Linear Stability of Mandal-Sengupta-Wadia Black Holes

H. Gürsel, G. Tokgöz, and I. Sakalli

Physics Department, Eastern Mediterranean University, Famagusta, Northern Cyprus, Mersin 10, Turkey
(Dated: March 19, 2018)

In this paper, the linear stability of static Mandal, Sengupta, and Wadia (MSW) black holes in (2 + 1)-dimensional gravity against spherical perturbations is studied. The associated fields are assumed to have small perturbations in these static backgrounds. We then consider the dilaton equation and specific components of the linearized Einstein equations. The resulting decoupled equation for growing modes is nothing but the radial Klein-Gordon equation. Schrödinger like wave equation with the associated effective potential is obtained. Finally, it is shown that MSW black holes are stable against the small spacetime dependent perturbations.

sectionIntroduction

One of the compelling problems in black hole physics is stability (for a review see [1] and references therein). The stability of a black hole presents an ideal theoretical test-bed for any gravity theory [2]. In the pioneering works on black hole stability [3–6], the linear stability of gravitational perturbations of the Schwarzschild black hole was comprehensively studied. The stability of the Kerr black hole was first proved in [7], which employed the Teukolsky equation [8]. Recent developments have shown that there is a conserved and positive definite energy in the axially symmetric linear gravitational perturbations of the extreme Kerr black hole [9]. This work stipulates the framework of linear stability in axial symmetry. On the other hand, the non-linear stability of black holes remains an open question. In fact, linear stability studies are expected to clarify the non-linear stability problem. But, this is conditional on techniques that can be suitably extended to the non-linear regime [10]. In general, no-hair theorems (NHTs) [11] are about the existence of black hole solutions and they do not consider the stability of black holes. However, we know from the literature that some black hole solutions have been rejected due to their instability under perturbation [12].

As a matter of fact, there is a sort of consensus among physicists that a new NHT is more likely to be accepted when the stability of a black hole solution is assured. On the other hand, many black holes that are well known in the literature and proved by NHTs have not been tested for stability. For example, according our literature knowledge, the linear stability of the famous three dimensional MSW black hole solution [13], which is a solution in three dimensions from the classical dilaton system of Chan and Mann [14], has not been studied before. Like the Banados-Teitelboim-Zanelli (BTZ) black hole model [15], which is the most famous toy model of four-dimensional black holes using general relativity theory, MSW black holes have attracted much attention. For instance, the problems of spectroscopy, thermodynamics, Hawking radiation, quasinormal modes, and scalar perturbations for this black hole have been extensively studied in [16–20]. It is worth noting that (2 + 1)-dimensional black holes have the potential to generate valuable insight into many conceptual issues that arise in (3 + 1)-dimensional black holes. In line with the study [21], the main aim of the present paper is to address the stability problem of MSW black holes, as hinted above. To this end, MSW dilaton black holes are examined to see if they are stable against small spacetime-dependent perturbations. The paper is laid out as follows. In Sect. II, we review MSW black holes and their characteristic features. The stability of the charged MSW static solutions against small perturbations is studied in Sect. III. Finally, in Sect. IV we draw conclusions. (Throughout the paper, natural units with \( c = G = k_B = \hbar = 1 \) are used).

I. MSW BLACK HOLES

MSW black holes redound to investigations in which the concepts of general relativity and string theory can be linked. Examining the actions of Einstein-Maxwell-dilaton and string theories [22], and using conformal field transformations to relate them to each other, the unification of these two theories can be achieved [14]. Throughout this section, we focus on MSW black hole solutions from the general relativity perspective only.

(2 + 1)-dimensional Einstein-Maxwell-dilaton action is given by [14]

\*Electronic address: huriye.gursel@cc.emu.edu.tr
†Electronic address: gulinihal.tokgoz@emu.edu.tr
‡Electronic address: izzet.sakalli@emu.edu.tr
in which $\phi$ and $F_{\mu\nu}$ represent the fields of concern (dilaton and Maxwell fields, respectively), $a$, $b$, and $B$ are arbitrary constants, and $\Lambda$ stands for the cosmological constant. If we perform variations in the metric, gauge, and dilaton field, we obtain the following equations of motions:

$$R_{\mu\nu} = \frac{B}{2} \nabla_\mu \phi \nabla_\nu \phi + \exp(-4a\phi) \left( -g_{\mu\nu} F^2 + 2F_\mu^\alpha F_\nu^\alpha \right) - 2g_{\mu\nu} \exp(b\phi) \Lambda,$$

(2)

$$\nabla^\mu \left( \exp(-4a\phi) F_{\mu\nu} \right) = 0,$$

(3)

$$\frac{B}{2} \left( \nabla^\mu \nabla_\mu \phi \right) + 2a \exp(-4a\phi) F^2 + b \exp(b\phi) \Lambda = 0.$$

(4)

The above field equations with $b = 4a = \frac{B^2}{2} = 4$ and $\phi = -\frac{1}{4} \ln \left( \frac{\beta}{\beta} \right) \{\beta : \text{constant}\}$ lead to the charged MSW black hole solutions, which can be stated as

$$ds^2 = -\left( 8\Lambda \beta r - 2m\sqrt{r} + 8Q^2 \right) dt^2 + \frac{dr^2}{\left( 8\Lambda \beta r - 2m\sqrt{r} + 8Q^2 \right)} + \sigma^2 r^2 d\theta^2,$$

(5)

where $\sigma^2 = \beta$. It is worthwhile to note that the corresponding event (outer) and inner horizon values for the charged MSW black holes read

$$r^+ = \frac{m^2 - 32\Lambda \beta Q^2 + m\sqrt{m^2 - 64\Lambda \beta Q^2}}{32\Lambda^2 \beta^2},$$

(6)

with $m^2 \geq 64\Lambda \beta Q^2$, whereas for the uncharged MSW black holes (at the limit of $Q \to 0$), they reduce to $r^+ = \frac{m^2}{16\Lambda^2 \beta}$ and $r^- = 0$.

Using Eq. (6), one can obtain the Hawking temperature of the MSW black holes via

$$T_H = \frac{1}{4\pi} \frac{d}{dr} \left( -g_{tt} \right) \left|_{r = r^+} \right.,$$

(7)

which results in

$$T_H = \frac{2\Lambda \beta}{\pi} - \frac{m}{32\pi Q^2} \left[ m - \sqrt{m^2 - 64\Lambda \beta Q^2} \right].$$

(8)

At the $Q = 0$ limit, the Hawking temperature of an uncharged MSW black hole is determined as $T_H|_{Q=0} = \frac{\Lambda \beta}{\pi}$, which means that having an ordinary black hole (with positive-valued temperature) is conditional on $\Lambda \beta \geq 0$. Moreover, it is obvious from Eq. (8) that for a real physical temperature of the charged MSW black hole, the condition of $m \geq 8Q\sqrt{\Lambda \beta}$ should also be stipulated. On the other hand, it is worth noting that a negative temperature is a well-known issue in spin systems with some upper energy level limits [23], and corresponds to a non-blackbody spectrum for exotic black holes [24].

II. STABILITY OF MSW BLACK HOLES

Our purpose is to explore the linear stability of the charged MSW black holes. To this end, we borrowed some ideas from [21].
The electrically charged circularly symmetric static solution of the Einstein-Maxwell-dilaton field equations can be described by

\[ ds^2 = \exp(2\gamma)dt^2 - \exp(2\alpha)dr^2 - \exp(2\eta)d\theta^2, \]

with the Maxwell tensor component

\[ F_{01} = q \exp(\alpha + \gamma - \eta + 4\phi), \]

where \(\gamma, \alpha, \) and \(\eta\) are \((r,t)\) dependent metric functions and \(q = Q\). The field equation (2) admits the following non-zero \(R_{tr}\) and \(R_{\theta\theta}\) components:

\[ R_{tr} = R_{rt} = 4\ddot{\phi}\phi', \]

\[ R_{\theta\theta} = -2 \left[ \Lambda - q^2 \exp(-2\eta) \right] \exp(4\phi), \]

where prime and dot denote derivatives with respect to \(r\) and \(t\), respectively. One can assume that the metric functions and the dilaton field have the following small perturbations:

\[ \gamma \equiv \gamma(r,t) = \gamma_0(r) + \varepsilon \gamma_1(r,t), \]

\[ \alpha \equiv \alpha(r,t) = \alpha_0(r) + \varepsilon \alpha_1(r,t), \]

\[ \eta \equiv \eta(r,t) = \eta_0(r) + \varepsilon \eta_1(r,t), \]

\[ \phi \equiv \phi(r,t) = \phi_0(r) + \varepsilon \phi_1(r,t). \]

Setting \(\eta_1(r,t) = 0\) gauge and comparing the metrics (5) and (9), one obtains the following identity: \(\exp(2\eta) = \sigma^2r\). Thus, after a straightforward calculation, \(R_{tr}\) and \(R_{\theta\theta}\) components of the Ricci tensor and the Klein-Gordon equation (4) become

\[ R_{tr} = \dot{\alpha}\eta', \]

\[ R_{\theta\theta} = \left[ \eta'\alpha' - \gamma' - (\gamma')^2 - \eta'' \right] \exp(-2\alpha), \]

\[ \exp(-2\alpha) \left[ \phi'' - \phi'\left( \alpha' - \gamma' - \eta' \right) \right] - \exp(-2\gamma) \left[ \dot{\phi} + \dot{\phi}(\alpha - \gamma) \right] + \]

\[ \exp(4\phi) \left[ \Lambda - q^2 \exp(-2\eta) \right] = 0. \]

Matching Eqs. (11) and (12) with Eqs. (17) and (18) and performing the perturbations considering Eqs. (13-16) with the gauge of \(\eta_1(r,t) = 0\), the linearized forms of the Einstein equations for the \(R_{tr}\) and \(R_{\theta\theta}\) components and the Klein Gordon equation (19) yield the following expressions:

\[ \dot{\alpha}_1 + 2\dot{\phi}_1 = 0, \]
\[ 4 \left( \Lambda \beta r - Q^2 \right) (\alpha_1 + 2 \phi_1) + r \left( 4 \Lambda \beta r - m \sqrt{r} + 4 Q^2 \right) (\alpha_1' - \gamma_1) = 0, \]  
(21)

\[ (\alpha_1' - \gamma_1' + 4 r \phi_1'')(4 \Lambda \beta r - m \sqrt{r} + 4 Q^2) + 4(\alpha_1 + 2 \phi_1) \left( \Lambda \beta \frac{Q^2}{r} \right) \]
\[ - \frac{r}{(4 \Lambda \beta r - m \sqrt{r} + 4 Q^2)} \dot{\phi}_1 + 4(6 \Lambda \beta r - m \sqrt{r} + 2 Q^2) \phi_1' = 0. \]  
(22)

From Eq. (20), one possesses the constraint \( \alpha_1 = -2 \phi_1 \), which leads Eq. (21) to reveal another constraint \( \alpha_1' = \gamma_1' \). Thus, combining those constraints with Eq. (22), the linearized Klein Gordon equation results in

\[ \phi_1'' + \left[ \frac{6 \Lambda \beta r - m \sqrt{r} + 2 Q^2}{r(4 \Lambda \beta r - m \sqrt{r} + 4 Q^2)} \right] \phi_1' - \frac{\ddot{\phi}_1}{(8 \Lambda \beta r - 2m \sqrt{r} + 8 Q^2)^2} = 0. \]
(23)

For the exponentially growing modes, one can introduce

\[ \phi_1 = \phi(r) \exp(kt), \]
(24)

where \( k \) represents the growth factor. Therefore, Eq. (23) reduces to the following effective Klein-Gordon equation:

\[ \phi''(r) + h \phi'(r) - j \phi(r) = 0, \]
(25)

where \( h \) and \( j \) are the functions of \( r \), which are provided as

\[ h = \frac{6 \Lambda \beta r - m \sqrt{r} + 2 Q^2}{r(4 \Lambda \beta r - m \sqrt{r} + 4 Q^2)}, \]
(26)

\[ j = \frac{k^2}{(8 \Lambda \beta r - 2m \sqrt{r} + 8 Q^2)^2}. \]
(27)

Let us introduce a tortoise coordinate-like new radial variable:

\[ du = \frac{dr}{\sqrt{r(8 \Lambda \beta r - 2m \sqrt{r} + 8 Q^2)}}, \]
(28)

which can be integrated to give

\[ u = \frac{1}{\sqrt{m^2 - 64 \Lambda \beta Q^2}} \ln \left[ \frac{8 \Lambda \beta \sqrt{r} - m + \sqrt{m^2 - 64 \Lambda \beta Q^2}}{8 \Lambda \beta \sqrt{r} - m - \sqrt{m^2 - 64 \Lambda \beta Q^2}} \right]. \]
(29)

Inserting Eq. (29) into Eq. (25) we readily obtain the one-dimensional wave equation from Eq. (25)

\[ \left[ \frac{d^2}{du^2} - V_{eff}(r) \right] \phi = 0, \]
(30)

where \( V_{eff}(r) \) is the effective potential, which is given by

\[ V_{eff}(r) = k^2 r. \]
(31)

The asymptotic limit of the effective potential (for real \( k \)) is

\[ \lim_{r \to \infty} V_{eff}(r) = \infty. \]
(32)

It is obvious that the effective potential is positive everywhere and this guarantees the absence of bounded solutions. Thus, one can conclude that MSW black holes (regardless of whether they are charged or uncharged) are linearly stable.
III. CONCLUSION

We applied linear stability analysis to MSW black holes. For this purpose, we used a semi-analytical method used in [12, 21], which is mainly based on the Fubini-Sturm theorem [25]. Then, we considered small spacetime-dependent perturbations in both $R_{t\tau}$ and $R_{\theta\theta}$ components of the Einstein equations and in the Klein Gordon equation (4). The key feature of the applied method was reduction of the linearized equations to a single one-dimensional Schrödinger type differential equation. As a result, the effective potential of the MSW black holes was derived. For all real growth factor values $k$, the effective potential was found to be positive everywhere. This means that there is no bounded solution to Eq. (30) . Thus, we concluded that the MSW black hole solutions are linearly stable. Our findings give support to [26], in which it was shown that MSW black holes are stable against to small time-dependent perturbation.

Our future plan is to extend our linear stability analysis to higher dimensional ($n \geq 3$) rotating black holes in Einstein-Maxwell-Dilaton gravity [27]. Through this, we aim to see the effects of rotation parameters and the number of dimensions on the stability of black holes.

[1] M. Dafermo and I. Rodnianski, *Lectures on black holes and linear waves* (2008); arXiv:0811.0354
[2] Berti et al., Class. Quantum Grav. 32, 243001 (2015).
[3] V. Moncrief, Phys. Rev. D 12, 1526 (1975).
[4] T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[5] F. J. Zerilli, Phys. Rev. Lett. 24, 737 (1970).
[6] F. J. Zerilli, Phys. Rev. D 9, 860 (1974).
[7] B. F. Whiting, J. Math. Phys. 30, 1301 (1989).
[8] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[9] S. Dain and I. G. de Austria, Class. Quantum Grav. 31, 195009 (2014).
[10] M. Dafermos and I. Rodnianski, *The black hole stability problem for linear scalar perturbations” in XVth International Congress on Mathematical Physics pp. 421-433* (P. Exner (ed.), World Scientific, London, 2009); arXiv:1010.5137
[11] N. Gállebeck, Phys. Rev. Lett. 114, 151102 (2015).
[12] P. Kanti, *Linear Stability of Dilatonic Black Holes* (1998); arXiv:hep-th/9804203.
[13] G. Mandal, A. M. Sengupta, and S. R. Wadia, Mod. Phys. Lett. A 6, 1685 (1991).
[14] K. C. K. Chan and R. B. Mann, Phys. Rev. D 50, 6385 (1994).
[15] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[16] S. Sebastian and V. C. Kuriakose, Mod. Phys. Lett. A 28, 1350149 (2013).
[17] S. Fernando and K. Arnold, Gen. Relativ. Gravit. 36, 1805 (2004).
[18] I. Sakalli, Mod. Phys. Lett. A 28, 1350109 (2013).
[19] I. Sakalli, Astrophys. Space Sci. 340, 317 (2012).
[20] S. Fernando, Gen. Relativ. Gravit. 37, 461 (2005).
[21] G. Clément, D. Gal’tsov, and C. Leygnac, Phys. Rev. D 67, 024012 (2003).
[22] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D 43, 3140 (1991); Erratum Phys. Rev. D 45, 3888 (1992).
[23] C. Kittel, *Elementary Statistical Physics* (John Wiley & Sons Inc., New York, 1967).
[24] M. Park, Phys. Lett. B 647, 472 (2007).
[25] G. Birkhoff and G-C. Rota, *Ordinary Differential Equations* (John Wiley & Sons Inc., New York, 1989).
[26] R. R. Hsu, G. Huang, W. F. Lin, and C. R. Lee , Class. Quantum Grav. 10, 505 (1993).
[27] A. Sheykhi, Phys. Rev. D 77, 104022 (2008).