Strongly symmetric spectral convex bodies are Jordan algebra state spaces

Howard Barnum\textsuperscript{*1} and Joachim Hilgert\textsuperscript{†2}

\textsuperscript{1}Los Alamos, New Mexico, USA
\textsuperscript{2}Department of Mathematics, University of Paderborn, Germany

April 9, 2019

Abstract

We show that the finite-dimensional convex compact sets having the properties of \textit{spectrality} and \textit{strong symmetry} are precisely the normalized state spaces of finite-dimensional simple Euclidean Jordan algebras and the simplices. Various assumptions are known characterizing complex quantum state spaces among the Jordan state spaces, which combine with this theorem to give simple characterizations of finite-dimensional quantum state space.

Spectrality and strong symmetry arose in the study of “general probabilistic theories” (GPTs), in which convex compact sets are considered as state spaces of abstractly conceivable physical systems, though not necessarily ones corresponding to actual physics. We discuss some implications of our result—which is purely convex geometric in nature—for such theories. A major concern in the study of such theories, and also in the theory of operator algebras, has been the characterization of the state spaces of finite and infinite-dimensional Jordan algebras, and the characterization of standard quantum theory over complex Hilbert spaces, or the state spaces of von Neumann or \(C^*\)-algebras, within the class of Jordan-algebraic state...
spaces. In the finite-dimensional case, our characterization of simple Jordan-algebraic state spaces and classical (i.e. simplicial) state spaces, can serve as an alternative to existing characterizations of Jordan-algebraic systems, for example via the properties of positive projections associated with faces [1,2].

While our result is purely convex-geometric in nature, it has strong implications for work relating geometric properties of the state spaces of systems to physical and information processing characteristics of GPT theories. It shows that some generalizations of important aspects of quantum and classical thermodynamics to theories satisfying natural postulates, e.g. in [3,4], apply to a narrower class of theories than might have been hoped, already relatively close to complex quantum theory since their systems are Jordan algebraic. Sorkin’s notion of irreducibly $k$-th order interference, involving $k$ or more possibilities, generalizing the interference between two possibilities in quantum theory, has been studied in the GPT framework and looked for in experiments. Our result shows that the assumption of no higher-order ($k \geq 3$) interference, used along with spectrality and strong symmetry to characterize the same class of Jordan-algebraic convex sets in [5], was superfluous. It also implies that [6]’s extension, on the assumption that interference is no greater than some fixed maximal degree $k$, of the important $\Omega(\sqrt{N})$ lower bound on the quantum black-box query complexity of searching $N$ possibilities for one having a desired property (which is achieved by Grover’s celebrated quantum algorithm), to a class of GPTs satisfying certain postulates allowing the formulation of a generalized notion of query algorithm actually applies in the Jordan-algebraic setting where higher-order interference ($k \geq 3$) is not possible.

On the other hand, our work suggests that whether one is interested in investigating the physical and information properties of possible probabilistic theories or in the geometric properties of convex compact sets, it is worth focusing on the implications of postulates weaker than the conjunction of strong symmetry and spectrality.

1 Introduction

In [5], the normalized state spaces of simple finite-dimensional Euclidean Jordan algebras, and the simplices, were characterized as the unique finite-dimensional compact convex sets satisfying three properties: spectrality, strong symmetry, and the absence of higher-order interference. In this paper, we show that the same class of convex compact sets is characterized by the first two of these properties:
Theorem 1.1. A finite-dimensional convex compact set is spectral and strongly symmetric if and only if it is a simplex or affinely isomorphic to the space of normalized states of a simple Euclidean Jordan algebra.

Simplices are the state spaces of particular nonsimple Euclidean Jordan algebras, namely products of several copies of the trivial (one-dimensional) Euclidean Jordan algebra, so this theorem implies the claim in our title.

Spectrality (in this convex sense) and strong symmetry are notions originating in an area of research, now often called “general probabilistic theories” (GPTs), that considers convex compact sets as an abstract notion of state space of a system, physical or otherwise, on which one can make measurements, attempt to control the dynamics, etc.; a state encodes the probabilities of the results of all measurements that it is possible to make on a system. Roughly speaking, a system specified by a convex compact state space $\Omega$ is spectral if every state is a convex combination of pure (i.e. extremal, in the convex sense) states that are perfectly distinguishable from each other via some measurement it is possible to perform on the system. A system is strongly symmetric if the symmetry group of its state space acts transitively on the set of lists (of a given length) of perfectly distinguishable states. These two properties are convex abstractions of properties that hold for quantum systems: in the quantum case, the state space is the set of density matrices on a Hilbert space, perfectly distinguishable sets of states are the sets of rank-one projectors (pure density matrices) $vv^\dagger$ corresponding to orthonormal sets of Hilbert space vectors $v$ (modulo phase factors), spectrality is the spectral decomposition of density matrices, and strong symmetry reflects the facts that any orthonormal basis can be taken to any other via a unitary operator and that conjugations by unitaries, $\rho \mapsto U\rho U^\dagger$, are symmetries of the set of density matrices. Although much of quantum physics is best represented in infinite-dimensional Hilbert spaces, quantum information theory has focused on degrees of freedom that can be represented in finite-dimensional Hilbert spaces, or on controlling finite-dimensional subalgebras of observables in larger systems, corresponding to effective finite-dimensional Hilbert spaces, e.g. ones describable as tensor products of two-dimensional complex Hilbert spaces (“qubits”). And most of the characteristically quantum phenomena such as interference, tradeoffs between information gain and disturbance in the measurement process, the existence of incompatible observables, and entanglement, that are associated with the cryptographic and computational advantages of quantum information processing over classical, are already present in finite-dimensional systems. So finite-dimensional results such as those of the present paper can help us understand the conceptual
and physical basis of such quantum phenomena. Indeed most recent research on general probabilistic theories has been done in finite-dimensional settings.

That simple Euclidean Jordan algebras and simplices are spectral and strongly symmetric is known (cf. [5]), so the new result here is the converse. It is obtained by showing that compact convex sets in finite dimension satisfying spectrality and strong symmetry are (up to affine isomorphism) a subset of the regular convex bodies, defined in [7] as bodies whose symmetry group acts transitively on maximal chains of faces, and using the classification of those convex bodies in [8] to verify that this subset consists precisely of the abovementioned classes of sets.

Several recent works have explored the consequences of the conjunction of spectrality and strong symmetry, or sets of principles that imply these, for the state space of a system in the GPT framework. In particular, in [6], the important lower bound of $\Omega(\sqrt{N})$ queries (due to Bennett, Brassard, Bernstein, and Vazirani [9]) on computing OR (that is, $x_1 \lor x_2 \lor \cdots \lor x_N$) via quantum queries to an $N$-bit-string $(x_1, x_2, \cdots, x_N) \in \{0, 1\}^N$, which shows that one cannot improve on the $\sqrt{N}$ queries used in Grover’s quantum algorithm for “searching” over $N$ possibilities for one having a desired property, was extended to systems satisfying a set of five postulates. These postulates include strong symmetry, and imply (by results in [10]) spectrality. In [3], the conjunction of strong symmetry and spectrality was shown to imply thermodynamical results generalizing important facts about quantum thermodynamics, and similar results were obtained in [4]. Our work implies that these recent results, as well as additional results on query complexity obtained in [11], apply to a much smaller set of GPT systems than might have been hoped. We discuss this further in the concluding section of the paper.

The paper is organized as follows. After this outline of the paper’s structure, Section 1.1 briefly describes some terminological and mathematical conventions used in the rest of the paper. Section 2 provides the necessary background on convex compact sets and associated structures, especially their groups of automorphisms, and defines the notion of measurement, which is used in formulating the notion of perfect distinguishability and the properties of spectrality and strong symmetry which depend on it. It also touches on the use of this framework to represent potential physical systems abstractly, which motivates notions like measurement, distinguishability, spectrality, and strong symmetry. Section 3 defines perfect distinguishability, spectrality and strong symmetry, and reviews important consequences (mostly from [3]) of the latter two properties that will be needed in proving the main theorem. Section 4 describes the main class of convex compact sets that are the target of the characterization via spectrality and strong symmetry in the main theorem: the normalized state spaces of simple Euclidean
Jordan algebras. It also reviews the already-known direction of the main theorem for these cases, explaining how known results in Jordan theory imply that these state spaces are spectral and strongly symmetric. Finally, it reviews the structure of these Jordan algebras considered as representations of the automorphism groups of their normalized state spaces, and of the automorphism groups of the cones over these state spaces. This structure will be used in the proof of the main theorem in Section 9 to compare them with the polar representations in which Madden and Robertson [8] embed all regular convex bodies in order to classify them. Section 5 briefly reviews the easily seen fact that simplices, the other part of the class of bodies we characterize, are spectral and strongly symmetric, which completes the proof of the “if” direction of the main theorem. Section 6 establishes, via a direct and simple geometric argument essentially from [12] (but also using strong symmetry which was not explicitly assumed there), the “only if” direction for the special case of strongly symmetric spectral systems in which no more than a pair of states can be perfectly distinguished at once: they are affinely isomorphic to balls. This enables us to avoid some case-checking later. Section 7 introduces the main tools we will use in the new part of our characterization theorem: the notion of regular convex body and associated theory, and Section 8 describes the classification by Madden and Robertson [8] of these bodies as convex hulls of particular orbits in polar representations of compact groups, along with some of the theory of polar representations used in the classification. The classification uses a one-to-one correspondence (which was described in Section 7) which associates each regular convex body $\Omega$ embedded as the convex hull of an orbit in a polar representation with a polytope $\pi(\Omega)$ given as the convex hull of a particular orbit of the “restricted Weyl group” of the representation, acting in a subspace $a$, with $\pi(\Omega) = a \cap \Omega$. Finally, Section 9 completes the proof of the characterization theorem by proving the new part: that all strongly symmetric spectral convex compact sets are normalized state spaces of simple Jordan algebras, or simplices. This is done by first proving that they are regular, and then that the associated polytope $\pi(\Omega)$ is a simplex. We then go through the list of representations in the Madden-Robertson classification of regular convex bodies, and verify that all of those (other than the simplices themselves) whose polytope is a simplex with three or more vertices occur in polar representations of automorphism groups of Euclidean Jordan algebras acting on those algebras. We compare the representation-theoretic description of the orbits leading to normalized Jordan state spaces from Section 4 to the description of the points whose orbits yield the regular convex body in the Madden-Robertson construction, and see that, up to an affine transformation, they are the same. Section 10 discusses the implica-
tions of the result for GPTs, including recent work assuming strong symmetry and spectrality; discusses several simple and physically or informationally meaningful ways of adding additional assumptions to further narrow the class of systems to that of standard complex quantum theory, and also discusses the relation of the present work to other characterizations of Jordan-algebraic state spaces, and Section [I] briefly concludes.

1.1 Some mathematical terminology

A few terminological conventions and definitions are worth noting. We will sometimes abuse notation by referring to a singleton set, \( \{x\} \), by the name of its element \( x \), especially when \( \{x\} \) is the face of a convex set consisting of the single extremal point \( x \). For \( S \) any subset of an inner product space \( V, (\cdot, \cdot) \) we define \( S^\perp := \{x \in V : \forall y \in S \ (x,y) = 0\} \). We also adopt the common practice whereby, when we substitute a set in a place in an expression where an element of the set is expected, the expression is taken to refer to the set of all its referents when elements of the set are substituted in that place. For example, for \( S \) a subset of an inner product space \( V, \{x : (x,S) = 0\} \) has the same meaning as \( \{x : \forall y \in S \ (x,y) = 0\} \), i.e. it refers to \( S^\perp \); for \( G \) a group acting on a set \( S, G.x \) for fixed \( x \in S \) is the orbit through \( x \), and so forth.

We use the notation \( \mathbb{E}^n \) to indicate \( n \)-dimensional Euclidean space, by which we mean a finite-dimensional real vector space equipped with a distinguished positive definite inner product (a concrete example is \( \mathbb{R}^n \) equipped with the dot product). When \( S \) is a subset of a real affine space or linear space, we use the notation \( \text{Conv} \ S \) for the convex hull of \( S \), i.e. the set of all convex combinations of elements of \( S \).

For any group \( G \) acting linearly on a vector space \( V \), and any subspace \( L \) of \( V \), we denote the subgroup that preserves the subspace \( L \) by \( G_L \), and the subgroup that fixes \( L \) pointwise by \( G^L \). (\( N_G(L) \) and \( Z_G(L) \) are also common notation for these.) We write \( G_0 \) for the connected identity component of a Lie group \( G \), and \( G_0^s \) for the semisimple part of \( G_0 \).

We write \( := \) or \( =: \) to indicate that the expression on the side with the colon is defined by the expression on the side with the equals sign, both when defining an expression for the first time and in occasional reminders. Finally, as is common in the literature, we use “positive” to mean “nonnegative”.

2 Background and framework

We study convex compact subsets \( \Omega \) of finite-dimensional real affine spaces \( A \).
From now on the term “convex compact set” refers to such sets. Unless otherwise noted, it refers to full-dimensional sets, i.e. ones whose affine span, Aff \( \Omega \), is \( A \).
We will be concerned with intrinsic properties of such a set, which are invariant under affine transformations. In this setting, unlike the setting of compact convex subsets of Euclidean space (defined as a finite-dimensional real vector space with positive definite inner product), there is no way of distinguishing a notion of rectangle from that of square or parallelogram, for example; a trapezoid, however, represents a distinct affine isomorphism class from the other three.

Some intrinsic properties of convex compact sets are best formulated by embedding Aff \( \Omega \) as an affine hyperplane, not containing the origin, in a real vector space \( V \) of dimension one greater than the dimension, which we’ll call \( n \), of Aff \( \Omega \). The properties we study are independent of the particular embedding. Such an embedding determines the convex, topologically closed, pointed, generating cone \( V^+ := \mathbb{R}^+ \Omega \), its dual cone \( V^+_* \subset V^* \), and a unique \( u \) in the interior of \( V^+_* \), defined by the property that \( u(\Omega) = 1 \) (meaning \( u(\omega) = 1 \) for all \( \omega \in \Omega \)).

**Definition 2.1.** Let a compact convex set \( \Omega \) of dimension \( n \) be embedded in an affine hyperplane in \( V \setminus \{0\} \) where \( V \) is a vector space of dimension \( n + 1 \), as above. We call \( V^+ := \mathbb{R}^+ \Omega \subset V \) the cone over \( \Omega \), the cone generated by \( \Omega \), or Cone \( (\Omega) \). We call the unique \( u \in \text{int} V^+_* \) defined by the property that \( u(\Omega) = 1 \), the order unit associated with \( \Omega \).

We also define the cone generated by \( S \), sometimes writing it as Cone \( (S) \), for general subsets \( S \) of a real vector space \( V \), as the set of nonnegative linear combinations of elements of \( S \); in the special case of \( S = \Omega \) in the setting of Definition 2.1 this agrees with the definition given there. For our purposes, a cone \( C \) is defined to be a subset of a real vector space, closed under addition and

---

1These concepts will be defined below.
2Note that \( V^+ \) by itself does not determine \( \Omega \); different choices of \( u \in \text{int} V^+_* \) give rise to different bases for \( V^+_* \), which need not in general be affinely isomorphic. For example, a cone with square base also has trapezoidal bases, obtained for example by swinging the hyperplane containing the square base around one edge of the square (which we can think of as a “hinge”). \( V^+_* \), on the other hand, is determined (up to affine isomorphisms) by \( \Omega \) in the above construction (independently of the particular embedding of Aff \( \Omega \) into \( V \setminus \{0\} \)).
3The term comes from the theory of order-unit and base-norm Banach spaces; see for example [13, 1].
nonnegative scalar multiplication. It is called generating if it spans the vector space, and pointed if \( C \cap -C = \{0\} \). The cones obtained from Definition 2.1 are pointed, generating, and topologically closed. For reasons that will become clearer later, sometimes \( V^* \) is referred to as the observable space.

An important interpretation of this formalism, which was the motivation and setting of \([5]\), views \( \Omega \) as the space of normalized states of an abstract physical system. This may or may not correspond to the state space of any system in physical theories currently in use. Because of this interpretation, we sometimes use the terms “state” for elements of \( \Omega \), and “pure state” for extremal elements of \( \Omega \). For instance, for finite-dimensional quantum systems corresponding to a Hilbert space of finite dimension \( d \), \( \Omega \), of affine dimension \( d^2 - 1 \), is the set of density matrices (i.e. unit-trace positive semidefinite \( d \times d \) complex Hermitian matrices). The pure states are the rank-one density matrices (which are the Hermitian projectors onto one-dimensional subspaces of the underlying Hilbert space). The cone \( V^+ \) is the positive semidefinite matrices.

An \((n-1)\)-simplex in an affine space is the convex hull of \( n \) affinely independent points; if \( x_1, \ldots, x_n \) are such points, we write \( \Delta(x_1, \ldots, x_n) \) for this simplex. The \((n-1)\)-simplex refers to its dimension, equal to the dimension of the affine space it spans. All \( k \)-simplices are affinely isomorphic; the abstract \( k \)-simplex, i.e. the affine equivalence class of such simplices, or an arbitrary such simplex considered only up to affine equivalence, is often referred to as \( \Delta_k \). In the “operational” or GPT interpretation, the \((n-1)\)-simplex is often thought of as a finite-dimensional classical state space. The associated vector space \( V \) of one higher dimension is usually taken to be \( \mathbb{R}^n \), with the extremal points \( x_i \) embedded as the unit coordinate vectors \( e_i = (0,0,\ldots,0,1,0,\ldots,0) \) with 1 in the \( i \)-th place, so that \( \Delta_{n-1} \) is identified with the standard probability simplex.

The \textit{barycenter} (also known as centroid) \( c(\Omega) \) of a full-dimensional compact convex set \( \Omega \subset A \), where \( \dim A = \dim \Omega = n \), is defined by introducing coordinates so as to identify \( A \) with \( \mathbb{R}^n \), and letting:

\[
c(\Omega) := \int_{\Omega} d\mu(x) x
\]

where \( d\mu \) is Lebesgue measure on \( \mathbb{R}^n \). One shows that \( c(\Omega) \) is independent of the particular coordinatization of \( A \) as \( \mathbb{R}^n \), and (what amounts to the same thing) that \( c(\Omega) \) is covariant under affine transformations, i.e. for any affine transformation \( T \), it holds that \( c(T(\Omega)) = T(c(\Omega)) \).

The automorphism group, \( \text{Aut} \Omega \), of \( \Omega \) is the set of affine transformations of \( \text{Aff} \Omega \) that take \( \Omega \) onto itself. It is a compact group. (This is often called \( \Omega \)'s
symmetry group but (mainly for consistency with [7, 8]) we’ll reserve that term for use in a slightly different context to be introduced later.) We’ll often denote it with the letter $K$. In the GPT literature, its elements are often called reversible transformations. Any affine space can be viewed as a vector space by choosing a point to be 0; it is natural to do this for $\text{Aff} \Omega$ by taking the barycenter of $\Omega$ as zero. (This should not be confused with the extension of $\text{Aff} \Omega$ to the vector space $V$ described above, which is of one greater dimension.) The transformations in $\text{Aut} \Omega$, because they are affine, fix the barycenter, so with respect to this vector space structure on $\text{Aff} \Omega$, they are linear. They also extend (as do any affine transformations on $\text{Aff} \Omega$) linearly to all of $V$. We may also refer to the extension as $\text{Aut} \Omega$, or as $K$ ($K$ and its extension are of course isomorphic as groups). In fact we have:

**Proposition 2.2.** Let $\Omega$ be a compact convex subset of $A \simeq \text{Aff} \Omega$. $A$ may be equipped with the structure of a Euclidean space $E$ in such a way that $\text{Aut} \Omega \subseteq O(E)$. In doing so, the barycenter of $\Omega$ becomes $0 \in E$, and $\text{Aut} \Omega$ is precisely the subgroup of $O(E)$ that preserves $\Omega$.

This follows from the well-known fact that if a compact group $K$ acts linearly on a real vector space $V$, there exists a $K$-invariant inner product on $V$ and the fact that $\text{Aut} \Omega$ is a compact group that fixes $c(\Omega)$ (Proposition 2.4).

**Definition 2.3.** We call the identification of $\text{Aff} \Omega$ with a Euclidean vector space such that $\text{Aut} \Omega \subseteq O(E) \equiv O(\text{Aff} \Omega)$ a canonical embedding of $\Omega$ into Euclidean space.

Recalling the construction of the cone $V_+ \subset V$ over $\Omega$ in Definition 2.1, we note that we can also equip $V$ with an inner product so that $K$’s extension to $V$ is a subgroup of $O(n + 1)$. This subgroup will fix the ray $\mathbb{R}_+ c(\Omega)$, i.e. the ray over the image of $0 := c(\Omega)$ under the embedding of $\text{Aff} \Omega$ into $V$. With this choice of inner product, and identifying $V^*$ with $V$ via the inner product, $u \in V$ is the embedded image of $0 \in E \simeq \text{Aff} \Omega$.

The following fact is very useful:

**Proposition 2.4.** Let $K := \text{Aut} \Omega$ act transitively on the extreme boundary $\partial_e \Omega$ of $\Omega$, and let $\omega \in \partial_e \Omega$. Then $\int_K d\mu(k)k \omega = c(\Omega)$, where $d\mu$ is Haar measure on $K$. $c(\Omega)$ is the unique $K$-invariant point in $\Omega$.

---

*One proves this by verifying that if $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} :: (x, y) \mapsto \langle x, y \rangle$ is an arbitrary inner product on $V$, then $(x, y) \mapsto \int_K dk(kx, ky)$, where $dk$ is normalized Haar measure on $K$, is a $K$-invariant inner product.*
The property in the premise of this proposition, transitive action of Aut Ω on ∂eΩ, is sometimes called *reversible transitivity on pure states*. A convex set with the property is also sometimes called an *orbitope*, following [14]. When the compact group and representation K are clear, we will sometimes refer to Conv K.ω as the *orbitope over ω* or the *orbitope generated by ω*. If the connected identity component of Aut Ω acts transitively on ∂eΩ, we say Ω has *continuous reversible transitivity*.

The next two definitions are essential for defining the central properties we will investigate: spectrality and strong symmetry.

**Definition 2.5.** An *effect* is an element of the interval [0,u] ⊆ V∗, with respect to the (partial) ordering of V∗ induced by V∗+ (namely, x ≥ y := x − y ∈ V∗+). A *measurement* is a finite sequence e𝑖, 𝑖 ∈ {1,...,m} of effects such that ∑m𝑖=1 e𝑖 = u.

An equivalent characterization of effects is as precisely the elements of V∗ that satisfy, for all ω ∈ Ω, e𝑖(ω) ∈ [0,1] ⊆ R. (While the first clause only requires them to be in V∗, the rest of the definition implies they are positive, i.e. in V∗+.) In the operational interpretation effects, since they give probabilities when evaluated on states, are used to mathematically represent the probabilities of outcomes of measurements. The definition of measurement guarantees that for every state ω ∈ Ω, the probabilities e𝑖(ω) for the outcomes 1,...,m of a measurement e1,...,em sum to 1: ∑𝑖=1 e𝑖(ω) = u(ω) = 1.

Since we are in finite dimension, a positive definite inner product (·, ·) : V × V → R induces (as does any nondegenerate bilinear form, positive definite or not) an isomorphism between V∗ and V. It is common to use such an inner product to represent the dual space “internally” in the primal space; then the dual cone becomes V∗internal := {y ∈ V : ∀x ∈ V+, (y,x) ≥ 0}. We say that a cone V+ is self-dual if there exists an inner product with respect to which V∗internal = V+. We will call such an inner product *self-dualizing*.

When, as in much of this paper, we deal with self-dual cones, we will normally fix a self-dualizing inner product, use it to identify V with V∗ and drop the superscript “internal”.

A *face* of a convex set C is a convex subset F such that whenever x = ∑𝑖pi𝑥𝑖, with 𝑝𝑖 > 0, is a finite convex decomposition of x ∈ F in terms of 𝑥𝑖 ∈ C, all the 𝑥𝑖 are in F. An *exposed face* is the intersection of C with a supporting hyperplane;

---

5This is a properly stronger property than affine isomorphism of V∗+ with V+, which is sometimes called “weak self-duality”. The cone with square base, for example, separates the two properties.
it is easily shown to be a face, but the two notions are not equivalent. Where necessary we modify these definitions so that (1) \( C \) itself is considered to be an exposed face as well as a face, and (2) the empty set is considered to be an exposed face, and a face, when \( C \) is compact, but not when \( C \) is a cone. This gives a bijection \( \varphi \) between the faces of the cone \( V_+ \equiv \mathbb{R}_+ \Omega \subset V \) over \( \Omega \), and the faces of a convex compact set \( \Omega \): for each face \( G \) of \( \mathbb{R}_+ \Omega \), \( \varphi(G) := G \cap \text{Aff} \Omega \) is a face of \( \Omega \). \( \varphi \)’s inverse takes each nonempty face \( H \) of \( \Omega \) to the face \( \mathbb{R}_+H \) of \( G \), and \( \emptyset \) to \( \{0\} \). There is an analogous bijection in the case of exposed faces. When ordered by inclusion, the set of faces and the set of exposed faces are each lattices, in which we write the least upper bound (“join”) of a pair \( x, y \) of elements as \( x \lor y \), and their greatest lower bound (“meet”) as \( x \land y \). In any lattice, meet and join are associative. For any subset \( S \) of a convex set \( C \), we define the face generated by \( S \), \( \text{face}(S) \) as the smallest face that contains \( S \). (Of course it must be shown unique for this to be an admissible definition; it is.) So for a pair of elements \( \text{face}(\{x, y\}) = x \lor y \), and by associativity of join, for a finite set of elements, \( \text{face}(\{x_1, \ldots, x_k\}) = x_1 \lor x_2 \lor \cdots \lor x_k \). A similar notion applies to the lattice of exposed faces: \( \text{ExpFace}(S) \), the exposed face generated by \( S \), is the smallest exposed face containing \( S \). When several convex sets may coexist in the same space, we may use the name of a set as a subscript to indicate which convex set we are generating a face in, e.g. \( \text{face}_C(S) \) is the face of \( C \) generated by \( S \).

Both the set of faces and the set of exposed faces of a compact convex set \( \Omega \) or of a (pointed, closed, generating) cone \( V_+ \), are bounded lattices. Their upper bounds are \( \Omega \) (resp. \( V_+ \)), and lower bound \( \emptyset \) (resp. \( \{0\} \)).

A cone \( V_+ \) is said to be perfect if \( V \) can be equipped with an inner product such that every face \( F \) of \( V_+ \), including \( V_+ \) itself, is equal to its dual with respect to the restriction of the inner product to \( \text{lin } F \).

In the literature on general probabilistic theories, systems are sometimes modeled using an explicit specification of a proper subset of the sets of effects and of measurements, or positive maps, subject to reasonable conditions, called the “allowed” or “physical” effects, measurements, or positive maps. When the full set of effects is allowed, the system is said to satisfy the “no-restriction hypothesis”. We are concerned with intrinsic properties of the compact convex set \( \Omega \) which cannot depend on such a choice, so we have no need of this possibility. We mention it

\[\text{face}_C(S)\]

\[\text{ExpFace}(S)\]

\[\text{lin } F\]

---

Footnotes:

6 The face lattice is an affine invariant of compact convex sets, but does not fully separate affine equivalence classes of compact convex sets, as the fact that a cone and its base have isomorphic face lattices, but the cone does not in general determine its base up to affine equivalence, illustrates.

7 The cone with pentagonal base (indeed, the cone over any regular \( n \)-gon with odd number of vertices greater than 3) illustrates that this is properly stronger than self-duality.
because the framework used in [5] does allow for the possibility of a restricted set of allowed effects, and the notions of spectrality and strong symmetry in [5] are defined as we define them here except that the notion of perfect distinguishability used is with respect to the set of allowed effects. In principle this could affect the notions of spectrality, and of strong symmetry, for a fixed state space. However, the conjunction of spectrality and strong symmetry, even with respect to a restricted set of effects, is shown in [5] to imply no-restriction. All of the results we use from [5] concern consequences of the conjunction of these two properties, so they hold equally for strongly symmetric spectral sets in our more specialized setting.

3 Distinguishability, spectrality, and strong symmetry

In this section we continue to use $\Omega$ to refer to an arbitrary finite-dimensional compact convex set.

Definition 3.1. A set of points $\omega_i, i \in \{1, \ldots, k\}$ in $\Omega$ is called perfectly distinguishable if there is a measurement $\{e_i\}, i \in \{1, \ldots, k\}$ such that $e_i(\omega_j) = \delta_{ij}$.

In the operational interpretation, perfect distinguishability means that if we are guaranteed that a physical system has been prepared in one of the states $\omega_1, \ldots, \omega_k$, but we do not know which one, we may ascertain which one was prepared with perfect certainty by doing the measurement $\{e_i\}$. If we get the result $j$, then the state that was prepared must have been $\omega_j$. Perfect distinguishability, and approximations to it, are central to considerations of the information storage and processing properties of abstract systems in this interpretation.

An equivalent characterization of distinguishability will also be useful to us. We call an indexed set of effects $E = \{e_i\}$ a submeasurement if $\sum_i e_i \leq u$. For each submeasurement $E = \{e_1, \ldots, e_r\}$, the indexed set $E' := \{e_1, \ldots, e_r, e_{r+1}\}$, where $e_{r+1} := u - \sum_{i=1}^r e_i$, is a measurement. We could have equally well defined perfect distinguishability of $\omega_1, \ldots, \omega_r$ as the existence of a submeasurement such that $e_i(\omega_j) = \delta_{ij}$, for it is easy to see that $e_{r+1}(\omega_i) = 0$ for all $i \in \{1, \ldots, r\}$, whence there are many ways of constructing an $r$-outcome measurement $\{e'_1, \ldots, e'_r\}$ such that $e_i(\omega_j) = \delta_{ij}$—for instance, let $e'_1 = e_1 + e_{r+1}$, and $e'_i = e_i$ for all $i \in \{2, \ldots, r\}$.

Definition 3.2. [5] A sequence $\omega_1, \ldots, \omega_k$ of perfectly distinguishable pure states is called a frame, or a $k$-frame if we wish to specify its cardinality.
In GPT models in which restricted sets of allowed effects are specified, as done in [5], distinguishability and frames are usually defined with respect to the *allowed* effects, so fewer sets of states may be distinguishable, and there may be fewer frames, than in the no-restriction case. Although we referenced [5] for the above definitions, we have defined distinguishability, and hence all concepts dependent on it, with respect to the set of *all* effects. So for us, these notions depend only the convex geometry of \( \Omega \).

**Definition 3.3.** A frame in \( \Omega \) is called *maximal* if it is not a subsequence of any other frame.

Although the cardinality of a frame can be no greater than \( n + 1 \) (recall that \( n := \dim(\text{Aff} \ \Omega) \)), for a given \( \Omega \) the maximal cardinality of a frame may be much less than \( n \).

We are now ready to introduce the two properties of convex sets that we will use in our characterization theorem.

**Definition 3.4 ([5]).** A convex compact set \( \Omega \) is called *spectral* if, for each point \( \omega \in \Omega \), there is some frame whose convex hull contains \( \omega \).

This is a very strong property of convex compact sets. Jordan algebraic state spaces satisfy a stronger property, called *unique spectrality*, which requires that for any two convex decompositions \( \omega = \sum_{i=1}^{k} p_i \omega_i = \sum_{i=1}^{r} q_i \tau_i \) of \( \omega \) into the elements \( \omega_i, \tau_i \) of two frames, \( k = r \) and there exists a permutation \( \sigma \) of \( \{1, \ldots, k\} \) such that \( q_{\sigma(i)} = p_i \). Unique spectrality was not assumed in [5] and will not be assumed in our characterization theorem either, but it follows [5] from the conjunction of spectrality and the next property, strong symmetry.

**Definition 3.5.** A convex compact set \( \Omega \) of dimension \( n \) is called *strongly symmetric* if, for each \( k \in \{1, \ldots, n+1\} \), \( \text{Aut} \ \Omega \) acts transitively on the set of \( k \)-frames.

---

8 An example of the latter phenomenon is the mixed-state space (the density matrices) of a \( d \)-dimensional quantum system, where as mentioned before, \( \dim(\text{Aff} \ \Omega) = d^2 - 1 \). The frames are just lists of mutually orthogonal rank-one projectors, and the cardinality (length) of a largest such list is \( d \), which is far from achieving the general upper bound of \( n + 1 \) which is here \( d^2 \). Modulo reorderings a simplex has a unique maximal frame, the set of vertices of the simplex; the frames of a simplex are just the ordered subsets of the vertices, i.e. finite sequences, without repetition, of vertices. In this case, the general upper bound is achieved.

9 There are other notions of spectrality for convex compact sets in the literature, for instance Alfsen and Shultz’s ([1], Definition 8.74). Although related, Alfsen and Shultz’s notion should not be confused with the one used here. However the conjunction of our weak notion of spectrality with strong symmetry also implies [5] that \( \Omega \) is spectral in Alfsen and Shultz’s sense. Riedel [15] also introduced a notion of spectral ordered linear space, but we will not use it here.
Note that the set of $k$-frames may well be empty for many values of $k$ (in which case, trivially, $\text{Aut} \, \Omega$ acts transitively on it).

“Strong symmetry” was the term introduced in [5]. “Frame-symmetric” or “frame-transitive” might have been a better choice. The notion of frame that is involved in the definition of strong symmetry is not itself obviously symmetry-related. And for sets having few frames, it is actually not a very strong symmetry requirement.

Since a $k$-frame is a finite sequence of perfectly distinguishable pure states, strong symmetry implies that one can arbitrarily permute the elements of any set of perfectly distinguishable pure states, via symmetries. It is therefore at least prima facie a stronger property than requiring that every set of perfectly distinguishable pure states (which we might call an unordered frame) can be mapped onto any other such set of the same size by a symmetry\(^{10}\).

We collect some more consequences of the conjunction of spectrality and strong symmetry in the following proposition.

**Proposition 3.6** (Mostly from [5]). For a convex compact set $\Omega$ that is spectral and strongly symmetric, the following hold:

1. Every face of $\Omega$ is generated by a frame. Any two frames that generate the same face $F$ have the same cardinality, which we call the rank, $|F|$, of the face. If the face $G$ is a proper subset of $F$, then $|G| < |F|$.

2. Every face of $\Omega$ is exposed.

3. The cone $V_+ := \mathbb{R}_+ \Omega$ over $\Omega$ is a perfect self-dual cone. The self-dualizing inner product $(\cdot, \cdot)$ can be chosen to be $\text{Aut} \, \Omega$-invariant, and such that $(\omega, \omega) = 1$ for all pure states (extremal points) $\omega$ of $\Omega$.

4. With respect to the self-dualizing inner product on $V$, the elements of any frame are an orthonormal set. The states of a frame, viewed as elements of the dual space via this inner product, are effects, and are therefore a distinguishing submeasurement for that frame. If $\omega_1, \ldots, \omega_n$ is a maximal frame, i.e. a frame for $\Omega$, then $\sum_{i=1}^n \omega_i$ is the order unit.

\(^{10}\)The pentagon (like any regular $n$-gon with $n > 3$) provides an example of a non-spectral but strongly symmetric convex compact set. Spectral but not strongly symmetric convex compact sets also abound: for example, any ball is easily smoothly deformed to another smooth, strictly convex set with trivial automorphism group; all smooth, strictly convex sets are spectral.
5. If $F = \omega_1 \lor \cdots \lor \omega_k$ for some frame $\omega_1, \ldots, \omega_k$ (equivalently, $F$ is the face generated by that frame) then the barycenter of $F$ is $\sum_{i=1}^{k} \omega_i / k$. (It follows that $\sum_{i=1}^{k} \omega_i$ does not depend on which frame $\omega_1, \ldots, \omega_k$ for $F$ is summed over.)

6. If $F$ is a face of $V_+$, (resp. $\Omega$) then $F' := F^\perp \cap V_+$ (resp. $F^\perp \cap \Omega$) is a face of $V_+$ (resp. $\Omega$) such that $F \land F' = \{0\}$ (resp. $F \land F' = \emptyset$) and $F \lor F' = V_+$ (resp. $F \lor F' = \Omega$). (Here all $\perp$'s and linear spans are taken in $V$, with respect to the self-dualizing, invariant inner product.) In a lattice, these two conditions define what it means for an element $F'$ to be a complement of $F$, so we call $F'$ the face complementary to $F$, or simply $F$’s complement.

7. The face generated by a maximal frame is $\Omega$ itself. Every frame $A$ of $\Omega$, generating a face $F$, extends to a maximal frame $M$, by appending a frame $B$ for $F'$. Similarly if $F < G$ (i.e. $F \subsetneq G$), every frame for $F$ extends to a frame for $G$.

8. The map $F \mapsto F'$ on the face lattice of $V_+$ (equivalently of $\Omega$) is an orthocomplementation, with respect to which the lattice is orthomodular, i.e. for $F \leq G$, $G = F \lor (F' \land G)$. The additional states appended to a frame on $F$ in order to extend it to a frame on $G$ (cf. item [7] above), are a frame for $F' \land G$.

We may think of orthomodularity as stating that $F' \land G$ behaves as a “relative orthocomplement” of $F$ in $G$, i.e. an orthocomplement in the sublattice consisting of elements below or equal to $G$.

Proof. Item [1] is Proposition 2 of [5]. Item [2] follows easily from item [1].

Item [5] is established in [5] in the course of proving Theorem 8 of that paper, which states that for every face $F$, the orthogonal projection (with respect to the self-dualizing Aut $\Omega$-invariant inner product) $P_F$ onto the linear span of a face $F$, is a positive map. Iochum [16] showed that for self-dual cones, positivity of all such projections with respect to a self-dualizing inner product is equivalent to perfection. The self-duality of the cones $\mathbb{R}_+ \Omega \subset V$ over strongly symmetric spectral convex sets, with respect to an inner product with the stated properties, was established as Proposition 3 of [5][11].

[11] In fact self-duality does not require spectrality, nor does it require the full strength of strong symmetry: in [17] it was shown that transitivity of Aut $\Omega$ on 2-frames (ordered pairs of perfectly distinguishable states) implies self-duality.
Item 4 is Proposition 6 of [5].
Item 6 is partly stated in Proposition 7 of [5], and the rest can be extracted from the proof of that Proposition.
Item 7 is part of Proposition 7 of [5].
Item 8 begins with the claim that the map \( \cdot \) is an orthocomplementation. A complementation is an involutive map of a bounded lattice such that \( F \vee F' = 1 \) and \( F \wedge F' = 0 \). It is called an orthocomplementation if it is order-reversing: \( F \leq G \iff G' \leq F' \). The involutiveness of \( \cdot \) is also part of Proposition 7 of [5]. The other two conditions on a complementation are part of item 6 above. The last part of orthocomplementation, order-reversingness, is shown as part of the proof of Theorem 9 in [5]; it follows directly from the extendibility of frames to maximal frames.

The second part of item 8, i.e. orthomodularity, is Theorem 9 of [5]. The crucial element in its proof (given that we have already established that \( \cdot \) is an orthocomplement) is the “relative frame extension property”, i.e. the last sentence in item 7. The last sentence of item 8 is a step in this proof from [5].

The only item we have not covered yet is item 5. It does not appear to be explicitly stated in [5], although it is likely known. We include a proof in Appendix B as part of a proof that the faces of strongly symmetric spectral sets are strongly symmetric and spectral.

4 A class of examples: strongly symmetric spectral convex sets from simple Euclidean Jordan algebras

Finite-dimensional Euclidean Jordan algebras were introduced by Pascual Jordan around 1932 [18], as a possible algebraic setting for the formalism of quantum theory. The notion abstracts properties of the complex Hermitian matrices, which are the observables\(^{12}\) of a finite-dimensional quantum system; in particular, properties of the symmetrized product \( A \bullet B := (AB + BA)/2 \) which, unlike the ordinary associative matrix product, preserves Hermiticity. Our main result asserts that we can identify, up to affine isomorphisms, the spaces of normalized states of a certain class of these algebras with the strongly symmetric spectral convex compact sets. So in this section, we will introduce these algebras, define their

\(^{12}\)As noted in Section 2 in the setting of general probabilistic theories, a reasonable generalization of the space of observables is the vector space \( V^\ast \).
normalized state spaces, and give their classification so that we can identify them when they appear in the classification of regular convex bodies (as defined in §7.6). We also give the proof, mainly consisting of references to known results, that they are strongly symmetric and spectral, since this is one direction (the already-known one) of our main theorem. Finally, we explain that the action of the automorphism group of the normalized state space on the subspace where it acts nontrivially, is via a polar representation, indeed a symmetric space representation, and identify a point in the representation, the convex hull of whose orbit is the normalized state space. This will be used in Section 9 to connect these state spaces with the Madden-Robertson classification of regular convex bodies, since the latter proceeds by showing that their symmetry groups’ actions can be induced by polar representations.

Our main reference for facts about Euclidean Jordan algebras and the associated cones will be Chapters I-V of the book of Faraut and Koranyi [19], which deals with symmetric cones in finite dimension. We also refer to Chapter I of [20].

A Jordan algebra is a real algebra (i.e., a real vector space $V$ equipped with a bilinear product $\cdot : V \times V \to V$) that is commutative and that, while not in general associative, satisfies a special case of associativity, the Jordan property: $a^2 \cdot (a \cdot b) = a \cdot (a^2 \cdot b)$, where we use the notation $a^2 := a \cdot a$. It need not have a unit, but one can always be adjoined if it is absent. We will consider only unital finite-dimensional Jordan algebras. A Jordan algebra is called formally real if $a^2 + b^2 = 0$ implies that $a = b = 0$. In finite dimension, formal reality coincides with another property, Euclideanity. A Jordan algebra $(V, \cdot)$ is said to be Euclidean if it is possible to introduce an inner product $(\cdot, \cdot) : V \times V \to \mathbb{R}$ that is “associative”, i.e. $(a \cdot b, c) = (a, b \cdot c)$.

The set of squares in a Jordan algebra is obviously closed under nonnegative scalar multiplication. In a formally real (equivalently Euclidean), Jordan algebra, it is also closed under addition, so it is a convex cone (cf. e.g. [19], Ch III §2),

---

[19] is the most elementary and the most focused on our concerns, but because it is very detailed, relevant material is occasionally somewhat scattered and one sometimes needs to chase chains of definitions and theorems. Satake’s Chapter I [20] is succinct and extremely well organized, but uses notions somewhat more general than we need (e.g. Jordan triple systems), and also uses some machinery from algebraic geometry and algebraic groups. Another good reference, but less focused than the other two on the concerns of this paper, and somewhat more involved because it also covers infinite-dimensional cases, is the first part of [1]. The spectral theorem is Theorem 2.20 on p. 46. Neither this nor Satake’s chapter is as explicit as [19] about frames and frame-transitivity.
which we call $V_+$. It is immediate from formal reality that $V_+$ is pointed. Since it is in addition topologically closed, it has a compact convex base, giving an example of the formalism of compact convex sets embedded as bases of regular cones, described in Section 2, and permitting an “operational” interpretation as the state space of a physical system. $V_+$ is self-dual with respect to the associative inner product.

Soon after Jordan introduced them, Jordan, von Neumann and Wigner [21] classified the finite-dimensional formally real Jordan algebras. They are precisely the $n \times n$ self-adjoint matrices with entries in $\mathbb{R}, \mathbb{C},$ or $\mathbb{H}$ and the $3 \times 3$ octonionic self-adjoint matrices, equipped in each case with symmetrized matrix multiplication $x \cdot y = (xy + yx)/2$ as Jordan product, and the spin factors $\mathbb{R}^n \oplus \mathbb{R}$ for every $n \geq 1$, equipped with the product

\[(x, s) \cdot (y, t) = (tx + sy, \langle x, y \rangle + st). \tag{2}\]

Here $x, y \in \mathbb{R}^n, s, t \in \mathbb{R}$. Self-adjoint (“Hermitian” is also used) means $M = M^\dagger$, where $M^\dagger := \overline{M}$, and $\overline{M}$’s entries are the conjugates of $M$’s with respect to the canonical conjugation on $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. The conjugation is the identity in the case of $\mathbb{R}$, thus the self-adjoint real matrices are just the real symmetric matrices.

Although we do not make direct use of them in obtaining our results, important facts about the positive cones of Euclidean Jordan algebras are (1) the Koecher-Vinberg theorem [22, 23] that the cones of squares in finite-dimensional formally real Jordan algebras are precisely the homogeneous self-dual cones (homogeneity being defined as transitive action of the automorphism group of the cone on the interior), and (2) the result that they are precisely the symmetric cones, i.e. the regular cones whose interiors are Riemannian symmetric spaces ([25]; see also [23]).

4.1 Normalized Jordan algebra state spaces: spectrality and strong symmetry

Now we define the normalized state spaces of Euclidean Jordan algebras (which we will henceforth sometimes abbreviate as “EJAs”), and explain how to show the known fact that they are spectral and strongly symmetric.

Recall that an idempotent in an algebra is an element $c$ such that $c^2 = c$, and it is called primitive if it is not a nontrivial sum of other idempotents.

\[^{14}\text{Proofs may also be found in [20], Ch. I Theorem 8.5, [19], Theorem III.3.1, and [24].}\]
Definition 4.1. A Jordan frame is defined to be a complete set of orthogonal primitive idempotents $c_i$ in a Euclidean Jordan algebra, where orthogonality means that $c_i \cdot c_j = \delta_{ij}c_i$ and completeness means $\sum_i c_i = e$, the unit of the algebra.

Theorem 4.2 (Spectral theorem for finite-dimensional Euclidean Jordan algebras). Every element $x$ of a Euclidean Jordan algebra has a decomposition

$$x = \sum_{i=1}^{r} \lambda_i c_i$$

where $\lambda_i \in \mathbb{R}$ and $c_i$ are a Jordan frame. When we rewrite this as

$$x = \sum_{\alpha} \lambda_{\alpha} c_{\alpha},$$

where $c_{\alpha} := (\sum_{i \in \alpha} c_i)$ and the sets $\alpha \subseteq \{1, \ldots, r\}$ are a partition of the indices into the largest subsets within which $\lambda_i =: \lambda_{\alpha}$ is constant, then the decomposition (4), into not-necessarily-primitive idempotents, is unique.

This is a combination of Theorems III.1.1 and III.1.3 in [19].

The values $\lambda_{\alpha}$ are the spectrum of $x$. We can define the trace of $x \in V$ as $\sum_i \lambda_i$, and the determinant as $\Pi_i \lambda_i$, where $\lambda_i$ are the coefficients in the spectral decomposition (3). Using the trace we define a bilinear form $(x, y) := \text{tr}(x \cdot y)$ on the Jordan algebra. Proposition III.1.5 of [19] states that the positive definiteness of this form (making it an inner product) is equivalent to Euclideanity. Indeed, in a simple Euclidean Jordan algebra every associative inner product is a positive scalar multiple of this one.

Definition 4.3. In a Euclidean Jordan algebra $V$, the squares satisfying $\text{tr} x = 1$ form a compact convex base $\Omega$ for the cone $V_+$ of squares. We call this the normalized state space of $V$.

Proposition 4.4 (e.g. [19], Corollary IV.3.2). The primitive idempotents are precisely the extremal points of $\Omega$.

In a Euclidean Jordan algebra $V$ the spectrum of a square is nonnegative. One usually represents the dual space internally using the trace inner product. Since $\text{tr} x = \text{tr}(e \cdot x) \equiv (e, x)$, we then have that $e$ is the order unit for this choice of base.

Proposition 4.5. The normalized state spaces of finite-dimensional Euclidean Jordan algebras are spectral.
**Proof.** The primitive idempotents of an EJA $V$ are precisely the extremal points of its normalized state space $\Omega$. Since the cone is self-dual, and all idempotents are below or equal to the order unit $e^{15}$ they are also effects. Orthogonality of primitive idempotents in the sense $e_i \bullet e_j = 0$ of Definition 4.1 implies orthogonality with respect to the inner product, $\text{tr} e_i \bullet e_j = 0$, so any subset of a Jordan frame, considered as a set of effects, is a submeasurement that perfectly distinguishes the same subset, considered as states. Consequently, an ordered subset of a Jordan frame (and in particular, an ordered Jordan frame itself) is a frame in the sense of Definition 3.2. So the spectral theorem implies spectrality.

In order to complete the proof of strong symmetry, we also show:

**Proposition 4.6.** All the frames in an EJA state space are ordered subsets of Jordan frames.

This is known, and implicitly assumed in $\text{5}$, but we give a proof.

**Proof.** Since $\partial e \Omega$ is the set of primitive idempotents, and $V_+$ is self-dual with respect to the inner product $\langle a, b \rangle = \text{tr} a \bullet b$, a frame is a sequence $c_i, i \in \{1, \ldots, s\}$ of primitive idempotents such that there exists a submeasurement $e_i$ for which $\langle e_i, c_j \rangle = \delta_{ij}$. In a general setting, not only in Jordan state spaces, it follows immediately from the condition $\langle e_i, \omega_j \rangle = \delta_{ij}$ on a frame that if $e_i = \sum_k p_k f_k$ is a convex decomposition of $e_i$ into effects $f_k$, each of the $f_k$ also has the property $f_k(\omega_j) = \delta_{kj}$. So the condition that $\omega_i$ is a frame may be restated as the existence of a submeasurement consisting of extremal (in the convex body $[0, u]$) effects.

Proposition 1.40 of $\text{1}$ states that the extreme points of the positive part $[0, u]$ of the unit ball of a JB-algebra are the idempotents. The finite-dimensional JB-algebras are the EJAs. It is also known (cf. $\text{1}$, Proposition 2.18) that for idempotents $p_i, \sum_{i=1}^k p_i \leq u$ implies that $p_i \perp p_j$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$. So in the condition for $c_i$ to be a frame, we may take the $e_i$ to be a mutually orthogonal set of idempotents. We show that $\langle e_i, c_i \rangle = 1$ for an idempotent $e_i$ and a primitive idempotent $c_i$ implies that $c_i \leq e_i$. To do so we use the fact, from $\text{1}$, that JB-algebras $V$ are equipped with normalized self-adjoint idempotent positive linear maps $P_p : V \rightarrow V$ called compressions, in bijection with the idempotents $p$, such that $p = P_{p}e$ and the exposed faces (all faces in finite dimension) are the positive parts of the images of compressions. We have $1 = \langle e_i, c_i \rangle = \langle P_{c_i}e, c_i \rangle = \langle e, P_{e} c_i \rangle$. Since it follows from Proposition 1.41 (in finite dimensions, where the dual space

---

$^{15}$An easy argument from the spectral theorem gives that every primitive idempotent is part of a Jordan frame, and it is part of the definition of Jordan frame that it sums to the order unit.
may be identified with the primal space) of \([1]\) that compressions on an EJA are neutral, i.e. \(||P\omega|| = ||\omega|| \implies P\omega = \omega\), we have that \(P_{ei}c_i = c_i\), hence \(c_i \in \text{im}_+P_{ei}\), where the latter is defined as \(\text{im}P_{ei} \cap V_+\). By Lemma 1.39 of \([1]\), \(\text{im}_+P_{ei} \cap [0,e] = [0, e_i]\), and since \(c_i \in [0,e]\), we have \(c_i \leq e_i\).

With \(c_i \leq e_i\), and \(e_i \perp e_j\) for all \(i \neq j\), it follows that \(c_i \perp c_j\) for all \(i \neq j\), and consequently that the \(c_i\) are a subsequence of an ordered Jordan frame. \(\square\)

**Proposition 4.7.** The normalized state space of a Euclidean Jordan algebra is strongly symmetric.

**Proof.** Corollary IV.2.7 of Theorem IV.2.5 in \([19]\) states that the compact group \(K\), defined as the subgroup of \(\text{Aut}_0(V_+)\) that fixes the Jordan unit \(e\), acts transitively on the set of Jordan frames. Since we’ve adopted a canonical inner product, this is a subgroup of \(\text{Aut}\Omega\). It is clear from the proof of Theorem IV.2.5 in \([19]\) that this transitive action is on ordered Jordan frames. Since we showed, in the proof of Proposition 4.5 and in Proposition 4.6, that the frames (in the sense of Definition 3.2) are precisely the ordered subsets of Jordan frames, the group \(K\), and hence \(\text{Aut}\Omega\), acts transitively on the set of \(k\)-frames for each \(k\). \(\square\)

In particular, we have the following:

**Corollary 4.8.** The normalized state space of a Euclidean Jordan algebra is an orbitope, i.e. \(\Omega = \text{Conv} K.\omega_0\), for any \(\omega_0 \in \partial\Omega\). In particular, for \(\text{Herm}(n, \mathbb{D})\), \(\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}\), \(\Omega = \text{Conv} K.e_{11}\), where \(e_{11}\) is the matrix unit \(\text{diag}(1, 0, 0, ..., 0)\).

This is so because extremal states are 1-frames, and the last sentence follows from Proposition 4.4 and the fact that \(e_{11}\) is a primitive idempotent.

We note here that the state space of a spin factor, \(V = \mathbb{R}^n \oplus \mathbb{R}\), is the unit \(n\)-ball \(\{(x,t) : t = 1, |x|^2 \leq 1\}\) in the affine subspace \(t = 1\).

### 4.2 Classification of Euclidean Jordan algebras, their cones, state spaces, and automorphism groups

In this section we give a more detailed description of the Euclidean Jordan algebras, their cones of squares, their normalized state spaces, and the automorphism groups of these objects. We also give a representation-theoretic description of normalized Jordan algebra state spaces that will allow us to identify them with the convex hulls of certain orbits in polar representations.

The automorphism group of the cone of squares of a Jordan algebra \(V\), like the automorphism group of any self-dual cone, is reductive \([26]\). If the algebra
is simple, then $\text{Aut } V_+ = G^s \times \mathbb{R}_+$, where $G^s$ is simple. Also, $\text{Aut}_0(V_+) = G^s_0 \times \mathbb{R}_+$\[^{16}\] We will also write $\text{Aut}^s(V_+)$ and $\text{Aut}^s_0(V_+)$ for the groups $G^s$ and $G^s_0$. $V$ is an irreducible representation space for $G^s$ (cf. e.g. \cite{[20]}, p. 42), and for its connected identity component $G^s_0$. Like any semisimple Lie group, $G^s$ has Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of its Lie algebra and, correspondingly, $G^s = K \exp \mathfrak{p}$ of the group. We may choose one such that $K$ is the subgroup of $G^s_0$ that fixes the identity $e$, since this is a maximal compact subgroup of the linear group $G^s$. $V$ is an irreducible spherical representation of $G^s$, and of its connected identity component $G^s_0$, which means that $G^s$'s (and also $G^s_0$'s) maximal compact subgroup, $K$, has a one-dimensional fixed-point space (in this case, $\mathbb{R}e$).

In Table 1 (essentially from \cite{[19]}) we list the finite-dimensional simple Euclidean Jordan algebras and associated data. $V$ is the algebra, with positive cone $V_+$, $\mathfrak{g}$ is the Lie algebra of $\text{Aut } V_+$, and $\mathfrak{t}$ the Lie algebra of $\text{Aut } \Omega$ (where $\Omega$ is the normalized state space). We use the notation $\text{Herm}(m, \mathbb{D})$ for the Jordan algebra of self-adjoint matrices over a classical division algebra $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $\text{PSD}(m, \mathbb{X})$ for the positive semidefinite matrices, and $\text{Lorentz}(1, n-1)$ for the Lorentz cone $\{(z, x) \in \mathbb{R} \oplus \mathbb{R}^{n-1} : |z|^2 \geq |x|^2, z \geq 0\}$. Also we write $\text{Sym}(m, \mathbb{D})$ for the space of symmetric matrices with entries in $\mathbb{D}$, and note in particular that $\text{Herm}(m, \mathbb{R}) \equiv \text{Sym}(m, \mathbb{R})$.

In the three infinite families of matrix cases, $\text{Aut}_0V_+$ is the group of transfor-

\[^{16}\]In the non-simple case, $\text{Aut}_0V_+ \simeq G^s_0 \times \mathbb{R}_+$, where $k$ is the number of simple factors of $V$, each copy of $\mathbb{R}_+$ acts as dilations on simple factors, and $G^s_0$ is a product of the groups $\text{Aut}^s_0 V_+$ acting on simple factors $V_i$ (in other words, $\text{Aut}_0V_+ = \Pi_i \text{Aut}_0 V^i$); the full (not necessarily connected) automorphism group allows, besides non-identity components in each factor, permutations of isomorphic factors.
This is dubbed SU. The identity-preserving subgroup, in the cases of the previous paragraph. In the octonionic case it is the compact real form of \(e\) that fixes the Jordan unit \(j\). One can show (cf. \([27]\)) that for arbitrary \(m\) \(\in \mathbb{N}\), there is a Jordan-algebraic definition of determinant, cf. \([19]\). Then one can deal with this by observing that the transformations \(X \mapsto AXA^\dagger\) for \(A \in SL(m, \mathbb{D})\), \(D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}\) are determinant-preserving, and symmetric octonionic matrices have well-defined determinant (indeed, there is a Jordan-algebraic definition of determinant, cf. \([19]\)). Then one can show (cf. \([27]\)) that for arbitrary \(m\) and \(D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}\), and for \(m = 3\) and \(D = \mathbb{O}\), the determinant-preserving linear transformations of \(V\) are \(Aut_0(V_+)\). The identity-preserving subgroup, in the cases \(D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}\), is as stated in the previous paragraph. In the octonionic case it is the compact real form of \(F_4\). (In \([27]\) this is dubbed \(SU(3, \mathbb{O})\).)

Returning to consideration of general EJAs, the Lie algebra of the subgroup that fixes the Jordan unit \(e\) is the compact part \(\mathfrak{k}\) in a Cartan decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) of the reductive group \(Aut V_+\), and \(V\) itself can be identified with \(\mathfrak{p}\) in this decomposition.\(^{19}\) As with any Cartan decomposition, there is a natural \(K\)-invariant\(^{18}\) identity component, \(Aut_0(\Omega)\), of the maximal compact subgroup consists of those transformations for which \(A\) is drawn from the subgroups \(SO(m), SU(m, \mathbb{C}), SU(m, \mathbb{H})\) respectively. This preserves the identity matrix in \(V \simeq \text{Herm}(m, \mathbb{D})\), which is the unit \(e\) of the Jordan algebra. The space orthogonal to this consists of the traceless self-adjoint matrices over \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) respectively, and it is an irreducible representation space for \(K\).

\(^{17}\)Aut_0(V_+) = Aut V_+ except in the following cases where Aut_0(V_-) is of index two, with a representative of the nonidentity component as stated: \(Herm(n, \mathbb{R})\) for \(n\) even, \(X \mapsto MXM^\dagger\) with \(M = \text{diag}(-1, 1, \ldots, 1)\); \(Herm(n, \mathbb{C})\), transpose; \(\mathbb{R} \oplus \mathbb{R}^{n-1}\) for \(n \geq 3\), \(x \mapsto Mx\) with \(M = \text{diag}(1, -1, 1, 1, \ldots, 1)\). The groups generated by \(X \mapsto MXM^\dagger\) with \(M \in SL(n, \mathbb{C}), SL(n, \mathbb{H})\) are not precisely \(SL(n, \mathbb{C}), SL(n, \mathbb{H})\) in general, although they have the same Lie algebras as those groups; they are homomorphic images.

\(^{18}\)Even though in the case of \(D = \mathbb{O}\) we can identify a subset of \(3 \times 3\) octonionic matrices \(M\) such that \(X \mapsto MXM^\dagger\) is determinant-preserving on octonionic-hermitian matrices \(X\), and this generates the group of determinant-preserving linear transformations, this group is not identical to the maps \(X \mapsto MXM^\dagger\) because of the nonassociativity of octonionic matrix multiplication (see \([27]\), where the group is dubbed \(SL(3, \mathbb{O})\)).

\(^{19}\)For example, this is part of Theorem 8.5 of \([20]\), with the Cartan decomposition given in
inner product on $V$, such that $\mathfrak{g}$ is represented by real antisymmetric matrices and $\mathfrak{p}$ by symmetric ones. Furthermore for a simple Jordan algebra $\mathfrak{p}$ decomposes as $\mathbb{R} \oplus \mathfrak{p}_0$, where $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}^s$, $\mathfrak{g}^s$ is the semisimple part of the Lie algebra of $\text{Aut} \, V_+ = \mathbb{R}_+ \times \mathfrak{g}^s$ (so $\mathfrak{g}^s = \text{Lie}\mathbb{G}_s$) and $\mathbb{R}$ is generated by the Jordan unit $e$. Although the actions of $\text{Aut} \, V_+$, and of $\mathfrak{g}^s$, on $V$ are not their adjoint actions (on, i.e. corestricted to, $\mathfrak{p}$), the restriction of these actions to $K$ does coincide (on $\mathfrak{p}$) with the restriction of the adjoint action. In other words, $\mathfrak{p}_0$ is the representation space of a polar representation (Definition 8.5), called a symmetric space representation (Definition 8.9), of the compact group $K$. The Jordan trace gives the component in the $\mathbb{R}$ factor of the Jordan algebra $V \simeq \mathfrak{p} = \mathbb{R} \oplus \mathfrak{p}_0$. Recall that the normalized state space, $\Omega := V_+ \cap \{x \in V : \text{tr} \, x = 1\}$, is also $\text{Conv} \, K \omega$ for any extremal $\omega \in \Omega$ (from strong symmetry). Everything in the affine plane $\{x \in V : \text{tr} \, x = 1\}$ has the form $c \oplus \omega_0$, where $c = e / \text{rank} \, V$ is the unit-trace element of the fixed-point space $\mathbb{R}e$, and $\omega_0 \in \mathfrak{p}_0$. Thus $\Omega$ is affinely isomorphic to $\text{Conv} \, K \omega_0$, an orbitope in the polar representation of $\mathfrak{g}$ on $\mathfrak{p}_0$. This is what will permit us, in Section 9, to identify these Jordan algebra state spaces with certain regular convex bodies, since regular bodies are described in the Madden-Robertson classification [8] as convex hulls of orbits in polar representations.

5 Simplices are strongly symmetric and spectral

In the previous section, we saw that the state spaces of simple Euclidean Jordan algebras are strongly symmetric and spectral, part of the “if” direction of our main theorem. In this section, we establish the other part of the “if” direction: the easy fact that simplices are strongly symmetric and spectral.

The standard presentation of an $n$-simplex as embedded in a vector space $V$ of one higher dimension takes the $n + 1$ vertices of the simplex as the unit vectors $e_i$ of $V = \mathbb{R}^{n+1}$ considered as a Euclidean space in the usual way, i.e. with “dot product” as inner product, and represents effects in the same space, with evaluation given by this inner product. $V_+$ is then the nonnegative orthant, $\mathbb{R}_+^n$, and it is manifestly self-dual with respect to this inner product. So the order unit $u$ is the

---

Lemma 8.6 of that book and the proof. It also follows from the conjunction of Theorem III.2.1 and Theorem III.3.1 in [19] (this conjunction is, roughly, Satake’s Theorem 8.5).

---

In the matrix cases with $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, this is manifested in the fact that while in general $A^\dagger \neq A^{-1}$, equality does hold when $A$ is respectively orthogonal, unitary, or quaternionic-unitary; this exemplifies the general fact that for compact groups, representations are isomorphic to their duals, which establishes the claim for the spin-factor and octonionic cases too.
all-ones vector \((1, 1, \ldots, 1)\), the dual cone \(V^*_+\) is equal to the primal cone, and the set of effects is the unit hypercube (the convex hull of the \(2^{n+1}\) vectors of length \(n + 1\) with entries in \(\{0, 1\}\)). The unit vectors are therefore not only the pure states, but are also effects, and they sum to \(u\), so they constitute a measurement. Every ordered subset of the unit vectors is a submeasurement, and perfectly distinguishes the same subset considered as states, so is a frame; since by definition frames are ordered subsets of the pure states, and in a simplex, the pure states are the unit vectors, these are all the frames. The full set of unit vectors is the unique maximal frame, up to order. Since the simplex is defined as its convex hull, the simplex is spectral. The group \(\text{Aut} \Omega\) is the symmetric group \(S_n\) acting to permute the vertices, so it is obviously transitive on \(k\)-frames for each \(k \in \{1, \ldots, n + 1\}\), i.e. the simplex is strongly symmetric.

It is relatively easy to show that the simplices are the only strongly symmetric spectral polytopes. For example, from item 3 of Proposition 3.6 we know that the cone over a strongly symmetric spectral body must be perfect. Theorem 1 of [28] states that the only self-dual polyhedral cones in \(E^n\) whose maximal faces are all self-dual in their spans (with respect to the restriction of the inner product) are isometric (hence also affinely isomorphic) to the simplicial cone \(R^n_+\).

### 6 Strongly symmetric spectral bits are balls

In this section, we prove a special case of our main theorem, for convex sets whose largest frame has cardinality 2. We show that they are all affinely isomorphic to balls, which are the normalized state spaces of the Euclidean Jordan algebras known as spin factors. This case can be handled without the Farran-Madden-Robertson theory [7, 8] of regular convex bodies, and we can use it in the proof of our main theorem, where it allows us to avoid some tedious case-checking involving the Madden-Robertson classification.

We call a convex body \(\Omega\) a **bit** if the largest frame it contains has cardinality 2. We say \(\Omega\) has **reversible transitivity** if \(\text{Aut} \Omega\) acts transitively on the set \(\partial_e \Omega\) of pure states of \(\Omega\). All strongly symmetric convex bodies have this property, since extremal states are 1-frames.

In [12] (Section IV) B. Dakić and C. Brukner claimed that spectral bits that have reversible transitivity on pure states (i.e. on 1-frames) are necessarily balls, and give an argument for this claim. The claim may well be true, but their argument makes an implicit assumption. When this assumption is made explicit, one immediately sees that it follows from transitivity on 2-frames, which in the
case of bits is identical to strong symmetry. Once this is observed, one sees that Dakic and Brukner’s argument establishes the following result, which is weaker than Dakic and Brukner’s claim.

**Theorem 6.1.** Let $\Omega$ be a strongly symmetric spectral compact convex body whose largest frame is of cardinality 2. Then $\Omega$ is affinely isomorphic to a ball.

The proof, which is essentially Dakic and Brukner’s argument done in slightly more detail and with an explicit assumption of transitivity on 2-frames, is in Appendix A.

### 7 Regular convex bodies and regular convex compact sets

In this section we will use some definitions and results from [7] and [8] concerning compact convex sets embedded in Euclidean space, which the authors of [7] and [8] call *convex solids* or, when they are full-dimensional, *convex bodies*. Some of the notions studied in [7] and [8] are sensitive to the particular embedding of a compact convex set in Euclidean space, and fail to be invariant under affine transformations, whereas our concern is with affine-invariant structure. Nevertheless their results are immediately applicable to our situation, because of the canonical embedding (Definition 2.3 of compact convex sets of dimension $n$ into $\mathbb{E}^n$). In the following, the group of *rigid transformations* of $\mathbb{E}^n$ is defined as the group generated by rotations, translations, and reflections.

**Definition 7.1 ([7]).** A solid is a compact convex subset of Euclidean space $\mathbb{E}^n$. It has affine dimension $m \leq n$, and if we wish to indicate its dimension, we may call it an $m$-solid in $\mathbb{E}^n$ or simply $m$-solid. A convex body is an $n$-solid in $\mathbb{E}^n$, i.e. a full-dimensional solid; it may also be called an $n$-body. The *symmetry group* $GB$ (sometimes $G$, for short) of a convex body $B$ is the subgroup of the group of rigid transformations of $\mathbb{E}^n$ consisting of those transformations that take $B$ into (so in fact, onto) itself.

The symmetry group $GB$ of $B$ consists of affine automorphisms, but in general it may be a proper subgroup of the group $\text{Aut} B$ of affine automorphisms of $B$. For example, a nonsquare rectangle has a smaller symmetry group than a square, and a parallelogram whose sides are not all the same length has a smaller symmetry group than a rectangle, whereas $\text{Aut} \Omega$ is the same in all three cases. Similarly, a
general solid ellipsoid has a smaller symmetry group than a (spherical) ball, but
the two are affinely isomorphic.

The notion of symmetry group, and the associated results of [7] and [8] are
nevertheless useful to us because of the following (which is immediate from
Proposition 2.2):

**Proposition 7.2.** Let $\Omega$ be canonically embedded in a Euclidean space $E$, in the
sense of Definition 2.3. Then $\text{Aut } \Omega$ is the symmetry group $G_\Omega$.

In [7] and [8] the term “face” is used to mean “exposed face”, that is, the
intersection of the set with a supporting hyperplane. Since we will use some
definitions and results from [7] and [8], and since all faces in strongly symmetric
spectral convex bodies are exposed,21 that is how we will use the term from now
on. It is a fairly standard convention to include $\emptyset$ and $\Omega$ as faces, and even as
exposed faces, of $\Omega$; they are called improper faces and the others are called proper faces.

**Definition 7.3.** A flag of a convex compact set $\Omega$ is a sequence $F_1, \ldots, F_r$ of distinct
nonempty exposed faces of $\Omega$ such that $F_0 \subset F_1 \subset \cdots \subset F_r$.

In other words it is a chain in the face lattice, not containing the empty face.22
Every convex body may be considered as a compact convex set relative to the
natural affine space structure of $\mathbb{E}^n$ (obtained by forgetting about the inner product
and $0$). So the notion of flag also applies to convex bodies.

**Definition 7.4.** A maximal flag is a flag that is not a subsequence of any other flag.23

---

21 In fact this is true of all regular convex bodies as well, by Proposition 8.7 and the fact [29, 30]
that all faces of orbitopes in polar representations are exposed. We suspect it would remain true if
the definition of regularity were changed to refer not to exposed faces, but just to faces.

22 This is almost identical to the definition in [7], except that they also exclude the face $\Omega$,
whereas we prefer to allow (but not require) its inclusion. When we need their notion we will use
the term “short flag”.

23 We define a maximal flag exactly as in [8], except that our flags contain the full set, whereas
theirs do not. Maximal flags of these two types are obviously in bijection with each other, by
deleting or appending $\Omega$ at the end of the sequence of faces. In [7] a slightly different definition
of maximal flag is given, but the definitions give rise to the same notion of regularity, and are
equivalent for regular convex bodies. Explicitly, $\sigma_\Omega \subset \mathbb{N}$, which we may call the “signature” of
$\Omega$, is the ordered (by restriction from the usual ordering of $\mathbb{N}$) set of integers $i$ such that there
is a proper face of $\Omega$ of dimension $i$. Farran and Robertson call a flag maximal if the sequence
dim $F_0, \ldots, \dim F_r$ of dimensions of faces in the flag is equal to $\sigma_\Omega$. Once regular convex bodies are
defined, it will be clear the two definitions coincide for such bodies (though not in general, cf. the
examples in Fig. 2 of [7]).
We have the following elementary fact.

**Proposition 7.5.** Let \( g \) be an automorphism of \( \Omega \). If \( F \) is a face of \( \Omega \), \( g.F \) is also a face of \( \Omega \), and \( g.c(F) = c(g.F) \). Let \( \Phi \) be a flag of \( \Omega \). Then \( g.\Phi \) is a flag of \( \Omega \); it is maximal if and only if \( \Phi \) is.

**Definition 7.6.** A convex body \( B \) is called *regular* if its symmetry group \( GB \) acts transitively on the set of maximal flags of \( B \). \(^{24}\)

We give an analogous definition for arbitrary convex compact sets:

**Definition 7.7.** A convex compact set \( \Omega \) is called *regular* if its affine automorphism group \( \text{Aut} \Omega \) acts transitively on the set of maximal flags of \( \Omega \).

Unlike the notion of regular compact convex set, the notion of regular convex body is not affine-invariant, because it is sensitive to the embedding in Euclidean space. But we have the following elementary relation between the notions of regular convex body (Definition 7.6) and of regular compact convex set (Definition 7.7).

**Proposition 7.8.** Every regular convex body \( B \), considered as a compact convex set, is regular. Not every convex body that is regular when considered as a compact convex set, is a regular convex body, but every canonically embedded regular compact convex set is a regular convex body.

**Proof.** The first sentence holds because because the symmetry group \( GB \) is a subgroup of \( \text{Aut} B \), and transitive action by a subgroup trivially implies transitive action by the group. The second holds because a regular compact convex set may be embedded in such a way that its symmetry group is too small a subgroup of the automorphism group to act transitively on frames (consider a triangular simplex, embedded as a non-equilateral triangle), but in its canonical embedding, the symmetry group is equal to the automorphism group (cf. Proposition 7.2).

The following is Farran and Robertson’s version (probably originating, for the case of groups generated by a finite number of reflections, with Coxeter or earlier), adapted to this setting, of a standard notion from the theory of group actions on topological spaces, that of a *fundamental set* for an action.

\(^{24}\)Once again, this is essentially the definition from Farran and Robertson [7], except that their notion of flag omits \( B \) itself. It obviously gives the same notion of regularity.
Definition 7.9. Let $B$ be a convex body of affine dimension $n$ in $V \simeq \mathbb{E}^n$, with barycenter 0, and let $G$ be a compact subgroup of $O(\mathbb{E}^n)$ that preserves $B$. A convex solid (not necessarily full-dimensional) $D \subset \mathbb{E}^n$, is called a fundamental region for the action of $G$ on $B$ if

1. $B \subseteq GD$, and

2. Every $G$-orbit in $B$ meets the relative interior of $B$ in at most one point.

Theorem 7.10 (Farran and Robertson (Theorem 7 of [7])). Let $B$ be a convex body embedded in $\mathbb{E}^n$. Suppose in addition that $B$ is regular. Let $\Phi = (F_1, \ldots, F_r)$ be a maximal flag of $B$, and let $c_i$ be the barycenter of $F_i$. Then the $(r-1)$-simplex $\triangle \Phi := \triangle(c_1, \ldots, c_{r-1}, c_r)$ is a fundamental region for the action of $B$'s symmetry group $G$ on $B$.

When we work with a fixed flag $\Phi$, we often write $\triangle$ for $\triangle \Phi$, and omit the subscripts indicating the dependence of other objects on $\Phi$ as well. The automorphism group, $\text{Aut} B = K$, of any convex compact set is a compact Lie group. If it has trivial Lie algebra ($\mathfrak{k} = \{0\}$), then it is a finite group. The same two statements hold for the symmetry group of a convex body $B$ as defined by Farran and Robertson (since their definition of convex body requires $B$ to affinely span the Euclidean space in which it is embedded). We write $\triangle$ for $\triangle \Phi$ and $\triangle'$ for the $r-2$-simplex $\triangle(c_1, \ldots, c_{r-1})$ in $\text{Aff} B \equiv \mathbb{R}^n$. This differs from the definition of $\triangle$ only by omitting the barycenter $c_r \equiv 0$ of $B$; its affine dimension is $r-2$, whereas $\triangle$'s is $r-1$.

The facial structure of the simplex $\triangle$ encodes important information about how generic the orbits are, via the following theorem.

Theorem 7.11 (Theorem 8 of [7]). Let $\triangle$ be the fundamental domain for the regular convex body $B$, as defined in Theorem 7.10, and let $G = GB$, $B$'s symmetry group. Let $F$ be a face of $\triangle$ of nonzero dimension. If $x$ is in the relative interior of $F$, and $y \in F$ is arbitrary, then $G_x \leq G_y$. In particular, if $x$ and $y$ are both in the relative interior, $G_x = G_y$.

We now assume without loss of generality that $B$ is canonically embedded in $\mathbb{E}^n$, so in particular $c_r = 0$. The points $c_1, \ldots, c_{r-1}$ linearly generate—equivalently, $\triangle'$ linearly generates—an $(r-1)$-dimensional vector space in $\text{Aff} B \equiv \mathbb{R}^n$, which we call $L$. (Of course, $\triangle$ also (both linearly and affinely) generates $L$.)

Following [7] we define a map $\pi : B \mapsto \pi(B) := B \cap L$. In [7], $\pi$ is called a projection, presumably because it is idempotent: if $B$ is a regular polytope, then $\pi(B) = B$. 
**Definition 7.12.** For a regular convex body $B$ embedded in Euclidean space $\mathbb{E}^n$, we call the subspace $L$ defined in the preceding paragraph a *Farran-Robertson section*, and the polytope $\pi(B) := L \cap B$ its *Farran-Robertson polytope*. We define the *Farran-Robertson polytope of a regular compact convex set* $\Omega$ as the Farran-Robertson polytope of $\Omega$ when $\Omega$ is considered as a convex body by canonically embedding it in Euclidean space.

As embedded in $\mathbb{E}^n$, the convex solid $\pi(B)$ depends on the choice of a maximal flag of $B$, but as we shall see in Propositions [8,7] and [8,8] all $\pi(B)$ are $G$-conjugate, hence isometric.

**Theorem 7.13 (Theorem 9 of [7]).** Let $B$ be a regular convex body with symmetry group $K$. The Farran-Robertson polytope $\pi(B)$ is a regular polytope, of affine dimension $k = \dim L$, whose symmetry group $W$ is a finite group generated by reflections, isomorphic to $K_L / K^L$.

As an example, consider a 3-dimensional ball $\Omega$ centered on the origin. $\text{Aut } \Omega \equiv G\Omega$ is $O(3)$. $L$ may be taken to be any line through the origin and $\pi(\Omega)$ the simplex $\triangle_1$ consisting of the diameter $L \cap \Omega$. $\text{Aut } \pi(\Omega) \simeq \mathbb{Z}_2$ is generated by the reflection through the plane orthogonal to this diameter.

Examples where $\Omega \neq \pi(\Omega)$ but with a more interesting $\pi(\Omega)$, for example where $\pi(\Omega)$ is the three-vertex simplex $\triangle_2$, are harder to visualize because the affine span of $\Omega$ will tend to be too large. In the lowest-dimensional example (as it will turn out) $\Omega$ is the unit-trace positive semidefinite real symmetric $3 \times 3$ matrices, with $L$ the diagonal unit-trace matrices (two dimensions) and $L \cap \Omega \simeq \triangle_2$. $\text{Aff } \Omega$, the unit-trace real symmetric matrices, is 5-dimensional in that case. $\Omega$ in this second example is affinely isomorphic to the example on pp. 378-379 of [7], where it is presented as the convex hull of the Veronese surface, an embedding of $\mathbb{P}_2(\mathbb{R})$ into $\mathbb{E}^6$. This surface is the extreme boundary of its convex hull $\Omega$, which spans a 5-dimensional subspace.

The following theorem allows us to understand the facial structure of $\Omega$ by understanding that of $\pi(\Omega)$. We will need it later to show that frames in $\pi(\Omega)$ correspond to frames in $\Omega$ in the strongly symmetric spectral case; it is of course also of intrinsic interest. This theorem is stated near the top of p. 369 of [8].

---

25 An essentially identical (except for its more restricted premise) theorem is stated for a more restricted setting (the subclass of polar representations occurring within adjoint representations of real semisimple groups) in [7].
Proposition 7.14 ([8]; see also [7], Theorem 10, and its proof). Let $B$ be a regular convex body with symmetry group $G$, and Farron-Robertson polytope $\pi(B)$. Let $F$ be a face of $\pi(B)$ with centroid $c(F)$, and write $G^c(F)$ for the isotropy subgroup of $G$ at $c(F)$. Then the orbit $G^c(F)F$ is a face of $B$, which we call $H_F$, and each face of $B$ is of the form $gH_F$ for some face $F$ of $\pi(B)$ and some $g \in G$. Moreover, if $F_1, F_2, \ldots, F_r$ is a maximal flag of $P$, then $H_{F_1}, H_{F_2}, \ldots, H_{F_r}$ is a maximal flag of $B$, and every maximal flag of $B$ arises from a flag of $\pi(B)$ in this way.

Since the symmetry group of a convex body is its automorphism group, we also have the analogous statement for convex compact sets and their automorphism groups.

8 Classification of regular convex bodies via polar representations

In this section, we sketch the classification of regular convex bodies from [8], via polar representations as defined, classified (in the irreducible case), and related to noncompact symmetric spaces in [31]. The results of this section will be used through the classification set out in [8], Tables 2, 3, and 4. Besides the tables themselves we need to understand which orbit, in the polar representation listed in the table, gives rise to the regular convex body.

Let us write $\rho$ for the representation of the action of the symmetry group $G\Omega$ on $\mathbb{R}^n$. Proposition 2.1 of [8] states that for a regular convex body $\Omega$, this representation is irreducible. Proposition 2.2 of [8] states that for regular convex bodies $\Omega$, $\rho$ is a polar representation, defined as one for which the subspace normal to a principal (i.e. highest-dimensional) orbit of the action meets every orbit orthogonally.

More formally, let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $\rho : G \to O(V)$ be a representation on a real inner product space. (As usual we often write $g$ for $\rho(g)$ where $g \in G$.) $v \in V$ is called regular if the orbit $G.v$ is of maximal dimension (i.e. no orbit has higher dimension). The tangent space at $v$ to an orbit

26In [8] the proof is attributed to M. Pinto in a paper “to appear” which we have not been able to find. It involves showing that regular convex bodies belong to the wider class of perfect (not to be confused with the notion of perfection of cones) convex bodies [7], and the observation that this implies $\rho$ is irreducible. It is not hard to reconstruct a simple proof along these lines. We note that Markus Müller [32] has shown more directly that the automorphism group of a strongly symmetric spectral convex body $\Omega$ acts irreducibly on Aff $\Omega$. 
G.v is g.v. It is a linear subspace of V. We define a linear cross-section of the set of G-orbits in V to be a subspace of V that intersects every G-orbit. Although this term is not explicitly defined in [31], this is the sense in which it is used there. We will usually omit the qualifier linear, since we make no use of any other kind of cross-section.

**Definition 8.1** (following [31]). For any \( v \in V \), define \( a_v \) to be the subspace of \( V \) normal to the G-orbit at \( v \), i.e. \( a_v := (g.v)^\perp \).

**Proposition 8.2** ([31], Lemma 1). For any \( v \), \( a_v \) is a cross-section of the set of G-orbits.

Note that by Definition 8.1 \( V \) itself is a cross-section. More interesting are the minimal-dimensional cross sections.

**Definition 8.3** ([31]). Cross-sections of the form \( a_v \) for regular \( v \) are called Cartan subspaces.

Such cross-sections are of minimal dimension.

**Proposition 8.4** ([31]). Let \( v_0 \) be regular. The following are equivalent:

1. For any regular \( v \in V \), there is a \( k \in G \) such that \( g.v = k.(g.v_0) \).
2. For any regular \( v \in V \), there is a \( k \in G \) such that \( a_v = k.a_{v_0} \).
3. For any \( u \in a_{v_0} \), \( (g.u, a_{v_0}) = 0 \).

Thus we have uniqueness (up to the action of G) of minimal cross-sections if and only if the orbits intersect one such cross-section orthogonally.

**Definition 8.5** ([31]). A representation satisfying any (and hence all) of the three equivalent conditions in Proposition 8.4 is called polar.

Thus the polar representations are ones for which the minimal dimensional cross-sections of the form \( a_v \) for regular \( v \), i.e. normals to maximal-dimensional orbits, are all G-conjugate (item 1 in Proposition 8.4), or equivalently ones for which, for every maximal-dimensional orbit, the subspace normal to that orbit (at any point) intersects every orbit orthogonally (item 3 in Proposition 8.4).

---

27Sometimes a definition is used in which a cross section must intersect each orbit finitely many times, which would rule out \( V \) except in the case of \( g = 0 \).

28It seems likely that in [31] this definition is meant to apply only in the polar case, but in the proof of Proposition 2.2 of [8] (Proposition 8.7 below), it is clearly meant to apply prior to establishing the representation is polar.
Remark 8.6. A cross-section that meets all orbits orthogonally is minimal. Hence an equivalent definition (cf. (33, 34, 35)) of polar representation is that it is a representation for which there exists a cross-section that meets all orbits orthogonally.

Proposition 8.7 ([8], Proposition 2.2 and its proof). Let $B$ be a regular $n$-solid in $E \simeq \mathbb{E}^n$, with centroid 0, and let $\pi$ be the inclusion of the symmetry group $G$ of $B$ in $O(n) \simeq O(E)$. Then the representation $\pi$ is polar, and for any maximal flag $\Phi$ of $B$, the centroids of the faces in $\Phi$ are a basis for a Cartan subspace, $L_\Phi$.

Proof (greatly expanded version of proof in [8]). Let $x$ and $v$ each be on principal orbits (not assumed to be the same orbit) of this $G$-action, and recall that $a_x := \{u \in \mathbb{E}^n : \langle u, g.x \rangle = 0\}$ and $a_v := \{u \in \mathbb{E}^n : \langle u, g.v \rangle = 0\}$. We will show:

Claim 1. There is a $g \in G$ such that $a_v = g.a_x$.

This will prove Proposition 8.7 because transitive action of $G$ on the set $\{a_y : y \in V\}$ of Cartan subspaces is one of the equivalent conditions that defines (via Proposition 8.4) Dadok’s notion of polar representation.

To do this we show that the Farran-Robertson sections $L_\Phi$ are Cartan subspaces; recall that $L_\Phi$ is the linear space spanned by the centroids of the faces in the maximal flag $\Phi$, and hence the linear span of the simplex $\triangle$ whose vertices are those centroids, and indeed of the simplex $\triangle'$ whose vertices are those of $\triangle$, with the exception of 0 (the centroid of $\Omega$). Since the topological boundary $\partial B$ contains representatives of all nonzero orbits, $L$ is a cross-section of the representation. In the proof of Theorem 9 in [7], it is established that “$G.y$ is perpendicular to $L$ for all $y \in L$.” In other words, for all $y \in L$, $L \subseteq a_y$. Since $L$ is a cross-section it contains representatives of principal orbits, i.e. regular elements; letting one such be $x$, we have $L \subseteq a_x$. But since $x$ is regular $a_x$ is a minimal cross-section by Lemma 1 of [31], so $L = a_x$, and $L$ is a Cartan subspace.

To show that $G$ acts transitively on the set of Cartan subspaces, we fix one such subspace $a_x = L_\Phi$. Since $\triangle_\Phi$ is a fundamental region for the action of $G$ on $B$, every nonzero $v$ on a principal orbit has the form $v = g.x$ for some $g \in G$ and some $x$ in an element of $\triangle_\Phi$. We have $x \in L_\Phi$. By Proposition 7.5 $g.\Phi$ is also a maximal flag, and (since by that Proposition $g$ takes centroids of face to centroids of faces) $g.L_\Phi = L_{g.\Phi}$. Of course $g.x \in L_{g.\Phi}$; moreover, by the same argument used to establish that $L_\Phi$ is a Cartan subspace, $L_{g.\Phi} = a_{g.x}$, and hence $L_{g.\Phi}$ is a Cartan subspace. Since $L_{g.\Phi} = g.L_\Phi$, this shows that $a_v = g.a_x$, establishing Claim 1. □
Proposition 8.8 (Corollary of Proposition 8.7). Let $\Omega$ be a regular compact convex set canonically embedded in a Euclidean space $E$. The representation of $\text{Aut} \, \Omega$ in $O(E)$ is polar. The centroids of a maximal flag of $\Omega$ are a basis for a Cartan subspace.

To study regular nonpolytopal convex compact sets $\Omega$, we may assume that the Lie algebra $\mathfrak{g}$ of $\text{Aut} \, \Omega$ is nontrivial, i.e. not equal to $\{0\}$. For compact Lie groups, this is equivalent to assuming the group is not finite. The next definitions and results concern situations where the Lie group is connected, and hence (except in the degenerate case of the trivial group) has nontrivial Lie algebra. If $\rho$ is a representation of a Lie group $G$, we write $d\rho$ for the derived representation of $G$’s Lie algebra.

Definition 8.9 ([31]). Let $G$ be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra. A representation $\rho : G \to SO(V)$ is called a symmetric space representation if there are a real semisimple Lie algebra $\mathfrak{h}$ with Cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, a Lie algebra isomorphism $A : \mathfrak{g} \to \mathfrak{k}$, and an isomorphism $L$ of linear spaces $V \to \mathfrak{p}$ such that $L \circ d\rho(X) \circ L^{-1}(y) = [A(X), y]$ for all $X \in \mathfrak{g}, y \in \mathfrak{p}$.

In other words, $L \circ d\rho(X) \circ L^{-1} = \text{ad}(A(X))$.

The symmetric spaces in the proposition are $H/K$, for groups such that $K$ is compact with Lie algebra $\mathfrak{k} \simeq \mathfrak{g}$, and the Lie algebra $\mathfrak{h}$ of the real semisimple group $H$ has Cartan decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$, as mentioned in the definition, can be identified with the representation space $V$. They are of noncompact type since one normally takes the term “Cartan decomposition” to imply that $\mathfrak{k} \oplus \mathfrak{p}$ is noncompact semisimple. However by the duality bijection between compact and noncompact symmetric spaces, the definition also includes the isotropy representations associated with symmetric spaces of compact type, because the compact duals (with Lie algebra decomposition $\mathfrak{k} \oplus i\mathfrak{p}$) give rise to equivalent polar representations $K \circlearrowright i\mathfrak{p}$.

Symmetric space representations are polar, with $\mathfrak{a}$, defined as usual as a maximal abelian subspace of $\mathfrak{p}$, a Cartan subspace in the terminology of [31].

---

29 The statement in [31] appears to have a couple of typographical errors which we have corrected: it has $\mathfrak{a}$ in place of $\mathfrak{g}$ in the last line, and omits the $L^{-1}$ that we’ve inserted on the left-hand side. Other than that, we quote [31] verbatim.

30 Of course $\mathfrak{a}$ is not a Cartan subalgebra of $\mathfrak{h}$, unless $\mathfrak{h}$ is a split real form of a complex Lie algebra. But $\mathfrak{a}$ is the intersection of $\mathfrak{p}$ with a particular type of Cartan subalgebra of $\mathfrak{h}$ (one that is “maximally noncompact”, i.e. whose intersection with $\mathfrak{p}$ is maximal). The term “Cartan subspace” was already standard in the symmetric space context.
action of $G$ on $p$ is often called the isotropy representation of $G$ in the context of symmetric space theory; a symmetric space representation in the above sense is just one that is equivalent to such an isotropy representation.

Dadok is able to use the symmetric space representations to classify the irreducible polar representations (up to orbit equivalence) by using the following theorem:

**Theorem 8.10** (Proposition 6 of [31]). Let $\rho : G \to SO(V)$ be a polar representation of a connected compact Lie group $G$. There is a connected Lie group $\tilde{G}$ with symmetric space representation $\tilde{\rho} : \tilde{G} \to SO(V)$ such that the $G$-orbits and the $\tilde{G}$-orbits in $V$ coincide.

The key point, from [8], is that “for an irreducible noncompact symmetric space $H/K$, knowledge of $K$ and the dimension of $H/K$ are sufficient to determine $H$. But these are already given by the symmetry group $G$ and the dimension $n$, respectively. Thus we have associated to the given $n$-solid $B$ a symmetric space $H/K$.” (The bracketed modification is necessary for the truth of the claim.)

The following incorporates and extends Proposition 8.7, giving more detail on the embedding of a regular convex body in polar representations of its symmetry group and some subgroups thereof, including consequences of Theorem 8.10, mostly extracted from the discussion on pp. 366-367 of [8], and from [31].

**Proposition 8.11.** Let $B$ (of dimension $n$) be a regular convex body in a Euclidean space $V \simeq \mathbb{E}^n$, with centroid $0$ and symmetry group $G$, and $\text{lie}(G) = g$. Fix a maximal flag $\Phi$ and a corresponding fundamental region $\triangle := \triangle_{\Phi}$ (cf. Theorem 7.10). Then

1. The Farran-Robertson section $L_{\Phi} \subseteq V$ is a linear cross-section of the form $a_x$ for regular $x$.

2. For every regular $x' \in V$, $a_{x'}$ is a linear cross-section and there is a $g \in G$ such that $a_{x'} = L_{g,\Phi}$.

3. Both the inclusions $G \to O(V)$ and $G_0 \to SO(V) \to O(V)$ are polar.

4. By Theorem 8.10 we may make the identification $V \simeq p$ where $p$ is the isotropy representation of $\tilde{G} \leq G$ associated with an irreducible noncompact symmetric space $H/\tilde{G}$. (So, $\mathfrak{h} = \mathfrak{g} \oplus p$ is a Cartan decomposition of $\mathfrak{h}$.) The representation $G \curvearrowright V \simeq p$ is the restriction of the representation $\tilde{G} \curvearrowright p$. We have $\tilde{\mathfrak{g}} := \text{lie}(\tilde{G}) = g := \text{lie}(G) = \text{lie}(G_0)$. Every maximal abelian subspace
\[ a \subseteq p \text{ is a Cartan subspace of the polar representations } \tilde{G} \curvearrowright p, G \curvearrowright p, \text{ and } G_0 \curvearrowright p, \text{ and is a Farran-Robertson section of } B, \text{ so } \pi(B) = a \cap B. \]

**Proof.** The polarity of the inclusion \( G \to O(V) \) in item 3 is Proposition 8.7, while items 1 and 2 are the key points in its proof. Item (iii) in Proposition 8.4 characterizes the polarity of a representation entirely in terms of the derived representation \( g \to \mathfrak{so}(V) \), whence the inclusion \( G_0 \to SO(V) \to O(V) \) is also polar since \( g \equiv \text{Lie}(G) = \text{Lie}(G_0) \).

Turning to item 4, Theorem 8.10 and its proof in [31] show that because \( \tilde{G} \) has the same orbits as \( G \), its Lie algebra \( \tilde{g} \) is a quotient of \( g \) and the derived symmetric space representation \( \tilde{g} \to \mathfrak{so}(V) \) factorizes \( g \to \mathfrak{so}(V) \). So \( g = \tilde{g} \oplus \mathfrak{z} \), where \( \mathfrak{z} \) is central in \( g \) and does not act on \( V \), i.e. \( \mathfrak{z} V = \{0\} \). Hence the connected subgroup \( Z \) of \( G_0 \) with \( \text{Lie}(Z) = \mathfrak{z} \) acts trivially on \( V \). Since \( G_0 \) is a subgroup of \( SO(V) \) this implies that \( Z \) is trivial, whence \( g = \tilde{g} \). But the Riemannian symmetric space doesn’t change if we replace the maximal compact subgroup \( \tilde{G} \) of \( H \) with a locally isometric one, so we can assume that \( G_0 = \tilde{G} / Z \), where \( Z := \{ g \in \tilde{G} : g|_V = \text{id} \} \). It follows that \( V = p, \mathfrak{h} = g \oplus p \). Since it is known from the theory of symmetric spaces that the subspaces \( (g, x) \perp \) for regular \( x \), i.e. the Cartan subspaces of Definition 8.3 of a symmetric space isotropy representation are precisely the maximal abelian subspaces of \( p \), we may choose any such subspace as a Cartan subspace. The Cartan subspaces of the polar representations of \( G, \tilde{G}, \text{ and } G_0 \) coincide because their Lie algebras do; consequently \( a \) is a Farran-Robertson section for \( G \to V \) (with respect to some choice of maximal flag of \( B \subset V \), and \( \pi(B) = a \cap B \). \[ \square \]

Madden and Robertson do not state the above Proposition (Proposition 8.11) as formally as we do. But their classification applies it to all of the irreducible polar representations that are symmetric space representations in the sense of Definition 8.9 which were classified by Cartan. Combined with the regularity of the Farran-Robertson polytope (Theorem 7.13), and Coxeter’s work [36] (building on the classification of regular polytopes by Schläfi [37] and a construction by Wythoff) classifying the embeddings of regular polytopes as orbitopes in finite groups generated by reflections, this enables them to classify regular convex bodies, a classification given in their Tables 2, 3, and 4. We formally summarize their classification as Theorem 8.12 below, and reproduce their tables, with minor changes and the addition of the rightmost column, indicating which symmetric spaces are associated with Euclidean Jordan algebras, as Tables 2, 3 and 4 below. We precede this with a brief sketch of their argument involving Coxeter’s work.

Groups generated by a finite set of reflections in finite-dimensional real affine space, or isomorphic to such a concrete group, are usually called *reflection groups*;
each finite (not merely finitely generated) reflection group has a canonical representation on Euclidean space $\mathbb{E}^n$, in which it is generated by a finite number of orthogonal reflections through linear hyperplanes and there is no subspace on which the action is trivial. The automorphism groups of regular polytopes are finite reflection groups; when the polytope is canonically embedded in Euclidean space, its automorphism group acts in this canonical representation. If $W$ is a finite reflection group, we use the term $W$-orbitope for the convex hull $\text{Conv} \ W.\nu$ of an orbit of $W$ in its canonical representation, and also call $\text{Conv} \ W.\nu$ the $W$-orbitope of $\nu$, or simply the orbitope of $\nu$ when $W$ is clear from context. Coxeter considered all finite reflection groups groups $W$, specified by what are now called Coxeter diagrams, and showed that the only regular $W$-orbitopes are, up to dilation by a positive scalar, convex hulls of orbits of particular points in the canonical representation, identified by marking a node of the Coxeter diagram.

With a finite reflection group $W$ fixed, a unit-length normal $\alpha \in \mathbb{E}^n$ to the fixed hyperplane $\alpha^\perp$ (the “mirror”) of a reflection that is an element of $W$ is called a root, and the associated reflection is denoted by $s_\alpha$. (A different normalization constant is sometimes chosen, and in the theory of Weyl groups of Lie groups, which are a subset of the finite reflection groups, a different normalization is often used, in which not all roots have the same length.) Thus each reflection is associated with two roots $\pm \alpha$, and the set $R$ of roots is finite. Since for any $g \in O(n)$, $gs_\alpha g^{-1}$ is the reflection $s_{g\alpha}$, in particular for any two roots $s_\alpha s_\beta s_\alpha$ is the reflection with root $s_\alpha \beta$, so $R$ is $W$-invariant, $WR = R$.

A set $\Phi$ of roots is called simple if every root $\beta \in R$ can be written $\beta = \sum_{\alpha \in \Phi} c_\alpha \alpha$, where the coefficients $c_\alpha$ are all of the same sign (which may of course depend on $\beta$), or zero. Simple sets of roots exist, and are bases for the canonical representation space $\mathbb{E}^n$; the associated reflections $s_\alpha$ are a minimal set of generators for $W$. The complement of the union of the mirrors of the reflections in $R$ is a dense subset of Euclidean space, and we call its connected components open Weyl chambers and their closures Weyl chambers.$^{31}$ The Weyl chambers are regular cones with simplicial base, and are fundamental regions for the group action. The elements of the basis dual to the simple roots (when the roots are normalized in a particular way) are called the fundamental weights. The cone generated by the fundamental weights is a Weyl chamber, called the positive Weyl chamber $C_+$; each fundamental weight generates an extremal ray of $C_+$, and each extremal ray of $C_+$ is generated by a fundamental weight.

$^{31}$The terminology is not uniform in the literature; sometimes “Weyl chamber” is used for the open Weyl chambers, and “closed Weyl chamber” for their closures.
For a group generated by a finite set of linear reflections in $\mathbb{E}^n$ to be finite, obviously the product $s_\alpha s_\beta$ of each pair of reflections must have finite order $n_{\alpha \beta}$. (It is a nontrivial fact that this is sufficient.) It follows from this and the fact that the product of two reflections is a rotation by twice the angle between them, that the angles between $\alpha$ and $\beta$ must be rational multiples $k/n$ of $\pi$. For $\alpha, \beta$ in a simple set $\Phi$, we will have $k = n_{\alpha \beta} - 1$, i.e. the angles are $\pi - \pi/n_{\alpha \beta}$, although we still say the lines generated by the roots have angle $\pi/n_{\alpha \beta}$. The Coxeter diagram of the group is determined by these angles for a simple set of roots: its nodes correspond to the simple roots and edges between nodes correspond to the angles other than $\pi/2$ between the lines generated by roots—these edges are labeled by the integer $n_{\alpha \beta}$ (less commonly, $n_{\alpha \beta}$ lines are used to link the pair of nodes, as in a Dynkin diagram). Usually the label $n_{\alpha \beta} = 3$ is omitted, so the angle $\pi/3$ is represented by an unlabeled edge; there is no edge between nodes corresponding to pairs of roots at angle $\pi/2$. It turns out that the relations $(s_\alpha s_\beta)^{n_{\alpha \beta}} = 1$ between pairs of generators encoded in such a diagram suffice to uniquely determine a finite reflection group (this nontrivial theorem immediately implies the nontrivial fact mentioned above).

For the Weyl groups of Lie groups, which includes the reflection groups relevant to the classification of nonpolytopal regular convex bodies, only the labels 4 and 6 appear, reflecting the fact that in this case the angles between lines through the simple roots are limited to $\pi/2, \pi/3, \pi/4,$ and $\pi/6$. In this case the Coxeter diagram, in the version with angles represented by edge multiplicities, is just the Dynkin diagram with the information about root lengths (associated with the different normalization usually used in this case) omitted.

Returning to the general case, the point identified by marking a node is, up to (strictly) positive scalar multiples, the fundamental weight dual to the marked root. As mentioned above, Coxeter showed that the regular polytopes arise as convex hulls of orbits (i.e. the orbitopes) of particular fundamental weights; every such polytope arises as the orbitope of the weight specified by marking an end node (node connected to only one other node) in some Coxeter diagram, although not all end nodes have such polytopes as orbitopes, and some non-end nodes give rise to

---

Coxeter diagrams are also used to specify not-necessarily-finite, but discrete, groups generated by a finite number of reflections in finite-dimensional real affine space, and further groups sharing algebraic properties with these—see e.g. [38]. In this case, the labels on edges are drawn from the set $\{4, 5, 6, \ldots\} \cup \{\infty\}$, the label indicating the order of the product of the two reflections associated with the linked pair of nodes, still with order 3 unlabeled and no edge for order 2. The full set of groups thus specified by Coxeter diagrams are known as Coxeter groups. For finite reflection groups, only the integer labels appear.
regular polytopes. To classify non-polytopal regular convex bodies, Madden and Robertson needed only to consider Coxeter’s classification of regular orbitopes for the case of finite reflection groups that are Weyl groups of irreducible symmetric spaces, cf. the last two paragraphs on p. 366 of [8].

**Theorem 8.12** (Madden-Robertson classification of regular convex bodies [8]).

1. Let $G/K$ be an irreducible noncompact symmetric space, with the compact group $K$ connected, and let $K \lhd \mathfrak{p}$ be the isotropy representation of $K$, i.e. $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g} := \{ \mathfrak{g} \in G \text{ and } K \lhd \mathfrak{p} \text{ is the restriction to } K \text{ of the corestriction to } \mathfrak{p} \text{ of the adjoint action } G \lhd \mathfrak{g} \}$. Let $\mathfrak{a}$ be a maximal abelian subspace (also called Cartan subspace) of $\mathfrak{p}$ and let $\mathfrak{v} \in \mathfrak{a}$ be such that $P := \text{Conv } W_K.\mathfrak{v} \subset \mathfrak{a}$ is a regular polytope, where $W_K := K_\mathfrak{a}/K^\mathfrak{a}$ is the Weyl group of $K$. Then $B := K.P = \text{Conv } K.\mathfrak{v}$ is a nonpolytopal regular convex body, $P$ is its Farran-Robertson polytope $\pi(B)$, and $P = \mathfrak{a} \cap B$. $B$ is completely determined, up to affine isomorphism, by the affine isomorphism class of $\pi(B)$ and the symmetric space.

2. Conversely, for every regular convex body $B$ that is not a polytope, there exists an irreducible noncompact symmetric space $G/K$ such that $B$ is an orbitope $\text{Conv } K.\mathfrak{v}$ in the isotropy representation $K \lhd \mathfrak{p}$ of the compact connected Lie group $K$. Its Farran-Robertson polytope $\pi(B)$ is $\mathfrak{a} \cap B$, and is a $W_K$-orbitope.

3. In the above items $\mathfrak{v}$ may be taken to be any strictly positive multiple of one of a certain set of fundamental weights in $\mathfrak{a}$. Any weight $\mathfrak{w}$ for which $\text{Conv } W_K.\mathfrak{w}$ is a regular polytope may be chosen. These weights were classified by Coxeter: fundamental weights are specified by marking a node of the Coxeter diagram, and Coxeter indicated which marked nodes correspond to weights giving rise to regular polytopes.

4. The list of irreducible noncompact symmetric space representations presented as $G/K$ for connected $K$, together with information on their dimensions, ranks, and the regular polytopes that occur as Weyl orbitopes in Cartan subspaces, is given in Tables 2, 3 and 4 which except for their last column are essentially Tables 2, 3, and 4 of [8].

Note that the same regular convex body may appear more than once in the tables describing the classification. These occurrences include the coincidences between noncompact irreducible symmetric spaces already noted by Cartan (cf.
40

[39], pp. 519-520), which are finite in number, plus some cases, including some infinite series, where the same invariant regular convex body $B$ appears in different symmetric spaces. For the spectral strongly symmetric bodies, the coincidences between symmetric spaces arising from EJAs are indicated in the tables by coincidences of EJAs in the rightmost column. The only other coincidences between spectral strongly symmetric convex bodies in the tables are the actions of proper subgroups of $SO(n)$ on balls $B_n$, which occur in type AIII and CII when $p = 1$. Appendix C contains more detail.

9 Classification of strongly symmetric spectral convex compact sets

In this section we apply the theory of the preceding sections to prove our main result, which is the “only if” direction of Theorem 1.1: that strongly symmetric spectral convex sets are normalized EJA state spaces or simplices. We proceed by way of several intermediate propositions.

Proposition 9.1. Let $\Omega$ be a strongly symmetric spectral convex compact set. Then $\Omega$ is regular.

In order to prove this proposition we first establish:

Lemma 9.2. Let $\Omega$ be a strongly symmetric spectral convex set. Let $\omega_1, \ldots, \omega_r$ be a maximal frame in $\Omega$. The sequence

$$F_i = \bigvee_{1 \leq j \leq i} \{\omega_j\}, \quad i \in \{1, \ldots, r\}$$

is a maximal flag. Conversely, let $(F_1, \ldots, F_r)$ be a maximal flag of $\Omega$. Then there exists a maximal frame $\omega_1, \omega_2, \ldots, \omega_r$ such that (5) holds.

In other words, the formula (5) gives a bijection between maximal frames in $\Omega$ and maximal flags of $\Omega$.

Proof. Let $X = [\omega_1, \ldots, \omega_r]$ be a maximal frame. It follows from item 1 of Proposition 3.6 that the initial segments of $X$ generate a sequence of faces, the $F_i$ of $\Delta_1$: $\text{Herm}(2, \mathbb{R}) \simeq \mathbb{R}^2 \oplus \mathbb{R}, \text{Herm}(2, \mathbb{C}) \simeq \mathbb{R}^3 \oplus \mathbb{R}$ and $\text{Herm}(2, \mathbb{H}) \simeq \mathbb{R}^5 \oplus \mathbb{R}$. (Although it only occurs once in the table, we note that $\mathbb{R}^9 \oplus \mathbb{R}$ may be interpreted as $\text{Herm}(2, \mathbb{O})$.)
Table 2: Classical noncompact symmetric spaces with associated isotropy representations and Farran-Robertson polytopes (adapted from [8], Table 2)

| Type | Symmetric space $G/K$ | Rank of symmetric space | Dimension of isotropy representation | Root space | Polytope | EIA |
|------|-----------------------|-------------------------|-------------------------------------|------------|----------|-----|
| AI   | $SL(n,\mathbb{H})/SO(n)$ | $n-1$ | $(n-1)(n+2)/2$ | $A_{n-1}$ | $\Delta_{n-1}$ | $\text{Herm}(n,\mathbb{R})$ |
| AII  | $SU^*(2n)/Sp(n)$ | $n-1$ | $(n-1)(2n+1)$ | $A_{n-1}$ | $\Delta_{n-1}$ | $\text{Herm}(n,\mathbb{H})$ |
| AIII | $SU(p,q)/(U_p \times U_q)$ | $q$ | $2pq$ | $\begin{cases} C_q & (q < p) \\ B_q & (q = p) \end{cases}$ | $\Box_q \diamond_q \mathbb{R}^2 \oplus \mathbb{R}$ | $(p = q = 1)$ |
| BI   | $SO_0(p,q)/(SO(p) \times SO(q))$ | $q$ | $pq$ | $B_q$ | $\Box_q \diamond_q \mathbb{R}^p \oplus \mathbb{R}$ | $(q = 1)$ |
| DI   | $SO_0(p,q)/(SO(p) \times SO(q))$ | $q$ | $pq$ | $\begin{cases} B_q & (q < p) \\ D_q & (q = p) \end{cases}$ | $\Box_q \diamond_q \mathbb{R}^p \oplus \mathbb{R}$ | $(q = 1)$ |
| DIII | $SO^*(2n)/U(n)$ | $\lfloor n/2 \rfloor = q$ | $n(n-1)$ | $\begin{cases} C_q & q \text{ odd} \\ BC_q & q \text{ even} \end{cases}$ | $\Box_q \diamond_q \mathbb{R}^2 \oplus \mathbb{R}$ | $(q = 1)$ |
| CI   | $Sp(n,\mathbb{R})/U(n)$ | $n = q$ | $n(n+1)$ | $C_q$ | $\Box_q \diamond_q \mathbb{R}^2 \oplus \mathbb{R}$ | $(q = 1)$ |
| CII  | $Sp(p,q)/(Sp(p) \times Sp(q))$ | $q$ | $4pq$ | $\begin{cases} C_q & (q = p) \\ BC_q & (q < p) \end{cases}$ | $\Box_q \diamond_q \mathbb{R}^4 \oplus \mathbb{R}$ | $(p = q = 1)$ |
Table 3: Exceptional noncompact symmetric spaces with associated isotropy representations and Farran-Robertson polytopes (adapted from [8], Table 3)

| Type   | Symmetric space $G/K$ presented by $\mathfrak{g}$, $\mathfrak{k}$ | Rank of symmetric space | Dimension of isotropy representation | Root space | Polytope | EJA |
|--------|---------------------------------------------------------------|-------------------------|--------------------------------------|------------|----------|-----|
| EIII   | $\mathfrak{e}_6(-14)$ $\mathfrak{so}(10) \oplus \mathbb{R}$ | 2                       | 32                                   | $B_2$      | $\Box_2$ |     |
| EIV    | $\mathfrak{e}_6(-26)$ $f_4$                          | 2                       | 26                                   | $A_2$      | $\Box_2$ | $\text{Herm}(3,0)$ |
| EVI    | $\mathfrak{e}_7(-5)$ $\mathfrak{so}(12) \oplus \mathfrak{su}(2)$ | 4                       | 64                                   | $F_4$      | 24-cell  |     |
| EVII   | $\mathfrak{e}_7(-25)$ $\mathfrak{e}_6 \oplus \mathbb{R}$ | 3                       | 54                                   | $C_3$      | $\Box_3, \Diamond_3$ |     |
| EIX    | $\mathfrak{e}_8(-24)$ $\mathfrak{e}_7 \oplus \mathfrak{su}(2)$ | 4                       | 112                                  | $F_4$      | 24-cell  |     |
| FI     | $f_4(4)$ $\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$     | 4                       | 28                                   | $F_4$      | 24-cell  |     |
| FII    | $f_4(-20)$ $\mathfrak{so}(9)$                           | 1                       | 16                                   | $A_1$      | $\Box_1$ |     |
| G      | $\mathfrak{g}_2(2)$ $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ | 2                       | 8                                    | $G_2$      | hexagon  |     |

Table 4: Noncompact symmetric spaces arising as $K^C/K$ for simple $K$, with associated isotropy representations and Farran-Robertson polytopes ([8], Table 4)

| Type and root space | Symmetric space $G/K$ | Dimension of isotropy representation | Polytope | EJA |
|---------------------|-----------------------|-------------------------------------|----------|-----|
| $A_n(n \geq 1)$     | $\text{SL}(n+1,\mathbb{C})/\text{SU}(n+1)$ | $n(n+2)$                           | $\Box_n$ | $\text{Herm}(n,\mathbb{C})$ |
| $B_n(n \geq 2)$     | $\text{SO}(2n+1,\mathbb{C})/\text{SO}(2n+1)$ | $n(2n+1)$                          | $\Box_n, \Diamond_n$ |     |
| $C_n(n \geq 3)$     | $\text{Sp}(n,\mathbb{C})/\text{Sp}(n)$ | $n(2n+1)$                          | $\Box_n, \Diamond_n$ |     |
| $D_n(n \geq 4)$     | $\text{SO}(2n,\mathbb{C})/\text{SO}(2n)$ | $n(2n-1)$                          | $\Diamond_n$ |     |
| $F_4$               | $F_4^C/F_4$           | 52                                  | 24-cell  |     |
| $G_2$               | $G_2^C/G_2$           | 14                                  | hexagon  |     |
each properly contained in the next. This is a flag, which we will call \( \Phi_X \).

In this sequence, the rank \(|F_i|\) of \( F_i \) is \( i \). Suppose this flag is not maximal. Then it can be enlarged, either by extending it before \( F_1 \), or by extending it after \( F_r \), or by inserting some face \( G \) with \( F_i \not\subset G \not\subset F_{i+1} \). It cannot be extended before \( F_i = \{ \omega_i \} \), because the only face below the pure state \( \omega_i \) is the improper face \( \emptyset \).

It must have \( F_r = \Omega \) by Proposition 3.6 (7), so it cannot be extended beyond \( F_r \).

So there must be an \( i \in \{1, \ldots, r\} \) and a face \( G \) such that \( F_i \not\subset G \not\subset F_{i+1} \). By Proposition 3.6 (1) \( F_i \not\subset G \not\subset F_{i+1} \) implies \(|F_i| < |G| < |F_{i+1}|\), which contradicts the fact, observed above, that \(|F_i| = i \) and \(|F_{i+1}| = i + 1 \) by construction. Since every way of extending the flag \( \Phi_X \) is inconsistent with the maximality of the frame \( X \), \( \Phi_X \) is a maximal flag.

Conversely, suppose \( \Phi = \{F_1, \ldots, F_r\} \) is a maximal flag. By Proposition 3.6 (1) each \( F_i \) is the join of (the singletons corresponding to) a frame, whose cardinality is \(|F_i|\). We will show (“Claim 1”) that \( F_1 = \{\omega_1\} \) for some extremal point \( \omega_1 \), and (“Claim 2”) that for each \( i \in \{2, \ldots, r\} \), there exists an extremal \( \omega_i \) distinguishable from every state in the frame \( F_{i-1} \), such that \( F_i = F_{i-1} \join \omega_i \). It follows from the associativity of join that (5) holds for each \( i \), and since \( F_r = \Omega \) for a maximal flag, \( \omega_1, \ldots, \omega_r \) generates \( \Omega \). Since \( \Omega \) is generated by a maximal frame, and by Proposition 3.6 (1), all frames generating the same face have the same cardinality, \( \omega_1, \ldots, \omega_r \) has the same cardinality as a maximal frame, whence it is a maximal frame.

To show Claim 2 we use the fact, which is part of Prop. 3.6 (7), that if \( F \not\subset G \), any frame for \( F \) extends to a frame for \( G \) by adjoining a frame for \( F' \cap G \). Consider \( F = F_{i-1}, G = F_i \), for \( i \in \{2, \ldots, r\} \). The frame for the face \( F' \cap F_i \), whose join with \( F_{i-1} \) is \( F_i \), is nonempty because \( F' \)’s containment in \( G \) is strict.34 Say it is \( \Sigma = [\sigma_1, \ldots, \sigma_m] \), for \( m \geq 1 \). We show that \( m = 1 \). If \( m > 1 \), then we can extend the flag \( \Phi \) to a flag \( \tilde{\Phi} \) defined by \( \tilde{F}_j = F_j \) for \( j \in \{1, \ldots, i-1\} \), \( \tilde{F}_{j} := \tilde{F}_{j-1} \join \sigma_{j-i+1} \) for \( j \in \{i, \ldots, i+m-1\} \), and \( \tilde{F}_j := F_{j-m+1} \) for \( j \in \{i+m, \ldots, r\} \). For \( m > 1 \), \( \Phi \) is a proper subflag of \( \tilde{\Phi} \), contradicting \( \Phi \)’s maximality. So we must have \( m = 1 \), and \( \Phi = \Phi_\Sigma \).

To show Claim 1, that \( F_1 = \{\omega_1\} \) for some extremal \( \omega_1 \), we use essentially the same argument: there is a frame \( [\eta_{F_1}] \) for \( F_1 \), nonempty because \( F_1 \not\subset \emptyset \), and if \(|F_1| \neq 1 \), then we can extend the flag by prefixing it with the nonempty sequence of subfaces \( [H_i := \bigvee_{k \in I} \{\eta_k\}]_{i \in \{1, \ldots, |F_1|-1\}} \), generated by the initial segments of that frame. Since the flag was maximal, the extension must be impossible, so

---

34 Were \( F' \cap G = 0 \) (i.e. \( \emptyset \)) then we’d have \( (F' \cap G) \cup F = F \), while orthomodularity says \( (F' \cap G) \cup F = G \).
Proposition 9.1 follows almost immediately.

Proof of Proposition 9.1. Let $\Phi_1 = \{F_1, \ldots, F_r\}$ and $\Phi_2 = \{G_1, \ldots, G_r\}$ be two maximal flags of $\Omega$. Then by Lemma 9.2 the faces of $\Phi_1$, resp. $\Phi_2$, are the sequences of faces generated by the initial segments of the maximal frames $\omega_1, \ldots, \omega_r$, $\eta_1, \ldots, \eta_r$ respectively, defined by the bijection (5). By strong symmetry, there exists $g \in \text{Aut} \Omega$ such that for all $i \in \{1, \ldots, r\}$, $g \omega_i = \eta_i$. It follows that $g \Phi_1 = \Phi_2$.

Next we determine the implications of the conjunction of strong symmetry and spectrality for the polytope $\pi(\Omega)$ that exists because of regularity.

Proposition 9.3. Let $\Omega$ be a strongly symmetric spectral convex compact set of rank $r$. The polytope $\pi(\Omega)$ is a simplex with $r$ vertices, which constitute a maximal frame.

Proof. Let the dimension of $\Omega$ be $n$; without loss of generality we view $\Omega$ as canonically embedded in $\mathbb{E}^n$, i.e. assume that the barycenter of $\Omega$ is $0 \in \text{Aff} \Omega \simeq \mathbb{E}^n$, viewing $\text{Aff} \Omega$ as a vector space as described earlier, and introduce an invariant inner product. Picking a maximal flag (it does not matter which one), $F_1, \ldots, F_r$, and letting $\omega_1, \ldots, \omega_r$ be the maximal frame corresponding to it via the bijection (5), we consider the simplex $\Delta'$ of Farran and Robertson (defined following Theorem 7.10 above), which, using the description of the barycenters of faces from Proposition 3.6 (5), is equal to $\Delta(\omega_1, (\omega_1 + \omega_2)/2, \ldots, (\omega_1 + \cdots \omega_{r-1})/(r-1))$. We next identify the linear subspace $L$ of $\text{Aff}(\Omega)$ generated by this simplex. Since the barycenters just listed are manifestly linearly independent in $\text{Aff}(\Omega)$ (as must be any set of barycenters of faces of a flag), $L$ is $(r-1)$-dimensional (as also noted following Theorem 7.10). It is easy to see that $\omega_1, \ldots, \omega_{r-1}$ are a (linear) basis for the space $L$.

So far in this proof, we have been working in the setting where $\text{Aff} \Omega$ is viewed as a Euclidean vector space $E$ with $c(\Omega)$ as its zero, and inner product chosen so that $\text{Aff} \Omega$ is a subgroup of the orthogonal group. It is now useful to go to the setting, described in Section 2 and used extensively in Proposition 3.6 in which this Euclidean vector space is embedded as an affine subspace in the vector space $V$ of dimension one greater, in such a way that the embedded version of $E \simeq \text{Aff} \Omega$ does not contain the origin of $V$, and hence the cone $V_+ = \mathbb{R}_+ \Omega \subset V$ has affine dimension one more than $\Omega$’s. Of course the barycenter of $\Omega$, which was $0$ in $E \simeq \text{Aff} \Omega$, embeds as a nonzero element $c$ of $V$. As described just
before Proposition 2.4, we extend the action of \( SO(E) \) to an action of \( SO(E) \) on \( V \), which must fix the ray over \( c \) (pointwise), and equip \( V \) with a corresponding invariant inner product such that this action of \( SO(E) \simeq SO(n) \) is as a subgroup of \( SO(V) \simeq SO(n+1) \). By Proposition 3.6 this invariant inner product can be chosen to be self-dualizing for the cone \( V \) and such that all pure states have unit Euclidean norm, and we do so.

Now \( L \), which was a linear subspace in \( E \simeq \text{Aff} \Omega \), is embedded in \( V \) as an affine subspace but–since it is an affine subspace of \( \text{Aff} \Omega \)–not a linear subspace. \( L \) was the linear space spanned by \( c_1, \ldots, c_{r-1}, c_r \); as an affine space, it is generated by \( c_1, \ldots, c_{r-1}, c_r \), where \( c_r = c(1) \).

The simplex \( \pi(\Omega) = L \cap \Omega \) is therefore embedded as the intersection of the affine subspace \( L \subset V \) with \( \Omega \). \( L \subset \text{Aff} \Omega \) is also affinely generated by \( \omega_1, \ldots, \omega_r \in \text{Aff} \Omega \). When viewed as vectors in \( V \) (the vector space of dimension one greater than \( \text{Aff} \Omega \)), \( \omega_1, \ldots, \omega_r \) are orthonormal. Defining \( \tilde{L} \) as the linear span of \( \omega_1, \ldots, \omega_r \) in \( V \), we have that \( L = \tilde{L} \cap \text{Aff} \Omega \). From this it follows that \( \pi(\Omega) := L \cap \Omega \) is just the intersection of \( \Omega \) with the subspace \( \tilde{L} = \text{lin} \{ \omega_1, \ldots, \omega_r \} \subseteq V \).

We will show that this intersection \( \tilde{L} \cap \Omega \) is the simplex \( \Delta(\omega_1, \ldots, \omega_r) \). Recall (Proposition 3.6 item [4]) that the extremal points \( \omega_1, \ldots, \omega_r \) are orthonormal in \( V \), and are therefore an orthonormal basis for their span \( \tilde{L} \). By the self-duality of \( V_+ \), \( \omega_1, \ldots, \omega_r \) are also extremal rays of the dual cone with respect to the inner product. So everything in \( V_+ \) has nonnegative inner product with each of \( \omega_1, \ldots, \omega_r \). These constraints impose in particular that \( \tilde{L} \cap V_+ \) lies in the closed positive halfspaces \( H_i^+ := \{ x \in \tilde{L} : (\omega_i, x) \geq 0 \}, i \in \{1, \ldots, r\} \) of each the hyperplanes \( H_i = \{ x \in \tilde{L} : (\omega_i, x) = 0 \} \) in \( \tilde{L} \). These constraints define a polyhedral cone which (using the mutual orthogonality of the \( \omega_i \)) is identical to the cone over the simplex \( \Delta(\omega_1, \ldots, \omega_r) \). Since \( \omega_1, \ldots, \omega_r \) are in \( V_+ \) and in \( L \), we know that \( L \cap V_+ \) contains this cone, and since we have just shown that \( \tilde{L} \cap V_+ \) is contained in this cone, we have that \( \tilde{L} \cap V_+ \) is equal to it, and hence that \( \pi(\Omega) := \tilde{L} \cap \Omega = \Delta(\omega_1, \ldots, \omega_r) \).

Since the states \( \omega_1, \ldots, \omega_r \) are the vertices of the Farren-Robertson polytope of \( \Omega \), the faces \( F_1, \ldots, F_r \) of the polytope defined by the formula (5) are a maximal flag of that polytope. Then by Proposition 7.14 the faces \( H_{F_i} \) of \( \Omega \) are also a maximal flag of \( \Omega \). Since \( H_{F_i} \) is \( G^{(F_i)}, F, c(F_i) \) is the centroid of \( H_{F_i} \). Since

\[35\] This was equivalent to its being linearly generated by \( c_1, \ldots, c_{r-1} \) when we had identified \( c_r \) with \( 0 \in E \).
\[ c(F_i) = \frac{1}{i} \sum_{j=1}^{i} \omega_j, \quad H_{F_i} \text{ is the face of } \Omega \text{ generated by } \omega_1, \ldots, \omega_i, \text{ i.e.} \]

\[ H_{F_i} = \bigvee_{i=1}^{i} \omega_i, \, i \in \{1, \ldots, r\}. \quad (6) \]

So by Lemma 9.2, \( \omega_1, \ldots, \omega_i \) are a frame in \( \Omega \), not merely in \( \pi(\Omega) \), and \( \omega_1, \ldots, \omega_r \) is a maximal frame.

We now have results sufficient to prove Theorem 1.1, which we reiterate here.

**Theorem 1.1.** A convex compact set is strongly symmetric and spectral if and only if it is a simplex or affinely isomorphic to the space of normalized states of a simple Euclidean Jordan algebra.

**Proof.** The “if” direction, that Euclidean Jordan algebra state spaces and simplices are strongly symmetric and spectral, was already known [5] based on known results about Euclidean Jordan algebras; an explicit proof was reviewed in Sections 4 and 5.

By Propositions 9.1 and 9.3, the strongly symmetric spectral convex compact sets are all to be found among the regular convex compact sets whose Farrar-Robertson polytope is a simplex. Since a regular polytope is its own Farrar-Robertson polytope, the only polytopes that are strongly symmetric and spectral are the simplices, of every dimension. To identify the nonpolytopal strongly symmetric spectral compact convex sets we will use the Madden-Robertson classification of nonpolytopal regular convex bodies \( B \) in \( \mathbb{E}^n \), which is given in Tables 2, 3 and 4 (Tables 2, 3 and 4 of [8]). Because each regular compact convex set admits a canonical embedding as a regular convex body in \( \mathbb{E}^n \), with symmetry group equal to its affine automorphism group (Prop. 7.2) all regular compact convex sets appear in the table. Conversely all table entries correspond to regular convex compact sets, because all regular convex bodies, considered as convex compact sets, i.e. forgetting about the Euclidean and vector space structure of \( \mathbb{E}^n \), are regular (see Proposition 7.3).

All possibilities for nonpolytopal strongly symmetric spectral convex bodies are found among the entries of Tables 2, 3 and 4; they are the cases whose Farrar-Robertson polytope is a simplex. For each such case, Theorem 8.12 tells us, the given data suffices to determine, up to isomorphism, the regular convex body as the convex hull of the orbit of a (positive scalar multiple of a) fundamental weight in \( \alpha \), corresponding to one of the ends of the Coxeter diagram associated with the Weyl group. The entries with simplicial polytope are those for which the polytope
is explicitly indicated to be a simplex, and the cases \( q = 1 \) of families where the polytope is \( \square_q \) or \( \Diamond_q \), since \( \square_1 = \Diamond_1 = \triangle_1 \), the closed line segment, which is the unique 1-dimensional polytope.

Theorem 6.1, which states that strongly symmetric spectral convex bodies whose maximal frames have size two must be balls, establishes that the \( \square_1 \) and \( \Diamond_1 \) cases in the tables in [8] (as well as the \( \triangle_1 \) cases) all have \( \Omega \) a ball of some dimension; these are Jordan-algebraic normalized state spaces (for the spin factors).[^36]

In each of the symmetric space representations in these tables, the regular convex body is determined, up to isomorphism, by the regular polytope \( \pi(B) \) and the symmetric space. A comparison of the cases with simplicial polytopes \( \triangle_n \) for \( n \geq 2 \) with our Table 1 (taken from [19]) shows that they all correspond to polar representations \( K \lhd p_0 \) of compact groups \( K \) coming from simple Euclidean Jordan algebras \( V \simeq p = p_0 \oplus \mathbb{R}e \), where \( e \) is the Jordan unit. These are indicated in the rightmost columns of Tables 2, 3 and 4. By Proposition 9.3 in the strongly symmetric spectral cases, \( \pi(B) \) is a simplex, and as embedded in \( V \), its vertices are a maximal frame. The maximal abelian subspaces of \( V \simeq p \) (viewed as subspaces of \( g = \text{Lie}(\text{Aut} V) \)) are of the form \( a_0 \oplus \mathbb{R}e \), where \( a_0 \) are the maximal abelian subspaces of \( p_0 \). In, for example, [19], Proposition VI.3.3, and the discussion preceding it, the Peirce decomposition \( \bigoplus_{i,j=1}^r V_{ij} \) of a simple EJA of rank \( r \) is used, and it is observed that \( a = \bigoplus_i V_{ii} \) where \( V_{ii} = \mathbb{R}c_i \), and the \( c_i, \ i \in \{1, \ldots, r\} \) are a Jordan frame. With \( \Omega \) defined as usual as the normalized state space embedded in \( V = p \), with the Jordan unit \( e \) as order unit, we have that \( a \cap \Omega \) is the simplex generated by the \( c_i \), which is affinely isomorphic to \( a_0 \cap \Omega_0 \), where \( \Omega_0 := \Omega - e/\text{tr} e = \Omega - e/r \) is the translation of the normalized Jordan state space into \( a_0 \). This exhibits the strongly symmetric compact convex set \( \Omega \) as affinely isomorphic to the \( K \)-orbit \( \Omega_0 \) of a Farran-Robertson polytope \( \Omega_0 \cap a_0 \), which is a simplex. Because according to the Madden-Robertson classification the Farran-Robertson polytope determines, up to isomorphism, the regular convex body obtained as its \( G \)-orbit in a symmetric space representation, and all regular convex bodies (up to isomorphism) are obtained in this way, all strongly symmetric convex bodies are isomorphic to Jordan state spaces. □

[^36]: Appendix C includes a case-by-case examination of the \( \square_1 \) and \( \Diamond_1 \) cases, but this is not necessary for the proof. It includes the cases in which the symmetric space is that associated with a spin factor (noted in the "EJA" column in Tables 2, 3, and 4), as well as the infinite series of symmetric spaces of type AIII and CII which illustrate the fact that balls \( B_d \) may have actions by affine automorphisms, transitive on the sphere \( \partial B_d \simeq S_{d-1} \), of proper subgroups of the obvious rotation group \( SO(d) \).
Remark 9.4. The reader may wonder how the irreducible Riemannian symmetric spaces $H/K$ relate to the cones of squares in simple EJAs associated with the polar representation, since the interiors of these cones are precisely the irreducible symmetric cones, meaning the irreducible open cones $V^+ / K$ that are noncompact Riemannian symmetric spaces $\text{Aut}_0(V^+)/K$ (where $K$ is a maximal compact connected subgroup) on which $\text{Aut}(V^+)$ acts via isometries. Writing $V_+ := V^+$, note that $\text{Aut} V^+ / \text{Aut} V_+$ has in general a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with respect to which $\mathfrak{p}$ can be identified with the Jordan algebra $V$, and then $V_+ = \exp \mathfrak{p}$ (cf. e.g. [19]). However, for simple Jordan algebras also $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathbb{R}$, where $\mathbb{R}$ generates the uniform dilations and the semisimple part $\mathfrak{g}_{ss}$ is simple, and when we further break down $\mathfrak{g}_{ss} = \mathfrak{t} \oplus \mathfrak{p}_0$, we see that $\mathfrak{p}_0$ is the traceless part of $\mathfrak{p}$, which goes under $\exp$ to the component of the unit-determinant part of $\mathfrak{p}$ that lies in $V_+$, which is in fact a symmetric space $H/K$ for $H \equiv G_{ss}$, the semisimple part of $\text{Aut} V_+$. The symmetric cone is in fact a reducible symmetric space, the product of $H/K$ with $R$. The $R$ factor gives the flat direction in the tangent space of $V_+$, i.e. the direction along a ray from the origin. The interior of $V_+$ is the cone over the irreducible symmetric space associated with the representation. A nice example is the Lorentz cone, in which $H/K$ is a paraboloid inside the cone, with focus on the axis of rotational symmetry, and is in fact an orbit of the (connected, also known as “orthochronous”) Lorentz group; the full cone is obtained by including dilations. When $V$ is not simple, everything still works roughly as before except that now the symmetric space is a product $\times_{i=1}^k (H_i/K_i \times \mathbb{R}) \simeq (\times_{i=1}^k H_i/K_i) \times \mathbb{R}^k$ of irreducible symmetric spaces, with $\mathbb{R}^k$ reflecting the possibility of independent dilations of the cones over the simple factors $H_i/K_i$. 

10 Discussion

We have characterized a particular subset of the finite-dimensional Jordan algebraic state spaces—those corresponding to simple Euclidean Jordan algebras, and to finite products of the one-dimensional Euclidean Jordan algebra—as those convex sets satisfying the properties of spectrality and strong symmetry. In this sec-

---

$^{37}$with pointed closure, i.e. proper cones rather than wedges.

$^{38}$We remind the reader that trace and determinant are defined for EJAs in general (cf. [19]), and coincide with the usual matrix notions in the matrix cases $\text{Herm}(n, \mathbb{C})$.

$^{39}$Concretely, the second factor is represented as $\mathbb{R}_+$, but it is equipped with the multiplicative group structure, isomorphic (via the exponential map) to the additive group structure on $\mathbb{R}$. 

---
tion, we consider extensions and implications of this result. First we discuss various known ways in which the two properties can be supplemented with additional ones to characterize precisely the state spaces of complex quantum theory. Then we discuss some other characterizations of this class, or more general classes, of Jordan-algebraic state spaces, and how these relate to our work. Finally we review work in the setting of general probabilistic theories that derives strong consequences from the conjunction of spectrality and strong symmetry, or from sets of postulates that can be shown to imply these two properties, emphasizing the new light our result throws on this work.

10.1 From Jordan algebra state spaces to complex quantum theory

Characterizations of quantum state space, whether in terms of postulates whose appeal is mathematical, physical, informational, or some combination of these, often proceed by first characterizing Jordan-algebraic state spaces, or some subset thereof, and then adding an additional postulate or set of postulates that narrows things down to standard, i.e. complex, quantum theory. In the following two subsections we describe two important classes of such postulates. These can, of course, be used in conjunction with Theorem 1.11 to characterize the irreducible complex quantum systems.

10.1.1 From Jordan state spaces to complex quantum theory via relations between continuous symmetries and observables

The important characterizations by Alfsen and Shultz ([40], cf. [1]) of the state spaces of two natural classes of Euclidean Jordan algebras, the JB-algebras and the JBW-algebras, which are Jordan analogues of the $C^*$-algebras and the von Neumann algebras, were extended by them to characterize the state spaces of $C^*$-algebras and von Neumann algebras (each of which reduces to direct sums of the state spaces of standard complex quantum theory, in the finite-dimensional setting), in several ways. One of these, by Proposition 10.27 of [1], is to postulate the existence of a “dynamical correspondence” ([1], Definition 6.10), on the JB-algebra $A$. The dynamical correspondence determines a unique $C^*$ product on $A + iA$. In the special case in which the JB-algebra is a JBW-algebra, the dynamical correspondences on $A$ are in bijection with the Connes orientations (Theorem 6.18 of [1]); the existence of either of these determines a unique $W^*$ product on $A + iA$,
and the normal state space of $A$ is isomorphic to the normal state space of this von Neumann algebra.

A dynamical correspondence $\psi$ on a JB-algebra $A$ is a linear map, $\psi : a \mapsto \psi_a$, of $A$ into the set of skew order-derivations of $A$, satisfying certain properties. It is called complete if it is surjective (note, also, that there is no requirement that it be injective). The order-derivations are the elements of the Lie algebra $\text{aut} V_+$ of the group of affine automorphisms of the positive cone of a Jordan algebra. This is the span of two complementary subspaces, the self-adjoint (also called symmetric) and skew-adjoint (also called skew) order derivations, which in the finite-dimensional case are the $-1$ and $+1$ eigenspaces, respectively, usually denoted $\mathfrak{p}$ and $\mathfrak{k}$, of the Cartan involution on $\text{aut} V_+$. The self-adjoint ones may be identified with the space of Jordan multiplication operators, $L_a : b \mapsto a \cdot b$. The skew order-derivations are precisely the generators of one-parameter groups of automorphisms of the Jordan algebra (cf. Lemma 2.81 of [1]). The Jordan automorphisms are also precisely the Jordan unit preserving automorphisms of the cone of unnormalized states, and hence in a manifestly self-dual representation in our finite-dimensional setting, they are also precisely the (linearized extensions of) affine automorphisms of $A$’s normalized state space (cf. [19]). The conditions defining a dynamical correspondence are (1) that the commutator of the images of Jordan algebra elements $a$ and $b$ is the negative of the commutator of the corresponding Jordan multiplication operators, that is, that $[\psi_a, \psi_b] = -[L_a, L_b]$, and (2) that the image of an element annihilate that element, $\psi_a a = 0$.

The first of these conditions requires Jordan structure for its formulation. As Alfsen and Shultz note, their notion of dynamical correspondence “axiomatizes the transition $h \mapsto L_\psi h$ from the self-adjoint part of a $C^\ast$-algebra to the set of skew order-derivations on the algebra.” The appearance of the minus sign in the commutator when one moves between commutators of Jordan algebra multiplication operators (which as noted above are the selfadjoint part of $\text{aut} V_+$) and the Lie bracket of generators of reversible transformations (the skew-adjoint part of $\text{aut} V_+$) reflects the close relation of this transition to the existence of a complex structure on $\text{aut} V_+$. Such a complex structure, compatible with Lie brackets and the Cartan involution, is what Alfsen and Shultz ([1], Definition 6.8) call a Connes orientation on a JB-algebra.

The second condition can be rephrased as saying that the one-parameter dynamical group generated by the image $\psi_a$ of an observable $a$ conserves that observable, so it can easily be reformulated, in our setting, in a way that refers only to convex, and not to Jordan, structure, using the abovementioned identification of the Jordan automorphisms with the order-unit-preserving affine order-
automorphisms of the cone over the effects.

In finite dimension, Alfsen and Shultz’ results (Theorem 6.15 or Proposition 10.27 of [1]) combined with the main result of the present paper imply that that the conjunction of spectrality, strong symmetry, and the existence of a dynamical correspondence, characterizes complex quantum theory and classical theory, that is, the normalized state spaces of Jordan algebras corresponding to the Hermitian parts of full matrix algebras over \( \mathbb{C} \), and simplices.

**Proposition 10.1.** Let \( \Omega \) be a finite dimensional convex compact set satisfying (1) spectrality, (2) strong symmetry, and (3) dynamical correspondence, or equivalently the existence of a Connes orientation. Then \( \Omega \) is affinely isomorphic to the normalized state space of the Jordan algebra \( \text{Herm}(n, \mathbb{C}) \), i.e. the set of density matrices of a finite-dimensional quantum system, or to a simplex, i.e. the state space of a finite-dimensional classical system.

Another assumption is then needed to rule out classical theory (i.e., the simplices). Many natural alternatives are known, and among these we mention: existence of a tradeoff between information gained about an unknown state, and disturbance to that state (a result reported in [41]); impossibility of universal cloning, or of universal broadcasting [42, 43], the existence of a state having two different convex decompositions into pure states (a more or less folkloric mathematical fact that is the finite-dimensional case of Choquet’s theorem); the lack of universal compatibility of measurements [44]; nontriviality of the connected identity component of the automorphism group of the normalized states (emphasized by Hardy [45, 46]).

The authors of [5] narrowed down the class of Jordan algebras characterized there to the complex quantum state spaces, using a principle they call *energy observability*.

**Definition 10.2.** A normalized state space \( \Omega \) is said to have *energy observability* ([5], Def. 30) if the Lie algebra \( \text{aut} \Omega \) of \( \text{Aut} \Omega \) is nontrivial and there exists an injective linear map \( \varphi \) from \( \text{aut} \Omega \) to the observable space \( V^* \) of the system, such that for each \( x \in \text{aut} \Omega \), \( \varphi(x) \) is conserved by the one-parameter subgroup generated by \( x \), and \( \varphi(x) = \lambda u \) (for some \( \lambda \in \mathbb{R} \)) if and only if \( x = 0 \).

Energy observability is closely related to dynamical correspondence, but it is formulated in the convex framework without reference to Jordan structure, incorporates nontriviality of the connected automorphism group of \( \Omega \), and differs
from the existence of a dynamical correspondence in other ways. The terminology is motivated by the idea that a continuous one-parameter subgroup of automorphisms is a potential dynamical time-evolution, and in quantum physics the generator of such an evolution is a Hermitian operator $H$ (the Hamiltonian) conserved by the evolution (identified with energy). Here “generator” is meant in the “physicists” sense that the evolution operator is $\omega \mapsto e^{iHt}$ (where $i = \sqrt{-1}$). The assumption that $\text{aut} \, \Omega$ is nontrivial is there because without it there is nothing that fits the intuitive notion of energy that inspired the definition. (And if we were to formally extend the above definition to that situation, energy would, logically speaking, be observable because anything is true of the empty set.) However, it should also be noted that this nontriviality assumption is doing the work of ruling out classical theory.

**Proposition 10.3.** Let $\Omega$ be a finite dimensional convex compact set satisfying (1) spectrality, (2) strong symmetry, and (3) energy observability. Then $\Omega$ is affinely isomorphic to the normalized state space of the Jordan algebra $\text{Herm}(n, \mathbb{C})$, i.e. the set of density matrices of a finite-dimensional quantum system.

The relation of energy observability to dynamical correspondence is, roughly, that a dynamical correspondence is a linear map (but not necessarily an injection) of observables into the Lie algebra, i.e. in the opposite direction from the map required by energy observability, and also that dynamical correspondence imposes some conditions of compatibility with the Jordan structure, while the notion of energy observability uses only the convex structure of the state space and does not need Jordan structure for its definition. The conservation conditions are essentially the same for the two notions (modulo the reversal of direction of the map). The possibility of noninjectivity of dynamical correspondences is needed in order to allow some non-simple Jordan algebras to have dynamical correspondences: for example, a finite product of one-dimensional Jordan algebras—which corresponds to a finite-dimensional classical system, has trivial (zero-dimensional) $\text{aut} \, \Omega$, but may still have a dynamical correspondence, because all observables can map to the unique element 0 of $\text{aut} \, \Omega$; more generally, for a product of nontrivial Jordan algebras, observables that are linear combinations of the Jordan units of the simple factors will map to zero.

---

40 In the usual mathematical terminology, the generator of this evolution is instead the anti-Hermitian operator $iH$; then the injection from the Lie algebra of generators of one-parameter subgroups of automorphisms (i.e. $\text{lie}(\text{Aut}(\Omega)) \equiv \text{aut} \, \Omega$) into the observables is just $X \mapsto -iX$.

41 In [5] (on p. 29) dynamical correspondences are mistakenly described as injections; they are injective in the simple case, which is the one under consideration there, but not in general.
We have already mentioned the close relation of the existence of a dynamical correspondence to Connes’ condition of the existence of an orientation. In [47], Connes defined an orientation in the setting of self-dual cones $V_+$ in (not necessarily finite-dimensional) Hilbert spaces. The Lie algebra of the automorphism group of such a cone is involutive, and an orientation on such a cone was defined as a complex structure on the Lie algebra of $\text{aut } V_+$ compatible with this involution. Connes showed that the existence of an orientation characterizes the positive cones of von Neumann algebras within the class of self-dual, facially homogeneous cones in Hilbert spaces ([47], Théorème 5.2). Since Bellissard and Iochum [48] showed that these are precisely the positive cones of JBW algebras, this provides a way of characterizing the von Neumann algebra state spaces within the class of JBW algebraic ones. Alfsen and Shultz explicitly transferred the definition of Connes orientation to JBW-algebraic state spaces and established, in Theorem 6.18 of [1], a bijection between Connes orientations in the sense of their Definition 6.8 ([1]) and dynamical correspondences in the JBW-algebraic case.

10.1.2 From Jordan state spaces to complex quantum theory via local tomography

A different approach to ruling out the Jordan algebraic systems other than complex quantum theory involves introducing an appropriate notion of composite system consisting of two or more “subsystems”. The existence of “tomographically local” Jordan-algebraic composites of Jordan-algebraic systems can then be used as a postulate to narrow things down to complex quantum systems. Tomographic locality can be mathematically formulated as the requirement that the ambient vector space $V_{AB}$ spanned by the cone of unnormalized states of a composite of systems $A, B$ whose ambient vector spaces are $V_A$ and $V_B$, be the real tensor product $V_A \otimes V_B$. In [24] it was shown that the existence of a locally tomographic Jordan-algebraic composite of a Jordan-algebraic system $A$ with a qubit (the lowest-dimensional nontrivial complex quantum system, whose state space is a three-dimensional ball), satisfying some other natural desiderata, implies that $A$ must be complex quantum. However, this does not rule out the possibility of

---

42The notion of “tomography” in this context is that of determining the state of a system by making various measurements on identically prepared copies of a system. “Local” tomography of a composite system is possible if one can estimate the state by making measurements on its parts, $A$ and $B$, and estimating the correlations between sufficiently many measurement results. If $V_{AB} = V_A \otimes V_B$, then products of effects $e_A \otimes f_B$ span the dual of the state space, so determining the probabilities of a spanning set allows one to determine the components of the state in a basis.
theories whose systems are spectral and strongly symmetric but in which qubits do not occur as a system type that must be composable with other systems.

In [49], L. Masanes and M. Müller formulated five postulates applicable to theories whose systems are described in the GPT framework, and showed that the only two theories satisfying them are finite-dimensional complex quantum theory and finite-dimensional classical theory. One of these postulates is that composite systems are locally tomographic. Their notion of theory is somewhat implicit in their arguments, rather than fully explicit, but it appears to require that for any two systems of the theory, there exists another system of the theory that is a locally tomographic composite of those systems. Since the four postulates not referring to composite systems are satisfied by all systems with simple Jordan algebraic state spaces, and by systems whose state spaces are simplices, we can combine their result with the main result of this paper to conclude that any collection of (finite-dimensional) systems satisfying strong symmetry and spectrality, and closed under the formation of composites, must consist either entirely of complex quantum systems, or entirely of classical systems.

That the tomographic locality of the assumed composites is necessary for these results is indicated, for example, by the constructions in [50] of theories in which some of the Jordan algebraic systems other than complex quantum ones can be combined to form composites that are not tomographically local; these theories even have the additional structure of dagger compact closed categories (the terminology of [51] for a notion earlier defined in [52]). In addition to the category of irreducible complex quantum systems, there is the category of real quantum systems, and another category that includes all irreducible real and quaternionic quantum systems. A third additional category allows the inclusion of real, quaternionic, and complex systems in a single category, at the price of a notion of composite that does not preserve irreducibility. Not all Jordan algebraic systems occur in these constructions, though: the spin factors whose state spaces are $d$-balls for $d \notin \{1, 2, 3, 5\}$ do not occur in these categories, nor does the exceptional Jordan-algebraic system.\footnote{The $d$-balls with $d \in \{1, 2, 3, 5\}$ are the classical, real quantum, complex quantum, and quaternionic quantum bits, respectively.} The latter, indeed, does not have Jordan-algebraic composites, tomographically local or not, with nonclassical Jordan-algebraic systems ([50], Corollary 4.10), and Example 6.2 in [50] suggests that there may be obstructions to Jordan-algebraic composites involving the spin factors whose state spaces are the $d$-balls for $d = 4$ and $d \geq 6$ as well. (References to [50] are to arXiv version v2.)
A similar argument involving tomographic locality can be made using the result of [53], in which it is shown that only for $d = 3$ does there exist a composite, satisfying tomographic locality and continuous reversible transitivity on pure states, of two systems each of which has a Euclidean $d$-ball as state space. Since by Theorem 1.1 (cf. Theorem 6.1), spectrality and strong symmetry imply that bits are balls, and also that nonclassical systems are simple Jordan-algebraic and hence have continuous reversible transitivity on pure states, it follows from Theorem 1.1 and [53] that no tomographically local composite of a bit with itself can preserve spectrality and strong symmetry, except in the complex quantum case (for which bits are 3-balls) or the classical case.

10.2 Relations with other characterizations of Jordan algebraic classes of state spaces

Jordan algebraic systems share with standard (i.e. complex) quantum theory many geometric properties of informational and physical significance in addition to spectrality and strong symmetry. These include the absence of higher-order interference [54, 55]; purity-preserving projectivity [56, 15]; homogeneity and self-duality. Obviously, Theorem 1.1 implies that systems with spectrality and strong symmetry have these other properties too. This raises the question of whether the theorem could be proved differently, by establishing a direct implication from spectrality and strong symmetry to combinations of these other properties sufficient to establish the Jordan algebraic nature of the state space. For example, as noted in Proposition 3.6 it is known [5] that spectrality and strong symmetry imply self-duality of the cone over the normalized state space. In light of the Koecher-Vinberg theorem [22, 23], which states that the homogeneous self-dual cones are precisely the Jordan-algebraic ones, one could therefore try to show directly that spectrality and strong symmetry imply homogeneity. Similarly, it is known [5] that the conjunction of spectrality and strong symmetry implies projectivity of the state space in the sense that each face is the positive part of the image of a projection that is the dual of (and hence, given self-duality of the cone, identical to) what Alfsen and Shultz call a compression. Since self-duality near-trivially implies Alfsen and Shultz’ postulate of symmetry of transition probabilities (STP), and the finite-dimensional case of their theorem characterizing the state spaces of a wide class of Euclidean Jordan algebras has projectivity, symmetry of transition probabilities, and one of several other properties (equivalent in the context of projectivity and STP) as premises, we would just need to establish a direct im-
lication from spectrality and strong symmetry to one of these other properties to obtain a direct proof of the Jordan structure. The most promising choice is perhaps the preservation of purity by the duals of compressions: that the image of a pure state under a compression is always a multiple of a pure state. In fact, the result in [5] was obtained by showing that the duals of compressions preserve purity—but the additional assumption of absence of higher-order interference was used in showing this.

Alternative properties capable of characterizing the Jordan-algebraic state spaces among those satisfying Alfsen and Shultz’s conditions of projectivity and symmetry of transition probabilities are (1) the Hilbert ball property, or (2) the satisfaction of the atomic covering law by the lattice of faces of the state space. A finite-dimensional convex set has Alfsen and Shultz’ Hilbert ball property if and only if for every pair of extreme points of Ω, the face they generate is affinely isomorphic to a Euclidean ball. (See [1], Def. 9.9, for additional technical conditions relevant in infinite dimension.) The atomic covering law for a lower-bounded lattice states that if a is an atom in the lattice, and b any element of the lattice, then either $a \lor b = b$, or $a \lor b$ covers $b$. Here “x covers y” means $x > y$ and there exists no w such that $x > w > y$, i.e. x is above y and there is nothing between them, and an atom is an element that covers 0. So an alternative proof of our result could also aim at establishing either one of these properties directly. By the main result of [5] a direct proof of the absence of higher-order interference (see the next subsection for a rough definition) from spectrality and strong symmetry would also do the job.

10.3 Implications for other results on general probabilistic theories having spectrality and strong symmetry

10.3.1 Higher order interference

In [57] Rafael Sorkin introduced a notion of a hierarchy of orders or degrees of probabilistic interference, one for each positive integer, for physical theories modeled in a “histories” framework. In [58, 55, 59] Sorkin’s notion was adapted to the GPT framework. Roughly speaking, interference of order $k$ represents the idea that there are processes in which $k$ or more mutually exclusive “paths” or “histories” are available to the system, such that there is a measurement whose probabilities, when measured on a system that has undergone such a process, cannot be determined from the probabilities in all situations in which $k - 1$ or fewer of the paths are available. Quantum theory has interference of order at most 2 (the
lowest order that intuitively represents interference, since theories with maximal interference order of 1 are classical). If a theory fails to have interference of order \( k \), then it can have no interference of order \( k + 1 \) or higher. We call interference of order 3 or above higher-order interference. As mentioned in the introduction, Theorem 1.1 improves on the main result of [5], which characterized the same class of theories, but in addition to spectrality and strong symmetry, used the additional assumption of no higher-order interference.

10.3.2 Query computation and query complexity

In query problem in the theory of computation, it is assumed that one has access to a sequence, parametrized by “instance size” \( n \), of “black boxes” capable of computing a function \( f_n \) between finite sets (the usual case is \( f : \{0, 1\}^n \rightarrow \{0, 1\} \)). \( f_n \) is not known, but a finite family \( F_n \) of possible functions is specified; sometimes a prior distribution over \( F_n \) is specified as well. One may then ask about the “query complexity” of computing some function \( g_n \) whose domain is the family of functions \( F_n \); roughly speaking, this is the number of times the black box appears in a circuit capable of computing, with high probability of success (either in the worst case or, in case a prior distribution on \( F_n \) has been specified, in expectation with respect to that distribution), the function \( g_n \). The circuit is realized in some background circuit model of computation, such as a classical or quantum circuit model, and the behavior of the black boxes may also be classical, quantum, or more general, giving rise to a variety of possible query models of computation. Typically one is concerned with how the number of queries scales with the input size \( n \). For example, in Grover’s “search problem” of “identifying a marked state”, for which the instance size parameter is usually written as \( N \), \( F_N \), of cardinality \( N \), is the set of functions \( \{1, ..., N\} \rightarrow \{0, 1\} \) that take the value 1 on exactly one element of its domain \( \{1, ..., N\} \), usually termed the “marked state”. In Grover’s problem \( g_N : F_N \rightarrow \{1, ..., N\} \) is the function that identifies which of these \( N \) possibilities for \( f_N \) is computed by the black box, i.e. which state is marked. Grover’s celebrated quantum query algorithm [60, 61] computes \( g_N \) with a number of queries of order \( \sqrt{N} \). In the closely related query problem of computing OR, \( F_N \) consists of all \( N^2 \) functions \( \{1, ..., N\} \rightarrow \{0, 1\} \) and \( g_N \) is OR of the values of \( f_N \) on all its inputs, i.e. \( g_N(f_N) \) is 1 if \( f_N \) is zero on all \( N \) inputs, otherwise it is 1. A variant of Grover’s algorithm computes OR, again with a number of queries of order \( \sqrt{N} \).

In [6] it was shown, in a reasonable query model generalizing the quantum query model, that five principles, of which the fifth is strong symmetry, imply that to have probability 1/2 or greater of correctly identifying the marked
state in Grover’s search problem, the number of queries must be at least \((3/2 - \sqrt{2}) \sqrt{N}/h\), where \(h\) is the maximal “order of interference” of the GPT theory. A lower bound of \(\Omega(\sqrt{N})\) was established in the quantum case in \([9]\); it is achieved by Grover’s algorithm. The bound in \([6]\) is also \(\Omega(\sqrt{N})\). This limits the potential gain from higher-order interference of degree \(h\) to at most a constant factor, \(c/\sqrt{h}\) compared to quantum. The authors of \([6]\) argued that their result is “somewhat surprising as one might expect more interference to imply more computational power”.

However, it can be shown (cf. \([10]\)) that the conjunction of the principles used in \([6]\) implies spectrality. Together with Theorem \([11]\) this implies that the GPT systems considered in \([6]\) are Jordan-algebraic, and hence that the order \(h\) of interference is at most 2 \([55, 54]\). The question of whether or not higher-order interference of some fixed maximal degree can give a non-constant asymptotic speedup (in terms of number of queries) over Grover’s quantum algorithm for the Grover search problem in some GPT with a reasonable query model remains open, but our results imply that if such a speedup is possible it will be in a setting not allowed by the postulates in \([6]\).

Besides strong symmetry, the assumptions in \([6]\) are causality (roughly, no signaling from the future, which is implicit in the setting of this paper, and most work on GPTs, because of the uniqueness of the order unit \(u\)), purification (every state on a system \(A\) arises as the marginal of a pure state on a possibly larger composite system \(AB\), and any two such purifications are related by an automorphism of \(\Omega_B\)), purity preservation under composition, and the existence of a pure sharp effect. It is not explicitly stated whether one must assume purity preservation under both parallel and sequential composition, but it appears that only parallel composition is used in the proofs. Even with strong symmetry omitted, and purity preservation required only under parallel composition, the conjunction of these conditions is an extremely strong assumption \([44]\) since if they are augmented with purity preservation under pure operations \([45]\) they would already imply the Jordan-algebraic nature of the state space (though not the restriction to simple Jordan algebras or simplices), as shown in \([62]\). However, since to the best of our knowledge purity

\[\text{References}\]

\[44\] Most of the strength of this conjunction lies in purification, purity preservation under parallel composition, and the existence of a pure sharp effect, since causality is part of essentially every definition of composite system in the GPT framework. The existence of a pure sharp effect would follow from the other conditions and the no-restriction hypothesis, which, although also quite strong in its way, is a natural and oft-made assumption in the GPT framework.

\[45\] Pure operations are linear maps that lie in extremal rays of the cone of “allowed” positive maps on the state space.
preservation under sequential composition of pure operations was not previously known to follow from the other assumptions, the possibility that the assumptions of [6] still allow for higher-order interference was open until it was excluded by the main result of the present paper.

In [11], a definition of query computation was formulated and two results were obtained concerning query computation in general probabilistic theories under nearly the same assumptions as [6]: the ubiquitous (and innocuous) causality, purification, purity preservation under parallel composition, and strong symmetry, as well as preservation of the maximally mixed state (centroid of the state space) under parallel composition. Pure sharpness was not explicitly assumed, but the results of the paper involve situations in which nontrivial sets of perfectly distinguishable states exist, which are needed for the function queries to be possible, and the existence of such sets can be shown to imply pure sharpness; indeed, in [11] the cone of unnormalized states is shown to be not only self-dual, but perfect.

The first main result of [11] was that if \( kn \) classical queries yield no information concerning a function to be computed, then in a general probabilistic model with maximal order of interference \( k \), \( n \) queries yield no information. This allows one to obtain, from classical zero-information lower bounds on the number of queries needed to compute or approximate properties of a black-box function \( f \), lower bounds in more general models, but it neither rules out nor definitively establishes the possibility of speedups over quantum query computation in more general GPT models. It generalizes a result of Meyer and Pommersheim [63], who studied the quantum case. The second main result of [11] was a confirmation that the generalization (from the classical and quantum cases) of the notion of black-box query used there is reasonable, in the sense that if there is a polynomial-size family of GPT circuits \( C_f \) for a family of functions \( f \), one can use them to simulate the black-box queries to the functions \( f \), with a polynomial family of circuits. Much of the interest in query algorithms, both quantum and classical, implicitly relies on this type of result, since they allow one to pass with at most polynomial cost from query algorithms to concrete circuit algorithms in cases where circuits for the function \( f \) exist—giving rise to efficient algorithms in cases where the amount of resources required by the query algorithm for the computation interleaved between the queries is also polynomial.

The absence of an explicit assumption of sequential purity preservation is the only thing distinguishing the assumptions (excluding strong symmetry but including pure sharpness) of [11] from those of sharp theories with purification (which

---

46This last may well follow from the others.
are known to have Jordan-algebraic state spaces. But without sequential purity preservation, it was still not clear that the systems of [11] are Jordan algebraic. However, they do have spectrality so, as in the case of [6], Theorem 1.1 implies that these theories, too, have Jordan-algebraic state spaces, and cannot exhibit higher-order interference. This should motivate attempts to extend these results to more general settings.

10.3.3 Entropic and thermodynamic aspects of probabilistic theories

In [3], spectrality and strong symmetry were used to obtain important properties of quantum entropy and entropy-like quantities in a more general context. Given spectrality, it is natural to investigate real-valued entropy-like functions on the space of states defined using Schur-concave functions. For each such function $f$, one defines a corresponding generalized entropy as the value of $f$ on the spectrum of the state. The von Neumann entropy of a quantum state, which is given by the Shannon entropy $H(p) := -\sum p_i \ln p_i$ of its spectrum $p = \{p_1, \ldots, p_n\}$, is one such entropy. In general theories, one can define the measurement entropy and the preparation entropy of states. The measurement entropy of state $\sigma$ is the minimum, over fine-grained measurements, of the Shannon entropy of the probabilities of the outcomes when the measurement is made on a system in state $\sigma$; the preparation entropy is the minimum entropy of probabilities $p_i$ such that $\sigma = \sum p_i \omega_i$, for pure states $\omega_i$. Analogous definitions can also be made for the generalized entropies determined by Schur-concave functions other than Shannon entropy. In quantum theory, the preparation and measurement entropies corresponding to a given $f$ are equal to each other and to the spectral entropy corresponding to $f$. In [3], it was shown, assuming spectrality and strong symmetry, that the outcome probabilities of any fine-grained measurement on $\sigma$ are majorized by those of

---

47 A function $f : \mathbb{R}^n \to \mathbb{R}$ is called Schur-concave if whenever $x$ majorizes $y$, $f(y) \geq f(x)$. $x$ is said to majorize $y$, for $x, y \in \mathbb{R}^n$, if for all $m \in \{1, \ldots, n\}$, it holds that $\sum_{i=1}^m x_i \geq \sum_{i=1}^m y_i$, where $x^i, y^i$ are the vectors whose elements are those of $x$ and $y$ respectively, arranged in decreasing order. Sometimes Schur-concave functions are considered to have as domain $\bigcup_{n \in \mathbb{N}} \mathbb{R}^n$, in which case majorization is defined by comparing two vectors with the shorter one padded out with zeros to the length of the longer one. “$x$ majorizes $y$” is generally interpreted as a formalization of the idea that $x$ is “more mixed” or “more random” than $y$, because of the Birkhoff-von Neumann theorem, which states that $x$ majorizing $y$ is equivalent to $y$ being a convex combination of vectors obtained from $x$ by permuting its entries. Schur-concave functions are often viewed as real-valued “measures of randomness” since they are precisely the real-valued functions that can never decrease under such operations. This accounts for the terminology “generalized entropies” and also for a relation of majorization to microcanonical thermodynamics (cf. e.g. [64]).
the spectral measurement (which are equal to $\sigma$’s spectrum), and hence that the measurement entropy determined by any Schur-concave function is equal to the corresponding spectral entropy.\footnote{48} In parallel work in \cite{4} the same conclusion was obtained using causality, purification, purity preservation under both parallel and sequential composition of pure operations, and strong symmetry. The first four of these assumptions together imply spectrality. So in light of the present paper, the setting of \cite{3,4} is no more general than that of simple Jordan-algebraic state spaces, and classical ones.\footnote{49} However, the same conclusions can also be obtained from different postulates, including or implying spectrality, but not strong symmetry. This was done, using different sets of assumptions, in \cite{10,65,66}. In light of the present work, it becomes even more interesting to determine whether the assumptions used in these works imply the Jordan algebraic structure of state space (even if not the simple structure or classicality of the Jordan algebra, which is enforced by strong symmetry but which does not follow from the assumptions of \cite{65,10,66}). The results in \cite{10,66} concern a class of theories they call \textit{sharp theories with purification}, which satisfy the four properties of causality, purification, purity preservation (under both parallel and sequential composition), and pure sharpness. In \cite{62} it was shown that all systems in this class of theories are Jordan-algebraic (although this class is not precisely simple Jordan algebras and classical theory, since some nonclassical nonsimple state spaces are definitely allowed, and to the best of our knowledge it is not known whether all simple Jordan algebras are). However, the assumptions of \cite{65} are just projectivity of the state space and symmetry of transition probabilities (equivalently, projectivity and self-duality of the state cone, which are in turn equivalent (\cite{2}, cf. also \cite{65}) to its perfection together with the normalization of the orthogonal projections onto the linear spans of faces). All Jordan algebraic state spaces have these properties, but it is an open question whether they are the only ones. Essentially these assumptions (in the guise of projectivity and symmetry of transition probabilities) appear in Alfsen and Shultz’s derivation \cite{56,1} of Jordan-algebraic structure, but there a choice of one of several additional assumptions, for instance that the “filters” that project onto faces of the state space take pure states to multiples of pure states or one of the other alternatives discussed above, is used. It is not known whether or not this assumption can be dropped in the characterization\footnote{48}The theorem was stated for the Renyi $\alpha$-entropies, but the proof uses Schur concavity and applies to arbitrary Schur concave functions.\footnote{49}It should nevertheless be noted that to the best of our knowledge, the conclusions obtained in \cite{3} and \cite{4} were not previously known for the non-quantum, non-classical simple Jordan algebras.
of Jordan-algebraic state spaces. So, the results of [65] add additional interest to the question of whether there are non-Jordan-algebraic state spaces satisfying projectivity and self-duality, while the results of the present paper show that such state spaces will not be found among those satisfying strong symmetry.

11 Conclusion

We have shown that the finite-dimensional compact convex sets satisfying two properties, spectrality and strong symmetry, are up to affine isomorphism precisely the sets of normalized states of simple finite-dimensional Euclidean Jordan algebras, and simplices, answering a question posed in [5] of whether the additional property of no higher-order interference used there in addition to spectrality and strong symmetry, is needed to characterize this set of state spaces. While this can be viewed purely as a result in convex geometry, it has important implications for the research program that studies general probabilistic theories, since significant results in that program were obtained under the assumption that the normalized state spaces of systems are spectral and strongly symmetric, or under assumptions that imply this; we described some of these implications in the preceding section. We also discussed the relation of our result to other characterizations of Jordan algebraic state spaces.

These Jordan-algebraic compact convex sets, and the cones over them, figure in many other areas of mathematics and applications, including complex analysis, symmetric spaces, optimization, and statistics, and the present result may have interesting implications in some of these areas as well. From the both the perspective of general probabilistic theories and that of the purely mathematical theory of convex sets, our results suggest exploring the consequences of assumptions weaker than spectrality and strong symmetry.

In [2] H. Araki gave a derivation of Jordan-algebraic structure in finite dimension inspired by Alfsen and Shultz’s, but using a notion of projection on the state space he calls “filter”, which is prima facie weaker than the notion of dual of a compression; he also assumed filters preserve purity, but conjectured the assumption could be dropped. Alfsen and Shultz [1], p. 354, state that “in the finite-dimensional context his axioms force the filters to be compressions”, but it is not clear whether this is meant to apply to the axioms without the purity-preservation assumption.
Acknowledgments

HB thanks the Department of Mathematical Sciences and the QMATH group at the University of Copenhagen for hosting him as Visiting Professor while investigations preliminary to the present work were undertaken, and the Villum Foundation for making his visit possible through its support of QMATH. Both authors thank Henrik Schlichtkrull for putting them in contact with each other.

A Proof of Theorem 6.1

In this section we establish Theorem 6.1 following [12] but bringing in the stronger assumption of 2-transitivity.

Theorem 6.1 Let $\Omega$ be a strongly symmetric spectral compact convex body whose largest frame is of cardinality 2. Then $\Omega$ is affinely isomorphic to a ball.

Proof. We view $\Omega$ as canonically embedded in the affine space $\text{Aff} \, \Omega \simeq E$ that it generates, of dimension $n$, equipped with a Euclidean inner product for which $\text{Aut} \, \Omega$ is a subgroup of $O(E)$, and recall that because $\text{Aut} \, \Omega$ acts transitively on $\partial_e \Omega$, we have that $\int_{\text{Aut} \, \Omega} d\mu(k) \, k \cdot \omega$ is a fixed point of the group action, and is in fact the barycenter, 0. Because it is an orbit of a subgroup of $O(V)$, the set $\partial_e \Omega$ of extremal points of $\Omega$ is contained in the sphere $S := \{x \in V : ||x|| = c\}$ with respect to the associated norm. We scale the inner product by a positive real number so that $c = 1$.

We now prove that every pair of perfectly distinguishable points in $\Omega$ are the endpoints of some diameter of $S$. We begin by showing (following Dakić and Brukner but with a bit more detail) that for every extremal $\omega \in \Omega$, the point $-\omega$ also belongs to $\Omega$, and $[\omega, -\omega]$ is a 2-frame.

A chord of a sphere is defined to be a closed line segment whose endpoints are two distinct points on the sphere. We will use the fact that the only chords of a sphere that contain its center are the diameters, i.e. the chords from $x$ to $-x$. Since $\Omega$ is spectral with maximal frame size 2, every nonextremal point in $\Omega$, in particular its center, 0, is a convex combination of two perfectly distinguishable extremal points of $\Omega$. Let $\omega_0$ and $\omega_1$ be extremal points of $\Omega$ such that 0 is a convex combination of them. Since we showed above that all extremal points of $\Omega$ lie on the sphere $S$, the set of convex combinations of $\omega_0$ and $\omega_1$ is a chord of $S$ containing its center, 0. Therefore it is a diameter, and $\omega_1 = -\omega_0$. Since $\Omega$ has reversible transitivity on pure states (i.e. transitivity of $\text{Aut} \, \Omega$ on 1-frames, which
are precisely the extremal points), every extremal point $\omega$ of $\Omega$ can be obtained from $\omega_0$ by acting with an element of $O(V)$, whence by linearity of the action, $-\omega$ is also in $\Omega$. So we have established that $\Omega$ is symmetric under coordinate inversion $x \mapsto -x$, and that every pair $\omega, -\omega$ is a maximal frame.

We still need to show that there are no other maximal frames in $\Omega$, i.e. no 2-frames that are not the endpoints of a diameter. If we have transitivity on 2-frames, we get this immediately: every 2-frame is an automorphic image of $(\omega_0, -\omega_0)$, and therefore of the form $(\omega, -\omega)$ for some extremal $\omega$.

Once we have this fact, we can use it as in [12] to establish that $\Omega$ is a ball. The centroid of a compact convex set, which is 0 in the case of $\Omega$, is in its relative interior. $\Omega$ is full-dimensional, so its relative interior is its interior, and there is an open ball around 0 contained in $\Omega$. So for any $x \in S$ there is $\lambda \in (0, 1]$ small enough that $\lambda x \in \Omega$. By spectrality, $\lambda x$ is a convex combination of two perfectly distinguishable extremal points of $\Omega$. Since we showed above, using 2-transitivity, that all such pairs are endpoints of diameters, $\lambda x$ must be a convex combination of the endpoints of a diameter. For $x \in S$ the only diameter containing $\lambda x \neq 0$ is the one between $x$ and $-x$. So we have shown that $x \in \Omega$; but $x$ was an arbitrary element of $S$. Since the entire sphere $S$ belongs to the extreme boundary of the convex set $\Omega$, and we earlier showed that all extremal points of $\Omega$ are in $S$, $\Omega$ is the convex hull of the $(n-1)$-sphere $S_{n-1}$, i.e. an $n$-dimensional ball.

**B**  

**Faces of strongly symmetric spectral sets are strongly symmetric and spectral**

In this section we prove the following theorem, which includes item 5 of Proposition 3.6.

**Theorem B.1.** Let $\Omega$ be a strongly symmetric spectral compact convex set. Then every face of $\Omega$ is a strongly symmetric spectral compact convex set; moreover if $F$ is a face of $\Omega$ and $K = \text{Aut} \, \Omega$, then $K(F) := K_F / K^F = \text{Aut} \, F$. Here $K_F$ is the subgroup that takes $F$ to itself; $K^F$ is the subgroup that fixes $F$ pointwise. Also, $K_F := K^{c(F)}$, where $c(F) = \sum_{i=1}^{\lvert F \rvert} \omega_i / \lvert F \rvert$, for any frame $\omega_i$ for $F$, is the centroid of $F$.

---

51 This is the main point at which we perceive a gap in the argument in [12] which we do not see how to easily bridge using only reversible transitivity and strong symmetry.
Proof. By Proposition 2.4 and the fact that \( u = \sum_{i=1}^{r} \omega_i \) in a strongly symmetric compact convex set, where \( \omega_i \) are a maximal frame, and the fact that \( u \) is \( \text{Aut}\Omega \)-invariant in our setup, the barycenter of \( \Omega \) is easily shown to be \( (\sum_{i=1}^{r} \omega_i) / r \), for any maximal frame \([\omega_1, \ldots, \omega_r]\). It is easily shown that any face of \( \Omega \) is spectral, since spectrality of \( \Omega \) asserts, for \( \omega \in F \), that \( \omega \) is a convex combination of perfectly distinguishable states, but these states must be in \( F \) by the definition of face, and by Proposition 3.6 they must be extendable to a frame for \( F \). Then one shows, using strong symmetry, that for each face \( F \) of \( \Omega \), there is a subgroup of \( K := \text{Aut}\Omega \) that preserves \( \text{lin}\ F \) and (necessarily or else it could not consist of automorphisms of \( \Omega \)) induces automorphisms of \( F \), and acts transitively on the maximal frames in \( F \). This is immediate from strong symmetry, since the maximal frames in \( F \) are \(|F|\)-frames in \( \Omega \), and strong symmetry says \( K \) can take any \(|F|\)-frame (whether in \( F \) or not) to any other. One has to show that frames in \( F \) are still frames for \( F \) viewed in its affine span, but that is so because the cone is perfect, and when the dual cone is represented internally via the self-dualizing inner product, the distinguishing effects are the states themselves (cf. item 4 of Proposition 3.6). Perfection of the cone \( V_+ \) implies that the cone over \( F \) is self-dual in its linear span according to the restriction of the inner product, so these effects are still in the relative dual cone of \( F \).

In fact, an element of \( K \) takes maximal frames of \( F \) to maximal frames of \( F \) if, and only if, it belongs to the subgroup \( K_{\text{lin}} F \), that preserves \( \text{lin}\ F \). The action of this group on \( \text{lin}\ F \) gives a faithful representation of the group \( K_{\text{lin}} F / K_{\text{lin}} F \) (equivalently \( K_F / K^F \)). Since we have shown that \( F \) is spectral and strongly symmetric, it follows from claim in the first sentence of this proof that the barycenter of \( F \) is \( \sum_{i=1}^{|F|} \omega_i / |F| \) for any maximal frame \( \omega_i \) for \( F \). Any automorphism of \( F \) preserves its barycenter, so the automorphism of \( F \) induced by any element of \( K_F \) must do so, i.e. \( K_F \subseteq K^{c(F)} \). Furthermore, \( K^{c(F)} \subseteq K_F \). To see this, note that \( c(F) \) is in the relative interior of \( F \) and therefore \( F = \text{face}(c(F)) \). Hence for \( \phi \in K^{c(F)} \) we have \( \phi(F) = \phi(\text{face}(c(F))) = \text{face}\phi(c(F)) = \text{face}(c(F)) = F \), i.e. \( \phi \in K_F \). Here the second equality is a general fact about automorphisms \( \phi \) (that \( \text{face}(\phi(x)) = \phi(\text{face}(x)) \)), and the third is from the assumption \( \phi \in K^{c(F)} \).
C  More detail on the classification of regular polytopes via symmetric space representations

C.1 Details of the symmetric space representations containing regular convex bodies as orbits of Farran-Robertson polytopes

In this subsection we go line by line through those entries in Tables 2, 3 and 4 that describe symmetric spaces associated with simplicial Farran-Robertson polytopes, along the way verifying that all cases where the simplex has 3 or more vertices ($\triangle_n, n \geq 2$), are ones associated with EJAs, and for the $\triangle_1$ cases (where the convex body must be a ball), noting which ones are spin factor Jordan algebra automorphism representations (i.e. $\text{lie}(K) = \text{lie}(\text{Aut } V) \equiv \text{der} V$) and which involve other transitive actions on balls.

The nomenclature for groups, symmetric spaces, and root systems is that used in [8], which is very close to that used in Helgason [39], Ch. X.

Tables 2 and 3 in [8], reproduced in the first columns of Tables 2 and 3 above, include those symmetric spaces from Table V in Chapter X of Helgason [39]. Table 2 being the series, Table 3 the exceptional cases. These are the Type I and Type III noncompact symmetric spaces, in Cartan’s nomenclature. The list of coincidences between different classes of symmetric spaces given in §6.4 of Ch. X of [39] (pp. 519-520) is helpful here too; we refer to these below by Helgason’s enumeration, e.g. as “coincidence (i)”, etc.

From Table 2: classical symmetric space representations.

AI is the symmetric space $SL(n, \mathbb{R})/SO(n)$ with root space $A_{n−1}$, whose polytope is the $n$-vertex simplex, $\triangle_{n−1}$. $\text{lie}(K) = \mathfrak{so}(n)$, and the associated polar representation $V_0 = \mathfrak{p}_0$, from the Cartan decomposition $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}_0$, is the traceless real symmetric matrices. As discussed in Section 4, this is the polar representation $V_0$ associated with the Jordan algebra $V$ consisting of the real symmetric matrices $V_0 \oplus \mathbb{R}I$ (where $I$ is the identity matrix). The regular convex body is then isomorphic to the unit-trace real symmetric positive semidefinite matrices.

AII is $SU^*(2n)/Sp(n)$, of rank $n−1$, and dimension $(n−1)(2n+1)$, with polytope $\triangle_{n−1}$. $SU^*(2n)$ is also known as $SL(n, \mathbb{H})$, and $Sp(n)$ as $SU(n, \mathbb{H})$.

---

52 Here $V$ is a Jordan algebra, and $\text{Aut } V$ is the compact group of Jordan algebra automorphisms, with Lie algebra $\text{der} V$.

53 Helgason combines $BI$ and $DI$ in the class $BDI$, whereas [8] keep them separate.
The Cartan decomposition is $\mathfrak{sl}(n, \mathbb{H}) = \mathfrak{su}(n, \mathbb{H}) \oplus \mathfrak{p}_0$ where $\mathfrak{p}_0$ is the traceless quaternionic-Hermitian matrices, so the associated polar representation has $SU(n, \mathbb{H}) \cong Sp(n)$ acting on the traceless quaternionic-Hermitian matrices. The regular convex compact body of the Madden-Robertson construction is therefore affinely isomorphic to the unit-trace positive semidefinite quaternionic-Hermitian matrices.

Type AIII, $SU(p,q)/SU(p) \times SU(q)$, with root system $C_q$ (for $p = q$) or $BC_q$ (for $p > q$). Two polytopes are listed here because when $q > 1$ the two end nodes of the Dynkin diagrams of $C_q$ are non-equivalent, as are those of $BC_q$, giving rise to two orbit types in each case. At any rate, only in the case $q = 1$ are these simplicial polytopes, $\triangle_1$. The symmetric spaces are then $SU(p,1)/SU(p) \times SU(1)$, the dimension is $2p$, and we know that the associated orbitopes are balls $B_{2p}$ by Theorem 6.1. For $p > 1$, the root system is of type $BC_1$, while for $p = 1$ it is of type $C_1$.

The $p = 1, q = 1$ case corresponds to the spin factor $\mathbb{R}^2 \oplus \mathbb{R}$, with positive cone a Lorentz cone with 2 space dimensions, whose base is the disc, by coincidence (i) of Helgason. (Note that $SU(1) \times SU(1) \cong U(1)$, with Lie algebra $\mathfrak{so}(2)$.)

In the $p > 1$ case, $SU(p,1) \cong U(p)$ acts transitively on $B_{2p}$. In, for instance, [53] the $U(p)$ and $SU(p)$ representations with transitive action on $B_{2p}$ are described; for $Q \in U(p)$ or $SU(p)$, we have the $2p \times 2p$ real matrix

$$\rho(Q) = \begin{pmatrix} \text{Re } Q & \text{Im } Q \\ -\text{Im } Q & \text{Re } Q \end{pmatrix}.$$  \hfill (7)

For Type BI the simplicial cases are the $q = 1$ cases of $SO(p,q)/(SO(p) \times SO(q))$ for $p + q$ odd, $q < p$, of type BI, with root space $B_q$. So we have $p > 1$ even, and the symmetric space is $SO(p,1)/(SO(p) \times SO(1)) \equiv SO(p,1)/SO(p)$. The Cartan decomposition is $\mathfrak{so}(p,1) = \mathfrak{so}(p) \oplus \mathbb{R}^p$, and we have $\mathfrak{so}(p)$ acting irreducibly on $\mathbb{R}^p$. The action is equivalent to the Lie algebra representation derived from the defining representation of $SO(p)$. This is the polar representation embedded in the spin factor Jordan algebra $\mathbb{R}^p \oplus \mathbb{R}$, where the $\mathbb{R}$ summand is spanned by the Jordan identity $e$, which is fixed by the full $SO(n)$ representation as explained in Section 6.1. Its positive cone is a Lorentz cone in one “time” dimension, and even “space” dimension $p$, with normalized state space $\Omega$ a $p$-ball. If we write $(x,t)$ for

\[54\] $SU(p \times U_q)$ is defined by the (complex) linear representation on $\mathbb{C}^p \oplus \mathbb{C}^q$ consisting of block-diagonal matrices $M$ with blocks in $U(p)$ and $U(q)$ with $\det M = 1$, so $SU(p \times U_1)$ is isomorphic to the matrices of the linear representation $V \otimes \det^*$ of $U(p)$, where $V$ is the defining representation and $\det^*$ the dual of the one-dimensional determinant representation. This is a faithful representation of $U(p)$.\]
a an element of $\mathbb{R}^p \oplus \mathbb{R}$, the convex hull of the orbit in the polar representation is that ball, translated to the plane $t = 0$.

Type DI with the same group quotient, but $p + q$ even, similarly gives the Lorentz cones for odd “space” dimension in the $q = 1$ cases, again as embedded in spin factors.

The simplicial cases $\triangle_1$ in these two lines (BI and DI) account for all spin factor isotropy group representations. A few of these will reappear below, due to coincidences noted in [39].

DIII has $q = \lfloor n/2 \rfloor$, which gives $q = 1$, and hence $\triangle_1$, only for $n = 2$ and $n = 3$. For $n = 2$, the dimension $n(n - 1)$ is 2, and the symmetric space is $SO^*(4)/U(2)$. Coincidence (xi) in Helgason’s list indicates that this coincides with type AI ($n = 1$), i.e. real symmetric $2 \times 2$ matrices $\text{Herm}(2, \mathbb{R}) \simeq \mathbb{R}^2 \oplus \mathbb{R}$, aka the “rebit”, also appearing as AIII ($p = q = 1$), DI ($p = 2, q = 1$) (Helgason’s BDI($p = 2, q = 1$)), and CI ($n = 1$).

For $n = 3, q = 1$ we have $d = 6$. By coincidence (vii) in Helgason, this is also AIII ($p = 3, q = 1$). $\Omega$ is the 6-ball. The DIII symmetric space is given by $SO^*(6)/U(3)$, while $p = 3, q = 1$ implies the AIII symmetric space is $SU(3, 1)/(S(U(3) \times U(1))$. The isomorphisms cited in the Helgason coincidence (vii) are $su(3, 1) \simeq so^*(6)$ and $su(4) \simeq so(6)$, corresponding to the “numerators” in the noncompact and compact forms of the symmetric space respectively. As for the denominators, $S(U_3 \times U_1) \simeq U(3)$. A transitive action of $U(3)$ on the 5-sphere (boundary of the 6-ball) is known (cf. Table I in [53]).

Type CI, for $n = q = 1$, gives $Sp(1, \mathbb{R})/U(1)$, with dimension 2, rank 1. This is again the 2-ball (cf. coincidence (i) in Helgason again). This is the only simplicial case.

For type CIII, the simplicial cases have $q = 1$, with polytope $\triangle_1$ and regular convex body a ball $B_{4p}$. The symmetric space is $Sp(p, 1)/(Sp(p) \times Sp(1))$. Since $sp(1) \simeq su(2)$, and $Sp(p) \times SU(2)$ acts transitively on the boundary of the $4p$-ball (cf. [53] once again) In dimension 4, i.e. $p = q = 1$, Helgason’s coincidence (iii) implies that we have an EJA $\mathbb{R}^4 \oplus \mathbb{R}$: we have $sp(1) \times sp(1) \simeq so(4)$.

From Table 3: exceptional symmetric space representations.

Table 3 concerns spaces derived from exceptional Lie groups. Of these, only two have simplicial polytopes.

EIV, with polytope $\triangle_2$, has symmetric space determined by the pair of Lie algebras $(g, \mathfrak{t}) = (e_6(-26), f_4)$. The dimension of the polar representation of $\mathfrak{t}$, hence of $\text{Aff } \Omega$, is 26, and the Cartan decomposition and the Lie algebra $f_4$ is that
of the automorphism group of the convex set of unit-trace $3 \times 3$ positive definite octonionic-Hermitian matrices, which is indeed 26-dimensional, corresponding to the 27-dimensional exceptional Jordan algebra (cf. Table 11; $p_0$ in the Cartan decomposition is the traceless octonionic-Hermitian matrices.

FII has polytope $\triangle_1$, and symmetric space determined by $\mathfrak{g} = \mathfrak{f}_4(-20)$, $\mathfrak{k} = \mathfrak{so}(9)$. Its dimension is 16, so $B$ is a 16-ball. $\text{Spin}(9)$, a double cover of $SO(9)$, is known to act transitively on the unit 15-sphere in its (16-dimensional) fundamental representation (cf. [53] or [67]). Since the other exceptional case is $\text{Herm}(3,\mathbb{O})$, one might be tempted to think this representation is the octonionic bit, $\text{Herm}(2,\mathbb{O})$. But $\text{Herm}(2,\mathbb{O})$ is 10-dimensional, with 9-dimensional traceless part, so the “octobit” is naturally identified with the spin factor $\mathbb{R}^9 \oplus \mathbb{R}$, with state space a 9-ball [55].

**From Table 4 representations from simple complex groups**

In Table 4, corresponding to Helgason’s Type IV symmetric spaces, which arise from quotients of complex semisimple groups (viewed as real groups) by their compact real forms, the only simplicial polytopes appear in the the first line, of type and root space $A_n$, for $n \geq 1$, and noncompact symmetric space given by $\text{SL}(n+1,\mathbb{C})/\text{SU}(n+1)$, with dimension $n(n+2)$. The Cartan decomposition is $\mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{su}(n+1,\mathbb{C}) \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the traceless complex Hermitian matrices; the convex hull, $\Omega$, of the orbit of the highest restricted weight state in the traceless positive semidefinite matrices is affinely isomorphic to the unit-trace positive semidefinite matrices, i.e. the “density matrices” of standard quantum theory over $\mathbb{C}$.

The remaining lines of Table 4 include some families with polytopes $\square_n, \diamond_n$, but the cases $n = 1$ are not included so there are no more simplicial cases.

**C.2 Actions by polar representations other than symmetric space representations**

In Dadok’s classification of polar representations of compact connected groups $K$ by symmetric space representations, usually the polar representation is isomorphic to a symmetric space representation, but sometimes it is necessary to pass, via Theorem 8.10 to a symmetric space representation of a larger connected compact group $\tilde{K}$, acting on the same space and having the same orbits as the original polar

---

[55] See [27] for an interpretation of $\text{Herm}(2,\mathbb{O})$ as acted on by double covers of $SO(9,1)$ and of $SO(9)$.
representation. The original representation is the restriction of the representation of $\tilde{K}$ to the subgroup $K$.

In the representations associated with simplicial Farran-Robertson polytopes, this only occurs in the $\triangle_1$ cases, those in which the regular body $B$ is a ball. These must be cases in which $K$ acts transitively (and linearly) on the sphere $\partial B \equiv \partial B$. Any representation of a compact group acting transitively on a sphere is polar\footnote{This is implied at the end of Case I on p. 133 of [31]. It is also fairly obvious: the Cartan subspace is 1-dimensional, generated by a diameter, and the sphere is the unique (up to dilation) orbit.}. So all such representations of compact connected groups must occur either as symmetric space isotropy representations in Tables 2, 3 and 4, or as the restrictions of such representations to subgroups via Theorem 8.10 of Dadok (for a more compact and slightly more detailed presentation, see the main theorem of [67] and the remarks following it).

In both cases, the subgroup necessarily acts transitively not only on the set of extremal points (1-frames), but on the set of 2-frames as well, since the 2-frames in a ball are just pairs $[x, -x]$. One motivation for interest in these possibilities comes from general probabilistic theories: we might be interested in theories in which the state space is strongly symmetric and spectral, but the group of allowed reversible transformations is a proper subgroup of $\text{Aut}_0\Omega$, yet we still want it to act transitively on frames. In [5], the condition of energy observability (formulated for this case to require an injection of the Lie algebra of the allowed subgroup into the observables with the properties described in [5], Def. 30) was shown to rule out not only all Jordan-algebraic state spaces with $\text{Aut}_0\Omega$ as the allowed transformations, but also the systems with $\Omega$ a ball $B_d$, but with only a proper subgroup of $SO(d)$ as the allowed reversible transformations.

### C.3 More detail on the Jordan-algebraic cases

For the symmetric space isotropy representations in which the compact group acts on the orthocomplement of the identity in a Jordan algebra, and has the Lie algebra of the group of Jordan automorphisms (call these Jordan isotropy representations), the main text already gave a general argument why the regular convex body is affinely isomorphic to the normalized Jordan algebra state space. In this subsection, we see a little more concretely how this works, by looking in more detail at those representations for which the Farran-Robertson polytopes are $\triangle_n$, $n \geq 2$, which is to say the “matrix” cases $\text{Herm}(m, \mathbb{D})$. We need to verify that,
possibly up to an affine isomorphism, the orbit whose convex hull is the regular convex body $B$ in the Madden-Robertson construction is the same orbit whose convex hull gives the normalized state space of $V$. For $\triangle_n$, $n \geq 2$, if $A = p \oplus \mathbb{R}$ is an EJA then it must be of the form $\text{Herm}(n, \mathbb{D})$, $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. The Coxeter/Dynkin diagram for the restricted Weyl group of $G_{ss}$ is that of $A_{n-1}$. The space $p$ in the Cartan decomposition $g = k \oplus p$ of the Lie algebra of $\text{Aut}_0 A_+$ is the traceless symmetric matrices over $\mathbb{D}$. A maximal abelian subspace $a$ of $p$ is given by the traceless diagonal matrices in each case; its dimension is $n - 1$. Its Weyl group $K_a/K^a$ (also written $N_K(a)/Z_K(a)$) is $S_n$. It is convenient to study $K$’s action $p$ by extending it to an action of $p \oplus \mathbb{R}$, where $\mathbb{R}$ in this case is generated by the identity matrix, so we have an action on the space of symmetric matrices, not just the traceless ones. $K$’s action fixes the identity matrix, and the traceless matrices are an invariant (in fact irreducible) subspace for the $K$-action. $W \simeq S_n$ acts as the group of all permutations of the unit diagonal matrices $e_{ii}$, for $i \in \{1, \ldots, n\}$ (or if we prefer to consider its action on the traceless matrices, it permutes the $n$ traceless matrices $e_{11} - I/n \equiv \text{diag}((n-1)/n, -1/n, \ldots, -1/n), e_{22} - I/n$, etc.). These matrices form the $n$ vertices of an $(n-1)$-simplex in the unit-trace (resp. traceless) matrices. A set of generators for this group is the reflections in the hyperplanes normal to the traceless matrices $e_{ii} - e_{jj}$, but these are not all necessary; a minimal set is $e_{ii} - e_{i+1,i+1}$. Thus $e_{ii} - e_{i+1,i+1}$ are a system of simple roots for $A_{n-1}$. For $A_{n-1}$, the diagram is linear with $n-1$ simple roots, and unmarked links because adjacent roots are all at angle $2\pi/3$. If we identify the unit matrices $e_{ii}$ with unit vectors $e_i$ in $\mathbb{R}^n$ (the unit-trace matrices) to make index manipulation easier, then $e_{11}$, projected into the traceless matrices (i.e. $e_{11} - I/n$) is $\lambda_1$, the first fundamental weight, which is the point in the Madden-Robertson construction that corresponds to the first node of the Coxeter diagram. The convex hull of its $W$-orbit is a simplex, the translation to the traceless matrices of the simplex $\triangle(e_1, \ldots, e_n)$ in the unit-trace matrices. This simplex in $a$ is therefore Madden and Robertson’s $\pi(B)$ in this polar representation, for the case of the first (leftmost) node of the Coxeter diagram. The last node gives (up to possible dilation) the polar body and the polar of a simplex is again a simplex; indeed since the diagram is symmetric the regular convex body associated with the last node will also be

\[57\]In the particular case of $A_{n-1}$, this is confirmed by the fact that $\lambda_{n-1}/(n-1)$ is the barycenter of a maximal face of the simplex $\pi(B)$, which is a negative multiple of a vertex of the simplex, so the convex hull of its orbit gives (up to dilation) the negative of the original simplex, which is an isomorphic simplex and (up to dilation) the polar of the original simplex.
affinely isomorphic to the one associated with the first.  

Since we saw that in this construction, we could identify the fundamental weight \( \lambda_1 \in \mathfrak{a} \) whose \( W \)-orbitope gives \( \pi(B) \) and whose \( K \)-orbitope gives \( B \) in the Madden-Robertson embedding in a polar representation, as the traceless part of the unit matrix \( e_{11} \), when that polar representation is one corresponding to a Euclidean Jordan algebra, we see that

\[
B = \text{Conv}\left(K,(e_{11} - I/n)\right) \equiv \text{Conv}\left(K.e_{11}\right) - I/n
\]

is affinely isomorphic to the normalized state space \( \text{Conv} K.e_{11} \) of the corresponding Euclidean Jordan algebra, the isomorphism being given by mapping the linear hyperplane of traceless symmetric matrices to the affine hyperplane of unit-trace ones, by adding \( I/n \).

References

[1] E. Alfsen and F. Shultz, *Geometry of State Spaces of Operator Algebras*. Birkhäuser, 2003.

[2] H. Araki, “On a characterization of the state space of quantum mechanics,” *Comm. Math. Phys.*, vol. 75, pp. 1–24, 1980.

[3] M. Krumm, H. Barnum, J. Barrett, and M. P. Müller, “Thermodynamics and the structure of quantum theory,” *New J. Phys.*, vol. 19, p. 043025, 2017. [arXiv:1608.04461](https://arxiv.org/abs/1608.04461)

The other nodes in the Coxeter diagram correspond, in general in the Madden-Robertson construction, to the barycenters of a maximal flag of faces of this convex body (the orbitope over \( \lambda_1 \)), and this is confirmed in the \( A_{n-1} \) case by the explicit formula for the fundamental weights: the weight \( \lambda_k \) is the projection of \( \sum_{i=1}^k e_i \) into subspace \( \sum_i x_i = 0 \), i.e. writing \( t_i \) for the projection to the subspace \( \sum_i x_i = 0 \) (the traceless diagonal matrices in our particular representation),

\[
\lambda_k = \sum_{i=1}^k t_i.
\]

(8)

For comparison, the barycenters of the unit \((n-1)\)-simplex in the unit-trace matrices are \( b_k := \sum_{i=1}^k e_i / k \), and their translations to the \((n-1)\)-dimensional subspace \( \sum_{i=1}^k x_i = 0 \), which are the barycenters of a maximal flag of of (proper) faces of the simplex \( \text{Conv} W.t_1 \) in that space, are just

\[
c_k := \sum_{j=1}^l t_j / k \equiv \lambda_k / k.
\]

(9)
[4] G. Chiribella and C. M. Scandolo, “Operational axioms for diagonalizing quantum states,” 2015. arXiv:1506.00380.

[5] H. Barnum, M. Müller, and C. Ududec, “Higher-order interference and single-system postulates characterizing quantum theory,” New J. Phys., vol. 16, p. 123029, 2014. Also: arXiv:1403.4147.

[6] C. M. Lee and J. H. Selby, “Deriving Grover’s lower bound from simple physical principles,” New J. Phys., vol. 18, p. 093047, 2016.

[7] H. R. Farran and S. Robertson, “Regular convex bodies,” J. London Math. Soc., ser. 2, vol. 49, pp. 371–384, 1994.

[8] T. Madden and S. Robertson, “The classification of regular solids,” Bull. London Math. Soc., vol. 27, pp. 363–370, 1995.

[9] C. H. Bennett, G. Brassard, E. Bernstein, and U. Vazirani, “Strengths and weaknesses of quantum computing,” SIAM J. Comp., pp. 1510–1523, 1997.

[10] G. Chiribella and C. M. Scandolo, “Entanglement as an axiomatic foundation for statistical mechanics,” 2016. arXiv:1608.0449.

[11] H. Barnum, C. M. Lee, and J. H. Selby, “Oracles and query lower bounds in generalised probabilistic theories,” Found. Phys., vol. 48, pp. 954–981, 2018. arXiv:1704.05043.

[12] B. Dakić and C. Brukner, “Quantum theory and beyond: Is entanglement special?,” in Deep Beauty: Understanding the Quantum World Through Mathematical Innovation (H. Halvorson, ed.), Cambridge: Cambridge University Press, 2011. Also: arXiv:0911.0695.

[13] E. Alfsen and F. Shultz, State Spaces of Operator Algebras. Birkhäuser, 2001.

[14] R. Sanyal, F. Sottile, and B. Sturmfels, “Orbitopes,” Mathematika, vol. 57, pp. 275–314, 2011. Also http://arxiv.org/abs/0911.5436.

[15] N. Riedel, “Spektraltheorie in geordneten Vektorraumen,” Rev. Roum. Math., vol. 28, pp. 33–79, 1983.

[16] B. Iochum, Cônes Autopolaires et Algèbres de Jordan. Lecture Notes in Mathematics, Vol. 1049, Springer, 1984.
[17] M. Müller and C. Ududec, “The structure of reversible computation determines the self-duality of quantum theory,” *Phys. Rev. Lett.*, vol. 108, p. 130401, 2012.

[18] P. Jordan, “Über ein Klasse nichtassoziativer hypercomplexer Algebren,” *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. I.*, vol. 33, pp. 569–575, 1933.

[19] J. Faraut and A. Koranyi, *Analysis on Symmetric Cones*. Oxford University Press, 1994.

[20] I. Satake, *Algebraic Structures of Symmetric Domains*, vol. 14 of *Publications of the Mathematical Society of Japan*. Iwanami Shoten and Princeton University Press, 1980.

[21] P. Jordan, J. von Neumann, and E. Wigner, “On an algebraic generalization of the quantum mechanical formalism,” *Ann. Math.*, vol. 35, pp. 29–64, 1934.

[22] M. Koecher, “Die Geodätischen von Positivitätsbereichen,” *Math. Annalen*, vol. 135, pp. 192–202, 1958.

[23] E. Vinberg, “Homogeneous cones,” *Dokl. Acad. Nauk. SSSR*, vol. 133, pp. 9–12, 1960. English translation: E. Vinberg (1961): Soviet Math. Dokl. 1, pp. 787–790.

[24] H. Barnum and A. Wilce, “Local tomography and the Jordan structure of quantum theory,” *Found. Phys.*, vol. 44, pp. 192–212, 2014. Also: arXiv:1202.4513.

[25] M. Koecher, “Positivitätsbereiche im $\mathbb{R}^n$,” *Amer. J. Math.*, vol. 79, pp. 575–596, 1957.

[26] G. Mostow, “Self-adjoint groups,” *Ann. Math.*, vol. 62, pp. 44–55, 1955.

[27] T. Dray and C. A. Manogue, “Octonionic Cayley spinors and $E_6$,” *Comment. Math. Univ. Carolin.*, vol. 51, pp. 193–207, 2010.

[28] G. P. Barker and J. Foran, “Self-dual cones in Euclidean spaces,” *Linear Algebra and its Applications*, vol. 13, no. 12, pp. 147–155, 1976.

[29] L. Biliotti, A. Ghighi, and P. Heinzner, “Polar orbitopes,” 2012. http://arxiv.org/abs/1206.5717.
[30] L. Biliotti, A. Ghigli, and P. Heinzner, “Invariant convex sets in polar representations,” Israel J. Math., vol. 213, pp. 423–441, 2016. E-print http://arxiv.org/abs/1411.6041.

[31] J. Dadok, “Polar coordinates induced by actions of compact Lie groups,” Trans. Am. Math. Soc., vol. 288, pp. 124–137, 1985.

[32] M. P. Müller personal communication to H. Barnum, 2016.

[33] V. M. Gichev, “Polar representations of compact groups and convex hulls of their orbits,” Differential Geometry and its Applications, vol. 28, pp. 608 – 614, 2010.

[34] F. Gozzi, “A note on polar representations.” [arXiv:1704.03129] 2017.

[35] J. Eschenburg and E. Heintze, “Polar representations and symmetric spaces,” Journal für die Reine und Angewandte Mathematik (Crelles Journal), vol. 1999, no. 507, pp. 93–106, 1999.

[36] H. S. M. Coxeter, Regular Polytopes. New York: Macmillan, 1963. Reprinted by Dover Press, 1973.

[37] L. Schläfi, Theorie der Vielfachen Kontinuität. Bern: J. H. Graf, 1901.

[38] J. E. Humphreys, Reflection Groups and Coxeter Groups. No. 29 in Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, 1992.

[39] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, vol. 34 of Graduate Studies in Mathematics. American Mathematical Society, 2001. Reprint of 1978 edition published by Academic Press, New York, vol. 80 of series Pure and Applied Mathematics.

[40] E. M. Alfsen and F. W. Shultz, “State spaces of Jordan algebras,” Acta Mathematica, vol. 140, pp. 155–190, 1978.

[41] J. Barrett, “Information processing in generalized probabilistic theories,” Phys. Rev. A, vol. 75, p. 032304, 2007. [arXiv:0508211]

[42] H. Barnum, J. Barrett, M. Leifer, and A. Wilce, “Generalized no-broadcasting theorem,” Phys. Rev. Lett., vol. 99, p. 240501, 2007.
[43] H. Barnum, J. Barrett, M. Leifer, and A. Wilce, “Cloning and broadcasting in generic probabilistic theories.” arXiv.org e-print quant-ph/0611295, 2006.

[44] M. Plávala, “All measurements in a probabilistic theory are compatible if and only if the state space is a simplex.” Phys. Rev. A, vol. 94, p. 042108, 2016. arxiv:1608.05614.

[45] L. Hardy, “Quantum theory from five reasonable axioms.” arXiv.org e-print quant-ph/0101012, 2001.

[46] L. Hardy, “Why quantum theory?.” arXiv.org e-print quant-ph/0111068. Contribution to NATO Advanced Research Workshop “Modality, Probability, and Bell’s Theorem”, Cracow, Poland 19–23.8.01, 2001.

[47] A. Connes, “Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann,” Annales de l’Institut Fourier (Grenoble), vol. 24, pp. 121–155, 1978.

[48] J. Bellissard and B. Iochum, “Homogeneous self-dual cones and Jordan algebras,” in Quantum Fields—Algebras, Processes (L. Streit, ed.), (Wien, New York), pp. 153–165, Springer, 1980.

[49] Ll. Masanes and M. P. Müller, “A derivation of quantum theory from physical requirements,” New J. Phys., vol. 13, no. 6, p. 063001, 2011. arXiv:1004.1483.

[50] H. Barnum, M. Graydon, and A. Wilce, “Composites and categories of Euclidean Jordan algebras.” arXiv:1606.09331, 2016.

[51] P. Selinger, “Dagger compact closed categories and completely positive maps (extended abstract),” in Electronic Notes in Theoretical Computer Science (Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005)), vol. 170, pp. 139–163, Elsevier, 2007.

[52] S. Abramsky and B. Coecke, “A categorical semantics of quantum protocols,” in Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS ’04), pp. 415–425, 2004.

[53] Ll. Masanes, M. P. Müller, R. Augusiak, and D. Pérez-Garcia, “Entanglement and the three-dimensionality of the Bloch ball,” J. Math. Phys., vol. 55, p. 122203, 2014. Also arxiv:1111.4060v4.
[54] G. Niestegge, “Conditional probability, three-slit experiments, and the Jordan algebra structure of quantum mechanics,” *Adv. Math. Phys.*, vol. 2012, p. 156573, 2012.

[55] C. Ududec, H. Barnum, and J. Emerson, “Probabilistic interference in operational models.” Unpublished, 2009.

[56] E. M. Alfsen and F. W. Shultz, “State spaces of Jordan algebras,” *Acta Mathematica*, vol. 140, pp. 155–190, 1978.

[57] R. Sorkin, “Quantum mechanics as quantum measure theory,” *Mod. Phys. Lett.*, vol. A9, pp. 3119–3128, 1994. arXiv:gr-qc/9401003.

[58] C. Ududec, H. Barnum, and J. Emerson, “Three slit experiments and the structure of quantum theory,” *Found. Phys.*, vol. 41, pp. 396–405, 2011.

[59] C. Ududec, *Perspectives on the formalism of quantum theory*. PhD thesis, University of Waterloo, 2012.

[60] L. Grover, “A fast quantum mechanical algorithm for database search,” *Proceedings of the 28th Annual ACM Symposium on the Theory of Computing (STOC)*, pp. 212–219, May 1996.

[61] L. Grover, “Quantum mechanics helps in searching for a needle in a haystack,” *Phys. Rev. Lett.*, pp. 325–328, July 1997.

[62] H. Barnum, C. M. Lee, C. M. Scandolo, and J. H. Selby, “Ruling out higher order interference from purity principles,” *Entropy*, vol. 19, p. 253, 2017. arXiv:1704.05106.

[63] D. A. Meyer and J. Pommersheim, “On the uselessness of quantum queries,” *Theoretical Computer Science*, vol. 412, no. 51, pp. 7068–7064, 2011.

[64] P. M. Alberti and A. Uhlmann, *Stochasticity and partial order*. Berlin: VEB Deutscher Verlag Wiss., 1981.

[65] H. Barnum, J. Barrett, M. Krumm, and M. Müller, “Entropy, majorization and thermodynamics in general probabilistic theories,” in *Electronic Proceedings in Theoretical Computer Science (Proceedings 12th International Workshop on Quantum Physics and Logic)* (C. Heunen, P. Selinger, and J. Vicary, eds.), vol. 195, pp. 43–58, 2015. Also: arXiv:1508.03107.
[66] G. Chiribella and C. M. Scandolo, “Microcanonical thermodynamics in
general physical theories,” New J. Phys., vol. 19, p. 123043, 2017.
\texttt{arXiv:1608.04459}

[67] J. Eschenburg and E. Heintze, “On the classification of polar representa-
tions,” Mathematische Zeitschrift, vol. 232, no. 3, pp. 391–398, 1999.