**

In this section, we prove that complex cone is a set of injectivity for certain weighted twisted spherical mean for the radial class of functions which satisfy some exponential growth condition.
Let $P \in H_{p,q}$ and denote $d\nu_r = P d\mu_r$. Let $C$ be a complex cone in $\mathbb{C}^n$. Then we have the following result.

**Theorem 4.1.** Let $f$ be a radial function on $\mathbb{C}^n$ such that $e^{\frac{1}{4} |z|^2} f(z) \in L^p(\mathbb{C}^n)$, for $1 \leq p < \infty$. Suppose $f \times \nu_r(z) = 0$, $\forall \ r > 0$ and $\forall \ z \in C$. Then $f = 0$ a.e. if and only if $C \not\subseteq P^{-1}(0)$.

Theorem 4.1 does not hold for $p = \infty$ as can be seen from the following weighted functional equations for the spherical function $\varphi_k^{n-1}$.

**Lemma 4.2.** [23] For $z \in \mathbb{C}^n$, let $P \in H_{p,q}$ and $d\nu_r = P d\mu_r$. Then

$$\varphi_k^{n-1} \times \nu_r(z) = (2\pi)^{-n} C(n, p, q) r^{2(p+q)} e^{n+p+q-1} P(z) \varphi_k^{n+p+q-1}(z),$$

if $k \geq q$ and $0$ otherwise.

**Remark 4.3.** From Lemma 4.2 it can be seen that Theorem 4.1 does not hold for $p = \infty$. For instance, take $P \in H_{0,1}$ and let $d\nu_r = P d\mu_r$. Then $\varphi_0^{n-1} \times \nu_r(z) = 0$, where $\varphi_0^{n-1}(z) = e^{-\frac{1}{4} |z|^2}$.

For the proof of Theorem 4.1 we need the following result from [20] which is related to the Weyl correspondence of bi-graded spherical harmonics. Let us consider the following invariant differential operators which arise in study of the twisted convolution on $\mathbb{C}^n$. For $\lambda \in \mathbb{R} \setminus \{0\}$, let

$$\tilde{Z}_{j,\lambda} = \frac{\partial}{\partial z_j} - \frac{\lambda}{4} z_j \quad \text{and} \quad \tilde{Z}_j^* = \frac{\partial}{\partial \bar{z}_j} + \frac{\lambda}{4} \bar{z}_j, \ j = 1, 2, \ldots, n.$$ 

Let $P$ be a bi-graded homogeneous harmonic polynomial on $\mathbb{C}^n$ with expression

$$P(z) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$ 

Then by using a result of Geller ([14], p. 616, Proposition 2.7) about Weyl correspondence of the spherical harmonics, the operator analogue of $P(z)$ can be expressed as

$$P(\tilde{Z}) = \sum_{|\alpha|=p} \sum_{|\beta|=q} c_{\alpha\beta} \tilde{Z}^\alpha \tilde{Z}^\beta.$$ 

The following result has been proved by the author in article [20].

**Theorem 4.4.** [20] Let $P \in H_{p,q}$. Then

$$P(\tilde{Z}_\lambda) \varphi_k^{n-1} = \begin{cases} (-2\lambda)^{-p-q} P \varphi_k^{n+p+q-1}, & \text{if } \lambda < 0, \ k \geq q; \\ (-2\lambda)^{-p-q} P \varphi_k^{n+p+q-1}, & \text{if } \lambda > 0, \ k \geq p. \end{cases}$$

Consider the following right invariant differential operators for the twisted convolution:

$$\tilde{A}_j = \frac{\partial}{\partial z_j} + \frac{1}{4} z_j \quad \text{and} \quad \tilde{A}_j^* = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} \bar{z}_j, \ j = 1, 2, \ldots, n.$$
Let $\varphi_{k}^{-1}(z) = L_{k}^{n-1}(\frac{1}{2}|z|^{2})e^{-\frac{1}{2}|z|^{2}}$. Then by Theorem 4.4 we get
\begin{equation}
(4.2) \quad P(\tilde{A})\varphi_{k}^{-1} = (2)^{-p-q}P\varphi_{k-q}^{n+p+q-1}, \text{ if } k \geq q.
\end{equation}

Suppose $f$ be a function on $\mathbb{C}^n$ such that $e^{\frac{1}{2}|z|^{2}}f(z) \in L^{p}(\mathbb{C}^n)$, for $1 \leq p < \infty$. Let $\varphi_{e}$ be a smooth, radial compactly supported approximate identity on $\mathbb{C}^n$. Then $f \times \varphi_{e} \in L^{1} \cap L^{\infty}(\mathbb{C}^n)$ and in particular $f \times \varphi_{e} \in L^{2}(\mathbb{C}^n)$. Let $dv_{r} = Pd\mu_{r}$. Suppose $f \times \nu_{r}(z) = 0, \forall r > 0$ and $\forall z \in C$. Then by polar decomposition $f \times P\varphi_{k-q}^{n+p+q-1}(z) = 0, \forall k \geq q$ and $\forall z \in C$. Since $\varphi_{e}$ is radial, we can write
\[f \times \varphi_{e} \times \nu_{r}(z) = \sum_{k \geq 0} B_{k}^{n} \langle \varphi_{e}, \varphi_{k}^{-1} \rangle f \times \varphi_{k-q}^{n+p+q-1} \times \nu_{r}(z) .\]
By Lemma 4.2 it follows that $f \times \varphi_{e} \times \nu_{r}(z) = 0, \forall k \geq q$ and $\forall z \in C$. Thus, without loss of generality, we can assume $f \in L^{2}(\mathbb{C}^n)$. Hence to prove the Theorem 4.1 it is enough to prove the following result.

**Proposition 4.5.** Let $f \in L^{2}(\mathbb{C}^n)$ be radial and $e^{\frac{1}{2}|z|^{2}}f(z) \in L^{p}(\mathbb{C}^n)$, for $1 \leq p < \infty$. Suppose $f \times \nu_{r}(z) = 0, \forall r > 0$ and $\forall z \in C$. Then $f = 0$ a.e. if and only if $C \notin P^{-1}(0)$.

**Proof.** Given that $f \times \nu_{r}(z) = 0, \forall r > 0$ and $\forall z \in C$. By polar decomposition $f \times P\varphi_{k-q}^{n+p+q-1}(z) = 0, \forall k \geq q$ and $\forall z \in C$. Using the identity (4.2), we can write $f \times P(\tilde{A})\varphi_{k}^{-1}(z) = 0, \forall k \geq q$ and $\forall z \in C$. Since $P(\tilde{A})$ is a right invariant operator, it follows that
\begin{equation}
(4.3) \quad P(\tilde{A}) \left(f \times \varphi_{k}^{-1}\right)(z) = 0.
\end{equation}
Since $f$ is radial, therefore, it can be expressed as
\[f = \sum_{k=0}^{\infty} B_{k}^{n} \langle f, \varphi_{k}^{-1} \rangle \varphi_{k}^{-1} .\]
Since the Laguerre functions satisfy the orthogonality conditions $\varphi_{k}^{-1} \varphi_{j}^{-1} = (2\pi)^{n}\delta_{jk} \varphi_{k}^{-1}$. Therefore, $f \times \varphi_{k}^{-1} = (2\pi)^{n} B_{k}^{n} \langle f, \varphi_{k}^{-1} \rangle \varphi_{k}^{-1}$. Hence, from (4.3) it implies that $\langle f, \varphi_{k}^{-1} \rangle P(\tilde{A})\varphi_{k}^{-1}(z) = 0$. Once again by applying the identity (4.2), we get
\[\langle f, \varphi_{k}^{-1} \rangle P(z)\varphi_{k-q}^{n+p+q-1}(z) = 0 .\]
That is,
\[\langle f, \varphi_{k}^{-1} \rangle P(z)\varphi_{k-q}^{n+p+q-1}\left(\frac{1}{2}|z|^{2}\right) = 0 ,\]
whenever, $k \geq q$ and $z \in C$. Since $C$ is closed under scaling, it follows that
\[\langle f, \varphi_{k}^{-1} \rangle P(z)\varphi_{k-q}^{n+p+q-1}\left(\frac{1}{2}r^{2}|z|^{2}\right) = 0 ,\]
for all $r > 0$. Thus, we get $\langle f, \varphi_{k}^{-1} \rangle P(z) = 0$, whenever, $k \geq q$ and $z \in C$. Hence $\langle f, \varphi_{k}^{-1} \rangle = 0$, for all $k \geq q$ if and only if $C \notin P^{-1}(0)$. Therefore, $f$ is a
finite linear combination of $\varphi_k^{n-1}$’s and by the given growth condition on $f$, it follows that $f = 0$. Thus, we infer that $f = 0$ if and only if $C \not\subset P^{-1}(0)$. □

Remark 4.6. Theorem 4.1 has been worked out only for radial class of functions. However, the question that a complex cone is set of injectivity for the weighted twisted spherical mean for those functions $f$ on $\mathbb{C}^n$ satisfying the growth condition $e^{\frac{1}{4}|z|^2}f(z) \in L^p(\mathbb{C}^n)$ with $1 \leq p < \infty$ is still unanswered.

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