Group manifold approach to higher spin theory

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Abstract

We consider the group manifold approach to higher spin theory. The deformed higher spin transformation is realized as the diffeomorphism transformation on group manifold $M$. With the suitable rheonomy condition and the torsion constraint imposed, the unfolded equation can be obtained from the Bianchi identity, by solving which, fields on $M$ is determined by the multiplet at one point, or equivalently, by $(W^{[a(s-1),b(0)]}_\mu, H)$ on $AdS_4 \subset M$. Although the space is extended to $M$ to get the geometrical formulation, the dynamical degrees of freedom is still in $AdS_4$. The 4d equations of motion for $(W^{[a(s-1),b(0)]}_\mu, H)$ is obtained by plugging the rheonomy condition into the Bianchi identity. The proper rheonomy condition allowing for the maximum on-shell degrees of freedom is given by Vasiliev equation. We also discuss the theory with the global higher spin symmetry, which is in parallel with the WZ model in supersymmetry.

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1 Introduction

Group manifold approach provides a natural geometrical formulation for supergravity [1, 2]. The starting point is the supergroup. Supergravity field and matter field are vielbein 1-form and 0-form on the group manifold $M$. Local super Poincare transformation is realized as the diffeomorphism transformation on $M$. Curvature for the 1-form can be defined, on which, the rheonomy condition is imposed. The condition ensures that fields on the whole $M$ are determined by fields on a four-dimensional (4d) submanifold $M_4$. So the final dynamics is still in $M_4$, on which the diffeomorphism transformation of $M$ reduces to the on-shell super Poincare transformation of the 4d fields. The equations of motion in $M_4$ are obtained by plugging the rheonomy condition into the Bianchi identity. Instead of imposing the rheonomy condition, one can also construct the extended action, which is the integration of some 4-form on a 4d submanifold $M_4$. Variation of the action with respect to both fields and $M_4$ gives the rheonomy condition as well as the 4d equations of motion.

In this paper, we will reformulate the group manifold method, adding an infinite number of auxiliary fields so that the final system is equivalent to the unfolded dynamics approach which is convenient for higher spin theory [3]. For higher spin theory in group manifold $M$ of the algebra $ho(1|2 : [3, 2])$, the fields are 1-form $W_\alpha^\alpha_M$ and 0-form $H$ with the corresponding curvature 2-form

$$dW^\alpha = \frac{1}{2}(f_\beta^\gamma + R_\beta^\gamma)W^\beta \wedge W^\gamma = \frac{1}{2}f_\beta^\gamma W^\beta \wedge W^\gamma \quad (1.1)$$

and the 1-form

$$dH = H_\alpha W^\alpha, \quad (1.2)$$

where $\tilde{M} = 1, 2, \ldots, \dim ho(1|2 : [3, 2])$, $\alpha \sim [a(s - 1), b(t)]$ is in the adjoint representation of $ho(1|2 : [3, 2])$, $f_\beta^\gamma$ is the structure constant of $ho(1|2 : [3, 2])$. The deformed higher spin transformation is the diffeomorphism transformation on $M$. The rheonomy condition is

$$\tilde{f}_\beta^\gamma = \tilde{f}_\beta^\gamma(R_{ab}^{[a(s - 1), b(s - 1)]}, R_{abc}^{[a(s - 1), b(s - 1)]}, \ldots, H, H_{c1}, \ldots),$$

$$H_\alpha = h_\alpha(R_{ab}^{[a(s - 1), b(s - 1)]}, R_{abc}^{[a(s - 1), b(s - 1)]}, \ldots, H, H_{c1}, \ldots), \quad (1.3)$$

where

$$R_{abc,c_1\ldots c_n}^{[a(s - 1), b(s - 1)]} = \partial_{c_n} \ldots \partial_{c_1} R_{ab}^{[a(s - 1), b(s - 1)]}, \quad H_{c_1\ldots c_n} = \partial_{c_n} \ldots \partial_{c_1} H, \quad (1.4)$$

$$\partial_c = W_{e=M}^c \partial_M, \quad a, b, c = 1, 2, 3, 4. \quad \text{With } (W_\alpha^\alpha, R_{ab}^{[a(s - 1), b(s - 1)]}, R_{abc}^{[a(s - 1), b(s - 1)]}, \ldots, H, H_{c1}, \ldots)$$

given at one point, its value on the whole $M$ can be determined via the Bianchi identity. On $AdS_4 \subset M$, we have the further relation

$$\left( R_{ab}^{[a(s - 1), b(s - 1)]}, R_{abc}^{[a(s - 1), b(s - 1)]}, \ldots, H, H_{c1}, \ldots; \right)$$

$$\sim \left( W_\mu^{[a(s - 1), b(0)]}, \partial_{c_1} W_\mu^{[a(s - 1), b(0)]}, \ldots, H, \partial_{c_1} H, \ldots, \right), \quad (1.5)$$

where $\partial_\mu$ is the derivative on $AdS_4$. So equivalently, with $(W_\mu^{[a(s - 1), b(0)]}, H)$ given on $AdS_4$, $(W_\alpha^\alpha_M, H)$ on the whole $M$ can be determined up to a gauge transformation. The dynamical 1-form fields are $W^{[a(s - 1), b(0)]}$, which is because in \([1,3]\), the torsion constraint is also
implicitly imposed: \( \tilde{f}_\alpha^\beta \) and \( H_\alpha \) do not depend on \( R_{[a(b(s-1))c(t)]}^{[a(s-1)b(t)]} \) with \( t \neq s - 1 \). For 0-form, the deformed higher spin transformation is \( \xi^M \partial_M = \epsilon^\alpha \partial_\alpha \), under which, the multiplet \( (R_{[a(b(s-1))c(t)]}^{[a(s-1)b(t)]}, H_{c_1} \cdots c_n, \cdots) \) forms the complete representation on-shell.

The whole dynamics is encoded in functions \((\tilde{f}_\beta^\gamma, h_\alpha)\), which should satisfy the Bianchi identity and also give the correct free theory limit. The Bianchi identity, with the unfolded equation plugged in, are polynomials of \((R_{[a(b(s-1))c(t)]}^{[a(s-1)b(t)]}, H_{c_1} \cdots c_n, \cdots)\), by solving which, \((\tilde{f}_\beta^\gamma, h_\alpha)\) is determined with the rest constraints on \((R_{[a(b(s-1))c(t)]}^{[a(s-1)b(t)]}, H_{c_1} \cdots c_n, \cdots)\) acting as the 4d equations of motion. The procedure is simple in supergravity but is extremely complicated in higher spin theory. Instead of fixing \((\tilde{f}_\beta^\gamma, h_\alpha)\) and getting the 4d equations of motion by solving the Bianchi identity, one can first identify the on-shell degrees of freedom, for example spin theory. Instead of fixing \((\tilde{f}_\beta^\gamma, h_\alpha)\), the deformed higher spin transformation is \( \{H_{c_1} \cdots c_n, n = 0, 1, \cdots \} \cup \{R_{[a(b(s-1))c(t)]}^{[a(s-1)b(t)]}, s = 2, 4, \cdots, n = 0, 1, \cdots \} \) (1.6). Written in terms of \( \Phi^\hat{\alpha} \), the unfolded equation becomes

\[
dW^\alpha = \frac{1}{2} \tilde{f}_\alpha^\beta (\Phi^\hat{\alpha}) W^\beta \wedge W^\gamma, \quad d\Phi^\hat{\alpha} = F_\beta^\gamma (\Phi^\hat{\alpha}) W^\beta \Phi^\hat{\gamma}, \quad (1.7)
\]

It is still difficult to find \((\tilde{f}_\beta^\gamma, F_\beta^\gamma)\) satisfying the Bianchi identity and also giving rise to the correct free theory limit\(^3\). The Vasiliev theory gives the elegant solution \([4][5][6]\). By solving the \( Z \) part of the Vasiliev equation order by order, one will finally get the consistent \((\tilde{f}_\beta^\gamma, F_\beta^\gamma)\) \([7]\).

In group manifold approach, \( ho(1|2 : [3,2]) \) has a subalgebra \( so(3,2) \), so higher spin theory is finally defined in a 4d submanifold with the topology of \( AdS_4 \). For supergravity, we may start from the uncontracted \( Osp(1/4) \) group with \( SO(3,2) \) a subgroup, or the contracted \( Osp(1/4) \) with Poincare group a subgroup. The corresponding supergravity theories are defined in \( AdS_4 \) and the Minkowski space \( M_4 \) respectively. Unfortunately, there is no contracted version of \( ho(1|2 : [3,2]) \). As a result, the flat space higher spin theory cannot be constructed here. This is consistent with the no-go theorem \([8][9]\) as well.

With \( H \) set to 0, we get the dynamics for gauge fields with \( s = 2, 4, \cdots \) like the pure supergravity system. For supersymmetry, it is also possible to study the dynamics of the 0-form matter on group manifold with the fixed background such as the WZ model. The allowed gauge transformation is the global super Poincare transformation, which is the diffeomorphism transformation on \( M \) preserving the background. For higher spin theory, one

\(^3\)As is shown in appendix C, there are \((\tilde{f}_\beta^\gamma, F_\beta^\gamma)\) satisfying the Bianchi identity but failing to give the correct free theory limit. It is unclear whether the two requirements can uniquely fix \((\tilde{f}_\beta^\gamma, F_\beta^\gamma)\) (up to a field redefinition) or not.
can similarly consider the 0-form \( H \) on \( M \) with
\[
dW_0^\alpha = \frac{1}{2} f_{\beta\gamma}^\alpha W_0^\beta \wedge W_0^\gamma, \quad dH = H_\alpha W_0^\alpha. \tag{1.8}
\]
The corresponding rheonomy condition, which is in parallel with the chiral constraint in WZ model, is
\[
H_\alpha = h_\alpha, \quad \partial_{[\alpha_1\ldots\alpha_s, b_1\ldots b_{s+1}]} H_{[\alpha_{d_1\ldots d_{t}, e_1\ldots e_{t+1}]} \rho = m_{[\alpha_1\ldots\alpha_s, b_1\ldots b_{s+1}], [\alpha_{d_1\ldots d_{t}, e_1\ldots e_{t+1}]}, (1.9)
\]
where \( h_\alpha \) and \( m_{[\alpha_1\ldots\alpha_s, b_1\ldots b_{s+1}], [\alpha_{d_1\ldots d_{t}, e_1\ldots e_{t+1}]}, \) are functions of
\[
(H, H_{c_1}, \ldots, H_{[\alpha_{a_1, b_2, b_3}, H_{[\alpha_{a_1, a_2, b_2, e_3}], \ldots, H_{[\alpha_{a_1, a_2, b_2, b_3}, H_{[\alpha_{a_1, a_2, b_3}, b_3}, \ldots}. \tag{1.10}
\]
\( (1.10) \) is in one-to-one correspondence with the spin \( s = 0, 2, 4, \ldots \) particles carrying the arbitrary on-shell momentum in \( AdS_4 \) and forms the complete higher spin multiplet. Again, in \( AdS_4, (1.10) \) and the 4\( d \) fields
\[
(H, H_{[\alpha_1 a_2, b_1 b_3], H_{[\alpha_1 a_2, a_3, b_5], \ldots}) \tag{1.11}
\]
are equivalent. We will give an explicit form for (1.9).

The rest of the paper is organized as follows. In Section 2, we construct a symmetric space \( M \) with the higher spin transformation group the isometry group. In Section 3, we discuss the theories with the global higher spin symmetry. In Section 4, we consider the theory with the local higher spin symmetry. The discussion and conclusion are given in Section 5.

## 2 Symmetric space from the higher spin algebra

We will consider the minimal bosonic higher spin theory in \( AdS_4 \) with the coordinate \( u^\mu, \mu = 1, 2, 3, 4 \). The corresponding algebra is \( ho(1|2 : [3, 2]) \) with the bases \( \{ t_\alpha \sim t_{A_1\ldots A_{s-1}, B_1\ldots B_{s-1}} \} \). \( t_{A_1\ldots A_{s-1}, B_1\ldots B_{s-1}} \) is in irreducible representations of \( SO(3, 2) \) characterized by two row rectangular Young tableaux, \( A_i, B_i = 0, 1, 2, 3, 4, s = 2, 4, \ldots \)
\[
\begin{align*}
t_{A_1\ldots A_{s-1}, B_1\ldots B_{s-1}} &= t_{\{A_1\ldots A_{s-1}, B_1\ldots B_{s-1}\}},

t_{A_1\ldots A_{s-1}, A_3} B_2\ldots B_{s-1} &= 0, \quad t_{A_1\ldots A_{s-3}CC}, B_2\ldots B_{s-1} = 0. \tag{2.1}
\end{align*}
\]
With \( a_i, b_i = 1, 2, 3, 4 \), bases of \( ho(1|2 : [3, 2]) \) can be rewritten as
\[
\{ t_\alpha \} = \{ t_{A_1\ldots A_{s-1}, B_1\ldots B_{s-1}} \} = \{ t_{0\ldots 0, b_1\ldots b_{s-1}}, t_{0\ldots 0, a_1, b_1\ldots b_{s-1}}, \ldots, t_{a_1\ldots a_{s-1}, b_1\ldots b_{s-1}} \}. \tag{2.2}
\]
Let
\[
\{ t_Q \} = \{ t_{0\ldots 0 a_1, b_1\ldots b_{s-1}}, t_{0\ldots 0 a_1 a_2 a_3, b_1\ldots b_{s-1}}, \ldots, t_{a_1\ldots a_{s-1}, b_1\ldots b_{s-1}} \} \tag{2.3}
\]
Indeed the same as (2.3).

There is a subgroup \( E \) of \( M \) where \( g \) be the bases of \( a[\mathcal{E}] \),

\[
\{ t_A \} = \{ t_{0\cdots 0,b_1\cdots b_{s-1}}, t_{0\cdots 0,a_2b_1\cdots b_{s-1}}, \cdots, t_{0a_1\cdots a_{s-2}b_1\cdots b_{s-1}} \}
\]  

(2.4)

be the bases of \( K \), \( ho(1|2 : [3, 2]) = a[\mathcal{E}] \oplus K \).

\[
[a[\mathcal{E}], a[\mathcal{E}]] \subset a[\mathcal{E}], \quad [a[\mathcal{E}], K] \subset K, \quad [K, K] \subset a[\mathcal{E}].
\]

With the group given, it is a standard procedure in mathematics to construct the group manifold \( M \) for \( G[ho(1|2 : [3, 2])] \) and the symmetric space \( M \) for \( G[ho(1|2 : [3, 2])]/E \). In the following, we will give a construction based on the operators and the conserved charges in the quantum higher spin theory in \( AdS_4 \). For earlier work on space with the tensor coordinates, see [10, 11].

In higher spin theory, there are conserved charges \( \{ Q_{A_1\cdots A_{s-1}, B_1\cdots B_{s-1}} \} \) in one-to-one correspondence with \( \{ t_{A_1\cdots A_{s-1}, B_1\cdots B_{s-1}} \} \). In particular, \( \{ Q_{A_1,B_1} \} \) are generators of \( SO(3,2) \). Suppose \( O^s(u) \) is the operator for the spin \( s \) field in \( AdS_4 \). We will focus on \( O^0(u) \equiv O(u) \) which is the operator for the spin 0 field. Let 0 be a point in the bulk of \( AdS_4 \), for example, \( (1, 0, 0, 0, 0) \) in \( x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 = 1 \), then the orbit generated by \( SO(3,2) \) gives the operators in \( AdS_4 \).

\[
\{ O(u) | u \in AdS_4 \} = \{ gO(0)g^{-1} | g \in SO(3,2) \},
\]

(2.6)

where \( g = e^{i\omega A_1,B_1 Q_{A_1,B_1}} \). Aside from \( AdS_4 \), the orbit generated by \( G[ho(1|2 : [3, 2])] \) gives the operators in an enlarged space \( M \).

\[
\{ O(z) | z \in M \} = \{ gO(0)g^{-1} | g \in G[ho(1|2 : [3, 2])] \},
\]

(2.7)

where \( g = e^{i\omega A_1\cdots A_{s-1},B_1\cdots B_{s-1} Q_{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}} \). \( G[ho(1|2 : [3, 2])] \) has a subgroup \( SO(3,2) \), so \( M \) has a fiber bundle structure with the fiber \( AdS_4 \) attached at each point of the base manifold. \( AdS_4 = SO(3,2)/SO(3,1) \), since \( gO(0)g^{-1} = O(0) \) for \( g \in SO(3,1) \). Similarly, there is a subgroup \( E(z) \subset G[ho(1|2 : [3,2])] \) for all \( e \in E(z) \), \( eO(z)e^{-1} = O(z) \). The higher spin algebra can be decomposed into:

\[
ho(1|2 : [3, 2]) = K(z) \oplus a[E(z)] = g(z)K(0)g(z)^{-1} \oplus g(z)a[\mathcal{E}(0)]g(z)^{-1}
\]

(2.8)

with \( K(z) \) the tangent space of \( M \) at \( z \). \( M \) is the coset space \( G[ho(1|2 : [3, 2])]/E \).

It remains to determine the subalgebra \( a[\mathcal{E}] \). Although the direct quantization of the higher spin theory in \( AdS_4 \) is still not available, its CFT dual is quite simple. In appendix A, the CFT realization of \( O(0) \), or more accurately, \( O^+(0) \), is given. It is shown that the charge \( Q_{0\cdots 0a_1\cdots a_{2k-1},b_1\cdots b_{s-1}} \) corresponding to \( (2.3) \) commutes with \( O(0) \). So \( a[\mathcal{E}] \) here is indeed the same as \( (2.3) \).

The metric on the coset space \( M = G[ho(1|2 : [3, 2])]/E \) is given in group theory. Alternatively, we can use the operator \( O(z) \) to give the same result. There is a one-to-one
correspondence between $T_z(M) = \{v^M \partial_M | M = 1, \cdots, \dim M\}$ and $K(z)$. For the given $\partial_M$, $\exists k_M(z) \in K(z)$ satisfying
\[ \partial_M O(z) = i[k_M(z), O(z)]. \tag{2.9} \]
\{k_M(z)\} compose the bases for $K(z)$, from which, one can define a special set of the coordinate on $M$
\[ O(z) = e^{ik_M(0)z^M} O(0) e^{-i k_M(0) z^M}. \tag{2.10} \]
The metric on $T_z(M)$ can be induced from $K(z)$, i.e.
\[ g_{MN}(z) = \langle k_M(z) | k_N(z) \rangle, \tag{2.11} \]
where $\langle k_M(z) | k_N(z) \rangle$ is the killing form. $g_{MN}$ is $G[ho(1|2 : [3, 2])$ invariant. Under the $G[ho(1|2 : [3, 2])$ transformation,
\[ O(z) \rightarrow gO(z)g^{-1} = O(z'). \tag{2.12} \]

$G[ho(1|2 : [3, 2])$ generates the isometric transformation $z \rightarrow z'$ on $M$.

The tangent space on the coset space $M$ is $\{t_A\}$. The group manifold of $ho(1|2 : [3, 2])$ is the manifold $M$ with the tangent space $\{t_A\}$, $\dim M = \dim ho(1|2 : [3, 2])$. The coordinate on $M$ is $Z_M$, $ik_M(Z) = \partial_M g(Z) g(Z)^{-1}$, $G_{MN}(Z) = \langle k_M(Z) | k_N(Z) \rangle$.
\[ \partial_M O(Z) = i[k_M(Z), O(Z)]. \tag{2.13} \]
When $k_M(Z) \in E(Z)$, $\partial_M O(Z) = 0$. The vielbein on $M$ is $W^\alpha_M$.
\[ W^\alpha_M W^\beta_N = \delta^\alpha_\beta, \quad W^\alpha_M W^\alpha_N = \delta^\beta_M, \quad \eta_{\alpha\beta} W^\alpha_M W^\beta_N = G_{MN}. \tag{2.14} \]
Let $k_\alpha(Z) = W^M_\alpha(Z) k_M(Z)$ and suppose $\partial_N k_M(Z) = \Gamma^L_{NM} k_L(Z)$, $\partial_N k_\alpha(Z) = \phi^\beta_\alpha k_\beta(Z)$, there will be
\[ \partial_N W^\alpha_M + \Gamma^L_{NM} W^\alpha_L - \phi^\beta_\alpha W^\beta_M = 0. \tag{2.15} \]
We will assume $k_\alpha(Z) = g(Z) k_\alpha g^{-1}(Z)$, which is always possible for the suitably selected $W^\alpha_M$. With the covariant derivative $\partial_M = \partial_M - \Gamma_M$ and $\partial_\alpha = W^M_\alpha (\partial_M - \phi_M) = \partial_\alpha - \phi_\alpha$,
\[ \mathcal{D}_{\alpha n} \cdots \mathcal{D}_{\alpha 1} \mathcal{D}_1 O(Z) = i^n [k_{M_1}(Z), [k_{M_2}(Z), \cdots, [k_{M_n}(Z), O(Z)] \cdots]], \tag{2.16} \]
\[ \mathcal{D}_{\alpha n} \cdots \mathcal{D}_{\alpha 2} \mathcal{D}_{\alpha 1} O(Z) = i^n [k_{\alpha_1}(Z), [k_{\alpha_2}(Z), \cdots, [k_{\alpha_n}(Z), O(Z)] \cdots]]. \tag{2.17} \]
As is shown in appendix A, for $[Q_{0, \cdots, 0a_1 \cdots, a_{s+k}, O(0)}]$ with $k = 1, 3, \cdots$
\[ [Q_{0, \cdots, 0a_1 \cdots, a_{s+k}, O(0)}] = \sum_{r=0, 2, \cdots, s} g_{0, \cdots, 0a_1 \cdots, a_{s+k}} c_1 \cdots c_r \cdots d_{s+k} [Q_{0c_1 \cdots c_r \cdots d_{s+k}, \cdots, [Q_{0d_{s+k}}, O(0)] \cdots]} \tag{2.18} \]
At the point $Z$, $O(Z) = g(Z)O(0)g^{-1}(Z)$, $Q_A(Z) = g(Z)Q_Ag^{-1}(Z)$,

\[
\begin{align*}
\{Q_{0\cdots 0a_1\cdots a_s b_1\cdots b_{s+k}}(Z), O(Z)\}
&= \sum_{r=0,2,\ldots,s} g_{0\cdots 0a_1\cdots a_s b_1\cdots b_{s+k}}^{c_1\cdots c_r d_1\cdots d_{r-1} d_{r+1}\cdots d_{s+k}} \{Q_{0c_1\cdots c_r d_1\cdots d_{r-1} d_{r+1}}(Z), \ldots [Q_{0, d_{r+1}, Z}, [Q_{0, d_{r+2}, Z}, O(Z)]\ldots\}.
\end{align*}
\]  
(2.19)

Since

\[
\begin{align*}
\mathcal{D}_{0, b_{s+k}} \mathcal{D}_{0, b_{s+k-1}} \cdots \mathcal{D}_{0, 1\cdots a_s b_1 b_{s+k}} O(Z)
&= i^{k} [Q_{0a_1\cdots a_s b_1 b_{s+k}}(Z), \ldots [Q_{0, b_{s+k}}(Z), O(Z)]\ldots],
\end{align*}
\]  
(2.20)

there will be

\[
\begin{align*}
\partial_{0\cdots 0a_1\cdots a_s b_1\cdots b_{s+k}} O(Z)
&= i^{-k} \sum_{r=0,2,\ldots,s} g_{0\cdots 0a_1\cdots a_s b_1\cdots b_{s+k}}^{c_1\cdots c_r d_1\cdots d_{r+1} d_{r+2}\cdots d_{s+k}} \mathcal{D}_{0, d_{r+1}} \mathcal{D}_{0, d_{r+2}} \cdots \mathcal{D}_{0, c_1\cdots c_r d_1\cdots d_{r+1}} O(Z).
\end{align*}
\]  
(2.21)

Just as the chiral constraint relates $\partial_{\bar{g}}$ with $\partial_{\mu}$, here, $\partial_{0\cdots 0a_1\cdots a_s b_1 \cdots b_{s+k}}$ is determined by $\mathcal{D}_{0, d_{r+1}} \mathcal{D}_{0, d_{r+2}} \cdots \mathcal{D}_{0, c_1\cdots c_r d_1\cdots d_{r+1}}$. This is because $[Q_{0\cdots 0a_1\cdots a_s b_1 \cdots b_{s+k}}(Z), O(Z)]\ldots]$ are all in the 1-particle Hilbert space of the higher spin theory, for which

\[
\begin{align*}
\{[Q_{0, b_{s+k}}(Z), \ldots [Q_{0, b_{s+2}}(Z), [Q_{0a_1\cdots a_s b_1 b_{s+k}}(Z), O(Z)]\ldots]\}
&\sim \{[Q_{0a_1\cdots a_s b_1 b_{s+1}}(Z), [Q_{0, b_{s+2}}(Z), \ldots, [Q_{0, b_{s+k}}(Z), O(Z)]\ldots]\}
\end{align*}
\]  
(2.22)

compose the complete bases. In [11], by considering the zeroth level of the unfolded equation for the 0-form $\Phi$ in $M$, the similar result is also obtained. $\Phi = \Phi[b(0), a(0)]$ is the lowest component of $\Phi[b(s), a(s+1)]$. Generically, one may expect

\[
\begin{align*}
\{Q_{0\cdots 0a_1^p\cdots a_s^p b_1^p\cdots b_{s+k}^p}(Z), \ldots [Q_{0\cdots 0a_1^p\cdots a_s^p b_1^p\cdots b_{s+k}^p}(Z), O(Z)]\ldots\}
&\sim \sum \alpha(a_1 \cdots a_s, b_1 \cdots b_{s+k}) [Q_{0a_1\cdots a_s b_1 \cdots b_{s+k}}(Z), \ldots, [Q_{0, b_{s+k-1}}(Z), [Q_{0, b_{s+k}}(Z), O(Z)]\ldots]\ldots]
\end{align*}
\]  
(2.23)

where $\alpha(a_1 \cdots a_s, b_1 \cdots b_{s+k})$ are constants to be determined.

\[
\begin{align*}
\mathcal{D}_{0\cdots 0a_1^p\cdots a_s^p b_1^p\cdots b_{s+k}^p}(Z)
&\sim \sum \alpha(a_1 \cdots a_s, b_1 \cdots b_{s+k}) \mathcal{D}_{0, b_{s+k}} \mathcal{D}_{0, b_{s+k-1}} \cdots \mathcal{D}_{0, a_1 b_1 \cdots b_{s+k}} O(Z).
\end{align*}
\]  
(2.24)

(2.24) is the $G[ho(1|2 ; [3,2])]$-invariant differential operators on $M$. 

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$^4$M($Z$) and $O(Z)$ are operators in [12], so $[M(Z), \ldots [M(Z), [M(Z), O(Z)]\ldots]]\ldots]$ are all in the 1-particle Hilbert space constructed in [12], where all states are physical with no redundancy to remove, i.e. the spin $s$ particle only has two helicity. Although the 0-form fields with spin 2, 4, $\cdots$ do not have the gauge symmetry to remove the extra degrees of freedom like that in gauge field situation, there is no redundant degrees of freedom to remove as well.
3 Theory with the global higher spin symmetry

We have constructed the space $M$ and the operator $O(z)$ on it. $\forall g \in G[ho(1|2 : [3, 2])]$, $gO(z)g^{-1} = O(z)$ induces the global higher spin transformation $z \rightarrow z_g$, which is the isometric transformation on $M$. This is exactly in parallel with supersymmetry. Consider a supersymmetric theory with the supersymmetry generators $P_{\mu}, Q$ and $\bar{Q}$.

\[ [P_{\mu}, P_{\nu}] = 0, \quad [P_{\mu}, Q] = [P_{\mu}, \bar{Q}] = 0, \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = 2\sigma^\mu_{\alpha\beta}P_\mu. \]  

(3.1)

For an operator $\Phi(0)$,

\[ \{\Phi(x, \theta, \bar{\theta})|\forall (x, \theta, \bar{\theta}) \in M\} = \{G(x, \theta, \bar{\theta})\Phi(0)G^{-1}(x, \theta, \bar{\theta})|\forall x, \theta, \bar{\theta}\} \]  

(3.2)

gives the operators in superspace $M$ with the coordinate $(x, \theta, \bar{\theta})$. Under the supersymmetry transformation

\[ G(a, \xi, \bar{\xi})\Phi(x, \theta, \bar{\theta})G^{-1}(a, \xi, \bar{\xi}) = \Phi(x + a + i\theta\sigma\xi - i\xi\sigma\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}) , \]  

(3.3)

\[ i(a^\mu P_\mu + \xi^\alpha Q_\alpha + \bar{\xi}^\dot{\alpha} \bar{Q}_{\dot{\alpha}}) = (a^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^\dot{\alpha} - i\xi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\xi^\dot{\alpha}) \frac{\partial}{\partial x^\mu} + \xi^\alpha \frac{\partial}{\partial \theta^\alpha} + \bar{\xi}^\dot{\alpha} \frac{\partial}{\partial \bar{\theta}^\dot{\alpha}} , \]  

(3.4)

\[ P_\mu = -i\frac{\partial}{\partial x^\mu}, \quad Q_\alpha = -i\frac{\partial}{\partial \theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^\dot{\alpha} \partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = -i\frac{\partial}{\partial \bar{\theta}^\dot{\alpha}} - \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\xi^\dot{\alpha} \partial_\mu . \]  

(3.5)

Superalgebra is realized as the killing vector fields in superspace, for which, the supersymmetry transformation is the isometric transformation.

Similarly, for higher spin theory, $\forall t_\sigma \in ho(1|2 : [3, 2])$, there is a corresponding killing field $\Lambda_\sigma$ in $M$. $\{\Lambda_\sigma\}$ compose a Lie algebra which is isomorphic to $ho(1|2 : [3, 2])$. Under the $G[ho(1|2 : [3, 2])]$ transformation,

\[ \delta_\epsilon O(z) = i[t_\sigma, O(z)]\epsilon = i\Lambda_\sigma O(z)\epsilon, \]  

(3.6)

where $\Lambda_\sigma = -iq^M_\sigma(z)\frac{\partial}{\partial z^M}$. In the given coordinate $O(z) = e^{ik_M z^M}O(0)e^{-ik_M z^M}$ with $k_M \in K(0)$, we can write down $\Lambda_\sigma$ explicitly. The detail is given in appendix B. Let $\{k_M\}$ and $\{e_m\}$ be the bases of $K(0)$ and $a[E(0)]$ respectively, $[k_M, O(z)] = \Lambda_M O(z)$, $[e_m, O(z)] = \Lambda_m O(z)$, then

\[ \Lambda_M = -iq^L_M(z)\partial_L, \quad \Lambda_m = -iC^{L}_{Nm}z^N\partial_L . \]  

(3.7)

where

\[ q^L_M(z) = \delta^L_M + \frac{1}{3}z^M_1z^M_2C^L_{M2n}C^m_{M1n} - \frac{1}{45}z^M_1z^M_2z^M_3z^M_4C^L_{M1n}C^m_{M2n}C^N_{M3n}C^m_{M4n} + \cdots , \]  

(3.8)

\[ [k_M, k_N] = iC^m_{MN}e_m, \quad [k_M, e_m] = iC^N_{Nm}k_N . \]  

(3.9)

One can also put the scalar field $H(z)$ on $M$, which is the higher spin counterpart of the superfield.

\[ \Lambda_\sigma = -iq^M_\sigma(z)\partial_M = -iq^A_\sigma(z)\partial_A \]  

(3.10)
where \( q^A_{\sigma} = E^A_{M} q^M_{\sigma} \). \( \bar{\partial}_A = E^M_A \partial_M \) should be distinguished from \( \partial_A = W^M_A \partial_M \) in group manifold. Under the global higher spin transformation, we have

\[
\delta_t H(z) = \epsilon^\sigma A_\sigma H(z) = -i \epsilon^\sigma q^A_{\sigma}(z) \bar{\partial}_A H(z) = -i \epsilon^\sigma q^A_{\sigma}(z) H_A(z),
\]

(3.11)

where \( \bar{\partial}_A H(z) = H_A(z) \). Generically,

\[
\delta_t H_{A_1 \cdots A_n}(z) = -i \epsilon^\sigma q^{A_{n+1}}_{\sigma}(z) \bar{\partial}_{A_{n+1}} H_{A_1 \cdots A_n}(z) = -i \epsilon^\sigma q^{A_{n+1}}_{\sigma}(z) H_{A_1 \cdots A_{n+1}}(z),
\]

(3.12)

with \( H_{A_1 \cdots A_n}(z) = \bar{\partial}_{A_1} \cdots \bar{\partial}_{A_n} H(z) \).

\((H, H_{A_1}, H_{A_1 A_2}, \cdots)\) forms the linear representation of the higher spin algebra. However, such representation is highly reducible. As is shown in the previous section, not all of \( \mathcal{D} \cdots \mathcal{D} O \) are linear independent. The linear independent basis can be selected as \( \mathcal{D}_{a_{b+k}} \cdots \mathcal{D}_{a_{b+k-1}} \mathcal{D}_{a_{b+k}} O(z) \). Correspondingly, among \( (H, H_{A_1}, H_{A_1 A_2}, \cdots) \), the independent fields can be chosen to be \( H_{[a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}]} \). One may impose the constraint

\[
\mathcal{D}_{0 \cdots 0^a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}} \cdots \mathcal{D}_{0 \cdots 0^a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}} H(z) \\
\sim \sum \alpha(a_1 \cdots a_s, b_1 \cdots b_{s+k}) \mathcal{D}_{0 \cdots 0^a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}} H(z)
\]

(3.13)

on \( H(z) \), which is invariant under the global higher spin transformation.

The imposing of the constraint \((3.13)\) is necessary. Although the space is extended from \( AdS_4 \) to \( M \), the dynamics is still in \( 4d \). The operators \( O(z) \) on \( M \) are all in the 1-particle Hilbert space of the 4d higher spin theory, so they are highly linear dependent. Rheonomy constraint simply picks out the independent degrees of freedom.

The coordinate \( z \) on \( M \) can be decomposed as \( z^M = (u^\mu, y^i) \) with \( u^\mu \) and \( y^i \) the coordinates on the \( AdS_4 \) fiber and the base space. In analogy with supersymmetry, from \( H(u, y) \), one can write the most generic action that is invariant under the global higher spin transformation

\[
S = \int_M du dy \sqrt{g(u, y)} [H(u, y) + a_2 H^2(u, y) + a_3 H^3(u, y) + \cdots]
\]

\[
= \int_M du' dy' \sqrt{g(u', y')} [H(u', y') + a_2 H^2(u', y') + a_3 H^3(u', y') + \cdots]
\]

\[
= \int_M du dy \sqrt{g(u, y)} [H'(u', y) + a_2 H'^2(u, y) + a_3 H'^3(u, y) + \cdots],
\]

(3.14)

where \( du dy \sqrt{g(u, y)} = du' dy' \sqrt{g(u', y')} \) because \( (u, y) \to (u', y') \) is the isometric transformation of \( M \). \( H(u, y) \to H(u', y') = H'(u, y) \) is the rigid higher spin transformation for the scalar field \( H(u, y) \). \( H(u, y) \) can be expanded in terms of \( H_{[a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}]}(u, 0) \), from which, the higher spin transformation law \( H_{[a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}]}(u, 0) \to H'_{[a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}]}(u, 0) \) can be induced. Integration over \( y \) gives the Lagrangian in \( AdS_4 \), which is invariant under \( H_{[a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}]}(u, 0) \to H'_{[a_{1} \cdots a_{s} b_{1} \cdots b_{s+k}]}(u, 0) \). However, unlike supersymmetry, it is not quite convenient to do this, since the \( y \) integration will give the infinity.
4 Theory with the local higher spin symmetry

In previous discussion, the background on $M$ is fixed to be the intrinsic geometry with $dW^{\alpha}_0 - \frac{1}{2} f^{\alpha}_{\beta\gamma} W^\beta_0 \wedge W^\gamma_0 = 0$. The allowed diffeomorphism transformation is the global higher spin transformation preserving $W^\alpha_0$. To have the local higher spin symmetry, the 1-form $W^\alpha$ on $M$ should be dynamical. We will study the dynamics of the 1-form $W^\alpha$ and the 0-form $H$ on $M$. With the suitable rheonomy condition and the torsion constraint imposed, $(W^\alpha, H)$ on the whole $M$ is determined by $(W^\alpha_{[a(s-1),b(0)]}, H)$ on $AdS_4$. We then discuss the relation between the unfolded equation in group manifold approach and the unfolded equation in Vasiliev theory.

4.1 Higher spin theory on group manifold and the rheonomy condition

The 1-form $W^\alpha_{\bar{M}}$ is the vielbein on $M$. $W^\alpha_{\bar{M}} W^\bar{\beta} = \delta^\alpha_{\beta}$, $W^\alpha_{\bar{M}} W^\bar{\alpha} = \delta^\alpha_{\bar{M}}$, $\eta_{\alpha\beta} W^\alpha_{\bar{M}} W^\beta_{\bar{N}} = G_{\bar{M}\bar{N}}$. The curvature 2-form is defined as

$$R^\alpha = dW^\alpha - \frac{1}{2} \hat{f}^{\alpha}_{\beta\gamma} W^\beta \wedge W^\gamma. \quad (4.1)$$

It is convenient to use the 0-form $R^\alpha_{\bar{\beta}\bar{\gamma}}$ to parameterize $R^\alpha_{\bar{M}\bar{N}}$, $R^\alpha_{\bar{M}\bar{N}} = R^\alpha_{\bar{\beta}\bar{\gamma}} W^\beta_{\bar{M}} W^\gamma_{\bar{N}}$, $R^\alpha_{\bar{\beta}\bar{\gamma}} = R^\alpha_{\bar{M}\bar{N}} W^\bar{\beta} W^\bar{\gamma}$.

$$dW^\alpha = \frac{1}{2} (f^{\alpha}_{\beta\gamma} + R^\alpha_{\bar{\beta}\bar{\gamma}}) W^\beta \wedge W^\gamma = \frac{1}{2} \hat{f}^{\alpha}_{\beta\gamma} W^\beta \wedge W^\gamma, \quad (4.2)$$

where $\hat{f}^{\alpha}_{\beta\gamma}$ is the deformed structure constant. The Bianchi identity is

$$\partial_{[\gamma} \hat{f}^{\alpha}_{\rho\sigma]} + \hat{f}^{\alpha}_{\beta[\gamma} \hat{f}^{\beta}_{\rho\sigma]} = 0, \quad (4.3)$$

where $\partial_{\gamma} = W^\bar{N}_{\gamma} \partial_{\bar{M}}$. In addition, we can add the 0-form matter field $H$ on $M$ as follows

$$dH = H_{\alpha} W^\alpha \Leftrightarrow \partial_{\alpha} H = H_{\alpha}, \quad (4.4)$$

$$\partial_{[\rho} H_{\sigma]} + H_{\alpha} \hat{f}^{\alpha}_{\rho\sigma} = 0. \quad (4.5)$$

The group manifold $M$ is necessarily involved in the definition of $R^\alpha_{\bar{\beta}\bar{\gamma}}$ and $H_{\alpha}$. (4.3) and (4.5) are defined in $M$ as well.

The definition (4.2) and (4.4) is invariant under the diffeomorphism transformation generated by $\xi^\bar{M}$,

$$\delta_{\xi} W^\alpha_{\bar{M}} = \xi^\bar{N} \partial_{\bar{N}} W^\alpha_{\bar{M}} + \partial_{\bar{M}} \xi^\bar{N} W^\alpha_{\bar{N}}, \quad \delta_{\xi} \hat{f}^{\alpha}_{\beta\gamma} = \xi^\bar{N} \partial_{\bar{N}} \hat{f}^{\alpha}_{\beta\gamma}, \quad \delta_{\xi} H = \xi^\bar{N} \partial_{\bar{N}} H, \quad \delta_{\xi} H_{\alpha} = \xi^\bar{N} \partial_{\bar{N}} H_{\alpha}. \quad (4.6)$$

With

$$e^\alpha = \xi^\bar{M} W^\alpha_{\bar{M}}, \quad \xi^\bar{M} = e^\alpha W^\alpha_{\bar{M}}, \quad (4.7)$$
Nevertheless, from (4.3) and (4.5), we have

\[ H \]

with

\[ M \]

deformed. However, at least in Vasiliev theory, in Section 2, we have discussed the coset space

\[ \text{effective reduces to the coset space} \]

\[ M = G[\text{ho}(1|2 : [3, 2])/SO(3, 1)]. \]

Recall that in Section 2, we have discussed the coset space \( M = G[\text{ho}(1|2 : [3, 2])/E]. \) For \( M \) to reduce to \( M \), there must be \( R_{Q \gamma} = 0 \) so that the local gauge transformation generated by \( \epsilon^\beta \) is undeformed. However, at least in Vasiliev theory, \( R_{(ab)\gamma} = 0 \) is valid but \( R_{Q \gamma} = 0 \) does not necessarily hold.

When \( \beta \neq (ab) \), \( \partial_\beta \hat{f}_\rho^\alpha \) and \( \partial_\beta H_\alpha \) cannot be uniquely determined by (4.3) and (4.5). Nevertheless, from (4.3) and (4.5), we have

\[ \partial_\gamma R_{ab}^\alpha = \partial_\beta R_{a\gamma}^\alpha + \hat{f}_{\beta[\gamma} f_{\alpha]}^\beta \]

\[ \partial_\gamma H_\alpha = \partial_\beta H_\gamma + H_\alpha f_{a\gamma}^\alpha \]

with \( H_\alpha = \partial_\alpha H \). Let \( R_{ab;c_1\ldots c_n} = \partial_{c_1} \ldots \partial_{c_n} R_{ab}^\alpha, H_{c_1\ldots c_n} = \partial_{c_1} \ldots \partial_{c_n} H \), if

\[ R_{b\gamma}^\alpha = r_{b\gamma}^\alpha (R_{ab}^\sigma, R_{ab;c_1\ldots c_n}^\sigma, H, H_{c_1\ldots c_n}) \]

\[ H_\gamma = h_\gamma (R_{ab}^\sigma, R_{ab;c_1\ldots c_n}^\sigma, H, H_{c_1\ldots c_n}) \]

with \( r_{b\gamma}^\alpha \) and \( h_\gamma \) the polynomials of \( R_{ab}^\sigma, R_{ab;c_1\ldots c_n}^\sigma, H, H_{c_1\ldots c_n} \) whose coefficients are constants on \( M \), then

\[ \partial_\gamma R_{ab}^\alpha = \partial_\beta R_{a\gamma}^\alpha + \hat{f}_{\beta[\gamma} f_{\alpha]}^\beta = r_{ab;\gamma}^\alpha (R_{ab}^\sigma, R_{ab;c_1\ldots c_n}^\sigma, H, H_{c_1\ldots c_n}) \]

\[ \partial_\gamma H_\alpha = \partial_\beta H_\gamma + H_\alpha f_{a\gamma}^\alpha = h_{a;\gamma} (R_{ab}^\sigma, R_{ab;c_1\ldots c_n}^\sigma, H, H_{c_1\ldots c_n}) \]

are also polynomials. Moreover, since

\[ (\partial_\beta \partial_\gamma - \partial_\gamma \partial_\beta) F = \hat{f}_{\gamma\beta}^\alpha \partial_\alpha F, \]

(4.6) can be rewritten as

\[ \delta W^\alpha = de^\alpha + \hat{f}_\beta^\alpha e^\beta W^\gamma, \quad \delta \hat{f}_\rho^\alpha = \epsilon^\beta \partial_\beta \hat{f}_\rho^\alpha, \quad \delta H = \epsilon^\beta H_\beta, \quad \delta_\gamma H_\alpha = \epsilon^\beta \partial_\beta H_\alpha, \]

which is the deformed local higher spin transformation.

\[ \delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2} = \delta_{[\epsilon_2, \epsilon_1]}, \quad [\epsilon_2, \epsilon_1]^\alpha = \hat{f}_{\beta\gamma}^\epsilon_2 \epsilon_1^\beta. \]

The algebra is closed with the deformed structure constant \( \hat{f}_{\beta\gamma}^\epsilon \).
\[
(\partial_r \partial_t - \partial_t \partial_r) R^{\alpha}_{ab} = \hat{f}_{\gamma}^\alpha \partial_{\sigma} R^{\alpha}_{ab} = \hat{f}_{\gamma}^\alpha \epsilon^{\alpha}_{ab,\sigma} (R^3_{ab}, R^3_{ab, c_1}, \ldots, H, H_{c_1}, \ldots), \tag{4.17}
\]
\[
(\partial_c \partial_{\gamma} - \partial_{\gamma} \partial_c) H_a = \hat{f}_{\gamma}^\alpha \partial_{\sigma} H_a = \hat{f}_{\gamma}^\alpha \epsilon^{\alpha}_{a,\gamma} (R^3_{ab}, R^3_{ab, c_1}, \ldots, H, H_{c_1}, \ldots), \tag{4.18}
\]
so
\[
\partial_\gamma R^{\alpha}_{ab,c} = \partial_\gamma \partial_c R^{\alpha}_{ab} = \partial_c R^{\alpha}_{ab} - \hat{f}^\alpha_{\gamma} \epsilon^{\alpha}_{ab,\gamma} = r^{\alpha}_{ab,c} \tag{4.19}
\]
and
\[
\partial_\gamma H_{ac} = \partial_\gamma H_{ac} = \partial_\gamma \partial_c H_a = \partial_c h_{a,\gamma} - \hat{f}^\alpha_{\gamma} h_{a,\gamma} = h_{a,\gamma} \tag{4.20}
\]
are again polynomials. Subsequently, one can prove for \(n = 0, 1, \ldots\), we have
\[
\partial_\gamma R^{\alpha}_{ab,c_1 \ldots c_n} = r^{\alpha}_{ab,c_1 \ldots c_n\gamma} (R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots), \tag{4.21}
\]
or equivalently,
\[
dR^{\alpha}_{ab,c_1 \ldots c_n} = r^{\alpha}_{ab,c_1 \ldots c_n\gamma} (R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots) W^\gamma, \tag{4.22}
\]
where \(r^{\alpha}_{ab,c_1 \ldots c_n\gamma}\) and \(h_{c_1 \ldots c_n\gamma}\) are polynomials of \(R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots\). Finally, we get the unfolded equation
\[
dW^\alpha = \frac{1}{2} (f^\alpha_{\beta\gamma} + r^{\alpha}_{\beta\gamma}) W^\beta \wedge W^\gamma, \\
dR^{\alpha}_{ab,c_1 \ldots c_n} = r^{\alpha}_{ab,c_1 \ldots c_n\gamma} W^\gamma, \\
dH_{c_1 \ldots c_n} = h_{c_1 \ldots c_n\gamma} W^\gamma, \tag{4.23}
\]
with \(n = 0, 1, \ldots\), \(r^{\alpha}_{\beta\gamma}, r^{\alpha}_{ab,c_1 \ldots c_n\gamma}\) and \(h_{c_1 \ldots c_n\gamma}\) the functions of \(R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots\). From \((R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots)\) at one point, \((W^\alpha, H)\) on the whole \(M\) can be determined up to a gauge transformation. \ref{4.23} is invariant under the local gauge transformation \ref{4.8} which can now be explicitly written as
\[
\delta_\epsilon W^\alpha = d\epsilon^\alpha + \hat{f}^\alpha_{\sigma\gamma} \epsilon^\sigma W^\gamma, \\
\delta_\epsilon R^{\alpha}_{ab,c_1 \ldots c_n} = \epsilon^\sigma r^{\alpha}_{ab,c_1 \ldots c_n\sigma}, \\
\delta_\epsilon H_{c_1 \ldots c_n} = \epsilon^\sigma h_{c_1 \ldots c_n\sigma}. \tag{4.24}
\]
\((R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots)\) forms a complete higher spin multiplet.

The parameterization \ref{4.13} should satisfy the Bianchi identity
\[
\partial_{[\gamma} R^{\alpha}_{\rho\sigma]} + \hat{f}^{\alpha}_{[\gamma\sigma]} \hat{f}^\beta_{\rho\sigma]} = 0, \quad \partial_{[\gamma} H^{\beta]} + H_{\alpha} \hat{f}^{\alpha}_{\gamma\beta} = 0. \tag{4.25}
\]
With \ref{4.21} and \ref{4.13} plugged in \ref{4.25}, we get
\[
F^\alpha_{[\gamma\rho\sigma]} (R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots) = 0, \quad F_{[\gamma\sigma]} (R^3_{ab}, R^3_{ab,c_1}, \ldots, H, H_{c_1}, \ldots) = 0, \tag{4.26}
\]
where $F_{\gamma\rho\sigma}^{\alpha}$ and $F_{\beta\gamma}^{\beta}$ are also polynomials of $R_{ab}^{\alpha}, R_{abc}^{\beta}, \ldots, H, H_{c_1}, \ldots$. \((4.26)\) gives the 4d equations of motion for \((R_{ab}^{\alpha}, R_{abc}^{\beta}, \ldots, H, H_{c_1}, \ldots)\). For the randomly selected \((r_{\beta\gamma}^{\alpha}, h_{\gamma})\), \((4.26)\) only has the trivial solution $R_{ab}^{\beta} = H = 0$. \((r_{\beta\gamma}^{\alpha}, h_{\gamma})\) should be chosen to allow as many on-shell degrees of freedom as possible. In this sense, \((4.26)\) determines both \((r_{\beta\gamma}^{\alpha}, h_{\gamma})\) and the 4d equations of motion.

To guarantee the local Lorentz invariance, in \((4.13)\), $R_{(ab)}^{\alpha\beta} = H_{(ab)} = 0$. Since

$$\partial_{(ab)} R_{\rho\sigma}^{\alpha} = \frac{\partial R_{\rho\sigma}^{\alpha}}{\partial R_{\de c_1 \cdots c_n}^{\beta}} \partial_{(ab)} R_{\de c_1 \cdots c_n}^{\beta} + \frac{\partial R_{\rho\sigma}^{\alpha}}{\partial H_{c_1 \cdots c_n}} \partial_{(ab)} H_{c_1 \cdots c_n},$$

$$\partial_{(ab)} H_{\alpha} = \frac{\partial H_{\alpha}}{\partial R_{\de c_1 \cdots c_n}^{\beta}} \partial_{(ab)} R_{\de c_1 \cdots c_n}^{\beta} + \frac{\partial H_{\alpha}}{\partial H_{c_1 \cdots c_n}} \partial_{(ab)} H_{c_1 \cdots c_n},$$

where $\partial_{(ab)} R_{\rho\sigma}^{\alpha}$, $\partial_{(ab)} H_{\alpha}$, $\partial_{(ab)} R_{\de c_1 \cdots c_n}^{\beta}$ and $\partial_{(ab)} H_{c_1 \cdots c_n}$ are all standard local Lorentz transformations, the coefficients in $r_{\rho\sigma}^{\alpha}$ and $h_{\alpha}$ must be Lorentz invariants. In fact, \((4.27)\) are also included in \((4.25)\), so the Lorentz invariance of $r_{\rho\sigma}^{\alpha}$ and $h_{\alpha}$ is also the requirement of the Bianchi identity if $R_{(ab)}^{\alpha\beta} = H_{(ab)} = 0$.

We only considered the equation \((4.23)\) on group manifold $M$, since in that space, the diffeomorphism transformation and the local gauge transformation are in one-to-one correspondence. As the universal property of the unfolded equation \((1.13), (4.23)\) is well-defined in space $m$ with $\dim m \geq 4$. If $\dim m < \dim M$, different diffeomorphism transformations may be realized as the same gauge transformation, i.e. there are flat directions with $\xi^M W_{\mu}^{\alpha} = 0$; if $\dim m > \dim M$, some gauge transformation does not have the diffeomorphism realization like that in $AdS_4$.

The initial value is $(R_{ab}^{\alpha}, R_{abc}^{\beta}, \ldots, H, H_{c_1}, \ldots)$ at one point, it is desirable to express it in terms of $(W_{\mu}^{\alpha}, H)$ as well as its 4d derivatives at that point.

\begin{align*}
R_{\mu\nu}^{\alpha} &= \xi_\mu W_{\nu}^{\beta} W_\alpha^{\gamma} \\
\partial_\lambda R_{\mu\nu}^{\alpha} &= (\frac{\partial r_{\rho\sigma}^{\beta}}{\partial R_{ab,c_1 \cdots c_n}^{\gamma}} r_{ab,c_1 \cdots c_n}^{\sigma,\rho} + \frac{\partial h_{c_1 \cdots c_n}}{\partial H_{c_1 \cdots c_n}}) W_{\lambda}^{\rho} W_{\mu}^{\beta} W_\nu^{\gamma} + \xi_\mu \partial_\lambda W_{\mu}^{\beta} W_\nu^{\gamma} \\
\vdots

H_{\mu} &= h_{\alpha} W_{\mu}^{\alpha} \\
\partial_\lambda H_{\mu} &= (\frac{\partial h_{\alpha}}{\partial R_{ab,c_1 \cdots c_n}^{\gamma}} r_{ab,c_1 \cdots c_n}^{\sigma,\rho} + \frac{\partial h_{c_1 \cdots c_n}}{\partial H_{c_1 \cdots c_n}}) W_{\lambda}^{\rho} W_{\mu}^{\gamma} + h_{\alpha} \partial_\lambda W_{\mu}^{\alpha} \\
\vdots
\end{align*}

(4.28)

$r$ and $h$ are functions of $(R_{ab,c_1 \cdots c_n}^{\alpha}, H_{c_1 \cdots c_n})$. In \((4.28)\), the unknowns are $(R_{ab,c_1 \cdots c_n}^{\alpha}, H_{c_1 \cdots c_n})$, while the number of equations is the same as the number of the degrees of freedom of $(R_{\mu\nu;\lambda_1 \cdots \lambda_n}^{\alpha}, H_{\lambda_1 \cdots \lambda_n})$, where $\mu, \nu = 1, 2, 3, 4$. \((4.11)\) and \((4.12)\) also impose constraints on the off-shell $(R_{\mu;\nu;\lambda_1 \cdots \lambda_n}^{\beta}, H_{\lambda_1 \cdots \lambda_n})$ to make it have the same number of degrees of freedom as $(R_{\mu;\nu;\lambda_1 \cdots \lambda_n}^{\beta}, H_{\lambda_1 \cdots \lambda_n})$, so in principle, from \((4.28)\), $(R_{ab,c_1 \cdots c_n}^{\alpha}, H_{c_1 \cdots c_n})$ can be solved in terms
of \((W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots)\).

\[
R^\alpha_{ab|\cdots|c_n} = g_{ab,|\cdots|c_n}^\alpha(W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots),
H_{c_1,\cdots,c_n} = q_{c_1,\cdots,c_n}(W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots).
\]

(4.29)

The local gauge transformation of \((W_\mu^\alpha, H)\) in \(AdS_4\) is

\[
\delta W_\mu^\alpha = \partial_\mu \epsilon^\alpha + \hat{f}_{\sigma\gamma}^\alpha(R^\beta_{ab}, R^\beta_{abc_1}, \cdots, H, H_{c_1}, \cdots) \epsilon^\sigma W_\mu^\gamma
\]

\[
\delta H = \epsilon^\sigma v_\sigma(W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots)
\]

(4.30)

With (4.29) plugged in (4.30),

\[
\delta W_\mu^\alpha = \partial_\mu \epsilon^\alpha + u_{\sigma\gamma}^\alpha(W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots) \epsilon^\sigma W_\mu^\gamma
\]

\[
\delta H = \epsilon^\sigma v_{\sigma}(W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots)
\]

(4.31)

gives the local gauge transformation rule of the matter-gravity coupled system \((W_\mu^\alpha, H)\) in \(AdS_4\). Since

\[
(R^\beta_{ab}, R^\beta_{abc_1}, \cdots, H, H_{c_1}, \cdots) \sim (W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots),
\]

(4.32)

\((W_\alpha, H)\) on the whole \(\mathbb{M}\) is determined by the on-shell \((W_\mu^\alpha, \partial_\nu W_\mu^\alpha, \cdots, H, \partial_\nu H, \cdots)\) at one point, or equivalently, the on-shell \((W_\mu^\alpha, H)\) in \(AdS_4\). This is the rheonomy in higher spin theory. As is shown in Section 2, although the space is \(\mathbb{M}\) with the infinite dimension, the physical Hilbert space is still the same as the 4d higher spin theory. Imposing the rheonomy condition is a way to project out the physical degrees of freedom.

### 4.2 Group manifold approach to supergravity

In this subsection, we will give a review of the group manifold approach for supergravity \(\mathcal{N} = 1\). Some modification is made so that supergravity is treated in the same way as the above discussed higher spin theory. For \(\mathcal{N} = 1\) supergravity in \(R^{5,1}\), the coordinate in the group manifold is \((x^\mu, x^{\mu\nu}, \theta^\alpha)\), the associated 1-form is \(\nu^A = (\omega^{ab}, e^a, \psi^\alpha)\) and the 0-form matter field is \(H\). We have

\[
d\nu^A = \frac{1}{2} \hat{f}_{BC}^A \nu^B \wedge \nu^C,
\]

\[
dH = H_A \nu^A,
\]

\[
\partial_{[E] \hat{f}_{BC}^A} + \hat{f}_{D[E] \hat{f}_{BC}^A} = 0,
\]

\[
\partial_{[A \hat{H}_B]} + \hat{H}_C \hat{f}_{AB}^C = 0,
\]

(4.33)

where \(\hat{f}_{BC}^A = f_{BC}^A + R_{BC}^A, H_B = (H_a, H_{(ab)}, H_\alpha)\). \(f_{BC}^A\) is the structure constant of the super Poincare group \(Osp(4|1)\). (4.33) is invariant under the diffeomorphism transformation in group manifold generated by \(\xi^M = (\xi^\mu, \xi^\nu, \xi^\lambda)\)

\[
\delta \nu^A_M = \xi^N \partial_N \nu^A_M + \partial_M \xi^N \nu^A_N,
\]

\[
\delta \hat{f}_{BC}^A = \xi^N \partial_N \hat{f}_{BC}^A,
\]

\[
\delta H = \xi^N \partial_N H,
\]

\[
\delta H_A = \xi^N \partial_N H_A,
\]

(4.35)

\(^5\)Here, \(\alpha\) represents the spinor index, which should be distinguished from \(\alpha\) in the rest sections, which represents the adjoint representation of \(ho(1|2 : [3, 2])\). Also, \(\alpha\) here is equivalent to the spinor index \((\alpha, \dot{\alpha})\) in section 3.
which, when written in terms of the components, are local Lorentz transformation, the 4d
diffeomorphism transformation and the supersymmetry transformation respectively. With
\( \epsilon^A = \xi^M \nu^A_M \), (4.35) can be rewritten as

\[
\delta_{\epsilon} \nu^A = d\epsilon^A + \hat{f}_{BC}^A \epsilon^B \nu^C, \quad \delta_{\epsilon} \hat{f}_{BC}^A = \epsilon^D \partial_D \hat{f}_{BC}^A, \quad \delta_{\epsilon} H = \epsilon^D H_D, \quad \delta_{\epsilon} H_A = \epsilon^D \partial_D H_A. \tag{4.36}
\]

Until now, no dynamics is involved at all. The dynamical information is brought by
imposing the suitable constraints on \( R_{BC}^A \) and \( H_A \). Here, the constraints that will be imposed are

(a) Factorization condition \( R_{(ab)c}^A = 0 = H_{(ab)} \);
(b) Rheonomy condition and the torsion constraint:

\[
(i) \quad R_{BC}^A = r_{BC}^A(R_{ab}^{cd}, R_{ab,c_1}^{cd}, \ldots, R_{ab}^{\alpha}, R_{ab,c_1}^{\alpha}, \ldots, H, H_{c_1}, \ldots, H_A, H_{\alpha,c_1} \ldots), \tag{4.37}
\]

or

\[
(ii) \quad R_{BC}^A = r_{BC}^A(R_{ab}^{cd}, R_{ab,c_1}^{cd}, \ldots, R_{ab}^{\alpha}, R_{ab,c_1}^{\alpha}, \ldots, H, H_{c_1}, \ldots)
\]

\[
H_A = h_A(R_{ab}^{cd}, R_{ab,c_1}^{cd}, \ldots, R_{ab}^{\alpha}, R_{ab,c_1}^{\alpha}, \ldots, H, H_{c_1}, \ldots). \tag{4.38}
\]

(a) is imposed so that the local Lorentz transformation is undeformed. In (b), the rheonomy
condition requires that the lower index of the independent fields can only contain \( a \) so that
the whole dynamics in group manifold is determined by that in a 4d submanifold; torsion
constraint requires that the upper index cannot be \( a \) so that \( \omega^{ab} \) can be solved in terms of
the rest fields. There are two possibilities. In (i), the final dynamical fields are \((e^a, \psi^a, H, H_A)\)
in \( M_4 \), which is the situation for \( \mathcal{N} = 1 \) supergravity coupled to the WZ matter. In (ii),
the dynamical fields are \((e^a, \psi^a, H)\) in \( M_4 \) like that in higher spin theory.

\( r_{BC}^A \) and \( h_A \) are polynomials, the coefficients of which should be selected so that some
scaling relation is respected \([\Pi]\). The weight of \( t^A \) is denoted as \( w(A) \), \( w(a) = 1 \), \( w(ab) = 0 \),
\( w(\alpha) = 1/2 \). The super Poincare algebra \([t_A, t_{A'}] = if_{A,A'}^A t_A \) is invariant under \( t_A \rightarrow \nu^{-w(A)} \). The 0-forms \( H_A \) and \( R_{BC}^A \) have the weight \(-w(A)\) and \( w(A) - w(B) - w(C)\) as follows

| \( H_a \) | \( H_\alpha \) | \( R_{ab}^{cd} \) | \( R_{aa}^{\alpha \beta} \) | \( R_{\alpha \beta}^{\alpha \beta} \) | \( R_{ab}^{\gamma} \) | \( R_{a}^{\gamma} \) | \( R_{\alpha \beta}^{\gamma} \) |
|---|---|---|---|---|---|---|---|
| \(-1\) | \(-1/2\) | \(-2\) | \(-3/2\) | \(-1\) | \(-1\) | \(-1/2\) | \(-2\) |

Especially, \((R_{ab}^{cd}, R_{ab}^{\alpha \beta}, H_A, H_{\alpha}, H_\alpha)\) have the weight \((-2, -3/2, 0, -1, -1/2)\). (ii) cannot satisfy
the scaling relation thus should be ruled out. For (i), with the \( H_A \) odd terms dropped, the
most general form of \( r_{BC}^A \) is

\[
R_{aa}^{bc} = r_{aa}^{bc} d e^{de} + r_{aa}^{bc} |^{\beta} d H_{\beta} H_D, \\
R_{ab}^{c} = r_{ab}^{c} |^{\alpha \beta} H_\alpha H_\beta, \\
R_{aa}^{\alpha \beta} = r_{aa}^{\alpha \beta} |^{\rho \sigma} H_\rho H_\sigma, \\
R_{\alpha \beta}^{cd} = r_{\alpha \beta}^{cd} |^{\rho \sigma} H_\rho H_\sigma, \\
R_{aa}^{c} = R_{\alpha \beta}^{c} = R_{\alpha \beta}^{\gamma} = 0. \tag{4.39}
\]
where \( r^*_a r^*_a = \frac{1}{2} r^*_a r^*_a (H) \) are functions of \( H \) since \( H \) has the weight 0. \( r^*_a |^* a = \) should be a Lorentz invariant to preserve the local Lorentz invariance. Although the torsion constraint is also imposed, \( R^a_{AB} \) does not need to vanish, see for example \( [14] \). However, if \( H_a = H_a = 0 \), \( R^a_{AB} = 0 \), and in pure supergravity case, we do have \( R^a_{AB} = 0 \). Due to the scaling relation, the rheonomy condition is greatly simplified. For supergravity in \( AdS_4 \) with the symmetry group \( Osp(4|1) \), a constant \( L \) with the weight 1 is involved. \( \lim_{L \to \infty} \) gives the flat space limit, so only the \( \lim_{L \to \infty} \) terms with \( n \geq 0 \) are allowed in rheonomy condition. \( (4.39) \) is then unmodified.

\( (4.39) \) should satisfy the Bianchi identity

\[
\partial_\alpha f^A_{BC} + f^A_{DF} f^D_{BC} + f^A_{D(E} f^D_{BC)] + R^A_{D[E} R^D_{BC]} = 0, \quad \partial_\alpha H_B + H_C f^C_{AB} + H_C R^C_{AB} = 0. \tag{4.40}
\]

In pure supergravity situation with \( H = 0 \), \( r^A_{BC} \) becomes \( R^a_{bc} = r^a_{bc |de} R^d_{de} \), \( \partial_\alpha f^A_{BC} = R^c_{\alpha \beta} = R^c_{\alpha \beta} = 0 \). \( (4.40) \) reduces to

\[
\partial_\alpha R^A_{BC} = f^A_{(ab)D} R^D_{BC} - f^A_{(ab)C} R^A_{BD} - f^A_{(ab)B} R^A_{DC}, \tag{4.41}
\]

\[
f^a_{(ef)[b} R^f_{\alpha c]} = 0, \quad \partial_\alpha f^a_{bc} = 0, \quad \partial_\alpha R^a_{bc} + f^a_{(ab)\beta} R^\beta_{bc} = 0, \quad \partial_\alpha f^a_{bc} = 0, \quad \partial_\beta R^a_{bc} + f^a_{(ab)\alpha} R^\alpha_{bc} = 0, \tag{4.42}
\]

\[
f^a_{(ab)|b} R^f_{e\alpha} = 0, \quad R^a_{ac} f^f_{\beta \gamma} + f^a_{(ab)\beta} R^f_{\gamma c} = 0, \quad \partial_\alpha f^a_{ab} + f^a_{(ab)\beta} R^\beta_{bc} = 0. \tag{4.43}
\]

\( (4.41) \) gives the Lorentz transformation of \( R^A_{BC} \), which can be preserved in \( R^A_{BC}(R^a_{bc}, R^\beta_{bc}) \) if \( r^a_{bc |de} \) is a Lorentz scalar. \( (4.42) \) are Bianchi identities in 4d. \( (4.43) \) gives the evolution of \( (R^a_{bc}, R^\beta_{bc}) \) along the \( \alpha \) direction. With \( (4.43) \) plugged in \( (4.44) \), \( r^a_{bc |de} \) can be fixed and the 4d equations of motion

\[
R^a_{bc} - \frac{1}{2} \delta_a^{(c} R^b_{d) = 0, \quad R^a_{bc} (\gamma_5 \gamma_\beta) R^b_{de} = 0 \tag{4.45}
\]

come out. If we use the on-shell \( \tilde{R}^a_{bc} \) and \( \tilde{R}^a_{bc} \) satisfying \( (4.45) \) to parameterize \( r^A_{BC} \). \( (4.44) \) will hold automatically. This is in analogy with the Vasiliev theory, with \( R^a_{BC} \) parametrized by the 0-form \( \Phi^a \) in the twisted-adjoint representation of the higher spin algebra, the Bianchi identity is satisfied for the arbitrary \( \Phi^a \).

Written as the unfolded equation,

\[
d\nu^A = \frac{1}{2} (f^A_{BC} + r^A_{BC}) \nu^B \wedge \nu^C, \tag{4.46}
\]

\[
d R^a_{abc1 \cdots c_n} = r^a_{abc1 \cdots c_n} \nu^A, \quad d R^a_{abc1 \cdots c_n} = r^a_{abc1 \cdots c_n} \nu^A, \quad d H_{c1 \cdots c_n} = h_{c1 \cdots c_n} \nu^A, \quad d H_{a1 \cdots c_n} = h_{a1 \cdots c_n} \nu^A,
\]

where \( r, h \) are all determined by \( r^A_{BC} = r^A_{BC}(R^a_{bc}, R^\alpha_{bc}, H_a, H_a) \) and are functions of \( (R^a_{abc1 \cdots c_n}, R^\alpha_{abc1 \cdots c_n}, H_{c1 \cdots c_n}, H_{a1 \cdots c_n}) \). With the on-shell \( (R^a_{abc1 \cdots c_n}, R^\alpha_{abc1 \cdots c_n}, H_{c1 \cdots c_n}, H_{a1 \cdots c_n}) \) given at one point, \( (\nu^A, R^a_{abc1 \cdots c_n}, R^\alpha_{abc1 \cdots c_n}, H_{c1 \cdots c_n}, H_{a1 \cdots c_n}) \) on the whole \( M \) can be solved. The local
gauge transformation is
\[ \delta_\nu \nu^A = d\epsilon^A + \tilde{f}_{BC}^A \epsilon^B \nu^C, \]
\[ \delta_\nu R_{abcd} = \epsilon_{\nu} R_{abcd}, \quad \delta_\nu \alpha R_{abc} = \epsilon^\alpha R_{abc} \alpha, \]
\[ \delta_\nu H_{c1} = \epsilon^A h_{c1}, \quad \delta_\nu H_{\alpha c} = \epsilon^A h_{\alpha c}. \]  
(4.47)

\( R_{ab}, R_{abc}, \cdots, R_{ab}, R_{abc}, \cdots, H, H_{c1}, \cdots, H_{\alpha c1}, \cdots \) compose the complete supersymmetry multiplet.

In addition to the 1-form \( \nu^A \), the 0-form multiplet is introduced, forming the representation of the deformed local super Poincare transformation. The physical interpretation of the 0-form is the curvature and the matter field plus their derivatives. This is in the same spirit as the higher spin theory. Different from the higher spin theory, the rheonomy condition only allows the trivial solution \( R_{ab} = H = 0 \) when the Bianchi identity is imposed, no matter how coefficients in \( (r_{\alpha}, h) \) are adjusted.

Again,
\[ (R_{ab}, R_{abc}, \ldots, R_{ab}, R_{abc}, \ldots, H, H_{c1}, \ldots, H_{\alpha c}, \cdots) \sim (\epsilon^\mu, \partial_\nu \epsilon^\mu, \ldots, \psi^\alpha, \partial_\nu \psi^\alpha, \ldots, H, \partial_\nu H, \ldots, H_{\alpha c}, \ldots). \]  
(4.48)

With the on-shell \( (\epsilon^\mu, \psi^\alpha, H, H_\alpha) \) given on \( M_4 \), \( (\nu^A, H) \) on the whole group manifold can be determined up to a gauge transformation.

The dynamics is entirely encoded in function \( r_{BC}^A (R_{bc}, R_{bc}', H, H_\alpha) \). By setting \( H = 0 \), we obtain the pure supergravity situation. Alternatively, one can consider the dynamics of the 0-form matter on the fixed supergravity background by setting \( r_{BC}^A \rightarrow 0 \). With \( \tilde{f}_{BC}^A = f_{BC}^A \), (4.33) and (4.34) reduce to
\[ d\nu_0^A = \frac{1}{2} f_{BC}^A \nu_0^B \wedge \nu_0^C, \quad dH = H_A \nu_0^A, \quad \partial_A H_B + H_C f_{AB}^C = 0. \]  
(4.49)

\( \nu_0^A \) describes the intrinsic geometry of the group manifold. The allowed gauge transformation parameter \( \epsilon_0^A \) should make \( \nu_0^A \) invariant
\[ \delta_\epsilon_0 \nu_0^A = \partial_M \epsilon_0^A + f_{BC}^A \epsilon_0^B \nu_0^C = 0. \]
(4.50)

\( \epsilon_0^A \) generates the global super Poincare transformation on group manifold.
\[ \delta_\epsilon_0 H = \xi_0^M \partial_M H = \epsilon_0^D H_D, \quad \delta_\epsilon_0 H_A = \xi_0^M \partial_M H_A = \epsilon_0^D \partial_D H_A. \]  
(4.51)

\( \xi_0^M \nu_0^D = \epsilon_0^D \). Still, \( H_{(ad)} = 0 \).
\[ \delta_{(ad)} H = 0, \]
\[ \delta_{(ad)} H_c + H_{\beta} f_{(ad)c}^\beta = 0, \]
\[ \delta_{(ad)} H_\alpha + H_{\beta} f_{(ad)\alpha}^\beta = 0. \]  
(4.52)
Evolution along \((ad)\) direction is the Lorentz transformation. One cannot assume \(H_\alpha\) is the function of \((H, H_{c_1}, H_{c_1 c_2}, \cdots)\), since the scaling relation is not respected. Let \(\alpha = (\lambda, \dot{\lambda})\), one can at most require \(H_\lambda = 0\), which is the chiral constraint for superfield.

### 4.3 Imposing the torsion constraint in higher spin theory

Back to the higher spin theory, a further reduction of (4.32) can be made by imposing the following torsion constraint

\[
R^\alpha_{\beta \gamma} = \gamma^\alpha_{\beta \gamma} \left( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} \right) + \cdots, H, H_{c_1}, \cdots,
\]

\[
H_{\gamma} = h_{\gamma} \left( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} \right) + \cdots, H, H_{c_1}, \cdots.
\]

Namely, in (4.13), \(\sigma\) is restricted to \([a(s - 1), b(s - 1)]\) with \(s = 2, 4, \cdots\). In (4.28), the number of equations is equal to the number of degrees of freedom of \((R^{a(s-1), b(s-1)}_{ab}, c_{1\cdots c_n})\) but the number of unknowns is equal to the degrees of freedom of \((R^{a(s-1), b(s-1)}_{ab}, c_{1\cdots c_n})\) now, so effectively, there will be some constraints imposed on \((W_\mu, H)\) in \(AdS_4\) whose number is equal to the degrees of freedom of \(R^{a(s-1), b(t)}_{ab}\) with \(0 \leq t \leq s - 2\). It is expected that by solving these constraints, \(W^{a(s-1), b(t+1)}_\mu\) can be expressed in terms of \((W^{a(s-1), b(0)}_\mu, H)\).

In fact, at least in free theory limit, imposing the torsion constraint \(R_{ab}^{[a(s-1), b(t)]} = 0\) for \(0 \leq t \leq s - 2\) can indeed make \(W^{a(s-1), b(t+1)}_\mu\) solved in terms of \(W^{a(s-1), b(0)}_\mu\) \([4]\). (4.32) is then reduced to

\[
( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} ) \sim ( W^{a(s-1), b(0)}_\mu, \partial_\nu W^{a(s-1), b(0)}_\mu ) + \cdots, H, \partial_\nu H, \cdots.
\]

With \(W^{a(s-1), b(t+1)}_\mu\) written in terms of \((W^{a(s-1), b(0)}_\mu, H)\), (4.31) becomes

\[
\delta_\epsilon W^{a(s-1), b(0)}_\mu = \partial_\mu \epsilon^{a(s-1), b(0)} + \epsilon^{a(s-1), b(0)} ( W^{a(r-1), b(0)}_\mu, \partial_\nu W^{a(r-1), b(0)}_\mu ) + \cdots, H, \partial_\nu H, \cdots,
\]

\[
\delta_\epsilon W^{a(r-1), b(0)}_\mu = \partial_\mu \epsilon^{a(r-1), b(0)} ( W^{a(r-1), b(0)}_\mu, \partial_\nu W^{a(r-1), b(0)}_\mu ) + \cdots, H, \partial_\nu H, \cdots.
\]

which is the local gauge transformation rule of \((W^{a(s-1), b(0)}_\mu, H)\) in \(AdS_4\). One may expect it is

\[
h^{a_1 \cdots a_s} = W^{a_1 \cdots a_s W^{\mu a_1 \cdots a_{s-1} a_{s+1}, 0, \cdots 0}}_a W^{a_1 \cdots a_{s-1}, 0, \cdots 0}
\]

that will finally appear in equations of motion and the gauge transformation. The frame-like formulation reduces to the metric-like formulation.

Altogether, the complete equations are

\[
\tilde{f}^\alpha_{\beta \gamma} = \tilde{f}^\alpha_{\beta \gamma} \left( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} \right) + \cdots, H, H_{c_1}, \cdots,
\]

\[
H_{\gamma} = h_{\gamma} \left( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} \right) + \cdots, H, H_{c_1}, \cdots,
\]

\[
r^{a(s-1), b(s-1)}_{ab} = r^{a(s-1), b(s-1)}_{ab} \left( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} \right) + \cdots, H, H_{c_1}, \cdots,
\]

\[
h^{a_1 \cdots a_{s+1}}_{c_1 \cdots c_n} = h^{a_1 \cdots a_{s+1}}_{c_1 \cdots c_n} \left( R^{a(s-1), b(s-1)}_{ab}, R^{a(s-1), b(s-1)}_{abc} \right) + \cdots, H, H_{c_1}, \cdots,
\]
The input is \((\hat{f}_{\beta\gamma}^\alpha, h_\alpha)\), from which all the rest equations are determined. The left hand side of the 4d equations of motion \((4.64)-(4.65)\) are polynomials of \((R_{c1...c_n}^{\alpha(a(s-1),b(s-1))}, H_{c1...c_n})\). For the randomly selected \((\hat{f}_{\beta\gamma}^\alpha, h_\alpha)\), \((4.64)-(4.65)\) only has the trivial solution \(R_{c1...c_n}^{\alpha(a(s-1),b(s-1))} = H_{c1...c_n} = 0\). A natural question is what might be the maximum on-shell degrees of freedom. If one can find such \((\hat{f}_{\beta\gamma}^\alpha, h_\alpha)\), for which, \((4.64)-(4.65)\) is satisfied for the arbitrary \((R_{c1...c_n}^{\alpha(a(s-1),b(s-1))}, H_{c1...c_n})\), then there are no 4d equations of motion. However, this is not quite likely to be the case. By solving \((4.64)-(4.65)\), one may determine \((\hat{f}_{\beta\gamma}^\alpha, h_\alpha)\), which, when plugged in \((4.64)-(4.65)\), gives the 4d equations of motion for \((R_{c1...c_n}^{\alpha(a(s-1),b(s-1))}, H_{c1...c_n})\). In supergravity situation, the procedure is quite simple as is demonstrated in Section 4.2. In higher spin theory, the more direct way is to first determine the on-shell degrees of freedom \(\Phi^\alpha\). Then with the off-shell \((R_{c1...c_n}^{\alpha(a(s-1),b(s-1))}, H_{c1...c_n})\) expressed in terms of the on-shell \(\Phi^\alpha\), we only need to find \((\hat{f}_{\beta\gamma}^\alpha, h_\alpha)\) satisfying the Bianchi identity for the arbitrary \(\Phi^\alpha\). From the on-shell \((R_{c1...c_n}^{\alpha(a(s-1),b(s-1))}, H_{c1...c_n})\) at one point, or the on-shell \((W_\mu^{\alpha(a(s-1),b(0))}, H)\) in AdS, \((W_\alpha, H)\) on \(\mathcal{M}\) can be determined via \((4.61)-(4.63)\). With \((W_\alpha, H)\) on \(\mathcal{M}\) solved, the finite local higher spin transformation is the finite diffeomorphism transformation on \(\mathcal{M}\), under which, \((W_\mu^{\alpha(a(s-1),b(0))}, H)\) in one AdS slice is moved to \((W_\mu^{\alpha(a(s-1),b(0))}, H)\) in another AdS slice. The higher spin symmetry is realized as an on-shell symmetry.

4.4 Relation with the unfolded equation in Vasiliev theory

With \(\Phi^\alpha\) representing the on-shell degrees of freedom of \((R_{c1...c_n}^{\alpha(a(s-1),b(s-1))}, H_{c1...c_n})\), where \(\alpha\) is some representation of the Lorentz group, the unfolded equation becomes

\[
\begin{align*}
\hat{f}_{\beta\gamma}^\alpha &= \hat{f}_{\beta\gamma}^\alpha(\Phi^\alpha), \\
F^\alpha &= F^\alpha(\Phi^\alpha), \\
dW^\alpha &= \frac{1}{2} \hat{f}_{\beta\gamma}^\alpha W^\beta \wedge W^\gamma, \\
d\Phi^\alpha &= F^\alpha, \\
\frac{\partial F^\alpha}{\partial \Phi^\beta} + \hat{f}_{\beta\gamma}^\alpha f_{\rho\sigma}^\gamma = 0, \\
\frac{\partial F^\alpha}{\partial \Phi^\beta} F^\beta + F^\alpha \hat{f}_{\beta\gamma}^\gamma F^\rho_{\rho\sigma} = 0.
\end{align*}
\]
In Vasiliev theory, $\Phi^\tilde{\alpha} \sim \Phi^{a(s),b(s+n)}$ is in the twisted-adjoint representation of the higher spin algebra. \[ (4.69) \] can be obtained by solving Vasiliev equation order by order. \[ (4.68) \] is then automatically satisfied for the arbitrary $\Phi^\tilde{\alpha}$. Let us make a comparison between

$$\{ H_{\epsilon_1 \cdots \epsilon_n} | n = 0, 1, \cdots \} \cup \{ R^{a(s-1),b(s-1)}_{\alpha_{\epsilon_1 \cdots \epsilon_n}} | s = 2, 4, \cdots , n = 0, 1, \cdots \}$$

(4.69)

and $\{ \Phi^{a(s),b(s+n)} | s = 0, 2, \cdots , n = 0, 1, \cdots \}$. The two have the same number of indices, but the former is the off-shell field while the latter is the on-shell field. With the 4$d$ equations of motion imposed on \[ (4.69) \], the two may contain the same number of degrees of freedom.

Fields in the twisted-adjoint representation and the adjoint representation are related via the action of the Klein operator

$$\Phi^\alpha t_\alpha \rightarrow \Phi^\alpha t_\alpha \ast \kappa = \Phi^\alpha \rho_\alpha ^\tilde{\alpha} t_\tilde{\alpha} = \Phi^\tilde{\alpha} t_\tilde{\alpha} ,$$

(4.70)

where $\Phi^\tilde{\alpha} = \rho^\tilde{\alpha} \Phi^\alpha$, $\Phi^{\alpha \tilde{\alpha}} = (\rho^{-1})^\alpha _\tilde{\alpha} \Phi^\tilde{\alpha}$, $\rho^\tilde{\alpha}$ is a constant matrix. For $\Phi$ is in the adjoint representation, i.e. $\Phi^\alpha \sim \Phi^{a(s-1),b(t)}$, one may let $F^\alpha _\beta = f^\alpha _\beta \Phi^\gamma$ \[ (4.66)-(4.68) \] reduce to

$$f^\alpha _{\beta \gamma} = f^\alpha _{\beta \gamma}(\Phi^\sigma),$$

(4.71)

$$dW^\alpha = \frac{1}{2} f^\alpha _{\beta \gamma} W^\beta \wedge W^\gamma ,$$

(4.72)

$$d\Phi^{\alpha \tilde{\alpha}} = \frac{1}{2} f^\alpha _{\beta \gamma} W^\beta \Phi^{\gamma \tilde{\alpha}},$$

(4.73)

$$- \Phi^{\nu \rho}_{\nu \rho} \frac{\partial f^\alpha _{\beta \gamma}}{\partial \Phi^{\beta \gamma}} + f^\alpha _{\beta \gamma} f^\beta _{\rho \sigma} = 0.$$  

With $\Phi^\alpha \rightarrow \Phi^\tilde{\alpha}$, \[ (4.66)-(4.68) \] is recovered with $F^\alpha _{\beta \gamma} = \tilde{k}^\beta _{\beta \gamma} \Phi^\tilde{\alpha}$. $\tilde{k}^\alpha _{\beta \gamma} = \rho^\alpha \rho^\gamma _\beta \Phi^\tilde{\alpha}$. $\Phi^{a(0),b(0)} \equiv \Phi = H$, $\partial_\beta \Phi = \partial_\beta H = H_\beta = \tilde{k}^\theta _{\beta \gamma} \Phi^\tilde{\theta}$. $\partial_\beta \Phi^\tilde{\alpha} = \tilde{k}^\beta _{\beta \gamma} \Phi^\tilde{\theta}$. With \[ (4.66)-(4.68) \] at hand, we have $R^\alpha _{\beta \gamma} = R^\alpha _{\beta \gamma}(\Phi^\tilde{\alpha}) = f^\alpha _{\beta \gamma}(\Phi^\tilde{\alpha}) - f^\alpha _{\beta \gamma},$ $H_\beta = \tilde{k}^\beta _{\beta \gamma} (\Phi^\tilde{\alpha}) \Phi^\tilde{\alpha}.$

Especially,

$$R^a_{\alpha \beta} = R^{a(s-1),b(s-1)}_{\alpha \beta}(\Phi^\tilde{\alpha}),$$

(4.74)

and subsequently,

$$R^{a(s-1),b(s-1)}_{\alpha \beta \gamma} = R^{a(s-1),b(s-1)}_{\alpha \beta \gamma}(\Phi^\tilde{\alpha}),$$

(4.75)

where $\partial_\gamma \Phi^\tilde{\alpha} = \tilde{k}^\alpha _{\beta \gamma}(\Phi^\tilde{\alpha}) \Phi^\tilde{\alpha}$ is used.

$$\dot{f}^\alpha _{\beta \gamma}(R^{a(s-1),b(s-1)}_{\alpha \beta \gamma \delta}, H_{\epsilon_1 \cdots \epsilon_n}) = f^\alpha _{\beta \gamma}[R^{a(s-1),b(s-1)}_{\alpha \beta \gamma \delta}, H_{\epsilon_1 \cdots \epsilon_n}](\Phi^\tilde{\alpha}),$$

(4.76)

$$h_\gamma (R^{a(s-1),b(s-1)}_{\alpha \beta \gamma}, H_{\epsilon_1 \cdots \epsilon_n}) = h_\gamma [R^{a(s-1),b(s-1)}_{\alpha \beta \gamma}, H_{\epsilon_1 \cdots \epsilon_n}](\Phi^\tilde{\alpha}),$$

Let us return to the discussion below \[ (4.65) \]. With $\dot{f}^\alpha _{\gamma \delta}$ and $h_\gamma$ determined from the Bianchi identity, \[ (4.64)-(4.65) \] may still have further constraints on $(R^{a(s-1),b(s-1)}_{\alpha \beta \gamma \delta}, H_{\epsilon_1 \cdots \epsilon_n})$, which are just the 4$d$ equations of motion. Alternatively, one may use the on-shell fields $\Phi^\tilde{\alpha}$ to parameterize the off-shell fields $(R^{a(s-1),b(s-1)}_{\alpha \beta \gamma \delta}, H_{\epsilon_1 \cdots \epsilon_n})$ as in \[ (4.75) \]. The 4$d$ equations of motion are then solved automatically. \[ (4.64)-(4.65) \] do not impose any constraints on $\Phi^\tilde{\alpha}$. The key step in group manifold approach is to get the rheonomy condition and the
4d equations of motion from the Bianchi identity. For higher spin theory, the on-shell degrees of freedom form the twisted-adjoint representation of the higher spin algebra, while the Vasiliev equation gives an elegant way to solve the Bianchi identity. The solution for \((W^\alpha, H)\) on \(M\) is characterized by the on-shell \((R_{ab}^{[a(s-1),b(s-1)]}, H_{1\ldots n})\) at one point, or by the arbitrary \(\Phi^\alpha\) at that point. Nevertheless, it is \((R_{ab}^{[a(s-1),b(s-1)]}, H_{1\ldots n})\) that has the physical meaning. We are free to make a change of the variables \(\varphi^\alpha = \varphi^\alpha(\Phi^\beta)\) to use \(\varphi^\alpha\) to parameterize \((R_{ab}^{[a(s-1),b(s-1)]}, H_{1\ldots n})\). The good variables are those which are as relevant to \((R_{ab}^{[a(s-1),b(s-1)]}, H_{1\ldots n})\) as possible.

For supergravity in \(AdS_4\), by a rescaling \(t_{[a,0]} \rightarrow Lt_{[a,0]}\) and taking the \(L \rightarrow \infty\) limit, the \(so(3,2)\) algebra reduces to the Poincare algebra. However, for higher spin algebra, the similar flat space limit is not well-defined. For \(t_{\alpha_1} = t_{[a(s_1-1),b(t_1)]}\), we have

\[
[t_{\alpha_1}, t_{\alpha_2}] = if_{\alpha_1\alpha_2}t_{\alpha_3}, \quad w(\alpha_i) = s_i - 1 - t_i. \tag{4.77}
\]

By a rescaling \(t_{\alpha_1} \rightarrow L w(\alpha_1) t_{\alpha_1}, f_{\alpha_1\alpha_2} \rightarrow L w(\alpha_3) - w(\alpha_1) - w(\alpha_2) f_{\alpha_1\alpha_2}, \) so in flat space limit \(L \rightarrow \infty\), \(f_{\alpha_1\alpha_2}\) is finite when \(w(\alpha_3) = w(\alpha_1) + w(\alpha_2)\), \(f_{\alpha_1\alpha_2}\) is zero when \(w(\alpha_3) < w(\alpha_1) + w(\alpha_2)\), \(f_{\alpha_1\alpha_2}\) is infinite when \(w(\alpha_3) > w(\alpha_1) + w(\alpha_2)\). It is only when \(f_{\alpha_1\alpha_2} = 0\) for \(w(\alpha_3) > w(\alpha_1) + w(\alpha_2)\) the algebra has the well-defined flat space limit. This is the case for \(so(3,2)\) subalgebra but does not hold in general in \(ho(1|2 : [3, 2])\). Unlike \(Osp(4|1)\) which has a contracted version \(Osp(4|1), ho(1|2 : [3, 2])\) does not have the well-defined contraction with \(so(3,2)\) reducing to the Poincare algebra. As a result, in group manifold approach, supergravity can be defined in \(AdS_4\) as well as the 4d Minkowski space, but the higher spin theory can only be constructed in \(AdS_4\). The theory has the unique vacuum with \(R^\alpha_{\beta\gamma} = 0\). The corresponding geometry is discussed in Section 2. \(M\) has a fiber bundle structure with the \(AdS_4\) fiber attached at each point of the base space.

\(R^\alpha_{\beta\gamma}, H_{\gamma}, R_{ab}^{[a(s-1),b(s-1)]}\) and \(H_{1\ldots n}\) have the weight \(w(\alpha) - w(\beta) - w(\gamma), -w(\gamma), -n - 2\) and \(-n\) respectively. In Section 4.2, the scaling relation makes the rheonomy condition for supergravity in \(R^{3,1}\) simplify a lot. For supergravity in \(AdS_4\), the weight 1 parameter \(L\) is involved. To have the well-defined flat space limit, only the \(L^{-n}\) terms are allowed, so the rheonomy condition remains the same. For higher spin theory in \(AdS_4\), if we require the well-defined flat space limit so that only the \(L^{-n}\) terms are included in (4.53), then with the Bianchi identity imposed, (4.53) may only have the trivial solution \(R_{ab}^{[a(s-1),b(s-1)]} = H = 0\) no matter how its coefficients are adjusted. To allow for the nontrivial on-shell degrees of freedom, \(L^n\) terms must be added which will then give an ill-defined flat space limit. Since both \(L^{-n}\) and \(L^n\) terms are allowed, the scaling relation is not so powerful to restrict the form of (4.53).

Finally, higher spin theory should have the proper free theory limit that is equivalent to Fronsdal theory [10] [17]. In free theory limit, the equations of motion in (4.57)-(4.65)
so that (4.78) becomes
\[ dW^\alpha - f^\alpha_{(ab)\gamma} W^{(ab)} \land W^\gamma - f^\alpha_{a\gamma} W^a \land W^\gamma = \frac{1}{2} R^\alpha_{ab} W^a \land W^b, \]  
(4.78)
\[ dR^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n} = r^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n(cd)} W^{cd} + R^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n c}, \]  
(4.79)
\[ dH_{c_1\cdots c_n} = h_{c_1\cdots c_n(cd)} W^{cd} + H_{c_1\cdots c_n c} W^{c+1}, \]  
(4.80)

Note that \( r^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n} = \partial_{(cd)} R^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n} \) and \( h_{c_1\cdots c_n(cd)} = \partial_{(cd)} H_{c_1\cdots c_n} \) give the local Lorentz transformation, \( (4.77)-(4.80) \) can be rewritten as
\[ DR^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n} = R^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n c+1} W^{c+1}, \]  
(4.81)
\[ DH_{c_1\cdots c_n} = H_{c_1\cdots c_n c+1} W^{c+1}, \]  
(4.82)
where \( D \) is the standard covariant derivatives for \( W^{cd} \) and especially, \( DH = dH = H c W^c \).

For the theory to have the correct free theory limit, there will be
\[ R^{[a(s-1),b(t)]}_{ab} = 0 \quad \text{for} \quad t \neq s - 1 \]  
(4.83)
so that \( (4.78) \) becomes
\[ DW^{[a(s-1),b(t)]}_{a[c(s-1),d(t+1)]} = f^{[a(s-1),b(t)]}_{a[c(s-1),d(t+1)]} W^a \land W^c \land W^d + f^{[a(s-1),b(t)]}_{a[c(s-1),d(t-1)]} W^a \land W^c \land W^d, \]  
(4.84)
where \( t < s - 1 \). \( (4.83) \) is also called the “central on-mass-shell theorem” \( [18] \). In Vasiliev theory, \( R^\alpha_{\beta\gamma} \) satisfies \( (4.83) \) at the first order of the \( \Phi^\alpha \) expansion.

Since the adjoint representation and the twisted-adjoint representation are related by a Klein transformation which is invertible, we may try to use \( \Phi^\alpha \) to parameterize \( f^\alpha_{\beta\gamma} \) as in \( (4.71) \). If we further make a restriction that \( (4.72) \) can be written as
\[ dW = H(W, \Phi), \quad d\Phi = F(W, \Phi) \]  
(4.85)
with \( H(W, \Phi) \) and \( F(W, \Phi) \) polynomials of \( W = W^\alpha t_\alpha \) and \( \Phi = \Phi^\alpha t_\alpha \), the solution for \( (4.73) \) can be easy fixed, which is given in appendix C. Although the Bianchi identity is satisfied for the arbitrary \( \Phi^\alpha \), \( (4.83) \) does not hold at the first order of the \( \Phi^\alpha \) expansion, so the theory does not have the correct free theory limit.

Satisfying the Bianchi identity for the on-shell \( (R^{[a(s-1),b(s-1)]}_{abc_1\cdots c_n}, H_{c_1\cdots c_n}) \) and giving rise to the correct free theory limit are two requirements for \( (f^\alpha_{\beta\gamma}, h_{\gamma}) \). It is unclear whether the requirements can uniquely fix \( (f^\alpha_{\beta\gamma}, h_{\gamma}) \) or not. Starting from the rheonomy condition \( (4.13) \) in Section 4.1, one may get \( (4.29) \) with no torsion constraint imposed on \( W^\alpha \). The torsion constraint is just \( (4.83) \), or concretely,
\[ R^{[a(s-1),b(t)]}_{ab} = g^{[a(s-1),b(t)]}_{ab} (W^\sigma_{\mu}, \partial_{\nu_1} W^\sigma_{\mu}, \cdots, H, \partial_{\nu_1} H, \cdots) = 0, \quad \text{for} \quad t \neq s - 1, \]  
(4.86)
which will make $W_\mu^\alpha$ reduce to $W_\mu^{[a(s-1),b(0)]}$ and also guarantee the correct free theory limit. In this case, having the correct free theory limit and satisfying the torsion constraint are the same thing. If there is such (4.13), for which the Bianchi identity on $(R_{ab; c_1...c_n}^\alpha, H_{c_1...c_n})$ reduces to the 4d equations of motion, then by setting $R_{ab; c_1...c_n}^{[a(s-1),b(t)]}$ to 0 for $t \neq s-1$, we will get (4.53) satisfying the Bianchi identity for the on-shell $(R_{ab; c_1...c_n}^{[a(s-1),b(s-1)]}, H_{c_1...c_n})$ and having the right free theory limit. (4.83) holds exactly in this situation.

4.5 The extended action principle for higher spin theory

In group manifold approach to supergravity, instead of imposing the rheonomy condition directly, one may construct the extended action whose variation gives both the rheonomy condition and the 4d equations of motion [1]. For example, in $\mathcal{N}=1$ supergravity, the extended action is of the form

$$S = S[\nu^A, M_4] = \int_{M_4 \subset M} L^{(4)}(\nu^A),$$

(4.87)

where $M_4$ is a 4d submanifold of the superspace $M^6$ and $L^{(4)}$ is a local Lorentz invariant 4-form in $M$ constructed from $\nu^A$ via the exterior differentiation and the exterior product. Variation of $S$ with respect to both $\nu^A$ and $M_4$ gives

$$\frac{\delta L^{(4)}}{\delta \nu^A} = K^{(3)}_A(z) = 0. \quad (4.88)$$

$K^{(3)}_A$ is a 3-form that should vanish all over $M$. $K^{(3)}_A(z) = 0$ contains both the rheonomy condition and the 4d equations of motion. The concrete form of $L^{(4)}$ is

$$L^{(4)} = \epsilon_{abcd} R^{ab} \wedge \nu^c \wedge \nu^d + 4 \bar{\psi} \gamma_5 \gamma_\rho \rho \wedge \nu^a,$$

(4.89)

where $\rho^\alpha_{MN} = R^\alpha_{MN}$.

For higher spin theory, if the extended action exists, it will take the form

$$S = S[W^\alpha, M_4] = \int_{M_4 \subset M} L^{(4)}(W^\alpha),$$

(4.90)

where $L^{(4)}$ is a 4-form invariant under the local Lorentz transformation.

$$K^{(3)}_\sigma = \frac{\delta L^{(4)}}{\delta W_\sigma} = K^{(3)}_{\sigma[a\beta\gamma]} W^\alpha \wedge W^\beta \wedge W^\gamma,$$

(4.91)

$$K^{(3)}_\sigma = 0 \iff K^{(3)}_{\sigma[a\beta\gamma]} = 0. \quad (4.92)$$

We need to find the configuration $W^\alpha$ on $M$ with $K^{(3)}_\sigma = 0$ everywhere. Still, the on-shell solution on $M$ is characterized by the on-shell solution on $M_4$. $M_4 \rightarrow M'_4$ is a diffeomorphism.

\footnote{We can use the group manifold $M$ instead of $M$, but the result is the same due to the factorization condition.}
transformation on $M$ that is equivalent to the deformed higher spin gauge transformation. The equation $K_{[3]}^\sigma = 0$ is on-shell gauge invariant. Off-shell higher spin invariance has the further requirement $dL^{(4)} = 0$. Although the on-shell gauge invariance is automatically guaranteed, for the generic $L^{(4)}$, $K_{[3]}^\sigma = 0$ only has the trivial solution $R_{\beta\gamma}^\alpha = 0$, so the question is whether there is $L^{(4)}$ for which, the related $K_{[3]}^\sigma = 0$ has the nontrivial solution or not. In supergravity, having the nontrivial solution also puts the severe constraint on $S$.

In the simplest situation, if

$$L^{(4)} = \kappa_{\alpha\beta} R^\alpha \wedge R^\beta + \kappa_{\alpha\beta\gamma} R^\alpha \wedge W^\beta \wedge W^\gamma + \kappa_{\alpha\beta\gamma\sigma} W^\alpha \wedge W^\beta \wedge W^\gamma \wedge W^\sigma$$

(4.93)

with $\kappa$ the constant, then

$$K_{[3\alpha\beta\gamma]} = -2\kappa_{\rho\sigma} f^\rho_{\chi[\alpha} R^\chi_{\beta\gamma]} - 2\kappa_{\rho\chi} f^\rho_{\sigma[\alpha} R^\chi_{\beta\gamma]} + \kappa_{\rho\sigma[\gamma} R^\rho_{\alpha]\beta\gamma] + 2\kappa_{\rho\sigma[\gamma} R^\rho_{\alpha]\beta\gamma] + \kappa_{\rho[\beta\gamma} f^\rho_{\alpha]\sigma] + 4\kappa_{\sigma[\alpha\beta\gamma]}.$$  (4.94)

(4.94) imposes a set of linear relations among $R_{\beta\gamma}^\alpha$, which, when plugged into the Bianchi identity, may only allow the trivial solution $R_{\beta\gamma}^\alpha = 0$. The more general form of $L^{(4)}$ is

$$L^{(4)} = f_{\rho\sigma\chi\eta}(R_{\beta\gamma}^\alpha, \partial_\lambda R_{\beta\gamma}^\alpha, \cdots) W^\rho \wedge W^\sigma \wedge W^\chi \wedge W^\eta$$

(4.95)

including an infinite number of derivatives. $K_{[3\alpha\beta\gamma]} = 0$ are functions of $(R_{\beta\gamma}^\alpha, \partial_\lambda R_{\beta\gamma}^\alpha, \cdots)$. With $R_{\beta\gamma}^\alpha = f_{\beta\gamma}^{\alpha}(\Phi^\beta) - f_{\beta\gamma}^{\alpha}$ plugged in, $K_{[3\alpha\beta\gamma]}$ should automatically vanish for the arbitrary $\Phi^\beta$ if it is the action from which, the Vasiliev equation come out. However, it is too complicated to fix the exact form of (4.95).

4.6 The dynamics of the 0-form matter on group manifold with the fixed background

(4.57)-(4.65) describes the coupling of the spin 0 matter $H$ and the spin 2, 4, · · · gravity fields $W^\alpha$. Under the local gauge transformation, which is the deformed higher spin transformation as well as the diffeomorphism transformation on $M$, spin 0, 2, 4, · · · fields mix with each other. The pure gravity system can be obtained by setting $H$ to 0. On the other hand, to describe the dynamics of the 0-form matter on $M$ with the fixed background, the matter-gravity coupling must be turned off and so, $r_{\beta\gamma}^\alpha$ should be modified. One may assume $r_{\beta\gamma}^\alpha = 0$, then $W_0^\alpha$ gives the intrinsic geometry of the group manifold $M$ discussed in Section 2. The equations of motion are simplified to

$$dW_0^\alpha - \frac{1}{2} f_{\beta\gamma}^\alpha W_0^\beta \wedge W_0^\gamma = 0, \quad dH = H_\alpha W_0^\alpha \Leftrightarrow \partial_\alpha H = H_\alpha, \quad \partial_\rho H_\sigma + H_\alpha f^\alpha_{\rho\sigma} = 0.$$  (4.96)

The allowed gauge transformation parameter $\epsilon_0^\alpha$ should satisfy

$$\delta_\epsilon W_0^\alpha = \partial_\rho \epsilon_0^\alpha + f_{\beta\gamma}^\alpha \epsilon_0^\beta W_0^\gamma = 0,$$

(4.97)

generating the global higher spin transformation on $M$

$$\delta_\epsilon H = \xi_0^\gamma \partial_\gamma H = \epsilon_0^\beta H_\beta, \quad \delta_\epsilon H_\alpha = \xi_0^\gamma \partial_\gamma H_\alpha = \epsilon_0^\beta \partial_\beta H_\alpha.$$  (4.98)
With $\epsilon_0$ satisfying (4.97), (4.96), is invariant under (4.98). $[\epsilon_0, \epsilon'_0] = f^{\alpha}_\beta \epsilon_0^\beta$. The structure constant is undeformed.

According to the previous decomposition $\alpha = (A, Q)$, we may further let $\partial_Q H = H_Q = 0$ and then

$$\partial_Q H_A = -f_{Q4}^B H_B.$$  \hspace{1cm} (4.99)

The evolution along the $Q$ direction is a gauge transformation. The rest Bianchi identity is

$$\partial_A H_B = \partial_B H_A.$$ \hspace{1cm} (4.100)

We will assume

$$H_A = h_A,$$ \hspace{1cm} (4.101)

$$\partial_{[a_1\ldots a_s} h_{b_1\ldots b_{s+1}]} H_{[d_1\ldots d_t e_1\ldots e_{t+1}]} = m_{[a_1\ldots a_s b_1\ldots b_{s+1}]} [d_1\ldots d_t e_1\ldots e_{t+1}].$$ \hspace{1cm} (4.102)

where $h_A$ and $m_{[a_1\ldots a_s b_1\ldots b_{s+1}]} [d_1\ldots d_t e_1\ldots e_{t+1}]$ are functions of $\{H_{c_1\cdot\cdot\cdot c_n}, H_{[a_1\ldots a_s b_1\ldots b_{s+1}]} |c_1\cdot\cdot\cdot c_n| n = 0, 1, \ldots ; s = 2, 4, \ldots \}$. It is easy to see

$$\partial_{\alpha} H_{[a_1\ldots a_s b_1\ldots b_{s+1}]} |c_1\cdot\cdot\cdot c_n = h_{[a_1\ldots a_s b_1\ldots b_{s+1}]} |c_1\cdot\cdot\cdot c_n \alpha;$$

$$\partial_{\alpha} H_{c_1\cdot\cdot\cdot c_n} = h_{c_1\cdot\cdot\cdot c_n \alpha}.$$ \hspace{1cm} (4.103)

are again the functions of $\{H_{c_1\cdot\cdot\cdot c_n}, H_{[a_1\ldots a_s b_1\ldots b_{s+1}]} |c_1\cdot\cdot\cdot c_n| n = 0, 1, \ldots ; s = 2, 4, \ldots \}$. So the value of $(H, H_{c_1}, \ldots, H_{[a_1\ldots a_2 a_3 b_2 b_3]}; H_{[a_1\ldots a_2 b_1 b_2 b_3]}; H_{[a_1\ldots a_4 b_1 b_3 b_5]}; H_{[a_1\ldots a_4 b_1 b_5]}; H_{[a_1\ldots a_4 b_1 b_5]; c_1, \ldots}]$ at one point determines its value on $M$. Alternatively, $(H, H_{[a_1 a_2 b_1 b_2 b_3]}; H_{[a_1 a_4 b_1 b_5]}; \ldots)$ on $AdS_4$ determines $(H, H_{[a_1 a_2 b_1 b_2 b_3]; H_{[a_1 a_4 b_1 b_5]; \ldots})$ on $M$.

The multiplet $(H, H_{c_1}, \ldots, H_{[a_1 a_2 b_1 b_2 b_3]}; H_{[a_1 a_2 b_1 b_2 b_3]}; H_{[a_1 a_4 b_1 b_3 b_5]}; H_{[a_1 a_4 b_1 b_5]); \ldots)$ on $AdS_4$ both form the complete representation of the higher spin algebra. \{H_{[a_1 a_2 a_3 b_1 b_2 b_3]}; H_{[a_1 a_2 a_3 b_1 b_2 b_3]}; H_{[a_1 a_4 b_1 b_3 b_5]}; H_{[a_1 a_4 b_1 b_5]); \ldots\}$ are in one-to-one correspondence with the spin $s$ particles carrying the arbitrary momentum on $AdS_4$ and the spin $s$ fields on $AdS_4$, which are indeed the representations of the higher spin algebra. A question is why in (4.101), we do not assume $h_A$ is merely the function of $H_{c_1\cdot\cdot\cdot c_n}$, then there is no need to impose (4.101). The reason is that $H$ on $AdS_4$ cannot form the nontrivial representation of the higher spin algebra. Similarly, in supersymmetry, $H_A$ cannot be expressed in terms of $(H, H_{c_1}, \ldots)$ not only due to the scaling relation, but also because $H$ on $M_4$ cannot form the nontrivial supersymmetry representation.

Although the discussion in higher spin theory is a direct extension of the supersymmetry theory, it looks strange there are global higher spin invariant theories for 0-form fields with spin 0, 2, 4, \ldots on the fixed $AdS$ background. A natural expectation is that the theory may be a gauge-fixed version of the traditional higher spin theory with fields $h_{\mu_1\cdot\cdot\cdot\mu_s}$ for $s = 0, 2, 4, \ldots$. The gauge-fixing makes the local higher spin symmetry reduce to the global one. In fact, $H_{[a_1 a_2 a_3 b_1 b_2 b_3]}$ is related with $\{Q_{[a_1 a_2 a_3 b_1 b_2 b_3]}; O\}$ which are indeed the operators in the gauge-fixed higher spin theory in $AdS_4$ \cite{17}. For supersymmetry, $N = 1$ supergravity multiplet with spin $(3/2, 2)$ can couple with WZ matter with spin $(0, 1/2)$. It is unclear
whether the higher spin version of the supergravity-WZ matter coupled system makes sense or not. The theory describing the coupling between the spin \((2, 4, \cdots)\) gauge fields and the spin \((0, 2, 4, \cdots)\) matter fields may contain two copies of fields with the same spin.

Finally, we need to determine the exact form of \(h_{A}\) and \(m_{[a_1 \cdots a_s, b_1 \cdots b_{s+1}], [0d_1 \cdots d_t, e_1 \cdots e_{t+1}]}\) satisfying \(\partial_A h_B = \partial_B h_A\). This is done in Section 2, where \((2.21)\) and \((2.24)\) are obtained. With \(O(Z)\) replaced by \(H(Z)\), from

\[
\partial_{0-0a_{1}\cdots a_{s},b_{1}\cdots b_{s+k}} H = i^{1-k} \sum_{r=0,2,\cdots,s} c_{1\cdots c_{r}, d_{1}\cdots d_{t+1}, \cdots, d_{r+1}, d_{r+1}}^{0-0a_{1}\cdots a_{s},b_{1}\cdots b_{s+k}} \mathcal{D}_{0,d_{r+1}} \mathcal{D}_{0,d_{r+1}} \cdots \mathcal{D}_{0c_{r}, d_{1}\cdots d_{r+1}} H, \quad (4.104)
\]

\[
\mathcal{D}_{0-0a_{1}\cdots a_{s},b_{1}\cdots b_{s+k}} \sum_{r=0,2,\cdots,s} a(a_1 \cdots a_s, b_1 \cdots b_{s+k}) \mathcal{D}_{0,b_{s+k}} \mathcal{D}_{0,b_{s+k-1}} \cdots \mathcal{D}_{0a_{1}\cdots a_{s},b_{1}\cdots b_{s+k}} H, \quad (4.105)
\]

we finally obtain

\[
h_{0-0a_{1}\cdots a_{s},b_{1}\cdots b_{s+k}} = \sum_{s} G_{0-0a_{1}\cdots a_{s},b_{1}\cdots b_{s+k}}^{[0d_1 \cdots d_t, c_1 \cdots c_n]} H_{[0d_1 \cdots d_t, e_1 \cdots e_{t+1}]}^{[c_1 \cdots c_n]}, \quad s \text{ even, } k \text{ odd, } t \text{ even},
\]

\[
m_{[a_1 \cdots a_s, b_1 \cdots b_{s+1}], [0d_1 \cdots d_t, e_1 \cdots e_{t+1}]} = \sum_{s} K_{[m_1 \cdots m_r, a_1 \cdots a_{n+1}]^{[c_1 \cdots c_n]}} H_{[0m_1 \cdots m_r, n_1 \cdots n_{r+1}]^{[c_1 \cdots c_n]}}, \quad s \text{ even, } t \text{ even, } r \text{ even.}
\]

The Bianchi identity is satisfied since \(O(Z)\) is a scalar field on \(M\).

The complete \(h_{\alpha}\) is exhausted by \(h_{Q} = 0\) and \((4.106)\). It is also reasonable to add \(h_{0-0a_{1}\cdots a_{s},b_{1}\cdots b_{t+1}}\) with \(s\) even, \(t\) even. \(h\) and \(\tilde{h}\) together form the twisted-adjoint representation of the higher spin algebra. One may assume

\[
h_{0a_{1}\cdots a_{s},b_{1}\cdots b_{t+1}} = \sum_{\{b_1 \cdots b_{s+1}\}} \partial_{b_{s+1}} \tilde{h}_{a_{1}\cdots a_{s},b_{1}\cdots b_{s}}. \quad (4.108)
\]

The relation \((4.106)\) is obtained from the operator \(O(Z)\) on \(M\). We may get the similar relation from Vasiliev theory. With \(\tilde{f}_{\beta\gamma}^{\alpha} = f_{\beta\gamma}^{\alpha}\), the unfolded equation in Vasiliev theory reduces to

\[
dW_{\alpha} = \frac{1}{2} f_{\beta\gamma}^{\alpha} W_{\beta} \wedge W_{\gamma}, \quad d\Phi = k_{\beta\gamma}^{\hat{\alpha}} \Phi \wedge W_{\beta}, \quad \Rightarrow \partial_{\beta} \Phi \hat{\alpha} = k_{\beta\gamma}^{\hat{\alpha}} \Phi \wedge \hat{\gamma},
\]

where \(k_{\beta\gamma}^{\hat{\alpha}} = \rho^{\hat{\alpha}}_{\beta\gamma} f_{\beta\gamma}^{\alpha}\). \(H = \Phi, \partial_{\beta} H = \partial_{\beta} \Phi = k_{\beta\gamma}^{\hat{\alpha}} \Phi \wedge \hat{\gamma} = H_{\beta}\). In particular,

\[
\partial_{A} H = \partial_{A} \Phi = k_{A\gamma}^{\hat{\alpha}} \Phi \wedge \hat{\gamma} = H_{A} = \Phi_{A}, \quad \partial_{Q} H = \partial_{Q} \Phi = k_{Q\gamma}^{\hat{\alpha}} \Phi \wedge \hat{\gamma} = H_{Q} = 0. \quad (4.110)
\]

\[
\partial_{b} \Phi^{[a_1 \cdots a_s, b_1 \cdots b_{s+t}]} = k_{b}^{[a_1 \cdots a_s, b_1 \cdots b_{s+t}]} \Phi^{[c_1 \cdots c_s, d_1 \cdots d_{s+t+1}]} + k_{b}^{[a_1 \cdots a_s, b_1 \cdots b_{s+t}]} \Phi^{[c_1 \cdots c_s, d_1 \cdots d_{s+t-1}]}.
\]

\[
(4.111)
\]
From (4.111), we have

\[
\begin{align*}
\partial_b \Phi^{[a_1\ldots a_s,b_1\ldots b_s]} &= k_{b}^{[a_1\ldots a_s,b_1\ldots b_s]} \Phi^{[c_1\ldots c_s,d_1\ldots d_s+1]}, \\
\partial_b \Phi^{[a_1\ldots a_s,b_1\ldots b_s+1]} &= k_{b}^{[a_1\ldots a_s,b_1\ldots b_s]} \Phi^{[c_1\ldots c_s,d_1\ldots d_s+2]} + k_{b}^{[a_1\ldots a_s,b_1\ldots b_s+1]} \Phi^{[c_1\ldots c_s,d_1\ldots d_s]}, \\
\partial_b \Phi^{[a_1\ldots a_s,b_1\ldots b_s+2]} &= k_{b}^{[a_1\ldots a_s,b_1\ldots b_s+2]} \Phi^{[c_1\ldots c_s,d_1\ldots d_s+3]} + k_{b}^{[a_1\ldots a_s,b_1\ldots b_s+2]} \Phi^{[c_1\ldots c_s,d_1\ldots d_s+1]}.
\end{align*}
\]

so

\[
\begin{align*}
\Phi^{[a(s),b(s+1)]} &\sim \partial^b \Phi^{[a(s),b(s)]} \sim \partial [a(s),b(s+1)] \Phi, \\
\Phi^{[a(s),b(s+2)]} &\sim \partial^b \partial^c \Phi^{[a(s),b(s)]} + \Phi^{[a(s),b(s)]} \sim \partial^b \partial^c \Phi^{[a(s),b(s+1)]} \Phi + \Phi^{[a(s),b(s+1)]} \Phi, \\
\Phi^{[a(s),b(s+3)]} &\sim \partial^b \partial^c \partial^d \Phi^{[a(s),b(s)]} + \partial^b \Phi^{[a(s),b(s)]} \sim \partial^b \partial^c \partial^d \Phi^{[a(s),b(s+1)]} \Phi + \partial^b \Phi^{[a(s),b(s+1)]} \Phi,
\end{align*}
\]

\tag{4.112}

\begin{align*}
\Phi^{[a(s),b(s+4)]} &\sim \partial^b \partial^c \partial^d \partial^e \Phi^{[a(s),b(s)]} + \partial^b \partial^c \Phi^{[a(s),b(s)]} \sim \partial^b \partial^c \partial^d \partial^e \Phi^{[a(s),b(s+1)]} \Phi + \partial^b \Phi^{[a(s),b(s+1)]} \Phi,
\end{align*}

\tag{4.113}

Compared with \( h \) and \( \hat{h} [^{a(s),b(s+2k-1)}] \sim \Phi^{[a(s),b(s+2k-1)]} \), \( \hat{h} [^{a(s),b(s+2k)}] \sim \Phi^{[a(s),b(s+2k)]} \), which is consistent with (4.110). The result is also obtained in (4.109) by considering the 0-th level unfolded equation of Vasiliev theory, which is just (4.109).

Just as (4.96), (4.109) is a consistent equation by itself. At the first sight, it describes the dynamics of the 0-form \( \Phi^\sigma \) on \( M \). However, as an unfolded equation with \( \Phi_A \sim \partial_A \Phi, \Phi_Q \sim \Phi_A \sim \partial_A \Phi, (4.109) \) is actually the equation for the 0-form \( \Phi \) on \( M \) with the rheonomy condition imposed, or equivalently, \( \{\Phi^{[a(s),b(s)]}\} \) on \( AdS_4 \). Since \( R^\alpha_{\beta\gamma} = 0, \Phi^\sigma \) are not related with the curvature any more. They just give a set of the 0-form fields with the spin \( s = 0,2,4 \ldots \) in \( AdS_4 \).

In the interacting theory, \( \partial_{\beta} \Phi^\sigma = \hat{k}^\sigma_{\beta\gamma} (\Phi^\sigma) \Phi^\gamma \)

\[
\begin{align*}
\partial_{b_1} \Phi^{[a(s),b(s)]} &= \hat{k}^{[a(s),b(s)]}_{b_1\gamma} \Phi^\gamma, \\
\partial_{b_2} \partial_{b_1} \Phi^{[a(s),b(s)]} &= \frac{\partial^2 [a(s),b(s)]}{\partial \Phi^\sigma} \hat{k}^{\sigma}_{b_2\gamma} \Phi^\gamma + \hat{k}^{[a(s),b(s)]}_{b_1\gamma} \hat{k}^{\gamma}_{b_2\mu} \Phi^\mu.
\end{align*}
\]

\tag{4.114}

From (4.114), \( \{\Phi^\sigma \sim \Phi^{[a(s),b(s+1)]}\} \) can be expressed in terms of \( \{\partial_{b_1} \ldots \partial_{b_1} \Phi^{[a(s),b(s)]}\} \), in a complicated way.

### 5 Discussion

In supergravity, the rheonomy condition is simply \( R^A_{BC} = r^A_{BC}(R^a_{bc}, R^a_{bd}, H, H_a) \). Nevertheless, the most generic rheonomy condition in group manifold approach takes the form of (4.13) and (4.37) with all orders of derivatives included. If we make a similar truncation \( R^\alpha_{\beta\gamma} = r^\alpha_{\beta\gamma}(H^{[a(s-1),b(s-1)]}, H) \) in higher spin theory, then with \( r^\alpha_{\beta\gamma} \) plugged into the Bianchi identity, we will get the 4d equations of motion, which, when expressed in terms of
(\(W_{\mu}^{[a(s-1),b(0)]}, H\)), do not contain the derivatives higher than two. However, it is quite likely that such equations may only have the trivial solution \(R_{ab}^{[a(s-1),b(s-1)]} = H = 0\) no matter how the functions \(r_\alpha\) are adjusted. To allow for the nontrivial on-shell degrees of freedom, higher derivatives must be included so that \(R_{ab}^{[a(s-1),b(s-1)]}\) at one point is effectively determined by \((W_{\mu}^{[a(s-1),b(0)]}, H)\) on the whole \(AdS_4\). The 4d equations of motion for \((W_{\mu}^{[a(s-1),b(0)]}, H)\) will also contain an infinite number of the higher derivative terms which makes the theory nonlocal.

To write the unfolded equations (4.46) and (4.61)-(4.63), the infinite 0-form multiplets are necessarily involved in both supergravity and higher spin theory, since the solution on the whole \(M\), including \(M_4/AdS_4\), is characterized by the on-shell 0-form multiplet at one-point. For higher spin theory, the on-shell \((R_{ab,c_1\cdots c_n}^{[a(s-1),b(s-1)]}, H_{c_1\cdots c_n})\) is equivalent to \(\{\Phi^{[a(s),b(s+n)]}\}\), so the solution on \(M\) is characterized by the arbitrary \(\{\Phi^{[a(s),b(s+n)]}\}\) at one-point. Merely based on the group manifold approach without the knowledge of the Vasiliev theory, we will finally arrive at (4.57)-(4.65) and then face the problem of finding the proper rheonomy condition that could solve the Bianchi identity, allow for the maximum on-shell degrees of freedom and have the correct free theory limit. It is the Vasiliev theory that gives the solution meeting all these requirements. A question is whether there are other kinds of the on-shell degrees of freedom. One may consider \(\Phi^\alpha\) in the adjoint representation, but \(\Phi^\alpha\) is related with \(\Phi^\delta\) by an invertible Klein transformation thus is equivalent to \(\Phi^\delta\). The rheonomy condition satisfying the Bianchi identity for the arbitrary \(\Phi^\alpha/\Phi^\delta\) is not unique. At least in appendix C, there is such an example (for the bosonic higher spin theory). However, the correct free theory limit is not recovered and the local Lorentz transformation is deformed there.

In superspace with the fixed background geometry, the local super Poincare invariance reduces to the global super Poincare invariance. With the chiral constraint imposed, the component expansion of the scalar superfield on superspace gives the WZ fields \((H, H^\alpha)\) on \(M_4\). For higher spin theory, one can fix the background of \(M\) and then study the scalar field on \(M\) with the global higher spin symmetry. It turns out that with the suitable rheonomy constraint imposed, the dynamics of the scalar on \(M\) is determined by \(s = 0, 2, 4, \cdots\) fields \((H, H_{[0a_1a_2,b_1b_2b_3]}, H_{[0a_1\cdots a_4,b_1\cdots b_5]}, \cdots)\) on \(AdS_4\). We get the global higher spin invariant theory for scalar fields with spin \(0, 2, 4, \cdots\) on the fixed \(AdS_4\) background, which may be related with the gauge-fixed version of the standard higher spin theory.

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A Technical Details I

For $AdS_4$ parameterized by $x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 = 1$,

$$[Q_{AB}, O(x)] = i(x_A \partial_B - x_B \partial_A)O(x),$$

(A.1)

where $A, B = 0, 1, 2, 3, 4$. Let 0 denote a point in the bulk of $AdS_4$ with the coordinate $x^0 = 1, x^1 = \ldots = x^4 = 0$, then

$$[Q_{mn}, O(0)] = [Q_{m4}, O(0)] = [K_m - iP_m, O(0)] = 0,$$

(A.2)

where $m, n = 1, 2, 3$. \{Q_{04}, Q_{0m}\} $\subset K(0)$ generates the tangent space along $AdS_4$. From (A.2), according to the operators constructed in [12], $O(0)$ is solved as

$$O(0) = \sum \frac{k!}{(2k + 1)!!} a^+_{i_1 \ldots i_k} a^+_{i_1 \ldots i_k}.$$  \hfill (A.3)

In [12], the generic elements of $ho(1|2 : [3, 2])$ can be written as

$$Q_{m_1 \ldots m_p, n_1 \ldots n_q} = i \sum g(l) a^+_{m_1 \ldots m_p i_1 \ldots i_l} a_{n_1 \ldots n_q i_1 \ldots i_l}.$$ \hfill (A.4)

with $m_k, n_k, i_k = 1, 2, 3$, so

$$[Q_{m_1 \ldots m_p, n_1 \ldots n_q}, O(0)] \sim i \sum g(l) a^+_{m_1 \ldots m_p i_1 \ldots i_l} a^+_{n_1 \ldots n_q i_1 \ldots i_l}.$$ \hfill (A.5)

One can choose the basis $\{Q\}$ of $ho(1|2 : [3, 2])$ with the definite conformal dimension.

$$[D, Q] = -i \Delta Q, \quad [D, Q^+] = i \Delta Q^+.$$ \hfill (A.6)

Let $H_Q = Q + Q^+$, $\bar{H}_Q = i(Q - Q^+)$, there will be

$$[H_Q, O(0)] = 0, \quad [\bar{H}_Q, O(0)] = 2i[Q, O(0)].$$ \hfill (A.7)

$a[E(0)] = \{H_Q\}, K(0) = \{\bar{H}_Q\}$. Moreover,

$$\{\bar{H}_Q\}, \{H_Q\} \subset \{H_Q\}, \quad \{\bar{H}_Q\}, \{Q\} \subset \{H_Q\}, \quad \{\bar{H}_Q\}, \{\bar{H}_Q\} \subset \{\bar{H}_Q\}.$$ \hfill (A.8)

$M$ is a symmetric space. $\forall Q, Q = Q(a, a^+), Q(a^+, a^+)$ could be mapped to $K(0)$, for some $Q(a, a^+), Q(a^+, a^+) = 0$ thus is projected out.

$$K(0) = \{Q(a^+, a^+) | \forall Q(a, a^+) \in ho(1|2 : [3, 2])\}.$$ \hfill (A.9)

For $ho(1|2 : [3, 2])$, there is an involution $\sigma$

$$\sigma(Q) = Q^+,$$ \hfill (A.10)

with $\sigma^2 = 1$. $\sigma$ has the eigenvalue 1 and $-1$ with $\{Q\}$ and $\{\bar{H}_Q\}$ defined above as the corresponding eigenspaces.

$$ho(1|2 : [3, 2]) = \{H_Q\} \oplus \{\bar{H}_Q\}.$$ \hfill (A.11)

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Under the Wick rotation, \( x^0 \rightarrow ix^0 \), the action of \( \sigma \) is then \( \sigma : ix^0 \rightarrow -ix^0 \), so

\[
\begin{align*}
\{E(0)\} &= \{\mathcal{H}_Q\} = \{t_{0\ldots 0 a_1 a_2 a_3}, t_{0\ldots 0 a_1 a_2 a_3 b_1 \ldots b_{s-1}}, \ldots, t_{a_1 \ldots a_{s-1} b_1 \ldots b_{s-1}}\}, \\
K(0) &= \{\tilde{\mathcal{H}}_Q\} = \{t_{0\ldots 0 b_1 \ldots b_s}, t_{0\ldots 0 a_1 a_2 b_1 \ldots b_{s-1}}, \ldots, t_{a_1 \ldots a_{s-2} b_1 \ldots b_{s-1}}\}. 
\end{align*}
\]

(A.12)

The decomposition is consistent with (2.3) and (2.4).

For \( m = 1, 2, 3 \), let us consider

\[
\begin{align*}
A^+_{1k} &= a^+_{i_1 \ldots i_k}a^+_{i_1 \ldots i_k}, \\
A^+_{mk} &= a^+_{mi_1 \ldots i_k}a^+_{i_1 \ldots i_k}, \\
A^+_{mn,k} &= a^+_{mn1 \ldots i_k}a^+_{i_1 \ldots i_k} + f_1 a^+_{mi_1 \ldots i_k}a^+_{i_1 \ldots i_k}, \\
A^+_{mn1,k} &= a^+_{mn1 \ldots i_k}a^+_{i_1 \ldots i_k} + f_1 a^+_{mi_1 \ldots i_k}a^+_{i_1 \ldots i_k} + f_2 a^+_{mi_1 \ldots i_k}a^+_{mi_1 \ldots i_k} + f_3 a^+_{mi_1 \ldots i_k}a^+_{mi_1 \ldots i_k} + \ldots
\end{align*}
\]

(A.13)

which are the most general \( q \)-tensor operators with the dimension \( q + 2k + 1 \). Assume

\[
\frac{1}{2} [Q^{AB}, [Q_{AB}, A^+_{m_1 \ldots m_q,k}]] = [C_2, A^+_{m_1 \ldots m_q,k}] = \lambda A^+_{m_1 \ldots m_q,k},
\]

(A.14)

where \( C_2 \) is the Casimir operator, then for even \( q \), \( \lambda = 2(s^2 - 1) \) with \( s = 0, 2, \ldots, q \), for odd \( q \), \( \lambda = 2(s^2 - 1) \) with \( s = 0, 2, \ldots, q - 1 \). For each \( s \),

\[
[C_2, A^+_{m_1 \ldots m_q,k}] = 2(s^2 - 1)A^+_{m_1 \ldots m_q,k}
\]

(A.15)

can uniquely fix \( f_1 \). The corresponding operator is denoted as \( A^{(s)+}_{m_1 \ldots m_q,k} \), which is totally symmetric and traceless.

\[
O^s_{m_1 \ldots m_q}(0) = \sum g(k)A^{(s)+}_{m_1 \ldots m_q,k}
\]

(A.16)

is a spin \( s \) operator at \( 0 \) forming the representation of \( SO(3) \). The complete \( SO(3,1) \) representation can be obtained by the successive action of \( Q_{4,m} \). \( g(k) \) is determined by the requirement that at some point, no new operators can be created. When \( s = 0 \), the solution of \( [Q_{4,m}, O^0(0)] = 0 \) is \( O(0) \) in (A.3). When \( s = 2 \), the minimal times for the action of \( Q_{4,m} \) is 3. The corresponding \( O^2_{m_1 m_2}(0) \) can be written as \( O^2_{m_1 m_2 444}(0) \), while the action of \( \{Q_{4,m}, Q_{4,m}\} \) gives the complete \( SO(3,1) \) representation \( O^2_{b_1 b_2 b_3 b_4}(0), b_i = 1, 2, 3, 4 \). Similarly, for spin \( s \) operator with \( s \geq 2 \), we have \( O^s_{m_1 \ldots m_4 \ldots}(0) \) with the \( SO(3,1) \) completion \( O^s_{a_1 \ldots a_{2s+1}}(0) \). The maximum number of 4 in \( O^s_{b_1 \ldots b_{2s+1}}(0) \) is \( s + 1 \). Successive action of \( Q_{0,4} \) gives \( [Q_{0,4}, O^s_{m_1 \ldots m_4 \ldots}(0)] = O^s_{m_1 \ldots m_4 \ldots 444}(0), [Q_{0,4}, O^s_{m_1 \ldots m_4 \ldots 444}(0)] = O^s_{m_1 \ldots m_4 \ldots 444}(0), \) etc. The \( SO(3,1) \) completion of \( O^s_{m_1 \ldots m_4 \ldots 444}(0) \) is \( O^s_{b_1 \ldots b_{2s+1}}(0) \). \( O^s_{b_1 \ldots b_{2s+1}}(0) \) is the descendant of \( O^s_{b_1 \ldots b_{2s+1}}(0) \) obtained by the successive application of \( \{Q_{4,0}, Q_{m,0}\} \) on \( O^s_{b_1 \ldots b_{2s+1}}(0) \) thus is a spin \( s \) operator as well.

\[
O^s_{b_1 \ldots b_{2s+1}}(0) = [Q_{0,b_{2s+1}}, \ldots, [Q_{0,b_{2s+3}}, [Q_{0,b_{2s+2}}, O^s_{b_1 \ldots b_{2s+1}}(0)]]] \ldots
\]

(A.17)

Still, the maximal number of 4 in \( O^s_{b_1 \ldots b_{2s+1}}(0) \) is \( s + k \).
Now consider \([Q_{0\cdots 0a_1\cdots a_s b_1\cdots b_{s+k}}, O(0)]\) with \(k = 1, 3, \cdots, a_i, b_i = 1, 2, 3, 4\), which can always be uniquely decomposed as

\[
[Q_{0\cdots 0a_1\cdots a_s b_1\cdots b_{s+k}}, O(0)] = O^0_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}(0) + \cdots + O^s_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}(0)
\]

(A.18)

with

\[
[C_2, O^0_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}(0)] = 2(p^2 - 1)O^0_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}(0).
\]

(A.19)

We can make the identification \(O^s_{0 a_1\cdots a_s b_1\cdots b_{s+1}}(0) = O^s_{a_1\cdots a_s b_1\cdots b_{s+1}}(0)\), both of which are the \((2s + 1)\)-tensor operator with spin \(s\). When \(s = 0\),

\[
[C_2, [Q_{0\cdots 0 b_1\cdots b_k}, O(0)]] = 0,
\]

(A.20)

\[
[Q_{0\cdots 0 b_1\cdots b_k}, O(0)] \sim \sum_{\{b_1\cdots b_k\}} [Q_{0,b_k}, \cdots [Q_{0,b_2}, [Q_{0,b_1}, O(0)]]],
\]

(A.21)

where the summation represents the symmetrization over \(a_1 \cdots a_k\). \([Q_{0\cdots 0 a_1\cdots a_k}, O(0)]\) is the descendant of \(O(0)\). Generically,

\[
\begin{align*}
[Q_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}, O(0)] &= \sum_{r=0,2,\ldots,s} f_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}^{d_1\cdots d_r+2,c_1\cdots c_r,d_1\cdots d_{r+1}} [Q_{0,d_{r+1}}, \cdots [Q_{0,d_{r+2}}, [Q_{0,c_1\cdots c_r,d_1\cdots d_{r+1}}, O(0)]]] \cdots ] \\
&= \sum_{r=0,2,\ldots,s} g_{0\cdots 0 a_1\cdots a_s b_1\cdots b_{s+k}}^{c_1\cdots c_r,d_1\cdots d_{r+1},d_{r+1-1}d_{r+2}} [Q_{0,c_1\cdots c_r,d_1\cdots d_{r+1}}, \cdots [Q_{0,d_{r+2}}, [Q_{0,d_{r+1}}, O(0)]]] \cdots ].
\end{align*}
\]

(A.22)

### B Technical Details II

\(O(z) = e^{ikzM}O(0)e^{-ikzM}z^M\), \(k_M \in K(0)\), \(\{k_M\}\) and \(\{e_m\}\) are bases of \(K(0)\) and \(a[E(0)]\) respectively,

\[
[k_M, O(z)] = e^{ikz} [e^{-ikz}k_M e^{ikz}, O(0)] e^{-ikz}
\]

\[
= e^{ikz} [(\delta_N^N + \frac{1}{2!} z^{M_1} \overline{z^{M_2}} C_N^{N_1} C_M^{m_1} C_{M_2 M}^{m_2} + \frac{1}{4!} z^{M_1} \overline{z^{M_2}} \overline{z^{M_3}} z^{M_4} C_N^{N_1} C_{M_2 M_1}^{m_1 m_2} C_{M_3 m_1}^{m_3} C_{M_4 M}^{m_4} + \cdots ) k_N, O(0)] e^{-ikz}
\]

\[
= u_N^M [e^{ikz} k_N e^{-ikz}, O(z)] = -i u_N^M D_N O(z),
\]

where

\[
u_N^M(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!!} z^{M_1} \overline{z^{M_2}} \cdots \overline{z^{M_{2k}}} C_{M_1 m_1}^{m_1} C_{M_2 m_1}^{m_2} C_{M_3 m_1}^{m_3} \cdots C_{M_{2k} M}^{m_k},
\]

(B.1)

\[
e_{m}, O(z) \]

\[
= e^{ikz} [e^{-ikz} e_{m} e^{ikz}, O(0)] e^{-ikz}
\]

\[
= e^{ikz} [(z^{M_1} C_{M_1 m} + \frac{1}{3!} z^{M_1} \overline{z^{M_2}} z^{M_3} C_{M_1 m_1}^{m_1} C_{M_2 m_2}^{m_2} C_{M_3 m_1}^{m_3} + \cdots ) k_N, O(0)] e^{-ikz}
\]

\[
= u_m^N [e^{ikz} k_N e^{-ikz}, O(z)] = -i u_m^N D_N O(z),
\]
where
\[ u^N_m(z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!!} z^{M_1} z^{M_2} \cdots z^{M_{k+1}} C^N_{M_1 m_1} C^m_{M_2 n_1} C^N_{M_3 m_2} \cdots C^N_{M_{k+1} m} , \] (B.2)

\[ [k_M, k_N] = i C^m_{MN} e_m, \quad [k_M, e_m] = i C^N_{M m} k_N. \] (B.3)

To determine \( D_N O(z) \),
\[ \partial_M O(z) = i f_M^N [e^{ikz} k_N e^{-ikz}, O(z)] = i f_M^N D_N O(z), \] (B.4)

where
\[ f_M^N(z) = \delta^N_M + \frac{1}{3!} z^{M_1} z^{M_2} C^N_{M_1 m_1} C^m_{M_2 m_2} + \frac{1}{5!} z^{M_1} z^{M_2} z^{M_3} z^{M_4} C^N_{M_1 m_1} C^m_{M_2 m_1} C^N_{M_3 m_2} C^m_{M_4 m} + \cdots, \] (B.5)

whose inverse is
\[ g_N^M(z) = \delta^L_N - \frac{1}{6} z^{M_1} z^{M_2} C^L_{M_1 m_1} C^m_{M_2 m_2} + \frac{7}{360} z^{M_1} z^{M_2} z^{M_3} z^{M_4} C^L_{M_1 m_1} C^m_{M_2 m_1} C^N_{M_3 m_2} C^m_{M_4 m} + \cdots, \] (B.6)

\[ f_M^N(z) g_L^M(z) = \delta^N_L, \] (B.7)

\[ D_M O(z) = g_M^N \partial_N O(z). \] (B.8)

So the final expression for \( \Lambda_M \) is
\[ \Lambda_M = -i u^N_M D_N = -i u^N_M g_N^L \partial_L = -i q^L_M \partial_L \] (B.9)

with
\[ q^L_M(z) = \delta^L_M + \frac{1}{3} z^{M_1} z^{M_2} C^L_{M_1 m_1} C^m_{M_2 m_2} - \frac{1}{45} z^{M_1} z^{M_2} z^{M_3} z^{M_4} C^L_{M_1 m_1} C^m_{M_2 m_1} C^N_{M_3 m_2} C^m_{M_4 m} + \cdots \] (B.10)

\( \Lambda_m \) takes a simpler form
\[ \Lambda_m = -i u^N_m D_N = -i u^N_m g_N^L \partial_L = -i C^L_{Nm} z^N \partial_L. \] (B.11)

## C Technical Details III

The equations of motion and the gauge transformation are
\[ dW^\alpha = \frac{1}{2} \tilde{f}_{\beta\gamma}^\alpha W^\beta \wedge W^\gamma, \quad d\Phi^\alpha = \tilde{f}_{\beta}^\alpha W^\beta \Phi^\gamma \] (C.1)

and
\[ \delta_\xi W^\alpha = d\epsilon^\alpha + \tilde{f}_{\beta\gamma}^\alpha \epsilon^\beta W^\gamma, \quad \delta_\xi \Phi^\alpha = \tilde{f}_{\beta}^\alpha \epsilon^\beta \Phi^\gamma \] (C.2)

with
\[ \tilde{f}_{\beta [\gamma}^\alpha \tilde{f}_{\rho \sigma]}^\beta - \Phi^\nu \tilde{f}_{\nu [\gamma}^\beta \frac{\partial \tilde{f}_{\rho \sigma]}^\alpha}{\partial \Phi^\beta} = 0. \] (C.3)
Expanding in terms of $\Phi^\alpha$,
\[
\bar{f}_\beta^\gamma = f_\beta^\gamma + f_\beta^\gamma|_{\sigma_1} \Phi^{\sigma_1} + f_\beta^\gamma|_{\sigma_1\sigma_2} \Phi^{\sigma_1} \Phi^{\sigma_2} + \cdots.
\] (C.4)

If we assume
\[
t_\alpha \bar{f}_\beta^\gamma|_{\sigma_1\ldots\sigma_n} = f(t_\beta, t_\gamma; t_{\sigma_1} \ldots t_{\sigma_n}),
\] (C.5)
where $f(t_\beta, t_\gamma; t_{\sigma_1} \ldots t_{\sigma_n})$ is the sum of the product of $t_\beta, t_\gamma, t_{\sigma_1}, \ldots, t_{\sigma_n}$ with $t_\beta$ and $t_\gamma$ anti-symmetric, $t_{\sigma_1} \ldots t_{\sigma_n}$ symmetric, then we will get
\[
t_\alpha \bar{f}_\beta^\gamma = f(t_\beta, t_\gamma) + f(t_\beta, t_\gamma; \Phi) + f(t_\beta, t_\gamma; \Phi, \Phi) + \cdots
\] (C.6)

With (C.4) plugged in (C.3), comparing the coefficients order by order, the solution can only be
\[
t_\alpha \bar{f}_\beta^\gamma = [t_\beta, t_\gamma] F(\Phi), \quad \text{or} \quad t_\alpha \bar{f}_\beta^\gamma = F(\Phi) [t_\beta, t_\gamma],
\] (C.7)
where $F(\Phi)$ is an arbitrary function of $\Phi$ with $F(0) = 1$. With (C.7) plugged in (C.3), we can see (C.3) is indeed satisfied. (C.1) and (C.2) become
\[
dW = [W, W] F(\Phi), \quad d\Phi = [W, \Phi] F(\Phi)
\] (C.8)
and
\[
\delta_\epsilon W = d\epsilon + [\epsilon, W] F(\Phi), \quad \delta_\epsilon \Phi = [\epsilon, \Phi] F(\Phi),
\] (C.9)
or
\[
dW = F(\Phi) [W, W], \quad d\Phi = F(\Phi) [W, \Phi]
\] (C.10)
and
\[
\delta_\epsilon W = d\epsilon + F(\Phi) [\epsilon, W], \quad \delta_\epsilon \Phi = F(\Phi) [\epsilon, \Phi].
\] (C.11)
\[
\bar{f}_\beta^\gamma = \langle [t_\beta, t_\gamma] F(\Phi) | t^\alpha \rangle = f_\beta^\gamma (t_\sigma F(\Phi) | t^\alpha \rangle \quad \text{or} \quad \bar{f}_\beta^\gamma = \langle F(\Phi) [t_\beta, t_\gamma] | t^\alpha \rangle = f_\beta^\gamma \langle F(\Phi) t_\sigma | t^\alpha \rangle.
\] (C.12)

Each $F(\Phi)$ gives a consistent deformation of the free theory with $\bar{f}_\beta^\gamma = f_\beta^\gamma$. With the field redefinition $\Phi' = f(\Phi)$, (C.8)--(C.11) become
\[
dW = [W, W] F[f^{-1}(\Phi')], \quad d\Phi' = [W, \Phi'] F[f^{-1}(\Phi')]
\] (C.13)
and
\[
\delta_\epsilon W = d\epsilon + [\epsilon, W] F[f^{-1}(\Phi')], \quad \delta_\epsilon \Phi' = [\epsilon, \Phi'] F[f^{-1}(\Phi')],
\] (C.14)
or
\[
dW = F[f^{-1}(\Phi')] [W, W], \quad d\Phi' = F[f^{-1}(\Phi')] [W, \Phi']
\] (C.15)
and
\[
\delta_\epsilon W = d\epsilon + F[f^{-1}(\Phi')] [\epsilon, W], \quad \delta_\epsilon \Phi' = F[f^{-1}(\Phi')] [\epsilon, \Phi'].
\] (C.16)

Especially, when $f = F$, $F[f^{-1}(\Phi')] = \Phi'$. All of the consistent deformations are related to the $F(\Phi) = \Phi$ case by a field redefinition.
Until now, we have not made any assumption on the algebra \( \{ t_\alpha \} \), so (C.12) holds for the arbitrary algebra which is also a ring. Consider the 4d bosonic higher spin theory with the spin \( s = 0, 1, 2, \cdots \) and the algebra \( g \), for \( t_\alpha \in g, \ t_\alpha \sim t_{a_{1} \cdots a_s b_1 \cdots b_{t_\alpha - s - 1}} \sim y^m \bar{y}^n \) with \( m + n = 2(s - 1), \ t = |m - n|/2 \). \( \forall t_\alpha, t_\beta \in g, t_\alpha t_\beta \in g \), so (C.8)-(C.11) are well defined, although the truncation to the minimal bosonic higher spin theory is not possible. The theory does not have the correct free theory limit since (4.83) is not satisfied. Also, \( R^{\alpha}_{(ab)\gamma} \neq 0 \), the local Lorentz transformation is deformed.

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