The optimal fidelity estimation operation for entanglement distribution networks with arbitrary noise

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Fidelity estimation is essential for the quality control of entanglement distribution networks. In this paper, we consider a setup in which nodes randomly sample a subset of the entangled qubit pairs for measurement and then estimate the fidelity of the unsampled pairs conditioned on the measurement outcome. The proposed estimation protocol, which performs local Pauli operators according to a predefined sequence, is implementation friendly. Despite its simplicity, this protocol achieves the lowest estimation error in the difficult scenario with arbitrary noise and no prior information. The analysis reveals the issue of excessive measurements, a counterintuitive phenomenon in which more measurements lead to less accurate estimation, when the sampling ratio exceeds 0.5 in scenarios with arbitrary noise.

Introduction — Entanglement distribution networks \([1]\) are a major stage of development towards a full-blown quantum Internet \([2, 4]\). Such networks enable device-independent protocols \([3, 6]\), thereby achieving the highest level of quantum security \([10]\). Entanglement distribution networks do not require nodes to have local quantum memory. Since efficient communication-compatible quantum memories are still under development \([11]\), entanglement distribution networks serve as a cornerstone for realizing trustworthy quantum applications with state-of-the-art quantum technologies \([12, 15]\).

Entanglement quality assessment is a key building block for entanglement distribution networks. To this end, fidelity estimation for entangled states is a promising candidate. Fidelity is a metric that indicates the quality of quantum states \([16, 19]\) and can be estimated with separable quantum measurements and classical post-processing. Several fidelity estimation protocols have been proposed \([20, 23]\), and fidelity estimation protocols of entangled states have been implemented in several recent experiments \([12, 13, 24]\). However, there are still two challenges that need to be addressed.

- **Excessive estimation error due to arbitrary noise:** Quantum networks often face heterogeneous and correlated noise. When distributing quantum keys and estimating channel capacities, this noise leads to excessive estimation errors \([23, 26]\). Therefore, designing low-error fidelity estimation protocols in the presence of arbitrary noise is an interesting area of research.

- **Efficiency loss due to separable operations:** Without quantum memory, nodes in entanglement distribution networks perform operations that are separable between all qubits. Such operations lead to significantly lower estimation efficiency \([27]\) compared to joint operations. Minimizing the efficiency loss due to separable operations is a major challenge for fidelity estimation.

This work focuses on fidelity estimation for entanglement distribution networks. Since measurements collapse quantum states, we consider a setup in which nodes randomly select a subset of qubit pairs for measurement and estimate the average fidelity of unsampled pairs conditioned on the measurement outcome.

The proposed estimation protocol performs only local Pauli measurements, standard operations that are implementable on a variety of quantum platforms \([12, 24, 28]\). Moreover, the protocol determines the basis of each Pauli measurement according to a predefined sequence, so no adaptive measurements are required. Therefore, this protocol is very implementation friendly.

Despite its simplicity, the proposed protocol achieves the lowest estimation error in the difficult scenario with arbitrary noise and no prior information, overcoming the aforementioned challenges. We characterize the accuracy of the estimated fidelity and uncover the issue of excessive measurements, a counterintuitive phenomenon in which more measurements lead to less accurate estimation, when the sampling ratio exceeds 0.5 in scenarios with arbitrary noise.

Protocol Description — Consider a scenario in which two nodes share \(N\) noisy qubit pairs and have no prior information of the noise. The nodes want to estimate the fidelity of the qubit pairs with respect to the target maximally entangled state. Since maximally entangled states are mutually convertible via local operations, we set the target state to \(|\Psi^\pm\rangle\) without loss of generality, where \(|\Psi^\pm\rangle = \frac{\left|01\right\rangle \pm \left|10\right\rangle}{\sqrt{2}}, |\Phi^\pm\rangle = \frac{\left|00\right\rangle \pm \left|11\right\rangle}{\sqrt{2}}\) are the four Bell states. Denote the set of all qubit pairs by \(\mathcal{N}\). The nodes randomly sample \(M(< N)\) number of qubit pairs for measurement. The set of sampled pairs, \(\mathcal{M}\), is drawn from all \(M\)-subsets of \(\mathcal{N}\) with equal probability.

Because measurements collapse quantum states, the nodes estimate the average fidelity of the unsampled pairs.
**Protocol 1 Measurement & Point estimation**

1: Preset measurement parameters. The nodes select the sample set $M$ completely at random and generate $M$ number of independent and identically distributed (i.i.d.) random variables $A_n \in \{x, y, z\}$, $n \in M$, with distribution $Pr[A_n = u] = \frac{1}{3}$, $u \in \{x, y, z\}$.

2: Perform measurements. For the qubit pair $n \in M$, both nodes measure the qubit in the $A_n$-basis. If the measurement results of the two nodes match, record measurement outcome $r_n = 1$, otherwise record $r_n = 0$.

3: Estimate fidelity. The number of errors and the quantum bit error rate (QBER) are expressed as $ε_M = \sum_{n \in M} r_n$, and $ε_M = \frac{ε_M}{M}$ respectively. The estimated fidelity is

$$\bar{f} = 1 - \frac{3}{2} ε_M,$$

(1)

conditioned on the measurement outcome $r$, i.e.,

$$\bar{f} = \frac{1}{N - M} \sum_{n \in N \setminus M} (\Psi^- | ρ_n^{(r)} | \Psi^-),$$

(2)

where $ρ_n^{(r)}$ is the state of the $n$-th qubit pair conditioned on $r$. The imperfection of the measurements is considered as part of the noise. The point and interval estimators are denoted by $\bar{f}$ and $\bar{C}$, respectively.

The proposed protocol consists of two parts, i.e., Protocol 1, which involves measurement and point estimation, and Protocol 2, which performs interval estimation that is reliable in scenarios with arbitrary noise.

To illustrate the performance of the proposed protocol in scenarios with non-i.i.d. noise, we consider an illustrative example with two types of noise, i.e., i.i.d. noise, with joint state of all qubit pairs

$$ρ_{all} = \bigotimes N (f_{all} | \Psi^- \rangle \langle \Psi^- | + (1 - f_{all}) | 00 \rangle \langle 00 |),$$

(3)

and heterogeneous noise, with joint state

$$ρ_{all} = \bigotimes N f_{all} | \Psi^- \rangle \langle \Psi^- | \otimes \bigotimes N (1 - f_{all}) | 00 \rangle \langle 00 |),$$

(4)

where the average fidelity of all qubit pairs $f_{all}$ is selected such that $N f_{all}$ is an integer.

As Fig. 1 shows, the actual fidelity lies within the standard error interval in cases with i.i.d. noise and heterogeneous noise with probability 0.68 and 0.29, respectively. This contrast shows that the standard error interval is no longer a reliable interval estimator in cases with non-i.i.d. noise. The credible interval obtained by Protocol 2 contains the $α$ probability intervals of the actual fidelity for both types of noise, demonstrating its reliability.

**Estimation Error Minimization** — The nodes target at minimizing the mean squared error of the estimated fidelity, under the condition that the estimation is unbiased. To ensure the robustness of the fidelity estimation protocol, we consider the scenario with arbitrary noise and no prior information. In such a scenario, the state of all qubits $ρ_{all} \in S_{arb}$, where $S_{arb}$ denotes the set of arbitrary $N$ qubit-pair states. To address the issue of having no prior information, the state $ρ_{all}$ that leads to the largest error is considered. This optimization problem is formulated as follows.

**$\mathcal{O}$-1:**

$$\text{minimize } _{\bar{C}, \mathcal{D}} \max _{α} \sum _{M} \mathbb{E}[R \left[ (\bar{F} - \bar{C})^2 \right]]$$

subject to $\sum _{M} \mathbb{E}[\bar{F} - \bar{C}] = 0$, $\forall ρ_{all} \in S_{arb}$

where $\bar{F}$, $\bar{C}$, and $R$ are the random variable form of the estimated fidelity $\bar{f}$, the average fidelity $\bar{f}$, and the measurement outcome $r$, respectively, the measurement operation $\mathcal{O}$ is a set of separable operators, i.e.,

$$\mathcal{O} = \left\{ \sum _{k} \bigotimes n \in M (M_{r,n,k}^{(A)} \otimes M_{r,n,k}^{(B)}) | r \in \mathcal{R} \right\},$$

(5)

in which the node index $X \in \{A, B\}$, qubit pair index $n \in M$, and operator index $r \in \mathcal{R}$, with $\mathcal{R}$ denoting the set of all possible measurement outcomes, and $\mathcal{D}$ is the estimator that maps the measurement outcome $r$ to the estimated fidelity $\bar{f}$, i.e.,

$$\bar{f} = \mathcal{D}(r).$$

(6)
For Bell states and then along the y qubits first along the qubit pairs. (13b) shows that $T$ transforms states in $S$ without changing the fidelity. This operation transforms eliminates the entanglement among different qubit pairs. Specifically, the probabilistic rotation $T$ simplifies one can simplify it to an equivalent problem with independent noise. This step consists of three substeps.

**A1.** This substep simplifies $P$ to an equivalent one with classical correlated noise, i.e., $P_1$ one can simplify it to an equivalent problem with independent noise. This step consists of three substeps.

**P-1** involves two major steps, namely problem simplification and operation construction. The formal theorems can be found in [30, Sec. II].

The problem simplification shows that to solve $P_1$ one can simplify it to an equivalent problem with independent noise. This step consists of three substeps.

**A2.** This substep shows that to solve $P_2$ it is sufficient to consider the special case with independent noise, i.e., $P_3$: Special case of $P_2$ with $S_{sp}$ replaced by $S_{id}$, i.e., the set of $\rho_{all}$ that are product states.

Denote $\bar{f}(r)$ as the estimated fidelity given the measurement outcome $r$, and denote $\bar{f}(\rho_{all}, \mathcal{M}, r)$ as the average fidelity given the state $\rho_{all}$, the sample set $\mathcal{M}$ and the measurement outcome $r$. Separable states $\rho_{all} \in S_{sp}$ are ensembles of product states, i.e., $\rho_{all} = \sum_k p_k \rho_{all}^{(k)}$, where $\rho_{all}^{(k)} \in S_{id}$, and ensemble probabilities $p_k$ satisfy $p_k \geq 0$, $\sum_k p_k = 1$. Consequently, for each set $\mathcal{M}$, the estimation error with a separable state $\rho_{all}$ and that with product states $\{\rho_{all}^{(k)}\}$ are connected by the following inequality.

$$E_{\mathcal{R}}[(\bar{f} - \bar{f})^2 | \rho_{all}]$$

$$= \sum_r \Pr(r|\mathcal{M}) (\bar{f}(r) - \bar{f}(\rho_{all}, \mathcal{M}, r))^2 \quad (14a)$$

$$= \sum_r \Pr(r|\mathcal{M}) (\bar{f}(r) - \sum_k \Pr(\rho_{all}^{(k)}|\mathcal{M}, r) \bar{f}(\rho_{all}^{(k)}, \mathcal{M}, r))^2 \quad (14b)$$

$$\leq \sum_{r,k} \Pr(r|\mathcal{M}) \Pr(\rho_{all}^{(k)}|\mathcal{M}, r) (\bar{f}(r) - \bar{f}(\rho_{all}^{(k)}, \mathcal{M}, r))^2 \quad (14c)$$

$$= \sum_k p_k \sum_r \Pr(r|\rho_{all}^{(k)}, \mathcal{M}) (\bar{f}(r) - \bar{f}(\rho_{all}^{(k)}, \mathcal{M}, r))^2 \quad (14d)$$

$$= \sum_k p_k E_{\mathcal{R}}[(\bar{f} - \bar{f})^2 | \rho_{all}^{(k)}]. \quad (14e)$$

where $\Pr(\cdot)$ denotes the probability, (14e) is true because $(\sum_k w_k x_k)^2 \leq \sum_k w_k x_k^2$, $\forall x_k \in \mathbb{R}, w_k \geq 0$, $\sum_k w_k = 1$, and (14d) holds because according to Bayes’ theorem

$$\Pr(r|\mathcal{M}) \Pr(\rho_{all}^{(k)}|\mathcal{M}, r) = \Pr(\rho_{all}^{(k)}|\mathcal{M}) \Pr(r|\rho_{all}^{(k)}, \mathcal{M})$$

$$= \Pr(\rho_{all}^{(k)}) \Pr(r|\rho_{all}^{(k)}, \mathcal{M})$$

$$= p_k \Pr(r|\rho_{all}^{(k)}, \mathcal{M}), \quad (15)$$

where the second equation holds because the sampling set $\mathcal{M}$ and the state $\rho_{all}^{(k)}$ are independent. Because $\rho_{all} \in S_{sp}$ and $\rho_{all}^{(k)} \in S_{id}, \forall k, (14)$ shows that the minimum estimation error of $P_2$ is upper bounded by that of $P_3$.

**Remark 1** (Neutralize the effect of correlation): The result of step A2 is unexpected because correlated noise usually leads to a higher estimation error compared to independent noise. This result occurs because the estimation target is evaluated conditioned on the measurement outcome. In scenarios with correlated noise, such conditional evaluation post-selects the qubit states according to the measurement outcome, thereby enhancing
the effectiveness of the measurements and neutralizing the adverse effect of correlation.

Specifically, the conditional distribution $\Pr(\rho^{(k)} | M, r)$ in (14c) represents the post-selection effect mentioned above. This conditional probability enables the application of Bayes’ theorem in (15), thereby bounding the estimation error of the states in $S_{\text{sp}}$ by that of the states in $S_{\text{id}}$.

- **A3.** This substep shows that to solve $\mathcal{P}_3$ it is sufficient to minimize the estimation error of the sampled pairs, i.e., $\mathcal{P}_4$:

$$\begin{align*}
\text{minimize} & \quad \max_{\mathcal{O}, \mathcal{D}} \max_{\rho_{\text{all}} \in S_{\text{id}}(\mathcal{S}_{\text{all}})} \left( \frac{N}{M} \right)^{-1} \sum_{\mathcal{M}} \mathbb{E}_R[(\bar{F} - \bar{f}_M)^2] \\
\text{subject to} & \quad \mathbb{E}_R[\bar{F} - \bar{f}_M] = 0, \forall \rho_{\text{all}} \in S_{\text{id}}(\mathcal{S}_{\text{all}}), \mathcal{M} \subset \mathcal{N}
\end{align*}$$

where $S_{\text{id}}(\mathcal{S}_{\text{all}}) = \{ \otimes_{n \in \mathcal{N}} \rho_n \}$ is the set of product states with fidelity composition $\mathcal{S}_{\text{all}} = \{ f_n, n \in \mathcal{N} \}$, in which $f_n$ is the fidelity of $\rho_n$, and $f_M$ is the average fidelity of the sampled qubit pairs.

In the case of independent noise, the measurement does not affect the fidelity of the unsampled qubit pairs. Consequently, the estimation error of $\mathcal{P}_3$ can be decomposed into two parts, namely the estimation error of the sampled qubit pairs,

$$\left( \frac{N}{M} \right)^{-1} \sum_{\mathcal{M}} \mathbb{E}_R[(\bar{F} - \bar{f}_M)^2],$$

and the sampling error, i.e., the deviation between the average fidelity of the sampled and the unsampled qubit pairs,

$$\left( \frac{N}{M} \right)^{-1} \sum_{\mathcal{M}} (\bar{f}_M - \bar{f})^2.$$  (17)

The sampling error (17) is not affected by the estimation protocol. Therefore, $\mathcal{P}_3$ can be simplified to $\mathcal{P}_4$ which minimizes (16).

The next step is to construct the optimal solution to $\mathcal{P}_4$ which involves the following two substeps.

- **B1.** This substep characterizes the minimum estimation error of $\mathcal{P}_4$.

The Cramér-Rao bound [31, 32] is the fundamental lower bound of the mean squared estimation error given the distribution of a measurement outcome. In $\mathcal{P}_4$ this distribution is determined by the state of the qubit pairs $\rho_{\text{all}}$ and the measurement operation $\mathcal{O}$. However, even with independent noise, the state of each qubit pair $\rho_n \in H^{1 \times 4}$ has several parameters other than fidelity. This fact complicates the expression of the measurement outcome and makes the analysis of the Cramér-Rao bound infeasible.

To overcome this challenge, we show that the minimum estimation error of general independent states is bounded below by that of independent Werner states, i.e., $\mathcal{P}_5$: Special case of $\mathcal{P}_4$ with $S_{\text{id}}(\mathcal{S}_{\text{all}})$ replaced by $S_{\text{id}}(\mathcal{S}_{\text{all}})$ via the separable operation $\mathcal{B}$, the minimum estimation error of $\mathcal{P}_4$ is no higher than that of $\mathcal{P}_5$. The states in $S_{\text{id}}(\mathcal{S}_{\text{all}})$ are fully parameterized by the fidelity composition $f_{\text{all}}$, which allows the Cramér-Rao bound analysis of $\mathcal{P}_5$.

The other factor affecting the distribution of measurement outcome is the separable measurement operation $\mathcal{O}$. Since the fidelity of any separable state w.r.t. a maximally entangled state lies in the interval $[0, \frac{1}{2}]$ [34], we have that for any separable operator $M$,

$$0 \leq \frac{\text{Tr}(\mathcal{P}_4)}{\text{Tr}(M)} \leq \frac{1}{2}.$$  (19)

For all separable operators, (19) limits the sensitivity of the measurement outcome to the changes in fidelity. Plugging this result into the Cramér-Rao bound, we lower bound the estimation error of $\mathcal{P}_5$ by

$$\sum_{n \in \mathcal{N}} \frac{(2f_n + 1)(1 - f_n)}{2MN}.$$  (20)

- **B2.** The second substep shows that Protocol 1 is the optimal solution to $\mathcal{P}_4$.

The analysis in step B1 indicates that to achieve the Cramér-Rao bound, each measurement operator must balance either of the two inequalities in (19). The measurement operation $\mathcal{O}^*$ of Protocol 1 is built according to this principle. For example, when nodes measure in the $z$-basis, the operators corresponding to $r = 0$ and $r = 1$ are respectively

$$M_{z,0} = |01\rangle \langle 01| + |10\rangle \langle 10| = |\Psi^-\rangle \langle \Psi^-| + |\Psi^+\rangle \langle \Psi^+|,$$

$$M_{z,1} = |00\rangle \langle 00| + |11\rangle \langle 11| = |\Phi^-\rangle \langle \Phi^-| + |\Phi^+\rangle \langle \Phi^+|.$$  (21)

According to (21), $M_{z,0}$ and $M_{z,1}$ balance the second and first inequalities of (19), respectively. Based on this property, we prove that operation $\mathcal{O}^*$ achieves the minimum estimation error given by (20) for states in $S_{\text{id}}$, and is thus optimal for $\mathcal{P}_4$. In this case, according to the problem simplification step, the composite operation $\mathcal{O}^* = \mathcal{O}^* \circ T$ is optimal for $\mathcal{P}_4$.
According to (13), the operation $\mathcal{T}$ on each qubit pair can be expressed by the following four Kraus operators
\[
T_\phi = |\phi\rangle\langle\phi|, \quad \text{where} \quad \phi \in \{\Phi^\pm, \Psi^\pm\}. \quad (22)
\]
Substituting (22) into (21) shows that the operation $\mathcal{T}$ does not change the operators of $\mathcal{O}^*$, e.g.,
\[
M_{z,r} = \sum_{\phi \in \{\Phi^\pm, \Psi^\pm\}} T_\phi^i M_{z,r} T_\phi, \quad r \in \{0,1\}, \quad (23)
\]
(23) shows that $\hat{O}^* = \mathcal{O}^* \circ \mathcal{T} = \mathcal{O}^*$, i.e., $\hat{O}^*$ and $\mathcal{O}^*$ are equivalent. Therefore, $\mathcal{O}^*$ is optimal for $\mathcal{P}_1$. 

**Interval Estimation** — In addition to the point estimate of fidelity given in Protocol 1, many applications also require an interval estimate of fidelity to indicate the precision of the estimation. The mean squared error analyzed above can, in principle, achieve this goal. However, the estimation error is unlikely to be Gaussian in scenarios with arbitrary noise. In this case, methods that exploit only the second-order moment lead to inaccurate interval estimates. Therefore, we design a Gedankenexperiment to enable Bayesian inference and use multiple moments of the posterior distribution to obtain an accurate interval estimate.

Consider a Gedankenexperiment in which all unsampled qubit pairs $n \in U = \mathcal{N} \backslash \mathcal{M}$ are also measured. Define the QBER of the unsampled pairs as
\[
\varepsilon_U = \frac{\sum_{n \in U} \varepsilon_n}{N - M}. \quad (24)
\]
Given that the set $\mathcal{M}$ is drawn completely at random, the posterior distribution of $\varepsilon_U$, $\mathcal{P}(\varepsilon_U | \mathcal{E}_U)$, can be derived.

With arbitrary noise, the likelihood function $\mathcal{P}(\varepsilon_M | \hat{f})$ of the average fidelity $\hat{f}$ is unknown. Consequently, the posterior distribution $\mathcal{P}(\hat{f} | \mathcal{E}_M)$ cannot be derived. To overcome this difficulty, we note that because the estimate given in (2) is unbiased, for any state $\rho_U$ of the unsampled qubit pairs with average fidelity $\hat{f}$, $\hat{f} = \mathbb{E}_U \left[ 1 - \frac{3}{2} \varepsilon_U | \rho_U \right]$, (25) where $\mathcal{E}_U$ is the random variable form of $\varepsilon_U$. Since $\hat{f}$ is the expected value of $1 - \frac{3}{2} \mathcal{E}_U$, the randomness of the former variable is no higher than that of the latter. Specifically, the even moments of $\mathcal{P}(\hat{f} | \mathcal{E}_M)$ are upper bounded by those of $\mathcal{P}(\varepsilon_U | \mathcal{E}_M)$, i.e.,
\[
M^{(2t)}_{\varepsilon_U} [\varepsilon_U | \mathcal{E}_M] \leq \left( \frac{3}{2} \right)^{2t} M^{(2t)}_{\mathcal{E}_U} [\mathcal{E}_U | \mathcal{E}_M], \quad \forall t \in \mathbb{Z}^+, \quad (26)
\]
where $M^{(2t)}_{\varepsilon_U} [\cdot]$ denotes the $2t$-th order central moment.

According to (26) and using Chebyshev’s inequality (extended to higher order), one can bound the tail probability of $\mathcal{P}(\hat{f} | \mathcal{E}_M)$ by any even moment $M^{(2t)}_{\varepsilon_U} [\mathcal{E}_U | \mathcal{E}_M]$, $t \in \mathbb{Z}^+$. These bounds lead to the credible interval defined in (4), in which the maximum computed moment $T^*$ is designed to achieve the narrowest interval with low computational cost. Please refer to [30, Sec. III] for the formal propositions.

**Impact of Arbitrary Noise** — The expression of the credible interval $\mathcal{C}_{\text{arb}}(T^*)$ in Protocol 2 is relatively complicated. To illustrate the effects of arbitrary noise on estimation accuracy, we analyze the expression of $\mathcal{C}_{\text{arb}}$, the credible interval that bounds the tail probability of $\mathcal{P}(\hat{f} | \mathcal{E}_M)$ only by the second-order moment. The radius of this interval is given by
\[
\delta^{(2)} = \sqrt{\frac{1}{1 - \alpha} \frac{N + 1}{N - M} \frac{(2(f - b) + 1)(1 - \hat{f} + b)}{2(M + 2)}} \quad (27)
\]
where $b = \frac{3}{4} \frac{1 - \varepsilon_M}{M + 1}$. According to (27), three factors determine the estimation accuracy in scenarios with arbitrary noise.

\(1\) Decay rate of tail probability: This term indicates that because only the second-order moment is used to obtain $\delta^{(2)}$, the tail probability decays quadratically. Considering higher order moments allows for faster decay, leading to more accurate estimates, especially when the credible probability $\alpha$ is close to 1.

\(2\) Error due to atypical unsampled qubit pairs: This term reflects the estimation error introduced by the deviation between the average fidelity of all qubit pairs $\bar{f}_{\text{all}}$ and that of the unsampled ones $\bar{f}$. In particular, $\bar{f}$ deviates more significantly from $\bar{f}_{\text{all}}$ as $M$ approaches $N$, so this factor is an increasing function of $M$.

\(3\) Information from measurements: This term, decreasing at $\mathcal{O}(M^{-2})$, represents the contribution of information from measurements. For large $M$, this factor approaches $\sqrt{\frac{2(f + 1)(f - 1)}{2M + 2}}$, which is the minimum standard deviation of $\hat{f}$ characterized by the Cramér-Rao bound in (20) in scenarios with i.i.d. noise.

Please see [30, Sec. IV] for a comparison with the estimation accuracy in scenarios with i.i.d. noise. The following remarks summarize the impact of arbitrary noise.

**Remark 2** (Same order of efficiency as the i.i.d. case): According to (27), when the sampling ratio $\frac{M}{N}$ remains constant, $\delta^{(2)}$ decreases at $\mathcal{O}(M^{-2})$, the same error convergence order as in scenarios with i.i.d. noise. This result is consistent with Remark 1 and confirms that the proposed protocol minimizes the adverse effects of arbitrary noise.

**Remark 3** (Excessive measurements): When $N$ remains a constant and $M$ increases, factors (2) and (3) have contradictory effects. Factor (3) makes $\delta^{(2)}$ smaller, while
prior information of the noise, the measurement ratio $M$ proposed credible interval [30, Sec. IV-C] shows that this criterion also holds for the measurements for the credible interval noise and should be avoided in practice. The accurate interval $(\alpha = 0.05)$ between the 100th and 100th percentiles of the posterior distribution of fidelity $\bar{F}$, is also plotted, where $d \in \{0, 0.5, 1\}$ represents the degree of heterogeneity and correlation in the noise. In particular, the noise is i.i.d. when $d = 0$. Please see [28] Sec. V] for details.

factor $2^2$ has the opposite effect. In particular, when

$$M > \left\lceil \frac{N}{2^2} \right\rceil,$$

(28)

$\delta^{(2)}$ is an increasing function of $M$, indicating that the error from atypical unsampled qubit pairs, factor $2^2$, dominates other factors, resulting in an undesirable phenomenon in which more measurements lead to less accurate estimation. This phenomenon of excessive measurements is a unique challenge associated with arbitrary noise and should be avoided in practice. [28] presents the criterion for avoiding excessive measurements for the credible interval $C_{\text{arb}}$. Further tests in [30] Sec. IV-C] shows that this criterion also holds for the proposed credible interval $C_{\text{arb}}^{(\alpha)}$, indicating that without prior information of the noise, the measurement ratio $M/N$ shall not exceed 0.5. Fig. 2 shows an example of excessive measurements.

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