U-folds as K3 fibrations

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Abstract

We study $\mathcal{N} = 2$ four-dimensional flux vacua describing intrinsic non-perturbative systems of 3 and 7 branes in type IIB string theory. The solutions are described as compactifications of a G(avity) theory on a Calabi Yau threefold which consists of a fibration of an auxiliary K3 surface over an $S^2$ base. In the spirit of F-theory, the complex structure of the K3 surface varying over the base codifies the details of the fluxes, the dilaton and the warp factors in type IIB string theory. We discuss in detail some simple examples of geometric and non-geometric solutions where the precise flux/geometry dictionary can be explicitly worked out. In particular, we describe non-geometric T-fold solutions exhibiting non-trivial T-duality monodromies exchanging 3- and 7-branes.
1 Introduction

F-Theory [1,2] describes fully non-perturbative solutions of type IIB theory (with a varying axio-dilaton field) in purely geometric terms. It links type IIB to M-Theory and elucidates the geometric origin of the $SL(2,\mathbb{Z})$ S-duality symmetry of type IIB theory. Besides the conceptual beauty, F-Theory is a powerful tool to build and analyse semi realistic gauge theories (F-theory GUT’s) with non-trivial strong coupling dynamics [3,6] and provides explicit D-brane set-ups where the gauge-gravity correspondence can be tested at the non-perturbative level [7,9].

In this paper we present a construction which, in a similar spirit, makes use of the
larger U-duality group of type IIB string theory compactified on a $K3$ surface. Type IIB supergravity compactified on a K3 surface is described by an effective $\mathcal{N} = (2,0)$ six-dimensional supergravity with 105 scalars and U-duality group $SO(5,21,\mathbb{Z})$. A class of supersymmetric solutions (vacua) preserving $\mathcal{N} = 2$ supersymmetries in this theory can be found by allowing a subset of the scalars spanning the moduli space submanifold

$$\mathcal{M}_{\text{BPS}} = O(\Gamma_{2,18}) \backslash \frac{O(2,18;\mathbb{R})}{O(2;\mathbb{R}) \times O(18;\mathbb{R})}$$

(1.1)

to vary holomorphically on a complex plane $[10]$. The scalars $\varphi^I = (\tau, \sigma, \beta^a)$, $a = 1,..16$, spanning (1.1) follow from the reduction along K3 of the axio-dilaton field, the four form, the warp factor and the NS-NS/R-R two form potentials, respectively. The solutions, that we dub U-folds, are specified by a set of holomorphic functions $\varphi^I(z)$ defined on a punctured complex plane (described by the coordinate $z$) up to non-trivial monodromies of the U-duality group $O(\Gamma_{2,18}) \sim O(2,18;\mathbb{Z})$.

In F-theory the positions of the punctures specify the locations of the 7-branes and the $SL(2,\mathbb{Z})$ monodromies give their $(p,q)$ charges. Analogously the location of punctures and monodromies in the U-fold solution encode the positions and type of a richer set of exotic branes in the $O(2,18;\mathbb{Z})$ U-duality orbit of a D7- or a D3-brane (see [11] for a recent discussion of exotic branes in string theory). Indeed, in the same way that $\tau$ is sourced by D7-branes, $\sigma$ is sourced by D3-branes and therefore the general U-fold solutions describe systems of 3- and 7-branes non-perturbatively completed by brane instantons. We remark that at generic points in the moduli space, the solutions are intrinsically non-perturbative so that only under very special circumstances one can give a perturbative D-brane description (see [10] for explicit choices of fibrations realizing systems of fractional D3 and D7 brane in type IIB theory). Like in F-theory, the presence of branes curves the plane $\mathbb{C}$ and when a maximal number is reached the plane is compactified to a sphere $S^2$ [12]. Here we limit ourselves to this compact case since we are interested in four-dimensional flux vacua.

It is important to observe that the space in which the scalars in the U-fold solution live, (1.1), is isomorphic to the moduli space of complex deformations of a K3 surfaces of elliptic type. This is not a coincidence. There is, in fact, an alternative way of building an $\mathcal{N} = 2$ vacuum by simply compactifying type IIB theory on a Calabi-Yau threefold given by a fibration of a K3 surface over $S^2$ with no fluxes. The U-duality group $SO(5,21,\mathbb{Z})$ of type IIB string theory on a K3 surface relates the two solutions, translating the geometry of the threefold into fluxes and vice versa. Indeed a point in the moduli space of compactifications of type IIB string theory on a K3 surface is specified by fixing a positive definite 5-plane $\Sigma_5$ in a 26-dimensional space with signature $(5,21)$. The orientation of this 5-plane determines the metric as well as the fluxes. In this language, both a K3 fibred Calabi-Yau threefold and the U-fold solution of [10] can be described by letting two of
the five vectors spanning \( \Sigma_5 \) vary within a subspace of signature \((2, 18)\). The orientation of the 2-plane is described by the Grassmannian \((1, 1)\). It is a pure matter of convention to identify the 2-plane with that specifying the geometry or fluxes.

In the spirit of F-theory, one can view the threefold (a K3 fibration over \( S^2 \)) underlying the dual geometric solution as the compactification manifold of a new theory, we dub as G-theory (with G standing for Gravity, or Geometry or Grandfather). Like in F-theory, the geometry of the threefold codifies the details of the flux solution. In particular, the holomorphic functions \( \varphi^I(z) \) describing how the complex structure of the K3 surface vary over the base will describe how the axio-dilaton field \( \tau(z) \), the warp factor/four-form field \( \sigma(z) \) and the two-form NS-NS and R-R potentials \( \beta^a(z) \) vary over \( S^2 \). The degenerations of the K3 fibre signal the presence of spacetime branes. Alternatively one can think of G-theory as a purely geometric lift of F-theory (or M-theory) with non-trivial \( G_4 \)-form fluxes \[13, 14\] and fivebrane charges. In figure [1] we display the general structure of the G-theory lift. We remark that, as in the case of F-theory, only the complex structure of the K3 fibre has a meaning in the realm of G-theory. The Kähler structure moduli can be thought of as being frozen to zero.

![Figure 1: G-theory lift. Coordinates \( z \) and \( w \) runs along the spacetime and the base of the auxiliary elliptic K3 surface respectively.](image)

Alternatively, one can think of the threefold as an elliptic fibration over a Hirzebruch surface \( F_n \) (a \( S^2 \) fibered over \( S^2 \) with winding number \( n \)). These geometries have been ex-
tensively studied in the past in the context of F-theory/heterotic duality in six-dimension. Indeed, the space (1.1) is also the moduli space of heterotic string theory on $T^2$. In this paper, we exploit the detailed knowledge of these geometries coming from the F-theory experience to produce U-fold solutions and explain how the 3- and 7-brane content of the solutions is codified in the threefold geometry. We discuss a number of examples where the explicit dictionary between geometry and fluxes can be worked out in detail. In particular, we find examples of non-geometric U-fold solutions exhibiting non-trivial T-duality monodromies exchanging 3- and 7-branes. These provide one of the simplest explicit realizations of T-folds in string theory (see [16–23] for previous works on T-folds). A related construction of string vacua where fluxes are codified into geometry has been recently developed in [24–28]. This approach has been pioneered in [29,30], see also [31,32] for an embedding of these ideas into a broader perspective. A more complete list of references can be found in [33–35].

The plan of the paper is the following: In Section 2.1 we review the construction of U-fold solutions in $\mathcal{N} = (2,0)$ supergravity and describe the details of the underlying threefold geometry. In Section 3 we discuss three examples of U-fold solutions associated to K3 fibrations with two or three active complex structure moduli. These geometries are very well studied in the context of F-theory/heterotic duality and are associated to K3 fibres with singularities of type $E_8 \times E_8$, $E_8 \times E_7$ and $D_4$ respectively. Appendix A collects some background material and details on elliptic K3 surfaces.

2 U-fold solutions

2.1 The supergravity solution

Here we review the U-fold construction in [10] of $\mathcal{N} = 2$ supersymmetric solutions of type IIB supergravity (see [36] for local versions of these solutions). We consider type IIB supergravity on a warped space with topology $\mathbb{R}^{1,3} \times Y$ a varying axio-dilaton field and non-trivial NS-NS and R-R fluxes. The ten-dimensional metric is given in the Einstein frame by

$$ds_E^2 = e^{2A} dx^\mu dx_\mu + e^{-2A} ds_Y^2$$

with $A$ a warp factor,

$$ds_Y^2 = (e^{-\phi}|h(z)|^2 dzd\bar{z} + ds_X^2)$$

$^4$J.F.M. thanks D. Waldram for an interesting discussion on this point.

$^5$Stringy set ups where D3- and D7-branes are identified have been realized in the past in terms of asymmetric orbifolds involving the quotient by the T-duality action on all four coordinates [15].
| generator | non-trivial action |
|-----------|--------------------|
| $S$       | $\tau \to -\frac{1}{\tau}$ $\sigma \to \sigma - \frac{1}{2\tau}\Delta_{ab}\beta^a\beta^b$ $\beta^a \to \frac{1}{\tau}\beta^a$ |
| $T$       | $\tau \to \tau + 1$ |
| $W_a$     | $\beta^b \to \beta^b + \delta_b^a$ |
| $R$       | $\tau \leftrightarrow \sigma$ |

Table 1: Generators of the $SO(2, 2 + n, \mathbb{Z})$ U-duality group

and $ds^2_X$ the Ricci flat metric on a K3 surface $X$. Here $z$ is a coordinate on the complex plane $\mathbb{C}$ with punctures at the positions of branes, $h(z)$ is a holomorphic function (away from the punctures) and $\phi$ is the dilaton field. The surface $Y$ admits complex, $\Omega$, and Kähler, $J$, structures

$$
\Omega = h(z)dz \wedge w \quad J = j - \frac{i}{2}e^{-\phi}\left|h(z)\right|^2dz \wedge d\bar{z}
$$

satisfying the Calabi-Yau conditions

$$
d\Omega = dJ = 0
$$

Here $\Omega$ and $J$ are the complex and Kähler structures of the K3 surface. As we explained before, the space $Y$ can be compact or non compact depending on the number of branes but here we limit ourselves to the compact case where the number of branes is maximal and $\mathbb{C}$ is compactified to a $S^2$.

NS-NS and R-R potentials are conveniently packed into a set of complex functions

$$\tau(z) = C_0 + ie^{-\phi}$$

$$\sigma(z) = \int_X{(C_4 - \frac{i}{2}e^{-4A}J \wedge J + B \wedge C_2 + \frac{1}{2}T B \wedge B)}$$

$$\beta^a(z) = \int_{C_a}(C_2 + \tau B)$$

varying holomorphically on $z$ (away from punctures), i.e.

$$\bar{\partial}\tau = 0, \quad \bar{\partial}\sigma = 0, \quad \bar{\partial}\beta^a = 0
$$

The solutions $\varphi^I = (\tau, \sigma, \beta^a)$, parametrize the moduli space $\{1.1\}$. $\Gamma_{2,18} \approx \Gamma_{2,2} \oplus \Gamma_{16}$ is the lattice orthogonal to the three plane $\Sigma \in \Gamma_{5,21}$ defined by a choice of a hyper Kähler structure $(J, \Omega, \bar{\Omega})$ on $K3$. The cycles $C_a$ in $\{2.5\}$ are associated to $\Gamma_{16}$ with intersection matrix $\Delta_{ab} = \int_X{[C_a] \wedge [C_b]}$ given in terms of the Poincaré dual forms $[C_a]$.

We summarize in Table $\{1\}$ the action of the U-duality group $O(\Gamma_{2,18}) \sim O(2, 18; \mathbb{Z})$ on the fields $\varphi^I$. U-fold solutions are defined in terms of a set of holomorphic functions $\varphi^I(z)$ on the (compactified) complex plane $\mathbb{C}$ with non-trivial U-duality monodromies around
the punctures. As we explained in the introduction, the functions \( \varphi^I(z) \) can be codified by the geometry of a Calabi-Yau threefold given by an elliptic K3 surface fibered over the complex plane, \( \mathbb{C} \) (or its compactification, \( S^2 \)). In the following we describe the details of these geometries and the flux/geometry dictionary.

### 2.2 U-folds as K3 fibrations

The geometry we have in mind is displayed in figure 1. It is made out of an auxiliary K3 manifold fibered over the space \( S^2 \) (or \( \mathbb{CP}^1 \)) in such a way as to produce a Calabi-Yau threefold. As we want the K3 fibre to be elliptic, this Calabi-Yau threefold can also be viewed as an elliptic fibration over a Hirzebruch surface, \( F_n \), made out of the sphere, \( S^2 \), and the base of K3 (another \( S^2 \)). As the axio-dilaton field is part of the complex structure moduli of the K3 fibre, F-theory compactifications will appear as a special subset of our solutions, where only the complex structure of the torus fibre inside the auxiliary K3 varies over the spacetime sphere.

The Calabi-Yau threefold can be described as a hypersurface (given by a homogeneous equation \( \Sigma \)) in a toric ambient space \( (\mathbb{C}^7 - Z) / \mathbb{C}^* \) described by the weight system

\[
\begin{array}{cccccccc}
\Sigma & y & x & u & w_0 & w_1 & z_0 & z_1 \\
6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\
12 & 6 & 4 & 0 & 1 & 1 & 0 & 0 \\
12 + 6n & 6 + 3n & 4 + 2n & 0 & 0 & n & 1 & 1 \\
\end{array}
\]

(2.7)

It is simply given by the Weierstrass equation

\[
\Sigma : \quad y^2 = x^3 + x f u^4 + g u^6,
\]

(2.8)

with

\[
f = f_{8,8+4n}(\vec{w}, \vec{z}) \quad g = g_{12,12+6n}(\vec{w}, \vec{z})
\]

(2.9)

polynomials of the indicated degrees with respect to the last two \( \mathbb{C}^* \)-actions in (2.7). The Weierstrass equation (2.8) defines a torus at each point \( u, \vec{w}, \vec{z} \). The coordinates \( \vec{w}, \vec{z} \) span two spheres fibered with winding number \( n \), i.e. a Hirzebruch surface \( F_n \). When we are in a patch where \( u, w_0, z_0 \) are non-vanishing, the \( \mathbb{C}^* \) identifications may be used to fix these coordinates to 1. We will always work in this chart, so that \( w_1 = w, z_1 = z \) become affine coordinates on \( F_n \) while the \( (x, y) \) coordinates satisfying (2.8) describe a torus.

Alternatively one can think of the threefold as a K3 surface fibered over a base \( S^2 \) parametrized by \( z \). Over any point \( z \) in the base of this fibration the complex structure of the K3 is determined by 18 complex parameters. This can be seen from (2.8) as follows.
For a fixed \( z \), \( f \) and \( g \) are polynomials in \( w \) of degree 8 and 12, respectively. These polynomials have \( 9 + 13 = 22 \) coefficients. We may use the \( SL(2, \mathbb{C}) \) and the rescaling symmetry to fix 4 of these coefficients to arbitrary values, so that only 18 independent moduli remain. In the Calabi-Yau threefold, these 18 complex parameters are given by the holomorphic functions \( \varphi^I(z) \) varying over the base. We identify these functions with the profiles of the NS-NS, R-R forms, warp factors and axio-dilaton, i.e. \( \varphi^I = (\sigma(z), \tau(z), \beta_a(z)) \).

The functions \( \varphi^I(z) \) are determined by the periods of the holomorphic two form

\[
\Omega = \frac{dx \wedge dw}{\sqrt{x^3 + fx + g}}. \tag{2.10}
\]

In practice, an explicit evaluation of \( \varphi^I(z) \) is in general technically involved, so that we limit our discussion to specific choices of \( f \) and \( g \) in the following. Still even for simple choices of \( f \) and \( g \), the physics of the corresponding U-fold solution is very rich and illustrates already the main features of the general solution. In particular, special choices of \( f \) and \( g \) correspond to a restriction to a smaller number of active moduli and therefore to a subgroup of the U-duality group.

In the following we will discuss in details some simple examples of Calabi-Yau threefolds where the moduli/flux dictionary can be worked out quite explicitly.

3 Examples

The elliptic K3 surfaces we introduce are characterized by (blow ups of) singularities of higher rank. We have collected some background on elliptic K3 surfaces in the appendix. We recall that the lattice of integral two-cycles of a K3 surface, \( \Gamma_{K3} \), can be always written as

\[
\Gamma_{K3} = U \oplus U \oplus U \oplus E_8 \oplus E_8 \tag{3.1}
\]

with \( U \) a two dimensional lattice with intersection matrix \((01)\). The intersection \( \text{Pic} \equiv H^{1,1}(K3) \cap H^2(K3, \mathbb{Z}) \) is known as the Picard lattice of the K3 surface. For elliptic K3 surfaces with a section (the case we have in mind) the Picard lattice always contains the \( U \) lattice (associated to the base and the fibre) and the lattice describing the blow up cycles at the singularity. The orthogonal complement of the Picard lattice is known as the transcendental lattice of the K3 surface and will be denoted by \( T_X \). It gives rise to the space of complex deformations of the K3 surface which leave the Picard lattice fixed. These moduli will be identified with the active six-dimensional scalars in the U-fold solution.
Table 2: The examples of K3 surfaces with few complex structure deformations considered. Here, $U(2)$ denotes a lattice with intersection matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $(n)$ denotes a one-dimensional lattice generated by a vector squaring to $n$.

Table 2 summarizes the main three working examples. The relevant geometries are discussed in detail in the appendix. The first and third cases are parametrized by two complex structure moduli denoted $(\tau, \sigma)$ that will be identified with the axio-dilaton and warp/four-form fields in type IIB theory. The second example includes also a non-trivial NS-NS/R-R three form flux.

In the case with two moduli, the space of the complex structure deformations of the K3 surface $X$ is

$$
\mathcal{M} = \frac{O(T_X) \backslash O(2, 2; \mathbb{R})}{O(2; \mathbb{R}) \times O(2; \mathbb{R})} \approx O(T_X) \backslash \left( \frac{SL(2; \mathbb{R})}{U(1)} \right)^2.
$$

(3.2)

with $O(T_X)$ a subgroup of the U-duality group

$$
O(\Gamma_{2,2}) = SO(2, 2, \mathbb{Z}) \approx \mathbb{Z}_2 \times SL(2, \mathbb{Z})_\tau \times SL(2, \mathbb{Z})_\sigma
$$

(3.3)

The $\mathbb{Z}_2$ acts by permuting the two factors and it is the generator $R$ in Table 1. Besides this $\mathbb{Z}_2$ action, the space (3.2) describes the complex structures of two factorized tori. In the case for which $\text{Pic} = U \oplus D_4^{\oplus 4}$, we will see that the $\mathbb{Z}_2$ holonomies are trivial. Therefore one can think of this fibration as a double elliptic fibration where at each point $z$ we have two factorized tori with complex structures $\sigma$ and $\tau$. On the other hand for $\text{Pic} = U \oplus E_8 \oplus E_8$, there are non-trivial $\mathbb{Z}_2$ monodromies around the singularities in the $z$-plane and therefore the 3- and 7-branes are consistently identified. Indeed, going around these points the 3-branes are exchanged with the 7-branes and the solution gets back to itself up to an action of the T-duality element $R$ of Table 1. As we discuss in detail later, the distinction between these two cases can be traced back to the different transcendental lattices of the corresponding K3 surfaces.

In the case of a K3 surface $X$ with three moduli the space of complex structure deformations is

$$
\mathcal{M} = \frac{O(T_X) \backslash O(2, 3; \mathbb{R})}{O(2; \mathbb{R}) \times O(3; \mathbb{R})}.
$$

(3.4)

with $O(T_X)$ a subgroup of the U-duality group

$$
O(\Gamma_{2,3}) = SO(2, 3, \mathbb{Z}) \sim Sp(4, \mathbb{Z}).
$$

(3.5)
Notice that $Sp(4,\mathbb{Z})$ is also the modular group of a genus two Riemann surface with period matrix $\Omega = \begin{pmatrix} \sigma \beta & 
abla \\
abla & \beta \sigma \end{pmatrix}$. In these cases, the elliptic fibration can be thought of as a fibration of a genus two Riemann surface over $S^2$ \cite{10}. The threefold now describes a system containing both regular and fractional 3- and 7-branes or equivalently non-trivial NS-NS $H_3$ and R-R $F_{1,3,5}$ fluxes. Again non trivial $R$-monodromies will occur signalling the non-geometric nature of the U-fold solution.

### 3.1 $E_8 \times E_8$ fibration: two moduli

Let us consider the following Calabi-Yau hypersurface defined by the Weierstrass equation (2.8) with

$$f = a_8(z)w^4$$

$$g = d_{12-n}(z)w^7 + b_{12}(z)w^6 + d'_{12+n}(z)w^5.$$  \hspace{1cm} (3.6)

Here, the subscripts denote the degree of the polynomials in the variable $z$. The discriminant is $\Delta = w^{10}P_4(w)$ with $P_4(w)$ a polynomial of order four in $w$. This K3 surface is singular at $w = 0$ and $w = \infty$ with both singularities of type $E_8$ according to the Kodaira classification (see Table 3 in the Appendix). The moduli space of complex structure deformations of the K3 surface (3.2) can be characterized by \cite{37,38}

$$-\frac{a^3}{27dd'} = j_1j_2$$

$$\frac{b^2}{4dd'} = (j_1 - 1)(j_2 - 1)$$  \hspace{1cm} (3.7)

with

$$j_i = \frac{j(\tau_i)}{1728} \quad \tau_i = (\sigma, \tau)$$  \hspace{1cm} (3.8)

and $j = \frac{E_3^3}{E_6}$. Equivalently one may write

$$j_{1,2} = \frac{P \pm \sqrt{P^2 + 1728 a^3 d d'}}{216 d d'}$$  \hspace{1cm} (3.9)

with $P = 108 dd' - 4a^3 - 27b^2$. The functions $\tau_i(z)$ can be determined from the $j_i(z)$ given by (3.9) up to U-duality frame rotations (3.3). It is important to notice that the functions $j_i(z)$ defined in (3.9) are not single valued. Indeed, going around the zeros of

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6 In the stable degeneration limit ($b_{12}$ and $a_8$ going to infinity keeping $a^3/b^2$ fixed), this geometry describes the F-Theory dual of the heterotic string compactified on $T^2$ in the absence of Wilson lines and at large volume \cite{39}. A fibration of this K3 manifold away from the stable degeneration limit has been used in \cite{23} to give a F-theory description of a non-geometric heterotic compactification.

7 Here $E_4 = \frac{1}{2}(\theta_2^8 + \theta_3^8 + \theta_4^8)$ and $E_6 = \frac{1}{2}(\theta_2^6 + \theta_3^6)(\theta_2^4 + \theta_3^4)(\theta_2^4 - \theta_3^4)$ satisfy the identity $E_4^3 - E_6^2 = 1728 \eta^{24}$. 

---
the polynomial under the square root, \( j_1(z) \) and \( j_2(z) \) get exchanged, i.e. \( \sigma \leftrightarrow \tau \). This \( \mathbb{Z}_2 \) monodromy is however part of the U-duality group and therefore the functions assuming values in the quotient space (3.2) are single valued. The U-fold solution may be then viewed as a T-fold where 3- and 7-branes are identified.

Finally, one can determine the locations of the branes by looking for those points in the \( z \)-plane where one of the torus fibres degenerates, let us say \( j_1 \to \infty \). This happens at the 24 zeros of the polynomial \( dd' \). At these points, writing \( dd' \sim z - z_0 \) one finds

\[
\tau_1(z) \approx -\frac{1}{2\pi i} \log j_1(z) \approx \frac{1}{2\pi} \log(z - z_0),
\]

so that \( \tau \to \tau + 1 \) while the \( \sigma \) modulus stays finite. We hence have 24 7-branes. We notice that an equivalent solution with only 3-branes can be written by flipping the signs in (3.9). The two solutions are clearly equivalent since 3- and 7-branes are identified.

We remark that monodromies can be alternatively extracted by looking at the geometry of the K3 fibre near its degeneration points. As explained in appendix A.2.1, the transcendental lattice of a K3 surface given by (3.6) is \( U \oplus 2 \). Choosing an integral basis of cycles \( \alpha_i \), we may write the holomorphic two-form as

\[
\Omega = \tau \alpha_1 + \sigma \alpha_2 + \alpha_3 - \tau \sigma \alpha_4.
\]

In particular, this lattice contains a cycle \( \alpha_1 - \alpha_2 \) which squares to \(-2\), i.e. it can be represented by an \( S^2 \). Whenever \( \tau = \sigma \), this cycle collapses. As can be seen from (3.9), this happens precisely when \( P^2 + 1728 a^3 d d' = 0 \). A collapsing \( S^2 \) gives rise to the Picard-Lefschetz monodromy (see appendix for details)

\[
\alpha_1 \leftrightarrow \alpha_2,
\]

i.e. the roles of \( \tau \) and \( \sigma \) are exchanged. The same result is obtained upon collapsing \( \alpha_3 - \alpha_4 \). This argument shows that the presence of the T-duality transformation is linked to the existence of a cycle squaring to \(-2\) in the transcendental lattice. Note that such a cycle is not necessarily present in any choice of transcendental lattice we can make. In fact, the example presented in section 3.3, where the transcendental lattice is \( U(2) \oplus 2 \), does not allow a Picard-Lefschetz monodromy. Correspondingly, there is no monodromy exchanging 7 with 3-branes there.

Summarizing, the \( E_8 \times E_8 \) K3 fibration codifies a non-geometric U-fold solution with non-trivial T-duality monodromies. Consistently, the resulting string vacuum does not admit a weak coupling description in terms of D-branes as expected from the underlying exceptional symmetry.
3.2 \(E_8 \times E_7\) fibration: three moduli

A deformation of the elliptic threefold presented in the last section is given by

\[
\begin{align*}
f &= a_8(z) w^4 + c_{8+n}(z) w^3 \\
g &= d_{12-n}(z) w^7 + b_{12}(z) w^6 + d'_{12+n}(z) w^5
\end{align*}
\tag{3.13}
\]

As before this threefold is given by a fibration of an elliptic K3 surface \(X\) over \(S^2\). This K3 surface is a one-parameter deformation of the K3 surface with two singularities of type \(E_8\) discussed in the last section, so that it has 3 complex structure moduli. It has two singularities of the types \(E_7\) and \(E_8\) at \(w = 0\) and \(w = \infty\), respectively. The moduli space of complex structure deformations is given by (3.4). We notice that this moduli space is isomorphic to the space of complex structures for a Riemann surface of genus two whose period matrix is

\[
\Pi = \begin{pmatrix}
\tau & \beta \\
\beta & \sigma
\end{pmatrix}.
\tag{3.14}
\]

The U-duality group is then identified with the genus two modular group \(Sp(4, \mathbb{R})\). The precise map between \(a, b, c, d, d'\) and the modular forms of the genus two surface has been worked out in [37]

\[
a_8 = -\frac{\psi_4}{3}, \quad b_{12} = \frac{2\psi_6}{27}, \quad c_{8+n} d_{12-n} = 2^{12} \chi_{10}, \quad d_{12-n} d'_{12+n} = 2^{12} \chi_{12}
\tag{3.15}
\]

with \(\psi_4, \psi_6\) the genus two Einstein series of weight 4, 6 and \(\chi_{10}, \chi_{12}\) the cusp forms of weight 10, 12 (see the Appendix in [10] for a self-contained review of genus two Riemann surfaces and modular forms). We notice that the degrees of the \(z\)-polynomials in the left hand side of (3.15) are twice the weight of the corresponding modular form in the right hand side as expected from the topological analysis in [10]. The case \(E_8 \times E_8\) correspond to the choice \(c = 0\) (or \(\beta = 0\)) where the genus two surface factorizes into a product of two tori

\[
\psi_4 \rightarrow E_4(\sigma) E_4(\tau) \quad \psi_6 \rightarrow E_6(\sigma) E_6(\tau) \\
\chi_{10} \rightarrow 0 \quad \chi_{12} \rightarrow \eta^{24}(\sigma) \eta^{24}(\tau)
\tag{3.16}
\]

and equations (3.15) reduce to (3.7). If one sets

\[
q_1 = e^{2\pi i \tau}, \quad q_2 = e^{2\pi i \sigma}, \quad y = e^{2\pi i \beta}
\tag{3.17}
\]

and uses the expansions (for small values of \(q_1, q_2\)) of the cusp forms

\[
\begin{align*}
\chi_{10} &= \frac{(1 - y)^2}{4y} q_1 q_2 \ldots \\
\chi_{12} &= \frac{(1 + 10y + y^2)}{12y} q_1 q_2 \ldots
\end{align*}
\tag{3.18}
\]

\[
\text{In fact, this K3 surface has a `Shioda-Inose structure' [40], i.e. it is the double cover of a Kummer surface which is constructed from a genus two curve [41].}
\]
one finds now \( q_1 q_2 y^{-1} \sim (z - z_0) \) near the 24 zeros of \( dd' \). Going around these points one finds that the combination \( \sigma + \tau - \beta \) is shifted by one, signalling for the presence of a brane of one of the three types. Locally, one can always choose a frame so that the brane is a D7 or a D3 brane but globally the total charge should add to zero so exotic branes should be always present. Similarly, singularities occur at the 20 zeros of \( cd \) where \( \chi_{10} \) vanishes. The structure of the singularity is more involved and a detailed description of the brane content for a general choice of the \( a, b, c, d, d' \) polynomials is a challenging task that goes beyond the scope of this paper.

Still some information can be extracted again from the geometry of the transcendental lattice \( U^{\oplus 2} \oplus (-2) \). We may write:

\[
\Omega = \tau \alpha_1 + \sigma \alpha_2 + \alpha_3 - (\tau \sigma - \beta^2) \alpha_4 + \alpha_5 \beta.
\]  

(3.19)

Note that we have adjusted the coefficient of \( \alpha_4 \) to maintain \( \int_{K^3} \Omega \wedge \Omega = 0 \). In particular we find again that there is a monodromy exchanging \( \tau \leftrightarrow \sigma \) coming from the collapsing cycle \( \alpha_1 - \alpha_2 \). Now, however, there are further cycles of self-intersection \( (-2) \), e.g. \( \alpha_5 \), giving rise to a Picard-Lefschetz monodromy. A systematic study of the richer geometry of this fibration would be very welcome.

### 3.3 \( D_4^4 \)-fibration: two moduli

#### 3.3.1 The \( F_0 \)-case

Next we consider an elliptic \( K3 \) surface given by a Weierstrass model at the sublocus of the moduli space where

\[
f_{8,8}(w, z) = \alpha_{2k}(z) h_{4,4-k}(w, z)^2
\]

\[
g_{12,12}(w, z) = \beta_{3k}(z) h_{4,4-k}(w, z)^3.
\]

(3.20)

The subscripts of the various polynomials here denote their degrees in the variables \( w, z \). At each point \( z \), the K3 fibre is singular in four points \( w_i \), the zeros of \( h \). According to Kodaira’s classification of Table \( \text{3} \) they correspond to four \( D_4 \) singularities.

The holomorphic two-form \( (2.10) \) factorizes into

\[
\Omega = \frac{dx'}{\sqrt{x'^3 + \alpha x' + \beta}} \wedge \frac{dw}{\sqrt{h}},
\]

(3.21)

with \( x \equiv x' h \). On the other hand for \( h(w, z) \) one may write

\[
h = a_4 w^4 + 4a_3 w^3 + 6a_2 w^2 + 4a_1 w + a_0,
\]

(3.22)
with the $a_i$’s being polynomials of degree $(4 - k)$ in $z$. Since $h$ is quartic in $w$, one can identify $\frac{dw}{\sqrt{h}}$ with the differential form of a torus with defining equation $\xi^2 = h$, or in its Weierstrass form

$$\xi^2 = w^3 + \bar{\alpha}w + \bar{\beta},$$

(3.23)

with

$$\bar{\alpha} = 4(a_1a_3 - a_4a_0 - 3a_2^2) \quad \bar{\beta} = 16(a_4a_1^2 + a_0a_3^2 + a_2^4 - a_4a_2a_0 - 2a_1a_2a_3).$$

(3.24)

Notice that $\bar{\alpha}$ and $\bar{\beta}$ are polynomials of degree $(8 - 2k)$ and $(12 - 3k)$ respectively. One can then identify $(\sigma, \tau)$ as the complex structures of two tori. Explicitly

$$j(\tau) = \frac{4(24\bar{\alpha})^3}{4\bar{\alpha}^3 + 27\bar{\beta}^2} \quad j(\sigma) = \frac{4(24\bar{\alpha})^3}{4\bar{\alpha}^3 + 27\bar{\beta}^2}.$$  

(3.25)

The discriminants of the two tori are given by

$$\Delta_\tau = 4\alpha^3 + 27\beta^2 \quad \Delta_\sigma = 4\bar{\alpha}^3 + 27\bar{\beta}^2.$$  

(3.26)

We notice that $\Delta_\tau$ and $\Delta_\sigma$ are polynomials of order $6k$ and $(24 - 6k)$ respectively with zeros at the positions of the 3- and 7-branes. We conclude then that there are 24 branes in total, $6k$ 7-branes and $24 - 6k$ 3-branes. Finally we observe that, when a 7- and a 3-branes collide, i.e. for $\Delta_\tau = \Delta_\sigma = 0$ the full K3 fibre degenerates.

The transcendental lattice is now given by $U(2)^{\oplus 2}$ with intersection matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. Choosing an integral basis $\alpha_i, i = 1, \cdots, 4$, we may write

$$\Omega = \tau \alpha_1 + \sigma \alpha_2 + \alpha_3 - \tau \sigma \alpha_4.$$  

(3.27)

At first sight, this looks very similar to the situation with two $E_8$ singularities given by (3.11). However, the inner form on $T_X$ is now twice the one we had there and no cycle of self-intersection $-2$ is found. In particular, $\alpha_1 - \alpha_2$ does not correspond to a sphere, but it is a reducible cycle of self-intersection $-4$. This means that the K3 fibre is not singular when $\tau = \sigma$ and therefore the solutions with 3- and 7-branes remain distinct.

### 3.3.2 The $F_n$-case

It is straightforward to generalize the previous analysis to the case when the base is an $F_n$ Hirzebruch surface with $n > 0$. One takes again

$$f = \alpha(z) h(w, z)^2$$

$$g = \beta(z) h(w, z)^3,$$  

(3.28)

---

9Here for simplicity we are referring as 3-branes to all exotic R-duals of $(p,q)$ 7-branes.

10A K3 is singular if $P = \partial_y P = \partial_x P = \partial_w P = 0$ with $P = y^2 - x^3 - \alpha h^2 x - \beta$ the defining equation. The vanishing of $P = \partial_y P = \partial_x P$ follows from $\Delta_\tau = 0$ while $\partial_w P \sim \partial_w h = 0$ is implied by $\Delta_\sigma = 0$. 

14
but now with
\[\alpha(z) = \sum_{i=0}^{2k} \alpha_i z^i \quad \beta = \sum_{i=0}^{3k} \beta_i z^i \quad h(w, z) = \sum_{i=0}^{4} \sum_{j=0}^{4-k+(2-i)n} h_{i,j} w^i z^j. \] (3.29)

The only difference with the $F_0$ case is that now the coefficients $a_i$ entering in the expansion of $h(w)$ (3.22) are polynomials of degree $(4 - k + (2 - i)n)$ in $z$. Still the resulting $\tilde{\alpha}$ and $\tilde{\beta}$ are again given by polynomials of degree $(8 - 2k)$ and $(12 - 3k)$ respectively and therefore the number of branes does not depend on $n$.

### 3.3.3 Weak coupling limit

Finally we consider the special limit of the geometry where the axio-dilaton or four-form warping fields become almost constant and large along the $z$-plane. In this limit one expects that the vacuum admits a perturbative description in terms of D3, D7-branes and O3-, O7-planes. Indeed if one sets $k = 4$, the function $h$ becomes $z$-independent and therefore the complex structure $\sigma$ of the $w$-torus is constant everywhere. The threefold becomes $T^4/\mathbb{Z}_2 \times T^2$ and describes the standard F-theory compactification on K3. The cases $k = 0, 1, 2, 3$ are new in the sense that some stacks of six 7-branes are commuted into stacks of 3-branes. In analogy with F-theory, one can ask whether the moduli of the Calabi-Yau can be tuned in such a way that $\sigma$ and $\tau$ are both constant and large almost everywhere in the $z$-plane allowing for a perturbative description in type IIB theory. The answer is clearly yes, since we have two factorized tori, each of which can be treated in the same way that the elliptic fibre of the more familiar $D_4$ F-theory geometry. For example, for the $\tau$ torus, the weak coupling limit corresponds to take \[42\]
\[
\alpha = (-3\gamma^2 + \epsilon \delta)
\]
\[
\beta = (-2\gamma^3 + \epsilon \gamma \delta - \frac{1}{12} \epsilon^2 \chi),
\] (3.30)
with $\epsilon$ a small constant and $\gamma, \delta, \chi$ two homogeneous functions of degrees $k$, $2k$ and $3k$ respectively. Plugging this into (3.25) one finds
\[
j(\tau) = \frac{(24)^4}{2} \frac{\gamma^4}{\epsilon^2(\delta^2 - h\chi)}. \] (3.31)

In the limit $\epsilon \to 0$, $\tau \to i\infty$ almost everywhere except at the zeros of $\gamma$. Writing
\[
\gamma = \prod_{i=1}^{k} (z - \zeta_i^O) \quad (\delta^2 - h\chi) = \prod_{m=1}^{4k} (z - \zeta_m^D), \] (3.32)
we can identify then $\zeta_i^O$ as the position of the $k$ O7-planes and $\zeta_m^D$ as the positions of the $4k$ D7-branes. Indeed, $j(\tau)$ has zeros of order four at $z = \zeta_i^O$ and therefore going around
these points $\tau \to \tau - 4$ indicating the presence of an O7 plane at this point. Similarly one finds the holonomy $\tau \to \tau + 1$ around $z = \zeta_m^D$ indicating the presence of a D7-brane.

The analysis here can be repeated for the $\sigma$-torus leading to identical conclusions. One finds $(4 - k)$ groups of O3-planes (each group with charge -4) and $16 - 4k$ D3-branes. Functions $\sigma(z), \tau(z)$ codified in the geometry describe the exact running of the couplings in the 7-brane and 3-brane world-volume theories. Remarkably, like in the F-theory case, the full tower of multi-instanton corrections to these couplings can be extracted from this functional dependence.

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A Elliptic K3 surfaces

In this appendix, we collect some background material on elliptic K3 surfaces needed in the main text. We will not attempt to provide a self-contained or through discussion and we will simply state the relevant facts. Further details can be found in [43–45] and references therein.

In this work, we restrict ourselves to elliptic K3 surfaces described by a Weierstrass equation. Such K3 surfaces can be described as hypersurfaces in an ambient toric space with the weight system

\[
\begin{array}{cccccc}
 y & x & u & w_0 & w_1 \\
 3 & 2 & 1 & 0 & 0 \\
 6 & 4 & 0 & 1 & 1 \\
\end{array}
\] (A.1)

by an equation of the type

\[
y^2 = x^3 + xu^4f_8 + u^6g_{12}.
\] (A.2)

Here, $f$ and $g$ are homogeneous functions of $(w_0, w_1)$ of the indicated degree. From the homogeneity degree of the above equation and the weight system it follows that the hypersurface is a complex two-dimensional Calabi-Yau manifold, i.e. a K3 surface.

The fact that we have an elliptic fibration can also be instantly verified: if we fix any $[w_0 : w_1]$, (A.2) describes an elliptic curve, i.e. a torus. Hence the elliptic K3 surface is
given by a torus sitting over every point of an $S^2$. Furthermore, there is a (holomorphic) section: we may embed the base $S^2$ of the elliptic fibration by mapping it to the point $[x : y : u] = [1, 1, 0]$ in the fibre.

As it is true for any Calabi-Yau manifold, a K3 surface has a non-vanishing holomorphic top-form $\Omega$. By the method of residues (see e.g. [46]), the holomorphic two-form $\Omega$ of a K3 surface embedded in an ambient toric variety map be written as (see [47] for a nice derivation)

$$\Omega = \frac{1}{2\pi i} \oint_{P=0} \frac{w}{P} \cdot \prod_a V_a$$

(A.3)

with $w/P$ an invariant top form in the ambient space

$$w = dx\, dy\, du\, dw_0\, dw_1 \quad \quad P = y^2 - (x^3 + fx + g)$$

(A.4)

and

$$V_1 = 3y \partial_y + 2x \partial_x + u \partial_u$$

$$V_2 = 6y \partial_y + 4x \partial_x + w_0 \partial_{w_0} + w_1 \partial_{w_1}$$

(A.5)

the generators of the $C^*$'s actions in (A.1). In the chart where $u = w_0 = 1$, $w_1 = w$, one can then write

$$\Omega = \frac{1}{2\pi i} \oint_{P=0} \frac{dx\, dy\, dw}{y^2 - (x^3 + fx + g)} = \frac{dx\, dw}{\sqrt{x^3 + fx + g}}$$

(A.6)

The fibre of the elliptic fibration is smooth for a generic point of the base $S^2$. Over the 24 points where the discriminant

$$\Delta = 4f_8^3 + 27g_{12}^2$$

(A.7)

vanishes, however, the fibre degenerates by pinching the one cycle $(p, q)$. If we encircle one of those points, the fibre undergoes the $SL(2, \mathbb{Z})$ monodromy transformation $\begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$

As long as these degenerate fibres stay separate, the K3 surface stays smooth. When the polynomials $f_8$ and $g_{12}$ are such that two or more of these singular fibres go on top of each other, i.e. $\Delta$ has a double (or higher) root, also the K3 surface becomes singular. The singularities that occur in this way are nothing but the ADE (or simple surface, Kleinian, duVal) singularities. This is not unexpected, as these are precisely the orbifold singularities for which the orbifold group is a finite subgroup of $SU(2)$, the holonomy group of a K3 surface. The types of singular fibres that can occur and the corresponding ADE singularities are displayed in Table 3.

\[\text{In F-Theory, such degenerations give the locations of $(p, q)$ 7-branes.}\]
Table 3: Kodaira’s classification of bad fibres in terms of the vanishing degree of $f$, $g$ and $\Delta$. Also given is the corresponding monodromy and the type of surface singularity. This table has already appeared in [39].

| $\text{ord}(f)$ | $\text{ord}(g)$ | $\text{ord}(\Delta)$ | fibre type | singularity type | monodromy |
|-----------------|-----------------|-----------------------|------------|-----------------|-----------|
| $\geq 0$        | $\geq 0$        | 0                     | smooth     | none            | $(1 \ 0 \ 1)$ |
| 0               | 0               | $n$                   | $I_n$      | $A_{n-1}$       | $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ |
| $2 \geq 3$      | $n + 6$         | $I^*_n$               | $D_{n+4}$  | $- \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ |
| $\geq 2$        | 3               | $n + 6$               | $I^*_n$    | $D_{n+4}$       | $- \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ |
| $\geq 1$        | 1               | 2                     | $II$       | none            | $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ |
| $\geq 4$        | 5               | 10                    | $II^*$     | $E_8$           | $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ |
| 1               | $\geq 2$        | 3                     | $III$      | $A_1$           | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ |
| 3               | $\geq 5$        | 9                     | $III^*$    | $E_7$           | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ |
| $\geq 2$        | 2               | 4                     | $IV$       | $A_2$           | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ |
| $\geq 3$        | 4               | 8                     | $IV^*$     | $E_6$           | $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ |
The Euler characteristic of a K3 surface is 24. Besides the 0- and 4-form there are 20 harmonic forms of type (1, 1) as well as the holomorphic (2, 0) form and its complex conjugate. In the following, however, we will be mostly interested in the integer (co)homology \( H^2(K3, \mathbb{Z}) \). One might think of the integer homology as being generated by (homology classes of) submanifolds of the K3 surface via the natural pairing between homology and cohomology given by the integration. The integer cohomology classes \( H^2(K3, \mathbb{Z}) \) are those ones for which the integral over any submanifold gives an integer number, i.e. it is the dual lattice. By Poincaré duality, the two lattices are isomorphic. This implies we have an inner form \( \alpha \cdot \beta \) on both of them. Thinking in terms of homology, this number counts the geometric intersections between two representatives, in terms of cohomology this number is found by wedging the two forms and integrating over the whole K3 surface:

\[
\alpha \cdot \beta = \alpha \cap \beta = \int_{K3} \alpha \wedge \beta . \tag{A.8}
\]

Here, we both denote an element of \( H^2(K3, \mathbb{Z}) \) and its Poincaré dual in \( H^2(K3, \mathbb{Z}) \) by the same letter. With this inner form,

\[
H^2(K3, \mathbb{Z}) = U^\oplus 3 \oplus E_8^\oplus 2 , \tag{A.9}
\]

where \( U \) is the lattice with inner form \((01) \) and \( E_8 \) is the root lattice of \( E_8 \). We will refer to elements of this lattice as cycles. Given an element \( \gamma \) in \( H^2(K3, \mathbb{Z}) \) which can be represented by a Riemann surface of genus \( g \), the self-intersection number is simply

\[
\gamma \cdot \gamma = 2g - 2 . \tag{A.10}
\]

Hence a sphere will correspond to a lattice point with self-intersection \(-2\) and a torus will have self-intersection zero. Notice that this means that a cycle of self-intersection smaller than \(-2\) can never correspond to an irreducible submanifold.

The Picard lattice of a K3 surface \( X \) is defined as

\[
\text{Pic}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X) , \tag{A.11}
\]

so that it contains only integral cycles of type (1, 1). Its dimension is called the Picard number \( \rho \). Clearly, any two-cycle given by an algebraic equation becomes a member of the Picard lattice. By the Lefschetz theorem on (1, 1) classes, this statement also has a converse: the Picard lattice is generated by algebraic cycles.

The transcendental lattice is defined as the orthogonal complement of the Picard lattice in \( H^2(X, \mathbb{Z}) \):

\[
T_X = \text{Pic}^\perp \subset H^2(X, \mathbb{Z}) . \tag{A.12}
\]

More precisely we denote by \( E_8 \) the lattice with intersection matrix given by minus the Cartan matrix of the \( E_8 \) Lie algebra.
As it is familiar from Calabi-Yau threefolds, the moduli space of a K3 surface defined in terms of algebraic equations consists of Kähler and complex structure deformations. Using the Picard and the transcendental lattices we may write the Kähler from $J$ and the holomorphic two-form $\Omega$ as

$$J = \sum_{\delta_i \in \text{Pic}(X)} j_i \delta_i, \quad j_i \in \mathbb{R}$$  \hspace{1cm} (A.13)

$$\Omega = \sum_{\gamma_i \in T_X} w_i \gamma_i, \quad w_i \in \mathbb{C},$$  \hspace{1cm} (A.14)

where

$$\int_X \Omega \wedge \Omega = 0$$  \hspace{1cm} (A.15)

puts an extra constraint on the $w_i$. The mutual orthogonality of the Picard lattice and the transcendental lattice ensures that $J \cdot \Omega = 0$, as it should be. From (A.13), the periods

$$\pi_i = \int_{C_i \in H_2(X, \mathbb{Z})} \Omega,$$  \hspace{1cm} (A.16)

are determined by using (A.9). Fixing the Picard lattice, the moduli space of complex structures becomes the Grassmannian

$$O(T_X) \setminus \frac{O(2, 20 - \rho; \mathbb{R})}{O(2; \mathbb{R}) \times O(20 - \rho; \mathbb{R})},$$  \hspace{1cm} (A.17)

where $\rho$ denotes the Picard number and $O(T_X)$ denotes the isometries of the transcendental lattice.

A K3 surface develops a singularity of type ADE when there is two cycle $\alpha$ with $\alpha^2 = -2$ for which

$$J \cdot \alpha = \Omega \cdot \alpha = 0.$$  \hspace{1cm} (A.18)

Intuitively, this means that $\alpha$, which correspond to an $S^2$, has collapsed to a point. For this reason, cycles of this type are referred to as vanishing cycles. The vanishing cycles generate a root lattice and the lattices obtained in this way precisely match the ADE type of the corresponding singularity. If the lattice of vanishing cycles decomposes into a direct sum of root lattices, the corresponding K3 surface has distinct singularities of the corresponding types.

We have discussed the Picard and the transcendental lattices for smooth K3 surfaces above. In the singular case, it is customary in the mathematics literature (and natural from several viewpoints) to include the vanishing cycles in the Picard lattice. We can rephrase this in the following way: for any ADE singularity, there is a unique resolution which corresponds to a Kähler deformation. Hence we can use the corresponding resolved K3 surface instead of the singular one to define the Picard and transcendental lattices. But this simply means grouping the vanishing cycles with the Picard lattice.
When we consider fibrations of K3 surfaces, over an $S^2$ parametrized by $[z_0 : z_1]$, say, the complex structure of the K3 fibre varies from point to point. We may think of this either as making the coefficients appearing in $f_8$ and $g_{12}$ functions of $[z_0 : z_1]$ or as making the periods, i.e. the coefficients $w_i$ in (A.13), functions of $[z_0, z_1]$. From the latter perspective, it is clear that there will be (in general) points for which the K3 surface develops a singularity of ADE type. Encircling such a locus, there will be a monodromy transformation acting on $H^2(K3, \mathbb{Z})$. Since the vanishing cycles at these points is a sphere, the monodromy action can be determined from the intersection form through the Picard-Lefschetz formula (see e.g. [48]). If a cycle $\gamma$ shrinks at the point we are encircling, the induced map on any other cycle $\alpha$ is given by

$$\alpha \mapsto \alpha + (\alpha \cdot \gamma) \gamma.$$  \hspace{1cm} (A.19)

Note that, for root lattices, this is nothing but a Weyl reflection, which is a symmetry of the root lattice. This fits with the structure of the moduli space (A.17), in which the isometries of the transcendental lattice are modded out. Note however, that not all the isometries of the transcendental lattice are Weyl reflections of the transcendental lattice and the vanishing of cycles with topology different from the sphere can (and indeed) occur. In the text, we mainly determine the monodromy action from the explicit expression for the periods.

### A.1 Picard and transcendental lattices of generic elliptic K3 surfaces

For an elliptic K3 surface (with $f_8$ and $g_{12}$ chosen generically), the Picard lattice is generated by the base and the fibre of the elliptic fibration. Let us denote the divisor classes associated to the basic hyperplanes $x_i = 0$ by $D_{x_i}$. The invariance under $\mathbb{C}^*$ of $y^2/x^3$, $y^2/(u^6w^{12})$, and $w_0/w_1$ imply the relations

$$D_{w_0} = D_{w_1} = D_w \quad 2D_y = 3D_x = 6D_u + 12D_w$$  \hspace{1cm} (A.20)

To find the intersection numbers is convenient to think of the ambient space $W$ as the $U(1)^2$ quotient of the hyperplane defined by the $U(1)^2$ moment maps

$$3|y|^2 + 2|x|^2 + |u|^2 = \xi_1$$
$$6|y|^2 + 4|x|^2 + |w_0|^2 + |w_1|^2 = \xi_1$$ \hspace{1cm} (A.21)

Then it is easy to see that equations $x = y = u = 0$ and $w_1 = w_2 = 0$ have no solution while $x = y = w = 0$ has a single solution. This implies

$$\int_W D_w^3 = \int_W D_w^2 D_u = \int_W D_x D_y D_u = 0 \quad \int_W D_x D_y D_w = 1$$ \hspace{1cm} (A.22)
with \( W \) the threefold. Combining (A.20) and (A.22) one finds

\[
\int_W D^2_w = \int_W D^2_w D_u = 0 \quad \int_W D^2_u D_w = 1/6 \quad \int_W D^3_u = -2/3 \quad (A.23)
\]

The intersection on \( K3 \) can be found from the adjunction formula

\[
\int_{K3} \alpha = \int_W \alpha \wedge (6D_u + 12D_w) \quad (A.24)
\]

leading to

\[
\int_{K3} D^2_w = 0 \quad \int_{K3} D^2_u = -2 \quad \int_{K3} D_u D_w = 1. \quad (A.25)
\]

This is as expected: (A.2) defines a fibration of the torus \( w = 0 \) over the sphere \( u = 0 \), which is also a section of the fibration, i.e. it intersects every fibre in a single point. If we choose a different integral basis for the Picard lattice,

\[
\alpha_1 = D_w, \quad \alpha_2 = D_u + D_w, \quad (A.26)
\]

its inner form becomes \((0_1, 1_0)\). From this is follows that the Picard lattice is \( U \) and the transcendental lattice of the generic elliptic K3 surface is \( T_X = U^\oplus 2 \oplus E_8^\oplus 2 \) (the orthogonal complement of \( U \) inside (A.9)).

### A.2 Picard and transcendental lattices of our examples

In the following we describe in details the transcendental lattices for the three examples we discuss in this paper. We can restrict the complex structure moduli space (A.17) by enlarging the Picard number \( \rho \). As the examples we are considering have 2 or 3 complex structure moduli, the Picard numbers are 18 or 17. This can be achieved by considering elliptic K3 surfaces given by the blow up of an ADE singularity of high rank (16 or 15). As the blowups of the singularities we consider are unique and well known, we do not have to carry them out explicitly. We will now consider our set of examples in turn. From what we have said, it is natural to label the examples by their singularity structure.

### A.2.1 The \( E_8 \oplus E_8 \) case

Let us consider the elliptic K3 surface

\[
y^2 = x^3 + x a w^4 + (d w^7 + b w^6 + d' w^5) \quad (A.27)
\]

with discriminant

\[
\Delta = w^{10} P_4(w). \quad (A.28)
\]
Here,

\[ P_4 = 27d'^2 + 54d'bw + (4a^3 + 27b^2 + 54d'd)w^2 + 54dbw^3 + 27d^2w^4, \tag{A.29} \]

is a polynomial of order 4 in \( w \). Using the classification of Table 3 one finds two \( E_8 \) singularities over \( w = 0 \) and \( w = \infty \) and four \( A_0 \) regular points associated to the zeros of \( P_4(w) \), see Figure 2.

![Figure 2: A cartoon depicting an elliptic K3 surface with two singularities of type \( E_8 \).](image)

As we have seen already, the sublattice of the Picard lattice generated by the fibre and section of the elliptic fibration is \( U \). Together with the vanishing cycles of the two \( E_8 \) singularities we hence find that (A.27) has the Picard lattice

\[ \text{Pic} = U \oplus E_8 \oplus E_8. \tag{A.30} \]

The transcendental lattice is \( U^\oplus 2 \), i.e. the orthogonal complement of the Picard lattice (A.30) in \( U^\oplus 3 \oplus E_8^\oplus 2 \). Following [49], see also [50–52], we may construct this lattice as follows. Let us denote the four roots of \( P_4 \) by \( p_i, i = 1..4 \). The one-cycles in the \( T^2 \) fibre which collapse over these four points are pairwise the same, for \( p_1 \) and \( p_3 \) a cycle \( \phi_1 \) shrinks and for \( p_2 \) and \( p_4 \) a one-cycle \( \phi_2 \) shrinks. We may then choose a basis such that \( \phi_1 \) and \( \phi_2 \) are as depicted in Figure 2. Hence we may construct a two-cycle \( \gamma_1 \) by fibring \( \phi_1 \) over the interval \( \beta_1 \) connecting \( p_1 \) and \( p_3 \). This cycle has the topology of a two-sphere
and therefore its self-intersection number is $-2$. Similarly, a second sphere is made of the fibration of $\phi_2$ over the path $\beta_2$ connecting $p_2$ and $p_4$. Hence we may suggestively write

$$\gamma_1 = \beta_1 \wedge \phi_1 \quad \gamma_2 = \beta_2 \wedge \phi_2.$$  \hspace{1cm} \text{(A.31)}

Furthermore, the $SL(2,\mathbb{Z})$ monodromies of this configuration are such that the monodromy map induced along the loop $\beta_3$ is trivial\footnote{This can be seen from the $\mathbb{Z}_2$ symmetry which exchanges the internal and external regions of the sphere surround by $\beta_3$. It implies that the monodromy around $\beta_3$ should coincide with its inverse.}. Hence we can fibre any one-cycle in the elliptic fibre over $\beta_3$ to obtain a two-cycle. Using the same basis of cycles in the fibre as before, we can hence form the two cycles

$$\alpha_1 = \beta_3 \wedge \phi_2 \quad \alpha_3 = \beta_3 \wedge \phi_1,$$  \hspace{1cm} \text{(A.32)}

so that

$$\alpha_1 \cdot \gamma_1 = 1 \quad \alpha_3 \cdot \gamma_2 = 1.$$  \hspace{1cm} \text{(A.33)}

As the fibration of the elliptic fibre along $\beta_3$ is trivial, these cycles are just $S_1 \times S_1$, i.e. they are two-tori, so that

$$\alpha_1^2 = \alpha_3^2 = 0.$$  \hspace{1cm} \text{(A.34)}

Finally, we may define $\alpha_2 = \gamma_1 + \alpha_1$, $\alpha_4 = \gamma_2 + \alpha_3$ so that the four two-cycles $\alpha_i$ have the intersection form

$$T_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = U^\oplus 2.$$  \hspace{1cm} \text{(A.35)}

For special values of $a, b, d, d'$, some of the roots of $P_4$ may come together, so that the $K3$ surface develops further singularities. For example when $d'$ goes to zero, $p_1$ and $p_3$ will go to zero in the $w$-plane and so the cycle $\beta_3$ is pinched. The same is true for $d$ going to zero where $p_2$ and $p_4$ go to infinity. On the other hand when $a = 0$, $P_4$ becomes a square, i.e. the $p_i$ will move pairwise on top of each other. When $p_1$ and $p_2$ or $p_3$ and $p_4$ coincide, $\gamma_1$ or $\gamma_2$ collapse. For instance when $\gamma_1 = \alpha_2 - \alpha_1$ collapses one finds the Picard-Lefschetz monodromy

$$\alpha \mapsto \alpha + (\alpha \cdot \gamma_1) \gamma_1.$$  \hspace{1cm} \text{(A.36)}

inducing the map

$$\alpha_1 \leftrightarrow \alpha_2.$$  \hspace{1cm} \text{(A.37)}

\textbf{A.2.2 The $E_8 \oplus E_7$ case}

Next, we consider the elliptic $K3$ surface

$$y^2 = x^3 + x(a w^4 + c w^3) + (d w^7 + b w^6 + d' w^5)$$  \hspace{1cm} \text{(A.38)}
with discriminant
\[ \Delta = w^9 P_5(w), \] (A.39)
where
\[ P_5 = -4c^3 - 12ac^2 w - 12a^2 cw^2 - w \left( 27d^2 + 54d'w(b + dw) + w^2(4a^3 + 27(b + dw)^2) \right) \] (A.40)
is a polynomial of order 5 in \( w \). According to Table (3) one finds one \( E_8 \) singularity at \( w = \infty \), one \( E_7 \) singularity at \( w = 0 \) and five regular points at the zeros of \( P_5(w) \). We have depicted the situation in figure 3.

![Diagram of K3 surface with one E8, one E7 singularity and five A0 regular points.](image)

**Figure 3:** K3 surface with one \( E_8 \), one \( E_7 \) singularity and five A0 regular points.

The Picard lattice now becomes
\[ \text{Pic} = U \oplus E_8 \oplus E_7 \] (A.41)
so that the transcendental lattice must be five dimensional.

This configuration may be obtained from the situation with two \( E_8 \) singularities by a one-parameter deformation. The four cycles \( \alpha_i \) constructed in the last section remain present and there is a single new cycle \( \alpha_5 \) in the transcendental lattice. From the explicit construction of the vanishing cycles of an \( E_8 \) singularity in an elliptic surface [50], one can infer that the cycle which is deformed to finite volume simply connects \( p_5 \) to the remaining \( E_7 \) singularity, i.e. it is a \( S^2 \) which is dual to \( \beta_4 \wedge \phi_1 \). Even though this cycle measures
the deformation of an $E_8$ singularity to an $E_7$, it is not part of the transcendental lattice since it intersects with the vanishing cycles of the $E_8$ singularity which are part of the Picard lattice.

On the other hand, $\gamma = \beta_5 \wedge \phi_1$ does not intersect any cycle of the Picard lattice, so that it is contained in the transcendental lattice. It has the following non-vanishing intersection numbers

$$\gamma^2 = -2, \quad \gamma \cdot \alpha_2 = 1.$$  

(A.42)

We can choose $\alpha_i, i = 1, 2, 3, 4$ and $\alpha_5 \equiv \gamma - \alpha_1$ as the generators of the transcendental lattice. In this basis its inner form is

$$T_X = U^\oplus 2 \oplus (-2).$$  

(A.43)

The K3 surface (A.38) has in this case a very rich pattern of degenerations. Whenever $c = 0$, we get back to the case with $E_8 \times E_8$ singularities, i.e. the cycle $\gamma$ collapses. The remaining singular point can be identified by studying the discriminant of (A.40). A systematic study is beyond the scope of this paper.

A.2.3 The $D_4$ case

A situation with four singularities of type $D_4$ is given by

$$y^2 = x^3 + x^2 h^2 + \beta h^3$$  

(A.44)

with $h(w)$ a polynomial of order 4 in $w$. The discriminant is

$$h^6 (4\alpha^2 + 27\beta^2)$$  

(A.45)

Using again Table (3), one can see that the K3 surface has singularities of type $D_4$ at the four roots of $h(w)$. Hence the Picard lattice contains $U \oplus D_4^\oplus 4$.

(A.46)

The transcendental lattice is then four-dimensional and can be constructed as follows (see [53] for a similar discussion). As the monodromy induced around the four $D_4$-singular points fibres is $-1$, any path in the base which encircles two such points will give trivial monodromy. A basis of such paths are, let us say $\beta_1$ and $\beta_2$, is depicted in Figure 4. A basis of two-cycles is built by taking an arbitrary one-cycle in the fibre $\phi_{1,2}$ and transporting it

\footnote{It turns out that this is not all of the Picard lattice, which also contains a few integral cycles which are linear combinations of the elements of the lattices $U$ and $D_4^\oplus 4$. As this subtlety is irrelevant for our analysis, we do not dwell on this further.}
along $\beta_1$ or $\beta_2$. Using dual forms instead of cycles (which be denote by the same letters) we can write
\[ \alpha_{ij} = \beta_i \wedge \phi_j \quad i, j = 1, 2 \] (A.47)

We notice that the cycles associated to $\alpha_{ij}$ have self-intersection zero since two homologous cycles differing by the choice of a different representative for $\phi_1$ or $\phi_2$ do not intersect. On the other hand $\alpha_{12} \cdot \alpha_{21} = \alpha_{11} \cdot \alpha_{22} = 2$. The associated cycles intersect twice as can be seen in Figure 4 ($\beta_1$ and $\beta_2$ meet in two points with opposite orientation but the orientations of $\phi_1$ and $\phi_2$ are also flipped). Hence the inner product on the transcendental lattice takes the form
\[ T_X = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \] (A.48)

The transcendental lattice is then $U(2)^{\oplus 2}$. As discussed e.g. in [53], the same result for the transcendental lattice is obtained by embedding the Picard lattice (A.46) into $H^2(K3, \mathbb{Z}) = U^{\oplus 3} \oplus E_8^2$ and computing its orthogonal complement.

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