Effective Potential and KK-Renormalization Scheme in a 5D Supersymmetric Theory

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Abstract

We calculate the 1-loop effective potential in a supersymmetric model in 5D with $S^1/(Z_2 \times Z_2)$ orbifold compactification. The procedure of calculation consists of evaluating first the integrals over four-momenta using the dimensional regularization and then the sum over Kaluza-Klein modes using the zeta-regularization. We show that both fermionic and bosonic contributions are separately finite and argue that, supersymmetry is not necessary for the finiteness of the theory at 1-loop. Also, some general arguments on the finiteness of the theory with arbitrary number of extra dimensions are presented.
1 Introduction

The Standard Model (SM) of electroweak and strong interactions, providing a description of the elementary particles and three of the fundamental interactions, has been rigorously tested at high energy colliders with an excellent agreement. However, there are reasons to believe that the SM is not a complete theory. One of the fundamental problems of the SM is to explain the origin of the electroweak symmetry breaking (EWSB) that leads to the known pattern of vector boson and fermion masses. The only known perturbative mechanism of the EWSB is the Higgs mechanism with the Higgs boson acquiring a vacuum expectation value. As it is well known, in the SM the radiative corrections to the Higgs boson mass are dominated by ultraviolet quadratic divergences. As a consequence, were the SM a fundamental theory up to the Planck scale, the tree level parameters of the theory would have to be fine tuned with a high precision (fine tuning problem). Contrary to this, if the SM is a part of a supersymmetric theory with a low energy (TeV scale) supersymmetry (SUSY) breaking mechanism the divergence is logarithmic and the fine tuning problem can be relaxed.

Much work has been done on five-dimensional (5D) generalizations of the SUSY SM with the extra dimension being compact and of large size \( R \sim 1/\text{TeV}^{-1} \) \cite{1} - \cite{7}. The main features of models of this kind are the following: (i) they provide a mechanism of SUSY breaking; (ii) the contribution of Kaluza-Klein (KK) modes to the mass term in the effective potential is negative, thus triggering the EWSB via radiative corrections; (iii) the 1-loop radiative correction to the effective potential is finite.

Such 5D theories may emerge as low energy limits of string theories \cite{8}, in which case they have a natural cutoff \( \Lambda \) which is of the order of the string scale. Because of their amazing softness the EWSB is not sensitive (at least at 1-loop order \cite{10}) to \( \Lambda \), i.e. the Higgs physics is independent of the physics at higher energies. In the present paper we will focus precisely on the finiteness of the 1-loop contribution to the effective potential.

Let us remind that the 1-loop Higgs effective potential in a 5D generalization of the SM is equal to the infinite sum of 1-loop contributions of KK modes, each of the contributions being given by an integral over four-momenta. The index \( k \), labelling the KK modes, is essentially the discrete momentum along the fifth direction (in units of \( R^{-1} \)). The contribution of an individual KK mode is ultraviolet divergent, therefore, in general, the integration and summation do not commute. In a number of articles the 1-loop effective potential was obtained by, firstly, calculating the sum over KK modes, and then integrating the result over the four-dimensional momentum \cite{3, 4}. Such procedure of calculation is often referred to as the KK-regularization. The validity of the calculation was questioned in some papers (see Ref. \cite{11, 12}), the main point of the discussion being the apparent non-commutativity of the integration and summation within this regularization. It was also argued there that from the physical point of view it did not make sense to take into account the whole infinite tower of KK modes and that rather only a finite number of modes with masses below some effective cutoff scale \( \Lambda \) should be included into the sum. Indeed, since a 5D theory is in general nonrenormalizable, it is natural to think that there exists a cutoff \( \Lambda \) such that for energies above this cutoff the physics is described by a more fundamental theory. For instance, in Ref. \cite{11} explicit cutoffs both in the KK sum

\footnote{Recently the 2-loop Higgs two-point function at zero external momenta was calculated and was shown to be finite \cite{9}; some general arguments on the absence of ultraviolet divergencies at all loops in such theories were presented in Ref. \cite{10}.}
and in the momentum integral were introduced, and it was shown that the finiteness of the result depends on whether the sum is taken over a finite or infinite number of KK modes. In fact, it is clear that cutting the sum explicitly breaks the 5D structure of the theory, so that it is not surprising to find different ultraviolet behaviors for different ways of dealing with the divergencies. Alternatively, in Ref. [13] both a momentum cutoff and an appropriate symmetry preserving KK cutoff are used. Within this framework contributions of high KK modes are exponentially suppressed and a consistent finite result is obtained.

For regularizations which do not violate essential symmetries of the theory, the result after renormalization should not depend on the order of integration and summation, as well as on the particular regularization used. Among such “good” regularizations some are more convenient from the practical point of view. In the present article we propose both a regularization of this kind and a renormalization scheme related to it. Our approach turns out to be quite effective, at least at 1-loop order, and provides a deeper understanding of the origin of the softness of the theory.

As a concrete example we study the 1-loop contribution to the effective potential in a 5D SUSY theory formulated in Ref. [4]. Our approach consists in calculating first the four-dimensional integrals using the dimensional regularization, and then evaluating the sum over the KK modes, using the ζ-function regularization [14]. The idea to calculate the sums over KK modes in multidimensional theories by means of the ζ-function regularization method is not new and was used in a number of papers (see for example [2, 15, 16, 17]). Divergencies associated with the momentum integration appear as simple poles in $2 - n/2 = \epsilon$. Here $n = 4 - 2\epsilon$ is the dimension of the momentum space in regularized integrals, and $n \to 4$ when the regularization is removed. The $1/\epsilon$-pole in the 1-loop contribution of a multidimensional field $X$ is multiplied by the sum of three terms: (1) the term $\propto m_{X,k=0}^4$, where $m_{X,k=0}$ is the mass of the zero mode of $X$; (2) a contribution of massive modes labelled by $k > 0$ (we denote this set of modes by $X^+$); (3) a contribution of massive modes with $k < 0$ (we denote this set of modes by $X^-$). We will show that the sum of these three terms is exactly zero provided that the contributions of the whole tower of KK modes are included.

Because of the finiteness of the 1-loop effective potential, in principle no further subtraction of ultraviolet divergences is needed. However, when using the result without subtraction one should bear in mind that a natural renormalization scheme, associated with the dimensional regularization of integrals and ζ-regularization of the sum, is implicitly assumed. We will refer to this renormalization scheme as the ”KK-renormalization”. The corresponding renormalization procedure consists basically in subtracting poles in $\epsilon$. In the model, considered here, no subtraction is actually needed because the coefficient of the pole $1/\epsilon$ turns out to be zero. The reason of this feature is the cancellation between the contributions of the three terms described above, namely the term $\propto m_{X,k=0}^4$, $X^+$ and $X^-$. 

Important properties of the KK-renormalization scheme are the following:

1. The bosonic and fermionic contributions to the 1-loop effective potential are finite on their own, and the cancellation between bosons and fermions due to SUSY in principle is not needed.

2. No special fine-tuning of parameters is needed; this is in contrast with the case of the KK-regularization (see [11]).
3. An important condition of the ultraviolet finiteness at 1-loop order is that the spectrum of the massive modes taken into account in the sum must be the complete KK tower, from $k = -\infty$ to $k = \infty$.

4. Finally, the KK-renormalization scheme respects all the symmetry of the theory; in the case under consideration it is the $\mathcal{N} = 2$ supersymmetry.

Our result confirms the one of Refs. [3, 4].

The structure of the paper is the following. In Sect. 2 we describe the model and give a brief review of previous calculations of the 1-loop effective potential. In Sect. 3 we calculate the potential using the dimensional and $\zeta$-regularizations. In Sect. 4 the limit of the zero size of the space of extra dimension is analyzed. Sect. 4 contains a discussion of the result and concluding remarks.

2 The model

We study a 5D SUSY extension of the SM formulated in Ref. [4] (see also [13]). The extra dimension is compactified on the orbifold $S^1/(Z_2 \times Z'_2)$ with the radius of the circle being equal to $R$. The orbifold is constructed by making two $Z_2$-identifications of points $y$ of the circle as follows: (1) $Z_2 : y \rightarrow -y$, and (2) $Z'_2 : y - \pi R/2 \rightarrow -y + \pi R/2$. All fields of the theory propagate in the bulk. Brane interactions are located at $y = 0, \pi R$ and $\pm \pi R/2$.

The SM matter and Higgs fields are described by a set of $\mathcal{N} = 2$ hypermultiplets $(\phi, \hat{\phi}, \Psi)_X$, where $\Psi_X$ is a Dirac spinor with components $(\psi, \psi^c)_X$; $\phi_X$ and $\hat{\phi}_X$ are two complex scalars and $X$ runs over three sets of multiplets $Q, U, D, L, E$, corresponding to the three generations of matter, and a single Higgs $H$. The gauge fields belong to the vector supermultiplet $V$ in the adjoint representation of the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$. The on-shell field content of $V$ is given by $V = (V_M, \lambda^i, \Sigma)$, where $M = 0, 1, 2, 3, 5, \Sigma$ is a real scalar field and $\lambda^i, i = \{1, 2\}$ are two gauginos. The latter reflects the presence of two supersymmetries in the theory obtained by the dimensional reduction to four dimensions. Namely, the hypermultiplets decompose into two four-dimensional $\mathcal{N} = 1$ chiral multiplets $X = (\phi_X, \psi_X^c)$ and $X^c = (\phi^c_X, \psi^c_X)$, where $\hat{\phi} = \phi^c$. The 5D vector supermultiplets can be decomposed into the four-dimensional vector supermultiplet $V = (V_\mu, \lambda^i)$ $(\mu = 0, 1, 2, 3)$ and the chiral multiplet $(\Sigma + iV_5, \lambda^2)$ in the adjoint representation of the gauge group. In accordance with the discrete $Z_2 \times Z'_2$-symmetry all fields in the theory have one of the following $(Z_2, Z'_2)$ quantum numbers (parities): $(+, +), (+, -), (-, +)$ or $(-, -)$.

Once both orbifoldings are imposed only the SM fields have the $(+, +)$ parity and, hence, the zero modes. The rest of the fields form infinite KK towers without zero modes. The Yukawa interactions are introduced only at the fixed points of the orbifold, i.e. at $y = 0, \pi R, \pi R/2$ and $-\pi R/2$. At low energies the effective theory is precisely the SM with the tree-level Higgs potential proportional to $|\phi_H|^4$ [4].

It is well known that the dominant radiative correction to the Higgs potential comes from the top and stop KK towers whereas the first and second generations and the gauge interactions give relatively negligible contributions. Since the initial theory is five-dimensional, all massive modes circulate inside 1-loop Feynman diagrams. The complete (all-orders) 1-loop effective potential
is equal to
\[ V_{\text{1-loop}} = \frac{1}{2} Tr \left[ \sum_{k=-\infty}^{\infty} \int \frac{d^4 p}{(2\pi)^4} \ln \left( \frac{p^2 + (m_k^B(H))^2}{p^2 + (m_k^F(H))^2} \right) \right], \]  
where \( H = <\phi_H> \) is the Higgs background field,
\[ m_k^B(H) = \left| \frac{\pm 2k + 1}{R} + m_t(H) \right| \]  
and
\[ m_k^F(H) = \left| \frac{\pm 2k}{R} + m_t(H) \right| \]  
\((k = 0, 1, 2, \ldots)\) are the mass eigenvalues of the bosonic (top squark) and fermionic (top quark) KK states respectively,
\[ m_t(H) = \frac{2}{\pi R} \arctan \frac{\pi y_t R H}{2} \]  
is the mass of the zero mode top quark, and \( y_t \) is the four-dimensional Yukawa coupling. The trace is taken over the degrees of freedom of the top hypermultiplet for a given \( k \), thus giving the factor \( N = 12 \).

Eqs (2) and (3) describe three classes of mass eigenvalues for both bosons and fermions, namely the zero mode with \( k = 0 \), \( k > 0 \) modes and \( k < 0 \) modes. Note that though the compactification is on the orbifold the sum over the KK modes goes from \( k = -\infty \) to \( k = \infty \). What happens is that after the orbifold compactification the net number of towers is divided by two but the positive non-zero modes of \( X \) replace the negative non-zero modes of \( X^c \), forming a single tower with \( k \) running from \( k = -\infty \) to \( k = \infty \). This turns out to be a rather generic property which takes place in a number of models (see Refs. [2], [19]).

As it was already mentioned in the Introduction, one of the procedures to obtain \( V_{\text{1-loop}}(H) \), Eq. (10), is to calculate first the sum and then the integral. Evaluating each term of the sum we get
\[ V_{\text{1-loop}} \propto \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\rho \rho \ln \left( \frac{1 + (m_k^B(H))^2/\rho}{1 + (m_k^F(H))^2/\rho} \right) \]
\[ = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\rho \left[ (m_k^B(H))^2 - (m_k^F(H))^2 + O \left( \frac{1}{\rho} \right) \right], \]  
where \( \rho = |p|^2 \). It is easy to see that the contribution of each mode is ultraviolet divergent. Because of this, in general, interchanging the summation and integration in Eq. (10) is not a well defined operation.

If one first sums over the modes and then integrate over the four-momentum the 1-loop effective potential becomes
\[ V_{\text{1-loop}} = N \int \frac{d^4 p}{(2\pi)^4} (W_B - W_F) \]  
where
\[ W_B(F) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \ln \left( R^2 \left[ p^2 + (m_k^B(F))^2 \right] \right) \]  
\[ W_B(F) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \ln \left( R^2 \left[ p^2 + (m_k^B(F))^2 \right] \right) \]
To perform summation one can calculate $dW_{B(F)}/d(R|p|)$ (the tadpole contribution) using the method of residues and then integrate with respect to $R|p|$. The result is

$$W_B = \frac{1}{2} \left[ \pi R|p| + \ln \left( 1 + re^{-\pi R|p|} \right) + \ln \left( 1 + \frac{1}{r} e^{-\pi R|p|} \right) \right]$$

$$W_F = \frac{1}{2} \left[ \pi R|p| + \ln \left( 1 - re^{-\pi R|p|} \right) + \ln \left( 1 - \frac{1}{r} e^{-\pi R|p|} \right) \right],$$

were $r = \exp(i\pi R\mu(H))$. The momentum integral of the first term in Eqs. (8), (9), the one $\propto \pi R|p|$, is divergent, but the divergencies cancel each other in the final expression, Eq. (6).

It is easy to see that the rest of the terms in Eqs. (8), (9) give finite contributions to the 1-loop potential. We would like to note that if the integral in Eq. (6) is calculated using the dimensional regularization, then integration of the term $\propto \pi R|p|$ gives zero, and $W_B$ and $W_F$ are separately finite.

Expanding (8) and (9) in powers of $r$ and integrating over the four-momenta one obtains the result of Ref. [4]:

$$V_{1\text{-loop}}(H) = \frac{3N}{2\pi^6 R^4} \sum_{k=0}^{\infty} \frac{\cos \left[(2k+1)\pi R\mu(H)\right]}{(2k+1)^5}.$$  

(10)

Note that in the above calculation the effective potential is finite (for any regularization which does not break SUSY) because the divergencies in the bosonic and fermionic contributions cancel each other exactly due to SUSY.

If the regularization is consistent with symmetries of the theory, then interchanging of summation and integration in Eq. (11) should not alter the result. In the next section we will use a regularization of this kind, namely a combination of the dimensional and $\zeta$-regularizations, to calculate $V_{1\text{-loop}}(H)$. We will first perform the integration in Eq. (11) and then the summation. It turns out that with such regularization the bosonic and fermionic contributions are separately finite, and the final result for the 1-loop effective potential is in agreement with (10).

### 3 K\:K\text{-Renormalization}

Let us write expression (11) for the 1-loop term of the bare effective potential as the difference of bosonic and fermionic contributions,

$$V_{1\text{-loop}}^{(bare)}(H) = V_B(H) - V_F(H),$$

and analyse first the bosonic contribution

$$V_B(H) = \frac{N}{2} \sum_{k=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \ln \left(R^2 \left[p^2 + (m_k^{(B)}(H))^2\right]\right).$$

(12)

Introducing the dimensional regularization one gets

$$V_B(H) = \frac{N}{2} \left(\mu^2\right)^{2-n/2} \sum_{k=-\infty}^{\infty} \int \frac{d^np}{(2\pi)^n} \ln \left(R^2 \left[p^2 + (m_k^{(B)}(H))^2\right]\right)$$

$$= \frac{N}{2} \left(\mu^2\right)^{2-n/2} \frac{\partial}{\partial \alpha} \sum_{k=-\infty}^{\infty} \int \frac{d^np}{(2\pi)^n} \left[1 \left.R^2 \left[p^2 + (m_k^{(B)}(H))^2\right]\right|_{\alpha=0}\right]$$

(13)
where \( n = 4 - 2\epsilon \) and \( \mu \) is a mass scale which appears due to the dimensional transmutation. Using standard formulas we integrate over momentum and obtain that

\[
V_B(H) = \frac{N (\mu^2)^{2-n/2}}{2} \frac{\partial}{\partial \alpha} \Gamma(\alpha - n/2) \frac{\Gamma(\alpha - n/2)}{4^\alpha R^{2\alpha} \Gamma(\alpha)} \sum_{k=-\infty}^{\infty} (m_k^{(B)}(H))^{n-2\alpha} \bigg|_{\alpha=0}
\]

\[
= \frac{N (\mu^2)^{2-n/2}}{2} \left(\frac{\Gamma(\alpha - 2 + \epsilon)}{4^\alpha R^{2\alpha \Gamma(\alpha)}} \sum_{k=-\infty}^{\infty} \frac{1}{\left(\frac{2k+1}{R} + m_k(H)\right)^{2\alpha-2+\epsilon}} \right)_{\alpha=0}.
\]

(14)

The sum in Eq. (14) is calculated with the help of the \( \zeta \)-regularization method [14]. The main idea is to treat the power \( 2\alpha - n \) in the r.h.s. of Eq. (14) as a complex parameter, represent the sum in terms of an appropriate \( \zeta \)-function assuming that its real part is large enough to make the sum finite, and then, at the end of calculation, continue it analytically to the value \( \alpha = 0, \epsilon = 0 \). For example, we can write \( V_B(H) \) in terms of the Epstein \( \zeta \)-function [20], which for \( \text{Re } \nu > 1 \) is defined by

\[
Z_1^{\nu^2}(\nu; w; a) = \sum_{k=-\infty}^{\infty} \left[w(k-a)^2 + v^2\right]^{-\nu}.
\]

(15)

The bosonic contribution to the effective potential is then equal to

\[
V_B(H) = \frac{N (\pi R^2 \mu^2)^{\epsilon}}{2} \frac{\partial}{\partial \alpha} \left[ \frac{\Gamma(\alpha - 2 + \epsilon)}{2^{2\epsilon} \Gamma(\alpha)} Z_1^0(\alpha + \epsilon - 2; 1; -u_B) \right]_{\alpha=0}
\]

\[
= \frac{N (\pi R^2 \mu^2)^{\epsilon}}{2} \Gamma(-2 + \epsilon) Z_1^0(-2 + \epsilon; 1; -u_B),
\]

(16)

where the notation

\[
u_B = \frac{1}{2} + \frac{1}{2} R m_t(H)
\]

was introduced. In the derivation of formula (16) we used the relation

\[
\frac{\partial}{\partial \alpha} F(\alpha) \bigg|_{\alpha=0} = F(0)
\]

valid for any function \( F(\alpha) \) analytic at \( \alpha = 0 \).

Expanding the r.h.s. of (16) in the Laurent series in \( \epsilon \) at \( \epsilon = 0 \) one gets

\[
V_B(H) = -\frac{N}{2 R^4 \pi^2} \left\{ \frac{1}{2\epsilon} Z_1^0(-2; 1; -u_B) + \frac{1}{2} \left( \ln(\pi \mu^2 R^2) + \frac{3}{2} - \gamma_E \right) Z_1^0(-2; 1; -u_B) \right. \\
+ \left. \frac{1}{2} Z_1^0(-2; 1; -u_B) \right\} + \mathcal{O}(\epsilon),
\]

(18)

where the prime denotes the derivative of the Epstein function with respect to the first argument and \( \gamma_E \) is the Euler constant.
For $s = 0$, the Hurwitz $\zeta$ function is defined by

$$
\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k + a)^s}, \quad \text{Re } s > 0, \quad 0 < a \leq 1.
$$

(c.f. (14), (16), (18)), where

$$
u > 1
$$

For further calculation of the effective potential we will need certain properties of the function $Z_1^0(\nu; 1; a)$. We are going to discuss them now. For $\text{Re } \nu > 1$

$$
Z_1^0(\nu; 1; -a) = \sum_{k=-\infty}^{\infty} \frac{1}{(k - a)^2} = -a^{-2\nu} + \sum_{k=0}^{\infty} \frac{1}{(k - a)^{2\nu}} + \sum_{k=0}^{\infty} \frac{1}{(k + a)^{2\nu}}
$$

(see [21], [24]). The function is analytic in $\nu$ in the whole complex plane except for the simple pole at $\nu = 1$. Two other functions which will be needed for our analysis are the Lerch function $\zeta(\nu, a)$ (also called the generalized Riemann $\zeta$-function) for $\text{Re } \nu > 1, 0 < a \leq 1$ is defined by

$$
\zeta(\nu, a) = \sum_{k=0}^{\infty} \frac{1}{(k + a)^\nu}
$$

(see [21], [24]).

The important functional relation for the Lerch function, which allows the analytical continuation of $\zeta(s, a)$ to negative values of $s$, is [21]

$$
\phi(x, a, 1-s) = (2\pi)^{-s} \Gamma(s) \left\{ e^{2\pi i(s/4-x)} \phi(-a, a, s) + e^{2\pi i(-s/4+a-x)} \phi(a, 1-x, s) \right\}. 
$$

(26)

It is valid for any $s, 0 < x < 1$ and $0 < a \leq 1$. Taking $a = 1$ in Eq. (26) we obtain the following formula:

$$
\psi(x, 1-s) = (2\pi)^{-s} \Gamma(s) \left\{ e^{i\pi s/2} \zeta(s, x) + e^{-i\pi s/2} \zeta(s, 1-x) \right\}, 
$$

(27)
where we have used the relations
\[
\phi(0, a, s) = \zeta(s, a), \quad \phi(x, 1, s) = e^{-2\pi i x} \psi(x, s)
\]
and the periodicity of the Lerch function in the first argument
\[
\phi(x + 1, a, s) = \phi(x, a, s).
\]
Calculating the real part of the both sides of Eq. (27) one gets
\[
\zeta(s, x) + \zeta(s, 1 - x) = 2\Gamma(1 - s) \pi^{-s} \sin(\pi s) \text{Re} \psi(x, 1 - s).
\]
Now we make the change of variable \(s \to 1 - s\), assume that \(\text{Re} s > 1\) and use representation (25) of the \(\psi\)-function in terms of the series. We obtain that
\[
\zeta(1 - s, x) + \zeta(1 - s, 1 - x) = 2\Gamma(s) \pi^{-s} \cos(\pi s) \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{k^s}.
\] (28)
The r.h.s. of this relation is valid for \(\text{Re} s > 0\). Taking into account the property
\[
\zeta(s, 1 + a) = \zeta(s, a) - a^{-s},
\] (29)
one can see from Eq. (22) that the l.h.s. of (28) is equal to \(Z_1^0(1 - s)/2; 1; x\). Evaluating it at \(s = 5 - 2\epsilon\) and \(x = -u\) one gets
\[
Z_1^0(-2 + \epsilon; 1; -u) = \zeta(-4 + 2\epsilon, u) + \zeta(-4 + 2\epsilon, 1 - u)
\]
\[
= \frac{2 \Gamma(5 - 2\epsilon)}{\pi (2\pi)^{4 - 2\epsilon}} \sin(\pi \epsilon) \sum_{k=1}^{\infty} \frac{\cos(2\pi k u)}{k^{5 - 2\epsilon}}.
\] (30)
From this it easy to derive the following formulas:
\[
Z_1^0(-2; 1; -u) = 0,
\]
\[
Z_1''(2; 1; -u) = \frac{3}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2\pi k u)}{k^5}.
\] (31, 32)
Substituting these relations into Eqs. (18) and (20) we obtain that the bosonic and fermionic contributions to the 1-loop effective potential in the limit \(\epsilon \to 0\) are finite and equal to
\[
V_B(H) = -\frac{3N}{4\pi^6 R^4} \sum_{k=1}^{\infty} (-1)^k \cos(\pi k R m_t(H)) \frac{1}{k^5},
\] (33)
\[
V_F(H) = -\frac{3N}{4\pi^6 R^4} \sum_{k=1}^{\infty} \cos(\pi k R m_t(H)) \frac{1}{k^5}.
\] (34)
The renormalization scale \(\mu\) does not enter into these expressions because its power is multiplied by the same factor as the pole \(1/\epsilon\) (see Eq. (16)). Substituting results (33), (34) into Eq. (11) we arrive at the final expression for the 1-loop effective potential
\[
V_{1\text{-loop}}(H) = V_{1\text{-loop}}^{(R)}(H) = \frac{3N}{2\pi^6 R^4} \sum_{k=0}^{\infty} \frac{\cos[(2k + 1)\pi R m_t(H)]}{(2k + 1)^5}.
\] (35)
The result in the KK-renormalization scheme is obtained by subtracting the \(1/\epsilon\) - pole terms. Since they vanish in the regularization used here no subtraction is actually needed, and the bare result is equal to the renormalized one, as indicated in Eq. (35). It coincides with expression (10).

We have shown that for the regularization used here the bosonic and fermionic contributions are separately finite. Therefore the supersymmetric cancellation is not necessary for getting the finite 1-loop effective potential. Clearly the main difference between our approach and the previous ones, namely the KK-regularization [2, 3, 4, 5] and the proper time regularization [13, 23], is in the way the ultraviolet divergencies reveal themselves in the regularized theory.

Let us analyze its ultraviolet properties in more detail. As it was already mentioned in the Introduction, the pole \(1/\epsilon\) is multiplied by the sum of three terms: the term with \(k = 0\) (zero mode contribution), the sum of contributions of modes with \(k > 0\) and the sum of contributions of modes with \(k < 0\) (see Eqs. (18), (20), (22)). For example, in the case of the fermionic contribution they correspond to top quark contribution, to the contribution of non-zero KK modes \(X^+ = \psi_1^+\) and to the contribution of non-zero KK modes \(X^- = \psi_i^-\) respectively. The divergent part \(V_F^{(\text{div})}(H)\) can be written as:

\[
V_F^{(\text{div})}(H) = -\frac{N}{64\pi^2\epsilon} \left[ m_t(H)^4 + \sum_{k=1}^{\infty} \left( \frac{2k}{R} + m_t(H) \right)^4 \right] + \sum_{k=1}^{\infty} \left( \frac{-2k}{R} + m_t(H) \right)^4 \]

(36)

where \(m_t(H)\) is the mass of the top quark (see Eq. (4)). Here the infinite sums are understood as being regularized by the \(\zeta\)-regularization, namely

\[
\sum_{k=1}^{\infty} \left( \frac{2k}{R} + m_t(H) \right)^4 = \frac{16}{R^4} \zeta(-4, u_F) - m_t^4(H),
\]

\[
\sum_{k=1}^{\infty} \left( \frac{-2k}{R} + m_t(H) \right)^4 = \frac{16}{R^4} \zeta(-4, -u_F) - m_t^4(H).
\]

We have shown that the three terms in Eq. (36) cancel each other and

\[
V_F^{(\text{div})}(H) = -\frac{N}{64\pi^2\epsilon} \left[ -m_t(H)^4 + \frac{16}{R^4} \zeta(-4, u_F) + \frac{16}{R^4} \zeta(-4, -u_F) \right] = 0.
\]

The cancellation occurs because the KK modes, contributing to the 1-loop effective potential, combine into a single tower whose spectrum is given by (3) with \(k\) running from \(k = -\infty\) to \(k = \infty\). In general such cancellation does not occur in models where higher KK modes are truncated.

A generalization of the Epstein \(\zeta\)-function to the case of \(d\) extra dimensions is given by

\[
Z_d^{\nu,2}(\nu, w_1, \ldots, w_d; u_1, \ldots, u_d) = \sum_{k_1, \ldots, k_d = -\infty}^{\infty} \left[ w_1(k_1 - u_1)^2 + \ldots + w_d(k_d - u_d)^2 + v^2 \right]^{-\nu}.
\]
The property of this function, which is important for us, is \[20\] (see also \[16\])

\[
Z_d^\nu (-2; w_1, \ldots, w_d; u_1, \ldots, u_d) = \begin{cases} 
0, & \text{for } d \text{ odd}, \\
2(-1)^{\frac{d}{2}} \pi^2 \nu \frac{d+4}{2} \nu \frac{d+4}{2}, & \text{for } d \text{ even}.
\end{cases}
\]  

(37)

One important observation is that, in fact, the 1-loop effective potential in a 5D generalization of the SM is finite for any particle independently of the concrete form of the spectrum. For example, KK modes in the gauge sector have the masses given by \(3\)

\[
m_{G,k}^2 = \frac{(k + q_G)^2}{R^2} + f_G(H)^2,
\]

(38)

where the lower index \(G\) stands for the gauge bosons and the gauginos, \(f_G(H)^2\) is some function of the Higgs field which is proportional to the gauge coupling, and \(q_G\) is a constant whose value is 0 (1/2) for the gauge bosons (gauginos) in the model of Ref. \[4\] and is arbitrary in models of Ref. \[3, 6\]. The contribution of the gauge sector in the Landau gauge to the divergent part of the effective potential is equal to

\[
V_G^{\text{div}}(H) = -\frac{3}{2R^4 \pi^2 \nu \frac{d}{2} + 2} Z_d^\nu (-2; 1; -q_G),
\]

(39)

where \(v_G^2 = R^2 f_G(H)^2\). From formula (37) we get \(Z_d^G(H)^2 (-2; 1; -q_G) = 0\) and, hence, the divergent part of the effective potential is zero both for the gauge boson and for the gaugino separately.

According to Eq. (37) the contribution to the 1-loop effective potential of an individual field is in general not finite if the number of extra dimensions is even. In this case the finiteness of the theory relies on the SUSY cancellation between bosons and fermions. It can be easily seen from Eq. (37) that, since \(Z_u^\nu (-2; w; u)\) is independent of the soft masses, entering into \(u_i\), the divergences cancel each other by the SUSY mechanism and the 1-loop effective potential is finite provided the bosons and fermions have the same Higgs field dependent part (analog of \(f_G(H)^2\) in Eq. (38)) in their spectrum. Also, using property (37) we conclude that as long as the spectra are of the form

\[
m_{k_1, \ldots, k_d}^2 \propto \frac{1}{R^2} \left[ w_1(k_1 - u_1)^2 + \ldots + w_d(k_d - u_d)^2 \right],
\]

(40)

the 1-loop effective potential is finite without supersymmetry in any number of extra dimensions. In particular, the linear spectrum of the model considered above, given by Eqs. (2), (3), is of this type. In Ref. \[19\], using the proper time regularization, the effective potential in a model with an arbitrary number of extra dimensions and with the spectrum of type (40) was calculated. It was shown there that without SUSY the effective potential had a divergent term associated with the cosmological constant. In our approach the field-independent divergence vanishes automatically.

4 \(R \to 0\) limit

A natural question to pose is whether the 1-loop effective potential of the four-dimensional SM is recovered in the limit \(R \to 0\). One can calculate the limit of the final expressions (30) - (35)
taking into account that
\[ Rm_t(H) \sim R(y_t H) \to 0, \quad u_B \sim \frac{1}{2} + \frac{1}{2} R(y_t H) \to \frac{1}{2}, \quad u_F \sim \frac{1}{2} R(y_t H) \to 0 \]
when \( R \to 0 \) and the properties
\[ \zeta \left(s; \frac{1}{2}\right) = (2^s - 1)\zeta(s), \quad \zeta(s, 1) = \zeta(s), \]
where \( \zeta(s) \) is the Riemann \( \zeta \)-function (see for example [21], [24]). Using Eqs. (29), (30) one gets
\[ Z^0_1(\nu; 1; -u_B) = \zeta \left(2\nu, \frac{1}{2} + \frac{1}{2} Rm_t\right) + \zeta \left(2\nu, \frac{1}{2} - \frac{1}{2} Rm_t\right) \to 2(2^{2\nu} - 1)\zeta(2\nu), \quad (41) \]
\[ Z^0_1(\nu; 1; -u_F) = \zeta \left(2\nu, 1 + \frac{1}{2} Rm_t\right) + \zeta \left(2\nu, 1 - \frac{1}{2} Rm_t\right) + \left(\frac{1}{2} Rm_t\right)^{-2\nu} \to 2\zeta(2\nu) + \left(\frac{1}{2} Rm_t\right)^{-2\nu}. \quad (42) \]

Substituting \( \nu = -2 - \epsilon \) into (41) one obtains the following formulas necessary for the calculation of the bosonic contribution in the limit \( R \to 0 \):
\[ Z^0_1(-2; 1; -\frac{1}{2}) = -\frac{15}{8}\zeta(-4) = 0, \]
\[ Z^0_1'(-2; 1; -\frac{1}{2}) = \frac{1}{4} \ln 2\zeta(-4) - \frac{15}{4}\zeta'(-4) = -\frac{3}{\pi^4} \frac{15}{16}\zeta(5). \quad (43) \]

Here we used the relations \( \zeta(-4) = 0 \) and \( \zeta'(-4) = 3\zeta(5)/(4\pi^4) \) [21], [24]. They follow from a functional equation for the Riemann \( \zeta \)-function which relates \( \zeta(s) \) to \( \zeta(1 - s) \).

The fermionic contribution in the limit \( R \to 0 \) is given by the Eq. (42) at \( \nu = -2 + \epsilon \)
\[ Z^0_1(-2; 1; 0) = 2\zeta(-4) = 0, \]
\[ Z^0_1'(-2; 1; 0) \sim 4\zeta'(-4) - \left(\frac{1}{2} Rm_t\right)^4 \ln \left. \frac{R^2 m_t^2}{4} \right|_{R \to 0} = \frac{3}{\pi^4}\zeta(5). \quad (44) \]

Substituting (13) and (14) into Eqs. (18) and (20) we obtain that in the limit \( R \to 0 \) the potential is divergent in \( R \) with the leading term being
\[ V^{(\text{bare})}_1(H) = V_B(H) - V_F(H) = V^{(R)}_{1\text{-loop}}(H) \sim \frac{3N}{4R^4\pi^6}\zeta(5). \quad (45) \]

As one could expect, the 1-loop contribution does not contain \( 1/\epsilon \)-pole. Note also that in the calculation of \( V^{(R)}_{1\text{-loop}}(H) \) it was basically assumed that \( RH \ll 1 \), i.e. Eq. (13) can be considered as the limit of weak field, \( H \ll 1/R \).

The limit \( R \to 0 \) can also be studied using Eqs. (14), (13) as the starting point. If we take the limit formally before summing over \( k \), then the only non-vanishing contribution is the one

\[ V^{(\text{bare})}_1(H) = V_B(H) - V_F(H) = V^{(R)}_{1\text{-loop}}(H) \sim \frac{3N}{4R^4\pi^6}\zeta(5). \quad (45) \]

This equation is actually a particular case of Eq. (26) for \( a = 1, \ x = 1/2 \).
of the fermionic mode with \( k = 0 \) in Eq. (15). By a straightforward calculation we arrive at the result

\[
V_{1\text{-loop}}^{(\text{bare})}(H) = -V_F(H) = -\frac{N}{2} \frac{\mu^{2\epsilon}(4\pi)^\epsilon}{16\pi^2} \Gamma(-2 + \epsilon)(y_t H)^{4-2\epsilon} + \mathcal{O}(R)
\]

\[
= \frac{N}{64\pi^2}(y_t H)^4 \left[ \frac{1}{\epsilon} + \left( \frac{3}{2} - \gamma_E - \ln \frac{y_t^2 H^2}{4\pi\mu^2} \right) + \mathcal{O}(\epsilon, R) \right].
\] (46)

This is the known expression for the 1-loop contribution to the effective potential in the SM.

In four dimensions the KK-renormalization scheme reduces to the \( \overline{\text{MS}} \) scheme. Subtracting \((1/\epsilon - \gamma_E + \log 4\pi)\) in Eq. (46) we get

\[
V_{1\text{-loop}}^{(R)}(H) = -\frac{N}{64\pi^2}(y_t H)^4 \left( \ln \frac{y_t^2 H^2}{\mu^2} - \frac{3}{2} \right) + \mathcal{O}(R).
\] (47)

Results (45) and (47) are obviously different. The reason is that Eq. (45) was calculated by first summing over the KK modes by means of the \( \zeta \)-regularization and then taking the limit \( RH \to 0 \). The summation was performed over the complete infinite KK tower of modes, including heavy modes with the masses \( m_k^{(B)}, m_k^{(F)} \sim k/R \gg 1 \).

Contrary to this, Eq. (47) was derived by first taking the limit \( R \to 0 \) in Eqs. (14), (19). After this the sums over KK modes reduce just to the term with \( k = 0 \) in \( V_F(H) \). The non-zero KK modes, i.e. all modes in (2) and all modes with \( k \neq 0 \) in (3), decouple and the four-dimensional results (46), (47) emerge.

The conclusion is that the summation over the KK modes by means of the \( \zeta \)-regularization and the limit \( R \to 0 \) do not commute within the KK-renormalization scheme. Analyzing the limit of the 5D expression for the renormalized 1-loop effective potential, Eq. (45), we see that the contribution of KK modes does not decouple. Therefore, the KK-renormalization scheme is not a scheme with explicit decoupling of heavy modes [25]. We would like to note that a renormalization scheme with explicit decoupling of KK modes, based on a certain momentum subtraction procedure, was developed and used for calculations in multidimensional theories [16], [26].

5 Conclusions

We have discussed a technique for calculation of the 1-loop effective potential and considered the problem of its finiteness in 5D SUSY theories. As a simple example the model proposed in Ref. [4] was used.

As we mentioned in the Introduction, recently there have been some discussion in the literature regarded a controversy in the calculation the effective potential [11, 12]. In particular it was pointed out that for each KK mode the 1-loop integral, contributing to the effective potential, is ultraviolet divergent and does not commute with the summation, hence it is unclear whether the KK-regularization is an appropriate procedure. It was also argued that KK modes above some “natural” scale of the theory can not contribute to the effective potential and, therefore, a cutoff of the sum over KK modes should be introduced.

The KK-renormalization proposed in the present paper overcomes successfully the first problem by combining the dimensional regularization to evaluate the integrals and the zeta-regularization to calculate the infinite sums. Since both regularizations are based on analytic
continuation of the integrand and the summand, no difficulty arises in interchanging the summation and integration. In general, with such regularization the result of calculation may contain a pole $1/e$ coming from the pole of the generalized Riemann $\zeta$-function (see Eq. (22) and the remark below Eq. (23)). The KK-renormalization procedure consists in subtraction of the pole terms, and is similar to the (MS) scheme. When the final result is finite, as in the model considered in the article, obviously no subtraction is needed. Our technique for calculation of the 1-loop effective potential reproduces the finite result obtained by using regularizations respecting the analytical symmetry of the contributions [4].

The KK-renormalization introduces no cutoff, neither in the integrals over four dimensional momentum, nor in the sums over KK modes. In this sense the sums over $k$ are treated in the same fashion as the integrals, thus making the whole procedure consistent. Recall that $k$, the number of a KK mode, plays the role of the (discrete) momentum in the fifth direction (in terms of $R^{-1}$).

In the model considered here both the KK-regularization and our approach give the finite 1-loop effective potential. However, in the KK-regularization the bosonic and fermionic contributions are separately divergent and cancel out due to the SUSY mechanism. Instead, in our approach the bosonic and fermionic contributions to the effective potential are separately finite, and the supersymmetry is not necessary for the finiteness of the theory at 1-loop. In Sect. 3 we have also shown that for the torus compactification the finiteness at 1-loop order is rather generic and independent of concrete details of the spectrum in the theory.

As we have seen, what is really important for the finiteness of the effective potential at 1-loop order is that the spectrum of KK modes effectively corresponds to compactification to the circle $S^1$, i.e. it is given by formulas of type (2), (3) with $k$ running from $k = -\infty$ to $k = \infty$. This assures that the contributions to the divergent part from the zero mode, positive modes and negative modes cancel each other exactly.

We also studied the 1-loop effective potential in the limit $RH \to 0$, i.e. when either the Higgs background field $H \ll 1/R$, or the space of extra dimension collapses to a point. We showed that within the KK-renormalization scheme the summation over the KK modes and the limit $R \to 0$ do not commute, i.e. the KK-renormalization scheme is not a scheme with explicit decoupling of heavy KK modes.

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