Quarkyonic Matter and Chiral Symmetry Breaking

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Abstract

The appearance of a new phase of QCD, Quarkyonic Matter in the limit of large number of colors is studied within Nambu-Jona-Lassinio effective chiral model coupled to the Polyakov loop. The interplay of this novel QCD phase with chiral symmetry restoration and color deconfinement is discussed. We find that at vanishing temperature and at large $N_c$, the quarkyonic transition occurs at densities only slightly lower than that expected for the chiral transition. This property is also shown to be valid at finite temperature if the temperature is less than that of deconfinement. The position and $N_c$-dependence of chiral critical end point is also discussed.

Key words: Dense quark matter, Chiral symmetry breaking, Large $N_c$ expansion
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1 Introduction

The conventional view on the phase diagram of QCD is that high density strongly interacting matter is divided into two phases: the confined and the deconfined\textsuperscript{12345678910111213}. The phase diagram of QCD as a function of baryon number chemical potential $\mu$ and temperature $T$ as shown in Fig. 1, was originally envisioned by Cabibbo and Parisi\textsuperscript{3} and has changed little conceptually since. Until very recently, the possible new physics in the QCD phase diagram were Color Superconducting phases which might be important at asymptotically high baryon number density and low temperatures\textsuperscript{14}. In addition, a combination of efforts of Lattice Gauge Theory (LGT)\textsuperscript{15} and effective model calculations\textsuperscript{161718192021} have given new insight on the
position, the order and the universal properties [22] of the QCD phase diagram.

It has been argued recently that there may be an additional phase, the Quarkyonic Phase, of dense QCD [23]. This phase was rigorously shown to exist in the limit of a large number of colors $N_c$. In this limit, both the exponential of the free energy of a heavy test quark added to the system

$$e^{-\beta F_q} = \frac{1}{N_c} \langle L \rangle,$$  \hspace{1cm} (1)

and the baryon number density are order parameters [23]. The baryon number is an order parameter since $\langle N_B \rangle \sim e^{-\beta (M_B - \mu_B)}$. Thus, for temperatures of $T \sim \Lambda_{QCD}$ and with $M_B \sim N_c$, the $\langle N_B \rangle \sim e^{-\kappa N_c} \rightarrow 0$ at large $N_c$, as long as the baryon number chemical potential is small compared to the baryon mass, i.e. $\mu \ll M_B$. When $\mu_B \geq M_B$, then baryons begin to populate the system and the baryon number density is non-zero. In Ref. [23], it was argued that there are at least three phases in QCD at large $N_c$: the mesonic-phase which is confined and has zero baryon number density, the de-confined phase which has finite baryon number density, and the quarkyonic-phase which has finite baryon number density and is confined. The role of the chiral phase transition was not established.

The reason for the existence of the quarkyonic world was because for any finite value of chemical potential for quarks, $\mu_Q = \mu_B/N_c$, quark loops do not affect the confining potential. The de-confinement temperature is at some $T_c$ and is independent of $\mu_Q$. Therefore when baryons are added to the system, one can compress the baryons to very high chemical potential compared to $\Lambda_{QCD}$, and the baryonic matter remains confined. When $\mu_Q \geq \sqrt{N_c} \Lambda_{QCD}$, then the effects of the quark loops are felt on the potential and there is de-confinement, but

\[ \text{Speculations about related phases of matter were made in early strong coupling lattice studies of QCD at high baryon number density [24].} \]
Fig. 2. The phase diagram of QCD in large $N_c$. We do not display either the Chiral or Color Superconducting phases on this plot.

Fig. 3. A hypothetical phase diagram including $1/\!N_c$ effects (again ignoring the effects of the Chiral transition and Color Superconductivity).

as $N_c \to \infty$, the density at which this occurs approaches infinity.

The mesonic world is confined and has an energy density which scales as $O(1)$ in powers of $N_c$. The de-confined energy density scales as $N_c^2$, due to unconfined gluons. The energy density of the quarkyonic world scales as $N_c$, since both for baryonic matter and quark matter, the energy density is of order $N_c$. The quarkyonic world may be visualized as a quasi free degenerate Fermi gas of quarks in a sea of thermally excited mesons and glueballs. The effects of confinement are important for quark interactions only near the Fermi surface. The bulk interactions deep inside the Fermi sea, even though in a confined phase, are described by perturbation theory. The name quarkyonic was chosen since it is a combination of baryonic and quark matter, and expresses the Yin-Yang nature of the matter. A hypothetical phase diagram of QCD in the large $N_c$ limit is shown in Fig. 2.

In this paper we outline a theory which allows for explicit computation in the context of the PNJL model of QCD [18,19]. This provides a concrete descrip-
Fig. 4. A guess for the phase diagram of QCD for realistic value of $N_c$ (without the Chiral and Color Superconducting phase transitions).

As we decrease $N_c$ from asymptotically large values, there is some point where the first order deconfinement transition weakens and begins to disappear, as shown in Fig. 3. Eventually the low density confinement-deconfinement part of the phase transition completely disappears, leaving a line of first order phase transitions and a critical point which is a remnant of the chiral-quarkyonic phase transition, as shown in Fig. 4. These distinct branches of the phase diagram were shown previously in the work within the PNJL model $[27,28]$, and by Miura and Ohnishi in strong coupling lattice gauge theory $[29]$. Strictly speaking, the quarkyonic phase transition becomes a sharp cross over at finite $N_c$. We find that in large $N_c$, the quarkyonic transition occurs at densities slightly lower than that of the chiral transition. This difference in density is however so small that it may be an artifact of the model. As shown on Fig. 4, the cross over associated with the deconfinement phase transition continues on to higher densities, and becomes a separate line. This will be explained in detail later.
2 The PNJL Model in Large $N_c$

In order to study the QCD phase diagram in large $N_c$ we construct a chiral model where constituent quarks couple to effective gluon degrees of freedom. Here we follow to introduce an extended Nambu-Jona-Lasinio model with Polyakov loops (PNJL model).

We take the Lagrangian for a constituent quark field $\psi$ as

$$L = \overline{\psi} (i\gamma_\mu D^\mu - m + i\mu\gamma_0) \psi + \frac{G}{2} \left\{ (\overline{\psi}\psi)^2 + (\overline{\psi}i\vec{\tau}\gamma_5\psi)^2 \right\} - U(\Phi, \overline{\Phi}),$$

(2)

where $m$ is the current quark mass, $\mu$ is the quark chemical potential and $\vec{\tau}$ are Pauli matrices. In the following we restrict our discussion to two quark flavors, $N_f = 2$. An extension of the model to $N_f > 2$ is straightforward.

The interaction between the quarks and the effective gluon field is implemented through a covariant derivative

$$D_\mu = \partial_\mu - iA_\mu, \quad A_\mu = \delta_{\mu0}A^0,$$

(3)

where $A_\mu = gA^a_\mu \lambda^a$. Here $g$ is the color SU(3) gauge coupling constant and $\lambda^a$ are the Gell-Mann matrices. In the PNJL model, the transverse components of $A$ are integrated out. If we assume that these reflect short distance degrees of freedom, the effective potential in terms of $\Phi$ and $\overline{\Phi}$ should be at most 6’th order in the fields $(\Phi, \overline{\Phi})$, since these are the relevant operators for the three dimensional space on which $\Phi$ and $\overline{\Phi}$ exist. The variable $\Phi$ is defined as a trace of the Wilson line $L = P \exp \left\{ i \int_0^\beta d\tau A_0(\vec{x}, \tau) \right\}$ in color space:

$$\Phi = \frac{1}{N_c} \text{Tr} L,$$

(4)

and $\overline{\Phi}$ is the complex conjugate.

We will work in mean field approximation for the fields $\Phi$. The potential $U$ must respect the $Z(N_c)$ symmetry. For large $N_c$, the most general potential is of the form\[16,18,19\],

$$\frac{U}{T^4} = C \left( \frac{N_c^2 - 1}{8} \right) \left( -\frac{b_2(T)}{2} \Phi \Phi + \frac{b_4}{4} \left( \Phi \Phi \right)^2 + \frac{b_6}{6} \left( \Phi \Phi \right)^3 \right),$$

(5)

$^2$ One should keep in mind that the PNJL model describes statistical suppression of colored one- and two-quarks contributions which imitates color confinement.

$^3$ When we have a kinetic energy term $(\partial \Phi)^2$ of a scalar field $\Phi$, we would have to rescale $\Phi$ in such a way that it has the correct canonical dimension $(d - 2)/2$. Thus, the highest order term with a renormalizable coupling involves the 6th order in scalar fields.
with an overall constant $C$. One factorizes the $N_c^2 - 1$ dependence to get at high $T$ the pressure of an ideal gluon gas. The coefficients $b_i$ must be chosen so that at high temperatures, there is spontaneous breaking of the $Z(N_c)$ symmetry, and at low $T$ the symmetry is restored. For $N_c \geq 3$, the pure gauge theory has a first order phase transition corresponding to the de-confinement at some temperature $T_0$. At $T = T_0$ one finds

$$b_2(T_0) = -\frac{3b_4^2}{16b_6}. \quad (6)$$

We assume that $b_2(T)$ has the following temperature dependence:

$$b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2, \quad (7)$$

with constant parameters $a_i$. The Polyakov loop expectation value $\langle \Phi \rangle$ must be unity at asymptotically high temperature $T \to \infty$. This leads to

$$a_0 = b_4 + b_6. \quad (8)$$

We fix the parameters $a_i, b_i$ and $C$ taking $b_6 = 1$ in such a way that (5) describes the LGT observations for $\Phi(T_0)$ and pressure $P(T) = -U(T)$ in $SU(N_c = 3)$ pure gauge theory [31]. We list the resulting parameters in Table 1. We assume that $T_0$ is approximately independent of $N_c$ in $U$. This is supported by a finding in lattice QCD for several $N_c$ [32] where the deconfinement transition temperature approaching from low temperature phase is parameterized as

$$T_0(N_c; \mu = 0) = \sqrt{\sigma} \left( 0.596 + \frac{0.453}{N_c^2} \right), \quad (9)$$

with the string tension $\sigma$ being independent of $N_c$. Thus at large $N_c$, $T_0$ is $O(1)$.

In this paper we are interested in structural features of the PNJL model. Here we simply assume that there is a first order transition in the absence of quarks, and then determine the effect of quarks on this transition. Note that in the large $N_c$ limit, to leading order, the quarks do not affect the effective potential $U$. We will therefore assume a rigid background of fields $\Phi$ and explore the consequences for fermion-induced phase transitions.

Note also, that we need to specify which root of $\Phi$ we take when the $Z(N_c)$

| $a_0$ | $a_1$ | $a_2$ | $b_4$ | $b_6$ | $C$ |
|------|------|------|------|------|------|
| 0.787 | 0.333 | -1.13 | -0.213 | 1.00 | 5.35 |

Table 1
Set of parameters for the Polyakov-loop effective potential.
symmetry is broken. We take the real positive root, the root which is selected if there is a small breaking of the $Z(N_c)$ symmetry induced by fermions.

Using bosonization, this Lagrangian can be re-expressed as

$$\mathcal{L} = -U - \frac{\sigma^2 + \pi^2}{2G} - i \text{Tr} \ln S^{-1}.$$  \hspace{1cm} (10)

with quark propagator

$$S^{-1} = i \gamma_\mu \partial^\mu - \gamma_0 A^0 - M,$$  \hspace{1cm} (11)

and dynamical quark mass

$$M = m - (\sigma + i \gamma_5 \tau \cdot \pi).$$ \hspace{1cm} (12)

The field $\sigma$ in the mean field approximation,

$$\langle \sigma \rangle = G \langle \bar{\psi} \psi \rangle,$$ \hspace{1cm} (13)

gets a nonzero expectation value from solving the gap equation.

### 3 The NJL sector

In the following we first consider the solution of the fermionic sector of the theory at large $N_c$ and at finite $T$ and $\mu$. There are two parameters we need to fix in vacuum: the 3-momentum cutoff $\Lambda$ and the 4-Fermion coupling $G$. With $f_\pi \sim 92 \text{MeV}$ for QCD, we have the relationship

$$\frac{f_\pi^2}{N_c} = M_Q^2 \int \frac{d^3 p}{(2\pi)^3} \frac{\Theta(\Lambda - |\vec{p}|)}{E_p} \sim \frac{M_Q^2}{2\pi^2} \left( \ln \left( \frac{2\Lambda}{M_Q} \right) - 1 \right),$$  \hspace{1cm} (14)

with the constituent quark energy $E_p = \sqrt{\vec{p}^2 + M_Q^2}$. On the right hand side of this equation, we have dropped all terms which vanish in the limit $\Lambda \to \infty$. Note that $f_\pi^2$ should be of order $N_c$ and $G \sim 1/N_c$ for large $N_c$. We also have

$$\frac{\langle \bar{\psi} \psi \rangle}{N_c} = 4M_Q \int \frac{d^3 p}{(2\pi)^3} \frac{\Theta(\Lambda - |\vec{p}|)}{E_p} \sim \frac{M_Q}{\pi^2} \left\{ \Lambda^2 - M_Q^2 \left( \ln \left( \frac{2\Lambda}{M_Q} \right) - \frac{1}{2} \right) \right\},$$  \hspace{1cm} (15)

where the dynamical quark mass $M_Q$ is obtained as the solution of the gap equation

$$M_Q^2 \left\{ \ln \left( \frac{2\Lambda}{M_Q} \right) - \frac{1}{2} \right\} = \Lambda^2 \left( 1 - \frac{2\pi^2}{N_c N_f G \Lambda^2} \right).$$  \hspace{1cm} (16)
For $N_f = 2$ taking $m = 5$ MeV with constituent quark mass $M_Q = 320$ MeV and $f_\pi = 92$ MeV $\times \sqrt{3/N_c}$ one gets: $\Lambda = 646$ MeV and $G = 10.2$ GeV$^{-2} \times (3/N_c)$.

At finite temperature and density, the thermodynamic potential becomes

$$\Omega(T, \mu) = U + \frac{(M-m)^2}{2G} - \frac{T}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} \text{Tr} \ln S^{-1}((2n+1)T, \vec{p})/T.$$  \hfill (17)

The trace in the above expression can be done explicitly leading to

$$\Omega = -2N_fT \int \frac{d^3p}{(2\pi)^3} \left[ \text{Tr} \ln \left\{ 1 + Le^{-(E_p-\mu)/T} \right\} + \left( L \rightarrow L^\dagger, \mu \rightarrow -\mu \right) \right]$$

$$- 2N_cN_f \int \frac{d^3p}{(2\pi)^3} E_p \Theta(\Lambda^2 - |\vec{p}|^2) + \frac{(M-m)^2}{2G} + U.$$ \hfill (18)

Taking $\Phi = 1$ in Eq. (18) reproduces a standard NJL potential

$$\Omega_{\text{NJL}} = -2N_cN_fT \int \frac{d^3p}{(2\pi)^3} \left[ \ln(1 + e^{-\beta(E_p-\mu)}) + \ln(1 + e^{-\beta(E_p+\mu)}) \right]$$

$$- 2N_cN_f \int \frac{d^3p}{(2\pi)^3} E_p \Theta(\Lambda^2 - |\vec{p}|^2) + \frac{(M-m)^2}{2G},$$ \hfill (19)

which quantifies the thermodynamics of interacting quarks in the deconfined phase.

4 Solving the Fermion Sector of the Theory at large $N_c$

The fermionic contribution to the thermodynamic potential is obtained from Eq. (18) as

$$\delta \Omega_f = -2N_fT \int \frac{d^3p}{(2\pi)^3} \left[ \text{Tr} \ln \left\{ 1 + Le^{-\beta(E_p-\mu)} \right\} + \left( L \rightarrow L^\dagger, \mu \rightarrow -\mu \right) \right].$$ \hfill (20)

Suppose we are at $\mu \leq M_Q$. In this case the exponential factors inside the logarithm are all less than one. Moreover, $L$ is a unitary matrix, $LL^\dagger = 1$, so that we can always expand the logarithm as a series in $L$ (To see this, work in a diagonal representation where each element of $L$ is a pure phase). Using $\text{Tr} \ln A = \ln \det A$ for a matrix $A$, one finds
\[ \begin{align*}
&\quad \text{Tr} \ln \left( 1 + L e^{-\beta(E_p-\mu)} \right) \\
&= \ln \left[ 1 + N_c \Phi e^{-\beta(E_p-\mu)} + F_2(\langle L^2 \rangle, \langle L \rangle) e^{-2\beta(E_p-\mu)} \\
&\quad + \cdots + F_p(\langle L^p \rangle, \langle L^{p-1} \rangle, \cdots, \langle L \rangle) e^{-p\beta(E_p-\mu)} + \cdots + e^{-N_c\beta(E_p-\mu)} \right].
\end{align*} \] (21)

The coefficient of the \( p \)-quark contribution \( F_p \) is a function that contains the trace of at most \( L^p \).

In the confined phase, the first nonzero contribution occurs when the determinant is expanded to order \( L^{N_c} \). This contribution is of order \( e^{-\kappa N_c} \) for temperatures of order \( \Lambda_{\text{QCD}} \) and \( \mu \) a finite amount below \( M \). Therefore quarks do not contribute to the effective potential in the confined phase, since they are exponentially suppressed. In the de-confined phase, all terms of order \( L^p \) contribute, so the fermions are important again. Note however, that to compute the contribution of the fermion determinant requires evaluating contributions of order \( \langle L^p \rangle \) which cannot be simply re-expressed in terms of \( \langle L \rangle \). In fact to really deal with the large \( N_c \) limit requires an effective potential for the Wilson line in the pure gauge sector which retains contributions such as \( \langle L^p \rangle \). Nevertheless, even when in the de-confined phase, where in large \( N_c \) there is an expectation value of \( L \), it should be a good approximation to expand out the fermion determinant to first order in \( L \). This is known for a free fermion theory where the Boltzman statistics result is accurate within 20\% – 30\%. Thus, to include the non-leading order effects in \( N_c \), we will therefore expand the determinant to first order in \( L \). Note that in the confined phases of the theory, this should always be a very good approximation since there, in leading order in \( 1/N_c \), \( \Phi = 0 \) and non-leading effects generate only a small expectation value for \( \Phi \). The terms of higher order in \( L \) are expected to be further suppressed.

Consequently, when expanding the fermion determinant in positive powers of \( e^{-\beta(E-\mu)} \) for \( \mu \leq M_Q \) we conclude that whenever we are confined then baryons are exponentially suppressed. In large \( N_c \), there is no affect of temperature on the boundary between the quarkyonic and confined phase. There is no contribution of fermions to the expectation value of the \( \sigma \) field, so that the chiral symmetry is unaffected by an increase in either density or temperature while in the confined phase. There is no feedback back in large \( N_c \) of the gluons onto the expectation values of the \( \sigma \) field, again in the confined phase. We expect that the above is changed when going to finite \( N_c \), and that the expectation value of the \( \sigma \) field will weaken as we go to higher temperatures. This is the conventional picture that increasing temperature destabilizes the chiral condensate. Note also that at finite \( N_c \), we should expect that the fermions will feed back upon the gluon potential. Assume \( N_c \) is large so that we can expand the fermion contribution and keep only the first term in \( \Phi \), then this term will act to destabilize the first order deconfinement transition. Its effect increases as the baryon number density increases. We are therefore led to a picture like that of Philipsen and de Forcrand: The confinement transition weakens with
increasing baryon number density. Therefore, if there is a critical endpoint for realistic $N_c$, it is the critical endpoint of the chiral phase transition, not that of confinement. We will discuss these non-leading effects in $N_c$ in a later section.

Now let us determine the properties of the system when $\mu \geq M_Q$. We assume we are in the confined phase. We have to rearrange the determinant whenever $\mu \geq E_p$. We first note that the

$$\text{Tr} \ln L = 0,$$  

because it is an element of the $SU(N_c)$ group so that we can rewrite

$$\text{Tr} \ln \left(1 + L e^{-\beta(E_p-\mu)}\right)$$

$$= \text{Tr} \ln Le^{-\beta(E_p-\mu)} \left(1 + L^\dagger e^{-\beta(\mu-E_p)}\right)$$

$$= \text{Tr} \ln L + \text{Tr} \ln e^{-\beta(E_p-\mu)} + \text{Tr} \ln \left(1 + L^\dagger e^{-\beta(\mu-E_p)}\right)$$

$$= \beta(\mu - E_p) + \text{Tr} \ln \left(1 + L^\dagger e^{-\beta(\mu-E_p)}\right).$$  

(23)

Thus, Eq. (20) becomes

$$\delta \Omega_f = -2N_f T \int \frac{d^3 p}{(2\pi)^3} \text{Tr} \left[ \Theta(E_p - \mu) \left\{ \ln(1 + Le^{-\beta(E_p-\mu)}) + \left( L, \mu \to L^\dagger, -\mu \right) \right\} 
+ \Theta(\mu - E_p) \left\{ \beta(\mu - E_p) + \ln(1 + L^\dagger e^{-\beta(\mu-E_p)}) + (\mu \to -\mu) \right\} \right].$$  

(24)

The second term in the above equation is part of the ideal gas contribution for a zero temperature degenerate gas of quarks. Note that it is no longer exponentially suppressed. Thus, non-interacting quarks contribute to the free energy. This contribution persists even in the confined phase. In the confined phase in large $N_c$, this is the only term present, so it represents the contribution of quarkyonic matter.

For large but finite $N_c$ if we are at low temperatures, far away from the deconfined phase, the expectation value of $\Phi$ is small. Therefore, we can expand the determinant to first order. This shows how we generate an explicit expectation value, and should show that its effects on the quarkyonic phase boundary are small.

So let’s assume that the expectation value of $\Phi$ is very small in cold matter. Then, the vacuum contribution plus the finite chemical potential contribution of the fermions are of the form
\[ \Omega_{\text{quark}} = -2N_c N_f \int \frac{d^3p}{(2\pi)^3} \left( \frac{E_p}{2\pi^3} \chi(\Lambda - E_p) - \chi(\mu - E_p) \right) \]
\[ -2N_c N_f \mu \int \frac{d^3p}{(2\pi)^3} \chi(\mu - E_p) . \]  

(25)

The extremization of this effective potential leads to the gap equation,

\[ \frac{\pi^2}{G N_c N_f} = \int_{p_F}^{\Lambda} dp \frac{p^2}{E_p} , \]  

(26)

where the integral is cut off at the Fermi momentum \( p_F = \sqrt{\mu^2 - M^2} \). The chiral phase transition point is determined by the vanishing of the chiral condensate, \( M \to 0 \). Using Eq. (16) and the above gives us the solution for the position of the chiral phase transition as

\[ \mu_{\text{chiral}}^2(T = 0) = M_Q^2 \left\{ \ln(2\Lambda/M_Q) - \frac{1}{2} \right\} , \]  

(27)

with the vacuum quark mass \( M_Q \). Putting in numbers, one finds that \( \mu_{\text{chiral}} = 1.04 M_Q \) in \( N_c = 3 \). Note, that this equation tells us that the chiral transition occurs at densities slightly greater than that where the quarkyonic transition occurs (\( \mu = M_Q \)). The cutoff \( \Lambda \) is independent of \( N_c \) for large \( N_c \) and thus the coefficient \( \ln(2\Lambda/M_Q) - 1/2 \) has only a weak \( N_c \)-dependence. The significance of this conclusion is subject to many uncertainties. One cannot therefore with certainty conclude nor rule out, that there is an intermediate phase of unconfined constituent quarks, as has been argued by Feinberg et al. [21]. In any case, the position of the chiral and the quarkyonic phase occur at numerically close, or perhaps identical, values of \( \mu \).

5 Gluon effects on the chiral phase transition in large \( N_c \)

Let us begin by considering the response of the Fermion determinant to a non-zero expectation value of \( \Phi \). If we are at high temperature in the deconfined phase, then it is a good approximation to set \( \Phi = 1 \). In this case, we have an ideal gas of quarks whose thermodynamics is described by a pure NJL model given in [19]. In the following, we will be interested in the restoration of chiral symmetry where near the restoration point, we may approximate the mass as small. In this limit, one determines the chiral critical line from the gap

\[ \mu_{\text{chiral}}^2(T = 0) = M_Q^2 \left\{ \ln(2\Lambda/M_Q) - \frac{1}{2} \right\} , \]

(27)

The model with present parameters shows a first order phase transition in low temperatures. Nevertheless, an actual numerical calculation with \( m = 5 \) MeV gives almost identical critical value for a first order transition, \( \mu_{\text{chiral}} = 1.05 M_Q \).
where the integration can be carried out analytically. Using Eq. (16) for the constituent quark mass $M_Q$ in vacuum, this equation becomes

$$\mu^2 + \frac{\pi^2}{3} T^2 = M_Q^2 \left\{ \ln\left(\frac{2\Lambda}{M_Q}\right) - \frac{1}{2} \right\}. \quad (29)$$

If we put in numbers, we see that everywhere in the de-confined phase, in the limit $L = 1$, chiral symmetry is restored for $T_{\text{chiral}} \sim 177$ MeV at $\mu = 0$ at $N_c = 3$.

If we include the effect of nonzero $\Phi$, by expanding the fermion determinant to first order in $\Phi$ (an approximation which is good to 20%-30% even for $\Phi = 1$), one obtains

$$\Omega = U + \frac{(M - m)^2}{2G} - 2N_f N_c \int \frac{d^3p}{(2\pi)^3} E_p \Theta(\Lambda - |\vec{p}|) + \delta\Omega_f, \quad (30)$$

$$\delta\Omega_f = -2N_c N_f T \int \frac{d^3p}{(2\pi)^3} \left[ \Theta(E_p - \mu) \Phi \left( e^{-\beta(E_p - \mu)} + e^{-\beta(E_p + \mu)} \right) \right.$$

$$\left. + \Theta(\mu - E_p) \left\{ \beta(\mu - E_p) + \Phi \left( e^{-\beta(\mu - E_p)} + e^{-\beta(\mu + E_p)} \right) \right\} \right], \quad (31)$$

where a difference between expectation values of $\Phi$ and $\bar{\Phi}$ at finite $\mu$ is neglected. This leads to the gap equations for $M$ and $\Phi$,

$$\Lambda \sqrt{\Lambda^2 + M^2} - M^2 \ln \left( \frac{\Lambda + \sqrt{\Lambda^2 + M^2}}{M} \right) - \left( 1 - \frac{m}{M} \right) \frac{2\pi^2}{N_c N_f G}$$

$$= 4\Phi M T \cosh(\mu/T) K_1(M/T) + \Theta(\mu - M) \left[ p_F \mu - M^2 \ln \left( \frac{p_F + \mu}{M} \right) \right.$$

$$\left. - 4\Phi \int_M^\mu dE \sqrt{E^2 - M^2} \cosh((E - \mu)/T) \right], \quad (32)$$

$$T^3 C \left( \frac{N_c^2 - 1}{8} \right) \left[ -b_2(T) + b_4 \Phi^2 + b_6 \Phi^4 \right] \Phi$$

$$= \frac{2N_c N_f}{\pi^2} \left[ \cosh(\mu/T) M^2 T K_2(M/T) \right.$$

$$\left. + \Theta(\mu - M) \int_M^\mu dE \sqrt{E^2 - M^2} \sinh((E - \mu)/T) \right]. \quad (33)$$

One can see that the gap equations take a simple form assuming a second order chiral transition,
Fig. 5. The phase diagram for $N_c = 3$ obtained in our model with the current quark mass $m = 5$ MeV. The solid lines indicate a first order phase transition while the dashed lines cross over transitions. The critical end point (CEP) is indicated by a dot on the chiral phase boundary lines.

\begin{align}
\mu^2 + 4\Phi T^2 &= M_Q^2 \left[ \ln \left( 2\Lambda/M_Q \right) - \frac{1}{2} \right], \\
\mu^2 + 2T^2 &= \frac{C\pi^2(N_c^2 - 1)}{16N_cN_f} T^2 \left[ -b_2(T) + b_4\Phi^2 + b_6\Phi^4 \right] \Phi. 
\end{align}

Thus, in the large $N_c$ limit they are two gap equations which describe the quark and gluon sectors without any interference. Eq. (34) coincides with Eq. (27) and thus the chiral phase transition is indicated by a straight line $\mu_{\text{chiral}}(T) = \mu_{\text{chiral}}(T = 0)$. This also dictates the order of phase transition for any $T$. Eq. (35) determines the first-order deconfinement transition described by $\Phi(T)$ and thermodynamics now depends only on $T$.

Fig. 5 shows the model phase diagram for $N_c = 3$. The chiral and deconfinement cross over lines are identified as a maximum of derivatives $\partial M/\partial T$ and $\partial \Phi/\partial T$, respectively. The chiral and deconfinement lines are almost on top in a wide range of $\mu$. Near the critical end point (CEP) the chiral and quarkyonic transitions are strongly coupled and the CEP appears near the intersection of those boundary lines. The chiral phase boundary is influenced the deconfinement transition, and may weaken the chiral transition and result in the appearance of a CEP.

The evolution of the phase boundaries with $N_c$ is shown in Fig. 6. The chiral transition lines move to larger $\mu$ and approach the vertical line. Correspondingly, the CEP is also shifted to the right with $N_c$ and eventually disappears in the $N_c \to \infty$ limit. The coincidence of the chiral and deconfinement transi-
Fig. 6. The deconfinement and chiral phase boundary for various $N_c$. The horizontal line describes the deconfinement phase boundary in $N_c \to \infty$. The vertical line indicates the chiral phase boundary in $N_c \to \infty$. The solid line indicates a first order phase transition while the dashed lines cross over transitions. The symbols represent the chiral CEP for corresponding $N_c$.

Fig. 7. The deconfinement transition lines for various $N_c$. The horizontal line describes the deconfinement phase boundary in $N_c \to \infty$. The solid lines indicate a first order phase transition while the dashed lines cross over transitions. The symbols represent the CEP associated with $Z(N_c)$ symmetry.

Increasing $N_c$ further, the cross over of deconfinement turns into a first order transition and a CEP associated with the $Z(N_c)$ symmetry appears at finite
Fig. 8. The phase diagram of our model for $N_c = 3$ and for $N_c \to \infty$. The horizontal and vertical solid lines indicate the deconfinement and chiral phase boundaries in $N_c \to \infty$. The vertical broken line indicates the second order quarkyonic phase transition in $N_c \to \infty$.

$\mu$. This behavior is indicated in Fig. 7. This CEP appears as a result of quark interactions which makes the transition weaken. Thus the CEP disappears again in the large $N_c$ limit since quarks do not affect deconfinement.

Fig. 8 shows the phase diagram for $N_c = 3$ and for $N_c = \infty$. The model describes three distinct phases in the large $N_c$ limit which agrees with the argument made in [23]. (There may or may not be a fourth phase in a narrow range of chemical potential where there is baryon number but chiral symmetry is not restored. However, such a phase might also be an artifact of the model used in our calculation.) For realistic $N_c = 3$ the order of phase transitions is changed due to the quark-gluon interference. Nevertheless, one sees a remnant of the phase structure in large $N_c$ along with a deformation of the boundaries including finite $N_c$ effects.

6 Summary and Conclusions

In this paper, we have shown, that the first order phase transition and associated critical end point are driven by the chiral rather then deconfinement transition. The lines of both these transitions sit nearly atop the cross over associated with the remnants of the first order quarkyonic phase transitions of the large $N_c$ limit. The critical end point itself may appear where the cross over from the quarkyonic and the deconfinement phase transitions intersect.

It is interesting that within this theory, chiral symmetry breaking may occur at
temperatures below that of deconfinement. This has been addressed in several papers, and might be associated with parity doubling [33,34,35].

Clearly, chiral symmetry restoration in QCD may be more complicated than that which appears in PNJL type models. The Fermi surface of the quarkyonic phase is presumably associated with confined particles, and it may be, that there is a chiral condensate associated with this Fermi surface. Therefore, although chiral symmetry may be approximately restored at or very near the quarkyonic transition, the full restoration of chiral symmetry might require much high energy density, and might ultimately be associated with the deconfinement transition.

We have also seen that the cross overs associated with deconfinement and the quarkyonic phase transition are remnants of the first order phase transition which occurred in the limit of large \( N_c \). Thus, also at finite \( N_c \), the lines of cross overs reflect the phase structure seen in the large \( N_c \) limit. It is interesting, that the chiral critical end point typically appears at the juncture of the confinement and quarkyonic cross overs. This property may be more general and may appear beyond the PNJL model analysis. This is because, the large change of baryon number density associated with the quarkyonic phase transition may drive a first order chiral phase transition until the increase in the number of degrees of freedom associated with deconfinement destabilizes it, resulting in a critical end point.

In summary, it is fair to say that the conclusions from the PNJL model analysis are at best suggestive of what the true structure of high baryon number density matter might be.

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