Partial Differential Equations

Lack of compactness in the 2D critical Sobolev embedding, the general case

Défaut de compacité de l'injection de Sobolev critique en deux dimensions d'espace

Hajer Bahouri\textsuperscript{a}, Mohamed Majdoub\textsuperscript{b}, Nader Masmoudi\textsuperscript{c}

\textsuperscript{a} LAMA UMR CNRS 8050, Université Paris-Est Créteil, 61, avenue du Général de Gaulle, 94010 Créteil cedex, France
\textsuperscript{b} Département de mathématiques, faculté des sciences de Tunis, 2092 Manar, Tunisia
\textsuperscript{c} The Courant Institute of Mathematical Sciences, NY University, 251 Mercer St., New York, NY 10012, USA

\textbf{Article Info}

\textbf{Article history:}
Received 18 November 2011
Accepted after revision 23 January 2012
Available online 11 February 2012
Presented by Haim Brezis

\textbf{Abstract}

This Note is devoted to the description of the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^2)$ in the critical Orlicz space $\mathcal{L}(\mathbb{R}^2)$. It turns out that up to cores our result is expressed in terms of the concentration-type examples derived by J. Moser (1971) in [16] as in the radial setting investigated in Bahouri et al. (2011) [5]. However, the analysis we used in this work is strikingly different from the one conducted in the radial case which is based on an $L^\infty$ estimate far away from the origin and which is no longer valid in the general framework. The strategy we adopted to build the profile decomposition in terms of examples by Moser concentrated around cores is based on capacity arguments and relies on an extraction process of mass concentrations.

© 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\textbf{Résumé}

Cette Note est consacrée à l’étude du défaut de compacité de l’injection de Sobolev de $H^1(\mathbb{R}^2)$ dans l’espace d’Orlicz critique $\mathcal{L}(\mathbb{R}^2)$. Nous démontrons que la description donnée dans Bahouri et al. (2011) [5] concernant le cas radial reste valable dans le cas général (à des translations près par des coeurs de concentration). La preuve utilise des arguments de capacité ainsi qu’un processus d’extraction de concentrations.

© 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\textbf{Version française abrégée}

Il est bien connu que $H^1(\mathbb{R}^2)$ s'injecte continûment dans tous les espaces $L^p(\mathbb{R}^2)$ pour $2 \leq p < \infty$, mais pas dans $L^\infty(\mathbb{R}^2)$. De plus, on sait d’aprèes les inégalités de Moser-Trudinger (voir [16,18]) que $H^1$ s'injecte dans l'espace d'Orlicz $\mathcal{L}$ (défini ci-dessous). Plus précisément, on a

$$\|u\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}. \quad (1)$$

Cette injection est critique dans le sens où on ne peut pas remplacer l'espace d'Orlicz $\mathcal{L}$ par un espace plus petit du type Orlicz (voir [5,7]). Il est aussi à remarquer que cette injection est non compacte pour, au moins, deux raisons. La première
est la perte de compacité à l’infini illustrée par l’exemple \( u_\varepsilon(x) = \varphi(x + x_\varepsilon) \) où \( \varphi \in \mathcal{D}'(\mathbb{R}) \) et \( |x_\varepsilon| \to \infty \). La seconde est due à un phénomène de concentration et un exemple typique est celui de P.-L. Lions [14] et J. Moser [16] :

\[
f_\alpha(x) = \begin{cases} 
0 & \text{si } |x| \geq 1, \\
-\frac{\log |x|}{\sqrt{2\pi \alpha}} & \text{si } e^{-\alpha} \leq |x| \leq 1, \\
\sqrt{\frac{\alpha}{2\pi}} & \text{si } |x| \leq e^{-\alpha},
\end{cases}
\]

où \( \alpha \to \infty \). Remarquons que \( f_\alpha \) peut-être écrite \( f_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}} L(-\frac{\log |x|}{\alpha}) \) avec

\[
L(t) = \begin{cases} 
0 & \text{si } t \leq 0, \\
t & \text{si } 0 \leq t \leq 1, \\
1 & \text{si } t \geq 1.
\end{cases}
\]

L’objectif principal de cette Note est de démontrer que le défaut de compacité de l’injection de Sobolev \( H^1 \hookrightarrow \mathcal{L} \) peut-être décrit à l’aide de l’exemple de Moser (à des translations près par des coeurs de concentration). Le résultat principal est le suivant :

**Théorème 1.** Soit \( (u_n) \) une suite de \( H^1(\mathbb{R}^2) \) telle que

\[
u_n \rightharpoonup 0, \quad \limsup_{n \to \infty} \|u_n\|_{\mathcal{L}} = A_0 > 0 \quad \text{et} \quad \lim_{K \to \infty} \limsup_{n \to \infty} \|u_n\|_{\mathcal{L}(|x| > K)} = 0. \tag{2}
\]

Alors, il existe une suite \( (\alpha_n^{(j)}) \) d’échelles, une suite \( (\chi_n^{(j)}) \) de coeurs et une suite \( (\psi_n^{(j)}) \) de profils tels que les triplets \( (\alpha_n^{(j)}, \chi_n^{(j)}, \psi_n^{(j)}) \) soient deux à deux orthogonaux et, à extraction près, on a pour tout entier \( \ell \geq 1 \),

\[
u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \psi_n^{(j)} \left( \frac{-\log |x - x_n^{(j)}|}{\alpha_n^{(j)}} \right) + r_n^{(\ell)}(x), \quad \limsup_{n \to \infty} \|r_n^{(\ell)}\|_{\mathcal{L}} \to 0, \tag{3}
\]

\[
\|\nabla u_n\|^2_{L^2} = \sum_{j=1}^{\ell} \|\psi_n^{(j)}\|^2_{L^2} + \|\nabla r_n^{(\ell)}\|^2_{L^2} + o(1), \quad n \to \infty. \tag{4}
\]

La preuve de ce résultat repose sur un procédé d’extraction diagonale : l’extraction des échelles utilise le cas radial [5] à travers le réarrangement de Schwarz \( u_\varepsilon^* \) de \( u_\varepsilon \) tandis que celle des coeurs de concentration se base essentiellement sur des arguments de capacité. Nous renvoyons à la version en anglais ci-dessous pour les définitions et les idées de la preuve de ce résultat et à [6] pour des détails et des compléments.

### 1. Introduction

It is well known that \( H^1(\mathbb{R}^2) \) is continuously embedded in all Lebesgue spaces \( L^p(\mathbb{R}^2) \) for \( 2 \leq p < \infty \), but not in \( L^\infty(\mathbb{R}^2) \). On the other hand, it is also known (see for instance [11]) that \( H^1(\mathbb{R}^2) \) embeds in \( BMO(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \), where \( BMO(\mathbb{R}^d) \) denotes the space of bounded mean oscillations which is the space of locally integrable functions \( f \) such that

\[
\|f\|_{BMO} \overset{\text{def}}{=} \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty \quad \text{with} \quad f_B \overset{\text{def}}{=} \frac{1}{|B|} \int_B f \, dx.
\]

The above supremum being taken over the set of Euclidean balls \( B \), \( |\cdot| \) denoting the Lebesgue measure.

For the sake of geometric problems and the understanding of features of solutions to nonlinear partial differential equations with exponential growth, we investigate in this paper the lack of compactness of Sobolev embedding of \( H^1(\mathbb{R}^2) \) in the Orlicz space \( \mathcal{L}(\mathbb{R}^2) \) (defined below) which turns out not to be comparable with \( BMO(\mathbb{R}^2) \) (for a proof of this fact, one can consult [5]). Remark that the lack of compactness of \( H^1(\mathbb{R}^2) \hookrightarrow BMO(\mathbb{R}^d) \) was characterized in [2] using a wavelet-based profile decomposition.

Note that in higher dimension, are available several works that highlight the role of the study of the lack of compactness in critical Sobolev embedding to the description of solutions of nonlinear partial differential equations. Among others, one can mention [3,4,10,12,13,15,19].

Let us introduce the so-called Orlicz spaces on \( \mathbb{R}^d \) (for a complete presentation and more details, we refer the reader to [17]).

**Définition 1.** Soit \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) une fonction convexe croissante telle que
\[
\phi(0) = 0 = \lim_{s \to 0^+} \phi(s), \quad \lim_{s \to \infty} \phi(s) = \infty.
\]

We say that a measurable function \( u : \mathbb{R}^d \to \mathbb{C} \) belongs to \( L^\phi \) if there exists \( \lambda > 0 \) such that \( \int \frac{|u(x)|}{\lambda} \, dx < \infty \). We denote then
\[
\|u\|_{L^\phi} = \inf\{\lambda > 0, \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) \, dx \leq \kappa\},
\]
where \( \kappa \) is the constant appearing in (7).

In what follows we shall fix \( d = 2 \), \( \phi(s) = e^{s^2} - 1 \) and denote the Orlicz space \( L^\phi \) by \( L^\alpha \) endowed with the norm \( \| \cdot \|_\alpha \).

The 2D critical Sobolev embedding in the Orlicz space \( L^\phi \) can be stated as follows:

**Proposition 1.**
\[
\|u\|_\alpha \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}.
\]

Let us point out that the embedding (6) derives immediately from the following Trudinger–Moser inequality proved in [18]:

**Proposition 2.**
\[
\sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left( e^{4\pi |u(x)|^2} - 1 \right) \, dx := \kappa < \infty,
\]
and this is false for \( \alpha > 4\pi \).

The embedding \( H^1 \hookrightarrow L^\phi \) is not compact at least for two reasons. The first reason is the lack of compactness at infinity. A typical example is given by \( u_n(x) = \psi(x + x_0) \) where \( 0 \neq \psi \in \mathcal{D} \) and \( |x_n| \to \infty \). The second reason is of concentration-type derived by P.-L. Lions [14] and J. Moser [16], and illustrated by the following fundamental example \( f_\alpha \) defined for \( \alpha > 0 \) (\( \alpha \to \infty \)) by:
\[
f_\alpha(x) = \begin{cases} 
0 & \text{if } |x| \geq 1, \\
\frac{-\log |x|}{\sqrt{2\alpha\pi}} & \text{if } e^{-\alpha} \leq |x| \leq 1, \\
\frac{\alpha}{2\pi} & \text{if } |x| \leq e^{-\alpha},
\end{cases}
\]
which satisfies \( \|f_\alpha\|_\alpha \to \frac{1}{\sqrt{4\pi}} \) as \( \alpha \to \infty \). The heart of this work is to prove that the lack of compactness of the Sobolev embedding
\[
H^1(\mathbb{R}^2) \hookrightarrow L^{\alpha}(\mathbb{R}^2)
\]
can be reduced up to cores to the example by Moser. Before coming to the statement of the main theorem, let us introduce some definitions as in [8] and [5].

**Definition 2.** A scale is a sequence \( \varrho := (\varrho_n) \) of positive real numbers going to infinity, a core is a sequence \( \chi := (\chi_n) \) of points in \( \mathbb{R}^2 \) and a profile is a function \( \psi \) belonging to the set
\[
\mathcal{P} := \{ \psi \in L^2(\mathbb{R}, e^{-2s} \, ds); \psi' \in L^2(\mathbb{R}), \psi|_{]-\infty,0]} = 0 \}.
\]
Given two scales \( \varrho, \tilde{\varrho} \), two cores \( \chi, \tilde{\chi} \) and two profiles \( \psi, \tilde{\psi} \), we shall say that the triplets \((\varrho, \chi, \psi)\) and \((\tilde{\varrho}, \tilde{\chi}, \tilde{\psi})\) are orthogonal (in short \((\varrho, \chi, \psi) \perp (\tilde{\varrho}, \tilde{\chi}, \tilde{\psi})\)) if
\[
\text{either } |\log(\tilde{\varrho}/\varrho_n)| \to \infty, \quad \text{or } \tilde{\varrho}_n = \varrho_n \text{ and } \\
-\frac{|\chi_n - \tilde{\chi}_n|}{\varrho_n} \to a \geq 0 \text{ with } \psi \text{ or } \tilde{\psi} \text{ null for } s < a.
\]

Our result is as follows:
Theorem 1. Let \((u_n)\) be a bounded sequence in \(H^1(\mathbb{R}^2)\) such that
\[
\lim_{n \to \infty} \sup_{|x| > R} \|u_n\|_{L^2} = A_0 > 0 \quad \text{and} \quad \lim_{R \to \infty} \lim_{n \to \infty} \sup_{|x| > R} \|u_n\|_{L^2} = 0. \tag{10}
\]
Then, there exist a sequence of scales \((\alpha^{(j)}_n)\), a sequence of cores \((x^{(j)}_n)\) and a sequence of profiles \((\psi^{(j)}_n)\) such that the triplets \((\alpha^{(j)}_n, x^{(j)}_n, \psi^{(j)}_n)\) are pairwise orthogonal and, up to a subsequence extraction, we have for all \(\ell \geq 1\),
\[
\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \left( \|\psi^{(j)}\|_{L^2}^2 + \|\nabla \psi_n^{(j)}\|_{L^2}^2 + o(1) \right), \quad n \to \infty. \tag{11}
\]
\[
\limsup_{n \to \infty} \|u_n^{(\ell)}\|_{L^2} = 0. \tag{12}
\]

The elements used in our orthogonal asymptotic decomposition are completely different from the ones involving in the decomposition derived by P. Gérard in [8] in the framework of Sobolev embedding in Lebesgue spaces or by S. Jaffard in [11] in the more general case of Riesz potential spaces.

2. Ideas of the proof

We just give some basic ideas of our strategy, and we refer the reader to [6] for more details and some complements. The proof of Theorem 1 is done in four steps. In the first step, according to Schwarz symmetrization, we are reduced to the study of \(u^*_n\) the symmetric decreasing rearrangement of \(u_n\). Since \(u^*_n \in H^1_{\text{rad}}(\mathbb{R}^2)\) and satisfies assumptions of Theorem 1.6 in [5], it can be written as an orthogonal asymptotic decomposition by means of elementary concentrations of type \(g^{(j)}_n(x) = \sqrt{\frac{\alpha^{(j)}_n}{2\pi}} \varphi^{(j)}(\frac{-\log |x|}{\alpha^{(j)}_n})\). Then in the second step, taking advantage of a cut-off argument, we reduce to \(g^{(1)}_n\) and extract the first core \((x^{(1)}_n)\) and the first profile \(\psi^{(1)}\) which leads to the extraction of the first element \(\sqrt{\alpha^{(1)}_n} \varphi^{(1)}(\frac{-\log |x|}{\alpha^{(1)}_n})\). This step constitutes the heart of the matter and relies on capacity arguments: the basic idea is to show that the mass does not scatter and is mainly concentrated around some point that will constitute the core. In the third step, we study the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same scatter and is mainly concentrated around some point that will constitute the second core \((x^{(2)}_n)\). Thereafter, we establish the property of orthogonality between the two first elementary concentrations and finally we prove that this process converges.

As it is mentioned above, one can reduce to studying the case of a non-negative sequence \(u_n\) satisfying
\[
\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \left( \|\psi^{(j)}\|_{L^2}^2 + \|\nabla \psi_n^{(j)}\|_{L^2}^2 + o(1) \right), \quad n \to \infty. \tag{13}
\]

Now, the heart of the matter is to extract cores and profiles and the key point in that process relies in the following lemma.

Lemma 1. There exist \(\delta_0 > 0\) and \(N \in \mathbb{N}\) such that for any \(n \geq N\) there exists \(x_n\) such that
\[
|E_n \cap B(x_n, e^{-b\alpha_n})| \geq \delta_0 A_0^2 |E_n|, \tag{14}
\]
where \(E_n := \{x \in \mathbb{R}^2 \mid |u_n(x)| \geq \sqrt{2\alpha_n (1 - \delta_0)} A_0\}\), \(B(x_n, e^{-b\alpha_n})\) designates the ball of center \(x_n\) and radius \(e^{-b\alpha_n}\) with \(b = 1 - 2\varepsilon_0\) and \(0 < \varepsilon_0 < \frac{1}{2}\).

Sketch of proof. We argue by contradiction. Hence, up to a subsequence extraction, we have for any \(\delta > 0\), \(n \in \mathbb{N}\) and \(x \in \mathbb{R}^2\)
\[
|E_n \cap B(x, e^{-b\alpha_n})| \leq \delta A_0^2 |E_n|. \tag{15}
\]
In particular, property (15) occurs for any ball centered at a point in \(T_n := (e^{-b\alpha_n} \mathbb{Z}) \times (e^{-b\alpha_n} \mathbb{Z})\). Since the balls \(B(x, e^{-b\alpha_n})\) constitute a covering of \(\mathbb{R}^2\) when the points \(x\) describe \(T_n\) and each point of \(\mathbb{R}^2\) belongs at most to four balls among \(B_n := \{B(x, e^{-b\alpha_n}) \mid x \in T_n\}\), we deduce that
\[
\|\nabla u_n\|_{L^2(\mathbb{R}^2)}^2 \geq \frac{1}{4} \sum_{B \in B_n} \|\nabla u_n\|_{L^2(B)}^2. \tag{16}
\]

Now, the idea is to get a contradiction by proving that for \(\delta\) small enough the sum \(\frac{1}{4} \sum_{B \in B_n} \|\nabla u_n\|_{L^2(B)}^2\) exceeds the energy of \(u_n\). For this purpose, making use of capacity arguments we estimate the energy of \(u_n\) on the set \(E_n\) of balls \(B \in B_n\).
satisfying $|E_n \cap B| \geq e^{-10a_n}$. To do so, we take advantage of the fact that the values of $|u_n|$ on $B$ vary at least from $\sqrt{2a_n(1 - 2|\alpha_n|)A_0}$ on $E_n \cap B$ to $\sqrt{2a_n(1 - 2|\alpha_n|)A_0}$ on a set of Lebesgue measure $\geq \frac{R}{2}$, for all $n \geq N_{\delta}$, where $N_{\delta}$ is an integer big enough which only depends on $\delta$. It follows by standard capacity arguments [9] that

$$\|\nabla u_n\|^2_{L^2(B)} \geq C \left( \left( \frac{E_0}{2} - \frac{E_0}{10} \sqrt{2a_n A_0} \right)^2 \frac{1}{\log \frac{e^{-\alpha_n(1 - 2|\alpha_n|)}}{\sqrt{2a_n A_0}}} \right) \geq C \varepsilon^2 \delta A_0^2.$$  

Hence,

$$4\|\nabla u_n\|^2_{L^2(B)} \geq \#(\tilde{B}_n) C \varepsilon^2 \delta A_0^2. \tag{17}$$

But by (15), the covering of $\mathbb{R}^2$ by $B_n$ and the fact that mass concentrated in $E_n$ is mainly due to balls of $\tilde{B}_n$, we necessarily have

$$\#(\tilde{B}_n) \geq \frac{1}{2\delta A_0^2}$$

which yields a contradiction for $\delta$ small enough in view of (17). $\square$

To extract the profile, we first consider $\psi_n(y, \theta) = \sqrt{2\pi} \nu_n(\alpha_n y, \theta)$ where

$$\nu_n(s, \theta) = (\tau, \alpha_n u_n)(e^{-s} \cos \theta, e^{-s} \sin \theta)$$

and $\alpha_n$ is given by Lemma 1. Using energy estimate, we deduce that

$$\partial_\theta \psi_n \to 0, \quad \partial_y \psi_n \to g \text{ in } L^2(y, \theta), \quad \text{as } n \to \infty,$$

where $g$ only depends of $y$ and is null for $y \leq 0$. Finally, we prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_n(y, \theta) d\theta \to \psi(y) = \int_0^y g(\tau) d\tau,$$

and that there exists an absolute constant $C$ so that $\|\psi\|^2_{L^2} \geq CA_0$.

References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften, Springer, 2011.

[2] H. Bahouri, A. Cohen, G. Koch, A general wavelet-based profile decomposition in the critical embedding of function spaces, Confluentes Mathematici 3 (3) (2011) 1–25.

[3] H. Bahouri, I. Gallagher, Weak stability of the set of global solutions to the Navier-Stokes equations, arXiv:1109.4043, 2011.

[4] H. Bahouri, P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, American Journal of Mathematics 121 (1999) 131–175.

[5] H. Bahouri, M. Majdoub, N. Masmoudi, On the lack of compactness in the 2D critical Sobolev embedding, Journal of Functional Analysis 260 (2011) 208–252.

[6] H. Bahouri, M. Majdoub, N. Masmoudi, Lack of compactness in the 2D critical Sobolev embedding, the general case, arXiv:1112.2998, 2011.

[7] H. Brezis, S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Communications in Partial Differential Equations 5 (1980) 773–789.

[8] P. Gérard, Description du défaut de compacité de l’injection de Sobolev, ESAIM: Control, Optimisation and Calculus of Variations 3 (1998) 213–233, http://www.emath.fr/cocv/(electronic).

[9] A. Henrot, M. Pierre, Variations et optimisation de formes, Mathématiques et Applications, vol. 48, Springer, 2005.

[10] S. Ibrahim, M. Majdoub, Comparaison des ondes linéaires et non linéaires à coefficients variables, Bulletin de la Société Mathématique de Belgique 10 (2003) 299–312.

[11] S. Jaffard, Analysis of the lack of compactness in the critical Sobolev embeddings, Journal of Functional Analysis 161 (1999) 384–396.

[12] C.K. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy critical focusing nonlinear wave equation, Acta Mathematica 201 (2008) 147–212.

[13] S. Keraani, On the defect of compactness for the Strichartz estimates of the Shrödinger equation, Journal of Differential Equations 175 (2001) 353–392.

[14] P.-L. Lions, The concentration–compactness principle in the calculus of variations. The limit case. I, Revista Matemática Iberoamericana 1 (1985) 145–201.

[15] M. Majdoub, Qualitative study of the critical wave equation with a subcritical perturbation, Journal of Mathematics Analysis and Applications 301 (2005) 354–365.

[16] J. Moser, A sharp form of an inequality of N. Trudinger, Indiana University Mathematics Journal 20 (1971) 1077–1092.

[17] M.-M. Rao, Z.-D. Ren, Applications of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 250, Marcel Dekker Inc., 2002.

[18] B. Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$, Journal of Functional Analysis 219 (2005) 340–367.

[19] M. Struwe, Critical points of embeddings of $H^1_0$ into Orlicz spaces, Annales de l’Institut Henri Poincaré Analyse Non Linéaire 5 (1988) 425–464.