Integrable Perturbations of $W_n$ and WZW Models

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Abstract

We present a new class of 2d integrable models obtained as perturbations of minimal CFT with $W$-symmetry by fundamental weight primaries. These models are generalisations of well known $(1,2)$-perturbed Virasoro minimal models. In the large $p$ (number of minimal model) limit they coincide with scalar perturbations of WZW theories. The algebra of conserved charges is discussed in this limit. We prove that it is noncommutative and coincides with twisted affine algebra $G$ represented in a space of asymptotic states. We conjecture that scattering in these models for generic $p$ is described by $S$-matrix of the $q$-deformed $G$ - algebra with $q$ being root of unity.
1 Introduction

It was pointed out in [1] that a large class of 2d quantum integrable field theories can be considered as certain perturbation of 2d CFT by an operator $\Phi(z, \bar{z})$ from the Hilbert space of a theory

$$S_{IM} = S_{CFT} + \lambda \int \Phi(z, \bar{z})d^2z$$  \hspace{1cm} (1)

If the conformal symmetry generated by holomorphic tensor $T(z)(\bar{T}(\bar{z}))$ is the only affine symmetry of a theory, the field $\Phi$ has to be completely degenerate primary with respect to this symmetry (see [2] for review). Complete degeneracy leads to a substantial lowering of dimensionality of irreducible highest weight representation. As a result, some holomorphic polynomials of $T(z)$ and its derivatives survive the perturbation. They become left components of infinitely many integrals of motion (IM). Strictly speaking, aside from dimensionality of irreducible representation of Virasoro algebra defined by perturbing field, one has to take into account renormalizability properties of a perturbed CFT. Renormalization generically brings about counterterms which are fields from a fusion algebra generated by $\Phi$ and their conformal descendants. These counterterms could destroy integrability. They do not appear if $\Phi$ is the most relevant primary in a theory, or at least in the fusion algebra generated by $\Phi$. But even if this is not the case one can argue [3] that counterterms (descendants of other primaries) do not contribute to renormalization of conservation laws unless some special "resonance" condition on conformal dimensions of primaries holds. We will keep in mind the case when "resonance" condition does not spoil integrability. An examination of the Virasoro characters of irreducible highest weight representations shows all of the possible relevant primary fields leading to integrable theories. These are $\Phi_{(1,2)}[\Phi_{(2,1)}], \Phi_{(1,3)}, \Phi_{(1,5)}$. The field $\Phi_{(1,5)}$ turns out to be relevant in nonunitary minimal models $(p,q)$ if $p < q/2, q > 6$. The field $\Phi_{(2,1)}$ is relevant for $q/2 < p < q$. This $(2,1)$ and $(1,5)$ duality is probably the simplest example of Affine Toda dualities which has been observed recently in several papers [1, 2, 3].

In this particular case we deal with $A_2^{(2)}$ selfduality. This does not mean that these are the only integrable perturbations for all the $(p,q)$ - models. For some of them perturbations not belonging to this list are integrable, but operators from the list have a special reason to be integrable, namely, they can be associated with quantum Affine Toda Theories (ATT) [4, 5, 6, 7, 8]. The field $\Phi_{(1,3)}$ corresponds to the complex $A_1^{(1)}$, or the Sine-Gordon model, $\Phi_{(1,2)}, \Phi_{(2,1)}, \Phi_{(1,5)}$ lead to the complex $A_2^{(2)}$ ATT [7]. The latter one is non-hermitian and admits a physical interpretation after an appropriate quantum group (QG) reduction [7].

The same argument applied to $W_3$ theories gives the following list of operators $F_{(22|11)}, F_{(21|11)}, F_{(11|21)}, F_{(41|11)}$, where we use notations introduced by Zamolodchikov, Fateev and Lukyanov [14, 15]. The first operator corresponds to $A_1^{(1)}$ and is well known $(1, 1, Adj)$- perturbation explicitly studied in [16]. It preserves $S_3$ symmetry of $W_3$ minimal model and is in a sense an analog of $(1,3)$ from Virasoro minimal model.

The second and the third operators correspond to $G_2^{(1)}$ ATT and are generalizations of $(1,2)$ and $(2,1)$. The last operator corresponds to $D_4^{(3)}$ which is dual to $G_2^{(1)}$ and
generalizes (1,5). $A_2$ finite Dynkin diagram enters as a subdiagram affine diagrams $A_2^{(1)}$, $G_2^{(1)}$ and $D_4^{(3)}$.

For $W_4$ algebra we have $(21|111), (12|111), (13|111)$ corresponding to $A_3^{(1)}, B_3^{(1)}$ and $A_5^{(2)}$.

These simple observations are just hints and possible integrability of listed perturbations needs to be proven rigorously. In Sections 2,3 we prove integrability of $(21|11), (11|21)$ perturbation of minimal $W_3 (p,p+1)$ models and $(12|111), (111|21)$ perturbations of $W_4$. In Section 4 we argue that in the $p \to \infty$ limit factorized scattering theory defined by these perturbations is one of $A_2^{(1)} k = 1$ WZW model perturbed by $(1,0) \otimes (0,1)$ operator and one of $A_3^{(1)} k = 1$ WZW model perturbed by $(0,1,0) \otimes (0,1,0)$ operator. These perturbations preserve diagonal $A_2$ and $A_3$ symmetries respectively. It appears to be possible to construct infinite set of conserved non-commutative charges in these perturbed WZW acting as $D_4^{(3)}$ and $A_5^{(2)}$ on asymptotic states. It is interesting what is the full algebra generated by these charges. The work on it is in progress now. It seems natural to assume that for finite $p$ these theories exhibit $q$-deformed $D_4^{(3)}$ and $A_5^{(2)}$ symmetries, i.e. fundamental kinks belong to vector representations of twisted affine Lie algebras.

## 2 Integrable perturbations of $W_3$ minimal models.

$(21|11)$ and $(12|11)$ operators form $S_3$-doublet and can be represented by free scalar fields:

$$F_{(21|11)} = e^{-i\alpha_+ \omega_1 \phi} = \Omega(z, \bar{z})$$  \hspace{1cm} (2)
$$F_{(12|11)} = e^{-i\alpha_+ \omega_2 \bar{\phi}} = \Omega^+(z, \bar{z})$$  \hspace{1cm} (3)

$\omega_1, \omega_2$ are fundamental weights of $su(3)$ algebra and $\alpha_+ = \sqrt{p \over p+1}$. In $(p,p+1)$ model their dimension equals

$$\Delta_{(21|11)}^{(p)} = \Delta_{(12|11)}^{(p)} = \frac{1}{3}(1 - \frac{4}{p+1})$$  \hspace{1cm} (4)

IM of CFT, i.e. normally ordered polynomials of Lorentz spin $s$ $P_s(T, W)$ are generically spoiled by $W$-descendants of $\Omega(z, \bar{z})$ but it may happen that for some of them this $W$-descendant is total derivative:

$$\bar{\partial} P_s(T, W) = \partial \Omega_{s-2}(z, \bar{z})$$  \hspace{1cm} (5)

That is always the case when a number of level $s$ descendants excluding total derivatives in a vacuum module of a theory is greater than a number of level $s - 1$ descendants of perturbing field. Comparison of $W_3$ characters $\chi_{(21|11)}^{(p)}$ and $\chi_{(11|11)}^{(p)}$ with total derivatives subtracted shows that it happens for $s = 6$ and any $p$. Formulae for $W_n^{(p)}$ characters obtained in [17] reads:

$$\chi^{(p)}(\Omega_1, \Omega_2) = \prod_{i=1}^{\infty} (1 - q^i)^{-r} \sum_{s \in W} \sum_{\lambda \in \Lambda_R} det(s) q^{[p s((\Omega_1) -(p+1)\Omega_2 + p(p+1)\lambda)]^2 / 2p(p+1)}$$  \hspace{1cm} (6)
where the sums run over the elements $s$ of the Weyl group $w$ and the root lattice $\Lambda_R$ of the Lie algebra $A_{n-1}$. Let us note that Verma module of completely degenerate highest weight of $W_n$ algebra contains $n$ independent null-vectors. The sum over Weyl group accounts for $(n-1)$ of them as long as the sum over root lattice accounts for $n$-th one. If $p$ is sufficiently large $n$-th null-vector does not affect multiplicities of first levels and is irrelevant for our considerations. So we can keep only the $\lambda = 0$ term. Sum over the Weyl group can be calculated explicitly. Simplified character formula for $W_3$ is

$$\tilde{\chi}^{(p)}_{(21|11)} = q^{\Delta} \prod_{i=1}^{\infty} (1 - q_i^*)^{-2}(1 - q - q^2 + q^4 + q^6)$$

(7)

where $\Delta$ means Virasoro dimension of primary field. Subtraction of total derivatives yields

$$(1 - q)\tilde{\chi}^{(p)}(\Omega_1, \Omega_2) = q^\Delta \sum_i D_i^{(\Omega_1, \Omega_2)} q^i$$

(8)

where $D_i$ are multiplicities we search for. For $p > 5$ (7) and (8) give

$$D_6^{(11|11)} = 4$$

$$D_5^{(21|11)} = 3$$

For $p = 4$ (critical $Z_3$ model) additional null-vectors are to be taken into account what lowers by one both dimensionalities

$$D_6^{(11|11)} = 3$$

$$D_5^{(21|11)} = 2$$

It means that there exists

$$P_6(z, \bar{z}) = a : T^3 : +b : (\partial T)^2 : +c : W^2 : +d : W\partial T :$$

(9)

obeying (5). Explicit expressions for $a, b, c, d$ can be obtained but they are not essential for the rest of the paper. Let us note nevertheless that $c$ vanishes for the first minimal model $W_3^{(4)}$. Appearance of the IM of spin 5

$$Q_5 = \int dz P_6(z) + \int d\bar{z} \Omega_4(z, \bar{z})$$

(10)

does not seem strange. The weight of perturbing operator together with two simple roots of $A_2$ algebra forms the root system of affine $G_2^{(1)}$ algebra with Coxeter exponents $1, 5 \pmod{6}$. So, the nontrivial IM we have found confirms $G_2^{(1)}$-origin of $(21|11)$ perturbation of $W_3$. The same reasoning is valid for $\Omega^+$- perturbation with exchange $d \rightarrow -d$. Let us note, that $s = 5$ conserved charge can be constructed for $(11|21)$ perturbation as well. But $G_2^{(1)}$-structure shows up only for sufficiently large $p$. For the lowest $p = 4$

$$(11|21) = \Psi(z)$$

$$\Omega^+(11|12) = \Psi^+(z)$$

these are Zamolodchikov-Fateev parafermions. The theory perturbed by hermitian combination of them possess IM with $s = 3$ due to additional degeneracy and coincides with restricted SG.
3 Integrable perturbation of $W_4$ minimal model

Besides operator $(1, 1, \text{Adj})$ in $W_4^{(p)}$ models exists one more defining integrable perturbation

$$F_{(121|111)} = e^{-i\alpha + \omega_2 \phi}$$  \hspace{1cm} (11)

$\omega_2$ is the second fundamental weight of $A_3$. In order to prove it we have to compare two characters $\chi_{(121|111)}^{(p)}$ and $\chi_{(111|111)}^{(p)}$. As before we will consider several lowest levels of Verma modules unaffected by fourth null-vector. Then the same approximation as in the previous case is applicable. It becomes exact for irrational values of central charge and for $c = n - 1 = 3$.

$$\tilde{\chi}_{(121|111)} = q^\Delta \prod_{i=1}^{\infty} (1 - q^i)^{-3}(1 - 2q + 2q^4 + 2q^5 - 3q^6 - 3q^7 + 2q^8 + 2q^9 - 2q^{12} + q^{13})$$  \hspace{1cm} (12)

where $\Delta$ means Virasoro dimension of primary field. Subtraction of total derivatives gives

$$D_{4}^{(111|111)} = 2$$
$$D_{3}^{(121|111)} = 1$$

So we discover a nontrivial IM of spin $s = 3$

$$P_3 = \int dz : T^2 : + kW_4 + \lambda \int d\bar{z} F_2(z, \bar{z})$$

The weight of perturbing operator along with three simple roots of $A_3$ algebra forms the root system of affine $B_3^{(1)}$ algebra with Coxeter exponents $1, 3, 5 \pmod{6}$. So, the nontrivial IM we have found confirms $B_3^{(1)}$-origin of $(121|111)$ perturbation of $W_4$. We conjecture that these integrable models are lowest in hierarchy $WD_n^{(p)} + F_{\text{vec}}$, where $F_{\text{vec}}$ is $(21...1|1...1)$-primary.

4 $p \to \infty$ and perturbed WZW

Let us consider $p \to \infty$ limit of perturbed $WX_n$ models. Minimal $WX_n$ CFT constructed out of simply-laced affine Lie algebra

$$WX_n^{(x(n)+p)} = \frac{X_n^{(1)} \times X_n^{(1)}}{X_{n,p+1}^{(1)}}$$

admit well known integrable $(1, 1, \text{Adj})$ perturbation. It coincides in this limit with $JJ$-perturbation of $X_n^{(1)}$ WZW model, or with complex ATT at some specific value of coupling constant. It enjoys $X_n$ symmetry which acts as diagonal $(J_0 + \bar{J}_0)$-algebra of zero modes. Solitonic particles appear as fundamental $n$-plet of $X_n$. As was argued in \cite{19, 20, 21, 22, 23} for $p < \infty$ this symmetry becomes $q$-deformed quantum group with

$$q^{x(n)+p} = 1,$$
where $x(n)$ is a dual Coxeter number of $X_n$. We argue that for fundamental weight perturbation we get some affine twisted symmetry with $X_n$ as zero mode subalgebra. Let $X_n$ be $A_1$. For $(1, 2)$ perturbation of Virasoro minimal models it was shown by F. Smirnov [13] that for $p \to \infty$ scattering theory is one of SG model at $\beta = \frac{1}{\sqrt{2}}$. Let us note that this model can be thought of as a fundamental weight perturbation of $A_1 \ k = 1$ WZW theory preserving diagonal $A_1$.

$$ S_{(1,2)}^{(p)} \big|_{p \to \infty} = S_{wzw} + \lambda \int d^2 z \left[ \phi^{1/2,1/2} \otimes \overline{\phi}^{1/2,-1/2} - \phi^{1/2,-1/2} \otimes \overline{\phi}^{1/2,1/2} \right] $$

One might think that being simply-laced SG model ”forgets” about it’s $A_2^{(2)}$ - origin, but amazingly it turns out to keep ”memory” of $A_2^{(2)}$. Namely, it exhibits noncommuting conserved charges which act on solitonic sector by vector representation of $A_2^{(2)}$. SG theory at generic point possess commuting IM with spins $s = 1, 3, 5 \ldots$. At $\beta = 1/\sqrt{2}$ (the point in consideration) there are additional IM. In order to classify them let us note that those are first of all

$$ L^\pm,0 = J_0^{\pm,0} + \bar{J}_0^{\pm,0} $$

Each IM $Q_s$ of odd (possibly negative) spin acquires an angular momentum with respect to this $A_1$ algebra. It turns out that

$$ j = 2 \quad \text{for } s = 6k + 3, $$

$$ j = 0 \quad \text{for } s = 6k \pm 1. $$

Besides it there are additional $j = 1$ charges of Lorentzian spin $s = 6k$, $k \in Z$. Non-zero $j$-momentum means that there is a $j$-multiplet of charges of a spin $s$. Let us prove this result for $s = 3$. We deal with perturbed WZW model, i.e. original space of holomorphic currents is given by polynomials of $A_1^{(1)}$ currents $P^{j,m}(J^{\pm,0}(\bar{z})$) and $\bar{P}^{j,m}_s(J^{\pm,0}(\bar{z}))$. $j, m$ are angular momentum and it’s 0- component of $P$-charges with respect to $A_1$ symmetry. After perturbation we get

$$ \bar{\partial} P^{j,m}_s = \lambda [\phi^{1,m+1/2} \otimes \overline{\phi}^{1/2,-1/2} - \phi^{1,m-1/2} \otimes \overline{\phi}^{1/2,1/2}] $$

$\phi^{j,m}_s$ is a field in Verma module of highest weight $1/2$ with given quantum numbers. $A_1$-momentum conservation yields

$$ j = l \pm 1/2 $$

Now we are going to apply counting argument invented by A.Zamolodchikov with some modification. Instead of counting of multiplicities of single operators we will count multiplicities of $A_1$-multiplets. Examining characters of affine $A_1^{(1)}$ algebra with $k = 1$ one notices that after subtraction of total derivatives in the vacuum module at $s = 4$ two multiplets ($j = 0, 2$) remain there. In 1/2-module at $s = 3$ there is only one $l = 1/2$ which fails to obey $j = l \pm 1/2$ selection rule for $j = 2$. So, we stay along with five conserved charges of spin $s = 3$!

$$ Q^{2,m}_3 = \int dz P^{2,m}_3 + \lambda \int d\bar{z} [\phi^{3/2, m+1/2} \otimes \overline{\phi}^{1/2,-1/2} + \phi^{3/2, m-1/2} \otimes \overline{\phi}^{1/2,1/2}] $$

6
In this formula we imply that \( \phi_{j \rightarrow m} = 0 \) if \( m > l \). \( P_{4}^{2 \cdot m} \) is the only \( j = 2 \)-plet in 0-module on level 4 and \( \varphi_{2 \cdot 3/2 \cdot m} \) is the only \( j = 3/2 \)-plet in 1/2-module on level 2. So, (15) unambiguously defines conserved charges. These five charges along with their right counterparts of \( s = -3 \) and three \( A_{1} \)-charges generate the whole algebra. Consider now the \( A_{2}^{(2)} \mapsto A_{2}^{(1)} \) embedding. \( A_{2} \) can be thought of as \( Z_{2} \)-graded algebra

\[
A_{2} = A_{2}^{+} + A_{2}^{-},
\]

where \( A_{2}^{+} = A_{1} = \text{so}(3) \) and \( A_{2}^{-} \) is \( j = 2 \) representation of \( \text{so}(3) \) adjoint action. Commutation relation in \( A_{2} \) can be schematically written down as follows

\[
[A_{2}^{+}, A_{2}^{+}] = A_{2}^{+}, \quad [A_{2}^{+}, A_{2}^{-}] = A_{2}^{-}, \quad [A_{2}^{-}, A_{2}^{-}] = A_{2}^{+}.
\]

Now we take the subalgebra of \( A_{2}^{(1)} \) generated by \( [(A_{2}^{+})_{2n}, (A_{2}^{-})_{2n+1}; n \in \mathbb{Z}] \)

This subalgebra coincides with \( A_{2}^{(2)} \) for \( k = 0 \). Conserved \( Q \)-charges can be identified with generators of \( A_{2}^{(2)} \)

\[
Q_{6n+3}^{2 \cdot m} = (A_{2}^{-})_{2n+1}, \quad Q_{6n}^{1 \cdot m} = (A_{2}^{+})_{2n}, \quad n \in \mathbb{Z}, \quad n \geq 0.
\]

If action of \( Q \)-charges on asymptotic triplet states (kink-first breather- antikink) \(|\theta, \pm >, |\theta, 0 >\) is considered, one can complete the identification by

\[
Q_{6n+3}^{2 \cdot m} = (A_{2}^{-})_{2n+1}, \quad Q_{6n}^{1 \cdot m} = (A_{2}^{+})_{2n}, \quad n \in \mathbb{Z}, \quad n < 0.
\]

This identification immediately follows if we recall that asymptotic states are eigenstates of canonical IM of SG model - \( Q_{s}^{2 \cdot 0}, \bar{Q}_{s}^{2 \cdot 0} \) in our notations - and form \( A_{1} \)-triplet. So the explicit action of conserved charges on asymptotic states after an appropriate normalization is given by \( 3 \times 3 \) matrices of \( A_{2}^{(2)} \) vector representation \( [A_{2}^{+} \tau^{2n}, A_{2}^{-} \tau^{2n+1}, n \in \mathbb{Z}], \tau = e^{3 \theta}, \theta - \text{particle rapidity.} \)

A very similar picture can be obtained for \((21\mid 11)\)-perturbed \( W_{3}^{(p)} \) minimal models.

In the \( p \rightarrow \infty \) limit it coincides with \( A_{2}^{(1)} k = 1 \) WZW model perturbed by \( A_{2} \)-scalar operator

\[
H = \Phi^{(1,0)} \otimes \bar{\Phi}^{(-1,0)} + \Phi^{(-1,1)} \otimes \bar{\Phi}^{(1,-1)} + \Phi^{(-1,-1)} \otimes \bar{\Phi}^{(1,1)}
\]

Upper indices indicate \( A_{2} \)-weights of operators in fundamental and conjugate representations. This model is nothing but complex \( A_{2} \) ATT at \( \beta = 1/\sqrt{3} \) and possess therefore a set of commuting IM of \( s = 1, 2 \) (mod 3). One can check that these IM are not singlets but rather multiplets of diagonal \( A_{2} \) symmetry preserved by \( H \)-perturbation. We will list the highest weights \((j_{1}, j_{2})\) of these multiplets

\[
(j_{1}, j_{2}) = (3, 0) \text{ for } s = 6n + 2,
\]

\[
(j_{1}, j_{2}) = (0, 3) \text{ for } s = 6n - 2,
\]

\[
(j_{1}, j_{2}) = (0, 0) \text{ for } s = 6n \pm 1,
\]

\[n \in \mathbb{Z}.\]
Besides these one, there are \( s = 6n \) charges absent in generic \( A_2^{(1)} \) ATT with \((j_1, j_2) = (1, 1)\). This result can be obtained by modified counting argument as in the previous case. We argue that conserved charges \( Q_{s}^{(j_1,j_2)} \) and \( \bar{Q}_{-s}^{(j_1,j_2)} \) for \( s = 2n \) form two subalgebras of \( D_4^{(3)} \) and give the full \( D_4^{(3)} \) if acting on solitonic sector of the theory. Let us define first \( D_4^{(3)} \) as a certain subalgebra of \( D_4^{(1)} \). \( D_4 \) can be decomposed into direct sum of three linear spaces:

\[
D_4 = A^{(1,1)} \oplus A^{(3,0)} \oplus A^{(0,3)}.
\]

These are spaces of irreducible representations of adjoint \( A_2 \)-action embedded in \( D_4 \) by \((1,1)\)-representation. Commutation relation in \( D_4 \) schematically look as follows

\[
[A^{(1,1)}, A^{(1,1)}] = A^{(1,1)}, \quad [A^{(1,1)}, A^{(3,0)}] = A^{(3,0)}, \quad [A^{(1,1)}, A^{(0,3)}] = A^{(0,3)},
\]

\[
[A^{(3,0)}, A^{(3,0)}] = A^{(0,3)}, \quad [A^{(3,0)}, A^{(0,3)}] = A^{(1,1)}, \quad [A^{(0,3)}, A^{(0,3)}] = A^{(3,0)}.
\]

Thereby \( Z_3 \)-graded Lie algebra structure is defined. Then \( D_4^{(3)} \) basis is subset of \( D_4^{(1)} \) one’s

\[
Q_{6n}^{(1,1)} = A_{3n}^{(1,1)}, \quad Q_{6n+2}^{(3,0)} = A_{3n+1}^{(3,0)}, \quad Q_{6n-2}^{(0,3)} = A_{3n-1}^{(0,3)}, \quad n \geq 0
\]

\[
\bar{Q}_{6n}^{(1,1)} = A_{3n}^{(1,1)}, \quad \bar{Q}_{6n+2}^{(3,0)} = A_{3n+1}^{(3,0)}, \quad \bar{Q}_{6n-2}^{(0,3)} = A_{3n-1}^{(0,3)}, \quad n \leq 0.
\]

The \( S \)-matrix of complex \( A_2^{(1)} \) ATT has been constructed recently by T. Hollowood [24]. In \( A_2^{(1)} \) ATT at \( \beta = 1/\sqrt{3} \) scattering bootstrap exhibits eight lightest particles of the same mass including 3 and 3-plets of fundamental kinks and antikinks along with two breathers. The mass of these breathers coincides with the mass of kinks precisely at that value of \( \beta \)! All these particles form vector representation of \( D_4^{(3)} \). It is curious that here we encounter kind of non-simply-laced duality observed recently in [4, 5]. Namely, starting from \( G_2^{(1)} \)-like perturbation of \( W_3 \) we get in the \( p \to \infty \) limit hidden \( D_4^{(3)} \) which can be obtained by inversion of the \( G_2^{(1)} \) Dynkin diagram arrows. The same is true for SG but \( A_2^{(2)} \) hidden symmetry is selfdual. Now it looks natural to expect that in \( W_4^{(p)} \) model perturbed by \( B_3^{(1)} \)-like (121|111)-operator we should find \( A_2^{(2)} \) dual to \( B_3^{(1)} \). Let us argue that it is true. As before in the \( p \to \infty \) limit we arrive at perturbed \( A_3 \) WZW model with \( k = 1 \):

\[
S = S_{wzw} + \lambda \int d^2z V(z, \bar{z}) \tag{16}
\]

where

\[
V(z, \bar{z}) = \sum_{\bar{\mu} \in \pi_{(0,1,0)}} \Psi_{\bar{\mu}}^{(0,1,0)} \otimes \bar{\Psi}_{-\bar{\mu}}^{(0,1,0)}
\]

(0, 1, 0) denotes 6-plet of \( A_3 \). It is a free massive theory of six Majorana fermions:

\[
S = \sum_{i=1}^{6} \int d^2z (\bar{\psi}_i \partial \psi_i + \bar{\psi}_i \partial \bar{\psi}_i + m \bar{\psi}_i \psi_i) \tag{17}
\]
A\(_4\) algebra is generated by 15 spinless charges

\[ Q_{ij}^0 = \int dz \psi_i \psi_j + \int d\bar{z} \bar{\psi}_i \bar{\psi}_j \]

belonging to (1,0,1)-representation. At spin one we have 21 conserved charges. One of them is momentum generator

\[ p = \sum_{i=1}^{6} \left( \int dz \psi_i \partial \psi_i - 2m \int d\bar{z} \bar{\psi}_i \psi_i \right) \]

which is \( A_3 \)-scalar. Another 20 charges form (0,2,0)-representation of \( A_3 \):

\[ Q_{ij}^1 (G) = \sum_{i,j=1}^{6} \left( \int dz G_{ij}^{ij} \psi_i \partial \psi_j - m \int d\bar{z} G_{ij}^{ij} [\psi_i \bar{\psi}_j + \psi_j \bar{\psi}_i] \right) \]

\( G \) is traceless symmetric 6×6 matrix. The charges \( Q_{ij}^0, Q_{ij}^1 \) and \( \bar{Q}_{ij}^{-1} \) and their pairwise commutator generate \( A_3^{(2)} \). Proof is absolutely similar to two previous examples if one notes that \( A_5 \) is decomposed as \((1,0,1) + (0,2,0)\) under adjoint action of \( A_3 \) embedded as \( so(6) \). This algebra has been discovered independently in [25]. Different twisted affine algebras appear as subalgebras of that symmetry. Obviously, this construction is valid for any \((21 \ldots 1 | 1 \ldots 1)\)-perturbed \( WD_n \).

5 Discussion

We considered quantum integrable models defined as relevant perturbation of simply-laced \( W \)-models \((W^{(p)}_3, WD^{(p)}_n, n \geq 3)\) by operator different from \((1,1,Adj)\) and their \( p \rightarrow \infty \) limit. All these models are generalizations of \((1,2)\)-perturbed minimal CFT. Scattering theory of \((1,1,Adj)\)-models is commonly believed to be described in terms of finite quantum group \((X_n)_q\). We conjecture that for fundamental weight perturbations \( q \)-deformation of some twisted affine Lie algebra is the key symmetry of a theory. This algebra is dual to affine algebra obtained as extension of \( X_n \) by a weight of perturbing operator. So we get \( A_2^{(2)} \) for \((1,2)\) perturbed CFT \((A_2^{(2)}_q)\) R-matrix was used by F.Smirnov in order to construct \( S \)-matrices for these models), \( D_4^{(3)} \) for \((2,1|1,1)\)-perturbed \( W_3 \) CFT and \( A_2^{(2)}_{2n-1} \) for vector \((21\ldots 1 | 1 \ldots 1)\)-perturbation of \( WD_n \). \( S \)-matrix construction for these algebras will be addressed in the future paper. In the \( p \rightarrow \infty \) limit \( q \)-symmetry becomes classical what is explicitly demonstrated. Generators of affine twisted symmetry appear as Noether charges of perturbed WZW models which arise in considered limit. This result seems to be similar to recent construction of A.LeClair [26]. The lowest models for \( n = 2,3 \) can be identified as critical Ashkin-Teller models (decoupled Isings and \( Z_4 \)-parafermions) in electric field.

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