Local well posedness of the Euler-Korteweg equations on \( T^d \)

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Dedicated to the memory of Walter Craig

Abstract

We consider the Euler-Korteweg system with space periodic boundary conditions \( x \in T^d \).

We prove a local in time existence result of classical solutions for irrotational velocity fields requiring natural minimal regularity assumptions on the initial data.

1 Introduction

In this paper we consider the compressible Euler-Korteweg (EK) system

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho \vec{u}) &= 0 \\
\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla g(\rho) &= \nabla \left( K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 \right),
\end{aligned}
\]

(1.1)

which is a modification of the Euler equations for compressible fluids to include capillary effects, under space periodic boundary conditions \( x \in T^d := (\mathbb{R}/2\pi\mathbb{Z})^d \). The scalar variable \( \rho(t,x) > 0 \) is the density of the fluid and \( \vec{u}(t,x) \in \mathbb{R}^d \) is the time dependent velocity field. The functions \( K(\rho), g(\rho) \) are defined on \( \mathbb{R}^+ \), smooth, and \( K(\rho) \) is positive.

The quasi-linear equations (1.1) appear in a variety of physical contexts modeling phase transitions [15], water waves [13], quantum hydrodynamics where \( K(\rho) = \kappa/\rho \) [3], see also [14].

Local well posedness results for the (EK)-system have been obtained in Benzoni-Gavage, Danchin and Descombes [7] for initial data sufficiently localized in the space variable \( x \in \mathbb{R}^d \). Then, thanks to dispersive estimates, global in time existence results have been obtained for small irrotational data by Audiard-Haspot [6], assuming the sign condition \( g'(\rho) > 0 \). The case of quantum hydrodynamics corresponds to \( K(\rho) = \kappa/\rho \) and, in this case, the (EK)-system is formally equivalent, via Madelung transform, to a semilinear Schrödinger equation on \( \mathbb{R}^d \).

Exploiting this fact, global in time weak solutions have been obtained by Antonelli-Marcati [3, 4] also allowing \( \rho(t,x) \) to become zero (see also the recent paper [5]).

In this paper we prove a local in time existence result for the solutions of (1.1), with space periodic boundary conditions, under natural minimal regularity assumptions on the initial datum in Sobolev spaces, see Theorem 1.1. Relying on this result, in a forthcoming paper [9], we shall
prove a set of long time existence results for the (EK)-system in 1-space dimension, in the same spirit of [10, 11].

We consider an initially irrotational velocity field that, under the evolution of (1.1), remains irrotational for all times. An irrotational vector field on \( \mathbb{T}^d \) reads (Helmholtz decomposition)

\[
\vec{u} = \vec{c}(t) + \nabla \phi, \quad \vec{c}(t) \in \mathbb{R}^d, \quad \vec{c}(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \vec{u} \, dx, \tag{1.2}
\]

where \( \phi : \mathbb{T}^d \to \mathbb{R} \) is a scalar potential. By the second equation in (1.1) and \( \text{rot} \, \vec{u} = 0 \), we get

\[
\partial_t \vec{c}(t) = -\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \vec{u} \cdot \nabla \vec{u} \, dx = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{2} \nabla (|\vec{u}|^2) \, dx = 0 \quad \Rightarrow \quad \vec{c}(t) = \vec{c}(0)
\]

is independent of time. Note that if the dimension \( d = 1 \), the average \( \frac{1}{2} \int_{\mathbb{T}} u(t,x) \, dx \) is an integral of motion for (1.1), and thus any solution \( u(t,x), x \in \mathbb{T} \), of the (EK)-system (1.1) has the form (1.2) with \( c(t) = c(0) \) independent of time, that is \( u(t,x) = c(0) + \vec{c}_x(t,x) \).

The (EK) system (1.1) is Galilean invariant: if \( (\rho(t,x), \vec{u}(t,x)) \) solves (1.1) then

\[
\rho \vec{c}(t,x) := \rho \vec{c}(t,x + \partial t), \quad \vec{u} \vec{c}(t,x) := \vec{u}(t,x + \partial t) - \vec{c}
\]

solve (1.1) as well. Thus, regarding the Euler-Korteweg system in a frame moving with a constant speed \( \vec{c}(0) \), we may always consider in (1.2) that

\[
\vec{u} = \nabla \phi, \quad \phi : \mathbb{T}^d \to \mathbb{R}.
\]

The Euler-Korteweg equations (1.1) read, for irrotational fluids,

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho \vec{c} \nabla \phi) &= 0, \\
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(\rho) &= K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2.
\end{aligned} \tag{1.3}
\]

The main result of the paper proves local well-posedness for the solutions of (1.3) with initial data \((\rho_0, \phi_0)\) in Sobolev spaces

\[
H^s(\mathbb{T}^d) := \left\{ u(x) = \sum_{j \in \mathbb{Z}^d} u_j e^{ij \cdot x} : \|u\|^2_s := \sum_{j \in \mathbb{Z}^d} |u_j|^2 < +\infty \right\}
\]

where \( \langle j \rangle := \max\{1,|j|\} \), under the natural mild regularity assumption \( s > 2 + (d/2) \). Along the paper, \( H^s(\mathbb{T}^d) \) may denote either the Sobolev space of real valued functions \( H^s(\mathbb{T}^d, \mathbb{R}) \) or the complex valued ones \( H^s(\mathbb{T}^d, \mathbb{C}) \).

**Theorem 1.1. (Local existence on \( \mathbb{T}^d \))** Let \( s > 2 + \frac{d}{2} \). For any initial data

\[
(\rho_0, \phi_0) \in H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R}) \quad \text{with} \quad \rho_0(x) > 0, \quad \forall x \in \mathbb{T}^d,
\]

there exists \( T := T(\|\rho_0, \phi_0\|_{\infty,2}, \min_x \rho_0(x)) > 0 \) and a unique solution \((\rho, \phi)\) of (1.3) such that

\[
(\rho, \phi) \in C^0 \left( [-T,T], H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R}) \right) \cap C^1 \left( [-T,T], H^{s-2}(\mathbb{T}^d, \mathbb{R}) \times H^{s-2}(\mathbb{T}^d, \mathbb{R}) \right)
\]

and \( \rho(t,x) > 0 \) for any \( t \in [-T,T] \). Moreover, for \( |t| \leq T \), the solution map \((\rho_0, \phi_0) \mapsto (\rho(t,\cdot), \phi(t,\cdot))\) is locally defined and continuous in \( H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R}) \).
We remark that it is sufficient to prove the existence of a solution of (1.3) on \([0,T]\) because system (1.3) is reversible: the Euler-Korteweg vector field \(X\) defined by (1.3) satisfies \(X \circ S = \sim S \circ X\), where \(S\) is the involution
\[
S \left( \left[ \begin{array}{c} \rho \\ \phi \end{array} \right] \right) := \left[ \begin{array}{c} \rho' \\ -\phi' \end{array} \right], \quad \rho'(x) := \rho(-x).
\]
Thus, denoting by \((\rho, \phi)(t, x) = \Omega^t(\rho_0, \phi_0)\) the solution of (1.3) with initial datum \((\rho_0, \phi_0)\) in the time interval \([0,T]\), we have that \(\sim S \circ \Omega^{-t}(S(\rho_0, \phi_0))\) solves (1.3) with the same initial datum but in the time interval \([-T,0]\).

Let us make some comments about the phase space of system (1.3). Note that the average \(H^s(T^d)\) of functions which differ only by a constant. For simplicity of notation we denote the equivalent class \([u] := \{ u + c, c \in \mathbb{R} \}\), just by \(u\). The homogeneous norm of \(u \in H^s(T^d)\) is \(\|u\|_s^2 := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} |u_j|^2 j^{2s}\). We shall denote by \(\| \cdot \|_s\) either the Sobolev norm in \(H^s\) or that one in the homogeneous space \(\tilde{H}^s\), according to the context.

Let us make some comments about the proof. First, in view of (1.5), we rewrite system (1.3) in terms of \(\rho \sim \mu + \rho\) with \(\rho \in H^s_0(T^d)\), obtaining
\[
\begin{aligned}
\partial_t \rho &= -\mu \Delta \rho - \text{div}(\rho \nabla \phi) \\
\partial_t \phi &= -\frac{1}{2} |\nabla \phi|^2 - g(\mu + \rho) + K(\mu + \rho)\Delta \rho + \frac{1}{2} K'(\mu + \rho)|\nabla \rho|^2.
\end{aligned}
\]

Then Theorem 1.1 follows by the following result, that we are going to prove

**Theorem 1.2.** Let \(s > \frac{2 + d}{2}\) and \(0 < m_1 < m_2\). For any initial data of the form \((\mu + \rho_0, \phi_0)\) with \((\rho_0, \phi_0) \in H^s_0(T^d) \times \tilde{H}^s(T^d)\) and \(m_1 < \rho + \rho_0(x) < m_2, \forall x \in T^d\), there exists \(T = T(||(\rho_0, \phi_0)||_{s+2}, \min, (\mu + \rho_0(x))) > 0\) and a unique solution \((\mu + \rho, \phi)\) of (1.1) such that
\[
(\rho, \phi) \in C^0\left( [0,T], H^s_0(T^d, \mathbb{R}) \times \tilde{H}^s(T^d, \mathbb{R}) \right) \cap C^1\left( [0,T], H^{s-2}_0(T^d, \mathbb{R}) \times \tilde{H}^{s-2}(T^d, \mathbb{R}) \right)
\]
and \(m_1 < \mu + \rho(t, x) < m_2\) holds for any \(t \in [0,T]\). Moreover, for \(|t| \leq T\), the solution map \((\rho_0, \phi_0) \mapsto (\rho(t, \cdot), \phi(t, \cdot))\) is locally defined and continuous in \(H^s_0(T^d) \times \tilde{H}^s(T^d)\).

We consider system (1.6) as a system on the homogeneous space \(\tilde{H}^s \times \tilde{H}^s\), that is we study
\[
\begin{aligned}
\partial_t \rho &= -\mu \Delta \rho - \text{div}(\mu \nabla \phi) \\
\partial_t \phi &= -\frac{1}{2} |\nabla \phi|^2 - g(\mu + \rho) + K(\mu + \rho)\Delta \rho + \frac{1}{2} K'(\mu + \rho)|\nabla \rho|^2.
\end{aligned}
\]
where $\Pi_0^\perp$ is the projector onto the Fourier modes of index $\neq 0$. For simplicity of notation we shall not distinguish between systems (1.7) and (1.6). In Section 3 we paralinearize (1.6), i.e. (1.7), up to bounded semilinear terms (for which we do not need Bony paralinearization formula). Then, introducing a suitable complex variable, we transform it into a quasi-linear type Schrödinger equation, see system (3.4), defined in the phase space

$$\dot{H}^s := \left\{ U = \frac{u}{\pi} : u \in \dot{H}^s(T^d, C) \right\}, \quad \|U\|_s^2 := \|\pi u\|_{\dot{H}^s}^2 = \|u\|_s^2 + \|\pi\|_s^2.$$  \hspace{1cm} (1.8)

We use paradifferential calculus in the Weyl quantization, because it is quite convenient to prove energy estimates for this system. Since (3.4) is a quasi-linear system, in order to prove local well posedness (Proposition 4.1) we follow the strategy, initiated by Kato [18], of constructing inductively a sequence of linear problems whose solutions converge to the solution of the quasilinear equation. Such a scheme has been widely used, see e.g. [20, 1, 7, 16] and reference therein.

The equation (1.3) is a Hamiltonian PDE. We do not exploit explicitly this fact, but it is indeed responsible for the energy estimate of Proposition 4.4. The method of proof of Theorem 1.1 is similar to the one in Feola-Iandoli [17] for Hamiltonian quasi-linear Schrödinger equations on $T^d$ (and Alazard-Burq-Zuily [1] in the case of gravity-capillary water waves in $\mathbb{R}^d$). The main difference is that we aim to obtain the minimal smoothness assumption $s > 2 + (d/2)$. This requires to optimize several arguments, and, in particular, to develop a sharp para-differential calculus for periodic functions that we report in the Appendix in a self-contained way. Some other technical differences are in the use of the modified energy (section 4.2), the mollifiers (4.17) which enables to prove energy estimates independent of $\varepsilon$ for the regularized system, the argument for the continuity of the flow in $H^s$. We expect that our approach would enable to extend the local existence result of [17] to initial data fulfilling the minimal smoothness assumptions $s > 2 + (d/2)$.

We now set some notation that will be used throughout the paper. Since $K : \mathbb{R}_+ \to \mathbb{R}$ is positive, given $0 < m_1 < m_2$, there exist constants $c_K, C_K > 0$ such that

$$c_K \leq K(\rho) \leq C_K, \quad \forall \rho \in (m_1, m_2).$$  \hspace{1cm} (1.9)

Since the velocity potential $\phi$ is defined up to a constant, we may assume in (1.6) that

$$g(m) = 0.$$  \hspace{1cm} (1.10)

From now on we fix $s_0$ so that

$$\frac{d}{2} < s_0 < s - 2.$$  \hspace{1cm} (1.11)

The initial datum $\rho_0(x)$ belongs to the open subset of $H_0^s(T^d)$ defined by

$$Q := \left\{ \rho \in H_0^s(T^d) : m_1 < m + \rho(x) < m_2 \right\}$$  \hspace{1cm} (1.12)

and we shall prove that, locally in time, the solution of (1.6) stays in this set.

We write $a \preceq b$ with the meaning $a \leq Cb$ for some constant $C > 0$ which does not depend on relevant quantities.

## 2 Functional setting and paradifferential calculus

The Sobolev norms $\|\cdot\|_s$ satisfy interpolation inequalities (see e.g. section 3.5 in [S]):

(i) for all $s \geq s_0 > \frac{d}{2}$, $u, v \in H^s$,

$$\|uv\|_s \preceq \|u\|_{s_0}\|v\|_s + \|u\|_s\|v\|_{s_0}. $$  \hspace{1cm} (2.1)
Similarly we denote by $\Gamma^2$ the
\begin{align}
\|uv\|_s & \lesssim \|u\|_{s_0}\|v\|_s. \quad (2.2) \\
\|u\|_{\theta s_1+(1-\theta)s_2} & \leq \|u\|^{\theta}_{s_1}\|u\|^{1-\theta}_{s_2}. \quad (2.3) \\
\|u\|_{a}\|v\|_{\beta} & \leq \|u\|_{a}\|v\|_{b} + \|u\|_{b}\|v\|_{a}. \quad (2.4)
\end{align}

\textbf{Paradifferential calculus.} We now introduce the notions of paradifferential calculus that will be used in the proof of Theorem 1.1. We develop the space of functions $\mathcal{W}$ that is more convenient to get the energy estimates of section 4. The main results are the continuity Theorem 2.4 and the composition Theorem 2.5, which require mild regularity assumptions of the symbols in the Appendix. This is needed in order to prove the local existence Theorem 1.1 with the natural minimal regularity on the initial datum $(\rho_0,\phi_0) \in H^s \times H^s$ with $s > 2 + \frac{d}{2}$.

Along the paper $\mathcal{W}$ may denote either the Banach space $L^{\infty}(\mathbb{T}^d)$, or the Sobolev spaces $H^s(\mathbb{T}^d)$, or the Hölder spaces $W^{0,\infty}(\mathbb{T}^d)$, introduced in Definition A.3. Given a multi-index $\beta \in \mathbb{N}_0^d$ we define $|\beta| := \beta_1 + \ldots + \beta_d$.

\textbf{Definition 2.1. (Symbols with finite regularity)} Given $m \in \mathbb{R}$ and a Banach space $\mathcal{W} \in \{L^{\infty}(\mathbb{T}^d), H^s(\mathbb{T}^d), W^{0,\infty}(\mathbb{T}^d)\}$, we denote by $\Gamma_m^{\mathcal{W}}$ the space of functions $a : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$, $a(x,\xi)$, which are $C^\infty$ with respect to $\xi$ and such that, for any $\beta \in \mathbb{N}_0^d$, there exists a constant $C_\beta > 0$ such that

$$
\|\partial_\xi^\beta a(\cdot,\xi)\|_{\mathcal{W}} \leq C_{\beta} (\langle \xi \rangle)^{m - |\beta|}, \quad \forall \xi \in \mathbb{R}^d. \quad (2.5)
$$

We denote by $\Sigma_m^{\mathcal{W}}$ the subclass of symbols $a \in \Gamma_m^{\mathcal{W}}$ which are spectrally localized, that is

$$
\exists \delta \in (0,1) : \quad \widehat{a}(j,\xi) = 0, \quad \forall |j| \geq \delta \langle \xi \rangle, \quad (2.6)
$$

where $\widehat{a}(j,\xi) := (2\pi)^{-d} \int_{\mathbb{T}^d} a(x,\xi)e^{-ij \cdot x} \, dx$, $j \in \mathbb{Z}^d$, are the Fourier coefficients of the function $x \mapsto a(x,\xi)$.

We endow $\Gamma_m^{\mathcal{W}}$ with the family of norms defined, for any $n \in \mathbb{N}_0$, by

$$
|a|_{m,\mathcal{W},n} := \max_{|\beta| \leq n} \|\langle \xi \rangle^{-m + |\beta|} \partial_\xi^\beta a(\cdot,\xi)\|_{\mathcal{W}}. \quad (2.7)
$$

When $\mathcal{W} = H^s$, we also denote $\Gamma_s^{\mathcal{W}} \equiv \Gamma_0^{\mathcal{W}}$ and $|a|_{m,s,n} \equiv |a|_{m,H^s,n}$. We denote by $\Gamma_m^{\mathcal{W}} \otimes \mathcal{M}_2(\mathbb{C})$ the $2 \times 2$ matrices $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ of symbols in $\Gamma_m^{\mathcal{W}}$ and $|A|_{m,\mathcal{W},n} := \max_{i=1,\ldots,4} \{ |u_i|_{m,\mathcal{W},n} \}$. Similarly we denote by $\Gamma_m^{\mathcal{W}} \otimes \mathbb{R}^d$ the $d$-dimensional vectors of symbols in $\Gamma_m^{\mathcal{W}}$.

Let us make some simple remarks:

- (i) a given function $a(x) \in \mathcal{W}$ then $a(x) \in \Gamma_0^{\mathcal{W}}$ and

$$
|a|_{0,\mathcal{W},n} = \|a\|_{\mathcal{W}}, \quad \forall n \in \mathbb{N}_0. \quad (2.8)
$$

- (ii) For any $s_0 > \frac{d}{2}$ and $0 \leq s' \leq s$, we have that

$$
|a|_{m,L^{\infty},n} \lesssim |a|_{m,W^{s',\infty},n} \lesssim |a|_{m,W^{s,\infty},n} \lesssim |a|_{m,H^{s_0+e},n}, \quad \forall n \in \mathbb{N}_0. \quad (2.9)
$$
• (iii) If $a \in \Gamma^{m}_{\mathbb{F}}$, then, for any $\alpha \in \mathbb{N}^d$, we have $\partial_{\xi}^{\alpha} a \in \Gamma^{m-|\alpha|}_{\mathbb{F}}$ and

$$|\partial_{\xi}^{\alpha} a|_{m-|\alpha|, \mathbb{F}, n} \lesssim |a|_{m, \mathbb{F}, n+|\alpha|}, \ \forall n \in \mathbb{N}_0.$$ (2.10)

• (iv) If $a \in \Gamma^{m}_{H^{\infty}}$, resp. $a \in \Gamma^{m}_{W^{s, \infty}}$, then $\partial_{\xi}^{\alpha} a \in \Gamma^{m}_{H^{\infty-|\alpha|}}$, resp. $\partial_{\xi}^{\alpha} a \in \Gamma^{m}_{W^{s-|\alpha|, \infty}}$, and

$$|\partial_{\xi}^{\alpha} a|_{m,s-|\alpha|, \mathbb{F}, n} \lesssim |a|_{m,s,n}, \ \text{resp.} \ |\partial_{\xi}^{\alpha} a|_{m,W^{s-|\alpha|, \infty}, n} \lesssim |a|_{m,W^{s, \infty}, n}, \ \forall n \in \mathbb{N}_0.$$ (2.11)

• (v) If $a, b \in \Gamma^{m}_{\mathbb{F}}$, then $ab \in \Gamma^{m}_{\mathbb{F}}$ with $|a|_{m+m', \mathbb{F}, n} \lesssim |a|_{m, \mathbb{F}, n} |b|_{m', \mathbb{F}, n}$ for any $n \in \mathbb{N}_0$. In particular, if $a, b \in \Gamma^{m}_{s}$ with $s > d/2$ then $ab \in \Gamma^{m+m'}_{s}$ and

$$|ab|_{m+m', s, n} \lesssim |a|_{m, s, n} |b|_{m', s, n} + |a|_{m, s, n} |b|_{m', s, n}, \ \forall n \in \mathbb{N}_0.$$ (2.12)

Let $\epsilon \in (0, 1)$ and consider a $C^{\infty}$, even cut-off function $\chi: \mathbb{R}^d \to [0, 1]$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1.1, \\ 0 & \text{if } |\xi| \geq 1.9, \end{cases} \ \chi(\xi) := \chi\left(\frac{\xi}{\epsilon}\right).$$ (2.13)

Given a symbol $a$ in $\Gamma^{m}_{\mathbb{F}}$ we define the regularized symbol

$$a_\chi(x, \xi) := \chi(\xi) (D) a(x, \xi) = \sum_{j \in \mathbb{Z}^d} \chi\left(\frac{j}{\epsilon}\right) \hat{a}(j, \xi) e^{ij \cdot x}.$$ (2.14)

Note that $a_\chi$ is analytic in $x$ (it is a trigonometric polynomial) and it is spectrally localized.

In order to define the Bony-Weyl quantization of a symbol $a(x, \xi)$ we first remind the Weyl quantization formula

$$\text{Op}^{W}(a)[u] := \sum_{j \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \hat{a}(j, k) \frac{j + k}{2} u_k \right) e^{ij \cdot x}. \ \text{(2.15)}$$

**Definition 2.2. (Bony-Weyl quantization)** Given a symbol $a \in \Gamma^{m}_{\mathbb{F}}$, we define the *Bony-Weyl paradifferential operator* $\text{Op}^{BW}(a) = \text{Op}^{W}(a_\chi)$ that acts on a periodic function $u$ as

$$(\text{Op}^{BW}(a)[u])(x) := \sum_{j \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \hat{a}(j, k) \frac{j + k}{2} u_k \right) e^{ij \cdot x} \ \rightarrow \ \sum_{j \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \hat{a}(j, k) \frac{j + k}{2} \chi\left(\frac{j + k}{\epsilon}\right) u_k \right) e^{ij \cdot x}.$$ (2.16)

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}$ is a matrix of symbols in $\Gamma^m_{s}$, then $\text{Op}^{BW}(A)$ is defined as the matrix valued operator $\begin{pmatrix} \text{Op}^{BW}(a_{11}) & \text{Op}^{BW}(a_{12}) \\ \text{Op}^{BW}(a_{31}) & \text{Op}^{BW}(a_{32}) \end{pmatrix}$.

Given a symbol $a(\xi)$ independent of $x$, then $\text{Op}^{BW}(a)$ is the Fourier multiplier operator

$$\text{Op}^{BW}(a) u = a(D) u = \sum_{j \in \mathbb{Z}^d} a(j) u_j e^{ij \cdot x}.$$ (2.17)

Note that if $\chi\left(\frac{k-j}{\epsilon}\right) \neq 0$ then $|k-j| \leq \epsilon(j+k)$ and therefore, for $\epsilon \in (0, 1),

$$\frac{1 - \epsilon}{1 + \epsilon} |k| \leq |j| \leq \frac{1 + \epsilon}{1 - \epsilon} |k|, \ \forall j, k \in \mathbb{Z}^d. \ \text{(2.17)}$$
This relation shows that the action of a para-differential operator does not spread much the Fourier support of functions. In particular $\text{Op}^\text{BW}(a)$ sends a constant function into a constant function and therefore $\text{Op}^\text{BW}_R(a)$ sends homogenous spaces into homogenous spaces.

**Remark 2.3.** Actually, if $\chi_\varepsilon(k/(k+1)) \neq 0$, $\varepsilon \in (0, 1/4)$, then $|j| \leq |j + k| \leq 3|j|$, for all $j, k \in \mathbb{Z}^d$.

Along the paper we shall use the following results concerning the action of a paradifferential operator in Sobolev spaces.

**Theorem 2.4. (Continuity of Bony-Weyl operators)** Let $a \in \Gamma^m_{s_0}$, resp. $a \in \Gamma^m_{s_0, \infty}$, with $m \in \mathbb{R}$. Then $\text{Op}^\text{BW}(a)$ extends to a bounded operator $\dot{H}^s \to \dot{H}^{s - m}$ for any $s \in \mathbb{R}$ satisfying the estimate, for any $u \in \dot{H}^s$,

$$\|\text{Op}^\text{BW}(a) u\|_{s - m} \lesssim |a|_{m,s_0,2(d+1)} \|u\|_s$$

Moreover, for any $q \geq 0$, $s \in \mathbb{R}$, $u \in \dot{H}^s(\mathbb{T}^d)$,

$$\|\text{Op}^\text{BW}(a) u\|_{s - m - \frac{q}{2}} \lesssim |a|_{m,s_0 - q,2(d+1)} \|u\|_s.$$  

**Proof.** Since $\text{Op}^\text{BW}(a) = \text{Op}^W(a_\chi)$, the estimate (2.18) follows by (A.35), (A.21) and $|a|_{m,L^\infty,N} \lesssim |a|_{m,s_0,N}$. Note that the condition on the Fourier support of $a_\chi$ in Theorem A.7 is automatically satisfied provided $\varepsilon$ in (2.13) is sufficiently small. To prove (2.19) we use also (A.22).

The second result of symbolic calculus that we shall use regards composition for Bony-Weyl paradifferential operators at the second order (as required in the paper) with mild smoothness assumptions for the symbols in the space variable $x$. Given symbols $a \in \Gamma^m_{s_0 + \varrho}$, $b \in \Gamma^{m'}_{s_0 + \varrho}$ with $m, m' \in \mathbb{R}$ and $\varrho \in (0, 2]$ we define

$$a \#_{\varrho} b := \begin{cases} a b, & \varrho \in (0, 1] \\ a b + \frac{\varrho}{2} \{a, b\}, & \varrho \in (1, 2], \end{cases} \quad \text{where} \quad \{a, b\} := \nabla_x a \cdot \nabla_x b - \nabla_x a \cdot \nabla_x b,$$

is the Poisson bracket between $a(x, \xi)$ and $b(x, \xi)$. By (2.10) and (2.12) we have that $ab$ is a symbol in $\Gamma^m_{s_0 + \varrho}$ and $\{a, b\}$ is in $\Gamma^{m + m'}_{s_0 + \varrho - 1}$. The next result follows directly by Theorem A.8 and (2.9).

**Theorem 2.5. (Composition)** Let $a \in \Gamma^m_{s_0 + \varrho}$, $b \in \Gamma^{m'}_{s_0 + \varrho}$ with $m, m' \in \mathbb{R}$ and $\varrho \in (0, 2]$. Then

$$\text{Op}^\text{BW}(a) \text{Op}^\text{BW}(b) = \text{Op}^\text{BW}(a \#_{\varrho} b) + R^{-\varrho}(a, b)$$

(2.21)

where the linear operator $R^{-\varrho}(a, b): \dot{H}^s \to \dot{H}^{s - (m + m')} + \varrho$, $s \in \mathbb{R}$, satisfies, for any $u \in \dot{H}^s$,

$$\|R^{-\varrho}(a, b) u\|_{s - (m + m') + \varrho} \lesssim \left( |a|_{m,s_0 + \varrho,N} |b|_{m',s_0,N} + |a|_{m,s_0,N} |b|_{m',s_0 + \varrho,N} \right) \|u\|_s$$

(2.22)

where $N \geq 3d + 4$.

A useful corollary of Theorems 2.4 and 2.5 (using also (2.10)-(2.12)) is the following:

**Corollary 2.6.** Let $a \in \Gamma^m_{s_0 + 2}$, $b \in \Gamma^m_{s_0 + 2}$, $c \in \Gamma^m_{s_0 + 2}$ with $m, m', m'' \in \mathbb{R}$. Then

$$\text{Op}^\text{BW}(a) \circ \text{Op}^\text{BW}(b) \circ \text{Op}^\text{BW}(c) = \text{Op}^\text{BW}(abc) + R_1(a, b, c) + R_0(a, b, c),$$

(2.23)

where

$$R_1(a, b, c) := \text{Op}^\text{BW}(\{a, c\} b + \{b, c\} a + \{a, b\} c)$$

(2.24)

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satisfies $R_1(a,b,c) = -R_1(c,b,a)$ and $R_0(a,b,c)$ is a bounded operator $\hat{H}^s \to \hat{H}^{s-(m+m'+m'')}+2$, $\forall s \in \mathbb{R}$, satisfying, for any $u \in \hat{H}^s$, 
\[ \|R_0(a,b,c)\|_{s-(m+m'+m'')+2} \lesssim |a|_{m,s_0+2,N} |b|_{m',s_0+2,N} |c|_{m'',s_0+2,N} \|u\|_s \]  
(2.25)  
where $N \geq 3d+5$.

We now provide the Bony-paraproduct decomposition for the product of Sobolev functions in the Bony-Weyl quantization. Recall that $\Pi_0^\perp$ denotes the projector on the subspace $H_0^s$.

**Lemma 2.7. (Bony paraproduct decomposition)** Let $u \in \hat{H}^s$, $v \in \hat{H}^r$ with $s+r \geq 0$. Then
\[ uv = \text{Op}^{\text{BW}}(u) + \text{Op}^{\text{BW}}(v) + R(u,v) \]  
(2.26)
where the bilinear operator $R : \hat{H}^s \times \hat{H}^r \to \hat{H}^{s+r}$ is symmetric and satisfies the estimate
\[ \|R(u,v)\|_{s+r-s_0} \lesssim \|u\|_s \|v\|_r. \]  
(2.27)
Moreover $R(u,v) = R(\Pi_0^\perp u, \Pi_0^\perp v) - u_0v_0$ and then
\[ \|\Pi_0^\perp R(u,v)\|_{s+r-s_0} \lesssim \|\Pi_0^\perp u\|_s \|\Pi_0^\perp v\|_r. \]  
(2.28)

**Proof.** Introduce the function $\theta_\epsilon(j,k)$ by
\[ 1 = \chi_\epsilon\left(\frac{j-k}{\epsilon} + k\right) + \chi_\epsilon\left(\frac{k}{2j-k}\right) + \theta_\epsilon(j,k). \]  
(2.29)
Note that $|\theta_\epsilon(j,k)| \leq 1$. Let $\Sigma := \{(j,k) \in \mathbb{Z}^d \times \mathbb{Z}^d : \theta_\epsilon(j,k) \neq 0\}$ denote the support of $\theta_\epsilon$. We claim that
\[ (j,k) \in \Sigma \implies |\epsilon - j| \leq C \min(|j-k|, |k|). \]  
(2.30)
Indeed, recalling the definition of the cut-off function $\chi$ in (2.13), we first note that\footnote{For $\delta$ sufficiently small, if $|j-k| \leq \delta (j+k)$ and $|k| \leq \delta (2j-k)$ then $(j,k) = (0,0)$.}
\[ \Sigma = \{(0,0)\} \cup \left\{(j-k) \geq \epsilon (j+k) , |k| \geq \epsilon (2j-k) \right\}. \]

Thus, for any $(j,k) \in \Sigma$,
\[ |j| \leq \frac{1}{2} |j-k| + \frac{1}{2} |j+k| \leq \left(\frac{1}{2} + \frac{1}{2\epsilon}\right) |j-k|, \quad |j| \leq \frac{1}{2} |2j-k| + \frac{1}{2} |k| \leq \left(\frac{1}{2} + \frac{1}{2\epsilon}\right) |k| \]
proving (2.30). Using (2.29) we decompose
\[ uv = \sum_{j,k} \hat{u}_{j-k} \chi_\epsilon\left(\frac{j-k}{\epsilon} + k\right) \hat{v}_k \epsilon^{ijx} + \sum_{j,k} \hat{v}_k \chi_\epsilon\left(\frac{k}{2j-k}\right) \hat{u}_{j-k} \epsilon^{ijx} + \sum_{j,k} \theta_\epsilon(j,k) \hat{u}_{j-k} \hat{v}_k \epsilon^{ijx} \]
\[ = \text{Op}^{\text{BW}}(u) + \text{Op}^{\text{BW}}(v) + R(u,v). \]

By (2.30), $s+r \geq 0$, and the Cauchy-Schwartz inequality, we get
\[ \|R(u,v)\|_{s+r-s_0}^2 \lesssim \sum_j \langle j \rangle^{2s+r-s_0} \left| \sum_k \theta_\epsilon(j,k) \hat{u}_{j-k} \hat{v}_k \right|^2 \]
\[ \lesssim \sum_j \langle j \rangle^{-2s_0} \left| \sum_k \langle j-k \rangle^s |\hat{u}_{j-k}| \langle k \rangle^r |\hat{v}_k| \right|^2 \lesssim \|u\|_s^2 \|v\|_r^2 \]
proving \((2.27)\). Finally, since on the support of \(\theta\) we have or \((j,k) = (0,0)\) or \(j-k \neq 0\) and \(k \neq 0\), we deduce that
\[
R(u, v) = \theta_x(0,0)\hat{u}_0 \hat{v}_0 + \sum_{j-k \neq 0, k \neq 0} \theta_x(j,k)\hat{u}_{j-k} \hat{v}_ke^{ijx} = -\hat{u}_0 \hat{v}_0 + R(\Pi_0 u, \Pi_0 v)
\]
and we deduce \((2.28)\).

Composition estimates. We will use the following Moser estimates for composition of functions in Sobolev spaces.

**Theorem 2.8.** Let \(I \subseteq \mathbb{R}\) be an open interval and \(F \in C^\infty(I; \mathbb{C})\) a smooth function. Let \(J \subseteq I\) be a compact interval. For any function \(u, v \in H^s(\mathbb{T}^d, \mathbb{R})\), \(s > \frac{d}{2}\), with values in \(J\), we have
\[
\begin{align*}
\|F(u)\|_s & \leq C(s,F,J) (1 + \|u\|_s), \\
\|F(u) - F(v)\|_s & \leq C(s,F,J) (\|u - v\|_s + (\|u\|_s + \|v\|_s)|u - v|_{L^\infty}) \quad (2.31) \\
\|F(u)\|_s & \leq C(s,F,J)\|u\|_s \quad \text{if} \quad F(0) = 0. 
\end{align*}
\]

**Proof.** Take an extension \(\tilde{F} \in C^\infty(\mathbb{R}; \mathbb{C})\) such that \(\tilde{F}|_I = F\). Then \(F(u) = \tilde{F}(u)\) for any \(u \in H^s(\mathbb{T}^d; \mathbb{R})\) with values in \(J\), and apply the usual Moser estimate, see e.g. [2], replacing the Littlewood-Paley decomposition on \(\mathbb{R}^d\) with the one on \(\mathbb{T}^d\) in \((A.12)\).

### 3 Paralinearization of (EK)-system and complex form

In this section we paralinearize the Euler-Korteweg system \((1.6)\) and write it in terms of the complex variable
\[
u := \frac{1}{\sqrt{2}} \left( \frac{\theta}{K(m)} \right)^{-1/4} \rho + \frac{i}{\sqrt{2}} \left( \frac{\theta}{K(m)} \right)^{1/4} \phi, \quad \rho, \phi \in \dot{H}^s, \phi \in \dot{H}^s. \tag{3.1}
\]
The variable \(\nu \in \dot{H}^s\). We denote this change of coordinates in \(\dot{H}^s \times \dot{H}^s\) by
\[
\begin{bmatrix}
\nu\\
\phi
\end{bmatrix} = C^{-1} \begin{bmatrix}
\rho\\
\phi
\end{bmatrix}, \quad C := \frac{1}{\sqrt{2}} \begin{bmatrix}
\left( \frac{\theta}{K(m)} \right)^{1/4} & \left( \frac{\theta}{K(m)} \right)^{-1/4} \\
-i \left( \frac{\theta}{K(m)} \right)^{-1/4} & i \left( \frac{\theta}{K(m)} \right)^{1/4}
\end{bmatrix}, \quad C^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix}
\left( \frac{\theta}{K(m)} \right)^{-1/4} & i \left( \frac{\theta}{K(m)} \right)^{-1/4} \\
-i \left( \frac{\theta}{K(m)} \right)^{1/4} & -i \left( \frac{\theta}{K(m)} \right)^{1/4}
\end{bmatrix}. \tag{3.2}
\]
We also define the matrices
\[
J := \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad \mathbb{J} := \begin{bmatrix}
-i & 0 \\
0 & i
\end{bmatrix}, \quad \mathbb{I} := \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}. \tag{3.3}
\]

**Proposition 3.1.** (Paralinearized Euler-Korteweg equations in complex coordinates)
The \((EK)\)-system \((1.6)\) can be written in terms of the complex variable \(U := \begin{bmatrix} \nu \\ \overline{\nu} \end{bmatrix}\) with \(\nu\) defined in \((3.1)\), in the paralinearized form
\[
\partial_t U = \mathbb{J} \left[ \text{Op}^{BW}(A_2(U; x, \xi) + A_1(U; x, \xi)) \right] + R(U) \tag{3.4}
\]
where, for any function \( U \in \dot{H}^{m+2} \) such that
\[
\rho(U) := \frac{1}{\sqrt{2}} \left( \frac{m}{K(m)} \right)^{1/4} \Pi_0^U(u + \overline{u}) \in \mathcal{Q} \quad \text{(see (1.12))},
\]
\[(3.5)\]

(i) \( A_2(U; x, \xi) \in \Gamma_{s_0+2}^2 \otimes \mathcal{M}_2(\mathbb{C}) \) is the matrix of symbols
\[
A_2(U; x, \xi) := \sqrt{m K(m)|\xi|^2} \begin{bmatrix} 1 + a_+(U; x) & a_-(U; x) \\ a_-(U; x) & 1 + a_+(U; x) \end{bmatrix}
\]
\[(3.6)\]
where \( a_{\pm}(U; x) \in \Gamma_{s_0}^0 \) are the \( \xi \)-independent functions
\[
a_{\pm}(U; x) := \frac{1}{2} \left( \frac{K(\rho + m) - K(m)}{K(m)} \pm \rho \right). \quad \text{(3.7)}
\]

(ii) \( A_1(U; x, \xi) \in \Gamma_{s_0+1}^1 \otimes \mathcal{M}_2(\mathbb{C}) \) is the diagonal matrix of symbols
\[
A_1(U; x, \xi) := \begin{bmatrix} b(U; x) \cdot \xi & 0 \\ 0 & -b(U; x) \cdot \xi \end{bmatrix}, \quad b(U; x) := \nabla \phi \in \Gamma_{s_0+1}^0 \otimes \mathbb{R}^d. \quad \text{(3.8)}
\]

Moreover for any \( \sigma \geq 0 \) there exists a non decreasing function \( C(\ ) : \mathbb{R}_+ \to \mathbb{R}_+ \) (depending on \( K \)) such that, for any \( U, V \in \dot{H}^{s_0} \) with \( \rho(U), \rho(V) \in \mathcal{Q} \), \( W \in \dot{H}^{s_0+2} \) and \( j = 1, 2 \), we have
\[
\| \text{Op}_{\mathbb{B}W}(A_j(U)) W \|_\sigma \leq C(\|U\|_{s_0}) \|W\|_{\sigma+2} \quad \text{(3.9)}
\]
\[
\| \text{Op}_{\mathbb{B}W}(A_j(U) - A_j(V)) W \|_\sigma \leq C(\|U\|_{s_0}, \|V\|_{s_0}) \|W\|_{\sigma+2} \|U - V\|_{s_0} \quad \text{(3.10)}
\]
where in (3.10) we denoted by \( C(\cdot, \cdot) := C(\max\{\cdot, \cdot\}) \).

(iii) The vector field \( R(U) \) satisfies the following “semilinear” estimates: for any \( \sigma \geq s_0 > d/2 \) there exists a non decreasing function \( C(\ ) : \mathbb{R}_+ \to \mathbb{R}_+ \) (depending also on \( g, K \)) such that, for any \( U, V \in \dot{H}^{\sigma+2} \) such that \( \rho(U), \rho(V) \in \mathcal{Q} \), we have
\[
\|R(U)\|_\sigma \leq C(\|U\|_{s_0+2}) \|U\|_\sigma, \quad \|R(U)\|_\sigma \leq C(\|U\|_{s_0}) \|U\|_{\sigma+2}, \quad \text{(3.11)}
\]
\[
\|R(U) - R(V)\|_\sigma \leq C(\|U\|_{s_0+2}, \|V\|_{s_0+2}) \|U - V\|_\sigma + C(\|U\|_\sigma, \|V\|_\sigma) \|U - V\|_{s_0+2} \quad \text{(3.12)}
\]
\[
\|R(U) - R(V)\|_{s_0} \leq C(\|U\|_{s_0+2}, \|V\|_{s_0+2}) \|U - V\|_{s_0}, \quad \text{(3.13)}
\]
where in (3.11) and (3.12) we denoted again by \( C(\cdot, \cdot) := C(\max\{\cdot, \cdot\}) \).

Proof. We first paralinearize the original equations (1.6), then we switch to complex coordinates.

**Step 1: paralinearization of (1.6).** We apply several times the paraproduct Lemma 2.7 and the composition Theorem 2.5. In the following we denote by \( R^g \) the remainder that comes from Lemma 2.7 and by \( R^{-g} \), \( g = 1, 2 \), the remainder that comes from Theorem 2.3. We shall adopt the following convention: given \( \mathbb{R}^d \)-valued symbols \( a = (a_j)_{j=1, \ldots, d}, b = (b_j)_{j=1, \ldots, d} \) in some class \( \Gamma_s \otimes \mathbb{R}^d \), we denote \( R^g(a, b) := \sum_{j=1}^d R^g(a_j, b_j) \),
\[
R^{-g}(a, b) := \sum_{j=1}^d R^{-g}(a_j, b_j) \quad \text{and} \quad \text{Op}_{\mathbb{B}W}(a) \cdot \text{Op}_{\mathbb{B}W}(b) := \sum_{j=1}^d \text{Op}_{\mathbb{B}W}(a_j) \text{Op}_{\mathbb{B}W}(b_j). \]
We paralinearize the terms in the first line of (1.6). We have \( \Delta \phi = -\text{Op}^{BW}(|\xi|^2) \phi \) and 
\[
\begin{align*}
\rho \Delta \phi &= -\text{Op}^{BW}(\rho|\xi|^2 + \nabla \rho \cdot i \xi) \phi \\
&\quad + \text{Op}^{BW}(\Delta \phi) \rho + R^\rho(\rho, \Delta \phi) + R^{-2}(\rho, |\xi|^2) \phi, \\
\nabla \rho \cdot \nabla \phi &= \text{Op}^{BW}(\nabla \rho \cdot i \xi) \phi + \text{Op}^{BW}(\nabla \phi \cdot i \xi) \rho \\
&\quad + R^\rho(\nabla \rho, \nabla \phi) + R^{-1}(\nabla \rho, i \xi) \phi + R^{-1}(\nabla \phi, i \xi) \rho.
\end{align*}
\]
(3.14)
(3.15)
Then we paralinearize the terms in the second line of (1.6). We have
\[
\begin{align*}
\frac{1}{2} |\nabla \phi|^2 &= \text{Op}^{BW}(\nabla \phi \cdot i \xi) \phi \\
&\quad + \frac{1}{2} R^\rho(\nabla \phi, \nabla \phi) + R^{-1}(\nabla \phi, i \xi) \phi.
\end{align*}
\]
(3.16)
Using (1.10) we regard the semilinear term
\[
g(m + \rho) = g(m + \rho) - g(m) =: R(\rho)
\]
directly as a remainder. Moreover, writing \( \Delta \rho = -\text{Op}^{BW}(|\xi|^2) \rho \), we get
\[
K(m + \rho) \Delta \rho = \text{Op}^{BW}(K(m + \rho)) \Delta \rho + \text{Op}^{BW}(\Delta \rho) K(m + \rho) + R^\rho(\rho, \Delta \rho, K(m + \rho)) \\
&= -\text{Op}^{BW}(K(m + \rho)|\xi|^2 + K'(m + \rho) \nabla \rho \cdot i \xi) \rho \\
&\quad + \text{Op}^{BW}(\Delta \rho) K(m + \rho) + R^\rho(\rho, \Delta \rho, K(m + \rho)) - R^{-2}(K(m + \rho), |\xi|^2) \rho.
\]
(3.18)
Finally, using for \( \frac{1}{2} |\nabla \phi|^2 \) the expansion (3.16) for \( \rho \) instead of \( \phi \), we obtain
\[
\begin{align*}
\frac{1}{2} K'(m + \rho) |\nabla \phi|^2 &= \frac{1}{2} \text{Op}^{BW}(K'(m + \rho)) |\nabla \rho|^2 + \frac{1}{2} \text{Op}^{BW}(|\nabla \rho|^2) K'(m + \rho) \\
&\quad + \frac{1}{2} R^\rho(|\nabla \rho|^2, K'(m + \rho)) = \text{Op}^{BW}(K'(m + \rho) \nabla \rho \cdot i \xi) \rho + R(\rho)
\end{align*}
\]
where
\[
R(\rho) := \frac{1}{2} \text{Op}^{BW}(|\nabla \rho|^2) K'(m + \rho) + \frac{1}{2} R^\rho(|\nabla \rho|^2, K'(m + \rho))
\]
(3.19)
\[
+ \frac{1}{2} \text{Op}^{BW}(K'(m + \rho)) R^\rho(\nabla \rho, \nabla \rho) + \frac{1}{2} \text{Op}^{BW}(K'(m + \rho)) R^{-1}(\nabla \rho, i \xi) \rho + R^{-1}(K'(m + \rho), i \nabla \rho \cdot i \xi) \rho.
\]
(3.20)
Collecting all the above expansions and recalling the definition of the symplectic matrix \( J \) in (3.3), the system (1.6) can be written in the paralinearized form
\[
\begin{align*}
\frac{\partial}{\partial t} \left( \begin{array}{c} \rho \\ \phi \end{array} \right) &= J \text{Op}^{BW} \left( \begin{array}{c} K(m + \rho)|\xi|^2 \\
\nabla \phi \cdot i \xi \\
- \nabla \phi \cdot i \xi \end{array} \right) \left( \begin{array}{c} \rho \\ \phi \end{array} \right) + R(\rho, \phi)
\end{align*}
\]
(3.22)
where we collected in \( R(\rho, \phi) \) all the terms in lines (3.14)–(3.21).

**Step 2: complex coordinates.** We now write system (3.22) in the complex coordinates \( U = C^{-1} \left( \begin{array}{c} \rho \\ \phi \end{array} \right) \). Note that \( C^{-1} \) conjugates the Poisson tensor \( J \) to \( \mathbb{J} \) defined in (3.3), i.e. \( C^{-1} J = \mathbb{J} C^* \) and therefore system (3.22) is conjugated to
\[
\begin{align*}
\frac{\partial}{\partial t} U &= J C^* \text{Op}^{BW} \left( \begin{array}{c} K(m + \rho)|\xi|^2 \\
\nabla \phi \cdot i \xi \\
- \nabla \phi \cdot i \xi \end{array} \right) C U + C^{-1} R(\mathbb{J} U).
\end{align*}
\]
(3.23)
Using (3.2), system (3.23) reads as system (3.24)-(3.8) with $R(U) := C^{-1}R(CU)$.

Indeed, we bound

We also claim that

Concerning the remainder $R(U)$. We now prove (3.11)-(3.13). Since $\|\rho\|_{\sigma}, \|\phi\|_{\sigma} \sim \|U\|_{\sigma}$ for any $\sigma \in \mathbb{R}$ by (3.2), the estimates (3.11)-(3.13) directly follow from those of $R(\rho, \phi)$ in (3.22). We now estimate each term in (3.11)-(3.13). In the sequel $\sigma \geq s_0 > d/2$.

**Step 3: Estimate of the remainder $R(U)$**. We now prove (3.11)-(3.13). Since $\|\rho\|_{\sigma}, \|\phi\|_{\sigma} \sim \|U\|_{\sigma}$ for any $\sigma \in \mathbb{R}$ by (3.2), the estimates (3.11)-(3.13) directly follow from those of $R(\rho, \phi)$ in (3.22). We now estimate each term in (3.11)-(3.13). In the sequel $\sigma \geq s_0 > d/2$.

**Estimate of the terms in line (3.14).** Applying first (2.18) with $m = 0$, and then (2.19) with $g = 2$, we have

By (2.27), the smoothing remainder in line (3.14) satisfies the estimates

and, by (2.22) with $g = 2$, and the interpolation estimate (2.4),

By (3.24), (3.26) and $\|\rho\|_{\sigma}, \|\phi\|_{\sigma} \sim \|U\|_{\sigma}$ we deduce that the terms in line (3.14), written in function of $U$, satisfy (3.11). Next we write

and applying (2.18) with $m = 0$, and (2.19) with $g = 2$ to $\text{Op}^{\text{BW}}(\Delta \phi_1 - \Delta \phi_2)\rho_2$, we get

Concerning the remainder $R^p(\rho, \Delta \phi)$, we write $R^p(\rho, \Delta \phi_1) - R^p(\rho, \Delta \phi_2) = R^p(\rho_1 - \rho_2, \Delta \phi_1) + R^p(\rho_2, \Delta \phi_2 - \Delta \phi_1)$ and, applying (2.27), we get

Finally we write $R^{-2}(\rho_1, [\xi]^2)\phi_1 - R^{-2}(\rho_2, [\xi]^2)\phi_2 = R^{-2}(\rho_1 - \rho_2, [\xi]^2)\phi_1 + R^{-2}(\rho_2, [\xi]^2)\phi_2$.

Using (2.22) we get

We also claim that

Indeed, we bound

and, to control $R^{-2}(\rho_1 - \rho_2, [\xi]^2)\phi_1$, we use that, by definition, it equals

$$\text{Op}^{\text{BW}}(\rho_1 - \rho_2)\text{Op}^{\text{BW}}([\xi]^2)\phi_1 - \text{Op}^{\text{BW}}((\rho_1 - \rho_2)[\xi]^2)\phi_1 - \text{Op}^{\text{BW}}(\nabla(\rho_1 - \rho_2) \cdot i\xi)\phi_1$$
and we estimate the first two terms using (2.19) with \( g = 0 \) and the last term with \( g = 1 \), by \( |R^{-2}(\rho_1 - \rho_2, \xi_0^2)\phi|_\sigma \lesssim |\rho_1 - \rho_2|_{s_0} \phi|_\sigma \), proving (3.30). By (3.27), (3.30) and \|\rho\|_\sigma, \|\phi\|_\sigma \sim \|U\|_\sigma\), we deduce that the terms in line (3.14), written in function of \( U \), satisfy (3.13).

The estimates (3.11)-(3.13) for the terms in lines (3.15), (3.16), (3.18) and (3.17), follow by similar arguments, using also (2.31).

**ESTIMATES OF R(\rho) DEFINED IN (3.19)-(3.21).**

Writing \( \text{Op}^{BW}_0 (\nabla \rho_1^2) K'(m + \rho) = \text{Op}^{BW}_0 (K'(m + \rho) - K'(m)) \) (in the homogeneous spaces \( \tilde{H}^s \)), we have, by (2.18), the fact that \( \rho \in \mathbb{Q} \), Theorem 2.8, (2.27), (2.2), (2.22) with \( g = 1 \),

\[ \|R(\rho)\|_\sigma \leq C(\|\rho\|_{s_0+2}) \|\rho\|_\sigma. \]

Thus \( R(\rho) \), written as a function of \( U \), satisfies (3.11). The estimates (3.12)-(3.13) follow by

\[ \|R(\rho_1) - R(\rho_2)\|_\sigma \leq C(\|\rho_1\|_{s_0+2}, \|\rho_2\|_{s_0+2}) \|\rho_1 - \rho_2\|_\sigma + C(\|\rho_1\|_\sigma, \|\rho_2\|_\sigma) \|\rho_1 - \rho_2\|_{s_0+2} \quad (3.31) \]

\[ \|R(\rho_1) - R(\rho_2)\|_{s_0} \leq C(\|\rho_1\|_{s_0+2}, \|\rho_2\|_{s_0+2}) \|\rho_1 - \rho_2\|_{s_0} \quad (3.32) \]

**PROOF OF (3.31).** Defining \( w := \nabla(\rho_1 + \rho_2), v := \nabla(\rho_1 - \rho_2) \), then we have, by (2.1),

\[ \|\nabla \rho_1^2 - \nabla \rho_2^2\|_{s_0} = \|w \cdot v\|_{s_0} \lesssim (\|\rho_1\|_{s_0+1} + \|\rho_2\|_{s_0+1}) \|\rho_1 - \rho_2\|_{s_0+1} \quad (3.33) \]

\[ \|\nabla \rho_1^2 - \nabla \rho_2^2\|_{s_0-1} = \|w \cdot v\|_{s_0-1} \lesssim (\|\rho_1\|_{s_0+1} + \|\rho_2\|_{s_0+1}) \|\rho_1 - \rho_2\|_{s_0}. \quad (3.34) \]

Let us prove (3.31) for the first term in (3.19). Remind that \( \rho_1, \rho_2 \) are in \( \mathbb{Q} \). We have

\[ \|\text{Op}^{BW}_0 (\nabla \rho_1^2) K'(m + \rho_1) - \text{Op}^{BW}_0 (\nabla \rho_2^2) K'(m + \rho_2)\|_\sigma \leq \|\text{Op}^{BW}_0 (w \cdot v) K'(m + \rho_1)\|_\sigma + \|\text{Op}^{BW}_0 (\nabla \rho_2^2) [K'(m + \rho_1) - K'(m + \rho_2)]\|_\sigma \quad (3.35) \]

In the same way the second term in (3.19) is bounded by (3.35). Regarding the term in (3.20), using that \( R^p(\cdot, \cdot) \) is bilinear and symmetric, we have

\[ \|\text{Op}^{BW}_0 (K'(m + \rho_1)) R^p(\nabla \rho_1, \nabla \rho_1) - \text{Op}^{BW}_0 (K'(m + \rho_2)) R^p(\nabla \rho_2, \nabla \rho_2)\|_\sigma \leq \|\text{Op}^{BW}_0 (K'(m + \rho_1) - K'(m + \rho_2)) R^p(\nabla \rho_1, \nabla \rho_1)\|_\sigma + \|\text{Op}^{BW}_0 (K'(m + \rho_2)) R^p(w, v)\|_\sigma \quad (3.36) \]

Also the terms in (3.21) are bounded by (3.35), proving that \( R(\rho) \) satisfies (3.31).

**PROOF OF (3.32).** Regarding the first term (3.19), we have

\[ \|\text{Op}^{BW}_0 (\nabla \rho_1^2) K'(m + \rho_1) - \text{Op}^{BW}_0 (\nabla \rho_2^2) K'(m + \rho_2)\|_{s_0} \leq \|\text{Op}^{BW}_0 (w \cdot v) K'(m + \rho_1)\|_{s_0} + \|\text{Op}^{BW}_0 (\nabla \rho_2^2) [K'(m + \rho_1) - K'(m + \rho_2)]\|_{s_0} \quad (3.37) \]

In the same way the second term in (3.19) is bounded by (3.35). Regarding the term in (3.20), using that \( R^p(\cdot, \cdot) \) is bilinear and symmetric, we have

\[ \|\text{Op}^{BW}_0 (K'(m + \rho_1)) R^p(\nabla \rho_1, \nabla \rho_1) - \text{Op}^{BW}_0 (K'(m + \rho_2)) R^p(\nabla \rho_2, \nabla \rho_2)\|_\sigma \leq \|\text{Op}^{BW}_0 (K'(m + \rho_1) - K'(m + \rho_2)) R^p(\nabla \rho_1, \nabla \rho_1)\|_\sigma + \|\text{Op}^{BW}_0 (K'(m + \rho_2)) R^p(w, v)\|_\sigma \quad (3.36) \]

Also the terms in (3.21) are bounded by (3.35), proving that \( R(\rho) \) satisfies (3.31).
Similarly we deduce that the second term in (3.19) is bounded as in (3.37). Regarding the term in (3.20), note that the bound (3.32) follows from (3.36) applied for \( \sigma = s_0 \). The estimate for last two terms in (3.21) follows in the same way so we analyze the last one. First we have

\[
\| R^{-1}(K'(m + p_1), i\nabla \rho_1 \cdot \xi) \rho_1 - R^{-1}(K'(m + p_2), i\nabla \rho_2 \cdot \xi) \rho_2 \|_{s_0} \\
\leq \| [R^{-1}(K'(m + p_1), \nabla \rho_1 \cdot i\xi) - R^{-1}(K'(m + p_2), \nabla \rho_2 \cdot i\xi)] \rho_1 \|_{s_0} \\
+ \| R^{-1}(K'(m + p_2), \nabla \rho_2 \cdot i\xi) (\rho_1 - \rho_2) \|_{s_0}
\]

On the other hand, by definition, we have

\[
\| R^{-1}(K'(m + p_1), \nabla \rho_1 \cdot i\xi) - R^{-1}(K'(m + p_2), i\nabla \rho_2 \cdot \xi) \| \rho_1 \|_{s_0} + C(\| \rho_2 \|_{s_0+2}) \| \rho_1 - \rho_2 \|_{s_0}.
\]

Thus (3.32) is proved.

4 Local existence

In this section we prove the existence of a local in time solution of system (3.4). For any \( s \in \mathbb{R} \) and \( T > 0 \), we denote \( L^2_T H^s := L^\infty([0, T], H^s) \). For \( \delta > 0 \) we also introduce

\[
Q_{\delta} := \{ \rho \in H^{m}_{0} : m + \delta \leq m + \rho(x) \leq m - \delta \} \subset Q
\]

where \( Q \) is defined in (1.12).

**Proposition 4.1.** (Local well-posedness in \( T^d \)) For any \( s > \frac{d}{2} + 2 \), any initial datum \( U_0 \in H^s \) with \( \rho(U_0) \in Q_{\delta} \) for some \( \delta > 0 \), there exist \( T := T(\| U_0 \|_{s_0+2}, \delta) > 0 \) and a unique solution \( U \in C^0([0, T], H^s) \cap C^1([0, T], H^{s+2}) \) of (3.4) satisfying \( \rho(U) \in Q \), for any \( t \in [0, T] \). Moreover the solution depends continuously with respect to the initial datum in \( H^s \).

**Proposition 4.1** proves Theorem 1.2 and thus Theorem 1.1.

The first step is to prove the local well-posedness result of a linear inhomogeneous problem.

**Proposition 4.2.** (Linear local well-posedness) Let \( \Theta \geq r > 0 \) and \( U \) be a function in \( C^0([0, T], H^{s+2}) \cap C^1([0, T], H^{\infty}) \) satisfying

\[
\| U \|_{L^2_T H^{s+2}} + \| \partial_t U \|_{L^2_T H^{s}} \leq \Theta, \quad \| U \|_{L^2_T H^{\infty}} \leq r, \quad \rho(U(t)) \in Q, \quad \forall t \in [0, T].
\]

Let \( \sigma > 0 \) and \( t \mapsto R(t) \) be a function in \( C^0([0, T], H^r) \). Then there exists a unique solution \( V \in C^0([0, T], H^s) \cap C^1([0, T], H^{s+2}) \) of the linear inhomogeneous system

\[
\partial_t V = J \text{Op}^{BW}(A_2(U(t); x, \xi) + A_1(U(t); x, \xi)) \dot{V} + R(t), \quad V(0, x) = V_0(x) \in H^s
\]

satisfying, for some \( C_{\Theta} \) satisfying \( C_{\Theta} > 0 \) and \( C_r := C_{r, \sigma} > 0 \), the estimate

\[
\| V \|_{L^2_T H^s} \leq C_{r} e^{C_{\Theta} T} \| V_0 \|_\sigma + C_{\Theta} e^{C_{\Theta} T} T R \|_{L^2_T H^{\infty}}.
\]
The following two sections are devoted to the proof of Proposition 4.2. The key step is the construction of a modified energy which is controlled by the $H^\sigma$-norm, and whose time variation is bounded by the $H^\sigma$ norm of the solution, as done e.g. in [1] and [19] for linear systems. In order to construct such modified energy, the first step is to diagonalize the matrix $JA_2$ in (4.3).

4.1 Diagonalization at highest order

We diagonalize the matrix of symbols $JA_2(U;\xi)$. The eigenvalues of the matrix

$$J \begin{bmatrix} 1 + a_+(U; x) & a_-(U; x) \\ a_-(U; x) & 1 + a_+(U; x) \end{bmatrix}$$

(4.5)

with $a_\pm(U; x)$ defined in (3.7) are given by $\pm i\lambda(U; x)$ with

$$\lambda(U; x) := \sqrt{(1 + a_+(U; x))^2 - a_-(U; x)^2} = \sqrt{\frac{(m + \rho(U)) K(m + \rho(U)) - m C_K(m)}{m K(m)}}.$$  

(4.6)

These eigenvalues are purely imaginary because $\rho(U) \in \mathcal{Q}$ (see (1.12)) and (1.9), which guarantees that $\lambda(U; x)$ is real valued and fulfills

$$0 < \lambda_{\text{min}} := \sqrt{\frac{m_1 C_K}{m K(m)}} \leq \lambda(U; x) \leq \sqrt{\frac{m_2 C_K}{m K(m)}} =: \lambda_{\text{max}}.$$  

(4.7)

A matrix which diagonalizes (4.5) is

$$F := \begin{pmatrix} f(U; x) & g(U; x) \\ g(U; x) & f(U; x) \end{pmatrix}, \quad f := \frac{1 + a_+ + \lambda}{\sqrt{(1 + a_+ + \lambda)^2 - a_-^2}}, \quad g := \frac{-a_-}{\sqrt{(1 + a_+ + \lambda)^2 - a_-^2}}.$$  

(4.8)

Note that $F(U; x)$ is well defined because

$$(1 + a_+ + \lambda)^2 - a_-^2 = \left( \frac{K(m + \rho(U))}{K(m)} + \lambda \right) \left( \frac{m + \rho(U)}{m} + \lambda \right) > \frac{(m + \rho(U)) K(m + \rho(U))}{m K(m)} \geq \frac{m_1 C_K}{m K(m)}.$$  

(4.9)

by (1.12) and (1.9). The matrix $F(U; x)$ has $\det F(U; x) = f^2 - g^2 = 1$ and its inverse is

$$F(U; x)^{-1} := \begin{pmatrix} f(U; x) & -g(U; x) \\ -g(U; x) & f(U; x) \end{pmatrix}.$$  

(4.10)

We have that

$$F(U; x)^{-1} J \begin{bmatrix} 1 + a_+(U; x) & a_-(U; x) \\ a_-(U; x) & 1 + a_+(U; x) \end{bmatrix} F(U; x) = J\lambda(U; x).$$  

(4.11)

By (2.31) and (4.9) we deduce the following estimates: for any $N \in \mathbb{N}_0$, $s \geq 0$ and $\sigma > \frac{d}{2}$,

$$||a_\pm(U)||_\sigma, ||f(U)||_\sigma, ||g(U)||_\sigma \leq C(||U||_\sigma),$$

$$|\lambda(U; x)|\xi^2|2\xi,\sigma,N \leq C_N(||U||_\sigma), \quad |b(U) \cdot \xi|_{1,\sigma,N} \leq C_N(||U||_{\sigma+1}).$$  

(4.12)
For any $\varepsilon > 0$, consider the regularized matrix function
\[ A^\varepsilon(U; x, \xi) := (A_2(U; x, \xi) + A_1(U; x, \xi)) \chi(\varepsilon\lambda(U; x)|\xi|^2), \]  
where $\chi$ is the cut-off function in (4.13) and $\lambda(U; x)$ is the function defined in (4.6). In what follows we will denote by $\chi_\varepsilon := \chi(\varepsilon\lambda(U; x)|\xi|^2)$. Note that, by (2.31), (4.7), and by the fact that the function $y \mapsto (|\xi|^2|\partial_x^2\chi(y)|^2)$ is bounded together with its derivatives uniformly in $\varepsilon \in (0, 1)$, $\xi \in \mathbb{R}^d$ and $y \in [\lambda_{\min}, \lambda_{\max}]$, the symbol $\chi_\varepsilon$ satisfies, for any $N \in \mathbb{N}$, $\sigma > d/2$
\[ |\chi_\varepsilon|_{0, \sigma, N} \leq C(|U||_\sigma), \quad \text{uniformly in } \varepsilon. \]  
(4.14)

The diagonalization (4.11) has the following operatorial consequence.

**Lemma 4.3.** We have
\[ \text{Op}^{\text{BW}}(F^{-1}) \text{Op}^{\text{BW}}(A^\varepsilon) \text{Op}^{\text{BW}}(F) = \text{Op}^{\text{BW}} \left( (\sqrt{\text{Op}^{\text{BW}}(\mathbb{R}|(U)|^2 - b \cdot \xi)} \right) + \mathcal{F}(U) \]  
where $\mathcal{F}(U) := J_x(\xi) : \mathbb{H} \rightarrow \mathbb{H}^{\sigma}$, $\forall \sigma \geq 0$, satisfies, uniformly in $\varepsilon$,
\[ \|\mathcal{F}(U)W\|_\sigma \leq C\|U\|_{s_0+2}\|W\|_\sigma, \quad \forall W \in \mathbb{H}^{\sigma}. \]  
(4.16)

**Proof.** We have that
\[ \text{Op}^{\text{BW}}(F^{-1}) \text{Op}^{\text{BW}}(A_2\chi_\varepsilon) \text{Op}^{\text{BW}}(F) = \text{Op}^{\text{BW}} \left( \sqrt{\text{Op}^{\text{BW}}(\mathbb{R}|(U)|^2 - b \cdot \xi)} \right) + \mathcal{F}_1(U), \]
\[ \mathcal{F}_1(U) := \sum_{i=1}^{n} \left( (f_i^2 + g_i^2)(1 + a_\pm) + 2f_i g_i a_{\pm} \right) \chi_\varepsilon(U) = \text{Op}^{\text{BW}} \left( \lambda(U)|\xi|^2 \right) + \mathcal{F}_2(U), \]
where $\mathcal{F}_1, \mathcal{F}_2$ satisfy (4.10) by (4.29), (4.12), and (4.14) and since, by the definition of $f$ and $g$ in (4.5) and $\lambda$ in (4.6), we have $(f^2 + g^2)(1 + a_\pm) + 2f g a_{\pm} = \lambda$ and $(f^2 + g^2)a_{\pm} + 2f g(1 + a_\pm) = 0$. Moreover
\[ \text{Op}^{\text{BW}}(F^{-1}) \text{Op}^{\text{BW}}(A_1\chi_\varepsilon) \text{Op}^{\text{BW}}(F) = \text{Op}^{\text{BW}} \left( \sqrt{\text{Op}^{\text{BW}}(\mathbb{R}|(U)|^2 - b \cdot \xi)} \right) - \mathcal{F}_2(U), \]
where
\[ D_1 = \text{Op}^{\text{BW}}(f) \text{Op}^{\text{BW}}(\xi \chi_\varepsilon) \text{Op}^{\text{BW}}(f) - \text{Op}^{\text{BW}}(g) \text{Op}^{\text{BW}}(g) \text{Op}^{\text{BW}}(g), \]
\[ B_1 = \text{Op}^{\text{BW}}(f) \text{Op}^{\text{BW}}(\xi \chi_\varepsilon) \text{Op}^{\text{BW}}(f) - \text{Op}^{\text{BW}}(g) \text{Op}^{\text{BW}}(g) \text{Op}^{\text{BW}}(g). \]

Applying Theorem 2.5, 4.12, 4.14, using that $f^2 - g^2 = 1$ we obtain $D_1 = \text{Op}^{\text{BW}}(b \cdot \xi \chi_\varepsilon) + \mathcal{F}_1(U)$ and $B_1 = \mathcal{F}_2(U)$ with $\mathcal{F}_1, \mathcal{F}_2$ satisfying (4.16). \qed
4.2 Energy estimate for smoothed system

We first solve (4.3) in the case $L(t) = 0$ and $V_0 \in \dot{H}^\infty \equiv \cap_{\sigma \in \mathbb{R}} \dot{H}^\sigma$. Consider the regularized Cauchy problem

$$\partial_t V^\varepsilon = J \text{Op}^{\text{BW}}(A^\varepsilon(U(t); x, \xi)) V^\varepsilon, \quad V^\varepsilon(0) = V_0 \in \dot{H}^\infty,$$

where $A^\varepsilon(U; x, \xi)$ is defined in (4.3). As the operator $\text{Op}^{\text{BW}}(A^\varepsilon(U; x, \xi))$ is bounded for any $\varepsilon > 0$, and $U(t)$ satisfies (4.12), the differential equation (4.17) has a unique solution $V^\varepsilon(t)$ which belongs to $C^2([0, T], \dot{H}^\sigma)$ for any $\sigma \geq 0$. The important fact is that it admits the following $\varepsilon$-independent energy estimate.

**Proposition 4.4. (Energy estimate)** Let $U$ satisfy (4.2). For any $\sigma \geq 0$, there exist constants $C_r, C_\sigma > 0$ (depending also on $\sigma$), such that for any $\varepsilon > 0$, the unique solution $V^\varepsilon(t)$ of $\varepsilon$-independent energy estimate.

$$\|V^\varepsilon(t)\|^2_2 \leq C_r \|V_0\|^2_\sigma + C_\sigma \int_0^t \|V^\varepsilon(\tau)\|^2_\sigma d\tau, \quad \forall \tau \in [0, T].$$

(4.18)

As a consequence, there are constants $C_r, C_\sigma$ independent of $\varepsilon$, such that

$$\|V^\varepsilon(t)\|_\sigma \leq C_r e^{C_\sigma t} \|V_0\|_\sigma, \quad \forall \tau \in [0, T].$$

(4.19)

In order to prove Proposition 4.4, we define, for any $\sigma \geq 0$, the modified energy

$$\|V\|_{\sigma, U}^2 := \langle \text{Op}^{\text{BW}}(\lambda^\sigma(U; x)) \text{Op}^{\text{BW}}(F^{-1}(U; x)) V, \text{Op}^{\text{BW}}(F^{-1}(U; x)) V \rangle,$$

where we introduce the real scalar product

$$\langle V, W \rangle := 2 \text{Re} \int_{\mathbb{R}^d} v(x) \overline{w(x)} \, dx, \quad V = \left[ \begin{array}{c} v \\ \mathbf{\tau} \end{array} \right], \quad W = \left[ \begin{array}{c} w \\ \mathbf{\tau} \end{array} \right].$$

**Lemma 4.5.** Fix $\sigma > 0$, $r > 0$. There exists a constant $C_r > 0$ (depending also on $\sigma$) such that for any $U \in \dot{H}^\infty$ with $\|U\|_{\sigma_0} \leq r$ and $\rho(U) \in Q$ we have

$$C_r^{-1} \|V\|_r^2 - \|V\|_{-2}^2 \leq \|V\|_{\sigma, U}^2 \leq C_r \|V\|_\sigma^2, \quad \forall V \in \dot{H}^\sigma.$$  

(4.21)

**Proof.** We first prove the upper bound in (4.21). We note that, by (4.12), $\lambda^\sigma(U; x)|\xi|^{2\sigma} \in \Gamma_{\sigma_0}$ and $F^{-1}(U; x) \in \Gamma_{\sigma_0} \otimes \mathcal{M}_2(\mathbb{C})$ and, by Theorem 2.1 and (4.12) we have

$$\|V\|_{\sigma, U}^2 \leq \|\text{Op}^{\text{BW}}(\lambda^\sigma(U; x)|\xi|^{2\sigma}) \text{Op}^{\text{BW}}(F^{-1}(U; x)) V\|_{-\sigma} \|\text{Op}^{\text{BW}}(F^{-1}(U; x)) V\|_\sigma \leq C_r \|V\|_\sigma^2.$$

In order to prove the lower bound, we fix $\delta \in (0, 1)$ such that $s_0 - \delta > \frac{d}{2}$ and, due to (2.7), we have $\lambda^\frac{d}{2} \in \Gamma_{s_0 - \delta}$. So, applying Theorem 2.3 and (4.12) with $s_0 - \delta$ instead of $s_0$ and with $\rho = \delta$, we have

$$\text{Op}^{\text{BW}}(\lambda^{\frac{d}{2}}) \text{Op}^{\text{BW}}(F) \text{Op}^{\text{BW}}(\lambda^{\frac{d}{2}}) \text{Op}^{\text{BW}}(F^{-1}) = I + F^{-\delta}(U),$$

where for any $\sigma' \in \mathbb{R}$ there exists a constant $C_{r, \sigma'} > 0$ such that

$$\|F^{-\delta}(U)f\|_{\sigma'} \leq C_{r, \sigma'} \|f\|_{\sigma' - \delta}, \quad \forall f \in \dot{H}^{\sigma' - \delta}.$$  

(4.23)

Again, applying Theorem 2.3 with $s_0 - \delta$ instead of $s_0$ and with $\rho = \delta$, we have also

$$\text{Op}^{\text{BW}}(\lambda^{\frac{d}{2}}) \text{Op}^{\text{BW}}(|\xi|^{2\sigma}) \text{Op}^{\text{BW}}(\lambda^{\frac{d}{2}}) = \text{Op}^{\text{BW}}(\lambda^{\sigma}|\xi|^{2\sigma}) + F^{2\sigma - \delta}(U),$$

(4.24)
where for any $\sigma' \in \mathbb{R}$ there exists a constant $C_{r, \sigma'} > 0$ such that

$$
\|F^{2\sigma - \delta}(U) f\|_{\sigma' - 2\sigma + \delta} \leq C_{r, \sigma'} \|f\|_{\sigma'}, \quad \forall f \in \mathcal{H}^{\sigma'}.
$$

(4.25)

By (4.22)–(4.25), Theorem 2.4 and 4.12 and using also that $\text{Op}^{BW}(\lambda \tilde{x})$ is symmetric with respect to $\langle \cdot, \cdot \rangle$, we have

$$
\|V\|_{\sigma}^2 \leq 2 \|\text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F) \text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F^{-1}) V\|_{\sigma}^2 + 2 \|\mathcal{F}^{-\delta}(U)V\|_{\sigma}^2.
$$

$$
C_r \left( \|\text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F^{-1}) V\|_{\sigma}^2 + \|V\|_{\sigma - \delta}^2 \right).
$$

$$
C_r \left( \|\text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F^{-1}) V\|_{\sigma}^2 + \|V\|_{\sigma - \delta}^2 \right).
$$

By (4.22)–(4.25), Theorem 2.4 and 4.12 and using also that $\text{Op}^{BW}(\lambda \tilde{x})$ is symmetric with respect to $\langle \cdot, \cdot \rangle$, we have

$$
\|V\|_{\sigma}^2 \leq 2 \|\text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F) \text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F^{-1}) V\|_{\sigma}^2 + 2 \|\mathcal{F}^{-\delta}(U)V\|_{\sigma}^2.
$$

(4.26)

$$
\|V\|_{\sigma}^2 \leq 2 \|\text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F^{-1}) V\|_{\sigma}^2 + 2 \|\mathcal{F}^{-\delta}(U)V\|_{\sigma}^2.
$$

(4.27)

$$
\|V\|_{\sigma}^2 \leq 2 \|\text{Op}^{BW}(\lambda \tilde{x}) \text{Op}^{BW}(F^{-1}) V\|_{\sigma}^2 + 2 \|\mathcal{F}^{-\delta}(U)V\|_{\sigma}^2.
$$

(4.28)

Now we use (4.20) and the asymmetric Young inequality to get, for any $\epsilon > 0$,

$$
\|V\|_{\sigma - \frac{\epsilon}{2}} \leq \|V\|_{\sigma} \leq \epsilon \frac{2^{(\sigma + 2) - \frac{\epsilon}{2}}}{\epsilon} \|V\|_{\sigma - \delta}^2 + \epsilon \frac{2^{(\sigma + 2)}}{\epsilon} \|V\|_{\sigma}^2.
$$

we choose $\epsilon$ so small so that $\epsilon \frac{2^{(\sigma + 2) - \frac{\epsilon}{2}}}{\epsilon} C_r = \frac{1}{2}$ and we get $\|V\|_{\sigma}^2 \leq 2C_r (\|V\|_{\sigma - \delta}^2 + \|V\|_{\sigma}^2)$. This proves the lower bound in (4.21).

**Proof of Proposition 4.4** The time derivative of the modified energy (4.20) along a solution $V^\varepsilon(t)$ of (4.17) is

$$
\frac{d}{dt} \|V^\varepsilon\|_{\sigma, U(t)}^2 = \langle \text{Op}^{BW}(\partial_t(\lambda^\sigma)) \|\xi\|_{2\sigma} \text{Op}^{BW}(F^{-1}) V^\varepsilon, \text{Op}^{BW}(F^{-1}) V^\varepsilon \rangle + 2 \|\text{Op}^{BW}(\lambda^\sigma \|\xi\|_{2\sigma}) \text{Op}^{BW}(\partial_t F^{-1}) V^\varepsilon, \text{Op}^{BW}(F^{-1}) V^\varepsilon \rangle + 2 \|\text{Op}^{BW}(\lambda^\sigma \|\xi\|_{2\sigma}) \text{Op}^{BW}(F^{-1}) \partial_t V^\varepsilon, \text{Op}^{BW}(F^{-1}) V^\varepsilon \rangle.
$$

(4.26)

$$
\text{By Theorem 2.4 and using that } \forall \sigma \geq 0, N \in \mathbb{N}, \quad \|\partial_t \lambda^\sigma(U)\|_{2\sigma, s_0, N} \leq C_N(\|U\|_{s_0}, \|\partial U\|_{s_0}).
$$

and the assumption (4.2), there exists a constant $C_\Theta > 0$ (depending also on $\sigma$) such that

$$
\epsilon \frac{2^{(\sigma + 2) - \frac{\epsilon}{2}}}{\epsilon} C_r = \frac{1}{2}.
$$

(4.29)

We now estimate (4.28). By Theorem 2.5 with $\sigma = 2$ and (4.2) we have

$$
\text{Op}^{BW}(F) \text{Op}^{BW}(F^{-1}) = 1 + \mathcal{F}^{-2}(U), \quad \text{Op}^{BW}(F^{-1}) \text{Op}^{BW}(F) = 1 + \mathcal{F}^{-2}(U).
$$

(4.30)

where $\mathcal{F}^{-2}(U)$ are bounded operators from $\mathcal{H}^{\sigma'}$ to $\mathcal{H}^{\sigma' + 2}$, $\forall \sigma' \in \mathbb{R}$, satisfying

$$
\|\mathcal{F}^{-2}(U) W\|_{\sigma' + 2} \leq C_{\Theta, \sigma'} \|W\|_{\sigma'}, \quad \forall W \in \mathcal{H}^{\sigma'}.
$$

(4.31)

Thus, denoting $\tilde{V}^\varepsilon := \text{Op}^{BW}(F^{-1}) V^\varepsilon$, by (4.30), we have

$$
\text{Op}^{BW}(F) \tilde{V}^\varepsilon = V^\varepsilon + \mathcal{F}^{-2}(U)V^\varepsilon.
$$

(4.32)
Recalling (4.17) we have

\[ (4.28) = 2\langle \text{Op}^{BW}(\lambda^{|\xi|^2\sigma}), \text{Op}^{BW}(F^{-1}) \rangle \text{Op}^{BW}(A^\varepsilon) V^{\varepsilon}, \tilde{V}^{\varepsilon} \]

and by Lemma 4.3 we get

\[ (4.30) = \langle J[\text{Op}^{BW}(\lambda^{|\xi|^2\sigma}), \text{Op}^{BW}(\sqrt{\pi K(m)} \lambda |\xi|^2 \chi)] \tilde{V}^{\varepsilon}, \tilde{V}^{\varepsilon} \rangle \]

where in line (4.30) the operator \( F(U) \) is the bounded remainder of Lemma 4.3. We estimate each contribution. First we consider line (4.33). Using Theorem 2.5 with \( \varrho = 2 \), the principal symbol of the commutator is

\[ i^{-1} \{ \lambda^{|\xi|^2\sigma}, \sqrt{\pi K(m)} \lambda |\xi|^2 \chi (\varepsilon \lambda |\xi|^2) \} = 0, \]

and, using (4.14), (4.12) and assumption (4.2), we get

\[ (4.33) \leq C_\varrho \| \tilde{V}^{\varepsilon} \|^2_\sigma. \]

Similarly, using Theorem 2.5 with \( \varrho = 1 \), Theorem 2.4 (4.12) and estimates (4.31) and (4.10), we obtain

\[ (4.34) \leq + (4.35) \leq C_\varrho \| V^{\varepsilon} \|^2_\sigma. \]

In conclusion, by (4.29), (4.37), (4.38), we deduce the bound \( \frac{d}{dt} \| V^{\varepsilon}(t) \|^2_\sigma, U(t) \leq C_\varrho \| V^{\varepsilon}(t) \|^2_\sigma \), that gives, for any \( t \in [0, T] \)

\[ \| V^{\varepsilon}(t) \|^2_\sigma, U(t) \leq \| V^{\varepsilon}(0) \|^2_\sigma, U(0) + C_\varrho \int_0^t \| V^{\varepsilon}(\tau) \|^2_\sigma d\tau \]

\[ \leq C_\varrho \| V^{\varepsilon}(0) \|^2_\sigma + C_\varrho \int_0^t \| V^{\varepsilon}(\tau) \|^2_\sigma d\tau. \]

Since \( V^{\varepsilon}(t) \) solves (4.17), by Theorem 2.4 (4.12), (4.14) there exists a constant \( C_\varrho \geq 0 \) (independent on \( \varepsilon \)) such that \( \| \partial_t V^{\varepsilon}(t) \|^2_2 \leq C_\varrho \| V^{\varepsilon}(t) \|^2_\sigma \leq C_\varrho \| V^{\varepsilon}(t) \|^2_\sigma \), and therefore

\[ \| V^{\varepsilon}(t) \|^2_2 \leq \| V^{\varepsilon}(0) \|^2_2 + C_\varrho \int_0^t \| V^{\varepsilon}(\tau) \|^2_\sigma d\tau, \quad \forall t \in [0, T]. \]

We finally deduce (4.18) by (4.39), the lower bound in (4.21) and (4.40). The estimate (4.19) follows by Gronwall inequality.

**Proof of Proposition 4.2** By Proposition 4.4, Ascoli-Arzela theorem ensures that, for any \( \sigma \geq 0 \), \( V^{\varepsilon} \) converges up to subsequence to a limit \( V \) in \( C^1([0, T], \dot{H}^\sigma) \), as \( \varepsilon \to 0 \) that solves (1.3) with \( R(t) = 0 \), initial datum \( V_0 \in C^\infty \), and satisfies \( \| V(t) \|^2_\sigma \leq C_{\text{ref}} C_{\text{ref}} \| V_0 \|^2_\sigma \), for any \( \sigma \geq 0 \).

The case \( V_0 \in \dot{H}^\sigma \) follows by a classical approximation argument with smooth initial data. This
shows that the propagator of $J\text{Op}^{BW}(A_2(U(t);x,\xi) + A_1(U(t);x,\xi))$ is, for any $\sigma \geq 0$, a well defined bounded linear operator

$$
\Phi(t) : \tilde{H}^s \mapsto \tilde{H}^s, \quad V_0 \mapsto \Phi(t)V_0 := V(t), \quad \forall t \in [0,T], \quad \text{satisfying} \quad \|\Phi(t)V_0\|_\sigma \leq C_r e^{C_\Theta t}\|V_0\|_\sigma.
$$

In the inhomogeneous case $R \neq 0$, the solutions of (4.3) is given by the Duhamel formula $V(t) = \Phi(t)V_0 + \Phi(t) \int_0^t \Phi^{-1}(\tau)R(\tau) d\tau$, and the estimate (4.1) follows.

### 4.3 Iterative scheme

In order to prove that the nonlinear system (3.4) has a local in time solution we consider the sequence of linear Cauchy problems

$$
P_1 := \begin{cases}
\partial_t U_1 = -J\sqrt{nK(m)} \Delta U_1 \\
U_1(0) = U_0,
\end{cases}
\quad P_n := \begin{cases}
\partial_t U_n = J\text{Op}^{BW}(A(U_{n-1};x,\xi)) U_n + R(U_{n-1}) \\
U_n(0) = U_0,
\end{cases}
$$

for $n \geq 2$, where $A := A_2 + A_1$, cfr. (3.6), (3.8). The strategy is to prove that the sequence of solutions $U_n$ of the approximated problems $P_n$ converges to a solution $U$ of system (3.4).

**Lemma 4.6.** Let $U_0 \in \tilde{H}^s$, $s > 2 + \frac{4}{\sigma}$, such that $\rho(U_0) \in Q_\delta$ for some $\delta > 0$ (recall (3.3) and (4.1)) and define $r := 2\|U_0\|_{s_0}$. Then there exists a time $T := T(\|U_0\|_{s_0} + \delta, \delta) > 0$ such that, for any $n \in \mathbb{N}$:

(S0)$_n$: The problem $P_n$ admits a unique solution $U_n \in C^0([0,T], \tilde{H}^s) \cap C^1([0,T], \tilde{H}^{s-2})$.

(S1)$_n$: For any $t \in [0,T]$, $\rho(U_n(t))$ belongs to $Q_\frac{\delta}{2}$.

(S2)$_n$: There exists a constant $C_r \geq 1$ (depending also on $s$) such that, defining $\Theta := 4C_r\|U_0\|_{s_0+2}$ and $M := 4C_r\|U_0\|_{s}$, for any $1 \leq m \leq n$ one has

$$
\|U_m\|_{L_\infty^r \tilde{H}^s} \leq r; \quad \|U_m\|_{L_\infty^r \tilde{H}^{s+2}} \leq \Theta; \quad \|\partial_t U_m\|_{L_\infty^r \tilde{H}^{s}} \leq C_r \Theta; \quad \|U_m\|_{L_\infty^r \tilde{H}^{s-2}} \leq C_r M.
$$

(S3)$_n$: For $1 \leq m \leq n$ one has

$$
\|U_1\|_{L_\infty^r \tilde{H}^{s}} = r/2, \quad \|U_m - U_{m-1}\|_{L_\infty^r \tilde{H}^{s}} \leq 2^{-m-r}, \quad m \geq 2.
$$

**Proof.** We prove the statement by induction on $n \in \mathbb{N}$. Given $r > 0$, we define

$$
C_r := \max\{1, C_{r,s_0}, C_{r,s_0+2}, C_{r,s}, 2C(r)\},
$$

where $C_{r,\sigma}$ is the constant in Proposition 4.2 (where we stress that it depends also on $\sigma$) and $C(\cdot)$ is the function in (3.9) and (3.11). In the following we shall denote by $C_{\Theta}$ all the constants depending on $\Theta$, which can vary from line to line.

**Proof of (S0)$_1$:** The problem $P_1$ admits a unique global solution which preserves Sobolev norms.

**Proof of (S1)$_1$:** We have $\rho(U_0) \in Q_\delta$. In addition

$$
\|\rho(U_1(t) - U_0\|_{L_\infty^r \tilde{H}^{s}} \lessapprox \|U_1(t) - U_0\|_{s_0} \lesssim T\|U_0\|_{s_0+2} \leq \delta/2
$$

for $T := T(\|U_0\|_{s_0+2}, \delta) > 0$ sufficiently small, which implies $\rho(U_1(t)) \in Q_{\frac{\delta}{2}}$, for any $t \in [0,T]$. 

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Proof of (S2)_1 and (S3)_1: The flow of \(\mathcal{P}_1\) is an isometry and \(M \geq \|U_0\|_{s_0}, \Theta \geq \|U_0\|_{s_0+2}\).

Suppose that \((S0)_{n-1}-(S3)_{n-1}\) hold true. We prove \((S0)_n-(S3)_n\).

Proof of (S0)_n: We apply Proposition 4.2 with \(s = s, U \rightarrow U_{n-1}\) and \(R(t) := R(U_{n-1}(t))\). By (S1)_{n-1} and (S2)_{n-1}, the function \(U_{n-1}\) satisfies assumption 4.2 with \(\Theta \rightarrow (1 + C_r)\Theta\). In addition \(R(U_{n-1}(t))\) belongs to \(C^0([0, T], \hat{H}^s)\) thanks to (3.12) and \(U_{n-1} \in C^0([0, T]; \hat{H}^s)\). Thus Proposition 4.2 with \(s = s_0\) implies \((S0)_n\). In particular \(U_n\) satisfies the estimate (4.4).

Proof of (S2)_n: We first prove (4.42). The estimate (4.4) with \(s = s_0 + 2\), the bound (3.11) of \(R(U_{n-1}(t))\) and (4.42) at the step \(n - 1\), imply

\[
\|U_n\|_{L_T^\infty \hat{H}^{s_0+2}} \leq C_{e} e^{C_{e} T} \|U_0\|_{s_0+2} + T C_{e} e^{C_{e} T} \Theta.
\]

As \(\Theta = 4C_{r} \|U_0\|_{s_0+2}\), we take \(T > 0\) small such that

\[
C_{e} T \leq 1, \quad T C_{e} e^{C_{e} T} \leq 1/4.
\]

which, by (4.44), gives \(\|U_n\|_{L_T^\infty \hat{H}^{s_0+2}} \leq \Theta\). This proves the first estimate of (4.42). Regarding the control of \(\partial_t U_n\), we use the equation \(\mathcal{P}_n\), the second estimate in (3.11) and (3.9) with \(s = s_0\) to obtain

\[
\|\partial_t U_n(t)\|_{s_0} \leq C (\|U_{n-1}(t)\|_{s_0}) \|U_n(t)\|_{s_0+2} + C (\|U_{n-1}(t)\|_{s_0}) \|U_{n-1}(t)\|_{s_0+2} \leq C_r \Theta
\]

which proves the second estimate of (4.42).

Next we prove \((4.43)\) with \(s = s, \\Theta\) we have

\[
\|U_n\|_{L_T^\infty \hat{H}^s} \leq C_{e} e^{C_{e} T} \|U_0\|_{s} + T C_{e} e^{C_{e} T} M \leq M
\]

for \(M = 4C_{r} \|U_0\|_{s}\) and since \(T > 0\) is chosen as in (4.45). The estimate for \(\|\partial_t U_n\|_{s_0+2}\) is similar to (4.46), and we omit it. Estimate (4.44) is a consequence of (S3)_n, which we prove below.

Proof of (S1)_n: We use estimate (4.40) to get

\[
\|\partial_t U_n(t)\|_{s_0} \leq C \|U_n(t)\|_{s} - \|U_0\|_{s} \leq C \int_{0}^{T} \|\partial_t U_n(t)\|_{s_0} dt \leq C_r T \Theta \leq \delta/2
\]

provided that \(T < \delta/(2C_r \Theta)\). This shows that \(\rho(U_n(t)) \in \mathcal{Q}_{\gamma}^+\).

Proof of (S3)_n: Define \(V_n := U_n - U_{n-1}\) if \(n \geq 2\) and \(V_1 := \hat{U}_1\). Note that \(V_n, n \geq 2\), solves

\[
\partial_t V_n = JOp^{BW}(A(U_{n-1})) V_n + f_n, \quad V_n(0) = 0,
\]

where \(A := A_2 + A_1\) and

\[
f_n := JOp^{BW}(A(U_{n-1}) - A(U_{n-2})) U_{n-1} - R(U_{n-1}) - R(U_{n-2}), \text{ for } n > 2,
\]

\[
f_2 := JOp^{BW}(A(U_1) - \sqrt{\text{mK}(\text{m})} |\xi|^2) U_1 + R(U_1).
\]

Applying estimates (3.13), (3.10), (3.11) and (4.42) we obtain, for \(n \geq 2\),

\[
\|f_n\|_{s_0} \leq C_{e} \|V_n-1\|_{s_0}, \quad \forall t \in [0, T].
\]

We apply Proposition 4.2 to (4.47) with \(s = s_0\). Thus by (4.3) and (4.48) we get

\[
\|V_n\|_{L_T^\infty \hat{H}^{s_0}} \leq C_{e} e^{C_{e} T} \|f_n\|_{L_T^\infty \hat{H}^{s_0}} \leq C_{e} e^{C_{e} T} \|V_n-1\|_{L_T^\infty \hat{H}^{s_0}} \leq \frac{1}{2} \|V_n-1\|_{L_T^\infty \hat{H}^{s_0}}
\]

provided \(C_{e} e^{C_{e} T} \leq \frac{1}{2}\). The proof of Lemma 4.0 is complete. \(\square\)
Corollary 4.7. With the same assumptions of Lemma 4.6, for any $s_0 + 2 \leq s' < s$:

(i) $(U_n)_{n \geq 1}$ is a Cauchy sequence in $C^0([0,T], \mathbf{H}^{s'}) \cap C^1([0,T], \mathbf{H}^{s'-2})$ given by Lemma 4.6. It converges to the unique solution $U(t)$ of (3.3) with initial datum $U_0$, $U(t)$ is in $C^0([0,T], \mathbf{H}^{s'}) \cap C^1([0,T], \mathbf{H}^{s'-2})$. Moreover $\rho(U(t)) \in Q$, $\forall t \in [0,T]$.

(ii) For any $t \in [0,T]$, $U(t) \in \tilde{\mathbf{H}}^s$ and $\|U(t)\|_s \leq 4C_r \|U_0\|_s$ where $C_r$ is the constant of (S2)$_n$.

Proof. (i) If $s' = s_0$ it is the content of (S3)$_n$. For $s' \in (s_0, s)$, we use interpolation estimate (2.3), (4.33) and (S3)$_n$ to get, for $n \geq 2$,

$$
\|U_n - U_{n-1}\|_{L^2_T \mathbf{H}^{s'}} \leq \|U_n - U_{n-1}\|_{L^2_T \mathbf{H}^{s_0}}^\theta \|U_n - U_{n-1}\|_{L^2_T \mathbf{H}^{s_0}}^{1-\theta} \leq 2^{-n\theta} C_M,
$$

where $\theta \in (0, 1)$ is chosen so that $s' = \theta s_0 + (1 - \theta) s$. Thus $(U_n)_{n \geq 1}$ is a Cauchy sequence in $C^0([0,T], \mathbf{H}^{s'})$; we denote by $U(t) \in C^0([0,T], \mathbf{H}^{s'})$ its limit. Similarly using that $\partial_t U_n$ solves $\mathcal{T}_n$, one proves that $\partial_t U_n$ is a Cauchy sequence in $C^0([0,T], \mathbf{H}^{s'-2})$ that converges to $\partial_t U$ in $C^0([0,T], \mathbf{H}^{s'-2})$. In order to prove that $U(t)$ solves (3.3), it is enough to show that

$$
\mathcal{R}(U, U_{n-1}, U_n) := \mathcal{J}_{\text{Op}}^{\text{BW}}(A(U_{n-1})) U_n - \mathcal{J}_{\text{Op}}^{\text{BW}}(A(U)) U + R(U_{n-1}) - R(U)
$$

converges to 0 in $L^2_T \mathbf{H}^{s'-2}$. This holds true because by estimates (2.13), (3.10), (3.12), (S2)$_n$, and the fact that $U(t) \in C^0([0,T], \mathbf{H}^{s'})$, we have

$$
\|\mathcal{R}(U, U_{n-1}, U_n)\|_{L^2_T \mathbf{H}^{s'-2}} \leq C_M \left( \|U - U_{n-1}\|_{L^2_T \mathbf{H}'} + \|U - U_{n-1}\|_{L^2_T \mathbf{H}^{s_0}} + \|U - U_n\|_{L^2_T \mathbf{H}'} \right)
$$

which converges to 0 as $n \to \infty$.

Let us now prove the uniqueness. Suppose that $V_1, V_2 \in C^0([0,T], \mathbf{H}^{s'}) \cap C^1([0,T], \mathbf{H}^{s'-2})$ are solutions of (3.3) with initial datum $U_0$. Then $W := V_1 - V_2$ solves

$$
\partial_t W = \mathcal{J}_{\text{Op}}^{\text{BW}}(A(V_1)) W + R(t), \quad W(0) = 0,
$$

where $R(t) := \mathcal{J}_{\text{Op}}^{\text{BW}}(A(V_1) - A(V_2)) V_2 + R(V_1) - R(V_2)$. Applying Proposition 4.2 with $\sigma = s_0$ and $\Theta, r$ defined by

$$
\Theta := \max_{j=1,2} \left( \|V_j\|_{L^2_T \mathbf{H}^{s_0}} + \|\partial_t V_j\|_{L^2_T \mathbf{H}^{s_0}} \right), \quad r := \max_{j=1,2} \|V_j\|_{L^2_T \mathbf{H}^{s_0}}
$$

together with estimates (3.13) and (3.10) we have, for any $t \in [0,T]$,

$$
\|W\|_{L^2_T \mathbf{H}^{s_0}} \leq C e^{C_{\text{Op}} t} \|R\|_{L^2_T \mathbf{H}^{s_0}} \leq C e^{C_{\text{Op}} t} \|W\|_{L^2_T \mathbf{H}^{s_0}}.
$$

Therefore, provided $t$ is so small that $C e^{C_{\text{Op}} t} < 1$, we get $V_1(t) = V_2(t)$ $\forall t \in [0,t]$. As (3.4) is autonomous, actually one has $V_1(t) = V_2(t)$ for all $t \in [0,T]$. This proves the uniqueness.

Finally, as $\rho(U_n(t)) \in Q$ and $U_n(t) \to U(t)$ in $\mathbf{H}^{s_0}$, then $\rho(U(t)) \in Q$.

(ii) Since $\|U_n\|_s \leq 4C_r \|U_0\|_s$ and $U_n(t) \to U(t)$ in $\mathbf{H}^s$ then $\|U(t)\|_s \leq 4C_r \|U_0\|_s$. \hfill $\square$

Let $\Pi_N U := \sum_{1 \leq j \leq N} U_{j,N} e^{i\omega_j x}$, $\sum_{1 \leq j \leq N} \mu_j e^{-i\omega_j x}$. We need below the following technical lemma.

Lemma 4.8. Let $U_0 \in \tilde{\mathbf{H}}^s$, $s > 2 + \frac{d}{2}$, with $\rho(U_0) \in Q$ for some $\delta > 0$. Then there exists a time $\tilde{T} := \tilde{T}(\|U_0\|_{s+2,2}, \delta) > 0$ and $N_0 > 0$ such that for any $N > N_0$.
(i) system \((3.3)\) with initial datum \(\Pi NU_0\) has a unique solution \(U_N \in C^0([0,T], \tilde{H}^{s+2})\).

(ii) Let \(U\) be the unique solution of \((3.3)\) with initial datum \(U_0\) defined in the time interval \([0,T]\) (which exists by Corollary \((4.7)\)). Then there is \(\hat{T} < \min(T,T)\), depending on \(\|U_0\|_s\), independent of \(N\), such that

\[
\|U - U_N\|_{L^2_T H^s} \leq C\|U_0\|_s \left(\|U_0 - \Pi NU_0\|_s + N^{s_0+2-s}\right).
\]

This proves item (i). In the following let \(\hat{T} \leq \min(\hat{T}, T)\).

Let us prove (ii). Let \(\Theta := \|U\|_{L^\infty_T H^s} + \|\partial_t U\|_{L^\infty_T H^{s+2}}\) and \(r := \|U\|_{L^\infty_T H^{s+2}}\). The function \(W_N(t) := U(t) - U_N(t)\) satisfies \(\|W_N(t)\|_s \leq \|U(t)\|_s + \|U_N(t)\|_s \leq C\|U_0\|_s\), \(\forall t \in [0,\hat{T}]\), by Corollary \((4.7)\) (ii). Moreover, \(W_N\) solves

\[
\partial_t W_N = J\text{Op}^\text{BW}(A(U)) W_N + R(t), \quad W_N(0) = U_0 - \Pi NU_0
\]

where \(R(t) := J\text{Op}^\text{BW}(A(U) - A(U_N)) U_N + R(U) - R(U_N)\). Applying Proposition \((4.2)\) with \(\sigma = s_0\) and estimates \((3.10)\), \((3.13)\), \((4.50)\) one obtains

\[
\|W_N\|_{L^\infty_T H^s} \leq C_r e^{C_\Theta T} \|U_0 - \Pi NU_0\|_{s_0} + \tilde{T}_r C e^{C_\Theta T} C(\|U_0\|_{s_0+2}) \|W_N\|_{L^\infty_T H^{s+2}},
\]

which, provided \(\hat{T}\) is so small that \(\tilde{T}_r C e^{C_\Theta T} C(\|U_0\|_{s_0+2}) \leq \frac{1}{2}\) (eventually shrinking it), gives

\[
\|W_N\|_{L^\infty_T H^{s+2}} \leq C_r \|U_0 - \Pi NU_0\|_{s_0} \leq C_r N^{s_0-\delta} \|U_0\|_s.
\]

Similarly one estimates \(\|W_N(t)\|_{H^r}\), getting

\[
\|W_N\|_{L^\infty_T H^s} \leq C_r e^{C_\Theta T} \|U_0 - \Pi NU_0\|_s + C e^{C_\Theta T} \tilde{T}_r C(\|U_0\|_s) \left(\|W_N\|_{L^\infty_T H^s} \|U_N\|_{L^\infty_T H^{s+2}} + \|W_N\|_{L^\infty_T H^r}\right)
\]

\[
\leq C_r e^{C_\Theta T} \|U_0 - \Pi NU_0\|_s + C \Theta e^{C_\Theta T} \tilde{T}_r C(\|U_0\|_s) \left(N^{s_0-\delta} + \|W_N\|_{L^\infty_T H^r}\right)
\]

from which \((4.49)\) follows provided \(\hat{T}\) (depending on \(\|U_0\|_s\)) is sufficiently small. \(\square\)

**Proof of Proposition \((4.1)\)** Given an initial datum \(U_0 \in \tilde{H}^s\) with \(\rho(U_0) \in \mathcal{Q}\), choose \(\delta > 0\) so small that \(\rho(U_0) \in \mathcal{Q}_{\delta}\). Then Corollary \((4.7)\) gives us a time \(T = T(\|U_0\|_{s_0+2}, \delta) > 0\) and a unique solution \(U \in C^0([0,T], \tilde{H}^{s'}) \cap C^1([0,T], \tilde{H}^{s'-2})\), \(\forall s_0 + 2 \leq s' < s\), of \((3.3)\) with initial datum
Consider now the term $\varepsilon > 0$ or any $\delta > 0$ well defined in sufficiently large. Denote by $\Omega^t$ and pick $\delta > 0$ and $\varepsilon = \varepsilon(t)$ such that $\Omega^t(\Omega_0) = U(t)$.

We first provide some preliminary technical results. Continuity of $t \mapsto U(t)$: We show that $U \in C^0([0, T], H^s)$ and $\lim_{\tau \to 0} \|U(t, \cdot) - \tilde{U}(t, \cdot)\|_{H^s} = 0$. Now take an open neighborhood $U \Subset \tilde{H}^s$ of $U_0$ such that $\forall V \in U$ one has $\rho(V) \in Q_\delta$ and $\|V\|_s \leq 2 \|U_0\|_s$. Then there exists $\tilde{T} \in (0, T]$ such that the flow map of $(3.4)$, $\Omega^t : U \to \tilde{H}^s \cap \{U \in \tilde{H}^s : \rho(U) \in Q_\delta\}$, $U_0 \mapsto \Omega^t(U_0) := U(t)$, is well defined for any $t \in [0, \tilde{T}]$, it satisfies the group property

$$\Omega^{t+\tau} = \Omega^t \circ \Omega^\tau, \quad \forall t, \tau, t + \tau \in [0, \tilde{T}],$$

and $\|\Omega^t(U_0)\|_s \leq C(\|U_0\|_s)$ for all $U_0 \in U$, $t \in [0, \tilde{T}]$. For simplicity of notation in the sequel we denote by $T$ a time, independent of $N$, smaller than $\tilde{T}$.

Continuity of the flow map: We shall follow the method by [12, 7]. Let $U_0^\varepsilon \to U_0 \in \tilde{H}^s$ and pick $\delta > 0$ such that $\rho(U_0^\varepsilon), \rho(U_0), \rho(\Pi_N U_0^\varepsilon), \rho(\Pi_N U_0) \in Q_\delta$, for any $n \geq n_0, N \geq N_0$. Denote by $U_n^\varepsilon, U \in C^0([0, T], \tilde{H}^s)$ the solutions of $(3.4)$ with initial data $U_0^\varepsilon$, respectively $U_0$, and $U_N(t) := \Omega^t(\Pi_N U_0), U_n^\varepsilon(t) := \Omega^t(\Pi_N U_0^\varepsilon)$. Note that these solutions are well defined in $\tilde{H}^s$ up to a common time $T^* \in (0, T]$, depending on $\|U_0\|_s$, thanks to Lemma 4.8. By triangular inequality we have, by (4.53), for any $n \geq n_0, N \geq N_0$,

$$\|U^\varepsilon - U\|_{L^2_T \tilde{H}^s} \leq \|U^\varepsilon - U_0\|_{L^2_T \tilde{H}^s} + \|U_n - U_N\|_{L^2_T \tilde{H}^s} + \|U_N - U\|_{L^2_T \tilde{H}^s} \leq C(\|U_0\|_s)(\|\Pi_N U_0\|_s + N^{n_0 + 2 - s}) \leq C(\|U_0\|_s)\|U_0\|_s + \|U_0\|_s \leq C(\|U_0\|_s)\|U_0\|_s + \|U_0\|_s + \|U_0\|_s + \|U_0\|_s.$$

For any $\varepsilon > 0$, since $s > s_0 + 2$, there exists $N_\varepsilon \in N$ (independent of $n$) such that

$$C(\|U_0\|_s)(\|\Pi_N U_0\|_s + N^{n_0 + 2 - s}) \leq \varepsilon/2.$$

Consider now the term $\|U_n^\varepsilon - U_n\|_{L^2_T \tilde{H}^s}$. As $\Pi_N U_0, \Pi_N U_0^\varepsilon \in \tilde{C}^\infty$, the solutions $U_n(t), U_n^\varepsilon(t)$ belong actually to $\tilde{H}^{s+2}$. By interpolation and by item (ii) of Corollary 4.7, applied with $s \to s + 2$ one has, for $s + 2 = \theta s_0 + (1 - \theta)(s + 2),$

$$\|U_n^\varepsilon - U_n\|_{L^2_T \tilde{H}^s} \leq C(\|\Pi_N U_0\|_s + 2, \|\Pi_N U_0^\varepsilon\|_s + 2)\|U_n^\varepsilon - U_n\|_{L^2_T \tilde{H}^s} \leq C(\|\Pi_N U_0\|_s, \|U_n^\varepsilon - U_n\|_{L^2_T \tilde{H}^s}.$$

Arguing in the same way of the proof of (4.52) one obtains

$$\|U_n^\varepsilon - U_n\|_{L^2_T \tilde{H}^s} \leq C(\|\Pi_N U_0\|_s + 2) \|\Pi_N (U_n^\varepsilon - U_0)\|_s.$$

By (4.53)-(4.57), we have $\limsup_{\varepsilon \to 0} \|U_n^\varepsilon - U\|_{L^2_T \tilde{H}^s} \leq \varepsilon, \forall \varepsilon \in (0, 1).$

A Bony-Weyl calculus in periodic Hölder spaces

In this Appendix we develop in a self-contained manner paradifferential calculus for space periodic symbols $a(x, \xi)$ which belong to the Banach scale of Hölder spaces $W^{\theta, \infty}(\mathbb{T}^d)$. The main results are the continuity Theorem A.1 and the composition Theorem A.3 which require mild regularity assumptions of the symbols in the space variable, and imply Theorems 2.4 and 2.5. We first provide some preliminary technical results.
Technical lemmas. In the following we denote by $\partial_m$, $m = 1, \ldots, d$ the discrete derivative, defined for functions $f: \mathbb{Z}^d \to \mathbb{C}$ as

$$(\partial_m f)(n) := f(n) - f(n - \vec{e}_m), \quad n \in \mathbb{Z}^d,$$ (A.1)

where $\vec{e}_m$ denotes the usual unit basis vector of $\mathbb{N}_0^d$ with 0 components expect the $m$-th one. Given a multi-index $\beta \in \mathbb{N}_0^d$, we set $\partial^\beta f := \partial_1^{\beta_1} \ldots \partial_d^{\beta_d} f$.

We shall use the Leibniz rule for finite differences in the following form: given $k \in \mathbb{N}$, $m = 1, \ldots, d$, there exist constants $C_{k_1, k_2}$ (binomial coefficients) such that

$$(\partial_m^k)(fg)(n) = \sum_{k_1 + k_2 = k} C_{k_1, k_2} (\partial_m^{k_1} f)(n - k_2)(\partial_m^{k_2} g)(n).$$ (A.2)

Moreover, when using discrete derivatives, the analogous of the integration by parts formula is given by the Abel resummation formula:

$$\sum_{n \in \mathbb{Z}^d} e^{i n \cdot \beta}(x, n) = \frac{1}{e^{i \vec{e}_m \cdot z} - 1} \sum_{n \in \mathbb{Z}^d} e^{i n \cdot \beta}(\partial_m \beta)(x, n), \quad \forall m = 1, \ldots, d.$$ (A.3)

Lemma A.1. Let $K: \mathbb{T}^d \to \mathbb{C}$ be a function satisfying, for constants $A$ and $B$, the estimate

$$|K(y)| \leq A^d B \min \left(1, \min_{1 \leq m \leq d} \frac{1}{|A 2 \sin \frac{\pi}{2}(d+1)|} \right), \quad \forall y \in \mathbb{T}^d.$$ (A.4)

Then

$$\int_{\mathbb{T}^d} |K(y)| \, dy \leq B.$$ (A.5)

Proof. If $A \leq 1$ the bound (A.5) follows trivially integrating the first inequality in (A.3). Then we suppose $A > 1$. We split the integral in (A.5) as

$$\int_{\mathbb{T}^d} |K(y)| \, dy = \int_{\mathbb{T}^d \cap \{|y| \leq \frac{1}{A}\}} |K(y)| \, dy + \int_{\mathbb{T}^d \cap \{|y| > \frac{1}{A}\}} |K(y)| \, dy.$$ (A.6)

We bound the first integral using the first inequality in (A.3), getting

$$\int_{\mathbb{T}^d \cap \{|y| \leq \frac{1}{A}\}} |K(y)| \, dy \leq A^d B \max \left(y \in [-\pi, \pi]^d : \|y\| \leq \frac{1}{A}\right) \leq B.$$ (A.7)

To bound the second integral in (A.6) we use that, for some $c > 0$, $\max_{1 \leq m \leq d} |\sin \left(\frac{\pi m}{2}\right)| \geq c|y|$, $\forall y \in [-\pi, \pi]^d$, and therefore the second inequality in (A.3) implies

$$\int_{\mathbb{T}^d \cap \{|y| > \frac{1}{A}\}} |K(y)| \, dy \leq A^d B \int_{\{|y| \geq \frac{1}{A}\}} \frac{dy}{|Ay|^{d+1}} \leq B \int_{\{|z| > 1\}} \frac{dz}{|z|^{d+1}} \leq B.$$ (A.8)

The bounds (A.7)-(A.8) and (A.6) imply (A.5). \qed

The next lemma represents a Fourier multiplier operator acting on periodic functions as a convolution integral on $\mathbb{R}^d$. The key step is the use of Poisson summation formula.

A BONY-WEYL CALCULUS IN PERIODIC HÖLDER SPACES
Lemma A.2. Let $\chi \in S(\mathbb{R}^d)$. Then the Fourier multiplier $\chi_\theta(D) := \chi(\theta^{-1}D)$, $\theta \geq 1$, acting on a periodic function $u \in L^1(\mathbb{T}^d, \mathbb{C})$ can be represented by

$$
\chi_\theta(D)u = \int_{\mathbb{R}^d} u(y)\psi_\theta(x-y)dy = \int_{\mathbb{R}^d} u(x-y)\psi_\theta(y)dy
$$

(A.9)

where $\psi_\theta(z) := \theta^d \hat{\psi}(\theta z)$ and $\hat{\psi}$ denotes the anti-Fourier transform of $\chi$ on $\mathbb{R}^d$.

Proof. For $\theta \geq 1$ we write

$$
\chi(D)u = \int_{\mathbb{T}^d} u(y) h_\theta(x-y)dy \quad \text{where} \quad h_\theta(z) := \frac{1}{(2\pi)^d} \sum_{j \in \mathbb{Z}^d} \chi\left(\frac{j}{\theta}\right) e^{ij\theta z}.
$$

(A.10)

Then the Fourier transform $\hat{\psi}_\theta(x) = \int_{\mathbb{T}^d} \theta^d \psi(\theta z) e^{-iz\cdot\xi}dz = \int_{\mathbb{T}^d} \psi(y) e^{-iz\cdot\hat{\psi}(\theta z)}dy = \hat{\psi}\left(\frac{x}{\theta}\right) = \chi\left(\frac{x}{\theta}\right)$, and, using Poisson summation formula, we write the periodic function $h_\theta(z)$ in (A.10) as

$$
h_\theta(z) = \frac{1}{(2\pi)^d} \sum_{j \in \mathbb{Z}^d} \hat{\psi}_\theta(j) e^{ij\theta z} = \sum_{j \in \mathbb{Z}^d} \psi_\theta(z + 2\pi j).
$$

Therefore the integral (A.10) is

$$
\chi(\theta^{-1}D)u = \sum_{j \in \mathbb{Z}^d} \int_{\mathbb{T}^d} u(y) \psi_\theta(x-y + 2\pi j)dy = \sum_{j \in \mathbb{Z}^d} \int_{[0,2\pi]^d + 2\pi j} u(y)\psi_\theta(x-y)dy
$$

$$
= \int_{\mathbb{T}^d} u(y)\psi_\theta(x-y)dy = \int_{\mathbb{R}^d} u(x-y)\psi_\theta(y)dy
$$

proving (A.9). \qed

We now give the definition and basic properties of the Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$.

Definition A.3. (Periodic Hölder spaces) Given $\varrho \in \mathbb{N}_0$, we denote by $W^{\varrho,\infty}(\mathbb{T}^d)$ the space of continuous functions $u : \mathbb{T}^d \to \mathbb{C}$, 2π-periodic in each variable $(x_1, \ldots, x_d)$, whose derivatives of order $\varrho$ are in $L^\infty$, equipped with the norm $\|u\|_{W^{\varrho,\infty}} := \sum_{|\alpha| \leq \varrho} \|\partial_\alpha^\varrho u\|_{L^\infty}$, $\alpha \in \mathbb{N}_0^d$. In case $\varrho > 0$, $\varrho \notin \mathbb{N}$, we denote $[\varrho]$ the integer part of $\varrho$, and we define $W^{\varrho,\infty}(\mathbb{T}^d)$ as the space of functions $u$ in $C^{[\varrho]}(\mathbb{T}^d, \mathbb{C})$ whose derivatives of order $[\varrho]$ are $(\varrho - [\varrho])$-Hölder-continuous, that is

$$
[\partial_\alpha^\varrho u]_{[\varrho]} := \sup_{x \neq y} \frac{|\partial_\alpha^\varrho u(x) - \partial_\alpha^\varrho u(y)|}{|x-y|^{[\varrho]}} < +\infty, \quad \forall |\alpha| = [\varrho],
$$

equipped with the norm

$$
\|u\|_{W^{\varrho,\infty}} := \sum_{|\alpha| \leq [\varrho]} \|\partial_\alpha^\varrho u\|_{L^\infty} + \sum_{|\alpha| = [\varrho]} [\partial_\alpha^\varrho u]_{[\varrho]}.
$$

For $\varrho = 0$ the norm $\| \|_{W^{0,\infty}} = \| \|_{L^\infty}$.

The Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$ can be described by the Paley-Littlewood decomposition of a function. Consider the locally finite partition on unity

$$
1 = \chi(\xi) + \sum_{k \geq 1} \varphi(2^{-k}\xi), \quad \varphi(z) := \chi(z) - \chi(2z),
$$

(A.11)
where $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ is the cut-off function defined in (2.13). It induces the decomposition of a distribution $u \in \mathcal{S}'(\mathbb{T}^d)$ as

$$u = \sum_{k \geq 0} \Delta_k u \quad \text{where} \quad \Delta_0 := \chi(D), \quad \Delta_k := \varphi(2^{-k}D) = \chi_{2^k}(D) - \chi_{2^{k+1}}(D), \quad k \geq 1. \quad (A.12)$$

We also set

$$S_k := \sum_{0 \leq j \leq k} \Delta_j = \chi_{2^k}(D). \quad (A.13)$$

The Paley-Littlewood theory of the Hölder spaces $W^{p,\infty}(\mathbb{T}^d)$ follows as in $\mathbb{R}^d$, see e.g. [20], once we represent the Fourier multipliers $\Delta_k$ as integral convolution operators on $\mathbb{R}^d$, by Lemma A.2. In particular the following smoothing estimates hold: for any $\alpha \in \mathbb{N}_0^d$, $\varrho \geq 0$,

$$\|\partial_\alpha^\varrho S_k u\|_{L^\infty} \lesssim 2^{k(\alpha \cdot \varrho)} \|u\|_{W^{p,\infty}}, \quad (A.14)$$

and, for any $\varrho > 0$,

$$\|u - \chi_\varrho(D)u\|_{L^\infty} \lesssim \varrho^{-\varrho} \|u\|_{W^{p,\infty}}. \quad (A.15)$$

In this way it results as in $\mathbb{R}^d$ that the Hölder norms $\|\cdot\|_{W^{p,\infty}}$ satisfy interpolation estimates. In particular we shall use that, given $\varrho_1, \varrho_2 \geq 0$,

$$\|uv\|_{W^{p,\infty}} \lesssim \|u\|_{W^{p,\infty}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{W^{p,\infty}},$$

$$\|u\|_{W^{p,\infty}} \lesssim \|u\|_{W^{p,\infty}} \|u\|_{W^{p,\infty}}^{1-\varrho} \|u\|_{W^{p,\infty}}^\varrho, \quad \varrho = \varrho_1 + (1 - \varrho) \varrho_2, \quad \varrho \in (0, 1). \quad (A.16)$$

Hölder estimates of regularized symbols. In order to prove estimates of the regularized symbol $a_\chi$ defined in (2.14) in Hölder spaces (Lemma A.5) we represent it as a convolution integral on $\mathbb{R}^d$, by Lemma A.2

$$a_\chi(x, \xi) = \int_{\mathbb{R}^d} a(x - y, \xi) \psi_\epsilon(\xi)(y) \, dy \quad (A.17)$$

where $\psi_\epsilon(z) = \theta^\epsilon \psi(\theta z)$ and $\psi$ is the anti-Fourier transform of $\chi$.

In the proof of Lemma A.5 we shall use the following estimate.

**Lemma A.4.** For any $\beta \in \mathbb{N}_0^d$, $u \in L^\infty(\mathbb{T}^d)$, we have

$$\|\partial_\beta^\xi \chi_\epsilon(\xi)(D)u\|_{L^\infty} \lesssim \langle \xi \rangle^{-|\beta|} \|u\|_{L^\infty}. \quad (A.18)$$

**Proof.** By (A.17) we have, for all $\beta \in \mathbb{N}_0^d$,

$$\partial_\beta^\xi \chi_\epsilon(\xi)(D)u = \int_{\mathbb{R}^d} u(x - y) \partial_\beta^\xi \psi_\epsilon(\xi)(y) \, dy \quad (A.19)$$

By the definition $\psi_\epsilon(\xi)(y) = (\epsilon \langle \xi \rangle)^d \psi(\epsilon \langle \xi \rangle y)$ and Faa di Bruno formula, we have that

$$\int_{\mathbb{R}^d} |\partial_\beta^\xi \psi_\epsilon(\xi)(y)| \, dy \lesssim \langle \xi \rangle^{-|\beta|}, \quad \forall \xi \in \mathbb{R}^d. \quad (A.20)$$

Then (A.18) follows by (A.19) and (A.20). \qed

The next lemma provides estimates of the regularized symbol $a_\chi$ in terms of the symbol $a$.

**Lemma A.5.** (Estimates on regularized symbols) Let $m \in \mathbb{R}$, $N \in \mathbb{N}_0$.
1. If \( a \in \Gamma^{m}_{L^{\infty}}, m \in \mathbb{R} \), then \( a_{\chi} \) defined in (2.14) belongs to \( \Sigma^{m}_{L^{\infty}} \) and
\[
|a_{\chi}|_{m,L^{\infty},N} \lesssim |a|_{m,L^{\infty},N}.
\] (A.21)

2. If \( a \in \Gamma^{m}_{H^{10^{-\epsilon}}} \), \( \theta \geq 0 \), \( s_{0} > \frac{1}{2} \), then \( a_{\chi} \) belongs to \( \Gamma^{m}_{L^{\infty}} \) and
\[
|a_{\chi}|_{m+\theta,L^{\infty},N} \lesssim |a|_{m,H^{10^{-\epsilon}},N}.
\] (A.22)

3. If \( a \in \Gamma^{m}_{W^{s,\infty}} \), \( \theta \geq 0 \), then, for any \( \beta \in \mathbb{N}^{d}_{0} \), \( \partial^{\beta}_{\xi} a_{\chi} = (\partial^{\beta}_{\xi} a)_{\chi} \in \Sigma^{m|\beta|^{-\theta}}_{L^{\infty}} \) and
\[
|\partial^{\beta}_{\xi} a_{\chi} = (\partial^{\beta}_{\xi} a)_{\chi}|_{m-|\beta|-\theta,L^{\infty},N} \lesssim |a|_{m,W^{s,\infty},N+|\beta|}.
\] (A.23)

4. If \( a \in \Gamma^{m}_{W^{s,\infty}} \), \( \theta \geq 0 \), then, for any \( \alpha \in \mathbb{N}^{d}_{0} \) with \( |\alpha| \geq \theta \), \( \partial^{\alpha}_{\xi} a_{\chi} = (\partial^{\alpha}_{\xi} a)_{\chi} \in \Sigma^{m+|\alpha|^{-\theta}}_{L^{\infty}} \) and
\[
|\partial^{\alpha}_{\xi} a_{\chi}|_{m+|\alpha|-\theta,L^{\infty},N} \lesssim |a|_{m,W^{s,\infty},N}.
\] (A.24)

5. If \( a \in \Gamma^{m}_{W^{s,\infty}} \), \( \theta \geq 0 \), then, \( a - a_{\chi} \in \Gamma^{m-\theta}_{L^{\infty}} \) and
\[
|a - a_{\chi}|_{m-\theta,L^{\infty},N} \lesssim |a|_{m,W^{s,\infty},N}.
\] (A.25)

**Proof.** PROOF OF (A.21). Differentiating (2.14) for any \( \beta \in \mathbb{N}^{d}_{0} \), we have
\[
\partial^{\beta}_{\xi} a_{\chi}(x,\xi) = \sum_{\beta_{1}+\beta_{2}=\beta} C_{\beta_{1},\beta_{2}} \partial^{\beta_{1}}_{\xi} \chi_{\epsilon}(\xi) (D) \partial^{\beta_{2}}_{\xi} a(\cdot,\xi).
\]

Then (2.7) and (A.18) directly imply (A.21).

PROOF OF (A.22). By the Cauchy-Schwartz inequality
\[
|a_{\chi}(x,\xi)| = \left| \sum_{n \in \mathbb{Z}^{d}} \chi_{\epsilon} \left( \frac{n}{\xi} \right) \hat{a}(n,\xi) e^{i \cdot x} \right| \leq \sum_{n \in \mathbb{Z}^{d}} \chi_{\epsilon} \left( \frac{n}{\xi} \right) |\hat{a}(n,\xi)| \lesssim \left( \sum_{n \in \mathbb{Z}^{d}} \chi_{\epsilon}^{2} \left( \frac{n}{\xi} \right) \left( \frac{n^{2}}{(n^{2} + \eta^{2})^{1/2}} \right) \right)^{1/2} \|a(\cdot,\xi)\|_{H^{10^{-\epsilon}}} \lesssim \langle \xi \rangle^{m+\theta} |a|_{m,H^{10^{-\epsilon}},0}.
\]
The case \( N \geq 1 \) follows in the same way.

PROOF OF (A.23). First, for any \( \xi \in \mathbb{R}^{d} \), we define \( k \in \mathbb{N} \) such that \( 2^{k-1} \leq \epsilon \langle \xi \rangle \leq 2^{k} \). Then, by the properties of the cut-off function \( \chi \) in (2.13) and the projector \( S_{k} \) in (A.13) we have
\[
\partial^{\beta}_{\xi} \chi_{\epsilon} \left( \frac{\eta}{\xi} \right) = \left( \partial^{\beta}_{\xi} \chi_{\epsilon} \left( \frac{\eta}{\xi} \right) \right) S_{k}, \quad \forall \eta \in \mathbb{R}^{d} \quad \forall \beta \in \mathbb{N}^{d}_{0}.
\] (A.26)

Differentiating (2.14) and using (A.26) we get
\[
\partial^{\beta}_{\xi} a_{\chi} - (\partial^{\beta}_{\xi} a)_{\chi} = \sum_{\beta_{1}+\beta_{2}=\beta, \beta_{1} \neq 0} C_{\beta_{1},\beta_{2}} \partial^{\beta_{1}}_{\xi} \chi_{\epsilon}(\xi) (D) S_{k} \partial^{\beta_{2}}_{\xi} a(\cdot,\xi),
\]
and, using (A.18) and (A.14)
\[
\| (\partial^{\beta}_{\xi} a_{\chi} - (\partial^{\beta}_{\xi} a)_{\chi}) \|_{L^{\infty}} \lesssim \sum_{\beta_{1}+\beta_{2}=\beta, \beta_{1} \neq 0} \langle \xi \rangle^{-|\beta_{1}|} 2^{-k\theta} \| \partial^{\beta_{2}}_{\xi} a(\cdot,\xi) \|_{W^{s,\infty}} \lesssim \langle \xi \rangle^{m-|\beta|} 2^{-k\theta} |a|_{m,W^{s,\infty},|\beta|} \lesssim \langle \xi \rangle^{m-|\beta|-\theta} |a|_{m,W^{s,\infty},|\beta|}.
\]
because $\langle \xi \rangle \lesssim 2^k$. This proves (A.23) for $N = 0$. For $N \geq 1$ the estimate is similar.

**Proof of** (A.24). For any $\xi \in \mathbb{R}^d$, we define $k \in \mathbb{N}$ such that $2^{k-1} \leq 2^k \langle \xi \rangle \leq 2^k$. By (2.14) and (A.26) with $\beta = 0$, we write $a_\chi(\cdot, \xi) = \chi_{\leq \xi}(D) a(\cdot, \xi) = \chi_{\leq \xi}(D) S_k a(\cdot, \xi)$, and then

$$\| \partial_\xi^\alpha a_\chi(\cdot, \xi) \|_{L^\infty} = \| \chi_{\leq \xi}(D) \partial_\xi^\alpha S_k a(\cdot, \xi) \|_{L^\infty} \lesssim \| \partial_\xi^\alpha S_k a(\cdot, \xi) \|_{L^\infty}$$

(A.19)

$$\lesssim 2^k |\alpha| - |\xi| \| a(\cdot, \xi) \|_{W^{s, \infty}} \lesssim \langle \xi \rangle |\xi|^{-s} \| a(\cdot, \xi) \|_{W^{s, \infty}} \lesssim \langle \xi \rangle^{m+|\alpha| - s} \| a_{m, W^{s, \infty}} \|_0$$

by (2.7). This proves (A.24) with $N = 0$. For $N \geq 1$ the estimate is similar.

**Proof of** (A.25). For any $\beta \in \mathbb{N}_0^d$ we write $\partial_\xi^\beta a(\cdot, \xi) = [\partial_\xi^\beta a - (\partial_\xi^\beta a)_\chi] + [(\partial_\xi^\beta a)_\chi - \partial_\xi^\beta a_\chi]$. The first term is bounded, using (A.19) with $\theta = \epsilon \langle \xi \rangle$, as

$$\| (\partial_\xi^\beta a - (\partial_\xi^\beta a)_\chi)(\cdot, \xi) \|_{L^\infty} \lesssim \langle \xi \rangle^{-s} \| \partial_\xi^\beta a(\cdot, \xi) \|_{W^{s, \infty}} \lesssim |a|_{m, W^{s, \infty}, |\beta|} \langle \xi \rangle^{m-|\beta|}$$

The second term satisfies the same bound by (A.23). This proves (A.25). \qed

**Change of quantization.** In order to prove the boundedness Theorem [A.4] and the composition Theorem [A.5], it is convenient to pass from the Weyl quantization of a symbol $a(x, \xi)$, defined in (2.14), to the standard quantization which is defined, given a symbol $b(x, \xi)$, as

$$\text{Op}(b)[u] := \sum_{j \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \hat{b}(j - k, k) u_k \right) e^{ij \xi} = \sum_{k \in \mathbb{Z}^d} b(x, k) u_k e^{ik \xi}. \quad \text{(A.27)}$$

We have the change of quantization formula

$$\text{Op}^W(a) = \text{Op}(b) \quad \Leftrightarrow \quad \hat{b}(n, \xi) := \hat{a}(n, \xi + \frac{n}{2}). \quad \text{(A.28)}$$

In the next lemma we estimate the norms of $b$ in terms of those of $a$. We remind that $\Sigma_{m, W}^\infty$ denotes the set of spectrally localized symbols, i.e. satisfying (2.6).

**Lemma A.6. (Change of quantization)** Let $a \in \Sigma_{m, W}^\infty$, $m \in \mathbb{R}$. If $\delta > 0$ in (2.6) is small enough, then (cfr. (A.23))

$$b(x, \xi) := \sum_{n \in \mathbb{Z}^d} \hat{a}(n, \xi + \frac{n}{2}) e^{in \xi} \quad \text{(A.29)}$$

is a symbol in $\Sigma_{m, W}^\infty$ satisfying

$$|b|_{m, L^\infty, N} \lesssim |a|_{m, L^\infty, N + d + 1}, \quad \forall N \in \mathbb{N}_0. \quad \text{(A.30)}$$

**Proof.** Since $a$ satisfies (2.6) with $\delta$ small enough, it follows that $b$ satisfies (2.6). In order to prove (A.30) we differentiate (A.29) obtaining that, for any $\beta \in \mathbb{N}_0^d$,

$$\partial_\xi^\beta b(x, \xi) = \sum_{n \in \mathbb{Z}^d} \partial_\xi^\beta a(n, \xi + \frac{n}{2}) e^{in \xi} = \sum_{n \in \mathbb{Z}^d} \partial_\xi^\beta a(n, \xi + \frac{n}{2}) \chi_\epsilon \left( \frac{n}{\langle \xi \rangle} \right) e^{in \xi}$$

for some $\epsilon = \epsilon(\delta') > 0$, where in the last equality we used that the sum is actually restricted over the indexes for which $|n| \leq \delta' \langle \xi \rangle$, $\delta' \in (0, 1)$. Then we represent $\partial_\xi^\beta b$ as the integral

$$\partial_\xi^\beta b(x, \xi) = \int_{T^d} K(x, y) dy, \quad K(x, y) := \frac{1}{(2\pi)^d} \sum_{n \in \mathbb{Z}^d} (\partial_\xi^\beta a)(x - y, \xi + \frac{n}{2}) \chi_\epsilon \left( \frac{n}{\langle \xi \rangle} \right) e^{in \xi}. \quad \text{(A.31)}$$
We are going to estimate the $L^1$-norm of $K(x, \cdot)$ using Lemma \[A.1\]. First note that, since $a \in \Sigma^m_{L^\infty}$, we have $\langle \xi + \frac{n}{2} \rangle \sim \langle \xi \rangle$ on the support of $\partial_{\xi}^{\alpha} a(n, \xi + (n/2))$, and then we bound \[A.31\] as

$$|K(x,y)| \lesssim \sum_{|n| \leq \delta'} |a|_{m, L^\infty, |\beta|} \langle \xi \rangle^{-|\beta|} \lesssim |a|_{m, L^\infty, |\beta|} \langle \xi \rangle^{d+|\beta|},$$

uniformly in $x$. Moreover, using Abel resummation formula \[A.3\] and the Leibniz rule \[A.2\] for finite differences, we get, for any $h = 1, \ldots, d$,

$$K(x,y) = \frac{1}{(e^{nh} - 1)^d} K_{k_1, k_2} \sum_{|n| \leq \delta} \partial_h^{k_1} \partial_{\xi}^{k_2} a(x - y, \xi + \frac{n}{2}) \partial_h^{k_2} \chi_{\langle \xi \rangle^{d+|\beta|}}(\xi) e^{in \cdot y}.$$

Then, using \[2.7\] and that $|\partial^h \chi_{\langle \xi \rangle^{d+|\beta|}}(\xi)| \lesssim \langle \xi \rangle^{-h}$, we estimate

$$|K(x,y)| \lesssim \langle \xi \rangle^{-|\beta|} \cdot \langle \xi \rangle^{d+|\beta|} |a|_{m, L^\infty, |\beta|} \sum_{|n| \leq \delta} 1 \lesssim |a|_{m, L^\infty, |\beta|} \langle \xi \rangle^{d+|\beta|}$$

uniformly in $x$. In view of \[A.32\], \[A.33\] we apply Lemma \[A.1\] with $A = \langle \xi \rangle$ and $B = \langle \xi \rangle^{m - |\beta|} |a|_{m, L^\infty, |\beta|} + 1$ obtaining

$$|\partial^2 b(x, \xi)| \leq \int_{\mathbb{R}^d} |K(x,y)| dy \lesssim \langle \xi \rangle^{m - |\beta|} |a|_{m, L^\infty, |\beta|} + 1, \quad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d,$$

that proves \[A.30\].

**Continuity.** We now prove boundedness estimates in Sobolev spaces of operators with spectrally localized symbols, requiring derivatives in $\xi$ of the symbol and no derivatives in $x$.

**Theorem A.7.** (Continuity) Let $a \in \Sigma^m_{L^\infty}$ with $m \in \mathbb{R}$. Then $\text{Op}(a)$ defined in \[A.27\] extends to a bounded operator from $H^s \rightarrow H^{s-m}$, for any $s \in \mathbb{R}$, satisfying

$$\|\text{Op}(a) u\|_{s-m} \lesssim |a|_{m, L^\infty, d+1} \|u\|_s.$$ \[A.34\]

Moreover, if $a$ fulfills \[2.10\] with $\delta > 0$ small enough, then the operator $\text{Op}^W(a)$ defined in \[2.15\] satisfies

$$\|\text{Op}^W(a) u\|_{s-m} \lesssim |a|_{m, L^\infty, 2(d+1)} \|u\|_s.$$ \[A.35\]

**Proof.** We first recall the Littlewood-Paley characterization of the Sobolev norm

$$\|u\|^2 \sim \sum_{k \geq 0} 2^{2ks} \|\Delta_k u\|^2_0$$ \[A.36\]

where $\Delta_k$ are defined in \[A.12\]. The norm $\|\|_0 = \|\|_{L^2}$. We first prove \[A.34\].

**Step 1:** according to \[A.11\], we perform the Littlewood-Paley decomposition of $\text{Op}(a)$,

$$\text{Op}(a) v = \sum_{k \geq 0} \text{Op}(a_k) v,$$ \[A.37\]

where

$$a_0(x, \xi) := a(x, \xi) \chi(\xi), \quad a_k(x, \xi) := a(x, \xi) \varphi(2^{-k} \xi), \quad k \geq 1.$$ \[A.38\]

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In order to prove (A.34), it is sufficient to prove that
\[ \| \text{Op} (a_k) v \|_0 \lesssim |a|_{m,L^{\infty,d+1}} 2^k m \| v \|_0, \quad \forall k \in \mathbb{N}_0, \quad \forall v \in L^2. \] (A.39)
Indeed, decomposing \( v \) in Paley-Littlewood packets as in (A.12),
\[ v = \sum_{j \geq 0} \Delta_j v, \quad \Delta_0 = \chi(D), \quad \Delta_j = \varphi(2^{-j} D), \] (A.40)
which are almost orthogonal in \( L^2 \) (namely \( \Delta_k \Delta_j = 0 \) for any \( |j - k| \geq 3 \)), using the fact that \( \text{Op} (a_k) v = \text{Op} (a) \Delta_k v \), and since the action of \( \text{Op} (a_k) \) does not spread much the Fourier support of functions being a spectrally localized, according to (2.17), we have
\[ \| \text{Op} (a) v \|_{s-m}^{2} \xrightarrow{\text{A.38}} \sum_{k \geq 0} 2^{2k(s-m)} \| \text{Op} (a_k) \Delta_j v \|_{s-m}^{2} \lesssim |a|_{m,L^{\infty,d+1}} 2^{2k} \| \Delta_j v \|_0^{2} \]
\[ \lesssim |a|_{m,L^{\infty,d+1}}^{2} \sum_{k \geq 0} 2^{2k} \| \Delta_k v \|_0^{2} \| v \|_{s}^{2}. \] (A.39)

Step 2: By (A.38) and (A.27) we write \( \text{Op} (a_k) \) as the integral operator
\[ (\text{Op} (a_k) v)(x) = \int_{\mathbb{T}^d} K_k (x, x - y) v(y) dy \] (A.41)
with kernel
\[ K_k (x, z) := \frac{1}{(2\pi)^d} \sum_{\ell \in \mathbb{Z}^d} e^{i\ell \cdot z} a(x, \ell) \varphi(2^{-k} \ell). \] (A.42)
We shall deduce (A.39) by applying the Schur lemma: if
\[ \sup_{x \in \mathbb{T}^d} \int_{\mathbb{T}^d} |K_k (x, x - y)| dy =: C_1 < +\infty, \quad \sup_{y \in \mathbb{T}^d} \int_{\mathbb{T}^d} |K_k (x, x - y)| dx =: C_2 < +\infty \] (A.43)
then Schur lemma guarantees that the integral operator (A.41) is bounded on \( L^2 (\mathbb{T}^d) \) and
\[ \| \text{Op} (a_k) v \|_0 \leq (C_1 C_2)^{1/2} \| v \|_0. \] (A.44)
Let us prove (A.43) and estimate the constants \( C_1, C_2 \). By (A.42) we have that
\[ |K_k (x, z)| \lesssim \sum_{\ell \in \mathbb{Z}^d} |a(x, \ell)| \varphi(2^{-k} \ell) \overset{2.7}{\lesssim} |a|_{m,L^{\infty,0}} \sum_{\ell \in \mathbb{Z}^d} |\ell|^m \varphi(2^{-k} \ell) \lesssim 2^{k(d+m)} |a|_{m,L^{\infty,0}}. \] (A.45)
Then, applying \((d + 1)\)-times Abel resummation formula (A.3) to (A.42), we obtain, for any \( h = 1, \ldots, d \),
\[ K_k (x, z) = \frac{1}{(2\pi)^d} \left[ \frac{1}{(e^{2\pi i \cdot z} - 1)^{d+1}} \sum_{\ell \in \mathbb{Z}^d} e^{i\ell \cdot z} \partial^{d+1}_h (a(x, \ell) \varphi(2^{-k} \ell)) \right] \]
and we deduce, using (2.7), (A.2), \[ |K_k (x, z)| \lesssim 2 \sin (z_h / 2) |z_h|^{-d-1} |a|_{m,L^{\infty,d+1}} 2^k |a|_{m,L^{\infty,d+1}} \]
for any \( h = 1, \ldots, d \), thus
\[ |K_k (x, z)| \lesssim 2^{k(d+m)} |a|_{m,L^{\infty,d+1}} \min_{h=1,\ldots,d} \frac{1}{(2k^2 |\sin (z_h / 2)|)^{d+1}}. \] (A.46)
By (A.45), (A.46) we apply Lemma A.1 with \( A = 2^k \) and \( B = 2^{km}|a|_{m,L^\infty,d+1} \), deducing that

\[
\int_{T^d} |K_k(x,y)|dy = \int_{T^d} |K_k(x,z)|dz \lesssim 2^{km} |a|_{m,L^\infty,d+1} \tag{A.47}
\]

uniformly for \( x \in T^d \). Similarly

\[
\int_{T^d} |K_k(x,y)|dy \lesssim 2^{km} |a|_{m,L^\infty,d+1} \tag{A.48}
\]

uniformly for \( y \in T^d \). Finally (A.47), (A.48), (A.44) prove (A.39) completing the proof of (A.34).

Proof of (A.35). By Lemma A.6 we have \( Op^{W}(a) = Op(b) \) for a spectrally localized symbol \( b \in \Sigma^m_{L^\infty} \) which fulfills estimate (A.30). Then (A.35) follows by (A.34). \( \square \)

Composition of paradifferential operators. We finally prove a composition result for paradifferential operators. The difference with respect to Theorem 6.1.1 and 6.1.4 in [20] is to have periodic symbols and the use of the Weyl quantization.

We shall use that, in view of the interpolation inequality (A.16), if \( a \in \Gamma^m_{W^\nu,\infty} \) and \( b \in \Gamma^m_{W^\nu,\infty} \) then \( ab \in \Gamma^m_{W^\nu,\infty} \) and, for any \( N \in \mathbb{N}_0 \), any \( 0 \leq q_1 \leq \alpha \leq \beta \leq q_2 \) such that \( q_1 + q_2 = \alpha + \beta \)

\[
|ab|_{m+m',W^\nu,\infty} \lesssim |a|_{m,W^\nu,\infty} |b|_{m',L^\infty,N} + |a|_{m,L^\infty,N} |b|_{m',W^\nu,\infty} , \tag{A.49}
\]

\[
|a|_{m,W^\alpha,\infty} |b|_{m',W^{\beta},\infty} \lesssim |a|_{m,W^{\nu},\infty} |b|_{m',W^{\nu},\infty} + |a|_{m,L^\infty,N} |b|_{m',W^{\nu},\infty} .
\]

Theorem A.8. (Composition) Let \( a \in \Gamma^m_{W^\nu,\infty} \), \( b \in \Gamma^m_{W^\nu,\infty} \) with \( m, m' \in \mathbb{R} \) and \( \varrho \in (0,2] \). Then

\[
Op^{BW}(a) Op^{BW}(b) = Op^{BW}(a,b) + R^{-\varrho}(a,b) \tag{A.50}
\]

where the linear operator \( R^{-\varrho}(a,b) : \dot{H}^s \rightarrow \dot{H}^{s-(m+m')-\varrho} \), \( \forall s \in \mathbb{R} \), satisfies

\[
||R^{-\varrho}(a,b)||_{s-(m+m')-\varrho} \lesssim \left( |a|_{m,W^\nu,\infty} |b|_{m',L^\infty,N} + |a|_{m,L^\infty,N} |b|_{m',W^\nu,\infty} \right) ||u||_s \tag{A.51}
\]

with \( N \geq 3d+4 \).

Proof. We give the proof in the case \( \varrho \in (1,2] \). We first compute \( Op^{BW}(a) Op^{BW}(b) \). Recalling the definition (2.16) we obtain

\[
Op^{BW}(a) Op^{BW}(b) u = Op^{W}(a,\chi) Op^{W}(b,\chi) = \sum_{j,k,\ell} \hat{a}_\chi \left( j-k, \frac{j+k}{2} \right) \hat{b}_\chi \left( k-\ell, \frac{k+\ell}{2} \right) u \xi^{j+k}. \tag{A.52}
\]

We now perform a Taylor expansion of \( \hat{a}_\chi \left( j-k, \frac{j+k}{2} \right) \) in the second variable, around the point \( \frac{j+k}{2} \). Writing \( j+k = j+\ell + (k-\ell) \), we obtain

\[
\hat{a}_\chi \left( j-k, \frac{j+k}{2} \right) = \hat{a}_\chi \left( j-k, \frac{j+\ell}{2} \right) + \frac{k-\ell}{2} \cdot \partial_\ell \hat{a}_\chi \left( j-k, \frac{j+\ell}{2} \right) + \sum_{\alpha \in \mathbb{N}_0, |\alpha|=2} \frac{k-\ell}{2}^\alpha \int_0^1 (1-t) \partial^\alpha_\ell \hat{a}_\chi \left( j-k, \frac{j+\ell+tl(k-\ell)}{2} \right) dt. \tag{A.53}
\]

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We expand analogously $\hat{b}_\chi \left(k-\ell, \frac{k+\ell}{2}\right)$ around the point $\frac{i j + \ell}{2}$. Writing $k + \ell = j + \ell - (j - k)$, we obtain
\[
\hat{b}_\chi \left(k-\ell, \frac{k+\ell}{2}\right) = \hat{b}_\chi \left(k-\ell, \frac{j+\ell}{2}\right) - \left(\frac{j - k}{2}\right) \partial_{\xi} \hat{b}_\chi \left(k-\ell, \frac{j+\ell}{2}\right) + \sum_{\beta \in N_0^{\alpha} |\beta| = 2} \left(\frac{k-j}{2}\right) \int_{0}^{1} (1-t) \partial_{\xi}^{\beta} \hat{b}_\chi \left(k-\ell, \frac{j+\ell + t(k-j)}{2}\right) dt.
\]
Moreover, recalling (2.20) and (2.15), we write $\text{Op}^B W (a \#_b \phi) u = \text{Op} W \left((ab + \frac{1}{21} \{a, b\})_\chi\right) u$ and, by the previous expansions,
\[
\left(\text{Op}^B W (a) \text{Op}^B W (b) - \text{Op}^B W \left(ab + \frac{1}{21} \{a, b\}\right)\right) u = \sum_{i=1}^{4} R_i(a, b) u
\]
where
\[
\begin{align*}
R_1(a, b) u &:= \text{Op} W \left(a x b - (ab)\chi + \frac{1}{21} \left(\{a, b\}_\chi - \{\{a, b\}\}_{\chi}\right)\right) u, \\
R_2(a, b) u &:= \sum_{j, k, \ell} \hat{b}_\chi \left(k-\ell, \frac{k+\ell}{2}\right) \left(\frac{k-j}{2}\right) \int_{0}^{1} (1-t) \partial_{\xi} \hat{a}_\chi \left(j-k, \frac{j+\ell + t(k-j)}{2}\right) dt u e^{ijx}, \\
R_3(a, b) u &:= \sum_{j, k, \ell} \hat{a}_\chi \left(j-k, \frac{j+\ell}{2}\right) \left(\frac{j-k}{2}\right) \int_{0}^{1} (1-t) \partial_{\xi} \hat{b}_\chi \left(k-\ell, \frac{j+\ell + t(k-j)}{2}\right) dt u e^{ijx}, \\
R_4(a, b) u &:= \sum_{j, k, \ell} \hat{a}_\chi \left(j-k, \frac{j+\ell}{2}\right) \left(\frac{j-k}{2}\right) \int_{0}^{1} (1-t) \partial_{\xi} \hat{b}_\chi \left(k-\ell, \frac{j+\ell + t(k-j)}{2}\right) dt u e^{ijx}.
\end{align*}
\]
We show now that the operators $R_i(a, b), i = 1, \ldots, 4$ fulfill estimate (A.51).

Estimate of $R_1(a, b)$. By exchanging the role of $a$ and $b$ it is enough to prove that the symbols
\[
\partial_{\xi}^{\alpha} a \partial_{\xi}^{\beta} b - (\partial_{\xi}^{\alpha} a) (\partial_{\xi}^{\beta} b) \chi, |\alpha| \leq 1,
\]
belong to $\Sigma_{K_{L_\infty}}^{1+m' - \rho}$ and then apply Theorem A.7. The spectral localization property follows because of the cut-off $\chi$ and $\epsilon$ small. As $\partial_{\xi}^{\alpha}$ commutes with the Fourier multiplier $\chi_{\epsilon}(\xi)$, we have that $\partial_{\xi}^{\alpha} b_{\chi} = (\partial_{\xi}^{\alpha} b)_{\chi}$ and we write $\partial_{\xi}^{\alpha} a_{\chi} \partial_{\xi}^{\beta} b_{\chi} - (\partial_{\xi}^{\alpha} a \partial_{\xi}^{\beta} b)_{\chi}$ as
\[
\begin{align*}
(\partial_{\xi}^{\alpha} a)_{\chi} [(\partial_{\xi}^{\beta} b)_{\chi} - (\partial_{\xi}^{\beta} b)] + [(\partial_{\xi}^{\alpha} a)_{\chi} - (\partial_{\xi}^{\alpha} a)] (\partial_{\xi}^{\beta} b) + [\partial_{\xi}^{\alpha} a \partial_{\xi}^{\beta} b - (\partial_{\xi}^{\alpha} a \partial_{\xi}^{\beta} b)_{\chi}]
\end{align*}
\]
\begin{align*}
(\partial_{\xi}^{\alpha} a)_{\chi} [(\partial_{\xi}^{\beta} b)_{\chi} - (\partial_{\xi}^{\beta} b)] + [(\partial_{\xi}^{\alpha} a)_{\chi} - (\partial_{\xi}^{\alpha} a)] (\partial_{\xi}^{\beta} b) + [\partial_{\xi}^{\alpha} a \partial_{\xi}^{\beta} b - (\partial_{\xi}^{\alpha} a \partial_{\xi}^{\beta} b)_{\chi}]
\end{align*}

Consider first the term in (A.57). By Lemma A.5, $\partial_{\xi}^{\alpha} a_{\chi} - (\partial_{\xi}^{\alpha} a)_{\chi} \in \Gamma_{L_\infty}^{m' - \rho - |\alpha|}$ and $(\partial_{\xi}^{\beta} b)_{\chi} \in \Gamma_{L_\infty}^{1+m' - |\alpha|}$ and by remark (v) after Definition 2.3, for any $n \in N_0$,
\[
\begin{align*}
| (\partial_{\xi}^{\alpha} a_{\chi} - (\partial_{\xi}^{\alpha} a)_{\chi}) (\partial_{\xi}^{\beta} b)_{\chi} |_{m + m' - \rho, L_\infty} & \lesssim |\partial_{\xi}^{\alpha} a_{\chi} - (\partial_{\xi}^{\alpha} a)_{\chi} |_{m - |\alpha| - \rho, L_\infty} |(\partial_{\xi}^{\beta} b)_{\chi} |_{m' + |\alpha|, L_\infty} \\
& \lesssim |\alpha|_{m, W^{\infty} + |\alpha|} |b|_{m', L_\infty}.
\end{align*}
\]
Next consider the terms in (A.56). By remarks (iii), (iv) after Definition 2.4, we have $\partial_{\xi}^{\alpha} a \in \Gamma_{W^{\infty} - |\alpha|}^{m - |\alpha|} \subset \Gamma_{W^{\infty} - |\alpha|}^{m' - |\alpha|}$, $\partial_{\xi}^{\beta} b \in \Gamma_{W^{\infty} - |\alpha|}^{m' - |\alpha|}$, so we can apply Lemma A.5 property (A.29) and (2.10).
where 

\[ |\text{A.50}| \leq |a|_{m, W^{s+|s|, \infty, n+|s|}} |b|_{m', W^{s+|s|, N}} + |a|_{m, L^{\infty, n+|s|}} |b|_{m', W^{s+|s|, n}} + |a|_{m, L^{\infty, n+|s|}} |b|_{m', W^{s+|s|, n+1}} \]  

where to pass from the first to the second line we used the second interpolation inequality in (A.49). Altogether we have proved that the symbol in (A.52) belongs to \( \Sigma_{\infty}^{s} \) and its semi-norms are bounded by (A.58). Then Theorem A.7 proves that \( R_1(a, b) \) fulfills estimate (A.51).

**Estimate of** \( R_2(a, b) \). First we rewrite (A.53) as

\[ R_2(a, b)u = \frac{1}{4} \sum_{j, \ell} \left( \int_0^1 (1-t) \sum_{|n|=2} \hat{f}_t^n(j - \ell, \ell) dt \right) u e^{i j \cdot x} \]

where

\[ \hat{f}_t^n(n, \ell) := \sum_{k \in \mathbb{Z}^d} \hat{D}^a \hat{b}_\ell(k - \ell, \ell + \ell \cdot \ell, k \cdot \ell) \hat{\partial}_\ell^t \hat{a}_\ell(n - \ell, \ell + \ell \cdot \ell, k \cdot \ell) \]

\[ j = -\ell \quad \sum_{k \in \mathbb{Z}^d} \hat{D}^a \hat{b}_\ell(j + \ell, j \cdot \ell, k \cdot \ell) \hat{\partial}_\ell^t \hat{a}_\ell(n - j, j \cdot \ell, k \cdot \ell) \]

and \( D_{x_n} := \partial_{x_n}/i \) and \( D_x^a := D_{x_1}^a \cdots D_{x_d}^a \). Then, recalling (A.27)

\[ R_2(a, b)u = \frac{1}{4} \int_0^1 (1-t) \sum_{|a|=2} \text{Op}(f_t^n) u dt \]

where

\[ f_t^n(x, \xi) := \sum_{n, j} \hat{D}^a \hat{b}_\ell(j + \ell, \ell \cdot \ell, k \cdot \ell) \hat{\partial}_\ell^t \hat{a}_\ell(n - j, j \cdot \ell, k \cdot \ell) e^{i n \cdot x}. \]  

(A.59)

We claim that \( f_t^n(x, \xi) \) is spectrally localized, namely

\[ \exists \delta \in (0, 1) : |n| \leq \delta |\xi|, \quad \forall (n, \xi) \in \text{supp} \hat{f}_t^n. \]  

(A.60)

In fact on the support of \( \hat{b}_\ell(j + \ell, \ell \cdot \ell, k \cdot \ell) \) we have, for some \( \delta' \in (0, 1) \),

\[ |j| \leq \delta'(\xi), \]  

(A.61)

whereas, on the support of \( \hat{\partial}_\ell^t \hat{a}_\ell(n - j, j \cdot \ell, k \cdot \ell) \), \( t \in [0, 1] \),

\[ |n - j| \leq \delta |\xi| + \delta |n| \]  

(A.62)

The estimates (A.61) - (A.62) then give \( |n| \leq |j| + |n - j| \leq \delta' |\xi| + \delta |\xi| + \delta |n| \), which implies (A.60).

In order to apply Theorem A.7 it remains to prove that, for any \( N \geq 3d + 4 \),

\[ |f_t^n(x, \xi)|_{m + m' + q, L^{\infty, d + 1}} \lesssim |b|_{m', W^{s+|s|, N}} |a|_{m, L^{\infty, N}} \]

(A.63)

which implies, for any \( s \in \mathbb{R} \), \( u \in \dot{H}^s \),

\[ \|R_2(a, b)u\|_{s-m-m'+q} \lesssim |b|_{m', W^{s+|s|, N}} |a|_{m, L^{\infty, N}} \|u\|_s. \]

Thus \( R_2(a, b) \) fulfills the estimate (A.51).
In order to prove (A.63) note that, differentiating (A.59), for any \( \beta \in \mathbb{N}_0^d \),
\[
\partial_\xi^\beta f_t^\alpha (x, \xi) = \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} \sum_{n,j} \partial_{\xi_1}^{\beta_1} D_2^{n} b_k (j, \xi + \frac{j}{2}) \partial_{\xi_2}^{\beta_2} a_k (n - j, \xi + \frac{n + t j}{2}) e^{in x}
\]
\[
= \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} \int_{\mathbb{T}^2d} K_1^{\beta_1, \beta_2} (x, y, z) \, dy \, dz
\]
where \( C_{\beta_1, \beta_2} \) are binomial coefficients and
\[
K_1^{\beta_1, \beta_2} (x, y, z) := \frac{1}{(2\pi)^d} \sum_{n,j} (\partial_{\xi_1}^{\beta_1} D_2^{n} b_k) (x - j y - z, \xi + \frac{j}{2}) \partial_{\xi_2}^{\beta_2} a_k (x - j z, \xi + \frac{n + t j}{2}) e^{i(n + j y).}
\]
By (A.60) and (A.61) the sum over \( n \) in (A.69) is restricted to indexes satisfying
\[|n| \ll \langle \xi \rangle, \quad |j| \ll \langle \xi \rangle,
\]
and therefore \( \langle \xi + \frac{j}{2} \rangle \sim \langle \xi + \frac{n + t j}{2} \rangle \sim \langle \xi \rangle\).

We deduce that the sum in (A.65) is bounded by
\[
|K_1^{\beta_1, \beta_2} (x, y, z)| \lesssim \langle \xi \rangle^{2d + m + m' - |\beta| - \varepsilon} \| \partial_{\xi_1}^{\beta_1} D_2^{n} b_k \|_{m' - |\beta|, L^\infty, 0} \| \partial_{\xi_2}^{\beta_2} a_k \|_{m - |\alpha| - |\beta_2|, L^\infty, 0}
\]
\[
\lesssim \langle \xi \rangle^{2d + m + m' - |\beta| - \varepsilon} \| b \|_{m', W^{\infty, |\beta|}} \| a \|_{m, L^\infty, 2d + |\beta|},
\]
(recalling that \( |\alpha| = 2 \)). We also estimate \( K_1^{\beta_1, \beta_2} (x, y, z) \) applying Abel resummation formula (A.3) in the sum (A.69), in the index \( n \) and in the index \( j \) separately, obtaining, using (A.24), (A.21), (A.2) and (2.10),
\[
|K_1^{\beta_1, \beta_2} (x, y, z)| \lesssim \langle \xi \rangle^{2d + m + m' - |\beta| - \varepsilon} \| b \|_{m', W^{\infty, 2d + |\beta|}} \| a \|_{m, L^\infty, 2d + 3|\beta|}
\]
\[
\times \min_{1 \leq b \leq d} \left( \left| \langle \xi \rangle 2 \sin \frac{y h}{2} \right|^{-(2d + 1)} , \left| \langle \xi \rangle 2 \sin \frac{2 h}{2} \right|^{-(2d + 1)} \right).
\]
In view of (A.66)-(A.67) and \( |\beta| \leq d + 1 \), we apply Lemma A.1 with \( d \sim 2d \), choosing \( A = \langle \xi \rangle \), \( B = \langle \xi \rangle^{m + m' - |\beta| - \varepsilon} \| b \|_{m', W^{\infty, 2d + |\beta|}} \| a \|_{m, L^\infty, 2d + 3|\beta|} \) and we obtain
\[
\| \partial_\xi^\beta f_t^\alpha (\cdot, \xi) \|_{L^\infty} \lesssim \int_{\mathbb{T}^2d} |K_1^{\beta_1, \beta_2} (x, y, z)| \, dy \, dz \lesssim \langle \xi \rangle^{m + m' - \varepsilon - |\beta|} \| b \|_{m', W^{\infty, 3d + 2}} \| a \|_{m, L^\infty, 3d + 4}
\]
proving (A.63).

The proof that \( R_3(a, b) \) and \( R_4(a, b) \) satisfy the estimate (A.51) follows similarly.

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