INTEGRAL-EINSTEIN HYPERSURFACES IN SPHERES

JIANQUAN GE+ AND FAGUI LI1,2

Abstract. Combining the intrinsic and extrinsic geometry, we generalize Einstein manifolds to Integral-Einstein (IE) submanifolds. A Takahashi-type theorem is established to characterize minimal hypersurfaces with constant scalar curvature (CSC) in unit spheres which are conjectured to be isoparametric in the Chern conjecture. For these hypersurfaces, we obtain some integral inequalities with the bounds characterizing exactly the totally geodesic hypersphere, the non-IE minimal Clifford torus \( S^1(\sqrt{\frac{1}{2}}) \times S^{n-1}(\sqrt{\frac{1}{2}}) \) and the IE minimal CSC hypersurfaces. Moreover, if further the third mean curvature is constant, then it is an IE hypersurface or an isoparametric hypersurface with \( g \leq 2 \) principal curvatures. In particular, all minimal isoparametric hypersurfaces with \( g \geq 3 \) principal curvatures are IE hypersurfaces. As applications, we obtain some spherical Bernstein theorems, including that any embedded closed minimal surface of genus no more than \( g \) inside a tubular neighborhood of constant radius \( r(g) > 0 \) around an equator in \( S^3 \) is an equator.

1. Introduction

In 1969, Lawson [21] gave a classification of minimal Einstein hypersurfaces in unit spheres, i.e., if \( M^n \subset S^{n+1} \) is Einstein, then either it is totally geodesic, or \( n = 2k \) and it is an open submanifold of

\[
M_{k,k} = S^k(\sqrt{\frac{1}{2}}) \times S^k(\sqrt{\frac{1}{2}}) \subset S^{n+1}.
\]

Meanwhile, Ryan [29] classified Einstein hypersurfaces in all space forms without the minimal condition. In particular, if \( M^n \subset S^{n+1} \) is a closed Einstein hypersurfaces, then \( M^n \) is either a totally umbilical hypersphere, or one of \( S^k(\sqrt{\frac{k-1}{n-2}}) \times S^{n-k}(\sqrt{\frac{n-k-1}{n-2}}) \), \( (2 \leq k \leq n-2) \). These Einstein hypersurfaces only consist of isoparametric hypersurfaces with no more than 2 principal curvatures (except \( S^1(r) \times S^{n-1}(t) \), \( t = \sqrt{1-r^2} \) and \( 0 < r < 1 \)) in \( S^{n+1} \). Recall that isoparametric hypersurfaces in unit

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* the corresponding author.

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spheres are hypersurfaces which have \( g \in \{1, 2, 3, 4, 6\} \) distinct constant principal curvatures. The classification problem was studied extensively, since late 1930s initiated by Cartan (for \( g \leq 3 \)) till to the year 2020 completed by Miyaoka [24] (for \( g = 6 \)) and finally by Cecil, Jenson and Chi [3, 11, 12] (for \( g = 4 \)) (see a number of important contributions in references of the excellent book [6] and the elegant survey [13]). In this paper, by combining the intrinsic and extrinsic geometry we introduce an extension of Einstein hypersurfaces so as to include these fascinating hypersurfaces.

A Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is called Einstein if it satisfies the pointwise intrinsic condition (cf. [1])

\[
\text{Ric} = \frac{R}{n} g,
\]

where \(\text{Ric}\) is the Ricci curvature tensor, and \(R\) is the scalar curvature which is constant by Schur’s theorem. By the famous Nash embedding theorem [23], \((M^n, g)\) is always realizable as a submanifold of a Euclidean space \(\mathbb{R}^N\). To relax the pointwise intrinsic condition of Einstein manifolds, we restrict the extrinsic geometry of the submanifold by taking an integral as follows.

**Definition 1.1.** Let \(M^n (n \geq 3)\) be a compact submanifold in the Euclidean space \(\mathbb{R}^N\). We call \(M^n\) an Integral-Einstein (IE) submanifold if for any unit vector \(a \in \mathbb{S}^{N-1}\),

\[
\int_M \left( \text{Ric} - \frac{R}{n} g \right) (a^T, a^T) = 0,
\]

where \(a^T \in \Gamma(TM)\) denotes the tangent component of the constant vector \(a\) along \(M^n\).

For noncompact submanifolds, one can also define the IE property by requiring the integral equation (1.1) over any geodesic ball \(B_R(p)\) of sufficiently large radius \(R\), or over certain exhausting compact domains (which might be useful for those noncompact manifolds as total spaces of vector bundles).

It is a natural problem whether there is a Nash-type embedding theorem for IE submanifolds, i.e., can any Riemannian manifold be embedded as an IE submanifold in a Euclidean space \(\mathbb{R}^N\)? More discreetly, one should allow the embedding to be IE up to some ambient transformations like Lie sphere transformations (cf. [4]) which include spherical parallel translations for submanifolds of spheres. It turns out that all isoparametric hypersurfaces of \(S^{n+1}\) except \(S^1(r) \times S^{n-1}(\sqrt{1-r^2})\) are IE hypersurfaces (up to spherical parallel translations), including those non-Einstein minimal isoparametric hypersurfaces with \(g \geq 3\) principal curvatures (see Corollary 2.4 and Theorem 4.4). The only left case \(S^1(r) \times S^{n-1}(\sqrt{1-r^2})\) provides a candidate of counterexample to the IE embedding problem above, since we do not know wether it can be embedded as a higher codimensional IE submanifold.

Another motivation comes from the study of the Chern conjecture [31] which asserts that a closed minimal hypersurface \(M^n\) with constant scalar curvature (CSC) in
$S^{n+1}$ is isoparametric. By the Simons inequality and the pinching rigidity \[10, 21, 32\], $M^n$ is either totally geodesic or a Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, which are isoparametric with $g \leq 2$, if the constant squared length of the second fundamental form $S := \|A\|^2 \leq n$. Hence, only the case of $S > n$ is left to be verified as minimal isoparametric hypersurfaces with $g \geq 3$ principal curvatures which have constant $S = (g-1)n > n$. The first nontrivial case when $n = 3$ was proven by Chang \[7\], while for higher dimensions it is still open in general (see various partial results in \[16, 26, 31\], and see a recent important progress in \[33, 34\] which generalized the 3-dimensional result of \[15\] to all dimensions).

During the study of such minimal CSC hypersurfaces, we find that the following position and normal position height functions $\varphi_a(x), \psi_a(x)$ take important roles as in Minkowski’s integral formula (cf. \[27\]). For any unit vector $a \in S^{n+1}$, the height functions are defined as

\[
\varphi_a(x) = \langle x, a \rangle, \quad \psi_a(x) = \langle \nu, a \rangle,
\]

where $\nu$ is the unit normal vector field along $x \in M^n$. There are many applications of minimal submanifolds in spheres by using the height functions recently, such as Solomon-Yau’s conjecture \[19\], Perdomo’s conjecture \[18\] and isoperimetric-type inequality \[22\]. The well known Takahashi theorem \[35\] states that $M^n$ is minimal if and only if there exists a constant $\lambda$ such that $\Delta \varphi_a = -\lambda \varphi_a$ for all $a \in S^{n+1}$. Analogously, we find that the same equation for $\psi_a$ is a sufficient and necessary condition for minimal CSC hypersurfaces (see Theorem 2.1). Similar characterization for constant mean curvature is also obtained. These lead us to study the uniform bounds of the $L^2$ squared norm of the position height function $\varphi_a$ on minimal CSC hypersurfaces. It turns out that the bounds characterize exactly the totally geodesic hypersphere, the non-IE minimal Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$ and the IE minimal CSC hypersurfaces (see Theorem 2.7). Here the Reilly formula \[28\] is applied and then an alternative characterization of IE hypersurfaces follows (see Theorem 2.2), which shows the non-IE property of $S^1(r) \times S^{n-1}(\sqrt{1-r^2}) \subset S^{n+1}$ and the IE property of minimal isoparametric hypersurfaces with $g \geq 3$ principal curvatures. Moreover, if $M^n$ is a closed minimal CSC hypersurface in $S^{n+1}$ with $S = \|A\|^2 > n$ and constant $f_3 = \text{Tr}(A^3)$, then $M^n$ is an IE hypersurface (see Corollary 2.4). Here the Cheng-Yau operator \[9\] is applied which greatly simplifies the proof for isoparametric hypersurfaces in Theorem 4.3. As applications of the integral inequalities about $\varphi_a$, we also obtain some spherical Bernstein theorems (see Theorem 2.11). Specifically, we demonstrate that the non-totally geodesic minimal hypersurfaces cannot curl up near an equator. In particular, if $n = 2$, the distance from the equator is only related to the Euler characteristic $\chi$ of the minimal surface $M^2$. Hence any embedded closed minimal surface of genus no more than $g$ inside a tubular neighborhood of constant radius $r(g) > 0$. 
around an equator in $\mathbb{S}^3$ is an equator. This can be compared to the result that there are infinitely many minimal surfaces in any neighborhood of an equator (cf. \cite{20, 30}).

2. Main results

Firstly, we give a generalization of the classical Takahashi theorem \cite{35} (case (i) and $H = 0$ in Theorem 2.1) for hypersurfaces in unit spheres.

**Theorem 2.1.** Let $M^n$ be a connected hypersurface immersed in $\mathbb{S}^{n+1}$ with mean curvature $H := \text{Tr}(A)/n$ and squared length of the second fundamental form $S := \|A\|^2$.

(i) $H$ is constant if and only if there exist some continuous function $\lambda$ and constant $\mu$ such that for all $a \in \mathbb{S}^{n+1}$,

$$\Delta \varphi_a = -\lambda \varphi_a + n\mu \psi_a,$$

in which case $\lambda = n$ and $\mu = H$ are constant. In particular, $H = 0$ if and only if there exists a continuous function $\lambda$ such that $\Delta \varphi_a = -\lambda \varphi_a$ for all $a \in \mathbb{S}^{n+1}$.

(ii) $H$ is constant if and only if there exist some continuous functions $\lambda$ and $\mu$ such that for all $a \in \mathbb{S}^{n+1}$,

$$\Delta \psi_a = -\lambda \psi_a + n\mu \varphi_a,$$

in which case $\lambda = S$, and $\mu = H$ is constant. In particular, $H = 0$ if and only if there exists a continuous function $\lambda$ such that $\Delta \psi_a = -\lambda \psi_a$ for all $a \in \mathbb{S}^{n+1}$.

(iii) $H$ and $S$ are both constant if and only if there exist some constant $\lambda$ and continuous function $\mu$ such that for all $a \in \mathbb{S}^{n+1}$,

$$\Delta \psi_a = -\lambda \psi_a + n\mu \varphi_a,$$

in which case $\lambda = S$ and $\mu = H$ are constant. In particular, $H = 0$ and $S = \text{Const}$ if and only if there exists a constant $\lambda$ such that $\Delta \psi_a = -\lambda \psi_a$ for all $a \in \mathbb{S}^{n+1}$.

Next we give a characterization of IE hypersurfaces in unit spheres.

**Theorem 2.2.** Let $M^n$ be a closed hypersurface immersed in $\mathbb{S}^{n+1}$. Then $M^n$ is IE if and only if for all $a \in \mathbb{S}^{n+1}$,

$$\int_M \left(1 - (n + 1)\varphi_a^2 - \psi_a^2\right) = \int_M \left(\rho - 1\right) \left(1 - \varphi_a^2 - (n + 1)\psi_a^2\right),$$

where $\rho = \frac{\nu^2 H^2 - S}{n(n-1)}$ and $\rho = \frac{R}{n(n-1)}$ is the normalized scalar curvature. In particular, we have the following special cases.

(A) If $M^n$ is minimal, then $M^n$ is IE if and only if

$$\int_M S \left(1 - \varphi_a^2 - (n + 1)\psi_a^2\right) = 0, \text{ for all } a \in \mathbb{S}^{n+1}.$$
If \( M^n \) is minimal and \( S > 0 \) is constant, then we have

\[
\int_M \left( \text{Ric} - \frac{R}{n} g \right)(a^T, a^T) = S \left( (n + 2) \int_M \varphi_a^2 = \text{Vol}(M^n) \right).
\]

In this case, \( M^n \) is IE if and only if one of the follows holds

(i)

\[
\int_M \varphi_a^2 = \frac{1}{n + 2} \text{Vol}(M^n), \quad \text{for all } a \in S^{n+1};
\]

(ii)

\[
\int_M \psi_a^2 = \frac{1}{n + 2} \text{Vol}(M^n), \quad \text{for all } a \in S^{n+1};
\]

(iii)

\[
\int_M \varphi_a^2 = \int_M \psi_a^2, \quad \text{for all } a \in S^{n+1};
\]

(iv)

\[
\int_M \varphi_a \psi_a f_3 = 0, \quad \text{for all } a \in S^{n+1},
\]

where \( f_3 = \text{Tr}(A^3) = 3(n^3) H_3 \) and \( H_3 \) is the third mean curvature.

**Remark 2.3.** For \( n = 2 \), the equation (1.1) for the definition of IE submanifolds is automatically satisfied and so is the integral formula (2.1), which is nontrivial and new to our best knowledge. Notice that (2.1) can be rewritten as

\[
\int_M \left( I_{n+2} - (n + 1) xx^t - \nu \nu^t \right) = \int_M (\rho - 1) \left( I_{n+2} - xx^t - (n + 1) \nu \nu^t \right),
\]

where \( I_{n+2} \) is the identity matrix, \( xx^t \) and \( \nu \nu^t \) are regarded as matrix-valued functions.

**Corollary 2.4.** A closed minimal CSC hypersurface in \( S^{n+1} \) with \( S > n \) and constant third mean curvature is an IE hypersurface. In particular, every minimal isoparametric hypersurface with \( g \geq 3 \) principal curvatures in \( S^{n+1} \) is an IE hypersurface. Moreover, the Clifford torus \( S^1(r) \times S^{n-1}(\sqrt{1-r^2}) \subset S^{n+1} \) is not IE.

**Remark 2.5.** It is natural to ask a weak Chern conjecture that a closed minimal CSC hypersurface in \( S^{n+1} \) with \( S > n \) must be an IE hypersurface.

**Remark 2.6.** Both of IE minimal CSC hypersurfaces and minimal isoparametric hypersurfaces with \( g \geq 3 \) share the same average-symmetric property (2.3), namely, the \( L^2 \) squared norm of any coordinate function equals the average \( \frac{1}{n+2} \text{Vol}(M^n) \).

The following inequalities imply a sharp gap as the Simons inequality mentioned before for minimal CSC hypersurfaces in \( S^{n+1} \). In particular, the equality cases characterize exactly the totally geodesic hypersphere, the IE minimal CSC hypersurfaces and the non-IE minimal Clifford torus \( S^1(\sqrt{\frac{1}{n^3}}) \times S^{n-1}(\sqrt{\frac{2-n}{n^3}}) \) (see other characterizations of this Clifford torus in [8], etc).
Theorem 2.7. Let $M^n$ be a closed minimal hypersurface immersed in $S^{n+1}$. Then

\begin{equation}
0 \leq \inf_{a \in S^{n+1}} \frac{\int_M \varphi_a^2}{\text{Vol}(M^n)} \leq \frac{1}{n+2} \leq \sup_{a \in S^{n+1}} \frac{\int_M \varphi_a^2}{\text{Vol}(M^n)} \leq \frac{1}{n+1}.
\end{equation}

\begin{enumerate}[(i)]
\item The first or last equality holds if and only if $M^n$ is totally geodesic.
\item In the case of minimal CSC hypersurfaces, the second or third equality holds if and only if $M^n$ is an IE, non-totally geodesic, minimal CSC hypersurface. Moreover, if $S > 0$, i.e., $M^n$ is non-totally geodesic, then
\end{enumerate}

\begin{equation}
\frac{1}{2n} \leq \inf_{a \in S^{n+1}} \frac{\int_M \varphi_a^2}{\text{Vol}(M^n)},
\end{equation}

where the equality holds if and only if $M^n$ is $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$.

In fact, the left three inequalities of (2.4) still hold without the minimal condition. Without the condition of constant scalar curvature in case (ii) of Theorem 2.7, we also have the following Simons-type gap.

Theorem 2.8. Let $M^n$ be a closed minimal hypersurface immersed in $S^{n+1}$. Then

\begin{enumerate}[(i)]
\item \[ \frac{1}{2n} \int_M S \leq \sup_{p \in M^n} S(p) \inf_{a \in S^{n+1}} \int_M \varphi_a^2. \]

The equality holds if and only if $S \equiv 0$ or $n$, and thus $M^n$ is either totally geodesic or the minimal Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$.
\item \[ \frac{n}{4n^2 - 3n + 1} \left( \int_M S \right)^2 \leq \int_M S^2 \inf_{a \in S^{n+1}} \int_M \varphi_a^2. \]

The equality holds if and only if $M^n$ is totally geodesic.
\end{enumerate}

Let $\text{Index}(M^n)$ denote the index of minimal hypersurfaces $M^n \subset S^{n+1}$, the number of negative eigenvalues associated with the Jacobi (second variation) operator.

Corollary 2.9. Let $M^n$ ($2 \leq n \leq 6$) be a closed, non-totally geodesic, embedded minimal hypersurface in $S^{n+1}$. Then there is a positive constant $C$ depending on $\text{Vol}(M^n)$ and $\text{Index}(M^n)$ such that

\[ \inf_{a \in S^{n+1}} \int_M \varphi_a^2 \geq C \text{Vol}(M^n). \]

In particular, if $n = 2$, the positive constant $C$ depends only on Euler characteristic $\chi$ of $M^2$.

Lastly, we apply these inequalities to give some spherical Bernstein theorems.

Definition 2.10. For any $a \in S^{n+1}$ and $0 < t < 1$, we define the spherical zone as

\[ S_{\text{zone}}^{n+1}(t) = \{ x \in S^{n+1} : |\langle x, a \rangle| < t \}. \]
It is well known that a closed minimal hypersurface lying in a closed hemisphere is totally geodesic (see Proposition 5.1). Similarly, we have the following results for spherical zones.

**Theorem 2.11.** Let $M^n$ be a closed minimal hypersurface immersed in $S^{n+1}$.

(i) If $M^n$ is CSC and the image of $M^n$ lying in $S^{n+1}_{\text{zone}}(\sqrt{\frac{1}{2n}})$ (or in $S^{n+1}_{\text{zone}}(\sqrt{n+1})$ when $M^n$ is IE), then it is totally geodesic.

(ii) If $M^n$ is non-totally geodesic, then the image of $M^n$ can not lie in $S^{n+1}_{\text{zone}}(\sqrt{r})$, where 

\[
r_1 = \frac{\int_MS(p)\sup_{p\in M^n} S(p)}{2n\text{Vol}(M^n)} \quad r_2 = \frac{n}{4n^2 - 3n + 1}\frac{\left(\int_MS\right)^2}{\text{Vol}(M^n)\int_MS^2}.
\]

(iii) If $M^n$ $(2 \leq n \leq 6)$ is embedded and non-totally geodesic, then there is a positive constant $\theta$ depending on $\text{Vol}(M^n)$ and $\text{Index}(M^n)$ such that the image of $M^n$ can not lie in $S^{n+1}_{\text{zone}}(\theta)$. In particular, if $n = 2$, $\theta$ depends only on Euler characteristic $\chi$ of $M^2$.

### 3. Takahashi-type theorem and characterizations for IE hypersurfaces

Firstly, we recall the basic properties of the height functions $\varphi_a$ and $\psi_a$ defined in (1.2), most of which were already in literature (cf. [25]).

Let $x : M^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ be a closed hypersurface immersed in the unit sphere. Let $\nabla$, $\widetilde{\nabla}$ and $D$ be the Levi-Civita connections on $M^n$, $S^{n+1}$ and $\mathbb{R}^{n+2}$, respectively. Observe that the gradients of the height functions are given by

\[
\nabla\varphi_a(x) = a^T, \quad \nabla\psi_a(x) = -A(a^T),
\]

where $a^T \in \Gamma(TM)$ denotes the tangent component of $a$ along $M^n$, and $A$ is the shape operator with respect to the unit normal vector field $\nu$, i.e., $A(X) = -\widetilde{\nabla}_X \nu$.

Clearly, we can decompose the unit vector $a \in S^{n+1}$ as

\[
a = a^T + \varphi_a(x)x + \psi_a(x)\nu, \quad |a^T|^2 + \varphi_a^2 + \psi_a^2 = 1.
\]

Since $D\varphi_a = a$, one deduces that the Hessian is

\[
\text{Hess}^\nabla \varphi_a(X, Y) = \text{Hess}^D \varphi_a(X, Y) + B(X, Y) \varphi_a = 0 + \langle B(X, Y), a \rangle
\]

for $X, Y \in \Gamma(TM)$ (the superscripts denote the connections). Here $B$ is the second fundamental form of $M^n$ as a submanifold in $\mathbb{R}^{n+2}$. That is, $D_X Y = \nabla_X Y + B(X, Y)$. Observe

\[
B(X, Y) = \langle B(X, Y), x \rangle x + \langle B(X, Y), \nu \rangle \nu = -\langle X, Y \rangle x + \langle AX, Y \rangle \nu.
\]

Thus

\[
\text{Hess}^\nabla \varphi_a(X, Y) = -\varphi_a(x)\langle X, Y \rangle + \psi_a(x)\langle AX, Y \rangle,
\]
which, regarding \(\text{Hess} \nabla \varphi_a\) as a \((1,1)\)-tensor, can be rewritten as
\[
\text{Hess} \nabla \varphi_a = -\varphi_a(x) \text{Id} + \psi_a(x) A.
\]
Hence
\[
\Delta \varphi_a(x) = -n\varphi_a(x) + nH\psi_a(x),
\]
where \(H := \text{Tr}(A)/n\) is the mean curvature.

On the other hand,
\[
\text{Hess} \nabla \psi_a(X) := \nabla X \nabla \psi_a = -\nabla X (A(a^T)) = -(\nabla X A)(a^T) - A(\nabla X a^T) = -(\nabla X a)(a^T) - A(\nabla \varphi_a(X)) = -(\nabla a^T A)(X) + \varphi_a(x) A(X) - \psi_a(x) A^2(X).
\]
Here the last equality follows from the Codazzi equation \((\nabla Y A)(X) = (\nabla X A)(Y)\).

Again we rewrite the Hessian as a \((1,1)\)-tensor
\[
\text{Hess} \nabla \psi_a = -\nabla a^T A + \varphi_a(x) A - \psi_a(x) A^2.
\]

Therefore
\[
\Delta \psi_a = -\text{Tr}(\nabla a^T A) + nH\varphi_a(x) - \|A\|^2\psi_a(x) = -n(\nabla H, a) + nH\varphi_a(x) - \|A\|^2\psi_a(x).
\]

In conclusion, we have shown

**Proposition 3.1.** For a hypersurface \(x : M^n \hookrightarrow S^{n+1} \subset \mathbb{R}^{n+2}\) with the height functions \(\varphi_a\) and \(\psi_a\) defined in (1.2), we have
\[
\begin{align*}
\nabla \varphi_a &= a^T, & \nabla \psi_a &= -Aa^T, \\
\Delta \varphi_a &= -n\varphi_a + nH\psi_a, & \Delta \psi_a &= -n(\nabla H, a) + nH\varphi_a - \|A\|^2\psi_a, \\
\text{Hess} \nabla \varphi_a &= -\varphi_a \text{Id} + \psi_a A, & \text{Hess} \nabla \psi_a &= -\nabla a^T A + \varphi_a A - \psi_a A^2.
\end{align*}
\]

Now we are ready to prove the Takahashi-type Theorem.

**Proof of Theorem 2.1.** Case (i). By Proposition 3.1, one has
\[
\Delta \varphi_a = -n\varphi_a + nH\psi_a,
\]
which proves the necessity for \(\lambda = n\) and \(\mu = H\). Conversely, if there exist some continuous function \(\lambda\) and constant \(\mu\) such that for all \(a \in S^{n+1}\),
\[
\Delta \varphi_a = -\lambda\varphi_a + n\mu\psi_a,
\]
then, combining the two equations above, we have
\[
-nx + nH\nu = -\lambda x + n\mu\nu,
\]
which shows \(\lambda = n\) and \(H = \mu\) are constant by the orthogonality of \(x\) and \(\nu\).

Case (ii). By Proposition 3.1, one has
\[
\Delta \psi_a = -n(\nabla H, a) + nH\varphi_a - \|A\|^2\psi_a,
\]
which proves the necessity for $\lambda = \|A\|^2 = S$, and $\mu = H$ is constant. Conversely, if there exist some continuous functions $\lambda$ and $\mu$ such that for all $a \in S^{n+1}$,

$$\Delta \psi_a = -\lambda \psi_a + n\mu \varphi_a,$$

then, combining the two equations above, we have

$$-n\nabla H + nH x - \|A\|^2 \nu = -\lambda \nu + n\mu x,$$

which shows $\nabla H = 0$, $\lambda = \|A\|^2$, and $\mu = H$ is constant by the orthogonality of $\nabla H$, $x$ and $\nu$. The same argument can prove the case (iii). □

Next we give the characterization of IE hypersurfaces in spheres.

**Proof of Theorem 2.2.** Firstly, we recall Reilly’s formula [28]

\[(3.2) \quad \int_M \left( (\Delta f)^2 - \|\text{Hess} \nabla f\|^2 \right) = \int_M \text{Ric}(\nabla f, \nabla f), \quad \text{for any } f \in C^\infty(M). \]

By Proposition 3.1, we have

\[
(\Delta \varphi_a)^2 = n^2 \varphi_a^2 + n^2 H^2 \psi_a^2 - 2n^2 H \varphi_a \psi_a,
\]

\[
\|\text{Hess} \nabla \varphi_a\|^2 = n \varphi_a^2 + \|A\|^2 \psi_a^2 - 2nH \varphi_a \psi_a,
\]

\[
\frac{1}{2} \Delta \varphi_a^2 = \varphi_a \Delta \varphi_a + \langle \nabla \varphi_a, \nabla \varphi_a \rangle = -n \varphi_a^2 + nH \varphi_a \psi_a + |a^T|^2 = 1 - (n+1) \varphi_a^2 - \psi_a^2 + nH \varphi_a \psi_a,
\]

where the last equality follows from (3.1) and it implies

\[(3.3) \quad \int_M \left( 1 - (n+1) \varphi_a^2 - \psi_a^2 + nH \varphi_a \psi_a \right) = 0. \]

Let $\rho = \frac{R}{n(n-1)}$ be the normalized scalar curvature. Then by the Gauss equation,

\[
\rho - 1 = \frac{n^2 H^2 - \|A\|^2}{n(n-1)},
\]

Set $f(x) = \varphi_a(x)$ in (3.2). By the preceding formulae and (3.1), we calculate the IE integral (1.1) as

\[(3.4) \quad \int_M \left( \text{Ric} - \frac{R}{n} g \right) (a^T, a^T)
= \int_M \left( (\Delta \varphi_a)^2 - \|\text{Hess} \nabla \varphi_a\|^2 - \frac{R}{n} \langle a^T, a^T \rangle \right)
= \int_M \left( n^2 \varphi_a^2 + n^2 H^2 \psi_a^2 - 2n^2 H \varphi_a \psi_a - (n \varphi_a^2 + \|A\|^2 \psi_a^2 - 2nH \varphi_a \psi_a) - \frac{R}{n} \|a^T\|^2 \right)
= \int_M \left( (n^2 - 1 + (n-1)(\rho - 1)) \varphi_a^2 + (n-1 + (n^2 - 1)(\rho - 1)) \psi_a^2 \right) - \int_M (2n(n-1)H \varphi_a \psi_a + (n-1)\rho) \]
\[(n - 1) \left( \int_M (1 - (n + 1)\varphi_a^2 - \psi_a^2) - \int_M (\rho - 1) \left( 1 - \varphi_a^2 - (n + 1)\psi_a^2 \right) \right) \].

This shows the equivalence between the IE equation (1.1) and (2.1).

For the proof of case (A), i.e., \(H = 0\), it follows from (3.3) that the left hand of (2.1) vanishes, namely,

\[(3.5) \int_M \left( 1 - (n + 1)\varphi_a^2 - \psi_a^2 \right) = 0.\]

Therefore, as \(\rho - 1 = -S/(n(n - 1))\), (2.1) is equivalent to

\[\int_M S (1 - \varphi_a^2 - (n + 1)\psi_a^2) = 0.\]

For the proof of case (B), i.e., \(H = 0\) and \(S = \|A\|^2 \equiv \text{Const}\), the formula (2.2) follows easily from (3.5) and (3.4). Then the subcases (i), (ii) and (iii) of case (B) follow directly from (2.2) and (3.5).

The last subcase (iv) of case (B) is intriguing but useful in deriving Corollary 2.4. In the following we give a simple proof by the self-adjoint operator of Cheng-Yau [9]. We briefly recall the Cheng-Yau operator as follows. For a \(C^2\)-function \(f\) on \(M^n\), the gradient \(\nabla f = \sum_i f_i e_i\) and the Hessian \(\text{Hess}^\nabla f = \sum_{i,j} f_{ij} \omega_i \otimes \omega_j\) of \(f\) under a local orthonormal frame \(\{e_i\}_{i=1}^n\) can be computed by

\[df = \sum_i f_i \omega_i, \quad \sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji},\]

where \(\{\omega_i\}_{i=1}^n\) is the coframe and \(\{\omega_{ji}\}\) are the connection forms. The covariant derivative \(\phi_{ijk}\) of a 2-tensor \(\phi_{ij}\) is defined by

\[\sum_k \phi_{ijk} \omega_k = df_{ij} + \sum_k \phi_{kj} \omega_{ki} + \sum_k \phi_{ik} \omega_{kj}.\]

Let \(\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j\) be a symmetric tensor on \(M^n\). The Cheng-Yau operator associated to \(\phi\) is defined by

\[\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \langle \phi, \text{Hess}^\nabla f \rangle.\]

Then if \(M^n\) is a closed manifold, by Stokes’ theorem, for any \(C^2\)-function \(u\) on \(M^n\),

\[\int_M (\Box f) u = \int_M \sum_{i,j} \phi_{ij} f_{ij} u = -\int_M \sum_{i,j} (\phi_{ij} u)_j f_i = -\int_M \sum_{i,j} (\phi_{ij} u + \phi_{ij} u_j) f_i = \int_M \sum_{i,j} \phi_{ij} u f_j + \int_M \sum_{i,j} \phi_{ij} (f u_i - u f_i) = \int_M f (\Box u) + \int_M \sum_{i,j} \phi_{ij} (f u_i - u f_i).\]
Thus, the operator $\Box$ is self-adjoint if and only if

$$\sum_j \phi_{ijj} = 0,$$

for all $i$ \([11]\). Cheng-Yau provided two symmetric tensors satisfying the preceding condition, namely,

$$\phi_{ij} = \frac{R}{2} \delta_{ij} - \text{Ric}_{ij}, \quad \text{or} \quad \phi_{ij} = (\text{Tr}\Psi)\delta_{ij} - \Psi_{ij},$$

where $\Psi$ is a symmetric Codazzi tensor. Now in our case (\(B\)), $R$ is constant, $\text{Tr}A = nH = 0$ and the shape operator $A$ is Codazzi. Then both of the following two tensors

$$\phi_{ij} = \frac{R}{n} \delta_{ij} - \text{Ric}_{ij}, \quad \text{or} \quad \phi = A,$$

give rise to a self-adjoint Cheng-Yau operator, either of which can help to prove the subcase (iv) of case (\(B\)). We proceed with the proof by the second for example.

By Proposition 3.1, we have

$$\int_M \psi_a(\Box \varphi_a) = \int_M \psi_a(A_a - \varphi_a \text{Id} + \psi_a A) = \int_M \psi^2_a S,$$

$$\int_M \varphi_a(\Box \psi_a) = \int_M \varphi_a(A_a - \nabla_a \text{T} A_a + \varphi_a A_a - \psi_a A^3) = \int_M (\varphi^2_a S - \varphi_a \psi_a \text{Tr}A^3),$$

which by the self-duality of the Cheng-Yau operator implies

$$\int_M (\varphi^2_a - \psi^2_a) S = \int_M \varphi_a \psi_a \text{Tr}A^3.$$

The proof is completed by taking use of the subcase (iii). \qed

**Proof of Corollary 2.4** For minimal CSC hypersurfaces in $\mathbb{S}^{n+1}$ with $S > n$, we know from Proposition 3.1 that $\varphi_a$ and $\psi_a$ are eigenfunctions of the Laplacian to the different eigenvalues $n$ and $S$ respectively. Therefore, they are orthogonal and thus the condition in the subcase (iv) of case (\(B\)) of Theorem 2.2 is satisfied, namely,

$$\int_M \varphi_a \psi_a \text{Tr}A^3 = (\text{Tr}A^3) \int_M \varphi_a \psi_a = 0,$$

if further $M^n$ has constant third mean curvature (and thus constant $\text{Tr}A^3$).

In particular, minimal isoparametric hypersurfaces with $g \geq 3$ principal curvatures have constant mean curvatures of each order and have constant $S = (g-1)n > n$, thus they are IE hypersurfaces in unit spheres.

The Einstein isoparametric hypersurfaces (with $g = 2$) $S^k(\sqrt{\frac{k-1}{n-2}}) \times S^{n-k}(\sqrt{\frac{n-k-1}{n-2}})$ \((2 \leq k \leq n-2)\) are automatically IE hypersurfaces in $\mathbb{S}^{n+1}$. However, the only left Clifford torus $M^n := S^1(r_1) \times S^{n-1}(r_2) \subset \mathbb{S}^{n+1}$ \((0 < r_1 < 1, r_1^2 + r_2^2 = 1)\) is not an IE hypersurface in $\mathbb{S}^{n+1}$. The proof is a long but straightforward calculation of the
integrals in both sides of (2.1). Here we leave the details to the reader and only give the final result of the calculation of the two sides of (2.1) as follows

\[
\text{LHS} = \left(1 - (n + 1)\frac{r_1^2 |a_1|^2}{2} + \frac{r_2^2 |a_2|^2}{n}ight) - \left(\frac{r_1^2 |a_1|^2}{2} + \frac{r_2^2 |a_2|^2}{n}\right)V,
\]

\[
\text{RHS} = \frac{n r_2^2 - 2}{r_2^n} \left(1 - \frac{r_1^2 |a_1|^2}{2} + \frac{r_2^2 |a_2|^2}{n}\right) - (n + 1)\left(\frac{r_2^2 |a_1|^2}{2} + \frac{r_2^2 |a_2|^2}{n}\right)V,
\]

where \( V = \text{Vol}(M^n) = r_1 r_2^{n-1}\text{Vol}(S^1)\text{Vol}(S^{n-1}) \) and \( a = (a_1, a_2) \in \mathbb{R}^2 \oplus \mathbb{R}^n \). Direct calculations can show that the equation (2.1), i.e., LHS = RHS does not hold for all \( a \in S^{n+1} \). This completes the proof by Theorem 2.2. □

4. INTEGRAL INEQUALITIES WITH EQUALITIES BY IE HYPERSURFACES

In this section, based on the previous arguments, we estimate uniformly the \( L^2 \) squared norm of the position height function \( \varphi_a \) of (1.2) by further considering the height functions \( \varphi_{a_j} \) with respect to an orthonormal frame \( \{a_j\}_{j=1}^{n+2} \) of \( \mathbb{R}^{n+2} \).

**Proof of Theorem 2.7.** The first inequality of (2.4) is obvious and attains equality only at the totally geodesic hyperspheres \( \{x \in S^{n+1} : \varphi_a(x) = 0\} \).

For the second and third inequality of (2.4), we consider the height functions \( \varphi_{a_j} \) with respect to an orthonormal frame \( \{a_j\}_{j=1}^{n+2} \) of \( \mathbb{R}^{n+2} \). It is easily seen that

\[
\sum_{j=1}^{n+2} \varphi_{a_j}^2 = 1, \quad \sum_{j=1}^{n+2} \int_M \varphi_{a_j}^2 = \text{Vol}(M^n),
\]

which directly shows

\[
(n + 2) \inf_{a \in S^{n+1}} \int_M \varphi_a^2 \leq \text{Vol}(M^n) \leq (n + 2) \sup_{a \in S^{n+1}} \int_M \varphi_a^2.
\]

In the case of minimal CSC hypersurfaces, the equalities above hold if and only if \( (n + 2) \int_M \varphi_a^2 = \text{Vol}(M^n) \) for all \( a \in S^{n+1} \), i.e., \( M^n \) is an IE (non-totally geodesic) minimal CSC hypersurface by the case (B) of Theorem 2.2.

The last inequality of (2.4) follows easily from (3.5) (when \( M^n \) is minimal), namely,

\[
(n + 1) \int_M \varphi_a^2 = \int_M \left(1 - \psi_a^2\right) \leq \text{Vol}(M^n),
\]

which attains equality if and only if \( \psi_{a_0} \equiv 0 \) for some \( a_0 \in S^{n+1} \), i.e., the Gauss image of \( M^n \) lies in an equator of \( S^{n+1} \), and in this case \( M^n \) is embedded as an equator in \( S^{n+1} \) by a theorem of Nomizu and Smyth [25].

Now we come to prove the inequality (2.5). When \( M^n \) is minimal, by Proposition 3.1 we have

\[
\frac{1}{2} \Delta \psi_a^2 = -S \psi_a^2 + |Aa^T|^2,
\]
and thus
\begin{equation}
\int_M S\psi_a^2 = \int_M |Aa^T|^2. \tag{4.1}
\end{equation}

Let \( \{\lambda_i\}_{i=1}^n \) be the eigenvalues of \( A \) with \( \lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2 \). Then we have
\[ \sum_{i=1}^n \lambda_i = 0, \quad \sum_{i=1}^n \lambda_i^2 = \|A\|^2 = S. \]
Thus
\[ 0 = \left( \sum_{i=1}^n \lambda_i \right)^2 = \lambda_1^2 + 2\lambda_1 \sum_{i=2}^n \lambda_i + \left( \sum_{i=2}^n \lambda_i \right)^2 = -\lambda_1^2 + \left( \sum_{i=2}^n \lambda_i \right)^2 \]
\[ \leq -\lambda_1^2 + (n-1) \sum_{i=2}^n \lambda_i^2 = (n-1)S - n\lambda_1^2. \]
Hence
\begin{equation}
\lambda_1^2 \leq \frac{n-1}{n} S, \tag{4.2}
\end{equation}
where the equality holds if and only if \( \lambda_1 = (1-n)\lambda_2 \) and \( \lambda_2 = \lambda_3 = \cdots = \lambda_n \).

It follows from (4.1) and (4.2) that
\begin{equation}
\int_M S\psi_a^2 = \int_M |Aa^T|^2 \leq \int_M \lambda_1^2 |a^T|^2 \leq \frac{n-1}{n} \int_M S|a^T|^2. \tag{4.3}
\end{equation}

On the other hand, by (3.5) and (3.1) we have
\begin{equation}
\int_M |a^T|^2 = n \int_M \varphi_a^2. \tag{4.4}
\end{equation}
Combining this with (4.3) and (4.1), we obtain
\begin{equation}
\int_M S = \int_M S(\varphi_a^2 + \psi_a^2 + |a^T|^2) \leq 2n \sup_{p \in M^n} S(p) \int_M \varphi_a^2, \tag{4.5}
\end{equation}
which proves the inequality (2.5) if \( S > 0 \) is constant. The equality of (4.5) holds for some \( a \) only if \( S \equiv \text{Const} \) and equalities hold in (4.2), (4.3), which thus implies that \( M^n \) is either totally geodesic or a minimal hypersurface with two constant distinct principal curvatures \( \lambda_1 \) and \( \lambda_2 \) of multiplicities 1 and \( n-1 \) respectively. Hence, when \( S > 0 \), \( M^n \) is the minimal Clifford torus \( M_{1,n-1} := S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}}) \). It is left to verify that there exists some \( a \) such that the equality of (4.5) holds on \( M_{1,n-1} \), i.e.,
\[ \text{Vol} (M_{1,n-1}) = 2n \int_{M_{1,n-1}} \varphi_a^2. \]
Let \( a = (a_1, a_2) \) with \( \|a\| = 1 \), where \( a_1 \in \mathbb{R}^2 \) and \( a_2 = 0 \in \mathbb{R}^n \). Write \( x = (x_1, x_2) \in S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}}) \). Then the verification can be done directly by

\[
2n \int_{M_{1,n-1}} \varphi^2_a = 2n \int_{M_{1,n-1}} \langle a, x \rangle^2
\]

\[
= 2n \int_{S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})} \langle a_1, x_1 \rangle^2
\]

\[
= 2n \text{Vol} \left( S^{n-1}(\sqrt{\frac{n-1}{n}}) \right) \int_{S^1(\sqrt{\frac{1}{n}})} \langle a_1, x_1 \rangle^2
\]

\[
= \text{Vol} \left( S^{n-1}(\sqrt{\frac{n-1}{n}}) \right) \text{Vol} \left( S^1(\sqrt{\frac{1}{n}}) \right)
\]

\[
= \text{Vol} (M_{1,n-1}).
\]

**Proof of Theorem 2.8.** The first case has been proven previously in (4.5).

For the second case, by (4.1, 4.3, 4.4) and the Cauchy inequality, we have

\[
\int_M S \leq \int_M S \left( \varphi^2_a + \frac{2n-1}{n} |a|^2 \right)
\]

\[
\leq \int_M \left( 2^2 \varphi^2_a + 2^n \frac{n-1}{n} |a|^2 \right)^{\frac{1}{2}} \left( \varphi^2_a + \left( \frac{2n-1}{n} \right)^2 |a|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \int_M S \left( \varphi^2_a + \left( \frac{2n-1}{n} \right)^2 |a|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_M S^2 \right)^{\frac{1}{2}} \left( \int_M 4n^2 - 3n + 1 \frac{n}{\varphi_a^2} \right)^{\frac{1}{2}}.
\]

The equality holds if and only if \( S \equiv 0 \), or \( |a|^2 = \psi_a \equiv 0 \) which implies also \( M^n \) is totally geodesic. \( \square \)

To prove Corollary 2.9 we need the following lemmas.

**Lemma 4.1** (Choi-Schoen [14]). Assume \( N^3 \) is a closed Riemannian manifold with positive Ricci curvature. If \( M^2 \) is a compact embedded minimal surface of \( N^3 \), then there exists a constant \( C_E \) depending only on \( N^3 \) and Euler characteristic \( \chi \) of \( M^2 \) such that

\[
\sup_{p \in M^2} S(p) \leq C_E.
\]

**Lemma 4.2** (Sharp [30]). Assume \( N^{n+1} \) (\( 2 \leq n \leq 6 \)) is a closed Riemannian manifold with positive Ricci curvature. If \( M^n \) is a compact embedded minimal hypersurface of \( N^{n+1} \), then there exists a constant \( C_1 \) depending only on \( N^{n+1} \), \( \text{Vol}(M^n) \) and
Index\((M^n)\) such that 
\[ \sup_{p \in M^n} S(p) \leq C_1. \]

**Lemma 4.3** (Ge-Li [18]). Let \(M^n\) be a closed embedded, non-totally geodesic, minimal hypersurface in \(S^{n+1}\). Then there is a positive constant \(C_2\), depending only on \(n\), such that 
\[ \int_M S \geq C_2 \text{Vol}(M^n). \]

**Proof of Corollary** 2.9 By Lemmas 4.2 - 4.3 and Theorem 2.8 (i), we have
\[ \inf_{a \in S^{n+1}} \int_M \varphi_a^2 \geq \frac{\int_M S}{2n \sup_{p \in M^n} S(p)} \geq \frac{C_2}{2nC_1} \text{Vol}(M^n). \]

In particular, by the Gauss equation and the Gauss-Bonnet theorem, for genus \(g\) minimal surface \(M^2 \subset S^3\), we have \(\chi = 2 - 2g\) and
\[ \int_M S = 8\pi (g - 1) + 2\text{Vol}(M^2). \]

If \(g = 0\), Calabi [3] proved that if \(S^2\) is minimally immersed in \(S^3\), then \(S^2\) is an equator (i.e., totally geodesic). If \(g = 1\), Brendle [2] verified Lawson’s Conjecture, i.e., the only embedded minimal torus in \(S^3\) is the Clifford torus. For \(g \geq 2\), by Lemma 4.1 and Theorem 2.8 (i), one has
\[ \inf_{a \in S^3} \int_M \varphi_a^2 \geq \frac{4\pi (g - 1) + \text{Vol}(M^2)}{2C_E} \geq \frac{4\pi + \text{Vol}(M^2)}{2C_E}. \]

In conclusion, for surface case, there is a positive constant \(C > 0\) depending only on the Euler characteristic \(\chi\) of \(M^2\) such that
\[ \inf_{a \in S^{n+1}} \int_M \varphi_a^2 \geq C\text{Vol}(M^n). \]

\[ \square \]

To conclude this section, we give the uniform bounds of the \(L^2\) squared norm of \(\varphi_a\) on minimal isoparametric hypersurfaces, which give another proof of the result of Corollary 2.4 minimal isoparametric hypersurfaces with \(g \geq 3\) are IE hypersurfaces.

**Theorem 4.4.** Let \(M^n\) be a minimal isoparametric hypersurface with \(g \geq 2\) distinct principal curvatures in \(S^{n+1}\).

(i) For \(g = 2\), i.e., \(M^n = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}), \ (1 \leq k \leq \lfloor \frac{n}{2} \rfloor)\), we have
\[ \inf_{a \in S^{n+1}} \frac{\int_M \varphi_a^2}{\text{Vol}(M^n)} = \frac{k}{n(k+1)}, \ \sup_{a \in S^{n+1}} \frac{\int_M \varphi_a^2}{\text{Vol}(M^n)} = \frac{n-k}{n(n-k+1)}. \]
(ii) For $g \geq 3$, we have
\[
\int_M \varphi_a^2 = \int_M \psi_a^2 = \frac{1}{n+2} \text{Vol}(M^n),
\]
for all $a \in S^{n+1}$, and thus $M^n$ is an IE minimal CSC hypersurface.

**Remark 4.5.** In fact, for $g = 4$, on each isoparametric hypersurface (not only minimal) we have $\int_M \varphi_a^2 = \int_M \psi_a^2$.

**Proof.** Case (i). Let $a = (a_1, a_2) \in S^{n+1}$ and $x = (x_1, x_2) \in S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, where $a_1 \in \mathbb{R}^{k+1}$ and $a_2 \in \mathbb{R}^{n+1-k}$. Then we have
\[
\int_M \varphi_a^2 = \int_M \langle a, x \rangle^2 = \int_M \langle a_1, x_1 \rangle^2 + \langle a_2, x_2 \rangle^2 = \text{Vol} \left( S^{n-k}(\sqrt{\frac{n-k}{n}}) \right) \int_{S^k(\sqrt{\frac{k}{n}})} \langle a_1, x_1 \rangle^2 + \text{Vol} \left( S^k(\sqrt{\frac{k}{n}}) \right) \int_{S^{n-k}(\sqrt{\frac{n-k}{n}})} \langle a_2, x_2 \rangle^2
\]
\[
= \left( \frac{k}{n(n+1)} ||a_1||^2 + \frac{n-k}{n(n-k+1)} ||a_2||^2 \right) \text{Vol} \left( S^{n+1} \right),
\]
which gives the uniform bounds immediately.

Case (ii). For $g \geq 4$, we observe that $\nu : M^n \hookrightarrow M^n$ is a diffeomorphism (if $M^n$ is minimal when $g = 6$) and $|\det d\nu| = 1$, thus $\int_M \varphi_a^2 = \int_M \psi_a^2$ by diffeomorphism invariance of integration. This shows that $M^n$ is IE if $M^n$ is minimal, by the subcase (iii) of case (B) in Theorem 2.7 (other than (iv) as in the proof of Corollary 2.4), or by the case (ii) of Theorem 2.7 since now $\int_M \varphi_a^2 = \text{Vol}(M^n)/(n+2)$ by (3.5).

In general, for $g \geq 3$ minimal isoparametric hypersurfaces in $S^{n+1}$, we give one more proof by using the isoparametric theory and the integral inequality (2.4) of Theorem 2.7. Recall (cf. [6], [17]) that now $M^n = M_{\theta_0} = f^{-1}(c_0)$ is the minimal level hypersurface of the Cartan-Münzner isoparametric function $f$ on $S^{n+1}$, where $c_0 = \frac{m_- - m_+}{m_- + m_+} = \cos(\theta_0)$, $0 < \theta_0 < \frac{\pi}{g}$, and $f(p) = \cos(g\theta(p))$ with $\theta(p)$ the distance of $p \in S^{n+1}$ to $M_\pm$ (one of the two focal submanifolds $M_{\pm} := f^{-1}(\pm 1)$ with codimensions $m_{\pm} + 1$). Moreover, the parallel level sets $M_\theta := f^{-1}(\cos(g\theta))$, $\theta \in [0, \frac{\pi}{g}]$ (with $M_0 = M_+ + M_\theta = M_-$), constitute a singular Riemannian foliation of $S^{n+1}$. Hence we have
\[
\int_0^{\frac{\pi}{g}} \int_{M_\theta} \varphi_a^2 = \int_{x \in S^{n+1}} \varphi_a^2(x) = \frac{1}{n+2} \text{Vol} \left( S^{n+1} \right).
\]
For $\theta \in (0, \frac{\pi}{g})$, the following spherical parallel translation is a diffeomorphism:
\[
\phi_{\theta} : M_{\theta_0} \to M_\theta,
\]
\[
x \mapsto \cos(\theta_0 - \theta)x + \sin(\theta_0 - \theta)\nu.
\]
It follows that (cf. [6])
\[
\phi_{\theta}^*(d\text{Vol}_{M_{\theta_0}}) = h(\theta)d\text{Vol}_{M_{\theta_0}},
\]
where \( h(\theta) = \prod_{i=1}^{n} (\cos(\theta_0 - \theta) - \sin(\theta_0 - \theta)\lambda_i) \), and \( \{\lambda_i\}_{i=1}^{n} \) are the constant principal curvatures of \( M_{\theta_0} \) with the \( g \) distinct values \( \{\cot(\theta_0 + \frac{(j-1)\pi}{g})\}_{j=1}^{g} \) of multiplicities \( m_+ \) and \( m_- \) alternately.

Since \( g \geq 3 \), \( S = n(g - 1) \) and by Proposition 3.1 on \( M_{\theta_0} \),

\[
\Delta \varphi_a = -n \varphi_a, \quad \Delta \psi_a = -n(g - 1) \psi_a,
\]

we have \( \int_{M_{\theta_0}} \psi_a \varphi_a = 0 \). Therefore

\[
\int_{M_{\theta_0}} \varphi_a^2 = \int_{M_{\theta_0}} (\cos(\theta_0 - \theta)x + \sin(\theta_0 - \theta)\nu, a)^2 |h(\theta)|
\]

\[
= \int_{M_{\theta_0}} (\cos^2(\theta_0 - \theta)\varphi_a^2 + \sin^2(\theta_0 - \theta)\psi_a^2 + \sin(\theta_0 - \theta)\varphi_a \psi_a) |h(\theta)|
\]

\[
= \int_{M_{\theta_0}} (\begin{array}{c} (1 - (n + 2) \sin^2(\theta_0 - \theta)) \varphi_a^2 + \sin^2(\theta_0 - \theta) \end{array}) |h(\theta)|,
\]

where the last equality follows from (3.3). By (4.6) and (4.7), we have

\[
\beta - (n + 2)\alpha \int_{M_{\theta_0}} \varphi_a^2 + \alpha \text{Vol}(M_{\theta_0}) = \frac{1}{n + 2} \text{Vol}(S^{n+1}) ,
\]

where \( \alpha = \int_{0}^{\pi} \sin^2(\theta_0 - \theta)|h(\theta)|d\theta \), and \( \beta = \int_{0}^{\pi} |h(\theta)|d\theta \). Analogous to (4.8), we have

\[
\beta \text{Vol}(M_{\theta_0}) = \text{Vol}(S^{n+1}) .
\]

It follows from (4.8) that either \( \beta - (n + 2)\alpha = 0 \) or \( \int_{M_{\theta_0}} \varphi_a^2 = \text{Vol}(M_{\theta_0})/(n + 2) \). So we are left with proving that the former equality is impossible for \( g \geq 3 \). Here we take the case \( g = 3 \) and \( m_\pm = 1 \) for example and leave the other cases to the reader. Now \( \theta_0 = \frac{\pi}{6}, \lambda_1 = \cot \frac{\pi}{6} = \sqrt{3}, \lambda_2 = \cot \frac{\pi}{3} = 0, \lambda_3 = \cot \frac{\pi}{\pi} = -\sqrt{3}, \) and thus

\[
\beta - (n + 2)\alpha = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1 - 5 \sin^2 t)(\cos^2 t - 3 \sin^2 t) \cos t dt = \frac{1}{2}.
\]

\( \square \)

5. Applications to spherical Bernstein theorems

In this section, we apply the integral inequalities of Theorems 2.7 and 2.8 to prove Theorem 2.11 for spherical zone domains. Firstly we recall the following classical spherical Bernstein theorem for hemispheres.

**Proposition 5.1.** Let \( M^n \) be a closed minimal hypersurface lying in a closed hemisphere \( S^{n+1}_+ := \{ x \in S^{n+1} : \varphi_a(x) \geq 0 \} \). Then \( M^n \) is an equator.
Proof. Since
\[ \Delta \varphi_a = -n \varphi_a, \]
if \( \varphi_a \geq 0 \) for some \( a \in S^{n+1} \), it implies that
\[ \Delta \varphi_a \leq 0. \]
But
\[ \int_M \Delta \varphi_a = 0, \]
one has \( \Delta \varphi_a \equiv 0 = \varphi_a \) and thus \( M^n \) is totally geodesic. \qed

Proof of Theorem ~\text{2.11}~ Case (i). If \( M^n \) lies in some spherical zone \( S_{\text{zone}}(\sqrt{\frac{1}{2n}}) \) completely, then there is some \( a_0 \in S^n \) such that
\[ |\varphi_{a_0}(x)| = |\langle x, a_0 \rangle| < \sqrt{\frac{1}{2n}}, \]
for all \( x \in M^n \). Then it follows from Theorem ~\text{2.7}~ the following contradiction
\[ \frac{1}{2n} \leq \inf_{a \in S^{n+1}} \frac{\int_M \varphi_a^2 \text{Vol}(M^n)}{\text{Vol}(M^n)} < \frac{1}{2n}. \]
Similarly, for IE minimal CSC hypersurfaces with \( |\varphi_{a_0}(x)| < \sqrt{\frac{1}{n+2}} \), we have
\[ \inf_{a \in S^{n+1}} \frac{\int_M \varphi_a^2 \text{Vol}(M^n)}{\text{Vol}(M^n)} < \frac{1}{n+2}, \]
which shows that \( M^n \) is totally geodesic by Theorem ~\text{2.7}~.

Applying Theorem ~\text{2.8}~ and Corollary ~\text{2.9}~, cases (ii) and (iii) can be proven similarly as for case (i). \qed

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1School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. CHINA.

Email address: jqge@bnu.edu.cn

1China Beijing International Center for Mathematical Research, Peking University, Beijing 100871, P.R. CHINA.

Email address: faguili@bicmr.pku.edu.cn

2School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. CHINA.

Email address: faguili@mail.bnu.edu.cn