Ground State H-Atom in Born-Infeld Theory

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Abstract

Within the context of Born-Infeld (BI) nonlinear electrodynamics (NED) we revisit the H-atom. Born’s integral expression for the electrostatic potential is simulated by the Morse type potential with remarkable fit. The Morse potential is employed to determine both the ground state energy of the electron and the BI parameter.
The first example of nonlinear electrodynamics (NED) was introduced in 1934 by Born and Infeld (BI) in order to eliminate divergences in the Coulomb problem. With the advent of quantum electrodynamics (QED), however, divergences were resolved by the well-established scheme of renormalization. Being popular enough, QED suppressed the efforts of BI to the extend of being forgotten until very recent decade. We observe now that BI theory gained recognition anew within string theory; a theory that aims to unify all forces of nature, including quantum gravity, under a common title. Once BI paved the way toward NED, various modifications emerged as alternative theories to the well-known linear electrodynamics of Maxwell. The common feature in all these NED theories is that in the linear limit it recovers the Maxwell’s electrodynamics, as it should. The NED Lagrangian is commonly constructed from the invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}^*F^{\mu\nu}$ ($\ast$ means dual) as non-polynomial functions. Since these invariants determine the vacuum polarization BI theory, or any version of NED serves to polarize the vacuum. Vanishing of these invariants, in plane waves for example, exempts the latter as polarizer of the vacuum. We note that beside string theory, NED and naturally BI, find applications in different areas, ranging from black holes and cosmology to the model of elementary particles. In this Letter we revisit the Born’s intricate solution and employ a Morse type of potential simulation to determine the ground level energy of the H-atom. We restrict ourselves entirely to the pure electrical potential (i.e. without magnetic fields) applied to the ground state of the non-relativistic Schrödinger equation. The electrostatic BI Lagrangian is given by

$$L(X) = \frac{4}{\beta^2} \left[ 1 - \sqrt{1 + \frac{\beta^2}{2} X} \right]$$

where $X = F^2 = F_{\mu\nu}F^{\mu\nu}$ and $\beta$ is the BI parameter. In the limit $\beta \to 0$ we recover the Maxwell Lagrangian $L = -X$. The electrostatic potential is given by

$$A_\mu = \delta^0_\mu \phi(r)$$

for an $r$ dependent function $\phi(r)$. The sourceless BI equation is

$$\partial_\mu \left( \sqrt{g} \frac{\partial L}{\partial X} F^{\mu\nu} \right) = 0$$

in which $\sqrt{g}$ refers to the square root of determinant for the spherically symmetric flat metric

$$ds^2 = -dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2\theta \, d\varphi^2 \right).$$
Born’s solution follows from (3) as an indefinite integral

$$\phi(r) = \frac{1}{\beta} \int \frac{dr}{\sqrt{1 + r^4}}$$

in which we note that the lower limit of integral must be changed according to the location of the electron, i.e. the location of delta function fixes the lower limit while the upper limit is infinite.

H. Carley and M. K.-H. Kiessling \[3\] have shown recently that BI effects \[3, 4, 5\] on the Schrödinger spectrum of the ideal H-atom (i.e. a spinless electron bound to an infinitely massive point proton) can be calculated. For this purpose they used the following modified Coulomb potential (Born-Infeld-Coulomb (BIC) potential)

$$V(r) = -\alpha \frac{Z(\frac{\rho}{\beta})}{r}$$

where $Z(\frac{\rho}{\beta})$ is only a function of $\frac{\rho}{\beta}$. From \[3, 4\], $Z(\frac{\rho}{\beta})$ is explicitly given by the following integral form

$$Z(\rho) = \rho^2 \int_0^{\frac{\sqrt{2}}{4}} dy \frac{2y\sqrt{1+y^2} - 2y^2 - 1}{\sqrt{1+4y^2} \sqrt{1+y^2} \sqrt{1+y^2} \sqrt{1+\rho^4 y^4} + \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{4}\right) \rho}$$

in which $B(., .)$ indicates Euler’s Beta function, $\rho = \frac{\rho}{\beta}$ and $\beta \in \mathbb{R}^+$ is Born’s electromagnetic vacuum constant. For simplicity, we also follow the units accepted in Ref. \[3\] i.e., units of $\hbar$ for both action and angular momentum, electron charge $e$ for charge, electron rest mass $m_e$ for mass, speed of light $c$ for velocity, and Compton wavelength of the electron $\lambda_C = \hbar/m_ec$ for both length and time. In these units, the Schrödinger equation of an electron which undergoes the potential (6), becomes \[3\]

$$-\frac{1}{2}\nabla^2 \Psi(r) - \alpha \frac{Z(r)}{r} \Psi(r) = \varepsilon \Psi(r)$$

where $\alpha = 1/137.036$ is the fine structure constant, $r$ is the distance of the electron from the origin (proton) in terms of Compton wavelength of electron $\lambda_C = 0.003861\,\text{Å}$ and $\varepsilon$ is eigenenergy of the system in terms of the electron rest energy. In Fig.1 we plot $Z(\rho)$ versus $\rho$ which shows that $\lim_{\rho \to \infty} Z(\rho) = 1$, $\lim_{\rho \to 0} Z(\rho) = 0$, a maximum occurs at $\rho = 2.139634$ and $Z(\rho = 0.654988) = 1$. The asymptotic behavior of $Z(\rho)$ manifests that either for a finite $\beta$ and extremely large $r$ or for a finite $r$ and extremely small $\beta$, Born-Infeld interaction gives the usual Coulomb potential.
Now let’s consider BIC’s potential as

\[ V_{BIC}(\rho) = -\frac{\alpha}{\beta} Z(\rho) = \frac{\alpha}{\beta} W(\rho) \]  

(9)

where \( W(\rho) = -\frac{Z(\rho)}{\rho} \) is only a function of \( \rho \) and explicitly

\[ W(\rho) = -\rho \int_0^{\sqrt{2}} \frac{2y\sqrt{1+y^2} - 2y^2 - 1}{\sqrt{1 + 4y^2 - 4y\sqrt{1 + y^2}} \sqrt{1 + \rho^4 y^4}} - \frac{1}{4} B \left( \frac{1}{4}, \frac{1}{4} \right). \]  

(10)

We plot \( W(\rho) \) as a function of \( \rho \) in Fig. 2 (this figure is also given by [3]) which clearly reveals that BIC’s potential is finite at the origin.

Here although the latter function may not be easily expressed in a closed form we simulate it into some familiar functions to represent \( W(\rho) \) in the process of analytical solution of the Schrödinger equation (8).

First we try to map \( W(\rho) \) into a Morse-type-potential \([6]\) in the form of

\[ W_s(\rho) = -\left[ G \left( 1 - e^{-\kappa(\rho-b)} \right)^2 + V_o + \frac{1}{4} B \left( \frac{1}{4}, \frac{1}{4} \right) \right] \]  

(11)

where \( G, \kappa, b \) and \( V_o \) are four adjusting parameters. In Fig. 3 we plot the original \( W(\rho) \) and a simulated version of \( W(\rho) \) (let’s call it \( W_s(\rho) \)) after setting

\[ G = -1.8300, \]  
\[ V_o = 0.09805 \]  
\[ \kappa = 0.58520 \]  
\[ b = -0.45720. \]  

(12)

One may notice that for the given interval of \( \rho \) in the Fig. 3 (i.e. \( \rho \leq 10 \)) this is a very well matched simulation. Furthermore, a simple calculation shows that within order of 10Å, for the size of H-atom, having \( \rho \leq 10 \) restricts \( \alpha\beta \) to be greater than one.

Continuing the solution of (8), after usual separation of variables, the radial part of the Schrödinger equation (8) reads, \((s-\text{state})\)

\[ -\frac{1}{2r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) R(r) + \frac{\alpha}{\beta} W_s(\rho) R(r) = \varepsilon R(r) \]  

(13)

where \( W_s(\rho) \) is given by (11). We impose the following change of variables

\[ r = \beta \rho \text{ and } u(\rho) = \rho R(\rho) \]  

(14)
to get
\[-\frac{1}{2} \frac{d^2 u}{d\rho^2} (\rho) - \alpha\beta \left[ G (1 - e^{-\kappa(\rho-b)})^2 + V_0 + \frac{1}{4} B \left( \frac{1}{4}, \frac{1}{4} \right) \right] u (\rho) = \frac{\varepsilon}{\alpha^2} (\alpha\beta)^2 u (\rho). \quad (15)\]

The latter equation, by introducing \( x = \kappa (\rho - b) \) yields
\[-\frac{1}{2} \frac{d^2 u}{dx^2} (x) - |A| (2e^{-x} - e^{-2x}) u (x) = Eu (x) \quad (16)\]

where
\[|A| = \frac{\alpha\beta}{\kappa^2} |G| > 0, \quad (17)\]
\[E = \frac{1}{\kappa^2} \left[ \frac{\varepsilon}{\alpha^2} (\alpha\beta)^2 + \alpha\beta \left( V_0 + \frac{1}{4} B \left( \frac{1}{4}, \frac{1}{4} \right) - |G| \right) \right] < 0. \quad (18)\]

We notice that since \( 0 < \rho < \infty \) then \( \kappa b < x < \infty \) in which infinity is in contrast with the dimension of the atom. Also we note that this is (an energy-dependent-potential) Schrödinger equation and therefore in the solution we identify \( V_0 \) (consequently \( \alpha\beta \)) and \( E \) simultaneously. One can show that the proper solution which satisfies the boundary conditions
\[\lim_{x \to \infty} u (x) = 0, \quad \lim_{x \to \kappa |b|} u (x) = 0 \quad (19)\]
is given by
\[u (x) = C \text{WhittakerM} (a, \nu, 2ae^{-x}), \quad (20)\]
\[a = \sqrt{2 |A|}, \quad \nu = \sqrt{2 |E|}, \quad (21)\]
in which \( 0 < \nu \in \mathbb{R} \) and \( \text{WhittakerM}(a, \nu, 2ae^{-\kappa|b|}) = 0 \). In fact here \( \nu \) is not a quantum number but it is a new parameter and we shall use it to adjust the results, and \( 2ae^{-\kappa|b|} = X_\nu \) is the first root of \( \text{WhittakerM}(a, \nu, 2ae^{-x}) = 0 \). It is remarkable to observe that once we choose \( \nu \), we identify both potential and energy of the system at the same time.

Hence one can show that
\[\frac{\varepsilon}{\alpha^2} = \frac{1}{(\alpha\beta)^2} \left[ \frac{\kappa^2 \nu^2}{2} - \alpha\beta \left( V_0 + \frac{1}{4} B \left( \frac{1}{4}, \frac{1}{4} \right) - |G| \right) \right], \quad (22)\]
\[\alpha\beta = \frac{\kappa^2 a^2}{2 |G|}. \quad (23)\]

This closed expression helps us to adjust \( \nu \) in order to set the ground-state energy of the H-atom in accordance with the empirical data and consequently to find the corresponding
value for Born’s parameter $\beta$. By setting $\nu = 2.89873$ one finds $X_\nu = 6.756270935$ and consequently $a = 4.414424$, $\alpha \beta = 1.823373498$ and $\frac{\varepsilon}{\alpha^2} = -0.4997331195$.

This value for $\alpha \beta$ is comparable with $\alpha \beta_B$ suggested by Born\cite{7} and the one reported in \cite{3}. In Tab.1 we compare our results with the usual Coulomb potential and numerical results of Ref.\cite{3}.

|                     | $-\frac{\varepsilon_{\text{up}}}{\alpha^2}$ | $\alpha \beta$ |
|---------------------|---------------------------------------------|-----------------|
| Our results(Morse-type-potential) | 0.4997331195 | 1.823373498   |
| Ref.\cite{3}        | 0.50000 | 1.83297   |
| Empirical           | 0.49973 | 0           |
| Born’s proposal     | $-$    | 1.2361$\alpha^2$ |

(Tab.(1))

In conclusion, we recall that our attempt was to show that one can always find a simulation for the functions whose closed forms are not known. This provides analytical approximations for the final results. Having such analytical solutions for physical systems always have some significant features consisting the application of the results for the similar systems. In particular we simulated the BIC’s potential into a Morse-type potential to find an estimation for the Born’s parameter. We have shown that in this approach, Born’s parameter has a value between the original value proposed by Born and the value found by H. Carley et. al.

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I. FIGURE CAPTIONS

Fig.(1): A plot of $Z(\rho)$ as a function of $\rho = r/\beta$. This plot displays that $Z(\rho)$ asymptotically approaches to 1.

Fig.(2): A plot of $W(\rho) = \frac{V(\rho)}{\alpha\beta}$ versus $\rho$. This BIC’s potential takes a finite value at the origin and vanishes as $\rho \to \infty$.

Fig.(3): A plot of the original BIC’s potential $W(\rho)$ and the simulated Morse-type potential $W_s(\rho)$ (Eq.(6)) for $G = -1.83000$, $V_o = 0.09805$, $\kappa = 0.58520$ and $b = -0.45720$. 

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