Bayesian AirComp with Sign-Alignment Precoding for Wireless Federated Learning

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Abstract—In this paper, we consider the problem of wireless federated learning based on sign stochastic gradient descent (signSGD) algorithm via a multiple access channel. When sending locally computed gradient's sign information, each mobile device requires to apply precoding to circumvent wireless fading effects. In practice, however, acquiring perfect knowledge of channel state information (CSI) at all mobile devices is infeasible. In this paper, we present a simple yet effective precoding method with limited channel knowledge, called sign-alignment precoding. The idea of sign-alignment precoding is to protect sign-flipping errors from wireless fadeings. Under the Gaussian prior assumption on the local gradients, we also derive the mean squared error (MSE)-optimal aggregation function called Bayesian over-the-air computation (BayAirComp). Our key finding is that one-bit precoding with BayAirComp aggregation can provide a better learning performance than the existing precoding method even using perfect CSI with AirComp aggregation.

I. INTRODUCTION

Federated learning (FL) is a class of distributed machine learning technique using locally generated heterogenous datasets at mobile devices. Communicating between mobile devices and a central server, it can train a model accurately, while maintaining the privacy of data present in mobile devices [1], [2]. Federated averaging (FedAvg) and federated stochastic gradient descent (FedSGD) are the representative algorithms for FL. In FedSGD, mobile devices send locally computed gradient information to the server, and the server aggregates the local gradients to update the global model parameters. To improve learning efficiency for FL, the variations of FedAvg and FedSGD have been proposed in [3]–[6].

Over-the-air computation (AirComp)-based FL has been recently proposed as a communication-bandwidth efficient aggregation method [7]–[12]. Using the superposition property of wireless medium, AirComp performs wireless analog aggregation of the local gradients on the fly. This approach can attain low-latency learning performance compared with the case of using orthogonal access techniques when implementing FedSGD in a wireless setup. In addition, AirComp enhances the security of individual data because it makes difficult to estimate individual local gradient information. In AirComp, precoding for aligning the local gradients is essential to circumvent heterogenous channel fading effects across mobile devices [7]–[12]. Several precoding strategies have been presented, including truncated-channel inversion precoding [7] and dithering-based precoding [13]. The underlying idea of the precoding strategies is to perform pre-equalization to mitigate fading effects; thereby, the server can receive a superposition of aligned local gradients on the fly. These alignment precoding methods, however, can be challenging to implement in wireless FL systems. Mobile devices located far from the server may be infeasible to consistently apply channel-inversion-based precoding, while satisfying the power constraint. Besides, it is challenging to acquire perfect channel state information (CSI) for uplink communications in practice.

In this paper, we consider a sign stochastic gradient descent (signSGD) algorithm [14] for FL over a shared wireless multiple access channel. Sign-SGD is a communication-efficient distributed learning algorithm. This algorithm can reduce the communication cost because it exploits the sign information of local gradients when updating the model. Besides, it can be implementable using simple binary digital modulated transmission techniques in wireless FL settings [8]. Each mobile device performs one-bit quantization of the locally computed gradient in every communication round to minimize uplink communication cost. Then, it transmits the sign of local gradient along with precoding to mitigate channel fadeings using a shared time-frequency resource. Then, the server receives a superposition of precoded local gradient signs. Using this received gradient information, the server updates the model parameters and shares them with the mobile devices for the next round iteration.

Our main contribution is to propose novel precoding called sign-alignment precoding. The idea of our precoding is to align the sign of the channel fading coefficient to avoid gradients’ sign-flipping errors by fadeings. This precoding requires one-bit CSI at transmitter (CSIT) information; thereby, it can significantly reduce the channel acquisition and feedback overheads for wireless FL compared to the conventional FL algorithm using channel-inversion based precoding, which requires full CSIT at mobile devices. We also present a novel Bayesian aggregation method for AirComp, referred to as BayAirComp. Inspired by our prior work in [15], the key idea of BayAirComp is to map the received signal to the estimate of the sum of local gradients to minimize the mean squared error (MSE) by harnessing the knowledge of prior distributions of local gradients as side-information. We present experimental results to show that sign-alignment precoding with BayAirComp can outperform the state-of-the-art one-bit broadband digital aggregation (OBDA) algorithm [8].
II. SYSTEM MODEL

In this section, we describe learning and communication models for a wireless FL system. The wireless FL system consists of \( K \) mobile devices and a server (or a base station) as depicted in Fig. 1. The server trains a neural network with a large number of hyper-parameters \( w \in \mathbb{R}^M \) by communicating with \( K \) mobile devices through a shared wireless channel.

A. Loss function

Let \( z_k^i \in \mathbb{R}^d \) and \( r_k^i \in \mathbb{R} \) be the \( i \)th pair of the training data example stored at mobile device \( k \in [K] \). Assuming, device \( k \) has \( N_k \) training examples, we define a set of training examples stored at device \( k \in [K] \) as \( \mathcal{D}_k = \{ z_k^i, r_k^i \}_{i=1}^{N_k} \). Therefore, a total number of training examples for learning becomes \( N = \sum_{k=1}^{K} N_k \). Given model parameter \( w \in \mathbb{R}^M \), we define a loss function with training pair \( (z_k^i, r_k^i) \) as \( \ell (z_k^i, r_k^i; w) : \mathbb{R}^M \times \mathbb{R} \rightarrow \mathbb{R} \). This loss can be either a cross entropy or a mean-squared error function according to machine learning applications. Using the sample average, the local loss function of device \( k \) is defined as

\[
    f_k (w) = \frac{1}{N_k} \sum_{i=1}^{N_k} \ell (z_k^i, r_k^i; w).
\]

Summing \( f_k (w) \) with weight \( \frac{N_k}{N} \) for \( k \in [K] \), the global loss function is given by

\[
    F (w) = \frac{1}{N} \sum_{k=1}^{K} \frac{N_k}{N} f_k (w) = \frac{1}{N} \sum_{k=1}^{K} \frac{N_k}{N} f_k (w) = \sum_{k=1}^{K} \frac{N_k}{N} f_k (w).
\]

B. Wireless federated learning via singSGD

The wireless FL system iteratively optimizes model parameter \( w \) over \( T \) communication rounds. Each communication round comprises 1) gradient computation and compression, 2) uplink transmissions, 3) model update, and 4) downlink transmission.

1) Gradient computation and compression: In communication round \( t \in [T] \), mobile device \( k \in [K] \) first computes local gradient information. Let \( g_k^t \triangleq \nabla f_k (w^t) \) be the local gradient evaluated using model knowledge \( w^t \) and local data set \( \mathcal{D}_k \) for \( k \in [K] \). Then, it compresses local gradient using one-bit quantizer to diminish the uplink communication cost as

\[
    \hat{g}_k^t = \text{sign} (g_k^t),
\]

where \( \text{sign} (x) = 1 \) for \( x \geq 0 \) and \( \text{sign} (x) = -1 \) otherwise.

2) Uplink communications with precoding via a MAC: After the compression, mobile device \( k \) transmits binary vector \( \hat{g}_k^t \) along with the precoding coefficient \( v_k^t \in \mathbb{R} \).

\[
    x_k^t = v_k^t \text{sign} (\hat{g}_k^t),
\]

where the precoder \( v_k^t \) satisfies the average power constraint as

\[
    \mathbb{E} \left[ |v_k^t|^2 \right] \leq P.
\]

3) Model update: In communication round \( t \in [T] \), the server performs the update of the model parameter using the gradient descent algorithm [1]. To perform the gradient descent algorithm, the server requires to estimate the sum of local gradients from the received signal \( y^t \), which is a noisy version of the sum of faded local gradients. Let \( U (\cdot) : \mathbb{R}^M \rightarrow \mathbb{R} \) be the sum gradient estimator. Then, the MSE-optimal gradient estimator is defined as

\[
    f^* (y^t) = \arg \min_{U (\cdot) : \mathbb{R}^M \rightarrow \mathbb{R}} \mathbb{E} \left[ \left\| \sum_{k=1}^{K} g_k^t - U (y^t) \right\|_2^2 \right].
\]

Using this MSE-optimal estimator, the server updates the model parameter with learning rate \( \gamma^t \in \mathbb{R}^+ \) at communication round \( t \) for the next round iteration:

\[
    w^{t+1} = w^t - \gamma^t f^* (y^t).
\]

4) Downlink communication: Using the broadcast nature of the wireless medium, the server multicasts the updated model parameter \( w^{t+1} \) to mobile devices using a shared...
downlink channel. We assume that all mobile devices can perfectly decode the updated model parameters over entire communication rounds for ease of exposition.

III. BayAirComp with Sign-Alignment Precoding

In this section, we present a novel wireless federated learning algorithm. The key idea of the proposed algorithm entails two operations: 1) sign-alignment precoding in the uplink transmission and 2) the Bayesian AirComp aggregation in the reception.

A. Local Gradient Parameter Estimation and Compression

To implement the Bayesian aggregation in [15], the server requires to know the prior distribution of local gradients \( g^t_k \in \mathbb{R}^M \). Unfortunately, it is infeasible to characterize the exact prior distribution of \( g^t_k \) because it depends on both the local data distribution and deep neural network structures. Instead, we model the prior distribution of \( g^t_k \) as Gaussian with the moment matching technique [15]. Specifically, let \( g^t_{k,m} \) be the \( m \)th entry of \( g^t_k \). We model that \( g^t_{k,m} \) follows IID Gaussian with mean \( \mu^t_k \) and variance \( (\nu^t_k)^2 \), i.e., \( g^t_{k,m} \sim \mathcal{N}(\mu^t_k, (\nu^t_k)^2) \). The mean and variance are estimated by taking the sample average estimator as

\[
\mu^t_k = \frac{1}{M} \sum_{m=1}^{M} g^t_{k,m} \quad \text{and} \quad (\nu^t_k)^2 = \frac{1}{M} \sum_{m=1}^{M} (g^t_{k,m} - \mu^t_k)^2.
\]

Although this Gaussian approximation on the prior distribution is not exact, it not only allows the Bayesian aggregation computationally tractable but also improves learning performance when training CNNs using MNIST datasets [15] in an orthogonalized multiple access channel environment. After computing the moments, each device normalizes the local gradient by subtracting its mean:

\[
\tilde{g}^t_{k,m} = g^t_{k,m} - \mu^t_k.
\]

Afterwards, mobile device \( k \) compresses the local gradient to diminish the uplink communication cost by using one-bit quantizer as

\[
\hat{g}^t_k = \text{sign}(\tilde{g}^t_k).
\]

Mobile device \( k \in [K] \) sends \( \hat{g}^t_k \in \{-1, +1\}^M \) with \( \mu^t_k \in \mathbb{R}^+ \) and \( \nu^t_k \in \mathbb{R}^+ \) to the server.

B. Sign-alignment precoding

When sending compressed local gradient, \( \hat{g}^t_k \), device \( k \) requires to use precoder to compensate for the effect of wireless fading \( h^t_k \). Unlike the prior approaches to invert the fading coefficient for precoding, we take a novel precoding strategy that requires one-bit CSI feedback from the BS. Our proposed precoding strategy is to align the signs of local gradients using one-bit precoding, i.e., \( \nu^t_k = \text{sign}(h^t_k) \) as

\[
x_k = \text{sign}(h^t_k) \hat{g}^t_k.
\]

The received signal at the BS becomes

\[
y^t = \sum_{k=1}^{K} |h^t_k| \hat{g}^t_k + n^t.
\]

This precoding strategy ensures to align the signs of local gradients. This sign alignment effect helps to estimate the sum-gradient accurately by avoiding the sign flipping errors due to the wireless channel fades. In addition, this precoding strategy requires only one-bit CSI overhead compared to the conventional precoding system which needs \( 6 \sim 12 \) bits for full CSI. Therefore, high efficiency can be obtained through this precoding strategy.

C. Bayesian AirComp

Using received signal \( y^t \), the BS requires estimating the sum of local gradients \( \sum_{k=1}^{K} g^t_k \) to accomplish the model update via a stochastic gradient descent algorithm. We present a novel aggregation method called BayAirComp. The key idea of BayAirComp is to estimate the sum of local gradients \( \sum_{k=1}^{K} g^t_k \) by jointly exploiting the knowledge of the prior distribution of \( g^t_k \) (i.e., \( \mu^t_k \) and \( \nu^t_k \), one-bit quantizer, and fading coefficient \( h^t_k \) to minimize the MSE. The aggregation function for BayAirComp \( f_{\text{BayAirComp}}(y^t_m) : \mathbb{R} \rightarrow \mathbb{R} \) aims to minimize MSE, i.e.,

\[
f_{\text{BayAirComp}}(y^t_m) = \arg \min_{f : \mathbb{R} \rightarrow \mathbb{R}} \mathbb{E} \left[ \left( f(y^t_m) - \sum_{k=1}^{K} g^t_{k,m} \right)^2 \right].
\]

The following theorem suggests the aggregation function for BayAirComp in closed form.

**Theorem 1.** Let \( h^t = [|h^t_1|, |h^t_2|, \ldots, |h^t_K|] \in \mathbb{R}^K_+ \) be the sign-aligned channel vector at communication round \( t \). We also denote \( B_K = \{-1, 1\}^K \) be a set with \( 2^K \) elements of binary vectors with length \( K \), i.e., \( b \in B_K \). We also define a subset \( B_{K,k} \subset B_K \) that contains binary vectors whose \( k \)th
The component is fixed to one. Then, the BayAirComp aggregation function is given by (9).

**Proof.** From the Bayesian principle, the MSE-optimal aggregation function is obtained by computing the conditional expectation as

\[
\hat{g}_{k,m} = \arg \min_{f: \mathbb{R} \to \mathbb{R}} \mathbb{E} \left[ \left( f(y_m) - \sum_{k=1}^{K} \bar{g}_{k,m} \right)^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{K} \bar{g}_{k,m} \mid y_m \right],
\]

where the last equality is the linearity of the expectation. The conditional expectation in (16) is computed as

\[
\mathbb{E} \left[ \bar{g}_{k,m} \mid y_m \right] = \frac{\int_{-\infty}^{\infty} \bar{g}_{k,m} P\left(y_m \mid \bar{g}_{k,m}\right) P\left(\bar{g}_{k,m}\right) \, d\bar{g}_{k,m}}{\int_{-\infty}^{\infty} P\left(y_m \mid \bar{g}_{k,m}\right) P\left(\bar{g}_{k,m}\right) \, d\bar{g}_{k,m}}.
\]

We define \( \bar{g}_{m} = [\bar{g}_{1,m}, \bar{g}_{2,m}, \ldots, \bar{g}_{K,m}] \). Applying the one-bit precoding for the sign alignment, the channel likelihood distribution is

\[
P\left(y_m \mid \bar{g}_{m,1}, \ldots, \bar{g}_{m,K}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_m - \sum_{k=1}^{K} \bar{g}_{k,m} \mathbb{I}(\bar{g}_{k,m}) \cdot h_k)^2}{2\sigma^2}}.
\]

To obtain \( P\left(y_m \mid \bar{g}_{m,k}\right) \), we need to marginalize (18) with respect to \( \bar{g}_{m,1}, \ldots, \bar{g}_{m,k-1}, \bar{g}_{m,k+1}, \ldots, \bar{g}_{m,K} \), where \( \bar{g}_{k,m} \sim \mathcal{N}\left(0, (\nu_k^2)^2\right) \). Then, the marginal distribution \( P\left(y_m \mid \bar{g}_{m,k}\right) \), we compute the numerator in (17) as

\[
\int_{-\infty}^{\infty} \bar{g}_{k,m} \mathbb{I}(\bar{g}_{k,m}) \mathbb{I}(\bar{g}_{k,m} \bar{g}) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_m - \sum_{j=1}^{K} \bar{g}_{j,m} \mathbb{I}(\bar{g}_{j,m}) \cdot h_j)^2}{2\sigma^2}\right] \, d\bar{g}_{k,m}.
\]

The estimated gradient of the \( k \)th device is derived by substituting (19) and (20) for (17), which arrives at the expression in (9).

It is instructive to consider special cases for a better understanding of the proposed aggregation function for BayAirComp.

**Example:** Suppose \( K = 5 \). We first assume that all channel fading coefficients are identical in the magnitude \( |h_k| = 1 \) for \( k \in [K] \). In this case, as depicted in Fig. 2 (left-side), the aggregation function becomes a uniform soft-step function with maximum and minimum values of \( \pm \sqrt{2} \). As SNR increases, the soft-step function tends to be sharp. In a heterogeneous fading environment, \( |h_k| = 5 \) and \( |h_k| = 1 \) for \( k \in \{2, 3, 4, 5\} \), the proposed aggregation function plays a role of a non-uniform quantizer as illustrated in depicted in Fig. 2 (right-side). As depicted in Fig. 2(a), our proposed BayAirComp clearly differs from the majority-voting based aggregation function. The proposed BayAirComp provides the magnitude information of \( \bar{g}_{k,m} \) in a quantized manner. Whereas, the majority-voting based aggregation keeps the sign of \( \bar{g}_{k,m} \).
Remark (Implementation): To implement BayAirComp, mobile device $k \in [K]$ requires to additionally send $\mu_k^t \in \mathbb{R}$ and $\nu_k^t \in \mathbb{R}^+$ to the server per communication round. As shown in our prior work [15], this information can be quantized with $B$-bit scalar quantizer and be transmitted to the server using orthogonal resources. Since this additional information bits are much smaller than the model size $M \sim 10^6$, i.e., $2B \ll M$, the additional overheads can be negligible.

IV. PERFORMANCE ANALYSIS

In this section, we provide the convergence analysis of the proposed FL algorithm in this paper. The analysis procedure is carried out in two steps. First, obtain the MSE bound between the true gradient and the gradient estimate by the aggregation function, and then show the gradient of the SGD-based FL algorithm converges to zero. The convergence proof is under the assumption that the global loss function $F(x)$ is $L$-Lipschitz smooth and has the least value in $x^*$. For ease of expression, let define $g^t_{true} = \nabla F(w^t) = \frac{1}{K} \sum_{k=1}^{K} g^t_k$.

A. MSE Bound

Theorem 2. Let $g^t_{k,m}$ be an IID Gaussian with mean $\mu_k^t$ and variance $(\nu_k^t)^2$, i.e., $g^t_{k,m} \sim \mathcal{N}(\mu_k^t, (\nu_k^t)^2)$, for $k \in [K]$ and $m \in [M]$. For the error $e^t = f_{BayAirComp}(y^t) - g^t_{true}$, the MSE bound $\sigma^2_{MSE}$ can be expressed as

$$\sigma^2_{MSE} = \frac{M}{K^2} \left( 1 + \frac{2}{\pi} \right) \sum_{k=1}^{K} (\nu_k^t)^2. \quad (21)$$

Proof. To reduce the complexity of expressing formulas, we simplify (9) as

$$f_{BayAirComp}(y^t_m) = \frac{1}{K} \sum_{k=1}^{K} \left[ \mu_k^t + \nu_k^t \sqrt{\frac{2}{\pi}} A_k(y^t_m) \right]. \quad (22)$$

Putting the aggregation function into the gradient error definition, we can obtain the formula as

$$\frac{1}{K} \sum_{k=1}^{K} \left[ \mu_k^t + \nu_k^t \sqrt{\frac{2}{\pi}} A_k(y^t_m) \right] = \frac{1}{K} \sum_{k=1}^{K} g^t_{k,m} + e^t_m$$

$$= \frac{1}{K} \sum_{k=1}^{K} g^t_{k,m} + e^t_{k,m}$$

$$= \frac{1}{K} \sum_{k=1}^{K} g^t_{k,m} + e^t_{k,m}, \quad (23)$$

where (23) is for the $m$th component of gradient, and $e^t_{k,m}$ is the error for the $k$th device in $e^t_m$. Then, we compute the MSE bound as follows.

$$E \left[ \|e^t\|_2^2 \right] = E \left[ \sum_{m=1}^{M} |e^t_{m}|^2 \right]$$

$$= \sum_{m=1}^{M} E \left[ |e^t_{m}|^2 \right]$$

$$= \sum_{m=1}^{M} E \left[ \left( \frac{1}{K} \sum_{k=1}^{K} e^t_{k,m} \right)^2 \right]$$

$$\leq \sum_{k=1}^{K} E \left[ \frac{1}{K^2} \sum_{k=1}^{K} |e^t_{k,m}|^2 \right]$$

$$= \frac{1}{K^2} \sum_{m=1}^{M} \sum_{k=1}^{K} E \left[ |e^t_{k,m}|^2 \right].$$

The inequality in (24) is reasonable according to the convexity. Using the assumption that the local gradients $g^t_{k,m}$ and $g^t_{k,m}$ is IID Gaussian, it is possible to compute the upper bound of MSE as below.

$$E \left[ \|e^t\|_2^2 \right]$$

$$\leq \frac{1}{K^2} \sum_{m=1}^{M} \sum_{k=1}^{K} E \left[ g^t_{k,m} - \mu_k^t - \nu_k^t \sqrt{\frac{2}{\pi}} A_k(y^t_m) \right]^2$$

$$= \frac{1}{K^2} \sum_{m=1}^{M} \sum_{k=1}^{K} E \left[ g^t_{k,m} - \mu_k^t - \nu_k^t \sqrt{\frac{2}{\pi}} A_k(y^t_m) \right]$$

$$\leq \frac{1}{K^2} \sum_{m=1}^{M} \sum_{k=1}^{K} E \left[ Var(g^t_{k,m}) + \frac{2}{\pi} (\nu_k^t)^2 \right]$$

$$= \frac{1}{K^2} \sum_{m=1}^{M} \sum_{k=1}^{K} \left( 1 + \frac{2}{\pi} \right) (\nu_k^t)^2.$$ (25)

The inequality in (25) is due to the property that

$$E_X [(X - a)^2] = \sigma_X^2 + (\mu_X - a)^2$$

$$\leq \sigma_X^2 + \max_a (\mu_X - a)^2,$$

and $-1 < A_k(y^t_m) < 1$ regardless of $y^t_m$. Therefore the upper bound of MSE $\sigma^2_{MSE}$ in (21) can be achieved.

B. Convergence Analysis

Theorem 3. For the L-Lipschitz smooth loss function $F(x)$, the proposed FL algorithm with the learning rate $\gamma$ for $\gamma > 0$ satisfies

$$E \left[ \frac{1}{T} \sum_{t=0}^{T} \|g^t_{true}\|_2^2 \right]$$

$$\leq \frac{1}{\sqrt{T}} \left[ \frac{F(w^0) - F(w^*)}{\gamma (1 - \frac{L\gamma}{T})} + \sigma^2_{MSE} (1 + \ln T) \frac{L^2}{1 - \frac{L\gamma}{T}} \right].$$ (27)
Proof. The proposed FL algorithm is based on GD, and the model parameter update formula is given as
\[
w^{t+1} = w^t - \gamma^t f_{\text{BayAirComp}} (y^t).
\]
Since the loss function is \(L\)-smooth, the convergence formula can be derived as
\[
F (w^{t+1}) \\
\leq F (w^t) + (g_{\text{true}})^T (w^{t+1} - w^t) + \frac{L}{2} \|w^{t+1} - w^t\|^2 \\
= F (w^t) - (g_{\text{true}})^T \gamma^t f_{\text{BayAirComp}} (y^t) \\
\quad + \frac{L}{2} (\gamma^t)^2 \|f_{\text{BayAirComp}} (y^t)\|^2 \\
= F (w^t) - \gamma^t (g_{\text{true}})^T (g_{\text{true}} + \epsilon^t) + \frac{L}{2} (\gamma^t)^2 \|g_{\text{true}} + \epsilon^t\|^2,
\]
where the last equation is obtained by the gradient error definition. By taking the expectation in (29), we can derive
\[
\mathbb{E} [F (w^{t+1}) - F (w^t)] \\
\leq - \left(\gamma^t - \frac{L}{2} (\gamma^t)^2\right) \mathbb{E} [\|g_{\text{true}}\|^2] \\
\quad - \left(\gamma^t - L (\gamma^t)^2\right) \mathbb{E} [(g_{\text{true}})^T \epsilon^t] + \frac{L}{2} (\gamma^t)^2 \mathbb{E} [\|\epsilon^t\|^2].
\]
Usually the learning rate \(\gamma^t < 1\), it seems reasonable that \(\gamma^t - L (\gamma^t)^2 > 0\). To continue the convergence analysis, we should find the lower bound of \(\mathbb{E} [(g_{\text{true}})^T \epsilon^t]\).

Corollary: If the components of true gradient is Gaussian with mean \(\mu_{\text{true}}\) and variance \((\nu_{\text{true}})^2\), i.e. \(g_{\text{true}, m} \sim (\mu_{\text{true}}^m, (\nu_{\text{true}}^m)^2)\) where \(m \in [M]\), \(\mathbb{E} [(g_{\text{true}})^T \epsilon^t]\) has a positive value in SNR \(\rightarrow 0\) and SNR \(\rightarrow \infty\).

Proof. Firstly, we can derive the mean and variance of \(g_{\text{true}, m}\) as
\[
\mu_{\text{true}}^m = \mathbb{E} [g_{\text{true}, m}] = \mathbb{E} \left[\frac{1}{K} \sum_{k=1}^{K} g_{k, m}^t\right] = \frac{1}{K} \sum_{k=1}^{K} \mu_{k}^m,
\]
\[
(\nu_{\text{true}}^m)^2 = \mathbb{E} [g_{\text{true}, m} - \mu_{\text{true}}^m]^2 \\
= \mathbb{E} \left[\frac{1}{K} \sum_{k=1}^{K} (g_{k, m}^t - \mu_k^t)^2\right] \\
= \mathbb{E} \left[\frac{1}{K^2} \sum_{k=1}^{K} (g_{k, m}^t - \mu_k^t)^2\right] = \frac{1}{K^2} \sum_{k=1}^{K} (\nu_k^m)^2.
\]
We can expressed \(\mathbb{E} [(g_{\text{true}})^T \epsilon^t]\) by summation of each component of the vector as below.
\[
\mathbb{E} [ (g_{\text{true}}^t)^T \epsilon^t ] \\
= \sum_{m=1}^{M} \mathbb{E} [ (g_{\text{true}, m}^t)^2 ] \\
= \sum_{m=1}^{M} \left[ (\nu_{\text{true}}^m)^2 - \frac{2}{\pi} \sum_{k=1}^{K} \frac{1}{\nu_k^m} \mathbb{E} [g_{\text{true}, m} A_k (y_m^t)] \right].
\]
We consider about the exact value of \(\mathbb{E} [g_{\text{true}, m} A_k (y_m^t)]\) in two SNR cases to obtain \(\mathbb{E} [(g_{\text{true}})^T \epsilon^t]:\) SNR \(\rightarrow 0\), and SNR \(\rightarrow \infty\).

1) SNR \(\rightarrow 0\): In this case, the all exponential terms in (33) goes to one as the noise variance \(\sigma^2\) goes to infinity, hence \(A_k (y_m^t) \rightarrow 0\). This derives \(\mathbb{E} [g_{\text{true}, m} A_k (y_m^t)] = 0\), so \(\mathbb{E} [(g_{\text{true}})^T \epsilon^t] = M (\nu_{\text{true}})^2 > 0\) can be achieved in (33).

2) SNR \(\rightarrow \infty\): The additive noise is assumed to be zero, so the only one exponential term where \(b\) got the whole correct signs of users’ gradients is non-zero, and the others are zeros in (33). Therefore we can obtain \(A_k (y_m^t) = \text{sign} (\bar{g}_{k, m})\).

Using this, we can represent the value of \(\mathbb{E} [g_{\text{true}, m} A_k (y_m^t)]\) as
\[
\mathbb{E} [g_{\text{true}, m} A_k (y_m^t)] \\
= \mathbb{E} \left[\frac{1}{K} \sum_{\ell=1}^{K} g_{\ell, m}^t \text{sign} (\bar{g}_{k, m})\right] \\
= \frac{1}{K} \left[ \mathbb{E} [g_{k, m}^t \text{sign} (\bar{g}_{k, m})] + \sum_{\ell \neq k} \mathbb{E} [g_{\ell, m}^t \text{sign} (\bar{g}_{k, m})] \right].
\]
We already know that \(g_{k, m}^t\) and \(g_{\ell, m}^t (\ell \neq k)\) are independent, so \(\bar{g}_{k, m}\) and \(\bar{g}_{\ell, m}\) are also independent. By the property that \(\mathbb{E} [X \cdot f(Y)] = \mathbb{E} [X] \cdot \mathbb{E} [f(Y)]\) where \(X\) and \(Y\) are independent, \(\sum_{\ell \neq k} \mathbb{E} [g_{\ell, m}^t \text{sign} (\bar{g}_{k, m})] = 0\) since \(\mathbb{E} [\text{sign} (\bar{g}_{k, m})] = 0\). This helps to obtain the value of \(\mathbb{E} [g_{\text{true}, m} A_k (y_m^t)]\) as
\[
\mathbb{E} [g_{\text{true}, m} A_k (y_m^t)] = \frac{1}{K} \mathbb{E} [g_{k, m}^t \text{sign} (\bar{g}_{k, m})] \\
= \frac{1}{K} \mathbb{E} [(g_{k, m}^t + \mu_k^t) \text{sign} (\bar{g}_{k, m})] \\
= \frac{1}{K} \mathbb{E} [(\bar{g}_{k, m} + \mu_k^t) \text{sign} (\bar{g}_{k, m})] \\
= \frac{1}{K} \mathbb{E} [\bar{g}_{k, m}^t] = \frac{1}{K} \sqrt{\frac{2}{\pi}} \nu_k^m.
\]
Put this into (33), we can get the exact value of $\mathbb{E} \left[ (g_{\text{true}}^t)^T e^t \right]$ as

$$
\mathbb{E} \left[ (g_{\text{true}}^t)^T e^t \right] = \sum_{m=1}^{M} \left( (\nu_{\text{true}}^m)^2 - \frac{2}{\pi} \left( \frac{1}{K} \sum_{k=1}^{K} (\nu_k^m)^2 \right) \right)
$$

$$
= M \left( 1 - \frac{2}{\pi} \right) (\nu_{\text{true}}^m)^2 > 0.
$$

Consequently, we can summarize that

$$
\mathbb{E} \left[ (g_{\text{true}}^t)^T e^t \right] = \left\{ \begin{array}{ll}
M (\nu_{\text{true}}^m)^2, & \text{SNR} \to 0 \\
M \left( 1 - \frac{2}{\pi} \right) (\nu_{\text{true}}^m)^2, & \text{SNR} \to \infty
\end{array} \right.,
$$

and all the values are positive. This completes the proof.

According to the corollary, (30) can be reduced in some particular cases as

$$
\mathbb{E} \left[ F (w_{t+1}^+) - F (w^*) \right] 
\leq - \left( \frac{\gamma}{\sqrt{t+1}} - L \frac{\gamma^2}{2} \right) \mathbb{E} \left[ (g_{\text{true}}^t)^2 \right] + \frac{L}{2} \frac{\gamma^2}{t+1} \sigma_{\text{MSE}}^2.
$$

Using the result $\mathbb{E} \left[ (e^t)^2 \right] \leq \sigma_{\text{MSE}}^2$ and the adaptive learning rate $\gamma_t = \frac{\gamma}{\sqrt{t+1}}$, we can simplify (38) as

$$
\mathbb{E} \left[ F (w_{t+1}^+) - F (w^*) \right] 
\leq - \left( \frac{\gamma}{\sqrt{t+1}} - L \frac{\gamma^2}{2} \right) \mathbb{E} \left[ (g_{\text{true}}^t)^2 \right] + \frac{L}{2} \frac{\gamma^2}{t+1} \sigma_{\text{MSE}}^2.
$$

If (39) is summed for all rounds $t \in [T]$, it can be organized as

$$
F (w^0) - F (w^*) 
\geq \mathbb{E} \left[ \sum_{t=0}^{T-1} (F (w^t) - F (w^{t+1})) \right] 
\geq \sum_{t=0}^{T-1} \left[ \frac{\gamma}{\sqrt{t+1}} \mathbb{E} \left[ (g_{\text{true}}^t)^2 \right] \left( 1 - \frac{L\gamma}{2} \right) - \frac{L}{2} \frac{\gamma^2}{t+1} \sigma_{\text{MSE}}^2 \right]
\geq \sqrt{T} \mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} (g_{\text{true}}^t)^2 \right] \left( 1 - \frac{L\gamma}{2} \right) - \frac{L}{2} \frac{\gamma^2}{t+1} \sigma_{\text{MSE}}^2.
$$

(37)

Through the MSE bound and convergence analysis in section [IV] we found that the expected value of the gradient norm decreases as the communication round $T$ increases in the order of

$$
\frac{c + c' \sigma_{\text{MSE}}^2 \ln T}{\sqrt{T}},
$$

for some positive constants $c$ and $c'$. If $\sigma_{\text{MSE}}^2 = 0$, there is no error between the true gradient and gradient estimate and the convergence rate of FL algorithm reduces to $O \left( \frac{1}{\sqrt{T}} \right)$. Hence the MSE $\sigma_{\text{MSE}}^2$ makes the convergence speed slower. We obtained that the MSE has the constant upper bound, so (41) decreases to zero as $T$ goes to infinity because $\lim_{T \to \infty} \frac{\ln T}{\sqrt{T}} = 0$. Finally, we can conclude that the proposed FL algorithm converges to a stationary point as the expected value of gradient goes to zero. Also this analysis can be extended to the algorithm based on SGD using a mini-batch size.

V. Simulation Results

This section provides numerical results to compare the test accuracy of the proposed algorithm and OBDA, a conventional wireless FL scheme. The OBDA system has features of the truncated channel-inversion precoding and majority-voting-based aggregation at the server [8].

Network model: We consider a hundred mobile devices, which are uniformly located in a cell with a radius of 1 km. We consider the COST-231 HATA model to take into account path-loss effects between mobile devices and the server and the Rayleigh fading model for small-scale fading effects.

Training model: We consider the task of image classification using MNIST and CIFAR10 datasets. We train a convolutional neural network (CNN) comprising two $5 \times 5$ convolutional layers (the first with 32 channels and the second with 64) in [8] using MNIST datasets. We also train ResNet44.
model using CIFAR10 datasets \cite{16}. To train the model, we assume that the server randomly selects ten mobile users. We also consider two orthogonal time-frequency resources, in which five mobile devices transmit their gradients using a shared time-frequency resource. For a heterogeneous data assumption, we assign only two distinct types of images to a mobile device. Each mobile device is assumed to compute the local gradient with the same batch size of 32 images. The maximum transmission power is set to be $P = 1$.

Effect of sign-alignment precoding: To see the effect of the proposed sign-alignment precoding, we train the models using the majority-voting based aggregation method as in OBDA, while chaining the precoding strategy from the channel-inversion precoding requiring infinite-resolution CSIT to our sign-alignment precoding using one-bit CSIT. As can be seen in Fig. 3, the both algorithms achieve over 95% and 70% test accuracies for MNIST and CIFAR10 datasets, respectively. It is remarkable that our sign-alignment precoding using one-bit CSIT is sufficient for wireless FL systems when the applying signSGD optimizer. This result shows that the sign-information for precoding degrade the learning performance when applying the channel-inversion precoding.

Effect of BayAirComp with sign-alignment precoding: For heterogeneous datasets, we train the models using our BayAirComp aggregator with the sign-alignment precoding. To improve the convergence speed, the server may harness an accelerated gradient descent algorithm by using a momentum term. To be specific, instead of \cite{4}, the server can update the model parameter as

$$w^{t+1} = w^t - \gamma^t \left[\delta f^*(y^{t-1}) + f^*(y^t)\right], \quad (42)$$

where $\delta \in (0, 1)$ is a constant for the moment term with initial value of $f^*(y^0) = 0$. As shown in Table I we can attain the highest accuracy performance for the proposed FL scheme when hyper-parameters are set to be $\gamma^t = 10^{-3}$ and $\delta = 0.9$. For OBDA, we set the hyper-parameters to be $\gamma^t = 10^{-3}$ and $\delta = 0$. Fig. 4 shows the test accuracy comparison between OBDA and the proposed algorithm. The proposed algorithm achieve 3.0% and 3.7% higher test accuracies than those attained by the OBDA for both MNIST and CIFAR10 dataset, respectively. This result demonstrates that BayAirComp aggregator is beneficial to improve the learning performance for heterogeneous datasets.

| Hyperparam. | Test Accuracy |
|-------------|---------------|
| $\gamma$ | $\delta$ |
| $10^{-2}$ | 0 | 93.70% |
| $10^{-2}$ | 0.9 | - |
| $10^{-3}$ | 0 | 81.81% |
| $10^{-3}$ | 0.9 | 94.63% |
| $10^{-4}$ | 0 | 71.32% |
| $10^{-4}$ | 0.9 | 71.32% |

VI. CONCLUSION

In this work, we studied the problem of wireless federated learning and presented novel sign-alignment precoding and BayAirComp aggregation method for signSGD. We derived MSE-optimal aggregation function under the IID Gaussian prior of the local gradients when sign-alignment precoding is applied. Our major finding is that one-bit CSIT for precoding suffices to improve the learning performance compared to the scheme using perfect CSIT. This implies that it is possible to reduce the signaling overheads considerably to implement the wireless FL systems.

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