Non-Commutative Geometry, Spin and Quarks

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Abstract

In this paper we use considerations of non-commutative geometry to deduce a model for QCD interactions. The model also explains within the same theoretical framework hitherto purely phenomenological characteristics of the quarks like their fractional charge, mass, handedness and confinement.

1 Introduction

In recent years it is being realized that an important assumption that has been uncritically taken for granted in much of twentieth century Physics is that space forms a differentiable manifold. This is the case, for example in the Riemannian General Relativity, and the Minkowski Space Time of relativistic Quantum Mechanics and Quantum Field Theory. It is only in recent years that space time manifolds where we cannot go down to arbitrarily small intervals have been considered in the context of, for example non-commutative space time models, SuperString Theory and Quantum Gravity and similar areas [1, 2, 3, 4].

Indeed it has been suggested by the author that once we discard the conventional space time geometries and consider a non-commutative space time then it is possible to reconcile the hitherto irreconcilable pillars of the twentieth century, namely General Relativity and Quantum Theory [5, 6, 7, 8].

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We will now argue that from the same non-commutativity, it is possible to also understand strong interactions and the quark picture.

## 2 Strong Interactions

Let us introduce the effect of a non-commutative space time into the usual metric, at some scale \((l, \tau)\)

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}
\]

If we split up the product \(dx^\mu dx^\nu\) into symmetric and non symmetric parts, \((1)\) will become

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2}
\]

In \((2)\), the first term on the right side represents the usual metric, while the second term represents the effect of non-commutativity of the space time coordinates viz.,

\[
[x, y] = 0(l^2), [x, p_x] = i\hbar[1 + l/\hbar^2]p_x^2 etc. \tag{3}
\]

Equation \((3)\) can be deduced from relations that were worked out by Snyder [9, 10], and subsequently by several other scholars over the past five decades and more. Incidentally if in \((3)\), we specialize to the Compton scale, i.e. \(l\) denotes the Compton wavelength (and similarly, \(\tau\), the Compton time), then, we can in fact deduce the Dirac equation of the electron \([11, 12]\). In other words the non-commutativity is an \(\sim 0(l^2)\) effect. Starting from \((2)\), it has been shown in detail that we can deduce the gravitational field equations

\[
\Box \phi^{\mu\nu} = -k T^{\mu\nu} \tag{4}
\]

where

\[
\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \hbar \tag{5}
\]

(Cf.[5, 6]for details).

Equations \((4)\) and \((5)\) represent the linearized equations of General Relativity \([13]\).

Starting from equation \((4)\), we have, as is well known, in suitable units,

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, h_{\mu\nu} = \int \frac{4T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \tag{6}
\]
It may be mentioned that in (6), velocities comparable to that of light are allowed, and at the same time the stresses $T_{ij}$ and momentum densities $T_{0j}$ can be comparable to the energy momentum density $T_{00}$.

It is well known that when $|\vec{x}'| << |\vec{x}|$, we have the equations

$$m = \int T^{00} d^3x$$

$$S_k = \int \epsilon_{klm} x^l T^{m0} d^3x$$

where $m$ is the mass and $S_k$ is the angular momentum. In any case, as is well known, (6) leads to the gravitational potential $\square$. It may be mentioned that integrals and derivatives within the non-commutative geometry are approximate, because, in any case, the effects are, as pointed out, $\sim 0(\ell^2)$.

We consider the integrals in (4) and (5) in a region bounded by the Compton wavelength, because in any case, the density of the particle vanishes outside this region, and hence, so also the energy momentum density $T_{\mu\nu}$. Remembering that the velocity at the Compton wavelength equals $c$, the velocity of light, we can easily deduce from (8) that the angular momentum $S_k$ is given by,

$$S_k = \frac{h}{2},$$

that is the Quantum Mechanical spin half. (Cf. [12] for details).

Indeed from an alternative viewpoint it has been shown that the non-commutative relations (3) imply spin and conversely [14].

Let us now consider the case when $|\vec{x}'| \sim |\vec{x}|$

Then we have from (5), expanding in a Taylor series about $t$,

$$h_{\mu\nu} = 4 \int \frac{T_{\mu\nu}(t, \vec{x})}{|\vec{x} - \vec{x}'|} d^3x' + (\text{terms independent of } \vec{x}') + 2 \int \frac{d^2}{dt^2} T_{\mu\nu}(t, \vec{x}), |\vec{x} - \vec{x}'| d^3x' + 0(|\vec{x} - \vec{x}'|^2)$$

(9)
To proceed further, we will need the relation,

$$|\frac{du_\nu}{dt}| = |u_\nu| \omega \quad (10)$$

where \( \omega \) is the frequency given by,

$$\omega = \frac{|u_\nu|}{R} = \frac{2mc^2}{\hbar}$$

Equation (10) can be derived in a simple way as follows: The non-commutative relations (3) imply

$$y \equiv \hbar p_x \left( \hbar = \frac{H}{\hbar}, H = 0(l^2) \right)$$

(Cf. ref. [7, 8]).

Whence we have

$$\dot{\mu} = \frac{1}{m} (\dot{\rho}_\mu) = \frac{1}{m} \frac{h}{H} \dot{x}_\nu = \frac{1}{m} \frac{h}{H} u_\nu,$$

which leads to (10). Interestingly, this is also true in the theory of the Dirac equation itself.

Using (10), we have,

$$d_{\tau} T_{\mu\nu} = \rho u^\nu \frac{du^\mu}{dt} + \rho u^\mu \frac{du^\nu}{dt} = 2\rho u^\mu u^\nu \omega,$$

so that,

$$d_{\tau}^2 T_{\mu\nu} = 4\rho u^\mu u^\nu \omega^2 = 4\omega^2 T_{\mu\nu}$$

where \( \omega \) is given in (11). Substitution in (9) now gives,

$$h_{\mu\nu} \approx \frac{\beta M}{r} + 8\beta M \left( \frac{Mc^2}{\hbar} \right)^2 \cdot r \quad (11)$$

\( \beta \) being a constant.

This resembles the QCD quark potential [13]. In any case these considerations suggest that we can get different interactions at different distances or scales in a unified picture, which can approximately at least represent quarks also.
We can further refine this argument. For this we will use the following relation already deduced (Cf.[12] and references therein for details):

\[ A_\sigma = \frac{1}{2} (\eta^{\mu\nu} h_{\mu\nu})_{,\sigma} , \]

It was then shown that the electromagnetic potential is given by,

\[
\frac{e^2}{r} = A_0 \approx \frac{2ch}{r} \int \eta^{\mu\nu} \frac{d}{d\tau} T_{\mu\nu} d^3x' = \frac{2ch}{r} \int \eta^{ij} \frac{d}{d\tau} T_{ij} d^3x',
\]

\[ = 2ch\left(\frac{mc^2}{\hbar}\right) \int \eta^{ij} \frac{T_{ij}}{r} d^3x', \quad (12) \]

outside the Compton wavelength.

As we approach the Compton wavelength however, we have to use equation (9), which after a division by \( m \), the mass of the particle to be identified with the quark, and taking \( \hbar = 1 = c \) to correspond to the usual theory, goes over to,

\[ -\frac{\alpha}{r} + \frac{\beta m_e}{r^2} , \quad (13) \]

which is essentially (11).

In (13) \( \alpha \sim 1 \) and \( \beta \sim \frac{1}{m} \) and \( m_e \) is the electron mass. This is the QCD potential with both the Coulumbic and confining parts (Cf.ref.[15]).

We now observe that the usual three dimensionality of space, as pointed out by Wheeler [13] arises due to the double connectivity or spinorial behaviour of Fermions, which takes place outside the Compton wavelength due to the fact that as has been seen elsewhere, while it is the negative energy components of the Dirac four-spinor which dominate inside, it is the positive energy components which predominate outside (cf.ref.[5, 12] for details). Such a three dimensionality can also be deduced using Penrose’s spin network theory [14]. Interestingly, if we consider the Dirac equation in two (or one dimension) [17, 18], we encounter handedness and the absence of an invariant mass - features which in the light of our considerations arise at the Compton wavelength. As we approach the Compton wavelength, we encounter mostly the negative energy component and the above double connectivity and therefore three dimensionality disappear: We have two or less dimensions. Indeed even in the purely classical case of a collection of relativistic particles, the various centres of mass form a two dimensional disk [19]. Such a conclusion has been
drawn alternatively at very small scales (cf. [20, 21]).
This leads to the following circumstance: We have to consider two and one spatial dimensions. We now use the fact that as is well known [22] for each dimension the $T_{ij}$ in (9) or (12) is given by $(1/3)\epsilon$, where $\epsilon$ is the energy density. In this case it follows from (12) that the particle would have the charge $(2/3)e$ or $(1/3)e$, in two or one dimensions. Incidentally, this provides an explanation for the remarkable and well known fact that one third of charge appears to be concentrated in a core of the size of the order of the proton Compton wavelength as was experimentally well established [23].
Using the fact that at the Compton wavelength the charge becomes $e/3$, and $d^3r \to l^2dr$ owing to the single dimensionality in equation (12), and also using equation (10) in (9) we get

$$\frac{1}{9 \times 137} \sim m_e \cdot l^2 \int \frac{T}{r} dr \quad \text{or} \quad \int \frac{T}{r} \sim \frac{1}{r \times 10^3 m_e \cdot m^2} \sim \frac{m}{r}$$

where the last step follows from a comparison with the Coloumb part of (11) or (13). Whence

$$m \sim 10^3 m_e$$

(14)
where $m$ is the quark mass, remembering that in the above natural units, $l = \frac{1}{m}$.
Equation (14) is of course correct. Infact using the quark mass given in (14) in (11) or (13), it is easy to see that the ratio of the coefficients of the confining and Coulumbic parts is $\sim (Gev)^2$ which is also true [15].
Thus the quark and QCD identification is complete, and also at the same time a theoretical rationale for the mass, fractional charges and handedness of the quark has now been obtained. As noted by Salam, these were hitherto not explained theoretically [24].
This would also automatically imply that these fractionally charged particles cannot be observed individually, as they by their very nature appear when confined to dimensions of the order of their Compton wavelength. This is expressed by the confining part of the QCD potential (11) or (13).

3 Remarks

We would like to reemphasize that our model gives the mass (order of magnitude), the fractional charge, the handedness and the confinement feature of
quarks correctly. Experimentally, a charge $e/3$ has been observed within the quark Compton wavelength, as pointed out. The correct phenomenological QCD potential also is obtained.

A final remark: In the above considerations, we had specialized to the Compton wavelength. We can see from an alternative and simple point of view, how the compton scale emerges. For this we use the fact that in the Dirac theory of the electron, we have (Cf.ref.\[14, 25\])

\[
\hat{p}(\approx \frac{\hbar}{\hat{x}}) = \frac{E^2}{\hbar c^2} \hat{x}
\]

whence

\[
\hat{x} \approx \frac{\hbar}{mc}
\]

where $\hat{x}, \hat{p}$ arise due to well known non-Hermitian effects. That is we recover the Compton scale, within which non-commutative and non-Hermitian effects come into play.

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