Quantum enhancement of randomness distribution

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Abstract—The capability of a given channel to communicate information is, a priori, distinct from its capability to distribute shared randomness. In this article we define randomness distribution capacities of quantum channels assisted by forward, back, or two-way classical communication and compare these to the corresponding communication capacities. With forward assistance or no assistance, we find that they are equal. We establish the mutual information of the channel as an upper bound on the two-way assisted randomness distribution capacity. This implies that all of the capacities are equal for classical-quantum channels. On the other hand, we show that the back-assisted randomness distribution capacity of a quantum-classical channel is equal to its mutual information. This is often strictly greater than the back-assisted communication capacity. We give an explicit example of such a separation where the randomness distribution protocol is noiseless.

Index Terms—Quantum Shannon theory, noisy channels, capacity, randomness

I. SUMMARY

If Alice can send a bit of her choosing to Bob over some channel then she is also able to use that channel to distribute one bit of shared randomness between herself and Bob: she just locally generates a random bit and sends a copy to Bob. More generally, if $\mathcal{E}$ is a quantum operation and $C(\mathcal{E})$ the classical capacity of the channel $\mathcal{E}$, we expect that the randomness distribution capacity $R(\mathcal{E})$ of $\mathcal{E}$ obeys $R(\mathcal{E}) \geq C(\mathcal{E})$. The HSW theorem states that $C(\mathcal{E}) = \lim_{n \to \infty} \chi(\mathcal{E}^n)/n$ where $\chi(\mathcal{E})$ is the Holevo information of $\mathcal{E}$. It follows from a typical proof of the converse part of this theorem that, in fact, $R(\mathcal{E}) = C(\mathcal{E})$ for any $\mathcal{E}$. But what happens if we allow some auxiliary classical communication resources?

We will consider the communication capacity achieved by communication protocols in which feedback $C_{\leftarrow}$, auxiliary forward communication $C_{\rightarrow}$, and two-way classical communication $C_{\leftrightarrow}$ are available. Since the auxiliary forward communication can be used to communicate by itself, one substracts the amount of auxiliary forward communication from the gross communication rates in the definitions of the later two quantities. We will similarly define randomness distribution protocols (RDPs) with various kinds of auxiliary communication and the associated capacities $R_{\leftarrow}$, $R_{\rightarrow}$, $R_{\leftrightarrow}$, but in this case we must substract both forward and backward auxiliary communication, as both of these may be used to establish shared randomness by themselves.

We give formal definitions of the various capacities in Section II and represent their relations in Figure I. Unsurprisingly, these satisfy the inequalities

$$ R(\mathcal{E}) \leq R_{\rightarrow}(\mathcal{E}) \leq R_{\leftarrow}(\mathcal{E}), \quad R_{\leftarrow}(\mathcal{E}) \leq R_{\leftrightarrow}(\mathcal{E}) \leq R_{\rightarrow}(\mathcal{E}), \quad C(\mathcal{E}) \leq C_{\leftarrow}(\mathcal{E}) \leq C_{\leftrightarrow}(\mathcal{E}) \leq C_{\rightarrow}(\mathcal{E}). $$(1)

Intuitively, one also expects that $C_{\rightarrow}$ be greater than $R_{\rightarrow}$ for arbitrary assistance, since randomness distribution seems easier than communication. While it is straightforward to turn this intuition into a proof for forward-assisted and unassisted protocols, it is not so straightforward when back-assistance is allowed because we regard this as “free” for communication protocols but account for it in RDPs. Nevertheless, in Section III we establish the expected relations:

Theorem 1. For any operation $\mathcal{E}$

$$ C(\mathcal{E}) \leq R(\mathcal{E}), \quad C_{\rightarrow}(\mathcal{E}) \leq R_{\rightarrow}(\mathcal{E}), \quad C_{\leftarrow}(\mathcal{E}) \leq R_{\leftarrow}(\mathcal{E}). \quad (2) $$

Theorem 2. For any operation $\mathcal{E}$

$$ C(\mathcal{E}) = R(\mathcal{E}) = C_{\rightarrow}(\mathcal{E}) = R_{\rightarrow}(\mathcal{E}). \quad (4) $$

Theorem 3. For any operation $\mathcal{E}$, $R_{\leftrightarrow}(\mathcal{E}) \leq I(\mathcal{E})$.

In Section IV we show that for forward-assisted protocols, and unassisted protocols, randomness distribution capacities are no greater than classical distribution capacities.

Theorem 4. If $\mathcal{E}$ is classical-quantum (cq) then $C(\mathcal{E}) = \chi(\mathcal{E}) = I(\mathcal{E})$, so a consequence of the results given so far is

Corollary 4. If $\mathcal{E}$ is classical-quantum then $R(\mathcal{E}) = R_{\rightarrow}(\mathcal{E}) = R_{\leftarrow}(\mathcal{E}) = C(\mathcal{E}) = C_{\rightarrow}(\mathcal{E}) = C_{\leftarrow}(\mathcal{E}) = C_{\leftrightarrow}(\mathcal{E})$.

In Section V we show that the mutual information $I(\mathcal{E})$ of $\mathcal{E}$ is an upper bound on $R_{\leftrightarrow}(\mathcal{E})$:

Theorem 5. For any quantum-classical (qc) operation $\mathcal{E}$, $R_{\leftrightarrow}(\mathcal{E}) = R_{\leftrightarrow}(\mathcal{E}) = I(\mathcal{E})$.

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In Section VI we establish the quantum enhancement of our title by showing that there are (qc) operations $\mathcal{E}$ such that $R_{\leftrightarrow}(\mathcal{E}) > C_{\leftrightarrow}(\mathcal{E})$. First, in VI-A we use a result of Devetak and Winter [3] to prove

Theorem 5. For any quantum-classical (qc) operation $\mathcal{E}$, $R_{\leftrightarrow}(\mathcal{E}) = R_{\leftrightarrow}(\mathcal{E}) = I(\mathcal{E})$.

On the other hand, a result of Bowen and Nagarajan [3] allows us to show (in subsection VI-B) that

1We mean the memoryless channel for which the operation describing $n$ channel uses is $\mathcal{E}^n$. 

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Proposition 6. For any entanglement-breaking operation $\mathcal{E}$
\[ C(\mathcal{E}) = C_{\rightarrow}(\mathcal{E}) = C_{\leftrightarrow}(\mathcal{E}) = C_{\rightarrow\leftarrow}(\mathcal{E}). \] (5)

Since qc operations are entanglement-breaking, any qc channel with $C(\mathcal{E}) < I(\mathcal{E})$ also demonstrates a separation $C_{\rightarrow}(\mathcal{E}) < R_{\rightarrow}(\mathcal{E})$. Holevo has shown that there are many such channels [4]. In subsection VI-D we give an explicit example

Proposition 7. There is a qc operation $\mathcal{F}$ such that $R_{\rightarrow}(\mathcal{F}) = \log(d)$ while $C_{\rightarrow}(\mathcal{F}) = \chi(\mathcal{F}) = \frac{1}{2} \log(d)$.

A. Previous work

The back-assisted communication capacity was studied in [12], where it was shown that there are random-phase coupling channels (informally called “rocket channels”) which exhibit a strict separation $C(\mathcal{E}) < C_{\leftrightarrow}(\mathcal{E})$.

A different definition of two-way assisted classical capacity, $C_2$, was given in [14]. In this definition, the back-communication is not subtracted to obtain the rate, but the two-way classical communication, taken as a whole, must be independent of the message being transmitted. In [14] it was shown that by concatenating an echo-correctable channel and a depolarising channel one can obtain an entanglement-breaking channel $\mathcal{E}$ such that $C_{\rightarrow}(\mathcal{E}) < C_2(\mathcal{E})$.

Using the independent two-way communication as an additional source of shared randomness shows that $C_2 \leq R_{\rightarrow\leftarrow}$, but it is not obvious to us what the relationship between $C_2$ and $C_{\leftrightarrow}$ is. It seems that the fact that we don’t subtract the auxiliary communication in the definition of $C_{\rightarrow}$ means that there are examples where $C_2 > C_{\leftrightarrow}$ but we leave the question open here.

A result similar in spirit to some of those given here is that forward communication over entanglement-breaking channels cannot increase the quantum capacity, which is the “ninth variation” studied by Kretschmann and Werner in [4].

As for randomness distribution, in the completely classical setting the tradeoff between the gross rate of randomness distribution and the rate of feedback allowed was characterised (among many other things) by Ahlswede and Csiszár in [2]. A corollary of this result is that for classical $\mathcal{E}$, $R_{\rightarrow}(\mathcal{E}) = C(\mathcal{E})$.

To our knowledge the only previous work studying specifically the generation of shared randomness in a quantum scenario was the work of Devetak and Winter [7] on the distillation of shared randomness from bipartite quantum states, which gave operational meaning to an information quantity proposed earlier by Henderson and Vedral [8] (see however the unpublished PhD thesis of Wilming [9]). That work considered a static scenario of distillation of randomness from a quantum state already shared between Alice and Bob, where in this manuscript we are interested on a dynamic scenario of randomness distribution over quantum channels.

II. Definitions

The completely dephasing operation $\mathcal{M}$ on a quantum system $Q$ is defined by $\mathcal{M} : \rho_Q \rightarrow \sum_{0 \leq i < d_Q} |i\rangle\langle i|\rho|i\rangle\langle i|$. An operation $\mathcal{E}$ is called classical-quantum (cq) if $\mathcal{M} = \mathcal{E}$, quantum-classical (qc) if $\mathcal{M} = \mathcal{E}$, and classical (cc) if it is both cq and qc.

When we have a random variable stored in the computational basis of a quantum system (a “classical register”) we will adopt the convention that the system has the same symbol as the variable, but in the sans serif font.

The mutual information $I(\mathcal{E})$ of an operation $\mathcal{E}^{X\rightarrow Y}$ is the maximum of $I(R : Y)_{\mathcal{E}^{X\rightarrow Y},\rho_{RX}}$ over all finite dimensional systems $R$ and density operators $\rho_{RX}$. We note that it was shown by Bennett, Shor, Smolin and Thapliyal [15], that the entanglement-assisted classical capacity $C_E(\mathcal{E})$ of a channel $\mathcal{E}$ is equal to $I(\mathcal{E})$.

The Holevo information $\chi(\mathcal{E})$ of an operation $\mathcal{E}^{X\rightarrow Y}$ is the maximum of $I(R : Y)_{\mathcal{E}^{X\rightarrow Y},\rho_{RX}}$ over all finite dimensional systems $R$ and density operators $\rho_{RX}$ such that $\mathcal{M}^{R} = \rho_{RX}$.

A. Randomness distribution protocols

Our definitions in this section are based on those used by Ahlswede and Csiszár in [2], and Devetak and Winter [7].

A two-way assisted randomness distribution protocol (RDP) for a channel $\mathcal{E}$ consists of local generation of random variables $A_0$ and $B_0$ followed by a finite number of steps, each consisting of communication followed by local processing. The communication is either (i) forward communication via one use of the noisy channel $\mathcal{E}$; (ii) noiseless auxiliary forward classical communication; (iii) noiseless auxiliary back classical communication.

Suppose we have a RDP of $n+m$ steps where $n$ of the steps are of type (i) and the other $m$ steps are of type (ii) or (iii). At the end of the protocol, Alice must produce $J$ and Bob must produce $K$ (by local processing) both of which take values in the same alphabet $\mathcal{A}_K$. An example of such a protocol with $n = m = 2$ is illustrated in Figure 2.

We require that
\[ |\mathcal{A}_K| \leq 2^{cn} \] (6)
for some constant $c$ independent of $n$ (but depending on the channel $\mathcal{E}$). We say that the protocol is $c$-good if
\[ \Pr(J \neq K) \leq \epsilon. \] (7)
By Fano’s inequality and (6), an \( \epsilon \)-good protocol has
\[
H(K|J) \leq \epsilon cn + 1 \quad (8)
\]

We denote the data transmitted in each instance of auxiliary communication (regardless of whether it is forward or backward) by \( Z_k \), where \( k \in \{1, \ldots, m\} \), in temporal order.

If the total auxiliary communication \( Z := (Z_1, \ldots, Z_m) \) has \( |A_Z| \) possible values (we require this number to be finite for any given protocol), then this alone would allow the parties to establish \( \log |A_Z| \) bits of perfect common randomness without using the channel \( \mathcal{E} \) at all! We therefore subtract \( \log |A_Z| \) from the final amount of common randomness established and hence define the net rate of the protocol by
\[
\frac{1}{n}(H(K) - \log |A_Z|).
\]

A forward-assisted RDP is one in which all steps are of type (i) or (ii). A back-assisted RDP is one in which all steps are of type (i) or (iii). An unassisted RDP is one in which all steps are of type (i).

**Definition 8.** We say a net rate \( r \) is achieved by two-way protocols for channel \( \mathcal{E} \) if for all \( \epsilon > 0 \) and all sufficiently large \( n \), there is an \( \epsilon \)-good protocol for \( n \) noisy channel uses with net rate no less than \( r \). We define \( R_+ (\mathcal{E}) \) to be the supremum of net rates achieved by two-way protocols; \( R_- (\mathcal{E}) \) to be the supremum of net rates achieved by forward-assisted protocols; \( R_{\leftrightarrow} (\mathcal{E}) \) to be the supremum of net rates achieved by back-assisted protocols; and \( R (\mathcal{E}) \) to be the supremum of net rates achieved by unassisted protocols.

**B. Communication protocols**

We define two-way assisted communication protocols in similar way, except for a few key differences. An example of such a protocol with \( n = m = 2 \) is illustrated in Figure 2. Now, Alice starts with a message \( M \) taking values in a set \( \mathcal{A}_M \) satisfying
\[
|\mathcal{A}_M| \leq 2^m \quad (9)
\]
where \( c \) is a constant which can depend on the channel \( \mathcal{E} \), and at the end of the protocol Bob produces an estimate \( \hat{M} \) of \( M \) which also takes values in \( \mathcal{A}_M \). We say that a communication protocol is \( \epsilon \)-good\(^2\) if
\[
\Pr(\hat{M} \neq M | M = m) \leq \epsilon \quad \forall m \in \mathcal{A}_M.
\]

The other important difference is how we define the net rate for these protocols. Since auxiliary communication from Bob to Alice is, by itself, useless for the communication task we do not subtract it to obtain the net rate. Letting \( F_1, \ldots, F_r \) be all of the forward auxiliary communications (just a relabelling of those \( Z_i \), which are in the forward direction) we define the net rate of a two-way assisted communication protocol as
\[
\frac{1}{n}(\log |\mathcal{A}_M| - \log |\mathcal{A}_F|)
\]
where \( F = (F_1, \ldots, F_r) \).

**Definition 9.** We say a net rate \( r \) is achieved by a two-way communication protocol for channel \( \mathcal{E} \) if for all \( \epsilon > 0 \) and all sufficiently large \( n \), there is an \( \epsilon \)-good protocol for \( n \) noisy channel uses with net rate less than \( r \). We define \( C_+ (\mathcal{E}) \) to be the supremum of net rates achieved by two-way protocols; \( C_- (\mathcal{E}) \) to be the supremum of net rates achieved by forward-assisted protocols; \( C_{\leftrightarrow} (\mathcal{E}) \) to be the supremum of net rates achieved by back-assisted protocols; and \( C (\mathcal{E}) \) to be the supremum of net rates achieved by unassisted protocols.

**III. TURNING COMMUNICATION PROTOCOLS INTO RANDOMNESS DISTRIBUTION PROTOCOLS**

In this section we prove Theorem 1. Suppose that we have an assisted communication protocol \( cp \) which can send a uniformly distributed message \( M \) (taking values in \( \mathcal{A}_M \)) with probability of error no more than \( \epsilon_0 \) (ie \( \Pr(\hat{M} \neq M) \leq \epsilon_0 \)) by making \( n_0 \) uses of the noisy channel \( \mathcal{E} \) and \( m \) auxiliary communication steps of which \( b \) are in the backwards direction. Let \( G_1, \ldots, G_b \) denote the \( b \) random variables representing the auxiliary communications from Bob to Alice in the order they occur in the protocol, and let \( F_1, \ldots, F_{m-b} \) denote the \( m-b \) RVs representing the auxiliary communications from Alice to Bob in the order they occur in the protocol. This is just a convenient relabelling of the random variables \( Z_i \).

![Fig. 2. An example of a two-way assisted RDP which makes two uses of the channel \( \mathcal{E} \). Time runs left to right. Classical systems are shown as double lines, quantum systems as solid lines. Empty boxes represent local processing.](image-url)
which were introduced in Section II Let $G := (G_1, \ldots, G_b)$ and $F := (F_1, \ldots, F_{m-b})$.

The net rate of communication achieved by $\mathbf{cp}$ is

$$r_0 = \frac{1}{n} (\log |A_M| - \log |A_F|)$$

(11)

where $A_F$ is the set of possible values of $F_1, \ldots, F_{m-b}$. Recall that we do not subtract the auxiliary backwords communication here because, by itself, it is useless for the forward communication task.

We will first describe an RDP, which we call $\mathbf{rdp}$, which uses $\ell$ parallel runs of $\mathbf{cp}$ followed by an extra round of back communication to do randomness distribution. The shared randomness consists of $\ell$ randomly chosen messages, generated by Alice and communicated by $\mathbf{cp}$, as well as all of the back communication used in the protocol. This doesn’t get us to the required result because the extra entropy from the back communication in the shared randomness might not be enough to make up for subtracting $\log |A_G|$ to get the net rate of $\mathbf{rdp}$. To get around this we define a modified version of $\mathbf{rdp}$ which uses the i.i.d. distribution of the parallel back communication to compress it, taking advantage of side information in Alice’s possession, so that it is approximately independent of the message, and thus reduce $\log |A_G|$ to a size which is compensated for by the extra shared randomness from the back communication. We call this modified version $\mathbf{rdp'}$.

The protocol $\mathbf{rdp}$ is as follows. Alice generates $\ell$ messages $M_i$ for $i \in \{1, \ldots, \ell\}$, each one uniformly distributed over $A_M$ and independent of the others. Alice and Bob perform the protocol $\mathbf{cp}$ $\ell$ times, which results in Bob producing an estimate $\hat{M} = (\hat{M}_1, \ldots, \hat{M}_\ell)$ of $M = (M_1, \ldots, M_\ell)$ such that the $\hat{M}_i$ are i.i.d. and

$$\Pr(\hat{M}_i \neq M_i) \leq \epsilon_0 \forall i.$$ 

(12)

This requires $\ell n_0$ uses of the noisy channel and $\ell m$ auxiliary communication steps. We can order these so that we do the first step of the run of $\mathbf{cp}$ which sends $M_1$, then the first step for the run of $\mathbf{cp}$ which sends $M_2$, and so on, completing step $j$ for message $M_j$ before moving on to step $j + 1$ for $M_j$. Letting $G_{j,i}$ denote the $j$-th step of auxiliary back communication in the run of $\mathbf{cp}$ to send $M_i$, this means that $G_{j,1}, \ldots, G_{j,\ell}$ are received by Alice before $G_{j+1,1}, \ldots, G_{j+1,\ell}$ for each $j \in \{1, \ldots, b\}$.

Once Bob has produced all $\ell$ estimates $\hat{M}_1, \ldots, \hat{M}_\ell$, he uses an extra step of back communication sending $v$ bits of compressed information about $\hat{M}$ such that decompression using side information $M$ allows Alice to make an estimate $\hat{M}$ of $M$ such that $\Pr(\hat{M} \neq M) \leq \epsilon_1$. After this, Alice sets her share $J$ of the randomness to $(\hat{M}, G)$, while Bob sets his share $K$ to $(\hat{M}, G)$.

These are all the essential parts of $\mathbf{rdp}$, but in order to define $\mathbf{rdp'}$ and compare it to $\mathbf{rdp}$ we will suppose that in $\mathbf{rdp}$ Alice also compresses $G_j := (G_{j,1}, \ldots, G_{j,\ell})$ to $v_j$ bits such that a decompressor with side information $M, G_1, \ldots, G_{j-1}$ can make an estimate $\hat{G}_j$ of $G_j$ from the compressed data with $\Pr(\hat{G}_j \neq G_j) \leq \epsilon_1$, and that Alice uses does produce this estimate. Note that this does not affect the amount of communication resources used by $\mathbf{rdp}$, its error probability, nor its rate.

For each $j$, $G_{j,1}, \ldots, G_{j,\ell}$ are i.i.d. as are $\hat{M}_1, \ldots, \hat{M}_\ell$. We know that, for any $\epsilon_1 > 0$ and any $\delta_1 > 0$ and all sufficiently large $\ell$, we can find compression schemes such that

$$\frac{v}{\ell} \leq H(\hat{M} | M) + \delta_1,$$

(13)

and

$$\frac{v_j}{\ell} \leq H(G_j | G_{j-1}, \ldots, G_1, M) + \delta_1 \ \forall j$$

(14)

which, by the chain rule for conditional entropy, implies that

$$\sum_{j=1}^b \frac{v_j}{\ell} \leq H(G | M) + b \delta_1.$$ 

(15)

where $G := (G_1, \ldots, G_1)$. Recall that $M$, $\hat{M}$, and the $G_i$ are random variables from the original communication protocol.

The protocol $\mathbf{rdp'}$ is exactly the same as $\mathbf{rdp}$ except that for each $j$ Bob, rather than Alice, does the compression for the $G_j$ on his side and just sends the $v_j$ bits of compressed data to Alice, who then uses her estimate $\hat{G}_j$ in place of $G_j$ in the remainder of the protocol. Consequently, at end of $\mathbf{rdp'}$ Alice sets her share $J$ of the randomness to $(\hat{M}, \hat{G})$, where $\hat{G} := (\hat{G}_b, \ldots, \hat{G}_1).

In the protocol $\mathbf{rdp}$, for $i \in \{1, 2, \ldots, b-1\}$ suppose that at the time when Alice receives $G_i$ she has, in addition to her record of $M$ and $G_{i-1}, \ldots, G_1$, quantum systems $A_i$ while Bob has quantum systems $B_i$. Starting from this time $t_i$, the overall process by which Bob produces $G_{i+1}$ given particular values $M = m, G_1 = g_1, \ldots, G_i = g_i$, may involve an arbitrary number of noisy channel and auxiliary forward information.
communication steps but it can be described as an instrument with elements

\[ \{ T_i( g_{i+1} | g_i, \ldots, g_1, m ) : g_{i+1} \} , \]

which are completely positive maps taking states of \( A_i B_i \) to states of \( A_{i+1} B_{i+1} \) whose sum is trace-preserving. Given that \( M = m, G_1 = g_1, \ldots, G_i = g_i \), if the state of \( A_i B_i \) at time \( t_i \) is \( \rho_{A_i B_i}^{(i)} \), then

\[
\Pr(G_{i+1} = g_{i+1} | M = m, G_1 = g_1, \ldots, G_i = g_i) = \text{Tr} T_i( g_{i+1} | g_i, \ldots, g_1, m ) \rho_{A_i B_i}^{(i)}
\]

and the state of \( A_{i+1} B_{i+1} \) at time \( t_{i+1} \), conditional on obtaining outcome \( G_{i+1} = g_{i+1} \) is

\[
T_i( g_{i+1} | g_i, \ldots, g_1, m ) \rho_{A_i B_i}^{(i)}
\]

Furthermore, denote by \( \hat{\rho}(m, g_1)_{A_i B_i} \) the density operator of \( A_i B_i \) at the time when Alice receives \( G_1 \), given that \( M = m \) and \( G_1 = g_1 \), multiplied by the probability \( \Pr(M = m, G_1 = g_1) \). For \( i \in \{1, \ldots, b\} \) let \( p_i( g_i | g_i, \ldots, g_1, m ) \) denote the probability that \( G_i = \hat{g}_i \) when \( G_i = g_i, \ldots, G_1 = g_1 \) and \( M = m \). Finally, let \( E(m | g_b, \ldots, g_1, m) \) denote the POVM element which gives the probability of outcome \( M = m \) as a function of the state of \( A_i B_i \) at the time when Alice receives the final back communication \( G_b \), given \( G_b = g_b, \ldots, G_1 = g_1 \) and \( M = m \). Then, in the protocol \( \text{rdp} \) we have

\[
\Pr(\hat{M} = \hat{m}, M = m, \hat{G} = \hat{g}, G = g \mid \text{rdp} ) = \text{Tr} E( m | g_b, \ldots, g_1, m ) \rho_{A_i B_i}^{(i)} \rho_{A_i B_i}^{(i)} \]

whereas in the protocol \( \text{rdp}' \)

\[
\Pr(\hat{M} = \hat{m}, M = m, \hat{G} = \hat{g}, G = g \mid \text{rdp}' ) = \text{Tr} E( m | g_b, \ldots, g_1, m ) \rho_{A_i B_i}^{(i)} \rho_{A_i B_i}^{(i)} \]

These two probabilities are equal whenever \( \hat{g} = g \). The sum of all these equalities is

\[
\Pr(\hat{M} = \hat{m}, M = m, \hat{G} = G \mid \text{rdp} ) = \Pr(\hat{M} = \hat{m}, M = m, \hat{G} = G \mid \text{rdp}' )
\]

and using this we find

\[
\Pr(J = K \mid \text{rdp}' ) = \Pr(\hat{M} = \hat{m}, \hat{G} = G \mid \text{rdp}' ) = \sum_{m, \hat{m}} \Pr(\hat{M} = \hat{m} | M = m, \hat{G} = G \mid \text{rdp}' ) \times \Pr(M = m, M = m, G = G \mid \text{rdp}' )
\]

\[
= \Pr(\hat{M} = \hat{m}, \hat{G} = G \mid \text{rdp}' ) = \Pr(\hat{M} = \hat{m} | M = m, \hat{G} = G \mid \text{rdp}' ) \times \Pr(\hat{M} = \hat{m}, M = m, G = G \mid \text{rdp}' )
\]

\[
\geq 1 - \epsilon_1 - (1 - (1 - \epsilon_1)^b).
\]

Using \([14], [13]\), Fano’s inequality, and

\[
\begin{align*}
H(\hat{M}, G) - H(G | M) & \geq H(\hat{M}, G) - H(\hat{M}, G | M) \\
& = H(M) - H(M | \hat{M}, G) \geq H(M) - H(M | \hat{M}),
\end{align*}
\]

the net rate of \( \text{rdp}' \) is

\[
\frac{1}{\ell} H(\hat{M}, G) - v - \sum_{i=1}^{b} \log | A_F |
\]

\[
\geq \frac{1}{n_0} (H(M, G) - H(M | M) - H(G | M) - \log | A_F | - (b + 1)\delta_1)
\]

\[
\geq \frac{1}{n_0} (H(M) - 2H(M | M) - (b + 1)\delta_1 - \log | A_F |)
\]

\[
\geq \frac{1}{n_0} (\log | A_M | - \log | A_F |) - 2\epsilon_1 c - \frac{1}{n_0} (2 + (b + 1)\delta_1)
\]

\[
= r_0 - 2\epsilon_1 c - \frac{2}{n_0} - \frac{b + 1}{n_0} \delta_1.
\]

We can now show that \( R_+(X) \geq C_+_{\epsilon}(X) \). Given any \( \epsilon > 0 \) and \( \delta > 0 \), for some sufficiently large \( n_0 \) we can choose a back-assisted communication protocol \( \text{cp} \) such that \( r_0 \geq C_+_{\epsilon}(X) - \frac{\epsilon}{2}, \epsilon_0 \leq \frac{\epsilon}{2} \), and \( 2/n_0 \leq \frac{\epsilon}{2} \). Fixing this \( \text{cp} \), there exists some \( \ell_0 \) such that for all \( \ell \geq \ell_0 \) we have \( \frac{\epsilon_1}{\ell} \delta_1 \leq \delta/4 \) and \( \epsilon_1 \) small enough that

\[
\Pr(J \neq K \mid \text{rdp}' ) \leq \epsilon.
\]

For each \( \ell \geq \ell_0 \) we have a RDP which makes \( n_0 \ell \) uses of the noisy channel, is \( \epsilon \)-good, and has net rate no less than \( r = C_+_{\epsilon}(X) - \delta \). To complete the proof we use an idea from \([6]\): Given any \( n \geq n_0 \ell_0 \) uses of the channel we may use the protocol which makes just \( n_0 \ell_0 \) uses of the channel, where \( \ell = \lceil n/n_0 \rceil \) and \( n = n_0 \ell_0 + q \) and achieve a rate of at least

\[
\frac{r n_0 \ell}{n_0 \ell_0 + q} \geq \frac{r n_0 \ell}{n_0 \ell_0} = \frac{r \ell}{\ell_0 + \ell}
\]

with error probability at most \( \epsilon \). Therefore, for any \( \epsilon > 0 \) and rate \( r < C_+_{\epsilon}(X) \) for all sufficiently large \( n \) there is an \( \epsilon \)-good back-assisted randomness distribution protocol which makes \( n \) uses of \( X \) and has rate no less than \( r \), which is to say \( R_+(X) \geq C_+_{\epsilon}(X) \). Almost exactly the same argument shows that \( R_{++}(X) \geq C_{++}(X) \).
IV. UNASSISTED AND FORWARD-ASSISTED CAPACITIES

In this section we prove Theorem 2 which says that for any operation $\mathcal{E}$, $\mathcal{C}(\mathcal{E}) = R(\mathcal{E}) = C_{\rightarrow}(\mathcal{E}) = R_{\rightarrow}(\mathcal{E})$. In light of the trivial inequalities \([1] \) and \([2]\) it is sufficient to prove that $R_{\rightarrow}(\mathcal{E}) \leq C(\mathcal{E})$.

Since Bob does not send anything back to Alice during a forward-assisted protocol, there is no loss of generality if Alice makes all $n$ uses of the noisy channel, sends all auxiliary classical communication, and produces $J$ (her part of the shared randomness) before Bob does anything, as illustrated in Figure 4.

Denote by $R$ all systems retained by Alice that she uses to produce her share of the common randomness. Let $X^n$ be the $n$ input systems, and $Y^n$ the $n$ output systems, for the $n$ uses of the operation $\mathcal{E}^\otimes n$. We introduce a register $Z$ which stores the value of the auxiliary forward communication, which can take one of $|A_Z|$ values. After Alice has made all her communication to Bob, the state of the $ZY^nR$ system is

$$\sigma_{ZY^nR} = \sum_z p(z) |z\rangle\langle z| \otimes (\mathcal{E}^\otimes n)^X \rightarrow Y^n \rho_{X^nR}^{(z)} (18)$$

where $\rho_{X^nR}^{(z)}$ is the state of the $X^nR$, conditioned on $Z = z$. Now, Alice performs a measurement (POVM) on the system $R$ to obtain her share $J$ of the common randomness, which is stored in register $J$. At this point the state of the system is

$$\tau_{ZY^n} = \sum_z q(j|z)p(z) |j\rangle\langle j| \otimes |z\rangle\langle z| \otimes (\mathcal{E}^\otimes n)^X \rightarrow Y^n \rho_{X^nR}^{(z,j)},$$

where, denoting by $E(j)_{R}$ the POVM element for the measurement outcome $J = j$,

$$q(j|z) = \text{Tr}_R E(j)_{R} \rho_{X^nR}^{(z)}$$
defines the states $\rho_{X^nR}^{(z,j)}$ and conditional distribution $q(j|z)$.

After this, Bob performs a measurement on the $ZY^n$ system to obtain his share of randomness $K$. We can bound the mutual information between the shares by

$$I(J : K) \leq I(J : Z^n)_\tau = I(J : Y^n)_\tau + I(J : Z^n | Y^n)_\tau$$
$$= I(J : Y^n)_\tau + H(Z) - I(J : Y^n)_\tau - H(Z | J, Y^n)_\tau \leq I(J : Y^n)_\tau + H(Z)_\tau \leq \chi(\mathcal{E}^\otimes n) + \log |A_Z| \leq \chi(\mathcal{E}^\otimes n) + \log |A_Z|$$

where (a) is data processing, (b) is because $\tau$ is separable with respect to the $Z : Y^n$ bipartition so $H(Z | J, Y^n) \geq 0$, and by positivity of mutual information, and (c) is because $I(J : Y^n)_\tau \leq \chi(\mathcal{E}^\otimes n)$. We use this to bound the net rate $r$ of the protocol thus

$$r = \frac{1}{n} (H(K) - \log |A_Z|) = \frac{1}{n} (I(J : K) + H(K | J) - \log |A_Z|) \leq \frac{1}{n} \chi(\mathcal{E}^\otimes n) + \log |A_Z| + H(K | J) - \log |A_Z|$$
$$\leq \frac{1}{n} \chi(\mathcal{E}^\otimes n) + \log |A_Z| + \log |A_Z| = \frac{1}{n} \chi(\mathcal{E}^\otimes n) + \log |A_Z| + H(K | J) - \log |A_Z| \leq \frac{1}{n} \chi(\mathcal{E}^\otimes n) + \log |A_Z| + H(K | J) - \log |A_Z|.$$

It follows that $R_{\rightarrow}(\mathcal{E}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \chi(\mathcal{E}^\otimes n) = C(\mathcal{E})$, where the equality is the Holevo-Schumacher-Westmoreland theorem \([10], [11]\).

V. MUTUAL INFORMATION UPPER BOUND

In this section we prove Theorem 3 which says that for any operation $\mathcal{E}$, $R_{\rightarrow}(\mathcal{E}) \leq I(\mathcal{E})$. Let us consider a protocol which makes $n$ uses of the channel $\mathcal{E}$ and $m$ auxiliary communication steps. For $k \in \{1, \ldots, n\}$, let $X_k$ denote the input system, and $Y_k$ the output system, for the $k$-th use of the noisy channel.

Initially, Alice and Bob have systems $A_0$ and $B_0$ which are uncorrelated in that $I(A_0 : B_0) = 0$. We may assume without loss of generality that any local randomness used in the protocol is already present in the state of these systems. We denote by $A_j$ Alice’s system, and by $B_j$ Bob’s system, immediately after step $j$. We may assume without loss of generality that at each step Alice and Bob have retained a full record of all auxiliary communication up to that step.

Suppose that at step $j$ of the protocol, Bob sends Alice $Z_k$ by auxiliary back communication. Then we may bound

$$I(A_j : B_j) \leq I(A_j - 1 Z_k) : B_j \leq I(A_j - 1 Z_k) : B_j$$
$$= H(Z_k | A_j - 1) + H(A_j - 1)$$
$$- H(A_j - 1 | B_j) - H(Z_k | A_j - 1 B_j)$$
$$\leq I(A_j - 1 : B_j) + H(Z_k | A_j - 1)$$
$$\leq I(A_j - 1 : B_j) + H(Z_k | Z^{(k-1)}$$

where (a) and (b) are data processing, (c) is because $Z_k A_j - 1 B_j - 1$ is in a separable state with respect to the partition between $Z_k$ and $A_j - 1 B_j - 1$ so $H(Z_k | A_j - 1 B_j - 1) \geq 0$, and (d) is because $A_j - 1$ includes $Z^{(k-1)} := (Z_1, \ldots, Z_{k-1})$. A similar argument establishes the same inequality when Alice sends Bob $Z_k$ by auxiliary forward communication, instead.

Now consider the case where Alice makes an input $X_k$ to the noisy channel $\mathcal{E}$ at step $j$, with Bob receiving output $Y_k$. Then

$$I(A_j : B_j) \leq I(A_j : Y_k) + I(A_j : B_j | Y_k)$$
$$= I(A_j : Y_k) + I(A_j Y_k : B_j - 1) - I(Y_k : B_j - 1)$$
$$\leq I(A_j : Y_k) + I(A_j Y_k : B_j - 1) - I(Y_k : B_j - 1)$$
$$\leq C(\mathcal{E}) + I(A_j - 1 : B_j - 1).$$

Here, (a) and (c) are by data processing, (b) is positivity of mutual information, and (d) is by the result of Bennett, Shor, Smolin and Thapliyal.

Recall that $Z := Z^{(m)}$ is the total record of auxiliary communication. Starting with $I(A_{n+m} : B_{n+m})$, and repeatedly invoking the inequality (19) or (20) depending on the type of
step, we obtain

\[ I(A_{n+m} : B_{n+m}) \leq I(B_0 : A_0) + nC(E) \]
\[ + \sum_{k=1}^{m} H(Z_k|Z^{(k-1)}) \]
\[ = nC(E) + H(Z) \]
\[ \leq nC(E) + \log |A_Z|, \]

where the equality is by the chain rule and \( I(B_0 : A_0) = 0 \).

Finally, we bound the net rate \( R \) of the protocol by

\[ R = \frac{1}{n} (H(K) - \log |A_Z|) \]
\[ = \frac{1}{n} (I(K : J) + H(K|J) - \log |A_Z|) \]
\[ \leq \frac{1}{n} (I(A_{n+m} : B_{n+m}) + H(K|J) - \log |A_Z|) \]
\[ \leq \frac{1}{n} (nI(E) + \log |A_Z| + n\epsilon + 1 - \log |A_Z|) \]
\[ = I(E) + \epsilon + 1/n \]

where (a) is data processing, (b) is by inequalities (21) and (8). Recalling the definition of \( R_{++} \), we have established that

\[ R_{++}(E) \leq I(E). \]

(22)

VI. QUANTUM SEPARATIONS

In this section, we give examples of quantum channels where the feedback or two-way assisted randomness distribution capacity is strictly greater than the corresponding capacity for communication.

A. Quantum-classical channels: separation \( C_{++}(E) < R_{++}(E) \)

Here we prove Theorem 5, which says that for any quantum-classical \( E \), \( R_{++}(E) = R_{++}(E) = I(E) \). For any \( E \), \( R_{++}(E) \leq R_{++}(E) \) and Theorem 3 tells us \( R_{++}(E) = I(E) \), so it remains to show that \( I(E) \leq R_{++}(E) \) when \( E \) is qc.

Any qc \( E^{X \rightarrow Y} \) can be written

\[ E^{X \rightarrow Y} : \rho_X \mapsto \sum_{y \in A_Y} |y \rangle \langle y| \psi_{RX}^{(y)} \]  

where \( \{E(y)_X : y \in A_Y\} \) is a POVM on \( X \). If Alice locally prepares a state \( \psi_{RX} \) and applies one use of the channel to \( X \) then the density operator for \( R_Y \) is

\[ \rho_{R_Y} := \sum_y p(y) \rho(y)_R \otimes |y \rangle \langle y|_{Y} = E^{X \rightarrow Y} \psi_{RX}. \]

(24)

where \( p(y) := \text{tr}_{RX} E(y)_X \psi_{RX} \) and \( \rho(y)_R := \text{tr}_X E(y)_X \psi_{RX}/p(y) \). If Alice does this for \( i \in \{1, \ldots, n\} \) with systems \( R_i X_i \) (isomorphic to \( R_X \) then the density operator for \( R_1 Y_1 \cdots R_n Y_n \) will be \( \otimes_{i=1}^{n} \rho_{R_i Y_i} \), where Bob holds the systems \( Y_i \) and Alice the systems \( R_i \).

This density operator represents the situation where each \( Y_i \) stores a random variable \( Y_i \) taking values in \( A_{Y_i} \) and the \( Y_i \) are distributed identically and independently according to the distribution \( p \) and conditional on \( Y_i = y_i \), the density operator for system \( R_i \) is \( \rho(y_i)_{R_i} \). Let \( Y^{(n)} := (Y_1, \ldots, Y_n) \). In the “coding” part of the proof of the classical-quantum Slepian-Wolf theorem of Devetak and Winter [5] it was shown that, for any \( 0 < \epsilon < 1/2 \) and \( \delta > 0 \), and all sufficiently large \( n \), we can find \( |A_{Z_n}| \) disjoint subsets \( \{C_z : z \in A_{Z_n}\} \) of \( A_{Y} \) such that

(i) the probability that \( Y^{(n)} \) fails to belong to one of the \( C_z \) is no more than \( 2\epsilon \),

(ii) given the knowledge that \( Y^{(n)} \in C_z \), Alice can perform a measurement with POVM \( E^{(z)} \) on \( R_1 \cdots R_n \) which produces an estimate \( Y^{(n)} \) of \( Y^{(n)} \) such that \( \Pr(Y^{(n)} \neq Y^{(n)'} \leq \epsilon \).

(iii) \( \frac{1}{n} \log |A_{Z_n}| \leq H(Y|R) + \delta \).

This suggests a back-assisted RDP whereby Bob takes \( K = Y^{(n)} \) as his share of the common randomness; Bob sends Alice \( Z \), such that the subset \( C_Z \) contains \( Y^{n} \), if such a subset exists and if not, he sends some arbitrary value from \( A_{Z} \). On receiving \( Z \), Alice measures \( E^{(Z)} \) on \( R_1 \cdots R_n \) to obtain an estimate \( J \) of \( Y^{(n)} \).

This protocol has \( \Pr(K \neq J) \leq 3\epsilon \) and, since \( H(K) = nH(Y) \), net rate

\[ \frac{1}{n}(H(K) - \log |A_Z|) \geq H(Y) - H(Y|R) - \delta \]
\[ = I(Y : R) - \delta, \]

so, by optimising over the choice of \( \psi_{XR} \) in the protocol, we
have established the inequality
\[ R_{\rightarrow}(\mathcal{E}) \geq \max_{\psi_{\mathbb{X}}} I(\mathcal{Y} : \mathcal{R})_{\psi_{\mathbb{X}}} = I(\mathcal{E}), \] (25)
which we needed to complete the proof.

B. Communication capacities of entanglement-breaking channels

Here we prove Proposition 6. We already established that 
\[ C(\mathcal{E}) = C_{\rightarrow}(\mathcal{E}) \] in Section 4. Now, note that we can write
\[ C_{\rightarrow}(\mathcal{E}) = \lim_{m \to \infty} \left\{ C_{\rightarrow}(\mathcal{E} \otimes \mathcal{A}_m) - \log m \right\} \]
where \( \mathcal{A}_m \) is a classical identity channel with \( m \) input symbols. Since \( \mathcal{E} \) and \( \mathcal{A}_m \) are both entanglement-breaking, we have
\[ C_{\rightarrow}(\mathcal{E} \otimes \mathcal{A}_m) = C(\mathcal{E} \otimes \mathcal{A}_m) = C(\mathcal{E}) + C(\mathcal{A}_m) = C(\mathcal{E}) + \log m \]
by Bowen-Nagarajan [3], the HSW theorem [10, 11], and the fact that the Holevo information is additive for entanglement breaking channels [13]. Therefore,
\[ C_{\rightarrow}(\mathcal{E}) = C_{\rightarrow}(\mathcal{E}) = C(\mathcal{E}) \]
for entanglement-breaking \( \mathcal{E} \).

C. Family of examples

Quantum-classical channels are entanglement breaking. It was shown by Bowen and Nagarajan [3] that classical feedback cannot increase the classical capacity of entanglement breaking channels, so we have \( C_{\rightarrow}(\mathcal{E}) = C(\mathcal{E}) \). Meanwhile, in [4], Holevo has given examples of quantum-classical channels with \( I(\mathcal{E}) > C(\mathcal{E}) \). By Theorem 5 and Bowen-Nagarajan, these channels also exhibit a separation \( R_{\rightarrow}(\mathcal{E}) > C_{\rightarrow}(\mathcal{E}) \). To be more specific, consider the case where the POVM elements determining \( \mathcal{E} \) are rank-one projectors onto pair-wise linearly independent subspaces. Then \( C(\mathcal{E}) \leq C(\mathcal{E}) = \log d \), and Holevo shows that the inequality is strict unless the the POVM is an orthonormal basis measurement [4].

D. Specific example

Finally, we construct the quantum-classical operation \( \mathcal{F} \), of Proposition 7 which has \( R_{\rightarrow}(\mathcal{F}) = \log(d) \) while \( C_{\rightarrow}(\mathcal{F}) = C(\mathcal{F}) = \chi(\mathcal{F}) = \frac{1}{2} \log d \).

Given two rank-1 projective measurements \( E^{(0)} \) and \( E^{(1)} \) with outcomes in \( \{1, \ldots, d\} \) on a \( d \)-dimensional system \( \mathcal{X} \), we may construct a quantum-classical operation \( \mathcal{F} \) whose input system is \( \mathcal{X} \) and whose output system \( \mathcal{Y} \) encodes a pair \( Y = (G, M) \) where \( G \) is a bit chosen uniformly at random, and \( M \) is the result of performing the measurement \( E^{(G)} \) on \( \mathcal{X} \). That is, \( G \) indicates which basis was measured and \( M \) is the result of that measurement. For our purposes, there is no loss of generality in taking \( E^{(0)} \) to be the computational basis measurement. Since the POVM corresponding to this classical-quantum operation has rank-one elements we already know that
\[ R_{\rightarrow}(\mathcal{F}) = \log(d). \] (26)

In Figure 5 we illustrate a protocol which distributes \( 1 + \log d \) bits of perfectly correlated randomness with one use of \( \mathcal{F} \) and a single bit of communication from Bob to Alice, thus attaining a net rate of \( \log d \) bits per channel use.

On the other hand, if \( E^{(1)} \) is chosen so that the two measurement bases are mutually unbiased, then \( C_{\rightarrow}(\mathcal{F}) = C(\mathcal{F}) = \chi(\mathcal{F}) = \frac{1}{2} \log d \). The first two equalities are because the channel is entanglement breaking. It remains to compute the Holevo information \( \chi(\mathcal{F}) \) by maximising
\[ H(\mathcal{M}, \mathcal{G})_\rho = \sum_w p(w) H(\mathcal{M}, \mathcal{G})_{\psi(w)} \] (27)
where \( \rho = \sum_k p(w) \psi(w) \) over all ensembles \( \{(p(w), \psi(w)) : w = 1, \ldots, k\} \). For any density operator \( \rho \) we have the trivial upper-bound
\[ H(\mathcal{M}, \mathcal{G})_\rho \leq 1 + \log d, \] (28)
which holds with equality when
\[ p(w) = 1/k, \quad \psi(w) = |w\rangle \langle w|. \] (29)
Using the chain rule and \( \Pr(G = 0) = 1/2 \) we have, for any density operator \( \psi \),
\[ H(\mathcal{M}, \mathcal{G})_\psi = 1 + \frac{1}{2} [H(\mathcal{M}|G = 0)_\psi + H(\mathcal{M}|G = 1)_\psi] \]
Since the bases are mutually unbiased, Maassen and Uffink’s entropic uncertainty relation [17] tells us that
\[ \frac{1}{2} [H(\mathcal{M}|G = 0)_\psi + H(\mathcal{M}|G = 1)_\psi] \geq \log d. \]
Therefore,
\[ H(\mathcal{G}, \mathcal{M})_{\psi(w)} \geq 1 + \frac{1}{2} \log d \] (30)
which is also an equality for the ensemble [29]. Combining the bounds (28) and (30) (and equality conditions) with (27), we have
\[ \chi(\mathcal{F}) = \frac{1}{2} \log(d). \]

VII. CONCLUSION

Despite being, a priori, different things, we have seen that the capacity for a classical-quantum channel with various kinds of classical assistance to distribute shared randomness and to send information are the same. For these channels, the optimal way of distributing randomness is to generate it locally and communicate it through the channel, and we don’t benefit from using the noisy channel as a source of randomness.

For quantum channels, we have shown that the mutual information capacity \( I(\mathcal{E}) \) is a general upper bound for \( R_{\rightarrow}(\mathcal{E}) \) and that this bound can be achieved using only back-communication for quantum-classical channels. Using this result we have established that strict separations \( C_{\rightarrow}(\mathcal{E}) < R_{\rightarrow}(\mathcal{E}) \) are possible for quantum-classical channels and gave an explicit example for which \( R_{\rightarrow}(\mathcal{E}) = \log(d) \) while \( C_{\rightarrow}(\mathcal{E}) = \frac{1}{2} \log(d) \). In these cases, back-communication is allowing us to extract additional randomness from the channel, resulting in a net gain in the amount of shared randomness generated.
Fig. 5. Sharing $1 + \log d$ bits of perfect randomness with one use of the channel $F$ (the contents of the dashed rectangle) and one bit of back communication: Alice locally prepares a maximally entangled state $\phi_{RX}$ and inputs $X$ to the channel. We can view the channel as performing a unitary controlled by the bit $G$ and then performing a computational basis measurement to yield $M$. Alice sets $Z = G$ and sends $Z$ to Bob, who performs $\bar{U}$ (the complex conjugate of $U$) iff $Z = 1$ and then performs a computational basis measurement on $R$ to yield a value $\hat{M}$. By the $U \otimes \bar{U}$ invariance of $\phi$, $\hat{M} = M$ with probability one, so if Alice sets $J = (\hat{M}, Z)$ and Bob sets $K = (M, G)$ then $\Pr(K = J) = 1$, and $K$ is uniformly distributed. Local operations are surrounded by dotted lines.

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