Inflation with teleparallelism: Can torsion generate primordial fluctuations without local Lorentz symmetry?

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Abstract

Arbitrary generalization to the teleparallel equivalent of general relativity loses local Lorentz invariance to reparametrize the orthonormal coordinate system and gives rise to asymmetry field equations. We investigate consequences of local Lorentz violation to primordial fluctuations in extended single field inflationary models based on the scalar-tensor formulation of the torsion scalar $T$ that effectively includes $f(T)$ gravity as a special case. We show that despite some asymmetry part of the field equations are removed in a spatially homogeneous and isotropic cosmic background, no subhorizon scalar-perturbation mode can survive by the time of horizon crossing. As a result, any scalar field mediated in torsion cannot generate enough primordial density inhomogeneity alone, even if it brings some de Sitter background solutions in generalized teleparallel gravity.

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I. INTRODUCTION

The generation of small density inhomogeneity in the primordial Universe is one of the most important predictions of cosmic inflation. In the single field inflationary paradigm one considers the inflaton to be a canonical scalar field and quantum fluctuations in the inflaton field may convert to classical density perturbations after crossing the Hubble horizon. These inflaton fluctuations simultaneously activate scalar-mode perturbations in the gravitational action, usually addressed by “curvature perturbations” in the language of Einstein’s general relativity (GR). The gauge invariant observable $\zeta$ for curvature perturbations is so far in well agreement with a Gaussian and nearly but not exactly flat spectrum [1].

Expanding the full action of the inflationary model in terms of small perturbations is a powerful approach for computing the spectrum or higher-order correlation functions of $\zeta$ [2]. In this approach it is convenient to perform a spacetime splitting with respect to a unit time-like vector $n_\mu = (-N, 0)$, where components of $n_\mu$ is determined by the metric in the Arnowitt-Deser-Misner (ADM) formalism [3]:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (1)$$

When choosing the uniform field gauge of the inflaton, one can reparametrize the induced metric of the spatial hypersurface as $h_{ij} = a^2(\delta_{ij} + \gamma_{ij})$, where $a(t)$ is the scale factor of the homogeneous background and $\gamma_{ij}$ parametrizes the transverse-traceless tensor fluctuation. Both $N$ and $N^i$ are treated as Lagrange multipliers and can be solved in terms of $\zeta$ and its time derivative from the Hamiltonian and momentum constraints.

It shall be noticed that the ADM decomposition [4] assumes no particular spacetime connection that defines the rules of parallel transformations of the spacetime vectors. From this perspective similar attempts have been made in [4] to compute the primordial fluctuations in single field inflationary models based on Einstein’s other description of GR constructed under teleparallelism [5] (see also [6–9]). In the teleparallel description of gravity, the dynamical variables are living inside the vierbein field instead of the metric, and the spacetime curvature tensor vanishes mandatorily so that only “torsion perturbations” can be invoked by the inflaton field.

Driving inflation without an explicit scalar field strongly motivates the study on modified Lagrangian density of gravity. In the curvature version of GR, zero-momentum de Sitter solutions are found to exist in the nonlinear extension of the Einstein-Hilbert action [10–11], which is now subject to a specific $f(R)$ gravity (see [12–13] for a review). It is well-known that $f(R)$ gravity can be cast into some particular types of the Brans-Dicke theory [14–15], where one may identify an auxiliary scalar field (also called “scalaron” [10]) to realize inflation and generate primordial fluctuations. By virtue of the scalar-tensor correspondence [12], we will refer $f(R)$ inflation to the extended single field inflationary scenario [16].

On the other hand, de Sitter background solutions are found to exist in the nonlinear modification to the teleparallel equivalent of GR [17–19] (or simply $f(T)$ gravity, see [20] and references therein) and in the extended single field model with a nonminimal coupling to the torsion scalar $T$ [21, 22]. Similarly, $f(T)$ gravity is recast into a specific class of scalar-tensor theory with respect to the torsion scalar [23], such that we can study the inflationary fluctuations in a unified manner based on the general scalar-tensor setup within teleparallelism. In this work we aim to recognize whether the torsion in-
duced scalaron is a suitable candidate for inflaton. We improve the parametrization for vierbein variables in [4] and perform a research independent to the metric perturbation approach [19, 24].

Due to the presence of local Lorentz violation in nonlinear teleparallelism [23, 29], generalized teleparallel gravity is essentially unhealthy unless closely recovers the limit of GR [31–33]. Even if a simple cosmic background is assumed, local Lorentz symmetry breaking will deny the coordinate reparametrization in the orthonormal frame from removing some dynamical variables, and thus additional field equations must arise to account for those non-vanishing degrees of freedom. However,consequences of additionally induced field equations in cosmological perturbations were not addressed in previous study [4, 19, 24, 27]. In this work, we scrutinize implications of these unusual equations of motion. We show

II. TELEPARALLEL INFLATION

Teleparallel gravity naturally defines vectors and tensors in the orthonormal frame [30] with bases $\hat{e}_A$ satisfying $\hat{e}_A \cdot \hat{e}_B = \eta_{AB}$, where $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$. The orthonormal bases are related to the general coordinate bases through the vierbein field $e^A_{\mu}$ as $\hat{e}_A = e^A_{\mu} \partial_\mu$. The spacetime metric

$$g_{\mu\nu} = \eta_{AB} e^A_{\mu} e^B_{\nu}$$

is given by the dual vierbein $e^A_{\mu}$, where $e_A^\mu e^A_{\nu} = \delta^\nu_\mu$ and $e^A_{\mu} e^B_{\nu} = \delta^B_\mu$. The absolute parallel (zero-curvature) condition $\nabla^a e^a_{\nu} = 0$ is achieved by the Weitzenböck connection $\Gamma^\lambda_{\nu\mu} = e^A_{\nu} \partial_\mu e_A^\lambda$. The Weitzenböck connection is related to the Levi-Civita connection $\bar{\Gamma}^\lambda_{\nu\mu}$ (the space-time connection in GR) via

$$\Gamma^\lambda_{\nu\mu} = \bar{\Gamma}^\lambda_{\nu\mu} + K^\lambda_{\nu\mu},$$

where $K^\lambda_{\nu\mu} = \tfrac{1}{2}(T^\lambda_{\nu\mu} + T^\lambda_{\mu\nu} - T^\lambda_{\nu\mu})$ is the contorsion and

$$T^\lambda_{\nu\mu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\nu\mu}$$

is the torsion tensor in teleparallel gravity.

We expect the standard inflationary perturbations [2] to be recovered in the teleparallel equivalent formulation of GR with the gravitational Lagrangian given by the torsion scalar [3, 8]

$$T = S^\lambda_{\mu\nu} T^\lambda_{\mu\nu},$$

where $S^\lambda_{\mu\nu} = \tfrac{1}{2}(K^\lambda_{\mu\nu} + \delta^\lambda_\mu T^\alpha_{\nu\alpha} - \delta^\lambda_\nu T^\alpha_{\mu\alpha})$. Taking the connection decomposition [33] into the torsion scalar, one finds $T = -\bar{R} - 2\nabla_\mu T^\alpha_{\mu\alpha}$, where $\bar{R}$ is nothing but the curvature scalar in GR and $\nabla_\mu T^\alpha_{\mu\alpha}$ is the covariant derivative with respect to the Levi-Civita connection. Therefore the action $S = \int d^4 x e T$ manifests the Einstein-Hilbert action up to a total divergence, where $e = \sqrt{-g}$.

To retain a general discussion, we consider the action based on the scalar-tensor formulation with respect to the torsion scalar as [22, 31]:

$$S = \int d^4 x e \left[ \frac{F(\phi)}{2} T + \frac{Z(\phi)}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right],$$

where $\phi$ shall serve as the inflaton field. Non-trivial coupling between torsion and a scalar field is also seen by the higher-dimensional teleparallel gravity [34, 35]. The action [6] can mimic the dynamics of $f(T)$ gravity according to the reformulation: $F(\phi) = f \equiv df/dT$, $Z(\phi) = 0$, and $V(\phi) = f(T(\phi) - \phi T(\phi))$, provided a sufficient condition $df/dT^2 \neq 0$ [23]. On the other hand, the model with a nonminimal coupling to the torsion scalar [28] is recovered by the choice $F(\phi) = 1 + 2C\phi^2$, $Z(\phi) = 1$.

We express the field equations in the general coordinate bases, and the components are given by

$$F(\phi) G_{\mu\nu} - 2F_{\phi} S^\lambda_{\mu\nu} \partial_\lambda \phi \equiv g_{\mu\nu} \left[ \frac{Z(\phi)}{2} (\partial_\phi)^2 + V(\phi) \right] - Z(\phi) \partial_\mu \partial_\nu \phi,$$

where $G_{\mu\nu}$ is the Einstein tensor defined in GR and $F_{\phi} \equiv df/d\phi$. In Eq. (7), the manipulation $G_{\mu\nu} = e^A_{\mu} e^A_{\nu} G_A$, where $G_A = 2e^{-1} \partial_\mu (e^2 e^A_S/\lambda) - 2e^A L e^2 S^\mu_{\rho} - e^A T_{/2}$ has been used. The equation of motion for $\phi$ is

$$Z(\phi) \Box \phi + \frac{1}{2} Z(\phi) (\partial_\phi)^2 - \frac{F_{\phi}}{2} T - V(\phi) = 0,$$

where $\Box \equiv \nabla^\mu \nabla_\mu$. The clear evidence of the local Lorentz violation in the action [9] is the existence of the antisymmetry part of [7] (see the discussion in [29]), which reads

$$F_{\phi} (S_{\mu\nu}^\lambda - S_{\nu\mu}^\lambda) \partial_\lambda \phi = 0.$$
assume that such transition does not happen during the epoch of inflation.

With the background choice (10), the antisymmetry equations (9) vanish identically and the field equations (7) are given by

\[ 3H^2 F = \frac{Z}{2} \dot{\phi}^2 + V, \]  
\[ -(2 \dot{H} + 3H^2) F = \frac{Z}{2} \dot{\phi}^2 - V + 2HF \phi \dot{\phi}, \]

where \( T = 6H^2 \). The background equation of motion for \( \phi \) is

\[ Z \left( \dot{\phi} + 3H \phi \right)^2 + \frac{Z}{2} \dot{\phi}^2 + 3H^2 F \phi + V_\phi = 0. \]  

(13)

It is not our goal to build up a new inflation model, but we can easily show the existence of de Sitter background solutions in the extended single-field theory (see also [17–19, 22]). For example, let us consider \( Z = c_1 \) and \( V = c_2 F \), where \( c_1 \) and \( c_2 \) are some constants. We then combine the Friedmann equations (11) and (12) to obtain

\[ (3H^2 - c_2) F = c_1 \dot{\phi}^2, \]  
\[ -2 \dot{H} F = (c_1 \dot{\phi} + 2HF \phi) \dot{\phi}, \]

and the equation of motion (13) may be rewritten as

\[ \ddot{\phi} + 3H \dot{\phi} + \frac{F \phi}{2F} \dot{\phi}^2 = 0. \]  

(16)

Assuming \( c_2 > 0 \), one simply check that \( (H, \phi) = (\pm \sqrt{c_2/3}, \phi_*) \) with an arbitrary constant \( \phi_* \) are solutions for the above equations (14), (15) and (16). In particular, it is straightforward to perform the perturbations \( H = H_0 + \delta H, \phi = \phi_* + \delta \phi \) around the fixed points to find that \( H_0 = \sqrt{c_2/3} \) is a stable de Sitter solution. Since \( F(\phi) \) approaches to a constant toward the fixed point, the stable de Sitter attractor \( (H, \phi) = (H_0, \phi_*) \) satisfies the viable conditions considered in [22].

III. PRIMORDIAL FLUCTUATIONS

Let us now compute the cosmological perturbations by virtue of the ADM formalism (1). To our purpose, we will parametrize the 16 variables in the vierbein field in a separate way. The first step is to identify a primary vierbein with respect to the homogeneous choice (10) that satisfies the condition (2) for the ADM metric (1). A suitable solution is found in [3] of the form

\[ e^{\bar{a}}_\mu = (N, 0), \quad e^a_\mu = (N^a, h^a), \]
\[ e^0_0 = (1/N, -N^i/N), \quad e^0_\mu = (0, h^i), \]

where \( N^a \equiv N h^0_a \). Here \( h_{ij} \) manifests the induced metric of the 3-surface, so that \( h^i \) is a possible choice of the induced vierbein (the dreibein hereafter). The representation of \( e^A_\mu \), in (17) is specially chosen such that \( e^A_\mu \) coincides with the unit vector \( n^\mu = (1/N, -N^i/N), \) where only dynamical variables responsible to the metric are shown. In particular, one can parametrize \( \zeta \) and \( \gamma \) in \( h^a_i \) to arbitrary order of interest.

The next step is to put together the components corresponding to the local Lorentz invariance. Schematically, we have decomposed the vierbein with respect to the infinitesimal Lorentz transformation \( \Lambda_B^A(x) = (e^c_\nu)^A_B = \delta^A_B + \omega^A_B \bar{\gamma}^c_\nu + \ldots \) as

\[ e^A_\mu = (e^c_\nu)^A_B e^B_\mu = (0) e^0_\mu + (1) e^A_\mu + \ldots, \]  

(18)

where \( \omega_{AB} = -\omega_{BA} \) and \( \bar{e}^A_\mu \) is given by (17).

If the local Lorentz symmetry is broken, then \( \bar{e}^0_\mu \) is no longer the transformation matrix but rather than the “Goldstone modes” of the symmetry breaking. We may parametrize these modes as

\[ \omega^0_B = (0, \omega_b), \quad \omega^a_B = (\omega^a, B^a_B), \]

(19)

where \( \omega^a = \eta^{ab} \omega_b \) and \( B_{ab} + B_{ba} = 0 \). We then define the spatial vector \( \omega^i = \omega^a h^a_i \) and the antisymmetric spatial tensor \( B_{ij} = B_{ab} h^a_i h^b_j \).

Since \( \omega^A_B \) satisfies \( \eta_{AB} = \eta_{CD} (e^C_\nu)^B_A (e^0_\nu)^D_B \), Eq. (2) implies that \( \omega^a \) and \( B^a_B \) cannot contribute to the metric and the curvature scalar, which is given by

\[ R = \bar{R}^3 + \Sigma_{ij} \bar{\nabla}_{ij} - \bar{S}^2 - 2\Sigma_{\mu \nu} (n^\mu \bar{\Sigma} - 2/N \bar{\Delta} N), \]  

(20)

where \( \Sigma_{ij} = \frac{1}{2N} (\bar{h}_{ij} - \bar{\nabla}_i \bar{N}_j - \bar{\nabla}_j \bar{N}_i) \) is defined in the same way as the extrinsic curvature in GR, while \( \Delta \equiv h^{ij} D_i D_j \) and \( D_i \) is the 3-covariant derivative with respect to the 3-Levi-Civita connection \( \bar{\Gamma}^i_{jk} \).

Using the projection tensor \( \perp_{\nu} \equiv \bar{h}^i_\nu \), where \( \bar{h}_{\mu \nu} = \eta_{\mu \nu} + \eta_{\mu \nu} n_{\nu} \), we can obtain the induced connection in the 3-surface through the definition \( D_i A_j = \perp_{\nu} \bar{\Gamma}^i_{\nu j} A_{\nu} \) for an arbitrary vector \( A_{\nu} \) lies in the 3-surface \((n^\nu A_\nu = 0)\). Taking the zeroth order vierbein (17) into calculation, we find that \( \bar{\Sigma}^0_{ij} = h^i_\nu \partial_\nu h^j_\nu \) is a 3-Weitzenböck connection so that the intrinsic curvature of the 3-surface is zero (namely it is a teleparallel hypersurface). Note that the zero-curvature condition of the 3-surface holds at any-order expansion of the vierbein with respect to \( \omega^A_B \). This is achieved through a redefinition of the dreibein \( \bar{h}^a_i \equiv (e^0_\nu)^a_i \), where one can check that \( h_{ij} = \eta_{ab} h^a_i h^b_j \).

In order to compute the quadratic action, we shall consider the expanded vierbein (18) up to second order in \( \omega^A_B \). In terms of the parametrization (19), the first order expansion reads

\[ e^0_\mu = (N^a \omega_a, \omega_i), \quad e^a_\mu = (N^a + N^b B^b_i, B^a_i), \]
\[ e^0_\mu = (0, -\omega^i), \quad e^a_\mu = (- \omega^a/N, B^a_i + N^i \omega_a). \]  

(21)
The second order expansion takes the form

\begin{align}
(2) e^0_0 &= \frac{1}{2} N \omega_\alpha \omega^\alpha + \frac{1}{2} N^b \omega_c B^b_c, \\
(2) e^i_0 &= \frac{1}{2} \omega_a B^a_i, \\
(2) e^0_i &= \frac{1}{2} N^b \omega^a \omega_b + \frac{1}{2} N B^a_b \omega^b + \frac{1}{2} N^b B^a_c B^c_b, \\
(2) e^i_i &= \frac{1}{2} \omega^a \omega_i + \frac{1}{2} B^a_e B^e_i,
\end{align}

where the dual field components are

\begin{align}
(2) e^0_0 &= \frac{1}{2N} \omega_\alpha \omega^\alpha, \\
(2) e^i_0 &= -\frac{N}{2N} \omega_\alpha \omega^a - \frac{1}{2} \omega_b B^b_i, \\
(2) e^i_0 &= -\frac{1}{2N} B_{ab} \omega^b, \\
(2) e^i_i &= \frac{1}{2} \omega_a \omega_i + \frac{1}{2} B_{ab} B^{bi} + \frac{N_i}{2N} B_{ac} \omega^c.
\end{align}

It is convenient to use \( D_i \omega_j = \partial_i \omega_j - 3 \Gamma^l_{ij} \omega^l = \delta_\alpha^i \partial_i \omega_\alpha \).

Together with (20), we can write down the second order action of (2) as

\begin{align}
(2) S &= \int dt d^3 x N \sqrt{\mathcal{h}} \left\{ \frac{F(\phi)}{2} T + Z(\phi) \left[ h^{ij} \partial_i \phi \partial_j \phi - (\mu^2 \partial_\mu \phi)^2 \right] + V(\phi) \right\},
\end{align}

where

\begin{align}
T &= -\ddot{R}^3 - \dddot{\Sigma}^2 - \dddot{\Sigma}^{ij} \Sigma_{ij} + \frac{2}{N} \dot{\Sigma} \dot{N} - \frac{2}{N} \dot{D}_i (h^{ij} N \Sigma^{\alpha j i}) \\
&\quad - 2 \nabla_{\mu} \left[ n^\mu D_i \omega^i + \frac{n^\mu}{N} D_i (N^b B^i_b) \right] \\
&\quad - \nabla_{\mu} \left[ n^\mu (B^{ij} D_j \omega_i + \omega_j D_i B^{ij}) \right] + \ldots
\end{align}

Those unlabeled terms above are higher-order contributions, keeping in mind that \( N^i \) and \( \partial_i N \) are at least first order perturbations.

Let us examine the first order solution of \( N \) and \( N^i \) in the uniform field gauge where \( \delta \phi = 0 \) with the parametrization of the dreibein \( h^a_i = a e^\xi \delta^a_i \) (the general coordinate is now completely fixed). For this purpose we only need to focus on the scalar and vector perturbations and keep \( \gamma^{ij} \) and the three modes in \( B^{ij} \) (which are pseudoscalar and pseudovector modes [27]) aside. The resulting hamiltonian and momentum constraints from [28] are

\begin{align}
F(\ddot{R}^3 + \dddot{\Sigma}^2 - \dddot{\Sigma}^{ij} \Sigma_{ij}) - \frac{Z}{N^2} \dot{\phi}^2 - 2V &= 0, \\
\dot{D}_i (\dot{\Sigma} \delta^i_j - \ddot{\Sigma}^{ij}) &= 0.
\end{align}

Taking \( N = 1 + N_1 \) and \( N^i = \partial^i \psi + N^i_\perp \) with \( \partial_i N^i_\perp = 0 \), we find that \( N^i_\perp = 0 \) and that

\begin{align}
N_1 = \dot{\psi} = \frac{-\xi}{2a^2 H} + \chi, \quad \partial^2 \chi = \frac{Z \dot{\phi}^2 \dot{\psi}}{2H^2 F},
\end{align}

We can now solve the equation of motion for \( \omega^i \) and \( B^{ij} \) by using the results [29]. To perform the variation with respect to \( \omega^i \) and \( B^{ij} \), it is convenient to use the relation \( D_i \omega^i = \dot{D}_i \omega^i + 3 K_j^i \omega^j \), where \( 3 K_j^i \) is the 3-torsion defined with respect to the 3-torsion \( 3 \Gamma^j_{ik} = 3 \Gamma^i_{kj} - 3 \Gamma^i_{jk} \) on the 3-surface. The resulting equations from variation of the action are

\begin{align}
3 \Gamma^j_{ik} \phi = h^i_\alpha (\partial_i h^a_\alpha - \partial_i h^a_\alpha) F \phi &= 0, \quad (27) \\
(\partial_i \omega) - (\partial_i \omega) F \phi &= 0, \quad (28)
\end{align}

at first order in the uniform field gauge. One can check that these equations are identical to the linearized equations of [30].

If \( F \phi = 0 \), both Eq. (27) and Eq. (28) vanish as the theory converge to the GR limit. This was already pointed out in Ref. [23, 20]. Since the kinetic term \( Z(\partial \phi)^2 \) can be canonicalized through a redefinition of \( \phi \), the solutions to result in a standard quadratic action of the single field inflation model [2] up to a constant \( F \). In general cases where \( F \phi \neq 0 \), the decomposition \( \omega^i = \partial^i \alpha + \omega^a_\perp \) with \( \partial^i \omega^a_\perp = 0 \) then leads to the solution \( \omega^a_\perp = 0 \) from (28). The gradient of Eq. (27) gives \( F \phi \partial^2 \zeta = 0 \), which implies \( k^2 \zeta = 0 \), where \( \zeta \) denotes a non-zero Fourier mode of \( \zeta \). Since the wavenumber \( k \) is arbitrary, we find an unexpected result \( \zeta = 0 \) for each \( k \in [0, k_s] \), where \( k_s \) denotes some cutoff scale well inside the horizon [37] (and the zero-mode solution \( \zeta = \zeta_0(t) \) is always rescaled into the scale factor). Taking \( \zeta = 0 \) into Eq. (8), one then finds that \( \alpha = 0 \). One can see an obvious discontinuity in the primordial fluctuations with the presence of an amazingly small \( F \phi \). To smooth the solution in the limit \( F \phi \rightarrow 0 \), we impose a possible choice of the cutoff as \( k_s = \sqrt{|F \phi|} k_{end} \), where \( k_{end} \) is the horizon scale at the end of inflation.

We remark that the 3-torsion \( 3 \Gamma^j_{ik} = h^i_\alpha (\partial_i h^a_\alpha - \partial_i h^a_\alpha) \) is indeed a tensor form of the dreibein field on the 3-surface so that any non-diagonal background choice given by a spatial rotation of the dreibein will not change the form of Eq. (27). As a result, the 3-torsion (and therefore the scalar perturbation \( \zeta \)) must vanish if \( F \phi \neq 0 \).

The other convenient gauge for computation is the spatially flat slicing in which the general coordinate is fixed such that \( h^a_i = a \delta^a_i \) and \( \delta \phi = \delta \phi(t, x) \). In this gauge \( N \) and \( N^i \) are solved as

\begin{align}
N_1 &= \frac{Z \dot{\phi}}{2HF} \delta \phi, \quad N^i_\perp = 0, \quad (29) \quad \partial^2 \psi = -\frac{Z \dot{\phi}^2}{2H^2 F} \frac{d}{dt} \left( \frac{H}{\phi} \delta \phi \right) + \frac{3HF \dot{\phi}^2}{H^2 F} \delta \phi.
\end{align}

Similarly, the linearized equation of motion for \( \omega^i \) and \( B^{ij} \) (or Eq. (9)) gives

\begin{align}
\partial_i (N \mu \partial^i F) &= 0, \quad (30) \\
(\partial_i \omega - \partial_i \omega) N \mu \partial^i F &= 0, \quad (31)
\end{align}

which indicates \( \delta \phi = \delta \phi_0(t) \), \( \omega^a_\perp = 0 \) if \( F \phi \neq 0 \). Therefore, any subhorizon mode of \( \delta \phi_0 \) at the onset of inflation
is eliminated when stretched toward the horizon scale unless $F_0 = 0$.

It is now clear that the gauge invariant variable

$$\zeta = \varphi - \frac{H}{F} \delta F,$$  \hspace{1cm} (32)

where $\varphi$ parametrizes the diagonal scalar mode in $h_{ij}$, is at best a homogeneous solution in either the uniform field gauge ($\delta \phi = 0$) or in the spatially flat gauge ($\varphi = 0$) unless $F$ is a constant, due to the constrain equation $\delta$. Indeed, this result is consistent with observations in the Newtonian gauge (that is to set $\psi = 0$ and $h^a_i = ae^\gamma \delta^a_i$ in our notation) where one obtains $\varphi = H \delta \phi / \phi$ for non-zero modes in the $f(T)$ gravity [38] as well as the model of nonminimal coupling $F(\phi) = 1 + 2\xi \phi^2$ [39].

### IV. CONCLUSIONS

In this work we considered the generation of primordial fluctuations in the extended single field inflationary scenario based on a scalar-tensor formulation of teleparallel gravity which effectively includes $f(T)$ theory as a special case. Let us summarize the fate of the $16 + 1$ perturbation variables reside in $e^\mu_A$ and $\phi$ under the idealized homogeneous and isotropic background [13]. We always remove 4 variables via the general coordinate invariance and we can apply the hamiltonian and momentum constraints to solve the other 4 variables in $N$ and $N\phi$. We have put aside the two tensor degrees of freedom for separate discussion. In fact, by taking the linear parametrization $h^a_i = a(\delta^a_i + \gamma^a_j/2)$ for the tensor variable, one can observe that $\gamma_{ij} = \eta_{ab}(\delta^a_j \delta^b_i + \delta^b_j \delta^a_i)/2$ only involves in $R^3$ and $\Sigma_{ij}$ in the action [33]. Therefore, the quadratic action for $\gamma$ is identical to that of the Einstein-Hilbert action upto the factor $F$.

The equation of motion [33] helps to replace one scalar mode by the others so that there are now 6 modes left for the 6 asymmetry field equations (9). If $F$ is evolving with time, three of the 6 remaining variables are eliminated by the asymmetry field equations (9), which are accounted by a scalar mode (that is $\zeta$ or $\delta \phi$, depending on the gauge) and a transverse vector mode ($\omega_j^i$). Nevertheless, owing to the background choice [10], we notice that the equation of motion for the other three variables inside $B_{ij}$ never shows up. Further investigation beyond linear perturbation level is required, given that those modes released by local Lorentz violation are found to be pathological in non-trivial spacetime background [31–33]. Therefore if there exists any higher-order coupling between $B_{ij}$ and other dynamical modes it would jeopardize the theory even if the sudden background transition has been omitted, as a priori assumption.

After clarifying all the equations of motion, we conclude that the general teleparallel scalar-tensor theory only admits zero-momentum solutions for the scalaron, and the extended single field inflationary model [7] exhibits no dynamical perturbation modes other than the tensor fluctuations unless converging to the GR limit. In particular, the modified Lagrangian $f(T)$ itself is not enough to explain the horizon problem, the flatness problem and the origin of the structure of the Universe at once. This is a drastically different prediction from the well-known result given by $f(R)$ (or $f(R)$ according to our notations) theories [10,13]. To realize a suitable initial state of the Universe, one may consider to restore the local Lorentz symmetry to get rid of the asymmetry field equations [40], or to introduce some extra matter to trigger the scalar perturbation $\zeta$ [10]. In the former case successful primordial fluctuations will be promised by using the proper covariant formulation of generalized teleparallel gravity [40], while in the latter case one would have to go beyond the single field inflationary paradigm. We remark that any additional matter will not involve in the asymmetry field equations (9), if it is protected by local Lorentz symmetry (as the scenario considered in [17,19]). As shows in the flat slicing, one can see that if Eq. (30) is unchanged then the solution $\delta \phi = 0$ persists.

Finally we notice that no trivial Einstein frame of the generalized teleparallel gravity [6] is available through a conformal transformation where the scalar field is fully decoupled with torsion. For instance, by taking $\hat{e}^A_\mu = \sqrt{F} e^A_\mu$ for the vierbein, we can reach a minimally coupled torsion scalar $\bar{T}$ in the new frame but the other scalar-torsion coupling $\bar{T} \hat{\rho} \hat{\gamma} \hat{\phi}$ at the same time [41], where $\delta \phi \equiv \sqrt{6} d\phi / (2F)$. This scalar-torsion coupling in the new frame leads to asymmetry field equations (3), and they once again restrict the generation of any non-zero scalar-perturbation mode during inflation.

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