Projective and Conformal Schwarzian Derivatives
and Cohomology of Lie Algebras Vector Fields
Related to Differential Operators

Sofiane Bouarroudj
Department of Mathematics, U.A.E. University, Faculty of Science
P.O. Box 15551, Al-Ain, United Arab Emirates.
e-mail:bouarroudj.sofiane@uaeu.ac.ae
Abstract

Let $M$ be either a projective manifold $(M, \Pi)$ or a pseudo-Riemannian manifold $(M, g)$. We extend, intrinsically, the projective/conformal Schwarzian derivatives that we have introduced recently, to the space of differential operators acting on symmetric contravariant tensor fields of any degree on $M$. As operators, we show that the projective/conformal Schwarzian derivatives depend only on the projective connection $\Pi$ and the conformal class $[g]$ of the metric, respectively. Furthermore, we compute the first cohomology group of $\text{Vect}(M)$ with coefficients into the space of symmetric contravariant tensor fields valued into $\delta$-densities as well as the corresponding relative cohomology group with respect to $\text{sl}(n+1, \mathbb{R})$.

1 Introduction

The investigation of invariant differential operators is a famous subject that have been intensively investigated by many authors. The well-known invariant operators and more studied in the literature are the Schwarzian derivative, the power of the Laplacian (see [13]) and the Beltrami operator (see [2]). We have been interested in studying the Schwarzian derivative and its relation to the geometry of the space of differential operators viewed as a module over the group of diffeomorphisms in the series of papers [4, 8, 9]. As a reminder, the classical expression of the Schwarzian derivative of a diffeomorphism $f$ is:

$$\frac{f'''}{f''} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2. \tag{1.1}$$

The two following properties of the operator (1.1) are the most of interest for us:

(i) It vanishes on the Möbius group $\text{PSL}(2, \mathbb{R})$ – here the group $\text{SL}(2, \mathbb{R})$ acts locally on $\mathbb{R}$ by projective transformations.

(ii) For all diffeomorphisms $f$ and $g$, the equality

$$S(f \circ g) = g^2 \cdot S(f) \circ g + S(g) \tag{1.2}$$

holds true.

The equality (1.2) seems to be known since Cayley; however, it was first reported by Kirillov and Segal (see [17, 18, 32]) that this property is nothing but a 1-cocycle property – it should be stressed that cocycles on the group are not easy to come up with, and only few explicit expressions are known (cf. [14]).

Our study has its genesis from the geometry of the space of differential operators acting on tensor densities, viewed as a module over the group of diffeomorphisms and also over the Lie algebra of smooth vector fields. In the one-dimensional case, this study have led to compute the (relative) cohomology group

$$H^1(\text{Diff}(\mathbb{R}), \text{PSL}(2, \mathbb{R}); \text{End}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)).$$
where $\mathcal{F}_\lambda$ is the space of tensor densities of degree $\lambda$ on $\mathbb{R}$.

It turns out that the Schwarzian derivative as well as new cocycles span the cohomology group above, as proved in [9]. These new 1-cocycles can also be considered as natural generalizations of the Schwarzian derivative (1.1), although they are only defined on an one-dimensional manifold.

The first step towards generalizing the Schwarzian derivative underlying the properties (i) and (ii) to multi-dimensional manifolds was a part of our thesis [5]. It was aimed at defining the projective Schwarzian derivatives as 1-cocycles on $\text{Diff}(\mathbb{R}^n)$ valued into the space of differential operators acting on contravariant twice-tensor fields, and vanish on $\text{PSL}(n+1, \mathbb{R})$. Later on, we constructed in [4] 1-cocycles on $\text{Diff}(\mathbb{R}^n)$ valued into the same space but vanish on the conformal group $O(p+1, q+1)$, where $p + q = n$. These $O(p + 1, q + 1)$-invariant 1-cocycles were interpreted as conformal Schwarzian derivatives. Moreover, these projectively/conformally invariant 1-cocycles were built intrinsically by means of a projective connection and a pseudo-Riemannian metric, thereby making sense on any curved manifold. As projective structures and conformal structures coincide in the one-dimensional case, these (projective/conformal) 1-cocycles are considered as natural generalizations of the Schwarzian derivative (1.1).

This paper is, first, devoted to extend these derivatives to the space of differential operators acting on symmetric contravariant tensor fields of any degree.

In virtue of the one-dimensional case, the (projective/conformal) Schwarzian derivatives should define cohomology classes belonging to

$$H^1(\text{Diff}(\mathbb{R}^n), \mathfrak{h}; \text{End}_{\text{diff}}(\mathcal{S}_\delta(\mathbb{R}^n), \mathcal{S}_\delta(\mathbb{R}^n))),$$

where $\mathcal{S}_\delta(\mathbb{R}^n)$ is the space of symmetric contravariant tensor fields on $\mathbb{R}^n$ valued into $\delta$-densities and $\mathfrak{h}$ is the Lie group $\text{PSL}(n+1, \mathbb{R})$ or $O(p+1, q+1)$.

The cohomology group above is not easy to handle; nevertheless, we compute in Theorem 6.11 the cohomology group

$$H^1(\text{Diff}(\mathbb{S}^n); \text{End}_{\text{diff}}(\mathcal{S}_\delta(\mathbb{S}^n), \mathcal{S}_\delta(\mathbb{S}^n))),$$

for the (two and three)-dimensional sphere.

Moreover, we compute in Theorem 6.5 the (relative) cohomology group

$$H^1(\text{Vect}(\mathbb{R}^n), \mathfrak{sl}(n+1, \mathbb{R}); \text{End}_{\text{diff}}(\mathcal{S}_\delta(\mathbb{R}^n), \mathcal{S}_\delta(\mathbb{R}^n))).$$

The computation being inspired from Lecomte-Ovsienko’s work [21], uses the well-known Weyl’s classical invariant theory [36]. It provides a proof – at least in the infinitesimal level – that the infinitesimal projective Schwarzian derivatives that we are introducing are unique.

Furthermore, we compute in Theorem 6.10 the cohomology group

$$H^1(\text{Vect}(M); \text{End}_{\text{diff}}(\mathcal{S}_\delta(M), \mathcal{S}_\delta(M))),$$
where $M$ is an arbitrary manifold.

According to the Neijenhuis-Richardson’s theory of deformation [24], the cohomology group above will measure all infinitesimal deformations of the $\text{Vect}(M)$-module $\mathcal{S}_\delta(M)$.

2 The space of symbols as modules over $\text{Diff}(M)$ and $\text{Vect}(M)$

Throughout this paper, $M$ is an (oriented) manifold of dimension $n$ endowed with an affine symmetric connection. We denote by $\Gamma$ the Christoffel symbols of this connection and by $\nabla$ the corresponding covariant derivative. It should be clear from the context whether the connection is arbitrary or a Levi-Civita one.

We use the Einstein convention summation over repeated indices.

Our symmetrization does not contain any normalization factor.

2.1 The space of tensor densities

The space of tensor densities of degree $\delta$ on $M$, denoted by $\mathcal{F}_\delta(M)$, is the space of sections of the line bundle: $| \wedge^n T^*M|^{\otimes \delta}$, where $\delta \in \mathbb{R}$. In local coordinates $(x^i)$, any $\delta$-density can be written as

$$\phi(x) |dx^1 \wedge \cdots \wedge dx^n|^\delta.$$

As examples, $\mathcal{F}_0(M) = C^\infty(M)$ and $\mathcal{F}_1(M) = \Omega^1(M)$.

The affine connection $\Gamma$ can be naturally extended to a connection that acts on $\mathcal{F}_\delta(M)$. The covariant derivative of a density $\phi \in \mathcal{F}_\delta(M)$ is given as follows. In local coordinates $(x^i)$, we have

$$\nabla_i \phi = \partial_i \phi - \delta \Gamma_{ir}^u \phi,$$

where $\partial_i$ stands for the partial derivative with respect to $x^i$.

2.2 The space of tensor fields as a module

Denote by $\mathcal{S}(M)$ the space of contravariant symmetric tensor fields on $M$. This space is naturally a module over the group $\text{Diff}(M)$ by the natural action. Moreover, it is isomorphic to the space of symbols, namely functions on the cotangent bundle $T^*M$ that are polynomial on fibers.

We are interested in defining a one-parameter family of $\text{Diff}(M)$-modules on $\mathcal{S}(M)$ by

$$\mathcal{S}_\delta(M) := \mathcal{S}(M) \otimes \mathcal{F}_\delta(M).$$

The action is defined as follows. Let $f \in \text{Diff}(M)$ and $P \in \mathcal{S}_\delta(M)$ be given. Then, in a local coordinates $(x^i)$, we have

$$f^*_\delta P = f^* P \cdot (J_{f^{-1}})^\delta,$$  (2.1)
where \( J_f = \left| Df / Dx \right| \) stands for the Jacobian of \( f \), and \( f^* \) stands for the natural action of \( \text{Diff}(M) \) on \( \mathcal{S}(M) \).

By differentiating the action (2.1) we get the infinitesimal action of \( \text{Vect}(M) \) : for all \( X \in \text{Vect}(M) \), and for all \( P \in \mathcal{S}(M) \) we have

\[
L_X P = L_X (P) + \delta \text{Div} X P,
\]  

where \( \text{Div} \) is the divergence operator associated with some orientation.

Denote by \( \mathcal{S}_s^k(M) \) the space of symmetric tensor fields of degree \( k \) on \( M \) endowed with the \( \text{Diff}(M) \)-module structure (2.1). We then have a graduation of \( \text{Diff}(M) \)-modules: \( \mathcal{S}_s(M) = \bigoplus_{k \geq 0} \mathcal{S}_s^k(M) \).

The actions (2.1) and (2.2) are of most interest of us. Throughout this paper, all actions will be referred to them.

### 3 A compendium on projective and conformal structures

We will collect, in this section, some gathers on projective and conformal structures. These notions are well-known in projective and conformal geometry. However, they are necessary to introduce here in order to write down explicit expressions of the Schwarzian derivatives.

#### 3.1 Projective structures

A **projective connection** is an equivalent class of symmetric affine connections giving the same non-parameterized geodesics.

Following [19], the symbol of the projective connection is given by the expression

\[
\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{n+1} \left( \delta_i^k \Gamma_{lj}^l + \delta_j^k \Gamma_{il}^l \right).
\]  

(3.1)

Two affine connections \( \Gamma \) and \( \tilde{\Gamma} \) are **projectively equivalent** if the corresponding symbols (3.1) coincide. Equivalently, if there exists a 1-form \( \omega \) such that

\[
\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k \omega_j + \delta_j^k \omega_i.
\]  

(3.2)

A projective connection on \( M \) is called **flat** if in a neighborhood of each point there exists a local coordinates such that the symbols \( \Pi_{ij}^k \) are identically zero (see [19] for a geometric definition).

A projective structure on \( M \) is given by a local action of the group \( \text{SL}(n+1, \mathbb{R}) \) on it. Every flat projective connection defines a projective structure on \( M \).

On \( \mathbb{R}^n \) with its standard projective structure, the Lie algebra \( \text{sl}(n+1, \mathbb{R}) \) can be embedded into the Lie algebra \( \text{Vect}(\mathbb{R}^n) \) by

\[
\frac{\partial}{\partial x^i}, \quad x^i \frac{\partial}{\partial x^j}, \quad x^i x^k \frac{\partial}{\partial x^k}, \quad i, j = 1, \ldots, n.
\]  

(3.3)
where \((x^i)\) are the coordinates of the projective structure. The first two vector fields form a Lie algebra isomorphic to the affine Lie algebra \(gl(n, \mathbb{R}) \ltimes \mathbb{R}^n\).

### 3.2 Conformal structures

A **conformal** structure on a manifold is an equivalence class of pseudo-Riemannian metrics \([g]\) that have the same direction.

If \(\Gamma^k_{ij}\) are the Levi-Civita connection associated with the metric \(g\), then the Levi-Civita connection, \(\tilde{\Gamma}^k_{ij}\), associated with the metric \(e^{2F} \cdot g\), where \(F\) is a function on \(M\), are related, in any local coordinates \((x^i)\), by

\[
\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + F_i \delta^k_j + F_j \delta^k_i - g_{ij} g^{kt} F_t, \tag{3.4}
\]

where \(F_i = \partial F / \partial x^i\).

A conformal structure on \((M, g)\) is called **flat** if in a neighborhood of each point there exists a local coordinate system such that the metric \(g\) is a multiple of \(g_0\), where \(g_0\) is the metric \(\text{diag}(1, \ldots, 1, -1, \ldots, -1)\) whose trace is \(p - q\).

It is well-known that the group of diffeomorphisms of \(\mathbb{R}^n\) that keep the standard metric \(g_0\) in the conformal class is the group \(O(p + 1, q + 1)\), where \(p + q = n\).

**Remark 3.1** The Lie algebra \(o(p + 1, q + 1)\) can also be embedded into \(\text{Vect}(\mathbb{R}^n)\) via formulas analogous to (3.3) but we do not need them here.

### 3.3 An intrinsic 1-cocycle and a Lie derivative of a connection

A connection itself is not a well-defined geometrical object. However, the difference between two connections is a well-defined tensor fields of type \((2, 1)\). Therefore, the following object

\[
\mathfrak{L}(f) := f^*\Gamma - \Gamma, \tag{3.5}
\]

where \(f\) is a diffeomorphism, is globally defined on \(M\).

It is easy to see that the map

\[
f \mapsto \mathfrak{L}(f^{-1})
\]

defines a 1-cocycle on \(\text{Diff}(M)\) with values into tensor fields of type \((2, 1)\).

The infinitesimal 1-cocycle associated with the tensor (3.5), denoted by \(l\), is called the **Lie derivative** of a connection; it can also be defined as follows. For all \(X \in \text{Vect}(M)\), the 1-cocycle \(l(X)\) is the map

\[
(Y, Z) \mapsto [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z]. \tag{3.6}
\]

We will use intensively, throughout this paper, the tensor (3.5) as well as the tensor (3.6).
4 Projectively invariant Schwarzian derivatives

Let Π and ˜Π be two projective connections on $M$. Then the difference $\Pi - ˜\Pi$ is a well-defined $(2,1)$-tensor field. Therefore, it is clear that a projective connection on $M$ leads to the following 1-cocycle on $\text{Diff}(M)$:

$$\mathfrak{T}(f^{-1})^{k}_{ij} := (f^{-1})^{*}\Pi^{k}_{ij} - \Pi^{k}_{ij}, \quad (4.1)$$

which vanishes on (locally) projective diffeomorphisms.

Remark 4.1 There is also an alternative approach in defining the 1-cocycle (4.1) by means of the tensor (3.5).

4.1 The main definitions

Definition 4.2 For all $f \in \text{Diff}(M)$ and for all $P \in S^{k}_{0}(M)$, we put

$$\mathfrak{U}(f)(P)^{i_1 \cdots i_{k-1}} = \sum_{s=1}^{k-1} \mathfrak{T}(f)^{i_s}_{ij} P^{i_1 \cdots \hat{i}_s \cdots i_{k-1}}, \quad (4.2)$$

where $\mathfrak{T}(f)$ is the tensor (4.1).

By construction, the operator (4.2) is projectively invariant, viz it depends only on the projective class of the connection.

Theorem 4.3 (i) For all $\delta \neq \frac{2k-1+n}{1+n}$, the map $f \mapsto \mathfrak{U}(f^{-1})$ defines a non-trivial 1-cocycle valued into $\mathcal{D}(S^{k}_{0}(M), S^{k-1}_{0}(M))$;
(ii) for $\delta = \frac{2k-1+n}{1+n}$, we have

$$\mathfrak{U}(f)(P)^{i_1 \cdots i_{k-1}} = (f^{-1} * \nabla_j - \nabla_j) P^{i_1 \cdots i_{k-1}}.$$

Proof. (i) The 1-cocycle property of the operator (4.2) follows immediately from the 1-cocycle property of the tensor (4.1). Let us prove the non-triviality. Suppose that there exists an operator $A$ such that

$$\mathfrak{U}(f) = f^{-1} * A - A. \quad (4.3)$$

As $\mathfrak{U}(f)$ is a zero-order operator, the operator $A$ is almost first-order. If $A$ is zero-order, namely a multiplication operator, its principal symbol, say $a$, transforms under coordinates change as a tensor fields of type $(2,1)$. The equality above implies that $\mathfrak{T}(f) = f^{-1} * a - a$ which is absurd, as $\mathfrak{T}$ is a non-trivial 1-cocycle. Suppose then that $A$ is a first-order operator, namely

$$A(P)^{i_1 \cdots i_{k-1}} = \nabla_j P^{i_1 \cdots i_{k-1}},$$
for all \( P \in S^k_\delta(M) \). It is a matter of direct computation to prove that

\[
(f^{-1} A - A)(P)^{i_1 \cdots i_{k-1}} = \sum_{s=1}^{k-1} \mathcal{L}(f)_{i_1}^s P^{ij_1 \cdots j_{k-1}} - (\delta - 1) \mathcal{L}(f)_{j} P^{j i_1 \cdots i_{k-1}},
\]

where \( \mathcal{L}(f)_{i} \) are the components of the tensor \( B \). We can easily seen that the equality (4.3) holds true if and only if \( \delta = \frac{2k-1+n}{1+n} \).

We will introduce a second 1-cocycle valued into \( \mathcal{D}(S^k(M), S^{k-2}(M)) \). But, at first, we start by giving its expression when \( k = 2 \).

**Definition 4.4** For all \( f \in \text{Diff}(M) \) and for all \( P \in S^2_\delta(M) \), we put

\[
\mathfrak{D}(f)(P) := \mathfrak{T}(f)_{ij}^k \nabla_k P^{ij} + \nabla_k \mathcal{L}(f)_{ij}^k P^{ij} - \frac{3 + n - \delta(1 + n)}{1 + n} \nabla_i \mathcal{L}(f)_{j} P^{ij} \tag{4.4}
\]

\[
+ (1 - \delta) \left( \mathcal{L}(f)^{u}_{ij} \mathcal{L}(f)_{u} - \frac{1}{1 + n} \mathcal{L}(f)_{i} \mathcal{L}(f)_{j} + \frac{1}{n - 1} \delta n (f^{s-1} R_{ij} - R_{ij}) \right) P^{ij},
\]

where \( \mathcal{L}(f)^{k}_{ij} \) are the components of the 1-cocycle \( B \), \( \mathfrak{T}^{k}_{ij}(f) \) are the components of the 1-cocycle \( f \), and \( R_{ij} \) are the components of the Ricci tensor.

**Theorem 4.5**

(i) For all \( \delta \neq \frac{n+2}{n+1} \), the map \( f \mapsto \mathfrak{D}(f^{-1}) \) defines a non-trivial 1-cocycle on \( \text{Diff}(M) \) with values into \( \mathcal{D}(S^2_\delta(M), S^0_\delta(M)) \).

(ii) For \( \delta = \frac{n+2}{n+1} \), we have

\[
\mathfrak{D}(f) = f^{-1} B - B,
\]

where \( B \) is the operator

\[
B := \nabla_i \nabla_j - \frac{1}{n - 1} R_{ij}. \tag{4.5}
\]

(ii) The operator \( \mathfrak{D}(f) \) depends only on the projective class of the connection. When \( M = \mathbb{R}^n \) (or \( M = S^n \)) and \( M \) is endowed with a flat projective structure, this operator vanishes on the projective group \( \text{PSL}(n + 1, \mathbb{R}) \).

**Remark 4.6**

(i) The operator \( \mathfrak{D}(f) \) in (4.4) enjoys the elegant expression:

\[
\mathfrak{T}(f)^{k}_{ij} \nabla_k - \frac{2 - \delta(n + 1)}{n - 1} \nabla_k (\mathfrak{T}(f)^{k}_{ij}) + \frac{(n + 1)(1 - \delta)}{n - 1} \mathfrak{T}(f)^{m}_{ij} \mathfrak{T}(f)^{n}_{mk}, \tag{4.6}
\]

which can be obtained through the relation

\[
f^{-1} R_{jk} - R_{jk} = - \nabla_i \mathcal{L}(f)^{i}_{jk} + \nabla_j \mathcal{L}(f)^{j}_{k} + \mathcal{L}(f)^{m}_{sk} \mathcal{L}(f)^{k}_{km} - \mathcal{L}(f)_{im} \mathcal{L}(f)^{m}_{jk}. \tag{4.7}
\]

(ii) We will retain the Ricci tensor into the explicit expression of the Schwarzian derivatives disregarding the equation (4.7), because it will be useful when we will study theirs relation to the well-known Vey cocycle.
For $k > 2$, we state the following definition.

**Definition 4.7** For all $f \in \text{Diff}(M)$, and for all $P \in S_{k}^{k}(M)$, we put

$$
\mathfrak{M}(f) (P)_{i_{1} \cdots i_{k-2}} = \sum_{s=1}^{k-2} \xi(f)_{tu} \nabla_{v} P_{uv\hat{s}_{i_{1}} \cdots \hat{s}_{i_{k-2}}} + \alpha_{1} \xi(f)_{uv} \nabla_{t} P_{uv\hat{i}_{1} \cdots \hat{i}_{k-2}}
$$

$$
+ \sum_{s=1}^{k-2} \left( \alpha_{2} \nabla_{t} \xi(f)_{uv} + \alpha_{3} \xi(f)_{wt} \xi(f)_{uv} + \alpha_{4} \xi(f)_{tu} \xi(f)_{v} \right) P_{uv\hat{i}_{1} \cdots \hat{i}_{k-2}}
$$

$$
+ \left( \alpha_{5} \nabla_{t} \xi(f)_{uv} + \alpha_{6} \nabla_{u} \xi(f)_{v} + \alpha_{7} \xi(f)_{u} \xi(f)_{v} + \alpha_{8} \xi(f)_{uv} \xi(f)_{w} \right) P_{uv\hat{i}_{1} \cdots \hat{i}_{k-2}}
$$

$$
+ \alpha_{9} \left( f^{-1} R_{uv} - R_{uv} \right) P_{uv\hat{i}_{1} \cdots \hat{i}_{k-2}} + e \sum_{1 \leq s < r \leq k-2} \xi(f)_{uv} \xi(f)_{pq} P_{uvpq_{1} \cdots \hat{s}_{r} \cdots \hat{s}_{i_{k-2}}},
$$

(4.8)

where $R_{uv}$ are the Ricci tensor components, $\xi(f)_{ij}^{k}$ are the components of the tensor (3.5) and $\xi(f)_{ij}^{k}$ are the components of the tensor (4.1). The constant $e = \begin{cases} 1 & \text{if } k \geq 4, \\ 0 & \text{otherwise} \end{cases}$ and the constants $\alpha_{1}, \ldots, \alpha_{9}$ are given by

$$
\alpha_{1} = \frac{1}{2} (3 - 2k + n(\delta - 1) + \delta); \quad \alpha_{5} = \frac{1}{2} (3 - 2k + n(\delta - 1) + \delta);
$$

$$
\alpha_{2} = \frac{1}{6} (2k + (1 - \delta) (1 + n)); \quad \alpha_{6} = \frac{1}{2} (\delta - 1)(1 - 2k + n(\delta - 1) + \delta);
$$

$$
\alpha_{3} = \frac{1}{3} (5 - 2k + n(\delta - 1) + \delta); \quad \alpha_{7} = \frac{1}{2} (\delta - 1)^{2};
$$

$$
\alpha_{4} = (1 - \delta); \quad \alpha_{8} = \frac{1}{2} (1 - \delta)(3 - 2k + n(\delta - 1) + \delta);
$$

$$
\alpha_{9} = \frac{11 + 4k^{2} + (\delta - 1)(2n(5 - 4k + 3\delta) + 3n^{2}(\delta - 1)) + 10\delta + 3\delta^{2} - 4k(3 + 2\delta)}{6 - 6n}.
$$

**Theorem 4.8** (i) For all $\delta \neq \frac{2k - 2 + n}{n + 1}$, the map $f \mapsto \mathfrak{M}(f^{-1})$ defines a non-trivial 1-cocycle on $\text{Diff}(M)$ with values into $\mathcal{D}(S_{k}^{k}(M), S_{k}^{k-2}(M))$.

(ii) The operator (4.8) depends only on the projective class of the connection. When $M = \mathbb{R}^{n}$ (or $M = S^{n}$) and $M$ is endowed with a flat projective structure, this operator vanishes on the projective group $\text{PSL}(n + 1, \mathbb{R})$.

We will prove Theorem (4.3) and Theorem (4.8) simultaneously.

**Proof Theorem (4.5) and Theorem (4.3).** To prove that the map $f \mapsto \mathfrak{M}(f^{-1})$ is a 1-cocycle we have to verify the 1-cocycle condition

$$
\mathfrak{M}(f \circ g) = g^{-1} \mathfrak{M}(f) + \mathfrak{M}(g) \quad \text{for all } f, g \in \text{Diff}(M),
$$

(4.10)
where \( g^* \) is the natural action on \( \mathcal{D}(S^k_\delta(M), S^{k-2}_\delta(M)) \). In order to prove this condition we will, first, remove the Ricci tensor from the expressions (4.1) and (4.8), because it is obviously a coboundary; secondly, we use the equalities

\[
\nabla_u f^*_\delta P^{i_1\ldots i_k} = f^*_\delta \nabla_u P^{i_1\ldots i_k} - \sum_{s=1}^k \left( \mathcal{L}(f^{-1})_{iu} f^*_\delta P^{u i_1\ldots i_s i_{s+1}\ldots i_k} + \delta \mathcal{L}(f^{-1})_u f^*_\delta P^{i_1\ldots i_k} \right),
\]

\[
\mathcal{L}(f \circ g)^u_{ij} = g^* \mathcal{L}(f)^u_{ij} + \mathcal{L}(g)^u_{ij},
\]

and the equality

\[
\nabla_u g^* \mathcal{L}(f)_{ij} = g^* \nabla_u \mathcal{L}(f)_{ij} + h^* \mathcal{L}(f)^t_{ij} \mathcal{L}(g^{-1})^k_{ut} + \text{Sym}_{ij} \left( g^* \mathcal{L}(f)^k_{it} \mathcal{L}(g^{-1})^t_{ju} \right),
\]

where \( \mathcal{L}(f)_{ij}^k \) are the components of the tensor (4.5). The 1-cocycle condition for the operator (4.8) can verified by a long and tedious computation. We will give a proof here only when \( k = 2 \). By using the equalities above we see that, for all \( P \in S^2_\delta(M) \), we have

\[
\mathfrak{A}(f \circ g)(P) = (g^* \mathcal{L}(f)_{ij}^k + \mathcal{L}(g)_{ij}^k) \nabla_k P^{ij} + \nabla_k \left( g^* \mathcal{L}(f)_{ij}^k + \mathcal{L}(g)_{ij}^k \right) P^{ij} - \frac{3 + n - \delta (1 + n)}{1 + n} \nabla_i \left( g^* \mathcal{L}(f)_{j}^k + \mathcal{L}(g)_{j}^k \right) P^{ij} + (1 - \delta) \left( \mathfrak{A}(f \circ g)_{ij} \mathfrak{A}(f \circ g)_{ij} - \frac{1}{n + 1} \mathfrak{A}(f \circ g)_{ij} \mathfrak{A}(f \circ g)_{ij} \right) P^{ij} = g^*_\delta \left( \mathfrak{A}(f) \mathfrak{A}(g)^{-1}(P) \right) + \mathfrak{A}(g)(P)
\]

Now we prove that the 1-cocycles (4.1) and (4.8) are not trivial. Suppose that there exists an operator \( A : S^k_\delta(M) \rightarrow S^{k-2}_\delta(M) \) such that

\[
\mathfrak{A}(f) = f^* A - A.
\]

Since the operators (4.1) and (4.8) are first-order, the operator \( A \) is at most second-order. If the operator \( A \) is first-order, its principal symbol should transforms under coordinates change as a tensor fields of type (2, 1). From the equality (4.12) one can easily seen that \( \mathcal{L}(f)_{ij} \) is a trivial 1-cocycle, which is absurd. If \( A \) is second-order, its principal symbol should be equal to the identity, otherwise the equality (4.12) does not hold true. Therefore, the operator \( A \) is given by

\[
A(P)^{i_1\ldots i_{k-2}} = \nabla_u \nabla_v P^{i_1\ldots i_{k-2}}
\]

for all \( P \in S^k_\delta(M) \). Now, an easy computation gives

\[
f^{-1} A - A = \sum_{s=1}^{k-2} \mathcal{L}(f)_{iu} \nabla_v P^{u i_1\ldots i_s i_{s+1}\ldots i_{k-2}} + (1 - \delta) \mathcal{L}(f)_{v} \nabla_u P^{u i_1\ldots i_{k-2}}
\]

\[
- \sum_{s=1}^{k-2} \left( f^{-1} \nabla_u \mathcal{L}(f^{-1})_{iu} P^{u i_1\ldots i_{s+1}\ldots i_{k-2}} + \mathcal{L}(f)_{iu} f^{-1} \nabla_v f^* P^{u i_1\ldots i_{k-2}} \right)
\]

\[
- (f^{-1} \nabla_u \mathcal{L}(f^{-1})_{uw} - (\delta - 1) f^{-1} \nabla_u \mathcal{L}(f^{-1})_{v} P^{u i_1\ldots i_s i_{s+1}\ldots i_{k-2}} + \mathcal{L}(f)_{uw} f^{-1} \nabla_v f^* P^{u i_1\ldots i_s i_{s+1}\ldots i_{k-2}}
\]

\[
\]
Therefore, the tensor 

\[ T_{\text{PSL}}(k) = \text{straightforward computation.} \]

possibility so that (4.12) holds true is when and only when \( \delta \).

Using the equation above and the equations (4.11) we can easily seen that the only

\[ T = \text{it is a matter of a direct computation to prove that} \]

As the operators (4.4) and (4.8) are projectively invariant, we can take \( \Gamma \)

\[
\tilde{\nabla} u\mathcal{P}^{i_1 \cdots i_k} = \nabla u\mathcal{P}^{i_1 \cdots i_k} + (2k - \delta(n + 1)) \mathcal{P}^{i_1 \cdots i_k} \omega_u + \sum_{s=1}^{k} \delta_{u}^{i_s} \omega_v \mathcal{P}^{v_1 \cdots i_{s+1} \cdots i_k},
\]

for all \( P \in \mathcal{S}_0^k(M) \), and

\[
\tilde{\nabla} u \mathcal{L}(f)^v_{ij} = \mathcal{L}(f)^v_{ij} + \text{Sym}_{i,j} \delta^v_i f^{-1} \omega_j - \text{Sym}_{i,j} \delta^v_i \omega_j,
\]

\[
\tilde{\nabla} u \tilde{\mathcal{L}}(f)^v_{ij} = \nabla u \mathcal{L}(f)^v_{ij} + \text{Sym}_{i,j} \delta^v_i \nabla u f^{-1} \omega_j - \text{Sym}_{i,j} \delta^v_i \nabla u \omega_j - \text{Sym}_{i,j} \omega_i \tilde{\mathcal{L}}(f)^v_{uj} + \delta^v_i \omega_i \tilde{\mathcal{L}}(f)^v_{ij} - \omega_u \tilde{\mathcal{L}}(f)^v_{ij},
\]

and finally

\[
\tilde{R}_{ij} = R_{ij} + (n - 1) (\nabla_i \omega_j - \omega_i \omega_j).
\]

By substituting these formulæ into (4.4) we obtain, after a long computation, that

\[ \mathfrak{V} \mathcal{V}(f) = \mathfrak{V} \tilde{\mathcal{V}}(f). \]

Suppose now \( M = \mathbb{R}^n \) (or \( M = \mathbb{S}^n \)) and \( M \) is endowed with a projective structure. Let \( f \) be a diffeomorphism belonging to \( \text{PSL}(n + 1, \mathbb{R}) \). Then there exist some constants \( a_j, b^i, c_l, d \), where \( i, j, l = 1 \ldots n \), such that

\[ f(x) = \left( \frac{a_1^i x^j + b^i}{c_l x^l + d}, \ldots, \frac{a_n^i x^j + b^n}{c_l x^l + d} \right). \]

As the operators (4.4) and (4.8) are projectively invariant, we can take \( \Gamma \equiv 0 \). Therefore, the tensor \( \mathfrak{T}(f)^v_{ij} \) will take the form

\[
\mathfrak{T}(f)^v_{ij} = \frac{\partial^2 f^r}{\partial x^i \partial x^j} \frac{\partial x^v}{\partial f^r} - \frac{1}{n + 1} \text{Sym}_{i,j} \delta^v_i \frac{\partial^2 f^r}{\partial x^j} \frac{\partial x^l}{\partial f^r}.
\]

It is a matter of a direct computation to prove that \( \mathfrak{T}(f)^v_{ij} \equiv 0 \), for all \( f \in \text{PSL}(n + 1, \mathbb{R}) \). Now, directly from the equation (4.6) we see that \( \mathfrak{V}(f) \equiv 0 \) when \( k = 2 \). For \( k > 2 \), we will use again the equation (4.7) and the proof is a long but straightforward computation.
4.2 A remark on the projective analogue of the Laplace-Beltrami operator

As a by-product of the formula (4.4) is the projective analogue of the well-known Laplace-Beltrami operator (see [2]). It has been shown in [4] that, for $k = 2$ and for a particular value of $\delta$, the conformal Schwarzian derivative is given by the coboundary $f^{s-1} \Delta - \Delta$,

where $\Delta$ is the Laplace-Beltrami operator. In Theorem (4.5), we have proved that, for $\delta = \frac{n + 2}{n + 1}$, the projective Schwarzian derivative is the coboundary $f^{s-1}B - B$,

where

$$B := \nabla_i \nabla_j - \frac{1}{n - 1} R_{ij}.$$  

The operator $B$ is indeed projectively invariant; in virtue of the conformal case, it can be then interpreted as the projective analogue of the Laplace-Beltrami operator.

4.3 Infinitesimal projective Schwarzian derivatives

**Definition 4.9**  (i) The infinitesimal operator associated with the operator (4.2) is the operator

$$t(X) (P)_{i_1 \ldots i_s} := \sum_{s=1}^{k-1} \left( t(X)_{i_s} - \frac{1}{n + 1} \text{Sym}_{i,j} \delta^i t(X)_j \right) \ n_i P^{i_1 \ldots \hat{i_s} \ldots i_{k-1}},$$  

where $t$ is the 1-cocycle (3.6).

(ii) The infinitesimal operator associated with the operator (4.4) and (4.8) are respectively the operators

$$u(X) (P) := \left( t(X)_{ij} - \frac{1}{n + 1} \text{Sym}_{i,j} \delta^i t(X)_j \right) \nabla_i P^{ij} + \nabla_i t(X)_{ij} P^{ij} + \frac{(1 + n)(1 - \delta)}{1 + n} (L_X R_{ij}) P^{ij},$$

and

$$u(X) (P)_{i_1 \ldots i_{k-2}} := \sum_{s=1}^{k-2} \left( t(X)_{i_s} - \frac{1}{n + 1} \text{Sym}_{i,j} \delta^i t(X)_j \right) \nabla_i P^{i_1 \ldots \hat{i_s} \ldots i_{k-2}}$$

$$+ \alpha_1 \left( t(X)_{ij} - \frac{1}{n + 1} \text{Sym}_{i,j} \delta^i t(X)_j \right) \nabla_i P^{i_1 \ldots i_{k-2}}$$

$$+ \alpha_5 \nabla_i t(X)_{ij} P^{i_1 \ldots \hat{i_s} \ldots i_{k-2}} + \alpha_2 \sum_{s=1}^{k-2} \nabla_i t(X)_{ij} P^{i_1 \ldots \hat{i_s} \ldots i_{k-2}}$$

$$+ \alpha_6 (\nabla_i t(X) + \alpha_9 L_X R_{ij}) P^{i_1 \ldots i_{k-2}},$$

where the constants $\alpha_1, \alpha_2, \alpha_5, \alpha_6$ and $\alpha_9$ are given as in (4.9).
The following Corollaries result from Theorems (4.3), (4.5) and (4.8).

**Corollary 4.10** For all \( \delta \neq \frac{2k-1+n}{1+n} \), the map \( X \mapsto \mathfrak{t}(X) \) defines a non-trivial 1-cocycle valued into \( \mathcal{D}(\mathcal{S}_0^k(M), \mathcal{S}_0^{k-1}(M)) \). Moreover, the operator (4.14) is projectively invariant, namely it depends only on the projective class of the connection.

**Corollary 4.11** For all \( \delta \neq \frac{2k-2+n}{1+n} \), the map \( X \mapsto \mathfrak{u}(X) \) defines a non-trivial 1-cocycle valued into \( \mathcal{D}(\mathcal{S}_0^k(M), \mathcal{S}_0^{k-2}(M)) \). Moreover, the operators (4.16) and (4.17) are projectively invariant.

5 Conformally invariant Schwarzian derivatives

Let \( (M, g) \) be a pseudo Riemannian manifold and let \( \Gamma \) be the Levi-Civita connection associated with the metric \( g \).

**Definition 5.1** For all \( f \in \text{Diff}(M) \) and for all \( P \in \mathcal{S}_0^k(M) \), we put

\[
\mathfrak{A}(f)(P)_{ji\ldots i_{k-1}} = \text{Coboundary} + c \sum_{s=1}^{k-1} \left( \mathfrak{L}(f)_{ij} - \frac{1}{n} \text{Sym}_{ij} \delta^{is}_i \mathfrak{L}(f)_j \right) P^{ji\ldots i_{k-1}},
\]

where \( \mathfrak{L}(f) \) is the tensor (3.5) and the constant

\[
c = 2 - \delta n.
\]

**Theorem 5.2** (i) For almost all values of \( \delta \), the map \( f \mapsto \mathfrak{A}(f^{-1}) \) defines a non-trivial 1-cocycle on \( \text{Diff}(M) \) with values into \( \text{End}_{\text{diff}}(\mathcal{S}_0^k(M), \mathcal{S}_0^{k-1}(M)) \);

(ii) The operator (5.1) depends only on the conformal class \([g]\) of the metric. When \( M = \mathbb{R}^n \) and \( M \) is endowed with a flat conformal structure, this operator vanishes on the conformal group \( O(p+1, q+1) \), where \( p + q = n \).

Now, we will introduce another conformally invariant 1-cocycle that takes values into \( \text{End}_{\text{diff}}(\mathcal{S}_0^k(M), \mathcal{S}_0^{k-2}(M)) \). We suppose that \( k > 2 \); for \( k = 2 \), the 1-cocycle have already been introduced in [4].

We denote by \( R_{ij} \) the Ricci tensor components and by \( R \) the scalar curvature associated with the metric \( g \).
**Definition 5.3** For all \( f \in \text{Diff}(M) \) and for all \( P \in S^k_\delta(M) \), we put

\[
\mathfrak{B}(f) (P)_{i_1 \cdots i_{k-2}} = \text{Coboundary} + \sum_{s=1}^{k-2} \left( \mathfrak{L}(f)^i_j - \frac{1}{n} \text{Sym}_{i,j} \delta^i_j \, \mathfrak{L}(f)_j \right) \nabla_t P^{ij_1 \cdots \hat{i} \cdots i_{k-2}} \\
+ \beta_1 \left( \mathfrak{L}(f)^t_j - \frac{1}{n} \text{Sym}_{i,j} \delta^i_j \, \mathfrak{L}(f)_j \right) \nabla_t P^{ij_1 \cdots \hat{i} \cdots i_{k-2}} \\
+ (\beta_2 \nabla_t \mathfrak{L}(f)^t_j + \beta_3 \nabla_t \mathfrak{L}(f)_j + \beta_4 \mathfrak{L}(f)_i \, \mathfrak{L}(f)_j + \beta_5 \mathfrak{L}(f)^u_l \, \mathfrak{L}(f)_u) \, P^{ij_1 \cdots \hat{i} \cdots i_{k-2}} \\
+ \sum_{s=1}^{k-2} \left( \beta_6 \nabla_t \mathfrak{L}(f)^i_j + \beta_7 \mathfrak{L}(f)^i_j \mathfrak{L}(f)_t + \beta_8 \mathfrak{L}(f)^u_l \, \mathfrak{L}(f)_u \right) P^{ij_1 \cdots \hat{i} \cdots i_{k-2}} \\
+ (\beta_9 (f^{*^{-1}}R_{ij} - R_{ij}) + \beta_{10} (f^{*^{-1}}R_{g_{ij}} - R_{g_{ij}})) \, P^{ij_1 \cdots \hat{i} \cdots i_{k-2}},
\]

(5.2)

where \( \mathfrak{L}(f)^i_j \) are the components of the tensor \([3.5]\). The coefficient \( e = \begin{cases} 1 & \text{if } k \geq 4, \\ 0 & \text{otherwise} \end{cases} \), and the coefficients \( \beta_1, \ldots, \beta_{10} \) are given by

\[
\beta_1 = \frac{1}{2} (4 - 2k + n(\delta - 1)); \quad \beta_5 = \frac{1}{2} (1 - \delta)(4 - 2k + n(\delta - 1)); \\
\beta_2 = \frac{1}{2} (4 - 2k + n(\delta - 1)); \quad \beta_6 = \frac{1}{6} (n + 2k - \delta n); \\
\beta_3 = \frac{1}{2} (\delta - 1)(2 - 2k + n(\delta - 1)); \quad \beta_7 = (1 - \delta); \\
\beta_4 = \frac{1}{2} (\delta - 1)^2 \quad \beta_8 = \frac{1}{3} (6 - 2k + n(\delta - 1)); \\
\beta_9 = \frac{1}{6} \left\{ \frac{4(6 - 5k + k^2) - 8(n - 2)(\delta - 1) + 3n^2(\delta - 1)^2}{n - 2} \right\}; \\
\beta_{10} = \frac{2(k - 2)(2k(2k - 5) + n(1 + 11\delta - k(12\delta - 7))) - 6n^2(\delta - 1)^2\delta}{12(n - 2)(n - 1)(2 - 2k + n(-1 + 2\delta))} - \frac{n^2(\delta - 1)(2 + 32\delta - k(22\delta - 5))}{12(n - 2)(n - 1)(2 - 2k + n(-1 + 2\delta))}.
\]

**Theorem 5.4**

(i) For almost values of \( \delta \), the map \( f \mapsto \mathfrak{B}(f^{-1}) \) defines a non-trivial 1-cocycle on \( \text{Diff}(M) \) with values into \( \mathcal{D}(S^k_\delta(M), S^{k-2}_\delta(M)) \);

(ii) The operator \([5.2]\) depends only on the conformal class \([g]\) of the metric.

When \( M = \mathbb{R}^n \) and \( M \) is endowed with a flat conformal structure, this operator vanishes on the conformal group \( \text{O}(p+1,q+1) \).
5.1 The Algorithm and the proof of Theorems (5.2) and (5.4)

The operator
\[ \sum_{s=1}^{k-1} \left( \mathcal{L}(f)_{ij}^s - \frac{1}{n} \text{Sym}_{ij} \delta_i^s \mathcal{L}(f)_{ji} \right) \]  

satisfies obviously the 1-cocycle property. However, it lacks the invariance property, in contradistinction with the operator \((4.1)\) which is projectively invariant. We will establish here an Algorithm to transform the operator above into a conformally invariant one.

Let us denote by \(C\) the 1-cocycle above written by means of a connection \(\nabla\) associated with the metric \(g\) and denote by \(\tilde{C}\) the same 1-cocycle written by means of a connection belong to the same conformal class as described in Section 3. Using the formula \((3.4)\), we get
\[ (\tilde{C}(f)_{ij}^s - C(f)_{ij}^s) P^{i_{j_1}\cdots j_{s-1}i_k} = (g_{ij} F^s - f^{s-1} g_{ij} f^{s-1} F^s) P^{i_{j_1}\cdots j_{s-1}i_k}. \]

In order to get ride the component \(g_{ij} F^s P^{i_{j_1}\cdots j_{s-1}i_k} - f^{s-1} g_{ij} f^{s-1} F^s P^{i_{j_1}\cdots j_{s-1}i_k}\), we adjust the 1-cocycle \((5.3)\) by incorporating the coboundary
\[ \gamma_1 (f^{s-1} B_1 - B_1), \]
where \(\gamma_1\) is a constant – to be determined – and \(B_1(P) := \sum_{s=1}^{k-1} g_{uv} g^{tis} \nabla_t P^{uvi_1\cdots i_{s-1}i_{k-1}}\).

A direct computation using \((3.4)\) proves that
\[ g_{uv} g^{tis} \tilde{\nabla}_t P^{uvi_1\cdots i_{s-1}i_{k-1}} = g_{uv} g^{tis} \nabla_t P^{uvi_1\cdots i_{s-1}i_{k-1}} + (k - \delta n) F^s g_{uv} P^{uvi_1\cdots i_{s-1}i_{k-1}} 
+ \sum_{t=1, t \neq s}^{k-1} g_{uv} \left( g^{tis} F^w P^{uvi_1\cdots i_{t-1}i_{t}i_{s-1}i_{k-1}} - F^{is} P^{uvi_1\cdots i_{t-1}i_{t}i_{s-1}i_{k-1}} \right) \]
(For \(k = 2\) the last two terms will not to be taken into account.)

If we collect the coefficient of the component \(F_i^s g_{uv} P^{uvi_1\cdots i_{s-1}i_{k-1}}\) and the component \(f^{s-1} F^s f^{s-1} g_{uv} P^{uvi_1\cdots i_{s-1}i_{k-1}}\) we will get the equation
\[ \gamma_1 (2 - \delta n) = c. \]

If \(k = 2\), this is the only equation we need. In that case, the coefficient \(c\) and \(\gamma_1\) are as in Table 1.
\[ c \gamma_1 + \mathfrak{A} = \begin{array}{ccc} \delta = \frac{2}{n} & 0 & 1 \quad \text{trivial} \\ \delta \neq \frac{2}{n} & 2 - \delta n & 1 \quad \text{not trivial} \end{array} \]

Table 1.

If \( k > 2 \), the 1-cocycle
\[ \gamma_1(f^{*} B - B) + c \mathcal{C}(f) \]
is still not conformally invariant. We have to incorporate, then, another coboundary
\[ \gamma_2(f^{*} B_2 - B_2), \]
where \( \gamma_2 \) is a constant and \( B_2(P) := \sum_{i=1}^{k-2} \sum_{t \neq s} g^{i st} g_{ij} \nabla_u P^{uij i_1 \ldots \hat{i}_s \ldots \hat{i}_t \ldots \hat{i}_{i-1}}. \]

Now, a direct computation using (3.4), we get
\[
\begin{align*}
g^{i st} g_{ij} \nabla_u P^{uij i_1 \ldots \hat{i}_s \ldots \hat{i}_t \ldots \hat{i}_{i-1}} &= g^{i st} g_{ij} \nabla_u P^{uij i_1 \ldots \hat{i}_s \ldots \hat{i}_t \ldots \hat{i}_{i-1}} \\
&+ (2k + n - 4 - \delta n) F_{w} g^{i st} g_{ij} P^{ijw i_1 \ldots \hat{i}_s \ldots \hat{i}_t \ldots \hat{i}_{i-1}} \\
&- \sum_{1 \leq l \leq k-1} g^{i st} g_{uv} g_{ij} F^{ij} P^{ijw i_1 \ldots \hat{i}_s \ldots \hat{i}_t \ldots \hat{i}_{i-1}}
\end{align*}
\]
(The last term should not be taken into account if \( k = 3 \).

Now, we collect the coefficient of the component \( F_{m} g^{i st} g_{ij} P^{ijm i_1 \ldots \hat{i}_s \ldots \hat{i}_t \ldots \hat{i}_{i-1}} \) we get the equation
\[ 2\gamma_1 + (2k - 4 + n(1 - \delta)) \gamma_2 = 0. \]

For \( k = 3 \), the coefficients \( c, \gamma_1, \) and \( \gamma_2 \) are given as in Table 2.

\[
\begin{array}{ccc|c}
\delta = \frac{2}{n} & 0 & 1 & -\frac{2}{n} \quad \text{trivial} \\
\delta = \frac{2 + n}{n} & 0 & 0 & 1 \quad \text{trivial} \\
\delta \text{ not like above} & 2 - \delta n & 1 & \frac{2}{2 + n(1 - \delta)} \quad \text{not trivial} \\
\end{array}
\]

Table 2.
If \( k > 3 \), the 1-cocycle
\[
\gamma_2(f^{*^{-1}}B_2 - B_2) + \gamma_1(f^{*^{-1}}B - B) + c \mathcal{C}(f)
\]
is still not conformally invariant. We have to incorporate then another coboundary
\[
\gamma_3(f^{*^{-1}}B_3 - B_3)
\]
where \( B_3 := \sum_s \sum_{\ell} \sum_p \tilde{g}^i_{i_1 \cdots i_{k-1}} \tilde{g}^{i_1 \cdots i_{k-1}} g_{uv} \nabla_i \nabla_j \nabla_l P_{i_1 \cdots i_{k-1} \cdots i_{k-2}}. \) Then we proceed as before to find the constant \( \gamma_3 \). We will continue the procedure of incorporating coboundaries up to the last coboundary:

\[
\gamma_k(f^{*^{-1}}B_k - B_k),
\]
where \( B_k \) is an operator defined as follows:

1. If \( k \) is even, then \( B_k := \text{Sym}_{i_1, \ldots, i_k} g^{i_1 \cdots i_k} g^{i_{k-1} \cdots i_2} \nabla_i P_{i_1 \cdots i_k}. \)

2. If \( k \) is odd, then \( B_k := \text{Sym}_{i_1, \ldots, i_k} g^{i_1 \cdots i_k} g^{i_{k-1} \cdots i_2} \nabla_i P_{i_1 \cdots i_k}. \)

The resulting 1-cocycle should be conformally invariant.

To prove that the operator \( \mathcal{L}_g \) satisfies the 1-cocycle property is a long but straightforward computation using the equations \( (4.11) \). It will determine the coefficients \( \epsilon, \beta_1, \ldots, \beta_k \) uniquely. In order to study the invariance property, we need some ingredients. Using the relation \( (3.4) \), we can prove that the following relations hold

\[
\nabla_u \mathcal{L}_g^{(k)} = \nabla_u \mathcal{L}_g^{(k)} - \text{Sym}_{i,j} F_j \mathcal{L}_g^{(k)} - F_u \mathcal{L}_g^{(k)} - \delta^k_u F_m \mathcal{L}_g^{(k)} + \text{Sym}_{i,j} g_{stu} F^l \mathcal{L}_g^{(k)} - g_{mu} F^k \mathcal{L}_g^{(k)}
\]

\[
\nabla_u P^{i_1 \cdots i_k} + (k - \delta n) F_u P^{i_1 \cdots i_k} + \sum_{s=1}^k \delta^i_u F_m P^{m i_1 \cdots \hat{i}_s \cdots i_k} - \sum_{s=1}^k g_{mu} F^i_s P^{m i_1 \cdots \hat{i}_s \cdots i_k}.
\]

Moreover,

\[
\tilde{R}_{ij} = R_{ij} - (n - 2) (\nabla_i F_j - F_i F_j) - (\nabla_i F_v + (6 - n) F_u F_v) g^{uv} g_{ij}
\]

\[
\tilde{R} = e^{-2F} (R - (2n - 2) \nabla_i F_v g^{uv} - (7n - 2 - n^2) F_u F_v g^{uv}),
\]

where the wide tilde on each tensor means that the tensor is written by means of a metric belonging to the conformal class.

In order to get a conformally invariant operator, we are required to add the coboundary

\[
\mu_1(f^{*^{-1}}B - B),
\]

where \( B := g^{uv} g_{ij} \nabla_u \nabla_v P^{i j_1 \cdots i_{k-2}} \). Now, we proceed as above, to find the constant \( \beta_0 \) and \( \beta_{10} \) as well as the constant \( \mu_1 \). We continue this process until we get a conformally invariant operator.
6 Schwarzian derivatives and Cohomology

Let us first recall the following classical result (see [11, 37]). Consider the space of Sturm-Liouville operators

\[ A := -2 \frac{d^2}{dx^2} + u(x) : \mathcal{F}_{-\frac{1}{2}}(\mathbb{R}) \to \mathcal{F}_{\frac{1}{2}}(\mathbb{R}), \]

where \( u(x) \in \mathcal{F}_2(\mathbb{R}) \) is the potential.

For all diffeomorphism \( f \in \text{Diff}(\mathbb{R}) \), the operator \( f^*A \) is still a Sturm-Liouville operator with potential \( u \circ f^{-1} \cdot (f^{-1})'' + S(f^{-1}) \), where \( S(f^{-1}) \) is the Schwarzian derivative (1.1).

According to the Neijenhuis-Richardson’s theory of deformation, the space of Sturm-Liouville operators viewed as a \( \text{Diff}(\mathbb{R}) \)-module (also as a \( \text{Vect}(\mathbb{R}) \)-module) is a non-trivial deformation of the quadratic differentials \( \mathcal{F}_2(\mathbb{R}) \), generated by the Schwarzian derivative (see [9]). More generally, the space of differential operators acting on densities of arbitrarily weights is a non-trivial deformation of a direct sum of densities of appropriate weights (see [9]). It is well-known that the problem of deformation is related to the cohomology group

\[ H^1(\text{Vect}(\mathbb{R}), \text{sl}(2, \mathbb{R}); \mathcal{D}(\mathcal{F}_\delta(\mathbb{R}), \mathcal{F}_{\delta'}(\mathbb{R}))) \]

It has been proved in [9] that the infinitesimal Schwarzian derivative as well as other 1-cocycles generate this cohomology group.

**Remark 6.1** The analogue cocycle on \( \text{Vect}(\mathbb{R}) \) associated with the Schwarzian derivative is the so-called Gelfand-Fuchs cocycle: \( X \frac{d}{dx} \mapsto X''' dx^2 \) (see e.g. [14, 16]).

Following these lines of thought, we believe that, in higher-dimension, the infinitesimal projective Schwarzian derivatives are classes belonging to the cohomology group

\[ H^1(\text{Vect}(\mathbb{R}^n), \text{sl}(n+1, \mathbb{R}); \mathcal{D}(S^k_\delta(\mathbb{R}^n), S^j_\delta(\mathbb{R}^n))). \]

In the next section we will compute this cohomology group, generalizing the result of [21] for \( \delta = 0 \).

### 6.1 The projectively equivariant cohomology

Consider \( \mathbb{R}^n \) with the standard \( \text{SL}(n+1, \mathbb{R}) \)-action as described in Section [3].

**Theorem 6.2** If \( n > 2 \), we have

\[ H^1(\text{Vect}(\mathbb{R}^n), \text{sl}(n+1, \mathbb{R}); \mathcal{D}(S^k_\delta(\mathbb{R}^n), S^j_\delta(\mathbb{R}^n))) = \]

\[ \begin{align*}
\mathbb{R}, & \quad \text{if } k - j = 1, j \neq 0 \text{ and } \delta \neq \frac{2k-1+n}{1+n}, \\
\mathbb{R}, & \quad \text{if } k - j = 2 \text{ and } \delta \neq \frac{2k-2+n}{1+n}, \\
0, & \quad \text{otherwise.}
\end{align*} \]

(6.2)
The 1-cocycles that span the cohomology group above are the operators (4.14), (4.16) and (4.17).

The following remark will play a central role in our proof; it has already been used in the papers [9, 21]. Let \( g \) be a Lie algebra, \( h \subset g \) be a subalgebra and \( M \) be a \( g \)-module. Any 1-cocycle \( c : g \to M \) that vanishes on the Lie sub-algebra \( h \) is automatically \( h \)-invariant. Indeed, the 1-cocycle property reads

\[
L_X c(Y, A) - L_Y c(X, A) = c([X, Y], A)
\]

for all \( X, Y \in g \) and for all \( A \in M \). Then

\[
L_X c(Y, A) = c([X, Y], A),
\]

which is nothing but the \( h \)-invariance property.

The strategy to proof Theorem (6.2) is as follows. We will classify all \( \mathfrak{sl}(n+1, \mathbb{R}) \)-invariant bilinear operators from \( \text{Vect}(\mathbb{R}) \otimes S^k_\delta(\mathbb{R}^n) \) to \( S^p_\delta(\mathbb{R}^n) \), then we will isolate among them 1-cocycles.

### 6.1.1 \( \mathfrak{sl}(n+1, \mathbb{R}) \)-invariant bilinear operators

To begin with, we recall a lemma that has been proved in [21] for \( \delta = 0 \) but the proof works well for any \( \delta \).

**Lemma 6.3** Every bilinear map from \( \text{Vect}(\mathbb{R}) \otimes S^k_\delta(\mathbb{R}^n) \) to \( S^p_\delta(\mathbb{R}^n) \) that is invariant with respect to the action of the affine Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \rtimes \mathbb{R}^n \) is differentiable; moreover, it is given by the divergence operator.

**Proof.** See [21]

**Remark 6.4** In fact, any 1-cocycle on \( \text{Vect}(M) \), where \( M \) is an arbitrary manifold, with values into \( \mathcal{D}(S^k_\delta(M), S^p_\delta(M)) \) is differentiable (cf. [21]).

**Proposition 6.5** The space of \( \mathfrak{sl}(n+1, \mathbb{R}) \)-equivariant bilinear operators form \( \text{Vect}(\mathbb{R}) \otimes S^k_\delta(\mathbb{R}^n) \) to \( S^{k-p}_\delta(\mathbb{R}^n) \) is as follows:

(i) for \( k > p \geq 2 \), it is 2-dimensional;

(ii) for \( k = p \), it is 1-dimensional;

(iii) for \( p = 1, k > 2 \), it is 1-dimensional;

(iv) for \( k = p = 1 \), there is no such operators.

**Proof.** According to Lemma (6.3), any such operator should have the expression

\[
c(X, P)^{i_1 \cdots i_{k-p}} = \sum_{s=1}^{p} \left( \sum_{t=1}^{k-p} \alpha_s \partial_{j_1} \cdots \partial_{j_{s+1}}(X^{i_t}) \partial_{j_{s+2}} \cdots \partial_{j_{p+1}}(P^{i_1 \cdots \hat{i_t} \cdots i_{k-p} i_{j_1} \cdots i_{j_p}}) \right. \\
+ \beta_s \partial_{j_1} \cdots \partial_{j_s} \partial_k(X^k) \partial_{j_{s+1}} \cdots \partial_{j_p}(P^{i_1 \cdots \hat{i_t} \cdots i_{k-p} j_1 \cdots j_p}) \\
+ \gamma_s \partial_{j_1} \cdots \partial_{j_s} \partial_k \partial_{j_{s+1}} \cdots \partial_{j_p}(P^{i_1 \cdots \hat{i_t} \cdots i_{k-p} i_{j_1} \cdots j_p}) \right)
\]

(6.3)
where $\alpha_s, \beta_s, \gamma_s$, for $s = 1, \ldots, p$, are real numbers.

We will use the expression above as an Ansatz in order to classify all $\text{sl}(n+1, \mathbb{R})$-invariant bilinear operators.

If we demand that the operator $c$ vanishes on the Lie algebra $\text{sl}(n+1, \mathbb{R})$ we will impose the conditions

$$\gamma_1 = 0,$$

and

$$2\alpha_1(k - p) + \beta_1(1 + n) + 2\gamma_2 = 0.$$  \hfill (6.4)

A straightforward computation but quite complicated, prove that the equivariance of the operator (6.3) with respect to the Lie algebra $\text{sl}(n+1, \mathbb{R})$ is equivalent to the following system

\begin{align*}
-s(s + 2)\alpha_{s+1} + \gamma_{s+1} + (p - s)(2k + n - p + s - \delta(1 + n))\alpha_s &= 0, \quad (6.5) \\
-s(s + 1)\beta_{s+1} + (s + 1)\gamma_{s+1} + (p - s)(2k + n - p + s - \delta(1 + n))\beta_s &= 0, \quad (6.6) \\
-(s^2 - 1)\gamma_{s+1} + (p - s)(2k + n - p + s - \delta(1 + n))\gamma_s &= 0, \quad (6.7) \\
(s + 1)\gamma_{s+1} + (k - p)(s + 1)\alpha_s + (k - p + s - \delta(1 + n))\gamma_s + (1 + n)\beta_s &= 0. \quad (6.8)
\end{align*}

where $s = 1, \ldots, p - 1$.

The outcome (6.5) should not be taken into account if $k = p$.

**Lemma 6.6** For all $\delta$, the system above is compatible.

**Proof.** For $s = 1$ the equation (6.8) is nothing but the equation (6.4). The proof follows by induction.

Now we are ready to prove Proposition (6.5).

(i) For $k > p \geq 2$, the space of solution is 2-dimensional spanned by $\alpha_1, \beta_1$.

(ii) For $p = k$, the constants $\alpha_s$ should be absence from the system (6.5). The space of solution is 1-dimensional.

(iii) For $p = 1$, and $k > 1$, all the constants $\gamma_s$ are zero. The space of solution is 1-dimensional generated by $\beta_1$.

(iv) For $k = p = 1$, all the constants $\gamma_s$ are zero and $\beta_2$ should be absence form the equation (6.4). There is no such operators.

(v) For $p = 0$, the space of solution is one-dimensional.

\[ \square \]

**6.1.2 Proof of Theorem (6.2)**

The 1-cocycle property of the operator (6.3) adds to the system above three other conditions:

\begin{align*}
2(p - 1) \alpha_1 - \gamma_{p-1} &= p \alpha_{p-1} \\
(p - 1) \beta_1 + \delta \gamma_2 &= \beta_2 \\
\beta_1 + \delta \gamma_p &= \beta_p \quad \hfill (6.9)
\end{align*}
By using Proposition (6.5), we get

(i) for \( k > p \geq 2 \), we distinguish two cases.

1. If \( p = 2 \), the system above together with the condition (6.9) admits (uniquely) a solution, independently on \( \delta \). The corresponding cocycle associated with this class is given in (4.17). This 1-cocycle turns into a trivial cocycle for \( \delta = \frac{2k-2+p}{1+n} \), as a consequence of Corollary (4.11).

2. If \( p > 2 \), the system above together with the conditions (6.9) admits (uniquely) a solution if and only if

\[
\delta = \frac{2k-p+n}{1+n}.
\]

(ii) for \( k = p \), one distinguishes two cases:

1. If \( k = p = 2 \), the system above together with the condition (6.9) admits (uniquely) a solution, independently on \( \delta \). The corresponding 1-cocycle associated with this class is given in (4.16). This 1-cocycle turns into a trivial cocycle for \( \delta = \frac{2+k+n}{1+n} \), as a consequence of Corollary (4.11).

2. if \( k = p > 2 \), the system above together with the conditions (6.9) admits (uniquely) a solution if and only if

\[
\delta = \frac{k+n}{1+n}.
\]

(iii) For \( p = 1 \), and \( k > 1 \), the unique 1-cocycle is given as in (4.14). This 1-cocycle turns into a trivial cocycle for \( \delta = \frac{2k-1+n}{1+n} \), as a consequence of Corollary (4.11).

To achieve the proof of Theorem (6.2) we are required to prove the following Lemma.

**Lemma 6.7** For \( \delta = \frac{2k-p+n}{1+n} \), any \( \text{sl}(n+1, \mathbb{R}) \)-invariant 1-cocycle from \( S^k_\delta(\mathbb{R}^n) \) to \( S^{k-p}_\delta(\mathbb{R}^n) \) is necessarily trivial.

**Proof.** The 1-cocycle conditions (6.9) turn the space of solution of the system above into a 1-dimensional space. We are led, then, to prove that any trivial 1-cocycle is necessarily \( \text{sl}(n+1, \mathbb{R}) \)-invariant for the particular value of \( \delta \). To do that, we consider the operator \( B \) defined as follows. For all \( P \in S^k_\delta \), we put

\[
B(P) = \partial_{j_1} \cdots \partial_{j_p} P^{j_1 \cdots j_p i_1 \cdots i_{k-p}}.
\]

(6.10)

Consider now the trivial 1-cocycle

\[
L_X \circ B - B \circ L_X.
\]

(6.11)

The order of the operator (6.11) is \((p-1)\), because the order of the operator (6.10) is \( p \). One can easily seen that the coefficients at any order less than \( p - 2 \) contain
expressions in which the component $X$ is differentiated at least three times. Thus, it vanishes on the Lie algebra $\text{sl}(n+1, \mathbb{R})$. Moreover, it is a matter of direct computation to prove that the principal symbol of the operator (6.11) vanishes on $\text{sl}(n+1, \mathbb{R})$ if and only if $\delta = \frac{2k-p+n}{1+n}$.

Theorem (6.2) is proven.

6.2 Cohomology of $\text{Vect}(M)$

We need to recall the following Theorem.

Theorem 6.8 [20]

$$H^1(\text{sl}(n+1, \mathbb{R}); D(S^k_\delta(\mathbb{R}^n), S^j_\delta(\mathbb{R}^n))) = \begin{cases} \mathbb{R}, & \text{if } k-j = 0 \\ \mathbb{R}^2, & \text{if } k-j > 0, \text{ and } \delta = \frac{k+j+n}{1+n} \\ 0, & \text{otherwise} \end{cases}$$

(6.12)

The 1-cocycles that span this cohomology group were given in [20]. These explicit expressions are as follows:

$$\tau_j(X)(P) = \partial_{i_1} X^{i_1} \cdots \partial_{i_k} X^{i_k} p^1 \cdots p^k, \quad (6.13)$$

$$\kappa_j(X)(P) = \partial_{i_1} \partial_{i_2} X^{i_1} \cdots \partial_{i_k} X^{i_k} p^1 \cdots p^k, \quad (6.14)$$

for all $P \in S^k_\delta(M)$.

For $k-j = 0$, the cohomology group above is spanned by $\tau_k$. For $k-j > 0$ and $\delta = \frac{k+j+n}{1+n}$, it is spanned by $\kappa_j$ and $\tau_j$.

**Proposition 6.9** (i) The 1-cocycles $\kappa_j$ can be extended uniquely as 1-cocycles on $\text{Vect}(\mathbb{R}^n)$ only for $k-j = 1, 2$.

(ii) The 1-cocycles $\tau_j$ can be extended uniquely to $\text{Vect}(\mathbb{R}^n)$ for $k-j = 0$, and for $(k,j) = (1,0)$ and $\delta = 1$.

**Proof.** (i) The 1-cocycles $\kappa_j$ can be extended to $\text{Vect}(\mathbb{R})$ for $k-j = 1, 2$ and the proof is just theirs explicit expressions given in (6.18), (6.19) and (6.20). Let us prove the uniqueness. Suppose that there are two 1-cocycles, say $c_1$ and $c_2$, that extend $\kappa_j$. This implies that $c_1 - c_2$ is zero on $\text{sl}(n+1, \mathbb{R})$. The 1-cocycle $c_1 - c_2$ is then projectively invariant. By using Theorem (6.2), the 1-cocycle $c_1 - c_2$ should be a coboundary, as $\delta = \frac{k+j+n}{1+n}$. Thus, $c_1 \equiv c_2$.

Now, we will prove that for $k-j > 2$, these 1-cocycles cannot be extended. Suppose without loosing generality that $k-j = 3$. Any 1-cocycles that extend the 1-cocycles $\kappa_j$ should retain a form as in (6.3) but we incorporate another term $\beta_1 \partial_{i_1} \partial_{i_2} X^{i_1}$. The fact that the 1-cocycles in question should coincide with the 1-cocycle $\kappa_j$, leads to the two conditions:

$$\gamma_1 = 0, \quad 2\gamma_2 + (1+n) \beta_1 + 2(k-3) \alpha_1 = 0. \quad (6.15)$$
The 1-cocycle property imposes the following conditions:

\[
\begin{align*}
3\alpha - 4\alpha_1 + \gamma_2 &= 0, \\
6\alpha - 3\alpha_2 + \gamma_2 &= 0, \\
\beta_3 - \delta \gamma_3 - \beta_1 + \tilde{\beta}_1 &= 0, \\
3\gamma_3 - 2\gamma_2 &= 0, \\
(\delta - 1)\gamma_2 + \beta_1 + \tilde{\beta}_1 &= 0,
\end{align*}
\]

The system above together with the outcomes (6.15) admits a solution if and only if \(\tilde{\beta}_1 = 0\) and \(\delta = \frac{2k-2+4n}{1+n}\). This means that the extended 1-cocycle is a coboundary and, moreover, vanishes on the Lie algebra \(\mathfrak{sl}(n+1, \mathbb{R})\), which is absurd. This implies that the 1-cocycle \(\kappa_j\) cannot be extended. Part (i) is proven.

(ii) The 1-cocycles \(\tau_j\) can be extended to \(\text{Vect}(\mathbb{R})\) for \(k-j = 0\) and for \((k, j) = (1, 0)\) and \(\delta = 1\). The proof is just theirs explicit expressions given in (6.17) and (6.21). For the uniqueness, we can easily proceed as in Part (i).

Suppose that the 1-cocycles \(\tau_j\) can be extended to \(\text{Vect}(\mathbb{R}^n)\) for the value of \(k-j\) different from those described above. Such 1-cocycles should retain a form as in (6.3) but we incorporate another term \(\beta_0 \partial_t X^t\). The fact that these 1-cocycles should coincide with the 1-cocycle \(\tau_j\) once restricted to \(\mathfrak{sl}(n+1, \mathbb{R})\), leads to the two conditions (6.15). Now, if we collect the coefficient of the term \(\partial_i Y^i \partial_j \partial_1 X^t \partial_j \cdots \partial_j^p\), we will get

\[
\begin{align*}
p k + j - 1 + n.
\end{align*}
\]

This last outcome does not vanish, except when \((k, j) = (1, 0)\), and therefore \(\delta = 1\). Part (ii) is proven.

Let \(M\) be any arbitrary manifold of dimension \(n\).

**Theorem 6.10** For all \(n > 1\), we have

\[
H^1(\text{Vect}(M); D(S^k_\delta(M), S^{ij}_\delta(M))) = \begin{cases} 
\mathbb{R} \oplus H^1_{\text{DR}}(M), & \text{if } k-j = 0 \\
\mathbb{R}, & \text{if } k-j = 1, j \neq 0 \\
\mathbb{R}^2 \oplus H^1_{\text{DR}}(M), & \text{if } (k, j) = (1, 0) \text{ and } \delta = 1 \\
\mathbb{R}, & \text{if } k-j = 2 \\
0, & \text{otherwise}
\end{cases}
\]

(6.16)

**Proof.** For the proof we proceed as follows. Firstly, we exhibit the 1-cocycles that span this cohomology group; secondly, we proof the theorem for \(\mathbb{R}^n\) then we extend the result to an arbitrarily manifold.

(i) For \(k-j = 0\), the 1-cocycles are already known (see [14]).

\[
\mathcal{a}_{\xi, \zeta}(X)(P) = (\xi \text{Div}(X) + \zeta \omega(X)) P,
\]

(6.17)
where $\omega$ is a 1-form, $\text{Div}(X)$ is the divergence operator associated to some orientation and $\xi, \zeta$ are real numbers.

(ii) For $k - j = 2$, and $\delta \neq \frac{2k - 2 + n}{1 + n}$, the 1-cocycle is given by the infinitesimal projective Schwarzian derivative $(4.17)$.

(iii) For $k - j = 2$, and $\delta = \frac{2k - 2 + n}{1 + n}$, we distinguish two cases:

1. For $k = 2$, the 1-cocycle in question is

$$\mathfrak{c}(X)(P) = i(X)_i \nabla_j P^{ij} + \nabla_i i(X)_j P^{ij}. \tag{6.18}$$

where $i(X)_i$ are the components of the tensor $(3.6)$.

2. For $k > 2$, the 1-cocycle is

$$\mathfrak{c}(X) (P)_{i_1 \cdots i_{k - 2}} = i(X)_i \nabla_j P^{ij i_1 \cdots i_{k - 2}}
\begin{aligned}
+ \gamma_1 \left( i(X)_{ij} - \text{Sym}_{ij} \frac{1}{n + 1} \delta_{ij} i(X)_j \right) \nabla_i P^{ij i_1 \cdots i_{k - 2}} \\
+ \gamma_2 \nabla_i i(X)_{ij} P^{ij i_1 \cdots i_{k - 2}} + \gamma_3 \sum_{s=1}^{k-2} \nabla_i i(X)_{ij} P^{ij i_1 \cdots \tilde{i_s} \cdots i_{k - 2}} \\
+ \gamma_4 \nabla_i i(X)_j P^{ij i_1 \cdots i_{k - 2}},
\end{aligned} \tag{6.19}$$

where the constants $\gamma_1, \ldots, \gamma_4$ are given by

$$\gamma_1 = \frac{1}{n + 1}; \quad \gamma_2 = \frac{1}{n + 1};
\gamma_3 = -\frac{1}{6} (1 + n); \quad \gamma_4 = -\frac{1}{2} (2k - 3).$$

(iv) For $k - j = 1$, $j \neq 1$ and $\delta \neq \frac{2k - 1 + n}{1 + n}$, the 1-cocycle is given by the infinitesimal projective Schwarzian derivative $(4.14)$.

(v) For $k - j = 1$, $j \neq 1$ and $\delta = \frac{2k - 1 + n}{1 + n}$, the 1-cocycle is given by

$$\mathfrak{c}(X) (P)_{i_1 \cdots i_{k - 1}} = i(X)_u P^{u i_1 \cdots i_{k - 1}}, \tag{6.20}$$

where $i(X)_i$ are the components of the tensor $(3.6)$.

(vi) For $(k, j) = (1, 0)$ and $\delta = 1$, the 1-cocycles are given by

$$\partial_{\varepsilon, \xi, \zeta} (X)(P) = \varepsilon i(X)_i P^i + (\xi \text{Div}(X) + \zeta \omega(X)) \nabla_i P^i. \tag{6.21}$$

### 6.2.1 Proof of Theorem (6.10) for the case $M = \mathbb{R}^n$

Let $\mathfrak{c}$ be any 1-cocycle on $\text{Vect}(\mathbb{R}^n)$ with values into $\mathcal{D}(S^0_k(\mathbb{R}^n), S^j_k(\mathbb{R}^n))$. The restriction of this 1-cocycle, say $\hat{\mathfrak{c}}$, to $\text{sl}(n + 1, \mathbb{R})$ is obviously a 1-cocycle on $\text{sl}(n + 1, \mathbb{R})$. We distinguish six cases:
(i) If \( k - j > 2 \), and \( \delta \neq \frac{k + j + n}{1 + n} \), then \( \hat{c} \) is trivial, by Theorem (6.8). It follows that there exists an operator, say \( B \), such that

\[
\hat{c}(X) = [L_X, B], \quad \text{for all } X \in \text{sl}(n + 1, \mathbb{R}).
\]

Now, for all \( X \in \text{Vect}(\mathbb{R}^n) \) the map \( X \mapsto c(X) - [L_X, B] \) is a 1-cocycle on \( \text{Vect}(\mathbb{R}^n) \) that vanishes on \( \text{sl}(n + 1, \mathbb{R}) \). Theorem (6.2) assures that such a 1-cocycle is trivial. A fortiori, \( c \equiv 0 \).

(ii) If \( k - j > 2 \), and \( \delta = \frac{k + j + n}{1 + n} \), then \( \hat{c} \) should be equal to zero by Proposition (6.9). It implies that the 1-cocycle \( c \) is vanishing on \( \text{sl}(n + 1, \mathbb{R}) \), and, thus, is trivial by Theorem (6.2).

(iii) If \( k = j \), then \( \hat{c} \) is cohomologous to \( a_{1,0} \), by Theorem (6.8). It follows that there exists an operator, say \( B \), such that

\[
\hat{c}(X) - \alpha a_{1,0}(X) = [L_X, B], \quad \text{for all } X \in \text{sl}(n + 1, \mathbb{R}).
\]

Now, for all \( X \in \text{Vect}(\mathbb{R}^n) \) the map \( X \mapsto c(X) - \alpha a_{1,0}(X) - [L_X, B] \) is a 1-cocycle on \( \text{Vect}(\mathbb{R}^n) \) that vanishes on \( \text{sl}(n + 1, \mathbb{R}) \). Theorem (6.2) assures that such a 1-cocycle is necessarily trivial. A fortiori, \( c \equiv a_{1,0} \).

(iv) If \( k - j = 1 \), and \( j \neq 1 \), we will prove that \( c \) is cohomologous to one of the 1-cocycles (4.1) or (6.20), depending on the value of \( \delta \).

1. For \( \delta \neq \frac{2k - 1 + n}{1 + n} \), the 1-cocycle \( \hat{c} \) should be trivial by Theorem (6.8). It follows that there exists an operator, say \( B \), such that

\[
\hat{c}(X) = [L_X, B], \quad \text{for all } X \in \text{sl}(n + 1, \mathbb{R}).
\]

Now, for all \( X \in \text{Vect}(\mathbb{R}^n) \) the map \( X \mapsto c(X) - [L_X, B] \) is a 1-cocycle on \( \text{Vect}(\mathbb{R}^n) \) that vanishes on \( \text{sl}(n + 1, \mathbb{R}) \). Theorem (6.2) assures that such a 1-cocycle is necessarily unique. A fortiori, \( c \equiv 0 \).

2. For \( \delta = \frac{2k - 1 + n}{1 + n} \), the 1-cocycle \( \hat{c} \) should be cohomologous to the 1-cocycle \( \alpha \kappa_{k-1} + \beta \tau_{k-1} \), by Theorem (6.8). Moreover, by using proposition (6.9) the 1-cocycle \( \kappa_{k-1} \) is the only 1-cocycle that can be extended. It follows that there exists an operator, say \( B \), such that

\[
\hat{c}(X) - \alpha \kappa_{k-1}(X) = [L_X, B], \quad \text{for all } X \in \text{sl}(n + 1, \mathbb{R}).
\]

Now, for all \( X \in \text{Vect}(\mathbb{R}^n) \), the map \( X \mapsto c(X) - \hat{c}(X) - [L_X, B] \) is a 1-cocycle on \( \text{Vect}(\mathbb{R}^n) \) that vanishes on \( \text{sl}(n + 1, \mathbb{R}) \). Theorem (6.2) assures that such a 1-cocycle is necessarily trivial. A fortiori, \( c \equiv \lambda \).

(v) If \( (k, j) = (1, 0) \) and \( \delta = 1 \). By using the same method as before, we can prove that \( c \) is cohomologous to the 1-cocycles \( \delta_{e, \xi, 0} \).

(vi) If \( k - j = 2 \), By using the same method as before, we can prove that \( c \) is cohomologous to the 1-cocycles (4.16) or (6.19).

Theorem (6.10) is proven for \( \mathbb{R}^n \).
6.2.2 Proof of Theorem (6.10) for the case of an arbitrary manifold

The techniques that we are going to use here have been already used in [21] for $\delta = 0$.

(i) For $k - j = 0$ we have

$$H^1(\text{Vect}(M); D(S^k_\delta(M), S^k_\delta(M))) \simeq H^1(\text{Vect}(M); C^\infty(M)).$$

The later cohomology group is well-known; it is isomorphic to $\mathbb{R} \oplus H^1_{\text{DR}}(M)$ (see, e.g., [14]).

(ii) For $(k, j) = (1, 0)$ and $\delta = 1$ we have

$$H^1(\text{Vect}(M); D(S^1_1(M), S^0_1(M))) \simeq H^1(\text{Vect}(M); \Omega^1(M)) \oplus H^1(\text{Vect}(M); C^\infty(M)).$$

For the proof we proceed as follows. Let $c$ be a 1-cocycle on $\text{Vect}(M)$ with values into $D(S^k_\delta(M), S^k_\delta(M))$. The fact that $M$ is endowed with a connection implies that the 1-cocycle $c$ can be written as $b(X)\nabla_i + a_i(X)$. The 1-cocycle condition of the 1-cocycle $c$ implies that the components $a_i$ should define a 1-cocycle belonging to the cohomology group $H^1(\text{Vect}(M); \Omega^1(M))$ and $b$ should define a 1-cocycle belonging to the cohomology group $H^1(\text{Vect}(M); C^\infty(M))$. Reciprocally, any two 1-cocycles in $H^1(\text{Vect}(M); \Omega^1(M))$ and $H^1(\text{Vect}(M); C^\infty(M))$ will define the 1-cocycle $c$, as it is given above. The cohomology group

$$H^1(\text{Vect}(M); \Omega^1(M))$$

is well-known; it is isomorphic to $\mathbb{R}$ (see, e.g., [34]).

(iii) For $k - j > 2$. Let $c$ be a 1-cocycle on $\text{Vect}(M)$ valued into $D(S^k_\delta(M), S^k_\delta(M))$. On a local chart $U$, the restriction $c|_U$ is trivial. Namely, it exists an operator, say $B|_U$, on $U$ such that

$$c|_U = L_X(B)|_U.$$

A local coordinates patching will be used to extend the operator $B|_U$. To do that, we should prove that $B|_U = B|_V$ on the intersection $U \cap V$. Indeed,

$$0 = c|_{U \cap V} - c|_{U \cap V} = L_X(B)|_U - L_X(B)|_V.$$

As there is no $\text{Vect}(M)$-invariant operators for $k - j > 2$, it implies that $B|_U = B|_V$ on $U \cap V$.

(iv) For $k - j = 2$ and $k > 2$. Let $c$ be a 1-cocycle on $\text{Vect}(M)$ valued into $D(S^k_\delta(M), S^k_\delta(M))$. On a local chart $U$, the restriction $c|_U$ is cohomologous to the 1-cocycle (4.16) or (6.19). Namely, it exists an operator, say $B|_U$, on $U$ such that

$$c|_U + \alpha|_U \mathbf{p}(X) = L_X(B)|_U.$$
where \( p \) is one of the two 1-cocycles (1.16) or (6.19). On the intersection \( U \cap V \), one has
\[
(\alpha_U - \alpha_V) p(X) = L_X(B)_{|U} - L_X(B)_{|V}.
\]
Thus, \( \alpha_U - \alpha_V = 0 \) because \( p \) is not a coboundary and, a fortiori, \( B_{|U} = B_{|V} \) on \( U \cap V \), as there is no \( \text{Vect}(M) \)-invariant operators for \( k - j = 2 \).

(v) For \( k = 2 \) and \( j = 0 \), the proof is the same as in (iii).

(vi) For \( k - j = 1 \), and \( j \neq 1 \), the proof is the same as in (iii).

### 6.3 Cohomology of \( \text{Diff}(S^n) \)

In order to compute the cohomology of the group of diffeomorphisms \( \text{Diff}(M) \), we deal with differential cohomology “Van Est Cohomology”; this means we consider only differential cochains (see [14]). The more general case – namely, the cohomology with also non-differentiable cochains – is an intricate problem, and even though no explicit cocycles are known in our situation.

Let \( S^n \) be the \( n \)-dimensional sphere. It is well-known that the maximal compact group of “rotations” of \( S^n \), \( SO(n+1) \), is a deformation retract of the group \( \text{Diff}^+(S^n) \), for \( n = 1, 2, 3 \) (see [33]). Since the space \( \text{Diff}^+(S^n)/SO(n+1) \) is acyclic, the Van Est cohomology of the Lie group \( \text{Diff}^+(S^n) \) can be computed using the isomorphism (see, e.g., [14, p. 298])
\[
H^1(\text{Diff}^+(S^n); \mathcal{D}(\mathcal{S}_\delta^k(S^n), \mathcal{S}_\delta^j(S^n))) \simeq H^1(\text{Vect}(S^n), SO(n+1); \mathcal{D}(\mathcal{S}_\delta^k(S^n), \mathcal{S}_\delta^j(S^n))).
\]

We state the following Theorem that generalizes the result of [3] for \( \delta = 0 \).

**Theorem 6.11** For \( n = 2, 3 \), the first-cohomology group
\[
H^1(\text{Diff}^+(S^n); \mathcal{D}(\mathcal{S}_\delta^k(S^n), \mathcal{S}_\delta^j(S^n))) = \begin{cases} 
\mathbb{R}, & \text{if } k - j = 0 \\
\mathbb{R}, & \text{if } k - j = 1, j \neq 0 \\
\mathbb{R}^2, & \text{if } (k, j) = (1, 0) \text{ and } \delta = 1 \\
\mathbb{R}, & \text{if } k - j = 2 \\
0, & \text{otherwise}
\end{cases}
\]

**Proof.** We will first give the explicit 1-cocycles that span the cohomology group above.

(i) For \( k - j = 0 \). Any diffeomorphism \( f \in \text{Diff}^+(S^n) \) preserves the volume form on \( S^n \) up to some factor. The logarithm function of this factor defines a 1-cocycle on \( \text{Diff}(S^n) \), say \( J(f) \), with values in \( C^\infty(S^n) \). Now, the 1-cocycle in question is just the multiplication operator by \( J(f) \).

(ii) For \( k - j = 1, j \neq 0 \) and \( \delta \neq \frac{2k-1+n}{1+n} \), the 1-cocycle in question is the Schwarzian derivative (4.2). For \( \delta = \frac{2k-1+n}{1+n} \), the 1-cocycle is
\[
\mathcal{L}(f)_u P^{ai_1...i_{k-1}}
\]
where \( \mathcal{L}(f)_u \) are the components of the trace of the tensor (4.1).

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In the cohomology group (6.2) is trivial since \( H^k_\text{DR}(4.8) \). For \( k \leq j \), we observe that the De Rham classes in the cohomology group (6.23). The isomorphism above shows that these 1-cocycles induce non-cohomologous classes in the cohomology group (6.2), which is absurd.

Remark 6.12 Theorem (6.11) remains true as far as the rotation group \( SO(n+1) \) is a deformation retract of the group \( \text{Diff}_+(S^n) \) for all \( n \). We do not know whether this statement is true or not.
6.4 Relation to the Vey Cocycle

Throughout this section, we will assume that \( \delta = 0 \). The main result is to give a relation between the projective Schwarzian derivative (1.8) and the well-known Vey cocycle, answering a question raised in [3].

Recall that the Vey cocycle is an object that is closely related to deformation quantization (see [35] for more details). It is, in fact, a cohomology class that spans the component \( \mathbb{R} \) of the cohomology group \( H^2(C^\infty(T^*M), C^\infty(T^*M)) \equiv H^2_{DR}(M) \oplus \mathbb{R} \) (see [35]). In order to write it down, we need to lift the connection to a connection on the cotangent bundle \( T^*M \) (see [38] for more details). We are mainly interested when its first component is restricted to \( \text{Vect}(M) \subset C^\infty(T^*M) \). The Vey cocycle reads accordingly as follows.

\[
S^3(X) := \text{Sym}_{j,i,k} \left( l(\tilde{X})^j_{ml} \cdot \omega^{im} \cdot \omega^{kl} \right) \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\nabla}_k. \tag{6.25}
\]

In the formula above, the subscript \( i \) runs from 1 to 2\( n \), and \( \omega \) stands for the standard symplectic structure on \( T^*M \), and \( \tilde{X} \) is the Hamiltonian lift of \( X \).

The following cocycle were introduced in [3], and interpreted as a group Vey cocycle:

\[
GS^3(f) := \text{Sym}_{j,i,k} \left( \mathcal{L}^j_{mi}(\tilde{f}) \cdot \omega^{im} \cdot \omega^{kl} \right) \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\nabla}_k 
- \frac{3}{2} \text{Sym}_{n,m,i} \left( \mathcal{L}^m_{ik}(\tilde{f}) \cdot \omega^{ml} \cdot \omega^{ik} \right) \cdot \mathcal{L}^j_{mn}(\tilde{f}) \tilde{\nabla}_i \tilde{\nabla}_j, \tag{6.26}
\]

where \( \tilde{f} \) is the symplectic lift of \( f \) to \( T^*M \) and \( \mathcal{L}(f)_{ij}^k \) are the components of the tensor (3.5) with respect to the lifted connection on \( T^*M \).

**Proposition 6.13** The relation between the Vey cocycle and the projective Schwarzian derivative is as follows:

(i) For all \( X \in \text{Vect}(M) \), we have

\[
\nu(X)^i_{i\cdots i_{k-2}} = \frac{1}{2} L_X(\nabla_i \nabla_j) + \frac{2 - 2k - n}{2} S^3(X)_{sk(M)} + \frac{11 + 4k^2 - 2n(5 - 4k) + 3n^2 - 12k}{6 - 6n} L_X(R_{ij}).
\]

(ii) For all \( f \in \text{Diff}(M) \), we have

\[
\mathfrak{W}(f)^i_{i\cdots i_{k-2}} = \frac{1}{2} f^{-1} \ast (\nabla_i \nabla_j) - \nabla_i \nabla_j + \frac{2 - 2k - n}{2} GS^3(f)_{sk(M)} + \frac{11 + 4k^2 - 2n(5 - 4k) + 3n^2 - 12k}{6 - 6n} (f^{-1} R_{ij} - R_{ij}). \tag{6.27}
\]

**Proof.** For the proof, we have to expound the formulas (6.26) and (6.25) once restricted to \( \mathcal{S}^k(M) \) and write these expressions in terms of the initial connection on \( M \). Then, the proof follows by a direct computation.
6.5 Conclusion and Open Problems

The programm for defining the projective and conformal multi-dimensional Schwarzian derivatives is achieved now in this paper. However, it would be interesting to investigate topological properties of these derivatives. For instance, it has recently been proved that the classical Schwarzian derivative of a diffeomorphism admits at least four zeros in \([27]\). According to Ghys-Ovsienko-Tabachnikov, this property is the \textit{four vertex Theorem} of a time-like curve on the torus endowed with a Lorentzian metric. It would be interesting to know whether a theorem of this type holds true for our multi-dimensional Schwarzian derivatives.

According to Theorem (6.11), the conformal Schwarzian derivatives are only the operators (5.1) and (5.2), except another cocycle may appear for the particular values \((k, j) = (1, 0)\) and \(\delta = 1\). But, we do not expect new cocycles other than those given here. More precisely, we are led to compute the cohomology group

\[ H^1(\text{Diff}(\mathbb{R}^n), O(p + 1, q + 1); \mathcal{D}(\mathcal{S}_0^k(\mathbb{R}^n), \mathcal{S}_0^j(\mathbb{R}^n))). \]

The computation of this cohomology group is more intricate, and even though for the cohomology of \(\text{Vect}(\mathbb{R}^n)\) the computation is still out of reach.

The conformal Schwarzian derivative is certainly related to the Vey cocycle and an analogue to the Proposition (6.13) is certainly true. We are required to incorporate the Vey cocycle an appropriate coboundary to get a formula analogous to that in (6.21). We recall that this coboundary has been added, as explained in section (5.1), in order to get the invariance property.

Recently, the author has investigated an analogue of the operator (5.1) to the (generic) Finsler structures in [7], using some connections associated with the Finsler structure. This operator has the property that it coincides with the operator (5.1) when the Finsler structure is Riemannian. It would be interesting to investigate Schwarzian derivatives in other geometry; for instance: CR structures, quaternionic structures...

It should be stressed that in the literature alternative approaches were developed in order to extend the classical Schwarzian derivative to a multi-dimensional manifold (see for example [1, 10, 15, 23, 25, 26, 30, 31]).

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