Time-weighted estimates for the Blackstock equation in nonlinear ultrasonics

Vanja Nikolić and Belkacem Said-Houari

Abstract. High frequencies at which ultrasonic waves travel give rise to nonlinear phenomena. In thermoviscous fluids, these are captured by Blackstock’s acoustic wave equation with strong damping. We revisit in this work its well-posedness analysis. By exploiting the parabolic-like character of this equation due to strong dissipation, we construct a time-weighted energy framework for investigating its local solvability. In this manner, we obtain the small-data well-posedness on bounded domains under less restrictive regularity assumptions on the initial conditions compared to the known results. Furthermore, we prove that such initial boundary-value problems for the Blackstock equation are globally solvable and that their solution decays exponentially fast to the steady state.

1. Introduction

Blackstock’s wave equation arises as a model of nonlinear propagation of ultrasonic waves through thermoviscous fluids, alternative to the Kuznetsov equation [22]. Originally derived by Blackstock [4], it later appeared independently in the works of Crighton [9] and Lesser and Seebass [25]. It is expressed in terms of the acoustic velocity potential $\psi = \psi(x, t)$ by

$$\psi_{tt} - c^2 (1 - 2k \psi_t) \Delta \psi - b \Delta \psi_t + 2\sigma \nabla \psi \cdot \nabla \psi_t = 0. \quad (1.1a)$$

Here $c > 0$ is the speed of sound in the fluid, $b > 0$ the sound diffusivity, and $k, \sigma \in \mathbb{R}$ nonlinear coefficients. Equation (1.1a) can be seen as an approximation of the compressible Navier–Stokes–Fourier system of governing equations of nonlinear sound motion. It was demonstrated in [7] that, in the small Mach number limit, the 1D Blackstock equation shows good agreement with the exact governing system based on the fully nonlinear theory. In the lossless case ($b = 0$), a comparison of different weakly nonlinear acoustic models performed in [6] singles out the Blackstock equation as the most consistent one.

The well-posedness and regularity analysis of nonlinear acoustic wave equations has gained a lot of interest in recent years; see [5,15,17,18,28,29,35] for a selection.

Keywords: Blackstock’s equation, Nonlinear acoustics, Time-weighted estimates.
of relevant results as well as the review paper [14]. One of the challenges in the well-posedness analysis of such models remains their solvability under reduced assumptions on data in terms of their smoothness and size.

In this work we consider Blackstock’s equation on smooth bounded domains \( \Omega \subset \mathbb{R}^d \), where \( d \in \{1, 2, 3\} \), and couple it with boundary and initial conditions:

\[
\psi|_{\partial \Omega} = 0, \quad (\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1). \tag{1.1b}
\]

A natural question arises: What is the minimal regularity of initial data is that ensures (at least) local existence and uniqueness of the solution to (1.1)? In answering this question, the aim of this work is threefold. First, we prove a large-time existence and uniqueness result in general three-dimensional domains for small data in

\[
(\psi_0, \psi_1) \in \left( H^2(\Omega) \cap H^1_0(\Omega) \right) \times H^1_0(\Omega), \tag{1.2}
\]

thereby improving upon the existing results in the literature which assume at least

\[
(\psi_0, \psi_1) \in \left( H^3(\Omega) \cap H^1_0(\Omega) \right) \times \left( H^2(\Omega) \cap H^1_0(\Omega) \right), \tag{1.3}
\]

see [13,20]. To this end, we exploit the strong damping present in the equation (with \( b > 0 \)) which contributes to its parabolic character. The parabolic nature of the problem will allow us to devise suitable (time-weighted) energy estimates under minimal regularity assumptions on the initial conditions.

Secondly, we address the question of existence of a global solution for small initial data satisfying (1.2). The proof is conducted by developing an energy method to arrive at suitable uniform estimates with respect to time for the solution of (1.1), and thus extend a local solution to be global. Thirdly, we prove the asymptotic stability as \( t \to \infty \) of the solution. More precisely, we show that the solution decays to the steady state with an exponential decay rate.

An alternative analysis framework to the time-weighted energy method is offered by a maximum regularity approach in temporally weighted \( L^p \) spaces. This method has been developed for abstract linear and quasilinear parabolic equations; see [3,8, 21,27,30,34] and the references contained therein. In comparison, the time-weighted framework developed in the present work is based on an energy method. As such, it offers a simple and robust alternative for handling different PDEs (not necessarily parabolic). We expect it may also be adapted more easily to a discrete setting to be employed in the numerical analysis of nonlinear ultrasonics.

The time-weighted energy method has been successfully used for problems related to the heat equation [10], the Navier–Stokes equations [11,26,33], where it allows gaining more regularity with minimal assumptions on the initial data. Time-weighted estimates have also been employed in the numerical analysis of strongly damped linear wave equations in [24]. Inspired by [11] and by exploiting the parabolic nature of (1.1) with \( b > 0 \), we use a maximal-regularity-type estimate (see (3.16) and (3.17) below) for a linearized problem combined with the time-weighted energy method to extract
higher regularity of the solution under the minimal assumption (1.2) on the initial data. More precisely, we prove that for any fixed final propagation time \(0 < T < \infty\) and for all \(t \in (0, T)\), the solution \(\psi\) satisfies

\[
\sqrt{t} \psi_{tt} \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{t} \nabla \psi_{tt} \in L^2(0, T; L^2(\Omega)), \\
\sqrt{t} \Delta \psi_t \in L^\infty(0, T; L^2(\Omega));
\]

(1.4)

see Theorem 4.1 for details. Without the time weight, regularity (1.4) would follow by an energy method only under additional smoothness assumption on the data. One of the key ideas in proving (1.4) is to write a linearization of (1.1) as a nonlocal heat equation for \(v = \psi_t\). The presence of the nonlocal term \(\Delta \psi\) in (3.2) makes the analysis more involved. The analysis of a linearization is then combined with Banach’s fixed-point theorem to arrive at the well-posedness of the nonlinear problem with small enough data, and arbitrary large final time \(T \in (0, \infty)\).

Although this result guarantees existence and uniqueness of the solution in very regular spaces and there is no restriction on the time of existence \(T\), we cannot take \(T = \infty\) since the estimates are time dependent. To obtain the estimates uniform in time and prove eventually the global existence (i.e., \(T = \infty\)), we apply a new method based on the construction of suitable compensating functions that encode the dissipation property of (1.1). More precisely, by restricting the regularity to the energy space and using a remarkably simple energy method performed directly on the nonlinear problem (1.1), we also show that for small initial data, the solution is global in time and decays to the steady state exponentially fast; see Theorem 5.1 below for details. It is important to note that the smallness assumption on the initial data seems necessary since solution for large initial data may blow up in finite time.

We note that we expect that the time-weighted energy framework developed in this work can be extended to more general (mixed) boundary conditions and that the ideas put forward here can be transferred to some extent to the study of suitable numerical discretizations of strongly damped nonlinear wave equations as well. We mention in passing that the local well-posedness of this problem in the hyperbolic case \((b = 0)\) follows by [18, Theorem 5.1], where (1.1a) is obtained in the limit of a fractionally damped wave equation for the vanishing sound diffusivity.

The rest of the paper is organized as follows. We begin in Sect. 2 by recalling useful interpolation inequalities that we often employ in the analysis. In Sect. 3 we devise time-weighted estimates for a linearization of (1.1a). Section 4 is dedicated to the analysis of the nonlinear problem which relies on a fixed-point argument under the assumption of small enough initial data. We conclude in Sect. 5 with investigation of the global solvability of the problem. Our main results are contained in Theorems 4.1 and 5.1.
2. Theoretical preliminaries

In this section, we collect certain helpful embedding results and inequalities that we will repeatedly use in the proofs. Throughout the paper, we assume that \( \Omega \subset \mathbb{R}^d \), where \( d \in \{1, 2, 3\} \), is a bounded and \( C^{1,1} \) regular or polygonal/polyhedral and convex domain. We denote by \( T > 0 \) the final propagation time. We make the following assumptions on the involved coefficients:

\[
\begin{align*}
c > 0, & \quad b > 0, & \quad k, \sigma \in \mathbb{R}. \quad (2.1)
\end{align*}
\]

**Notation.** Below we write \( x \lesssim y \) to denote \( x \leq Cy \) where \( C \) is a generic positive constant that does not depend on \( T \). We write \( \lesssim T \) when the hidden constant depends on \( T \) in such a manner that it tends to \( +\infty \) as \( T \to +\infty \). We often omit the spatial and temporal domain when writing norms; for example, \( \| \cdot \|_{L^p(\mathbb{R}^d)} \) denotes the norm in \( L^p(0, T; L^q(\Omega)) \).

In upcoming proofs, we will often use the continuous embeddings [1, Theorem 5.4]:

\[
W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad p \leq q < \infty \quad \text{if} \quad d \leq kp
\]

\[
W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad p \leq q \leq \frac{dp}{d-kp} \quad \text{if} \quad d > kp.
\]

We will also rely on the following application of Hölder’s inequality:

\[
\|u\|_{L^q(0,T;L^p(\Omega))} \lesssim \|u\|_{L^q_1(0,T;L^{p_1}(\Omega))}^{1-\gamma} \|u\|_{L^q_2(0,T;L^{p_2}(\Omega))}^\gamma, \quad \gamma \in [0, 1],
\]

with integers \( p, q, q_1, q_2, p_1, p_2 \in [1, \infty] \), such that

\[
\frac{1}{q} = \frac{1-\gamma}{q_1} + \frac{\gamma}{q_2}, \quad \frac{1}{p} = \frac{1-\gamma}{p_1} + \frac{\gamma}{p_2}.
\]

2.1. Interpolation inequalities

We will also need Agmon’s interpolation inequality [2, Ch. 13] for functions in \( H^2(\Omega) \):

\[
\|u\|_{L^\infty(\Omega)} \leq C_A \|u\|_{H^2(\Omega)}^{d/4} \|u\|_{L^2(\Omega)}^{1-d/4}, \quad (2.2)
\]

Let \( \alpha \in L^2(0, T; H^2(\Omega)) \). Using Agmon’s and Hölder’s inequalities, it follows that

\[
\|\alpha\|_{L^2(0,T;L^\infty(\Omega))} \lesssim \|\alpha(t)\|_{H^2(\Omega)}^{d/4} \|\alpha(t)\|_{L^2(\Omega)}^{1-d/4} \|\alpha(t)\|_{L^{2/(1-d/4)}(0,T)}^{d/4} \|\alpha(t)\|_{L^{2/(1-d/4)}(0,T)}^{1-d/4} \|\alpha\|_{L^2(0,T;L^2(\Omega))} \|\alpha\|_{L^2(0,T;L^2(\Omega))}. \quad (2.3)
\]

We also have the following helpful inequality.
Lemma 2.1. (See p. 74 in [23]) Let $q \in [2, \frac{2d}{d-2}]$ if $d > 2$ and $2 \leq q < \infty$ for $d = 2$ and $2 \leq q \leq \infty$ for $d = 1$. Let $u \in H^1(\Omega)$. Then
\[ \|u\|_{L^q(\Omega)} \leq C \|u\|_{H^1(\Omega)}^{\frac{d-d}{q}} \|u\|_{L^2(\Omega)}^{1-\frac{d}{q}}, \]
where $C$ is a constant which depends only on $d$ and $q$.

Particularly useful for the upcoming analysis will be cases $q = 3$ and $q = 4$:
\[ \|u\|_{L^3(\Omega)} \lesssim \|u\|_{H^1(\Omega)}^{\frac{d}{6}} \|u\|_{L^2(\Omega)}^{1-\frac{d}{6}}, \]
\[ \|u\|_{L^4(\Omega)} \lesssim \|u\|_{H^1(\Omega)}^{\frac{d}{4}} \|u\|_{L^2(\Omega)}^{1-\frac{d}{4}}. \]

2.2. A generalization of Gronwall’s inequality

Finally, we state the following result, which will be needed in the proof of the global solvability and exponential decay of the solution.

Lemma 2.2. (See Lemma 4.5 in [31]) Assume that $u \in C([0, \infty); \mathbb{R}_+)$ satisfies the following inequality
\[ u(t) \leq c_1 e^{at} u(0) + c_2 \int_0^t e^{a(t-s)} u(s)^{1+\kappa} \, ds, \quad \forall t \geq 0, \]
for some constants $c_1 > 1$, $c_2$, $\kappa > 0$, and $a < 0$. Then, under the smallness assumption
\[ a + (1 + 1/\kappa) c_2 2^\kappa c_1^\kappa u(0)^\kappa < 0, \]
it holds
\[ u(t) \leq \left( 1 + \frac{c_2 c_1^\kappa u(0)^\kappa}{a \kappa + (1 + \kappa) c_2 2^\kappa c_1^\kappa u(0)^\kappa} \right) c_1 e^{at} u(0). \]

3. Time-weighted estimates for a linearized problem

We first analyze a linearization of (1.1a) given by
\[ \psi_{tt} - c^2 (1 - 2k\alpha(x,t)) \Delta \psi - b \Delta \psi_t + 2\sigma \nabla \psi \cdot \nabla \alpha(x,t) = \tilde{f} \quad (3.1) \]
supplemented by initial and boundary conditions (1.1b). The results of this section will play a key role when applying the fixed-point argument to the nonlinear problem later in Sect. 4. Indeed, the variable coefficient $\alpha = \alpha(x,t)$ in (3.1) serves as a placeholder for the previous fixed-point iterate of $\psi_t$.

To exploit the parabolic character of (3.1) for $b > 0$, we define a new unknown $v = \psi_t$ so that
\[ \psi(x, t) = \psi_0(x) + \int_0^t v(x, s) \, ds. \]
Consequently, we recast the linearization of (1.1) as
\[
\begin{aligned}
&v_t - b\Delta v - c^2\Delta \psi = -2kc^2\alpha(x, t)\Delta\psi - 2\sigma\nabla\psi \cdot \nabla\alpha(x, t) \quad \text{in } \Omega \times (0, T), \\
v|_{t=0} = \psi_1, \\
v = 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]  
(3.2)

We note that the estimates below can be made rigorous using a Faedo–Galerkin procedure with smooth approximations of the solution in space combined with uniform energy estimates and compactness arguments; see, e.g. [12, Ch. 7]. As this is by now a rather standard procedure also in the context of nonlinear acoustic models (see, e.g., [13,19]), we omit the semi-discretization details in this work and focus on the main energy arguments in the presentation below.

3.1. Estimates for the nonlocal heat equation

We derive first the bounds for the solution of
\[
\begin{aligned}
&v_t - b\Delta v - c^2\Delta \psi = f \quad \text{in } \Omega \times (0, T), \\
v|_{t=0} = \psi_1, \\
v = 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]  
(3.3)

where we have in mind that \( f \) serves as a placeholder for
\[
f = -2kc^2\alpha(x, t)\Delta\psi - 2\sigma\nabla\psi \cdot \nabla\alpha(x, t)
\]  
(3.4)

and should be further estimated later on.

**Proposition 3.1.** Given a final time \( T > 0 \), let \( f \in L^2(0, T; L^2(\Omega)) \) and \((\psi_0, \psi_1) \in H_0^1(\Omega) \times H_0^1(\Omega)\).

Then the following estimate holds:
\[
\| (v, \nabla\psi, \nabla v) \|_{L^\infty(L^2)} + \| (\nabla v, v_t) \|_{L^2(L^2)} \lesssim \| \psi_0 \|_{H^1} + \| \psi_1 \|_{H^1} + \| f \|_{L^2(L^2)}. \]
(3.5)

Additionally,
\[
\| (v, \sqrt{t}\nabla v) \|_{L^\infty(L^2)} + \| (\nabla v, \sqrt{t}v_t) \|_{L^2(L^2)} \lesssim_T \| \psi_0 \|_{H^1} + \| \psi_1 \|_{H^1} + \| f \|_{L^2(L^2)} + \| \sqrt{t}f \|_{L^2(L^2)}. \]
(3.6)

If also \( \sqrt{t}f_t \in L^2(0, T; H^{-1}(\Omega)) \), then
\[
\| \sqrt{t}v_t \|_{L^\infty(L^2)}^2 + \| \sqrt{t}\nabla v_t \|_{L^2(L^2)}^2 \lesssim_T \| \psi_0 \|_{H^1}^2 + \| \psi_1 \|_{H^1}^2 + \| f \|_{L^2(L^2)}^2 + \| \sqrt{t}f_t \|_{L^2(H^{-1})}^2. \]
(3.7)
Proof. By testing the heat equation in (3.3) by $v$, integrating by parts, and using $v = \psi_t$, we obtain
\[
\frac{1}{2} \frac{d}{dt} (\|v\|^2_{L^2} + c^2 \|\nabla \psi\|^2_{L^2}) + b \int _\Omega |\nabla v|^2 \, dx = \int _\Omega f v \, dx. \tag{3.8}
\]
Integrating (3.8) in time and using Young’s $\varepsilon$-inequality together with Poincaré’s inequality, yields
\[
\|v(t)\|^2_{L^2} + \|\nabla \psi(t)\|^2_{L^2} + \int _0^t \|\nabla v(s)\|^2_{L^2} \, ds \lesssim \|\nabla \psi_0\|^2_{L^2} + \|\psi_1\|^2_{L^2} + \|f\|^2_{L^2(L^2)} \tag{3.9}
\]
for all $t \in [0, T]$. Testing the heat equation in (3.3) instead by $v_t$ results in
\[
b \frac{d}{dt} \int _\Omega |\nabla v|^2 \, dx + \int _\Omega v_t^2 \, dx + c^2 \frac{d}{dt} \int _\Omega \nabla \psi \cdot \nabla \psi_t \, dx - c^2 \|\nabla \psi_t\|^2_{L^2} = \int _\Omega v_t f \, dx. \tag{3.10}
\]
Integrating in time and using Young’s inequality leads to
\[
\|\nabla v(t)\|^2_{L^2} + \int _0^t \|v_t(s)\|^2_{L^2} \, ds \lesssim \|\psi_0\|^2_{H^1} + \|\psi_1\|^2_{H^1} + \int _0^t \|f(s)\|^2_{L^2} \, ds + C(\varepsilon) \|\nabla \psi(t)\|^2_{L^2}
\]
\[+ \varepsilon \|\nabla v(t)\|^2_{L^2} + \int _0^t \|\nabla v(s)\|^2_{L^2} \, ds. \tag{3.11}
\]
By multiplying (3.9) by $\lambda > 0$, adding the result to (3.11) and selecting $\varepsilon > 0$ small enough and $\lambda$ large enough, we obtain (3.5).

We prove estimate (3.6) next. To introduce the time weights, we multiply (3.10) by $s \in (0, t)$, which leads to
\[
b \frac{d}{ds} \left( s \|\nabla v\|^2_{L^2} \right) + s \int _\Omega v_t^2 \, dx + c^2 \frac{d}{ds} \left( s \int _\Omega \nabla \psi \cdot \nabla \psi_t \, dx \right) = b \frac{1}{2} \|\nabla v\|^2_{L^2} + s \int _\Omega v_t f \, dx + c^2 s \|\nabla v\|^2_{L^2} + c^2 \int _\Omega \nabla \psi \cdot \nabla \psi_t \, dx.
\]
Integrating the above equality over $s \in (0, t)$ for $t \in (0, T)$ yields
\[
b \frac{1}{2} t \|\nabla v(t)\|^2_{L^2} + \int _0^t \|\sqrt{s} v_t(s)\|^2_{L^2} \, ds
\]
\[\lesssim b \frac{1}{2} \int _0^t \|\nabla v(s)\|^2_{L^2} \, ds + \int _0^t (s v_t f) \, dx \, ds + \int _0^t \|\sqrt{s} \nabla v\|^2_{L^2} \, ds
\]
\[+ \int _0^t \int _\Omega |\nabla \psi \cdot \nabla v| \, dx \, ds + \int _\Omega |t \nabla \psi(t) \cdot \nabla v(t)| \, dx. \tag{3.12}
\]
We can then estimate
\[
\left| \int_0^t \int_\Omega s f v_t \, dx \, ds \right| \leq \varepsilon \int_0^t \| \sqrt{s} v_t(s) \|^2_{L^2} \, ds + C(\varepsilon) \int_0^t \| \sqrt{s} f(s) \|^2_{L^2} \, ds. \tag{3.13}
\]
We also have
\[
\int_0^t \int_\Omega |\nabla \psi \cdot \nabla v| \, dx \, ds \lesssim \int_0^t \| \nabla \psi \|^2_{L^2} \, ds + \int_0^t \| \nabla v \|^2_{L^2} \, ds.
\]
Furthermore, we can use the derived bounds (3.5) on \( \nabla \psi \) and \( \nabla v \) to find
\[
\int_\Omega |t \nabla \psi(t) \cdot \nabla v(t)| \, dx \lesssim T \left( \| \nabla \psi(t) \|^2_{L^2} + \| \nabla v(t) \|^2_{L^2} \right)
\lesssim T \left( \| \psi_0 \|^2_{H^1} + \| \psi_1 \|^2_{H^1} + \| f \|^2_{L^2(L^2)} \right). \tag{3.14}
\]
The first term on the right of (3.13) will be absorbed by the left-hand side of (3.12) as long as \( \varepsilon \) is small enough. We thus infer from (3.12) by using estimates (3.13)–(3.14) that
\[
t \| \nabla v(t) \|^2_{L^2} + \int_0^t \| \sqrt{s} v_t(s) \|^2_{L^2} \, ds
\lesssim \int_0^T \| \sqrt{s} f(s) \|^2_{L^2} \, ds + \int_0^t \| \nabla v(s) \|^2_{L^2} \, ds + \int_0^t \| \nabla \psi(s) \|^2_{L^2} \, ds
+ \int_0^t \| \sqrt{s} \nabla v \|^2_{L^2} \, ds + T \left( \| \nabla \psi(t) \|^2_{L^2} + \| \nabla v(t) \|^2_{L^2} \right).
\]
Combining this estimate with (3.9) and applying Gronwall’s inequality yields (3.6), where the hidden constant has the form \( C(1 + T) e^{CT} \).

It remains to prove estimate (3.7). To this end, we take the time derivative of the heat equation and multiply it by \( \sqrt{t} \):
\[
\partial_t (\sqrt{t} v_t) - \frac{1}{2 \sqrt{t}} v_t - b \Delta \sqrt{t} v_t = \sqrt{t} f_t + c^2 \Delta \sqrt{t} v. \tag{3.15}
\]
Multiplying (3.15) by \( \sqrt{t} v_t \) and integrating over \( \Omega \) (keeping in mind that \( v_t|_{\partial \Omega} = 0 \)) then yields
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{t} v_t \|^2_{L^2} + b \| \sqrt{t} \nabla v_t \|^2_{L^2}
\lesssim \| v_t \|^2_{L^2} + \varepsilon \| \sqrt{t} \nabla v_t \|^2_{L^2} + \| \sqrt{t} f_t \|^2_{H^{-1}} + \sqrt{T} \| \nabla v \|^2_{L^2},
\]
where we have used the estimate \( \langle \sqrt{t} v, \sqrt{t} f_t \rangle_{H^{-1}, H^1} \leq \| \sqrt{t} \nabla v \|_{L^2} \| \sqrt{t} f_t \|_{H^{-1}} \).

For small enough \( \varepsilon > 0 \), by integrating over \( t \in (0, T) \) and using (3.5) to bound \( \| v_t \|_{L^2(L^2)} \) and \( \| \nabla v \|_{L^2(L^2)} \), we obtain (3.7), thus completing the proof. \( \square \)
Our aim now is to show that we can gain one spatial derivative in terms of regularity of $\psi_t$ with respect to the initial condition $\psi_1$, provided we pay the price of a time weight. To this end, we will establish sufficient conditions under which the solution of (3.3) satisfies

$$\sqrt{t} \Delta v \in L^\infty(0, T; L^2(\Omega)).$$

The corresponding bound on $\|\sqrt{t} \Delta v\|_{L^\infty(L^2)}$ will be crucial in the later analysis of the nonlinear problem.

**Proposition 3.2.** Given a final time $T > 0$, let the initial conditions be

$$(\psi_0, \psi_1) \in \left( H^2(\Omega) \cap H^1_0(\Omega) \right) \times H^1_0(\Omega),$$

and the source term $f \in L^2(0, T; L^2(\Omega)).$ Then the following bound holds for the solution of (3.3):

$$\|\psi\|_{L^\infty(H^2)} + \|v\|_{L^2(H^2)} \lesssim_T \|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} + \|f\|_{L^2(L^2)}.$$  \hspace{1cm} (3.16)

If additionally $\sqrt{t} f \in L^\infty(0, T; L^2(\Omega))$, $\sqrt{t} f_t \in L^2(0, T; H^{-1}(\Omega))$ for all $t \in (0, T)$, then

$$\|\sqrt{t} \Delta v\|_{L^\infty(L^2)} \lesssim_T \|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} + \|f\|_{L^2(L^2)} + \|\sqrt{t} f\|_{L^\infty(L^2)} + \|\sqrt{t} f_t\|_{L^2(H^{-1})}.$$  \hspace{1cm} (3.17)

**Proof.** We conduct the proof by bootstrapping the regularity obtained in Proposition 3.1. To estimate $\Delta v$, we write the nonlocal heat equation in (3.3) in the form

$$-\Delta v - \frac{c^2}{b} \Delta \psi = -\frac{1}{b} v_t + \frac{1}{b} f.$$  \hspace{1cm} (3.18)

We then multiply it by $-\Delta v$ and use $v = \psi_t$ to arrive at

$$\|\Delta v\|_{L^2} + \frac{c^2}{2b} \frac{d}{dr} \|\Delta \psi\|_{L^2} = -\frac{1}{b} \int_\Omega v_t \Delta v \, dx + \frac{1}{b} \int_\Omega f \Delta v \, dx.$$  \hspace{1cm} (3.19)

Young’s inequality with $\varepsilon > 0$ small enough yields, after integration in time,

$$\|\Delta \psi\|_{L^\infty(L^2)} + \|\Delta v\|_{L^2(L^2)} \lesssim \|\Delta \psi_0\|_{L^2} + \|v_t\|_{L^2(L^2)} + \|f\|_{L^2(L^2)}.$$ 

Taking into account the estimate of $\|v_t\|_{L^2(L^2)}$ in (3.5), we obtain (3.16).

To prove estimate (3.17), we multiply (3.18) by $\sqrt{t}$:

$$-\sqrt{t} \Delta v = -\frac{\sqrt{t}}{b} v_t + \frac{\sqrt{t}}{b} f + \frac{c^2}{b} \sqrt{t} \Delta \psi.$$ 

From here we immediately have

$$\|\sqrt{t} \Delta v\|_{L^\infty(L^2)} \lesssim \|\sqrt{t} f\|_{L^\infty(L^2)} + \|\sqrt{t} v_t\|_{L^\infty(L^2)} + \sqrt{T} \|\Delta \psi\|_{L^\infty(L^2)}.$$ 

Combining this bound with (3.7) and (3.16) to estimate the last two terms on the right yields (3.17). \qed
We observe from the last proof that the assumption \(\psi_0 \in H^2(\Omega)\) in the statement of Proposition 3.2 above is due to the having the nonlocal term \(-c^2 \Delta \psi\) in the heat equation. A bound on \(\|\Delta \psi\|_{L^\infty(L^2)}\) will also be needed to estimate \(f\) further using (3.4) and, in turn, tackle the nonlinear problem.

Motivated by the previous analysis, let us introduce the time-weighted space \(X^v_t \subset X^v\) to which \(v = \psi_t\) belongs:

\[
X^v_t = \{ v \in X^v : \|\sqrt{t} v_t\|_{L^\infty(L^2)} + \|\sqrt{t} \nabla v_t\|_{L^2(L^2)} + \|\sqrt{t} \Delta v_t\|_{L^\infty(L^2)} < \infty \}
\]

with the weight-independent contribution

\[
X^v = \{ v \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) : v_t \in L^2(0, T; L^2(\Omega)) \}.
\]

The corresponding norm is denoted by \(\|\cdot\|_{X^v_t}\). According to Propositions 3.1 and 3.2, we then have

\[
\|v\|_{X^v_t} \lesssim_T \|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} + \|f\|_{L^2(L^2)} + \|\sqrt{t} f\|_{L^\infty(L^2)} + \|\sqrt{t} f_t\|_{L^2(H^{-1})}. \tag{3.20}
\]

3.2. Estimates for the linearized Blackstock equation

Our next aim is to derive time-weighted bounds for (3.2) by relying on the obtained estimates for the nonlocal heat equation but now using the form of \(f\) given in (3.4).

The solution space for the acoustic velocity potential will be \(X^\psi_t \subset X^\psi\), defined by

\[
X^\psi_t = \left\{ \psi \in X^\psi : \|\sqrt{t} \psi_{tt}\|_{L^\infty(L^2)} + \|\sqrt{t} \nabla \psi_{tt}\|_{L^2(L^2)} + \|\sqrt{t} \Delta \psi_t\|_{L^\infty(L^2)} < \infty \right\}
\]

with the weight-independent contribution

\[
X^\psi = \{ \psi : \psi \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \psi_t \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \psi_{tt} \in L^2(0, T; L^2(\Omega)) \}.
\]

We next prove well-posedness of the linearized Blackstock problem in \(X^\psi_t\).

**Proposition 3.3.** Let \(T > 0\) and let assumption (2.1) on the medium coefficients hold. Assume that

\[(\psi_0, \psi_1) \in \left( H^2(\Omega) \cap H_0^1(\Omega) \right) \times H_0^1(\Omega)\]

and let

\[
\tilde{f} \in \{ \tilde{f} \in L^2(0, T; L^2(\Omega)) : \|\sqrt{t} \tilde{f}\|_{L^\infty(L^2)} + \|\sqrt{t} \tilde{f}_t\|_{L^2(H^{-1})} < \infty \}.
\]
Furthermore, assume that there exists $R > 0$, such that

$$||\alpha||_{L^1_t} \leq R.$$  

Then there exists $m = m(R, T) > 0$, such that if the coefficient $\alpha$ is sufficiently small in the sense of

$$|k|(||\alpha||_{L^\infty(L^2)} + \sqrt{t} ||\alpha||_{L^2(L^3)}) + |\sigma|||\nabla\alpha||_{L^2(L^2)} + \sqrt{t} ||\nabla\alpha||_{L^2(L^2)} \leq m,$$

then there is a unique $\psi \in X_t^\psi$ which solves

$$\begin{cases}
\psi_{tt} - c^2(1 - 2k\alpha(x, t))\Delta\psi - b\Delta\psi_t + 2\sigma \nabla\psi \cdot \nabla\alpha(x, t) = \tilde{f} \text{ in } \Omega \times (0, T), \\
(\psi, \psi_t)|_{t=0} = (\psi_0, \psi_1), \\
\psi|_{\partial\Omega} = 0.
\end{cases}$$

(3.22)

This solution satisfies the following bound:

$$||\psi||_{X_t^\psi} \lesssim_T ||\psi_0||_{H^2} + ||\psi_1||_{H^1} + ||\tilde{f}||_{L^2(L^2)} + \sqrt{t} ||\tilde{f}||_{L^\infty(L^2)} + ||\sqrt{t} \tilde{f}_t||_{L^2(H^{-1})}. $$

(3.23)

**Proof.** By combining estimates (3.16) and (3.20), we obtain

$$||\psi||_{X_t^\psi} \lesssim_T ||\Delta\psi||_{L^\infty(L^2)} + ||\psi||_{X_t^\psi} \lesssim_T ||\psi_0||_{H^2} + ||\psi_1||_{H^1(\Omega)} + ||\tilde{f}||_{L^2(L^2)} + ||\sqrt{t} \tilde{f}||_{L^\infty(L^2)} + ||\sqrt{t} \tilde{f}_t||_{L^2(H^{-1})}. $$

(3.24)

Thus the proof boils down to estimating the $f$ terms on the right-hand side above. Recall that

$$f = -2k c^2 \alpha(x, t) \Delta\psi - 2\sigma \nabla\psi \cdot \nabla\alpha(x, t) + \tilde{f}.$$  

Hölder’s inequality and interpolation estimates (2.3) allow us to conclude that

$$||f||_{L^2(L^2)} \lesssim |k| ||\alpha||_{L^2(L^\infty)} ||\Delta\psi||_{L^\infty(L^2)} + |\sigma|||\nabla\psi||_{L^\infty(L^4)} ||\nabla\alpha||_{L^2(L^4)} + ||\tilde{f}||_{L^2(L^2)} \lesssim |k| ||\Delta\alpha||_{L^2(L^2)} ||\alpha||_{L^2(L^2)} ||\Delta\psi||_{L^\infty(L^2)} + |\sigma|||\nabla\alpha||_{L^2(H^1)} ||\nabla\psi||_{L^\infty(L^4)} + ||\tilde{f}||_{L^2(L^2)}. $$

(3.25)

Employing additionally Poincaré’s inequality and the embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^4(\Omega)$ together with elliptic regularity yields

$$||f||_{L^2(L^2)} \lesssim m^{1-d/4} ||\Delta\alpha||_{L^2(L^2)} ||\Delta\psi||_{L^\infty(L^2)} + ||\tilde{f}||_{L^2(L^2)}. $$

(3.26)
We next estimate $\|\sqrt{t}f\|_{L^\infty(\mathbb{R}^2)}$ in (3.24). Hölder’s and Agmon’s inequalities imply
\[
\|\sqrt{t}f(t)\|_{L^2} \lesssim |k| \|\sqrt{t}\alpha(t)\|_{L^\infty} \|\Delta \psi(t)\|_{L^2} + |\sigma| \|\sqrt{t}\nabla \alpha(t)\|_{L^4} \|\nabla \psi(t)\|_{L^4} + \|\sqrt{t} f(t)\|_{L^2}
\]
\[
\lesssim_T |k| \|\sqrt{t} \alpha(t)\|_{H^{d/4}} \|\alpha(t)\|_{L^2}^{-d/4} \|\Delta \psi(t)\|_{L^2} + |\sigma| \|\sqrt{t}\nabla \alpha(t)\|_{L^4} \|\nabla \psi(t)\|_{L^4}
+ \|\sqrt{t} \tilde{f}(t)\|_{L^2}.
\]

(3.27)

Above in the last line we have used
\[
\|\sqrt{t} \alpha(t)\|_{L^2} \leq \sqrt{T} \|\alpha(t)\|_{L^2}.
\]

Using Lemma 2.1 with $q = 4$ together with Hölder’s inequality in time, we obtain
\[
|\sigma| \|\sqrt{t}\nabla \alpha\|_{L^\infty(\mathbb{R}^4)} \lesssim |\sigma| \|\sqrt{t}\nabla \alpha\|_{L^\infty(\mathbb{R}^2)} \|\sqrt{t}\nabla \alpha\|_{L^\infty(\mathbb{R}^4)} \lesssim m^{1-d/2} \|\sqrt{t} \Delta \alpha\|_{L^\infty(\mathbb{R}^2)}.
\]

These estimates employed in (3.27) yield
\[
\|\sqrt{t} f\|_{L^\infty(\mathbb{R}^2)} \lesssim_T m^{1-d/4} \|\sqrt{t} \Delta \alpha\|_{L^\infty(\mathbb{R}^2)} \|\Delta \psi\|_{L^\infty(\mathbb{R}^2)} + \|\sqrt{t} \tilde{f}\|_{L^\infty(\mathbb{R}^2)}.
\]

Next we estimate $\|\sqrt{t} f_t\|_{L^2(\mathbb{R}^{-1})}$. To this end, we rely on the following inequality:
\[
\|ab\|_{H^{-1}} \lesssim \|ab\|_{L^{6/5}} \lesssim \|a\|_{L^2} \|b\|_{L^3} \quad a \in L^2(\Omega), \quad b \in L^3(\Omega). \tag{3.28}
\]

Since
\[
f_t = -2kc^2 \alpha \psi - 2\sigma \nabla \psi_t \cdot \nabla \alpha(x, t) - 2kc^2 \alpha(x, t) \Delta \psi_t - 2\sigma \nabla \psi \cdot \nabla \alpha_t(x, t) + \tilde{f}_t,
\]
the use of estimate (3.28) together with Hölder’s inequality implies
\[
\|\sqrt{t} f_t\|_{L^2(\mathbb{R}^{-1})}
\lesssim |k| \|\sqrt{t} \alpha_t\|_{L^2(\mathbb{R}^3)} \|\Delta \psi\|_{L^\infty(\mathbb{R}^2)} + |\sigma| \|\sqrt{t}\nabla \psi_t\|_{L^\infty(\mathbb{R}^2)} \|\nabla \alpha\|_{L^2(\mathbb{R}^2)}
+ |k| \|\alpha\|_{L^2(\mathbb{R}^3)} \|\sqrt{t} \Delta \psi_t\|_{L^\infty(\mathbb{R}^2)} + |\sigma| \|\nabla \psi\|_{L^\infty(\mathbb{R}^3)} \|\sqrt{t}\nabla \alpha_t\|_{L^2(\mathbb{R}^2)}
+ \|\sqrt{t} \tilde{f}_t\|_{L^2(\mathbb{R}^{-1})}.
\]

We have by using Lemma 2.1 together with the elliptic regularity
\[
|\sigma| \|\nabla \alpha\|_{L^2(\mathbb{R}^3)} \lesssim |\sigma| \|\nabla \alpha\|_{L^2(\mathbb{R}^2)} \|\Delta \alpha\|_{L^2(\mathbb{R}^2)} \lesssim m^{1-d/6} \|\Delta \alpha\|_{L^2(\mathbb{R}^2)}. \tag{3.29}
\]

Similarly,
\[
|k| \|\alpha\|_{L^2(\mathbb{R}^3)} \lesssim |k| \|\alpha\|_{L^2(\mathbb{R}^2)} \|\nabla \alpha\|_{L^2(\mathbb{R}^2)} \lesssim m^{1-d/6} \|\nabla \alpha\|_{L^2(\mathbb{R}^2)}
\]
and
\[
|k| \|\sqrt{t} \alpha_t\|_{L^2(\mathbb{R}^3)} \lesssim |k| \|\sqrt{t} \alpha_t\|_{L^2(\mathbb{R}^2)} \|\sqrt{t} \alpha_t\|_{L^2(\mathbb{R}^{-1})} \lesssim m^{1-d/6} \|\sqrt{t}\nabla \alpha_t\|_{L^2(\mathbb{R}^2)}. \tag{3.30}
\]
Thus we have by using (3.29)–(3.30) and elliptic regularity,
\[
\|\sqrt{t}f_t\|_{L^2(H^{-1})} \lesssim m^{1-\frac{d}{6}} \|\sqrt{t}\nabla \alpha_t\|_{L^2(L^2)} \|\Delta \psi\|_{L^\infty(L^2)} + m^{1-\frac{d}{6}} \|\Delta \alpha\|_{L^2(L^2)} \|v\|_{X^\psi_t} + \|\sqrt{t}f_t\|_{L^2(H^{-1})}.
\]
Inserting all the derived bounds on \(f\) terms into (3.24) yields
\[
\|\Delta \psi\|_{L^\infty(L^2)} + \|v\|_{X^\psi_t} \lesssim T \|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} + \Lambda[\alpha, m](\|\Delta \psi\|_{L^\infty(L^2)} + \|v\|_{X^\psi_t}) + \|\sqrt{t}f_t\|_{L^\infty(L^2)} + \|f_t\|_{L^2(L^2)} + \|\sqrt{t}f_t\|_{L^2(H^{-1})} + \|\sqrt{t}f_t\|_{L^2(H^{-1})},
\]
with
\[
\Lambda[\alpha, m] = \max\{m^{1-d/4}, m^{d/4}, m^{1-d/6}\} \|\alpha\|_{X^\psi_t}.
\]
Thus, from (3.31) for sufficiently small \(m = m(\|\alpha\|_{X^\psi_t}, T) > 0\), we obtain
\[
\|\Delta \psi\|_{L^\infty(L^2)} + \|\psi_t\|_{X^\psi_t} \lesssim T \|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} + \|\sqrt{t}f_t\|_{L^\infty(L^2)} + \|f_t\|_{L^2(L^2)} + \|\sqrt{t}f_t\|_{L^2(H^{-1})},
\]
from which (3.23) follows. We note that if \(\sigma = 0\), a smallness assumption on \(\|\nabla \alpha\|_{L^2(L^2)} + \|\sqrt{t}\nabla \alpha_t\|_{L^2(L^2)}\) is not needed. Of course, if both \(k = \sigma = 0\), the smallness condition in the statement is trivially satisfied. \(\square\)

4. A fixed-point argument

To relate the previous analysis to the nonlinear problem, we employ the Banach fixed-point theorem under the assumption of small enough data.

**Theorem 4.1.** (Local solvability of the Blackstock equation) Let \(T > 0\) and
\[
(\psi_0, \psi_1) \in \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times H^1_0(\Omega).
\]
Let the medium coefficients satisfy (2.1). There exists \(\delta = \delta(T) > 0\), such that if data is sufficiently small in the sense of
\[
\|\psi_0\|_{H^2} + \|\psi_1\|_{H^1} \leq \delta,
\]
then there is a unique \(\psi \in X^\psi_t\) which solves
\[
\begin{cases}
\psi_{tt} - c^2(1 - 2k\psi_t)\Delta \psi - b\Delta \psi_t + 2\sigma \nabla \psi \cdot \nabla \psi_t = 0 \text{ in } \Omega \times (0, T), \\
(\psi, \psi_t) = (\psi_0, \psi_1), \\
\psi|_{\partial \Omega} = 0,
\end{cases}
\]
with \(X^\psi_t \subset X^\psi\) defined in (3.21). The solution depends continuously on the initial data with respect to the \(\|\cdot\|_{X^\psi_t}\) norm.
Before moving onto the proof, we briefly discuss the statement made above.

– Theorem 4.1 guarantees solvability under weaker regularity assumptions on initial conditions than those available in the literature [13, 20, 35], where the initial data is assumed to have at least the regularity given in (1.3).

– Although the final time $T$ is fixed, there are no restrictions on its size.

– The presence of the time weights yields the additional higher regularity of the solution so that $\psi \in X^\psi_t$ and not only $\psi \in X^\psi$. Without the developed time-weighted framework, such a regularity cannot be shown for initial data satisfying (4.1).

**Proof.** As announced, we set up a fixed-point mapping

$$T : \mathcal{B} \ni \psi^* \mapsto \psi,$$

where

$$\mathcal{B} = \{\psi^* \in X^\psi_t : \|\psi^*\|_{X^\psi_t} \leq R, (\psi^*_0, \psi^*_1) = (\psi_0, \psi_1),$$

$$\|k\|_{L^\infty(L^2)} + \|\sqrt{t} \psi^*_{tt}\|_{L^2(L^2)} + \|\sqrt{t} \psi^*_t\|_{L^2(L^2)} \leq m\}.$$  \hspace{1cm} (4.3)

and $\psi$ solves the linear problem (3.22) with $\tilde{f} = 0$ and the variable coefficient $\alpha = \psi^*_t$:

$$\begin{align*}
  \psi_{tt} - c^2 (1 - 2k \psi^*_t) \Delta \psi - \sqrt{t} \psi_t - b \Delta \psi_t + 2\sigma \nabla \psi \cdot \nabla \psi^*_t &= 0 \text{ in } \Omega \times (0, T), \\
  (\psi, \psi_t) &= (\psi_0, \psi_1), \\
  \psi|_{\partial \Omega} &= 0.
\end{align*}$$

It is suffices to find a (unique) fixed point of the mapping $T(\psi^*) = \psi$. We choose $m > 0$ in (4.3) according to Proposition 3.3 which guarantees that the mapping is well-defined (and $\mathcal{B}$ non-empty).

Take $\psi^* \in \mathcal{B}$. To prove the self-mapping property, we rely on Proposition 3.3. We choose $R > 0$ so that

$$R \geq C_{\text{lin}}(T)(\|\psi_0\|_{H^2} + \|\psi_1\|_{H^1}) \geq \|\psi\|_{X^\psi_t},$$  \hspace{1cm} (4.4)

where $C_{\text{lin}}(T)$ is the hidden constant in (3.23). To prove that $\psi$ satisfies the $m$ bound within (4.3), we note that

$$\|k\|_{L^\infty(L^2)} + \|\sqrt{t} \psi_{tt}\|_{L^2(L^2)} + \|\sqrt{t} \psi_t\|_{L^2(L^2)} \lesssim \|\psi\|_{X^\psi_t}.$$  \hspace{1cm} (4.5)

Thus, energy bound (3.23) for the linearized problem guarantees that

$$\|\psi\|_{X^\psi_t} \leq C_{\text{lin}}(T)(\|\psi_0\|_{H^2} + \|\psi_1\|_{H^1}) \lesssim C_{\text{lin}}(T) \delta \leq m$$  \hspace{1cm} (4.6)

by reducing the size of data $\delta$. Hence, (4.4) together with (4.6) shows that $\psi \in \mathcal{B}$. 
In the second part of the proof, we prove strict contractivity. Take \( \varphi^*, \phi^* \in \mathcal{B} \) and let \( T(\varphi^*) = \varphi, \ T(\phi^*) = \phi \). We also introduce the differences
\[
\tilde{\psi} = \varphi - \phi, \quad \tilde{\psi}^* = \varphi^* - \phi^*.
\]
Then \( \tilde{\psi} \in \mathcal{B} \) solves
\[
\tilde{\psi}_{tt} - c^2 (1 - 2k\varphi^*_t) \Delta \tilde{\psi} - b \Delta \tilde{\psi}_t + 2\sigma \nabla \tilde{\psi} \cdot \nabla \varphi^*_t = -2k c^2 \tilde{\psi}^*_t \Delta \psi - 2\sigma \nabla \psi \cdot \nabla \tilde{\psi}^*_t
\]
with homogeneous boundary and initial conditions. We can thus employ estimate (3.23) with zero initial data, that is
\[
\| \tilde{\psi} \|_{X_t^\psi} \lesssim_T \| \sqrt{T} \tilde{f} \|_{L^\infty(L^2)} + \| \tilde{f} \|_{L^2(L^2)} + \| \sqrt{T} \tilde{f} \|_{L^2(H^{-1})} + \| \sqrt{T} \tilde{f}_t \|_{L^2(H^{-1})},
\]
where
\[
\tilde{f} = -2k c^2 \tilde{\psi}^*_t \Delta \psi - 2\sigma \nabla \psi \cdot \nabla \tilde{\psi}^*_t.
\]
It remains to estimate the \( \tilde{f} \) terms, which we can do similarly to the estimates of \( f \) terms in (3.24) in the proof of Proposition 3.3. We have
\[
\| \tilde{f} \|_{L^2(L^2)} \lesssim \| \tilde{\psi}^* \|_{L^2(L^\infty)} \| \Delta \psi \|_{L^\infty(L^2)} + \| \nabla \psi \|_{L^\infty(L^4)} \| \nabla \tilde{\psi}^*_t \|_{L^2(L^4)}
\]
\[
\lesssim \| \Delta \psi \|_{L^\infty(L^2)} \| \tilde{\psi}^* \|_{X_t^\psi}
\]
\[
\lesssim R \| \tilde{\psi}^* \|_{X_t^\psi}.
\]
Next,
\[
\| \sqrt{T} \tilde{f} \|_{L^\infty(L^2)} \lesssim \| \sqrt{T} \tilde{\psi}^* \|_{L^\infty(L^2)} \| \Delta \psi \|_{L^\infty(L^2)} + \| \nabla \psi \|_{L^\infty(L^4)} \| \sqrt{T} \nabla \tilde{\psi}^*_t \|_{L^\infty(L^2)}
\]
\[
\lesssim R \| \tilde{\psi}^* \|_{X_t^\psi}.
\]
Additionally,
\[
\| \sqrt{T} \tilde{f}_t \|_{L^2(H^{-1})} = \| \sqrt{T} (-2k c^2 \tilde{\psi}^*_t \Delta \psi - 2k c^2 \tilde{\psi}^*_t \Delta \phi_t - 2\sigma \nabla \phi_t \cdot \nabla \tilde{\psi}^*_t
\]
\[
- 2\sigma \nabla \psi \cdot \nabla \tilde{\psi}^*_t) \|_{L^2(H^{-1})}
\]
\[
\lesssim \| \sqrt{T} \tilde{\psi}^*_t \|_{L^2(L^3)} \| \Delta \psi \|_{L^\infty(L^2)} + \| \tilde{\psi}^*_t \|_{L^2(L^3)} \| \sqrt{T} \Delta \phi_t \|_{L^\infty(L^2)}
\]
\[
+ \| \sqrt{T} \nabla \phi \|_{L^\infty(L^2)} \| \nabla \tilde{\psi}^*_t \|_{L^2(L^3)} + \| \nabla \phi \|_{L^\infty(L^3)} \| \sqrt{T} \nabla \tilde{\psi}^*_t \|_{L^2(L^2)}
\]
\[
\lesssim R \| \tilde{\psi}^* \|_{X_t^\psi}.
\]
Therefore, we can guarantee strict contractivity of \( T \) with respect to the \( \| \cdot \|_{X_t^\psi} \) norm by reducing the radius \( R \), which in turn requires sufficient smallness of \( \delta \). By Banach’s fixed-point theorem, we obtain a unique \( \psi \in \mathcal{B} \), which solves (4.2). \( \square \)
5. Global existence

To conclude, we discuss the global solvability of the nonlinear problem (1.1). Our goal is to control the solution of (1.1) uniformly as $t \to \infty$ in a suitable energy norm. In addition, we accurately describe the asymptotic behavior of the solution of (1.1) as $t \to \infty$. More precisely, we show that the solutions decays exponentially fast in time. To state the global result, we introduce the energy $E(t)$ and the corresponding dissipation $D(t)$ at time $t \in (0, T)$ as follows:

$$E(t) = \frac{1}{2} \| \psi_t(t) \|^2_{L^2} + \frac{c^2}{2} \| \nabla \psi(t) \|^2_{L^2} + \frac{c^2}{2b} \| \Delta \psi(t) \|^2_{L^2} + \| \nabla \psi_t(t) \|^2_{L^2}$$

and

$$D(t) = \int_0^t \left( \| \nabla \psi(s) \|^2_{L^2} + \| \Delta \psi(s) \|^2_{L^2} + \| \nabla \psi_t(s) \|^2_{L^2} + \| \Delta \psi(s) \|^2_{L^2} + \| \psi_{tt}(s) \|^2_{L^2} \right) ds.$$

**Theorem 5.1.** (Global solvability of the Blackstock equation) Assume that

$$(\psi_0, \psi_1) \in \left( H^2(\Omega) \cap H^1_0(\Omega) \right) \times H^1_0(\Omega) .$$

There exists $\epsilon_0 > 0$, such that if the data is sufficiently small so that

$$\| \psi_0 \|_{H^2} + \| \psi_1 \|_{H^1} \leq \epsilon_0 ,$$

then there is a unique global solution $\psi$ of (1.1), such that

$$\psi \in L^\infty(0, \infty; H^2(\Omega) \cap H^1_0(\Omega)) ,$$
$$\psi_t \in L^\infty(0, \infty; H^1_0(\Omega)) \cap L^2(0, \infty; H^2(\Omega) \cap H^1_0(\Omega)) ,$$
$$\psi_{tt} \in L^2(0, \infty; L^2(\Omega)) .$$

In addition, there exists a constant $\xi > 0$, such that for all $t \geq 0$, we have

$$E(t) \leq C E(0) e^{-\xi t} ,$$

where $C > 0$ does not depend on time.

**Proof.** The proof relies on the construction of suitable compensating functions $F_i = F_i(t)$ for $i = 1, 2, 3$ that can capture the dissipation properties of problem (1.1). A Lyapunov function $L = L(t)$ can then be constructed as a linear combination of these functionals (with appropriate weights) and of the total energy $E = E(t)$. As the function $L$ is equivalent to the energy, it allows recovering the optimal dissipation of the Blackstock equation. In addition, it satisfies a differential inequality that facilitates the exponential decay of the energy norm of the solution. Below $C > 0$ denotes a generic constant independent of time. Let

$$E_1(t) = \frac{1}{2} \| \psi_t(t) \|^2_{L^2} + \frac{c^2}{2} \| \nabla \psi(t) \|^2_{L^2} .$$
Recall from (3.8) that multiplying (1.1a) by $\psi_t$, integrating over $\Omega$, and using integration by parts yields

\[
\frac{d}{dt} E_1(t) + b \| \nabla \psi_t \|_{L^2}^2 = \int_\Omega f \psi_t \, dx,
\]

where

\[
f = -2k c^2 \psi_t \Delta \psi - 2\sigma \nabla \psi \cdot \nabla \psi_t.
\] (5.1)

Thus by Young’s and Poincaré’s inequalities, we have

\[
\frac{d}{dt} E_1(t) + \frac{b}{2} \| \nabla \psi_t \|_{L^2}^2 \lesssim \| f \|_{L^2}^2.
\]

Let

\[
E_2(t) = \frac{c^2}{2b} \| \nabla \psi(t) \|_{L^2}^2.
\]

We have from (3.19),

\[
\frac{d}{dt} E_2(t) + \| \Delta \psi_t \|_{L^2}^2 \leq C (\| \psi_{tt} \|_{L^2}^2 + \| f \|_{L^2}^2).
\]

Next we introduce

\[
F_1 = \int_\Omega \left( \psi \psi_t + \frac{1}{2} b |\nabla \psi|^2 \right) \, dx.
\]

By testing (1.1a) by $\psi$, we immediately have

\[
\frac{d}{dt} F_1(t) + c^2 \| \nabla \psi \|_{L^2}^2 = \| \psi_t \|_{L^2}^2 + \int_\Omega f \psi \, dx.
\]

Hence by Young’s and Poincaré’s inequalities we have

\[
\frac{d}{dt} F_1(t) + \frac{c^2}{2} \| \nabla \psi \|_{L^2}^2 \lesssim \| \psi_t \|_{L^2}^2 + \| f \|_{L^2}^2 \lesssim \| \nabla \psi_t \|_{L^2}^2 + \| f \|_{L^2}^2.
\]

We further introduce the functional

\[
F_2(t) = \int_\Omega \left( -\Delta \psi \psi_t + \frac{b}{2} |\Delta \psi|^2 \right) \, dx.
\]

By testing (1.1a) by $-\Delta \psi$, we can see that

\[
\frac{d}{dt} F_2(t) + c^2 \| \Delta \psi \|_{L^2}^2 = \| \nabla \psi_t \|_{L^2}^2 - \int_\Omega \Delta \psi f \, dx,
\]

which yields

\[
\frac{d}{dt} F_2(t) + \| \Delta \psi \|_{L^2}^2 \lesssim \| \nabla \psi_t \|_{L^2}^2 + \| f \|_{L^2}^2.
\]
To capture further dissipation terms, we also introduce

\[ F_3 = c^2 \int_{\Omega} \nabla \psi \cdot \nabla \psi_t \, dx + \frac{b}{2} \int_{\Omega} |\nabla \psi_t|^2 \, dx. \]

Then from (3.10) we know that

\[ \frac{d}{dt} F_3(t) + \int_{\Omega} \psi_{tt}^2 \, dx = c^2 \| \nabla \psi_t \|_{L^2}^2 + \int_{\Omega} \psi_{tt} f \, dx \]

and thus

\[ \frac{d}{dt} F_3(t) + \int_{\Omega} \psi_{tt}^2 \, dx \lesssim \| \nabla \psi_t \|_{L^2}^2 + \| f \|_{L^2}^2. \]

Let \( \gamma_i \) for \( i \in \{1, 2, 3\} \) be small positive constants. We define the Lyapunov functional

\[ L(t) = E_1(t) + \gamma_1 E_2(t) + \gamma_2 F_1(t) + \gamma_2 F_2(t) + \gamma_3 F_3(t), \quad (5.2) \]

which we will show is equivalent to the energy \( E \). We have by Poincaré’s inequality

\[ \left| L(t) - E_1(t) - \gamma_1 E_2(t) - \gamma_3 \frac{b}{2} \| \nabla \psi_t \|_{L^2}^2 \right| \]

\[ \leq \gamma_2 (|F_1(t)| + |F_2(t)|) + \gamma_3 c^2 \int_{\Omega} \nabla \psi \cdot \nabla \psi_t \, dx \]

\[ \leq C \gamma_2 (\| \psi_t \|_{L^2}^2 + \| \nabla \psi \|_{L^2}^2 + \| \Delta \psi \|_{L^2}^2) + \gamma_3 \frac{b}{4} \| \nabla \psi_t \|_{L^2}^2 + C \gamma_3 \| \nabla \psi \|_{L^2}^2 \]

\[ \leq C \gamma_2 (E_1(t) + E_2(t)) + \gamma_3 \frac{b}{4} \| \nabla \psi_t \|_{L^2}^2. \]

Hence, this estimate yields

\[ (1 - C \gamma_2 - C \gamma_3) E_1(t) + (\gamma_1 - C \gamma_2) E_2(t) + \gamma_3 \frac{b}{4} \| \nabla \psi_t \|_{L^2}^2 \leq L(t) \lesssim C E(t). \]

We fix \( \gamma_2 > 0 \) and \( \gamma_3 > 0 \) small enough so that

\[ \gamma_2 + \gamma_3 < 1/C \]

and \( \gamma_1 \) large enough so that

\[ \gamma_1 > C \gamma_2. \]

Then for all \( t \geq 0 \) we have the equivalence

\[ C_1 E(t) \leq L(t) \leq C_2 E(t) \quad (5.3) \]

for some \( C_1, C_2 > 0 \), independent of time. From (5.2) and the derived bounds, we conclude that

\[ \frac{d}{dt} L(t) + b \| \nabla \psi_t \|_{L^2}^2 + \gamma_1 \| \Delta \psi_t \|_{L^2}^2 + \gamma_2 \frac{c^2}{2} \| \nabla \psi \|_{L^2}^2 + \gamma_2 c^2 \| \Delta \psi \|_{L^2}^2 + \gamma_3 \| \psi_{tt} \|_{L^2}^2 \]

\[ \leq C (\gamma_1 \| \psi_{tt} \|_{L^2}^2 + (\gamma_2 + \gamma_3) \| \nabla \psi_t \|_{L^2}^2 + \| f \|_{L^2}^2). \]
Using Poincaré’s inequality and choosing 
\[ \gamma_2 + \gamma_3 < b/C, \quad \gamma_1 < \gamma_3/C, \]
we obtain
\[ \frac{d}{dt} L(t) + \| \nabla \psi_t \|_{L^2}^2 + \| \Delta \psi_t \|_{L^2}^2 + \| \nabla \psi \|_{L^2}^2 + \| \Delta \psi \|_{L^2}^2 + \| \psi_{tt} \|_{L^2}^2 \lesssim \| f \|_{L^2}^2. \quad (5.4) \]
Integrating (5.4) with respect to time and using equivalence (5.3) leads to
\[ \sup_{t \in (0,T)} E(t) + \sup_{t \in (0,T)} D(t) \lesssim E(0) + \int_0^t \| f(s) \|_{L^2}^2 \, ds. \quad (5.5) \]
Recalling the definition of \( f \) in (5.1), we have
\[ \int_0^t \| f(s) \|_{L^2}^2 \, ds \lesssim \int_0^t \| \psi_t(s) \|_{L^\infty} \| \Delta \psi(s) \|_{L^2}^2 \, ds + \int_0^t \| \nabla \psi(s) \|_{L^4} \| \nabla \psi_t(s) \|_{L^4}^2 \, ds \]
\[ \lesssim \int_0^t \| \Delta \psi_t(s) \|_{L^2} \| \Delta \psi(s) \|_{L^2}^2 \, ds \]
\[ \lesssim \sup_{t \in (0,T)} E(t) D(t). \]
Plugging this into (5.5) yields
\[ \sup_{t \in (0,T)} E(t) + \sup_{t \in (0,T)} D(t) \lesssim E(0) + \sup_{t \in (0,T)} E(t) D(t). \]
Hence, if \( E(0) \) is small enough, a bootstrap argument leads to
\[ \sup_{t \in (0,T)} E(t) + \sup_{t \in (0,T)} D(t) \lesssim C. \]
We next prove the exponential decay of the energy. Using (2.2), we have by applying Agmon’s and Young’s inequalities,
\[ \| \psi_t \|_{L^\infty} \| \Delta \psi \|_{L^2} \leq C_A \| \psi_t \|_{L^2(\Omega)}^{d/4} \| \psi_t \|_{L^2(\Omega)}^{1-d/4} \| \Delta \psi \|_{L^2} \]
\[ \lesssim \varepsilon \| \psi_t \|_{H^2(\Omega)} + C(\varepsilon) \left( \| \nabla \psi_t \|_{L^2(\Omega)} \| \Delta \psi \|_{L^2} \right)^{4/(4-d)}. \]
Applying Young’s inequality yields
\[ \| \psi_t \|_{L^\infty} \| \Delta \psi \|_{L^2} \, ds \leq \varepsilon^2 \| \psi_t \|_{H^2(\Omega)}^2 + C(\varepsilon)(E(t))^{1+\kappa} \quad (5.6) \]
for some \( \kappa > 0 \). Similarly, we have by Lemma 2.1
\[ \| \nabla \psi \|_{L^4} \| \nabla \psi_t \|_{L^2} \lesssim \| \nabla \psi_t \|_{L^2(\Omega)}^{2(1-d/4)} \| \psi_t \|_{H^2(\Omega)}^{d/2} \| \Delta \psi \|_{L^2} \]
\[ \leq \varepsilon^2 \| \psi_t \|_{H^2(\Omega)}^2 + C(\varepsilon)(E(t))^{1+\kappa} \quad (5.7) \]
Inserting (5.6) and (5.7) into (5.4), and selecting $\varepsilon$ small enough leads to
\[
\frac{d}{dt} L(t) + \| \nabla \psi_t \|_{L^2}^2 + \| \Delta \psi_t \|_{L^2}^2 + \| \nabla \psi \|_{L^2}^2 + \| \Delta \psi \|_{L^2}^2 + \| \psi_{tt} \|_{L^2}^2 \lesssim (E(t))^{1+\kappa}.
\]
From the equivalence (5.3), we deduce that there exists a positive constant $\zeta > 0$, such that
\[
\frac{d}{dt} L(t) + \zeta L(t) \lesssim (L(t))^{1+\kappa}.
\]
By integrating (5.8) with respect to time, we obtain
\[
L(t) \leq c_1 e^{-\zeta t} L(0) + c_2 \int_0^t e^{-\zeta (t-s)} (L(s))^{1+\kappa} \, ds.
\]
Applying Lemma 2.2 then with
\[
-\zeta + (1 + 1/\kappa)c_2 2^\kappa c_1^\kappa L(0)^\kappa < 0
\]
gives
\[
L(t) \leq \left(1 + \frac{c_2 c_1^\kappa L(0)^\kappa}{-\zeta \kappa + (1 + \kappa)c_2 2^\kappa c_1^\kappa L(0)^\kappa}\right) c_1 e^{-\zeta t} L(0).
\]
Finally, employing the equivalence (5.3) yields the desired result. \qed

Remark 1. (On the Kuznetsov equation) Blackstock’s equation can be viewed as an alternative model to the Kuznetsov equation [22] given by
\[
(1 + 2k \psi_t) \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t + 2\sigma \nabla \psi \cdot \nabla \psi_t = 0.
\]
(5.9)
Although the developed theoretical framework can be transferred to (5.9) as well, we do not expect a gain in terms of the regularity assumptions compared to the available results in the literature in [20,32]. The reason is that the right-hand side nonlinearity $f$ in (5.1) would contain $\psi_t \psi_{tt}$. Then $\| f \|_{L^2(\Omega)}$ would involve $\| \psi_t \psi_{tt} \|_{L^2(\Omega)}$, which cannot be controlled by $E(t)D(t)$ in their present form. Therefore, having a higher-order energy functional and assuming $(\psi_0, \psi_1) \in H^3(\Omega) \times H^2(\Omega)$ in the global well-posedness analysis of (5.9) seems necessary within the present framework. We note, however, that (5.9) also appears in the pressure (or pressure–velocity) form in the literature, which allows for weaker regularity assumptions on the data; see [16,17,29].

Remark 2. (The use of other time weights) The use of the time weight in the present work has been crucial in gaining more regularity of the solution under minimal assumptions on the initial data; see (1.4). It might be possible to reduce the regularity of the initial data further to $\psi_1 \in H^\mu(\Omega)$, $\mu > 0$ by considering different time weights (i.e., $t^\beta$ for a suitably chosen $\beta > 0$). Within the energy method framework, this approach has been successfully employed for the Navier–Stokes equations in [36] to
improve the data regularity assumption in [11]. However, it does not seem straight-
forward to obtain such a result for the nonlocal equation for \( v \) in (3.2). The main
obstacle here is the estimate of the right-hand side term in (3.2), which involves \( \psi \);
see, for instance, the estimate (3.26), where the use of (3.16) is crucial to handle the
term \( \| \Delta \psi \|_{L^\infty(L^2)} \). One way of going around this would be to include a time weight
in (3.26). However, this poses significant challenges since there is a mismatch of the
time weights, and it would be difficult to “close” the nonlinear estimates.

Acknowledgements

We would like to thank the reviewer for the helpful comments and suggestions that
have led to an improved version of the manuscript.

Data availability Data sharing is not applicable to this article as no datasets were
generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License,
which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long
as you give appropriate credit to the original author(s) and the source, provide a link to the Creative
Commons licence, and indicate if changes were made. The images or other third party material in this
article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line
to the material. If material is not included in the article’s Creative Commons licence and your intended use
is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission
directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/
by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims
in published maps and institutional affiliations.

REFERENCES

[1] R. A. Adams and J. J. Fournier, Sobolev spaces, Elsevier, 2003.
[2] S. Agmon, Lectures on elliptic boundary value problems, vol. 369, American Mathematical Soc.,
2010.
[3] H. Amann et al., Linear and quasilinear parabolic problems, vol. 1, Springer, 1995.
[4] D. T. Blackstock, Approximate equations governing finite-amplitude sound in thermoviscous fluids,
tech. rep., General Dynamics/Electronics Rochester NY, 1963.
[5] M. Bongarti, S. Charoenphon, and I. Lasiecka, Vanishing relaxation time dynamics of the Jordan–
Moore–Gibson–Thompson equation arising in nonlinear acoustics, Journal of Evolution Equations,
21 (2021), pp. 3553–3584.
[6] I. Christov, C. Christov, and P. Jordan, Modeling weakly nonlinear acoustic wave propagation,
Quarterly Journal of Mechanics and Applied Mathematics, 60 (2007), pp. 473–495.
[7] I. C. Christov, P. Jordan, S. Chin-Bing, and A. Warn-Varnas, Acoustic traveling waves in thermo-
viscous perfect gases: Kinks, acceleration waves, and shocks under the Taylor–Lighthill balance,
Mathematics and Computers in Simulation, 127 (2016), pp. 2–18.
[8] P. Clément and G. Simonett, *Maximal regularity in continuous interpolation spaces and quasilinear parabolic equations*, Journal of Evolution Equations, 1 (2001), pp. 39–67.

[9] D. G. Crighton, *Model equations of nonlinear acoustics*, Annual Review of Fluid Mechanics, 11 (1979), pp. 11–33.

[10] R. Danchin and P. B. Mucha, *New maximal regularity results for the heat equation in exterior domains, and applications*, in Studies in Phase Space Analysis with Applications to PDEs, Springer, 2013, pp. 101–128.

[11] R. Danchin and P. B. Mucha, *The incompressible Navier-Stokes equations in vacuum*, Communications on Pure and Applied Mathematics, 72 (2019), pp. 1351–1385.

[12] L. C. Evans, *Partial Differential Equations*, vol. 2, Graduate Studies in Mathematics, AMS, 2010.

[13] M. Fritz, V. Nikolić, and B. Wohlmuth, *Well-posedness and numerical treatment of the Blackstock equation in nonlinear acoustics*, Mathematical Models and Methods in Applied Sciences, 28 (2018), pp. 2557–2597.

[14] B. Kaltenbacher, *Mathematics of nonlinear acoustics*, Evolution Equations & Control Theory, 4 (2015), p. 447.

[15] B. Kaltenbacher and I. Lasiecka, *Global existence and exponential decay rates for the Westervelt equation*, Discrete & Continuous Dynamical Systems-S, 2 (2009), p. 503.

[16] B. Kaltenbacher and I. Lasiecka, *An analysis of nonhomogeneous Kuznetsov’s equation: Local and global well-posedness; exponential decay*, Mathematische Nachrichten, 285 (2012), pp. 295–321.

[17] B. Kaltenbacher, I. Lasiecka, and S. Veljović, *Well-posedness and exponential decay for the Westervelt equation with inhomogeneous Dirichlet boundary data*, in Parabolic problems, Springer, 2011, pp. 357–387.

[18] B. Kaltenbacher, M. Meliani, and V. Nikolić, *Limiting behavior of quasilinear wave equations with fractional-type dissipation*, arXiv preprint arXiv:2206.15245, (2022).

[19] B. Kaltenbacher and V. Nikolić, *Parabolic approximation of quasilinear wave equations with applications in nonlinear acoustics*, SIAM Journal on Mathematical Analysis, 54 (2022), pp. 1593–1622.

[20] S. Kawashima and Y. Shibata, *Global existence and exponential stability of small solutions to nonlinear viscoelasticity*, Communications in mathematical physics, 148 (1992), pp. 189–208.

[21] M. Köhne, J. Prüss, and M. Wilke, *On quasilinear parabolic evolution equations in weighted $L^p$ spaces*, Journal of Evolution Equations, 10 (2010), pp. 443–463.

[22] V. P. Kuznetsov, *Equations of nonlinear acoustics*, Soviet Physics: Acoustics, 16 (1970), pp. 467–470.

[23] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Uralceva, *Linear and Quasi-Linear Equations of Parabolic Type*, vol. 23, American Mathematical Society, Providence, 1968.

[24] S. Larsson, V. Thomée, and L. B. Wahlbin, *Finite-element methods for a strongly damped wave equation*,IMA journal of numerical analysis, 11 (1991), pp. 115–142.

[25] M. B. Lesser and R. Seebass, *The structure of a weak shock wave undergoing reflexion from a wall*, Journal of Fluid Mechanics, 31 (1968), pp. 501–528.

[26] J. Li, *Local existence and uniqueness of strong solutions to the Navier–Stokes equations with nonnegative density*, Journal of Differential Equations, 263 (2017), pp. 6512–6536.

[27] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Springer Science & Business Media, 2012.

[28] S. Meyer and M. Wilke, *Optimal regularity and long-time behavior of solutions for the Westervelt equation*, Applied Mathematics & Optimization, 64 (2011), pp. 257–271.

[29] S. Meyer and M. Wilke, *Global well-posedness and exponential stability for Kuznetsov’s equation in $L^p$-spaces*, Evolution Equations and Control Theory, 2 (2013), pp. 365–378.

[30] M. Meyries and R. Schnaubelt, *Maximal regularity with temporal weights for parabolic problems with inhomogeneous boundary conditions*, Mathematische Nachrichten, 285 (2012), pp. 1032–1051.

[31] S. Mischler, *An introduction to evolution PDEs*, Lecture notes, Paris Dauphine University, 2020.

[32] K. Mizohata and S. Ukai, *The global existence of small amplitude solutions to the nonlinear acoustic wave equation*, Journal of Mathematics of Kyoto University, 33 (1993), pp. 505–522.

[33] M. Paicu, P. Zhang, and Z. Zhang, *Global unique solvability of inhomogeneous Navier-Stokes equations with bounded density*, Communications in Partial Differential Equations, 38 (2013), pp. 1208–1234.
[34] J. Prüss and G. Simonett, *Moving interfaces and quasilinear parabolic evolution equations*, vol. 105, Springer, 2016.

[35] A. Tani, *Mathematical analysis in nonlinear acoustics*, in AIP Conference Proceedings, vol. 1907, AIP Publishing LLC, 2017, p. 020003.

[36] J. Zhang, W. Shi, and H. Cao, *Global unique solvability of inhomogeneous incompressible Navier–Stokes equations with nonnegative density*, Nonlinearity, 35 (2022), p. 4795.

Vanja Nikolić
Department of Mathematics
Radboud University
Heyendaalseweg 135
6525 AJ Nijmegen
The Netherlands
E-mail: vanja.nikolic@ru.nl

Belkacem Said-Houari
Department of Mathematics, College of Sciences
University of Sharjah
P.O. Box: 27272 Sharjah
United Arab Emirates
E-mail: bhouari@sharjah.ac.ae

Accepted: 18 July 2023