ABSOLUTELY CONTINUOUS MAPPINGS
ON DOUBLING METRIC MEASURE SPACES

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Abstract. Following Malý’s [Ma99] definition of absolutely continuous functions of several variables, we consider $Q$-absolutely continuous mappings $f: X \to V$ between a doubling metric measure space $X$ and a Banach space $V$. The relation between these mappings and Sobolev mappings $f \in N^{1,p}(X; V)$ for $p \geq Q$ is investigated. In particular, a locally $Q$-absolutely continuous mapping on an Ahlfors $Q$-regular space is a continuous mapping in $N^{1,Q}_{\text{loc}}(X; V)$, as well as differentiable almost everywhere in terms of Cheeger derivatives provided $V$ satisfies the Radon-Nikodym property. Conversely, though a continuous Sobolev mapping $f \in N^{1,Q}_{\text{loc}}(X; V)$ is generally not locally $Q$-absolutely continuous, this implication holds if $f$ is further assumed to be pseudomonotone. It follows that pseudomonotone mappings satisfying a relaxed quasiconformality condition are also $Q$-absolutely continuous.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain. The definition of absolutely continuous functions on the real line is generalized by Malý [Ma99] to mappings $f: \Omega \to \mathbb{R}^d$: $f$ is called $n$-absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite family of disjoint balls $\{B_i\}$ in $\Omega$ we have

$$\sum_i \mathcal{H}^n(B_i) < \delta \Rightarrow \sum_i (\text{diam}(f(B_i)))^n < \varepsilon.$$ 

Properties of $n$-absolutely continuous mappings including weak differentiability, area and coarea formulas, and the relation with Sobolev mappings $W^{1,p}(\Omega, \mathbb{R}^d)$ for $p \geq n$ are studied in [Ma99]. We say that the mapping $f: \Omega \to \mathbb{R}^d$ is $n$-absolutely continuous in measure or satisfies the Lusin property if $\mathcal{H}^n(f(E)) = 0$ whenever $\mathcal{H}^n(E) = 0$. One can show that $f$ is absolutely continuous in measure provided $f$ is $n$-absolutely continuous.

This notion of absolutely continuous functions gives a unified approach to study the above mentioned properties implied by conditions including monotonicity, finite dilatation and higher integrability of the gradient. There are several works generalizing this definition in Euclidean spaces [Hen02, Bon05], and extensions of this notion in metric spaces are

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considered by Marola and Ziemer [MaZi15]. Using a rather different definition, absolutely continuous functions in a class of one-dimensional metric spaces $X$ have been studied by the second author in [Zh19].

Consider a mapping $f: X \to Y$ between metric measure spaces $(X, d, \mu)$ and $(Y, d_Y, \nu)$. We can define the class of absolutely continuous mappings by replacing the dimension $n$ with an appropriate dimension of $X$ and replacing $H^n$ with $\mu$. We say a measure $\mu$ is doubling if there exists $C > 1$ such that $\mu(B_{2r}(x)) \leq C\mu(B_r(x))$ for all $x \in X$ and $r > 0$. For every doubling metric space, we call the exponent $Q \geq 1$ in the lower mass bound defined in (2.1) the homogeneous dimension of $X$. We define the class of $Q$-absolutely continuous mappings on $X$ and denote the collection as $AC^Q(X; Y)$. A $Q$-absolutely continuous mapping $f: X \to Y$ is $Q$-absolutely continuous in measure, that is, $H^Q(f(E)) = 0$ if $\mu(E) = 0$. We can extend the definition of $Q$-absolutely continuous mappings to a class of mappings with bounded $Q$-variation and call it $BV^Q(X; Y)$. One can verify that $AC^Q_{\text{loc}}(X; Y) \subset BV^Q_{\text{loc}}(X; Y)$, see Remark 2.16.

Sobolev mappings $f: X \to Y$ between metric spaces are introduced by Heinonen, Koskela, Shanmugalingam and Tyson [HKST01]. The Sobolev spaces are denoted by $N^{1,p}(\Omega; Y)$, and the corresponding Dirichlet spaces by $D^p(\Omega; Y)$. If the space $X$ is Ahlfors $Q$-regular, we obtain the following result.

**Theorem 1.1.** Let $X$ be a complete Ahlfors $Q$-regular space with $Q > 1$. If $\Omega \subset X$ is open and bounded and $f \in BV^Q(\Omega; Y)$, then by choosing a suitable $\mu$-representative of $f$, we have $f \in D^Q(\Omega; Y)$.

We will always understand $Y$ to be isometrically embedded into a Banach space $V$. Cheeger [Ch99] shows that if $\mu$ is doubling and admits a $p$-Poincaré inequality for $p \geq 1$, then the space $X$ acquires a measurable differentiable structure, and for every $u \in N^{1,p}(X)$ one can define almost everywhere a gradient $x \mapsto Du(x) \in \mathbb{R}^N$, whose length is comparable to the minimal $p$-weak upper gradient of $u$. Thanks to the Stepanov theorem proved by Wildrick and Zürcher [WiZu15], we can obtain that mappings of bounded $Q$-variation into $V$ are almost everywhere Cheeger differentiable, provided that $V$ satisfies the Radon-Nikodym property.

**Proposition 1.2.** Let $X$ be an Ahlfors $Q$-regular space with $Q \geq 1$. Assume that there exists a measurable differentiable structure $\{(U_\alpha, x_\alpha)\}$ for $(X, d, \mu)$, and that $V$ satisfies the Radon-Nikodym property. If $\Omega \subset X$ is open and $f \in BV^Q_{\text{loc}}(\Omega; V)$, then $f$ is differentiable almost everywhere with respect to the structure $\{(U_\alpha, x_\alpha)\}$.

In the Euclidean case $\Omega \subset \mathbb{R}^n$, every mapping $f \in N^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^n)$ for $p > n$ is immediately seen to be $n$-absolutely continuous by a combination of the Sobolev embedding theorem that gives Hölder continuity, and Young’s inequality [Ma99, Theorem 4.1]. Assume the metric measure space $X$ with homogeneous dimension $Q \geq 1$ supports a $p$-Poincaré inequality, with $p > Q$. Due to the dilation factor in balls on the right-hand side of the
Sobolev embedding theorem, the Euclidean argument does not apply to achieve $Q$-absolute continuity of $f \in N_{1,p}^{1,p}(\Omega; Y)$. However, by additionally using the Hardy-Littlewood maximal function, the same result can still be achieved.

**Theorem 1.3.** Assume that $\mu$ is doubling, that $p > Q$ where $Q \geq 1$ is the homogeneous dimension of $X$, and that $X$ supports a $p$-Poincaré inequality. Let $\Omega \subseteq X$ be a bounded open set. Then every Sobolev mapping $f \in N_{1,p}^{1,p}(\Omega; Y)$, by choosing a suitable pointwise representative, satisfies the Rado–Reichelderfer condition (RR) locally in $\Omega$, and thus is locally $Q$-absolutely continuous, as well as absolutely continuous in measure.

A continuous Sobolev mapping $f \in W^{1,n}$ does not necessarily satisfy the Lusin property in the Euclidean space, see for example [HenKo14, Theorem 4.3] for a continuous mapping $f \in W^{1,n}([-1,1]^n; [-1,1]^n)$ such that $f([-1,1] \times \{0\}^{n-1}) = [-1,1]^n$. More pathological examples can be found in [HaZh16] where a Sobolev homeomorphism $f: S^n \to \mathbb{R}^{n+1}$ maps a set of Hausdorff dimension zero to Cantor set with positive $n+1$-dimensional measure. With some additional conditions including being a homeomorphism, or being continuous and open, or being a quasiregular mapping, one can obtain the Lusin property of a Sobolev mapping $f \in W^{1,n}$ in Euclidean spaces, see for example [MaMa95, KMZ12] and the references therein. As pointed out in [KMZ12, Page 458], the essential qualitative information leading to the validity of the Lusin property is the pseudomonotone condition, and a continuous pseudomonotone Sobolev mapping $f \in W_{1,loc}^{1,n}(\Omega)$ is shown to be locally $n$-absolutely continuous in [Ma99, Theorem 4.3]. A mapping $f : X \to Y$ is called pseudomonotone if there is $C_m \geq 1$ such that

$$\text{diam } f(B_r(x)) \leq C_m \text{diam } f(S_r(x))$$

for all $x \in X$ and $r > 0$, where $S_r(x)$ denotes the sphere. The following theorem recovers the above result in metric measure spaces.

**Theorem 1.4.** Suppose $\mu$ is doubling with homogeneous dimension $Q > 1$, and $X$ supports a $Q$-Poincaré inequality. Let $\Omega \subseteq X$ be an open set and let $f \in D^Q(\Omega; Y)$ be a continuous pseudomonotone mapping. Then $f$ satisfies the condition (RR) with exponent $Q$ locally, and hence is locally $Q$-absolutely continuous, as well as absolutely continuous in measure.

**Remark 1.5.** Similar results as Theorem 1.3 and Theorem 1.4 are obtained by Marola and Ziemer in [MaZi15, Proposition 5.1]. Compared to their result, our Theorem 1.3 is stronger in the sense that it implies the condition (RR) instead of weak (RR) in the case $p > Q$.

Another proof for every continuous pseudomonotone Sobolev mapping $f \in N_{1,loc}^{1,Q}(X; V)$ being absolutely continuous in measure is given by Heinonen, Koskela, Shanmugalingam and Tyson [HKST01, Theorem 7.2].

In [HKST01], also the absolute continuity in measure of quasiconformal mappings in metric spaces with appropriate structure assumptions is discussed. In Euclidean spaces, it
has been known since Gehring [Ge60, Ge62] that the definition of quasiconformality can be relaxed to allow certain exceptional sets in the conditions on the distortion \( H_f \) or \( h_f \). In a proper Ahlfors \( Q \)-regular spaces \( X, Y \), with \( X \) supporting a 1-Poincaré inequality, such a relaxed definition is shown to imply Sobolev \( N^{1,Q}_{1,Q}(X;Y) \)-regularity in [BKR07].

In the recent work of the authors [LaZh21], the conditions on \( h_f \) needed to obtain Sobolev \( N^{1,Q}_{1,Q}(X;Y) \)-regularity are further weakened by considering a weight function \( w_Y \). Combining this with Theorem 1.4, we obtain the following corollary. Due to the weight function, the corollary gives something new already when \( X = Y \) is the unweighted Euclidean space.

**Corollary 1.6.** Let \( Q > 1 \). Suppose \( (X,d,\mu) \) and \( (Y,d_Y,\nu) \) are Ahlfors \( Q \)-regular, and \( X \) supports a 1-Poincaré inequality. Suppose \( w_Y \in L^1_{\text{loc}}(Y) \) with \( w_Y > 0 \) represented by (4.4). Let \( \Omega \subset X \) be a bounded open set, and let \( E \subset \Omega \) be a set of \( \sigma \)-finite \((Q-1)\)-dimensional Hausdorff measure. Assume \( f : \Omega \to f(\Omega) \subset Y \) is a pseudomonotone homeomorphism such that \( f(\Omega) \) is open, \( \int_{f(\Omega)} w_Y \, d\nu < \infty \), \( h_f(x) < \infty \) for every \( x \in \Omega \setminus E \), and \( \frac{h_f(x)^Q}{w_Y(f(x))} \in L^\infty(\Omega) \). Then \( f \) is locally \( Q \)-absolutely continuous.

The paper is organized as follows: we review definitions and some preliminary theorems in Section 2, and in Section 3 we prove Theorem 1.1 and Proposition 1.2, which give Sobolev regularity and almost everywhere differentiability of mappings of bounded \( Q \)-variation. In Section 4 we prove Theorems 1.3 and 1.4 and Corollary 1.6, which give \( Q \)-absolute continuity of certain Sobolev and quasiconformal mappings.

**2. Preliminaries**

**2.1. Basic definitions.** Let \( (X,d,\mu) \) and \( (Y,d_Y,\nu) \) be metric measure spaces, equipped with Borel regular outer measures \( \mu, \nu \). We assume that \( Y \) is separable.

We say that the measure \( \mu \) is doubling if there exists a constant \( C_d \geq 1 \) such that

\[
0 < \mu(B_{2r}(x)) \leq C_d \mu(B_r(x)) < \infty
\]

for all balls \( B_r(x) \) with \( x \in X \) and \( r > 0 \). We understand balls to be open. Given a ball \( B = B_r(x) \), we sometimes denote \( 2B = B_{2r}(x) \); in a metric space a ball (as a set) may not have a unique center point and radius, but we will understand these to be prescribed whenever using this notation. If \( \mu \) is doubling, then there exist an exponent \( Q \geq 1 \) and a constant \( C \geq 1 \), both depending only on the doubling constant, such that

\[
\frac{\mu(B_r(x))}{\mu(B_{r_0}(x_0))} \geq \frac{1}{C} \left( \frac{r}{r_0} \right)^Q
\]

(2.1)

for every ball \( B_{r_0}(x_0) \) and every \( x \in B_{r_0}(x_0) \), \( r \leq 2r_0 \). If such \( Q \) and \( C \) exist, we say that the measure \( \mu \) satisfies the lower mass bound (2.1), and that \( X \) has homogeneous dimension \( Q \).
We say that a space is proper if every closed and bounded set is compact. If \( \mu \) is doubling, then \( X \) is separable, and if additionally \( X \) is complete, then it is proper, see e.g. [HKST15, p. 102–103].

The \( s \)-dimensional Hausdorff content is denoted by \( \mathcal{H}^s_R \), with \( R > 0 \) and \( s \geq 0 \), and then the \( s \)-dimensional Hausdorff measure \( \mathcal{H}^s \) is obtained as the limit when \( R \to 0 \). These definitions extend automatically to metric spaces.

We say that the space \( X \) is Ahlfors \( Q \)-regular if there exists a constant \( C \geq 1 \) such that
\[
C^{-1} r^Q \leq \mu (B_r(x)) \leq C r^Q
\]
whenever \( x \in X \) and \( 0 < r < \text{diam}(X) \). If \( X \) is Ahlfors \( Q \)-regular, then the measure \( \mu \) and the \( Q \)-dimensional Hausdorff measure \( \mathcal{H}^Q \) are comparable.

A continuous mapping \( \gamma : [a,b] \to X \) is said to be a rectifiable curve if it has finite length. A rectifiable curve always admits an arc-length parametrization (see e.g. [Ha02, Theorem 3.2]), so that we obtain a curve \( \gamma : [0, \ell_\gamma] \to X \). Then if \( g : \gamma([0, \ell_\gamma]) \to [0, \infty) \) is a Borel function, we define
\[
\int g \, ds := \int_0^{\ell_\gamma} g(\gamma(s)) \, ds.
\]

We always consider \( 1 \leq p < \infty \), and by \( \Omega \subset X \) we denote an open set. As usual, a mapping \( f : \Omega \to Y \) is said to be \( \mu \)-measurable if for every open set \( W \subset Y \) we have that \( f^{-1}(W) \) is a \( \mu \)-measurable set. By the Kuratowski embedding theorem, every metric space, in particular \( Y \), can be isometrically embedded into a Banach space, for example \( V = L^\infty(Y) \). See e.g. [HKST15, p. 100]. We denote by \( | \cdot | \) the norm on the real line as well as in the Banach space \( V \). By a simple mapping \( f : \Omega \to V \) we mean a finite sum \( f = \sum_i v_i \chi_{E_i} \), with \( v_i \in V \) and the sets \( E_i \subset \Omega \) are pairwise disjoint and \( \mu \)-measurable with \( \mu(E_i) < \infty \). The integral of a simple mapping is given by
\[
\int_{\Omega} f \, d\mu = \sum_i v_i \mu(E_i).
\]

**Definition 2.2 (Integrable mappings).** We say that a measurable mapping \( f : \Omega \to V \) is (Bochner) integrable if there exists a sequence of simple mappings \( \{f_j\}_{j=1}^{\infty} \) such that
\[
\int_{\Omega} |f_j - f| \, d\mu \to 0.
\]
Then the integral of \( f \) is defined by
\[
\int_{\Omega} f \, d\mu = \lim_{j \to \infty} \int_{\Omega} f_j \, d\mu.
\]
Moreover, we say that \( f \in L^p(\Omega; Y) \) if \( |f| \in L^p(\Omega) \).

If \( f \) is \( \mu \)-measurable and \( |f| \in L^1(\Omega) \), it follows that \( f \) is integrable, see [HKST15, Proposition 3.2.7].
Definition 2.3 (Upper gradient). Let \( f : \Omega \to Y \). We say that a Borel function \( g : \Omega \to [0, \infty] \) is an upper gradient of \( f \) in \( \Omega \) if
\[
d_Y(f(\gamma(\ell_\gamma)), f(\gamma(0))) \leq \int_\gamma g \, ds
\]
for every curve \( \gamma : [0, \ell_\gamma] \to \Omega \). We use the conventions \( \infty - \infty = \infty \) and \( -\infty - (-\infty) = -\infty \). If \( g : X \to [0, \infty] \) is a \( \mu \)-measurable function and (2.4) holds for \( p \)-almost every curve, we say that \( g \) is a \( p \)-weak upper gradient of \( u \). A property is said to hold for \( p \)-almost every curve if there exists a nonnegative Borel function \( \rho \in L^p(X) \) such that \( \int_\gamma \rho \, ds = \infty \) for every curve \( \gamma \) for which the property fails.

The Sobolev class \( N^{1,p}(\Omega; Y) \) consists of those mappings \( f \in L^p(\Omega; Y) \) for which there exists an upper gradient \( g \in L^p(\Omega) \). The Dirichlet class \( D^p(\Omega; Y) \) consists of those mappings \( f : \Omega \to Y \) for which there exists an upper gradient \( g \in L^p(\Omega) \). We also denote \( N^{1,p}(X) = N^{1,p}(X, \mathbb{R}) \).

The integral average of \( f : \Omega \to Y \) over a ball \( B \subset \Omega \) is defined by
\[
f_B := \frac{1}{\mu(B)} \int_B f \, d\mu
\]
if the integral exists; note that \( f_B \) might not be in \( Y \), but it is in the Banach space \( V \supset Y \).

Definition 2.5 (Space supporting Poincaré inequality). Let \( 1 \leq p < \infty \). The space \( X \) is said to support a \( p \)-Poincaré inequality if there exist constants \( C_P > 0 \) and \( \lambda \geq 1 \) such that the following holds for every pair of functions \( u : X \to \mathbb{R} \) and \( g : X \to [0, \infty] \), where \( u \in L^1(X) \) and \( g \) is an upper gradient of \( u \):
\[
\int_{B_r(x)} |u - u_{B_r(x)}| \, d\mu \leq C_P \left( \int_{B_{\lambda r}(x)} g^p \, d\mu \right)^{1/p}
\]
for every ball \( B_r(x) \).

The Poincaré inequality is defined by considering extended real-valued functions, but we will be interested in mappings. If \( X \) supports a \( p \)-Poincaré inequality, by [HKST15, Theorem 8.1.49] we know that (2.6) holds also when \( u \) is replaced by a Banach-space valued-mapping \( f : X \to V \).

By \( C \geq 1 \) we will often denote a constant depending only on the doubling, Ahlfors regularity, and Poincaré inequality constants, and the value of \( C \) may change at each occurrence.

We recall the following embedding theorem on spheres for Sobolev mappings, from [HaKo00, Theorem 7.1]. It is only proved for real-valued functions, but the proof works also for Banach-space valued mappings, since the Lebesgue differentiation theorem holds also for such mappings, see [HKST15, p. 77].
Theorem 2.7. Suppose \( \mu \) is doubling and that \( p > Q - 1 \), where \( Q \geq 1 \) is the lower mass bound from (2.1). Let \( \Omega \subset X \) be open, and suppose that a continuous mapping \( f: \Omega \rightarrow V \) and a nonnegative function \( g \) in \( \Omega \) satisfy the \( p \)-Poincaré inequality (2.6) for every ball \( B \subset \lambda B \subset \Omega \). Consider a ball \( B_{r_0}(x_0) \subset 5\lambda B_{r_0}(x_0) \subset \Omega \). Then for almost all \( r \in (0, r_0) \), \( f \) is Hölder continuous on the sphere \( \{ x \in X : d(x, x_0) = r \} \). Moreover, there exists a constant \( C \) depending only on the doubling constant and the constants in the Poincaré inequality, and \( r \in (\frac{p}{2}, r_0) \) such that

\[
dy(f(x), f(y)) \leq Cd(x, y)^{1-\frac{Q-1}{p}} \frac{Q-1}{r_0^p} \left( \int_{5\lambda B_{r_0}(x_0)} g^p \, d\mu \right)^{\frac{1}{p}}
\]

whenever \( d(x, x_0) = d(y, x_0) = r \).

Definition 2.8 (Cheeger differentiable structure). A measurable differentiable structure on \((X, d, \mu)\) is a finite or countable collection of pairs \( \{U_\alpha, x_\alpha\}_{\alpha \in I} \) called coordinate patches with the following properties:

(i) Each \( U_\alpha, \alpha \in I \), is a \( \mu \)-measurable subset of \( X \) with positive measure, and the complement of \( \bigcup_{\alpha \in I} U_\alpha \) has zero \( \mu \)-measure.

(ii) Each \( x_\alpha: X \rightarrow \mathbb{R}^{N(\alpha)}, \alpha \in I \), is a Lipschitz map on \( X \), with \( N(\alpha) \in \mathbb{N} \) bounded above independently of \( \alpha \in I \).

(iii) For every Lipschitz function \( f: X \rightarrow \mathbb{R} \) and every \( \alpha \in I \) there exists an \( L^\infty \)-map \( Df^\alpha: X \rightarrow \mathbb{R}^{N(\alpha)} \) such that for \( \mu \)-a.e. \( x \in U_\alpha \) we have

\[
\lim_{y \to x} \frac{1}{d(x, y)} \left| f(y) - f(x) - Df^\alpha(x) \cdot (x_\alpha(y) - x_\alpha(x)) \right| = 0. \tag{2.9}
\]

For those \( x \in X \) for which a \( Df^\alpha(x) \) exists so that (2.9) holds, we say that \( f \) is differentiable at \( x \). Cheeger proved the existence of a measurable differentiable structure, and thus a Rademacher differentiation theorem for Lipschitz functions on metric spaces with a doubling measure and supporting a Poincaré inequality. The Rademacher theorem was later extended by Cheeger and Kleiner [ChK109] to mappings into a Banach space satisfying the Radon-Nikodym property; the Banach space \( V \) satisfies the Radon-Nikodym property if every Lipschitz function \( f: \mathbb{R} \rightarrow V \) is differentiable almost everywhere with respect to Lebesgue measure. We recall the differentiability of such mappings below.

Definition 2.10. Let \( \{U_\alpha, x_\alpha\}_{\alpha \in I} \) be a measurable differentiable structure on \((X, d, \mu)\). Given a measurable subset \( S \) of \( X \), a mapping \( f: X \rightarrow V \) is differentiable almost everywhere in \( S \) if there exists a collection of measurable functions \( \{\partial f/\partial x_\alpha^n: U_\alpha \cap S \rightarrow V\}_{\alpha \in I, n \in \{1, 2, \ldots, N(\alpha)\}} \) such that

\[
\lim_{y \to x} \frac{1}{d(x, y)} \left| f(y) - f(x) - \sum_{n=1}^{N(\alpha)} ((x_\alpha^n(y) - x_\alpha^n(x)) \frac{\partial f}{\partial x_\alpha^n(x)}) \right| = 0
\]
for each $\alpha \in I$ and for almost every point in $S \cap U_\alpha$, and the collection \( \{ \partial f / \partial x_n^\alpha \} \) is determined uniquely up to sets of measure zero in $S$.

### 2.2. Absolutely continuous mappings.

We always consider $Q \geq 1$. Also recall that $\Omega \subset X$ is always an open set. We will use the notation

\[
\text{diam}_B f = \sup \{ d_Y(f(x), f(y)) : x, y \in B \}.
\]

**Definition 2.11.** A $\mu$-measurable mapping $f : \Omega \to Y$ is $Q$-absolutely continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that for each finite family of disjoint balls $\{ B_i \}$ in $\Omega$ satisfying $\sum_i \mu(B_i) < \delta$, we have

\[
\sum_i \left( \text{diam}_{B_i} f \right)^Q < \varepsilon.
\]

We denote the collection of $Q$-absolutely continuous mappings on $\Omega$ by $\text{AC}^Q(\Omega; Y)$.

The above then clearly holds also for countable collections $\{ B_i \}$.

We say that $f : \Omega \to Y$ is absolutely continuous in measure if $\mathcal{H}^Q(f(N)) = 0$ whenever $\mu(N) = 0$. We have that a $Q$-absolutely continuous mapping is absolutely continuous in measure.

**Lemma 2.12.** Suppose $\mu$ is doubling, with homogeneous dimension $Q \geq 1$. If $f \in \text{AC}^Q(\Omega; Y)$, then $\mu(N) = 0$ implies $\mathcal{H}^Q(f(N)) = 0$.

**Proof.** Let $N \subset \Omega$ with $\mu(N) = 0$. Since $f$ is $Q$-absolutely continuous in $\Omega$, there exists $\delta > 0$ such that for any countable collection of pairwise disjoint balls $B_j \subset \Omega$ satisfying $\sum_j \mu(B_j) < \delta$, we have

\[
\sum_j \left( \text{diam}_{B_j} f \right)^Q < \varepsilon.
\]

Take an open set $W \subset \Omega$ containing $N$, with $\mu(W) < \delta/C_d^3$. Then for every $x \in N$ there exists $r_x \in (0, 1)$ such that $B_{5r_x}(x) \subset W$. Consider the covering $\{ B_{r_x}(x) \}_{x \in N}$. By the 5-covering theorem, see e.g. [HKST15, p. 60], we can select an at most countable collection of balls $B_{r_i}(x_i) \subset W$ such that $N \subset \bigcup_i B_{r_i}(x_i)$ and the balls $B_{r_i/5}(x_i)$ are disjoint. Thus also

\[
\sum_i \mu(B_{r_i}(x_i)) \leq \sum_i C_d^3 \mu(B_{r_i/5}(x_i)) \leq C_d^3 \mu(W) < \delta.
\]

Let $\rho_i = \text{diam}_B(f(B_{r_i}(x_i)))$. It follows that

\[
f(B_{r_i}(x_i)) \subset B_{\rho_i}(f(x_i)) \quad \text{and} \quad f(N) \subset \bigcup_i B_{\rho_i}(f(x_i)).
\]

Note that the numbers $\rho_i$ are necessarily uniformly bounded from above, because otherwise $Q$-absolute continuity would be violated. Thus we can apply the 5-covering theorem, and
select an at most countable collection of pairwise disjoint balls \( \{B_{\rho_i}(f(x_i))\}_{i \in I} \) such that 

\[
f(N) \subset \bigcup_{i \in I} 5B_{\rho_i}(f(x_i)).
\]

It is clear that \( B_{\rho_i}(x_i) \cap B_{\rho_j}(x_j) = \emptyset \) whenever \( B_{\rho_i}(f(x_i)) \cap B_{\rho_j}(f(x_j)) = \emptyset \). Hence, we get a countable collection of pairwise disjoint balls \( \{B_{\rho_i}(x_i)\}_{i \in I} \) with \( \sum_{i \in I} \mu(B_{\rho_i}(x_i)) < \delta \). Denoting the constant appearing in the definition of the Hausdorff measure by \( C_Q \), we get 

\[
H^Q_{\epsilon_i/Q}(f(N)) \leq C_Q \sum_{i \in I} (5\rho_i)^Q = 5^Q C_Q \sum_{i \in I} \left( \operatorname{diam} B_{\rho_i} f \right)^Q < 5^Q C_Q \epsilon.
\]

Letting \( \epsilon \to 0 \), we get the result. \( \square \)

**Remark 2.13.** Note that when \( X = \mathbb{R}^n \), we could deduce absolute continuity in measure from \( Q \)-absolute continuity by applying the Besicovitch covering theorem in the space \( X \). In general metric spaces, for example in the first Heisenberg group equipped with the Korányi distance, we may not have such a covering theorem. Instead, we applied the 5-covering theorem in the space \( Y \).

Furthermore, if \( Y \) is Ahlfors \( Q \)-regular, then a \( Q \)-absolutely continuous mapping \( f \) satisfies the condition: if \( \mu(N) = 0 \) then \( \nu(f(N)) = 0 \).

Besides \( AC^Q \), we consider a more general class defined as follows.

**Definition 2.14.** Let \( \Omega \subset X \) be an open set. Given a mapping \( f: \Omega \to Y \), define

\[
V_Q(f, \Omega) = \sup \left\{ \sum_{i} \left( \operatorname{diam} B_{\rho_i} f \right)^Q : \{B_i\} \text{ is a finite family of disjoint balls in } \Omega \right\}.
\]

We define the seminorm

\[
\|f\|_{BV^Q(\Omega; Y)} = (V_Q(f, \Omega))^{1/Q}
\]

and say that \( f \) is of bounded \( Q \)-variation if \( \|f\|_{BV^Q(\Omega; Y)} < \infty \), denoted by \( f \in BV^Q(\Omega; Y) \).

We say that a mapping \( f: \Omega \to Y \) satisfies a property locally if for every \( x \in \Omega \), there is \( r > 0 \) such that the property holds in \( B_r(x) \subset \Omega \).

**Lemma 2.15.** Suppose \( \mu \) is doubling and \( X \) is connected. Then every \( f \in AC^Q_{\text{loc}}(\Omega; Y) \) is continuous.

*Proof.* Suppose \( f \in AC^Q_{\text{loc}}(\Omega; Y) \) is not continuous at \( x \in \Omega \). Then there exists \( \varepsilon > 0 \) such that for all \( r > 0 \), we have \( \operatorname{diam} f > \varepsilon \) for all \( r > 0 \). On the other hand, by the fact that \( X \) is connected, we have \( \mu(B_r(x)) \to 0 \) as \( r \to 0 \), see [BjBj11, Corollary 3.9]. This contradicts \( f \in AC^Q_{\text{loc}}(\Omega; Y) \). \( \square \)

**Lemma 2.16.** Suppose \( \mu \) is doubling and \( X \) is connected. Then \( AC^Q_{\text{loc}}(\Omega; Y) \subset BV^Q_{\text{loc}}(\Omega; Y) \).
Proof. Let \( f \in AC^Q_{\text{loc}}(\Omega; Y) \) and let \( x \in \Omega \). By Lemma 2.15, we find a ball \( B_0 \subset \Omega \) containing \( x \) such that \( \sup_{B_0} |f| < \infty \). There exists \( \delta > 0 \) such that \( \sum_i (\text{diam } f(B_i))^Q < 1 \) if \( \{B_i\}_i \) is a finite family of disjoint balls in \( B_0 \) with \( \sum_i \mu(B_i) < 2\delta \). Let \( n \in \mathbb{N} \) be such that \( \mu(B_0) < n\delta \). Consider an arbitrary finite collection of pairwise disjoint balls \( \{B_j\} \) in \( B_0 \). Let \( I_1 \) consist of those indices \( i \) for which \( \mu(B_i) \geq \delta \), and \( I_2 \) consist of the remaining indices. Now \( \sum_{i \in I_1} (\text{diam } f(B_j))^Q \leq 2n \sup_{B_0} |f| < \infty \). Divide the indices \( I_2 \) into at most \( n \) subcollections such that each subcollection satisfies \( \sum_i \mu(B_i) < 2\delta \). Summing over all of these subcollections, we obtain \( \sum_{i \in I_2} (\text{diam } f(B_j))^Q < n \). Thus \( V_Q(f, B_0) \leq 2n \sup_{B_0} |f| + n < \infty \). \( \square \)

**Definition 2.17.** A mapping \( f : \Omega \to Y \) is said to satisfy the Rado-Reichelderfer (RR) condition with exponent \( Q \) if there is a function \( h \in L^1(\Omega) \) such that

\[
(\text{diam } f)_{B_r(x)}^Q \leq \int_{B_r(x)} h \, d\mu \tag{RR}
\]

for every ball \( B_r(x) \subset \Omega \).

A mapping satisfying the (RR) condition with exponent \( Q \) is easily shown to be \( Q \)-absolutely continuous, by using the absolute continuity of the integral.

Given \( x \in X \) and \( r > 0 \), denote the sphere by \( S_r(x) = \{ y \in X : d(y, x) = r \} \). Note that in metric spaces, we always have \( \partial B_r(x) \subset S_r(x) \), where the inclusion can be strict.

**Definition 2.18.** We say that a mapping \( f : \Omega \to Y \) is pseudomonotone if there exists a constant \( C_m \geq 1 \) such that

\[
\text{diam } f_{B_r(x)} \leq C_m \text{diam } f_{S_r(x)}
\]

for each ball \( B_r(x) \subset \overline{B_r(x)} \subset \Omega \).

*Throughout this paper we assume that \((X, d, \mu)\) and \((Y, d_Y, \nu)\) are metric spaces equipped with Borel regular outer measures \( \mu \) and \( \nu \), and that \( Y \) is separable. We always understand \( Y \) to be embedded in a Banach space \( V \).*

3. **Sobolev regularity and almost everywhere differentiability of \( BV^Q_{\text{loc}} \)**

3.1. **Sobolev regularity of \( BV^Q_{\text{loc}} \).** In this subsection, we always assume that \( X \) is complete and \( \mu \) is doubling, with doubling constant \( C_d \geq 1 \). Hence \( X \) is separable and proper. Moreover, here \( \Omega \subset X \) is always a bounded open set.

We will show that a mapping \( f \in BV^Q(\Omega; Y) \) belongs to the Dirichlet space \( D^Q(\Omega; Y) \). To achieve this, we construct locally Lipschitz mappings \( f_\varepsilon \) and their upper gradients \( g_\varepsilon \) such that \( f_\varepsilon \to f \) in \( L^Q_{\text{loc}}(\Omega; Y) \) and the \( g_\varepsilon \)'s converge weakly to \( g \in L^Q(\Omega) \), from which it follows that \( f \in D^Q(\Omega; Y) \).

We first recall a Lipschitz partition of unity of \( \Omega \). For any \( \varepsilon > 0 \), consider a Whitney-type covering \( \{B^\varepsilon_i\}_i \) of \( \Omega \), with \( B^\varepsilon_i = B_{r_{\varepsilon,i}}(x_{\varepsilon,i}) \) where \( r_{\varepsilon,i} \leq \varepsilon \), and \( 4B^\varepsilon_i \subset \Omega \). If \( 4B^\varepsilon_i \cap \Omega \neq \emptyset \), then define \( u_{\varepsilon,i} \) such that \( u_{\varepsilon,i} \in C^0(\overline{B_{r_{\varepsilon,i}}(x_{\varepsilon,i})}) \) with \( \text{supp } u_{\varepsilon,i} \subset B_{r_{\varepsilon,i}/2}(x_{\varepsilon,i}) \) and \( u_{\varepsilon,i}(x_{\varepsilon,i}) = 1 \). Let \( f_{\varepsilon,i} \) be a \( \varepsilon \)-Lipschitz mapping on \( \overline{B_{r_{\varepsilon,i}}(x_{\varepsilon,i})} \) such that \( f \to f_{\varepsilon,i} \) on \( \overline{B_{r_{\varepsilon,i}}(x_{\varepsilon,i})} \). We then define \( f_\varepsilon \) on \( \Omega \) by \( f_\varepsilon = \sum_i u_{\varepsilon,i} f_{\varepsilon,i} \).
4B_j^\varepsilon \neq \emptyset$, then $r_{\varepsilon,i} \leq 2r_{\varepsilon,j}$. Moreover, there exists a constant $M = M(C_d)$ determined by the doubling constant $C_d$ such that $1 \leq \sum_i \chi_{12B_i^\varepsilon}(x) \leq M$ for all $x \in \Omega$. For such a construction, see e.g. [HKST15, Lemma 4.1.15].

Define $\psi_i = \min\{\varepsilon / \text{dist}(\cdot, X \setminus 2B_i^\varepsilon), 1\}$ and $\phi_i = \frac{\psi_i}{\sum_k \psi_k}$. We can verify that $\phi_i: X \to [0, 1]$ satisfy

(i) $\sum_i \phi_i(x) = 1$ and the number of nonzero terms in this sum is bounded by the constant $M$;
(ii) $\phi_i = 0$ in $X \setminus 2B_i^\varepsilon$;
(iii) $\phi_i$ is $M/r_{\varepsilon,i}$-Lipschitz continuous.

Given $f \in \mathcal{L}_1^1(\Omega; Y)$, we define the discrete convolution approximation $f_\varepsilon: \Omega \to V$ as

$$
f_\varepsilon(x) = \sum_i f_{B_i^\varepsilon} \phi_i(x), \quad x \in \Omega.
$$

Note that $f$ is assumed to take values in $Y$, but $f_\varepsilon$ takes values in $V \supset Y$. The following basic properties hold for $f_\varepsilon$:

1. $f_\varepsilon \in \text{Lip}_\text{loc}(\Omega; V)$;
2. $f_\varepsilon \to f$ as $\varepsilon \to 0$ uniformly on every compact subset of $\Omega$ if $f$ is continuous;
3. if $f \in \mathcal{L}_p^p(\Omega; Y)$, then $\|f_\varepsilon\|_{\mathcal{L}_p(\Omega; V)} \leq C_0\|f\|_{\mathcal{L}_p(\Omega; V)}$, where $C_0 \geq 1$ depends only on $C_d$.

The proofs for the above properties are similar to Lemma 8.1, Lemma 8.2 in [HKST01] and [HKT07, Lemma 5.3(2)].

**Lemma 3.1.** Let $f \in \mathcal{L}_\text{loc}^p(\Omega; V)$ for some $1 \leq p < \infty$. Then $f_\varepsilon \to f$ in $\mathcal{L}_\text{loc}^p(\Omega; V)$ as $\varepsilon \to 0$.

**Proof.** By using a cutoff function, we can assume that $f \in \mathcal{L}_p^p(\Omega; V)$. Fix $\varepsilon > 0$. By [HKST15, Proposition 3.3.52], there exists $\hat{f} \in C_c(\Omega; V)$ with $\|\hat{f} - f\|_{\mathcal{L}_p(\Omega; V)} < \varepsilon/3C_0$.

We can assume that $\mu(\text{supp} \hat{f}) > 0$. Denote the discrete convolution of $\hat{f}$ at scale $\delta > 0$ by $\hat{f}_\delta$. Denote the $\delta$-neighborhood of a set $A$ by $N_\delta(A)$. By property (2), we find $\delta > 0$ sufficiently small so that

$$
\|\hat{f}_\delta - \hat{f}\|_{\mathcal{L}_\infty(\Omega; V)} < \frac{\varepsilon}{3 \cdot \mu(N_{3\delta}(\text{supp} \hat{f}))^{1/p}},
$$

and so

$$
\|\hat{f}_\delta - \hat{f}\|_{\mathcal{L}_p(\Omega; V)} \leq \|\hat{f}_\delta - \hat{f}\|_{\mathcal{L}_\infty(\Omega; V)} \mu(N_{3\delta}(\text{supp} \hat{f}))^{1/p} < \varepsilon/3.
$$

Using also property (3), we get

$$
\|f_\varepsilon - f\|_{\mathcal{L}_p(\Omega; V)} \leq \|f_\varepsilon - \hat{f}_\delta\|_{\mathcal{L}_p(\Omega; V)} + \|\hat{f}_\delta - \hat{f}\|_{\mathcal{L}_p(\Omega; V)} + \|\hat{f} - f\|_{\mathcal{L}_p(\Omega; V)}
\leq \|(f - \hat{f})_\delta\|_{\mathcal{L}_p(\Omega; V)} + \varepsilon/3 + \|\hat{f} - f\|_{\mathcal{L}_p(\Omega; V)}
\leq C_0\|f - \hat{f}\|_{\mathcal{L}_p(\Omega; V)} + \varepsilon/3 + \|\hat{f} - f\|_{\mathcal{L}_p(\Omega; V)}
< \varepsilon.
$$
Next, define a function \( g_\varepsilon \) by
\[
g_\varepsilon(x) = \sum_i \frac{\text{diam } f(4B_\varepsilon^i)}{r_{\varepsilon,i}} \chi_{B_\varepsilon^i}(x), \quad x \in \Omega. \tag{3.2}
\]

Moreover, let \( L(f_\varepsilon, x, r) = \sup\{|f_\varepsilon(x) - f_\varepsilon(y)| : d(x, y) \leq r\} \), and define the local Lipschitz constant by
\[
\text{Lip } f_\varepsilon(x) = \limsup_{r \to 0} \frac{L(f_\varepsilon, x, r)}{r}.
\]

We have the following lemma.

**Lemma 3.3.** There exists a constant \( C \) depending on \( C_d \) such that \( \text{Lip } f_\varepsilon \leq Cg_\varepsilon \). In particular, \( Cg_\varepsilon \) is an upper gradient of \( f_\varepsilon \) in \( \Omega \).

**Proof.** Let \( x \in \Omega \). Choose a ball \( B_\varepsilon^j \) containing \( x \). Let \( I \) denote the collection of indexes for which \( 2B_\varepsilon^i \cap B_\varepsilon^j \neq \emptyset \) for \( i \in I \). Then \( \text{card}(I) \leq M \). Now for every \( y \in B_\varepsilon^j \),
\[
\sum_i (\phi_i(x) - \phi_i(y)) f_{2B_\varepsilon^i} = \sum_{i \in I} (\phi_i(x) - \phi_i(y)) f_{2B_\varepsilon^i},
\]
and \( B_\varepsilon^i \subset 4B_\varepsilon^j \) if \( i \in I \).

It follows that
\[
|f_\varepsilon(x) - f_\varepsilon(y)| = |f_\varepsilon(x) - f_\varepsilon(y) - f_{B_\varepsilon^i} \sum_i (\phi_i(x) - \phi_i(y))|
\leq \sum_{i \in I} |\phi_i(x) - \phi_i(y)||f_{B_\varepsilon^i} - f_{B_\varepsilon^j}|
\leq 2M^2 d(x, y) \text{diam } f(4B_\varepsilon^j),
\]
where the last inequality follows from the fact that \( \phi_i \) is \( M/r_{\varepsilon,i} \)-Lipschitz and thus \( 2M/r_{\varepsilon,j} \)-Lipschitz, and \( \text{card}(I) \leq M \). Hence, we obtain
\[
\limsup_{r \to 0} \frac{L(f_\varepsilon, x, r)}{r} \leq \frac{2M^2}{r_{\varepsilon,j}} \text{diam } f(4B_\varepsilon^j) \leq Cg_\varepsilon(x).
\]

**Proof of Theorem 1.1.** Let \( f \in BV^Q(\Omega; Y) \). From the definition of \( BV^Q(\Omega; Y) \), we see that \( f \in L^\infty_{\text{loc}}(\Omega; Y) \subset L^Q_{\text{loc}}(\Omega; Y) \). By Lemma 3.1, \( f_\varepsilon \to f \) in \( L^Q_{\text{loc}}(\Omega; V) \) as \( \varepsilon \to 0 \). Recall that \( \sum_i \chi_{12B_\varepsilon^i}(x) \leq M \), and that if \( 4B_\varepsilon^i \cap 4B_\varepsilon^j \neq \emptyset \), then \( r_{\varepsilon,i} \leq 2r_{\varepsilon,j} \). Thus we can divide the collection of balls \( \{4B_\varepsilon^i\}_i \) into at most \( M + 1 \) collections of disjoint balls \( \{4B_\varepsilon^i\}_{i \in I_1}, \ldots, \{4B_\varepsilon^i\}_{i \in I_{M+1}} \). Otherwise, there would be a ball \( 4B_\varepsilon^i \) whose center is in
$M + 1$ balls $4B_i^\varepsilon$. Defining $g_\varepsilon$ as in (3.2), we have by the triangle inequality, and then using Ahlfors $Q$-regularity,

$$\left( \int_{\Omega} g_\varepsilon^Q \, d\mu \right)^{1/Q} = \left( \int_{\Omega} \left( \sum_{i} \frac{\text{diam } f(4B_i^\varepsilon)}{r_{\varepsilon,i}} \chi_{2B_i^\varepsilon} \right)^Q \, d\mu \right)^{1/Q}$$

$$\leq \left( \sum_{i \in I_1} \left( \frac{\text{diam } f(4B_i^\varepsilon)}{r_{\varepsilon,i}} \right)^Q \mu(2B_i^\varepsilon) \right)^{1/Q} + \ldots + \left( \sum_{i \in I_{M+1}} \left( \frac{\text{diam } f(4B_i^\varepsilon)}{r_{\varepsilon,i}} \right)^Q \mu(2B_i^\varepsilon) \right)^{1/Q}$$

$$\leq C \left( \sum_{i \in I_1} (\text{diam } f(4B_i^\varepsilon))^Q \right)^{1/Q} + \ldots + C \left( \sum_{i \in I_{M+1}} (\text{diam } f(4B_i^\varepsilon))^Q \right)^{1/Q}$$

$$\leq (M + 1) CV_Q(f, \Omega) < \infty.$$ 

Then by reflexivity, there exists a $g \in L^Q(\Omega)$ such that $g_\varepsilon \to g$ weakly in $L^Q(\Omega)$. Combining this with the fact that $f_\varepsilon \to f$ in $L^Q_{\text{loc}}(\Omega)$, and the fact that $Cg_\varepsilon$ is an upper gradient of $f_\varepsilon$ in $\Omega$ by Lemma 3.3, we deduce that $Cg$ is a $Q$-weak upper gradient of $f$ in $\Omega$ by choosing an appropriate $\mu$-representative of $f$ [HKST15, Theorem 7.3.8]. Denoting this representative still by $f$, we have $f \in D^Q(\Omega; Y)$, and the proof is complete. 

3.2. Almost everywhere differentiability of $BV_{\text{loc}}^Q$. We recall that a Banach space $V$ satisfies the Radon-Nikodym property if every Lipschitz function $f : \mathbb{R} \to V$ is differentiable almost everywhere with respect to Lebesgue measure. Wildrick and Zürcher [WiZu15, Theorem 4.2] proved the following Stepanov differentiability theorem.

**Theorem 3.4.** Let $(X, d, \mu)$ be a complete metric measure space with $\mu$ doubling. Let $V$ be a Banach space with the Radon-Nikodym property. Assume there is a measurable differentiable structure $\{(U_\alpha, x_\alpha)\}$ on $(X, d, \mu)$. Then a measurable function $f : X \to V$ is $\mu$-a.e. differentiable in $S(f) = \{x \in X : \text{Lip } f(x) < \infty\}$ with respect to $\{(U_\alpha, x_\alpha)\}$.

Hence, to complete the proof of almost everywhere Cheeger differentiability of mappings $f \in BV^Q(\Omega; V)$ for $V$ with Radon-Nikodym property, it suffices to show that the local Lipschitz constant $\text{Lip } f$ is finite $\mu$-almost everywhere. The argument in $\mathbb{R}^n$ by Malý [Ma99, Theorem 3.3] can be extended to metric spaces as below.

**Proof of Proposition 1.2.** As before, for $L(f, x, r) = \sup \{|f(x) - f(y)| : d(x, y) \leq r\}$, we have

$$\text{Lip } f(x) = \limsup_{r \to 0} \frac{L(f, x, r)}{r}.$$ 

We will show that $\int_\Omega (\text{Lip } f)^Q \, d\mu \leq CV_Q(f, \Omega)$. Note that by the definition of the Lebesgue integral, we have

$$\int_{\Omega} (\text{Lip } f)^Q \, d\mu = \sup \left\{ \int_{\Omega} g \, d\mu : g \text{ is a simple function and } 0 \leq g \leq (\text{Lip } f)^Q \chi_{\Omega} \right\}.$$
Simple functions are finite sums $\sum_i a_i \chi_{E_i}$, with $a_i \geq 0$, and by an inner regularity property of the Borel regular outer measure $\mu$ we can assume that the sets $E_i$ are closed, see e.g. [HKST15, Proposition 3.3.37]. We conclude that it suffices to consider upper semicontinuous functions $g$. Assume $g$ is such a function and let $E = \{g > 0\} \subset \Omega$. For $x \in E$, denote by $k(x)$ the integer such that

$$2^{k(x)} \leq g(x) < 2^{k(x)+1}.$$ 

By the fact that $g$ is upper semicontinuous and $g(x) \leq (\text{Lip } f(x))^Q$, for each $x \in E$ we can find a radius $r = r(x)$ small enough such that $B_r(x) \subset \Omega$,

$$B_{5r}(x) \subset \{g < 2^{k(x)+1}\},$$

and

$$2^{k(x)r} \leq g(x)r^Q < 2L(f,x,r)^Q < 2 \text{ diam } f(B_r(x))^Q.$$ 

By the 5-covering theorem, see e.g. [HKST15, p. 60], from the collection $\{B_r(x)\}_{x \in E}$ we can select an at most countable disjoint subcollection $\{B_i = B_{r_i}(x_i)\}_{i}$ such that

$$E \subset \bigcup_i 5B_i.$$ 

Let $I_k = \{i: k(x_i) \geq k\}$ and

$$E_k = \{x \in E: 2^k \leq g(x) < 2^{k+1}\}.$$ 

Since we chose $r(x_i)$ such that $B_{5r_i}(x_i) \subset \{g < 2^{k(x_i)+1}\}$, we know that $E_k \subset \bigcup_i 5B_i$.

Using also Ahlfors $Q$-regularity, it follows that

$$\mu(E_k) \leq \sum_{i \in I_k} \mu(5B_i) \leq C \sum_{i \in I_k} r_i^Q \leq C \sum_{i \in I_k} 2^{-k(x_i)} \text{ diam } f(B_{r_i}(x_i))^Q.$$ 

Hence, we have

$$\int g \, d\mu = \sum_{k = -\infty}^{\infty} \int_{E_k} g \, d\mu \leq \sum_{k = -\infty}^{\infty} 2^{k+1} \mu(E_k) \leq C \sum_{k = -\infty}^{\infty} \sum_{i \in I_k} 2^{k+1-k(x_i)} \text{ diam } f(B_{r_i}(x_i))^Q = C \sum_{i} \sum_{k \leq k(x_i)} 2^{k+1-k(x_i)} \text{ diam } f(B_{r_i}(x_i))^Q \leq CV_Q(f, \Omega) < \infty.$$ 

We deduce that $\text{Lip } f \in L^Q(\Omega)$. From Theorem 3.4 we deduce that $f$ is differentiable almost everywhere with respect to the given measurable differentiable structure. \qed
4. Absolute continuity of Sobolev mappings

In the previous section we considered properties of the classes $AC^Q$ and $BV^Q$. In this section we consider the converse question of when a mapping can be shown to be an $AC^Q$-mapping. We show this for Sobolev mappings $N^{1,p}$ in three cases: when $p > Q$, when $p = Q$ and the mapping is pseudomonotone, and finally when the Sobolev property of the mapping is obtained from a suitable quasiconformality condition.

We define the non-centered Hardy–Littlewood maximal function of a locally integrable nonnegative function $g \in L^1_{\text{loc}}(X)$ by

$$
\mathcal{M}g(x) := \sup_{B \ni x} \int_B g \, d\mu, \quad x \in X,
$$

where the supremum is taken over all balls containing $x$.

4.1. Sobolev mapping $f \in N^{1,p}$ with $p > Q$ satisfies (RR). We show that a Sobolev mapping $f \in N^{1,p}(\Omega; Y)$ with $p > Q$ satisfies the Rado-Reichelderfer condition (RR) of Definition 2.17, which implies $Q$-absolute continuity.

Proof of Theorem 1.3. By Keith–Zhong, see [BjBj19, Theorem 5.1] as well as [KeZh08, Theorem 1.0.1] for the original result in complete spaces, $X$ supports a $q$-Poincaré inequality for some $q \in (Q,p)$. Let $f \in N^{1,p}(\Omega; V) \subset N^{1,q}(\Omega; V)$ and let $g \in L^p(\Omega)$ be an upper gradient of $f$ in $\Omega$. The Sobolev embedding theorem (see e.g. [HKST15, Theorem 9.1.36(iii)]) states that $f$ is locally Hölder continuous on $\Omega$ by a suitable choice of representative, which we still denote by $f$. The Sobolev embedding theorem also states for some constant $\sigma \geq 1$, we have

$$
\diam_{B_r(x)} f \leq C r \left( \int_{B_{\sigma r}(x)} g^q \, d\mu \right)^{\frac{1}{q}}
$$

for every ball $B_r(x) \subset B_{\sigma r}(x) \subset \Omega$. Consider a fixed ball $B_0 = B_{R_0}(x_0)$ with $B_0 \subset 3\sigma B_0 \subset \Omega$. Then consider a ball $B_r(x) \subset B_0$; note that $B_{\sigma r}(p) \subset \Omega$. First note that by (2.1), we have

$$
\frac{\mu(B_r(x))}{\mu(B_0)} \geq \frac{1}{C} \left( \frac{r}{R_0} \right)^Q. \tag{4.1}
$$

Let $\rho$ be the zero extension of $g^q$ from $\Omega$ to the whole space. Now for all $z \in B_r(x)$,

$$
\left( \diam_{B_r(x)} f \right)^q \leq C r^q \int_{B_{\sigma r}(x)} g^q \, d\mu \leq C r^q \mathcal{M}\rho(z).
$$

Integrating both sides with respect to $z$ over $B_r(x)$, by (4.1) we obtain

$$
\left( \diam_{B_r(x)} f \right)^q \leq \frac{C r^q}{\mu(B_r(x))} \int_{B_r(x)} \mathcal{M}\rho \, d\mu \leq C r^{q-Q} \frac{r^Q}{\mu(B_0)} \int_{B_r(x)} \mathcal{M}\rho \, d\mu.
$$
Finally, taking power \( \frac{Q}{q} \) on both sides and applying Young’s inequality, it follows that

\[
\left( \text{diam}_{B_r(x)} f \right)^Q \leq C \left( \frac{R_0^Q}{\mu(B_0)} \right)^{Q/q} r^{Q(1 - \frac{Q}{q})} \left( \int_{B_r(x)} \mathcal{M} \rho \, d\mu \right)^{\frac{Q}{q}} \\
\leq C \left( \frac{R_0^Q}{\mu(B_0)} \right)^{Q/q} \left( r^Q + \int_{B_r(x)} \mathcal{M} \rho \, d\mu \right) \\
\leq C \left( \frac{R_0^Q}{\mu(B_0)} \right)^{Q/q} \int_{B_r(x)} \left[ \frac{R_0^Q}{\mu(B_0)} + \mathcal{M} \rho \right] \, d\mu
\]

again by (4.1). Since \( \rho \in L^\frac{Q}{q}(X) \) with \( \frac{Q}{q} > 1 \), the Hardy-Littlewood theorem implies that \( \mathcal{M} \rho \in L^\frac{Q}{q}(X) \subset L^1(B_0) \), see e.g. [HKST15, Theorem 3.5.6]. Let \( h = 1 + \mathcal{M} \rho \in L^1(B_0) \). Thus \( f \) satisfies the condition (RR) locally with exponent \( Q \). As a result, \( f \) is locally \( Q \)-absolutely continuous in \( \Omega \).

Since \( X \) is separable, we can cover \( \Omega \) by countably many balls \( B_i \subset 3\lambda B_i \subset \Omega \), and applying Lemma 2.12 in each \( B_i \), we obtain absolute continuity in measure on \( \Omega \). \( \square \)

### 4.2. Continuous pseudomonotone Sobolev mapping \( f \in N^{1,Q} \) satisfies (RR).

**Proof of Theorem 1.4.** There is \( p \in (Q - 1, Q) \) such that \( X \) supports a \( p \)-Poincaré inequality, see [BjBj19, Theorem 5.1]. Consider a fixed ball \( B_0 = B_{R_0}(x_0) \) with \( B_0 \subset 21\sigma B_0 \subset \Omega \), where \( \sigma \) is the dilation factor from the \( p \)-Poincaré inequality. Then consider a ball \( B_r(x) \subset B_0 \). First note that by (2.1), we have

\[
\frac{\mu(B_r(x))}{\mu(B_0)} \geq \frac{1}{C} \left( \frac{r}{r_0} \right)^Q.
\]

(4.2)

By the Sobolev embedding theorem on spheres, Theorem 2.7, for \( f \) and its upper gradient \( g \in L^Q(\Omega) \) we have for some \( R \in (r, 2r) \)

\[
\left( \text{diam}_{S_R(x)} f \right)^p \leq C \frac{r^p}{\mu(B_{2r}(x))} \int_{10\sigma B_r(x)} g^p \, d\mu.
\]

Let \( \rho \) be the zero extension of \( g^p \) from \( \Omega \) to the whole space. Since \( f \) is pseudomonotone, we have

\[
\left( \text{diam}_{B_r(x)} f \right)^p \leq \left( \text{diam}_{S_R(x)} f \right)^p
\]

\[
\leq C^p_{m} \left( \text{diam}_{S_R(x)} f \right)^p
\]

\[
\leq CC^p_{m} \frac{r^p}{\mu(B_{2r}(x))} \int_{10\sigma B_r(x)} g^p \, d\mu
\]

\[
\leq C r^p \mathcal{M} \rho(z)
\]
for each $z \in B_r(x)$. Integrating both sides with respect to $z$ over $B_r(x)$ and applying Hölder’s inequality as well as (4.2), we get
\[
(diam\, f)^{p} \leq \frac{C_{p}^{p}}{\mu(B_r(x))} \int_{B_r(x)} \mathcal{M}\rho\, d\mu
\leq C \frac{R_{0}^{p}}{\mu(B_0)^{p/q}} \left( \int_{B_r(x)} (\mathcal{M}\rho)^{q/p} \, d\mu \right)^{\frac{q}{p}}.
\]

The fact that $\frac{q}{p} > 1$ and the Hardy-Littlewood theorem imply that $h = (\mathcal{M}\rho)^{q/p} \in L^1(X)$, and now we have
\[
(diam\, f)^{q} \leq C \frac{R_{0}^{q}}{\mu(B_0)} \|h\|_{L^1(B_r(x))}.
\]
Thus $f$ satisfies the condition (RR) with exponent $Q$ in the ball $B_0$, and thus locally in $\Omega$. The local $Q$-absolute continuity follows, and then the absolute continuity in measure on $\Omega$ again follows. \hfill \Box

4.3. Absolute continuity of quasiconformal mappings. Given a mapping $f: \Omega \rightarrow Y$, we define for every $x \in \Omega$ and $r > 0$
\[
L_f(x,r) := \sup\{d_Y(f(y), f(x)) : d(y, x) \leq r\},
\]
\[
l_f(x,r) := \inf\{d_Y(f(y), f(x)) : d(y, x) \geq r\},
\]
and
\[
H_f(x,r) := \frac{L_f(x,r)}{l_f(x,r)}.
\]
We interpret this as infinite if the denominator is zero. We say that a homeomorphism $f: \Omega \rightarrow f(\Omega) \subset Y$ is quasiconformal if there is a number $1 \leq H < \infty$ such that
\[
H_f(x) := \limsup_{r \rightarrow 0} H_f(x,r) \leq H
\]
for all $x \in \Omega$. In Euclidean spaces, it is proved by Gehring [Ge60, Ge62] that for an exceptional set $E$ of $\sigma$-finite $(n-1)$-dimensional Hausdorff measure and another exceptional set $F$ of zero Lebesgue measure, in the definition it suffices to assume $H_f(x) < \infty$ for all $x \in \Omega \setminus E$ and $H_f(x) \leq H < \infty$ for all $x \in \Omega \setminus F$. Heinonen and Koskela [HeKo95] show that $H_f(x)$ in the definition can be relaxed to
\[
h_f(x) := \liminf_{r \rightarrow 0} H_f(x,r) \leq H.
\]
Moreover, Kallunki and Koskela [KaKo00] show that similar exceptional sets are also admissible with the relaxed limit $h_f$.

The seminal work of Heinonen and Koskela [HeKo98] develops the foundation of the theory of quasiconformal mappings in metric spaces that satisfy certain bounds on their mass and geometry. Mappings $f: X \rightarrow Y$ between two locally Ahlfors $Q$-regular metric measure spaces supporting a $Q$-Poincaré inequality are carefully investigated. A homeomorphism $f: X \rightarrow Y$ is called quasiconformal if there is a constant $H < \infty$ such that
$H_f(x) \leq H$ for all $x \in X$ and $f$ is called quasisymmetric if $H_f(x, r) \leq H$ for all $x \in X$ and $r > 0$. Furthermore, quasisymmetric mappings are shown to be absolutely continuous in measure in [HeKo98, Corollary 7.13], [HKST01, Theorem 8.12].

The equivalence of quasiconformal mappings and quasisymmetric mappings between metric spaces of locally $Q$-bounded geometry is obtained in [HKST01, Theorem 9.8]. The absolute continuity in measure of quasiconformal mappings in metric spaces with appropriate structure assumptions follows as a corollary.

Balogh, Koskela and Rogovin further show that if $X$ supports a 1-Poincaré inequality, and $h_f < \infty$ on $X \setminus E$ where $E$ is a set of $\sigma$-finite $H^{Q-1}$-measure, and $h_f \leq H < \infty \mu$-almost everywhere on $X$, then $f \in N_{\text{loc}}^{1,Q}(X; Y)$ [BKR07, Theorem 5.1]. A recent result by the authors shows that the condition on $h_f$ can be relaxed by introducing an appropriate weight in the target space $Y$; we state a special case of Corollary 1.2 of [LaZh21] below.

By a weight we simply mean a nonnegative locally integrable function, and we consider the pointwise representative

$$w_Y(y) = \liminf_{r \to 0} \frac{1}{\nu(B_r(y))} \int_{B_r(y)} w_Y \, dv, \quad y \in Y.$$ (4.4)

**Theorem 4.5.** Let $(X, d, \mu)$ and $(Y, d_Y, \nu)$ be Ahlfors $Q$-regular spaces with $Q > 1$, and $X$ supports a 1-Poincaré inequality. In the space $Y$, let $w_Y > 0$ be a weight represented by (4.4). Let $\Omega \subset X$ be a bounded open set, and let $f: \Omega \to f(\Omega)$ be a homeomorphism such that $f(\Omega)$ is open and $\int_{f(\Omega)} w_Y \, d\nu < \infty$. Assume $E \subset \Omega$ is a set with $\sigma$-finite $(Q-1)$-Hausdorff measure such that $h_f(x) < \infty$ for $x \in \Omega \setminus E$ and $\frac{h_f(x)^Q}{w_F(f(x))} \in L^\infty(\Omega)$. Then $f \in D_Q^{\text{loc}}(\Omega; Y)$.

**Proof of Corollary 1.6.** From Theorem 4.5 we obtain $f \in D^Q(\Omega; Y)$. Now the result follows from Theorem 1.4. \hfill \Box

**Remark 4.6.** Note that in Corollary 1.6 we are able, perhaps unexpectedly, to think of $Y$ as a weighted space in order to show that $f \in AC^Q(\Omega; Y)$, even though $Q$-absolute continuity is defined by essentially thinking of $Y$ as an unweighted space.

Another observation is that we obtain $Q$-absolute continuity, which is defined by considering finite collections of arbitrary disjoint balls, from a condition on the quantity $h_f$ that only gives pointwise, asymptotic control of $f$.

A simple demonstration of applying the above corollary is given in the following example, which uses a similar construction as [LaZh21, Example 6.3].

**Example 4.7.** Consider the square $\Omega := (-1, 1) \times (-1, 1)$ on the unweighted plane $X = Y = \mathbb{R}^2$. Define the homeomorphism $f: \Omega \to f(\Omega) = (-2/3, 2/3)$:

$$f(x_1, x_2) := \begin{cases} (x_1, \frac{2}{3}x_2^{3/2}), & x_2 \geq 0, \\ (x_1, -\frac{2}{3}|x_2|^{3/2}), & x_2 \leq 0. \end{cases}$$
We have $|Df| = \sqrt{1 + |x_2|^2}$. Thus $|Df| \in L^2(\Omega)$ and then clearly $f \in N^{1,2}(\Omega; f(\Omega))$. It is also easy to verify that $f$ is pseudomonotone, since $f$ is monotone in both $x_1$ and $x_2$. From the classical area formula, we can obtain directly that $f$ is in $AC^2(\Omega; f(\Omega))$.

Overall, for such a regular mapping $f$, all of these properties are easy to verify directly, but it is also interesting to check what we can obtain from considering the quantity $h_f$.

We easily find that $h_f(x_1, x_2) = |x_2|^{-1/2} - 1/2$. Now $h_f \notin L^\infty(\Omega)$, and so previous results on quasiconformal mappings, such as those of [BKR07] or [Wi14], do not give $f \in N^{1,2}(\Omega; f(\Omega))$, let alone $f \in AC^2(\Omega; f(\Omega))$. However, by considering the weight $w_f(y_1, y_2) = |y_2|^{-2/3}$, we get $\frac{h_f(y_1)}{w_f(f(y_1))} \in L^\infty(\Omega)$. We choose $E = (-1, 1) \times \{0\}$. Now Corollary 1.6 gives $f \in AC^2(\Omega; f(\Omega))$.

References

[BKR07] Z. Balogh, P. Koskela, and S. Rogovin, Absolute continuity of quasiconformal mappings on curves, Geom. Funct. Anal. 17 (2007), no. 3, 645-664.

[BjBj11] A. Björn and J. Björn, Nonlinear potential theory on metric spaces, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.

[BjBj19] A. Björn and J. Björn, Poincaré inequalities and Newtonian Sobolev functions on noncomplete metric spaces, J. Differential Equations 266 (2019), no. 1, 44–69.

[Bon05] D. Bongiron, Absolutely continuous functions in $\mathbb{R}^n$, J. Math. Anal. Appl. 303 (2005), no. 1, 119-134.

[Ch99] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999), 428–517.

[ChKl09] J. Cheeger and B. Kleiner, Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon-Nikodym property, Geom. Funct. Anal. 19 (2009), no. 4, 1017-1028.

[Ge60] F. W. Gehring, The definitions and exceptional sets for quasiconformal mappings, Ann. Acad. Sci. Fenn. Ser. A I No. 281 1960 28 pp.

[Ge62] F. W. Gehring, Rings and quasiconformal mappings in space, Trans. Amer. Math. Soc. 103 (1962), 353–393.

[Ha02] P. Hajlasz, Sobolev spaces on metric-measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces, (Paris, 2002), 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[HaKo00] P. Hajlasz and P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.

[HaZh16] P. Hajlasz and X. Zhou, Sobolev embedding of a sphere containing an arbitrary Cantor set in the image, Geom. Dedicata 184 (2016), 159–173.

[HKT07] T. Heikkinen, P. Koskela, and H. Tuominen, Sobolev-type spaces from generalized Poincaré inequalities, Studia Math. 181 (2007), no. 1, 1–16.

[HeKo95] J. Heinonen and P. Koskela, Definitions of quasiconformality, Invent. Math. 120 1995, no. 1, 61–79.

[HeKo98] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, Acta Math. 181 (1998), no. 1, 1–61.

[HKST01] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, Sobolev classes of Banach space-valued functions and quasiconformal mappings, J. Anal. Math. 85 (2001), 87–139.
[HKST15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, Sobolev spaces on metric measure spaces, An approach based on upper gradients, New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015. xii+434 pp.

[Hen02] S. Hencl, On the notions of absolute continuity for functions of several variables, Fund. Math. 173 (2002), no. 2, 175–189.

[HenKo14] S. Hencl and P. Koskela, Lectures on mappings of finite distortion, Lecture Notes in Mathematics, 2096. Springer, Cham, 2014. xii+176 pp.

[KaKo00] S. Kallunki and P. Koskela, Exceptional sets for the definition of quasiconformality, Amer. J. Math. 122 (2000), no. 4, 735–743.

[KeZh08] S. Keith and X. Zhong, The Poincaré inequality is an open ended condition, Ann. of Math. (2) 167 (2008), no. 2, 575–599.

[KMZ12] P. Koskela, J. Malý, and T. Zürcher, Luzin’s condition (N) and Sobolev mappings, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 23 (2012), no. 4, 455–465.

[LaZh21] P. Lahti and X. Zhou, Quasiconformal and Sobolev mappings in non-Ahlfors regular metric spaces when $p > 1$, preprint 2021, https://arxiv.org/abs/2109.01260v1

[Ma99] J. Malý, Absolutely continuous functions of several variables, J. Math. Anal. Appl. 231 (1999), no. 2, 492–508.

[MaMa95] J. Malý and O. Martio, Luzin’s condition (N) and mappings of the class $W^{1,n}$, J. Reine Angew. Math. 458 (1995), 19-36.

[MaZi15] N. Marola and W. Ziemer, Aspects of area formulas by way of Luzin, Radó, and Reichelderfer on metric measure spaces, J. Math. Soc. Japan 67 (2015), no. 2, 561–579.

[WiZu15] K. Wildrick and T. Zürcher, Sharp differentiability results for the lower local Lipschitz constant and applications to non-embedding, J. Geom. Anal. 25 (2015), no. 4, 2590-2616.

[Wi14] M. Williams, Dilatation, pointwise Lipschitz constants, and condition N on curves, Michigan Math. J. 63 (2014), no. 4, 687–700.

[Zh19] X. Zhou, Absolutely continuous functions on compact and connected 1-dimensional metric spaces, Ann. Acad. Sci. Fenn. Math. 44 (2019), no. 1, 281–291.

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