HIROTA QUADRATIC EQUATIONS FOR THE EXTENDED
TODA HIERARCHY

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ABSTRACT. The Extended Toda Hierarchy (shortly ETH) was introduce by E. Getzler [Ge] and independently by Y. Zhang [Z] in order to describe an integrable hierarchy which governs the Gromov–Witten invariants of \(\mathbb{C}P^1\). The Lax type presentation of the ETH was given in [CDZ]. In this paper we give a description of the ETH in terms of tau-functions and Hirota Quadratic Equations (known also as Hirota Bilinear Equations). A new feature here is that the Hirota equations are given in terms of vertex operators taking values in the algebra of differential operators on the affine line.

1. Introduction

We begin with a description of the KdV hierarchy of integrable systems which will be used as a model for introducing tau-functions and Hirota Quadratic Equations (shortly HQE) of the ETH. Both the KdV and the ETH hierarchies will be presented in the form convenient for the applications to Gromov–Witten theory. In particular, we work over the field \(\mathbb{C}((\epsilon))\), where \(\epsilon\) corresponds to the genus expansion parameter.

The KdV hierarchy can be described in the Lax form [GD] as a sequence of commuting flows on the space of Lax operators

\[ L = \frac{\epsilon^2 \partial_x^2}{2} + u(x, \epsilon), \]

where \(u = u_0(x) + u_1(x)\epsilon + u_2(x)\epsilon^2 + \ldots\) is a power \(\epsilon\)-series with infinitely differentiable coefficients. By definition, the flows have the form

\[ \frac{\partial}{\partial q_n} L = \epsilon^{-1} \left[ \left( \frac{(2L)^{n+1/2}}{(2n+1)!!} \right) + L \right], \quad n = 0, 1, 2, \ldots, \]

where \(B_+\) means the differential part \(\sum_{m \geq k \geq 0} w_k(x)\partial_x^k\) of a pseudo-differential operator \(B = \sum_{m \geq k > -\infty} w_k(x)\partial_x^k\), and \((2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)\).

Given a Lax operator \(L\) there exists an integral operator \(P = 1 + w_1 \partial_x^{-1} + w_2 \partial_x^{-2} + \ldots\), called dressing operator such that \(L = P(\epsilon^2 \partial_x^2/2)P^{-1}\). A Lax operator \(L\) is a solution to the KdV hierarchy if and only if there exists a dressing operator \(P\) satisfying the equation

\[ \epsilon \frac{\partial}{\partial q_n} P = - \left( \frac{(2L)^{n+1/2}}{(2n+1)!!} \right)_- P. \]

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Such a dressing operator is called \textit{wave operator of the KdV hierarchy}. The wave operator is uniquely determined by $L$ up to multiplication from the right by an operator of the form $1 + a_1 \partial_x^{-1} + a_2 \partial_x^{-2} + \ldots$ where $a_i$ are constants independent of $x$ and the time variables $q_n$, $n \geq 0$.

An equivalent description of the KdV hierarchy can be given in terms of tau-functions and Hirota quadratic equations (see e.g. [K]). A non-vanishing function $\tau(q_0, q_1, \ldots; \epsilon)$, is called a \textit{tau-function} of the KdV hierarchy if there exists a wave operator $P = 1 + w_1 \partial_x^{-1} + w_2 \partial_x^{-2} + \ldots$, such that

$$1 + \frac{w_1}{\sqrt{2\lambda}} + \frac{w_2}{\sqrt{2\lambda}}^2 + \ldots = \exp \left( - \sum_{n \geq 0} \frac{(2n-1)!!}{(2\lambda)^{1/2+n}} \epsilon \frac{\partial}{\partial q_n} \right) \frac{\tau(q_0 + x, q_1, \ldots; \epsilon)}{\tau(q_0 + x, q_1, \ldots; \epsilon)}$$

The tau-function is uniquely determined by $P$ up to multiplication by a constant (see e.g. [K] or [PvM]).

According to [K], $\tau$ is a tau-function of the KdV hierarchy if and only if it satisfies the following Hirota equation. Put $\partial_n = \partial/\partial q_n$ and consider the Heisenberg Lie algebra spanned by the operators of differentiation $\epsilon \partial_n$ and multiplication $q_n/\epsilon$. The following \textit{vertex operators} $\Gamma^\pm$ represent the action on the Fock space of certain elements of the Heisenberg group:

$$\Gamma^\pm := \exp \left( \pm \sum_{n \geq 0} \frac{(2\lambda)^{n+1/2} q_n}{(2n+1)!!} \epsilon \right) \exp \left( \mp \sum_{n \geq 0} \frac{(2n-1)!!}{(2\lambda)^{1/2+n}} \epsilon \partial_n \right).$$

We remark that

$$\frac{(2n-1)!!}{(2\lambda)^{1/2+n}} = \left( - \frac{d}{d\lambda} \right)^n \frac{1}{\sqrt{2\lambda}} \frac{(2\lambda)^{1/2+n}}{(2n+1)!!} = \left( \frac{d}{d\lambda} \right)^{-n} \frac{1}{\sqrt{2\lambda}}.$$

According to [G], the Hirota equation of the KdV hierarchy can be stated this way:








(1.1) $\frac{d\lambda}{\sqrt{\lambda}} \left( \Gamma^+ \otimes \Gamma^- - \Gamma^- \otimes \Gamma^+ \right) \ (\tau \otimes \tau) \ \text{is regular in } \lambda.$

Here $\tau \otimes \tau$ means the function $\tau(q'; \epsilon)\tau(q''; \epsilon)$ of the two copies of the variable $q = (q_0, q_1, \ldots)$, and the vertex operators in $\Gamma^\pm \otimes \Gamma^\mp$ preceding (respectively — following) $\otimes$ act on $q'$ (respectively — on $q''$). The expression in (1.1) is in fact single-valued in $\lambda$ near $\lambda = \infty$. Passing to the variables $x = (q' + q'')/2$ and $y = (q' - q'')/2$ and using Taylor’s formula one can expand (1.1) into a power series in $y$ with coefficients which are Laurent series in $\lambda^{-1}$ (whose coefficients are polynomials in $\tau(x)$ and its partial derivatives). The regularity condition in (1.1) means, by definition, that all the Laurent series in $\lambda^{-1}$ are polynomials in $\lambda$.

We describe the Lax form of the ETH following [CDZ].\footnote{In the papers [CC] [Z] the ETH is defined implicitly via bihamiltonian recursion relations} Introduce a \textit{Lax operator}








(1.2) $L = \Lambda + u + Q e^u \Lambda^{-1},$
where $\Lambda = e^{\partial x}$ is the shift operator, i.e. $\Lambda a(x) = \sum_{k \geq 0} a^{(k)}(x)e^k/k!$, $u$ and $v$ are formal series

$$u = u_0(x) + u_1(x)\epsilon + u_2(x)\epsilon^2 + \ldots$$

$$v = v_1(x)\epsilon + v_2(x)\epsilon^2 + \ldots$$

with coefficients which are infinitely differentiable functions of $x$, and $Q$ is a nonzero constant.

Let $L$ be a Lax operator. A pair of dressing operators consists of two operators

$$P_L = 1 + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \ldots,$$

$$P_R = w_0 + \Lambda^{-1}w_1 + \Lambda^{-2}w_2 + \ldots,$$

such that $L = P_L\Lambda P_L^{-1} = (P_R^{-1}\Lambda P_R)^\#$, where # is an antiinvolution acting on the space of Laurent series in $\Lambda$ by $x^\# = x$ and $\Lambda^\# = Q\Lambda^{-1}$. The pair is unique up to multiplying $P_L$ from the right and $P_R$ from the left by operators of the form respectively $1 + a_1\Lambda^{-1} + a_2\Lambda^{-2} + \ldots$ and $\tilde{a}_0 + \tilde{a}_1\Lambda + \tilde{a}_2\Lambda^2 + \ldots$ with coefficients independent of $x$. Using $P_L$ and $P_R$ one expresses the logarithm of the Lax operator

$$\log L = \frac{1}{2}(P_L\epsilon\partial_x P_L^{-1} + (P_R^{-1}\epsilon\partial_x P_R)^\#) =$$

$$= \frac{1}{2}\left(\log Q + \epsilon\left(P_R^{-1}\frac{\partial P_R}{\partial x}\right)^\# - \epsilon\frac{\partial P_L}{\partial x}P_L^{-1}\right),$$

as a Laurent series in $\Lambda$ possibly infinite in both directions.

By definition, the ETH is the following sequence of flows with time variables $q_{n,0}$ and $q_{n,1}$, $n = 0, 1, 2 \ldots$:

$$\partial_{n,1}L = \epsilon^{-1}\left[\left(\frac{L^{n+1}}{(n+1)!}\right)_+ L\right],$$

$$\partial_{n,0}L = 2\epsilon^{-1}\left[\left(\frac{L^n}{n!}(\log L - C_n)\right)_+ L\right].$$

where $\partial_{n,i} = \partial/\partial q_{n,i}$, $C_n$ are the harmonic numbers defined by: $C_0 = (1/2)\log Q$, $C_n = C_{n-1} + 1/n$, and for $B = \sum B_k\Lambda^k$ we put $B_+ = \sum_{k \geq 0} B_k\Lambda^k$. According to [CDZ], the flows of ETH commute pairwise and preserve the class of Lax operators with a fixed $Q$.

We will prove that $L$ is a solution to the ETH if and only if there is a pair of dressing operators $P_L$ and $P_R$, which satisfies the differential equations:

$$\partial_{n,1}P_L = -\left(\frac{L^{n+1}}{\epsilon(n+1)!}\right)_- P_L,$$

$$\partial_{n,0}P_L = -\left(\frac{2L^n}{\epsilon n!}(\log L - C_n)\right)_- P_L,$$

$$\partial_{n,1}P_R^\# = \left(\frac{L^{n+1}}{\epsilon(n+1)!}\right)_+ P_R^\#.$$
(1.10) \[ \partial_{n,0} \mathcal{P}^\#_R = \left( \frac{2 \mathcal{L}_n}{\epsilon n!} (\log \mathcal{L} - C_n) \right) \mathcal{P}^\#_R. \]

We call such a pair wave operators of the ETH. It is unique up to multiplying \( \mathcal{P}_L \) from the right and \( \mathcal{P}_R \) from the left by operators of the form respectively \( 1 + a_1 \Lambda^{-1} + a_2 \Lambda^{-2} + \ldots \) and \( \tilde{a}_0 + \tilde{a}_1 \Lambda^{-1} + \tilde{a}_2 \Lambda^{-2} + \ldots \), where \( a_i \) and \( \tilde{a}_j \) are independent of \( x \) and \( q \).

Given a non-vanishing function \( \tau(q; \epsilon) \) associate to it a pair of operators

\[ \mathcal{P}_L = 1 + w_1 \Lambda^{-1} + w_2 \Lambda^{-2} + \ldots, \]
\[ \mathcal{P}_R = \tilde{w}_0 + \Lambda^{-1} \tilde{w}_1 + \Lambda^{-2} \tilde{w}_2 + \ldots \]

defined by

\[ P_L := 1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \ldots := \frac{\exp \left( -\sum_{n \geq 0} n! \lambda^{-n-1} \epsilon \partial_{n,1} \right) \tau(q_{0,0} + x - \frac{\epsilon}{2}, q_{0,1}, \ldots; \epsilon)}{\tau(q_{0,0} + x - \frac{\epsilon}{2}, q_{0,1}, \ldots; \epsilon)} \]
\[ P_R := \tilde{w}_0 + \frac{\tilde{w}_1}{\lambda} + \frac{\tilde{w}_2}{\lambda^2} + \ldots := \frac{\exp \left( \sum_{n \geq 0} n! \lambda^{-n-1} \epsilon \partial_{n,1} \right) \tau(q_{0,0} + x + \frac{\epsilon}{2}, q_{0,1}, \ldots; \epsilon)}{\tau(q_{0,0} + x + \frac{\epsilon}{2}, q_{0,1}, \ldots; \epsilon)} \]

We call \( \tau \) a tau-function of the ETH if \( \mathcal{P}_L \) and \( \mathcal{P}_R \) are wave operators of the ETH. For a given pair of wave operators the tau-function is unique up to multiplication by a non-vanishing function independent of \( q_{0,0} \) and \( q_{0,1} \) with all \( n \geq 0 \). In [CDZ] the tau-functions of the ETH were introduced via the Lax operator \( \mathcal{L} \). The two definitions should agree, up to a factor depending on \( q_{n,0} \), \( n > 1 \) and up to a shift by \( \epsilon/2 \) of the translation variable \( x \). However, we could not prove this fact.

Introduce the vertex operators

\[ \Gamma^{\pm \alpha} := \exp \left\{ \pm \frac{1}{2} \sum_{n \geq 0} \left[ \frac{\lambda^{n+1}}{(n+1)!} \frac{q_{n,1}}{\epsilon} + \frac{2 \lambda^n}{n!} (\log \lambda - C_n) \frac{q_{n,0}}{\epsilon} \right] \right\} \]
\[ \times \exp \left\{ \mp \frac{\epsilon}{2} \partial_{0,0} \mp \sum_{n \geq 0} \frac{n!}{\lambda^{1+n} \epsilon \partial_{n,1}} \right\}. \]

(1.11)

We remark that the involved functions of \( \lambda \) are derivatives and anti-derivatives of \( \frac{\lambda^{n+1}}{(n+1)!} \frac{q_{n,1}}{\epsilon} + \frac{2 \lambda^n}{n!} (\log \lambda - C_n) \frac{q_{n,0}}{\epsilon} \) and \( \frac{1}{2} \partial_{0,0} \pm \sum_{n \geq 0} \frac{n!}{\lambda^{1+n} \epsilon \partial_{n,1}} \).

Because of the logarithmic terms, the vertex operators \( \Gamma^{\pm \alpha} \otimes \Gamma^{\mp \alpha} \) under the analytic continuation around \( \lambda = \infty \) are multiplied by the monodromy factors

\[ \exp \left\{ \pm \frac{2 \pi i}{\epsilon} \sum_{n \geq 0} \frac{\lambda^n}{n!} (q_{n,0} \otimes 1 - 1 \otimes q_{n,0}) \right\}. \]

(1.12)

As a result, the expressions similar to (1.11) do not expand into Laurent series near \( \lambda = \infty \). To offset the complication we need to generalize the concept of vertex operators.

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2by an anti-derivative of a function \( f \) we mean one of its primitives, i.e. a function \( F \) such that \( F' = f \).
We will allow vertex operators with coefficients in the algebra $\mathcal{A}$ of differential operators $\sum a_i(x, \epsilon) \partial_x^i$, where each $a_i(x, \epsilon)$ is a formal Laurent series in $\epsilon$ with coefficients infinitely differentiable functions in $x$. Let $\# \!$ be the antinvolution on $\mathcal{A}$ which acts on the generators $\partial_x$ and $x$ as follows:

\[(\epsilon \partial_x)^\# = -\epsilon \partial_x + \log Q, \quad x^\# = x.\]

Introduce the vertex operator (note that $n = 0$ is excluded from the summation range)

\[
\Gamma^\delta = \exp \left( - \sum_{n>0} \frac{\lambda^n}{n!} (\epsilon \partial_x - \log \sqrt{Q}) q_{n,0} \right) \exp (x \partial_{0,0}).
\]

We will say that $\tau$ satisfies the Hirota Quadratic Equation of the ETH (shortly — satisfies the HQE) if

\[
\frac{d\lambda}{\lambda} \left( \Gamma^\delta \otimes \Gamma^\delta \right) \left( \Gamma^\alpha \otimes \Gamma^{-\alpha} - \Gamma^{-\alpha} \otimes \Gamma^\alpha \right) (\tau \otimes \tau)
\]

computed at $q_{0,0}' - q_{0,0}'' = m$ is regular in $\lambda$ for each $m \in \mathbb{Z}$.

The expression (1.14) is interpreted as taking values in the algebra $\mathcal{A}$ of differential operators with coefficients depending on $q', q'', \epsilon$ and $\lambda$. Note that

\[
\left( \Gamma^\delta \otimes \Gamma^\delta \right) M = M \exp(x \partial_{0,0}') \exp \left( \pm \frac{2\pi i}{\epsilon} x \right)
\]

\[
\exp \left( \sum_{n>0} \frac{\lambda^n}{n!} (\epsilon \partial_x - \log \sqrt{Q}) (q_{n,0}' - q_{n,0}'') \right) \exp \left( \pm \frac{2\pi i}{\epsilon} x \right) \exp (x \partial_{0,0}') =
\]

\[
M \exp(x \partial_{0,0}') \exp \left( \sum_{n>0} \frac{\lambda^n}{n!} (\epsilon \partial_x - (\pm 2\pi i / \epsilon)) - \log \sqrt{Q}) (q_{n,0}' - q_{n,0}'') \right) \exp (x \partial_{0,0}') =
\]

\[
M \exp \left( \pm \frac{2\pi i}{\epsilon} \sum_{n>0} \frac{\lambda^n}{n!} (q_{n,0}' - q_{n,0}'') \right) \left( \Gamma^\delta \otimes \Gamma^\delta \right) = e^{\pm 2\pi i (q_{0,0}' - q_{0,0}'')} \left( \Gamma^\delta \otimes \Gamma^\delta \right).
\]

Thus when $q_{0,0}' - q_{0,0}'' \in \mathbb{Z}$, the expression (1.14) is single-valued near $\lambda = \infty$. After the change $y = (q' - q'')/2, x = (q' + q'')/2$ it expands (for each $m$) as a power series in $y$ ($y_{0,0} = me$ excluded) with coefficients which are Laurent series in $\lambda^{-1}$ (whose coefficients are differential operators in $x$ depending on $x$ via $\tau$, its translations and partial derivatives).

**Theorem 1.1.** A non-vanishing function $\tau$ is a tau-function of the Extended Toda Hierarchy if and only if it satisfies the Hirota Quadratic Equations (1.14).
As a by-product we obtain a Hirota equation for the (unextended) Toda Lattice Hierarchy \([UT]\) described in the Lax form by \([ES]\). The concept of tau-functions easily carries over to this case. Introduce the vertex operators 

$$\Gamma^{\pm \beta} = \exp \left\{ \pm \frac{1}{2} \sum_{n \geq 0} \frac{\lambda^{n+1}}{(n+1)!} \frac{q_{n+1}}{\epsilon} \right\} \exp \left\{ \mp \frac{\epsilon}{2} \partial_{0,0} + \sum_{n \geq 0} \frac{n!}{\lambda^{1+n}} \epsilon \partial_{n,1} \right\}.$$ 

**Corollary 1.2.** A non-vanishing function \(\tau\) of \((q_{0,0}; q_{1,0}, q_{1,1}, \ldots; \epsilon)\) is a tau-function of the Toda Lattice Hierarchy \([ES]\) if and only if for each \(m \in \mathbb{Z}\)

$$\frac{d\lambda}{\lambda} \left\{ \left( \frac{\lambda}{\sqrt{Q}} \right)^m \Gamma^\beta \otimes \Gamma^{-\beta} - \left( \frac{\lambda}{\sqrt{Q}} \right)^{-m} \Gamma^{-\beta} \otimes \Gamma^\beta \right\} (\tau \otimes \tau)$$

computed at \(q_{0,0} - q_{0,0}' = m\epsilon\) is regular in \(\lambda\).

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# 2. Wave operators and tau-functions of the ETH

The arguments in this section are parallel to the ones in \([UT]\).

## 2.1. Wave operators.

For any Lax operator \(\mathcal{L}\) put

$$A_{n,0} = \left( \frac{2\mathcal{L}^n}{n!\epsilon} (\log \mathcal{L} - C_n) \right)_+, \quad A_{n,1} = \left( \frac{\mathcal{L}^{n+1}}{(n+1)!\epsilon} \right)_+.$$ 

The compatibility of \([1.7]–[1.10]\) is equivalent to the Zakharov-Shabat equations:

**Lemma 2.1.** If \(\mathcal{L}\) satisfies the equations of the ETH then

$$\partial_{n,\alpha} A_{m,\beta} - \partial_{m,\beta} A_{n,\alpha} = [A_{n,\alpha}, A_{m,\beta}],$$

where \(m, n \geq 0\) and \(\alpha, \beta = 0, 1\).

**Proof.** The proof that a solution to the Lax equations implies the Zakharov-Shabat equations is standard: see for example \([PVX]\) Theorem 1.1, where this is done for the KP hierarchy. However, we remark that in our case one should use the following non-trivial formula, which is proven in \([CDZ]\):

$$\partial_{n,\alpha} \log \mathcal{L} = [A_{n,\alpha}, \log \mathcal{L}], \quad \alpha = 0, 1.$$ 

As a corollary we find that the Cauchy problem for the system of differential equations \([1.7]–[1.10]\) has a unique solution.

**Proposition 2.2.** A Lax operator \(\mathcal{L}\) satisfies the equations of the ETH if and only if there is a pair of dressing operators \(\mathcal{P}_L\) and \(\mathcal{P}_R\), which satisfies \([1.7]–[1.10]\).
Proof. Fix some sequence of times \( q^0 \). Consider the Lax operator \( L^0 := L(q^0) \). There exist dressing operators \( P_L^0 \) and \( P_R^0 \), i.e.
\[
L^0 = P_L^0 \Lambda (P_L^0)^{-1} = ((P_R^0)^{-1} \Lambda P_R^0)^{\#},
\]
Let \( P_L \) and \( P_R \) be solutions to the system of equations (1.7)–(1.10), satisfying the initial conditions \( P_L|q=q^0 = P_L^0 \) and \( P_R|q=q^0 = P_R^0 \). One checks that the operators
\[
L P_L - P_L \Lambda \quad \text{and} \quad L P_R^\# - Q P_R^\# \Lambda^{-1}
\]
also satisfy (1.7)–(1.10). Since at \( q = q^0 \) both operators are 0 we obtain that they are identically zero. This proves that \( P_L \) and \( P_R \) are wave operators.

In the other direction we need to use only that if \( L = P_L \Lambda P_L^{-1} \) then
\[
\partial_{n, \alpha} L = [(\partial_{n, \alpha} P_L) P_L^{-1}, L], \quad n \geq 0, \quad \alpha = 0, 1.
\]

\( \square \)

2.2. Characterization of the wave operators of the ETH. Let \( \mathcal{A}[[q]] \) be the algebra of formal power series in \( q \) with coefficients in the algebra \( \mathcal{A} \) of differential operators. We identify the Lax and the corresponding dressing operators with vectors in the space \( \mathcal{A}[[q]][[\Lambda^{\pm 1}]] \) of formal series in \( \Lambda \), where the translation operator \( \Lambda \) is identified with a formal symbol which commutes with the elements of \( \mathcal{A}[[q]] \) in the same ways as \( e^{\epsilon \partial_x} \) does, i.e. if \( \sum_{k \geq 0} a_k(x, q; \epsilon) \partial_x^k \in \mathcal{A}[[q]] \) then
\[
\Lambda^{\pm 1} \left( \sum_{k \geq 0} a_k(x, q; \epsilon) \partial_x^k \right) = \left( \sum_{k \geq 0} a_k(x \pm \epsilon, q; \epsilon) \partial_x^k \right) \Lambda^{\pm 1}.
\]

Let
\[
Q = \sum_{k \in \mathbb{Z}} b_k \Lambda^k = \sum_{k \in \mathbb{Z}} \Lambda^k b_k \in \mathcal{A}[[q]][[[\Lambda^{\pm 1}]].
\]

Then the series
\[
\sum_{k \in \mathbb{Z}} b_k \lambda^k \quad \text{and} \quad \sum_{k \in \mathbb{Z}} b_k \lambda^k
\]
will be called respectively left and right symbols of \( Q \).

Let \( P_L \) and \( P_R \) be two arbitrary operator series of the form respectively (1.3) and (1.4). We will assume that the coefficients of \( P_L \) and \( P_R \) are formal series in \( x + q_{0,0}, \ q_{0,1}, \ q_{n,i}, \ n > 0, \ i = 0, 1 \). Note that the wave operators of the ETH also have this form. Our goal is to see what further restrictions should be imposed on \( P_L \) and \( P_R \) so that they become wave operators of the ETH. Introduce the following two series in \( \mathcal{A}[[q]][[\Lambda^{\pm 1}]] \):

\[
(2.1) \quad W_L(x, q, \Lambda) = P_L \exp \left\{ \sum_{n \geq 0} \frac{\Lambda^{n+1}}{2\epsilon(n+1)!} q_{n,1} + \sum_{n > 0} \frac{\Lambda^n}{\epsilon n!} (\epsilon \partial_x - C_n) q_{n,0} \right\}.
\]

\[
(2.2) \quad W_R(x, q, \Lambda) = \exp \left\{ -\sum_{n \geq 0} \frac{\Lambda^{n+1}}{2\epsilon(n+1)!} q_{n,1} - \sum_{n > 0} \frac{\Lambda^n}{\epsilon n!} (\epsilon \partial_x - C_n) q_{n,0} \right\} P_R.
\]
Finally, denote by $W_L$ the left symbol of $W_L$ and by $W_R$ the right symbol of $W_R$.

**Proposition 2.3.** Let $q'$ and $q''$ be such that $q'_{0,0} = q''_{0,0}$. The following conditions are equivalent:

(a) $\mathcal{P}_L$ and $\mathcal{P}_R$ are wave operators of the ETH.

(b) $\mathcal{P}_L$ and $\mathcal{P}_R$ satisfy the following identities:

\[
\mathcal{P}_L \Lambda \mathcal{P}_L^{-1} = (\mathcal{P}_R^{-1} \Lambda \mathcal{P}_R)^\#.
\]

\[
W_L(x, q', \Lambda)W_R(x, q'', \Lambda) = \{W_L(x, q'', \Lambda)W_R(x, q', \Lambda)\}^\#.
\]

(c) For all integers $r \geq 0$ the following identity holds:

\[
W_L(x, q', \Lambda)\Lambda^r W_R(x, q'', \Lambda) = \{W_L(x, q'', \Lambda)\Lambda^r W_R(x, q', \Lambda)\}^\#.
\]

(d) For all integers $m$ and $r \geq 0$ the following identity holds:\

\[
\text{Res}_{\Lambda = \infty} \left\{ \lambda^r \left( \frac{\lambda}{\sqrt{Q}} \right)^m W_L(x, q', \lambda)W_R(x - m\epsilon, q'', \lambda) \right\} \frac{d\lambda}{\lambda} = \text{Res}_{\Lambda = \infty} \left\{ \lambda^r \left( \frac{\lambda}{\sqrt{Q}} \right)^{-m} (W_R(x, q', \lambda))^\# (W_L(x - m\epsilon, q'', \lambda))^\# \right\} \frac{d\lambda}{\lambda}.
\]

Proof. \((d) \Leftrightarrow (c)\) Let $m$ and $r \geq 0$ be some arbitrary integers and $q'_{0,0} = q''_{0,0}$. Put

\[
W_L(x, q, \Lambda) = \sum_{i \in \mathbb{Z}} a_i(x, q, \partial_x) \Lambda^i \quad \text{and} \quad \Lambda^j b_j(x, q, \partial_x)
\]

and compare the coefficients in front of $\Lambda^{-m}$ in (2.5):

\[
\sum_{i+j=m-r} a_i(x, q', \partial_x)b_j(x - m\epsilon, q'', \partial_x) = \sum_{i+j=m-r} Q^m b_j^\#(x, q', \partial_x)a_i^\#(x - m\epsilon, q'', \partial_x).
\]

This equality can be written also as

\[
\text{Res}_{\Lambda = \infty} \left\{ \lambda^r \left( \frac{\lambda}{\sqrt{Q}} \right)^m W_L(x, q', \lambda)W_R(x - m\epsilon, q'', \lambda) \right\} \frac{d\lambda}{\lambda} = \text{Res}_{\Lambda = \infty} \left\{ \lambda^r \left( \frac{\lambda}{\sqrt{Q}} \right)^{-m} (W_R(x, q', \lambda))^\# (W_L(x - m\epsilon, q'', \lambda))^\# \right\} \frac{d\lambda}{\lambda}.
\]

The R.H.S. of the last equality is the coefficient in front of $\lambda^0$ in the series inside the \{ \}-brackets. Replacing $\lambda$ with $Q/\Lambda$ inside the \{ \}-brackets we get (2.6).

\((c) \Rightarrow (b)\) Assume first that (2.5) holds. The identity (2.4) is obtained by letting $r = 0$. To prove (2.3), put $r = 1$ and $q' = q''$ in (2.5). We get the equality $\mathcal{P}_L \Lambda \mathcal{P}_L^{-1} = (\mathcal{P}_L \Lambda \mathcal{P}_L)^\#$ which can be written also as:

\[
\mathcal{P}_L \Lambda \mathcal{P}_L^{-1} (\mathcal{P}_L \mathcal{P}_R) = (\mathcal{P}_R^{-1} \Lambda \mathcal{P}_R)^\# (\mathcal{P}_L \mathcal{P}_R)^\#
\]

It remains only to notice that $\mathcal{P}_L \mathcal{P}_R = (\mathcal{P}_L \mathcal{P}_R)^\#$. Indeed, in (2.5) let $r = 0$ and $q' = q''$.\footnote{The residue here is interpreted as the coefficient in front of $\lambda^{-1}$.}
Put $L = P_L \Lambda P_L^{-1} = (P_R^{-1} \Lambda P_R)^\#$. Note that $L$ should have the form $\Lambda + a_0 + a_1 \Lambda^{-1}$. Thus $L$ is a Laurent polynomial in $\Lambda$ and it makes sense to multiply by $L$ any $\Lambda$-series (possibly infinite in both directions). In particular multiply (2.4) by $L^r$, $r \geq 0$, and use
\[ LW_L = W_L \Lambda, \quad W_R \Lambda^# = \Lambda W_R \]
to obtain (2.5).

Note that (2.3) implies that $L := P_L \Lambda P_L^{-1}$ is a Lax operator. Let’s prove (1.10) and (1.8). When $n = 0$ we need to show that $\partial_{0,0} P_L = \partial_x P_L$ and $\partial_{0,0} P_R^# = \partial_x P_R^#$. Both are satisfied by definition. Assume that $n > 0$. Differentiate (2.4) with respect to $q'_{0,0}$ and then put $q' = q''$:
\[
(\partial_{n,0} P_L) P^{-1}_L + P_L \Lambda^n_{en!} (\epsilon \partial_x - \log(Q - C_n)) P^{-1}_L =
(\partial_{n,0} P_L) P^{-1}_L \bigg( \partial_{n,0} P_R^# - (P^{-1}_R \Lambda^{n}_{en!} (\epsilon \partial_x - \log(Q - C_n)) P_R)^# \bigg)
\]
Using the definition of log $L$ and (2.3) the last identity simplifies to
\[
(\partial_{n,0} P_L) P^{-1}_L + 2L^n_{en!} (\log L - C_n) = (P^{-1}_R \partial_{n,0} P_R)^#
\]
Since $(\partial_{n,0} P_L) P^{-1}_L$ contains only negative powers of $\Lambda$ and $(P^{-1}_R \partial_{n,0} P_R)^#$ – non-negative, we get (1.10) and (1.8) by separating the negative and the positive part of the equation.

To check that (1.9) and (1.7) hold: differentiate (2.5) with respect to $q'_{0,1}$, put $q' = q''$ and then apply a similar argument. Thus $P_L$, $P_R$ is a pair of wave operators.

Let $\alpha = (\alpha_{0,1}, \alpha_{1,0}, \alpha_{1,1}, \alpha_{2,0}, \alpha_{2,1}, \ldots)$ be a multindex with only finitely many non-zero components. Put
\[
\partial^\alpha : = \partial_{0,1}^{\alpha_{0,1}} \partial_{1,0}^{\alpha_{1,0}} \partial_{1,1}^{\alpha_{1,1}} \partial_{2,0}^{\alpha_{2,0}} \partial_{2,1}^{\alpha_{2,1}} \cdots,
\]
where $\partial_{n,i} = \partial / \partial q_{n,i}$ (note that the differentiation $\partial / \partial q_{0,0}$ is not involved). First, using induction on $|\alpha|$, we prove that
\[
(\partial^\alpha W_L(x, q, \Lambda)) W_R(x, q, \Lambda) = (W_L(x, q, \Lambda) \partial^\alpha W_R(x, q, \Lambda))^#,
\]
When $\alpha = 0$ we need to prove that
\[
W_L(x, q, \Lambda) W_R(x, q, \Lambda) = (W_L(x, q, \Lambda) W_R(x, q, \Lambda))^#,
\]
which is equivalent to $P_L P_R = (P_L P_R)^#$. We claim that the last identity holds for any pair $(P_L, P_R)$ of dressing operators corresponding to the Lax operator $L = \Lambda + u + Q e^{\nu} \Lambda^{-1}$. Note that $L^# = \tilde{w}_0^{-1} L \tilde{w}_0$, where $\tilde{w}_0$ is such that $e^{\nu(x)} = \tilde{w}_0(x)/\tilde{w}_0(x - \epsilon)$. Thus $(P_L, P_R := P^{-1}_L \tilde{w}_0)$ is another pair of dressing operators for $L$. By the uniqueness of the dressing operators we get that there is a series $K = a_0 + a_1 \Lambda^{-1} + \ldots$.
with coefficients $a_i$ independent of $x$ such that $\mathcal{P}_R = K\tilde{P}_R$. Thus

$$\mathcal{P}_L\mathcal{P}_R = \mathcal{P}_L(a_0 + a_1\Lambda^{-1} + \ldots)\mathcal{P}_L^{-1}\tilde{w}_0 = (a_0 + a_1\mathcal{L}^{-1} + \ldots)\tilde{w}_0 =
$$

$$\tilde{w}_0(a_0 + a_1(\mathcal{L}^\#)^{-1} + \ldots) = ((a_0 + a_1\mathcal{L}^{-1} + \ldots)\tilde{w}_0)^\# = (\mathcal{P}_L\mathcal{P}_R)^\#.
$$

Assume now that (2.7) is true for some $\alpha$ and differentiate in $\partial_{n,0}$. We need only to check that

$$(2.8) \quad \partial^n\mathcal{W}_L\partial_{n,0}\mathcal{W}_R = \{\partial_{n,0}\mathcal{W}_L\partial^n\mathcal{W}_R\}^\#.
$$

Note that

$$\partial_{n,0}\mathcal{W}_L = \left((\partial_{n,0}\mathcal{P}_L)\mathcal{P}_L^{-1} + \mathcal{P}_L\frac{\Lambda^n}{\epsilon n!}(\epsilon\partial_x - \log\sqrt{Q} - C_n)\mathcal{P}_L^{-1}\right)\mathcal{W}_L,$n
$$

$$\partial_{n,0}\mathcal{W}_R = \mathcal{W}_R\left(\mathcal{P}_R^{-1}\partial_{n,0}\mathcal{P}_R - \mathcal{P}_R^{-1}\frac{\Lambda^n}{\epsilon n!}(\epsilon\partial_x - \log\sqrt{Q} - C_n)\mathcal{P}_R\right).
$$

Now (2.8) follows from (1.8), (1.10) and the inductive assumption (2.7). Thus if we increase $\alpha_{n,0}$ by 1 then (2.7) still holds. Similarly if we increase $\alpha_{n,1}$ by 1 then (2.7) still holds. The induction is completed.

Using the Taylor’s formula expand both sides of (2.4) about $q = q'$. Then the coefficients in front of $(q' - q'')^\alpha$ are equal exactly when (2.7) holds. □

### 2.3. Tau-functions.

Let

$$\mathcal{P}_L(x, q, \Lambda) = 1 + w_1(x, q; \epsilon)\Lambda^{-1} + w_2(x, q; \epsilon)\Lambda^{-2} + \ldots,
$$

$$\mathcal{P}_R(x, q, \Lambda) = \tilde{w}_0(x, q; \epsilon) + \Lambda^{-1}\tilde{w}_1(x, q; \epsilon) + \Lambda^{-2}\tilde{w}_2(x, q; \epsilon) + \ldots
$$

be a pair of wave operators of the ETH. Denote by:

$$P_L(x, q, \lambda) := 1 + w_1(x, q; \epsilon)\lambda^{-1} + w_2(x, q; \epsilon)\lambda^{-2} + \ldots
$$

$$P_R(x, q, \lambda) := \tilde{w}_0(x, q; \epsilon) + \tilde{w}_1(x, q; \epsilon)\lambda^{-1} + \tilde{w}_2(x, q; \epsilon)\lambda^{-2} + \ldots
$$

the left and the right symbol respectively of $\mathcal{P}_L$ and $\mathcal{P}_R$. For shortness denote by $[\lambda^{-1}]$ the sequence of times with components

$$[\lambda^{-1}]_{n,0} = 0, \ [\lambda^{-1}]_{n,1} = n!\lambda^{-n-1}\epsilon.
$$

**Lemma 2.4.** The following identities hold

$$(2.9) \quad P_L(x, q, \lambda)P_R(x - \epsilon, q - [\lambda^{-1}], \lambda) = \tilde{w}_0(x - \epsilon, q - [\lambda^{-1}]),
$$

$$(2.10) \quad P_L(x, q, \lambda_1)P_R(x - \epsilon, q - [\lambda^{-1}] - [\lambda_2^{-1}], \lambda_1) =
$$

$$= P_L(x, q, \lambda_2)P_R(x - \epsilon, q - [\lambda^{-1}] - [\lambda_2^{-1}], \lambda_2),
$$

$$(2.11) \quad P_L(x, q, \lambda)P_R(x, q - [\lambda^{-1}], \lambda) = \tilde{w}_0(x, q).
$$

**Proof.** Let $q'$ and $q''$ be two sequences of time variables such that $q'_{n,0} = q''_{n,0}$, $n \geq 0$. The identities (2.9)–(2.11) are consequence of the following one:

$$(2.12) \quad \mathcal{P}_L(x, q', \Lambda)\exp\left(\sum_{n \geq 0} \frac{\Lambda^{n+1}}{\epsilon(n+1)!}(q'_{n,1} - q''_{n,1})\right)\mathcal{P}_R(x, q'', \Lambda) =
$$

$$\mathcal{P}_R(x, q', \Lambda)\mathcal{P}_L^\#(x, q'', \Lambda).
$$
The proof of (2.12) is completely analogous to the argument in the implication a) \(\Rightarrow\) b) in Proposition 2.3 and it will be omitted.

To prove (2.3): in (2.12) put \(q'' = q' - [\lambda^{-1}]\). The exponential factor turns into

\[ \exp \left( \sum_{n \geq 0} \frac{(\lambda^{-1}\Lambda)^n}{n+1} \right) = (1 - \lambda^{-1}\Lambda)^{-1} = \sum_{N \geq 0} (\lambda^{-1}\Lambda)^N. \]

Comparing the coefficients in front of \(\Lambda^{-1}\) we find

\[
\sum_{N \geq 0, j+k=N+1} w_j(x, q') \lambda^{-N} \Lambda^{-1} \hat{w}_k(x, q'') = 0,
\]

\[
\sum_{N \geq 0, j+k=N+1} w_j(x, q') \lambda^{-j} \Lambda^{-k} \hat{w}_k(x - \epsilon, q'') = 0,
\]

\[ P_L(x, q', \lambda) P_R(x - \epsilon, q' - [\lambda^{-1}], \lambda) - \hat{w}_0(x - \epsilon, q' - [\lambda^{-1}]) = 0. \]

To prove (2.10): in (2.12) put \(q'' = q' - [\lambda_1^{-1}] - [\lambda_2^{-1}]\). The exponential factor turns into

\[ (1 - \lambda_1^{-1}\Lambda)^{-1} (1 - \lambda_2^{-1}\Lambda)^{-1} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left( (1 - \lambda_1^{-1}\Lambda)^{-1} - (1 - \lambda_2^{-1}\Lambda)^{-1} \right) \Lambda^{-1}. \]

Comparing the coefficients in front of \(\Lambda^{-1}\) we arrive at (2.10).

To prove (2.11): in (2.12) put \(q'' = q' - [\lambda^{-1}]\) and compare the coefficients in front of \(\Lambda^{0}\).

The main result in this subsection is the following proposition:

**Proposition 2.5.** Given a pair of wave operators \(P_L\) and \(P_R\) of the ETH there exists a corresponding tau-function, which is unique up to multiplication by a non-vanishing function independent of \(q_{0,0}\) and \(q_{n,1}\), \(n \geq 0\).

**Proof.** We need to prove that the system of equations \(^4\)

\begin{align}
(2.13) & \quad P_L(x, q, \lambda) = \frac{\tau(x, q - [\lambda^{-1}])}{\tau(x, q)}, \\
(2.14) & \quad P_R(x, q, \lambda) = \frac{\tau(x + \epsilon, q + [\lambda^{-1}])}{\tau(x, q)}, \\
(2.15) & \quad \partial_{0,0} \tau(x, q) = \partial_x \tau(x, q).
\end{align}

has a solution \(\tau(x, q)\) unique up to multiplication by a non-vanishing function independent of \(q_{0,0}\) and \(q_{n,1}\), \(n \geq 0\). The system is equivalent to:

\begin{align}
(2.16) & \quad \log P_L = \left( e^{- \sum n! \lambda^{-n-1} \epsilon \partial_{n,1}} - 1 \right) \log \tau, \\
(2.17) & \quad \log P_R = \left( e^{\epsilon \partial_x + \sum n! \lambda^{-n-1} \epsilon \partial_{n,1}} - 1 \right) \log \tau, \\
(2.18) & \quad \partial_{0,0} \log \tau(x, q) = \partial_x \log \tau(x, q).
\end{align}

\(^4\)By definition, the tau-functions corresponding to the wave operators \(P_L\) and \(P_R\) are \(\tau(x - \epsilon/2, q)\), where \(\tau(x, q)\) is a solution to the system (2.10), (2.15).
Expanding in the powers of $\lambda$ we find:

$$\log P_L = \sum_{N \geq 1} b_N(x, q) \lambda^{-N},$$

$$\log P_R = \sum_{N \geq 0} \tilde{b}_N(x, q) \lambda^{-N}$$

and

$$\exp \left( -\sum_{i=0}^{n} i! \lambda^{-n-1} \epsilon \partial_n, 1 \right) - 1 = \sum_{N \geq 1} a_N(\partial_{0,1}, \partial_{1,1}, \ldots) \lambda^{-N}.$$  \hspace{1cm} (2.19)

Comparing the coefficients in front of the powers of $\lambda$ we get the following system of partial differential equations (note that (2.17) is equivalent to (2.20) and (2.22)):

$$a_N(-\partial_{0,1}, -\partial_{1,1}, \ldots) \log \tau = e^{-\epsilon \partial_x} \tilde{b}_N(x, q), \ N \geq 1, \hspace{1cm} (2.20)$$

$$a_N(\partial_{0,1}, \partial_{1,1}, \ldots) \log \tau = b_N(x, q), \ N \geq 1, \hspace{1cm} (2.21)$$

$$\epsilon \partial_x \log \tau = \frac{\epsilon \partial_x}{e^{\epsilon \partial_x} - 1} \tilde{b}_0, \hspace{1cm} (2.22)$$

$$\partial_{0,0} \log \tau = \partial_x \log \tau, \hspace{1cm} (2.23)$$

where

$$\frac{\epsilon \partial_x}{e^{\epsilon \partial_x} - 1} = \sum_{k=0}^{\infty} B_k \frac{(\epsilon \partial_x)^k}{k!}, \hspace{0.5cm} B_k \text{ are the Bernoulli numbers.}$$

Let us exclude the first equation. Later on we will see that it is a consequence from the rest. Note that the differential operators $a_1(\partial), a_2(\partial), \ldots$ generate polynomially the ring $C[\partial_{0,1}, \partial_{1,1}, \ldots]$. Thus the system (2.21)–(2.23) can be written in the form

$$\begin{cases} 
\partial_{n,1} \log \tau = \beta_n(x, q), \ n \geq 1, \\
\epsilon \partial_x \log \tau = \frac{\epsilon \partial_x}{e^{\epsilon \partial_x} - 1} \tilde{b}_0, \\
\partial_{0,0} \log \tau = \partial_x \log \tau,
\end{cases} \hspace{1cm} (2.24)$$

where $\beta_n(x, q)$ depend polynomially on $b_n(x, q)$ and their derivatives. We need to show that the system is compatible.

Since $P_L$ and $P_R$ depend on $x + q_{0,0}$ the equation on the third line of (2.24) is compatible with the rest of the equations.

The equation on the second line of (2.24) is compatible with the equations on the first line if and only if

$$a_i(\partial) \frac{\epsilon \partial_x}{e^{\epsilon \partial_x} - 1} \tilde{b}_0 = \epsilon \partial_x b_i, \ i \geq 1,$$

which is equivalent to

$$a_i(\partial) \tilde{b}_0 = (e^{\epsilon \partial_x} - 1)b_i, \ i \geq 1.$$ 

Let us write a generating series for these identities

$$\sum_{i \geq 1} a_i(\partial) \tilde{b}_0 \lambda^{-i} = \sum_{i \geq 1} (e^{\epsilon \partial_x} - 1)b_i \lambda^{-i}.$$
Comparing with the expansions of $\log P_L$, (2.19) and letting $\tilde{b}_0 = \log \tilde{w}_0$ we get
\[ \log \frac{\tilde{w}_0(x, q - [\lambda^{-1}])}{\tilde{w}_0(x, q)} = \log \frac{P_L(x + \epsilon, q, \lambda)}{P_L(x, q, \lambda)}, \]
which is equivalent to
\[ \tilde{w}_0(x, q - [\lambda^{-1}]) P_L(x, q, \lambda) = \tilde{w}_0(x, q) P_L(x + \epsilon, q, \lambda). \]
Similarly, the compatibility between the equations on the first line of (2.24) is equivalent to (2.25) and (2.26).
\[ P_L(x, q, \lambda_1) P_L(x, q - [\lambda_1^{-1}], \lambda_2) = P_L(x, q, \lambda_2) P_L(x, q - [\lambda_2^{-1}], \lambda_1). \]
Thus the compatibility of (2.24) is equivalent to (2.25) and (2.26).

Thus the compatibility of (2.24) is equivalent to (2.25) and (2.26).

We will show that (2.25) and (2.26) follow from the identities in Lemma 2.4. To prove (2.25), divide (2.10) by $\tilde{w}_0(x - \epsilon, q - [\lambda_1^{-1}] - [\lambda_2^{-1}])$ and then apply (2.9). To prove (2.26), substitute $x$ with $x - \epsilon$ in (2.9) and use (2.11).

Thus the system (2.24) is compatible. Let $\tau$ be a solution – it is unique up to multiplication by a constant independent of $x, q_{00}$ and $q_{n1}$, $n \geq 0$. Then we have:
\[ P_R(x, q, \lambda) = \frac{\tau(x + \epsilon, q + [\lambda^{-1}])}{\tau(x, q)}, \quad \tilde{w}_0 = \frac{\tau(x + \epsilon, q)}{\tau(x, q)}. \]
Using (2.11) we get
\[ P_L(x, q, \lambda) = \frac{\tau(x, q - [\lambda^{-1}])}{\tau(x, q)}. \]
\[ \square \]

2.4. Proof of Theorem 1.1. Assume that $\tau$ is a non-vanishing function and let $\mathcal{P}_L$ and $\mathcal{P}_R$ be the corresponding operators (see the introduction). It is enough to prove that the HQE are equivalent to condition (d) in Proposition 2.3.

After a straightforward computation one finds
\[ \Gamma^{\delta \# \Gamma} \alpha_{\tau} = \tau(x - \epsilon/2, q) \left( \frac{\lambda}{\sqrt{Q}} \right)^{q_{00}/\epsilon} W_L(x, q, \lambda) \left( \frac{\lambda}{\sqrt{Q}} \right)^{x/\epsilon} \]
\[ \Gamma^{\delta \# \Gamma} \alpha_{-\tau} = \tau(x - \epsilon/2, q) \left( \frac{\lambda}{\sqrt{Q}} \right)^{-q_{00}/\epsilon} \left( W_R(x, q, \lambda) \right)^{\#} \left( \frac{\lambda}{\sqrt{Q}} \right)^{-x/\epsilon} \]
\[ \Gamma^{\delta \Gamma} \alpha_{-\tau} = \left( \frac{\lambda}{\sqrt{Q}} \right)^{-q_{00}/\epsilon} W_R(x, q, \lambda) \left( \frac{\lambda}{\sqrt{Q}} \right)^{x/\epsilon} \]
\[ \Gamma^{\delta \Gamma} \alpha_{\tau} = \left( \frac{\lambda}{\sqrt{Q}} \right)^{q_{00}/\epsilon} \left( \frac{\lambda}{\sqrt{Q}} \right)^{x/\epsilon} W_L(x, q, \lambda) \left( \frac{\lambda}{\sqrt{Q}} \right)^{\#} \tau(x - \epsilon/2, q). \]

Let us prove the first identity. The other three are derived in a similar way.
\[ \Gamma^{\delta \# \Gamma} \alpha_{\tau} = \exp(x \partial_{00}) \exp \left( \sum_{n>0} \frac{\lambda^n}{e \sqrt{Q}} (\epsilon \partial_x - \log \sqrt{Q}) q_{n0} \right) \times \]
Then we move the operator \( \exp(\text{line}) \) (note that the third line commutes with the preceding two exponential factors). We move the third line of the last formula between the two exponents on the first line (note that the third line commutes with the preceding two exponential factors). Then we move the operator \( \exp(x\partial_{0,0}) \) from left to right. We get

\[
\Gamma^\delta\#\Gamma^\alpha_{-\tau} = \tau(x - \epsilon/2, q)P_L(x, q, \lambda) \exp \left( \sum_{n>0} \frac{\lambda^n}{en!} \left( \epsilon(\partial_x - \partial_{0,0}) - \log \sqrt{Q} \right) q_{n,0} \right)
\]

\[
\exp \left\{ \frac{1}{2} \sum_{n>0} \frac{\lambda^{n+1}}{(n+1)!} \frac{q_{n,1}}{\epsilon} + \frac{2\lambda^n}{n!} \left( \log \lambda - C_n \right) \frac{q_{n,0}}{\epsilon} \right\} + \left( \log \lambda - \log \sqrt{Q} \right) q_{0,0}/\epsilon = \tau(x - \epsilon/2, q)P_L(x, q, \lambda) \exp \left( \sum_{n>0} \frac{\lambda^n}{en!} \left( \epsilon(\partial_x - \log \sqrt{Q}) q_{n,0} \right) \right)
\]

\[
\exp \left\{ \frac{1}{2} \sum_{n>0} \frac{\lambda^{n+1}}{(n+1)!} \frac{q_{n,1}}{\epsilon} + \frac{2\lambda^n}{n!} \left( \log \lambda - C_n \right) \frac{q_{n,0}}{\epsilon} \right\} + \left( \log \lambda - \log \sqrt{Q} \right) \left[ q_{0,0} - \left( \sum_{n>0} \frac{\lambda^n}{n!} q_{n,0} \right) \right] /\epsilon = \tau(x - \epsilon/2, q)P_L(x, q, \lambda) \times
\]

\[
\exp \left\{ \frac{1}{2} \sum_{n>0} \frac{\lambda^{n+1}}{(n+1)!} \frac{q_{n,1}}{\epsilon} + \frac{2\lambda^n}{n!} \left( \log \lambda - \log \sqrt{Q} \right) \frac{q_{n,0}}{\epsilon} \right\} \left( \frac{\lambda}{\sqrt{Q}} \right)^{(q_{0,0} + x)/\epsilon}.
\]

After substituting formulas (2.27) - (2.30) into the HQE we find:

\[
\left\{ \Gamma^\delta\#\Gamma^\alpha_{-\tau} \otimes \Gamma^\delta\#\Gamma^{-\alpha}_{-\tau} - \Gamma^\delta\#\Gamma^{-\alpha}_{-\tau} \otimes \Gamma^\delta\#\Gamma^\alpha_{-\tau} \right\} \frac{d\lambda}{\lambda} =
\]

\[
\tau(x - \epsilon/2, q') \left( \frac{\lambda}{\sqrt{Q}} \right)^{(q_{0,0} - q''_{0,0})/\epsilon} W_L(x, q', \lambda)W_R(x, q'', \lambda) \tau(x - \epsilon/2, q'') -
\]

\[
\tau(x - \epsilon/2, q') \left( \frac{\lambda}{\sqrt{Q}} \right)^{- (q_{0,0} - q''_{0,0})/\epsilon} \left( W_R(x, q', \lambda) \right)^{\#} \left( W_L(x, q'', \lambda) \right)^{\#} \tau(x - \epsilon/2, q'') \frac{d\lambda}{\lambda}.
\]

Let \( q_{0,0} - q''_{0,0} = m\epsilon \) and use that

\[
W_{L/R}(x, q_{0,0}' - m\epsilon, q''_{0,1}, \ldots, \lambda) = W_{L/R}(x - m\epsilon, q_{0,0}', q''_{0,1}, \ldots, \lambda).
\]
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