Vacuum Fluctuations, Geometric Modular Action and Relativistic Quantum Information Theory

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Abstract. A summary of some lines of ideas leading to model-independent frameworks of relativistic quantum field theory is given. It is followed by a discussion of the Reeh-Schlieder theorem and geometric modular action of Tomita-Takesaki modular objects associated with the quantum field vacuum state and certain algebras of observables. The distillability concept, which is significant in specifying useful entanglement in quantum information theory, is discussed within the setting of general relativistic quantum field theory.

1 Introduction

About 100 years ago, new insights into the physical world were gained which at that time had a new quality to them. The new feature was that certain phenomena could successfully be described by means of concepts which have little in common with the behaviour of physical objects familiar from everyday experience. The first of these insights we are referring to was Planck’s quantum hypothesis in his account of black-body radiation. The second was Einstein’s theory of special relativity. (See, e.g., [45] for a historical presentation of these developments.)

It took a while — more or less, two decades — until quantum theory reached the form of (non-relativistic) quantum mechanics which is nowadays taught in courses at universities. A further step was the combination and unification of the principles of quantum mechanics and special relativity. The endeavours to accomplish this step took still longer — and, rigorously speaking, they haven’t come to an end even today. And the synthesis of quantum mechanics and general relativity into some form of a quantum theory of gravity lies still well ahead of us.

The theory unifying the principles of quantum mechanics and special relativity has come to be called relativistic quantum field theory, or QFT, for short. To delineate the basic characteristics of QFT, let us recall first the basic features of quantum mechanics, which provides a conceptual foundation for describing physical processes at small scales (in space and time), and is therefore relevant in the microscopic domain and accounts for the stability of atoms and molecules. Moreover, its (experimentally testable) predictions are of statistical nature, with the characteristic feature of uncertainty relations.

Special relativity, on the other hand, can be viewed as providing a conceptual foundation for the description of space and time, relevant in particular in the context
of processes involving very high energies and momenta. Among its principal features are the absence of preferred inertial frames (observers), i.e. Poincaré-covariance, the speed of light as maximal velocity of signal propagation, and matter (mass)-energy equivalence.

The fundamental aspects of both quantum mechanics and special relativity find a unification in the form of

**Quantum field theory**, which consequently provides a theoretical framework for the description of processes with very high energy/momentum exchange at very small time/length scales; it is therefore relevant in the sub-microscopic domain and accounts for the properties and the stability of elementary particles, predicts annihilation and creation of particles, new types of charges, anti-charges, PCT and spin-statistics theorems, fluctuations and long-range correlations.

While this is not the place to give a review of the historical development of QFT and its interplay with the development of elementary particle physics, involving also new concepts such as renormalization, internal group symmetries, gauge theory, spontaneous symmetry breaking, Higgs mechanism etc., there are some comments to be made at this point about the various sub-branches of QFT and its status as a physical theory, as well as its status as concerns mathematical consistency of the framework.

Let us begin by mentioning the by far largest branch of QFT, which we refer to as **perturbative QFT**. The idea here is to look at concrete quantum field models, mostly in the form of a Lagrangean for an — initially — classical field theory model involving certain types of matter and gauge fields. Typically, the fields interact in some way and this leads to the occurrence of multilinear (polynomial) expressions of the fields in the field equations. One would then like to have “quantized” solutions of the field equations. It is not a priori clear what this means, but the pragmatic way to proceed is as follows. One starts with the interaction-free part of the field equation (neglecting the multilinear, interacting parts of the field equations) and constructs “quantized” solutions for that in the form of “free” quantum fields — where it is in most of the relevant cases known what this means. Then one regards the interacting expressions of the (now quantized, free) fields as a perturbation of the free dynamics, and tries to construct solutions to the full dynamics by means of a perturbation series in the parameter specifying the strength of the interaction (the coupling parameter). At this point there arises the difficulty that the various multilinear expressions in the fields appearing in the perturbation series are not well defined at the level of (free) quantized fields, and that they need to be “renormalized”. If this is possible systematically to all polynomial orders upon introducing only finitely many parameters (to be determined experimentally), one calls the quantum field model under consideration (perturbatively) renormalizable. Once the renormalization parameters are determined experimentally, predictions of the quantum field model can be compared with experimental data e.g. obtained in scattering experiments with elementary particles — up to a given order in the coupling parameter of the perturbation series.

The successes of perturbative quantum field theory in comparison with experiment are truly impressive. The numerical agreement of theoretical predictions and experimental data is in many cases of the in the range of 8 significant figures or better, and also properties of particles whose existence was predicted by QFT prior to observation, like in the case of the $W^\pm$ and $Z^0$ bosons in the electroweak interactions, are in excellent agreement with experimental findings. (See [68, 34] for the various aspects of perturbative QFT.)

However, from a more fundamental point of view, perturbative QFT is not fully
satisfactory. The perturbation series by which one attempts to approximate the full interacting quantum field dynamics won’t converge, and then it is unclear if there is a solution to the quantized field equations at all. This provokes the question at which order in the coupling parameter the perturbation series ought to be truncated to yield acceptable agreement with experiment, and this question remains so far unanswered within perturbative quantum field theory. Moreover, the number of renormalization parameters which have to be determined by experiment and are not derivable within perturbative theory are quite large for physically realistic quantum field models (of the order of about 20 in the case of the standard model), and this is regarded as a considerable drawback as concerns the predictive power of perturbative QFT.

Hence, there clearly is room for approaches to QFT (and elementary particle physics) other than by perturbative QFT. Let me point out three basic branches. One idea is that theories such as the standard model are simply not rich enough and/or do not include all interactions (such as gravity), and that a richer theory should be considered in the first place (first at the level of a “classical field theory” then quantized, maybe at the level of perturbative QFT), with the hope that the richer symmetry structure constrains the amount of free parameters considerably. Grand unified theories, and string theory, can be seen in this light.

The next branch is constructive quantum field theory, where one attempts to construct solutions to the quantized, interacting field equations mathematically rigorously. This branch of QFT is much smaller than those mentioned previously, but has had quite impressive successes which are partly documented in 

The mathematical difficulties one is faced with in constructive QFT are immense, not least by the circumstance that it is often not entirely clear what is actually meant by a solution to a quantized, interacting field equation (we will soon come back to this point). Nevertheless, interacting quantum field models have been rigorously constructed in spacetime dimensions 2 and 3. The case of a rigorous solution to quantized field equations for models regarded as physically relevant remains open in 4 spacetime dimensions and is still an area of active research. The Clay Institute of Mathematics awards a million dollars for the solution of this problem. There is also a branch of QFT which is known as lattice gauge theory, and which can be placed somewhere between perturbative QFT and constructive QFT. The interested reader is referred to 

Finally, there is yet another branch of QFT, commonly called axiomatic quantum field theory, although this labelling is to some degree misleading. The basic idea is that one wishes to formulate and analyze the properties which are thought to be common to all physically realistic quantum field models. This is on one hand indispensable to make the problem of rigorous construction of interacting quantum field models a mathematically well-defined problem, on the other hand it is also difficult in the absence of rigorously constructed interacting quantum field models in 4 spacetime dimensions as a guidance. To begin with, the task is to find a mathematical structure which encodes the basic principles of quantum mechanics and special relativity, and which subsumes the known rigorously constructed quantum field models where these principles are implemented (e.g. for free quantum fields, or for interacting quantum fields in lower spacetime dimension). This task was taken up initially by Wightman and others (see 

The present contribution is, in fact, placed within the framework of axiomatic
QFT. In the next section, we will sketch how one can combine the principles of quantum mechanics and of special relativity in a mathematical structure which more or less is “common to all quantum field models”. Then we will present the “Reeh-Schlieder-theorem” and discuss some of aspects of it. The Reeh-Schlieder-theorem is a strong mathematical statement about the ubiquity and complexity of vacuum fluctuations in quantum field theory, regardless of the particular quantum field theoretical model considered: It is a consequence of first principles such as locality (causal propagation), stability of the vacuum, and covariance. Then we will discuss a mathematical structure arising in connection with the Reeh-Schlieder-theorem: Geometric modular action. While discovered already in 1975 by Bisognano and Wichmann [7], this mathematical structure has in the recent years given rise to many new insights into quantum field theory which we will briefly discuss. In a sense, it unifies the mathematical domains of quantum mechanics — operator algebras — and of special relativity — affine geometry — completely. Moreover, it opens very interesting new perspectives.

We will then proceed to another topic where the Reeh-Schlieder-theorem plays again a prominent role: In discussing aspects of entanglement in the framework of relativistic QFT. This part of the present contribution is essentially a summary of parts of a recent joint work with R. Werner [67]. We will present a variant of the distillability concept of bipartite systems in quantum field theory. Furthermore, we will quote our result stating that the vacuum state (as well as any relativistic thermal equilibrium state) is distillable over arbitrary spacelike distances.

Taking up a line of thought mentioned at the very beginning of this introduction, we should like to point out that also in the realm of phenomena described by quantum field theory one encounters theoretical propositions which at first sight appear implausible because of their highly counterintuitive character. The Reeh-Schlieder-theorem serves as an example, as well as distillability of the vacuum state. However, careful statement of the concepts and careful analysis of their consequences, together with proper use of adequate mathematical methods, will bring us closer to an understanding of these novel situations and, ultimately, their experimental testing. Thus, we will need to collect also some mathematical concepts and results which are not necessarily in every theoretical physicist’s toolbox. Nevertheless, we have tried to keep the amount of formalities at a minimum and to make this contribution as self-contained as possible, hoping that everyone familiar with quantum mechanics, special relativity and the rudiments of quantum field theory will be able to follow this contribution without undue strain.

2 From Quantum Mechanics and Special Relativity to Quantum Field Theory

Let us once more recall the basic features of quantum mechanics, this time at a more formal level. The theory of quantum mechanics says that a quantum mechanical system is described by:

- $\mathcal{H}$: a Hilbert space
- $\mathcal{R} \subset B(\mathcal{H})$: a $*$-algebra of operators, where:
  - $A = A^* \in \mathcal{R}$ is interpreted as an observable
  - For $\psi \in \mathcal{H}$ with $||\psi|| = 1$, the quantity
    $$\langle A \rangle_\psi = \langle \psi, A\psi \rangle$$
is interpreted as the expectation value of the observable \( A \) in the state given by \( \psi \).

More generally: For \( \rho \) = trace-class operator on \( \mathcal{H} \) with \( \rho \geq 0 \), \( \text{trace}(\rho) = 1 \), we interpret \( \langle A \rangle_\rho = \text{trace}(\rho A) \) as expectation value of \( A \) in the state given by \( \rho \).

We need to explain some notation and terminology appearing here. First note that by Hilbert space we mean a complex-linear Hilbert space. The scalar product of two vectors \( \psi, \phi \in \mathcal{H} \) is denoted \( \langle \psi, \phi \rangle \), and \( ||\psi||^2 = \langle \psi, \psi \rangle \). By \( B(\mathcal{H}) \) we denote the set of all bounded linear operators \( A : \mathcal{H} \to \mathcal{H} \). A subset \( \mathcal{R} \) of \( B(\mathcal{H}) \) (which may, but need not, coincide with \( B(\mathcal{H}) \)) is a \( * \)-algebra if, given \( A \) and \( B \) in \( \mathcal{R} \) and \( \lambda, \mu \in \mathbb{C} \), the operators \( \lambda A + \mu B \), \( AB \) and \( A^* \) are again contained in \( \mathcal{R} \), where \( A^* \) is the adjoint operator. Hence, a quantum mechanical system is described by specifying its state Hilbert space \( \mathcal{H} \) and its algebra of observables \( \mathcal{R} \).

There are a few remarks to be made:

(2.1) One might take the point of view that the description of a quantum mechanical system requires also the specification of dynamics, e.g. in the form of a Hamiltonian operator \( H \) acting in \( \mathcal{H} \). Furthermore, one may also require that the quantum system admits states of lowest energy for \( H \) ("ground states") [or that the spectrum of \( H \) is bounded below], or thermal equilibrium states, since the sudden decay of matter which would otherwise occur (for quantum systems not having these properties) is not observed in real systems. We shall ignore aspects of dynamics for the moment, but will come back to this point later in the discussion of quantum field theory.

(2.2) It is tacitly assumed that \( \mathcal{R} \) is non-abelian, i.e. that \( AB \neq BA \) holds for some \( A \) and \( B \) in \( \mathcal{R} \), as otherwise there are no uncertainty relations which are characteristic of quantum theory.

(2.3) One may wonder if the setting presented here is general enough since \( \mathcal{R} \) contains only bounded operators, while in quantum mechanics of single particles observables like position or momentum are represented by unbounded operators as a consequence of the canonical commutation relations. Employing the functional calculus, however, one may pass e.g. from the unbounded operator \( P \) representing the observable "momentum" to the bounded operator \( f(P) \), which is bounded when \( f \) is a bounded real function on \( \mathcal{R} \), and which represents the observable "\( f \) (momentum)". This shows that it is in general no loss of physical information to work only with bounded operators as observables; moreover, unbounded operators can be regarded as suitable limits of sequences of bounded operators. Working with bounded operators has considerable advantages as far as the mathematical analysis is concerned, since subtle domain problems that plague the rigorous manipulation of unbounded operators are avoided.

(2.4) One may also wonder why we have not simply taken \( \mathcal{R} = B(\mathcal{H}) \), the standard case in quantum mechanics of a single particle. The reason is that we would like to allow greater flexibility, making it possible to consider also subsystems of a larger, ambient system. An example, occuring often in quantum information theory, is the case \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) with \( \mathcal{R} = B(\mathcal{H}_1) \otimes 1 \) modelling a subsystem of the full system whose algebra of observables is given by \( B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) \). We will encounter a similar situation later. In discussions of model-independent properties of quantum field theories, \( \mathcal{R} \) often means the algebra of observables measurable — and in this sense, localized — in a proper subregion of Minkowski spacetime, as we will discuss below.

Having thus collected the basics of the formal framework of quantum mechanics, we turn now to special relativity. We will be very brief in recalling its basic formal ingredients. The theory of special relativity states that all physical events can
be collected in a catalogue which has the structure of a 4-dimensional affine space $M$, where each point in $M$ represents a (possible) event. There is a metric $\eta$ of Lorentzian signature on $M$; that is, one can choose identifications of $M$ with $\mathbb{R}^4$ in such a way that, with respect to the standard coordinates of $\mathbb{R}^4$, $\eta$ is represented by the diagonal matrix $\text{diag}(1, -1, -1, -1)$. The choice of such an identification is also referred to as fixing of an inertial frame. With respect to a fixing of an inertial frame (inducing an orientation and a time-orientation), one can introduce the proper orthochronous Poincaré group $\mathfrak{P}_+^\uparrow$, which is the unit connected component of the full Poincaré group $\mathfrak{P}$, defined as the group of all invertible affine transformations of $M$ leaving $\eta$ invariant. We assume from now on that an inertial frame has been fixed. Any $L \in \mathfrak{P}$ (or $\mathfrak{P}_+^\uparrow$) decomposes as a semidirect product of $\Lambda \in \mathfrak{L}$ (or $\mathfrak{L}_+^\uparrow$), the Lorentz group (or its unit connected component) and $a \in \mathfrak{T} \equiv \mathbb{R}^4$, the group of translations, according to $Lx = (\Lambda, a)x = \Lambda x + a, \quad x \in M \equiv \mathbb{R}^4$.

The reader is referred to the contribution by Domenico Giulini in this volume for a full discussion of special relativity, Minkowski spacetime and the Poincaré group (alternatively, see e.g. [59]).

The theory of special relativity states that the description of a physical system is equivalent for all inertial observers, i.e. in arbitrary inertial frames. Put differently, the description of physical processes should be covariant with respect to proper, orthochronous Poincaré transformations. More formally, this means:

Suppose a quantum system is modelled by $(\mathcal{R}, \mathcal{H})$. Let $\rho$ be a density matrix and $A$ an observable with respect to a given inertial frame. If $L \in \mathfrak{P}_+^\uparrow$, then there corresponds, with respect to the $L$-transformed inertial frame, a density matrix $\rho_L$ and observable $A_L$ to $\rho$ and $A$, respectively, such that

$$\langle A_L \rangle_{\rho_L} = \langle A \rangle_{\rho}. \quad (2.5)$$

One can add some mathematical precision, requiring that the maps taking $A$ to $A_L$ and $\rho$ to $\rho_L$ are one-to-one and onto, i.e. bijective. Following Wigner, one may think of elementary systems where $\mathcal{R} = B(\mathcal{H})$, and then one can conclude:

There is a unitary representation

$$\tilde{\mathfrak{P}}_+^\uparrow \ni \tilde{L} \mapsto \tilde{U}(\tilde{L})$$

of the universal covering group of $\mathfrak{P}_+^\uparrow$ on $\mathcal{H}$, such that

$$A_L = \tilde{U}(\tilde{L})A\tilde{U}(\tilde{L})^*, \quad \rho_L = \tilde{U}(\tilde{L})\rho\tilde{U}(\tilde{L})^*,$$

where $\tilde{\mathfrak{P}}_+^\uparrow \ni \tilde{L} \mapsto L \in \mathfrak{P}_+^\uparrow$ is the canonical projection. Moreover, if suitable assumptions about the continuity of the maps $A \mapsto A_L$, $\rho \mapsto \rho_L$ are made — and we tacitly make this assumption — then one can conclude that the unitaries $\tilde{U}(\tilde{L})$ depend continuously on $\tilde{L}$.

This result is known as the Wigner-Bargmann-theorem, which actually holds under somewhat weaker assumptions than expressed in (2.5); it is sufficient to consider as observables 1-dimensional projections $A = \vert\psi\rangle\langle\psi\vert$ and likewise, 1-dimensional projections $\rho = \vert\phi\rangle\langle\phi\vert$ as density matrices, and to replace (2.5) by the weaker requirement

$$\Vert\langle A_L \rangle_{\rho_L} \Vert = \Vert\langle A \rangle_{\rho} \Vert.$$

We refer to the original articles by Wigner [70] and Bargmann [2] and to [28, 60, 55] for considerable further discussion.
The Wigner-Bargmann-theorem states that, in the case of an (elementary) quantum system compatible with the covariance principle of special relativity, the state Hilbert space $\mathcal{H}$ carries a unitary representation of $\Pi^+_1$, the universal covering group of the proper orthochronous Poincaré group, implementing the change of inertial frames. The appearance of a unitary representation of the universal covering group instead of the proper orthochronous Poincaré group itself is due to the fact that (2.5) fixes only a unitary representation of $\tilde{P}$ instead of the proper orthochronous Poincaré group itself is due to the fact that frames. The appearance of a unitary representation of the universal covering group, implementing the change of inertial frames compatible with the principles of special relativity, but not all, in particular the aspect of a “quantized field” hasn’t appeared yet. To see how this aspect comes into play, one usually takes a complementary route: Consider a typical $\Pi^+_1$-covariant classical system; i.e. a classical field subject to a linear wave-equation. The electromagnetic field provides the prime and archetypical example, but let us consider here a much simpler example, the scalar Klein-Gordon field $\varphi(x)$, $x \in M = \mathbb{R}^4$, obeying the following equation of motion:

$$(\gamma^{\mu \nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + m^2)\varphi(x) = 0$$

where $m \geq 0$ is a constant. Such a classical field can be viewed as a Hamiltonian system with infinitely many degrees of freedom, and one may therefore try and quantize it by regarding it as a “limit” of a Hamiltonian system with $N$ degrees of freedom as $N \to \infty$, and taking as its quantized version the “limit” of the quantized systems with $N$ degrees of freedom as $N \to \infty$. In the case of the Klein-Gordon field, the classical field $\varphi(x^0, \mathbf{x})$, $x = (x^0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$, at time-coordinate $x^0$ (with respect to an arbitrary but fixed inertial frame) can be approximated by a discrete lattice of coupled harmonic oscillators with canonical coordinates $q_{\lambda \mu \nu}(x^0)$ at the lattice site

$$\mathbf{x}(\lambda, \mu, \nu) = a \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} \in \mathbb{R}^3, \quad \lambda, \mu, \nu \in \mathbb{Z}, \quad |\lambda|, |\mu|, |\nu| \leq \frac{1}{a^2},$$

where $a > 0$ is the lattice spacing. To the discrete lattice system one can associate the quantum system of coupled harmonic oscillators (at lattice spacing $a$, there are $N \sim 1/a^6$ of them), where the canonical classical coordinates $q_{\lambda \mu \nu}(x^0)$ and conjugate momenta $p_{\lambda \mu \nu}(x^0)$ become operators $Q_{\lambda \mu \nu}(x^0)$ and $P_{\lambda \mu \nu}(x^0)$ obeying the canonical commutation relations. In the limit as $a \to 0$ and $N \to \infty$, one obtains for each $f, h \in C^\infty_0(\mathbb{R}^3)$ the field operators

$$\Phi(x^0, f) = \lim_{a \to 0, N \to \infty} \sum_{\lambda, \mu, \nu} Q_{\lambda \mu \nu}(x^0)f(\mathbf{x}(\lambda, \mu, \nu))a^3,$$

$$\Pi(x^0, f) = \lim_{a \to 0, N \to \infty} \sum_{\lambda, \mu, \nu} P_{\lambda \mu \nu}(x^0)f(\mathbf{x}(\lambda, \mu, \nu)).$$

For a detailed discussion of this construction, cf. [33]. To summarize, we find the following formal correspondences (where we use the shorthand $j$ or $\ell$ for the index triple $\lambda \mu \nu$, and occasionally drop the time-argument $x^0$):
Let us denote this representation by \( \tilde{U} \) and \( \tilde{\Pi} \), respectively. It is quite useful to introduce, for test-functions \( f \) and \( h \) in \( C_0(\mathbb{R}^3) \), the \( \tilde{\Pi}(x^0, h) \), whence the equal-time canonical commutation relations are often written in the form

\[
\{\varphi(x^0, f), \pi(x^0, h)\} = i\hbar f(x) h(x)
\]

So far we have introduced field operators \( \Phi(x^0, f) \) and their canonically conjugate momenta \( \Pi(x^0, h) \), at fixed inertial frame-coordinate time \( x^0 \). They are “smeared” against the spatial argument \( x \) with test-functions \( f \) and \( h \) in \( C_0(\mathbb{R}^3) \). Without smearing, the density-like quantities \( \Phi(x^0, x) \) and \( \Pi(x^0, x) \) cannot be interpreted as operators on a Hilbert space as a consequence of the canonical commutation relations — the entry in the lower right corner of the just tabled scheme — but only as quadratic forms. This is due to the distributional character of the \( \Phi(x^0, x) \) and \( \Pi(x^0, x) \), whence the equal-time canonical commutation relations are often written in the form

\[
[\Phi(x^0, x), \Pi(x^0, x')] = i\hbar \delta(x - x').
\]

It is quite useful to introduce, for test-functions \( F \in C_0^\infty(\mathbb{R}^4) \) distributed over open subsets of Minkowski spacetime, the field operators

\[
\Phi(F) = \int d^4x F(x^0, x) \Phi(x^0, x) = \int dx^0 \Phi(x^0, f_{x^0}), \quad f_{x^0}(x) = F(x^0, x).
\]

These field operators can be rigorously interpreted as unbounded (and for real-valued \( F \), selfadjoint) operators on a suitable domain of a Hilbert space \( \mathcal{H} \) which arises as the bosonic Fock space over the one-particle space of solutions to the Klein-Gordon equation with positive energy. This one-particle space carries an irreducible, unitary representation of \( \mathfrak{g}_+^\dagger \), which lifts to a unitary representation of \( \mathfrak{g}_+^\dagger \) on \( \mathcal{H} \). Let us denote this representation by \( U \), since it is actually a representation of \( \mathfrak{g}_+^\dagger \) in this case, as for every linear field equation of integer spin. Then one finds that covariance holds in the form of

\[
U(L) \Phi(F) U(L)^* = \Phi(F \circ L^{-1}), \quad L \in \mathfrak{g}_+^\dagger, \quad F \in C_0^\infty(\mathbb{R}^4);
\]

moreover, one also has

\[
\Phi((\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + m^2) F) = 0, \quad F \in C_0^\infty(\mathbb{R}^4),
\]

and there holds also the covariant form of the canonical commutation relations,

\[
[\Phi(F_1), \Phi(F_2)] = i\hbar G(F_1, F_2), \quad F_1, F_2 \in C_0^\infty(\mathbb{R}^4),
\]
with the “causal Green’s function”

\[ G(F_1, F_2) = \operatorname{Im} \int_{\mathbb{R}^3} d^3 p \frac{\tilde{F}_1(\omega(p), -p)\tilde{F}_2(\omega(p), -p)}{\omega(p)} \]

\[ \omega(p) = \sqrt{p^2 + m^2}, \quad \tilde{F} = \text{Fourier-transform of } F \]

which vanishes whenever the supports of \( F_1 \) and \( F_2 \) are causally separated.

We shall not elaborate on the mathematical details related to the Fock space operators \( \Phi(F) \) since this is all well-documented in the literature (see, e.g., [49, 8]. Rather we should make the remark at this point that the properties of the operators \( \Phi(F) \), interpreted as Fock space operators, may serve as a blueprint of a general concept of a (in this case, scalar) “quantum field”, as soon as they are abstracted from properties pertaining to the model of the Klein-Gordon field, i.e., the equation of motion (2.6). The ensuing conceptual framework for a general scalar quantum field are represented by the “Wightman axioms”, which we list now, not paying too much attention to full mathematical rigor (see [60, 8, 37] for a more detailed exposition of these matters).

i) \( \exists \) a Hilbert space \( \mathcal{H} \) with a dense domain \( D \subset \mathcal{H} \), so that all \( \Phi(F) \) are well-defined operators on \( D \), and \( \Phi(F)^* = \Phi(F) \)

ii) \( F \mapsto \Phi(F) \) is complex linear and suitably continuous

iii) **Covariance:** There is on \( \mathcal{H} \) a unitary representation

\[ \mathcal{P}_L \ni L \mapsto U(L), \text{ with } U(L)D \subset D, \text{ so that} \]

\[ U(L)\Phi(F)U(L)^* = \Phi(F \circ L^{-1}) \quad (\Phi(x)_L = \Phi(L(x))) \]

iv) **Locality, or relativistic causality:**

If the supports of the test-function \( F_1 \) and \( F_2 \) are causally separated, the corresponding field operators commute:

\[ [\Phi(F_1), \Phi(F_2)] = 0 \]

v) **Spectrum condition/positivity of the total energy:**

Writing \( U(1, a) = e^{iP_0 a_0^2} \), it holds (in the sense of expectation values) that

\[ P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0, \quad P_0 \geq 0 \]

vi) **Existence (and uniqueness) of the vacuum:**

\( \exists \Omega \in D, ||\Omega|| = 1, \text{ so that } U(L)\Omega = \Omega \text{ and this vector is uniquely determined up to a phase factor.} \)

vii) **Cyclicity of the vacuum:**

The domain \( D \) is spanned by vectors of the form

\[ \Omega, \ \Phi(F)\Omega, \ \Phi(F_1)\Phi(F_2)\Omega, \ldots, \ \Phi(F_1) \cdots \Phi(F_n)\Omega, \ldots \]

As indicated above, the just given collection of conditions tries to capture the essential properties of a “quantum field”. We notice that, compared to the properties of the Klein-Gordon field, the commutation relations (2.7) have been generalized to the condition of spacelike commutativity, and the reference to a specific field equation has been dropped. Spacelike commutativity says that there should be no uncertainty relations between observables measured at causal separation from each other, and thus gives expression to the principle that there is no operational signal propagation faster than the speed of light. It should be remarked here that there is
no difficulty in generalizing the above stated conditions to fields of general spinor-
tensor-type \[60, 8, 37\]. The basic difference is that for fields of half-integer spin, spacelike commutativity of the field operators must be replaced by spacelike anti-
commutativity in order to ensure consistency with the other conditions: This is, basically, the content of the spin-statistics theorem. In this sense, a field carrying half-integer spin does not have the character of an observable — typically, it also transforms non-trivially under gauge transformations. Observable quantities, and related quantum field operators fulfilling spacelike commutation relations, can be built from half-integer spin quantum fields by forming suitable bilinear expressions in those fields. Once more, we must refer to the literature for a fuller discussion of these matters \[8\].

Furthermore, it is worth noting that the type of Poincaré covariance \(\text{iii}\), implemented by a unitary representation of \(P^+_\uparrow\), makes an explicit appearance here, completely in the spirit of the Wigner-Bargmann theorem. (For fields of half-integer spin type, this must be replaced by a unitary representation of \(\tilde{P}^+_\uparrow\), in keeping with the circumstance that such fields are not directly observable.)

Some new aspect appears here which we have already alluded to in remark (2.1) and which made an implicit appearance elsewhere when we referred to irreducible unitary representations of \(\tilde{P}^+_\uparrow\) having positive energy. This is the aspect that the time-translations which the unitary representation \(\tilde{U}\) of \(\tilde{P}^+_\uparrow\) implements on the Hilbert space \(\mathcal{H}\) are interpreted also as dynamical evolutions of the system, and that these dynamical evolutions be stable in the sense that their corresponding energy is always non-negative and that there should be a common state of lowest energy, the vacuum state. This state is “void of stable particles” but, as we shall see later, not void of correlations, and these have actually a rich structure.

It is the subtle interplay of dynamical stability in the form of the spectrum condition together with locality (or spacelike anti-commutativity in the case of quantum fields carrying half-integer spin) which is responsible for this richness. The condition of cyclicity is mainly made for mathematical convenience; it says that all state vectors of the theory can be approximated by applying polynomials of all field operators on the vacuum. In case of the presence of a vacuum vector, this property could be sharpened to irreducibility, i.e. that already all observables can be approximated by polynomials in the field operators. This is actually equivalent to clustering of vacuum expectation values \[60, 8, 37\]. However, in a more general situation where there is no vacuum state for all time-evolutions (time-shifts), but e.g., a thermal equilibrium state, irreducibility doesn’t hold in general.

While the Wightman framework captures apparently many essential aspects of (observable) quantum fields and is so far not in obvious conflict with experiences gained in constructive quantum field theory, there are some points which lead one to trade this framework for a still more abstract approach. Let me try to illustrate some of these points. The first is of a more technical nature: In handling the — in general — unbounded field operators \(\Phi(F)\), subtle domain questions come into play whose physical significance is often not entirely clear. More seriously, it might happen that the field operators \(\Phi(F)\) do not correspond to directly observable quantities, and then it is doubtful why they should be regarded as the basic objects of the formal description of a physical theory, at least from an operational point of view. Somehow related to this shortcoming, the \(\Phi(F)\) aren’t invariants of the experimentally accessible quantities in the following sense: In general, one can find for a given Wightman field \(F \mapsto \Phi(F)\) other Wightman fields \(F \mapsto \tilde{\Phi}(F)\), subject to different field equations and commutation relations, which yield the same \(S\)-matrix as the field \(F \mapsto \Phi(F)\) \(\text{[10]}\), see also \[51\] for a more recent instance of this fact). Apart from that, gauge-carrying quantum fields do not fit completely into the framework. Assuming them to be local fields in the same sense as described
above often leads to difficulties with Hilbert space positivity, as e.g. in quantizing free electrodynamics. This difficulty can be cured symptomatically by allowing \( \mathcal{H} \) to carry an inner product that is not positive definite \[8\]. However, such a complication makes technical issues, such as domain questions, even much worse.

Hence, there is considerable motivation to base the description of a relativistic quantum system on observable quantities and to abandon the mainly classically inspired concept of (a quantized version of) a field. In the case that \( F \mapsto \Phi(F) \) is an observable quantum field, one can pass to a description of this system which emphasizes the localization of observables in space and time rather than their arrangement into “field strengths”: One can form, for each open subset \( O \) of Minkowski spacetime \( M \equiv \mathbb{R}^4 \), a \( * \)-algebra of bounded operators

\[
\mathcal{R}(O) = \{ * \text{-algebra generated by all } A = f(\Phi(F)), \quad f : \mathbb{R} \to \mathbb{R} \text{ bounded}, \quad F = \mathcal{F} \text{ has support in } O \} \quad (2.8)
\]

The properties that one finds for the family of \( * \)-algebras \( \mathcal{R}(O) \), \( O \) ranging over the bounded subsets of \( \mathbb{R}^4 \), form the conditions of the operator algebraic approach to general quantum field theories according to Haag and Kastler \[29, 28\]. These conditions read as follows.

a) **Isotony:** \( O_1 \subset O_2 \Rightarrow \mathcal{R}(O_1) \subset \mathcal{R}(O_2) \)

b) **Covariance:** \( A \in \mathcal{R}(O) \iff U(L)A(U(L))^* \in \mathcal{R}(L(O)), \)
   or \( U(L)\mathcal{R}(O)U(L)^* = \mathcal{R}(L(O)) \)

c) **Locality:** If the space-time regions \( O_1 \) and \( O_2 \) are causally separated, then the corresponding operator algebras \( \mathcal{R}(O_1) \) and \( \mathcal{R}(O_2) \) commute elementwise:

\[ A \in \mathcal{R}(O_1), \quad B \in \mathcal{R}(O_2) \Rightarrow [A, B] = 0 \]

d) **Spectrum condition and existence of the vacuum:**
As before in v) and vi)

e) **Cyclicity of the vacuum:**
\( \{ A\Omega : A \in \bigcup_{O} \mathcal{R}(O) \} \) is dense in \( \mathcal{H} \)

f) **Weak additivity:** If \( \bigcup_i O_i \) contains \( O \), then the algebra generated by the \( \mathcal{R}(O_i) \) contains \( \mathcal{R}(O) \)

We should emphasize that, adopting this framework as basis for a description of a special relativistic quantum system, the crucial structural ingredient is the assignment of not just a single operator algebra to the system but of operator algebras \( \mathcal{R}(O) \) to the individual sub-regions \( O \) of Minkowski spacetime. Each \( \mathcal{R}(O) \) is generated by the observables which can be measured at times and locations in \( O \), and therefore one refers to the observables in \( \mathcal{R}(O) \) as those localized in \( O \), and to the \( \mathcal{R}(O) \) as local observable algebras. If actually there is a quantum field \( F \mapsto \Phi(F) \) generating the local observable algebras as in \[23\], then one may view it as a “coordinatization” of the family \( \{ \mathcal{R}(O) \}_{O \subset M} \), the latter being the “invariant” object, in analogy to a manifold built up from coordinate systems.

A set of data \( \{ \{ \mathcal{R}(O) \}_{O \subset M}, U, \Omega \} \) fulfilling the conditions just listed is called a quantum field theory in vacuum representation. One can consider other representations of a quantum field theory, e.g. thermal representations, where the spectrum condition imposed on \( U \) and the vacuum vector condition imposed on \( \Omega \) are replaced by the condition that the state \( \langle \Omega | \Omega \rangle \) be a thermal equilibrium state. We will encounter such a situation later.
The reader might wonder at this point how charge carrying quantum fields fit into this operator algebraic version of quantum field theory where up to now only observable quantities have been mentioned. The answer is that charge carrying field operators arise in connection with yet other Hilbert space representations of the quantum field field theory, i.e., of the family of operator algebras \( \{ \mathcal{R}(O) \}_{O \subset \mathcal{M}} \). States in these Hilbert space representations cannot be coherently superposed with any state in the vacuum representation. These charged representations are therefore called superselection sectors. The analysis of superselection sectors and the full reconstruction of a compact gauge group and of associated charge carrying quantum field operators from the structure of superselection sectors can be regarded as being one of the greatest achievements in axiomatic quantum field theory so far, but we shall not pause to explore these matters and refer the reader to [28, 21, 54] for further information.

It should be pointed out that all quantum fields obeying linear equations of motion provide examples for the operator algebraic framework, by the relation (2.8) [for integer spin fields; for half-integer spin fields, one must instead define \( \mathcal{R}(O) \) by first constructing suitable bilinear expressions in the fields]. Moreover, there are examples of interacting quantum fields in 2 and 3 spacetime dimensions and these are compatible with the operator algebraic framework via (2.8).

The interplay between the spectrum condition and locality puts non-trivial constraints on quantum field theories and leads to interesting general results about their structure. Prime examples are the PCT theorem, the spin-statistics relation (cf. [60, 8, 28]) and geometric modular action. About the latter, perhaps less familiar, but highly fascinating issue we have more to report in the following section.

3 The Reeh-Schlieder Theorem and Geometric Modular Action

In 1961, Helmut Reeh and Siegfried Schlieder showed that the conditions for a quantum field theory of Wightman type, given above, lead to a remarkable consequence [50]. Namely, let \( O \) be any non-void open region in Minkowski spacetime, and denote by \( \mathcal{P}(O) \) the \(*\)-algebra generated by all quantum field operators \( \Phi(F) \) where the test-functions are supported in \( O \). Then the set of vectors \( \mathcal{R}(O) \Omega = \{ A \Omega : A \in \mathcal{R}(O) \} \) is dense in the Hilbert space \( \mathcal{H} \). In other words, given an arbitrary vector \( \psi \in \mathcal{H} \), and \( \epsilon > 0 \), there is a polynomial

\[
\lambda_0 1 + \sum_{j,k \leq N} \Phi(F_{1,j}) \cdots \Phi(F_{k,j})
\]

(3.1)

in the field operators, with \( \lambda_0 \in \mathbb{C} \) and \( F_{\ell,j} \in C_0^\infty(O) \), such that

\[
\| \psi - (\lambda_0 1 + \sum_{j,k \leq N} \Phi(F_{1,j}) \cdots \Phi(F_{k,j})) \Omega \| < \epsilon .
\]

(3.2)

In the operator algebraic setting of Haag and Kastler, the analogous property states that the set of vectors \( \mathcal{R}(O) \Omega = \{ A \Omega : A \in \mathcal{R}(O) \} \) is dense in \( \mathcal{H} \) whenever \( O \) is a non-void open set in \( \mathcal{M} \); equivalently, given \( \psi \in \mathcal{H} \) and \( \epsilon > 0 \), there is some \( A \in \mathcal{R}(O) \) fulfilling

\[
\| \psi - A \Omega \| < \epsilon .
\]

(3.3)

This result by Reeh and Schlieder appears entirely counter-intuitive since it says that every state of the theory can be approximated to arbitrary precision by acting with operators (operations) localized in any arbitrarily given spacetime region on the vacuum. To state it in a rather more drastic an provocative way (which I
learned from Reinhard Werner): By acting on the vacuum with suitable operations in a terrestrial laboratory, an experimenter can create the Taj Mahal on (or even behind) the Moon!

One might thus be truly concerned that this unusual behaviour of relativistic quantum field theory potentially entails superluminal signalling. However, despite the fact that such propositions have been made, this is not the case (see [50, 32, 17] for some clarifying discussions). We will also turn to aspects of this below in Sec. 4. A crucial point is that the operator \( A = A_\epsilon \) in (3.3) depends on \( \epsilon \) (and likewise, the polynomial \( \delta \lambda \) in (3.2) depends on \( \epsilon \)), and while \( \| A, \Omega \| \) will be bounded (in fact, close to 1) for arbitrarily small \( \epsilon \) (as follows from (3.3)), it will in general (in particular, with our drastic Taj Mahal illustration) be the case that \( A_\epsilon \) doesn’t stay bounded as \( \epsilon \to 0 \), in other words, \( \| A_\epsilon \| \) diverges as \( \epsilon \) tends to 0.

In keeping with the standard operational interpretation of quantum theory [41], \( \| A_\epsilon \|/\| A, \Omega \| \) is to be viewed as the ratio of cost vs. effect in the attempt to create a given state (Taj Mahal on the Moon) by local operations (in a laboratory on Earth, say) [28]. In other words, upon testing for coincidence with the “Taj Mahal state \( \psi \)”, it takes on average an ensemble of \( \| A_\epsilon \|/\| A, \Omega \| \) samples failing in the coincidence test to find a single successful coincidence. And in our illustration, the ratio \( \| A_\epsilon \|/\| A, \Omega \| \) will be an enormous number. A rough estimate can be based on the decay of vacuum correlations in quantum field theory. The order of magnitude of that decay is approximately given by \( e^{-d/\lambda c} \), where \( d \) denotes the spatial distance of the correlations and \( \lambda_c \) is the Compton wave length of the stable particles under consideration; then \( 1/e^{-d/\lambda c} \) is a rough measure for \( \| A_\epsilon \|/\| A, \Omega \| \) (when \( \epsilon \) is very small compared to 1). Taking for instance electrons as stable particles, and the distance Earth-Moon for \( d \), one obtains an order of magnitude of about \( 10^{-1020} \) for \( e^{-d/\lambda c} \). This shows that one can hardly construe a contradiction to special relativity on account of the Reeh-Schlieder theorem.

Nevertheless, for distances that are comparable to the Compton wavelength, the Reeh-Schlieder-theorem does predict a behaviour of the correlations in the vacuum state which is in principle experimentally testable, and is of truly quantum nature in the sense that they entail quantum entanglement over subsystems, as will be seen later in Sec. 4.

We will complement the previous discussion by a couple of remarks.

(3.4) The mathematical cause for the Reeh-Schlieder theorem lies in the spectrum condition, which entails that, for each \( \psi \) in the Hilbert space of a quantum field theory’s vacuum representation, the function

\[
(a_1, \ldots, a_n) \mapsto \langle \psi, U(a_1)A_1 U(a_2)A_2 \cdots U(a_n)A_n \Omega \rangle, \quad A_j \in \mathcal{R}(O), \quad \lambda_j \in \mathbb{R}^4, \quad (3.4)
\]

is the continuous boundary value of a function which is analytic in a conical subregion of \( \mathbb{C}^4 \). Hence, if the expression (3.3) vanishes when the \( a_j \) are in an arbitrarily small open subset of \( \mathbb{R}^4 \), then it vanishes for all \( a_j \in \mathbb{R}^4 \). Together with weak additivity one can conclude from this that any vector \( \psi \) which is orthogonal to \( \mathcal{R}(O)\Omega \) is actually orthogonal to \( \bigcup O, \mathcal{R}(O)\Omega \) and hence, by cyclicity of the vacuum vector, \( \psi \) must be equal to 0.

(3.5) There are many other state vectors \( \xi \in \mathcal{H} \) besides the vacuum vector for which the Reeh-Schlieder theorem holds as well, i.e. for which \( \mathcal{R}(O)\xi = \{ A\xi : A \in \mathcal{R}(O) \} \) is a dense subset of \( \mathcal{H} \) whenever \( O \subset M \) is open and non-void. In fact, one can show that there is a dense subset \( \mathcal{X} \) of \( \mathcal{H} \) so that every \( \xi \in \mathcal{X} \) has the property that \( \mathcal{R}(O)\xi \) is dense in \( \mathcal{H} \) as soon as \( O \subset M \) is a non-void open set [20]. Now, every element \( \xi \in \mathcal{X} \) (assumed to be normalized) induces a state (expectation value functional

\[
\omega_\xi(A) = \langle \xi, A\xi \rangle, \quad A \in \mathcal{R}(\mathbb{R}^4),
\]
and owing to the Reeh-Schlieder property of the vectors $\xi \in X$, $\omega_\xi$ will have long-range correlations, meaning that generically

$$\omega_\xi(AB) \neq \omega_\xi(A)\omega_\xi(B)$$

for $A \in \mathcal{R}(O_A)$ and $B \in \mathcal{R}(O_B)$ even if the spacetime regions $O_A$ and $O_B$ are separated by an arbitrarily large spacelike distance. However, even though the set of vectors $\xi$ inducing such long-range correlations is dense in the set of all state vectors in $\mathcal{H}$, there are in general also very many uncorrelated states. In fact, under very general conditions it could be shown that, as soon as a pair of (finitely extended) spacetime regions $O_A$ and $O_B$ separated by a non-zero spacelike distance is given, together with a pair of vectors $\xi_A$ and $\xi_B$ in $\mathcal{H}$ inducing states $\omega_{\xi_A}$ and $\omega_{\xi_B}$ on the local observable algebras $\mathcal{R}(O_A)$ and $\mathcal{R}(O_B)$, respectively, there is a state vector $\eta \in \mathcal{H}$ inducing a state $\omega_\eta$ on $\mathcal{R}(\mathbb{R}^4)$ with the property

$$\omega_\eta(AB) = \omega_{\xi_A}(A)\omega_{\xi_B}(B), \quad A \in \mathcal{R}(O_A), \quad B \in \mathcal{R}(O_B).$$

That is to say, in restriction to the algebra of observables associated to the region $O_A \cup O_B$ the state $\omega_\eta$ coincides with the (prescribed) product state induced by the pair of states $\omega_{\xi_A}$ and $\omega_{\xi_B}$ which has no correlations between the subsystems $\mathcal{R}(O_A)$ and $\mathcal{R}(O_B)$. We should like to refer the reader to [16, 64] for considerable discussion on this issue.

(3.6) There are states $\xi \in X$ for which the Reeh-Schlieder correlations are much stronger than in the vacuum $\Omega$, and in such states the correlations are sufficiently strong so that they can be used for quantum teleportation over macroscopic distances as has been demonstrated experimentally [26]. While this is perhaps intuitively less surprising than for the case of the vacuum state since the states $\xi$ have some “material content” to which one could ascribe the storage of correlation information, it should be kept in mind that also here the correlations are non-classical, i.e. they manifestly exemplify quantum entanglement.

(3.7) In the Haag-Kastler setting, local commutativity and the Reeh-Schlieder theorem together imply that any local operator $A \in \mathcal{R}(O)$, $O$ open and bounded, which annihilates the vacuum: $A\Omega = 0$, must in fact be equal to the zero operator, $A = 0$. As a consequence, for the vacuum vector $\Omega$ (as well as for any other $\xi \in X$ having the Reeh-Schlieder property) it holds that

$$\langle \Omega, A^* A \Omega \rangle > 0$$

for all $A \in \mathcal{R}(O)$ with $A \neq 0$, $O$ open and bounded. This may be interpreted as the generic presence of vacuum fluctuations; every local counting instrument will give a non-zero expectation value in the vacuum state. This is, actually, a situation where relativistic quantum field theory deviates from quantum mechanics. (Quantum mechanics needs to postulate the existence of fluctuations as e.g. in the semiclassical theory of radiation to account for spontaneous emission.)

A related mathematical argument shows that quantities like the energy density will fail to be pointwise positive in the quantum field setting, in contrast to their classical behaviour. Yet, the spectrum condition puts limitations to the failure of positivity. For this circle of questions, we recommend that the reader consults the review article [24].

Now, in order to turn to the discussion of “geometric modular action”, we need to introduce some notation. We consider a generic von Neumann algebra $\mathcal{R}$ acting on a Hilbert space $\mathcal{H}$, together with a unit vector $\Omega \in \mathcal{H}$ which is assumed to be cyclic and separating for $\mathcal{R}$. To explain the terminology, $\mathcal{R}$ is a von Neumann algebra.
state \( \omega \) system, then a density matrix state \( \omega \) continuous) group of automorphisms of condition with respect to the adjoint action of \( \Delta \) it modelling the observables of a quantum system and \( \beta = 1 \).

One says that \( \Omega \in \mathcal{H} \) is cyclic for \( \mathcal{R} \) if \( \mathcal{R}\Omega \) is dense in \( \mathcal{H} \)—in view of our previous discussion, this is the same as saying that the Reeh-Schlieder property holds for \( \Omega \), with respect to the algebra \( \mathcal{R} \). Moreover, one says that \( \Omega \) is separating for \( \mathcal{R} \) if \( A \in \mathcal{R} \) and \( A\Omega = 0 \) imply \( A = 0 \), and this is equivalent to \( \langle \Omega, A^*A\Omega \rangle > 0 \) for all \( A \in \mathcal{R} \) different from 0. One can in fact show that \( \Omega \) is cyclic for \( \mathcal{R} \) if and only if \( \Omega \) is separating for \( \mathcal{R}' \), and vice versa.

Given a von Neumann algebra \( \mathcal{R} \) on a Hilbert space \( \mathcal{H} \) and a cyclic and separating unit vector, \( \Omega \in \mathcal{H} \), for \( \mathcal{R} \), there is a canonical anti-linear operator

\[
S : \mathcal{R}\Omega \rightarrow \mathcal{R}\Omega, \quad A\Omega \mapsto S(A\Omega) := A^*\Omega
\]

associated with these data. By cyclicity of \( \Omega \) for \( \mathcal{R} \), the set \( \mathcal{R}\Omega = \{ A\Omega : A \in \mathcal{R} \} \) is a dense linear subspace of \( \mathcal{H} \), so the operator is densely defined; furthermore, to assign the value \( A^*\Omega \) to the vector \( A\Omega \) in the domain of \( S \) is a well-defined procedure in view of the assumption that \( \Omega \) is separating for \( \mathcal{R} \). The anti-linearity of \( S \) is then fairly obvious. What is less obvious is the circumstance that the operator \( S \) is usually unbounded (provided \( \mathcal{H} \) is infinite-dimensional). Nevertheless, one can show that \( S \) is a closable operator and thus the closure of \( S \) (which we denote here again by \( S \)) possesses a polar decomposition, i.e. there is a unique pair of operators \( J \) and \( \Delta \) so that \( S \) can be written as

\[
S = J\Delta^{1/2}
\]

and where \( J : \mathcal{H} \rightarrow \mathcal{H} \) is anti-linear and fulfills \( J^2 = 1 \) while \( \Delta = S^*S \) is positive (and selfadjoint on a suitable domain, and usually unbounded). This is nothing but the usual polar decomposition of a closable operator, with the slight complication that the operator \( S \) is, by definition, anti-linear instead of linear. See, e.g., [11] for further information.

The operators \( J \) and \( \Delta \) are called the modular conjugation, and modular operator, respectively, corresponding to the pair \( \mathcal{R}, \Omega \). Often, \( J \) and \( \Delta \) are also referred to as the modular objects of \( \mathcal{R}, \Omega \). Their properties have been investigated by the mathematicians Tomita and Takesaki and hence they appear also under the name Tomita-Takesaki modular objects. The important properties of \( J \) and \( \Delta \) which were discovered by Tomita and Takesaki (see, e.g., [25] [11] [9] for a full survey of the mathematical statements which we make in what follows) are, first, that the adjoint action of \( J \) maps \( \mathcal{R} \) onto its commutant \( \mathcal{R}' \): \( A \in \mathcal{R} \Leftrightarrow JAJ \in \mathcal{R}' \). This is written in shorter notation as \( J\mathcal{R}J = \mathcal{R}' \). One also has that \( J\Omega = \Omega \). Secondly, since \( \Delta \) is an invertible non-negative selfadjoint operator, \( \ln(\Delta) \) can be defined as a selfadjoint operator by the functional calculus, and hence one can define a one-parametric unitary group \( \Delta^t = e^{it\ln(\Delta)}, t \in \mathbb{R}, \) on \( \mathcal{H} \), called the modular group of \( \mathcal{R} \) and \( \Omega \). It has the property that its adjoint action leaves \( \mathcal{R} \) invariant, i.e. \( A \in \mathcal{R} \Leftrightarrow \Delta^tA\Delta^{-it} \in \mathcal{R} \), or simply \( \Delta^t\mathcal{R}\Delta^{-it} = \mathcal{R} \). Moreover, \( \Delta^t\Omega = \Omega \) holds for all \( t \in \mathbb{R} \). A third property relates to the spectral behaviour of the unitary group \( \{ \Delta^t \}_{t \in \mathbb{R}} \). Namely, the state \( \omega_\beta(A) = \langle \Omega, A\Omega \rangle \) on \( \mathcal{R} \) fulfills the KMS (Kubo-Martin-Schwinger) boundary condition with respect to the adjoint action of \( \Delta^t \), \( t \in \mathbb{R} \), at inverse temperature \( \beta = 1 \).

Let us explain the terminology used here. If \( \mathcal{R} \) is a von Neumann algebra modelling the observables of a quantum system and \( \{ \sigma_t \}_{t \in \mathbb{R}} \) is a one-parametric (continuous) group of automorphisms of \( \mathcal{R} \) modelling the dynamical evolution of the system, then a density matrix state \( \omega_\beta(A) = \text{trace}(\rho A) \) on \( \mathcal{R} \) is said to fulfil the
KMS boundary condition with respect to \( \{ \sigma_t \}_{t \in \mathbb{R}} \) (shorter: is a KMS state for \( \{ \sigma_t \}_{t \in \mathbb{R}} \)) at inverse temperature \( \beta > 0 \) provided that the following holds: Given any pair of elements \( A, B \in \mathcal{R} \), there exists a function \( F_{AB} \) which is analytic on the complex strip \( \mathcal{S}_\beta = \{ t + i\eta : t \in \mathbb{R} \, ; \, 0 < \eta < \beta \} \), and is continuous on the closure of the strip \( \mathcal{S}_\beta \), with the boundary values

\[
F_{AB}(t) = \omega_\rho(\sigma_t(A)B), \quad F_{AB}(t + i\beta) = \omega_\rho(B\sigma_t(A)), \quad t \in \mathbb{R}.
\]

For a quantum mechanical system with a Hamilton operator \( H \) such that \( e^{-\beta H} \) is a trace-class operator \((\beta > 0)\), one can form the density matrices

\[
\rho^\beta = \frac{1}{\text{trace}(e^{-\beta H})}e^{-\beta H}
\]

and one can show that the corresponding Gibbs states \( \omega_{\rho^\beta} \) are KMS states at inverse temperature \( \beta \) for the dynamical evolution given by \( \sigma_t(A) = e^{itH}Ae^{-itH} \).

Haag, Hugenholtz and Winnink [30] have shown that states of an infinite quantum system — being modelled by \( \mathcal{R} \) and \( \{ \sigma_t \}_{t \in \mathbb{R}} \) — which are suitably approximated by Gibbs states of finite subsystems, are under very general conditions also KMS states, and thus the KMS boundary condition is viewed as being characteristic of thermal equilibrium states.

Therefore, if \( \omega_\Omega \) is a KMS state with respect to the (adjoint action of the) modular group \( \{ \Delta^u \}_{u \in \mathcal{R}} \) of \( \mathcal{R}, \Omega \), this signalizes that there is some relation to physics provided that \( \{ \Delta^u \}_{u \in \mathcal{R}} \) can be interpreted as dynamical evolution of a quantum system. This is not always the case, but the converse always holds true: Suppose that a quantum system dynamical system consisting of \( \mathcal{R} \) and \( \{ \sigma_t \}_{t \in \mathbb{R}} \) and a KMS state \( \omega_\rho \) at inverse temperature \( \beta > 0 \) is given. Then one can pass to the GNS (Gelfand-Naimark-Segal) representation associated with \( \mathcal{R} \) and \( \omega_\rho \). This is a triple \( (\pi^\rho, \mathcal{H}^\rho, \Omega^\rho) \) where \( \mathcal{H}^\rho \) is a Hilbert space, \( \pi^\rho \) is a representation of \( \mathcal{R} \) by bounded linear operators on \( \mathcal{H}^\rho \) (which may differ from the “defining” representation of \( \mathcal{R} \) that is pre-given since the elements of \( \mathcal{R} \) act as bounded linear operators on a Hilbert space \( \mathcal{H} \) and \( \Omega^\rho \) is a unit vector in \( \mathcal{H}^\rho \) which is cyclic for \( \pi^\rho(\mathcal{R}) \) and with \( \omega^\rho(A) = \langle \Omega^\rho, \pi^\rho(A)\Omega^\rho \rangle \). In this GNS representation, \( \{ \sigma_t \}_{t \in \mathbb{R}} \) is implemented by the (rescaled) modular group \( \{ \Delta^{ui/\beta} \}_{u \in \mathcal{R}} \) corresponding to \( \pi^\rho(\mathcal{R}) \) and \( \Omega^\rho \): \( \pi^\rho(\sigma_t(A)) = \Delta^{ui/\beta}\pi^\rho(A)\Delta^{-ui/\beta} \).

Tomita-Takesaki theory has had a considerable impact on the development of operator algebra theory. Owing to its relation to thermal equilibrium states, it has also found applications in quantum statistical mechanics. It took longer, however, until a connection between Tomita-Takesaki modular objects and the action of the Poincaré group was revealed in the context of relativistic quantum field theory. Such a connection was established in the seminal work of Bisognano and Wichmann [4]. To explain their result, let \( (x^0, x^1, x^2, x^3) \) denote the coordinates of points in Minkowski spacetime in some Lorentzian frame. Then let \( W = \{ x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 : x^1 > 0, \, -x^1 < x^0 < x^1 \} \) denote the “right wedge region” with respect to the chosen coordinates. Moreover, we shall introduce the following maps of Minkowski spacetime:

\[
\mathbf{j} : (x^0, x^1, x^2, x^3) \mapsto (-x^0, -x^1, x^2, x^3)
\]

which is a reflection about the spatial \( x^2-x^3 \) plane together with a time-reflection, and

\[
\Lambda_1(\theta) = \begin{pmatrix}
\cosh(\theta) & -\sinh(\theta) & 0 & 0 \\
-\sinh(\theta) & \cosh(\theta) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \theta \in \mathbb{R},
\]
the Lorentz boosts along the $x^1$-axis, which map $W$ onto itself.

Now consider a quantum field theory of the Haag-Kastler type (in vacuum representation), where it is also assumed that the local algebras of observables $\mathcal{R}(O)$ are generated by a Wightman-type quantum field $F \mapsto \Phi(F)$ as in (2.8). It will also be assumed that the $\mathcal{R}(O)$ are actually von Neumann algebras, so that one has $\mathcal{R}(O) = \mathcal{R}(O)''$ for open, bounded regions. Then one can also built the algebra of observables located in the wedge region $W$,

$$\mathcal{R}(W) = \{ A \in \mathcal{R}(O) : O \subset W \}''.$$

We will denote by $J$ the modular conjugation and by $\{ \Delta^it \}_{t \in \mathbb{R}}$ the modular group, respectively, associated with $\mathcal{R}(W)$ and the vacuum vector $\Omega$. These are well-defined since the vacuum vector is, by the Reeh-Schlieder theorem, cyclic and separating for $\mathcal{R}(W)$. With these assumptions, Bisognano and Wichmann [7] found the following remarkable result.

(3.8) **Theorem**

The following relations hold:

$$\Delta^it = U(\Lambda_1(2\pi t))$$
$$J\mathcal{R}(O)J = R(\mathfrak{j}(O))$$
$$J\Phi(F)J = \Phi(F \circ \mathfrak{j})$$
$$JU(L)J = U(\mathfrak{j} \circ L \circ \mathfrak{j}), \quad L \in \mathfrak{P}_+^{++}$$

Here, $U$ denotes the unitary representation of the Poincaré group belonging to the quantum field theory under consideration, and we have written $U(\Lambda_1(2\pi t))$ for the unitary representation of the Lorentz boost $\Lambda_1(2\pi t)$.

The remarkable point is that by this theorem, the modular conjugation and modular group associated with $\mathcal{R}(W)$ and $\Omega$ acquire a clear-cut geometric meaning. Moreover, since the adjoint action of $J$ involves, in its geometric meaning, a time and space reflection, it induces a PCT symmetry in the following way:

The rotation $D_{(2,3)}$ by $\pi = 180^\circ$ in the $(x^2, x^3)$ plane is contained in the proper, orthochronous Poincaré group, and

$$\mathfrak{j} \circ D_{(2,3)} = D_{(2,3)} \circ \mathfrak{j} = PT : x \mapsto -x$$

is the total inversion.

Then $\Theta = JU(D_{(2,3)})$ is a **PCT operator**: $\Theta$ is anti-unitary and fulfills $\Theta^2 = 1$, and

$$\Theta\Omega = \Omega$$
$$\Theta\mathcal{R}(O)\Theta = \mathcal{R}(PT(O))$$
$$\Theta\Phi(F)\Theta = \Phi(F \circ PT)$$
$$\Theta U(L)\Theta = U(PT \circ L \circ PT).$$

Because of the geometric significance of the modular objects $J$ and $\{ \Delta^it \}_{t \in \mathbb{R}}$ one also says that the Bisognano-Wichmann theorem is an instance of “geometric modular action” (although this term is actually used also in a wider context). The concept of “geometric modular action” has been used quite fruitfully in the analysis of general quantum field theories over the past years and has led to remarkable progress and insights. We cannot get into this matter in any depth and instead we refer the reader to the comprehensive review by Borchers [9]; we will only comment on a few aspects of geometric modular action by way of a couple of remarks.

(3.9) Because of $\Delta^it = U(\Lambda_1(2\pi t))$, the vacuum state functional $\langle \Omega, , \Omega \rangle$ restricted
to $\mathcal{R}(W)$ is a KMS state, i.e. a thermal equilibrium state. More precisely, an observer following the trajectory

$$\gamma_a(t) = \Lambda_1(t) \begin{pmatrix} 0 & 1/a \\ 0 & 0 \end{pmatrix}$$

will register the (restriction of the) vacuum state along his or her trajectory as a thermal equilibrium state at absolute temperature

$$T_a = \frac{\hbar a}{2\pi kc}$$

where here we have explicitly inserted $\hbar$, Boltzmann’s constant $k$ and the velocity of light $c$. This is called the Fulling-Unruh-effect \[25, 66\]. It has been noted by Sewell \[58, 61\] that a similar form of geometric modular action for quantum fields on the Schwarzschild-Kruskal spacetime can be viewed as a variant of the Hawking effect.

(3.10) The relation of Tomita-Takesaki objects to the action of the Poincaré group which is displayed by the Bisognano-Wichmann theorem is only realized if the observable algebras with respect to which the Tomita-Takesaki objects those belonging to wedge regions — i.e. any Poincaré-transform of $W$. For observable algebras $\mathcal{R}(O)$ belonging to bounded regions, the corresponding modular objects have in general no clear geometric meaning. An exception is the case of a conformal quantum field theory when $O$ is a double cone (see \[9\] and literature cited there).

(3.11) If $a$ is a lightlike vector parallel to the future lightlike boundary of $W$, let

$$J_a = \text{modular conjugation of } \mathcal{R}(W + a), \Omega$$

Then one can show that

$$U(-2a) = J_0 J_a,$$

i.e. the modular conjugations encode the translation group — together with the spectrum condition. Since the modular group of $\mathcal{R}(W)$ induces the boosts, it appears that the complete unitary action of the Poincaré group can be retrieved from the modular objects of observable algebras belonging to a couple of wedge regions in suitable position to each other, with a common vacuum vector. And indeed, a careful analysis has shown that this is possible under general conditions \[38, 9\]. This opens the possibility to approach the problem of constructing (interacting) quantum field theories in a completely novel manner, where one starts with a couple of von Neumann algebras together with a common cyclic and separating vector, and where the associated modular objects fulfil suitable relations so that they induce a representation of the Poincaré group. See \[57, 14\] for perspectives, first steps and results around this circle of ideas.

(3.12) It should also be pointed out that geometric modular action can be understood in a more general sense than above where the modular objects associated with the vacuum and algebras of observables located in wedge-regions induce point-transformations on the manifold — in our present discussion, always Minkowski spacetime — on which the quantum field theory under consideration lives. A more general criterion of geometric modular action would, e.g., be the following: Given a family of observable (von Neumann) algebras $\{\mathcal{R}(O)\}_{O \subset M}$ indexed by the open (and bounded) subsets of a spacetime manifold $M$, and a vector $\Omega$ in the Hilbert space representation of that family, one can try to find a sub-family $\{\mathcal{R}(\tilde{O})\}_{\tilde{O} \subset \tilde{K}}$ (where $\tilde{K}$ is a collection of subsets of $M$, sufficiently large so that a base of the topology of $M$ can be generated by countable intersections and unions of members
in \( \tilde{K} \), say) with the property that the adjoint action of the modular conjugation \( J_{\hat{O}} \) of \( \mathcal{R}(\hat{O}), \Omega \), where \( \hat{O} \) is any element of \( \tilde{K} \), maps the family \( \{ R(\hat{O}) \}_{\hat{O} \in \tilde{K}} \) onto itself. This would be a generalized form of geometric modular action. In the light of the Bisognano-Wichmann theorem, for the case of Minkowski spacetime one would take the collection of wedge regions as \( \tilde{K} \) and the vacuum vector as \( \Omega \). But there are instances where precisely such a generalized form of geometric modular action is realized when taking for \( M \) e.g. Robertson-Walker spacetimes. For more discussion on this intriguing generalization of geometric modular action, see [15].

4 Relativistic Quantum Information Theory: Distillability in Quantum Field Theory

The final section of this contribution is devoted to a subject which seems to be of growing interest nowadays [4, 22, 47, 52]: The attempt to bring together the flourishing discipline of quantum information theory with the principles of special relativity. Since quantum information theory is based on the principles of quantum mechanics and since quantum field theory is the theory which unifies quantum mechanics and special relativity, it appears entirely natural to discuss issues of relativistic quantum information theory in the setting of quantum field theory.

There are, of course, several foundational issues one might wish to discuss when studying a prospective merging of quantum information theory and special relativity even in the established setting of quantum field theory. One of them might be the so far omitted discussion on quantum measurement theory within quantum field theory. In view of the Reeh-Schlieder theorem, one may suspect delicate problems at this point — in fact, there are numerous discussions on the nature of locality/nonlocality in quantum (information) theory, where sometimes the various authors don’t agree on precisely what sort of locality is attributed to which object or structure within a particular theoretical framework. Our approach here is operational, and we refer to works already cited [56, 32] for some discussion on measurement in quantum field theory.

This said, we limit ourselves here to studying a very particular concept which has been developed and investigated in non-relativistic quantum information theory in the context of relativistic quantum field theory: The concept of distillability of quantum states. Very roughly speaking, one can say that distillable quantum states contain “useful” entanglement that can be enhanced, at least theoretically, so that it can be used as a resource for typical telecommunication tasks such as quantum cryptography or quantum teleportation [5, 23, 26]. (For a more detailed exposition of the formal apparatus of quantum information theory and important references, we recommend the review by M. Keyl [40].) To make this more precise, we will now have to specify our setting at a more formal level. Everything what follows is taken from a joint publication with R. Werner [67].

First, we will say that a bipartite system is a pair of mutually commuting \(*\)-subalgebras \( \mathcal{A}, \mathcal{B} \) of \( \mathcal{B}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \). Usually, we will in fact assume that both \( \mathcal{A} \) and \( \mathcal{B} \) are von Neumann algebras; one could also generalize the setting by only requiring that \( \mathcal{A} \) and \( \mathcal{B} \) are \(*\)-subalgebras of a common \( \mathcal{C}^* \)-algebra. In the quantum field theoretical context, \( \mathcal{A} \) will be identified with \( \mathcal{R}(O_A) \) and \( \mathcal{B} \) with \( \mathcal{R}(O_B) \) for a pair of (bounded) spacetime regions \( O_A \) and \( O_B \) which are causally separated. Quite generally, \( \mathcal{A} \) represents the algebra of observables in a laboratory controlled by a physicist named ‘Alice’ and \( \mathcal{B} \) represents the algebra of observables in a laboratory controlled by another physicist called ‘Bob’. The prototypical example of a bipartite system in (non-relativistic) quantum information theory is the situation where \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \), and where \( \mathcal{A} = B(\mathcal{H}_A) \otimes 1 \) and where
\[ \mathcal{B} = \mathbb{1} \otimes B(\mathcal{H}_B). \] The situation in relativistic quantum field theory can be a bit more complicated.

Let \( A, \mathcal{B} \subset B(\mathcal{H}) \) form a bipartite quantum system, and let \( \omega(X) = \text{trace}(\rho X) \), for some density matrix \( \rho \) on \( \mathcal{H} \), be a state on \( B(\mathcal{H}) \). We say that the state \( \omega \) is a product state on the bipartite system if \( \omega(AB) = \omega(A)\omega(B) \) holds for all \( A \in \mathcal{A} \) and all \( B \in \mathcal{B} \). Moreover, \( \omega \) is called separable on the bipartite system if it is a limit (in the sense of convergence of expectation values) of convex combinations of product states. Then, \( \omega \) is called entangled on the bipartite system if it is not separable.

Entanglement of states on bipartite systems is a typical quantum phenomenon with no counterpart in classical physics. As is well known, the Einstein-Podolsky-Rosen paradox really centers about entangled states, as has been clarified and formalized by John Bell (see the reprint collection \([69]\) for the relevant references and comments, and the textbook \([46]\) for a more modern and simpler discussion). As mentioned, nowadays entanglement is viewed as a resource for tasks of quantum communication, and this circumstance has motivated several studies on the “degree” or “quality” of entanglement that a state may have (see, again, the review \([40]\) for discussion and references). One possible measure of “entanglement strength” is provided by the Bell-CHSH correlation \([13] \([63]\). This is a number, \( \beta(\omega) \), which is assigned to any state \( \omega \) of a bipartite system \( \mathcal{A, B} \subset B(\mathcal{H}) \) as

\[ \beta(\omega) = \sup_{A, A', B, B'} \omega(A(B' + B) + A'(B' - B)) \]

where the supremum is taken over all hermitean \( A, A' \in \mathcal{A} \) and \( B, B' \in \mathcal{B} \) whose operator norm is bounded by 1. Separable states always have \( \beta(\omega) \leq 2 \). This case is referred to by saying that \( \omega \) fulfills the Bell-CHSH inequalities. States \( \omega \) for which \( \beta(\omega) > 2 \) are said to violate the Bell-CHSH inequalities; such states are entangled. The maximal number which \( \beta(\omega) \) can assume is \( 2\sqrt{2} \) \([63]\), and states for which \( \beta(\omega) = 2\sqrt{2} \) are said to violate the Bell-CHSH inequalities maximally. In a sense, one may view a state \( \omega_1 \) more strongly entangled than a state \( \omega_2 \) if \( \beta(\omega_1) > \beta(\omega_2) \).

Let us consider a particularly simple system where \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \), with \( \mathcal{A} = B(\mathbb{C}^2) \otimes \mathbb{1} \) and \( \mathcal{B} = \mathbb{1} \otimes B(\mathbb{C}^2) \), where \( B(\mathbb{C}^2) \) is a perhaps slightly unusual way to denote the algebra of complex \( 2 \times 2 \) matrices. A state violating the Bell-CHSH inequalities maximally is given by the singlet state \( \omega_\text{singlet}(X) = \langle \psi_\text{singlet} | X \psi_\text{singlet} \rangle \), \( X \in B(\mathbb{C}^2 \otimes \mathbb{C}^2) \), where

\[ \psi_\text{singlet} = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle); \]

here, \(|0\rangle\) and \(|1\rangle\) denote the two orthonormalized eigenvectors of the Pauli-matrix \( \sigma_z \). There are, in fact, experimental situations in quantum optics where the singlet state can be realized to a high degree of accuracy. In these situations, one identifies \(|0\rangle\) and \(|1\rangle\) with the two orthonormal polarization states of photons which are linearly polarized with respect to chosen coordinates perpendicular to the direction of propagation. One can prepare a source (state) producing an ensemble of pairs of polarized photons in the singlet state and send — e.g. through optical fibres over long distances — one member of each ensemble pair to the laboratory of Alice (whose observables, regarding the polarization of the photons, are represented by \( \mathcal{A} \)) and the other member of the same pair to the laboratory of Bob (whose polarization observables are represented by \( \mathcal{B} \)). In this way, Alice and Bob have access to a common entangled state \( \omega_\text{singlet} \) which they may use for carrying out tasks of quantum communication. The singlet state (or rather, any singlet-type state) is, in this sense, the best suited state owing to its “maximal” entanglement which is reflected by its maximal violation of the Bell-CHSH inequalities. Some experimental realizations and applications can be found e.g. in \([26]\).
There are entangled states $\omega$ which are not as strongly entangled as $\omega_{\text{singlet}}$, but contain still enough entanglement so that a sub-ensemble of photon pairs can be “distilled” from $\omega$ which coincides with $\omega_{\text{singlet}}$ to high accuracy and may then be used for carrying out quantum communication tasks. To make such a “distillability” an attribute of the given state $\omega$, one must ensure that the distillation process only enhances the entanglement already present in the given state $\omega$, and doesn’t induce previously non-existing entanglement. One tries to capture this requirement by demanding that the process of distillation involves only local operations and classical communication (LOCC) [6, 48, 40].

The idea behind LOCC is best illustrated by a simple example. We assume that both Alice and Bob operate a two-valued instrument in each of their laboratories. A two-valued instrument (i) takes an incoming state, (ii) puts out either of two classical values (“readings”), say “+” or “−” and (iii) changes the state into a new output state depending on the values of the classical readings, i.e. the values “+” or “−”. Thus, if the source (represented by the state $\omega$) produces a pair of polarized photons, then the pair member running to Alice passes her instrument while the other pair member travels to Bob and passes his instrument. The pair members are then subjected to state changes — operations — taking place individually at the sites of the laboratories of Alice and Bob, respectively, and are thus local (assuming that the operations are active at mutual spacelike separation); put differently, Alice’s instrument operates only on the pair member in her laboratory and likewise Bob’s instrument operates only on the pair member in his laboratory. We further suppose that Alice and Bob agree to discard all photon pairs except those which on passing their instruments have yielded in both cases the “+” reading. Since they don’t now beforehand what the values of these readings will be, they have to inform each other about the readings’ values of their instruments after both members of each photon pair have passed through. This requires “two-way classical communication” between Alice and Bob. Then, after a large number of photon pairs (corresponding, in idealization, to the original ensemble of the state $\omega$) has passed the instruments, and having discarded all the pairs not giving the “+” reading, Alice and Bob hold (in each lab, members of) a smaller number (a subensemble) of photon pairs which have been subjected to local operations mediated by the instruments. This new subensemble may correspond to a state with stronger entanglement, and if, in this way, a subensemble can be produced which approximates the singlet state $\omega_{\text{singlet}}$ to arbitrary precision, then the original state $\omega$ is called distillable. Strictly speaking, we should call the state 1-distillable, the qualifier “1” referring to only “1 round” of instrument application and classical communication for each photon pair, since one can envisage more complicated schemes of using localized (multi-valued) instruments and classical communication between Alice and Bob that are still in compliance with the idea of local operations and classical communication. But then, any state which is 1-distillable will also be distillable according to a more general scheme, so that 1-distillability is in this sense the most stringent criterion.

Now we need to give a mathematical description of 1-distillability of a state $\omega$. In the present simple case, the mathematical image of a two-valued instrument in Alice’s laboratory is given by two completely positive maps $T_{\pm} : A \to A$ with $T_{+}(1) + T_{-}(1) = 1$. Likewise, in Bob’s laboratory, his two-valued instrument is given by a pair of completely positive maps $S_{\pm} : B \to B$ with $S_{+}(1) + S_{-}(1) = 1$. The subensemble that Alice and Bob select from the original state $\omega$ corresponds to the positive functional $A \otimes B \ni x \otimes y \mapsto \omega(T_{+}(x)S_{+}(y))$, which is turned into a state, $A \otimes B \ni x \otimes y \mapsto \omega(T_{+}(x)S_{+}(y))/\omega(T_{+}(1)S_{+}(1))$, upon normalization. Let us denote this new state by $\omega^{T:S}$, identifying $T$ with $T_{+}$ and $S$ with $S_{+}$. To say that $\omega$ is 1-distillable now amounts to requiring that one can choose $S$ and $T$ in such a way that $\omega^{T:S}$ approximates $\omega_{\text{singlet}}$ to arbitrary precision.

All this applies as yet to the case that $A$ and $B$ are copies of $B(C^{2}).$ However, it
is not too difficult to generalize everything to the case of a generic bipartite quantum system. All that needs to be done is to ensure that the input state $\omega$, defined on the algebra generated by $A$ and $B$, yields an output state $\omega^{T,S}$ on $B(C^2 \otimes C^2)$ which can be compared to $\omega_{\text{singlet}}$. The formal definition of 1-distillability is then:

(4.1) Definition Let $\omega$ be a state on a general bipartite quantum system $A,B \subset B(H)$. The state $\omega$ is called 1-distillable if one can find completely positive maps $T : B(C^2) \rightarrow A$ and $S : B(C^2) \rightarrow B$ so that the state

$$\omega^{T,S}(x \otimes y) = \omega(T(x)S(y))/\omega(T(1)S(1)), \quad x \otimes y \in B(C^2 \otimes C^2),$$

on $B(C^2 \otimes C^2)$ approximates $\omega_{\text{singlet}}$ to arbitrary precision. That is to say, given $\epsilon > 0$, there are such $T = T_\epsilon$ and $S = S_\epsilon$ so that

$$|\omega^{T,S}(X) - \omega_{\text{singlet}}(X)| < \epsilon \|X\|, \quad X \in B(C^2 \otimes C^2).$$

This criterion for 1-distillability is now completely general and can, in particular, be applied in the context of relativistic quantum field theory. This is what we will do now.

As in Sec. 2, let $(\mathcal{R}(O))_{O \subset M}, U, \Omega)$ be a quantum field theory in vacuum representation. We quote following result, taken from [12].

(4.3) Theorem Let $A = \mathcal{R}(O_A)$ and $B = \mathcal{R}(O_B)$ be a bipartite quantum system formed by algebras of local observables localized in spacetime regions $O_A$ and $O_B$ which are separated by a non-zero spacelike distance. Then the vacuum state $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$ is 1-distillable on this bipartite system. Moreover, there is a dense set $X \subset H$ so that the vector states $\omega_{\chi}(\cdot) = \langle \chi, \cdot \chi \rangle, \|\chi\| = 1$, are 1-distillable on the bipartite system. (In fact, $X$ can be chosen so that this holds for all spacelike separated regions $O_A$ and $O_B$.)

We will add a couple of remarks.

(4.4) The conclusion of the theorem remains valid if one considers the quantum field theory in a relativistic thermal equilibrium representation instead of a vacuum representation. Representations of this kind have been introduced by Bros and Buchholz [10]. The distinction from the vacuum prerepresentation is as follows: The spectrum condition is dropped, and it is assumed that $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$ fulfills the relativistic KMS condition at some inverse temperature $\beta > 0$. Following [10], one says that a state $\omega$ on $\mathcal{R}(R^4)$ satisfies the relativistic KMS condition at inverse temperature $\beta > 0$ (with respect to the adjoint action of the translation group $U(a), a \in \mathbb{R}^4$) if there exists a timelike vector $e$ in $V_+$, the open forward light cone, so that $e$ has unit Minkowskian length, and so that for each pair of operators $A, B \in \mathcal{R}(R^4)$ there is a function $F = F_{AB}$ which is analytic in the domain $\mathcal{T}_{\beta e} = \{ z \in C^4 : \text{Im} z \in V_+ \cap (\beta e - V_+) \}$, and continuous at the boundary sets determined by $\text{Im} z = 0, \text{Im} z = \beta e$ with the boundary values $F(a) = \langle \Omega, AU(a)B\Omega \rangle, F(a + i(\beta e)) = \langle \Omega, BU(-a)A\Omega \rangle$ for $a \in \mathbb{R}^4$. Upon comparison with the non-relativistic KMS-condition of the previous section, one may get an idea in which way this is a relativistic generalization of thermal equilibrium states.

(4.5) It is the Reeh-Schlieder theorem which is responsible for the distillability result; we briefly sketch the argument. In fact, one can show that each non-abelian von Neumann algebra contains an isomorphic copy of $B(C^2)$. In the particular case considered in the situation of Theorem (4.3), one can use the Reeh-Schlieder theorem to prove that there are algebraic morphisms $\tau : B(C^2) \rightarrow A$ and $\sigma : B(C^2) \rightarrow B$ so that $\pi : B(C^2 \otimes C^2) \rightarrow B(H)$ given $\pi(x \otimes y) = \tau(x)\sigma(y)$ is a faithful algebraic embedding. Then there is a unit vector $\chi$ in $H$ so that $\omega_{\chi}(\pi(X)) = \omega_{\text{singlet}}(X)$ for all $X \in B(C^2 \otimes C^2)$. According to the Reeh-Schlieder theorem, there is for any
$\epsilon > 0$ some $A = A_\epsilon \in \mathcal{A}$ with $||A|| = 1$ so that $||(A\Omega)||^{-1}A\Omega - \chi|| < \epsilon$. Thus we choose $T(x) = A^*\tau(x)A$ and $S(y) = \sigma(y)$ to obtain that the state $\omega^{T,S}$ fulfills the required estimate. The Reeh-Schlieder theorem for relativistic thermal equilibrium states has been proved in [35].

(4.6) The normalization factor $\omega(T(1)S(1))$ equals, in the previous remark, the quantity $\langle \Omega, A^*\tau(1)A\Omega \rangle$ which, in turn, is equal to $||A\Omega||^2$ up to a term of at most order $\epsilon$. Since we have taken $||A||$ to be equal to 1 (which made the occurrence of the normalization factor $||A\Omega||^{-1}$ necessary in the approximation of $\chi$), the quantity $||A\Omega||$ here coincides in fact with $||A\Omega||/||A||$, i.e. the effect vs. cost ratio which made its appearance in our discussion of the Reeh-Schlieder theorem. Thus, the factor $\omega(T(1)S(1))$ (compared to 1) is a rough measure for the efficiency of the distillation process, or put differently, the fraction of members in the subensemble corresponding to $\omega^{T,S}$ distilled from the members of the original ensemble $\omega$. As we have seen before, this will be a very small number when $\epsilon$ is small and the spatial distance between the regions $O_A$ and $O_B$ is macroscopic for $\omega$ the vacuum state.

(4.7) We should like to mention that there are many related works addressing the issue of long-range correlations in quantum field theory. In fact, Bell-correlations in quantum field theory have been investigated before quantum information theory was established; see the refs. [63, 62, 42, 43], and they have contributed to understand quantum entanglement in a mathematically rigorous form applicable to general quantum systems. More recent works in this direction prove that in the bipartite situation $A = \mathcal{R}(O_A), B = \mathcal{R}(O_B)$, for a relativistic quantum field theory, there is dense set of states violating the Bell-CHSH inequalities [31, 36, 52]. In this sense, they are quite closely related to the result of the theorem above, which however gives also information about the distillability of specific states, such as the vacuum or relativistic thermal equilibrium states, over arbitrarily spacelike subsystems of a relativistic quantum field theory.

I think that, in the light of the theoretical developments summarized in this contribution, it is fair to say that the interplay between special relativity and quantum physics is holding a significant position at the frontier of current research. Thus I am quite confident that special relativity will live well through the next 100 years.

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