Ancient solutions to the Ricci flow in dimension 3

Simon Brendle, Columbia University

May 26, 2021
What is an ancient solution, and why do we study them?

A solution to a geometric evolution equation is called **ancient** if it is defined on \((-\infty, T]\).

Ancient solutions to parabolic PDE are analogous to entire solutions of elliptic PDE.

Ancient solutions arise as limits of rescalings in geometric flows (just like entire solutions arise as limits of rescalings in elliptic problems).

Basic mechanism: Suppose that \((M, g(t))\) is a solution of Ricci flow which goes singular at time \(T < \infty\). Assume \((x_j, t_j)\) is a sequence of points such that \(\sup_{M \times [0, t_j]} |Rm(x, t)| \leq 2 |Rm(x_j, t_j)|\). Dilate around \((x_j, t_j)\) to make \(|Rm|\) equal to 1. Then the rescaled flows converge to an ancient solution.
Noncollapsing

Definition (Perelman): A solution to the Ricci flow is $\kappa$-noncollapsed if $\text{vol}(B_r(p)) \geq \kappa r^n$ whenever $\sup_{B_r(p)} |\text{Rm}| \leq r^{-2}$.

Theorem (Perelman): Any ancient solution that arises as a blow-up limit at a finite time singularity is $\kappa$-noncollapsed.

Definition (Perelman): We say that $(M, g(t))$ is an ancient $\kappa$-solution if it is defined on $(-\infty, T]$; is complete; non-flat; has bounded and non-negative curvature; and is $\kappa$-noncollapsed.
Ancient solutions to the Ricci flow in dimension 2

Shrinking spheres: \( M = S^2, \ g(t) = -2t g_{S^2}, \ t \in (-\infty, 0). \) Noncollapsed.

Cigar soliton: \( M = \mathbb{R}^2, \ g(t) = \frac{4}{e^{t+|x|^2}} g_{\text{eucl}}, \ t \in (-\infty, \infty). \) Steady soliton (moves by diffeomorphisms). Asymptotic to cylinder at spatial infinity. Collapsed.

Rosenau solution: Consider the metrics \( g(t) = \frac{8 \sinh(-t)}{1 + 2 \cosh(-t) |x|^2 + |x|^4} g_{\text{eucl}}, \ t \in (-\infty, 0). \) This defines a solution to the Ricci flow on \( \mathbb{R}^2 \) which can be extended smoothly to \( S^2. \) Not self-similar. For \( t \to -\infty, \) it looks like two cigar solitons joined together. Collapsed.

Theorem (Perelman 2002): Up to scaling, every ancient \( \kappa \)-solution in 2D is isometric to a family of shrinking spheres (or a \( \mathbb{Z}_2 \) quotient thereof).
Examples of ancient solutions to the Ricci flow in dimension 3

Shrinking spheres: $M = S^3$, $g(t) = -4t g_{S^3}$, $t \in (-\infty, 0)$. Noncollapsed.

Shrinking cylinders: $M = S^2 \times \mathbb{R}$, $g(t) = -2t g_{S^2} + dz \otimes dz$, $t \in (-\infty, 0)$. Noncollapsed.

Cigar soliton crossed with a line. Collapsed.

Bryant soliton (3D analogue of the cigar soliton). Steady soliton with rotational symmetry. Opens up like a paraboloid. Noncollapsed.

Perelman’s ancient solution (3D analogue of the Rosenau solution). Rotationally symmetric, but not a soliton. For $t \to -\infty$, it looks like two Bryant solitons joined together. Noncollapsed.
Structure of ancient $\kappa$-solutions in dimension 3

Theorem (Perelman): Let $(M, g(t))$ be a non-compact ancient $\kappa$-solution in 3D which is not locally isometric to a family of shrinking cylinders. Then $M$ is diffeomorphic to $\mathbb{R}^3$. Moreover, $(M, g(t))$ has the structure of an infinitely long $\varepsilon$-tube with a cap attached on one side.

Theorem (Perelman): Let $(M, g(t))$ be a compact ancient $\kappa$-solution in 3D which is simply connected and is not isometric to a family of shrinking round spheres. Then $M$ is diffeomorphic to $S^3$. Moreover, for $-t$ sufficiently large, $(M, g(t))$ has the structure of a finite $\varepsilon$-tube with caps attached on each side.

Here, an $\varepsilon$-tube is a set obtained by gluing $\varepsilon$-necks. The caps have "controlled geometry" up to scaling.
Classification of ancient $\kappa$-solutions in dimension 3

Theorem (B. 2018, conjectured by Perelman): Up to scaling, every noncompact ancient $\kappa$-solution in 3D is isometric to the shrinking cylinders (or a quotient thereof), or it is isometric to the Bryant soliton.

Theorem (B.-Daskalopoulos-Šešum 2020): Up to scaling and translation in time, every compact ancient $\kappa$-solution in 3D is isometric to the shrinking spheres (or a quotient thereof), or it is isometric to Perelman’s ancient solution (or a quotient thereof).
Overview of the proof in the noncompact case

The proof consists of two main steps:

Theorem A: Every noncompact ancient $\kappa$-solution in dimension 3 which has positive sectional curvature and is rotationally symmetric must be isometric to the Bryant soliton, up to scaling.

Theorem B: Every noncompact ancient $\kappa$-solution in dimension 3 which has positive sectional curvature is rotationally symmetric.

In the following, we explain the main ingredients in the proof of Theorem B (proof of rotational symmetry).

The proof of Theorem B uses Theorem A in a crucial way.
Classification of steady solitons

Definition: A steady gradient Ricci soliton is a triplet \((M, g, f)\) such that \(\text{Ric} = D^2 f\).

Steady gradient Ricci solitons move by diffeomorphisms when evolved by the Ricci flow. They are special cases of ancient solutions.

Theorem (B. 2012): Let \((M, g, f)\) be a 3D steady gradient Ricci soliton which has positive sectional curvature and is \(\kappa\)-noncollapsed. Then \((M, g)\) is isometric to the Bryant soliton up to scaling.

Idea of proof: By Perelman’s work, every point outside some compact set lies on an \(\varepsilon\)-neck. Moreover, \((M, g)\) opens up like a paraboloid at infinity. Therefore, \((M, g)\) admits approximate Killing vector fields \(U^{(1)}, U^{(2)}, U^{(3)}\) near infinity. By solving a linear elliptic PDE, we can construct exact Killing vector fields \(V^{(1)}, V^{(2)}, V^{(3)}\) which are asymptotic to \(U^{(1)}, U^{(2)}, U^{(3)}\). Thus, \((M, g)\) is rotationally symmetric.
Existence of Type II blow-up sequences

Let \((M, g(t))\) be a noncompact ancient \(\kappa\)-solution which has positive sectional curvature. Let \(R_{\text{max}}(t)\) denote the supremum of the scalar curvature at time \(t\).

Yongjia Zhang proved \(\limsup_{t \to -\infty} (-t) R_{\text{max}}(t) = \infty\). The proof uses, inter alia, the Neck Stability Theorem of Kleiner-Lott.

Combining this fact with Hamilton’s Harnack inequality we conclude that there exists a sequence of points \(\hat{p}_k\) and times \(\hat{t}_k \to -\infty\) such that, after rescaling around \((\hat{p}_k, \hat{t}_k)\), the solution converges to a steady gradient Ricci soliton. By the classification of solitons, the limit must be the Bryant soliton.

Upshot: There exists a sequence of times \(\hat{t}_k \to -\infty\) such that, at time \(\hat{t}_k\), the cap looks like a piece of the Bryant soliton up to scaling.

Problem: We do not know what happens in between those times.
Preservation of symmetry

Key Proposition: Let \((M, g(t))\) be a solution of Ricci flow. Let \(V(t)\) be a time-dependent vector field such that
\[
\frac{\partial}{\partial t} V = \Delta V + \text{Ric}(V),
\]
and let \(h(t) := \mathcal{L}_{V(t)}(g(t))\). Then \(\frac{\partial}{\partial t} h(t) = \Delta_{L,g(t)} h(t)\), where \(\Delta_L\) is the Lichnerowicz Laplacian:
\[
\Delta_L h_{ik} = \Delta h_{ik} + 2R_{ijkl}h^{jl} - \text{Ric}_i h_{kl} - \text{Ric}_k h_{il}.
\]

Note that \(\Delta V + \text{Ric}(V) = \text{div} h - \frac{1}{2} \nabla \text{tr}(h)\). In particular, if \(V\) is a Killing vector field, then \(\Delta V + \text{Ric}(V) = 0\) and \(V\) is unchanged under the evolution.

Remark: The PDE \(\frac{\partial}{\partial t} V = \Delta V + \text{Ric}(V)\) is the linearization of the harmonic map heat flow.

The PDE \(\frac{\partial}{\partial t} h = \Delta_L h\) is the linearization of the Ricci-DeTurck flow. We call it the parabolic Lichnerowicz equation.
Analysis of the parabolic Lichnerowicz equation on shrinking cylinders

Let $\bar{g}(t) = (-2t) g_{S^2} + dz \otimes dz$ denote the standard metrics on $S^2 \times \mathbb{R}$.

Proposition: Let $h(t)$ be a solution of $\frac{\partial}{\partial t} h(t) = \Delta_{L, \bar{g}(t)} h(t)$ defined on $S^2 \times [-\frac{L}{2}, \frac{L}{2}]$ and for $t \in [-\frac{L}{2}, -1]$. Assume that $|h(t)|_{\bar{g}(t)} \leq 1$ for $t \in [-\frac{L}{2}, -\frac{L}{4}]$, and $|h(t)|_{\bar{g}(t)} \leq L^{10}$ for $t \in [-\frac{L}{4}, -1]$. Then there exists a scalar function $\psi$ lying in the span of the first spherical harmonics on $S^2$ and a rotationally invariant tensor of the form $\bar{\omega}(z, t) g_{S^2} + \bar{\beta}(z, t) dz \otimes dz$ such that

$$|h(t) - \bar{\omega}(z, t) g_{S^2} - \bar{\beta}(z, t) dz \otimes dz - (-t) \psi g_{S^2}|_{\bar{g}(t)} \leq C L^{-\frac{1}{2}}$$

on $S^2 \times [-1000, 1000]$ and for $t \in [-1000, -1]$.

Idea of proof: Decompose $h(t)$ into components, and perform a mode decomposition in spherical harmonics. This leads to a system of one-dimensional linear heat equations.
Definition of an evolving $\varepsilon_0$-neck

Let $(M, g(t))$ be a solution to the Ricci flow in dimension 3.

Definition: Let $(\bar{x}, \bar{t})$ be a point in space-time with $R(\bar{x}, \bar{t}) = r^{-2}$. We say that $(\bar{x}, \bar{t})$ lies at the center of an evolving $\varepsilon_0$-neck if, after rescaling by the factor $r^{-1}$, the parabolic neighborhood $B_{g(\bar{t})}(\bar{x}, \varepsilon_0^{-1}r) \times [\bar{t} - \varepsilon_0^{-1}r^2, \bar{t}]$ is $\varepsilon_0$-close in $C[\varepsilon_0^{-1}]$ to a family of shrinking cylinders.

Theorem (Hamilton): At each point in time, a neck admits a canonical foliation by constant mean curvature (CMC) spheres. The CMC foliation depends on $t$. 
A notion of $\varepsilon$-symmetry on an evolving $\varepsilon_0$-neck

Definition: Suppose that $(\bar{x}, \bar{t})$ lies at the center of an evolving $\varepsilon_0$-neck for some small number $\varepsilon_0$, let $R(\bar{x}, \bar{t}) = r^{-2}$, and let $\varepsilon \leq \varepsilon_0$. We say that $(\bar{x}, \bar{t})$ is $\varepsilon$-symmetric if there exist smooth, time-independent vector fields $U^{(1)}, U^{(2)}, U^{(3)}$ such that:

(i) $\sum_{l=0}^{2} \sum_{a=1}^{3} r^{2l} |D^l(\mathcal{L}_{U^{(a)}}(g(t)))|^2 \leq \varepsilon^2$ on $\bar{B}_{g(t)}(\bar{x}, 100r) \times [\bar{t} - 100r^2, \bar{t}]$.

(ii) $\sum_{a=1}^{3} r^{-2} |\langle U^{(a)}, \nu \rangle|^2 \leq \varepsilon^2$ on $\bar{B}_{g(t)}(\bar{x}, 100r) \times [\bar{t} - 100r^2, \bar{t}]$, where $\nu$ denotes the unit normal vector to the CMC foliation in $(M, g(t))$.

(iii) If $t \in [\bar{t} - 100r^2, \bar{t}]$ and $\Sigma \subset \bar{B}_{g(t)}(\bar{x}, 100r)$ is a leaf of the CMC foliation of $(M, g(t))$, then

$$|\delta_{ab} - \text{area}_{g(t)}(\Sigma)^{-2} \int_{\Sigma} \langle U^{(a)}, U^{(b)} \rangle_{g(t)} d\mu_{g(t)}|^2 \leq \varepsilon^2.$$
Improvement of symmetry on a neck

Neck Improvement Theorem: There exist a large constant $L$ and small positive constant $\varepsilon_1$ with the following property. Suppose that $(x_0, t_0)$ lies at the center of an evolving $\varepsilon_1$-neck and write $R(x_0, t_0) = r^{-2}$. Moreover, suppose that every point in $B_{g(t_0)}(x_0, Lr) \times [t_0 - Lr^2, t_0)$ is $\varepsilon$-symmetric, where $\varepsilon \leq \varepsilon_1$. Then $(x_0, t_0)$ is $\frac{\varepsilon}{2}$-symmetric.

Note: $L$ and $\varepsilon_1$ are independent of $\varepsilon$!

The strategy is to choose $L$ large, and then choose $\varepsilon_1$ small depending on $L$. The proof uses in a crucial way the results on the parabolic Lichnerowicz equation on a cylindrical background.
**Proof of Theorem B**

Assume that \((M, g(t))\) is an ancient \(\kappa\)-solution which has positive sectional curvature and is not rotationally symmetric.

Perelman’s work \(\Rightarrow\) For each \(t\), the flow looks like an \(\varepsilon_1\)-tube, with a cap attached on one side.

**Definition:** Let \(\varepsilon \leq \varepsilon_1\). We say that the flow is \(\varepsilon\)-symmetric if every point on the tube is \(\varepsilon\)-symmetric in the sense of the definition above, and in addition the cap is \(\varepsilon\)-symmetric in a suitable sense.

We can find a sequence \(\hat{t}_k \to -\infty\) and a sequence \(\hat{\varepsilon}_k \to 0\) such that the flow is \(\hat{\varepsilon}_k\)-symmetric at time \(t\) for each \(t \in [\hat{t}_k - \hat{\varepsilon}_k^{-2} R_{\max}(\hat{t}_k)^{-1}, \hat{t}_k]\).
Proof of Theorem B (continued)

Choose a sequence $\epsilon_k \to 0, \epsilon_k \gg \hat{\epsilon}_k$.

Define

$$t_k = \inf \{ t \in [\hat{t}_k, 0] : \text{The flow is not } \epsilon_k\text{-symmetric at time } t \}.$$  

By definition of $t_k$, the flow is $\epsilon_k$-symmetric at time $t$ for each $t \in [\hat{t}_k - \hat{\epsilon}^{-2}_k R_{\text{max}}(\hat{t}_k)^{-1}, t_k)$.

Note that $t_k \to -\infty$. (If $\limsup_{k \to \infty} t_k > -\infty$, then the flow is rotationally symmetric for $t < \limsup_{k \to \infty} t_k$, contrary to our assumption.)

Choose a sequence of points $q_k$ so that

$$\liminf_{k \to \infty} \frac{\lambda_1(q_k, t_k)}{R(q_k, t_k)} > 0.$$  

Rescale around $(q_k, t_k)$ and take the limit in the Cheeger-Gromov sense. The limit $(M^\infty, g^\infty(s))$ is an ancient $\kappa$-solution with positive sectional curvature.
Proof of Theorem B (continued)

Recall that the flow \((M, g(t))\) is \(\varepsilon_k\)-symmetric at time \(t\) for each \(t \in [\hat{t}_k - \frac{\varepsilon_k}{\bar{\varepsilon}_k^2} R_{\text{max}}(\hat{t}_k)^{-1}, \hat{t}_k]\).

\[ \Rightarrow \] The limit flow \((M^\infty, g^\infty(s))\) is rotationally symmetric.

By Theorem A, the limit flow \((M^\infty, g^\infty(s))\) must be the Bryant soliton.

In other words, at time \(t_k\) (and for long period of time before \(t_k\)) the original flow is close to the Bryant soliton.

Using the Neck Improvement Theorem, we can show that the symmetry in the tube region improves.

Using a maximum principle estimate due to Anderson-Chow, Choi, we can show that the symmetry in the cap region improves.

Upshot: The flow is \(\frac{\varepsilon_k}{2}\)-symmetric at time \(t_k\). This contradicts the definition of \(t_k\).
Conclusion

To summarize, we have proved Theorem B. Combining this with Theorem A (the classification of noncompact ancient $\kappa$-solutions with rotational symmetry), Perelman’s conjecture follows: Up to scaling, translation in time, and diffeomorphisms, the only noncompact ancient $\kappa$-solutions in dimension 3 are shrinking cylinders and the Bryant soliton.