Persistence of Manifolds in Nonequilibrium Critical Dynamics

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We study the persistence $P(t)$ of the magnetization of a $d'$ dimensional manifold (i.e. the probability that the manifold magnetization does not flip up to time $t$, starting from a random initial condition) in a $d$-dimensional spin system at its critical point. We show analytically that there are three distinct late time decay forms for $P(t)$: exponential, stretched exponential and power law, depending on a single parameter $\zeta = (D - 2 + \eta)/z$ where $D = d - d'$ and $\eta, z$ are standard critical exponents. In particular, our theory predicts that the persistence of a line magnetization decays as a power law in the $d = 2$ Ising model at its critical point. For the $d = 3$ critical Ising model, the persistence of the plane magnetization decays as a power law, while that of a line magnetization decays as a stretched exponential. Numerical results are consistent with these analytical predictions.

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Following a rapid quench of a spin system from infinite temperature to zero temperature, domains of competing ground states form and grow with time [1]. In the standard nonconserved dynamics a given spin inside the sample flips only when a domain wall passes through it, which happens rather rarely. As a result, persistence, i.e. the probability that the spin remains unflipped up to time $t$ decays slowly as a power law, $P(t) \sim t^{-\theta}$ at late times [2]. In contrast, if the spin system is quenched to its critical temperature $T_c$, any given spin fluctuates rapidly due to the finite temperature and the persistence of a single spin has an exponential tail at late times. However, for a quench to $T_c$, the persistence of the total magnetization of the sample (as opposed to that of a single spin) again decays as a power law, $P(t) \sim t^{-\theta'}$, where the exponent $\theta'$ has been argued to be a new nonequilibrium critical exponent [3]. The global persistence has since been studied in a wide variety of systems [4].

In a $d$-dimensional sample, a single spin is a 0-dimensional manifold. On the other hand, the global magnetization corresponds to summing over all the spins of the full sample which is a $d$-dimensional manifold. To interpolate between these two limits, it is natural to study the persistence of the magnetization of a $d'$-dimensional manifold with $0 \leq d' \leq d$. In the two limits $d' = 0$ and $d' = d$, the persistence has very different asymptotic decay, respectively exponential and power law. The following question then naturally arises: if one tunes the manifold dimension $d'$ from $d' = 0$ to $d' = d$, how does the asymptotic behavior of persistence changes from an exponential ($d' = 0$) to a power law decay ($d' = d$)? Does this change occur abruptly at some intermediate value of $d'$, or is there an intermediate regime of $d'$ where the behavior is neither exponential nor power law, but something in between? In this Letter we address this interesting issue and we show analytically that indeed there is a novel intermediate regime of $d'$ where the persistence has a stretched exponential tail. Our main results can be summarized in terms of a single number $\zeta = (D - 2 + \eta)/z$, where $D = d - d'$ is the co-dimension of the manifold and $\eta$ and $z$ are the standard critical exponents: $z$ is the dynamical exponent that describes the temporal growth of the correlation length $\xi(t) \sim t^{\eta/z}$ and $\eta$ describes the power law decay of the equilibrium spin-spin correlation function at $T_c$, $(\phi(0)\phi(r)) \sim r^{-(d - 2 + \eta)}$ for large separation $r$. Depending on the value of $\zeta$, the persistence of the magnetization of a $d'$-dimensional manifold at $T_c$ has the asymptotic behavior,

$$ P(t) \sim t^{-\theta(d',d)}, \qquad \zeta < 0 $$

$$ \sim \exp(-a_1 t^\zeta), \quad 0 \leq \zeta \leq 1 $$

$$ \sim \exp(-b_1 t^\zeta), \quad \zeta > 1, $$

(1)

where the exponent $\theta(d',d)$ depends on $d'$ and $d$ and $a_1, b_1$ are constants. Strictly speaking, in the intermediate regime $0 \leq \zeta \leq 1$ we will show that, $\exp(-a_2 t^\zeta \ln t) \leq P(t) \leq \exp(-a_1 t^\zeta)$ for large $t$. We will derive the results in Eq. (1) within the mean-field theory (valid for $d > 4$), in the $n \to \infty$ limit of the $O(n)$ model, followed by a general scaling theory.

Two specific applications of our general results are as follows: (i) Consider the persistence of the line magnetization ($d' = 1$) in the $d = 2$ Ising model at $T_c$. Using $d' = 1, d = 2, \eta = 1/4$ and $z \approx 2.172$, one gets $\zeta = (d - d' - 2 + \eta)/z \approx -0.3453 < 0$. Hence Eq. (1) predicts a power law decay for the persistence of the line magnetization; (ii) For the $d = 3$ Ising model, using $\eta \approx 0.032$ and $z \approx 2$, one finds that while for the plane magnetization ($d' = 2$), $\zeta \approx -0.484 < 0$, for the line magnetization ($d' = 1$), $0 < \zeta \approx 0.016 < 1$. Our Eq. (1) then predicts a power law decay for the persistence of the plane magnetization, but a stretched exponential decay for the line magnetization. Numerical simulations for small samples in 2 and 3 dimensions are consistent with these analytical predictions.

Our starting point is the standard Langevin equation for the vector order parameter $\vec{\phi} = (\phi_1, \ldots, \phi_n)$,
\[ \partial_t \phi_i = \nabla^2 \phi_i - r \phi_i - \left( u/n \right) \phi_i \sum_{j=1}^{n} \phi_j^2 + \eta_i, \]  

(2)

where \( \eta(x,t) \) is a Gaussian white noise with mean zero and correlator \( \langle \eta_i(x,t) \eta_j(x',t') \rangle = 2 \delta_{ij} \delta^d(x-x') \delta(t-t'). \) The magnetization of a \( d \)-dimensional manifold will be defined by the vector field, \( \psi_i(x,d+1, \ldots, x_d, t) = \int \phi_i(\vec{x}, t) \prod_{i=1}^{d} \frac{dx_i}{\sqrt{2\pi}}, \) obtained by integrating the order parameter over the \( d' \) directions. Here \( L \) denotes the length of the sample along any direction. For a vector order parameter, we define the persistence \( P(t) \) of the manifold to be the probability that any given component of the manifold magnetization, say \( \psi_i \), does not change sign up to time \( t \). Since all the components of the spin are equivalent, henceforth we will drop the subscript \( i \) of \( \psi_i \) for convenience. Note that the magnetization \( \psi \) is a field over the remaining \( D = d - d' \) dimensional space whose coordinates \( (x_{d+1}, \ldots, x_d) \) will be relabelled by the vector \( \vec{r} = (r_1, r_2, \ldots, r_D) \) for convenience. The observation that allows us to make analytical predictions for the persistence of the magnetization \( \psi(\vec{r}, t) \) is that it is a Gaussian variable at all finite times. This follows simply from the fact that \( \psi(\vec{r}, t) \) is a sum of \( L^d \) random variables which are correlated but only over a finite correlation length \( \xi(t) \sim t^{1/2} \). Thus in the thermodynamic limit where \( t^{1/2} \ll L \), the central limit theorem asserts that \( \psi(\vec{r}, t) \) is a Gaussian field. Hence its persistence is determined by the autocorrelation function \( C(t_1, t_2) = \langle \psi(\vec{r}, t_1) \psi(\vec{r}, t_2) \rangle \) [2].

We start with the mean-field theory, valid for \( d \geq 4 \), where we set \( u = 0 \), and also \( r = 0 \) (at the critical point) in Eq. (2). Next we integrate the Langevin equation over the \( d' \) space directions and then solve the resulting linear equation in the Fourier space. Defining \( \hat{\psi}(\vec{k}, t) = \int \psi(\vec{r}, t) e^{i \vec{k} \cdot \vec{r}} d^D r \), it is easy to compute the two-time correlation function,

\[
\langle \hat{\psi}(\vec{k}, t_1) \hat{\psi}(\vec{k}, t_2) \rangle = \Delta(\vec{k}) e^{-k^2(t_1 + t_2)} + \frac{1}{2} \left( e^{-k^2|t_1 - t_2|} - e^{-k^2(t_1 + t_2)} \right),
\]

(3)

where \( \Delta(\vec{k}) = \langle \hat{\psi}(\vec{k}, 0) \hat{\psi}(\vec{k}, 0) \rangle \) is taken to be a constant (for uncorrelated initial condition). At late times, the initial condition dependent term becomes negligible and it is sufficient to retain only the second term on the right-hand side of Eq. (3). The autocorrelation function is then obtained by integrating over the \( k \) space, \( C(t_1, t_2) = \int d^D k \ e^{-k^2 a^2 \Delta(\vec{k})} \langle \hat{\psi}(\vec{k}, t_1) \hat{\psi}(\vec{k}, t_2) \rangle \) where we have introduced a soft ultraviolet cut-off \( a \). One finds

\[
C(t_1, t_2) = \beta A \left[ (t_1 + t_2 + a^2)^{2\beta} - (|t_1 - t_2| + a^2)^{2\beta} \right],
\]

(4)

where \( 2\beta = (2 - D)/2 \) and \( A \) is an unimportant constant.

Consider first the case \( D < 2 \), i.e., \( \beta > 0 \). In this case there is no need for the ultraviolet cut-off since \( \langle \psi^2(\vec{r}, t) \rangle = C(t, t) \) does not diverge at any finite time even if one puts \( a = 0 \). Setting \( a = 0 \) in Eq. (4) we find that the correlation function \( C(t_1, t_2) \) is still non-stationary. However, one can render it stationary by employing a well known trick [2], where one introduces a normalized Gaussian process, \( X = \psi/\sqrt{\langle \psi^2 \rangle} \) which, when observed in the logarithmic time \( T = \ln t \), becomes a stationary Gaussian variable with the correlator, \( C(T) = \langle X(0)X(T) \rangle = \cosh(T/2)^{2\beta} - |\sinh(T/2)|^{2\beta} \). Interestingly, the same Gaussian correlator also appears in the context of the persistence of the Edwards-Wilkinson type rough interfaces [6]. It is well known [5,2] that for such a stationary Gaussian correlator (decaying exponentially for large \( T \)), the persistence of the process also decays exponentially for large \( T \), \( P(T) \sim \exp(-\beta T) \). In terms of the real time \( t = e^T \), this indicates a power law decay, \( P(t) \sim t^{-\theta} \), with persistence exponent \( \theta \). For this particular correlator, the exponent \( \theta \) has been studied in great detail in the context of the interface problem [6] and the exponent is known to depend continuously on the roughness parameter \( \beta = (2 - D)/4 > 0 \).

For \( D > 2 \), on the other hand, one needs the ultraviolet cut-off explicitly in order to keep \( \langle \psi^2(\vec{r}, t) \rangle = C(t, t) \) finite. In that case, the appropriate scaling limit is \( t_1, t_2 \rightarrow \infty \), but keeping their difference \( |t_1 - t_2| \) fixed. In this limit, Eq. (4) reduces to a stationary correlator in the original time variable, \( C(t_1, t_2) \sim (|t_1 - t_2| + a^2)^{-(D-2)/2} \) that decays as a power law for large \( |t_1 - t_2| \). To calculate the persistence of Gaussian stationary processes with an algebraically decaying correlator is nontrivial. However, there exists a powerful theorem due to Newell and Rosenblatt [7], which states that if the stationary correlator decays as \( C(t) \sim t^{-\alpha} \) with \( \alpha > 0 \) for large time difference \( t = |t_1 - t_2| \), then the persistence \( P(t) \) (probability of no zero crossing between \( t_1 \) and \( t_2 \)) of such a process has the following asymptotic behaviors: (i) \( P(t) \sim \exp(-K_1 t) \) if \( \alpha > 1 \) and (ii) \( \exp(-K_2 t^{\alpha}) \) for \( \alpha < 1 \), where the \( K_i \)'s are constants. Applying this theorem to our problem, we find that \( P(t) \sim \exp(-K_1 t) \) for \( D > 4 \) and \( \exp(-K_2 t^{(D-2)/2}) \) for \( 2 < D < 4 \). In the borderline case \( D = 4 \), there will be an additional logarithmic correction.

Combining these results for \( D < 2 \) and \( D > 2 \) and noting that \( z = 2 \) and \( \eta = 0 \) within the mean-field theory, we find that the explicit exact results for the mean-field theory derived above are just the special cases of the general result in Eq. (1) provided one uses the mean-field value \( \zeta = (D - 2)/2 \) in Eq. (1).

The mean-field theory is valid for \( d \geq 4 \). In order to access the physically relevant dimensions \( d \leq 4 \), we now consider another solvable limit, namely the \( n \rightarrow \infty \) limit where Eq. (2) becomes

\[ \partial_t \phi_i = \nabla^2 \phi_i - [r + S(t)] \phi_i + \eta_i, \]

(5)
where \( S(t) = u \langle \phi^2 \rangle \) has to be determined self-consistently. The critical point corresponds to \( r + S(\infty) = 0 \). This self-consistent determination of \( S(t) \) can be done using standard techniques [3] and one finds that at late times, \( S(t) \to S(\infty) - (4 - d)/4t \) for \( 2 < d \leq 4 \). Substituting this result into Eq. (5), summing over the \( d' \) directions, solving the resulting equation in the Fourier space and finally integrating over the \( \vec{k} \) space (as in the mean-field theory), we finally arrive at the following autocorrelation function for the manifold magnetization \( \psi(\vec{r}, t) \) in \( 2 < d \leq 4 \),

\[
C(t_1, t_2) = A_1(t_1 t_2)^{(4-d)/4} \int_0^{t_1} \frac{dt'}{(t_1 + t_2 - 2t' + a^2)^{D'/2}},
\]

where \( t_1 \leq t_2 \), \( A_1 \) is a constant and \( a \) represents the soft ultraviolet cut-off as before.

For \( D < 2 \), as in the mean-field theory, one can set the cut-off \( a = 0 \) and the resulting nonstationary correlator in Eq. (6) can be reduced to a stationary correlator for the normalized process \( X = \psi/\sqrt{\psi^2} \) in the logarithmic time \( T = \ln t \),

\[
A(T) = [\cosh(T/2)^{d-D/2} B[\mu, 2\beta, 2/(1 + e^T)] B[\mu, 2\beta]],
\]

where \( \mu = (d-2)/2, 2\beta = (2-D)/2 \), \( B[m, n] \) is the standard Beta function and \( B[m, n, x] = \int_0^x dy y^{m-1}(1-y)^{n-1} \). Since the stationary correlator in Eq. (7) decays exponentially for large \( T \), \( A(T) \sim \exp[-(d+D-2)T/4] \), one concludes [5, 2] that the corresponding persistence also decays exponentially for large \( T \), \( P(T) \sim \exp(-\theta T) \) and hence as a power law in the original \( t = e^T \) variable, \( P(t) \sim t^{-\theta} \). Determining the exponent \( \theta \) analytically is still a challenging task. However one can make progress in the limit of small co-dimension \( D \to 0 \). Note that for \( D = 0 \), i.e. for the global persistence, Eq. (7) becomes a pure exponential \( A(T) = \exp[-(d-2)T/4] \) for all \( T \), indicating that the process is Markovian [3]. One then finds \( P(T) \sim \exp(-\theta_0 T) \) with \( \theta_0 = (d-2)/4 \) [3]. For \( D \) nonzero but small, one can expand the correlator in Eq. (7) around the Markov process \( (D = 0) \) and then use a perturbation theory result [8] to calculate \( \theta_0 \) to first order in \( D \). We get \( \theta = \theta_0 + D \theta_0 L_d/\pi + O(D^2) \) for all \( 2 < d \leq 4 \), where \( L_d \) is a complicated d-dependent integral. For special values of \( d \), this integral simplifies [9]. For example, for \( d = 4 \), we get, \( \theta = 1/2 + (2\sqrt{2} - 1)D/4 + O(D^2) \) and for \( d = 3 \), \( \theta = 1/4 + 0.183615 \ldots D + O(D^2) \).

For \( D > 2 \), it is evident from Eq. (6) that one needs to keep a nonzero cut-off \( a \) in order that the integral does not diverge at \( t_1 = t_2 \). In this case, for large \( t_1 \), the dominant contribution to the integral comes from the regime \( t' \to t_1 \). It is easy to see that in the limit when \( t_1, t_2 \) are both large with their difference \( |t_1 - t_2| \) fixed, the autocorrelator in Eq. (6) reduces to a stationary one,

\[
C(t_1, t_2) \approx B_1(|t_1 - t_2| + a^2)^{-(D-2)/2}, \quad B_1 \text{ is an unimportant constant.}
\]

One can then invoke the Newell-Rosenblatt theorem [7] once again to conclude that for large \( t \), the persistence \( P(t) \sim \exp(-\kappa_1 t) \) if \( D > 4 \) and \( \exp(-\kappa_2 t^{(D-2)/2} \ln t) \leq P(t) \leq \exp(-\kappa_3 t^{(D-2)/2}) \) for \( 2 < D < 4 \), where the \( \kappa_i \)’s are constants. Combining these results we thus find that the \( n \to \infty \) results for the persistence of manifold magnetization in \( 2 < d \leq 4 \) are also compatible with our general results in Eq. (1) on noting that \( \zeta = (D-2)/2 \) since \( \eta = 0 \) and \( z = 2 \) within the large \( n \) limit.

Taking hints from the two solvable cases above we now construct a general scaling theory valid for all \( d \geq 2 \). The two-point correlation function of the order parameter, at the critical point, has the generic scaling form, \( \langle \phi(0, t_1)\phi(x, t_2) \rangle \sim x^{-(d-2+\eta)}F(xt_1^{1/z}, t_2/t_1) \) for large distance \( x \) and large times \( t_1, t_2 \), where \( \eta = z = 2 \).

\[
\langle \phi(0, t_1)\phi(x, t_2) \rangle \sim k^{-(2-\eta)}G(Kt_1^{1/z}, t_2/t_1), \quad \text{where} \quad K \quad \text{is a d-dimensional vector conjugate to} \ x.
\]

The manifold magnetization \( \psi \) is obtained by summing the order parameter \( \phi \) over \( d' \) directions. This is equivalent to putting \( K_i = 0 \) along the \( i = 1, \ldots, d' \) directions. One then obtains the scaling behavior of the two point correlator of the manifold magnetization, \( \langle \phi(0, t_1)\phi(x, t_2) \rangle \sim x^{-(d-2+\eta)} \langle \phi(0, t)\phi(x, t) \rangle \sim x^{-(d-2+\eta)}F(xt_1^{1/z}, t_2/t_1) \), the \( k \) now is a \( D - d' \) dimensional vector. For example, within the mean-field theory, the scaling function \( g(x, y) = \exp[-x^2(1 + y)] - \exp[-x^2(1 + y)] \), as evident from Eq. (3) after dropping the \( \Delta \) dependent term at late times. The autocorrelation function \( C(t_1, t_2) = \langle \psi(\vec{r}, t_1)\psi(\vec{r}, t_2) \rangle \) is then obtained by integrating over \( \vec{k} \),

\[
C(t_1, t_2) = \int \frac{d^Dk}{k^{2-\eta}} g(kt_1^{1/z}, t_2/t_1)e^{-k^2a^2}, \quad a \text{ is the soft ultraviolet cut-off as before.}
\]

Consider first the case when \( D - 2 + \eta < 0 \). One can then set the cut-off \( a = 0 \) (since the integral in Eq. (8) is convergent at the upper limit), and one obtains \( C(t_1, t_2) \approx t_1^{-(D-2+\eta)/2}f(t_2/t_1) \) in the limit \( t_1, t_2 \to \infty \) with \( t_2/t_1 \) arbitrary. Note that the function \( f(x) \sim x^{-\lambda_1/z} \) for large \( x \) such that \( C(t_1, t_2) \sim t_2^{\lambda_1/z} \) for \( t_2 \gg t_1 \) where \( \lambda_1 \) is the standard autocorrelation exponent \( [10, 11, 1] \). This nonstationary Gaussian correlator can then be reduced, as before, to a stationary one for the normalized variable \( X = \psi/\sqrt{\psi^2} \) in the logarithmic time, \( T = \ln t \) and one gets, \( A(T) = \langle X(0)X(T) \rangle = \exp[(D-2+\eta)/2]f(1) \). Since \( A(T) \sim \exp[-(D-2+\eta)/2]f(1) \) for large \( T \), it follows, as before, that the persistence \( P(T) \sim \exp(-\theta T) \) for large \( T \). This means that the persistence decays as a power law in the original time variable \( t = e^T \), \( P(t) \sim t^{-\theta} \) for large \( t \).
In the complementary case $D - 2 + \eta > 0$, the integral in Eq. (8) is, for $t_1 = t_2$, divergent near the upper limit without the cut-off. Hence one needs to keep $\alpha$ nonzero and then the appropriate scaling limit is obtained by taking $t_1, t_2$ both large keeping their difference $|t_1 - t_2|$ fixed but arbitrary. Then one can replace the scaling function $g(kt_1^{1/\eta}, t_2/t_1)$ in Eq. (8) by another function $g_1(k) = (|t_1 - t_2|^{1/\eta})$ of a single scaling variable, as in the two previous solvable cases. Performing the integral, one then finds $C(t_1, t_2) \sim |t_1 - t_2|^{-(D - 2 + \eta)/z}$ for $|t_1 - t_2| \gg a^2$.

This correlator is stationary and decays as a power law. Invoking the Newell-Rosenblatt theorem once more, we find that $P(t)$ decays exponentially for $(D - 2 + \eta)/z > 1$ and as a stretched exponential for $0 < (D - 2 + \eta)/z < 1$. Combining this with the result for $D - 2 + \eta < 0$ outlined in the previous paragraph, gives our general result in Eq. (1) on defining $\zeta = (D - 2 + \eta)/z$.

\[ \ln[P(t)] \sim -10 \ln(t) \quad \text{for} \quad \theta = 0.72. \]

Similarly the persistence for the plane magnetization in $d = 3$ shows a power law decay, $P(t) \sim t^{-\theta}$ with $\theta \approx 0.88$, estimated from small lattice sizes $L = 15$ and 31. The estimates of these exponents are only rough and may shift a little with larger lattices. The persistence of the line magnetization in $d = 3$, in contrast, has a much faster decay. Note that our theory predicts a decay, $P(t) \sim \exp(-a_1 t^\zeta)$ where $\zeta \approx 0.016$. Such a small stretching exponent is difficult to determine from the small size lattices due to the strong finite-size effects at late times. All we can say is that the present data for the line magnetization in $d = 3$ is consistent with a decay of persistence that is faster than a power law but slower than an exponential. More extensive simulations and a somewhat sophisticated finite size scaling analysis are required to pin down the precise value of the stretching exponent [9].

In summary, we have shown that the persistence of submanifolds in a critical system decays with time in a manner that depends on the dimensions $d$ and $d'$ of the system and manifold respectively. The crossover between the power-law decay observed for the ‘global’ persistence ($d' = d$) and the exponential decay of the ‘local’ persistence ($d' = 0$), occurs via an intermediate regime of stretched exponential decay with stretched exponent $\zeta = (d - d' - 2 + \eta)/z$, for $0 \leq \zeta \leq 1$. This latter behavior is predicted for the line magnetization of the $d = 3$ critical Ising model. Numerical simulations are consistent with the analytical predictions.

\[ \text{FIG. 1. The three outer curves represent the persistence of the line magnetization in } d = 2 \text{ critical Ising model as a function of time for system sizes } L = 63, 95 \text{ and 127. The two curves in the middle represent the persistence of the plane magnetization in } d = 3 \text{ critical Ising model for system sizes } L = 15 \text{ and 31. For the same system sizes, the two leftmost curves represent the persistence of the line magnetization in } d = 3. \]

To test the analytical predictions (i) and (ii), we have done preliminary Monte Carlo simulations for $d = 2$ and $d = 3$ Ising model for small size lattices with periodic boundary conditions. The results are summarized in Fig. 1. The systems were evolved using heat-bath Monte Carlo dynamics at their bulk critical couplings: $K_c = [\ln(1 + \sqrt{2})]/2$ in $d = 2$ and $K_c \approx 0.221656$ in $d = 3$. For $d = 2$, the data for the persistence of the line magnetization for system sizes $L = 63, 95$ and 127 (in each case the data was averaged over 1000 samples) shows a power law decay $P(t) \sim t^{-\theta}$ for $t \ll L^2$ and crosses over to a faster decay for $t \gg L^2$. A fit to the linear part of the data on the log-log plot gives an estimate $\theta \approx 0.72$.

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