Can We Derive Quantum Mechanics as an Emergent Theory from a Modified Relativity?

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October 9, 2019

Abstract

A new model of Deformed Special Relativity (DSR) was proposed to derive a modified General Relativity at the sub-quantum level towards the Planck scale. This was used as a basis for a new approach to Emergent Quantum Mechanics from which the standard mathematical formalism of Relativistic Quantum Mechanics was derived. Lastly, a new interpretation for the so-called quantum wave was suggested.

Keyword: Deformed Special Relativity, Emergent Quantum Mechanics, Unification Theory

1 Introduction

1.1 Deformed Special Relativity

The seminal work done by Amelino-Camelia [1, 2], Smolin–Maguiejo [3] among others [4, 5, 6, 7] on what now became as "Deformed Special Relativity" or "Doubly Special Relativity" (DSR), is an attempt to modify Lorentz Invariance at the region near the Planck Scale. It was initially motivated by the idea that the Planck energy and the Planck Length are fundamentally invariant. It modifies the momentum space via an introduction of another invariant quantity in addition with the speed of light, thus the term "doubly". Here, a new model of DSR was proposed that does not involve any of dimensionful quantities like the Planck energy or the Planck length as additional invariant quantity at the sub-quantum level. What had been done is a fundamental transformation of the Lorentz Boost via an introduction of a unitless quantity χ thru a complex function ϕ. This approach of transforming the Lorentz Boost to modify Relativity is not something new as it is already in the literature [8]. However, most transformation matrices that were use to transform Λ are not complex but real and usually expressed in terms of momentum like in the work of Heuson [9]. In general case, the modification was done in energy dispersion relation and expressed as follows [10]

\[ E^2 = p^2 + m^2 + \eta L_p^2 p^2 E^n + O(L_p^{n+1} p E^{n+3}) \] (1)

where \( L_p \) is the Planck Length and \( n \) is a whole number. In simple terms, Salesi et. al. [8], expressed the equation above as

\[ E^2 = p^2 + m^2 + p^2 f \left( \frac{p}{M} \right) \] (2)

as a general form of the natural deformation of the standard dispersion law where \( f \) is a function of \( M \) which indicates a mass scale characterizing the Lorentz breakdown. The last term, they suggested, can be rewritten as a series expansion which they called "Lorentz-Violating power terms" [8]. Another approach mentioned in [8] was categorized to be written in terms of "form factors" as

\[ g^2(p)E^2 - f^2(p)p^2 = m^2 \] (3)

Another approach is in terms of "momentum-dependent metric" which became the basis of Rainbow Gravity Theory

\[ ds^2 = g^{-2}(p)dt^2 - f^{-2}(p)dl^2 \] (4)

Finally, they categorized those DSR theories that are defined by what they called "deformation function" \( F \), i.e.,

\[ F(E^2 - p^2) = m^2 \] (5)

where the function \( F \) is expressed in terms of momentum. They derived \( F \) to be

\[ F = A \lambda^2 (p_x - \nu E) + p_y^2 + p_z^2 \] (6)

for some real function \( A \) and the quantity \( \lambda \) is the energy-momentum cut-off parameter. They compared it in the work of Lee and Smolin [3] with \( F \) was defined as

\[ F = (1 + \lambda^2 p^2)^{-1} \] (7)
Here, the modification was done by generalizing the so-called Improper Lorentz transformation which then applied, not only on momentum space, but also on position space. Also, the new DSR theory modifies Special Relativity not by introducing another invariant dimensionful quantities in addition with the speed of light. What was introduced is a new dimensionless arbitrary quantity \( \chi \) through a complex function \( \varphi = e^{2\pi i \chi} \). Since the Lorentz Boost is essentially a matrix of dimensionless constants which relates observed physical quantities from one frame of reference to another, then according to Duff [11][12], such transformation of dimensionless constants is operationally meaningful. On the other hand, mainstream approaches in DSR modifies the Lorentz Boost by using dimensionful quantities like the Planck Energy and the Planck Length. Though both approaches have the same goal one approach might be advantageous than the other and would give more results. It is suggested here that an emphasis should be given on the crucial difference between the two approaches where one uses dimensionful quantities and the other uses dimensionless quantities. Dimensionful quantities are the result of asymmetries between two orthogonal axes of description or measurements of quantities. That is why the use of dimensionful constants are known to be arbitrary since they are conventionally used as conversion factors [11][12]. On the other hand, dimensionless quantities are the result of what appears to be the symmetry between measured quantities. Thus, if symmetry alone would be the basis, the latter type of quantities seems to be more fundamental and appropriate to use in modifying Special Relativity at the fundamental level.

1.2 Complex Lorentz Boost

Consider two collinear frames of reference in uniform motion along x-axis with no rotation, the Lorentz Transformation in matrix form as presented in [13] is typically expressed as follows:

\[
X' = \Lambda X \quad P' = \Lambda P
\]  
(8)

where

\[
X' = \begin{pmatrix} t' \\ x' \end{pmatrix} \quad X = \begin{pmatrix} t \\ x \end{pmatrix} \quad P' = \begin{pmatrix} E' \\ p'_x \end{pmatrix} \quad P = \begin{pmatrix} E \\ p_x \end{pmatrix}
\]  
(9)

and the Pure Lorentz Boost \( \Lambda \) is given by

\[
\Lambda = \begin{pmatrix} \cosh \xi & -\sinh \xi \\ -\sinh \xi & \cosh \xi \end{pmatrix}
\]  
(10)

in terms of rapidity \( \xi \) and the convention \( c = 1 \) was used. Transforming \( X' \) into a square matrix by multiplying it with the Minkowski metric

\[
\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]  
(11)

and then multiplying the transpose \( X'^T \), i.e.,

\[
X'^T \eta X' = (X'^T \Lambda^T) \eta (AX) = X^T (\Lambda^T \eta \Lambda) X
\]  
(12)

implies Lorentz Invariance since \( \Lambda^T \eta \Lambda = \eta \). Another way in showing Lorentz Invariance without the use of the Minkowski metric is using Pauli matrices \( \sigma_i \) via the coordinate transformation \( X \rightarrow \sigma^T X = X = i \sigma_0 + x \sigma_1 = \begin{pmatrix} t \\ x \\ i \end{pmatrix} \) for both primed and unprimed coordinates where \( \sigma \) is a column vector \( (\sigma_0 \sigma_1) \). Solving for the determinants and multiplying with \( \det \Lambda \) gives us

\[
(\det \bar{X})^2 = (\det \Lambda)^2 (\det \bar{X})^2 = (\det \bar{X})^2
\]  
(13)

where \( (\det \Lambda)^2 = 1 \), thus, \( \det \Lambda = \pm \sqrt{1} \). This shows two possible cases where the first case is \( \det \Lambda = +1 \) for Proper Lorentz Transformation and the other case is \( \det \Lambda = -1 \) for the so-called Improper Lorentz Transformation. Notice that for the latter case \( s'^2 = -s^2 \), it implies \( t' = \pm it \), \( x' = \pm ix \), and \( s' = \pm i s \). Similarly, it gives us \( m'^2 = -m^2 \) and yield us \( m' = \pm im \). The result is the so-called ’t Hooft and Nobbenhuis space-time complex transformation (tHNCT) which is an imaginary transformation of space-time: \( x' \rightarrow ix^a \). It was introduced by ’t Hooft and Nobbenhuis as an attempt to solve the cosmological constant problem using a symmetry argument [14]. It was then extended by Arbab and Widatallah by including the mass [15]:

\[
t \rightarrow it, \quad x \rightarrow ix, \quad m \rightarrow im
\]  
(14)

in which they incorporated Quantum Mechanics’ Operator Correspondence that equate momentum and energy to a complex differential operator. In his succeeding papers, Arbab incorporated the transformation above with his “Quaterionic Quantum Mechanics” [16][17]. A more generalized form of ’t Hooft-Nobbenhuis complex transformation can be derived by modifying Special Relativity where the value of \( (\det \Lambda)^2 \) is not just limited to 1. This can be done by considering an imaginary transformation of the rapidity \( \xi \rightarrow i \xi \). It gives us a complex form of the Pure Lorentz Boost

\[
\Lambda \rightarrow L = \begin{pmatrix} \cos(\xi) & -i \sin(\xi) \\ -i \sin(\xi) & \cos(\xi) \end{pmatrix}
\]  
(15)

This can be generalized further by transforming \( L \) as follows

\[
L \rightarrow \Sigma L A^{-1} = \begin{pmatrix} \cos(\xi) & -i \sin(\xi) \\ -ia \sin(\xi) & \cos(\xi) \end{pmatrix}
\]  
(16)

for some transformation matrix \( \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \). Using a matrix generalization of Euler’s Identity: \( e^{ia} = \cos \theta + i \sin \theta \), as suggested in [18], the equation above can be written as

\[
\Sigma \Phi = e^{-\xi \Phi} = \cos(\xi) I - \frac{1}{\alpha} \sin(\xi) \Phi
\]  
(17)

The matrix \( I \) is the identity matrix and \( \Phi \) is defined as follows;

\[
\Phi = \begin{pmatrix} 0 & i \\ ia^2 & 0 \end{pmatrix}
\]  
(18)
that acts like an imaginary unit matrix. Lorentz invariance is still preserved since \( \det \Omega^2 = 1 \). Following Argentini in [13], the \( 2 \times 2 \) matrices can be set into a \( 1 \times 1 \) matrices, i.e., \( \Phi = [i] \) and \( I = [1] \), and \( \alpha = 1 \), to yield us the usual Euler Identity:

\[
e^{-i\xi} = \cos \xi - i \sin \xi = \varphi^*\varphi.
\]

This will give us a generalized form of 't Hooft-Nobbenhuis Complex Transformation as follows:

\[
t \rightarrow \varphi^*t = \tilde{t} \\
x \rightarrow \varphi^*x = \tilde{x}
\]

(19)

from a modified Lorentz Transformation \( X' = \varphi^*X \). In order to violate Lorentz Symmetry, one approach is to set \( \alpha = e^{\varphi x} \) to be complex number and not as a real number. By transforming the complex Lorentz Boost \( L \) via the transformation \( L \rightarrow G = \text{ALA}^T \) such that \( \det G = \alpha^2 = \varphi \), will give us

\[
t \rightarrow \varphi t = \tilde{t} \\
x \rightarrow \varphi x = \tilde{x}
\]

(20)

using a modified Lorentz Transformation \( \tilde{X} = G\tilde{X} \) where \( \varphi = e^{2\varphi x} \). The function \( \varphi \) introduces a unitless quantity \( \varphi \) by which the energy and spacetime at the sub-quantum level down to the Planck Scale are to be defined by it. Using the result above, a postulate can be formalized on the nature of what had been called here as the 'sub-quantum region', which is define here to be such region outside the quantum scale towards the Planck Scale, i.e.,

**Postulate I:**

The Lorentz symmetry is violated at the sub-quantum region.

The violation is suggested to be mathematically expressed via the complex transformation of the Lorentz Transformation which implies a Weyl metric tensor transformation given by

\[
g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \alpha^2 g_{\mu\nu}
\]

(21)

for some complex function \( \alpha^2 \).

### 1.3 Metric Fluctuation

It is well-known that under Weyl metric transformation, the geometry changes. It transforms the metrical condition of the Riemannian Geometry i.e.,

\[
\bar{\nabla}g_{\mu\nu} = 0 \rightarrow \bar{\nabla}(\alpha^2g_{\mu\nu}) = (\bar{\nabla}\alpha^2)g_{\mu\nu} + \alpha^2(\bar{\nabla}g_{\mu\nu}) = 0
\]

(22)

which gives us the non-metric condition by which we set the covariant derivative to become the usual derivative \( \partial \). Studies on spacetime at the Planck Scale were known to include the notion of a metric tensor fluctuation.

The works of Frederick [19] [20], for example, give emphasis on metric tensor fluctuation and used it as basis of his idea of "stochasticity" of spacetime and his formulation of a chaotic and deterministic model of an emergent Quantum Mechanics. Also the idea of quantum fluctuation of spacetime had been around for some time now [21]. The present study, in essence, is a continuation of such earlier works in which a new mathematical tool is incorporated as an attempt to have a deterministic description of the so-called metric tensor fluctuation. Studies on the so-called "metric fluctuation" or "spacetime fluctuation" at the Planck Scale region use an approximation scheme in order to describe the spacetime at/near the Planck Scale. Possible effects that can be observed were described as follows: modified inertial mass in violation of weak equivalence principle and Lorentz Invariance [22, 23, 24], quantum systems suffering decoherence [25, 26, 27] and in the context of the propagation of light, fluctuating light cones and angular blurring [28]. Also discussed were modified dispersion relations [29, 30, 31, 32]. These results were derived by using a Minkowskian background on which small spacetime dependent metrical fluctuations are imposed. It is usually described in the weak field approximation, where the metric, \( g_{\mu\nu} \) is expressed with a small perturbation \( h_{\mu\nu} \) of the flat Minkowski spacetime, \( g_{ab} \), i.e

\[
g_{\mu\nu}(x, t) = g_{ab}(x, t) + h_{\mu\nu}(x, t)
\]

(25)

Thus, they literally split the metric by having a background metric \( g_{ab} \) and having \( h_{\mu\nu}(x, t) \) as the field that is set to be quantized. The mathematical approaches utilized to most quantization schemes were mainly algebraic and geometric. The goal of the theory presented here is to find a more revealing, if not exact description, of the fluctuation of the metric tensor at the Planck Scale and Quantum Level using the ideas in Differential Topology like the Ricci Flow [33, 34, 35, 36, 37, 38, 39, 40] and Conformal Symmetry [41, 42, 43, 44]. Both approaches are currently used to solve foundational issues in Quantum Mechanics. The so-called Emergent Quantum Mechanics, meanwhile, posits the notion that Quantum Mechanics can emerge from a deterministic model and becomes probabilistic via a dissipative mechanism that implement information loss [45]. It was first introduced by 't Hooft [46] [47] and later developed by others (See [48, 49] and references therein). Other deterministic approaches much earlier than all of these are the geometric interpretations of Quantum Mechanics. For example, J.T. Wheeler suggested a geometric picture of quantum mechanics using Weyl Geometry [50]. Other geometric approach to Quantum Mechanics was suggested by Wood and Papini [51] in which they incorporate modified Weyl-Dirac theory with particle aspects of matter and Weyl symmetry breaking. Sidharth [52], on the other hand, try to
geometric Quantum Mechanics using a non-commutative non-integrable geometry. Most recently, tools in Differential Topology like the Ricci Flow had been used. One of it is the work of Dzhunushaliev in 2006 where he suggested two major ideas which is the impetus of the present work. First is the notion that, Ricci Flow is a statistical system that can be use to describe the topology change at Quantum Gravity or at Planck Scale. Second is the idea that, the metric tensor can be considered as a microscopical state in a statistical system. In the work of Dzhunushaliev the probability density is proportional to Perelman’s functional \[ \mathcal{F} \] here the probability density was shown to be proportional to the metric tensor itself. This is the cornerstone of the theory presented here that the conformal metric \( g_{\mu\nu} \) and the quantum probability density \( |\psi|^2 \) or the wave function \( \psi \) are connected with each other and that both variables must be used for the description of spacetime at the microscopic world.

Also two major results in the work of Isidro et. al. [54, 45] were justified here. First, Schrödinger Equation can be derived from a conformally flat metric under Ricci Flow. Second, the State Vector or the wave function is related to the conformal term. In both references [53, 54], the use of Perelman’s functional had been a key ingredient. In this study, the use of Perelman’s functional was reserved for some future studies. What was used here is the most generalized functional as the conformal term for the metric tensor. To end this section, a second postulate is formalize here and stated as follows:

**Postulate II:** The metric tensor fluctuates as the Law of Conservation of Energy is violated at sub-quantum region.

The violation of the Law of Conservation of Energy can be expressed by the condition \( \nabla T_{\mu\nu} \neq 0 \) where the energy tensor \( T_{\mu\nu} \) represents the external energy that enters a system.

## 2 Formalism

Postulate I as expressed by the generalized tHNC theory gives us the following transformation:

\[
dt \rightarrow \int adt = \widetilde{t} \quad dx \rightarrow \int adx = \tilde{x}
\]

(26)

where \( \tilde{x} \) and \( \tilde{t} \) as new space and time scale defined by the conformal factor \( \alpha \) as a normalization factor to preserve the volume. This is done by setting the normalized volume into a unit volume i.e.,

\[
\int_M \rho \, dV = \int_M a^n \rho \, dV = 1
\]

(27)

in \( n \)-dimensional complete Reimannian manifold \( M \). By postulate II, metric tensor fluctuation can be set to vary in time, within such volume of space. The fluctuation in time is suggested to be described by the equation below

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}
\]

(28)

The equation above is known as the Hyperbolic Geometric Flow (HGF) in unnormalized form. It is an equation first introduced by DeXing Kong and Kefeng Lui in 2006 as a second order version of the Ricci Flow [55]. It is suggested, that HGF can be used to describe the fundamental metric tensor fluctuation that is happening at the sub-quantum level via the inherent dynamics and curvature of spacetime due to the presence of the so-called vacuum energy. According to Kong and Lui [55], the condition set by equation (27), i.e., \( \int_M dV = \alpha^{-n} \), gives us the normalized form of HGF

\[
a_{ij}\frac{\partial}{\partial t} g_{ij} + b_{ij} \partial_r g_{ij} + c_{ij} g_{ij} = -2R_{ij}
\]

(29)

where \( a_{ij}, b_{ij} \) and \( c_{ij} \) are certain smooth functions in \( M \) which may depend on \( t \). Notice that for the case where \( a_{ij} = b_{ij} = 0 \) and \( c_{ij} = \gamma \) is a constant, we have

\[
R_{ij} = \frac{1}{2} \gamma \partial_t \rho_{ij}
\]

(30)

which is the vacuum case for Einstein Field Equation. A modified Einstein Field Equation can be expressed as follows

\[
G_{\mu\nu} = -(a_{ij} \frac{\partial}{\partial t} g_{ij} + b_{ij} \partial_r g_{ij})
\]

(31)

On the other hand, the case where \( a_{ij} = c_{ij} = 0 \) and \( b_{ij} = -1 \), gives us the famous Ricci Flow

\[
\frac{\partial}{\partial r} g_{ij} = -2R_{ij}
\]

(32)

which is similar to a “heat equation”. It was first introduced by Hamilton [56] in 1980’s and then used by Perelman to prove Poincare’s conjecture [57] using the earlier work of Thurston [59]. The normalized form of the Ricci Flow was also derived by Hamilton [56] given by

\[
\frac{\partial}{\partial r} g_{ij} = \frac{2}{n} R_{ij} - 2R_{ij}
\]

(33)

where \( r = \int_{V(r)} \frac{R_{ij} \, dV}{\int_{V(r)} \rho \, dV} \) is the average scalar curvature. Note that equation (33) can be derived from equation (29) by setting \( b_{ij} = -1, a_{ij} = 0 \) and \( c_{ij} = \frac{2}{n} r \). Now, by putting in an external energy source, we can further modify equation (29) as follows

\[
a_{ij}\frac{\partial}{\partial t} g_{ij} + b_{ij} \partial_r g_{ij} + c_{ij} g_{ij} + 2R_{ij} = 2kT_{ij}
\]

(34)

where \( a_{ij}, b_{ij} \) and \( c_{ij} \) can assume values that are constants. The equation above was first proposed by Kong and Lui [55]. It suggests that even in the absence of an external energy, an inherent energy still exists and cause for the spacetime to fluctuate. Notice also that for \( a_{ij} = b_{ij} = c_{ij} = 0 \), it gives us Einstein’s initial field equation based on the work of Nordström

\[
R_{ij} = kT_{ij}
\]

(35)
where it satisfies the condition for violation of Law of Conservation of Energy since $\nabla T_{ij} = \nabla R_{ij} \neq 0$. Kong and Lui also described HGF as the “Wave Equation of the Metric” [55]. Its normalized form written in [23] is a wave equation with extra damping term. To show this explicitly, an approximation for the Ricci Tensor $R_{ij}$ in terms of the metric tensor [60] can be used, i.e.,

$$ R_{ij} \approx -\frac{1}{2} \nabla^2 g_{ij} $$

(36)

it will transform equation [28] into a wave equation;

$$ (\partial_t^2 - \nabla^2 )g_{ij} = 0 $$

(37)

In this form, it is appropriate to use greek indices instead of using latin indices to indicate 4-dimensional consideration. Hence, for $\alpha_i = 1$, the normalized Hyperbolic Geometric Flow expressed in equation (29) can be rewritten as follows:

$$ 0 = \partial_t g_{\mu\nu} - \nabla^2 g_{\mu\nu} + b_\mu \partial_t g_{\nu\mu} + c g_{\mu\nu} $$

(38)

$$ = \partial^2_t g_{\mu\nu} - \nabla^2 g_{\mu\nu} + F = 0 $$

(39)

where $\Box = \partial^2_t - \nabla^2 + b_\mu \partial_t + c g_{\mu\nu}$ is a modified d’Alembert operator with two additional terms and the function $F = F(\partial_t g_{\mu\nu}, g_{\mu\nu})$ acts as a single damping term of the wave equation. The equation, in form, is a Telegraphy Equation but instead of electrical signals, it is the metric tensor that is fluctuating or oscillating. The most general form of it as suggested in [55], involves additional higher-order terms, and on inserting the Planck-de Broglie equation in Quantum Mechanics

$$ E = \hbar f \quad p = \hbar \lambda $$

(46)

This gives us equations that are strikingly similar to the famous Planck-de Broglie equation in Quantum Mechanics

$$ E = \hbar f \quad p = \hbar \lambda $$

(46)

3.2 Deriving Operator Correspondence, Wave Function and Born Rule

From (43) and (44), it will also yield us;

$$ E = \varphi E \quad p = -i\hbar \nabla \varphi $$

(49)

from which we can derived an “Operator Correspondence”:

$$ E = \hbar \partial_t \quad p = -i\hbar \nabla \varphi $$

(50)

This can be interpreted as the energy and momentum at the “rest frame”. On the other hand, for an observer in another

3 Emergent Quantum Mechanics

(Kinematical Part)

3.1 Deriving de Broglie-Planck Equation

Applying now the generalized THNCT on energy and momentum, we can define the following transformation:

$$ E \rightarrow \varphi E = \tilde{E} \quad p \rightarrow \varphi p = \tilde{p} $$

(42)

where $\varphi = e^{i2\pi \psi}$. Now, since $i2\pi \varphi = \frac{\partial \varphi}{\partial t}$, $E = -\frac{\partial S}{\partial t}$ and $p_x = \frac{\partial S}{\partial x}$, the energy $\tilde{E}$ and momentum $\tilde{p}$, upon inserting the imaginary number $i$, can be written as follows,

$$ \tilde{E} = -\frac{i}{2\pi} (i2\pi \varphi) E = \frac{i}{2\pi} \frac{\partial \varphi}{\partial t} \frac{\partial S}{\partial \tilde{t}} = \frac{i}{2\pi} \frac{\partial \varphi}{\partial \tilde{t}} \left( \frac{1}{f} \frac{\partial S}{\partial \tilde{t}} \right) $$

(43)

$$ \tilde{p}_x = -\frac{i}{2\pi} (i2\pi \varphi) p_x = -\frac{i}{2\pi} \left( \frac{\partial \varphi}{\partial x} \right) \left( \frac{\partial S}{\partial x} \right) = -\frac{i}{2\pi} \left( \frac{\partial \varphi}{\partial x} \right) \left( \frac{\partial S}{\partial x} \right) $$

(44)

where two variables are define as $\frac{1}{\tilde{t}} = \frac{\partial \varphi}{\partial t}$ and $\lambda = \frac{\partial \varphi}{\partial x}$ that shows space and time being both dependent on a unitless variable $\chi$. For energy being dependent on $\chi$, another variable can be defined as follows:

$$ \hbar = 1 \frac{\partial S}{\partial \tilde{t}} = \lambda \frac{\partial S}{\partial x} = \frac{\partial S}{\partial \chi} $$

(45)
frame of reference outside of the rest frame, the energy and momentum are given by

\[ E = E\varphi = E\varphi \quad \hat{p} = \hat{p}\varphi = p\varphi \]  

(51)

such that the mass is given by

\[ m^2 = |E|^2 - |p|^2 = |\varphi|^2 (E^2 - p^2) = g_{\mu\nu} p^\mu p^\nu \]  

(52)

where \( g_{\mu\nu} = |\varphi|^2 g_{\mu\nu} \). To evaluate the function \( \varphi \), consider equation \( 45 \), where \( S = S(\chi) \), then integrating and setting the new variable \( \hat{h} = \frac{d}{d\chi} \) constant or its reduced form, i.e., \( \hat{h} = \hat{h} \), for some constant \( \hat{h} \), it will yield us

\[ 2\pi\chi = S(\hat{h}) + k \]  

(53)

for some integration constant \( k \). Thus, \( \chi \) is a unitless quantity that is energy-dependent. This will transform \( \varphi \) as follows:

\[ \varphi \rightarrow \psi = Ce^{iS/\hat{h}} \]  

(54)

Note that the function \( \psi \), in form, is the Wave Function \( 53 \). The function \( C = e^{i\chi} \) can be set as a normalization constant and the propagator \( e^{iS/\hat{h}} \) is Feynman’s Probability Amplitude. Thus, the metric tensor

\[ \bar{g}_{\mu\nu} = |\psi|^2 g_{\mu\nu} \]  

(55)

is suggested here to apply at the quantum level. The quantity given by \( |\psi|^2 \) is suggested here to be a conformal factor that defines the metric tensor of the so-called “Quantum Spacetime”. The complex function \( \psi \) is considered here to be a complexified Lorentz Boost for a generalized Improper Lorentz Transformation which allow for violation of Lorentz symmetry, space inversion, and time reflection. Each or the combined effect of these properties is suggested here to be the root cause of the so-called “non-classical” behaviour of a quantum particle like nonlocality, quantum entanglement and indeterminacy. Also, since \( |\varphi|^2 \) is a scalar function, the covariant derivative can be set as the usual derivative, i.e., \( \nabla |\varphi|^2 = \partial |\varphi|^2 \). Since \( |\varphi|^2 \) is constant, \( \partial |\varphi|^2 = 0 \), which is equivalent to the continuity equation of quantum probability density by integrating the derivative of the volume

\[ \partial \left( \int |\psi|^2 dV \right) = 0 \]  

(56)

This is consistent with equation \( 27 \) which gives us

\[ \int |\psi|^2 dV = 1 \]  

(57)

where \( |\psi|^2 \) served as the normalization factor. The equation above implies the presence of a quantum particle in a given volume of space as Born Rule suggests. Note that a quantum particle or a quantum field is considered here to be a region of confined metric fluctuation or spacetime fluctuation which is described by a wave equation. To meet the condition that the fluctuation is to be confined within a given volume of space, it is necessary for the wave equation to be normalized. This was done by normalizing the volume into a unit volume of space by setting the condition given by \( 57 \). If one is to consider equation \( 57 \) as a probability equation that indicates the presence of a particle at a given volume of space, then the quantity \( |\psi|^2 \) can be interpreted as a probability density of finding a quantum particle at a given region in space at a given time.

### 3.3 Deriving Correspondence Principle

In this section, it is shown that Relativity at the macroscopic level can emerge from Quantum Mechanics that had been shown in the previous sections to be emergent from a new model of DSR. To recap, the new DSR theory being proposed here, as shown in the previous sections, defined the Physics at the sub-quantum region or the Planck Scale. The Physics can be described as follows:

1. There is a breakdown of Relativity as
   
   (a) Lorentz Symmetry is violated and the speed of light is no longer the quantity maintained to be invariant and
   
   (b) The metricity condition and the Law of Conservation of Energy are violated as the metric tensor fluctuates.

2. There is a breakdown of Quantum Mechanics as
   
   (a) The constancy of Planck “constant” is violated as it becomes a varying quantity and
   
   (b) The Law of Conservation of Quantum Probability Density is violated.

The first list served as the postulates of the theory while the second list served as the implications of the first list and used as the basis for the emergent nature of Quantum Mechanics. Hence the theory still regards spacetime as a fundamental entity. Spacetime can still be regarded to be continuous and smooth at the sub-quantum level since Quantum Mechanical Laws break down. Spacetime however becomes complex and the metric tensor fluctuates. The fluctuation is defined by a modified Lorentz Boost \( G \) where the \( \det G = a^2 \). The function \( a^2 \) is a function in terms of a new dimensionless quantity \( \chi \). In comparison, Special Relativity at macroscopic level is defined by the Lorentz Boost \( \Lambda \) which gives us a determinant necessary to establish Lorentz symmetry

\[ \det \Lambda = \gamma^2 - \beta^2 \gamma^2 = f^2 = 1 \]  

(58)

where \( f = f(\beta^2) = \gamma \sqrt{1 - \beta^2} \) is a function in terms of the velocity ratio \( \beta \). The connection between \( \Lambda \) and \( G \) is established
by the new DSR theory via the complexification of \( \Lambda \) through the following transformations

\[
\Lambda \rightarrow L \quad L \rightarrow A \Lambda^\dagger = G
\]  

\( (59) \)

where

\[
\text{det} G = \alpha^2 = e^{i\Omega_y} = \varphi = C \exp(i2\pi S/\hbar) = Ce^{S/\hbar} \rightarrow \psi
\]

\( (60) \)

This is for \( \alpha = e^{i\Omega_y}, \chi = S/\hbar \) and \( \psi = \psi(S/\hbar) \). The function \( \psi \) is a special form of the function \( \varphi \) for the case \( \hbar \rightarrow 0 \). Hence, at the macroscopic level, the Lorentz Boost is a function in terms of the (square of the) velocity ratio \( \beta = v/c \) while at the Quantum level, the modified Lorentz Boost is represented by \( \psi \) which is a function in terms of the ratio \( \chi = S/\hbar \). Both are equivalent at different scale of application. The quantity \( \chi \) can be shown to be related to the square of the velocity ratio \( \beta \) via an approximation as shown below

\[
\chi = \frac{S}{\hbar} = \frac{H \hbar}{h} = \frac{H}{\hbar} \approx \frac{m^2 \gamma^2}{mc^2} = \beta^2
\]

\( (61) \)

where we consider the argument of Ellman in [64] that in order to use the hypothesis used by de Broglie to derive his famous equation that the Quantum energy is equal to the total energy \( \hbar f = mc^2 \), and do not encounter inconsistencies with velocities of matter waves, one must consider the fact that the total energy \( H \) must be expressed in terms of the energy in "Kinetic Form" plus the energy in "Rest Form", i.e.,

\[
H = m\gamma^2 + (mc^2 - \gamma^2)
\]

\( (62) \)

where \( m = m_0 \gamma, m_0 \) is the rest energy and \( \gamma \) is the Lorentz factor such that for \( v \approx c, H \approx m\gamma^2 \). The series of transformations below shows the correspondence of one scale from another scale in terms of the quantity or conformal factor that defines the violation or preservation of Lorentz Symmetry

\[
\begin{align*}
\alpha^2 &\rightarrow \vert \varphi \vert^2 \rightarrow \vert \psi \vert^2 \rightarrow (\det \Lambda)^2 \\
&\text{(Planck Scale)} \quad \text{(Sub-quantum Scale)} \quad \text{(Quantum Scale)} \quad \text{(Macroscopic Scale)}
\end{align*}
\]

This suggests the existence of "hierarchy of the metrics" that define all boundaries of observation, i.e.,

\[
\begin{align*}
\alpha^2 &\rightarrow \vert \varphi \vert^2 \rightarrow \vert \psi \vert^2 &\rightarrow (\det \Lambda)^2 \\
&\text{(Planck Scale)} \quad \text{(Sub-quantum Scale)} \quad \text{(Quantum Scale)} \quad \text{(Macroscopic Scale)}
\end{align*}
\]

Thus all are unified under one general metric tensor

\[
\hat{g}_{\mu\nu} = \begin{cases} 
\alpha^2 g_{\mu\nu} & \text{Kähler metric (Planck Scale)} \\
\vert \varphi \vert^2 g_{\mu\nu} & \text{Conformally Invariant metric (Quantum/Sub-Quantum Scale)} \\
(\det \Lambda)^2 g_{\mu\nu} & \text{Real metric (Macroscopic Scale)}
\end{cases}
\]

\( (63) \)

Transitions from Macroscopic Scale to Quantum Scale then to the Sub-Quantum region towards the Planck Scale involve topological changes. However, the topological change at the Planck Scale is one of the things that had not yet been fully clarified in hypothetical quantum gravity. The problem is that on how to describe the change of space topology that take place at (and beyond) the microscopic level towards the Planck Scale. There had been a number of investigations that attempt to describe the topology change at Planck Scale [65] [66] [67] [68] [69] [70] [71] [72] [73] but still far from being conclusive. The prevailing consensus among physicists today is that the spacetime at the Planck Scale is very much different from the macroscopic spacetime. Others [74] [75] suggested that the spacetime in that region is define by a non-commutative geometry which define the Hilbert space in Quantum mechanics. Wheeler suggested that the scenario can be considered as a continuous generation of microscopic wormhole in which within it the Laws of Physics break down as the values of fundamental constants vary [76]. He also suggested that curvature (the gravitational field) might arise as a kind of "averaging" over very complicated topological phenomena at very small scales, the so-called "spacetime foam". Spacetime is thought to have a foamylike structure that exhibits metric fluctuations. Today, the mainstream approaches suggest granular structure of the spacetime in terms of "loops" [77], "spin networks" [78] and "causal sets" [79]. Here, the idea of granular structure of spacetime is not considered as far as the description of the Physics of spacetime at Planck Scale is concern. While some may contend that the Physics at such fundamental level can never be described as one may encounter infinities, it is suggested here that such description is possible.

4 Emergent Quantum Mechanics (Dynamical Part)

4.1 Deriving Dirac Equation

Applying Postulate I, which implies Weyl metric tensor transformation, on the normalized Hyperbolic Geometric Flow as required by Postulate II, it yields us

\[
\tilde{\nabla} g_{\mu\nu} = 0
\]

\( (64) \)

where \( \tilde{\nabla} = \vert \psi \vert^2 g_{\mu\nu} \). This will give us three equations:

\[
\tilde{\nabla} \psi = 0; \quad \tilde{\nabla} \psi^* = 0; \quad \tilde{\nabla} g_{\mu\nu} = 0
\]

\( (65) \)

The last equation is the normalized Hyperbolic Geometric Flow with the usual real metric tensor. Expanding the first equation,

\[
a \partial_\mu \psi - \nabla^2 \psi + b \partial_\mu \psi - c \psi = 0
\]

\( (66) \)

where \( a_{\mu\nu}, b_{\mu\nu} \) and \( c_{\mu\nu} \) are set to be constants \( a, b \) and \( c \). Setting the values of constants \( a, b \) and \( c \) as follows:

\[
a = 1 \quad b = 2 \left( \frac{mc^2 \beta}{\hbar} \right) \quad c = \left( \frac{i mc^2 \alpha}{\hbar} \right)^2
\]

\( (67) \)
where \( \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \) (in terms of Pauli matrices \( \sigma \)) and \( \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), such that \( \psi \) can be written as bispinor, and will gives us the following equation

\[
\partial_t^2 \psi - \nabla^2 \psi + 2 \left( \frac{m_o c^2 \beta}{\hbar} \right) \partial_t \psi - \left( \frac{m_o c^2}{\hbar} \right)^2 \psi = 0 \tag{68}
\]

Equation (68) is called by Arbab as the ‘‘Unified Quantum Wave Equation’’ as he was able to show in [81] that Dirac Equation, Klein-Gordon Equation, and Schrödinger Equation can all be derived from such single unifying equation. From (68), it yields us a second-order operator

\[
(\alpha \cdot \nabla)^2 = \partial_t^2 + 2 \left( \frac{m_o c^2 \beta}{\hbar} \right) \partial_t - \left( \frac{m_o c^2}{\hbar} \right)^2 
\tag{69}
\]

where \( \alpha^2 = \beta^2 = 1 \). Factoring and getting the square root, will give us the following linear operator;

\[
\alpha \cdot \nabla = \partial_t + \frac{m_o c^2 \beta}{\hbar} \tag{70}
\]

Arranging and putting back the function \( \psi \), it will give us

\[
\partial_t \psi - \alpha \cdot \nabla \psi + \frac{m_o c^2 \beta}{\hbar} \psi = 0 \tag{71}
\]

which is the Dirac Equation as written in [81].

### 4.2 Deriving Proca Equation

Since \( \psi = \psi(\chi) = e^{i2\chi} \), where \( \chi = S / h + k \) and \( S = \int H dt \), in the presence of an electric charge \( q \), it yields us a phase transformation since the classical action \( S \) will transform as follows:

\[
S = \int \left( H - \frac{q}{c} A^\nu \right) dt \tag{72}
\]

where \( A^\mu = (cA, \phi) \) is the electromagnetic 4-potential. This implies another wave equation, i.e.:

\[
\Box A^\mu = 0 \tag{73}
\]

since \( \Box \psi = i \hbar^{-1} (\Box S) \psi = 0 \), where the modified’ Alembrt operator is given by

\[
\Box \equiv \partial_t^2 - \nabla^2 + 2 \left( \frac{im_o c^2 \beta}{\hbar} \right) \partial_t - \left( \frac{m_o c^2}{\hbar} \right)^2 \tag{74}
\]

Putting in a source-charge,

\[
\Box A^\nu + \partial^\nu (\partial_\nu A^\lambda) - \left( \frac{m_o c^2}{\hbar} \right)^2 A^\mu = \frac{J^\mu}{c \epsilon} \tag{75}
\]

where \( J^\mu = (J, cp) \) is the 4-current density and \( b = \frac{m_o c^2}{\hbar} \neq 0 \). Setting

\[
b \partial_\nu A^\nu = \partial^\nu (\partial_\nu A^\lambda) \tag{76}
\]

it gives us

\[
\Box A^\mu + \partial^\mu (\partial_\nu A^\nu) + \left( \frac{m_o c^2}{\hbar} \right)^2 A^\mu = \frac{J^\mu}{c \epsilon} \tag{77}
\]

which is the Proca equation as written in [52]. If there are no sources, i.e., \( J^\nu = 0 \), the equation becomes;

\[
\Box A^\mu + \partial^\mu (\partial_\nu A^\nu) - \left( \frac{m_o c^2}{\hbar} \right)^2 A^\mu = 0 \tag{78}
\]

Note that, if the Lorenz gauge conditon is preserved \( (\partial_\nu A^\nu = 0) \), the equation gets simplified. If Lorenz Gauge condition is violated \( (\partial_\nu A^\nu \neq 0) \), there are two ways to express the violation of Lorenz Gauge condition. Either \( \partial_\nu A^\nu = bA^\nu \) for \( \partial_\nu A^\nu = 0 \) which implies that \( A^\nu \) is constant or \( \partial_\nu A^\nu = bV(\partial_\nu A^\lambda) \) for \( \partial_\nu A^\nu \) varies in space and time uniformly. This is for \( b \neq 0 \) which implies that \( A_{0} \neq 0 \) for both cases.

### 5 Metric Solutions

In this section, two possible solutions of the normalized Hyperbolic Geometric Flow (nHGF) given by equation (79) will be discussed. Kong and Lui in [55] suggested a special case of nHGF which they called as ‘‘Einstein’s Hyperbolic Geometric Flow’’ given by;

\[
\partial_t g_{ij} = -2R_{ij} + \frac{1}{2} \left( g^{pq} \partial_j g_{p(i} \partial_t g_{q)j} + g^{pq} \partial_i g_{p(j} \partial_j g_{q)i} \right) \tag{79}
\]

This is for the Lorentzian metric

\[
ds^2 = -dt^2 + g_{ij}(x, t)dx^i dx^j \tag{80}
\]

In comparison, Shu and Shen in [83], were able to show that nHGF satisfies Birkhoff’s Theorem and an exact metric solution is given by

\[
ds^2 = u(r) \left[ 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right] c^2 dt^2 - \left[ 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} \right] dr^2 + r^2 d\Omega^2 \tag{81}
\]

where \( u(r) = (r - r_o)^{\sigma_0} (r - r_i)^{\sigma_i} \), \( r_o, r_i, \sigma_0, \sigma_i \) corresponds to the surface gravity of the black hole and \( \Lambda \) is a constant. Shu and Shen then showed that the solution applies for a black hole which has multiple horizons. For \( \Lambda = 0 \), the solution gives a black hole with Schwarzchild metric while for \( \Lambda \neq 0 \), they derived a black hole solution which comes in two types. The first type is the case for \( \frac{\Lambda}{r_o^2} \geq 1 \) with two horizons one of them is the event horizon. The second type is for the case \( \frac{\Lambda}{r_o^2} < \frac{1}{2} r_o^{-2} \) with \( r_o = 2m \) is the Schwarzchild radius and only have one event horizon. These results if applied to quantum level, implies that a quantum field can be
interpreted as a kind of a sub-microscopic black holes where energy and spacetime fluctuation are confined in a given volume of space such that it will appear as a point particle at low-energy approximation at microscopic level. This idea of quantum particle being a black hole at the fundamental level is not something new as it was already suggested by others [83][85]. This is due to striking similarities between a black hole and a quantum particle in terms of its properties. Thus, a quantum particle can be viewed as a submicroscopic black hole with multiple horizons as internal structure. The boundary brought about by the multiple horizons can be viewed as the mechanism that gave a quantum particle an apparent rigidity for it to have an appearance of a point particle. It could also be the reason why there is a difficulty on finding the exact radius of a particle like in the case of Proton Radius Puzzle [86][87][88][89][90].

6 On Quantum Interpretation

In this section, it is not intended to come up with a full-blown quantum interpretation. The aim is simply to outline what possible interpretation can be derived out of the theory that was presented here. In retrospect, the root cause of all the confusions that surround the interpretational problem of Quantum Mechanics is due to the fact that no one really knows what is the true nature of the so-called “quantum wave”. Also, in experiment like the double-slit experiment, though indirectly prove the physical existence of the quantum wave, still, its true nature cannot be known in such experiment. Without the full grasp of the physical nature of a quantum wave, some resort to suggest that there is really no physical wave and the apparent observable wave nature of a quantum system is just an illusion brought about either by a classical way of thinking or the instrument used by an observer at the macroscopic level. The pragmatic approach of Copenhagen Interpretation (CI) suggested that there is no need to think of a real quantum wave that physically exists since the Wave Function is a complex quantity anyway. For CI, the wave function \( \psi \) and its corresponding wave equation, should not be postulated to describe a “quantum wave” that physically exists as what de Broglie and Bohm suggested. Wave Function for CI is a more mathematical tool to describe a quantum state or to predict the possible outcome of an experiment. Here, the so-called quantum wave or quantum field was suggested to be an ensemble of three kinds of wave that physically exists. The three kinds of wave are represented by the following wave equations:

\[
\Box \psi = 0 \quad \Box g_{\mu\nu} = 0 \quad \Box A^\mu = 0 \quad (82)
\]

The first kind is the so-called “matter wave” expressed by the second order Dirac Equation,

\[
\partial_t^2 \psi - \nabla^2 \psi + 2 \left( \frac{imc^2}{\hbar} \right) \partial_t \psi - \left( \frac{mc^2}{\hbar} \right)^2 \psi = 0 \quad (83)
\]

It is related to the energy fluctuation of the system since \( \psi \) is a mathematical tool that is used to derive the energy and momentum of a quantum system. The wave equation above can be considered to represent a fluctuating energy that continuously interact with the vacuum energy. Using a modified Einstein’s Field Equation, the consequence of such energy fluctuation is to cause for the spacetime to curve and to fluctuate, i.e., for the metric tensor to oscillate. Such metric fluctuation is the second wave and represented by the nHGF,

\[
\Box g_{\mu\nu} = \partial_t^2 g_{\mu\nu} - \nabla^2 g_{\mu\nu} + b \partial_t g_{\mu\nu} + c g_{\mu\nu} = 0 \quad (84)
\]

It is a wave equation that has a damping term describing a soliton-like structure that may explain the particle-like behaviour of the matter wave. Shu and Shen, as discussed in the previous section, derived a black hole solution of it. This can be inferred to be the origin of the localization of a quantum field in a given volume of space and not just because it is acting like a wave packet of energy. For the third type of wave, it is observed in the famous Aharonov-Bohm effect that the wave function \( \psi \) is modified by the presence of the electromagnetic vector and scalar potential \( (A, \phi) \) that vary in space and time. The variation can be described by the equation given below

\[
\Box A^\mu + \left( \frac{imc^2}{\hbar} \right) \partial_\nu A^\nu - \left( \frac{im\phi}{\hbar} \right)^2 A^\mu = 0 \quad (85)
\]

This wave is known in the literature as the electromagnetic scalar wave. At present, the nature and properties of the scalar wave are not yet fully explored and its detection is still very much debated [21][22]. It could be that it is the “guiding wave” as postulated by de Broglie and Bohm [23] along with the spacetime fluctuation as both types of wave can affect the energy fluctuation. What seems to be the striking similarity of all types of wave that were mentioned here is the need for an abstract quantity to describe the different types of wave. The first wave as described by the first equation in (82) is in terms of the function \( \psi \). However, the wave function \( \psi \) does not represent a physically observable variable of the system as it is just a tool to compute the energy and the momentum. The second wave is represented by the second wave equation in (82) in terms of the metric tensor \( g_{\mu\nu} \). It represents an oscillating or fluctuating spacetime but it is not like the gravitational wave at macroscopic level that exhibit the usual gravitational effects. Its black hole solution suggests a particle-like configuration for a quantum field with its internal part hidden by multiple event horizons. Note also that the metric tensor \( g_{\mu\nu} \) does not represent any physically observable property of a particle but simply an abstract mathematical tool for the computation of the length and the mass. The third wave equation represents the scalar wave. It is a wave that physically exists as proven by the Aharonov-Bohm effect, but \( A^\mu \), just like \( \psi \) and \( g_{\mu\nu} \), remains an abstract mathematical tool which does not represent a physically observable property of the system.
It is just a tool to get the electric and magnetic field. All three wave equations represent three types of wave that physically exist but described in terms of $\psi$, $A^\mu$, and $g_{\mu\nu}$. All of which are just abstract mathematical tools that are needed to compute the value of an observable properties of a quantum field. However, the fluctuations of all those abstract variables, expressed in a set of wave equations, are suggested here to represent something physical. All known properties of a quantum system are encapsulated, not in the wave function $\psi$ alone, but in a modified metric tensor $\tilde{g}_{\mu\nu} = |\psi|^2 g_{\mu\nu}$. This is to consider the metric tensor fluctuation that is also happening at the quantum scale. In the standard quantum formalism, the metric tensor $g_{\mu\nu}$ is usually excluded because the role of gravity was considered negligible at the microscopic world. Here, the role of the metric tensor is naturally integrated with the energy fluctuation that is happening at the quantum scale. While the energy fluctuation is represented by the wave equation in terms of $\psi$, the metric tensor fluctuation is described by nHGF which is a modified Einstein Field Equation. In order to have a unified description of all the fluctuations as each type of wave must be concomitantly propagated, the quantum formalism must be modified. It is suggested here that the modification includes the use of new mathematical tools such as: the generalized ‘t Hooft-Nobbenhuis Complex Transformation which implies the Weyl metric transformation and the normalized Hyperbolic Geometric Flow which describes the metric tensor fluctuation. The latter is a consequence of the assumption that the Lorentz symmetry needs to be modified while the latter is brought about by idea that the spacetime fluctuates at the quantum level.

### 7 Summary and Recommendation

A theory was presented based on the idea that at the sub-quantum level, the Laws of Quantum Mechanics break down. As the Heisenberg Uncertainty Principle collapses, spacetime can be approximated to be smooth such that Relativity can still be applied but in a modified form by which its known postulates also collapse. The collapse of the postulates of Relativity does not mean a granular structure of spacetime. It simply means that the Lorentz symmetry is violated, the speed of light is no longer maintained to be invariant and the metric tensor fluctuates. Using these properties, the so-called Quantum Spacetime was defined and used as a background spacetime where Quantum Mechanics can be derived. It is recommended that a full-blown quantum interpretation and application of the theory to Standard Model should be developed from the results given here. Also tetrad formalism and Weyl Geometry should be integrated in the mathematical framework of the theory.

### Acknowledgments

The author is very much grateful to the NEU Board of Trustees, the NEU Administration, the Office of the Vice President for Research, NEU-TECH and the University Research Center.

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