### VIRTUAL BILLIARDS IN PSEUDO–EUCLIDEAN SPACES: DISCRETE HAMILTONIAN AND CONTACT INTEGRABILITY

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**Abstract.** The aim of the paper is to unify the efforts in the study of integrable billiards within quadrics in flat and curved spaces and to explore further the interplay of symplectic and contact integrability. As a starting point in this direction, we consider virtual billiard dynamics within quadrics in pseudo–Euclidean spaces. In contrast to the usual billiards, the incoming velocity and the velocity after the billiard reflection can be at opposite sides of the tangent plane at the reflection point. In the symmetric case we prove noncommutative integrability of the system and give a geometrical interpretation of integrals, an analog of the classical Chasles and Poncelet theorems and we show that the virtual billiard dynamics provides a natural framework in the study of billiards within quadrics in projective spaces, in particular of billiards within ellipsoids on the sphere $\mathbb{S}^{n-1}$ and the Lobachevsky space $\mathbb{H}^{n-1}$.

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1. Introduction. It is well known that the billiards within ellipsoids are the only known integrable billiards with smooth boundary in constant curvature spaces \cite{3, 7, 8, 11, 20, 33, 38, 34, 35}. The elliptical billiards in pseudo-Euclidean spaces are also integrable \cite{24, 12}. We will try to present all these integrable models through a unified perspective, within the framework of the virtual billiard dynamic (see \cite{23}).

A pseudo–Euclidean space $\mathbb{E}^{k,l}$ of signature $(k, l)$, $k, l \in \mathbb{N}$, $k + l = n$, is the space $\mathbb{R}^n$ endowed with the scalar product

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i - \sum_{i=k+1}^{n} x_i y_i \quad (x, y \in \mathbb{R}^n).$$

Two vectors $x, y$ are orthogonal, if $\langle x, y \rangle = 0$. A vector $x \in \mathbb{E}^{k,l}$ is called space–like, time–like, light–like, if $\langle x, x \rangle$ is positive, negative, or $x$ is orthogonal to itself, respectively. Denote by $(\cdot, \cdot)$ the Euclidean inner product in $\mathbb{R}^n$ and let

$$E = \text{diag}(\tau_1, \ldots, \tau_n) = \text{diag}(1, \ldots, 1, -1, \ldots, -1),$$

where $k$ diagonal elements are equal to $1$ and $l$ to $-1$. Then $\langle x, y \rangle = (Ex, y)$, for all $x, y \in \mathbb{R}^n$.

We consider a $n-1$–dimensional quadric

$$Q^{n-1} = \{ x \in \mathbb{E}^{k,l} | (A^{-1}x, x) = 1 \},$$

where

$$A = \text{diag}(a_1, \ldots, a_n), \quad a_i \neq 0, \quad i = 1, \ldots, n.$$

A point $x \in Q^{n-1}$ is singular, if a normal $EA^{-1}x$ at $x \in Q^{n-1}$ is light–like: $(EA^{-2}x, x) = 0$, or equivalently, the induced metric is degenerate at $x$.

In the case that $A$ is positive definite, following Khesin and Tabachnikov \cite{24} and Dragović and Radnović \cite{12}, we define a billiard flow inside the ellipsoid \cite{11} in $\mathbb{E}^{k,l}$ as follows. Between the impacts, the motion is uniform along the straight lines. If $x \in Q^{n-1}$ is non–singular, then the normal $EA^{-1}x$ is transverse to $T_xQ^{n-1}$ and the incoming velocity vector $w$ can be decomposed as $w = t + n$, where $t$ is its tangential and $n$ the normal component in $x$. The velocity vector after reflection is $w_1 = t - n$. If $x \in Q^{n-1}$ is singular, the flow stops.

Let $\phi : (x_j, y_j) \mapsto (x_{j+1}, y_{j+1})$ be the billiard mapping, where $x_j \in Q^{n-1}$ is a sequence of non–singular impact points and $y_j$ is the corresponding sequence of outgoing velocities (in the notation we follow \cite{38, 36, 16}, which slightly differs from the one given in \cite{30}, where $y_j$ is the incoming velocity). As in the Euclidean case (see \cite{38, 36, 16}), the billiard mapping $\phi$ is given by:

$$x_{j+1} = x_j + \mu_j y_j,$$

$$y_{j+1} = y_j + \nu_j EA^{-1}x_{j+1},$$

where the multipliers

$$\mu_j = -2 \frac{(A^{-1}x_j, y_j)}{(A^{-1}y_j, y_j)}, \quad \nu_j = 2 \frac{(A^{-1}x_{j+1}, y_{j+1})}{(EA^{-2}x_{j+1}, x_{j+1})}.$$
are determined from the conditions
\[(A^{-1}x_{j+1}, x_{j+1}) = (A^{-1}x_j, x_j) = 1, \quad \langle y_{j+1}, y_{j+1} \rangle = \langle y_j, y_j \rangle.\]

From the definition, the Hamiltonian \(H = \frac{1}{2} \langle y_j, y_j \rangle\) is an invariant of the mapping \(\phi\). Therefore, the lines \(l_k = \{x_k + sy_k \mid s \in \mathbb{R}\}\) containing segments \(x_kx_{k+1}\) of a given billiard trajectory are of the same type: they are all either space–like (\(H > 0\)), time–like (\(H < 0\)) or light–like (\(H = 0\)). Also, the function \(J_j = (A^{-1}x_j, y_j)\) is an invariant of the billiard mapping (see Lemma 3.1 in [23]).

Note that the billiard mapping (3), (4) is well defined for arbitrary quadric \(Q^{n-1}\) given by (1) and not only for ellipsoids. In that case, the outgoing velocity (directed from \(x_k\) to \(x_{k+1}\)) is either \(y_k\) or \(-y_k\), while the segments \(x_{k-1}x_k\) and \(x_kx_{k+1}\) determined by 3 successive points of the mapping (3), (4) may be:

(i) on the same side of the tangent plane \(T_{x_k}Q^{n-1}\);
(ii) on the opposite sides of the tangent plane \(T_{x_k}Q^{n-1}\).

![Figure 1. A segment of a virtual billiard trajectory within hyperbola \((a_1 > 0, a_2 < 0)\) in the Euclidean space \(E^{2,0}\). The caustic is an ellipse.](image)

In the case (i) we have a part of the usual pseudo–Euclidean billiard trajectory, while in the case (ii) the billiard reflection corresponds to the points \(x_{k-1}x_kx'_k\), where \(x'_k\) is the symmetric image of \(x_k\) with respect to \(x_k\). In the three-dimensional Euclidean case, Darboux referred to such reflection as the virtual reflection (e.g., see [9] and [11], Ch. 5). In Euclidean spaces of arbitrary dimension, such configurations were introduced by Dragović and Radnović in [9]. It appears that a multidimensional variant of Darboux’s 4–periodic virtual trajectory with reflections on two quadrics, refereed as double–reflection configuration [11], is fundamental in the construction of the double reflection nets in Euclidean spaces (see [13]) and in pseudo-Euclidean spaces (see [14]). They also played a role in a construction of the billiard algebra in [10]. The 4–periodic orbits of real and complex planar billiards with virtual reflections are also studied in [15].

**Definition 1.1.** [23] Let \(Q^{n-1}\) be a quadric in the pseudo–Euclidean space \(E^{k,l}\) defined by (1). We refer to (3), (4) as the virtual billiard mapping, and to the sequence of points \(x_k\) determined by (3), (4) as the virtual billiard trajectory within \(Q^{n-1}\).
The system is defined outside the singular set

\[ \Sigma = \{ (x, y) \in \mathbb{R}^n \mid (EA^{-2}x, x) = 0 \lor (A^{-1}x, y) = 0 \lor (A^{-1}y, y) = 0 \} \] (5)

and it is invariant under the action of a discrete group \( \mathbb{Z}_2^n \) generated by the reflections

\[ (x_i, y_i) \mapsto (-x_i, -y_i), \quad i = 1, \ldots, n. \] (6)

We can interpret (5), (11) in the case of non–light–like billiard trajectories as the equations of a discrete dynamical system (see [36, 30, 38]) on \( \mathbb{Q}^{n-1} \) described by the discrete action functional:

\[ S[x] = \sum_{k} \mathbf{L}(x_k, x_{k+1}), \quad \mathbf{L}(x_k, x_{k+1}) = \sqrt{|(x_{k+1} - x_k, x_{k+1} - x_k)|}, \]

where \( x = (x_k), k \in \mathbb{Z} \) is a sequence of points on \( \mathbb{Q}^{n-1} \). Note that the virtual billiard dynamics on \( \mathbb{Q}^{n-1} \) can have both virtual and real reflections.

Motivated by the Lax representation for elliptical billiards with the Hooke’s potential (Fedorov [16], see also [20, 32]), we proved in [23] that the trajectories \((x_j, y_j)\) of (3, 4) outside the singular set (5) satisfy the matrix equation

\[ \mathcal{L}_{x_{j+1}, y_{j+1}}(\lambda) = \mathcal{A}_{x_j, y_j}(\lambda) \mathcal{L}_{x_j, y_j}(\lambda) \mathcal{A}_{x_{j+1}, y_{j+1}}^{-1}(\lambda), \] (7)

with 2 × 2 matrices depending on the parameter \( \lambda \)

\[ \mathcal{L}_{x_j, y_j}(\lambda) = \begin{pmatrix} q_\lambda(x_j, y_j) & q_\lambda(y_j, y_j) \\ -1 - q_\lambda(x_j, x_j) & -q_\lambda(x_j, y_j) \end{pmatrix}, \]

\[ \mathcal{A}_{x_j, y_j}(\lambda) = \begin{pmatrix} I_j \lambda + 2J_j \nu_j & -J_j \nu_j \\ -J_j \lambda & I_j \lambda \end{pmatrix}, \]

where \( q_\lambda \) is given by

\[ q_\lambda(x, y) = ((\lambda E - A)^{-1} x, y) = \sum_{i=1}^{k} \frac{x_i y_i}{\lambda - a_i} - \sum_{i=k+1}^{n} \frac{x_i y_i}{\lambda + a_i}, \] (8)

and

\[ J_j = (A^{-1}x_j, y_j), \quad I_j = - (A^{-1}y_j, y_j), \quad \nu_j = 2J_j/(EA^{-2}x_{j+1}, x_{j+1}). \] (9)

For a non–symmetric case \((\tau_i a_i \neq \tau_j a_j)\) the matrix representation is equivalent to the system up to the \( \mathbb{Z}_2^n \)–action [36]. Further, from the expression

\[ \det \mathcal{L}_{x, y}(\lambda) = q_\lambda(y, y)(1 + q_\lambda(x, x)) - q_\lambda(x, y)^2 = \sum_{i=1}^{n} f_i(x, y), \] (10)

one can derive the integrals \( f_i \) in the form

\[ f_i(x, y) = \tau_i y_i^2 + \sum_{j \neq i} \frac{(x_j y_i - x_i y_j)^2}{\tau_j a_i - \tau_i a_j} \quad (i = 1, \ldots, n). \] (11)

Outline and results of the paper. In Section 2 we describe discrete symplectic (Theorem 2.1) and contact integrability in the light–like case (Theorem 2.2) of the virtual billiard dynamics directly, by the use of the Dirac–Poisson bracket. This is slightly different from the construction within the framework of the symplectic reduction given by Khesin and Tabachnikov [24, 25].

In the symmetric case, when \( a_i \tau_i = a_j \tau_j \) for some indexes \( i, j \), we further develop the analysis from [23] of geodesic flows on \( \mathbb{Q}^{n-1} \) and elliptical billiards. We prove noncommutative integrability of the system (Theorem 3.2, Section 3) and, by a subtle estimate of the number of real zeros in the spectral parameter \( \lambda \) of the rational
function \( \det \mathcal{L}_{x,y}(\lambda) \), give a geometrical interpretation of integrals - an analog of the classical Chasles and Poncelet theorems for symmetric quadrics (Theorems 4.2 – 4.6 Section 4). The Poncelet theorem is based on a noncommutative variant of the description of Liouville integrable symplectic correspondences given by Veselov [38, 39] (Theorem 3.1, Section 3).

Further, in Section 5 we show that the virtual billiard dynamics provides a natural framework in the study of billiards within quadrics in projective spaces, in particular the billiards within ellipsoids on the sphere \( \mathbb{S}^{n-1} \) and the Lobachevsky space \( \mathbb{H}^{n-1} \). It is well known that the ellipsoidal billiards on \( \mathbb{S}^{n-1} \) and \( \mathbb{H}^{n-1} \) are completely integrable [7, 37, 34, 8]. The “big” \( n \times n \)-matrix representation of the ellipsoidal \( \mathbb{H}^{n-1} \)-billiard, together with the integration of the flow is obtained in [37]. In this paper we provide a “small” \( 2 \times 2 \)-matrix representation (Theorem 5.2), a modification of [7], as well as the Chasles theorem (Theorem 5.4).

2. Symplectic and contact properties of the virtual billiard dynamics.

2.1. Hamiltonian description. In the pseudo-Euclidean case it is convenient to use the following symplectic form on \( \mathbb{R}^{2n} = T\mathbb{E}^{k,l}(x, y) \) (see [24]):

\[
\omega = Edy \wedge dx = \sum_{i=1}^{k} dy_i \wedge dx_i - \sum_{i=k+1}^{n} dy_i \wedge dx_i,
\]

obtained after identification \( T^*\mathbb{E}^{k,l}(x, p) \cong T\mathbb{E}^{k,l}(x, y) \) using the scalar product \( \langle \cdot, \cdot \rangle \). The corresponding Poisson bracket is

\[
\{f, g\} = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \sum_{i=k+1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \sum_{i=1}^{k} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} + \sum_{i=k+1}^{n} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i},
\]

(12)

Consider a \((2n - 2)\)-dimensional submanifold \( M_h \) of \( \mathbb{R}^{2n} \) defined by

\[
M_h = \{ (x, y) \in \mathbb{R}^{2n} \setminus \Sigma \mid \phi_1 = (A^{-1} x, x) = 1, \ \phi_2 = 2H = \langle y, y \rangle = h \}
\]

\[
= (Q^{n-1} \times S^{n-1}) \setminus \Sigma,
\]

where \( \Sigma \) is given by [5] and \( S^{n-1}_h = \{ y \in \mathbb{R}^n \mid \langle y, y \rangle = h \} \) is a pseudosphere \((h \neq 0)\) or a light–like cone \((h = 0)\).

Due to \( \{\phi_1, \phi_2\} = 4(A^{-1} x, y) \neq 0 \) on \( M_h \), it follows that \( M_h \) is a symplectic submanifold of \((\mathbb{R}^{2n}, \omega)\). Recall, for \( F_1, F_2 \in C^\infty(M_h) \), the Hamiltonian vector field \( X_{F_1} \) is defined by \( i_{X_{F_1}} \omega_{M_h} = -dF_1 \), while the Poisson bracket is given by \( \{F_1, F_2\}_{M_h} = X_{F_1}(F_2) \).

Alternatively, we can define the Poisson bracket in redundant variables by the use of Dirac’s construction (e.g., see [29, 33]). Let \( F_1 = f_1|_{M_h}, \ F_2 = f_2|_{M_h}, \ f_1, f_2 \in C^\infty(R^{2n}) \). Then

\[
\{F_1, F_2\}_{M_h} = \{f_1, f_2\}_{M_h} - \frac{\{\phi_1, f_1\}\{\phi_2, f_2\} - \{\phi_2, f_1\}\{\phi_1, f_2\}}{\{\phi_1, \phi_2\}}.
\]

(13)

The bracket is characterized by

\[
\{x_i, x_j\}_{M_h} = 0, \ \{x_i, y_j\}_{M_h} = \tau_i \delta_{ij} - \frac{x_j y_i \tau_j a_{ij}^{-1}}{(A^{-1} x, y)}, \ \{y_i, y_j\}_{M_h} = 0.
\]

(14)

Theorem 2.1. (i) The mapping \( \phi : M_h \rightarrow M_h, \ \phi(x_k, y_k) = (x_{k+1}, y_{k+1}) \) given by [6], [4] is symplectic, \( \phi^* \omega_{M_h} = \omega_{M_h} \).
At the beginning let’s show that

\[
\{ A^{-1}x, y \} = (A^{-1}x, y). \tag{15}
\]

Notice also that

\[
(A^{-1}x, \tilde{y}) = -(A^{-1}x, y). \tag{16}
\]

Indeed, due to \( \tilde{y} + y \in T\tilde{\mathbb{Q}}^{n-1} \), we have

\[
(A^{-1}x, \tilde{y}) = (EA^{-1}x, \tilde{y}) = -(EA^{-1}x, y) = -(A^{-1}x, y).
\]

According to (14) it suffices to prove that

\[
\{ \tilde{x}_i, \tilde{x}_j \} = 0, \quad \{ \tilde{x}_i, \tilde{y}_j \} = \tau_i \delta_{ij} - \frac{\tilde{x}_j \tilde{y}_i a_j^{-1}}{(A^{-1}x, y)} \quad \{ \tilde{y}_i, \tilde{y}_j \} = 0. \tag{17}
\]

The proofs of the first and the third relation in (17) are tedious and we will omit them here. Assuming that \( \{ \tilde{x}_i, \tilde{x}_j \} = 0 \), we will prove only the second relation. At the beginning let’s show that

\[
\{ \tilde{x}_i, y_j \} M_h = \tau_i \delta_{ij} - \frac{\tilde{x}_j y_i a_j^{-1}}{(A^{-1}x, y)}. \tag{18}
\]

First, owing to \( \{ y_i, y_j \} M_h = 0 \) it is

\[
\{(A^{-1}x, y), y_j \} M_h = \sum_{i=1}^{n} y_i a_i^{-1} \{ x_i, y_j \} M_h
\]

\[
= \sum_{i=1}^{n} y_i a_i^{-1} \left( \tau_i \delta_{ij} - \frac{x_j y_i a_j^{-1}}{(A^{-1}x, y)} \right)
\]

\[
= y_j \tau_j a_j^{-1} - \frac{x_j y_i a_j^{-1}}{(A^{-1}x, y)} (A^{-1}y, y).
\]

Consequently, from (14), (15), (16), we have

\[
\{ \tilde{x}_i, y_j \} M_h = \{ x_i - 2 \frac{(A^{-1}x, y)}{(A^{-1}y, y)} y_i, y_j \} M_h
\]

\[
= \{ x_i, y_j \} M_h - 2 \frac{y_i}{(A^{-1}y, y)} \{ (A^{-1}x, y), y_j \} M_h
\]

\[
= \tau_i \delta_{ij} - \frac{x_j y_i a_j^{-1}}{(A^{-1}x, y)} - 2 \frac{y_i y_j a_j^{-1}}{(A^{-1}y, y)} + 2 \frac{x_j y_i a_j^{-1}}{(A^{-1}y, y)}
\]

\[
= \tau_i \delta_{ij} - \frac{y_i \tau_j a_j^{-1}}{(A^{-1}x, y)} (x_j - 2 \frac{(A^{-1}x, y)}{(A^{-1}y, y)} y_j)
\]

\[
= \tau_i \delta_{ij} - \frac{\tilde{x}_j y_i a_j^{-1}}{(A^{-1}x, y)}.
\]
Now, using (18) and (16) we obtain

\[
\{\tilde{x}_i, \nu\}_{M_h} = \{\tilde{x}_i, 2\left(\frac{A^{-1}\tilde{x}, \tilde{y}}{EA^{-2}\tilde{x}, \tilde{x}}\right)\}_{M_h}
\]

\[
= -\frac{2}{(EA^{-2}\tilde{x}, \tilde{x})}\{\tilde{x}_i, (A^{-1}\tilde{x}, y)\}_{M_h}
\]

\[
= -\frac{2}{(EA^{-2}\tilde{x}, \tilde{x})}\sum_{l=1}^{n} \tilde{x}_l a_l^{-1}\{\tilde{x}_i, y_l\}_{M_h}
\]

\[
= -\frac{2}{(EA^{-2}\tilde{x}, \tilde{x})}\sum_{l=1}^{n} \tilde{x}_l a_l^{-1}\left(\tau_l \delta_{il} - \frac{\tilde{x}_l y_l \tau_l a_l^{-1}}{(A^{-1}\tilde{x}, y)}\right)
\]

\[
= -\frac{2\tau_l a_l^{-1}\tilde{x}_i}{(EA^{-2}\tilde{x}, \tilde{x})} + \frac{2y_i}{(A^{-1}\tilde{x}, y)}.
\]

Therefore,

\[
\{\tilde{x}_i, \tilde{y}_j\}_{M_h} = \{\tilde{x}_i, y_j + \nu \tau_j a_j^{-1}\tilde{x}_j\}_{M_h}
\]

\[
= \{\tilde{x}_i, y_j\}_{M_h} + \tau_j a_j^{-1}\tilde{x}_j\{\tilde{x}_i, \nu\}_{M_h}
\]

\[
= \tau_i \delta_{ij} - \frac{\tilde{x}_j y_j \tau_j a_j^{-1}}{(A^{-1}\tilde{x}, y)} - 2\frac{\tilde{x}_i \tilde{x}_j \tau_i a_i^{-1} \tau_j a_j^{-1}}{EA^{-2}\tilde{x}, \tilde{x})} + 2\frac{\tilde{x}_j y_j \tau_j a_j^{-1}}{(A^{-1}\tilde{x}, y)}
\]

\[
= \tau_i \delta_{ij} - \frac{\tilde{x}_j \tau_j a_j^{-1}}{(A^{-1}\tilde{x}, y)}(y_j + \nu \tau_j a_j^{-1}\tilde{x}_j)
\]

\[
= \tau_i \delta_{ij} - \frac{\tilde{x}_j \tilde{y}_j \tau_j a_j^{-1}}{(A^{-1}\tilde{x}, y)}.
\]

(ii) Note that the only relation between the integrals on \(M_h\) is

\[
f_1 + \cdots + f_n = (y, y) = h. \tag{19}
\]

Similarly as in the Euclidean space, we have \(\{f_i, f_j\} = 0\) (see [24, 25]). Further \(\{\phi_2, f_i\} = \{2H, f_i\} = \{f_1 + \cdots + f_n, f_i\} = 0\), and therefore

\[
\{f_i, f_j\}_{M_h} = 0, \quad i, j = 1, \ldots, n.
\]

\[
\square
\]

Remark 1. Observe that \(\{\tilde{x}_i, \tilde{x}_j\}_{M_h} = 0\), [18], and \(\{y_i, y_j\}_{M_h} = 0\) imply that the mapping \((x, y) \mapsto (\tilde{x}, y)\) is also symplectic on \(M_h\).

Remark 2. Note that in the virtual billiard mapping [3, 4] we allow the trajectories both with \(J > 0\) and \(J < 0\) (\(J = (A^{-1}x, y) = 0\) defines the tangent space \(T_xQ^{n-1}\)). For example, in the ellipsoidal case when \(A\) is positive definite, \(J > 0\) means that \(y\) is directed outward \(Q^{n-1}\). It is also natural to consider the dynamics of lines

\[
l_k = \{x_k + sy_k \mid s \in \mathbb{R}\}, \quad k \in \mathbb{Z},
\]

described by Khesin and Tabachnikov within the framework of the symplectic reduction for \(A\) being positive definite [21]. In our notation, in the space–like and time–like cases, the dynamics of lines corresponds to the virtual billiard dynamics on \(M_h/\pm 1\) with identified \(y\) and \(-y\), while in the light–like case it corresponds to the induced dynamics on \(\hat{M} = M_0/\mathbb{R}^*\), where we take the projectivization of the light–like cone \(S_0^{n-1}\). The latter case will be studied in details below.
2.2. Contact description. In the light–like case $h = 0$ we show the existence of a contact structure associated to $M_0$. Let us introduce an action of $\mathbb{R}^* = \mathbb{R}\backslash \{0\}$ on $M_0$ by
\[ g_\lambda(x, y) = (x, \lambda y), \quad \lambda \in \mathbb{R}^*. \]
The action is evidently free and proper, from which we conclude that the orbit space $\bar{M} := M_0/\mathbb{R}^*$ is a smooth manifold of dimension $\dim \bar{M} = \dim M_0 - 1 = 2n - 3$ and the projection $\pi : M_0 \to \bar{M}, \pi(x, y) = (x, [y])$ is a surjective submersion.

With the notation above, $(M_0, \omega_{M_0})$ is a symplectic Liouville manifold:
\[ g^*_\lambda \omega_{M_0} = \lambda \omega_{M_0}. \]
The associated Liouville vector field and the Liouville 1-form are given by
\[ Z(x, y) = \frac{d}{d\lambda} g_\lambda(x, y) \bigg|_{\lambda = 1} = (0, y) \]
and
\[ \tilde{\beta} := i_Z \omega_{M_0} = Ey \cdot dx|_{M_0}, \]
respectively. Then $d\tilde{\beta} = \omega_{M_0}$ and $g_\lambda^* \tilde{\beta} = \lambda \tilde{\beta}$ (e.g., see [27]). It is well known that the orbit space $\bar{M}$ carries the natural contact structure induced by $\tilde{\beta}$ (Proposition 10.3, Ch. V, [27]). We describe this contact structure below.

Let $\beta := \frac{1}{J} \tilde{\beta} = \frac{1}{(A^{-1}x, y)} \tilde{\beta}$.

Theorem 2.2. (i) There exists a unique 1-form $\bar{\beta}$ on $\bar{M}$, such that $\beta = \pi^* \bar{\beta}$. Furthermore, the form $\bar{\beta}$ is contact and $\bar{R} := \pi^* X_J$ is the Reeb vector field on $(\bar{M}, \bar{\beta})$, where $X_J$ is the Hamiltonian vector field of the function $J = (A^{-1}x, y)$ on $M_0$.

(ii) The mapping $\bar{\phi} : \bar{M} \to \bar{M}$ defined by $\bar{\phi}(x, [y]) := \pi(\phi(x, y))$ is contact:
\[ (\bar{\phi})^* \bar{\beta} = \bar{\beta}. \]

(iii) Assume that the quadric is not symmetric. The functions $f_i/J^2$ descend to the commutative integrals $\bar{f}_i$,
\[ [\bar{f}_i, \bar{f}_j] = 0, \quad i, j = 1, \ldots, n, \]
of the contact mapping $\bar{\phi}$, where $[\cdot, \cdot]$ is the Jacobi bracket on $(\bar{M}, \bar{\beta})$. Further, $\bar{f}_i$ are preserved by the Reeb vector field $\bar{R}$ of $(\bar{M}, \bar{\beta})$
\[ \bar{R}(\bar{f}_i) = 0 \iff [1, \bar{f}_i] = 0, \quad i = 1, \ldots, n, \]
and the contact mapping $\bar{\phi}$ is contact completely integrable: the manifold $\bar{M}$ is almost everywhere foliated on $(n - 1)$–dimensional pre-Legendrian invariant manifolds.

Proof. (i) We have,
\[ \ker \pi_* = \text{span} \{ Z \}. \quad (20) \]
As a consequence of $g_\lambda^* \beta = \lambda \tilde{\beta}$ and $g_\lambda^* J = \lambda J$ we conclude that $\beta$ is $\mathbb{R}^*$–invariant, $g_\lambda^* \beta = \beta$. By definition of $\beta$ it is $\beta(Z) = 0$, which in view of (20) implies that $\beta$ is basic (e.g., see [27], Ch. II) and there exists a unique 1-form $\bar{\beta}$ on $\bar{M}$, such that $\beta = \pi^* \bar{\beta}$. 

Further note
\[ \beta \wedge (d\beta)^{n-2} = \frac{1}{J} \tilde{\beta} \wedge \left( \frac{1}{J} d\tilde{\beta} - \frac{1}{J^2} dj \wedge \tilde{\beta} \right)^{n-2} \]
\[ = \frac{1}{J^{n-1}} \tilde{\beta} \wedge (d\beta)^{n-2} \]
\[ = \frac{1}{J^{n-1}} (iz\omega_{M_0} \wedge \omega_{M_0}^{n-2}) \]

Taking into account that \( iz\omega_{M_0}^{n-1} = (n-1)(iz\omega_{M_0}) \wedge \omega_{M_0}^{n-2} \), we obtain that
\[ \beta \wedge (d\beta)^{n-2} = \frac{1}{(n-1)J^{n-1}} iz\omega_{M_0}^{n-1}. \]  

(21)

Let \( \gamma_1, \ldots, \gamma_{2n-3} \in T_{x,y} M \) be arbitrary linearly independent tangent vectors. Since \( \pi \) is a submersion, there exist \( \gamma_1, \ldots, \gamma_{2n-3} \in T_{x,y} M_0 \), such that \( \pi_* \gamma_i = \gamma_i \), for all \( i = 1, \ldots, 2n-3 \). According to (20), the vectors \( Z, \gamma_1, \ldots, \gamma_{2n-3} \) are linearly independent. Because \( \omega_{M_0}^{n-1} \) is a volume form on \( M_0 \), from (21) we have
\[ \tilde{\beta} \wedge (d\tilde{\beta})^{n-2}(\gamma_1, \ldots, \gamma_{2n-3}) = \beta \wedge (d\beta)^{n-2}(\gamma_1, \ldots, \gamma_{2n-3}) \]
\[ = \frac{1}{(n-1)J^{n-1}} \omega_{M_0}^{n-1}(Z, \gamma_1, \ldots, \gamma_{2n-3}) \neq 0. \]

Hence, \( \tilde{\beta} \) is a contact form on \( \tilde{M} \).

Now, let \( X_J \) be the Hamiltonian vector field of \( J \) on \( M_0 \). We have
\[ \tilde{\beta}(X_J) = \omega_{M_0}(Z, X_J) = dJ(Z) = \sum_{i=1}^{n} a_i^{-1}(x_i dy_i + y_idx_i)(Z) = J. \]

Consequently,
\[ \tilde{\beta}(\tilde{R}) = \tilde{\beta}(\pi_* X_J) = \beta(X_J) = \frac{1}{J} \tilde{\beta}(X_J) = 1 \]
and \( \tilde{R} := \pi_* X_J \) is the Reeb vector field on \( \tilde{M} \).

(ii) Evidently, \( g_\lambda \circ \tilde{\phi} = \phi \circ g_\lambda \) for all \( \lambda \in \mathbb{R}^* \) and \( \tilde{\phi} \) is well defined. Taking derivative in \( \lambda = 1 \), we get \( \phi \circ Z = Z \) and \( iz\phi^* \omega_{M_0} = \phi^*(iz\omega_{M_0}) \). According to Theorem 2.1, the symplectic form \( \omega_{M_0} \) is \( \phi \)-invariant, \( \phi^* \omega_{M_0} = \omega_{M_0} \), and consequently,
\[ \phi^* \tilde{\beta} = \phi^*(iz\omega_{M_0}) = iz\phi^* \omega_{M_0} = iz\omega_{M_0} = \tilde{\beta}. \]

Dividing the last equation by \( J \) and using \( \phi^* J = J \), we get \( \phi^* \beta = \beta \). This implies that
\[ \pi^* (\tilde{\phi})^* \tilde{\beta} = (\tilde{\phi} \circ \pi)^* \tilde{\beta} = (\pi \circ \phi)^* \tilde{\beta} = \phi^* \pi^* \tilde{\beta} = \phi^* \beta = \beta = \pi^* \tilde{\beta}. \]

Using the fact that \( \pi \) is a submersion, we finally obtain \( (\tilde{\phi})^* \tilde{\beta} = \tilde{\beta} \).

(iii) The Jacobi brackets \( [\tilde{f}_i, \tilde{f}_j] \) are given by
\[ [\tilde{f}_i, \tilde{f}_j] = \tilde{Y}_{\tilde{f}_i} \tilde{f}_j - \tilde{f}_j \tilde{Y}_{\tilde{f}_i}, \quad i, j = 1, \ldots, n, \]
where \( \tilde{R} \) is the Reeb vector field on \( (M, \tilde{\beta}) \), \( \tilde{\beta}(\tilde{R}) = 1, i\tilde{R}d\tilde{\beta} = 0 \), and
\[ \tilde{Y}_{\tilde{f}_i} = \tilde{f}_i \tilde{R} + \tilde{H}_i, \quad i = 1, \ldots, n, \]
is the contact Hamiltonian vector field of \( \tilde{f}_i \). Here, \( \tilde{H}_i \) are the horizontal vector fields, \( \tilde{\beta}(\tilde{H}_i) = 0 \), satisfying
\[ d\beta(\tilde{H}_i, \tilde{X}) = -(d\tilde{f}_i(\tilde{X}) - \tilde{R}\tilde{f}_i(\tilde{X})), \quad i = 1, \ldots, n, \]
for all tangent vectors \( \tilde{X} \) on \( \tilde{M} \).
In addition, having in mind that each tangent vector \( \bar{X} \) on \( \bar{M} \) has the form \( \bar{X} = \pi_* X \) for some vector field \( X \) on \( M_0 \), we have
\[
d\bar{\beta}(\bar{X}, \bar{R}) = d\bar{\beta}(\pi_* X, \pi_* X_J) = d\beta(X, X_J)
\]
\[
= \frac{1}{J} \omega_{M_0}(X, X_J) - \frac{1}{J^2} (dJ \wedge \hat{\beta})(X, X_J)
\]
\[
= \frac{1}{J} [dJ(X) - \frac{1}{J} dJ(X) \hat{\beta}(X_J) + \frac{1}{J} dJ(X_J) \hat{\beta}(X)]
\]
\[
= \frac{1}{J} [dJ(X) - \frac{1}{J} dJ(X) J] = 0.
\]

Next, we prove that \( \bar{f}_i \) are integrals of the Reeb vector field \( \bar{R} \). As the first step we need the assertion
\[
\{ J, f_i \}_{M_0} = 0, \tag{23}
\]
for all integrals \( f_i \), which, for example, follows from (28). Using this, from the definition \( \frac{f_i}{J^2} = \pi^* \bar{f}_i \), we have
\[
\bar{R} \bar{f}_i = d\bar{f}_i(\pi_* X_J)
\]
\[
= d(\frac{f_i}{J^2})(X_J)
\]
\[
= \frac{1}{J^2} df_i(X_J) - \frac{2}{J^3} dJ(X_J)
\]
\[
= \frac{1}{J} \{ J, f_i \}_{M_0} = 0. \tag{24}
\]

There exist, at least locally, vector fields \( H_i \) that project to horizontal vector fields \( \bar{H}_i \): \( \pi_* H_i = \bar{H}_i \). If we substitute \( \bar{X} = \pi_* X_{f_j} \) in (24), we obtain
\[
d\beta(H_i, X_{f_j}) = -d\left( \frac{f_i}{J^2} \right)(X_{f_j}). \tag{25}
\]

Our aim is to prove that
\[
df_j(H_i) = \frac{2f_i}{J} dJ(H_i). \tag{26}
\]

Due to
\[
\frac{1}{J^2} df_i(X_{f_j}) = \frac{1}{J^2} df_i(X_{f_j}) - \frac{2}{J^3} dJ(X_{f_j})
\]
\[
= \frac{1}{J^2} \{ f_i, f_j \}_{M_0} - \frac{2}{J^3} \{ J, f_j \}_{M_0} = 0,
\]
the relation (26) becomes \( d\beta(H_i, X_{f_j}) = 0 \), or equivalently,
\[
d\hat{\beta}(H_i, X_{f_j}) = \frac{1}{J} (dJ \wedge \hat{\beta})(H_i, X_{f_j}). \tag{27}
\]

Owing to
\[
\hat{\beta}(X_{f_j}) = \omega_{M_0}(Z, X_{f_j}) = df_j(Z) = 2f_j,
\]
and using (28), we obtain
\[
\frac{1}{J} (dJ \wedge \hat{\beta})(H_i, X_{f_j}) = \frac{1}{J} [dJ(H_i)\hat{\beta}(X_{f_j}) - dJ(X_{f_j})\hat{\beta}(H_i)] = \frac{2f_i}{J} dJ(H_i).
\]

On the other hand
\[
d\hat{\beta}(H_i, X_{f_j}) = \omega_{M_0}(H_i, X_{f_j}) = df_j(H_i),
\]
which together with (27) yields (26). In the end, thanks to (24), (26) we have

$$[\bar{f}_i, \bar{f}_j] = Y_{\bar{f}_i} \bar{f}_j - \bar{f}_j Y_{\bar{f}_i} = d \bar{f}_j (\bar{H}_i) = d \left( \frac{\bar{f}_j}{f^2} \right) (H_i) = \frac{1}{f^2} df_j (H_i) - \frac{2f_j}{f^3} dJ(H_i) = 0.$$ 

Finally note that the integrals $f_i$ and $J$ on $M_h$ are related by

$$J^2 = \sum_{i=1}^{n} \tau_i a_i^{-1} f_i,$$ 

which together with (19) imply that among the integrals $\bar{f}_i$ we have two relations,

$$\bar{f}_1 + \cdots + \bar{f}_n = 0, \quad \tau_1 a_1^{-1} \bar{f}_1 + \cdots + \tau_n a_n^{-1} \bar{f}_n = 1,$$

and that the number of the independent ones is $n - 2$. According to the theorem on contact integrability, their invariant level-sets almost everywhere define $(n-1)$-dimensional pre-Legendrian manifolds, which have an additional $(n-2)$-dimensional Legendrian foliation (see 25, 19). $\square$

3. Noncommutative integrability and symmetric quadrics.

3.1. Discrete noncommutative integrability. Recall that a Hamiltonian flow on a $2n$-dimensional symplectic manifold $(M^{2n}, \omega)$ (respectively, a contact flow on a $2n+1$-dimensional contact manifold $(M^{2n+1}, \beta)$) is noncommutatively integrable, if it has a complete set of integrals $\mathcal{F}$. The set $\mathcal{F}$ closed under the Poisson bracket (respectively, the Jacobi bracket) is complete, if one can find $2n - r$ almost everywhere independent integrals $F_1, F_2, \ldots, F_{2n-r} \in \mathcal{F}$, such that $F_1, \ldots, F_r$ Poisson commute with all integrals 31 28 (respectively, $F_1, \ldots, F_r$ commute with respect to the Jacobi bracket with all integrals, and the functions in $\mathcal{F}$ are integrals of the Reeb flow, as well 19).

Regular compact connected invariant manifolds of the system are $r$-dimensional isotropic tori generated by the Hamiltonian flows of $F_1, \ldots, F_r$, i.e., $r+1$-dimensional pre-isotropic tori generated by the Reeb vector field and the contact Hamiltonian flows of $F_1, \ldots, F_r$. Here, a submanifold $N \subset M^{2n+1}$ is pre-isotropic, if it transversal to the contact distribution $\mathcal{H} = \ker \beta$ and if $G_x = T_x N \cap \mathcal{H}_x$ is an isotropic subspace of the symplectic linear space $(\mathcal{H}_x, d\beta)$, for all $x \in N$. The last condition is equivalent to the condition that distribution $\mathcal{G} = \bigcup_x G_x$ defines a foliation 19.

In a neighborhood of a regular torus there exist canonical generalized action–angle coordinates 31 (generalized contact action–angle coordinates 19), such that integrals $F_i$, $i = 1, \ldots, r$ depend only on the actions and the flow is a translation in the angle coordinates. If $r = n$ we have the usual Liouville integrability described in the Arnold-Liouville theorem 1, i.e., contact integrability described in 4 25.

If instead of the continuous flow we consider the symplectic mapping $\Phi : M^{2n} \to M^{2n}$, $\Phi^* \omega = \omega$ (the contact mapping $\Phi : M^{2n+1} \to M^{2n+1}$, $\Phi^* \beta = \beta$) having the complete set of integrals $\mathcal{F}$, as above, compact connected components of an invariant regular level set

$$M_c = \{ F_1 = c_1, F_2 = c_2, \ldots, F_{2n-r} = c_{2n-r} \}$$  

(29)
are \( r \)-dimensional isotropic tori \((r + 1\)-dimensional pre-isotropic tori) and in their
neighborhoods there exist canonical generalized (contact) action–angle coordinates.

By the same argumentation as given by Veselov [35, 39] for the Liouville inte-
grable symplectic correspondences, we have the following description of the dynam-
ics.

**Theorem 3.1.** Let \( M_c = T_1 \cup T_2 \cup \cdots \cup T_p \) be a compact regular level set \([29]\). If
the torus \( T_i \cong \mathbb{R}^{r(+1)}/\Lambda_i \) is \( \Phi \)-invariant, then the restriction of the mapping \( \Phi \) to
\( T_i \) is the shift by a constant vector \( a_i \in \mathbb{R}^{r(+1)} \)

\[
\Phi ([x]) \equiv x + a_i, \quad [x] \in T_i.
\]

Otherwise, if

\[
\Phi (T_{ik}) = T_{ik+1}, \quad k = 1, \ldots, q \leq p, \quad i = i_q = i_{q+1}, \quad T_{ik} \cong \mathbb{R}^{r(+1)}/\Lambda_{ik},
\]

define tori \( T_{ik+1} = \mathbb{R}^{r(+1)}/\Lambda_{ik+1} \) by the lattices

\[
\Lambda_{ik+1} = \{ b \in \mathbb{R}^{r(+1)} | \Phi ([x + b]) \equiv \Phi ([x]) = \{ b \in \mathbb{R}^{r(+1)} | \Phi ([x]) \equiv \Phi ([x]) + b \},
\]

\([x] \in T_{ik}, \Phi ([x]) \in T_{ik+1}, \) containing \( \Lambda_{ik} \) and \( \Lambda_{ik+1} \) as sublattices. Then we have the
following commutative diagrams

\[
T_{ik} \xrightarrow{\Phi} T_{ik+1} \quad \pi_k \downarrow \quad \pi_{k+1} \downarrow
\]

\[
T_{ik+1} \quad \tau_{ik+1} \quad T_{ik+1}
\]

where \( \tau_{ik} \) are the shifts by constant vectors \( a_{ik} \in \mathbb{R}^{r(+1)} \). The \( q \)-th iteration
of \( \Phi \) is given by

\[
\Phi^q ([x]) \equiv x + a_{ik}, \quad [x] \in T_{ik},
\]

for some vectors \( a_{ik} \in \mathbb{R}^{r(+1)} \). In particular, if a point \([x] \in T_{ik} \) is periodic with a
period \( mq \), then all points of \( T_{ik}, T_{ik} \cup T_{ik+1}, \) and \( \cdots \cup T_{ik} \) are periodic with the same period.

### 3.2. Symmetric quadrics

We turn back to the virtual billiard dynamics and consider the case when the quadric \( \mathbb{Q}^{n-1} \) is symmetric. Define the sets of indices \( I_s \subset \{1, \ldots, r\} \) \( s = 1, \ldots, r \) by the conditions

1. \( \tau_i a_j = \alpha_j a_j = \alpha_s \) for \( i, j \in I_s \) and for all \( s \in \{1, \ldots, r\} \),

2. \( \alpha_s \neq \alpha_t \) for \( s \neq t \).

Let

\[
\mathbb{E}^{k,l} = \mathbb{E}^{k_1,l_1} \oplus \cdots \oplus \mathbb{E}^{k_r,l_r}
\]

be the associated decomposition of \( \mathbb{E}^{k,l} \), where \( \mathbb{E}^{k_s,l_s} \) are pseudo–Euclidean sub-
spaces of the signature \( (k_s,l_s) \) with

\[
k_s = \{ |\tau_i| = 1, i \in I_s \}, \quad l_s = \{ |\tau_i| = -1, i \in I_s \}, \quad k_s + l_s = |I_s|.
\]

By \( \langle \cdot, \cdot \rangle_s \) we denote the restriction of the scalar product to the subspace \( \mathbb{E}^{k_s,l_s} \)

\[
\langle x, x \rangle_s = \sum_{i \in I_s} \tau_i x_i^2, \quad x \in \mathbb{E}^{k,l}.
\]

1To simplify the notation, we omitted the projection operator \( \pi_s : \mathbb{E}^{k,l} \to \mathbb{E}^{k_s,l_s} \) at the left
hand side of (31).
Let $SO(k_s,l_s)$ be the special orthogonal group of $\mathbb{E}^{k_s,l_s}$. The quadric, as well as the virtual billiard flow, is $SO(k_1,l_1) \times \cdots \times SO(k_r,l_r)$–invariant. The integrals
\[ \Phi_{s,ij} := y_ix_j - x_iy_j, \quad i,j \in I_s \]
are proportional to the components of the corresponding momentum mapping
\[ \Phi: M_h \rightarrow so(k_1,l_1)^* \times \cdots \times so(k_r,l_r)^*. \]
On the other hand, the determinant $\det L_{x,y}(\lambda)$ is an invariant of the flow, and by expanding it in terms of $1/(\lambda - \alpha_s), 1/(\lambda - \alpha_s)^2$, we get
\[ \det L_{x,y}(\lambda) = (1 + q_\lambda(x,x))q_\lambda(y,y) - q_\lambda(x,y)^2 \]
where the integrals $F_s, P_s$ are given by:
\[ F_s = \sum_{i \in I_s} (\tau_i y_i^2 + \sum_{j \notin I_s} (x_iz_j - x_iz_j)^2), \]
\[ P_s = \sum_{i,j \in I_s, i < j} \tau_i \tau_j \Phi_{s,ij}^2 \quad \text{for} \quad |I_s| \geq 2 \quad (P_s \equiv 0, \quad \text{for} \quad |I_s| = 1). \]
The Hamiltonian is equal to the sum $H = \frac{1}{2} \sum_{s=1}^r F_s$, that is, among integrals $F_s$ we have the relation $\sum_s F_s = 2h$ on $M_h$.

For $h = 0$, by $F_s, P_s, \Phi_{s,ij}$, we denote the functions on $M$ obtained from $\mathbb{R}^*$–invariant integrals $F_s/J^2, P_s/J^2, \Phi_{s,ij}/J$.

**Theorem 3.2.** (i) The virtual billiard flow within symmetric quadric $\mathbb{E}^{k_s,l_s}$ is completely integrable in a noncommutative sense by means of integrals $\mathcal{F} = \{F_s, \Phi_{s,ij}\}$. The functions $F_s, P_s = \sum_{i < j} \tau_i \tau_j \Phi_{s,ij}^2$ are central within the algebra of integrals generated by $\mathcal{F}$:
\[ \{F_s, F_t\}_{M_h} = 0, \quad \{F_s, P_t\}_{M_h} = 0, \quad \{P_s, P_t\}_{M_h} = 0, \]
\[ \{F_s, \Phi_{t,ij}\}_{M_h} = 0, \quad \{P_s, \Phi_{t,ij}\}_{M_h} = 0, \]
and their Hamiltonian vector fields generate $N - 1$–dimensional isotropic manifolds, regular level sets of the integrals $\mathcal{F}$, where
\[ N = r + |\{s \in \{1, \ldots, r\} : |I_s| \geq 2\}|. \]
(ii) In the light–like case, the mapping $\phi$ is contact completely integrable in a noncommutative sense by means of integrals $\tilde{\mathcal{F}} = \{\tilde{F}_s, \tilde{\Phi}_{s,ij}\}$. The integrals are invariant with respect to the Reeb flow
\[ [1, \tilde{F}_s] = 0, \quad [1, \tilde{P}_s] = 0, \quad [1, \tilde{\Phi}_{s,ij}] = 0, \]
and the functions $\tilde{F}_s, \tilde{P}_s$ are central within the algebra of integrals generated by $\tilde{\mathcal{F}}$:
\[ [\tilde{F}_s, \tilde{F}_t] = 0, \quad [\tilde{F}_s, \tilde{P}_t] = 0, \quad [\tilde{P}_s, \tilde{P}_t] = 0, \]
\[ [\tilde{F}_s, \tilde{\Phi}_{t,ij}] = 0, \quad [\tilde{P}_s, \tilde{\Phi}_{t,ij}] = 0. \]
Among central functions $\tilde{F}_s, \tilde{P}_s$ there are $(N - 2)$–independent ones and their contact Hamiltonian vector fields, together with the Reeb vector field $\tilde{R}$, generate $N - 1$–dimensional pseudo–isotropic manifolds – regular levels sets of the integrals $\tilde{\mathcal{F}}$.

---

2In [23] the term $\tau_i \tau_j$ is omitted in the formula for $P_s$. This misprint, however, does not affect the results in [23].
The first statement is an analog of Theorems 5.1, 5.2 for the Jacoby-Rosochatius problem [20] and Theorem 4.1 for geodesic flows on quadrics in pseudo–Euclidean spaces [23], where the Dirac construction is applied for the constraints

\((A^{-1} x, x) = 1, \quad (A^{-1} x, y) = 0.\)

The second statement follows from the same considerations as in the proof of Theorem 2.2. For example, similarly as in (24), we have

\[
\bar{R} \bar{\Phi}_{s,ij} = d \bar{\Phi}_{s,ij}(\pi_{*} X_{J}) = \frac{1}{J} \Phi_{s,ij} J d(X_{J}) = \frac{1}{J} \{ J, \Phi_{s,ij} \}_{M_0} = 0.
\]

The last equality follows from the commuting relations \(\{ J, \phi_{2} \} = 0, \{ \Phi_{s,ij}, \phi_{2} \} = 0, \) and \(\{ J, \Phi_{s,ij} \} = 0.\)

Note that the relation (33) for \(\lambda = 0\) implies \(J^2 = \sum_{s} (\alpha_{s}^{-1} F_{s} - \alpha_{s}^{-2} P_{s}),\) whence the relations

\[
\sum_{s} F_{s} = 0, \quad \sum_{s} (\alpha_{s}^{-1} \bar{F}_{s} - \alpha_{s}^{-2} \bar{P}_{s}) = 1
\]

among the integrals \(\bar{F}_{s}, \bar{P}_{s}\) on \(\bar{M}.\)

**Remark 3.** An example of noncommutatively integrable multi-valued symplectic correspondence is a recently constructed discrete Neumann system on a Stiefel variety [17]. Another example of a discrete integrable contact system is the Heisenberg model in pseudo–Euclidean spaces [21]. We shall discuss relationship between the Heisenberg model and virtual billiard dynamics in a forthcoming paper.

4. The Chasles and Poncelet theorems for symmetric quadrics.

4.1. **Pseudo–confocal quadrics.** There is a nice geometric manifestation of integrability of elliptical billiards in pseudo–Euclidean spaces given by Khesin and Tabachnikov [24]. Consider the following “pseudo–confocal” family of quadrics in \(E^{k,l}\)

\[
Q_{\lambda}: \quad ((A - \lambda E)^{-1} x, x) = \sum_{i=1}^{n} \frac{x_{i}^2}{\alpha_{i} - \tau_{i} \lambda} = 1, \quad \lambda \neq \tau_{i} \alpha_{i}, \quad i = 1, \ldots, n. \quad (34)
\]

For a nonsymmetric ellipsoid, the lines \(l_{k}, k \in \mathbb{Z}\) determined by a generic space–like or time–like (respectively light–like) billiard trajectory are tangent to \(n - 1\) (respectively \(n - 2\)) fixed quadrics from the pseudo–confocal family \(Q_{\lambda}\) (pseudo–Euclidean version of the Chasles theorem, see Theorem 4.9 in [24] and Theorem 5.1 in [12]). A related geometric structure of the set of singular points for the pencil \(Q_{\lambda}\) is described in [12, 14].

Here we consider the case of symmetric quadrics and further develop the analysis given in [24], where \(A\) had been positive definite.

Without loss of generality we assume in the section that

\[
\alpha_{1} > \alpha_{2} > \cdot \cdot \cdot > \alpha_{r}. \quad (35)
\]

The equation (34) has \(r\) solutions in the complex plane for a generic \(x\). The following lemma estimates the number of real solutions in certain cases.
Lemma 4.1. (i) Through points \( x \in \mathbb{E}^{k,l} \) that satisfy
\[
\begin{align*}
\text{sign}(x, x)_s &= \kappa_1 \neq 0, \quad s = 1, \ldots, g, \\
\text{sign}(x, x)_s &= \kappa_2 \neq 0, \quad s = g + 1, \ldots, r,
\end{align*}
\]
for some index \( g \) pass either \( r \) quadrics (when \( \kappa_1 = -1, \kappa_2 = +1 \), \( \kappa_1 = \kappa_2 = +1 \) or \( \kappa_1 = \kappa_2 = -1 \), or \( r \) resp. \( r - 2 \) quadrics (when \( \kappa_1 = +1, \kappa_2 = -1 \)) from the pseudo–confocal family \([24]\). Similarly, if
\[
\begin{align*}
\text{sign}(x, x)_s &= \kappa_1, \quad s = 1, \ldots, g_1, g_2, \ldots, r, \\
\text{sign}(x, x)_s &= \kappa_2, \quad s = g_1 + 1, \ldots, g_2 - 1, \quad \kappa_1 \cdot \kappa_2 = -1,
\end{align*}
\]
for some indexes \( g_1, g_2, g_1 < g_2 \), through \( x \) pass either \( r \) or \( r - 2 \) quadrics from the pseudo–confocal family \([24]\).

(ii) The quadrics passing through arbitrary point \( x \) are mutually orthogonal at \( x \).

Proof. (i) We slightly modify the proof of the corresponding Khesin and Tabachnikov statement given for non-symmetric ellipsoids (Theorem 4.5 \([24]\)). Consider the function
\[
S(\lambda) = ((A - E\lambda)^{-1}x, x) = \sum_{s=1}^{r} \frac{\langle x, x \rangle_s}{\alpha_s - \lambda}
\]
We have
\[
\begin{align*}
S(\lambda) &\sim -1/\lambda \langle x, x \rangle, \quad \lambda \to \pm \infty, \\
S(\lambda) &\sim \frac{\langle x, x \rangle_s}{\alpha_s - \lambda}, \quad \lambda \to \alpha_s, \quad s = 1, \ldots, r,
\end{align*}
\]
implying
\[
\lim_{\lambda \to \pm \infty} S = 0 \quad \lim_{\lambda \to \alpha_s} S = \text{sign}(x, x)_s \cdot \infty, \quad \lim_{\lambda \to \alpha_s +} S = -\text{sign}(x, x)_s \cdot \infty.
\]
Therefore, if \([36]\) holds, the equation \( S(\lambda) = 1 \) has real solutions in the \( r - 2 \) intervals \((\alpha_{s+1}, \alpha_s), s = 1, \ldots, r - 1, \; s \neq g \). In addition, we also have 2 real solutions for \( \kappa_1 = -1, \kappa_2 = +1 \) (in the intervals \((\infty, \alpha_1), (\alpha_1, \infty)\) and in the case when all signs are equal (in the intervals \((\infty, \alpha_r), (\alpha_{g+1}, \alpha_g), \) for \( \kappa_1 = \kappa_2 = +1 \), and in the intervals \((\alpha_1, \infty), (\alpha_{g+1}, \alpha_g), \) for \( \kappa_1 = \kappa_2 = -1 \)).

In the case when \([37]\) holds, the equation \( S(\lambda) = 1 \) always has real solutions in the \( r - 3 \) intervals \((\alpha_{s+1}, \alpha_s), s = 1, \ldots, r - 1, \; s \neq g, g_2 - 1 \), and an additional solution in the interval \((\alpha_1, \infty)\) for \( \kappa_1 = -1, \kappa_2 = +1 \), i.e., in the interval \((\infty, \alpha_r)\) for \( \kappa_1 = +1, \kappa_2 = -1 \).

(ii) The second statement has the same proof as in the case when \( A \) is positive definite (Theorem 4.5 \([24]\)).

Example 1. From Lemma 4.1 it follows that in the Euclidean space \( \mathbb{E}^{n,0} \) through a generic point pass \( r \) quadrics, while through a generic point in the Lorentz–Poincaré–Minkowski space \( \mathbb{E}^{n-1,1} \) pass \( r \) or \( r - 2 \) quadrics from the pseudo–confocal family \([24]\) for arbitrary symmetric quadratic \( \mathbb{Q}^{n-1} \) (Figures 2 and 3).

Example 2. If \( A \) is positive definite, then \( \alpha_1 > \cdots > \alpha_g > 0 > \alpha_{g+1} > \cdots > \alpha_r \) for some index \( g \). At a generic point \( x \in \mathbb{E}^{k,l} \) we have \( \langle x, x \rangle_s > 0, \; s = 1, \ldots, g, \; \langle x, x \rangle_s < 0, \; s = g + 1, \ldots, r \). Therefore, through a generic point \( x \in \mathbb{E}^{k,l} \) pass either \( r \) or \( r - 2 \) quadrics from the pseudo–confocal family \([24]\) (see \([24] [24]\)).
Figure 2. Families of pseudo-confocal quadrics for $a_1 > 0, a_2 < 0$ in $\mathbb{E}^{1,1}$ (with $\alpha_1 = -a_2 > \alpha_2 = a_1$) and $\mathbb{E}^{2,0}$, respectively.

Figure 3. Family of pseudo-confocal quadrics for $a_1 > 0, a_2 < 0$ in $\mathbb{E}^{1,1}$, where $\alpha_1 = a_1 > \alpha_2 = -a_2$.

Example 3. Suppose that
\[
\max\{a_1, \ldots, a_k\} < \min\{-a_{k+1}, \ldots, -a_n\}. \tag{38}
\]
Then there is an index $g$, such that
\[
\mathbb{E}^{k, l_s} = \mathbb{E}^{0, l_s}, \quad s = 1, \ldots, g, \quad \mathbb{E}^{k, l_s} = \mathbb{E}^{k, 0}, \quad s = g + 1, \ldots, r, \tag{39}
\]
and through a generic point $x \in \mathbb{E}^{k, l}$ pass $r$ quadrics from the confocal family (Figure 2). On the other hand, if
\[
\max\{-a_{k+1}, \ldots, -a_n\} < \min\{a_1, \ldots, a_k\}, \tag{40}
\]
then there is an index $g$, such that
\[
\mathbb{E}^{k, l_s} = \mathbb{E}^{k, 0}, \quad s = 1, \ldots, g, \quad \mathbb{E}^{k, l_s} = \mathbb{E}^{0, l_s}, \quad s = g + 1, \ldots, r, \tag{41}
\]
and through a generic point $x \in \mathbb{E}^{k, l}$ pass $r$ or $r - 2$ quadrics.
4.2. Geometrical interpretation of integrals. The condition
\[ \det L_{x,y}(\lambda) = q_\lambda(y,y)(1 + q_\lambda(x,x)) - q_\lambda(x,y)^2 = 0 \]  
(42)
is equivalent to the geometrical property that the line
\[ l_{x,y} = \{ x + sy : s \in \mathbb{R} \} \]
is tangent to the quadric \( Q_\lambda \) (see [29, 12]).

Therefore, if the line \( l_k \) determined by the segment \( x_k x_{k+1} \) of the virtual billiard trajectory within \( Q^{n-1} \) is tangent to a quadric \( Q_{\lambda^*} \), then \( \det L_{x_k,y_k}(\lambda^*) = 0 \), implying \( \det L_{x,y}(\lambda^*) = 0 \) for all \( k \). Also note that \( \det L_{x,y}(\lambda) \) is \( SO(k_1,l_1) \times \cdots \times SO(k_r,l_r) \)-invariant function.

As a result we have:

**Theorem 4.2.** [23] If a line \( l_k \) determined by the segment \( x_k x_{k+1} \) of the virtual billiard trajectory within \( Q^{n-1} \) is tangent to a quadric \( Q_{\lambda^*} \) from the pseudo-confocal family \( \{ Q_\nu \} \), then it is tangent to \( Q_\lambda \) for all \( k \). In addition, \( R(x_k) \) is a virtual billiard trajectory tangent to the same quadric \( Q_\lambda \) for all \( R \in SO(k_1,l_1) \times \cdots \times SO(k_r,l_r) \).

From (33) follows that for a symmetric quadric \( (30) \) we have
\[ P(\lambda) = (\lambda - \alpha_1)^{\delta_1} \cdots (\lambda - \alpha_r)^{\delta_r} \det L_{x,y}(\lambda) \]
\[ = \sum_{s=1}^r (\lambda - \alpha_s)^{\delta_s - 1} \prod_{i \neq s} (\lambda - \alpha_i)^{\delta_i} P_s + \prod_{i \neq s} (\lambda - \alpha_i)^{\delta_i} P_s \]
\[ = \lambda^{N-1} K_{N-1} + \cdots + \lambda K_1 + K_0, \]
where
\[ \delta_s = 2 \text{ for } |I_s| \geq 2, \quad \delta_s = 1 \text{ for } |I_s| = 1, \quad N = \delta_1 + \cdots + \delta_r. \]

In particular, \( K_{N-1} = 2H = (y,y) \). Thus, the degree of \( P(\lambda) \) is \( N - 1 \) for a space-like or time-like vector \( y \), or \( N - 2 \) for a light-like \( y \), and for a general point \( (x,y) \in M_h \), the equation \( \det L_{x,y}(\lambda) = 0 \) has either \( N - 1 \) \((h \neq 0)\) or \( N - 2 \) \((h = 0)\) complex solutions. As in the lemma above, the number of real solutions can be estimated in certain cases. In [23] we proved:

**Theorem 4.3.** [23] Suppose that \( A \) is positive definite or the signature of the space is \((n,0)\). The lines determined by space-like or time-like (respectively light-like) billiard trajectories passing through generic points \((x,y) \in M_h \) are tangent to \( N - 1 \) (respectively \( N - 2 \)) fixed quadrics from the pseudo-confocal family \( \{ Q_\nu \} \).

We proceed with the cases mentioned in the Example 8.

**Theorem 4.4.** (i) Suppose that the condition (35) is satisfied. If \( EA \) is positive or negative definite, that is \( \alpha_r > 0 \) or \( \alpha_1 < 0 \), the lines determined by space-like or time-like (respectively light-like) billiard trajectories passing through generic points \((x,y) \in M_h \) are tangent to \( N - 1 \) (respectively \( N - 2 \)) fixed quadrics from the pseudo-confocal family \( \{ Q_\nu \} \).

(ii) In the case when the condition (40) is satisfied and \( EA \) is positive (negative) definite, the lines determined by generic time-like, light-like, space-like billiard trajectories are tangent to at least \( N - 1 \) \((N - 3)\), \( N - 2 \), \( N - 3 \) \((N - 1)\) quadrics from the pseudo-confocal family \( \{ Q_\nu \} \), respectively.
Proof. The proof is a modification of the idea used in [2, 12] and [23] for an analogous assertion in the case of nonsymmetric ellipsoids and symmetric ellipsoids, respectively. We have

\[ q_\lambda(y, y) = -\sum_{i=1}^{n} \frac{y_i^2}{a_i - \tau_i\lambda} = -\sum_{s=1}^{r} \frac{\langle y, y \rangle_s}{\alpha_s - \lambda} = -\frac{R(\lambda)}{\prod_{s=1}^{r} (\alpha_s - \lambda)}, \]  

(44)

where

\[ R(\lambda) = \sum_{s=1}^{r} \langle y, y \rangle_s \prod_{t \neq s} (\alpha_t - \lambda) = (-1)^{r-1} \cdot \sum_{s=1}^{r} \langle y, y \rangle_s \prod_{t \neq s} (\lambda - \alpha_t). \]

From the definition of \( R(\lambda) \) we obtain

\[ \text{sign} R(\alpha_s) = \text{sign} \ (y, y)_s (-1)^{s+r}, \quad s = 1, \ldots, r \]  

(45)

and for a space–like or a time–like vector \( y \):

\[ \text{sign} R(-\infty) = \text{sign} \ (y, y) = \text{sign} h, \]
\[ \text{sign} R(\infty) = (-1)^{r-1} \text{sign} (y, y) = (-1)^{r-1} \text{sign} h. \]

Thus, for a space–like or a time–like vector \( y \), we have

\[ \text{sign} R(-\infty) \text{sign} R(\alpha_r) = \text{sign} h \text{sign} (y, y)_r \]
\[ \text{sign} R(\infty) \text{sign} R(\alpha_1) = \text{sign} h \text{sign} (y, y)_1. \]

(46)

Assume the relation \( \alpha_r > 0 \). The proof for the case \( 0 > \alpha_1 \) is the same.

(i) From [58], for a generic \((x, y) \in M_h \) we have

\[ \text{sign}(y, y)_1 = \cdots = \text{sign}(y, y)_g = -1, \]
\[ \text{sign}(y, y)_{g+1} = \cdots = \text{sign}(y, y)_r = 1, \]  

(47)

for a certain index \( g \).

From the relations [45], [46], [47], we obtain that the equation \( R(\lambda) = 0 \) has \( r - 2 \) solutions \( \zeta \in (\alpha_{s+1}, \alpha_s) \) for \( s \in \{1, \ldots, r - 1\} \setminus \{g\} \) and another solution \( \zeta_r \in (-\infty, \alpha_r) \) (if \( h < 0 \)) or \( \zeta_0 \in (\alpha_1, \infty) \) (if \( h > 0 \)).

Further, since \((x, y) \in M_h \), it follows that \( 1 + q_0(x, x) = 1 - (A^{-1}x, x) = 0, \)
\[ q_0(x, y) = -(A^{-1}x, y) \neq 0. \]  

Whence,

\[ \det L_{x,y}(0) = -q_0(x, y)^2 < 0. \]

Thus, the left hand side of

\[ \det L_{x,y}(\lambda) = q_\lambda(y, y)(1 + q_\lambda(x, x)) - q_\lambda(x, y)^2 = \frac{P(\lambda)}{\prod_{s=1}^{r} (\lambda - \alpha_s)^{\delta_s}}. \]  

(48)

takes negative values at the ends of each of the \( r - 2 \) intervals

\[ (0, \zeta_{r-1}), (\zeta_{r-1}, \zeta_{r-2}), \ldots, (\zeta_{g+2}, \zeta_{g+1}), (\zeta_{g+1}, \zeta_{g-1}), (\zeta_{g-1}, \zeta_{g-2}), \ldots, (\zeta_2, \zeta_1), \]

and

\[ \alpha_r \in (0, \zeta_{r-1}), \alpha_{r-1} \in (\zeta_{r-1}, \zeta_{r-2}), \ldots, \alpha_{g+2} \in (\zeta_{g+2}, \zeta_{g+1}), \]
\[ \alpha_{g+1}, \alpha_g \in (\zeta_{g+1}, \zeta_{g-1}), \alpha_{g-1} \in (\zeta_{g-1}, \zeta_{g-2}), \ldots, \alpha_2 \in (\zeta_2, \zeta_1). \]
From \(\delta\), we have that \(\tau_i a_i = \tau_j a_j\) only if \(a_i = a_j\) and \(\tau_i = \tau_j\). Hence, generically \(P_s > 0\) for \(\delta_s = 2\). Now, from

\[
\lim_{\lambda \to \alpha_s} \frac{F_s}{\lambda - \alpha_s} + \frac{P_s}{(\lambda - \alpha_s)^2} = \infty, \quad \lim_{\lambda \to \alpha_s + \lambda - \alpha_s} + \frac{P_s}{(\lambda - \alpha_s)^2} = \infty,
\]

(49) and \(\delta\), it follows that in the interval containing \(\alpha_s, s \in \{2, 3, \ldots, r\}\), there are at least two zeros of \(\det L_{x,y}(\lambda)\) for \(\delta_s = 2\) or at least one zero in the case \(\delta_s = 1\). Similarly, in \((\zeta_{g+1}, \zeta_{g-1})\) there are at least \(\delta_g + \delta_{g+1} - 2\) zeros of \(\det L_{x,y}(\lambda)\).

As a result, we get that in \((0, \zeta_1)\) there are

\[
\delta_2 + \cdots + \delta_r - 1 = N - \delta_1 - 2
\]

roots of \(P(\lambda)\).

In the space–like case \(h > 0\), due to \(\delta\), we have a root \(\zeta_0 \in (\alpha_1, \infty)\) of \(R(\lambda)\) and so there are additional \(\delta_1\) roots of \(P(\lambda)\) in \((\zeta_1, \zeta_0)\). Also, according to

\[
\det L_{x,y}(\lambda) = \frac{P(\lambda)}{\prod_{s=1}^{r} (\lambda - \alpha_s)^{s_r}} \sim \frac{\langle y, y \rangle}{\lambda}, \quad \lambda \to \pm \infty,
\]

(50) we have a zero of \(\det L_{x,y}(\lambda)\) in \((\zeta_0, \infty)\) as well. Therefore, the number of real roots of \(P(\lambda)\) is \(N - 1\).

If \(h < 0\), thanks to \(\delta\), there is a zero of \(\det L_{x,y}(\lambda)\) in \((-\infty, 0)\). Consequently, for \(\delta_1 = 1\) we have at least \(N - \delta_1 - 2 + 1 = N - 2\) real roots of \(P(\lambda)\). However, since the polynomial \(P(\lambda)\) is of degree \(N - 1\), it must have \(N - 1\) real roots. By a similar argument there are \(N - 2\) real roots for \(\delta_1 = 1\) and \(h = 0\). If \(\delta_1 = 2\) and \(h < 0\) or \(h = 0\), there is an additional zero of \(\det L_{x,y}(\lambda)\) in \((\zeta_1, \alpha_1)\) and we can proceed as in the \(\delta_1 = 1\) case.

(ii) From \(\delta\), for a generic \((x, y) \in M_h\) we have

\[
\text{sign}(y, y)_1 = \cdots = \text{sign}(y, y)_g = 1, \\
\text{sign}(y, y)_{g+1} = \cdots = \text{sign}(y, y)_r = -1,
\]

(51)

for a certain index \(g\). As above, we obtain that in \((0, \zeta_1)\) there are \(\delta_2 + \cdots + \delta_r - 1 + \delta_r = N - \delta_1 - 2\) roots of \(P(\lambda)\).

From \(\delta\), \(\delta_g = 1\), we have a root \(\zeta_0 \in (\alpha_1, \infty)\) of \(R(\lambda)\) for \(h < 0\). Hence additional \(\delta_1\) roots of \(P(\lambda)\) in \((\zeta_1, \zeta_0)\). Also, according to \(\delta\), we have a zero of \(\det L_{x,y}(\lambda)\) in \((-\infty, 0)\) as well. Therefore, the number of real roots of \(P(\lambda)\) is \(N - 1\).

On the other hand, the analysis above in the space–like case \(h > 0\) implies at least \(N - 3\) real roots of \(P(\lambda)\). The analysis for the light–like case \(h = 0\) is the same as in the proof of (i). \(\square\)

**Remark 4.** In the previous proof we considered the case when \(1 < g < r\). The borderline cases \(g = 1\) and \(g = r\) have similar analysis. Moreover, we have better estimates of the number of quadrics for the assumptions \(\delta\) and \(\delta_g = 1\): if \(EA\) is positive (negative) definite and \(g = 1\) \((g = r)\), then the signature of the space is \((1, n-1)\) \((n-1, 1)\) respectively and there are \(N - 1\) caustics for billiard trajectories with \(h \neq 0\) and \(N - 2\) caustics for \(h = 0\). This situation appears in Theorem 4.7.

**Example 4.** Let us consider \(E_1^{1,1}\) and a nonsymmetric conic defined by \(A = \text{diag}(a_1, a_2), a_1 > 0 > a_2, -a_2 > a_1\) (see Figure 4). Then \(\delta_1 = \delta_2 = 1, \alpha_1 = -a_2 > a_2 = a_1 > 0\) and from Lemma 4.1 through the points \(x = (x_1, x_2)\) outside
the coordinate axes \((x_1 \cdot x_2 \neq 0)\) pass 2 quadrics from the family \[33\]. In the non-light–like case \((h = F_1 + F_2 \neq 0)\), the polynomial
\[
P(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2) \left( \frac{F_1}{\lambda - \alpha_1} + \frac{F_2}{\lambda - \alpha_2} \right) = \lambda h - \alpha_1 \alpha_2 J^2
\]
has the real root \(\lambda = (\alpha_1 \alpha_2 J^2)/h\). This is a root also in the case \(\alpha_1 = a_1 > \alpha_2 = -a_2 > 0\), as well (see Figure 5).

\textbf{Example 5.} Next, we take \(\mathbb{E}^{2,1}\) and a nonsymmetric quadric defined by \(A = \text{diag}(a_1, a_2, a_3)\), \(a_1 = -a_3 > a_2 = a_2 > a_3 = a_1 > 0\). According to Lemma \[4.1\] through the points \(x = (x_1, x_2, x_3)\) outside the coordinate planes \((x_1 \cdot x_2 \cdot x_3 \neq 0)\) pass 3 quadrics from the pseudo–confocal family \[33\]. The discriminant of the polynomial
\[
P(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)(\lambda - \alpha_3) \left( \frac{F_1}{\lambda - \alpha_1} + \frac{F_2}{\lambda - \alpha_2} + \frac{F_3}{\lambda - \alpha_3} \right)
\]
\[
= \lambda^2 h - \lambda ((\alpha_2 + \alpha_3) F_1 + (\alpha_3 + \alpha_1) F_2 + (\alpha_1 + \alpha_2) F_3) + a_1 a_2 a_3 J^2
\]
equals \(D = ((\alpha_2 + \alpha_3) F_1 + (\alpha_3 + \alpha_1) F_2 + (\alpha_1 + \alpha_2) F_3)^2 - 4 a_1 a_2 a_3 h J^2\). It is obvious that in the time–like case the discriminant is positive and we always have two real
roots. From Theorem 4.4 (i) follows that $D > 0$ in the space-like case, too. In the light-like case, the real root is

$$
\alpha_1 \alpha_2 \alpha_3 J^2 / ((\alpha_2 + \alpha_3) F_1 + (\alpha_3 + \alpha_1) F_2 + (\alpha_1 + \alpha_2) F_3).
$$

Let us consider the signature $(n-1,1)$ in general situation. Suppose (35) and let $g \in \{1, \ldots , r\}$ be the index, such that $n \in I_g$. In order to simplify the formulation of the theorem we additionally assume that $\delta_g = 1$, i.e., $I_g = \{n\}$.

**Theorem 4.5.** Consider the lines determined by billiard trajectories in the Lorentz–Poincaré–Minkowski space $\mathbb{E}^{n-1,1}$ passing through generic points $(x,y) \in M_h$.

(i) If $1 < g < r$, the number of their caustics from the pseudo–confocal family \((34)\) is at least $N - 3$ ($h > 0$), $N - 1$ ($h < 0$) or $N - 4$ ($h = 0$). If $\alpha_1 < 0$ and $h > 0$, the number of caustics is $N - 1$.

(ii) Assuming $g = 1$, there are $N - 1$ quadrics ($h \neq 0$) or at least $N - 4$ quadrics ($h = 0$). In addition, if $\alpha_r > 0$ or $\alpha_1 < 0$, there are $N - 2$ tangent quadrics for $h = 0$.

(iii) In the case $g = r$ the minimal number of quadrics is $N - 3$, $N - 1$ and $N - 2$ for $h > 0$, $h < 0$ and $h = 0$, respectively. If we suppose $0 \in (\alpha_r, \alpha_{r-1})$, then there are $N - 1$ quadrics for $h > 0$, as well. If $\alpha_r > 0$ ($\alpha_1 < 0$) and $h > 0$, $h < 0$, $h = 0$, the number of caustics is at least $N - 3$ ($N - 1$), $N - 1$ ($N - 1$), $N - 2$ ($N - 2$), respectively.

**Proof.** Let us prove the item (i). The proof of the other statements is similar.

Since generically $(y,y)_s > 0$ for all $s \neq g$ and $(y,y)_g < 0$, from (43) we have that there exist $r - 3$ solutions $x_s \in (\alpha_{s+1}, \alpha_s)$, $s \in \{1, \ldots , r-1\} \setminus \{g-1, g\}$ of the equation $R(\lambda) = 0$. Note that generically also $P_s > 0$ for $s \geq 2$. Therefore, there are at least

$$
\delta_r - 1 + \cdots + \delta_{g+2} + \delta_{g-2} + \cdots + \delta_2 = N - \delta_1 - \delta_{g-1} - \delta_g - \delta_{g+1} - \delta_r
$$

zeros of $\det L_{x,y}(\lambda)$ in the union $\left(\zeta_{r-1}, \zeta_{g+1}\right) \cup \left(\zeta_{g-2}, \zeta_1\right)$. By considering all the cases when $\delta_{g+1}, \delta_{g-1} \in \{1,2\}$, one concludes that the interval $\left(\zeta_{g+1}, \zeta_{g-2}\right)$ contains at least $\delta_{g+1} + \delta_g + \delta_{g-1} - 2$ zeros, hence there are at least $N - 2 - \delta_1 - \delta_r$ zeros of $\det L_{x,y}(\lambda) = 0$ within the interval $\left(\zeta_{r-1}, \zeta_1\right)$.

In the space-like case $h > 0$, from (50) it follows that there exists $\zeta_r < \alpha_r$, such that $\det L_{x,y}(\zeta_r) < 0$, whence additional $\delta_r$ zeros in $\left(\zeta_r, \zeta_{r-1}\right)$. On the other hand, in $\left(\zeta_1, \infty\right)$ lie at least $\delta_1 - 1$ zeros. In particular, if $\alpha_1 < 0$, we have $\delta_1$ zeros in $\left(\zeta_1, 0\right)$ and, thanks to (50), an additional zero in $\left(0, \infty\right)$.

If $h < 0$, due to (10), there are roots $\zeta_1 > \alpha_1$ and $\zeta_r < \alpha_r$ of $R(\lambda)$ and, consequently, $(\zeta_r, \zeta_0)$ has at least $N - 2$ zeros of $\det L_{x,y}(\lambda) = 0$. Further, from (50) it follows that $(-\infty, \zeta_r)$ also has an additional zero of $\det L_{x,y}(\lambda) = 0$.

Finally, for the light-like trajectories, by considering all the cases when $\delta_r, \delta_1 \in \{1,2\}$, the intervals $(-\infty, \zeta_{r-1})$ and $(\zeta_1, \infty)$ have at least $\delta_r - 1$ and $\delta_1 - 1$ zeros, respectively.

**Remark 5.** Note that, if $g = r$, $\delta_g = 1$ and $0 \in (\alpha_r, \alpha_{r-1})$, then $A$ is positive definite. On the other hand, if $g = 1$ and $0 \in (\alpha_2, \alpha_1)$, then in the case $\delta_1 = 1$, it is $\mathbb{Q}^{n-1} = \emptyset$, since $a_i < 0$ for all $i$.

4.3. The Poncelet porism. Here, we suppose that one of the following conditions holds:

(i) The signature is arbitrary, $A$ is positive definite.
(ii) The signature is \((n, 0)\), \(A\) is arbitrary.

(iii) The signature is arbitrary, \(EA\) is positive or negative definite and the assumption (53) is satisfied.

Then \(\tau_{ai} = \tau_{aj}\) only if \(ai = aj\), \(\tau_i = \tau_j\), and the symmetry group is

\[G = SO(|I_1|) \times \cdots \times SO(|I_r|).\]  

From Theorems 4.3, 4.4 we get that, in the space-like and the time-like cases, given a point \((x, y)\) \(\in M_h\) in a generic position, we have \(N - 1\) caustics

\[Q_{\lambda_1}, \ldots, Q_{\lambda_{N-1}}\]  

determined by the real zeros \(\lambda_1, \ldots, \lambda_{N-1}\) of \(\det L_{x,y}(\lambda)\). They uniquely define the values of the commuting integrals \(F_s, P_s\) on \(M_h\). Similarly, in a light-like case, caustics

\[Q_{\lambda_1}, \ldots, Q_{\lambda_{N-2}}\]  

determined by the real zeros \(\lambda_1, \ldots, \lambda_{N-2}\) of \(\det L_{x,y}(\lambda)\), uniquely define the values of the commuting integrals \(\bar{F}_s, \bar{P}_s\) on \(\bar{M} = M_0/\mathbb{R}\), for a generic \((x, [y]) \in M_0\).

Furthermore, all invariant isotropic tori in \(M_h\) with the same values of \(F_s, P_s\), i.e., all invariant pre-isotropic tori in \(\bar{M}\) with the same values of \(\bar{F}_s, \bar{P}_s\), are related by the action of the groups of symmetries \(\mathbb{Z}_n^2\) (see (6)) and \(G\) (see (52)).

Therefore, by combining Theorems 3.1, 3.2, 4.3, and 4.4, we obtain:

**Theorem 4.6.** If a billiard trajectory \((x_k)\) is periodic with a period \(m\) and if the lines \(l_k\) determined by the segments \(x_kx_{k+1}\) are tangent to \(N - 1\) quadrics (53) (in the space-like or the time-like case) or to \(N - 2\) quadrics (54) (in the light-like case), then any other billiard trajectory within \(Q^{n-1}\) with the same caustics is also periodic with the same period \(m\).

Similarly, Theorem 4.6 applies also in all cases described in Theorem 4.4 (ii) and Theorem 4.5 with maximal number of caustics.

5. **Pseudo–Euclidean billiards in projective spaces.**

5.1. **Billiards on sphere and Lobachevsky space.** It is well-known that the billiards within an ellipsoid \(E^{n-2}\) on the sphere \(S^{n-1}\) and the Lobachevsky space \(H^{n-1}\) are completely integrable [7, 37, 34, 8]. The ellipsoid \(E^{n-2}\) can be defined as an intersection of a cone

\[K^{n-1} : (A^{-1}x, x) = 0,\]  

where

\[A = \text{diag}(a_1, \ldots, a_n), \quad 0 < a_1, a_2, \ldots, a_{n-2}, a_{n-1} < -a_n,\]  

with the Euclidean sphere

\[S^{n-1} = \{(x, x) = 1\} \subset \mathbb{E}^{n,0},\]  

or a connected component of a pseudosphere in the Lorentz–Poincare–Minkowski space \(\mathbb{E}^{n-1,1}\)

\[H^{n-1} = \{(x, x) = -1, \quad x_n > 0\} \subset \mathbb{E}^{n-1,1},\]  

respectively. The induced metrics on \(S^{n-1}\) and \(H^{n-1}\) (a model of the Lobachevsky space) are Riemannian with constant curvatures +1 and −1, while geodesic lines are simply intersections of \(S^{n-1}\) and \(H^{n-1}\) with two–dimensional planes through the origin.
Lemma 5.1. Assume that the signature of the pseudo–Euclidean space $\mathbb{R}^{k,l}$ is $(n,0)$ or $(n-1,1)$, respectively. Let \((x_j, y_j)\) be a trajectory of the billiard mapping $\phi$ given by (61), (62), where $A$ is given by (60). Then the intersections $z_j$ of the sequence of the lines span \(\{x_j\}\) with the ellipsoid $\mathbb{E}^{n-2}$ determine the billiard trajectory within $\mathbb{E}^{n-2}$ on the sphere $\mathbb{S}^{n-1}$ and the Lobachevsky space $\mathbb{H}^{n-1}$, respectively.

**Proof.** Firstly, we prove that the virtual billiard mapping $\phi$ defines the dynamics of the lines span \(\{x_j\}\), i.e., the dynamics of the 2-planes $\pi_j = \text{span} \{x_j, y_j\}$ through the origin.

Consider the transformation

\[
\begin{align*}
x_j' &= \alpha x_j, \\
y_j' &= \beta x_j + \gamma y_j, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \alpha, \gamma \neq 0.
\end{align*}
\]

Let \((x_{j+1}, y_{j+1})\) and \((x_{j+1}', y_{j+1}')\) be respectively the images of \((x_j, y_j)\) and \((x_j', y_j')\) with respect to the mapping $\phi$. Since $x_j, y_j$ and $x_j', y_j'$ determine the same 2-plane $\pi_j$, it follows that $x_{j+1}$ and $x_{j+1}'$ are proportional and belong to $\pi_j \cap \mathbb{K}^{n-1}$. Thus, the tangent planes $T_{x_{j+1}} \mathbb{K}^{n-1}$ and $T_{x_{j+1}'} \mathbb{K}^{n-1}$ are equal and the corresponding billiard Reflections coincide.

Further, the incoming velocities $y_j$ and $y_j'$ also can be related by $y_j' = \beta' x_{j+1} + \gamma' y_j$, for certain $\beta', \gamma' \in \mathbb{R}$. Since $x_{j+1}$ belongs to the tangent plane $T_{x_{j+1}} \mathbb{K}^{n-1}$, after the reflections

\[
y_j \mapsto y_{j+1}, \quad y_j' \mapsto y_{j+1}' = \beta' x_{j+1} + \gamma' y_{j+1},
\]

we get the unique 2-plane

\[
\pi_{j+1} = \text{span} \{x_{j+1}, y_{j+1}\} = \text{span} \{x_{j+1}', y_{j+1}'\}.
\]

Concerning the singular set (61), note that the equation $(A^{-1}x, y) = 0$ is invariant of the mapping $\phi$ and, under condition (60), the only solution of the equations $(EA^{-2}x, y) = 0, (A^{-1}x, y) = 0$ is $x = 0$. Also, if $(A^{-1}y_j, y_j) = 0$, then we can apply the transformation (62) to obtain $(A^{-1}y_j', y_j') = \beta' y_j, y_j)$.

On the other hand, let $z_j, z_{j+1}, z_{j+2} \in \mathbb{E}^{n-2}$ be 3 successive points of the billiard trajectory within $\mathbb{E}^{n-2}$ and let $x = z_j, y_j = z_{j+1} - z_j$. Then

\[
\text{span} \{z_{j+1}, z_{j+2}\} = \text{span} \{x_{j+1}, y_{j+1}\},
\]
where \((x_{j+1}, y_{j+1}) = \phi(x_j, y_j)\), which completes the proof.

In [8], Cayley’s type conditions for periodical trajectories of the ellipsoidal billiard on the Lobachevsky space \(H^{n−2}\) are derived using the “big” \(n \times n\)-matrix representation obtained by Veselov [37]. Here, as a simple modification of the Lax representation [4], we obtain the following “small” \(2 \times 2\) representation obtained by Veselov [37]. Here, as a simple modification of the Lax representation [4], we obtain the following “small” \(2 \times 2\) representation obtained by Veselov [37]. Here, as a simple modification of the Lax representation [4], we obtain the following “small” \(2 \times 2\) representation obtained by Veselov [37]. Here, as a simple modification of the Lax representation [4], we obtain the following “small” \(2 \times 2\) representation obtained by Veselov [37].

Theorem 5.2. The trajectories of the mapping \([55]\), \([60]\) satisfy the matrix equation

\[
\hat{L}_{x_{j+1}, y_{j+1}}(\lambda) = \hat{A}_{x_j, y_j}(\lambda) \hat{L}_{x_j, y_j}(\lambda) \hat{A}_{x_j, y_j}^{-1}(\lambda),
\]

with \(2 \times 2\) matrices depending on the parameter \(\lambda\),

\[
\hat{L}_{x_j, y_j}(\lambda) = \begin{pmatrix}
q_\lambda(x_j, y_j) & q_\lambda(y_j, x_j) \\
-q_\lambda(x_j, x_j) & -q_\lambda(x_j, y_j)
\end{pmatrix},
\]

\[
\hat{A}_{x_j, y_j}(\lambda) = \begin{pmatrix}
I_\lambda + 2J_\lambda \nu_j & -I_\lambda \nu_j \\
-2I_\lambda \nu_j & I_\lambda \nu_j
\end{pmatrix},
\]

where \(q_\lambda\) is given by \([8]\) and \(J_\lambda, I_\lambda, \nu_j\) by \([4]\).

5.2. Billiards in projective spaces. Next, we consider the mapping \([55]\), \([60]\) in the pseudo–Euclidean spaces \(E^{k,l}\) of arbitrary signature and without the assumption \([50]\). We also suppose the symmetries \([50]\). Note that Theorem 5.2 still applies and from the expression

\[
\det \hat{L}_{x_j, y_j}(\lambda) = \sum_{s=1}^{r} \frac{\hat{P}_s}{\lambda - \alpha_s} + \frac{\hat{P}_s}{(\lambda - \alpha_s)^2},
\]

we get the integrals:

\[
\hat{F}_s = \sum_{i \in I_s, j \notin I_s} \frac{(x_iy_j - x_jy_i)^2}{\tau_i a_i - \tau_i a_j},
\]

\[
\hat{P}_s = \sum_{i, j \in I_s, i < j} \tau_i \tau_j (x_iy_j - x_jy_i)^2.
\]

They satisfy the relation

\[
\hat{F}_1 + \cdots + \hat{F}_r = 0.
\]

Further, as in the proof of Lemmas \([54]\) if \((x'_j, y'_j)\) is the image of \((x_j, y_j)\) by the transformation \([62]\) and \((x'_{j+1}, y'_{j+1}) = \phi(x_j, y_j), (x'_{j+1}, y'_{j+1}) = \phi(x'_j, y'_j)\), then the 2–planes spanned by \(x_{j+1}, y_{j+1}\) and \(x'_{j+1}, y'_{j+1}\) coincide. Also, the part of the singular set \((EA^{-2}x, x) = 0\) \(\cup\) \((A^{-1}x, y) = 0\) in \([64]\) is invariant with respect to the transformation \([62]\). If \((A^{-1}y_1, y_1) = 0\), then we can apply the transformation \([62]\) to obtain \((A^{-1}y'_1, y'_1) = \beta \gamma (A^{-1}x_1, y_1) \neq 0\). Thus, if necessary, we can replace \(y_j\) by \(y'_j\) in order to determine \(x_{j+1}\).

Therefore, the dynamics \([55]\), \([60]\) induces a well defined dynamics of the lines span \(\{x_j\}\), i.e., the points of the \((n − 1)\)-dimensional projective space \(\mathbb{P}(E^{k,l})\)

\[
z_j = [x_j] \in \mathbb{Q}^{n−2}
\]
outside the singular set
\[ \mathcal{Z} = \{ [x] \in \mathbb{P}(\mathbb{E}^k) \mid (EA^{-2}x, x) = 0 \}, \]
where \( \mathbb{Q}^{n-2} \) is the projectivisation of the cone \( \mathcal{E} \) within \( \mathbb{P}(\mathbb{E}^k) \).

**Definition 5.3.** We refer to a sequence of the points \( (z_j) \) as a *billiard trajectory* within the quadric \( \mathbb{Q}^{n-2} \) in the projective space \( \mathbb{P}(\mathbb{E}^k) \) with respect to the metric induced from the pseudo–Euclidean space \( \mathbb{E}^k \).

In particular, for signatures \((n, 0)\) and \((n-1, 1)\) with the condition \(\mathcal{E}\), we obtain ellipsoidal billiards on the sphere \(\mathbb{S}^n\) and the Lobachevsky space \(\mathbb{L}^n\), respectively.

Now we consider the following pseudo–confocal family of cones (see [37])
\[ \mathcal{K}_\lambda : (A - E\lambda)^{-1}x, x) = \sum_{i=1}^{n} \frac{x_i^2}{a_i - \tau_i \lambda} = 0, \quad \lambda \neq \tau_i a_i, \quad i = 1, \ldots, n, \] (66)
and the corresponding projectivisation, the pseudo–confocal family of quadrics \( \mathcal{P}_\lambda \).

**Theorem 5.4.** Let \((z_k)\) be a sequence of the points of a billiard trajectory within quadric \(\mathbb{Q}^{n-2}\) in the projective space \(\mathbb{P}(\mathbb{E}^k)\). If a projective line
\[ l_k = z_kz_{k+1} \]
is tangent to a quadric \(\mathcal{P}_\lambda\), then it is tangent to \(\mathcal{P}_\lambda\) for all \(k \in \mathbb{Z}\).

**Proof.** Let \(\pi_I, I = (i_1, \ldots, i_k)\), \(1 \leq i_1 < i_2 < \cdots < i_k \leq n\) be the Plücker coordinates of a \(k\)-plane \(\pi\) passing through the origin in \(\mathbb{R}^n\). Then \(\pi\) is tangent to the nondegenerate cone \(\{ (x, Bx) = 0 \}\), \(B = \text{diag}(b_1, \ldots, b_n)\) if and only if (see Fedorov [15])
\[ \sum_{I} |B|_I \pi_I^2 = 0, \quad |B|_I = b_{i_1} \cdots b_{i_k}. \] (67)

Now, let \(z_k = [x_k]\), \(z_{k+1} = [x_{k+1}]\) and define \(y_k = x_{k+1} - x_k\), \(\pi_k = \text{span} \{ x_k, y_k \}\).

The condition that the plane \(\pi_k\) is tangent to the cone \(\mathcal{K}_\lambda\) from the confocal family \(\mathcal{E}\) is given by the similar invariant expression as in the case of virtual billiards within quadric \(\mathbb{Q}^{n-1}\).
\[ \det \hat{\mathcal{L}}_{x_k, y_k}(\lambda^*) = q_{\lambda^*}(y_k, y_k)q_{\lambda^*}(x_k, x_k) - q_{\lambda^*}(x_k, y_k)^2 = 0. \] (68)

Further, if \(\det \hat{\mathcal{L}}_{x_k, y_k}(\lambda^*) = 0\) for a given \((x_k, y_k)\), it will be zero for all \(k \in \mathbb{Z}\) under the mapping \(\phi\) (Theorem 5.2), while from the description of the billiard dynamics, the projectivisation of \(\pi_k = \text{span} \{ x_k, y_k \}\) equals \(l_k\) for all \(k \in \mathbb{Z}\).

To obtain \(\mathcal{E}\) we set
\[ B = \text{diag} \left( \frac{1}{a_1 - \lambda^* \tau_1}, \ldots, \frac{1}{a_n - \lambda^* \tau_n} \right). \]

Then, in view of \(\mathcal{E}\), the set of the 2-planes \(\pi = \text{span} \{ x, y \}\) that are tangent to \(\mathcal{K}_\lambda\) is described by the following quadratic equation in terms of the Plücker coordinates \(\pi_{i,j} = x_iy_j - x_jy_i, 1 \leq i < j \leq n\) of \(\pi\)
\[ 0 = \sum_{1 \leq i < j \leq n} \frac{1}{(a_i - \lambda^* \tau_j)(a_j - \lambda^* \tau_i)} (x_iy_j - x_jy_i)^2 \]
\[ = \sum_{1 \leq i < j \leq n} \frac{1}{(a_i - \lambda^* \tau_j)(a_j - \lambda^* \tau_i)} (x_i^2y_j^2 - x_ix_jy_iy_j) \]
\[ = \sum \frac{x_i^2}{a_i - \lambda^* \tau_i} \sum \frac{y_i^2}{a_i - \lambda^* \tau_i} - \left( \sum \frac{x_iy_i}{a_i - \lambda^* \tau_i} \right)^2 = \det \hat{\mathcal{L}}_{x,y}(\lambda^*). \]
In order to determine the number of caustics one should provide an additional analysis. The following situation leads to the statement analogous to Theorems 4.3 and 4.4.

As in the case of the ellipsoidal billiards on a sphere $S^{n-1}$ and a Lobachevsky space $H^{n-1}$, we assume the relation (56). Then $\tau_i a_i = \tau_j a_j$ only if $a_i = a_j$, $\tau_i = \tau_j$, $i, j < n$. As above, let $\delta_s = 2$ for $|I_s| \geq 2$, $\delta_s = 1$ for $|I_s| = 1$, and $N = \delta_1 + \cdots + \delta_r$.

**Theorem 5.5.** The lines $l_k = z_k z_{k+1}$ determined by a generic billiard trajectory within $Q^{n-2}$ are tangent to $N-2$ fixed quadrics from the projectivisation of the confocal family (66). In particular, the trajectories of billiards within ellipsoid $E^{n-2}$, with the above symmetry, on the sphere (57) and the Lobachevsky space (58) are tangent to $N-2$ fixed cones from the confocal family (66).

**Proof.** From (64), (65), we get

$$\hat{P}(\lambda) = (\lambda - \alpha_1)^{\delta_1} \cdots (\lambda - \alpha_r)^{\delta_r} \det \hat{L}_{x,y}(\lambda) = \lambda^{N-2} \hat{K}_{N-2} + \cdots + \lambda \hat{K}_1 + \hat{K}_0.$$ 

In addition, under the assumption (56), we can take representatives $x_k, x_{k+1}$ of $z_k, z_{k+1}$, such that the last components are equal to 1. Then, if we denote $x = x_k$ and $y = x_{k+1} - x_k$, we have

$$x = (x_1, \ldots, x_{n-1}, 1), \quad y = (y_1, \ldots, y_{n-1}, 0).$$

From (65) we have $\det \hat{L}_{x,y}(0) < 0$ and following the lines of the proof of Theorem 4.4, it can be proved that the equation $\hat{P}(\lambda) = 0$ has $N-2$ real solutions, for a generic $(x, y)$. \qed

Theorem 5.5 for a nonsymmetric ellipsoid $E^{n-2}$ $(N = n)$ on the Lobachevsky space $H^{n-1}$ is well known (Theorem 3, [37]).

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**REFERENCES**

[1] V. I. Arnol’d, *Matematicheskie Metody Klassicheskoy Mehaniki*, Moskva, Nauka 1974 (Russian). English translation: V. I. Arnol’d, *Mathematical Methods of Classical Mechanics*, Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.

[2] M. Audin, *Courses algébriques et systèmes intégrables: Géodésiques des quadriques*, (French) *Expo. Math.*, 12 (1994), 193–226.

[3] A. Avila, J. De Simoi and V. Kaloshin, An integrable deformation of an ellipse of small eccentricity is an ellipse, *Annals of Mathematics*, 184 (2016), 527–558, arXiv:1412.2853.

[4] A. Banyaga and P. Molino, Géométrie des formes de contact complètement intégrables de type torique, *Séminaire Gaston Darboux*, Montpellier (1991-92), 1–25 (French). English translation: Complete integrability in contact geometry, Penn State preprint PM 197, 1996.

[5] M. Bialy and A. E. Mironov, Angular billiard and algebraic Birkhoff conjecture *Adv. Math.*, 313 (2017), 102–126, arXiv:1601.03196.

[6] M. Bialy and A. E. Mironov, Algebraic Birkhoff conjecture for billiards on Sphere and Hyperbolic plane, *Journal of Geometry and Physics*, 115 (2017), 150–156, arXiv:1602.05698.
V. Dragović and M. Radnović, Billiard algebra, integrable line congruences, and double reflection nets.

Yu. N. Fedorov, Ellipsoidal billiards in quadratic potentials.

Yu. N. Fedorov and B. Jovanović, Continuous and discrete Neumann systems on Stiefel varieties.

B. Jovanović, Contact flows and integrable systems.

B. Jovanović and V. Jovanović, Geodesic and billiard flows on quadrics in pseudo–Euclidean spaces.

B. Jovanović, Heisenberg model in pseudo–Euclidean spaces.

B. Jovanović, The Jacobi–Rosochatius problem on an ellipsoid: The Lax representations and associated geodesic hierarchies.

B. Jovanović, Contact complete integrability.

B. Jovanović and V. Jovanović, Contact flows and integrable systems.

B. Jovanović and V. Jovanović, Geodesic and billiard flows on quadrics in pseudo–Euclidean spaces: L–A pairs and Chasles theorem.

B. Khesin and S. Tabachnikov, Contact complete integrability.

B. Khesin and S. Tabachnikov, Contact flows and integrable systems.

B. Khesin and S. Tabachnikov, Contact flows and integrable systems.

B. Khesin and S. Tabachnikov, Contact flows and integrable systems.

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B. Khesin and S. Tabachnikov, Contact flows and integrable systems.

B. Khesin and S. Tabachnikov, Contact flows and integrable systems.

B. Khesin and S. Tabachnikov, Contact flows and integrable systems.

B. Khesin and S. Tabachnikov, Contact flows and integrable systems.
[31] N. N. Nekhoroshev, Peremennye deystvie–ugol i ih oboshcheniya Tr. Mosk. Mat. O.-va., 26 (1972), 181–198 (Russian). English translation: N. N. Nekhoroshev, Action-angle variables and their generalization, Trans. Mosc. Math. Soc., 26 (1972), 180–198.

[32] M. Radnović, Topology of the elliptical billiard with the Hooke’s potential, Theoretical and Applied Mechanics, 42 (2015), 1–9, arXiv:1508.01025.

[33] Yu. B. Suris, The Problem of Integrable Discretization: Hamiltonian Approach, Progress in Mathematics, 219, Birkhäuser Verlag, Basel, 2003.

[34] S. L. Tabachnikov, Ellipsoids, complete integrability and hyperbolic geometry, Mosc. Math. J., 2 (2002), 183–196.

[35] S. Tabachnikov, Geometry and Billiards, volume 30 of Student Mathematical Library. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2005.

[36] A. P. Veselov, Integriruemye sistemy s diskretnym vremenem i raznostnye operatory Funk. Analiz i ego Prilozh. 22 (1988), 1–13 (Russian); English translation: A. P. Veselov, Integrable systems with discrete time, and difference operators, Funct. Anal. Appl. 22 (1988), 83–93.

[37] A. P. Veselov, Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space, J. Geom. Phys., 7 (1990), 81–107.

[38] A. P. Veselov, Integriruemye otoobrazheniya, Uspekhi Mat. Nauk, 46 (1991), 3–45 (Russian); English translation: A. P. Veselov, Integrable mappings, Russ. Math. Surv., 46 (1991), 1–51.

[39] A. P. Veselov, Growth and integrability in the dynamics of mappings, Comm. Math. Phys., 145 (1992), 181–193.

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