ON SETS OF ZERO STATIONARY HARMONIC MEASURE

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Abstract. In this paper, we prove that any subset with an appropriate sub-linear horizontal growth has a non-zero stationary harmonic measure. On the other hand, we also show any subset with linear horizontal growth will have a 0 stationary harmonic measure at every point. This result is fundamental to the study of stationary DLA.

1. Introduction

In this paper, we present conditions for an infinite subset in the upper half plane to have non-zero stationary harmonic measure. Stationary harmonic measure is first introduced in [4], and plays a fundamental role in the study of diffusion limit aggregation (DLA) models on non-transitive graphs with absorbing boundary conditions. Roughly speaking, the stationary harmonic measure of a subset is the expected number of random walks hitting each of its points, when we drop “infinite” number of random walks from a horizontal line “infinitely high” and stop once they first hit the subset or the $x$–axis, and it plays an equivalent role as the harmonic measure in $\mathbb{Z}^d$ used to construct the ordinary DLA model, see [2], [3], and [1] for example.

For the precise discussions, we first refer to several definitions in our previous paper [4]. Let $\mathcal{H} = \{(x,y) \in \mathbb{Z}^2, y \geq 0\}$ be the upper half plane (including $x$–axis), and $S_n, n \geq 0$ be a 2-dimensional simple random walk. For any $x \in \mathbb{Z}^2$, we will write

$$x = (x_1, x_2)$$

with $x_i$ denoting the $i$th coordinate of $x$. Then let the subsets $L_n, D_n \subset \mathbb{Z}^2$ be defined as follows: for each nonnegative integer $n$, define

$$L_n = \{(x,n), \ x \in \mathbb{Z}\},$$

$$V_n = \{(0,k), \ 0 \leq k \leq n\},$$

and

$$U_n = L_0 \cup V_n.$$  

I.e., $L_n$ is the horizontal line of height $n$ while $U_n$ is $x$–axis plus the vertical line segment between $(0,0)$ and $(0,n)$. And let $y_n = (0,n)$ be the “end point” of $V_n$. Moreover, we use $\mathcal{P}_n \subset \mathcal{H}$ for an arbitrary finite path in the upper half plane connecting $y_n$ and the $x$–axis. One can immediately see that $V_n$ is one of such paths.
And for each subset \( A \subset \mathbb{Z}^2 \) we define stopping times
\[
\bar{\tau}_A = \min\{n \geq 0, S_n \in A\}
\]
and
\[
\tau_A = \min\{n \geq 1, S_n \in A\}.
\]
For any subsets \( A_1 \subset A_2 \) and \( B \) and any \( y \in \mathbb{Z}^2 \), by definition one can easily check that
\[
P_y (\tau_{A_1} < \tau_B) \leq P_y (\tau_{A_2} < \tau_B),
\]
and that
\[
P_y (\tau_B < \tau_{A_1}) \leq P_y (\tau_B < \tau_{A_2}).
\]
Now we define the stationary harmonic measure on \( \mathcal{H} \). For any connected \( B \subset \mathcal{H} \), any edge \( \vec{e} = x \to y \) with \( x \in B \), \( y \in \mathcal{H} \setminus B \) and any \( N \), we define
\[
H_{B,N}(\vec{e}) = \sum_{z \in \mathcal{L}_N \setminus B} P_z (S_{\bar{\tau}_B \cup \mathcal{L}_0} = x, S_{\bar{\tau}_B \cup \mathcal{L}_0 - 1} = y)
\]
By definition, \( H_{B,N}(\vec{e}) > 0 \) (although at this point we have not yet ruled out the possibility it equals to infinity) only if \( y \in \partial \text{out} B \) and \( |x - y| = 1 \).

And for all \( x \in B \), we can also define
\[
H_B(x) = \sum_{\vec{e} \text{ starting from } x} H_{B,N}(\vec{e}) = \sum_{z \in \mathcal{L}_N \setminus B} P_z (S_{\bar{\tau}_B \cup \mathcal{L}_0} = x).
\]

And for each point \( y \in \partial \text{out} B \), we can also define
\[
\hat{H}_{B,N}(y) = \sum_{\vec{e} \text{ starting in } B \text{ ending at } y} H_{B,N}(\vec{e}) = \sum_{z \in \mathcal{L}_N \setminus B} P_z (\tau_B \leq \tau_{\mathcal{L}_0}, S_{\bar{\tau}_B \cup \mathcal{L}_0 - 1} = y).
\]

In [4] we prove that,

**Proposition 1** (Proposition 1 in [4]). For any \( B \) and \( \vec{e} \) above, there is a finite \( H_B(\vec{e}) \) such that
\[
\lim_{N \to \infty} H_{B,N}(\vec{e}) = H_B(\vec{e}).
\]

And we call \( H_B(\vec{e}) \) the stationary harmonic measure of \( \vec{e} \) with respect to \( B \). Thus we immediately have the limits \( H_B(x) = \lim_{N \to \infty} H_{B,N}(x) \) and \( \hat{H}_B(y) = \lim_{N \to \infty} \hat{H}_{B,N}(y) \) also exists and we call them the stationary harmonic measure of \( x \) and \( y \) with respect to \( B \).

For a finite subset \( B \), it is shown in [4] that there must be an \( x \in B \) such that \( H_B(x) > 0 \), see Theorem 3 for details. However, for infinite \( B \), it is possible that \( H_B(\cdot) \) can be uniformly 0. The intuitive reason for such phenomena is that when \( B \) is infinite, each point \( x \in B \) may live in the shadow of other much higher points, which will block the random walk starting from “infinity” to visit the former first. In the following counterexample, we see that there can be a uniformly 0 harmonic measure
even when the height of $B$ is finite for each $x$–coordinate. We encourage
the reader to check the subset here has zero stationary harmonic measure
before reading the proof of the main results.

**Counterexample 1.** Let

$$B^0 = \bigcup_{n=-\infty}^{\infty} \{ (n,k), k = 0, 1, \cdots, 2^n \}.$$ 

Then $H_{B^0}(x) = 0$ for all $x \in B^0$.

**Remark 1.** The monotonicity in Proposition 2 in [4] does not contradict
with Counterexample 1. The reason is that in the proof of Proposition 2 we
need to interchange the order of a (finite) summation and a limit, which is
not true for the infinite summation in Counterexample 1.

In this paper, we actually prove a much stronger statement: for any
(infinite) $B \subset \mathcal{H}$, and any $x_1 \in \mathbb{Z}$, define

$$h_{x_1} = \sup \{ x_2 \geq 0, \ x_1 \times [1, x_2] \subset B \}.$$ 

**Definition 1.** We say that $B$ has a **horizontal linear growth** if there
are constants $c > 0$, $C < \infty$, and $M < \infty$ such that

$$h_{x_1} \geq |cx_1|$$

for all $|x_1| \geq M$, and that

$$|B \setminus \{ x \in \mathcal{H}, \ x_2 \leq C|x_1| \}| < \infty.$$ 

**Remark 2.** In this paper, we use $C$ and $c$ as constants in $(0, \infty)$ indepen-
dent to the change of variables like $N$ or $n$. But their exact values can be
different from place to place.

Then we have

**Theorem 1.** For any $B$ which has a horizontal linear growth and any
$x \in B$

$$H_B(x) = 0.$$ 

With Theorem 1, Counterexample 1 is immediate. Now, we prove that
for $B$’s of which the spatial growth rate has some sub-linear upper bound,
$H_B(\cdot)$ cannot be 0 everywhere:

**Theorem 2.** There is integer $n > 1$. For any $B$ such that there exists a
finite constant $C$ where

$$|B \setminus \{ x \in \mathcal{H}, \ x_2 \leq |x_1|^{1/n} \}| < \infty,$$

there must be some $x \in B$ such that $H_B(x) > 0$.

**Remark 3.** With exactly the same argument shown in this paper, one can
actually generalize Theorem 2 to any sub-linear polynomial growth.

**Remark 4.** In [4], Theorem 2 is used to show non degeneracy of a growth
process with rates proportional to square root of the height of any point,
which bounds the growth of stationary DLA in $\mathcal{H}$. 
2. Proof of Theorem 1

For any $B$ with a horizontal linear growth and any $x = (x_1, x_2) \in B$, recall Definition 1 and let

$$n_1 = \max\{|x_1|, M, [x_2/c]\}$$

and

$$D_1 = [-n_1, n_1] \times [-\lceil cn_1 \rceil, \lceil cn_1 \rceil]$$

Then $x \in D_1$, and by Definition 1,

(7) $\hat{W}_c \setminus D_1 \subset B \setminus D_1$

where

$$\hat{W}_c = \{x \in \mathcal{H}, x_2 < c|x_1|\}.$$

Moreover, it is not hard to check that for any $N > \lceil cn_1 \rceil$ and $y \in L_N \setminus B$, a simple random walk starting from $y$ hits $x$ before hitting any other point in $B$ only if it hits $l_{n_1} = [-n_1, n_1] \times \lceil cn_1 \rceil = L_{\lceil cn_1 \rceil} \setminus \hat{W}_c$ before hitting $\hat{W}_c$. I.e.,

(8) $P_y(\tau_x = \tau_B) \leq P_y(\tau_{l_{n_1}} < \tau_{\hat{W}_c}).$

Thus by (7) and (8)

$$H_{B, N}(x) = \sum_{y \in L_N \setminus B} P_y(\tau_x = \tau_B)$$

$$\leq \sum_{y \in L_N \setminus \hat{W}_c} P_y(\tau_{l_{n_1}} < \tau_{\hat{W}_c})$$

$$= \sum_{w \in l_{n_1} \cup \hat{W}_c, N} H_{l_{n_1} \cup \hat{W}_c, N}(w).$$

Figure 1. Escaping probability for each step
Then by the Proof of Proposition 1 in [4], for any \( w \in l_{n_1} \)

\begin{align}
H_{l_{n_1} \cup \hat{W}_c, N}(w) = \sum_{y \in L_N \setminus \hat{W}_c} P_w \left( \tau_{L_N} < \tau_{l_{n_1} \cup \hat{W}_c}, S_{\tau_{L_N}} = y \right) E_y \left[ \text{number of visits to } L_N \text{ in } [0, \tau_{l_{n_1} \cup \hat{W}_c}] \right] \\
\leq \sum_{y \in L_N \setminus \hat{W}_c} P_w \left( \tau_{L_N} < \tau_{\hat{W}_c}, S_{\tau_{L_N}} = y \right) E_y \left[ \text{number of visits to } L_N \text{ in } [0, \tau_{L_0}] \right] \\
= 4N \cdot P_w \left( \tau_{L_N} < \tau_{\hat{W}_c} \right).
\end{align}

Now let \( N_1 = [cn_1] \) and \( N_2 = 2N_1 \). For any \( z \in l_{n_1} \) define the rectangular region

\[ D_{1,z} = [z - 4[N_2/c], z + 4[N_2/c]] \times [0, N_2]. \]

Moreover, we define

\[
\begin{align*}
\partial^1 D_{1,z} &= [z - 4[N_2/c], z + 4[N_2/c]] \times N_2, \\
\partial^2 D_{1,z} &= (z + 4[N_2/c]) \times [0, N_2], \\
\partial^3 D_{1,z} &= [z - 4[N_2/c], z + 4[N_2/c]] \times 0, \\
\partial^4 D_{1,z} &= -(z + 4[N_2/c]) \times [0, N_2]
\end{align*}
\]

as the four sides on the boundary of \( D_{1,z} \). Note that if a random walk starting at \( z \) hits \( \partial^2 D_{1,z} \cup \partial^3 D_{1,z} \cup \partial^4 D_{1,z} \) before hitting \( \partial^1 D_{1,z} \), it must have already hit \( \hat{W}_c \) before reaching \( L_{N_2} \). Thus

\[
P_z \left( \tau_{L_{N_2}} < \tau_{\hat{W}_c} \right) \leq P_z \left( \tau_{\partial^1 D_{1,z}} = \tau_{\partial D_{1,z}} \right).
\]

Then by translation invariance we have

\[
P_z \left( \tau_{\partial^1 D_{1,z}} = \tau_{\partial D_{1,z}} \right) = P_0 \left( \tau_{\partial^1 D_{1,0}} = \tau_{\partial D_{1,0}} \right).
\]

And by symmetry

\begin{align}
P_{(0,N_1)} \left( \tau_{\partial^1 D_{1,0}} = \tau_{\partial D_{1,0}} \right) = \frac{1}{2} - \frac{1}{2} \cdot P_0 \left( \tau_{\partial^2 D_{1,0}} \land \tau_{\partial^4 D_{1,0}} < \tau_{\partial^1 D_{1,0}} \land \tau_{\partial^4 D_{1,0}} \right).
\end{align}

Note that the last term in (11) is the probability a random walk first reaches the two vertical sides of \( D_{1,z} \) before the horizontal sides. By invariance principle, there is a constant \( c > 0 \) independent to \( N_1 \) such that

\begin{align}
P_{(0,N_1)} \left( \tau_{\partial^1 D_{1,0}} = \tau_{\partial D_{1,0}} \right) \leq \frac{1 - c}{2}.
\end{align}

In general, define \( N_k = 2^{k-1} N_1 \) for all \( k \geq 2 \), and let

\[
l_{N_k} = L_{N_k} \setminus \hat{W}_c, \\
D_{k,z} = [z - 4[N_{k+1}/c], z + 4[N_{k+1}/c]] \times [0, N_{k+1}],
\]
with $\partial^1 D_{k,z} - \partial^4 D_{k,z}$ as its four sides defined as before. Using exactly the same argument as for $k = 1$, we have for any $z \in l_{N_k}$,

$$P_z \left( \tau_{L_{N_k} + 1} < \tau_{\hat{W}_{c}} \right) \leq P_{(0,N_k)} \left( \tau_{\partial^1 D_{k,0} - \partial^4 D_{k,0}} \right) \leq \frac{1 - c}{2}. \tag{12}$$

Noting that the upper bound in (12) is uniform for all $z \in L_{N_k} \setminus \hat{W}_c$, by strong Markov property we have for any $w \in l_{n_1}$

$$P_w \left( \tau_{L_{N_k}} < \tau_{\hat{W}_c} \right) \leq \left( 1 - \frac{c}{2} \right)^{k-1} = 2^{-\gamma(k-1)N_1} + N_1 \to 0 \tag{13}$$

as $k \to \infty$. Thus the proof of Theorem 1 is complete. \hfill \Box

3. PROOF OF THEOREM 2

Recall that for $B$ in Theorem 2 there exist an $n > 1$ such that

$$\hat{B} = \left| B \setminus \{ x \in \mathcal{H}, \ x_2 \leq |x_1|^{1/n} \} \right| < \infty.$$ 

For any $N > h_1 = \max_{x \in B} x_2$ it is easy to see that

$$\{ (i, N), \ |i| \leq [N^n] \} \cap B = \emptyset.$$ 

Here we use the convention $\max\{ \emptyset \} = 0$. Moreover, since $B$ is nonempty, let

$$h_2 = \inf \{ h : B \cap [-h^n, [h^n]] \times [0, h] \neq \emptyset \}.$$ 

Then $h_2$ is also finite. Then for $h_0 = \max\{ h_1, h_2 \}$,

$$D_0 = [-h_0^n, [h_0^n]] \times [0, h_0],$$

and

$$B_0 = B \cap D_0,$$

we have $1 < |B_0| < \infty$ and $\hat{B} \subset B_0$. To see the reason for the second equation, for any $x \in \hat{B}$, by definition of $h_1$, we have $x_2 \leq h_1$. At the same time since $x \in \hat{B}$, $|x_1| < x_2^n \leq h_1^n \leq [h_0^n]$.

Now to prove Theorem 2, we show that there is a constant $c > 0$ such that for all sufficiently large $N$,

$$H_{B,N}(B_0) \geq c. \tag{15}$$

Let $l_{h_0} = [-h_0^n, [h_0^n]] \times h_0$. We first prove that

**Lemma 3.1.** There is a constant $c > 0$ such that for any sufficiently large $N$

$$H_{B,N}(l_{h_0}) \geq c.$$
Proof. Now define
\[ U_0 = \{ x \in \mathcal{H}, \ |x_1| \geq \lceil h_0^n \rceil, x_2 \leq |x_1|^{1/n} \} \]
and
\[ \hat{W} = D_0 \cup U_{h_0}, \ W = \mathcal{H} \setminus (D_0 \cup U_{h_0}). \]
Note that \((\pm \lceil h_0^n \rceil, h_0) \in U_{h_0}\). Since \(\hat{W}\) is connected and \(B \subset \hat{W}\), we have that
\[ \tau_B \geq \hat{\tau}_W, \ \bar{\tau}_B \geq \bar{\tau}_W \]
for a random walk from any initial location. Thus
\[ H_{B,N}(x_0) = H_{\hat{W},N}(x_0) \]
where \(x_0 = (0, h_0)\). Then again by the Proof of Proposition 1 in [4],
\[ H_{\hat{W},N}(x_0) = \sum_{z=(i,N), |i|\leq[N^n]} P_z(\tau_{x_0} \leq \hat{\tau}_W) \]
\[ = \sum_{z \in l_N} P_{x_0}(\tau_{L_N} < \hat{\tau}_W, S_{\tau_{L_N}} = z) E_z[\text{number of visits to } l_N \text{ in } [0, \hat{\tau}_W]]. \]
Then let
\[ k_0 = \inf \{ k \in \mathbb{Z}^+, \ 2^k > 2h_0^{(2+n)/3} \}, \]
and consider the event
\[ A_0 = \left\{ \tau_{L_{k_0}} < \tau_{W_0}, \ S_{\tau_{L_{k_0}}} = (0, 2^{k_0}) \right\}. \]
One can immediately have
\[ P_{x_0}(A_0) = c > 0, \]
where \(c\) is a constant independent to \(N\). Now for \(X_1 = (0, 2^{k_0})\), consider a new wedge

\[ W_0 = \left\{ (x_1, x_2) \in \mathcal{H}, \ x_2 - \lceil 2^{2k_0/3} \rceil \geq |x_1| \cdot 2^{-nk_0/2} \right\}. \]
Note that for any $x \in W_1$, $x_2 \geq \lceil 2^{2k_0/3} \rceil > h_0$ and that for all $n \geq 4$
\[
x^n_2 = \left(\lceil 2^{2k_0/3} \rceil + |x_1| \cdot 2^{-nk_0/2}\right)^n
\geq n2^{2k_0(n-1)/3-nk_0/2} |x_1| > |x_1|.
\]
We have $W_0 \subset W$. Define
\[
p_0 = P_{X_1}(\tau_{L_2^{k_0+1}} \leq \tau_{W_0}).
\]
Then let $k_1 = k_0 + 1$. We have
\[
W_0 \cap L_2^{k_1} \subset \{(x_1, 2^{k_1}), |x_1| \leq 2^{k_1(1+n/2)}\}.
\]
And for all $y \in \{(y_1, 2^{k_1}), |y_1| \leq 2^{(1+n/2)k_1}\}$, we define wedge
\[
W_{1,y} = \{(x_1, x_2) \in \mathcal{H}, x_2 - \lceil 2^{2k_1/3} \rceil \geq |x_1 - y_1| \cdot 2^{-nk_1/2}\}.
\]
Again, we have for any $x \in W_{1,y}$
\[
x^n_2 = \left(\lceil 2^{2k_1/3} \rceil + |x_1 - y_1| \cdot 2^{-nk_1/2}\right)^n
\geq 2^{n2nk_1/3} + n2^{2k_1(n-1)/3-nk_2/2} |x_1 - y_1|
\geq 2^{n2nk_1/3} + n2^{2k_1(n-1)/3-nk_2/2} + |x_1| - |y_1| - 1
\geq |x_1| + 2^{n2nk_1/3} + n2^{2k_1(n-1)/3-nk_2/2} - 2^{(1+n/2)k_1} - 1
> |x_1|.
\]

**Remark 5.** Without loss of generality we assume here and for all other $k_1$’s $x_1 \neq y_1$, one can easily check the inequality holds when equal.

Thus $W_{1,y} \subset W$. Then for each $y \in s_1 = \{(y_1, 2^{k_1}), |y_1| \leq 2^{(1+n/2)k_1}\}$ define
\[
p_{1,y} = P_{x_2}(\tau_{L_2^{k_1+1}} \leq \tau_{W_{1,y}}).
\]
By translation invariance, we have $p_{2,y} = p_2$ for all such $y$’s. Now let $k_2 = k_1 + 1$. We have
\[
\left(\bigcup_{y \in s_1} W_{1,y}\right) \cap L_2^{k_2} \subset s_2 = \{(x_1, 2^{k_2}), |x_1| \leq 2^{k_1(1+n/2)} + 2^{k_2(1+n/2)}\}.
\]
And for all $y \in s_2$, we further define wedge
\[
W_{2,y} = \{(x_1, x_2) \in \mathcal{H}, x_2 - \lceil 2^{2k_2/3} \rceil \geq |x_1 - y_1| \cdot 2^{-nk_2/2}\}
\]
and have
\[
x^n_2 = \left(\lceil 2^{2k_2/3} \rceil + |x_1 - y_1| \cdot 2^{-nk_2/2}\right)^n
\geq 2^{n2nk_2/3} + n2^{2k_2(n-1)/3-nk_2/2} |x_1 - y_1|
\geq 2^{n2nk_2/3} + n2^{2k_2(n-1)/3-nk_2/2} + |x_1| - |y_1| - 1
\geq |x_1| + 2^{n2nk_2/3} + n2^{2k_2(n-1)/3-nk_2/2} - 2^{(1+n/2)k_2} - 2^{(1+n/2)k_2} - 1
\geq |x_1| + 2^{n2nk_2/3} + n2^{2k_2(n-1)/3-nk_2/2} - 2^{(1+n/2)k_2} - 1
> |x_1|.
Thus $W_{2,y} \subset W$. Then for all $y \in s_2$ define

$$p_2 = P_y(\tau_{L_{2^{k_2+1}}} \leq \tau_{W_{2,y}}).$$

In general, for all $i \geq 2$ let $k_i = k_0 + i$. And for all

$$y \in s_i = \left\{ (y_1, 2^{k_i}), |y_1| \leq \sum_{j=1}^{i} 2^{(1+n/2)k_j} \right\}$$

we define wedge

$$W_{i,y} = \left\{ (x_1, x_2) \in \mathcal{H}, x_2 - \left\lceil \frac{2^{k_i/3}}{2} \right\rceil \geq |x_1 - y_1| \cdot 2^{-nk_i/2} \right\}$$

and have

$$x_2^n = \left( \left[ 2^{2k_i/3} + |x_1 - y_1| \cdot 2^{-nk_i/2} \right]^n \right.$$

$$\geq 2^{2nk_i/3} + n2^{2k_i(n-1)/3-nk_i/2}|x_1 - y_1|$$

$$\geq 2^{2nk_i/3} + n2^{2k_i(n-1)/3-nk_i/2} + |x_1| - |y_1| - 1$$

$$\geq |x_1| + 2^{2nk_i/3} + n2^{2k_i(n-1)/3-nk_i/2} - \sum_{j=1}^{i} 2^{(1+n/2)k_j} - 1$$

$$\geq |x_1| + 2^{2nk_i/3} + n2^{2k_i(n-1)/3-nk_i/2} - 2^{(1+n/2)k_i+1} - 1$$

$$> |x_1|,$$

which implies that $W_{i,y} \subset W$. We also have

$$\left( \bigcup_{y \in s_i} W_{i,y} \right) \cap L_{2^{k_{i+1}}} \subset s_{i+1} = \left\{ (y_1, 2^{k_i}), |y_1| \leq \sum_{j=1}^{i+1} 2^{(1+n/2)k_j} \right\}.$$

And for all $y \in s_i$ define

$$p_i = P_y(\tau_{L_{2^{k_{i+1}}} \leq \tau_{W_{1,y}}}).$$

With the constructions above, one can see that for each $i$

$$(17) \quad P_{x_0} \left( \tau_{L_{2^{k_i}}} < \tau_{\mathcal{W}} \right) \geq P_{x_0} \left( \tau_{S} < \tau_{\mathcal{W}} \right) \geq c \prod_{j=0}^{i-1} p_j.$$

Now we need to the following simple lemma showing that it is highly unlikely for a simple random walk starting from the middle of a very wide but short rectangular box to exit from the vertical sides:

**Lemma 3.2.** For any integers $n, k \geq 1$, let rectangle

$$R_{k,n} = [-nk, nk] \times [-k, k]$$

with

$$l^v_{k,n} = [-nk, nk] \times [-k, k]$$

as its two vertical sides and

$$l^h_{k,n} = [-nk, nk] \times \{-k, k\}$$
as its two horizontal sides. Then there is a $\delta \in (0, 1)$ such that for any $n, k \geq 1$ and any integer $x \in \{0\} \times [-k, k]$,

$$P_x \left( \tau_{k,n}^e < \tau_{k,n}^b \right) \leq (1 - \delta)^n.$$

With Lemma 3.2 we can control the lower bound on the probabilities $p_i$. Recalling that by translation invariance, for each $i$, $y_{i,0} = (0, 2^{ki})$ and

$$W_{i,0} = \{(x_1, x_2) \in \mathcal{H}, x_2 - \lfloor 2^{2ki/3} \rfloor \geq |x_1| \cdot 2^{-nk_i/2}\},$$

we have

$$p_i = P_{y_{i,0}} \left( \tau_{L2^{ki+1}} \leq \tau_{W_{i,0}}^e \right).$$

Then consider the rectangle

$$R_i = \left[-\lfloor 2^{nk_i/2} \rfloor, \lfloor 2^{nk_i/2} \rfloor\right] \times \left[2\lfloor 2^{2ki/3} \rfloor, 2^{k_i+1}\right] \subset W_{i,0}$$

and

$$\text{top}_i = \left[-\lfloor 2^{nk_i/2} \rfloor, \lfloor 2^{nk_i/2} \rfloor\right] \times 2^{k_i+1},$$

$$\text{bottom}_i = \left[-\lfloor 2^{nk_i/2} \rfloor, \lfloor 2^{nk_i/2} \rfloor\right] \times 2\lfloor 2^{2ki/3} \rfloor,$$

$$\text{left}_i = -\lfloor 2^{nk_i/2} \rfloor \times 2\lfloor 2^{2ki/3} \rfloor, 2^{k_i+1},$$

$$\text{right}_i = 2\lfloor 2^{nk_i/2} \rfloor \times 2\lfloor 2^{2ki/3} \rfloor, 2^{k_i+1},$$

as its four sides. And for any $n \geq 6$, note that $2^{nk_i/2} \geq 2^{2ki+1}$. Let

$$\hat{R}_i = \left[-2^{2k_i+1}, 2^{2k_i+1}\right] \times \left[0, 2^{k_i+1}\right].$$

Note that for a random walk starting at $y_{i,0}$, if it already visits the top and bottom of $\hat{R}_i$ before hitting its two vertical sides, it will have no chance to hit $\text{left}_i \cup \text{right}_i$ before hitting $\text{top}_i \cup \text{bottom}_i$. Thus by Lemma 3.2

$$P_{y_{i,0}} \left( \tau_{\text{left}_i \cup \text{right}_i} < \tau_{\text{top}_i \cup \text{bottom}_i} \right) \leq (1 - \epsilon)^{2k_i}. \tag{18}$$

Moreover, by the gambler’s ruin problem, we have

$$P_{y_{i,0}} \left( \tau_{\text{top}_i} < \tau_{\text{bottom}_i} \right) = \frac{2^{k_i} - 2\lfloor 2^{2ki/3} \rfloor}{2^{k_i+1} - 2\lfloor 2^{2ki/3} \rfloor} \geq \frac{1}{2} - 2^{-k_i/3+1}.$$

Thus

$$P_{y_{i,0}} \left( \tau_{\text{top}_i} = \tau_{\partial R_i} \right) \geq P_{y_{i,0}} \left( \tau_{\text{top}_i} < \tau_{\text{bottom}_i} \right) - P_{y_{i,0}} \left( \tau_{\text{left}_i \cup \text{right}_i} < \tau_{\text{top}_i} < \tau_{\text{bottom}_i} \right) \geq P_{y_{i,0}} \left( \tau_{\text{top}_i} < \tau_{\text{bottom}_i} \right) - P_{y_{i,0}} \left( \tau_{\text{left}_i \cup \text{right}_i} < \tau_{\text{top}_i \cup \text{bottom}_i} \right) \geq \frac{1}{2} - 2^{-k_i/3+1} - (1 - \epsilon)^{2k_i}. \tag{19}$$

Finally, note that since $R_i \subset W_{i,0}$, under event $\{\tau_{\text{top}_i} = \tau_{\partial R_i}\}$, the random walk has escaped to $L^{2^i+1}$ before exiting $W_{i,0}$. Thus

$$p_i \geq \frac{1}{2} - 2^{-k_i/3+1} - (1 - \epsilon)^{2k_i}. \tag{20}$$
Now recalling (17), we have

\[ P_{x_0} \left( \tau_{s_i} < \tau_W \right) \geq c \prod_{j=0}^{i-1} p_j \geq c 2^{-k_i} \prod_{j=0}^{i-1} \left[ 1 - 2^{-k_j/3+2} - 2(1 - \varepsilon)^{2k_j} \right]. \]

Noting that

\[ \sum_{i=0}^{\infty} 2^{-k_i/3+2} + 2(1 - \varepsilon)^{2k_i} < \infty \]

we have

(21) \[ P_{x_0} \left( \tau_{s_i} < \tau_W \right) \geq c 2^{-k_i}. \]

Now Recall that

\[ s_i = \left\{ (y_1, 2^{k_i}), |y_1| \leq \sum_{j=1}^{i} 2^{(1+n/2)k_j} \right\} \subset L_{2^{k_i}}. \]

For each \( y = (y_1, 2^{k_i}) \in s_i \), define

\[ N^y_i = [y_1 - 2^{nk_i/2}, y_1 + 2^{nk_i/2}] \times [2^{k_i} - 2^{k_i-1}, 2^{k_i} + 2^{k_i-1}]. \]

Note that \( |y_1| \leq 2^{1+(1+n/2)k_i} \),

\[ 2^{1+(1+n/2)k_i} + 2^{nk_i/2} \leq 2^{3+(1+n/2)k_i} < 2^{n(k_i-1)} \]

for all \( n \geq 6 \) and sufficiently large \( i \). We have \( N^y_i \subset W \) such that

\( E_y[\text{number of visits to } l_{2^{k_i}} \text{ in } [0, \tau_W]] \geq E_y[\text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N^y_i}]]. \)

Now let \( \Gamma_{i,1} = \tau_{L_{2^{k_i}}} \) and for each \( j \)

\[ \Gamma_{i,j} = \inf \{ n > \Gamma_{i,j-1}, S_n \in L_{2^{k_i}} \} \]

be the \( j \)th time a random walk returns to \( L_{2^{k_i}} \). We have

\[ E_y \left[ \text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N^y_i}] \right] = \sum_{j=1}^{\infty} \frac{P_y \left( \Gamma_{i,j} \leq \tau_{\partial N^y_i} \right)}{P_y \left( \Gamma_{i,j} \leq \tau_{\partial N^y_i} \right)}. \]

Then let

\( \text{top}_{y,i} = [y_1 - 2^{nk_i/2}, y_1 + 2^{nk_i/2}] \times (2^{k_i} + 2^{k_i-1}), \)

\( \text{bottom}_{y,i} = [y_1 - 2^{nk_i/2}, y_1 + 2^{nk_i/2}] \times (2^{k_i} - 2^{k_i-1}), \)

\( \text{left}_{y,i} = (y_1 - 2^{nk_i/2}) \times [2^{k_i} - 2^{k_i-1}, 2^{k_i} + 2^{k_i-1}], \)

\( \text{right}_{y,i} = (y_1 + 2^{nk_i/2}) \times [2^{k_i} - 2^{k_i-1}, 2^{k_i} + 2^{k_i-1}] \)

as the four sides of \( N^y_i \). Note that for any \( j \)

\[ P_y \left( \Gamma_{i,j} \leq \tau_{\text{top}_{y,i}} \land \tau_{\text{bottom}_{y,i}} \right) \geq P_y \left( \Gamma_{i,j} \leq \tau_{L_{2^{k_i}} \cup \text{top}_{y,i}} \land \tau_{L_{2^{k_i}} \setminus \text{right}_{y,i}} \right) = (1 - 2^{-k_i})^j. \]
At the same time,

\[ P_y \left( \Gamma_{i,j} \leq \tau_{\partial N_y} \right) = P_y \left( \Gamma_{i,j} \leq \tau_{\text{top}_{y,i}} \wedge \tau_{\text{bottom}_{y,i}} \right) \]

\[ - P_y \left( \tau_{\text{left}_{y,i}} \wedge \tau_{\text{right}_{y,i}} \leq \Gamma_{i,j} \leq \tau_{\text{top}_{y,i}} \wedge \tau_{\text{bottom}_{y,i}} \right) \]

\[ \geq (1 - 2^{-k_i})^j - P_y \left( \tau_{\text{left}_{y,i}} \wedge \tau_{\text{right}_{y,i}} < \tau_{\text{top}_{y,i}} \wedge \tau_{\text{bottom}_{y,i}} \right). \]

And again by Lemma 3.2, we have

\[ P_y \left( \tau_{\text{left}_{y,i}} \wedge \tau_{\text{right}_{y,i}} < \tau_{\text{top}_{y,i}} \wedge \tau_{\text{bottom}_{y,i}} \right) \leq (1 - \delta)^{2^{(1-n/2)k_i+1}}. \]

Thus

\[ E_y \left[ \text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N_y}] \right] \]

\[ \geq \sum_{j=1}^{2^{k_i}} P_y \left( \Gamma_{i,j} \leq \tau_{\partial N_y} \right) \]

\[ \geq \sum_{j=1}^{2^{k_i}} (1 - 2^{-k_i})^j - 2^{k_i}(1 - \delta)^{2^{(1-n/2)k_i+1}} \]

\[ \geq 2^{k_i} \left( 1 - 2^{-k_i} \right) \left[ 1 - (1 - 2^{-k_i})^{2^{k_i}} \right] - C \]

\[ \geq c 2^{k_i} \]

for some \( c > 0 \) independent to \( i \) and \( y \in s_i \).

Now combining (16), (21) and (22),

\[ H_{W,2^{k_i}}(x_0) = \sum_{z \in L_{2^{k_i}}} P_{x_0} \left( \tau_{L_{2^{k_i}}} < \tau_W, S_{\tau_{L_{2^{k_i}}} = z} = z \right) E_z \left[ \text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_W] \right] \]

\[ \geq \sum_{y \in s_i} P_{x_0} \left( \tau_{L_{2^{k_i}}} < \tau_W, S_{\tau_{L_{2^{k_i}}} = y} = y \right) E_y \left[ \text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N_y}] \right] \]

\[ \geq P_{x_0} \left( \tau_{L_{2^{k_i}}} < \tau_W \right) \inf_{y \in s_i} E_y \left[ \text{number of visits to } L_{2^{k_i}} \text{ in } [0, \tau_{\partial N_y}] \right] \]

\[ \geq c. \]

And thus the proof of Lemma 3.1 is complete. \( \square \)

Now back to finish the proof of Theorem 2, note that both \( l_0 \) and \( B_0 \) are finite and not depending on \( N \). There is a \( c > 0 \) such that for any \( z \in l_0 \),

\[ P_z(\bar{\tau}_{B_0} = \bar{\tau}_B) \geq c. \]
Thus by strong Markov property,
\[ H_{B,2^{k_i}}(B_0) = \sum_{z \in L_{2^{k_i}} \setminus B} P_z(\tau_{B_0} = \tau_B) \]
\[ \geq H_{B,2^{k_i}}(l_0) \inf_{z \in l_0} P_z(\bar{\tau}_{B_0} = \bar{\tau}_B) \]
\[ \geq c. \]

Then taking \( i \to \infty \), Proposition 1 completes the proof. \( \square \)

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