A spline interpretation of Eulerian numbers

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Abstract

In this paper, we explore the interrelationship between Eulerian numbers and B splines. Specifically, using B splines, we give the explicit formulas of the refined Eulerian numbers, and descents polynomials. Moreover, we prove that the coefficients of descent polynomials $D_n^d(t)$ are log-concave. This paper also provides a new approach to study Eulerian numbers and descent polynomials.

Keywords: B splines, Eulerian numbers, Log-concavity.

1 Introduction

Denote by $C^d$ the $d$-dimensional unit cube, i.e.,

$$C^d = \{ x \in \mathbb{R}^d : 0 \leq x_i \leq 1, \ i = 1, \ldots, d \}.$$

For a real number $\lambda$, the dilation of $C^d$ by $\lambda$ is the set

$$\lambda C^d = \{ \lambda \cdot x : x \in C^d \}.$$

Throughout this paper, all the volumes and mixed volumes will be normalized so that the volume of $C^d$ is given by $V(C^d) = d!$, where $V(\cdot)$ denotes the volume.
function (cf. [3]). Denoted by \( T^d_k \) and \( X^d_{n,k} \), the \( k \)th slices of \( C^d \) and \( n \cdot C^d \) respectively:

\[
T^d_k = \{ x \in C^d : k - 1 \leq \sum_{i=1}^{d} x_i \leq k \},
\]

\[
X^d_{n,k} = \{ x \in nC^d : (k - 1)n + 1 \leq \sum_{i=1}^{d} x_i \leq kn + 1 \},
\]

where \( n \) is a positive integer.

The Eulerian number \( A^d_{d,k} \) is the number of permutations in the symmetric group \( S_d \) that have exactly \( k - 1 \) descents. It seems that Laplace first expressed the volume of \( T^d_k \) in terms of \( A^d_{d,k} \) as follows [4]:

\[
A^d_{d,k} = V(T^d_k).
\]

(1.1)

The refinement of the Eulerian number \( A^d_{d,k,j} \) is the number of permutations in the symmetric group \( S_d \) with \( k \) descents and ending with the element \( j \). A result due to Ehrenborg, Readdy and Steingrímsson [3] generalized Laplace’s result to express the mixed volumes of two adjacent slices from the unit cube in terms of the refinement of the Eulerian numbers \( A^d_{d+1,k,d+1-j} \), i.e.,

\[
A^d_{d+1,k,d+1-j} = V(T^d_k, j; T^d_{k+1}, d-j).
\]

(1.2)

The number \( V(T^d_k, j; T^d_{k+1}, d-j) \) is the \( (d-j) \)-th mixed volume of the convex bodies \( T^d_k \) and \( T^d_{k+1} \) (cf. [5]). For \( \lambda, \mu \geq 0 \), define the Minkowski sum

\[
\lambda T^d_k + \mu T^d_{k+1} = \{ \lambda \alpha + \mu \beta : \alpha \in T^d_k, \beta \in T^d_{k+1} \}.
\]

It was shown essentially by Minkowski (though he treated only the case \( d \leq 3 \)) that there are real numbers \( V(T^d_k, d-j; T^d_{k+1}, j) \geq 0 \) satisfying

\[
V(\lambda T^d_k + \mu T^d_{k+1}) = \sum_{j=0}^{d} \binom{d}{j} V(T^d_k, d-j; T^d_{k+1}, j) \lambda^{d-j} \mu^j,
\]

(1.3)

for all \( \lambda, \mu \geq 0 \). The number \( V(T^d_k, d-j; T^d_{k+1}, j) \) is called the \( j \)-th mixed volume of \( T^d_k \) and \( T^d_{k+1} \).
Descent polynomials, denoted by $D_n^d(t)$, are defined as
\[
\sum_{k=0}^{d} D(d, n, k) t^k,
\]
where $D(d, n, k)$ is the number of permutations in indexed permutation $S_n^d$ with $k$ descents. The indexed permutation $S_n^d$ of length $d$ and with indices in $\{0, 1, \ldots, n-1\}$ is an ordinary permutation in the symmetric group $S_d$ where each letter has been assigned an integer between 0 and $n-1$ (see [3, 6]). Steingrímsson [6] first investigated the relationship between $D(d, n, k)$ and the volume of the slice of $nC^d$. To give a combinatorial interpretation for the mixed volumes of two adjacent slices from the unit cube, Ehrenborg, Readdy and Steingrímsson [3] found the refinement of the Eulerian numbers are associated with $D(d, n, k)$:
\[
D(d, n, k) = V(X_{n,k}^d) = \sum_{j=0}^{d} \binom{d}{j} A_{d+1,k,d+1-j}(n-1)^j.
\]
Steingrímsson [6] gave the following recurrence relations of $D(d, n, k)$,
\[
D(d, n, k) = (nk+1)D(d-1, n, k) + (n(d-k) + (n-1))D(d-1, n, k-1).
\]

We now turn to the definition of B splines, and show the relation among Eulerian numbers, descent polynomials and B splines. B splines with order $d$, denoted by $B_d(\cdot)$, is defined by the induction as follows
\[
B_1(x) = \begin{cases} 
1 & \text{if } x \in [0, 1), \\
0 & \text{otherwise},
\end{cases}
\]
and for $d \geq 2$
\[
B_d(x) = \int_0^1 B_{d-1}(x-t)dt.
\]
A well known explicit formula for $B_d(\cdot)$ is
\[
B_d(x) = \frac{1}{(d-1)!} \sum_{i=0}^{d} \binom{d}{i} (-1)^i (x-i)^{d-1}. 
\]
There is a recurrence relationship
\[
B_d(x) = \frac{x}{d-1} B_{d-1}(x) + \frac{d-x}{d-1} B_{d-1}(x-1).
\]
From the viewpoint of the discrete geometry, $B_d(x)$ equals the volume of slice of unit cubes (see [1]), i.e.,

$$B_d(x) = (d - 1)! \cdot V(H \cap C^d), \quad (1.8)$$

where $H = \{y : y_1 + \cdots + y_d = x\}$.

Based on (1.8), we can derive the relationship between B splines and Eulerian numbers. To state conveniently, we use $[\lambda^j] f(\lambda)$ to denote the coefficient of $\lambda^j$ in $f(\lambda)$ for any given power series $f(\lambda)$. Then the following theorem shows the relationship between B splines and Eulerian numbers.

**Theorem 1.1.**

(i) $A_{d,k} = d! \cdot B_{d+1}(k)$;

(ii) $A_{d+1,k,d-j+1} = d! \cdot [\lambda^j] \left((\lambda + 1)^d B_{d+1}(k + \frac{1}{\lambda + 1})\right) / \binom{d}{j}$, $\lambda \geq 0$;

(iii) $D(d, n, k) = d! \cdot n^d \cdot B_{d+1}(k + \frac{1}{n})$.

We can also present the explicit expressions for $A_{d+1,k,d-j+1}$ and $D(d, n, k)$:

**Corollary 1.1.**

(i) $A_{d+1,k,d-j+1} = \sum_{i=0}^{k} \binom{d+1}{i} (-1)^i (k - i)^d (k - i + 1)^{d-j}$;

(ii) $D(d, n, k) = \sum_{i=0}^{k} \binom{d+1}{i} (-1)^i (n(k - i) + 1)^d$.

In [6], Steingrímsson proved that $D(d, n, k)$ for $k = 0, \ldots, d$ are unimodal. Using B splines, we have

**Corollary 1.2.** For any $d$ and $n$, the sequence $D(d, n, k)$ for $k = 0, \ldots, d$ is log-concave.

The two-scale property of B splines also implies the two-scale equations of $A_{d,k}$ and $D(d, n, k)$. 
Corollary 1.3. The Eulerian numbers $A_{d,k}$ and $D(d,n,k)$ satisfy the two-scale equations

$$A_{d,k} = \sum_{j=0}^{d+1} 2^{-d} \binom{d+1}{j} A_{d,2k-j},$$

and

$$D(d,2n,k) = \sum_{j=0}^{d+1} \binom{d+1}{j} D(d,n,2k-j),$$

respectively.

Remark 1.1. In [6], the recurrence formula for $D(d,n,k)$ is presented. Based on Theorem 1.1, the recurrence relation (1.7) of B splines implies that the recurrence relations of $A_{d,k}$ and $D(d,n,k)$.

2 Proofs of Main Results

To prove the main results, we need introduce some properties on B splines.

Lemma 2.1. (2)

(i) The function $\log B_d(\cdot)$ is concave on the open interval $(0,d)$;

(ii) The B spline $B_d(x)$ satisfies the following two-scale equation

$$B_d(x) = \sum_{j=0}^{d-1} 2^{-d-j} \binom{d-j}{j} B_d(2x-j).$$

(2.1)

We are now ready to prove our results.

Proof of Theorem 1.1. We firstly prove (i). Noting that (1.1) and (1.8), we have

$$A_{d,k} = V(T^d_k) = d! \cdot \int_{k-1}^{k} B_d(x)dx = d! \cdot \int_{0}^{1} B_d(k-t)dt = d! \cdot B_{d+1}(k).$$

We now prove (iii) for convenience. Let

$$Y_k = \{y \in \mathbb{R}^d : (k-1) + \frac{1}{n} \leq \sum_{i=1}^{d} y_i \leq k + \frac{1}{n}\}.$$

In view of $X_{n,k}^d = nY_k$, there holds $V(X_{n,k}^d) = n^d \cdot V(Y_k)$. From (1.8), we have

$$V(X_{n,k}^d) = n^d \cdot V(Y_k) = d! \cdot n^d \int_{(k-1)+\frac{1}{n}}^{(k+\frac{1}{n})} B_d(x)dx.$$
Then (iii) follows from the integral evaluation:
\[
D(d, n, k) = V(X^d_{n,k}) = d! \cdot n^d \int_{(k-1)+\frac{1}{n}}^{k+\frac{1}{n}} B_d(x) dx = d! \cdot n^d \int_0^1 B_d(k + \frac{1}{n} - t) dt = d! \cdot n^d B_{d+1}(k + \frac{1}{n}).
\]

To prove (ii), recall the equation (cf. [3])
\[
X^d_{\lambda+1,k} = \lambda T^d_k + T^d_{k+1}.
\]

From the argument above, one has
\[
V(X^d_{\lambda+1,k}) = d! \cdot (\lambda + 1)^d \int_0^1 B_d(k + \frac{1}{\lambda + 1} - t) dt.
\]

Then, we have
\[
A_{d+1,k,d-j+1} = V(T^d_{k,d}; T^d_{k+1}, d - j) = [\lambda^j] \left( V(\lambda T^d_k + T^d_{k+1}) \right) / \binom{d}{j} = [\lambda^j] \left( V(X^d_{\lambda+1,k}) \right) / \binom{d}{j} = [\lambda^j] \left( d! \cdot (\lambda + 1)^d \int_0^1 B_d(k + \frac{1}{\lambda + 1} - t) dt \right) / \binom{d}{j} = [\lambda^j] \left( d! \cdot (\lambda + 1)^d B_{d+1}(k + \frac{1}{\lambda + 1}) \right) / \binom{d}{j}.
\]

**Proof of Corollary 1.1.** To prove (i), using (ii) in Theorem 1 and (1.6), for any \( \lambda \geq 0 \), we have
\[
A_{d+1,k,d-j+1} = V(T^d_{k,d}; T^d_{k+1}, d - j) = [\lambda^j] \left( d! \cdot (\lambda + 1)^d B_{d+1}(k + \frac{1}{\lambda + 1}) \right) / \binom{d}{j} = [\lambda^j] \left( \sum_{i=0}^{d+1} \binom{d+1}{i} (-1)^i ((\lambda + 1)(k - i) + 1)^d \right) / \binom{d}{j} = [\lambda^j] \left( \sum_{i=0}^{d} \binom{d}{i} \lambda^i \sum_{t=0}^{k} \binom{k}{i} (-1)^i (k - i)^d (k - i + 1)^{d-t} \right) / \binom{d}{j} = \sum_{i=0}^{k} \binom{d+1}{i} (-1)^i (k - i)^d (k - i + 1)^{d-j}. \]
To prove (ii), combining (iii) in Theorem 1 with \(1.6\), we obtain that

\[
D(d, n, k) = d! \cdot n^d B_{d+1}(k + \frac{1}{n}) = \sum_{i=0}^{k} \binom{d+1}{i} (-1)^i (n(k - i) + 1)^d.
\]

\[
\square
\]

**Proof of Corollary 1.2.** According to Theorem 1, we have

\[
D(d, n, k) = d! \cdot n^d B_{d+1}(k + \frac{1}{n}).
\]

From Lemma 2.1, \(B_{d+1}(x)\) is log-concave, which implies that \(D(d, n, k), k = 0, \ldots, d\), is log-concave for each fixed \(d\) and \(n\).

\[
\square
\]

**Proof of Corollary 1.3.** By Theorem 1, we have

\[
A_{d,k} = d! \cdot B_{d+1}(k),
\]

\[
D(d, n, k) = d! \cdot n^d \cdot B_{d+1}(k + \frac{1}{n}).
\]

Since B-splines \(B_d(x)\) satisfy the following two-scale equation

\[
B_d(x) = \sum_{j=0}^{d} 2^{-d+1} \binom{d}{j} B_d(2x - j),
\]

then we have

\[
d! \cdot B_{d+1}(k) = \sum_{j=0}^{d+1} 2^{-d} \binom{d+1}{j} d! \cdot B_{d+1}(2k - j) \quad (2.2)
\]

and

\[
d! \cdot n^d \cdot B_{d+1}(k + \frac{1}{2n}) = \sum_{j=0}^{d+1} 2^{-d} \binom{d+1}{j} d! \cdot n^{d} \cdot B_{d+1}(2k + \frac{1}{n} - j). \quad (2.3)
\]

The equations (2.2) and (2.3) imply that the Eulerian numbers \(A_{d,k}\) and \(D(d, n, k)\) satisfy the following two-scale equations

\[
A_{d,k} = \sum_{j=0}^{d+1} 2^{-d} \binom{d+1}{j} A_{d,2k-j}
\]
and

$$D(d, 2n, k) = \sum_{j=0}^{d+1} \binom{d+1}{j} D(d, n, 2k - j),$$

respectively.

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