Quasi-degenerate neutrinos
from an abelian family symmetry

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\textsuperscript{4}Supported in part by the United States Department of Energy under grant DE-FG05-86-ER40272.
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Abstract

We show that models with an abelian family symmetry which accounts for the observed hierarchies of masses and mixings in the quark sector may also accommodate quasi-degeneracies in the neutrino mass spectrum. Such approximate degeneracies are, in this context, associated with large mixing angles. The parameters of this class of models are constrained. We discuss their phenomenological implications for present and foreseen neutrino experiments.
1 Introduction

Family symmetries which might help to understand the observed pattern of quark and lepton masses and mixings have received lately an increasing interest. Even in the simplest example of an abelian family symmetry [1, 2, 3] some general properties seem to emerge, such as an anomalous behavior [4] which could be traced back to an underlying superstring theory. In the sector of quarks and charged leptons, one is until now limited to “postdictions”, that is, one tries to explain the already observed spectrum of masses and Cabibbo-Kobayashi-Maskawa mixing angles. The neutrino sector on the other hand represents a frontier where one could in principle make predictions based on the constraints obtained from the quark and charged lepton sectors. This would represent a test of these ideas and models.

Of course, the situation is not as straightforward. For example, the models constructed until now seem to give a hierarchical pattern of neutrino masses [5, 6, 7, 8, 9], typically masses of the order of a small parameter \(< \theta > /M\) to some power \(n\), which decreases with the family index \(i\). It therefore seems that one has to invoke a non-abelian family symmetry in order to have some sort of degeneracy between some of the light neutrino masses. We will show in what follows that this is not necessary and that some abelian family symmetries not only yield neutrino mass degeneracies but also predict at which level the degeneracy is lifted.

In the context of abelian family symmetries, the presence of degenerate light neutrinos is directly related with large mixing angles. The type of mass spectrum that we consider may thus be of some use in all scenarios where large mixing angles are needed. One may mention the large angle branch in the MSW interpretation of the solar neutrino deficit [10] or the angles needed in order to account for the present atmospheric neutrino data. We will return to a discussion of the possible phenomenological uses of such models in Section 4 but we would like to keep the discussion as general as possible for the next two sections. In Section 2, we review how to use an abelian family symmetry in order to generate mass and mixing hierarchies. We also describe the class of models we will consider in the following. Section 3 gives explicit models with two or three quasi-degenerate neutrinos.

\footnote{Schemes with three massive neutrinos two of which or all three are quasi-degenerate in mass, and their phenomenology have been discussed in recent articles [11, 12] and [13], respectively. However, the models considered in refs. [11, 12] are very different from those we study here.}
2 Neutrino masses and family symmetry

The class of models we consider are extensions of the Minimal Supersymmetric Standard Model (MSSM) with:

1. an additional abelian family symmetry, $U(1)_X$. We assume that this symmetry is gauged and that its anomalies are compensated for by the Green-Schwarz mechanism. One could be less restrictive and allow for example discrete symmetries. We prefer to stick to continuous gauge symmetries since the anomaly cancellation provides constraints which seem to go in the right direction for phenomenology [4, 9].

2. a MSSM singlet field $\theta$ with $U(1)_X$ charge $X_{\theta} = -1$. This singlet is used to break $U(1)_X$ and to generate fermion masses,

3. three right-handed neutrinos, $\tilde{N}_i$ ($i$ is a family index), in addition to the MSSM spectrum. The light neutrino masses are then generated by the seesaw mechanism [14]. Of course, the number of right-handed neutrinos is not constrained by experiment and could be larger, or smaller, than three. We choose to have one right-handed neutrino per family mainly for illustration purpose.

In the following, we will concentrate on the lepton sector. We denote the lepton fields by $L_i$ (lepton doublets, with their $I_W = +1/2$ components $\nu_i$), $E_i$ (charged lepton singlets) and $\tilde{N}_i$ (right-handed neutrinos), and their charges under $U(1)_X$ respectively by $l_i$, $e_i$ and $n_i$. We also note $h_u$ and $h_d$ the $U(1)_X$ charges of the two Higgs doublets $H_u$ and $H_d$.

2.1 Dirac and Majorana matrices

Let us recall briefly how the Dirac ($M_D$) and Majorana ($M_S$) matrices, which determine the light neutrino spectrum through the seesaw mechanism, are constrained by the family symmetry. Each Dirac mass term $L_i \tilde{N}_j H_u$ carries an $X$-charge $p_{ij} = l_i + n_j + h_u$. If $p_{ij} \neq 0$, the coupling is forbidden by $U(1)_X$, and the corresponding entry of $M_D$ is zero. However, if the excess charge $p_{ij}$ is positive, one can write non-renormalizable interactions involving the chiral singlet $\theta$:

$$L_i \tilde{N}_j H_u \left( \frac{\theta}{M} \right)^{p_{ij}}$$

where $M$ is a large scale characteristic of the underlying theory (typically $M \sim M_{\text{Planck}}$ or $M_{\text{GUT}}$). When $\theta$ acquires a vev, $U(1)_X$ is spontaneously broken and effective Dirac masses are generated:

$$\langle M_D \rangle_{ij} \sim v_2 \left( \frac{\langle \theta \rangle}{M} \right)^{p_{ij}}$$
where $v_2 = \langle H_u \rangle$. Since $U(1)_X$ is broken below the scale $M$, $\epsilon \equiv \langle \theta \rangle / M$ is a small parameter. Thus the Dirac matrix obtained has a hierarchical structure, with the order of magnitude of its entries fixed by their excess charges $p_{ij}$ under $U(1)_X$.

Indeed the same type of analysis has proved to be successful in the quark sector where it was shown that $U(1)_X$ charges can be found which account for the observed pattern of Dirac masses and mixing angles. Constraints are numerous and the success of the procedure is not guaranteed from the start. Indeed it was shown in that, in a large class of models, the observed masses of the charge $(-1/3)$ quarks and charge $(-1)$ leptons constrain the $U(1)_X$ symmetry to be anomalous. This anomaly must be cancelled through a Green-Schwarz mechanism, which in turn imposes constraints on the theory and suggests a superstring origin to the model. Among the constraints is the value of the Weinberg angle which turns out to be $\sin^2 \theta_W = 3/8$ and thus surprisingly a successful prediction. If one takes seriously the superstring nature of the model, then the parameter $\langle \theta \rangle / M$ is typically of order $10^{-2}$ to $10^{-1}$, that is of the order of the sine of the Cabibbo angle, the basic mixing angle in the CKM matrix.

The entries of the Majorana matrix $M_S$ are generated in the same way, with non-renormalizable interactions of the form:

$$M \bar{N}_i N_j \left( \frac{\theta}{M} \right)^{q_{ij}}$$

(3)

giving rise to effective Majorana masses

$$(M_S)_{ij} \sim M \left( \frac{\langle \theta \rangle}{M} \right)^{q_{ij}}$$

(4)

provided that $q_{ij} = n_i + n_j$ is a positive integer (otherwise $(M_S)_{ij} = 0$).

The structure of the light neutrino mass matrix

$$M_\nu = -M_D M_S^{-1} M_D^T$$

(5)

is therefore fixed by the charges of the leptons under $U(1)_X$. Note, however, that each of the entries is determined only up to an arbitrary factor of order one by the family symmetry. Irrespective of these factors, the Majorana mass matrix $M_S$ and therefore the light neutrino mass matrix $M_\nu$ are automatically symmetric. The presence of these unknown factors of order one is certainly the most unwelcome feature of this type of approach when we come to detailed predictions since we will then take the parameter $\epsilon$ to be $\sin \theta_c \sim 0.22$, which is not such a small number. The only way to avoid this type of problem would

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7 The two mass parameters which appear here and are, for simplicity, both denoted by $M$, may be different. We will not discuss here in more details the nature of these scales and how they might give rise after seesaw to the right scale of neutrino mass.
be to go to a specific model. Since we want to advertise the general features of such a class of models, we will refrain from doing so. But, keeping this in mind, all the constraints that we will obtain below on the $U(1)_X$ lepton charges will be understood up to one unit.

### 2.2 Light neutrino spectrum

If all entries of $M_D$ and $M_S$ are nonzero (i.e. $p_{ij}, q_{ij} \geq 0$), one obtains:

\[(M_\nu)_{ij} \sim \frac{v^2}{M} \epsilon^{l_i + l_j + 2h_u}\]  

which leads to the following light neutrino masses and mixings:

\[m_{\nu_i} \sim \frac{v^2}{M} \epsilon^{2l_i + 2h_u}, \quad (R_\nu)_{ij} \sim \epsilon^{|l_i - l_j|}\]  

Here $R_\nu$ is the matrix that diagonalizes $M_\nu$: $M_\nu = R_\nu \text{Diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) R_\nu^T$. The physical mixing angles are given by the lepton mixing matrix $U = R_L R_\nu R^T$, which also includes a contribution from the charged lepton sector: $R_L^T M_e R_e R^T$.

When the excess charges of the charged lepton Yukawa couplings $L_i \bar{E}_j H_d$ are positive, $R_L^T$ has the same structure as $R_\nu$, and $U$ is given by:

\[U_{ij} = (R_L^T R_\nu)_{ij} \sim \epsilon^{|l_i - l_j|}\]

Then one has the relation between neutrino mass ratios and mixing angles:

\[U_{ij}^2 \sim \frac{m_{\nu_i}}{m_{\nu_j}}, \quad m_{\nu_i} < m_{\nu_j}\]

A general feature of such models is that the neutrino mass spectrum is naturally hierarchical, with small mixing angles. Mass degeneracies occur when some lepton charges are equal (for example, if $l_2 = l_3$, one obtains $m_{\nu_2} \sim m_{\nu_3}$). Unfortunately, the model is less predictive in this case. Indeed, because of the presence of arbitrary factors of order one in each entry of $M_\nu$, the squared mass difference between almost degenerate neutrinos cannot be related to the parameters of the family symmetry. An accurate mass degeneracy would then require fine-tuning. This problem could be solved by going to a larger non-abelian family symmetry, which would fix the factors of order one. We will however follow a different path here: keep abelian family symmetries but assume zeroth order (in $\epsilon$) relations among the Yukawa couplings which ensure degeneracies.

Indeed, if $M_D$ and $M_S$ contain some zeros, the above results can be modified. One can show that, if there are simultaneous and correlated zeros in $M_D$ and $M_S$, the mixing of neutrinos having different flavour was considered first in ref. [21], while a Majorana mass term for the left-handed flavour neutrinos was discussed as a possible source of lepton mixing first in ref. [22].

9 Also, in presence of degeneracies, these factors could upset the formulae [4].
and \( M_S \), some entries of \( M_\nu \) become zero, the other entries being still given by (5). Of course, the neutrino spectrum then deviates from (7). This raises the hope that the presence of correlated zeroes in \( M_D \) and \( M_S \) can explain mass degeneracies and large mixing angles.

3 Models of quasi-degenerate neutrinos

In order to find abelian family symmetries leading to quasi-degenerate neutrinos, we proceed in the following way. We start from a very simple pattern for the light neutrino matrix \( M_\nu \) (here we assume that the seesaw mechanism has already been performed), with a little number of nonzero entries and two or three exactly degenerate eigenvalues. Then, assuming a \( U(1)_X \) symmetry, we identify the most general \( U(1)_X \) charges compatible with this pattern. The breaking of \( U(1)_X \) fills in the zero entries in \( M_\nu \) with powers of the small parameter \( \epsilon \). This lifts the degeneracy between the neutrinos and allows us to relate the squared mass differences to the \( X \)-charges. Note that this method is similar to the one used in the quark sector where, in order to account for the strong hierarchy of the quark mass spectrum, one starts from an up quark matrix with all entries zero except the (3,3) entry.

3.1 Patterns of neutrino mass matrices with degenerate eigenvalues

Consider the following three symmetric matrices [23, 24]:

\[
\begin{align*}
(i) & \quad M_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \\
(ii) & \quad M_\nu = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & a \\ b & a & 0 \end{pmatrix} \\
(iii) & \quad M_\nu = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}
\end{align*}
\]

where \( a \) and \( b \) are arbitrary numbers of order one. These matrices have two degenerate eigenvalues [23] and their corresponding diagonalizing matrices \( R_\nu \) have at least one large mixing angle. Explicitly:

\[
\begin{align*}
(i) & \quad D_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \\
R_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\end{align*}
\]

The negative values in the diagonal mass matrix \( D_\nu \) imply that the corresponding Majorana neutrinos with positive definite mass have a CP-parity equal to \((-i)\), while the CP-parity of the other definite mass Majorana neutrinos is \((+i)\) [24]. Two mass-degenerate Majorana neutrinos having opposite CP-parities are equivalent to a Dirac neutrino.
\( D_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{a^2+b^2} & 0 \\ 0 & 0 & \sqrt{a^2+b^2} \end{pmatrix} \)

\( R_\nu = \frac{1}{\sqrt{2(a^2+b^2)}} \begin{pmatrix} -\sqrt{2}a & -b & b \\ \sqrt{2}b & -a & a \\ 0 & \sqrt{a^2+b^2} & \sqrt{a^2+b^2} \end{pmatrix} \) \tag{12} \)

\( \text{(iii) } D_\nu = \begin{pmatrix} b & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \quad R_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \) \tag{13} \)

where \( D_\nu = \text{Diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) = R_\nu^T \mathcal{M}_\nu R_\nu \). It can be easily seen that the previous three patterns are, up to permutations in family space, the only matrices with arbitrary nonzero numbers having at least two degenerate eigenvalues (up to a sign). Three degenerate eigenvalues cannot be obtained with arbitrary entries. One needs for example to impose \( a = b \) in pattern (iii), which requires an additional symmetry.

It is not difficult to find the most general \( X \)-charge reproducing pattern (i), (ii) or (iii) when unbroken. The charge operator acting on lepton doublets, \( X_L \), can be identified with a generalized Zeldovich-Konopinsky-Mahmoud \(^{26}\) combination of lepton numbers \(^{23}\): patterns (ii) and (iii) correspond respectively to \( X_L = l(L_e + L_\mu - L_\tau) \) and \( X_L = l(L_\mu - L_\tau) \), where \( l \) is a rational number. Pattern (i) corresponds to \( X_L = l(L_e + l' L_\mu - L_\tau) \) with \( l' \neq -l, l \), which means that \( L_e \) and \( L_\mu - L_\tau \) are separately conserved.

The breaking of the family symmetry fills in the zero entries with powers of the small parameter \( \epsilon \), namely \( (\mathcal{M}_\nu)_{ij} \sim \epsilon^{l_i + l_j} \) when \( p_{ij}, q_{ij} \geq 0 \). If all powers are positive, this slightly modifies the previous results. In particular, the degeneracy between \( m_{\nu_2} \) and \( m_{\nu_3} \) is broken \(^{11}\) and \( m_{\nu_1} \) becomes nonzero in cases (i) and (ii). If, on the other hand, some powers are negative, the previous patterns are destabilized. This happens in fact in all three cases. Assuming \( l \geq 0 \), one obtains \( (\mathcal{M}_\nu)_{33} \sim \epsilon^{-2l} \gg 1 \), thus the (3,3) entry is the dominant one after breaking of the family symmetry, and the mass spectrum becomes hierarchical. One can easily check that the only way to avoid such effect is to assume the presence of simultaneous and correlated zeroes in \( \mathcal{M}_D \) and \( \mathcal{M}_S \), which forces \( (\mathcal{M}_\nu)_{33} \) to vanish.

### 3.2 Explicit models with 2 degenerate neutrinos

In this section, we study an explicit model with 2 degenerate neutrinos based on the pattern (i). We will briefly comment on models based on patterns (ii) and (iii) at the end of the section.

\(^{11}\) Thus transforming the corresponding Dirac neutrino into a pseudo-Dirac one.
In order to reproduce pattern (i), we choose the following assignment of lepton charges:

\[
\begin{align*}
  l_1 &= l' \\
  l_2 &= -l_3 = l \\
  0 < l < l' \\
  0 < n_1 < (-n_3) \leq l \leq n_2
\end{align*}
\]

and, for simplicity, we assume \( h_u = h_d = 0 \).

The model thus contains five parameters \( l, l', n_1, n_2 \), and \( n_3 \). With this charge assignment, the Dirac and Majorana matrices have correlated zeroes:

\[
M_D \sim \begin{pmatrix}
  \epsilon^{l'+n_1} & \epsilon^{l'+n_2} & \epsilon^{l'+n_3} \\
  \epsilon^{l+n_1} & \epsilon^{l+n_2} & \epsilon^{l+n_3} \\
  0 & \epsilon^{-l+n_2} & 0
\end{pmatrix}
\]

\[
M_S \sim \begin{pmatrix}
  \epsilon^{2n_1} & \epsilon^{n_1+n_2} & 0 \\
  \epsilon^{n_1+n_2} & \epsilon^{2n_2} & \epsilon^{n_2+n_3} \\
  0 & \epsilon^{n_2+n_3} & 0
\end{pmatrix}
\]

which results in the following structure for the light neutrino matrix:

\[
\mathcal{M}_\nu \sim \begin{pmatrix}
  \epsilon^{2l'} & \epsilon^{l'+l} & \epsilon^{l'-l} \\
  \epsilon^{l'+l} & \epsilon^{2l} & 1 \\
  \epsilon^{l'-l} & 1 & 0
\end{pmatrix}
\]

where we have left aside the overall mass scale set by the seesaw mechanism.

Note that, due to the simultaneous presence of zeroes in \( M_D \) and \( M_S \), the structure of \( \mathcal{M}_\nu \) follows pattern (i) to zeroth order in the small parameter \( \epsilon \). The main effect of the family symmetry breaking is to break slightly the degeneracy between \( m_{\nu_2} \) and \( m_{\nu_3} \). We thus obtain a pattern with two almost degenerate neutrinos

\[
m_{\nu_1} \sim \epsilon^{2l} \ll m_{\nu_2} \simeq m_{\nu_3},
\]

their squared mass difference being determined by the family symmetry:

\[
\Delta m_{12}^2 \sim \epsilon^{2l}
\]

Actually, one should take into account the presence of nondiagonal kinetic terms in the Kähler potential which are allowed by the symmetry \([\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2]\):

\[
\sum_{i,j} L_i^\dagger L_j \left[ H(l_i - l_j) \left( \frac{\theta^1}{M} \right)^{l_i - l_j} + H(l_j - l_i) \left( \frac{\theta^1}{M} \right)^{l_j - l_i} \right].
\]

where \( H \) is the Heaviside function \((H(x) = x \text{ if } x \geq 0, \ H(x) = 0 \text{ otherwise})\).

The lepton fields have to be redefined in order to bring the Kähler potential into its canonical form:

\[
L_i \rightarrow (W_L)_{ij} L_j
\]
where the matrix elements of $W_L$ are constrained in order of magnitude by the lepton charges:

$$(W_L)_{ij} \sim \epsilon^{|i_l-j_l|}.$$  

(22)

The light neutrino mass matrix, expressed in the new basis, is then:

$$\widehat{\mathcal{M}}_\nu = W^T_L M_\nu W_L$$  

(23)

The effect of this redefinition is to fill in the zero entries in $M_\nu$, the other entries remaining of the same order of magnitude. For the model under consideration:

$$\widehat{\mathcal{M}}_\nu \approx \begin{pmatrix} b \epsilon^{2l'} & c \epsilon^{l'+l} & e \epsilon^{l'-l} \\ c \epsilon^{l'+l} & d \epsilon^{2l} & a \\ e \epsilon^{l'-l} & a & f \epsilon^{2l} \end{pmatrix}$$  

(24)

where only the dominant term in each entry is given. We have restored in (24) arbitrary coefficients of order one $a$, $b$, $c$, $d$, $e$ and $f$.

The eigenvalues are not significantly modified by the filling of previously vanishing entries. New subleading terms (which we will omit for the sake of simplicity) appear in their expressions, but their orders of magnitude remain the same:

$$\begin{cases} m_{\nu_1} & \approx \frac{a^2b+de^2-2ace}{a} \epsilon^{2l'} \\
m_{\nu_2} & \approx a \\
m_{\nu_2} + m_{\nu_3} & \approx (d+f) \epsilon^{2l} \end{cases}$$  

(25)

Thus, the mass splitting between $m_{\nu_2}$ and $m_{\nu_3}$ remains of the same order:

$$\Delta m_{32}^2 \approx 2a(d+f) \epsilon^{2l}$$  

(26)

The other squared mass differences are:

$$\begin{cases} \Delta m_{31}^2 \approx [a^2 + \ldots] - [a(d+f) \epsilon^{2l} + \ldots] \\
\Delta m_{31}^2 \approx [a^2 + \ldots] + [a(d+f) \epsilon^{2l} + \ldots] \end{cases}$$  

(27)

The diagonalization matrix $R_\nu$ is:

$$R_\nu \approx \begin{pmatrix} 1 & \frac{e}{\sqrt{2a}} \epsilon^{l'-l} & \frac{e}{\sqrt{2a}} \epsilon^{l'-l} \\ -\frac{b}{a} \epsilon^{l'-l} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{de-ace}{a} \epsilon^{l'+l} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$  

(28)

where only the dominant term in each entry is given. As expected, the mixing between the two degenerate neutrinos remains large after breaking of $U(1)_X$ and zero filling.

\footnote{The other terms introduced by the lepton field redefinition have been taken into account in the diagonalization of $\mathcal{M}_\nu$.}
Since the physical mixing angles, i.e. the mixing angles which are relevant for neutrino oscillations, are given by the lepton mixing matrix \( U = R^L_e R_\nu \), we must study the charged lepton sector too. Like for the neutrinos, the order of magnitude of the charged lepton Yukawa couplings \( L_i \bar{E}_j H_d \) are determined by the charges of the fields \( L_i \) and \( \bar{E}_i \) under \( U(1)_X \):

\[
(M_e)_{ij} \sim v_1 \left( \frac{< \theta >}{M} \right)^{n_{ij}}
\]

(29)

where \( v_1 = < H_d > \) and \( n_{ij} = l_i + e_j \) (we have assumed \( h_d = 0 \)). The charges of the lepton doublets have already been fixed. Obviously, one cannot choose the same type of charge assignment for the charged lepton singlets \( \bar{E}_i \), otherwise there would be two degenerate charged leptons. The most natural structure for \( M_e \) is a hierarchical structure, as for the quark mass matrices, with dominant entry in position (3,3). When all \( n_{ij} \) are positive, the charged lepton mass matrix has the following form:

\[
M_e \sim \begin{pmatrix}
e^{l+e_1} & e^{l'+e_2} & e^{l'+e_3} \\
e^{l+e_1} & e^{l+e_2} & e^{l+e_3} \\
e^{-l+e_1} & e^{-l+e_2} & e^{-l+e_3}
\end{pmatrix}
\]

(30)

When some \( n_{ij} \) are negative, the corresponding entries of \( M_e \) are modified (after zero filling). \( M_e \) is in general not hermitian, so it is diagonalized by two unitary matrices: \( \text{Diag}(m_1, m_2, m_3) = R^L_e M_e R^L_e \). Assuming simply \( (M_e)_{33} \geq (M_e)_{ij} \) and \( (M_e)_{33} \gg (M_e)_{ij} \) for \( i, j = 1, 2 \) (in order to obtain a hierarchical mass spectrum), one can show that the diagonalization matrix \( R^L_e \), which enters the lepton mixing matrix, is given by:

\[
R^L_e \sim \begin{pmatrix}1 & e^{l'-l} & e^{l'+l} \\
e^{l'-l} & 1 & e^{2l} \\
e^{l'+l} & e^{2l} & 1
\end{pmatrix}
\]

(31)

except in the very particular case where \( e_1 \geq e_3 > e_2 \) and \( l' - l \geq e_3 - e_2 > 2l \).

The lepton mixing matrix is then:

\[
U \sim \begin{pmatrix}1 & A e^{l'-l} & A e^{l'-l} \\
-\sqrt{2} A^* e^{l'-l} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
B e^{l'+l} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\]

(32)

where \( A \) and \( B \) are functions of the arbitrary order one factors entering \( M_\nu \) and \( M_e \). Unless unnatural cancellations occur, \( A \) and \( B \) should not be much different from 1, say \( 0.2 < A, B < 5 \).

\(^{13}\)If \( e_2 = -e_3 = l \), the charged lepton matrix \( M_e \) has its dominant entries in positions (2,3) and (3,2).
Note that due to the smallness of the $R^L_\nu$ entries, $U$ has the same structure as $R^L_\nu$. As for the charged lepton masses, their experimental values constrain not only the singlet lepton charges $e_1$, $e_2$ and $e_3$, but also the doublet charges $l$ and $l'$.

Let us now briefly comment on models based on patterns (ii) and (iii). Pattern (ii) can be obtained by setting $l' = l$ in the charge assignment (14). Thus $l_1 = l_2 = -l_3 = l$, and $M_\nu$ depends on one charge parameter only:

$$M_\nu = \begin{pmatrix} b \epsilon^{2l} & c \epsilon^{2l} & e \\ c \epsilon^{2l} & d \epsilon^{2l} & a \\ e & a & 0 \end{pmatrix}$$

(33)

where $a$, $b$, $c$, $d$, $e$ are arbitrary coefficients of order one. There are still two degenerate neutrinos ($m_{\nu_1} \ll -m_{\nu_2} \simeq m_{\nu_3}$) with a squared mass difference of order $\epsilon^{2l}$, and a light neutrino with mass $m_{\nu_1} \sim \epsilon^{2l}$. The diagonalization matrix $R_\nu$ has only one small entry:

$$R_\nu \simeq \begin{pmatrix} -\frac{a}{\sqrt{a^2+e^2}} & -\frac{e}{\sqrt{2(a^2+e^2)}} & \frac{e}{\sqrt{2(a^2+e^2)}} \\ -\frac{c}{\sqrt{a^2+e^2}} & -\frac{\epsilon}{\sqrt{2(a^2+e^2)}} & \frac{\epsilon}{\sqrt{2(a^2+e^2)}} \\ \frac{d}{(a^2+e^2)^{1/2}} & \frac{\epsilon}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(34)

Pattern (iii) can be obtained by setting $l' = 0$ in the charge assignment (14). Thus $l_1 = 0$ and $l_2 = -l_3 = l$, and again $M_\nu$ depends on one charge parameter only:

$$M_\nu = \begin{pmatrix} b & c \epsilon^l & 0 \\ c \epsilon^l & d \epsilon^{2l} & a \\ 0 & a & 0 \end{pmatrix}$$

(35)

where $a$, $b$, $c$, $d$ are arbitrary coefficients of order one. The three masses lie in the same range, two of them being quasi-degenerate ($m_{\nu_1} \simeq -m_{\nu_2} \simeq m_{\nu_3}$), with a squared mass difference of order $\epsilon^{2l}$. The diagonalization matrix is:

$$R_\nu \simeq \begin{pmatrix} -\frac{1}{\sqrt{2(a+b)}} \epsilon^l & -\frac{c}{\sqrt{2(a+b)}} \epsilon^l & \frac{c}{\sqrt{2(a-b)}} \epsilon^l \\ -\frac{\sqrt{2(a+b)}}{a^2-b^2} \epsilon^l & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2(a-b)}}{a^2-b^2} \epsilon^l & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(36)

### 3.3 Case of 3 degenerate neutrinos

As mentioned above, pattern (iii) with $a = b$ leads to three degenerate eigenvalues. Unfortunately, this cannot be obtained from a U(1) family symmetry alone. One needs an additional symmetry to explain why $a = b$. Let us assume that such a symmetry exists. With the charge assignment of previous section ($l_1 = 0$ and $l_2 = -l_3 = l$), we get:

$$M_\nu = \begin{pmatrix} a_0 & c_0 \epsilon^l & 0 \\ c_0 \epsilon^l & d_0 \epsilon^{2l} & a_0 \\ 0 & a_0 & 0 \end{pmatrix}$$

(37)
Taking into account the presence in the Lagrangian of the theory of nondiagonal kinetic terms allowed by the symmetry we obtain:

\[
\tilde{M}_\nu \simeq \begin{pmatrix}
  a & c \epsilon^l & e \epsilon^l \\
  c \epsilon^l & d \epsilon^{2l} & a \\
  e \epsilon^l & a & f \epsilon^{2l}
\end{pmatrix}
\] (38)

The calculation of the eigenvalues up to order \( \epsilon^l \) gives:

\[
\begin{align*}
  m_{\nu_1} & \simeq a - \frac{c+e}{\sqrt{2}} \epsilon^l \\
  m_{\nu_2} & \simeq -a \\
  m_{\nu_3} & \simeq a + \frac{c+e}{\sqrt{2}} \epsilon^l
\end{align*}
\] (39)

We obtain three almost degenerate neutrinos \((m_{\nu_1} \simeq m_{\nu_2} \simeq m_{\nu_3})\), with all squared mass differences of the same order \(\epsilon^l\):

\[
\begin{align*}
  \Delta m^2_{21} & \simeq \sqrt{2} a(c+e) \epsilon^l \\
  \Delta m^2_{31} & \simeq 2\sqrt{2} a(c+e) \epsilon^l \\
  \Delta m^2_{32} & \simeq \sqrt{2} a(c+e) \epsilon^l
\end{align*}
\] (40)

The diagonalization matrix and correspondingly the lepton mixing matrix have only one small entry

\[
R_\nu \simeq \begin{pmatrix}
  \frac{1}{\sqrt{2}} & -\frac{c-e}{2\sqrt{2}a} \epsilon^l & \frac{1}{\sqrt{2}} \\
  -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\
  \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\] (41)

\[
U \simeq \begin{pmatrix}
  \frac{1}{\sqrt{2}} & A \epsilon^l & \frac{1}{\sqrt{2}} \\
  -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\
  -\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}
\end{pmatrix}
\] (42)

The corrections in the entries of order one in the matrices \(R_\nu\) and \(U\) are of the order of \(\epsilon^l\).

4 Phenomenological analysis

Experimental data yield strong indications of neutrino oscillations. Most convincing is the solar neutrino deficit [27] and its MSW interpretation [10]. The solar neutrino problem admits also a vacuum oscillation solution [28]. Present data implies (see, e.g., [29]) two possible sets of fundamental parameters in the

\[14\] Obviously, the same pattern of neutrino masses (i.e., three quasi-degenerate neutrinos) will arise also if instead of the charge \(X_L = l(L_\mu - L_\tau)\) the charge \(X_L = l(L_\mu - L_e)\) is conserved before the spontaneous breaking of the \(U(1)_X\) symmetry and the two different large entries in \(M_\nu\) are equal.
MSW case: the adiabatic branch requires large mixing, and the non-adiabatic branch requires small mixing of the electron neutrino with another species of neutrino. The vacuum oscillation interpretation of the data is possible only if the indicated mixing is large \[29\]. Several experiments (Super-Kamiokande, SNO, BOREXINO, ICARUS, HELLAZ) should in the future allow to distinguish between these solutions \[32\]. Measurements of the fluxes of electron and muon neutrinos resulting from cosmic ray interactions in the Earth atmosphere indicate a deficiency of muon neutrinos \[30, 31, 32\]. This anomaly can be caused by oscillations of the atmospheric muon neutrinos into another species of neutrino with a large mixing angle \[30, 32\]. The region of the parameter space the neutrino oscillation solution of the atmospheric neutrino problem implies will soon be tested in accelerator experiments. Lastly, the LSND group has reported \[33\] evidence for $\bar{\nu}_{\mu} - \bar{\nu}_{e}$ oscillations with a small mixing angle. Of course, there remains to be seen how all these experimental results will stand the test of time.

There is also some cosmological rationale for one or several neutrinos with masses of a few $eV$, to add the right amount of hot dark matter to cold dark matter in order to reproduce structure formation in the Universe \[34\].

As stressed in the introduction, if one wants to explain all mass hierarchies with abelian family symmetries, there is a one-to-one correspondence between large mixing angles and mass degeneracies. Indeed, unless some fine-tuning occurs in the mass matrix $M_{\nu}$, quasi-degenerate neutrinos automatically have a large mixing, while neutrinos well separated in mass have a small mixing. This implies that the MSW adiabatic solution (AS) or the vacuum oscillation solution (resp. the interpretation of the atmospheric neutrino data in terms of oscillations) requires the electron neutrino (resp. the muon neutrino) to be degenerate in mass \[16\] with another kind of neutrino, while the MSW non-adiabatic solution (NAS) as well as the oscillation explanation of the LSND result involve neutrinos with a substantial mass splitting. The vacuum oscillation solution of the solar neutrino problem can be realized only if the massive neutrinos are highly degenerate in mass ($\Delta m^2 \approx (10^{-10} - 5.10^{-12})$ eV$^2$ \[29\]), which in turn requires huge and seemingly unrealistic values of the charge $l$. For this reason we shall not consider it further.

Keeping the above in mind, one can show that only six neutrino mass patterns are compatible with at least two of the three possible “unconventional” interpretations of the three indicated experimental results, namely the MSW transitions of solar neutrinos ($\nu_{\odot}$), the atmospheric neutrino oscillations and the small angle $\bar{\nu}_{\mu} - \bar{\nu}_{e}$ oscillations. These mass patterns are:

1. $m_{\nu_{\tau}(e)} \ll m_{\nu_{\mu}(\tau)} \ll m_{\nu_{\mu}}$ ($\nu_{\odot}$-problem (MSW NAS) and LSND effect)

\[15\] The Super-Kamiokande experiment began operation on April 1, 1996, and the completion of SNO is scheduled for the beginning of 1997.

\[16\] More precisely, we mean the massive neutrino whose state vector is the dominant component in the electron neutrino state vector when expressed (with the help of the lepton mixing matrix $U$) as linear combination of the state vectors of the neutrinos $\nu_{i}$, $i=1,2,3$, with definite mass, etc.
2. $m_{\nu_e} \simeq m_{\nu_{\tau}} \ll m_{\nu_{\mu}}$ ($\nu_\odot$-problem (MSW AS) and LSND effect)
3. $m_{\nu_{\mu}} \ll m_{\nu_e} \simeq m_{\nu_{\tau}}$ ($\nu_\odot$-problem (MSW AS) and LSND effect)
4. $m_{\nu_e} \ll m_{\nu_{\mu}} \simeq m_{\nu_{\tau}}$ (atmospheric-$\nu$ problem and LSND effect)
5. $m_{\nu_e} \simeq m_{\nu_{\mu}} \simeq m_{\nu_{\tau}}$ ($\nu_\odot$-problem (MSW AS) and atmospheric-$\nu$ problem)
6. $m_{\nu_{\tau}}, m_{\nu_{\mu}} \ll m_{\nu_e}$ (atmospheric-$\nu$ problem and LSND effect)

Patterns 3, 4 and 5 can account for the hot dark matter of the Universe, while in patterns 1,2 and 6, the heavier neutrino may be too light for this purpose. Note that it is not possible to accommodate all present data simultaneously with only three species of light neutrinos, if one ever wanted to.

We will adopt in this section the following strategy. We start with either of the two problems whose neutrino physics solutions involve large lepton mixing angles (atmospheric neutrino anomaly or the solar neutrino problem (MSW adiabatic solution)) and try to accommodate them in the framework of the models presented earlier. This in turn constrains the lepton charges. A further important constraint arises from the charged lepton sector where one wants to reproduce the observed mass hierarchy. We will see that, generally, these constraints, together with the hot dark matter one, sufficiently restrict the range of family charges so that one tends to fall in the region of parameter space favored by the LSND experiment.

### 4.1 Atmospheric neutrino problem

We start with the atmospheric neutrino problem which, if it remains, requires large mixing angles, and thus in our framework, degenerate neutrinos. In the model of Subsection 3.2, the mass spectrum contains two heavy quasi-degenerate neutrinos with a large mixing angle, and a light neutrino which mixes weakly with the other ones ($m_{\nu_1} \ll m_{\nu_2} \simeq m_{\nu_3}$). In the model of Subsection 3.3 there are three quasi-degenerate neutrinos concomitant with large lepton mixing. Such patterns can account for the hot dark matter of the Universe, and simultaneously explain the atmospheric neutrino deficit in terms of $\nu_\mu - \nu_\tau$ oscillations [11, 30, 13].

Consider first the model with two quasi-degenerate neutrinos. The parameters of the model are $l$ and $l'$ (with $0 < l < l'$), to which one should add $m_{\nu_3}$, whose value depends on the mass scale of the heavy RH neutrinos $\bar{N}_i$ involved in the seesaw. Leaving out the arbitrary factors of order one and keeping only the orders of magnitude, one can express all masses in units of $m_{\nu_3}$ and in terms of $l$, $l'$:

$$\frac{m_{\nu_1}}{m_{\nu_3}} \sim \epsilon^{2l'}, \quad \frac{m_{\nu_2}}{m_{\nu_3}} \sim -1$$  \quad (43)
\[ \frac{\Delta m^2_{21}}{m^2_{\nu_3}} \simeq \frac{\Delta m^2_{31}}{m^2_{\nu_3}} \simeq 1, \quad \frac{\Delta m^2_{32}}{m^2_{\nu_3}} \sim \epsilon^2, \]  

\[ \frac{|m_{\nu_{eff}}|}{m_{\nu_3}} \sim \epsilon^{2l} \]  

where \( m_{\nu_{eff}} = \sum_i m_{\nu_i} |U_{ei}|^2 \) is the effective Majorana neutrino mass measured in neutrinoless double beta decay experiments. As for the mixing angles, their orders of magnitude are given by \( \theta \).

When \( CP \) violation in the lepton sector is neglected, the neutrino oscillation probability from one flavour \( \alpha \) to another \( \beta \) reads:

\[ P(\nu_{\alpha} \rightarrow \nu_{\beta}) = P(\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta}) = \delta_{\alpha\beta} - 4 \sum_{i<j} U_{\alpha i} U_{\beta i} U_{\alpha j} U_{\beta j} \sin^2 \left( \frac{\Delta m^2_{ji} L}{4E} \right) \]  

where \( E \) is the neutrino energy, \( L \) the distance travelled by the neutrino between the source and the detector, and \( U_{\alpha i} \) where the index \( \alpha = e, \mu, \tau \) labels the weak eigenstates, while \( i = 1, 2, 3 \) labels the mass eigenstates – denote the entries of the lepton mixing matrix \( U \). In the case of two heavy quasi-degenerate neutrinos, \( \Delta m^2_{32} \ll \Delta m^2_{21} \approx \Delta m^2_{31} \) and (46) reduces to:

\[ P(\nu_{\alpha} \rightarrow \nu_{\beta}) = \delta_{\alpha\beta} - 4 U_{\alpha 1} U_{\beta 1} (\delta_{\alpha\beta} - U_{\alpha 1} U_{\beta 1}) \sin^2 \left( \frac{\Delta m^2_{21} L}{4E} \right) \]

\[ -4 U_{\alpha 2} U_{\beta 2} U_{\alpha 3} U_{\beta 3} \sin^2 \left( \frac{\Delta m^2_{32} L}{4E} \right) \]  

Due to the presence of two terms oscillating with very different frequencies (\( \Delta m^2_{32} \ll \Delta m^2_{21} \)), one must distinguish between “short-distance” and “long-distance” oscillations. Actually \( P(\nu_{\alpha} \rightarrow \nu_{\beta}) \) depends on the ratio \( L/E \), so “short-distance” regime means that \( \Delta m^2_{32} L \ll 1 \). This is the case for accelerator and reactor experiments. In this regime, \( \nu_e - \nu_\mu, \nu_\mu - \nu_\tau \) and \( \nu_e - \nu_\tau \) oscillations are characterized by one and the same \( \Delta m^2 \approx \Delta m^2_{21} \), but with different oscillation amplitudes:

\[ P_{sd}(\nu_{\alpha} \rightarrow \nu_{\beta}) \approx 4 U_{\alpha 1}^2 U_{\beta 1}^2 \sin^2 \left( \frac{\Delta m^2_{21} L}{4E} \right) \]

\[ = \sin^2 2\theta_{e\beta} \sin^2 \left( \frac{\Delta m^2_{21} L}{4E} \right), \alpha \neq \beta, \]  

where in our scheme

\[ \sin^2 2\theta_{e\mu} \approx 8 A^2 \epsilon^{-2l}, \quad \sin^2 2\theta_{\mu\tau} \approx 8 A^2 B^2 \epsilon^{4l}, \quad \sin^2 2\theta_{e\tau} \approx 4 B^2 \epsilon^{2l+2l} \]  

17 In our notations, \( m_{\nu_i} \) is the mass of the \( i \)th Majorana neutrino “signed” by its \( CP \)-parity.
However, if \( \sin^2 2\theta_{\nu e}^{sd} \) and \( \sin^2 2\theta_{\mu}^{sd} \) are very small, as the above expressions suggest, the corrections in the probabilities \( P_{sd}(\nu_e \rightarrow \nu_\tau) \) and \( P_{sd}(\nu_\mu \rightarrow \nu_\tau) \) due to the long wavelength oscillation term in eq. (45) can be important: they are given respectively by the terms 
\[
2 \left( \frac{\Delta m^2_{32} L}{4 E} \right)^2 A^2 \epsilon^{2l' - 2l} \quad \text{and} \quad \left( \frac{\Delta m^2_{32} L}{4 E} \right)^2.
\]
Actually, as it follows from eq. (46), the expression for the relevant correction and the fact that \( l' > l \), the “long distance” term is always dominant in \( P_{sd}(\nu_\mu \rightarrow \nu_\tau) \).

In the "long-distance" regime \( \frac{\Delta m^2_{32} L}{4 E} \gg 1 \) and the oscillations corresponding to the first term in (47) are averaged out. The probabilities relevant for atmospheric neutrinos are:

\[
P_{ld}(\nu_\mu \rightarrow \nu_e) \simeq 4 A^2 \epsilon^{2l' - 2l} \left[ 1 - \frac{1}{2} \sin^2 \left( \frac{\Delta m^2_{32} L}{4 E} \right) \right] \tag{50}
\]
\[
P_{ld}(\nu_\mu \rightarrow \nu_\tau) \simeq \sin^2 \left( \frac{\Delta m^2_{32} L}{4 E} \right) + 4 A^2 B^2 \epsilon^{2l'} \tag{51}
\]

Clearly \( \nu_\mu - \nu_e \) oscillations are suppressed compared with \( \nu_\mu - \nu_\tau \) oscillations.

Let us now show that the model considered can account for the atmospheric neutrino anomaly and the dark matter problem simultaneously. The requirement that the mu and the tau neutrinos constitute the hot dark matter fixes the mass scale \( m_{\nu_3} \) to be in the few eV range, typically \( m_{\nu_3} = 2 - 3 \text{ eV} \)[34].

The solution of the atmospheric neutrino problem in terms of \( \nu_\mu - \nu_\tau \) oscillations requires \( 5 \times 10^{-3} \text{ eV}^2 \leq \Delta m^2_{32} \leq 3.10^{-2} \text{ eV}^2 \). We must also take into account the experimental limits on \( \nu_e - \nu_\mu \) oscillations. In the few \( \text{eV}^2 \) region, the experimental upper bound on \( \sin^2 2\theta_{\nu e}^{sd} \) from E776 is \( 2 - 3 \times 10^{-3} \). These data strongly constrain the parameters of the model. Assuming that \( \epsilon \) is the Cabbibo angle (\( \epsilon \simeq 0.22 \)), as suggested by the quark sector, we obtain:

\[
l = 2 \pm 1/2 \quad \text{and} \quad l' - l \geq 3 \tag{52}
\]

With such values of \( l \) and \( l' \), only a few charge assignments for the charged lepton singlets \( \bar{E}_i \) reproduce the observed mass hierarchy in the charged lepton sector. Indeed, under our hypotheses, we have

\[
\frac{m_e}{m_\tau} = \epsilon^{l' + l + e_1 - e_3} \tag{53}
\]

For example, the following choice:

\[
\begin{align*}
l' &= 5 \\
l &= 2
\end{align*}
\]

\[
\begin{align*}
e_1 &= e_2 = e_3 - 2 \\
2 &\leq e_3 < 4
\end{align*}
\]

leads to \( m_\mu/m_\tau \sim \epsilon^2 \) and \( m_e/m_\tau \sim \epsilon^5 \) (which deviates slightly from the geometrical hierarchy \( 1 : \epsilon^2 : \epsilon^4 \)). The features of the model are then:
1. the neutrino mass spectrum contains two quasi-degenerate neutrinos in the few eV range, which can account for the hot dark matter:

\[ m_{\nu_e} \simeq m_{\nu_1} \simeq (5 - 8) \times 10^{-7} \text{eV} \quad m_{\nu_\mu} \simeq m_{\nu_3} \simeq 2 - 3 \text{eV} \] (55)

2. the atmospheric neutrino anomaly is explained by \( \nu_\mu - \nu_\tau \) oscillations with parameters:

\[ \Delta m_{\mu\tau}^2 \simeq \Delta m_{32}^2 \sim (0.9 - 2.1) \times 10^{-2} \text{eV}^2 \quad \text{and} \quad \sin^2 2\theta_{\mu\tau} \simeq 1 \] (56)

3. one finds that \( \nu_\mu - \bar{\nu}_e \) oscillations may be compatible with the positive result announced by the LSND collaboration and can be observed by KARMEN (58):

\[ \Delta m_{e\mu}^2 \simeq \Delta m_{21}^2 = 4 - 9 \text{eV}^2 \quad \text{and} \quad \sin^2 2\theta_{e\mu} \simeq 0.9 A^2 10^{-3} \] (57)

with \( 0.2 < A < 5 \).

4. \( \nu_e - \nu_\tau \) and \( \nu_\mu - \nu_\tau \) oscillations are below the sensitivity of CHORUS and NOMAD (37), but \( \nu_\mu - \nu_\tau \) oscillations may be observable in future long-baseline experiments.

5. neutrinoless double beta decay rate is far below the sensitivity of current and planned experiments (see, e.g., [38]):

\[ |m_{\nu_{ee}}| \ll 10^{-2} \text{eV} \] (58)

the solar neutrino problem cannot be solved by this model, unless a sterile neutrino \( \nu_s \) is added (39), since the squared mass difference required by the MSW solution (\( \Delta m^2 \sim 10^{-5} \) eV\(^2\)) or the vacuum oscillation solution (\( \Delta m^2 \sim 10^{-11} - 10^{-10} \) eV\(^2\)) is much less than \( \Delta m_{32}^2 \) and \( \Delta m_{21}^2 \).

Let us note that the neutrino phenomenology of such a model is somewhat close to the one encountered in the Zee model [40], as discussed recently by Smirnov and Tanimoto [41] (see also [42]). Although the models and their physical motivations are very different, it would not be so easy to distinguish between the two, except if a signal for \( \nu_e - \nu_\tau \) oscillations is found in CHORUS and NOMAD.

Finally, let us comment briefly on the solution of the atmospheric neutrino problem in the model of Subsection 3.3 with three quasi-degenerate neutrinos. In this case there are essentially two parameters: the mass \( |m_{\nu_2}| \simeq a \equiv m > 0 \) and the charge \( l \). One has to leading order in \( \epsilon^4 \):

\[ m_{\nu_1} \simeq |m_{\nu_2}| \simeq m_{\nu_3} \simeq m, \] (59)

\[ \Delta m_{31}^2 \simeq 2 \Delta m_{21}^2 \simeq 2 \Delta m_{32}^2 \simeq 2m^2 \epsilon^4, \] (60)
\[ |m_{\nu_{eff}}| \simeq m, \quad (61) \]

where we have used \((c + e) \sim a\) in obtaining eq. (60).

The massive neutrinos can constitute the hot component in the two component “hot + cold” dark matter theory provided \(m \simeq (1.5 - 2.0) \text{ eV}\). This implies in our model \(|m_{\nu_{eff}}| \simeq (1.5 - 2.0) \text{ eV}\), which is by approximately a factor of 2 larger than the most stringent upper limit on \(|m_{\nu_{eff}}|\), quoted in the literature on the subject and derived from the negative results of the searches for neutrinoless double-beta decay of \(^{76}\text{Ge}\) \([43]\). However, in view of the uncertainties in the calculations of the nuclear matrix elements entering into the expression for the neutrinoless double-beta decay amplitude, values of \(|m_{\nu_{eff}}| \simeq 1.5 \text{ eV}\) cannot be ruled out by the presently existing data (see, e.g., \([44]\)). Ongoing and future experiments will test relatively soon this possibility \([18]\).

The atmospheric neutrino problem can be solved in the model under consideration in terms of three-neutrino oscillations of the \(\nu_\mu\) and \(\bar{\nu}_\mu\), characterized by two \(\Delta m^2\) which differ by a factor of two. A three-neutrino oscillation analysis of the Kamiokande data suggests \([45]\) \(\Delta m^2_{31} \simeq 2\Delta m^2_{21} \simeq (2 - 4) \times 10^{-2} \text{ eV}^2\). With \(m\) in the cosmologically relevant region this implies \(l = 3\) or \(4\). The \(\nu_\mu - \nu_e\) and \(\nu_\mu - \nu_\tau\) oscillation probabilities have a very simple form:

\[
P(\nu_\mu \rightarrow \nu_e) \simeq \frac{1}{2} \sin^2 \left( \frac{2\Delta m^2_{31} L}{4E} \right),
\]

\[
P(\nu_\mu \rightarrow \nu_\tau) \simeq \sin^4 \left( \frac{\Delta m^2_{31} L}{4E} \right) \quad (62)
\]

Note that the amplitude of the \(\nu_\mu - \nu_\tau\) oscillations is larger than the amplitude of the \(\nu_\mu - \nu_e\) oscillations.

The model predicts neutrino energy-independent suppression of the different components of the solar neutrino flux by the factor 0.5. Such a suppression is not favoured by the current solar neutrino data \([10]\).

### 4.2 Adiabatic MSW effect

We now address the possibility that the model of Subsection 3.2 be used to solve the solar neutrino problem along the lines of the MSW interpretation in the large angle branch. The scale pattern is then: \(m_{\nu_\mu} \ll m_{\nu_\tau} \simeq m_{\nu_\mu}\). Due to the inverted hierarchy in the electron-muon sector, the model has to be slightly modified. The relevant charge assignment is now:

\[
\begin{align*}
l_2 &= l' \\
l_1 &= -l_3 = l \\
0 &< l' < l
\end{align*}
\]

\[
\begin{align*}
l_1 &= -l_3 = l \\
0 &< l' < l
\end{align*}
\]

The neutrinoless double-beta decay rate will be suppressed if the quasi-degeneracy of the masses of the three neutrinos is a result of the conservation of the charge \(X_L = l (L_\mu - L_e)\) (instead of the charge \(X_L = l (L_\mu - L_\tau)\) in the example we are considering) before the breaking of the \(U(1)_X\) symmetry.
corresponding to the following pattern for $\hat{M}_\nu$:

$$
\hat{M}_\nu \simeq \begin{pmatrix}
\epsilon^2 l' + l & \epsilon^{l' + l} & 1 \\
\epsilon^{l' + l} & \epsilon^{2l'} & \epsilon^{l - l'} \\
1 & \epsilon^{l - l'} & \epsilon^{2l}
\end{pmatrix}
$$

(64)

where again only the dominant term in each entry is given. This yields mass ratios:

$$
\frac{m_{\nu_2}}{m_{\nu_3}} \sim \epsilon^{2l'} \quad \frac{m_{\nu_1}}{m_{\nu_3}} \simeq -1
$$

(65)

$$
\frac{\Delta m_{31}^2}{m_{\nu_3}^2} \sim \epsilon^{2l}
$$

(66)

The requirement that the model account for the solar neutrino deficit by adiabatic MSW transitions and solve the dark matter problem simultaneously, together with the experimental constraints on $\nu_e - \nu_\mu$ oscillations and the measured values of the charged lepton masses, leads to the following choice of parameters:

$$
\begin{cases}
 l' = 3/2 \\
 l = 9/2
\end{cases}
\quad\text{and}\quad
\begin{cases}
 e_1 = e_2 = e_3 - 4 \\
 9/2 \leq e_3 < 17/2
\end{cases}
$$

(67)

Again, the mass hierarchy in the charged lepton sector slightly departs from the geometrical scheme: $\frac{m_\mu}{m_\tau} \sim \epsilon^2$ and $\frac{m_\mu}{m_\tau} \sim \epsilon^5$. The features of the model are:

1. the electron and the tau neutrinos are quasi-degenerate in mass and can account for the hot dark matter:

$$
m_{\nu_\mu} \simeq m_{\nu_2} \sim 0.03 \text{ eV} \quad m_{\nu_e} \simeq m_{\nu_\tau} \simeq m_{\nu_3} = 2 - 3 \text{ eV}
$$

(68)

2. the solar neutrino deficit is explained by adiabatic MSW $\nu_e - \nu_\mu$ transitions with parameters:

$$
\Delta m_{e\mu}^2 \simeq \Delta m_{31}^2 \sim (0.5 - 1.1) \times 10^{-5} \text{ eV}^2 \quad \text{and} \quad \sin^2 2\theta_{e\mu} \simeq 1
$$

(69)

3. $\bar{\nu}_\mu - \bar{\nu}_e$ oscillations may be in the domain of sensitivity of LSND and KARMEN:

$$
\Delta m_{e\mu}^2 \simeq \Delta m_{32}^2 = 4 - 9 \text{ eV}^2 \quad \text{and} \quad \sin^2 2\theta_{e\mu} \simeq A^2 \times 10^{-3}
$$

(70)

with $0.2 < A < 5$.

4. $\nu_e - \nu_\tau$ and $\nu_\mu - \nu_\tau$ oscillations are below the sensitivity of current and planned experiments:

$$
\begin{cases}
 P_{sd}(\nu_e \to \nu_\tau) \simeq \sin^2(\Delta m_{e\tau}^2 L/4E) \\
P_{sd}(\nu_\mu \to \nu_\tau) \leq 5 \times 10^{-8}
\end{cases}
$$

(71)
5. non-leading terms in the lepton mixing matrix can yield a contribution to $|m_{\nu_{e\tau}}|$ of the order of $m_{\nu_e}e^{2i\theta}$, in the border range of sensitivity of planned neutrinoless double-beta decay experiments.

Note, however, that the solar neutrino data tends to favour smaller mixings between the electron and the tau neutrino than predicted by the model, namely $0.2 \leq \sin^2 2\theta_{e\mu} \leq 0.9$.

In the model with three quasi-degenerate neutrinos of Subsection 3.3 the solar neutrino problem can be solved in terms of three-neutrino MSW transitions. Neutrinos can have masses in the cosmologically relevant range of (1.5 - 2.0) eV and in this case the neutrinoless double-beta decay rate may not be suppressed. Obviously, this model cannot provide an explanation of the atmospheric neutrino anomaly and/or of the LSND result if $\Delta m^2_{21}$ (and therefore $\Delta m^2_{31}$) has a value required by the MSW solution of the solar neutrino problem.

5 Conclusions

It certainly follows a natural path to try and apply the recent ideas on family symmetries – developed to explain the observed hierarchies of masses and mixings in the sector of quarks and charged leptons – to the poorly known neutrino sector. There, the promises of new experimental results in the near future make it a natural ground for theorists to make predictions. Also the models considered put the problem of neutrino masses in the perspective of a more general framework which deals with the masses and mixings of all the low-energy particles.

We have considered in this paper abelian family gauge symmetries, because of both their simplicity and their attractive features in the quark sector. In the case where the third family, including the superheavy right-handed neutrino, is the heaviest of the three, one obtains predictions in the low-energy neutrino sector which strikingly resemble those of the quark sector. In particular, neutrino masses have a hierarchical structure and, much like the CKM matrix, the neutrino mixing angles can be, to a first approximation, expressed as ratios of neutrino masses $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}$. We tend to favor such a case, although it cannot be reconciled, as is, with all the present neutrino experimental results – in particular because the mixing angles are small.

In this paper, we have on the other hand considered the situation where one, or more, mixing angle are large. We have showed that this is perfectly compatible with an abelian gauge family symmetry. The spectrum of light neutrinos then involves some level of degeneracy. The models in this class are rather constrained. They can be used to address more particularly the atmospheric neutrino problem or the solar neutrino problem in its large angle MSW interpretation. We have studied the phenomenological consequences of such models for present and foreseen neutrino experiments and conclude that they are within the range of such experiments.
Acknowledgments.

P.B. and P.R. would like to thank the Aspen Center for Physics where part of this work was completed. They also acknowledge a CNRS-NSF fund INT-9512897 for this research. The work of S.P. was supported in part by the EEC grant ERBCHRX CT930132 and by the grant PH-510 from the Bulgarian Science Foundation.

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