Symplectic Method for the Thin Piezoelectric Plates

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Abstract: The symplectic method for a thin piezoelectric plate problem is developed. The Hamiltonian canonical equation of thin piezoelectric plate is given by using the variational principle. By applying the separation of variables method, we can obtain symplectic orthogonal eigensolutions. As an application, the problem of a thin piezoelectric plate with full edges simply supported under a uniformly distributed load is discussed, and analytical solutions of the deflection and potential of a piezoelectric thin plate are obtained. A numerical example shows that the solutions converge very rapidly. The advantage of this method is that it does not need to assume the predetermined function in advance, so it has better universality. It may also be applied to the problem of thin piezoelectric plate buckling and vibrating.

Keywords: symplectic method; Hamilton canonical equations; thin piezoelectric plate; analytical solutions

1. Introduction

Recent years have seen an increase in the use of piezoelectric materials in structural shape control, vibration and noise control, shape detection, and other fields due to their good electromechanical coupling characteristics [1–5]. At present, there are many models to study the piezoelectric effect. Dunn and Wienecke [6] applied Green’s function to transversely isotropic piezoelectric solids. Hill [7] applied the boundary element method to the problems of 3D piezoelectricity. Ding et al. [8] obtained some exact solutions for the 3D piezoelectric problem. The above research are based on the Lagrange system.

The symplectic method for solving the elastic theory was proposed by Zhong [9,10]. The characteristic of this method is that it does not assume the trial function in advance but introduces the problem into the Hamiltonian system and uses the separated variable method to solve the elastic problem under specific boundary conditions. This method differs from the traditional semi-inverse method, it does not require a trial function, so it is more universal. The symplectic method can easily be adapted to various thin plate problems, thick plate problems, or a combination of vibration and buckling. Therefore, many researchers use the symplectic method to study problems in mechanics and engineering science. The 2D and 3D symplectic structures of transversely isotropic piezoelectric media have been derived by Gu [11–14]. Xu et al. [15] studied the problem of interfacial cracks in non-ideal interfacial magnetoelctroelastic bimaterials. Some exact solutions for the bending and vibration of rectangular plates were also obtained by Li and Zhong [16–18]. In a study by Zhang and Deng et al. [19], non-local Timoshenko beams with gradational materials were studied for free vibration. Xu and Deng et al. [20] analyzed the free vibration and buckling of natural-fiber-reinforced composite plates with partial or internal cracks supported by corners. Leung and Mao [21] presented the symplectic Galerkin method for non-linear vibrations of beams and plates. Jia et al. [22] calculated the free vibration of orthotropic and isotropic cylindrical shells whose thicknesses were both uniform and stepped under general boundary conditions.
The above reflects the wide application of the symplectic method in various fields. Therefore, in the present study we have further developed the symplectic method for thin piezoelectric plates in the Hamiltonian system. The basic equation of the thin piezoelectric plate is transformed into a Hamiltonian canonical equation. By applying the separation of variables method, we can obtain symplectic orthogonal eigensolutions. In this paper, the analytical solutions for a thin piezoelectric plate with full edges simply supported are obtained.

2. Hamilton Canonical Equations

Figure 1 shows the coordinate system of a thin piezoelectric plate subjected to a uniformly distributed load $q$ and applied voltage $V$, where $0 \leq x \leq a, 0 \leq y \leq b$ and $h$ is the thickness of the plate.

![Figure 1. Coordinates, load, and applied voltage of thin piezoelectric plate.](image)

The constitutive equations of a transverse-isotropic piezoelectric plate are as follows [23]

$$
\begin{align*}
\sigma_{xx} &= C_{11} \varepsilon_{xx} + C_{12} \varepsilon_{yy} + C_{13} \varepsilon_{zz} - e_{31} E_z, \\
\sigma_{yy} &= C_{12} \varepsilon_{xx} + C_{11} \varepsilon_{yy} + C_{13} \varepsilon_{zz} - e_{31} E_z, \\
\sigma_{zz} &= C_{13} \varepsilon_{xx} + C_{12} \varepsilon_{yy} + C_{33} \varepsilon_{zz} - e_{31} E_z, \\
\sigma_{xy} &= 2C_{66} \varepsilon_{xy}, \\
\sigma_{xz} &= 2C_{44} \varepsilon_{xz} - e_{15} E_x, \\
\sigma_{yz} &= 2C_{44} \varepsilon_{yz} - e_{15} E_y, \\
D_x &= 2e_{15} \varepsilon_{xz} + \kappa_{11} E_x, \\
D_y &= 2e_{15} \varepsilon_{yz} + \kappa_{11} E_y, \\
D_z &= e_{31} (\varepsilon_{xx} + \varepsilon_{yy}) + e_{33} \varepsilon_{zz} + \kappa_{33} E_z.
\end{align*}
$$

The strain-displacement relations are given as below:

$$
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \\
\varepsilon_{xy} &= \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right),
\end{align*}
$$

(1)
and the equilibrium equations (if the body forces and bulk charge density are neglected) can be written as:

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0, \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0, \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0,
\end{align*}
\]

(3)

\[
\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0,
\]

(4)
in which \( C_{66} = \frac{(C_{11} - C_{12})}{2} \). \( \sigma_{ij}, \varepsilon_{ij}, C_{ij} \) are the stress, strain, and elastic constants, respectively. \( D_i, E_i, \epsilon_{ij}, \kappa_{ij} \) respectively represent the electric displacements, the electric field intensity, and the piezoelectric and the dielectric constants.

According to Kirchhoff’s hypothesis [24], i.e., \( \frac{\partial w}{\partial z} = 0, \varepsilon_{xz} = 0, \varepsilon_{yz} = 0 \), the displacement components \( u, v \) and \( w \) in the \( x, y \) and \( z \) directions are described as follows:

\[
\begin{align*}
\varepsilon_{xx} &= -z \frac{\partial^2 w(x,y)}{\partial x^2}, \\
\varepsilon_{yy} &= -z \frac{\partial^2 w(x,y)}{\partial y^2}, \\
\varepsilon_{xy} &= -z \frac{\partial^2 w(x,y)}{\partial x \partial y}.
\end{align*}
\]

(6)

Substituting Equation (5) into the Equation (2), we have:

\[
\begin{align*}
\varepsilon_{xx} &= -z \frac{\partial^2 w(x,y)}{\partial x^2}, \\
\varepsilon_{yy} &= -z \frac{\partial^2 w(x,y)}{\partial y^2}, \\
\varepsilon_{xy} &= -z \frac{\partial^2 w(x,y)}{\partial x \partial y}.
\end{align*}
\]

(6)

Given that the thin piezoelectric plate has a large in-plane size to thickness ratio and it was subjected to the potential across its thickness, the electric field component in the length direction can be ignored [25,26]. Therefore, it is assumed that the electric field exists only along the \( z \) direction and the electric field intensity vector can be expressed as follows:

\[
E_z = -\frac{\partial \phi}{\partial z},
\]

(7)

where \( \phi \) is electric potential and \( D_x \) and \( D_y \) are approximately assumed to be zero. Thus, the electric boundary conditions are as follows:

\[
\phi(h/2) = V, \quad \phi(-h/2) = 0.
\]

(8)

Hence, the stress and electric potential displacement of the thin piezoelectric plate can be deduced from Equation (1):

\[
\begin{align*}
\sigma_{xx} &= C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} - e_{31}E_z, \\
\sigma_{yy} &= C_{12}\varepsilon_{xx} + C_{11}\varepsilon_{yy} - e_{31}E_z, \\
\sigma_{xy} &= 2C_{66}\varepsilon_{xy}, \\
D_z &= e_{31}(\varepsilon_{xx} + \varepsilon_{yy}) + \kappa_{33}E_z.
\end{align*}
\]

(9)

According to Equations (4) and (7)–(9), the expressions for the electric potential and electric field can be obtained as:

\[
\begin{align*}
\phi &= -\frac{c_{31}}{2\varepsilon_{33}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \left( z^2 - \frac{h^2}{4} \right) + \frac{V}{\varepsilon_\pi} z + \frac{V}{\varepsilon_\pi}, \\
E_z &= \frac{c_{31}}{\varepsilon_{33}} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) z - \frac{V}{\varepsilon_\pi}.
\end{align*}
\]

(10)
According to the principle of virtual work, the expressions of the axial force and bending moments can be obtained as [26]:

\[
\begin{align*}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= 0, \\
\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} &= 0, \\
\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_x &= 0, \\
\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xx}}{\partial x} - Q_y &= 0, \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_{xx} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{yy} \frac{\partial w}{\partial y} + N_{yx} \frac{\partial w}{\partial x} \right) + q &= 0,
\end{align*}
\]

(11)

where \( M_{ij}, Q_i \) are the bending moments and shear forces, respectively. \( N_{ij} \) are the internal forces (axial forces) caused by the electrical coupling effect. The bending moments, shear forces, and axial forces are related to stresses by:

\[
M_{ij} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{ij} zdz, \quad Q_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{iz} dz, \quad N_{ij} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_{ij} dz.
\]

Then, we have

\[
\begin{align*}
M_{xx} &= -D_{11} \frac{\partial^4 w}{\partial x^4} - D_{12} \frac{\partial^4 w}{\partial x^2 \partial y^2} - D_{66} \frac{\partial^2 w}{\partial x \partial y}, \\
M_{yy} &= -D_{12} \frac{\partial^4 w}{\partial x^2 \partial y^2} - D_{11} \frac{\partial^4 w}{\partial y^4}, \\
M_{xy} &= -2D_{66} \frac{\partial^2 w}{\partial x \partial y}, \\
N_{xx} &= N_{yy} = e_{31} V, \\
N_{yx} &= N_{xy} = e_{31} V.
\end{align*}
\]

(12)

where

\[ D_{11} = \frac{h^3}{12} \left( C_{11} + \frac{e_{31}^2}{k_{33}} \right), \quad D_{12} = \frac{h^3}{12} \left( C_{12} + \frac{e_{31}^2}{k_{33}} \right), \quad D_{66} = \frac{h^3}{12} C_{66}. \]

By Equations (11) and (12), the bending equation of the piezoelectric thin plate is as follows:

\[
-D_{11} \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) - 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + e_{31} V \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + q = 0.
\]

(13)

The plate functional is [27]:

\[
\Pi = \iint_{\Omega} \left[ -M_{xx} \frac{\partial^4 w}{\partial x^4} - 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} - M_{yy} \frac{\partial^4 w}{\partial y^4} - DE + N_{xx} \left( \frac{\partial w}{\partial x} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + N_{yy} \left( \frac{\partial w}{\partial y} \right)^2 - qw \right] dxdy
\]

(14)

where \( DE = \int_{-\frac{b}{2}}^{\frac{b}{2}} D_{xx} E_x dz. \)

Set the derivative of \( w \) over \( y \) be \( \theta \)

\[
\frac{\partial w}{\partial y} = \tilde{w} = \theta
\]

(15)

where the dot indicates \( \partial / \partial y \). In combination with Equation (12) and introducing the Lagrange multiplier by \( V_y \), Equation (14) is modified as follows:

\[
\Pi' = \iint_{\Omega} \left[ \frac{M_{xx}}{2D_{11}} - M_{yy} \theta - \frac{D_{12}}{2D_{11}} M_{xy} \frac{\partial^2 w}{\partial x^2} + \frac{D_{12} - D_{66}}{2D_{11}} \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 + 2D_{66} \left( \frac{\partial w}{\partial x} \right)^2 + V_y (\tilde{w} - \theta) + e_{31} V \left( \frac{\partial w}{\partial x} \right)^2 - qw \right] dxdy.
\]

(16)
The variation $\delta \Pi' = 0$ results in
\[
\frac{\partial w}{\partial y} = \theta, \\
\frac{\partial \theta}{\partial y} = \frac{-D_{12}}{D_{11}} \frac{\partial^2 w}{\partial x^2} - \frac{1}{D_{11}} M_{yy}, \\
\frac{\partial V_y}{\partial y} + e_{31} V_y \frac{\partial \theta}{\partial y} = \frac{D_{12}^2 - D_{11}^2}{D_{11}} \frac{\partial^4 w}{\partial x^4} + e_{31} V \frac{\partial^2 w}{\partial x^2} - \frac{D_{12}}{D_{11}} \frac{\partial^2 M_{yy}}{\partial x^2} - q, \\
\frac{\partial M_{yy}}{\partial y} = 4D_{66} \frac{\partial^4 \theta}{\partial x^4} + V_y.
\] (17)

Set $T = -V_y - e_{31} V \theta$, by Equation (17), we have:
\[
\frac{\partial T}{\partial y} = \frac{D_{12}^2 - D_{11}^2}{D_{11}} \frac{\partial^4 w}{\partial x^4} + e_{31} V \frac{\partial^2 w}{\partial x^2} + \frac{D_{12}}{D_{11}} \frac{\partial^2 M_{yy}}{\partial x^2} + q, \\
\frac{\partial M_{yy}}{\partial y} = 4D_{66} \frac{\partial^4 \theta}{\partial x^4} - e_{31} V \theta - T.
\]

Then, Equation (17) can be expressed in the following matrix form:
\[
\dot{Z} = HZ + f
\] (18)
where
\[
H = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{-D_{12}}{D_{11}} \frac{\partial^2}{\partial x^2} & 0 & 0 & -\frac{1}{D_{11}} \\
\frac{D_{12}^2 - D_{11}^2}{D_{11}} \frac{\partial^4}{\partial x^4} + e_{31} V \frac{\partial^2}{\partial x^2} & 0 & 0 & \frac{D_{12}}{D_{11}} \frac{\partial^2}{\partial x^2} \\
0 & 4D_{66} \frac{\partial^4}{\partial x^4} - e_{31} V & -1 & 0
\end{pmatrix},
Z = \begin{pmatrix} w \\ \theta \\ T \\ M_{yy} \end{pmatrix},
f = \begin{pmatrix} 0 \\ 0 \\ q \\ 0 \end{pmatrix}.
\]

It is obvious that $H^T = JHJ$, where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, is the symplectic matrix and $I_2$ is the $2 \times 2$ unit matrix. Therefore, $H$ is a Hamiltonian operator matrix, and Equation (18) is the Hamilton canonical equations for the thin piezoelectric plate.

3. Symplectic Analysis

The homogeneous equation of Equation (18) is [9]:
\[
\dot{Z} = HZ.
\] (19)

In order to separate the variables in Equation (19), we let:
\[
Z = X(x) Y(y),
\] (20)
where $X(x) = [w(x), \theta(x), T(x), M_{yy}(x)]^T$. Substituting Equation (20) into Equation (19) gives
\[
\frac{dY(y)}{dy} = \mu Y(y),
\] (21)
\[
HX(x) = \mu X(x),
\] (22)
where $\mu$ is the eigenvalue and $X(x)$ is the corresponding eigenvector.
Equation (22) has the following characteristic equation:

\[
\begin{vmatrix}
-\mu & 1 & 0 & 0 \\
-\frac{D_{12}}{D_{11}}\lambda^2 & -\mu & 0 & -\frac{1}{D_{11}} \\
\frac{D_{12}^2-D_{11}^2}{D_{11}}\lambda^4 + e_{31}V\lambda^2 & 0 & -\mu & \frac{D_{12}^2\lambda^2}{D_{11}} \\
0 & 4D_{66}\lambda^2 - e_{31}V & -1 & -\mu \\
\end{vmatrix} = 0,
\]

(23)

the eigenvalue equation is derived by extending the determinant

\[D_{11}\lambda^4 + (2\mu^2D_{11} - e_{31}V)\lambda^2 + D_{11}\mu^4 - e_{31}V\mu^2 = 0,
\]

(24)

which has roots:

\[\lambda_{1,2} = \pm \mu_1 i, \lambda_{3,4} = \pm \mu_2 i,
\]

where \(\mu_1 = \sqrt{\mu_2^2 - \frac{e_{31}V}{D_{11}}}\). The eigenvector \(X(x)\) of \(H\) satisfies:

\[
X(x) = \begin{pmatrix}
A_1 \cos(\mu_1 x) + B_1 \sin(\mu_1 x) + F_1 \cos(\mu_2 x) + G_1 \sin(\mu_2 x) \\
A_2 \cos(\mu_1 x) + B_2 \sin(\mu_1 x) + F_2 \cos(\mu_2 x) + G_2 \sin(\mu_2 x) \\
A_3 \cos(\mu_1 x) + B_3 \sin(\mu_1 x) + F_3 \cos(\mu_2 x) + G_3 \sin(\mu_2 x) \\
A_4 \cos(\mu_1 x) + B_4 \sin(\mu_1 x) + F_4 \cos(\mu_2 x) + G_4 \sin(\mu_2 x)
\end{pmatrix},
\]

(25)

where \(A_i, B_i, F_i, G_i (i = 1, 2, 3, 4)\) are undetermined constants, and these constants are usually not independent. Combined with the boundary conditions, the above constants can be determined, then the analytical solutions can be obtained.

4. Symplectic Solutions of a Thin Piezoelectric Plate with Full Edges Simply Supported

In order to illustrate the application of the symplectic method in the thin piezoelectric plate problem, consider a thin piezoelectric plate whose full edges are simply supported as is shown in Figure 2.

![Figure 2. A thin piezoelectric plate with full edges simply supported.](image-url)
The full edges simply supported satisfies the boundary conditions:

\[ w|_{x=0,a} = 0, \quad M_{xx}|_{x=0,a} = 0. \quad (26) \]
\[ w|_{y=\pm \frac{b}{2}} = 0, \quad M_{yy}|_{y=\pm \frac{b}{2}} = 0. \quad (27) \]

### 4.1. Eigensolutions of a Thin Piezoelectric Plate

The substitution of general solution (25) into Equation (22) gives the following relations:

\[
A_2 = \mu_2^2 A_1, \quad A_3 = 2D_{66}\mu_2^2 A_1, \quad A_4 = (D_{12}\mu_1^2 - D_{11}\mu_2^2) A_1, \\
B_2 = \mu_2 B_1, \quad B_3 = 2D_{66}\mu_2^2 B_1, \quad B_4 = (D_{12}\mu_1^2 - D_{11}\mu_2^2) B_1, \\
F_2 = \mu_2 F_1, \quad F_3 = (2D_{66}\mu_2^2 + e_{31}V\mu_2) F_1, \quad F_4 = -2D_{66}\mu_2^2 F_1, \\
G_2 = \mu_2 G_1, \quad G_3 = (2D_{66}\mu_2^2 + e_{31}V\mu_2) G_1, \quad G_4 = -2D_{66}\mu_2^2 G_1. \quad (28)
\]

After substituting Equations (25) and (28) into Equation (26), we have \( A_i = F_i = 0 \) \((i = 1, 2, 3, 4)\). Then, when the coefficient determinant is zero, we get the following transcendental equation:

\[
\sin(a\mu_{n1}) \sin(a\mu_{n2}) = 0. \quad (29)
\]

Solve the equation above, we have:

\[
\mu_{n1} = \frac{n\pi}{a}, \quad \mu_{-n1} = -\frac{n\pi}{a}, \quad \mu_{n2} = \frac{n\pi}{a}, \quad \mu_{-n2} = -\frac{n\pi}{a}.
\]

When \( \mu_{n1} = \pm \frac{n\pi}{a} \), then \( B_1 = 1, \ G_1 = 0 \). By Equations (25) and (28), the corresponding basic eigenvectors of \( \mu_{n1} \) and \( \mu_{-n1} \) are:

\[
X_{n1}(x) = \sin(\mu_{n1}x) \begin{pmatrix} 1 \\ \mu_{n2} \\ 2D_{66}\mu_2^2 \mu_{n1} \\ D_{12}\mu_1^2 - D_{11}\mu_2^2 \end{pmatrix}, \quad X_{-n1}(x) = \sin(\mu_{n1}x) \begin{pmatrix} -1 \\ \mu_{n2} \\ 2D_{66}\mu_2^2 \mu_{n1} \\ -D_{12}\mu_1^2 + D_{11}\mu_2^2 \end{pmatrix}. \quad (30)
\]

When \( \mu_{n2} = \pm \frac{n\pi}{a} \), then \( B_1 = 0, \ G_1 = 1 \). By Equations (25) and (28), the corresponding basic eigenvectors of \( \mu_{n2} \) and \( \mu_{-n2} \) are:

\[
X_{n2}(x) = \sin(\mu_{n2}x) \begin{pmatrix} 1 \\ \mu_{n2} \\ 2D_{66}\mu_2^3 + e_{31}V\mu_{n2} \\ -2D_{66}\mu_2^2 \end{pmatrix}, \quad X_{-n2}(x) = \sin(\mu_{n2}x) \begin{pmatrix} -1 \\ \mu_{n2} \\ 2D_{66}\mu_2^3 + e_{31}V\mu_{n2} \\ 2D_{66}\mu_2^2 \end{pmatrix}. \quad (31)
\]

Up to here, all eigenvectors of the thin piezoelectric plate with boundary conditions (26) are obtained. These eigenvectors constitute a set of conjugate symplectic orthogonal bases, where \( X_{n1}(x), \ X_{-n1}(x) \) and \( X_{n2}(x), \ X_{-n2}(x) \) are symplectically conjugate, respectively, and the others are symplectically orthogonal.

### 4.2. Expansion of Eigenfunctions

Assume the solution of the inhomogeneous Equation (18) with the expression as:

\[
Z = X(x) \ Y(y), \quad (32)
\]

where

\[
X(x) = [X_{n1}(x), \ X_{n2}(x), \ X_{-n1}(x), \ X_{-n2}(x)], \quad (n = 1, 2, 3, \ldots) \\
Y(y) = [Y_{n1}(x), \ Y_{n2}(x), \ Y_{-n1}(x), \ Y_{-n2}(x)]^T.
\quad (33)
\]
Substituting Equation (32) into Equation (22), we have:

\[ X(x) \frac{dY(y)}{dy} = HX(x)Y(y) + f. \tag{34} \]

By the Equation (22), we find:

\[ HX = XM, \tag{35} \]

where \( M = \text{diag}(\mu_{n1}, \mu_{n2}, -\mu_{n1}, -\mu_{n2}) \), \text{diag} represents the diagonal matrix.

Let,

\[ f = X(x) \, G, \tag{36} \]

where \( G = (g_{n1}, g_{n2}, -g_{n1}, -g_{n2})^T \) is a column matrix for the expansion coefficients. Substituting Equations (35) and (36) into Equation (34) gives:

\[ \frac{dY(y)}{dy} = MY(y) + G, \tag{37} \]

we obtain a group of equations

\[
\begin{align*}
\frac{dY_{n1}(y)}{dy} - \mu_{n1}Y_{n1}(y) &= g_{n1}, \\
\frac{dY_{n2}(y)}{dy} - \mu_{n2}Y_{n2}(y) &= g_{n2}, \\
\frac{dY_{-n1}(y)}{dy} - \mu_{-n1}Y_{-n1}(y) &= g_{-n1}, \\
\frac{dY_{-n2}(y)}{dy} - \mu_{-n2}Y_{-n2}(y) &= g_{-n2}.
\end{align*} \tag{38}\]

The inhomogeneous terms can be expanded by eigenvectors according to adjoint symplectic orthogonality. Multiplying both sides of Equation (37) by \( X(x)^T J dx \) and integrating from 0 to \( a \), we obtain:

\[ \int_0^a X(x)^T J X(x) G dx = \int_0^a X(x)^T [q, 0, 0, 0]^T dx. \tag{39} \]

Expanding Equation (39), we have:

\[
\begin{align*}
g_{n1} &= g_{-n1} = \frac{-q(1 - \cos(n\pi))}{c_{33} n^2 \pi V \xi}, \\
g_{n2} &= g_{-n2} = \frac{q(1 - \cos(n\pi))}{c_{33} n^2 \pi V},
\end{align*} \tag{40}\]

where \( \xi = \sqrt{\frac{n^2 \pi^2}{a^2} + \frac{c_{34} V}{D_{11}}} \).

Substituting Equation (40) into Equation (38) can obtain:

\[
\begin{align*}
Y_{n1}(y) &= C_1 e^{\xi y} + \frac{2q \sin^2(n\pi/2)}{c_{33} n^2 \pi V \xi^2}, \\
Y_{n2}(y) &= C_2 e^{\frac{\pi}{2} y} - \frac{2q \sin^2(n\pi/2)}{c_{33} n^2 \pi V}, \\
Y_{-n1}(y) &= C_3 e^{-\xi y} - \frac{2q \sin^2(n\pi/2)}{c_{33} n^2 \pi V \xi^2}, \\
Y_{-n2}(y) &= C_4 e^{-\frac{\pi}{2} y} + \frac{2q \sin^2(n\pi/2)}{c_{33} n^2 \pi V},
\end{align*} \tag{41}\]

where \( C_1, C_2, C_3, C_4 \) are undetermined constants. These constants can be determined by the boundary conditions of the plate in the \( y \) direction.
4.3. Analytical Solutions of Thin Piezoelectric Plate

By Equations (32) and (33), we have the solution of the state vector $Z$, which is:

$$Z = \sum_{n=1}^{\infty} \left[ Y_{n1}(y) X_{n1}(x) + Y_{n2}(y) X_{n2}(x) + Y_{-n1}(y) X_{-n1}(x) + Y_{-n2}(y) X_{-n2}(x) \right]. \quad (42)$$

The constants $C_1, C_2, C_3, C_4$ can be solved by the boundary conditions (27) as follows:

$$C_1 = -\frac{2q\text{sech}(b\xi/2) \sin^2(n\pi/2)}{\varepsilon_3 n\pi V h^2},$$
$$C_2 = \frac{2q^2\text{sech}(b\pi/2a) \sin^2(n\pi/2)}{\varepsilon_3 n\pi V h^2},$$
$$C_3 = \frac{2q\text{sech}(b\pi/2a) \sin^2(n\pi/2)}{\varepsilon_3 n\pi V h^2},$$
$$C_4 = -\frac{2q^2\text{sech}(b\pi/2a) \sin^2(n\pi/2)}{\varepsilon_3 n\pi V h^2}. \quad (43)$$

Finally, the bending deflection of the thin piezoelectric plate with full edges simply supported under a uniformly distributed load is:

$$w = \frac{qa^2}{e_{31} V} \sum_{n=1}^{\infty} \frac{4}{n^3 \pi^3} \sin^2 \left( \frac{n\pi}{2} \right) \sin \left( \frac{n\pi}{a} x \right) S_n(y), \quad (44)$$

where

$$S_n(y) = 1 - \frac{\cosh(n\pi y/a)}{\cosh(n\pi b/(2a))} - \frac{n^2\pi^2}{\xi^2 a^2} \left( 1 - \frac{\cosh(\xi y)}{\cosh(\xi b/2)} \right). \quad (45)$$

By Equation (10), the electric potential can be expressed as:

$$\phi = \frac{V}{2} + \frac{V}{h} \sum_{n=1}^{\infty} \frac{2h_n(y)}{n\pi \xi^2 a^2} \sin \left( \frac{n\pi}{2} x \right) \sin \left( \frac{n\pi}{a} x \right), \quad (46)$$

where

$$h_n(y) = 1 - \frac{\cosh(\xi y)}{\cosh(\xi b/2)}. \quad (47)$$

When an applied voltage is not considered, Equations (44) and (46) can be rewritten as:

$$w = \frac{qa^2}{D_{11}} \sum_{n=1}^{\infty} \frac{4 \sin^2(n\pi/2)}{n^3 \pi^3} \tilde{S}_n(y) \sin \left( \frac{n\pi}{a} x \right), \quad (48)$$

$$\phi = \frac{qa^2 e_{31}}{D_{11} \kappa_{33}} \sum_{n=1}^{\infty} \frac{2 \sin^2(n\pi/2)}{\pi^3 n^3} \sin \left( \frac{n\pi}{a} x \right) \tilde{h}_n(y), \quad (49)$$

where

$$\tilde{S}_n(y) = 1 - \frac{2 + \frac{b\pi}{2a} \tanh \left( \frac{b\pi}{2a} \right)}{2 \cosh(\frac{b\pi}{2a})} \cosh \left( \frac{n\pi}{a} y \right) + \frac{\frac{b\pi}{2a} y}{2 \cosh(\frac{b\pi}{2a})} \sinh \left( \frac{n\pi}{a} y \right), \quad (50)$$

$$\tilde{h}_n(y) = \left( 1 - \cosh \left( \frac{n\pi}{a} y \right) \right) \text{sech} \left( \frac{b\pi}{2a} \right). \quad (51)$$

Especially, it is worth mentioning that when the piezoelectric material factor is not considered, Equation (48) is exactly the same as the classical plate bending solution [24].
5. Numerical Results

In this section, the numerical results are discussed in detail to validate the effectiveness of the proposed method.

The dimensionless bending deflections \( w \left( qa^4/D_{11} \right) \) at the midpoint \((a/2,0)\) of the classical thin plate are calculated as the special case of the present results, shown in Table 1. A comparison with the known results of ref. [24] displays excellent agreement.

Table 1. The dimensionless bending deflection of midpoint.

| \( a/b \) | \( n \) | \( w \mid_{x=a/2,y=0,V=0}\left( qa^4/D_{11} \right) \) |
|----------|--------|-----------------------------------------------|
| Present  | 1.0    | 0.00410935                                   |
|          | 3      | 0.0040588                                    |
|          | 5      | 0.00406297                                   |
| ref. [24]| 1      | -                                            |
|          |        | 0.004062                                    |

PZT-4 material is selected as an example and the material properties are \( C_{11} = 139 \text{ GPa}, C_{12} = 77.8 \text{ GPa}, \varepsilon_{31} = -6.98 \text{ N/Vm}, \kappa_{33} = 5.47 \times 10^{-9} \text{ F/m} \) [28]. We assume that the length of the thin plate is \( a = 1 \text{ m} \), the width is \( b = 1 \text{ m} \), the thickness is \( h = 20 \text{ mm} \), the uniformly distributed load is \( q = -2 \text{ MPa} \), \( V = 20 \text{ V} \). Table 2 presents \( w \) of thin piezoelectric plates with full edges simply supported \( n \) is the expanded term. The results show that the solutions converge very rapidly. The bending deflection diagram of a thin piezoelectric plate with full edges simply supported under a uniformly distributed load is shown in Figure 3.

Table 2. The bending deflections for a uniformly loaded thin plate.

| \( n \) | \( w \mid_{x=a/2,y=0} \text{(mm)} \) |
|--------|---------------------------------|
| 1      | -8.44573                        |
| 3      | -8.43419                        |
| 5      | -8.43454                        |
| 7      | -8.43450                        |
| 9      | -8.43450                        |
| 11     | -8.43450                        |

6. Conclusions

The symplectic method for the thin piezoelectric plate problem is established in this paper. The analytical solutions are obtained for the bending deflection and the electric potential. The accuracy, validity, and reliability of the proposed symplectic method are verified by comparison studies with respect to the published literature. Compared with the traditional methods, the implementation of the symplectic method is simple. It can also be applied to other problems with piezoelectric materials.
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