Quiver Invariants from Intrinsic Higgs States

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Abstract

In study of four-dimensional BPS states, quiver quantum mechanics plays a central role. The Coulomb phases capture the multi-centered nature of such states, and are well understood in the context of wall-crossing. The Higgs phases are given typically by F-term-induced complete intersections in the ambient D-term-induced toric varieties, and the ground states can be far more numerous than the Coulomb phase counterparts. We observe that the Higgs phase BPS states are naturally and geometrically grouped into two parts, with one part given by the pulled-back cohomology from the D-term-induced ambient space. We propose that these pulled-back states are in one-to-one correspondence with the Coulomb phase states. This also leads us to conjecture that the index associated with the rest, intrinsic to the Higgs phase, is a fundamental invariant of quivers, independent of branches. For simple circular quivers, these intrinsic Higgs states belong to the middle cohomology and thus are all angular momentum singlets, supporting the single-center black hole interpretation.
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1 Conjectures

There are many stringy realizations of $D = 4$, $N = 2$ supersymmetric theories, and accordingly, many realizations of BPS states \[1\] thereof. Among the latter, the most powerful and all-encompassing picture appears to be the quiver realization due to Denef \[2\]. The simplest way to motivate the quiver quantum mechanics is to consider the BPS states as wrapped D3 branes in Calabi-Yau-compactified type IIB theories. The BPS condition then translates to Special Lagrange (SL) property of the 3-cycle wrapped by D3; when the 3-cycle is a sum of simpler SL 3-cycles, this induces a low energy quantum mechanics of the constituent BPS particles of quiver type with four supercharges.

The Coulomb phase of such quiver quantum mechanics has been studied in many guises, but in the end is always related to wall-crossing phenomena \[3, 4\]. Natural description of a BPS state in this phase is as a multi-center bound state where centers of relatively non-local charges are held fixed relative to each other via balance of classical forces \[5\]. Celebrated wall-crossing phenomena are simple consequences of this multi-center picture, when the size of a bound state diverges as one approaches a wall in the vacuum moduli space or the parameter space \[6, 7, 8, 9\]. Ground state counting in the Coulomb phase has been understood extensively \[10, 11, 12, 13, 14, 15, 16, 17, 18\], from what the right index problem is to how such indices are related to those of $D = 4$, $N = 2$ field theory. The resulting wall-crossing formula \[14, 16, 18\] has been proved \[19\] to be consistent with abstract wall-crossing formulæ \[20, 21, 22\] as well, including that of Kontsevich and Soibelman.

The Higgs phase is of entirely different character and is represented by a complete intersection, call it $M$, in an ambient projective variety, call it $X$. This is so because D-term conditions associated with the gauge symmetries of the quiver produce a projective toric variety, $X$, whereas the F-term conditions reduce the phase to the zero-locus of a set of polynomials, say $\partial W = 0$, when a superpotential $W$ exists. We mean, by Higgs phase, this complete intersection manifold $M$ assuming a generic superpotential $W$. Then the supersymmetric ground states are represented naturally by the cohomology ring, $H(M)$, which gives a simple way to count the index thereof,

$$
\chi(M) = \sum_n (-1)^n \dim H^n(M),
$$

as the alternative sum of cohomology group dimensions.

For simple quivers, such as loop-less ones, the Higgs phase counting is found to agree with the Coulomb phase. For more generic cases, however, in particular for those quivers that contain a loop and also allow, in the Coulomb phase, certain

\[1\] See Ref. \[10\] for a general review from field theory perspective.
“scaling” solutions associated with the loops, it is known that the Higgs counting is often much larger than the Coulomb one, and can even be exponentially so at that. As the Coulomb phase is supposed to capture multi-center states, it is then natural to expect that these extra states in Higgs phase, which we will call “intrinsic Higgs states”, are intrinsic to a single-centered black hole and the entropy thereof.

No matter what the correct physical interpretations of these numerous states are, however, we must first understand exactly how the intrinsic Higgs states are characterized in the Higgs phase of quiver quantum mechanics.\(^\#2\) For this, we note that the Higgs phase as a complete intersection inside a projective toric space suggests an interesting dichotomy of \(H(M)\) as follows,

\[
H(M) = i_M^*(H(X)) \oplus [H(M)/i_M^*(H(X))],
\]

where \(i_M^*\) is the pull-back onto \(M\) using the embedding map \(i_M : M \hookrightarrow X\). We may further define \(\chi(i_M^*(H(X)))\) as the usual alternating sum of Betti numbers. Note that \(\chi(i_M^*(H(X))) = \dim i_M^*(H(X))\) if the ring \(H(X)\) only consists of even cohomology groups, as is the case for the type of quivers we study in this note. Hence, we use the two alternatively.

With this dichotomy of Higgs phase ground states, we now conjecture the following:

- The pull-back \(i_M^*(H(X))\) of the ambient cohomology is in one-to-one correspondence with the Coulomb phase states. In particular,

\[
\Omega_{\text{Coulomb}} = \dim i_M^*(H(X)),
\]

where \(\Omega_{\text{Coulomb}}\) is the Coulomb phase index, and hence, according to our terminology, the states not in \(i_M^*(H(X))\) are intrinsic Higgs states.

- These intrinsic Higgs states in \(H(M)/i_M^*(H(X))\) are inherent to the quiver quantum mechanics in that their counting given by

\[
\chi(M) - \chi(i_M^*(H(X))),
\]

does not change as we cross walls by adjusting Fayet-Illiopoulos (FI) constants.

We will be checking the conjectures for the simplest class of quivers made of a circular loop. Simple generalizations, such as connecting one or more such circular quivers

\(^\#2\) One might be tempted to search for another type of index that only captures the intrinsic Higgs states. An obvious candidate, the signature of \(M\), can be seen not to work at all when complex dimension of \(M\) is odd, since it vanishes then, and typically too small in the even cases also.
with trees, follow immediately as long as we do not create new loops in the process. The second conjecture appears to be quite a nontrivial statement, especially when taken as a purely geometric statement. It would be most interesting to understand it from mathematical perspectives, but this is beyond the scope of this note.

We stated the conjectures as two separate ones, but they are of course linked to each other, once we believe that the underlying wall-crossing behavior is entirely captured by the Coulomb phase ground states. The latter has been convincingly demonstrated [19] when the quiver has no loops\(^3\) but general expectation is that the same will hold for quivers with loops, since the fundamental physics underlying wall-crossing is in the multi-center nature, as captured faithfully by the Coulomb phase.

There is one immediate evidence for the first conjecture from the angular momentum consideration. As we will see in section 3, for simple circular quivers, the intrinsic Higgs phase states are all angular momentum singlets. This fact alone goes a long way to support the conjecture since, if the single-center black hole interpretation for intrinsic Higgs states is correct, we naturally expect no angular momentum associated with them\(^4\).

In the next section, we review geometry of the Higgs branches for a quiver with a single circular loop, and comment on the Coulomb counterpart. In particular, we identify the ambient space \(X\) as a product of projective spaces, \(\mathbb{CP}^n\)'s, and present basic topology of the Higgs phase \(M\) as a complete intersection manifold embedded therein. In section 3, we explore both \(H(M)\) and \(i^*_M(H(X))\). In section 4, we take the case of simplest circular quiver, with three nodes, and show validity of these conjectures explicitly. In section 5, we make further checks of the conjectures numerically for quivers with four or more nodes. In section 6, we close with comments.

## 2 Higgs Phases of Quivers with a Loop

Let us start with a cyclic \((n + 1)\)-gon quiver, and denote the bifundamental fields \(Z_{i,i+1}\) by \(Z_i\) for \(i = 1, \ldots, n\), and \(Z_{n+1,1}\) by \(Z_{n+1}\). For each node \(i\), there are \(a_i\) arrows from the \(i\)-th node to the \((i+1)\)-th node, where all the linking numbers \(a_i\) are positive. This means that \(Z_i\) are actually \(a_i\)-dimensional complex vectors,

\[
Z_i = (Z_i^{(1)}, \ldots, Z_i^{(a_i)}), \tag{2.1}
\]

\(^3\)Most rigorous check of which was for the Coulomb phase wall-crossing against Kontsevich-Soibelman algebra when all charge vectors involved are on a single plane passing through origin.

\(^4\)Vanishing angular momentum for single-center states has been conjectured and checked for \(N = 4\) black holes recently [24, 25].
as depicted by Figure 2.1. Because the quiver has a loop, with the linking numbers of the same sign, we should expect a generic superpotential of the type,

\[ W(Z_1, Z_2, \cdots, Z_{n+1}) = \sum_{\beta_1=1}^{a_1} \cdots \sum_{\beta_{n+1}=1}^{a_{n+1}} \gamma_{\beta_1 \beta_2 \cdots \beta_{n+1}} Z_1^{(\beta_1)} Z_2^{(\beta_2)} \cdots Z_{n+1}^{(\beta_{n+1})}, \quad (2.2) \]

whose F-term supersymmetric vacuum conditions are

\[ \partial_{Z_i^{(\beta_i)}} W = 0 \quad (\beta_i = 1, 2, 3, \cdots, a_i), \quad (2.3) \]

for \( i = 1, \cdots, n+1 \). We show that the solutions to \( \partial W = 0 \) split into branches, where one of the \( Z_i \) complex vectors is identically zero.

Figure 2.1: A cyclic \( (n+1) \)-gon quiver consists of \( n+1 \) nodes, cyclically connected by directed arrows. Associated to each node is a \( U(1) \) gauge group, whose FI constant is denoted by \( \zeta_i \), and the \( a_i \) arrows from the \( i \)-th node to the \( (i+1) \)-th node correspond to the \( a_i \) bifundamental fields \( Z_i = (Z_i^{(1)}, \cdots, Z_i^{(a_i)}) \).

Let us first show that there is no solution to F-term conditions with all the complex vectors \( Z_i \) nontrivial. If there were such a solution, it has to be a discrete solution since the number of F-term equations equals the total number of complex variables
Z’s. (We always assume that the coefficients in \( W \) are generic, so the algebraic equations \( \partial W = 0 \) are also generic.) However, the above F-term conditions have \( n + 1 \) scaling symmetries under \( Z_i \to \lambda_i Z_i \) for any complex numbers \( \lambda_i \), and one can actually generate \( n + 1 \) complex dimensional family of solutions. This contradicts the expected discreteness of the solution, so we cannot generically expect to find solutions of this type.

The preceding argument may look rather prohibiting, but it can be evaded easily by setting \( Z_j = 0 \) for some \( j \). Because of the homogeneous form of \( \partial W \)’s, \( Z_j = 0 \) renders all \( \partial_{Z_{i \neq j}} W \) to vanish identically, leaving behind only \( a_j \) number of equations, \( \partial_Z_j W = 0 \). In this situation, the number of remaining variables \( Z_{i \neq j} \) are typically much larger than the surviving F-term conditions. Therefore, we conclude that, for generic \( W \), F-term conditions have to be solved by setting one of \( Z_i \) to be identically zero.

This key observation also helps us to solve D-term conditions almost trivially. With \( U(1)^{n+1} \) gauge groups, of which a trace part \( U(1) \) decouples, we have \( n \)-independent D-term conditions

\[
\begin{align*}
|Z_{n+1}|^2 - |Z_1|^2 &= \zeta_1, \\
|Z_1|^2 - |Z_2|^2 &= \zeta_2, \\
|Z_2|^2 - |Z_3|^2 &= \zeta_3, \\
&\vdots \\
|Z_n|^2 - |Z_{n+1}|^2 &= \zeta_{n+1},
\end{align*}
\]

which require as usual

\[
\zeta_1 + \cdots + \zeta_{n+1} = 0.
\]

Thanks to the above F-term discussion, we learned that the Higgs phase has \( n + 1 \) generic branches with one of \( Z_i = 0 \) identically. The \( k \)-th branch is realized when

\[
\sum_{i=1}^{k} \zeta_i > 0, \quad \sum_{i=k+1}^{J} \zeta_i < 0
\]

for consecutive and mutually exclusive sets of \( I \)'s and \( J \)'s, where the cyclic nature of the indices are understood. In this \( k \)-th branch, \( Z_k = 0 \), and the remaining D-term conditions are solved entirely by

\[
X_k = \mathbb{C}P^{a_1-1} \times \cdots \times \mathbb{C}P^{a_{k-1}-1} \times \mathbb{C}P^{a_{k+1}-1} \times \cdots \times \mathbb{C}P^{a_n-1}.
\]
At the boundary of this branch, one or more of $\mathbb{CP}^1$'s get squashed to zero size, and wall-crossing may occur.

Combining the two lines of thoughts, we conclude that the $k$-th Higgs branch, $M_k$, that emerges in a particular domain of FI constants, has the form of a complete intersection,

$$M_k = X_k \bigg|_{\partial z_k W = 0},$$

(2.7)
given by the zero-locus of the $a_k$ F-term conditions; All other F-term conditions are trivially met by $Z_k = 0$.

Since these phases are selected by a choice of domain in the FI constant space, each Higgs branch has a Coulomb counterpart, determined as a subspace of $\mathbb{R}^{3(n+1)}$ by 1-loop zero energy conditions,

$$a_{n+1} \frac{|\vec{x}_{n+1} - \vec{x}_1|}{|\vec{x}_{n+1} - \vec{x}_1|} - a_1 \frac{|\vec{x}_1 - \vec{x}_2|}{|\vec{x}_1 - \vec{x}_2|} = \zeta_1,$$

$$a_1 \frac{|\vec{x}_1 - \vec{x}_2|}{|\vec{x}_1 - \vec{x}_2|} - a_2 \frac{|\vec{x}_2 - \vec{x}_3|}{|\vec{x}_2 - \vec{x}_3|} = \zeta_2,$$

$$
\vdots
$$

$$a_n \frac{|\vec{x}_n - \vec{x}_{n+1}|}{|\vec{x}_n - \vec{x}_{n+1}|} - a_{n+1} \frac{|\vec{x}_{n+1} - \vec{x}_1|}{|\vec{x}_{n+1} - \vec{x}_1|} = \zeta_{n+1},$$

with the same sign choices on FI constants. $\vec{x}_i$ represents the position of the $i$-th charge center in real space $\mathbb{R}^3$. In this note, the corresponding Coulomb phase index will be denoted by $\Omega_{\text{Coulomb}}$, with appropriate labels for the choice of branch.

From the Coulomb phase perspective, this quiver falls into two big regimes, depending on the linking numbers $a_i$. In one regime, so-called non-scaling regime, one finds that classical Coulomb vacuum manifolds are such that $|\vec{x}_i - \vec{x}_{i+1}|$'s are all fixed. In the other, so-called scaling regime, however, one finds classical configurations of arbitrarily short distances among the charge centers. The criteria between the two take a geometric character. The scaling regime occurs when the $a_i$'s are such that they can be lengths of a geometric $(n + 1)$-gon in $\mathbb{R}^3$, i.e.,

$$a_j < \sum_{i \neq j} a_i$$

(2.9)

for all $j$. The Coulomb index is expected to agree with Higgs phase Euler number in the non-scaling cases, but known to be typically smaller in the scaling cases.
3 Basics on $H(M)$ and $i^*_M(H(X))$ for Cyclic $(n+1)$-Gons

Let us characterize the pull-back of the ambient cohomology for a cyclic $(n+1)$-gon, and, as before, denote the linking numbers as $(a_1, a_2, \ldots, a_{n+1})$, all positive. We may take, without loss of generality, the Higgs branch where $(n+1)$-th bifundamentals are forced to vanish. Thus, the ambient toric space is simply

$$X_{n+1} = \mathbb{CP}^{a_1 - 1} \times \mathbb{CP}^{a_2 - 1} \times \cdots \times \mathbb{CP}^{a_n - 1}. \quad (3.1)$$

The superpotential $W \sim Z_1 Z_2 \cdots Z_{n+1}$ then yields total of $a_{n+1}$ number of F-term conditions. This defines the Higgs phase $M_{n+1}$, where the subscript emphasizes the fact that this manifold differs as we choose different branches of the Higgs phase. The complex dimension of $M_{n+1}$ is then

$$d_{n+1} = a_1 + a_2 + \cdots + a_n - a_{n+1} - n. \quad (3.2)$$

The Higgs phase index is counted by the Euler number which is computed as integral of the top Chern class of $M_{n+1}$. Now, the adjunction formula relates the Chern class of $M_{n+1}$ to that of the ambient space $X_{n+1}$, leading to

$$c(M_{n+1}) = \frac{(1 + J_1)^{a_1} (1 + J_2)^{a_2} \cdots (1 + J_n)^{a_n}}{(1 + J_1 + J_2 + \cdots + J_n)^{a_{n+1}}}, \quad (3.3)$$

where $J_i$'s are the Kähler class of the $\mathbb{CP}^{a_i - 1}$ factor. Evaluation of $\chi(M_{n+1})$ proceeds equally simply in the ambient space $X_{n+1}$ as

$$\chi(M_{n+1}) = \int_{X_{n+1}} \frac{(1 + J_1)^{a_1} (1 + J_2)^{a_2} \cdots (1 + J_n)^{a_n} (J_1 + J_2 + \cdots + J_n)^{a_{n+1}}}{(1 + J_1 + J_2 + \cdots + J_n)^{a_{n+1}}}, \quad (3.4)$$

which is the same as extracting the coefficient of the top form $J_1^{a_1-1} \cdots J_n^{a_n-1}$ of the integrand. Alternatively and more explicitly, the Euler number can also be shown to be

$$\chi(M_{n+1}) = a_1 a_2 \cdots a_n - \int_0^\infty ds \, e^{-s} L_{a_1-1}^1(s) L_{a_2-1}^1(s) \cdots L_{a_{n+1}-1}^1(s) \quad (3.5)$$

$$- \sum_{s_1=0}^{a_1-1} \sum_{s_2=0}^{a_2-1} \cdots \sum_{s_{n+1}=0}^{a_{n+1}-1} (s_1 + s_2 + \cdots + s_{n+1})! \frac{n+1}{i=1} \left( \frac{a_i}{s_i + 1} \right) \frac{(-1)^{s_i}}{s_i!}$$

with Laguerre polynomials $L_{a_i-1}^1$, following the same procedure as in [12]. All of these apply to $M_k$ straightforwardly.
As we are splitting the cohomology into two parts, one of which comes from pull-back of $H(X)$, the Lefschetz hyperplane theorem (see appendix A) comes in handy. When used iteratively, the theorem implies for the circular quivers that the pull-back of $H(X)$ is isomorphic to $H(M)$ for the lower-half cohomologies up to $H^{d-1}$, where $d$ is the complex dimension of $M$. Also, the map is injective for $H^d$. Combined with the Poincaré duality, this tells us that

$$H^n(M) \simeq H^{2d-n}(M) \simeq H^n(X), \quad n < d,$$

or equivalently,

$$H(M) = i^*_M(H(X)) \oplus [H^d(M)/i^*_M(H^d(X))].$$

Thus, we also learned that the intrinsic Higgs states all belong to the middle cohomology of $M$.

In turn, this translates physically to the statement that all intrinsic Higgs states are angular momentum singlets, as $(n - d)/2$ can be understood as the helicity; since $M$ is a Kähler manifold, the following natural actions on $n$-forms

$$L_3 = (n - d)/2, \quad L_+ = J\wedge, \quad L_- = J \cdot ,$$

constitute an $SU(2)$ algebra on $H(M)$ [23], which inherits spatial rotation of the underlying four-dimensional theory.

As we already noted in the first section, this fact is consistent with expectations that intrinsic Higgs states have something to do with single-center black holes. When we consider more general quivers, this statement must be generalized a bit, because additional tree-like component of the quiver could add extra charge centers, orbiting around the single-center black hole of the loop, and carry additional angular momentum. In the Higgs phase description, this will manifest as a factorization in the ambient space $X$, say $X = X' \times Y$, such that

$$H(M) = H(M') \otimes H(Y),$$

and our discussion above will apply to $M' \hookrightarrow X'$.

One immediate classification among these arises depending on whether the dimension $d_k$ is even or odd. When $d_k$ is even, the cohomology is entirely of even-dimensional, again thanks to the hyperplane theorem, so that

$$H(M_k) = i^*_M(H(X_k)) \oplus [H^{d_k}(M_k)/i^*_M(H^{d_k}(X_k))],$$

and the Euler number counts the number of ground states faithfully. When $d_k$ is odd, we have instead

$$H(M_k) = i^*_M(H(X_k)) \oplus H^{d_k}(M_k),$$
since \( H^{d_k}(X_k) \) is empty. In this case the Euler number counts the difference between the dimension of pull-back \( i^*_M(H(X_k)) \) and that of the middle cohomology \( H^{d_k}(M_k) \).

With this in mind, let us consider properties of \( i^*_M(H(X_k)) \). A simple method to recover this is to truncate the well-known Poincaré polynomial of \( X_k \),

\[
P[X_k] = \prod_{i \neq k} (1 + x^2 + x^4 + \cdots + x^{2(a_i - 2)}) = \sum b_{2l}(X_k) \cdot x^{2l} \quad (3.12)
\]

up to the order \( d_k \), and to complete it by inverting the lower half to the upper half. This defines the “Poincaré polynomial” of the pulled-back cohomology,

\[
P[i^*_M(H(X_k))] \equiv b_{d_k}(X_k) \cdot x^{d_k} + \sum_{0 \leq 2l < d_k} b_{2l}(X_k) \cdot (x^{2l} + x^{2d_k-2l}) \quad (3.13)
\]

where, as we noted above, the first term on the right exists only when \( d_k \) is even. It is not difficult to see that \( b_{2l}(X_k) \) has three different regimes, increasing for small \( l \), a plateau in the middle, and decreasing for large \( l \). Thus, counting of \( i^*_M(H(X_k)) \) will give different type of behaviors depending on the value \( d_k \) relative to the linking numbers.

A pair of useful numbers, \( 2N_k \) and \( 2L_k \), defined, for example for \( k = n + 1 \),

\[
\begin{align*}
L_{n+1} &= a_1 + \cdots + a_n - n - N_{n+1} , \\
N_{n+1} &= \max \left\{ a_1 - 1, a_2 - 1, \cdots, a_n - 1, \left[ \frac{a_1 + \cdots + a_n - n + 1}{2} \right] \right\} ,
\end{align*}
\quad (3.14, 3.15)
\]

are such that we have

\[
b_0 < b_2 < \cdots < b_{2L_{n+1}} = b_{2L_{n+1}+2} = \cdots = b_{2N_{n+1}} > b_{2N_{n+1}+2} > \cdots > b_{2(a_1+\cdots+a_n-n)} .
\]

One obvious category is \( d_k \leq 2L_k + 1 \), which in turn translates to the condition \(^{#5}\)

\[
d_k \leq 2L_k + 1 \iff a_j + (n - 3) \leq \sum_{i \neq j} a_j , \quad j \neq k . \quad (3.17)
\]

The second category arises when \( 2L_k + 1 < d_k \leq 2N_k + 1 \), whereby one finds size of \( i^*_M H^{2l}(X_k) \) plateaus through a symmetric range of middle dimensions \( 2l \sim d_k \).

\(^{#5}\)This \( d_k \leq 2L_k + 1 \) condition is similar to those for a scaling solution to exist,

\[
a_m < \sum_{i \neq m} a_j , \quad \text{for } m = 1, \ldots, n + 1,
\]

but not quite.
Physically the corresponding ground states are such that the angular momentum multiplets have some minimum spin. We believe that, in this case, no intrinsic Higgs states exist at all.

The third apparent possibility $2N_k + 1 < d_k$ is never realized, because it is inconsistent with the Lefschetz $SU(2)$ symmetry. This can also be seen directly from the numbers as

$$2N_{n+1} \geq 2 \left[ \frac{a_1 + \cdots + a_n - n + 1}{2} \right] > d_{n+1} = a_1 + a_2 + \cdots + a_n - a_{n+1} - n$$ (3.18)

with positive $a$’s. Although this is, from purely geometric viewpoint, a trivial consequence of the Kählerian property of $M_k$’s, it does provide a simple consistency check on the first conjecture that $i_{M_k}^{*}(H(X_k))$ is a faithful image of the Coulomb ground states in the Higgs phase. The Coulomb phase states are expected to be organized as a sum of angular momentum multiplets of all integer or all half-integer spins. Degeneracy of a given helicity is then always maximized at 0 or $\pm 1/2$, with non-increasing behavior as we increase the absolute value of the total helicity.

4 Analytical Check: 3-Gons

The simplest example is a quiver with three nodes ($n = 2$). Before we consider actual quivers, let us first observe that, for $X = \mathbb{CP}^{a-1} \times \mathbb{CP}^{b-1}$ and a complete intersection embedding $i_M : M \hookrightarrow X$ by $c$ F-term constraints, the dimension

$$D \equiv \dim i_M^{*}(H(X)) ,$$ (4.1)

of the pulled-back cohomology is given by

$$D = \mathcal{D}^{(1)}(a, b; c) \equiv \frac{1}{4} \left( (a + b - c)^2 - r \right) ,$$ (4.2)

with $r = 0, 1$ for $a + b + c$ even and odd, respectively, provided that $a, b,$ and $c$ obey

$$a \leq b + c + 1 ,$$
$$b \leq c + a + 1 ,$$
$$c \leq a + b + 1 .$$ (4.3)

Otherwise, we have

$$D = \mathcal{D}^{(2)}(a, b; c) \equiv \begin{cases} (a - c) \cdot b & \text{if } a > b + c + 1 , \\ (b - c) \cdot a & \text{if } b > c + a + 1 , \\ 0 & \text{if } c > a + b + 1 , \end{cases}$$ (4.4)
Note that only one of the three inequalities (4.3) can be violated at a time. Here, the superscript for $D$ is shown as a reminder of the class of the corresponding quiver, to which we turn next.

Let us classify the 3-gon quivers with linking numbers $a_1, a_2, a_3$ as follows. We say that the quiver belongs to the first class if the three inequalities,

\begin{align*}
    a_1 &\leq a_2 + a_3 + 1 , \\
    a_2 &\leq a_3 + a_1 + 1 , \\
    a_3 &\leq a_1 + a_2 + 1 ,
\end{align*}

are obeyed. This is reminiscent of the criteria (4.3) (and hence, the superscript of $D^{(1)}$ for cohomology counting in Eq. (4.2), for instance). The second class is defined to be those quivers that violate one of the three inequalities (4.5). The violation can happen only for the largest of three linking numbers, and without loss of generality, we may label it $a_2$. So for the second class we effectively assume

\begin{equation}
    a_2 > a_3 + a_1 + 1 .
\end{equation}

This implies failure of the triangle condition, which is necessary for a scaling solution to exist. Expectation is that in such cases the Euler number of Higgs phase equals the Coulomb index, which we will be verifying along the way.

For simplicity, let us denote, in each branch $k$, the complex dimension of $M_k$ as

\begin{equation}
    d_k \equiv \dim \mathbb{C} M_k = a_1 + a_2 + a_3 - 2a_k - 2 ,
\end{equation}

and the dimension of the pulled-back cohomology as

\begin{equation}
    D_k \equiv \dim i^*_{M_k} (H(X_k)) .
\end{equation}

We start with the second class of quivers, and the branch $M_3$ thereof. Here, $a_3$ plays the role of $c$ in Eq. (4.4), so we find

\begin{equation}
    D_3 = D^{(2)}(a_1, a_2; a_3) = (a_2 - a_3) \cdot a_1 ,
\end{equation}

which agrees with the known Coulomb phase index in this parameter regime

\begin{equation}
    \Omega_{\text{Coulomb}}(a_1, a_2; a_3) = (a_2 - a_3) \cdot a_1 ,
\end{equation}

as our first conjecture asserts. On the other hand, Denef and Moore also observed that, under a weaker condition $a_2 + 1 \geq a_3 + a_1$, the Euler number $\chi(M_3)$ of the Higgs phase agrees with this Coulomb phase index,

\begin{equation}
    \Omega_{\text{Coulomb}}(a_1, a_2; a_3) = \chi(M_3) .
\end{equation}

\# See Appendix B for details.
Taking these two facts together, we find
\[ \chi(M_3) - D_3 = 0. \]  
(4.12)

Computations for \( M_1 \) proceeds verbatim, with \( a_1 \) and \( a_3 \) exchanged, so we have
\[ D_1 = \Omega_{\text{Coulomb}}(a_3, a_2; a_1) = \chi(M_1), \]  
(4.13)

and in particular,
\[ \chi(M_1) - D_1 = 0. \]  
(4.14)

The remaining branch \( M_2 \) is empty, because the number of F-term constraints, \( a_2 \), is larger than the complex dimension \( d_2 = a_1 + a_3 - 2 \) of the ambient space. Therefore, we have
\[ \chi(M_2) - D_2 = 0 - 0 = 0, \]  
(4.15)

which is still consistent with the second conjecture. The final check (for the first conjecture) is to show that \( \Omega_{\text{Coulomb}}(a_1, a_3; a_2) \) also vanishes in this branch; it follows trivially from (2.8) with \( a_2 > a_1 + a_3 + 1 \) taken into account.\(^\#7\)

Next, we turn to the quivers in the first class. Recall that in each of the three branches \( M_k \), the pulled-back cohomology \( D_k \) is counted by \( D^{(1)} \) in Eq. (4.2). For example, we have
\[ D_3 = D^{(1)}(a_1, a_2; a_3) = \frac{1}{4} \left( (a_1 + a_2 - a_3)^2 - r \right), \]  
(4.16)

where, again, \( r = 1, 0 \) for odd and even \( d_3 \), respectively. On the other hand, under a slightly stronger condition of triangle inequalities, \( a_i < a_j + a_k \), the Coulomb index has been computed as \([13, 16]\)
\[ \Omega_{\text{Coulomb}}(a_1, a_2; a_3) = \frac{1}{4} \left( (a_1 + a_2 - a_3)^2 - r \right), \]  
(4.17)

which agrees with Eq. (4.16). For \( a_i = a_j + a_k \) and \( a_i = a_j + a_k + 1 \), where the Coulomb phase should be counted by Eq. (4.10) instead of Eq. (4.17), it so happens that the two expressions coincide. Therefore, Eq. (4.17) works actually for the entire regime, \( a_i \leq a_j + a_k + 1 \), so that
\[ D_3 = \Omega_{\text{Coulomb}}(a_1, a_2; a_3), \]  
(4.18)

\(^\#7\)The corresponding Coulomb phase should be given by, with \( r_{ij} = |\vec{x}_i - \vec{x}_j| \) and \( \vec{x}_k \in \mathbb{R}^3 \),
\[ a_1/r_{12} - a_2/r_{23} > 0, \quad a_2/r_{23} - a_3/r_{31} < 0 \]
which can be shown to have no solution when \( a_2 > a_1 + a_3 \). Thus, the Coulomb phase is also empty when its counterpart \( M_2 \) is empty.
again, and generally,
\[ D_k = \Omega_{\text{Coulomb}}(a_i, a_j; a_k), \quad i, j, k \text{ distinct}, \]
which shows that the first conjecture holds true for the quivers in the second class too. Now, the second conjecture demands that
\[ \chi(M_k) - D_k, \]
is invariant under the choice of \( k \). Starting with Eq. (3.5) for \( \chi(M) \) in three-node cases [12],
\[ \chi(M_k) = \frac{a_1a_2a_3}{a_k} - \int_0^\infty ds \, e^{-s}L^1_{a_1-1}(s)L^1_{a_2-1}(s)L^1_{a_3-1}(s), \]
we find
\[ \chi(M_k) - D_k = \chi(M_k) - D^{(1)}(a_i, a_j; a_k) \]
\[ = -\frac{1}{4} \left( a_1^2 + a_2^2 + a_3^2 - 2a_1a_2 - 2a_2a_3 - 2a_3a_1 - r \right) \]
\[ - \int ds \, e^{-s}L^1_{a_1-1}(s)L^1_{a_2-1}(s)L^1_{a_3-1}(s), \]
with \( r = 1, 0 \) respectively, for odd and even \( a_1 + a_2 + a_3 \), which is clearly invariant under cyclic rotations among \( a_1, a_2, a_3 \). That is, \( \chi(M_k) - D_k \) is independent of \( k = 1, 2, 3 \), as conjectured.

So far, we checked the two conjectures against the relevant degeneracies. However, our analytical check actually goes beyond this. For the type of quivers in this section, the relevant angular momentum information of Coulomb phase states can be extracted from the proposal of Ref. [16]. For the first class of quivers, one finds exactly one angular momentum multiplet for each spin \( d/2, \ldots, d/2 - 1, \ldots \) down to 0 or \( 1/2 \), respectively, depending on whether \( d \) is even or odd. For the second class, this descending series of angular momentum multiplets also starts with \( d/2, d/2 - 1, \ldots \), but is cut off at some positive spin with no angular momentum multiplet below it. If our first conjecture holds, the same information should be encoded in the Poincaré polynomials (3.13) of the pulled-back cohomology in the Higgs phase. For all 3-gon cases, this comparison has been made and found to agree with each other completely. In terms of this Poincaré polynomial, the two classes distinguish themselves by either having a monotonically increasing \( b_{2l} = l + 1 \) between \( 2l = 0 \) and \( 2l = 2[d/2] \) or reaching a plateau at \( 2l = 2L \) till \( 2l = 2[d/2] \).

This concludes the explicit and analytical demonstration that the two conjectures hold separately for all possible three-node cyclic quivers.
5 Numerical Evidences: 4-Gons, 5-Gons, and 6-Gons

| $(a_1, a_2, a_3, a_4)$ | $k$ | $\chi(M_k)$ | $D_k$ | $\chi(M_k) - D_k$ |
|------------------------|-----|-------------|-------|-------------------|
| $(2, 3, 4, 11)$        | 1   | 108         | 108   | 0                 |
|                        | 2   | 64          | 64    | 0                 |
|                        | 3   | 42          | 42    | 0                 |
|                        | 4   | 0           | 0     | 0                 |
| $(2, 3, 4, 9)$         | 1   | 84          | 84    | 0                 |
|                        | 2   | 48          | 48    | 0                 |
|                        | 3   | 30          | 30    | 0                 |
|                        | 4   | 0           | 0     | 0                 |
| $(4, 5, 6, 7)$         | 1   | -653458     | 110   | -653568           |
|                        | 2   | -653500     | 68    | -653568           |
|                        | 3   | -653528     | 40    | -653568           |
|                        | 4   | -653548     | 20    | -653568           |
| $(5, 7, 11, 13)$       | 1   | -28895778010| 656   | -28895778666      |
|                        | 2   | -28895778296| 370   | -28895778666      |
|                        | 3   | -28895778556| 110   | -28895778666      |
|                        | 4   | -28895778626| 40    | -28895778666      |
| $(11, 12, 13, 14)$     | 1   | -7025159641580583958| 908 | -7025159641580584866 |
|                        | 2   | -7025159641580584140| 726 | -7025159641580584866 |
|                        | 3   | -7025159641580584294| 572 | -7025159641580584866 |
|                        | 4   | -7025159641580584426| 440 | -7025159641580584866 |

Table 1: Computation of indices $\chi(M_k)$ and $D_k \equiv \dim i_{M_k}^* (H(X_k))$ for five 4-gon quivers, whose edges labeled by $(a_1, a_2, a_3, a_4)$. For each quiver the indices are computed in four different branches, but $\chi(M_k) - D_k$ listed in the last column is clearly insensitive to the choice of the branch, and in particular uniformly zero for the case that violates geometric 4-gon condition and thus no scaling solution in the Coulomb phase. The second example is a marginal case in this sense.

Let us now turn to more involved examples with many nodes. As before, we will consider distinct branches, labeled by $k = 1, 2, \ldots, n + 1$, and compute the indices.
\[
\begin{array}{|c|c|c|c|}
\hline
(a_1, a_2, a_3, a_4, a_5) & k & \chi(M_k) & D_k & \chi(M_k) - D_k \\
\hline
(1, 2, 3, 4, 11) & 1 & 240 & 240 & 0 \\
 & 2 & 108 & 108 & 0 \\
 & 3 & 64 & 64 & 0 \\
 & 4 & 42 & 42 & 0 \\
 & 5 & 0 & 0 & 0 \\
\hline
(2, 3, 4, 5, 6) & 1 & 173100 & 259 & 172841 \\
 & 2 & 172980 & 139 & 172841 \\
 & 3 & 172920 & 79 & 172841 \\
 & 4 & 172884 & 43 & 172841 \\
 & 5 & 172860 & 19 & 172841 \\
\hline
(2, 3, 4, 7, 11) & 1 & 3650745 & 951 & 3649794 \\
 & 2 & 3650360 & 566 & 3649794 \\
 & 3 & 3650052 & 258 & 3649794 \\
 & 4 & 3649920 & 126 & 3649794 \\
 & 5 & 3649800 & 6 & 3649794 \\
\hline
(3, 6, 9, 11, 16) & 1 & -10110744325279026 & 7852 & -10110744325286878 \\
 & 2 & -10110744325283778 & 3100 & -10110744325286878 \\
 & 3 & -10110744325285362 & 1516 & -10110744325286878 \\
 & 4 & -10110744325285938 & 940 & -10110744325286878 \\
 & 5 & -10110744325286748 & 130 & -10110744325286878 \\
\hline
\end{array}
\]

Table 2: Computation of indices \(\chi(M_k)\) and \(D_k \equiv \dim i_{M_k}^* (H(X_k))\) for four 5-gon quivers, whose edges labeled by \((a_1, a_2, a_3, a_4, a_5)\). For each quiver the indices are computed in five different branches, but \(\chi(M_k) - D_k\) listed in the last column is clearly insensitive to the choice of the branch, and in particular uniformly zero for the case that violates geometric 5-gon condition.

\(\chi(M_k)\) and \(D_k \equiv \chi(i_{M_k}^*(H(X_k))) = \dim i_{M_k}^* (H(X_k))\) individually. We show then that the intrinsic Higgs states, counted by the difference, \(\chi(M_k) - D_k\), are invariant as we move from one branch to another. Tables 1, 2 and 3 each illustrate this, for 4-gon, 5-gon and 6-gon quivers, respectively.

While we do not have general formulae for \(\Omega_{\text{Coulomb}}\), there are a couple of important checks we can perform. First, general expectation from physical considerations is that,
Table 3: Computation of indices $\chi(M_k)$ and $D_k \equiv \dim i_{M_k}^*(H(X_k))$ for three 6-gon quivers, whose edges labeled by $(a_1, a_2, a_3, a_4, a_5, a_6)$. For each quiver the indices are computed in six different branches, but $\chi(M_k) - D_k$ listed in the last column is clearly insensitive to the choice of the branch, and in particular uniformly zero for the case that violates geometric 6-gon condition.

when no scaling solution exists in the Coulomb phase, Higgs and Coulomb should agree with each other [2, 19]. When the geometric $(n + 1)$-gon condition fails, the first conjecture combined with the expectation $\chi = \Omega_{\text{Coulomb}}$ implies that

$$\chi(M) - \dim i_M^*(H(X)) = 0,$$

regardless of the branch. This can be seen to hold with the example at the top of each table.

More generally, we show by explicit examples that this difference which can be typically very large is independent of the choice of branch, consistent with the second
conjecture. As noted in the first section, the two conjectures are physically linked to each other as long as the wall-crossing behavior is entirely captured by the Coulomb phase ground states. In this indirect sense, we are effectively testing validity of the first conjecture as well.

6 Comments

In this note we conjectured that the difference between Coulomb phase and Higgs phase indices can be given a purely Higgs phase characterization, namely the difference between the full cohomology of Higgs phase itself $M$ and the pull-back to $M$ of cohomology of the ambient D-term vacuum manifold $X$. The statement that Coulomb phase states correspond to the pulled-back cohomology of ambient D-term toric variety is well motivated by the fact that the two sides agree when no loops are present and thus no F-term constraints arise either. Introducing loops and thus F-terms tends to make Higgs phase relatively more complicated, with the known result that the intrinsic Higgs states begin to show and in fact can be exponentially more numerous. In some crude sense, the Coulomb states are already known to D-term-induced ambient space in the Higgs side, to which F-terms add more states.

Since wall-crossing physics arises inherently from Coulombic physics of multi-centered nature, this leads to the second conjecture that the difference is in fact an invariant of the quiver itself. Traditional invariants, such as $\Omega_{\text{Coulomb}}$ that directly enters the wall-crossing formulae, and the Euler number $\chi$ of Higgs phase, experience wall-crossing and thus change discontinuously as we deform FI constants. The difference

$$\chi - \Omega_{\text{Coulomb}}$$

according to our first conjecture, becomes a geometric object entirely of Higgs phase,

$$\chi(M_k) - \chi(i^*_{M_k}H(X_k)),$$

which, under the second conjecture, is an invariant of the quiver itself. This is one very intriguing consequence of this work. Equality of this number among different Higgs branches implies an invariant of the quiver diagram itself, rather than individual branches. This aspect may be further explored in purely mathematical terms also.

We proved the conjectures for the simplest case of all 3-gon quivers, for which ground states in Coulomb and Higgs phases are catalogued explicitly. We also checked the conjectures numerically for a large number of 4-gon, 5-gon and 6-gon quivers, and presented a few typical examples in the note. More thorough and analytical treatments of the conjectures, including further proofs and consistency checks, will appear elsewhere [27].
As this work was drawing to conclusion, Ref. [26] appeared with some partially overlapping results.

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A Lefschetz Hyperplane Theorem

Let \( X \) be a complex compact manifold and let \( \lambda \) be a holomorphic section of a positive holomorphic line bundle \( L \) over \( X \). The section \( \lambda \) can be thought of as a defining polynomial for the hypersurface \( M = \lambda^{-1}(0) \subset X \). We then have, from Lefschetz hyperplane theorem [23], that the natural map \( H^p(X, \mathbb{Z}) \to H^p(M, \mathbb{Z}) \) is an isomorphism for \( p < \dim_{\mathbb{C}} M \) and is an injection for \( p = \dim_{\mathbb{C}} M \).

A straightforward generalisation to complete intersection cases can be made by iteration: Let \( \lambda_1, \lambda_2, \ldots, \lambda_m \) be the sections of positive line bundles \( L_1, L_2, \ldots, L_m \) over \( X \), and let \( M^{(r)} \), for \( r \leq m \), be the complete intersections of the first \( r \) hypersurfaces \( \lambda_1^{-1}(0), \ldots, \lambda_r^{-1}(0) \) to \( X \). We then have the following chain of hypersurfaces

\[
M^{(m)} \subset M^{(m-1)} \subset \cdots \subset M^{(1)} \subset M^{(0)} \equiv X ,
\]

where the ambient space \( X \) is denoted by \( M^{(0)} \). We begin with the first line bundle \( L_1 \) and consider the hypersurface \( M^{(1)} \subset X \). Since the \( r \)-th line bundle \( L_r \) is positive over \( X \), it is also positive over the subspace \( M^{(r-1)} \), which is the complete intersection of the previous \( r - 1 \) hypersurfaces. Now, by applying Lefschetz hyperplane theorem to the hypersurface \( M^{(r)} \subset M^{(r-1)} \), we have the isomorphisms

\[
H^p(M^{(r-1)}, \mathbb{Z}) \cong H^p(M^{(r)}, \mathbb{Z}) , \quad p < \dim_{\mathbb{C}} M^{(r)} ,
\]

which leads, when iteratively applied, to

\[
H^p(X, \mathbb{Z}) \cong H^p(M^{(m)}, \mathbb{Z}) , \quad p < \dim_{\mathbb{C}} M^{(m)} ,
\]

for all the lower-half cohomologies but for the middle one. Similarly, we also see that the natural map \( H^p(X, \mathbb{Z}) \to H^p(M^{(m)}, \mathbb{Z}) \) for \( p = \dim_{\mathbb{C}} M^{(m)} \), to the middle cohomology of \( M^{(m)} \), is injective.
An Exercise in Cohomology Counting

Let us recall that the Poincaré polynomial for \( X = \mathbb{CP}^{a-1} \times \mathbb{CP}^{b-1} \), with \( b \geq a \), is given by

\[
P[X] = 1 + \cdots + a \cdot x^{2(a-1)} + a \cdot x^{2a} + \cdots + a \cdot x^{2(b-1)} + (a-1) \cdot x^{2b} + \cdots + x^{2(a+b-2)},
\]

which naturally splits into three parts, in accord with the general structure (3.16), with \( 2L+1 \equiv 2a-1 \). We embed a complete intersection \( M \) using \( c \) F-term constraints, and ask what the pulled-back cohomology looks like. The answer should depend on where the \( x^d \)-term, with \( d \equiv \dim_{\mathbb{C}} M = a + b - c - 2 \), sits in \( P[X] \) relative to the power \( 2L + 1 \).

When \( d > 2L + 1 \) (or equivalently, \( b > a + c + 1 \)), the \( x^d \)-term sits in the middle plateau of \( P[X] \), and

\[
\dim i^*_M(H(X)) - \dim i^*_M(H^d(X)) = 2 \cdot \left( 1 + 2 + \cdots + a \right) + a \cdot \left[ \frac{d - 2a + 1}{2} \right]
\]

\[
= \begin{cases} 
(b - c) \cdot a & \text{if } d \text{ odd,} \\
(b - c - 1) \cdot a & \text{if } d \text{ even.}
\end{cases}
\]

Since \( \dim i^*_M(H^d(X)) = 0 \), for \( d \) odd and even, respectively, we have

\[
\dim i^*_M(H(X)) = (b - c) \cdot a,
\]

(B.2)

independent of the parity of \( d \). On the other hand, when \( d \leq 2L + 1 \) (or equivalently, \( b \leq a + c + 1 \)), the \( x^d \)-term now sits in the first part of \( P[X] \), and the cohomology counting in this case is given by

\[
\dim i^*_M(H(X)) - \dim i^*_M(H^d(X)) = 2 \cdot \left( 1 + 2 + \cdots + \left[ \frac{d + 1}{2} \right] \right)
\]

\[
= \begin{cases} 
\frac{1}{4} ((a + b - c)^2 - 1) & \text{if } d \text{ odd,} \\
\frac{1}{4} ((a + b - c)^2 - (\frac{d}{2} + 1)) & \text{if } d \text{ even.}
\end{cases}
\]

Since \( \dim i^*_M(H^d(X)) = 0, d/2 + 1 \), for \( d \) odd and even, respectively, we have

\[
\dim i^*_M(H(X)) = \frac{1}{4} \left( (a + b - c)^2 - r \right),
\]

(B.3)

where \( r = 1, 0 \) for \( d \) odd and even, respectively.
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