GROUND STATE SOLUTIONS FOR FRACTIONAL SCALAR FIELD EQUATIONS UNDER A GENERAL CRITICAL NONLINEARITY

CLAUDIANOR O. ALVES
Universidade Federal de Campina Grande,
58429-970, Campina Grande - PB - Brazil

GIOVANY M. FIGUEIREDO
Universidade de Brasília, Campus Universitário Darcy Ribeiro
70910-900, Brasília - DF - Brazil

GAETANO SICILIANO*
Universidade de São Paulo, Departamento de Matemática - IME
Rua do Matão 1010, 05508-090, São Paulo - SP - Brazil

(Communicated by Enrico Valdinoci)

Abstract. In this paper we study existence of ground state solution to the following problem
\[(−Δ)^α u = g(u) \text{ in } \mathbb{R}^N, \quad u \in H^α(\mathbb{R}^N)\]
where \((−Δ)^α\) is the fractional Laplacian, \(α \in (0, 1)\). We treat both cases \(N \geq 2\) and \(N = 1\) with \(α = 1/2\). The function \(g\) is a general nonlinearity of Berestycki-Lions type which is allowed to have critical growth: polynomial in case \(N \geq 2\), exponential if \(N = 1\).

1. Introduction. In the present paper, we are interested in the existence of ground state solution for a class of nonlocal problem of the following type
\[(-\Delta)^α u = g(u), \quad \text{in } \mathbb{R}^N \quad (P)\]
where \(N \geq 1\), \(α \in (0, 1)\), \((-\Delta)^α\) denotes the fractional Laplacian operator and \(g\) is a \(C^1\)-function verifying some conditions which will be mentioned later on.

The main motivation for this paper comes from the papers Berestycki and Lions [6] and Berestycki, Gallouet and Kavian [7] which have studied the existence of solution for \((P)\) in the local case \(α = 1\), that is, for a class of elliptic equations like
\[-\Delta u = g(u), \quad \text{in } \mathbb{R}^N, \quad (1)\]

2000 Mathematics Subject Classification. Primary: 35J20; Secondary: 35A15, 35B33.
Key words and phrases. Fractional Laplacian, Berestycki-Lions type nonlinearity, critical growth.

The first author is supported was partially supported by CNPq/Brazil Proc. 304036/2013-7; the second author was partially supported by CNPq, Capes and FAPDF, Brazil; the third author was partially supported by CNPq, Capes and FAPESP, Brazil.

* Corresponding author.
where \( N \geq 2 \), \( \Delta \) denotes the Laplacian operator and \( g \) is a continuous function verifying some conditions. In [6], Berestycki and Lions have assumed \( N \geq 3 \) and the following conditions on \( g \):

\[
-\infty < \liminf_{s \to 0^+} \frac{g(s)}{s} \leq \limsup_{s \to 0^+} \frac{g(s)}{s} \leq -m < 0,
\]

\[
\limsup_{s \to +\infty} \frac{g(s)}{s^{2^* - 1}} \leq 0,
\]

there is \( \xi > 0 \) such that \( G(\xi) > 0 \), where

\[
G(s) = \int_0^s g(t) \, dt.
\]

In [7], Berestycki, Gallouet and Kavian have studied the case where \( N = 2 \) and the nonlinearity \( g \) possesses an exponential growth of the type

\[
\limsup_{s \to +\infty} g(s) e^{\beta s^2} = 0, \quad \forall \beta > 0.
\]

In the two papers above mentioned, the authors have used the variational method to prove the existence of solution for (P). The main idea is to solve the minimization problem

\[
\min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}
\]

and

\[
\min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 0 \right\}
\]

for \( N \geq 3 \) and \( N = 2 \) respectively. After that, the authors showed that the minimizer functions of the above problem are in fact ground state solutions of (1). By a ground state solution, we mean a solution \( u \in H^1(\mathbb{R}^N) \) which satisfies

\[
E(u) \leq E(v) \quad \text{for all nontrivial solution } v \text{ of (1)},
\]

where \( E : H^1(\mathbb{R}^N) \to \mathbb{R} \) is the energy functional associated to (P) given by

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx.
\]

After, Jeanjean and Tanaka in [19] showed that the mountain pass level of \( E \) is a critical level and it is indeed the lowest critical level.

In the above mentioned papers, the nonlinearity does not have critical growth. Motivated by this fact, Alves, Montenegro and Souto in [1] have studied the existence of ground state solution for (P) by supposing that \( g(s) = f(s) - s \) and that \( f \) may have critical growth, more precisely, the following condition were considered:

\[
\lim_{s \to 0} \frac{f(s)}{s} = 0
\]

\[
\limsup_{s \to +\infty} \frac{f(s)}{s^{2^* - 1}} \leq 1, \quad \text{if } N \geq 3
\]

\[
\lim_{s \to +\infty} \frac{f(s)}{s^{2^*}} = 0 \quad (+\infty) \quad \text{if } \beta > \beta_0 \quad (\beta < \beta_0) \quad \text{when } N = 2
\]

\[
H(s) = f(s) s - 2F(s) \geq 0 \quad \forall s > 0 \quad \text{where } F(s) = \int_0^s f(t) \, dt,
\]

there is \( \tau > 0 \) and \( q \in (2, 2^*) \) if \( N \geq 3 \) and \( q \in (2, +\infty) \) if \( N = 2 \) such that

\[
f(s) \geq \tau s^{q-1}, \quad \forall s \geq 0.
\]
By using the variational method, the authors in [1] give a unified approach in order to deal with subcritical and critical case. However, we would like to point out that the Concentration Compactness Principle of Lions [21] was crucial for the case \( N \geq 3 \). For the case \( N = 2 \), as in the previous references, a Trudinger-Moser inequality due to Cao [9] was the main tool used. A similar study was made for the critical case and \( N \geq 3 \) in Zhang and Zou [26].

After a bibliographic review, we have observed that there is no version of the paper [1] for the fractional Laplacian operator. Indeed the interest in fractional semilinear equations has grown in these recent years due to many applications, e.g. in physics and economy; we refer the reader to the recent monograph [8].

Motivated by this fact, we have decide to study this class of problem. However, we would like point out that some estimates made in [1] are not immediate for fractional Laplacian operator. For example, there is some restriction to use Concentration Compactness Principle of Lions [21], see [12] and Palatucci and Pisante [23] for the dimension \( N \geq 2 \) and \( \alpha \in (0, 1) \). To overcome this difficulty, we use a new approach which do not use the Concentration Compactness Principle of [23]. For the dimension \( N = 1 \) and \( \alpha = \frac{1}{2} \), we use a Trudinger-Moser inequality due to Ozawa [24] which also permits to apply variational methods in this case. Here, it is very important to mention that Zhan, do ´O and Squassina [27, Theorem 4.1] studied the existence of ground state solution for \((P)\) for \( N \geq 2 \), by supposing that \( g \) satisfies

\[
\lim_{s \to +\infty} \frac{g(s)}{s^{2^*_\alpha - 1}} = b > 0.
\]

These condition is not assumed in our paper, and so, our results complete the study made in that paper.

Before stating our main results, we must fix some notations. We will look for weak solutions of \((P)\) hence the natural setting involves the fractional Sobolev spaces \( H^\alpha(\mathbb{R}^N) \) defined as

\[
H^\alpha(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{\alpha/2} u \in L^2(\mathbb{R}^N) \}
\]

endowed with scalar product and (squared) norm given by

\[
(u, v) = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} v \, dx + \int_{\mathbb{R}^N} uv \, dx,
\]

\[
\| u \|^2 = \| (-\Delta)^{\alpha/2} u \|^2_L + |u|^2.
\]

It is well known that \( H^\alpha(\mathbb{R}^N) \) is a Hilbert space with the above scalar product. We are denoting with \( |u|_p = (\int_{\mathbb{R}^N} |u|^p \, dx)^{1/p} \) the \( L^p \)-norm of \( u \), and by \((-\Delta)^\alpha\) the fractional Laplacian, which is the pseudodifferential operator defined via the Fourier transform in the following way

\[
\mathcal{F}((-\Delta)^\alpha u) = |\cdot|^{2\alpha} \mathcal{F} u.
\]

It is known that \( H^\alpha(\mathbb{R}^N) \) has continuos embedding into \( L^q(\mathbb{R}^N) \) for suitable \( q \) depending on \( N \): we will denote by \( C_q > 0 \) the embedding constant.

It is useful to introduce also the homogeneous fractional Sobolev space

\[
D^{\alpha, 2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx < \infty \right\}
\]

where hereafter \( 2^*_\alpha = \frac{2N}{N-2\alpha} \) for \( N \geq 2 \). It is well known that the following inequality holds

\[
S \left( \int_{\mathbb{R}^N} |u|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx \quad \text{for all } u \in D^{\alpha, 2}(\mathbb{R}^N)
\]

(2)
for some positive $S > 0$. For these facts and the relation between the fractional Laplacian and the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$, we refer the reader to classical books on Sobolev space, and to the monograph [11].

We will study (P) by variational methods: its solutions will be found as critical points of a $C^1$ functional $I : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$. Actually, following some ideas of [1], our results concern the existence of ground state solutions, that is a solution $u \in H^\alpha(\mathbb{R}^N)$ such that $I(u) \leq I(v)$ for every nontrivial solution $v \in H^\alpha(\mathbb{R}^N)$ of (P). In view of this, we make some assumptions on the nonlinearity. More precisely, we assume that $g(s) = f(s) - s$ where $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$-function satisfying

\begin{enumerate}[\textbf{(f1)}]
  \item $\displaystyle \lim_{s \to 0^+} f(s)/s = 0$;
  \item $\displaystyle \limsup_{s \to +\infty} f(s)/s^{2\alpha - 1} \leq 1$;
  \item $f(s)s - 2F(s) \geq 0$ for $s > 0$, where $F(s) = \int_0^s f(t) \, dt$;
  \item $f(s) \geq \tau s^{q-1}$, $s \in \mathbb{R}$ with $s \geq 0$, where
    \begin{itemize}
      \item If $N \geq 2$, we assume $q \in (2, 2_\ast^\alpha)$ and
        \[ \tau > \tau^* := \left(2(2\alpha-N)/2\alpha \right) g^{-N/2\alpha N} \left(2N/(N-2\alpha)\right)^{(N-2\alpha)/2\alpha} \left(q-2\right)^{(q-2)/2} C_{q/2}^{q/2}, \]
      \item If $N = 1$, we assume $q > 2$ and
        \[ \tau > \tau^* := \left(q-2\right)^{(q-2)/2} C_{q/2}^{q/2}. \]
    \end{itemize}
\end{enumerate}

In order to introduce the next assumption to deal with the case $N = 1$, we recall an important result due to T. Ozawa.

**Theorem 1.1** (see [24]). There exists $0 < \omega \leq \pi$ such that, for all $r \in (0, \omega)$, there exists $H_r > 0$ satisfying

\[ \int_\mathbb{R} (e^{ru^2} - 1) \, dx \leq H_r |u|_{2}^2, \]

for all $u \in H^{1/2}(\mathbb{R})$ with $|(-\Delta)^{1/4} u|_2 \leq 1$.

Our next assumption is then the following:

\begin{enumerate}[\textbf{(f5)}]
  \item $\displaystyle \lim_{s \to +\infty} f(s) e^{\beta s^2} = 0$, $\forall \beta > \beta_0$, and $\displaystyle \lim_{s \to +\infty} f(s) e^{\beta s^2} = +\infty$, $\forall \beta < \beta_0$.
\end{enumerate}

As we can see, a critical growth for the function $f$ is allowed. Note also that a weaker condition than the usual Ambrosetti-Rabinowitz condition is imposed on $f$, see condition (f3).

Our main results are the following one.

**Theorem 1.2.** Suppose that $N \geq 2$, $\alpha \in (0, 1)$ and $f$ satisfies (f1) – (f4). Then problem (P) admits a ground state solution which is non-negative, radially symmetric and decreasing.
Theorem 1.3. Suppose that $N = 1, \alpha = \frac{1}{2}$ and $f$ satisfies (f1), (f4) and (f5). Then problem (P) admits a ground state solution which is non-negative, radially symmetric and decreasing.

The plan of the paper is the following. In Section 2 we study the case $N \geq 2$. We first introduce the variational framework, then give some preliminary results and Lemmas which will be useful to prove Theorem 1.2. In Section 3 we consider the case $N = 1$ and $\alpha = \frac{1}{2}$, where again, after some preliminaries, the proof of Theorem 1.3 is given.

Before concluding this introduction, we would like to cite some papers involving the fractional Laplacian operator where the problem is related to the problem (P) in some sense, see for example, Alves, do Ó and Squassina [2], Ambrosio [3], Barrios, Colorado, Servadei and Soria [5], Dipierro, Medina, Peral and Valdinoci [12, 13], do Ó, Miyagaki and Squassina [14], Frank and Lenzmann [16], Felmer, Quass and Tan [17], Iannizzotto and Squassina [18], Zhang, do Ó and Squassina [27] and their references. We finally remark that the case $N \geq 2$ with a subcritical nonlinearity has been treated in [10] with a different approach.

Notations As a matter of notations, we will use in all the paper the letter $C, C, C'$, ... to denote suitable positive constants whose exact value is insignificant for our purpose.

2. The case $N \geq 2$.

2.1. The variational framework. The energy functional $I : H^\alpha(\mathbb{R}^N) \to \mathbb{R}$ associated to equation (P) is defined as follows

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(u) \, dx.$$ 

Under assumptions (f1) and (f2), $I \in C^1(H^\alpha(\mathbb{R}^N), \mathbb{R})$ with Frechét derivative given by

$$I'(u)[v] = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/2} u(-\Delta)^{\alpha/2} vdx + \int_{\mathbb{R}^N} uvdx - \int_{\mathbb{R}^N} f(u)vdx, \quad \forall u, v \in H^\alpha(\mathbb{R}^N).$$

Hence the critical points are easily seen to be weak solutions to (P).

We remark two inequalities which will be frequently used in the sequel. From (f1) and (f2), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon |s| + C_\varepsilon |s|^{2^*_\alpha - 1} \quad \text{for all } s > 0 \quad (3)$$

and, then by integration,

$$|F(s)| \leq \frac{\varepsilon}{2} s^2 + C_\varepsilon |s|^{2^*_\alpha} \quad \text{for all } s > 0. \quad (4)$$

Once we intend to find nonnegative solution, we will assume that $f(s) = 0$ for every $s \leq 0$. Let us consider the set of non-zero critical points of $I$, that is non trivial solution of (P),

$$\Sigma = \{u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} : I'(u) = 0\},$$

and define

$$m = \inf_{u \in \Sigma} I(u)$$

the so called ground state level.
Now, denoting with \( G(u) = F(u) - \frac{u^2}{2} \), the primitive of \( g(u) = f(u) - u \), let us introduce the set
\[
\mathcal{M} = \left\{ u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}
\] (5)
and
\[
T(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx,
\]
\[D = \inf_{u \in \mathcal{M}} T(u).
\] (6)
In particular
\[2D = \inf_{u \in \mathcal{M}} \left\{ \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx \right\}.
\]
It is worth to point out that if we define the \( C^1 \) functional
\[
J(u) := \int_{\mathbb{R}^N} G(u) \, dx - 1,
\]
it holds from (f3):
\[u \in \mathcal{M} \implies J'(u)[u] = \int_{\mathbb{R}^N} (f(u)u - u^2) \, dx = \int_{\mathbb{R}^N} (f(u)u - 2F(u)) \, dx + 2 \int_{\mathbb{R}^N} G(u) \geq 2.
\] (7)
The last information will be used later on.

In addition, we define the min-max level associated to the functional \( I \)
\[b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))
\] (8)
where \( \Gamma = \{ \gamma \in C([0,1], H^\alpha(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \} \)
which is not empty since \( I \) has a Mountain Pass Geometry.

Let us define also the set, usually called Pohozaev manifold,
\[
\mathcal{P} = \left\{ u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} : \frac{N - 2\alpha}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx = N \int_{\mathbb{R}^N} G(u) \, dx \right\}
\]
which, according to [10, Proposition 4.1], contains any weak solution of (P). If we denote by
\[p = \inf_{u \in \mathcal{P}} I(u),
\]
from [20, Lemma 2.4] it holds that
\[p = \frac{\alpha}{N} \left( \frac{N - 2\alpha}{2N} \right)^{(N-2\alpha)/2\alpha} (2D)^{N/2\alpha}.
\] (9)

2.2. Some preliminary stuff. At this point we establish some preliminary results which will be useful in order to prove Theorem 1.2.

Lemma 2.1. It holds
\[\frac{\alpha}{N} \left( \frac{N - 2\alpha}{2N} \right)^{(N-2\alpha)/2\alpha} (2D)^{N/2\alpha} \leq b
\]
where \( b \) is the min-max level of \( I \) defined in (8).
Proof. Indeed, from [20, Lemma 2.3], for each $\gamma \in \Gamma$ with
\[ \Gamma = \{ \gamma \in C([0,1], H^\alpha(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \} \]
it results $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$. Then, there exists $t_0 \in [0,1]$ such that $\gamma(t_0) \in \mathcal{P}$. So
\[ p \leq I(\gamma(t_0)) \leq \max_{t \in [0,1]} I(\gamma(t)) \]
from where it follows that $p \leq b$ and the result follows from (9). \qed

The next result is standard. We recall the proof for the reader’s convenience.

**Lemma 2.2.** The set $\mathcal{M}$ defined in (5) is not empty and a $C^1$ manifold.

**Proof.** Observe that, fixed $0 \neq \varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$ the function $h(t) = \int_{\mathbb{R}^N} G(t\varphi) \, dx$ is strictly negative for small $t$ and $h'(t) > 0$ for $t$ large; this implies that there exists some $t > 0$ such that $t\varphi \in \mathcal{M}$. Moreover $\mathcal{M}$ is a $C^1$ manifold in virtue of (7). \qed

The next steps consist in proving the boundedness of the minimizing sequences in $H^\alpha(\mathbb{R}^N)$ for the problem
\[ \min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u|^2 \, dx : u \in H^\alpha(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}. \tag{10} \]

**Lemma 2.3.** Any minimizing sequence $\{u_n\} \subset \mathcal{M}$ for $T$ is bounded in $H^\alpha(\mathbb{R}^N).

**Proof.** Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for $T$, then
\[ T(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u_n|^2 \, dx \to D \quad \text{as } n \to +\infty \]
and
\[ \int_{\mathbb{R}^N} G(u_n) \, dx = 1, \quad \text{that is, } \int_{\mathbb{R}^N} \left( F(u_n) - \frac{1}{2} u_n^2 \right) \, dx = 1. \]
Then
\[ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u_n|^2 \, dx \leq C \quad \text{for all } n \in \mathbb{N} \text{ and for some constant } C > 0 \tag{11} \]
and
\[ \int_{\mathbb{R}^N} F(u_n) \, dx = 1 + \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \, dx. \]
By using (3) with $\varepsilon = 1/2$, we get
\[ 1 + \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \, dx \leq \frac{1}{4} \int_{\mathbb{R}^N} u_n^2 \, dx + C_{1/2} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx. \]
Then, for every $n \in \mathbb{N}$, by using (11), it follows
\[ \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \, dx \leq C_{1/4} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq C_{1/4}C \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u_n|^2 \, dx \leq \mathcal{C}. \]
Consequently $\{u_n\}$ is bounded also in $L^2(\mathbb{R}^N)$ and this ensures its boundedness in $H^\alpha(\mathbb{R}^N)$.

By the Ekeland Variational Principle (see [15]), we can assume that the minimizing sequence $\{u_n\}$ is also a Palais-Smale sequence, that is, there exists a sequence of Lagrange multipliers $\{\lambda_n\} \subset \mathbb{R}$ such that
\[ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u_n|^2 \, dx \to D \quad \text{as } n \to +\infty \tag{12} \]
and
\[ T'(u_n) - \lambda_n J'(u_n) \longrightarrow 0 \text{ in } (H^\alpha(\mathbb{R}^N))^{-1} \quad \text{as } n \to +\infty. \] (13)

In the remaining part of this section, \( \{\lambda_n\} \) will be the associated sequence of Lagrange multipliers. At this point it is useful to establish some properties of the levels \( D \) and \( b \).

**Lemma 2.4.** The number \( D \) given by (6) is positive, namely, \( D > 0 \).

**Proof.** Clearly by definition \( D \geq 0 \). Suppose, by contradiction, that \( D = 0 \). If \( \{u_n\} \) is a minimizing sequence for \( D = 0 \), then
\[
\frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx \to 0 \quad \text{as } n \to +\infty
\]
and
\[
1 = \int_{\mathbb{R}^N} G(u_n) \, dx = \int_{\mathbb{R}^N} \left( F(u_n) - \frac{1}{2} u_n^2 \right) \, dx.
\]
Then, for any \( \varepsilon > 0 \), see (4),
\[
1 + \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \, dx = \int_{\mathbb{R}^N} F(u_n) \, dx \leq \varepsilon \int_{\mathbb{R}^N} u_n^2 \, dx + C \varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx
\]
so that
\[
1 + \frac{1}{2} (1 - \varepsilon) \int_{\mathbb{R}^N} u_n^2 \, dx \leq C \varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq C \varepsilon \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx.
\]
By choosing \( \varepsilon = 1/2 \), we obtain
\[
1 \leq C_{1/2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx \longrightarrow 0 \quad \text{as } n \to +\infty.
\]
This contradiction concludes the proof. \( \square \)

**Lemma 2.5.** The sequence of Lagrange multipliers \( \{\lambda_n\} \) associated to the minimizing sequence \( \{u_n\} \) is bounded. More precisely, we have that
\[
0 < \liminf_{n \to +\infty} \lambda_n \leq \limsup_{n \to +\infty} \lambda_n \leq D.
\]

Hence, for some subsequence, still denoted by \( \{\lambda_n\} \), we can assume that \( \lambda_n \to \lambda^* \), for some \( \lambda^* \in (0, D) \).

**Proof.** By (13),
\[
2T(u_n) = T'(u_n)u_n = \lambda_n J'(u_n)u_n + o_n(1).
\] (14)

Then, from (7)
\[
2T(u_n) \geq 2\lambda_n + o_n(1)
\]
which implies, taking into account (12),
\[
\limsup_{n \to +\infty} \lambda_n \leq \frac{1}{2} \limsup_{n \to +\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx = D.
\]
Since \( \{u_n\} \) is a bounded minimizing sequence, it is easy to see that \( |J'(u_n)|u_n| = |\int_{\mathbb{R}^N} g(u_n)u_n| \leq C \), and then by (14) and the fact that \( 2T(u_n) \to 2D > 0 \), we infer that
\[
\liminf_{n \to +\infty} \lambda_n > 0.
\]
The proof is thereby completed. \( \square \)
In the sequel, we will show that a minimizing sequence for (10) can be chosen nonnegative and radially symmetric around the origin. Note that for our proof we do not need to consider the “odd extension” of the nonlinearity, as it is usually done in the literature to show that the minimizing sequence can be replaced by the sequence of the absolute values. In fact we will prove that the minimizing sequence can be replaced, roughly speaking, with the sequence of the positive parts.

**Lemma 2.6.** Any minimizing sequence \( \{u_n\} \) for (10) can be assumed radially symmetric around the origin and nonnegative.

**Proof.** To begin with, we recall that \( F(s) = 0 \) for all \( s \leq 0 \). Thus, \( F(u_n) = F(u_n^+) \) for all \( n \in \mathbb{N} \) with \( u_n^+ = \max\{0, u_n\} \). From this, the equality

\[
\int_{\mathbb{R}^N} G(u_n) \, dx = 1, \quad \forall n \in \mathbb{N}
\]

leads to

\[
\int_{\mathbb{R}^N} G(u_n^+) \, dx \geq 1, \quad \forall n \in \mathbb{N}.
\]

Defining the function \( h_n : [0, 1] \to \mathbb{R} \) by

\[
h_n(t) = \int_{\mathbb{R}^N} G(tu_n^+) \, dx
\]

the conditions on \( f \) yield that \( h \) is continuous with \( h_n(1) \geq 1 \). Once \( u_n^+ \neq 0 \) for all \( n \in \mathbb{N} \), the condition \((f1)\) ensures that \( h_n(t) < 0 \) for \( t \) close to 0. Thus there is \( t_n \in (0, 1] \) such that \( h_n(t_n) = 1 \), that is,

\[
\int_{\mathbb{R}^N} G(t_n u_n^+) \, dx = 1, \quad \forall n \in \mathbb{N},
\]

implying that \( t_n u_n^+ \in \mathcal{M} \). On the other hand, we also know that

\[
\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n^+|^2 \, dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx.
\]

Once \( t_n \in (0, 1] \), the last inequality gives

\[
D \leq T(t_n u_n^+) \leq T(u_n) = D + o_n(1)
\]

that is,

\[
t_n u_n^+ \in \mathcal{M} \quad \text{and} \quad T(t_n u_n^+) \to D,
\]

showing that \( \{t_n u_n^+\} \) is a minimizing sequence for \( T \). Thereby, without loss of generality, we can assume that \( \{u_n\} \) is a nonnegative sequence.

Moreover, by noticing that

\[
\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n^+|^2 \, dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx, \quad \forall n \in \mathbb{N}
\]

(see [4, Theorem 3]) and

\[
\int_{\mathbb{R}^N} G(u_n^+) \, dx = \int_{\mathbb{R}^N} G(u_n) \, dx, \quad \forall n \in \mathbb{N}
\]

where \( u_n^+ \) is the Schwartz symmetrization of \( u_n \), any minimizing sequence can be assumed radially symmetric, non-negative and decreasing in \( r = |x| \).
In what follows, we will use that the embedding
\[ H^{\sigma}_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \]  
(15)
is compact for all \( p \in (2, 2^*_\alpha) \), see Lions [22] for more details.

Due to the boundedness in \( H^\alpha(\mathbb{R}^N) \) of the (non-negative and radial symmetric) minimizing sequence \( \{u_n\} \) (see Lemma 2.3) we can assume that \( \{u_n\} \) has a weak limit in \( H^\alpha(\mathbb{R}^N) \) denoted hereafter with \( u \). Observe also that, by the boundedness in \( L^2(\mathbb{R}^N) \) we have the uniform decay \( |u_n(x)| \leq C|x|^{-N/2} \), see [3, Lemma 1]. Therefore, passing to a subsequence, if necessary, we deduce that the weak limit \( u \) is non-negative, radially symmetric and decreasing.

It turns out that the weak limit \( u \) is a solution of the minimizing problem (6) we were looking for. Before seeing this, some preliminary lemmas are required in order to recover some compactness.

**Lemma 2.7.** Assume that \( v_n := u_n - u \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}v_n|^2 \, dx \to L > 0 \). Then
\[ D \geq 2^{-2\alpha/N} S. \]

**Proof.** First of all, we recall the limit \( T'(u_n) - \lambda_n J'(u_n) \to 0 \) as \( n \to +\infty \) gives
\[ T'(u_n)[u_n] - \lambda_n J'(u_n)[u_n] = o_n(1). \]
Using standard arguments, it is possible to prove that
\[ T'(u_n)[u_n] - \lambda_n J'(u_n)[u_n] = T'(v_n)[v_n] - \lambda_n J'(v_n)[v_n] + T'(u)[u] - \lambda^* J'(u)[u] + o_n(1) \]
and
\[ T'(u) - \lambda^* J'(u) = 0 \quad \text{in} \quad (H^\alpha(\mathbb{R}^N))^{-1}, \]
where \( \lambda^* \) is the same which appear in Lemma 2.5. Then \( T'(v_n)[v_n] - \lambda_n J'(v_n)[v_n] = o_n(1) \), or equivalently,
\[ \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}v_n|^2 \, dx = \lambda_n \int_{\mathbb{R}^N} f(v_n) v_n \, dx - \lambda_n \int_{\mathbb{R}^N} v_n^2 \, dx + o_n(1). \]
Using the growth conditions on \( f \), fixed \( q \in (2, 2^*_\alpha) \) and given \( \varepsilon > 0 \), there exists \( C = C(\varepsilon, q) > 0 \) such that
\[ f(t)t \leq \varepsilon t^2 + C|t|^q + (1 + \varepsilon)|t|^{2^*_\alpha}, \quad \forall t \in \mathbb{R}. \]
From this,
\[ \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}v_n|^2 \, dx \]
\[ \leq \lambda_n \left( \varepsilon \int_{\mathbb{R}^N} v_n^2 \, dx + C \int_{\mathbb{R}^N} |v_n|^q \, dx + (1 + \varepsilon) \int_{\mathbb{R}^N} |v_n|^{2^*_\alpha} \, dx \right) + o_n(1). \]
Now, using the definition of \( S \), see (2), we get
\[ \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}v_n|^2 \, dx \]
\[ \leq \lambda_n \left( \varepsilon \int_{\mathbb{R}^N} v_n^2 \, dx + C \int_{\mathbb{R}^N} |v_n|^q \, dx + (1 + \varepsilon) \left( \frac{1}{S} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}v_n|^2 \, dx \right)^{\frac{2^*_\alpha}{2}} \right) + o_n(1). \]
(16)
Passing to the limit in (16), recalling that \( \{v_n\} \) is bounded,
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} v_n|^2 \, dx \longrightarrow L
\]  
and that \( v_n \to 0 \) in \( L^q(\mathbb{R}^N) \) (see (15)), we find
\[
L \leq \lambda^* \left( \varepsilon C_1 + (1 + \varepsilon) \left( \frac{L}{S} \right)^{2^*_\alpha/2} \right).
\]

By the arbitrariness of \( \varepsilon \) and the fact that \( \lambda^* \leq D \) (recall Lemma 2.5), we derive
\[
L \leq \frac{2}{N} \frac{\alpha}{N} \left( \frac{N - 2\alpha}{2\alpha} \right)^{(N-2\alpha)/2\alpha} 2^{(N-2\alpha)/2\alpha} S^{N/2\alpha}.
\]

On the other hand (17) implies that \( L = 2D - \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx \leq 2D \). Hence (18) becomes
\[
S^{2^*_\alpha/2} \leq \frac{2}{N} \frac{\alpha}{N} \left( \frac{N - 2\alpha}{2\alpha} \right)^{(N-2\alpha)/2\alpha} 2^{(N-2\alpha)/2\alpha}, \quad \text{i.e.} \quad D \geq 2^{-2\alpha/N} S
\]
and the proof is finished.

In the next result the condition \( \tau > \tau^* \) given in (f4) plays a crucial role.

**Lemma 2.8.** It holds
\[
b < \frac{\alpha}{N} \left( \frac{N - 2\alpha}{2N} \right)^{(N-2\alpha)/2\alpha} 2^{(N-2\alpha)/2\alpha} S^{N/2\alpha}.
\]

**Proof.** Take \( \varphi \in H^\alpha(\mathbb{R}^N) \) such that \( \|\varphi\| = 1 \) and \( |\varphi|^2_q = C_q^{-1} \). From definition of \( b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \) and (f4)
\[
b \leq \max_{t \geq 0} I(t \varphi) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} - \frac{t^q}{q} \int_{\mathbb{R}^N} |\varphi|^q \, dx \right\}
\]
\[
= \max_{t \geq 0} \left\{ \frac{t^2}{2} - \frac{t^q}{q} C_q^{-q/2} \right\}
\]
\[
= \frac{q - 2}{2q} \frac{C_q^{q/(q-2)}}{\tau^{2/(q-2)}}.
\]
This gives (by the definition of \( \tau^* \)) exactly the conclusion.

**Lemma 2.9.** If \( u_n \to u \) in \( H^\alpha(\mathbb{R}^N) \), then \( u_n \to u \) in \( D^{\alpha,2}(\mathbb{R}^N) \). In particular, \( u_n \to u \) in \( L^{2^*_\alpha}(\mathbb{R}^N) \).

**Proof.** Of course \( v_n = u_n - u \to 0 \) in \( H^\alpha(\mathbb{R}) \). Suppose by contradiction that \( u_n \not\to u \) in \( D^{\alpha,2}(\mathbb{R}^N) \). Thereby, \( \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} v_n|^2 \, dx \to L > 0 \) for some subsequence. Then, by Lemma 2.7,
\[
D \geq 2^{-2\alpha/N} S.
\]  
On the other hand, from Lemma 2.1
\[
\frac{\alpha}{N} \left( \frac{N - 2\alpha}{2N} \right)^{(N-2\alpha)/2\alpha} (2D)^{N/2\alpha} \leq b,
\]
from which, using (19), it follows that
\[
\frac{\alpha}{N} \left( \frac{N - 2\alpha}{2N} \right)^{(N-2\alpha)/2\alpha} 2^{(N-2\alpha)/2\alpha} S^{N/2\alpha} \leq b.
\]
This contradicts Lemma 2.8 and finishes the proof. \hfill \Box

2.3. **Proof of Theorem 1.2.** At this point we wish to show that $D$ is attained by $u$, where $u$ is the weak limit of $\{u_n\}$. First of all, we know that

$$T(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx \leq \liminf_{n \to +\infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u_n|^2 \, dx = D \tag{20}$$

so we just need to prove that $u \in \mathcal{M}$.

By [3, Lemma 1], there is $R > 0$ such that

$$\frac{1}{2} u_n^2 - F(u_n) \geq 0 \quad \forall n \in \mathbb{N} \quad \text{in } \mathbb{R}^N \setminus B_R,$$

$B_R$ being the ball of radius $R$ centered in 0. Since $u_n \to u$ in $L^{2^*_\alpha}(B_R)$, up to subsequence, $u_n \to u$ a.e. in $B_R$ and there exists $v \in L^{2^*_\alpha}(B_R)$ such that $|u_n(x)| \leq v(x)$. Moreover we have $F(u_n(x)) \to F(u(x))$ a.e. and $|F(u_n)| \leq 2^{-1} \varepsilon u_n^2 + C_\varepsilon |u_n|^{2^*_\alpha} \leq 2^{-1} \varepsilon u_n^2 + C_\varepsilon |v|^{2^*_\alpha}$ which, joint with the arbitrariness of $\varepsilon$ and the Dominated Convergence Theorem, give $\int_{B_R} F(u_n) \, dx \to \int_{B_R} F(u) \, dx$. Then by

$$\int_{B_R} F(u_n) \, dx = \frac{1}{2} \int_{B_R} u_n^2 \, dx + \int_{\mathbb{R}^N \setminus B_R} \left( \frac{1}{2} u_n^2 - F(u_n) \right) \, dx + 1,$$

taking into account the above considerations and the Fatous’ Lemma we infer

$$\int_{B_R} F(u) \, dx \geq \frac{1}{2} \int_{B_R} u^2 \, dx + \int_{\mathbb{R}^N \setminus B_R} \left( \frac{1}{2} u^2 - F(u) \right) \, dx + 1$$

which leads to

$$\int_{\mathbb{R}^N} G(u) \, dx \geq 1.$$

Suppose by contradiction that

$$\int_{\mathbb{R}^N} G(u) \, dx > 1$$

and define $h : [0,1] \to \mathbb{R}$ by $h(t) = \int_{\mathbb{R}^N} G(tu) \, dx$. The growth conditions on $f$ ensure that $h(t) < 0$ for $t$ close to 0 and $h(1) = \int_{\mathbb{R}^N} G(u) \, dx > 1$. Then, by the continuity of $h$, there exists $t_0 \in (0,1)$ such that $h(t_0) = 1$. Then,

$$\int_{\mathbb{R}^N} G(t_0 u) \, dx = 1 \iff t_0 u \in \mathcal{M}.$$

Consequently, by (20)

$$D \leq T(t_0 u) = \frac{t_0^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx = t_0^2 T(u) \leq t_0^2 D < D$$

which is absurd. Thus $\int_{\mathbb{R}^N} G(u) \, dx = 1$, i.e. $u \in \mathcal{M}$. The fact that the solution $u$ of the minimizing problem gives rise to a ground state solution, follows by standard arguments; indeed, since $u$ is a solution of the minimizing problem (6), i.e. $D = T(u) = \inf_{w \in \mathcal{M}} T(w)$, then there exists an associated Lagrange multiplier $\lambda$ such that, in a weak sense,

$$(-\Delta)^{\alpha/2} u = \lambda g(u).$$

Now by testing the previous equation on the same minimizer $u$, we deduce that

$$2T(u) = \lambda \int_{\mathbb{R}^N} g(u) \, dx = \lambda J'(u)[u] \geq 2\lambda.$$
so that it has to be, by Lemma 3.4, $T(u) \geq \lambda > 0$. Setting $u_\sigma(x) := u(\sigma x)$ for $\sigma > 0$, we easily see that

$$\mathcal{L}_u = \lambda \sigma^{2\alpha} g(u).$$

Choosing $\sigma = \lambda^{1/2\alpha}$ we obtain a solution of (P). Arguing as in [6, Theorem 3], $u_\sigma$ is a ground state solution.

3. The case $N = 1$ and $\alpha = 1/2$.

3.1. The variational framework. As for the previous case, let us consider the set of nontrivial solutions of (P), namely

$$\Sigma = \{ u \in H^{1/2}(\mathbb{R}) \setminus \{0\} : I'(u) = 0 \},$$

and let

$$m = \inf_{u \in \Sigma} I(u).$$

Denoting with $G(u) = F(u) - \frac{u^2}{2}$, the primitive of $g(u) = f(u) - u$, we introduce the set

$$\mathcal{M} = \left\{ u \in H^{1/2}(\mathbb{R}) \setminus \{0\} : \int_{\mathbb{R}} G(u) \, dx = 0 \right\},$$

(21)

the functional $J(u) = \int_{\mathbb{R}} G(u) \, dx$, and let

$$T(u) = \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx, \quad D = \inf_{u \in \mathcal{M}} T(u).$$

(22)

It is, again as before,

$$2D = \inf_{u \in \mathcal{M}} \left\{ \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx \right\}.$$

We point out here that, since we will deal with minimizing sequences $\{u_n\}$ for the minimization problem (22), as in the previous Section we suppose that $u_n$ is non-negative and radially symmetric. Moreover, we again define the min-max level associated to the functional $I$

$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

(23)

where

$$\Gamma = \left\{ \gamma \in C\left([0,1], H^{1/2}(\mathbb{R})\right) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \right\}.$$

3.2. Some preliminary stuff. At this point we establish some preliminary results which will be useful in order to prove Theorem 1.3.

Lemma 3.1. The set $\mathcal{M}$ defined in (21) is not empty and a $C^1$ manifold.

Proof. Consider $w \in C^\infty_0(\mathbb{R})$ with $w(x) > 0$ and define a function

$$h(t) = \int_{\mathbb{R}} G(tw) \, dx = \int_{\mathbb{R}} F(tw) \, dx - \frac{t^2}{2} \int_{\mathbb{R}} w^2 \, dx.$$

From (f1) for $t > 0$ small we have

$$h(t) \leq \frac{\varepsilon - 1}{2} t^2 \int_{\mathbb{R}} w^2 \, dx.$$

For $\varepsilon < 1$, we get $h(t) < 0$ for $t > 0$ small.
Now using (f4) we obtain
\[ h(t) \geq \frac{t^q}{q} \int_{\mathbb{R}}^{} w^q dx - \frac{t^2}{2} \int_{\mathbb{R}}^{} w^2 dx \quad \text{and} \quad h'(t) \geq \lambda t^{q-1} \int_{\mathbb{R}}^{} w^q dx - t \int_{\mathbb{R}}^{} w^2 dx. \]

Then, \( h(t) > 0 \) for \( t > 0 \) large and \( h'(t) > 0 \) for \( t > 0 \) large. Then there is a \( \bar{t} > 0 \) such that
\[ \int_{\mathbb{R}} G(\bar{t}w) \, dx = h(\bar{t}) = 0. \]

Now we prove that \( \mathcal{M} \) is a manifold. Indeed, if \( w \in \mathcal{M} \), then \( w \neq 0 \). Then, from (f1) and the fact that \( \lim_{s \to \infty} w(x) = 0 \), there exists \( x_0 \in \mathbb{R} \) such that \( g(w(x_0)) < 0 \). Thereby, by continuity, there is an open interval \( B_{\delta}(x_0) \) such that \( g(w(x)) < 0, \forall x \in B_{\delta}(x_0) \).

As a consequence we can always find a \( \phi \in C_0^\infty(\mathbb{R}) \subset H^{1/2}(\mathbb{R}) \) such that \( J'(w)[\phi] = \int_{\mathbb{R}} g(w)\phi \, dx < 0 \), showing that \( J'(w) \neq 0 \).

**Lemma 3.2.** Assume that \( f \) satisfies (f1), (f4) and (f5). Let \( \{v_n\} \subset H^{1/2}(\mathbb{R}) \) be a sequence of radial functions such that
\[ v_n \rightharpoonup v \quad \text{in} \quad H^{1/2}(\mathbb{R}) \]
and
\[ \sup_n |(-\Delta)^{1/4}v_n|^2 = \rho \leq \frac{1}{2} \quad \text{and} \quad \sup_n |v_n|^2 = M < \infty. \]

Then,
\[ \int_{\mathbb{R}} F(v_n) dx \to \int_{\mathbb{R}} F(v) dx. \]

**Proof.** Without loss of generality, we can assume that there is \( v \in H^{1/2}(\mathbb{R}) \), radial, such that
\[ v_n \to v \quad \text{in} \quad H^{1/2}(\mathbb{R}), \quad v_n(x) \to v(x) \quad \text{a.e in} \quad \mathbb{R} \]
and
\[ \lim_{|x| \to +\infty} v_n(x) = 0, \quad \text{uniformly in} \quad n. \]

Using the Theorem of Ozawa recalled in the Introduction, we know that there exists \( \omega \in (0, \pi) \) such that for every \( r \in (0, \omega) \) there exists \( H_r > 0 \) satisfying
\[ \sup_{u \in B} \int_{\mathbb{R}} (e^{-ru^2} - 1) \, dx \leq H_r 2M, \tag{24} \]
where \( B = \{ u \in H^{1/2}(\mathbb{R}) : |(-\Delta)^{1/4}u|^2 \leq 1 \quad \text{and} \quad |u|^2 \leq 2M \} \).

Now consider \( \sqrt{2}v_n \in B \) and let \( t := \frac{4}{3} \omega > \omega \geq \beta_0 \), where \( \omega, \beta_0 \) are the same appearing in hypothesis (f5). Then
\[ \int_{\mathbb{R}} (e^{t v_n^2} - 1) \, dx = \int_{\mathbb{R}} (e^{\frac{4}{3}\omega(\sqrt{2}v_n)^2} - 1) \, dx, \]
and consequently, by using (24) with \( r = 2\omega/3 \),
\[ \int_{\mathbb{R}} (e^{t v_n^2} - 1) \, dx \leq \sup_{u \in B} \int_{\mathbb{R}} (e^{\frac{2}{3}\omega u^2} - 1) \, dx \leq H_r 2M. \]

Now, setting \( P(s) = F(s) \) and \( Q(s) = e^{s^2} - 1 \), from (f1), (f4) and the last inequality, we get
\[ \lim_{s \to 0} \frac{P(s)}{Q(s)} = \lim_{s \to +\infty} \frac{P(s)}{Q(s)} = 0, \]
\[
\sup_n \int_{\mathbb{R}} Q(v_n) dx < \infty
\]
and
\[
P(v_n(x)) \rightarrow P(v(x)) \quad \text{a.e in } \mathbb{R}.
\]

Consequently the hypotheses of the Compactness Lemma of Strauss [6, Theorem A.I] are fulfilled. Hence \(P(v_n)\) converges to \(P(v)\) in \(L^1(\mathbb{R})\), and then
\[
\int_{\mathbb{R}} F(v_n) dx \rightarrow \int_{\mathbb{R}} F(v) dx
\]
concluding the proof.

\(\square\)

The relation between the ground state level and the minimax level defined in (23) is given in the following

**Lemma 3.3.** The numbers \(D\) and \(b\) satisfy the inequality \(D \leq b\).

**Proof.** Arguing as in Lemma 3.1, given \(v \in H^{1/2}(\mathbb{R})\) with \(v^+ = \max\{v, 0\} \neq 0\), there is \(t_0 > 0\) such that \(t_0 v^+ \in \mathcal{M}\). Then,
\[
D \leq \frac{t_0^2}{2} \int_{\mathbb{R}} |(-\Delta)^{1/4} v^+|^2 dx = I(t_0 v^+) \leq \max_{t \geq 0} I(t v^+).
\]

On the other hand, since \(f(s) = 0\) for \(s \leq 0\), if \(v \in H^{1/2}(\mathbb{R})\), \(v \neq 0\) with \(v^+ = 0\), then \(\max_{t \geq 0} I(t v) = \infty\). Hence in any case \(D \leq b\).

\(\square\)

**Lemma 3.4.** The number \(D\) given by (22) is positive, namely, \(D > 0\).

**Proof.** By definition \(D \geq 0\). Assume by contradiction that \(D = 0\) an let \(\{u_n\}\) be a (non-negative and radial) minimizing sequence in \(H^{1/2}(\mathbb{R})\) for \(T\), that is,
\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} u_n|^2 dx \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}} G(u_n) dx = 0.
\]

For each \(\mu_n > 0\), the function \(v_n(x) := u_n(x/\mu_n)\) satisfies
\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} v_n|^2 dx = \int_{\mathbb{R}} |(-\Delta)^{1/4} u_n|^2 dx \quad \text{and} \quad \int_{\mathbb{R}} G(v_n) dx = 0.
\]

Since
\[
\int_{\mathbb{R}} v_n^2 dx = \mu_n^2 \int_{\mathbb{R}} u_n^2 dx,
\]
we choose \(\mu_n^2 = |u_n|_2^{-2}\) to obtain
\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} v_n|^2 dx \rightarrow 0, \quad \int_{\mathbb{R}} v_n^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}} G(v_n) dx = 0.
\]

and we can assume that there exists \(v \in H^{1/2}(\mathbb{R})\), radial, such that \(v_n \rightharpoonup v\) in \(H^{1/2}(\mathbb{R})\). From Lemma 3.2 we get
\[
\int_{\mathbb{R}} F(v_n) dx \rightarrow \int_{\mathbb{R}} F(v) dx.
\]

Note that \(\int_{\mathbb{R}} G(v_n) dx = 0\) implies \(\int_{\mathbb{R}} F(v_n) dx = \frac{1}{2}\) and \(\int_{\mathbb{R}} F(v) dx = \frac{1}{2}\). Then \(v \neq 0\). But
\[
\int_{\mathbb{R}} |(-\Delta)^{1/4} v|^2 dx \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}} |(-\Delta)^{1/4} v_n|^2 dx \rightarrow 0,
\]
implies \(v = 0\) which is an absurd.

\(\square\)

**Lemma 3.5.** We have \(b < 1/2\).
Proof. It is sufficient to repeat the same argument of the Lemma 2.8, recalling that now by (f4) it is
\[
\tau^* = \left(\frac{q-2}{q}\right)^{(q-2)/2} C_q^{q/2}
\]
concluding the proof. \(\square\)

3.3. Proof of Theorem 1.3. At this point we will show that \(D\) is attained by \(u\), where \(u\) is the weak limit of \(\{u_n\}\). Indeed, since \(u_n \rightharpoonup u\) in \(H^{1/2}(\mathbb{R})\) we have
\[
T(u) = \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx \leq \liminf_{n \to +\infty} \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{1/4} u_n|^2 \, dx = D. \tag{25}
\]
Moreover, by Lemma 3.2 we have
\[
\int_{\mathbb{R}} F(u) \, dx = \liminf_{n \to \infty} \int_{\mathbb{R}} F(u_n) \, dx \geq \frac{1}{2} \int_{\mathbb{R}} u^2 \, dx
\]
leading to
\[
\int_{\mathbb{R}} G(u) \, dx \geq 0.
\]
As in the previous case \(N \geq 2\), we just need to prove that \(u \in \mathcal{M}\), i.e. \(\int_{\mathbb{R}^N} G(u) \, dx = 0\).

We again argue by contradiction by supposing that
\[
\int_{\mathbb{R}} G(u) \, dx > 0.
\]
As in the previous section, we set \(h : [0, 1] \to \mathbb{R}\) by \(h(t) = \int_{\mathbb{R}} G(tu) \, dx\). Using the growth condition of \(f\) we have \(h(t) < 0\) for \(t\) close to 0 and \(h(1) = \int_{\mathbb{R}} G(u) \, dx > 0\). Then, by the continuity of \(h\), there exists \(t_0 \in (0, 1)\) such that \(h(t_0) = 0\), that is \(t_0 u \in \mathcal{M}\). Consequently, by (25)
\[
D \leq T(t_0 u) = \frac{t_0^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx = t_0^2 T(u) \leq \frac{t_0^2}{2} D < D
\]
which is absurd.

As for the case \(N \geq 2\), one show that the minimizer \(u\) of (22) gives rise to a ground state solution of (P).

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Received June 2017; Revised November 2017.
E-mail address: coalves@dme.ufcg.edu.br
E-mail address: giovany@unb.br
E-mail address: sicilian@ime.usp.br