Constraining the $K\pi$ vector form factor by $\tau \to K\pi\nu_\tau$ and $K_{13}$ decay data

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Abstract

A subtracted dispersive representation of the $K\pi$ vector form factor, $F_{+}^{K\pi}$, is used to fit the Belle spectrum of $\tau \to K\pi\nu_\tau$ decays incorporating constraints from results on $K_{13}$ decays. Through the use of three subtractions, the slope and curvature of $F_{+}^{K\pi}$ are obtained directly from the data yielding $\lambda'_s = (25.49 \pm 0.31) \times 10^{-3}$ and $\lambda''_s = (12.22 \pm 0.14) \times 10^{-4}$. The phase-space integrals relevant for $K_{13}$ analyses are calculated. Additionally, from the pole position on the second Riemann sheet the mass and width of the $K^*(892)^+$ are found to be $m_{K^*(892)^+} = 892.0 \pm 0.5$ MeV and $\Gamma_{K^*(892)^+} = 46.5 \pm 1.1$ MeV. Finally, we study the $P$-wave isospin-1/2 $K\pi$ phase-shift and its threshold parameters.

Keywords: $\tau$ decays, Kaon decays, dispersion relations

1. Introduction

The non-perturbative physics of $K \to \pi l\nu_l$ ($K_{13}$) and $\tau \to K\pi\nu_\tau$, decays is governed by two Lorentz-invariant $K\pi$ form factors, namely the vector, denoted $F_{+}^{K\pi}(q^2)$, and the scalar, $F_{0}^{K\pi}(q^2)$. A good knowledge of these form factors paves the way for the determination of many parameters of the Standard Model, such as the quark-mixing matrix element $|V_{ut}|$ obtained from $K_{13}$ decays [1], or the strange-quark mass $m_s$ determined from the scalar QCD strange spectral function [2].

Until recently, the main source of experimental information on $K\pi$ form factors have been $K_{13}$ decays. Latterly, five experiments have collected data on semileptonic and leptonic $K$ decays: BNL-E865, KLOE, KTeV, ISTRA+, and NA48. The results from these analyses yielded an important amount of information on the form factors as well as stringent tests of QCD at low-energy and of the Standard Model itself (for a recent review on theoretical and experimental aspects of kaon physics we refer to Ref. [3]). Additional knowledge on the $K\pi$ form factors can be gained from the dominant Cabibbo-suppressed $\tau$ decay: the channel $\tau \to K\pi\nu_\tau$. Presently, the $B$ factories have become a superior source of high-statistics data for this reaction by virtue of the important cross-section for $e^+e^- \to \tau^+\tau^-$ around the $\Gamma(4S)$ peak. A detailed spectrum for $\tau \to K_S\pi^-\nu_\tau$ produced and analysed by Belle was published in 2007 [4]. Also, preliminary BaBar spectra with similar statistics have appeared recently in conference proceedings [5] and, finally, BESIII should produce results for this decay in the future [6]. The new data sets provide the substrate for up-to-date theoretical analyses of the $K\pi$ form factors. In Ref. [7] we have performed a reanalysis of the $\tau \to K\pi\nu_\tau$ spectrum of [4]. More recently, we carried out an analysis with restrictions from $K_{13}$ experiments [8].

On the theory side, the knowledge of these form factors consists of two tasks. The first of them is to determine their value at the origin, $F_{+,0}(0)$, crucial in order to disentangle the product $|V_{ut}|F_{+,0}(0)$. Historically, chiral perturbation theory has been the main tool to study $F_{+,0}(0)$, but recently lattice QCD collaborations have produced more accurate results for this quantity [9]. Second, one must know the energy dependence of the form factors, which is required when calculating phase-space integrals for $K_{13}$ decays or when analysing the detailed shape of the $\tau \to K\pi\nu_\tau$ spectrum. In our work we concentrate on the latter aspect of the problem and therefore it is convenient to introduce form factors normalised to one at the origin

$$F_{+,0}(q^2) = F_{+,0}(q^2)/F_{+,0}(0).$$

A salient feature of the form factors in the kinematical region relevant for $K_{13}$ decays, i.e. $m_{\tau}^2 < q^2 < m_{\tau}^2$
\((m_K - m_\pi)^2\), is that they are real. Within the allowed phase-space they admit a Taylor expansion and the energy dependence is customarily translated into constants \(\lambda_{+,0}^{(n)}\) defined as

\[
\tilde{F}_{+,0}(q^2) = 1 + \lambda_{+,0}^{(n)} q^2 + \frac{1}{2} \lambda_{+,0}^{(n)} \left( \frac{q^2}{m_\pi^2} \right)^2 + \cdots. \tag{2}
\]

In \(\tau \to K\pi\nu_\ell\) decays, however, since \((m_K + m_\pi)^2 < q^2 < m_\pi^2\), one deals with a different kinematical regime in which the form factors develop imaginary parts, rendering the expansion of Eq. (2) inadmissible. One must then resort to more sophisticated treatments. Moreover, in order to fully benefit from the available experimental data, it is desirable to employ representations of the form factors that are valid for both \(K_0\) and \(\tau \to K\pi\nu_\ell\) decays. Dispersive representations of the form factors provide a powerful tool to achieve this goal.

From general principles, the form factors must satisfy a dispersion relation. Supplementing this constraint with unitarity, the dispersion relation has a well-known closed-form solution within the elastic approximation referred to as the Omnès representation \([10]\). Although simple, this solution requires the detailed knowledge of the phase of \(F_+(s)\) up to infinity, which is unrealistic. An advantageous strategy to circumvent this problem is to treat them as free parameters that capture our ignorance of the higher energy part of the integral. The constants \(\lambda'\) and \(\lambda''\) can then be determined through the fit. The main advantage of this procedure, advocated for example in Refs. \([7, 11, 12]\), is that the subtraction constants turn out to be less model dependent as they are determined by the best fit to the data. It is important to stress that Eq. (3) remains valid beyond the elastic approximation provided \(\delta(s)\) is the phase of the form factor, instead of the corresponding scattering phase. But, of course, in order to employ it in practice we must have a model for the phase. As described in detail in Ref. \([7]\), we take a form inspired by the RChT approximation of Refs. \([13]\) with two vector resonances. For the detailed expressions we refer to the original works. With Eq. (3), the transition from the kinematical region of \(\tau \to K\pi\nu_\ell\) to that of \(K_0\) decays is straightforward and the dominant low-energy behaviour of \(\tilde{F}_+(s)\) is encoded in \(\lambda'_\ell\). The cut-off \(s_{cut}\) in the dispersion integral is introduced to quantify the suppression of the higher energy part of the integral. The stability of the results is checked varying this cut-off in a wide range from 1.8 GeV < \(s_{cut}\) < \(\infty\). As a final comment, since \(\alpha_{1,2}\) are determined by the data, in the limit \(s \to \infty\) the asymptotic behaviour of \(\tilde{F}_+(s)\) cannot be satisfied. This is so because a perfect cancellation between terms containing \(\alpha_1\) and \(\alpha_2\) with polynomial terms coming from the dispersion integral must occur in order to guarantee that \(\tilde{F}_+(s)\) vanishes as \(1/s\). We have checked that our form factor, within the entire range where we apply it (and beyond), is indeed a decreasing function of \(s\) which renders our approach credible.

In \(\tau \to K\pi\nu_\ell\) decays, the scalar form factor is suppressed kinematically. Although marginal, the contribution from \(F_0\) cannot be neglected in the lower energy part of the spectrum. Here, we keep this contribution fixed using the results for \(F_0\) from the coupled-channel dispersive analysis of Refs. \([2, 13]\).

2. Fits to \(\tau \to K\pi\nu_\ell\) with constraints from \(K_0\)

The analysis of the spectrum for \(\tau \to K\pi\nu_\ell\) produces a wealth of physical results, many of them with great accuracy, e.g., the mass and width of the \(K'(892)\). We have advocated by means of Monte Carlo simulations that a joined analysis of \(\tau \to K\pi\nu_\ell\) and \(K_0\) spectra further constrains the low-energy part of the vector form-factor yielding results with a better precision \([7]\). This idea was pursued in our recent work \([8]\).

In order to include the experimental information available from \(K_0\) decays—and for the want of true un-
folded data sets from these experiments—we adopt the following strategy. In our fits, the $\chi^2$ that to be minimised contains a standard part from the $\tau \to K\pi\nu$ spectrum and a piece which constrains the parameters $\lambda'_{\pi}^{\rm th}$ using information from $K_{l3}$ experiments. This is realized in practice as

$$\chi^2 = \sum_{i=1}^{90} \left( \frac{(N_i^{\rm th} - N_i^{\rm exp})^2}{\sigma_i^{\rm th}} + \frac{(B_{K\pi}-B_i^{\exp})^2}{\sigma_{B_{K\pi}}^2} \right) + (\lambda'_{\pi}^{\rm th} - \lambda'_{\pi}^{\rm exp})^T V^{-1}(\lambda'_{\pi}^{\rm th} - \lambda'_{\pi}^{\rm exp}),$$

where the first two terms on the r.h.s. are those of a fit to the spectrum of $\tau \to K\pi\nu$, and the third one encodes the information from $K_{l3}$ analyses and acts as a sort of prior (in the Bayesian sense) for the parameters $\lambda'_{\pi}$ and $\lambda''_{\pi}$. In the last equation the theoretical number of events $N_i^{\rm th}$ in the $i$-th bin is taken to be (as explained in Ref. [13])

$$N_i^{\rm th} = N_{\ell} \prod_{j=1}^{2} \frac{2}{3} \Delta_i^j \frac{1}{\Gamma_{B_{K\pi}}} \frac{d\Gamma_{B_{K\pi}}(s_j)}{ds_j},$$

where $N_{\ell}$ is the total number of events, the factor $\frac{1}{\Gamma_{B_{K\pi}}}$ and $\frac{2}{3}$ account for the fact that the $K_{l3}$ (or $K_{\ell3}$) channel was analysed, $\Delta_i^j$ is the width of the $i$-th bin, $\Gamma_{B_{K\pi}}$ is the total $\tau$ decay width, $B_{K\pi}$ is a normalisation constant that, for a perfect description of the spectrum, should be the $\tau \to K\pi\nu$, branching ratio, and, finally, $s_j$ is the centre of the $i$-th bin. Furthermore, $N_i^{\rm exp}$ and $\sigma_i^{\rm exp}$ are, respectively, the experimental number of events in the Belle spectrum and the corresponding uncertainty in the $i$-th bin. The prime in the symbol of sum indicates that bins 5, 6, and 7 are excluded from the minimisation.

The second term on the right-hand side of Eq. (4) introduces an additional restriction that allows us to treat the normalisation $B_{K\pi}$ of Eq. (5) as a free parameter.

In the last term of Eq. (4), the vectors $\lambda'^{\rm th}_{\pi}$ are given by

$$\lambda'^{\rm th}_{\pi} = \left( \begin{array}{c} \lambda'^{\rm th}_{\pi,1} \\ \lambda'^{\rm th}_{\pi,2} \end{array} \right),$$

and the $2 \times 2$ matrix $V$ is the experimental covariance for $\lambda'^{\rm th}_{\pi}$ such that

$$V_{ij} = \rho_{ij} \sigma_i \sigma_j,$$

where the indices refer to $\lambda'_{\pi}$ and $\lambda''_{\pi}$, $\rho_{ij}$ is the correlation coefficient ($\rho_{ij} = 1$ if $i = j$), and $\sigma_i$ the experimental errors on $\lambda'_i$ and $\lambda''_i$. For the experimental values we employ the results of the compilation of $K_{l3}$ analyses performed by Antonelli et al. for the FlaviaNet Working Group on Kaon Decays in Ref. [13]. $\lambda'^{\exp}_{\pi} = (24.9 \pm 1.1) \times 10^{-3}$, $\lambda''_{\pi}^{\exp} = (16 \pm 5) \times 10^{-3}$, and $\rho_{\lambda'_{\pi},\lambda''_{\pi}} = -0.95$.

### 2.1. Results

From the minimisation of the $\chi^2$ of Eq. (4) a collection of physical results can be derived. Some of them are obtained directly from the fit, such as $\lambda'_{\pi}$ and $\lambda''_{\pi}$ and the mass and width of the $K^{*}(892)$. With the form factor under control, one can then obtain other results such as the phase-space integrals for $K_{l3}$ decays. In order to control the uncertainties and the consistency of the results one must check the stability of the fit with respect to the cut-off $s_{\text{cut}}$ of Eq. (3). The detailed tables of Ref. [8] attest that the results are indeed rather stable, but in some cases a residual $s_{\text{cut}}$ dependence contributes to the final uncertainty we quote. Here, we present the main results of Ref. [8]. A careful comparison with other results found in the literature can be found in that reference.

We start by quoting our final results for the mass and the width of the $K^{*}(892)^+$

$$m_{K^{*-}(892)^+} = 892.03 \pm (0.19)_{\text{stat}} \pm (0.44)_{\text{sys}} \text{ MeV},$$

$$\Gamma_{K^{*-}(892)^+} = 46.53 \pm (0.38)_{\text{stat}} \pm (1.0)_{\text{sys}} \text{ MeV}. \quad (8)$$

These results are obtained from the complex pole position on the second Riemann sheet, $s_{\text{cut}}$, following the definition $\sqrt{s_{\text{cut}}} = m_{K^{*}} - (i/2)\Gamma_{K^{*}}$. It is important to stress that the mass and width thus obtained are rather different from the parameters that enter our description of the phase of $F_{\pi}(s)$. When comparing results from different works one must always be sure that the same definition is used in all cases. In Ref. [8], we showed that our results are compatible with others provided the pole position prescription is employed for all the analyses.

The final results for the parameters $\lambda'_{\pi}$ and $\lambda''_{\pi}$ read

$$\lambda'_{\pi} \times 10^3 = 25.49 \pm (0.30)_{\text{stat}} \pm (0.06)_{\text{sys}},$$

$$\lambda''_{\pi} \times 10^4 = 12.22 \pm (0.10)_{\text{stat}} \pm (0.10)_{\text{sys}}. \quad (9)$$

In this case, the uncertainty from the variation of $s_{\text{cut}}$ contributes as indicated. From the expansion of Eq. (3) we can calculate the third coefficient of a Taylor series of the type of Eq. (4). We find

$$\lambda''_{\pi} \times 10^5 = 8.87 \pm (0.08)_{\text{stat}} \pm (0.05)_{\text{sys}}. \quad (10)$$

These results are in good agreement with other analyses but have smaller uncertainties since our fits are constrained by $\tau \to K\pi\nu$ and $K_{l3}$ experiments.

Once the low-energy behaviour of the vector form-factor is obtained from the fit, we can compare it to the equivalent chiral expansion in order to determine the low-energy constant $L''_{\pi}$. Using the $O(p^3)$ expressions of Ref. [14] with $F_0 = F_2$ we obtain

$$L''_{\pi}(m_{K^{*}})(p^2 = 0) \times 10^3 = 5.19 \pm (0.07)_{\text{stat}}. \quad (11)$$
It is well known that the dominant uncertainty is given by the truncation of the series at $O(p^4)$. As an estimate of $O(p^6)$ effects we can employ $F^2_0 = F_a F_K$ which gives

$$L_a^2(m_K^2) |_{F_0=F_a F_K} \times 10^3 = 6.29 \pm 0.08)(\text{stat}) \quad (12)$$

In the extraction of $|V_{ud}|$ from the $K_{\bar{3}}$ decay widths, one must perform phase-space integrals where the form-factors play the central role. The integrals are defined in Ref. [1, 3]. From our form-factors we obtain the following results

$$I_{K_{\bar{3}}} = 0.15466(17), \quad I_{K_{\bar{3}}} = 0.10276(10), \quad I_{K_{\bar{3}}} = 0.15903(17), \quad I_{K_{\bar{3}}} = 0.10575(11). \quad (13)$$

The uncertainties were calculated with a MC sample of parameters obeying the results of our fits with the correlations properly included. The final uncertainties are competitive if compared with the averages of [3] and the central values agree.

Another interesting result that can be extracted from the $\tau \to K\pi\gamma$ spectrum is the $K\pi$ isospin-1/2 $P$-wave scattering phase. The decay in question is indeed a very clean source of information about $K\pi$ interactions, since the hadrons are isolated in the final state. Below inelastic thresholds, the phase of the form-factor is the scattering phase, as dictated by Watson’s theorem. The result of our $P$-wave phase is shown in Fig. 1 where we compare it with the results of two hadronic experiments [16, 17].

![Figure 1: Phase of the form factor $F_a(s)$ and results from LASS [18] and Estabrooks et al. [19]. The opening of the first inelastic channel, $K^+\pi^-$, is indicated by the dashed vertical line. The gray band represents the uncertainty due to $\delta_{\text{stat}}$.](image)

From the expansion of the corresponding partial-wave $T$-matrix element in the vicinity of the $K\pi$ threshold one can determine the $K\pi P$-wave threshold param-

etters. With our results, the first three read

$$m_{\pi}^2 \frac{1}{2} a_1 = 0.166(4), \quad m_{\pi}^2 \frac{1}{2} b_1 = 0.258(9), \quad m_{\pi}^2 \frac{1}{2} c_1 = 0.90(3). \quad (14)$$

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