Jet geometrical extension of the KCC-invariants

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Abstract

In this paper we construct the jet geometrical extensions of the KCC-invariants, which characterize a given second-order system of differential equations on the 1-jet space $J^1(\mathbb{R}, M)$. A generalized theorem of characterization of our jet geometrical KCC-invariants is also presented.

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1 Geometrical objects on 1-jet spaces

We remind first several differential geometrical properties of the 1-jet spaces. The 1-jet bundle

$$\xi = (J^1(\mathbb{R}, M), \pi_1, \mathbb{R} \times M)$$

is a vector bundle over the product manifold $\mathbb{R} \times M$, having the fibre of type $\mathbb{R}^n$, where $n$ is the dimension of the spatial manifold $M$. If the spatial manifold $M$ has the local coordinates $(x^i)_{i=1}^n$, then we shall denote the local coordinates of the 1-jet total space $J^1(\mathbb{R}, M)$ by $(t, x^i, x^i_1)$; these transform by the rules

$$\begin{align}
\tilde{t} &= \tilde{t}(t) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^i_1 &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{dt} \cdot x^j_1.
\end{align}$$

(1.1)

In the geometrical study of the 1-jet bundle, a central role is played by the distinguished tensors ($d-$tensors).

Definition 1.1. A geometrical object $D = (D_{1k(1)(i)})$ on the 1-jet vector bundle, whose local components transform by the rules

$$D_{1k(1)(i)}^{1(j)(1)...} = \tilde{D}_{1r(1)(s)}^{1(p)(m)(n)...} \frac{dt}{dt} \frac{\partial x^i}{\partial \tilde{x}^p} \left( \frac{\partial x^j}{\partial \tilde{x}^m} \frac{dt}{dt} \frac{\partial \tilde{x}^r}{\partial x^k} \left( \frac{\partial x^u}{\partial x^v} \frac{dt}{dt} \right) \right) $$

(1.2)

is called a $d-$tensor field.
Remark 1.2. The use of parentheses for certain indices of the local components
\[ D^{(j)(1)}_{(1)(l)} \]
of the distinguished tensor field \( D \) on the 1-jet space is motivated by the fact that the pair of indices \((j)\) or \((1)\) behaves like a single index.

Example 1.3. The geometrical object
\[ C = C^{(i)}_{(1)} \frac{\partial}{\partial x^i} \]
where \( C^{(i)}_{(1)} = x^i \), represents a \( d \)-tensor field on the 1-jet space; this is called the canonical Liouville \( d \)-tensor field of the 1-jet bundle and is a global geometrical object.

Example 1.4. Let \( h = (h_{11}(t)) \) be a Riemannian metric on the relativistic time axis \( \mathbb{R} \). The geometrical object
\[ J_h = J^{(i)}_{(1)1} \frac{\partial}{\partial x^i} \otimes dt \otimes dx^j, \]
where \( J^{(i)}_{(1)1} = h_{11} \delta^i_j \) is a \( d \)-tensor field on \( J^1(\mathbb{R}, M) \), which is called the \( h \)-normalization \( d \)-tensor field of the 1-jet space and is a global geometrical object.

In the Riemann-Lagrange differential geometry of the 1-jet spaces developed in [11], [12] important rôles are also played by geometrical objects as the temporal or spatial semisprays, together with the jet nonlinear connections.

Definition 1.5. A set of local functions \( H = \left( H^{(j)}_{(1)1} \right) \) on \( J^1(\mathbb{R}, M) \), which transform by the rules
\[ 2\tilde{H}^{(k)}_{(1)1} = 2H^{(j)}_{(1)1} \left( \frac{dt}{dt} \right)^2 \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{dt}{dt} \frac{\partial \tilde{x}^k}{\partial t}, \quad (1.3) \]
is called a temporal semispray on \( J^1(\mathbb{R}, M) \).

Example 1.6. Let us consider a Riemannian metric \( h = (h_{11}(t)) \) on the temporal manifold \( \mathbb{R} \) and let
\[ H^1_{11} = \frac{h^{11}}{2} \frac{dt}{dt}, \]
where \( h^{11} = 1/h_{11} \), be its Christoffel symbol. Taking into account that we have the transformation rule
\[ \tilde{H}^1_{11} = H^1_{11} \frac{dt}{dt} + \frac{\tilde{d}t}{dt} \frac{d^2t}{dt^2}, \quad (1.4) \]
we deduce that the local components
\[ \tilde{H}^{(j)}_{(1)1} = -\frac{1}{2} H^1_{11} x^j \]
define a temporal semispray \( \hat{H} = \left( \hat{H}^{(j)}_{(1)1} \right) \) on \( J^1(\mathbb{R}, M) \). This is called the \textit{canonical temporal semispray associated to the temporal metric} \( h(t) \).

**Definition 1.7.** A set of local functions \( G = \left( G^{(j)}_{(1)1} \right) \), which transform by the rules

\[
2\tilde{G}^{(k)}_{(1)1} = 2G^{(j)}_{(1)1} \left( \frac{dt}{dt} \right)^{2} \frac{\partial x^{k}}{\partial x^{j}} - \frac{\partial x^{m}}{\partial x^{j}} \frac{\partial x^{k}}{\partial x^{m}},
\]

is called a \textit{spatial semispray} on \( J^1(\mathbb{R}, M) \).

**Example 1.8.** Let \( \varphi = (\varphi_{ij}(x)) \) be a Riemannian metric on the spatial manifold \( M \) and let us consider its Christoffel symbols. Taking into account that we have the transformation rules

\[
\tilde{\gamma}_{pq}^{i} = \gamma_{jk} \frac{\partial \tilde{x}^{p}}{\partial x^{i}} \frac{\partial \tilde{x}^{j}}{\partial x^{k}} + \frac{\partial \tilde{x}^{p}}{\partial \tilde{x}^{i}} \frac{\partial^{2} \tilde{x}^{j}}{\partial \tilde{x}^{i} \partial \tilde{x}^{r}},
\]

we deduce that the local components

\[
\tilde{G}^{(j)}_{(1)1} = \frac{1}{2} \gamma^{i} x^{i}_{j} m_{1}
\]

define a spatial semispray \( \tilde{G} = \left( \tilde{G}^{(j)}_{(1)1} \right) \) on \( J^1(\mathbb{R}, M) \). This is called the \textit{canonical spatial semispray} associated to the spatial metric \( \varphi(x) \).

**Definition 1.9.** A set of local functions \( \Gamma = \left( M^{(j)}_{(1)1}, N^{(j)}_{(1)1} \right) \) on \( J^1(\mathbb{R}, M) \), which transform by the rules

\[
\tilde{M}^{(k)}_{(1)1} = M^{(j)}_{(1)1} \left( \frac{dt}{dt} \right)^{2} \frac{\partial \tilde{x}^{k}}{\partial x^{j}} - \frac{\partial x^{m}}{\partial x^{j}} \frac{\partial \tilde{x}^{k}}{\partial x^{m}},
\]

and

\[
\tilde{N}^{(k)}_{(1)1} = N^{(j)}_{(1)1} \left( \frac{dt}{dt} \right)^{2} \frac{\partial \tilde{x}^{k}}{\partial x^{j}} - \frac{\partial x^{m}}{\partial x^{j}} \frac{\partial \tilde{x}^{k}}{\partial x^{m}},
\]

is called a \textit{nonlinear connection} on the 1-jet space \( J^1(\mathbb{R}, M) \).

**Example 1.10.** Let us consider that \( (\mathbb{R}, h(t)) \) and \( (M, \varphi_{ij}(x)) \) are Riemannian manifolds having the Christoffel symbols \( H_{11}^{1}(t) \) and \( \gamma_{jk}^{i}(x) \). Then, using the transformation rules \( (1.1), (1.4) \) and \( (1.6) \), we deduce that the set of local functions

\[
\tilde{\Gamma} = \left( \tilde{M}^{(j)}_{(1)1}, \tilde{N}^{(j)}_{(1)1} \right),
\]

where

\[
\tilde{M}^{(j)}_{(1)1} = -H_{11}^{1} x^{j} \quad \text{and} \quad \tilde{N}^{(j)}_{(1)1} = \gamma^{i} m_{1} x^{i}
\]

represents a nonlinear connection on the 1-jet space \( J^1(\mathbb{R}, M) \). This jet nonlinear connection is called the \textit{canonical nonlinear connection attached to the pair of Riemannian metrics} \( (h(t), \varphi(x)) \).
In the sequel, let us study the geometrical relations between temporal or spatial semisprays and nonlinear connections on the 1-jet space $J^1(\mathbb{R}, M)$. In this direction, using the local transformation laws (1.3), (1.7) and (1.1), respectively the transformation laws (1.5), (1.8) and (1.1), by direct local computation, we find the following geometrical results:

**Theorem 1.11.**

a) The temporal semisprays $H = (H^{(j)}_{(1)1})$ and the sets of temporal components of nonlinear connections $\Gamma_{\text{temporal}} = (M^{(j)}_{(1)1})$ are in one-to-one correspondence on the 1-jet space $J^1(\mathbb{R}, M)$, via:

$$M^{(j)}_{(1)1} = 2H^{(j)}_{(1)1}, \quad H^{(j)}_{(1)1} = \frac{1}{2}M^{(j)}_{(1)1}.$$  

b) The spatial semisprays $G = (G^{(j)}_{(1)1})$ and the sets of spatial components of nonlinear connections $\Gamma_{\text{spatial}} = (N^{(j)}_{(1)1})$ are connected on the 1-jet space $J^1(\mathbb{R}, M)$, via the relations:

$$N^{(j)}_{(1)1} = \frac{\partial G^{(j)}_{(1)1}}{\partial x^i}, \quad G^{(j)}_{(1)1} = \frac{1}{2}N^{(j)}_{(1)1}x^m_1.$$  

### 2 Jet geometrical KCC-theory

In this Section we generalize on the 1-jet space $J^1(\mathbb{R}, M)$ the basics of the KCC-theory ([1], [4], [6], [13]). In this respect, let us consider on $J^1(\mathbb{R}, M)$ a second-order system of differential equations of local form

$$\frac{d^2 x^i}{dt^2} + F^{(i)}_{(1)1}(t, x^k, x^k_1) = 0, \quad i = 1, n, \quad (2.1)$$

where $x^k_1 = dx^k/dt$ and the local components $F^{(i)}_{(1)1}(t, x^k, x^k_1)$ transform under a change of coordinates (1.1) by the rules

$$\tilde{F}^{(i)}_{(1)1} = F^{(i)}_{(1)1} \left(\frac{dt}{d\tilde{t}}\right)^2 \frac{\partial \tilde{x}^i}{\partial x^j} - \frac{dt}{d\tilde{t}} \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \frac{\partial x^m}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^i}{\partial x^m}. \quad (2.2)$$

**Remark 2.1.** The second-order system of differential equations (2.1) is invariant under a change of coordinates (1.1).

Using a temporal Riemannian metric $h_{11}(t)$ on $\mathbb{R}$ and taking into account the transformation rules (1.3) and (1.5), we can rewrite the SODEs (2.1) in the following form:

$$\frac{d^2 x^i}{dt^2} - H_{11} x^i_1 + 2G^{(i)}_{(1)1}(t, x^k, x^k_1) = 0, \quad i = 1, n,$$

where

$$G^{(i)}_{(1)1} = \frac{1}{2}F^{(i)}_{(1)1} + \frac{1}{2}H_{11} x^i_1.$$
are the components of a spatial semispray on \(J^1(\mathbb{R}, M)\). Moreover, the coefficients of the spatial semispray \(G^{(i)}_{(1)j}\) produce the spatial components \(N^{(i)}_{(1)j}\) of a nonlinear connection \(\Gamma\) on the 1-jet space \(J^1(\mathbb{R}, M)\), by putting

\[
N^{(i)}_{(1)j} = \frac{\partial G^{(i)}_{(1)1}}{\partial x^j_1} = \frac{1}{2} \frac{\partial F^{(i)}_{(1)1}}{\partial x^j_1} + \frac{1}{2} H^{11}_{11} \delta^j_1.
\]

In order to find the basic jet differential geometrical invariants of the system \((2.1)\) (see Kosambi [10], Cartan [8] and Chern [9]) under the jet coordinate transformations \((1.1)\), we define the \(h-KCC\)-covariant derivative of a \(d\)-tensor of kind \(T^{(i)}_{(1)}(t, x^k, x^k_1)\) on the 1-jet space \(J^1(\mathbb{R}, M)\) via

\[
\frac{h}{dt} T^{(i)}_{(1)} = \frac{dT^{(i)}_{(1)}}{dt} + N^{(i)}_{(1)r} T^{(r)}_{(1)} - H^{11}_{11} T^{(i)}_{(1)} = \frac{dT^{(i)}_{(1)}}{dt} + \frac{1}{2} \frac{\partial F^{(i)}_{(1)1}}{\partial x^j_1} T^{(r)}_{(1)} - \frac{1}{2} H^{11}_{11} T^{(i)}_{(1)},
\]

where the Einstein summation convention is used throughout.

**Remark 2.2.** The \(h-KCC\)-covariant derivative components \(\frac{h}{dt} T^{(i)}_{(1)}\) transform under a change of coordinates \((1.1)\) as a \(d\)-tensor of type \(T^{(i)}_{(1)1}\).

In such a geometrical context, if we use the notation \(x^i_1 = dx^i/dt\), then the system \((2.1)\) can be rewritten in the following distinguished tensorial form:

\[
\frac{h}{dt} x^i_1 = -F^{(i)}_{(1)1}(t, x^k, x^k_1) + N^{(i)}_{(1)r} x^r_1 - H^{11}_{11} x^i_1 = -F^{(i)}_{(1)1} + \frac{1}{2} \frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1} x^r_1 - \frac{1}{2} H^{11}_{11} x^i_1.
\]

**Definition 2.3.** The distinguished tensor

\[
\frac{h}{dt} x^i_1 = -F^{(i)}_{(1)1} + \frac{1}{2} \frac{\partial F^{(i)}_{(1)1}}{\partial x^r_1} x^r_1 - \frac{1}{2} H^{11}_{11} x^i_1
\]

is called the first \(h-KCC\)-invariant on the 1-jet space \(J^1(\mathbb{R}, M)\) of the SODEs \((2.1)\), which is interpreted as an external force [1], [0].

**Example 2.4.** It can be easily seen that for the particular first order jet rheonomic dynamical system

\[
\frac{dx^i}{dt} = X^{(i)}_{(1)}(t, x^k) \Rightarrow \frac{d^2 x^i}{dt^2} = \frac{\partial X^{(i)}_{(1)}}{\partial t} + \frac{\partial X^{(i)}_{(1)}}{\partial x^m} x^m_1,
\]

(2.3)
where \( X^{(i)}_1(t, x) \) is a given \( d \)-tensor on \( J^1(\mathbb{R}, M) \), the first \( h \)-KCC-invariant has the form
\[
\xi^{(i)}_1 = \frac{\partial X^{(i)}_1}{\partial t} + \frac{1}{2} \frac{\partial X^{(i)}_1}{\partial x^r} x^r_1 - \frac{1}{2} H^{1}_{11} x^i_1.
\]

In the sequel, let us vary the trajectories \( x^i(t) \) of the system (2.1) by the nearby trajectories \( (x^i(t, s))_{s \in (-\varepsilon, \varepsilon)} \), where \( x^i(t, 0) = x^i(t) \). Then, considering the variation \( d \)-tensor field
\[
\xi^i(t) = \frac{\partial x^i}{\partial s} \bigg|_{s=0},
\]
we get the variational equations
\[
\frac{d^2 \xi^i}{dt^2} + \frac{\partial F^{(i)}(1)}{\partial x^j} \xi^j + \frac{\partial F^{(i)}(1)}{\partial x^r_1} \frac{d x^r}{dt} = 0.
\]

In order to find other jet geometrical invariants for the system (2.1), we also introduce the \( h \)-KCC-covariant derivative of a \( d \)-tensor of kind \( \xi^i(t) \) on the 1-jet space \( J^1(\mathbb{R}, M) \) via
\[
\frac{h}{d} \frac{D \xi^i}{dt} = \frac{d \xi^i}{dt} + H^{(i)}(1)_{m} \xi^m = \frac{d \xi^i}{dt} + \frac{1}{2} \frac{\partial F^{(i)}(1)}{\partial x^m_1} \xi^m + \frac{1}{2} H^{1}_{11} \xi^i.
\]

**Remark 2.5.** The \( h \)-KCC-covariant derivative components \( \frac{h}{d} \frac{D \xi}{dt} \) transform under a change of coordinates (1.1) as a \( d \)-tensor \( T^{(i)}_n \).

In this geometrical context, the variational equations (2.4) can be rewritten in the following distinguished tensorial form:
\[
\frac{h}{d} \frac{D \xi^i}{dt} = \frac{h}{d} T^{i}_{m11} \xi^m,
\]
where
\[
\frac{h}{d} T^{i}_{j11} = - \frac{\partial F^{(i)}(1)}{\partial x^j} + \frac{1}{2} \frac{\partial^2 F^{(i)}(1)}{\partial x^r_1} x^r_1 + \frac{1}{2} \frac{\partial^2 F^{(i)}(1)}{\partial x^r_1} x^r_1 - \frac{1}{2} \frac{\partial^2 F^{(i)}(1)}{\partial x^r_1} x^r_1 + \frac{1}{4} \frac{d H^{1}_{11}}{d t} \delta^i_j - \frac{1}{4} H^{1}_{11} H^{1}_{11} \delta^i_j.
\]

**Definition 2.6.** The \( d \)-tensor \( \frac{h}{d} T^{i}_{j11} \) is called the second \( h \)-KCC-invariant on the 1-jet space \( J^1(\mathbb{R}, M) \) of the system (2.1), or the jet \( h \)-deviation curvature \( d \)-tensor.
Example 2.7. If we consider the second-order system of differential equations of the harmonic curves associated to the pair of Riemannian metrics \((h_{11}(t), \varphi_{ij}(x))\), system which is given by (see the Examples 1.6 and 1.8)

\[
\frac{d^2 x^i}{dt^2} - H_{11}^i(t) \frac{dx^i}{dt} + \gamma_{jk}^i(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0,
\]

where \(H_{11}^i(t)\) and \(\gamma_{jk}^i(x)\) are the Christoffel symbols of the Riemannian metrics \(h_{11}(t)\) and \(\varphi_{ij}(x)\), then the second \(h-KCC\)-invariant has the form

\[
P_{j11}^i = -R_{pqj}^i x_1^p x_1^q,
\]

where

\[
R_{pqj}^i = \frac{\partial \gamma_{pq}^i}{\partial x^j} - \frac{\partial \gamma_{pj}^i}{\partial x^q} + \gamma_{pq}^r \gamma_{rj}^i - \gamma_{pj}^r \gamma_{rq}^i
\]

are the components of the curvature of the spatial Riemannian metric \(\varphi_{ij}(x)\). Consequently, the variational equations (2.4) become the following jet Jacobi field equations:

\[
\frac{h}{dt} \left[ \frac{h}{dt} \xi^i \right] + R_{pqm}^i x_1^p x_1^q \xi^m = 0,
\]

where

\[
\frac{h}{dt} \xi^i = \frac{d \xi^i}{dt} + \gamma_{jm}^i \xi^j.
\]

Example 2.8. For the particular first order jet rheonomic dynamical system (2.3) the jet \(d\)-deviation curvature \(d\)-tensor is given by

\[
P_{j11}^i = \frac{1}{2} \frac{\partial^2 X_{(1)}^{(i)}}{\partial t \partial x^j} + \frac{1}{2} \frac{\partial^2 X_{(1)}^{(i)}}{\partial x^j \partial x^j} + \frac{1}{4} \frac{\partial X_{(1)}^{(i)}}{\partial x^j} \frac{\partial X_{(1)}^{(r)}}{\partial x^j} + \frac{1}{2} \frac{d H_{11}^i}{dt} \delta_j^i - \frac{1}{4} H_{11}^i H_{11}^j \delta_j^i.
\]

Definition 2.9. The distinguished tensors

\[
P_{j11}^i = \frac{1}{3} \left[ \frac{h}{\partial x_1^1} \frac{\partial P_{j11}^i}{\partial x_1^1} - \frac{h}{\partial x_1^1} \frac{\partial P_{j11}^i}{\partial x_1^1} \right], \quad F_{jkm} = \frac{\partial h}{\partial x_1^1}
\]

are called the third, fourth and fifth \(h-KCC\)-invariant on the 1-jet vector bundle \(J^1(\mathbb{R}, M)\) of the system (2.1).

Remark 2.10. Taking into account the transformation rules (2.2) of the components \(P_{j11}^i\), we immediately deduce that the components \(D_{jkm}^i\) behave like a \(d\)-tensor.
Example 2.11. For the first order jet rheonomic dynamical system \( \mathcal{L} \) the third, fourth and fifth \( h-\text{KCC-invariants} \) are zero.

Theorem 2.12 (of characterization of the jet \( h-\text{KCC-invariants} \)). All the five \( h-\text{KCC-invariants} \) of the system \( \mathcal{L} \) cancel on \( J^1(\mathbb{R}, M) \) if and only if there exists a flat symmetric linear connection \( \Gamma^i_{jk}(x) \) on \( M \) such that

\[
F^{(i)}_{(1)1} = \Gamma_{pq}^i(x)x_1^px_1^q - H_1^1(t)x_1^i.
\] (2.5)

Proof. \( \Leftarrow \) By a direct calculation, we obtain

\[
h_{\varepsilon}^{(i)} = 0,
\]

\[
P_{j11}^i = -\mathfrak{R}^i_{pqj}x_1^px_1^q = 0
\]

and

\[
D_{ijkl}^1 = 0
\]

where \( \mathfrak{R}^i_{pqj} = 0 \) are the components of the curvature of the flat symmetric linear connection \( \Gamma^i_{jk}(x) \) on \( M \).

\( \Rightarrow \) By integration, the relation

\[
D_{ijkl}^1 = \frac{\partial^3 F^{(i)}_{(1)1}}{\partial x_1^j \partial x_1^k \partial x_1^l} = 0
\]

subsequently leads to

\[
\frac{\partial^2 F^{(i)}_{(1)1}}{\partial x_1^j \partial x_1^k} = 2\Gamma_{jk}(t, x) \Rightarrow \frac{\partial F^{(i)}_{(1)1}}{\partial x_1^i} = 2\Gamma_{jp}x_1^p + \mathcal{U}^{(i)}_{(1)j}(t, x) \Rightarrow
\]

\[
\Rightarrow F^{(i)}_{(1)1} = \Gamma_{pq}x_1^px_1^q + \mathcal{U}^{(i)}_{(1)p}x_1^p + \mathcal{V}^{(i)}_{(1)1}(t, x),
\]

where the local functions \( \Gamma^i_{jk}(t, x) \) are symmetrical in the indices \( j \) and \( k \).

The equality \( \varepsilon_{(1)1}^i = 0 \) on \( J^1(\mathbb{R}, M) \) leads us to

\[
\mathcal{V}^{(i)}_{(1)1} = 0
\]

and to

\[
\mathcal{U}^{(i)}_{(1)j} = -H_1^1\delta^i_j.
\]

Consequently, we have

\[
\frac{\partial F^{(i)}_{(1)1}}{\partial x_1^i} = 2\Gamma_{jp}x_1^p - H_1^1\delta^i_j
\]

and

\[
F^{(i)}_{(1)1} = \Gamma_{pq}x_1^px_1^q - H_1^1x_1^i.
\]

The condition \( P_{j11}^i = 0 \) on \( J^1(\mathbb{R}, M) \) implies the equalities \( \Gamma^i_{jk} = \Gamma^i_{jk}(x) \) and

\[
\mathfrak{R}^i_{pqj} + \mathfrak{R}^i_{qjp} = 0,
\]

and

\[
\mathcal{U}^{(i)}_{(1)p} = 0.
\]
where
\[ R^i_{pqj} = \frac{\partial \Gamma^i_{pq}}{\partial x^j} - \frac{\partial \Gamma^i_{pj}}{\partial x^q} + \Gamma^r_{pq} \Gamma^i_{rj} - \Gamma^r_{pj} \Gamma^i_{rq}. \]

It is important to note that, taking into account the transformation laws (2.2), (1.3) and (1.1), we deduce that the local coefficients \( \Gamma^i_{jk}(x) \) behave like a symmetric linear connection on \( M \). Consequently, \( R^i_{pqj} \) represent the curvature of this symmetric linear connection.

On the other hand, the equality \( h R^i_{jk} = 0 \) leads us to \( R^i_{qjk} = 0 \), which infers that the symmetric linear connection \( \Gamma^i_{jk}(x) \) on \( M \) is flat. \( \square \)

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