AN \(E_\infty\)-EXTENSION OF THE ASSOCIAHEDRA AND THE TAMARKIN CELL MYSTERY

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Abstract. In this note based on the author’s communication with M. Batanin, we study a cofibrant \(E_\infty\)-operad generated by the Fox-Neuwirth cells of the configuration space of points in the Euclidean space. We show that, below the ‘critical dimensions’ in which ‘bad cells’ exist, this operad is modeled by the geometry of the Fulton-MacPherson compactification of this configuration space. We analyze the Tamarkin bad cell and calculate the differential of the corresponding generator. We also describe a simpler, four-dimensional bad cell. We finish the paper by proving an auxiliary result giving a characterization, over integers, of free Lie algebras.

Conventions. All algebraic objects will be considered over the ring of integers \(\mathbb{Z}\). Terminology regarding operads and related constructions follows [30]. With few exceptions, all operads in this paper live in the monoidal category \(\text{Chain}\) of non-negatively graded chain complexes of abelian groups. If we consider cofibrant operads, we refer to the model structure of the category of operads in \(\text{Chain}\) considered in [8, Example 3.3.3]. The central operad of this paper, \(J = (J, \partial)\), will in fact be special cofibrant in a more specific sense of [26, page 143].

Motivations and historical overview. Cofibrant resolutions of the operad \(\text{Com}\) for commutative associative algebras play a key rôle in (co)homology theory of commutative or \(E_\infty\)-algebras, applications to the iterated bar construction, infinite loop spaces, and many other areas of homological algebra and topology. Surprisingly, not many explicit and combinatorially accessible cofibrant resolutions are known.

- There is the simplicial Barratt-Eccles operad \([1]\) whose chain version \(\mathcal{E}\) was introduced in \([7]\). The surjection operad \(\mathcal{X}\) was described as a quotient of \(\mathcal{E}\) in \([8]\); it appeared independently in \([27]\).
- One also has the Eilenberg-Zilber operad \(\mathcal{Z}\) of all natural operations \(N_\ast(S) \to N_\ast(S)^{\otimes n}\) on the normalized chain complex \(N_\ast(S)\) of a simplicial set \(S\), see \([19, 33]\). As proved in \([1]\), these three operads are related by the maps

\[
\mathcal{E} \xrightarrow{\text{TR}} \mathcal{X} \xrightarrow{\text{AW}} \mathcal{Z},
\]

where \(\text{TR}\) is the table reduction map and \(\text{AW}\) abbreviates Alexander-Whitney.

Unfortunately, none of the operads in \([1]\) is cofibrant. But, at least, the components of the operads \(\mathcal{E}(n)\) and \(\mathcal{X}(n)\) are free modules over the symmetric group \(\Sigma_n\), so one can get cofibrant resolutions \(W(\mathcal{E})\) and \(W(\mathcal{X})\) of \(\text{Com}\) by applying the cellular version \(W(\text{--})\) of Boardman-Vogt’s \(W\)-construction \([10]\). The \(W\)-construction in fact coincides with the operadic double bar construction and substantially inflates the size of the operad to which it is applied.

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Another chain operads resolving $\mathcal{C}om$ are the chain condensation of the lattice path operad $|\mathcal{L}|$ and the normalization $N(\mathcal{K})$ of the complete graph operad $\mathcal{K}$. The surjection operad $\mathcal{X}$ is a suboperad of $|\mathcal{L}|$ via the whiskering map, and C. Berger informed us that there is a simple zigzag between $|\mathcal{L}|$ and $\mathcal{K}$. As in the previous item, the operads $|\mathcal{L}|$ and $\mathcal{K}$ are only $\Sigma$-free, not cofibrant, so the $W(-)$-functor must be applied to obtain cofibrant resolutions.

In [16], the existence of cofibrant models of $E_n$-operads of the form $B^c(s^{-n}E_n^\vee)$, where $B^c(-)$ is the cobar construction of a cooperad, $s^{-n}$ the $n$-fold operadic desuspension and $E_n^\vee$ is the dual cooperad of an explicit filtration layer of the Barratt-Eccles operad, was proved for each $n \geq 0$. The colimit $\text{colim}_n B^c(s^{-n}E_n^\vee)$ is a combinatorial cofibrant model for $\mathcal{C}om$. While its operad structure is explicit, there is no simple formula for the augmentation to $\mathcal{C}om$.

The sizes of the cofibrant resolutions $W(\mathcal{E})$, $W(\mathcal{X})$, $W(N(\mathcal{K}))$, $W(|\mathcal{L}|)$ and $\text{colim}_n B^c(s^{-n}E_n^\vee)$ of $\mathcal{C}om$ recalled above are huge. There has been, however, a long-standing candidate for a small cofibrant chain resolution proposed by E. Getzler and J.D.S. Jones in [18], whose space of generators is of the size of the Fox-Neuwirth cell decomposition of the configuration space. It unfortunately turned out that, due to the existence of ‘bad cells,’ the proposed combinatorial formula for the differential $\partial$ did not work above certain dimensions.

The purpose of this paper is two-fold. The first one is to show that the differential $\partial$ can be modified to a correct one that coincides with the one proposed in [18] below the dimensions of the bad cells. The second aim is to show that explicit calculations of the values of $\partial$ on some bad cells are possible. The methods and results of the paper are described in more detail in the introduction below where also the two main results of the paper, Theorems A and B, are formulated.

Our results imply that, in the applications mentioned in the first paragraph of this subsection, Getzler-Jones’ formula can be used in dimensions less than the dimension of the bad cells, and that some explicit results can be obtained also in dimensions containing bad cells. A closed formula for $\partial$ is still, however, a challenging open question. We would like to note that the initial pieces of the cellular Getzler-Jones operad have in fact already been calculated in the proceedings [23] of the Winter School ‘Geometry and Physics,’ Zdíkov, Bohemia, January 1993.

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References

Glossary

[July 31, 2021]
1. Introduction

We describe an operad $J$ that can be viewed as an $E_\infty$-analog of the minimal model $A_\infty$ of the operad $Ass$ for associative algebras, governing Stasheff’s $A_\infty$-algebras [24]. Our $J$ lives in the monoidal category of differential graded abelian groups. It is of the form $J = (\mathbb{F}(G), \partial)$ where

- $\mathbb{F}(G)$ is the free operad generated by the graded $\Sigma$-module $G = \{G_*(n)\}_{n \geq 2}$ specified in Definition 3, and

- the differential $\partial$ is the sum $\partial = \partial_{\text{lin}} + \partial_{\text{prt}}$ of the ‘linear’ part $\partial_{\text{lin}}$ induced from a differential (denoted by the same symbol) $\partial_{\text{lin}}$ on the $\Sigma$-module $G$ introduced in Definition 3, and the ‘perturbed’ part $\partial_{\text{prt}}$ that maps $G$ into the decomposables of the free operad $\mathbb{F}(G)$. Below the ‘critical dimension’ in which the ‘bad’ cells exist, explicitly specified in Definition 6, $J$ is determined by the cell structure of the configuration operad $\mathbb{F}$ induced from the Fox-Neuwirth decomposition of the configuration space. Moreover,

- the operad $J$ is equipped with a dg-operad homomorphism $\rho : J \to \operatorname{Com}$ that makes $J$ a cofibrant resolution of the operad $\operatorname{Com} = \mathcal{E}nd_Z$ for commutative associative algebras.

The $n$th piece $(G_*(n), \partial_{\text{lin}})$ of the generating $\Sigma$-module $G$ is, for each $n \geq 2$, the colimit of the (shifted) cellular chain complexes of the one-point compactifications of the configuration spaces $\text{Cnf}(\mathbb{R}^h, n)$ of $n$ distinct labeled points in the $h$-dimensional Euclidean space $\mathbb{R}^h$, with the Fox-Neuwirth cell structure. Alternative descriptions of the right $\Sigma_n$-dg-abelian group $(G_*(n), \partial_{\text{lin}})$ are given in Section 2. Summing up, we prove

Theorem A. There exist a cofibrant resolution $\rho : J = (\mathbb{F}(G), \partial) \to \operatorname{Com}$ such that the linear part $\partial_{\text{lin}}$ of the differential is as in Definition 3 and the restriction of $\partial$ to generators below the dimension of bad cells is determined by the cell structure of the configuration operad $\mathbb{F}$.

Theorem A is proved in Subsection 3.2. Since the image of the canonical embedding $K \hookrightarrow \mathbb{F}$ of the Stasheff’s associahedron $K$ into the configuration operad $\mathbb{F}$ does not contain bad cells (see Figure 3), the operad $J$ is an extension of the $A_\infty$-operad $\mathcal{A}_\infty$ of cellular chains of $K$. This explains the title of the paper.

Let $G_*(n) \subset G_*(n)$ be a graded abelian group\footnote{As in [30], underlining indicates the non-$\Sigma$ version of an object.} generating $G_*(n)$ as a free graded $\Sigma_n$-module – one such a specific generating space will be described on page 4. It follows from standard facts that $J(n)$ is the free $\Sigma_n$-module generated by the $n$th piece $\mathbb{F}(G)(n)$ of the free non-$\Sigma$ operad $\mathbb{F}(G)$ generated by $G$. In particular, $(J(n), \partial)$ is, for each $n \geq 1$, a $\Sigma_n$-free resolution of the trivial $\Sigma_n$-module $\mathbb{Z} = \operatorname{Com}(n)$.

Since $J$ is cofibrant, for an arbitrary dg $E_\infty$-operad $\mathcal{E}$ (as the Barratt-Eccles operad, surjection operad, Eilenberg-Zilber operad, see [4, 28], etc.), there exist an operadic morphism $J \to \mathcal{E}$ that lifts the identity endomorphism of the operad $\operatorname{Com}$. If the ground ring is a field of characteristic
zero, \( J \) contains as its deformation retract the minimal model \( C_{\infty} \) of the operad \( \text{Comm} [24] \) (operad \( C_{\infty} \) describes \( C_{\infty}, \) also called commutative or balanced \( A_{\infty}, \) algebras).

In Section 4 we analyze two particular bad cells and calculate the differential of the corresponding generators of \( G. \) The first one is the famous 6-dimensional Tamarkin cell, the second is a simpler 4-dimensional bad cell whose existence was a surprise for us. The dimension of these two bad cells is precisely the critical one, i.e. they are bad cells of the lowest dimension in a given arity. Although we did not give a general formula, it will be clear that our construction of the differential applies to any bad cell in the critical dimension. Our second main theorem, proved in Subsection 4.3, says that this formula can always be extended to a differential on \( \mathbb{F}(G). \)

**Theorem B.** Any formula for the differential of (one or more) bad cells in the critical dimension extends to a differential with the properties stated in Theorem A.

General bad cells are analyzed in Section 5. There is an obvious question about finding a closed formula for the differential of Theorem A that would apply to bad cells of arbitrarily high dimensions. It is clear from our analysis of bad cells that this formula should involve cellular approximations of the images of the source-target conditions. Theorem 6.1 of [29] shows that any such an approximation is associative only up to a hierarchy of higher homotopies. All these homotopies must be build in the formula for the differential as appropriate ‘correction terms.’ Writing such a formula is far beyond our abilities.

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**Glossary of notation** is given on page 40.

### 2. Trees, barcodes and the \( E_{\infty} \)-operad \( J = (\mathbb{F}(G), \partial) \)

In this section we describe the graded \( \Sigma \)-module \( G = \{G_*(n)\}_{n \geq 2} \) generating the operad \( J, \) together with the linear part \( \partial_{\text{lin}} \) of the differential. Let us start by recalling some definitions of [4]. We denote, as usual, by \([k]\) the ordered set \( 1 < 2 < \cdots < k. \)

**Definition 1.** Let \( h \geq 1. \) A tree of height \( h \) (or tree with \( h \) levels, or \( h \)-tree) is a sequence of order preserving maps

\[
T = [k_h] \xrightarrow{\rho_{h-1}} [k_{h-1}] \xrightarrow{\rho_{h-2}} \cdots \xrightarrow{\rho_0} [1].
\]

We are not going to consider degenerate trees, so we assume that all \( k_m \geq 1, \) for \( 0 \leq m \leq h. \)
A vertex of height $m$, $0 \leq m \leq h$, is an element of $[k_m]$. One may imagine that each vertex $i \in [k_m]$, $m \geq 1$, determines the oriented edge that starts at $i$ and ends at the vertex $\rho_{m-1}(i) \in [k_{m-1}]$. With this intuition, one may indeed interpret objects of Definition 1 as planar directed trees with vertices arranged at $h + 1$ horizontal lines shown in Figure 1. A leaf of height $m$ is a vertex $i \in [k_{m}]$ which is not in the image of $\rho_{m}$. A tip is a leaf of maximal height $h$. The arity of $T$ is then the number of tips. The tree is pruned if all its leaves are tips. These definitions should be clear from Figure 1.

\[ \begin{array}{cccc}
[4] & \vdots & \cdots & [2] \\
[2] & \vdots & \cdots & [3] \\
[1] & \vdots & \cdots & [1] \\
\end{array} \]

**Figure 1.** Example of trees of height two. The left tree is pruned, the right one is not. Arity of the left tree is 4, arity of the right one is 2.

We say that a tree $T$ as in (2) has a trunk if $k_m = 1$ for some $m \geq 1$. A trunk of $T$ is then everything ‘below’ $k_m$; see Figure 2.

\[ \begin{array}{cccc}
& \vdots & \cdots & \\
& \vdots & \cdots & \\
& \vdots & \cdots & \\
\end{array} \]

**Figure 2.** The left tree has a trunk (bold edge) and is not pruned. The right tree is its maximal reduced subtree.

We say that a tree is reduced if it is pruned and if it has no trunk. Obviously, for each $T$ there exists a unique maximal reduced subtree $r(T)$ of maximal height. See again Figure 2—the right tree is obtained from the left one by first cutting off the trunk and then pruning the remaining shrub. So pruning is for us cutting out branches that do not end in tips as opposed to what one does in garden, namely cutting of those that stick up too high. Finally, for a tree $T$ as in (2) define its dimension $\dim(T)$ as

\[ \dim(T) := e(T) - h - 1, \]

where $e(T)$ is the number of edges and $h$ the height.

The terminal tree $U_h$ is the tree with all $k_m = 1$. Terminal trees play a very special rôle and, unless stated otherwise, we will not consider them. If necessary, we set $\dim(U_h) := 0$ (formula (3) would give $\dim(U_h) = -1$). We also define $r(U_h) := U_h$.

\[ ^3\text{Our terminology is not a standard one – reduced usually means no vertices of arity 1.} \]

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Notation 2. Denote by $\text{Tree}^h(n) = \bigcup_{i \geq 0} \text{Tree}^i(n)$ the graded set whose $i$th component consists of pruned $h$-trees of dimension $i$, with $n$ tips, $h, n \geq 1$. We also denote $\text{Tree}^h(n) = \bigcup_{i \geq 0} \text{Tree}^i(n)$ the graded set of labeled pruned trees of height $h$. Elements of $\text{Tree}^i(n)$ are couples $T = (T, \ell)$, where $T \in \text{Tree}^h(n)$ is as in \([3]\) and $\ell$ an isomorphism (labeling) $\ell : [k_h] \xrightarrow{\sim} [n]$. The symmetric group $\Sigma_n$ freely acts on $\text{Tree}^h(n)$ by relabeling the tips.

One has the inclusion $\text{Tree}^h(n) \subset \text{Tree}^h(n)$ given by numbering the legs of an unlabeled planar tree from the left to the right. The subset $\text{Tree}^h(n)$ freely generates $\text{Tree}^h(n)$ as a right graded $\Sigma_n$-set.

There are the suspensions $s : \text{Tree}^h(n) \hookrightarrow \text{Tree}^{h+1}(n)$ resp. $s : \text{Tree}^h(n) \hookrightarrow \text{Tree}^{h+1}(n)$ that adjoin to a (labeled) tree a trunk of height one. The graded sets $\text{Tree}(n) := \varprojlim \text{Tree}^h(n)$ resp. $\text{Tree}(n) := \varprojlim \text{Tree}^h(n)$ clearly consist of (labeled) reduced trees of an arbitrary height.

The first step towards our definition of the operad $J$ is:

Definition 3. The $n$th component $G_\ast(n)$ of the graded $\Sigma$-module $G_\ast = \{G_\ast(n)\}_{n \geq 2}$ generating the operad $J$ is the free graded abelian group $\text{Span}(\text{Tree}_\ast(n))$ spanned by the graded $\Sigma_n$-set $\text{Tree}(n)$ of labeled reduced trees with $n$ tips.

Each $G_\ast(n)$ is clearly a free $\Sigma_n$-module $\Sigma_n$-generated by the graded abelian group $G_\ast(n) = \text{Span}(\text{Tree}_\ast(n))$ spanned by (unlabeled) reduced trees. A complete list of unlabeled reduced trees up to dimension 3 is given in Figure \([3]\); there are exactly $2^d$ reduced trees of dimension $d$.

Observe that $G_\ast(n) = \varinjlim G_\ast^h(n)$, where $G_\ast^h(n) := \text{Span}(\text{Tree}_\ast^h(n))$. There is a convenient "barcode" notation for the reduced labeled trees (and therefore also the Fox-Neuwirth cells recalled in Section \([3]\) introduced in \([18]\):

Definition 4. The barcode of a reduced labeled tree is the list of labels of its tips, separated by the vertical bars whose number equals the depth of the gaps between the tips.

Since the tips of an unlabeled tree can be labeled by $1, 2, \ldots$ in the increasing order from the left to the right, the barcodes can be used for unlabeled trees as well. See again Figure \([3]\). The height of the corresponding reduced tree is the maximal number of the adjacent bars, and the dimension is the number of vertical bars minus 1.

The shortest way to describe the differential $\partial_{\text{lin}}$ on the $\Sigma$-module $G = \{G(n)\}_{n \geq 2}$ is to identify this $\Sigma$-module to a suitable dg-submodule of an iterated bar construction. Let $B(A)$ denote the bar construction of an associative algebra $A$, i.e. the tensor algebra $\mathbb{T}(\uparrow A)$ generated by the suspension $\uparrow A$ of the abelian group $A$, with the degree $-1$ differential $\partial_B$ induced by the multiplication of $A$. It is classical \([20], \text{X.12}\) that, if $A$ is commutative, the shuffle product of the tensor algebra makes $B(A) = (\mathbb{T}(\uparrow A), \partial_B)$ a commutative associative algebra, thus the bar construction can be iterated.

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Let us denote, for \( h \geq 1 \), by \( \mathcal{B}^h(A) \) the \( h \)-th iterate of \( \mathcal{B}(-) \) applied to \( A \). Since the natural inclusion \( i^h : \mathcal{B}^h(A) \hookrightarrow \mathcal{B}^{h+1}(A) \) is a degree +1 map, to have a natural grading on the direct limit we need to regrade by putting \( \hat{\mathcal{B}}^h(A) := \lim_{\rightarrow}^{h+1} \mathcal{B}^h(A) \). The induced inclusion \( \hat{\mathcal{B}}^h(A) \hookrightarrow \hat{\mathcal{B}}^{h+1}(A) \) is degree-preserving so one may take \( \hat{\mathcal{B}}^\infty(A) = (\hat{\mathcal{B}}^\infty(A), \partial^\infty) \), the direct limit \( \hat{\mathcal{B}}^h(A) \) with the induced differential.

Consider the free abelian group \( V \) spanned by \( x_1, \ldots, x_n \), interpreted as a commutative algebra with the trivial multiplication. Denote by \( \hat{\mathcal{B}}^i_{1,\ldots,1}(V) \) the sub-dg abelian group of \( \hat{\mathcal{B}}^i(V) \) spanned by monomials that contain each basic element \( x_1, \ldots, x_n \) exactly once, with the obvious right \( \Sigma_n \)-action given by relabeling. Finally, let \( \hat{\mathcal{B}}^\infty_{1,\ldots,1}(V) := \lim_{\rightarrow} \hat{\mathcal{B}}^i_{1,\ldots,1}(V) \).

As observed in [13] and, in more general setting, also in [13], the graded abelian group \( \hat{\mathcal{B}}^\infty_{1,\ldots,1}(V) \) is isomorphic to the graded abelian group \( G_*(n) \) of Definition 3. The isomorphism \( \omega : G(n) \cong \hat{\mathcal{B}}^\infty_{1,\ldots,1}(V) \) has the following inductive description.

Let \( T \in \text{Tree}^h(n) \) and \( g_T \) the corresponding generator of \( G(n) \). If \( h = 1 \), then \( T \) is the \( n \)-corolla \( *_n \), i.e. the 1-tree with the barcode \([1] \ldots [n] \). In this case we put

\[
\omega(g_T) := \downarrow^2 (\uparrow x_1 \otimes \cdots \otimes \uparrow x_n) \in \hat{\mathcal{B}}^1_{1,\ldots,1}(V) \subset \hat{\mathcal{B}}^\infty_{1,\ldots,1}(V).
\]
For example, \( \omega \) reduced, \( (h-1) \)-trees \( T_1, \ldots, T_u \) at the tips of the \( u \)-corolla \( *_u \):

\[
T = \begin{array}{c}
n_1 \\
T_1 \\
T_2 \\
\vdots \\
T_u \\
n_u 
\end{array}
\]

where \( n_1, \ldots, n_u \geq 1 \) with \( n_1 + \cdots + n_u = n \) are the arities of the trees \( T_1, \ldots, T_u \). For \( i, 1 \leq i \leq u \), we denote

\[
V_i := \text{Span}\{x_j; \ n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_i\}.
\]

We distinguish two cases.

(a) \( n_i \geq 2 \). Then let \( \omega(g_i) \in \text{Tree}^h(n_i) \) be the maximal reduced subtree of \( T_i \). By induction, \( \omega(g_i) \in \hat{B}^h_{1\ldots1}(V_i) \) is defined and we put \( \omega_i \in \hat{B}^{h-1}_{1\ldots1}(V_i) \) the image of \( \omega(g_i) \) under the natural inclusion \( \hat{B}^h_{1\ldots1}(V_i) \hookrightarrow \hat{B}^{h-1}_{1\ldots1}(V_i) \).

(b) \( n_i = 1 \). In this case, let \( j := n_1 + \cdots + n_{i-1} + 1 \) and define \( \omega_i \in \hat{B}^{h-1}_{1\ldots1}(V_i) \) the image of \( \uparrow^2(\uparrow x_j) \in \hat{B}^1_{1\ldots1}(V_i) \) under the natural inclusion \( \hat{B}^1_{1\ldots1}(V_i) \hookrightarrow \hat{B}^{h-1}_{1\ldots1}(V_i) \).

Observe that, in both cases, \( \uparrow^{h+1} \omega_i \in \hat{B}^h_{1\ldots1}(V_i) \). Finally, let

\[
\omega(g_T) := \downarrow^{h+1} \left( \uparrow^{h+1} \omega_1 \otimes \cdots \otimes \uparrow^{h+1} \omega_u \right) \in \hat{B}^h_{1\ldots1}(V) \subset \hat{B}^\infty_{1\ldots1}(V).
\]

For example,

\[
\omega(g_{\varphi}) = \downarrow^2 \left( \uparrow x_1 \otimes \uparrow x_2 \otimes \uparrow x_3 \right) \in \hat{B}^1_{1\ldots1}(V) \subset \hat{B}^\infty_{1\ldots1}(V),
\]

\[
\omega(g_{\varphi}) = \downarrow^3 \left( \uparrow (\uparrow x_1) \otimes \uparrow (\uparrow x_2 \otimes \uparrow x_3) \right) \in \hat{B}^2_{1\ldots1}(V) \subset \hat{B}^\infty_{1\ldots1}(V), \quad \&c.
\]

**Definition 5.** The differential \( \partial_{\text{lin}} \) on \( G_\ast(n) \) is defined by \( \partial_{\text{lin}} := \omega^{-1} \circ \partial_B \circ \omega \). Thus \( \partial_{\text{lin}} \) is the unique differential such that \( \omega : (G_\ast(n), \partial_{\text{lin}}) \rightarrow (\hat{B}^\infty_{1\ldots1}(V), \partial_B) \) is an isomorphism of dg-abelian groups.

It is easy to see that \( (G_\ast(2), \partial_{\text{lin}}) \) is the cellular chain complex of the ‘globular’ decomposition of the infinite sphere \( S^\infty \). The piece \( G_\ast(3) \) contains Stasheff’s associator, two Mac Lane’s hexagons (right and left), etc.

### 3. Relation to the Compactification of the Configuration Space

Let \( \hat{F}_h(n) := \text{Cnf}(\mathbb{R}^h, n)/\text{Aff}(\mathbb{R}^h) \) be the moduli space of configurations of \( n \) distinct points in the \( h \)-dimensional Euclidean plane \( \mathbb{R}^h \), modulo the action of the affine group of dilatations and translations. Getzler and Jones in [18] described a compactification \( F_h(n) \) of \( \hat{F}_h(n) \) such that the \( \Sigma \)-space \( F_h := \{ F_h(n) \}_{n \geq 1} \) is an operad in the category of manifolds with corners, see also [25].

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Will will in fact be interested in the colimit \( \mathcal{F} := \lim_{\to} \mathcal{F}_h \) which inherits an operad structure from its constituents \( \mathcal{F}_h \). As observed in [18], the Fox-Neuwirth cells (recalled below) of the open part \( \mathcal{F}_h(n) \) induce a decomposition of the colimit \( \mathcal{F} := \lim_{\to} \mathcal{F}_h \) which in turn induces a cell decomposition of the compactification \( \mathcal{F} \) compatible with the operad structure.

There was a belief that the operad \( J \) can be easily read off from the combinatorics of this cell decomposition, including a formula for the differential. This turned out not to be the case because, due to the existence of ‘bad’ cells, the cell structure of \( \mathcal{F} \) is not regular. We identify, in Propositions 14 and 15, the dimensions in which bad cells exist and analyze them further in Section 5. We start with:

**Definition 6.** The critical dimension \( d_{\text{crit}}(n) \) is defined as

\[
d_{\text{crit}}(n) = \begin{cases} 
\infty, & \text{if } n = 2, 3 \text{ and } \\
n, & \text{if } n \geq 4.
\end{cases}
\]

Let us denote by \( \mathcal{G}^{\text{reg}} \) the graded sub-\( \Sigma \)-module of \( \mathcal{G} \) spanned by the cells whose dimension is less than the critical one. The results of this section are summarized in the following statement which is a combination of Propositions 14, 16 and the results of Subsection 3.1.

**Proposition 7.** The combinatorics of the Fox-Neuwirth decomposition of the compactification \( \mathcal{F} \) determines, by formula (10) on page 17, a differential \( \partial \) on the free operad \( \mathcal{F}(\mathcal{G}^{\text{reg}}) \).

We start by recalling, following closely [18], a correspondence between pruned trees and flags of pre-orders. This point of view will be useful in describing decompositions of configuration spaces.

**Definition 8.** A pre-order \( \pi \) on a non-empty set \( S \) is a reflexive transitive relation \( \leq \) such that if \( a, b \in S \), either \( a \leq b \) or \( b \leq a \).

A pre-order defines an equivalence \( \sim \) on \( S \) by \( a \sim b \) if and only if \( a \leq b \) and \( b \leq a \), and induces a total order on the quotient \( S/\sim \). We denote \( |\pi| \) the number of equivalence classes. A pre-order \( \pi \) is trivial if \( a \leq b \) of all \( a, b \in S \) or, equivalently, if \( |\pi| = 1 \). Pre-orders on \( S \) form a poset: \( \pi_1 \prec \pi_2 \) if \( a \leq_2 b \) implies \( a \leq_1 b \) for all \( a, b \in S \). The maximal elements of this poset are the total orders of \( S \), the unique minimal element is the trivial pre-order.

**Definition 9.** A flag of pre-orders on the set \( S \) of height \( h \geq 1 \) is a sequence \( (\pi_1 \prec \cdots \prec \pi_h) \) of pre-orders on \( S \) such that \( \pi_h \) is a total order of \( S \). Such a flag is reduced if \( \pi_1 \) is not the trivial pre-order.

Let \( \text{Flag}^h(n) = \bigcup_{i \geq 0} \text{Flag}^h_i(n) \) denote the graded set whose \( i \)th component is formed by flags of preorders of height \( h \) on the set \( \{1, \ldots, n\} \) satisfying \( i = \sum_{s=1}^{h} |\pi_s| - h - 1 \). We also denote \( \text{Flag}^\text{reg}^h(n) = \bigcup_{i \geq 0} \text{Flag}^\text{reg}^h_i(n) \) the graded subset of flags of preorders \( (\pi_1 \prec \cdots \prec \pi_h) \) such that \( \pi_h \) is the standard linear order of \( \{1, \ldots, n\} \).
The suspension $s : \text{Flag}^h(n) \hookrightarrow \text{Flag}^{h+1}(n)$ resp. $g : \text{Flag}^h(n) \rightarrow \text{Flag}^{h+1}(n)$ extends a given flag from the left by the trivial preorder. The graded sets $\text{Flag}(n) := \varprojlim \text{Flag}^h(n)$ resp. $\text{Flag}(n) := \varprojlim \text{Flag}^h(n)$ consist of reduced flags of an arbitrary height. The next proposition relies on Notation 2.

**Proposition 10.** For each $n, h \geq 1$, there are natural isomorphisms of graded sets \( \text{Tree}^h(n) \cong \text{Flag}^h(n) \) and \( \overline{\text{Tree}}(n) \cong \text{Flag}(n) \) which induce isomorphisms of the colimits \( \overline{\text{Tree}}(n) \cong \text{Flag}(n) \).

**Proof.** Let $T = (T, \ell) \in \text{Tree}^h(n)$ be a labeled tree, i.e. $T \in \text{Tree}^h(n)$ is as in (2) and $\ell : [k_h] \cong [n]$ a labeling. Such a tree $T$ defines a flag of preorders $(\pi_1 \prec \cdots \prec \pi_h)$ as follows.

For $i, j \in \{1, \ldots, n\}$ we put $i \leq_h j$ if and only if $\ell^{-1}(i) \leq \ell^{-1}(j)$. In other words, $\pi_h$ is the image of the natural order on $\{1, \ldots, k_h\}$ under the isomorphism $\ell$. For $1 \leq s < h$ we write $i \leq_s j$ if and only if

$$
\rho_s \rho_{s+1} \cdots \rho_{h-1}(\ell^{-1}(i)) \leq \rho_s \rho_{s+1} \cdots \rho_{h-1}(\ell^{-1}(j)).
$$

It is easy to prove that the above correspondence is one to one, induces an isomorphism of the colimits and restricts to isomorphisms of the ‘underlined’ versions $\overline{\text{Tree}}(n) \cong \text{Flag}(n)$. It is also clear that for the flag $(\pi_1 \prec \cdots \prec \pi_h)$ corresponding to a tree $T = (T, \ell)$ one has $e(T) = \sum_{s=1}^{h} |\pi_s|$, therefore, by (3),

$$
\dim(T) = \sum_{s=1}^{h} |\pi_s| - h - 1,
$$

so the above isomorphism are compatible with the gradings. \qed

**Convention 11.** Given Proposition 11, we will make no difference between pruned trees and the corresponding flags of preorders. Thus, for instance, a boldfaced $T$ will denote both a labeled tree $(T, \ell)$ and the corresponding flag $(\pi_1 \prec \cdots \prec \pi_h)$.

Recall that $\text{Cnf}(\mathbb{R}^h, n)$ denotes the configuration space of $n$ distinct labeled points $p_1, \ldots, p_n$ in the $h$-dimensional affine space $\mathbb{R}^h$, $n, h \geq 1$. It is an $hn$-dimensional oriented smooth non-compact manifold whose points are monomorphisms $f : \{1, \ldots, n\} \rightarrow \mathbb{R}^h$ given as $f(k) := p_k$, $1 \leq k \leq n$. For such an $f$ and $1 \leq s \leq h$, denote by $f_s$ the composition of $f$ with the projection $\mathbb{R}^h \rightarrow \mathbb{R}^s$, $(x_1, \ldots, x_h) \mapsto (x_1, \ldots, x_s)$, to the first $s$ coordinates. We finally denote $\pi'_f$ the preorder on the set $\{1, \ldots, n\}$ given by the pullback of the lexicographic order of $\mathbb{R}^s$ via $f_s$. In this way, each monomorphism $f : \{1, \ldots, n\} \rightarrow \mathbb{R}^h \in \text{Cnf}(\mathbb{R}^h, n)$ determines a flag of preorders

$$
T_f = (\pi'_1 \prec \cdots \prec \pi'_h).
$$

Conversely, for a given tree $\overline{T} = (T, \ell) \in \overline{\text{Tree}}^h(n)$ define $[T] := \{ f \in \text{Cnf}(\mathbb{R}^h, n); T_f = T \}$. It is clear that $\text{Cnf}(\mathbb{R}^h, n)$ is the disjoint union

$$
\text{Cnf}(\mathbb{R}^h, n) = \bigcup_{T \in \overline{\text{Tree}}^h(n)} [T].
$$

---

4We are already using Convention 11.

[July 31, 2021]
Each $[T]$ is an open ball of dimension $e(T) = \sum_{h=1}^{n} |\tau_s|$, therefore a tree $T \in \text{Tree}^h(n)$ determines a cell of dimension $i + h + 1$. For $h = 2$, $[T]$ describes the classical Fox-Neuwirth decomposition [13] generalized in [18] to arbitrary $h \geq 2$. One may assign an orientation to $[T]$ taking first the coordinates $x_1$ of the equivalence class $\tau_1$ in the increasing order, next the coordinates $x_2$ of the equivalence class $\tau_2$, also in the increasing order, &c. An example can be found on page 36 of Section 2.

As we already indicated, we will need the moduli space $\hat{F}_h(n) := \text{Cnf}([\mathbb{R}^h, n])/\text{Aff}(\mathbb{R}^h)$, where the affine group $\text{Aff}(\mathbb{R}^h) = \mathbb{R}^h \times \mathbb{R}_+^\times$ acts by translations and dilatations in the obvious manner. We denote the quotient of the cell $[T]$ modulo $\text{Aff}(\mathbb{R}^h)$ by $\mu[T]$. It is clear that $\mu[T]$ is an open ball of dimension $e(T) - h - 1$. This explains formula (3) for the dimension of a tree. One has the disjoint decomposition

$$\bigcup_{T \in \text{Tree}^h(n)} \mu[T].$$

Let us denote $\hat{F}_h$ the $\Sigma$-space $\hat{F}_h = \{\hat{F}(n)\}_{n \geq 2}$. Fulton and MacPherson construct in [17] a compactification $F_h = \{F_h(n)\}_{n \geq 1}$ of $\hat{F}_h$ such that $F_h(n)$ is, for $n \geq 2$, a smooth manifold with corners containing $\hat{F}_h(n)$ as its unique open stratum. The $\Sigma$-space $F_h$ is obtained by gluing the free operad $F(\hat{F}_h)$. In particular, decomposition (5) of $\hat{F}_h$ induces a decomposition of the free operad $F(\hat{F}_h)$ which in turn induces a CW-structure of $F_h$ via the gluing map $F(\hat{F}_h) \to F_h$.

This implies that the cells of $F_h$ are indexed by the free set-operad $F(\text{Tree}^h)$. Since the pieces $\text{Tree}^h(n)$ of the generating $\Sigma$-set $\text{Tree}^h = \{\text{Tree}^h(n)\}_{n \geq 2}$ are freely $\Sigma_n$-generated by the subset $\text{Tree}^h(n) \subset \text{Tree}^h(n)$, the natural inclusion $F(\text{Tree}^h) \to F(\text{Tree}^h)$ induces, for each $n \geq 2$, the isomorphism of right $\Sigma_n$-sets

$$F(\text{Tree}^h)(n) \cong \text{Ind}_{\Sigma_n} \rightarrow F(\text{Tree}^h)(n) = F(\text{Tree}^h)(n) \times \Sigma_n,$$

where $1_n$ denotes the trivial representation of $\Sigma_n$ and $\rightarrow$ the free non-$\Sigma$ operad functor. We will abbreviate the above display by

$$F(\text{Tree}^h) \cong F(\text{Tree}^h) \times \Sigma.$$

It follows from (5) and the structure theorem for free operads [30, Section II.1.9] that

$$F(\text{Tree}^h)(n) = \bigcup_{\tau \in \text{PTree}(n)} \tau(\text{Tree}^h), \quad n \geq 2,$$

where $\text{PTree}(n)$ denotes the set of planar rooted trees whose each vertex has at least two input edges$^5$ with leaves labeled by an isomorphism $\omega : \text{Leaf}(\tau) \to \{n\}$, see [30, Sect. II.1.5] for the terminology and notation. The trees in $\text{PTree}(n)$ are different than the trees in Section 2 in that they do not have levels. The set $\tau(\text{Tree}^h)$ in (5) is the cartesian product

$$\tau(\text{Tree}^h) := \times_{\nu \in \text{Vert}(\tau)} \text{Tree}^h(\text{ar}(\nu)),$$

---

$^5$This is the usual meaning of being reduced, compare the footnote on page 3.
where $\text{Vert}(\tau)$ is the set of vertices of $\tau$ and $\text{ar}(v)$ the number of input edges (= the arity) of a vertex $v$. Informally, $\text{Tree}_{\Sigma}^\text{reg}$ means that the cells of $F_h$ are indexed by planar leaf-labeled rooted trees whose vertices are decorated by the graded set $\text{Tree}_h^\Sigma$ of pruned non-labeled $h$-trees.

The inclusion $\mathbb{R}^h \hookrightarrow \mathbb{R}^{h+1}$, $(x_1, \ldots, x_h) \mapsto (0, x_1, \ldots, x_h)$, induces an inclusion of $\Sigma$-spaces $\hat{\mathcal{F}}_h \hookrightarrow \hat{\mathcal{F}}_{h+1}$ so one can take the colimit $\hat{\mathcal{F}} := \varinjlim \hat{\mathcal{F}}_h$. Decomposition (5) induces the decomposition

$$\hat{\mathcal{F}}(n) = \bigcup_{T \in \text{Tree}(n)} \mu[T]$$

with the cells indexed by reduced trees. The colimit $\mathcal{F} = \varinjlim \mathcal{F}_h$ is again obtained by gluing the free operad $\mathbb{F}(\hat{\mathcal{F}})$, so (8) gives a decomposition of $\mathcal{F}$ with cells indexed by the free set-operad $\mathbb{F}(\text{Tree})$.

At this stage we need to extend Definition 6 of the critical dimension for finite $h$ by

$$d^h_{\text{crit}}(n) := \begin{cases} \infty, & \text{if } n = 2, 3, \text{or } n = 4, 5 \text{ and } h \leq 2, \text{ or } n \geq 6 \text{ and } h = 1, \\ n, & \text{in the remaining cases.} \end{cases}$$

Clearly, $d_{\text{crit}}(n) = \lim_{h \to \infty} d^h_{\text{crit}}(n)$. Let $\text{Tree}_{\Sigma}^{reg, h}$ be, for $h \geq 1$, the graded $\Sigma$-subset of the graded $\Sigma$-set $\text{Tree}_h^{\Sigma}$ consisting of reduced trees of dimension less that the critical one, i.e. the graded $\Sigma$-set such that

$$\text{Tree}_{i, \Sigma}^{reg, h}(n) := \begin{cases} \text{Tree}_i^h(n), & \text{if } i < d^h_{\text{crit}}(n), \text{ and} \\ \emptyset, & \text{if } i \geq d^h_{\text{crit}}(n). \end{cases}$$

Observe that $\text{Tree}^{reg, 1}_{\Sigma} = \text{Tree}^1_{\Sigma}$. We will also need the direct limit $\text{Tree}^{reg} := \varinjlim \text{Tree}^{reg, h}_{\Sigma}$. Clearly $\text{Tree}^{reg}_{\Sigma}(n) = \text{Tree}_n(n)$ if $n = 2, 3$ while, for $n \geq 4$,

$$\text{Tree}_{i, \Sigma}^{reg}(n) := \begin{cases} \text{Tree}_i(n), & \text{if } i < n, \text{ and} \\ \emptyset, & \text{if } i \geq n. \end{cases}$$

We will call, just for the purposes of this section, the trees in $\text{Tree}^{reg, h}_{\Sigma}$ or $\text{Tree}^{reg}$ the regular trees. We also denote by $\text{Tree}^{reg, h}_{\Sigma}$ (resp. $\text{Tree}^{reg}$) the $\Sigma$-subset of $\text{Tree}^{reg, h}_{\Sigma}$ (resp. $\text{Tree}^{reg}$) of unlabeled regular trees, i.e. $\text{Tree}^{reg, h}_{\Sigma} := \text{Tree}^{reg, h}_{\Sigma} \cap \text{Tree}$ (resp. $\text{Tree}^{reg} := \text{Tree}^{reg} \cap \text{Tree}$).

**Definition 12.** The regular skeleton $F^{reg}$ of $F$ is the union of the cells of the CW-complex $F$ indexed by the suboperad $\mathbb{F}(\text{Tree}^{reg}) \subset \mathbb{F}(\text{Tree})$. The regular skeleton $F^{reg}_h$ of $F_h$ is the intersection $F^{reg} \cap F_h$.

For $\mathbb{F}(\text{Tree}^{reg})$ we have a formula similar to (5), i.e.

$$\mathbb{F}(\text{Tree}^{reg})(n) = \bigcup_{\tau \in \text{Tree}^{reg}(n)} \tau(\text{Tree}^{reg}), \quad n \geq 2,$$

thus the cells of $\mathbb{F}^{reg}$ are indexed by planar rooted labeled trees with vertices decorated by unlabeled reduced regular trees. It is clear that $\mathbb{F}^{reg}_h$ is the union of the cells indexed by the suboperad $\mathbb{F}(\text{Tree}^{reg, h})$. It is equally obvious that the sub-$\Sigma$-modules $F^{reg}_h \subset F_h$ and $F^{reg} \subset F$ are suboperads and that $F^{reg}_1 = F_1$. Let us recall the following standard

**Definition 13.** A CW-complex is regular if (i) the attaching maps are homeomorphisms and (ii) the boundary of each cell is a union of cells.

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We have the following correction to [13, Lemma 5.11] whose proof we postpone to Section 3.

**Proposition 14.** The CW-structures of $F_h$, $h \geq 1$, and of $F$ are compatible with the operad structures and the symmetric group acts freely on the cells. The spaces $F(n)$ are regular cell complexes if and only if $n = 2$ or $3$. The complexes $F_h(n)$ are regular if and only if

(i) $n = 2, 3$ and $h$ arbitrary, or

(ii) $n \leq 5$ and $h \leq 2$, or

(iii) $n$ arbitrary and $h = 1$.

The spaces $F_h(n)$ and $F(n)$ satisfy condition (i) of Definition 13 for arbitrary $n$ and $h$. The CW-subcomplexes $F^{reg}_h \subset F_h$ and $F^{reg} \subset F$ are regular, with the cell structures compatible with the operad structures and the symmetric group acting freely on the cells.

Observe that, by the definition of the critical dimension, Proposition 14 says that the spaces $F_h(n)$ (resp. $F(n)$) are regular if and only if $d_{crit}^h(n) = \infty$ (resp. $d_{crit}(n) = \infty$). We call cells violating (ii) of Definition 13 the bad cells. The following statement provides a ‘coordinate-free’ definition of the regular skeleta.

**Proposition 15.** For each $n \geq 4$, $h \geq 3$ or $n \geq 6$, $h \geq 2$, there exists a bad cell $e_n^h$ in the open stratum $\tilde{F}_h(n)$ whose dimension $\dim(e_n^h)$ equals $d_{crit}^h(n)$. Likewise, for each $n \geq 4$ there exists a bad cell $e_n \subset \tilde{F}(n)$ such that $\dim(e_n) = d_{crit}(n)$. The regular skeleta are therefore the maximal regular subcomplexes closed under the operad structure.

The first bad cell was found by D. Tamarkin. We will call this particular bad cell the Tamarkin cell and recall its definition in Section 4 in which we also prove Propositions 14 and 15. The case $h = 1$ is special; $F_1$ is the Stasheff’s operad of the associahedra [34] which indeed forms a regular cell complex. The dimensions/arities in which the bad cells of the open strata sit are shown in Figures 4 and 5.

Let us define the increasing filtration $\mathcal{F}_*(n) = \cdots \mathcal{F}_0(n) \subset \mathcal{F}_1(n) \subset \mathcal{F}_2(n) \cdots$ of $F^{reg}(n)$ by

$$\mathcal{F}_p(n) := \bigcup \{e \text{ a cell of } F^{reg}(n); \dim(e) \leq p\}.$$  

Since the cells of $F^{reg}$ are indexed by the free operad $F(\text{Tree}^{reg})$, this filtration is manifestly operadic, i.e. each $\mathcal{F}_p(n)$ is $\Sigma_n$-invariant and $\mathcal{F}_p(m) \circ_i \mathcal{F}_q(n) \subset \mathcal{F}_{p+q}(m+n-1)$ for $m, n \geq 1$, $1 \leq i \leq m$. Each layer $(\mathcal{E}_*^{r}(n), \partial^r)$ of the induced spectral sequence determines a dg-operad $\mathcal{E}^r := \{(\mathcal{E}_*^{r}(n), \partial^r)\}_{n \geq 1}$ with $\mathcal{E}_*^{r}(n) := \bigoplus_{s = p+q} \mathcal{E}_{pq}^{r}(n)$.

The dg-operad $\mathcal{E}^2$ will be of a particular importance for us. As usual, the abelian group $\mathcal{E}_{pq}^{2}(n)$ equals the reduced homology $\overline{H}_{p+q}(\mathcal{F}_p(n)/\mathcal{F}_{p-1}(n))$, so the regularity of the CW-structure established in Proposition 14 implies that

$$\mathcal{E}_{pq}^{2}(n) = \begin{cases} \text{Span(set of } p\text{-dimensional cells of } F(n)), & \text{if } q = 0 \text{ and } \\ 0, & \text{if } q \neq 0. \end{cases}$$
In particular, $e^2_{p_0}$ contains, for each $T \in \text{Tree}^\text{reg}_r(n)$, the generator $c_T$ corresponding to the cell $\mu[T]$.

Recall that $G^\text{reg}_s = \text{Span}(\text{Tree}^\text{reg}_s)$ is the graded $\Sigma$-submodule of the $\Sigma$-module $G_s$ from Definition 3 spanned by the generators $g_T$ indexed by regular trees $T \in \text{Tree}^\text{reg}$. We have a natural map of graded operads $j : F(G^\text{reg}) \to e^2$ given by $j(g_T) := c_T$ for $T \in \text{Tree}^\text{reg}$.

**Proposition 16.** There is a unique differential $\partial$ on the free operad $F(G^\text{reg})$ such that the map $j : (F(G^\text{reg}), \partial) \to (e^2, \partial^2)$ is a map of dg-operads. Moreover, $\partial$ is the sum $\partial_{\text{lin}} + \partial_{\text{prt}}$ where $\partial_{\text{lin}}$ is as in Definition 3.

**Proof.** It is clear from the description of the cell structure of $F^\text{reg}$ via the free set-operad $F(\text{Tree}^\text{reg})$ that the map $j$ is an isomorphism of graded operads, which implies the existence and uniqueness of the differential $\partial$. The fact that $\partial$ constructed in this way is a perturbation of the linear part $\partial_{\text{lin}}$ of Definition 3 will follow from explicit calculations given below and Proposition 18. 

[July 31, 2021]
3.1. **The differential \( \partial \) in sub-critical dimensions.** Proposition [4] translates the description of the differential operad \( \mathbb{F}(G^{\text{reg}}, \partial) \) into the standard task of calculating the second term of the spectral sequence associated to the regular cell complex \( \mathbb{F}^{\text{reg}} \). Given an \( i \)-dimensional cell \( e \) of \( \mathbb{F}^{\text{reg}} \), one needs first to identify cells forming the boundary of \( e \). The differential of the generator corresponding to \( e \) then is then the sum of the generators corresponding to \( (i - 1) \)-dimensional cells in the boundary of \( e \), with the signs determined by the orientations.

In our particular case, the compatibility of the differential with the operad structure implies that it suffices to describe the boundaries of the cells \( \mu[T] \) corresponding to the operadic generators in \( G^{\text{reg}} \), indexed by unlabeled regular reduced trees \( T \in \text{Tree}^{\text{reg}} \). This was in fact already done in [4], so we only need to recall the necessary notions. Let us recollect the notation first.

**Notation 17.** We introduced the following objects indexed by trees \( T \in \text{Tree} \) (resp. the unlabeled versions \( T \in \text{Tree}^{\text{reg}} \)): the corresponding generator \( g_T \) (resp. \( g_T \)) of \( G = \text{Span}(\text{Tree}) \) (resp. of \( G = \text{Span}(\text{Tree}^{\text{reg}}) \)), the Fox-Neuwirth cell \( \mu[T] \) (resp. \( \mu[T] \)) of \( \mathbb{F} \), and \( c_T \) (resp. \( c_T \)) -- the corresponding generator of \( \mathcal{E}^2 \). We will also denote by \( E_T \) the corresponding generator of the free set-operad \( \mathbb{F}(\text{Tree}) \) and, for an element \( C \in \mathbb{F}(\text{Tree}) \), by \( \mu[C] \) the corresponding cell of \( \mathbb{F} \).

Let us return to our task of describing the differential \( \partial \). According to [4, Definition 2.2], a morphism of \( h \)-trees

\[
T = [k_h] \xrightarrow{\rho_{h-1}} [k_{h-1}] \xrightarrow{\rho_{h-2}} \cdots \xrightarrow{\rho_0} [1]
\]

and

\[
S = [s_h] \xrightarrow{\xi_{h-1}} [s_{h-1}] \xrightarrow{\xi_{h-2}} \cdots \xrightarrow{\xi_0} [1]
\]

is given by a sequence \( \sigma = (\sigma_h, \ldots, \sigma_0) \) of not necessary order preserving maps \( \sigma_m : [k_m] \to [s_m] \), \( 0 \leq m \leq h \), with the property that for each \( m \) and each \( j \in [k_{m-1}] \), the restriction of \( \sigma_m \) to \( \rho_{m-1}^{-1}(j) \) preserves the induced order.\(^6\)

Let \( T \in \text{Tree}^{\text{reg}} \) be a reduced unlabeled \( h \)-tree as above. We will consider maps \( \sigma : T \to S \) of \( h \)-trees such that

(i) the tree \( S \) is pruned, but possibly with a trunk, and

(ii) the map \( \sigma \) induces an epimorphism of tips, that is, \( \sigma_h : [k_h] \to [s_h] \) is onto.

We will call such a map \( \sigma \) a **face** of the tree \( T \). Observe that \( \sigma \) is determined by the values \( \sigma_h(i), i \in [k_h] \). Let us explain how a face \( \sigma \) determines a cell of \( \mathbb{F} \) in the boundary of \( \mu[T] \). We need first to describe, following again M. Batanin’s [4], faces \( \sigma \) in terms of fibers.

Let \( \sigma : T \to S \) be a face of \( T \) as above. For each tip \( j \in [s_h] \), let \( S_j \) be the path in \( S \) connecting \( \{j\} \) with the root of \( S \). Then the \( j \)th **fiber** of \( \sigma \) is the subtree \( F_j := \sigma^{-1}(S_j) \) of \( T \). We believe that Figure [4] elucidates this definition.

Each such a \( \sigma : T \to S \) is characterized by its **fiber diagram**, obtained by drawing fibers \( F_j \) over the corresponding tips of \( S \). Some examples of fiber diagrams can be found in Figure [4].

\(^6\)We believe the same implicit notation for a permutation and a morphism of trees will not confuse the reader.
Figure 6. Fiber $F_2$ (shown in bold lines) of the map $\sigma : T \to S$ given by $\sigma_2(1) = \sigma_2(3) = 1$ and $\sigma_2(2) = 2$.

Figure 7. Examples of fiber diagrams.

Diagram $D_1$ is the fiber diagram of face $\sigma$ from Figure 3. Diagram $D_2$ is the fiber diagram of the same trees as in Figure 3, but with $\sigma$ determined by $\sigma_2(1) = \sigma_2(2) = 1$ and $\sigma_2(3) = 2$. Diagram $D_3$ is the diagram of the map $\sigma : \mathcal{U} \to \mathcal{V}$ given by $\sigma_2(1) = 1$, $\sigma_2(2) = 3$ and $\sigma_2(3) = 2$. Diagram $D_4$ describes the map $\sigma : \mathcal{U} \to \mathcal{U}$ given by $\sigma_2(1) = \sigma_2(2) = 1$ and $\sigma_2(3) = 2$.

We sometimes decorate the tips of fibers by numbers that indicate to which tip of $S$ they are mapped, see again Figure 3. Other examples of fiber diagrams can be found in Figures XV and XVI of [1].

We are ready to describe the element $C_\sigma \in \mathbb{F}(\text{Tree})$ indexing the cell $\mu[\sigma] := \mu[C_\sigma]$ of $\mathbb{F}$ corresponding to the face $\sigma$. We take the fiber diagram of $\sigma$ and replace all trees of this diagram by their maximal reduced subtrees. We obtain a tree-shaped diagram of reduced trees in Tree which are, by definition, the generators of the free operad set operad $\mathbb{F}(\text{Tree})$. The terminal tree represents the identity $\mathbb{1} \in \mathbb{F}(\text{Tree})(1)$. We then interpret this reduced fiber diagram as the indicated composition of elements in the free operad $\mathbb{F}(\text{Tree})$ using the direct limit of isomorphisms (3)

$$\mathbb{F}(\text{Tree}) \cong \mathbb{F}(\text{Tree}) \times \Sigma.$$  

Let us denote this composition by $C_\sigma$. We believe that the construction of $C_\sigma$ is clear from Figure 3 which relies on Notation 4.

It will be convenient to extend the barcode notation of Definition 1 to elements of the free operad $\mathbb{F}(\text{Tree})$. For example, the extended barcode $[1\mathbb{1}|3\mathbb{1}|2\mathbb{1}|]_\mathbb{2}$ of the element $E_{\mathcal{U}} \circ (E_{\mathcal{U}} \times \mathbb{1}) \circ (132)$ is obtained by inserting the barcode $[1\mathbb{1}|2\mathbb{1}|]$ for the tree $E_{\mathcal{U}}$ into the first position in the barcode $[1\mathbb{1}|2\mathbb{1}|]$ for $E_{\mathcal{U}}$ and permuting the labels according to the permutation $(132)$. See Figure 8 for more examples of the extended barcodes.

The last step is counting $\deg(C_\sigma)$ by adding up the degrees of generators that constitute $C_\sigma$. For example, in Figure 8 all $C_\sigma$’s are of degree 1 except the one corresponding to $D_2$ which is of degree 0. Let $\iota : \mathbb{F}(\text{Tree}) \hookrightarrow \mathbb{F}(G)$ be the monomorphism induced by the inclusion $\text{Tree} \hookrightarrow G = [July 31, 2021]$. 

$C_\sigma = E \circ (E \times 1) \circ (132) = [1|3|2]$  

\[\begin{array}{c}
\begin{array}{cccc}
D_1: & 1 & \downarrow & 3 \\
& & & 2
\end{array}
\end{array}\]

$C_\sigma = E \circ (132) = [1|3|2]$  

\[\begin{array}{c}
\begin{array}{cccc}
D_3: & 1 & \downarrow & 3 \\
& & & 2
\end{array}
\end{array}\]

$C_\sigma = E \circ (E \times 1) = [1|2|3]$  

\[\begin{array}{c}
\begin{array}{cccc}
D_2: & 1 & \downarrow & 2 \\
& & & 3
\end{array}
\end{array}\]

$C_\sigma = E \circ (E \times 1) = [1|2|3]$  

\[\begin{array}{c}
\begin{array}{cccc}
D_4: & 1 & \downarrow & 2 \\
& & & 3
\end{array}
\end{array}\]

**Figure 8.** Reduced fiber diagrams and elements $C_\sigma \in F(\text{Tree}) \cong F(\text{Tree}) \times \Sigma$ they determine. The symbol $\mathbb{1}$ denotes the identity and $(132) \in \Sigma_3$ the permutation $(1,2,3) \mapsto (1,3,2)$. For $T \in \text{Tree}$, $E_T$ is the corresponding generator of $F(\text{Tree})$.

Span(\text{Tree}), $T \mapsto g_T$, of graded $\Sigma$-sets. For $T \in \text{Tree}$ and $g_T$ the corresponding generator of $G$, put

$$\partial(g_T) := \sum_{\sigma} \pm \iota(C_\sigma),$$

with the sum taken over faces $\sigma$ of $T$ such that $\dim(C_\sigma) = \dim(T) - 1$. The signs are determined by the orientation of the cells. While it is possible to determine the signs for each particular $g_T$, we do not know a reasonable general formula.

**Proposition 18.** Formula (10) extends to a differential on $F(G^\text{reg}) \subset F(G)$ having the form

$$\partial = \partial_{\text{lin}} + \partial_{\text{prt}},$$

where $\partial_{\text{lin}}$ is as in Definition 4.

**Proof.** The first part of the proposition follows from the fact that (10) calculates the cellular differential of the regular cell complex $F^\text{reg}$. Let us prove that the linear part of (10) coincides with $\partial_{\text{lin}}$.

It is obvious that, for $T \in \text{Tree}^\text{reg}$, $\partial_{\text{lin}}(g_T)$ is given by the sum (10) restricted to the faces $\sigma : T \rightarrow S$ with trivial reduced fibers. Equivalently, we restrict to $\sigma$’s that induce isomorphisms $\sigma_h : [k_h] \cong [s_h]$ of the tips. Batanin calls, in 3, such maps $\sigma$ *quasibijections*. The reduced fiber diagram of a quasiisomorphism $\sigma$ is simply $S$ with the tips labeled by the permutation $\sigma_h$, see $D_3$ in Figure 8 for an example. If we denote this labeled tree by $S_\sigma := (S, \sigma_h)$, then $C_\sigma = S_\sigma \in \text{Tree} \subset F(\text{Tree})$. Condition $\dim(S) = \dim(T) - 1$ means that the tree $S$ has one edge less than $T$. Each such $S$ is obtained from $T$ by the following procedure.

Assume that $T$ is as in (2) and choose $1 \leq m < h$ such that there exists $u \in [k_m]$ satisfying $\rho_{m-1}(u) = \rho_{m-1}(u+1)$. Let $b_1, \ldots, b_s$ (resp. $b_{s+1}, \ldots, b_{s+t}$) be the branches of $T$ over $u$ (resp. $u+1$). By a *branch* over $u$ we mean a subtree determined by a vertex $\tilde{u} \in [k_{m+1}]$ satisfying $\rho_m(\tilde{u}) = u$. 

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The corresponding branch is the maximal subtree of $T$ of height $h - m$ whose trunk is the edge connecting $\bar{u}$ and $u$. Branches over $u + 1$ are defined analogously. The situation is shown in Figure 9.

Choose finally an $(s,t)$-unshuffle $\tau \in \Sigma_{s,t}$ and denote by $S$ the labeled tree obtained from $T$ by identifying the edge $e'$ starting from $u$ with the edge $e''$ starting at $u + 1$, and permuting the branches $b_1, \ldots, b_s, b_{s+1}, \ldots, b_{s+t}$ according the shuffle $\tau$, see again Figure 9. Let $\sigma : T \to S$ be the projection. It is clear that all codimension-one faces $\sigma$ of $T$ are of this form and that $g_\sigma \in G$ corresponds, under the isomorphism $\omega : G \to B_{1,\ldots,1}^\infty(V)$ defined on page 8, to a component of the top-level differential in $B^{m+1}(V) \subset B^\infty(V)$ applied to $\omega(g_T)$. \hfill $\Box$

3.2. Proof of Theorem A. Recall that the cobar construction on a coaugmented cooperad $\mathcal{C}$ is the dg-operad $\Omega(\mathcal{C})$ of the form $\Omega(\mathcal{C}) = (F(s_! \mathcal{C}), \partial_\Omega)$, where $\mathcal{C}$ is the co-augmentation co-ideal of $\mathcal{C}$ and $s_! \mathcal{C}$ the $\Sigma$-module defined by

$$s_! \mathcal{C}(n) := \text{sgn}_n \otimes \tau^{n-2} \mathcal{C}(n), \ n \geq 2,$$

the product of the signum representation and the suspension of $\mathcal{C}$ iterated $(n - 2)$-times. The differential $\partial_\Omega$ is induced in the standard manner from the structure operations of the cooperad $\mathcal{C}$, see [3], Definition II.3.9].

Let $\mathcal{Lie} = \{ \mathcal{Lie}(n) \}_{n \geq 1}$ be the operad for Lie algebras. It is well-known that each component $\mathcal{Lie}(n)$ of this operad is a finite-dimensional free abelian group. Denote by $\mathcal{Lie}' = \{ \mathcal{Lie}(n)' \}_{n \geq 1}$ the component-wise linear dual of $\mathcal{Lie}$ with the induced cooperad structure. It follows from Theorem 6.7, Fact 6.2 and Proposition 5.2.12 of [14] that the natural morphism

$$\alpha : \Omega(\mathcal{Lie}') \to \mathcal{Com}$$

of dg-operads is a homology isomorphism over $\mathbb{Z}$. This can also be expressed by saying that the operads $\mathcal{Com}$ and $\mathcal{Lie}$ are Koszul dual to each other, and Koszul over $\mathbb{Z}$. The last step is based on Lemma 20 below, formulated in generality that allows to use it in our proof of Theorem B as well. For related results, see Propositions 5.2.13 and 3.2.6 of [14]. Before we formulate the lemma, we explain what precisely we mean by a perturbation.

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Definition 19. Let \( (M, \partial_{\text{lin}}) \) = \( \{(M(n), \partial_{\text{lin}})\}_{n \geq 2} \) be a dg-\( \Sigma \)-module. A perturbation of \( \partial_{\text{lin}} \) is a degree \(-1\) map \( \partial_{\text{prt}} : M \to \mathbb{F}^{\geq 2}(M) \) from \( M \) to the decomposables in the free operad \( \mathbb{F}(M) \) such that \( \partial := \partial_{\text{lin}} + \partial_{\text{prt}} : M \to \mathbb{F}(M) \) extends to a differential of \( \mathbb{F}(M) \).

Let us state the assumptions of the lemma. As everywhere in the paper, we will use the same symbols for \( \Sigma \)-module maps \( M \to \mathbb{F}(M) \) and for their extensions to derivations of \( \mathbb{F}(M) \). Suppose we are given a dg-operad \( (\mathbb{F}(E), \vartheta) \), for some \( \Sigma \)-module \( E = \{E(n)\}_{n \geq 2} \), such that \( E(n) \) is concentrated in degree \( n - 2 \) and \( \vartheta(E) \) consist of decomposable elements in \( \mathbb{F}(E) \). Assume we also have, for each \( n \geq 2 \), a \( \Sigma_n \)-projective resolution \( \gamma_n : (M(n), \partial_{\text{lin}}) \to (E(n), 0) \) such that \( M(n) \) is trivial in degrees \( < n - 2 \). We denote by \( \gamma : \mathbb{F}(M) \to \mathbb{F}(E) \) the map of free operads induced by \( \{\gamma_n\}_{n \geq 2} \).

Suppose there are, for each \( n \geq 4 \), \( \Sigma_n \)-modules \( A(n) \) and \( B(n) \) such that \( M_n(n) = A(n) \oplus B(n) \) (\( A(n) \) and \( B(n) \) are then projective, too). Let us denote by \( \overline{M} = \{\overline{M}(n)\}_{n \geq 2} \) the graded \( \Sigma \)-submodule of \( M \) defined by

\[
\overline{M}_i(n) := \begin{cases} 
M_i(n), & \text{if } n < 4 \text{ or } n \geq 4 \text{ and } i < n, \\
A(n), & \text{if } n \geq 4 \text{ and } i = n, \text{ and} \\
0, & \text{in the remaining cases.}
\end{cases}
\]

Finally, suppose we are given a perturbation \( \overline{\partial}_{\text{prt}} \) of the restricted differential \( \overline{\partial}_{\text{lin}} := \partial_{\text{lin}}|_{\overline{M}} \) such that the restricted map \( \gamma : (\mathbb{F}(\overline{M}), \overline{\partial}) \to (\mathbb{F}(E), \vartheta) \), where \( \overline{\partial} := \overline{\partial}_{\text{lin}} + \overline{\partial}_{\text{prt}} \), is a morphism of dg-operads.

Lemma 20. In the above situation, there exists a perturbation \( \partial_{\text{prt}} \) of the differential \( \partial_{\text{lin}} \) on \( \mathbb{F}(M) \) that extends the given restricted perturbation \( \overline{\partial}_{\text{prt}} \) such that \( \gamma : (\mathbb{F}(M), \partial) \to (\mathbb{F}(E), \vartheta) \), where \( \partial := \partial_{\text{lin}} + \partial_{\text{prt}} \), is a map of dg-operads inducing an isomorphism of homology.

Proof. We start by stating three simple facts.

(i) From degree reasons, for an arbitrary extension \( \partial_{\text{prt}} \) of the perturbation \( \overline{\partial}_{\text{prt}} \), the extended differential \( \partial = \partial_{\text{lin}} + \partial_{\text{prt}} \) will always commute with the homomorphism \( \gamma : (\mathbb{F}(M), \partial) \to (\mathbb{F}(E), \vartheta) \). We therefore do not need to check the compatibility with \( \gamma \).

(ii) By an easy spectral sequence argument, for every extension \( \partial \) as in (i), the map \( \gamma : (\mathbb{F}(M), \partial) \to (\mathbb{F}(E), \vartheta) \) is a homology isomorphism, i.e. the last property of the lemma is automatic.

(3) Since the free operad functor is a tensor functor, \( H_*(\mathbb{F}(M), \partial_{\text{lin}}) \cong \mathbb{F}(H_*(M, \partial_{\text{lin}})) \cong \mathbb{F}(E) \). Again from simple degree reasons, \( H_i(\mathbb{F}^{\geq 2}(M), \partial_{\text{lin}})(n) = 0 \) for \( i \geq n - 2 \).

By (i), the only equation for the extended differential \( \partial = \partial_{\text{lin}} + \partial_{\text{prt}} \) which has to be checked is \( \partial^2 = 0 \), that is

\[
(\partial_{\text{lin}} + \partial_{\text{prt}})(\partial_{\text{lin}} + \partial_{\text{prt}}) = \partial_{\text{lin}}\partial_{\text{lin}} + \partial_{\text{lin}}\partial_{\text{prt}} + \partial_{\text{prt}}\partial_{\text{lin}} + \partial_{\text{prt}}\partial_{\text{prt}} = 0
\]

which, since \( \partial_{\text{lin}}\partial_{\text{lin}} = 0 \), reduces to

\[
\partial_{\text{lin}}\partial_{\text{prt}} + \partial_{\text{prt}}\partial_{\text{lin}} + \partial_{\text{prt}}\partial_{\text{prt}} = 0.
\]
Decomposing \( \partial_{\text{prt}} = \sum_{k \geq 2} \partial^k_{\text{prt}} \) (locally finite sum), in which \( \partial^k_{\text{prt}} : M \to \mathbb{F}^k(M) \) is the component that sends the generators \( M \) to the sub-\( \Sigma \)-module \( \mathbb{F}^k(M) \subset \mathbb{F}(M) \) spanned by trees with \( k \) vertices, one can expand the above equation into the system

\[
\partial_{\text{lin}} \partial_{\text{prt}}^l + \partial_{\text{lin}}^l \partial_{\text{lin}} + \sum_{i+j=l+1} \partial_{\text{lin}}^i \partial_{\text{lin}}^j = 0
\]

which has to be satisfied for each \( l \geq 2 \). The extension \( \partial_{\text{lin}} \) will be constructed by induction on \( l \), degree \( i \) and arity \( n \) using standard obstruction theory.

Let us start by extending \( \partial_{\text{lin}} \) to \( B(4) \). Let \( k \geq 2 \) and suppose we already constructed, for each \( 2 \leq l < k \), \( \partial^k_{\text{lin}} : B(4) \to \mathbb{F}^l(M)(4) \) satisfying (12). Let us denote

\[
o^k := -\partial^k_{\text{lin}} \partial_{\text{lin}} - \sum_{i+j=k+1} \partial^i_{\text{lin}} \partial^j_{\text{lin}} : B(4) \to \mathbb{F}^k_2(M)(4).
\]

Observe that the term \( \partial^k_{\text{lin}} \partial_{\text{lin}} \) in the above display has already been defined. It is simple to verify that \( \partial_{\text{lin}} o^k = 0 \), that is, \( o^k \) is in fact a map from \( B(4) \) to 2-dimensional \( \partial_{\text{lin}} \)-cycles in \( \mathbb{F}^{\geq 2}(M)(4) \). It follows from the \( \partial_{\text{lin}} \)-acyclicity (iii) and the projectivity of \( B(4) \) that there exists a map \( \partial_{\text{lin}}^k : B(4) \to \mathbb{F}^k(M)(4) \) such that \( o^k = \partial_{\text{lin}} \partial_{\text{lin}}^k \) which is the same as

\[
\partial_{\text{lin}}^l \partial_{\text{lin}}^l + \partial_{\text{lin}}^l \partial_{\text{lin}} + \sum_{i+j=k+1} \partial_{\text{lin}}^i \partial^j_{\text{lin}} = 0.
\]

Repeating this process we extend \( \partial_{\text{lin}} \) onto \( B(4) \). In exactly the same way, we extend \( \partial_{\text{lin}} \) onto \( M(n) \) for \( n > 4 \) is the same.

The existence of the operad \( J = (\mathbb{F}(G), \partial) \), \( \partial = \partial_{\text{lin}} + \partial_{\text{lin}} \), as in Theorem A now follows from Lemma [20] in which we take \( E \) the \( \Sigma \)-module whose \( n \)th component equals \( \text{sgn}_n \otimes \uparrow^{n-2} \text{Lie}(n)' \) and \( \vartheta \) the cobar differential, i.e., \( \mathbb{F}(E), \vartheta := \Omega(\text{Lie}') \). As \( (M, \partial_{\text{lin}}) \) we take the \( \Sigma \)-module \( (G, \partial_{\text{lin}}) \) and set \( A(n) \) to be trivial for each \( n \geq 4 \), so \( M = \overline{M} \). In place of \( \partial = \partial_{\text{lin}} + \partial_{\text{lin}} \) we take the differential from Proposition [15]. Since the \( \Sigma \)-module \( M \) is, by Theorem [33], a component-wise \( \Sigma \)-free resolution of the collection \( E \) defined above, the assumptions of the lemma are fulfilled.

The operad \( J \) resolves \( \text{Com} \) via the composition

\[
J = (\mathbb{F}(G), \partial) \xrightarrow{\gamma} \Omega(\text{Lie}') \xrightarrow{\alpha} \text{Com},
\]

of the map \( \gamma \) of Lemma [20] and \( \alpha \) in (11). It remains to prove that \( J \) is cofibrant. To this end we show that there exists a totally ordered set \( \Lambda \) such that \( G \) decomposes as \( G = \bigoplus_{\lambda \in \Lambda} G_{\lambda} \) and

\[
\partial(G_{\lambda}) \subset \mathbb{F}(G)_{<\lambda}, \text{ for each } \lambda \in \Lambda,
\]

where \( \mathbb{F}(E)_{<\lambda} \) denotes the suboperad of \( \mathbb{F}(E) \) generated by \( \bigoplus_{\lambda' < \lambda} E_{\lambda'} \). It follows from [26, Lemma 20] that (13) guarantees the lifting property of \( J \) with respect to trivial fibrations, so \( J \) is cofibrant in the standard model structure of the category of operads [8, Example 3.3.3].

Let \( \Lambda := \mathbb{N} \times \mathbb{N} \) be the cartesian product of two copies of natural numbers with the lexicographic order and, for \( (i, n) \in \Lambda \), \( G_{(i, n)} := G_i(n) \) (the subspace of arity \( n \) and degree \( i \)). While obviously \( \partial_{\text{lin}}(G_{(i, n)}) \subset G_{i-1}(n) \), it follows from simple combinatorics that \( \partial_{\text{lin}}(G(n)) \) consists of compositions of elements of arities \( < n \). This establishes (13) for the sum \( \partial = \partial_{\text{lin}} + \partial_{\text{lin}} \) and finishes our proof of Theorem A.
3.3. **Some formulas.** In this subsection we calculate the differential of some low-dimensional generators of the operad $J = \mathcal{F}(G)$. Recall that, for a tree $T \in \text{Tree}(n)$, we denoted by $g_T$ the corresponding generator of $G(n)$. We will denote a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \in \Sigma_n$ by the $n$-tuple $(\sigma_1, \ldots, \sigma_n)$, where $\sigma_i := \sigma^{-1}(i)$ for $1 \leq i \leq n$.

The degree 0 generator $g_{\psi}$ is mapped to the commutative associative multiplication in $\text{Com}$. Of course, $\partial(g_{\psi}) = 0$. The degree one generator $g_{\psi}$ is the ‘associator’ and

$$\partial(g_{\psi}) = g_{\psi} \circ (g_{\psi} \otimes \mathbb{1} - \mathbb{1} \otimes g_{\psi}).$$

The second degree one generator $g_{\Upsilon}$ represents the homotopy for the commutativity of $g_{\psi}$:

$$\partial(g_{\Upsilon}) = g_{\psi}(1 \otimes (21)).$$

The degree two generator $g_{\psi}$ is the Stasheff/Mac Lane pentagon and we all know the formula

$$\partial(g_{\psi}) = g_{\psi}(g_{\psi} \otimes \mathbb{1}^{\otimes 2}) - g_{\psi}(\mathbb{1} \otimes g_{\psi} \otimes \mathbb{1}) + g_{\psi}(\mathbb{1}^{\otimes 2} \otimes g_{\psi}) - g_{\psi}(g_{\psi} \otimes \mathbb{1}) - g_{\psi}(\mathbb{1} \otimes g_{\psi})$$

from kindergarten, see Figure 10. The degree two generator $g_{\Upsilon}$ is the left hexagon whose differential is given by

$$\partial(g_{\Upsilon}) = g_{\psi} \circ (\mathbb{1} - (132) + (312)) - g_{\psi} \circ (g_{\psi} \otimes \mathbb{1}) + g_{\psi} \circ (\mathbb{1} \otimes g_{\psi}) + g_{\psi} \circ (g_{\psi} \otimes \mathbb{1})(132),$$

see Figure 10. The formula for the right hexagon $g_{\Upsilon}$ is a similar:

**Figure 10.** Pentagon, left hexagon and disk.
\[ \partial(g_{\cup}) = g_{\cup} \circ (\mathbb{1} - (213) + (231)) + g_{\cup} \circ (\mathbb{1} \otimes g_{\cup}) - g_{\cup} \circ (\mathbb{1} \otimes g_{\cup})(213) - g_{\cup} \circ (g_{\cup} \otimes \mathbb{1}). \]

The last degree two generator \( g_{\cup} \) is the homotopy for the anticommutativity of \( g_{\cup} \):
\[ \partial(g_{\cup}) = g_{\cup}(\mathbb{1} + (21)), \]
see again Figure 10. The differential of the degree three generator \( g_{\Psi} \) is
\[ \partial(g_{\Psi}) = g_{\Psi} \circ (\mathbb{1} - (213)) - g_{\Psi} \circ (\mathbb{1} - (132)) - g_{\Psi} \circ (g_{\Psi} \otimes \mathbb{1} - \mathbb{1} \otimes g_{\Psi}). \]
Let us give also a formula for \( \partial(g_{\cup}) \):
\[ \partial(g_{\cup}) = g_{\cup} - g_{\cup} \circ (312) + g_{\cup} \circ (g_{\cup} \otimes \mathbb{1}) - g_{\cup} \circ (\mathbb{1} \otimes g_{\cup}) - g_{\cup} \circ (g_{\cup} \otimes \mathbb{1})(132). \]

3.4. \( E_\infty \)-algebras. As explained in the Introduction, algebras over the differential graded operad \( J = (J, \partial) \) are particular realizations of \( E_\infty \)-algebras. A structure of this \( E_\infty \)-algebra on a dg-abelian group \( V = (V, d) \) is given by multilinear maps \( \mu : V^{\otimes n} \rightarrow V \) indexed by reduced trees. The degree of \( \mu_T \) equals \( \text{dim}(T) \) and the arity \( n \) equals the number of the tips of \( T \). The axioms could be read off from the formulas for differential \( \partial \) given in Subsection 3.3. One gets
\[
\begin{align*}
(14a) \quad & \delta \mu_{\cup} (a, b) = 0, \\
(14b) \quad & \delta \mu_{\cup} (a, b, c) = \mu_{\cup} (\mu_{\cup} (a, b, c)) - \mu_{\cup} (a, \mu_{\cup} (b, c)), \\
(14c) \quad & \delta \mu_{\cup} (a, b) = \mu_{\cup} (a, b) - \mu_{\cup} (b, a), \\
(14d) \quad & \delta \mu_{\cup} (a, b, c) = \mu_{\cup} (a, b, c) - \mu_{\cup} (a, b, c) + \mu_{\cup} (c, a, b) - \mu_{\cup} (\mu_{\cup} (a, b, c)) + (-1)^{\text{deg}(a)} \mu_{\cup} (a, \mu_{\cup} (b, c)) + \mu_{\cup} (\mu_{\cup} (a, c), b), \\
(14e) \quad & \delta \mu_{\cup} (a, b, c) = \mu_{\cup} (a, b, c) - \mu_{\cup} (b, a, c) + \mu_{\cup} (b, c, a) - \mu_{\cup} (\mu_{\cup} (a, b, c)) + (-1)^{\text{deg}(a)} \mu_{\cup} (a, \mu_{\cup} (b, c)) - (-1)^{\text{deg}(b)} \mu_{\cup} (b, \mu_{\cup} (a, c)), \\
(14f) \quad & \delta \mu_{\cup} (a, b, c, d) = \mu_{\cup} (\mu_{\cup} (a, b, c), d) - \mu_{\cup} (a, \mu_{\cup} (b, c), d) + \mu_{\cup} (a, b, \mu_{\cup} (c, d)) - \mu_{\cup} (\mu_{\cup} (a, b, c), d) - (-1)^{\text{deg}(a)} \mu_{\cup} (a, \mu_{\cup} (b, c, d)), \\
(14g) \quad & \delta \mu (a, b) = \mu (a, b) + \mu (b, a), \\
& \quad \delta \mu (a, b, c) = \mu (a, b, c) - \mu (b, a, c) - \mu (a, b, c) + \mu (a, c, a) + \mu (a, c, b) - \mu (\mu (a, b, c)) + (-1)^{\text{deg}(a)} \mu (a, \mu (b, c)) - (-1)^{\text{deg}(b)} \mu (b, \mu (a, c)), \\
& \quad \delta \mu (a, b, c) = \mu (a, b, c) - \mu (c, a, b) + \mu (\mu (a, b, c)) - \mu (a, \mu (b, c)) - \mu (\mu (a, c, b)), \text{ &c.}
\end{align*}
\]

In the above formulas, \( a, b, c, d \) are homogeneous elements of \( V \), and \( \delta \) the induced differential in the endomorphism complex of \( V = (V, d) \). For example
\[ \delta \mu_{\cup} (a, b) := d \mu_{\cup} (a, b) - \mu_{\cup} (da, b) - (-1)^{\text{deg}(a)} \mu_{\cup} (a, db). \]

Some of the above axioms have obvious interpretations. Axiom (14b) says that \( \mu_{\cup} \) is a (chain) homotopy for the multiplication \( \mu_{\cup} \), axiom (14c) means that \( \mu_{\cup} \) is \( -1 \) for \( \mu_{\cup} \) and (14g) means [July 31, 2021]
that $\mu_\bigcup$ is $\sim_2$ for $\mu_\bigcup$. Axioms (14d) and (14e) are algebraic versions of the left and right hexagons. Axiom (14f) is an algebraic version of the pentagon.

More generally, if $\mu_n := \mu_\bigstar^n$ with $\bigstar^n \in \text{Tree}^1(n)$ the $n$-corolla with the barcode $[1|\cdots|n]$, then $(V, d, \mu_2, \mu_3, \ldots)$ is an $A_\infty$-algebra, with (14a), (14b) and (14f) Axiom (1) of [21] for $n = 2, 3$ and 4. This justifies calling the operad $J$ an extension of Stasheff’s operad. Axioms (14a)–(14g) were already obtained in Example 4.8 of the 1996 paper [22].

4. The Tamarkin cell mystery

The first example of a cell violating the regularity of the CW-complex $F$ was found by Dimitri Tamarkin. It is a 6-dimensional cell $T \subset F(6)$ which actually lives in the subcomplex $F_2(6)$ of compactified configurations of six points in $\mathbb{R}^2$. Surprisingly, there exist even a simpler, 4-dimensional ‘bad’ cell $\mathcal{M} \subset F(4)$ living in the subcomplex $F_3(4)$ of compactified configurations of 4 points in $\mathbb{R}^3$. It will be clear from Section 3 that $\mathcal{M}$ has the smallest possible dimension.

In this section we analyze the above two cells and show how to construct the differential of the corresponding generators of $G$. Theorem B proved at the end of this section shows that any such a ‘partial’ differential extends to a ‘global’ one.

4.1. The Tamarkin cell. The Tamarkin cell $T$ corresponds to the tree $T$ with the barcode $[1|2||3|4||5|6]$ shown in Figure 11 (left). Consider the tree $S := [1|2]$ and the map $\nu : T \to S$ that

![Figure 11. The Tamarkin tree $T$ (left), the tree $S$ (right bottom), the map $\nu : T \to S$ and its fiber diagram.](image)

sends the tips of $T$ labeled 1,3,5 (resp. labeled 2,4,6) to the tip 1 (resp. tip 2) of $S$, see Figure 11. We easily read off from the fiber diagram of $\nu$ that $\mu[\nu] = \mu[[1|3][5]|2|4|6]]$. Simple degree count shows that $\dim(\mu[\nu]) = 6$ i.e. it is the same as the dimension of the Tamarkin cell $T$!

The explanation is that the face $\mu[\nu]$ is not a subset of the boundary $\partial T$ of $T$ but that $\partial T$ intersect $\mu[\nu]$ in a 5-dimensional subspace which is not a union of 5-dimensional cells. This violates Definition 13(ii). An “ideological” picture of this situation is shown in Figure 12.
Let us analyze this phenomenon in detail. If we denote by \( c : \mathbb{R}^2 \to \mathbb{R} \) the projection to the first coordinate, then each point \( x = (x_1, \ldots, x_6) \) in the interior of \( \mathcal{T} \) satisfies
\[
\frac{c(x_3) - c(x_1)}{c(x_5) - c(x_3)} = \frac{c(x_4) - c(x_2)}{c(x_6) - c(x_4)}.
\]
Observe that both sides are invariant under the affine group action. The same condition is satisfied also by the points in the intersection \( \partial \mathcal{T} \cap \mu[\nu] \). More precisely, \( \mu[\nu] \) consists of two ‘microscopic’ configurations \( x_u \) (resp. \( x_b \)) of points \((x_2, x_4, x_6)\) (resp. \((x_1, x_2, x_3)\)) in \( F(3) \) arranged at the vertical line (Figure 13 middle). Since the points in \( \partial \mathcal{T} \cap \mu[\nu] \) are the limits of the points in the interior of \( \mathcal{T} \), the configurations \( x_u \) and \( x_b \) are still tied by (15), see Figure 13 (right). Therefore the intersection \( \partial \mathcal{T} \cap \mu[\nu] \) is a codimension-one subspace of \( \mu[\nu] \). Loosely speaking, when the point \( x \in \mathcal{T} \) moves towards the boundary, it still ‘remembers’ that its coordinates were lined up at tree vertical lines parametrized by a point in \( K(3) = F_1(3) \). This is a particular instance the ‘source-target’ condition of in a globular category \( \mathcal{E} \), see also formula (17) in Section 3.

In the rest of this subsection \( T, S \) and \( \nu : T \to S \) will have the same meaning as above. Let us try to determine the value \( \partial(g_T) \) of the differential on the generator \( g_T \in G(6) \) corresponding to the Tamarkin tree. Inspired by (14), define
\[
\partial_{\text{reg}}(g_T) := \sum_{\sigma} \pm \epsilon(C_{\sigma}),
\]
with the sum taken over all ‘regular’ faces $\sigma$ of $T$, i.e. faces $\sigma$ such that $\dim(C_\sigma) = \dim(T) - 1$. In Figure 12, the union of these faces is denoted $\partial_{\text{reg}} \mathcal{T}$. Since $\partial_{\text{reg}}(g_T)$ is of dimension $5 < d_{\text{crit}}(6)$, the value $\partial(\partial_{\text{reg}}(g_T))$ is determined by calculations of Subsection 3.1 and equals the sum of elements of $\mathbb{F}(G)$ corresponding to the 4-dimensional cells in the intersection $\partial_{\text{reg}} \mathcal{T} \cap \partial_{\text{sng}} \mathcal{T}$ marked by two bullets $\bullet$ in Figure 13. In particular, $\partial(\partial_{\text{reg}}(g_T)) \neq 0$. We shall find a ‘counterterm’ $\partial_{\text{sng}}(g_T)$ such that $\partial(\partial_{\text{reg}}(g_T)) = -\partial(\partial_{\text{sng}}(g_T))$ and put

$$\partial(g_T) := \partial_{\text{reg}}(g_T) + \partial_{\text{sng}}(g_T).$$

The idea of finding such a counterterm is clear from Figure 12; $\partial_{\text{sng}}(g_T)$ shall correspond to an union of 5-dimensional cells $U$ such that $\partial U = \partial_{\text{reg}} \mathcal{T} \cap \partial_{\text{sng}} \mathcal{T}$. In the ideological Figure 13, there are two such unions, the ‘upper’ $U'$ and the ‘lower’ $U''$ which indicates that the choice of $U$ need not be unique.

The first step is to identify 4-dimensional cells in the intersection $\partial_{\text{reg}} \mathcal{T} \cap \partial_{\text{sng}} \mathcal{T}$. The extended barcodes of these cells will be of the form $[b_1|b_2]$ for some barcodes $b_1, b_2$. To shorten the formulas, we use an ‘additive’ notation for the corresponding cells, so that $\mu[b_1' \cup b_2''|b_2] = \mu[b_1'|b_2] \cup \mu[b_2''|b_2]$, &c. With this notation, one easily expresses $\partial_{\text{reg}} \mathcal{T} \cap \partial_{\text{sng}}$ as the union of 4-dimensional cells:

$$\partial_{\text{reg}} \mathcal{T} \cap \partial_{\text{sng}} \mathcal{T} = \mu \left[ \bigcup_{\tau \in \Sigma_{1,3,5}} \left[ \tau \_1 \| \tau_3 \| \tau_5 \right] \cup \bigcup_{\tau \in \Sigma_{1,3,5}} \left[ \tau_1 \| \tau_3 \| \tau_6 \right] \cup \bigcup_{\tau \in \Sigma_{1,3,5}} \left[ \left[ \{1\} \| \{3\} \| \{5\} \right] \cup \bigcup_{\tau \in \Sigma_{2,4,6}} \left[ \tau_2 \| \tau_4 \| \tau_6 \right] \cup \bigcup_{\tau \in \Sigma_{2,4,6}} \left[ \{2\} \| \{4\} \| \{6\} \right] \cup \bigcup_{\tau \in \Sigma_{1,3}} \left[ \{1\} \| \{3\} \| \{5\} \right] \cup \bigcup_{\tau \in \Sigma_{2,4}} \left[ \{1\} \| \{3\} \| \{5\} \right] \cup \bigcup_{\tau \in \Sigma_{1,3}} \left[ \{2\} \| \{4\} \| \{6\} \right] \cup \bigcup_{\tau \in \Sigma_{1,6}} \left[ \{2\} \| \{4\} \| \{6\} \right] \right].$$

In the above display, $\Sigma_{1,3,5}$ is the group of permutations of the set $\{1, 3, 5\}$, and the symbols $\Sigma_{2,4,6}, \Sigma_{1,3}, \Sigma_{2,4}, \Sigma_{3,5}$ and $\Sigma_{4,6}$ have the obvious similar meanings. Moreover,

$$\Sigma_{1,3,5} := \{ \tau \in \Sigma_{1,3,5}; \tau_3 < \tau_5 \}, \Sigma_{1,3,5} := \{ \tau \in \Sigma_{1,3,5}; \tau_1 < \tau_3 \},$$

$$\Sigma_{2,4,6} := \{ \tau \in \Sigma_{2,4,6}; \tau_4 < \tau_6 \} \text{ and } \Sigma_{2,3,6} := \{ \tau \in \Sigma_{2,3,6}; \tau_2 < \tau_4 \}.$$

The structure of the right hand side should be clear from Figure 14 which shows, without specifying the labels, generic points of the corresponding configurations.

The four boxes of this figure correspond to the four lines of the display. One of the possible choices for the set $U$ is then

$$U' := \mu \left[ \{1\} \| \{3\} \| \{5\} \cup \bigcup_{\tau \in \Sigma_{1,3,5}} \left[ \left[ \{1\} \| \{3\} \| \{5\} \right] \cup \bigcup_{\tau \in \Sigma_{2,4}} \left[ \{1\} \| \{3\} \| \{5\} \right] \cup \bigcup_{\tau \in \Sigma_{1,3}} \left[ \{2\} \| \{4\} \| \{6\} \right] \cup \bigcup_{\tau \in \Sigma_{2,4}} \left[ \{2\} \| \{4\} \| \{6\} \right] \right].$$
Another choice is the ‘diagonal image’

\[ U'' := \mu \left[ \left[ 1 \| 3 \| 5 \right] \cup \bigcup_{\tau \in \Sigma_{1,3}} [\tau_1 \| 3 \| 5] \bigcup \left[ 2 \| 4 \| 6 \right] \cup \mu \left[ \left[ 1 \| 3 \| 5 \right] \bigcup \bigcup_{\tau \in \Sigma_{4,6}} [2 \| 4 \| 6] \right] \right]. \]

Generic points of the corresponding cells are shown in Figure 13. In both cases, the counterterm

\[ \partial_{\text{sing}}(g_T) \]

is the sum of 6 terms corresponding to the six 4-cells of \( U' \) resp. \( U'' \).

4.2. A 4-dimensional bad cell. It is the cell \( \mathscr{U} := \mu[T] \) indexed by the reduced 3-tree \( T := [1]2\|3\|4 \) shown in Figure 14. A generic point of this cell is presented in Figure 14. Consider the 3-tree \( S := [1]2 \) and the map \( \nu : T \to S \) that sends the tips of \( T \) labeled 1,3 (reps. 2,4) to [July 31, 2021]
Figure 16. The tree $T$ (left), the tree $S$ (right bottom), the map $\nu : T \to S$ and its fiber diagram.

Figure 17. A generic point of the cell $\mathcal{M}$. The points in the ambient $\mathbb{R}^3$ lie on two lines parallel to the 3rd coordinate. The 2nd coordinate of the points labeled 1 and 2 is less than the 2nd coordinate of the points labeled 3 and 4.

the tip 1 (resp. tip 2) of $S$, see again Figure 16. It is clear that the corresponding face $\mu[\nu]$ of $\nu$ equals $\mu[[1||3][2||4]]$ and that $\dim(\mu[\nu]) = \dim(\mathcal{M}) = 4$.

Let us determine the value $\partial(g_T)$ of the differential on the 4-dimensional generator $g_T \in G_4(4)$ corresponding to $T$. As in Subsection 4.1, define

$$
\partial_{\text{reg}}(g_T) := \sum_{\sigma} \pm \nu(C_{\sigma}),
$$

with the sum over all faces $\sigma$ such that $\dim(C_{\sigma}) = 3$. One easily sees that these faces form the union

$$
\partial_{\text{reg}} \mathcal{M} = \mu[1|2||3|4] \cup \mu[3|4||1|2] \cup \mu[1|2||3|4] \cup \mu[[1|2]|3|4] \\
\cup \bigcup_{\tau \in \Sigma_{1,3}} \mu[\tau_1|\tau_3][2||4] \cup \mu[1][2||3|4] \cup \mu[3][1||4|2] \cup \mu[1||3][\tau_2|\tau_4].
$$

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Observe that the first two terms give the linear part of $\partial_{\text{lin}}(g_T)$. As in Subsection 4.1 we need to describe 2-dimensional cells in the intersection $\partial_{\text{reg}} M \cap \partial_{\text{sng}} M$. The result is:

\[
\partial_{\text{reg}} M \cap \partial_{\text{sng}} M = \mu \left( \bigcup_{\tau \in \Sigma_{1,3}} [\tau_1|\tau_3][2||4] \bigcup [1||3] \bigcup [\tau_2|\tau_4] \right) \\
\cup \mu \left( [1||3][2||4] \bigcup [3||1][4||2] \right),
\]

where $\Sigma_{1,3}$ (resp. $\Sigma_{2,4}$) is the group of permutations of the set $\{1,3\}$ (resp. $\{1,3\}$). One of the possible choices for the set $U$ of 3-cells generating the counterterm $\partial_{\text{sng}}(g_T)$ is then

\[
U' := \mu \left( [1||3][4||2] \bigcup [1||3][2||4] \right).
\]

The second one is the ‘diagonal image’

\[
U'' := \mu \left( [1||3][2||4] \bigcup [3||1][2||4] \right).
\]

In both cases, the counterterm $\partial_{\text{sng}}(g_T)$ has 2 terms corresponding to two 3-cells of $U'$ resp. $U''$. The differential $\partial(g_T)$ is the sum of 2 linear terms, 8 regular decomposable terms and 2 singular terms. As in Subsection 4.1, it helps to represent the cells entering the above calculations by depicting their generic points. We leave it to the reader as an exercise.

4.3. **Proof of Theorem B.** Let $\mathcal{Bad}$ be a set of unlabeled trees indexing (one or more) cells in the critical dimension (such as $\mathcal{T}$ or $\mathcal{M}$ above). Suppose that, for each $T \in \mathcal{Bad}$, we found an element $\partial_{\text{prt}}(g_T) \in \mathbb{F}^{\geq 2}(G)$ such that

\[
\partial(\partial_{\text{lin}} + \partial_{\text{prt}})(g_T) = 0.
\]

Since $T$ has the critical dimension, $(\partial_{\text{lin}} + \partial_{\text{prt}})(g_T) \subset \mathbb{F}(G^{\text{reg}})$ and the above formula makes sense. Examples of $\partial_{\text{prt}}(g_T)$ are given in Subsections 4.1 and 4.2.

Now we take, in Lemma 11, $(\mathbb{F}(E), \theta) := \Omega(Lie')$, $(M, \partial_{\text{lin}}) := (G, \partial_{\text{lin}})$ and as $\overline{\partial} = \overline{\partial}_{\text{lin}} + \overline{\partial}_{\text{prt}}$ we take the differential from Proposition 16. Finally, let $A(n) := \Sigma_n[\mathcal{Bad}(n)]$ be, for $n \geq 4$, the free $\Sigma_n$-module generated by trees $T \in \mathcal{Bad}$ of arity $n$. With these choices, the assumptions of the lemma are clearly satisfied and Theorem B follows.

5. **BAD CELLS**

In the first part of this section we analyze the ‘source-target’ conditions responsible for the existence of bad cells. In the second part we prove Propositions 14 and 15.
5.1. **The source-target conditions.** Assume we are given a pruned unlabeled $h$-tree $T \in \text{Tree}^h(n)$ as in (2). Suppose that there is an $s \geq 2$ and natural numbers

$$1 \leq a_1 < b_1 < a_2 < b_2 \cdots < a_s < b_s \leq k_h = n$$

and some $1 \leq m < h$ such that

$$\rho_m \circ \cdots \circ \rho_{h-1}(a_i) = \rho_m \circ \cdots \circ \rho_{h-1}(b_i),$$

for all $1 \leq i \leq s$. We also assume that the common values of the expression in (16) form a strictly increasing sequence of $s$ elements of $[k_m]$. Suppose there is a $h$-tree $S \in \text{Tree}^h(k)$, $k < n$, and a map $\nu : T \to S$ for which there exist $1 \leq u < v \leq k$ such that $\nu_h(a_i) = u$ and $\nu_h(b_i) = v$ for all $1 \leq i \leq s$.

In this situation, denote $A$ (resp. $B$) the fiber of $\nu$ over $u$ (resp. $v$) and $A'$ (resp. $B'$) the maximal pruned subtree of $A$ (resp. $B$) with the tips $a_1, \ldots, a_s$ (resp. $b_1, \ldots, b_s$). Denote finally $R \in \text{Tree}^m(s)$ the pruned unlabeled $m$-tree obtained from $A'$ by amputating everything above level $m$. Observe that instead of amputating $A'$ we could have amputated $B'$ with the same result. The situation is visualized in Figure 18.

![Figure 18. The origin of the source-target conditions – schematic picture.](image-url)

The tree $R$ determines a cell $\mu[R] \subset F_m(s)$. The **source and target maps** $\pi_s, \pi_t : \mu[C_v] \to \mu[R]$ are defined as follows. Since $\mu[C_v]$ is the cartesian product of the cell $\mu[S]$ with the cells indexed by the fibers of $\nu$, one has the projections $\pi_A : \mu[C] \to \mu[A]$ resp. $\pi_B : \mu[C] \to \mu[B]$. One also has the ‘forgetful’ projections $\pi'_A : \mu[A] \to \mu[A']$ (resp. $\pi'_B : \mu[B] \to \mu[B']$) given by forgetting all points of the configurations in $\mu[A]$ (resp. $\mu[B]$) except those with labels in $\{a_1, \ldots, a_s\}$ (resp. $\{b_1, \ldots, b_s\}$).

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Let finally $\pi'_A : \mu[A'] \to \mu[R]$ (resp. $\pi''_B : \mu[B'] \to \mu[R]$) be the projection induced by the projection $\mathbb{R}^h \to \mathbb{R}^m$ to the first $m$ coordinates. The maps $\pi_s, \pi_t$ are the compositions

$$\pi_s : \mu[C_v] \to \mu[A] \to \mu[A'] \to \mu[R] \text{ and } \pi_s : \mu[C_v] \to \mu[B] \to \mu[B'] \to \mu[R].$$

**Definition 21.** The source-target condition is the following equality of points of $\mu[R]$

$$\pi_s(x) = \pi_t(x) \quad (17)$$

satisfied by each point $x \in \mu[C_v] \cap \partial \mu[T]$.

**Example 22.** In the Tamarkin case analyzed in Subsection 4.1, $h = 2$, $n = 6$, $k = 2$, $m = 1$, the trees $T \in \text{Tree}^2(6)$, $S \in \text{Tree}^2(2)$ and the map $\nu : T \to S$ are as in Figure 11. Moreover, $s = 3$, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $b_1 = 1$, $b_2 = 4$, $b_3 = 6$. The amputated tree $R$ equals $[1|2|3]$ so $\mu[R]$ is the open interval $\mathbb{F}_1(3)$.

For the 4-dimensional bad cell of Subsection 4.2, $h = 3$, $n = 4$, $k = 2$, $m = 2$, the trees $T \in \text{Tree}^3(4)$, $S \in \text{Tree}^3(2)$ and the map $\nu : T \to S$ are as in Figure 16. Moreover, $s = 2$, $a_1 = 1$, $a_2 = 3$, $b_1 = 2$, and $b_2 = 4$. The amputated tree $R$ equals $[1|2|3]$, so $\mu[R]$ is the open half-circle in $\mathbb{F}_2(2) = \mathbb{S}^1$.

### 5.2. Proofs of Propositions 14 and 15

Let us introduce the following terminology. We call a cell of $F$ *regular* if its boundary is an union of cells. For subsets $a \subset F(m)$, $b \subset F(n)$ and $1 \leq i \leq m$ we write, as expected,

$$a \circ_i b := \{x \circ_i y \in F(m + n - 1); \; x \in a, \; y \in b\}.$$

We call $a \circ_i b$ the $\circ_i$-composition of the sets $a$ and $b$.

**Lemma 23.** Let $\sigma \in \Sigma_m$. A cell $e \subset F(m)$ is a regular if and only if the cell $e \nu := \{x \sigma; x \in e\}$ is regular. The $\circ_i$-composition of cells is regular if and only if and only if both factors are regular.

**Proof.** The first part of the lemma is obvious. The second part follows from the equality $\partial(e' \circ_i e'') = \partial e' \circ_i e'' \cup e' \circ_i \partial e''$.

**Theorem 24.** For $n \geq 2$ and $h \geq 1$, each cell of dimension $< d_{\text{crit}}(n)$ in the open part $\mathcal{F}_h(n)$ is regular and all its faces are regular, too. Similarly, each cell of $\mathcal{F}(n)$ of dimension $< d_{\text{crit}}(n)$ is regular along with all its faces.

**Proof.** The theorem can be proved by case-studying cells of the indicated dimensions. We, however, prefer a conceptual approach based on the analysis of the source-target conditions given in Subsection 5.1. In particular, we observe that condition (17) is nontrivial only if the dimension of the cell $\mu[R]$ is at least one. This implies that

- (i) either $s \geq 2$ and $m \geq 2$, or
- (ii) $s \geq 3$ and $m \geq 1$.

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In case (i), one has \( n \geq 4 \) and \( h \geq 3 \), while in case (ii) one has \( n \geq 6 \) and \( h \geq 2 \). Let us prove that each \( e \subset \tilde{\mathcal{F}}_h(n) \) with \( \dim(e) < d^h_{\text{crit}}(n) \) is regular. We again distinguish two cases:

(a) \( d^h_{\text{crit}}(n) = \infty \), which means \( n = 2, 3 \), or \( n = 4, 5 \) and \( h \leq 2 \), or \( n \geq 6 \) and \( h = 1 \),
(b) \( d^h_{\text{crit}}(n) = n \), which means \( n = 4, 5 \) and \( h \geq 3 \), or \( n \geq 6 \) and \( h \geq 2 \).

Case (a) is complementary to the cases (i) and (ii) above, therefore the source-target conditions are trivial and \( e \) is a regular cell. The faces of the cell \( e \) are \( \sigma \)-compositions of cells \( e' \) from \( \tilde{\mathcal{F}}_h(n') \) for some \( 2 \leq n' \leq n \). Since, by the definition of the critical dimension, \( d^h_{\text{crit}}(n') \geq d^h_{\text{crit}}(n) \), one has \( d^h_{\text{crit}}(n') = \infty \). As we already established, this implies that each such an \( e' \) is regular, therefore, by Lemma 23, each face of \( e \) is regular, too.

Let us assume (b), i.e. \( d^h_{\text{crit}}(n) = n \). Since, by Lemma 23, the symmetric group action preserves the regularity, we may assume that \( e = \mu[T] \) for a reduced tree \( T \in \text{Tree}^h(n) \) as in (2) with \( \dim(T) < n \). By simple combinatorics,

\[ (\alpha) \text{ either } h = 1 \text{ or } (\beta) \text{ } h = 2 \text{ and } k_1 = 2. \]

In the first case the cell \( \mu[T] \) belongs to the Stasheff polytope \( K(n) = \mathcal{F}_1(n) \subset \mathcal{F}(n) \), so \( \mu[T] \) and all its faces are regular cells. In case (\( \beta \)), the amputated tree \( R \) must be \([1|2]\), thus the corresponding source-target condition is trivial, so \( \mu[T] \) is a regular cell. It is not difficult to verify that if \( T \) satisfies (\( \beta \)), \( S \) is an arbitrary tree and \( \nu : T \to S \) a map, then each reduced fiber of \( \nu \) also satisfies (\( \alpha \)) or (\( \beta \)) above. This implies that all faces of \( \mu[T] \) are regular cells, too.

This finishes the proof of the first part of the theorem. Since each cell \( e \subset \tilde{\mathcal{F}}(n) \) belongs to some \( \tilde{\mathcal{F}}_h(n), h \geq 1 \), the second part follows from the first part and the inequality \( d_{\text{crit}}(n) \leq d^h_{\text{crit}}(n) \). \( \square \)

**Proof of Proposition 14.** The compatibility of the CW-structures of the \( \Sigma \)-modules \( \mathcal{F}_h, h \geq 1 \), and \( \mathcal{F} \) with the operad structures and the freeness of the symmetric group action on the cells follows from the very definition of the cell structure reviewed on page 31. Recall 3, 32 that there is the canonical embedding

\[ \iota : \mathcal{F}_h(n) \hookrightarrow \bigtimes_{1 \leq i,j \leq n, i \neq j} S^{h-1} \times \bigtimes_{1 \leq i,j,k \leq n, i \neq j, j \neq k, k \neq i} [0, \infty]. \]

It follows from the analysis of the images of the cells of \( \mathcal{F}_h(n) \) under \( \iota \) given in Section 6 of [3], namely from Proposition 6.1 of that section, that the spaces \( \mathcal{F}_h(n) \) satisfy condition (i) of Definition 13 for arbitrary \( n \) and \( h \). The analogous claim for \( \mathcal{F}(n) \) stems from the fact that each cell of \( \mathcal{F}(n) \) belongs to the subcomplex \( \mathcal{F}_h(n) \) for some \( h \geq 1 \).

Since condition (i) of Definition 13 has already been established, the space \( \mathcal{F}_h(n) \) is regular if and only if each its cell is regular in the sense introduced at the beginning of this subsection. Since the cells of \( \mathcal{F}_h(n) \) are iterated \( \sigma \)-compositions of the cells from \( \tilde{\mathcal{F}}_h(n') \) with \( n' \leq n \), the complex \( \mathcal{F}_h(n) \) is, by Lemma 23, regular if and only if all cells of \( \tilde{\mathcal{F}}_h(n') \) are regular, for each \( 2 \leq n' \leq n \). Since clearly \( d^h_{\text{crit}}(n) = \infty \) implies \( d^h_{\text{crit}}(n') = \infty \) for each \( n' \leq n \), the space \( \mathcal{F}_h(n) \) is, by Theorem 24, regular if \( d^h_{\text{crit}}(n) = \infty \).
The non-regularity of \( F_h(n) \) if \( d_{\text{crit}}^h(n) \) is finite follows from Proposition 13 proved below. This, by the remark following Proposition 14, proves the characterization of the regularity of the spaces \( F_h(n) \). The similar obvious analysis applies to \( F(n) \) as well.

The regularity of the suboperads \( F^\text{reg}_h \) and \( F^\text{reg} \) follows from the fact that they are generated by the regular cells in \( \hat{F}_h \) resp. \( \hat{F} \), from Lemma 23 and the fact that cells in the boundary of regular cells are regular established in Theorem 24. This finishes the proof.

Proof of Proposition 13. We put \( e_4^h := \mathcal{M} = \mu[1|2||3|4] \), the cell introduced in Subsection 4.2. The cell \( e_4^h \) for \( h \geq 4 \) is defined as the image of \( e_4^3 \) under the natural inclusion \( F_3(4) \hookrightarrow F_h(4) \). Likewise, we put \( e_5^3 := \mu[1|2||3|4|5] \) and \( e_6^h \) for \( h \geq 4 \) defined similarly.

The cell \( e_6^2 \) is the Tamarkin cell \( \mathcal{T} = \mu[1|2||3|4|5|6] \), and \( e_5^2 := \mu[1|2||3|4|5|6|\cdots|n] \), for \( n \geq 7 \). The cells \( e_6^h \) for \( h \geq 3 \) and \( n \geq 6 \) are the images of \( e_5^n \) under the natural inclusions \( F_5(n) \hookrightarrow F_h(n) \). The cells \( e_n, n \geq 4 \), are the images of \( e_5^3 \) under the inclusions \( F_3(n) \hookrightarrow F(n) \). We leave to the reader to verify that the cells \( e_n^h \) defined in this way are bad. \( \square \)

6. Free Lie algebras and configuration spaces

In this section we prove integral variants of some results whose characteristic zero versions are known. Therefore, all algebraic objects will be considered over the ring \( \mathbb{Z} \) of integers. The results below easily generalize to an arbitrary integral domain with unit.

Let \( T(x_1, \ldots, x_n) \) be the tensor algebra with generators \( x_1, \ldots, x_n, n \geq 1 \), and \( \mathbb{L}(x_1, \ldots, x_n) \) the free Lie algebra on the same set of generators, considered in the standard way as a subspace of \( T(x_1, \ldots, x_n) \). We denote by \( T_{1,\ldots,1}(x_1, \ldots, x_n) \subset T(x_1, \ldots, x_n) \) the linear subspace spanned by words containing each of the generators \( x_1, \ldots, x_n \) precisely once, and \( \mathbb{L}_{1,\ldots,1}(x_1, \ldots, x_n) := \mathbb{L}(x_1, \ldots, x_n) \cap T_{1,\ldots,1}(x_1, \ldots, x_n) \). We will sometimes simplify the notation and denote \( \text{Lie}(n) := \mathbb{L}_{1,\ldots,1}(x_1, \ldots, x_n) \).

Each space above has a natural right \( \Sigma_n \)-module action permuting the generators. Take another set of generators \( \alpha_1, \ldots, \alpha_n \) and denote by

\[
\Phi : T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n) \rightarrow T_{1,\ldots,1}(x_1, \ldots, x_n)', \tag{18}
\]

where \( T_{1,\ldots,1}(x_1, \ldots, x_n)' \) is the linear dual of \( T_{1,\ldots,1}(x_1, \ldots, x_n) \), the isomorphism defined by

\[
\Phi(\rho_{\alpha(1)} \otimes \cdots \otimes \rho_{\sigma(n)})(x_{\omega(1)} \otimes \cdots \otimes x_{\omega(n)}):= \begin{cases} 1, & \text{if } \rho = \omega, \\
0, & \text{otherwise}, \end{cases}
\]

where \( \rho, \omega \in \Sigma_n \).

For \( s, t \geq 1 \) denote by \( \Sigma_{s,t} \) the set of all \((s, t)\)-unshuffles, i.e. permutations \( \tau \in \Sigma_n, n = s + t \), such that

\[
\tau(1) < \cdots < \tau(s) \text{ and } \tau(s + 1) < \cdots < \tau(s + t).
\]

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Let, finally, $U\sh_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \subset T_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ be the linear span of the expressions
\[
\sum_{\tau \in \Sigma_n} \alpha_{\rho \tau(1)} \otimes \cdots \otimes \alpha_{\rho \tau(n)},
\]
for $\rho \in \Sigma_n$ and $s, t \geq 1$ such that $s + t = n$.

**Theorem 25.** The map (18) induces a $\Sigma_n$-equivariant isomorphism
\[
\Phi : T_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) / U\sh_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \to L_{1,\ldots,1}(x_1,\ldots,x_n)'.
\]

Theorem 25 will follow from a sequence of claims. The first one is probably known, but we were unable to find a reference.

**Claim 26.** For each $n \geq 1$, $\mathcal{L}ie(n) = L_{1,\ldots,1}(x_1,\ldots,x_n)$ is the free abelian group with basis
\[
b_\lambda := [x_{\lambda(1)},[x_{\lambda(2)},\ldots,[x_{\lambda(n-1)},x_n]]], \ \lambda \in \Sigma_{n-1}.
\]

**Proof.** It is known that $L(x_1,\ldots,x_n)$ is torsion free (see, for instance, the overview in [9]), so its subspace $\mathcal{L}ie(n)$ is torsion-free as well. Let us prove, by induction on $n$, that the elements in (19) span $\mathcal{L}ie(n)$.

This statement is obvious for $n = 1,2$. Let $n > 2$. Since $\mathcal{L}ie(n)$ is spanned by elements of the form $[F_1,F_2]$, $F_1 \in \mathcal{L}ie(s)$, $F_2 \in \mathcal{L}ie(t)$, $s + t = n$, $s,t \geq 1$, it suffices to prove that each such $[F_1,F_2]$ is a linear combination of elements of the basis (19). We may clearly assume that $F_1$ contains $x_1,\ldots,x_s$ and $F_2$ contains $x_{s+1},\ldots,x_n$. By induction, $F_2$ is a linear combination of iterated commutators as in (19), with $x_n$ at the extreme right position.

Now we proceed by induction on $s$. If $s = 1$, $[F_1,F_2]$ is an element of (19). If $s \geq 2$ we may assume that $F_1 = [A,B]$, for some $A \in \mathcal{L}ie(a)$, $B \in \mathcal{L}ie(b)$, with $a + b = s$, $a,b \geq 1$. In that case
\[
[F_1,F_2] = [A,[B,F_2]] + [B,[F_2,A]].
\]

By induction on $n$, both $[B,F_2]$ and $[F_2,A]$ are combinations of the commutators as in (19), with $x_n$ at the rightmost place, therefore both $[A,[B,F_2]]$ and $[B,[F_2,A]]$ are combinations of basic elements (19), by induction on $s$.

The linear independence of the elements (19) follows from the well-known fact that the dimension of $\mathcal{L}ie(n)$ is $\left(n - 1\right)!$ [9], which is the number of elements (19). \hfill $\square$

There is a straightforward way to verify the linear independence of the elements (19) based on Claim 27 below which will be useful also for other purposes. Recall that $T_{1,\ldots,1}(x_1,\ldots,x_n)$ is the free abelian group with basis
\[
e_\sigma := x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \ \sigma \in \Sigma_n.
\]
Claim 27. In a (unique) decomposition of \( b_\lambda, \lambda \in \Sigma_{n-1} \), into a linear combination of the basis \( \{ e_\sigma \}_{\sigma \in \Sigma_n} \), the element

\[ e_{\lambda \times 1} = x_{\lambda(1)} \otimes \cdots \otimes x_{\lambda(n-1)} \otimes x_n \]

appears with coefficient 1 and \( b_\lambda \) is the only basis element \( \{17\} \) whose decomposition contains \( e_{\lambda \times 1} \).

Proof. A simple induction on \( n \).

The following proposition is based on famous Theorem 2.2 of [31] that, however, assumes the existence of a solution \( \xi \in R \) of the equation \( n\xi = \alpha \) for each natural \( n \geq 1 \) and each \( \alpha \in R \), in the ground ring \( R \). We show that this assumption is not necessary when this theorem is applied to the subspace \( T_{1,...,1}(x_1, \ldots, x_n) \) not to the whole \( T(x_1, \ldots, x_n) \).

Proposition 28. An element \( F = \sum_{\sigma \in \Sigma_n} a_\sigma \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \in T_{1,...,1}(x_1, \ldots, x_n), a_\sigma \in \mathbb{Z}, \) belongs to the subspace \( L_{1,...,1}(x_1, \ldots, x_n) \) if and only if

\[ \sum_{\tau \in \Sigma_s,t} a_{\rho \tau} = 0, \]

for each permutation \( \rho \in \Sigma_n \) and each \( s, t \geq 1 \) such that \( s + t = n \).

Proof. By analyzing the proof of [31] Theorem 2.2, one sees that (21) in fact implies \( nF \in L_{1,...,1}(x_1, \ldots, x_n) \). By Claim 26 this means that \( nF = \sum_{\lambda \in \Sigma_{n-1}} \beta_\lambda \cdot b_\lambda \), for some \( \beta_\lambda \in \mathbb{Z} \) and \( b_\lambda \) the commutators in (19). On the other hand, it follows from Claim 27 that, in the expression

\[ nF = \sum_{\sigma \in \Sigma_n} na_\sigma \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \]

one has \( na_{\lambda \times 1} = \beta_\lambda \), for each \( \lambda \in \Sigma_{n-1} \). This means that \( F = \sum_{\lambda \in \Sigma_{n-1}} a_{\lambda \times 1} \cdot b_\lambda \), so \( F \) is a Lie element. \( \square \)

Another important piece of the proof of Theorem 27 is

Claim 29. The restriction \( r: T_{1,...,1}(x_1, \ldots, x_n)' \to L_{1,...,1}(x_1, \ldots, x_n)' \) is an epimorphism.

Proof. We need to show that an arbitrary linear map \( \varphi: L_{1,...,1}(x_1, \ldots, x_n) \to \mathbb{Z} \) extends into a linear map \( \tilde{\varphi}: T_{1,...,1}(x_1, \ldots, x_n) \to \mathbb{Z} \). Let \( \{ b_\lambda \}_{\lambda \in \Sigma_{n-1}} \) be the basis (19) of \( L_{1,...,1}(x_1, \ldots, x_n) \), \( \{ e_\sigma \}_{\sigma \in \Sigma_n} \) the basis (20) of \( T_{1,...,1}(x_1, \ldots, x_n) \) and put

\[ \tilde{\varphi}(e_\sigma) := \begin{cases} \phi(b_\lambda), & \text{if } \sigma = \lambda \times 1 \text{ for some } \lambda \in \Sigma_{n-1}, \text{ and} \\ 0, & \text{otherwise}. \end{cases} \]

By Claim 27, \( \tilde{\varphi} \) defined in this way extends \( \varphi \). \( \square \)

Let \( K \) denote the kernel of the composition

\[ T_{1,...,1}(\alpha_1, \ldots, \alpha_n) \xrightarrow{\Phi} T_{1,...,1}(x_1, \ldots, x_n)' \xrightarrow{r} L_{1,...,1}(x_1, \ldots, x_n)', \]

where \( \Phi \) is as in (18) and \( r \) the restriction.

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Claim 30. An element \( x \in T_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n) \) belongs to the kernel \( K \) if and only if there exists a natural \( N \) such that \( N \cdot x \in \text{Ush}_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n) \).

Proof. It is an elementary consequence of Proposition 28 that
\[
K = (\text{Ush}_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n)^\perp)^\perp \supset \text{Ush}_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n),
\]
where \( ^\perp \) denotes the annihilator in the dual space. It is another elementary fact that, for any subspace \( A \) of a finite-dimensional vector space \( V \), one has \( (A^\perp)^\perp \cong A \), therefore, after extending the scalars to \( \mathbb{Q} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \), the inclusion in the above display becomes an isomorphism. The claim follows.

Claim 31 implies that, in the composition
\[
\Phi : T_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n)/\text{Ush}_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n) \to T_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n)/K \xrightarrow{\cong} L_{1, \ldots, 1}(x_1, \ldots, x_n)',
\]
the kernel of the projection \( \pi \) consists of torsion elements. The second map, induced by \( r \), is an isomorphism by Claim 29. Theorem 25 will thus be established if we prove

Claim 31. The abelian group \( T_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n)/\text{Ush}_{1, \ldots, 1}(\alpha_1, \ldots, \alpha_n) \) is torsion-free.

To prove the claim we need some properties of configuration spaces. We do not know a purely algebraic proof. As in Section 3, let \( \text{Cnf}(\mathbb{R}^h, n) \) be the configuration space of \( n \) distinct labeled points in the \( h \)-dimensional Euclidean space \( \mathbb{R}^h \), \( h \geq 1 \). It is known \[36\] that \( [3] \) induces a cell decomposition
\[
\text{Cnf}(\mathbb{R}^h, n) = \bigcup_{T \in \text{Tree}^h(n)} [\hat{T}],
\]
of the one-point compactification \( \text{Cnf}^\bullet(\mathbb{R}^h, n) \) of \( \text{Cnf}(\mathbb{R}^h, n) \), where \([\hat{T}]\) is the closure of \([T]\) in \( \text{Cnf}^\bullet(\mathbb{R}^h, n) \). Let \( F_p \text{Cnf}(\mathbb{R}^h, n) := \bigcup\{[\hat{T}] ; \dim([\hat{T}]) \leq p\} \). We have an increasing bounded filtration
\[
\emptyset \subset F_{h+n-1} \text{Cnf}(\mathbb{R}^h, n) \subset \cdots \subset F_{hn} \text{Cnf}(\mathbb{R}^h, n) = \text{Cnf}(\mathbb{R}^h, n)
\]
which induces a spectral sequence converging to the reduced homology \( \overline{H}_*(\text{Cnf}(\mathbb{R}^h, n)) \). Let us denote by \( (G^h_*(n), \partial^h) \) the \( E^1 \)-term of this spectral sequence desuspended \((h + 1)\)-times\[37\]
\[
G^h_*(n) := \bigoplus_{p+q = h+1} \overline{H}_{p+q}(F_p \text{Cnf}(\mathbb{R}^h, n)/F_{p-1} \text{Cnf}(\mathbb{R}^h, n)),
\]
with the induced differential. The quotient \( F_p \text{Cnf}(\mathbb{R}^h, n)/F_{p-1} \text{Cnf}(\mathbb{R}^h, n) \) is isomorphic to the cluster of \( p \)-dimensional spheres indexed by trees in \( \text{Tree}^h(n) \) with \( p \) edges, therefore \( G^h_d(n) \) is spanned by labeled pruned \( h \)-trees with \( d + h + 1 \) edges, \( G^h_d(n) = \text{Span}(\text{Tree}^h_d(n)) \). So \( G^h_*(n) \) agrees with the graded abelian group introduced under the same name on page \[3\]. Our spectral sequence degenerates at the \( E^2 \)-level, therefore,
\[
H_*(G^h_*(n), \partial^h) \cong \overline{H}_{*+h+1}(\text{Cnf}(\mathbb{R}^h, n)),
\]
\[7\]The number \( h + 1 \) equals the dimension of the affine group of \( \mathbb{R}^h \).
while, by the Poincaré-Lefschetz duality [35, Section 13.3],
\[ \overline{H}_*(\text{Cnf}(\mathbb{R}^h, n)) \cong H^{h-n-*}(\text{Cnf}(\mathbb{R}^h, n)). \]
The cohomology in the right hand side of the above display is known [12, Theorem 1.6]; for the purpose of this paper, it is enough to recall that \( H^*(\text{Cnf}(\mathbb{R}^h, n)) \) is torsion-free and nontrivial only in degrees \( i(h-1), 0 \leq i \leq n-1 \). The above results combine into

**Claim 32.** The homology \( H_*(G_*^h(n), \partial^h) \) is torsion-free and concentrated in degrees \( (n-2) + i(h-1), 0 \leq i \leq n-1 \).

There is a natural degree zero dg-monomorphism \( \iota : (G_*^h(n), \partial^h) \hookrightarrow (G_*^{h+1}(n), \partial^{h+1}) \) that sends the generator \( e_T \) indexed by \( T \in \text{Tree}^h(n) \) into the generator \( e_{s(T)} \) indexed by the suspension \( s(T) \in \text{Tree}^{h+1}(n) \). By simple combinatorics, \( G_*^h(n) = G_*^{h+1}(n) \) whenever \( d \leq h(n-1) - 1 \). Let us denote
\[ (G_*(n), \partial) := \lim \overrightarrow{\text{lin}} (G_*^h(n), \partial^h). \]
It is clear that \( G_*(n) \) is the span of the graded set \( \text{Tree}_*(n) \) so it coincides with the graded abelian group introduced in Definition [3]. It is not difficult to see that the differential \( \partial \) is the differential \( \partial_{\text{lin}} \) of Definition [3] but we will not use this fact. Observe that \( G_*(n) = 0 \) for \( d < n-2 \). The above results imply

**Claim 33.** One has \( H_*(G_*(n), \partial) = 0 \) for \( d \neq n-2 \) while \( H_{n-2}(G_*(n), \partial) \) is torsion-free.

Let us calculate \( H_{n-2}(G_*(n), \partial) \). Since \( G_*^h(n) = G_*^2(n) \) for \( d \leq n-1 \), clearly \( H_{n-2}(G_*(n), \partial) \cong H_{n-2}(G_*^2(n), \partial^2) \). The space \( G_*^{n-2}(n) \) is spanned by labeled corollas of height two
\[
\sigma(1) \sigma(2) \sigma(n) \\
\vdots \vdotswithin{\vdots} \vdotswithin{\vdots} \vdotswithin{\vdots} \\
\sigma(n) \\
\]
therefore \( G_*^{n-2}(n) \cong \mathbb{T}_{\sigma, \ldots, \sigma} \), with the \( \Sigma_n \)-equivariant isomorphism sending the above corolla into the generator \( \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)} \). The corolla (22) corresponds to the cell of \( \text{Cnf}(\mathbb{R}^2, n) \) whose generic point is shown in Figure [19] (left).

The little arrows numbered \( 1, \ldots, n+1 \) indicate a frame in the tangent bundle determining the orientation. Likewise, \( G_*^{n-1}(n) \) is spanned by trees of height two
\[
\rho(1) \rho(2) \rho(s) \rho(s+1) \rho(n) \\
\vdots \\
\rho(n) \\
\]
representing the cell whose generic point is shown in Figure [19] (right).

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By imagining how a generic point of the cell corresponding to the tree $\{23\}$ moves to the boundary, one sees that the differential $\partial$ sends this tree into the element that, under the isomorphism $G_{n-2}(n) \cong T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$, equals

$$\sum_{\tau \in \Sigma_n} \text{sgn}(\tau) \cdot \alpha_{\rho(1)} \otimes \cdots \otimes \alpha_{\rho(n)}$$

(observe the $\text{sgn}(\tau)$-factor), therefore

$$H_{n-1}(G_*, \partial) \cong T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)/\widetilde{Ush}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n),$$

where $\widetilde{Ush}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$ denotes the span of elements in (24). We, however, have

**Claim 34.** Let $\text{sgn}$ denote the signum representation. There is a $\Sigma_n$-equivariant isomorphism

$$T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)/\widetilde{Ush}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n) \cong \text{sgn} \otimes T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)/Ush_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n).$$

**Proof.** The isomorphism $\Psi : T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n) \rightarrow \text{sgn} \otimes T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$ given by

$$\Psi(\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}) := \text{sgn}(\sigma) \otimes \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}, \quad \sigma \in \Sigma_n,$$

clearly restricts to an isomorphism $\widetilde{Ush}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n) \cong Ush_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$ and induces the isomorphism of the claim. 

By Claim 33 and isomorphism (25), $T_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)/\widetilde{Ush}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$ is torsion-free, which, by Claim 34, proves Claim 31 and therefore also Theorem 25. Recall that the space $G_*(n)$ inherits a natural free $\Sigma_n$-action given by relabeling the spanning trees. As a combination of the above results we get

**Theorem 35.** The tree complex $(G_*(n), \partial)$ is a $\Sigma_n$-free resolution of the $\Sigma_n$-module $\text{sgn} \otimes \mathcal{L}ie(n)'.$

Tensoring $(G_*(n), \partial)$ with the signum representation therefore leads to a free resolution of $\mathcal{L}ie(n)'.$ The following example shows that $(G_*(n), \partial)$ is not the smallest possible.

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Example 36. Inspecting Figure 3, one sees that $G_1(3)$ has one $\Sigma_3$-generator $[1|2|3]$, $G_2(3)$ two $\Sigma_3$-generators $[1|2|3]$ and $[1|2|3]$, and $G_3(3)$ three $\Sigma_3$-generators $[1||2|3],[1|2||3]$ and $[1|2|3]$. In general, the number of generators of the free $\mathbb{Z}[\Sigma_3]$-module $G_d(3)$ equals $d$. One has
\[
\partial([1|2|3]) = [1|2|3] - [1|3|2] + [3|1|2] = \partial([3|1|2])
\]
and
\[
\partial([1|2||3]) = [1|2|3] - [3|1|2].
\]
This shows that one of the generators of $G_2(3)$ is superfluous, so there is a resolution with only one $\Sigma_3$-generator in degree 2. We believe in the existence of a free $\Sigma_3$-resolution $(\tilde{G}_d(3), \partial)$ of $\text{Lie}(3)'$ in which $\tilde{G}_d(3)$ has $[\frac{d-1}{2}] + 1$ $\Sigma_3$-generators, where $[-]$ denotes the integral part, $d \geq 1$.

Notice that the group ring $R[\Sigma_n]$ does not have good properties even for $R$ a characteristic zero field. For the augmentation ideal $\mathcal{I}$ of $R[\Sigma_n]$ one has $\mathcal{I}/\mathcal{I}^2 = 0$, so there is no good notion of minimality of $\Sigma_n$-resolutions of $\Sigma_n$-modules.

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Glossary

$F(-)$, free operad functor, page 3

$\mathcal{F}(-)$, free non-$\Sigma$ operad functor, page 3

$\text{Tree}^h(n)$, set of pruned $h$-trees with $n$ tips, page 5

$\text{Tree}^h(n)$, set of labeled pruned $h$-trees with $n$ tips, page 5

$\text{Tree}(n)$, set of reduced trees of arbitrary height, equals $\lim \text{Tree}^h(n)$, page 6

$\text{Tree}(n)$, set of reduced labeled trees of arbitrary height, equals $\lim \text{Tree}^h(n)$, page 6

$G^h(n)$, right $\Sigma_n$-module spanned by $\text{Tree}^h(n)$, page 9

$G(n)$, right $\Sigma_n$-module spanned by $\text{Tree}(n)$, equals $\lim G^h$, page 6

$B(A)$, bar construction of an associative algebra $A$, page 6

$B^h(A)$, $h$th iterate of the bar construction of an ass. comm. algebra, page 7

$\hat{B}^h(A)$, desuspension $\downarrow_{h+1} B^h(A)$, page 7

$\hat{B}^\infty(A)$, direct limit $\lim \hat{B}^h(A)$, page 7

$\text{Cnf}(\mathbb{R}^h, n)$, configuration space of distinct labeled points in $\mathbb{R}^n$, page 3

$\hat{F}_h(n)$, moduli space $\text{Cnf}(\mathbb{R}^h, n)/\text{Aff}(\mathbb{R}^h)$, page 8

$F_h(n)$, Fulton-MacPherson compactification of $\hat{F}_h(n)$, page 8

$F(n)$, direct limit $\lim F_h(n)$, page 9

$[T]$, cell of $\text{Cnf}(\mathbb{R}^h, n)$ indexed by a labeled tree $T = (T, \ell)$, page 10

$\mu[T]$, quotient $[T]/\text{Aff}(\mathbb{R}^h)$, page 11

$\text{Tree}^\text{reg}_h(n)$, subset of $\text{Tree}^h(n)$ of trees of dimension $< d^h_{\text{crit}}(n)$, page 12

$\text{Tree}^\text{reg}(n)$, subset of $\text{Tree}(n)$ of trees of dimension $< d_{\text{crit}}(n)$, equals $\lim \text{Tree}^\text{reg}_h$, page 12

$F^\text{reg}$, $F^\text{reg}_h$ regular skeleton of the configuration operad $F_h$ resp. $F$, page 12

$\mu[\sigma]$, cell of $F$ corresponding to a face $\sigma : T \to S$ of $T$, page 10

$(\sigma_1, \ldots, \sigma_n)$, notation for a permutation $\sigma \in \Sigma_n$, $\sigma_i := \sigma^{-1}(i)$, $1 \leq i \leq n$, page 21

$\mathcal{T}$, Tamarkin cell, equals $\mu[1|2||3|4|5|6]$, page 24

$\mathcal{M}$, a 4-dimensional bad cell, equals $\mu[1|2||3|4|5|6]$, page 24