Formulation of the Spinor Field in the Presence of a Minimal Length Based on the Quesne-Tkachuk Algebra

S. K. Moayedi \textsuperscript{a*}, M. R. Setare \textsuperscript{b†} and H. Moayeri \textsuperscript{a‡}

\textsuperscript{a} Department of Physics, Faculty of Sciences, Arak University, Arak 38156-8-8349, Iran
\textsuperscript{b} Department of Science, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran

Abstract

In 2006 Quesne and Tkachuk (J. Phys. A: Math. Gen. \textbf{39}, 10909, 2006) introduced a (D+1)-dimensional $(\beta, \beta')$-two-parameter Lorentz-covariant deformed algebra which leads to a nonzero minimal length. In this work, the Lagrangian formulation of the spinor field in a (3+1)-dimensional space-time described by Quesne-Tkachuk Lorentz-covariant deformed algebra is studied in the case where $\beta' = 2\beta$ up to first order over deformation parameter $\beta$. It is shown that the modified Dirac equation which contains higher order derivative of the wave function describes two massive particles with different masses. We show that physically acceptable mass states can only exist for $\beta < \frac{1}{8m^2c^2}$. Applying the condition $\beta < \frac{1}{8m^2c^2}$ to an electron, the upper bound for the isotropic minimal length becomes about $3 \times 10^{-13} m$. This value is near to the reduced Compton wavelength of the electron ($\lambda_c = \frac{\hbar}{mc} = 3.86 \times 10^{-13} m$) and is not incompatible with the results obtained for the minimal length in previous investigations.

Keywords: Quantum gravity; Generalized uncertainty principle; Minimal length; Spinor field; Dirac equation

\textsuperscript{*}E-mail: s-moayedi@araku.ac.ir
\textsuperscript{†}E-mail: rezakord@ipm.ir
\textsuperscript{‡}E-mail: h-moayeri@phd.araku.ac.ir
1 Introduction

The unification between general relativity and quantum mechanics is one of the major subjects of recent studies in theoretical physics. The most interesting consequence of the above unification is that in quantum gravity there is a minimal observable distance on the order of the Planck length, \( l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-35}\text{m} \), where \( G \) is the Newton’s constant. The existence of this minimal observable length which is motivated from various theories of quantum gravity (such as perturbative string theory) and black hole gedanken experiments, leads to a generalization of Heisenberg uncertainty principle. This generalized or gravitational uncertainty principle (GUP) can be written as

\[
\Delta X \geq \frac{\hbar}{2\Delta P} + \frac{\hbar}{2}\beta\Delta P, 
\]

where \( \beta \) is a positive parameter [1-19]. At high energies, the second term on the right-hand side of (1) becomes significant and leads to important deviations from the usual quantum mechanics. With the GUP even at high momenta \( \Delta X \) is limited in resolution because of quantum gravitational effects. In other words, independent of momentum, \( \Delta X \) is always larger than a minimal observable length \((\Delta X)_{\text{min}} = \hbar\sqrt{\beta}\). Nowadays, physicists are trying to reformulate the quantum field theory in the presence of a minimal observable length and there is hoping that this approach causes unwanted divergencies can be eliminated or modified in quantum field theory [7]. In Ref. [10], the real Klein-Gordon field in the presence of a minimal observable length was studied and the authors estimated the minimal observable length must be in the range \( 10^{-17}\text{m} < (\Delta X)_{\text{min}} < 10^{-15}\text{m} \). SI units are used throughout this work.

This paper is organized as follows. In Sect. 2, the (3+1)-dimensional \((\beta,\beta')\)-two-parameter Lorentz-covariant deformed algebra introduced by Quesne and Tkachuk is reviewed and it is shown that the above algebra leads to a minimal observable length [5,6]. In Sect. 3, the Lagrangian formulation of the Dirac spinor field in a (3+1)-dimensional space-time described by Quesne-Tkachuk algebra is presented in the case where \( \beta' = 2\beta \) up to first order over deformation parameter \( \beta \). In Sect. 4, the solutions of the modified spinor field equation for free motion of a Dirac particle are obtained and it is shown that these solutions are associated with two different mass states. We find that the minimal observable length in the modified spinor theory is of the order of about \( 3 \times 10^{-13}\text{m} \). Finally, Sect. 5 is devoted to the conclusions.

2 A Brief Review of the Quesne-Tkachuk Algebra

Let us start with a quick review of the Quesne-Tkachuk algebra, which is a Lorentz-covariant deformed algebra that describes a \((D+1)\)-dimensional quantized space-time [5,6]. The (3+1)-
The dimensional Quesne-Tkachuk algebra is characterized by the following modified commutation relations

\[ [X^\mu, P^\nu] = -i\hbar (g^{\mu\nu}(1 - \beta P_\rho P^\rho) - \beta' P^\mu P^\nu), \]  
\[ [X^\mu, X^\nu] = i\hbar \frac{2\beta - \beta' - (2\beta + \beta')\beta P_\rho P^\rho}{1 - \beta P_\rho P^\rho} (P^\mu X^\nu - P^\nu X^\mu), \]  
\[ [P^\mu, P^\nu] = 0, \]

where \( \mu, \nu, \rho = 0, 1, 2, 3 \) and \( \beta, \beta' \) are two deformation parameters which are assumed non-negative \( (\beta, \beta' \geq 0) \). In terms of length \( (L) \), mass \( (M) \), and time \( (T) \) the deformation parameters \( \beta \) and \( \beta' \) have the same dimensions \( M^{-2}L^{-2}T^2 \), i.e., \( [\beta] = [\beta'] = (\text{momentum})^{-2} \). Also, \( X^\mu \) and \( P^\mu \) are deformed position and momentum operators and \( g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \). Using (2) and the Schwarz inequality for a quantum state, the uncertainty relation for position and momentum by assuming that \( \Delta P^i \) is isotropic \( (\Delta P^i = \Delta P, \; i = 1, 2, 3) \) becomes

\[ \Delta X^i \Delta P \geq \frac{\hbar}{2} \left| 1 - \beta \left\{ \langle (P^0)^2 \rangle - 3(\Delta P)^2 - \sum_{j=1}^{3} \langle P^j \rangle^2 \right\} + \beta' \left[ (\Delta P)^2 + \langle P^i \rangle^2 \right] \right|. \]

Hence, we arrive at an isotropic absolutely smallest uncertainty in position given by

\[ (\Delta X^i)_0 = \hbar \sqrt{(3\beta + \beta') \left[ 1 - \beta \langle (P^0)^2 \rangle \right]}, \; \; i \in \{1, 2, 3\}. \]

In [8,11], Samar and Tkachuk introduced a representation which satisfies the modified commutation relations (2)-(4) up to first order in \( \beta, \beta' \). The Samar-Tkachuk representation is given by

\[ X^\mu = x^\mu - \frac{2\beta - \beta'}{4} (x^\mu p^2 + p^2 x^\mu), \]
\[ P^\mu = (1 - \frac{\beta'}{2} p^2) p^\mu, \]

where \( x^\mu, p^\mu = i\hbar \frac{\partial}{\partial x^\mu} = i\hbar \partial^\mu \) are position and momentum operators in ordinary relativistic quantum mechanics, and \( p^2 = p_\alpha p^\alpha = (p^0)^2 - \mathbf{p} \cdot \mathbf{p} \).

In this paper, we only consider the special case \( \beta' = 2\beta \), wherein the position operators commute with each other in linear approximation over the deformation parameters, i.e., \( [X^\mu, X^\nu] = 0 \). In such a linear approximation, the Quesne-Tkachuk algebra reads

\[ [X^\mu, P^\nu] = -i\hbar (g^{\mu\nu}(1 - \beta P_\rho P^\rho) - 2\beta P^\mu P^\nu), \]
\[ [X^\mu, X^\nu] = 0, \]
\[ [P^\mu, P^\nu] = 0. \]
It is easy to show that the following representations satisfy (9)-(11), at the first order in $\beta$,

$$X^\mu = x^\mu,$$  \hspace{1cm}  (12)

$$P^\mu = (1 - \beta p^2) p^\mu.$$  \hspace{1cm}  (13)

It should be noted that the representations (7),(8) and (12),(13) coincide when $\beta' = 2\beta$.

3 Lagrangian Formulation of the Spinor Field Based on the Quesne-Tkachuk Algebra

The Dirac Lagrangian density for a spinor (spin $-\frac{1}{2}$) field is [20]

$$L(\Psi, \overline{\Psi}, \partial_\mu \Psi, \partial_\mu \overline{\Psi}) = \frac{i\hbar c}{2} \left\{ \overline{\Psi} \gamma^\mu (\partial_\mu \Psi) - (\partial_\mu \overline{\Psi}) \gamma^\mu \Psi \right\} - mc^2 \overline{\Psi} \Psi,$$  \hspace{1cm}  (14)

where $\Psi$ is a Dirac spinor, $\gamma^\mu$ are the Dirac matrices, and $\overline{\Psi} := \Psi^\dagger \gamma^0$ is the adjoint spinor. The Euler-Lagrange equation for $\overline{\Psi}$ is

$$\frac{\partial L}{\partial \overline{\Psi}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \overline{\Psi})} \right) = 0.$$  \hspace{1cm}  (15)

If we substitute the Lagrangian density (14) into the Euler-Lagrange equation (15), we will obtain the Dirac equation as follows

$$(i\hbar \gamma^\mu \partial_\mu - mc) \Psi = 0.$$  \hspace{1cm}  (16)

If we apply the Euler-Lagrange equation to $\Psi$, we obtain

$$i\hbar \left( \partial_\mu \overline{\Psi} \right) \gamma^\mu + mc \overline{\Psi} = 0,$$  \hspace{1cm}  (17)

which is the adjoint of the Dirac equation. Now we want to obtain the Lagrangian density for the spinor field in the presence of a minimal length based on the Quesne-Tkachuk algebra. For such a purpose, let us write the Lagrangian density by using the representations (12) and (13), i.e.,

$$x^\mu \rightarrow x^\mu,$$  \hspace{1cm}  (18)

$$\partial^\mu \rightarrow (1 + \beta \hbar^2 \Box) \partial^\mu,$$  \hspace{1cm}  (19)

where $\Box := \partial_\mu \partial^\mu$ is the d’Alembertian operator.

The result reads

$$L = \frac{i\hbar c}{2} \left\{ \overline{\Psi} \gamma^\mu (\partial_\mu \Psi) - (\partial_\mu \overline{\Psi}) \gamma^\mu \Psi + \beta \hbar^2 \left\{ \overline{\Psi} \gamma^\mu (\Box \partial_\mu \Psi) - (\Box \partial_\mu \overline{\Psi}) \gamma^\mu \Psi \right\} \right\} - mc^2 \overline{\Psi} \Psi + O(\beta^2).$$  \hspace{1cm}  (20)
The term $\beta \hbar^2 \left[ \bar{\Psi} \gamma^\mu (\square \partial_\mu \Psi) - (\square \partial_\mu \bar{\Psi}) \gamma^\mu \Psi \right]$ in (20) can be considered as a minimal length effect. The generalized Euler-Lagrange equation for the adjoint spinor $\bar{\Psi}$ is [21,22]

$$\frac{\partial L}{\partial \bar{\Psi}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \bar{\Psi})} \right) + \partial_\mu \partial_\nu \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \bar{\Psi})} \right) - \partial_\mu \partial_\nu \partial_\lambda \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \partial_\lambda \bar{\Psi})} \right) + \cdots = 0. \quad (21)$$

If we substitute the Lagrangian density (20) into the generalized Euler-Lagrange equation (21) and neglecting terms of order $\beta^2$, we will obtain the modified Dirac equation as follows

$$[i \hbar \gamma^\mu (1 + \beta \hbar^2 \square) \partial_\mu - mc] \Psi = 0. \quad (22)$$

The term $i \hbar^3 \gamma^\mu \square \partial_\mu \Psi$ in (22) shows the minimal length effects. The wave equation (22) is a third order relativistic wave equation that in the limit of $\beta \rightarrow 0$ turns in to the ordinary Dirac equation. Applying the generalized Euler-Lagrange equation to $\Psi$, i.e.,

$$\frac{\partial L}{\partial \Psi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \Psi)} \right) + \partial_\mu \partial_\nu \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \Psi)} \right) - \partial_\mu \partial_\nu \partial_\lambda \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \partial_\lambda \Psi)} \right) + \cdots = 0, \quad (23)$$

and neglecting terms of order $\beta^2$, we find

$$i \hbar [(1 + \beta \hbar^2 \square) \partial_\mu \bar{\Psi}] \gamma^\mu + mc \bar{\Psi} = 0, \quad (24)$$

which is the adjoint of the modified Dirac equation.

4 Plane-Wave Solutions of the Modified Dirac Equation

In this section, the notation and conventions are the same as in Greiner, Relativistic Quantum Mechanics: Wave Equations, 3rd edn (Springer 2000) [23]. Now, we will obtain the plane-wave solutions of the modified Dirac equation (22). The modified Dirac equation (22) can be written as

$$\left[ i \hbar (1 + \beta \hbar^2 \square) \frac{\partial}{\partial t} + i \hbar c (1 + \beta \hbar^2 \square) \hat{\alpha} \cdot \nabla - mc^2 \beta \right] \Psi = 0, \quad (25)$$

where

$$\beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \hat{\alpha}^i = \beta \gamma^i. \quad (26)$$

In equation (26) $\sigma_i$ are the $2 \times 2$ Pauli matrices, $I$ is the $2 \times 2$ unit matrix, and 0 is the $2 \times 2$ null matrix.

To solve equation (25), we try the following ansatz

$$\Psi(r, t) = \psi(r) e^{i \frac{\gamma i}{\hbar} \varepsilon t}, \quad (27)$$
where $\varepsilon$ describes the time evolution of the stationary state $\psi(r)$. If we substitute (27) into (25), we will obtain

$$[(1 - \beta \frac{\varepsilon^2}{c^2} - \beta \hbar^2 \nabla^2)(\varepsilon + ihc\hat{\alpha}.\nabla) - mc^2 \hat{\beta}]\psi(r) = 0. \quad (28)$$

The four-component spinor $\psi(r)$ splits up into two two-component spinors $\phi$ and $\chi$, i.e.,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad (29)$$

with

$$\phi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \quad (30)$$

Using the explicit form (26) for the $\hat{\alpha}^i$ and $\hat{\beta}$ matrices (28) can be written as

$$(1 - \beta \frac{\varepsilon^2}{c^2} - \beta \hbar^2 \nabla^2)\varepsilon \phi = (1 - \beta \frac{\varepsilon^2}{c^2} - \beta \hbar^2 \nabla^2)c\sigma^i \frac{\hbar}{i} \nabla \chi + mc^2 \phi, \quad (31)$$

$$\varepsilon \chi = (1 - \beta \frac{\varepsilon^2}{c^2} - \beta \hbar^2 \nabla^2)c\sigma^i \frac{\hbar}{i} \nabla \phi - mc^2 \chi. \quad (32)$$

If we substitute the following ansatz

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix} \exp\left(\frac{i}{\hbar}\mathbf{p}.\mathbf{r}\right) \quad (33)$$

into the equations (31) and (32), we will obtain

$$[\varepsilon(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2) - mc^2]\phi_0 - c(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2)(\sigma^i \mathbf{p})\chi_0 = 0, \quad (34)$$

$$-c(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2)(\sigma^i \mathbf{p})\phi_0 + [\varepsilon(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2) + mc^2]\chi_0 = 0. \quad (35)$$

So we have a linear homogeneous system of equations for $\phi_0$ and $\chi_0$, and it has nontrivial solutions only in the case of a vanishing determinant of the coefficients, that is

$$\begin{vmatrix} [\varepsilon(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2) - mc^2]I & -c(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2)(\sigma^i \mathbf{p}) \\ -c(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2)(\sigma^i \mathbf{p}) & [\varepsilon(1 - \beta \frac{\varepsilon^2}{c^2} + \beta \mathbf{p}^2) + mc^2]I \end{vmatrix} = 0. \quad (36)$$

Using the identity

$$(\sigma^i \mathbf{A})(\sigma^j \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} I + i\sigma^i (\mathbf{A} \times \mathbf{B}), \quad (37)$$
equation (36) transforms into

\[(\varepsilon^2 - c^2 p^2)(1 - \beta \varepsilon^2 + \beta p^2)^2 - m^2 c^4 = 0.\]  

(38)

We observe that for \(\beta \rightarrow 0\), equation (38) leads to the conventional result

\[\varepsilon^2 = c^2 p^2 + m^2 c^4,\]  

(39)

from which follows

\[\varepsilon = \pm E_p, \quad E_p = c\sqrt{p^2 + m^2 c^2},\]  

(40)

as it should be. The two signs of the time evolution factor \(\varepsilon\) in (40) correspond to positive and negative energy solutions of the conventional Dirac equation, respectively. But for the case \(\beta \neq 0\), neglecting terms of order \(\beta^2\) provides us with two sets of results

\[\varepsilon_- = \pm E_p^(-), \quad E_p^(-) = c\sqrt{p^2 + m_-^2 c^2},\]  

(41)

\[\varepsilon_+ = \pm E_p^(+), \quad E_p^(+) = c\sqrt{p^2 + m_+^2 c^2},\]  

(42)

where the non-degenerate effective masses \(m_-\) and \(m_+\) are defined as

\[m_- = \frac{1}{2\sqrt{2}\beta c} \left[ \sqrt{1 + 2\sqrt{2\beta mc} - \sqrt{1 - 2\sqrt{2\beta mc}}} \right],\]  

(43)

\[m_+ = \frac{1}{2\sqrt{2}\beta c} \left[ \sqrt{1 + 2\sqrt{2\beta mc} + \sqrt{1 - 2\sqrt{2\beta mc}}} \right].\]  

(44)

From the standpoint of quantum mechanics, (43) and (44) indicate that our modified spinor field is associated with particles having the effective masses \(m_-\) and \(m_+\). To avoid particles of complex mass, (43) and (44) require that

\[\beta < \frac{1}{8m^2 c^2}.\]  

(45)

It should be noted that at \(\beta = \frac{1}{8m^2 c^2}\) both effective masses are equal, i.e., \(m_- = m_+ = m\sqrt{2}\). From equation (35) we obtain

\[\chi_0 = \frac{c(1 - \beta \varepsilon^2 + \beta p^2)(\boldsymbol{\sigma} \cdot \mathbf{p})}{\varepsilon(1 - \beta \varepsilon^2 + \beta p^2) + mc^2}\phi_0.\]  

(46)
If we denote the two-component spinor \( \phi_0 \) in the form

\[
\phi_0 = U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},
\]

with the normalization condition \( U^\dagger U = U_1^* U_1 + U_2^* U_2 = 1 \) and using (27), (29), (33) and (46), we will obtain two complete sets of positive and negative energy solutions of the modified Dirac equation as

\[
\Psi_{p\lambda}^{(\pm)}(r, t) = N^{(\pm)} \left( \frac{\sqrt{\lambda E^{(\pm)}_p (1 - \beta m^2 c^2) + mc^2}}{\lambda E^{(\pm)}_p (1 - \beta m^2 c^2) + mc^2 + c^2 p^2 (1 - \beta m^2 c^2)} \right) \exp \left( i \frac{\hbar}{\hbar} (p.r - \lambda E^{(\pm)}_p t) \right). \quad (48)
\]

Here \( \lambda = \pm 1 \) characterizes the positive and negative energy solutions with the time evolution factors \( \varepsilon^{\pm} = \lambda E^{(\pm)}_p \). The normalization factors \( N^{(\pm)} \) in (48) are determined from the conditions

\[
\int \Psi_{p\lambda}^{(\pm)*}(r, t) \Psi_{p'\lambda'}^{(\pm)}(r, t) d^3r = \delta_{\lambda\lambda'} \delta(p - p'). \quad (49)
\]

Using equations (48) and (49) together with identity (37), the normalization factors \( N^{(\pm)} \) will be determined as

\[
N^{(\pm)} = \left\{ \frac{\left[ \lambda E^{(\pm)}_p (1 - \beta m^2 c^2) + mc^2 \right]^2}{\left[ \lambda E^{(\pm)}_p (1 - \beta m^2 c^2) + mc^2 + c^2 p^2 (1 - \beta m^2 c^2) \right]^2 + c^2 p^2 (1 - \beta m^2 c^2)} \right\}^{\frac{1}{2}}. \quad (50)
\]

If we expand the mass parameter \( m_- \) in (43) to first order in \( \beta \) we will obtain

\[
m_- = m + \beta m^3 c^2. \quad (51)
\]

Inserting (51) into (41) we find the following relation

\[
E^{(-)^2}_p = m^2 c^4 + c^2 p.p + 2 \beta m^4 c^6, \quad (52)
\]

which is a modification of Einstein relation for a free particle in special relativity. After simplification, the generalized Dirac spinor \( \Psi_{p\lambda}^{(-)}(r, t) \) in (48) to first order in \( \beta \) can be written as

\[
\Psi_{p\lambda}^{(-)}(r, t) = \frac{1}{(2\pi \hbar)^{\frac{3}{2}}} \sqrt{\frac{\lambda E^{(-)}_p (1 - \beta m^2 c^2) + mc^2}{2\lambda E^{(-)}_p (1 - \beta m^2 c^2)}} \left( \frac{U}{\lambda E^{(-)}_p (1 - \beta m^2 c^2) + mc^2} \right) \exp \left( i \frac{\hbar}{\hbar} (p.r - \lambda E^{(-)}_p t) \right), \quad (53)
\]
where $E_p^{(-)}$ was given in (52).

It is clear that for $\beta \to 0$ the generalized Dirac spinor $\Psi_{p\lambda}^{(-)}(r, t)$ in (53) will be converted into the conventional Dirac spinor $\Psi_{p\lambda}(r, t)$ for a free particle in relativistic quantum mechanics [23], i.e.,

$$\Psi_{p\lambda}(r, t) = \lim_{\beta \to 0} \Psi_{p\lambda}^{(-)}(r, t) = \frac{1}{(2\pi \hbar)^{\frac{3}{2}}} \sqrt{\frac{\lambda E_p + mc^2}{2\lambda E_p}} \left( \frac{U}{\lambda E_p + mc^2} U \right) \exp\left( \frac{i}{\hbar} (p \cdot r - \lambda E_p t) \right).$$

(54)

For small $\beta$ the effective mass $m_+$ in (44) reduces to

$$m_+ = \frac{1}{\sqrt{2\beta c}} - \frac{m^2}{2} \sqrt{2\beta c},$$

(55)

which diverges for $\beta \to 0$.

Thus we have two massive particles in our theory, one with the usual mass $m$ ($\lim_{\beta \to 0} m_-$) and the other a heavy-mass particle of mass $\frac{m}{\sqrt{2\beta c}} (\lim_{\beta \to 0} m_+)$ which, leads to an indefinite metric in our model. Until all the energies of the system are kept below the production threshold of the $\frac{m}{\sqrt{2\beta c}}$ mass particle, the indefinite metric does not enter and the theory obeys all physical requirements such as unitarity. The generalized Dirac spinor $\Psi_{p\lambda}^{(+)}(r, t)$ in (48) which describes a particle with effective mass $m_+$ is entirely new and does not have a counterpart in the conventional Dirac equation. The essential feature that is responsible for the existence of the generalized Dirac spinor $\Psi_{p\lambda}^{(+)}(r, t)$ is that the modified Dirac equation (22) is third order in space-time derivatives in the presence of a minimal length, whereas the ordinary Dirac equation is only first order in space-time derivatives.

Now, let us estimate the numerical value of the minimal length in our work. By putting $\beta' = 2\beta$ into (6) and neglecting terms of order $\beta^2$, the isotropic minimal length becomes $(\Delta X^i)_0 \simeq \hbar \sqrt{5\beta}$. The upper bound for deformation parameter $\beta$ ($\beta_{\text{upper-bound}} \simeq 1 \times 10^{-31}$) together with isotropic minimal length $\hbar \sqrt{5\beta_{\text{upper-bound}}}$ for an electron ($m_e = 9.11 \times 10^{-31}$ kg) are respectively $\beta_{\text{upper-bound}} \simeq 1.67 \times 10^{-42}$ kg$^2$m$^2$ and $(\Delta X^i)_0 \simeq 3 \times 10^{-13}$ m.

5 Conclusions

Heisenberg in 1938 wrote an article about the significance of a minimal length in physics [24]. He believed that every theory of elementary particles must contain a minimal length besides two fundamental constants, $c$ and $\hbar$. The hope was that the introduction of such a minimal length would eliminate the ultraviolet divergencies from quantum field theory. This minimal length scale leads to a GUP. An immediate consequence of the GUP is a generalization of momentum operator according to (13) for $\beta' = 2\beta$. This generalized form of momentum
operator leads to a modified Dirac equation. We have shown that our modified Dirac equation which contains higher order derivative of the wave function describes two massive particles, one particle with the effective mass $m_-$ and the other a very heavy particle with the effective mass $m_+$ according to (43) and (44). From (43) and (44) the restriction on the deformation parameter $\beta$ becomes $\beta < \frac{1}{8m^2c^2}$. This restriction leads to an isotropic minimal length $(\Delta X^i)_{\theta} \simeq \frac{\sqrt{\hbar} \bar{\lambda}_c}{mc}$. In [13-18] considering the Lamb shift the authors estimated $(\Delta X^i)_{\theta} \leq 10^{-16} - 10^{-17} m$, analysis of electron motion in a Penning trap also gives $(\Delta X^i)_{\theta} \leq 10^{-16} m$ [19]. The obtained value for upper bound of the isotropic minimal length in this work is $(\Delta X^i)_{\theta} \simeq 3 \times 10^{-13} m$. Although the above value for the isotropic minimal length is about 2 orders of magnitude larger than that was proposed by Heisenberg ($10^{-15} m$) [25], this value is near to the reduced Compton wavelength of the electron ($\lambda_c = \frac{\hbar}{mc} = 3.86 \times 10^{-13} m$).

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