Extensions of Bougerol’s identity in law and the associated anticipative path transformations

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Abstract

Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion and denote by $A_t$, $t \geq 0$, the quadratic variation of the geometric Brownian motion $e^{B_t}$, $t \geq 0$. Bougerol’s celebrated identity (1983) asserts that, if $\beta = \{\beta(t)\}_{t \geq 0}$ is another Brownian motion independent of $B$, then $\beta(A_t)$ is identical in law with $\sinh B_t$ for every fixed $t > 0$. In this paper, we extend Bougerol’s identity to an identity in law for processes up to time $t$, which exhibits a certain invariance of the law of Brownian motion. The extension is described in terms of anticipative transforms of $B$ involving $A_t$ as an anticipating factor. A Girsanov-type formula for those transforms is shown. An extension of a variant of Bougerol’s identity is also presented.

1 Introduction

Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion. For every $\mu \in \mathbb{R}$, we denote by $B^{(\mu)} = \{B_t^{(\mu)} := B_t + \mu t\}_{t \geq 0}$ the Brownian motion with drift $\mu$, to which we associate the exponential additive functional $A_t^{(\mu)}$, $t \geq 0$, defined by

$$A_t^{(\mu)} := \int_0^t e^{2B_s^{(\mu)}} \, ds;$$

when $\mu = 0$, we simply write $A_t$ for $A_t^{(0)}$. This additive functional is the quadratic variation of the geometric Brownian motion $e^{B_t^{(\mu)}}$, $t \geq 0$, and appears in a number of fields such as mathematical finance, diffusion processes in random environments, stochastic analysis of Laplacians on hyperbolic spaces, and so on; see the detailed surveys \cite{14, 15} by Matsumoto and Yor.

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Although the law of $A_t$, as well as of $A_t^{(μ)}$, has a rather complicated expression (refer to the beginning of the Subsection 4.2 in this respect), there are alternative ways to understand it, one of which is provided by Bougerol’s celebrated identity (3) asserting that, if $β = \{β(t)\}_{t≥0}$ denotes another one-dimensional standard Brownian motion and if it is independent of $B$, then for every fixed $t > 0$,

$$β(A_t) \overset{(d)}{=} \sinh B_t,$$  \hspace{1cm} (1.1)

or, more generally, for every fixed $t > 0$ and $x ∈ \mathbb{R}$,

$$e^{B_t}\sinh x + β(A_t) \overset{(d)}{=} \sinh(x + B_t);$$  \hspace{1cm} (1.2)

the latter is originally due to Alili–Gruet [1, Proposition 4]. For a straightforward derivation of (1.1) by means of stochastic differential equations, which works for (1.2) as well, we refer the reader to an inventive argument [4, Appendix] by Alili and Dufresne. From the former identity (1.1) in particular, we easily obtain the moment formula

$$E[(A_t)^ν] = \frac{\sqrt{π}}{2ν\Gamma(ν + 1/2)}E[|\sinh B_t|^{2ν}], \hspace{1cm} ν > -1/2,$$

where $Γ$ is the gamma function. The expression of the right-hand side is fairly tractable.

For a detailed account of Bougerol’s identity and recent progress in its study such as extensions to other processes, see the survey [19] by Vakeroudis.

Define $C([0, ∞); \mathbb{R})$ to be the space of continuous functions $φ : [0, ∞) → \mathbb{R}$. We set

$$A_t(φ) := \int_0^t e^{2φ_s} ds, \hspace{1cm} t ≥ 0,$$  \hspace{1cm} (1.3)

so that $A_t^{(μ)} = A_t(B^{(μ)})$ and $A_t = A_t(B)$, $t ≥ 0$. In this paper, we show that Bougerol’s identity may be extended to an identity in law for processes. Unless otherwise stated, we fix $t > 0$ and for every $z ∈ \mathbb{R}$, consider the path transformation $T_z$ defined by

$$T_z(φ)(s) := φ_s - \log \left\{1 + \frac{A_s(φ)}{A_t(φ)} (e^z - 1)\right\}, \hspace{1cm} 0 ≤ s ≤ t,$$  \hspace{1cm} (1.4)

which maps the space $C([0, t]; \mathbb{R})$ of real-valued continuous functions over $[0, t]$ to itself. We also denote by

$$\text{Argsh } x \equiv \log \left(x + \sqrt{1 + x^2}\right), \hspace{1cm} x ∈ \mathbb{R},$$

the inverse function of the hyperbolic sine function. One of the main results of the paper is stated as follows:

**Theorem 1.1.** Suppose $B$ and $β$ are independent. Then, for every fixed $x ∈ \mathbb{R}$, the process

$$T_{x + B_t} \text{Argsh}(e^{B_t}\sinh x + β(A_t))(B)(s), \hspace{1cm} 0 ≤ s ≤ t,$$  \hspace{1cm} (1.5)

is identical in law with $\{B_s\}_{0 ≤ s ≤ t}$.  


To see that the above theorem extends Bougerol’s identity, we evaluate (1.5) at \( s = t \). Then the theorem entails
\[-x + \text{Argsh} \left( e^{Bt} \sinh x + \beta(A_t) \right) \overset{[d]}{=} B_t,
\]
which is nothing but (1.2).

**Remark 1.1.** A notable thing about the theorem is that, in view of (1.2), the difference of the two random variables with same law, \( x + B_t \) and \( \text{Argsh} \left( e^{Bt} \sinh x + \beta(A_t) \right) \), is substituted into \( z \) of the transform \( T_z(B) \), and the resulting process is distributed as \( T_0(B) \) as if those two random variables were identical.

In Section 3, we prove Theorem 1.1 in an extended form. See Theorem 3.1.

Following the notation used in a series of papers [10, 11, 13, 15] by Matsumoto and Yor, we set
\[ Z^{(\mu)}_t := e^{-B^{(\mu)}_t} A^{(\mu)}_t, \quad t \geq 0, \]
for \( \phi \in C([0, \infty); \mathbb{R}) \), so that \( Z^{(\mu)}_t = Z_t(B^{(\mu)}_t), \) \( t \geq 0 \). With slight abuse of notation, we simply write \( Z_t \) for \( Z_t(B) \) as well. In this paper, we also prove an absolutely continuous relationship for the anticipative transforms \( T_z(B), \) \( z \in \mathbb{R} \), of the Brownian motion \( B \), and reveal how Theorem 1.1 is connected to it.

**Theorem 1.2.** Fix \( z \in \mathbb{R} \). For every nonnegative measurable functional \( F \) on \( C([0, t]; \mathbb{R}) \), we have
\[ \mathbb{E} \left[ F(T_z(B)(s), s \leq t) \right] = \mathbb{E} \left[ \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} F(B_s, s \leq t) \right], \quad (1.7) \]
or, equivalently,
\[ \mathbb{E}[F(B_s, s \leq t)] = \mathbb{E} \left[ \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} F(T_{-z}(B)(s), s \leq t) \right]. \quad (1.8) \]

A link between the above theorem and the Malliavin calculus will be mentioned at the end of the paper.

For each \( \phi \in C([0, \infty); \mathbb{R}) \) such that \( A_\infty(\phi) := \lim_{t \to \infty} A_t(\phi) < \infty \), we define
\[ T^*_z(\phi)(s) := \phi_s - \log \left\{ 1 + \frac{A_s(\phi)}{A_\infty(\phi)} (e^z - 1) \right\}, \quad s \geq 0. \quad (1.9) \]
By applying the Cameron–Martin relation in Theorem 1.2 and passing to the limit as \( t \to \infty \), we immediately obtain
Corollary 1.1. Let \( \mu > 0 \) and fix \( z \in \mathbb{R} \). For every nonnegative measurable functional \( F \) on \( C([0, \infty); \mathbb{R}) \), we have
\[
\mathbb{E}[F(T^*_{\mu}(B^{(-\mu)})(s), s \geq 0)] = e^{-\mu z} \mathbb{E}\left[ \exp\left( \frac{1-e^{-z}}{2A^{(-\mu)}_\infty} \right) F(B^{(-\mu)}_s, s \geq 0) \right],
\]
or, equivalently,
\[
\mathbb{E}[F(B^{(-\mu)}_s, s \geq 0)] = e^{-\mu z} \mathbb{E}\left[ \exp\left( \frac{1-e^{-z}}{2A^{(-\mu)}_\infty} \right) F(T^*_{-\mu}(B^{(-\mu)})(s), s \geq 0) \right].
\]

When \( \mu > 0 \), we know
\[
A^{(-\mu)}_\infty = \lim_{t \to \infty} A^{(-\mu)}_t < \infty \text{ a.s.; in fact, it is shown by Dufresne [6, Proposition 4.4.4(b)] that}
\]
\[
A^{(-\mu)}_\infty \overset{(d)}{=} \frac{1}{2\gamma_\mu},
\]
where \( \gamma_\mu \) is a gamma random variable with parameter \( \mu \):
\[
\mathbb{P}(\gamma_\mu \in du) = \frac{1}{\Gamma(\mu)} u^{\mu-1} e^{-u} du, \quad u > 0.
\]

Remark 1.2. Taking \( F \equiv 1 \) and \( z = -\log(1 + \alpha) \) for \( \alpha > -1 \), we may deduce Dufresne’s identity (1.12) from Corollary 1.1 in such a way that, for any \( \alpha > -1 \),
\[
\mathbb{E} \left[ \exp \left( -\frac{\alpha}{2A^{(-\mu)}_\infty} \right) \right] = \left( \frac{1}{1 + \alpha} \right)^\mu
\]
\[= \frac{1}{\Gamma(\mu)} \int_0^\infty du u^{\mu-1} e^{-\alpha u},
\]
which implies (1.12) thanks to the injectivity of the Laplace transform.

An intriguing fact which may be deduced from Corollary 1.1 is the following invariance of the law of \( B^{(-\mu)} \):
\[
\left\{ T^*_{\log(2\gamma_\mu A^{(-\mu)}_\infty)}(B^{(-\mu)})(s) \right\}_{s \geq 0} \overset{(d)}{=} \left\{ B^{(-\mu)}_s \right\}_{s \geq 0}.
\]
See Proposition 4.1. Here \( B \) and \( \gamma_\mu \) are independent on the left-hand side. Noting
\[
\log(2\gamma_\mu A^{(-\mu)}_\infty) = \log(2\gamma_\mu) - \log(1/A^{(-\mu)}_\infty)
\]
and remembering Dufresne’s identity (1.12), we may compare the above invariance with Remark 1.1. Our research is also inspired by Donati-Martin–Matsumoto–Yor [5], in which a family of path transformations on \( C([0, \infty); \mathbb{R}) \), denoted by \( T_\alpha, \alpha > 0 \), is introduced and its properties are investigated. We discuss its connection with transformations (1.9) and Corollary 1.1 as well as with the above-mentioned invariance; see Subsection 4.2.

Let \( \varepsilon \) denote a Rademacher (or symmetric Bernoulli) random variable taking values \( \pm 1 \) with equal probability. The following variant of Bougerol’s identity is known (see, e.g., [19, Corollary 2.2]): for every fixed \( t > 0 \),
\[
\beta(A_t^{(1)}) \overset{(d)}{=} \sinh B_t^{(\varepsilon)},
\]
(1.14)
where, on the left-hand side, \( \beta = \{ \beta(s) \}_{s \geq 0} \) is a one-dimensional standard Brownian motion independent of \( B \) as before, and on the right-hand side, \( \varepsilon \) is independent of \( B \) as before. In the sequel, given a real-valued process \( X = \{ X_s \}_{s \geq 0} \) and a point \( a \in \mathbb{R} \), we denote by \( \tau_a(X) \) the first hitting time of \( X \) to the level \( a \):

\[
\tau_a(X) := \inf\{ s \geq 0; X_s = a \}
\]

with the convention \( \inf \emptyset = \infty \).

**Theorem 1.3.** Let \( \beta = \{ \beta(s) \}_{s \geq 0} \) and \( \hat{B} = \{ \hat{B}_s \}_{s \geq 0} \) be one-dimensional standard Brownian motions. The following identity in law holds:

\[
\left( \left\{ T_{B_t^{(1)} - \text{Argsh} \beta(A_t^{(1)})} (B^{(1)}(s)) \right\}_{0 \leq s \leq t}, e^{-2B_t^{(1)} A_t^{(1)}} \right) \overset{(d)}{=} \left( \left\{ B_s^{(\varepsilon)} \right\}_{0 \leq s \leq t}, \tau_1(\hat{B}(\cosh B_t^{(\varepsilon)} / Z_t^{(\varepsilon)})) \right),
\]

where, on the left-hand side, \( B \) and \( \beta \) are independent, and on the right-hand side, \( \hat{B} \) and \( \varepsilon \) are independent. In particular, we have

\[
\left( \beta(A_t^{(1)}), e^{-2B_t^{(1)} A_t^{(1)}}, Z_t^{(1)} \right) \overset{(d)}{=} \left( \sinh B_t^{(\varepsilon)}, \tau_1(\hat{B}(\cosh B_t^{(\varepsilon)} / Z_t^{(\varepsilon)})), Z_t^{(\varepsilon)} \right).
\]

The last identity may be regarded as a counterpart to [7, Theorem 1.2] with \( x = 0 \) and \( \tau = t \) therein, in the presence of the drift \( \mu = 1 \); in addition, the identity between the third coordinates is consistent with the fact [10, Theorem 1.6(ii)] that, for each \( \mu \in \mathbb{R} \),

\[
\{ Z_t^{(\mu)} \}_{t \geq 0} \overset{(d)}{=} \{ Z_t^{(-\mu)} \}_{t \geq 0}.
\]

The rest of the paper is organized as follows. In Section 2 we summarize some properties, such as a semigroup property, of the path transformations \( T_z \), which will be referred to throughout the paper. Theorem 1.1 is proven in Section 3 by extending it to Theorem 3.1 we state and prove Theorem 3.1 in Subsection 3.1 by preparing Lemma 3.1 whose proof is given in Subsection 3.2. In Section 4 we prove Theorem 1.2 and its Corollary 1.1 after discussing how Theorems 1.1 and 3.1 are connected to Theorem 1.2; the proof of Theorem 1.2 is given in Subsection 4.1 while Corollary 1.1 is proven in Subsection 4.2. In Section 5, we give a proof of Theorem 1.3. The paper is concluded with Section 6 in which two remarks are given: one is about a possible extension of Theorems 1.1, 3.1 and 1.2 to the situation where the terminal time \( t \) therein is replaced by any positive and finite stopping time of the process \( \{ Z_s \}_{s \geq 0} \); and the other a plausible derivation of Theorem 1.2 in the framework of the Malliavin calculus.

In the sequel, when we say a Brownian motion, it always refers to a one-dimensional and standard one. We equip the spaces \( C([0, t]; \mathbb{R}) \) and \( C([0, \infty); \mathbb{R}) \) with topologies of uniform convergence and local uniform convergence, respectively; in particular, functionals on these spaces are said to be measurable if they are Borel-measurable with respect to those topologies.
2 Properties of $T_z$

In this section, we investigate properties of the transformations $T_z$, $z \in \mathbb{R}$, defined by (1.4) with $t > 0$ fixed. Accordingly, the two transformations $A$ and $Z$ defined by (1.3) and (1.6), respectively, are restricted on $C([0, t]; \mathbb{R})$. These are related via

$$
\frac{d}{ds} \frac{1}{A_s(\phi)} = -\left\{ \frac{1}{Z_s(\phi)} \right\}^2, \quad 0 < s \leq t,
$$

for any $\phi \in C([0, t]; \mathbb{R})$. We also consider the time reversal which we denote by $R$:

$$
R(\phi)(s) := \phi_{t-s} - \phi_t, \quad 0 \leq s \leq t, \quad \phi \in C([0, t]; \mathbb{R}).
$$

Proposition 2.1. We have the following properties (i)–(v).

(i) For every $z \in \mathbb{R}$ and $\phi \in C([0, t]; \mathbb{R})$, $T_z(\phi)(t) = \phi_t - z$.

(ii) For every $z \in \mathbb{R}$ and $\phi \in C([0, t]; \mathbb{R})$,

$$
\frac{1}{A_s(T_z(\phi))} = \frac{1}{A_s(\phi)} + e^z - 1 \frac{1}{A_t(\phi)}, \quad 0 < s \leq t.
$$

In particular, $A_t(T_z(\phi)) = e^{-z}A_t(\phi)$.

(iii) (Semigroup property) $T_z \circ T_{z'} = T_{z+z'}$ for any $z, z' \in \mathbb{R}$. In particular,

$$
T_z \circ T_{-z} = T_0 = I \quad \text{for any } z \in \mathbb{R}.
$$

Here $I : C([0, t]; \mathbb{R}) \to C([0, t]; \mathbb{R})$ is the identity map.

(iv) $Z \circ T_z = Z$ for any $z \in \mathbb{R}$.

(v) For every $z \in \mathbb{R}$, $T_z \circ R \circ T_z = R$, and hence

$$
R \circ T_z = T_{-z} \circ R.
$$

Proof. (i) This follows immediately from the definition (1.4) of $T_z$.

(ii) This follows by a direct computation: for $z \neq 0$, we have

$$
A_s(T_z(\phi)) = \int_0^s du \frac{e^{2\phi_u}}{\left\{ 1 + \frac{A_s(\phi)}{A_t(\phi)}(e^z - 1) \right\}^2} = \frac{A_t(\phi)}{e^z - 1} \left\{ \frac{1}{1 + \frac{A_s(\phi)}{A_t(\phi)}(e^z - 1)} \right\} = \frac{A_s(\phi)}{1 + \frac{A_s(\phi)}{A_t(\phi)}(e^z - 1)},
$$
which entails the claim. The case $z = 0$ is obvious since $T_0(\phi) = \phi$.

(iii) It suffices to show that, for each $\phi \in C([0, t]; \mathbb{R})$,

$$A_s((T_z \circ T_{z'})(\phi)) = A_s(T_{z+z'}(\phi)), \quad 0 \leq s \leq t; \quad (2.3)$$

indeed, once this identity is shown, then taking the derivative with respect to $s$ on both sides leads to the conclusion. To this end, for every $0 < s \leq t$, repeated use of (ii) yields

$$\frac{1}{A_s((T_z \circ T_{z'})(\phi))} = \frac{1}{A_s(T_z(\phi))} + \frac{e^{z'} - 1}{A_t(T_z(\phi))}$$

$$= \frac{1}{A_s(\phi)} + \frac{e^z - 1}{A_t(\phi)} + \frac{e^{z'} - 1}{e^z A_t(\phi)}$$

$$= \frac{1}{A_s(T_{z+z'}(\phi))},$$

which shows (2.3).

(iv) Noting (2.1) and taking the derivative with respect to $s$ on both sides of the displayed identity in property (ii), we arrive at the conclusion.

(v) Since the case $z = 0$ is obvious, we let $z \neq 0$. Pick $\phi \in C([0, t]; \mathbb{R})$ arbitrarily and set

$$\psi_s = (R \circ T_z)(\phi)(s), \quad 0 \leq s \leq t.$$

A direct computation shows that

$$A_s(\psi) = e^{-2\phi_t + 2z} \frac{A_t(\phi)}{e^z - 1} \left\{ \frac{1}{1 + \frac{A_{t-s}(\phi)}{A_t(\phi)}(e^z - 1)} - e^{-z} \right\},$$

and in particular, $A_t(\psi) = e^{-2\phi_t + z} A_t(\phi)$, from which we have

$$1 + \frac{A_s(\psi)}{A_t(\psi)}(e^z - 1) = \frac{e^z}{1 + \frac{A_{t-s}(\phi)}{A_t(\phi)}(e^z - 1)}.$$

Therefore, by the definition of $T_z$,

$$T_z(\psi)(s) = \psi_s - \log \left\{ 1 + \frac{A_s(\psi)}{A_t(\psi)}(e^z - 1) \right\}$$

$$= \phi_{t-s} - \log \left\{ 1 + \frac{A_{t-s}(\phi)}{A_t(\phi)}(e^z - 1) \right\} - \phi_t + z$$

$$- \log \left\{ \frac{e^z}{1 + \frac{A_{t-s}(\phi)}{A_t(\phi)}(e^z - 1)} \right\}$$

$$= \phi_{t-s} - \phi_t,$$

which proves the former assertion. The latter follows from the former and (iii). □
We end this section with a lemma concerning the time reversal $R$ for later use.

**Lemma 2.1.** It holds that, for all $\phi \in C([0, t]; \mathbb{R})$,

$$Z_t(R(\phi)) = Z_t(\phi).$$

**Proof.** By the definition of $Z$,

$$Z_t(R(\phi)) = e^{-R(\phi)(t)} \int_0^t e^{2R(\phi)(s)} \, ds = e^{\phi_t} \int_0^t e^{2(\phi_t - s)} \, ds = e^{-\phi_t} \int_0^t e^{2\phi_s} \, ds,$$

which shows the claim. \qed

## 3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1; we do this by providing an extension of the theorem. We keep $t > 0$ fixed, and $T_z, z \in \mathbb{R}$, are the transformations (1.4) acting on $C([0, t]; \mathbb{R})$.

### 3.1 Extension of Theorem 1.1 and its proof

Theorem 1.1 is part of the statement of the following theorem:

**Theorem 3.1.** Let $\beta = \{\beta(s)\}_{s \geq 0}$ and $\tilde{B} = \{\tilde{B}_s\}_{s \geq 0}$ are Brownian motions independent of $B$. For each fixed $x \in \mathbb{R}$, we denote

$$\zeta = \text{Argsh} \left( e^{B_t \sinh x + \beta(A_t)} \right) - (x + B_t), \quad T = \tau_{\cosh (x + B_t)}(\tilde{B}^{(\cosh x / Z_t)}),$$

for simplicity. Then the following identities in law hold:

\begin{align*}
\{B_s\}_{0 \leq s \leq t}, \quad \zeta &\overset{(d)}{=} \left( \{T_{\log(A_t / T)}(B)(s)\}_{0 \leq s \leq t}, \log(A_t / T) \right), \\
\{T_{-\zeta}(B)(s)\}_{0 \leq s \leq t}, \quad A_t &\overset{(d)}{=} \{B_s\}_{0 \leq s \leq t}, \quad T.
\end{align*}

In particular, both

$$\{T_{\log(A_t / T)}(B)(s)\}_{0 \leq s \leq t} \quad \text{and} \quad \{T_{-\zeta}(B)(s)\}_{0 \leq s \leq t}$$

are Brownian motions.
It is our finding ([7, Theorem 1.2]) that, for every fixed \(x \in \mathbb{R}\),

\[
(e^{B_t} \sinh x + \beta(A_t), A_t, Z_t) \\
\overset{(d)}{=} \left( \sinh(x + B_t), \tau_{\cosh(x+B_t)}(\tilde{B}^{(\cosh x/Z_t)}), Z_t \right).
\]

(3.3)

The starting point of our proof of Theorem 3.1 is to observe that identity (3.3) may be extended in such a way that

\[
(e^{B_t} \sinh x + \beta(A_t), A_t, \{Z_s\}_{0 \leq s \leq t}) \\
\overset{(d)}{=} \left( \sinh(x + B_t), \tau_{\cosh(x+B_t)}(\tilde{B}^{(\cosh x/Z_t)}), \{Z_s\}_{0 \leq s \leq t} \right).
\]

(3.4)

Lemma 3.1. Identity (3.4) holds.

We postpone the proof of Lemma 3.1 to Subsection 3.2. By using this lemma, the proof of Theorem 3.1 proceeds in the following way:

Proof of Theorem 3.1. By noting (2.1) and by Lemma 3.1,

\[
\left( e^{B_t} \sinh x + \beta(A_t), A_t, \left\{ \frac{1}{A_s} - \frac{1}{A_t} \right\}_{0 \leq s \leq t} \right) \\
\overset{(d)}{=} \left( \sinh(x + B_t), T, \left\{ \frac{1}{A_s} - \frac{1}{A_t} \right\}_{0 \leq s \leq t} \right),
\]

from which it follows that

\[
\left( \zeta + B_t, \left\{ \frac{1}{A_s} \right\}_{0 \leq s \leq t} \right) \\
\overset{(d)}{=} \left( B_t, \left\{ \frac{1}{A_s} - \frac{1}{A_t} + \frac{1}{T} \right\}_{0 \leq s \leq t} \right).
\]

Note that, on the right-hand side, we have for every \(0 < s \leq t\),

\[
\frac{1}{A_s} - \frac{1}{A_t} + \frac{1}{T} = \frac{1}{A_s} + \frac{1}{A_t} \left( \frac{A_t}{T} - 1 \right) \\
= \frac{1}{A_s(\mathbb{T}_{\log(A_t/T)}(B))},
\]

(3.5)

where the second line follows from Proposition 2.1(ii). Therefore the last identity in law entails that

\[
(\zeta + B_t, \{A_s\}_{0 \leq s \leq t}) \\
\overset{(d)}{=} \left( B_t, \left\{ A_s(\mathbb{T}_{\log(A_t/T)}(B)) \right\}_{0 \leq s \leq t} \right),
\]

and hence, by differentiating the second coordinate with respect to \(s\) on both sides,

\[
(\zeta + B_t, \{B_s\}_{0 \leq s \leq t}) \\
\overset{(d)}{=} \left( B_t, \left\{ \mathbb{T}_{\log(A_t/T)}(B)(s) \right\}_{0 \leq s \leq t} \right).
\]

This proves (3.1) because \(B_t - \mathbb{T}_{\log(A_t/T)}(B)(t) = \log(A_t/T)\) by Proposition 2.1(i). To prove (3.2), observe from (3.1) the identity

\[
\left( \{\mathbb{T}_{-\zeta}(B)(s)\}_{0 \leq s \leq t}, \zeta \right) \\
\overset{(d)}{=} (\{B_s\}_{0 \leq s \leq t}, \log(A_t/T)),
\]

(3.6)
which may be seen as a consequence of Proposition 2.1(iii) (see also Remark 4.1). Or, more convincingly, by (3.1) and as noted in (3.5),

\[
\left\{ \frac{1}{A_s} \right\}_{0<s\leq t}, \frac{1}{A_t}, \zeta \right\} \overset{(d)}{=} \left\{ \frac{1}{A_s} - \frac{1}{A_t} + \frac{1}{T} \right\}_{0<s\leq t}, \frac{1}{T}, \log \frac{A_t}{T}, \log \frac{A_t}{T}
\]

from which we have, by Proposition 2.1(ii),

\[
\left\{ \frac{1}{A_s}(T - \zeta B(s)) \right\}_{0<s\leq t}, e^{-\zeta} - 1, \zeta \right\} \overset{(d)}{=} \left\{ \frac{1}{A_s} \right\}_{0<s\leq t}, \log \frac{A_t}{T}, \log \frac{A_t}{T}
\]

Then the same argument as used in the first half of the proof leads to (3.6). It now follows from (3.6) that, by Proposition 2.1(i),

\[
\left\{ T - \zeta B(s) \right\}_{0<s\leq t}, B_t + \zeta, e^{-\zeta} \right\} \overset{(d)}{=} \left\{ B_s \right\}_{0<s\leq t}, B_t, T/A_t
\]

and hence, by the definition \( Z_t = e^{-B_t}A_t \) of \( Z_t \),

\[
\left\{ T - \zeta B(s) \right\}_{0<s\leq t}, e^{B_t} \right\} \overset{(d)}{=} \left\{ B_s \right\}_{0<s\leq t}, T/Z_t
\]

We multiply the second coordinates on the respective sides by \( Z_t(T - \zeta B(B)) \) and \( Z_t \), respectively. Then the right-hand side turns into that of the claimed identity (3.2) and so does the left-hand side because \( Z_t(T - \zeta B(B)) = Z_t \) by virtue of Proposition 2.1(iv). The proof of Theorem 3.1 is completed.

**Remark 3.1.** As for identity (3.1), notice that, by the definition of \( Z_t \),

\[
\log(A_t/T) = B_t - \log(T/Z_t)
\]

Identity (3.3) reveals that

\[
e^{B_t} \overset{(d)}{=} \tau_{\cosh(x+B_t)}(\hat{B}(\cosh x/Z_t))/Z_t,
\]

and hence \( B_t \overset{(d)}{=} \log(T/Z_t) \) by recalling \( T = \tau_{\cosh(x+B_t)}(\hat{B}(\cosh x/Z_t)) \). Therefore the same remark as Remark 1.1 applies to the identity in law between the first coordinates in identity (3.1) as well: the difference of the two random variables \( B_t \) and \( \log(T/Z_t) \) with same law is substituted into \( z \) of \( T_z(B) \), which results in the process identical in law with \( T_0(B) \).
3.2 Proof of Lemma 3.1

In this subsection, we give a proof of Lemma 3.1. We denote by \( \{Z_s\}_{s \geq 0} \) the natural filtration of the process \( \{Z_s\}_{s \geq 0} \). The proof hinges upon the following proposition due to Matsumoto and Yor, which is a consequence of the diffusion property of \( \{Z_s\}_{s \geq 0} \) investigated in detail in a pair of their papers \cite{10,11}; as they are dealing with Brownian motion with drift, we present here a particular case without drift.

**Proposition 3.1** \cite{10}, Proposition 1.7. Conditionally on \( Z_t \) with \( 1/Z_t = u > 0 \), \( B_t \) is distributed as a random variable \( z_u \) whose law is given by

\[
P(z_u \in dx) = \frac{1}{2K_0(u)} e^{-u \cosh x} dx, \quad x \in \mathbb{R}.
\]

Here \( K_0 \) is the modified Bessel function of the third kind (Macdonald function) of order 0.

The proof of Lemma 3.1 is done by combining the above proposition with Proposition 2.3(ii) of \cite{7} asserting that, for every \( x \in \mathbb{R} \) and \( u > 0 \),

\[
(e^{z_u} \sinh x + \beta(e^{z_u}/u), e^{z_u}) \stackrel{(d)}{=} \left( \sinh(x + z_u), u \tau \cosh(x + z_u)(\hat{B}(u \cosh x)) \right),
\]

where \( \beta \) and \( \hat{B} \) are Brownian motions independent of \( z_u \).

**Proof of Lemma 3.1** Let \( f : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) be a bounded measurable function and \( F : C([0,t];\mathbb{R}) \to \mathbb{R} \) a bounded measurable functional. By conditioning on \( Z_t \),

\[
\mathbb{E}\left[ f(e^{B_t} \sinh x + \beta(e^{z_u}/u), e^{z_u}) \right] = \mathbb{E}\left[ \mathbb{E}\left[ f(e^{B_t} \sinh x + \beta(A_t), e^{B_t}) \mid Z_t \right] F(Z_s, s \leq t) \right].
\]

Notice that, by Proposition 3.1 and the relation \( A_t = e^{B_t} Z_t \), the above conditional expectation given \( Z_t \) is equal a.s. to

\[
\mathbb{E}\left[ f(e^{z_u} \sinh x + \beta(e^{z_u}/u), e^{z_u}) \right] \bigg|_{u=1/Z_t},
\]

supposing \( \beta \) is independent of \( z_u \). Owing to (3.7), the above expression is further rewritten as

\[
\mathbb{E}\left[ f(\sinh(x + z_u), u \tau \cosh(x + z_u)(\hat{B}(u \cosh x))) \right] \bigg|_{u=1/Z_t}.
\]

Therefore, using Proposition 3.1 again, we have

\[
\mathbb{E}\left[ f(e^{B_t} \sinh x + \beta(A_t), e^{B_t}) \right] F(Z_s, s \leq t) = \mathbb{E}\left[ f(\sinh(x + B_t), \tau \cosh(x + B_t)(\hat{B}(\cosh x/Z_t))/Z_t)F(Z_s, s \leq t) \right],
\]

\( \hat{B} \) being assumed to be independent of \( B \). As \( f \) and \( F \) are arbitrary, the last equality entails (3.4). \( \square \)
4 Proofs of Theorem 1.2 and Corollary 1.1

In this section, we prove Theorem 1.2 and its Corollary 1.1. Before proceeding to the proof of Theorem 1.2, we explain how Theorem 1.1 and its extension Theorem 3.1 are connected to Theorem 1.2. We keep the notation used in Theorem 3.1.

Recall the fact that, for $a > 0$ and $\mu \geq 0$,

$$
\mathbb{P}(\tau_a(B(\mu)) \in du) = \frac{a}{\sqrt{2\pi u^3}} \exp\left\{-\frac{(a - \mu u)^2}{2u}\right\} du, \quad u > 0
$$

(see, e.g., [2, p. 301, formula 2.2.0.2]). Suppose that $F : C([0, t]; \mathbb{R}) \rightarrow [0, \infty)$ is bounded and continuous for the time being. Thanks to the independence of $B$ and $\hat{B}$, the finite measure

$$
\mathbb{E}[F(T_{\log(A_t/T)}(B)(s), s \leq t); \log(A_t/T) \in dz]
$$
on $\mathbb{R}$, admits a density $f_1$ with respect to the Lebesgue measure $dz$ expressed as

$$
f_1(z) = \mathbb{E} \left[ \frac{\cosh(x + B_t)}{\sqrt{2\pi e^{-z}A_t}} \exp\left\{-\frac{(\cosh(x + B_t) - e^{B_t - z} \cosh x)^2}{2e^{-z}A_t}\right\} F(T_z(B)(s), s \leq t) \right].
$$

Indeed, for every $z \in \mathbb{R}$, we have, by (4.1) and the independence of $B$ and $\hat{B}$,

$$
\mathbb{E}[F(T_{\log(A_t/T)}(B)(s), s \leq t); \log(A_t/T) \leq z]
$$

$$
= \mathbb{E} \left[ \int_{e^{-z}A_t}^{\infty} du \frac{\cosh(x + B_t)}{\sqrt{2\pi u^3}} \exp\left\{-\frac{(\cosh(x + B_t) - u \cosh x/Z_t)^2}{2u}\right\}\right. \\
\times F(T_{\log(A_t/u)}(B)(s), s \leq t) \right],
$$

which is equal, by changing the variables inside the expectation with $u = e^{-y}A_t$, $y \in \mathbb{R}$, and by Fubini’s theorem, to

$$
\int_{-\infty}^{z} f_1(y) dy.
$$

This verifies the above expression of $f_1$. Notice that $f_1$ is continuous owing to the boundedness and continuity of $F$. In the same manner, we have

$$
\mathbb{E}[F(B_s, s \leq t); \zeta \in dz] = f_2(z) dz, \quad z \in \mathbb{R},
$$

with

$$
f_2(z) = \mathbb{E} \left[ \frac{\cosh(x + z + B_t)}{\sqrt{2\pi A_t}} \exp\left\{-\frac{(\sinh(x + z + B_t) - e^{B_t - z} \sinh x)^2}{2A_t}\right\} F(B_s, s \leq t) \right].
$$
See also the proof of Lemma 5.1 as to the reasoning of the above derivation. It is clear that \( f_2 \) is continuous as well by the boundedness of \( F \). By identity (3.1) in Theorem 3.1 we have

\[
\tag{4.2}
f_1(z) = f_2(z)
\]

for a.e. \( z \), and hence for all \( z \) by the continuity of \( f_1 \) and \( f_2 \). By the monotone convergence theorem, (4.2) holds true if \( F \) is not bounded. Notice that, by (i) and (ii) of Proposition 2.1,

\[
f_1(z) = \mathbb{E}\left[ \frac{\cosh(x + z + T_z(B)(t))}{\sqrt{2\pi A_t(T_z(B))}} \exp\left\{ -\frac{\left(\cosh(x + z + T_z(B)(t)) - e^{T_z(B)(t)} \cosh x\right)^2}{2A_t(T_z(B))} \right\} \times F(T_z(B)(s), s \leq t) \right].
\]

Therefore, if we replace \( F \) by a functional of the form

\[
\frac{\sqrt{2\pi A_t(\phi)}}{\cosh(x + z + \phi_t)} \exp\left\{ \frac{\left(\cosh(x + z + \phi_t) - e^{\phi_t} \cosh x\right)^2}{2A_t(\phi)} \right\} F(\phi, s \leq t)
\]

for \( \phi \in C([0, t]; \mathbb{R}) \), then we have, from (1.2),

\[
\mathbb{E}\left[ F(T_z(B)(s), s \leq t) \right]
= \mathbb{E}\left[ \exp\left\{ \frac{\left(\cosh(x + z + B_t) - e^{B_t} \cosh x\right)^2 - \left(\sinh(x + z + B_t) - e^{B_t} \sinh x\right)^2}{2A_t} \right\} \times F(B, s \leq t) \right]
\]

Since

\[
\left(\cosh(x + z + B_t) - e^{B_t} \cosh x\right)^2 - \left(\sinh(x + z + B_t) - e^{B_t} \sinh x\right)^2 = 1 - 2e^{B_t} \cosh(z + B_t) + e^{2B_t},
\]

we arrive, by the definition of \( Z_t \), at relation (1.7) when \( F \) is nonnegative and continuous. Then successive use of density and monotone class arguments extends \( F \) to the class of all nonnegative measurable functionals as claimed in Theorem 1.2. In the subsequent subsection, we give a direct proof of (1.7) via Proposition 3.1.

Remark 4.1. Thanks to Proposition 2.1(iii), identity (3.6) in the proof of Theorem 3.1 may also be proven by replacing \( F \) by \( F \circ T_{-z} \) in (4.2).
4.1 Proof of Theorem 1.2

In what follows, we fix \( z \in \mathbb{R} \). We begin with

**Lemma 4.1.** It holds that, for every bounded measurable function \( f : \mathbb{R} \to \mathbb{R} \) and for every bounded measurable functional \( F : C([0, t]; \mathbb{R}) \to \mathbb{R} \),

\[
\mathbb{E}[f(B_t - z)F(Z_s, s \leq t)] = \mathbb{E}\left[ \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} f(B_t)F(Z_s, s \leq t) \right].
\]

**Proof.** We start from the right-hand side, by conditioning on \( Z_t \), to rewrite it as

\[
\mathbb{E}\left[ \mathbb{E}\left[ \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} f(B_t) \bigg| Z_t \right] F(Z_s, s \leq t) \right].
\]

Then, by Proposition 3.1, the conditional expectation in the integrand is equal a.s. to

\[
\mathbb{E}\left[ \exp \left\{ u(\cosh z_u - \cosh(z + z_u)) \right\} f(z_u) \bigg|_{u=1/Z_t} \right],
\]

which is computed as

\[
\frac{1}{2K_0(u)} \int_{\mathbb{R}} dx \ e^{-u \cosh x} \exp \left\{ u(\cosh x - \cosh(z + x)) \right\} f(x)
\]

\[
= \frac{1}{2K_0(u)} \int_{\mathbb{R}} dx \ e^{-u \cosh(z + x)} f(x)
\]

\[
= \mathbb{E}[f(z_u - z)],
\]

with \( 1/Z_t \) inserted into \( u \). Therefore we see that the right-hand side of the claimed equality turns into

\[
\mathbb{E}[\mathbb{E}[f(B_t - z) | Z_t] F(Z_s, s \leq t)]
\]

and hence into the left-hand side. \( \square \)

Now that we are convinced that

\[
\mathbb{P}_{t,z} := \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} \mathbb{P}
\]

determines a probability measure, we may translate Lemma 4.1 into the following identity in law:

\[
(B_t - z, \{Z_s\}_{0 \leq s \leq t}) \overset{(d)}{=} \mathbb{P}_{t,z} (B_t, \{Z_s\}_{0 \leq s \leq t}). \tag{4.3}
\]

Here the notation \( \overset{(d)}{=} \mathbb{P}_{t,z} \) indicates that the law of the right-hand side of a claimed identity is considered under \( \mathbb{P}_{t,z} \). With this notation and from (4.3), the proof of Theorem 1.2 proceeds along the same lines as in the proof of Theorem 3.1.
Proof of Theorem 1.2. By (4.3) and (2.1), we have
\[
(B_t - z, Z_t, \left\{ \frac{1}{A_s} - \frac{1}{A_t} \right\}_{0 < s \leq t}) \overset{d}{=} \mathbb{P}_{t, z} (B_t, Z_t, \left\{ \frac{1}{A_s} - \frac{1}{A_t} \right\}_{0 < s \leq t}),
\]
and hence, by the relation \( A_t = e^{B_t} Z_t \),
\[
(e^{-z} A_t, \left\{ \frac{1}{A_s} - \frac{1}{A_t} \right\}_{0 < s \leq t}) \overset{d}{=} \mathbb{P}_{t, z} (A_t, \left\{ \frac{1}{A_s} - \frac{1}{A_t} \right\}_{0 < s \leq t}).
\]
Therefore we have
\[
\left\{ \frac{1}{A_s} - \frac{1}{A_t} + e^z \right\}_{0 < s \leq t} \overset{d}{=} \mathbb{P}_{t, z} \left\{ \frac{1}{A_s} \right\}_{0 < s \leq t},
\]
from which it follows that, thanks to Proposition 2.1(ii),
\[
\{ A_s(T_z(B))(s) \}_{0 \leq s \leq t} \overset{d}{=} \mathbb{P}_{t, z} \{ A_s \}_{0 \leq s \leq t}.
\]
This proves relation (1.7). By virtue of Proposition 2.1(iii), relation (1.8) follows by replacing \( F \) by \( F \circ T_{-z} \) in (1.7). The equivalence between (1.7) and (1.8) is also clear. \( \square \)

We give a comment on the consistency of relation (1.7) under the time reversal (2.2). Recall that the law of \( \{B_s\}_{0 \leq s \leq t} \) is invariant under \( R \):
\[
\{ R(B)(s) \}_{0 \leq s \leq t} \overset{d}{=} \{ B_s \}_{0 \leq s \leq t}.
\] (4.4)

Remark 4.2. By (1.7), and by \( R(B)(t) = -B_t \) and \( Z_t(R(B)) = Z_t \) because of Lemma 2.1, we have, for every \( z \in \mathbb{R} \),
\[
\mathbb{E} \left[ F(T_{-z}(B)(s), s \leq t) \right]
= \mathbb{E} \left[ \exp \left\{ \frac{\cosh R(B)(t) - \cosh(-z + R(B)(t))}{Z_t(R(B))} \right\} F(R(B)(s), s \leq t) \right]
= \mathbb{E} \left[ \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} F(R(B)(s), s \leq t) \right],
\]
which is equal to
\[
\mathbb{E} \left[ F(R(T_z(B))(s), s \leq t) \right]
\]
by (1.7) again. The above observation entails that
\[
T_{-z}(B) \overset{d}{=} R(T_z(B)),
\]
which is indeed the case since, by Proposition 2.1(v) and (4.4),
\[
R(T_z(B)) = T_{-z}(R(B)) \overset{d}{=} T_{-z}(B).
\]
4.2 Proof of Corollary 1.1

In this subsection, we prove Corollary 1.1 as well as explore some related facts. We begin with the following lemma.

**Lemma 4.2.** For every $\mu \in \mathbb{R}$ and $t > 0$, we have

$$
\mathbb{E} \left[ \exp \left( \frac{\theta}{2A_t^{(\mu)}} \right) \right] < \infty \quad \text{for all } \theta < 1.
$$

In order to prove the lemma, we recall from [1, equation (1.5)] that the law of $A_t$ admits the density function expressed as

$$
\exp \left( \frac{\pi^2}{8t} \right) \mathbb{E} \left[ \cosh \frac{B_t}{\sqrt{2\pi \nu}} \exp \left( -\frac{\cosh^2 B_t}{2\nu} \right) \cos \left( \frac{\pi}{2t} B_t \right) \right], \quad \nu > 0.
$$

We refer the reader to [8] for an account of explicit expressions of the laws of the functionals $A_t^{(\mu)}$; see also [13] and references therein.

**Proof of Lemma 4.2.** From (4.5), it follows readily that

$$
\mathbb{P}(1/A_t \in dv) \leq \frac{C}{\sqrt{v}} \exp \left( -\frac{v}{2} \right) \quad \text{for all } v > 0,
$$

for some positive constant $C$ depending only on $t$. Pick $\theta < 1$ arbitrarily and let $p > 1$ be such that $p\theta < 1$. Then, by the Cameron–Martin relation and Hölder’s inequality,

$$
\mathbb{E} \left[ \exp \left( \frac{\theta}{2A_t^{(\mu)}} \right) \right] = \mathbb{E} \left[ \exp \left( \frac{\theta}{2A_t} \right) e^{\mu B_t} \right] e^{-\mu^2 t/2}
$$

$$
\leq \mathbb{E} \left[ \exp \left( \frac{p\theta}{2A_t} \right) \right]^{1/p} \exp \left\{ \frac{\mu^2 t}{2(p - 1)} \right\},
$$

which is finite in view of (4.6).

Using the above lemma, we prove Corollary 1.1.

**Proof of Corollary 1.1.** Fix $\mu > 0$. Since there is nothing to prove when $z = 0$, we let $z \neq 0$. It suffices to prove the claim for functionals $F$ of the form

$$
F(\phi_s, s \geq 0) = f(\phi_{t_1}, \ldots, \phi_{t_n}), \quad \phi \in C([0, \infty); \mathbb{R}),
$$

with a positive integer $n$, $0 \leq t_1 \leq \cdots \leq t_n$, and a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. We pick such an $F$ and let $t \geq t_n$. Then, by replacing $F$ in (1.7) by $F(\phi_s, s \geq 0)e^{-\mu \phi_s - \mu^2 t/2}$, the Cameron–Martin relation entails that

$$
\mathbb{E} \left[ F(T_z B^{(-\mu)}(s), s \geq 0) \right] e^{\mu z}
$$

$$
= \mathbb{E} \left[ \exp \left( \frac{\cosh B^{(-\mu)}_t - \cosh(z + B^{(-\mu)}_t)}{Z^{(-\mu)}_t} \right) \right] F(B^{(-\mu)}_t, s \geq 0).
$$
Here the factor $e^{\mu z}$ on the left-hand side comes from property (i) in Proposition 2.1. Because of the continuity and boundedness of $f$ determining the functional $F$, the bounded convergence theorem entails that, as $t \to \infty$, the left-hand side converges to

$$\mathbb{E}[F(T_z^*(B^{(-\mu)})(s), s \geq 0)] e^{\mu z}.$$  

(Recall (1.4) and (1.9) of the definitions of $T_z$ and $T_z^*$, respectively; in particular, the definition of the former involves parameter $t$.) We turn to the right-hand side. By the definition of $Z_t^{(-\mu)}$, the integrand is rewritten as

$$\exp\left\{\frac{e^{2B_t^{(-\mu)}}(1 - e^z) + 1 - e^{-z}}{2A_t^{(-\mu)}} \right\} F(B_t^{(-\mu)}, s \geq 0),$$  

which, as $t \to \infty$, converges a.s. to

$$\exp\left(\frac{1 - e^{-z}}{2A_\infty^{(-\mu)}} \right) F(B_\infty^{(-\mu)}, s \geq 0).$$  

(4.9)

Therefore, if we know that the family of random variables (4.8), indexed by $t \geq t_n$, is uniformly integrable, then the right-hand side of the last equality converges to the expectation of (4.9) and hence relation (1.10) follows. To this end, we denote by $X_t$ the random variable given in (4.8). To see the uniform integrability of $\{X_t\}_{t \geq t_n}$, we divide the case into two cases. In the case $z > 0$, $|X_t| \leq M \exp\left(\frac{1 - e^{-z}}{2A_t^{(-\mu)}} \right)$ for any $t \geq t_n$, where $M$ is the supremum of $|f|$. Therefore, by Lemma 4.2, there exists $p > 1$ such that

$$\sup_{t \geq t_n} \mathbb{E}[|X_t|^p] < \infty,$$

from which the desired uniform integrability follows. In the case $z < 0$, the same estimate holds as well; indeed,

$$|X_t| \leq M \exp\left\{\frac{e^{2B_t^{(-\mu)}}(1 - e^z)}{2A_t^{(-\mu)}} \right\} = M \exp\left\{\frac{1 - e^z}{2A_t(R(B^{(-\mu)}))} \right\},$$

with $R$ the time reversal on $C([0,t];\mathbb{R})$ in (2.2), and hence picking $p > 1$ such that $p(1 - e^z) < 1$, we have, for any $t \geq t_n$,

$$\mathbb{E}[|X_t|^p] \leq M^p \mathbb{E}\left[\exp\left\{\frac{p(1 - e^z)}{2A_t^{(\mu)}} \right\}\right] \leq M^p \mathbb{E}\left[\exp\left\{\frac{p(1 - e^z)}{2A_{tn}^{(\mu)}} \right\}\right] < \infty.$$
by Lemma 4.2. Here, for the first line, we used the fact that
\[ \{ R(B^{(-\mu)})(s) \}_{0 \leq s \leq t} \overset{(d)}{=} \{ B_s^{(-\mu)} \}_{0 \leq s \leq t}, \]
which follows from (4.4). The proof of relation (1.10) completes. Relation (1.11) is immediate from (1.10) and the fact that, for any \( z \in \mathbb{R} \) and for any \( \phi \in C([0, \infty); \mathbb{R}) \) with \( A_\infty(\phi) < \infty \),
\[ T^*_z(T^*_z(\phi)) = \phi, \]
proof of which is done in the same way as that of Proposition 2.1(iii), hence omitted. The equivalence of the two relations (1.10) and (1.11) is also clear from the above fact.

**Remark 4.3.** We have not used Dufresne’s identity (1.12) in the above proof as pointed out in Remark 1.2 that it can be deduced from Corollary 1.1. If we have Dufresne’s identity at our disposal, the interchangeability of the order of limit and expectation seen above may be verified by Scheffé’s lemma (see, e.g., [20, p. 55]) since, for any \( t > 0 \),
\[ \mathbb{E} \left[ \exp \left\{ \frac{\cosh B_t^{(-\mu)} - \cosh (z + B_t^{(-\mu)})}{Z_t^{(-\mu)}} \right\} \right] = \mathbb{E} \left[ \exp \left( \frac{1 - e^{-z}}{2A_\infty^{(-\mu)}} \right) \right] \]
(in fact, both sides are equal to \( e^{\mu z} \)), which, together with their a.s. convergence, ensures the \( L^1 \)-convergence of the integrands.

Recall from [5] a family \( \{ T_\alpha \}_{\alpha > 0} \) of path transformations defined by
\[ T_\alpha(\phi)(s) := \phi_s - \log \{ 1 + \alpha A_s(\phi) \}, \quad s \geq 0, \quad (4.10) \]
for \( \phi \in C([0, \infty); \mathbb{R}) \). The following identity concerning the laws of \( B^{(\pm \mu)} \) is shown in Matsumoto–Yor [12, Theorem 2.1]: when \( \mu > 0 \),
\[ \{ B_s^{(-\mu)} \}_{s \geq 0} \overset{(d)}{=} \{ T_{2\gamma_{\mu}}(B^{(\mu)})(s) \}_{s \geq 0}. \quad (4.11) \]
Here \( \gamma_{\mu} \) is a gamma random variable with parameter \( \mu \) independent of \( B \). We will show that Corollary 1.1 may also be obtained by (4.11). First we observe the following relationship between the two transformations (1.9) and (4.10):

**Lemma 4.3.** It holds that, for every \( z \in \mathbb{R} \) and \( \alpha > 0 \),
\[ T^*_z(T_\alpha(\phi))(s) = T_{\alpha e^z}(\phi)(s), \quad s \geq 0, \]
for any \( \phi \in C([0, \infty); \mathbb{R}) \) such that \( A_\infty(\phi) = \infty \).

**Proof.** By the definition of \( T_\alpha \), we have, for every \( \phi \in C([0, \infty); \mathbb{R}) \),
\[ A_s(T_\alpha(\phi)) = \frac{A_s(\phi)}{1 + \alpha A_s(\phi)}, \quad s \geq 0, \]
and hence when \( A_\infty(\phi) = \infty \),

\[
A_\infty(T_\alpha(\phi)) \equiv \lim_{s \to \infty} A_s(T_\alpha(\phi)) = \frac{1}{\alpha},
\]

which is finite. Therefore, by the definition of \( T_\alpha^* \), we have for every \( s \geq 0 \),

\[
T^*_\alpha(T_\alpha(\phi))(s) = \phi_s - \log \{1 + \alpha A_s(\phi)\} - \log \left\{ \frac{1 + A_s(\phi)}{1 + \alpha A_s(\phi)}(e^s - 1) \right\}
\]

\[
= \phi_s - \log \{1 + e^s A_s(\phi)\}.
\]

which is the claim. \( \square \)

**Proof of Corollary 1.1 via (4.11).** Here we prove the latter relation (1.11). By identity (4.11) and Lemma 4.3, the right-hand side of (1.11) is written as

\[
e^{-\mu z} \mathbb{E} \left[ \exp \left\{ \frac{1 - e^{-z}}{2A_\infty(T_{2\gamma_\mu}(B(\mu)))} F(T_{2e^{-z}\gamma_\mu}(B(\mu))(s), s \geq 0) \right\} \right]
\]

\[
e^{-\mu z} \mathbb{E} \left[ \exp \left\{ (1 - e^{-z})\gamma_\mu F(T_{2e^{-z}\gamma_\mu}(B(\mu))(s), s \geq 0) \right\} \right],
\]

where the equality is due to (4.12). By the independence of \( B \) and \( \gamma_\mu \), and by Fubini’s theorem, the above expression is computed as

\[
e^{-\mu z} \frac{1}{\Gamma(\mu)} \int_0^\infty du u^{\mu-1} e^{-u} \exp \left\{ (1 - e^{-z})u \right\} \mathbb{E} \left[ F(T_{2e^{-z}\gamma_\mu}(B(\mu))(s), s \geq 0) \right]
\]

\[
= \frac{1}{\Gamma(\mu)} \int_0^\infty dv v^{\mu-1} e^{-v} \mathbb{E} \left[ F(T_{2\gamma_\mu}(B(\mu))(s), s \geq 0) \right],
\]

where we have changed the variables with \( u = e^z v \) for the second line. Owing to (4.11), the last expectation coincides with the left-hand side of (1.11). \( \square \)

**Remark 4.4.** On the other hand, applying the same reasoning as above to the right-hand side of (1.10) leads to the identity

\[
\left\{ T^*_\alpha(B^{-\mu})(s) \right\}_{s \geq 0} \overset{(d)}{=} \left\{ T_{2e\gamma_\mu}(B^{-\mu})(s) \right\}_{s \geq 0},
\]

with \( \gamma_\mu \) independent of \( B \). The case \( z = 0 \) is identity (4.11); letting \( z \to -\infty \) leads to

\[
\left\{ B^{-\mu}_s - \log \left( 1 - \frac{A^{(-\mu)}_s}{A^{(-\mu)}_\infty} \right) \right\}_{s \geq 0} \overset{(d)}{=} \left\{ B^{(\mu)}_s \right\}_{s \geq 0},
\]

which is found in [12, Proposition 3.1].

Corollary 1.1 enables us to obtain identity (1.13).
Proposition 4.1. Let \( \mu > 0 \) and suppose \( B \) and \( \gamma_\mu \) are independent. Then we have identity (1.13); that is, the process

\[
B_s^{(-\mu)} - \log \left\{ 1 + A_s^{(-\mu)} \left( 2\gamma_\mu - \frac{1}{A_s^{(-\mu)}} \right) \right\}, \quad s \geq 0,
\]

is identical in law with \( B^{(-\mu)} \).

Proof. In what follows, we write \( X \) for \( 1/(2A_\infty^{(-\mu)}) \) for simplicity. We suppose that \( F : C([0, \infty); \mathbb{R}) \to \mathbb{R} \) is bounded and continuous, and we take a bounded measurable function \( f : (0, \infty) \to \mathbb{R} \) arbitrarily. By noting (cf. Proposition 2.1(ii)) that

\[
\frac{1}{A_s(T_s^z(B^{(-\mu)}))} = \frac{1}{A_s^{(-\mu)}} + \frac{e^z - 1}{A_\infty^{(-\mu)}} \quad \text{as} \quad s \to \infty \quad A_\infty^{(-\mu)},
\]

it follows from (1.10) that

\[
\mathbb{E} \left[ F(T_s^z(B^{(-\mu)})(s), s \geq 0) f(X) \right] = e^{-\mu z} \mathbb{E} \left[ \exp \left\{ (1 - e^{-z})X \right\} F(B_s^{(-\mu)}, s \geq 0) f(e^{-z}X) \right].
\]

By the fact that \( X \overset{(d)}{=} \gamma_\mu \), the left-hand side is disintegrated as

\[
\frac{1}{\Gamma(\mu)} \int_0^\infty du u^{\mu-1} e^{-u} f(u) \mathbb{E} \left[ F(T_s^z(B^{(-\mu)})(s), s \geq 0) \mid X = u \right].
\]

On the other hand, by conditioning on \( X = u \), and by using the same change of the variables as in the last proof, the right-hand side is expressed as

\[
\frac{1}{\Gamma(\mu)} \int_0^\infty dv v^{\mu-1} e^{-v} f(v) \mathbb{E} \left[ F(B_s^{(-\mu)}, s \geq 0) \mid X = e^z v \right].
\]

Since the last two expressions agree for any \( f \), we conclude that

\[
\mathbb{E} \left[ F(T_s^z(B^{(-\mu)})(s), s \geq 0) \mid X = u \right] = \mathbb{E} \left[ F(B_s^{(-\mu)}, s \geq 0) \mid X = e^z u \right]
\]

for a.e. \( u > 0 \). By the continuity of \( F \), we may assume that each side admits a continuous version (see Remark 4.6) and hence that the above relation holds for all \( u > 0 \). Given \( v > 0 \), we insert log\((v/u)\) into \( z \) to obtain

\[
\mathbb{E} \left[ F(T_s^z(B^{(-\mu)})(s), s \geq 0) \mid X = u \right] = \mathbb{E} \left[ F(B_s^{(-\mu)}, s \geq 0) \mid X = v \right],
\]

which is valid for all \( u, v > 0 \). Integrating both sides with respect to the measure \( \mathbb{P}(X \in du) \mathbb{P}(X \in dv) \) over \((0, \infty)^2\), we reach the conclusion owing to the arbitrariness of \( F \). \( \square \)
Remark 4.5. Let \( h \) be an arbitrary positive measurable function on \((0, \infty)^2\) and replace in the last displayed equation \( F \) by a functional of the form
\[
F \left( \mathbb{T}^*_s \log(u h(u,v)) (\phi) (s), s \geq 0 \right)
\]
for \( \phi \in C([0, \infty); \mathbb{R}) \) with \( A_\infty(\phi) < \infty \). Then, using a semigroup property as in Proposition 2.1(iii) of \( \mathbb{T}_z, z \in \mathbb{R} \), we may have the following generalization of Proposition 4.1:

the two processes
\[
B_{s}^{(-\mu)} - \log \left\{ 1 + \frac{A_{s}^{(-\mu)}}{A_{\infty}^{(-\mu)}} \left( \gamma_{\mu} h \left( \frac{1}{2A_{\infty}^{(-\mu)}}, \frac{1}{\gamma_{\mu}} \right) - 1 \right) \right\}, \quad s \geq 0,
\]
are identical in law; taking either \( h(u,v) = 1/u \) or \( h(u,v) = 1/v \) yields Proposition 4.1.

The above identity in law may also be verified by means of identity (4.11). Indeed, if we let \( \hat{\gamma}_{\mu} \) be a copy of \( \gamma_{\mu} \) such that \( B_{\gamma_{\mu}} \) and \( \hat{\gamma}_{\mu} \) are independent, and if we replace \( B_{s}^{(-\mu)} \) by \( T_{2\gamma_{\mu}}(B_{(\mu)}) \) in the above two processes, then we obtain the process \( T_{2\gamma_{\mu}}(B_{(\mu)}) \) for the former and likewise \( T_{2\gamma_{\mu}}(B_{(\mu)}) \) for the latter, and these two are clearly identical in law because of the independence of the three elements involved.

Remark 4.6. To see that there exists a continuous version for a function of the form
\[
\mathbb{E} \left[ F \left( B_{s}^{(-\mu)}, s \geq 0 \right) \left| A_{\infty}^{(-\mu)} = u \right. \right], \quad u > 0,
\]
with \( F : C([0, \infty); \mathbb{R}) \to \mathbb{R} \) a bounded continuous functional, we recall Lamperti’s relation for the exponential functionals of \( B^{(-\mu)} \): there exists a \( 2(1 - \mu) \)-dimensional Bessel process \( \rho \) starting from \( 1 \), such that
\[
e^{B_{s}^{(-\mu)}} = \rho_{A_{\infty}^{(-\mu)}}, \quad s \geq 0.
\]
See, e.g., [12, Section 3]. In particular, letting \( s \to \infty \) on both sides yields the relation \( A_{\infty}^{(-\mu)} = \tau_{0}(\rho) \). By setting
\[
\alpha_{s}(\rho) = \inf \left\{ t \geq 0; \int_{0}^{t} \frac{dv}{\rho_{v}^{2}} 1_{\{v < \tau_{0}(\rho)\}} > s \right\}, \quad s \geq 0,
\]
Lamperti’s relation (4.14) entails the representation of \( B^{(-\mu)} \) in terms of \( \rho \):
\[
B_{s}^{(-\mu)} = \log \rho_{\alpha_{s}(\rho)}, \quad s \geq 0.
\]
Note that, conditionally on \( \tau_{0}(\rho) = u > 0 \), the process \( \rho \) is a \( 2(\mu + 1) \)-dimensional Bessel bridge of duration \( u \) ending at 0 (see, e.g., [18, Exercises XI.1.23 and XI.3.12]), and hence identical in law with
\[
\left( 1 - \frac{s}{u} \right) \hat{\rho}_{u-s}, \quad 0 \leq s < u,
\]
where \( \hat{\rho} \) is a \( 2(\mu + 1) \)-dimensional Bessel process starting from 1 (see, e.g., [18, Exercise XI.3.6]). These facts amount to the identity between the law of \( B^{(-\mu)} \) conditioned on \( A^{(\infty)} = u \) and that of the process

\[
\log \left\{ \frac{u}{u + \alpha_s(\hat{\rho})} \hat{\rho}_s \right\}, \quad s \geq 0,
\]

with \( \alpha_s(\hat{\rho}) \) defined through (4.15), \( \hat{\rho} \) replacing \( \rho \) (in this case, \( \tau_0(\hat{\rho}) = \infty \) a.s.). It is clear that the law of (4.16) depends on \( u \) continuously. Therefore the function in (4.13) admits a continuous version.

**Remark 4.7.** Lamperti’s relation for the exponential functionals of \( B^{(\mu)} \) reveals that (4.16) is identical in law with the process \( B^{(\mu)} - \log \left( \frac{1}{1 + A^{(\mu)}s/u} \right) \), \( s \geq 0 \), which, together with \( 1/A^{(\infty)} \overset{(d)}{=} 2\gamma \mu \), entails (4.11). A proof of identity (4.11) along this line may also be found in [12, Section 3].

## 5 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. For each \( z \in \mathbb{R} \), we denote by \( b^x = \{ b^x_s \}_{0 \leq s \leq t} \) a Brownian bridge of duration \( t \), starting from 0 and ending at \( x \). Let \( F : C([0, t]; \mathbb{R}) \to [0, \infty) \) be a bounded continuous functional and \( f : (0, \infty) \to [0, \infty) \) a bounded continuous function.

**Lemma 5.1.** Fix \( z \in \mathbb{R} \) and let \( \hat{B} = \{ \hat{B}_s \}_{s \geq 0} \) be a Brownian motion independent of \( b^x \). Then it holds that, for any \( x \in \mathbb{R} \),

\[
\mathbb{E} \left[ F(\max_{t \geq s \leq t} (\hat{B}_s - z) \hat{B}_t) f(A_t) \exp \left\{ \frac{1}{\sqrt{2\pi t}} \left( \frac{(\sinh(x + z) - e^{\hat{B}_t} \sinh x)^2}{2A_t} \right) \right\} \right]
\]

\[
= \mathbb{E} \left[ F(b^x_s, s \leq t) f(\tau_{\cosh(x+z)}(\hat{B}^{(\cosh(x/|b^x|))})) \right] \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{z^2}{2t} \right).
\]

**Proof.** For each fixed \( x \in \mathbb{R} \), we denote by the left-hand and right-hand sides of the claimed equality by \( g_1(z) \) and \( g_2(z) \), respectively. Notice that \( g_1 \) and \( g_2 \) are continuous in \( z \) by the assumptions on \( F \) and \( f \) (see Remark 5.1). We keep the notation of Theorem 3.1. By identity (3.2), we know that the two finite measures

\[
\mathbb{E}[F(\max_{t \geq s \leq t} (B_s - z)) \mid B_t + \zeta \in dz], \quad \mathbb{E}[F(B_s, s \leq t) \mid B_t \in dz]
\]

on \( \mathbb{R} \), agree. It is clear that the latter admits the density function \( g_2 \) with respect to the Lebesgue measure. On the other hand, for an arbitrary \( z \in \mathbb{R} \), noting that the
event that $B_t + \zeta \leq z$ is equal to the event that $e^{B_t} \sinh x + \beta(A_t) \leq \sinh(x + z)$ by the definition of $\zeta$, we see that, by the independence of $B$ and $\beta$, and by Fubini’s theorem,

$$
\mathbb{E}[F(T_{B_t - (B_t + \zeta)}(B), s \leq t) f(A_t); B_t + \zeta \leq z] = \int_{-\infty}^{\sinh(x+z)} dy \mathbb{E}
\left [ F(T_{B_t - \arg\sinh y + z}(B), s \leq t) f(A_t)
\times \frac{1}{\sqrt{2\pi A_t}} \exp \left \{ -\frac{(y - e^{B_t} \sinh x)^2}{2A_t} \right \} \right ].
$$

Differentiating the last expression with respect to $z$ entails that the former measure in (5.1) admits the density $g_1$. Therefore $g_1(z) = g_2(z)$ for a.e. $z$, and hence for all $z$ by the continuity of $g_1$ and $g_2$. This proves the lemma. □

**Remark 5.1.** As for the continuity of $g_2$ mentioned above, notice that, in view of formula (4.1), the expectation in the definition of $g_2$ may be expressed as

$$
\int_{0}^{\infty} du \frac{\cosh(x + z)}{\sqrt{2\pi u^3}} f(u) \mathbb{E}
\left [ F(b^\varepsilon_s, s \leq t) \exp \left \{ -\frac{(\cosh(x + z) - u \cosh x/Z_t(b^\varepsilon))^2}{2u} \right \} \right ],
$$

by the independence of $b^\varepsilon$ and $\hat{B}$, and that $b^\varepsilon \overset{(d)}{=} \{b_s^\varepsilon + (z/t)s\}_{0 \leq s \leq t}$.

**Proof of Theorem 4.3.** Since $x$ is arbitrary, we may take $x = -z$ in Lemma 5.1 which entails the relation

$$
\mathbb{E}
\left [ F(T_{B_t - z}(B), s \leq t) f(A_t) \frac{1}{\sqrt{2\pi A_t}} \exp \left ( -\frac{e^{2B_t} \sinh^2 z}{2A_t} \right ) \right ] = \mathbb{E}
\left [ F(b^\varepsilon_s, s \leq t) f(\tau_1(\hat{B}^{(\cosh z/Z_t(b^\varepsilon))})) \frac{1}{\sqrt{2\pi t}} \exp \left ( -\frac{z^2}{2t} \right ) \right ].
$$

Multiplying both sides by $\cosh z$, we integrate both sides with respect to $z$ over $\mathbb{R}$. Then the left-hand side turns into

$$
\mathbb{E}
\left [ F(T_{B_t - \arg\cosh \beta(e^{-2B_t}A_t)}(B), s \leq t) f(A_t) e^{-B_t} \right ] = e^{1/2} \mathbb{E}
\left [ F(T_{B_t^{(-1)} - \arg\cosh \beta(e^{-2B_t^{(-1)}A_t^{(-1)}})}(B^{(-1)}), s \leq t) f(A_t^{(-1)}) \right ],
$$

with $\beta$ independent of $B$, where the second line is due to the Cameron–Martin relation. On the other hand, the right-hand side turns into

$$
e^{1/2} \mathbb{E}
\left [ F(B_s^{(\varepsilon)}, s \leq t) f(\tau_1(\hat{B}^{(\cosh B_s^{(\varepsilon)})/Z_t^{(\varepsilon)}})) \right ]
$$

by the Cameron–Martin relation. Here $B$, $\hat{B}$ and are $\varepsilon$ independent. Since the last two expressions agree for any $F$ and $f$, we obtain the identity in law

$$
\left \{ \left \{ T_{B_t^{(-1)} - \arg\cosh \beta(e^{-2B_t^{(-1)}A_t^{(-1)}})(B^{(-1)})(s) \right \}_{0 \leq s \leq t}, A_t^{(-1)} \right \}
\overset{(d)}{=}
\left \{ \left \{ B_s^{(\varepsilon)} \right \}_{0 \leq s \leq t}, \tau_1(\hat{B}^{(\cosh B_s^{(\varepsilon)})/Z_t^{(\varepsilon)}}) \right \},
$$

(5.2)
By observing that
\[ \left( \{ B_i^{(-1)} \}_{0 \leq i \leq t}, e^{-2B_i^{(-1)}A_i^{(-1)}} \right) \overset{(d)}{=} \left( \{ R(B^{(1)})(s) \}_{0 \leq s \leq t}, A_i^{(1)}, e^{-2B_i^{(1)}A_i^{(1)}} \right) \]
due to (4.4), the left-hand side of (5.2) is identical in law with
\[ \left( \{ T_{-B_i^{(1)} - \text{Argsh} \beta(A_i^{(1)})}(R(B^{(1)}))(s) \}_{0 \leq s \leq t}, e^{-2B_i^{(1)}A_i^{(1)}} \right) \]
where we used Proposition 2.1(v) for the second line. Therefore, by observing that
\[ (R \circ R)(\phi) = \phi - \phi_0 \text{ for every } \phi \in C([0,t];\mathbb{R}), \]
we have, from (5.2),
\[ \left( \{ T_{B_i^{(1)} + \text{Argsh} \beta(A_i^{(1)})}(B^{(1)})(s) \}_{0 \leq s \leq t}, e^{-2B_i^{(1)}A_i^{(1)}} \right) \overset{(d)}{=} \left( \{ R(B^{(1)})(s) \}_{0 \leq s \leq t}, \tau_1(\hat{B}^{(cosh B^{(1)/Z_s^{(1)}}}) \right). \]

Note that \( B_i^{(e)} \) and \( Z_i^{(e)} \) in the second coordinate on the right-hand side may be expressed respectively as \(-R(\hat{B}^{(e)})(t)\) and \( R(\hat{B}^{(e)})(t) \) by Lemma 2.1. Moreover,
\[ \left( \{ R(B^{(e)})(s) \}_{0 \leq s \leq t}, \tau_1(\hat{B}^{(cosh B^{(e)/Z_s^{(e)}})} \right). \]
because of (4.4) and the independence of \( B \) and \( \varepsilon \). Consequently, the right-hand side of (5.3) is identical in law with that of the claimed identity (1.15). As for the left-hand side of (5.3), we use the symmetry \( \beta \overset{(d)}{=} -\beta \) and the independence of \( B \) and \( \beta \) to see that it is identical in law with the left-hand side of (1.15). Therefore identity (1.15) is proven. The latter identity in the theorem is shown by evaluating the first coordinates on both sides of (1.15) at \( s = t \) and noting property (iv) in Proposition 2.1; in fact, we have
\[ \left( \beta(A_i^{(1)}), e^{-2B_i^{(1)}A_i^{(1)}}, Z_i^{(1)} \right)_{0 \leq s \leq t} \overset{(d)}{=} \left( \sinh B_i^{(e)}, \tau_1(\hat{B}^{(cosh B_i^{(e)/Z_s^{(e)}})}, \{ Z_s^{(e)} \}_{0 \leq s \leq t} \right), \]
the identity between the third coordinates of which is consistent with (1.16). The proof of Theorem 1.3 completes.

6 Concluding remarks

We conclude this paper with the following two remarks.

1. Theorems 1.1 and 3.1, as well as Theorem 1.2, may be extended to the case where the deterministic time \( t \) therein is replaced by any \( \{ Z_s \} \)-stopping time \( \tau \) satisfying \( P(0 < \tau < \infty) = 1 \). This is due to the fact that, as stated in [10, Proposition 1.7],
the assertion of Proposition 3.1 holds true with the above replacement because of the diffusion property of \( \{Z_s\}_{s \geq 0} \), as well as to the invariance of the path transformation \( Z \) under the composition with \( T_z \) as in Proposition 2.1(iv), where we also consider a slight modification of the definition (1.4) of \( T_z \) in accordance with replacement of \( t \) by \( \tau \).

2. (1) The path transformations \( T_z \) provide an example of anticipative transformations studied by a number of researchers in the framework of the Malliavin calculus; see [17], [9] and [21], to name a few. Of concern in their studies is a Girsanov-type formula for those transformations, in which the density with respect to the underlying Wiener measure involves a Carleman–Fredholm determinant. Although we have not pursued it in this paper, we expect that Theorem 1.2 will furnish an example for which the associated Carleman–Fredholm determinant is explicitly calculated.

(2) Instead of going into details, here we content ourselves with showing a link between Theorem 1.2 and the Malliavin calculus. For simplicity, let \( F \) be a cylindrical functional on \( C([0,t];\mathbb{R}) \) as of the form (4.7), with \( f \) a bounded smooth function with bounded first derivatives which we denote by \( \partial_i f, i = 1, \ldots, n \). Observe that

\[
\frac{d}{dz} F(T_z(B))(s), s \leq t) = -\left( (DF)(T_z(B)), \int_0^t e^{2T_z(B)(s)} ds \right)_H, \quad z \in \mathbb{R},
\]

where \( \langle \cdot, \cdot \rangle_H \) stands for the inner product on the Cameron–Martin subspace \( H \) of \( C([0,t];\mathbb{R}) \), and \( DF \) denotes the Malliavin derivative of \( F \), namely,

\[
DF(\phi) = \sum_{i=1}^n \partial_i f(\phi_1, \ldots, \phi_n) \int_0^t 1_{[0,t]}(s) ds, \quad \phi \in C([0,t];\mathbb{R}).
\]

Then, taking the derivative at \( z = 0 \) on both sides of (1.7) yields

\[
\mathbb{E} \left[ DF(B), \int_0^t e^{2B_s} ds \right]_H = \mathbb{E} \left[ \sinh B_t Z_t F(B_s, s \leq t) \right]. \tag{6.1}
\]

We denote by \( \delta \) the Skorokhod integral, namely the adjoint of the operator \( D \); see [16, Definition 1.3.1]. Equation (6.1) may be explained in terms of \( \delta \). Indeed, by the definition of \( \delta \), the left-hand side of (6.1) is rewritten as

\[
\mathbb{E} \left[ F(B_s, s \leq t) \delta \left( \int_0^t e^{2B_s} ds \right) \right], \tag{6.2}
\]

in which, by [16 Proposition 1.3.3], the Skorokhod integral is expanded into

\[
\frac{1}{A_t} \delta \left( \int_0^t e^{2B_s} ds \right) - \left( D \left( \frac{1}{A_t} \right), \int_0^t e^{2B_s} ds \right)_H. \tag{6.3}
\]
Notice that

\[
\delta \left( \int_0^t e^{2B_s} \, ds \right) = \int_0^t e^{2B_s} \, dB_s \\
= \frac{1}{2} \left( e^{2B_t} - 1 \right) - A_t,
\]

where the second line is due to Itô’s formula. On the other hand, since the directional derivative of the functional \(1/A_t\) along each \(h \in H\) is

\[
-\frac{2}{(A_t)^2} \int_0^t h_s e^{2B_s} \, ds,
\]

the \(H\)-inner product in (6.3) is equal to \(-1\). These amount to

\[
\delta \left( \frac{\int_0^t e^{2B_s} \, ds}{A_t} \right) = \frac{e^{2B_t} - 1}{2A_t}
\]

\[
= \frac{\sinh B_t}{Z_t}
\]

by the definition of \(Z_t\), which, together with (6.2), verifies (6.1). Similarly, differentiation of both sides of (1.7) at any fixed \(z\), with relation (1.7) applied to, also yields

\[
\mathbb{E} \left[ \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} \left\langle DF(B), \frac{\int_0^t e^{2B_s} \, ds}{A_t} \right\rangle_H \right]
\]

\[
= \mathbb{E} \left[ \frac{\sinh(z + B_t)}{Z_t} \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} F(B_s, s \leq t) \right].
\]

This equality may be verified in the same manner as above; indeed, a direct computation shows that

\[
\delta \left( \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\} \frac{\int_0^t e^{2B_s} \, ds}{A_t} \right)
\]

\[
= \frac{\sinh(z + B_t)}{Z_t} \exp \left\{ \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right\}.
\]

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