On the Existence of Proper Nearly Kenmotsu Manifolds

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Abstract. We prove that a nearly Kenmotsu manifold is locally isometric to the warped product of a real line and a nearly Kähler manifold. As consequence, a normal nearly Kenmotsu manifold is Kenmotsu. Furthermore, we show that there do not exist nearly Kenmotsu hypersurfaces of nearly Kähler manifolds.

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1. Introduction

Nearly Kähler manifolds were defined and studied by Gray in [7–9]. Nearly Sasakian manifolds were introduced by Blair et al. in [3]. Afterwards, Olszak studied nearly Sasakian manifolds in [11]. He gave properties of five-dimensional nearly Sasakian non-Sasakian manifolds [12]. In parallel with Olszak’s works [11,12], Endo investigated the geometry of nearly cosymplectic manifolds [6]. Recently, Cappelletti Montano and Dileo studied nearly Sasakian geometry [4]. While several similarities between nearly Sasakian manifolds and nearly cosymplectic manifolds are emphasized, and properties of these manifolds are investigated, nearly Kenmotsu manifolds are ignored. The notion of nearly Kenmotsu manifolds was introduced in [13]. In this paper, we want to fill this gap in the study of nearly Kenmotsu manifolds. In the literature we did not come across proper nearly Kenmotsu manifold examples. So one can ask the following question. Do there exist proper nearly Kenmotsu manifolds? In this paper, we give a positive answer to the question for dimension >5. Our work is structured as follows: in Sect. 2, we report some basic information about nearly Kenmotsu manifolds. In Sect. 3, we give some curvature identities on nearly Kenmotsu manifolds and we prove that a nearly Kenmotsu manifold is locally isometric to the warped product of a real line and a nearly Kähler manifold. In Sect. 4, we show that a normal nearly Kenmotsu manifold is a Kenmotsu manifold and there do not exist nearly Kenmotsu hypersurfaces of nearly Kähler manifolds.
2. Preliminaries

In this paper, all objects are to be considered as $C^\infty$-class, manifolds are assumed to be connected. We accept the following convention that $X, Y, Z, W$ etc., will denote vector fields, if it is not otherwise stated.

Let $M$ be a $(2n + 1)$-dimensional differentiable manifold, endowed with a $(1, 1)$-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$. Then $(\phi, \xi, \eta, g)$ is called an almost contact metric structure on $M$, if

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

$M$ is said to be an almost contact metric manifold with structure $(\phi, \xi, \eta, g)$ [2,15]. This structure satisfies:

$$\eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad (2.1)$$

$$g(X, \phi Y) + g(Y, \phi X) = 0. \quad (2.2)$$

The tensor field $\Phi$ defined by $\Phi(X, Y) = g(X, \phi Y)$ is called the fundamental 2-form. In this paper, we refer to $\xi$ as the Reeb vector field and to $\eta$ as the Reeb form. By $[\phi, \phi]$, we denote Nijenhuis torsion tensor of $\phi$, by definition

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \quad (2.3)$$

where $[X, Y]$ denotes the Lie bracket of vector fields.

An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is called nearly Kenmotsu manifold [13], if

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \quad (2.4)$$

where $\nabla$ is the Levi–Civita connection of $g$. Moreover, if $M$ satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (2.5)$$

then it is called Kenmotsu manifold [10]. Every Kenmotsu manifold is a nearly Kenmotsu manifold but the converse is not true, which in fact will be proved in this paper. If $M$ is nearly Kenmotsu but non Kenmotsu, we will call the manifold proper nearly Kenmotsu manifold.

Let $M$ be a nearly Kenmotsu manifold. We define the skew-symmetric $(1,1)$-tensor field $H$, by $d\eta(X, Y) = g(HX, Y)$. Later on we will show that $H = 0$.

**Proposition 1.** For a nearly Kenmotsu manifold, we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2g(\phi X, \phi Y), \quad \nabla_X \xi = -\phi^2 X + HX, \quad (2.6)$$

$$\nabla_\xi \phi = \phi H, \quad \phi H + H \phi = 0, \quad H\xi = 0, \quad \nabla_\xi \xi = 0. \quad (2.7)$$

**Proof.** By (2.4), $(\nabla_\xi \phi)\xi = -\phi(\nabla_\xi \xi) = 0$, hence $\nabla_\xi \xi = 0$, and $\nabla_\xi \eta = 0$. Now, $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, yields

$$0 = g((\nabla_\xi \phi)X, \phi Y) + g((\nabla_\xi \phi)Y, \phi X) = -g((\nabla_X \phi)\xi, \phi Y) - g((\nabla_Y \phi)\xi, \phi X) - 2g(\phi X, \phi Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) - 2g(\phi X, \phi Y).$$

With help of definition of $H$, $\nabla_X \xi = -\phi^2 X + HX$. By $\phi \xi = 0$, and $\eta(\phi X) = 0$

$$0 = (\nabla_X \phi)\xi + \phi \nabla_X \xi = -(\nabla_\xi \phi)X + \phi HX, \quad (2.8)$$
0 = \eta((\nabla_X \phi)Y) + \eta((\nabla_Y \phi)X) = -g((\nabla_X \phi)\xi, Y) - g((\nabla_Y \phi)\xi, X)
= g((\nabla_\phi X)Y) + g((\nabla_\phi Y)X) = g(\phi H X, Y) + g(\phi H Y, X)
= g((\phi H + H \phi)X, Y). \quad (2.9) 

\[\square\]

3. Structure of Nearly Kenmotsu Manifolds

In this section, we prove curvature relations for nearly Kenmotsu manifold. Let \( R \) be the Riemannian curvature tensor given by

\[\begin{align*}
R(X, Y, Z, W) &= (\nabla^2_{X,Y} Z) - (\nabla^2_{Y,X} Z) = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.
\end{align*}\]

By the same letter we denote the \((0,4)\)-tensor field

\[R(X, Y, Z, W) = g(R(X, Y) Z, W).\]

**Theorem 1.** Let \((M, \phi, \xi, \eta, g)\) be a nearly Kenmotsu manifold. We have following curvature relations

\[R(\phi X, \phi Y, Z, W) + R(\phi X, \phi Z, W, Y) + R(X, Y, \phi Z, W) + R(X, Y, Z, \phi W) = 0, \quad (3.1)\]

\[R(\xi, X, Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y), \quad (3.2)\]

\[R(\phi X, \phi Y, Z, W) = R(X, Y, \phi Z, \phi W), \quad (3.3)\]

\[R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W)
- \eta(X)R(\xi, Y, Z, W) + \eta(Y)R(\xi, X, Z, W). \quad (3.4)\]

**Proof.** Let \( T \) be the \((1,3)\)-tensor field defined by (cf. (2.4))

\[\begin{align*}
(\nabla^2_{X,Y} \phi)Z + (\nabla^2_{Y,X} \phi)Y = T(X, Y, Z),
\end{align*}\]

which satisfies \( T(X, Y, Z) = T(X, Z, Y) \). For simplicity \( T \) will also denote the \((0,4)\)-tensor field defined by

\[T(X, Y, Z, W) = g(T(X, Y, Z), W).\]

From the Ricci identity,

\[0 = R(X, Y, Z, \phi W) - R(X, Y, W, \phi Z) - g((\nabla^2_{X,Y} \phi) Z, W) + g((\nabla^2_{X,Y} \phi) Z, W).\]

Applying (3.5) and the first Bianchi identity, we have

\[\begin{align*}
R(X, Y, Z, \phi W) &= R(X, Y, W, \phi Z) + g((\nabla^2_{X,Y} \phi) Z, W) - g((\nabla^2_{Y,X} \phi) Z, W)
= R(X, Y, W, \phi Z) - g((\nabla^2_{Y,Z} \phi) Y, W) + g((\nabla^2_{Y,Z} \phi) X, W)
+ T(X, Z, Y, W) \quad - T(Y, Z, X, W),
\end{align*}\]

\[R(X, Y, Z, \phi W) = R(X, Z, Y, \phi W) - R(Y, Z, X, \phi W) = R(X, Z, Y, \phi W)
- R(Y, Z, W, \phi X) - g((\nabla^2_{Y,Z} \phi) X, W) + g((\nabla^2_{Z,Y} \phi) X, W); \quad (3.7)\]
comparing right hand sides of these equations, we obtain
\[
R(X, Z, Y, \phi W) - R(Y, Z, W, \phi X) - R(X, Y, W, \phi Z) + g((\nabla^2_{Z,Y} \phi) X, W) \\
+ g((\nabla^2_{X,Z} \phi) Y, W) + T(Y, Z, X, W) - T(X, Z, Y, W) = 2g((\nabla^2_{Y,Z} \phi) X, W). \\
(3.8)
\]

We note that
\[
g((\nabla^2_{Z,Y} \phi) X, W) + g((\nabla^2_{X,Z} \phi) Y, W) = R(X, Z, Y, \phi W) \\
-R(X, Z, W, \phi Y) + T(Z, X, Y, W), \\
(3.9)
\]
which being taken into account in (3.8), imply that
\[
2R(X, Z, Y, \phi W) - R(X, Y, W, \phi Z) - R(Y, Z, W, \phi X) - R(X, Z, W, \phi Y) \\
+ T(Y, Z, X, W) + T(Z, X, Y, W) - T(X, Y, Z, W) \\
+ 2T(Y, W, Z, X) = 2g((\nabla^2_{Y,W} \phi) Z, X). \\
(3.11)
\]

Applying (2.4), the following equation holds:
\[
T(X, Y, Z, W) = (-g(Y, X + HX) + \eta(X)\eta(Y))g(\phi Z, W) \\
+ (-g(Z, X + HX) + \eta(X)\eta(Z))g(\phi Y, W) \\
- \eta(Y)g((\nabla X \phi) Z, W) - \eta(Z)g((\nabla X \phi) Y, W)
\]
and a straightforward computation gives
\[
T(Y, Z, X, W) + T(Z, X, Y, W) - T(X, Y, Z, W) + 2T(Y, W, Z, X) \\
= C(X, Y, Z, W) + 2g(\phi Y, W)g(HX, Z) + 2g(\phi X, Z)g(HY, W) \\
+ 2g(\phi X, W)g(HY, Z) + 2g(\phi Z, W)g(HX, Y) \\
+ 2g(\phi Z, X)g(Y, \phi^2 W) + \eta(X)\eta(Y)g(\phi Z, W) - \eta(Z)\eta(Y)g(\phi X, W), \\
(3.12)
\]
where
\[
C(X, Y, Z, W) = -\eta(Y)g((\nabla Z \phi) X, W) + \eta(Y)g((\nabla X \phi) Z, W) \\
- 2\eta(W)g((\nabla Y \phi) Z, X). \\
(3.13)
\]
The anti-symmetrization of (3.11) in \( Y \) and \( W \), and the first Bianchi identity, imply
\[
3R(\phi X, Z, Y, W) + 3R(\phi Z, Y, W) + 3R(X, Z, \phi Y, W) + 3R(X, Z, Y, \phi W) \\
+ 4g(\phi Y, W)g(HX, Z) + 4g(\phi X, Z)g(HY, W) + 2g(\phi X, W)g(HY, Z) \\
- 2g(\phi X, Y)g(HW, Z) + 2g(\phi Z, W)g(HX, Y) - 2g(\phi Z, Y)g(HX, W) = 0,
\]
which implies Eq. (3.1) if one assumes \( H = 0 \). We shall focus on the proof that \( H = 0 \).

For \( X = \xi \) (\( H\xi = \phi \xi = 0 \)), we obtain
\[
R(\xi, \phi Z, Y, W) + R(\xi, Z, \phi Y, W) + R(\xi, Z, Y, \phi W) = 0, \\
(3.14)
\]
\[
- R(\xi, Z, \phi Y, W) - R(\xi, \phi Z, Y, W) + \eta(Y)R(\xi, \phi Z, \xi, W) \\
+ R(\xi, \phi Z, \phi Y, \phi W) = 0, \\
(3.15)
\]
hence
\[ R(\xi, Z, Y, \phi W) + R(\xi, \phi Z, \phi Y, \phi W) + \eta(Y)R(\xi, \phi Z, \xi, W) = 0, \] (3.16)
and
\[ -R(\xi, Z, \phi Y, W) + R(\xi, \phi Z, Y, W) = \eta(W)R(\xi, Z, \xi, \phi Y) - \eta(W)R(\xi, \phi Z, \xi, Y) + \eta(Y)R(\xi, \phi Z, \xi, W). \] (3.17)
By (3.14) and (3.17), we have
\[ 2g(R(\xi, \phi Z)Y, W) + g(R(\xi, Z)Y, \phi W) = \eta(W)(g(R(\xi, Z)\xi, \phi Y) - g(R(\xi, \phi Z)\xi, W)). \] (3.18)
and changing \( Z \) by \( \phi Z \) and \( W \) by \( \phi W \) in (3.18), we get
\[ -2g(R(\xi, Z)Y, \phi W) - g(R(\xi, \phi Z)Y, W) = \eta(W)g(R(\xi, \phi Z)\xi, Y) - \eta(Y)g(R(\xi, Z)\xi, \phi W). \] (3.19)
The sum of the last two above equations is
\[ g(R(\xi, \phi Z)Y, W) - g(R(\xi, Z)Y, \phi W) = \eta(W)g(R(\xi, Z)\xi, \phi Y) + \eta(Y)(g(R(\xi, \phi Z)\xi, W) - g(R(\xi, Z)\xi, \phi W)). \] (3.20)
In the virtue of (3.17) and (3.20), (3.14) is taking form
\[ 3R(\xi, \phi Z, Z, W) = 2\eta(Y)R(\xi, \phi Z, \xi, W) + 2\eta(W)R(\xi, Z, \xi, \phi Y) - \eta(W)R(\xi, \phi Z, \xi, Y) - \eta(Y)R(\xi, Z, \xi, \phi W). \] (3.21)
Applying \( \nabla \xi = -\phi^2 + H \), we have
\[ R(Y, Z, \xi, X) = -g((\nabla_Y \phi^2)X, Z) + g((\nabla_Z \phi^2)X, Y) - g((\nabla_Y H)X, Z) + g((\nabla_Z H)X, Y). \] (3.23)
Taking the cyclic sum over \( X, Y, Z \) and applying the Bianchi identity, we get
\[ g((\nabla_Z H)X, Y) + g((\nabla_X H)Y, Z) - g((\nabla_Y H)X, Z) = 0, \]
\[ R(Y, Z, \xi, X) = -g((\nabla_Y \phi^2)X, Z) + g((\nabla_Z \phi^2)X, Y) - g((\nabla_X H)Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y) + \eta(Y)g(X, HZ) - \eta(Z)g(X, HY) - 2\eta(X)g(Z, HY) - g((\nabla_X H)Y, Z), \] (3.24)
\[ 0 = R(\xi, X, \phi Y, \phi Z) = 2\eta(X)g(H\phi Y, \phi Z) - g((\nabla_X H)\phi Y, \phi Z). \] (3.25)
Let us take a unit eigenvector field \( Y \) such that \( \eta(Y) = 0 \) and \( H^2Y = \lambda Y \); note that \( H^2\phi Y = \lambda \phi Y \), as \( \phi H + H\phi = 0 \). Then
\[ 0 = R(\xi, X, \phi Y, \phi HY) = 2\lambda \eta(X) - g((\nabla_X H)\phi Y, \phi HY) - \frac{1}{2}g((\nabla_X H^2)\phi Y, \phi Y) = 2\lambda \eta(X) - \frac{1}{2}d\lambda(X), \] (3.26)
so that \( d\lambda = -4\lambda \eta \), as \( X \) is arbitrary. Consequently, \( \lambda = 0 \) or \( d\eta = 0 \), implying \( H = 0 \). Hence from (3.24), we get (3.2).

To prove (3.3), let us denote the left hand side of (3.1) by \( R_l \). Then, applying (3.2), we get

\[
0 = \mathcal{R}_l(\phi X, Y, Z, W) + \mathcal{R}_l(X, \phi Y, Z, W) - \mathcal{R}_l(X, Y, \phi Z, W) - \mathcal{R}_l(X, Y, Z, \phi W) = 2R(\phi X, \phi Y, Z, W) - 2R(X, Y, \phi Z, \phi W),
\]

(3.27)

now (3.4) is immediate. \( \square \)

**Proposition 2.** For a nearly Kenmotsu manifold, we have

\[
(\nabla_{\phi X} \phi)\phi Y + (\nabla_X \phi)Y - 2g(\phi X, Y)\xi + \eta(Y)\phi X = 0.
\]

**Proof.** By \( \phi^2 = -I + \eta \otimes \xi, \)

\[
g((\nabla_X \phi)Y, Z) = g((\nabla_X \phi)Y, \phi Z) + \eta(Z)g(X, Y)
\]

\[
+ \eta(Y)g(X, Z) - 2\eta(X)\eta(Y)\eta(Z),
\]

(3.28)

taking into account (2.4), we obtain

\[
g((\nabla_{\phi X} \phi)Y, Z) = g((\nabla_X \phi)Y, \phi Z) + 2\eta(Y)g(X, Z)
\]

\[
- \eta(Z)g(X, Y) - \eta(X)\eta(Y)\eta(Z),
\]

(3.29)

and the above identities imply the requested equation. \( \square \)

**Proposition 3.** For a nearly Kenmotsu manifold, we have the following relations

\[
\text{Ric}(X, \xi) = -2n\eta(X), \quad (3.30)
\]

\[
\text{Ric}(\phi Y, \phi Z) = \text{Ric}(Y, Z) + 2n\eta(Y)\eta(Z), \quad (3.31)
\]

\[
\text{Ric}(Z, \phi Y) + \text{Ric}(\phi Z, Y) = 0, \quad (3.32)
\]

where \( \text{Ric} \) denotes the Ricci tensor and \( Q \) the Ricci operator, defined by \( \text{Ric}(X, Y) = g(QX, Y). \)

**Proof.** Let \( (E_0 = \xi, E_1, \ldots, E_n, E_{n+1}, \ldots, E_{2n}), \) \( \dim M = 2n + 1, \) denote an orthonormal \( \phi \)-frame, which satisfies \( \phi E_i = E_{i+n}, \phi E_{i+n} = -E_i, \) \( i = 1, \ldots, n. \) Taking into account \( \phi \)-basis and (3.2), \( \text{Ric}(X, \xi) \) can be given by (3.30). Then from (3.4),

\[
\text{Ric}(X, Y) = \sum_{i=1}^{n} \left( R(E_i, X, Y, E_i) + R(E_{i+n}, X, Y, E_{i+n}) \right) + R(\xi, X, Y, \xi) \]

(3.33)

\[
= \text{Ric}(\phi X, \phi Y) + \eta(X)\text{Ric}(\xi, Y) - R(\xi, \phi X, \phi Y, \xi) + R(\xi, X, Y, \xi)
\]

\[
= \text{Ric}(\phi X, \phi Y) + \eta(X)\text{Ric}(\xi, Y) = \text{Ric}(\phi X, \phi Y) - 2n\eta(X)\eta(Y),
\]

(3.34)

where we applied (3.30). The last identity is now direct consequence of (3.31). \( \square \)
Proposition 4. The fundamental form of a nearly Kenmotsu manifold satisfies
\[ 3d\Phi(X, Y, Z) = -3g((\nabla_X \phi)Y, Z) - \eta(Y)g(\phi X, Z) + \eta(Z)g(\phi X, Y) \]
\[ -2\eta(X)g(\phi Y, Z). \] (3.35)
\[ d\Phi(X, Y, Z) = \frac{1}{4}g([\phi, \phi](X, Y), \phi Z) + 2(\eta \wedge \Phi)(X, Y, Z). \] (3.36)

Proof. From identities
\[ 3d\Phi(X, Y, Z) = (\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y), \] (3.37)
\[ [\phi, \phi](X, Y) = -\phi(\nabla_X \phi)Y + \phi(\nabla_Y \phi)X + (\nabla_{\phi X} \phi)Y - (\nabla_{\phi Y} \phi)X. \] (3.38)
we obtain
\[ 3d\Phi(X, Y, Z) = -g((\nabla_X \phi)Y, Z) - g((\nabla_Z \phi)X, Y) + g((\nabla_Y \phi)X, Z) \]
\[ -2\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi Z, X) \]
\[ -\eta(Z)g(\phi Y, X), \] (3.39)
\[ \frac{1}{2}[\phi, \phi](X, Y) = -\phi(\nabla_X \phi)Y + \phi(\nabla_Y \phi)X + \eta(Y)X - \eta(X)Y. \] (3.40)
Hence
\[ 6d\Phi(X, Y, Z) = -3g((\nabla_X \phi)Y - (\nabla_Y \phi)X, Z) + \eta(Y)g(\phi X, Z) \]
\[ -\eta(X)g(\phi Y, Z) + 2\eta(Z)g(\phi X, Y) \]
\[ = \frac{3}{2}g([\phi, \phi](X, Y), \phi Z) + 4\eta(X)g(Y, \phi Z) + 4\eta(Y)g(Z, \phi X) \]
\[ + 4\eta(Z)g(X, \phi Y) \]
\[ = \frac{3}{2}g([\phi, \phi](X, Y), \phi Z) + 12(\eta \wedge \Phi)(X, Y, Z). \]
\[ \square \]

An almost Hermitian manifold $(\tilde{N}, J, \tilde{g})$ is called nearly Kähler if $(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0$, $\tilde{\nabla}$ denotes the Levi–Civita connection of $\tilde{g}$, (see for more details [7, 9]).

Once we know that $d\eta = 0$, we are able to describe completely the local structure of a nearly Kenmotsu manifold.

Theorem 2. Let $(M, \phi, \xi, \eta, g)$ be a nearly Kenmotsu manifold. Then
(a) The distribution $D = \ker \eta$ is completely integrable, and maximal integral submanifolds of $D$ are totally umbilical hypersurfaces,
(b) The maximal integral submanifolds of $D$ inherit nearly Kähler structures,
(c) A nearly Kenmotsu manifold is locally isometric to the warped product of a real line and a nearly Kähler manifold.

Proof. (a) As the Reeb form is closed, it is clear that $D = \ker \eta$ is completely integrable. If $\tilde{M}$ denotes a maximal integral submanifold of $D$ then the dimension of $\tilde{M}$ is $2n$ and the restriction $\xi|_{\tilde{M}}$ is normal vector field, and with respect to such choice of normal, the Weingarten map is $A : \tilde{X} \mapsto \nabla_{\tilde{X}} \xi = -\tilde{X}$, hence $\tilde{M}$ is umbilical.
(b) Let $J$ be the $(1,1)$-tensor field on $\tilde{M}$ defined by $J\tilde{X} = \phi \tilde{X}$. This definition is correct, as $D$ is $\phi$-invariant. $J$ is almost complex structure. We verify 
\begin{align*}
(\nabla_{\tilde{X}}J)\tilde{X} = \nabla_{\tilde{X}}J\tilde{X} - J\nabla_{\tilde{X}}\tilde{X} = (\nabla_{\tilde{X}}\phi)(\tilde{X}) = 0,
\end{align*}
as $\eta(\tilde{X}) = 0$, and $\tilde{M}$ is totally umbilical.

(c) For all the concepts considering Riemannian submersions, O’Neill’s tensors $A, T, T^0$ and $N$ we refer to chapter 9 of Besse’s book [1].

We can choose coordinate neighborhood $U = I \times U'$, where $I = (-\epsilon, \epsilon) \subset \mathbb{R}$ is a non-empty interval, and $U' \subset \mathbb{R}^{2n}$ is a disk. Submanifolds of the form $t \times U'$ are simply leaves of distribution $D$, and $\xi = \partial/\partial t$, $\eta = dt$. The projection $\pi : I \times U' \to I$ onto the first factor is a Riemannian submersion with respect to the metric $dt \otimes dt$ on $I$, with fibers $\pi^{-1}(t) = t \times U'$. The horizontal distribution is the distribution spanned by $\xi$, and $D$ is the vertical distribution. From the properties of $A$, being $A_{\xi} = 0$, it follows that $A = 0$.

From $\nabla X \xi = X - \eta(X)\xi$, we obtain $T_U \xi = U$, for vertical vector fields $U, V$, using symmetries of $T$, we find $T_U V = T_V U = -g(U, V)\xi$, and $N = -2n\xi$, therefore 
\begin{align*}
T^0_U V &= T_U V - g(U, V) - g(U, V)\xi + g(U, V)\xi = 0, \\
T^0_U \xi &= T_U \xi + \frac{g(N, \xi)}{2n} U = U - U = 0.
\end{align*}
By [1, Prop. 9.104] $(U, g|_U)$ is warped product $(I \times U', dt^2 + f^2(t)h)$, where as metric $h$ we may take $h = t_0^* g$, $t_0 : t' \mapsto p = (0, t')$, and $\pi_+(N) = -2n \tilde{D}f$, where $\tilde{D}f$ is the gradient of $f$ for the metric $dt^2$. Hence, $\pi_+(\xi) = \frac{1}{f} \tilde{D}f$, and as $\pi_+$ maps isometrically $\xi$ into $\partial/\partial t$, a function $f$ must satisfy $f^2 = Ce^{2t}$ and $C = 1$, by our choice of $h$. $
$ $\square$

Remark 1. In [5], Dileo and Pastore described examples of almost Kenmotsu manifolds locally isometric to the warped product of a real line and an almost Kähler manifold.

Below, we provide direct local construction of nearly Kenmotsu manifolds. Let $\mathcal{D} \subset \mathbb{R}^{2n}$, $n \geq 3$, be an open disk, $p \in \mathcal{D}$, $p = (x^1, \ldots, x^{2n})$. Assume that $(J, \bar{g})$ is a nearly Kähler structure on $\mathcal{D}$, so $(\mathcal{D}, J, \bar{g})$ is nearly Kähler manifold. Now, let $\mathcal{M} = \mathbb{R} \times \mathcal{D}$, $q = (t, p) = (t, x^1, \ldots, x^{2n})$, and we define an almost contact metric structure $(\phi, \xi, \eta, g)$ on $\mathcal{M}$ as follows: by $X_i = \partial/\partial x^i$, $i = 1, \ldots, 2n$, we denote coordinate vector fields and $X_0 = \partial/\partial t$, then 
\begin{align}
\phi X_i &= JX_i, \quad \xi = X_0, \quad \eta = dt, \quad (3.41) \\
g &= dt \otimes dt + f^2 \bar{g}, \quad (3.42)
\end{align}
where $f = f(t) > 0$ is a function on $\mathbb{R}$. Let us denote by $\nabla$ and $\nabla$, Levi–Civita connections of $g$ and $\bar{g}$ respectively. We recall well-known formulas for the connection of warped product metric 
\begin{align}
\nabla_{X_0} X_0 &= 0, \quad \nabla_{X_i} X_0 = \nabla_{X_0} X_i = d \ln f(X_0) X_i, \quad i = 1, \ldots, 2n, \quad (3.43)
\end{align}
\[ \nabla_{X_i}X_j = \bar{\nabla}_{X_i}X_j - g(X_i, X_j)\tilde{D} f, \]  
(3.44)

where \( \tilde{D} f \) is gradient of \( \ln f \). Note \( d \ln f = u\eta \), and \( \bar{\nabla} f = u\xi \), for a function \( u = \xi \ln f \).

Let us compute \( \nabla \phi \). From the definition of \( \phi \) and by assumption that \((J, \bar{g})\) is nearly Kähler, we obtain

\[ (\nabla_{X_i}\phi)X_j + (\nabla_{X_j}\phi)X_i = (\bar{\nabla}_{X_i}J)X_j + (\bar{\nabla}_{X_j}J)X_i - u(g(X_i, \phi X_j) + g(X_j, \phi X_i))\xi = 0, \]
(3.45)
i, j = 1, \ldots, 2n, therefore

\[ (\nabla_{X_i}\phi)X_j + (\nabla_{X_j}\phi)X_i = (\bar{\nabla}_{X_i}J)X_j + (\bar{\nabla}_{X_j}J)X_i - u\phi X_i = 0, \]
(3.46)
i = 1, \ldots, 2n, and

\[ (\nabla_{X_i}\phi)\xi = -\phi \nabla_{X_i}\xi = -u\phi X_i. \]
(3.48)

In conclusion, we obtain that \( \phi \) satisfies

\[ (\nabla_{X}\phi)Y + (\nabla_{Y}\phi)X = -u(\eta(Y)\phi X + \eta(X)\phi Y), \]
(3.49)
hence, \( (\phi, \xi, \eta, g) \) is nearly Kenmotsu if \( u = 1, f = e^t \).

4. Some Theorems About Nearly Kenmotsu Manifolds

In this section, we will show that a normal nearly Kenmotsu manifold is Kenmotsu. Moreover, a nearly Kenmotsu manifold can be never realized as hypersurface of a nearly Kähler manifold.

**Theorem 3.** Every normal nearly Kenmotsu manifold is Kenmotsu.

**Proof.** We know that \( d\eta = 0 \). On the other hand, the structure is normal iff \( N = 0 \). In view of Proposition 4, in the case \( N = 0 \), we have

\[ d\Phi = 2\eta \wedge \Phi, \]

which means that \( M \) is almost Kenmotsu. Now we use the fact that a normal almost Kenmotsu manifold is Kenmotsu. \( \square \)

**Theorem 4** [9]. Let \( M \) be a nearly Kähler manifold with \( \dim M \leq 4 \). Then \( M \) is Kählerian.

Tashiro [14] proved that a Riemannian hypersurface \((M, g)\) of an almost Hermitian manifold \((\tilde{N}, J, \tilde{g})\) inherits an almost contact metric structure \((\phi, \xi, \eta, g)\), where \( (\phi, \xi, \eta) \) are defined by

\[ JX = \phi X + \eta(X)\nu, \quad J\nu = -\xi, \]
(4.1)
where \( \nu \) is normal vector field.
**Theorem 5.** There do not exist nearly Kenmotsu hypersurfaces in nearly Kähler manifolds.

**Proof.** Let $A = -\bar{\nabla}_\nu$ be the Weingarten map and let $\nabla$ be the Levi–Civita connection on $M$. From $(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0$ and the Gauss–Weingarten equations, it follows that

$$
(\nabla_X \phi)Y + (\nabla_Y \phi)X - \eta(Y)AX - \eta(X)AY + 2h(X,Y)\xi = 0,
$$

(4.2)

$$
g(Y,\nabla_X \xi) + g(X,\nabla_Y \xi) = -h(Y,\phi X) - h(X,\phi Y),
$$

(4.3)

if $M$ is nearly Kenmotsu, then

$$
- \eta(Y)\phi X - \eta(X)\phi Y = \eta(Y)AX + \eta(X)AY - 2h(X,Y)\xi,
$$

(4.4)

For $X = Y = \xi$ in (4.4), we have

$$
h(\xi,\xi)\xi = A\xi
$$

(4.5)

Taking the inner product of the Eq. (4.4) with $\xi$, we obtain

$$
2h(X,Y) = \eta(Y)g(AX,\xi) + \eta(X)g(AY,\xi).
$$

Since $A$ is symmetric, applying (4.5) in the last equation, we get

$$
h(X,Y) = h(\xi,\xi)\eta(X)\eta(Y).
$$

In consequence $g(\nabla_X \xi,Y) + g(\nabla_Y \xi,X) = 0$, which contradicts with Proposition 1. □

Using Theorem 4, we can give the following corollary.

**Corollary 1.** There do not exist proper nearly Kenmotsu manifolds for dimension 3 and 5.

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