Families of solutions of differential equations that are defined by contour integrals

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Abstract

We present and derive properties of two families of solutions of linear differential equations that are defined by contour integrals. One family includes the Airy integral and Airy’s equation as a particular case, and the derived properties of the family are generalizations of known properties of the Airy integral, which include exponential decay growth in a certain sector. The second family includes a known example and is shown to be related to another known example by a specific identity.

Keywords: Contour integral solutions, linear differential equations, polynomial coefficients, Airy integral, Airy’s equation.

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1 Introduction

Many people have expressed that it would be useful to have more examples of solutions of complex differential equations that have concrete properties. The authors share this sentiment and the purpose of this paper is to make a contribution to this topic which includes a generalization of the Airy integral.

The solutions of the linear differential equation

\[ f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_1(z)f' + P_0(z)f = 0 \]  

(1.1)

with polynomial coefficients \( P_0(z), \ldots, P_{n-1}(z) \) are entire functions. We assume that \( n \geq 2 \) and \( P_0(z) \neq 0 \). A transcendental solution \( f \) of equation (1.1) has a finite rational order \( d > 0 \) and finite type \( c > 0 \), such that

\[ \log M(r, f) = (c + o(1))r^d \]  

(1.2)
as $r \to \infty$, where $M(r, f)$ is the maximum modulus function; see [18, p. 108]. For possible values of $d$, see [7, Theorem 1].

Equations of the form (1.1) that possess solutions with interesting properties are used to (a) illustrate results, (b) give examples on the “boundary line” of theorems, (c) provide counterexamples, and (d) exhibit possibilities that can occur. A classical example is Airy’s equation $f'' - zf = 0$, which possesses the Airy integral solution $\text{Ai}(z)$, which has been used for the purposes (a)-(d), e.g., see [3], [4], [10], [11], [14], [15], [16]. The Airy integral has a well-known representation as a contour integral, see Example 1 below.

We present and derive properties of two families of transcendental solutions of equations of the two respective forms

$$f^{(n)} + (-1)^{n+1}bf^{(k)} + (-1)^{n+1}zf = 0, \quad 0 < k < n, \quad b \in \mathbb{C},$$

$$f^{(n)} - zf^{(k)} - f = 0, \quad 1 < k < n,$$

where the solutions are defined by contour integrals.

The first family includes the Airy integral $\text{Ai}(z)$ and Airy’s equation, and the derived properties of the family generalize well-known properties of $\text{Ai}(z)$, which include exponential decay growth in a certain sector, see Theorems 1-5. Since the Airy integral is a special function and Airy’s equation has several applications (e.g., theory of diffraction, dispersion of water waves, turning point problem; see [13], [17]), this paper may be of interest in the areas of special functions and applied mathematics.

The second family includes a known example and is connected to another known example. The derived properties of the second family include particular features of a subclass of the family and a separate analysis of three pairwise linearly independent contour integral solutions of the equation $f^{(4)} - zf''' - f = 0$ which are related by a specific identity, see Theorems 6-11.

Although there seem to be relatively few investigations involving contour integral solutions of differential equations in recent years, there are worthwhile questions to consider on this classical topic. It is hoped that this paper and possible future studies on contour integral solutions can be used for the above purposes (a)-(d).

The two families are discussed in Sections 3-4 and Sections 5-6, respectively. The next section contains some known examples.

## 2 Known solutions from contour integrals

Solutions of (1.1) that can be represented by contour integrals often have interesting properties. Throughout the paper, $\rho(f)$ denotes the order of an entire function $f$. 
Example 1 [5], [12], [13], [14] The Airy integral \( f(z) = \text{Ai}(z) \) is a solution of the Airy differential equation

\[
f'' - zf = 0
\]

that is defined by the contour integral

\[
\text{Ai}(z) = \frac{1}{2\pi i} \int_C \exp \left\{ \frac{1}{3}w^3 - wz \right\} \, dw,
\]

where the contour \( C \) runs from \( \infty \) to 0 along \( \arg w = -\pi/3 \) and then from 0 to \( \infty \) along \( \arg w = \pi/3 \). The Airy integral \( \text{Ai}(z) \) has the following properties: (i) \( \rho(\text{Ai}) = 3/2 \) and \( |\text{Ai}(z)| \) has exponential decay growth of order \( \exp\{-|z|^{3/2}\} \) in the sector \(-\pi/3 < \arg z < \pi/3\), and (ii) \( \text{Ai}(z) \) has an infinite number of negative real zeros and no other zeros, plus \( \text{Ai}(z) \) has less than the usual overall frequency of zeros. We generalize property (i) in Theorem 3.

If \( \beta_1, \beta_2, \beta_3 \) are the three distinct cube roots of unity, and if we set

\[
f_j(z) = \text{Ai}(\beta_j z), \quad j = 1, 2, 3,
\]

then the functions \( f_1, f_2, f_3 \) are three pairwise linearly independent solutions of (2.1). Thus, every solution of (2.1) can be expressed as a contour integral function. We generalize property (2.3) in Theorem 4.

In Sections 3-4 we present and discuss a family of equations and solutions which includes Airy’s equation and the Airy integral.

Example 2 [6] A generalization of the Airy integral is as follows. For each positive integer \( q \), there exist contour integral functions of the form

\[
g(z) = \frac{1}{2\pi i} \int_D e^{P_q(z,w)} \, dw,
\]

where \( P_q(z,w) \) is a polynomial in \( z \) and \( w \), and where the contour \( D \) can be chosen to consist of two rays that are connected in a similar way as in Example 1, such that \( f = g(z) \) is a solution of the equation

\[
f'' - z^q f = 0.
\]

When \( q = 1 \), (2.5) is Airy’s equation (2.1) and the contour \( D \) can be chosen so that \( g(z) \) in (2.4) reduces to the Airy integral \( \text{Ai}(z) \) in (2.2). Generalizing Example 1 for each \( q \), the functions \( g(z) \) in (2.4) have the properties that \( |g(z)| \) has exponential decay growth in a certain sector and \( g(z) \) has less than the usual frequency of zeros.
Remark. Example 2 gives a generalization of the Airy integral in one way and this paper gives a generalization of the Airy integral in another way.

Example 3 [7, Example 1] For any \( R > 0 \), the contour integral function \( G(z) \) defined by

\[
G(z) = \int_{|w|=R} \exp \left\{ \frac{z}{w} - \frac{1}{2w^2} - w \right\} \, dw
\]

is a solution of the equation

\[
f''' - zf'' - f = 0,
\]

where \( \rho(G) = 1/2 \). Since it is known [7, Corollary 2] that every transcendental solution \( f \) of (1.1) satisfies \( \rho(f) \geq 1/(n-1) \), the contour integral solution \( G(z) \) is on the "boundary line" of this result, since \( \rho(G) = 1/2 = 1/(n-1) \) when \( n = 3 \). In Section 5 we present and discuss a family of equations and solutions that includes this example as a special case.

Example 4 [7, Example 7] Let the function \( H(z) \) be defined by the contour integral

\[
H(z) = \int_K \exp \left\{ \frac{z}{\sqrt{2w}} - \frac{1}{4w} - w \right\} \, dw,
\]

where the contour \( K \) is defined by \( K = K_1 + K_2 + K_3 \) with

- \( K_1 \): \( w = re^{i\pi/4} \), \( r \) goes from \( +\infty \) to 1,
- \( K_2 \): \( w = e^{i\theta} \), \( \theta \) goes from \( \pi/4 \) to \( 7\pi/4 \),
- \( K_3 \): \( w = re^{i7\pi/4} \), \( r \) goes from 1 to \( +\infty \),

and where \( \sqrt{2w} \) is defined by the branch

\[
\sqrt{\zeta} = \exp \left\{ \frac{1}{2} \log |\zeta| + i \frac{1}{2} \arg \zeta \right\}, \quad 0 < \arg \zeta < 2\pi.
\]

Then \( f = H(z) \) is a solution of the equation

\[
f^{(4)} - zf''' - f = 0,
\]

where \( \rho(H) = 2/3 \). In Section 5 we discuss another contour integral function \( U(z) \) that solves this equation, and show that \( U(z), H(z), H(-z) \) are three pairwise linearly independent solutions that satisfy a particular identity, see Theorem 11. In this process, a more convenient form for \( H(z) \) is derived.
3 A family that includes the Airy integral

Consider a linear differential equation of the form
\[ f^{(n)} + (-1)^{n+1}bf^{(k)} + (-1)^{n+1}zf = 0, \quad 0 < k < n, \quad b \in \mathbb{C}, \] (3.1)
and let \( \varphi = \varphi(n,k,b) \) denote the contour integral function
\[ \varphi(z) = \frac{1}{2\pi i} \int_C \exp \{ -wz + b\alpha w^{k+1} + \beta w^{n+1} \} \, dw, \] (3.2)
where the contour \( C \) runs from \( \infty \) to 0 along \( \arg w = -\pi/(n + 1) \) and then from 0 to \( \infty \) along \( \arg w = \pi/(n + 1) \), and where
\[ \alpha = \frac{(-1)^{k+1}}{k + 1} \quad \text{and} \quad \beta = \frac{1}{n + 1}. \]

In the particular case when \( n = 2 \) and \( b = 0 \), \( \varphi(z) \) in (3.2) is the Airy integral \( \text{Ai}(z) \) in (2.2) and (3.1) is Airy’s equation (2.1). In the general case, we will show that \( \varphi = \varphi(n,k,b) \) in (3.2) is a transcendental solution of (3.1) with concrete properties that generalize known properties of the Airy integral. These properties include (i) exponential decay growth of \( |\varphi(z)| \) in a certain sector (Theorem 3), and, (ii) when \( b = 0 \), all solutions of (3.1) can be expressed in terms of contour integral functions that have a similar form to (3.2) (Theorem 4).

The next section contains the aforementioned properties of \( \varphi \). Most of this section is devoted to showing that \( \varphi(z) \neq 0 \).

**Theorem 1** The function \( f = \varphi(z) \) in (3.2) is a nontrivial solution of (3.1).

We use the following lemma in our proof of Theorem 1.

**Lemma 1** Let \( a \geq 0 \) and \( \eta > 0 \) be real constants. Suppose that \( E(x) \) is a continuous real-valued function on \([0, \infty)\) for which there exists an \( x_0 > 0 \) such that \( E'(x) > \eta \) for \( x \geq x_0 \). Then
\[ \int_0^\infty x^a E(x) \sin x \, dx = -\infty. \] (3.3)

**Proof.** Set \( v(x) = x^a E(x) \). For each integer \( j \geq 0 \), we have
\[ \int_{2j\pi}^{(2j+2)\pi} v(x) \sin x \, dx = \int_{2j\pi}^{(2j+1)\pi} v(x) \sin x \, dx + \int_{(2j+1)\pi}^{(2j+2)\pi} v(x) \sin x \, dx \]
\[ = \int_{2j\pi}^{(2j+1)\pi} (v(x) - v(x + \pi)) \sin x \, dx. \]
Choose a positive integer $j_0$ such that $2\pi j_0 \geq x_0$, and set $D = -\eta\pi(2\pi j_0)^a$. Then for each $j \geq j_0$, when $x$ satisfies $2j\pi < x < (2j + 1)\pi$, it follows from the mean value theorem that there exists a point $\xi_j \in (x, x + \pi)$ such that

$$v(x) - v(x + \pi) \leq x^aE(x) - x^aE(x + \pi) = -\pi E'(\xi_j)x^a < D < 0.$$ 

Therefore,

$$\int_{2\pi j_0}^{\infty} x^aE(x) \sin x \, dx = \sum_{j=j_0}^{\infty} \int_{2j\pi}^{(2j+1)\pi} (v(x) - v(x + \pi)) \sin x \, dx < \sum_{j=j_0}^{\infty} D \int_{2j\pi}^{(2j+1)\pi} \sin x \, dx = \sum_{j=j_0}^{\infty} 2D = -\infty.$$ 

It follows that (3.3) holds. \[\square\]

**Proof of Theorem 1.** Since $0 < k < n$, we obtain from (3.2),

$$2\pi i \left\{ \varphi^{(n)}(z) + (-1)^{n+1}b\varphi^{(k)}(z) + (-1)^{n+1}z\varphi(z) \right\}$$

$$= \int_C \left\{ (-w)^n + (-1)^{n+1}b(-w)^k + (-1)^{n+1}z \right\} \exp(-wz + b\alpha w^{k+1} + \beta w^{n+1}) \, dw$$

$$= (-1)^n \int_C \frac{\partial}{\partial w} \exp(-wz + b\alpha w^{k+1} + \beta w^{n+1}) \, dw$$

$$= (-1)^n \left[ \exp(-wz + b\alpha w^{k+1} + \beta w^{n+1}) \right]_C = 0.$$ 

It follows that $f = \varphi(z)$ is a solution of (3.1). To complete the proof of Theorem 1, it suffices to show that $\varphi(z) \neq 0$.

Consider the value of $\varphi^{(p)}(0)$, where $p \geq 0$ is an integer. Set

$$\gamma = \frac{k + 1}{n + 1}\pi \quad \text{and} \quad b = |b|e^{i\lambda},$$

where $\lambda \in \mathbb{R}$. From (3.2),

$$\varphi^{(p)}(0) = \frac{(-1)^p}{2\pi i} \int_C w^p \exp \left\{ b\alpha w^{k+1} + \beta w^{n+1} \right\} \, dw$$

$$= \frac{(-1)^p}{2\pi i} \int_0^0 r^p \exp \left\{ |b|\alpha r^{k+1}e^{(\lambda-\gamma)i} - \beta r^{n+1} - \frac{p + 1}{n + 1}\pi i \right\} \, dr$$

$$+ \frac{(-1)^p}{2\pi i} \int_0^\infty r^p \exp \left\{ |b|\alpha r^{k+1}e^{(\lambda+\gamma)i} - \beta r^{n+1} + \frac{p + 1}{n + 1}\pi i \right\} \, dr.$$
By taking real parts, we obtain

\[ \text{Re}(\varphi(0)) = \frac{(-1)^p}{2\pi} \int_0^\infty r^p e_1(r) \sin \left\{ |b|\alpha r^{k+1}\sin(\gamma - \lambda) + \frac{p + 1}{n + 1}\pi \right\} \, dr + \frac{(-1)^p}{2\pi} \int_0^\infty r^p e_2(r) \sin \left\{ |b|\alpha r^{k+1}\sin(\gamma + \lambda) + \frac{p + 1}{n + 1}\pi \right\} \, dr, \]

(3.4)

where

\[ e_1(r) = \exp \left\{ |b|\alpha r^{k+1}\cos(\gamma - \lambda) - \beta r^{n+1} \right\}, \]
\[ e_2(r) = \exp \left\{ |b|\alpha r^{k+1}\cos(\gamma + \lambda) - \beta r^{n+1} \right\}. \]

Suppose first that \( b = 0 \). Then from (3.4),

\[ \text{Re}(\varphi(0)) = \frac{1}{\pi} \sin \left( \frac{\pi}{n + 1} \right) \int_0^\infty \exp \left\{ -\beta r^{n+1} \right\} \, dr > 0, \]

which proves that \( \varphi(z) \not\equiv 0 \).

Next suppose that \( b \neq 0 \). We now make the assumption that \( \varphi(z) \equiv 0 \). Then \( \text{Re} \left( \varphi^{(p)}(0) \right) = 0 \) for every \( p \geq 0 \). By choosing \( p = n + j(n + 1) \) for \( j = 0, 1, 2, \ldots \), it follows from (3.4) and \( \text{Re} \left( \varphi^{(p)}(0) \right) = 0 \) that

\[ \int_0^\infty r^n r^{j(n+1)} u(r) \, dr = 0, \quad j = 0, 1, 2, \ldots, \]

(3.5)

where

\[ u(r) = e_1(r) \sin \left\{ |b|\alpha r^{k+1}\sin(\gamma - \lambda) \right\} + e_2(r) \sin \left\{ |b|\alpha r^{k+1}\sin(\gamma + \lambda) \right\}. \]

Let

\[ \exp \left\{ 2\beta r^{n+1} \right\} = \sum_{j=0}^\infty c_j r^{j(n+1)} \]

(3.6)

represent the power series expansion of \( \exp \left\{ 2\beta r^{n+1} \right\} \) where the constants \( c_j \) are the Taylor coefficients. From (3.5) and (3.6), we obtain

\[ \int_0^\infty r^n \exp \left\{ 2\beta r^{n+1} \right\} u(r) \, dr = \sum_{j=0}^\infty c_j \int_0^\infty r^n r^{j(n+1)} u(r) \, dr = 0, \]

which yields

\[ \int_0^\infty r^n \exp \left\{ |b|\alpha r^{k+1}\cos(\gamma - \lambda) + \beta r^{n+1} \right\} \sin \left\{ |b|\alpha r^{k+1}\sin(\gamma - \lambda) \right\} \, dr \]
\[ + \int_0^\infty r^n \exp \left\{ |b|\alpha r^{k+1}\cos(\gamma + \lambda) + \beta r^{n+1} \right\} \sin \left\{ |b|\alpha r^{k+1}\sin(\gamma + \lambda) \right\} \, dr = 0. \]
Noting that the sum of these two integrals equals zero, we have three cases.

Case 1. Suppose that \( \sin(\gamma - \lambda) \sin(\gamma + \lambda) > 0 \). Then we have
\[
\int_{0}^{\infty} r^n \exp \left\{ |b| r^{k+1} \cos(\gamma - \lambda) + \beta r^{n+1} \right\} \sin \left( |b\alpha \sin(\gamma - \lambda)| r^{k+1} \right) \, dr \\
+ \int_{0}^{\infty} r^n \exp \left\{ |b| r^{k+1} \cos(\gamma + \lambda) + \beta r^{n+1} \right\} \sin \left( |b\alpha \sin(\gamma + \lambda)| r^{k+1} \right) \, dr = 0.
\]
By using the change of variables
\[
x = |b\alpha \sin(\gamma - \lambda)| r^{k+1} \quad \text{and} \quad y = |b\alpha \sin(\gamma + \lambda)| r^{k+1},
\]
we deduce that
\[
L \int_{0}^{\infty} x^a \exp \left\{ cx + dx^\mu \right\} \sin x \, dx + M \int_{0}^{\infty} y^a \exp \left\{ sy + ty^\mu \right\} \sin y \, dy = 0,
\]
where
\[
\mu = \frac{n+1}{k+1}, \quad L = \frac{1}{|\sin(\gamma - \lambda)|^\mu}, \quad a = \frac{n-k}{k+1},
\]
\[
c = \frac{\alpha \cos(\gamma - \lambda)}{|\alpha \sin(\gamma - \lambda)|^{\mu/2}}, \quad d = \frac{\beta}{|b\alpha \sin(\gamma - \lambda)|^\mu}, \quad M = \frac{1}{|\sin(\gamma + \lambda)|^\mu},
\]
\[
s = \frac{\alpha \cos(\gamma + \lambda)}{|\alpha \sin(\gamma + \lambda)|^{\mu/2}}, \quad t = \frac{\beta}{|b\alpha \sin(\gamma + \lambda)|^\mu}.
\]
Since \( \mu > 1 \), \( d > 0 \) and \( t > 0 \), the two integrands in (3.8) both satisfy the conditions of the integrand in (3.3). Hence, from Lemma 1,
\[
L \int_{0}^{\infty} x^a \exp \left\{ cx + dx^\mu \right\} \sin x \, dx + M \int_{0}^{\infty} y^a \exp \left\{ sy + ty^\mu \right\} \sin y \, dy = -\infty,
\]
which contradicts (3.8). Therefore, Case 1 cannot occur.

Case 2. Suppose that \( \sin(\gamma - \lambda) \sin(\gamma + \lambda) < 0 \). Then we have
\[
\int_{0}^{\infty} r^n \exp \left\{ |b| r^{k+1} \cos(\gamma - \lambda) + \beta r^{n+1} \right\} \sin \left( |b\alpha \sin(\gamma - \lambda)| r^{k+1} \right) \, dr \\
= \int_{0}^{\infty} r^n \exp \left\{ |b| r^{k+1} \cos(\gamma + \lambda) + \beta r^{n+1} \right\} \sin \left( |b\alpha \sin(\gamma + \lambda)| r^{k+1} \right) \, dr.
\]
As in Case 1, we use the change of variables (3.7), and deduce that
\[
L \int_{0}^{\infty} x^a \exp \left\{ cx + dx^\mu \right\} \sin x \, dx = M \int_{0}^{\infty} y^a \exp \left\{ sy + ty^\mu \right\} \sin y \, dy,
\]
which can be rewritten as
\[ \int_0^\infty x^a F(x) \sin x \, dx = 0, \quad (3.9) \]
where
\[ F(x) = L \exp \{ cx + dx^\mu \} - M \exp \{ sx + tx^\mu \}. \]

Suppose that \( c = s \) and \( d = t \) both hold. Since \( \sin(\gamma - \lambda) \sin(\gamma + \lambda) < 0 \),
we would then obtain
\[ \cos(\gamma - \lambda) = \cos(\gamma + \lambda) \quad \text{and} \quad \sin(\gamma - \lambda) = -\sin(\gamma + \lambda). \quad (3.10) \]
It can be verified that \((3.10)\) leads to a contradiction. Therefore, we cannot
have both \( c = s \) and \( d = t \).

Then from Lemma 1, it can be deduced that either
\[ \int_0^\infty x^a F(x) \sin x \, dx = -\infty \quad \text{or} \quad \int_0^\infty x^a F(x) \sin x \, dx = +\infty \]
must hold, which contradicts \((3.9)\). Thus, Case 2 cannot occur.

**Case 3.** Suppose that \( \sin(\gamma - \lambda) \sin(\gamma + \lambda) = 0 \). Assume first that
\( \sin(\gamma - \lambda) = 0 \). Then it can be verified that \( \sin(\gamma + \lambda) \neq 0 \). We have
\[ \int_0^\infty r^n \exp \{ |b| |\alpha r^{k+1} \cos(\gamma + \lambda) + \beta r^{n+1} \} \sin \left( |b\alpha \sin(\gamma + \lambda)| r^{k+1} \right) \, dr = 0. \]
By using the change of variable \( x = |b\alpha \sin(\gamma + \lambda)| r^{k+1} \), we obtain
\[ \int_0^\infty x^a \exp \{ sx + tx^\mu \} \sin x \, dx = 0. \quad (3.11) \]
But it follows from Lemma 1 that
\[ \int_0^\infty x^a \exp \{ sx + tx^\mu \} \sin x \, dx = -\infty, \]
which contradicts \((3.11)\). By using the analogous argument, we also get a
contradiction when \( \sin(\gamma + \lambda) = 0 \). Hence, Case 3 cannot occur.

Since Cases 1, 2 and 3 cannot occur, we have a contradiction. Hence,
our original assumption \( \varphi(z) \equiv 0 \) is false. Thus, \( \varphi(z) \neq 0 \). The proof of
Theorem 1 is complete. \( \square \)
4 Properties of $\varphi(z)$

From Theorem 1 the contour integral function $\varphi(z)$ is a nontrivial solution of equation (3.1). In this section we derive properties of $\varphi(z)$ which generalize properties of the Airy integral $\text{Ai}(z)$.

**Theorem 2** The order of $\varphi$ in (3.2) satisfies $\rho(\varphi) = 1 + 1/n$.

*Proof.* Observe first that (3.1) cannot possess a nontrivial polynomial solution. Therefore, it follows from Theorem 1 that $\varphi(z)$ must be transcendental. The assertion now follows from [7, Theorem 1].

Since the Airy integral $\text{Ai}(z)$ satisfies $\rho(\text{Ai}) = 3/2$ and has only negative real zeros, where the three critical rays of equation (2.1) are $\arg z = \pi, \pm \pi/3$, it follows from the asymptotic theory of integration that $|\text{Ai}(z)|$ has exponential decay growth of order $\exp\{-|z|^{3/2}\}$ on any ray in the sector $-\pi/3 < \arg z < \pi/3$, see [1 Ch. 7.4], [14]. This property of $\text{Ai}(z)$ is a particular case of the following general result.

**Theorem 3** Let $n \geq 2$. If $z = |z|e^{i\theta}$, where $\theta$ satisfies

$$-\frac{n\pi}{2n+2} < \theta < \frac{n\pi}{2n+2},$$

then there exist positive constants $K = K(\theta)$ and $M = M(\theta)$ that depend only on $\theta$, such that $\varphi$ in (3.2) satisfies

$$|\varphi(z)| \leq M \exp\{-K|z|^{1+1/n}\}.$$  \hfill (4.2)

*Remark.* Two numbers in Theorem 3 are best possible. Since $\rho(\varphi) = 1+1/n$ from Theorem 2, it follows from (1.2) and the Phragmén-Lindelöf principle [2] that (i) the exponent $1+1/n$ in (4.2) could not be larger and (ii) the length $n\pi/(n+1)$ of the interval (4.1) could not be larger.

*Proof of Theorem 3.* We will use Cauchy’s theorem twice to show that, for any $\theta$ satisfying (4.1), the contour $C$ in (3.2) can be replaced by two other contours (which depend on $\theta$) without changing the value of $\varphi(z)$ when $\arg z = \theta$.

Let $\theta$ be a fixed constant satisfying (4.1). Then let $\mu = \mu(\theta)$ and $\tau = \tau(\theta)$ denote the constants

$$\mu = -\frac{\pi}{n+1} - \frac{\theta}{n} \quad \text{and} \quad \tau = \frac{\pi}{n+1} - \frac{\theta}{n}. \hfill (4.3)$$
From (4.1) and (4.3), we obtain the following inequalities:
\[-\frac{3\pi}{2} < (n+1)\mu < -\frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \mu + \theta < \frac{\pi}{2}, \tag{4.4}\]
\[\frac{\pi}{2} < (n+1)\tau < \frac{3\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \tau + \theta < \frac{\pi}{2}. \tag{4.5}\]

Next let \( z \) be a point satisfying \( \arg z = \theta \), i.e., set \( z = |z|e^{i\theta} \). Then let \( C(\mu) \) be the arc of the circle \( |w| = |z| \) from \( w = |z|e^{-i\pi/(n+1)} \) to \( w = |z|e^{i\mu} \). From (4.1), (4.3), (4.4), it can be deduced that
\[
\cos((n+1)\xi) \leq \cos((n+1)\mu) < 0
\]
holds for any point \( w = |z|e^{i\xi} \) lying on \( C(\mu) \). Therefore, with \( \alpha \) and \( \beta \) as in (3.2), we obtain
\[
\left| \frac{1}{2\pi i} \int_{C(\mu)} \exp \left\{ -wz + b\alpha w^{k+1} + \beta w^{n+1} \right\} \, dw \right| \\
\leq \frac{|z|}{2\pi} \int_{C(\mu)} \exp \left\{ |z|^2 + |b\alpha||z|^{k+1} + \beta|z|^{n+1}\cos(n+1)\xi \right\} \, |d\xi| \\
\leq |z| \exp \left\{ |z|^2 + |b\alpha||z|^{k+1} + \beta|z|^{n+1}\cos(n+1)\mu \right\}.
\]

Since \( 2 \leq k+1 < n+1, \beta > 0 \) and \( \cos(n+1)\mu < 0 \), we obtain
\[
\frac{1}{2\pi i} \int_{C(\mu)} \exp \left\{ -wz + b\alpha w^{k+1} + \beta w^{n+1} \right\} \, dw \to 0 \quad \text{as} \quad |z| \to \infty. \tag{4.6}\]

Similarly, by using (4.1), (4.3), (4.5), we obtain
\[
\frac{1}{2\pi i} \int_{C(\tau)} \exp \left\{ -wz + b\alpha w^{k+1} + \beta w^{n+1} \right\} \, dw \to 0 \quad \text{as} \quad |z| \to \infty, \tag{4.7}\]

where \( C(\tau) \) denotes the arc of the circle \( |w| = |z| \) from \( w = |z|e^{i\pi/(n+1)} \) to \( w = |z|e^{i\tau} \).

Now we observe from Cauchy’s theorem that
\[
\frac{1}{2\pi i} \int_{E_j} \exp \left\{ -wz + b\alpha w^{k+1} + \beta w^{n+1} \right\} \, dw = 0, \quad j = 1, 2,
\]

where \( E_1 \) is the closed curve consisting of the line segment from 0 to \( |z|e^{-i\pi/(n+1)} \), the arc \( C(\mu) \), and the line segment from \( |z|e^{i\mu} \) to 0, and \( E_2 \) is the closed curve consisting of the line segment from 0 to \( |z|e^{i\pi/(n+1)} \), the arc \( C(\tau) \), and the
line segment from $|z|e^{i\tau}$ to 0. Then by letting $|z| \to \infty$, it can be deduced from (3.2), (4.6) and (4.7), that

$$\varphi(z) = \frac{1}{2\pi i} \int_L \exp \{-wz + b\alpha w^{k+1} + \beta w^{n+1}\} \, dw, \quad (4.8)$$

where the contour $L = L(\theta)$ runs from $\infty$ to 0 along arg $z = \mu$ and then from 0 to $\infty$ along arg $z = \tau$. Note that $L = C$ when $\theta = 0$.

![Figure 1: Contours L, C and closed curves $E_1, E_2$](image)

We next apply Cauchy’s theorem to (4.8), and obtain further that for any constant $R > 0$,

$$\varphi(z) = \frac{1}{2\pi i} \int_J \exp \{-wz + b\alpha w^{k+1} + \beta w^{n+1}\} \, dw, \quad (4.9)$$

where $J$ is the contour defined by $J = J_1 + J_2 + J_3$ with

- $J_1 : w = re^{i\mu}$, $r$ goes from $\infty$ to $R$,
- $J_2 : w = Re^{i\xi}$, $\xi$ goes from $\mu$ to $\tau$,
- $J_3 : w = re^{i\tau}$, $r$ goes from $R$ to $\infty$. 
Solutions defined by contour integrals

From (4.9),

\[ \varphi(z) = \frac{1}{2\pi i} \int_{\gamma} \exp \left\{ -r|z|e^{i(\mu + \theta)} + b \alpha r^{k+1} e^{i(k+1)\mu} + \beta r^{n+1} e^{i(n+1)\mu} \right\} e^{i\mu} dr \]

\[ + \frac{1}{2\pi} \int_{\mu R}^{\tau} \exp \left\{ -R|z|e^{i(\xi + \theta)} + b \alpha R^{k+1} e^{i(k+1)\xi} + \beta R^{n+1} e^{i(n+1)\xi} \right\} R e^{i\xi} d\xi \]

\[ + \frac{1}{2\pi i} \int_{\gamma}^{\infty} \exp \left\{ -r|z|e^{i(\tau + \theta)} + b \alpha r^{k+1} e^{i(k+1)\tau} + \beta r^{n+1} e^{i(n+1)\tau} \right\} e^{i\tau} dr \]

\[ = : I_1 + I_2 + I_3. \]

Using (4.11) and (4.12), let \( A = A(\theta) \) denote the constant

\[ A = \min \{ \cos(\mu + \theta), \cos(\tau + \theta) \} > 0. \quad (4.10) \]

Then let \( B = B(\theta) \) be any fixed constant that satisfies

\[ 0 < B < \{(n + 1)A\}^{1/n}. \quad (4.11) \]

Choose \( R = B|z|^{1/n} \). Since \( \cos(\mu + \theta) > 0 \) and \( |\alpha| < 1 < 2\pi \),

\[ |I_1| < \int_{B|z|^{1/n}}^{\infty} \exp \left\{ -r|z| \cos(\mu + \theta) + |b|r^{k+1} + \beta r^{n+1} \cos(n + 1)\mu \right\} dr \]

\[ \leq \exp \left\{ -B|z|^{1+1/n} \cos(\mu + \theta) \right\} \int_{B|z|^{1/n}}^{\infty} \exp\{|b|r^{k+1} + \beta r^{n+1} \cos(n + 1)\mu \} dr. \]
We have \( \cos(n+1)\mu < 0 \) from (4.4), and since \( \beta > 0 \) and \( k < n \),

\[
|I_1| < P \exp \left\{ -B|z|^{1+1/n} \cos(\mu + \theta) \right\},
\]

where \( P = P(\theta) \) is the following positive constant:

\[
P = \int_0^\infty \exp\{b r^{k+1} + \beta r^{n+1} \cos(n+1)\mu\} \, dr.
\]

Similarly, since \( \cos(\tau + \theta) > 0 \) and \( \cos(n+1)\tau < 0 \) from (4.5), we obtain

\[
|I_3| < Q \exp \left\{ -B|z|^{1+1/n} \cos(\tau + \theta) \right\},
\]

where \( Q = Q(\theta) \) is the following positive constant:

\[
Q = \int_0^\infty \exp\{b r^{k+1} + \beta r^{n+1} \cos(n+1)\tau\} \, dr.
\]

Last, for \( |I_2| \), we have

\[
|I_2| \leq \frac{B|z|^{1/n}}{2\pi} \int_\mu^\tau \exp \left\{ B(\beta B^n - \cos(\xi + \theta))|z|^{1+1/n} + |b|B^{k+1}|z|^{(k+1)/n} \right\} \, d\xi.
\]

From (4.4) and (4.5),

\[
-\frac{\pi}{2} < \mu + \theta \leq \xi + \theta \leq \tau + \theta < \frac{\pi}{2}, \quad \mu \leq \xi \leq \tau.
\]

Thus, from (4.10), we see that

\[
\cos(\xi + \theta) \geq A, \quad \mu \leq \xi \leq \tau.
\]

Hence, we obtain

\[
|I_2| \leq B|z|^{1/n} \exp \left\{ B(\beta B^n - A)|z|^{1+1/n} + |b|B^{k+1}|z|^{(k+1)/n} \right\},
\]

where \( B(\beta B^n - A) < 0 \) from \( \beta = 1/(n+1) \) and (4.11). Since \( \varphi(z) = I_1 + I_2 + I_3 \) when \( \arg z = \theta \), we deduce that (4.2) holds from an observation of (4.12), (4.13), (4.14). This proves Theorem 3.\(\square\)

We now generalize another property of the Airy integral. Let \( u(z) \) be the function \( \varphi(z) \) in (3.2) when \( b = 0 \), that is,

\[
u(z) = \frac{1}{2\pi i} \int_C \exp \left\{ -wz + \frac{1}{n+1}w^{n+1} \right\} \, dw,
\]

where \( C \) is a suitable contour.
where the contour \( C \) runs from \( \infty \) to 0 along \( \arg w = -\pi/(n+1) \) and then from 0 to \( \infty \) along \( \arg w = \pi/(n+1) \). Then from Theorems 1 and 2, \( u(z) \) is a transcendental solution of the equation

\[
f^{(n)} + (-1)^{n+1}zf = 0. \tag{4.16}
\]

When \( n = 2 \), \( u(z) \) is the Airy integral \( \text{Ai}(z) \) in (2.2) and (4.16) is Airy’s equation (2.1).

If \( \beta_1, \beta_2, \ldots, \beta_{n+1} \) are the \( n+1 \) distinct roots of unity, and if \( f_j(z) \) denotes the function

\[
f_j(z) = u(\beta_j z), \quad j = 1, 2, \ldots, n+1, \tag{4.17}
\]

then \( f_1, f_2, \ldots, f_{n+1} \) are solutions of equation (4.16). The following result is a generalization of the known property of the Airy integral \( \text{Ai}(z) \) in (2.3).

**Theorem 4** The functions \( f_1, f_2, \ldots, f_{n+1} \) in (4.17) are solutions of (4.16) of order \( 1 + 1/n \), and any \( n \) of these \( n+1 \) functions are linearly independent.

Theorem 4 shows that every solution of equation (4.16) can be expressed as a contour integral function.

**Proof of Theorem 4** Since \( u(z) \) in (4.15) is the function \( \varphi(z) \) in (3.2) when \( b = 0 \), it follows from Theorem 1, Theorem 2 and (4.17), that each \( f_j \) is a solution of (4.16) of order \( 1 + 1/n \). It remains to show that any \( n \) of the \( n+1 \) functions \( f_1, f_2, \ldots, f_{n+1} \) are linearly independent.

First we show that the \( n \) functions \( f_1, f_2, \ldots, f_n \) are linearly independent. Since \( u \) is the function \( \varphi \) in (3.2) when \( b = 0 \), it follows from Theorem 3 below that \( u \) is real on the real axis. Hence from (3.4), we obtain for \( 0 \leq p \leq n-1 \),

\[
u^{(p)}(0) = \frac{(-1)^p}{\pi} \int_0^\infty r^p \exp \left(-\frac{1}{n+1}r^{n+1}\right) \sin \left(\frac{p+1}{n+1}\pi\right) dr \neq 0. \tag{4.18}
\]

Using (4.17) and the classical Vandermonde determinant, we calculate the Wronskian of \( f_1, f_2, \ldots, f_n \) at \( z = 0 \):

\[
W(f_1, f_2, \ldots, f_n)(0) = \begin{vmatrix}
u(0) & \nu(0) & \cdots & \nu(0) \\
\beta_1 u'(0) & \beta_2 u'(0) & \cdots & \beta_n u'(0) \\
\vdots & \vdots & & \vdots \\
\beta_1^{n-1} u^{(n-1)}(0) & \beta_2^{n-1} u^{(n-1)}(0) & \cdots & \beta_n^{n-1} u^{(n-1)}(0)
\end{vmatrix}
\]

\[
= u(0)u'(0) \cdots u^{(n-1)}(0) \begin{vmatrix}1 & 1 & \cdots & 1 \\
\beta_1 & \beta_2 & \cdots & \beta_n \\
\vdots & \vdots & & \vdots \\
\beta_1^{n-1} & \beta_2^{n-1} & \cdots & \beta_n^{n-1}
\end{vmatrix}
\]

\[
= u(0)u'(0) \cdots u^{(n-1)}(0) \prod_{1 \leq i < j \leq n} (\beta_j - \beta_i).
\]
Since (4.18) holds and since the $\beta_j$ are all distinct, we obtain

$$W(f_1, f_2, \ldots, f_n)(0) \neq 0.$$ 

Hence, the $n$ functions $f_1, f_2, \ldots, f_n$ are linearly independent. Similarly, the same argument will show that any $n$ of the $n+1$ functions $f_j$ in (4.17) are linearly independent. 

**Remark.** By varying the contour $C$ in (4.15), we can express the functions $f_1, f_2, \ldots, f_{n+1}$ in (4.17) in a different form as follows. Let the functions $u_1, u_2, \ldots, u_{n+1}$ be defined by

$$u_j(z) = \frac{1}{2\pi i} \int_{C_j} \exp \left\{ -wz + \frac{1}{n+1}w^{n+1} \right\} dw, \quad (4.19)$$

where the contour $C_j$ runs from $\infty$ to 0 along arg $w = -\frac{\pi}{n+1} + \frac{2\pi(j-1)}{n+1}$ and then from 0 to $\infty$ along arg $w = \frac{\pi}{n+1} + \frac{2\pi(j-1)}{n+1}$. We assume that $\beta_j = \exp\left\{ \frac{2\pi(j-1)}{n+1} \right\}$ for $j = 1, 2, \ldots, n+1$. By using the change of variable $w = \beta_j \zeta$ in (4.19), we obtain from (4.15) and (4.17) that

$$u_j(z) = \frac{\beta_j}{2\pi i} \int_C \exp \left\{ -\beta_j \zeta z + \frac{1}{n+1}\zeta^{n+1} \right\} d\zeta = \beta_j u(\beta_j z) = \beta_j f_j(z)$$

for $j = 1, 2, \ldots, n+1$. Thus for each $j$, the contour integral function $u_j(z)$ in (4.19) is a non-zero constant multiple of the function $f_j(z)$ in (4.17).

Regarding solutions of (4.16) that are defined by contour integrals, see page 414 of [8]. By using a change of variable, it can be seen that, for each $n$, formula (53) in [8] can be transformed into a non-zero constant multiple of (4.19) above, where the constant multiple depends on $n$.

It is well known that the Airy integral $Ai(z)$ is real on the real axis. More generally, the next result shows that this property holds for $\varphi(z)$ whenever $b$ is real.

**Theorem 5** When $b \in \mathbb{R}$, the function $\varphi(z)$ in (3.2) is real on the real axis.

**Proof.** Let $\varphi(z)$ be given by (3.2), where $b \in \mathbb{R}$. For convenience, set

$$\gamma = \frac{k+1}{n+1} \pi \quad \text{and} \quad \eta = \frac{\pi}{n+1}.$$
Let \( x \in \mathbb{R} \). From (3.2),
\[
\varphi(x) = \frac{1}{2\pi i} \int_C \exp \left(-wx + b\omega w^{k+1} + \beta w^{n+1}\right) dw \\
= \frac{1}{2\pi i} \int_0^\infty \exp \left(-rxe^{-\eta i} + b\alpha r^{k+1}e^{-\gamma i} - \beta r^{n+1} - \eta i\right) dr \\
+ \frac{1}{2\pi i} \int_0^\infty \exp \left(-rxe^{\eta i} + b\alpha r^{k+1}e^{\gamma i} - \beta r^{n+1} + \eta i\right) dr \\
= -\frac{1}{2\pi i} \int_0^\infty \exp (A + iB) dr + \frac{1}{2\pi i} \int_0^\infty \exp (A - iB) dr,
\]
where
\[
A = -rx \cos \eta + b\alpha r^{k+1} \cos \gamma - \beta r^{n+1}, \\
B = rx \sin \eta - b\alpha r^{k+1} \sin \gamma - \eta.
\]
Therefore, we obtain
\[
\varphi(x) = \frac{1}{\pi} \int_0^\infty \exp(A) \sin(-B) dr,
\]
which is real. This proves the assertion.

\[\Box\]

5 Another family of contour integrals

Consider a linear differential equation of the form
\[
f^{(n)} - zf^{(k)} - f = 0, \quad 1 < k < n, \tag{5.1}
\]
and let \( \psi = \psi(n, k) \) denote the contour integral function
\[
\psi(z) = \frac{1}{2\pi i} \int_{|w|=1} w^{k-2} \exp \left\{ \frac{z}{w} - \frac{1}{(n-k+1)w^{n-k+1}} - \frac{w^{k-1}}{k-1} \right\} dw. \tag{5.2}
\]

**Theorem 6** The function \( f = \psi(z) \) is a transcendental solution of (5.1).

**Proof.** From (5.2), we obtain
\[
(2\pi i) \left\{ \psi^{(n)}(z) - z\psi^{(k)}(z) - \psi(z) \right\} \\
= \int_{|w|=1} \left( \frac{1}{w^{n-k+2}} - \frac{z}{w^2} - w^{k-2} \right) \exp \left\{ \frac{z}{w} - \frac{1}{(n-k+1)w^{n-k+1}} - \frac{w^{k-1}}{k-1} \right\} dw \\
= \int_{|w|=1} \frac{\partial}{\partial w} \exp \left\{ \frac{z}{w} - \frac{1}{(n-k+1)w^{n-k+1}} - \frac{w^{k-1}}{k-1} \right\} dw \\
= \left[ \exp \left\{ \frac{z}{w} - \frac{1}{(n-k+1)w^{n-k+1}} - \frac{w^{k-1}}{k-1} \right\} \right]_{|w|=1} = 0,
\]
\[\Box\]
since \(|w| = 1\) is a closed curve. Thus, \(\psi(z)\) satisfies (5.1).

Next we show that \(\psi \neq 0\). From (5.2) and \(w = e^{i\theta}\), we obtain

\[
\psi^{(k-1)}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(k-1)\theta} \left( \frac{e^{-i\theta}}{n-k+1} + \frac{1}{k-1} \right) d\theta. \tag{5.3}
\]

Let \(L = L(n, k, \theta)\) denote

\[L = -e^{i(k-1)\theta} \left\{ \sigma e^{-in\theta} + \eta \right\}, \tag{5.4}\]

where

\[
\sigma = \frac{1}{n-k+1} \quad \text{and} \quad \eta = \frac{1}{k-1}.
\]

From (5.4), we have

\[L = P(n, k, \theta) + iQ(n, k, \theta), \tag{5.5}\]

where \(P(n, k, \theta)\) and \(Q(n, k, \theta)\) are the real-valued functions

\[
P(n, k, \theta) = -\sigma \cos((n-k+1)\theta) - \eta \cos((k-1)\theta),
\]

\[
Q(n, k, \theta) = \sigma \sin((n-k+1)\theta) - \eta \sin((k-1)\theta).
\]

From (5.3), (5.4), (5.5),

\[
\psi^{(k-1)}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{P(n,k,\theta)} \{ \cos Q(n, k, \theta) + i \sin Q(n, k, \theta) \} d\theta. \tag{5.6}
\]

Taking real parts of (5.6) gives

\[
\Re \left( \psi^{(k-1)}(0) \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{P(n,k,\theta)} \cos Q(n, k, \theta) d\theta. \tag{5.7}
\]

Since \(1 < k < n\), we obtain for all \(\theta\),

\[
|Q(n, k, \theta)| \leq \frac{|\sin((n-k+1)\theta)|}{n-k+1} + \frac{|\sin(k-1)\theta|}{k-1}
\]

\[
\leq \frac{1}{n-k+1} + \frac{1}{k-1} \leq \frac{1}{2} + 1 < \frac{\pi}{2}.
\]

Hence, \(\cos Q(n, k, \theta) \geq \cos(3/2) > 0\) for all \(\theta\). Then from (5.7), it follows that \(\Re \left( \psi^{(k-1)}(0) \right) > 0\). Thus, \(\psi \neq 0\). Furthermore, \(\psi\) must be transcendental because it can be observed that (5.1) cannot possess a nontrivial polynomial solution. This proves Theorem 6.

\[\Box\]

**Remark.** Observe that Example 3 is a particular case of Theorem 6 because when \(n = 3\) and \(k = 2\) in (5.2), we obtain \(2\pi i\psi(z) \equiv G(z)\), where \(G(z)\) is the function in (2.6). Example 4 is related to the particular case of Theorem 6 when \(n = 4\) and \(k = 3\), see Section 6 below.

The following result shows that the order of \(\psi\) lies in \((0, 1)\).
Theorem 7  The order of \( \psi \) in (5.2) satisfies \( \rho(\psi) = 1 - 1/k \).

Proof. From (5.2) and Cauchy’s theorem, it follows that

\[
\psi(z) = \frac{1}{2\pi i} \int_{|w|=R} w^{k-2} \exp \left\{ \frac{z}{w} - \frac{1}{(n-k+1)w^{n-k+1}} - \frac{w^{k-1}}{k-1} \right\} \, dw
\]

holds for any \( R > 0 \). Since \( 1 < k < n \), we have

\[
|\psi(z)| \leq \frac{1}{2\pi} \int_{|w|=R} |w|^{k-2} \exp \left\{ \left| \frac{z}{w} \right| + \frac{1}{|w|^{n-k+1}} + |w|^{k-1} \right\} \, |dw|. \quad (5.8)
\]

Let \( z \) be a point satisfying \( |z| \geq 1 \) and set \( R = |z|^{1/k} \). Then from (5.8),

\[
|\psi(z)| \leq |z|^{1-1/k} \exp \left\{ 2|z|^{1-1/k} + 1 \right\},
\]

which gives \( \rho(\psi) \leq 1 - 1/k \). Since \( \psi \) is transcendental from Theorem 6, it follows from [7, Theorem 1] that the only possible orders of \( \psi \) are \( 1 + 1/(n-k) \) and \( 1 - 1/k \). Since \( n > k \) and \( \rho(\psi) \leq 1 - 1/k \), we obtain \( \rho(\psi) = 1 - 1/k \). \( \blacksquare \)

The proofs of the next two results use the following observation. If \( s \) is a non-negative integer, then from (5.2),

\[
\psi^{(s)}(0) = \frac{1}{2\pi i} \int_{|w|=1} w^{k-2-s} \exp \left\{ \frac{1}{(n-k+1)w^{n-k+1}} - \frac{w^{k-1}}{k-1} \right\} \, dw.
\]

Thus, from the residue theorem, we obtain

\[
\psi^{(s)}(0) = \text{Res} \left\{ w^{k-2-s} \sum_{j=0}^{\infty} \frac{(-1)^j w^{j(k-1)}}{j!(k-1)^j} \left( \frac{k-1}{(n-k+1)w^{n+1}} \right)^j, \, 0 \right\}. \quad (5.9)
\]

Theorem 8  When \( n \) is even and \( k \) is odd, then \( \psi \) is an even function that is unbounded on every ray from the origin.

Proof. Since \( n \) is even and \( k \) is odd, for any odd positive integer \( s \) and any non-negative integer \( j \), the two integers \( k - 2 - s \) and \( j(n-k+1) \) are both even integers. It follows that all the powers of \( w \) in (5.9) are even powers, which means that the residue must be zero. Therefore, from (5.9), \( \psi^{(s)}(0) = 0 \) for all odd integers \( s \). Hence, from the Maclaurin series, \( \psi(z) \) is an even function.

Now suppose that \( \psi(z) \) is bounded on one particular ray \( \text{arg} \, z = \theta_0 \). Since \( \psi(z) \) is an even function, it follows that \( \psi(z) \) is also bounded on the ray \( \text{arg} \, z = \theta_0 + \pi \). Since \( \psi \) is bounded on these two rays where \( (\theta_0 + \pi) - \theta_0 = \pi \), and since \( \rho(\psi) \in (0,1) \) from Theorem 7, it follows from the Phragmén-Lindelöf theorem that \( \psi \) must be bounded in the whole complex plane. Hence, from Liouville’s theorem, \( \psi \) is a constant, which is a contradiction. Thus our assumption is false, and \( \psi \) must be unbounded on every ray from the origin. \( \blacksquare \)
Theorem 9 The function $\psi$ is real on the real axis. If $n$ is even and $k$ is odd, then $\psi$ is also real on the imaginary axis.

Proof. From (5.9), $\psi^{(s)}(0)$ is a real number for every non-negative integer $s$. Thus, from the Maclaurin series, $\psi$ is real on the real axis.

When $n$ is even and $k$ is odd, then $\psi(z)$ is an even function from Theorem 8. Then from the reflection principle, it follows that $\psi$ is real on the imaginary axis. $\blacksquare$

6 Solutions of $f^{(4)} - zf''' - f = 0$

In this section we discuss contour integral functions that are solutions of the particular differential equation

$$f^{(4)} - zf''' - f = 0. \quad (6.1)$$

One such solution is when $n = 4$ and $k = 3$ in (5.2), and for convenience, we denote this function by $U(z)$. Then from Theorem 6,

$$U(z) = \frac{1}{2\pi i} \int_{|w|=1} w \exp \left\{ \frac{z}{w} - \frac{1}{2w^2} - \frac{w^2}{2} \right\} dw \quad (6.2)$$

satisfies equation (6.1). Another contour integral function that satisfies (6.1) is $H(z)$ in (2.7) from Example 4.

First, properties of $U(z)$ will be discussed. Next we derive an alternate form for $H(z)$ that is more convenient than (2.7). This alternate form is then used to show that $U(z)$, $H(z)$, $H(-z)$ are three pairwise linearly independent solutions of (6.1) that satisfy a specific identity, see Theorem 11.

From Theorems 7, 8 and 9 $U(z)$ has order 2/3, is an even function, is unbounded on every ray from the origin, and is real on both the real and imaginary axis. The next result gives further properties of $U(z)$ on the imaginary axis.

Theorem 10 The following statements hold for the function $U(z)$ in (6.2).

(a) $U(z)$ has its largest growth in both directions of the imaginary axis, and we have

$$|U(ir)| = |U(-ir)| = M(r, U) = (c_0 + o(1))r^{2/3} \quad (6.3)$$

as $r \to \infty$, where $c_0 > 0$ is some constant.

(b) $U(z)$ does not have any zeros on the imaginary axis.
We use the following lemma in our proof of Theorem 10.

**Lemma 2** The Maclaurin series of the function \( U(z) \) in (6.2) is given by

\[
U(z) = \sum_{\nu=0}^{\infty} a_{2\nu} z^{2\nu}, \quad (6.4)
\]

where

\[
a_0 = -\frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{4^m m!(m+1)!}
\]

and

\[
a_{2\nu} = \frac{(-1)^{\nu+1}}{2^{\nu-1} (2\nu)!} \sum_{m=0}^{\infty} \frac{1}{4^m m!(m+\nu-1)!}, \quad \nu \geq 1.
\]

**Proof.** Applying the residue theorem to (6.2) gives

\[
U(z) = \text{Res} \left\{ \sum_{j=0}^{\infty} \frac{1}{j!2^j} \left( \frac{2z}{w} - \frac{1}{w^2} - w^2 \right)^j, \ 0 \right\}.
\]

For convenience, we rewrite this as follows:

\[
U(z) = \text{Res} \left\{ \sum_{j=0}^{\infty} \frac{(1 + (w^4 - 2zw)^j)}{(-2)^j j! w^{2j-1}}, \ 0 \right\}. \quad (6.5)
\]

Contributions to this residue can only come from terms in \((1 + (w^4 - 2zw))^j\) that contain a power \(w^{2j-2}\). From the binomial expansions, we obtain

\[
(1 + (w^4 - 2zw))^j = \sum_{m=0}^{j} A_m (w^4 - 2zw)^m, \quad \text{where} \quad A_m = \frac{j!}{m!(j-m)!},
\]

and

\[
(w^4 - 2zw)^m = \sum_{p=0}^{m} B_p z^{p} w^{4m-3p}, \quad \text{where} \quad B_p = (-2)^p \frac{m!}{p!(m-p)!}.
\]

From the above, we see that the residue contributions in (6.5) can only occur when \(p, m, j\) are integers satisfying

\[
4m - 3p = 2j - 2 \quad \text{and} \quad 0 \leq p \leq m \leq j. \quad (6.6)
\]

The contribution \(L(p, m, j)\) to the residue from three such integers \(p, m, j\) is

\[
L(p, m, j) = \frac{A_m B_p z^p}{(-2)^j j!} = \frac{z^p}{(-2)^{j-p}(j-m)! p!(m-p)!}. \quad (6.7)
\]
From (6.6), \( p \) must be an even integer. It follows that only even powers of \( z \) can appear in the Maclaurin series of \( U(z) \), which is also known from the fact that \( U(z) \) is an even function.

Suppose that \( p = 0 \). Then from (6.6),
\[
j = 2m + 1, \quad m \geq 0,
\]
and from (6.7),
\[
L(0, m, 2m + 1) = \frac{1}{(-2)^{2m+1}(m + 1)!m!}.
\]
It follows that
\[
a_0 = -\frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{4^m(m + 1)!m!}, \quad \tag{6.8}
\]
Now suppose that \( p = 2\nu > 0 \), where \( \nu \geq 1 \). Then from (6.6),
\[
j = 2m + 1 - 3\nu, \quad m \geq 3\nu - 1,
\]
and from (6.7),
\[
L(2\nu, m, 2m + 1 - 3\nu) = \frac{z^{2\nu}}{(-2)^{2m+1-5\nu}(m + 1 - 3\nu)!(2\nu)!(m - 2\nu)!}.
\]
It follows that
\[
a_{2\nu} = (-1)^{\nu+1} \frac{2^{5\nu-1}}{(2\nu)!} \sum_{m=3\nu-1}^{\infty} \frac{1}{4^m(m + 1 - 3\nu)!(m - 2\nu)!}, \quad \nu \geq 1.
\]
By setting \( q = m + 1 - 3\nu \), we obtain
\[
a_{2\nu} = \frac{(-1)^{\nu+1}}{2^{\nu-1}(2\nu)} \sum_{q=0}^{\infty} \frac{1}{4^q q!(q + \nu - 1)!}, \quad \nu \geq 1. \quad \tag{6.9}
\]
Lemma 2 follows from (6.8) and (6.9).

Proof of Theorem 10. If \( \lambda \) is a real number, then from (6.4), we obtain
\[
U(i\lambda) = -\sum_{\nu=0}^{\infty} |a_{2\nu}| \lambda^{2\nu},
\]
which proves both part (b) and the first two equalities in (6.3). The last equality in (6.3) follows from \( \rho(U) = 2/3 \) and (1.2).
Remark. Although it is known that the order of $U(z)$ is $2/3$, it is not yet known what the type of $U(z)$ is. Below, $\tau(U)$ denotes the type of $U(z)$ with respect to $M(r, U)$. We show how $\rho(U)$ and $\tau(U)$ can be computed by using Lemma 2 with well-known formulas for the Maclaurin coefficients \[1, \text{ Ch. 2}\].

Using Stirling’s formula,

$$\nu! = (1 + o(1))\sqrt{2\pi\nu} \left(\frac{\nu}{e}\right)^\nu$$

as $\nu \to \infty$, we obtain for all $\nu$ large enough,

$$|a_{2\nu}| = \frac{1}{2^{\nu-1}(2\nu)!} \sum_{m=0}^{\infty} \frac{1}{4^m m!(m+\nu-1)!} \leq \frac{2}{2^\nu (2\nu)! (\nu-1)!} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m \leq \frac{3\nu}{2^\nu (2\nu)! \nu!} \leq \left(\frac{e}{2\nu}\right)^{3\nu}.$$

To get a lower bound for $|a_{2\nu}|$, we again use Stirling’s formula, and obtain for all $\nu$ large enough,

$$|a_{2\nu}| \geq \frac{1}{2^{\nu-1}(2\nu)! (\nu-1)!} = \frac{2\nu}{2^\nu (2\nu)! \nu!} \geq \frac{1}{5} \left(\frac{e}{2\nu}\right)^{3\nu}.$$

Therefore, it can be seen that

$$\rho(U) = \limsup_{\nu \to \infty} \frac{(2\nu) \log(2\nu)}{-\log|a_{2\nu}|} = \frac{2}{3},$$

which agrees with the known result. Then, using the notation $\rho = \rho(U) = 2/3$ and the fact that $\lim_{\nu \to \infty} \nu^{\frac{1}{3\nu}} = 1$, we obtain

$$\tau(U) = (e\rho)^{-1} \limsup_{\nu \to \infty} (2\nu)^{\frac{1}{3\nu}} |a_{2\nu}| \frac{\nu!}{\nu^{\frac{3}{2}}} = \frac{3}{2} \limsup_{\nu \to \infty} (2\nu)^{\frac{1}{3\nu}} |a_{2\nu}| \frac{\nu!}{\nu^{\frac{3}{2}}} = \frac{3}{2}.$$

As discussed earlier, the contour integral function $H(z)$ in \[(2.7)\] also satisfies equation \[(6.1)\] and Theorem 11 below exhibits relationships between the three solutions $U(z)$, $H(z)$, $H(-z)$ of \[(6.1)\]. In order to prove Theorem 11, we first derive a more convenient form of $H(z)$ that does not involve a branch of the logarithm as in Example 4.
Lemma 3  The function $H(z)$ in (2.7) can be written in the form
\[
H(z) = \int_A w \exp\left\{ \frac{z}{w} - \frac{1}{2w^2} - \frac{w^2}{2} \right\} dw,
\]
where the contour $A$ is defined by $A = A_1 + A_2 + A_3$ with
\begin{align*}
A_1 : w = r, & \quad r \text{ goes from } +\infty \text{ to } 1, \\
A_2 : w = e^{i\theta}, & \quad \theta \text{ goes from } 0 \text{ to } \pi, \\
A_3 : w = -r, & \quad r \text{ goes from } 1 \text{ to } +\infty.
\end{align*}

Proof. By making the change of variable $v = \sqrt{2}w$ in (2.7), we obtain
\[
H(z) = \int_J v \exp\left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv,
\]
where $J$ is the contour defined by $J = J_1 + J_2 + J_3$ with
\begin{align*}
J_1 : v = re^{\pi i/8}, & \quad r \text{ goes from } +\infty \text{ to } \sqrt{2}, \\
J_2 : v = \sqrt{2}e^{i\theta}, & \quad \theta \text{ goes from } \pi/8 \text{ to } 7\pi/8, \\
J_3 : v = re^{7\pi i/8}, & \quad r \text{ goes from } \sqrt{2} \text{ to } +\infty.
\end{align*}

From Cauchy’s theorem, it can be seen that the curve $J$ can be replaced by the curve $L$ below as follows:
\[
H(z) = \int_L v \exp\left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv,
\]
where $L$ is the contour defined by $L = L_1 + L_2 + L_3$ with
\begin{align*}
L_1 : v = re^{\pi i/8}, & \quad r \text{ goes from } +\infty \text{ to } 1, \\
L_2 : v = e^{i\theta}, & \quad \theta \text{ goes from } \pi/8 \text{ to } 7\pi/8, \\
L_3 : v = re^{7\pi i/8}, & \quad r \text{ goes from } 1 \text{ to } +\infty.
\end{align*}

Figure 3: Contour $L$
We now show that the curve $L$ in (6.11) can be replaced by the curve $A$ in (6.10), which will prove Lemma 3. First we address the right half-plane. For any fixed $R > 1$, it follows from Cauchy’s theorem that

\[
\int_{D(R)} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv = 0,
\]  

(6.12)

where $D(R)$ is the closed contour defined by

\[
D(R) = D_1 + D_2 + D_3 + D_4
\]

with

\[
D_1 : v = re^{\pi i/8}, \ r \text{ goes from } R \text{ to } 1,
\]
\[
D_2 : v = e^{i\theta}, \ \theta \text{ goes from } \pi/8 \text{ to } 0,
\]
\[
D_3 : v = r, \ r \text{ goes from } 1 \text{ to } R,
\]
\[
D_4 : v = Re^{i\theta}, \ \theta \text{ goes from } 0 \text{ to } \pi/8.
\]

We now show that the contribution to the integral in (6.12) from the arc $D_4$ goes to zero as $R \to \infty$. We have

\[
\left| \int_{D_4} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv \right| \leq R^2 \int_0^{\pi/8} \exp \left\{ \frac{|z|}{R} + \frac{1}{2R^2} - \frac{R^2}{2} \cos 2\theta \right\} d\theta.
\]

Since $\cos 2\theta \geq \cos \pi/4 = \sqrt{2}/2$ for $0 \leq \theta \leq \pi/8$, we obtain

\[
\left| \int_{D_4} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv \right| \leq \frac{\pi R^2}{8} \exp \left\{ \frac{|z|}{R} + \frac{1}{2R^2} - \frac{\sqrt{2}R^2}{4} \right\}.
\]

Therefore,

\[
\int_{D_4} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv \to 0 \quad \text{as} \quad R \to \infty.
\]  

(6.13)

By letting $R \to \infty$ in (6.12) and using (6.13), we deduce that

\[
\int_{L_1} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv = \int_{E_2+E_3} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv,
\]  

(6.14)

where $L_1$ (above), $E_2$, $E_3$ are defined by

\[
L_1 : v = re^{\pi i/8}, \ r \text{ goes from } +\infty \text{ to } 1,
\]
\[
E_2 : v = r, \ r \text{ goes from } +\infty \text{ to } 1,
\]
\[
E_3 : v = e^{i\theta}, \ \theta \text{ goes from } 0 \text{ to } \pi/8.
\]
We now use the same reasoning in the left half-plane. From Cauchy’s theorem, for any fixed \( R > 1 \),

\[
\int_{S(R)} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv = 0, \tag{6.15}
\]

where \( S(R) \) is the closed contour defined by \( S(R) = S_1 + S_2 + S_3 + S_4 \) with

\begin{align*}
S_1 : v &= re^{7\pi/8}, \ r \text{ goes from } 1 \text{ to } R, \\
S_2 : v &= Re^{i\theta}, \ \theta \text{ goes from } 7\pi/8 \text{ to } \pi, \\
S_3 : v &= -r, \ r \text{ goes from } R \text{ to } 1, \\
S_4 : v &= e^{i\theta}, \ \theta \text{ goes from } \pi \text{ to } 7\pi/8.
\end{align*}

\[\text{Figure 4: Contours } D(R) \text{ and } S(R)\]

From similar reasoning to that used to get (6.13), we will obtain

\[
\int_{S_2} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv \to 0 \quad \text{as} \quad R \to \infty.
\]

Thus, by letting \( R \to \infty \) in (6.15), we deduce that

\[
\int_{L_3} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv = \int_{T_2 + T_3} v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} dv, \tag{6.16}
\]
Solutions defined by contour integrals

where $L_3$ (above), $T_2$, $T_3$ are defined by

\[
L_3 : v = re^{7\pi i/8}, \ r \text{ goes from } 1 \text{ to } +\infty,
\]

\[
T_2 : v = e^{i\theta}, \ \theta \text{ goes from } 7\pi/8 \text{ to } \pi,
\]

\[
T_3 : v = -r, \ r \text{ goes from } 1 \text{ to } +\infty.
\]

By combining (6.11), (6.14) and (6.16), we obtain (6.10).

The following result exhibits relationships between $U(z)$ in (6.2), $H(z)$ in (6.10), and $H(-z)$. Since $U(z)$ is an even function, we have $U(z) \equiv U(-z)$, which is confirmed again by (6.18) below.

**Theorem 11** The three contour integral functions $U(z)$, $H(z)$, $H(-z)$ are pairwise linearly independent solutions of (6.1) which satisfy

\[
\rho(U(z)) = \rho(H(z)) = \rho(H(-z)) = 2/3.
\]  

Moreover, we have

\[
2\pi i U(z) \equiv H(z) + H(-z).
\]  

**Proof.** Since $H(z)$ is a known solution of (6.1) from Example 4, it can be verified that $H(-z)$ is also a solution of (6.1). Since $U(z)$ in (6.2) is a solution of (6.1), all three functions $U(z)$, $H(z)$, $H(-z)$ are solutions of (6.1). We have $\rho(U(z)) = 2/3$ from Theorem 7. Since $\rho(H(z)) = 2/3$ from Example 4, it follows that $\rho(H(-z)) = 2/3$. Thus, (6.17) holds.

Next, we have

\[
H(-z) = \int_A w \exp \left\{ -\frac{z}{w} - \frac{1}{2w^2} - \frac{w^2}{2} \right\} \, dw,
\]

where $A$ is the contour in (6.10). From the substitution $v = -w$, we obtain

\[
H(-z) = \int_P v \exp \left\{ \frac{z}{v} - \frac{1}{2v^2} - \frac{v^2}{2} \right\} \, dv,
\]  

where the contour $P$ is defined by $P = P_1 + P_2 + P_3$ with

\[
P_1 : v = -r, \ r \text{ goes from } +\infty \text{ to } 1,
\]

\[
P_2 : v = e^{i\theta}, \ \theta \text{ goes from } \pi \text{ to } 2\pi,
\]

\[
P_3 : v = r, \ r \text{ goes from } 1 \text{ to } +\infty.
\]

The identity (6.18) can be deduced from (6.2), (6.10), and (6.19).

It remains to prove that the three functions $U(z)$, $H(z)$, $H(-z)$ are pairwise linearly independent. First we show that $U(z)$ and $H(z)$ are linearly
independent. To this end, we will compute the Wronskian $W(U(z), H(z))(0)$. Since $U(z)$ is an even function, it follows that $U'(0) = 0$. Hence,

$$W(U(z), H(z))(0) = U(0)H'(0). \quad (6.20)$$

We compute $U(0)$ and $H'(0)$. From (6.2),

$$U(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{2i\theta} \exp \left\{-\frac{1}{2}e^{-2i\theta} - \frac{1}{2}e^{2i\theta} \right\} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos 2\theta e^{-\cos 2\theta} d\theta + \frac{i}{2\pi} \int_0^{2\pi} \sin 2\theta e^{-\cos 2\theta} d\theta.$$

The second integral equals zero because $U(z)$ is real on the real axis from Theorem 7. By using a computer calculation on the first integral, we get the approximation

$$U(0) \approx -0.5652. \quad (6.21)$$

From (6.10), we have

$$H'(z) = \int_A \exp \left\{ \frac{z}{w} - \frac{1}{2w^2} - \frac{w^2}{2} \right\} dw$$

$$= \int_1^{\infty} \exp \left\{ \frac{z}{r} - \frac{1}{2r^2} - \frac{1}{2} \right\} dr$$

$$+ i \int_0^{\pi} e^{i\theta} \exp \left\{ ze^{-i\theta} - \frac{1}{2}e^{-2i\theta} - \frac{1}{2}e^{2i\theta} \right\} d\theta$$

$$- \int_{1}^{\infty} \exp \left\{ \frac{-z}{r} - \frac{1}{2r^2} - \frac{1}{2} \right\} dr.$$

This gives

$$H'(0) = i \int_0^{\pi} e^{i\theta} \exp \left\{ -\cos 2\theta \right\} d\theta - 2 \int_1^{\infty} \exp \left\{ -\frac{1}{2r^2} - \frac{1}{2} \right\} dr,$$

which reduces to

$$H'(0) = - \int_0^{\pi} \sin \theta e^{-\cos 2\theta} d\theta - 2 \int_1^{\infty} \exp \left\{ -\frac{1}{2r^2} - \frac{1}{2} \right\} dr,$$

because

$$\int_0^{\pi} \cos \theta e^{-\cos 2\theta} d\theta = - \int_{-\pi/2}^{\pi/2} \sin \theta e^{\cos 2\theta} d\theta = 0.$$

Hence, $H'(0) \neq 0$, which when combined with (6.20) and (6.21), gives

$$W(U(z), H(z))(0) \neq 0.$$

It follows that $U(z)$ and $H(z)$ are linearly independent. Thus, since (6.18) holds, it can be deduced that the three functions $U(z), H(z), H(-z)$ are pairwise linearly independent. This completes the proof of Theorem 11. □
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