A LECTURE ON KAC–MOODY LIE
ALGEBRAS OF THE ARITHMETIC TYPE

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Abstract. We name an indecomposable symmetrizable generalized Cartan matrix $A$ and the corresponding Kac–Moody Lie algebra $g'(A)$ of the arithmetic type if for any $\beta \in Q$ with $(\beta|\beta) < 0$ there exist $n(\beta) \in \mathbb{N}$ and an imaginary root $\alpha \in \Delta^{\text{im}}$ such that $n(\beta)\beta \equiv \alpha \mod \text{Ker}(.,.)$ on $Q$. Here $Q$ is the root lattice. This generalizes "symmetrizable hyperbolic" type of Kac and Moody.

We show that generalized Cartan matrices of the arithmetic type are divided in 4 types: (a) finite, (b) affine, (c) rank two, and (d) arithmetic hyperbolic type. The last type is very closely related with arithmetic groups generated by reflections in hyperbolic spaces with the field of definition $\mathbb{Q}$.

We apply results of the author and É.B. Vinberg on arithmetic groups generated by reflections in hyperbolic spaces to describe generalized Cartan matrices of the arithmetic hyperbolic type and to show that there exists a finite set of series of the generalized Cartan matrices of the arithmetic hyperbolic type. For the symmetric case all these series are known.

0. Introduction.

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I am grateful to Professor V. Chari for useful discussions. I am grateful to Professor É.B. Vinberg for his interest to this subject.

We want to pay attention to one class of Kac–Moody Lie algebras which is very closely related with arithmetic reflection groups in hyperbolic spaces.

1. Reminding on symmetrizable Kac–Moody Lie algebras.

Here we recall results on symmetrizable Kac–Moody Lie algebras which we need. One can find them in the book by Victor Kac [Ka1].

(1.1) An $n \times n$-matrix $A = (a_{ij})$ is called a generalized Cartan matrix if

(C1) $a_{ii} = 2$ for $i = 1, ..., n$;

(C2) $a_{ij}$ are non-positive integers for $i \neq j$;

(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

We denote by $l$ the rank of $A$ and by $k = n - l$ the dimension of the kernel of $A$.

For simplicity, below we suppose that $A$ is indecomposable which means that there does not exist a decomposition $I = \{1, ..., n\} = I_1 \cup I_2$ such that both $I_1$ and $I_2$ are non-empty and $a_{ij} = 0$ for any $i \in I_1$ and any $j \in I_2$.

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A generalized Cartan matrix $A$ is called *symmetrizable* if there exists an invertible diagonal matrix $D = \text{diag}(\epsilon_1, \ldots, \epsilon_n)$ and a symmetric matrix $B = (b_{ij})$, such that

\[
A = DB; \quad \text{or} \quad (a_{ij}) = (\epsilon_i b_{ij})
\]

One always can suppose that

\[
(1.3') \quad \epsilon_i \in \mathbb{Q}, \quad \epsilon_i > 0 \text{ for any } 1 \leq i \leq n, \quad b_{ij} \in \mathbb{Z} \text{ and } B.C.D.({b_{ij} \mid 1 \leq i, j \leq n}) = 1.
\]

By (C1)—(C2), this is equivalent to

\[
(1.3) \quad b_{ij} \in \mathbb{Z}, \quad b_{ii} > 0, \quad b_{ij} \leq 0 \text{ for } i \neq j \text{ and } B.C.D.({b_{ij} \mid 1 \leq i, j \leq n}) = 1.
\]

Then the matrices $D$ and $B$ are defined uniquely. Later we always suppose that these conditions (1.3) are satisfied.

One formally defines the *root lattice* and the *root semigroup*

\[
(1.4) \quad Q = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i, \quad Q_+ = \sum_{i=1}^{n} \mathbb{Z}_+\alpha_i \subset Q
\]

where $\mathbb{Z}_+$ denotes non-negative integers. Similarly, one defines the *coroot lattice* $Q^\vee$ and the *coroot semigroup* $Q_+^\vee$.

\[
(1.5) \quad Q^\vee = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i^\vee, \quad Q_+^\vee = \sum_{i=1}^{n} \mathbb{Z}_+\alpha_i^\vee \subset Q^\vee.
\]

One has a natural pairing

\[
(1.6) \quad \langle ., . \rangle : Q^\vee \times Q \to \mathbb{Z}, \quad \langle \alpha_i^\vee, \alpha_j \rangle = a_{ij} \quad (i, j = 1, \ldots, n)
\]

and an integral symmetric bilinear form

\[
(1.7) \quad (., .) : Q \times Q \to \mathbb{Z}, \quad (\alpha_i | \alpha_j) = b_{ij} = a_{ij}/\epsilon_i.
\]

The symmetric bilinear form $(., .)$ is called *canonical*. Pairings (1.6) and (1.7) are connected by the formula

\[
(1.8) \quad a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = 2(\alpha_i | \alpha_j)/(\alpha_i | \alpha_i).
\]

*Kac–Moody Lie algebra* $g^\prime(A)$ is the complex Lie algebra defined by $3n$ generators $e_1, \ldots, e_n, f_1, \ldots, f_n, \alpha_1^\vee, \ldots, \alpha_n^\vee$ and defining relations

\[
(1.9) \quad [\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [e_i, f_j] = \delta_{ij}\alpha_i^\vee, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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for any $1 \leq i, j \leq n$, where $\delta_{ij}$ is the Kroneker symbol. The main property of $g'(A)$ is that $h' = Q^\vee \otimes \mathbb{C}$ is the maximal commutative subalgebra of $g'(A)$; the center of $g'(A)$ is equal to the

\begin{equation}
\mathfrak{c} = \{ x \in h' \mid \langle x, Q \rangle = 0 \},
\end{equation}

and the Lie algebra $g'(A)/\mathfrak{c}$ is simple. The Lie algebra $g'(A)$ has so called root space decomposition

\begin{equation}
g'(A) = \bigoplus_{\alpha \in Q^+} g_{-\alpha} \oplus h' \oplus \bigoplus_{\alpha \in Q^+} g_\alpha
\end{equation}

where for a non-zero $\alpha \in \pm Q_+$, one set

$$
g_\alpha = \{ x \in g'(A) \mid [h, x] = \langle h, \alpha \rangle x, \forall h \in h' \}.
$$

It is known that $\dim g_\alpha < \infty$. An element $0 \neq \alpha \in \pm Q_+$ is called a root if $\dim g_\alpha > 0$; the dimension $\dim g_\alpha$ is called the multiplicity of the root $\alpha$. The set of roots is denoted by $\Delta$. It is a disjoint union $\Delta = \Delta_+ \cup -\Delta_+$ where $\Delta_+ = \Delta \cap Q_+$. A description of roots and their multiplicities is an important problem for the theory of Kac–Moody Lie algebras.

Due to Victor Kac, we have the following description of the set of roots $\Delta$. Evidently, $\alpha_1, ..., \alpha_n$ are roots (they are called simple roots). One defines fundamental reflections $r_{\alpha_i} \in GL(Q)$ by the formula

\begin{equation}
r_{\alpha_i}(x) = x - \frac{2(\alpha_i | x)}{(\alpha_i | \alpha_i)} \alpha_i = x - \langle \alpha_i^\vee, x \rangle \alpha_i, \quad x \in Q,
\end{equation}

and Weyl group $W$ generated by all reflections $r_{\alpha_i}, 1 \leq i \leq n$. Evidently, $W$ preserves the canonical symmetric bilinear form $(\cdot, \cdot)$. The set of roots $\Delta$ is invariant with respect to $W$ and is divided on two parts: real roots $\Delta^{re}$ and imaginary roots $\Delta^{im}$. Here, by definition,

\begin{equation}
\Delta^{re} = W(\alpha_1) \cup ... \cup W(\alpha_n).
\end{equation}

Let

\begin{equation}
K = \{ \alpha \in Q_+ - \{0\} \mid (\alpha | \alpha_i) \leq 0 \text{ for all } \alpha_i, 1 \leq i \leq n, \text{ and supp } \alpha \text{ is connected} \}.
\end{equation}

Here, for $\alpha = \sum_{i=1}^n k_i \alpha_i \in Q_+$, the subset supp $\alpha \subset \{ \alpha_1, ..., \alpha_n \}$ is the set of all $\alpha_i$ such that $k_i > 0$. The supp $\alpha$ is called connected if there does not exist a decomposition supp $\alpha = A_1 \cup A_2$ such that $(A_1|A_2) = 0$, with non-empty $A_1$ and $A_2$. One has

\begin{equation}
\Delta^{im}_+ = W(K).
\end{equation}

Since $(\cdot, \cdot)$ is $W$-invariant, from these results, it follows that

\begin{equation}
(\alpha | \alpha) > 0, \text{ if } \alpha \in \Delta^{re},
\end{equation}

and

\begin{equation}
(\alpha | \alpha) \leq 0, \text{ if } \alpha \in \Delta^{im}.
\end{equation}

Moreover, if $\alpha \in \Delta^{im}$, then $n\alpha \in \Delta^{im}$ for any $n \in \mathbb{N}$. 
2. Kac–Moody Lie algebras of the arithmetic type.

Definition 2.1. A generalized Cartan matrix $A$ and the corresponding Kac–Moody Lie algebra $g'(A)$ have the arithmetic type if $A$ is symmetrizable indecomposable and for the corresponding canonical symmetric bilinear form $(.,.)$ on the root lattice $Q$ one has: for each $\beta \in Q$ with the property $(\beta|\beta) < 0$ there exists $n(\beta) \in \mathbb{N}$ and an imaginary root $\alpha \in \Delta^{im}$ such that

$$n(\beta)\beta \equiv \alpha \mod Ker (.,.) \text{ on } Q.$$

Thus, the inequality $(\beta|\beta) < 0$ on the root lattice $Q$ should define imaginary roots up to the kernel of $(.,.)$ on $Q$ and multiplying by natural numbers.

To formulate results, let us denote by

$$S: M \times M \to \mathbb{Z}$$

the induced by $(.,.)$ canonical non-degenerate integral symmetric bilinear form on the free $\mathbb{Z}$-module $M = Q/Ker (.,.)$. We denote by $\pi: Q \to M$ the corresponding factorization map. To be shorter, we sometimes denote $\tilde{x} = \pi(x)$. In particular, we denote $\tilde{W} \subset O(S)$ the image of $W$ by $\pi$. Thus, $S$ is the symmetric bilinear form defined by the integral symmetric matrix $B$ (see (1.2)) modulo its kernel. We denote by $(t_+, t_-, t_0)$ the signature of a symmetric matrix. Thus, $t_+, t_-$ and $t_0$ are equal to numbers of positive, negative and zero "squares" respectively.

We have the following basic result which is well-known for the first cases (a), (b) and (c).

Theorem 2.1. A symmetrizable indecomposable generalized Cartan matrix $A$ and the corresponding Kac–Moody Lie algebra $g'(A)$ have the arithmetic type if and only if $A$ has one of the types (a), (b), (c) or (d) below:

(a) The finite type case: $B > 0$ (equivalently, $B$ has the signature $(l, 0, 0)$).

(b) The affine type case: $B \geq 0$ and $B$ has a 1-dimensional kernel (equivalently, $B$ has the signature $(l, 0, 1)$).

(c) The rank two hyperbolic case: $B$ is hyperbolic of the rank 2 (equivalently, $B$ or $S$ have the signature $(1, 1, 0)$).

(d) The arithmetic hyperbolic case: $B$ is hyperbolic of the rank $> 2$ (equivalently, $B$ has the signature $(l - 1, 1, k)$ where $l \geq 3$, or $S$ has the signature $(l - 1, 1)$ where $l \geq 3$) and the index $[O(S): \tilde{W}]$ is finite.

Proof. Let $(t_+, t_-, t_0)$ be the signature of $B$. If $t_- = 0$, we get cases (a) and (b), this is well-known (see [Ka1]). If $t_- = 1$ and $t_+ = 1$, we get the case (c). This is also well-known (see [Ka1]).

Thus, we assume that $t_- \geq 1$ and $l = t_+ + t_- \geq 3$. Let

$$\mathbb{R}Q_+ \subset Q \otimes \mathbb{R}$$

be the corresponding real cone in $Q \otimes \mathbb{R}$ generated by $Q_+$ with the origin at 0, and

$$\mathbb{R}+\mathbb{Q}_+ = \mathbb{R}_+\tilde{\alpha}_1 + \cdots + \mathbb{R}_+\tilde{\alpha}_n$$

its projection by $\pi$. We claim that

$$\mathbb{R}+\mathbb{Q}_+ \cap \mathbb{R}+\mathbb{Q}_- = \{0\}.$$
Otherwise, there are real \( \lambda_i \geq 0, 1 \leq i \leq n \), which are not all equal to 0 such that \( s = \lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n \in \text{Ker} \langle ., . \rangle \). By [Ka1, Theorem 5.6], there exists \( \alpha = \sum_{i=1}^{n} k_i \alpha_i \) such that \( k_i > 0 \) and \( (\alpha | \alpha_i) < 0 \) for any \( 1 \leq i \leq n \). Then evidently, \( s|\alpha) < 0 \), since all \( \lambda_i \geq 0 \) and not all of them are 0. We get a contradiction with \( s \in \text{Ker} \langle ., . \rangle \).

Let

\[
V(S) = \{ x \in M \otimes \mathbb{R} \mid S(x, x) < 0 \}.
\]

Let us suppose that \( A \) is of the arithmetic type. Since \( M \otimes \mathbb{Q} \) is everywhere dense in \( M \otimes \mathbb{R} \) and the set of roots \( \Delta^{im} \subset Q_+ \cup -Q_+ \), we get that

\[
V(S) \subset \widehat{\mathbb{R}_+Q_+} \cup -\widehat{\mathbb{R}_+Q_+}
\]

has at least two connected components by (2.3). It is well known and very easy to see that \( V(S) \) is connected if \( t_- > 1 \), and has two connected components if \( t_- = 1 \). Thus, we have proven that \( t_- = 1 \). Further, we suppose that this is the case.

Thus, further we suppose that \( t_+ \geq 2 \) and \( t_- = 1 \). Equivalently, the \( S \) is hyperbolic (i.e. of the signature \( (t_+, 1, k) \)) of the rank \( \geq 3 \). Then \( V(S) \) is a cone which is the union of two convex half-cones: \( V(S) = V^+(S) \cup -V^+(S) \) (later we will choose the half-cone \( V^+(S) \) canonically). We denote by

\[
L(S) = V^+(S)/\mathbb{R}_+
\]

the corresponding hyperbolic space. Its point is a ray \( \mathbb{R}_+x, x \in V^+(S) \). Any element \( \delta \in M \otimes \mathbb{R} \) with \( S(\delta, \delta) > 0 \) defines the half-space

\[
\mathcal{H}_{\delta}^+ = \{ \mathbb{R}_+x \in L(S) \mid S(x, \delta) \leq 0 \}
\]

bounded by the hyperplane

\[
\mathcal{H}_\delta = \{ \mathbb{R}_+x \in L(S) \mid S(x, \delta) = 0 \}.
\]

Then the \( \delta \) is called the vector which is orthogonal to the half-space and the hyperplane. This is defined uniquely up to multiplication \( \lambda \delta, \lambda > 0 \). Let

\[
\widehat{\mathbb{R}_+Q_+}^* = \{ x \in M \otimes \mathbb{R} \mid S(x, \tilde{\alpha}_i) \leq 0 \}
\]

be the dual cone to \( \widehat{\mathbb{R}_+Q_+} \).

**Lemma 2.2.** We have:

\[
Q_+\tilde{K} = (\widehat{\mathbb{R}_+Q_+} \cap \widehat{\mathbb{R}_+Q_+}^*) \cap M \otimes \mathbb{Q}.
\]

**Proof.** We first prove that

\[
\pi(Q_+\tilde{K}) = \pi(Q_+\{ \alpha \in Q_+ - \{0\} \mid (\alpha|\alpha_i) \leq 0 \text{ for all } \alpha_i, 1 \leq i \leq n \}).
\]

Let \( 0 \neq \beta \in Q_+ \) and \( (\beta|\alpha_j) \leq 0 \) for all \( 1 \leq j \leq n \). If \( \text{supp} \beta \) is not connected, there exists the canonical decomposition \( \beta = \beta_1 + \cdots + \beta_k \) with \( \beta_i \in Q_+ - \{0\} \), \( \supp \beta_1 \cap \supp \beta_2 = \emptyset \), \( (\beta_i|\beta) = 0 \) if \( i \neq j \), and \( k \geq 2 \). Then evidently \( \beta \in K \).
and \((\beta_i|\beta_j) \leq 0\). Since \(t_- = 1\) and \(k \geq 2\), it follows that \((\beta_i|\beta_j) = 0\) and there are \(\lambda_i \in \mathbb{N}\) such that \(\lambda_1 \beta_1 \equiv \lambda_2 \beta_2 \equiv \cdots \equiv \lambda_k \beta_k \mod \text{Ker}(.,.)\). Here we also use (2.3).

It follows (2.4). On the other hand, one can easily check that

\[
\pi(Q_+\{\alpha \in Q_+ - \{0\}|(\alpha|\alpha_i) \leq 0 \text{ for all } \alpha_i, \ 1 \leq i \leq n\}) = (\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*) \cap M \otimes \mathbb{Q}.
\]

It follows the proof of Lemma.

Using Lemma 2.2 and (1.15), we get that

\[
Q_+\Delta_{im}^{\mathbb{Q}} = \mathbb{W}(\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*) \cap M \otimes \mathbb{Q}.
\]

Here \(\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*\) is a convex connected cone which is evidently contained in \(V^+(S)\) since for \(x \in \mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*\) we have \(S(x,x) \leq 0\). It follows that we can choose the half-cone \(V^+(S)\) by the condition \(\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^* \subset V^+(S)\). (In fact, when we proved (2.3), we have mentioned that the cone \(\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*\) is non-degenerate, i.e. it contains a non-empty open subset of \(M \otimes \mathbb{R}\).)

Since \(M \otimes \mathbb{Q}\) is everywhere dense in \(M \otimes \mathbb{R}\), then we get that \(A\) is of the arithmetic type if and only if

\[
\mathbb{W}(\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*) = V^+(S).
\]

Let

\[
O_+(S) = \{\phi \in O(S) \mid \phi(V^+(S)) = V^+(S)\}
\]

a subgroup of the index two of the automorphism group of \(S\). From the arithmetic of integral symmetric bilinear forms, it is known that \(O_+(S)\) is discrete in the hyperbolic space \(L(S) = V^+(S)/\mathbb{R}_+\) and has a fundamental domain of a finite volume. The subgroup \(\mathbb{W} \subset O_+(S)\) is generated by reflections in hyperplanes \(H_{\alpha_i}\), \(1 \leq i \leq n\), orthogonal to \(\alpha_i\). By (1.3), we have \(S(\tilde{\alpha}_i, \tilde{\alpha}_j) \leq 0\) if \(i \neq j\). From the theory of groups generated by reflections in hyperbolic spaces (see [V1], [V2], for example), it follows that

\[
M = \bigcap_i H_{\alpha_i}^+ = \mathbb{H}^+(S)
\]

is a fundamental polyhedron of \(\mathbb{W}\) in \(L(S)\). Evidently,

\[
M = V^+(S) \cap \mathbb{R}_+Q_+^*/\mathbb{R}_+.
\]

By (2.6), we then get that \(A\) is of the arithmetic type if and only if the embedding

\[
(\mathbb{R}_+Q_+ \cap \mathbb{R}_+Q_+^*)/\mathbb{R}_+ \subset (V^+(S) \cap \mathbb{R}_+Q_+^*)/\mathbb{R}_+ = M
\]

is equality.

Let us suppose that \(A\) is of the arithmetic type. Thus, we have the equality for the embedding (2.9). It follows that \(M\) has finite volume since it is a convex envelope of a finite set of points in \(F(S)\) corresponding to edges of the finite polyhedral cone.
It follows that the index \( [O(S) : \widetilde{W}] < \infty \) because \( O_+(S) \) has a fundamental domain of a finite volume.

Now suppose that \( \mathcal{M} = V^+(S) \cap \mathbb{R}_+ Q_+^* / \mathbb{R}_+ \) has finite volume. Since \( \mathbb{R}_+ Q_+^* \) is a finite polyhedral cone, it is true if and only if \( \mathbb{R}_+ Q_+^* \subset V^+(S) \). Considering dual cones, we then get \( V^+(S)^* = V^+(S) \subset \mathbb{R}_+ Q_+^* \). Thus, finally we get the sequence of embedded cones:

\[
\mathbb{R}_+ Q_+^* \subset V^+(S) \subset \mathbb{R}_+ Q_+^*.
\]

It follows that the embedding (2.9) is equality, and that \( A \) is of the arithmetic type.

3. Invariants of a Kac–Moody Lie algebra of the arithmetic hyperbolic type.

Let \( A \) be a generalized Cartan matrix (or the corresponding Kac–Moody Lie algebra \( g'(A) \)) of the arithmetic hyperbolic type. In fact, above, we have defined for \( A \) several invariants which we describe more precisely below.

The main invariant is the isomorphism class of the reflective primitive hyperbolic non-degenerate integral symmetric bilinear form of the rank \( l \geq 3 \) (using (1.3) and (2.2)):

\[
S : M \times M \to \mathbb{Z}.
\]

Here \( hyperbolic \) means that \( S \) has signature \((l - 1, 1)\); \( primitive \) means that \( S/k \) is not integral for any integral \( k > 1 \); \( reflective \) means that \( [O(S) : W(S)] < \infty \). Here \( W(S) \) denote a subgroup of \( O_+(S) \) generated by all reflections. Here we consider elements of \( O_+(S) \) as motions of the hyperbolic space \( L(S) \). An automorphism \( \phi \in O_+(S) \) is called \( reflection \) if \( \phi \) acts in \( L(S) \) as a reflection with respect to a hyperplane of \( L(S) \). One can easily see that every reflection \( \phi \in O(S) \) is equal to \( r_\delta \) for some \( \delta \in M \) with the property \( S(\delta, \delta) > 0 \), where

\[
r_\delta : x \to x - (2S(x, \delta)/S(\delta, \delta))\delta, \quad x \in M,
\]

and \( r_\delta \in O(S) \) if and only if

\[
(2S(x, \delta)/S(\delta, \delta))\delta \in M \text{ for any } x \in M.
\]

(The automorphism \( r_\delta \) of \( L(S) \) is the reflection in the hyperplane \( \mathcal{H}_\delta \) which is orthogonal to \( \delta \).) The reflection \( r_\delta \) will not change if one replaces \( \delta \) by \( \lambda \delta, \lambda \in \mathbb{Q}^* \). Thus, we can take \( \delta \) to be primitive in \( M \). Then (3.3) is equivalent to

\[
S(\delta, \delta)|2S(M, \delta), \text{ for primitive } \delta \in M.
\]

The next invariant is:

\[
A \text{ finite index subgroup generated by reflections } \widetilde{W} \subset W(S).
\]

This subgroup is defined up to automorphisms of \( O(S) \). Let \( \mathcal{M} \) be a fundamental polyhedron of \( \widetilde{W} \) and

\[
\mathcal{M} = \bigcap_{\delta \in O(S)} \mathcal{H}_\delta^+.
\]
where \( P(\mathcal{M})_{pr} \) is the set of primitive vectors of \( M \) which are orthogonal to codimension 1 faces of \( \mathcal{M} \). The subgroup \( \tilde{W} \) evidently has the property:

\[
(3.7) \quad \text{the set } P(\mathcal{M})_{pr} \text{ generates } M.
\]

This property (3.7) is important since not every subgroup of \( W(S) \) of a finite index and generated by reflections has this property.

The elements \( \tilde{\alpha}_1, ..., \tilde{\alpha}_n \) above (see (2.7)) are orthogonal to faces of \( M \). Thus,

\[
\tilde{\alpha}_i = \lambda_i \delta_i \quad \text{where } \delta_i \in P(\mathcal{M})_{pr}, \lambda_i \in \mathbb{N}.
\]

Evidently, here \( \delta_i \mapsto \tilde{\alpha}_i = \lambda_i \delta_i \) is the isomorphism between sets \( P(\mathcal{M})_{pr} \) and \( \tilde{\alpha}_1, ..., \tilde{\alpha}_n \). Moreover, here we get another invariant of \( A \) which is a function

\[
(3.8) \quad \lambda : P(\mathcal{M})_{pr} \to \mathbb{N}, \quad \delta_i \mapsto \lambda_i.
\]

This function satisfies two important properties:

\[
(3.9) \quad \{ \lambda(\delta) \delta \mid \delta \in P(\mathcal{M})_{pr} \} \text{ generates } M
\]

(in particular, \( B.C.D. (\{ \lambda(\delta) \mid \delta \in P(\mathcal{M}) \}) = 1 \)) and

\[
(3.10) \quad S(\lambda(\delta) \delta, \lambda(\delta) \delta) | 2S(\lambda(\delta') \delta', \lambda(\delta) \delta) \quad \text{for any } \delta, \delta' \in P(\mathcal{M}).
\]

The last property follows from (1.8) and axioms (C1), (C2).

Thus, we correspond to a generalized Cartan matrix \( A \) and the Kac–Moody Lie algebra \( g'(A) \) of the arithmetic hyperbolic type the triplet of invariants:

\[
(3.11) \quad (S, \tilde{W} \subset W(S), \lambda : P(\mathcal{M})_{pr} \to \mathbb{N})
\]

satisfying the conditions (3.1)—(3.10). We evidently have

**Theorem 3.1.** Invariants (3.11) define the generalized Cartan matrix \( A \) and Kac–Moody Lie algebra \( g'(A) \) of the arithmetic hyperbolic type by the formula

\[
(3.12) \quad A = (2S(\lambda(\delta') \delta', \lambda(\delta) \delta) / S(\lambda(\delta) \delta, \lambda(\delta) \delta)), \quad \delta, \delta' \in P(\mathcal{M})_{pr}.
\]

**Proof.** It follows at once from our construction and (1.8).

4. On the classification of Kac–Moody Lie algebras of the arithmetic hyperbolic type.

The basic fact here is the following result. Its first part (a) was proved by the author [N4], [N5], and the second part by É.B. Vinberg [V3].

**Theorem 4.1.** (V.V. Nikulin, [N4], [N5], and É.B. Vinberg, [V3]) (a) For a fixed rank \( r \kern -0.2em \langle \kern 0.2em S = l \geq 3 \), the set of all isomorphism classes of reflective primitive hyperbolic integral symmetric bilinear forms \( S \) is finite.

(b) If \( S \) is a reflective hyperbolic integral symmetric bilinear form, then \( r \kern -0.2em \langle \kern 0.2em \text{rk } S \leq 30 \).

In particular, the whole set of isomorphism classes of reflective primitive hyperbolic integral symmetric bilinear forms \( S \) of the rank \( r \kern -0.2em \langle \kern 0.2em \text{rk } S \geq 3 \) is finite.

Thus, by Theorem 4.1, in principle, it is possible to describe all reflective primitive hyperbolic integral symmetric bilinear forms \( S \) of the rank \( r \kern -0.2em \langle \kern 0.2em \text{rk } S \geq 3 \).

Now let us fix one of reflective primitive hyperbolic integral symmetric bilinear forms \( S : M \times M \to \mathbb{Z} \).

The next invariant of the triplet (3.11) is a subgroup of a finite index \( \tilde{W} \subset W(S) \) generated by reflections. We evidently have...
Proposition 4.2. For a fixed index \( N = [W(S) : \tilde{W}] \), the set of generated by reflections subgroups \( \tilde{W} \subset W(S) \) is finite.

Proof. A fundamental polyhedron \( M \) of \( \tilde{W} \) is a convex polyhedron which is a union of \( N \) fundamental polyhedra of \( W(S) \). It follows that the number of possibilities for \( M \) is finite up to the action of \( \tilde{W} \).

Now, let us choose a subgroup \( \tilde{W} \subset W(S) \) generated by reflections and of a finite index. Let us choose a fundamental polyhedron \( M \) of \( \tilde{W} \), and let \( P(M)_{pr} \) be the set of primitive elements of \( M \) which are orthogonal to codimension 1 faces of \( M \) and directed outside (i.e. \( P(M)_{pr} \) is a minimal set of primitive elements of \( M \) with the property (3.6)). Let us additionally suppose that we have the property (3.7) for \( P(M)_{pr} \). Thus, we require that

\[(4.1) \quad P(M)_{pr} \text{ generates } M. \]

This gives some additional restrictions on \( \tilde{W} \) and even on the form \( S \) itself because a fundamental polyhedron \( M_0 \) of the \( W(S) \) should then have the same property:

\[(4.2) \quad P(M_0)_{pr} \text{ generates } M. \]

By (3.3), we have:

\[(4.3) \quad S(\delta, \delta) | 2S(\delta', \delta), \; \delta, \delta' \in P(M)_{pr}. \]

By (4.3), Theorem 2.1 and our construction, it evidently follows

**Theorem 4.3.** A reflective primitive hyperbolic integral symmetric bilinear form

\[ S : M \times M \to \mathbb{Z} \]

and a subgroup of a finite index generated by reflections

\[ \tilde{W} \subset W(S) \]

satisfying the condition (4.1) (and (4.2)) for a fundamental polyhedron \( M \) of \( \tilde{W} \) canonically define a generalized Cartan matrix of the arithmetic hyperbolic type

\[ A(S, \tilde{W}) = (2S(\delta', \delta)/S(\delta, \delta)), \; \delta, \delta' \in P(M)_{pr}, \]

with the first two invariants (3.11) equal to the \((S, \tilde{W} \subset W(S))\).

In particular, for \( \tilde{W} = W(S) \), the generalized Cartan matrix \( A(S) = A(S, W(S)) \) is defined canonically by the reflective form \( S \) itself.

Now, let us consider functions \( \lambda : P(M)_{pr} \to \mathbb{N} \) satisfying the conditions (3.9), (3.10). Thus, we require:

\[(4.4) \quad S(\lambda(\delta)\delta, \lambda(\delta)\delta) | 2S(\lambda(\delta')\delta', \lambda(\delta)\delta), \; \delta, \delta' \in P(M)_{pr}. \]

and

\[(4.5) \quad \{\lambda(\delta) \delta \mid \delta \in P(M)_{pr}\} \text{ generates } M \]

(in particular, \( B.C.D. (\{\lambda(\delta) \mid \delta \in P(M)_{pr}\}) = 1 \)).

We have
Proposition 4.4. The set of functions $\lambda : P(M) \to \mathbb{N}$ which satisfy the conditions (4.4) and (4.5) is finite.

Proof. Since the polyhedron $M$ has finite volume, for any two elements $\delta, \delta' \in P(M)_{pr}$, there exists a sequence $\delta = \delta_1, ..., \delta_k = \delta' \in P(M)_{pr}$ such that

$$S(\delta_i, \delta_{i+1}) \neq 0, \quad i = 1, ..., k - 1$$

(this property is well-known, see [V1]).

Using this property and (4.3), one can easily prove that up to replacing $\lambda$ by $t\lambda$ where $t \in \mathbb{Q}$, there exists only a finite set of functions $\lambda : P(M)_{pr} \to \mathbb{N}$ satisfying the condition (4.4). By the condition (4.5), the set of possible $t$ is also finite.

Now let us choose one of functions $\lambda : P(M) \to \mathbb{N}$ satisfying the conditions (4.4) and (4.5).

Then by our construction and Theorem 2.1, we have

Theorem 4.5. A reflective primitive hyperbolic integral symmetric bilinear form

$$S : M \times M \to \mathbb{Z},$$

a subgroup of a finite index generated by reflections

$$\tilde{W} \subset W(S)$$

satisfying the condition (4.1) (and (4.2)) for a fundamental polyhedron $M$ of $\tilde{W}$, and a function

$$\lambda : P(M)_{pr} \to \mathbb{N}$$

satisfying the conditions (4.4) and (4.5) canonically define a generalized Cartan matrix of the arithmetic hyperbolic type

$$A(S, \tilde{W}, \lambda) = (2S(\lambda(\delta')\delta, \lambda(\delta)\delta)/S(\lambda(\delta)\delta, \lambda(\delta)\delta)), \quad \delta, \delta' \in P(M)_{pr},$$

with the invariants (3.1) equal to the $(S, \tilde{W}, \lambda)$.

In particular, the case $A(S, \tilde{W})$ of the Theorem 4.3 corresponds to the case $\lambda = 1$, and the case $A(S)$ corresponds to the case $\tilde{W} = W(S)$ and $\lambda = 1$.

Thus, we have described all possible invariants (3.11) of generalized Cartan matrices of the arithmetic hyperbolic type and by Theorem 3.1 gave the description of all generalized Cartan matrices of the arithmetic hyperbolic type.

There exists a finite set of series of these matrices corresponding to a finite set of reflective primitive hyperbolic integral symmetric bilinear forms of Theorem 4.1. These algebras almost canonically correspond to these forms since they are constructed by a subgroup $\tilde{W} \subset O(S)$ of a finite index (with a finite set of additional data — $\lambda$). Generalized Cartan matrices $A(S)$ and $A(S, \tilde{W})$ canonically correspond to $S$ and a choice of the subgroup $\tilde{W}$. Thus, in principle, all information about these Kac—Moody Lie algebras one can get from the arithmetic of the reflective primitive hyperbolic integral symmetric bilinear forms $S$ which are described by Theorem 4.1.
In [Ka1], there was considered a very particular case of generalized Cartan matrices $A$ and corresponding Kac–Moody Lie algebras $g'(A)$ of the arithmetic hyperbolic type. They are called hyperbolic. In our notation, symmetrizable hyperbolic case is exactly the case when the fundamental polyhedron $\tilde{\mathcal{M}}$ of $\tilde{W}$ is a simplex. There exists only a finite list of these $A$. These $A$ are characterized by the property: $0 \neq \delta \in Q$ is an imaginary root if and only if $(\delta|\delta) \leq 0$.

Unfortunately, the complete list of the reflective forms $S$ of Theorem 4.1 is not known yet.

5. Symmetric case.

Let us consider symmetric generalized Cartan matrices $A$ of the arithmetic hyperbolic type. Then we put $B = A$, and the subgroup $\widetilde{W} \subset O(S)$ is a subgroup of the group $W^{(2)}(S)$ generated by reflections in vectors $\delta \in \mathcal{M}$ such that $S(\delta, \delta) = 2$. Thus, the hyperbolic integral symmetric bilinear form $S$ should be 2-reflective, which means that $[O(S) : W^{(2)}(S)] < \infty$. All these 2-reflective forms $S$ and fundamental polyhedra $\mathcal{M}_0$ for $W^{(2)}(S)$ are found (see [N1], [N2], [N3] and [N6]). For these forms, the maximum $rk S = 19$. Thus, one has a description of all series of symmetric generalized Cartan matrices $A$ of the arithmetic hyperbolic type.
References

[Ka1] V. Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, 1985.

[N1] V. V. Nikulin, *On factor groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections*, Dokl. Akad. Nauk SSSR 248 (1979), 1307–1309; English transl. in Soviet Math. Dokl. 20 (1979), 1156-1158.

[N2] ———, *On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections*, Algebraic-geometric applications, Current Problems in Math. Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow (1981), 3-114; English transl. in J. Soviet Math. 22 (1983), 1401-1476.

[N3] ———, *Surfaces of type K3 with finite automorphism group and Picard group of rank three*, Proc. Steklov. Math. Inst. 165 (1984), 113-142; English transl. in Trudy Inst. Steklov 3 (1985).

[N4] ———, *On arithmetic groups generated by reflections in Lobachevsky spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), 637 – 669; English transl. in Math. USSR Izv. 16 (1981).

[N5] ———, *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981), no. 1, 113 – 142; English transl. in Math. USSR Izv. 18 (1982).

[N6] ———, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, Proc. Int. Congr. Math. Berkeley 1986, vol. 1, pp. 654-669.

[V1] E.B. Vinberg, *Discrete groups generated by reflections in Lobachevsky spaces*, Math USSR Sb. 1 (1967).

[V2] ———, *The absence of crystallographic reflection groups in Lobachevsky spaces of large dimension*, Trudy Moscow. Mat. Obshch. 47 (1984), 68 – 102; English transl. in Trans. Moscow Math. Soc. 47 (1985).

[V3] ———, *The absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension*, Trans. Moscow Math. Soc. (1985), 75–112.

[V4] ———, *Discrete reflection groups in Lobachevsky spaces*, Proc. Int. Congr. Math. Warsaw 1983, pp. 593-601.