WEAK SOLUTIONS TO THE CONTINUOUS COAGULATION MODEL WITH COLLISIONAL BREAKAGE

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Abstract. A global existence theorem on weak solutions is shown for the continuous coagulation equation with collisional breakage under certain classes of unbounded collision kernel and distribution functions. The model describes the dynamics of particle growth when binary collisions occur to form either a single particle via coalescence or two/more particles via breakup with possible transfer of mass. Each of these processes may take place with a suitably assigned probability depending on the volume of particles participating in the collision.

1. Introduction. Coagulation and breakage processes arise in the different fields of science and engineering, for instance, chemistry (when a matter (water vapor) changes from its gas phase to a liquid phase by condensation process, the molecules in the gas start to come together to form bigger and bigger droplets of the liquid phase), astrophysics (formation of the planets), atmospheric science (raindrop breakup), biology (aggregation of red blood cells) etc. The basic reactions between particles taken into account are the coalescence of a pair of particles to form bigger particles and the breakage of particles into smaller pieces. In general, coagulation event is always a nonlinear process. However, the breakage process may be divided into two different categories on the basis of breakage behaviour of particles, (i) linear breakage and (ii) collisional or nonlinear breakage. Due to external forces or spontaneously (that depends on the nature of particles), linear breakage occurs whereas the collisional breakage happens due to the collision between a pair of particles. It is worth to mention that the smaller particles are only produced due to the linear breakage process while the collisional breakage allows some transfer of mass between a pair of particles and might produce particles of mass larger than
one of each colliding particles. Here, the volume (or size) of each particle is denoted by a positive real number. Now, let us turn to the mathematical formulation of the model considered in this work. We first take a closed system of particles undergoing binary collisions such that any number of particles is produced by the collision, subject to the constraint that the sum of the volumes of the product particles is equal to the sum of the volumes of the two original particles. The following three possible outcomes may arise in such a process:

- if only one particle is produced by the collision, then a coagulation event occurs,
- if the collision process gives two particles, the collision was either elastic or volume (or mass) was exchanged between the original particles,
- and if three or more particles emerge from the collision, then a breakage event takes place.

The continuous coagulation and collisional breakage model has been studied in [6, 19, 22, 23] to describe the evolution of raindrops in clouds. If the particle size distribution is represented by the number density \( g = g(z, t) \) for volume \( z \in \mathbb{R}_+ := (0, \infty) \) at time \( t \in [0, \infty) \), the continuous coagulation equation with collisional breakage reads as

\[
\frac{\partial g}{\partial t} = C(g) + B(g),
\]

where the coalescence term \( C(g) := C_1(g) - C_2(g) \),

\[
C_1(g)(z, t) := \frac{1}{2} \int_0^z E(z - z_1, z_1) \Phi(z - z_1, z_1) g(z - z_1, t) g(z_1, t) dz_1,
\]

\[
C_2(g)(z, t) := \int_0^\infty E(z, z_1) \Phi(z, z_1) g(z, t) g(z_1, t) dz_1,
\]

and the breakup term \( B(g) := B_1(g) - B_2(g) \),

\[
B_1(g)(z, t) := \frac{1}{2} \int_z^\infty \int_0^{z_1} P(z|z_1 - z_2; z_2) [1 - E(z_1 - z_2, z_2)] \\
\times \Phi(z_1 - z_2, z_2) g(z_1 - z_2, t) g(z_2, t) dz_2 dz_1,
\]

\[
B_2(g)(z, t) := \int_0^\infty [1 - E(z, z_1)] \Phi(z, z_1) g(z, t) g(z_1, t) dz_1.
\]

Adding \( C_2 \) and \( B_2 \), we obtain

\[
B_3(g)(z, t) := \int_0^\infty \Phi(z, z_1) g(z, t) g(z_1, t) dz_1.
\]

Hence, (1) can also be written in the following equivalent form

\[
\frac{\partial g}{\partial t} = C_1(g) - B_3(g) + B_1(g),
\]

with the following initial data

\[
g(z, 0) = g_0(z) \geq 0.
\]

Here, \( \Phi(z, z_1) \) denotes the collision kernel, which describes the rate at which particles of volumes \( z \) and \( z_1 \) are colliding and \( E(z, z_1) \) is the probability that the two colliding particles aggregate to form a single one. If they do not (an event which occurs with probability \( 1 - E(z, z_1) \)) they undergo breakage with possible transfer of mass. In
addition, both the collision kernel $\Phi$ and the collision probability $E$ are symmetric in nature, i.e. $\Phi(z, z_1) = \Phi(z_1, z)$ and $E(z, z_1) = E(z_1, z)$ with $0 \leq E(z, z_1) \leq 1$, $\forall(z, z_1) \in \mathbb{R} \times \mathbb{R}$. Next, $P(z|z_1; z_2)$ is a distribution function describing the expected number of particles of volume $z$ produced from the breakage event arising from the collision of particles of volumes $z_1$ and $z_2$.

The first integral $\mathcal{C}_1(g)$ and the second integral $\mathcal{C}_2(g)$ of (1) represent the formation and disappearance, respectively, of particles of volume $z$ due to coagulation events. On the other hand, the third integral $\mathcal{B}_1(g)$ represents the birth of particles of volumes $z_1$ and $z_2$, and the last integral $B_2(g)$ describes the decay of particles of volume $z$ due to the collisional breakage between a pair of particles of volumes $z_1$ and $z_2$. The factor $1/2$ appears in the integrals $\mathcal{C}_1(g)$ and $\mathcal{B}_1(g)$ to avoid the double counting of formation of particles due to coagulation and collisional breakage processes.

The distribution function $P$ has the following properties:

(i) $P$ is a non-negative and symmetric with respect to $z_1$ and $z_2$, i.e. $P(z|z_1; z_2) = P(z|z_2; z_1) \geq 0$,

(ii) The total number of particles resulting from the collisional breakage event is given by

$$
\int_0^{Z_1 + Z_2} P(z|z_1; z_2) dz = N, \text{ for all } z_1 > 0 \text{ and } z_2 > 0, \text{ } P(z|z_1; z_2) = 0 \text{ for } z > z_1 + z_2,
$$

(4)

where $N \geq 2$ is a finite constant and denotes the total number of daughter particles.

(iii) A necessary condition for mass conservation during collisional breakage events is

$$
\int_0^{Z_1 + Z_2} zP(z|z_1; z_2) dz = z_1 + z_2, \text{ for all } z_1 > 0 \text{ and } z_2 > 0.
$$

(5)

From the condition (5), the total volume $z_1 + z_2$ of particles remains conserved during the collisional breakage of particles of volumes $z_1$ and $z_2$.

Next, let us mention some particular cases of the continuous coagulation and collisional breakage equation. When $E \equiv 1$, then equation (2) becomes the continuous Smoluchowski coagulation equation $[3, 4, 12, 13]$. Another case taken into consideration is the collision between a pair of particles of volumes $z$ and $z_1$ that results in either the coalescence of both into of volumes $(z + z_1)$ or into an elastic collision leaving the incoming clusters unchanged. In both cases $P(z|z; z_1) = P(z_1|z; z_1) = 1$ and $P(z^*|z; z_1) = 0$ if $z^* \notin \{z, z_1\}$ which again reduces (2) into the continuous Smoluchowski coagulation equation with $(E(z, z_1)\Phi(z, z_1))$ as the coagulation rate. Now, by substituting $E \equiv 0$ and $P(z|z_1; z_2) = \chi_{|z, \infty|}(z_1)B(z|z_1; z_2) + \chi_{|z, \infty|}(z_2)B(z|z_2; z_1)$ into (2), it can easily be seen that (2) becomes the pure nonlinear breakage model which has been extensively studied in many articles, $[7, 8, 10, 14, 17]$. In these articles, the authors have taken into consideration that when a pair of particles collide, one of them fragments into smaller pieces without transfer of mass from another one. The continuous nonlinear breakage equation reads as

$$
\frac{\partial g(z,t)}{\partial t} = - \int_0^\infty \Psi(z,z_1)g(z,t)g(z_1,t)dz_1
$$
\[ + \int_z^\infty \int_0^\infty B(z|z_1; z_2)\Psi(z_1, z_2)g(z_1, t)g(z_2, t)dz_2dz_1, \]  
(6)

where \( \Psi(z, z_1) = \Psi(z_1, z) \geq 0 \) is the collisional kernel and \( B(z|z_1; z_2) \) denotes the breakup kernel or breakage function, which represents the particle of volume \( z \) is obtained by collision between particles of \( z_1 \) and \( z_2 \) and satisfies the following property

\[ \int_0^{z_1} zB(z|z_1; z_2)dz = z_1, \quad z < z_1 \in \mathbb{R}_+ \quad \text{and} \quad z_2 \in \mathbb{R}_+. \]

Let us now discuss about the total number of particles and the total mass of particles in the coagulation and collisional breakage processes. In collisional breakage events, the total number of particles, i.e. \( \int_0^\infty g(z, t)dz \), increases whereas \( \int_0^\infty g(z, t)dz \) decreases during coagulation events. In addition, it is expected that the total mass \( \int_0^\infty zg(z, t)dz \) of the system remains constant for each time \( t \geq 0 \) during these events. However, the mass conserving property fails due to the appearance of infinite gel in the system if the growth rate of coagulation kernel, \( (E\Phi) \), is very fast compared to the breakage kernel \( ([1 - E]\Phi) \). This process is known as gelation and the time at which this process takes place is called gelling time, see [18].

In this work, we mainly address the issue on the existence of weak solutions to the continuous coagulation and collisional breakage equation (2)–(3). The existence and uniqueness of solutions to the classical coagulation-fragmentation equations have been discussed in several articles by applying various techniques, see [2, 3, 4, 12, 20, 21]. However, best to our knowledge, the mathematical theory on the continuous coagulation and collisional breakage equation has not been rigorously studied. Although there are a few articles available which are devoted to (2)–(3), see [6, 17, 19, 22, 23]. This model has been described in [19, 23]. In particular, in [17], the existence of mass conserving weak solutions to the discrete version of (2)–(3) has been shown by using a weak \( L^1 \) compactness method. Moreover, they have also studied the uniqueness of solutions, long time behaviour in some particular cases and the occurrence of gelation transition. In [6], the structural stability of the continuous coagulation and collisional breakage model is studied by applying both analytical method and numerical experiment. Later in [22], the partial analytical solutions to the discrete (2)–(3) are studied for the constant collisional kernel. Moreover, this solution is also compared with Monte-Carlo simulation. In addition, there are a few articles in which analytical solutions to the continuous nonlinear breakage equations have been investigated for some specific collision kernels only, see [7, 8, 10, 14]. However, in general, it is quite delicate to handle the continuous nonlinear breakage equation mathematically because here the small sized particles have the tendency to fragment further into very small sized clusters which leads to the formation of an infinite number of clusters in a finite time. In order to overcome this situation, we consider a fully nonlinear continuous coagulation and collisional breakage model (2). Best to our knowledge, this is the first attempt to show the existence of global weak solutions to the continuous coagulation and collisional breakage equation (2)–(3) for large classes of unbounded collision kernels and distribution function.

The paper is organized in the following manner: In Section 2, we state some definitions, assumptions and lemmas, which are essentially required in subsequent sections. The statement of the main existence theorem is also given at the end of this section. Section 3 contains the rigorous proof of the existence theorem which relies on a weak \( L^1 \) compactness method.
2. Definitions and results. Let us define the following Banach space $S$ as

$$S := \{ g \in L^1(\mathbb{R}^+, dz) : \|g\|_{L^1(\mathbb{R}^+, (1+z)dz)} < \infty \},$$

where

$$\|g\|_{L^1(\mathbb{R}^+, (1+z)dz)} := \int_0^\infty (1+z)|g(z)|dz.$$  

We can also define the norms in the following way:

$$\|g\|_{L^1(\mathbb{R}^+, zdz)} := \int_0^\infty z|g(z)|dz,$$

and set $S^+ := \{ g \in S : g \geq 0 \text{ a.e.} \}$.

Next, we formulate weak solutions to (2)–(3) through the following definition:

**Definition 2.1.** Let $T \in (0, \infty)$. A solution $g$ of (2)–(3) is a non-negative continuous function $g : [0, T) \rightarrow S^+$ such that, for a.e. $z \in \mathbb{R}^+$ and all $t \in [0, T)$,

(i) the following integrals are finite

$$\int_0^t \int_0^\infty \Phi(z, z_1)g(z_1, s)dz_1ds < \infty, \text{ and } \int_0^t B_1(g)(z, s)ds < \infty,$$

(ii) the function $g$ satisfies the following weak formulation to (2)–(3)

$$g(z, t) = g_0(z) + \int_0^t \{ C_1(g)(z, s) - B_3(g)(z, s) + B_1(g)(z, s) \} ds.$$

Now, throughout the paper, we assume the following conditions on the collision kernel $\Phi$, the distribution function $P$, and the probability function $E$:

$(\Gamma_1)$ let $\Phi$ be a non-negative measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ such that $\Phi(z, z_1) = k_1(z^\alpha z_1^\beta + z_1^\alpha z^\beta)\leq 1,$

$\forall(z, z_1) \in (0, 1) \times (0, 1),$  

where $N$ is given in (4), and $0 \leq E(z, z_1) \leq 1,$  

$\forall(z, z_1) \in \mathbb{R}_+ \times \mathbb{R}_+,$

$(\Gamma_3)$ recalling that $\alpha$ is defined in $(\Gamma_1)$, for each $W > 0$ and $z_1 \in (0, W)$, and any measurable subset $U$ of $[0, W]$ with $|U| \leq \delta$, we have

$$\int_0^{z_1} \chi_U(z)P(z|z_1 - z_2; z_2)dz \leq \Omega_1(W, \delta)z_1^{-\alpha}, \text{ where } \lim_{\delta \to 0} \Omega_1(W, \delta) = 0.$$

Here, $|U|$ denotes the Lebesgue measure of $U$ and $\chi_U$ is the characteristic function of $U$ given by

$$\chi_U(z) := \begin{cases} 1, & \text{if } z \in U, \\ 0, & \text{if } z \notin U, \end{cases}$$

$(\Gamma_4)$ for $z_1 + z_2 > W$, we have $P(z|z_1; z_2) \leq k(W)z^{-\tau_2}$ for $z \in (0, W)$, where $\tau_2 \in (0, 1 - \alpha)$ and $k(W) > 0$.

Let us take the following example of distribution function $P$ as

$$P(z|z_1; z_2) = (\nu + 2)\frac{z^\nu}{(z_1 + \nu + 1)^{\nu + 1}}, \text{ where } -2 < \nu \leq 0 \text{ and } z < z_1 + z_2.$$  

Repeating 1 by $z_1 - z_2$ and $z_2 = z_2$ in above value of $P$, we have

$$P(z|z_1 - z_2; z_2) = (\nu + 2)\frac{z_1^\nu}{z_1^\nu + 1}, \text{ where } -2 < \nu \leq 0 \text{ and } z < (z_1 - z_2) + z_2 = z_1.$$
Substituting the above distribution function into (4) and then for \( \nu = 0 \), this leads to the case of binary breakage and for \(-1 < \nu \leq 0\), we get the finite number of particles, which is denoted by \( N \) and written as \( N = \frac{\nu^2}{\nu + 1} \). But, for \(-2 < \nu \leq -1\), an infinite number of daughter particles are produced. In this article, the existence theory is developed for the example given in (7) only when \( \nu \in (\alpha - 1, 0] \).

Now, (\( \Gamma_3 \)) is checked in the following way: for \( z_1 \in (0, W) \) and \( W > 0 \) is fixed, \( \int_{0}^{z_1} \chi_U(z) P(z|z_1 - z_2; z_2)dz = (\nu + 2) \int_{0}^{z_1} \chi_U(z) \frac{z^\nu}{z_1^\nu+1}dz \).

For \( \alpha < 1 \), and applying Hölder’s inequality, we get
\[
\int_{0}^{z_1} \chi_U(z) P(z|z_1 - z_2; z_2)dz \leq (\nu + 2)z_1^{-\nu - 1}|U|^\alpha \left( \int_{0}^{z_1} \frac{z^{\nu+1}}{z_1^\nu}dz \right)^{1-\alpha} \\
= (\nu + 2)z_1^{-\nu - 1}|U|^\alpha \left( \frac{z_1^{\nu+1}}{\nu+1} \right)^{1-\alpha}, \text{ for } \nu - \alpha > -1
\]

This implies that
\[
\int_{0}^{z_1} \chi_U(z) P(z|z_1 - z_2; z_2)dz \leq \Omega_1(W, \delta)z_1^{-\alpha}
\]

with \( \Omega_1(W, \delta) = C(\alpha, \nu)\delta^\alpha \).

In order to verify the (\( \Gamma_4 \)), for \( z_1 + z_2 > W \) and \( W > 0 \) is fixed, we have
\[
P(z|z_1; z_2) = (\nu + 2) \frac{z^\nu}{(z_1 + z_2)^{\nu+1}} \leq (\nu + 2) \frac{z^\nu}{W^{\nu+1}} \leq k(W)z^{-\tau_2},
\]
where \( \alpha - 1 < \nu \leq 0, \tau_2 = -\nu \in [0, 1 - \alpha) \) and \( k(W) \geq \frac{\nu+2}{W^{\nu+1}} \).

Now we are in the position to state the following existence result:

**Theorem 2.2.** Suppose that (\( \Gamma_1 \))–(\( \Gamma_4 \)) hold and assume that the initial value \( g_0 \in \mathcal{S}^+ \). Then, (2)–(3) has a weak solution \( g \) on \([0, \infty)\) in the sense of Definition 2.1. Moreover, \( g \in \mathcal{S}^+ \) and \( \|g(t)\|_{L^1(\mathbb{R}^+, zdz)} \leq \|g_0\|_{L^1(\mathbb{R}^+, zdz)} \).

3. Existence of weak solutions. In order to construct weak solutions to (2)–(3), we follow a weak \( L^1 \) compactness method introduced in the classical work of Stewart [20].

To prove theorem 2.2, we first write (2)–(3) into the limit of a sequence of truncated equations obtained by replacing the collision kernel \( \Phi \) by their cut-off kernels \( \Phi_n \) [20], where
\[
\Phi_n(z, z_1) := \Phi(z, z_1)\chi_{(0,n)}(z + z_1), \quad (8)
\]
for \( n > 1 \) and \( n \in \mathbb{N} \). Inserting the value of \( \Phi_n(z, z_1) \) stated in (8) into (2), then we have the corresponding sequence of truncated equations as
\[
\frac{\partial g^n}{\partial t} = \mathcal{C}_1^\nu(g^n) - \mathcal{B}_3^\nu(g^n) + \mathcal{B}_4^\nu(g^n), \quad (9)
\]
where
\[ C^n_1(g^n)(z,t) := \frac{1}{2} \int_0^z E(z - z_1, z_1) \Phi_n(z - z_1, z_1) g^n(z - z_1, t) g^n(z_1, t) \, dz_1, \]
\[ B^n_3(g^n)(z,t) := \int_0^{z-n} \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) \, dz_1, \]
\[ B^n_1(g^n)(z,t) := \frac{1}{2} \int_z^n \int_0^{z_1} P(z|z_1 - z_2; z_2) [1 - E(z_1 - z_2, z_2)] \times \Phi_n(z_1 - z_2, z_2) g^n(z_1 - z_2, t) g^n(z_2, t) \, dz_2 \, dz_1, \]
with the truncated initial data
\[ g^n_0(z) := g_0(z) \chi_{(0,n)}(z) \geq 0. \] (10)

**Proposition 1.** Consider \((\Gamma_1) - (\Gamma_4)\) hold and \(g_0 \in S^+\). Then, for each \(n > 1\) and \(n \in \mathbb{N}\), the initial value problem (9)–(10) has a unique non-negative solution \(g^n \in C^1([0, \infty); L^1((0,n), dz))\). In addition, it satisfies the truncated version of mass conservation, i.e.
\[ \int_0^n z g^n(z,t) \, dz = \int_0^n z g^n_0(z) \, dz, \forall t \geq 0. \] (11)

**Proof.** Here, we provide a short proof of Proposition 1. From (8) and (\(\Gamma_1\)), we have
\[ \Phi_n(z, z_1) \leq 2k_1 n^{\alpha + \beta}, \text{ for } n > 1. \] (12)

In order to show the existence of a unique solution to (9), we need to apply the Picard-Lindelöf theorem or the Cauchy-Lipschitz theorem [5, Theorem 7.3] in the Banach space \(L^1((0,n), dz)\). Thus, it is sufficient to show that each term in the right-hand side of (9) is locally Lipschitz continuous in the space \(L^1((0,n), dz)\).

Let \(f_1 \in L^1((0,n), dz)\) and \(f_2 \in L^1((0,n), dz)\). Then, from (12), Fubini’s theorem, symmetry of \(E\) and \(\Phi_n\), (\(\Gamma_2\)) and (4), we estimate
\[ \|C^n_1(f_1) - C^n_1(f_2)\|_{L^1((0,n), dz)} \leq k_1 n^{\alpha + \beta} (\|f_1\|_{L^1((0,n), dz)} + \|f_2\|_{L^1((0,n), dz)}) \|f_1 - f_2\|_{L^1((0,n), dz)}, \] (13)
\[ \|B^n_3(f_1) - B^n_3(f_2)\|_{L^1((0,n), dz)} \leq 2k_1 n^{\alpha + \beta} (\|f_1\|_{L^1((0,n), dz)} + \|f_2\|_{L^1((0,n), dz)}) \|f_1 - f_2\|_{L^1((0,n), dz)}, \] (14)
\[ \|B^n_1(f_1) - B^n_1(f_2)\|_{L^1((0,n), dz)} \leq k_1 n^{\alpha + \beta} (\|f_1\|_{L^1((0,n), dz)} + \|f_2\|_{L^1((0,n), dz)}) \|f_1 - f_2\|_{L^1((0,n), dz)}. \] (15)

From (13), (14) and (15), respectively, we have \(C^n_1, B^n_3\) and \(B^n_1\) are locally Lipschitz continuous functions in \(L^1((0,n), dz)\). Thus (9)–(10) has a unique solution \(g^n \in C^1([0,T^*); L^1((0,n), dz))\) defined on a maximal interval \(t \in [0, T^*)\), \(T^* \in (0, \infty]\), and either \(T^* = \infty\) or
\[ T^* < \infty \quad \text{and} \quad \lim_{t \to T^*} \|g_n(t)\|_{L^1((0,n), dz)} = \infty. \] (16)

Next, we have to show the positivity of \(g^n\). For this, since \(C^n_1\) and \(B^n_3\) are locally Lipschitz continuous functions in \(L^1((0,n), dz)\), hence the positive part \([C^n_1]_+ := \]
max\{C_1^n, 0\} of C_1^n, and \([B^n_1]_+ := \max\{B^n_1, 0\}\) of \(B^n_1\) are also locally Lipschitz continuous.

Next consider the following initial value problem, as

\[
\frac{\partial g^n}{\partial t} = [C^n_1(g^n)]_+ - B^n_3(g^n) + [B^n_1(g^n)]_+,
\]

with the same initial data given in (10). Similarly to the previous argument, it is clear that (17)–(10) also has a unique solution. Let \(sgn_+(x) = 1\), for \(x \geq 0\) and \(sgn_+(x) = 0\), for \(x < 0\). Then the chain rule gives

\[
\frac{\partial}{\partial t}(-f)_+ = -sgn_+(-f) \frac{\partial f}{\partial t}.
\]

Thus, (21) guarantees that \(g^n(\cdot, t) \geq 0 \forall t \in [0, T^*]\). Hence, both equations (9) and (17) are equal. Moreover, \(g^n\) satisfies (11) which can easily be shown by multiplying \(z\) into (9) and then using Fubini’s theorem and the symmetry of \(\Phi_n\) for \(t \in [0, T^*]\).

Now, let us evaluate the following integral, by using (9), the repeated applications of Fubini’s theorem, the transformation \(z - z_1 = z' \& z_1 = z'_1\), the symmetry of \(\Phi_n\) and (4), as

\[
\frac{d}{dt} \int_0^n g^n(z, t)dz = -\frac{1}{2} \int_0^n \int_0^{n-z} E(z, z_1)\Phi_n(z, z_1)g^n(z, t)g^n(z_1, t)dz_1dz + \left(\frac{N}{2} - 1\right) \int_0^n \int_0^{n-z} (1 - E(z, z_1))\Phi_n(z, z_1)g^n(z, t)g^n(z_1, t)dz_1dz,
\]

and

\[
\int_0^n g^n(z, t)dz = \frac{1}{2} \int_0^n \int_0^{n-z} E(z, z_1)\Phi_n(z, z_1)g^n(z, t)g^n(z_1, t)dz_1dz + \left(\frac{N}{2} - 1\right) \int_0^n \int_0^{n-z} (1 - E(z, z_1))\Phi_n(z, z_1)g^n(z, t)g^n(z_1, t)dz_1dz.
\]
Using the non-negativity of $g^n \forall t \in [0, T^*], (\Gamma_1), (\Gamma_2), (12)$ and Fubini’s theorem to (22), we get
\[
\frac{d}{dt} \|g^n(t)\|_{L^1((0,n),dz)} \leq Nk_1n^{\alpha+\beta} \int_0^1 \int_1^n g^n(z,t)g^n(z_1,t)dz_1dz
+ Nk_1n^{\alpha+\beta} \int_1^n \int_0^n g^n(z,t)g^n(z_1,t)dz_1dz
\leq 2Nk_1n^{\alpha+\beta} \int_1^n \int_0^n zg^n(z,t)g^n(z_1,t)dz_1dz
\leq 2Nk_1n^{\alpha+\beta} \left( \int_0^n zg^n(z)dz \right) \|g^n(t)\|_{L^1((0,n),dz)}.
\]
(23)

Applying Gronwall’s inequality to (23), we obtain
\[
\|g^n(t)\|_{L^1((0,n),dz)} \leq \|g_0^n\|_{L^1((0,n),dz)}e^{2Nk_1T^*n^{\alpha+\beta} \int_0^n zg^n(z)dz}, \quad \forall t \in [0, T^*).
\]
(24)

From (24), one can see that if $T^* < \infty$, then
\[
\lim_{t \to T^*} \|g^n(t)\|_{L^1((0,n),dz)} = \|g_0^n\|_{L^1((0,n),dz)}e^{2Nk_1T^*n^{\alpha+\beta} \int_0^n zg^n(z)dz} = \infty.
\]

Thus, it does not satisfy (16). Consequently, $T^* = \infty$. This completes the proof of Proposition 1.

Now, we extend the truncated solution $g^n$ by zero in $\mathbb{R}_+ \times [0, T)$, as
\[
g^n(z,t) := \begin{cases} g^n(z,t), & \text{if } 0 < z < n, \\ 0, & \text{if } z \geq n, \end{cases}
\]
(25)

for $n > 1$ and $n \in \mathbb{N}$.

Next, we wish to establish suitable bounds to apply the Dunford-Pettis theorem [9, Theorem 4.21.2] and then the equicontinuity of the sequence $(g^n)_{n \geq 1}$ in time to use the Arzelà-Ascoli Theorem [1, Appendix A8.5]. This is the aim of the coming subsections.

3.1. Weak compactness.

**Lemma 3.1.** Assume that $(\Gamma_1)$–$(\Gamma_4)$ hold and fix $T > 0$. Let $g_0 \in \mathcal{S}^+$ and $g^n$ be a solution to (9)–(10). Then, the following holds true:
(i) there is a constant $\mathcal{G}(T) > 0$ (depending on $T$) such that
\[
\int_0^n (1+z)g^n(z,t)dz \leq \mathcal{G}(T) \quad \text{for } n > 1 \text{ and all } t \in [0, T],
\]

(ii) for any given \( \epsilon > 0 \), there exists \( W_\epsilon > 1 \) (depending on \( \epsilon \)) such that, for all \( t \in [0, T] \)
\[
\sup_{n>1} \left\{ \int_{W_\epsilon} g^n(z, t) dz \right\} \leq \epsilon,
\]
(iii) for a given \( \epsilon > 0 \), there exists \( \delta_\epsilon > 0 \) (depending on \( \epsilon \)) such that, for every measurable set \( U \) of \( \mathbb{R}_+ \) with \( |U| \leq \delta_\epsilon \), \( n > 1 \) and \( t \in [0, T] \),
\[
\int_U g^n(z, t) dz < \epsilon.
\]

Proof. (i) Let \( n > 1 \), \( n \in \mathbb{N} \) and \( t \in [0, T] \), where \( T > 0 \) is fixed. Integrating (9) with respect to \( z \) from 0 to 1, we obtain
\[
\frac{d}{dt} \int_0^1 g^n(z, t) dz = \int_0^1 C^n_1(g^n)(z, t) dz - \int_0^1 B_3^n(g^n)(z, t) dz + \int_0^1 B_1^n(g^n)(z, t) dz.
\]

The first term on the right-hand side of (26) can be simplified, by using the Fubini theorem and the transformation \( z - z_1 = z' \) and \( z_1 = z_1' \), as
\[
\int_0^1 C^n_1(g^n)(z, t) dz = \frac{1}{2} \int_0^1 \int_0^{1-z_1} E(z, z_1) \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1.
\]

Using Fubini’s theorem, the third term on the right-hand side of (26) can be written as
\[
\int_0^1 B_1^n(g^n)(z, t) dz = \frac{1}{2} \int_0^1 \int_0^{z_1} \int_0^{z_2} P(z \mid z_1 - z_2; z_2) [1 - E(z_1 - z_2, z_2)]
\times \Phi_n(z_1 - z_2, z_2) g^n(z_1 - z_2, t) g^n(z_2, t) dz_2 dz_1 dz_2 dz_1,
\]
\[
+ \frac{1}{2} \int_0^1 \int_0^{z_1} \int_0^{z_2} P(z \mid z_1 - z_2; z_2) [1 - E(z_1 - z_2, z_2)]
\times \Phi_n(z_1 - z_2, z_2) g^n(z_1 - z_2, t) g^n(z_2, t) dz_2 dz_1 dz_2 dz_1 =: I^n_1 + I^n_2.
\]

Let us manipulate \( I^n_1 \), by changing the order of integrations, using (4) and the transformation \( z_1 - z_2 = z'_2 \) & \( z_2 = z'_2 \), and then replacing \( z'_2 \) to \( z \) & \( z'_2 \) to \( z_1 \), as
\[
I^n_1 = \frac{N}{2} \int_0^1 \int_0^{z_1} [1 - E(z_1 - z_2, z_2)] \Phi_n(z_1 - z_2, z_2) g^n(z_1 - z_2, t) g^n(z_2, t) dz_2 dz_1 dz_2 dz_1
\]
\[
= \frac{N}{2} \int_0^1 \int_0^{1-z} [1 - E(z, z_1)] \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz_1 dz.
\]

Next, simplifying \( I^n_2 \), by using Fubini’s theorem and (4), as
\[
I^n_2 = \frac{1}{2} \int_1^1 \int_0^{z_1} \int_0^{z_2} P(z \mid z_1 - z_2; z_2) [1 - E(z_1 - z_2, z_2)]
\times \Phi_n(z_1 - z_2, z_2) g^n(z_1 - z_2, t) g^n(z_2, t) dz_2 dz_1 dz_2 dz_1
\]
\[
= \frac{N}{2} \int_1^{z} [1 - E(z, z_1)] \Phi_n(z - z_1, z_1) g^n(z, t) g^n(z_1, t) dz_1 dz.
\]
Again using the Fubini theorem and applying the transformation \( z - z_1 = z' \) and \( z_1 = z_1' \) into (29), we get

\[
I_2^n \leq \frac{N}{2} \int_0^1 \int_1^n \left[ 1 - E(z - z_1, z_1) \right] \Phi_n(z - z_1, z_1) g^n(z - z_1, t) g^n(z_1, t) dz dz_1
+ \frac{N}{2} \int_1^n \int_{z}^{1} \left[ 1 - E(z - z_1, z_1) \right] \Phi_n(z - z_1, z_1) g^n(z - z_1, t) g^n(z_1, t) dz dz_1
= \frac{N}{2} \int_0^1 \int_{1-z}^{1} \left[ 1 - E(z, z_1) \right] \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 d z
+ \frac{N}{2} \int_1^n \int_{z}^{1} \left[ 1 - E(z, z_1) \right] \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz.
\]

Substituting the estimates on \( I_1^n \) and \( I_2^n \) into (28), we evaluate

\[
\int_0^1 B_i^n(g^n(z, t)) dz \leq \frac{N}{2} \int_0^1 \int_{1-z}^{1} \left[ 1 - E(z, z_1) \right] \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz
+ \frac{N}{2} \int_1^n \int_{z}^{1} \left[ 1 - E(z, z_1) \right] \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz.
\]

Inserting the estimates in (27) and (30) into (26) and then using (Γ_2) with a few manipulations, we obtain

\[
\frac{d}{dt} \int_0^1 g^n(z, t) dz
\leq - \frac{1}{2} \int_0^1 \int_{1-z}^{1} [2 - E(z, z_1) - N(1 - E(z, z_1))] \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz
- \int_0^1 \int_{1-z}^{1} \left( 1 - \frac{N}{2} (1 - E(z, z_1)) \right) \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz
+ \int_0^1 \int_{1-z}^{1} \frac{N}{2} (1 - E(z, z_1)) - 1 \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz
+ \frac{N}{2} \int_1^n \int_0^1 \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz.
\]

Applying (Γ_2) to the first and the second integrals and then using the negativity of the first and second terms on the right-hand side of (31), we have

\[
\frac{d}{dt} \int_0^1 g^n(z, t) dz \leq N \int_0^1 \int_1^n \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz
+ \frac{N}{2} \int_1^n \int_1^n \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz dz_1 dz.
\]

Using (Γ_1), (11) and \( g_0 \in S^+ \) into (32), we obtain

\[
\frac{d}{dt} \int_0^1 g^n(z, t) dz \leq N k_1 \int_0^1 \int_1^n [z^n z_1 + z^n z_1^n] g^n(z, t) g^n(z_1, t) dz dz_1 dz
+ N k_1 \int_1^n \int_1^n z z_1 g^n(z, t) g^n(z_1, t) dz dz_1 dz.
\]
\[ \leq 2Nk_t \| g_0 \|_{L^1(\mathbb{R}^+,zdz)} \int_0^1 g^n(z,t)dz + Nk_t \| g_0 \|_{L^1(\mathbb{R}^+,zdz)}^2. \tag{33} \]

Finally, applying the Gronwall inequality to (33), we find
\[ \int_0^1 g^n(z,t)dz \leq \mathcal{G}(T), \quad \text{for } t \in [0,T], \tag{34} \]
where
\[ \mathcal{G}(T) := (\| g_0 \|_{L^1(\mathbb{R}^+,zdz)} + Nk_t \| g_0 \|_{L^1(\mathbb{R}^+,zdz)}^2)e^{2Nk_tT\| g_0 \|_{L^1(\mathbb{R}^+,zdz)}}. \]

Now, using (34), (11) and (10), we estimate the following integral as
\[ \int_0^n (1+z)g^n(z,t)dz = \int_0^1 g^n(z,t)dz + \int_1^n g^n(z,t)dz + \int_0^n zg^n(z,t)dz \leq \int_0^1 g^n(z,t)dz + 2\| g_0 \|_{L^1(\mathbb{R}^+,zdz)} \leq \mathcal{G}(T), \]
where \( \mathcal{G}(T) := \mathcal{G}(T) + 2\| g_0 \|_{L^1(\mathbb{R}^+,zdz)}. \) This completes the proof of the first part of Lemma 3.1.

(ii) The second part of Lemma 3.1 can be easily proved for \( W_\varepsilon > 1 \) (depending on \( \varepsilon \)), by using (11), as
\[ \int_{W_\varepsilon}^\infty g^n(z,t)dz \leq \frac{1}{W_\varepsilon} \int_{W_\varepsilon}^\infty zg_0(z)dz \leq \frac{\| g_0 \|_{L^1(\mathbb{R}^+,zdz)}}{W_\varepsilon} < \varepsilon. \]

(iii) Choose \( \varepsilon > 0 \) and let \( U \subset \mathbb{R}^+ \). Using Lemma 3.1 (ii), we can choose \( W = W_{\varepsilon/2} \in (1,n) \) such that for all \( 1 < n \in \mathbb{N} \) and \( t \in [0,T] \),
\[ \int_{W_{\varepsilon/2}}^\infty \chi_U(z)g^n(z,t)dz \leq \int_{W_{\varepsilon/2}}^\infty g^n(z,t)dz < \frac{\varepsilon}{2}. \tag{35} \]

For \( n > 1, \delta \in (0,1) \) and \( t \in [0,T] \), we define
\[ r^n(\delta,t) := \sup \left\{ \int_0^W \chi_U(z)g^n(z,t)dz : U \subset (0,W) \text{ and } |U| \leq \delta \right\}. \]

For \( n > 1, t \in [0,T] \) and \( U \subset \mathbb{R}^+ \), it follows from the non-negativity of \( g^n \), Fubini’s theorem, (9)–(10) and the non-negativity of the second integral of (9) that
\[ \int_0^W \frac{\partial}{\partial t} \left( \chi_U(z)g^n(z,t) \right)dz \leq \int_0^W \chi_U(z)C_1^n(g^n)(z,t)dz + \int_0^W \chi_U(z)P_1^n(g^n)(z,t)dz \leq J_1^n + J_2^n. \tag{36} \]

Then, by using (\( \Gamma_1 \)), Fubini’s theorem and applying the transformation \( z - z_1 = z' \) and \( z_1 = z_1', J_1^n \) can be estimated as
\[ J_1^n \leq k_1 \int_0^W \int_0^{W-z_1} \chi_U(z+z_1)(1+z)^\beta (1+z_1)^\beta g^n(z,t)g^n(z_1,t)dzdz_1, \]
which can be further evaluated as
\[ J_1^n \leq k_1 (1+W)^\beta \int_0^W \int_0^{W-z_1} \chi_{(-z_1+U)\cap(0,W-z_1)}(z)(1+z_1)^\beta g^n(z,t)g^n(z_1,t)dzdz_1. \tag{37} \]
Since \((-z_1 + U) \cap (0, W - z_1) \subset (0, W)\) and \(|(-z_1 + U) \cap (0, W - z_1)| \leq | - z_1 + U | = |U| \leq \delta\), then from (37) and Lemma 3.1 (i), we obtain

\[ J^n_1 \leq k_1 G(T) (1 + W)r^n(\delta, t). \]

Next, by applying Fubini’s theorem twice, (\(\Gamma_3\)), (\(\Gamma_4\)) and Hölder’s inequality for \(p > 1\) such that \(p'r_2 < 1\), we estimate \(J^n_2\) as

\[
J^n_2 \leq \frac{1}{2} k_1 \Omega_1 (W, \delta) \int_0^W \int_0^{W-z} (z + z_1)^{-\alpha} (z_1^\beta \frac{z_2}{z_1} + z_1^\alpha) g^n(z, t) g^n(z_1, t) dz_1 dz \]

\[
+ \frac{1}{2} k_1 k(W) \delta^{\frac{n-1}{p'}} \left( \frac{W^{1-\tau_2p}}{1-\tau_2p} \right)^{\frac{1}{p}} \int_0^W \int_0^{W-z} (z_1^\beta + z_1^\alpha) g^n(z, t) g^n(z_1, t) dz_1 dz \]

\[
+ \frac{1}{2} k_1 k(W) \delta^{\frac{n-1}{p'}} \left( \frac{W^{1-\tau_2p}}{1-\tau_2p} \right)^{\frac{1}{p}} \int_0^W \int_0^{W-z} (z_1^\beta + z_1^\alpha) g^n(z, t) g^n(z_1, t) dz_1 dz \]

\[
\leq \frac{1}{2} k_1 \Omega_1 (W, \delta) \int_0^W \int_0^{W-z} (z + z_1)^{-\alpha} (z_1^\beta \frac{z_2}{z_1} + z_1^\alpha) g^n(z, t) g^n(z_1, t) dz_1 dz \]

\[
+ 2k_1 k(W) \delta^{\frac{n-1}{p'}} \left( \frac{W^{1-\tau_2p}}{1-\tau_2p} \right)^{\frac{1}{p}} G^2(T) \]

\[
\leq k_1 \Omega_1 (W, \delta) G^2(T) + 2k_1 k(W) \delta^{\frac{n-1}{p'}} \left( \frac{W^{1-\tau_2p}}{1-\tau_2p} \right)^{\frac{1}{p}} G^2(T). \]

Gathering the above estimates on \(J^n_1\), \(J^n_2\) and inserting them into (36), we obtain

\[
\frac{d}{dt} \int_0^W \chi_U(z) g^n(z, t) dz \leq k_1 G(T) (1 + W)r^n(\delta, t) + k_1 \Omega_1 (W, \delta) G^2(T) \]

\[
+ 2k_1 k(W) \delta^{\frac{n-1}{p'}} \left( \frac{W^{1-\tau_2p}}{1-\tau_2p} \right)^{\frac{1}{p}} G^2(T). \]
Integrating the above inequality with respect to $t$ and taking the supremum over all $U$ such that $U \subset (0, W)$ with $|U| \leq \delta$, we estimate
\[ r^n(\delta, t) \leq r^n(\delta, 0) + k_1 G(T)(1 + W) \int_0^t r^n(\delta, s) ds + k_1 \Omega_1(W, \delta) G^2(T) T \]
\[ + 2k_1 k(W) T \delta \frac{W^{1-\tau p}}{1-\tau p} \frac{\delta}{r_1^p} G^2(T), \quad t \in [0, T]. \]

An application of Gronwall’s inequality finally gives
\[ r^n(\delta, t) \leq C^*(W, \delta) \exp(k_1 G(T)T(1 + W)), \quad t \in [0, T], \]
where
\[ C^*(W, \delta) := \sup_{n>1} \{ r^n(\delta, 0) \} + k_1 \Omega_1(W, \delta) G^2(T) T \]
\[ + 2k_1 k(W) T \delta \frac{W^{1-\tau p}}{1-\tau p} \frac{\delta}{r_1^p} G^2(T). \]

Since $\sup_{n>1} \{ r^n(\delta, 0) \} \to 0$ as $\delta \to 0$ due to the integrability of $g_0$, thus we have
\[ \sup_{n>1} \{ r^n(\delta, t) \} \to 0 \text{ as } \delta \to 0. \] (38)

Combining (35) and (38), we thus obtain the required result.

Hence, from the Dunford-Pettis theorem, we have a sequence $\{g^n\}_{n>1}$ which lies in a weakly compact subset of $L^1(\mathbb{R}_+, dz)$ with respect to the volume variable, for each $t \in [0, T]$.

### 3.2. Equicontinuity with respect to time in weak sense

By showing the following lemma, we check the time equicontinuity in weak sense of the family $\{g^n(t), t \in [0, T]\}$ in $L^1(\mathbb{R}_+, dz)$.

#### Lemma 3.2

Let $\psi \in L^\infty(\mathbb{R}_+)$ and the initial data $g_0 \in S^+$. Assume $(\Gamma_1)$–$(\Gamma_4)$ hold true. Then, we have
\[
\lim_{h \to 0} \sup_{t \in [0, T-h]} \int_0^\infty \{ g^n(z, t + h) - g^n(z, t) \} \psi(z) dz = 0.
\]

**Proof.** Let $\psi \in L^\infty(\mathbb{R}_+)$, $h \in (0, T)$ with $h < 1$ and $t \in [0, T-h]$. Next, from (9), we can estimate the following integral as
\[
\left| \int_0^\infty \{ g^n(z, t + h) - g^n(z, t) \} \psi(z) dz \right|
\leq \| \psi \|_{L^\infty(\mathbb{R}_+)} \int_t^{t+h} \int_0^\infty \{ C^n_1(g^n)(z, s) + B^n_3(g^n)(z, s) + B^n_4(g^n)(z, s) \} dz ds. \] (39)

On the one hand, by using Fubini’s theorem, (25), $(\Gamma_1)$, $(\Gamma_2)$, and Lemma 3.1 (i),
\[
\int_0^\infty B^n_3(g^n)(z, t) dz \leq \int_0^n \int_0^n \Phi_n(z, z_1) g^n(z, t) g^n(z_1, t) dz_1 dz
\leq 2k_1 \int_0^n \int_0^n (1 + z)(1 + z_1) g^n(z, t) g^n(z_1, t) dz_1 dz
\leq 2k_1 G^2(T). \] (40)
On the other hand, by using Fubini’s theorem, applying the transformation \( z - z_1 = z' \) and \( z_1 = z'_1 \), \((4)\) and \((40)\), we obtain

\[
\int_0^\infty c_n^\alpha(g^n)(z,t)dz \leq \frac{1}{2} \int_0^\infty b_n^\alpha(g^n)(z,t)dz \leq k_1 g^2(T),
\]

(41)

and

\[
\int_0^\infty b_n^\alpha(g^n)(z,t)dz \leq \frac{N}{2} \int_0^\infty b_n^\alpha(g^n)(z,t)dz \leq Nk_1 g^2(T).
\]

(42)

Inserting estimates \((40)\), \((41)\) and \((42)\) into \((39)\), we thus obtain

\[
\left| \int_0^\infty \{g^n(t+h) - g^n(t)\} \psi(z)dz \right| \leq k_1(N + 3)\|\psi\|_{L^\infty(\mathbb{R}_+)} g^2(T)h,
\]

(43)

where \( h \) is arbitrary. This completes the proof of Lemma 3.2.

Then according to a refined version of the Arzelà-Ascoli theorem, see [20, Theorem 2.1] or Arzelà-Ascoli theorem, see Ash [1, page 228], we conclude that there exists a subsequence \( \{g^n\} \) (not relabeled) such that

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \int_0^\infty \{g^n(z,t) - g(z,t)\} \psi(z)dz \right| = 0,
\]

(44)

for all \( T > 0, \psi \in L^\infty(\mathbb{R}_+) \) and some \( 0 \leq g \in C_w([0,T]; L^1(\mathbb{R}_+,dz)) \), where \( C_w([0,T]; L^1(\mathbb{R}_+,dz)) \) is the space of all weakly continuous functions from \([0,T]\) to \( L^1(\mathbb{R}_+,dz)\). Next, we claim that

\[
g \in C([0,T]; L^1(\mathbb{R}_+,dz)).
\]

(45)

For \( t \geq 0, h > 0 \) and \( \psi \in L^\infty(\mathbb{R}_+) \), since \( \{g^n(t+h) - g^n(t)\} \) converges weakly to \( \{g(t+h) - g(t)\} \) in \( L^1(\mathbb{R}_+,dz) \), then we can pass to the limit \( n \to \infty \) in \((43)\) and obtain that \( g \) also satisfies \((43)\) from which we infer that

\[
\|g(t+h) - g(t)\|_{L^1(\mathbb{R}_+,dz)} = \sup_{\psi \in L^\infty(\mathbb{R}_+)} \left\{ \frac{1}{\|\psi\|_{L^\infty(\mathbb{R}_+)}} \left| \int_0^\infty \{g(z,t+h) - g(z,t)\}\psi(z)dz \right| \right\}
\]

\[
\leq k_1(N + 3) g^2(T)h.
\]

(46)

Taking \( h \to 0 \) into \((46)\), we obtain \((45)\). The proof of strong continuity argument is motivated from [16].

We can also further improve the weak convergence \((44)\) to

\[
g^n(t) \rightharpoonup g(t) \quad \text{in} \quad L^1(\mathbb{R}_+, (1 + z)^\beta dz)
\]

as \( n \to \infty \),

(47)

by using Lemma 3.1, \((44)\) and \( \beta < 1 \).

Next, for any \( m > 0, t \in [0,T] \), since we have \( g^n \rightharpoonup g \) in \( L^1(\mathbb{R}_+,dz) \), then we obtain

\[
\int_0^m zg(z,t)dz = \lim_{n \to \infty} \int_0^m zg^n(z,t)dz \leq \|g_0\|_{L^1(\mathbb{R}_+,dz)} < \infty.
\]

Using \((11)\), the non-negativity of each \( g^n \) and \( g \), then as \( m \to \infty \) implies that \( \|g(t)\|_{L^1(\mathbb{R}_+,dz)} \leq \|g_0\|_{L^1(\mathbb{R}_+,dz)} \) and \( g(t) \in S^+ \).
3.3. Convergence of approximated integrals. Now, we prove that the limit function \( g \) obtained in (44) is indeed a weak solution to (2)–(3). We then have the following result:

**Lemma 3.3.** Consider that \((\Gamma_1)–(\Gamma_4)\) hold and the initial data \(g_0 \in S^+\). Let \((g^n)_{n \geq 1}\) be a bounded sequence in \(S^+\) and \(g \in S^+\), where \(\|g^n\|_{L^1((\mathbb{R}_+,1+z)dz)} \leq \mathcal{G}(T)\) and \(g^n \rightharpoonup g\) in \(L^1(\mathbb{R}_+,1+z)dz\) as \(n \to \infty\). Then, for each \(W > 1\), we have \(C^n_i(g^n) \to C_i(g), B^n_3(g^n) \to B_3(g)\) and \(B^n_4(g^n) \to B_4(g)\) in \(L^1((0,W),dz)\) as \(n \to \infty\).

**Proof.** Choose \(W > 1\), where \(z \in (0,W)\) and let \(\psi \in L^\infty(0,W)\). The first two convergence result in (48), i.e. \(C^n_i(g^n) \to C_i(g)\) and \(B^n_3(g^n) \to B_3(g)\) in \(L^1((0,W),dz)\) as \(n \to 0\), are similar as that in [11, 13, 20] to which we refer. For \(B^n_4(g^n) \to B_4(g)\) in \(L^1((0,W),dz)\) as \(n \to \infty\), it is sufficient to prove that the following integral tends to zero as \(n \to \infty\):

\[
\left\lfloor \int_0^W \psi(z)\left|B^n_4(g^n)(z,t) - B_4(g)(z,t)\right|dz \right\rfloor \leq \frac{1}{2} \int_0^\infty \int_0^\infty \mathcal{H}(z_1, z_2)\left|g^n(z_1,t)g^n(z_2,t) - g(z_1,t)g(z_2,t)\right|(1 + z_1)^\beta(1 + z_2)^\beta dz_2dz_1
\]

(51)

where

\[
\mathcal{H}(z_1, z_2) = \left[1 - E(z_1, z_2)\right] \frac{\Phi(z_1, z_2)}{(1 + z_1)^\beta(1 + z_2)^\beta} \int_0^{\min\{W,z_1+z_2\}} \psi(z)P(z|z_1; z_2)dz.
\]

Here, the integrals on the right-hand side of (49) are simplified, by using the repeated applications of Fubini’s theorem and the transformation \(z_1 - z_2 = z'_1 \& z_2 = z'_2\). By applying (\(\Gamma_1\)) and (4), one can infer that

\[
\mathcal{H} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+).
\]

(50)

Hence, using [15, Lemma 2.9] and (50), one can show that

\[
\lim_{n \to \infty} \frac{1}{2} \int_0^W \int_0^W \mathcal{H}(z_1, z_2)\left|g^n(z_1,t)g^n(z_2,t) - g(z_1,t)g(z_2,t)\right|(1 + z_1)^\beta(1 + z_2)^\beta dz_2dz_1 = 0.
\]

(51)

Next, by using Lemma 3.1 and (4), the second and third integrals on the right-hand side to (49) can be estimated together, as

\[
\leq \frac{3N}{(1 + W)^{1-\beta}} \|\Psi\|_{L^\infty} k_1 \mathcal{G}(T) \int_W^\infty g^n(z_2,t)(1 + z_2)^\beta dz_2 \\
\leq 3k_1N \|\Psi\|_{L^\infty} \mathcal{G}^2(T).
\]

(52)
Similarly, we can show that
\[
\left| \left\{ 2 \int_0^W \int_0^\infty \int_0^\infty \mathcal{H}(z_1, z_2)g(z_1, t)g(z_2, t)(1 + z_1)\beta(1 + z_2)\beta \, dz_2 \, dz_1 \right\} \right| \leq 3k_1N\|\Psi\|_{L^\infty G^2(T)}.
\] (53)

Using (51)–(53) into (49), we have
\[
\lim_{n \to \infty} \left| \int_0^W \psi(z)\{\mathcal{B}_n^n(g^n)(z, t) - \mathcal{B}_1(g)(z, t)\}\, dz \right| \leq 6k_1N\|\Psi\|_{L^\infty G^2(T)}.
\] (54)

As we have considered \( W > 1 \), therefore above estimate is valid for \( W \in (1, \infty) \). Finally, from (52)–(54), we obtain
\[
\left| \int_0^W \psi(z)\{\mathcal{B}_n^n(g^n)(z, t) - \mathcal{B}_1(g)(z, t)\}\, dz \right| \to 0 \text{ as } n \to \infty.
\]

Thus, this completes the proof of Lemma 3.3.

Now, we are in a position to prove Theorem 2.2, by employing above results.

**Proof of Theorem 2.2.** Fix \( T > 0 \) and let us consider \((g^n)_{n \geq 1}\) be a weakly convergent subsequence of the approximating solutions obtained from (44). Hence, from (44) and for \( t \in [0, T] \), we get
\[
g^n(z, t) \to g(z, t) \text{ in } L^1(\mathbb{R}_+, dz) \text{ as } n \to \infty.
\] (55)

Let \( \psi \in L^\infty(\mathbb{R}_+) \). Then from Lemma 3.3, we have for each \( s \in [0, t] \)
\[
\int_0^\infty \psi(z)\{(C_1^n - \mathcal{B}_3^n + \mathcal{B}_1^n)(g^n)(z, s) - (C_1 - \mathcal{B}_3 + \mathcal{B}_1)(g)(z, s)\}\, dz \to 0 \text{ as } n \to \infty.
\] (56)

In order to apply the Lebesgue dominated convergence theorem, the boundedness of the following integral is shown, by using the estimates on \( g^n \) from (40)–(42) and similar bounds, that also holds, for \( g \), as
\[
\left| \int_0^\infty \psi(z)\{(C_1^n - \mathcal{B}_3^n + \mathcal{B}_1^n)(g^n)(z, s) - (C_1 - \mathcal{B}_3 + \mathcal{B}_1)(g)(z, s)\}\, dz \right| 
\leq 2k_1G^2(T)\|\psi\|_{L^\infty(\mathbb{R}_+)}(3 + N) < \infty.
\] (57)

Since the left-hand side of (57) is in \( L^1((0, t), ds) \), then from (56), (57) and the Lebesgue dominated convergence theorem, we obtain
\[
\int_0^t \int_0^\infty \psi(z)\{(C_1^n - \mathcal{B}_3^n + \mathcal{B}_1^n)(g^n)(z, s) - (C_1 - \mathcal{B}_3 + \mathcal{B}_1)(g)(z, s)\}\, dz\, ds \to 0
\text{ as } n \to \infty.
\] (58)

Since \( \psi \) is arbitrary and (58) holds for \( \psi \in L^\infty(\mathbb{R}_+) \) as \( n \to \infty \), hence, by applying the Fubini theorem, we get
\[
\int_0^t (C_1^n - \mathcal{B}_3^n + \mathcal{B}_1^n)(g^n)(z, s)\, ds \to \int_0^t (C_1 - \mathcal{B}_3 + \mathcal{B}_1)(g)(z, s)\, ds \text{ in } L^1(\mathbb{R}_+, dz)
\text{ as } n \to \infty.
\] (59)

Then, by the definition of \((C_1^n - \mathcal{B}_3^n + \mathcal{B}_1^n)\), we obtain
\[
g^n(z, t) = \int_0^t (C_1^n - \mathcal{B}_3^n + \mathcal{B}_1^n)(g^n)(z, s)\, ds + g_0^n(z), \text{ for } t \in [0, T]
\] (60)
and thus, it follows from (59), (55) and (60) that
\[
\int_{0}^{\infty} \psi(z)g(z,t)dz = \int_{0}^{t} \int_{0}^{\infty} \psi(z)(C_1 - B_3 + B_1)(g)(z,s)dzds + \int_{0}^{\infty} \psi(z)g_0(z)dz,
\]
for any \( \psi \in L^\infty(\mathbb{R}_+) \). Hence for the arbitrariness of \( T \), and for all \( \psi \in L^\infty(\mathbb{R}_+) \), we have \( g \) is a solution to (2)–(3). This implies that for almost any \( z \in \mathbb{R}_+ \), we have
\[
g(z,t) = g_0(z) + \int_{0}^{t} (C_1 - B_3 + B_1)(g)(z,s)ds.
\]
We conclude that \( g \) is a solution to (2)–(3) on \([0, \infty)\). This completes the proof of the existence Theorem 2.2.

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