TOTALISTIC TWO-DIMENSIONAL CELLULAR AUTOMATA
EXHIBITING GLOBAL PERIODIC BEHAVIOR

N. BOCCARA* and M. ROGER
DRECAM SPEC Centre d’Études de Saclay,
91191 Gif-sur-Yvette Cedex, France

Received
Revised

We have determined families of two-dimensional deterministic totalistic cellular automaton rules whose stationary density of active sites exhibits a period two in time. Each family of deterministic rules is characterized by an “average probabilistic totalistic rule” exhibiting the same periodic behavior.

Many natural populations of plants and animals exhibit large fluctuations of density with a roughly cyclic behavior. A well-known example is the oscillatory behavior of the Canadian lynx population as documented in the data compiled by the Hudson Bay Company over the period 1735-1940. Oscillations with an approximate period of about 10 years are observed with large amplitude fluctuations, which could, actually, correspond to a chaotic behavior. Good introductions to population dynamics may be found in May and Murray. Most models in population dynamics are formulated in terms of differential equations or difference equations, which means that the local character of the interactions between prey and predators, for example, is not taken into account. In order to describe more correctly the local character of the predation process, it would be better to formulate predator-prey models in terms of cellular automata (CAs), which are fully discrete dynamical systems evolving in time according to local rules. This paper does not deal with any specific model. Its purpose is to understand under which circumstances CAs exhibit nontrivial global behaviors. It has been argued that, in spatially isotropic systems with short-range interactions evolving in discrete time, periodic states with periods larger than 2, are never stable under generic conditions. Despite these arguments, Chaté and Manneville gave several examples of high-dimensional deterministic CAs exhibiting a global (quasi)period-3 behavior. In order to verify that the corresponding states were not metastable Gallas et al. performed a detailed study of the Chaté–Manneville CAs, and confirmed that the nontrivial behavior found by these authors extends to much longer times and to much larger lattices. Since the chance to find such a behavior should increase with the dimension of the CA, it is of interest to find low-dimensional CAs exhibiting the same behavior. A three-dimensional deterministic CA exhibiting a quasiperiodic behavior has been

*Permanent address: University of Illinois at Chicago, Dept. of Physics, Chicago, IL 60607-7059, USA
found by Hemmingson[10], and shortly after its nontrivial collective behavior has been confirmed through large-scale simulations (Chaté et al.[11]).

In this note we present families of two-dimensional totalistic class-3 CAs which exhibit a period-2 behavior. The class-3 requirement eliminates CAs whose attractor consists of only two points in the configuration space. No arguments have been given against such a collective behavior, but these systems might be of interest since they are not so frequent. Moreover, for each neighborhood defining a CA family, there exists a well-defined “average probabilistic CA rule” which exhibits the same collective behavior.

Two-dimensional deterministic totalistic CAs may be defined as follows: Let

\[ s : \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} \rightarrow \{0, 1\} \]

be a function that satisfies the equation

\[ s(i_1, i_2, t + 1) = f \left( \sum_{(j_1, j_2) \in V_r} s(i_1 + j_1, i_2 + j_2, t) \right), \]

for all \((i_1, i_2) \in \mathbb{Z}^2\) and all \(t \in \mathbb{N}\), where \(\mathbb{Z}\) is the set of all integers and \(\mathbb{N}\) the set of nonnegative integers. \(V_r\) defines a neighborhood of site \((i_1, i_2)\) which is either of the “von Neuman-type” with

\[ |j_1| + |j_2| \leq r \]

or of the “Moore-type” with

\[ |j_1| \leq r \quad \text{and} \quad |j_2| \leq r \]

The integer \(r\) characterizes the range of the neighborhood. For a given neighborhood, each totalistic rule is, therefore, determined by a function

\[ S \mapsto f(S), \]

where

\[ S \in \{0, 1, 2, \ldots, S_{\max}\} \quad \text{and} \quad f(S) \in \{0, 1\}, \]

with \(S_{\max} = (2r + 1)^2\) for a Moore-type neighborhood and \(S_{\max} = 2r^2 + 2r + 1\) for a von Neuman-type neighborhood.

For a given neighborhood, the number of rules is \(2^{S_{\max}}\). For relatively small neighborhoods \(S_{\max} \leq 13\), a systematic search for rules exhibiting nontrivial collective behavior is possible. For the simplest Moore and von Neuman neighborhoods, with \(r = 1\) we did not find any nontrivial behavior. The smallest neighborhood for which such a behavior has been found is the von Neuman neighborhood with \(r = 2\), and the corresponding family of \(2^{14} = 16384\) rules has been systematically investigated. Systematic investigations of families of CA rules with range larger than 2 is impossible. In this case, for each family, we studied a randomly selected sample of about 50000 rules.

**von Neuman neighborhood with \(r = 2\).**

Each rule has been first studied on a \(128 \times 128\)-lattice over 200 time steps. The average concentrations \(c_e\) and \(c_o\) of nonzero sites over the 50 last even and odd time steps were evaluated. We only retained rules satisfying the condition \(|c_e - c_o| > 0.1\). A few hundred rules satisfying this criterion have been thoroughly investigated on a \(512 \times 512\)-lattice over 20000 time steps. Several behaviors have been observed.

- Some of them are “similar” to Conway’s *Game of life*, in the sense that only periodic small structures, with various periods, remain on a period-2 background (see Figure 1). In this case, the attractor consists of a very few number of points in the configuration space.
Some converge to a fixed number of nonzero sites after large transient time during which the concentration oscillates. Their patterns are inhomogeneous (see Figure 2). The “local” concentration of nonzero sites oscillates; however, at large times, in the infinite-size limit, the concentration of nonzero sites tends to a fixed point (see Figure 2). Such a behavior has already been observed in some probabilistic cellular automata.

The 54 rules listed in Table I have a nontrivial collective period-2 behavior for the concentration of nonzero sites. They appear by pairs, rule \( g \) defined by
\[
g(S) = 1 - f(S_{\text{max}} - S)
\]
has the same properties as \( f \). A typical example (Rule 2077) is shown in Figures 4 and 5. Note that all these rules are such that \( f(0) = 1 \).

In order to characterize a family of rules \( \{ f_k \mid k \in K \} \), which exhibit a period two in the infinite-time limit, we define the function \( f_{\text{average}} : [0, 1] \to [0, 1] \) by
\[
f_{\text{average}}(S/S_{\text{max}}) = \frac{1}{|K|} \sum_{k \in K} f_k(S),
\]
where \(|K|\) denotes the number of rules belonging to the \( K \) family (see Figure 6).

**Rules with larger neighborhoods.**

We have studied three other families characterized by the following neighborhoods.

- Range-3 von Neuman neighborhood: \( S_{\text{max}} = 25 \), \( i.e. \) \( 2^{26} = 67108864 \) rules.
- Range-2 Moore neighborhood: \( S_{\text{max}} = 25 \), \( i.e. \) \( 2^{26} = 67108864 \) rules.
- Range-4 Moore neighborhood: \( S_{\text{max}} = 81 \), \( i.e. \) \( 2^{82} \approx 4.836 \times 10^{24} \) rules.
Fig. 2. Rule 2158: evolution of the concentration $c$ of nonzero sites on a $512 \times 512$-lattice.

Fig. 3. Spatial patterns of Rule 2158 at two consecutive time steps after $2 \times 10^4$ iterations. The lattice has $512 \times 512$ sites. There are short range correlations over a length scale which is much larger than the range $r = 2$ of the rule. However the concentration of nonzero sites tends to a fixed point on an infinite lattice.
Fig. 4. Rule 2077: evolution of the concentration of nonzero sites as a function of time on a 512 × 512-lattice. A period-2 behavior is clearly observed.

Fig. 5. Patterns of the Rule 2077 at two consecutive times after $2 \times 10^4$ iterations.
Table 1. Deterministic totalistic CA rules exhibiting period-2 collective behavior with a range-2 von-Neuman neighborhood.

| Rule | Initial State | Period | Final State |
|------|---------------|--------|-------------|
| 00000100111110  | 347  | 01000011011111  | 4219  |
| 00000110111110  | 445  | 01000011011111  | 4255  |
| 00001000111110  | 573  | 01000011011111  | 4335  |
| 00001001111110  | 637  | 01000011011111  | 4207  |
| 00010000111110  | 1079 | 00010001111111  | 1271  |
| 00010001111110  | 1083 | 00010001111111  | 2295  |
| 00010000111110  | 1085 | 01000011111111  | 4343  |
| 00010001111110  | 1143 | 01000011111111  | 4343  |
| 00010001111110  | 1147 | 01000011111111  | 4215  |
| 00010001111110  | 1149 | 01000011111111  | 4215  |
| 00010101111110  | 1275 | 01000011111111  | 4215  |
| 00100000111110  | 2061 | 01000111111111  | 2103  |
| 00101000011110  | 2075 | 01000111111111  | 2103  |
| 00101000111110  | 2077 | 01000111111111  | 2103  |
| 00101001111110  | 2107 | 01000111111111  | 2103  |
| 00101001111110  | 2109 | 01000111111111  | 2103  |
| 00101001111110  | 2171 | 01000111111111  | 2103  |
| 00101001111110  | 2173 | 01000111111111  | 2103  |
| 00101001111110  | 2177 | 01000111111111  | 2103  |
| 00101001111110  | 2189 | 01000111111111  | 2103  |
| 00101001111110  | 2219 | 01000111111111  | 2103  |
| 00101001111110  | 2237 | 01000111111111  | 2103  |
| 00101001111110  | 2301 | 01000111111111  | 2103  |
| 00101001111110  | 2329 | 01000111111111  | 2103  |
| 00101001111110  | 2557 | 01000111111111  | 2103  |
| 01000000111110  | 4109 | 01000111111111  | 2103  |
| 01000000111110  | 4125 | 01000111111111  | 2103  |
| 01000000111110  | 4157 | 01000111111111  | 2103  |
| 01000001111110  | 4189 | 01000111111111  | 2103  |
| 01000001111110  | 4221 | 01000111111111  | 2103  |
Fig. 6. Graph of the function $f_{\text{average}}$ characterizing the family of rules defined by a range-2 von Neumann neighborhood.
As mentioned above, a systematic investigation of such a large number of rules is impossible. Therefore, for each family, we only studied a sample of about 50,000 rules selected at random. We found that a small proportion of these samples did exhibit a period-2 behavior, i.e., about 0.7% of the rules with a neighborhood containing 25 sites, and 1% of the rules with a neighborhood containing 81 sites. In contrast with the family of rules with a range-2 von Neuman neighborhood, about 40 to 50% of them were such that $f(0) = 0$. For these three families, we determined the functions $f_{\text{average}}$ whose graphs are represented in Figure 7. These functions are well approximated by a simple combination of two sine functions of the form:

$$f_{\text{average}}(x) = 0.5 - A \sin(2\pi(x - 0.5)) - B \sin(2\pi(x - 0.5))$$

where $x = S / S_{\text{max}}$. Fitting the data, we found:

- for the range-3 von Neuman neighborhood, $A \approx 0.291$, $B \approx 0.008$;
- for the range-2 Moore neighborhood, $A \approx 0.304$, $B \approx 0.003$;
- for the range-4 Moore neighborhood, $A \approx 0.138$, $B \approx 0.074$.

![Graph of functions $f_{\text{average}}$ characterizing families of rules defined by: (a) range-3 von Neuman neighborhood (diamonds and dashed line), (b) range-2 Moore neighborhood (open circles and solid line), and (c) range-4 Moore neighborhood (bullets and solid line).](image)

To a function $f_{\text{average}}$ we associate a probabilistic totalistic law defined by

$$f(S) = \begin{cases} 
1, & \text{with probability } p = f_{\text{average}}(S / S_{\text{max}}); \\
0, & \text{with probability } 1 - p.
\end{cases}$$
Probabilistic totalistic CAs evolving in time according to rules of this type exhibit period-2 behaviors. As an example, the evolution of the concentration $c$ of nonzero sites for the probabilistic rule associated to the family defined by the range-4 Moore neighborhood is represented in Figure 8. The concentration oscillates between $c_1 = 0.337$ and $c_2 = 0.663$. The mean-field approximation predicts the amplitude of the oscillations with a very good accuracy. Within this approximation, it is found that the concentration oscillates between $c_{mfa}^1 = 0.335$ and $c_{mfa}^2 = 0.665$.

![Fig. 8. Concentration of nonzero sites of a probabilistic totalistic CA whose rule is determined by the function $f_{average}$ characterizing the family of rules defined by the range-4 Moore neighborhood.](image)

The proportion of deterministic totalistic class-3 CAs whose density of nonzero sites is periodic in time increases with the range of the neighborhood characterizing the evolution rule. In agreement with an argument due to Grinstein, no period larger than 2 has been observed. To each family of rules exhibiting a period-2 behavior, it is possible to associate a well-defined mapping from $[0, 1]$ to $[0, 1]$. Such a mapping defines a probabilistic totalistic CA rule, and the density of nonzero sites of a CA evolving according to such a rule also exhibits a period-2 behavior. For these probabilistic CAs, the mean-field approximation predicts the values of the density with a good accuracy.

**References**

1. W. M. Schaffer *Am. Nat.* **124**, 798 (1984).
2. W. M. Schaffer *Ecology* **66**, 93 (1985).
3. R. M. May *Proc. R. Soc. B* **228**, 241 (1985).
4. J. D. Murray
5. Mathematical Biology, Springer-Verlag (1989).
6. G. Grinstein *J. Stat. Phys.* **51**, 803 (1988).
7. H. Chaté and P. Manneville *Europhys. Lett.* **14**, 409 (1991).
8. H. Chaté and P. Manneville *Prog. Theor. Phys.* **87**, 1 (1992).
9. J. A. C. Gallas, P. Grassberger, H. J. Herrmann and P. Ueberholz *Physica A* **180**, 19 (1992).
10. J. Hemmingson *Physica A* **183**, 255 (1992).
11. H. Chaté, Grinstein and L. H. Tang *Phys. Rev. Lett.* **74**, 912 (1995).
12. N. Boccara, O. Roblin and M. Roger *J. Phys. A* **27**, 8039 (1994).