Harmonic aspects in an $\eta$-Ricci soliton

Adara M. Blaga

Abstract

We characterize the $\eta$-Ricci solitons $(g, \xi, \lambda, \mu)$ for the special cases when the 1-form $\eta$, which is the $g$-dual of $\xi$, is harmonic or Schrödinger-Ricci harmonic form. We also provide necessary and sufficient conditions for $\eta$ to be a solution of the Schrödinger-Ricci equation and point out the relation between the three notions in our context. In particular, we apply these results to a perfect fluid spacetime and using Bochner-Weitzenböck techniques, we formulate more conclusions for the case of gradient solitons and deduce topological properties of the manifold and its universal covering.

1 Introduction

Self-similar solutions to the Ricci flow, the Ricci solitons [19] have been studied in the different geometrical contexts on complex, contact and paracontact manifolds. The more general notion of $\eta$-Ricci soliton was introduced by J. T. Cho and M. Kimura [11] on real hypersurfaces in a Kähler manifold and treated in complex space forms [10] and paracontact geometries [2], [3], [7], [8].

A particular case of solitons arise when they evolve by diffeomorphisms generated by a gradient vector field, namely when the potential vector field is the gradient of a smooth function. The gradient vector fields play a central rôle in the Morse-Smale theory [24] and some aspects of gradient $\eta$-Ricci solitons were discusses by the author in [1], [4], [5], [6].

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In Section 2, after we point out the basic properties of an $\eta$-Ricci $(g, \xi, \lambda, \mu)$, we provide necessary and sufficient conditions for the $g$-dual 1-form of the potential vector field $\xi$ to be a solution of the Schrödinger-Ricci equation, a harmonic or a Schrödinger-Ricci harmonic form and characterize the 1-forms orthogonal to $\eta$. We end these considerations by discussing the case of a perfect fluid spacetime. In Section 3 we formulate the results for the special case of gradient solitons.

2 Geometrical aspects of $\eta$

Let $(M, g)$ be an $n$-dimensional Riemannian manifold, $n > 2$, and denote by $b : TM \to T^*M$, $b(X) := i_X g$, $\sharp : T^*M \to TM$, $\sharp := b^{-1}$. Consider the set $T^0_{2,s}(M)$ of symmetric $(0, 2)$-tensor fields on $M$ and for $Z \in T^0_{2,s}(M)$, denote by $Z^\sharp : TM \to TM$ and $Z^* : T^*M \to T^*M$ the maps defined by:

$$g(Z^\sharp(X), Y) := Z(X, Y), \quad Z^\sharp(\alpha)(X) := Z(\sharp(\alpha), X).$$

We also denote by $Z^\sharp$ by the map $Z^\sharp : T^*M \times T^*M \to C^\infty(M)$:

$$Z^\sharp(\alpha, \beta) := Z(\sharp(\alpha), \sharp(\beta))$$

and can identify $Z^\sharp$ with the map also denoted by $Z^\sharp : T^*M \times TM \to C^\infty(M)$:

$$Z^\sharp(\alpha, X) := Z(\sharp(\alpha), X).$$

Given a vector field $X$, its $g$-dual 1-form $X^b := b(X)$ is said to be a solution of the Schrödinger-Ricci equation if it satisfies:

(1) $$\text{div}(L_X g) = 0,$$

where $L_X g$ denotes the Lie derivative along the vector field $X$.

It is known that [13]:

(2) $$\text{div}(L_X g) = (\Delta + S^\sharp(X^b)) + d(\text{div}(X)),$$

where $\Delta$ denotes the Laplace-Hodge operator on forms w.r.t. the metric $g$ and $S$ the Ricci curvature tensor field. Denoting by $Q$ the Ricci operator defined by $g(QX, Y) := S(X, Y)$, for any vector fields $X$ and $Y$, by a direct computation we deduce that $S^\sharp(\gamma) = i_{Q^*\gamma} g$, for any 1-form $\gamma$. 

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We are interested to find the necessary and sufficient conditions for the $g$-dual 1-form $\eta$ of the potential vector field $\xi$ in an $\eta$-Ricci soliton to be a solution of the Schrödinger-Ricci equation, a harmonic or Schrödinger-Ricci harmonic form.

Consider the equation:

$$L_\xi g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0$$

where $g$ is a Riemannian metric, $S$ its Ricci curvature tensor field, $\xi$ a vector field, $\eta$ a 1-form and $\lambda$ and $\mu$ are real constants. The data $(g, \xi, \lambda, \mu)$ which satisfy the equation (3) is said to be an $\eta$-Ricci soliton on $M$ \cite{11}; in particular, if $\mu = 0$, $(g, \xi, \lambda)$ is a Ricci soliton \cite{19} and it is called shrinking, steady or expanding according as $\lambda$ is negative, zero or positive, respectively \cite{14}. If the potential vector field $\xi$ is of gradient type, $\xi = \text{grad}(f)$, for $f$ a smooth function on $M$, then $(g, \xi, \lambda, \mu)$ is called gradient $\eta$-Ricci soliton.

Taking the trace of the equation (3) we obtain:

$$\text{div}(\xi) + \text{scal} + \lambda n + \mu |\xi|^2 = 0.$$  

From a direct computation we get:

$$\text{div}(\eta \otimes \eta) = \text{div}(\xi) \eta + \nabla_\xi \eta.$$  

Now taking the divergence of (3) and using (2) we obtain:

$$\text{div}(L_\xi g) + d(\text{scal}) + 2\mu [\text{div}(\xi) \eta + \nabla_\xi \eta] = 0.$$  

Schrödinger-Ricci solutions

We say that a 1-form $\gamma$ is a solution of the Schrödinger-Ricci equation if

$$\Delta + S_\xi(\gamma) + d(\text{div}(\gamma^2)) = 0.$$  

**Theorem 2.1.** Let $(g, \xi, \lambda, \mu)$ be an $\eta$-Ricci soliton on the n-dimensional manifold $M$ with $\eta$ the $g$-dual of $\xi$. Then $\eta$ is a solution of the Schrödinger-Ricci equation if and only if

$$d(\text{scal}) = 2\mu [(\text{scal} + \lambda n + \mu |\xi|^2) \eta - \nabla_\xi \eta].$$

Moreover, in this case, $\text{scal}$ is constant if and only if $\mu = 0$ (which yields a Ricci soliton) or $(\text{scal} + \lambda n + \mu |\xi|^2) \eta = \nabla_\xi \eta.$
Proof. From (3), (4), (5) and 
\[ 2\text{div}(S) = d(\text{scal}) \]
it follows that \( \eta \) is a solution of the Schrödinger-Ricci equation if and only if (7) holds.

Remark 2.2. Under the hypotheses of Theorem 2.1 if the potential vector field is of constant length \( k \), then from (7) we deduce that the scalar curvature is constant if either the soliton is a Ricci soliton or, \((\text{scal} + \lambda n + \mu k^2)\eta = \nabla_{\xi} \eta \) which implies \( \text{scal} = -\lambda n - \mu k^2 \).

Corollary 2.3. Let \((g, \xi, \lambda, \mu)\) be an \( \eta \)-Ricci soliton on the \( n \)-dimensional manifold \( M \) with \( \eta \) the \( g \)-dual of \( \xi \) and assume that \( \eta \) is a nontrivial solution of the Schrödinger-Ricci equation. If \( \text{scal} \) is constant and \( \mu \neq 0 \), then \( \frac{1}{2|x|^2} \xi(|x|^2) - \mu |x|^2 = \text{scal} + \lambda n \) (constant).

Proof. Under the hypotheses conditions, from (7) we obtain:
\[(\text{scal} + \lambda n + \mu |\xi|^2)\eta = \nabla_{\xi} \eta,\]
applying \( \xi \) and taking into account that \( (\nabla_{\xi} \eta)\xi = \frac{1}{2} \xi(|\xi|^2) \),
we deduce that \((\text{scal} + \lambda n + \mu |\xi|^2)|\xi|^2 = \frac{1}{2} \xi(|\xi|^2)\).

For the case of Ricci solitons, from Theorem 2.1 we have:

Corollary 2.4. If \((g, \xi, \lambda)\) is a Ricci soliton on the \( n \)-dimensional manifold \( M \) and \( \eta \) is the \( g \)-dual of \( \xi \), then \( \eta \) is a solution of the Schrödinger-Ricci equation if and only if the scalar curvature of the manifold is constant.

Schrödinger-Ricci harmonic forms

We say that a 1-form \( \gamma \) is \emph{Schrödinger-Ricci harmonic} if
\[(\Delta + S_2)(\gamma) = 0.\]

From (6), (4) and (5) we deduce:

Theorem 2.5. Let \((g, \xi, \lambda, \mu)\) be an \( \eta \)-Ricci soliton on the \( n \)-dimensional manifold \( M \) with \( \eta \) the \( g \)-dual of \( \xi \). Then \( \eta \) is Schrödinger-Ricci harmonic form if and only if \( \mu = 0 \) (which yields a Ricci soliton) or
\[(\text{scal} + \lambda n + \mu |\xi|^2)\eta = \nabla_{\xi} \eta - \frac{1}{2} d(|\xi|^2).\]
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Remark 2.6. Under the hypotheses of Theorem 2.5 if $\mu \neq 0$, then from (8) we deduce that the scalar curvature is constant if and only if the potential vector field is of constant length.

Harmonic forms

We know that on a Riemannian manifold $(M, g)$, a 1-form $\gamma$ is harmonic (i.e. $\Delta(\gamma) = 0$) if and only if it is closed and divergence free.

Remark that on an $\eta$-Ricci soliton, a harmonic 1-form $\gamma$ is Schrödinger-Ricci harmonic if and only if

$$\gamma \circ \nabla \xi + \lambda \gamma + \mu \gamma(\xi) \eta = 0$$

which implies (using the fact that $(\nabla_X \gamma)^\sharp = \nabla_X \gamma^\sharp$, for any vector field $X$ and any 1-form $\gamma$):

$$\gamma^\sharp \in \ker[\nabla_{\xi} \eta + (\lambda + \mu |\xi|^2) \eta].$$

From (2) and (5) we deduce:

Theorem 2.7. Let $(g, \xi, \lambda, \mu)$ be an $\eta$-Ricci soliton on the $n$-dimensional manifold $M$ with $\eta$ the $g$-dual of $\xi$. Then $\eta$ is harmonic form if and only if

$$(9) \quad iQ \xi g = \mu \{2[(\text{scal} + \lambda n + \mu |\xi|^2) \eta - \nabla_{\xi} \eta] + d(|\xi|^2)\}.$$ 

For the case of Ricci solitons, from Theorem 2.7 we have:

Corollary 2.8. If $(g, \xi, \lambda)$ is a Ricci soliton on the $n$-dimensional manifold $M$ and $\eta$ is the $g$-dual of $\xi$, then $\eta$ is harmonic form if and only if $\xi \in \ker Q$.

From (1), (8) and (9) we deduce:

Corollary 2.9. Let $(g, \xi, \lambda, \mu)$ be an $\eta$-Ricci soliton on the $n$-dimensional manifold $M$ with $\eta$ the $g$-dual of $\xi$. If $\eta$ is harmonic form, then i) $\xi \in \ker Q$ and ii) the scalar curvature is constant if and only if the potential vector field $\xi$ is of constant length.

The relation between the cases when $\eta$ is a solution of the Schrödinger-Ricci equation, harmonic or the Schrödinger-Ricci harmonic form is stated in the following result:
Lemma 2.10. Let \((g, \xi, \lambda, \mu)\) be an \(\eta\)-Ricci soliton on the \(n\)-dimensional manifold \(M\) with \(\eta\) the \(g\)-dual of \(\xi\).

i) If \(\eta\) is a solution of the Schrödinger-Ricci equation, then \(\eta\) is:
   a) Schrödinger-Ricci harmonic form if and only if \(\text{scal} + \mu |\xi|^2\) is constant;
   b) harmonic form if and only if \(i_Q\xi = \text{d}(\text{scal} + \mu |\xi|^2)\); also \(\eta\) harmonic implies \(\xi \in \ker Q\).

ii) If \(\eta\) is Schrödinger-Ricci harmonic form, then \(\eta\) is:
   a) a solution of the Schrödinger-Ricci equation if and only if \(\text{scal} + \mu |\xi|^2\) is constant;
   b) harmonic form if and only if \(\xi \in \ker Q\).

iii) If \(\eta\) is harmonic form, then \(\eta\) is:
   a) a solution of the Schrödinger-Ricci equation if and only if \(\xi \in \ker Q\);
   b) Schrödinger-Ricci harmonic form if and only if \(\xi \in \ker Q\).

We can synthetise:

i) if \(\text{scal} + \mu |\xi|^2\) is constant, then \(\eta\) is Schrödinger-Ricci harmonic if and only if it is a solution of the Schrödinger-Ricci equation;

ii) if \(\xi \in \ker Q\), then \(\eta\) is Schrödinger-Ricci harmonic if and only if it is harmonic.

1-forms orthogonal to \(\eta\)

We say that two 1-forms \(\gamma_1\) and \(\gamma_2\) are orthogonal if \(g(\gamma_1^\sharp, \gamma_2^\sharp) = 0\) (i.e. \(\langle \gamma_1, \gamma_2 \rangle = 0\), where \(\langle \gamma_1, \gamma_2 \rangle := \sum_{i=1}^{n} \gamma_1(E_i)\gamma_2(E_i)\), for \(\{E_i\}_{1 \leq i \leq n}\) a local orthonormal frame field).

Remark that \(\gamma_1\) and \(\gamma_2\) are orthogonal if and only if

\[
\gamma_1^\sharp \in \ker \gamma_2 \quad \text{or} \quad \gamma_2^\sharp \in \ker \gamma_1.
\]

Theorem 2.11. Let \((g, \xi, \lambda, \mu)\) be an \(\eta\)-Ricci soliton on the \(n\)-dimensional manifold \(M\) with \(\eta\) the \(g\)-dual of \(\xi\) and \(\mu \neq 0\). If \(\gamma\) is 1-form, then \(\gamma\) is orthogonal to \(\eta\) if and only if

\[
(10) \quad \nabla_{\gamma^\sharp}\xi + Q\gamma^\sharp + \lambda \gamma^\sharp \in \ker \gamma.
\]

Proof. Observe that computing the soliton equation in \((\gamma^\sharp, \gamma^\sharp)\) and using the orthogonality condition we obtain:

\[
(11) \quad g(\nabla_{\gamma^\sharp}\xi, \gamma^\sharp) + g(Q\gamma^\sharp, \gamma^\sharp) + \lambda |\gamma^\sharp|^2 = 0
\]

which is equivalent to the condition (10).
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Example

We end these considerations by discussing the case of a perfect fluid spacetime $(M, g, \xi)$ \cite{6}. If we denote by $p$ the isotropic pressure, $\sigma$ the energy-density, $\lambda$ the cosmological constant, $k$ the gravitational constant, $S$ the Ricci curvature tensor field and $\text{scal}$ the scalar curvature of $g$, then \cite{6}:

\begin{equation}
S = -(\lambda - \frac{\text{scal}}{2} - kp)g + k(\sigma + p)\eta \otimes \eta,
\end{equation}

and the scalar curvature of $M$ is:

\begin{equation}
\text{scal} = 4\lambda + k(\sigma - 3p).
\end{equation}

From Theorem 2.1 we deduce that if $(g, \xi, a, b)$ is an $\eta$-Ricci soliton on $(M, g, \xi)$, then $\eta$ is a solution of the Schrödinger-Ricci equation if and only if

\[ kd(\sigma - 3p) = 2b\{[4(a + \lambda) - b + k(\sigma - 3p)]\eta - \nabla_\xi \eta\}. \]

Moreover, the fluid is a radiation fluid (i.e. $\sigma = 3p$) if and only if $b = 0$ (which yields the Ricci soliton) or $[4(a + \lambda) - b]\eta = \nabla_\xi \eta$ which implies $b = 4(a + \lambda)$.

From Theorem 2.5 we deduce that if $(g, \xi, a, b)$ is an $\eta$-Ricci soliton on $(M, g, \xi)$, then $\eta$ is Schrödinger-Ricci harmonic form if and only if

\[ b\{[4(a + \lambda) - b + k(\sigma - 3p)]\eta = \nabla_\xi \eta \]

which implies $b = 4(a + \lambda) + k(\sigma - 3p)$.

From Theorem 2.7 we deduce that if $(g, \xi, a, b)$ is an $\eta$-Ricci soliton on $(M, g, \xi)$, then $\eta$ is harmonic form if and only if

\[ \{4b[4(a + \lambda) - b + k(\sigma - 3p)] - 2\lambda + k(\sigma + 3p)\} \eta = 4b\nabla_\xi \eta. \]

For the case of Ricci soliton $(g, \xi, a)$ in a radiation fluid we obtain the constant pressure $p = \frac{\lambda}{3k}$.

3 Applications to gradient solitons

Let $f \in C^\infty(M)$, $\xi := \text{grad}(f)$, $\eta := \xi^\flat$ and $\lambda$ and $\mu$ real constants. Then $\eta = df$ and

\begin{equation}
\nabla_X \xi, Y = g(\nabla_X \xi, Y),
\end{equation}

\[ g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \]
for any $X, Y \in \mathfrak{X}(M)$. Also [3]:

\begin{align}
(15) \quad \text{trace}(\eta \otimes \eta) &= |\xi|^2, \\
(16) \quad \text{div}(\eta \otimes \eta) &= \text{div}(\xi)\eta + \frac{1}{2}d(|\xi|^2)
\end{align}

and

\begin{equation}
(17) \quad \nabla_\xi \eta = \frac{1}{2}d(|\xi|^2).
\end{equation}

For the gradient $\eta$-Ricci solitons we have:

**Proposition 3.1.** If $(g, \xi := \text{grad}(f), \lambda, \mu)$ is a gradient $\eta$-Ricci soliton on the $n$-dimensional manifold $M$ and $\eta = df$ is the $g$-dual of $\xi$, then $\eta$ is a solution of the Schrödinger-Ricci equation if and only if

\begin{equation}
(18) \quad d(\text{scal}) = 2\mu[(\text{scal} + \lambda n + \mu|\xi|^2)df - \frac{1}{2}d(|\xi|^2)].
\end{equation}

Moreover, in this case, scal is constant if and only if $\mu = 0$ (which yields a gradient Ricci soliton) or $(\text{scal} + \lambda n + \mu|\xi|^2)df = \frac{1}{2}d(|\xi|^2)$.

**Remark 3.2.** Under the hypotheses of Proposition 3.1 if the potential vector field is of constant length $k$, then (18) becomes:

\begin{equation}
(19) \quad d(\text{scal}) = 2\mu(\text{scal} + \lambda n + \mu k^2)df,
\end{equation}

so the scalar curvature is constant if either the soliton is a gradient Ricci soliton or $\text{scal} = -\lambda n - \mu k^2$.

**Remark 3.3.** i) Taking into account that for a gradient vector field $\xi$ [5]:

\begin{equation}
(20) \quad \text{div}(L_\xi g) = 2d(\text{div}(\xi)) + 2iQ_\xi g,
\end{equation}

the condition for the $g$-dual $\eta = df$ of the potential vector field $\xi := \text{grad}(f)$ of a gradient $\eta$-Ricci soliton $(g, \xi, \lambda, \mu)$ to be a solution of the Schrödinger-Ricci equation is:

\begin{equation}
(21) \quad d(\text{scal} + \mu|\xi|^2) = iQ_\xi g.
\end{equation}

In this case, $\text{scal} + \mu|\xi|^2$ is constant if and only if $\xi \in \ker Q$ and from the $\eta$-Ricci soliton equation we obtain $\nabla_\xi \xi = - (\lambda + \mu|\xi|^2)\xi$. Applying $\eta$ we get $\lambda + \mu|\xi|^2 = -\frac{1}{2|\xi|^2}\xi(|\xi|^2)$.
therefore, if the length of $\xi$ is constant (also, the scalar curvature will be constant), then $|\xi|^2 = -\frac{\lambda}{\mu}$, hence $\xi$ is a geodesic vector field.

ii) If $\xi$ is an eigenvector of $Q$ (i.e. $Q\xi = a\xi$, with $a$ a smooth function), then $\eta$ is a solution of the Schrödinger-Ricci equation if and only if $scal + \mu|\xi|^2 - af$ is constant. In particular, if $\xi \in \ker Q$, then $\eta$ is a solution of the Schrödinger-Ricci equation if and only if $\eta$ is harmonic form.

iii) If $\eta$ is Schrödinger-Ricci harmonic form, then $d(scal + \mu|\xi|^2) = 2iQ\xi g$. In this case, $scal + \mu|\xi|^2$ is constant if and only if $\xi \in \ker Q$ and using the same arguments as in i) we deduce that $\xi$ is a geodesic vector field.

Also, an exact 1-form $df$ is harmonic if and only if the function $f$ is harmonic. In the case of a gradient $\eta$-Ricci soliton, for $\eta$ harmonic form, denoting by $\Delta f := \Delta - \nabla_{\text{grad}(f)}$ the $f$-Laplace-Hodge operator, the result stated in Theorem 3.2 from [5] becomes:

**Theorem 3.4.** Let $(g, \xi := \text{grad}(f), \lambda, \mu)$ be a gradient $\eta$-Ricci soliton on the $n$-dimensional manifold $M$ with $\eta = df$ the $g$-dual of $\xi$. If $\eta$ is harmonic form, then:

$$\frac{1}{2} \Delta f(|\xi|^2) = |\text{Hess}(f)|^2 + \lambda|\xi|^2 + \mu|\xi|^4.$$  

Using Corollary 2.9 we get:

**Corollary 3.5.** Under the hypotheses of Theorem 3.4, if $M$ is of constant scalar curvature, then at least one of $\lambda$ and $\mu$ is non positive.

As a consequence for the case of gradient Ricci soliton, we have:

**Proposition 3.6.** Let $(g, \xi := \text{grad}(f), \lambda)$ be a gradient Ricci soliton on the $n$-dimensional manifold $M$ of constant scalar curvature, with $\eta = df$ the $g$-dual of $\xi$. If $\eta$ is harmonic form, then the soliton is shrinking.

**Proof.** From Theorem 2.9 and Theorem 3.4 we obtain $|\text{Hess}(f)|^2 + \lambda|\xi|^2 = 0$, hence $\lambda < 0$.

**Remark 3.7.** i) Assume that $\mu \neq 0$. If $\lambda \geq -\mu|\xi|^2$, then $\Delta f(|\xi|^2) \geq 0$ and from the maximum principle follows that $|\xi|^2$ is constant in a neighborhood of any local maximum. If $|\xi|$ achieve its maximum, then $M$ is quasi-Einstein. Indeed, since $\text{Hess}(f) = 0$, from the soliton equation we have $S = -\lambda g - \mu df \otimes df$. Moreover, in this case, $|\xi|^2(\lambda + \mu|\xi|^2) = 0$, which implies either $\xi = 0$ or $|\xi|^2 = -\frac{\lambda}{\mu} \geq 0$. Since $scal + \lambda n + \mu|\xi|^2 = 0$ we get $scal = \lambda(1 - n)$.

ii) For $\mu = 0$, we get the Ricci soliton case [22].
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Computing the gradient soliton equation in $(\gamma^\sharp, X), \ X \in \mathfrak{X}(M)$, we obtain:

$$g(\nabla_{\gamma^\sharp} \xi, X) + g(Q\gamma^\sharp, X) + \lambda g(\gamma^\sharp, X) + \mu \eta(\gamma^\sharp) \eta(X) = 0$$

and taking $X := \xi$ we get:

$$\frac{1}{2} \gamma^\sharp(|\xi|^2) + \gamma(Q\xi) + (\lambda + \mu |\xi|^2) \eta(\gamma^\sharp) = 0.$$  

Therefore:

**Proposition 3.8.** Let $(g, \xi, \lambda, \mu)$ be an $\eta$-Ricci soliton on the $n$-dimensional manifold $M$ with $\eta$ the $g$-dual of $\xi$ and $\mu \neq 0$. If $\gamma$ is 1-form, then $\gamma$ is orthogonal to $\eta$ if and only if

$$(23) \quad \nabla_{\gamma^\sharp} \xi + Q\gamma^\sharp + \lambda \gamma^\sharp = 0,$$

hence:

$$(24) \quad \frac{1}{2} \gamma^\sharp(|\xi|^2) = -\gamma(Q\xi).$$

Some results concerning the harmonic 1-forms on gradient $\eta$-Ricci solitons are further presented.

For two $(0,2)$-tensor fields $T_1$ and $T_2$, denote by $\langle T_1, T_2 \rangle := \sum_{1 \leq i,j \leq n} T_1(E_i, E_j) T_2(E_i, E_j)$, for $\{E_i\}_{1 \leq i \leq n}$ a local orthonormal frame field.

**Theorem 3.9.** Let $M$ be a compact and oriented $n$-dimensional manifold $M$, $(g, \xi := \text{grad}(f), \lambda, \mu)$ a gradient $\eta$-Ricci soliton with $\eta = df$ the $g$-dual of $\xi$ and $\gamma$ a 1-form.

1. If $\gamma$ is orthogonal to $\eta$ and $\mu \neq 0$, then $\gamma^\sharp \in \ker(\nabla_\xi \eta + \eta \circ Q)$.

2. If $\gamma$ is harmonic, then either we have a Ricci soliton or $\nabla_\xi \gamma^\sharp \in \ker \eta$.

3. If $\gamma$ is exact with $\gamma = du$, then:

$$(25) \quad \int_M \langle S, \text{div}(du) \rangle = -\int_M \langle \text{Hess}(f), \text{Hess}(u) \rangle - \mu(\eta(\nabla_{\text{grad}(f)} \text{grad}(u))).$$

Moreover, if $\gamma$ is harmonic, the relation $(22)$ becomes:

$$(26) \quad \int_M \langle \text{Hess}(f), \text{Hess}(u) \rangle = -\mu(\eta(\nabla_{\text{grad}(f)} \text{grad}(u))).$$
Proof. From (24) and using (14) we get:

\[ 0 = g(\nabla_\gamma \xi, \xi) + g(Q\xi, \gamma^z) = \xi(\eta(\gamma^z)) - \eta(\nabla_\xi \gamma^z) + g(\xi, Q\gamma^z) = (\nabla_\xi \eta)\gamma^z + \eta(Q\gamma^z) \]

and hence 1.

Let \( \{E_i\}_{1 \leq i \leq n} \) be a local orthonormal frame field with \( \nabla E_i E_j = 0 \) in a point. For any symmetric \((0,2)\)-tensor field \( Z \) and any 1-form \( \gamma \):

\[ \langle Z, L_\gamma g \rangle = \sum_{1 \leq i,j \leq n} Z(E_i, E_j)(L_\gamma g)(E_i, E_j) = 2 \sum_{1 \leq i,j \leq n} Z(E_i, E_j)g(\nabla E_i \gamma^z, E_j) = 2 \sum_{1 \leq i,j \leq n} Z(E_i, E_j)E_i(\gamma(E_j)) = 2\langle Z, \text{div}(\gamma) \rangle. \]

Also:

\[ \langle g, L_\gamma g \rangle = \sum_{i=1}^n (L_\gamma g)(E_i, E_i) = 2 \sum_{i=1}^n g(\nabla E_i \gamma^z, E_i) = 2\text{div}(\gamma^z) \]

and

\[ \langle df \otimes df, L_\gamma g \rangle = \sum_{1 \leq i,j \leq n} df(E_i)df(E_j)(L_\gamma g)(E_i, E_j) = 2 \sum_{1 \leq i,j \leq n} df(E_i)df(E_j)g(\nabla E_i \gamma^z, E_j) = 2g(\nabla \text{grad}(f)\gamma^z, \text{grad}(f)) = 2g(\nabla \text{grad}(f)\gamma^z, (df)^z). \]

Computing \( \langle S, \text{div}(\gamma) \rangle \) by replacing \( S \) from the \( \eta \)-Ricci soliton equation, we obtain:

\[ \langle S, \text{div}(\gamma) \rangle = -\frac{1}{2}\langle \text{Hess}(f), L_\gamma g \rangle - \lambda \text{div}(\gamma^z) - \mu g((\nabla \text{grad}(f)\gamma^z, (df)^z). \]

For 2. we use \( \text{div}(\gamma) = 0 = \text{div}(\gamma^z) \) and for 3. we use the fact that \( \gamma^z = \text{grad}(u) \), hence \( L_\gamma g = 2\text{Hess}(u) \) and apply the divergence theorem.

Since

\[ \eta(\nabla_\xi \xi) = \frac{1}{2}\xi(|\xi|^2) \]

and for \( \eta \) harmonic:

\[ \int_M |\text{Hess}(f)|^2 = -\mu \int_M df(\nabla_\xi \xi), \]

we get:

**Corollary 3.10.** Under the hypotheses of Theorem 3.9, if \( \eta \) is harmonic form, then either we have a Ricci soliton or the potential vector field \( \xi \) is of constant length. In the second case, \( \eta \) is a solution of the Schrödinger-Ricci equation and \( M \) is quasi-Einstein manifold.
We know that a Bochner-type formula for an arbitrary vector field $\xi$ states:

$$\frac{1}{2}\Delta(|\xi|^2) = |\nabla\xi|^2 + S(\xi, \xi) + \xi(div(\xi))$$

and taking into account that the $g$-dual 1-form $\eta$ of $\xi$ satisfies

$$|\xi| = |\eta|, \quad |\nabla\xi| = |\nabla\eta|, \quad S(\xi, \xi) = S^\sharp(\eta, \eta), \quad \xi(div(\xi)) = \langle \Delta(\eta), \eta \rangle,$$

we have the corresponding relation for $\eta$:

\begin{equation}
\frac{1}{2}\Delta(|\eta|^2) = |\nabla\eta|^2 + S^\sharp(\eta, \eta) + \langle \Delta(\eta), \eta \rangle.
\end{equation}

Let $\gamma$ be a 1-form and writing the previous relation for $\eta + \gamma$ we obtain:

$$\frac{1}{2}\Delta(\langle \eta, \gamma \rangle) = \langle \nabla\eta, \nabla\gamma \rangle + S^\sharp(\eta, \gamma) + \frac{1}{2}(\langle \Delta(\eta), \gamma \rangle + \langle \Delta(\gamma), \eta \rangle).$$

**Theorem 3.11.** Let $M$ be an $n$-dimensional manifold $M$, $(g, \xi := \text{grad}(f), \lambda, \mu)$ a gradient $\eta$-Ricci soliton with $\eta = df$ the $g$-dual of $\xi$ and $\gamma$ a 1-form. Then:

\begin{equation}
\frac{1}{2}\Delta(\langle df, \gamma \rangle) = \langle \text{Hess}(f), \nabla\gamma \rangle - \mu\Delta(f)\langle df, \gamma \rangle + \frac{1}{2}\langle df, \Delta(f) \rangle.
\end{equation}

**Proof.** From (4), (16), (20) and $2\text{div}(S) = d(\text{scal})$, we get:

$$S^\sharp(\eta, \gamma) = S(\xi, \eta^\sharp) = -\frac{1}{2}d(\Delta(f))(\gamma^\sharp) - \mu\Delta(f)\langle df, \gamma^\sharp \rangle = -\frac{1}{2}\langle \Delta(df), \gamma \rangle - \mu\Delta(f)\langle df, \gamma \rangle,$$

hence (28). \qed

**Proposition 3.12.** Let $M$ be an $n$-dimensional manifold $M$, $(g, \xi := \text{grad}(f), \lambda, \mu)$ a gradient $\eta$-Ricci soliton with $\eta = df$ the $g$-dual of $\xi$ and $\gamma$ a 1-form.

1. If $\gamma$ is orthogonal to $\eta$, then $\langle \text{Hess}(f), \nabla\gamma \rangle = -\frac{1}{2}\langle df, \Delta(\gamma) \rangle$.

2. If $\gamma$ is harmonic, then $\frac{1}{2}\Delta(\langle df, \gamma \rangle) = \langle \text{Hess}(f), \nabla\gamma \rangle - \mu\Delta(f)\langle df, \gamma \rangle$. In this case, $\langle df, \gamma \rangle$ is harmonic if and only if $\mu\Delta(f)\langle df, \gamma \rangle = \langle \text{Hess}(f), \nabla\gamma \rangle$.

Moreover, if $\gamma$ is orthogonal to $\eta$, then $\nabla\gamma$ is orthogonal to $\nabla\eta$. 

**$L^2_f$ harmonic 1-forms**

Endow the Riemannian manifold $(M, g)$ with the weighted volume form $e^{-f}dV$ and define $L^2_f$ forms those forms $\gamma$ satisfying $\int_M |\gamma|^2 e^{-f}dV < \infty$.

The most natural operator of Laplacian type associated to the weighted manifold $(M, g, e^{-f}dV)$ is the $f$-Laplace-Hodge operator

$$\Delta_f := \Delta - \nabla_{\text{grad}(f)}$$

which is self-adjoint with respect to this measure.

We say that a 1-form $\gamma$ is $f$-harmonic if

$$\Delta_f(\gamma) = 0.$$  

Remark that $\gamma$ is $f$-harmonic if and only if

$$\Delta(\gamma_0) = i_{\nabla^*_{\gamma_0} \xi} g.$$  

From (4) and (17) we deduce:

**Proposition 3.13.** Let $(g, \xi := \text{grad}(f), \lambda, \mu)$ be a gradient $\eta$-Ricci soliton on the $n$-dimensional manifold $M$ with $\eta = df$ the $g$-dual of $\xi$. Then $\eta$ is $f$-harmonic form if and only if $\text{scal} + (\mu + \frac{1}{2})|\xi|^2$ is constant.

In terms of $\Delta_f$, the relation (27) can be written [21]:

(29) \[ \frac{1}{2} \Delta_f(|\gamma|^2) = |\nabla \gamma|^2 + S^g_f(\gamma, \gamma) + \langle \Delta_f(\gamma), \gamma \rangle, \]

where $S_f := \text{Hess}(f) + S$ is the Bakry-Émery Ricci tensor.

Using a Reilly-type formula involving the $f$-Laplacian, an interesting result was obtained in [17], namely, if the manifold $M$ is the boundary of a compact and connected Riemannian manifold and has non negative $m$-dimensional Bakry-Émery Ricci curvature and non negative $f$-mean curvature, then either $M$ is connected or it has only two connected components, in the later case, being totally geodesic.

Another interesting topological property will be stated in the next theorem:

**Theorem 3.14.** Let $(\mathbb{M}^n, g, e^{-f}dV)$ be a complete, non compact smooth metric measure space and $(g, \xi := \text{grad}(f), \lambda, \mu)$ a gradient $\eta$-Ricci soliton with $\eta = df$ the $g$-dual of $\xi$. If there exists a non trivial $L^2_f$ harmonic 1-form $\gamma_0$ such that $\lambda |\gamma_0|^2 + \mu (\gamma_0(\xi))^2 \leq 0$, then $M$ has finite volume and its universal covering splits isometrically into $\mathbb{R} \times \mathbb{N}^{n-1}$. 
Proof. The condition \( \lambda |\gamma_0|^2 + \mu (\gamma_0 (\xi))^2 \leq 0 \) is equivalent to \( S_f^2 (\gamma_0, \gamma_0) \geq 0 \). From \( \gamma \) and Lemma 3.2 from [25]:
\[
|\gamma_0| \Delta_f (|\gamma_0|) \geq 0.
\]

Following the same steps as in [25], we obtain the conclusion.

Remark 3.15. i) Under the hypothesis of Theorem 3.14 in particular, we deduce that \( \gamma_0 \) is \( \nabla \)-parallel and of constant length. Also, \( \lambda \leq 0 \) since in [23] was shown that \( \lambda > 0 \) implies \( M \) compact.

ii) In the Ricci soliton case, the hypothesis of Theorem 3.14 requires that the space of \( L_f^2 \) harmonic 1-forms to be nonempty and the Ricci soliton to be shrinking in order to get the same conclusion.

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Department of Mathematics
West University of Timișoara
Bld. V. Pârvan nr. 4, 300223, Timișoara, România
adarablaga@yahoo.com