Quantum black holes and resolution of the singularity

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Abstract

We present a quantum description of black holes with a matter core given by coherent states of gravitons. The expected behaviour in the weak-field region outside the horizon is recovered, with arbitrarily good approximation, but the classical central singularity cannot be resolved because the coherent states do not contain modes of arbitrarily short wavelength. Ensuing quantum corrections both in the interior and exterior are also estimated by assuming the mean-field approximation continues to hold. These deviations from the classical black hole geometry could result in observable effects in the gravitational collapse of compact objects and both astrophysical and microscopic black holes.

1 Introduction and motivation

The gravitational collapse of compact objects generates geodesically incomplete spacetimes in general relativity if a trapping surface appears [1] and eternal point-like sources are not mathematically compatible with Einstein’s field equations [2]. We expect the quantum theory will fix this inconsistent classical picture of the gravitational interaction, like quantum mechanics explains the stability of atoms by not admitting quantum states corresponding to the classical ultraviolet catastrophe. At the same time, the quantum state of a macroscopic black hole must reproduce the phenomenology of spacetime we detect experimentally [3,4].

There are many quantum models of black holes in the literature (for a very partial list, see Refs. [5–7]). In particular, the corpuscular picture [5] belongs to the class of approaches for which geometry should only emerge at suitable (macroscopic) scales from the underlying (microscopic) quantum field theory of gravitons [8,9]. The key idea is that the constituents of black holes are soft gravitons marginally bound in their own potential and forming a condensate [5] with characteristic Compton-de Broglie wavelength

\[ \lambda_G \sim R_H , \] (1.1)

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where the gravitational (or Schwarzschild) radius of the black hole of Arnowitt-Deser-Misner (ADM) mass \[ M \] is given by

\[ R_H = 2 G_N M . \]  

(1.2)

The energy scale of the gravitons is correspondingly given by \( \epsilon_G \sim \hbar / \lambda_G \) and, if one assumes that the total mass of the black hole \( M \approx N_G \epsilon_G \), there immediately follows the scaling relation

\[ N_G \sim \frac{M^2}{m_p^2} \sim \frac{R_H^2}{\ell_p^2} , \]  

(1.3)

which reproduces Bekenstein’s conjecture for the horizon area quantisation [11].

The nonlinearity of the gravitational interaction plays a crucial role in this picture. This can be seen by considering that the (negative) gravitational energy of a source of mass \( M \) localised inside a sphere of radius \( R_s \) is given by \( U_N \sim M V_N(R_s) \), where

\[ V_N = -\frac{G_N M}{r} \]  

(1.4)

is the Newtonian potential. This potential can be obtained as the expectation value of a scalar field on a coherent state, whose normalisation then yields the graviton number (1.3) for any values of \( R_s \gtrsim R_H \) [12–14]. In addition to that, assuming most gravitons have the same wavelength \( \lambda_G \), the binding energy of each graviton is given by

\[ \epsilon_G \sim \frac{U_N}{N_G} \sim -\frac{\ell_p m_p}{R_s} , \]  

(1.5)

which yields the typical Compton-de Broglie length \( \lambda_G \sim R_s \). The graviton self-interaction energy hence reproduces the (positive) post-Newtonian energy,

\[ U_{GG} \sim N_G \epsilon_G V_N(R_s) \sim \frac{G_N^2 M^3}{R_s^2} , \]  

(1.6)

and the fact that gravitons in a black hole are marginally bound [5], that is \( U_N + U_{GG} \approx 0 \), finally yields the scaling (1.1) with \( \lambda_G \sim R_s \approx R_H \) [12, 14].

A key feature of the above scenario is that, like for all bound states in quantum physics, it does not contain modes of arbitrarily small wavelength and the classical central singularity is therefore not realised. However, viewing a black hole as a quantum state made of only gravitons with one typical wavelength (1.1) cannot reproduce the gravitational field in the accessible outer spacetime, even in the simple Newtonian approximation. In this work, we precisely consider how to address this (conceptually and phenomenologically) important issue. Our present argument will solely make use of the quantum description for the (Newtonian) potential (1.4) in the vacuum by means of a suitable coherent state, albeit with the important difference that the size \( R_s \) of the matter source will be explicitly accounted for in the definition of this state. In the original corpuscular picture, as briefly reviewed above, baryonic matter sourcing the gravitational field and triggering the gravitational

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1 Units with \( c = 1 \) are used throughout and the Newton constant \( G_N = \ell_p / m_p \), where \( \ell_p \) is the Planck length and \( m_p \) the Planck mass, so that \( \hbar = \ell_p m_p \).

2 This assumption can be viewed as a manifestation of the classicalization of gravity [15]. Independent arguments in support of this condition, based on the quantum nature of the source, were also given in Refs. [16, 17]. For other similar considerations, see Refs. [18].
collapse is argued to become essentially irrelevant after the black hole forms [5]. We shall instead follow the point of view of Refs. [12,14,16,17,19], according to which matter inside the black hole might still play a very significant role in defining the structure of the interior of astrophysical black holes (and possibly even microscopic ones). For the sake of simplicity and generality, the only piece of information regarding this matter source we will use here is given by its characteristic size represented by the radius $R_s$.  

Beside obtaining a more refined description of corpuscular black holes with a material core, aim of this work is also to start investigating deviations from the unique classical black hole geometries in the near-horizon region, where experimental bounds are not particularly strong and possibly sizeable quantum effects are therefore not yet ruled out. In fact, we will refer to Newtonian physics in the main text mostly for the sake of keeping the presentation simple. However, we remark that the same potential (1.4) appears in the fully general relativistic geodesic equation in the Schwarzschild spacetime [17,20] and its quantum version can therefore be employed in order to reconstruct a quantum corrected complete metric. From the latter, one can then analyse how the material core affects the black hole thermodynamics and test particle motion (see Appendix A for some preliminary results).

In the next section, we will briefly review how coherent states of a massless scalar field on a reference flat spacetime can be used to reproduce a classical potential, in general, and the gravitational potential (1.4) in particular; quantum corrections due to the material core will then be introduced and analysed in Section 3 and Appendix A; final remarks, outlook and connections with other works will be given in Section 4.

2 Quantum coherent state for Newtonian sources

We will first review how to describe a generic static potential $V = V(r)$ as the mean field of the coherent state of a free massless scalar field (for more details, see Ref. [14]). We first rescale the potential $V$ so as to obtain a canonically normalised real scalar field $\Phi = \sqrt{m_p} / \ell_p V$, and then quantise $\Phi$ as a massless field satisfying the free wave equation

$$\left[ -\frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \right] \Phi(t,r) \equiv (-\partial_t^2 + \Delta) \Phi = 0 .$$  

(2.1)

Solutions to Eq. (2.1) can be conveniently written as

$$u_k(t,r) = e^{-ikt} j_0(kr) ,$$  

(2.2)

where $k > 0$ and $j_0 = \sin(kr)/kr$ are spherical Bessel functions satisfying the orthogonality relation

$$4\pi \int_0^\infty r^2 \, dr \, j_0(kr) \, j_0(pr) = \frac{2\pi^2}{k^2} \delta(k-p) .$$  

(2.3)

\[3\]In a fully quantum picture, one expects that $R_s$ be at best the expectation value of some emergent operator in the relevant state of matter fields. For more considerations on the scale $R_s$ for quantum black holes, see the concluding remarks and Ref. [17].

\[4\]The two interpretations are respectively recovered by considering the radial coordinate $r$ as “harmonic” (in the Newtonian approximation) or areal (for the fully relativistic case).
The quantum field operator and its conjugate momentum read

\[
\hat{\Phi}(t, r) = \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} \sqrt{\frac{\hbar}{2k}} \left[ \hat{\alpha}_k u_k(t, r) + \hat{\alpha}_k^\dagger u_k^*(t, r) \right] 
\]

(2.4)

\[
\hat{\Pi}(t, r) = i \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} \sqrt{\frac{\hbar k}{2}} \left[ \hat{\alpha}_k u_k(t, r) - \hat{\alpha}_k^\dagger u_k^*(t, r) \right] ,
\]

(2.5)

which satisfy the equal time commutation relations,

\[
\left[ \hat{\Phi}(t, r), \hat{\Pi}(t, s) \right] = i \hbar \frac{4}{4\pi r^2} \delta(r - s) ,
\]

(2.6)

provided the creation and annihilation operators obey the commutation rules

\[
\left[ \hat{\alpha}_k, \hat{\alpha}_p^\dagger \right] = \frac{2 \pi^2}{k^2} \delta(k - p) .
\]

(2.7)

The Fock space of quantum states is then built from the vacuum defined by \( \hat{\alpha}_k |0\rangle = 0 \) for all \( k > 0 \).

Classical configurations of the scalar field must be given by suitable states in this Fock space, and a natural choice is given by coherent states \( |g\rangle \) such that

\[
\hat{\alpha}_k |g\rangle = g_k \exp{i \chi_k(t)} |g\rangle .
\]

(2.8)

In particular, we are interested in those \( |g\rangle \) for which the expectation value of the quantum field \( \hat{\Phi} \) reproduces the classical potential, namely

\[
\sqrt{\frac{\ell_p}{m_p}} \langle g | \hat{\Phi}(t, r) | g \rangle = V(r) .
\]

(2.9)

From the expansion (2.4), we obtain

\[
\langle g | \hat{\Phi}(t, r) | g \rangle = \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} \sqrt{\frac{2 \ell_p m_p}{k}} g_k \cos[\chi_k(t) - k t] j_0(k r) .
\]

(2.10)

If we now write the classical potential as

\[
V = \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} \tilde{V}(k) j_0(k r) ,
\]

(2.11)

we immediately obtain \( \chi_k = k t \) and

\[
g_k = \sqrt{\frac{\tilde{V}(k)}{2 \ell_p}} .
\]

(2.12)

The coherent state finally reads

\[
|g\rangle = e^{-N_G/2} \exp \left\{ \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} g_k \hat{\alpha}_k^\dagger \right\} |0\rangle ,
\]

(2.13)

where

\[
N_G = \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} g_k^2 .
\]

(2.14)
is the total occupation number.\(^5\) Another quantity of interest is given by
\[ \langle k \rangle = \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} k^2 g_k^2, \]  
(2.15)
from which one obtains the “average” wavelength \( \lambda_G = N_G / \langle k \rangle \).

### 2.1 Newtonian potential

All expressions can be explicitly computed from the coefficients \( g_k \). By substituting Eq. (2.11) into the Poisson equation
\[ \triangle V = 4 \pi \frac{\ell_p}{m_p} \rho, \]  
(2.16)
and writing
\[ \rho(r) = \int_0^\infty \frac{k^2 \, dk}{2 \pi^2} \tilde{\rho}(k) j_0(kr), \]  
(2.17)
we obtain
\[ \tilde{V} = -4 \pi \frac{\ell_p}{m_p} \frac{\tilde{\rho}(k)}{k^2}, \]  
(2.18)
which, together with Eq. (2.12), leads to
\[ g_k = -\frac{4 \pi \tilde{\rho}(k)}{\sqrt{2} k^3 m_p}. \]  
(2.19)

The mathematically simplest case for computing the \( g_k \) is given by a point-like source,
\[ \rho = \frac{M}{4 \pi r^2} \delta(r), \]  
(2.20)
which generates the Newtonian potential \( V_N \) in Eq. (1.4) for all \( r > 0 \). We then find
\[ \tilde{V}_N = -4 \pi G_N \frac{M}{k^2} \]  
(2.21)
and the coefficients
\[ g_k = -\frac{4 \pi M}{\sqrt{2} k^3 m_p}. \]  
(2.22)

For such a coherent state, we obtain
\[ N_G = 4 \frac{M^2}{m_p^2} \int_0^\infty \frac{dk}{k} \]  
(2.23)

\(^5\)We note that the value of \( N_G \) measures the “distance” of \( |g\rangle \) from the vacuum \( |0\rangle \) corresponding to \( N_G = 0 \).
The number of quanta \( N_G \) contains a logarithmic divergence both in the infrared (IR) and the ultraviolet (UV), whereas \( \langle k \rangle \) only diverges in the UV.

The meaning of such divergences was explored in details in previous works \[14,16\]. In particular, we recall that the UV divergences are due to the vanishing size of the source (2.20), hence they would not be present if the density profile \( \rho = \rho(r) \) were regular. Instead of smoothing out the source (2.20), we notice that the UV divergences can also be regularised by introducing a cut-off \( k_{UV} \sim 1/R_s \), where \( R_s \) can be interpreted as the finite radius of a would-be-regular matter source. Likewise, we introduce a IR cut-off \( k_{IR} \sim 1/R_\infty \) to account for the necessarily finite life-time \( \tau \sim R_\infty \) of a realistic source \[14\] and rewrite

\[
N_G = \frac{4 M^2}{m_p^2} \int_{k_{IR}}^{k_{UV}} dk = \frac{4 M^2}{m_p^2} \ln \left( \frac{R_\infty}{R_s} \right)
\]

and

\[
\langle k \rangle = \frac{4 M^2}{m_p^2} \int_{k_{IR}}^{k_{UV}} dk = \frac{4 M^2}{m_p^2} \left( \frac{1}{R_s} - \frac{1}{R_\infty} \right).
\]

The corpuscular scaling (1.3) for the number \( N_G \) with the square of the energy \( M \) of the system already appears at this stage, whereas the crucial result (1.1), or

\[
\lambda_G = \frac{N_G}{\langle k \rangle} \sim \frac{\ell_p M}{m_p},
\]

requires a relation between the cut-offs, namely

\[
\ln \left( \frac{R_\infty}{R_s} \right) \simeq \frac{R_H}{R_s}.
\]

Assuming \( R_s \ll R_H \ll R_\infty \), the above yields (see Fig. 1 for a graphical comparison with the exact solution)

\[
R_s \simeq \frac{R_H}{\ln \left( R_\infty/R_H \right)}.
\]

so that the size of the inner source and the radius of the outer Newtonian region appear connected in the quantum description. We will further comment about possible consequences of this result in the concluding Section 4.

\[\text{Of course, any disturbance of the source will propagate (at most) at the speed of light. Alternatively, we could take for } R_\infty \text{ the present size of our observable Universe as an upper bound. In any case, modes with wavelength } k^{-1} \text{ many orders of magnitude larger than } R_H \text{ do not contribute significantly to the determination of } V = V(r) \text{ in Eq. (2.9) (for the details, see Ref. [16,21]).}\]
3 A fully quantum picture

The condition in Eq. (2.9) demands that the coherent state $|g⟩$ reproduces the classical potential everywhere. We further recall that the IR cut-off $k_{\text{IR}} = 1/R_{\infty}$ does not significantly affect the resolution of the classical potential if $R_{\infty} \gg R_s$, as was shown in Ref. [16], following a general argument from Ref. [21]. In this Section, we shall therefore assume $k_{\text{IR}} \to 0$ whenever possible.

We next observe that, for a black hole, the mean field needs only reproduce the classical potential with sufficient accuracy to comply with experimental bounds at most in the region of outer communication outside the horizon. For the simple Newtonian description of the previous Section, this means that the coherent state $|g_{\text{BH}}⟩$ representing a (Newtonian) black hole must give

$$\sqrt{\ell_p/m_p} \langle g_{\text{BH}}| \hat{\Phi}(t,r)|g_{\text{BH}}⟩ \simeq V_N(r)$$

for $r \gg R_H$, (3.1)

where we recall that $V_N(R_H) = -1/2$ and the approximate equality is subject to experimental precision. In practice, this weaker condition means that $|g_{\text{BH}}⟩$ does not need to contain the modes of infinitely short wavelength that are necessary to resolve the classical singularity at $r = 0$.

In fact, Eq. (3.1) can be satisfied by building the coherent state $|g_{\text{BH}}⟩$ according to Eq. (2.13) with modes of wavelength $k^{-1}$ larger than some fraction of the size of the gravitational radius $R_H$ of the source, which we can further identify with the UV cut-off $R_s$. By momentarily considering also the IR scale $k_{\text{IR}}$, we thus have that only the modes $k$ satisfying

$$R_{\infty}^{-1} \sim k_{\text{IR}} \lesssim k \lesssim k_{\text{UV}} \sim R_s^{-1}$$

are populated in the quantum state $|g_{\text{BH}}⟩$. This yields an effective quantum potential

$$V_Q \simeq \int_{k_{\text{IR}}}^{k_{\text{UV}}} \frac{k^2 dk}{2\pi^2} \tilde{V}_N(k) j_0(k r)$$

$$\simeq -\frac{2\ell_p M}{\pi m_p r} \int_0^{r/R_s} dz \frac{\sin z}{z} ,$$

(3.3)
Figure 2: Quantum potential $V_Q$ in Eq. (3.4) (solid line) compared to Newtonian solution $V_N$ (dashed line) for $R_s = G_N M = R_H/2$ (left panel) and $R_s = R_H/20$ (right panel). The horizontal thin line marks the location of the horizon for $V = -1/2$.

Figure 3: Oscillations of the quantum potential $V_Q$ in Eq. (3.4) around the Newtonian solution $V_N$ for $R_s = G_N M = R_H/2$ (solid line) and $R_s = R_H/20$ (dashed line) in the region outside the horizon $R_H = 2G_N M$.

where we defined $z = k r$ and let $k_{IR} = 1/R_\infty \rightarrow 0$ as mentioned above. We thus find

$$V_Q \simeq -\frac{2G_N M}{\pi r} \text{Si} \left( \frac{r}{R_s} \right)$$

$$\simeq V_N \left\{ 1 - \left[ 1 - \frac{2}{\pi} \text{Si} \left( \frac{r}{R_s} \right) \right] \right\},$$

(3.4)

where Si denotes the sine integral function.

Fig. 2 shows that the quantum potential appears regular and finite everywhere, including the origin at $r = 0$, for any finite value of $R_s$. Moreover, $\partial_r V_Q = 0$ at $r = 0$, so that tidal forces diverge nowhere inside the black hole (see also Appendix A). It is therefore consistent to assume that the inner matter source does not need to collapse to a singularity, similarly to what is found in the bootstrapped Newtonian approach [22,23]. The latter could therefore provide a compatible effective description of the quantum black hole interior including (some of the) nonlinearities. Moreover, the term inside square brackets in Eq. (3.4) induces oscillations around the classical potential $V_N$ whose effects on test bodies could be potentially observed at $r > R_H$. Such oscillations are determined
by the (quantum) size $R_s$ of the matter source inside the quantum black hole. Another important observation is that the amplitude of these fluctuations around $V_N$ in the outer region (for $r > R_H$) decreases for decreasing values of $R_s/R_H$ (see Fig. 3). Therefore, one can always choose (finite) values of $R_s/R_H$ so that the oscillations are too small to be measured by a distant observer. On the other hand, one could interpret this effect as a damping of transients as $R_s$ shrinks and the collapse proceeds inside the horizon.

It is important to remark that different shapes for the deviation from the classical potential $V_N$ would be obtained if one employed a different UV cut-off in the integral (3.3). In particular, one could consider a smooth function rather than the hard cut-off $k < R_s^{-1}$. Assuming such a smooth function is physically related to the distribution of matter in the black hole interior, one could in principle detect different matter profiles from analysing (test particle motion in) the black hole exterior. However, we shall not endeavour in the survey of other possibilities in the present work, as very little can be accomplished analytically.

4 Conclusions and outlook

In this work we started from the description of a static and spherically symmetric gravitational potential in terms of the coherent state of a massless scalar field on a reference flat spacetime, as previously employed, e.g. in Refs. [12–14,16], and relaxed the condition that its mean field reproduces the classical behaviour inside the horizon of a black hole. In fact, it remains questionable whether the mean field approximation should make sense at all in the interior of a black hole as a (macroscopic) quantum object [6,7,17], but here we took the less drastic viewpoint that a mean field exists, although it does not have to reproduce the classical potential.

We found that the weaker requirement in Eq. (3.1) that the coherent state yields a Newtonian behaviour in the exterior can be achieved by removing modes of wavelength shorter than a UV cut-off $R_s \lesssim R_H$, where $R_s$ can be physically interpreted as the size of the (quantum) matter source. The expectation value of the scalar field then displays oscillations around the Newtonian potential, whose amplitudes decrease for shrinking $R_s$. We already commented that the specific shape of such deviations will be affected by the form of the UV cut-off, but the qualitative dependence on the scale $R_s$ should be fairly well captured by the simplest choice of a sharp cut-off in Eq. (3.3). One could then interpret this effect as a decay in time due to the relative collapse of the inner material core of size $R_s$ with respect to the outer boundary $R_\infty$ of the Newtonian region. Eq. (2.28) then tells us that a core that shrinks down to $R_s \simeq R_H/65$ roughly corresponds to an increase by 26 orders of magnitude in the outer Newtonian region of size $R_\infty$. Assuming the last signals generated by the black hole formation (when $R_s$ crosses $R_H$) are approximately located at the edge of the outer Newtonian potential, and $R_\infty$ expands at the speed of light, for a solar mass black hole with $R_H \sim 1$ km, this means a time of around $10^{10}$ years, the present age of the Universe. Such a consideration could be relevant for scenarios which predict a semiclassical bounce [24]. It is also tempting to push the model towards smaller and smaller black hole masses and conjecture that the oscillations shown in Fig. 3 grow significantly and totally disrupt the classical picture for $R_s \sim R_H \sim \ell_p$. This would exclude the existence of stable remnants of Planckian size at the end of the Hawking evaporation, in agreement with other approaches among those in Refs. [7].

The simple analysis we presented here appears to neglect both the quantum nature of the matter source and the nonlinearity of the gravitational interaction, not to mention any explicit time dependence. A rather straightforward improvement would thus be obtained by replacing the New
tonian potential with the full Schwarzschild potential in harmonic coordinates, or the bootstrapped Newtonian potential in the vacuum [14,22,23]. Since all such potentials are qualitatively very close to the Newtonian behaviour in the region outside the horizon, one does not expect any drastic departure from what we found here. Of course, for a full phenomenological analysis of geodesics and signal propagation in the region of outer communication, one should first proceed to reconstruct the complete (effective) metric. We remark that the potential in the radial geodesic equation written in the usual areal coordinate of the Schwarzschild manifold is formally equal to the Newtonian potential (1.4), which makes it possible to apply the results presented here to Schwarzschild black holes straightforwardly. \(^7\) The behaviour of geometric scalars near the centre and the quantum corrected radius of the horizon can thus be computed. These results are briefly described in Appendix A, along with remarks about the ensuing dependence of black hole thermodynamics on the scale \(R_s\).

For a quantum source, the size \(R_s\) needs not be sharply defined and one should include the effects of its uncertainty onto the gravitational state, like it was done by considering the bootstrapped Newtonian potential generated by a source of finite size in Ref. [16], or like in Ref. [17], where a bound on the compactness was obtained from the quantisation of the geodesic motion of the surface of a sphere of dust. The latter result in particular provides a clear motivation for including the UV scale \(R_s \sim G_N M\) in the definition (3.1) of the coherent state of gravitons, although the use of a sharp cut-off in Eq. (3.3) is justified mostly by the simplicity of the calculation. Finally, we note that the natural time evolution of quantum states would imply that the oscillations around the expected classical potential could make the region around the horizon rather fuzzy [7,26], with possibly relevant implications for the causal structure of astrophysical black holes and their interaction with infalling matter.

To conclude, all of the results presented here and in Refs. [7,17,26] should be regarded as suggestive of the kind of new effects that one can expect when properly describing with quantum physics the formation of black holes from the collapse of matter, and possibly inspire further developments along that direction. It would certainly be interesting to unify all of the various aspects in one consistent picture.

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A Geometry and thermodynamics

If we assume that \(r\) is the usual areal radial coordinate (see footnote 4), the potential (3.4) can be used to reconstruct the full spacetime metric straightforwardly as

\[
\text{d}s^2 = -(1 + 2V_Q)\text{d}t^2 + \frac{dr^2}{1 + 2V_Q} + r^2d\Omega^2,
\]

(A.1)

where the dependence of \(V_Q\) on \(R_s\) therefore results in a quantum violation of the no-hair theorem.

From the metric (A.1), one can directly compute the usual geometric scalars. For example, one finds that the Ricci scalar \(R \sim r^{-2}\) and the Kretschmann scalar \(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \sim R^2 \sim r^{-4}\). Hence,\(^7\) Reconstructing the metric from a weak field approximation is usually a much more complicated task [25].
tidal forces remain finite all the way into the centre of the system, which technically makes the point at \( r = 0 \) an integrable singularity.

The horizon radius \( r = r_H \) is given by the solution of \( 2V_Q = -1 \), which can be computed numerically for different values of \( R_s \) (see left panel in Fig. 4). In particular, one can see that \( r_H = R_H \) for very specific values of \( R_s \), the largest one being \( R_s \approx 0.53 R_H \), which is very close to the preferred case obtained in Ref. [17]. A remarkable result is that the horizon radius \( r_H \) shrinks for \( R_s \) approaching \( R_H \). In fact, \( r_H = R_s \) for \( R_s \approx 0.6 R_H \) and \( r_H \) would further vanish for \( R_s \approx 0.7 R_H \). This means that the material core cannot be too close in size to the classical gravitational radius \( R_H \) in order for it to lie inside the actual horizon. The shaded regions in Fig. 4 correspond to values of \( r_H < R_s \), which are therefore not black holes.

The above departure of \( r_H \) from the Schwarzschild radius \( R_H = 2G_NM \) will also give rise to corrections for the horizon area \( A_H \) and Bekenstein-Hawking entropy [11],

\[
S_{BH} = \frac{A_H}{4 \ell_p^2} = \frac{\pi r_H^2}{\ell_p^2} \tag{A.2}
\]

and for the black hole temperature [27]

\[
T_Q = \frac{h \kappa}{2 \pi} = \left. \frac{h}{2 \pi} \frac{\partial V_Q}{\partial r} \right|_{r=r_H}, \tag{A.3}
\]

where \( \kappa \) is the surface gravity at the horizon. The quantum corrected Hawking temperature can also be computed only numerically and is displayed in the right panel of Fig. 4. One can then see that quantum corrected black holes are colder than their classical counterpart, that is \( T_Q < T_H \), if \( R_s \gtrsim R_H/2 \). In fact, \( T_Q \) would vanish for \( R_s \approx 0.7 R_H \), which precisely correspond to \( r_H = 0 \) but falls outside the range \( r_H > R_s \) representing proper black holes. One could argue that the shaded regions in Fig. 4 still represent some intermediate stage in the gravitational collapse or will perhaps
become relevant at the end of the Hawking evaporation, but we shall not further pursue this topic here.  

References

[1] S. W. Hawking and G. F. R. Ellis, "The Large Scale Structure of Space-Time," (Cambridge University Press, Cambridge, 1973)

[2] R. P. Geroch and J. H. Traschen, Phys. Rev. D 36 (1987) 1017 [Conf. Proc. C 861214 (1986) 138].

[3] C. Goddi et al. [EHT], The Messenger 177 (2019) 25 [arXiv:1910.10193 [astro-ph.HE]].

[4] B. P. Abbott et al. [LIGO Scientific and Virgo], Phys. Rev. X 9 (2019) 031040 [arXiv:1811.12907 [astro-ph.HE]].

[5] G. Dvali and C. Gomez, Fortsch. Phys. 61 (2013) 742 [arXiv:1112.3359 [hep-th]]; G. Dvali, C. Gomez and S. Mukhanov, "Black Hole Masses are Quantized," arXiv:1106.5894 [hep-ph].

[6] K. V. Kuchar, Phys. Rev. D 50 (1994) 3961 [arXiv:gr-qc/9403003 [gr-qc]]; G. ’t Hooft, Nucl. Phys. B 256 (1985) 727; A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Phys. Rev. Lett. 80 (1998) 904 [arXiv:gr-qc/9710007 [gr-qc]].

[7] R. Casadio and F. Scardigli, Eur. Phys. J. C 74 (2014) 2685 [arXiv:1306.5298 [gr-qc]]; R. Casadio, O. Micu and F. Scardigli, Phys. Lett. B 732 (2014) 105 [arXiv:1311.5698 [hep-th]].

[8] R. P. Feynman, F. B. Morinigo, W. G. Wagner and B. Hatfield, "Feynman lectures on gravitation," (Addison-Wesley Pub. Co., 1995).
[9] S. Deser, Gen. Rel. Grav. 1 (1970) 9 [gr-qc/0411023]; Gen. Rel. Grav. 42 (2010) 641 [arXiv:0910.2975 [gr-qc]].
[10] R.L. Arnowitt, S. Deser and C.W. Misner, Phys. Rev. 116 (1959) 1322.
[11] J.D. Bekenstein, Phys. Rev. D 7 (1973) 2333.
[12] R. Casadio, A. Giugno and A. Giusti, Phys. Lett. B 763 (2016) 337 [arXiv:1606.04744 [gr-qc]].
[13] W. Mück, Can. J. Phys. 92 (2014) 973 [arXiv:1306.6245 [hep-th]].
[14] R. Casadio, A. Giugno, A. Giusti and M. Lenzi, Phys. Rev. D 96 044010 (2017) [arXiv:1702.05918 [gr-qc]].
[15] G. Dvali, G. F. Giudice, C. Gomez and A. Kehagias, JHEP 1108 (2011) 108 [arXiv:1010.1415 [hep-ph]]; G. Dvali and D. Pirtskhalava, Phys. Lett. B 699 (2011) 78 [arXiv:1011.0114 [hep-ph]]; G. Dvali, C. Gomez and A. Kehagias, JHEP 1111 (2011) 070 [arXiv:1103.5963 [hep-th]]; R. Percacci and L. Rachwal, Phys. Lett. B 711 (2012) 184 [arXiv:1202.1101 [hep-th]].
[16] R. Casadio, M. Lenzi and A. Ciarfella, Phys. Rev. D 101 (2020)124032 [arXiv:2002.00221 [gr-qc]].
[17] R. Casadio, “A quantum bound on the compactness,” [arXiv:2103.14582 [gr-qc]].
[18] S. Hofmann and M. Schneider, Phys. Rev. D 91 (2015) 125028 [arXiv:1504.05580 [hep-th]]; Phys. Rev. D 95 (2017) 065033 [arXiv:1611.07981 [hep-th]].
[19] R. Casadio and A. Giusti, Phys. Lett. B 797 (2019),134915 [arXiv:1904.12663 [gr-qc]].
[20] S. Weinberg, “Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity,” (Wiley & Sons, 1972)
[21] G. Dvali, C. Gomez, L. Gruending and T. Rug, Nucl. Phys. B 901 (2015) 338 [arXiv:1508.03074 [hep-th]].
[22] R. Casadio, M. Lenzi and O. Micu, Phys. Rev. D 98 (2018) 104016 [arXiv:1806.07639 [gr-qc]].
[23] R. Casadio, M. Lenzi and O. Micu, Eur. Phys. J. C 79 (2019) 894 [arXiv:1904.06752 [gr-qc]].
[24] D. Malafarina, Universe 3 (2017) 48 [arXiv:1703.04138 [gr-qc]]; R. Casadio, Int. J. Mod. Phys. D 9 (2000) 511 [arXiv:gr-qc/9810073 [gr-qc]]; H. M. Haggard and C. Rovelli, Phys. Rev. D 92 (2015) 104020 [arXiv:1407.0989 [gr-qc]]; W. Piechocki and T. Schmitz, Phys. Rev. D 102 (2020) 046004 [arXiv:2004.02939 [gr-qc]]; T. Schmitz, Phys. Rev. D 103 (2021) 064074 [arXiv:2012.04383 [gr-qc]].
[25] R. Casadio, A. Giusti, I. Kuntz and G. Neri, Phys. Rev. D 103 (2021) 064001 [arXiv:2101.12471 [gr-qc]].
[26] R. Casadio, “Localised particles and fuzzy horizons: A tool for probing Quantum Black Holes,” [arXiv:1305.3195 [gr-qc]]; Eur. Phys. J. C 75 (2015) 160 [arXiv:1411.5848 [gr-qc]].
[27] S. W. Hawking, Commun. Math. Phys. 43 (1975) 199-220 [erratum: Commun. Math. Phys. 46 (1976), 206].