The KP approximation under a weak Coriolis forcing

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Abstract

In this paper, we study the asymptotic behavior of weakly transverse water-waves under a weak Coriolis forcing in the long wave regime. We derive the Boussinesq-Coriolis equations in this setting and we provide a rigorous justification of this model. Then, from these equations, we derive two other asymptotic models. When the Coriolis forcing is weak, we fully justify the rotation-modified Kadomtsev-Petviashvili equation (also called Grimshaw-Melville equation). When the Coriolis forcing is very weak, we rigorously justify the Kadomtsev-Petviashvili equation. This work provides the first mathematical justification of the KP approximation under a Coriolis forcing.

1 Introduction

We consider the motion of an inviscid, incompressible fluid under the influence of the gravity $g = -ge_z$ and the rotation of the Earth with a rotation vector $f = \frac{\Omega}{2} e_z$. We assume that the fluid has a constant density $\rho$ and that no surface tension is involved. We assume that the surface is a graph above the still water level and that the seabed is flat. We denote by $X = (x, y) \in \mathbb{R}^2$ the horizontal variable and by $z \in \mathbb{R}$ the vertical variable. The fluid occupies the domain $\Omega_t := \{(X, z) \in \mathbb{R}^3, -H < z < \zeta(t, X)\}$. We denote by $U = (V, w)^t$ the velocity in the fluid. Notice that $V$ is the horizontal component of $U$ and $w$ its vertical component. Finally, we assume that the pressure $P$ is constant at the surface of the fluid. The equations governing such a fluid are the free surface Euler-Coriolis equations

$$\begin{cases}
\partial_t U + (U \cdot \nabla_{X,z}) U + f \times U = -\frac{1}{\rho} \nabla_{X,z} P - ge_z \quad \text{in } \Omega_t, \\
\text{div } U = 0 \quad \text{in } \Omega_t,
\end{cases}
$$

with the boundary conditions

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1The centrifugal potential is assumed to be constant and included in the pressure term.
\[\begin{align*}
\{ & P_{\mid z=\zeta} = P_0, \\
& \partial_t \zeta - \mathbf{U} \cdot \mathbf{N} = 0, \\
& w_b = 0,
\end{align*}\]

where \(P_0\) is constant, \(\mathbf{N} = \left(\begin{array}{c} -\nabla \zeta \\ 1 \end{array}\right)\), \(\mathbf{U} = \left(\begin{array}{c} \mathbf{V} \\ \mathbf{w} \end{array}\right) = \mathbf{U}_{\mid z=\zeta}\) and \(\mathbf{U}_b = \left(\begin{array}{c} \mathbf{V}_b \\ w_b \end{array}\right) = \mathbf{U}_{\mid z=-H}\).

In this work, we do not directly work on the free surface Euler-Coriolis equations. We rather consider another formulation called the Castro-Lannes formulation (see [4]). This formulation generalizes the well-known Zakharov/Craig-Sulem formulation ([22, 6]) to a fluid with a rotational component. In [4], Castro and Lannes shown that we can express the free surface Euler equations thanks to the unknowns \((\zeta, \mathbf{U}_f, \mathbf{w}_b)\)^{(2)} where \(\omega = \text{Curl} \, \mathbf{U}\) is the vorticity of the fluid and
\[\mathbf{U}_f = \nabla + \mathbf{w} \nabla \zeta.\]

Then, they provide a system of three equations on these unknowns. In [15], a similar work has been done to take into account the Coriolis forcing. It leads to the following system, called the Castro-Lannes system or the water waves equations with vorticity,
\[\begin{align*}
\{ & \partial_t \zeta - \mathbf{U} \cdot \mathbf{N} = 0, \\
& \partial_t \mathbf{U}_f + \nabla \zeta + \frac{1}{2} \nabla |\mathbf{U}_f|^2 - \frac{1}{2} \nabla \left[\left(1 + |\nabla \zeta|^2\right) \mathbf{w}^2\right] + \left(\nabla^\perp \cdot \mathbf{U}_f\right) \mathbf{V}^\perp + f \mathbf{V}^\perp = 0, \\
& \partial_t \omega + (\mathbf{U} \cdot \nabla_{X,z}) \omega = (\omega \cdot \nabla_{X,z}) \mathbf{U} + f \partial_z \mathbf{U},
\end{align*}\]

where \(\mathbf{U} = \left(\begin{array}{c} \mathbf{V} \\ \mathbf{w} \end{array}\right) = \mathbf{U}[\zeta](\mathbf{U}_f, \omega)\) is the unique solution in \(H^1(\Omega_t)\) of the following Div-Curl equation
\[\begin{align*}
\{ & \text{curl} \, \mathbf{U} = \omega \text{ in } \Omega_t, \\
& \text{div} \, \mathbf{U} = 0 \text{ in } \Omega_t, \\
& (\nabla + \mathbf{w} \nabla \zeta)_{\mid z=\zeta} = \mathbf{U}_f, \\
& w_b = 0.
\end{align*}\]

The main goal of this paper is to study weakly transverse long waves. Therefore, we consider a nondimensionalization of the previous equations. Five physical parameters are involved in this work: the typical amplitude of the surface \(a\), the typical longitudinal scale \(L_x\), the typical transverse scale \(L_y\), the characteristic water depth \(H\) and the typical Coriolis frequency \(f\). We introduce four dimensionless parameters
\[\mu = \frac{H^2}{L_x^2}, \quad \varepsilon = \frac{a}{H}, \quad \text{Ro} = \frac{a \sqrt{gH}}{H f L_x}, \quad \text{and} \quad \gamma = \frac{L_x}{L_y}.\]

\(^2\)Notice that Castro and Lannes used the unknowns \((\zeta, \frac{\partial}{\partial_{X,z}} \cdot \mathbf{U}_f, \omega)\). However, as noticed in [10], the unknowns \((\zeta, \mathbf{U}_f, \omega)\) are better to derive shallow water asymptotic models.
The parameter \( \mu \) is called the shallowness parameter. The parameter \( \varepsilon \) is called the nonlinearity parameter. The parameter \( R_0 \) is the Rossby number and finally the parameter \( \gamma \) is called the transversality parameter. Then, we can nondimensionalize the Euler equations (1) and the Castro-Lannes equations (2) (see Part 1.2). In this work, we study the following asymptotic regime

\[ A_{\text{boussi}} = \{ (\mu, \varepsilon, \gamma, R_0), 0 \leq \mu \leq \mu_0, \varepsilon = \mathcal{O}(\mu), \gamma \leq 1, \frac{\varepsilon}{R_0} = \mathcal{O}(\sqrt{\mu}) \} , \]

This regime corresponds to a long wave regime (\( \varepsilon = \mathcal{O}(\mu) \)) under a weak Coriolis forcing \( \frac{\varepsilon}{R_0} = \mathcal{O}(\sqrt{\mu}) \). For an explanation of the first assumption, we refer to [12]. The second assumption is standard in oceanography. Rewriting \( \frac{\varepsilon}{R_0} = \frac{f\ell}{\sqrt{gH}} \), this assumption means that the rotation period is assumed to be much smaller than the time scale of the waves.

We refer to [9, 7] for more explanations about this assumption (see also [10, 17, 8, 14]).

We organize this paper in four parts. In Section 1.2, we explain how we nondimensionalize the equations and we provide a local wellposedness result. In Section 2, we derive and justify the Boussinesq-Coriolis equations in the asymptotic regime \( A_{\text{boussi}} \). The Boussinesq-Coriolis equations are a system of three equations on the surface \( \zeta \) and the vertical average of the horizontal velocity denoted \( \overline{V} \) (defined in (9)). They correspond to a \( \mathcal{O}(\mu^2) \) approximation of the water waves equations. These equations are

\[
\begin{align*}
\partial_t \zeta + \nabla \gamma \cdot (1 + \epsilon \zeta) \overline{V} &= 0, \\
\left( 1 - \frac{\mu}{3} \nabla \gamma \cdot \nabla \right) \partial_t \overline{V} + \nabla \gamma \zeta + \epsilon \overline{V} \cdot \nabla \overline{V} + \frac{\varepsilon}{R_0} \overline{V}_\perp &= 0.
\end{align*}
\]

Then, in Section 3, we study the KP approximation which corresponds to the asymptotic regime \( A_{\text{boussi}} \) with \( \varepsilon = \mu \) and \( \gamma = \sqrt{\mu} \). This second assumption corresponds to weakly transverse effects (see for instance [12]). In this regime, we derive two other asymptotic models. When the Coriolis forcing is weak \( \left( \frac{\varepsilon}{R_0} = \mathcal{O}(\mu) \right) \), we rigorously justify the modified-rotation Kadomtsev-Petviashvili equation (see Subsection 3.1), also called Grimshaw-Melville equation in the physics literature,

\[
\partial_k \left( \partial_t k + \frac{3}{2} k \partial_k k + \frac{1}{6} \partial^3_k k \right) + \frac{1}{2} \partial_{yy} k = \frac{1}{2} k.
\]

Then, when the Coriolis forcing is very weak \( \left( \frac{\varepsilon}{R_0} = \mu \right) \), we fully justify the KP equation (see Subsection 3.2)

\[
\partial_k \left( \partial_t k + \frac{3}{2} k \partial_k k + \frac{1}{6} \partial^3_k k \right) + \frac{1}{2} \partial_{yy} k = 0.
\]

Finally, in Section 4, we compare the scalar asymptotic models we derive in Section 3 with the ones derived in [16]: the Ostrovsky equation and the KdV equation.
1.1 Notations/Definitions

- If $A \in \mathbb{R}^3$, we denote by $A_h$ its horizontal component.
- If $V = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, we define the orthogonal of $V$ by $V^\perp = \begin{pmatrix} -v \\ u \end{pmatrix}$.
- In this paper, $C(\cdot)$ is a nondecreasing and positive function whose exact value has no importance.
- Consider a vector field $A$ or a function $w$ defined on $\Omega$. Then, we denote $A|_{z = \epsilon \zeta}$, $w|_{z = \epsilon \zeta}$ and $A_b = A|_{z = -1}$, $w_b = w|_{z = -1}$.
- If $N \in \mathbb{N}$ and $f$ is a function on $\mathbb{R}^2$, $|f|_{H^N}$ is its $H^N$-norm, $|f|_2$ is its $L^2$-norm and $|f|_{L^\infty}$ its $L^\infty$-norm. We denote by $(\cdot, \cdot)_2$ the $L^2(\mathbb{R}^2)$ inner product.
- If $f$ is a function defined on $\mathbb{R}^2$, We use the notation $\nabla^\gamma f = (\partial_x f, \gamma \partial_y f)^t$.
- If $u = u(X, z)$ is defined in $\Omega$, we define $u(X) = 1 + \frac{1}{1 + \epsilon \zeta} \int_{-1}^{\epsilon \zeta(X)} u(X, z)dz$ and $u^* = u - \overline{u}$.
- For $N \geq 0$, we define the Hilbert spaces $\partial_x H^N(\mathbb{R}^2)$
- Similarly, for $N \geq 0$, we define the Hilbert spaces $\partial_x^2 H^N(\mathbb{R}^2)$.
- In the following definition, we recall the notion of consistence (see for instance \cite{12}).

**Definition 1.1.** We say that the Castro-Lannes equations (7) are consistent of order $O(\mu^2)$ with a system of equations $S$ for $\zeta$ and $\nabla$ if for any smooth solutions $(\zeta, U^{\mu, \gamma}, \omega)$ of the Castro-Lannes equations (7), the pair $(\zeta, \nabla[\epsilon \zeta] \left( U^{\mu, \gamma}, \omega \right))$ (defined in (9)) solves $S$ up to a residual of order $O(\mu^2)$.

1.2 Nondimensionalization

We recall the four dimensionless parameters

$$\mu = \frac{H^2}{L_x^2}, \quad \varepsilon = \frac{a}{H}, \quad \text{Ro} = \frac{a \sqrt{gH}}{H f L_x} \quad \text{and} \quad \gamma = \frac{L_y}{L_x}.$$  

We nondimensionalize the variables and the unknowns. We introduce (see \cite{12} or \cite{13})
\[
\begin{aligned}
\begin{cases}
x' = \frac{x}{L_x},
y' = \frac{y}{L_y},
z' = \frac{z}{H},
\zeta' = \frac{\zeta}{a},
t' = \frac{\sqrt{gH}}{L_x} t,
\end{cases}
\end{aligned}
\]
\[
V' = \sqrt{\frac{H}{g}} \frac{V}{a},
\text{and}
P' = \frac{\rho g H}{\rho g H}.
\]

In the following, we use the following notations

\[
\nabla^\gamma = \nabla^\gamma_X = \left( \frac{\partial x'}{\partial y'} \right),
\nabla^\mu,\gamma_X = \left( \sqrt{\mu} \nabla^\gamma_X \right),
curl^\mu,\gamma = \nabla^\mu,\gamma \times ,
\text{div}^\mu,\gamma = \nabla^\mu,\gamma \cdot .
\]

We also define

\[
U^\mu = \left( \sqrt{\mu} V' \right),
\omega' = \frac{1}{\mu} \text{curl}^\mu,\gamma U^\mu,
\tag{6}
\]

and

\[
U^\mu = \left( \sqrt{\mu} V' \right) |_{z'=\epsilon \zeta'},
U^\mu_0 = U^\mu |_{z'=-1},
N^\mu,\gamma = \left( -\frac{C^\mu}{1} \right).
\]

**Remark 1.2.** Notice that the nondimensionalization of the vorticity presented in \([6]\) corresponds to weakly sheared flows (see \([3], [20], [18]\)).

The nondimensionalized fluid domain is

\[
\Omega'_t := \{ (X', z') \in \mathbb{R}^3, \ -1 < z' < \epsilon \zeta'(t', X') \}.
\]

Finally, the Euler-Coriolis equations \([1]\) become

\[
\begin{cases}
\partial_t U^\mu + \epsilon \left( U^\mu \cdot \nabla^\mu,\gamma_X \right) U^\mu + \frac{\epsilon}{\text{Ro}} \left( V'^\perp \right) = -\frac{1}{\epsilon} \nabla^\mu,\gamma \zeta' - \frac{1}{\epsilon} e_z \text{ in } \Omega'_t,
\text{div}^\mu,\gamma_X U^\mu = 0 \text{ in } \Omega'_t,
\end{cases}
\]

with the boundary conditions

\[
\begin{cases}
\partial_t \zeta' - \frac{1}{\mu} U^\mu \cdot N^\mu,\gamma = 0,
w_b' = 0.
\end{cases}
\]

In the following, we omit the primes. We can proceed similarly to nondimensionalize the Castro-Lannes formulation. We define the quantity

\[
U_{\gamma}^\mu = \mathbf{V} + \epsilon w \nabla^\gamma \zeta.
\]

Then, the Castro-Lannes formulation becomes (see \([4]\) or \([15]\) when \(\gamma = 1\),

\[5\]
\[
\begin{align*}
&\partial_t \zeta - \frac{1}{\mu} U^\mu \cdot N^{\mu,\gamma} = 0, \\
&\partial_t U^{\mu,\gamma}_{\parallel} + \nabla \gamma \zeta + \frac{\varepsilon}{2} \nabla \left[ \left( 1 + \varepsilon^2 \mu |\nabla \gamma \zeta|^2 \right) w^2 \right] + \varepsilon \left( \nabla^\perp \cdot U^{\mu,\gamma}_{\parallel} \right) \mathbf{V}^\perp + \frac{\varepsilon}{\mu \Ro} \mathbf{V}^\perp = 0, \quad (7)
\end{align*}
\]

where \( U^\mu = \left( \frac{\sqrt{\mu} \mathbf{V}^w}{w} \right) = U^\mu[\varepsilon \zeta](U^{\mu,\gamma}_{\parallel}, \omega) \) is the unique solution in \( H^1(\Omega_t) \) of

\[
\begin{align*}
&\text{curl}^{\mu,\gamma} U^\mu = \mu \omega \text{ in } \Omega_t, \\
&\text{div}^{\mu,\gamma} U^\mu = 0 \text{ in } \Omega_t, \\
&(\mathbf{V} + \varepsilon \mathbf{w} \nabla \gamma \zeta)_{\parallel \epsilon = \epsilon \zeta} = U^{\mu,\gamma}_{\parallel}, \\
&w_b = 0.
\end{align*}
\]

In order to rigorously derive asymptotic models, we need an existence result for the Castro-Lannes formulation (7). We recall that the existence of solutions to the water waves equations is always obtained under the so-called Rayleigh-Taylor condition that assumes the positivity of the Rayleigh-Taylor coefficient \( a \) (see Part 3.4.5 in [12] for the link between \( a \) and the Rayleigh-Taylor condition or [15]) where

\[
a := a[\varepsilon \zeta](U^{\mu,\gamma}_{\parallel}, \omega) = 1 + \varepsilon \left( \partial_t + \varepsilon \mathbf{V}[\varepsilon \zeta](U^{\mu,\gamma}_{\parallel}, \omega) \cdot \nabla \right) \mathbf{w}[\varepsilon \zeta](U^{\mu,\gamma}_{\parallel}, \omega).
\]

We explain in [15] how we can define the Rayleigh-Taylor coefficient \( a \) at \( t = 0 \). We also assume that the water depth is bounded from below by a positive constant

\[
\exists h_{\text{min}} > 0, 1 + \varepsilon \zeta \geq h_{\text{min}}.
\]

The following theorem can be found in [15] and provide a local wellposedness result of the Castro-Lannes formulation (7) (see also Theorem 1.5 in [14]).

**Theorem 1.3.** Let \( A > 0 \) and \( N \geq 5 \). We suppose that \((\mu, \varepsilon, \gamma, \Ro) \in \mathcal{A}_{\text{boussi}} \). We assume that

\[
\left( \zeta_0, (U^{\mu,\gamma}_{\parallel})_0, \omega_0 \right) \in H^{N+\frac{1}{2}}(\mathbb{R}^2) \times H^N(\mathbb{R}^2) \times H^{N-1}(\Omega_0),
\]

with \( \nabla^{\mu,\gamma} \cdot \omega_0 = 0 \) and \( \nabla^{\perp} \cdot (U^{\mu,\gamma}_{\parallel})_0 = \omega_0 \cdot \begin{pmatrix} -\varepsilon \sqrt{\mu} \nabla \gamma \zeta_0 \\ 1 \end{pmatrix} \). Finally, we assume that

\[
\exists h_{\text{min}}, a_{\text{min}} > 0, 1 + \varepsilon \zeta_0 \geq h_{\text{min}} \text{ and } a[\varepsilon \zeta_0]((U^{\mu,\gamma}_{\parallel})_0, \omega_0) \geq a_{\text{min}},
\]

and that

\[
|\zeta_0|_{H^{N+\frac{1}{2}}} + \frac{1}{\sqrt{1 + \sqrt{\mu}|D|}}(U^{\mu,\gamma}_{\parallel})_0 \left|_{H^N} + ||\omega_0||_{H^{N-1}} \leq A.
\]

6
Then, there exists $T > 0$ and a unique classical solution $(\zeta, U^\mu_\gamma, \omega)$ to the Castro-Lannes (7) on $[0, T]$ with initial data $(\zeta_0, (U^\mu_\gamma)_0, \omega_0)$. Moreover,

$$T = \frac{T_0}{\max (\varepsilon, \frac{1}{\text{Ro}})}, \quad \frac{1}{T_0} = c^1,$$

$$\max_{[0,T]} \left( |\zeta(t, \cdot)|_{HN} + \frac{1}{\sqrt{1 + \sqrt{\mu}|D|}} \left| U^\mu_\gamma(t, \cdot) \right|_{H^{N-\frac{1}{2}}} + ||\omega(t, \cdot)||_{H^{N-1}} \right) = c^2,$$

with $c^j = C \left( A, \mu_0, \frac{1}{h_{\min}}, \frac{1}{a_{\min}} \right)$.

**Remark 1.4.** Notice that thanks to Theorem 1.3 together with Part 5.5.1 in [4], the quantities $\zeta, U^\mu_\gamma, \omega, \nabla, U, \partial_t \zeta, \partial_t U^\mu_\gamma, \partial_t \omega$ and $\partial_t U$ remain bounded uniformly with respect to the small parameters during the time evolution of the flow.

## 2 The Boussinesq-Coriolis equations

In this part, we derive and fully justify the Boussinesq-Coriolis equations (3) under a weak Coriolis forcing $\varepsilon \text{Ro} = O(\sqrt{\mu})$. We recall the corresponding asymptotic regime

$$\mathcal{A}_{\text{boussi}} = \{(\mu, \varepsilon, \gamma, \text{Ro}) : 0 \leq \mu \leq \mu_0, \varepsilon = O(\mu), \gamma \leq 1, \frac{\varepsilon}{\text{Ro}} = O(\sqrt{\mu})\}. \quad (8)$$

Notice that no assumption on $\gamma$ is made in this part. The Boussinesq equations correspond to an order $O(\mu^2)$ approximation of the water waves equations. Motivated by [16], we use the Castro-Lannes equations (7) to derive this asymptotic model. We introduce the water depth

$$h(t, X) = 1 + \varepsilon \zeta(t, X),$$

and the vertical average of the horizontal velocity

$$\nabla = \nabla [\varepsilon \zeta](U^\mu_\gamma, \omega)(t, X) = \frac{1}{h(t, X)} \int_{z=-1}^{\varepsilon \zeta(t, X)} V[\varepsilon \zeta, \beta b](U^\mu_\gamma, \omega)(t, X, z)dz. \quad (9)$$

In the following we denote $V = (u, v)^t$. More generally, if $u$ is a function defined in $\Omega$, we denote by $\overline{u}$ its vertical average and $u^* = u - \overline{u}$. We also have to introduce the "shear" velocity

$$V_{\text{sh}} = V_{\text{sh}} [\varepsilon \zeta](U^\mu_\gamma, \omega)(t, X) = \int_{z=-1}^{\varepsilon \zeta} \omega_{\scriptscriptstyle{\overline{h}}}(t, X, z')dz'$$

and its vertical average
As noticed in [4], these quantities appear when one wants to obtain an expansion with respect to \( \mu \) of the velocity. We recall that

\[
\mathbf{U}^{\mu, \gamma} = \mathbf{V} + \varepsilon \mathbf{w} \nabla \gamma \zeta.
\]

### 2.1 Asymptotic expansions with respect to \( \mu \)

In this part, we give an expansion of different quantities with respect to \( \mu \). These expansions will help us to derive the Boussinesq-Coriolis equations (3) in Section 2.2. The following proposition gives a link between \( \mathbf{V} \) and \( \mathbf{U}^{\mu, \gamma} \cdot \mathbf{N}^{\mu, \gamma} \) (the proof is a small adaptation of Proposition 4.2 in [15]).

**Proposition 2.1.** If \( (\zeta, \mathbf{U}^{\mu, \gamma} / \mathbf{sh}, \omega) \) satisfy the Castro-Lannes system (7), we have

\[
\mathbf{U}^{\mu} / \mathbf{N}^{\mu, \gamma} = -\mu \nabla \gamma / (h \mathbf{V}).
\]

Then we get the first equation of the Boussinesq-Coriolis system from the first equation of (7). We also need an expansion of \( \mathbf{V} \) and \( \mathbf{w} \) with respect to \( \mu \). We introduce the following operators

\[
T [\varepsilon \zeta] f = \int_{-1}^{\varepsilon \zeta} \nabla \gamma \nabla \gamma / \int_{-1}^{\varepsilon \zeta} f \text{ and } T^* [\varepsilon \zeta] f = (T [\varepsilon \zeta] f)^*.
\]

In the following, we denote \( T = T [\varepsilon \zeta] \) and \( T^* = T^* [\varepsilon \zeta] \) when no confusion is possible.

**Proposition 2.2.** In the Boussinesq regime \( A_{\text{boussi}} \), if \( (\zeta, \mathbf{U}^{\mu, \gamma} / \mathbf{sh}, \omega) \) satisfy the Castro-Lannes system (7), we have

\[
\mathbf{V} = \nabla + \sqrt{\mu} \mathbf{V}^* + \mu T^* \nabla + \mu^2 T^2 \mathbf{V}^* + O (\mu^2),
\]

\[
\mathbf{V} = \nabla - \sqrt{\mu} \mathbf{Q} + \mu T^* \nabla - \mu^2 T^2 \mathbf{V}^* + O (\mu^2),
\]

where

\[
T^* \nabla = \frac{1}{2} \left( \frac{h^2}{3} - [z + 1]^2 \right) \nabla \gamma \nabla \gamma \cdot \nabla \text{ and } T^* \nabla = -\frac{h^2}{3} \nabla \gamma \nabla \gamma \cdot \nabla.
\]

We also have

\[
w = -\mu (z + 1) \nabla \gamma \nabla + \mu^2 \int_{-1}^{\varepsilon} \nabla \gamma \cdot \mathbf{V}^* + O (\mu^2),
\]

\[
\mathbf{w} = -\mu h \nabla \gamma \cdot \nabla + O (\mu^2),
\]
Proof. This proof is an adaptation of part 2.2 in [3], Part 4.2 in [15] and Section 2.1 in [16]. First, using curl $\mu,\gamma$ $U^\mu = \mu \omega$, we obtain that

$$\sqrt{\mu} \omega_h = \partial_z V^\perp - \nabla^\perp w.$$

Then, we consider the ansatz $V = \nabla + \sqrt{\mu} V_1$. By integrating the previous equation, we obtain

$$\sqrt{\mu} \partial_z V_1 = -\sqrt{\mu} \omega^\perp_h + \nabla^\perp w.$$

Since $\nabla V_1 = 0$, we get

$$V_1 = \left( \int_z \omega^\perp_h \right)^* - \frac{1}{\sqrt{\mu}} \left( \int_z \nabla^\perp w \right)^*.$$

Secondly, using Proposition 2.1 and the divergence-free assumption, we get

$$w = -\mu \nabla \gamma \cdot \left( \int_{z_1}^z V \right). \tag{10}$$

Then, gathering the previous two equality, we obtain

$$V = \nabla + \sqrt{\mu} V^*_sh + \mu T^\gamma V. \tag{11}$$

Finally, the expansion of $\partial_t V$ follows by applying the operator $Id +\mu T^\gamma$ to the previous equality. For the second equality, we notice that $T^\gamma V^*_sh = -T V^*_sh$. The third and fourth equalities follow from the fact that $\nabla$ does not depend on $z$. The fifth equality are a consequence of Equalities (10) and (11). Finally, the sixth equality follows from the fact that $\nabla V^*_sh = 0$ and that $\varepsilon = O(\mu)$. \qed

We can also get an expansion of $\partial_t V$ and $\partial_t w$.

Proposition 2.3. In the Boussinesq regime $A_{bouss}$, if $\left( \zeta, U^\mu, \omega \right)$ satisfy the Castro-Lannes system (7), we have

$$\partial_t \left( V - \nabla - \sqrt{\mu} V^*_sh - \mu T^\gamma V - \mu^2 T^\gamma V^*_sh \right) = O(\mu^2),$$

$$\partial_t \left( \nabla V + \sqrt{\mu} Q - \mu T^\gamma \nabla + \mu^2 T^\gamma V^*_sh \right) = O(\mu^2),$$

$$\partial_t \left( W + \mu h \nabla^\gamma V \right) = O(\mu^2).$$

Proof. The result follows from Proposition 2.1 and the equality

$$V = (1 - \mu T^\gamma) \left( \nabla + \sqrt{\mu} V^*_sh \right) + \mu^2 T^\gamma T^\gamma V.$$

Since we can not express $Q$ and $V^*_sh$ with respect to $\zeta$ and $V$, we need an evolution equation at order $O(\mu^3)$ of these quantities. \qed
Proposition 2.4. In the Boussinesq regime $\mathcal{A}_{\text{boussi}}$, if $\left(\zeta, U^\mu, \nu^\gamma, \omega\right)$ satisfy the Castro-Lannes system \cite{Lannes}, then $Q$ satisfies the following equation

$$
\partial_t Q + \varepsilon \nabla \cdot \nabla^\gamma Q + \varepsilon Q \cdot \nabla \nabla + \frac{\varepsilon}{\text{Ro} \sqrt{\mu}} (\nabla - \nabla) = O \left( \mu^\frac{3}{2} \right),
$$

and $V_{sh}^*$ satisfies the equation

$$
\partial_t V_{sh}^* + \varepsilon \nabla \cdot \nabla^\gamma V_{sh}^* + \varepsilon V_{sh}^* \cdot \nabla \nabla - \varepsilon [1 + z] \left( \nabla^\gamma \cdot \nabla \right) \partial_z V_{sh}^* + \frac{\varepsilon}{\text{Ro} \sqrt{\mu}} (\nabla - \nabla) = O \left( \mu^\frac{3}{2} \right).
$$

Proof. This proof is an adaptation of Part 2.3 in \cite{Lannes} and Part 2.2 in \cite{Castro}. Thanks to the horizontal component of the vorticity equation of the Castro-Lannes formulation \cite{Castro}, we get

$$
\partial_t \omega_h + \varepsilon \nabla \cdot \nabla^\gamma \omega_h + \frac{\varepsilon}{\mu} \varepsilon \nabla \omega_h = \varepsilon \omega_h \cdot \nabla^\gamma V + \frac{\varepsilon}{\sqrt{\mu}} \omega \partial_z V + \frac{\varepsilon}{\text{Ro} \sqrt{\mu}} \partial_z V.
$$

Furthermore, since $\text{curl}^\mu \cdot U^\mu = \mu \omega$, we have

$$
\partial_t \V = -\sqrt{\mu} \omega_h^\perp + O (\mu) \quad \text{and} \quad \omega_z = \nabla^\gamma \cdot \nabla + O (\sqrt{\mu}).
$$

Then, using Proposition 2.2 we obtain

$$
\partial_t \omega_h + \varepsilon \nabla \cdot \nabla^\gamma \omega_h - \varepsilon [1 + z] \left( \nabla^\gamma \cdot \nabla \right) \partial_z \omega_h - \varepsilon \omega_h \cdot \nabla^\gamma \nabla - \varepsilon \left( \nabla^\gamma \nabla \right) \omega_h^\perp - \frac{\varepsilon}{\text{Ro} \sqrt{\mu}} \partial_z V = O \left( \mu^\frac{3}{2} \right),
$$

Then, integrating with respect to $z$, using the fact that $\partial_t \zeta + \nabla^\gamma \cdot (h \nabla) = 0$, $V_{sh} = \int_z^\xi \omega_h^\perp$ and $Q_x = \nabla_{sh}$ we get (see the computations in Part 2.3 in \cite{Lannes})

$$
\partial_t V_{sh} + \varepsilon \nabla \cdot \nabla^\gamma V_{sh} + \varepsilon V_{sh} \cdot \nabla^\gamma \nabla + \frac{\varepsilon}{\text{Ro} \sqrt{\mu}} (\nabla - \nabla) = \varepsilon [1 + z] \left( \nabla^\gamma \cdot \nabla \right) \partial_z V_{sh} + O \left( \mu^\frac{3}{2} \right),
$$

and

$$
\partial_t Q + \varepsilon \nabla \cdot \nabla^\gamma Q + \varepsilon Q \cdot \nabla^\gamma \nabla + \frac{\varepsilon}{\text{Ro} \sqrt{\mu}} (\nabla - \nabla) = O \left( \mu^\frac{3}{2} \right).
$$

Finally, the second equation follows from the fact that $V_{sh}^* = V_{sh} - Q$. \hfill \Box

2.2 Full justification of the Boussinesq-Coriolis equations

We can now establish the Boussinesq-Coriolis equations under a weak Coriolis forcing. The Boussinesq-Coriolis equations \cite{Lannes} are the following system

$$
\begin{cases}
\partial_t \zeta + \nabla^\gamma \cdot h \nabla = 0, \\
\left( 1 - \frac{\mu}{3} \nabla^\gamma \nabla \right) \partial_t V + \nabla^\gamma \zeta + \varepsilon \nabla \cdot \nabla^\gamma \nabla + \frac{\varepsilon}{\text{Ro}} V^\perp = 0.
\end{cases}
$$

First, we show a consistency result.
Proposition 2.5. In the Boussinesq regime $A_{\text{Boussi}}$, the Castro-Lannes equations \((7)\) are consistent at order $O(\mu^2)$ with the Boussinesq-Coriolis equations \((3)\) in the sense of Definition 1.1.

Proof. The first equation of the Boussinesq-Coriolis equations is always satisfied for a solution of the Castro-Lannes formulation by Proposition 2.1. We recall that the second equation of the Castro-Lannes formulation is

$$\partial_t U^{\mu,\gamma} + \nabla \gamma \zeta + \frac{\varepsilon}{2} \nabla \zeta \parallel U^{\mu,\gamma} \parallel^2_{\psi} - \frac{\varepsilon}{2\mu} \nabla \zeta \left[ \left( 1 + \varepsilon^2 \mu |\nabla \gamma \zeta|^2 \right) \psi^2 \right] + \varepsilon \left( \nabla \perp \cdot U^{\mu,\gamma} \right) \nabla \perp + \frac{\varepsilon}{\text{Ro}} \nabla \perp = 0.$$  

Thanks to Proposition 2.2, we know that ($\varepsilon = O(\mu)$)

$$U^{\mu,\gamma} = V + \varepsilon \nabla \gamma \zeta = V + O(\mu^2) = \nabla \sqrt{\mu} Q + \mu \nabla \gamma V - \mu^{3/2} T \nabla \perp_{\text{sh}} + O(\mu^2),$$

and

$$\frac{\varepsilon}{2} \nabla \zeta \parallel U^{\mu,\gamma} \parallel^2_{\psi} = \varepsilon U^{\mu,\gamma} \cdot \nabla \gamma U^{\mu,\gamma} - \varepsilon \left( \nabla \perp \cdot U^{\mu,\gamma} \right) U^{\mu,\gamma} \perp = \varepsilon \nabla \cdot \nabla \gamma V - \varepsilon \sqrt{\mu} Q \cdot \nabla \gamma V - \varepsilon \sqrt{\mu} \nabla \gamma Q - \varepsilon \left( \nabla \perp \cdot U^{\mu,\gamma} \right) V \perp + O(\mu^2).$$

Furthermore, thanks to Proposition 2.4 and Proposition 2.2, we get ($\frac{\mu^3}{\text{Ro}} = O(\sqrt{\mu})$)

$$\mu^{3/2} \partial_t T \nabla \perp_{\text{sh}} = \mu^{3/2} T \partial_t V \perp_{\text{sh}} + O(\mu^2) = -\mu^2 \frac{\varepsilon}{\text{Ro}} T \nabla \perp_{\text{sh}} + O(\mu^2) = O(\mu^2).$$

Finally, using Proposition 2.2, Proposition 2.4, Proposition 2.3 and Remark 1.4, we obtain from the second equation of the Castro-Lannes formulation that

$$\left( 1 - \frac{\mu^3}{3} \nabla \gamma \nabla \gamma \cdot \right) \partial_t \nabla \gamma + \nabla \gamma \zeta + \varepsilon \nabla \cdot \nabla \gamma V + \frac{\varepsilon}{\text{Ro}} \nabla \perp = O(\mu^2).$$

Notice that all the terms that involve $Q$ disappear (this fact was pointed out in [3] and [15]).

Remark 2.6. In [16], the author points out the fact that under a strong Coriolis forcing ($\frac{\text{Ro}}{\mu^2} \leq 1$), a new term appears in the Boussinesq-Coriolis equations. We would like to emphasize that this term is not present in this setting since we only study a weak Coriolis forcing ($\frac{\text{Ro}}{\mu^2} = O(\sqrt{\mu})$).

The purpose of this part is to fully justify the Boussinesq-Coriolis equations \((3)\). First, we give a local wellposedness result of the Boussinesq-Coriolis equations. We define the energy space

$$X^N_\mu(\mathbb{R}^2) = H^N(\mathbb{R}^2) \times H^N(\mathbb{R}^2) \times H^N(\mathbb{R}^2),$$

endowed with the norm

$$\|(\zeta, V)\|^2_{X^N_\mu} = |\zeta|^2_{H^N} + |V|^2_{H^N} + \mu |\nabla \gamma \cdot V|^2_{H^N}.$$
Proposition 2.7. Let $N \geq 3$ and $\left( \zeta_0, \nabla \zeta_0 \right) \in X_N^\mu (\mathbb{R}^2)$. Suppose that $(\mu, \epsilon, \gamma, \mathcal{R}_0) \in \mathcal{A}_{\text{boussi}}$. Assume that

$$\exists h_{\text{min}} > 0, \ 1 + \epsilon \zeta_0 \geq h_{\text{min}}.$$ 

Then, there exists an existence time $T > 0$ and a unique solution $(\zeta, \nabla \zeta)$ on $[0,T]$ to the Boussinesq-Coriolis equations (3) with initial data $\left( \zeta_0, \nabla \zeta_0 \right)$. Moreover, $(\zeta, \nabla \zeta) \in C([0,T]; X_N^\mu (\mathbb{R}^2))$ and

$$T = \frac{T_0}{\mu}, \ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} \left| (\zeta, \nabla \zeta) (t, \cdot) \right|_{X_N^\mu} = c^2,$$

with $c^j = C \left( \mu_0, \frac{h_{\text{min}}}{2}, \left| h \right|_{L^\infty} \right)$.

Proof. This proof is a small adaptation of the proof of Proposition 6.7 in [12]. We only give the energy estimates. We assume that $(\zeta, \nabla \zeta)$ solves (3) on $[0, T_0/\mu]$ and that $1 + \epsilon \zeta = \frac{h_{\text{min}}}{2}$ on $[0, T_0/\mu]$.

We denote $U = (\zeta, \nabla \zeta)$. We introduce the symmetric matrix operator

$$S(U) = \begin{pmatrix} 1 & 0 \\ 0 & hI_2 - \mu \frac{1}{3} \nabla \gamma (h \nabla \cdot \nabla \gamma) \end{pmatrix}$$

and the associated energy

$$\mathcal{E}^N(U) = \frac{1}{2} \sum_{|\alpha| \leq N} (S(U) \partial^\alpha U, \partial^\alpha U)_2.$$ 

Remark that there exists $c_1, c_2 = C \left( \frac{1}{h_{\text{min}}}, \left| h \right|_{L^\infty} \right)$ such that

$$c_1 \left| \nabla \gamma \cdot \nabla \zeta \right|_2 \leq \left( -\frac{1}{3} \nabla \gamma (h \nabla \gamma \cdot \nabla \gamma), \nabla \gamma \right)_2 \leq c_2 \left| \nabla \gamma \cdot \nabla \zeta \right|_2.$$ 

We also notice that for $|\alpha| = N$,

$$\frac{d}{dt} (S(U) \partial^\alpha U) = \left( h(1 - \frac{\mu}{3} \nabla \gamma \nabla \gamma) \partial^\alpha \partial_t \zeta - \frac{1}{3} \nabla \gamma (h \nabla \gamma \cdot \nabla \gamma) \nabla \gamma \zeta + \epsilon \text{l.o.t.} \right) + \epsilon l.o.t.$$ 

and that, denoting $\Delta \gamma = \nabla \gamma \cdot \nabla \gamma$,

$$\mu \left| \nabla \gamma \cdot \partial_t \nabla \gamma \right|_{H^{N+1}} \leq \left| (1 - \frac{\mu}{3} \Delta \gamma)^{-1} \mu \nabla \gamma \cdot (\nabla \gamma \zeta + \epsilon \nabla \gamma \cdot \nabla \gamma \nabla \gamma + \frac{\epsilon}{\mathcal{R}_0} \nabla \gamma) \right|_{H^{N+1}} \leq C \left( \mu_0, \mathcal{E}^N(U) \right).$$

Then, after some computations we obtain ($\epsilon = \mathcal{O}(\mu)$)
\[
\frac{d}{dt} \mathcal{E}^N(U) \leq \mu C \left( \mathcal{E}^N(U) \right) \mathcal{E}^N(U)
\]
and the result follows from Grönwall’s inequality.

We also have a stability result for the Boussinesq-Coriolis equations (3).

**Proposition 2.8.** Let the assumptions of Proposition 2.7 be satisfied. Suppose that there exists \((\tilde{\zeta}, \tilde{V}) \in C \left( \left[ 0, \frac{T_0}{\mu} \right]; X^N \left( \mathbb{R}^2 \right) \right)\) satisfying

\[
\begin{cases}
\partial_t \tilde{\zeta} + \nabla \cdot \hat{h} \tilde{V} = R_1, \\
\left( 1 - \frac{\mu}{3} \nabla \cdot \nabla \right) \partial_t \tilde{V} + \nabla \tilde{\zeta} + \tilde{V} \cdot \nabla \tilde{V} + \frac{\varepsilon}{Ro} \tilde{V} = R_2.
\end{cases}
\]

where \(\hat{h} = 1 + \varepsilon \tilde{\zeta}\) and with \(R = (R_1, R_2) \in L^\infty \left( \left[ 0, \frac{T_0}{\mu} \right]; H^{N-1} (\mathbb{R}^2) \right)\). Then, if we denote \(e = (\zeta, V) - (\tilde{\zeta}, \tilde{V})\) where \(U = (\zeta, V)\) is the solution given in Proposition 2.7, we have

\[
|e(t)|_{X^N} \leq c_1 \left( |e|_{t=0} \right)_{X^N} + t |R|_{L^\infty \left( \left[ 0, \frac{T_0}{\mu} \right]; H^{N-1} \right)}.
\]

where

\[
c_1 = C \left( \mu_0, \frac{1}{h_{\min}}, |(\zeta_{0}, \nabla_0)|_{X^N}, \left| (\tilde{\zeta}, \tilde{V}) \right|_{L^\infty \left( \left[ 0, \frac{T_0}{\mu} \right]; X^N \right)} \right).\]

**Proof.** We proceed as in Proposition 2.7 We define the energy

\[
\mathcal{F}^{N-1}(\varepsilon) = \frac{1}{2} \sum_{|\alpha| \leq N-1} (S(U) \partial^\alpha \varepsilon, \partial^\alpha \varepsilon).
\]

After some computations, we get

\[
\frac{d}{dt} \mathcal{F}^{N-1}(\varepsilon) \leq \left( |R|_{H^{N-1}} + \mu C \left( \mu_0, \frac{1}{h_{\min}}, |U|_{X^N}, \left| (\tilde{\zeta}, \tilde{V}) \right|_{X^N}, |R|_{L^\infty \left( \left[ 0, \frac{T_0}{\mu} \right]; H^{N-1} \right)} \right) \right) |e|_{X^N}.
\]

Then the result follows from Gronwall’s inequality.

We can now rigorously justify the Boussinesq-Coriolis equations. We recall that the operator \(\nabla [\varepsilon \zeta](U^{\mu, \gamma}_0, \omega)\) is defined in (9).

**Theorem 2.9.** Let \(N \geq 12\) and \((\mu, \varepsilon, \gamma, Ro) \in A_{\text{Boussi}}\). We assume that we are under the assumptions of Theorem 1.3. We define the following quantities

\[
\nabla_0 = \nabla [\varepsilon \zeta](U^{\mu, \gamma}_0, \omega_0) ; \nabla = \nabla [\varepsilon \zeta](U^{\mu, \gamma}_0, \omega).
\]
Then, there exists a time $T > 0$ such that

(i) $T$ has the form

$$T = \frac{T_0}{\max(\mu, \frac{\mu}{\sqrt{\epsilon}})}$$

and $\frac{1}{T_0} = c^1$.

(ii) There exists a unique classical solution $(\zeta_B, \nabla_B)$ of $\mathcal{C}$ with the initial data $(\zeta_0, \nabla_0)$ on $[0, T]$.

(iii) There exists a unique classical solution $\left(\zeta, U^{\mu, \gamma}, \omega\right)$ of System $\mathcal{D}$ with initial data $\left(\zeta_0, (U^{\mu, \gamma})_0, \omega_0\right)$ on $[0, T]$.

(iv) The following error estimate holds, for $0 \leq t \leq T$,

$$\left| (\zeta, \nabla) - (\zeta_B, \nabla_B) \right|_{L^\infty([0, t] \times \mathbb{R}^2)} \leq \mu^2 t c^2,$$

with $c^j = C \left( A, \mu_0, \frac{1}{h_{\text{min}}}, \frac{1}{\alpha_{\text{min}}} \right)$.

Therefore, in the Boussinesq regime $A$ a solution of the water waves system $\mathcal{C}$ remains close to a solution of the Boussinesq-Coriolis equations $\mathcal{D}$ over a time $\mathcal{O}\left( \frac{1}{\sqrt{\mu}} \right)$ with an accuracy of order $\mathcal{O}\left( \mu^\frac{3}{2} \right)$.

**Remark 2.10.** Notice that if one considers a solution of a system and wants to show that this solution remains close to a solution of the waves equations over times $\mathcal{O}\left( \frac{1}{\sqrt{\mu}} \right)$ with an accuracy of order $\mathcal{O}\left( \mu^\frac{3}{2} \right)$, it is sufficient to compare this solution with a solution of the Boussinesq-Coriolis equations $\mathcal{D}$. We use this strategy in the following.

### 3 The KP approximation

In this section, we consider the KP (Kadomtsev-Petviashvili) approximation under a weak Coriolis forcing. We assume that $\epsilon = \mu$ (long waves) and $\gamma = \sqrt{\mu}$ (weakly transverse effects). We consider two different regimes. First, if $\frac{\mu}{\sqrt{\epsilon}} = \sqrt{\mu}$ (weak rotation), we derive the rotation-modified KP equation $\mathcal{E}$. Then, if $\frac{\mu}{\sqrt{\epsilon}} = \mu$ (very weak rotation), we derive the KP equation $\mathcal{F}$. We refer to $\mathcal{I}$ for more physical explanations about these two models (see also $\mathcal{J}$).

#### 3.1 Weak rotation, the rotation-modified KP equation

In the irrotational setting, the KP equation provides a good approximation of the water waves equation under the assumption that $\epsilon$ and $\mu$ have the same order and that $\gamma$ and $\sqrt{\mu}$ have the same order (see $\mathcal{L}$ or Part 7.2 in $\mathcal{M}$). When a Coriolis forcing is taken into account, Grimshaw and Melville ($\mathcal{H}$) derived an equation for long waves, which is an adaptation of the KP equation.
\[ \partial_t \left( \partial_r k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial^3_\xi k \right) + \frac{1}{2} \partial_{yy} k = \frac{1}{2} k. \] (12)

This equation is called the rotation-modified KP equation or Grimshaw-Melville equation in the physics literature. Notice that this equation was originally derived for internal water waves \([10, 7]\). In this part, we fully justify this equation. Inspired by \([10, 7]\), we consider the asymptotic regime

\[ A_{\text{RKP}} = \left\{ (\mu, \varepsilon, \gamma, Ro), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \sqrt{\mu}, \frac{\varepsilon}{Ro} = \sqrt{\mu} \right\}. \]

Then, the Boussinesq-Coriolis equations become \((\gamma = \sqrt{\mu})\)

\[ \begin{cases} \partial_t \zeta + \nabla \gamma \cdot \left( 1 + \mu \zeta \right) V = 0, \\ \left( 1 - \frac{\mu}{3} \nabla \gamma \nabla \gamma \right) \partial_t V + \nabla \gamma \zeta + \mu \nabla \zeta + \sqrt{\mu} \nabla V = 0. \end{cases} \] (13)

In the following, we denote \(V = (u, v)^t\). Our strategy is similar to the one used in \([16]\) to fully justify the Ostrovsky equation. We consider an expansion of \((\zeta, V)\) with respect to \(\mu\). Inspired by \([13]\) or Part 7.2 in \([12]\), we seek an approximate solution \((\zeta_{\text{app}}, u_{\text{app}}, v_{\text{app}})\) of (13) in the form

\[ \begin{align*} 
\zeta_{\text{app}}(t, x, y) &= k(x - t, y, \mu t) + \mu \zeta(1)(t, x, y, \mu t), \\
u_{\text{app}}(t, x, y) &= k(x - t, y, \mu t) + \mu u(1)(t, x, y, \mu t), \\
v_{\text{app}}(t, x, y) &= \sqrt{\mu} v^{(1/2)}(t, x, y, \mu t) 
\end{align*} \] (14)

where \(k = k(\xi, \tau)\) is a traveling water wave modulated by a slow time variable and the others terms are correctors. In the following, we denote by \(\tau\) the variable associated to the slow time variable \(\mu t\). Plugging the ansatz into System (13), we obtain

\[ \begin{cases} \partial_t \zeta_{\text{app}} + \nabla \gamma \cdot \left( 1 + \mu \zeta_{\text{app}} \right) V_{\text{app}} = \mu R^1_{(1)} + \mu^2 R_1, \\
\left( 1 - \frac{\mu}{3} \nabla \gamma \nabla \gamma \right) \partial_t V_{\text{app}} + \nabla \zeta_{\text{app}} + \mu \nabla_{\text{app}} V_{\text{app}} + \sqrt{\mu} V_{\text{app}} = \sqrt{\mu} R^2_{(1)} + \mu R^1_{(2)} + \mu^2 R_2. \end{cases} \] (15)

where

\[ \begin{align*} 
R^1_{(1)} &= \partial_t \zeta(1) + \partial_x u(1) + \partial_r k + 2k \partial_\xi k + \partial_y v^{(1/2)}, \\
R^2_{(1/2)} &= \begin{pmatrix} 0 \\ \partial_{y} v^{(1/2)} + \partial_y k + k \end{pmatrix} \quad \text{and} \quad R^2_{(1)} = \begin{pmatrix} \partial_t u(1) + \partial_x \zeta(1) + \partial_r k + \frac{1}{2} \partial^3_\xi k + 2k \partial_\xi k - v^{(1/2)} \end{pmatrix}, 
\end{align*} \]

and

\[ \begin{align*} 
R_1 &= \partial_r \zeta(1) + \partial_x (k u(1) + k \zeta(1) + \mu \zeta(1) u(1)) + \partial_y ((k + \mu \zeta(1)) v^{(1/2)}), \\
R_2 &= (\sqrt{\mu} R_{2.1}, R_{2.2}) \end{align*} \] (16)

with
Then, using the fact that 

\[ \partial_x k - \frac{1}{3} \partial_x^3 u(1) - \frac{1}{3} \partial_x^2 \partial_x u(1) + \partial_x (ku(1)) + \mu u(1) \partial_x u(1) \]

\[ - \frac{1}{3} \partial_x^3 v(1/2) - \frac{\mu}{3} \partial_x^3 \tau v(1/2) + v(1/2) \partial_y (k + \mu u(1)), \]

\[ R_{2,2} = \partial_x v(1/2) + \partial_y \zeta(1) + k \partial_x v(1/2) + u(1) + \frac{1}{3} \partial_y \partial_x^2 k + \mu u(1) \partial_x v(1/2) + \mu v(1/2) \partial_y v(1/2) \]

\[ - \frac{\mu}{3} \left( \partial_y^3 k + \partial_y^3 \tau u(1) + \frac{3}{2} \partial_y \partial_x v(1/2) + \mu \partial_y \partial_x \tau u(1) + \mu \partial_y^2 \partial_x v(1/2). \right) \]

Then, the strategy is to choose \((k, v(1/2))\) such that, for all \((x, y) \in \mathbb{R}^2, t \in \left[0, \frac{T}{\mu}\right]\) and \(\tau \in [0, T]\),

\[ R^1_{(1)}(t, x, y, \tau) = 0 \text{ and } R^2_{(1)}(t, x, y, \tau) = R^2_{(1)}(t, x, y, \tau) = 0. \]

**Remark 3.1.** As noticed in Part 7.2.2 in [13], we should a priori add \(\sqrt{\mu} \zeta(1/2)(t, x, y, \mu t), \sqrt{\mu} u(1/2)(t, x, y, \mu t), v(0)(t, x, y, \mu t), \) and \(\mu v(1)(t, x, y, \mu t)\) to the ansatz [14] for \(\zeta_{app}, u_{app}\) and \(v_{app}\) respectively. But, it leads to \(\zeta(1/2) = u(1/2) = v(0) = v(1) = 0\) if these quantities are initially zero.

We focus first on the condition \(R^2_{(1)}(t, x, y, \tau) = 0\). Assuming that \(v(1/2)\) and \(k\) vanish at \(x = \infty\), this condition is equivalent to the equation

\[ \partial_t \partial_x v(1/2)(t, x, y, \tau) + \partial_x k(x - t, y, \tau) + \frac{\partial_x^3 k(x - t, y, \tau) = 0. \]

Then, using the fact that \(\partial_t (k(x - t, y, \tau)) = -\partial_x k(x - t, y, \tau),\) we can integrate with respect to \(t\) and we get

\[ \partial_x v(1/2)(t, x, y, \tau) = \partial_x v(0)(x, y) + k(x - t, y, \tau) + \partial_y k(x - t, y, \tau) - k^0(x, y) - \partial_y k^0(x - t, y, \tau), \]

where \(k^0\) and \(v(0)(1/2)\) are the initial data of \(k\) and \(v(1/2)\) respectively. Then, assuming that \(k(\cdot, \tau) \in \partial_x H^{N}(\mathbb{R}^2)\) for all \(\tau \in [0, T]\) (see [14]), we obtain

\[ v(1/2)(t, x, y, \tau) = v(0)(1/2)(x, y) + \partial_x^{-1} k(x - t, y, \tau) + \partial_y^{-1} \partial_y k(x - t, y, \tau) \]

\[ - \partial_x^{-1} k^0(x, y) - \partial_y^{-1} \partial_y k^0(x - t, y, \tau), \]

Secondly, we study the conditions \(R^1_{(1)} = R^2_{(1)} = 0\). Denoting \(w_{\pm} = \zeta(1) \pm u(1)\), we obtain

\[ (\partial_t + \partial_x) w_+ + \left( 2 \partial_t k + 3k \partial_t k + \frac{1}{3} \partial_t^3 k + \partial_x^{-1} \partial_t^2 k - \partial_x^{-1} k \right)(x - t, \tau) + F_1^1 = 0, \]

\[ (\partial_t - \partial_x) w_- + \left( k \partial_t k - \frac{1}{3} \partial_t^2 k + \partial_x^{-1} \partial_t^2 k + 2 \partial_x^{-1} \partial_y k + \partial_x^{-1} k \right)(x - t, \tau) + F_2^1 = 0, \]

where
\[\begin{align*}
F_0^1 &= \partial_y v_{(1/2)}^0 - v_{(1/2)}^0 + \partial_x^{-1} k^0 - \partial_x^{-1} \partial_y^2 k^0, \\
F_0^2 &= \partial_y v_{(1/2)}^0 + v_{(1/2)}^0 - \partial_x^{-1} k^0 - \partial_x^{-1} \partial_y^2 k^0 - 2\partial_x^{-1} \partial_y k^0.
\end{align*}\]

The following Lemma (see Lemma 7.20 in [12] or Lemma 2 in [13]) gives us a Condition to control \(\zeta(t)\) and \(u(t)\).

**Lemma 3.2.** Let \(c_1 \neq c_2\). Let \(k_1, k_2 \in H^2(\mathbb{R}^2)\) with \(k_2 = \partial_x K_2\) and \(K_2 \in H^3(\mathbb{R}^2)\). We consider the unique solution \(k\) of

\[
\begin{cases}
(\partial_t + c_1 \partial_x)k = k_1(x - c_1 t, y) + k_2(x - c_2 t, y), \\
k_{t=0} = 0.
\end{cases}
\]

Then, \(\lim_{t \to \infty} \|k(t, \cdot)\|_{H^2} = 0\) if and only if \(k_1 \equiv 0\) and in that case

\[|k(t, \cdot)|_{H^2} \leq C \frac{t}{1 + t} |K_2|_{H^3}.
\]

Hence, since we want to avoid a linear growth of the solution of (17), we must impose

\[\partial_t k + \frac{3}{2} k \partial_x k + \frac{1}{6} \partial_x^3 k + \frac{1}{2} \partial_x^{-1} \partial_y^2 k - \frac{1}{2} \partial_x^{-1} k = 0 \quad (18)
\]

which is the the rotation-modified KP equation defined in (12). In the following, we provide a local existence result for this equation. This proposition generalizes Theorem 1.1 in [3].

**Proposition 3.3.** Let \(N \geq 4\) and \(k_0 \in \partial_x H^N(\mathbb{R}^2)\). Then, there exists a time \(T > 0\) and a unique solution \(k \in C([0, T]; \partial_x H^N(\mathbb{R}^2))\) to the rotation-modified KP equation (18) and one has

\[\left|\partial_x^{-1} k(t, \cdot)\right|_{H^N} \leq C \left(T, \left|\partial_x^{-1} k_0\right|_{H^N}\right).
\]

Furthermore, if \(k_0, \partial_y^2 k_0 \in \partial_x^2 H^{N-2}(\mathbb{R}^2)\), then \(k \in C([0, T]; \partial_x^2 H^{N-2}(\mathbb{R}^2))\) and one has

\[\left|\partial_x^{-2} k(t, \cdot)\right|_{H^{N-2}} \leq C \left(T, \left|\partial_x^{-2} k_0\right|_{H^{N-2}}, \left|\partial_x^{-2} \partial_y^2 k_0\right|_{H^{N-2}}, \left|\partial y k_0\right|_{H^{N}}\right).
\]

Finally, if \(N \geq 6\) and \(k_0, \partial_y^2 k_0 \in \partial_x^2 H^{N-4}(\mathbb{R}^2)\), then \(\partial_y^2 k \in C([0, T]; \partial_x^2 H^{N-4}(\mathbb{R}^2))\) and one has

\[\left|\partial_x^{-2} k(t, \cdot)\right|_{H^{N-4}} \leq C \left(T, \left|\partial_x^{-2} k_0\right|_{H^{N-4}}, \left|\partial_x^{-2} \partial_y^2 k_0\right|_{H^{N-4}}, \left|\partial_x^{-1} k_0\right|_{H^{N}}\right).
\]
Proof. The first point follows from Theorem 1.1 in [5]. We only have to prove the second and the third points. This proof is similar to the proof of Lemma 7.22 in [12] for the KP equation and the proof of Proposition 3.8 in [16] for the Ostrovsky equation. In the following, we denote by \( S(t) \) the semi-group of the linearized rotation-modified KP equation

\[
\partial_{\tau} k + \frac{1}{6} \partial^3_{\xi} k + \frac{1}{2} \partial^{-1}_{\xi} \partial^2_{\eta} k - \frac{1}{2} \partial^{-1}_{\xi} k = 0.
\]

One can check that this semi-group acts unitary on \( H^N(\mathbb{R}^2) \). We also define \( \tilde{k} = \partial_{\tau} k \).

Using the Duhamel’s formula we obtain

\[
\partial^{-1}_{\xi} \tilde{k}(\tau) = S(t)\partial^{-1}_{\xi} \tilde{k}_0 - \frac{3}{2} \int^\tau_0 S(t-s) \left( k(s) \tilde{k}(s) \right) ds.
\]

We can see by product estimates that \( \partial^{-1}_{\xi} \tilde{k}_0, k(s) \tilde{k}(s) \in H^{N-4}(\mathbb{R}^2) \) and then that \( \tilde{k} \in C([0,T]; \partial_x H^{N-3}(\mathbb{R}^2)) \). Then, we consider the following equality

\[
\frac{1}{2} (1 - \partial^2_{\eta}) \partial^{-2}_{\xi} k = \partial^{-1}_{\xi} \tilde{k} + \frac{3}{4} k^2 + \frac{1}{6} \partial^2_{\eta} k,
\]

For the second point, we get that \((1 - \partial^2_{\eta}) \partial^{-2}_{\xi} k \in H^{N-4}(\mathbb{R}^2) \) and the result follows easily. For the third point, we obtain from the second point that \( \partial^2_{\eta} \partial^{-2}_{\xi} k \in H^{N-4}(\mathbb{R}^2) \).

We can now rigorously justify the rotation-modified KP equation. The following theorem is the main theorem of this part.

**Theorem 3.4.** Let \( k^0 \in \partial^2_{\eta} H^{12}(\mathbb{R}^2) \) such that \( 1 + \varepsilon k^0 \geq h_{\min} > 0 \) and \( v^0 \in \partial_x H^8(\mathbb{R}^2) \). Suppose that \((\mu, \varepsilon, \gamma, \text{Ro}) \in A_{\text{RKP}} \). Then, there exists a time \( T_0 > 0 \), such that we have

(i) a unique classical solution \((\zeta_B, u_B, v_B)\) of \((13)\) with initial data \((k^0, k^0, \sqrt{\text{Ro}} v^0)\) on \([0, T_0 / \mu] \).

(ii) a unique classical solution \( k \) of \((13)\) with initial data \( k^0 \) on \([0, T_0] \).

(iii) If we define \((\zeta_{\text{RKP}}, u_{\text{RKP}})(t, x, y) = (k(x-t, y, \mu t), k(x-t, y, \mu t))\) we have the following error estimate for all \( 0 \leq t \leq T_0 / \mu \),

\[
|(|\zeta_B; u_B) - (\zeta_{\text{RKP}}, u_{\text{RKP}})|_{L^\infty([0,t] \times \mathbb{R}^2)}| \leq C |\frac{\mu t}{1 + t} (1 + \sqrt{\mu t})|
\]

where \( C = C \left( \frac{1}{h_{\min}}, \mu_0, |\partial^2_{\eta} k^0|_{H^{12}}, |\partial^{-1}_{\xi} v^0|_{H^8} \right) \).
Proof. In order to simplify the technicality of this proof, $C$ is a constant of the form

$$ C = C \left( \frac{1}{\eta_{\text{min}}}, \mu_0, |\partial_x^{-2}k^0|_{H^{1/2}}, |\partial_x^{-1}v^0|_{H^3} \right) $$

The first and second point follow from Proposition 2.7 and Proposition 3.3. Then, from System (17) and Lemma 3.2, we obtain

$$ |\zeta(1)|_{H^2} + |u(1)|_{H^2} \leq C \frac{t}{1+t}. $$

We also notice that we can control all the derivatives with respect to $x$, $y$ or $\tau$ of $u$ and $v$ by differentiating (17). Hence, we get a control for the remainders $R_1$ and $R_2$ and we obtain, for $0 \leq t \leq \frac{T}{\mu}$,

$$ |R_1(t)|_{H^3} + |R_2(t)|_{H^3} \leq C. $$

Then, using Proposition 2.8, one can have

$$ |(\zeta_B, u_B, v_B) - (\zeta_{\text{app}}, u_{\text{app}}, v_{\text{app}})|_{L^\infty([0,t] \times \mathbb{R}^2)} \leq C \mu^2 t. $$

Finally, from the ansatz (14) and Lemma 3.2 we have

$$ |(\zeta_{\text{app}}, u_{\text{app}}) - (\zeta_{\text{RKP}}, v_{\text{RKP}})|_{L^\infty([0,t] \times \mathbb{R}^2)} \leq \mu \frac{t}{1+t}, $$

and the result follows easily.

\[\square\]

This theorem provides the first mathematical justification of the rotation-modified KP equation. Notice that the condition $k^0 \in \partial^2_x H^{10}(\mathbb{R}^2)$ is quite restrictive. As noted in [12] Part 7.2.1 and in [16] for the Ostrovsky equation, using the strategy developed in [1], we can hope to weaken the assumption on $k^0$ into $k^0 \in \partial_x H^{9}(\mathbb{R}^2)$.

### 3.2 Very weak rotation, the KP equation

In this part we study the situation of a very weak Coriolis forcing. We derive and fully justify the KP equation. We show that if $\frac{\varepsilon}{\text{Ro}}$ is small enough, we can derive the KP equation

$$ \partial_\xi \left( \partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial^2_\xi k \right) + \frac{1}{2} \partial_{yy} k = 0. \quad (19) $$

Inspired by [10], we consider the following asymptotic regime

$$ A_{\text{KP}} = \left\{ (\mu, \varepsilon, \gamma, \text{Ro}) \mid 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \sqrt{\mu}, \frac{\varepsilon}{\text{Ro}} = \mu \right\}. $$

The Boussinesq-Coriolis equations become $(\gamma = \sqrt{\mu})$
\[
\left\{ \begin{array}{l}
\partial_t \zeta + \nabla^\gamma \cdot ([1 + \mu \zeta] \nabla) = 0, \\
\left( 1 - \frac{\mu}{3} \nabla^\gamma \right) \partial_t \nabla + \nabla^\gamma \zeta + \mu \nabla \cdot \nabla \nabla + \mu \nabla^\perp = 0.
\end{array} \right.
\] (20)

Proceeding as in the previous part, we denote \( \mathbf{V} = (u, v)^T \) and we seek an approximate solution \( (\zeta_{\text{app}}, u_{\text{app}}, v_{\text{app}}) \) of (20) in the form
\[
\zeta_{\text{app}}(t, x) = k(x - t, y, \mu t) + \mu \zeta_1(t, x, y, \mu t),
\]
\[
u_{\text{app}}(t, x) = k(x - t, y, \mu t) + \mu \nu_1(t, x, y, \mu t),
\]
\[
v_{\text{app}}(t, x) = \sqrt{\mu v_0(t, x, y, \mu t) + \mu v_1(t, x, y, \mu t)}.
\] (21)

Then, we plug the ansatz into System (20) and we get
\[
\left\{ \begin{array}{l}
\partial_t \zeta_{\text{app}} + \nabla^\gamma \cdot ([1 + \mu \zeta_{\text{app}}] \nabla_{\text{app}}) = \mu R_{1(1)} + \mu^2 \partial_y v_1 + \mu^2 R_1, \\
\left( 1 - \frac{\mu}{3} \nabla^\gamma \right) \partial_t \nabla_{\text{app}} + \nabla^\gamma \zeta_{\text{app}} + \mu \nabla_{\text{app}} \cdot \nabla^\gamma \nabla_{\text{app}} + \mu \nabla_{\text{app}}^\perp = \sqrt{\mu v_0^2} + \mu R_{1(1)} + \mu \tilde{R}_2.
\end{array} \right.
\]

where
\[
R_{1(1)} = \partial_t \zeta_1 + \partial_x u_1 + \partial_x k + 2k \partial_k k + \partial_y v_{1/2},
\]
\[
R_{1(2)} = \left( \partial_t v_{1/2} + \partial_k k \right) \quad \text{and} \quad R_{2(1)} = \left( \begin{array}{c}
\partial_t u_1 + \partial_x \zeta_1 + \partial_x k + \frac{1}{3} \partial_k^3 k + k \partial_k k \\\n\partial_t v_{1/2} + \partial_k k
\end{array} \right),
\]

and
\[
R_1 = \partial_x \zeta_1 + \partial_x (ku_1 + k \zeta_1 + \mu \zeta_1 u_1) + \partial_y ((k + \mu \zeta_1)(v_{1/2} + \mu v_1))
\]
\[
R_2 = \left( -(v_{1/2} + \sqrt{\mu v_1}) + \sqrt{\mu} \tilde{R}_{2,1}, R_{2,2} \right).
\]

with
\[
\tilde{R}_{2,1} = \partial_x u_1 - \frac{1}{3} \partial^2 \partial_x \partial_x k - \frac{1}{3} \partial^2 \partial_k u_1 - \mu \frac{1}{3} \partial^2 \partial_x (ku_1) + \partial_x (ku_1) + \mu u_1 \partial_x (u_1)
\]
\[
- \frac{1}{3} \partial^2 \partial^2 \partial_x (u_1) + \frac{1}{3} \partial^2 \partial_k (v_{1/2}) + \frac{1}{3} \partial^2 \partial^2 \partial_k (v_{1/2}) + \mu u_1 \partial_x (v_{1/2}) + \mu v_1 \partial_y (v_{1/2})
\]
\[
R_{2,2} = \partial_x v_{1/2} + \partial_x \zeta_1 + k \partial_x (v_{1/2}) + \sqrt{\mu v_1}) + u_1 + \frac{1}{3} \partial_y \partial^2 \partial_x (v_{1/2}) + \mu u_1 \partial_x (v_{1/2}) + \mu v_1 \partial_y (v_{1/2})
\]
\[
- \mu \frac{1}{3} \partial_x \partial^2 \partial_x (v_{1/2}) + \mu \partial_y \partial^2 \partial_x (v_{1/2}) + \mu \partial_y \partial^2 \partial_x (v_{1/2}) + \mu \partial_y \partial^2 \partial_x (v_{1/2})
\]
\[
+ \mu (v_{1/2} + \sqrt{\mu v_1}) \partial_y (v_{1/2}) + \sqrt{\mu} v_1.)
\]

Then, we choose \( (k, v_{1/2}, v_1) \) such that, for all \( (x, y) \in \mathbb{R}^2, t \in \left[ 0, \frac{T}{\mu} \right] \) and \( \tau \in [0, T] \),
\[
R_{1(1)}(t, x, y, \tau) = 0 \quad \text{and} \quad R_{1(2)}(t, x, y, \tau) = 0.
\]

First, we obtain that
Let

\[ v_{(1/2)} = \partial_x^{-1} \partial_y k + v_{(1/2)}^0 - \partial_x^{-1} \partial_y k^0, \]
\[ v_{(1)} = \partial_x^{-1} k + v_{(1)}^0 - \partial_x^{-1} k^0. \]

Then, denoting \( w_\pm = \zeta_{(1)} \pm u_{(1)} \), we get

\[
(\partial_t + \partial_x) w_+ = \left( 2 \partial_t k + 3 k \partial_t k + \frac{1}{3} \partial_x^3 k + \partial_x^{-1} \partial_y^2 k \right) (x-t, \tau) + F_0 = 0,
\]
\[
(\partial_t - \partial_x) w_- = \left( k \partial_t k - \frac{1}{3} \partial_x^3 k + \partial_x^{-1} \partial_y^2 k \right) (x-t, \tau) + F_0 = 0,
\]

where

\[ F_0 = \partial_y v_{(1/2)} - \partial_x^{-1} \partial_y^2 k^0. \]

Therefore, in order to avoid a linear growth (see Lemma 3.2), \( k \) must satisfies the KP equation (19). The following Lemma is a local wellposedness result for the KP equation (see Lemma 7.22 in [12] or [19] 2 21).

**Proposition 3.5.** Let \( N \geq 5 \) and \( k_0 \in \partial_x H^N(\mathbb{R}^2) \). Then, there exists a time \( T > 0 \) and a unique solution \( k \in C \left( [0, T]; \partial_x H^N(\mathbb{R}^2) \right) \) to the KP equation (19) and one has

\[
\left| \partial_x^{-1} k(t, \cdot) \right|_{H^N} \leq C \left( T, \left| \partial_x^{-1} k_0 \right|_{H^N} \right).
\]

Furthermore, if \( N \geq 6 \) and \( \partial_y^2 k_0 \in \partial_x^2 H^{N-4}(\mathbb{R}^2) \), then \( \partial_y^2 k \in C \left( [0, T]; \partial_x^2 H^{N-4}(\mathbb{R}^2) \right) \) and one has

\[
\left| \partial_y^2 k_0 \partial_x^{-2} k(t, \cdot) \right|_{H^{N-4}} \leq C \left( T, \left| \partial_x^{-2} \partial_y^2 k_0 \right|_{H^{N-4}}, \left| \partial_x^{-1} k_0 \right|_{H^N} \right).
\]

We can now establish a rigorous justification of the KP equation.

**Theorem 3.6.** Let \( k^0 \in \partial_x^2 H^{12}(\mathbb{R}^2) \) such that \( 1 + \varepsilon k^0 \geq \varepsilon \min > 0 \) and \( v_{(1/2)}^0 \in \partial_x H^8(\mathbb{R}^2) \), \( v_{(1)}^0 \in H^7(\mathbb{R}^2) \). Suppose that \((\mu, \varepsilon, \gamma, \text{Ro}) \in A_{\text{KP}}. \) Denote \( v^0 = \sqrt{\mu v_{(1/2)}^0} + \mu v_{(1)}^0 \). Then, there exists a time \( T_0 > 0 \), such that we have

(i) a unique classical solution \((\zeta_B, u_B, v_B)\) of (13) with initial data \((k^0, k^0, v^0)\) on \([0, T_0/\mu]\).

(ii) a unique classical solution \( k \) of (19) with initial data \( k^0 \) on \([0, T_0]\).

(iii) If we define \((\zeta_{KP}, u_{KP})\) \((t, x) = (k(x-t, y, \mu t), k(x-t, y, \mu t))\) we have the following error estimate for all \( 0 \leq t \leq T_0/\mu \),

\[
\left| (\zeta_B, u_B) - (\zeta_{KP}, u_{KP}) \right|_{L^\infty([0, t] \times \mathbb{R}^2)} \leq C \frac{\mu t}{1 + t} (1 + \sqrt{\mu t})
\]

where \( C = C \left( \varepsilon \min, \mu_0, \left| \partial_x^{-2} k^0 \right|_{H^{12}}, \left| \partial_x^{-1} v_{(1/2)}^0 \right|_{H^8}, \left| v_{(1)}^0 \right|_{H^7} \right). \)
Proof. The proof is very similar to the proof of Theorem 3.4.

Remark 3.7. Contrary to the justification of the KP equation in the irrotational setting (see Part 7.2 in [12] or [13]), the transverse part of the horizontal velocity \( v \) must contain an order \( O(\mu) \) contribution. Notice that if one considers a weaker Coriolis forcing, for instance \( \hat{\nu}_0 = \mu^{3/2} \), this assumption is no more necessary.

4 Which equation for which asymptotic regime?

4.1 The Ostrovsky and KdV equations

In Section 3 we derived two asymptotic models in the long wave regime \( (\varepsilon = \mu) \). First, if \( \gamma = \sqrt{\mu} \) and \( \hat{\nu}_0 = \sqrt{\mu} \), we derived the rotation-modified KP equation

\[
\partial_t k \left( \partial_x k + \frac{3}{2} k \partial_y k + \frac{1}{6} \partial^3_y k \right) + \frac{1}{2} \partial_{yy} k = \frac{1}{2} k.
\]

Then, if \( \gamma = \sqrt{\mu} \) and \( \hat{\nu}_0 = \mu \), we obtained the KP equation

\[
\partial_t k \left( \partial_x k + \frac{3}{2} k \partial_y k + \frac{1}{6} \partial^3_y k \right) + \frac{1}{2} \partial_{yy} k = 0.
\]

In [10], we performed a similar derivation in the long wave regime under the assumption that \( \gamma = \mathcal{O}(\mu^2) \). When \( \hat{\nu}_0 = \sqrt{\mu} \), we derived the Ostrovsky equation

\[
\partial_t k \left( \partial_x k + \frac{3}{2} k \partial_y k + \frac{1}{6} \partial^3_y k \right) = \frac{1}{2} k,
\]

and when \( \hat{\nu}_0 = \mu \), we derived the KdV equation

\[
\partial_t k + \frac{3}{2} k \partial_y k + \frac{1}{6} \partial^3_y k = 0.
\]

We would like to emphasize that we can weaken the assumption \( \gamma = \mathcal{O}(\mu^2) \) into \( \gamma = \mu \). In the following, we show this fact on the Ostrovsky equation. We consider the asymptotic regime

\[ A_{ostrov} = \{ (\mu, \varepsilon, \gamma, \text{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \mu, \frac{\varepsilon}{\text{Ro}} = \sqrt{\mu} \}. \]

Then we seek an approximate solution \((\zeta_{app}, u_{app}, v_{app})\) of the Boussinesq-Coriolis equations in the form

\[
\zeta_{app}(t, x, y) = k(x - t, y, \mu t) + \mu \zeta_{(1)}(t, x, y, \mu t),
\]

\[
u_{app}(t, x, y) = k(x - t, y, \mu t) + \mu v_{(1)}(t, x, y, \mu t),
\]

\[
u_{app}(t, x, y) = \sqrt{\mu} v_{(1/2)}(t, x, y, \mu t) + \mu v_{(1)}(t, x, y, \mu t)
\]

Plugging the ansatz into the Boussinesq-Coriolis equations, we obtain
\[
\begin{aligned}
\partial_t \zeta_{\text{app}} + \nabla^\gamma \cdot (1 + \mu \zeta_{\text{app}} \nabla \zeta_{\text{app}}) &= \mu R^1_{(1)} + \mu^2 R_1, \\
\left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \right) \partial_t \nabla \zeta_{\text{app}} + \nabla^\gamma \zeta_{\text{app}} + \mu \nabla \zeta_{\text{app}} \cdot \nabla \nabla \zeta_{\text{app}} + \sqrt{\mu} \nabla \zeta_{\text{app}} &= \sqrt{\mu} R^2_{(1/2)} + \mu R^2_{(1)} + \mu^2 R_2.
\end{aligned}
\]

where

\[
R^1_{(1)} = \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_x k + 2k \partial_k k,
\]

\[
R^2_{(1/2)} = \begin{pmatrix} 0 \\ \partial_t v_{(1/2)} + k \end{pmatrix}
\]

and \[R^2_{(1)} = \begin{pmatrix} \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_x k + \frac{k}{3} \partial_k^3 k + k \partial_k k - v_{(1/2)} \\ \partial_t v_{(1)} + \partial_y k \end{pmatrix},\]

and where \(R_1, R_2\) are remainders similar to the ones found in Sections 3.1 and 3.2. Then, using the same strategy than before, we impose that \(R^1_{(1)} = 0\) and \(R^2_{(1/2)} = R^2_{(1)} = 0\). We obtain

\[
v_{(1/2)} = \partial_t^{-1} k + v^0_{(1/2)} - \partial_t^{-1} k^0,
\]

\[
v_{(1)} = \partial_{-1}^{-1} \partial_y k + v^0_{(1)} - \partial_t^{-1} \partial_y k^0,
\]

and, denoting \(w_\pm = \zeta_{(1)} \pm u_{(1)}\), we get

\[
(\partial_t + \partial_x) w_+ + \left(2 \partial_t k + 3k \partial_k k + \frac{1}{3} \partial_k^3 k - \partial_t^{-1} k \right) (x - t, \tau) - F_0 = 0,
\]

\[
(\partial_t - \partial_x) w_- + \left(k \partial_t k - \frac{1}{3} \partial_k^3 k + \partial_t^{-1} k \right) (x - t, \tau) + F_0 = 0,
\]

where \(F_0 = v^0_{(1/2)} - \partial_t^{-1} k^0\). In order to avoid a linear growth (see Lemma 3.2), \(k\) must satisfies the Ostrovsky equation (22). Proceeding as in [16], we can generalize Theorem 3.9 in [16] to the asymptotic regime \(A_{\text{ostrov}}\). A solution of the Ostrovsky equation provides a \(O(\sqrt{\mu})\) approximation of the Boussinesq-Coriolis equations over a time \(O\left(\frac{1}{\mu}\right)\). We can proceed similarly for the KdV equation (23). Under the asymptotic regime

\[
A_{\text{KdV}} = \left\{ (\mu, \varepsilon, \gamma, \text{Ro}) \mid 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \mu, \frac{\varepsilon}{\text{Ro}} = \mu \right\}.
\]

and with the ansatz

\[
\zeta_{\text{app}}(t, x, y) = k(x - t, y, \mu t) + \mu \zeta_{(1)}(t, x, y, \mu t),
\]

\[
u_{\text{app}}(t, x, y) = k(x - t, y, \mu t) + \mu \zeta_{(1)}(t, x, y, \mu t),
\]

\[
v_{\text{app}}(t, x, y) = \mu v_{(1)}(t, x, y, \mu t),
\]

we can generalize Theorem 3.12 in [16] to the asymptotic regime \(A_{\text{KdV}}\). A solution of the KdV equation provides a \(O(\mu)\) approximation of the Boussinesq-Coriolis equations over a time \(O\left(\frac{1}{\mu}\right)\).
4.2 Conclusion

We summarize Section 3 and Subsection 4.1 by the following table. Notice that all of these models provide a $O(\sqrt{\mu})$ approximation (at least) in the long wave regime ($\varepsilon = \mu$) of the Boussinesq-Coriolis equations over a time $O\left(\frac{1}{\mu}\right)$.

| $\gamma$ | $\sqrt{\mu}$ | $\mu$ |
|----------|---------------|-------|
| $\sqrt{\mu}$ | Rotation-modified KP equation | KP equation |
| $\mu$ | Ostrovsky equation | KdV equation |

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