A GENERALIZED TETRAHEDRAL PROPERTY FOR SPACES WITH CONICAL SINGULARITIES

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Abstract. In this article we extend the definition of Sormani’s Tetrahedral Property so that conical metric spaces satisfy our new definition. We prove that our generalized definition retains all the properties of the original tetrahedral property proven by Portegies-Sormani: it provides a lower bound on the sliced filling volume and Gromov filling volume of spheres, and a lower bound on the volumes of balls. Thus, sequences with uniform bounds on our generalized tetrahedral property also have subsequences which converge in both the Gromov-Hausdorff and Sormani-Wenger Intrinsic Flat sense to the same limit space.

1. Introduction

We start by recalling the tetrahedral property which was previously defined in [6] and [7]. Let \((X, d)\) be a metric space. Here, \(B_r(p)\) denotes the open metric ball of radius \(r\) centered at \(p\) and \(\bar{X}\) the metric completion of \(X\). Let \(S(p; r) = \{x \in X | d(x, p) = r\}\) and \(S(x_1, \ldots, x_j; t_1, \ldots, t_j) = \bigcap_{i=1}^j S(x_i; t_i)\). If \(A\) is a set we denote its cardinality by \(|A|\).

Definition 1.1. [Sormani] Let \(C > 0\) and \(\beta \in (0, 1)\). A metric space \((X, d)\) has the \(n\)-dimensional \((C, \beta)\)-tetrahedral property at a point \(p\) for radius \(r\) if there exist points \(p_1, \ldots, p_{n-1} \in \bar{X}\) such that \(d(p, p_i) = r\) and for all \((t_1, \ldots, t_{n-1}) \in [(1-\beta)r, (1+\beta)r]^{n-1}\)

\[
h(p, r, t_1, \ldots, t_{n-1}) \geq Cr,
\]

where

\[
h(p, r, t_1, \ldots, t_{n-1}) = \begin{cases} \inf \{d(x, y) | x \neq y, x, y \in S\} & |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
S = S(p, p_1, \ldots, p_{n-1}; r, t_1, \ldots, t_{n-1})
\]
We note that, in general, $C, \beta$ and $r$ depend on $p \in X$. Whenever this might cause confusion we will make explicit mention of the dependence.

We say that $X$ satisfies the tetrahedral property if it satisfies the $(C, \beta)$-tetrahedral property at every point for the same $C$, $\beta$ and radius $r$.

The only explicit examples given in the literature that satisfy the tetrahedral property are Example 2.1, Example 2.2 and Example 2.3 in [7]. They are: the three dimensional Euclidean space, tori of the form $S^1 \times S^1 \times S^1_{\varepsilon}$ where the radius for which the property is satisfied goes to zero as $\varepsilon$ goes to zero, and two copies of Euclidean space with a large collection of tiny necks between corresponding points. (The first two also appear in [6] as Examples 3.31 and Example 3.32).

Cones over Alexandrov spaces might not satisfy the $(C, \beta)$-tetrahedral property at the tip (see Proposition 3.6). By modifying Definition 1.1, manifolds with conical singularities and cones over Alexandrov spaces which failed to satisfy the $(C, \beta)$-tetrahedral property at the tip might satisfy the $(C, \alpha, \beta)$-tetrahedral property at their tips (see Example 3.2). We present the aforementioned modification.

**Definition 1.2** ($(C, \alpha, \beta)$-tetrahedral property). Let $C > 0$ and $\alpha, \beta \in (0, 2)$, $\alpha < \beta$. A metric space $(X, d)$ satisfies the $n$-dimensional $(C, \alpha, \beta)$-tetrahedral property at a point $p$ for radius $r$ if there exist points $p_1, \ldots, p_{n-1} \in \overline{X}$ such that $d(p, p_i) = r$ and for all $(t_1, \ldots, t_{n-1}) \in [\alpha r, \beta r]^{n-1}$ the following holds
\[ h(p, r, t_1, \ldots, t_{n-1}) \geq C r, \]
where $h$ is given as in Definition 1.1.

With this new definition, the lower bound estimate on the volume of balls and the Gromov-Hausdorff and intrinsic flat convergence theorems proven by Portegies-Sormani for manifolds that uniformly satisfy the $(C, \beta)$-tetrahedral property also hold (See Theorem 3.39, Theorem 3.41 and Theorem 5.2 in [6], cf. Theorem 2.3 and Theorem 2.7).

**Theorem A.** Let $M$ be a compact oriented Riemannian manifold possibly with boundary. Suppose that $p \in M$ and $B_R(p) \cap \partial M = \emptyset$. For almost every $r \in (0, R)$, if the $n$-dimensional $(C, \alpha, \beta)$-tetrahedral property at a point $p$ for radius $r$ holds then
\[ \text{Vol}(B_r(p)) \geq C(\beta - \alpha)^{n-1} r^n. \]

**Theorem B.** Let $r_0 > 0$, $0 < \alpha < \beta < 2$, $C > 0, V_0 > 0$ and $M_i$ a sequence of $n$-dimensional compact oriented and connected Riemannian manifolds (with no boundary) that satisfy
\[ \text{Vol}(M_i) \leq V_0 \]
and the $M_i$ satisfy the $n$-dimensional $(C, \alpha, \beta)$-tetrahedral property at all $p \in M_i$ for all radius $r$, $r \leq r_0$. Then a subsequence of the $M_i$ converge in Gromov-Hausdorff sense. In particular, there exists $D_0(C, \alpha, \beta, r_0, V_0) > 0$ such that $\text{diam}(M_i) \leq D_0$. 

Before stating Theorem C we recall that the Intrinsic Flat distance was developed by Sormani-Wenger in [9]. The Intrinsic Flat distance between two $n$-dimensional integral current spaces $(X_i, d_i, T_i)$ is an analogue of the Gromov-Hausdorff distance between metric spaces. That is, it is the infimum of the flat distances of isometric orienting preserving images of $(X_i, d_i, T_i)$. Here, an $n$-dimensional integral current space $(X, d, T)$ consist of an $H^n$ countably rectifiable metric space $(X, d)$ and an $n$-dimensional integral current structure $T$ on $X$ as defined by Ambrosio-Kirchheim in [1]. We recall that these are the generalization to metric spaces of the currents used by Federer and, Federer and Fleming, [3] [4].

Neither the Gromov-Hausdorff, nor the Intrinsic Flat convergences imply the other one, but when a sequence converges with both distances, then the intrinsic flat limit is either the zero integral current space or it is contained in the Gromov-Hausdorff limit space [9]. We note that the class of $n$-dimensional precompact integral current spaces with uniform lower bound on their mass and the mass of their boundaries and, an upper bound on the diameter is compact under the intrinsic flat distance [11].

**Theorem C.** Let $r_0 > 0$, $0 < \alpha < \beta < 2$, $C, V_0 > 0$ and $M_i$ be a sequence of $n$-dimensional compact oriented Riemannian manifolds that satisfy $$\text{Vol}(M_i) \leq V_0$$ and the $n$-dimensional $(C, \alpha, \beta)$-tetrahedral property at all $p \in M$ for all radius $r$, $r \leq r_0$. Then $M_i$ has a Gromov-Hausdorff and Intrinsic Flat convergent subsequence whose limits agree.

The paper is organized as follows. In Section 2 we state results concerning the tetrahedral property, integral current spaces and intrinsic flat distance. We skip many details from this topic, for a thorough treatment we suggest [1], [6] and [7]. Since Gromov-Hausdorff convergence is well known this manuscript does not cover it but we refer the reader to [2]. In Section 3 we analize the tetrahedral property at the vertices of Euclidean cones over metric spaces, $o \in K(X)$. First we see that if diam$(X) \leq \pi/3$ then the $(C, \beta)$-tetrahedral property cannot be satisfied at $o$. Then we show that the cone over the 2-dimensional projective space satisfies the 3-dimensional $(C, \beta)$-tetrahedral property. We finish the section providing sufficient conditions to achieve that the $(C, \beta)$-tetrahedral property is satisfied at the vertex of a Euclidean cone over an arbitrary metric space. For that purpose we prove that the slices of cones satisfying the $n$-dimensional tetrahedral property also satisfy the $n$-dimensional tetrahedral property. In Section 4 we define the $(C, \alpha, \beta)$-tetrahedral property, Definition 4.1. We notice that the $(C, \beta)$-tetrahedral property implies the $(C, 1 - \beta, 1 + \beta)$-tetrahedral property, Remark 4.2. The converse does not hold as can be seen in Example 4.3, Example 4.4 and Example 4.5. In this section we also prove a volume estimate and convergence results for integral current spaces satisfying the $(C, \alpha, \beta)$-tetrahedral property; Theorem 4.9, Theorem 4.10 and Theorem...
These are the analogues of Theorem 3.38, Theorem 3.42 and Theorem 5.2 in [6] proven by Portegies-Sormani for the $(C, \beta)$-tetrahedral property. We notice that from them we deduce Theorem A, Theorem B and Theorem C.

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## 2. Background

In this section we briefly recall the results that we will use in subsequent sections. In Subsection 2.1 we go over results concerning the tetrahedral property for Riemannian manifolds proven by Portegies-Sormani [6] and Sormani [7]. In Subsection 2.2 we give a brief introduction to integral current spaces and intrinsic flat distance following [1], [6] and [9].

### 2.1. Tetrahedral Property.

Here we state the integral tetrahedral property and see that it implies the tetrahedral property.

**Definition 2.1** (Sormani). Given $C > 0$ and $\beta \in (0, 1)$, a metric space $(X, d)$ is said to have the $n$-dimensional $(C, \beta)$-integral tetrahedral property at a point $p \in X$ for radius $r$ if there exist points $p_1, \ldots, p_{n-1} \in \bar{X}$, with $d(p, p_i) = r$, such that

$$
\int_{(1-\beta)r}^{(1+\beta)r} \cdots \int_{(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, \ldots, t_{n-1}) dt_1 \cdots dt_{n-1} \geq C(2\beta)^{n-1} r^n.
$$

The tetrahedral property implies the integral tetrahedral property.

**Proposition 2.2** (Portegies–Sormani). Let $(X, d)$ be a metric space that satisfies the $n$-dimensional $(C, \beta)$-tetrahedral property at $p$ for radius $r$, then it satisfies the $n$-dimensional $(C, \beta)$-integral tetrahedral property at $p$ for radius $r$.

#### 2.1.1. Volumes of Balls.

When $X = M$ is a Riemannian manifold that satisfies the $(C, \beta)$-tetrahedral property at $p$ for radius $r$ the following volume estimate for balls is obtained.

**Theorem 2.3** (Portegies-Sormani). Let $M$ be a Riemannian manifold. If $M$ satisfies the $n$-dimensional $(C, \beta)$-integral tetrahedral property at a point $p$ for radius $r$, then

$$
\text{Vol}(B_r(p)) \geq C(2\beta)^{n-1} r^n.
$$

In the next two examples we see that the metric spaces satisfying the tetrahedral property do not need to be Riemannian manifolds. In fact, such spaces do not need to be dimensionally homogeneous.
Example 2.4. Let \((\mathbb{R}^3, ||||)\) be the Euclidian metric space with the standard metric. Let \(X \subset \mathbb{R}^3\) be the union of the \(xy\)-plane and the \(yz\)-plane with the induced intrinsic metric, \(d\). Then \((X,d)\) satisfies the 2-dimensional \((C, \beta)\)-tetrahedral property.

To prove the claim it is enough to consider points on the \(xy\)-axis. Let \(p = (x,y,0)\). If \(x \neq 0\), take \(p_1\) to be the point in the line that passes trough \((0,y,0)\) and \(p\) that satisfies \(||p - p_1|| = r \leq ||p_1||\). Explicitly, \(p_1 = (x, y, 0)\).

For \(r \leq \text{dist}(p, y \text{-axis}) = |x| \neq 0\), \((p; r)\) equals the circle of radius \(r\) around \(p\) in the \(xy\)-plane. When \(t_1 \in (0, 2r)\), \((p_1; t_1)\) equals the circle of radius \(t_1\) around \(p_1\) in the \(xy\)-plane and intersects \((p; r)\) in exactly two points. Hence, \((X,d)\) satisfies the 2-dimensional \((C, \beta)\)-tetrahedral property at \(p\) for radius \(r \leq |x|\) for any \(0 < \beta < 1\).

If \(r > \text{dist}(p, y \text{-axis}) = |x|\) then \((p; r)\) equals the circle of radius \(r\) around \(p\) in the \(xy\)-plane union an ellipse in the \(yz\)-plane with center \((0,y,0)\), major axis of length \(2\sqrt{r^2 - |x|^2}\) on the \(y\)-axis and minor axis of length \(2(r - |x|)\) on the \(z\)-axis. Hence, when \(t_1 \in (0, 2r)\) the set \((p_1; t_1)\) intersects \((p; r)\) in exactly two points in the \(xy\)-plane. In the \(yz\)-plane, \((p_1; t_1)\) does not intersect \((p; r)\) for \(t_1 < \sqrt{2r^2 + 2r|x|}\). For \(t_1 \in [\sqrt{2r^2 + 2r|x|}, 2r)\), \((p_1; t_1)\) intersects \((p; r)\) in exactly two points in the \(yz\)-axis. Thus, if \(t_1 = \sqrt{2r^2 + 2r|x|}\) then \((p, p_1; r, t_1)\) consists of two points and of four points if \(t_1 \in (\sqrt{2r^2 + 2r|x|}, 2r)\). In the latter case,

\[
\lim_{t_1 \to \sqrt{2r^2 + 2r|x|}} \min\{d(x,y) | x \neq y, x, y \in (p, p_1; r, t_1)\} = 0.
\]

Thus, for \(p\) such that \(r > \text{dist}(p, y \text{-axis}) = |x|\) we have to choose \(0 < t_1 < \sqrt{2r^2 + 2r|x|}\). That implies, \(0 < \beta < \sqrt{2r^2 + 2r|x|}/r - 1\). Note that \(\sqrt{2} - 1 \leq \sqrt{2r^2 + 2r|x|}/r - 1\). Hence, \((X,d)\) satisfies the 2-dimensional \((C, \beta)\)-tetrahedral property at \(p\) for radius \(r > 0\) for \(0 < \beta < \sqrt{2} - 1\).

Example 2.5. Let \((\mathbb{R}^3, ||||)\) be the Euclidian metric space with the standard metric. Let \(X \subset \mathbb{R}^3\) be the union of the \(xy\)-plane and the non-negative part of the \(z\)-axis with the induced intrinsic metric, \(d\). Then \((X,d)\) satisfies the 2-dimensional \((C, \beta)\)-tetrahedral property at \(p\) in the \(xy\)-axis for all \(r > 0\) and at \(p\) in the positive part of the \(z\)-axis only for \(r > 2||p||\).

Let \(p\) be contained in the \(xy\)-plane and \(r > 0\). If \(p = 0\), take \(p_{1}\) to be the point in the line that passes trough \(0\) and \(p\), and that satisfies \(||p - p_{1}|| = r \leq ||p_{1}||\). Explicitly, \(p_{1} = p + r/||p||\). If \(p \neq 0\), take any point \(p_{1}\) in the \(xy\)-plane that satisfies \(r = ||p - p_{1}||\). If \(||p|| \leq r\) then \((p; r)\) equals the circle of radius \(r\) around \(p\) in the \(xy\)-plane union the point \(z = (0,0,r-||p||)\) in the non-negative part of the \(z\)-axis. Hence, in the \(xy\)-plane \((p_1; t_1)\) intersects \((p; r)\) in exactly two points when \(t_1 \in (0, 2r)\). Now, \(z \notin (p_1; t_1)\) for \(t_1 \in (0, 2r)\) since \(d(z, p_1) = ||z|| + ||p_1|| = r - ||p|| + ||p|| = r + 2r\). If \(||p|| > r\) then \((p; r)\) equals the circle of radius \(r\) around \(p\) in the \(xy\)-plane and does not intersect the \(z\)-axis. Hence, \((p_1; t_1)\) intersects \((p; r)\) in exactly two
points when \( t_1 \in (0, 2r) \). Thus, the \((C, \beta)\) tetrahedral property is satisfied at \( p \) for \( r > 0 \) for any \( \beta \in (0, 1) \).

If \( p \) is in the positive part of the \( z \)-axis and \( r \leq ||p|| \) then \( S(p; r) \) contains only two points. Then for \( p_1 \in S(p; r) \) we get that the cardinality of \( S(p, p_1; r, t) \) is less or equal than 1. Hence, \((X, d)\) cannot satisfy the 2-dimensional tetrahedral property at those points with that \( r \). Suppose that \( r > ||p|| \) and pick \( p_1 \) in the \( xy \)-plane such that \( ||p_1|| = r - ||p|| \). Then, \( S(p; r) \) equals the circle of radius \( r - ||p|| \) around 0 in the \( xy \)-plane union the point \( z = p + r(0, 0, 1) \) on the \( z \)-axis. Hence, in the \( xy \)-plane \( S(p_1; t_1) \) intersects \( S(p; r) \) in exactly two points only when \( t_1 \in (0, 2(r - ||p||)) \). For \( t_1 \in (0, 2(r - ||p||)) \), \( z \in S(p_1; t_1) \) since \( d(z, p_1) = ||z|| + ||p_1|| = 2r > 2(r - ||p||) \). By definition, \( t_1 \) has to be contained in an interval of the form \([1 - \beta)r, (1 + \beta)r\]. This means that, \((1 + \beta)r < 2(r - ||p||)\). Solving for \( \beta \), \( \beta < 1 - 2||p||/r \). Moreover, \( \beta \) must lie on the the interval \((0, 1)\). Thus, \( r > 2||p|| \). Hence, the \((C, \beta)\)-tetrahedral property is satisfied at \( p \) in the positive part of the \( z \)-axis only for \( r > 2||p|| \) with \( \beta \in (0, 1 - 2||p||/r) \).

**Remark 2.6.** Portegies-Sormani proved Theorem 2.6 for integral current spaces as well. There, the volume is replaced by a measure coming from the current structure. Since the theory behind these spaces requires several definitions we omit the general statement but prove a similar result in Section 4. Theorem 4.3.

2.1.2. Convergence Theorems. One of the main applications of the tetrahedral property is the following Gromov-Hausdorff compactness theorem.

**Theorem 2.7** (Sormani). Let \( r_0 > 0 \), \( \beta \in (0, 1) \), \( C > 0 \), \( V > 0 \) and \( \{M_i\}_{i=1}^{\infty} \) be a sequence of \( n \)-dimensional compact Riemannian manifolds such that for all \( i \)

1. \( \text{Vol}(M_i) \leq V \)
2. \( M_i \) satisfies the \( n \)-dimensional \((C, \beta)\)-(integral) tetrahedral property for all \( r \leq r_0 \).

Then a subsequence converges in Gromov-Hausdorff sense.

Note that the metric space given in Example 2.3 does not satisfy condition (2) of Theorem 2.7. There, the tetrahedral property is not satisfied at \( p \) on the positive part of the \( z \)-axis for any \( r \leq ||p|| \).

Using the intrinsic flat distance which we briefly define at the end of Subsection 2.2, Sormani’s Gromov-Hausdorff compactness theorem, Theorem 2.7 can be improved in the following way.

**Theorem 2.8** (Portegies-Sormani). Let \( r_0 > 0 \), \( 0 < \beta < 1 \), \( C, V_0 > 0 \) and \( \{M_i\} \) be a sequence of \( n \)-dimensional compact oriented Riemannian manifolds such that for all \( i \)

1. \( \text{Vol}(M_i) \leq V \)
2. \( M_i \) satisfies the \( n \)-dimensional \((C, \beta)\)-(integral) tetrahedral property for all \( r \leq r_0 \).
Then \( \{ M_i \} \) has a subsequence that converges in Gromov-Hausdorff and intrinsic flat sense such that the limits spaces agree. Hence, the limit space is \( \mathcal{H}^n \) countably rectifiable.

### 2.2. Integral Current Spaces and Intrinsic Flat Distance

In this subsection we give a brief introduction to integral current spaces and the intrinsic flat distance defined by Sormani-Wenger [9]. We recall that the definition of an integral current space is based upon work on currents in metric spaces by Ambrosio-Kirchheim [1].

An \( n \)-dimensional integral current space \((X, d, T)\) consists of a metric space \((X, d)\) and an \( n \)-dimensional integral current defined on the completion of \( X \), \( T \in I_n(\bar{X}) \), such that set \((T) = X\). Recall that \( T \) endows \( \bar{X} \) with a finite Borel measure, \( \| T \| \), called the mass measure of \( T \) and that set \((T)\) is defined as

\[
(2.3) \quad \text{set}(T) = \left\{ x \in \bar{X} \mid \liminf_{r \downarrow 0} \frac{\| T \|(B_r(x))}{\omega_n r^n} > 0 \right\}.
\]

The mass of \( T \) is defined as \( M(T) = \| T \|(X) \). Ambrosio-Kirchheim proved that \( \text{set}(T) \) is \( \mathcal{H}^n \)-rectifiable. That is, there exist Borel sets \( A_i \subset \mathbb{R}^n \) and Lipschitz functions \( \varphi_i : A_i \to X \) such that

\[
(2.4) \quad \mathcal{H}^n \left( \text{set}(T) \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i) \right) = 0.
\]

An \( n \)-dimensional compact oriented Riemannian manifold \( M \) has a canonical current given by integration of top forms:

\[
(2.5) \quad T(\omega) = \int_M \omega.
\]

With this current, \((X, d, T)\) is an \( n \)-dimensional integral current space, the mass measure of \( T \) equals the Riemannian volume and set \((T)\) is \( \mathcal{H}^n \).

Let \( B_r(p) \subset X \) be a ball of radius \( r \) and center \( p \). To obtain Gromov-Hausdorff and intrinsic flat convergence results we are interested in calculating a lower bound for \( \| T \|(B_r(p)) \). Thus, we consider the triple

\[
(2.6) \quad S(p, r) = (\text{set}(T \llcorner B_r(p)), d, T \llcorner B_r(p)),
\]

where \( T \llcorner B_r(p) \) denotes the restriction of \( T \) to \( B_r(p) \). Portegies-Sormani proved [Lemma 3.1 in [6]] that if \((X, d, T)\) is an \( n \)-dimensional integral current space, then for almost every \( r > 0 \), \( S(p, r) \) is an \( n \)-dimensional integral current space. Furthermore,

\[
(2.7) \quad B_r(p) \subset \text{set}(S(p, r)) \subset \bar{B}_r(p) \subset X.
\]

When \( M \) is a compact oriented Riemannian manifold (with or without boundary) endowed with the canonical current then (Lemma 3.2 in [6]) \( S(p, r) \) is an integral current space for all \( r \) and set \((S(p, r)) = B_r(p) \).

Lower bounds for \( \text{M}(S(p, r)) \) are obtained by studying the sliced filling volume and the \( k \)-th sliced filling. For the definition of FillVol(\( \partial \text{Slice}(S(p, r), F, t) \)) see Portegies-Sormani [6]. Let \( F_1, F_2, \ldots, F_k : X \to \mathbb{R}^k \) be Lipschitz functions
with $k \leq n-1$, the $k$-th sliced filling volume of $\partial S(p,r) \in I_{n-1}(X)$ is defined as

\begin{equation}
\mathbf{SF}(p,r,F_1,...,F_k) = \int_{t \in A_r} \text{FillVol}(\partial \text{Slice}(S(p,r),F,t)) \, d\mathcal{L}^k,
\end{equation}

where $F = (F_1,...,F_k)$,

$$A_r = \min F_1, \max F_1 \times \min F_2, \max F_2 \times \cdots \times \min F_k, \max F_k$$

$$\min F_j = \min \{ F_j(x) \mid d(x, p) = r \}$$

$$\max F_j = \max \{ F_j(x) \mid d(x, p) = r \}.$$ 

Given $q_1,...,q_k \in X$, set $\rho_i(x) = d(q_i, x)$ and define

\begin{equation}
\mathbf{SF}(p,r,q_1,...,q_k) = \mathbf{SF}(p,r,\rho_1,...,\rho_k)
\end{equation}

and,

\begin{equation}
\mathbf{SF}_k(p,r) = \sup \{ \mathbf{SF}(p,r,\rho_1,...,\rho_k) \mid q_i \in X \, d(p,q_i) = r \}.
\end{equation}

We now state the mass measure estimate and the Gromov-Hausdorff and intrinsic flat theorem obtained by Portegies-Sormani where bounds on $\mathbf{SF}$ and $\mathbf{SF}_k$ are considered.

**Theorem 2.9** (Portegies-Sormani, Theorem 3.25 in [6]). Let $(X,d,T)$ be an $n$-dimensional integral current space and $p_1,...,p_{n-1} \in X$. If $\overline{B}_R(p) \cap \text{set}(\partial T) = \emptyset$ then for almost every $r \in (0,R)$,

\begin{equation}
\mathbf{M}(S(p,r)) \geq \mathbf{SF}(p,r,p_1,...,p_{n-1}) \geq \int_{s_1-r}^{s_1+r} \cdots \int_{s_{n-1}-r}^{s_{n-1}+r} h(p,r,t_1,...,t_{n-1}) \, dt_1 \, dt_2 \cdots \, dt_{n-1},
\end{equation}

where $s_i = d(p_i,p)$,

\begin{equation}
h(p,r,t_1,...,t_{n-1}) = \begin{cases} 
\inf \{ d(x,y) \mid x \neq y, x,y \in S \} & |S| \geq 2 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

and $S = S(p,r) \cap S(p_1;t_1) \cap \cdots \cap S(p_{n-1};t_{n-1})$.

**Theorem 2.10** (Portegies-Sormani, Theorem 5.1 in [6]). Let $(X_i,d_i,T_i)$ be a sequence of compact $n$-dimensional integral current spaces, $V,A,D,C,r_0>0$ and $k \in \{0,...,n-1\}$ such that for all $i$, $p \in X_i$ and $r \leq r_0$:

$$\mathbf{M}(T_i) \leq V, \, \mathbf{M}(\partial T_i) \leq A, \, \text{diam}(X_i) \leq D, \, \mathbf{SF}_k(p,r) \geq Cr^n.$$

Then there is a subsequence of $(X_i,d_i,T_i)$ that converges in the Gromov-Hausdorff and intrinsic flat sense to the same metric space.

We recall that for $S,T \in I_n(X)$, the flat distance between $S$ and $T$ in $X$ is defined as

\begin{equation}
d_F^k(S,T) = \inf \{ \mathbf{M}(U) + \mathbf{M}(V) \mid S - T = U + \partial V, U \in I_n(X), V \in I_{n+1}(X) \}.
\end{equation}
The intrinsic flat distance between two integral current spaces \((X, d_X, T)\) and \((Y, d_Y, S)\) is given by

\[
d_{\mathcal{F}}((X, d_X, T), (Y, d_Y, S)) = \inf \left\{ d_{\mathcal{F}}(\varphi\#T, \psi\#S) \mid Z \text{ complete metric space, } \varphi : X \to Z, \psi : Y \to Z \text{ isometric embeddings} \right\}.
\]

We point out that \(d_{\mathcal{F}}((X, d_X, T), (Y, d_Y, S)) = 0\) if and only if there exists a current-preserving isometry \(\varphi : X \to Y\), that is, \(\varphi\) is an isometry of metric spaces such that \(\varphi\#T = S\).

3. Vertices and Slices of Cones

In this section we will study what happens at the vertex \(o\) of a Euclidean cone \(K(X)\) over a metric space \(X\). In Example 3.2 we see that if \(\text{diam}(X) \leq \pi/3\) then the \((C, \beta)\)-tetrahedral property cannot be satisfied at \(o\). Then in Example 3.4 we show that the cone over the 2-dimensional projective space satisfies the 3-dimensional \((C, \beta)\)-tetrahedral property as a means of showing that the existence of topological singularities is not an obstruction to the tetrahedral property. Note that, equipped with the usual round metric, it holds that \(\text{diam}(\mathbb{RP}^2) = \pi/2 > \pi/3\). We finish the section with Proposition 3.6 that provides sufficient conditions to ensure that the \((C, \beta)\)-tetrahedral property is satisfied at the vertex of a Euclidean cone over an arbitrary metric space. For that purpose we include Lemma 3.5 that shows that the slices of cones satisfying the \(n\)-dimensional tetrahedral property also satisfy the \(n\)-dimensional tetrahedral property.

We begin by recalling the definition of a Euclidean cone over a metric space.

**Definition 3.1.** Let \((Y, d_Y)\) be a metric space. Then \(K(Y) := Y \times [0, \infty) / Y \times \{0\}\) endowed with the metric

\[
d_K((x, t), (y, s)) := \sqrt{t^2 + s^2 - 2st \cos d_Y(x, y)}.
\]

is called the Euclidean cone over \(Y\) (cf. \[2\] Definition 3.6.12]). The vertex of \(Y\), that is the point corresponding to \(Y \times \{0\}\), is denoted by \(o\).

**Example 3.2.** Let \((X, d)\) be a metric space with \(\text{diam}(X) < \pi/3\). Then for all \(n \in \mathbb{N}, C > 0, \beta \in (0, 1)\) and \(r > 0\), the cone \(K(X)\) over \(X\) does not satisfy the \(n\)-dimensional \((C, \beta)\)-tetrahedral property at its vertex, \(o\), for radius \(r\). This follows from the following calculation

\[
d_K((x, r)(y, r)) = 2r^2 - 2r^2 \cos (d(x, y)) = 2r^2 (1 - \cos (d(x, y))) < r^2,
\]

which shows that for any \((x, r) \in S(o; r) = X \times \{r\}, S((x, r); r) = \emptyset\). Therefore, \(h(o, (x_1, r), ..., (x_{n-1}, r); r, r, ..., r) = 0\) for all \(x_i \in S(o; r)\).

**Remark 3.3.** The tetrahedral property can be used indirectly to estimate from below the diameter of a space as the previous example suggests.
We point out that in general the existence of topologically singular points, in the sense of Alexandrov geometry, is not a priori an obstruction to the tetrahedral property as the following example shows. Let us recall that a point $p$ in an $n$-dimensional Alexandrov space is topologically singular if the space of directions at $p$ is not homeomorphic to an $(n - 1)$-dimensional sphere. We refer the reader to [2].

**Example 3.4.** Let us consider $K(\mathbb{R}P^2)$ and let $o$ be the vertex. We claim there exists $r > 0$ and constants $C = C(o) > 0$, $\beta = \beta(o) \in (0, 1)$ such that $X$ satisfies the 3-dimensional $(C, \beta)$-tetrahedral property at $o$ for radius $r$. There exists an isometric involution $\iota$ on a 3-ball $B_R(0) \subset \mathbb{R}^3$, such that $B_R(0)/\iota$ is isometric to $K(\mathbb{R}P^2)$. The involution is given as the conification of the action induced by the antipodal map on $\mathbb{S}^2$.

Let $r > 0$ and $p_1, p_2$ be the points of Example 2.1 in [7]. It is clear that $p_i$ and the set $S(o, p_1, p_2; r, s_1, s_2)$ are contained in a fundamental domain of the antipodal map action. Furthermore, the position of the $p_i$ and the radii $s_i$ can be perturbed so that

$$\pi(S(o, p_1, p_2; r, s_1, s_2)) = S(o, p_1, p_2; r, s_1, s_2),$$

where $\pi : \mathbb{R}^3 \to K(\mathbb{R}P^2)$ is the canonical projection. Then the 3-dimensional $(C, \beta)$-tetrahedral property is satisfied for radius $r$ at $o$.

The aim of the remaining of this section is to present sufficient conditions to ensure the tetrahedral property at the vertex of a cone. More explicitly, if $X$ satisfies $\text{diam}(X) \geq \pi/3$ and a strengthened version of the tetrahedral property at a point for radius $r = \pi/3$, then the tetrahedral property is satisfied at the vertex of $K(X)$. Let’s start with the following lemma that shows that the slices of Euclidean cones over metric spaces satisfying the $n$-dimensional tetrahedral property also satisfy the $n$-dimensional tetrahedral property.

**Lemma 3.5.** Let $(X, d)$ be a metric space with $\text{diam} X \leq \pi$ satisfying the $n$-dimensional $(C, \beta)$-tetrahedral property for radius $r$. Let $K(X)$ be the cone of $X$ with the cone metric $d_K$. Then for $s > 0$, the slice $X \times \{s\} \subset K(X)$ with the restriction of $d_K$ satisfies the $n$-dimensional $(C, \beta_r)$-tetrahedral property for radius $r'$, where

$$C_r = \sqrt{\frac{1 - \cos(Cr)}{1 - \cos(r)}},$$

$$0 < \beta_r \leq \min \left\{ 1 - \sqrt{\frac{1 - \cos((1 - \beta)r)}{1 - \cos(r)}}, \sqrt{\frac{1 - \cos((1 + \beta)r)}{1 - \cos(r)}} - 1 \right\}$$

and

$$r' = \sqrt{2s^2(1 - \cos(r))}.$$

**Proof.** We begin by considering a point $p \in X$ and noting that, since $X$ satisfies the $n$-dimensional $(C, \beta)$-tetrahedral property for radius $r$ at every
point, there exist points \( p_1, \ldots, p_{n-1} \in S(p; r) \) such that
\[
S(p, p_1, \ldots, p_{n-1}; r, t_1, \ldots, t_{n-1}) \neq \emptyset
\]
for \( t_i \in [(1 - \beta)r, (1 + \beta)r] \). We claim that for \( t'_i \in [(1 - \beta_r)r, (1 + \beta_r)r] \),
\[
S((p, s), (p_1, s), \ldots, (p_{n-1}, s); r, t'_1, \ldots, t'_{n-1}) = S(p, p_1, \ldots, p_{n-1}; r, t_1, \ldots, t_{n-1}) \times \{s\}
\]
In particular, \( S((p, s), (p_1, s), \ldots, (p_{n-1}, s); r, t'_1, \ldots, t'_{n-1}) \neq \emptyset \).

To prove the claim, we first note that, \( \beta_r \) is well defined, that is, \( \beta_r \in (0, 1) \).
This follows from the fact that the cosine is decreasing in \([0, \pi]\) and therefore,
\( \cos((1 + \beta)r) < \cos(r) \cos((1 - \beta)r) \). Now, we observe that by the inequality defining \( \beta_r \) and the fact that \( t'_i \in [(1 - \beta_r)r, (1 + \beta_r)r] \) we have that
\[
\sqrt{2s^2(1 - \cos((1 - \beta)r))} \leq t'_i \leq \sqrt{2s^2(1 - \cos((1 + \beta)r))}
\]
It follows that \( \cos((1 + \beta)r) \leq 1 - \frac{(t'_i)^2}{2s^2} \leq \cos((1 - \beta)r) \) and therefore \( t_i = \arccos(1 - \frac{(t'_i)^2}{2s^2}) \) satisfies that \( t_i \in [(1 - \beta)r, (1 + \beta)r] \).

Finally we take \( x \in S((p, s), (p_1, s), \ldots, (p_{n-1}, s); r, t'_1, \ldots, t'_{n-1}) \). We proceed to show
that \( (x, s) \in S((p, s), (p_1, s), \ldots, (p_{n-1}, s); r, t'_1, \ldots, t'_{n-1}) \). Indeed:
\[
d_K((p, s), (x, s)) = \sqrt{2s^2 - 2s^2 \cos(d(p, x))} = r'
\]
\[
d_K((p_i, s), (x, s)) = \sqrt{2s^2(1 - \cos(d(p_i, x)))}
= \sqrt{2s^2(1 - \cos(t_i))}
= \sqrt{2s^2 \left(1 - \cos \left( \arccos \left(1 - \frac{(t'_i)^2}{2s^2}\right)\right)\right)}
= \sqrt{2s^2 \left(1 - \left(1 - \frac{(t'_i)^2}{2s^2}\right)\right)}
= \sqrt{(t'_i)^2} = t'_i.
\]
This shows our claim.

Now let \( (x, s), (y, s) \in S((p, s), (p_1, s), \ldots, (p_{n-1}, s); r, t'_1, \ldots, t'_{n-1}) \), \( (x, s) \neq (y, s) \). Using that the cosine is decreasing in \([0, \pi]\) and that the square root is increasing in \([0, \infty)\) we have that
\[
d_K((x, s), (y, s)) = \sqrt{2s^2 \left(1 - \cos(d(x, y))\right)}
\geq \sqrt{2s^2 \left(1 - \cos(Cr)\right)} = C_r r'.
\]

We stress that the previous result states that for each \( s > 0 \), the slice
\( X \times \{s\} \subset K(X) \) with the restriction of \( d_K \) satisfies the \( n \)-dimensional \((C, \beta)\)-tetrahedral property for some radius, but we do not claim that \( K(X) \) satisfies the \((n + 1)\)-dimensional \((C, \beta)\)-tetrahedral property. Now we address what happens at the vertex.
Proposition 3.6. Let \((X,d)\) be a metric space satisfying the \((n-1)\)-dimensional \((C,\beta)\)-tetrahedral property at a point \(p_0\) for radius \(r = \pi/3\). Let \(\{p_i\}_{i=1}^{n-2} \subset X\) be the configuration of points satisfying the property at \(p_0\) for radius \(r = \pi/3\). Assume that there exists \(\tilde{\beta} \in (0,1)\) such that for every \(x \neq y \in S(p_0,p_1, \ldots, p_{n-2}; \tilde{t}, t_1, \ldots, t_{n-2})\),

\[x + y \in S(p_0,p_1, \ldots, p_{n-2}; \tilde{t}, t_1, \ldots, t_{n-2}),\]

\[\tilde{t} \in [(1-\tilde{\beta})\frac{n}{3}, (1+\tilde{\beta})\frac{n}{3}] \text{ and } t_i \in [(1-\beta)\frac{n}{3}, (1+\beta)\frac{n}{3}] \text{ it holds } d(x, y) \geq C\frac{n}{3}.\]

Then for \(s > 0\), \(K(X)\) satisfies the \(n\)-dimensional \((C_s, \min(\tilde{\beta}, \beta_s))\)-tetrahedral property at the vertex for radius \(s\), where

\[
C_s = \sqrt{2(1 - \cos(C_s))},
\]

\[
\beta_s = \sqrt{2(1 - \cos((1 - \beta)s))},
\]

\[
\tilde{\beta}_s = \sqrt{2(1 - \cos((1 - \tilde{\beta})s))}.
\]

Proof. We claim that the configuration of points \(\{(p_0, s), (p_1, s), \ldots, (p_{n-2}, s)\} \subset X \times \{s\} \subset K(X)\) satisfies the tetrahedral property at \(o\) with radius \(s\).

First we observe that, by Lemma 3.5, the \((n-1)\)-dimensional \((C_s, \beta_s)\)-tetrahedral property of radius \(s\) is satisfied at \((p_0, s)\) via the configuration of points \(\{(p_1, s), \ldots, (p_{n-2}, s)\}\). By analogous computations to those of the proof of the same Lemma, we have that for every \(\tilde{t}' \in [(1 - \tilde{\beta})s, (1 + \tilde{\beta})s]\), every \(t'_i \in [(1 - \beta_s)s, (1 + \beta_s)s]\) and every

\[(x, s) \neq (y, s) \in S((p_0, s), (p_1, s), \ldots, (p_{n-2}, s); \tilde{t}', t'_1, \ldots, t'_{n-2})\]

it holds that \(d_{K(X)}((x, s), (y, s)) \geq C_ss\).

Recall that \(S(o; r) = X \times \{s\} \subset K(X)\), and therefore, for every \(\tilde{t}' \in [(1 - \tilde{\beta}_s)s, (1 + \tilde{\beta}_s)s]\) and every \(t'_i \in [(1 - \beta_s)s, (1 + \beta_s)s]\)

\[
S(o, (p_0, s), (p_1, s), \ldots, (p_{n-2}, s); s, \tilde{t}', t'_1, \ldots, t'_{n-2})
\]

\[= S((p_0, s), (p_1, s), \ldots, (p_{n-2}, s); \tilde{t}', t'_1, \ldots, t'_{n-2}) \cap (X \times \{s\})\]

Hence, it follows that for every \(t'_i \in [(1 - \min(\tilde{\beta}_s, \beta_s))s, (1 + \min(\tilde{\beta}_s, \beta_s))s]\) and every

\[(x, s) \neq (y, s) \in S(o, (p_0, s), (p_1, s), \ldots, (p_{n-2}, s); \tilde{t}', t'_1, \ldots, t'_{n-2})\]

it holds that \(d_{K(X)}((x, s), (y, s)) \geq C_ss\). \(\square\)

4. \((C, \alpha, \beta)\)-Tetrahedral Property

In this section we define the \((C, \alpha, \beta)\)-tetrahedral property, Definition 4.1. It is easy to see that the \((C, \beta)\)-tetrahedral property implies the \((C, 1 - \beta, 1 + \beta)\)-tetrahedral property, Remark 4.2. The converse does not hold as can be seen in the three examples we present. These examples also evidence the improvement over the \((C, \beta)\)-tetrahedral property: The \((C, \alpha, \beta)\)-tetrahedral property may be satisfied in spaces in which the \((C, \beta)\)-tetrahedral property fails, e.g. at the tips of metric cones. Furthermore it preserves all the main
properties of the \((C, \beta)\)-tetrahedral property such as those related to the Gromov-Hausdorff and Intrinsic Flat convergences.

Example 4.2 deals with the Euclidean cone of a sphere of small diameter. We show that it does not satisfy the \((C, \beta)\)-tetrahedral property, but it does satisfy the \((C, \alpha, \beta)\)-tetrahedral property for adequate choices of the parameters. In Example 4.4 we show a metric space \((X, d)\) that satisfies the \((C, \alpha, \beta)\)-tetrahedral property at some points \(p\) only for \(r > ||p||\). Meanwhile, it satisfies the \((C, \beta)\)-tetrahedral property at those points \(p\) only for \(r > 2||p||\). In Example 4.5 we present a metric space \((X, d)\) that satisfies the \((C, \beta)\)-tetrahedral property at \(p\) for \(r > 0\) and \(\beta < c(p, r)\), where \(c(p, r)\) denotes a function that depends on \(p\) and \(r\). Hence, \((X, d)\) satisfies the \((C, 1-\beta, 1+\alpha)\)-tetrahedral property. Moreover, \((X, d)\) satisfies the \((C, \alpha, \beta)\) property for \(c(p, r) < \alpha < \beta < 2\).

For simplicity, in Section 2 we stated Portegies-Sormani’s volume estimate (Theorem 2.3) and convergence results for Riemannian manifolds satisfying the \((C, \beta)\)-tetrahedral property (Theorem 2.7 and Theorem 2.8). We remark that those results also hold for integral current spaces (Theorem 3.38, Theorem 3.42 and Theorem 5.2 in [6]). In Subsection 4.2 and Subsection 4.3 we prove the analogue volume estimate and convergence results for integral current spaces satisfying the \((C, \alpha, \beta)\)-tetrahedral property; Theorem 4.9, Theorem 4.10 and Theorem 4.11. As corollaries we get the theorems for manifolds stated in the introduction, that is, Theorem A, Theorem B and Theorem C.

4.1. \((C, \alpha, \beta)\)-Tetrahedral Property and Examples. Let \((X, d)\) be a metric space, recall that \(S(p; r) = \{x \in X | d(x, p) = r\}\) and that

\[
S(x_1, \ldots, x_j; t_1, \ldots, t_j) = \bigcap_{i=1}^j S(x_i; t_i).
\]

Furthermore, the metric completion of \(X\) is denoted by \(\bar{X}\) and the cardinality of a set \(S\) by \(|S|\).

**Definition 4.1** ((\(C, \alpha, \beta\))-tetrahedral property). Let \(C > 0\) and \(\alpha, \beta \in (0, 2)\), \(\alpha < \beta\). A metric space \((X, d)\) satisfies the \(n\)-dimensional \((C, \alpha, \beta)\)-tetrahedral property at a point \(p\) for radius \(r\) if there exist points \(p_1, \ldots, p_{n-1} \in \bar{X}\) such that \(d(p, p_i) = r\) and for all \((t_1, \ldots, t_{n-1}) \in [\alpha r, \beta r]^{n-1}\) the following holds

\[
h(p, r, t_1, \ldots, t_{n-1}) \geq Cr,
\]

where

\[
h(p, r, t_1, \ldots, t_{n-1}) = \begin{cases} \inf \{d(x, y) | x \neq y, x, y \in S\} & |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}
\]

and \(S = S(p, p_1, \ldots, p_{n-1}; r, t_1, \ldots, t_{n-1})\).

**Remark 4.2.** Let \((X, d)\) be a metric space that satisfies the \(n\)-dimensional \((C, \beta)\)-tetrahedral property at \(p\) for radius \(r\). By definition, \((X, d)\) satisfies the \(n\)-dimensional \((C, 1-\beta, 1+\beta)\)-tetrahedral property at \(p\) for radius \(r\).
Example 4.3. Let $S^2(r)$ be a sphere of radius $r \leq 1/3$ (so that its diameter is less than $\pi/3$) and consider $K(S^2(r))$ with the cone metric. Then, as pointed out in Example 3.2 the cone $K(S^2(r))$ does not satisfy the 3-dimensional $(C,\beta)$-tetrahedral property at its vertex for any radius. However, $K(S^2(r))$ does satisfy the 3-dimensional $(C,\alpha,\beta)$-tetrahedral property at its vertex as we now show.

Let us denote the points on $K(S^2(r))$ by equivalence classes of the form $[q,t]$, where $t \in \mathbb{R}_{\geq 0}$ and $q \in S^2(r)$. Now we consider the slice $S^2(r) \times \{t\}$. We now let $(p_1,t)$ and $(p_2,t)$ be any two points on $S^2(r) \times \{t\}$ such that

$$D := d((p_1,t),(p_2,t)) \leq \frac{1}{C} t \sqrt{2(1 - \cos 2\pi r)}$$

where $C$ is a constant such that $\frac{1}{C} t \sqrt{2(1 - \cos 2\pi r)}$ is very small with respect to $t \sqrt{2(1 - \cos 2\pi r)}$. In the interest of providing explicit computations, we point out that it is enough to take $C = 4$ for example.

Now, it is enough to let $\alpha = 4D/6r$ and $\beta = 5D/6r$, since for every $t_i \in [\alpha,\beta] = [4D/6,5D/6]$, the set $S(o,p_1,p_2; t_1,t_2)$ consists of two points. Therefore $\{d(x,y) \mid x \neq y \in S(o,p_1,p_2; t_1,t_2)\}$ is bounded below by a positive constant and the result follows.

Example 4.4. Recall Example 2.5 in which $X \subset \mathbb{R}^3$ consisted of the union of the $xy$-plane and the non-negative part of the $z$-axis with the induced intrinsic metric, $d$. We showed that $X$ satisfies the 2-dimensional $(C,\beta)$-tetrahedral property at $p \in X$ contained in the $xy$-plane for all $r > 0$ and at $p$ on the positive part of the $z$-axis only for $r > 2||p||$. Here we prove that $X$ satisfies the 2-dimensional $(C,\beta,\alpha)$-tetrahedral property at points $p$ on the positive part of the $z$-axis for $r \in (||p||,2||p||]$ but not for $r \in (0,||p||]$. If $p$ is in the positive part of the $z$-axis, take $r \in (||p||,2||p||]$ and pick $p_1$ in the $xy$-plane such that $||p_1|| = r - ||p||$. Note that $S(p;r)$ equals the circle of radius $r - ||p||$ around 0 in the $xy$-plane union the point $z = p + r(0,0,1)$ on the $z$-axis. Hence, when $t_1 \in (0,2(r - ||p||)) S(p_1; t_1)$ intersects $S(p;r)$ in exactly two points in the $xy$-plane. For $t_1 \in (0,2(r - ||p||))$, $z \notin S(p_1; t_1)$ since $d(z,p_1) = ||z|| + ||p|| = 2r > 2(r - ||p||)$ and $r \in (||p||,2||p||]$ implies $2(r - ||p||) \leq r$. Thus, $(X,d)$ satisfies the 2-dimensional $(C,\alpha,\beta)$-tetrahedral property at $p$ for $r \in (||p||,2||p||]$ and $0 < \alpha < \beta \leq 2(r - ||p||)/r$.

If $p$ is in the positive part of the $z$-axis and $r \leq ||p||$ then $S(p;r)$ contains only two points. Then for $p_1 \in S(p;r)$ the cardinality of $S(p,p_1;r,t)$ is less or equal than 1. Hence, $(X,d)$ cannot satisfy the 2-dimensional tetrahedral property at those points with that $r$.

Example 4.5. Recall Example 2.4 where $X \subset \mathbb{R}^3$ equals the union of the $xy$-plane and the $yz$-plane with the induced intrinsic metric, $d$. We showed that $(X,d)$ satisfies the 2-dimensional $(C,\beta)$-tetrahedral property at $p \in X$ for radius $r \leq \max\{\text{dist}(p,xy-plane),\text{dist}(p,yz-plane)\}$ for any $0 < \beta < 1$. We also showed that $(X,d)$ satisfies the 2-dimensional $(C,\beta)$-tetrahedral
property at \( p \in X \) for radius \( r \geq \max\{\text{dist}(p, xy \text{- plane}), \text{dist}(p, yz \text{- plane})\} \)
for \( 0 < \beta < \sqrt{2r^2 + 2r|x|} = r - 1 \).

With the new definition, \((X, d)\) satisfies the 2-dimensional \((C, \alpha, \beta)\)-tetrahedral property at \( p \in X \) for radius \( r \leq \max\{\text{dist}(p, xy \text{- plane}), \text{dist}(p, yz \text{- plane})\} \) for any \( 0 < \alpha < \beta < 2 \) and, at \( p \in X \) for radius \( r \geq \max\{\text{dist}(p, xy \text{- plane}), \text{dist}(p, yz \text{- plane})\} \) for \( 0 < \alpha < \beta < \sqrt{2r^2 + 2r|x|}/r < \alpha < \beta < 2 \).

The \( n \)-dimensional \((C, \alpha, \beta)\)-integral tetrahedral property also has an integral version (c.f. Definition 2.1).

**Definition 4.6** \((C, \alpha, \beta)\)-integral tetrahedral property). Let \( C > 0 \) and \( \alpha, \beta \in (0, 2), \alpha < \beta \). A metric space \((X, d)\) satisfies the \( n \)-dimensional \((C, \alpha, \beta)\)-integral tetrahedral property at a point \( p \) for radius \( r \) if there exist points \( p_1, \ldots, p_{n-1} \in X \) such that \( d(p, p_i) = r \) and for all \((t_1, \ldots, t_{n-1}) \in [\alpha r, \beta r]^{n-1}\) the following estimate holds

\[
\int_{t_1=\alpha r}^{\beta r} \cdots \int_{t_{n-1}=\alpha r}^{\beta r} h(p, r, t_1, \ldots, t_{n-1}) dt_1 dt_2 \cdots dt_{n-1} \geq C(\beta - \alpha)^{n-1} r^n.
\]

Portegies-Sormani proved that the tetrahedral property implies the integral tetrahedral property. We prove a similar statement.

**Proposition 4.7.** If \((X, d)\) is a metric space that satisfies the \( n \)-dimensional \((C, \alpha, \beta)\)-tetrahedral property at a point \( p \) for radius \( r \) then it also satisfies the \( n \)-dimensional \((C, \alpha, \beta)\)-integral tetrahedral property at the point \( p \) for radius \( r \).

**Proof.** It follows immediately from the definitions:

\[
\int_{t_1=\alpha r}^{\beta r} \cdots \int_{t_{n-1}=\alpha r}^{\beta r} h(p, r, t_1, \ldots, t_{m-1}) dt_1 dt_2 \cdots dt_{m-1} \geq \\
\geq \int_{t_1=\alpha r}^{\beta r} \cdots \int_{t_{n-1}=\alpha r}^{\beta r} C r dt_1 dt_2 \cdots dt_{n-1} \\
= C(\beta - \alpha)^{n-1} r^n.
\]

\(\square\)

4.2. **Masses of Balls.** Now we will deal with integral current spaces. We recommend the reader to check Subsection \( \ref{subsec:masses} \) for a brief introduction to the subject and \[1, 6, 9\] for a complete treatment. In this section we prove Theorem \[\ref{thm:masses}\] and its analog for integral current spaces, Theorem \[\ref{thm:masses-int}\]

**Theorem 4.8.** Suppose \((X, d, T)\) is an \( n \)-dimensional integral current space and \( p \in X \) such that \( B_R(p) \cap \text{int} \partial T = 0 \). Then for almost every \( r \in (0, R) \), if \((X, d)\) satisfies the \( n \)-dimensional \((C, \alpha, \beta)\)-integral tetrahedral property at \( p \) for radius \( r \) then

\[
M(S(p, r)) \geq SF_{n-1}(p, r) \geq C(\beta - \alpha)^{n-1} r^n.
\]
Proof. Let \( q_1, ..., q_{n-1} \in X \), by Portegies-Sormani’s Theorem \( \text{2.9} \) we know that
\[
 \mathcal{M}(S(p, r)) \geq \mathbf{SF}(p, r, q_1, ..., q_{n-1})
\]
(4.4)
\[
 \geq \int_{t_1=d(p,q_1)}^{d(p,q_1)+r} \ldots \int_{t_{n-1}=d(p,q_{n-1})-r}^{d(p,q_{n-1})+r} h(p,r,t_1,...t_{n-1}) dt_1 dt_2 ... dt_{n-1}.
\]
To get the first inequality recall that
\[
 \mathbf{SF}_{n-1}(p,r) = \sup \{ \mathbf{SF}(p,r,q_1, ..., q_{n-1}) | d(p,q_i) = r \}.
\]
Since \( S(p,r) \) does not depend on \( q_1, ..., q_{n-1} \), we obtain from the previous series of inequalities
\[
 \mathcal{M}(S(p,r)) \geq \mathbf{SF}_{n-1}(p,r).
\]

For the second inequality notice that since \((\bar{X}, d)\) satisfies the \(n\)-dimensional \((C, \alpha, \beta)\)-integral tetrahedral property at \( p \) for radius \( r \), there exist \( p_1, ..., p_{n-1} \in X \) such that
\[
 \int_{t_1=\alpha r}^{\beta r} \ldots \int_{t_{n-1}=\alpha r}^{\beta r} h(p,r,t_1,...t_{n-1}) dt_1 dt_2 ... dt_{n-1} \geq C(\beta - \alpha)^{n-1}r^n.
\]
Now \( 0 < \alpha < \beta < 2 \) implies that \( d(p,q_i) - r < \alpha r < \beta r < d(p,q_i) + r \). Thus, recalling Equation (4.3) we obtain
\[
 \mathbf{SF}(p,r,p_1,...,p_{n-1}) \geq C(\beta - \alpha)^{n-1}r^n.
\]

An immediate consequence of Theorem \( \text{4.8} \) is Theorem \( \text{A} \)

Proof of Theorem \( \text{A} \). It follows from the previous theorem since in this case \( \mathcal{M}(S(p,r)) = \text{Vol}(B_r(p)) \) (See Lemma 3.2 in [6], cf. Section \( \text{2.2} \)). \( \square \)

Theorem 4.9. Suppose \((X,d,T)\) is an \(n\)-dimensional integral current space and \( p \in X \) such that \( B_R(p) \cap \text{set} \partial T = \emptyset \). If \((\bar{X},d)\) satisfies the \(n\)-dimensional \((C,\alpha,\beta)\)-integral tetrahedral property at \( p \) for all radius \( r, r \leq r_0 \), then for almost every \( r \in (0, \min \{r_0, R\}) \)
\[
 \|T\|(B_r(p)) \geq C(\beta - \alpha)^{n-1}r^n.
\]
(4.5)

Proof. By Theorem \( \text{4.8} \) we can choose \( \delta_i \to 0 \) such that \( \mathcal{M}(S(p, r + \delta_i/2)) \geq \mathbf{SF}_{n-1}(p, r + \delta_i/2) \geq C(\beta - \alpha)^{n-1}(r + \delta_i/2)^n \). Thus, using equation (2.7),
\[
 \|T\|(B_{r+\delta_i}(p)) \geq \|T\|(\bar{B}_{r+\delta_i/2}(p)) \geq \mathcal{M}(S(p, r + \delta_i/2)) \geq \mathbf{SF}_{n-1}(p, r + \delta_i/2) \geq C(\beta - \alpha)^{n-1}(r + \delta_i/2)^n.
\]
(4.6)
\[
 \|T\|(B_r(p)) \geq \|T\|(B_{r+\delta_i}(p)) \geq C(\beta - \alpha)^{n-1}(r + \delta_i/2)^n.
\]
(4.7)

Taking the limit as \( i \to \infty \), we get our estimate
\[
 \|T\|(B_r(p)) \geq C(\beta - \alpha)^{n-1}r^n.
\]
(4.8)
4.3. Convergence Theorems. Applying the mass measure estimates for balls from the previous subsection we show that the Gromov-Hausdorff and intrinsic flat convergence theorems proven by Portegies-Sormani for the \((C, \beta)-(\text{integral})\) tetrahedral property in [6] also hold for the \((C, \alpha, \beta)-(\text{integral})\) tetrahedral property, Theorem 4.10 and Theorem 4.11. We get as corollaries Theorem B and Theorem C.

**Theorem 4.10.** Let \(C > 0, 0 < \alpha < \beta < 2, r_0 > 0, V_0 > 0\) and \((X_i, d_i, T_i)\) be a sequence of \(n\)-dimensional integral current spaces that satisfy
\[
M(T_i) \leq V_0, \quad \partial T_i = 0.
\]
Suppose that \((X_i, d_i)\) are compact length metric spaces that satisfy the \(n\)-dimensional \((C, \alpha, \beta)-(\text{integral})\) tetrahedral property at all \(p \in X_i\) for all radius \(r, r \leq r_0\). Then a subsequence of the \((X_i, d_i)\) converges in Gromov-Hausdorff sense. In particular, there exists \(D_0(C, \alpha, \beta, r_0, V_0) > 0\) for which \(\text{diam}(X_i) \leq D_0(C, \alpha, \beta, r_0, V_0)\).

*Proof.* Given \(\varepsilon \in (0, r_0)\), apply Theorem 4.8 to get
\[
\text{SF}_{n-1}(p, r) \geq C(\beta - \alpha)^{n-1} r^n.
\]
This bound together with \(M(T_i) \leq V_0\) allows us to uniformly bound the maximal number of disjoint balls of radius \(\varepsilon\) contained in \(X_i\). Hence, precompactness follows from Gromov’s Compactness Theorem. Since \((X_i, d_i)\) are length metric spaces, the uniform bound on the maximal number of disjoint balls of radius \(r_0/2\) provides a uniform upper bound on \(\text{diam}(X_i)\) that only depends on \(C, \alpha, \beta, r_0, V_0\). □

Theorem B follows from Theorem 4.10.

**Theorem 4.11.** Let \(r_0 > 0, 0 < \alpha < \beta < 2, C, V_0 > 0\) and \((X_i, d_i, T_i)\) be a sequence of \(n\)-dimensional integral current spaces with
\[
M(T_i) \leq V_0, \quad \partial T_i = 0.
\]
Suppose that \((X_i, d_i)\) are compact length metric spaces that satisfy the \(n\)-dimensional \((C, \alpha, \beta)-(\text{integral})\) tetrahedral property at all \(p \in X_i\) for all radius \(r, r \leq r_0\). Then \((X_i, d_i)\) has a Gromov-Hausdorff and Intrinsic Flat convergent subsequence whose limits agree.

*Proof.* We just need to show that the hypotheses of Theorem 2.10 are satisfied. First, since we are considering integral current spaces \(X_i\) with \(\partial T_i = 0, M(\partial T_i) = 0\). Second, by Theorem 4.10 there exists a uniform upper bound on the diameter of the \(X_i\). Finally, since for all \(p \in X_i\) and \(r \leq r_0\), \((X_i, d_i)\) satisfies the \(n\)-dimensional \((C, \alpha, \beta)-(\text{integral})\) tetrahedral property at \(p \in X_i\) for radius \(r\), Theorem 4.8 implies that
\[
\text{SF}_{n-1}(p, r) \geq C(\beta - \alpha)^{n-1} r^n.
\]
Now we can apply Theorem 2.10 and get the result. □

Theorem C follows from Theorem 4.11.
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