1. Introduction

In this paper, we study the $K$-theory on higher modules in spectral algebraic geometry. We relate the $K$-theory of an $\infty$-category of finitely generated projective modules on certain $E_\infty$-rings with the $K$-theory of an ordinary category of finitely generated projective modules on ordinary rings. We introduce earlier studies and state the main theorem (Theorem 1.2).

1.1. Background of this paper. In 1990s, Elmendorf, Kriz, Mandell and May introduced a certain symmetric monoidal category of spectra called $S$-modules [15]. For an algebra object $R$ in the category of $S$-modules, they also defined a certain model category of spectra called $R$-modules, which we denote by $\mathcal{M}_R$.

Let $\mathcal{E}_R$ be the full subcategory of $\mathcal{M}_R$ such that $\mathcal{E}_R$ consists of cofibrant-fibrant $R$-modules with only finitely many non-zero homotopy groups which are finitely generated over the 0th homotopy group $\pi_0 R$ of $R$. Blumberg and Mandell showed the following theorem.

**Theorem 1.1** ([10]). Assume further that $\pi_0 R$ is a Noetherian ring. The $K$-theory of the category $\mathcal{E}_R$ is equivalent to the $K$-theory of the ordinary category of finitely generated $\pi_0 R$-modules.

The main theorem is an analogy of Theorem 1.1 generalized to the setting of $\infty$-categories as follows.

1.2. Main theorem of this paper. Although there are a lot of languages of higher category theory, we use the same notation in Lurie’s book [18] and paper [19].

Let $\mathcal{S}_\ast$ be the $\infty$-category of pointed spaces [18, Definition 7.2.2.1]. The *stable $\infty$-category of spectra*, $\mathbf{Sp}$, is obtained by the stabilization of $\mathcal{S}_\ast$ (cf. [19, Section 6.2.2]), which has a canonical symmetric monoidal structure induced from the cartesian products.
on $S$ [19, Example 6.2.4.13, 6.2.4.17]. We denote by $S$ the sphere spectrum. We say that a spectrum $E$ is connective if $\pi_n E \simeq 0$ for $n < 0$.

Let $R$ be an $E_\infty$-ring (cf. [19, Definition 2.1.2.7]). We also have an $\infty$-category $\text{Mod}_R$, which is called the $\infty$-category of $R$-module of $\text{Sp}$ [19, Section 4.2]. Since the tensor product on $\text{Sp}$ is compatible with the geometric realizations [19, Corollary 4.8.2.19], $\text{Mod}_R$ becomes the symmetric monoidal $\infty$-category by [19, Theorem 4.5.2.1]. We denote by $\otimes_R$ the tensor product on $\text{Mod}_R$. We define an $\infty$-category of perfect $R$-modules by the smallest stable full $\infty$-subcategory of $\text{Mod}_R$ which contains $R$ and is closed under retracts, and let us denote this $\infty$-category by $\text{Mod}^\text{perf}_R$ [19, Definition 7.2.5.1]. We say that an $R$-module $M$ is perfect if it belongs to $\text{Mod}^\text{perf}_R$.

Let $R$ be a connective $E_\infty$-ring and $M$ an $R$-module. We say that $M$ is finitely generated projective if it is a retract of a finitely generated free $R$-module [19, Proposition 7.2.2.18]. We denote by $\text{Mod}^\text{proj}_R$ the $\infty$-category of finitely generated projective $R$-modules.

Let $R$ be an ordinary category of finitely generated projective $\pi_0 R$-modules. Barwick and Lawson [7] showed that, if $R$ is a regular $E_\infty$-ring with only finitely many non-zero homotopy groups, $K(\text{Mod}^\text{perf}_R)$ is equivalent to the $K$-theory of the ordinary category $\mathcal{P}_{\pi_0 R}$. Here we recall the notion of regularity on $R$ in Definition 6.1.

We obtain the following theorem for $K(\text{Mod}^\text{proj}_R)$. The statement is an analogy of the result of Barwick and Lawson for $K(\text{Mod}^\text{perf}_R)$.

**Theorem 1.2** (cf. Theorem 6.4). Let $R$ be a regular $E_\infty$-ring with only finitely many non-zero homotopy groups. Then, there is a weak equivalence $K(\text{Mod}^\text{proj}_R) \simeq K(\mathcal{P}_{\pi_0 R})$.

**1.3. Remarks for Theorem 1.2.** As a key lemma for Theorem 1.2, we show the equivalence $K(\text{Mod}^\text{proj}_R) \simeq K(\text{Mod}^\text{perf}_R)$. The difficulty in showing the equivalence is that the resolution theorem is not established in the $K$-theory of $\infty$-categories.

Recently, Mochizuki [22] proved a resolution theorem in Waldhausen $K$-theory with certain assumptions which is explained in Theorem 5.7. We construct the relation between a sequence of certain subcategories in $\text{M}_R$ and that in $\text{Mod}^\text{perf}_R$ respectively, and apply his resolution theorem to the sequence of subcategories in $\text{M}_R$.

We remark that Lurie also shows the equivalence $K(\text{Mod}^\text{proj}_R) \simeq K(\text{Mod}^\text{perf}_R)$ in the completely $\infty$-categorical setting for a connective $E_\infty$-ring $R$ in his lecture [20]. It is a different kind of proof from ours.

**1.4. Outline of this paper.** This paper is organized as follows. In Section 2, we introduce the terminology of relative categories and Dwyer-Kan localization, and show Lemma 2.5 together with the functorial factorization and homotopically full condition. In Section 3, we have the existence of mapping cylinders and mapping path spaces in
the categories of spectra which we mainly use in this paper. We demonstrate the compatibility of simplicialicity and monoidality on the category of symmetric spectra, and have the correspondence between the category of $R$-modules in the sense of Elmendorf-Kriz-Mandell-May and the $\infty$-category of $R$-modules as Proposition 3.16. In Section 4, we define the notion of perfect $R$-modules in the category of $R$-modules in the sense of Elmendorf-Kriz-Mandell-May, and prove the correspondence in Lemma 4.6 between the perfect $R$-modules in the sense of Elmendorf-Kriz-Mandell-May and the $\infty$-categorical perfect $R$-modules. By using these consequences, we show an equivalence between the algebraic $K$-theory of perfect $R$-modules and the algebraic $K$-theory of finitely generated projective $R$-modules in Proposition 5.13 in Section 5. In Section 6, by using the consequence of Barwick and Lawson, we prove Theorem 6.4.

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2. Relative categories and Dwyer-Kan localization

A relative category is a pair $(\mathcal{C}, W)$ consisting of a category $\mathcal{C}$ and a subcategory $W \subset \mathcal{C}$ whose objects are the objects of $\mathcal{C}$ and whose class of morphisms is a certain class of morphisms in $\mathcal{C}$. The morphisms in $W$ are called weak equivalences. For relative categories $(\mathcal{C}, W)$ and $(\mathcal{C}', W')$, a functor $(\mathcal{C}, W) \to (\mathcal{C}', W')$ of relative categories is defined by a functor $\mathcal{C} \to \mathcal{C}'$ which sends $W$ in $W'$.

Recall that a monoidal category is an ordinary category $\mathcal{C}$ equipped with an associative product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $1$ [18, A. 1.3].

Let $\text{Set}_\Delta$ denote the category of simplicial sets. In this paper, we use the term “a simplicial category” as a $\text{Set}_\Delta$-enriched category. (For the definition of enrichment, see [18, A. 1.4].) Let $\text{Cat}_\Delta$ denote the category of simplicial categories such that the objects are small simplicial categories, and a morphism is a $\text{Set}_\Delta$-enriched functor.

We assume that $\text{Cat}_\Delta$ is endowed with the Bergner model structure (cf. [18, Definition A.3.2.16]) and call these weak equivalences Dwyer-Kan equivalences. The study of Bergner proved that a fibrant object of this model category is a fibrant simplicial category i.e. a simplicial category whose mapping spaces are Kan complexes (for a proof, see [18, Theorem A 3.2.2.4]).

There is a simplicial nerve functor $N_\Delta : \text{Cat}_\Delta \to \text{Set}_\Delta$ which is the right Quillen functor of model categories,

$$
\mathcal{C} : \text{Set}_\Delta \rightleftarrows \text{Cat}_\Delta : N_\Delta,
$$
where the model structure on $\text{Set}_\Delta$ is Joyal model structure and that on $\text{Cat}_\Delta$ is Bergner model structure. They induce the Quillen equivalence [18, Theorem 2.2.5.1].

**Definition 2.1** (Relative nerve [19] Definition 1.3.4.1). Let $\mathcal{C}$ be an $\infty$-category, and $W$ a collection of morphisms in $\mathcal{C}$.

(i) We say that a morphism $f : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as an $\infty$-category obtained from $\mathcal{C}$ by inverting the set of morphisms $W$ if, for every $\infty$-category $\mathcal{E}$, the composition with $f$ induces a fully faithful embedding $\text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})$ whose essential image is the collection of functors $\mathcal{C} \to \mathcal{E}$ which carry each morphism in $W$ to an equivalence in $\mathcal{E}$.

(ii) In the case of (i), the $\infty$-category $\mathcal{D}$ is determined uniquely up to equivalence by $\mathcal{C}$ and $W$. If $\mathcal{C}$ is an ordinary category and $W$ is a collection of morphisms in $\mathcal{C}$, we proceed the above construction for $N_\Delta(\mathcal{C})$ and denote the $\infty$-category $N_\Delta(\mathcal{C})$ as $N_\Delta(\mathcal{C})[W^{-1}]$. We call $N_\Delta(\mathcal{C})[W^{-1}]$ the relative nerve of $(\mathcal{C}, W)$.

**Remark 2.2.** For a relative category $(\mathcal{C}, W)$, the condition that the subcategory $W$ contains the every object in $\mathcal{C}$ implies the condition that the morphisms in $N_\Delta(\mathcal{C})$ spanned by $W$ contains all the degenerate edges.

Let $(\mathcal{C}, W)$ be a relative category. Let $W_{N_\Delta(\mathcal{C})}$ is a collection of morphisms which is generated by the image of $W$ in $N_\Delta(\mathcal{C})$. If we regard $N_\Delta(\mathcal{C})[W_{N_\Delta(\mathcal{C})}^{-1}]$ as a marked simplicial set $N_\Delta(\mathcal{C})[W_{N_\Delta(\mathcal{C})}^{-1}]^\ast$, then it is a fibrant replacement of $(N_\Delta(\mathcal{C}), W_{N_\Delta(\mathcal{C})})$ in $(\text{Set}_\Delta)^+ / \ast$ [19, Remark 1.3.4.2]. It follows from that the morphism $N_\Delta(\mathcal{C}) \to \ast$ factors through $N_\Delta(\mathcal{C})[W_{N_\Delta(\mathcal{C})}^{-1}] \to \ast$ by the definition of relative nerve.

### 2.1. Localization of relative categories.

Recall that, for a relative category $(\mathcal{C}, W)$, there is a construction of a simplicial category from a relative category, which is called a Dwyer-Kan localization [13].

A hammock localization [13, Section 2.1] of $(\mathcal{C}, W)$ is one of the explicit construction of a simplicial category from a relative category [13, Proposition 2.2], [12]. For an ordinary relative category $(\mathcal{C}, W)$, we denote by $L^H(\mathcal{C}, W)$ its hammock localization.

Recall that the objects of $L^H(\mathcal{C}, W)$ are the objects of $\mathcal{C}$. For $X, Y \in \mathcal{C}$, we denote by $L^H(\mathcal{C}, W)(X, Y)$ the mapping space of $X, Y \in L^H(\mathcal{C}, W)$. Note that $L^H(\mathcal{C}, W)(-,-)$ satisfies the axiom of identity and associitative composition of enrichment [18, A 1.4] by concatenating the horizontal morphisms in the hammock diagrams .

Note that the homotopy category $hL^H(\mathcal{C}, W)$ of the hammock localization $L^H(\mathcal{C}, W)$ is categorically equivalent to $\mathcal{C}[W^{-1}]$ [13, Proposition 3.1].

For a relative category $(\mathcal{C}, W)$, let $L^H(\mathcal{C}, W))^{fib}$ be the hammock localization, where $(-)^{fib}$ denotes a fibrant replacement with respect to the Bergner model structure. Then,
we have an equivalence $N_\Delta((\mathcal{C})[W^{-1}] \simeq N_\Delta((L^H(\mathcal{C}, W))^{fib})$ of $\infty$-categories [16, Proposition 1.2.1].

For the terminology of model category in this paper is as follows.

In terminology of Lurie, which we adopt in this paper, the model categories are required the existence of small limits and colimits. (On the other hand, Quillen [23] required the existence of only finite limits and colimits). However, since the model categories which will appear in this paper have small limits and colimits, we use the notation “a model category” in the sense of Lurie in this paper.

We remark that the terminology of Quillen [23] is used in Elmendorf-Kriz-Mandell-May [15], Mandell-May-Schwede-Shipley [21], Dwyer-Kan [12] [13] [14] and Barwick-Kan [5] [6].

For a model category $\mathcal{C}$, let $\mathcal{C}^c$, $\mathcal{C}^f$ and $\mathcal{C}^{c\cdot f}$ be the subcategory of cofibrant, fibrant and cofibrant-fibrant objects respectively. We denote by $W_\mathcal{C}$ the subcategory of weak equivalences in $\mathcal{C}$.

**Definition 2.3** ([6] Section 1.2). For a model category $\mathcal{C}$, let $(\mathcal{C}, W_\mathcal{C})$ be a relative category arising from $\mathcal{C}$. For a relative subcategory $(\mathcal{D}, W_\mathcal{D})$ of $(\mathcal{C}, W_\mathcal{C})$, we say that $(\mathcal{D}, W_\mathcal{D})$ is a homotopically full relative subcategory if it is a relative category of the form $(\mathcal{D}, W_\mathcal{C} \cap \mathcal{D})$, where $\mathcal{D}$ is a full subcategory of $\mathcal{C}$, which has the following property: if an object $C \in \mathcal{C}$ has a zig-zag of weak equivalences to an object $D \in \mathcal{D}$, then $C \in \mathcal{D}$.

**Definition 2.4** ([11] Example 5.2). Let $(\mathcal{C}, W)$ be a relative category, $A$ and $B$ objects of $\mathcal{C}$. We define a category $W^{-1}\mathcal{C}(A, B)$ by the following data.

(i) An object is a pair $\{X, f_1, f_2\}$, where $X \in \mathcal{C}$, $f_1 : B \to X$ is a morphism in $W$ and $f_2 : A \to X$ is a morphism in $\mathcal{C}$.

(ii) A morphism $\{X, f_1, f_2\} \to \{X', f'_1, f'_2\}$ is defined as the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f_2} & X \\
\downarrow^{f'_2} & \phi & \downarrow^{f_1} \\
X' & \xleftarrow{f'_1} & B
\end{array}
\]

where $\phi : X \to X'$ is a morphism in $W$.

In the case of homotopically full subcategories of model categories whose subcategory of cofibrants admits the functorial factorization [11, Section 2], we will describe the mapping space $L^H(-, -)$ more simply. In general, a combinatorial model category admits the functorial cofibrant and fibrant replacement. Therefore, its subcategory of cofibrants and its opposite of subcategory of fibrants especially admit the functorial factorization. Note that, in terminology of Quillen, a model category is not assumed to have the functorial cofibrant and fibrant replacement.
Lemma 2.5. Let \( \mathcal{C} \) be a model category whose subcategory of cofibrants admits the functorial factorization in the sense of [11, Section 2]. Let \((\mathcal{C}, W_{\mathcal{C}})\) be a relative category arising from \( \mathcal{C} \), and \((\mathcal{D}, W_{\mathcal{D}})\) a homotopically full relative subcategory of \((\mathcal{C}, W_{\mathcal{C}})\).

(i) If \( A \) and \( B \) are cofibrant-fibrant objects in \( \mathcal{D} \) (resp. \( \mathcal{C} \)), \( N_{\Delta}(W_{\mathcal{D}}^{-1}\mathcal{D}(A, B)) \simeq L^{H}(\mathcal{D}, W_{\mathcal{D}})(A, B) \) (resp. \( N_{\Delta}(W_{\mathcal{C}}^{-1}\mathcal{C}(A, B)) \simeq L^{H}(\mathcal{C}, W_{\mathcal{C}})(A, B) \)). Here, we regard a category \( W_{\mathcal{D}}^{-1}\mathcal{D}(A, B) \) (resp. \( W_{\mathcal{C}}^{-1}\mathcal{C}(A, B) \)) as a trivial simplicial category.

(ii) We have \( L^{H}(\mathcal{D}, W_{\mathcal{C}} \cap \mathcal{D})(A, B) \simeq L^{H}(\mathcal{C}, W_{\mathcal{C}})(A, B) \) for cofibrant-fibrant objects \( A, B \in \mathcal{D} \).

Proof. We prove (i). By the assumption, \( \mathcal{C}^{e} \) satisfies the assumption of [11, Proposition 5.4] by [9, Section 2.2]. Therefore, we apply [11, Proposition 5.4] to \( \mathcal{C}^{e} \). Since \( \mathcal{D} \) is the homotopically full subcategory of \( \mathcal{C} \), we also apply [11, Proposition 5.4] to \( \mathcal{D}^{e} \), so that we have \( N_{\Delta}(W_{\mathcal{C}}^{-1}\mathcal{C}(−, −)) \simeq L^{H}(\mathcal{C}, W_{\mathcal{C}})(−, −) \) and \( N_{\Delta}(W_{\mathcal{D}}^{-1}\mathcal{D}(−, −)) \simeq L^{H}(\mathcal{D}, W_{\mathcal{D}})(−, −) \). By [14, Proposition 5.2], we have \( L^{H}(\mathcal{C}, W_{\mathcal{C}}) \simeq L^{H}(\mathcal{C}, W_{\mathcal{C}}) \) and \( L^{H}(\mathcal{D}, W_{\mathcal{D}}) \simeq L^{H}(\mathcal{D}, W_{\mathcal{D}}) \).

For (ii), let \( A \) and \( B \) be objects in \( \mathcal{D} \). By comparing the diagram \( W_{\mathcal{D}}^{-1}\mathcal{D}(A, B) \) with \( W_{\mathcal{C}}^{-1}\mathcal{C}(A, B) \), we have the weak homotopy equivalence \( W_{\mathcal{D}}^{-1}\mathcal{D}(A, B) \simeq W_{\mathcal{C}}^{-1}\mathcal{C}(A, B) \). By applying (i), we have \( L^{H}(\mathcal{D}, W_{\mathcal{C}} \cap \mathcal{D})(A, B) \simeq L^{H}(\mathcal{C}, W_{\mathcal{C}})(A, B) \).

\( \square \)

3. Relation between classical and \( \infty \)-categorical spectra

3.1. \( R \)-modules in the sense of Elmendorf-Kriz-Mandell-May. Throughout this paper, we denote by \( \wedge \) the smash products in the sense of topology (defined on spaces and spectra).

Let \( R \) be an \( S \)-algebra, and \( \mathcal{M}_{R} \) the category of \( R \)-modules in the sense of Elmendorf-Kriz-Mandell-May [15]. Recall that \( \mathcal{M}_{R} \) admits the symmetric monoidal structure [15, II, Section 3] by the smash product over \( R \), which we denote by \( \wedge_{R} \).

The category \( \mathcal{M}_{R} \) admits a model structure which is defined in [15, VII Section 4]. Let us denote the subcategory of weak equivalences in \( \mathcal{M}_{R} \) by \( W_{\mathcal{M}_{R}} \).

Remark 3.1 (cf. [25]). Note that every object in \( \mathcal{M}_{R} \) is fibrant. Therefore, we have \( \mathcal{M}_{R}^{f} = \mathcal{M}_{R} \) and \( \mathcal{M}_{R}^{e} = \mathcal{M}_{R}^{o} \).

Let \( I \) denote the unit interval \([0, 1]\), where we regard \( \{0\} \) as the base point.

Definition 3.2. Let \( A \) and \( B \) be objects in \( \mathcal{M}_{R} \).
(i) A mapping cylinder for \( f : A \to B \), denoted by \( Mf \), is defined by the following pushout:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\partial_1} & & \downarrow{j_f} \\
A \land I & \xrightarrow{\pi_f} & Mf,
\end{array}
\]

where \( A \land I \) is a cylinder.

(ii) Dually, a mapping path space for \( f : A \to B \), denoted by \( Nf \), is defined by the following pullback:

\[
\begin{array}{ccc}
Nf & \xrightarrow{\pi_f'} & B' \\
\downarrow{j_f'} & & \downarrow{d_1} \\
A & \xrightarrow{f} & B.
\end{array}
\]

Lemma 3.3. Mapping cylinders and mapping path spaces exist in the model category \( \mathcal{M}_R \).

Proof. By [15, III, Lemma 3.2], the tensor product on \( \mathcal{M}_R \) has the right adjoint, so that \( \mathcal{M}_R \) has a path space. Then, the assertion follows from [15, III, Theorem 1.1]. □

Let \( A \) and \( B \) be cofibrant objects in \( \mathcal{M}_R \). Let \( f : A \to B \) be a morphism. A mapping cylinder gives a functorial factorization for \( \mathcal{M}_R^f \), i.e., every map \( f : A \to B \) factors through \( A \to Mf \to B \), where the map \( A \to Mf \) is a cofibration and \( Mf \to B \) has a natural section \( B \to Mf \), which is a weak equivalence. Moreover, \( Mf \) is a cofibrant object.

Dually, a mapping path object gives a functorial factorization for the opposite category \( (\mathcal{M}_R)^{op} \) (Recall that every object in \( \mathcal{M}_R \) is fibrant.), i.e., every map \( f : A \to B \) factors through \( A \to Nf \to B \) of the map \( A \to Nf \) has a natural projection \( Nf \to A \), which is a weak equivalence, and \( Nf \to B \) is a fibration.

Definition 3.4. Let \( R \) be a commutative \( S \)-algebra, \( A \) and \( X_R \)-modules in the sense of Elmendorf-Kriz-Mandell-May. Let \( i : A \to X \) a morphism of \( R \)-modules. We say that \( i \) is a Hurewicz cofibration if \( Mi = X \cup_i (A \land I) \) has the cylinder \( X \land I \) as a retract. Here, a mapping cylinder is defined in Definition 3.2.

Remark 3.5. Note that, by [15, VII, Theorem 4.15], the Hurewicz cofibrations include the cofibrations in \( \mathcal{M}_R \). Especially, on the cofibrant-fibrant objects, the weak equivalences are homotopy equivalences and the Serre fibrations are Hurewicz fibrations, so that the Hurewicz cofibrations is the cofibrations.

3.2. The category of symmetric spectra. According to [17], we recall the notion of symmetric spectra as follows.

Let \( \mathcal{M}^{sym} \) be a category of symmetric spectra built from the simplicial sets [17, Section 2.2]. We regard the category \( \mathcal{M}^{sym} \) as a simplicial category by [17, Definition 2.1.10]. The
category $\mathcal{M}^{sym}$ is a symmetric monoidal category with respect to tensor products [17, Definition 2.1.3] which is induced from the smash product. A (commutative) algebra object $R \in \mathcal{M}^{sym}$ with respect to the tensor product on $\mathcal{M}^{sym}$ is called a symmetric (commutative) ring spectrum.

For a symmetric ring spectrum $R$, an $R$-module object in $\mathcal{M}^{sym}$ with respect to the tensor product on $\mathcal{M}^{sym}$ is called a symmetric $R$-module spectrum. Let $\mathcal{M}^{sym}_R$ be a category which consists symmetric $R$-module spectra in $\mathcal{M}^{sym}$ and the morphisms compatible with the $R$-module structure. We also denote by $\wedge_R$ the smash product on $\mathcal{M}^{sym}_R$.

There exists a left proper combinatorial model structure on $\mathcal{M}^{sym}$ defined in [21, Definition 9.1], which is called the stable model structure.

### 3.3. Compatibility of simpliciality and monoidality on $\mathcal{M}^{sym}$

We say that $\mathcal{A}$ is a simplicial model category if it is a Set$_{\Delta}$-enriched model category in the sense of [18, Definition A.3.1.5].

**Remark 3.6 ([19], Example 4.1.3.6).** Let $\mathcal{M}$ be an ordinaly symmetric monoidal category. Then, the $\infty$-category $N_{\Delta}(\mathcal{M}^\circ)[W^{-1}]$ inherits the symmetric monoidal structure.

The following definition is a special case of [19, Definition 4.1.3.7].

**Definition 3.7.** Let $\mathcal{C}$ be a simplicial category.

(i) We say that a (symmetric) monoidal structure on $\mathcal{C}$ is weakly compatible with the simplicial structure on $\mathcal{C}$ if the (symmetric) monoidal operation $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a simplicial functor, and the associativity and unit condition (and symmetricity) is given by a natural transformation of simplicial functors.

(ii) Furthermore, assume that the monoidal structure on $\mathcal{C}$ satisfies the following conditions: for every $X, Y \in \mathcal{C}$, there exists an object $X^Y \in \mathcal{C}$ and evaluation morphism $e : X^Y \otimes Y \to X$ such that the morphism induces a bijection $\text{Hom}_C(Z, X^Y) \cong \text{Hom}_C(Z \otimes Y, X)$ for any $Z \in \mathcal{C}$.

(iii) We say that a symmetric monoidal structure on $\mathcal{C}$ is compatible with the simplicial structure on $\mathcal{C}$ if it is weakly compatible and the evaluation morphism $e$ induces an isomorphism $\text{Map}_C(Z, X^Y) \cong \text{Map}_C(Z \otimes Y, X)$ of simplicial sets for any $Z \in \mathcal{C}$.

**Definition 3.8** (cf. [19] Definition 4.1.3.8). A simplicial symmetric monoidal model category is a symmetric monoidal model category which is also equipped with the structure of a simplicial model category such that the simplicial structure and the (symmetric) monoidal structure are compatible in the sense of Definition 3.7.

Now, let $\mathcal{M}^{sym}$ be the category of symmetric spectra endowed with the stable model structure.
Proposition 3.9. (i) The category $\mathcal{M}^{sym}$ is a simplicial model category.
(ii) The category $\mathcal{M}^{sym}$ is a symmetric monoidal model category.

Proof. To prove that $\mathcal{M}^{sym}$ is symmetric monoidal model category, we check the conditions in [18, Definition A.3.1.2]. The condition (i) follows from [17, Proposition 3.4.2 (4), Theorem 5.3.7 (5)]. The symmetric sphere is a cell, so that it is cofibrant. The symmetric monoidal structure of $\mathcal{M}^{sym}$ is closed by [17, Theorem 5.5.2].

To prove that $\mathcal{M}^{sym}$ is a simplicial model category, what we prove is the conditions in [18, Definition A.3.1.1] and [18, Definition A.3.1.5].

The condition (i) of [18, Definition A.3.1.5] follows from [17, Proposition 1.3.1]. For the condition (ii) of [18, Definition A.3.1.5], we check the conditions in [18, Definition A.3.1.1]. The condition (i) of [18, Definition A.3.1.1] follows from [17, Proposition 3.4.2 (4), Theorem 5.3.7 (5)]. The condition (ii) of [18, Definition A.3.1.1] follows from [17, Proposition 1.2.10].

Proposition 3.10. $\mathcal{M}^{sym}$ is a simplicial symmetric monoidal model category in the sense of Definition 3.8

Proof. We check the compatibility in Definition 3.7(ii). The assumption of Definition 3.7 is satisfied by the definition of symmetric monoidal structure in $\mathcal{M}^{sym}$ [17, Definition 2.2.1]. By [17, Theorem 2.1.11], the condition Definition 3.7(ii) holds.

3.4. Relation between classical and $\infty$-categorical spectra. Let $\text{Kan}_*$ be a category of pointed Kan complexes. Here, a pointed simplicial set is a simplicial set together with a morphism $*: \to X$ from a point. Let $W_{\text{Kan}}$ be the category of weak homotopy equivalences in $\text{Kan}_*$. We can identified $S_*$ with the relative nerve $n: N_\Delta(\text{Kan}_*) \to N_\Delta(\text{Kan}_*)[W_{\text{Kan}}^{-1}]$, where we regard $\text{Kan}_*$ as an ordinary category and $n$ is induced from the inclusion which becomes the functor associated with the relative nerve in Definition 2.1. Then, we have the following functor induced from $n$

$$n': \text{Kan}_* \to S_*.$$

Let $\Omega = \text{Map}(S^1, -)$ be the endofunctor on $\text{Kan}_*$. Note that the endofunctor $\Omega = \text{Map}(S^1, -)$ on $\text{Kan}_*$ induces the derived functor on $N_\Delta(\text{Kan}_*)$ defined by homotopy cartesian. Since $N_\Delta$ is the right adjoint, it preserves small limits. Therefore, $n'$ commutes with $\Omega$.

We have the following commutative diagram

$$\cdots \xrightarrow{\Omega} S_* \xrightarrow{\Omega} S_* \xrightarrow{n'} \cdots \xrightarrow{\Omega} \text{Kan}_* \xrightarrow{\Omega} \text{Kan}_*,$$

where $\ldots$ denotes the identity functor.
where $S_{\ast}$ is an $\infty$-category of pointed spaces.

The bottom sequence in the diagram (3.1) gives the classical spectra built from the simplicial set and the upper sequence in the diagram gives the spectrum objects determined. The functor $n'$ gives an assignment between them.

Consequently, this assignment gives rise to the following equivalence in the proposition which is a special case of [1, Proposition B.3].

**Proposition 3.11.** Let $M^{\text{sym}}$ be a category of symmetric spectra endowed with the stable model structure. Then, the assignment in (3.1) induces an equivalence of $\infty$-categories

$$N_\Delta((M^{\text{sym}})^c)[W_{M^{\text{sym}}}^{-1}] \simeq \text{Sp}.$$ 

**Proof.** Since the category $\text{Set}_{\Delta}$ endowed with the Kan model structure is a left proper cellular simplicial model category, we can apply [1, Proposition B.3] to $M^{\text{sym}}$. Take $\mathcal{E} = \text{Set}_{\Delta}$. Then, the left hand side is a model category of symmetric spectra endowed with the stable model category and the right hand side is the stabilization of the $\infty$-category $S$, which is equivalent to $\text{Sp}$. □

By Proposition 3.9 and Proposition 3.10, $M^{\text{sym}}$ satisfies the assumption [19, Proposition 4.3.3.15], so that we have the following proposition.

**Proposition 3.12.** Let $R$ be a commutative cofibrant symmetric ring spectrum and $M^{\text{sym}}_R$ the category of symmetric $R$-module spectra.

Then, there is a combinatorial model structure on $M^{\text{sym}}_R$ defined as follows:

(i) A morphism of $R$-modules is a weak equivalence (resp. a fibration) if it is a weak equivalence (resp. a fibration) as a morphism of spectra in $M^{\text{sym}}$ endowed with the stable model structure.

(ii) If we regard $M^{\text{sym}}$ as simplicial monoidal model category, $M^{\text{sym}}_R$ becomes a simplicial model category.

In terminology of [21, Theorem 12.1] and [17, Corollary 5.4.3], the model structure on $M^{\text{sym}}_R$ which is defined in Proposition 3.12 is called the stable model structure. We denote by $W_{M^{\text{sym}}_R}$ the subcategory of weak equivalences. □

Since an $\infty$-category of $(A, A)$-bimodules is equivalent to $\text{Mod}_A$, we have the following proposition by applying Proposition 3.12 and [19, Theorem 4.3.3.17] to $M^{\text{sym}}$ endowed with the stable model structure.

**Proposition 3.13 ([19], Theorem 4.3.3.17).** Let $R$ be a commutative cofibrant symmetric ring spectrum and $M^{\text{sym}}_R$ the category of symmetric $R$-module spectra endowed with the
stable model structure. There is an equivalence of $\infty$-categories

$$N_\Delta((\mathcal{M}_R^{sym})^c)[W_{\mathcal{M}_R^{sym}}^{-1}] \simeq \text{Mod}_{R'},$$

where $R'$ is an object in $\text{Sp}$ corresponding to $R$ under the equivalence in Proposition 3.11, which becomes an $E_\infty$-ring.

\[Q.E.D.\]

We introduce another model structure on $\mathcal{M}_R^{sym}$.

Let $R$ be a symmetric ring spectrum. Let $\mathcal{M}_R^{sym}$ be the category of symmetric $R$-module. Assume that $\mathcal{M}_R^{sym}$ is endowed with the positive model structure which is defined in [21, Section 14]. By [21, Proposition 14.6], the positive model structure is Quillen equivalent to the stable model structure obtained in Proposition 3.12.

Since $\mathcal{M}_R^{sym}$ is built via the sequences of simplicial sets and has the set of generating cofibrations and acyclic cofibrations by [21, Theorem 14.1], it is a combinatorial model category for suitable cardinal [17, Proposition 3.2.3.13]. Therefore, we can take a cofibrant replacement functorially [18, Proposition 1.2.5].

Let $A$ be an $S$-module in the sense of Elmendorf-Kriz-Mandell-May. We say that a cofibrant $S$-module $A^{-1}$ is a cofibrant desuspension of $A$ if $A^{-1}$ is endowed with a weak equivalence $A^{-1} \wedge_S S^1 \to A$, where $\wedge_S$ is the smash product over $S$. Let $\mathcal{M}_S$ be the category of $S$-modules. By virtue of [15, II, 1.7], there exists a cofibrant desuspension of $S$ in $\mathcal{M}_S$. We denote by $(S^{-1})^c$ a cofibrant desuspension of $S$. Set $(S^0)^c = S$. We define a functor $\Phi : \mathcal{M}_S \to \mathcal{M}^{sym}$ by sending $M$ to a symmetric spectra $\Phi(M)$ whose $n$-th space is given by

$$\Phi(M)_n = \mathcal{M}_S(((S^{-1})^c)^n, M).$$

It was proved in [24] that $M$ and $\Phi(M)$ have the same homotopy groups.

Schwede [24] constructed a Quillen equivalence between the spectra, algebras and modules in $\mathcal{M}^{sym}$ and $\mathcal{M}_S$.

**Theorem 3.14 ([24]).** Let $R$ be a cofibrant-fibrant commutative $S$-algebra. Let $Q$ be a cofibrant replacement functor on $\mathcal{M}_{\Phi(R)}^{sym}$ [17, Proposition 3.2.3.13]. The functor $\Phi$ has a left adjoint denoted by $\Lambda$, and they induce a Quillen equivalence

$$\mathcal{M}_{\Phi(R)}^{sym} \rightleftarrows \mathcal{M}_R,$$

where $\mathcal{M}_R$ is endowed with the model structure and $\mathcal{M}_{\Phi(R)}^{sym}$ is endowed with the positive stable model structure.

\[Q.E.D.\]

Recall that the class of weak equivalences in $\mathcal{M}_R^{sym}$ with respect to stable model structure (resp. with respect to positive stable model structure) is denoted by $W_{\mathcal{M}_R^{sym}}$ (resp. $W_{\mathcal{M}_R^{sym}}'$) and the class of weak equivalences in $\mathcal{M}_R$ is denoted by $W_{\mathcal{M}_R}$. 

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Lemma 3.15. There is an equivalence $N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}] \simeq N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}]$ of $\infty$-category induced by the identity functor on $\mathcal{M}_{\Phi(R)}^{sym}$, where the left hand side is obtained by the positive stable model structure on $\mathcal{M}_{\Phi(R)}^{sym}$ and the right hand side is obtained by the stable model structure on $\mathcal{M}_{\Phi(R)}^{sym}$.

Proof. Since the model structure defined in Proposition 3.12 (resp. with respect to positive model structure) is combinatorial, we apply [19, Lemma 1.3.4.2.1] to the Quillen equivalence between the stable and positive stable model structures. Then, we obtain that the identity functor on $\mathcal{M}_{\Phi(R)}^{sym}$ induces an equivalence $N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}] \simeq N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}]$ of $\infty$-categories.

Proposition 3.16. Let $R$ be a cofibrant-fibrant commutative $S$-algebra. Let $Q$ be a cofibrant replacement functor on $\mathcal{M}_{\Phi(R)}^{sym}$ [17, Proposition 3.2.3.13].

Then, there is an equivalence of $\infty$-categories

\begin{equation}
N_\Delta(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}] \simeq \text{Mod}_{R'}.
\end{equation}

Proof. By Proposition 3.13 and Lemma 3.15, it is sufficient to show the equivalence $N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}] \simeq N_\Delta(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons [W_{\mathcal{M}}}^{-1}]$, where $\mathcal{M}_{\Phi(R)}^{sym}$ is endowed with positive stable model structure.

We regard $(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$ and $\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$ as ordinary categories with weak equivalences. $\Lambda$ and $\Phi$ the left and right Quillen adjoint functor given in Theorem 3.14.

Since $\Lambda(*) = *$, $\Lambda$ preserves cofibrant objects. Since every object in $\mathcal{M}_{\Phi}$ is fibrant, we take a functor $F : (\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c \rightarrow \mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$ as the composition $(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c \subset \mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$ with $\Lambda$. Since $\Lambda$ is a Quillen equivalence, $F$ preserves weak equivalences on $(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$. We also take $G : \mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c \rightarrow (\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$ as the composition $\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c \subset \mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c$ with $Q \circ \Phi$. Since $\Phi$ preserves weak equivalences on fibrant objects, $G$ preserves weak equivalences. Thus, we obtain the adjunction $N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c) \rightleftharpoons N_\Delta(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c)$ of simplicial set marked by weak equivalences. Since $(\Lambda, \Phi)$ is a Quillen equivalence and we have a cartesian equivalence between $N_\Delta(\mathcal{C})[\tilde{W}_{\mathcal{N}}]^{-1}$ and the marked simplicial set $(N_\Delta(\mathcal{C}), W_{\mathcal{N}})$, this adjunction induces an equivalence $N_\Delta((\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c)[W_{\mathcal{M}}^{-1}] \simeq N_\Delta(\mathcal{M}_{\Phi(R)}^{sym} \rightleftharpoons c)$ of $\infty$-categories by [18, Proposition 3.1.3.5 (2)].

4. Subcategories of perfect $R$-modules

Definition 4.1. A connective ring spectrum $R$ is coherent if $\pi_0 R$ is coherent (i.e. every finitely generated ideal is finitely presented as $\pi_0 R$-module) and $\pi_n R$ is finitely presented $\pi_0 R$-module for $n \geq 0$. 12
**Definition 4.2.** Let $R$ be a connective $\mathbb{E}_\infty$-ring.

(i) We say that $R$-module $M$ in $\text{Mod}_R$ (resp. in $\mathcal{M}_R$) is a discrete $R$-module if its homotopy group $\pi_n M$ vanishes if $n$ is not equal to 0.

(ii) We say that $R$-module $M$ in $\text{Mod}_R$ (resp. in $\mathcal{M}_R$) is Tor-amplitude $\leq n$ if, for all $i > n$, $\pi_i (M \otimes_R N) = 0$ for any discrete $R$-module $N$ (resp. any cofibrant discrete $R$-module $N$).

(iii) For a coherent $\mathbb{E}_\infty$-ring $R$, we define an $\infty$-category $\text{Mod}_{n,p}^R$ by a full $\infty$-subcategory of $\text{Mod}_{perf}^R$ consisting of the objects which is connective and have Tor-amplitude $\leq n$.

(iv) For a coherent ring spectrum $R$, a full subcategory $\mathcal{M}_{n,p}^R \subset \mathcal{M}_R$ is defined by those $R$-modules such that $\pi_n M = 0$ for sufficiently small $n$, $\pi_m M$ is finitely presented $\pi_0 R$-modules for every $m \in \mathbb{Z}$ and there exists $n$ such that $M$ has Tor-amplitude $\leq n$.

(v) For a coherent ring spectrum $R$, we define a category $\mathcal{M}_{n,p}^R \subset \mathcal{M}_{n,p}^R$ by a full subcategory of those connective $R$-modules of Tor-amplitude $\leq n$ for fixed $n$.

**Remark 4.3.** Note that, if $R$ is a connective coherent $\mathbb{E}_\infty$-ring, by [19, Proposition 7.2.5.23 (4), Proposition 7.2.5.17], the condition of perfect is described by the condition on homotopy groups as Definition 4.2.

We denote by $(\mathcal{M}_{n,p}^R)^{\circ}$ full subcategory of cofibrant-fibrant objects in $\mathcal{M}_{n,p}^R$ with respect to the model structure of $\mathcal{M}_R$.

**Lemma 4.4.** (i) The subcategory $\mathcal{M}_{n,p}^R \subset \mathcal{M}_R$ is closed under weak equivalences.

(ii) Mapping cylinders and mapping path spaces exist in $\mathcal{M}_{n,p}^R$.

**Proof.** The assertion (i) follows from the direct calculation of homotopy groups since the cofibrant replacement is a weak equivalence.

Then, (ii) follows from Lemma 3.3. (We remark that the first assertion also follows from [15, III, Theorem 3.8].) \(\square\)

**Lemma 4.5.** Let us regard the categories $\mathcal{M}_{n,p}^R$ and $\mathcal{M}_R$ as the relative categories with respect to the weak equivalences. We have an embedding $L^H(\mathcal{M}_{n,p}^R)^{\circ} \subset L^H(\mathcal{M}_R)^{\circ}$ such that it induces the weak homotopy equivalence on mapping spaces.

**Proof.** By the construction of hammock localization $L^H$, $\mathcal{M}_{n,p}^R \subset \mathcal{M}_R$ induces an embedding $L^H \mathcal{M}_{n,p}^R \subset L^H \mathcal{M}_R$. Since $\mathcal{M}_{n,p}^R \subset \mathcal{M}_R$ is a homotopically full subcategory by Lemma 4.4(i), it follows from Lemma 2.5 and the fact that $L^H(\mathcal{M}_{n,p}^R)^{\circ} \simeq L^H \mathcal{M}_{n,p}^R$ and $L^H(\mathcal{M}_R)^{\circ} \simeq L^H \mathcal{M}_R$ by [14, Proposition 5.2]. \(\square\)

**Lemma 4.6.** We have $N_\Delta((\mathcal{M}_{n,p}^R)^{\circ})[W_{\mathcal{M}_R}^{-1}] \simeq \text{Mod}_{n,p}^R$. 

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Proof. By Lemma 4.4 and Lemma 3.3, the inclusion $\mathcal{M}_{n,p}^R \subset \mathcal{M}^R$ satisfies the assumption of Lemma 2.5.

By Lemma 4.5, we take the inclusion of $L^H(\mathcal{M}_{n,p}^R)^o \subset L^H(\mathcal{M}^R)^o$ of simplicial categories, and we replace their mapping spaces by the associated simplicial sets defined as Definition 2.4. In this proof, we denote them by $L(\mathcal{M}_{n,p}^R)^o \subset L(\mathcal{M}^R)^o$.

Note that $L(\mathcal{M}_{n,p}^R)^o$ (resp. $L(\mathcal{M}^R)^o$) is Dwyer-Kan equivalent to its hammock localization by Lemma 2.5. Therefore, $N_\Delta((\mathcal{M}_{n,p}^R)^o)[W_{\mathcal{M}_{n,p}^R}^{-1}] \subset N_\Delta(\mathcal{M}^R)[W_{\mathcal{M}^R}^{-1}]$.

Step (i) We consider the diagram of simplicial sets,

\[
\begin{array}{ccc}
N_\Delta((\mathcal{M}_{n,p}^R)^o)[W_{\mathcal{M}_{n,p}^R}^{-1}] & \to & N_\Delta(\mathcal{M}^R)[W_{\mathcal{M}^R}^{-1}] \\
\downarrow & & \downarrow \\
N_\Delta((h\mathcal{M}_{n,p}^R)^o) & \to & N_\Delta(h\mathcal{M}^R),
\end{array}
\]

where the horizontal morphisms are induced by the inclusions of full subcategories.

We already have the equivalence $N_\Delta((\mathcal{M}_{n,p}^R)^o)[W_{\mathcal{M}_{n,p}^R}^{-1}] \simeq N_\Delta((L^H(\mathcal{M}_{n,p}^R)^o)^{fib})$. Note that the homotopy category of $L(\mathcal{M}_{n,p}^R)^o$ (resp. $L(\mathcal{M}^R)^o$) is a homotopy category $h\mathcal{M}_{n,p}^R$ (resp. $h\mathcal{M}^R$). Since $N_\Delta$ is the right adjoint, to check the right vertical morphism is a fibration with respect to the Joyal model structure, it suffices to prove that the projection $LM_R \to h\mathcal{M}_R$ is a fibration with respect to Bergner model structure, which follows from the axiom of model category.

Step (ii) According the definition of an $\infty$-subcategory [18, 1.2.11], we show that the diagram (4.1) is cartesian of simplicial sets. By Lemma 4.4(i) and Lemma 2.5, it suffices to show that the cartesian for objects and morphisms in $\mathcal{M}_R$. (Note that the higher simplices of ordinary nerve is determined from 0-simplices and 1-simplices.)

Take $X \in \mathcal{M}_R^o$ which is isomorphic to $Y \in h(\mathcal{M}_{n,p}^R)^c$. Then, there is an object $\tilde{Y} \in (\mathcal{M}_{n,p}^R)$ which is weakly equivalent to $X$. Since $\mathcal{M}_{n,p}^R$ is closed under weak equivalences, we have $X \in \mathcal{M}_{n,p}^R$. For an arbitrary simplices, by Lemma 4.5, the upper horizontal morphism is inclusion of simplicial fullsubsets, and we have representatives of the simplices in $N_\Delta((\mathcal{M}_{n,p}^R)^o)[W_{\mathcal{M}_{n,p}^R}^{-1}]$. Note that the composition law is well-defined by two out of three property. Thus, it is cartesian.

Step (iii) Next, according to the definition of an $\infty$-subcategory, we will consider the diagram of simplicial sets,

\[
\begin{array}{ccc}
\text{Mod}_{n,p}^R & \to & \text{Mod}_R \\
\downarrow & & \downarrow \\
N_\Delta((h\mathcal{M}_{n,p}^R)^o) & \to & N_\Delta(h\mathcal{M}_R^o),
\end{array}
\]
where the horizontal morphisms are the inclusions of full $\infty$-subcategories and the right vertical morphism is obtained by the identification of $h\mathcal{M}_R \cong h\text{Mod}_R$ under (3.2) and the unit map for the adjoint functors

$$h \circ \mathcal{C} : \text{Set}_\Delta \rightleftarrows \text{Cat} : N_\Delta \circ i,$$

where the functor $h$ is given by taking the homotopy category and $i$ is the inclusion of the category $\text{Cat}$ of ordinary categories into the category $\text{Cat}_\Delta$ of simplicial categories. Note that the right vertical morphism is a fibration.

Next step, by using that the objects in $\text{Mod}^{n,p}_R$ is characterized by their homotopy groups, we will see that the right vertical morphism induces the left vertical morphism. Then, it automatically follows that (4.2) is cartesian.

**Step (iv)** Note that the category equivalence of stable homotopy categories preserves the smash products since the smash products on a stable homotopy category is determined up to isomorphisms.

Take $\tilde{X} \in \text{Mod}_R$. Assume that the image of $\tilde{X}$ in $h\mathcal{M}_R^c$ is in $h(\mathcal{M}_R^{n,p})$ under the right vertical morphism. For a discrete cofibrant-fibrant $R$-module $\tilde{N}$ in $\text{Mod}_R$, we have its image in $h(\mathcal{M}_R^{n,p})$. Then, the tensor product $\tilde{N} \otimes_R \tilde{X}$ in $\text{Mod}_R$ is sent to the object in $h(\mathcal{M}_R^{n,p})$ under the right vertical morphism, and they have the same homotopy groups, so that we conclude that $\tilde{X} \in \text{Mod}^{n,p}_R$. Since the upper horizontal morphism is inclusion of simplicial sets, this construction shows that the diagram (4.2) is cartesian.

**Step (v)** Since the weak equivalences on cofibrant-fibrant objects in a model category is the homotopy equivalences, e.g., invertible morphisms. Therefore, by recalling Definition 2.4, all 1-simplices in the mapping space of $LM^0_R$ (resp. $L(\mathcal{M}_R^{n,p})^c$) is invertible, so that it is a Kan complex. Therefore, $LM^0_R$ (resp. $L(\mathcal{M}_R^{n,p})^c$) is a fibrant object with respect to the Bergner model structure.

Since $N_\Delta$ is the right adjoint, it preserves the fibrations and fibrant objects. By applying Cogluing lemma (cf. [18] A.2.4.3) to the diagrams (4.1) and (4.2), we have $N_\Delta((\mathcal{M}_R^{n,p})^c)[W^{-1}_{\mathcal{M}_R^{n,p}}] \simeq \text{Mod}^{n,p}_R$. □

5. **The proof of $K(\text{Mod}^{\text{proj}}_R) \simeq K(\text{Mod}^{\text{perf}}_R)$**

5.1. **Terminology of w-cofibrations and w-fibrations.** Let $\mathcal{C}$ be a pointed category. We fix a zero object of $\mathcal{C}$ and denote by *. Recall that we say that $\mathcal{C}$ is a Waldhausen category if it has two subcategories denoted by $\text{co}(\mathcal{C})$ and $W$, where morphisms in $\text{co}(\mathcal{C})$ are called w-cofibrations and the morphisms in $W$ is called weak equivalences, which satisfy the axiom of Waldhausen category in [26].
Remark 5.1. In terminology of [26], a w-cofibration in a Waldhausen category is called a cofibration. Note that we use the term “cofibration” as a certain class of morphisms in a model category.

We define a class of morphisms, called the w-fibrations, on a pointed category $\mathcal{C}$ with a zero object $\ast$ as follows: a class of w-fibrations is a class of morphisms in $\mathcal{C}$ whose image in the opposite category $\mathcal{C}^{\text{op}}$ satisfies the axiom of a class of w-cofibrations. We say that a diagram in a Waldhausen category $\mathcal{C}$ with weak equivalences $W$ is homotopy cocartesian if it gives a homotopy cocartesian in $L^H(\mathcal{C}, W)$.

Let $\mathcal{C}$ be a pointed category endowed with w-fibrations. A map $f : A \to B$ in $\mathcal{C}$ is said to be a weak w-fibration if it is the composition of a w-fibration with a zig-zag of weak equivalences. It is the dual notion of weak w-cofibrations defined in [10, Definition 2.2].

We say that a pointed category $\mathcal{C}$ defined above admits the functorial factorization of w-fibrations if any weak w-fibration is factored functorially as a weak equivalence followed by a w-fibration in $\mathcal{C}$. We call the following condition saturated: a morphism is a weak equivalence if and only if it is an isomorphism in the homotopy category [11, Theorem 6.4]. Note that a model category is saturated.

5.2. The proof of $K(\text{Mod}^{\text{proj}}_R) \simeq K(\text{Mod}^{\text{per}}_R)$. Now, recall that the notion of cofibrants and fibrants in $\mathcal{M}_R$ from Remark 3.1.

We show the several properties of $(\mathcal{M}_R^n)^o$ (resp. $\mathcal{M}_R^n$).

Lemma 5.2. Let $M' \to M \to M''$ be a fiber sequence of $R$-modules in $(\mathcal{M}_R^n)^o$ (resp. $\mathcal{M}_R^n$).

(i) Assume that $M'$ and $M''$ have Tor-amplitude $\leq n$, and $M$ has Tor-amplitude $\leq n-1$. Then, $M'$ has Tor-amplitude $\leq n-1$.

(ii) Assume that $M'$ and $M''$ have Tor-amplitude $\leq n$. Then, $M$ has Tor-amplitude $\leq n$.

(iii) $(\mathcal{M}_R^n)^o$ (resp. $\mathcal{M}_R^n$) is closed under extension, has a direct sum.

Proof. Let $N$ be a cofibrant discrete $R$-module. We prove that $\pi_k(N \otimes_R M') \simeq 0$ for $k \geq n$. We have an exact sequence of homotopy groups

$$\pi_{k+1}(N \otimes_R M'') \to \pi_k(N \otimes_R M') \to \pi_k(N \otimes_R M).$$

We prove the assertion (i). If $k \geq n$, $\pi_{k+1}(N \otimes_R M'')$ and $\pi_k(N \otimes_R M)$ vanish by assumption that $M''$ has Tor-amplitude $\leq n$ and $M$ has Tor-amplitude $\leq n-1$.

If $k \geq n + 1$, $\pi_k(N \otimes_R M')$ and $\pi_k(N \otimes_R M'')$ vanish in the above exact sequence of homotopy groups. Therefore the assertion (ii) is proved.

For (iii), the assertion (ii) shows $(\mathcal{M}_R^n)^o$ (resp. $\mathcal{M}_R^n$) is closed under extension. Since $(\mathcal{M}_R^n)^o$ (resp. $\mathcal{M}_R^n$) is a full subcategory of the stable category $\mathcal{M}_R$, its homotopy category is additive category. \qed
Note that the forgetful functor from $\mathcal{M}_R$ to the category of $S$-modules preserves the Hurewicz cofibrations [21, Theorem 12.1] and the extension of coefficients of modules preserves the Hurewicz cofibrations [21, Lemma 12.2].

**Definition 5.3.** We define a w-cofibration (resp. a w-fibration) in $\mathcal{M}_{n,p}^R$ as follows.

(i) We define a w-cofibration $X \to Y$ in $(\mathcal{M}_{n,p}^R)^{\circ}$ if it is a Hurewicz cofibration in $\mathcal{M}_{n,p}^p$ and its cofiber lies in $(\mathcal{M}_{n,p}^p)^{\circ}$.

(ii) We also define a w-fibration $Y \to Z$ in $\mathcal{M}_{n,p}^R$ if it is a fibration in $\mathcal{M}_R$ and its fiber lies in $\mathcal{M}_{n,p}^R$.

Then, $(\mathcal{M}_{n,p}^n)^{\circ}$ is a Waldhausen category with the w-cofibrations in $\mathcal{M}_{n,p}^n$, and $(\mathcal{M}_{n,p}^n)^{\text{op}}$ is a Waldhausen category with the w-fibrations in $\mathcal{M}_{n,p}^n$.

Note that, once we define the w-cofibrations and w-fibrations of $\mathcal{M}_{n,p}^R$, the weak w-cofibrations and weak w-fibrations in $\mathcal{M}_{n,p}^R$ are automatically determined.

5.3. **Resolution theorem for $\mathcal{M}_{n,p}^R \subset \mathcal{M}_{n+1,p}^R$.**

**Definition 5.4.** Let $\mathcal{C}$ be a full subcategory of a pointed model category. Let us take a cofibration between cofibrants such that its cofiber lies in $\mathcal{C}_c$ (resp. fibration between fibrants such that its fiber lies in $\mathcal{C}_f$) as a w-cofibration (resp. a w-fibration). We say that $\mathcal{C}$ satisfies the assumption (A) if it satisfies the following conditions:

(i) $\mathcal{C}$ is closed under extensions.

(ii) The w-cofibrations make $\mathcal{C}_c$ into a Waldhausen category.

(iii) The category $(\mathcal{C}_f)^{\text{op}}$ becomes a Waldhausen category by the w-fibrations in $\mathcal{C}$.

(iv) $\mathcal{C}$ is saturated.

(v) $\mathcal{C}_c$ (resp. $\mathcal{C}_f$) has functorial factorizations of w-cofibrations (resp. w-fibrations) respectively.

(vi) Let $W$ denote the category of weak equivalences in $\mathcal{C}$. Then, the homotopy category $hL^H(\mathcal{C}, W)$ is additive.

**Lemma 5.5.** The category $\mathcal{M}_{n,p}^R$ satisfies the assumption (A) in Definition 5.4.

**Proof.** We check the conditions of (A) in Definition 5.4. For (i) of (A), it follows from Lemma 5.2(i).

The assertion (ii), (iii) follows from Definition 5.3.

Since $(\mathcal{M}_{n,p}^n)^{\circ}$ is the full subcategory of the model category $\mathcal{M}_R$, (iv) is automatically satisfied. The assertion (v) of (A) follows from Lemma 4.4. For (vi) of (A), it follows from Lemma 5.2(iii).

□

**Remark 5.6** (cf. [4] Section 5.10). Let $\mathcal{C}$ be a Waldhausen category that admits functorial factorization of w-cofibrations and saturated. We define $wS^W_n\mathcal{C}$ (resp. $wS^W_n\mathcal{C}$) by
the category of weak equivalences in $S^W_n \mathcal{C}$ (resp. by the nerve of the category of weak equivalences in $S'_n \mathcal{C}$). We denote by $wS^W_n \mathcal{C}$ (resp. $wS'_n \mathcal{C}$) the bisimplicial set which sends $[n] \in \Delta^{op}$ to $wS^W_n \mathcal{C}$ (resp. $wS'_n \mathcal{C}$). Then, the inclusion $wS^W_n \mathcal{C} \to wS'_n \mathcal{C}$ induces a weak equivalence $wS^W_n \mathcal{C} \to wS'_n \mathcal{C}$ of bisimplicial set [10, Theorem 2.9].

We remark that the following assertion is well-known in the $K$-theory. Let $\mathcal{C}$ be a full subcategory of a pointed model category with the assumption (A) in Definition 5.4. Let us regard $(\mathcal{C}_f)^{op}$ as a Waldhausen category with the w-fibrations in $\mathcal{C}$. Then, we have a functorial equivalence $K(\mathcal{C}_c) \cong K((\mathcal{C}_f)^{op})$.

We will apply the following resolution theorem to $M^{n,p}_R \subset M^{n+1,p}_R$. By the duality of Remark 5.6, we state the following resolution theorem, which is due to Mochizuki, by the term of w-fibrations.

**Theorem 5.7** (cf. [22] Theorem 1.13). Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of full subcategories of a pointed model categories. Assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (A) in Definition 5.4.

Let $\mathcal{A}(m, w)$ (resp. $\mathcal{B}(m, w)$) be a full subcategory of the functor category $\text{Fun}([m], \mathcal{A})$ (resp. $\text{Fun}([m], \mathcal{B})$) which consists of the functors taking values in the category of weak equivalences $w\mathcal{A}$ in $\mathcal{A}$ (resp. $w\mathcal{B}$ in $\mathcal{B}$). Assume that, for each $m \geq 0$, $\mathcal{A}(m, w) \subset \mathcal{B}(m, w)$ satisfies the following conditions called the resolution condition:

(i) Closed under extensions,

(ii) For any $B \in \mathcal{B}(m, w)$, there exists $A \in \mathcal{A}(m, w)$ and a w-fibration $A \to B$,

(iii) For any fibration sequence $A' \to A \to B$ in $\mathcal{B}(m, w)$, $A'$ is an object in $\mathcal{A}(m, w)$ if $A \in \mathcal{A}(m, w)$.

Then, $\mathcal{A} \subset \mathcal{B}$ induces an equivalence $K(\mathcal{A}_c) \cong K(\mathcal{B}_c)$ for Waldhausen categories $\mathcal{A}_c$ and $\mathcal{B}_c$ obtained by (A)-(ii). \hfill \Box

**Proposition 5.8.** Let $\mathcal{A} \subset \mathcal{B} \subset M_R$ be an inclusion of full subcategories of a pointed model categories such that $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption (A) in Definition 5.4.

Assume that if $A \in \mathcal{A}$ is weakly equivalent to a cofibrant object $M$ in $M_R$, $M$ is the object of $\mathcal{A}$. Then, if $\mathcal{A} \subset \mathcal{B}$ satisfies the resolution condition in Theorem 5.7 for $m = 0$, $\mathcal{A} \subset \mathcal{B}$ satisfies the assumption of Theorem 5.7.

**Proof.** Take the inclusion $\mathcal{A}(m, w) \to \mathcal{B}(m, w)$ and the fiber sequence $x \to y \to z$ in $\mathcal{B}(m, w)$. Assume that, if $m = 0$, the conditions are satisfied. If $x$ and $z$ are objects in $\mathcal{A}(m, w)$, for each $0 \leq i \leq m$, $x_i$ and $z_i$ are in $\mathcal{A}$. By the assumption, $y_i$ is in $\mathcal{A}$ for each $i$. Thus, the condition $(i)$ in Theorem 5.7 is obvious. The condition $(iii)$ follows by the same argument. We will check the condition in Theorem 5.7(ii) by induction.
We proceed by induction on $m$. The condition (ii) is valid for $m = 0$. For $m \geq 0$, an object in $z \in B(m, w)$ is written in the form of a sequence of weak equivalences $z_0 \rightarrow \cdots \rightarrow z_m$ and a morphism between them is given by a diagram.

Take $z' = (z_0 \rightarrow \cdots \rightarrow z_{m-1}) \in B(m - 1, w)$. By the induction hypothesis, we have a fibration $y' \rightarrow z'$ in $B(m - 1, w)$ represented by the following diagram:

$$
\begin{array}{ccccccc}
& & y_0 & \rightarrow & \cdots & \rightarrow & y_{m-1} \\
& & \downarrow & & & & \downarrow \\
& & \cdots & \rightarrow & \cdots & \rightarrow & y_m \\
& & \downarrow & & & & \downarrow \\
& & z_0 & \rightarrow & \cdots & \rightarrow & z_{m-1} \\
& & \downarrow & & & & \downarrow \\
& & z_1 & \rightarrow & \cdots & \rightarrow & z_m,
\end{array}
$$

where the vertical morphisms are fibration. For the composition $y_{m-1} \rightarrow z_{m-1} \rightarrow z_m$, we apply the factorization of w-fibrations. Then, there is an object $y_m$ such that $y_{m-1} \simeq y_m$ and $y_m \rightarrow z_m$ is a fibration. By the assumption (ii), we have $y_m \in A$. Thus, we can proceed the induction. □

**Proposition 5.9.** Let $R$ be a connective coherent $E_\infty$-ring. Let $M_R^{n,p} \rightarrow M_R^{n+1,p}$ be the inclusion of $\infty$-categories. Then, the induced morphism $K((M_R^{n,p})^\circ) \simeq K((M_R^{n+1,p})^\circ)$ is an equivalence.

**Proof.** The category $M_R^{n,p}$ satisfies the assumptions (i) and (ii) of Proposition 5.8. Therefore, by Proposition 5.8, it suffices to check that the inclusion $M_R^{n,p} \rightarrow M_R^{n+1,p}$ satisfies the condition in Theorem 5.7 for $m = 0$. The inclusion satisfies the condition (i) and (iii) for $m = 0$ by Lemma 5.2(i) and (iii). Take $M \in M_R^{n+1,p}$.

For an $S$-algebra $R$, let $F_R S$ denote a sphere $R$-module $R \wedge \mathbb{L} S$ [15, III, Proposition 1.3]. Note that, for an object $A$ in $M_R$, the $n$-th homotopy group of $A$ is defined by the hom-set $\text{Hom}_{M_R}(F_R S^n, A)$, where $F_R S^n$ is the $n$-times shift of the sphere $R$-module.

Since $\pi_0 M$ is not empty, we choose a morphism $F_R S \rightarrow M$ corresponding to an object of $\pi_0 M$, where $F_R S$ is sphere $R$-module. Note that $F_R S$ is an object of $M_R^{n,p}$ for every $n \geq 0$, and the fiber of a morphism $F_R S \rightarrow M$ is obviously connective. The condition (ii) is satisfied by a fibrant replacement of $F_R S \rightarrow M$ since $M_R^{n,p}$ is closed under equivalences for every $n \geq 0$. Thus, we have an equivalence $K((M_R^{n,p})^\circ) \simeq K((M_R^{n+1,p})^\circ)$. □

5.4. **Comparison between the $K$-theory of $(M_R^{n,p})^\circ$ and $\text{Mod}_{R}^{n,p}$.** Let $\mathcal{C}$ be a pointed $\infty$-category with $w_{\infty}$-cofibrations. Let $K(\mathcal{C})$ be the algebraic $K$-theory of $\infty$-category in the sense of [9, Section 2.5]. We obtain the following comparison of algebraic $K$-theory spaces, which is functorial with respect to Waldhausen categories. Note that Barwick's construction of $K$-theory (cf. [4, Theorem 7.6, Section 10]) is equivalent to Lurie's construction.
Theorem 5.10 ([8] Corollary 7.12). Let $C_0$ be a Waldhausen category which admits functorial factorization of $w$-cofibrations and is saturated. Let $W \subset C_0$ be the category of weak equivalences.

Then there is an zigzag of equivalences
$$K((C_0, W)) \simeq K(N_{\Delta}(C_0)[W^{-1}]),$$
where the left hand side is the Waldhausen $K$-theory and the right hand side is the algebraic $K$-theory of $\infty$-category.

Now, we define a $w^\infty$-cofibration $X \to Y$ of $\text{Mod}^n_{R,p}$ if it is a $w^\infty$-cofibration in $\text{Mod}_R$ and its cofiber lies in $\text{Mod}^{n,p}_R$.

Note that we have $\text{Map}_{\text{Mod}^n_{R,p}}(A, B) = \text{Map}_{\text{M}^n_{R,p}}(A, B)$ for every cofibrant-fibrant objects $A$ and $B$ in $\text{M}^n_{R,p}$.

Lemma 5.11. Let us regard $(\text{M}^n_{R,p})^o$ as the Waldhausen category with $w$-cofibrations obtained by Definition 5.3. Then, the Waldhausen $K$-theory $K((\text{M}^n_{R,p})^o)$ is equivalent to the algebraic $K$-theory $K(\text{Mod}^n_{R,p})$.

Proof. The equivalence in Lemma 4.6 is a weak exact functor since a diagram in $(\text{M}^n_{R,p})^o$ is homotopy cocartesian if and only if its image in $\text{Mod}^{n,p}_R$ under the equivalence is a pullback. Since the category $(\text{M}^n_{R,p})^o$ admits a functorial factorizations of $w$-cofibrations, the assertion follows from Theorem 5.10. □

Proposition 5.12. Let $R$ be a connective coherent $\mathcal{E}_\infty$-ring. Let $\text{Mod}^{n,p}_R \subset \text{Mod}^{n+1,p}_R$ be the inclusion of $\infty$-categories. Then, the induced morphism $K(\text{Mod}^{n,p}_R) \rightarrow K(\text{Mod}^{n+1,p}_R)$ is an equivalence.

Proof. Since we have $K(\text{Mod}^{n,p}_R) \simeq K((\text{M}^n_{R,p})^c)$ by Lemma 5.11, the assertion follows from Lemma 5.9. □

Proposition 5.13. For a connective coherent $\mathcal{E}_\infty$-ring $R$, $K(\text{Mod}^{proj}_R) \simeq K(\text{Mod}^{perf}_R)$.

Proof. Note that $\text{Mod}^{0,p}_R \simeq \text{Mod}^{proj}_R$ by [19, Proposition 7.2.5.20, Remark 7.2.5.22]. Let us denote the $\infty$-category of connective perfect $R$-modules by $(\text{Mod}^{perf}_R)^{cn}$. Then, we have an equivalence $\text{colim}_n \text{Mod}^{n,p}_R \simeq (\text{Mod}^{perf}_R)^{cn}$ from Definition 4.2. Since the $K$-theory commutes with filtered colimits by [3, Section 7], we have $K(\text{Mod}^{proj}_R) \simeq K((\text{Mod}^{perf}_R)^{cn})$. Thus, it suffices to show that $K((\text{Mod}^{perf}_R)^{cn}) \simeq K(\text{Mod}^{perf}_R)$.

Consider the following colimit
$$\text{(Mod}^{perf}_R)^{cn} \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \text{(Mod}^{perf}_R)^{cn} \xrightarrow{\Sigma} \cdots .$$

By [3, Proposition 4.4], this filtered colimit exists as an $\infty$-category with $w$-cofibrations. From this filtered colimit, $\text{colim}_\Sigma K((\text{Mod}^{perf}_R)^{cn}) \simeq K(\text{colim}_\Sigma (\text{Mod}^{perf}_R)^{cn})$. 

We will show that the following equivalences
\[ \text{colim}_\Sigma K((\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{perf}}) \simeq K((\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{cn}}) \]
and
\[ K(\text{colim}_\Sigma (\text{Mod}_{\text{perf}}^{\text{cn}})) \simeq K((\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{perf}}). \]
Since we have the following cofiber sequence in \((\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{cn}}\)
\[
\begin{array}{c}
\text{id} \\
\downarrow \\
0 \\
\downarrow \\
\Sigma
\end{array}
\]
and \(\Sigma\) induces \(-\text{id}\) on \(K\)-theory, \(\text{colim}_\Sigma K((\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{cn}})\) is equivalent to \(K((\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{cn}})\). We show that \(\text{colim}_\Sigma (\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{cn}}\) is a stable \(\infty\)-category. Indeed, it has cofibers, and the endofunctor \(\Sigma\) is an equivalence. By \([19, \text{Lemma 1.1.3.3}]\), \(\text{colim}_\Sigma (\text{Mod}_{\text{perf}}^{\text{cn}})^{\text{cn}}\) is stable. \(\square\)

6. Proof of Theorem 1.2

We say that an \(R\)-module \(M\) is \textit{truncated} if \(\pi_n M = 0\) for sufficiently large \(n\) \([18, \text{Definition 5.5.6.1}]\). Let \(R\) be a coherent \(E_\infty\)-ring, which we recall in Definition 4.1. An \(R\)-module \(M\) is \textit{coherent} \(R\)-module if it is truncated, \(\pi_0 M = 0\) for sufficiently small \(n\) and \(\pi_n M\) is finitely presented \(\pi_0 R\)-module for \(n \geq 0\).

\textbf{Definition 6.1} ([4] Definition 8.4, [7] Proposition 1.3). Let \(R\) be a coherent \(E_\infty\)-ring defined in Definition 4.1.

(i) \(R\) is \textit{almost regular} if any coherent \(R\)-module has Tor-amplitude \(\leq n\) for some \(n \in \mathbb{Z}_{\geq 0}\).
(ii) \(R\) is \textit{regular} if \(\pi_0 R\) is regular and \(R\) is almost regular.

\textbf{Definition 6.2.} Let \(R\) be a coherent \(E_\infty\)-ring.

(i) We say that \(M\) is almost perfect if \(\pi_m M = 0\) for sufficiently small \(m\) and \(\pi_n M\) is finitely presented over \(\pi_0 R\) for \(n \in \mathbb{Z}\) \([19, \text{Proposition 7.2.5.17}]\).
(ii) Let \(\text{Coh}_R\) be a full \(\infty\)-subcategory of \(\text{Mod}_R\), which consists of almost perfect and truncated objects. It is a stable \(\infty\)-category.

\textbf{Lemma 6.3.} Let \(R\) be a connective \(E_\infty\)-ring. We define \((\text{Mod}^{\text{proj}}_R)^b\) by an \(\infty\)-category of finitely generated projective truncated \(R\)-modules. We define \((\text{Mod}^{\text{perf}}_R)^b\) by an \(\infty\)-category of perfect truncated \(R\)-modules.

(i) If a connective \(E_\infty\)-ring \(R\) has only finitely many non-zero homotopy groups, any finitely generated projective \(R\)-module has only finitely many non-zero homotopy groups.
(ii) Suppose that $R$ is an almost regular $E_{\infty}$-ring. Then $(\text{Mod}^\text{perf}_R)^b$ coincides with $\text{Coh}_R$.

(iii) Let $R$ be a connective $E_{\infty}$-ring with only finitely many non-zero homotopy groups. Then, the natural inclusion $(\text{Mod}^\text{perf}_R)^b \to \text{Mod}^\text{perf}_R$ is an equivalence.

Proof. If $R$ has only finitely many non-zero homotopy groups, so is a finite copies of $R$. Since a finitely generated projective $R$-module is a retract of direct summand of finite copies of $R$, the assertion (i) holds. The assertion (ii) is the consequence of [4, Proposition 8.6]. To show (iii), it suffices to show that the perfect $R$-module $M$ is truncated. By applying [2, Proposition 2.13 (7)] inductively, and by [2, Proposition 2.13 (6)], it comes down to the case of a shift of finitely generated projective $R$-module $\Sigma^n P$. It is truncated. Therefore if $R$ is truncated, so is $M$. □

Theorem 6.4. Let $R$ be a connective regular $E_{\infty}$-ring with only finitely many non-zero homotopy groups.

(i) The inclusion $\text{Mod}^\text{proj}_R \to \text{Mod}^\text{perf}_R$ induces an equivalence of $K$-theory spaces;

\[ K(\text{Mod}^\text{proj}_R) \simeq K(\text{Mod}^\text{perf}_R) \simeq K(\text{Coh}_R). \]

(ii) Let $\mathcal{P}_{\pi_0 R}$ be an ordinary category of finitely generated projective $\pi_0 R$-modules. Then, $K(\text{Mod}^\text{proj}_R) \simeq K(\mathcal{P}_{\pi_0 R})$.

Proof. The first part of the assertion follows from Proposition 5.13 and Lemma 6.3. By the first part of the assertion and the main theorem of [7] with the regularity of $R$, we obtain $K(\text{Mod}^\text{proj}_R) \simeq K(\mathcal{P}_{\pi_0 R})$. □

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