OFF-POLICY EVALUATION OF BANDIT ALGORITHM FROM DEPENDENT SAMPLES UNDER BATCH UPDATE POLICY

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ABSTRACT

The goal of off-policy evaluation (OPE) is to evaluate a new policy using historical data obtained via a behavior policy. However, because the contextual bandit algorithm updates the policy based on past observations, the samples are not independent and identically distributed (i.i.d.). This paper tackles this problem by constructing an estimator from a martingale difference sequence (MDS) for the dependent samples. In the data-generating process, we do not assume the convergence of the policy, but the policy uses the same conditional probability of choosing an action during a certain period. Then, we derive an asymptotically normal estimator of the value of an evaluation policy. As another advantage of our method, the batch-based approach simultaneously solves deficient support problem. Using benchmark and real-world datasets, we experimentally confirm the effectiveness of the proposed method.

1 Introduction

As an instance of sequential decision-making problems, the multi-armed bandit (MAB) algorithms have attracted significant attention in various applications, such as ad optimization, personalized medicine, search engines, and recommendation systems. Recently, various methods for evaluating a new policy using historical data obtained via the MAB algorithms (Beygelzimer & Langford, 2009; Li et al., 2010) have emerged. The goal of off-policy evaluation (OPE) is to evaluate a new policy by estimating the expected reward obtained from the new policy (Dudík et al., 2011; Wang et al., 2017; Narita et al., 2019; Bibaut et al., 2019; Kallus & Uehara, 2019; Oberst & Sontag, 2019). Although an OPE algorithm estimates the expected reward from a new policy, most existing studies presume that the samples are independent and identically distributed (i.i.d.). However, the MAB algorithm policy updates the probability of choosing an action based on past observations, and samples are not i.i.d. owing to this update. In this case, such existing studies do not guarantee that their estimators have asymptotic normality and $\sqrt{T}$-consistency for a sample size $T$. Therefore, there is a strong motivation to establish a novel method for OPE from dependent samples.

Several pioneering studies address OPE from dependent samples (van der Laan, 2008; Hahn et al., 2011; van der Laan & Lendle, 2014; Luedtke & van der Laan, 2016; Hadad et al., 2019; Kato et al., 2020a). We can group the methods for deriving asymptotic normality into the following three groups: (a) van der Laan (2008), van der Laan & Lendle (2014), Hadad et al. (2019), and Kato et al. (2020a) derive the asymptotic normality with the central limit theorem (CLT) of a martingale difference sequence (MDS) by assuming that the probability of choosing an action converges to a time-invariant probability; (b) Luedtke & van der Laan (2016) derives the asymptotic normality by standardizing a MDS, which is also used in the first group; (c) Hahn et al. (2011) apply asymptotic theory for the batched probability update process.

This paper focuses on an approach of the third group; that is, there are sufficiently large sample sizes in each batch. Compared with Hahn et al. (2011), our proposed method is more general and applicable in practical applications. Our method has the following three advantages compared with existing studies: (i) it does not assume convergence of the...
probability of choosing an action; (ii) it allows the probability of choosing an action to be 0 for some actions in some batches; (iii) we can also use non-Donsker nuisance estimators as well as van der Laan & Lendle (2014).

This paper has three main contributions. First, we provide a solution for OPE from dependent samples obtained via the MAB algorithms. Second, under the batch update policy, the proposed estimator achieves the asymptotic normality with fewer assumptions. Third, the estimator also experimentally shows a lower mean squared error (MSE) in some cases.

2 Problem Setting

Here, we formulate OPE under a batch update.

2.1 Date-Generating Process

Let $A_t$ be an action taking variable in $A = \{1, 2, \ldots, K\}$, $X_t$ the covariate observed by the decision maker when choosing an action, and $X$ the space of covariate. Let us denote a random variable of a reward at period $t$ as $Y_t = \sum_{a=1}^{K} \mathbb{1}[A_t = a] Y_t(a)$, where $Y_t : A \rightarrow \mathbb{R}$ is a potential outcome\(^1\). In this paper, we have access to a dataset $\{(X_t, A_t, Y_t)\}_{t=1}^{T}$ with the following data-generating process (DGP):

\[
\{(X_t, A_t, Y_t)\}_{t=1}^{T} \sim p(x) \pi_t(a \mid x, \Omega_{t-1}) p(y \mid a, x),
\]

where $\Omega_{t-1} \in M_{t-1}$ denotes the history until $t-1$ period defined as $\Omega_{t-1} = \{X_{t-1}, A_{t-1}, Y_{t-1}, \ldots, X_1, A_1, Y_1\}$ with the space $M_{t-1}$, $p(x)$ denotes the density of the covariate $X_t$, $\pi_t(a \mid x, \Omega_{t-1})$ denotes the probability of choosing an action $A_t$ conditioned on $X_t$ and $\Omega_{t-1}$, and $p(y \mid a, x)$ denotes the density of an outcome $Y_t$ conditioned on $A_t$ and $X_t$. We assume that $p(x)$ and $p(y \mid a, x)$ are invariant across periods, but $\pi_t(a \mid x, \Omega_{t-1})$ can take different values across periods. Let us call a policy inducing $\pi_t(a \mid x, \Omega_{t-1})$ a behavior policy.

2.2 Off-Policy Evaluation

This paper considers estimating the value of an evaluation policy $\pi^e : A \times X \rightarrow [0, 1]$ by using samples obtained under the behavior policy. Let an evaluation policy $\pi^e : A \times X \rightarrow [0, 1]$ be a probability of choosing an action $A_t$ conditioned on a covariate $X_t$. We are interested in estimating the expected reward from any pre-specified evaluation policy.

Assumption 1. There exists a constant $C_1$ such that $0 \leq \frac{\pi^e(a \mid x)}{\pi_t(a \mid x, \Omega_{t-1})} \leq C_1$.

Assumption 2. There exists a constant $C_2$ such that $|Y| \leq C_2$.

Remark 1 (Existing Methods for OPE). We review three types of standard estimators of $R(\pi^e)$ under the case where $\pi_t(a \mid x, \Omega_0) = \pi_t(a \mid x, \Omega_1) = \cdots = \pi_T(a \mid x, \Omega_{T-1}) = p(a \mid x)$ in the DGP defined in (1). The first estimator is an inverse probability weighting (IPW) estimator given by $\frac{1}{T} \sum_{t=1}^{T} \sum_{a=1}^{K} \frac{\pi^e(a \mid X_t) \mathbb{1}[A_t = a]}{p(a \mid X_t)} \left( Y_t - f_T(a, X_t) \right)$ (Rubin, 1987; Hirano et al., 2003; Swaminathan & Joachims, 2015). Although this estimator is unbiased when the behavior policy is known, it suffers from high variance. The second estimator is a direct method (DM) estimator $\frac{1}{T} \sum_{t=1}^{T} \sum_{a=1}^{K} \left( \pi^e(a \mid X_t) \mathbb{1}[A_t = a] \left( Y_t - f_T(a, X_t) \right) \right)$ and $\pi^e(a \mid X_t) f_T(a, X_t)$ (Hahn, 1998). This estimator is known to be weak against model misspecification for $f^*(a, X_t)$. The third estimator is an augmented IPW (AIPW) defined as $\frac{1}{T} \sum_{t=1}^{T} \sum_{a=1}^{K} \left( \pi^e(a \mid X_t) \mathbb{1}[A_t = a] \left( Y_t - f_T(a, X_t) \right) \right)$ (Robins et al., 1994; Chernozhukov et al., 2018). Under certain conditions, it is known that this estimator achieves the efficiency bound (a.k.a semiparametric lower bound), which is the lower bound of the asymptotic MSE of OPE among regular $\sqrt{T}$-consistent estimators (van der Vaart, 1998).

Remark 2 (Semiparametric Lower Bound). The lower bound of the variance is defined for an estimator of $\theta_0$ under some posited models of the DGP. If this posited model is a parametric model, it is equal to the Cramér-Rao lower bound. When this posited model is a semiparametric model, we can define a corresponding Cramér-Rao lower bound (Bickel et al., 1998). Narita et al. (2019) gives the semiparametric lower bound of the DGP (1) under $\pi_1(a \mid x, \Omega_0) = \cdots = \pi_T(a \mid x, \Omega_{T-1}) = p(a \mid x)$ as $E \left[ \sum_{a=1}^{K} \frac{\pi^e(a \mid X_t)^2}{p(a \mid X_t)} \right] + \left( \sum_{a=1}^{K} \pi^e(a \mid X_t) f^*(a, X_t) - \theta_0 \right)^2$.

\(^1\)We can express the DGP without using the potential outcome variable (Kato et al., 2020b).
Notations: Let us denote $E[Y_t(a) \mid x]$ and $\text{Var}(Y_t(a) \mid x)$ as $f^*(a, x)$ and $v^*(a, x)$, respectively. Let $\mathcal{F}$ be the class of $f^*(a, x)$. Let $\hat{f}_t(a, x \mid \Omega_{t-1})$ be an estimator of $f^*(a, x)$ constructed from $\Omega_{t-1}$, respectively. Let $N(\mu, \text{var})$ be the normal distribution with the mean $\mu$ and the variance var. For a random variable $Z$ and function $\mu$, let $\|\mu(Z)\|_2 = \int |\mu(z)|^2p(z)dz$ be the $L^2$-norm.

2.3 Patterns of Probability Update

In this paper, based on the update of $\pi_t(a \mid x, \Omega_{t-1})$, we classify the policies into two patterns, sequential update policy and batch update policy. For the sequential update policy, the policy updates $\pi_t(a \mid x, \Omega_{t-1})$ at each period (van der Laan, 2008; van der Laan & Lendle, 2014; Hadad et al., 2020a). Under the batch update policy, after the policy continues using a fixed probability $\pi_1(a \mid x, \Omega_{t-1})$ for some periods without updates, the policy updates $\pi_t(a \mid x, \Omega_{t-1})$ (Hahn et al., 2011; Narita et al., 2019). Although the sequential update is standard in the MAB problem, we often apply batch updates in industrial applications such as ad-optimization (Narita et al., 2019). For OPE under the sequential update, van der Laan (2008), van der Laan & Lendle (2014), Luedtke & van der Laan (2016), Hadad et al. (2019), and Kato et al. (2020a) proposed estimators with asymptotic normality. For instance, an adaptive AIPW (A2IPW) estimator (van der Laan & Lendle, 2014; Hadad et al., 2019; Kato et al., 2020a) has asymptotic normality if the behavior policy converges. On the other hand, we consider OPE under a batch update. Let $M$ denote the number of updates and $\tau \in I = \{1, 2, \ldots, M\}$ denotes the batch index. For $\tau \in I$, the probability is updated at a period $t_\tau$, where $t_\tau - t_{\tau-1} = T\tau$, using samples $\{(X_t, Y_t, A_t)\}_{t=t_{\tau-1}}^{t_{\tau}}$, where $r_1 + r_2 + \cdots + r_M = 1$ and $t_0 = 0$. Thus, in addition to the DGP (1), we assume

\[
\{(X_t, A_t, Y_t)\}_{t=t_{\tau-1}}^{t_{\tau}} \overset{i.i.d.}{\sim} \pi_\tau(a \mid x, \Omega_{t_{\tau-1}}),
\]

where $\pi_\tau(a \mid x, \Omega_{t_{\tau-1}})$ denotes the assignment probability updated based on samples until the period $t_{\tau-1}$.

2.4 Related Work

For the sequential update, van der Laan (2008), van der Laan & Lendle (2014), Hadad et al. (2019), and Kato et al. (2020a) assume that the probability of choosing an action converges to a time-invariant function almost certainly; that is, $\pi_t(a \mid x, \Omega_{t-1}) \overset{p}{\to} \alpha(a \mid x)$, where $\alpha : \mathcal{X} \to (0, 1)$. This assumption enables us to apply the CLT for MDS. van der Laan & Lendle (2014) proposed constructing step-wise nuisance estimators, which enables us to derive asymptotic normality without Donsker’s conditions of nuisance estimators. This technique is a generalization of sample-splitting, which is also called cross-fitting in a context of double/debiased machine learning (Klaassen, 1987; Zheng & van der Laan, 2011; Chernozhukov et al., 2018). For the A2IPW estimator, Hadad et al. (2019) proposed using an adaptive weight for stabilizing the behavior, and Kato et al. (2020a) derived concentration inequality based on the law of iterated logarithms. On the other hand, we also construct an MDS and apply the CLT, but do not assume $\pi_t(a \mid x, \Omega_{t-1}) \overset{p}{\to} \alpha(a \mid x)$ by using batch update policy. Instead, we assume a sufficient sample size for each batch.

For such a non-stationary setting, Luedtke & van der Laan (2016) also proposed an estimator with asymptotically normality for sequential update policy without using batch update policy. For deriving the asymptotic normality, Luedtke & van der Laan (2016) used standardization for an MDS. Although the method enables us to construct an asymptotically normal estimator for various estimators, the proposed estimator only has $\sqrt{T - \ell}$-consistency for another sample size $\ell > 0$, not $\sqrt{T}$, to estimate the variances of the MDS.

As other related work, in the MAB problem, Perchet et al. (2016) considered the setting of batch policy updates. In OPE, Narita et al. (2019) also discuss a similar problem setting, but they assume that samples are i.i.d. Independently, Zhang et al. (2020) provided a method for deriving a confidence interval of an ordinary least squares estimator, which is a different parameter of what we want to estimate.

3 OPE under Batch Update Policy

This section introduces a concept for conducting OPE under the batch update policy and a method based on the concept with its theoretical properties.

3.1 Strategy for OPE

For OPE under batch update policy, we consider asymptotic properties based on the assumption of $t_\tau - t_{\tau-1} \to \infty$ as $T \to \infty$ for fixed $\tau$. Because $\{(X_t, A_t, Y_t)\}_{t=t_{\tau-1}}^{t_{\tau}}$ is i.i.d., we can use the standard limit theorems for the partial sum.
of the samples to obtain an asymptotically normal estimator of \( \theta_0 = R(\pi^*) \). However, we also have the motivation to use all samples together to increase the efficiency of the estimator. Therefore, based on the idea of generalized method of moments (GMM), we propose an estimator of \( \theta_0 \) considering the sample averages of each block as an empirical moment conditions. The main difference from the standard GMM is the assumption that the samples are not i.i.d. However, for the case under the batch update, we can apply the central limit theorems (CLT) for the martingale difference sequences (MDS) by appropriately constructing an estimator. We describe the proposed method as follows.

### 3.2 Estimator for OPE

We propose an estimator of \( \theta_0 \) based on a idea of GMM. For an index of batch \( \tau \in I \), a function \( f \in \mathcal{F} \) such that \( f : \mathcal{A} \times \mathcal{X} \to \mathbb{R} \) and an evaluation policy \( \pi^* \in \Pi \), we define \( h^{\text{OPE}}_t : \mathcal{A} \times \mathcal{X} \to \mathbb{R} \) as

\[
\bar{h}^{\text{OPE}}_t(x, k, y; \tau, \theta, f, \pi^*) = \frac{1}{r_\tau} \eta_t(x, k, y; \tau, \theta, f, \pi^*) \mathbb{1}_{[t_{t-1} < t \leq t_\tau]},
\]

where \( \eta_t(x, k, y; \tau, \theta, f, \pi^*) := \phi_t(x, k, y; \tau, f, \pi^*) - \theta \) and \( \phi_t(x, k, y; \tau, f, \pi^*) := \sum_{a=1}^K \pi^*(a | x) \left\{ \frac{1[k = a]}{\pi^*(a | x, \Omega_{t_{t-1}})} y - f(a, x) \right\} \).

Let us note that, for \( \tau \in I, \theta_0 \in \Theta, f_{t-1} \in \mathcal{F}, \) and \( \pi^* \in \Pi \), the sequence \( \{h^{\text{OPE}}_t(X_t, A_t, Y_t; \tau, \theta_0, f_{t-1}, \pi^*)\}_{t=1}^T \) is an MDS: for \( h^{\text{OPE}}_t(X_t, A_t, Y_t; \tau, \theta_0, f_{t-1}, \pi^*) \), by \( \mathbb{E}[1[A_t = a] | X_t, \Omega_{t-1}] = \pi^*(a | X_t, \Omega_{t-1}) \), we have

\[
\mathbb{E}\left[ h^{\text{OPE}}_t(X_t, A_t, Y_t; \tau, \theta_0, f_{t-1}, \pi^*) | \Omega_{t-1} \right] = \mathbb{E}\left[ \sum_{a=1}^K \pi^*(a | x) \left\{ \frac{1[k = a]}{\pi^*(a | x, \Omega_{t_{t-1}})} y - f(a, x) \right\} \right] = 0.
\]

Let us also define \( h^{\text{OPE}}_t(X_t, A_t, Y_t; \tau, f_{t-1}, \pi^*) := \begin{pmatrix} h^{\text{OPE}}_t(X_t, A_t, Y_t; 1, \tau, f_{t-1}, \pi^*) \\ h^{\text{OPE}}_t(X_t, A_t, Y_t; 2, \tau, f_{t-1}, \pi^*) \\ \vdots \\ h^{\text{OPE}}_t(X_t, A_t, Y_t; M, \tau, f_{t-1}, \pi^*) \end{pmatrix} \) and then the sequence \( \left\{h^{\text{OPE}}_t(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^*)\right\}_{t=1}^T \) is an MDS with respect to \( \{\Omega_t\}_{t=0}^{T-1} \), that is,

\[
\mathbb{E}\left[h^{\text{OPE}}_t(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^*) | \Omega_{t-1} \right] = 0.
\]

Using the sequence \( \left\{h^{\text{OPE}}_t(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^*)\right\}_{t=1}^T \), we define an estimator of OPE as \( \hat{R}^{\text{BA2IPW}}_{\text{OPE}}(\pi^*) := \arg\min_{\theta \in \Theta} \left( q^{\text{OPE}}_T(\theta) \right)^\top \hat{W}_T \left( q^{\text{OPE}}_T(\theta) \right), \)

where \( q^{\text{OPE}}_T(\theta) = \frac{1}{T} \sum_{t=1}^T h^{\text{OPE}}_t(X_t, A_t, Y_t; \theta, f_{t-1}, \pi^*) \) and \( \hat{W}_T \) is a data-dependent \((M \times M)\)-dimensional positive semi-definite matrix. Let us note that the estimator defined in Eq. (2) is an application of GMM with the moment condition \( q^{\text{OPE}}_T(\theta_0) = \mathbb{E}\left[ \frac{1}{T} \sum_{t=1}^T h^{\text{OPE}}_t(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^*) \right] = 0 \). For the minimization problem defined in Eq. (2), we can analytically calculate the minimizer as \( \hat{R}^{\text{BA2IPW}}_{\text{OPE}}(\pi^*) = w_T^\top D_T(\pi^*), \) where \( w_T = (w_{T,1}, \cdots, w_{T,M})^\top \) is an \( M \)-dimensional vector such that \( \sum_{\tau=1}^M w_{T,\tau} = 1 \) and \( D_T(\pi^*) \) is

\[
\begin{pmatrix}
\frac{1}{T-t_{t-1}} \sum_{t=t_{t-1}+1}^{t_t} \phi_t(X_t, A_t, Y_t; 1, f_{t-1}, \pi^*) \\
\frac{1}{T-t_{t-1}} \sum_{t=t_{t-1}+1}^{t_t} \phi_t(X_t, A_t, Y_t; 2, f_{t-1}, \pi^*) \\
\vdots \\
\frac{1}{T-t_{M-1}} \sum_{t=t_{M-1}+1}^{t_t} \phi_t(X_t, A_t, Y_t; M, f_{t-1}, \pi^*)
\end{pmatrix}.
\]
We call the estimator a Batch-based Adaptive AIPW (BA2IPW) estimator. In Appendix B, we discuss the GMM perspective in more detail.

### 3.3 Asymptotic Properties

Here, we show the consistency and asymptotic normality of the proposed BA2IPW estimator $\hat{R}_{T}^{BA2IPW}(\pi^e)$.

#### Theorem 1 (Consistency of the BA2IPW Estimator)

Suppose that there exists a constant $C_f > 0$ such that $|f_{t-1}(a, x)| < C_f$ for $\tau \in I$. Then, under Assumptions 1 and 2, $\hat{R}_{T}^{BA2IPW}(\pi^e) \xrightarrow{p} \theta_0$.

**Proof.** We use the law of large numbers for an MDS from the boundedness of $h_{t}^{OPE}(X_t, A_t, Y_t; \theta_0, \hat{f}_{t-1}, \pi^e)$, and we have $\frac{1}{T} \sum_{t=1}^{T} h_{t}^{OPE}(X_t, A_t, Y_t; \theta_0, \hat{f}_{t-1}, \pi^e) \xrightarrow{p} 0$ (Proposition 4 in Appendix A). This result means that $D_{T}(\pi^e) \xrightarrow{p} I\theta_0$, where $I = (1 \ldots 1)^T$ is an M-dimensional vector. Therefore, $\hat{R}_{T}^{BA2IPW}(\pi^e) = w_{T} D_{T}(\pi^e) \xrightarrow{p} w_{T} I\theta_0 = \theta_0$.

**Theorem 2 (Asymptotic Distribution of the BA2IPW Estimator).** Suppose that (i) $w_T = (w_{T,1}, \ldots, w_{T,M})^\top \xrightarrow{d} w = (w_1, \ldots, w_M)^\top$; (ii) $w_{T,\tau} > 0$ and $\sum_{\tau=1}^{M} w_{T,\tau} = 1$; (iii) $\hat{f}_{t-1}(a, x) \xrightarrow{d} f^*(a, x)$ for all $a \in A$ and $x \in X$; (iv) There exists a constant $C_f > 0$ such that $|\hat{f}_{t-1}(a, x)| < C_f$. Then, under Assumptions 1 and 2, $\sqrt{T}(\hat{R}_{T}^{BA2IPW}(\pi^e) - \theta_0) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, where $\sigma^2 = \sum_{\tau=1}^{M} w_{\tau} \sigma_{\tau}^2$ and $\sigma_{\tau}^2 = \frac{1}{\tau} \mathbb{E} \left[ \sum_{a=1}^{K} \left( \frac{\pi^e(a | X)}{\pi_{\tau}(a | X, \theta_{t-1})} \right)^2 \mathbb{E}[Y^T_{\tau} | X] \right] + \left( \sum_{a=1}^{K} \pi^e(a | X) f^*(a, X) - \theta_0 \right)^2$.

The proof is shown in Appendix C. Readers might consider that the use of MDS for deriving the asymptotic normality is unnecessary. We discuss the necessity of MDS in Appendix D. We can also define a corresponding Batch-based Adaptive IPW (BAAdaIPW). For the BAAdaIPW estimator, the variance of a batch $\tau$ is $\sigma_{IPW, \tau}^2 = \frac{1}{\tau} \mathbb{E} \left[ \sum_{a=1}^{K} \left( \frac{\pi^e(a | X)}{\pi_{\tau}(a | X, \theta_{t-1})} \right)^2 \mathbb{E}[Y^T_{\tau} | X] \right] - \theta_0^2$.

**Remark 3** (Construction of $f_{t \tau}$ and Donsker Condition). As well as the cross-fitting of double/debiased machine learning proposed by Klaassen (1987), Zheng & van der Laan (2011), van der Laan & Lendle (2014), and Chernozhukov et al. (2018), the proposed estimator does not require Donsker’s condition for asymptotic normality. This property comes from the MDS as pointed by van der Laan & Lendle (2014). On the other hand, because the samples are not independent, we cannot use the standard regression to obtain a consistent estimator of $f^*$. For example, Yang & Zhu (2002) propose a nonparametric method for the bandit process under some mild conditions.

### 3.4 Weight of the Proposed Estimator

Next, we discuss the choice of weight $w_T$.

**Equal Weight:** A naive choice is weighting the moment conditions equally; that is, $w_{T,\tau} = \frac{1}{M}$. In this case, the proposed estimator boils down to $I^\top D_T(\pi^e)$, which is almost the same as the A2IPW estimator. Although the estimator itself is similar to the A2IPW estimator, the theoretical guarantee for the asymptotic normality is different. While the A2IPW estimator uses the assumption that the policy converges to a time-invariant policy, the proposed BA2IPW estimator uses the assumption of the batch update. We call the BA2IPW estimator with the equal weight a Plain BA2IPW (PBA2IPW) estimator.

**Efficient Weight:** First, we consider an efficient weight $w_T$ that minimizes the asymptotic variance of $\hat{R}_{T}^{BA2IPW}(\pi^e)$. As well as the standard GMM, the $\tau$-th element of the efficient weight is given as $w_{\tau}^* = \frac{1}{\tau} / \sum_{\tau=1}^{M} \frac{1}{\sigma_{\tau}^2}$ (Hamilton, 1994). Here, we use the orthogonality among moment conditions; that is, zero covariance. In this case, the asymptotic variance becomes $1 / \sum_{\tau=1}^{M} \frac{1}{\sigma_{\tau}^2}$. Therefore, for gaining efficiency, we use a weight $\hat{w}_{T,\tau} = \frac{1}{\hat{\sigma}_{T,\tau}^2} / \sum_{\tau=1}^{M} \frac{1}{\sigma_{\tau}^2}$, where $\hat{\sigma}_{T,\tau}^2$ is an estimator of $\sigma_{\tau}^2$. We call the BA2IPW estimator with the efficient weight an Efficient BA2IPW (EBA2IPW) estimator.
Based on these properties, we propose two-step estimation. For instance, for stabilization, we define a weight described above. The second term reflects the deviation between \( f_t \) and \( f_T \), which would be more accurate because it uses more samples.

### 3.5 Main Algorithm

As discussed in Section 3.4, we can minimize the asymptotic variance of the proposed estimator \( \hat{R}^{BA2IPW}(\pi^e) \) by choosing \( w_T \) appropriately. However, to obtain \( w_{T, \tau} \overset{p}{\rightarrow} \frac{1}{\tau} \sum_{\tau' = 1}^{M} \frac{1}{\sigma_{\tau'}} \), which is the optimal weight that minimizes the asymptotic variance, we need a consistent estimator of \( \theta_0 \), which is what we want to estimate. On the other hand, we have a consistent estimator of \( \theta_0 \) without using an optimal weight matrix \( w_{T, \tau} \overset{p}{\rightarrow} \frac{1}{\tau} \sum_{\tau' = 1}^{M} \frac{1}{\sigma_{\tau'}} \).

Based on these properties, we propose two-step estimation. First, using an arbitrary positive definite weight \( w^{(0)}_T \), such as the identity matrix, we obtain an initial estimate \( \hat{R}^{BA2IPW, (1)}(\pi^e) \). Then, using \( \hat{R}^{BA2IPW, (1)}(\pi^e) \), we construct \( w^{(1)}_{T, \tau} \overset{p}{\rightarrow} \frac{1}{\tau} \sum_{\tau' = 1}^{M} \frac{1}{\sigma_{\tau'}} \). We can obtain an efficient estimator \( \hat{R}^{BA2IPW, (2)}(\pi^e) \) of \( \theta_0 \), as discussed.

More generally, we consider an algorithm with iteration such that after obtaining \( \hat{R}^{BA2IPW, (i-1)}(\pi^e) \), we estimate \( u^{(i-1)}_{T, \tau} \overset{p}{\rightarrow} \frac{1}{\tau} \sum_{\tau' = 1}^{M} \frac{1}{\sigma_{\tau'}} \) and obtain a next estimator \( \hat{R}^{BA2IPW, (i)}(\pi^e) \) by using \( w^{(i-1)}_{T, \tau} \). We refer to this algorithm with \( N \)-iterations as \( N \)-step BA2IPW estimation. We can use sufficiently large \( N \) because, at each iteration, we only calculate the weighted average of the moment conditions using \( \hat{R}^{BA2IPW, (i)}(\pi^e) \), which is not time-consuming. Although the asymptotic properties of the iterated estimator are the same as those of the two-step estimator, we report that the iteration improves the empirical performance in some cases. For \( N \geq 2 \), we summarize the \( N \)-Step BA2IPW Estimation in Algorithm 1. We can use any method to construct \( f_t \) and \( \pi^e \) as long as they are consistent for bandit data and satisfy some regularity conditions needed for Theorem 4.

### 4 Deficient Support Problem

As an application of BA2IPW, we consider an OPE without Assumption 1, which assumes that there exists \( C_1 \) such that \( 0 \leq \frac{\sigma^2(t, a)}{\rho(t, a)} \leq C_1 \). Instead of Assumption 3, we consider a situation in which we are allowed to change the support of actions in each batch. For example, in the first batch, we choose an action from a set \( \{1, 2, 3\} \) with a probability larger than 0, but we choose an action from a set \( \{1, 2, 4\} \) with a probability larger than 0 in the second batch. In this
case, the probability of choosing the action 4 is 0 in the first batch, while the probability of choosing the action 3 is 0 in the second batch. This situation is a common in practice and called deficient support problem (Sachdeva et al., 2020). For this problem, instead of Assumption 1, we use the following assumption.

**Assumption 3.** For $a \in \{1, 2, \ldots, K\}$, there exist $\tau \in \{1, 2, \ldots, M\}$ and $C_1$ such that $0 \leq \frac{\pi^*(a | x, \Omega_{\tau-1})}{\pi^*(a | x, \Omega_{\tau-1})} \leq C_1$.

Under this assumption, if $\pi^*(a | x, \Omega_{\tau-1}) > 0$ for at least one batch, we are allowed to use $\pi^*(a | x, \Omega_{\tau-1}) = 0$ for $\tau' \neq \tau$. With this assumption, we derive the asymptotic normality in Appendix E. Thus, our approach provides a new solution to this problem.

### 5 Estimation of the Behavior Policy

In the proposed BA2IPW method, we assume that the true behavior policy is known. However, in many real-world applications, the assumption does not hold. To solve this problem, by using an estimator $\hat{g}_{t-1}$ of $\pi^*$, which is constructed from $\Omega_{t-1}$ as well as $\hat{f}_{t-1}$, we also propose a Batch-based Adaptive Doubly Robust (BADR) as $\hat{R}_{T}^{BADR}(\pi^*) = w_T^\top D_T(\pi^*)$, where $w_T = (w_{T,1} \cdots w_{T,M})^\top$ is an $M$-dimensional vector such that $\sum_{T=1}^{M} w_{T,\tau} = 1$, $D_T(\pi^*) = \left(\begin{array}{c} \frac{1}{T} \sum_{t=1}^{T} \hat{\phi}_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \hat{g}_{t-1}, \pi^*) \\ \frac{1}{T} \sum_{t=1}^{T} \hat{\phi}_t(X_t, A_t, Y_t; 2, \hat{f}_{t-1}, \hat{g}_{t-1}, \pi^*) \\ \vdots \\ \frac{1}{T} \sum_{t=M+1}^{T} \hat{\phi}_t(X_t, A_t, Y_t; M, \hat{f}_{t-1}, \hat{g}_{t-1}, \pi^*) \end{array}\right)$, and

$\hat{\phi}_t(x, k, y; \tau, f, g, \pi^*) := \frac{1}{K} \sum_{a=1}^{K} \pi^*(a | x) \left\{ \frac{\mathbb{I}[k = a]}{g(a | x)} y - f(a, x) \right\}.$

For the BADR estimator, we show the asymptotic normality as follows. The proof is shown in Appendix F.

**Theorem 3** (Asymptotic Distribution of the BADR Estimator). Suppose that (i) $w_T = (w_{T,1} \cdots w_{T,M})^\top \overset{P}{=} w = (w_1 \cdots w_M)^\top$; (ii) $w_T > 0$ and $\sum_{T=1}^{M} w_{T,\tau} = 1$; (iii) for $\alpha \beta = \alpha \beta (t - t_r - 1)^{-1/2}$, $\alpha = \alpha_0(1)$, $\beta = \beta_0(1)$, and each $\tau \in I$, the nuisance estimators satisfy $|\hat{g}_{t-1}(a | \Omega_{t-1}) - \pi^*(a | \Omega_{t-1})| \leq C_2$, and $|\hat{f}_{t-1}(a, X_t) - f^*(a, X_t)| \leq \beta$, where the expectation of the norm is over $X_t$; (iv) there exist constants $C_f$ and $C_g$ such that $|\hat{f}_{t-1}(a, x)| \leq C_f$ and $0 < \frac{\pi^*(a | x)}{\hat{g}_{t-1}(a | \Omega_{t-1})} \leq C_g$ for all $a \in A$ and $x \in X$. Then, under Assumptions 1 and 2, $\sqrt{T} (\hat{R}_{T}^{BADR}(\pi^*) - \theta_0) \overset{d}{\rightarrow} N(0, \sigma^2)$. 

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Table 1: Results of OPE under the RW policy. We highlight in bold the best two estimators in each dataset.

| Datasets       | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|---------------|----------|-----------|-------|--------|------------|-----------|
|               | MSE SD   | MSE SD    | MSE SD | MSE SD | MSE SD     | MSE SD    |
| PHA2IPW       | 0.038 0.020 | 0.129 0.047 | 0.173 0.104 | 0.331 0.547 | 0.146 0.089 | 0.021 0.021 |
| EBA2IPW       | 0.050 0.006 | 0.190 0.030 | 0.191 0.025 | 0.398 0.062 | 0.182 0.028 | 0.025 0.022 |
| EBA2IPW*      | 0.044 0.003 | 0.182 0.029 | 0.102 0.012 | 0.389 0.064 | 0.177 0.027 | 0.025 0.024 |
| BAdaIPW       | 0.077 0.010 | 0.178 0.082 | 0.200 0.111 | 0.333 0.337 | 0.160 0.093 | 0.027 0.027 |
| AdaDM         | 0.141 0.010 | 0.493 0.034 | 0.434 0.036 | 0.476 0.023 | 0.413 0.031 | 0.142 0.024 |
| APIW          | 0.032 0.001 | 0.110 0.030 | 0.244 0.028 | 0.254 0.216 | 0.128 0.064 | 0.055 0.022 |
| DM            | 0.099 0.004 | 0.452 0.025 | 0.282 0.028 | 0.459 0.023 | 0.395 0.025 | 0.086 0.018 |

Table 2: Results of OPE under the UCB policy. We highlight in bold the best two estimators in each dataset.

| Datasets       | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|---------------|----------|-----------|-------|--------|------------|-----------|
|               | MSE SD   | MSE SD    | MSE SD | MSE SD | MSE SD     | MSE SD    |
| PHA2IPW       | 0.000 0.005 | 0.088 0.015 | 0.240 0.415 | 0.205 0.088 | 0.162 0.087 | 0.032 0.024 |
| EBA2IPW       | 0.014 0.000 | 0.029 0.002 | 0.240 0.070 | 0.434 0.036 | 0.239 0.054 | 0.030 0.022 |
| EBA2IPW*      | 0.036 0.005 | 0.081 0.017 | 0.004 0.000 | 0.422 0.032 | 0.202 0.038 | 0.037 0.033 |
| BAdaIPW       | 0.087 0.037 | 0.122 0.030 | 0.275 0.406 | 0.219 0.090 | 0.183 0.067 | 0.057 0.057 |
| AdaDM         | 0.076 0.002 | 0.230 0.008 | 0.372 0.020 | 0.451 0.018 | 0.327 0.021 | 0.071 0.023 |
| APIW          | 0.036 0.003 | 0.058 0.012 | 0.136 0.007 | 0.170 0.043 | 0.134 0.038 | 0.022 0.012 |
| DM            | 0.009 0.000 | 0.049 0.003 | 0.161 0.019 | 0.371 0.019 | 0.214 0.013 | 0.019 0.010 |
Table 3: The coverage ratios (CRs) are shown. The left graph shows the results with the RW policy. The right graph shows the results with the UCB policy.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| PBA2IPW  | 1.00     | 0.96      | 1.00  | 0.88   | 0.94       | 1.00      |
| EBA2IPW  | 0.66     | 0.22      | 0.15  | 0.02   | 0.19       | 0.88      |
| BAdaIPW  | 0.93     | 0.90      | 0.90  | 0.85   | 0.91       | 0.99      |
| EBA2IPW' | 0.85     | 0.27      | 0.47  | 0.02   | 0.19       | 0.89      |
|          |          |           |       |        |            |           |

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| PBA2IPW  | 1.00     | 1.00      | 1.00  | 0.94   | 0.98       | 0.98      |
| EBA2IPW  | 0.92     | 0.74      | 0.01  | 0.00   | 0.02       | 0.78      |
| BAdaIPW  | 0.91     | 0.88      | 0.91  | 0.89   | 0.82       | 0.93      |
| EBA2IPW' | 0.90     | 0.58      | 0.17  | 0.00   | 0.03       | 0.67      |

Table 4: Results of OPL under the RW policy. We highlight in bold the best two estimators in each dataset.

| Metrics | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|---------|----------|-----------|-------|--------|------------|-----------|
| PBA2IPW | 0.812    | 0.020     | 0.060 | 0.072  | 0.493      | 0.298     |
| EBA2IPW | 0.813    | 0.022     | 0.717 | 0.063  | 0.519      | 0.313     |
| BAdaIPW | 0.815    | 0.023     | 0.697 | 0.089  | 0.515      | 0.312     |
| AdaDM   | 0.777    | 0.033     | 0.478 | 0.076  | 0.191      | 0.151     |
| AIPW    | 0.819    | 0.020     | 0.698 | 0.062  | 0.515      | 0.312     |
| DM      | 0.791    | 0.034     | 0.544 | 0.071  | 0.247      | 0.174     |

6 Off-Policy Learning

An important application of OPE is Off-Policy Learning (OPL), which attempts to determine the optimal policy maximizing the expected reward. Let us define the optimal policy \( \pi^* \) as \( \pi^* = \arg \max_{\pi \in \Pi} R(\pi) \), where \( \Pi \) is a policy class. By applying each OPE estimator, we estimate the optimal policy as \( \hat{\pi} = \arg \max_{\pi \in \Pi} \hat{R}_{BA2IPW}(\pi) \).

7 Experiments

Using benchmark datasets and real-world logged data, we demonstrate the effectiveness of the BA2IPW estimator with an equal weight (PBA2IPW) and efficient weight using variance estimators \( \hat{s}_{BA2IPW} \) (EBA2IPW) and \( \hat{s}_{BA2IPW'} \) (EBA2IPW'), and BAdaIPW estimator with an equal weight (BAdaIPW). Note that although the forms of the several estimators are the same as the existing studies, the theoretical guarantees are different.

7.1 Experiments with Benchmark Dataset

Following Dudík et al. (2011) and Farajtabar et al. (2018), we evaluate the proposed estimators using classification datasets by transforming them into contextual bandit data. From the LIBSVM repository, we use the satimage, pendigits, mnist, letter, sensorless, and connect-4 datasets. For a batched update behavior policy, we use the random walk (RW) and LinearUCB (UCB) (Sutton & Barto, 1998; Li et al., 2010; Chu et al., 2011) policies. When using the RW policy, we first decide the probability of choosing an action from uniform distribution. Then, we add a noise \( 0.01N(0, 1) \) at each batch, i.e., the policy is a random walk. At each batch, we standardize the values of random walk to be probability, i.e., all values are positive and the sum is 1. When using the UCB policy, we choose an estimated best arm \( A_1 \) firstly. Then, we create an adaptive policy. Then, we construct a behavior policy as a policy that chooses \( A_1 \) with probability 0.8 and the other arms with equal probability. While the probability of choosing an action of the UCB policy converges, that of the RW policy does not converge.

For each dataset, we compare the performances of the following estimators of policy value: PBA2IPW, EBA2IPW, EBA2IPW', BAdaIPW, Adaptive DM estimator (AdaDM) defined as \( \frac{1}{T} \sum_{t=1}^{T} \sum_{a=1}^{K} \pi^*(a \mid X_t) f_t(a, X_t) \), an AIPW defined as \( \frac{1}{T} \sum_{t=1}^{T} \sum_{a=1}^{K} \frac{\pi^*(a \mid X_t)}{\pi_t(a \mid X_t, M_{t-1})} f_t(a, X_t) \), and DM estimator (DM) defined as \( \frac{1}{T} \sum_{t=1}^{T} \sum_{a=1}^{K} \pi^*(a \mid X_t) f_t(a, X_t) \). When estimating \( f^* \), we use the Nadaraya-Watson regression (NW) estimator (Yang & Zhu, 2002).

MSEs: To construct an evaluation policy, we create a deterministic policy \( \pi^d \) by training a logistic regression classifier on historical data and set the output as \( \pi^d \). Through experiments, the behavior policy \( \pi_b \) is assumed to be known. More details are in Appendix G.1. Let us construct the evaluation policy \( \pi^e \) as a mixture of \( \pi^d \) and the uniform random policy \( \pi^u \), defined as \( \pi^e = 0.9\pi^d + 0.1\pi^u \). We construct the evaluation policy \( \pi^e \) as a mixture of \( \pi^d \) and the uniform random policy \( \pi^u \), defined as \( \pi^u = 0.9\pi^d + 0.1\pi^u \). We compare the MSEs of six estimators, the PBA2IPW, EBA2IPW, EBA2IPW', BAdaIPW, AdaDM, AIPW, and DM estimators. For estimating the weights of EBA2IPW and EBA2IPW' estimators, we iteratively estimate the wights and the value \( \theta_0 \) 10 times. In each experiment, we have historical data.

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2https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
with a sample size $T = 1500$. When estimating $f^*$, we use the Nadaraya-Watson regression (NW) estimator (Yang & Zhu, 2002).

The resulting MSEs and their standard deviations (SDs) over 100 replications of each experiment are shown in Tables 1 and 2. In many cases, the proposed methods show the preferable the existing methods. When using the RW policy, the policy does not converge to a time-invariant policy. Therefore, the proposed method is theoretically preferable for the situation. More importantly, we can construct confidence intervals from the proposed methods, but cannot construct it from the A2IPW estimator. On the other hand, when using the UCB policy, the policy converges to a time-invariant policy, but the proposed methods still show higher performance for various datasets in some cases. We consider that this result is based on the fact that, even though the policy approaches to a time-invariant policy, the update is only allowed in batch, and does not converge sufficiently. In Appendix G.1, we show the additional results.

**Coverage Ratio of Confidence Interval:** In Table 3, we show the coverage ratio of the confidence intervals derived in the previous experiments together with the MSEs. The coverage ratio of the confidence interval is a percentage at which it covers the true value $\theta_0$ in the confidence interval. For the 100 trials of the previous experiment, we calculate the coverage ratio (CR) of 95% confidence interval, which is constructed as

$$\bar{\theta}_T - 1.96 \sqrt{\frac{\hat{\sigma}^2}{1500}}, \bar{\theta}_T + 1.96 \sqrt{\frac{\hat{\sigma}^2}{1500}},$$

where $\bar{\theta}_T$ is an estimator of $\theta_0$ and $\hat{\sigma}^2$ is its estimated asymptotic variance. In the results, the PBA2IPW and BAdaIPW estimator shows CR close to 0.95 in many cases. The BAdaIPW estimator does not require an estimator of $f^*$. Therefore, compared with the other estimators, it shows more preferable performances. EBA2IPW estimator requires several variance estimators, and their estimation error worsen the results compared with the PBA2IPW and BAdaIPW estimators.

**OPL:** In the experiments of OPL, we compare the performances of estimated policy maximizing expected reward obtained from the PBA2IPW, EBA2IPW, BAdaIPW, AdaDM, AIPW, and DM estimators. We conducted 5 trials for each experiment with $T = 1500$. The resulting expected rewards over the evaluation data (RWDs) and the SDs are shown in Table 4, where we highlight in bold the best two estimators.

### 7.2 Experiments with Real-World Data

We apply our estimators to evaluate a policy using the real-world dataset in CyberAgent Inc., which is the second-largest Japanese advertisement company with about 7 billion USD market capitalization (as of August 2020). This company simultaneously runs Thompson sampling and uniformly random sampling to determine the design of advertisements. The Thompson sampling updates the parameter every 30 minute; therefore, there are batches with samples obtained during the 30 minute. We use the logged data produced by the algorithms to confirm the empirical performance of the proposed estimators. To check the performance, we calculate the estimation error between the estimates of the value of the uniformly random sampling policy estimated from the dataset obtained by the Thompson sampling and the observed average reward of the uniformly random policy. More details are shown in Appendix G.2. We apply the PBA2IPW, EBA2IPW, EBA2IPW’, BAdaIPW, and AdaDM estimators. The results are shown in Table 5 and Figure 1. We show the Bias, MSE, and averaged confidence intervals. While the PBA2IPW and BAdaIPW estimators suffer the high variance, the EBA2IPW, EBA2IPW’, and AdaDM estimators show the preferable performances. Although the AdaDM estimator also shows the effectiveness for this dataset, the BAdaIPW estimator is theoretically more robust because it is consistent even if $\hat{f}_t$ does not converge to $f^*$.

**Table 5: Results of the CyberAgent dataset.**

|          | Bias   | MSE     | 95% confidence | sample size |
|----------|--------|---------|----------------|-------------|
| PBA2IPW  | -0.01766 | 8.25830e-02 | 8.25830e-02    |
| EBA2IPW  | 0.00019  | 4.52574e-05 | 3.58261e-03    |
| EBA2IPW’ | 0.00095  | 4.03248e-05 | 2.92494e-03    |
| BAdaIPW  | 0.14650  | 5.70524e-02 | 7.68610e-02    |
| AdaDM    | 0.00714  | 2.26107e-04 | 2.23604e-05    |

### 8 Conclusion

This study presented solutions for causal inference from dependent samples obtained via a batch-based bandit algorithm. By using the asymptotic property in batch, we applied the CLT for an MDS to obtain the asymptotic normality without requiring the convergence of the assignment probability. Additionally, we showed that the proposed method is
applicable when the support of arms is incomplete, which is a notorious problem in OPE. In experiments, the proposed batch-based estimators showed theoretically expected performances for benchmark and real-world datasets.

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A Preliminaries

A.1 Mathematical Tools

Proposition 1. [Slutsky Theorem, Greene (2003), Theorem D. 16 1, p. 1117] If \( a_n \xrightarrow{d} a \) and \( b_n \xrightarrow{p} b \), then
\[
a_n b_n \xrightarrow{d} ba.
\]

Definition 1. [Uniformly Integrable, Hamilton (1994), p. 191] A sequence \( \{A_t\} \) is said to be uniformly integrable if for every \( \epsilon > 0 \) there exists a number \( c > 0 \) such that
\[
\mathbb{E}[|A_t| \cdot 1(|A_t| \geq c)] < \epsilon
\]
for all \( t \).

Proposition 2. [Sufficient Conditions for Uniformly Integrable, Hamilton (1994), Proposition 7.7, p. 191] (a) Suppose there exist \( r > 1 \) such that \( \mathbb{E}[|A_t|^r] < M \) for all \( t \). Then \( \{A_t\} \) is uniformly integrable. (b) Suppose there exist \( r > 1 \) and \( M < \infty \) such that \( \mathbb{E}[|b_t|^r] < M \) for all \( t \). If \( A_t = \sum_{j=-\infty}^{\infty} h_j b_{t-j} \) with \( \sum_{j=-\infty}^{\infty} |h_j| < \infty \), then \( \{A_t\} \) is uniformly integrable.

Proposition 3 \((L^r Convergence Theorem, Loeve (1977))\). Let \( 0 < r < \infty \), suppose that \( \mathbb{E}[|a_n|^r] \xrightarrow{n \to \infty} \infty \) for all \( n \) and that \( a_n \xrightarrow{p} a \) as \( n \to \infty \). The following are equivalent:
(i) \( a_n \xrightarrow{a} a \) in \( L^r \) as \( n \to \infty \);
(ii) \( \mathbb{E}[|a_n|^r] \xrightarrow{n \to \infty} \infty \);
(iii) \( \{ |a_n|^r : n \geq 1 \} \) is uniformly integrable.

A.2 Martingale Limit Theorems

Proposition 4. [Weak Law of Large Numbers for Martingale, Hall et al. (2014)] Let \( \{S_n = \sum_{t=1}^{n} X_t, \Omega_t, t \geq 1\} \) be a martingale and \( \{b_n\} \) a sequence of positive constants with \( b_n \to \infty \) as \( n \to \infty \). Then, writing \( X_{ni} = X_t \mathbb{1}[|X_t| \leq b_n] \), \( 1 \leq i \leq n \), we have that \( b_n^{-1} S_n \xrightarrow{P} 0 \) as \( n \to \infty \) if
(i) \( \sum_{i=1}^{n} P(|X_i| > b_n) \to 0 \);
(ii) \( b_n^{-1} \sum_{i=1}^{n} \mathbb{E}[X_{ni}|\Omega_{t-1}] \xrightarrow{P} 0 \), and;
(iii) \( b_n^2 \sum_{i=1}^{n} \left( \mathbb{E}[X_{ni}^2] - \mathbb{E}[\mathbb{E}[X_{ni}|\Omega_{t-1}]^2] \right) \to 0 \).

Remark 5. The weak law of large numbers for martingale holds when the random variable is bounded by a constant.

Proposition 5. [Central Limit Theorem for a Martingale Difference Sequence, Hamilton (1994), Proposition 7.9, p. 194] Let \( \{B_t\}_{t=1}^{\infty} \) be an \( n \)-dimensional vector martingale difference sequence with \( B_T = \frac{1}{T} \sum_{t=1}^{T} B_t \). Suppose that (a) \( \mathbb{E}[B_t | B_T] = \Omega_t \), a positive definite matrix with \( (1/T) \sum_{t=1}^{T} \Omega_t \to \Omega \), a positive definite matrix; (b) \( \mathbb{E}[B_{it}B_{jt}, B_{il}B_{ml}] < \infty \) for all \( t \) and all \( i, j, l, \) and \( m \) (including \( i = j = l = m \)), where \( B_{it} \) is the \( i \)-th element of vector \( B_t \); and (c) \( (1/T) \sum_{t=1}^{T} B_t B_t^\top \xrightarrow{P} \Omega. \) Then \( \sqrt{T} \overline{B}_T \xrightarrow{d} N(0, \Omega) \).

B Generalized Method of Martingale Difference Moments

In this section, using martingale difference sequences, we establish a frameworks of GMM with samples with dependency. Let \( s_t \) be a sample with the domain \( S, \Theta \) be the space of a parameter \( \theta \), and \( M \) be the number of moment conditions. For a vector-valued function \( h_t : S \times \Theta \to \mathbb{R}^M \) and a parameter \( \theta_0 \in \Theta \), let \( \{h_t(s_t; \theta_0)\}_{t=1}^{T} \) be a martingale difference sequence, i.e.,
\[
\mathbb{E}[h_t(S_t; \theta_0) | \Omega_{t-1}] = 0.
\]
(3)

Then, let \( q(\theta) \) be the moment condition defined as \( q(\theta) = \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} h_t(S_t; \theta) \right] \). For the parameter \( \theta_0 \in \Theta \), the moment condition \( q(\theta_0; S_t) \) is zero from (3), i.e.,
\[
q(\theta_0) = \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^{T} h_t(S_t; \theta_0) \right] = 0.
\]
Using samples \( \{s_t\}_{t=1}^T \in \mathcal{S} \), we approximate the moment condition by \( \hat{q}(\theta) = \frac{1}{T}h_t(s_t; \theta) \). Then, we define a GMM-like estimator \( \hat{\theta}_T \) as follows:

\[
\hat{\theta}_T = \arg\min_{\theta \in \Theta} (\hat{q}(\theta))^\top \hat{W}_T (\hat{q}(\theta)),
\]

where \( \hat{W}_T \) is a positive definite weight matrix constructed from \( T \) samples. Compared with the standard GMM, we do not assume that the samples are not i.i.d. However, from the assumption that \( \{h_t(s_t; \theta_0)\}_{t=1}^T \) is a martingale difference sequence, we can derive the following results on the consistency and asymptotic normality of \( \hat{\theta}_T \) under appropriate regularity conditions. We refer this method as Generalized Method of Martingale Difference Moments (GMMMDM).

For brevity, let us denote \( \hat{R}^{BA2IPW}_T (\pi^e) \) as \( \hat{\theta}^{OPE}_T \). Using the sequence \( \{h_t^{OPE}(X_t, A_t, Y_t; \theta, \hat{f}_{t-1}, \pi^e)\}_{t=1}^T \), we define an estimator of OPE as

\[
\hat{\theta}^{OPE}_T = \arg\min_{\theta \in \Theta} (\hat{q}^{OPE}(\theta))^\top \hat{W}_T (\hat{q}^{OPE}(\theta)),
\]

where \( q^{OPE}(\theta) = \frac{1}{T} \sum_{t=1}^T h_t^{OPE}(X_t, A_t, Y_t; \theta, \hat{f}_{t-1}, \pi^e) \). Note that we can consider that the estimator defined in (2) is an application of GMMMDM with the moment condition

\[
q^{OPE}(\theta_0) = \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T h_t^{OPE}(X_t, A_t, Y_t; \theta_0, \hat{f}_{t-1}, \pi^e) \right] = 0.
\]

For the minimization problem defined in (2), we can analytically calculate the minimizer as

\[
\hat{\theta}^{OPE}_T = \left( I^\top \hat{W}_T I \right)^{-1} I^\top \hat{W}_T D_T (\pi^e),
\]

where \( I \) is an \( M \)-dimensional vector such that \( I = (1 \ 1 \ \cdots \ 1)^\top \) and \( D_T(\pi^e) \) is

\[
\begin{pmatrix}
\frac{1}{t_1} \sum_{t=t_1+1}^{t_1+1} \eta_t(x, k, y; \tau, \theta, f, \pi^e) \\
\frac{1}{t_2-t_1} \sum_{t=t_1+1}^{t_2} \eta_t(x, k, y; \tau, \theta, f, \pi^e) \\
\vdots \\
\frac{1}{t_M-t_{M-1}} \sum_{t=t_{M-1}+1}^T \eta_t(x, k, y; \tau, \theta, f, \pi^e)
\end{pmatrix}.
\]

\section{Proof of Theorem 4}

Instead of \( \hat{\theta}^{OPE}_T = w_T D_T(\pi^e) \), from the original formulation Eq. (2), we consider an estimator \( \hat{\theta}^{OPE}_T = (I^\top \hat{W}_T I)^{-1} I^\top \hat{W}_T D_T(\pi^e) \), where \( W_T \) is a \( (M \times M) \)-dimensional positive-definite matrix. Let us note that \( w_T = (I^\top \hat{W}_T I)^{-1} I^\top \hat{W}_T \). We prove the following theorem, which is a generalized statement of Theorem C.

\begin{theorem}[Asymptotic Distribution the BA2IPW Estimator]
Suppose that

(i) \( \hat{W}_T \xrightarrow{d} W \);

(ii) \( W \) is a positive definite;

(iii) \( \hat{f}_{t-1}(a, x) \xrightarrow{d} f^*(a, x) \) for \( m > 0 \);

(iv) There exists a constant \( C_f > 0 \) such that \( |\hat{f}_{t-1}(a, x)| < C_f \) for \( \tau \in I \).

Then, under Assumptions 1 and 2,

\[
\sqrt{T} \left( \hat{R}^{BA2IPW}_T - R(\pi^e) \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2),
\]

where \( \sigma^2 = (I^\top W I)^{-1} I^\top W \Omega W^\top I (I^\top W I)^{-1} \) and \( \Omega \) is a \( (M \times M) \) diagonal matrix such that the \( (\tau \times \tau) \)-element is \( \frac{1}{r_T^2} \mathbb{E} \left[ \sum_{a=1}^K \left( \frac{\pi^e(a | X)}{\pi^e(a | X, f_{t-1}^{-1})} \right)^2 + \left( \pi^e(a | X)f^*(a, X) - R(\pi^e) \right)^2 \right] \).
Proof. For
\[
\sqrt{T}(\hat{\theta}_T^{\text{OPE}} - \theta_0) = (I^T \hat{W}_T I)^{-1} I^T \hat{W}_T \sqrt{T} \hat{q}_T^{\text{OPE}}(\theta_0),
\]
where
\[
\hat{q}_T^{\text{OPE}}(\theta_0) = \left( \frac{1}{T} \frac{1}{r_1} \sum_{t=1}^{T} \sum_{a=1}^{K} \left\{ \frac{\pi^m(a \mid X_t)\mathbb{I}[A_t = a]\{Y_t - \hat{f}_t(a, X_t)\}}{\pi_t(a \mid X_t, \Omega_{t-1})} + \pi^o(a \mid X_t)\hat{f}_t(a, X_t) - \theta_0 \right\} \mathbb{I}[t_0 = 0 < t \leq t_1] \right)
\]
\[
+ \left( \frac{1}{T} \frac{1}{r_2} \sum_{t=1}^{T} \sum_{a=1}^{K} \left\{ \frac{\pi^m(a \mid X_t)\mathbb{I}[A_t = a]\{Y_t - \hat{f}_t(a, X_t)\}}{\pi_2(a \mid X_t, \Omega_{t-1})} + \pi^o(a \mid X_t)\hat{f}_t(a, X_t) - \theta_0 \right\} \mathbb{I}[t_1 < t \leq t_2] \right)
\]
\[
+ \cdots
\]
\[
+ \left( \frac{1}{T} \frac{1}{r_M} \sum_{t=1}^{T} \sum_{a=1}^{K} \left\{ \frac{\pi^m(a \mid X_t)\mathbb{I}[A_t = a]\{Y_t - \hat{f}_t(a, X_t)\}}{\pi_M(a \mid X_t, \Omega_{t-1})} + \pi^o(a \mid X_t)\hat{f}_t(a, X_t) - \theta_0 \right\} \mathbb{I}[t_{M-1} < t \leq t_M = T] \right),
\]
we show that
\[
\sqrt{T} \hat{q}_T^{\text{OPE}}(\theta_0) \overset{d}{\to} \mathcal{N}(0, \Omega),
\]
where \(\Omega\) is a diagonal matrix such that the \((\tau, \tau)\)-element is
\[
\frac{1}{r_{\tau}} \mathbb{E} \left[ \sum_{a=1}^{K} \frac{(\pi^o(a \mid X))^2 \text{Var}(Y(a) \mid X)}{\pi^o(a \mid X, \Omega_{\tau-1})} + \left( \sum_{a=1}^{K} \pi^o(a \mid X) \mathbb{E}[Y(a) \mid X] - \theta_0 \right)^2 \right].
\]
Then, from Slutsky Theorem (Proposition 1 in Appendix), we can show that
\[
(I^T \hat{W}_T I)^{-1} I^T \hat{W}_T \sqrt{T} \hat{q}_T^{\text{OPE}}(\theta_0) \overset{d}{\to} \mathcal{N}(0, (I^T W I)^{-1} I^T W \Omega W^T I (I^T W I)^{-1}).
\]
To show this result, we use the central limit theorem for martingale difference sequences (Proposition 5 in Appendix A) by checking the following conditions:

(a) \((1/T) \sum_{t=1}^{T} \Omega_t \to \Omega\), where \(\Omega_t = \mathbb{E} \left[ (h_t^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^o)) (h_t^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^o))^\top \right];

(b) \(\mathbb{E}[\hat{h}_t(i, \theta_0, f_{t-1}, \pi^e)\hat{h}_t(j, \theta_0, f_{t-1}, \pi^o)\hat{h}_t(l, \theta_0, f_{t-1}, \pi^e)] < \infty\) for \(i, j, k, l \in I\), where \(\hat{h}_t(a, \theta_0, f_{t-1}, \pi^o) = h_t^{\text{OPE}}(X_t, A_t, Y_t; k, \theta_0, f_k, \pi^o)\) for \(k \in I\);

(c) \((1/T) \sum_{t=1}^{T} (h_t^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^o)) (h_t^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^o))^\top \overset{P}{\to} \Omega,\)

**Step 1: Condition (a)**

From
\[
\Omega_t = \mathbb{E} \left[ (h_t^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^o)) (h_t^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^o))^\top \right],
\]
the matrix \((1/T) \sum_{t=1}^{T} \Omega_t\) becomes a diagonal matrix such that the \((\tau, \tau)\)-element is
\[
\frac{1}{r_{\tau}^2 T} \sum_{t=1}^{T} \mathbb{E} \left[ \left\{ \sum_{a=1}^{K} \frac{\pi^o(a \mid X_t)\mathbb{I}[A_t = a]\{Y_t - \hat{f}_t(a, X_t)\}}{\pi_t(a \mid X_t, \Omega_{\tau-1})} + \pi^o(a \mid X_t)\hat{f}_t(a, X_t) - \theta_0 \right\} \mathbb{I}[t_{\tau-1} < t \leq t_{\tau}] \right]^2.
\]
For \( \tau \in I \) and \( t \) such that \( t_{\tau-1} < t \leq t_{\tau} \),

\[
\mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) f_{t-1}(a, X_t) \right)^2 \right] - \mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - \mathbb{E}[Y_t(a) \mid X_t]\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) \mathbb{E}[Y(a) \mid X_t] \right)^2 \right] \]

\[
\leq \mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) f_{t-1}(a, X_t) \right)^2 \right] - \mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - \mathbb{E}[Y_t(a) \mid X_t]\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) \mathbb{E}[Y(a) \mid X_t] \right)^2 \right] \]

Because \( \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) \), there exists a constant \( \gamma_0 > 0 \) such that

\[
\leq \gamma_0 \mathbb{E}\left[ \sum_{a=1}^{K} \left[ f_{t-1}(a, X_t) - \mathbb{E}[Y(a) \mid X_t] \right] \right].
\]

Here, from the assumption that \( f_{t-1}(a, x) - \mathbb{E}[Y(a) \mid X] \xrightarrow{P} 0 \) for \( \tau = 2, 3, \ldots, M \), and \( f_{t_{\tau-1}}(a, x) \) is bounded for \( \tau \in I \), we can use \( L^r \) convergence theorem (Proposition 3 in Appendix A). First, to use \( L^r \) convergence theorem, we use boundedness of \( f_{t_m} \) and Proposition 2 to derive the uniform integrability of \( f_{t_m} \) for \( m = 0, 1, \ldots, \tau - 1 \). Then, from \( L^r \) convergence theorem, we have \( \mathbb{E}[|f_{t_m}(a, X) - \mathbb{E}[Y(a) \mid X]|] \to 0 \) as \( t_m \to \infty \). Using this results, we can show that, as \( t_{\tau-1} \to \infty \) (this also means \( T \to \infty \)),

\[
\gamma_1 \sum_{a=1}^{K} \mathbb{E}\left[ f_{t-1}(a, X_t) - \mathbb{E}[Y(a) \mid X_t] \right] \to 0.
\]

Therefore, as \( t_{\tau-1} \to \infty \) \((T \to \infty)\),

\[
\mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) f_{t-1}(a, X_t) \right)^2 \right] - \mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - \mathbb{E}[Y_t(a) \mid X_t]\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) \mathbb{E}[Y(a) \mid X_t] \right)^2 \right] \]

Then, by using \( \mathbb{I}[A_t = a] \mathbb{I}[A_t = l] = 0 \), \( \mathbb{E} \left[ \frac{1}{\pi_T(a \mid X_t, \Omega_{t-1})} \right] = \mathbb{E} \left[ \frac{\mathbb{E}[Y^2(a) \mid X_t]}{\pi_T(a \mid X_t, \Omega_{t-1})} \right] \), and \( \frac{1}{\tau_T} \sum_{t=1}^{\tau} \mathbb{I}[t_{\tau-1} < t \leq t_{\tau}] = 0 \),

\[
\mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - \mathbb{E}[Y_t(a) \mid X_t]\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) \mathbb{E}[Y(a) \mid X_t] \right)^2 \right] - \mathbb{E}\left[ \left( \sum_{a=1}^{K} \frac{\pi^{\epsilon}(a \mid X_t) \mathbb{I}[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi_T(a \mid X_t, \Omega_{t-1})} + \pi^{\epsilon}(a \mid X_t) f_{t-1}(a, X_t) \right)^2 \right] \]

\[
= \mathbb{E}\left[ \left( \frac{\pi^{\epsilon}(a \mid X_t)}{\pi_T(a \mid X_t, \Omega_{t-1})} \right)^2 \text{Var}(Y_t(a) \mid X_t) \right] + \left( \frac{\pi^{\epsilon}(a \mid X_t)}{\pi_T(a \mid X_t, \Omega_{t-1})} \right) \mathbb{E}[Y_t(a) \mid X_t] - \mathbb{E}[Y(a) \mid X_t] \]

\[
+ \left( \frac{\pi^{\epsilon}(a \mid X_t)}{\pi_T(a \mid X_t, \Omega_{t-1})} \right) \mathbb{E}[Y(a) \mid X_t] - \mathbb{E}[Y(a) \mid X_t] \]

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In addition, the variance does not depend on \( t \). We represent the independence by omitting the subscript \( t \), i.e.,
\[
E \left[ \sum_{a=1}^{K} \left( \frac{(\pi^a(\cdot | X_t))^2 \text{Var}(Y_t(\cdot | X_t))}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \left( \pi^a(\cdot | X_t) E[Y_t(\cdot | X_t) - \theta_0]^2 \right) \right] \\
= E \left[ \sum_{a=1}^{K} \left( \frac{(\pi^a(\cdot | X))^2 \text{Var}(Y(\cdot | X))}{\pi^a(\cdot | X, \Omega_{t-1})} + \left( \pi^a(\cdot | X) E[Y(\cdot | X) - \theta_0]^2 \right) \right] .
\]

Therefore, we have
\[
\frac{1}{T^2} \sum_{t=1}^{T} E \left[ \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) f_{t-1}(a, X_t) \right) - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t] \right] \\
\approx \frac{1}{r^2} E \left[ \sum_{a=1}^{K} \left( \frac{(\pi^a(\cdot | X))^2 \text{Var}(Y(\cdot | X))}{\pi^a(\cdot | X, \Omega_{t-1})} + \left( \pi^a(\cdot | X) E[Y(\cdot | X) - \theta_0]^2 \right) \right] .
\]

Thus, the matrix \((1/T) \sum_{t=1}^{T} \Omega_t\) converges to a diagonal matrix \( \Omega \) as \( T \to \infty \), where the \((\tau, \tau)\)-element of \( \Omega \) is
\[
\frac{1}{r^2} \sum_{a=1}^{K} \left( \frac{(\pi^a(\cdot | X))^2 \text{Var}(Y(\cdot | X))}{\pi^a(\cdot | X, \Omega_{t-1})} + \left( \pi^a(\cdot | X) E[Y(\cdot | X) - \theta_0]^2 \right) \right]
\]

**Step 2: Condition (b)**

Because we assume that all variables are bounded, this condition holds.

**Step 3: Condition (c)**

Here, we check that \((1/T) \sum_{t=1}^{T} (h^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^a)) \cdot (h^{\text{OPE}}(X_t, A_t, Y_t; \theta_0, f_{t-1}, \pi^a))^T \to \Omega\). The \((\tau, \tau)\)-element of the matrix is
\[
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) f_{t-1}(a, X_t) \right) - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t] \\
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) f_{t-1}(a, X_t) \right) - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t] \\
- \frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - E[Y(\cdot | X_t)]\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) E[Y(\cdot | X_t)] - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t] \\
+ \frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - E[Y(\cdot | X_t)]\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) E[Y(\cdot | X_t)] - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t].
\]

The part
\[
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - f_{t-1}(a, X_t)\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) f_{t-1}(a, X_t) \right) - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t] \\
- \frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - E[Y(\cdot | X_t)]\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) E[Y(\cdot | X_t)] - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t]
\]
converges in probability to 0 because \( f_{t-1}(a, X_t) \xrightarrow{p} E[Y(\cdot | X_t)] \). The term
\[
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{r^2} \left( \sum_{a=1}^{K} \left( \frac{\pi^a(\cdot | X_t) I[A_t = a] \{Y_t - E[Y(\cdot | X_t)]\}}{\pi^a(\cdot | X_t, \Omega_{t-1})} + \pi^a(\cdot | X_t) E[Y(\cdot | X_t)] - \theta_0 \right)^2 I[t_{t-1} < t \leq t_t].
\]
converges in probability to
\[
\frac{1}{r_T} \sum_{a=1}^{K} \left\{ \left( \frac{\pi^o(a \mid X)}{\pi_T(a \mid X, \Omega_{t+1})} \right)^2 \text{Var}(Y(a) \mid X) + \left( \frac{\pi^o(a \mid X) \mathbb{E}[Y(a) \mid X] - \theta_0}{\pi_T(a \mid X, \Omega_{t+1})} \right)^2 \right\}.
\]
from the weak law of large numbers for i.i.d. samples as \( t_{r-1} - t_r \to \infty \) because the samples are i.i.d. between \( t_{r-1} \) and \( t_r \).

D Necessity of Martingale Difference Sequences

In the proposed method, we construct a moment condition using martingale difference sequences. On the other hand, for some readers, using martingale difference sequences may look unnecessary because samples are i.i.d in each block between \( t_{r-1} \) and \( t_r \). Therefore, such readers also might feel that we can use \( f_T(a, x) \), which is an estimator of \( \mathbb{E}[Y(a) \mid x] \) using samples until \( T \)-th period, without going through constructing several estimators \( \{ f_{t_r} \}_{t_r=0}^{M-1} \). However, in that case, it is difficult to guarantee the asymptotic normality of the proposed estimator. For example, we can consider Cramér-Wold theorem, which is stated as follows.

**Proposition 6.** Let \( R_T \) and \( R \) be \( k \)-dimensional random vectors. Then, for all \( v \in \mathbb{R}^k \),

\[
R_T \overset{d}{\to} R \iff \langle v, R_T \rangle \overset{d}{\to} \langle v, R \rangle.
\]

Let \( \tilde{h}_t(\theta_0) \) be

\[
\left( \begin{array}{c}
\frac{1}{r_T} \left( \eta_t(x, k, y; \tau, \theta, f, \pi^o) - \theta_0 \right) \mathbb{1}[t_{r-1} < t \leq t_1] \\
\frac{1}{r_T} \left( \eta_t(x, k, y; \tau, \theta, f, \pi^o) - \theta_0 \right) \mathbb{1}[t_1 < t \leq t_2] \\
\vdots \\
\frac{1}{r_T} \left( \eta_t(x, k, y; \tau, \theta, f, \pi^o) - \theta_0 \right) \mathbb{1}[t_{M-1} < t \leq t_M]
\end{array} \right),
\]

where \( f_T(a, x) \) is an estimator of \( \mathbb{E}[Y(a) \mid x] \) using samples until \( T \)-th period, i.e., all samples. Then, we consider the asymptotic property of \( \tilde{q}_T \) as \( \frac{1}{T} \sum_{t=1}^{T} \tilde{h}_t(\theta_0) \). From Cramér-Wold theorem, there exists a random vector \( R \) such that \( \tilde{q}_T \overset{d}{\to} R \) if and only if \( \langle v, \tilde{q}_T \rangle \overset{d}{\to} \langle v, R \rangle \). Here, for \( v = (1 \cdots 1)^T \), we can calculate \( \langle v, \tilde{q}_T \rangle \) as

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{r_T} \left( \begin{array}{c}
\pi^o(a \mid X_t) \mathbb{1}[A_t = a] \{ Y_t - f_T(a, X_t) \} \\
\sum_{\tau \in \mathcal{T}} \pi_T(a \mid X_t, \Omega_{t-1}) \mathbb{1}[t_{r-1} < t \leq t_r] \\
\vdots \\
\pi^o(a \mid X_t) f_T(a, X_t)
\end{array} \right) - \theta_0.
\]

Because of the existence of \( f_T \), the samples in the sum of the above equation have correlation each other. Therefore, in general, it is difficult to derive the asymptotic distribution of \( \langle v, \tilde{q}_T \rangle \). As far as we know, it is not guaranteed that there exists the asymptotic distribution of \( \langle v, \tilde{q}_T \rangle \), and it is an open problem. If there does not exist the asymptotic distribution of \( \langle v, \tilde{q}_T \rangle \), we also cannot derive the asymptotic distribution of \( \tilde{q}_T \).

More intuitively, even though random variables \( B_1 \) and \( B_2 \) follow normal distribution and they are uncorrelated, it does not guarantee \( B_1 + B_2 \) follows normal distribution when they are dependent.

E BA2IPW with Incomplete Support of Actions

As an application of BA2IPW, we consider an OPE without Assumption 1, which assumes that there exists \( C_1 \) such that \( 0 \leq \pi^o(a \mid x) \leq C_1 \). In stead of Assumption 1, we consider a situation in which we are allowed to change the support of actions in each batch. For example, in the first batch, we choose an action from a set \( \{1, 2, 3\} \) with probability larger than 0, but we choose an action from a set \( \{1, 2, 4\} \) with probability larger than 0 in the second batch. In this case, the probability of an action 4 is 0 in the first batch while the probability of an action 3 is 0 in the second batch. We often face such a kind of situation in practice. For dealing with this situation, instead of Assumption 1, we put Assumption 3. Under this assumption, if \( \pi_T(a \mid x, \Omega_{t+1}) > 0 \) for at least one batch, we are allowed to use \( \pi_T'(a \mid x, \Omega_{t+1}) = 0 \) for \( \tau' \neq \tau \).
For ease of discussion, we assume Assumption 1 for defining an estimator. The following estimator is a generalization of a BA2IPW estimator. Then, we explain the estimator still works only with Assumption 3 instead of Assumption 1. We call the estimator BA2IPW with Incomplete Support (BA2IPWIS). First, let us define

\[ g_{a,\tau,T} = \frac{1}{T} \sum_{t=1}^{T} \tilde{\phi}_{a,\tau}(X_t, A_t, Y_t; f, \pi) \text{, and} \]

\[ \tilde{\phi}_{a,\tau}(x, k, y; f, \pi) := \frac{\pi(a \mid x)}{\tau} \left\{ \mathbb{I}[k = a] \{ y - f(a, x) \} + f(a, x) \right\} \times \mathbb{I}[t_{\tau-1} < t \leq t_{\tau}] . \]

Then, let us define

\[ \tilde{h}_T = \begin{pmatrix} g_{1,1,T} - \theta_0^1 \\ g_{1,2,T} - \theta_0^2 \\ \vdots \\ g_{1,M,T} - \theta_0^1 \\ g_{2,1,T} - \theta_0^2 \\ \vdots \\ g_{K-1,M,T} - \theta_0^{K-1} \\ g_{K,1,T} - \theta_0^K \\ g_{K,M,T} - \theta_0^K \end{pmatrix} \quad \text{and} \quad \tilde{D}_T = \begin{pmatrix} g_{1,1,T} \\ g_{1,2,T} \\ \vdots \\ g_{1,M,T} \\ g_{2,1,T} \\ \vdots \\ g_{K-1,M,T} \\ g_{K,1,T} \\ g_{K,M,T} \end{pmatrix} , \]

As well as Theorem 4, we have

\[ \sqrt{N} \tilde{h}_T \overset{d}{\to} \mathcal{N}(0, \tilde{\Sigma}) , \]

where \( \tilde{\Sigma} \) is the \((KM \times KM)\) variance covariance matrix of

\[ \left( \tilde{\phi}_{1,1}(x, k, y; f, \pi) \tilde{\phi}_{1,2}(x, k, y; f, \pi) \cdots \tilde{\phi}_{K,M}(x, k, y; f, \pi) \right)^\top . \]

Then, for \( \tilde{D} \), let us define the estimator as

\[ \tilde{\theta}_T^{BA2IPWIS} = \zeta_T^\top \tilde{D}_T , \]

where \( \zeta_T \) is a data-dependent \( KM \)-dimensional vector such that \( \sum_{\tau=1}^{M} \zeta_{1,\tau} = 1, \sum_{\tau=1}^{M} \zeta_{2,\tau} = 1, \ldots \sum_{\tau=1}^{M} \zeta_{K,\tau} = 1 \). As well as Theorem 4, we have the following corollary.

**Corollary 1.** Under the same assumptions of Theorem 4,

\[ \sqrt{N} (\zeta_T^\top \tilde{D}_T - \theta_0) \overset{d}{\to} \mathcal{N} \left( 0, \zeta_T^\top \tilde{\Sigma} \zeta_T \right) . \]

**Proof.** From the constraint of \( \zeta_T \), we have

\[ \zeta_T^\top \tilde{h}_T = \sum_{a=1}^{K} \sum_{\tau=1}^{M} \zeta_{a,\tau} g_{a,\tau,T} - \theta_0 . \]

Then,

\[ \sqrt{N} \zeta_T^\top G = \sqrt{N} \left( \sum_{a=1}^{K} \sum_{\tau=1}^{M} \zeta_{a,\tau} \tilde{z}_{a,\tau} - \theta_0 \right) \overset{d}{\to} \mathcal{N} \left( 0, \zeta_T^\top \tilde{\Sigma} \zeta_T \right) . \]

\[ \square \]
Next, we consider an efficient weight that minimizes \(\zeta^T \Sigma T\). The efficient weight \(\zeta^*\) can be defined as the solution of the following constraint optimization problem:

\[
\min_{\zeta \in \mathbb{R}^{K \times M}} \zeta^T \Sigma T \\
\text{s.t. } \sum_{\tau=1}^M \zeta^{1, \tau} = 1, \sum_{\tau=1}^M \zeta^{2, \tau} = 1, \ldots \sum_{\tau=1}^M \zeta^{K, \tau} = 1.
\]

When we put Assumption 3 instead of Assumption 1, we can construct an estimator by removing an element \(g_{a, \tau, t}\) such that \(\pi_T(a \mid x, \Omega_{t-1}) = 0\) from \(\tilde{D}_T\).

F Proof of Theorem 3

Proof. Because the BADR estimator can be decomposed as

\[
\hat{F}_{BADR}^{\pi} = \frac{1}{M} \sum_{\tau=1}^M \sum_{t=t_{\tau-1}+1}^{t_{\tau}} \phi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \hat{g}_{t-1}, \pi^e)
\]

The last term \(\sum_{\tau=1}^M \sum_{t=t_{\tau-1}+1}^{t_{\tau}} \phi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \pi^e)\) is a special case of the BA2IPW estimator and has the asymptotic normality if \(w_T = (w_{T, 1} \cdots w_{T, M})^\top\) and Assumptions 1 and 2 hold. Then, we want to show that

\[
\sum_{\tau=1}^M \sum_{t=t_{\tau-1}+1}^{t_{\tau}} \phi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \pi^e)
\]

\[
= o_p(1/\sqrt{T}).
\]

Because the \(M\) is fixed, we only have to consider

\[
\frac{1}{t_{\tau} - t_{\tau-1}} \sum_{t=t_{\tau-1}+1}^{t_{\tau}} \phi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \pi^e) - \frac{1}{t_{\tau} - t_{\tau-1}} \sum_{t=t_{\tau-1}+1}^{t_{\tau}} \phi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \pi^e)
\]

\[
= o_p(1/\sqrt{T}).
\]

Here, for

\[
 \psi_1(X_t; f, g) = \sum_{a=1}^K \frac{\pi^e(a \mid X_t) \mathbb{1}[A_t = a](Y_t - f(a, X_t))}{g(a \mid X_t)},
\]

\[
 \psi_2(X_t; f) = \sum_{a=1}^K \pi^e(a \mid X_t) f(a, X_t),
\]
we have

\[
\frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \psi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \hat{g}_{t-1}, \tau, \pi^e) - \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \psi_t(X_t, A_t, Y_t; 1, \hat{f}_{t-1}, \pi, \pi^e)
= \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \mid \Omega_{t-1} \right] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \mid \Omega_{t-1} \right] \left\}
\]

In the following parts, we separately show that

\[
\sqrt{T} \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \mid \Omega_{t-1} \right] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \mid \Omega_{t-1} \right] \left\} = o_p(1)
\]

and

\[
\frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) \mid \Omega_{t-1} \right] + \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) \mid \Omega_{t-1} \right] \\
- \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \mid \Omega_{t-1} \right] - \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \mathbb{E} \left[ \psi_2(X_t; f^*) \mid \Omega_{t-1} \right] = o_p(1/\sqrt{T}).
\]

F.1 Proof of (4)

For any \( \varepsilon > 0 \), to show that

\[
\mathbb{P} \left( \left\| \sqrt{T} \frac{1}{r_T T} \sum_{t = t_{r-1} + 1}^{t_r} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \mid \Omega_{t-1} \right] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \mid \Omega_{t-1} \right] \left\} \right\| > \varepsilon \right) \\
\rightarrow 0,
\]

we show that the mean is 0 and the variance of the component converges to 0. Then, from the Chebyshev’s inequality, this result yields the statement.
The mean is calculated as

$$\sqrt{T} \frac{1}{r_T} \sum_{t=t_{r}-1+1}^{t_{r}} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right| \Omega_{t-1} \biggr] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \right| \Omega_{t-1} \biggr] \right] \\
= \sqrt{T} \frac{1}{r_T} \sum_{t=t_{r}-1+1}^{t_{r}} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right| \Omega_{t-1} \biggr] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \right| \Omega_{t-1} \biggr] \right] \\
= 0$$

Because the mean is 0, the variance is calculated as

$$\text{Var} \left( \sqrt{T} \frac{1}{r_T} \sum_{t=t_{r}-1+1}^{t_{r}} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right| \Omega_{t-1} \biggr] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \right| \Omega_{t-1} \biggr] \right) \\
= \frac{1}{r_T^2} \mathbb{E} \left[ \left( \sum_{t=t_{r}-1+1}^{t_{r}} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right. \right. \right. \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right| \Omega_{t-1} \biggr] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \right| \Omega_{t-1} \biggr] \right) \right)^2.$$
Then, we can decompose the variance as

\[
\begin{align*}
&= \frac{1}{r^2T} \sum_{t=t_1+1}^{t_r} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right] \\
&\quad - \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \mid \Omega_{t-1} \right] \\
&\quad + \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \mid \Omega_{t-1} \right] \\
&\quad + \frac{2}{r^2T} \sum_{t=1}^{t_r} \sum_{s=t+1}^{t_r} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{s-1}, \hat{f}_{s-1}) - \psi_1(X_t, A_t, Y_t; \pi_{s-1}, f^*) \right] \\
&\quad - \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{s-1}, \hat{f}_{s-1}) - \psi_1(X_t, A_t, Y_t; \pi_{s-1}, f^*) \mid \Omega_{s-1} \right] \\
&\quad + \psi_2(X_s; \hat{f}_{s-1}) - \psi_2(X_s; f^*) - \mathbb{E} \left[ \psi_2(X_s; \hat{f}_{s-1}) - \psi_2(X_s; f^*) \mid \Omega_{s-1} \right].
\end{align*}
\]

For \( s > t \), we can vanish the covariance terms as

\[
\begin{align*}
&\mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{s-1}, \hat{f}_{s-1}) - \psi_1(X_t, A_t, Y_t; \pi_{s-1}, f^*) \right] \\
&\quad - \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{s-1}, \hat{f}_{s-1}) - \psi_1(X_t, A_t, Y_t; \pi_{s-1}, f^*) \mid \Omega_{s-1} \right] \\
&\quad + \psi_2(X_t; \hat{f}_{s-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{s-1}) - \psi_2(X_t; f^*) \mid \Omega_{s-1} \right] \\
&\quad + \frac{2}{r^2T} \mathbb{E} \left[ \psi_1(X_s, A_s, Y_s; \hat{g}_{s-1}, \hat{f}_{s-1}) - \psi_1(X_s, A_s, Y_s; \pi_{s-1}, f^*) \right] \\
&\quad - \mathbb{E} \left[ \psi_1(X_s, A_s, Y_s; \hat{g}_{s-1}, \hat{f}_{s-1}) - \psi_1(X_s, A_s, Y_s; \pi_{s-1}, f^*) \mid \Omega_{s-1} \right] \\
&\quad + \psi_2(X_s; \hat{f}_{s-1}) - \psi_2(X_s; f^*) - \mathbb{E} \left[ \psi_2(X_s; \hat{f}_{s-1}) - \psi_2(X_s; f^*) \mid \Omega_{s-1} \right] \\
&\quad = 0,
\end{align*}
\]
where $U = \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right) - \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \middle| \Omega_{t-1} \right]$. Therefore, the variance is calculated as

$$\text{Var} \left( \sqrt{T} \frac{1}{r_T} \sum_{t=t_{r-1}+1}^{t_r} \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right) + \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \middle| \Omega_{t-1} \right] \right)$$

Then, by considering the conditional expectation,

$$= \frac{1}{r_T^2} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right)^2 \right]$$

$$\leq \frac{1}{r_T^2} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right)^2 \right].$$
Then, we want to show
\[
\mathbb{E} \left[ \mathbb{E} \left[ \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) + \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \right)^2 \mid \Omega_{t-1} \right] \right] \rightarrow 0.
\]

Here, we can use
\[
\mathbb{E} \left[ \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \mathbb{I}[A_t = a] \left( Y_t - \hat{f}_{t-1}(a, X_t) \right) \frac{\hat{g}_{t-1}(a \mid X_t)}{\pi_{t-1}(a \mid X_t, \Omega_{t-1})} \right] = o_p(1),
\]
and
\[
\mathbb{E} \left[ \left( \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \hat{f}_{t-1}(a, X_t) - \sum_{a=1}^{K} \pi^\circ(a \mid X_t) f^*(a, X_t) \right)^2 \mid \Omega_{t-1} \right] = o_p(1).
\]

The first equation (6) is proved by
\[
\mathbb{E} \left[ \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \mathbb{I}[A_t = a] \left( Y_t - \hat{f}_{t-1}(a, X_t) \right) \frac{\hat{g}_{t-1}(a \mid X_t)}{\pi_{t-1}(a \mid X_t, \Omega_{t-1})} \right] 
\leq 2 \mathbb{E} \left[ \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \mathbb{I}[A_t = a] \left( Y_t - \hat{f}_{t-1}(a, X_t) \right) \frac{\hat{g}_{t-1}(a \mid X_t)}{\pi_{t-1}(a \mid X_t, \Omega_{t-1})} \right] 
+ 2 \mathbb{E} \left[ \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \mathbb{I}[A_t = a] \left( Y_t - f^*(a, X_t) \right) \frac{\hat{g}_{t-1}(a \mid X_t)}{\pi_{t-1}(a \mid X_t, \Omega_{t-1})} \right] 
\leq 2C \| f^* - \hat{f}_{t-1} \|_2^2 + 2 \times 4C \| \hat{g}_{t-1} - \pi_{t-1} \|_2^2 = o_p(1),
\]
where $C > 0$ is a constant. Here, we have used a parallelagram law from the second line to the third line. We have use $|\hat{f}_{t-1}| < C_3$, and $0 < \frac{\pi^\circ}{\hat{g}_{t-1}} < C_4$ and convergence rate conditions, from the third line to the fourth line. The second equation (7) is proved by Jensen’s inequality.

Besides, we can also use
\[
\mathbb{E} \left[ \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \mathbb{I}[A_t = a] \left( Y_t - \hat{f}_{t-1}(a, X_t) \right) - \sum_{a=1}^{K} \pi^\circ(a \mid X_t) \mathbb{I}[A_t = a] \left( Y_t - f^*(a, X_t) \right) \right] 
= o_p(1)
\]
This is proved by
\[
\mathbb{E} \left\{ \sum_{a=1}^{K} \frac{\pi^\alpha(a \mid X_t) 1[A_t = a]}{\tilde{g}_{t-1}(a \mid X_t)} \left( Y_t - \hat{f}_{t-1}(a, X_t) \right) - \sum_{a=1}^{K} \frac{\pi^\alpha(a \mid X_t) 1[A_t = a]}{\pi_{t-1}(a \mid X_t, \Omega_{t-1})} (Y_t - f^*(a, X_t)) \right\} \\
\leq C \mathbb{E} \left\{ \sum_{a=1}^{K} \left( \pi_{t-1}(a \mid X_t, \Omega_{t-1}) - \tau_{t-1}(a \mid X_t, \Omega_{t-1}) \right) \right\} \\
\times \left\{ \sum_{a=1}^{K} \left( \pi^\alpha(a \mid X_t) \hat{f}_{t-1}(a, X_t) - \pi^\alpha(a \mid X_t) f^*(a, X_t) \right) \right\} | \Omega_{t-1} | \\
= o_p(1),
\]
where \( C > 0 \) is a constant. Here, we used Hölder’s inequality \( \|fg\|_1 \leq \|f\|_2 \|g\|_2 \) and
\[
\leq C \left\| \sum_{a=1}^{K} \left( \pi_{t-1}(a \mid X_t, \Omega_{t-1}) - \tau_{t-1}(a \mid X_t, \Omega_{t-1}) \right) \right\|_2 \left\| \sum_{a=1}^{K} \left( f^*(a, X_t) - \hat{f}_{t-1}(a, X_t) \right) \right\|_2 \\
= o_p(1)
\]
Therefore, from the \( L^r \) convergence theorem (Proposition 3) and the boundedness of the random variables, we can show that as \( t \to \infty \),
\[
\mathbb{E} \left[ \mathbb{E} \left[ \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right)^2 \right| \Omega_{t-1} \right] \right] \\
\to 0.
\]
Therefore, for any \( \epsilon > 0 \), there exists a constant \( \tilde{C} > 0 \) such that
\[
\frac{1}{r_T} \sum_{t=r_{t-1}+1}^{r_t} \mathbb{E} \left[ \left( \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) + \psi_2(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right)^2 | \Omega_{t-1} \right] \\
\leq \tilde{C}/T + \epsilon.
\]
Thus, the variance also converges to 0. Then, from Chebyshev’s inequality,
\[
\mathbb{P} \left( \sqrt{T} \frac{1}{r_T} \sum_{t=r_{t-1}+1}^{r_t} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right\} \right) \\
- \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right| \Omega_{t-1} \right] \\
+ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) - \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) - \psi_2(X_t; f^*) \right| \Omega_{t-1} \right] > \epsilon \\
\leq \mathbb{E} \left[ \left( \sqrt{T} \frac{1}{r_T} \sum_{t=r_{t-1}+1}^{r_t} \left\{ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) - \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \right\} \right) / \epsilon^2 \\
\to 0.
\]
F.2 Proof of (5)

\[
\frac{1}{r_t T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \hat{g}_{t-1}, \hat{f}_{t-1}) \mid \Omega_{t-1} \right] + \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \psi_2(X_t; \hat{f}_{t-1}) \mid \Omega_{t-1} \right]
\]

\[
- \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \psi_1(X_t, A_t, Y_t; \pi_{t-1}, f^*) \mid \Omega_{t-1} \right] - \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \psi_2(X_t; f^*) \mid \Omega_{t-1} \right]
\]

\[
= \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \sum_{a=1}^{K} \frac{\pi^a(a \mid X_t) \mathbb{I}[A_t = a]}{\hat{g}_{t-1}(a \mid X_t)} \left( Y_t - \hat{f}_{t-1}(a, X_t) \right) \right] \mid \Omega_{t-1}
\]

\[
+ \frac{1}{T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \sum_{a=1}^{K} \pi^a(a, X_t) \hat{f}_{t-1}(a, X_t) \right] \mid \Omega_{t-1} \]

\[
- \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \mathbb{E} \left[ \sum_{a=1}^{K} \pi^a(a, X_t) \left( f^*(a, X_t) - \hat{f}_{t-1}(a, X_t) \right) \right] \mid \Omega_{t-1}
\]

\[
= \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \sum_{a=1}^{K} \sum_{t_{r-1}+1}^{t_r} \mathbb{E} \left[ \frac{\pi^a(a \mid X_t) \pi_{t-1}(a \mid X_t, \Omega_{t-1}) \left( f^*(a, X_t) - \hat{f}_{t-1}(a, X_t) \right)}{\hat{g}_{t-1}(a \mid X_t)} \right] \mid \Omega_{t-1}
\]

\[
- \frac{1}{r_T T} \sum_{t=t_{r-1}+1}^{t_r} \sum_{a=1}^{K} \sum_{t_{r-1}+1}^{t_r} \mathbb{E} \left[ \frac{\pi^a(a \mid X_t) \left( \pi_{t-1}(a \mid X_t, \Omega_{t-1}) \left( f^*(a, X_t) - \hat{f}_{t-1}(a, X_t) \right) \right.}{\hat{g}_{t-1}(a \mid X_t)} \right] \mid \Omega_{t-1}
\]

Because (9) is 0,
By using Hölder’s inequality $\|fg\|_1 \leq \|f\|_2 \|g\|_2$, for a constant $C > 0$, we have

\[
\leq \frac{C}{r_\tau T} \sum_{t=t_{\tau-1}+1}^{T} \|\pi_{t-1}(a \mid X_t, \Omega_{t-1}) - \hat{g}_{t-1}(a \mid X_t)\|_2 \|f^*(a, X_t) - \hat{f}_{t-1}(a, X_t)\|_2
\]

\[
= \frac{C}{r_\tau T} \sum_{t=t_{\tau-1}+1}^{T} \alpha \beta
\]

\[
= \frac{C}{r_\tau T} \sum_{t=t_{\tau-1}+1}^{T} o_p((t - t_{\tau-1})^{-1/2})
\]

\[
= o_p(1/\sqrt{T}).
\]

G Experiments

G.1 Additional Experimental Results

In this section, we show the additional experimental results using different sample sizes, nonparametric estimator, and the numbers of batches.

In Table 6, we show the results of OPE with 1,000 samples under 10 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

In Table 7, we show the results of OPE with 2,000 samples under 10 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

In Table 8, we show the results of OPE with 1,500 samples under 10 batches using k-nearest neighbor (k-NN) regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

In Table 9, we show the results of OPE with 1,500 samples under 5 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

In Table 10, we show the results of OPE with 1,500 samples under 20 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

In Table 11, we show the results of OPE with 1,500 samples under the UCB policy. The best two methods are highlighted in bold.

G.2 Details of Experiments with CyberAgent Dataset

We use a logged dataset of advertisement selection. In the Cyberagent, the Thompson sampling and random sampling are simultaneously used as the behavior policies for selecting a video advertisement. An advertisement is selected as follows. First, after receiving a bid request for a user $i$ with the covariate $X_i$ in an online advertisement auction, we choose an advertisement campaign, which contains several video advertisements. Then, we choose a behavior policy from Thompson sampling and random sampling. At each period, the Thomson sampling is chosen with 5% ~ 20% probability; otherwise, the random sampling is chosen. Following the chosen behavior policy, we select a video advertisement ($A_i$) from the candidates to maximize the click rate, $Y_i$. The Thomson sampling is updated for every 30 minute. Each batch consists of about 500 ~ 1,000 samples.

For the experiment, we create a new dataset from the original dataset. By combining some batches, we make 15 datasets with 10,000 samples and about 10 ~ 15 batches in each dataset. We apply the OPE estimators to the dataset generated from the Thompson sampling and estimate the policy value of the random sampling. We regard the observed result of random sampling as the true policy value. The estimation error between the estimated policy value and the observed (true) policy value is reported.
Table 6: Results of OPE with 1,000 samples under 10 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| MSE      | SD       | MSE       | SD    | MSE    | SD         | MSE       | SD         |
| PBAIPW   | 0.062    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| EBAIPW   | 0.065    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| EBAIPW'  | 0.065    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| BAdaIPW  | 0.161    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| AdaDM    | 0.163    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| AIPW     | 0.046    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| DM       | 0.101    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |

Table 7: Results of OPE with 2,000 samples under 10 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| MSE      | SD       | MSE       | SD    | MSE    | SD         | MSE       | SD         |
| PBAIPW   | 0.088    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| EBAIPW   | 0.092    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| EBAIPW'  | 0.092    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| BAdaIPW  | 0.188    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| AdaDM    | 0.203    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| AIPW     | 0.288    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| DM       | 0.323    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |

Table 8: Results of OPE with 1,500 samples under 10 batches using nonparametric k-NN regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| MSE      | SD       | MSE       | SD    | MSE    | SD         | MSE       | SD         |
| PBAIPW   | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| EBAIPW   | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| EBAIPW'  | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| BAdaIPW  | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| AdaDM    | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| AIPW     | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |
| DM       | 0.354    | 0.030     | 0.030 | 0.128  | 0.035      | 0.265     | 0.019      |

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Table 9: Results of OPE with 1, 500 samples under 5 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| Metrics  | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD |
| PBA2IPW  | 0.046 | 0.005 | 0.127 | 0.059 | 0.136 | 0.064 | 0.243 | 0.184 | 0.141 | 0.085 | 0.022 | 0.022 |
| EBA2IPW  | 0.049 | 0.006 | 0.123 | 0.018 | 0.140 | 0.023 | 0.241 | 0.055 | 0.122 | 0.018 | 0.023 | 0.022 |
| EBA2IPW  | 0.048 | 0.005 | 0.117 | 0.017 | 0.113 | 0.019 | 0.233 | 0.055 | 0.136 | 0.047 | 0.023 | 0.022 |
| AdaIPW   | 0.138 | 0.015 | 0.496 | 0.040 | 0.439 | 0.044 | 0.479 | 0.021 | 0.413 | 0.030 | 0.143 | 0.026 |
| AIPW     | 0.034 | 0.003 | 0.100 | 0.016 | 0.249 | 0.026 | 0.172 | 0.038 | 0.107 | 0.050 | 0.061 | 0.023 |
| DM       | 0.098 | 0.007 | 0.460 | 0.032 | 0.289 | 0.030 | 0.465 | 0.022 | 0.397 | 0.025 | 0.090 | 0.019 |

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| Metrics  | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD |
| PBA2IPW  | 0.061 | 0.007 | 0.143 | 0.082 | 0.135 | 0.045 | 0.276 | 0.222 | 0.154 | 0.065 | 0.073 | 0.073 |
| EBA2IPW  | 0.023 | 0.001 | 0.068 | 0.021 | 0.168 | 0.034 | 0.401 | 0.079 | 0.185 | 0.048 | 0.028 | 0.028 |
| EBA2IPW  | 0.047 | 0.011 | 0.080 | 0.022 | 0.117 | 0.015 | 0.384 | 0.072 | 0.171 | 0.036 | 0.032 | 0.031 |
| AdaIPW   | 0.081 | 0.018 | 0.198 | 0.129 | 0.199 | 0.082 | 0.290 | 0.258 | 0.149 | 0.042 | 0.101 | 0.100 |
| AIPW     | 0.108 | 0.006 | 0.286 | 0.012 | 0.409 | 0.021 | 0.464 | 0.019 | 0.357 | 0.026 | 0.100 | 0.030 |
| DM       | 0.037 | 0.004 | 0.060 | 0.009 | 0.155 | 0.008 | 0.229 | 0.140 | 0.127 | 0.052 | 0.017 | 0.017 |

Table 10: Results of OPE with 1, 500 samples under 20 batches using nonparametric NW regression. The upper graph shows the results with the RW policy. The lower graph shows the results with the UCB policy.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| Metrics  | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD |
| PBA2IPW  | 0.045 | 0.004 | 0.139 | 0.056 | 0.163 | 0.017 | 0.376 | 0.329 | 0.162 | 0.076 | 0.058 | 0.058 |
| EBA2IPW  | 0.051 | 0.005 | 0.274 | 0.035 | 0.248 | 0.029 | 0.467 | 0.044 | 0.265 | 0.044 | 0.032 | 0.024 |
| EBA2IPW  | 0.051 | 0.006 | 0.267 | 0.035 | 0.162 | 0.077 | 0.460 | 0.041 | 0.258 | 0.037 | 0.032 | 0.031 |
| AdaIPW   | 0.089 | 0.013 | 0.193 | 0.089 | 0.213 | 0.103 | 0.399 | 0.396 | 0.188 | 0.100 | 0.076 | 0.075 |
| AdaIPW   | 0.144 | 0.012 | 0.492 | 0.033 | 0.434 | 0.037 | 0.476 | 0.022 | 0.412 | 0.027 | 0.139 | 0.023 |
| AIPW     | 0.057 | 0.021 | 0.109 | 0.024 | 0.234 | 0.018 | 0.282 | 0.127 | 0.150 | 0.062 | 0.073 | 0.057 |
| DM       | 0.101 | 0.003 | 0.447 | 0.024 | 0.265 | 0.022 | 0.457 | 0.021 | 0.389 | 0.021 | 0.083 | 0.015 |

Table 11: Results of OPL under the UCB policy. We highlight in bold the best two estimators in each dataset.

| Datasets | satimage | pendigits | mnist | letter | sensorless | connect-4 |
|----------|----------|-----------|-------|--------|------------|-----------|
| Metrics  | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD | MSE | SD |
| PBA2IPW  | 0.086 | 0.024 | 0.656 | 0.145 | 0.743 | 0.028 | 0.156 | 0.048 | 0.317 | 0.062 | 0.645 | 0.030 |
| EBA2IPW  | 0.810 | 0.026 | 0.463 | 0.100 | 0.653 | 0.054 | 0.099 | 0.046 | 0.312 | 0.075 | 0.668 | 0.030 |
| EBA2IPW  | 0.811 | 0.021 | 0.511 | 0.107 | 0.656 | 0.070 | 0.126 | 0.038 | 0.275 | 0.071 | 0.665 | 0.024 |
| AdaIPW   | 0.786 | 0.029 | 0.407 | 0.053 | 0.191 | 0.078 | 0.055 | 0.023 | 0.207 | 0.039 | 0.653 | 0.030 |
| AIPW     | 0.819 | 0.015 | 0.419 | 0.099 | 0.644 | 0.101 | 0.120 | 0.040 | 0.290 | 0.057 | 0.666 | 0.028 |
| DM       | 0.798 | 0.026 | 0.387 | 0.035 | 0.212 | 0.080 | 0.089 | 0.036 | 0.220 | 0.045 | 0.653 | 0.030 |