ADDITIVE $\rho$-FUNCTIONAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we solve the additive $\rho$-functional equations
\begin{equation}
    f(x + y + z) - f(x) - f(y) - f(z) = \rho \left( 2f \left( \frac{x + y + z}{2} \right) - f(x) - f(y) - f(z) \right),
\end{equation}
where $\rho$ is a fixed number with $\rho \neq 1$, and
\begin{equation}
    f(x + y + z) - f(x) - f(y) - f(z) = \rho \left( 2f \left( \frac{x + y}{2} + z \right) - f(x) - f(y) - 2f(z) \right),
\end{equation}
where $\rho$ is a fixed number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive $\rho$-functional equations (0.1) and (0.2) in Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [5] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [3] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [2] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

In Section 2, we solve the additive functional equation (0.1) and prove the Hyers-Ulam stability of the additive functional equation (0.1) in Banach spaces.

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In Section 3, we solve the additive functional equation (0.2) and prove the Hyers-Ulam stability of the additive functional equation (0.2) in Banach spaces. Throughout this paper, assume that $X$ is a normed space and that $Y$ is a Banach space.

2. Additive $\rho$-functional Equation (0.1)

Let $\rho$ be a number with $\rho \neq 1, 2$.

We solve and investigate the additive $\rho$-functional equation (0.1) in normed spaces.

**Lemma 2.1.** If a mapping $f : X \rightarrow Y$ satisfies

$$f(x + y + z) - f(x) - f(y) - f(z) = \rho \left( 2f \left( \frac{x+y+z}{2} \right) - f(x) - f(y) - f(z) \right)$$

(2.1)

for all $x, y, z \in X$, then $f : X \rightarrow Y$ is additive.

**Proof.** Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we get $-2f(0) = -\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ and $z = 0$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

(2.2) $$f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$f(x + y + z) - f(x) - f(y) - f(z) = \rho \left( 2f \left( \frac{x+y+z}{2} \right) - f(x) - f(y) - f(z) \right) = \rho (f(x + y + z) - f(x) - f(y) - f(z))$$

and so $f(x + y + z) = f(x) + f(y) + f(z)$ for all $x, y, z \in X$. Since $f(0) = 0$,

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

We prove the Hyers-Ulam stability of the additive $\rho$-functional equation (2.1) in Banach spaces.
Theorem 2.2. Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and

\[ \Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty, \]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[ \|f(x) - A(x)\| \leq \frac{1}{2} \Psi(x, x, 0) \]

for all \( x \in X \).

Proof. Letting \( y = x \) and \( z = 0 \) in (2.4), we get

\[ \|f(2x) - 2f(x)\| \leq \varphi(x, x, 0) \]

for all \( x \in X \). So

\[ \|f(x) - 2f \left( \frac{x}{2} \right) \| \leq \varphi \left( \frac{x}{2}, \frac{y}{2}, 0 \right) \]

for all \( x \in X \). Hence

\[ \left\| 2^j f \left( \frac{x}{2^j} \right) - 2^m f \left( \frac{x}{2^m} \right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f \left( \frac{x}{2^j} \right) - 2^{j+1} f \left( \frac{x}{2^{j+1}} \right) \right\| \]

\[ \leq \sum_{j=l}^{m-1} 2^j \varphi \left( \frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0 \right) \]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.7) that the sequence \( \{2^k f \left( \frac{x}{2^k} \right) \} \) is Cauchy for all \( x \in X \). Since \( Y \) is a Banach space, the sequence \( \{2^k f \left( \frac{x}{2^k} \right) \} \) converges. So one can define the mapping \( A : X \to Y \) by

\[ A(x) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right) \]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.7), we get (2.5).
Now, let $T : X \to Y$ be another additive mapping satisfying (2.5). Then we have
\[
\|A(x) - T(x)\| = \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\|
\leq \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\|
\leq 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, 0\right),
\]
which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $A$.

It follows from (2.3) and (2.4) that
\[
\left\|A(x + y + z) - A(x) - A(y) - A(z) - \rho\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z)\right)\right\|
\leq \lim_{n \to \infty} 2^n \left\|f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) + \rho\left(2f\left(\frac{x+y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right)\right\|
\leq \lim_{n \to \infty} 2^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, 0\right) = 0
\]
for all $x, y, z \in X$. So
\[
A(x + y) - A(x) - A(y) - A(z) = \rho\left(2A\left(\frac{x+y+z}{2}\right) - A(x) - A(y) - A(z)\right)
\]
for all $x, y, z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive.

**Corollary 2.3.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and
\[
\left\|f(x+y+z) - f(x) - f(y) - f(z) - \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right)\right\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]  
for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that
\[
\|f(x) - A(x)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r
\]
for all $x \in X$.

**Proof.** Letting $\phi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.2, we get the desired result. \[\square\]
Theorem 2.4. Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \), (2.4) and

\[
\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{1}{2} \Psi(x, x, 0) \tag{2.9}
\]

for all \( x \in X \).

Proof. It follows from (2.6) that

\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2} \varphi(x, x, 0)
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{2^n} f(2^n x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|
\]

\[
\leq \sum_{j=l}^{m-1} \frac{1}{2^j} \varphi(2^j x, 2^j x, 0) \tag{2.10}
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.10) that the sequence \( \{ \frac{1}{2^n} f(2^n x) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{2^n} f(2^n x) \} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Corollary 2.5. Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and (2.8). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r
\]

for all \( x \in X \).

Proof. Letting \( \varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r) \) in Theorem 2.4, we get the desired result. \( \square \)
3. ADDITIVE $\rho$-FUNCTIONAL EQUATION (0.2)

Let $\rho$ be a number with $\rho \neq 1$.
We solve and investigate the additive $\rho$-functional equation (0.2) in normed spaces.

**Lemma 3.1.** If a mapping $f : X \to Y$ satisfies

$$f(x + y + z) - f(x) - f(y) - f(z) = \rho \left( 2f \left( \frac{x + y}{2} + z \right) - f(x) - f(y) - 2f(z) \right)$$

(3.1)

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

**Proof.** Assume that $f : X \to Y$ satisfies (3.1).
Letting $x = y = z = 0$ in (2.1), we get $-2f(0) = -2\rho f(0)$. So $f(0) = 0$.
Letting $y = x$ and $z = 0$ in (2.1), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$
for all $x \in X$. Thus

$$f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)$$

(3.2)

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$f(x + y) - f(x) - f(y) = \rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right)$$

$$= \rho (f(x + y) - f(x) - f(y))$$

and so $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. \[\Box\]

We prove the Hyers-Ulam stability of the additive $\rho$-functional equation (3.1) in Banach spaces.

**Theorem 3.2.** Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\Psi(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j} \right) < \infty,$$

$$\left\| f(x + y + z) - f(x) - f(y) - f(z) - \rho \left( 2f \left( \frac{x + y}{2} + z \right) - f(x) - f(y) - 2f(z) \right) \right\| \leq \varphi(x, y, z)$$

(3.3)
for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{1}{2} \Psi(x, x, 0)
\]

for all \( x \in X \).

**Proof.** Letting \( y = x \) and \( z = 0 \) in (3.3), we get

\[
\|f(2x) - 2f(x)\| \leq \varphi(x, x, 0)
\]

for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.3.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and

\[
\|f(x + y + z) - f(x) - f(y) - f(z) - \rho \left( 2f\left(\frac{x + y}{2} + z\right) - f(x) - f(y) - 2f(z) \right) \| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{2\theta}{2^r - 2} \|x\|^r
\]

for all \( x \in X \).

**Proof.** Letting \( \varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r) \) in Theorem 3.2, we get the desired result. □

**Theorem 3.4.** Let \( \varphi : X^3 \to [0, \infty) \) be a function and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \), (3.3) and

\[
\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) < \infty
\]

for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{1}{2} \Psi(x, x, 0)
\]

for all \( x \in X \).
Proof. It follows from (3.4) that
\[ \left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2} \varphi(x, x, 0) \]
for all \( x \in X \).

The rest of the proof is similar to the proofs of Theorems 2.2 and 2.4. \( \square \)

**Corollary 3.5.** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and (3.5). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[ \| f(x) - A(x) \| \leq \frac{2\theta}{2 - 2r} \| x \|^r \]
for all \( x \in X \).

**Proof.** Letting \( \varphi(x, y, z) := \theta(|x|^r + |y|^r + |z|^r) \) in Theorem 3.4, we get the desired result. \( \square \)

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