On the von Neumann rule in quantization

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Abstract

We show that any linear quantization map into the space of self-adjoint operators in a Hilbert space violates the von Neumann rule about post-composition with real functions.

1 Introduction and main results

Physics has the ambition to be entirely mathematically derivable from two fundamental theories, gravity and the standard model of particle physics. Whereas the former is a classical field theory with only mildly paradoxical features such as black holes, the latter is a toolkit full of complex algorithms, ill-defined objects and philosophical mysteries, such as the measurement problem. Nevertheless, it is very successful if used by experts insofar as its predictions are in accordance with a large class of experiments to an unprecedented precision. It uses, via canonical quantization, a quantization map $Q : C^0(C) \supset G \to \text{LSA}(H)$ from a space $G$ of classical observables, where $C$ is the classical phase space (usually diffeomorphic to the space of solutions) and LSA$(H)$ the space of linear self-adjoint maps of a Hilbert space $H$ to itself. There is an established list of desirable properties such a map going back to Weyl, von Neumann and Dirac ([22], [17], [6]):

1. $Q$ is $\mathbb{R}$-linear (in particular, $G$ is a real vector space);

2. $Q$ is unit-preserving, i.e. $Q(1) = I_H$ where $I_H$ is the identity in $H$;

3. von Neumann rule: $Q$ is invariant under postcomposition with smooth maps $\mathbb{R} \to \mathbb{R}$, i.e. for all $f \in G, \psi \in C^0(\mathbb{R}, \mathbb{R})$ we have $\psi \circ f \in G$ and $Q(\psi \circ f) = \psi(Q(f))$ in the sense of functional calculus;

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4. \( \exists p, q \in G \exists c \in i \mathbb{R} : [Q(p), Q(q)] = cI_H \) (canonical commutation).

The last item is weaker than the assignment used in canonical quantization where \( C \) is a space of sections of a bundle \( \pi : E \to N \) whose fiber is the (co-)tangent space of a manifold with local adapted coordinates \((x_i, p_i)\), \( H \) is some space of complex (polarized) functions on \( C \), and for a function \( u \) on \( N \), \( Q(u \cdot x_i) \) is the operator of multiplication with \( ux_i \) whereas \( Q(u \cdot p_i) \) is the closure of \( u \cdot \partial_i \) (modulo the correspondence between vector fields along a function \( f \) and vectors at a function within the space of functions). Often, this assignment is first defined in the context of quantum mechanics, i.e., for \( N \) being a point, and in the limit of \( u \) tending to a delta distribution, and only later transferred to quantum field theory. A related requirement is that \( Q \) should be a Poisson representation in the sense that it takes the Poisson bracket to an imaginary multiple of the commutator.

The motivation for the von Neumann rule is that measuring \( f \) is the same as measuring \( \psi \circ f \), and the effect of \( \psi \) amounts to a mere relabelling of the scale of the measuring apparatus, if we recall that measuring a quantity simply means coupling a macroscopic quantity homeomorphically to it. If someone changes the scale of a measurement apparatus, applying to it a map \( \phi : \mathbb{R} \to \mathbb{R} \). The modified apparatus should still extract the same exact amount of information from the system, and this is precisely what is encoded in the von Neumann rule, at least if \( \phi \) is a homeomorphism.

To show how deeply the Neumann property is rooted in the axioms of quantum theory, we include the observation that the Neumann property follows from the Born rule (which in turn, via Gleason’s theorem, follows from the probabilistic interpretation of Hilbert space geometry, where projections correspond to ‘yes/no’-questions with ‘and’ corresponding to the intersection, ‘or’ to the closed linear span, ‘not’ to the orthogonal complement):

Assuming the Born rule (which, as a physical statement, contains the mathematically undefined term ‘measurement’), the probability \( p(f, \lambda, v) \) of measuring \( \lambda \) for a classical observable \( f \), if the system is in the state \( v \), is \( \langle v, P_{Q(f), \lambda}v \rangle \), where, for an operator \( A \), \( P_{A, \lambda} \) is the orthogonal projection onto the eigenspace \( E_{A, \lambda} := \ker(A - \lambda I_H) \) of \( A \) to the eigenvalue \( \lambda \). Now, because measuring \( f \) is the same as measuring \( \psi \circ f \), we have \( p(f, \lambda, v) = p(\psi \circ f, \psi(\lambda), v) \), thus, for all \( v \in H \),

\[
\langle v, P_{Q(f), \lambda}v \rangle = \langle v, P_{Q(\psi \circ f), \psi(\lambda)}v \rangle,
\]

so \( P_{Q(f), \lambda} = P_{Q(\psi \circ f), \psi(\lambda)} \) by polarization, and the Neumann rule follows.

Of course, one should additionally ask for other properties such as continuity and functoriality of \( Q \) in an appropriate category. But unfortunately, already
the four properties above cannot be satisfied at once by the same map. The proof of that fact goes back to Arens and Babbitt [2] and Folland [12], see also the excellent review article by Ali and Englisch [1]. Englis also obtained the remarkable result [9] that with canonical quantization as above (i.e., where $Q$ maps $x_j$ to the operator of multiplication with $x_j$ and $p_j$ to a multiple of the closure of $i \cdot \partial_j$), there is no Neumann map $Q$ which is a Poisson representation, without assuming linearity of $Q$ or even of $Q(f)$!

To the best of the author’s knowledge, all quantization schemes so far try to satisfy the von Neumann property only approximately, e.g., modulo higher orders of $\hbar$. But if we assume quantum theory to be a fundamental theory, the exact validity of the Neumann rule is central, as explained above. One could hope that it is possible to conversely satisfy the von Neumann property exactly at the expense of the canonical commutation relation, which then can be satisfied only approximately. This note shows that this kind of approach is doomed to failure. We first note that the von Neumann rule implies that the domain $G$ of $Q$ is a representation space for $LD_0$, where for $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, $LD_k$ is the monoid $\{f \in C^k(\mathbb{R},\mathbb{R})| f$ is a $C^k$-diffeomorphism onto its image$\}$. Here, a $C^0$‐diffeomorphism is just a homeomorphism. Conversely, for a representation space $G$ of $LD_k$ let us call a map $Q : G \to LSA(H)$

- **$LD_k$-Neumannian** iff for all $f \in G$ and all $\psi \in LD_k$ we have $Q(\psi \circ f) = \psi(Q(f))$ in the sense of functional calculus;

- **Abelian** iff $Q(G)$ is an Abelian subalgebra of $LSA(H)$;

- **local** iff $H$ is a Sobolev space of sections of a Hermitean bundle $\pi$ over a Fréchet manifold $F$ equipped with a Borel measure and $Q(f)|_{\Gamma_{C^\infty}(\pi)}$ does not increase supports for all $f \in G$.

Clearly any $LD_0$-Neumannian map is $LD_\infty$-Neumannian and thus $LD_\omega$-Neumannian. The motivation for locality is the guiding idea of Geometric Quantization and other quantization schemes to interpret a quantum state as a superposition, more precisely, a polarized complex probability distribution on the set of classical states, so that in this case $F = C$. This anchoring in spacetime is an aspect sometimes neglected by the abstract operator algebra formulation, but as there is only one isomorphism class of separable Hilbert spaces, in this case all physical information is not in the space itself but in its identification with probability densities located in spacetime. However, this notion of locality is stronger than the spacetime notion of locality linked to functoriality of quantization as in, for example, [3] or [10].
The results of this article are:

**Theorem 1** Every $\mathbb{R}$-linear $LD_\omega$-Neumannian map is Abelian.

(of course, $\mathbb{R}$-linearity of $Q$ presupposes that $G$ is a real vector space), and

**Theorem 2** If $F$ is a Hilbert manifold, then any local (not necessarily linear) $LD_0$-Neumannian map is Abelian.

**Remark.** The article [1] gives a similar statement as Theorem 1 without proof, referring apparently to [8], where a proof is given on the additional basis of Assumption 4 of our list above (existence of two quantum operators satisfying the canonical commutation relation).

As noncommutativity is fundamental for every quantum theory in the sense that the order of measurements changes the result in a statistically reliable way and taking into account the importance of the Neumann property, these theorems mean that any physically exactly valid quantization map is neither linear nor local on a Hilbert manifold. One possible way to construe the results is the view that quantization as in the Copenhagen interpretation should not be over stretched but seen as a merely heuristic device.

## 2 Is quantization the right concept?

The concept of quantization, despite its success in the standard model of particle physics, is sometimes subjected to the criticism that a truly fundamental structure should rather be a map in the reverse direction. This goes under the name ‘dequantization’. Even in several quantization schemes, inverses of the respective quantization map play a certain role, e.g. the Wigner transform in Weyl quantization ([13], [3]) and the Berezin symbol in Berezin-Toeplitz quantization ([8], [20]) (note that in Geometric Quantization, a simple computation shows that for the quantization map $Q$ of Geometric Quantization and for $\sigma$ being the principal symbol of a differential operator, we get $\sigma \circ Q(f) = sgrad(f)$, the symplectic gradient of $f$). More generally one could even allow for quantization relations instead of quantization maps. But this would make a difference only if there were two measurement devices ‘measuring the same classical quantity’ (in the classical decoherence regime) but could be represented by two different operators in a Hilbert space in a systematical way. Whether this is the case seems to be unknown at present ([7]). The canonical commutation relations suggest the opposite: that the commutator of *every* measurement apparatus
associated to classical momentum and every measurement apparatus associated to classical position is a multiple of the identity. In the context of quantization relations, for the same reasons as before, a relational Neumann property should hold, stating that for any apparatus with quantum operator $A$ related to an observable $f$ and any $\phi \in \text{LD}_0$ there is an apparatus with quantum operator $\phi(A)$ related to the classical quantity $\phi \circ f$. Now, if two momentum measurement devices that yield identical results in the classical decoherence regime but are represented by two different operators $P$ and $\tilde{P}$, assume that $[P, \tilde{P}] = 0$ and $[P, X] = c I_H = [\tilde{P}, X]$. Let us consider $R := \arctan(P - \tilde{P})$. Then $R$ is bounded and $[R, X] = 0 = [R, P]$. The relational Neumann property implies that $R$ commutes with a family of operators related to every classical quantity (by well-known Weierstraß-like theorems, see e.g. [3]), which by the usual assumption of irreducibility means that $R$ is a constant, i.e. $\tilde{P} = P + k I_H$, contradicting the assumption that $P$ and $\tilde{P}$ coincide classically. Thus anyone in favor of replacing quantization with dequantization should try to find two momentum measurement devices either not commuting with each other or at least one of which does not have commutator $c I_H$ with position. This effect should appear in any quantum theory without a quantization map, e.g. in objective collapse theories.

3 Proof of the main results

Proof of Theorem 1. Any linear Neumann map is unit-preserving, as for $1$ being the constant unit observable and $\phi \in \text{LD}_\omega$ with $\phi(\mathbb{R}) \subset (1/2, \infty)$ and $\phi(1) = 1$ we have $Q(1) = Q(\phi \circ 1) = \phi(Q(1))$, thus the spectrum of $Q(1)$ is in $(1/2; \infty)$. We would like to use squaring of operators, which is not represented by postcomposition with an injective map. However, we can extend $\{(x, x^2) | x \in [1/2; \infty)\}$ to some diffeomorphism $q \in C^\infty(\mathbb{R}, \mathbb{R})$ onto its image. Thus the spectrum condition allows to conclude $Q(1) = Q(q \circ 1) = Q(1) \circ Q(1)$, thus $Q(1)$ is a projection, which together with the condition of positive spectrum means that $Q(1) = I_H$. Now we pick two observables $a, b \in G$ whose quantizations $Q(a_0) =: A_0$ and $Q(b_0) =: B_0$ do not commute. First of all, we replace $a_0$ with $a := (\arctan + \pi) \circ a_0$ and correspondingly for $b$, obtaining two operators $A := Q(a)$ and $B := Q(b)$ with spectrum in $(\pi/2; \infty)$. We still have $[A, B] \neq 0$: One can e.g. invoke von Neumann’s theorem on the generating operator stating that if $A$ is a set $K$ of self-adjoint operators on a Hilbert space that commute with each other, there is a self-adjoint operator $A$ such that $B = f(A)$ for all $B \in K$. Or else the statement also follows e.g. from the useful formulas in [19] for commutators.

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with functions of operators that in turn follow from the Helffer-Sjöstrand formula. As above, the condition on the spectrum of $a$ and $b$ allows for an application of the Neumann rule for squaring of $a$, $b$ and $a + b$. Now for $a, b$, as for any other pair of observables, we have arithmetically

$$\left(\frac{(a + b)^2 - a^2 - b^2}{2}\right)^2 = (ab)^2 = a^2b^2 = \frac{(a^2 + b^2)^2 - a^4 - b^4}{2}, \quad (1)$$

thus if we apply to both sides the linearity of $Q$ and the von Neumann rule applied to $q$, we obtain $\frac{1}{2}(AB + BA)^2 = \frac{1}{2}(A^2B^2 + B^2A^2)$, so we have

$$S := (AB + BA)^2 - 2(A^2B^2 + B^2A^2) = 0, \quad (2)$$

Now, to get an idea of the proof, let us first assume the existence of an eigenvector $v$ of $B$ to the eigenvalue $\lambda$ (keep in mind however that there are bounded self-adjoint operators without eigenvalues, e.g. the multiplication with the function $x \mapsto x$ in $L^2([0, 1])$). If we assume $A$ to be diagonalizable, then, as $[A, B] \neq 0$, there is an eigenvector $v$ of $B$ with $Av \notin \ker(B - \lambda)$. Using self-adjointness of $A$ and $B$ and writing $w := Av$, we get

$$\langle Sv, v \rangle = \langle Bw, Bw \rangle + 2\lambda\langle Bw, w \rangle - 3\lambda^2\langle w, w \rangle,$$

and this can be made nonzero by replacing $B$ with $\phi(B)$ for $\phi: \mathbb{R} \to \mathbb{R}$ with $\phi(\lambda) = \lambda$, which does not change $w$ or $\lambda$ in the calculation above.

In the general case, we examine the operator norm of $P \circ S \circ P$ for $P := \mu_B(U)$.

We have $P \circ B = P \circ B \circ P = B \circ P$. Let $B_t := (Id + t \cdot \chi_{\mathbb{R} \setminus U})B$, then $B_0 = B$ and $P \circ B_t = P \circ B \circ P = B \circ P$ for all $t \in \mathbb{R}$. With the same argument as above, we get $S_t := (AB_t + B_tA)^2 - 2(A^2B_t^2 + B_t^2A^2) = 0$, but

$$|PS_tP| = |PAB_tAB_tP + PAB_tB_tA^2BP + PAB_tB_tA^2BP + PAB_tB_tA^2BP - 2PA^2B_t^2P - 2PAB_t^2A^2P|$$

$$= |PAB_tAB_tP + B_tPABtAP + PAB_tB_tA^2BP + PAB_tB_tA^2BP + B_tPABtAP - 2PA^2PB_t^2B - 2B_t^2PA^2B|$$

$$\geq \frac{|PABtB_tA^2P| + 2|B_tAP| \cdot |B_tPA| - 3|B_tPA|^2}{|B_tAP|^2} \to_{t \to \infty} \infty$$

where we use self-adjointness of $B_t$, $A$, $P$, the formulas $|W| = |W^\dagger|$, $|WW^\dagger| = |W|^2$, $|B_tPA| = |BPA|$. Finally, $\lim_{t \to \infty} |B_tAP| = \infty$, as there is some Borel

\[^{1}\text{Paralleling the proof above more strictly by considering not }|PS_tP| \text{ but the quantity } \sup\{|(P \circ S_t \circ P)(v), v) : v \in H, |v| = 1 \} \text{ would be a bit more complicated.}\]
$U \subset \mathbb{R}$ such that $P^\perp AP \neq 0$ for $P := \mu_B(U)$: Assume the opposite, then due to self-adjointness we have $P_0 P U = 0 = P_U A P_U^\perp$ and thus $[P_U, A] = 0$ for all $U$. Therefore $[A, B] = 0$ as $B = \int \mathbb{R} I_n(x) d\mu_B(x)$. □

As an auxiliary theorem for our second result, we will need an infinite-dimensional version of Peetre’s theorem. As it seems to be difficult to find a written reference in the literature, we include a full proof in the next section, which, nevertheless, goes very closely along the lines of the original proof.

**Theorem 3 (Peetre’s Theorem for Hilbert manifolds)** Let $M$ be a manifold modelled on a Hilbert space and let $\pi : E \to M$ and $\psi : F \to M$ be smooth Banach vector bundles over $M$. Let $L : \Gamma_{C^\infty}(\pi) \to \Gamma_{C^\infty}(\pi)$ be a morphism of sheaves that is support-nonincreasing, i.e. $\text{supp}(Ls) \subset \text{supp}(s)$ for all $s \in \Gamma_{C^\infty}(\pi)$. Then for all $p \in M$ there is an open neighborhood $U$ of $p$ and there is $k \in \mathbb{N}$ such that $L|_U$ is a differential operator of order $k$, i.e. there is a vector bundle homomorphism $u : J^k \pi \to \psi$ with $L|_U = u \circ \partial^k \circ r_U$, where $r_U$ is the restriction of a section to $U$.

**Proof of Theorem** Let $f \in G$, then Peetre’s Theorem above implies that in a small neighborhood $U$, $Q(f)$ is a differential operator of, say, order $k$. The odd root $x \mapsto 2k+1\sqrt{x}$ is in $\text{LD}_0$. As the order of a differential operator is multiplicative under taking powers, $Q(2k+1\sqrt{f})$ is a $(2k+1)$-th root of $Q(f)$ and so cannot be a differential operator, not even in a smaller neighborhood, contradiction. □

Considering the arguments given above in favor of an exact validity of the von Neumann rule, it appears worthwhile to think of alternatives to linearity of the quantization map, e.g. in the spirit of the proposals of Kibble [16] and Weinberg [21] elaborated further by Polchinski [18] and Jordan [14] (however, in those approaches not only $Q$ is nonlinear, but also the $Q(f)$ are, and there does not seem to be a good suggestion for how to replace the Born rule in this context. It is interesting that already decades ago Wigner [23] concluded from a gedanken experiment (in a certain double sense) in the context of the measurement problem that any truly fundamental quantum theory cannot be linear, without considering the von Neumann property.

\footnote{Another direction would be to discard the fundamental role of the — bosonic—commutator and replace them with — fermionic — anticommutators, trying to treat bosonic degrees of freedom in the quantized picture as secondary, emergent object (a possible limitation of this approach is the result by Kapustin [15] for finite-dimensional systems). For a non-quantization version of this idea, see [11].}
4 Peetre’s theorem for Hilbert manifolds

Proof of Theorem. As hypothesis and conclusion of the theorem are invariant under composition with trivializations (being local diffeomorphisms), it suffices to show the statement for $M$ an open subset of a Hilbert space $Z$ and trivial Banach vector bundles of fibers $V$ resp. $W$. Let us, for $x \in M$, denote by $N_x$ the set of open neighborhoods of $x$.

Lemma 1 Assume the hypothesis of the theorem, then:

$$\forall x \in M \forall C > 0 \exists U \in N_x \exists k \in \mathbb{N} \forall y \in U \setminus \{x\} \forall s \in C^\infty(U, V) : (j^k s)(y) = 0 \Rightarrow |Ls(y)| < C.$$ 

Proof of the Lemma. Assume the opposite, then there is a sequence $y \in M$ in $M$ with $\lim_{n \to \infty} (y_n) = x$ and a sequence $r \in (0; \infty)$ of radii such that, for $B_k := B(y_k, r_k)$, we have $\text{cl}(B_k) \cap \text{cl}(B_l) = \emptyset \forall k \neq l$, and there are $s_k \in C^\infty(M, V)$ with $(j^k s_k)(y_k) = 0$ and $|Ls_k(y_k)| \geq C > 0$. We want to produce a contradiction by evaluating separately at the even and at the odd points the image under the operator of a carefully chosen section. Let $a \in C^\infty(Z, [0; 1])$ with $a(B(0, 1/2)) = \{1\}$ and $a(Z \setminus B(0, 1)) = \{0\}$ with $\sum_{j=0}^{k} \sup\{|d^j a(x)| : x \in Z\} =: E_k < \infty$; such an $a$ can easily be constructed by radial invariance. For all $k \in \mathbb{N}$ we have $(j^{2k} s_{2k})(y_{2k}) = 0$, and the mean value theorem applied to $|d^j s_{2k}| \circ c$ for a radial curve $c$ implies that there is $\rho_{2k} \in (0; r_{2k})$ such that for all $\delta \in (0; \rho_{2k})$ we have

$$\sum_{|j| < k} \sup\{|d^j s_{2k}(y)| : y \in B(y_{2k}, \delta)\} \leq \frac{1}{N_k} \left[\frac{\delta}{2}\right]^k.$$ 

With $a_{2k, \delta} : Z \to [0; 1], a_{2k, \delta}(z) := a\left(\frac{z - y_{2k}}{\delta}\right)$ we get

$$\max_{j \leq k} \sup\{|d^j a_{2k, \delta}(z)| : y \in B(y_{2k}, \delta)\} \leq 2^{-k}.$$ 

By comparison with the geometric series and uniform convergence we see that $q : z \mapsto \sum_{k=0}^{\infty} a_{2k}(z) \cdot s_{2k}(z)$ is a smooth function from $Z$ to $V$. As $s_{2k}|_{B(y_{2k}, \delta/2)} = a_{2k, \delta} \cdot s_{2k}|_{B(y_{2k}, \delta/2)}$, we get $\lim_{k \to \infty} |Lq(y_{2k})| \geq C$, and continuity of $Lq$ implies

$$|Lq(x)| \geq C > 0. \quad (3)$$
On the other hand, tracing the odd points we get \( Lq(y_{2k+1}) = 0 \) as \( q|_{B_{2k+1}} = 0 \) and \( \text{supp} Lq \subset \text{supp} q \subset \mathbb{Z} \setminus B_{2k+1} \). Continuity of \( Lq \) implies \( Lq(x) = 0 \), in contradiction to Eq. 3.

\[ \square \]

Lemma 2  Assume the hypothesis of the theorem, then:

\[
\forall x \in M \exists \exists k \in \mathbb{N} \forall y \in U \forall s \in C^\infty(U, V) : 
(j^k s)(y) = 0 \Rightarrow Ls(y) = 0.
\]

Proof of the Lemma: Fix \( x \in M \) and \( C > 0 \), then there are \( U \) and \( k \) as in Lemma 1. Assume that there is a \( y \in U \setminus \{x\} \) with \( j^k s(y) = 0 \) and \( |Ls(y)| = b > 0 \). Then consider \( \tilde{s} := \frac{2C}{b} \cdot s \in C^\infty(U, V) \), then \( j^k \tilde{s}(y) = 0 \) and \( |L\tilde{s}(y)| = 2C > C \), in contradiction to Lemma 1. Finally, \( Ls(x) = 0 \) holds by continuity of \( Ls \).

\[ \square \]

Proof of the theorem, ctd.: Now, for \( U, k \) as in Lemma 2, \( y \in U \) and \( b \in J^k \pi y \), there is a map \( s \in C^\infty(U, V) \) with \( b = j^k s(y) \), and we define \( u((j^k s)(y)) := Ls(y) \), which is well-defined due to Lemma 2.

\[ \square \]

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