NONSMOOTH CONVEX FUNCTIONALS AND FEEBLE VISCOSITY SOLUTIONS OF SINGULAR EULER-LAGRANGE EQUATIONS

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Abstract. Let $F = F(A)$ be nonnegative, convex and in $C^2(\mathbb{R}^n \setminus K)$ with $K \subseteq \mathbb{R}^n$ a closed set. We prove that local minimisers in $(C_0^0 \cap W^{1,1}_{\text{loc}}(\Omega))$ of

$$(1) \quad E(u, \Omega) := \int_{\Omega} F(Du), \quad \Omega \subseteq \mathbb{R}^n,$$

are “very weak” viscosity solutions on $\Omega$ in the sense of Juutinen-Lindqvist [JL] of the highly singular Euler-Lagrange equation of (1) expanded:

$$(2) \quad F_{AA}(Du) : D^2u = 0.$$ 

The hypotheses on $F$ do not guarantee existence of minimising weak solutions and include the singular $p$-Laplacian for $p \in (1, 2)$. A much deeper converse is also true, if $K = \{0\}$ and extra natural assumptions are satisfied. Our main advance is that we introduce systematic “flat” sup-convolution regularisations which apply to general singular nonlinear PDE in order to cancel the strong singularity of $F$. As an application we extend a classical theorem of Calculus of Variations regarding existence for the Dirichlet problem. These results extends previous work of Julin-Juutinen [JJ] and Juutinen-Lindqvist-Manfredi [JLM].

1. Introduction

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative convex function. Consider the functional $E : W^{1,1}_{\text{loc}}(\Omega) \to [0, \infty]$ defined by

$$(1.1) \quad E(u, B) := \int_B F(Du), \quad \text{when } F(Du) \in L^1(B), \quad B \subseteq \Omega \text{ Borel},$$

with $E(u, B) := \infty$ otherwise. In this paper we establish the equivalence between continuous local minimisers of $E$ in the space $W^{1,1}_{\text{loc}}(\Omega)$ and of (appropriately defined) viscosity solutions in the sense of Crandall-Ishii-Lions [CIL] of the Euler-Lagrange equation corresponding to (1.1) expanded:

$$(1.2) \quad F_{AA}(Du) : D^2u = 0.$$ 

The notation is either self-explanatory or otherwise standard: $F_A$ and $F_{AA}$ stand for the 1st and 2nd derivatives of $F$, “$\cdot$” is the Euclidean inner product in $\mathbb{R}^{n \times n}$ and “;” is the inner product in $\mathbb{R}^n$. In this work a continuous local minimiser $u \in W^{1,1}_{\text{loc}}(\Omega)$ is meant in the sense that

$$(1.3) \quad E(u, \Omega') \leq E(u + \psi, \Omega'), \quad \text{for } \psi \in W^{1,1}_0(\Omega'), \quad \Omega' \Subset \Omega.$$ 

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In order to derive the PDE for local minimisers, we assume that
\begin{equation}
F \text{ is convex, } F \geq F(0) = 0 \text{ and } F \in C^2(\mathbb{R}^n \setminus K), \quad K \subseteq \mathbb{R}^n \text{ closed.}
\end{equation}

For the opposite direction, we will also need that
\begin{equation}
K = \{ F = 0 \} = \{ 0 \}, \quad F \in C^1(\mathbb{R}^n) \quad \text{and} \quad \int_0^1 \int_0^t \sup_{t<|a|<1} \Delta F(a) \, dt \, ds < \infty
\end{equation}
and either that
\begin{equation}
\text{the viscosity solution is locally Lipschitz continuous on } \Omega,
\end{equation}
or that
\begin{equation}
\begin{cases}
(F_A(b) - F_A(a)) \cdot (b - a) \geq c|b - a|^r,
|F_A(a)| \leq \frac{1}{c}|a|^{r-1},
\limsup_{|a| \to \infty} \Delta F(a) < \infty,
\end{cases}
\end{equation}
for some $c > 0$, $r > 1$ and $a, b \in \mathbb{R}^n$. We note that the standard example of $p$-Dirichlet density $F(A) = |A|^p$, $p > 1$, satisfies all the above assumptions, even in the singular range of exponents $1 < p < 2$.

The primary advance in this paper compared to results already existing in the literature is that (1.1) is a general nonsmooth functional and (1.2) is highly singular, since the Hessian of $F$ is undefined on $K$ and could even be unbounded near $K$. Under our assumption (1.4), the Euler-Lagrange equation can not be treated in the classical setting of weak solutions. For, there is no way to infer that a function $u$ satisfying (1.3) is a distributional weak solution of
\begin{equation}
\text{Div}(F_A(Du)) = 0
\end{equation}
since (1.1) may not be Gateaux differentiable and $F_A(Du)$ may not even be measurable if $F \not\in C^1(\mathbb{R}^n)$. The appropriate extension of “very weak” viscosity solutions which works in the case (1.2) has already been implicitly introduced in [JLM] in the special case of the $p$-Laplacian and has been subsequently formalised in [JL] as “feeble viscosity solutions” (see also [IS] by Ishii-Souganidis). This is nontrivial because if $F \not\in C^2(\mathbb{R}^n)$, extra caution is required since the PDE does not make sense on the set $\{ Du \in K \}$. We introduce this definition, appropriate adapted to our case, later in Section 2. Roughly, the idea is to view (1.2) as the free boundary problem
\begin{equation}
\begin{cases}
F_A(A) : D^2u = 0, & \text{on } \Omega \setminus \Omega(u),
 f(Du) = 0, & \text{on } \Omega(u),
\end{cases}
\end{equation}
where $f = \text{dist}(\cdot, K)$ and $\Omega(u) = \{ Du \in K \}$. This leads to “ordinary” viscosity solutions on the set $\Omega \setminus \Omega(u)$, coupled by a 1st order differential inclusion “$Du(x) \in K$” for $x \in \Omega(u)$.

The results herein extend recent work of Julin-Juutinen [JJ] done in the special case of the $p$-Laplacian, that is for $F(A) = |A|^p$ and respective equation
\begin{equation}
\Delta_p u = \text{Div}(|Du|^{p-2}Du) = 0,
\end{equation}
for $p > 1$. This paper provides a simplified proof of the original result due to Juutinen-Lindqvist-Manfredi [JLM] who proved the equivalence among three different notions of solution for the $p$-Laplacian, that of weak solutions based on
integration-by-parts, that of viscosity solutions based on the maximum principle for the expanded version of the PDE

\begin{equation}
|Du|^{p-2} \left( I + (p-2) \frac{Du \otimes Du}{|Du|^2} \right) : D^2 u = 0
\end{equation}

and an other based on nonlinear potential theory, introduced by Lindqvist in [L] as an extension of the classical idea of Riesz for the Laplacian. Also, Juutinen-Lindqvist-Manfredi observed that the standard idea of semicontinuous envelopes does not work in the singular case when $1 < p < 2$ because we then allow for "false" solutions. The primary advances of [JJ] is that they bypass the heavy uniqueness machinery of Viscosity Solution theory which was employed in [JLM], and also consider the inhomogeneous case of $\Delta_p u = f$.

Except for the more general setting that we consider in this paper, the main technical advance herein is that in the generality of (1.2) appropriate approximations have to be introduced in order to circumvent the strong singularities of $F_{AA}$. To this end, we introduce systematic regularisations of feeble viscosity solutions which we call "flat sup/inf convolutions". The "flat" counterparts $(u_\varepsilon)^{\varepsilon > 0}$ of the classical sup convolutions (see e.g. [CIL, S]) are semiconvex approximations which satisfy additional estimates of the type

\begin{equation}
D^2 u_\varepsilon \geq - \frac{\Phi(|Du_\varepsilon|)}{\varepsilon} I
\end{equation}

for $\Phi$ is sufficiently "flat" in order to cancel the singularity of $F_{AA}$ near $K = \{0\}$. The flatness property allows to show that if $u_\varepsilon$ is a strong subsolution of (1.2) a.e. on $\Omega \setminus \{Du_\varepsilon = 0\}$, then $u_\varepsilon$ is weak subsolution of (1.8) not only on $\Omega \setminus \{Du_\varepsilon = 0\}$ but on the whole of $\Omega$. Roughly, this is achieved by choosing as $\Phi$

\begin{equation}
\Phi(t) \approx \inf_{|a| > t} \frac{1}{\Delta F(a)} , \text{ for } t > 0
\end{equation}

(for details see Sections 4 and 5). We believe that the tools developed in Section 4 are flexible and useful for nonlinear singular elliptic and parabolic equations in general, so this section is written independently of the PDE we are handling. Moreover, herein (as in [JJ]) we do not appeal to the heavy uniqueness machinery of Viscosity Solutions. Our results and techniques extend to more general situations and to the case of nontrivial right-hand-side for (1.2), but at the expense of added technical complexity. Hence, we decided to keep things as simple as possible, illustrating the main ideas.

Let us now state the main result of this paper.

**Theorem 1.** Let $F$ be a convex function on $\mathbb{R}^n$ satisfying (1.4). Fix also $\Omega \subseteq \mathbb{R}^n$ open. Then:

(a) Continuous local minimisers of (1.1) in $W^{1,1}_\text{loc}(\Omega)$ are Feeble Viscosity Solutions of (1.2) on $\Omega$ (Definition 3).

(b) Conversely, if in addition $F$ satisfies (1.5) and also either (1.6) or (1.7), then, Feeble Viscosity Solutions of (1.2) on $\Omega$ which are in $W^{1,1}_\text{loc}(\Omega)$ (Definition 3), are continuous weak solutions of (1.8). Moreover, they are local minimisers of the functional (1.1) in $W^{1,r}_\text{loc}(\Omega)$ (under (1.6) we have $r = 1$ and under (1.7) we have $r > 1$ as in the assumption).
We note that the conclusion of (a) above remains true under the weaker assumption that \( u \) is a \textit{spherical local minimiser in} \( W_{loc}^{1,1}(\Omega) \), namely when we use test functions supported on small balls and with small \( W_1^{1,1} \) norm. As an application of (a) above, in Corollary 6 we obtain an extension of the classical theorem of Calculus of Variations regarding existence of solution to the Dirichlet problem

\[
\begin{aligned}
F_{AA}(Du) : D^2u = 0, & \quad \text{in } \Omega, \\
u = b, & \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \in \mathbb{R}^n \), \( b \in W_{loc}^{1,1}(\Omega) \cap C^0(\overline{\Omega}) \) has finite energy on \( \Omega \) (that is \( E(b,\Omega) < \infty \)) and \( F \) satisfies (1.1) together with the strengthened coercivity

\[
F(A) \geq c|A|^s - \frac{1}{c},
\]

for some \( s > n \) and \( c > 0 \). These assumptions are much weaker than those guaranteeing the existence of weak solutions. In the case of (b), things are trickier and the extra assumptions (1.5)-(1.7) are required. They however provide the stronger conclusion that viscosity solutions of (1.2) are locally minimising weak solutions.

We conclude this introduction by noting some very interesting papers which relate to the results herein. In [I], Ishii considered the question of equivalence between weak and viscosity solutions of linear (degenerate elliptic) PDEs, in [JLP] Juutinen-Lukkari-Parvianen consider the same question for the \( p(x) \)-Laplacian and in [SV] Servadei and Valdinoci consider the same question for the fractional Laplacian. In [JL], Juutinen and Lindqvist consider the problem of the removability of level sets, which in the present case of our singular PDE (1.2) is relavant to the removability of \( \{Du \in K\} \). Finally, we note that one further interesting non-smooth convex hamiltonian is

\[
F(A) = \max \{|A| - 1, 0\}^p, \quad p > 1,
\]

and relates to the problem of traffic congestion ([SaV, CF]). This example was brought to our attention by one of the referees. Since the singular set \( K \) here is a sphere (and not \( \{0\} \)), only (a) of Theorem 1 and Corollary 6 as they stand apply to this case. However, the convex hull of \( K \) is the unit ball. Possible extensions of our results to this interesting case may be investigated in future work.

2. \textbf{Feeble Viscosity Solutions}

In this section we consider the appropriate adaptation of the definition of viscosity solutions for the singular PDE (1.2). We will use as primary definition the version based on weak pointwise generalised derivatives (jets), rather than test functions.

We begin by recalling from [CIL] that for \( u \in C^0(\Omega) \), \( \Omega \subseteq \mathbb{R}^n \), the standard 2nd order subjets and superjets \( J^{2,\pm}u(x) \) of \( u \) at \( x \in \Omega \) are defined as

\[
J^{2,+}u(x) := \left\{ (p, X) \in \mathbb{R}^n \times S(n) \mid u(z + x) \leq u(x) + p \cdot z + \frac{1}{2} X : z \otimes z + o(|z|^2), \quad \text{as } z \to 0 \text{ in } \Omega \right\},
\]

\[
J^{2,-}u(x) := \left\{ (p, X) \in \mathbb{R}^n \times S(n) \mid u(z + x) \geq u(x) + p \cdot z + \frac{1}{2} X : z \otimes z + o(|z|^2), \quad \text{as } z \to 0 \text{ in } \Omega \right\}
\]
where $S(n) := \{ A \in \mathbb{R}^{n \times n} : A_{ij} = A_{ji} \}$ denotes the symmetric $n \times n$ matrices.

**Definition 2** (Feeble Jets). Let $K \subseteq \mathbb{R}^n$ be a closed set. For $u \in C^0(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, the 2nd order Feeble Subjet $J^2_{K, +}u(x)$ relative to $K$ of $u$ at $x \in \Omega$ is defined as

$$J^2_{K, +}u(x) := \{(p, X) \in J^2u(x) \mid p \in \mathbb{R}^n \setminus K\}$$

Similarly, the 2nd order Feeble Superjet $J^2_{K, -}u(x)$ relative to $K$ of $u$ at $x \in \Omega$ is defined as

$$J^2_{K, -}u(x) := \{(p, X) \in J^2u(x) \mid p \in \mathbb{R}^n \setminus K\}$$

**Definition 3** (Feeble Viscosity Solutions). Let $K \subseteq \mathbb{R}^n$ be a closed set and let $G \in C^0((\mathbb{R}^n \setminus K) \times S(n))$. Let also $u$ be in $C^0(\Omega)$. We say that $u$ is a Feeble Viscosity Solution of

$$(2.6) \quad G(Du, D^2u) \geq 0$$
on $\Omega$ (or, subsolution of $G(Du, D^2u) = 0$), when

$$(2.7) \quad \inf_{(p, X) \in J^2_{K, +}u(x)} G(p, X) \geq 0,$$

for all $x \in \Omega$. Similarly, we say that $u$ is a Feeble Viscosity Solution of

$$(2.8) \quad G(Du, D^2u) \leq 0$$
on $\Omega$ (or, supersolution of $G(Du, D^2u) = 0$), when

$$(2.9) \quad \sup_{(p, X) \in J^2_{K, -}u(x)} G(p, X) \leq 0,$$

for all $x \in \Omega$. We say that $u$ is a Feeble Viscosity Solution of $G(Du, D^2u) = 0$ on $\Omega$ when both (2.7) and (2.9) hold.

In the case of (1.2) we consider in this paper, we have $G(p, X) = F_{AA}(p) : X$. We also note the obvious identity $\Delta F = F_{AA} : I = F_{AA} : I$.

**Remark 4.** For the sake of clarity, let us state also the equivalent point of view of viscosity solutions via touching test functions. We say that $u$ is a Feeble Viscosity Solution of (2.6) on $\Omega$, when for all $x \in \Omega$ and $\psi \in C^2(\mathbb{R}^n)$ for which $u - \psi$ has a vanishing local maximum at $x$ and $D\psi(x) \notin K$, we have

$$(2.10) \quad G(D\psi(x), D^2\psi(x)) \geq 0.$$  

Similarly, we say that $u$ is a Feeble Viscosity Solution of (2.8) on $\Omega$, when for all $x \in \Omega$ and $\psi \in C^2(\mathbb{R}^n)$ for which $u - \psi$ has a vanishing local minimum at $x$ and $D\psi(x) \notin K$, we have

$$(2.11) \quad G(D\psi(x), D^2\psi(x)) \leq 0.$$  

We say that $u$ is a Feeble Viscosity Solution on $\Omega$ when both (2.7) and (2.9) hold.
3. Local minimisers are Feeble Viscosity Solutions.

In this section we establish the one half of Theorem 1, packed in the following

Proposition 5. Fix $\Omega \subseteq \mathbb{R}^n$ open and let $F$ be a convex function on $\mathbb{R}^n$ which satisfies (1.4) for some $K \subseteq \mathbb{R}^n$ closed.

Then, continuous local minimisers of (1.1) in $W^{1,1}_{\text{loc}}(\Omega)$ are Feeble Viscosity Solutions of (1.2) on $\Omega$ (Definition 3).

As we have already mentioned, it suffices $u$ to be a spherical local minimiser in the above result, but we will not use this generality.

Proof of Proposition 5. The argument utilised here follows an idea of Barron and Jensen from [BJ]. Assume for the sake of contradiction that there is a $u \in (C^0 \cap W^{1,1}_{\text{loc}})(\Omega)$ which satisfies (1.3) but for some $x \in \Omega$ (2.7) fails. Then, in view of Remark 4 and standard arguments, there exists a smooth $\psi \in C^2(\mathbb{R}^n)$ and an $r > 0$ such that $D\psi(x) \notin K$ and

\begin{equation}
\tag{3.1}
 u - \psi < 0 = (u - \psi)(x)
on \mathbb{B}_r(x) \setminus \{x\}, \text{ while there is a } c > 0 \text{ such that }
\end{equation}

\begin{equation}
\tag{3.2}
 F_{AA}(D\psi(x)) : D^2\psi(x) \leq -2c < 0.
\end{equation}

In the standard way, $\mathbb{B}_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$. Since $\mathbb{R}^n \setminus K$ is open and $\psi \in C^2(\mathbb{R}^n)$, we can decrease $r$ further to achieve

\begin{equation}
\tag{3.3}
 D\psi(\mathbb{B}_r(x)) \subseteq \mathbb{R}^n \setminus K.
\end{equation}

Hence, the map $F(D\psi)$ is in $C^2(\mathbb{B}_r(x))$ and as such we have

\begin{equation}
\tag{3.4}
 F_{AA}(D\psi) : D^2\psi = \text{Div}(F_A(D\psi))
on \mathbb{B}_r(x). \text{ By restricting } r \text{ even further, } (3.2) \text{ gives }
\end{equation}

\begin{equation}
\tag{3.5}
 - \text{Div}(F_A(D\psi)) \geq c,
on \mathbb{B}_r(x). \text{ By } (3.1), \text{ strictness of the maximum of } u - \psi \text{ implies that there is a } k > 0 \text{ small such that by sliding } \psi \text{ downwards to some } \psi - k, \text{ we have }
\end{equation}

\begin{equation}
\tag{3.6}
 \Omega^+ := \{u - \psi + k > 0\} \subseteq \mathbb{B}_r(x) \subseteq \Omega
\end{equation}

and also $u = \psi - k$ on $\partial\Omega^+$. By multiplying (3.5) by $u - \psi + k \in W^{1,1}_{\text{loc}}(\Omega^+)$ and integrating by parts, we obtain

\begin{equation}
\tag{3.7}
 \int_{\Omega^+} F_A(D\psi) \cdot (Du - D\psi) \geq c \int_{\Omega^+} |u - \psi + k|.
\end{equation}

Since $F$ is convex on $\mathbb{R}^n$, the elementary inequality

\begin{equation}
\tag{3.8}
 F_A(a) \cdot (b - a) \leq F(b) - F(a)
\end{equation}

implies that (3.7) gives

\begin{equation}
\tag{3.9}
 c \int_{\Omega^+} |u - \psi + k| \leq \int_{\Omega^+} F(Du) - \int_{\Omega^+} F(D(\psi - k)) = E(u, \Omega^+) - E(\psi - k, \Omega^+).
\end{equation}

In view of (1.3) and (3.6), we have $E(u, \Omega^+) - E(\psi - k, \Omega^+) \leq 0$ and hence $\Omega^+ = \emptyset$, which is a contradiction. Hence, $u$ is a Feeble Viscosity Solution of

\begin{equation}
\tag{3.10}
 F_{AA}(Du) : D^2u \geq 0
\end{equation}
on Ω. The supersolution property follows in the similar way and so does the proposition. □

Proposition 5 implies the following existence theorem:

**Corollary 6 (Existence for the Dirichlet Problem).** Assume that the convex function F satisfies (1.4) together with (1.15) for some s > n, Ω ⊆ \( \mathbb{R}^n \) is bounded and \( b \in W^{1,1}_{\text{loc}}(\Omega) \cap C^0(\overline{\Omega}) \) has finite energy on Ω i.e. \( E(b, \Omega) < \infty \) where E is given by (1.1).

Then, the Dirichlet Problem (1.14) has a Feeble Viscosity Solution (Definition 3), which is (globally) minimising for \( E \) in \( W^{1,s}_b(\Omega) \).

**Proof of Corollary 6.** The argument is a simple implementation of the direct method of Calculus of Variations, which we include it for the sake of completeness.

Since \( E(b, \Omega) < \infty \), by (1.15) it follows that
\[
C_2 \int_{\Omega} |Db|^p \leq C_1 |\Omega| + \int_{\Omega} F(Db) \leq C_1 |\Omega| + E(b, \Omega)
\]
and hence by Poincaré inequality \( b \in W^{1,s}_b(\Omega) \). Thus, the infimum of \( E \) in the affine space \( W^{1,s}_b(\Omega) \) is finite:
\[
0 \leq e := \inf_{W^{1,s}_b(\Omega)} E \leq E(b, \Omega) < \infty.
\]
Let \( (u^m)_m^\infty \) be a minimising sequence. Since \( E(u^m, \Omega) \to e \) as \( m \to \infty \), by (1.15), (3.11), (3.12) and Poincaré inequality we have the uniform bound \( \|u^m\|_{W^{1,s}_b(\Omega)} \leq C \). Hence, there exists a subsequence along which we have \( u^m \rightharpoonup u \) as \( m \to \infty \) weakly in \( W^{1,s}_b(\Omega) \). In view of assumption (1.4), the functional (1.1) is weakly lower-semicontinuous in \( W^{1,s}_b(\Omega) \) (see e.g. Dacorogna [D], p. 94). Hence,
\[
E(u, \Omega) \leq \liminf_{m \to \infty} E(u^m, \Omega) = e < \infty
\]
and as a result \( u \) is minimiser of \( E \) in \( W^{1,s}_b(\Omega) \). Since \( s > n \), By Morrey estimate we have \( u \in C^0(\Omega) \). By Proposition 5, \( u \) is a feeble viscosity solution of (1.2) and also \( u = b \) on \( \partial \Omega \). Hence \( u \) solves (1.15) and the corollary follows. □

### 4. Flat Sup-Convolution Approximations.

In this section we introduce the appropriate regularisations of viscosity solutions that allow to handle singular equations. For the reader’s benefit, this section is independent of the rest of the paper since these regularisations are fairly general and might be useful in other contexts as well.

**Definition 7 (Flat sup/inf convolutions).** Fix \( \Theta \in C^2(0, \infty) \) strictly increasing with
\[
\Theta(0^+) = \Theta'(0^+) = \lim_{s \to 0^+} \frac{\Theta''(s)}{s} = 0.
\]
Given \( u \in C^0(\overline{\Omega}), \Omega \subseteq \mathbb{R}^n \) and \( \varepsilon > 0 \), for \( x \in \Omega \) we define
\[
u^\varepsilon(x) := \sup_{y \in \Omega} \left\{ u(y) - \frac{\Theta(|x - y|^2)}{2\varepsilon} \right\},
\]
\[
u^\varepsilon(x) := \inf_{y \in \Omega} \left\{ u(y) + \frac{\Theta(|x - y|^2)}{2\varepsilon} \right\}.
\]
We call \( u^\varepsilon \) the flat sup-convolution of \( u \) and \( u^\varepsilon \) the flat inf-convolution of \( u \).
The nomenclature “flat” owes to the fact that the approximations satisfy flatness estimates of the type of (1.12) with \( \Phi \) sufficiently flat at zero in order to cancel the singularity in the gradient variable of a nonlinear coefficient \( G(Du, D^2 u) \), when \( u \) is a viscosity solution of such a PDE. The next result collects the main properties of \( u^\varepsilon \) and \( u_\varepsilon \).

**Lemma 8** (Basic Properties). Assume that

\( \Theta \in C^2(0, \infty) \) is strictly increasing & \( \Theta(0^+) = \Theta'(0^+) = \lim_{s \to 0^+} \frac{\Theta''(s)}{s} = 0. \)

If \( u^\varepsilon \) and \( u_\varepsilon \) are given by (4.1) and (4.2) respectively, for \( \varepsilon > 0 \) we have:

(i) \( u^\varepsilon = -(-u)^\varepsilon \) and \( u^\varepsilon \geq u \) on \( \Omega \).

(ii) If we set

\[
\begin{align*}
    u^\varepsilon (x) &= u(y) - \frac{\Theta(|x - y|^2)}{2\varepsilon},
    \\
    X(\varepsilon) &= \{ y \in \overline{\Omega} \mid u^\varepsilon (x) = u^\varepsilon (y) \},
    \\
    |x - x^\varepsilon| &\leq \sqrt{\Theta^{-1}(4\|u\|_{C^0(\Omega)} \varepsilon)} =: \rho(\varepsilon).
\end{align*}
\]

That is, \( X(\varepsilon) \subseteq \overline{B}_{\rho(\varepsilon)}(x) \). Moreover,

(iii) \( u^\varepsilon \searrow u \) in \( C^0(\overline{\Omega}) \) as \( \varepsilon \to 0 \).

(iv) For each \( \varepsilon > 0 \), \( u^\varepsilon \) is semiconvex and hence twice differentiable a.e. on \( \Omega \). Moreover, if \( (Du^\varepsilon, D^2 u^\varepsilon) \) denote the pointwise derivatives, we have the estimate

\[
D^2 u^\varepsilon \geq -\frac{1}{\varepsilon} \left( \sup_{0 < t < d} \{ 2\Theta''(t^2)^2 + \Theta'(t^2) \} \right) I
\]

a.e. on \( \Omega \), where \( d := \text{diam}(\Omega) \).

(v) We set

\[
\Omega^\varepsilon := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \rho(\varepsilon) \}.
\]

If \( u \) is a (Feeble) Viscosity Solution of

\[
G(Du, D^2 u) \geq 0\tag{4.10}
\]

on \( \Omega \) (Definition 3), then \( u^\varepsilon \) is a (Feeble) Viscosity Solution of

\[
G(Du^\varepsilon, D^2 u^\varepsilon) \geq 0\tag{4.11}
\]

on \( \Omega^\varepsilon \). Moreover, \( u^\varepsilon \) is a strong solution, a.e. on \( \Omega^\varepsilon \setminus \{ Du^\varepsilon \in \mathcal{K} \} \).

(vi) (Magic properties) Assume in addition that

\[
\Theta(t) := \Theta'(t^2) t \text{ is strictly increasing on } (0, d), \ d = \text{diam}(\Omega).
\]

Then,

\[
(p, X) \in J^{2,+} u^\varepsilon (x) \quad \Rightarrow \quad (p, X) \in J^{2,+} u(x^\varepsilon),
\]

where

\[
X(\varepsilon) \ni x^\varepsilon = x + T^{-1}(\varepsilon p)
\]
and in (4.14) $T$ stands for the extension of $T$ on $\mathbb{R}^n$, that is $T(z) := \Theta'(|z|^2)z$.

(vii) If (4.12) holds, then for a.e. $x \in \Omega$, the set $X(\varepsilon)$ of (4.5) is a singleton \{x$^\varepsilon$\}, and

\[(4.15)\quad x^\varepsilon = x + T^{-1}(\varepsilon Du^\varepsilon(x)).\]

(viii) If (4.12) holds, then for a.e. $x \in \Omega$, we have the estimate

\[(4.16)\quad |Du^\varepsilon(x)| \geq \frac{1}{\varepsilon} T(|x - x^\varepsilon|),\]

where $x^\varepsilon$ is as in (4.15) and $T$ is as in (4.12).

(ix) If (4.12) holds and $u \in W^{1,\infty}_0(\Omega)$, then for any $\Omega' \Subset \Omega$, we have

\[(4.17)\quad \|Du^\varepsilon\|_{L^\infty(\Omega')} \leq \|Du\|_{L^\infty(\Omega')}\]

The proof is an extension of standard results in the literature for the “ordinary” sup-convolutions corresponding to $\Theta(t) = t$, but we provide it for the sake of completeness and for convenience of the reader. In particular, Lemma 8 extends results of [33] in the special case of $\Theta(t) = t^{p/2}(p - 1)$ corresponding to the regularisation method used for the singular $p$-Laplacian for $1 < p < 2$.

**Proof of Lemma 8.** (i) is obvious.

(ii) By (i), (4.1) and (4.4), (4.5), we have

\[(4.18)\quad u^\varepsilon(x) = u(x^\varepsilon) - \frac{\Theta(|x^\varepsilon - x|^2)}{2\varepsilon} \geq u(y) - \frac{\Theta(|y - x|^2)}{2\varepsilon},\]

for all $y \in \Omega$. By choosing $y := x$, we get

\[(4.19)\quad \Theta(|x^\varepsilon - x|^2) \leq 4\|u\|_{C^0(\Omega)}\varepsilon.\]

(iii) By assumption, $u$ is uniformly continuous on $\Omega$. Hence, there is an increasing $\omega \in C^0[0, \infty)$ with $\omega(0) = 0$ such that $|u(x) - u(y)| \leq \omega(|x - y|)$, for all $x, y \in \Omega$. Hence, by (ii) and (4.18), for any $x \in \Omega$,

\[(4.20)\quad u^\varepsilon(x) = u(x^\varepsilon) \leq u(x) + \omega(|x - x^\varepsilon|) \leq u(x) + \omega(\rho(\varepsilon)),\]

while by (i) we have $u(x) \leq u^\varepsilon(x)$. In addition, it can be easily seen that $0 < \varepsilon' < \varepsilon''$ implies $u^{\varepsilon''} \geq u^{\varepsilon'}$.

(iv) By (4.4), we have

\[(4.21)\quad Du^\varepsilon(y) = -\frac{\Theta'(|x - y|^2)}{\varepsilon} (x - y),\]

\[(4.22)\quad D^2u^\varepsilon(y) = -\frac{1}{\varepsilon} \left( \Theta'(|x - y|^2) I + 2\Theta''(|x - y|^2)(x - y) \otimes (x - y) \right).\]

By (4.4) and (4.7) we have $u^\varepsilon(x) = \sup_{y \in X(\varepsilon)} \{u^\varepsilon(x)\}$, while (4.22) readily implies

\[(4.23)\quad D^2u^\varepsilon(y) \geq -\frac{1}{\varepsilon} \left\{ \sup_{y \in X(\varepsilon)} \left\{ \Theta'(|x - y|^2) + 2\Theta''(|x - y|^2)|x - y|^2 \right\} I \right\}
\geq -\frac{1}{\varepsilon} \left\{ \sup_{0 < t < \rho} \left\{ \Theta'(t^2) + 2\Theta''(t^2)t^2 \right\} I \right\},\]

for all $y \in X(\varepsilon)$. Hence, all $u^\varepsilon$ are semiconvex, uniformly in $y$. Since $u^\varepsilon$ is a supremum of semiconvex functions, the conclusion follows by invoking Alexandroff’s theorem.
We immediately have that $J^{2,+}u^{\varepsilon,z}(x) = J^{2,+}u(x+z)$. Hence, for $|z| \leq \rho(\varepsilon)$, each $u^{\varepsilon,z}$ is a Feeble Viscosity Solution of

$$
(4.25) \quad G(Du^{\varepsilon,z}, D^2u^{\varepsilon,z}) \geq 0
$$
on $\Omega^{\varepsilon}$, if $u$ is a Feeble Viscosity Solution of $G(Du, D^2u) \geq 0$ on $\Omega$. Moreover, the classical result that pointwise suprema of Viscosity Subsolutions are Viscosity Subsolutions ([CIL], p. 23) extends to the Feeble case as well. For, it suffices to observe that by (4.24) and (4.1) we have

$$
(4.26) \quad u^{\varepsilon}(x) = \sup_{|z| \leq \rho(\varepsilon)} u^{\varepsilon,z}(x)
$$

and that since $\mathcal{K}$ is closed, if $\psi$ touches $u^\varepsilon$ from above at $x \in \Omega$ and $D\psi(x) \notin \mathcal{K}$, then there is a $\delta > 0$ such that $D\psi(B_\delta(x)) \notin \mathcal{K}$.

(vi) Let $(p, X) \in J^{2,+}u^\varepsilon(x)$. Then, there is a $\psi \in C^2(\mathbb{R}^n)$ such that $u^\varepsilon - \psi \leq (u^\varepsilon - \psi)(x)$ with $D\psi(x) = p$ and $D^2\psi(x) = X$. Hence, for all $z, y \in \Omega$ and $x^\varepsilon \in X(\varepsilon)$,

$$
(4.27) \quad u(y) - \frac{\Theta(|y - z|^2)}{2\varepsilon} - \psi(z) \leq u(x^\varepsilon) - \frac{\Theta(|x^\varepsilon - z|^2)}{2\varepsilon} - \psi(x).
$$

For $y := x^\varepsilon$, we get

$$
(4.28) \quad \Theta(|z - x^\varepsilon|^2) + 2\varepsilon \psi(z) \geq \Theta(|x - x^\varepsilon|^2) + 2\varepsilon \psi(x)
$$

and hence the function $z \mapsto \Theta(|z - x^\varepsilon|^2) + 2\varepsilon \psi(z)$ has minimum at $x$ which implies that its gradient vanishes there. Consequently,

$$
(4.29) \quad \Theta'(|x^\varepsilon - x|^2)(x^\varepsilon - x) = \varepsilon p.
$$

By (4.12), we have that the operator $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(z) := \Theta'(|z|^2)z$ is injective and hence (4.29) gives $x^\varepsilon - x = T^{-1}(\varepsilon p)$, which is (4.14). By (4.27) for $z := y - x^\varepsilon + x$, we obtain

$$
(4.30) \quad u(y) - \psi(y - x^\varepsilon + x) \leq u(x^\varepsilon) - \psi(x)
$$

which implies that $(p, X) \in J^{2,+}u(x + T^{-1}(\varepsilon p))$, as desired.

(vii) Since $u^\varepsilon$ is semiconvex, for a.e. $x \in \Omega$, we have $(Du^\varepsilon(x), D^2u^\varepsilon(x)) \in J^{2,+}u^\varepsilon(x)$. The conclusion follows by (4.14).

(viii) Fix an $x \in \Omega$ such that $Du^\varepsilon(x)$ exists. For any $e \in \mathbb{R}^n$ with $|e| = 1$, we have

$$
|Du^\varepsilon(x)| \geq e \cdot Du^\varepsilon(x)
$$

$$
= \frac{d}{dt} \bigg|_{t=0} u^\varepsilon(x + te)
$$

$$
= \frac{d}{dt} \bigg|_{t=0} \left\{ \max_{y \in B_{\rho(\varepsilon)}(x)} \left( u(y) - \frac{\Theta(|x + te - y|^2)}{2\varepsilon} \right) \right\}.
$$
By Danskin’s theorem ([Da]) and (vii), we obtain
\[
|Du^\varepsilon(x)| \geq \max_{y \in X(\varepsilon)} \left\{ \frac{d}{dt} \bigg|_{t=0} \left( u(y) - \frac{\Theta(|x + te - y|^2)}{2\varepsilon} \right) \right\}
\]
\[
= \max_{y \in X(\varepsilon)} \left\{ \frac{1}{\varepsilon} \Theta'(|x - y|^2)(x - y) \cdot e \right\}
\]
\[
= -\frac{1}{\varepsilon} \Theta'(|x - x^\varepsilon|^2)(x - x^\varepsilon) \cdot e.
\]
If \( x = x^\varepsilon \), (4.16) follows immediately since \( T(0) = 0 \). If \( x \neq x^\varepsilon \), we choose \( e := \frac{x - x^\varepsilon}{|x - x^\varepsilon|} \) and then estimate (4.31) implies (4.16).

(ix) Let \( p \in J^{1,+}u^\varepsilon(x) \). Then, for a.e. \( x \in \Omega \), we have \( p = Du^\varepsilon(x) \) and also \( p \in J^{1,+}u(x^\varepsilon) \), where \( |x^\varepsilon - x| \leq \rho(\varepsilon) \). Fix \( \Omega' \Subset \Omega \), \( \varepsilon > 0 \) small and \( x \in \Omega^\varepsilon \) and choose \( \delta > 0 \) small such that \( B_\delta(x^\varepsilon) \subseteq \Omega' \). Since \( u(x^\varepsilon + z) - u(x^\varepsilon) \leq p \cdot z + o(|z|) \) as \( z \to 0 \), for \( z := -\delta e \) with \( |e| = 1 \), we have
\[
|p| = \max_{|e|=1} p \cdot e \leq o(1) + \max_{|e|=1} \frac{u(x^\varepsilon) - u(x^\varepsilon - \delta e)}{\delta} \leq o(1) + \sup_{x,y \in B_\delta(x^\varepsilon)} \frac{|u(y) - u(x)|}{|y - x|} \leq o(1) + \|Du\|_{L^\infty(\Omega')},
\]
as \( \delta \to 0 \). Hence, for a.e. \( x \in \Omega^\varepsilon \), we have \( |Du^\varepsilon(x)| \leq \|Du\|_{L^\infty(\Omega')} \). \( \square \)

The following result contains the main new property of the approximations (4.1), (4.2), which fails for the standard sup/inf convolutions and justifies the necessity of a general \( \Theta \) function.

**Lemma 9** (Flatness Estimates). Let \( \Phi \in C^0(0, \infty) \) be a strictly increasing function with \( \Phi(0^+) = 0 \) such that
\[
\int_0^1 \frac{dt}{\Phi(t)} < \infty.
\]
Fix a domain \( \Omega \Subset \mathbb{R}^n \) and set \( d := \text{diam}(\Omega) \).

Then, there exists a strictly increasing function \( \Theta \in C^2(0, \infty) \) satisfying
\[
\Theta(0^+) = \Theta'(0^+) = \lim_{s \to 0^+} \frac{\Theta''(s)}{s} = 0
\]
and such that, if we set \( T(t) := \Theta'(t^2)t \), the function \( T' \) is positive and increasing on \((0, d)\).

Moreover, the sup-convolution operator given for any \( u \in C^0(\overline{\Omega}) \) by (4.1) satisfies the properties (i)-(ix) of Lemma 8 together with the estimate
\[
D^2u^\varepsilon \geq -\frac{\Phi(|Du^\varepsilon|)}{\varepsilon} I
\]
a.e. on \( \Omega \), for any \( \varepsilon > 0 \).

**Proof of Lemma 9. Step 1.** We define a function \( T \in C^1(0, \infty) \) as follows: consider the initial value problem
\[
\begin{cases}
T'(t) = \Phi(T(t)), & t > 0, \\
T(0) = 0.
\end{cases}
\]
In view of our assumption (4.32), Osgood’s non-uniqueness criterion of ODE theory implies that (4.35) has a nontrivial solution $T$, which is positive and strictly increasing on $(0, \infty)$, since $\Phi > 0$ on $(0, \infty)$. Moreover, the composition $T' = \Phi \circ T$ is strictly increasing as well and $T'(0^+) = \Phi(0^+) = 0$.

**Step 2.** We define $\Theta \in C^2(0, \infty)$ by

\[
(4.36) \quad \Theta(t) := 2 \int_0^{\sqrt{t}} T(s) ds.
\]

Obviously, $\Theta(0^+) = 0$ and since $T' \geq T'(0^+) = 0$, we get

\[
(4.37) \quad \Theta(t) \leq 2 \sqrt{t} T' = o(t),
\]
as $t \to 0$, which implies $\Theta'(0^+) = 0$. By differentiating (4.36), we have the identity

\[
(4.38) \quad \Theta'(t^2) t = T'(t)
\]

which implies $\Theta > 0$ and $\Theta' > 0$ on $(0, \infty)$. Moreover, the identity

\[
(4.39) \quad 2 \Theta''(t^2) t^2 + \Theta'(t^2) = T'(t)
\]

implies $\lim_{s \to 0^+} \Theta''(s)/s = 0$ and hence (4.33) ensues.

**Step 3.** Let $\Theta$ be defined by (4.36), (4.35). Then, $\Theta$ as well as $T$ (given by (4.38)) satisfy all the assumption of Lemma 8. Hence, the sup-convolution operator defined by (4.1) for this $\Theta$ satisfies the properties (i)-(ix) of Lemma 8.

**Step 4.** We now establish (4.34). Fix $u \in C^0(\overline{\Omega})$ for an $\Omega \Subset \mathbb{R}^n$ and $0 < \varepsilon < 1$. For a.e. $x \in \Omega$, $u^\varepsilon$ is twice differentiable at $x$. Fix such an $x \in \Omega$. Then, by (4.16), we have

\[
(4.40) \quad |x - x^\varepsilon| \leq T^{-1}(\varepsilon |Du^\varepsilon(x)|).
\]

On the other hand, (4.39) and (4.8) imply

\[
D^2 u^\varepsilon(x) \geq -\frac{1}{\varepsilon} \left(2 \Theta''(|x - x^\varepsilon|^2) |x - x^\varepsilon|^2 + \Theta'(|x - x^\varepsilon|^2) \right) I
\]

\[
(4.41) \quad = -\frac{1}{\varepsilon} T'(|x - x^\varepsilon|) I.
\]

Since $T'$ is increasing, by (4.41) and (4.40) we obtain

\[
(4.42) \quad D^2 u^\varepsilon \geq -\frac{1}{\varepsilon} T' \left( T^{-1}(\varepsilon |Du^\varepsilon|) \right) I,
\]
a.e. on $\Omega$. Finally, since $T' \circ T^{-1}$ is increasing, by (4.42), (4.35) and by using that $\varepsilon |Du^\varepsilon| \leq |Du^\varepsilon|$, we have

\[
(4.43) \quad D^2 u^\varepsilon \geq -\frac{1}{\varepsilon} T' \left( T^{-1}( |Du^\varepsilon|) \right) I
\]

\[
= -\frac{1}{\varepsilon} \Phi \left( T \left( T^{-1}( |Du^\varepsilon|) \right) \right) I
\]

\[
= -\frac{1}{\varepsilon} \Phi \left( |Du^\varepsilon| \right) I,
\]
a.e. on $\Omega$. The lemma ensues. □

**Remark 10.** Our assumption (1.5) on the radial integrability the Laplacian $\Delta F$ near the origin will guarantee that the function $\Phi$ (roughly given by (1.13)) has the “Osgood property” (4.32) which is needed for the construction of the flat sup-convolutions.
5. Feeble viscosity solutions are weak locally minimising solutions.

In this section we utilise the systematic approximations of Section 4 to establish the second half of Theorem 1. The first half has been established in Proposition 5.

Motivation of the method. Roughly, the idea of the usage of flat sup-convolutions is the following: choose the function
\[
\Phi(t) = \inf_{|a| > t} \frac{1}{\Delta F(a)},
\]
and consider the flat sup-convolution \(u_\varepsilon\) for the respective \(\Theta\). If \(u_\varepsilon\) is a strong subsolution a.e. on \(\Omega \setminus \{Du_\varepsilon = 0\}\) of
\[
F_{AA}(Du_\varepsilon) : D^2u_\varepsilon \geq 0,
\]
the flatness estimate
\[
D^2u_\varepsilon \geq \frac{-\Phi(|Du_\varepsilon|)}{\varepsilon} I
\]
gives the lower bound
\[
F_{AA}(Du_\varepsilon) : D^2u_\varepsilon \geq \frac{-\Phi(|Du_\varepsilon|)}{\varepsilon} F_{AA}(Du_\varepsilon) : I \geq -\frac{1}{\varepsilon}
\]
and application of Fatou lemma allows to infer
\[
-\int_{\Omega} F_A(Du_\varepsilon) : D\psi \geq \int_{\Omega \setminus \{Du_\varepsilon = 0\}} \psi F_{AA}(Du_\varepsilon) : D^2u_\varepsilon \geq 0,
\]
for non-negative test functions. However, the above reasoning is too simplistic to apply exactly as it stands. Several regularisations are required in order to make this work, and this causes substantial complications. For the simpler case of the \(p\)-Laplacian, see [JJ].

The main result here is

**Proposition 11.** Let \(K \subseteq \mathbb{R}^n\) be closed and \(F\) a convex function on \(\mathbb{R}^n\) satisfying (1.4), (1.5) and also either (1.6) or (1.7). Fix also \(\Omega \subseteq \mathbb{R}^n\) open.

Then, if \(u \in W^{1,1}_{loc}(\Omega)\) is a Feeble Viscosity Solution of (1.2) on \(\Omega\) (Definition 3), then \(u\) is a continuous weak solution of (1.8) and also local minimiser of the functional (1.1) in \(W^{1,1}_{loc}(\Omega)\). Moreover, under (1.6) we have \(r = 1\) and under (1.7) we have \(r > 1\) as in the assumption.

The following lemma is the first step towards the proof of Proposition 11.

**Lemma 12.** Let \(F : \mathbb{R}^n \to \mathbb{R}\) be convex and satisfy assumptions (1.4) and (1.5).

(i) There exists a positive strictly decreasing function \(\rho \in C^0(0, \infty)\) with \(\rho(0^+) = \infty\) and \(\rho(\infty) > 0\) which depends only on \(F\) such that, for each \(R > 1\), the function \(\Phi^R \in C^0(0, R)\) given by
\[
(5.1) \quad \Phi^R(t) := \inf_{t < |a| < R} \left\{ \frac{1}{\rho(|a|)} + \Delta F(a) \right\},
\]
is positive, strictly increasing and satisfies \(\Phi^R(0^+) = 0\) and
\[
(5.2) \quad \int_0^1 \frac{dt}{\Phi^R(t)} < \infty.
\]
If in addition $F$ satisfies
\[ \limsup_{|a| \to \infty} \Delta F(a) < \infty, \]
then we may take $R = \infty$ and $\Phi^\infty \in C^0(0, \infty)$ is also positive, strictly increasing and satisfies the same properties.

(ii) Fix a domain $\Omega \subseteq \mathbb{R}^n$. Assume that $v \in C^0(\Omega)$ is a semiconvex strong subsolution of
\[ F_{AA}(Dv) : D^2v \geq 0 \]
a.e. on $\Omega \setminus \{Dv = 0\}$. Moreover, suppose that $\|Dv\|_{L^\infty(\Omega)} < R$ and that for some $\varepsilon > 0$, $v$ satisfies the flatness estimate
\[ D^2v \geq -\frac{\Phi^R(\|Dv\|)}{\varepsilon} I \]
a.e. on $\Omega$. Then, it follows that $v$ is a weak subsolution of
\[ \text{Div}(F_A(Dv)) \geq 0 \]
on $\Omega$.
If $\|Dv\|_{L^\infty(\Omega)} = \infty$, then the same conclusion follows if in addition we have $\limsup_{|a| \to \infty} \Delta F(a) < \infty$ and (5.4) is satisfied for $\Phi^\infty$.

The function $\rho$ is explicitly constructed in the proof and is a correction term which arises because we need to mollify $F$ near the singularity at $\{0\}$. The above lemma has a symmetric counterpart for supersolutions which we refrain from stating explicitly.

**Proof of Lemma 12.** Proof of (i):

**Step 1.** We begin by utilising the assumption (1.5) in order to construct an explicit modulus of differentiability for $F$ in terms of $\Delta F$. Note that by assumption $F \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$, zero is a strict global minimum for $F$ and $\Delta F \geq 0$ on $\mathbb{R}^n \setminus \{0\}$. We set
\[ \omega(t) := \sqrt{t} + \int_0^t \left\{ \sup_{s < |a| < 1} \Delta F(a) \right\} ds. \]
The “$\sqrt{t}$” term is needed in order to avoid small technical complications which arise if $\Delta F$ is not unbounded and strictly radially decreasing near the origin.

**Claim.** The function $\omega$ given by (5.6) is strictly increasing, concave and in $C^1(0,1)$. Moreover, $\omega(0^+) = 0$ and also satisfies
\[ t \mapsto \frac{\omega(t)}{t} \text{ is strictly decreasing, } \int_0^1 \frac{\omega(t)}{t} dt < \infty. \]
Moreover, for any $a \in \mathbb{R}^n$ with $0 < |a| < 1$, we have
\[ 0 \leq F(a) \leq \omega(|a|)|a|, \quad |F_A(a)| \leq C\omega(|a|), \]
where $C = C(n) > 0$ depends only on the dimension.
Proof of Claim. By (1.5), there is a $C > 0$ such that
\[
\infty > C \geq \int_{\frac{1}{2}}^{1} \int_{0}^{t} \left\{ \sup_{s < |a| < 1} \Delta F(a) \right\} ds \, dt
\]
\[
\geq \frac{1}{2} \int_{0}^{t_{0}} \left\{ \sup_{s < |a| < 1} \Delta F(a) \right\} ds
\]
for some $t_{0} \in [1/2, 1]$. Hence, we have the estimate
\[
\int_{\frac{1}{2}}^{1} \left\{ \sup_{s < |a| < 1} \Delta F(a) \right\} ds \leq 2t_{0}C \leq 2C
\]
which implies that the positive strictly decreasing function
\[
\delta(s) := \frac{1}{2\sqrt{s}} + \sup_{s < |a| < 1} \Delta F(a)
\]
is in $(L^{1} \cap C^{0})(0, 1)$. As a result, $\omega$ is concave, strictly increasing, $\omega(0^{+}) = 0$ and $\omega'(t) = \delta(t) > 0$ for $0 < t < 1$. Moreover,
\[
\left( \frac{\omega(t)}{t} \right)' = \frac{1}{t^{2}} \left[ t \delta(t) - \int_{0}^{t} \delta(s) ds \right] < 0,
\]
for $0 < t < 1$, and also (by assumption (1.5))
\[
\int_{0}^{1} \frac{\omega(t)}{t} \, dt = \int_{0}^{1} \frac{1}{t} \int_{0}^{t} \left\{ \frac{1}{2\sqrt{s}} + \sup_{s < |a| < 1} \Delta F(a) \right\} ds \, dt < \infty.
\]
Thus, (5.7) has been established. Finally, by Taylor’s theorem, for any $a \in \mathbb{R}^{n}$ with $0 < |a| < 1$ and $0 < \varepsilon < 1$,
\[
|F(a) - F(\varepsilon a) - (1 - \varepsilon)F_{A}(\varepsilon a) \cdot a| = \left| \int_{\varepsilon}^{1} (1 - t)F_{AA}(ta) : a \otimes a \, dt \right|
\]
\[
\leq |a|^{2} \int_{\varepsilon}^{1} (1 - t)\left| F_{AA}(ta) : \frac{a \otimes a}{|a|^{2}} \right| dt
\]
\[
\leq |a|^{2} \int_{\varepsilon}^{1} (1 - t)\Delta F(ta) dt.
\]
Hence,
\[
|F(a) - F(\varepsilon a) - (1 - \varepsilon)F_{A}(\varepsilon a) \cdot a| \leq |a|^{2} \int_{\varepsilon}^{1} \left\{ \sup_{t < |a| < 1} \Delta F(sa) \right\} dt
\]
\[
= |a| \int_{\varepsilon|a|}^{|a|} \left\{ \sup_{t < |A| < |a|} \Delta F(A) \right\} dt
\]
\[
\leq |a| \int_{0}^{|a|} \left\{ \sup_{t < |A| < |a|} \Delta F(A) \right\} dt,
\]
which gives
\[
|F(a) - F(\varepsilon a) - (1 - \varepsilon)F_{A}(\varepsilon a) \cdot a| \leq |a| \omega(|a|),
\]
for $0 < |a| < 1$ and $0 < \varepsilon < 1$. By passing to the limit as $\varepsilon \to 0$, we obtain the desired estimate $0 \leq F(a) \leq \omega(|a|)|a|$. Similarly, for any $e \in \mathbb{R}^n$, $|e| = 1$, we have

$$
(F_a(a) - F_a(\varepsilon a)) \cdot e \leq |a| \int_0^1 |F_{AA}(ta) : \frac{a}{|a|} \otimes e| dt.
$$

By norm equivalence on $S(\mathbb{R}^n) \subseteq \mathbb{R}^{n \times n}$, there is a $C = C(n) > 0$ such that, for any non-negative symmetric $n \times n$ matrix $X$,

$$
|X : \frac{a}{|a|} \otimes e| \leq \max_{|E| = 1} \{X : E\}
$$

$$
= |X| 
\leq C(n) \max_{|e| = 1} \{X : e \otimes e\} 
\leq C(n) \text{tr}(X).
$$

In view of the estimates (5.9) and (5.10) and by arguing as before, we conclude that

$$
|F_a(a) - F_a(\varepsilon a)| \leq C(n) \omega(|a|),
$$

for $0 < |a| < 1$ and $0 < \varepsilon < 1$. Hence by letting $\varepsilon \to 0$ we see that (5.8) has been established and the proof of the claim is complete.

**Step 2.** We begin by introducing appropriate $C^2$ approximations of $F$ in $C^1(\mathbb{R}^n)$. Fix $0 < \delta < 1$ and choose $\zeta \in C^\infty([0, \infty))$ such that $\zeta \equiv 0$ on $[0, 1/2]$ and $\zeta \equiv 1$ on $[1, \infty)$. Set

$$
\zeta^\delta(a) := \zeta\left(\frac{|a|}{\delta}\right), \quad a \in \mathbb{R}^n.
$$

Then, $\zeta^\delta \in C^\infty(\mathbb{R}^n)$, $\zeta^\delta \equiv 0$ on $\mathbb{B}_{\delta/2}(0)$ and $\zeta^\delta \equiv 1$ on $\mathbb{R}^n \setminus \mathbb{B}_{\delta}(0)$ and

$$
|D\zeta^\delta| \leq \frac{C}{\delta} \chi_{\mathbb{B}_\delta(0)}, \quad |D^2\zeta^\delta| \leq \frac{C}{\delta^2} \chi_{\mathbb{B}_\delta(0)},
$$

for some universal $C > 0$. We set $F^\delta := \zeta^\delta F$. Then, we have that $F^\delta \in C^2(\mathbb{R}^n)$, $F^\delta \equiv 0$ on $\mathbb{B}_{\delta/2}(0)$ and $F^\delta \equiv F$ on $\mathbb{R}^n \setminus \mathbb{B}_\delta(0)$. For $|a| \leq \delta$, we have

$$
|F(a) - F^\delta(a)| = \left|(1 - \zeta^\delta(a))F(a)\right| \leq \omega(|a|)|a| \leq \omega(\delta)|a|,
$$

and also

$$
|F_a(a) - F_a^\delta(a)| \leq |D\zeta^\delta(a)||F(a)| + |F_a(a)||1 - \zeta^\delta(a)|
\leq \frac{C}{\delta} \omega(|a|)|a| + C\omega(|a|)
\leq C\omega(\delta),
$$

as $\delta \to 0$. Hence, $F^\delta \to F$ in $C^1(\mathbb{R}^n)$ as $\delta \to 0$, as desired.
**Step 3.** Now we construct the function $\rho$ of the statement. By (5.12) and (5.8), for $0 < |a| < 1$ and $0 < \delta < 1$ we have

\[
|\Delta F^\delta(a)| = |F(a)\Delta \zeta^\delta(a) + 2DF^\delta(a) \cdot F_A(a) + \zeta^\delta(a)\Delta F(a)| \\
\leq \omega(|a|)|a| C_\delta \chi_{B_\delta(0)}(a) + 2C_\delta \chi_{B_\delta(0)}(a) \omega(|a|) + \Delta F(a) \\
\leq C_\delta \omega(|a|) + \Delta F(a) \\
\leq \frac{C_\delta \omega(|a|)}{|a|} + \Delta F(a).
\]

The last inequality owes to that $t \mapsto \omega(t)/t$ is strictly decreasing on $(0, 1)$. We set

\[
(5.16) \quad \rho(t) := \frac{C_\delta \omega(t)}{t}, \quad 0 < t < 1,
\]

and extend it on $[1, \infty)$ as a strictly decreasing positive function (for example, set

\[
\rho(t) := \frac{C_\delta \omega(1)}{t} (e^{1-t} + 1) \quad \text{for } t \geq 1.
\]

By (5.15) and (5.16) we have

\[
(5.17) \quad \sup_{0 < \delta < 1} \left| \Delta F^\delta(a) \right| \leq \rho(|a|) + \Delta F(a),
\]

for $a \neq 0$. Finally, we employ (5.1) to define for $R > 1$ the function $\Phi^R \in C^0(0, R)$ which is positive and strictly increasing with $\Phi^R(0^+) = 0$. By utilising (5.7), we obtain the estimate

\[
\int_0^1 \frac{dt}{\Phi^R(t)} = \int_0^1 \left\{ \sup_{0 < |a| < R} \left[ \rho(|a|) + \Delta F(a) \right] \right\} dt \\
\leq \int_0^1 \left\{ \sup_{0 < |a| < R} \rho(|a|) \right\} dt + \int_0^1 \left\{ \sup_{0 < |a| < R} \Delta F(a) \right\} dt \\
\leq C \int_0^1 \frac{\omega(t)}{t} dt + \omega(1) + \sup_{1 < |a| < R} \Delta F(a) \\
< \infty.
\]

Hence, (5.2) has been established as well. If $\lim \sup_{|a| \to \infty} \Delta F(a) < \infty$, then for $R = \infty$ the function $\Phi^\infty \in C^0(0, \infty)$ is also positive, increasing with $\Phi^\infty(0^+) = 0$ and satisfies the same estimate.

**Proof of (ii):**

**Step 1.** We now establish that the semiconvex function $v$ satisfies the inequality

\[
(5.18) \quad -\int_\Omega D\psi \cdot F_A^\delta(Dv) \geq \int_\Omega \psi F_A^\delta(Dv) : D^2v,
\]

for all $\psi \in C_c^\infty(\Omega)$, $\psi \geq 0$ and $0 < \delta < 1$. We note that this is not a trivial application of integration by parts due to the existence of the singular part of the full Hessian measure of $v$ and equality in (5.18) may fail. Since $v$ is semiconvex on $\Omega$, $(Dv, D^2v)$ exist a.e. on $\Omega$ and also there is $C > 0$ such that $D^2v \geq -\frac{C}{2} I$ a.e. on $\Omega$. To prove (5.18), we regularise $v$ further in the standard way by convolution, that is for $\sigma > 0$ we consider the mollifier $v * \eta^\sigma$, which is also semiconvex uniformly in $\sigma > 0$. 

**Step 2.** Now we establish that the semiconvex function $v$ satisfies the inequality

\[
(5.18) \quad -\int_\Omega D\psi \cdot F_A^\delta(Dv) \geq \int_\Omega \psi F_A^\delta(Dv) : D^2v,
\]

for all $\psi \in C_c^\infty(\Omega)$, $\psi \geq 0$ and $0 < \delta < 1$. We note that this is not a trivial application of integration by parts due to the existence of the singular part of the full Hessian measure of $v$ and equality in (5.18) may fail. Since $v$ is semiconvex on $\Omega$, $(Dv, D^2v)$ exist a.e. on $\Omega$ and also there is $C > 0$ such that $D^2v \geq -\frac{C}{2} I$ a.e. on $\Omega$. To prove (5.18), we regularise $v$ further in the standard way by convolution, that is for $\sigma > 0$ we consider the mollifier $v * \eta^\sigma$, which is also semiconvex uniformly in $\sigma > 0$. 

**Step 3.** Now we construct the function $\rho$ of the statement. By (5.12) and (5.8), for $0 < |a| < 1$ and $0 < \delta < 1$ we have

\[
|\Delta F^\delta(a)| = |F(a)\Delta \zeta^\delta(a) + 2DF^\delta(a) \cdot F_A(a) + \zeta^\delta(a)\Delta F(a)| \\
\leq \omega(|a|)|a| C_\delta \chi_{B_\delta(0)}(a) + 2C_\delta \chi_{B_\delta(0)}(a) \omega(|a|) + \Delta F(a) \\
\leq C_\delta \omega(|a|) + \Delta F(a) \\
\leq \frac{C_\delta \omega(|a|)}{|a|} + \Delta F(a).
\]

The last inequality owes to that $t \mapsto \omega(t)/t$ is strictly decreasing on $(0, 1)$. We set

\[
(5.16) \quad \rho(t) := \frac{C_\delta \omega(t)}{t}, \quad 0 < t < 1,
\]

and extend it on $[1, \infty)$ as a strictly decreasing positive function (for example, set

\[
\rho(t) := \frac{C_\delta \omega(1)}{t} (e^{1-t} + 1) \quad \text{for } t \geq 1.
\]

By (5.15) and (5.16) we have

\[
(5.17) \quad \sup_{0 < \delta < 1} \left| \Delta F^\delta(a) \right| \leq \rho(|a|) + \Delta F(a),
\]

for $a \neq 0$. Finally, we employ (5.1) to define for $R > 1$ the function $\Phi^R \in C^0(0, R)$ which is positive and strictly increasing with $\Phi^R(0^+) = 0$. By utilising (5.7), we obtain the estimate

\[
\int_0^1 \frac{dt}{\Phi^R(t)} = \int_0^1 \left\{ \sup_{0 \leq |a| \leq R} \left[ \rho(|a|) + \Delta F(a) \right] \right\} dt \\
\leq \int_0^1 \left\{ \sup_{0 < |a| < R} \rho(|a|) \right\} dt + \int_0^1 \left\{ \sup_{0 < |a| < R} \Delta F(a) \right\} dt \\
\leq C \int_0^1 \frac{\omega(t)}{t} dt + \omega(1) + \sup_{1 < |a| < R} \Delta F(a) \\
< \infty.
\]

Hence, (5.2) has been established as well. If $\lim \sup_{|a| \to \infty} \Delta F(a) < \infty$, then for $R = \infty$ the function $\Phi^\infty \in C^0(0, \infty)$ is also positive, increasing with $\Phi^\infty(0^+) = 0$ and satisfies the same estimate.

**Proof of (ii):**

**Step 1.** We now establish that the semiconvex function $v$ satisfies the inequality

\[
(5.18) \quad -\int_\Omega D\psi \cdot F_A^\delta(Dv) \geq \int_\Omega \psi F_A^\delta(Dv) : D^2v,
\]

for all $\psi \in C_c^\infty(\Omega)$, $\psi \geq 0$ and $0 < \delta < 1$. We note that this is not a trivial application of integration by parts due to the existence of the singular part of the full Hessian measure of $v$ and equality in (5.18) may fail. Since $v$ is semiconvex on $\Omega$, $(Dv, D^2v)$ exist a.e. on $\Omega$ and also there is $C > 0$ such that $D^2v \geq -\frac{C}{2} I$ a.e. on $\Omega$. To prove (5.18), we regularise $v$ further in the standard way by convolution, that is for $\sigma > 0$ we consider the mollifier $v * \eta^\sigma$, which is also semiconvex uniformly in $\sigma > 0$. 

Since by semiconvexity we have that $v \in W^{1,\infty}_{\text{loc}}(\Omega)$ (e.g. [EG], p.236), by Dominated Convergence it follows that
\begin{equation}
D\psi \cdot F^\delta_A(Dv * \eta^\sigma) \to D\psi \cdot F^\delta_A(Dv),
\end{equation}
in $L^1(\Omega)$, as $\sigma \to 0$. Moreover,
\begin{equation}
F^\delta_{AA}(Dv * \eta^\sigma) : (D^2v * \eta^\sigma) \to F^\delta_{AA}(Dv) : D^2v,
\end{equation}
a.e. on $\Omega$ as $\sigma \to 0$, and also for $\sigma > 0$ small we have the $L^1$ lower bound
\begin{equation}
\psi F^\delta_{AA}(Dv * \eta^\sigma) : D^2v \eta^\sigma \geq -\frac{C}{\varepsilon} \psi \max_{||a|| \leq ||Dv||_{L^\infty(\sup\psi)}} |\Delta F^\delta(a)|.
\end{equation}
Hence, by (5.19), (5.20), (5.21) and Fatou Lemma, we have
\begin{equation}
- \int_\Omega D\psi \cdot F^\delta_A(Dv) = -\lim_{\sigma \to 0} \int_\Omega D\psi \cdot F^\delta_A(Dv * \eta^\sigma)
\geq \liminf_{\sigma \to 0} \int_\Omega \psi F^\delta_{AA}(Dv * \eta^\sigma) : (D^2v * \eta^\sigma)
= \int_\Omega \psi F^\delta_{AA}(Dv) : D^2v.
\end{equation}
Hence, (5.18) follows.

**Step 2.** Since by (i) we have $F^\delta \to F$ in $C^1(\mathbb{R}^n)$ and also $F^\delta \equiv 0$ on $B_{\delta/2}(0)$, we may use Dominated Convergence theorem to pass in the limit as $\delta \to 0$ in (5.18):
\begin{equation}
- \int_\Omega D\psi \cdot F_A(Dv) = -\lim_{\delta \to 0} \int_\Omega D\psi \cdot F_A^\delta(Dv)
\geq \liminf_{\sigma \to 0} \int_\Omega \psi F^\delta_{AA}(Dv) : D^2v
= \liminf_{\sigma \to 0} \int_{\Omega \setminus \{Dv=0\}} \psi F^\delta_{AA}(Dv) : D^2v.
\end{equation}
We now utilise the assumption (5.4) in order to pass to the limit as $\delta \to 0$. Let $\Phi^R$ be given by (5.1). The choice of $R > 0$ depends on our assumptions on $F$ and $v$:
\begin{equation}
\begin{cases}
R := ||Dv||_{L^\infty(\Omega)} + 1, & \text{if } Dv \in L^\infty(\Omega), \\
R := \infty, & \text{if } \limsup_{|a| \to \infty} |\Delta F(a)| < \infty.
\end{cases}
\end{equation}
Then, by (5.17) we have
\begin{equation}
F^\delta_{AA}(Dv) : D^2v \geq -\frac{1}{\varepsilon} \Phi^R(|Dv|) F^\delta_{AA}(Dv) : I
\geq -\frac{1}{\varepsilon} \Phi^R(|Dv|) \Delta F^\delta(Dv)
\geq -\frac{1}{\varepsilon} \Phi^R(|Dv|) \left(\rho(|Dv|) + \Delta F(Dv)\right)
\geq -\frac{1}{\varepsilon}.
\end{equation}
a.e. on Ω, and this gives an $L^1$ lower bound in order to use Fatou Lemma in (5.23). Hence, we have
\begin{equation}
-\int_\Omega \nabla \psi \cdot F_A(Dv) \geq \int_{\Omega \setminus \{Dv=0\}} \liminf_{\delta \to 0} \psi F^\delta_{AA}(Dv) : D^2v
\geq \int_{\Omega \setminus \{Dv=0\}} \psi F_{AA}(Dv) : D^2v.
\end{equation}
By our assumption on $v$, the right hand side of (5.26) vanishes. Hence,
\begin{equation}
-\int_\Omega \nabla \psi \cdot F_A(Dv) \geq 0
\end{equation}
for any $\psi \in C_\infty^0(\Omega)$, $\psi \geq 0$. The lemma follows. $\square$

Remark 13. The $L^1$ lower bound estimate (5.25) is the main reason for the necessity to introduce the sup-convolution approximations in Section 4 which satisfy flatness properties that cancel the singularity of the coefficients. See also [JJ] where this idea has already been implicitly utilised in the special case of the singular $p$-Laplacian when $1 < p < 2$ and $F(A) = |A|^p$.

We may now prove the proposition.

Proof of Proposition 11. Let $u \in (C^0 \cap W^{1,1}_{loc})(\Omega)$ be a Feeble Viscosity Solution of $F_{AA}(Du) : D^2u \geq 0$ on $\Omega \subseteq \mathbb{R}^n$. Fix $\phi \in C_\infty^\infty(\Omega)$ with $\phi \leq 0$ and $\Omega' \Subset \Omega$ such that $\text{supp}(\phi) \subseteq \Omega'$. Obviously, $u \in C^0(\overline{\Omega})$.

1st case under assumption (1.6). Since $u \in W^{1,\infty}_{loc}(\Omega)$, we have $\|Du\|_{L^\infty(\Omega')} < \infty$. Consider the increasing function $\Phi_R \in C^0(0, \infty)$ with $\Phi_R(0) = 0$ defined by (5.1) in Lemma 12, where as $R$ we take $\|Du\|_{L^\infty(\Omega')} + 1$. We then extend $\Phi_R$ on $(0, \infty)$ by setting
\begin{equation}
\Phi(t) := \begin{cases}
\Phi_R(t), & 0 < t < R, \\
\Phi_R(R) \frac{t}{R}, & t \geq R.
\end{cases}
\end{equation}
Consider for $\varepsilon > 0$ the flat sup-convolution $u^\varepsilon$ of $u$ restricted on $\Omega'$, as given by (4.1), where as $\Theta \in C^2(0, \infty)$ we take the function given by Lemma 9 for the selected $\Phi$. Then $u^\varepsilon$ satisfies all the properties of Lemmas 8, 9. Hence the flatness estimate (4.34) holds. Since $u \in W^{1,\infty}_{loc}(\Omega)$, we have
\begin{equation}
\|Du^\varepsilon\|_{L^\infty(\Omega')} \leq \|Du\|_{L^\infty(\Omega')}
\end{equation}
(\Omega' as in Lemma 8). Moreover, by Lemma 8, $u^\varepsilon$ is a strong subsolution of
\begin{equation}
F_{AA}(Du^\varepsilon) : D^2u^\varepsilon \geq 0,
\end{equation}
a.e. on $\Omega^\varepsilon \setminus \{Du^\varepsilon = 0\}$. Since $u^\varepsilon$ is semiconvex, it satisfies the assumptions of Lemma 12. Hence, $u^\varepsilon$ is a locally Lipschitz weak subsolution on $\Omega^\varepsilon$, which implies
\begin{equation}
\int_\Omega \nabla \psi \cdot F_A(Du^\varepsilon) \geq 0 \quad \text{for } \psi \leq 0, \psi \in W^{1,1}_0(\Omega^\varepsilon).
\end{equation}
For $\phi$ as in the beginning of the proof, choose $\delta > 0$ small such that $\text{supp}(\phi) \subseteq \Omega^\delta$ and restrict $\varepsilon \leq \delta$. Since $\Omega^\delta \subseteq \Omega^\varepsilon$, by the elementary inequality (3.8) we have
\begin{equation}
0 \leq \int_{\Omega^\delta} F(Du^\varepsilon + D\psi) - \int_{\Omega^\delta} F(Du^\varepsilon)
\end{equation}
for $\psi \in W^{1,1}_0(\Omega^\delta)$, $\psi \leq 0$ and all $0 < \varepsilon \leq \delta$. Hence, $u^\varepsilon$ is a local subminimiser of (1.1) on $\Omega^\delta$. Moreover, by Lemma 8 and (5.29), we have $u^\varepsilon \rightharpoonup u$ weakly* in $W^{1,\infty}(\Omega^\delta)$ as $\varepsilon \to 0$. By assumption 1.4 and standard semicontinuity results (Dacorogna [D], p. 94), we have
\begin{equation}
E(u, \Omega^\delta) \leq \liminf_{\varepsilon \to 0} E(u^\varepsilon, \Omega^\delta).
\end{equation}
We now choose $\sigma > 0$ small and define the cut-off function
\begin{equation}
\zeta^\sigma := \min \left\{ \frac{1}{\sigma} \text{dist}(\cdot, \partial \Omega^\delta), 1 \right\}.
\end{equation}
Then $\zeta^\sigma \in W^{1,\infty}_0(\Omega^\delta)$, $\zeta^\sigma \equiv 1$ on the inner $\sigma$-neighborhood of $\Omega^\delta$
\begin{equation}
\{ x \in \Omega^\delta \mid \text{dist}(x, \partial \Omega^\delta) > \sigma \}
\end{equation}
and $|D\zeta^\sigma| \leq C/\sigma$ for a $C > 0$ independent of $\sigma$. We select as $\psi$ the function
\begin{equation}
\psi := \zeta^\sigma (u - u^\varepsilon) + \phi,
\end{equation}
which is admissible since by Lemma 8 we have $u - u^\varepsilon \leq 0$. We set
\begin{equation}
M := 2\|Du\|_{L^\infty(\Omega')} + \frac{C}{\sigma} \|u - u^\varepsilon\|_{C^0(\Omega')} + \|D\phi\|_{L^\infty(\Omega')}.
\end{equation}
Then, since $F \in C^1(\mathbb{R}^n)$, by (5.32) we have
\begin{align}
\int_{\Omega^\delta} F(Du^\varepsilon) &\leq \int_{\Omega^\delta} F\left( (1 - \zeta^\sigma)Du^\varepsilon + \zeta^\sigma Du + (u - u^\varepsilon)D\zeta^\sigma + D\phi \right) \\
&\leq \int_{\Omega^\delta} F\left( \zeta^\sigma Du + D\phi \right) \\
&\quad + \max_{B_M(0)} |F_A| \int_{\Omega^\delta} \left| (1 - \zeta^\sigma)Du^\varepsilon + (u - u^\varepsilon)D\zeta^\sigma \right| \\
&\leq \int_{\Omega^\delta} F\left( \zeta^\sigma Du + D\phi \right) \\
&\quad + \max_{B_M(0)} |F_A| \left\{ \|Du\|_{L^\infty(\Omega')} \int_{\Omega^\delta} (1 - \zeta^\sigma) + |\Omega'| \frac{C}{\sigma} \|u - u^\varepsilon\|_{C^0(\Omega')} \right\}.
\end{align}
By (5.37), $M$ is independent of $\varepsilon$. Moreover, we have that $\zeta^\sigma \not\nearrow 1$ a.e. on $\Omega^\delta$ as $\sigma \to 0$ and also the bound
\begin{equation}
F\left( \zeta^\sigma Du + D\phi \right) \leq \max_{B_M(0)} F, \quad N := \|Du\|_{L^\infty(\Omega')} + \|D\phi\|_{L^\infty(\Omega')}.
\end{equation}
By letting $\varepsilon \to 0$ and then $\sigma \to 0$ in (5.38), the Dominated Convergence theorem implies
\begin{equation}
\liminf_{\varepsilon \to 0} \int_{\Omega^\delta} F(Du^\varepsilon) \leq \int_{\Omega^\delta} F(Du + D\phi).
\end{equation}
By combining (5.40) with (5.33) and letting $\delta \searrow 0$ we get
\begin{equation}
\int_{\Omega} F(Du) \leq \int_{\Omega} F(Du + D\phi).
\end{equation}
for any $\phi \in C_c^\infty(\Omega')$, $\Omega' \Subset \Omega$, $\phi \leq 0$. On the other hand, by following the same method for $u$ which is a Feeble Viscosity Supersolution and utilising the flat inf-convolution (4.2), we deduce that (5.41) holds for $\phi \geq 0$ in $\phi \in C_c^\infty(\Omega')$. By
Claim 14. Let \( \phi \) be a general result with complicated assumptions, we prefer to give a short direct proof.

(5.42) \[
\int_{\Omega'} F_A(Du) \cdot D\phi = 0
\]

and since \( |F_A(Du)| \in L^\infty(\Omega') \), we have that (5.42) holds for all \( \phi \in W^{1,1}_0(\Omega') \). By inequality (3.8), \( u \) is a local minimiser as well and satisfies (1.3). The proposition under assumption (1.6) follows.

2nd case under assumption (1.7). We begin with the next

Claim 15. Suppose that \( \mathcal{F} \in C^1(\mathbb{R}^n) \) satisfy (1.7) with \( r > 1 \) as in (1.7) and assume that \( v \in (W^{1,r}_0 \cap L^\infty_0)(\Omega) \) is a weak solution of

(5.43) \[
\text{Div}(\xi F_A(Dv)) \geq 0
\]
on \( \Omega \subseteq \mathbb{R}^n \), where \( \xi \in \{1, -1\} \). Then, for any ball \( B_R \) such that \( B_{2R} \subseteq \Omega \), we have

(5.44) \[
\|Dv\|_{L^r(B_R)} \leq C\|v\|_{L^\infty(B_{2R})},
\]

where \( C > 0 \) depends only \( F, R \).

The proof of the claim is rather standard (see e.g. [GT]), but instead of quoting general results with complicated assumptions, we prefer to give a short direct proof.

Proof of Claim 14. Fix \( B_{2R} \subseteq \Omega \) and \( \zeta \in C_c(\mathbb{R}^n) \) with \( \chi_{B_R} \leq \zeta \leq \chi_{B_{2R}} \).

(5.45) \[
\psi := \zeta^r(\xi v + \|v\|_{L^\infty(B_{2R})}).
\]

Then, \( \psi \in W^{1,r}_0(B_{2R}) \), \( \psi \geq 0 \). Since \( |F_A(Dv)| \in L^\infty_0(\Omega) \), by (5.43) and (5.45) we have

(5.46) \[
- \int_{B_{2R}} \xi F_A(Dv) \cdot [\xi \zeta^r Dv + r(\xi v + \|v\|_{L^\infty(B_{2R})}) \zeta^{r-1} D\zeta] \geq 0.
\]

Since \( \xi^2 = 1 \), in view of (1.7) and Young inequality, for \( \varepsilon > 0 \) we obtain

(5.47) \[
\int_{B_{2R}} \zeta^r |Dv|^r \leq C\|v\|_{L^\infty(B_{2R})} \int_{B_{2R}} |Dv|^{r-1} \zeta^{r-1} |D\zeta|
\]

\leq C\|v\|_{L^\infty(B_{2R})}\left\{ \varepsilon \int_{B_{2R}} (|Dv|^{r-1} \zeta^{r-1})^\frac{r}{r-1} + \frac{1}{\varepsilon^{r-1}} \int_{B_{2R}} |D\zeta|^r \right\}.

For \( \varepsilon := 1/(2C\|v\|_{L^\infty(B_{2R})}) \) we immediately get \( \|\zeta Dv\|_{L^r(B_{2R})} \leq C\|v\|_{L^\infty(B_{2R})} \) which implies (5.44).

Claim 15. Suppose that \( F \in C^1(\mathbb{R}^n) \) satisfies (1.7) with \( r > 1 \) as in (1.7). Assume that \( (v^i)_1^\infty \subseteq (W^{1,r}_0 \cap L^\infty_0)(\Omega) \) is a sequence of uniformly bounded in (the Fréchet topology of) \( L^\infty_0(\Omega) \) weak solutions to

(5.48) \[
\text{Div}(F_A(Dv)) \geq 0
\]
on \( \Omega \subseteq \mathbb{R}^n \). Then, if \( v^{i+1} \leq v^i \) a.e. on \( \Omega \), the sequence \( (v^i)_1^\infty \) is strongly precompact in (the Fréchet topology of) \( W^{1,r}_0(\Omega) \). Moreover, if \( v^i \to v \) along a subsequence, then the limit solves (5.48) as well.
There is also a symmetric result for the compactness of increasing sequences of weak supersolutions, which we refrain from stating.

**Proof of Claim 15.** The idea is taken from [L]. Fix $\mathbb{B}_{2R}$ and $\zeta$ as in the proof of Claim 14. By (5.48) and our assumption, $(v^i)_i^\infty$ is weakly bounded in $W^{1,r}_{\text{loc}}(\Omega)$ and hence there is $v$ such that $v^i \rightharpoonup v$ weakly in $W^{1,r}_{\text{loc}}(\Omega)$ along a subsequence as $i \to \infty$. For each $i \in \mathbb{N}$, consider the integral

$$I^i := \int_{\mathbb{B}_{2R}} \left( F_A(Dv^i) - F_A(Dv) \right) \cdot D(\zeta(v^i - v)).$$

Since $\psi := \zeta(v^i - v)$ is in $W^{1,r}_{\text{loc}}(\mathbb{B}_{2R})$ and also $\psi \geq 0$, by utilising that $v^i$ is a weak solution of (5.48) and that $|F_A(Dv^i)| \in L^\infty_{\text{loc}}(\Omega)$, we have

$$I^i \leq \int_{\mathbb{B}_{2R}} F_A(Dv) \cdot (D(\zeta v^i) - D(\zeta v)).$$

Since $D(\zeta v^i) \rightharpoonup D(\zeta v)$ in $L^r(\mathbb{B}_{2R})$ as $i \to \infty$, by (5.50) we have that $\limsup_{i \to \infty} I^i \leq 0$.

By (1.7), (5.49) and Young inequality we have

$$I^i = \int_{\mathbb{B}_{2R}} (v^i - v) D\zeta \cdot (F_A(Dv^i) - F_A(Dv))$$

$$+ \int_{\mathbb{B}_{2R}} \zeta \left( F_A(Dv^i) - F_A(Dv) \right) \cdot D(v^i - v)$$

$$\geq -\|D\zeta\|_{L^\infty(\mathbb{B}_{2R})} \|v^i - v\|_{L^r(\mathbb{B}_{2R})} \left( \|Dv^i\|_{L^r(\mathbb{B}_{2R})} + \|Dv\|_{L^r(\mathbb{B}_{2R})} \right)$$

$$+ C \int_{\mathbb{B}_{2R}} \zeta |Dv^i - Dv|^r.$$}

Hence, by (5.51) we get $\limsup_{i \to \infty} \|Dv^i - Dv\|_{L^r(\mathbb{B}_{2R})} = 0$, as desired. \(\square\)

We may now complete the proof of Proposition 11 under assumption (1.7). Suppose $u \in (C^0 \cap W^{1,1}_{\text{loc}})(\Omega)$ is a Feeble Viscosity Subsolution of (1.2) on $\Omega$. Fix $\Omega' \subseteq \Omega$ and $\psi \in W^{1,1}_{\text{loc}}(\Omega')$ with $\psi \geq 0$ where $r > 1$ is as in (1.7) and observe that $u \in C^0(\Omega')$. Consider the increasing function $\Phi := \Phi^\infty \in C^0(0, \infty)$ with $\Phi(0^+) = 0$ defined by (5.1) in Lemma 12, where we take $R := \infty$ since by assumption (1.7) the Laplacian of $F$ is bounded at infinity.

Consider for $\varepsilon > 0$ the flat sup-convolution $u^\varepsilon$ of $u$ restricted on $\Omega'$, as given by (4.1), where as $\Theta \in C^2(0, \infty)$ we take the function given by Lemma 9 for the selected $\Phi$. Then $u^\varepsilon$ satisfies all the properties of Lemmas 8, 9. Hence the flatness estimate (4.34) holds. Moreover, by Lemma 8, $u^\varepsilon$ is a strong subsolution of

$$F_{AA}(Du^\varepsilon) : D^2u^\varepsilon \geq 0,$$

a.e. on $\Omega^\varepsilon \setminus \{Du^\varepsilon = 0\}$. Since $u^\varepsilon$ is semiconvex, it satisfies the assumptions of Lemma 12. Hence, by Lemma 12 $u^\varepsilon$ is a locally Lipschitz weak subsolution of $\text{Div}(F_A(Du^\varepsilon)) \geq 0$ on $\Omega^\varepsilon$, and hence for $\varepsilon$ small such that $\text{supp}(\psi) \subseteq \Omega^\varepsilon$, we have

$$-\int_{\Omega'} F_A(Du^\varepsilon) \cdot D\psi \geq 0.$$

By Lemma 8 we have $u^\varepsilon \rightharpoonup u$ in $C^0(\Omega')$ as $\varepsilon \to 0$ and hence by Claims 14, 15, we have that $u^\varepsilon \rightharpoonup u$ in $W^{1,1}_{\text{loc}}(\Omega')$ as $\varepsilon \to 0$, along a sequence. By passing to the limit
as $\varepsilon \to 0$ in (5.53) and utilising (3.8), we obtain that $u$ is a weak subolution of (1.8) in $W^{1,r}_{loc}(\Omega)$ and also satisfies
\begin{equation}
E(u, \Omega') \leq E(u + \phi, \Omega'), \quad \Omega' \Subset \Omega, \quad \phi \in W^{1,r}_0(\Omega'), \quad \phi \leq 0.
\end{equation}
Since $u$ is a Feeble Viscosity Supersolution as well, by arguing symmetrically for the flat inf-convolution, we obtain that $u$ is a weak supersolution as well, and (5.54) holds for $\phi \geq 0$ as well. Hence, the proposition follows and so does Theorem 1. □

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