STATISTICAL INFERENCE
WITH
DATA AUGMENTATION
AND
PARAMETER EXPANSION

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Summary

Statistical pragmatism embraces all efficient methods in statistical inference. Augmentation of the collected data is used herein to obtain representative population information from a large class of non-representative population’s units. Parameter expansion of a probability model is shown to reduce the upper bound on the sum of error probabilities for a test of simple hypotheses, and a measure, $R$, is proposed for the effect of activating additional component(s) in the sufficient statistic.

*Some key words:* Collected Data Augmentation; Parameter Expansion; Representative Information; Statistical Pragmatism
1 Introduction

In recent years, response rates in polls have decreased further and a representative sample may not always become available. For example, using random digit dialing (RRD) the response rate decreased from 36% in 1997 to 9% in 2012 (Kohut et al. 2012). According to Wang et al. (2015), it is convenient and cost-effective to collect a very large non-representative sample via online surveys and obtain with statistical adjustments accurate election forecasts, on par with those based on traditional representative polls.

We study initially the problem of obtaining representative information for the population, from a large number of non-representative population’s units with a common attribute, $\mathcal{A}$. This attribute could be, e.g., an account in Facebook or following a celebrity in a Social Network; the latter occurred in practice with Social Voting Advice Applications (Katakis et al., 2014).

Under assumptions occurring also in practice, it is expected that units with attribute $\mathcal{A}$ can provide each additional, accurate information for one of the remaining units in the population without attribute $\mathcal{A}$. Due to the large number of units with attribute $\mathcal{A}$, the so-augmented data-information from all strata can be used to obtain information equivalent to that from a representative sample (Kruskal and Mosteller, 1979).

The second problem studied is model parameter expansion (PX), which is shown to reduce the upper bound on the sum of error probabilities, when testing two simple hypotheses with a test introduced by Kraft (1955). The proof confirms that parameter expansion is “activating” a sufficient statistic with additional component(s) (Rubin, 1997) and the effect of the activation is clarified with the introduced measure, $R$, obtained with a one-parameter expansion. These results explain why the PX-EM algorithm (Liu et al., 1998) converges faster than the EM algorithm (Dempster et al., 1977) and its many variations.

Fisher (1922) introduced in statistical inference the use of a model that is not updated. Essentially all models are wrong but some are very useful (attributed to George Box in Rubin, 2005). The need for improvement of estimation procedures led statisticians to relax the use of the “one and only” assumed model by adopting, for example, the Bayesian model.
averaging approach (see e.g., Hoeting et al., 1999). Expansion of a probability model or the augmentation of collected data improve, respectively, the data’s fit and the estimates of the model’s parameters. Artificial data augmentation has been used in missing value problems (e.g. see Rubin, 1987), to improve the convergence of the EM algorithm (see, e.g., Meng and van Dyk, 1997) and to reduce the mean squared error of $U$-statistics, in particular the unbiased estimates of variance and covariance (Yatracos, 2005). Parameter expansion (Meng and van Dyk, 1997, Liu et al., 1998) and model updated maximum likelihood estimate (MUMLE, Yatracos, 2015) are also examples of deviations from the one-model approach.

Kass (2011) introduced statistical pragmatism. It is a new philosophy which is inclusive of the Bayesian, frequentist and all other efficient approaches in inference. Statistical pragmatism emphasizes the assumptions that connect statistical models with the observed data and it is implemented herein with parameter expansion and augmentation of the collected data. The additional contribution of the latter is that: $a)$ it introduces a new component in the statistical inference set-up: each unit in the population provides, in addition, data for other units, and $b)$ it includes as goal to obtain representative population information when representative units/sample are not available.

## 2 Representative Information from Non-Representative Units with Augmentation of the Collected Data

Accessible units in the population with attribute $\mathcal{A}$ responding to a questionnaire are not representative, e.g., when their age is between 25 and 35, and the collected information is not comparable to that of a random sample. Assumptions are made for attribute $\mathcal{A}$ and the population, allowing to obtain representative population information from these non-representative units.

*Assumption 1-Common Attribute:* Attribute $\mathcal{A}$ is *common* in the population: the number of respondents with attribute $\mathcal{A}$ is large, each respondent has in its immediate environment one associated unit without attribute $\mathcal{A}$, and collectively the associated units come
from all strata.

*Assumption 2- Effective Information:* A large percentage of units with attribute $\mathcal{A}$ have each accurate, questionnaire related information, for its associated unit without attribute $\mathcal{A}$.

**Definition 2.1** Representative population information is equivalent to that obtained from a representative sample from the population.

The Common Attribute and the Efficient Information assumptions guarantee that a large number of units responding to the questionnaire will provide each information for its associated unit, thus information will be obtained from all strata. A representative sample, e.g. random sample, from the so-obtained information will provide representative information for the population.

The degree of accuracy of the information given from the respondent for a unit without attribute $\mathcal{A}$ may be clarified from the respondent’s answers to the questionnaire. Given the large number of respondents, only the most accurate information for associated units without attribute $\mathcal{A}$ will be used. The additional information provided by some of the units may introduce bias, thus bias reduction methods will guarantee the best result.

The proposed method of data augmentation will be used in Voting Advice Applications before the general elections in Spain (December 20, 2015). Note that collected data augmentation differs from the notion of data augmentation introduced in Tanner and Wong (1987) and Gelman (2004).

### 3 Parameter Expansion in Statistical Procedures

For the EM-algorithm and its variations, the observed data model $f(x_{\text{obs}}|\theta)$ and the augmented (called also complete) data model $f(x_{\text{com}}|\theta)$ have the same parameter $\theta$. For the PX-EM algorithm (Liu et al., 1998), $f(x_{\text{com}}|\theta)$ contains another parameter with known value $\eta_0$ and is expanded to a larger model $f_X(x_{\text{com}}|\theta_*,\eta))$ with $\theta_*$ playing the role of $\theta$. 
The complete data model is preserved when \( \eta = \eta_0 \), that is,

\[
f_X(x_{\text{com}} | (\theta_*, \eta_0)) = f(x_{\text{com}} | \theta = \theta_*).\]

In several examples in the same paper it is observed that the PX-EM algorithm is faster than the EM-algorithm and its variations.

Testing simple hypotheses is an elementary but fundamental problem in statistical inference. For example, families of tests of simple hypotheses allow to obtain a consistent estimate, with calculation of rates of convergence of this estimate in Hellinger distance (LeCam, 1986). Improved tests will increase the accuracy of the so-obtained estimate. It will be shown that a test of simple hypotheses is improved with parameter expansion.

In the sequel, omitted domains of integration are determined by the integrands-densities.

**Definition 3.1** For densities \( f, g \) defined on \( \mathbb{R} \) the Hellinger distance, \( H(f, g) \), is given by

\[
H^2(f, g) = \int [f^{1/2}(x) - g^{1/2}(x)]^2 \, dx.
\]

The affinity of \( f, g \) is

\[
\rho(f, g) = \int f^{1/2}(x)g^{1/2}(x) \, dx.
\]

For \( H(f, g) \) and \( \rho \) it holds:

\[
H^2(f, g) = 2(1 - \rho(f, g)) \leq 2; \tag{1}
\]

\( H^2(f, g) = 2 \) if and only if \( \rho(f, g) = 0 \), i.e., if \( f(x)g(x) = 0 \) almost surely.

For a sample \( X_1, \ldots, X_n \) with density either \( f^n \) or \( g^n \), the larger \( H^2(f^n, g^n) \) is the easier it is to determine either the true density of the data or, in parametric models, the true parameter. Consistent testing is guaranteed because \( \lim_{n \to +\infty} H^2(f^n, g^n) = 2 \).

Assume that sample \( X_1, \ldots, X_n \) has density \( f(x | \theta) \) and that a different model parameter, \( \eta \), with known value \( \eta_0 \) is already included in the model. It is shown that the upper bound on the sum of error probabilities of a consistent test introduced by Kraft (1955) for hypotheses

\[
H_0 : \theta = \theta_0 \text{ and } H_1 : \theta = \theta_1, \tag{2}
\]
is reduced with parameter expansion.

Let \( t_{n,1} = t_1 \) be the sufficient statistic for \( \theta \) with density \( g(t_1|\theta) \). Kraft (1955) provided the consistent test
\[
\phi_n = I\left(\frac{\sqrt{g(t_1|\theta_1)}}{\sqrt{g(t_1|\theta_0)}} > 1\right);
\]
where \( I \) denotes the indicator function. Note that the criticisms of the classical \( \alpha \)-level test for the preferential treatment of \( H_0 \) and the predetermined level of the test do not hold for test (3).

The error probabilities when using \( \phi_n \) are
\[
E_{H_0}\phi_n \leq \int_{\sqrt{g(t_1|\theta_1)} > \sqrt{g(t_1|\theta_0)}} \sqrt{g(t_1|\theta_1)g(t_1|\theta_0)} dt_1,
\]
\[
E_{H_1}(1 - \phi_n) \leq \int_{\sqrt{g(t_1|\theta_0)} \geq \sqrt{g(t_1|\theta_1)}} \sqrt{g(t_1|\theta_1)g(t_1|\theta_0)} dt_1.
\]
Then, the sum of error probabilities
\[
E_{H_0}\phi_n + E_{H_1}(1 - \phi_n) \leq \int \sqrt{g(t_1|\theta_1)g(t_1|\theta_0)} dt_1.
\]

Lower values of the affinity \( \int \sqrt{g(t_1|\theta_1)g(t_1|\theta_0)} dt_1 \) indicate an increase in the separation of the densities \( g(t_1|\theta_0) \) and \( g(t_1|\theta_1) \), making easier to distinguish between the two hypotheses.

Consider the expanded model \( f(x|\theta, \eta) \) with the new sufficient statistics \( (t_1, t_{2,n} = t_2) \) that have joint density
\[
h(t_1, t_2|\theta, \eta) = g(t_1|\theta, \eta)\tilde{g}(t_2|t_1, \theta, \eta).
\]
Note that \( g(t_1|\theta, \eta = \eta_0) = g(t_1|\theta) \) and that \( \tilde{g} \) is the conditional density of \( t_2 \) given \( t_1 \).

For the expanded model (5) and the hypotheses testing problem
\[
H'_0 : \theta = \theta_0, \eta = \eta_0 \text{ and } H'_1 : \theta = \theta_1, \eta = \eta_0,
\]
statistic \( t_2 \) is “activated” and a consistent test for these hypotheses similar to \( \phi_n \) is
\[
\psi_n = I\left(\frac{\sqrt{g(t_1|\theta_1)\tilde{g}(t_2|t_1, \theta_1, \eta_0)}}{\sqrt{g(t_1|\theta_0)\tilde{g}(t_2|t_1, \theta_0, \eta_0)}} > 1\right).
\]
For the sum of error probabilities of $\psi_n$ it holds
\[ E_{H_0} \psi_n + E_{H_1} (1 - \psi_n) \leq \int \int \sqrt{g(t_1 | \theta_1, \eta_0) g(t_1 | \theta_0, \eta_0)} \sqrt{\tilde{g}(t_2 | t_1, \theta_1, \eta_0) \tilde{g}(t_2 | t_1, \theta_0, \eta_0)} dt_2 dt_1. \quad (7) \]

When $\tilde{g}$ depends on $\theta$ and $t_1$ is fixed, it follows that $\tilde{g}(t_2 | t_1, \theta_1, \eta_0)$ and $\tilde{g}(t_2 | t_1, \theta_0, \eta_0)$ are not equal a.s. $t_2$ and from Cauchy-Schwarz inequality for integrals
\[ \int \sqrt{\tilde{g}(t_2 | t_1, \theta_1, \eta_0)} \sqrt{\tilde{g}(t_2 | t_1, \theta_0, \eta_0)} dt_2 < 1, \quad (8) \]
for every $t_1$.

Using Fubini’s theorem in the right side of (7) and (8) it follows that
\[ \int \int \sqrt{g(t_1 | \theta_1, \eta_0) g(t_1 | \theta_0, \eta_0)} \sqrt{\tilde{g}(t_2 | t_1, \theta_1, \eta_0) \tilde{g}(t_2 | t_1, \theta_0, \eta_0)} dt_2 dt_1 < \int \sqrt{g(t_1 | \theta_1) g(t_1 | \theta_0)} dt_1. \quad (9) \]

From (4), (8) and (9) it follows that the upper bound on the sum of error probabilities in (7) for the expanded model is smaller than that of the original model.

**Definition 3.2** The effect of activating component $t_2$ in the sufficient statistic of the expanded model (5) with respect to model (2) is measured by the difference of the affinities,
\[ R = \int \sqrt{g(t_1 | \theta_1) g(t_1 | \theta_0)} dt_1 - \int \int \sqrt{h(t_1, t_2 | \theta_1, \eta_0) h(t_1, t_2 | \theta_0, \eta_0)} dt_1 dt_2; \quad (10) \]

$R$ is proportional to the difference of Hellinger distances of the models in (2) and in (5).

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