Hybrid H-infinity synchronization for uncertain continuous chaotic systems based on digital redesign approach

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Abstract
In this paper, the design of hybrid H-infinity synchronization control for continuous chaotic systems based on sliding mode control (SMC) is considered. H-infinity discrete sliding mode controllers integrated with the digital redesign approach are newly designed to achieve robust chaos synchronization. By the proposed design procedure, an H-infinity discrete-time SMC can be easily obtained to guarantee the robustness of synchronization even if the system is disturbed with unmatched perturbations. Besides, since the saturation function is adopted to eliminate the unexpected chattering phenomenon, this paper also discusses the effect of saturation function in multi-input multi-output (MIMO) SMC and the upper bounds of sliding mode trajectories are obtained which is not indicated in the literature. Finally, we perform the simulation to verify the effectiveness of the proposed controller.

Keywords
Sliding mode control, digital redesign, synchronization, H-infinity control

Introduction
In recent decades, chaotic systems have been widely discussed and studied. Chaotic systems are the systems producing nonlinear characteristics with complex dynamic behavior. The chaotic systems have many characteristics such as extreme sensitivity to the initial conditions (butterfly effect), aperiodicity, the broadband random responses, and so on.\textsuperscript{1,2} Special constellations of feedback loops in a nonlinear chaotic system can result in saddle-node and Hopf bifurcations and induce particular strange attractors.\textsuperscript{3} The synchronization of the chaotic system is the most common topic in science and engineering, especially in the secure communication application. Nowadays, there have many control techniques proposed to solve the problem of synchronization. He and Li,\textsuperscript{4} a synchronization control strategy is developed and applies to the considered linearly coupled complex network. The adaptive synchronization control is proposed to attain the purpose of synchronization and this method requires lenient conditions for quantized parameters.\textsuperscript{5} Jahanshahi et al.,\textsuperscript{6} the SMC combined with the fuzzy control to achieve the synchronization for the systems with fractional order. Wang et al.\textsuperscript{7} consider finite-time synchronization and finite-time adaptive synchronization for the impulsive neural networks are also proposed. Yao et al.\textsuperscript{8} is to present the synchronization control in parallel electrocoating conveyor. However, for systems with external disturbances, $H_\infty$ (H-infinity) and hybrid controllers are not considered in above mentioned papers.

In physical systems, there always exist some noise and external disturbances which might cause instability and influence the performance of controlled systems. It will be more challenge to deal with the synchronization of the chaotic system considering the unmatched disturbances. H-infinity control and the linear matrix inequality methods are frequently introduced to obtain a feasible solution and solve the robust control problem. It is a powerful control method which provides the stability and guarantees the controlled performance for systems containing the external disturbances. Zhang et al.,\textsuperscript{9} by using the Riccati equations, a robust H-infinity control is proposed for dealing with the problem of the underwater vehicle. To have a solution

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for the control problem of automated vehicles, the H-infinity control method is also adopted to deal with the unexpected disturbances.\cite{10} Refer to Nan and Wu,\cite{11} the H-infinity approach is also utilized to handle the filter issue. Wang et al.,\cite{12} the authors introduced H-infinity control to solve the non-fragile synchronization problem for a class of discrete-time T-S fuzzy systems. Therefore, the H-infinity control approach has been widely introduced to ensure satisfied robustness performances in many engineering systems with unmatched disturbances. However, the hybrid control design is not well solved in the papers mentioned above.

For designing a robust system, SMC is an outstanding approach due to the good transient performance, fast response and the robustness to external disturbances.\cite{13} The SMC uses discontinuous control signals to drive the system trajectory to hit the switching surface and the controlled systems can be robustly stabilized while in the sliding manifold. Most of all, restricting disturbances effectively is the advantage of the SMC.\cite{14–17} However, there also exist some negative phenomena when designing SMC. The discontinuous sign function is often applied in designing of SMC and it causes the high-frequency oscillation, which is called the chattering. To reduce the chattering, some alternative plans are developed, such as higher-order SMC,\cite{18–20} second-order SMC,\cite{21,22} scalar sign function,\cite{23} and saturation function.\cite{24} In this paper, to solve the undesired chattering, the saturation function is adopted and the effect of saturation function to sliding mode controller is well discussed. Furthermore, we design the disturbance estimator to predict the unknown but bounded disturbances and combine with the SMC to suppress the effect of the disturbances. Fang et al.\cite{25} established an analog tracker by using a disturbance estimator and utilize SMC to deal with the effect of the unknown disturbances. The perturbation estimation is also mentioned in Yan et al.\cite{26} for suppressing disturbances by applying a discrete-time SMC. To achieve the disturbance estimation, a reduced-order disturbance observer is discussed for the discrete-time linear system with unexpected disturbance.\cite{27} However, by using SMC, the unmatched disturbance is still difficult to be perfectly suppressed\cite{28} even in the above mentioned papers\cite{15–28} even with the disturbance estimators. Therefore, in this paper, the H-infinity control and control performance index are used to cope with the unmatched disturbance and suppress the influence of disturbances in the sliding mode manifold as possible.

Although many approaches and techniques have been proposed for chaotic system stability and synchronization, there still have some problems needed to discuss. For example, the synchronization of master-slave systems is frequently designed by using continuous-time controllers.\cite{29,30} In the literature, most papers regarding the design of discrete sliding mode control (DSMC) only consider the discrete-time chaotic systems.\cite{31,32} However, the DSMC is difficult to establish for continuous-time systems when the hybrid control is considered. However, due to the better reliability and lower cost of digital devices, using digital microcontroller to implement the controllers for continuous systems has become an important trend. Therefore, we will consider the hybrid synchronization control for continuous-time chaotic systems by using digital-redesign-based controllers. Due to the hybrid control structure in sample-data systems, the continuous-time controller cannot be utilized directly; hence, the controller should be discretized to the discrete-time domain. A sampled-data system is a control system in which a continuous-time plant is controlled by a digital device. Therefore, digital redesign approach is a type of sampled-data control. However, this design approach for sampled-data systems is easier than other design methods which directly discuss the discrete controller design in discrete time domain. The digital-redesign method can directly transform the designed analog controller to the discrete type and also ensure the control performance of the hybrid controlled systems. Morais et al.,\cite{33} by using the digital redesign, the designed continuous-time controllers are redesigned to the discrete-time controllers. The discretization of the continuous controller by utilizing the digital-redesign method for the controlled system considering the time delay is investigated.\cite{34} Tsai et al.\cite{35} presented a new digital-redesign control scheme to transform analog high-gain controllers to digital low-gain controllers. Thus, it verifies that the digital redesign approach is an effective approach for hybrid control design.

In this paper, we will consider the hybrid H-infinity synchronization control for continuous chaotic systems based on digital redesigned SMC. A novel sliding mode controller integrated with the digital redesign method is proposed to achieve hybrid control for chaotic system synchronization. This proposed design procedure is easily implemented and the robustness of synchronization can be ensured even if the controlled system is disturbed with unmatched perturbations. To avoid the undesired chattering, the saturation function is introduced in our controller and the influence due to the saturation function is well discussed.

The layout for this study is as follows. In section II, we present the master-slave system model and problem formulation for synchronization control. In section III, we design the continuous SMC. Then we apply the digital redesign approach to get the discrete SMC controller in section IV. In section V, we present the simulation results of the hybrid controlled system to verify the feasibility of the designed approach. Finally, we have the conclusion in section VI.

**Notation**

\( w^T \) is the transport of a matrix \( w \). \( \| w \| \) is the Euclidean norm of the vector \( w \). \( I_n \) denotes the identity matrix with size \( n \times n \). \( | w | \) represents the absolute value. \( w^d \) denotes to the pseudo inverse for a matrix \( w \).
the sliding mode function is chosen as \( x = [\text{sgn}(x_1), \text{sgn}(x_2), \ldots, \text{sgn}(x_m)]^T \in \mathbb{R}^m \) and \( \text{sgn}(x) \) denotes the nonlinear function of \( x \), when \( x > 0 \), \( \text{sgn}(x) = 1 \); when \( x < 0 \), \( \text{sgn}(x) = -1 \).

**System description**

Considering chaos synchronization, the slave system with external disturbances is described by

\[
\dot{x} = Ax(t) + B_u(g(x(t))) + f(x(t), t) + u_c(t) + B_d d(t)
\]

in (1), \( A \in \mathbb{R}^{n \times n}, B_u \in \mathbb{R}^{n \times m}, B_d \in \mathbb{R}^{n \times r}, \) and \( x(t) \in \mathbb{R}^n, u_c(t) \in \mathbb{R}^m, g(x(t)) \in \mathbb{R}^m, f(x(t), t) \in \mathbb{R}^m, \) and \( d(t) \in \mathbb{R}^r \) are the state vector, the control input vector, the known internal nonlinear vector and external bounded disturbances, respectively. \( \|d(t)\| \leq \alpha_1 \) is assumed. According to the slave system, the master system is given as

\[
\dot{y}(t) = Ay(t) + B_u g(y(t)),
\]

(2)

In (2), \( A \in \mathbb{R}^{n \times n}, B_u \in \mathbb{R}^{n \times m}, \) and \( y(t) \in \mathbb{R}^n, \) \( g(y(t)) \in \mathbb{R}^m, f(x(t), t) \in \mathbb{R}^m, \) \( f(x(t), t) \in \mathbb{R}^m, \) and \( d(t) \in \mathbb{R}^r \) are the state and known internal nonlinear vectors of system (2), respectively. Besides, \( f(x(t), t) = f_1(x(t), t) + f_2(t) \), which includes the term with the state and the term without the state, respectively, are bounded by \( \|f_1(x(t), t)\| \leq \alpha_2 \|x(t)\| \) and \( \|f_2(t)\| \leq \alpha_3 \), respectively. Under this condition, we can easily determine the parameters of the controller designed later.

Considering (1) and (2), we have the error state vector defined by \( e(t) = x(t) - y(t) \), and then differentiate \( e(t) \) to have the error dynamics as

\[
\dot{e}(t) = \dot{x}(t) - \dot{y}(t)
\]

\[
= \dot{x}(t) - \dot{y}(t)
\]

\[
= Ae(t) + B_u(g_c(t) + u_c(t) + f(x(t), t)) + B_d d(t),
\]

(3)

where \( g_c(t) = g(x(t)) - g(y(t)) \). Once we have the error dynamics between the slave system and the master system, we continue to design the controller to achieve the chaotic synchronization.

**Sliding mode control design**

In this section, we will use SMC to suppress the nonlinear disturbances and utilize the digital redesign method to discretize the controller for hybrid control. Here we firstly aim to design a controller \( u_c(t) \) in (3) to ensure the states of controlled slave systems (1) can be forced to track the state trajectories of systems (2) with a satisfying performance even with unmatched disturbances. To complete the SMC design, it is necessary to select a proper sliding mode function \( s(t) \) and ensure that \( s(t) \) can hit to the sliding mode (i.e. \( s(t) = 0 \)). First, the sliding mode function is chosen as

\[
s(t) = C_s e(t) + \int (-C_s A e(t) + K_s e(t)) \, dt,
\]

(4)

where \( C_s = B_u^T \) leads to \( C_s B = I_m \) and \( K_s \) is the designed controller gain which will be computed later. Differentiate (4), then we obtain

\[
\dot{s}(t) = C_s \dot{e}(t) - C_s A e(t) + K_s e(t)
\]

\[
= C_s (A e(t) + B_u (g_c(t) + u_c(t) + f(x(t), t)) + B_d d(t))
\]

\[
= C_s A e(t) + K_s e(t)
\]

(5)

From (5), the equivalent controller \( u_{eq}(t) \) in the sliding manifold can be obtained by using \( s(t) = \dot{s}(t) = 0 \).

Therefore, we have the equivalent control \( u_{eq}(t) \) as follows

\[
u_{eq}(t) = -K_s e(t) - g_c(t) - C_s B d(t) - f(x(t), t).
\]

(6)

Substituting (6) into (3), we obtain the equivalent error dynamics in the sliding manifold as follows

\[
\dot{e}(t) = A e(t) + B_u (e_c(t) - C_s B d(t)) + B_d d(t)
\]

\[
= A e(t) + B_u (e_c(t) - C_s B d(t)) + B_d d(t),
\]

(7)

where \( B_u = B_d - B_u C_s B_d \).

Based on (7), we can calculate \( K_s \) with LMI to ensure that the controlled system in the sliding manifold is with the \( H_\infty \) performance. Hence, we discuss the design of controller gain \( K_s \) in Theorem 1.

**Theorem 1**

If the following LMIs are satisfied with specified matrices \( Y \) and \( Z \)

\[
Y > 0,
\]

\[
\begin{bmatrix}
YA^T + AY - Z Z^T B_u^T - B_d Z & B_d & Y \\
B_d^T & -I_n & 0 \\
Y & 0 & -\gamma I_m
\end{bmatrix}
\]

(8)

where \( Y \in \mathbb{R}^{n \times n} \) is a symmetric matrix and \( Z \in \mathbb{R}^{m \times n} \), then the controller gain can be obtained as \( K_s = Z Y^{-1} \).

The equivalent error dynamics in the sliding manifold is robust stable in \( H_\infty \) sense with a pre-selected positive parameter \( \gamma \).

**Proof.**

Consider the Lyapunov function as

\[
V_e(t) = e^T(t) P e(t),
\]

where \( P \) is symmetric and positive definite. Then, we take the derivative of \( V_e(t) \) and obtain the following equation

\[
\dot{V}_e(t) = e^T(t) P (A - B_u K_s) e(t) + e^T(t) (A - B_u K_s)^T P e(t)
\]

\[
+ e^T(t) P B_d d(t) + d^T(t) B_u^T P e(t).
\]

(9)
To analyze the $H_\infty$ performance of the synchronization between the slave system and master systems, the following $H_\infty$ performance index is given

$$J_\infty = \int_0^\infty \gamma^2 e^T(\tau) e(\tau) - d^T(\tau) d(\tau) \, d\tau, \quad (10)$$

where $\gamma$ is a pre-specified positive parameter. Then, one has the following inequality

$$J_\infty = \int_0^\infty (\gamma^2 e^T(\tau) e(\tau) - d^T(\tau) d(\tau) + \dot{V}_e(\tau)) \, d\tau - \lim_{\tau \to \infty} V_e(\tau), \quad (11)$$

For any zero initial condition, we have

$$J_\infty \leq \int_0^\infty (\gamma^2 e^T(\tau) e(\tau) - d^T(\tau) d(\tau) + \dot{V}_e(\tau)) \, d\tau - \lim_{\tau \to \infty} V_e(\tau), \quad (12)$$

then one has the following inequality

$$\gamma^2 e^T(\tau) e(\tau) - d^T(\tau) d(\tau) + \dot{V}_e(\tau) \leq 0. \quad (13)$$

As long as the equation (13) is satisfied with the pre-specified parameter $\gamma$, the $H_\infty$ performance can be achieved. Therefore, we replace (9) into (13) and utilize Schur complement to obtain the linear matrix inequality (LMI) as follows

$$\begin{bmatrix} P(A - B_e K_e) + (A - B_e K_e)^T P & PB_d & I_n \\ B_d^T P & -I_n & 0 \\ I_n & 0 & -\gamma^2 I_m \end{bmatrix} < 0. \quad (14)$$

We set $T = \text{diag}\{Y, I, I\}$, where $Y = P^{-1}$, then we can establish a congruence transformation and rewrite (14) as

$$T^T \begin{bmatrix} P(A - B_e K_e) + (A - B_e K_e)^T P & PB_d & I_n \\ B_d^T P & -I_n & 0 \\ I_n & 0 & -\gamma^2 I_m \end{bmatrix} T < 0.$$  

Therefore, one can get

$$\begin{bmatrix} YA^T + AY - Z^T B_d^T - B_d Z & B_d Y & 0 \\ B_d^T Y & -I_n & 0 \\ Y & 0 & -\gamma^2 I_m \end{bmatrix} < 0, \quad (15)$$

where $Y \in \mathbb{R}^{n \times n}$ and $Z = K_e Y \in \mathbb{R}^{n \times n}$ are defined in LMI. From (15), it reveals that when having proper $Y$ and $Z$ with the pre-specified positive parameter $\gamma$, then the controller gain $K_e$ can be found. Therefore, for the error dynamics in the sliding manifold, one can ensure $H_\infty$ tracking performance with $||e(t)||^2 \leq \gamma ||d(t)||^2$.

After completing the discussion of controller gain $K_e$, to guarantee the state trajectory of sliding mode function in (4) can be operated in the sliding manifold as expected, we design $u_c(t)$ as

$$u_c(t) = -K_e e(t) - \gamma_1 s(t) - (\gamma_2 + \gamma_3) \text{sgn}(s(t)). \quad (16)$$

By (16), (5) can be rewritten as

$$\dot{s}(t) = g_e(t) + f(x(t), t) + C_e B_d d(t) - \gamma_1 s(t) - (\gamma_2 + \gamma_3) \text{sgn}(s(t)), \quad (17)$$

where $\gamma_1$ and $\gamma_2$ are positive parameters and $\gamma_3 = ||g_e(t)|| + \alpha_1 ||C_e|| ||B_d|| + \alpha_2 ||x(t)|| + \alpha_3$.

To prove the existence of the sliding manifold for the controlled system, a Lyapunov equation is given as

$$V(s(t)) = \frac{1}{2} s^T(t)s(t). \quad (18)$$

Differentiating (18), one has

$$\dot{V}(s(t)) = s^T(t)(g_e(t) + f(x(t), t) + C_e B_d d(t) - \gamma_1 s(t) - (\gamma_2 + \gamma_3) \text{sgn}(s(t))), \quad (19)$$

where $||d(t)|| \leq \alpha_1$ and then $||C_e B_d d(t)|| \leq \alpha_1 ||C_e|| ||B_d||$, $f(x, t) = f_1(x(t), t) + f_2(t)$, $||f_1(x(t), t)|| \leq \alpha_2 ||x(t)||$, and $||f_2(t)|| \leq \alpha_3$. Therefore, we design $\gamma_3$ as $\gamma_3 = ||g_e(t)|| + \alpha_1 ||C_e|| ||B_d|| + \alpha_2 ||x(t)|| + \alpha_3$ and replace the designed $\gamma_3$ into (19), the following result is obtained as

$$\dot{V}(s(t)) \leq (||g_e(t)|| + \alpha_1 ||C_e|| ||B_d|| + \alpha_2 ||x(t)|| + \alpha_3) s(t)|| - (\gamma_2 + \gamma_3) ||s(t)||^2 = - (\gamma_1 ||s(t)|| + \gamma_2) ||s(t)|| \leq 0. \quad (20)$$

According to the Lyapunov theorem, (20) indicates that $s(t)$ will converge to zero; therefore, we conclude that $u_c(t)$ can equal to $u_{e\in}(t)$ since $s(t) = 0$ and $\dot{s}(t) = 0$ are ensured.

**Remark 1.** In the traditional SMC, due to the discontinuous sign function, the undesired chattering often appears in SMC. To reduce the unexpected chattering, we can substitute the sign function with the saturation function. The saturation function is a continuous and smooth function that can be adopted to minimize or reduce the chattering phenomenon. The saturation function is described as

$$\text{sat}(s(t)) = \begin{bmatrix} s_1(t) \\ \vdots \\ s_m(t) \end{bmatrix} = \begin{bmatrix} \frac{s_1(t)}{|s_1(t)| + \epsilon} \\ \vdots \\ \frac{s_m(t)}{|s_m(t)| + \epsilon} \end{bmatrix}, \quad (21)$$

in (21), the parameter $\epsilon$ is a small positive selected constant.

Thus, according to **Remark 1**, we redesign (16) with the saturation function as follows

$$u_c(t) = -K_e e(t) - \gamma_1 s(t) - \gamma_2 \text{sat}(s(t)) - \gamma_3 s(t). \quad (22)$$
While the saturation function is utilized in (22), the Lyapunov equation (19) can be rewritten as

\[
\dot{V}(s(t)) = s^T(t)(g_s(t) + f(x(t), t) + C_sB_d(t) - \gamma_1s(t) - (\gamma_2 + \gamma_3)s_t^\text{sat}(s(t))) \\
\leq \|g_s(t)\| + \alpha_1\|C_s\|\|B_d\| + \alpha_2\|x(t)\| + \alpha_3)\|s(t)\| \\
- \gamma_1\|s(t)\|^2 - (\gamma_2 + \gamma_3)s_t^\text{sat}(s(t)).
\] (23)

We can rewrite (23) as follows

\[
\dot{V}(t) \leq \sum_{i=1}^m \left[\|g_s(t)\| + \alpha_1\|C_s\|\|B_d\| + \alpha_2\|x(t)\| + \alpha_3)s_t^\text{sat}(s(t))\right].
\]

\[
- \gamma_1\|s(t)\|^2 - (\gamma_2 + \gamma_3)\left(1 - \frac{e}{|s(t)| + e}\right)\|s(t)\|.
\]

Since \( \gamma_3 = \|g_s(t)\| + \alpha_1\|C_s\|\|B_d\| + \alpha_2\|x(t)\| + \alpha_3 \) and then one can further have

\[
\dot{V}(t) \leq \sum_{i=1}^m \left[-((s(t)) - e)^2 \gamma_1 - 2\gamma_1 e|s(t)| + \gamma_1 e^2.
\]

\[
- \gamma_2|s(t)| + (\gamma_2 + \max(\gamma_3))e
\]

\[
= \sum_{i=1}^m \left[-((s(t)) - e)^2 \gamma_1 - (2\gamma_1 e + \gamma_2)|s(t)|.
\]

\[
+ (\gamma_2 + \max(\gamma_3) + \gamma_1)e\right].
\] (24)

which implies \( \dot{V}(t) \leq 0 \) when \( |s(t)| \geq (\gamma_2 + \max(\gamma_3) + \gamma_1)e/(2\gamma_1 e + \gamma_2) \). Therefore, the sliding mode trajectories are bounded by \( (\gamma_2 + \max(\gamma_3) + \gamma_1)e/(2\gamma_1 e + \gamma_2) \), that is,

\[
|s(t)| \leq \frac{(\gamma_2 + \max(\gamma_3) + \gamma_1)e}{(2\gamma_1 e + \gamma_2)}. \] (25)

Based on the above discussion, the norm bound of sliding mode trajectories can be found from (25) and represented as

\[
\|s(t)\| \leq \frac{m(\gamma_2 + \max(\gamma_3) + \gamma_1)e}{2\gamma_1 e + \gamma_2}. \] (26)

Hence, we can ensure the desired performance in the SMC by selecting the proper parameters \( (\gamma_1, \gamma_2, \text{and} \ e) \) in (26). For example, a small enough parameter \( e \) and big enough parameters \( \gamma_1 \) and \( \gamma_2 \) can be designed for \( s(t) \) approximating to zero closely.

**Digital redesign of SMC**

In this section, the occurrence of the sliding manifold will be discussed and ensured in the discrete-time domain firstly. After completing the continuous-time SMC design, we continue to apply the digital-redesign method to obtain the discrete-time controller. First, we discretize (17) by using Euler method\(^{36} \) with sampling time \( T_s \) and we can get

\[
s(kT_s + T_s) - s(kT_s) = T_s(g_s(kT_s) + f(x(kT_s), kT_s) + C_sB_d(kT_s) - \gamma_1s(kT_s) - (\gamma_2 + \gamma_3)s_t^\text{sat}(s(kT_s))).
\] (27)

To ensure the existence of the sliding manifold in discrete-time domain, the reaching conditions are introduced in Lemma 1.

**Lemma 1**

The following conditions are proposed to prove the convergence of the sliding mode function.\(^{37} \)

\[
\Delta s(kT_s) = s(kT_s + T_s) - s(kT_s) \\
\leq - \gamma_1T_s(s(kT_s) - \gamma_2T_s\text{sgn}(s(kT_s)), \text{ for } s(kT_s) > 0.
\]

\[
\Delta s(kT_s) = s(kT_s + T_s) - s(kT_s) \\
\geq - \gamma_1T_s(s(kT_s) - \gamma_2T_s\text{sgn}(s(kT_s)), \text{ for } s(kT_s) < 0.
\] (28)

From (27) and Lemma 1, we can get

\[
\Delta s(kT_s) = - \gamma_1T_s(s(kT_s) + T_s(g_s(kT_s) + f(x(kT_s), kT_s) + C_sB_d(kT_s) - \gamma_1s(kT_s) - (\gamma_2 + \gamma_3)s_t^\text{sat}(s(kT_s))) \\
\leq - \gamma_1T_s(s(kT_s) - \gamma_2T_s\text{sgn}(s(kT_s)), \text{ for } s(kT_s) > 0.
\]

\[
\Delta s(kT_s) = - \gamma_1T_s(s(kT_s) + T_s(g_s(kT_s) + f(x(kT_s), kT_s) + C_sB_d(kT_s) - \gamma_1s(kT_s) - (\gamma_2 + \gamma_3)s_t^\text{sat}(s(kT_s))) \\
\geq - \gamma_1T_s(s(kT_s) - \gamma_2T_s\text{sgn}(s(kT_s)), \text{ for } s(kT_s) < 0.
\] (29)

where \( \gamma_3 = \|g_s(kT_s)\| + \alpha_1\|C_s\|\|B_d\| + \alpha_2\|x(kT_s)\| + \alpha_3 \).

Therefore, according to Lemma 1, \( s(kT_s) \) can converge to zero. In other words, the sliding mode manifold is ensured.

By utilizing the digital-redesign method\(^{21,22} \) and Remark 1, we can obtain the digital-redesign-based SMC as

\[
u_d(kT_s) = u_c^\text{d}(kT_s) - \gamma_1s(kT_s) \\
- \gamma_2\text{sgn}(s(kT_s)) - \gamma_3\text{sgn}(s(kT_s)),
\] (30)

where \( u_c^\text{d}(t) = -K_e e(t) \). According to (28) and (29), while the sliding manifold occurs \( s(kT_s) = 0 \), the control law \( u_c^\text{d}(t) \) in (30) can be transferred to the discrete-time control law \( u_c^\text{d}(kT_s) \) by using the digital-redesign method. Therefore, when the controlled system is in the sliding manifold, that is, \( u_c(t) = u_c^\text{eq}(t) \), (7) can be re-described as
\[ \dot{e}(t) = Ae(t) + B_d(-K_c e(t)) + B_n d(t) \]
\[ = Ae(t) + B_n u'_e(t) + B_n d(t), \]  
and it can be discretized as
\[ e(kT_s + T_s) = Ge(kT_s) + Hu'_e(kT_s) + H_n d(kT_s), \]
(32)
where \( G = e^{AT_s}, H = (G - I)A^{-1}B, \) and \( H_n = (G - I)A^{-1}B_n. \) Due to the zero-order-hold (Z.O.H.), the controller \( u'_e(t) \) can be represented as
\[ u'_e(t) = u'_e(kT_s) \approx u'_e(kT_s) + u'_e(kT_s + T_s) \]
\[ - u'_e(kT_s + T_s), \text{ for } kT_s \leq t < kT_s + T_s. \]  
(33)
Therefore, we utilize the above condition to achieve the digital redesign. Besides, Lemma 2 is introduced for completing the proposed digital redesign approach.

**Lemma 2**

If a function \( f(kT_s) \) is bounded and changed slowly, we can define it in the order of \( O(T_s) \) and have the following properties.\(^{38}\)

**Property.** \( f(kT_s) = O(T_s), \) and
\[ f(kT_s) - f(kT_s - T_s) = O(T_s^2). \]

To achieve the designed control law, the designed control input should be bounded and changed slowly. According to Lemma 2, we assume
\[ O_1(T_s) = u'_e(kT_s) - u'_e(kT_s + T_s), \]  
(34)
then we utilize the digital-redesign method and have the following equation according to (32) and (33)
\[ u'_e(kT_s) \approx u'_e(kT_s) + u'_e(kT_s + T_s) - u'_e(kT_s + T_s) \]
\[ = -K_c(ge(kT_s) + Hu'_e(kT_s) + H_n d(kT_s)) + O_1(T_s^2). \]  
(35)
Let \( d_a = H_n d, \) and according to Lemma 2, \( d_a(kT_s) \) belongs to \( O_2(T_s). \) Then we can rewrite (35) as
\[ \left(I_m + K_c H\right)u'_e(kT_s) \approx -K_c ge(kT_s) \]
\[ - K_c \hat{d}_a(kT_s) - K_c d_a(kT_s) - \hat{d}_a(kT_s) + O_1(T_s^2), \]  
(36)
where \( d_a(kT_s) \) belongs to \( O_2(T_s). \) Since \( d_a(kT_s) \) is difficult to acquire, we design an estimator \( \hat{d}_a(kT_s) \) in Theorem 2 to estimate \( d_a(kT_s) \) for disturbance rejection.

**Theorem 2**

With the bounded and slowly time-varying disturbance, the estimator \( \hat{d}_a(kT_s) \) is designed as
\[ \hat{d}_a(kT_s) = K_g e(kT_s) - \eta(kT_s), \]  
(37)
Where the estimator gain \( K_g \) is selected satisfying \( |\text{eig}(I - K_g)| < 1 \) and
\[ \eta(kT_s + T_s) = \eta(kT_s) \]
\[ + K_g \left[(G - I)e(kT_s) + Hu'_e(kT_s) + \hat{d}_a(kT_s)\right]. \]  
(38)
Then the proposed estimator (37) with (38) can estimate the undesirable disturbance \( d_a(kT_s), \) and the estimation error is smaller than \( K_g^{-1}a, \) where \( a \) is a small positive value.

**Proof.**

The error dynamics of (37) can be represented as
\[ d_a(kT_s + T_s) - d_a(kT_s + T_s) = d_a(kT_s + T_s) \]
\[ - d_a(kT_s) + d_a(kT_s) - \eta(kT_s + T_s) - K_g e(kT_s + T_s), \]  
(39)
which implies
\[ \hat{d}_a(kT_s + T_s) = d_a(kT_s) - K_g \left(d_a(kT_s) - \hat{d}_a(kT_s)\right) \]
\[ - \hat{d}_a(kT_s) + (d_a(kT_s + T_s) - d_a(kT_s)) \]
\[ = (I - K_g)\hat{d}_a + O_2(T_s^2), \]  
(40)
where \( \hat{d}_a(kT_s) = d_a(kT_s) - \hat{d}_a(kT_s), \) \( O_2(T_s^2) = d_a(kT_s + T_s) - d_a(kT_s), \) and \( |\text{eig}(I - K_g)| < 1. \)

First, we assume \( |O_2(T_s^2)| \leq \alpha, \) where \( \alpha \) is a small positive value. Hence, from (40), we have
\[ |\hat{d}_a(kT_s + T_s)| \leq (I - K_g)|\hat{d}_a(kT_s)| + \alpha \]  
(41)
and
\[ |\hat{d}_a(kT_s + nT_s)| \leq (I - K_g)^n|\hat{d}_a(kT_s)| + \sum_{j=0}^{n} (I - K_g)^j \alpha. \]  
(42)
Therefore, when \( n \to \infty, \) we have
\[ \lim_{n \to \infty} |\hat{d}_a(kT_s + nT_s)| \]
\[ \lim_{n \to \infty} |\hat{d}_a(kT_s + nT_s)| \]
\[ \leq \lim_{n \to \infty} (I - K_g)^n|\hat{d}_a(kT_s)| + \lim_{n \to \infty} \sum_{j=0}^{n} (I - K_g)^j \alpha \]
\[ = [(I - K_g)^0 + (I - K_g)^1 + \ldots + (I - K_g)^n] \alpha \]
\[ = K_g^{-1} \alpha. \]  
(43)
Since the proposed estimator gain \( K_g \) is selected to satisfy the condition \( |\text{eig}(I - K_g)| < 1. \) We can conclude that \( \hat{d}_a(kT_s) = d_a(kT_s) - \hat{d}_a(kT_s) \) is bounded by \( K_g^{-1} \alpha \) when \( n \to \infty, \) which implies that the difference between \( d_a(kT_s) \) and \( \hat{d}_a(kT_s) \) converges to zero closely and belongs to \( O(T_s^2) \) and \( \hat{d}_a(kT_s) \) belongs to \( O(T_s) \). In other words, the proposed estimator (37) and (38) can
estimate the undesirable disturbance \(d_e(kT_s)\), and the estimation error is smaller than \(K_e^{-1}\alpha_0\).

According to Lemma 2 and Theorem 2, we obtain control law from (35) which can be designed as
\[
\begin{align*}
\hat{u}_d(kT_s) &= -K_de(kT_s) - (I + K_eH)^{-1}K_ee(kT_s),
\end{align*}
\]

where \(K_d = (I_m + K_eH)^{-1}K_eG\). Substituting (44) into (32), one can get
\[
\begin{align*}
e(kT_s + T_s) &= Ge(kT_s) + H(-K_de(kT_s)) \\
 &- (I + K_eH)^{-1}K_ed(kT_s) + H_d(kT_s)
\end{align*}
\]
and then one has
\[
\lim_{k \to \infty} e(kT_s + T_s)
\]
\[
\leq (1 - \|G_e\|^{-1}) \frac{H_d(kT_s) - H(I + K_eH)^{-1}K_ee(kT_s))}{O(T_s)},
\]
where \(G_e = G - HK_d\). According to above results, one can infer that the tracking error \(e(kT_s)\) belongs to \(O(T_s)\) and the proposed controller (44) also belongs to \(O(T_s)\) such that we can conclude \(u_d(kT_s) - \hat{u}_d(kT_s) - K_eG_e(kT_s)\). Also, to achieve the digital-redesign-based SMC, we discretize (4) to implement the discrete-time sliding mode function as follows
\[
s(kT_s) = C_s(e(kT_s) + s_T(kT_s)),
\]
where
\[
s_T(kT_s) = s_T(kT_s) + T_s(-C_sAe(kT_s) + K_ee(kT_s)).
\]
Finally, by applications of the disturbance estimation approach and saturation function, the digital-redesign-based SMC controller \(u_d(kT_s)\) can be obtained as follows
\[
\begin{align*}
u_d(kT_s) &= \hat{u}_d(kT_s) - \gamma_1 \hat{s}(kT_s) - (\gamma_2 + \gamma_1)\text{sat}(s(kT_s)).
\end{align*}
\]

**Simulation results**

To verify the robustness of the proposed controller designed above, we apply our approach to Chen’s chaotic system and the dynamic equations are shown as follows
\[
\begin{align*}
\dot{x}_1(t) &= a(x_2(t) - x_1(t)), \\
\dot{x}_2(t) &= (c-a)x_1(t) - x_1(t)x_2(t) + cx_2(t), \\
\dot{x}_3(t) &= x_1(t)x_2(t) - bx_3(t),
\end{align*}
\]
where \(a = 35\), \(b = 3\), \(c = 28\) and the chaotic strange attractor is shown in Figure 1.

To perform the simulation, we rearrange Chen’s chaotic system as
\[
\dot{x}(t) = Ax(t) + B_u(u(t) + g(x(t))),
\]
where \(x(t) = [x_1(t) \ x_2(t) \ x_3(t)]^T\). The above system is assumed to be with disturbances, it can be rewritten as follows
\[
\dot{x}(t) = Ax(t) + B_u[g(x(t)) + f(x(t), t) + u(t)] + B_dd(t),
\]
where
\[
A = \begin{bmatrix}
-35 & 35 & 0 \\
-7 & 28 & 0 \\
0 & 0 & -3
\end{bmatrix}, \quad B_u = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\]
\[
B_d = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \text{and} \quad g(x(t)) = \begin{bmatrix}
-x_1(t)x_3(t) \\
x_1(t)x_2(t)
\end{bmatrix},
\]
and the master system is given by
\[
\dot{y}(t) = Ay(t) + B_dg(y(t)).
\]
To get the response of the synchronization, by utilizing the proposed controller (47) and the parameter \(\gamma\) is selected as 0.3 in LMI (8) and the controller parameters are selected as \(\gamma = 35\), \(\alpha_1 = 0.3\sqrt{3}\), \(\alpha_3 = 0.15\), and \(\kappa = 0.15\), \(\kappa_g = 0.85\), and \(\varepsilon = 0.0015\) in the saturation function, then we can solve the controller gain, which is \(K_e = \begin{bmatrix}
56.7572 & 17.6339 & 0 \\
0 & 0 & 5.8480
\end{bmatrix}\).

The sampling time is given by \(10^{-3}\) s. To complete the digital-redesign control method, we discretize \(A, B, K_e\) to obtain \(G = \begin{bmatrix}
0.9655 & 0.0349 & 0 \\
-0.0070 & 1.0283 & 0 \\
0 & 0 & 0.9970
\end{bmatrix}\),
\[
H = \begin{bmatrix}
1.746 \times 10^{-5} & 0 & 0 \\
0.00101 & 0 & 9.985 \times 10^{-4} \\
0 & 53.6625 & 19.7387
\end{bmatrix},
\]
\[
K_d = \begin{bmatrix}
53.6625 & 19.7387 & 0 \\
0 & 0 & 5.7966
\end{bmatrix}.
\]
The initial conditions of the slave system and master system are, respectively, selected as $X(0) = \frac{1}{2C_0}$ and $Y(0) = \frac{1}{2C_0}$ respectively. The disturbances are assumed as $d(t) = 0.3 \cos \left( \frac{\pi t}{15} \right)$ and $f(x(t), t) = 0.1 \|x(t)\|$ respectively. The simulation results are included in Figures 2 to 7. In Figure 2, we can see that the designed digital-redesign SMC-based controller can ensure that the states of the controlled slave system track the state trajectories of the master system very closely. The error vector between $x(t)$ and $y(t)$ is shown in Figure 3. Figure 4 shows the control history. In Figure 5, the bound of $s(t)$ is computed from (26). As the simulation results, $|s(t)|$ converge to the neighborhood of zero with selecting proper parameters in (25). From Figures 6 and 7, it is observed that the error between the disturbance estimator and the disturbances is approaching zero. Figure 8 shows the control history with the sign function. Compared with Figure 4, the chattering can be obviously reduced with the saturation function. After calculation, we obtain that

$$\gamma^2 = 0.09 > \int_0^\infty (g_e^T(\tau)g_e(\tau)) d\tau / \int_0^\infty (d^T(\tau)d(\tau)) d\tau$$

and

$$\gamma^2 = 0.09 > \int_0^\infty (g_e^T(\tau)g_e(\tau)) d\tau / \int_0^\infty (d^T(\tau)d(\tau)) d\tau = 0.0161$$

which means the $H_\infty$ synchronization performance is satisfied with the proposed digital-redesign SMC-based controller.

**Conclusion**

This paper proposes a discrete-time SMC integrated with the digital-redesign approach to deal with unpredicted unmatched disturbance and achieve the hybrid $H_\infty$ chaotic synchronization as well. The proposed digital-redesign method can directly transform the continuous analog controller to the discrete type. This proposed design procedure not only can easily obtain a discrete-time $H_\infty$ SMC but also guarantee the robust performance of synchronization even if the system is with unmatched perturbations. Furthermore, the sliding mode trajectories are bounded within the predicted bounds when a saturation function is introduced to reduce the chattering. The simulation results demonstrate the robustness synchronization of the chaotic systems by using the proposed hybrid $H$-
Figure 3. The errors between master and slave systems.

Figure 4. The control response with the saturation function.
Figure 5. The response of the sliding mode function.

Figure 6. The disturbance estimator $\hat{d}_a(kT_s)$ and $d_a(kT_s)$. 
infinity SMC. In the future research, we can extend the proposed digital redesign approach for other control problems such that it will become easy to implement the designed discrete controller by using a digital microprocessor controller for better reliability, lower cost, smaller size, more flexibility, and better performance.

Declaration of conflicting interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was financially supported by the Ministry of Science and Technology, Taiwan, under grant MOST-109-2221-E-167-017.

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Figure 7. The error between $\hat{d}_c(kT_s)$ and $d_c(kT_s)$.

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