MOMENT ESTIMATIONS OF NEW
SZÁSZ-MIRAKYAN-DURRMEYER OPERATORS

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Abstract. In [10] Jain introduced the modified form of the Szász-Mirakjan operator, based on certain parameter $0 \leq \beta < 1$. Several modifications of the operators proposed and available in the literature. Here we consider actual Durrmeyer variants of the operators due to Jain. It is observed here that the Durrmeyer variant have nice properties and one need not to take any restriction on $\beta$ in order to obtain convergence. We establish moments using the Tricomi’s confluent hypergeometric function and Stirling numbers of first kind, and also estimate some direct results.

Key words Szász-Mirakjan operator, confluent hypergeometric function, Stirling numbers, direct results, modulus of continuity.

1. Introduction

The well known Szász-Mirakyan operators are defined as

$$B_n^0(f, x) = \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \quad x \in [0, \infty).$$

In order to generalize the Szász-Mirakyan operators Jain, [10], introduced the following operators

$$B_n^\beta(f, x) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)} f \left( \frac{k}{n} \right), \quad x \in [0, \infty)$$

(1)

where $0 \leq \beta < 1$ and the basis function is defined as

$$L_{n,k}^{(\beta)}(x) = \frac{nx(nx + k\beta)^{k-1}}{k!} e^{-(nx+k\beta)}$$

where it is seen that $\sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) = 1$. As a special case when $\beta = 0$, the operators (1) reduce to the Szász-Mirakyan operators. Umar and Razi, [12], used a Kantorovich type modification of $L_{n,k}^{(\beta)}(x)$ in order to approximate integrable functions, where some direct estimates were considered. Recently Farcaș, [7], studied the operators (1) and estimated a Voronovskaja type asymptotic formula. While a review of Farcaș’ work was undertaken it
was found that minor errors were given in Lemma 2.1. These errors have been corrected and are given in Lemma 1.

In 1967 Durrmeyer, [6], introduced the integral modification of the well known Bernstein polynomials, which were later studied by Derriennic [3], Gonska-Zhou [4], [5] and Agrawal-Gupta [1]. The Durrmeyer type modification of the operators (1), with different weight functions, have been proposed by Tarabie [11] and Gupta et al [9]. In approximations by linear positive operators, moment estimations play an important role. So far no standard Durrmeyer type modification of the operators (1) have been discussed due to its complicated form in finding moments and this problem has not been discussed in the last four decades. Here we overcome this difficulty and we consider the following Durrmeyer variant of the operators (1) in the form

\[ D^\beta_{\mathbf{n}}(f, x) = \sum_{k=0}^{\infty} \left( \int_{0}^{\infty} L_{n,k}^{(\beta)}(t) \, dt \right)^{-1} L_{n,k}^{(\beta)}(x) \int_{0}^{\infty} L_{n,k}^{(\beta)}(t) f(t) \, dt \]

\[ = \sum_{k=0}^{\infty} \frac{\langle L_{n,k}^{(\beta)}(t), f(t) \rangle}{\langle L_{n,k}^{(\beta)}(t), 1 \rangle} L_{n,k}^{(\beta)}(x) \]

where \( \langle f, g \rangle = \int_{0}^{\infty} f(t) g(t) \, dt \). For the special case of \( \beta = 0 \) these operators reduce to the Szász-Mirakyan-Durrmeyer operators (see [8] and references therein). It has been observed that these operators have interesting convergence properties. In the original form of the operators (1) and its other integral modifications, one has to consider the restriction that \( \beta \to 0 \) as \( n \to \infty \), in order to obtain convergence. For these actual Durrmeyer variants, (2), we need not to take any restrictions on \( \beta \). Because of this beautiful property it is of worth to study these operators. Here we find moments using Stirling numbers of first kind and confluent hypergeometric function and estimate some basic direct results.

2. Moments

Lemma 1. [10], [7] For the operators defined by (1) the moments are as follows:

\[ B^\beta_{\mathbf{n}}(1, x) = 1, \quad B^\beta_{\mathbf{n}}(t, x) = \frac{x}{1 - \beta} \]

\[ B^\beta_{\mathbf{n}}(t^2, x) = \frac{x^2}{(1 - \beta)^2} + \frac{x}{n(1 - \beta)^3}, \]

\[ B^\beta_{\mathbf{n}}(t^3, x) = \frac{x^3}{(1 - \beta)^3} + \frac{3x^2}{n(1 - \beta)^4} + \frac{(1 + 2\beta)x}{n^2(1 - \beta)^5} \]

\[ B^\beta_{\mathbf{n}}(t^4, x) = \frac{x^4}{(1 - \beta)^4} + \frac{6x^3}{n(1 - \beta)^5} + \frac{(7 + 8\beta)x^2}{n^2(1 - \beta)^6} + \frac{(6\beta^2 + 8\beta + 1)x}{n^3(1 - \beta)^7} \]

\[ B^\beta_{\mathbf{n}}(t^5, x) = \frac{x^5}{(1 - \beta)^5} + \frac{10x^4}{n(1 - \beta)^6} + \frac{5(4\beta + 5)x^3}{n^2(1 - \beta)^7} + \frac{15(2\beta^2 + 4\beta + 1)x^2}{n^3(1 - \beta)^8} + \frac{(24\beta^3 + 58\beta^2 + 22\beta + 1)x}{n^4(1 - \beta)^9} \]
Lemma 2. For $0 \leq \beta < 1$, we have

$$< L_{n,k}^{(\beta)}(t), t^r > = P_r(k; \beta)$$

(4)

where $< f, g > = \int_0^\infty f(t)g(t)dt$ and $P_r(k; \beta)$ is a polynomial of order $r$ in the variable $k$. In particular

$$P_0(k; \beta) = 1$$
$$P_1(k; \beta) = \frac{1}{n} \left[ (1 - \beta)k + \frac{1}{1 - \beta} \right]$$
$$P_2(k; \beta) = \frac{1}{n^2} \left[ (1 - \beta)^2 k^2 + 3(1 - \beta)k + 2! \frac{1}{1 - \beta} \right]$$
$$P_3(k; \beta) = \frac{1}{n^3} \left[ (1 - \beta)^3 k^3 + 6(1 - \beta)k^2 + \frac{3!}{1 - \beta} \right]$$
$$P_4(k; \beta) = \frac{1}{n^4} \left[ (1 - \beta)^4 k^4 + 10(1 - \beta)^2 k^3 + 5(7 - 4\beta)k^2 + \frac{4!}{1 - \beta} \right]$$
$$P_5(k; \beta) = \frac{1}{n^5} \left[ (1 - \beta)^5 k^5 + 15(1 - \beta)^3 k^4 + 5(1 - \beta)(17 - 8\beta)k^3 + \frac{5!}{1 - \beta} \right]$$

Proof. First, we consider the integral:

$$< L_{n,k}^{(\beta)}(t), t^r > = \int_0^\infty L_{n,k}^{(\beta)}(t) t^r dt$$
$$= \frac{n}{k!} \int_0^\infty e^{-(nt+k\beta)} t^{r+1} (nt + k\beta)^{k-1} dt$$

We use Tricomi’s confluent hypergeometric function:

$$U(a, b, c) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} (1 + t)^{b-a-1}, a > 0, c > 0$$

we have

$$< L_{n,k}^{(\beta)}(t), t^r > = \frac{n}{k!} \int_0^\infty e^{-(nt+k\beta)} t^{r+1} (nt + k\beta)^{k-1} dt$$
$$= \frac{1}{k!} \int_0^\infty (x + k\beta)^{k-1} e^{-(x+k\beta)} \left( \frac{x}{n} \right)^{r+1} dx$$
$$= \frac{(k\beta)^{k+r+1}}{k!n^{r+1}} \int_0^\infty e^{-k\beta t} (1 + t)^{k-1} t^{r+1} dt$$
$$= \frac{(k\beta)^{k+r+1}}{k!n^{r+1}} e^{-k\beta (r+1)} U(r + 2, k + r + 2, k\beta).$$

(6)
The evaluation of \( \langle L_{n,k}^{(\beta)}(t), t^r \rangle \) can also be seen in the form

\[
\langle L_{n,k}^{(\beta)}(t), t^r \rangle = \frac{(r + 1)!}{kn^{r+1}} e^{-k\beta} \sum_{s=0}^{k-1} \frac{f^{k+r-s}}{s!} \left( \frac{k}{r+1} \right) \frac{(k\beta)^s}{s!}
\]

where \( x = \beta k \) and \( \phi_r(s) \) is given by

\[
\phi_r(s) = (k - s)^{r+1} \sum_{j=0}^{r+1} s(r+1, r-j+1)(k-s)^{r-j+1}
\]

where \( s(n,k) \) are the Stirling numbers of the first kind. The first few may be written as

\[
\begin{align*}
\phi_0 &= k - s \\
\phi_1 &= (k-s)^2 + (k-s) \\
\phi_2 &= (k-s)^3 + 3(k-s)^2 + 2(k-s) \\
\phi_3 &= (k-s)^4 + 6(k-s)^3 + 11(k-s)^2 + 6(k-s)
\end{align*}
\]

It can now be determined that

\[
\langle L_{n,k}^{(\beta)}(t), t^r \rangle = \frac{e^{-x}}{kn^{r+1}} \sum_{j=0}^{r+1} s(r+1, r-j+1)\theta_{r-j}(x)
\]

where

\[
\theta_m(x) = \sum_{s=0}^{k-1} (k-s)^{m+1} \frac{x^s}{s!}
\]

For the case of \( r = 0 \), (8) becomes

\[
\langle L_{n,k}^{(\beta)}(t), 1 \rangle = \frac{e^{-x}}{kn} \theta_0(x) = \frac{e^{-x}}{kn} \sum_{s=0}^{k-1} (k-s) \frac{x^s}{s!}
\]

and for the case \( r = 1 \),

\[
\langle L_{n,k}^{(\beta)}(t), t \rangle = \frac{e^{-x}}{kn^2} \theta_1(x) + \frac{1}{n} \langle L_{n,k}^{(\beta)}(t), 1 \rangle.
\]

Dividing both sides by \( \langle L_{n,k}^{(\beta)}(t), 1 \rangle \) leads to the expression

\[
P_1(k; \beta) = \frac{1}{n} \left( 1 + \frac{\theta_1(x)}{\theta_0(x)} \right) = \frac{1}{n} \left( 1 + S_1(x) \right)
\]

where \( S_r(x) \) is defined by

\[
S_r(x) = \frac{\theta_r(x)}{\theta_0(x)} = \frac{\sum_{s=0}^{k-1} (k-s)^{r+1} \frac{x^s}{s!}}{\sum_{s=0}^{k-1} (k-s)^{r+1} \frac{x^s}{s!}}.
\]
The general form of $P_r(k; \beta)$ is given by

$$P_r(k; \beta) = \frac{1}{n^r} \sum_{j=0}^{r+1} s(r+1,j)S_{j-1}(x). \tag{10}$$

What remains is to obtain calculations for $S_r(x)$. From (2) it is seen that $S_0(x) = 1$ and

$$S_1(x) = k - \left( \frac{k-1}{k} \right) x + \left( \frac{x}{k} \right)^2 + \left( \frac{x}{k} \right)^3 + \left( \frac{x}{k} \right)^4 + \cdots = k - x + \frac{x}{k - x},$$

$$S_2(x) = x^2 - (2k - 3)x + k^2 - \frac{x}{k - x}, \tag{11}$$

$$S_3(x) = k^3 - 3k - 1 - (3k^2 - 6k + 7)x + 3(k - 2)x^2 - x^3 + \frac{k(3k + 1)}{k - x},$$

$$S_4(x) = k^4 - (4k^3 - 10k^2 + 10k - 15)x + (6k^2 - 20k + 25)x^2 - 2(2k - 5)x^3 + x^4 - \frac{(10k + 1)x}{k - x}.$$  

Since $x = \beta k$ then the first few $S_r(\beta k)$ are seen to be

$$S_1(\beta k) = (1 - \beta)k + \frac{\beta}{1 - \beta},$$

$$S_2(\beta k) = (1 - \beta)^2k^2 + 3\beta k - \frac{\beta}{1 - \beta}, \tag{12}$$

$$S_3(\beta k) = (1 - \beta)^3k^3 + 6\beta(1 - \beta)k^2 + \frac{\beta(7\beta - 4)k}{1 - \beta} + \frac{\beta}{1 - \beta},$$

$$S_4(\beta k) = (1 - \beta)^4k^4 + 10\beta(1 - \beta)^2k^3 + 5\beta(5\beta - 2)k^2 + \frac{5\beta(1 - 3\beta)k}{1 - \beta} - \frac{\beta}{1 - \beta}$$

which are polynomials of order $r$ in the variable $k$. Using the resulting expressions of $S_r(\beta k)$, provided in (12), in (10) lead to the $P_r(k; \beta)$ polynomials of (5). It is now sufficient to conclude that

$$\frac{< L^{(\beta)}_{n,k}(t), t^r >}{< L^{(\beta)}_{n,k}(t), 1 >} = P_r(k; \beta)$$

are polynomials of order $r$ in the variable $k$. $\square$

**Lemma 3.** For $0 \leq \beta < 1$, $r \geq 0$, the polynomials $P_r(k; \beta)$ satisfy the recurrence relationship

$$n^2 P_{r+2}(k; \beta) = n[(1 - \beta)k + r + 2]P_{r+1}(k; \beta) + (r + 2)\beta k P_r(k; \beta). \tag{13}$$

**Proof.** By utilizing the recurrence relation,

$$U(a, b; z) = (a + 1)zU(a + 2, b + 2; z) + (z - b)U(a + 1, b + 1; z),$$

for the Tricomi confluent hypergeometric functions, (6) becomes

$$n^2 < L^{\beta}_{n,k}(t), t^{r+1} > = n[(1 - \beta)k + r + 1] < L^{\beta}_{n,k}(t), t^r > + (r + 1)\beta k < L^{\beta}_{n,k}(t), t^{r-1} >.$$
Now dividing by $<L_{n,k}^{(\beta)}(t), 1>$ leads to the desired relationship for the polynomials $P_r(k; \beta)$ given by (13).

\[ \square \]

**Lemma 4.** If the $r$-th order moment with monomials $e_r(t) = t^r$, $r = 0, 1, \cdots$ of the operators (2) be defined as

\[ T_{n,r}^{(\beta)}(x) : D_n^{(\beta)}(e_r, x) = \sum_{k=0}^{\infty} \left( \int_0^\infty L_{n,k}^{(\beta)}(t) \, dt \right)^{-1} L_{n,k}^{(\beta)}(x) \int_0^\infty L_{n,k}^{(\beta)}(t)t^r \, dt \]

or

\[ T_{n,r}^{(\beta)}(x) = \sum_{k=0}^{\infty} P_r(k; \beta) L_{n,k}^{(\beta)}(x). \] (14)

The first few are:

\[ T_{n,0}^{(\beta)}(x) = 1, \quad T_{n,1}^{(\beta)}(x) = x + \frac{1}{n(1 - \beta)}; \]
\[ T_{n,2}^{(\beta)}(x) = x^2 + \frac{4x}{n(1 - \beta)} + \frac{2!}{n^2(1 - \beta)}; \]
\[ T_{n,3}^{(\beta)}(x) = x^3 + \frac{9x^2}{n(1 - \beta)} + \frac{6(3 - \beta)x}{n^2(1 - \beta)^2} + \frac{3!}{n^3(1 - \beta)}; \]
\[ T_{n,4}^{(\beta)}(x) = x^4 + \frac{16x^3}{n(1 - \beta)} + \frac{12(6 - \beta)x^2}{n^2(1 - \beta)^2} + \frac{12(3\beta^2 - 6\beta + 8)x}{n^3(1 - \beta)^3} + \frac{4!}{n^4(1 - \beta)}. \] (15)

\[ T_{n,5}^{(\beta)}(x) = x^5 + \frac{25x^4}{n(1 - \beta)} + \frac{20(10 - \beta)x^3}{n^2(1 - \beta)^2} + \frac{120(\beta^2 - 2\beta + 5)x^2}{n^3(1 - \beta)^3} + \frac{120(5 - 6\beta + 6\beta^2 - \beta^3)x}{n^4(1 - \beta)^4} + \frac{5!}{n^5(1 - \beta)}. \]

**Proof.** Obviously by (2), we have $T_{n,0}^{(\beta)}(x) = 1$. Next by definition of $T_{n,r}^{(\beta)}(x)$, we have

\[ T_{n,r}^{(\beta)}(x) = \sum_{k=0}^{\infty} <L_{n,k}^{(\beta)}(t), t^r> L_{n,k}^{(\beta)}(x) = \sum_{k=0}^{\infty} P_r(k; \beta) L_{n,k}^{(\beta)}(x). \]

Using Lemma 1 and Lemma 2, we have

\[ \begin{align*}
T_{n,1}^{(\beta)}(x) &= \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x)P_1(k; \beta) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) \frac{1}{n} \left[ (1 - \beta)k + \frac{1}{1 - \beta} \right] \\
&= (1 - \beta)B_{n}^{(\beta)}(t, x) + \frac{1}{n(1 - \beta)} B_{n}^{(\beta)}(1, x) \\
&= x + \frac{1}{n(1 - \beta)}. 
\end{align*} \]
Lemma 5. For \( T \) a continuation of this process will provide where the first few coefficients \( A \)

\[
\begin{align*}
T^\beta_{n,2}(x) &= \sum_{k=0}^{\infty} L^{(\beta)}_{n,k}(x)P_2(k; \beta) = \sum_{k=0}^{\infty} L^{(\beta)}_{n,k}(x) \frac{1}{n^2} \left[ (1 - \beta)^2 k^2 + 3k + \frac{2}{1 - \beta} \right] \\
&= (1 - \beta)^2 B^\beta_n(t^2, x) + \frac{3}{n} B^\beta_n(t, x) + \frac{2}{n^2(1 - \beta)} \\
&= x^2 + \frac{4x}{n(1 - \beta)} + \frac{2}{n^2(1 - \beta)}. \\
T^\beta_{n,3}(x) &= \sum_{k=0}^{\infty} L^{(\beta)}_{n,k}(x)P_3(k; \beta) \\
&= \sum_{k=0}^{\infty} L^{(\beta)}_{n,k}(x) \frac{1}{n^3} \left[ (1 - \beta)^3 k^3 + 6(1 - \beta)k^2 + \frac{11 - 8\beta}{1 - \beta} k + \frac{3!}{1 - \beta} \right] \\
&= (1 - \beta)^3 B^\beta_n(t^3, x) + \frac{6(1 - \beta)}{n} B^\beta_n(t^2, x) \\
&\quad + \frac{(11 - 8\beta)}{n^2(1 - \beta)} B^\beta_n(t, x) + \frac{3!}{n^3(1 - \beta)} B^\beta_n(1, x) \\
&= x^3 + \frac{9x^2}{n(1 - \beta)} + \frac{6(3 - \beta) x}{n^2(1 - \beta)^2} + \frac{3!}{n^3(1 - \beta)}.
\end{align*}
\]

A continuation of this process will provide \( T^\beta_{n,r}(x) \) for cases of \( r \geq 4 \). \( \square \)

Lemma 5. For \( r \geq 1 \) the polynomials \( T^\beta_{n,r}(x) \) satisfy the relation

\[
T^\beta_{n,r}(x) = \left(x + \frac{2r - 1}{n(1 - \beta)}\right) T^\beta_{n,r-1}(x) - \sum_{j=0}^{r-2} \frac{(-1)^j A^r_{j-2}}{n^{j+2}(1 - \beta)^{j+2}} T^\beta_{n,r-j-2}(x) \tag{16}
\]

where the first few coefficients \( A^r_j \) are given by

\[
A^0_0 = 1 + 2\beta \\
A^0_1 = 4 + 4\beta \\
A^0_2 = 9 + 6\beta \\
A^1_0 = 16 + 8\beta \\
A^2_0 = -12\beta^2 + 48\beta^3 + 120\beta^4
\]

\[
A^1_1 = 2\beta + 6\beta^2 \\
A^1_2 = 6\beta + 30\beta^2 \\
A^2_1 = 12\beta^2 + 24\beta^3 \\
A^2_2 = 60\beta^2 + 96\beta^3
\]

Proof. Making use of (13) and (14) leads to the relation

\[
n^2 T^\beta_{n,r}(x) - n(r + 1) T^\beta_{n,r}(x) = \sum_{k=0}^{\infty} k \left[ n(1 - \beta) P_r(k; \beta) + (r + 1)\beta P_{r-1}(k; \beta) \right] L^{(\beta)}_{n,k}(x).
\]

Now making use of (3) the summation can be reformed into the desired relation. This can be verified by considering \( T^\beta_{n,r}(x) \) as a linear combination of \( T^\beta_{n,j}(x) \) for \( 0 \leq j \leq r - 1 \). \( \square \)
Remark 1. If we denote the central moment as \( \mu^3_{n,r}(x) = D_n^3((t - x)^r, x) \), then
\[
\begin{align*}
\mu^3_{n,1}(x) &= \frac{1}{n(1 - \beta)}, & \mu^3_{n,2}(x) &= \frac{2x}{n(1 - \beta)} + \frac{2!}{n^2(1 - \beta)}, \\
\mu^3_{n,3}(x) &= \frac{12x}{n^2(1 - \beta)^2} + \frac{3!}{n^3(1 - \beta)}, \\
\mu^3_{n,4}(x) &= \frac{12x^2}{n^2(1 - \beta)^2} + \frac{12(6 - 2\beta + \beta^2)x}{n^3(1 - \beta)^2} + \frac{4!}{n^4(1 - \beta)}.
\end{align*}
\]

In general using the similar approach, one can show that:
\[
\mu^3_{n,r}(x) = O \left(n^{-[(r+1)/2]}\right),
\]
where \([\alpha]\) denotes the integral part of \(\alpha\).

3. Direct Estimates

In this section, we establish the following direct result:

**Proposition 1.** Let \( f \) be a continuous function on \([0, \infty)\) for \( n \to \infty \), the sequence \( \{D_n^2(f, x)\} \) converges uniformly to \( f(x) \) in \([a, b] \subset [0, \infty)\).

**Proof.** For sufficiently large \( n \), it is obvious from Lemma 4 that \( D_n^2(e_0, x) \), \( D_n^2(e_1, x) \) and \( D_n^2(e_2, x) \) converges uniformly to 1, \( x \) and \( x^2 \) respectively on every compact subset of \([0, \infty)\). Thus the required result follows from Bohman-Korovkin theorem.

**Theorem 1.** Let \( f \) be a bounded integrable function on \([0, \infty)\) and has second derivative at a point \( x \in [0, \infty) \), then
\[
\lim_{n \to \infty} n[D_n^2(f, x) - f(x)] = \frac{1}{1 - \beta}f'(x) + \frac{x}{1 - \beta}f''(x).
\]

**Proof.** By the Taylor’s expansion of \( f \), we have
\[
f(t) = f(x) + f'(x)(t - x) + \frac{1}{2}f''(x)(t - x)^2 + r(t, x)(t - x)^2,
\]
where \( r(t, x) \) is the remainder term and \( \lim_{n \to \infty} r(t, x) = 0 \). Operating \( D_n^2 \) to the equation (18), we obtain
\[
D_n^2(f, x) - f(x) = D_n^2(t - x, x)f'(x) + D_n^2((t - x)^2, x) \frac{f''(x)}{2} + D_n^2(r(t, x)(t - x)^2, x)
\]
Using Cauchy-Schwarz inequality, we have
\[
D_n^2(r(t, x)(t - x)^2, x) \leq \sqrt{D_n^2(r^2(t, x), x)} \sqrt{D_n^2((t - x)^4, x)}.
\]
As \( r^2(x, x) = 0 \) and \( r^2(t, x) \in C_2^*[0, \infty) \), we have
\[
\lim_{n \to \infty} D_n^2(r^2(t, x), x) = r^2(x, x) = 0
\]
uniformly with respect to $x \in [0, A]$. Now from (19), (20) and from Remark 1, we get
\[
\lim_{n \to \infty} nD_n^\beta (r (t, x) (t - x)^2, x) = 0.
\]

Thus
\[
\lim_{n \to \infty} n \left( D_n^\beta (f, x) - f(x) \right) = \lim_{n \to \infty} n \left[ D_n^\beta (t - x, x) f'(x) + \frac{1}{2} f''(x) D_n^\beta ((t - x)^2, x) 
\right.
\]
\[
+ \left. D_n^\beta (r (t, x) (t - x)^2, x) \right] = \frac{1}{1 - \beta} f'(x) + \frac{x}{1 - \beta} f''(x).
\]

□

By $C_B[0, \infty)$, we denote the class on real valued continuous bounded functions $f(x)$ for $x \in [0, \infty)$ with the norm $||f|| = \sup_{x \in [0, \infty)} |f(x)|$. For $f \in C_B[0, \infty)$ and $\delta > 0$ the $m$-th order modulus of continuity is defined as
\[
\omega_m (f, \delta) = \sup_{0 \leq h \leq \delta} \sup_{x \in [0, \infty)} |\Delta^m_h f(x)|,
\]
where $\Delta$ is the forward difference. In case $m = 1$ we mean the usual modulus of continuity denoted by $\omega(f, \delta)$. The Peetre’s $K$-functional is defined as
\[
K_2 (f, \delta) = \inf_{g \in C_B^2[0, \infty)} \{ ||f - g|| + \delta ||g''|| : g \in C_B^2[0, \infty) \},
\]
where
\[
C_B^2[0, \infty) = \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \}.
\]

Theorem 2. Let $f \in C_B[0, \infty)$ and $\beta > 0$, then
\[
|D_n^\beta (f, x) - f(x)| \leq C \omega_2 \left( f, \sqrt{\frac{2x}{n(1 - \beta)} + \frac{2}{n^2(1 - \beta)} + \frac{1}{n^2(1 - \beta)^2}} \right)
\]
\[
+ \omega \left( f, \frac{1}{n(1 - \beta)} \right)
\]
where $C$ is a positive constant.

Proof. We introduce the auxiliary operators $\bar{D}_n^\beta : C_B[0, \infty) \to C_B[0, \infty)$ as follows
\[
\bar{D}_n^\beta (f, x) = D_n^\beta (f, x) - f \left( x + \frac{1}{n(1 - \beta)} \right) + f(x), \tag{21}
\]
These operators are linear and preserves the linear functions in view of Lemma 4. Let $g \in C_B^2[0, \infty)$ and $x, t \in [0, \infty)$. By Taylor’s expansion
\[
g(t) = g(x) + (t - x) g'(x) + \int_x^t (t - u) g''(u) du,
\]
we have

\[ |\tilde{D}_n^\beta(g, x) - g(x)| \leq \tilde{D}_n^\beta \left( \left| \int_x^t (t-u)g''(u)du \right|, x \right) \]

\[ \leq D_n^\beta \left( \left| \int_x^t (t-u)g''(u)du \right|, x \right) \]

\[ + \left| \int_x^{x + \frac{1}{n(1-\beta)}} \left( x + \frac{1}{n(1-\beta)} - u \right)g''(u)du \right| \]

\[ \leq D_n^\beta ((t-x)^2, x) \|g''\| + \left| \int_x^{x + \frac{1}{n(1-\beta)}} \left( \frac{1}{n(1-\beta)} \right) du \|g''\| \right| \]

Next, using Remark 1, we have

\[ |\tilde{D}_n^\beta(g, x) - g(x)| \leq \left[ D_n^\beta((t-x)^2, x) + \left( \frac{1}{n(1-\beta)} \right)^2 \right] \|g''\| \]

\[ \leq \left[ D_n^\beta((t-x)^2, x) + \left( \frac{1}{n(1-\beta)} \right)^2 \right] \|g''\| \]

\[ \leq \left[ \frac{2x}{n(1-\beta)} + \frac{2}{n^2(1-\beta)} + \frac{1}{n^2(1-\beta)^2} \right] \|g''\| = \delta_n \|g''\|. \quad (22) \]

Since

\[ |D_n^\beta(f, x)| \leq \sum_{k=0}^{\infty} \left( \int_0^\infty L_{n,k}^{(\beta)}(t) dt \right)^{-1} L_{n,k}^{(\beta)}(x) \int_0^\infty L_{n,k}^{(\beta)}(t)|f(t)| dt \leq \|f\|. \]

Now by (19), we have

\[ \|\|\tilde{D}_n^\beta(f, x)\|\| \leq \|D_n^\beta(f, x)\| + 2\|f\| \leq 3\|f\|, \quad f \in C_B[0, \infty). \quad (23) \]

Using (21), (22) and (23), we have

\[ |D_n^\beta(f, x) - f(x)| \leq |\tilde{D}_n^\beta(f - g, x) - (f - g)(x)| + |\tilde{D}_n^\beta(g, x) - g(x)| \]

\[ + \left| f \left( x + \frac{1}{n(1-\beta)} \right) - f(x) \right| \]

\[ \leq 4\|f - g\| + \delta_n \|g''\| + \left| f \left( x + \frac{1}{n(1-\beta)} \right) - f(x) \right| \]

\[ \leq C \{\|f - g\| + \delta_n \|g''\|\} + \omega \left( \frac{1}{n(1-\beta)} \right). \]

Taking infimum over all \( g \in C_B^2[0, \infty) \), and using the inequality \( K_2(f, \delta) \leq C\omega_2(f, \sqrt{\delta}), \delta > 0 \) due to [2], we get the desired assertion. \( \square \)

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