RESEARCH ARTICLE

Slightly supercritical percolation on non-amenable graphs I: The distribution of finite clusters

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Abstract
We study the distribution of finite clusters in slightly supercritical \((p \downarrow p_c)\) Bernoulli bond percolation on transitive non-amenable graphs, proving in particular that if \(G\) is a transitive non-amenable graph satisfying the \(L^2\) boundedness condition \((p_c < p_{2\rightarrow 2})\) and \(K\) denotes the cluster of the origin and then there exists \(\delta > 0\) such that if \(p \in (p_c - \delta, p_c + \delta)\), then

\[
P_p( n \leq |K| < \infty ) \asymp n^{-1/2} \exp \left[ -\Theta \left( |p - p_c|^2 n \right) \right]
\]

and

\[
P_p( r \leq \text{Rad}(K) < \infty ) \asymp r^{-1} \exp \left[ -\Theta \left( |p - p_c| r \right) \right]
\]

for every \(n, r \geq 1\), where all implicit constants depend only on \(G\). We deduce in particular that the critical exponents \(\gamma'\) and \(\Delta'\) describing the rate of growth of the moments of a finite cluster as \(p \downarrow p_c\) take their mean-field values of 1 and 2, respectively. These results apply in particular to Cayley graphs of non-elementary hyperbolic groups, to products with trees, and to transitive graphs of spectral radius \(\rho < 1/2\). In particular, every finitely generated non-amenable group has a Cayley graph to which these results apply. They are new for graphs that are not trees. The corresponding facts are yet
to be understood on $\mathbb{Z}^d$ even for $d$ very large. In a second paper in this series, we will apply these results to study the geometric and spectral properties of infinite slightly supercritical clusters in the same setting.

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1 | INTRODUCTION

In Bernoulli bond percolation, each edge of a countable graph $G = (V, E)$ is either deleted (closed) or retained (open) independently at random with retention probability $p \in [0, 1]$ to obtain a random subgraph $\omega$ of $G$. The connected components of $\omega$ are referred to as clusters. We will be primarily interested in the case that $G$ is transitive, that is, that the automorphism group of $G$ acts transitively on $V$, or more generally that $G$ is quasi-transitive, that is, that the action of the automorphism group of $G$ on $V$ has at most finitely many orbits. When $G$ is infinite, the critical probability $p_c = p_c(G)$ is defined by

$$p_c = \sup\{ p \in [0, 1] : \text{ every cluster is finite} \ P_p\text{-a.s.}\},$$

where we write $P_p = \mathbf{P}_p^G$ for the law of Bernoulli-$p$ bond percolation on $G$. It is now known that the phase transition is non-trivial (i.e., that $0 < p_c < 1$) for every infinite quasi-transitive graph with superlinear volume growth [8, 17, 49].

Percolation theorists are primarily interested in the geometry of clusters, and how this geometry changes as $p$ is varied. The theory is naturally decomposed into several regimes according to the relationship between $p$ and $p_c$. One possible taxonomy is as follows:

1. The subcritical regime, in which $0 < p < p_c$.
2. The slightly subcritical regime, in which $0 < p - p_c < 1$.
3. The critical regime, in which $p = p_c$.
4. The slightly supercritical regime, in which $0 < p - p_c < 1$.
5. The supercritical regime, in which $p_c < p < 1$.

(It is sometimes desirable to further differentiate the very subcritical regime $p \ll 1$ and very supercritical regime $1 - p \ll 1$; these regimes are often much easier to understand.) Among all of these regimes, the most difficult to study is usually the slightly supercritical regime. A central difficulty in the study of this regime, and in supercritical percolation more generally, is that one is interested in the probability of highly non-monotone events for the percolation configuration, such as $\{ n \leq |K| < \infty \}$ where $K$ is the cluster of the origin, while many of the tools that have been developed in the study of the other regimes are either mostly or exclusively suited to the analysis of monotone events and functions.

Indeed, there are essentially only two examples in which slightly supercritical percolation is reasonably well understood: trees and site percolation on the triangular lattice. In both cases,

\[ q(1 - q)^{k-2} = p(1 - p)^{k-2} \]
there are exact duality relations, developed extensively in the Euclidean setting by Kesten [41], which allow us to convert questions about slightly supercritical percolation into questions about slightly subcritical percolation. In the case of trees these slightly subcritical questions can then be answered with the classical theory of branching processes (see, e.g., [26, Chapter 10]), while for site percolation on the triangular lattice Smirnov and Werner [62] showed that they can be answered by combining the aforementioned work of Kesten [41] with the theory of conformally invariant scaling limits and SLE as developed in the landmark works of Schramm [60], Smirnov [61], and Lawler, Schramm, and Werner [46–48]. This methodology is very specific to planar graphs, and does not give any indication of how these problems should be approached in higher-dimensional examples.

In particular, slightly supercritical percolation on \( \mathbb{Z}^d \) remains poorly understood even when \( d \) is very large and all other regimes are now understood rather thoroughly. Highlights of the literature regarding the other regimes include [3, 21, 51] for the subcritical regime, [5, 28, 44, 45] for the critical and slightly subcritical regimes, and [4, 16, 27, 42] for the supercritical regime. See for example, [13, 26, 31] for overviews of this literature and of open problems in high-dimensional percolation, and [18] for some interesting recent partial progress on slightly supercritical percolation. Let us also mention that a good understanding of slightly supercritical percolation appears to be a prerequisite to the solution of several important open problems regarding invasion percolation and minimal spanning forests, see [31, Section 16.1] and references therein.

The primary purpose of this series of two papers is to study slightly supercritical percolation in the “infinite-dimensional” setting of non-amenable (quasi-)transitive graphs. Here, we recall that a connected, locally finite graph is said to be non-amenable if its Cheeger constant

\[
\Phi(G) = \inf \left\{ \frac{|\partial_E W|}{\sum_{w \in W} \deg(w)} : W \text{ a finite set of vertices} \right\}
\]

is positive, where \( \partial_E W \) denotes the set of edges with one endpoint in \( W \) and one endpoint not in \( W \); \( G \) is said to be amenable if it is not non-amenable, that is, if its Cheeger constant is zero. Background on percolation in the non-amenable context may be found in, for example, [50]. We prove our results under the additional hypothesis that \( G \) satisfies the \( L^2 \) boundedness condition, which was introduced in [35] and studied further in [36]. Let us now briefly introduce this condition. Given a countable graph \( G = (V, E) \), we write \( T_p(u, v) = P_p(u \leftrightarrow v) \) for the two-point matrix, and define

\[
p_{2\rightarrow 2} = p_{2\rightarrow 2}(G) = \sup\{p \in [0, 1] : \|T_p\|_{2\rightarrow 2} < \infty\},
\]

This duality is a consequence of the fact that every finite connected subgraph of a \( k \)-regular tree containing \( n \) edges also touches exactly \((k - 2)n + k\) edges that it does not contain. As \( p \downarrow p_c \), this dual probability \( q \) satisfies \( p_c - q \sim p - p_c \). Thus, all questions concerning the distribution of finite clusters in slightly supercritical percolation can immediately be converted into questions concerning slightly subcritical percolation, which are much easier. This property is very specific to trees, and these arguments do not generalize to other non-amenable transitive graphs. Let us note, however, that slightly more involved duality arguments should also allow one to understand slightly supercritical percolation on transitive non-amenable proper plane graphs with locally finite planar dual; to our knowledge such an analysis has not been carried out in the literature. Note that such graphs are always Gromov hyperbolic [23] and therefore have \( p_c < p_{2\rightarrow 2} \) by the results of [35]. Thus, the results of this paper are always applicable to them.
where we recall that if $M \in [0, \infty)^{V^2}$ is a $V$-indexed matrix with non-negative entries, then the $L^2(V) \to L^2(V)$ operator norm $\|M\|_{2\to2} \in [0, \infty]$ is defined by

$$\|M\|_{2\to2} = \sup\left\{ \|Mf\|_2 : f \in L^2(V), \|f\|_2 = 1 \right\}.$$  

We say that $G$ satisfies the $L^2$ boundedness condition if $p_c(G) < p_{2\to2}(G)$. This condition is conjectured to hold for every connected, locally finite, non-amenable quasi-transitive graph [36, Conjecture 1.3], and is now known to hold for several classes of examples, including Gromov hyperbolic graphs [35], highly non-amenable graphs [52, 56, 59], and graphs admitting a quasi-transitive non-unimodular subgroup of automorphisms [39]. In particular, it can be deduced by the methods of [56] that every non-amenable, finitely generated group has a Cayley graph for which $p_c < p_{2\to2}$. (On the other hand, we always have that $p_c = p_{2\to2}$ in the amenable case.) See [36] for an overview. See also [7, 37] and references therein for an overview of what is known regarding critical and near-critical percolation on general non-amenable transitive graphs without this assumption.

The main results of this paper apply the $L^2$ boundedness condition to establish a very precise understanding of the distribution of finite clusters in critical and near critical percolation. In a forthcoming pair of sequel to this paper [33, 34], we apply these results to study the large-scale geometry of infinite clusters in slightly supercritical percolation. All of our results regarding slightly supercritical percolation are new when the graph in question is not a tree.

The results of both papers build upon the methods of our recent work with Hermon [30], which established related, non-quantitative results for supercritical percolation on non-amenable transitive graphs (which do not necessarily satisfy the $L^2$ boundedness condition). Making these arguments quantitative in a sharp way in order to get the correct behavior as $p \downarrow p_c$ is a surprisingly delicate matter, and our proofs are, unfortunately, substantially more technical than those of [30].

Besides the intrinsic interest of our results, we are also hopeful that some of the tools we develop will be useful for approaching the high-dimensional Euclidean case; some perspectives on the remaining challenges in this case are presented in Section 5. It would also be very interesting (and seemingly highly non-trivial) to extend our methods to other infinite-dimensional settings, such as hypercubes or expander graphs (which are finite analogs of non-amenable graphs). Critical and slightly subcritical percolation on these graphs has been studied in many works, surveyed in [63], the highlights of which include [10–12, 32, 64]. (The analogous results for the complete graph are classical, see [9] and references therein.)

1.1 Statement of results

We now state our results concerning the distribution of finite clusters in near critical percolation. While the supercritical aspects of these results are the most novel, it seems that they also improve slightly upon the best existing estimates for slightly subcritical percolation. We write $K_v$ for the cluster of $v$ and $|K_v|$ for the number of vertices it contains.

**Theorem 1.1** (Volume of finite clusters). Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph such that $p_c(G) < p_{2\to2}(G)$. Then there exists a constant $\delta = \delta(G) > 0$ such as
that
\[
\mathbb{P}_p(n \leq |K_v| < \infty) \asymp n^{-1/2} \exp \left[ -\Theta(|p - p_c|^2 n) \right]
\]
for every \( n \geq 1, v \in V, \) and \( p \in (p_c - \delta, p_c + \delta) \), where all implicit constants depend only on \( G \).

Here and below, we write \( \asymp, \geq, \) and \( \leq \) to denote equalities and inequalities that hold up to positive multiplicative constants depending only on the graph \( G \). Thus, for example, \( "f(n) \asymp g(n)" \) for every \( n \geq 1" \) means that there exist positive constants \( c \) and \( C \) such that \( c g(n) \leq f(n) \leq C g(n) \) for every \( n \geq 1 \). We use Landau’s asymptotic notation similarly, so that, for example, \( f(n) = \Theta(g(n)) \) if and only if \( f \asymp g \), and \( f(n) \leq g(n) \) if and only if \( f(n) = O(g(n)) \). In particular, Theorem 1.1 is equivalent to the assertion that there exist positive constants \( c_1, c_2, C_1, C_2, \) and \( \delta \) such that

\[
c_1 n^{-1/2} \exp \left[ -C_1 |p - p_c|^2 n \right] \leq \mathbb{P}_p(n \leq |K_v| < \infty) \leq \frac{C_2}{c_2} n^{-1/2} \exp \left[ -c_2 |p - p_c|^2 n \right]
\]

for every \( v \in V, p \in (p_c - \delta, p_c + \delta), \) and \( n \geq 1 \).

Our next theorem establishes a similar result for the radius of a finite supercritical cluster. We write \( \text{Rad}_{\text{int}}(K_v) \) and \( \text{Rad}_{\text{ext}}(K_v) \) for the intrinsic and extrinsic radii of \( K_v \), that is, the maximum distance from \( v \) to another point of \( K_v \) in the graph metric on \( K_v \) and in the graph metric on \( G \), respectively. Note that we trivially have \( \text{Rad}_{\text{ext}}(K_v) \leq \text{Rad}_{\text{int}}(K_v) \).

**Theorem 1.2** (Radii of finite clusters). Let \( G = (V, E) \) be a connected, locally finite, quasi-transitive graph such that \( p_c(G) < p_{2\to2}(G) \). Then there exists a constant \( \delta = \delta(G) > 0 \) such that

\[
\mathbb{P}_p(r \leq \text{Rad}_{\text{int}}(K_v) < \infty) \asymp r^{-1} \exp \left[ -\Theta\left(|p - p_c| r\right) \right]
\]

and

\[
\mathbb{P}_p(r \leq \text{Rad}_{\text{ext}}(K_v) < \infty) \asymp r^{-1} \exp \left[ -\Theta\left(|p - p_c| r\right) \right]
\]

for every \( r \geq 1, v \in V, \) and \( p \in (p_c - \delta, p_c + \delta) \), where all implicit constants depend only on \( G \).

The parts of these results concerning critical percolation were already known, and are applied as a component of the proof. Indeed, a connected, locally finite, quasi-transitive graph \( G \) is said to satisfy the triangle condition if

\[
\nabla_{p_c}(v) := \sum_{u, w \in V} T_{p_c}(v, u) T_{p_c}(u, w) T_{p_c}(w, v) < \infty
\]

for every \( v \in V \). The triangle condition was introduced by Aizenman and Newman [5] and proven to hold on \( \mathbb{Z}^d \) with \( d \) large in the groundbreaking work of Hara and Slade [28]. It is conjectured to hold if and only if \( d > 6 \), and is now known to hold for all \( d \geq 11 \) [24]. It is known that if a connected, locally finite, quasi-transitive graph \( G \) satisfies the triangle condition then

\[
\mathbb{P}_{p_c}(|K_v| \geq n) \asymp n^{-1/2}
\]

for every \( n \geq 1 \) and \( v \in V \), and that

\[
\mathbb{P}_{p_c}(\text{Rad}_{\text{int}}(K_v) \geq r) \leq r^{-1}
\]

for every \( r \geq 1 \) and \( v \in V \).
so that, in particular, every cluster is finite $P_{p_c}$-almost surely. Note that the triangle condition is equivalent to the assertion that $T^3_p(v, v) < \infty$ for every $v \in V$ (powers of non-negative infinite matrices indexed by $V$ always being well defined as elements of $[0, \infty]^{V \times V}$), and is therefore implied by the $L^2$ boundedness condition since $T^3_p(v, v) \leq \|T^3_p\|_{2 \to 2} \leq \|T^3_p\|_{2 \to 2}$. The upper and lower bounds of (1.4) follow from the work of Aizenman and Newman [5] and Aizenman and Barsky [3], respectively, while (1.5) follows from the work of Kozma and Nachmias [44]. A simple proof of the complementary lower bound $P_{p_c}(\text{Rad}_{\text{int}}(K_v) \geq r) \geq r^{-1}$, which holds on every connected, locally finite, quasi-transitive graph, is given in Proposition 4.2. Moreover, in [36] it is shown that the $L^2$ boundedness condition allows one to compare intrinsic and extrinsic distances, which allows one to prove in particular that

$$P_{p_c}(\text{Rad}_{\text{ext}}(K_v) \geq r) \asymp r^{-1}$$

for every $r \geq 1$ and $v \in V$. (1.6)

Let $E(K_v)$ be the set of edges that touch (i.e., have at least one endpoint in) $K_v$, and define

$$\zeta(p) = -\limsup_{n \to \infty} \frac{1}{n} \log P_p(n \leq |E(K_v)| < \infty)$$

(1.7)

to be the exponential rate of decay of the probability that $v$ belongs to a large finite cluster (which is easily seen not to depend on the choice of $v$). It is a consequence of the sharpness of the phase transition that $\zeta(p) > 0$ for every connected, locally finite, quasi-transitive graph $G$ and every $0 \leq p < p_c$. This was first proven by Aizenman and Barsky [3] and Aizenman and Newman [5] (see also the closely related work of Menshikov [51]), and several alternative proofs are now available [20, 21, 38]. On the other hand, for supercritical percolation on quasi-transitive graphs, it is shown in [30] that $p_c < 1$ and $\zeta(p) > 0$ for some $p_c < p < 1$ if and only if $p_c < 1$ and $\zeta(p) > 0$ for every $p_c < p < 1$, if and only if $G$ is non-amenable. (In contrast, finite supercritical clusters in $\mathbb{Z}^d$ have stretched-exponential volume tails [27, 42]!) Thus, for connected, locally finite, non-amenable quasi-transitive graphs, we have that $\zeta(p) > 0$ if and only if $p \neq p_c$. (See [57] for counterexamples regarding extensions of this result to dependent percolation models.) Note, however, that these arguments do not give any quantitative control on the manner in which $\zeta(p) \to 0$ as $p \to p_c$.

Theorem 1.1 provides such a quantitative understanding, and yields in particular the following immediate corollary.

**Corollary 1.3.** Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph such that $p_c(G) < p_{2 \to 2}(G)$. Then there exists $\delta > 0$ such that $\zeta(p) \asymp |p - p_c|^2$ for every $p \in (p_c - \delta, p_c + \delta)$.

Of course, Theorems 1.1 and 1.2 tell us rather more than this: they show us the precise manner in which the polynomial tail at $p_c$ is gradually transformed into the exponential tail away from $p_c$. In particular, they make the following natural heuristic picture precise: There is a scaling window of order $|p - p_c|^{-1}$ such that within the scaling window percolation behaves in essentially the same way as critical percolation, whereas outside the scaling window the off-critical effects begin to become apparent. Moreover, roughly speaking, these off-critical effects manifest themselves in a way that is proportional to how much larger our cluster is than a cluster that is at the edge of the
scaling window (i.e., than a cluster that has radius \(|p - p_c|^{-1}\) or volume \(|p - p_c|^{-2}\)). This intuitive picture will be an important motivation to many of our proofs: We will often prove estimates by separate analyses of the “inside-window” and “outside-window” cases. Note that the restriction to a neighborhood of \(p_c\) is necessary as \(\zeta(p) \to \infty\) as \(p \downarrow 0\) or \(p \uparrow 1\).

Finally, we note that Theorem 1.1 also permits immediate computation of the slightly supercritical scaling exponents \(\gamma'\) and \(\Delta'\). It is believed that for every connected, locally finite, quasi-transitive graph \(G = (V, E)\) there exist \(\gamma, \gamma', \Delta, \Delta'\) such that if \(k\) is a positive integer then

\[
\mathbb{E}_p \left[ |K_v|^k \right] \asymp_k |p - p_c|^{-\gamma - (k-1)\Delta + o_k(1)} \quad \text{as } p \uparrow p_c \quad \text{and} \quad (1.8)
\]

\[
\mathbb{E}_p \left[ |K_v|^k \mathbf{1}(|K_v| < \infty) \right] \asymp_k |p - p_c|^{-\gamma' - (k-1)\Delta' + o_k(1)} \quad \text{as } p \downarrow p_c, \quad (1.9)
\]

where the \(k\) subscripts mean that the implicit constants may depend on \(k\). See [26, Chapters 9 and 10] for background on this conjecture. It is known that if \(G\) satisfies the triangle condition, then \(\gamma\) and \(\Delta\) are well defined and take their mean-field values of 1 and 2, respectively [5, 54] (see also [38]). Theorem 1.1 implies a similar result for \(\gamma'\) and \(\Delta'\) for graphs satisfying the \(L^2\) boundedness condition.

**Corollary 1.4.** Let \(G\) be a connected, locally finite, quasi-transitive graph such that \(p_c(G) < p_{2\to2}(G)\). Then there exist positive constants \(\delta = \delta(G), c = c(G),\) and \(C = C(G)\) such that

\[
c^k|p - p_c|^{-2k+1}k! \leq \mathbb{E}_p \left[ |K_v|^k \mathbf{1}(|K_v| < \infty) \right] \leq c^k|p - p_c|^{-2k+1}k!
\]

for every \(k \geq 1, p \in (p_c - \delta, p_c) \cup (p_c, p_c + \delta),\) and \(v \in V\). In particular, the exponents \(\gamma = \gamma' = 1\) and \(\Delta = \Delta' = 2\) are well defined and take their mean-field values.

### 1.2 About the proofs and organization

Let us now outline the content of the rest of the paper, and in particular how the strategy we pursue here builds upon that of [30].

1. In Section 2, we prove some estimates on percolation “inside the scaling window,” which in particular establish the upper bounds of Theorems 1.1 and 1.2 in the cases \(n = O(|p - p_c|^{-2})\) and \(r = O(|p - p_c|^{-1})\), respectively. These estimates are straightforward applications of what is known about critical percolation under the triangle condition, and will be very useful in the remainder of our analysis.

2. In Section 3, we complete the proofs of the upper bounds of Theorems 1.1 and 1.2. This section takes up most of the paper, and is both the most technical and the most novel part of the paper. We pursue a similar strategy to that of [30], but apply the assumption \(p_c < p_{2\to2}\) to obtain sharp quantitative versions of every estimate along the way.

   (a) In Section 3.1, we recall some basic ideas and notation from [30] which allow us to express the derivative of, say, the truncated \(k\)th moment \(\mathbb{E}_{p,n}[K]^k := \mathbb{E}_p[|K|^k \mathbf{1}(|K| \leq n)]\) of the cluster volume as the difference of two terms: a “positive term” \(\mathbf{U}_{p,n}[|K|^k]\) which accounts for the effect of a finite cluster growing but remaining smaller than the truncation threshold \(n\) and a “negative term” \(-\mathbf{D}_{p,n}[|K|^k]\) which accounts for the effect of finite clusters growing to break the truncation threshold \(n\) (possibly by becoming infinite).
Very roughly speaking, our goal in the remainder of the section will be (i) to lower bound the absolute value of the negative term; (ii) to write down an inequality of the form

\[ U_{p,n}[|K|^k] \leq \frac{1}{2} D_{p,n}[|K|^k] + \text{something we can hope to bound without yet understanding the truncated moments} \]

so that

\[
\frac{d}{dp} E_{p,n}[|K|^k] \\
\leq -\frac{1}{2} D_{p,n}[|K|^k] + \text{something we can hope to bound without yet understanding the truncated moments};
\]

(iii) to prove an upper bound on the second term on the right-hand side of (1.10) that is good enough to push through the final stage of the analysis. In the above formulation, this would mean an upper bound of the form \( k!C_k(p - p_c)^{-2k+1} \) for \( p \) slightly larger than \( p_c \); (iv) to analyze the resulting differential inequality (1.10) for \( E_{p,n}|K|^k \).

(b) In Section 3.2, we carry out step (i) of this strategy, applying the \( L^2 \) boundedness condition to prove a lower bound on the magnitude of the negative term when \( p \) is slightly larger than \( p_c \), proving in particular that \( D_{p,n}[|K|^k] \geq (p - p_c)E_{p,n}[|K|^{k+1}] \) for \( p \) slightly larger than \( p_c \). This strengthens [30, Proposition 2.4], which established a similar but non-quantitative inequality for all transitive non-amenable graphs; the method of proof here is quite different.

(c) In Section 3.3, we carry out the remainder of the strategy but for the radius rather than the volume, which is much easier. In this case, the analog of the second term on the right-hand side of (1.10) is expressed in terms of the probability that the origin is in a large skinny cluster, whose radius is large but whose volume is smaller than it ought to be given this large radius. An important part of the analysis is to obtain a sharp quantitative upper bound on the probability of this event, which we will also apply many more times throughout the paper. This inequality can be thought of as a strengthening of [30, Lemma 2.8] under the assumption that \( \nabla p_c < \infty \).

(d) In Section 3.4 and Section 3.5, we carry out the remainder of the strategy in the more difficult case of the volume. Here, the second term in (1.10) is expressed in terms of clusters that satisfy a certain “higher-order” variation of the skininess constraint considered above, related to the size of the tree of geodesics connecting \( k + 1 \) points. In order to bound the resulting quantities, we introduce in Section 3.4 a sequence of multivariate generating functions and prove that these generating functions satisfy a family of recursive differential inequalities relating the partial derivatives of \( k \)th function in the sequence to the value of the first \( k \) functions in the sequence. In the following Section 3.5 we analyze this family of differential inequalities and then apply the resulting bounds to conclude the proof of the upper bounds of Theorem 1.1 in the slightly supercritical case, that is, to carry out step (iv) above.

While these sections are based on similar high-level ideas to [30, Section 2.3], a much more delicate and technical implementation of these ideas was required to obtain sharp quantitative estimates. Indeed, while the methods developed in [30, Section 2.3] are quantitative, they are not sharp, and eventually lead to estimates of the form \( \zeta(p) \geq (p - p_c)^4 \) rather than \( \zeta(p) \geq (p - p_c)^2 \) when fed the estimates of Sections 3.2 and 3.3 as inputs. In
particular, while the family of differential inequalities between generating functions we derive here is closely related to [30, Lemma 2.9], the analysis it requires is completely different.

3. In Section 4 we complete the proofs of Theorems 1.1 and 1.2 by proving lower bounds in the slightly supercritical regime as well as both upper and lower bounds in the critical and slightly subcritical regimes. While several of these estimates are fairly similar to things that are already known, a careful treatment is required to establish optimal quantitative forms of all the required estimates, and some of the results we prove here improve upon what was already known about slightly subcritical percolation under the triangle condition. Several of these sharp quantitative bounds are obtained with the help of the bounds on skinny clusters that are proven in Section 3.3.

4. In Section 5 we give some concluding remarks, including a discussion of the challenges that remain to adapt our methods to the high-dimensional Euclidean case and some potential approaches to tackle them.

A glossary of notation is given at the end of the paper.

Remark 1.5. If the reader is familiar with [30], they may notice that we do not use one of the key ideas of that paper. In that paper, we wrote down a second formula for the derivative of the truncated $k$th moment in terms of the fluctuation of the number of open and closed edges in the cluster, the absolute value of which can be bounded via martingale methods. By comparing these bounds to those derived via Russo’s formula as above, we were able to bound the truncated moments directly without actually analyzing the resulting differential inequalities. The reason we do not use this method here is that the bounds they yield are not sharp, but rather contain various unwanted polylogarithmic errors. In order to circumvent this issue we must actually analyze the differential inequalities we establish for the truncated moments.

2 UPPER BOUNDS INSIDE THE SCALING WINDOW

The purpose of this section is to prove the following lemma, which establishes the upper bounds of Theorems 1.1 and 1.2 in the case that $n$ and $r$ are inside the scaling window and gives a weak bound for arbitrary $n$ and $r$. Both estimates are simple consequences of the results of [44, 58], which establish the analogous bounds for critical percolation. Throughout the paper it will be important to establish bounds that hold not only for $G$ but also for arbitrary subgraphs of $G$. This is done to circumvent non-monotonicity issues in inductive analyses of percolation, with similar arguments first appearing in [44]. (For example, the family of partial differential inequalities on generating functions established in Section 3.4 applies not to the generating function associated to $G$ but to the minimum of the generating functions associated to the subgraphs of $G$, so that our inductive analysis of these generating functions will require uniform control over subgraphs.)

Let $G$ be a countable graph, let $H$ be a subgraph of $G$, let $v$ be a vertex of $H$, and consider Bernoulli bond percolation on $H$. We write $R_v = R_v(H)$ for the intrinsic radius of the cluster of $v$ in $H$, and write $E_v = E_v(H)$ for the number of edges of $H$ that touch the cluster of $v$ in $H$, that is, have at least one endpoint in the cluster of $v$ in $H$. We write $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

Lemma 2.1. Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph, and let $p_c = p_c(G)$. Suppose that $\nabla_{p_c} < \infty$. Then there exists a positive constant $C$ such that the bounds

\begin{align*}
\end{align*}
\[
\mathbf{P}^H_p(R_v \geq r) \leq C\left(\frac{1}{r} \lor (p - p_c)\right)
\quad \text{and} \quad \mathbf{P}^H_p(E_v \geq n) \leq C\left(\frac{1}{n^{1/2}} \lor (p - p_c)\right)
\]

hold for every \( r, n \geq 1 \), every \( p \in [0, 1] \), every subgraph \( H \) of \( G \), and every vertex \( v \) of \( H \).

We stress that in the statement and proof of this estimate, \( p_c \) always refers to \( p_c(G) \). The proof will make use of Russo’s formula \([26, \text{Theorem 2.32}]\), which states that if \( X : \{0, 1\}^E \to \mathbb{R} \) depends on at most finitely many edges then \( E_p[X(\omega)] \) is a polynomial in \( p \) with derivative

\[
\frac{d}{dp} E_p[X(\omega)] = \sum_{e \in E} E_p[X(\omega^e) - X(\omega_e)] = \frac{1}{p} \sum_{e \in E} E_p[\mathbf{1}(\omega(e) = 1)(X(\omega) - X(\omega_e))]
\]

for every \( p \in (0, 1] \), where we let \( \omega^e = \omega \cup \{e\} \) and \( \omega_e = \omega \setminus \{e\} \). We write \( B_{\text{int}}(v, n) \) for the intrinsic ball of radius \( n \) around \( v \) in \( K_v \), and write \( \partial B_{\text{int}}(v, n) = B_{\text{int}}(v, n) \setminus B_{\text{int}}(v, n - 1) \) for the set of vertices at intrinsic distance exactly \( n \) from \( v \).

**Proof of Lemma 2.1.** Fix \( H \) and \( v \) and for each \( r \geq 0 \) let \( B_{\text{int}}(v, r) \) be the intrinsic ball of radius \( r \) around \( v \) in \( H \), that is, the set of vertices that can be reached from \( v \) by open paths of length at most \( r \) in \( H \). We know by the results of \([44, 58]\) (see also \([39, \text{Section 6}]\)) that

\[
\mathbf{P}^H_p(R_v \geq r) \leq r^{-1} \quad \text{and} \quad E_p^H[#B_{\text{int}}(v, r)] \leq r
\]

for every \( 0 \leq p \leq p_c \). Observe that, for each \( r \geq 1 \), if \( K_v \) has intrinsic radius at least \( r \) and \( e \) is such that \( K_v(\omega_e) \) does not have intrinsic radius at least \( r \), then \( e \) must lie on every intrinsic geodesic of length \( r \) starting at \( v \) in \( K_v \). There are clearly at most \( r \) such edges, and it follows from Russo’s formula that

\[
\frac{d}{dp} \mathbf{P}^H_p(R_v \geq r) \leq \frac{r}{p_c} - \frac{r}{p_c} \mathbf{P}^H_p(R_v \geq r)
\]

for every \( p_c \leq p \leq 1 \) and \( r \geq 1 \). This inequality may be written equivalently as

\[
\frac{d}{dp} \log \mathbf{P}^H_p(R_v \geq r) \leq \frac{r}{p_c}.
\]

Integrating this bound between \( p \) and \( p_c \) yields that

\[
\mathbf{P}^H_p(R_v \geq r) \leq \mathbf{P}^H_{p_c}(R_v \geq r) \exp\left[\frac{(p - p_c)r}{p_c}\right] \leq \frac{1}{r} \exp\left[\frac{(p - p_c)r}{p_c}\right]
\]

for every \( p_c \leq p \leq 1 \) and \( r \geq 1 \). Since \( \mathbf{P}^H_p(R_v \geq r) \) is decreasing in \( r \), it follows that

\[
\mathbf{P}^H_p(R_v \geq r) \leq \min\left\{\frac{1}{\ell} \exp\left[\frac{(p - p_c)\ell}{p_c}\right] : 1 \leq \ell \leq r\right\}
\]

for every \( p_c \leq p \leq 1 \) and \( r \geq 1 \). The claimed bound on the tail of the intrinsic radius follows by taking \( \ell = r \lor [(p - p_c)^{-1}] \).

Now, a similar argument to above yields that

\[
\frac{d}{dp} \log E_p^H[#B_{\text{int}}(v, r)] \leq \frac{r}{p_c}
\]
for every $p_c \leq p \leq 1$ and $r \geq 1$, and hence that

$$E_p^H[\#B_{\text{int}}(v, r)] \leq r \exp \left[ \frac{p - p_c}{p_c} r \right]$$  \hspace{1cm} (2.2)

for every $p_c \leq p \leq 1$ and $r \geq 1$. It follows by the union bound and Markov’s inequality that

$$P_p^{H}(E_v \geq n) \leq \frac{1}{n} E_p^H[\#B_{\text{int}}(v, r)] + P_p^{H}(R_v \geq r) \leq \frac{r}{n} \exp \left[ \frac{p - p_c}{p_c} r \right] + \left[ \frac{1}{r} \vee (p - p_c) \right]$$

for every $n, r \geq 1$. The claim follows by taking $r = \lceil n^{1/2} \wedge (p - p_c)^{-1} \rceil$. □

3 \quad UPPER BOUNDS OUTSIDE THE SCALING WINDOW

In this section we prove the upper bounds of Theorems 1.1 and 1.2 in the case $p > p_c$.

3.1 \quad Setting up the main differential inequalities

Most of the work to prove Theorems 1.1 and 1.2 will concern the case that $p > p_c$ is slightly supercritical and $n$ and $r$ are outside the scaling window, so that either $n \gg |p - p_c|^{-2}$ or $r \gg |p - p_c|^{-1}$. As discussed above, we follow the basic strategy of [30], but apply the assumption that $p_c < p_{2 \to 2}$ to make the proof quantitative. We begin by recalling some notation from [30]. Let $G = (V, E)$ be a connected, locally finite, transitive, non-amenable graph, and let $v$ be a vertex of $G$. Let $K_v$ denote the cluster of $v$, and let $E_v = |E(K_v)|$ be the number of edges touching $K_v$. Define $\mathcal{H}$ to be the set of all finite connected subgraphs of $G$, and let $\mathcal{H}_v$ be the set of all finite connected subgraphs of $G$ containing $v$. Given a function $F : \mathcal{H}_v \to \mathbb{R}$, we write

$$E_p,n[F(K_v)] := E_p[F(K_v)1(E_v \leq n)] \quad \text{and} \quad E_p,\infty[F(K_v)] := E_p[F(K_v)1(E_v < \infty)]$$

for every $p \in [0, 1]$ and $n \geq 1$.

Given $F : \mathcal{H}_v \to \mathbb{R}$ and $n \geq 1$, Russo’s formula allows us to express the derivative of the truncated expectation $E_{p,n}[F(K_v)]$, which is a polynomial in $p$, in terms of pivotal edges and obtain that

$$\frac{d}{dp} E_{p,n}[F(K_v)] = U_{p,n}[F(K_v)] - D_{p,n}[F(K_v)],$$  \hspace{1cm} (3.1)

where we write

$$U_{p,n}[F(K_v)] := \frac{1}{p} \sum_{e \in E} E_{p,n}[F[K_v] - F[K_v(\omega_e)]1(\omega(e) = 1)]$$

and

$$D_{p,n}[F(K_v)] := \frac{1}{1 - p} \sum_{e \in E} E_p[F(K_v)1(\omega(e) = 0, E_v \leq n < E_v(\omega^e))].$$

See [30, Section 2] for further details. Intuitively, in the $n \to \infty$ limit, the term $D_{p,\infty}[F(K_v)]$ accounts for the effect of finite clusters becoming infinite, while the term $U_{p,\infty}[F(K_v)]$ accounts
for the effect of finite clusters growing while remaining finite. (Note, however, that the above formulas are only \textit{a priori} valid for finite \( n \).) Note that \( \mathbf{U}_{p,n}[F(K_v)] \) is non-negative if \( F \) is increasing and that \( \mathbf{D}_{p,n}[F(K_v)] \) is non-negative if \( F \) is non-negative. Note also that \( \mathbf{U}_{p,n}[F(K_v)] \) and \( \mathbf{D}_{p,n}[F(K_v)] \) both depend linearly on the function \( F \).

In order to prove Theorems 1.1 and 1.2, we will need to prove lower bounds on \( \mathbf{D}_{p,n}[F(K_v)] \) and upper bounds on \( \mathbf{U}_{p,n}[F(K_v)] \) for appropriate choices of \( F \). The two quantities will often have roughly the same order, making the analysis of their difference rather delicate.

### 3.2 Bounding the negative term

In this section we prove a lower bound on \( \mathbf{D}_{p,n}[F(K_v)] \) for non-negative \( F \). In [30, Proposition 2.1], it is shown via an ineffective argument that if \( G \) is transitive and non-amenable, then for every \( p_c < p_0 \leq 1 \) there exists a positive constant \( c_{p_0} \) such that

\[
\mathbf{D}_{p,n}[F(K_v)] \geq c_{p_0} \mathbf{E}_{p,n}[|K_v| \cdot F(K_v)]
\]

for every \( p_0 \leq p \leq 1 \) and every increasing function \( F : \mathcal{H}_v \to [0, \infty) \). A key ingredient to the proof of our main theorems is the following proposition, which allows us to take \( c_{p_0} \) of order \( (p_0 - p_c) \) under the assumption that \( p_c < p_{2 \to 2} \). We write \( \hat{\theta}^*_+(p) = \inf_{v \in V} P_p(v \to \infty) \) and \( \hat{\theta}^*_-(p) = \sup_{v \in V} P_p(v \to \infty) \) and write \( M \) for the maximum degree of \( G \).

**Proposition 3.1.** Let \( G \) be a countable graph. Then

\[
\mathbf{D}_{p,n}[F(K_v)] \geq \frac{\hat{\theta}^*_-(p)}{p^2(1-p)M\hat{\theta}^*_+(p)||T_p||_{2 \to 2}^2} \theta^*_+(p) \mathbf{E}_{p,n}[|K_v| \cdot F(K_v)]
\]

for every non-negative function \( F : \mathcal{H}_v \to [0, \infty) \), every \( n \geq 1 \), and every \( p \in [0, 1) \). Consequently, if \( G \) is connected, locally finite, and quasi-transitive with \( p_c(G) < p_{2 \to 2}(G) \), then there exist positive constants \( \delta > 0 \) and \( c > 0 \) such that

\[
\mathbf{D}_{p,n}[F(K_v)] \geq c(p - p_c) \mathbf{E}_{p,n}[|K_v| \cdot F(K_v)]
\]

for every non-negative function \( F : \mathcal{H}_v \to [0, \infty) \), every \( n \geq 1 \), and every \( p \in (p_c, p_c + \delta) \).

The precise form of the argument given below was suggested to us by Antoine Godin; a similar argument will appear in his forthcoming PhD thesis [25]. We thank him for sharing this argument with us, which substantially simplified our proof.

**Remark 3.2.** Note that this is the only stage in our argument in which the \( L^2 \)-boundedness condition (as opposed to the triangle condition) is used directly.

The proof makes use of the notion of the BK inequality and the associated notion of the \textit{disjoint occurrence} \( A \circ B \) of two events \( A \) and \( B \). We refer the unfamiliar reader to [26, Chapter 2.3] for background.

**Proof of Proposition 3.1.** Let \( G \) be a countable graph. For each vertex \( v \) of \( G \), let \( E_v^- \) denote the set of oriented edges \( e \) of \( G \) with \( e^- = v \). We first claim that for each deterministic finite set of vertices
$S \subseteq V$ we have that

$$\Psi_p(S) := \frac{1}{1-p} \sum_{u \in S} \sum_{e \in E_{u}^-} 1(e^+ \notin S) \mathbb{P}_p(e^+ \to \infty \text{ off } S) \geq \left[ \frac{\theta_*(p)}{p^2(1-p)M \theta^*(p) \|T_p\|_{2\to2}^2} \right] \theta_*(p) |S|$$

(3.4)

for every $0 < p < p_{2\to2}$. (Note that we have written the expression on the right in this way as the bracketed term is of constant order in cases of interest.) The deduction of (3.2) from (3.4) is identical to the proof of [30, Proposition 2.1] and is omitted. Indeed, the proof of [30, Proposition 2.1] shows more generally that

$$D_{p,n}[F(K_v)] \geq E_{p,n}[F(K_v)\Psi_p(K_v)]$$

for every $p \in [0,1)$, $n \geq 1$, and every non-negative $F : \mathcal{H}_v \to [0,\infty)$.

Let $S$ be a deterministic finite set of vertices. Let $\partial^- S$ denote the set of oriented edges of $G$ with $e^- \in S$ and $e^+ \notin S$. Observe that for each $u \in S$ we have that

$$\{|K_u| = \infty\} \subseteq \bigcup_{e \in \partial^- S} \{u \leftrightarrow e^-\} \circ \{e \text{ open}\} \circ \{e^+ \to \infty \text{ off } S\}.$$

Indeed, suppose that $u \in S$ is in an infinite cluster, and let $\gamma$ be an infinite simple open path starting at $u$. Since $S$ is finite, there is some last vertex $v$ of $S$ that is visited by $\gamma$. Let $e$ be the edge of $\partial^- S$ that is crossed by $\gamma$ as it leaves $v$, which is necessarily open. Then the pieces of $\gamma$ before and after crossing $e$ are disjoint witnesses for the events $\{u \leftrightarrow e^-\}$ and $\{e^+ \to \infty \text{ off } S\}$, both of which are disjoint from the edge $e$. Thus, applying the BK inequality and the union bound yields that

$$\mathbb{P}_p(u \to \infty) \leq p \sum_{v \in S} T_p(u,v) \sum_{e \in E^-_v} 1(e^+ \notin S) \mathbb{P}_p(e^+ \to \infty \text{ off } S)$$

for every $u \in S$, where we write “$e^+ \to \infty \text{ off } S$” to mean that there is an infinite open path starting at $e^+$ that does not visit any vertex of $S$. Summing over $u$ we obtain that

$$|S| \theta_*(p) \leq p \sum_{v \in S} \sum_{u \in S} T_p(u,v) \sum_{e \in E^-_v} 1(e^+ \notin S) \mathbb{P}_p(e^+ \to \infty \text{ off } S).$$

(3.5)

Define $f : V \to \mathbb{R}$ by

$$f_p(v) = \frac{1}{1-p} 1(v \in S) \sum_{e \in E^-_v} 1(e^+ \notin S) \mathbb{P}_p(e^+ \to \infty \text{ off } S).$$

Rewriting the above inequality (3.5) in terms of $f$ and applying Cauchy–Schwarz, we obtain that

$$\frac{\theta_*(p) |S|}{p(1-p)} \leq \langle T_p 1_S, f \rangle \leq \|T_p\|_{2\to2} \|1_S\|_2 \|f\|_2 \leq \|T_p\|_{2\to2} |S|^{1/2} \|f\|_1^{1/2} \|f\|_\infty^{1/2},$$

and since we clearly have that $\|f\|_\infty \leq M \theta^*(p)/(1-p)$, it follows that

$$\frac{1}{1-p} \sum_{v \in S} \sum_{e \in E^-_v} 1(e^+ \notin S) \mathbb{P}_p(e^+ \to \infty \text{ off } S) = \|f\|_1 \geq \left[ \frac{\theta_*(p)}{p^2(1-p)M \theta^*(p) \|T_p\|_{2\to2}^2} \right] \theta_*(p) |S|$$

as claimed.
The deduction of (3.3) from (3.2) follows by standard arguments: Indeed, if \( G \) is connected and quasi-transitive, then there exists \( C \) such that \( \vartheta_*(p) \geq p^C \vartheta_*(p) \) for every \( p \in [0,1] \), while if \( p_c(G) < p_{2\to2}(G) \), then \( \|T_p\|_{2\to2} \) is bounded on a neighborhood of \( p_c \). On the other hand, for quasi-transitive graphs there always exists a positive constant \( c \) such that \( \vartheta_*(p) \geq c(p - p_c) \) for all \( p_c \leq p \leq 1 \) [21]. Together these observations allow us to deduce (3.3) from (3.2).

**Remark 3.3.** The proof of [30, Proposition 2.1] can also be made quantitative under the assumption that \( p_c < p_{2\to2} \), since in this case we know that the density of trifurcations is of order \( (p - p_c)^3 \) [36, Corollary 5.6]. Note, however, that the resulting bound is not sharp.

An easy corollary of Proposition 3.1 is the following weak version of the first moment estimate from Corollary 1.4. This weak estimate will nevertheless be useful to us as boundary data when we analyze a certain differential inequality later in the paper.

**Corollary 3.4.** Let \( G = (V, E) \) be a connected, locally finite, quasi-transitive graph such that \( p_c < p_{2\to2} \). Then there exist positive constants \( \delta \) and \( C \) such that

\[
\inf \left\{ (p - p_c)E_{p,\infty}[K_v] : p \in (p_c + \epsilon, p_c + 2\epsilon) \right\} \leq C
\]

for every \( v \in V \) and \( 0 < \epsilon \leq \delta \).

**Proof.** Fix \( v \in V \). Since \( G \) is quasi-transitive and satisfies the triangle condition, there exists a constant \( C \) such that \( P_p(|K_v| = \infty) \leq C(p - p_c) \) for every \( p_c \leq p \leq 1 \) [6] (this also follows from Lemma 2.1). On the other hand, Proposition 3.1 implies that there exist positive constants \( c \) and \( \delta \) such that

\[
\frac{d}{dp} P_p(|K_v| > n) = -\frac{d}{dp} E_{p,n}[1] = D_{p,n}[1] \geq c(p - p_c)E_{p,n}[K_v]
\]

for every \( n \geq 1 \) and \( p \in (p_c, p_c + \delta] \). Integrating this differential inequality yields that

\[
\int_{p_c + \epsilon}^{p_c + 2\epsilon} c(p - p_c)E_{p,n}[K_v]dp \leq P_{p_c + 2\epsilon}(|K_v| > n) - P_{p_c + \epsilon}(|K_v| > n) \leq P_{p_c + 2\epsilon}(|K_v| > n)
\]

for every \( 0 < \epsilon \leq \delta/2 \). Using the monotone convergence theorem to take the limit as \( n \to \infty \), we obtain that

\[
\int_{p_c + \epsilon}^{p_c + 2\epsilon} c(p - p_c)E_{p,\infty}[K_v]dp \leq P_{p_c + 2\epsilon}(|K_v| = \infty) \leq 2C\epsilon
\]

for every \( 0 < \epsilon \leq \delta/2 \), where the final inequality follows from Lemma 2.1. This is easily seen to imply the claim.

### 3.3 Skinny clusters and the intrinsic radius

The goal of this section is to prove the following proposition, which establishes the upper bounds of Theorem 1.2 in the slightly supercritical regime. This is substantially easier than the corresponding upper bounds on the tail of the volume.
**Proposition 3.5.** Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph such that $p_c < p_{2 \rightarrow 2}$. Then there exist positive constants $\delta, c,$ and $C$ such that

$$P_p(r \leq \text{Rad}_{\text{ext}}(K_v) < \infty) \leq P_p(r \leq \text{Rad}_{\text{int}}(K_v) < \infty) \leq Cr^{-1}e^{-c(p-p_c)r}$$

for every $r \geq 1$, $v \in V$, and $p \in [p_c, p_c + \delta)$.

We begin with the following proposition, which upper bounds the probability of having a large skinny cluster, whose radius is large but whose volume is smaller than it should be given the large radius. In particular, this proposition applies the assumption $\nabla p_c < \infty$ to give a quantitative improvement to [30, Lemma 2.8]. This proposition will be extremely useful to us, and will be applied many times throughout the paper. Again, it will be important for these future applications to have bounds that hold in arbitrary subgraphs of our fixed transitive graph $G$.

**Proposition 3.6 (Skinny clusters).** Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph such that $\nabla p_c < \infty$. There exist positive constants $\delta, c,$ and $C$ such that the bound

$$P^H_p(r \leq R_v < \infty \text{ and } E_v \leq \alpha R_v) \leq C \inf \left\{ \left( \frac{1}{r} + \lambda \right) \exp \left[ -ce^{-C\lambda} \lambda r \right] : 0 \vee (p - p_c) \leq \lambda \leq \delta \right\}$$

holds for every $0 \leq p \leq p_c + \delta$, $\alpha \geq 1$, $r \geq 0$, subgraph $H$ of $G$, and vertex $v$ of $H$.

Here, we recall that $E_v$ denotes the number of edges of $H$ touched by the percolation cluster of $v$ in $H$, and $R_v$ denotes the intrinsic radius of this cluster. Again, we stress that in the statement and proof of this proposition, $p_c$ will always denote $p_c(G)$ and all implicit constants will depend only on $G$.

**Proof of Proposition 3.6.** Fix a subgraph $H$ of $G$, a vertex $v$ of $H$, $0 \leq p \leq 1$, $r \geq 1$ and $\alpha \geq 1$. Let $\lambda \geq (p - p_c) \vee 0$. It suffices to prove that there exist positive constants $\delta, c,$ and $C$ depending only on $G$ such that if $\lambda \leq \delta$, then

$$P^H_p(r \leq R_v < \infty \text{ and } E_v \leq \alpha R_v) \leq \left( \frac{1}{r} + \lambda \right) \exp \left[ -ce^{-C\lambda} \lambda r \right]. \quad (3.6)$$

Let $n = \lceil 1/\lambda \rceil + 2$. The case $r = O(n)$ of (3.6) may be deduced easily from Lemma 2.1: Indeed, it follows from Lemma 2.1 that there exists $\delta_1 > 0$ such that if $0 \leq p \leq p_c + \delta_1$ and $r \leq 4n$, then

$$P^H_p(r \leq R_v < \infty \text{ and } E_v \leq \alpha R_v) \leq P^H_p(R_v \geq r) \leq \frac{1}{r}. \quad (3.7)$$

The bound (3.7) is already of the desired order when $r \leq 4n$, since the quantity in the exponential on the right-hand side of (3.6) is bounded in this regime. Thus, it suffices to prove that there exist positive constants $\delta_2, c,$ and $C$ depending only on $G$ such that if $\lambda \leq \delta_2$, then

$$P^H_p(r \leq R_v < \infty \text{ and } E_v \leq \alpha R_v) \leq \lambda \exp \left[ -ce^{-C\lambda} \lambda r \right] \quad (3.8)$$

for every $r \geq 4n$. 
To this end, suppose that \( r \geq 4n \), so that \( k := \lfloor r/2n \rfloor - 1 \geq 1 \). Suppose further that \( 0 \leq p \leq p_c + \delta_1 \) and that \( 0 \leq \lambda \leq \delta_1 \). Consider exploring the cluster of \( v \) as follows: at stage \( i \), expose the value of those edges that touch \( \partial B_{\text{int}}(v, i-1) \), the set of vertices with intrinsic distance exactly \( i-1 \) from \( v \), and have not yet been exposed. Stop when \( \partial B_{\text{int}}(v, i) = \emptyset \). Here, by intrinsic distance we always mean the graph distance on the percolation cluster (which is a subgraph of \( H \)). See Figure 1 for an illustration of this exploration process. For each \( i \geq 0 \) let \( X_i \) be the set of edges whose status is queried at stage \( i+1 \), so that \( X_i \) is determined by the process run up to time \( i \) and \( |X_i| > 0 \) for every \( 0 \leq i \leq r-1 \) on the event that \( R_v \geq r \). Define a sequence of stopping times \( (T_j)_{j \geq 0} \) for this exploration process by setting \( T_0 = 0 \) and recursively setting
\[
T_{j+1} = \inf \{ i \geq T_j + n : 0 < |X_i| \leq 4\alpha \},
\]
letting \( T_{j+1} = \infty \) if the set on the right-hand side is empty. Recalling that \( k := \lfloor r/2n \rfloor - 1 \), we claim that \( T_k < \infty \) on the event that \( r < R_v < \infty \) and \( E_v \leq \alpha R_v \). Indeed, suppose that this event holds. Let \( k' = k'(K_v) = \lfloor R_v/2n \rfloor - 1 \), so that \( k' \geq k \geq 1 \) and \( k' \geq R_v/4n \). We trivially have that \( 2nk' + n - 1 = 2n([R_v/2n] - 1) + n - 1 \leq R_v - 1 \) and that
\[
\sum_{a=1}^{2k'} \sum_{b=0}^{n-1} |X_{an+b}| \leq \sum_{i=n}^{R_v-1} |X_i| \leq E_v \leq \alpha R_v,
\]
and it follows that there exists \( 0 \leq b = b(K_v) \leq n - 1 \) such that \( \sum_{a=1}^{2k'} |X_{an+b}| \leq \alpha R_v/n \). Applying Markov’s inequality, we deduce that there exists a subset \( A = A(K_v) \) of \( \{1, \ldots, 2k'\} \) such that \( |A| \geq k' \) and \( |X_{an+b(K_v)}| \leq \alpha R_v/nk' \leq 4\alpha \) for every \( a \in A \). If we enumerate \( A \) in increasing order as \( A = \{a_1, a_2, \ldots\} \), then an easy induction shows that \( T_i \leq a_in + b < \infty \) for every \( i \leq k' \) and hence for every \( i \leq k \) as claimed.
Sequentially exploring the clusters of the boundary edges $X_{T_i}$. Left: A cluster in $\mathbb{Z}^2$ (bottom left) and the three vertices in the set $Y_3$ (top left), enumerated clockwise. Right: The subgraphs $H_0, \ldots, H_3$ determined by sequentially exploring the clusters of the edges $v_1, \ldots, v_3$ in the complement of the explored region $H_0$. At each step, revealed open edges are blue, revealed closed edges are red, and the vertices of $K_i$ are marked with black squares.

Let $\mathcal{F}_i$ be the $\sigma$-algebra generated by the first $i$ steps of the exploration process, and let $\mathcal{F}_{T_i}$ be the stopped $\sigma$-algebra associated to the stopping time $T_i$. We clearly have that

$$P^H_p(T_1 < \infty | \mathcal{F}_{T_0}) = P^H_p(T_1 < \infty) \leq P^H_p(R_v \geq n) \leq \lambda,$$

where the final inequality follows from (3.7). Now let $i \geq 1$ and condition on $\mathcal{F}_{T_i}$. If $T_i = \infty$, then we trivially have that $T_{i+1} = \infty$ also. Now suppose that $T_i < \infty$. Let $Y_{T_i}$ be the set of vertices of $\partial B_{\text{int}}(v, T_i)$ that have an edge of $X_{T_i}$ incident to them, so that $|Y_{T_i}| \leq 2|X_{T_i}|$. Enumerate the edges of $Y_{T_i}$ by $Y_{T_i} = \{w_1, \ldots, w_\ell\}$. Let $H_0$ be the subgraph of $H$ spanned those edges that have not been queried by time $T_i$ (i.e., those edges not in $\bigcup_{j=0}^{T_i-1} X_j$). Let $K_1$ be the cluster of $w_1$ in $H_0$ and let $H_1$ be the subgraph of $H_0$ defined by deleting every edge that touches $K_1$ from $H_0$. (Recall that we say an edge of $H_0$ touches $K_1$ if it has at least one endpoint in the vertex set of $K_1$.) Inductively, for each $2 \leq j \leq \ell$, let $K_j$ be the cluster of $w_j$ in $H_{j-1}$ and let $H_j$ be the subgraph of $H_{j-1}$ formed by deleting every edge that touches $K_j$ from $H_{j-1}$ (Figure 2). In order for $T_{i+1}$ to be finite, we must have that there exists $1 \leq j \leq \ell$ such that $K_j$ is non-empty and has intrinsic radius at least $n-1$ from the perspective of $w_j$. It follows from Lemma 2.1 applied to the subgraph $H_{j-1}$ that there exists a constant $C$ such that for each $1 \leq j \leq \ell$, the conditional probability that $K_j$ has intrinsic radius at least $n-1$ given $\mathcal{F}_{T_i}$ and the clusters $K_1, \ldots, K_{j-1}$ is at most $C\lambda$, and hence that

$$P^H_p(T_{i+1} < \infty | \mathcal{F}_{T_i}) \leq 1(T_i < \infty) \left[1 - |1 - 1 \wedge C\lambda|^{Y_{T_i}}\right] \leq 1(T_i < \infty) \left[1 - |1 - 1 \wedge C\lambda|^{8\alpha}\right].$$

(Note that at this stage it was very important that the bounds of Lemma 2.1 held for arbitrary subgraphs of $H$, not just for $H$ itself.) Taking products and using the bound $1 - x \leq e^{-x}$, we obtain that

$$P^H_p(T_i < \infty) \leq \lambda \left[1 - |1 - 1 \wedge C\lambda|^{8\alpha}\right]^{i-1} \leq \lambda \exp \left[-|1 - 1 \wedge C\lambda|^{8\alpha}(i - 1)\right].$$
for every $i \geq 1$ and hence that
\[ \mathbb{P}^H_p(R_v \geq r, E_v \leq \alpha R_v) \leq \mathbb{P}^H_p(T_k < \infty) \leq \lambda \exp \left[-|1 - 1 \wedge C\lambda|^{8\alpha}(k - 1)\right]. \]

Now, since $1 - x \geq e^{-2x}$ for small non-negative values of $x$, it follows that there exist positive constants $\delta_2, c,$ and $C'$ such that if $\lambda \leq \delta_2$ and $r \geq 4n$, then
\[ \mathbb{P}^H_p(R_v \geq r, E_v \leq \alpha R_v) \leq \lambda \exp \left[-ce^{-C'\lambda \alpha r}\right]. \quad (3.9) \]

The proof may be concluded by combining the bounds (3.7) and (3.9), which hold for $r \leq 4n$ and $r \geq 4n$, respectively. \[ \square \]

Remark 3.7. The expression $e^{-C\alpha \lambda}$ is maximized by $\lambda = 1/C\alpha$. In particular, taking $\alpha = rs$ and $\lambda = 1/rs$, it follows from Proposition 3.6 and Proposition 4.2 that, under the hypotheses of those results, there exist constants $c$ and $C$ such that
\[ \mathbb{P}_{p,\infty}(E_v \leq s^{-1}r^2 | R_v \geq r) \leq Ce^{-cs} \quad (3.10) \]
for every $v \in V$, $r \geq 1$, and $s \geq 1$.

We now apply Propositions 3.1 and 3.6 to prove Proposition 3.5.

Proof of Proposition 3.5. The first inequality is trivial, so it suffices to prove the second. It follows from Proposition 3.1 that there exist positive constants $\delta_1$ and $c_1$ such that
\[ \frac{d}{dp} \mathbb{P}_{p,n}(R_v \geq r) \leq -c_1(p - p_c)E_{p,n}[E_v1(R_v \geq r)] + U_{p,n}[1(R_v \geq r)] \]
for every $r \geq 1$, $n \geq 1$ and $p \in [p_c, p_c + \delta_1)$. As in the proof of Lemma 2.1, we can bound $pU_{p,n}[1(R_v \geq r)]$ by the expected number of open edges $e$ such that the cluster of $v$ has intrinsic radius at least $r$ in $\omega$ and strictly less than $r$ in $\omega_e$. Since any such open edge must lie on every intrinsic geodesic of length $r$ starting from $v$ in $\omega$, we deduce that
\[ \frac{d}{dp} \mathbb{P}_{p,n}(R_v \geq r) \leq -c_1(p - p_c)E_{p,n}[E_v1(R_v \geq r)] + \frac{r}{p_c} \mathbb{P}_{p,n}(R_v \geq r) \quad (3.11) \]
for every $r \geq 1$, $n \geq 1$ and $p \in [p_c, p_c + \delta_1)$.

On the other hand, it follows from Proposition 3.6 that there exists positive constants $\delta_2, c_2, c_3$, $C_1$, and $C_2$ such that
\[ \mathbb{P}_{p,\infty}(R_v \geq r \geq \frac{1}{2}c_1p_c(p - p_c)E_v) \leq C_1 \left[\frac{1}{r} + (p - p_c)\right] \exp \left[-c_2e^{-2C_1(p - p_c)[c_1p_c(p - p_c)]^{-1}}(p - p_c)r\right] \]
\[ \leq C_1 \left[\frac{1}{r} + (p - p_c)\right] \exp \left[-2c_3(p - p_c)r\right] \]
\[ \leq \frac{C_2}{r} \exp \left[-c_3(p - p_c)r\right] \]
for every $r \geq 1$ and $p \in [p_c, p_c + \delta_2)$, where we used that $xe^{-2x} \leq e^{-x-1}$ for every $x \geq 0$ in the final inequality. It follows that

$$c_1(p - p_c)E_{p,n}[E_v 1(R_v \geq r)] \geq \frac{2r}{p_c}P_{p,n}(R_v \geq r, c_1(p - p_c)E_v \geq 2r)$$

$$\geq \frac{2r}{p_c}P_{p,n}(R_v \geq r) - \frac{2r}{p_c}P_{p,\infty}(R_v \geq r) \geq \frac{1}{2} c_1 p_c (p - p_c)E_v$$

$$= \frac{2r}{p_c}P_{p,n}(R_v \geq r) - \frac{2C_2}{p_c} \exp[ -c_3(p - p_c)r ]$$

for every $r \geq 1$ and $p \in [p_c, p_c + \delta_2)$. Letting $\delta_3 = \delta_1 \wedge \delta_2$, we deduce from this and (3.11) that

$$\frac{d}{dp} P_{p,n}(R_v \geq r) \leq - \frac{r}{p_c} P_{p,n}(R_v \geq r) + \frac{2C_2}{p_c} \exp[ -c_3(p - p_c)r ]$$

for every $r \geq 1$ and $p \in [p_c, p_c + \delta_3)$. Letting $c_4 = c_3 \wedge (1/p_c)$ and $C_3 = 2C_2/p_c$, it follows that

$$\frac{d}{dp} \left[ e^{c_4(p - p_c)r} P_{p,n}(R_v \geq r) \right] \leq \left[ c_4 r - \frac{r}{p_c} \right] e^{c_4(p - p_c)r} P_{p,n}(R_v \geq r) + \frac{2C_2}{p_c} e^{(c_4 - c_3)(p - p_c)r}$$

$$\leq \frac{2C_2}{p_c} e^{(c_4 - c_3)(p - p_c)r} \leq C_3.$$

Integrating this bound yields that there exist constants $C_4$ and $C_5$ such that

$$P_{p,n}(R_v \geq r) \leq P_{p_c,n}(R_v \geq r) e^{-c_4(p - p_c)r} + C_3(p - p_c) e^{-c_4(p - p_c)r}$$

$$\leq C_4 \left( \frac{1}{r} + (p - p_c) \right) e^{-c_4(p - p_c)r} \leq \frac{C_5}{r} e^{-c_4(p - p_c)r/2}$$

for every $1 \leq r, n < \infty$ and $p \in [p_c, p_c + \delta_3)$. The claim follows by taking $n \to \infty$. □

### 3.4 Bounding the positive term I: Derivation of the auxiliary differential inequality

The goal of the following two subsections is to prove the upper bound of Theorem 1.1 in the slightly supercritical regime. This is the most technical part of the paper.

**Proposition 3.8.** Let $G = (V, E)$ be a connected, locally finite, quasi-transitive graph such that $p_c < p_{2→2}$. Then there exist positive constants $\delta, c$, and $C$ such that

$$P_p(n \leq |K_v| < \infty) \leq Cn^{-1/2} \exp \left[ -c(p - p_c)^2 n \right]$$

(3.12)

for every $n \geq 1, v \in V$, and $p \in (p_c, p_c + \delta)$. 
To prove this proposition, it suffices to prove that there exist positive constants \(c, C,\) and \(\delta\) such that

\[
\mathbb{E}_{p,\infty}[|K_v| \exp \left(c(p - p_c)^2|K_v|\right)] \leq \frac{C}{p - p_c} \quad (3.13)
\]

for every \(p \in (p_c, p_c + \delta)\). Indeed, Markov’s inequality will then imply that

\[
\mathbb{P}_p(n \leq |K_v| < \infty) \leq \frac{C}{(p - p_c)n} \exp \left[-c(p - p_c)^2n\right]
\]

for every \(p \in (p_c, p_c + \delta)\) and \(n \geq 1\), which is of the correct order when \(n \geq (p - p_c)^{-2}\). On the other hand, if \(n \leq (p - p_c)^{-2}\), then a bound of the correct order is already provided by Lemma 2.1.

The primary remaining obstacle we must overcome in order to prove (3.13) is to establish upper bounds on \(U_{p,n}[|K_v|]\), the positive part of the derivative of the truncated \(k\)th moment. Our approach will follow a similar philosophy to that of [30, Section 2.3]. Unfortunately, while the methods developed in that paper are quantitative, they are not sharp, and eventually lead to a factor of order \((p - p_c)^4\) rather than of order \((p - p_c)^2\) in the exponent of (3.12) when combined with our sharp control of skinny clusters, Proposition 3.6. Obtaining optimal bounds requires a rather more delicate and technical approach. In this subsection, we derive a differential inequality which we will use to bound these quantities; the analysis of this differential inequality is then performed in the next subsection. We refer to this differential inequality as the auxiliary differential inequality to distinguish it from the other differential inequalities we have been interested in.

As in [30], we begin by expressing \(U_{p,n}[|K_v|^k]\) geometrically in terms of bridges. We first recall the relevant definitions. Let \(H\) be a connected graph. Recall that two vertices \(u\) and \(v\) of \(H\) are said to be 2-connected if \(u\) and \(v\) remain connected when any edge is deleted from \(H\). (In particular, every vertex is 2-connected to itself.) Equivalently, by Menger’s theorem, \(u\) and \(v\) are 2-connected if there exist a pair of edge-disjoint paths each connecting \(u\) to \(v\). This defines an equivalence relation on the vertices of \(H\), the pieces of which are referred to as the 2-connected components of \(H\). We write \([v]\) for the 2-connected component of the vertex \(v\) in \(H\). An edge \(e\) of \(H\) is said to be a bridge of \(H\) if the graph formed by deleting \(e\) from \(H\) is disconnected. Equivalently, \(e\) is a bridge of \(H\) if its endpoints are in distinct 2-connected components of \(H\). We define \(\text{Tr}(H)\) to be the tree whose vertices are the 2-connected components of \(H\) and whose edges are the bridges of \(H\). Given a graph \(H\) and a sequence of vertices \(v_1, \ldots, v_k\) of \(H\), let \(\text{Br}(v_1, \ldots, v_k; H)\) be the number of edges in the subtree of \(\text{Tr}(H)\) spanned by the union of the geodesics between the vertices \([v_1], \ldots, [v_k]\) in the tree of 2-connected components \(\text{Tr}(H)\).

Let \(G\) be a connected, locally finite, and quasi-transitive; let \(p \in [0,1]\); and let \(v \in V\). We have by Proposition 3.1 that there exist positive constants \(c\) and \(\delta\) such that if \(p_c < p \leq p_c + \delta\), then the \(p\)-derivative of \(E_{p,n}[|K_v|e^{u|K_v|}]\) satisfies

\[
\partial_p E_{p,n}[|K_v|e^{u|K_v|}] \leq -c(p - p_c)E_{p,n}[|K_v|^2e^{u|K_v|}] + U_{p,n}[|K_v|e^{u|K_v|}] = -c(p - p_c)E_{p,n}[|K_v|^2e^{u|K_v|}] + \sum_{k=0}^{\infty} \frac{u^k}{k!} U_{p,n}[|K_v|^{k+1}] \quad (3.14)
\]

for every \(u \geq 0\) and \(n \geq 1\). Observe that by definition of the relevant quantities, we may express \(U_{p,n}[|K_v|^k]\) as
\[ \mathbf{U}_{p,n}[|K_v|^k] = \sum_{x_1, \ldots, x_k \in V(G)} \mathbf{U}_{p,n}[1(x_1, \ldots, x_k \in K_v)] \]

\[ = \frac{1}{p} \sum_{x_1, \ldots, x_k \in V(G)} \mathbf{E}_{p,n}[1(x_1, \ldots, x_k \in K_v) \text{Br}(v, x_1, \ldots, x_k; K_v)] , \]

where the second equality follows from the fact that if \( x_1, \ldots, x_k \) all belong to \( K_v \), then the quantity \( \text{Br}(v, x_1, \ldots, x_k; K_v) \) is equal to the number of edges that are open pivotal for this event. Writing \( \text{Br}(v, x_1, \ldots, x_k; K_v) = \text{Br}(v, x_1, \ldots, x_k) \), this can be written more succinctly as

\[ \mathbf{U}_{p,n}[|K_v|^k] = \frac{1}{p} \mathbf{E}_{p,n} \left[ \sum_{x_1, \ldots, x_k \in K_v} \text{Br}(v, x_1, \ldots, x_k) \right] \]

for every \( n, k \geq 1 \).

Summing over \( k \) it follows that

\[ \mathbf{U}_{p,n}[|K_v|e^{u|K_v|}] = \frac{1}{p} \sum_{k=0}^{\infty} \frac{u^k}{k!} \mathbf{E}_{p,n} \left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} \text{Br}(v, x_1, \ldots, x_{k+1}) 1(\text{Br}(v, x_1, \ldots, x_{k+1}) \geq \frac{1}{2} c p (p - p_c)|K_v|) \right] \]

\[ + \frac{1}{p} \sum_{k=0}^{\infty} \frac{u^k}{k!} \mathbf{E}_{p,n} \left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} \text{Br}(v, x_1, \ldots, x_{k+1}) 1(\text{Br}(v, x_1, \ldots, x_{k+1}) \leq \frac{1}{2} c p (p - p_c)|K_v|) \right] \]

for every \( n \geq 1 \) and \( u \geq 0 \), from which we deduce that

\[ \mathbf{U}_{p,n}[|K_v|e^{u|K_v|}] \]

\[ \leq \frac{1}{p} \sum_{k=0}^{\infty} \frac{u^k}{k!} \mathbf{E}_{p,n} \left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} \text{Br}(v, x_1, \ldots, x_{k+1}) 1(\text{Br}(v, x_1, \ldots, x_{k+1}) \geq \frac{1}{2} c p (p - p_c)|K_v|) \right] \]

\[ + \frac{1}{2} c (p - p_c) \mathbf{E}_{p,n}[|K_v|^2e^{u|K_v|}] \quad (3.15) \]

for every \( u \geq 0 \) and \( n \geq 1 \). We have split the equation up this way precisely so that the second term can be absorbed into the negative term in (3.14). Indeed, the inequalities (3.14) and (3.15) together imply that if \( G \) is a connected, locally finite, quasi-transitive graph with \( p_c < p_{2 \to 2} \), then there exist constants \( \delta, c_1, \) and \( c_2 \) such that
\[
\partial_p E_{p,n}[|K_v|e^{|K_v|}] \leq -c_1(p - p_c)E_{p,n}[|K_v|^2e^{|K_v|}]
\]

\[
+ \frac{1}{p} \sum_{k=0}^{\infty} \frac{u^k}{k!} E_{p,n}\left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} \mathrm{Br}(v, x_1, \ldots, x_{k+1})1\left( \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v) \geq c_2(p - p_c)|K_v| \right) \right]
\]

(3.16)

for every \(v \in V\), \(u \geq 0\), \(p \in (p_c, p_c + \delta)\) and \(1 \leq n < \infty\). Intuitively, the constraint that \(\mathrm{Br}(v, x_1, \ldots, x_{k+1}) \geq c_2(p - p_c)|K_v|\) can be thought of as a higher-order version of the skinniness constraint which we studied in Proposition 3.6.

We will control the summands on the right-hand side of (3.16) by an inductive analysis of certain generating functions, which we now introduce. Let \(G\) be a countable, locally finite, quasi-transitive graph, let \(p \in [0,1]\), and let \(v\) be a vertex of \(G\). For each \(k \geq 1\) and \(n \in \mathbb{N}_\infty = \{1, 2, \ldots\} \cup \{\infty\}\) we define \(G_{k,n}(\cdot, \cdot; G, v, p) : \mathbb{R}^2 \to [0, \infty]\) by

\[
G_{k,n}(s, t; G, v, p) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{x_1, \ldots, x_k \in V(G)} P_{p,n}^{G}\left(x_1, \ldots, x_k \in K_v, E_v = a, \mathrm{Br}(v, x_1, \ldots, x_k; K_v) = b\right) e^{sa+tb},
\]

which is a sort of multivariate generating function, and also define

\[
\mathcal{T}_{k,n}(s, t; G, p) := \sup \left\{ G_{k,n}(s, t; H, u, p) : H \text{ a subgraph of } G, u \text{ a vertex of } H \right\}.
\]

Finally, for each \(n \in \mathbb{N}_\infty\) define \(\mathcal{M}_n(\cdot, \cdot, \cdot; G, v, p) : \mathbb{R}^2 \times [0, \infty) \to [0, \infty]\) by

\[
\mathcal{M}_n(s, t, u; G, p) := \sum_{k=0}^{\infty} \frac{u^k}{k!} \mathcal{T}_{k+1,n}(s, t; G, p).
\]

(3.17)

Note that if \(G\) is a connected, locally finite, quasi-transitive graph with \(p_c < p_{2\to 2}\) and \(0 \leq s \leq c_2(p - p_c)t\), then we have trivially that \(1\left( \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v) \geq c_2(p - p_c)|K_v| \right) \leq \exp(-s|K_v| + t \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v))\) for every \(x_1, \ldots, x_{k+1} \in K_v\) and hence that the expression appearing on the right-hand side of (3.16) can be bounded

\[
\frac{1}{p} \sum_{k=0}^{\infty} \frac{u^k}{k!} E_{p,n}\left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v)1\left( \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v) \geq c_2(p - p_c)|K_v| \right) \right]
\]

\[
\leq \frac{1}{p} \sum_{k=0}^{\infty} \frac{u^k}{k!} E_{p,n}\left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v) e^{-s|K_v| + t \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v)} \right]
\]

\[
\leq \frac{e}{tp} \sum_{k=0}^{\infty} \frac{u^k}{k!} E_{p,n}\left[ \sum_{x_1, \ldots, x_{k+1} \in K_v} e^{-s|K_v| + 2t \mathrm{Br}(v, x_1, \ldots, x_{k+1}; K_v)} \right] \leq \frac{e}{tp} \mathcal{M}_n(-s, 2t, u; G, p)
\]
for every \( u \geq 0, n \geq 1 \), and \( 0 \leq s \leq c_2(p - p_c)t \), where we used the elementary bound \( xe^{tx} \leq et^{-1}e^{2tx} \) in the second inequality. It follows from this and (3.16) that if \( G \) is a connected, locally finite, quasi-transitive graph with \( p_c < p_{2 \rightarrow 2} \), then there exist positive constants \( \delta, c_1, c_2, \) and \( C_1 \) such that

\[
\partial_p E_{p,n}[|K_v|e^{tu|K_v|}] \leq -c_1(p - p_c)E_{p,n}[|K_v|^2e^{tu|K_v|}] + \frac{C_1}{t} \mathcal{M}_n(-c_2(p - p_c)t, t, u; G, p)
\]

(3.18)

for every \( v \in V, u \geq 0, p \in (p_c, p_c + \delta), t \geq 0, \) and \( 1 \leq n < \infty \).

In order to apply the inequality (3.18), we will need to bound the generating function \( \mathcal{M}_n \). To do this, we derive a family of recursive differential inequalities, Lemma 3.9, which in the next subsection we will use to bound the functions \( \mathcal{F}_{k,n} \) by an inductive argument.

When \( n < \infty \), all but finitely many terms of the sum defining \( g_{k,n}(s, t; G, v, p) \) are zero, so that \( g_{k,n}(s, t; G, v, p) \) is a differentiable function of \( (s, t) \) with \( t \)-derivative

\[
\partial_t g_{k,n}(s, t; G, v, p) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{x_1, \ldots, x_k \in V(G)} \mathbf{P}_p^G(x_1, \ldots, x_k \in K_v, E_v = a, Br(v, x_1, \ldots, x_k; K_v) = b) b \mathbb{e}^{sa + tb}.
\]

The following lemma can be thought of as a sharp form of [30, Lemma 2.9].

**Lemma 3.9.** Let \( G \) be a countable graph with degrees bounded by \( M \), let \( v \) be a vertex of \( G \), and let \( p \in (0, 1) \). Then

\[
\partial_t g_{k,n}(s, t; G, v, p) \leq \frac{Mp te^{t}}{1 - p} \sum_{\ell=0}^{k-1} \binom{k}{\ell} g_{\ell+1,n}(s, t; G, v, p) \mathcal{F}_{k-\ell,n}(s, t; G, p)
\]

for every \( k, n \geq 1 \), and \( s, t \in \mathbb{R} \).

Intuitively, this lemma encodes the fact that we can break up the exploration of a cluster at a bridge edge, first exploring the part of the cluster on the same side of the bridge as the root vertex \( v \) then exploring the part of the cluster on the other side of the bridge. Note that the more complicated form of this inequality will, unfortunately, make it rather more difficult to analyze than that of [30, Lemma 2.9].

**Proof of Lemma 3.9.** Fix \( k, n \geq 1, p \in (0, 1), v \in V, \) and \( s, t \in \mathbb{R} \). For each \( a, b \geq 0 \) let

\[
R_{k,n}(a, b; G, v, p) = \sum_{x_1, \ldots, x_k \in V(G)} b \mathbf{P}_p^G(x_1, \ldots, x_k \in K_v, E_v = a, Br(v, x_1, \ldots, x_k; K_v) = b),
\]

so that

\[
\partial_t g_{k,n}(s, t; G, v, p) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} e^{sa + tb} R_{k,n}(a, b; G, v, p).
\]
For each oriented edge $e$ of $G$, let $K^-_e$ and $K^+_e$ be the connected components of $e^-$ and $e^+$ in the subgraph of $G$ spanned by the open edges of $G$ other than $e$. Thus, $K^-_e \neq K^+_e$ if and only if $e^-$ and $e^+$ are not connected to each other by an open path not containing $e$. Let $E^-_e$ be the number of edges of $G$ that touch $K^-_e$, and let $E^+_e$ be the number of edges of $G$ that touch $K^+_e$ but do not touch $K^-_e$. For each oriented edge $e$ of $G$ and each $x_1, \ldots, x_k \in V$, let $\mathcal{A}_e(x_1, \ldots, x_k)$ be the event that $x_1, \ldots, x_k \in K_v$, that $e$ is open, that $v \in K_e^-$, and that there exists $1 \leq i \leq k$ such that $x_i \in K_e^+ \setminus K_e^-$. For each $x_1, \ldots, x_k \in K_v$, the number of oriented edges $e$ such that $\mathcal{A}_e(x_1, \ldots, x_k)$ holds is precisely $\text{Br}(v, x_1, \ldots, x_k; K_v)$ (see Figure 3), so that we can write

$$R_{k,n}(a, b; G, v, p) = \sum_{e \in E^+} \sum_{x_1, \ldots, x_k \in V(G)} \mathbb{P}_{p,n}^{G} \left( \mathcal{A}_e(x_1, \ldots, x_k), E_v = a, \text{Br}(v, x_1, \ldots, x_k; K_v) = b \right).$$

(3.19)

For each strict (possibly empty) subset $A$ of $\{1, \ldots, k\}$, let $\mathcal{B}_e(x_1, \ldots, x_k; A)$ be the event that $\mathcal{A}_e(x_1, \ldots, x_k)$ holds and that $x_i \in K^-_e$ if and only if $i \in A$ for each $1 \leq i \leq k$. Then we can expand

$$R_{k,n}(a, b; G, v, p) = \sum_{e \in E^+} \sum_{x_1, \ldots, x_k \in V(G)} \sum_{A \subset \{1, \ldots, k\}} \mathbb{P}_{p,n}^{G} \left( \mathcal{B}_e(x_1, \ldots, x_k; A), E_v = a, \text{Br}(v, x_1, \ldots, x_k; K_v) = b \right).$$

(3.19)

Since the value of this sum does not change when we permute the values of $x_1, \ldots, x_k$, each set $A$ of a given size $\ell$ contributes the same amount to this sum, so that

$$R_{k,n}(a, b; G, v, p) = \sum_{e \in E^+} \sum_{x_1, \ldots, x_k \in V(G)} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \mathbb{P}_{p,n}^{G} \left( \mathcal{B}_e(x_1, \ldots, x_k; \{1, \ldots, \ell\}), E_v = a, \text{Br}(v, x_1, \ldots, x_k; K_v) = b \right),$$

(3.20)

where we interpret $\{1, \ldots, \ell\}$ as the empty set when $\ell = 0$. 

For each oriented edge $e$ of $G$, let $K^-_e$ and $K^+_e$ be the connected components of $e^-$ and $e^+$ in the subgraph of $G$ spanned by the open edges of $G$ other than $e$. Thus, $K^-_e \neq K^+_e$ if and only if $e^-$ and $e^+$ are not connected to each other by an open path not containing $e$. Let $E^-_e$ be the number of edges of $G$ that touch $K^-_e$, and let $E^+_e$ be the number of edges of $G$ that touch $K^+_e$ but do not touch $K^-_e$. For each oriented edge $e$ of $G$ and each $x_1, \ldots, x_k \in V$, let $\mathcal{A}_e(x_1, \ldots, x_k)$ be the event that $x_1, \ldots, x_k \in K_v$, that $e$ is open, that $v \in K_e^-$, and that there exists $1 \leq i \leq k$ such that $x_i \in K_e^+ \setminus K_e^-$. For each $x_1, \ldots, x_k \in K_v$, the number of oriented edges $e$ such that $\mathcal{A}_e(x_1, \ldots, x_k)$ holds is precisely $\text{Br}(v, x_1, \ldots, x_k; K_v)$ (see Figure 3), so that we can write

**Figure 3** Left: A percolation cluster with a root $v$ (black disc) and five marked points $x_1, \ldots, x_5$ (black squares). The two-connected components of the cluster are represented by blue shaded regions. Edges for which the event $\mathcal{A}_e(x_1, \ldots, x_5)$ holds (for one of the possible orientations of the edge) are drawn as double black lines. Right: The tree of two-connected components of the cluster. Edges contributing to $\text{Br}(v, x_1, \ldots, x_5)$ are drawn with double black lines.
For each $e \in E^-$, $0 \leq \ell \leq k - 1$, each $y_1, \ldots, y_{\ell} \in V(G)$, each $z_1, \ldots, z_{k-\ell} \in V(G)$, and each $a_1, a_2, b_1, b_2 \geq 0$, consider the events

\[ C_{e, \ell}(y_1, \ldots, y_{\ell}; a_1, b_1) \]

\[ := \left\{ v \in K_e^-, y_i \in K_e^- \text{ for every } 1 \leq i \leq \ell, E_e^- = a_1, \text{ and } \text{Br}(v, y_1, \ldots, y_{\ell}, e^-; K_e^-) = b_1 \right\} \]

and

\[ D_{e, \ell}(z_1, \ldots, z_{k-\ell}; a_2, b_2) \]

\[ := \left\{ z_i \in K_e^+ \setminus K_e^- \text{ for every } 1 \leq i \leq k - \ell, E_e^+ = a_2, \text{ and } \text{Br}(e^+, z_1, \ldots, z_{k-\ell}; K_e^+) = b_2 \right\}, \]

where we recall that $E_e^-$ is the number of edges of $G$ that touch $K_e^-$ and $E_e^+$ is the number of edges of $G$ that touch $K_e^+$ but do not touch $K_e^-$. Observe that the event $B_e(x_1, \ldots, x_k; \{1, \ldots, \ell\}) \cap \{E_v = a, |\text{Br}(v, x_1, \ldots, x_k; K_v)| = b\}$ can be rewritten as the disjoint union

\[ B_e(x_1, \ldots, x_k; \{1, \ldots, \ell\}) \cap \{E_v = a, |\text{Br}(v, x_1, \ldots, x_k; K_v)| = b\} = \bigcup_{a_1=0}^{a} \bigcup_{b_1=0}^{b-1} (\{e \text{ open}\} \cap C_{e, \ell}(y_1, \ldots, y_{\ell}; a_1, b_1) \cap D_{e, \ell}(z_1, \ldots, z_{k-\ell}; a_2, b_2)). \]

Indeed, this follows from the observation that if $B_e(x_1, \ldots, x_k; \{1, \ldots, \ell\})$ holds, then $E_v = E_e^- + E_e^+$ and $\text{Br}(v, x_1, \ldots, x_k; K_v) = \text{Br}(v, x_1, \ldots, x_{\ell}, e^-; K_e^-) + \text{Br}(e^+, x_{\ell+1}, \ldots, x_k; K_e^+) + 1$. Noting that the random variable $\omega(e)$ is independent of the pair of random variables $(K_e^-, K_e^+)$, we deduce from (3.20) and (3.21) that

\[ R_{k,n}(a, b; G, v, p) = p \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{e \in E^-} \sum_{a_1=0}^{a} \sum_{b_1=0}^{b-1} \sum_{y_1, \ldots, y_{\ell}} \sum_{z_1, \ldots, z_{k-\ell}} \sum_{e \in V(G)} \sum_{e \in V(G)} e^{a_1+ib_1} e^{a_2+ib_2} \]

\[ \times P_{p,n}^G \left( C_{e, \ell}(y_1, \ldots, y_{\ell}; a_1, b_1) \cap D_{e, \ell}(z_1, \ldots, z_{k-\ell}; a_2, b_2) \right). \]

Let $\mathcal{F}_e^-$ be the $\sigma$-algebra generated by the random variable $K_e^-$ and let $H_e^+$ be the random subgraph of $G$ spanned by those edges of $G$ that do not touch $K_e^-$. The conditional distribution of $K_e^+ \setminus K_e^-$ given $\mathcal{F}_e^-$ coincides with that of the cluster of $e^+$ in Bernoulli-$p$ bond percolation on $H_e^+$,
so that

\[
\begin{align*}
&\sum_{a_2=0}^{\infty} \sum_{b_2=0}^{\infty} \sum_{z_1, \ldots, z_{k-\ell} \in V(G)} P^G_{p,n}(\mathcal{D}_{e,\ell} (z_1, \ldots, z_{k-\ell}; a_2, b_2) \mid \mathcal{F}^-_e) e^{sa_2+tb_2} \\
&= \sum_{z_1, \ldots, z_{k-\ell} \in V(H^+_e)} P^H_{p,n-E_e}(z_1, \ldots, z_{k-\ell} \in K_{e+}, E_{e+} = a_2, Br(e^+, z_1, \ldots, z_{k-\ell}; K_{e+}) = b_2) e^{sa_2+tb_2} \\
&= \mathcal{F}_{k-\ell,n-E_u}^-(s, t; H^+_e, e^+, p) \leq \mathcal{F}_{k-\ell,n-E_u}^- (s, t; G, p) \leq \mathcal{F}_{k-\ell,n}^- (s, t; G, p)
\end{align*}
\]

almost surely. Since \( \mathcal{C}_{e,\ell} (y_1, \ldots, y_{\ell}; a_1, b_1) \) is \( \mathcal{F}_e^- \)-measurable, it follows that

\[
\begin{align*}
\partial_i \mathcal{G}_{k,n}(s, t; G, v, p) &\leq pe^{t} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \mathcal{F}_{k-\ell,n}^-(s, t; G, p) \\
&\times \sum_{e \in E^{-}} \sum_{a_1=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{y_1, \ldots, y_{\ell} \in V(G)} P^G_{p,n}(\mathcal{C}_{e,\ell} (y_1, \ldots, y_{\ell}; a_1, b_1)) e^{sa_1+tb_1}. 
\end{align*}
\]  

(3.23)

On the other hand, since \( \omega(e) \) is independent of \( \mathcal{F}_e^- \), we have that

\[
\begin{align*}
P^G_{p,n}(\mathcal{C}_{e,\ell} (y_1, \ldots, y_{\ell}; a_1, b_1)) &= \frac{1}{1-p} P^G_{p,n}(\mathcal{C}_{e,\ell} (y_1, \ldots, y_{\ell}; a_1, b_1) \cap \{e \text{ closed}\}) \\
&= \frac{1}{1-p} P^G_{p,n}(\{e \text{ closed}, y_1, \ldots, y_{\ell}, e^- \in K_v, E_v = a_1, \text{ and } Br(v, y_1, \ldots, y_{\ell}, e^-; K_v) = b_1\}) \\
&\leq \frac{1}{1-p} P^G_{p,n}(\{y_1, \ldots, y_{\ell}, e^- \in K_v, E_v = a_1, \text{ and } Br(v, y_1, \ldots, y_{\ell}, e^-; K_v) = b_1\})
\end{align*}
\]

and hence that

\[
\begin{align*}
\sum_{e \in E^{-}} \sum_{a_1=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{y_1, \ldots, y_{\ell} \in V(G)} P^G_{p,n}(\mathcal{C}_{e,\ell} (y_1, \ldots, y_{\ell}; a_1, b_1)) e^{sa_1+tb_1} \\
&\leq \frac{M}{1-p} \sum_{a_1=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{y_1, \ldots, y_{\ell+1} \in V(G)} P^G_{p,n}(y_1, \ldots, y_{\ell}, y_{\ell+1} \in K_v, E_v = a_1, \text{Br}(v, y_1, \ldots, y_{\ell+1}; K_v) = b_1) e^{sa_1+tb_1} \\
&= \frac{M}{1-p} \mathcal{G}_{e+1,n}^-(s, t; G, v, p).
\end{align*}
\]  

(3.24)

Substituting (3.24) into (3.23) completes the proof.

\([\blacksquare]\)

We will use Lemma 3.9 in the following integral form.
Corollary 3.10. Let $G$ be a countable graph with degrees bounded by $M$ and let $p \in (0, 1)$. Then

$$\mathcal{F}_{k,n}(s, t_2; G, p) - \mathcal{F}_{k,n}(s, t_1; G, p) \leq Mp \frac{k-1}{1-p} \sum_{\ell=0}^{k-1} \left( \begin{array}{c} k \\ \ell \end{array} \right) \int_{t_1}^{t_2} e^t \mathcal{F}_{\ell+1,n}(s, t; G, p) \mathcal{F}_{k-\ell,n}(s, t; G, p) dt$$

(3.25)

for every $k, n \geq 1$ and $s, t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$.

Proof of Corollary 3.10. We have trivially that

$$\mathcal{F}_{k,n}(s, t_2; G, p) - \mathcal{F}_{k,n}(s, t_1; G, p) = \sup_{H,v} \mathcal{G}_{k,n}(s, t_2; H, v, p) - \sup_{H,v} \mathcal{G}_{k,n}(s, t_1; H, v, p)$$

and applying Lemma 3.9 yields that

$$\mathcal{F}_{k,n}(s, t_2; G, p) - \mathcal{F}_{k,n}(s, t_1; G, p) \leq \sup_{H,v} \left[ \mathcal{G}_{k,n}(s, t_2; H, v, p) - \mathcal{G}_{k,n}(s, t_1; H, v, p) \right]$$

and applying Lemma 3.9 yields that

$$\mathcal{F}_{k,n}(s, t_2; G, p) - \mathcal{F}_{k,n}(s, t_1; G, p) \leq \sup_{H,v} \int_{t_1}^{t_2} \frac{Mp}{1-p} \sum_{\ell=0}^{k-1} \left( \begin{array}{c} k \\ \ell \end{array} \right) \mathcal{G}_{\ell+1,n}(s, t; H, v, p) \mathcal{F}_{k-\ell,n}(s, t; H, p) dt.$$

The claim follows since $\mathcal{G}_{\ell+1,n}(s, t; H, v, p) \mathcal{F}_{k-\ell,n}(s, t; H, p) \leq \mathcal{G}_{\ell+1,n}(s, t; G, p) \mathcal{F}_{k-\ell,n}(s, t; G, p)$ for every subgraph $H$ of $G$. □

3.5 Bounding the positive term II: Analysis of the auxiliary and main differential inequalities

In this subsection we complete the proof of Proposition 3.8. The main step will be to prove the following proposition via an analysis of the recursive differential inequality provided by Lemma 3.9. This proposition (or, more accurately, the intermediate inequality (3.34)) serves as a sharp quantitative version of [30, Equation 2.21] under the additional assumption that $\nabla_{p_c} < \infty$.

The generating function $M_n$ was defined in (3.17).

Proposition 3.11. Let $G$ be an infinite, connected, locally finite, quasi-transitive graph such that $\nabla_{p_c} < \infty$, and let $\alpha \geq 0$. Then there exist positive constants $c_1 = c_1(G, \alpha), c_2 = c_2(G, \alpha), C = C(G, \alpha)$, and $\delta = \delta(G, \alpha)$ such that

$$M_n(-c_1 \varepsilon^2, \alpha c_1 \varepsilon, c_2 \varepsilon^2; G, p_c + \varepsilon) \leq C \varepsilon^{-1}$$

for every $n \geq 1$ and $0 < \varepsilon \leq \delta$.

Before proceeding further, let us see how Proposition 3.11 can be used to complete the proof of Proposition 3.8.
Proof of Proposition 3.8 given Proposition 3.11. Fix $v \in V$. By (3.18) there exist positive constants $c_1$, $c_2$, $\delta_1$, and $C_1$ such that

$$\partial_p E_{p,n}[|K_v|e^{u|K_v|}] \leq -c_1(p - p_c)E_{p,n}[|K_v|^2e^{u|K_v|}] + \frac{C_1}{t}M_n(-c_2(p - p_c)t, t, u; G, p)$$

$$= -c_1(p - p_c)\partial_u E_{p,n}[|K_v|e^{u|K_v|}] + \frac{C_1}{t}M_n(-c_2(p - p_c)t, t, u; G, p)$$

for every $u \geq 0$, $p \in (p_c, p_c + \delta_1)$, $t \geq 0$ and $1 \leq n < \infty$. On the other hand, applying Proposition 3.11 with $\alpha = c_2^{-1}$ yields that there exist positive constants $c_3$, $c_4$, $\delta_2$, and $C_2$ such that

$$M_n(-c_2c_3(p - p_c)^2, c_3(p - p_c), u; G, p) \leq \frac{C_2}{c_2(p - p_c)^{-1}}$$

for every $p \in (p_c, p_c + \delta_2)$ and $0 \leq u \leq c_4(p - p_c)^2$. It follows that there exists a constant $C_3$ such that

$$\partial_p E_{p,n}[|K_v|e^{u|K_v|}] \leq -c_1(p - p_c)\partial_u E_{p,n}[|K_v|e^{u|K_v|}] + C_3(p - p_c)^{-2}$$

for every $p \in (p_c, p_c + \delta_1 \land \delta_2)$ and $0 \leq u \leq c_4(p - p_c)^2$.

By Corollary 3.4, there exists $\delta_3 > 0$ and $C_4 < \infty$ such that for every $0 < \varepsilon \leq \delta_3$ there exists $p_0 = p_0(\varepsilon) \in (p_c + \varepsilon/4, p_c + \varepsilon/2)$ such that $E_{p_0,\infty}|K_v| \leq C_4\varepsilon^{-1}$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $c_5 = (c_1 \land c_4)/2$. Let $0 < \varepsilon \leq \delta$ and let $p_0 = p_0(\varepsilon)$. It follows by the chain rule that

$$\frac{d}{dp} E_{p,n}[|K_v| \exp[c_5(p - p_0)^2|K_v|]] \leq C_3(p - p_c)^{-2}$$

for every $n \geq 1$ and $p_0 \leq p \leq p_c + \varepsilon$. Integrating this differential inequality between $p_0$ and $p_c + \varepsilon$ and noting that $\varepsilon/2 \leq p_c + \varepsilon - p_0 \leq \varepsilon$ yields that

$$E_{p_c + \varepsilon,n}[|K_v| \exp[\frac{1}{4}c_5\varepsilon^2|K_v|]] \leq E_{p_c + \varepsilon,n}[|K_v| \exp[c_5(p_c + \varepsilon - p_0)^2|K_v|]]$$

$$= E_{p_0}[|K_v|] + \int_{p_0}^{p_c + \varepsilon} \frac{d}{dp} E_{p,n}[|K_v| \exp[c_5(p - p_0)^2|K_v|]] dp$$

$$\leq C_4\varepsilon^{-1} + \int_{p_0}^{p_c + \varepsilon} C_3(p - p_c)^{-2} dp \leq (C_4 + 16C_3)\varepsilon^{-1}$$

for every $n \geq 1$ and $0 < \varepsilon \leq \delta$ as required.

We now begin to work toward the proof of Proposition 3.11, which will rely on an inductive analysis of the integral inequality of Corollary 3.10. This analysis will require the following two lemmas as input: The first applies Lemma 2.1 to analyze $\hat{F}_{k,\infty}$ when $t = 0$ and $s < 0$, and the second applies Proposition 3.6 to establish the $k = 1$ base case.
Lemma 3.12 (Boundary conditions). Let $G$ be an infinite, connected, locally finite quasi-transitive graph such that $\nabla_{p_c} < \infty$. Then there exist positive constants $C$, and $\delta$ such that

$$\mathcal{F}_{k,\infty}(-\lambda \varepsilon^2,0;G,p_c + \varepsilon) \leq k! C^k \lambda^{-k} \varepsilon^{-2k+1}$$

(3.27)

for every $k \geq 1$, $0 < \varepsilon \leq \delta$, and $0 < \lambda \leq 1$.

Lemma 3.13 (Base case). Let $G$ be an infinite, connected, locally finite, quasi-transitive graph such that $\nabla_{p_c} < \infty$. Then there exist positive constants $c$, $C$, and $\delta$ such that

$$\mathcal{F}_{1,\infty}(-\lambda \varepsilon^2,\alpha \lambda \varepsilon;G,p_c + \varepsilon) \leq C \lambda^{-1} \varepsilon^{-1}$$

(3.28)

for every $\alpha \geq 0$, $0 < \varepsilon \leq \delta$, and $0 < \lambda \leq 1 \land c \alpha^{-1} e^{-C \alpha}$.

The proofs of both lemmas will use the fact that if $X$ is a non-negative random variable, then

$$\mathbb{E}[X^k e^{sX}] = \int_0^\infty (k + st) t^{k-1} e^{-st} \mathbb{P}(X \geq t) dt$$

for every $k \geq 1$ and $s \in \mathbb{R}$, where it is possible that both sides are equal to $+\infty$ when $s \geq 0$. This identity is a standard consequence of the integration-by-parts formula.

Proof of Lemma 3.12. Let $p_c = p_c(G)$. We have by Lemma 2.1 that there exist positive constants $C_1$ and $\delta$ such that

$$\mathbb{P}_{H_{p_c + \varepsilon}}(1 + E_v \geq u) \leq C_1 [u^{-1/2} + \varepsilon]$$

for every subgraph $H$ of $G$, every vertex $v$ of $H$, every $u \geq 0$, and every $0 < \varepsilon \leq \delta$. Since $|K_v| \leq 1 + E_v$, we deduce by standard calculations that

$$\mathcal{F}_{k,\infty}(-s,0;H,v,p_c + \varepsilon) = \sum_{a=0}^\infty \sum_{x_1,\ldots,x_k \in V(G)} \mathbb{P}_G(x_1,\ldots,x_k \in K_v,E_v = a) e^{-sa} = \mathbb{E}_{p_c + \varepsilon} [ |K_v|^k e^{-sE_v} ]$$

$$\leq e^s \mathbb{E}_{p_c + \varepsilon} \left[ (1 + E_v)^k e^{-s(1+E_v)} \right]$$

$$= e^s \int_0^\infty (k - su) u^{k-1} e^{-su} \mathbb{P}_p(1 + E_v \geq u) du$$

$$\leq C_1 e^s \left[ \int_0^\infty k u^{k-3/2} e^{-su} du + \varepsilon \int_0^\infty k u^{k-1} e^{-su} du \right]$$

for every $s > 0$. Using the identities $\int_0^\infty u^{a-1} e^{-su} du = s^{-a} \int_0^\infty y^{a-1} e^{-y} dy = s^{-a} \Gamma(a)$ and $\Gamma(k) = (k - 1)!$ we obtain that

$$\mathcal{F}_{k,\infty}(-s,0;H,v,p_c + \varepsilon) \leq C_1 e^s \left[ k s^{k+1/2} \Gamma(k - 1/2) + k \varepsilon s^{-k} \Gamma(k) \right] \leq C_1 e^s k! \left[ s^{-k+1/2} + \varepsilon s^{-k} \right]$$
for every subgraph $H$ of $G$, every vertex $v$ of $H$, every $n \geq 0$, every $0 < \varepsilon \leq \delta$, and every $s \geq 0$. The claim follows by taking $s = \lambda \varepsilon^2$.

Proof of Lemma 3.13. Fix $\alpha \geq 0$, a subgraph $H$ of $G$ and a vertex $v$ of $H$. Letting $R_v$ denote the intrinsic radius of the cluster of $v$ in $H$ (i.e., the maximal intrinsic distance of a vertex of $K_v$ from $v$), we trivially have that $\text{Br}(v, x; K_v) \leq R_v$ for every $x \in K_v$, so that

$$\mathcal{G}_{1, \infty}(-\lambda \varepsilon^2, \alpha \lambda \varepsilon; H, v, p_c + \varepsilon) = E^H_{p_c + \varepsilon} \left[ e^{-\lambda \varepsilon^2 E_v} \sum_{x \in K_v} e^{\alpha \lambda \varepsilon \text{Br}(v, x; K_v)} \right] \leq E^H_{p_c + \varepsilon} \left[ |K_v| e^{-\lambda \varepsilon^2 E_v} e^{\alpha \lambda \varepsilon R_v} \right]$$

and hence that

$$\mathcal{G}_{1, \infty}(-\lambda \varepsilon^2, \alpha \lambda \varepsilon; H, v, p_c + \varepsilon) \leq E^H_{p_c + \varepsilon} \left[ |K_v| e^{-\lambda \varepsilon^2 E_v} e^{\alpha \lambda \varepsilon R_v} \right] \leq E^H_{p_c + \varepsilon} \left[ |K_v| e^{-\lambda \varepsilon^2 E_v} e^{\alpha \lambda \varepsilon R_v/2} \right] \leq E^H_{p_c + \varepsilon} \left[ |K_v| e^{-\lambda \varepsilon^2 E_v/2} \right] \leq \mathcal{G}_{1, \infty}(-\lambda \varepsilon^2/2, 0; G, p_c + \varepsilon) \leq C_1 \lambda^{-1} \varepsilon^{-1}$$

for every $0 < \varepsilon \leq \delta_1$ and $0 < \lambda \leq 1$. For the first term, Lemma 3.12 implies that there exist positive constants $\delta_1$ and $C_1$ such that

$$E^H_{p_c + \varepsilon} \left[ |K_v| e^{-\lambda \varepsilon^2 E_v/2} \right] \leq E^H_{p_c + \varepsilon} \left[ |K_v| e^{-\lambda \varepsilon^2 E_v/2} \right] \leq \mathcal{G}_{1, \infty}(-\lambda \varepsilon^2/2, 0; G, p_c + \varepsilon) \leq C_1 \lambda^{-1} \varepsilon^{-1}$$

for every $0 < \varepsilon \leq \delta_1$ and $0 < \lambda \leq 1$.

For the second term, we first decompose further

$$E^H_{p_c + \varepsilon} \left[ |K_v| e^{\alpha \lambda \varepsilon R_v} \right] = E^H_{p_c + \varepsilon} \left[ |K_v| e^{\alpha \lambda \varepsilon R_v} \right] + E^H_{p_c + \varepsilon} \left[ |K_v| e^{\alpha \lambda \varepsilon R_v} \right] \leq C_2 \varepsilon^{-1} e^{\alpha \lambda},$$

where the second inequality means that we write $I$ and $II$ for the first and second terms appearing on the right-hand side of the first equality. To bound the term $I$, we apply (2.2) to deduce that there exist positive constants $\delta_2$ and $C_2$ such that

$$I \leq e^{\alpha \lambda} E^H_{p_c + \varepsilon} \left[ |K_v| 1(R_u < \varepsilon^{-1}) \right] \leq e^{\alpha \lambda} E^H_{p_c + \varepsilon} \left[ \#B_{\text{int}}(v, \varepsilon^{-1}) \right] \leq C_2 \varepsilon^{-1} e^{\alpha \lambda},$$

for every $0 \leq \varepsilon \leq \delta_2$ and $0 < \lambda \leq 1$. Finally, to bound the term $II$, we note that

$$II \leq \frac{4\alpha}{\varepsilon} E^H_{p_c + \varepsilon} \left[ R_v e^{\alpha \lambda \varepsilon R_v} \right]$$
(where we used the fact that \(|K_v| \leq E_v + 1 \leq 2E_v\) when \(R_v \geq \varepsilon^{-1} > 0\)) and hence by (3.29) that there exists a constant \(C_3\) such that

\[
II \leq C_3 \frac{\alpha}{\varepsilon} \int_{\varepsilon^{-1}}^{\infty} (1 + \alpha \lambda \varepsilon t) e^{\alpha \lambda \varepsilon t} \mathbf{P}^H_{\rho_c + \varepsilon} \left( t \leq R_v < \infty, E_v < \frac{2\alpha}{\varepsilon} R_v \right) dt.
\]

We then apply Proposition 3.6 to obtain that there exist positive constants \(c_1, C_4, C_5,\) and \(\delta_3\) such that

\[
II \leq C_5 \frac{\alpha}{\varepsilon} \int_{\varepsilon^{-1}}^{\infty} (\varepsilon + \alpha \lambda \varepsilon^2 t) \exp \left[-2c_1 e^{-2C_4 \alpha} \varepsilon t + \alpha \lambda \varepsilon t\right] dt
\]

for every \(0 < \varepsilon \leq \delta_3\) and \(0 < \lambda \leq 1\). We deduce in particular that if \(0 < \varepsilon \leq \delta_3\) and \(\alpha \lambda \leq c_1 e^{-2C_4 \alpha}\), then

\[
II \leq C_5 \frac{\alpha}{\varepsilon} \int_{\varepsilon^{-1}}^{\infty} (\varepsilon + \alpha \lambda \varepsilon^2 t) \exp \left[-c_1 e^{-2C_4 \alpha} \varepsilon t\right] dt.
\]

Similarly to the proof of Lemma 3.12, using the identities \(\int_0^\infty e^{-st} dt = s^{-1}\) and \(\int_0^\infty te^{-st} dt = s^{-2}\) yields that there exists a constant \(C_6\) such that

\[
II \leq C_6 \frac{\alpha}{\varepsilon} \left[ e^{2C_4 \alpha} + \alpha \lambda e^{4C_4 \alpha} \right] \leq C_6 \alpha (1 + \alpha \lambda)e^{4C_4 \alpha} \varepsilon^{-1}
\]

for every \(0 < \varepsilon \leq \delta_3\) and \(0 < \lambda \leq 1 \land c_1 \alpha^{-1} e^{-2C_4 \alpha}\).

Putting together all the estimates (3.30), (3.31), (3.32), and (3.33), we obtain that there exist positive constants \(\delta_4 = \delta_1 \land \delta_2 \land \delta_3\) and \(C_7\) such that

\[
\mathcal{G}_{1,\infty}(-\lambda \varepsilon^2, \alpha \lambda \varepsilon; H, v, p_c + \varepsilon) \leq C_7 \left[ \lambda^{-1} + e^{\alpha \lambda} + \alpha (1 + \lambda \varepsilon) e^{4C_4 \alpha} \varepsilon^{-1} \right]
\]

for every \(0 < \varepsilon \leq \delta_4\) and \(0 < \lambda \leq 1 \land c_1 \alpha^{-1} e^{-2C_4 \alpha}\). It follows that there exists a constant \(C_8\) such that if \(0 < \varepsilon \leq \delta_4\) and \(0 < \lambda \leq 1 \land c_1 \alpha^{-1} e^{-4C_4 \alpha} \land e^{-\alpha}\), then

\[
\mathcal{G}_{1,\infty}(-\lambda \varepsilon^2, \alpha \lambda \varepsilon; H, v, p_c + \varepsilon) \leq C_8 \lambda^{-1} \varepsilon^{-1}.
\]

Since \(H, v,\) and \(\alpha \geq 0\) were arbitrary, this is easily seen to imply the claim. \(\square\)

We are now ready to prove Proposition 3.11.

**Proof of Proposition 3.11.** Let \(G\) be an infinite, connected, locally finite, quasi-transitive graph, let \(p_c = p_c(G)\), and suppose that \(V_{\rho_c} < \infty\). Let \(\alpha \geq 0\). It suffices to prove that there exist positive constants \(c = c(G, \alpha), C = C(G, \alpha),\) and \(\delta = \delta(G, \alpha)\) such that

\[
\mathcal{F}_{k,\infty}(-c \varepsilon^2, \alpha c \varepsilon; G, p_c + \varepsilon) \leq (k - 1)! C^k \varepsilon^{-2k+1}
\]

for every \(k \geq 1\) and \(0 < \varepsilon \leq \delta\). Indeed, the claim will then follow by noting that if (3.34) holds, then

\[
\mathcal{M}_{\infty}(-c \varepsilon^2, \alpha c \varepsilon, \frac{1}{2C} \varepsilon^2; G, p_c(G) + \varepsilon) \leq \sum_{k=0}^{\infty} \frac{\varepsilon^{2k}}{k! 12^{k} C^k} k! C^{k+1} \varepsilon^{-2(k+1)+1} = 2C \varepsilon^{-1}
\]

for every \(0 < \varepsilon \leq \delta\). The case \(\alpha = 0\) is handled by Lemma 3.12, so we may suppose that \(\alpha > 0\).
All the constants appearing in this proof will depend only on $\alpha$ and $G$. Throughout the proof we will also use the convention that $(-1)! = 1$. We begin by noting that, with this convention, there exists a finite constant $C_1$ such that

$$\frac{1}{(k-1)!} \sum_{\ell=1}^{k-1} \binom{k}{\ell} (\ell-1)! (k-\ell-2)! = \frac{k}{\ell(k-\ell)(k-1)} + \frac{k}{k-1} \leq C_1$$

for every $k \geq 2$. By Lemma 3.13, there exist positive constants $c_1, C_2,$ and $\delta_1$ such that

$$F_{1,\infty}(-\lambda\varepsilon^2, \alpha\lambda\varepsilon; G, p_c + \varepsilon) \leq C_2\varepsilon^{-1}$$

for every $0 \leq \lambda \leq c_1$ and $0 < \varepsilon \leq \delta_1$. Define

$$c = \min \left\{ 1, c_1, \frac{1 - p_c}{4\alpha C_2 e M}, \frac{1 - p_c}{4\alpha C_1 e M} \right\},$$

where $M$ is the maximum degree of $G$. For each $k, n \geq 1$ and $0 < \varepsilon \leq \delta_1$ we define increasing functions $f_{k,n,\varepsilon} : [0, 1] \to [0, \infty)$ and $f_{k,\varepsilon} : [0, 1] \to [0, \infty]$ by

$$f_{k,n,\varepsilon}(\theta) = F_{k,n}(-ce^2, \alpha\varepsilon\theta; G, p_c(G) + \varepsilon) \quad \text{and} \quad f_{k,\varepsilon}(\theta) = F_{k,\infty}(-ce^2, \alpha\varepsilon\theta; G, p_c(G) + \varepsilon),$$

so that $f_{k,\varepsilon}(\theta) = \sup_{n \geq 1} f_{k,n,\varepsilon}(\theta)$. Thus, (3.35) implies that $f_{1,\varepsilon}(\theta) \leq f_{1,\varepsilon}(1) \leq C_2\varepsilon^{-1}$ for every $\varepsilon \leq \delta_1$ and $\theta \in [0, 1]$. Meanwhile, Lemma 3.12 easily implies that there exist positive constants $C_3$ and $\delta_2$, such that

$$f_{k,\varepsilon}(0) \leq C_3^k e^{-2k+1} (k-2)!$$

for every $0 < \varepsilon \leq \delta_2$ and $k \geq 1$.

Let $\delta = \min\{\delta_1, \delta_2, (1 - p_c)/2\}$ and let $C_4 = 4(C_2 \lor C_3) > 0$. It suffices to prove that

$$f_{k,\varepsilon}(\theta) \leq C_4^k e^{C_4(k-1)\theta} e^{-2k+1} (k-2)!$$

for every $k, n \geq 1, \theta \in [0, 1]$ and $0 < \varepsilon \leq \delta$. We will do this by induction on $k$. The base case $k = 1$ follows immediately from (3.35). Suppose that $k \geq 2$ and that the claim holds for all $1 \leq k' < k$. Fix $n \geq 1$ and $0 < \varepsilon \leq \delta$. It follows from Corollary 3.10 that

$$f_{k,n,\varepsilon}(\theta) \leq f_{k,n,\varepsilon}(0) + \frac{M(p_c + \varepsilon)}{1 - (p_c + \varepsilon)} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \int_{0}^{\theta} e^{\varepsilon\varphi} f_{\ell+1,n,\varepsilon}(\varphi) f_{k-\ell,n,\varepsilon}(\varphi) \alpha e d\varphi,$$

where the $\alpha e$ term comes from changing variables in the integral from $t$ to $\varphi$. Our choice of $c$ therefore yields that
\[ f_{k,n,\varepsilon}(\theta) \leq f_{k,\varepsilon}(0) + \frac{eMa\varepsilon}{1 - p_c - \delta} \int_0^\theta f_{k,n,\varepsilon}(\varphi)f_{1,\varepsilon}(\varphi)\,d\varphi + \frac{eMa\varepsilon}{1 - p_c - \delta} \sum_{\ell=1}^{k-1} \binom{k}{\ell} \int_0^\theta f_{\ell+1,\varepsilon}(\varphi)f_{k-\ell,\varepsilon}(\varphi)\,d\varphi \]
\[ \leq f_{k,\varepsilon}(0) + \frac{\varepsilon}{4C_2} \int_0^\theta f_{k,n,\varepsilon}(\varphi)f_{1,\varepsilon}(\varphi)\,d\varphi + \frac{\varepsilon}{4C_1} \sum_{\ell=1}^{k-1} \binom{k}{\ell} \int_0^\theta f_{\ell+1,\varepsilon}(\varphi)f_{k-\ell,\varepsilon}(\varphi)\,d\varphi \] (3.38)

for every \( \theta \in [0, 1] \), where we used that \( e^{e\varepsilon\varphi} \leq e^e \leq e \) in the first line. The first two terms are easily bounded by using (3.35), (3.36), and the definition of \( C_4 \) to obtain that
\[ f_{k,\varepsilon}(0) + \frac{\varepsilon}{4C_2} \int_0^\theta f_{k,n,\varepsilon}(\varphi)f_{1,\varepsilon}(\varphi)\,d\varphi \leq C_3^k \varepsilon^{-2k+1}(k - 2)! \]
\[ + \frac{\varepsilon}{2C_2} \int_0^\theta f_{k,n,\varepsilon}(\theta)C_2 \varepsilon^{-1}\,d\varphi \]
\[ \leq \frac{1}{4} C_4^k \varepsilon^{-2k+1}(k - 2)! \] (3.39)

for every \( \theta \in [0, 1] \). For the final term, we apply the induction hypothesis and the definition of \( C_1 \) to obtain that
\[ \frac{\varepsilon}{4C_1} \sum_{\ell=1}^{k-1} \binom{k}{\ell} \int_0^\theta f_{\ell+1,\varepsilon}(\varphi)f_{k-\ell,\varepsilon}(\varphi)\,d\varphi \]
\[ \leq \frac{\varepsilon}{4C_1} \sum_{\ell=1}^{k-1} \binom{k}{\ell} \int_0^\theta C_4^{\ell+1} e^{C_4 \ell \varphi} \varepsilon^{-2\ell+1}(\ell - 1)! C_4^{k-\ell} e^{C_4(k-\ell-1)\varphi} \varepsilon^{-2k+2\ell+1}(k - \ell - 2)!\,d\varphi \]
\[ = \frac{C_4^{k+1} \varepsilon^{-2k+1}}{4C_1} \sum_{\ell=1}^{k-1} \binom{k}{\ell} (\ell - 1)!(k - \ell - 2)! \int_0^\theta e^{C_4(k-1)\varphi}\,d\varphi \]
\[ \leq \frac{C_4^{k+1} \varepsilon^{-2k+1}}{4C_1} \sum_{\ell=1}^{k-1} \binom{k}{\ell} (\ell - 1)! (k - \ell - 2)! \frac{e^{C_4(k-1)\theta}}{k-1} \leq \frac{1}{4} \frac{C_4^k e^{C_4(k-1)\theta} \varepsilon^{-2k+1}(k - 2)!}{e^{C_4(k-1)\theta}} \] (3.40)

for every \( \theta \in [0, 1] \). Putting together the estimates (3.38), (3.39), and (3.40) yields that
\[ f_{k,n,\varepsilon}(\theta) \leq \frac{1}{4} C_4^k \varepsilon^{-2k+1}(k - 2)! + \frac{1}{2} f_{k,n,\varepsilon}(\theta) + \frac{1}{4} C_4^k e^{C_4(k-1)\theta} \varepsilon^{-2k+1}(k - 2)! \]

for every \( \theta \in [0, 1] \), which rearranges to give that
\[ f_{k,n,\varepsilon}(\theta) \leq C_4^k e^{C_4(k-1)\theta} \varepsilon^{-2k+1}(k - 2)! \]

for every \( \theta \in [0, 1] \) as desired. Since \( n \geq 1 \) and \( 0 < \varepsilon \leq \delta \) were arbitrary, taking \( n \to \infty \) completes the induction step and hence also the proof. \( \square \)

**Remark 3.14.** The correct form of the induction hypothesis (3.37) needed to make this argument work was not at all obvious to us, and was found by extensive trial and error. We would be interested to know if someone is aware of a more systematic way of approaching similar problems.
Remark 3.15. Consider the generating function

\[ \mathcal{N}_n(s, t, u) = \mathcal{N}_n(s, t, u; G, p) = \sum_{k=1}^{\infty} \frac{u^k}{k!} F_{k,n}(s, t; G, p), \]

which satisfies \( \delta_u \mathcal{N}_n = \mathcal{N}_n \). Summing the differential inequality given by Lemma 3.9 over \( k \geq 1 \) implies the partial differential inequality

\[ \partial_t \mathcal{N}_n \leq \frac{M p e^t}{1 - p} \mathcal{N}_n \partial_u \mathcal{N}_n \quad \forall s, t \in \mathbb{R}, u \geq 0, n \geq 1. \]

(Note that while \( \mathcal{N}_n \) need not be differentiable, it is locally Lipschitz and hence differentiable almost everywhere.) See, for example, the discussion of exponential generating functions in [65]. This point of view may be a useful starting point for further analysis. (It appeared to us to be ill-suited to our present aims, however.)

4 | COMPLETING THE PROOF

In this section we complete the proof of Theorems 1.1 and 1.2. It remains to establish lower bounds in the slightly supercritical regime, as well as both upper and lower bounds in the critical and slightly subcritical regimes. Several of these bounds are closely related to estimates that have already been proven in the literature, but still require a delicate treatment to establish in the desired sharp form.

We begin by proving upper bounds in the critical and slightly subcritical regimes under the assumption that \( \nabla \rho_c < \infty \).

Proposition 4.1 (Subcritical upper bounds). Let \( G \) be an infinite, connected, locally finite, quasi-transitive graph, and suppose that \( \nabla \rho_c < \infty \). Then there exists positive constants \( c \) and \( C \) such that

\[ P_p(\text{Rad}_{\text{ext}}(K_v) \geq r) \leq P_p(\text{Rad}_{\text{int}}(K_v) \geq r) \leq C \frac{r}{r} \exp(-c|p - p_c|r) \]

and

\[ P_p(|K_v| \geq n) \leq C \frac{n^{1/2}}{n} \exp(-c|p - p_c|^2 n) \]

for every \( n, r \geq 1, 0 \leq p \leq p_c \), and \( v \in V \).

Recall that we write \( \asymp, \preceq, \text{ and } \succeq \) for equalities and inequalities that hold to within multiplication by a positive constant depending only on \( G \).

Proof. Fix \( v \in V \) and write \( R_v = \text{Rad}_{\text{int}}(K_v) \). As discussed in the introduction, it is known that if \( G \) is an infinite, connected, locally finite, quasi-transitive graph with \( \nabla \rho_c < \infty \), then

\[ P_{p_c}(|K_v| \geq n) \asymp n^{-1/2} \quad \text{and} \quad P_{p_c}(R_v \geq r) \asymp r^{-1} \]
for every \( n, r \geq 1 \), and
\[
\mathbf{E}_p[|K_v|] \asymp (p - p_c)^{-1} \tag{4.4}
\]
for every \( 0 \leq p \leq p_c \). These results essentially follow from the works of Barsky and Aizenman [6], Kozma and Nachmias [44], and Aizenman and Newman [5]. These papers all dealt with the case \( G = \mathbb{Z}^d \), see [39, Section 7] for a discussion of how to generalize these results to arbitrary quasi-transitive graphs with \( \nabla p_c < \infty \). It follows from (4.4) and the tree-graph method of Aizenman and Newman [5] (see also [26, Chapter 6.3]) that there exists a constant \( C_1 \) such that
\[
\mathbf{E}_p[|K_v|^k] \leq k!C_1^k |p - p_c|^{-2k+1}
\]
for every \( k \geq 1 \) and \( p < p_c \) and hence that there exists a constant \( c_1 = 1/2C_1 \) such that
\[
\mathbf{P}_p(|K_v| \geq e^{c_1 |p - p_c|^2}n) \leq \frac{1}{n(p - p_c)} \exp \left[ -c_1 |p - p_c|^2n \right] \tag{4.5}
\]
for every \( 0 \leq p < p_c \), and together with (4.3) this implies the desired bound (4.2). (Indeed, simply use the bound (4.5) if \( n \geq (p - p_c)^{-2} \) and the bound (4.3) otherwise, noting that \( \mathbf{P}_p(|K_v| \geq n) \) is increasing in \( p \).) See also [38] for an alternative derivation of the inequality (4.5) from (4.3).

It remains to prove (4.1). The case \( r \leq |p - p_c|^{-1} \) is already handled by Lemma 2.1, so it suffices to consider the case \( r \geq |p - p_c|^{-1} \). We have by the union bound that
\[
\mathbf{P}_p(R_v \geq r) \leq \mathbf{P}_p(|K_v| \geq |p - p_c|^{-1}r) + \mathbf{P}_p(R_v \geq r \text{ and } |K_v| \leq |p - p_c|^{-1}r).
\]
Using (4.5) to bound the first term and Proposition 3.6 with \( \lambda = |p - p_c| \) to bound the second yields that there exist positive constant \( c_2 \) such that
\[
\mathbf{P}_p(R_v \geq r) \leq \frac{1}{r} \exp [-c_1|p - p_c|r] + \left( \frac{1}{r} + |p - p_c| \right) \exp [-c_2|p - p_c|r],
\]
which is easily seen to be of the required order (since \( xe^{-xr} \leq 2e^{-r}e^{-xr/2} \) for every \( x \in \mathbb{R} \)).

We next study the intrinsic radius in the subcritical case. This is our only bound that holds for all quasi-transitive graphs.

**Proposition 4.2.** Let \( G \) be an infinite, connected, locally finite, quasi-transitive graph. Then there exist positive constants \( c, C, \) and \( \delta \) such that
\[ \mathbb{P}_p (\text{Rad}_{\text{int}}(K_v) \geq r) \geq \frac{c}{r} \exp \left( -C|p - p_c|r \right) \]

for every \( p \in (p_c - \delta, p_c], \ r \geq 1, \) and \( v \in V. \)

**Proof.** A similar argument to that of Lemma 2.1 establishes that

\[ \mathbb{P}_p (\text{Rad}_{\text{int}}(K_v) \geq r) \geq \mathbb{P}_q (\text{Rad}_{\text{int}}(K_v) \geq r) \exp \left( -\frac{q-p}{p} r \right) \]

for every \( 0 \leq p \leq q \leq 1 \) and \( r \geq 1. \) It follows in particular that

\[ \mathbb{P}_p (\text{Rad}_{\text{int}}(K_v) \geq r) \geq \exp \left( -\frac{(1-p)}{p} r \right) \]

for every \( 0 \leq p \leq 1 \) and \( r \geq 1, \) which implies the claim in the case \( p_c = 1. \) On the other hand, if \( p_c < 1, \) then we have that there exists a constant \( c \) such that \( \mathbb{P}_p (v \to \infty) \geq c(p - p_c) \) for every \( p_c \leq p \leq 1 \ [3, 21, 38], \) so that

\[ \mathbb{P}_p (\text{Rad}_{\text{int}}(K_v) \geq r) \geq c(q - p_c) \exp \left( -\frac{q-p}{p} r \right) \]

for every \( 0 \leq p \leq p_c \leq q \leq 1. \) Taking \( q - p_c = (p_c - p) \land r^{-1} \) implies the claim. \( \square \)

We next prove sharp lower bounds on the tail of the volume under the assumption that \( \nabla p_c < \infty. \)

**Proposition 4.3.** Let \( G \) be an infinite, connected, locally finite, quasi-transitive graph, and suppose that \( \nabla p_c < \infty. \) Then there exist positive constants \( c, C, \) and \( \delta \) such that

\[ \mathbb{P}_p (n \leq |K_v| < \infty) \geq cn^{-1/2} \exp \left( -C|p - p_c|^2n \right) \quad (4.6) \]

for every \( p \in (p_c - \delta, p_c + \delta), r \geq 1, \) and \( v \in V. \)

**Remark 4.4.** Nguyen [54] proved that, under the same conditions as Proposition 4.3, there exist constants \((C_k)_{k \geq 1}\) such that

\[ \mathbb{E}_p [|K_v|^k] \geq C_k |p - p_c|^{-2k+1} \]

for every \( 0 \leq p < p_c \) and \( k \geq 1. \) This is sufficient to determine the value of the gap exponent \( \Delta = 2. \) However, it seems that the argument of [54] does not give sharp \((C_k \geq k!e^{-O(k)}))\) control of the value of the constant \( C_k, \) and therefore does not establish the subcritical case of the bound (4.6). Similarly, classical arguments of Durrett and Nguyen [22] and Newman [53] can be used to prove related inequalities for the truncated \( k \) th moment \( \mathbb{E}_{p, \infty}[|K_v|^k] \) in the slightly supercritical regime. Again, however, it appears that these estimates are not sharp, and lose various logarithmic factors compared to our estimate (4.6).

**Proof of Proposition 4.3.** Write \( R_v = \text{Rad}_{\text{int}}(K_v). \) First suppose that \( p \leq p_c. \) Taking \( \lambda = \alpha|p - p_c| \) in Proposition 3.6, we obtain that that there exist positive constants \( c_1 \) and \( C_1 \) such that
\[ P_p(|K_v| \leq n, R_v \geq \alpha |p - p_c|n) \leq C_1 \left( \frac{1}{\alpha |p - p_c|n} + \alpha |p - p_c| \right) \exp \left[ -c_1 \alpha^2 |p - p_c|^2 n \right] \]

for every \( 0 \leq p \leq p_c, n \geq 1 \), and \( \alpha \geq 1 \). Letting \( c_2, C_2, \) and \( \delta_1 \) be the constants from Proposition 4.2, it follows that

\[
P_p(|K_v| \geq n) \geq P_p(R_v \geq \alpha |p - p_c|n) - P_p(|K_v| \leq n, R_v \geq \alpha |p - p_c|n)
\]

\[
\geq \frac{c_2}{\alpha |p - p_c|n} \exp \left[ -C_2 \alpha |p - p_c|^2 n \right]
\]

\[
-C_1 \left( \frac{1}{\alpha |p - p_c|n} + \alpha |p - p_c| \right) \exp \left[ -c_1 \alpha^2 |p - p_c|^2 n \right]
\]

for every \( p_c - \delta_1 \leq p \leq p_c, r \geq 1, \) and \( \alpha \geq 1 \). Taking \( \alpha = 1 \vee (2C_1/c_1) \) we deduce that there exist positive constants \( c_3, C_3, \) and \( C_4 \) such that

\[
P_p(|K_v| \geq n) \geq \frac{c_3}{|p - p_c|n} \exp \left[ -C_3 |p - p_c|^2 n \right]
\]

\[
-C_4 \left( \frac{1}{|p - p_c|n} + |p - p_c| \right) \exp \left[ -2C_3 |p - p_c|^2 n \right]
\]

for every \( p \in (p_c - \delta, p_c) \) and \( n \geq 1 \). It follows readily that there exist positive constants \( c_4 \) and \( C_5 \) such that

\[
P_p(|K_v| \geq n) \geq \frac{c_4}{\sqrt{n}} \exp \left[ -C_3 |p - p_c|^2 n \right]
\]

for every \( p \in (p_c - \delta, p_c) \) and \( n \geq C_5 |p - p_c|^{-2} \). Since \( P_p(|K_v| \geq n) \) is decreasing in \( n \), it follows that

\[
P_p(|K_v| \geq n) \geq \frac{c_4}{\sqrt{n \vee C_5 |p - p_c|^{-1}}} \exp \left[ -C_3 |p - p_c|^2 (n \vee C_5 |p - p_c|^{-1}) \right]
\]

(4.7)

for every \( p \in (p_c - \delta, p_c) \) and \( n \geq 1 \).

We now handle the case that \( p \leq p_c \) and \( n \) is of order at most \( |p - p_c|^{-2} \). It follows from the proof of [36, Proposition 3.6] that

\[
\sup_{u \in \mathcal{V}} \mathbb{E}_{p_c} \left[ \sum_{\ell \neq r} 2r \# \partial B_{\text{int}}(u, \ell) \right] \geq r + 1
\]

for every \( r \geq 1 \), and an argument similar to that performed in the proof of Lemma 2.1 shows that there exist constants \( \delta_4 \leq p_c/2 \) and \( C_6 \) such that

\[
\sup_{u \in \mathcal{V}} \mathbb{E}_p \left[ \sum_{\ell = r}^{2r} \# \partial B_{\text{int}}(u, \ell) \right] \geq (r + 1) \exp \left[ -\frac{2(p - p_c)}{p} r \right] \geq (r + 1) \exp \left[ -C_6 |p - p_c| r \right]
\]

(4.8)
for every $p \in (p_c - \delta_4, p_c]$ and $r \geq 1$. Applying (4.3), it follows that

$$
\sup_{u \in V'} \mathbb{E}_p \left[ \sum_{\ell = r}^{2r} \# B_{\text{int}}(u, \ell) \mid R_u \geq r \right] \geq C_5 (r + 1)^2 \exp \left[ -C_6 |p - p_c| r \right]
$$

(4.9)

for every $p \in (p_c - \delta_4, p_c]$ and $r \geq 1$. On the other hand, since $\nabla_{p_c} < \infty$, it is known [44, 58] that there exists a constant $C_7$ such that

$$
\mathbb{E}_p \left[ \# B_{\text{int}}(u) \right] \leq C_7 (r + 1)
$$

for every $u \in V$, $0 \leq p \leq p_c$ and $r \geq 0$. A straightforward and well-known variation on the tree-graph inequality method of Aizenman and Newman [5] (see, e.g., [43, Lemma 2]) gives that

$$
\mathbb{E}_p \left[ (\# B_{\text{int}}(u, 2r))^2 \right] \leq \sup_{u \in V} \mathbb{E}_p \left[ (\# B_{\text{int}}(u, 2r))^2 \right]
$$

for every $u \in V$ and $r \geq 1$, and hence that

$$
\mathbb{E}_p \left[ (\# B_{\text{int}}(u, 2r))^2 \right] \leq C_7^2 (r + 1)^3
$$

for every $r \geq 1, u \in V$ and $0 \leq p \leq p_c$. It follows from the Paley-Zygmund inequality that

$$
\mathbb{P}_p \left( |K_u| \geq \frac{1}{2} \mathbb{E}_p \left[ \sum_{\ell = r}^{2r} \# B_{\text{int}}(u, \ell) \mid R_u \geq r \right] \right) \geq \frac{\mathbb{E}_p \left[ \sum_{\ell = r}^{2r} \# B_{\text{int}}(u, \ell) \right]^2}{4 \mathbb{E}_p \left[ (\# B_{\text{int}}(u, 2r))^2 \right]}
$$

(4.10)

for every $u \in V$, $0 < p \leq p_c$, and $r \geq 1$. Applying this inequality together with (4.8), (4.9), and (4.10) and maximizing over $u$, it follows that

$$
\sup_{u \in V} \mathbb{P}_p \left( |K_v| \geq \frac{c_5}{2} (r + 1)^2 e^{-C_6 |p - p_c| r} \right) \geq \frac{e^{-2C_6 |p - p_c| r}}{4C_7^3 (r + 1)}
$$

for every $p \in (p_c - \delta_4, p_c]$ and $r \geq 1$. Since $G$ is connected and quasi-transitive, it follows straightforwardly that there exist constants $c_6, c_7$, and $c_8$ such that

$$
\mathbb{P}_p (|K_v| \geq n) \geq c_6 \sup_{u \in V} \mathbb{P}_p (|K_u| \geq n) \geq \frac{c_7}{\sqrt{n}}
$$

for every $p \in (p_c - \delta_4, p_c]$ and $1 \leq n \leq c_8 |p - p_c|^{-2}$. The claimed bound (4.6) follows in the case $p \in (p_c - \delta_4, p_c]$ from this together with (4.7).

We now consider the case $p \geq p_c$. Let $\omega_p$ and $\omega_{p_c}$ be Bernoulli-$p$ and Bernoulli-$p_c$ percolation on $G$ coupled in the standard monotone way, so that, conditional on $\omega_{p_c}$, every $\omega_{p_c}$-open edge is $\omega_p$ open and every $\omega_{p_c}$-closed edge is chosen to be either $\omega_p$-open or $\omega_p$-closed independently at random with probability $(p - p_c)/(1 - p_c) = O(p - p_c)$ to be $\omega_p$-open. Let $K^p_{v_c}$ and $K^p_v$ denote the clusters of $v$ in $\omega_{p_c}$ and $\omega_p$, respectively. By (4.3), there exist constants $c_9$ and $C_8$ such that

$$
\mathbb{P} (n \leq |K^p_{v_c}| \leq C_8 n) \geq \frac{c_9}{\sqrt{n}} - \frac{C_8}{\sqrt{\alpha n}}
$$
for every $n \geq 1$ and $\alpha \geq 1$. Taking $\alpha = C_9 := 1 \lor (2C_8/c_9)^2$, it follows that there exists a positive constant $c_{10}$ such that

$$\mathbb{P}(n \leq |K_{\text{p}}^{\text{p}_c}| \leq C_9 n) \geq \frac{c_{10}}{\sqrt{n}} \tag{4.11}$$

for every $n \geq 1$. Let $A_n$ be the event that $n \leq |K_{\text{p}}^{\text{p}_c}| \leq C_9 n$ and let $B_n$ be the event that $n \leq |K_{\text{p}}^{\text{p}_c}| < \infty$. If $A_n$ occurs but $B_n$ does not, then there must exist an $\omega_{\text{p}_c}$-closed edge in the boundary of $K_{\text{p}}^{\text{p}_c}$ that is $\omega_{\text{p}}$-open and whose other endpoint is connected to infinity in $\omega_{\text{p}}$ by an open path that does not visit any vertex of $K_{\text{p}}^{\text{p}_c}$. Conditional on $K_{\text{p}}^{\text{p}_c}$, the probability that any particular edge in the boundary of $K_{\text{p}}^{\text{p}_c}$ has this property is bounded by $(p - p_c)\theta^*(p)/(1 - p_c) = O((p - p_c)^2)$, and it follows by the FKG inequality that there exists a constant $C_{10}$ such that

$$\mathbb{P}(B_n | K_{\text{p}}^{\text{p}_c}) \geq \mathbb{P}(A_n) [1 - 1 \lor (p - p_c)\theta^*(p)/1 - p_c] \geq 1(n \leq |K_{\text{p}}^{\text{p}_c}| \leq C_9 n)e^{-C_{10}(p - p_c)^2 n}, \tag{4.12}$$

where $M$ is the maximum degree of $G$. The claimed bound follows from (4.11) and (4.12) by taking expectations over $K_{\text{p}}^{\text{p}_c}$. \qed

Finally, we prove a lower bound on the tail of the radius of a finite cluster in the supercritical regime under the assumption that $p_c < p_{2\rightarrow 2}$.

**Proposition 4.5.** Let $G$ be an infinite, connected, locally finite, quasi-transitive graph, and suppose that $p_c < p_{2\rightarrow 2}$. Then there exist positive constants $c$ and $C$ such that

$$\mathbb{P}_p(\text{Rad}_{\text{int}}(K_v) \geq r) \geq \mathbb{P}_p(\text{Rad}_{\text{ext}}(K_v) \geq r) \geq \frac{c}{r} \exp(-C|p - p_c|r) \tag{4.13}$$

for every $r \geq 1$ and $p \in (p_c - \delta, p_c + \delta)$.

**Proof of Proposition 4.5.** By Proposition 4.2 there exist positive constants $c_1, C_1$, and $\delta_1$ such that

$$\mathbb{P}_p(\text{Rad}_{\text{int}}(K_v) \geq r) \geq \frac{c_1}{r} \exp(-C_1|p - p_c|r) \tag{4.14}$$

for every $r \geq 1$ and $p \in (p_c - \delta_1, p_c)$. On the other hand, it follows from [36, Proposition 3.2] that

$$\mathbb{P}_p(\text{Rad}_{\text{int}}(K_v) \geq r \text{ and } \text{Rad}_{\text{ext}}(K_v) \leq \ell') \leq 3\|T_p\|_{2\rightarrow 2} \exp\left[-\frac{r}{e\|T_p\|_{2\rightarrow 2}}\right] |B(v, \ell')|^{1/2}$$

for every $0 \leq p < p_{2\rightarrow 2}$ and $r, \ell' \geq 1$. Since $p_c < p_{2\rightarrow 2}$, it follows that there exist constants $c_2, c_3, C_2$, and $\delta_2$ such that

$$\mathbb{P}_p(\text{Rad}_{\text{int}}(K_v) \geq r \text{ and } \text{Rad}_{\text{ext}}(K_v) \leq c_2 r) \leq C_2 e^{-c_3 r} \tag{4.15}$$

for every $0 \leq p \leq p_c + \delta_2$ and $r \geq 1$. It follows in particular that there exist positive constants $c_4$ and $r_0$ such that
\[ P_p(\text{Rad}_{\text{ext}}(K_v) \geq c_2 r) \geq P_p(\text{Rad}_{\text{int}}(K_v) \geq r) - P_p(\text{Rad}_{\text{int}}(K_v) \geq r \text{ and } \text{Rad}_{\text{ext}}(K_v) \leq c_2 r) \geq \frac{c_1}{r} \exp(-C_1 |p-p_c| r) - C_2 e^{-c_3 r} \geq \frac{c_4}{r} \exp(-C_1 |p-p_c| r) \quad (4.16) \]

for every \( p \in (p_c - \delta_2, p_c] \) and \( r \geq r_0 \). This is easily seen to imply (4.14) in the case \( p \in (p_c - \delta_3, p_c] \).

We now treat the supercritical case. Combining the inequality (4.16) with (3.12), an easy argument similar to that of the previous paragraph shows that there exist positive constants \( c_5, C_3, \) and \( C_4 \) such that

\[ P_p(\text{Rad}_{\text{ext}}(K_v) \geq r \text{ and } |K_v| \leq C_3 |p-p_c|^{-1} r) \geq \frac{c_5}{r} \exp(-C_4 |p-p_c| r) \]

for every \( p \in (p_c - \delta_2, p_c) \). We apply a similar coupling argument to the end of the proof of Proposition 4.3, with the important difference that we compare \((p_c + \epsilon)\)-percolation to \((p_c - \epsilon)\)-percolation rather than to \(p_c\)-percolation. Let \( 0 < \epsilon \leq \delta_2, \) let \( p = p_c + \epsilon, \) and let \( q = p_c - \epsilon. \) Let \( \omega_q \) be the Bernoulli-\( p \) and \( \omega_q \) percolation on \( G \) coupled in the standard monotone way, so that, conditional on \( \omega_q \), every \( \omega_q \)-open edge is \( \omega_q \)-open and every \( \omega_q \)-closed edge is chosen to be either \( \omega_q \)-open or \( \omega_q \)-closed independently at random with probability \((p-q)/(1-q) = O(\epsilon)\) to be \( \omega_q \)-open. Let \( K^q_v \) and \( K^p_v \) denote the clusters of \( v \) in \( \omega_q \) and \( \omega_p \), respectively. Let \( \mathcal{K}_r \) be the event that \( K^q_v \) has extrinsic radius at least \( r \) and volume at most \( C_3 \epsilon^{-1} r \), and let \( \mathcal{S}_r \) be the event that \( K^q_v \) is finite and has extrinsic radius at least \( r \). If \( \mathcal{S}_r \) occurs but \( \mathcal{K}_r \) does not, then there must exist an \( \omega_q \)-closed edge in the boundary of \( K^q_v \) that is \( \omega_p \)-open and whose other endpoint is connected to infinity in \( \omega_p \) by an open path that does not visit any vertex of \( K^q_v \). Conditional on \( K^q_v \), the probability that any particular edge in the boundary of \( K^q_v \) that is \( \omega_p \)-open has property bounded by \((p-q)\Theta^*(p)/(1-q) = O(\epsilon^2)\), and it follows by the FKG inequality that there exists a constant \( C_5 \) such that

\[ \mathbb{P}(\mathcal{S}_r \mid K^q_v) \geq 1 - \mathbb{P}(\mathcal{K}_r) \left[ 1 - \left( \frac{p-q}{1-q} \right) \Theta^*(p) \right]^{\frac{M|K^q_v|}{1-q}} \geq 1(\mathcal{S}_r) e^{-C_5 \epsilon r}, \]

where \( M \) is the maximum degree of \( G \) and where we used that \( |K^q_v| \leq C_3 \epsilon^{-1} r \) on the event \( \mathcal{S}_r \) in the second inequality. Taking expectations, it follows that

\[ \mathbb{P}(\mathcal{S}_r) \geq \mathbb{P}(\mathcal{S}_r) e^{-C_5 \epsilon r} \geq \frac{c_5}{r} e^{-(C_4+C_5) \epsilon r} \]

and hence that there exists a constant \( C_6 \) such that

\[ P_p(r \leq \text{Rad}_{\text{int}}(K_v) < \infty) \geq P_p(r \leq \text{Rad}_{\text{ext}}(K_v) < \infty) \geq \frac{c_5}{r} \exp(-C_6 |p-p_c| r) \quad (4.17) \]

for every \( p \in (p_c, p_c + \delta_2) \) and \( r \geq 1 \). This completes the proof. (Note that this argument cannot be applied directly to the intrinsic radius as written due to non-monotonicity issues.)

We now have all the ingredients required to conclude the proofs of our main theorems.

Proof of Theorem 1.1. The upper bound follows from Propositions 3.8 and 4.1, while the lower bound follows from Proposition 4.3.
Proof of Theorem 1.2. The upper bound follows from Propositions 3.5 and 4.1, while the lower bound follows from Proposition 4.5. □

5 | PERSPECTIVES ON THE EUCLIDEAN CASE

In this subsection we discuss the (apparently rather substantial) challenges that remain to extend our analysis from non-amenable graphs to the high-dimensional Euclidean setting, and give some perspectives on how these challenges might be overcome.

Let us begin by stating what is conjectured to be the case. Let $d \geq 7$ and consider the hypercubic lattice $\mathbb{Z}^d$. The conjectured analog of Theorem 1.1 is that there exists $\delta > 0$ such that

$$P_p(n \leq |K| < \infty) \asymp \begin{cases} n^{-1/2} \exp \left[ -\Theta \left( |p - p_c|^2 n \right) \right] & p \in (p_c - \delta, p_c) \\ n^{-1/2} & p = p_c \\ n^{-1/2} \exp \left[ -\Theta \left( \left( |p - p_c|^2 n \right)^{(d-1)/d} \right) \right] & p \in (p_c, p_c + \delta), \end{cases}$$

(5.1)

while the conjectured analog of Theorem 1.2 is that

$$P_p(r \leq \text{Rad}_{\text{int}}(K) < \infty) \asymp n^{-1} \exp \left[ -\Theta \left( |p - p_c| n \right) \right]$$

(5.2)

and

$$P_p(r \leq \text{Rad}_{\text{ext}}(K) < \infty) \asymp n^{-2} \exp \left[ -\Theta \left( |p - p_c|^{1/2} n \right) \right]$$

(5.3)

for all $p \in (p_c - \delta, p_c + \delta)$. Further related questions of interest include the behavior of the truncated two-point function $\hat{\tau}_p(x, y) = P(x \leftrightarrow y, x \leftrightarrow \infty)$, which is conjectured to satisfy

$$\hat{\tau}_p(x, y) \asymp \|x - y\|^{-d+2} \exp \left[ -\Theta \left( |p - p_c|^{1/2} \|x - y\| \right) \right]$$

(5.4)

for all $p \in (p_c - \delta, p_c + \delta)$ and $x, y \in \mathbb{Z}^d$. In particular, it is conjectured that the correlation length $\xi(p)$ satisfies

$$\xi(p)^{-1} := -\lim_{n \to \infty} \frac{1}{n} \log \sup \left\{ P_p(0 \leftrightarrow x, 0 \leftrightarrow \infty) : x \in \mathbb{Z}^d, \|x\| \geq n \right\} \asymp |p - p_c|^{-1/2}$$

(5.5)

for $p \in (p_c - \delta, p_c + \delta)$. (Note that (5.4) would trivially imply (5.5).) Besides their intrinsic interest, a solution to these conjectures may be a necessary prerequisite to understanding invasion percolation, the minimal spanning forest, and random walks on slightly supercritical clusters. See [31, Part IV] for an overview.

At present, the state of these conjectures can be summarized as follows: The $p = p_c$ cases of (5.1) and (5.2) were proven to hold for all quasi-transitive graphs satisfying the triangle condition by Barsky and Aizenman [6] and Kozma and Nachmias [44], respectively. We showed how these statements imply the subcritical cases of the same statements in Propositions 4.1–4.3. Hara and Slade [28] proved via the lace expansion that the triangle condition holds on $\mathbb{Z}^d$ for sufficiently large $d$, as well as for "spread out" models in dimension $d \geq 7$. Around the same time, Hara built upon the methods of [28] to prove the $p \leq p_c$ case of (5.5) under the same hypotheses, that is, that
\(\xi(p) \approx (p - p_c)^{-1/2}\) as \(p \uparrow p_c\). Later, Hara, van der Hofstad, and Slade [29] performed a “physical space” version of the lace-expansion that allowed them to prove the \(p = p_c\) case of (5.4) under the same hypotheses. Kozma and Nachmias [45] then applied this result to prove the \(p = p_c\) case of (5.3). The subcritical case of (5.3) has very recently been established in the independent works [15, 40], with [40] also establishing the upper bound of (5.4) in the subcritical case. In contrast, almost no progress has been made on the slightly supercritical cases of these conjectures.

As we stated in the introduction, we are optimistic that some of the techniques we have developed in this paper will be prove useful to the eventual solution of these conjectures. We now outline some ideas about what such a solution might look like. Note that several of the challenges one would need to overcome to adapt our methods to the high-dimensional Euclidean setting are of a similar nature to those one would need to overcome to solve the more qualitative problems stated in [30, Section 5.3].

1. A good first step would be to find a sharp bound on the negative part of the derivative \(D_{p,n}|K|^k\) for \(p\) slightly supercritical. Such a bound would need to be of order \(C^k(k!)^d/(d-1)|p - p_c|^{-2k}\), but it is unclear what form it should take, presumably being written in terms of some higher truncated moment. A potentially serious difficulty is that it seems that one cannot rely on a worst-case analysis of the expected number of edges connecting some deterministic set \(S\) to infinity off of \(S\), as we did in the proof of Proposition 3.1. Indeed, heuristically, if \(\Lambda_n = [-n,n]^d\) is a box with \(n = \Omega(\xi(p)) = \Omega((p - p_c)^{-1/2})\), then the typical number of edges in the boundary of \(\Lambda_n\) whose other endpoint is connected to \(\infty\) off of \(\Lambda_n\) should be of order \((p - p_c)^{3/2}|\Lambda_n|^{(d-1)/d}\), where \((p - p_c)^{3/2}\) is conjectured to be the order of the probability that the origin is connected to infinity inside a half-space. See [14] for various related rigorous results. This (presumably) worst case bound would be too small to lead to a proof of (5.1), even if one did not have the positive term to contend with. Thus, to bound \(D_{p,n}\) via this approach, one would need to somehow understand how the geometry of large finite clusters in slightly supercritical percolation leads them to have a greater number of pivotal connections to infinity in their boundary than a box of comparable volume would. The techniques developed to understand phenomena such as Wulff crystals in supercritical percolation may be relevant [13].

An alternative approach may be to use the OSSS inequality, due to O’Donnel, Saks, Schramm, and Servedio [55], which has recently been recognized as a powerful tool in the study of percolation and other models following the breakthrough work of Duminil-Copin, Raoufi, and Tassion [19, 20]; see also [38, 40] for applications to the critical behavior of Bernoulli percolation. Briefly, this inequality lets us prove differential inequalities by finding randomized algorithms that determine the value of the function whose expectation we are interested in but which have a low maximum \(\text{revealment}\), that is, a low maximum probability of querying whether any particular edge is open or closed. While this inequality is most powerful as a tool for studying monotone functions, it can also be used to bound the expected total number of pivotal for non-monotone functions, which would mean bounding the sum \(D_{p,n} + U_{p,n}\) in our context. Such a bound would in fact be just as viable in the remainder of our strategy as a bound on \(D_{p,n}\) itself. The difficulty with this approach is to find, say, a low-revealment algorithm determining whether or not the origin is in a large finite cluster. It is unclear how this might be done. One possibility is to use invasion percolation, but this may be putting the cart before the horse; it seems that invasion percolation should be even harder to analyze than slightly supercritical percolation itself.
2. Even if one is able to get good bounds on $D_{p, n}$ or $D_{p, n} + U_{p, n}$, there remains the substantial challenge of getting good upper bounds on $U_{p, n}$ in the manner of (1.10). It is possible that this could be done by methods that are rather similar to what we have done in Sections 3.3–3.5. However, it is likely that, due to the different form of the lower bound on the negative term, one would need to initiate this analysis by proving a version of our skinny clusters estimate in which one could profitably take the radius to be at least a power of the radius rather than a small multiple as we have done here. Bounds of this form are known for Galton–Watson trees [1, 2], but it seems unclear what one could hope to be true for high-dimensional lattices, or how such an estimate might be proven. If such a bound on skinny clusters were found, we are hopeful that an analysis very similar to that performed in Sections 3.4 and 3.5 could be used to derive the higher-order variants of this bound needed to bound $U_{p, n}|K|^k$.

Finally, we remark that, by analogy with our setting, it may be substantially easier to obtain the correct behavior for the intrinsic radius than for the volume.

**GLOSSARY OF NOTATION**

$\simeq$, $\leq$, $\geq$ Equalities and inequalities holding to within positive multiplicative constants depending only on the choice of graph.

$P_p = P^2_p$ The law of Bernoulli-$p$ bond percolation on $G$. Defined in Section 1.

$p_c, p_{2 \to 2}$ The critical probability and $L^2$ boundedness threshold. The critical probability $p_c$ always refers to $p_c(G)$, rather than $p_c$ of any subgraph $H$ of $G$, unless specified otherwise. Defined in Section 1.

$T_p$ The two-point matrix $T_p(u, v) = P_p(u \leftrightarrow v)$. Defined in Section 1.

$K_u$ The cluster of $u$. When $v$ is fixed we often write $K = K_v$. Defined in Section 1.1.

$\text{Rad}_{\text{in}}(K_v)$ The intrinsic radius of $K_v$, that is, the maximum intrinsic distance from $v$ to another point of $K_v$. Defined in Section 1.1.

$\text{Rad}_{\text{ex}}(K_v)$ The extrinsic radius of $K_v$, that is, the maximum extrinsic distance from $v$ to another point of $K_v$. Defined in Section 1.1.

$\nabla_p(v)$ The triangle sum at $v$. Defined in Section 1.1.

$R_v$ Shorthand for $\text{Rad}_{\text{in}}(K_v)$. Defined in Section 2.

$E_v$ The number of edges touching $K_v$. Defined in Section 2.

$B_{\text{in}}(v, n)$ The set of vertices of $K_v$ of intrinsic distance at most $n$ from $v$. Defined in Section 2.

$\partial B_{\text{in}}(v, n)$ The set of vertices of $K_v$ of intrinsic distance exactly $n$ from $v$. Defined in Section 2.

$E_{p, n}$ The expectation truncated when the cluster of $v$ touches more than $n$ edges. Defined in Section 3.1.

$U_{p, n}$ The positive part of the derivative of $E_{p, n}$. Defined in Section 3.1.

$-D_{p, n}$ The negative part of the derivative of $E_{p, n}$. Defined in Section 3.1.

$\text{Br}(v_1, ..., v_k; K_v)$ The number of edges in the subtree of the tree of two-connected components of $K_v$ spanned by the union of the geodesics between the vertices in this tree corresponding to $v_1, ..., v_k$. Defined in Section 3.4.

Definitions of generating functions (all defined in Section 3.4):

$g_k(s, t; G, v, p) = \sum_{d=0}^{\infty} \sum_{b=0}^{\infty} \sum_{x_1, ..., x_k \in V(G)} \sum_{E_v = a, \text{Br}(v, x_1, ..., x_k; K_v) = b} P^2_{p, n}(x_1, ..., x_k \in K_v, E_v = a, \text{Br}(v, x_1, ..., x_k; K_v) = b) e^{sa+tb}.$
\[ F_{k,n}(s, t; G, p) = \sup \{ G_{k,n}(s, t; H, u, p) : H \text{ a subgraph of } G, u \text{ a vertex of } H \}. \]

\[ M_n(s, t, u; G, p) = \sum_{k=0}^{\infty} \frac{u^k}{k!} F_{k+1,n}(s, t; G, p). \]

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