PERIODIC CYCLIC HOMOLOGY AS SHEAF COHOMOLOGY

by Guillermo Cortiñas*
Departamento de Matemática
Facultad de Cs. Exactas y Naturales
Universidad de Buenos Aires
Argentina

0. Introduction. This paper continues the study of the noncommutative infinitesimal cohomology we introduced in [3]. This is the cohomology of sheaves on a noncommutative version of the commutative infinitesimal site of Grothendieck ([8]). Grothendieck showed that, for a smooth scheme $X$ of characteristic zero, the cohomology of the structure sheaf on the infinitesimal site gives de Rham cohomology:

$$H^{*}_{\text{inf}}(X, O) \cong H^{*}_{\text{dR}}(X)$$

Here we prove that, for any associative, not necessarily unital algebra $A$ over $\mathbb{Q}$, the noncommutative infinitesimal cohomology of the structure sheaf modulo commutators gives periodic cyclic homology (Theorem 3.0):

$$H^{*}_{\text{inf}}(A, O/[O, O]) \cong H^{*}_{\text{HP}}(A)$$

We view (2) as a noncommutative affine version of (1). Indeed, the noncommutative analogue of smoothness is quasi-freedom, and for quasi-free algebras $H^{*}_{\text{HP}}$ agrees with (Karoubi’s definition of) noncommutative de Rham cohomology, i.e. we have:

$$H^{*}_{\text{inf}}(R, O/[O, O]) \cong H^{*}_{\text{dR}}(R) := H^{*}(\Omega(R)/[\Omega(R), \Omega(R)], d)$$

for $R$ quasi-free. Here $\Omega(R)$ is the DGA of noncommutative forms. Grothendieck’s theorem includes also a description of the cohomology of the de Rham complex $(E \otimes_{O_X} \Omega_X, \nabla)$ associated with a bundle $E$ with a flat connection as the cohomology of a certain module on the infinitesimal site; in fact (1) is a obtained from this by setting $E = O_X$, $\nabla = d$. A noncommutative version of Grothendieck’s theorem for the de Rham cohomology of flat bundles is proved in Theorem 2.3. It expresses the cohomology of the complex $H^{*}(E \otimes_{R} \Omega(R), \nabla)$ associated with a right module over the augmented algebra $\tilde{R}$ with a flat connection $\nabla$ as sheaf cohomology; for $(E, \nabla) = (R, d)$ it gives:

$$H^{*}_{\text{inf}}(R, O) \cong H^{*}(\Omega(R), d) = 0$$

(*) Partially supported by grant BID802/OC-AR-PICT 2260 of the Agencia Nacional de Promoción Científica de La Argentina and by grants UBACyT EX014 and TW79 of the Universidad de Buenos Aires.
We also compute infinitesimal (hyper)cohomology with coefficients in algebraic $K$-theory. We show that, for any $\mathbb{Q}$-algebra $A$, we have a natural isomorphism:

$$(4) \quad H^n_{inf}(A, K_1) \cong HP_n(A) \quad (n \geq 2)$$

In section 4, we use the case $n = 2$ of (4) to give a natural, sheaf theoretic construction for the Chern-Connes-Karoubi character:

$$(5) \quad ch_0 : K_0(A) \to HP_0(A) \cong H^2(A, K_1)$$

The cohomology groups of $K_1$ not listed in (4) appear in an exact sequence:

$$(6) \quad K_2(A) \hookrightarrow HN_2(A) \rightarrow H_0^{inf}(A, K_1) \rightarrow K_1(A) \hookrightarrow HN_1(A) \rightarrow H^1(A, K_1) \rightarrow 0$$

The map $c$ above is the Jones-Goodwillie character to negative cyclic homology. The sequence (6) is obtained as the lower terms of the long exact sequence of homotopy groups of a homotopy fibration sequence (Th. 5.0):

$$(7) \quad \mathbb{H}_{inf}(A, K) \to K(A) \to SCN_{\geq 1}(A)$$

Here $K$ is a (functorial, fibrant) simplicial version of the $K$-theory space $BGl^+$ and $SCN_{\geq 1}$ is the simplicial set the Dold-Kan correspondence associates with the negative cyclic complex truncated below dimension 1.

As stated above, this paper is a continuation of [3]. In op.cit. the case $* = 0$ of (2) was proved, and the case $* = 1$ was conjectured. It was further conjectured that the higher infinitesimal cohomology groups with coefficients in $O/[O, O]$ and $K_1$ vanished; (3) and (4) disprove both conjectures. In [3] the sequence (6) was shown to be rationally exact everywhere except at $H^1_{inf}(A, K_1)$. The fibration sequence (7) was not proved in [3]–not even rationally– and is therefore new.

The rest of this paper is organized as follows. In section 1 we give the basic definitions and notations. Section 2 concerns flat connections; no assumption on the characteristic of the ground field $k$ is made in this section. We show that for a quasi-free algebra $R$, the category of pairs $(E, \nabla)$ of a right $\tilde{R}$-module and a flat connection is equivalent to a subcategory of the category of modules on the infinitesimal site (Proposition 2.1) and that the sheaf cohomology of the infinitesimal module corresponding to $(E, \nabla)$ is the cohomology of the complex $(E \otimes_R \Omega(R), \nabla)$ (Theorem 2.4). The isomorphism (2) is proved in section 3. Section 4 is devoted to the sheaf theoretic interpretation of the Chern character (5). The isomorphism (4) is proved in Proposition 4.0. In 4.1 we construct a character $ch_{inf} : K_0(A) \to H^2_{inf}(A, K_1)$ for algebras $A$ over an arbitrary field $k$. In theorem 4.4, we prove that in characteristic zero, the character of 4.1 agrees with that of Connes-Karoubi up to a normalization factor of 2. The fibration sequence (7) and the exact sequence (6) are the subject of section 5 (Theorem 5.0 and Corollary 5.1.)
1. Notations and definitions.

Basic notations and conventions 1.0. We fix a field \( k \) which we call the ground field. We make no assumptions on the characteristic of \( k \), except as stated. All tensor products are over \( k \) except as indicated. Algebras are nonunital associative \( k \)-algebras. We use the letters \( A, B \ldots \) for general algebras and the letters \( R, S \ldots \) for quasi-free algebras. We write \( \hat{A} \) for the augmented unital algebra \( A \oplus k \). Ideals are always two-sided. Modules over unital algebras are right modules and are always assumed unital (i.e. \( 1 \cdot m = m \) for \( m \in M \)). Curly brackets \{ \} are used to denote pro-objects. Thus \( \{ A_n \} \) is a pro-algebra having \( A_n \) at level \( n \in \mathbb{N} \) where the structure maps \( A_n \to A_{n-1} \) are understood from the context.

Cosimplicial cylinder 1.1. We write \( \Delta \) for the category of the finite ordered sets \( [n] = \{0 < \cdots < n\} \) and nondecreasing monotone maps. Recall a cosimplicial object in a category \( C \) is a functor \( \Delta \to C \). Any object \( A \) in a category \( C \) with coproducts gives rise to a cosimplicial object \( \Delta \to C \), by \( [n] \mapsto \coprod_{i=0}^{n} A \); we call this the coproduct cosimplicial object associated to \( A \). We write * for the coproduct in the category of algebras and \( Q^*A : [n] \mapsto Q^n(A) := A \ast \cdots \ast A \) \((n+1 \text{ factors})\) for the coproduct cosimplicial algebra. We write \( \partial^i \) and \( \mu^i \) for the cofaces and codegeneracies of this and of most cosimplicial objects appearing in this paper. The maps \( \mu^0 \mu^1 \cdots \mu^n : Q^n(A) \to A \) \((n \geq 0)\) assemble to give a cosimplicial map \( \mu \) to the constant cosimplicial algebra. We write \( q^*(A) \) for its kernel and set:

\[
Cyl^*(A) := Q^*(A)/q^*(A)^\infty
\]

Here the exponent \( \infty \) has the same meaning as in [7]; thus \( Cyl^*(A) \) is a cosimplicial pro-algebra or more precisely an inverse system of cosimplicial algebras having \( Cyl^*(A)_n := Q^*A/q^*(A)^n \) at level \( n \). We call the pro-algebra \( Cyl^1(A) \) the cylinder of \( A \) and \( Cyl^*(A) \) the cosimplicial cylinder. The name comes from the role these pro-algebras play in the notion of nil-homotopy developed in [2] and [3]. Roughly one thinks of \( Cyl^1(A) \) as two copies of \( A \) with the ‘contractible space’ \( qA/qA^\infty \) stuck in between. Next we shall show that for a quasi-free algebra \( R \), \( Cyl^*(R) \) can be described in terms of noncommutative differential forms. By the tubular neighborhood theorem ([4, Th. 2]; see also [4, 8.7] and [2, Theorem 2.1]), we have an isomorphism of inverse systems \( \alpha : Cyl^1(R)_n = Q^1(R)/q^1(R)^n \to \Omega(R)/\Omega^1(R)^n \cong \oplus_{i=0}^{n-1} \Omega^i(R) \), where \( \Omega^1(R) \) is the ideal generated by \( \Omega^1(R) \). The map \( \alpha \) is natural on pairs \( (R, \phi) \) of a quasi-free algebra and a 1-cochain \( \phi : R \to \Omega^2(R) \) satisfying \( \delta(\phi) = -d \cup d \), and is such that \( \alpha \partial^1(a) = a \) and \( \alpha \partial^0(a) = a + da + \phi(a) + \cdots \) \((a \in R)\). This isomorphism generalizes to higher codimension as follows. Write \( W \) for the coproduct cosimplicial vectorspace on \( k \); thus \( W^m = \bigoplus_{i=0}^{m} ke_i \), and cofaces and codegeneracies are as follows: \( \partial^i(e_j) = e_j \) if \( j < i \) and \( \partial^i(e_j) = e_{j+1} \) if \( j \geq i \), \( \mu^i(e_j) = e_j \) if \( j \leq i \) and \( \mu^i(e_j) = e_{j-1} \) if \( j > i \). We write \( V \) for the kernel of the canonical map to the constant cosimplicial vectorspace \( k \). Thus \( V^m = \bigoplus_{i=1}^{m} kv_i \), where \( v_i = e_i - e_{i-1} \) \((i = 1, \ldots, m)\). We have an isomorphism:

\[
(8) \quad \alpha^m : Cyl^m(R)_n \to \bigoplus_{i=0}^{n-1} \Omega^i(R) \otimes T^i(V^m)
\]

The right hand side of (8) has the algebra structure of the quotient of the unaugmented tensor algebra \( \Omega_m(R) := \ker(T_R(\bigoplus_{i=1}^{m} \Omega^i(R)) \to k) \) by the \( n \)-th power of the ideal generated by \( \bigoplus_{i=1}^{m} \Omega^1(R) = \bigoplus_{i=1}^{m} \Omega^1(R) \otimes V_m \). The isomorphism \( \alpha^0 \) is the
identity map, and \( \alpha^1 \) is the map \( \alpha \) defined above composed with the canonical identification induced by \( \Omega^i(R) \otimes T^i(V^1) = \Omega^i(R) \). To explain the isomorphism \( \alpha^m \) for higher \( m \), we need some more notation. Let us use the provisional notation \( Cyl^m(R)_n \) for the right hand side of (8); later on we shall identify both sides and eliminate the **s from our notation. For \( m \geq 1 \) and \( 1 \leq i \leq m + 1 \), let \( \partial'_i : Cyl^m(R) \to Cyl^{m+1}(R), \partial'_i(\omega \otimes x) = \omega \otimes \partial'^i(x), (\omega \otimes x \in \Omega^i(R) \otimes T^i(V^m)). \) Also let \( \partial'_i : R \to Cyl^i(R) \partial'_i = \alpha^1 \partial^i, i = 0,1. \) For \( m \geq 2 \), \( \alpha^m \) is the map determined by \( \alpha^n \partial^m \ldots \partial^{i+1} \partial^{i-1} \ldots \partial^0 = \partial'_m \ldots \partial'_{i+1} \partial'_{i-1} \ldots \partial'_0 \) \( (i = 0, \ldots, m). \) The same argument as in the proof of [2, Theorem 2.1] shows that \( \alpha^m \) is an isomorphism for all \( m \geq 0. \) Thus we may –and do– identify \( Cyl^m(R) \) with \( Cyl^m(R) \) and the coface and codegeneracy maps with their images through \( \alpha^* \). For \( m \geq 1 \) and \( 1 \leq i \leq m + 1 \) and for \( m = 0 \) and \( i = 0,1 \), the \( i \)-th coface map is identified with the map \( \partial'_i \) defined above. For \( m \geq 1 \) the coface map \( \partial_0 : Cyl^m(R) \to Cyl^{m+1}(R) \) can be written as follows:

\[
\partial^0 = \sum_{n=0}^{\infty} P_n := \{ \sum_{n=0}^r P_n : Cyl^m(R)_r \to Cyl^{m+1}(R)_r \}
\]

Here \( P_n(\Omega^i(R) \otimes T^i(V^*)) \subset \Omega^{r+n}(R) \otimes T^{r+n}(V^{*+1}). \) The 0-th component is simply the map \( P_0(\omega \otimes x) = \omega \otimes \partial^0(x); \) for \( \omega = a_0 da_1 \ldots da_p \in \Omega^p(R) \) and \( x_1 \ldots x_p \in T^p(V^m) \), we have:

\[
(9) \quad P_1(\omega \otimes x) = dw \otimes v_1 \partial^1(x) + \sum_{i=1}^p a_0 da_1 \ldots da_{i-1} \phi(a_i) da_{i+1} \ldots da_p \otimes \\
\partial^0(x_1) \ldots \partial^0(x_{i-1})(\partial^1(x_i)v_1 + v_1 \partial^1(x_i) - 2v_i^2) \partial^0(x_{i+1}) \ldots \partial^0(x_p)
\]

The formulae above are derived as follows. By definition of the canonical isomorphism \( \alpha^* \) we have \( a + da \otimes v_i + \phi(a) \otimes v_i^2 + \ldots = \partial^0 \ldots \partial^{i+1} \partial^{i-1} \ldots \partial^0(a) \) \( (i \geq 1) \); putting this together with the formula for \( \alpha \partial_0 \) and the cosimplicial identity \( \partial^0 \partial^m \ldots \partial^{i+1} \partial^{i-1} \ldots \partial^0 = \partial^{m+1} \ldots \partial^{i+2} \partial^i \ldots \partial^0 \) we obtain:

\[
(10) \quad \partial^0(a + da \otimes v_i + \phi(a) \otimes v_i^2 + \ldots) = a + da \otimes v_{i+1} + \phi(a) \otimes v_{i+1}^2 \ldots \quad (i \geq 1) \\
\partial_0(a) = a + da \otimes v_1 + \phi(a) \otimes v_1^2 + \ldots
\]

The formula for \( P_0 \) is immediate from these identities; the formula for \( P_1 \) is immediate from that for \( P_0 \) and (10). It is possible to iterate this process to obtain a general formula for \( P_m \) in terms of the homogeneous parts of degrees \( \leq m \) of \( \partial_0 : R \to Cyl^1(R) \), but we shall have no occasion for this. The description of \( Cyl^*(R) \) is completed as one checks that the codegeneracy maps are induced by those of the cosimplicial vectorspace \( V \); thus \( \mu^i(\omega \otimes x) = \omega \otimes \mu^i(x) \) \( (\omega \otimes x \in \Omega^i(R) \otimes T^i(V^*)) \).

**Adic filtration 1.2.** Let \( R \) be a quasi-free algebra, \( I \subset R \) an ideal, \( A = R/I \), and \( \pi : R \to A \) the projection map. We also write \( \pi \) for the map \( Q^*R \to A \) to the constant cosimplicial algebra defined as either of the equal composites \( Q^*R \to R \xrightarrow{\pi} A, Q^*(\pi), Q^*A, \mu^i, A. \) But \( \phi(R/I) = \ker \pi, Cyl^*(R/I) = Q^*(R/\phi(R/I)) \infty \)
and \( \text{Cyl}^\ast (R, I)_n := Q^\ast (R)/q^\ast (R, I)^n \). We call this the relative cosimplicial cylinder. It is easy to see that the isomorphism \( \alpha \) defined in 1.1 induces \( \text{Cyl}^m (R, I)_n \cong \Omega_m (R)/<I + \Omega^+_m (R)>^n \). For \( m = 1 \) we get \( \text{Cyl}^1 (R, I)_n = \Omega (R)/\mathcal{F}_n \) where

\[
\mathcal{F}_n = \mathcal{F}_1^n := <I + \Omega^+(R)>^n = \bigoplus_{i=0}^{\infty} F^{n-i} \Omega^i(R)
\]

Here \( F^p \Omega^q (R) = F^p_1 \Omega^q (R) := \Omega^q (R) \) if \( p \leq 0 \) and is \( \sum I^{i_0} d \Lambda^i_1 \ldots I^{i_{q-1}} d \Lambda^i_q \) for \( p \geq 1 \), with the sum taken over all multi-indices \( i = (i_0, \ldots, i_q) \) such that \( |i| := \sum i_j \geq p \). For all \( m \geq 0, n \geq 1 \), we have:

\[
\text{Cyl}^m (R, I)_n = \bigoplus_{i=0}^{n-1} \frac{\Omega^i(R)}{F^{n-i} \Omega^i(R)} \otimes T^i(V^m)
\]

For future use we also record here the behavior of the filtration \( \mathcal{F} \) with respect to the Karoubi operator and the Connes, de Rham and Hochschild boundary maps:

\[
k_{\mathcal{F}^n} \subset \mathcal{F}^n \quad B \mathcal{F}^n \subset \mathcal{F}^n \quad d \mathcal{F}^n \subset \mathcal{F}^n \quad b \mathcal{F}^n \subset \mathcal{F}^{n-1}
\]

**Noncommutative infinitesimal cohomology 1.3.** Let \( \mathcal{G} \) be a covariant functor going from the category of \( k \)-algebras to the category of abelian groups, \( R \) a quasi-free algebra, \( I \subset R \) an ideal, and \( A = R/I \). Consider the cosimplicial abelian group \( \mathcal{G} \text{Cyl}^\ast (R, I)^{\circ} : [n] \mapsto \lim_m \mathcal{G} \text{Cyl}^\ast (R, I)_m \). We write \( H^\ast_{inf}(A, \mathcal{G}) \) for the cohomology of \( \mathcal{G} \text{Cyl}^\ast (R, I)^{\circ} \); we call this the infinitesimal cohomology of \( A \) with values in \( \mathcal{G} \). Thus \( H^\ast_{inf}(A, \mathcal{G}) \) can be computed as the cohomology of the cochain complex:

\[
\text{C}(R, I, \mathcal{G}) : \text{lim}_n \mathcal{G}(R/I^n) \xrightarrow{\partial} \text{lim}_n \mathcal{G}(\text{Cyl}^1(R, I)_n) \xrightarrow{\partial} \text{lim}_n \mathcal{G}(\text{Cyl}^2(R, I)_n) \xrightarrow{\partial} \ldots
\]

where \( \partial = \sum_{i=0}^{m+1} (-1)^i \partial^i : C^m = C^m (R, I, \mathcal{G}) \rightarrow C^{m+1} \) or equivalently – by the Dold-Kan correspondence – as the cohomology of the normalized complex \( \tilde{C}^m = C^m / \sum_{i=0}^{m+1} \partial^i (C^{m+1}) \) with as coboundary map the map induced – and still denoted by – \( \partial^0 \). It is implicit in our notation that the cohomology of the complex (13) depends only on \( A \) rather than on its presentation as a quotient of a quasi-free algebra. In fact the groups \( H^\ast (A, \mathcal{G}) \) have the following interpretation in terms of sheaf cohomology. Consider the category \( \text{inf} \text{Alg} / A \) having as objects the surjective algebra maps \( p : B \twoheadrightarrow A \) with nilpotent kernel and as morphisms \( (p : B \twoheadrightarrow A) \rightarrow (p' : B' \twoheadrightarrow A) \) the homomorphisms \( B \rightarrow B' \) making the obvious diagram commute. Regard \( \text{inf} \text{Alg} / A^{op} \) as a site with the indiscrete Grothendieck topology; i.e. the coverings are the families \( \{ B \cong B' \} \) consisting of a single isomorphism; this site is the noncommutative infinitesimal site over \( A \). Any covariant functor \( \text{inf} \text{Alg} / A \rightarrow Ab \) to abelian groups is a sheaf on \( \text{inf} \text{Alg} / A^{op} \). It was shown in [3, §5], that the groups \( H^\ast_{inf}(A, \mathcal{G}) \) as defined above are the sheaf cohomology groups of the sheaf obtained by restriction of \( \mathcal{G} \) to \( \text{inf} \text{Alg} / A \). In particular they are independent of the choice of the presentation \( A = R/I \). We also consider the hypercohomology of sheaves of (cochain) complexes and of simplicial sets. If \( \mathcal{G} : \ldots \xrightarrow{\delta} \mathcal{G}^n \xrightarrow{\delta} \mathcal{G}^{n+1} \xrightarrow{\delta} \ldots \) is a cochain complex of sheaves, we write \( H^\ast_{inf}(A, \mathcal{G}) \) for the cohomology of the total complex:

\[
\text{Tot}^n = \prod \text{C}^p (R, I, \mathcal{G}^q)
\]
of the bicomplex \((p,q) \mapsto C^p(R,I,G^n)\). For a sheaf \(X\) of fibrant simplicial sets, we put \(\mathbb{H}^\inf_{n}(A, X) = \pi_{-r} \operatorname{holim}_\Delta \times \mathbb{N}^op((m,n) \mapsto XCyl^m(R,I)_n)\); the definition can be extended to Kan spectra as in [14].

Other variants of infinitesimal cohomology 1.4. For a unital algebra \(A\), we can consider, in addition to the infinitesimal site defined above, the indiscrete site on \(inf(\operatorname{Alg}_1/A)^op\), where \(inf(\operatorname{Alg}_1/A) \subset inf(\operatorname{Alg}/A)\) is the subcategory of unit preserving maps of unital algebras. However if \(G : inf(\operatorname{Alg}/A) \to Ab\) is a functor, then the sheaf cohomology groups \(H^*(inf(\operatorname{Alg}_1/A), G)\) agree with those defined above. Indeed, both \(H^*(inf(\operatorname{Alg}_1/A), G)\) and \(H^*(inf(\operatorname{Alg}/A), G)\) can be computed as the cohomology of \(C(R,I,G)\) where \(A = R/I\) is any presentation of \(A\) as a quotient of a unital quasi-free algebra. Similarly if \(A\) is unital and commutative, the cohomology of a functor \(G : inf(\operatorname{Comm}/A) \to Ab\) is the same whether considered as a sheaf on \(inf(\operatorname{Comm}/A)^op\) or \(inf(\operatorname{Comm}_1/A)^op\); they are both computed as the cohomology of the cosimplicial abelian group \([n] \mapsto \lim_m \ker(S^\otimes n+1 \to A)^m\) where \(S \to A\) is any surjective map from a smooth unital algebra \(S\), and \(S^\otimes n \to A\) is the induced map. On the other hand, it follows from the discussion on page 337 of [8] that, for \(A\) commutative and unital, the cohomology of the additive group functor \(inf(\operatorname{Comm}_1/A) \ni (B \to A) \mapsto B \in Ab\) agrees with the cohomology of the structure sheaf on Grothendieck’s infinitesimal site \(\operatorname{Spec}(A)^{\inf}\). Thus all three commutative infinitesimal cohomologies of a commutative unital algebra agree. However these do not agree with the noncommutative infinitesimal cohomology of the same algebra. In fact, we shall show further below that for any algebra \(A\), the cohomology of the sheaf \(O/[O,O] : inf(\operatorname{Alg}/A) \to Ab\), \((B \to A) \mapsto B/[B,B]\) is periodic cyclic homology \(HP_* (A)\). Since on the other hand, for a finite type algebra over a field we have \(HP_* (A) = \prod_{n=0}^\infty H^{2n+*}(\operatorname{Spec}A^{\inf}, O)\) (cf. [9]), we can see that the commutative and noncommutative infinitesimal cohomologies do not agree in general. Herefrom, we shall always use \(H_{\inf}\) to refer to the cohomology on the noncommutative site.

2. Flat connections.

Infinitesimal modules, crystals and stratifications 2.0. Let \(A\) be an algebra. By a right, infinitesimal module over \(A\) or \(\mathcal{O}_{inf}(\operatorname{Alg}/A)^{op}\)-module we mean a functor \(M : inf(\operatorname{Alg}/A) \to Ab\) such that \(M_p \in \operatorname{Mod} \mathcal{B}\) for each \(inf(\operatorname{Alg}/A) \ni p : B \to A\). We say that the infinitesimal module \(M\) is a crystal on the infinitesimal site if for each map \(\alpha : (p : B \to A) \to (p' : B' \to A) \in \operatorname{inf}(\operatorname{Alg}/A)\) the \(B'\)-homomorphism \(\rho_\alpha : \alpha(M_p) := M_p \otimes_{\tilde{\alpha}} B' \to M_{p'}\) induced by \(M(\alpha)\) is an isomorphism. A stratification on a right \(A\)-module \(E\) is a sequence of isomorphisms \(\theta_n : \partial_0^n(E) \to \partial_1^n(E) \in \operatorname{Mod} \mathcal{Cyl}^1(A)_n\) \((n \geq 0)\); here \(\partial_0^n\) is the \(i\)-th coface in the cosimplicial algebra \(\mathcal{Cyl}^1(A)_n\). The \(\theta\)'s are subject to the following prescriptions: we must have \(\theta_1 = 1\), the identity of \(E\); the sequence \(\{\theta_n\}\) must be compatible with the projection maps \(\mathcal{Cyl}^1(A)_{n+1} \to \mathcal{Cyl}^1(A)_n\), and finally, each \(\theta_n\) must satisfy the cocycle condition:

\begin{equation}
\partial_0^2(\theta_n) \partial_0^0(\theta_n) = \partial_1^1(\theta_n)
\end{equation}

For example if \(M\) is a crystal then the sequence \(\theta_n(M) = \rho_{\partial_1^n}^{-1} \rho_{\partial_0^n}\) is a stratification on \(E(M) = M_{\mathcal{O}_{inf}}\). For a quasi-free algebra \(R\), the assignment \(M \mapsto (\theta_n(M))\) becomes a functor.


(E(M), θ(M)), is an equivalence of categories between the category of crystals of \( \hat{R} \)-modules and natural transformations and the category of pairs \((E, θ)\) of an \( \hat{R} \)-module together with a stratification \( θ \), where a map \((E, θ) \rightarrow (E', θ')\) is a module homomorphism \( f : E \rightarrow E' \) such that \( θ_n \partial_n^0(f) = \partial_n^1(f) θ_n \) for all \( n ≥ 1 \). An inverse for this functor was constructed in [3,7.2]; here we write \( E_{crys} \) for this inverse equivalence. If \( M \) is a crystal, then the pro-cosimplicial group \( MC_{yl}^*\)(R) of 1.3 above can be described in terms of \( E \) and \( θ \) as follows. We have \( MC_{yl}^0\)(R) = \( E \) and \( MC_{yl}^m\)(R) = \( \partial^m \cdots \partial^1(E) \) for \( m ≥ 1 \). Codegeneracies are the maps \( id_E \otimes α^i : \partial^m \cdots \partial^1(E) \rightarrow \partial^{m-1} \cdots \partial^1(E) \) (0 ≤ \( i \) ≤ \( m \)). Similarly, for 1 ≤ \( j \) ≤ \( m+1 \), \( id_E \otimes ∂^j \) is the \( j \)-th coface map \( ∂^m \cdots ∂^1(E) \rightarrow ∂^{m+1} \cdots ∂^1(E) \). The 0-th coface map \( E \rightarrow ∂^1(E) \) is the composite \( ∂_0^0 : E \rightarrow ∂_0^0(E) \rightarrow ∂^1(E) \). For \( m ≥ 1 \) and \( e \otimes ω \in ∂^m_n \cdots ∂^1_n(E) \) we have \( ∂_0^0(e \otimes ω) = ∂_0^0(e) \otimes ∂^0(ω) \). Upon completion we obtain a complex \( C(E, θ) \) whose cohomology is the sheaf cohomology of the crystal \( M \) with which we started.

We have shown how one can go from crystals on a quasi-free algebra to stratifications and interpret the sheaf cohomology of the former in terms of the latter. Below we shall show how one can go from stratifications to flat connections. First we recall the definition of curvature given in [11]. Given a connection \( ∇ : E \rightarrow E \otimes_\hat{A} Ω^1(A) \) on a \( \hat{A} \)-module \( E \), extend it to a map \( ∇ : E \otimes_\hat{A} Ω^*(A) \rightarrow E \otimes_\hat{A} Ω^{*+1}(A) \) by the rule \( ∇(e \otimes ω) = ∇(e) ω + edω \). The curvature of \( ∇ \) is the map \( ∇^2 : E \otimes_\hat{A} Ω^*(A) \rightarrow E \otimes_\hat{A} Ω^{*+2}(A) \). We say that \( ∇ \) is flat if \( ∇^2 = 0 \). One sees immediately that flatness is equivalent to the vanishing of the degree 2 component \( ∇^2 : E \rightarrow E \otimes_\hat{A} Ω^2(A) \).

The pairs \((E, ∇)\) of a right \( \hat{A} \) module and a flat connection form a category, where a map \((E, ∇) \rightarrow (F, D)\) is a homomorphism of modules \( α : E \rightarrow F \) with \( Dα = α \otimes id_{Ω^1(A)} ∇ \).

**Proposition 2.1.** Let \( R \) be a quasi-free algebra. With the definitions of 2.0 above, let \((E, θ)\) be a stratified \( \hat{R} \)-module and \( ∂_0^0 : E \rightarrow ∂^1(E) \) the coface map. Then the composite \( ∇ : E \xrightarrow{∂_0^0} ∂^1(E) = E \otimes Ω^1(\hat{A}) \rightarrow E \otimes Ω^1(\hat{A}) = A \) is a flat connection. Moreover the assignment \((E, θ) \mapsto (E, ∇), f \mapsto f \) is an isomorphism between the categories of stratified modules and of flat connections.

**Proof.** That \( ∇ \) is a connection is immediate, as is that a homomorphism of stratifications is a morphism of modules with a connection. The proof that \( ∇ \) is flat will be easy once we have established the identity (15) below; to prove the latter, proceed as follows. Consider the map \( ∂_0^0 : E \rightarrow E \otimes_{∂_1} Cy^l(R) \cong E \otimes R Ω^+(R)^\infty \); write \( ∂_0^0 = \sum_{n=0}^∞ D_n \) where \( D_n(E) \subset E \otimes R Ω^n(R) \). Thus \( D_0 = 1 \) and \( D_1 = ∇ \). Similarly, the map \( ∂_0^0 : E \rightarrow ∂^2 Ω^1(E) \), decomposes as \( ∂_0^0 = ∑_{n=0}^∞ ∑_{i=0}^n D_i \otimes P_{n-i} \), where the \( P \)'s are the components of the map \( ∂_0^0 : Cy^l(R) \rightarrow Cy^l(R) \). The identity \( ∂_0^0 ∂_0^0 = ∂^2 Ω^1 \) is equivalent to the sequence of identities:

\[
D_n(e) \otimes v^0_2 - D_n(e) \otimes v^0_1 - D_n(e) \otimes (v_2 - v_1)^n = ∑_{i+j+k=n} (D_i \otimes P_j)(D_k(e) \otimes v^i_j)
\]

where \( e \in E, n ≥ 1 \), and the sum is taken over all \( i ≥ 0 \) and \( j, k ≥ 1 \) such that \( i + j + k = n \). The case \( n = 1 \) of the identity (15) is tautological. That \( ∇ \) is flat follows straightforwardly from the case \( n = 2 \) of the identity above, using the formula (9). Thus \((E, θ) \rightarrow (E, ∇)\) is a functor from stratified modules to modules with a flat connection. Next we construct a functor going in the opposite direction.
direction. Consider the map \( t : \Omega^1(R) \to Cy^1(R), \) \( adx \mapsto a(\partial^0(x) - x) \). Define a map \( \partial^0_e : E \to \partial^1(E), \partial^0_e(e) = e + (id \otimes t)\nabla(e) \), and put \( \theta : \partial^0(E) \to \partial^1(E), \) \( \theta(e \otimes \omega) = \partial^0_e(e)\partial^0(\omega) \). We must see now that \( \theta \) is a stratification. One checks that \( \partial^0_e(ea) = \partial^0_e(e)\partial^0(a) \) \( (e \in E, a \in R) \), from which it follows that \( \theta \) is a well-defined homomorphism. Of the conditions for a stratification only the cocycle condition is not immediate. To prove the latter, one does as follows. One checks that \((14)\) is equivalent to the cosimplicial identity \( \partial^0_1 : \partial^0_0 = \partial^1_0 \). In turn, since \( \partial^0_0 = \partial^0_e : E \to \partial^1(E) \), the latter condition is equivalent to the sequence of identities \((15)\) with as \( D_n \) the \( n \)-th component of \( \partial^0_e \). A straightforward induction shows that \((15)\) holds. On the other hand it is clear from the definition of \( D_n \) that if \( f : (E, \nabla) \to (E', \nabla') \) is a homomorphism of modules with a connection, and \( \theta \) and \( \theta' \) are the stratifications constructed from them, then \( \partial^1(f)\partial^0_0 = \partial^0_0 f \). It follows that \( f \) is a map of stratified modules \( (E, \theta) \to (E', \theta') \). Thus we have a functor from flat connections to stratifications. It is clear than if we start off with a flat connection, then take the stratification constructed from it and then go back by the functor of the proposition, we end up with the same connection we started with. That the other composition is the identity is a consequence of the recursive nature of the formula \((15)\); indeed, the latter formula says that all of the \( D_n \) – and therefore the whole stratification – are determined by \( D_1 = \nabla \). \( \square \)

**Corollary 2.2.** For a quasi-free algebra \( R \), the categories of \( R \) modules with a flat connection and of infinitesimal crystals are equivalent.

**Proof.** It follows from the proposition and the equivalence between the categories of stratified modules and crystals discussed in 2.0 above. \( \square \)

**Theorem 2.3.** Let \( R \) be a quasi-free algebra, \( E \) a right \( R \)-module, \( \nabla : E \to E \otimes_R \Omega^1(R) \) a flat connection in the sense of 2.0, and \( E_{crys} \) the infinitesimal crystal corresponding to \( (E, \nabla) \) under the category equivalence of corollary 2.2. Then there is an isomorphism:

\[
H^*_\inf(R, E_{crys}) \cong H^*(E \otimes_R \Omega(R), \nabla)
\]

The proof of the theorem above uses the following variant of the perturbation lemma of [12]:

**Lemma 2.4.** Let \( C = (\{C^n\}_{n \geq 0}, \delta) \) be a nonnegatively graded cochain complex. Assume there is a decomposition \( C^n = \bigoplus_{p=0}^\infty C^n_p \) \( (n \geq 0) \) and that \( \delta \) preserves the associated filtration, i.e. \( \delta(\bigoplus_{p \geq q}^\infty C^*_p) \subset \bigoplus_{p \geq q}^\infty C^*_p \) for all \( q \geq 0 \). Write \( \delta = \sum_{n=0}^\infty \delta_n \) where \( \delta_n(C^*_p) \subset (C^{*+1}_p)_n \). Suppose \( h_0 : C^*_p \to C^{*_+1}_p \) is a contracting homotopy for \( \delta_0 \) in the sense that \( 1 = h_0 + \delta_0 h_0 \). Define recursively:

\[
h_n = - \sum_{i=0}^{n-1} h_0 \delta_{n-i} h_i
\]

Then \( h = \sum_{n=0}^\infty h_n \) is a contracting homotopy for \( \delta \).

**Proof.** Straightforward induction. \( \square \)
Proof of Theorem 2.3. Consider the complex $C = C(R, 0, E_{crys})$ of (13); this is the cochain complex associated to the completion of the pro-cosimplicial abelian group $C(E, \theta)$ of 2.0. We shall define a map going from the normalized complex $\tilde{C}$ to the complex $(E \otimes_{\tilde{R}} \Omega(R), \nabla)$ and show it is a quasi-isomorphism. First we need a better description of the complex $\tilde{C}$. We have a decomposition $C^m = \prod_{n=0}^\infty C^m_n$, where $\tilde{C}^m_n := E \otimes_{\tilde{R}} \Omega^n(R) \otimes T^n(V^m)$; cofaces and codegeneracies are as described in 2.0 above. Thus all but the 0-th coface act only on the $T(V)$ part, i.e. are of the form $id \otimes \partial_i$ where $\partial_i$ is the coface of the completed tensor product cosimplicial vectorspace $T(V) : m \mapsto \prod_{n=0}^\infty T^n(V^m)$. The 0-th coface map decomposes as an infinite sum of homogeneous components of nonnegative degrees of which the 0-th degree component is of the form $\Delta$ where $\Delta$ is the coface of $T(V)$. Hence the decomposition above is preserved by normalizing and we have $\tilde{C}^m = \prod_{n=0}^\infty \tilde{C}^m_n$ where $\tilde{C}^m_n = E \otimes_{\tilde{R}} \Omega^n(R) \otimes \bar{T}^n(V^m)$. Here $\bar{T}^n(V^m)$ is the normalization of the cosimplicial tensor product of the cosimplicial vectorspace of 1.1. By definition, $\bar{T}^n(V^m)$ is the quotient of $T^n(V)$ by the subspace generated by the sum of the images of all cofaces but the 0-th coface; one checks this subspace is generated by all those words of length $n$ on the basis elements $v_1, \ldots, v_m$ of $V^m$ in which at least one of the $v_i$ is missing. Hence each $\bar{T}^n(V^m)$ is isomorphic to the free vectorspace on the set of all surjective maps going from a set of $n$ elements to a set of $m$ elements. In particular $\bar{T}^n(V^m) = 0$ for $m > n$, and $\bar{T}^n(V^m) \cong k[S_n]$, the group algebra of the symmetric group. The coboundary in $\tilde{C}$ is the map $\delta$ induced by $\partial^0$; it admits a decomposition as that of the lemma above, where $\delta_0$ is of the form $id \otimes \partial'_0$, with $\partial'_0$ the map induced by the 0-th coface of $T(V)$ on $\bar{T}(V)$. Consider the map $p : k[S_n] \to k$, $\sigma \mapsto sg(\sigma)$ where $sg$ is the permutation sign. One checks that $\ker p = \partial'_0(\bar{T}^n(V^{n-1}))$; thus if we regard $p$ as a map of cochain complexes $\bar{T}^n \to k[-n]$, we see it induces an isomorphism at the $H^n$ level. I claim it is in fact a quasi-isomorphism, i.e. we have $H^q(\bar{T}^n(V)) = 0$ for $q < n$. The claim follows from the fact that $\bar{V} = k[-1]$, the Künneth formula for the cohomology of cochain complexes and the fact that $\bar{V}$ is contractible. The claim follows from the fact that $\bar{V} = k[-1]$, the Künneth formula for the cohomology of cochain complexes and the fact that $\bar{V}$ is contractible. The claim follows from the fact that $\bar{V} = k[-1]$, the Künneth formula for the cohomology of cochain complexes and the fact that $\bar{V}$ is contractible.

3. Periodic homology v. infinitesimal cohomology. The purpose of this section is to prove the following:

Theorem 3.0. Let $k$ be a field of characteristic zero, $A$ a $k$-algebra. Then there is an isomorphism:

$$HP^*(A) \cong H^s_{inf}(A, O/[O, O])$$

Proof. Write $A = R/I$ with $R$ quasi-free. The proof has two parts. First we prove...
that $H_{s_m}^*(A, O/[O, O])$ equals the cohomology of the complex:

\[
(\lim_n \left( \frac{\Omega^*}{F^n + b\Omega^{*+1}} \right)_\kappa, d)
\]

where $\Omega^* = \Omega^*(R), F$ is the adic filtration of the ideal $I$, as defined in 1.2 above, and $\kappa$ is the Karoubi operator. The map $\kappa$ verifies $\kappa^m \equiv 1$ on $\Omega^m \mod (F^n + b\Omega^{m+1})$; the subscript in (16) indicates coinvariants with respect to this action of $\mathbb{Z}/m$. The second part consists of proving that the latter complex computes $HP_*(A)$. Now to the first part. Consider the inverse systems of normalized cochain complexes $\{\tilde{C}_n : n \geq 0\}$ and $\{\tilde{D}_n : n \geq 0\}$ associated with the pro-cosimplicial groups $\{Cyl(R, I)_{n}/[R, Cyl(R, I)_{n}] : n \geq 0\}$ and $\{Cyl(R, I)_{n}/[Cyl(R, I)_{n}, Cyl(R, I)_{n}] : n \geq 0\}$. Thus, by (11), $\bar{C}_n = \bigoplus_{i=m}^n \bar{C}_{n,i}$, where, for $\bar{T}$ as in the proof of Theorem 2.3, we write $\delta_n = \sum_{i=0}^n \delta_{n,i}$ for the coboundary of $\bar{C}_n$. A similar argument as that in the proof of assertion c) on page 30 of [11] shows that $\bar{D}_n = \bigoplus_{i=m}^n \bar{D}_{n,i}$ where $\bar{D}_{n,i} = \bar{C}_{n,i}/Z/i$ is the coinvariants with respect to the action given on a generator $\zeta$ of $\mathbb{Z}/i$ by $\zeta(\omega \otimes x) = \kappa^i(w) \otimes (x)$. Here $\kappa^i(a_0da_1 \ldots da_i) = da_i d_0 da_1 \ldots da_{i-1}$ is Karoubi’s operator without sign, and $\zeta(x_1 \ldots x_i) = x_i x_1 \ldots x_{i-1}$. Proceeding as in the proof of theorem 2.3 we see there is an inverse system of cochain maps $\pi_n: \bar{C}_n \rightarrow \Omega^* / F^n$ which are 0 in codimension $> n$ and are given in codimension $\leq n$ by the composite of the projection $\bar{C}_n \rightarrow \bar{C}_{n,m} = \frac{\Omega^m}{F^n + b\Omega^{m+1}} \otimes k[S_m]$ and the map $(1 \otimes p)$, where $p$ is as in the proof of theorem 2.3. Thus the restriction $p_i$ of $p$ to each of the complexes $T^i(V)$ is a quasi-isomorphism of $k[\mathbb{Z}/i]$-modules. Since $\text{char}(k) = 0$, we therefore have that $\ker p_i$ is contractible as a complex of $k[\mathbb{Z}/i]$-modules, i.e. there is a contracting homotopy which commutes with the action of $\mathbb{Z}/i$. This contracting homotopy is preserved upon tensoring with $\frac{\Omega^m}{F^n + b\Omega^{m+1}}$ and gives a map of inverse systems $\ker(\pi_n)^* \rightarrow \ker(\pi_n)^*-1$ which commutes with the action of the cyclic groups and is a contracting homotopy for $\delta_{n,0}$. Now apply Lemma 2.4 to obtain an inverse system of action preserving contracting homotopies for $\ker \pi_n$. Taking coinvariants and then inverse limit, we obtain a contracting homotopy for $\ker \lim_n s_n'$, where $s_n'$ is the map induced by $\pi_n$ at the coinvariant level. To finish the first part of the proof it suffices to show that $\pi' = \lim_n s_n'$ is surjective. To see this, consider the map $s_n : \Omega^m / F^n \rightarrow \bar{C}_n, \omega \mapsto \omega \otimes (1/m! \sum_{\sigma \in S_m} \sigma)$. We see that $s_n$ is a right inverse for $\pi_n$, that it commutes with the action of the cyclic groups –although not with coboundaries– and that the family $\{s_n\}$ is a map of inverse systems. Hence if $s_n'$ is the map induced by $s_n$ at the coinvariant level and $s' = \lim_n s_n'$, then $\pi' s' = 1$ and $\pi'$ is surjective.

Next we turn to the second part of the proof. First of all, note that, because the map $1 - \kappa$ is invertible on the second sumand of the harmonic decomposition $\Omega = P\Omega \oplus P^+ \Omega$ of [6], we have an isomorphism of cochain complexes $(P\Omega / F^n, d) \cong ((\frac{\Omega^m}{F^n - m\Omega^{m+1}}), d)$ for each $n \geq 0$. Here $P\Omega^m / F^n = P(\frac{\Omega^m}{F^{n+1} - m\Omega^{n+1}})$; we abuse notation and write $d$ for the map it induces at the $\kappa$ level. We want to compare the latter cochain complex with the supercomplex $X$ of [5], which is known to compute the periodic cyclic groups. Since these complexes live in different categories we introduce $\tau X$, a nonnegatively graded cochain complex version of the supercomplex $X$. We put $\tau X^0 = \Omega^0$, and for $n \geq 1, \tau X^{2n} = \Omega^0 = X^{even}, \tau X^{2n-1} = \Omega^1 = X^{odd}$. Note that $\tau X$ is a periodic cyclic group $X$.
\(X^{\text{odd}}\). Abusing notation and writing \(F^{\infty}\) for the image of the filtration (12), we have \(H^*(\tau X/F^{\infty}) = H^*(X/F^{\infty})\). It follows from [2, 2.3] that the filtration \(F\) is equivalent to the filtration induced by \(K^n = \ker(\Omega \to \Omega(R/I^n))\); in turn it was shown in [5] that for \(R = TA\) and \(I = JA\), the latter filtration is equivalent to the Cuntz-Quillen filtration. Thus \(H^*(X/F^{\infty}) = HP_*(A)\). We shall presently define inverse homotopy equivalences of cochain complexes \(P\Omega_q \rightleftarrows \tau X\). First we introduce the following two operators; let \(h : \Omega^* \to \Omega^{*-1}\), \(h(\omega da) = (-1)\omega^1\omega a\), and \(\nabla : \Omega^* \to \Omega^{*-1}\), \(\nabla(a_0da_1 \ldots a_n) = -a_0\phi(a_1)da_2 \ldots da_n\) where \(\phi\) is as in 1.1 above. One checks the following properties of \(h\) and \(\nabla\). We have:

\[
\nabla b + b\nabla)\omega = \omega, \quad \nabla(uv) = \nabla(u)v + (-1)^{|u|}udv, \quad \nabla F^n \subset F^n
\]

for \(\omega \in \Omega^{2*}\), \(u, v \in \Omega^{1*}\) and also:

\[
h^2 = 0 \quad hd + dh = 1 \quad hF^n \subset F^{n-1}
\]

Below we define maps \(\alpha : P\Omega_q \to \tau X\), \(\beta : \tau X \to P\Omega_q\), \(\gamma : \tau X \to \tau X[-1]\), and \(\epsilon : P\Omega_q \to P\Omega_q[-1]\). One checks, using (17), (18), (12) and induction, that all these maps are continuous in the adic topology, that \(\alpha\) and \(\beta\) are cochain maps and that \(\gamma\) and \(\epsilon\) are homotopies \(\alpha\beta \to 1\) and \(\beta\alpha \to 1\). For \(j = 0, 1, 1 \leq i \leq q\), and \(n = 2q + j\), we write \(c_{n,i} = n(n-2)(n-4) \ldots (2i+j)\), and \(c_n = c_{n,1-j}\). Let \(\alpha^0\), \(\alpha^1\), \(\beta^0\) and \(\beta^1\) be identity maps,

\[
\alpha^n = c_n(P\phi)^q, \quad \beta^{2q+1} = \frac{1}{(2q+1)!}((P(\nabla B + B\nabla)q^{-1}P, \quad \beta^{2q} = (2q+1)\beta^{2q+1}\nabla B
\]

Here and below the superscripts on expressions between parenthesis denote powers; thus for instance \((P\phi)^q = Phb \circ \cdots \circ Phb\) (\(q\) times). Put also \(\gamma_i = 0\) for \(i \leq 1\), \(\epsilon_l = 0\) for \(l \leq 2\), and:

\[
\gamma^{2n+1} = hP(b\nabla - 1) - \sum_{i=1}^{n} \alpha^{2i}(PhP\nabla b)\beta^{2i+1}, \quad \gamma^{2n} = -\sum_{i=1}^{n} \alpha^{2i-1}PhP\nabla b\beta^{2i}
\]

\[
\epsilon^{2q+1} = (2q+1)\sum_{i=1}^{q} \frac{1}{c_{2q,i}}(P(\nabla B + B\nabla))^{q-i}P\nabla(P\phi)^{q-i+1}
\]

\[
\epsilon^{2q} = \sum_{i=2}^{q} \frac{1}{c_{2q-1,i-1}}(P(\nabla B + B\nabla))^{q-i}P\nabla(P\phi)^{q-i+1}. \quad \square
\]

Remark 3.1. The cohomology isomorphism \(H^n(P\Omega_{F^{\infty}q}) \cong HP_n(A) \cong H^{n+2}(P\Omega_{F^{\infty}q})\) \((n \geq 0)\) is induced by the cochain map \(s' : PhbN : P\Omega_q \to P\Omega_q[-2]\), where \(N(\omega) = |\omega|\omega\). This follows from the fact that, for the homotopy equivalence of the proof above, we have \(\alpha^n s'^{n+2} = \alpha^{n+2}\).
4. A new presentation of the Chern character. The title of this section refers to the Connes-Karoubi map:

\[(19) \quad K_0(A) \xrightarrow{\text{cho}} HP_0(A)\]

We shall demonstrate here how this character fits into the infinitesimal cohomology framework. In 4.1 below we shall construct a character \(ch^{\text{inf}} : K_0(A) \to H^2_{\text{inf}}(A, K_1)\), where \(K_1\) is Bass’s. This character is defined independently of the characteristic of \(k\); we show in 4.4 that if \(\text{char}(k) = 0\), then our character is the same as (19), up to a factor of 2. The key to proving this agreement result is provided by the following:

**Proposition 4.0.** Let \(A\) be an algebra over a field \(k\) of characteristic zero; then there is a natural isomorphism:

\[\nu : H^n_{\text{inf}}(A, K_1) \cong HP_n(A) \quad (n \geq 2)\]

*Note:* The groups \(H^n_{\text{inf}}(A, K_1)\) for \(n = 0, 1\) are discussed in section 5.

**Proof.** Write \(R/I = A\) with \(R\) quasi-free. By virtue of (11) we have, for each \(n \geq 1\), an exact sequence of cosimplicial algebras:

\[0 \to \oplus_{i=1}^{n-1} \frac{\Omega^i(R)}{F^n \Omega^i(R)} \otimes T^i(V^*) \to \text{Cyl}^*(R, I)_n \xrightarrow{\pi_n} \frac{R}{I^n} \to 0\]

here \(\frac{R}{I}\) is the constant cosimplicial algebra. We note that we also have a natural inclusion map \(\frac{R}{I} \subset \text{Cyl}^*(R, I)_n\) which is an algebra homomorphism (although not a cosimplicial map), and also a right inverse for \(\pi_n\) and a map of inverse systems. Thus if \(\mathcal{G}\) is any functor going from algebras to abelian groups, the sequence

\[(20) \quad 0 \to \lim_n \ker \mathcal{G}(\pi_n^*) \to \lim_n \mathcal{G}(\text{Cyl}^*(R, I)_n) \to \lim_n \mathcal{G}(\frac{R}{I^n}) \to 0\]

is exact. The proposition follows from the isomorphism \(\ker K_1(\pi_n^*) \cong \ker HC_0(\pi_n^*)\) of [13], theorem 3.0 above, and the long cohomology sequences associated to (20) for \(\mathcal{G} = K_1\), and for \(\mathcal{G} = HC_0 = O/O\). \(\square\)

**The infinitesimal Chern character 4.1.** The construction has two parts; first we define, for any algebra \(A\), a character \(ch^{\text{strat}} : K_0(A) \to H^2_{\text{strat}}(A, K_1)\) taking values in the cohomology of the stratifying site of [3,5.6], i.e. in the cohomology of the complex \(\lim K_1(\text{Cyl}(A)_n)\). Note that for \(A\) quasi-free this suffices, as \(H^*_{\text{strat}}(A, -) = H^*_{\text{inf}}(A, -)\) in this case. The second part consists of lifting the construction above to the infinitesimal topology of an arbitrary algebra. Now, to the first part. Suppose a finitely generated projective right \(\tilde{A}\)-module \(E\) is given; then I claim that there exists, for each \(n \geq 1\), an isomorphism of \(\text{Cyl}(A)_n\)-modules \(\theta_n : \partial^i_n(E) \to \partial^i_n(E)\) with \(\theta_1 = 1\) which is compatible with the projection maps \(\partial^i_{n+1}(E) \to \partial^i_n(E)\) \((i = 0, 1)\). This is best seen in terms of idempotent matrices. Let \(p \in M_n(\tilde{A})\) be an idempotent matrix with \(p\tilde{A}^n = E\); we have to show that there exists a compatible family of invertible matrices \(\psi_n \in \text{Cl}(\text{Cyl}(A)_n)\) with
thing as a homomorphism $k \to M_n(A)$, it suffices to show the above for $A = k$ and $n = 1$. In this particular case, a formula for $u$ is given in Lemma 4.2 below. The claim is thus proved. Next we consider the map $\partial_n(\theta_n) := \partial^2_n(\theta_n)\partial^0_n(\theta_n)\partial^1_n(\theta_n)^{-1}$; we observe that this is an automorphism of the $\text{Cyl}^2(A)_n$-module $\partial_2\partial_1(E)_n$, and therefore has a class in $K_1(\text{Cyl}^2(A)_n)$. It is immediate from a well-known form of Whitehead’s lemma—which we have included below as lemma 4.3—that the class we have constructed is a 2-cocycle of the complex $K_1(\text{Cyl}^*_{n})(A)_n)$. Because the $\theta_n$ are compatible, so is the sequence $\partial(\theta) = \{\partial_n(\theta_n)\}$ whence it is a 2-cocycle of the completed cochain complex $\lim K_1(\text{Cyl}^*_{n})(A)_n)$. We write $ch^{\text{strat}}(E)$ for its cohomology class in $H^2_{\text{strat}}(A,K_1 \circ \sim) = H^2_{\text{strat}}(A,K_1(-) \oplus K_1(k)) = H^2_{\text{strat}}(A,K_1)$. It follows by lemma 4.3 that if $\theta'$ is another inverse system of isomorphisms as above, then the automorphisms $\partial_n(\theta_n)\partial_n(\theta'_n)^{-1}$ of $\partial^2_n\partial^0_n(E)$ and $\partial^2_n(\theta_n)\partial^0_n(\theta'_n)^{-1} \oplus \partial^2_n(\theta'_n)^{-1} \oplus \partial^0_n(\theta_n)\partial^1_n(E)$ have the same class in $K_1(\text{Cyl}^1_{n})(A)_n)$. Hence the definition of $ch^{\text{strat}}(E)$ is independent of the choice of $\theta$. Next, note that if $f : E \to F$ and $\alpha : \partial^0(E) \to \partial^1(E)$ are isomorphisms, then so is $\beta = \partial^1(f)\partial^0(f)^{-1}$; one checks, using lemma 4.3, that $\partial_n(\beta) = \partial_n(\alpha)$ in $K_1(\text{Cyl}^1_{n})(A)_n)$. Hence $ch^{\text{strat}}$ is well-defined on isomorphism classes of finitely generated projective $\tilde{A}$ modules. Furthermore it is a monoid homomorphism, because if $\alpha : \partial^0(E) \to \partial^1(E)$ and $\beta : \partial^0(F) \to \partial^1(F)$ are the isomorphisms used to define $ch^{\text{strat}}(E)$ and $ch^{\text{strat}}(F)$, then we can use $\alpha \oplus \beta$ for $E \oplus F$, and $\partial(\alpha \oplus \beta) = \partial(\alpha) \oplus \partial(\beta)$. Hence we have a well-defined group homomorphism $K_0(\tilde{A}) \xrightarrow{ch^{\text{strat}}_{\text{strat}}} H^2_{\text{strat}}(A,K_1)$, and by composition with $K_0(A) \to K_0(\tilde{A})$ or $\sim$-sense, as one sees immediately, $ch^{\text{strat}}(\tilde{A}) = 0$—by passage to the quotient mod $K_0(k)$, also a homomorphism $K_0(A) \xrightarrow{ch^{\text{strat}}_{\text{strat}}} H^2_{\text{strat}}(A,K_1)$. This homomorphism is natural, as follows from the fact that lemma 4.2 gives a formula for an isomorphism $\theta : \partial^0(E) \to \partial^1(E)$, whence also for $ch^{\text{strat}}(E)$.

Now we lift this construction to the infinitesimal site as follows. Write $A = R/I$ where $R$ is quasi-free. Given a projective right $\tilde{A}$-module lift it successively to an inverse system $\pi_n : E_n \to E_{n-1}$ where $E_1 = E$, and each $\pi_n$ is an $\tilde{R}/I^n$ homomorphism. That this can be done is a consequence of the fact that $k$ is quasi-free; explicit formulas for the succesive idempotents $e_n$ for each of the $E_n$ in terms of that of $E$ are given in [5] for $\text{char}(k) = 0$. Using as $\theta_n : \partial^0_n(E_n) \to \partial^1_n(E_n)$ the image through the homomorphism induced by $e_n$ of the truncation of the element $u$ of Lemma 4.2, we obtain an element $\{\partial_n(e_n(u(n)))\} \in \lim Gl(\partial_n^0,e_n^1)$. Taking the class of this element at each level, we obtain a 2-cocycle $\partial(e(u))$ of the complex $\lim K_1(\text{Cyl}^*_{n})(R/I)_n) \cong \lim K_1(\text{Cyl}^*_{n})(R/I)_n)$. The last isomorphism comes from the fact that, by [2, 23], the filtrations $(q^*(R)+<I>)^\infty < \ker(Q^*(R) \to Q^*(R/I^n))_n$ are equivalent. We write $ch^{\text{inf}}(E)$ for the cohomology class of $\partial(e(u))$. After what we have already seen, to show that $E \mapsto ch^{\text{inf}}(E)$ is well-defined on isomorphism classes of projective modules and gives a group homomorphism $ch^{\text{inf}} : K_0(A) \to H^2_{\text{inf}}(A,K_1)$, it suffices to show that any isomorphism of projective modules $\alpha_1 : E \to F$ can be lifted to an inverse system of isomorphisms $\alpha_n : E_n \to F_n$ of the chosen liftings of $E$ and $F$. But this is immediate from Lemma 4.2 and the fact that the algebra $h[t,t^{-1}]$ is quasi-free.
Lemma 4.2. Let $k$ be the ground field; write $e$ for its unit element. Then:

1) There is an isomorphism of inverse systems $\text{Cyl}^1(k) \rightarrow \frac{\Omega(k)}{\Omega^*(k)}$ such that the canonical coface maps $\partial^i : k \rightarrow \text{Cyl}^1(k)$ $i = 0, 1$ are $\partial^1(e) = e$ and

$$\partial^0(e) = e + de + \sum_{n=1}^{\infty} C_n(1 - 2e)de^{2n}$$

Here $C_1 = 1$ and $C_n = \sum_{i=1}^{n-1} C_i C_{n-i}$ is the $n + 1$-th Catalan number (cf. [10]).

2) The following element $u \in \frac{\Omega(k)}{\Omega^*(k)}$ verifies the identity $u\partial^0(e) = \partial^1(e)u$.

$$u = 1 + \sum_{n=1}^{\infty} C_n(2e - 1)de^{2n-1}$$

Proof. One checks that the 1-cocycle $\phi : k \rightarrow \Omega^2(k)$, $\phi(e) = (1 - 2e)de^2$ satisfies $-\delta(f) = d \cup d$. Part 1) of the lemma follows from this and the recursive formula (5') of [2]. Write $D_n(e)$ for the term of degree $n$ of (21); for $u = 1 + \sum_{n=1}^{\infty} \omega_n$, the condition $u\partial^0(e) = \partial^1(e)u$ is equivalent to the sequence of identities:

$$[e, \omega_n] = D_n(e) + \sum_{i=1}^{n-1} \omega_i D_{n-i}(e)$$

One checks that for $\omega_{2n-1} = C_n(2e - 1)de^{2n-1}$, $\omega_{2n} = 0$, the identity above is satisfied. \square

The following well-known version of Whitehead’s Lemma was used in 4.1 above.

Whitehead’s Lemma 4.3. Let $E$ and $F$ be right $\tilde{A}$-modules, and let $\phi : E \rightarrow F$ and $\psi : F \rightarrow E$ be module isomorphisms. Then the following automorphisms:

$$\left( \begin{array}{cc} \psi \phi & 0 \\ 0 & 1_{F \oplus E \oplus F} \end{array} \right) \sim \left( \begin{array}{cc} 1_{E \oplus F \oplus E} & 0 \\ 0 & \phi \psi \end{array} \right)$$

are in the same commutator class of the group $\text{Aut}(E \oplus F \oplus E \oplus F)$.

Theorem 4.4. Let $k$ be a field of characteristic zero, $A$ an algebra over $k$ and $\nu$ and $\text{ch}^{\inf}$ be the homomorphisms of 4.0 and 4.1 above. Then:

$$\nu \circ \text{ch}^{\inf} = 2\text{ch}_0$$

Proof. We keep the notations of 4.1. We shall show that, for $E = e_1 \tilde{A}^*$, the sequence of isomorphisms:

$$H^2_{\text{inf}}(A, K_1) \cong H^2_{\text{inf}}(A, HC_0) \quad \text{by 4.0}$$

$$\cong H^2(\lim_n \frac{P\Omega^*}{P(F^n + b\Omega^{**+1})}) \quad \text{by the proof of 3.0}$$

$$\cong H^0(\lim_n \frac{P\Omega^*}{P(F^n + b\Omega^{**+1})}) \quad \text{via 3.1}$$

$$\cong H^0(\hat{X}(R, I)) \quad \text{by the proof of 3.0}$$

$$\cong H_0(R, A) \quad \text{by [5]}$$
maps $ch^{inf}(E)$ to twice the class of $\{e_n\}$ in $H^0(\hat{X}(R, I))$; by [5, 12.4], the latter class coincides with $ch_0(E)$. First we need a more explicit description of $ch^{inf}$ in terms of $Cyl^*(R, I)$. We have a natural map $Cyl^*(R/I^n) \to Cyl^*(R, I)_n$ and, by [3, 3.7] also a natural map $Cyl^*(R, I)_{2(n^2+n-1)} \to Cyl^*(R/I^n)$. Thus, if $\{u_n\}$ is a sequence of elements $u_n \in G^1_n = GL_4(Cyl^*(R, I)_n)$ satisfying $u_1 = 1$, $u_n\partial^0_{n}(e_n) = \partial^1_{n}(e_n)u_n$, and compatible with the projections $G^1_n \to G^1_{n-1}$, then the sequence $\{[\delta_n]\}$ of the $K_1$-classes of the compatible elements:

$\delta_n = \partial^2_{n}(u_n)\partial^0_{n}(u_n)\partial^1_{n}(u_n)^{-1}e_n + 1 - e_n$

represents $ch^{inf}(E)$. Write $\Omega = \Omega(R)$, $\partial^0 = 1 + \sum_{i=1}^{\infty} D_i : R \to Cyl(R) \cong \Omega/\Omega^{+\infty}$ with $D_1 = d$, $D_2 = \phi$, $D_i(R) \subset \Omega^i$. Thus $\partial^0_{n} = 1 + \sum_{i=1}^{n-1} D_{i,n} : R/I^n \to Cyl^1(R, I)_n$ where the $D_{i,n}$ are the induced maps. As in the proof of 4.2 above, one checks that, for $u_n = 1 + \sum_{i=1}^{n-1} \omega_{i,n}$, $(\omega_{i,n} \in \frac{\Omega^1}{\Omega^{+\infty}})$, the condition that $u_n\partial^0_{n}(e_n) = \partial^1_{n}(e_n)u_n$ is equivalent to the sequence of identities:

$[e_n, \omega_{p,n}] = D_{p,n}(e_n) + \sum_{i=1}^{p-1} \omega_{i,n} D_{p-i,n}(e)$ \quad (1 \leq p \leq n - 1)

Next one verifies that $\omega_{1,n} = (2e_n - 1)de_n$ satisfies the identity for $p = 1$. Then one uses induction to show that the map $\psi_{p,n} : k \to \Omega^p/F_{n-p}\Omega^p$, $e \mapsto D_{p,n}(e_n) + \sum_{i=1}^{p-1} \omega_{i,n} D_{p-i,n}$ is a derivation. It follows that if $\bar{\psi}_{p,n} : \Omega^1k \to \Omega^p/F_{n-p}\Omega^p$ is the induced homomorphism, then $\omega_{p,n} = \bar{\psi}_{p,n}(\omega_{1,n})$ satisfies (25). Hence $\{u_n = 1 + \sum_{i=1}^{n-1} \omega_{i,n}\}$ is a compatible sequence satisfying the requirements above. A formula for $\omega_{1,n}$ has been given already; we have $\omega_{2,n} = \nabla(\omega_{1,n})$, where $\nabla(xdy) = x\phi(y)$. We shall have no use for the terms of higher degree. From now on we fix $n$ and omit it as a subscript. One uses (9) to calculate that the element (24) is:

$\delta = 1 + e(de)^2 \otimes (v_1v_2 + v_1^2) + \ldots$

Here and below, the $\ldots$ stand for higher degree terms. The isomorphism:

$H^2_{inf}(A, K_1) \cong H^2_{inf}(A, HC_0)$

of 4.0 is induced by the logarithm map $log : \ker(Gl(Cyl^2(R, I)_n) \to Gl(R/I^n)) \to \ker(HC_0(Cyl^2(R, I)_n) \to HC_0(R/I^n))$. Applying this map to (26) we still get $log(\delta) = e(de)^2 \otimes (v_1v_2 + v_1^2) + \ldots$ where again the $\ldots$ are higher degree terms, but not necessarily the same as in (26). The composite of the projection to the normalized complex and the map $1 \otimes p$ of the proof of 3.0 above applied to $log(\delta)$ yields the element $e(de)^2$. Mapping the latter through the periodicity map of 4.1 we obtain the element $2e$. \(\square\)

Remark 4.5. Here is an explanation for the normalization factor appearing in theorem 4.4. For simplicity we shall restrict our attention to the quasi-free case. Let $R$ be quasi-free, $E$ a finitely generated right projective $\hat{R}$-module and $\nabla$ a connection on $E$. Write $\nu = s'\nu'$ where $s' : H^2_{dR}(R) \cong H^0_{dR}(R) \cong H^0_P(R)$ is the periodicity isomorphism of 3.1. The proof above shows that:

$s'(ch^{inf}(E)) = [\nu'(\Sigma^2)] \in H^2_{dR}(R)$.
Here $tr$ is the trace and $[\ ]$ denotes cohomology class. Thus the factor of $1/2$ needed to normalize $\text{ch}^{\inf}(E)$ is the same as that appearing in Karoubi’s definition of the Chern characters to de Rham cohomology ([11, 1.17]):

$$
\text{ch}_{2q}(E) = (1/2q!)[tr(\nabla^{2q})] \in H^{2q}_{dR}(R)
$$

5. The Jones-Goodwillie character.

The title of this section refers to the homomorphism:

$$(27) \quad c_n : K_n(A) \to HN_n(A)$$

going from $K$-theory to negative cyclic homology ($n \geq 1$). The map (27) was defined for arbitrary unital rings in [13], where it was proved it induces an isomorphism at the level of the rational relative groups associated to a nilpotent ideal. Here we shall restrict our attention to the particular case when $A$ is a $\mathbb{Q}$-algebra. Because for algebras over $\mathbb{Q}$, relative $K$ and cyclic homology groups are $\mathbb{Q}$-spaces, (cf. [15]), it follows from [13] that if $A$ is a unital $\mathbb{Q}$-algebra and $I \subset A$ is nilpotent then:

$$(28) \quad c_n : K_n(A, I) \to HN_n(A, I)$$

is an isomorphism. It is easy to extend the map (27) to nonunit algebras and to show that (28) holds nonunitally also; see [3, 4.2] for details. Also in [3, 4.2] it is shown that (27) is the homomorphism induced at the level of homotopy groups by a map of fibrant simplicial sets:

$$(29) \quad c : \mathbb{Z}_\infty N\text{Gl}(A) \to SCN_{\geq 1}(A)$$

Here $\mathbb{Z}_\infty$ is the Bousfield-Kan completion [1], $N\text{Gl}$ is the nerve of the general linear group, and $SCN_{\geq 1}$ is the simplicial abelian group the Dold-Kan correspondence associates with the truncation of the negative cyclic chain complex. In general, if $(C_n, \delta)_{n \in \mathbb{Z}}$ is any chain complex, and $m \in \mathbb{Z}$, we write $C_{\geq m}$ for the complex which is $C_n$ in each degree $n \geq m$, $\delta C_m$ in degree $m - 1$ and $0$ in degrees $< m - 1$. The purpose of this section is to prove the following:

**Theorem 5.0.** Let $A$ be a $\mathbb{Q}$-algebra. Then the Jones-Goodwillie map (29) fits into a homotopy fibration sequence of fibrant simplicial sets:

$$
\mathbb{H}_{\inf}(A, \mathbb{Z}_\infty N\text{Gl}) \to \mathbb{Z}_\infty N\text{Gl}(A) \xrightarrow{c} SCN_{\geq 1}(A)
$$

Here $\mathbb{H}_{\inf}(A, \mathbb{Z}_\infty N\text{Gl})$ is the infinitesimal hypercohomology simplicial set, as defined in 1.3 above.
Corollary 5.1. Let \( A = R/I \) be a presentation of \( A \) as a quotient of a quasi-free algebra \( R \). For \( n \geq 2 \), write \( LK_n(A) = \pi_n(\text{holim}_m \mathbb{Z}_\infty NGl(R/I^m)) \). Then there is an exact sequence:

\[
\ldots \rightarrow HN_{n+1}(A) \rightarrow LK_n(A) \rightarrow K_n(A) \rightarrow HN_n(A) \rightarrow \ldots \rightarrow LK_2(A) \rightarrow K_2(A) \rightarrow HN_2(A) \rightarrow H^0_{inf}(A, K_1) \rightarrow K_1(A) \rightarrow HN_1(A) \rightarrow H^1_{inf}(A, K_1) \rightarrow 0
\]

Proof of Corollary 5.1. The sequence of the corollary is the long exact sequence of hypercohomology groups of the chain complex \( CN \). Thus it suffices to compute the hypercohomology groups \( \mathbb{H}^n_{inf}(A, \mathbb{Z}_\infty NGl) \). It follows from [3, 5.5] and from [1, IX.3.1] that \( LK_n(A) = H^0_{inf}(A, \mathbb{Q}_\infty NE) \) where \( E \) is the elementary group. By [1, 4.4] we have a functorial homotopy fibration sequence:

\[
\mathbb{Z}_\infty NE \rightarrow \mathbb{Z}_\infty NGl \rightarrow N(K_1)
\]

By [1, 7.2], \( \mathbb{H}^n_{inf}(A, N(K_1)) = H^{1-n}_{inf}(A, K_1) \) for \( n = 0, 1 \) and is zero otherwise. On the other hand, because \( \mathbb{H}_{inf}(A, -) \) is a holim, it preserves homotopy fibration sequences of fibrant simplicial sets. The groups \( \mathbb{H}^n_{inf}(A, \mathbb{Z}_\infty NGl) \) for \( n \geq 0 \) are now easily calculated from the long exact sequence of homotopy associated to the fibration sequence obtained from applying the functor \( \mathbb{H}_{inf}(A, -) \) to the sequence above. \( \Box \)

Proof of Theorem 5.0. I claim that:

\[
(30) \quad \mathbb{H}^n_{inf}(A, SCN_{\geq 1}) = 0 \quad (n \leq 0)
\]

Assume the claim holds. Let \( F \) be the homotopy fiber of the map (29). We know from [3, 5.2] that we have a map \( H_{inf}(-, X) \rightarrow X \) which is natural for functors going from rings to fibrant simplicial sets, and that the functor \( X \mapsto H_{inf}(-, X) \) preserves homotopy fibration sequences. Using these facts, one obtains a commutative diagram where both rows are homotopy fibrations:

\[
\begin{array}{ccc}
\mathbb{H}_{inf}(A, F) & \longrightarrow & \mathbb{H}_{inf}(A, \mathbb{Z}_\infty NGl) \\
\downarrow & & \downarrow \\
F & \longrightarrow & \mathbb{Z}_\infty NGl(A) \\
\end{array}
\quad (31)
\quad \longrightarrow \quad \mathbb{H}_{inf}(A, SCN_{\geq 1})
\quad \longrightarrow \quad SCN_{\geq 1}(A)
\]

From the claim we deduce that the first map in the top row of (31) is a homotopy equivalence. Next recall from [3] that for functors \( X \) which map surjections with nilpotent kernel into homotopy equivalences the map \( H_{inf}(-, X) \rightarrow X \) is a weak equivalence. By (28), this applies to \( F \). It follows that the first vertical map of (31) is an equivalence. We have shown that the claim implies the theorem. Next note that, by [3, 5.4.1] the claim is equivalent to the assertion that the nonpositive hypercohomology groups of the chain complex \( CN_{\geq 1} \) are zero. We shall actually show that \( \mathbb{H}^n_{inf}(A, CN_{\geq 1}) = 0 \) for \( n \leq 0 \), and equals \( HP_n(A) \) for \( n \geq 1 \). It is proven below (Corollary 5.3) that \( CN_{\geq 1} \) has the same hypercohomology as the complex having \( \Omega^1_z \) in degree 1, \( b\Omega^1 \) in degree zero, and zero elsewhere. A standard spectral sequence argument shows that the hypercohomology of this complex is
The latter is calculated by means of the long exact sequence of cohomology groups associated with the following exact sequences of sheaves:

\[ 0 \rightarrow HH_1 \rightarrow \Omega_2^1 \rightarrow b\Omega^1 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow b\Omega^1 \rightarrow O \rightarrow O/[O,O] \rightarrow 0 \]

By Lemma 5.5 below and the first of the sequences above, we get that \( H^0_{inf}(A, HH_1) = 0 \) and \( H^n_{inf}(A, HH_1) = H^{n-1}(A, b\Omega^1) \). From Theorem 2.3 and (18), we obtain \( H^*_{inf}(A, O) = 0 \); from Theorem 3.0 and what we have just seen, we obtain that \( H^1_{inf}(A, HH_1) = 0 \) and \( H^n_{inf}(A, HH_1) = H^P_n(A) \). This completes the proof. \( \square \)

**Lemma 5.2.** Let \( R \) be a quasi-free \( \mathbb{Q} \)-algebra, \( \Omega^* = \Omega^*(R) \), \( \nabla \) as in (17), \( X = (\Omega^0 \xrightarrow{\nabla} \Omega_2^1) \) the 2-periodic de Rham complex of [5], \( CP^* = CP^*(R) \) the \((b,B)\) periodic cyclic complex. Define a homogeneous map \( f : X \rightarrow CP^* \):

\[
f(\omega) = \{(-\nabla B)^n(1-b\nabla)\omega\}_{n \geq 0}, \quad f(a) = \{(-\nabla B)^n(a)\}_{n \geq 0} \quad (\omega \in \Omega_2^1, a \in \Omega^0.)
\]

Then:

i) The map \( f \) is a well-defined supercomplex homomorphism.

ii) For the natural projection \( \pi : CP \rightarrow X \), we have \( \pi f = 1 \).

iii) Put \( h_m : \prod_{r \geq 0} \Omega^r \rightarrow \prod_{r \geq 0} \Omega^{r+2m+1} \),

\[
h_m\left(\sum_{r=0}^\infty \omega_r\right) = \sum_{r=1}^\infty (-1)^m \nabla(B\nabla)^m \omega_r \quad m \geq 0
\]

Then for \( h = \sum_{m=0}^\infty h_m \), we have \( 1 - f\pi = (B+b)h(1-f\pi) + h(1-f\pi)(B+b) \).

iv) For the filtration \( F \) of 1.2 above, we have:

\[
f_1(b\Omega^2 + F_m\Omega^1) \subset \prod_{p \geq 0} F_{m-(2p+1)}\Omega^{2p+1}, \quad f_0(F_m\Omega^0) \subset \prod_{p \geq 0} F_{m-2p}\Omega^{2p}
\]

\[
\text{and} \quad h(1-f\pi)\left(\prod_{p \geq 0} F_{m-p}\Omega^p\right) \subset \prod_{p \geq 0} F_{m-1-p}\Omega^p.
\]

**Proof.** Part i) is straightforward; part ii) is immediate. By 2.4, the map \( h \) of iii) is a contracting homotopy for \( \ker \pi \); the assertion of iii) is immediate from this. Item iv) is immediate from (12).

**Corollary 5.3.** Let \( A \) be any algebra. Then the infinitesimal hypercohomology of each of the Hochschild, cyclic, negative cyclic and periodic cyclic chain complexes associated to the functorial mixed complex \( \Omega^* \xrightarrow{B} \Omega^{*+1} \) equals the hypercohomology of the corresponding chain complex associated to the functorial mixed complex \( X \).

**Remark 5.4.** The same argument as in the proof of Theorem 5.0 shows that, for any ring \( A \) –not necessarily a \( \mathbb{Q} \)-algebra– we have a fibration sequence:

\[ H^*(A \otimes \mathbb{Q}, NC_\ell) \rightarrow \Omega \otimes \mathbb{Q}, GL_{\ell} \xrightarrow{\otimes \mathbb{Q}} SCN_{\ell} \rightarrow \mathbb{Q} \]
Moreover the analogue of the exact sequence of Corollary 5.1 holds, with the same proof. The lower terms of this rational sequence give the exact sequence of [3, 6.2], except that now we know the map $HN_1(A \otimes \mathbb{Q}) \to H^1(A, K_1 \otimes \mathbb{Q})$ is surjective. The surjectivity of this map was conjectured in [3]; this true conjecture was derived from the wrong conjecture that $H^*_{inf}(A, HN_1) = 0$. In fact it is not hard to show, using 5.3, that $H^*_{inf}(A, HN_1) = H^*_{inf}(A, HH_1)$ which is as calculated in the proof of 5.0, and is nonzero in general.

**Lemma 5.5.** Let $A$ be a $\mathbb{Q}$-algebra. Then:

$$H^*_{inf}(A, \Omega^1_\omega) = 0$$

**Proof.** Let $A = R/I$ be a presentation of $A$ as a quotient of a quasi-free algebra. We have to show that the cohomology of the complex $C(R, I, \Omega^1_\omega)$ is trivial. By definition the latter complex is the inverse limit of the complexes $\Omega^1_\omega Cyl(R, I)_n$. The proof has two main parts. The first part consists of showing that there is an isomorphism of pro-vectorspaces:

$$\Omega^1_\omega \text{Cyl}^m(R, I) \cong \{ \oplus_{r=1}^n \frac{\Omega^{r+1} \oplus \Omega^r \otimes T^r(V^m)}{M^r_{T^r} + \mathcal{G}^{m,n}} \}$$

Here $\Omega = \Omega(R)$, $M^r_{T^r} = \{(b\omega \otimes x + u \otimes y + (-1)^r \kappa(u) \otimes \zeta(y), (-1)^{r+1}b(u) \otimes y) : \omega \in \Omega^{r+1}, u \in \Omega^r, x, y \in T^r(V^m)\}$, and:

$$\mathcal{G}^{m,n} = \sum_{i_0 + \ldots + i_{r+1} + j \geq n-r} I_0dA^i_1 \ldots I_i^r d(I^r) I_{r+1}^r \oplus F^{n-r} \Omega^r \otimes T^r(V^m)$$

Now to the first part. We consider first the case when the ideal $I = 0$. We have:

$$\Omega^1(Cyl^m(R)) = \{ \Omega^1(Cyl^m(R)_n) \} = \{ \oplus_{r=0}^{n-1} \oplus_{p+q=r} \tilde{\Omega}^p \otimes \Omega^q \otimes T^r(V^m) \} = \{ \Omega^1(\Omega_m)_{deg<n} \}$$

Here $\tilde{\Omega}^0 = \tilde{R}$, $\tilde{\Omega}^p = \Omega^p$ if $p \geq 1$, $\Omega_m = \Omega_m(R) = \oplus_{p \geq 0} \Omega^p \otimes T^p(V^m)$ is the graded algebra of 1.1 above, and $deg<n$ indicates a truncation of the graded $\Omega_m$-module $\Omega^1(\Omega_m)$. Because the commutator subspace is graded, we have $\{ \Omega^1_\omega(Cyl^m(R)_n) \} \cong \{ \Omega^1_\omega(\Omega_m)_{deg<n} \}$. We write $\delta$ for the de Rham differential of $\Omega_m$. Fix a degree $r$. By repeated application of the Leibniz formula satisfied by $\delta$, we see that every form of degree $r$ can be written as a sum of elements of the following types:

$$\text{Type}(p, 0) : a_0da_1 \ldots da_p \delta a_{p+1}da_{p+2} \ldots da_{r+1} \otimes x$$

$$\text{Type}(p, 1) : a_0da_1 \ldots da_{p-1}\delta(da_p)da_{p+1} \ldots da_r \otimes x$$

Here $x \in T^r(V^m)$, $a_i \in R$, and if $\epsilon = 0, 1$, then for elements of type $(p, \epsilon)$, the index $p$ runs between $\epsilon$ and $r$. One checks furthermore that the sum of the subspaces generated by the elements of each type is direct, i.e. that we have an isomorphism:

$$\left( \oplus_{\epsilon=0}^r \Omega^{r+1}_\omega \oplus \oplus_{\epsilon=0}^r \Omega^r_\omega \right) \otimes T^r(V^m) \cong \Omega^1(\Omega_m)_{deg<n}$$
Here we write $\omega e_p$ for the element $\omega$ in the $p$-th summand; the isomorphism maps $a_0da_1 \ldots da_{r+e}e_p$ to the element of type $(p, e)$ of (33) ($e = 0, 1$), and $r\Omega^1(\Omega_m)$ is the homogenous summand of degree $r$. To compute the degree $r$ part of $\Omega^1(\Omega)^2_\epsilon$, we must divide $r\Omega^1(\Omega_m)$ by the subspace generated by the elements of the form $[a, \omega \otimes x]$ and $[da \otimes v, u \otimes y]$ for $a \in R$, $\omega \in \Omega^1(\Omega_1)$, $u \in \Omega_{r-1}(\Omega_1)$, $x \in T^r(V^m)$, $v \in V^m$ and $y \in T^{r-1}(V^m)$. One calculates each of these commutators for each of the types of $\omega$ and $u$, and then pulls them back through the isomorphism (34) to obtain that $r\Omega^1(\Omega_m)_2$ is isomorphic to the quotient of the left hand side of (34) by the subspace $M'_r$ generated by the following elements:

$$[\omega, a]e_p \otimes x, \ (u, a)e_q + (-1)^{q+1}(uda_{e_q} + udae_{q-1})) \otimes x,$$

$$\eta \otimes xe_s - (-1)^{r-\epsilon}\kappa(\eta) \otimes \zeta(x)e_{s+1}$$

Here $0 \leq p \leq r$, $1 \leq q \leq r$, $\epsilon \leq s \leq r - 1$, $\omega \in \Omega^{r+1}$, $u \in \Omega^r$, $\eta \in \Omega^{r+1-\epsilon}$, $\epsilon = 0, 1$, $x \in T^r(V^m)$, and $\zeta$ is as in the proof of theorem 3.0 above. Thus $\eta \otimes xe_s \equiv (-1)^{r-\epsilon}\kappa(\eta) \otimes \zeta(x)e_{s+1}$ (mod $M'_r$) whenever every element in $r\Omega^1(\Omega_m)_2$ is the class of some element in $(\Omega^r e_r \oplus \Omega^{r+1} e_{r+1}) \otimes T^r(V^m)$, or in other words there is a surjective homomorphism going from the latter to the former. One has to check that, upon identifying $\Omega^{r+1} \oplus \Omega^r$ with $\Omega^{r+1} e_{r+1} \oplus \Omega^r e_r$, we have $M'_r = M''_r \cap \left((\Omega^{r+1} e_r \oplus \Omega^r e_r) \otimes T^r(V^m)\right)$. The inclusion $\subset$ is immediate. To prove the other inclusion, let $\omega^0 e_r + \omega^1 e_r \in ((\Omega^{r+1} e_r \oplus \Omega^r e_r) \otimes T^r(V^m)) \cap M''_r$. Then there exist elements $u^0_i \in \Omega^{r+2} \oplus T^r(V^m)$, $u^1_j \in \Omega^{r+1} \oplus T^r(V^m)$, $v^0_k \in \Omega^{r+1} \oplus T^r(V^m)$, and $v^1_l \in \Omega^r \oplus T^r(V^m)$, $0 \leq i \leq r$, $1 \leq j \leq r$, $0 \leq k \leq r - 1$, $1 \leq l \leq r - 1$, such that:

$$\omega^1 e_r = \sum_{i=1}^r (b \otimes 1)(u^1_i e_i) + \sum_{i=1}^{r-1} (u^1_i e_i + (-1)^r (\kappa \otimes \zeta)(v^1_i) e_{i+1})$$

and:

$$\omega^0 e_r = \sum_{i=0}^r (b \otimes 1)(u^0_i e_i) + \sum_{i=1}^r (-1)^{i+1}(u^1_i e_i + u^1_i e_{i-1}) + \sum_{i=0}^{r-1} (v^0_i + (-1)^{r+1} (\kappa \otimes \zeta)(v^0_i))$$

In the first identity, for each $i \neq r$ the coefficient of $e_i$ in the right hand side must be zero; thus if we apply $((-1)^{r+1} \kappa \otimes \zeta)^{r-i}$ to it and then take the sum over all $i$, we obtain the identity:

$$\omega^1 = b \otimes 1 \sum_{i=1}^r \frac{((-1)^{r+1} \kappa \otimes \zeta)^{r-i}(u^1_i)}{(1 + (-1)^r (\kappa \otimes \zeta))(1 - (-1)^{r+1} (\kappa \otimes \zeta))}$$

By a similar procedure, the second identity yields:

$$\omega^0 = (b \otimes 1) \sum_{i=0}^r \frac{((-1)^{r+1} \kappa \otimes \zeta)^{r-i}(u^0_i)}{(1 - (-1)^{r} (\kappa \otimes \zeta) + (-1)^{r+1} (\kappa \otimes \zeta))}$$

This proves the inclusion $M'_r \cap (\Omega^{r+1} e_r \oplus \Omega^r e_r) \subset M_r$. Thus the proof of (32) for the case $I = 0$ is complete. Now let $I \subset R$ be any ideal. We have:

$$\Omega^1(Cyl^m(R, I)_n) = \oplus_{0 \leq p, q \leq n} \frac{\Omega^p \delta \Omega^q}{\Theta_{p+q}(V^m)}$$
One checks that the filtrations
\[ \Omega^1(\Omega_m) \supset G^{m,n} = \sum_{p,q \geq 0} (F^{n-p} \Omega^p \delta \Omega^q + \Omega^p \delta F^{m-q} \Omega^q) \otimes T^{p+q}(V^m) \]
and
\[ G^{m,n} = \sum_{n \leq p+q+i+j} (F^i \Omega^p \delta F^j \Omega^q) \otimes T^{p+q}(V^m) \]
are equivalent, whence \( \Omega^1(Cyl^m(R, I)) \cong \frac{\Omega^1(\Omega_m)}{G^{m,n}} \) as pro-vectorspaces. For the degreewise identification (34), we have: \( G^{m,n} \cap (\Omega^r e_s \otimes T^r(V^m)) = F^{n-r} \Omega^r e_s \otimes T^r(V^m) \) and
\[ G^{m,n} \cap \Omega^{r+1} e_s \otimes T^r(V^m) = \sum I_{i_0} dAI_{i_1} \ldots I_{i_{s-1}} d(I^j) I_{i_s} \ldots dAI_{i_{r+1}} \]
We note that \( G^{m,n} \) is closed under \( \kappa \otimes \zeta \), and if \( u \otimes x \in G^{m,n} \cap \Omega^{r+1} e_s \) (\( s \geq 1 \)), then \( b \otimes 1(u \otimes x) \in G^{m,n} \cap (\Omega^r e_s \otimes T^r(V^m)) \). From these observations, we see that the same argument as that given in the proof of (32) for the case \( I = 0 \) can be applied here, to prove now that \( (\Omega^{r+1} \oplus \Omega^r e_s) \otimes T^r(V^m) \cap (M_r + G^{m,n}) = M_r + G^{m,n} \). This finishes the first part of the proof. For the second part, consider the map \( p \) of the proof of theorem 3.0 above. Using the calculation of the first part of the current proof and the arguments of the proof of 3.0, one deduces that \( 1 \otimes p \) induces a homotopy equivalence between the pro-complex \( \Omega^1(Cyl(R, I)) \) and the pro-complex having \( \frac{\Omega^{m+1} \oplus \Omega^m}{N^{m+2} \oplus \Omega^{m+1}} \) in codimension \( m \), with \( N^m = \{ (b \omega + u - \kappa(u), (-1)^{m+1} b(u)) : \omega \in \Omega^{m+2}, u \in \Omega^{m+1} \} \) and with as coboundary map that induced by \( \Omega^{m+1} \oplus \Omega^m \to \Omega^{m+2} \oplus \Omega^{m+1}, (\omega, u) \mapsto (d\omega, (-1)^m \omega + du) \). The latter complex is contractible: a contracting homotopy is given by \( (\omega, u) \mapsto (-1)^{|\omega|} u, 0) \). We see that this homotopy maps \( N^* \) to \( N^{*-1} \) and is continuous for the topology of the filtration \( G^1 \). This concludes the second and last part of the proof. \( \square \)

Acknowledgements. Although most of the research for this paper was carried out in Buenos Aires, the first draft was written during a four week visit to Münster, where I was a guest of Prof. J. Cuntz. I am grateful to him and to the people of his team for their hospitality. My visit to Münster was made possible by a special travel grant from the University of Buenos Aires, which covered part of my transportation expenses and by SFB’s 478 which supported my stay. I am indebted to my colleague Prof. J.J. Guccione from whom I learned Lemma 2.4. Special thanks to Dr. C. Valqui of Münster who made me aware of the fact that the coefficients in the formula (21) receive the name of Catalan numbers and appear in various parts of mathematics.

References
1. Bousfield, Kan, Homotopy limits, completions and localizations, Springer Lecture Notes in Math. 304.
2. G. Cortiñas, On the derived functor analogy in the Cuntz-Quillen framework for cyclic homology, Algebra Colloquium 5 (1998), 305-319.
3. ______, Infinitesimal K-theory, J. reine angew. Math. 503 (1998).
4. J. Cuntz, D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8 (1995), 251-289.
5. ______, Cyclic homology and nonsingularity, J. Amer. Math. Soc. 8 (1995), 373-442.
6. ______, Operators in noncommutative differential forms, Geometry, Topology and Physics (For Raoul Bott), S.-T. Yau, Editor, International Press, Boston, 1995, pp. 77-111.
7. ______, Excision in bivariant periodic cyclic cohomology, Invent. Math. 27 (1997), 67-98.
8. A. Grothendieck, Crystals and the de Rham cohomology of schemes, Dix exposés sur la cohomologie des schémas (A. Grothendieck, N.H. Kuiper, eds.), Masson & Cie, North-Holland, 1968, pp. 306-358.
9. B.L. Feigin, B.L. Tsygan, Additive K-theory and crystalline homology, Functional Anal. Appl. 19 (1985), 124-132.
10. Hilton, Pederson, Catalan numbers, their generalizations and their uses, Math. Intelligencer 13 (1991), 64-75.
11. M. Karoubi, Homologie cyclique et K-théorie, Asterisque 149 (1987).
12. C. Kassel, Homologie cyclique, caractère de Chern et lemme de perturbation, J. reine angew. Math. 408 (1990), 159-180.
13. T. Goodwillie, Relative algebraic K-theory and cyclic homology, Annals of Math. 124 (1986), 347-402.
14. R. W. Thomason, Algebraic K-theory and étale cohomology, Ann. Sci. Éc. Norm. Sup. 13 (1985), 437-552.
15. C. Weibel, Nil K-theory maps to cyclic homology, Trans. Amer. Math. Soc. 303 (1987), 541-558.

Departamento de Matemática, Ciudad Universitaria Pabellón 1, (1428) Buenos Aires, Argentina.
E-mail address: gcorti@dm.uba.ar