We consider Weyl symmetric structure of the classical vacuum in quantum chromodynamics. In the framework of formalism of gauge invariant Abelian projection we show that classical vacuums can be constructed in terms of Killing vector fields on the group SU(3). Consequently, homotopic classes of Killing vector fields determine the topological structure of the vacuum. In particular, the second homotopy group \( \pi_2(SU(3)/U(1) \times U(1)) \) describes all topologically non-equivalent vacuums which are classified by two topological numbers. Starting with a given Killing vector field one can construct vacuums forming a Weyl sextet representation. An interesting feature of SU(3) gauge theory is that it admits a Weyl symmetric vacuum represented by a linear superposition of the vacuums from the Weyl vacuum sextet. A non-trivial manifestation of the Weyl symmetry is demonstrated on monopole solutions. We construct a family of finite energy monopole solutions in Yang-Mills-Higgs theory which includes the Weyl monopole sextet. We conjecture that a similar Weyl symmetric vacuum structure can be realized at quantum level in quantum chromodynamics.

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I. INTRODUCTION

The mechanism of confinement in quantum chromodynamics (QCD) based on the Meissner effect in dual color superconductor is very attractive [1–3], and many features of quark confinement are described in numerous approaches to low energy QCD in agreement with experimental data. Despite on this the origin of color confinement remains much less known up to now. Formally, from the mathematical point of view the color confinement is manifestation of the fact that color symmetry represents an exact symmetry of strong interaction. This raises a simple, but fundamental question: why SU(3) color symmetry in QCD is preserved, whereas SU(2) gauge symmetry of weak interaction is spontaneously broken? A possible answer to that question can be related to features of the groups SU(2) and SU(3), namely, to Weyl symmetry and its physical implications in classical and quantum vacuum structures.

In the present paper we study the structure of the classical vacuum and related issues on monopole solutions in SU(3) QCD. In a standard approach the classical vacuum configurations are described by pure gauge potentials classified by the third homotopy group \( \pi_3(SU(N)) = Z \), i.e., by the topological Chern-Simons number (see, for ex., a review [4] and refs. there in). Such a non-trivial topological vacuum structure is manifested through the vacuum tunneling effect realized by means of instantons [5–7]. Our approach to study of the vacuum structure is based on the gauge invariant Abelian projection proposed originally in [8–10] and developed further in [11–14]. An essential feature of the formalism of Abelian projection is that it allows to describe the topological properties of the vacuum fields in terms of a more simple geometric object than the gauge potential, namely, in terms of Killing vector field \( \bar{m}_i \), \( i = 1, ..., N - 1 \), on the group SU(N). In the case of SU(2) gauge theory the classical vacuums can be classified by the knot number (Hopf number) corresponding to the third homotopy group \( \pi_3(SU(2)/U(1)) \) [12, 16]. Due to equivalence of the homotopy groups \( \pi_3(SU(2)) \cong \pi_2,3(SU(2)/U(1)) \) one has one to one correspondence between topological non-equivalent classes for the gauge potential and the Killing vector field. The case of SU(3) gauge theory reveals a more rich topological content of field configurations. Even though, the third homotopy groups \( \pi_3(SU(N)) = Z \) for \( N = 2 \) and \( N = 3 \) are the same, the second homotopy groups \( \pi_2(SU(2)/U(1)) = Z \) and \( \pi_2(SU(3)/U(1) \times U(1)) = Z \times Z \) describing homotopic classes of Killing vector fields are essentially different.

This implies important consequences: (i) topological classical vacuum structure in SU(3) QCD is determined by two topological numbers; (ii) topologically non-equivalent vacuums in SU(3) case form a Weyl sextet of degenerated vacuums and a non-trivial Weyl symmetric vacuum
singlet.

To find physical implications of the non-trivial topological vacuum structure we first demonstrate that SU(3) topologically non-equivalent vacuums form representations of the Weyl symmetry group. Starting with a given Killing vector field one can construct a Weyl vacuum sextet representation. A remarkable feature of SU(3) QCD is that there exists a Weyl symmetric vacuum. It is important to stress that such a non-trivial vacuum does not exist in SU(2) gauge theory. Another interesting fact is that singular Wu-Yang type monopole solutions are classified due to Weyl representation theory as well [17, 19]. We consider Weyl structure of finite energy monopole solutions in Yang-Mills-Higgs theory. Introducing a more general one-parameter family of monopole solutions in the BPS limit we show that different topological classes of monopoles are separated by infinite energy barrier.

The presence of Weyl symmetric structure of the classical vacuum and monopole solutions indicates to possible existence of a similar vacuum structure with monopole condensation in quantum theory. There are some indications that two-loop effective potential in QCD may admit the Weyl sextet of degenerated vacuums [20, 21]. This may shed light on the origin of color confinement phenomenon in QCD. We conjecture that to preserve the color symmetry in QCD against spontaneous symmetry breaking there must exist a non-trivial Weyl symmetric vacuum in addition to the Weyl vacuum sextet in the full quantum theory. In one-loop approximation such a Weyl symmetric vacuum does exist [22, 23], this provides a possible stable monopole condensation [24, 25].

As another application of our approach based on gauge invariant Abelian projection we consider a general ansatz for searching essentially SU(3) instanton and monopole solutions.

The paper is organized as follows. In Section II we overview briefly Cho-Duan-Ge gauge invariant Abelian projection in SU(3) gauge theory. We propose an alternative parametrization for Killing vectors in terms of two complex triplet fields which allows to describe the geometric origin of the Killing vectors and dual magnetic symmetry. In Section III we describe the topological vacuum structure in SU(3) QCD and provide an explicit construction of Weyl representations for topologically non-equivalent vacuums. A detailed analysis of instanton solution with a general ansatz including two topological numbers is presented in Section IV. Section V is devoted to Weyl symmetric structure of singular and finite energy monopole solutions. The last section contains conclusions and discussion of quantum vacuum structure in QCD.

II. CHO-DUAN-GE GAUGE INARIANT
ABELIAN PROJECTION

Let us start with main outlines of Cho-Duan-Ge gauge invariant Abelian projection in SU(3) QCD [8–10]. A principal role in the construction of the Abelian projection belongs to Killing vector fields \( \hat{m}_i^a \), \( (a = 1, \ldots, 8, \ i = 3, 8) \) which describe all mappings from the base space-time to the homogeneous coset space \( \mathcal{M}^6 = SU(3) / U(1) \times U(1) \).

The Abelian decomposition of SU(3) gauge connection is given by

\[
\begin{align*}
\hat{A}_i^a &= \hat{A}_i^a + \hat{X}_i, \\
\hat{A}_i^a &= \hat{A}_i^a + \hat{C}_i^a, \\
\hat{C}_i^a &= -i f^{abc} \hat{m}_i^b \partial_\mu \hat{m}_i^c = -i (\hat{m}_i \times \partial_\mu \hat{m}_i)^a, (1)
\end{align*}
\]

where \( \hat{A}_i^a \) is a restricted potential, \( \hat{A}_i^a \) is Abelian "photon" gauge potential, \( \hat{C}_i^a \) is a magnetic potential, and \( \hat{X}_i \) represents off-diagonal (valence) gluons which are orthogonal to \( \hat{m}_i \), \( (i = 3, 8) \). One can define a projectional operator which projects any color vector \( \hat{V}^a \) onto the Cartan plane formed by Killing vectors \( \hat{m}_i \).

\[
\begin{align*}
\hat{P}^{ab} &= \delta^{ab} - f^{abc} f^{def} \hat{m}_i^d \hat{m}_i^e, \\
\hat{P}^{ab} \hat{V}^b &= \hat{m}_i^i (\hat{m}_i \cdot \hat{V}). (2)
\end{align*}
\]

Notice, the projectional operator is defined properly only if the vectors \( \hat{m}_i \) satisfy the orthonormality condition. Using this projectional operator one can easily verify that the vectors \( \hat{m}_i \) are covariant constant

\[
\hat{D}_\mu \hat{m}_i = (\partial_\mu + \hat{A}_i^a) \hat{m}_i = 0, (3)
\]

i.e., \( \hat{m}_i \) represent Killing vectors on SU(3).

Let us consider the vector magnetic field strength \( \hat{H}_{\mu\nu} \) constructed from the magnetic gauge potential

\[
\hat{H}_{\mu\nu} = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu + \hat{C}_\mu \times \hat{C}_\nu. (4)
\]

Straightforward calculation shows that vector magnetic field strength \( \hat{H}_{\mu\nu} \) belongs to the Cartan plane

\[
\hat{H}_{\mu\nu} = H_{\mu\nu}^i \hat{m}_i. (5)
\]

One can check that two differential 2-forms \( H^i = dx^\mu \wedge dx^\nu H_{\mu\nu}^i \) are closed [8, 24]

\[
d H^i = 0. (6)
\]

Due to Poincare lemma the closed magnetic field two-forms \( H^i \) are locally exact. So that the magnetic fields \( H_{\mu\nu}^i \) can be expressed explicitly in terms of dual Abelian magnetic potential \( \hat{C}_\mu \)

\[
H_{\mu\nu}^i = \partial_\mu \hat{C}_\nu^i - \partial_\nu \hat{C}_\mu^i, (7)
\]

The definition of the magnetic fields \( H_{\mu\nu}^i \) implies the existence of the dual magnetic symmetry \( \hat{U}(1) \times \hat{U}'(1) \) which is an essential ingredient in the dual Meissner mechanism of confinement

\[
\begin{align*}
\delta_{\hat{U}(1)} \hat{C}_\mu^3 &= -i \partial_\mu \hat{A}, \\
\delta_{\hat{U}'(1)} \hat{C}_\mu^8 &= -i \partial_\mu \hat{A}'. (8)
\end{align*}
\]
The gauge invariant Abelian decomposition (1) leads to the following split of the gauge field strength into Abelian and off-diagonal parts
\[
\tilde{F}_{\mu
u} = (F^i_{\mu
u} + H^i_{\mu
u})\hat{m}_i^a + \bar{D}_\mu \bar{X}_\nu - \bar{D}_\nu \bar{X}_\mu + \bar{X}_\mu \times \bar{X}_\nu,
\]
where \(F^i_{\mu
u} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu\) is an Abelian field strength component, and \(\bar{X}_\mu\) can be treated as a color source [8,9].

Let us consider an alternative parametrization for the Killing vectors in terms of complex fields which have a simple geometric meaning of complex projective coordinates on the coset \(M^6 = SU(3)/U(1) \times U(1)\). The Cartan algebra of \(SU(3)\) Lie algebra is generated by two vectors \(m_3 = \hat{m}_3 t_3\), \(m_8 = \hat{m}_8 t_8\) with \(t_{3,8}\) as the generators in adjoint representation. In the case of \(SU(2)\) gauge theory it is known that corresponding Killing vector can be expressed in terms of a complex \(SU(2)\) vector in fundamental representation [13,24,27] which can be treated as a projective coordinate on \(SU(2)/U(1) \simeq S^2\). In \(SU(3)\) gauge theory since the homogeneous space \(M^6\) possesses a global complex structure one can define complex projective coordinates on it by introducing two complex triplet fields \(\Psi, \Phi\). Let us first express the lowest weight vector \(\hat{m}_8\) in terms of the complex triplet field \(\Psi\) that parameterizes the coset \(CP^2 \simeq SU(3)/SU(2) \times U(1)\)
\[
\hat{m}_8^a = -\frac{3}{2} \bar{\Psi}\lambda^a \Psi, \quad \bar{\Psi} \Psi = 1.
\]
The definition for the vector \(\hat{m}_8\) is consistent with the normalization condition and symmetric \(d-\)product operation in the Lie algebra of \(SU(3)\)
\[
\hat{m}_8^2 = 1, \quad d^{abc}\hat{m}_8^a \hat{m}_8^b = -\frac{1}{\sqrt{3}} \hat{m}_8^2.
\]
To construct a second Cartan vector \(\hat{m}_3\) orthogonal to \(\hat{m}_8\) it is convenient to define projectional operators
\[
P^\parallel_{ab} = \hat{m}_8^a \hat{m}_8^b, \quad P^\perp_{ab} = \delta_{ab} - \hat{m}_8^a \hat{m}_8^b.
\]
With this the vector \(\hat{m}_3\) can be parameterized as follows
\[
\hat{m}_3^a = P^\perp_{ab} \bar{\Phi} \lambda^b \Phi = \bar{\Phi} \lambda^a \Phi + \frac{1}{2} \bar{\Psi} \lambda^a \Psi,
\]
where we have introduced a second complex triplet field \(\Phi\). The definition of Killing vectors \(\hat{m}_3, \hat{m}_8\) by Eqs. (10)-(13) is invariant under the dual \(U(1) \times U(1)\) local transformations
\[
\Psi \rightarrow \exp[i\hat{a}(x)]\Psi, \quad \Phi \rightarrow \exp[i\hat{a}'(x)]\Phi,
\]
which represent explicitly the dual magnetic symmetry [8]. The dual magnetic potentials \(\hat{C}_\mu^3\) can be expressed through the complex fields as follows
\[
\hat{C}_\mu^3 = 2i(\hat{\Phi}\partial_\mu \Phi + \frac{1}{2} \bar{\Psi}\partial_\mu \Psi),
\]
\[
\hat{C}_\mu^8 = 2i(-\frac{\sqrt{3}}{2} \bar{\Psi}\partial_\mu \Psi).
\]
One can verify that \(\hat{m}_i\) satisfy the following relations
\[
d^{abc}\hat{m}_i^b \hat{m}_j^c = d_{ijk}\hat{m}_k^a,
\]
which imply the orthogonality condition for the complex fields \(\bar{\Psi}\Phi = 0\). One should notice, in general, it is not necessary to impose the orthogonality condition for Killing vectors, so that the complex fields \(\Phi, \Psi\) can be treated as arbitrary independent fields. This implies an interesting interpretation of the Killing vector fields as composite fields made of quarks, i.e., the complex fields \(\Phi, \Psi\) can be treated as a flavor \(SU(2)\) quark doublet \((u, d)\). A similar idea of composite chiral solitons in QCD is considered in [28]. The definition of the vectors \(\hat{m}_i\) in terms of the complex fields \(\Psi, \Phi\) provides a minimal set of fields with six independent degrees of freedom needed to parameterize the homogeneous space \(M^6 = SU(3)/U(1) \times U(1)\). In the present paper for our purpose to study the Weyl symmetric structure of the vacuum we will treat the Killing vectors \(\hat{m}_i\) as independent geometric objects following the original works [8,10].

III. WEYL SYMMETRIC VACUUM STRUCTURE

A standard approach to topological classification of field configurations in terms of the gauge potential \(\tilde{A}_\mu\) is based on the third homotopy group \(\pi_3(SU(N)) = \pi_3(SU(2)) = Z\) which describes characteristic Chern classes numerated by the topological Pontryagin number. In particular, instantons solutions represent field configurations with the minimal Euclidean action in each such a topological class. Classical vacuum in a pure Yang-Mills theory is defined by the equation \(\tilde{F}_{\mu\nu} = 0\) which is satisfied by an arbitrary pure gauge potential \(\tilde{A}_\mu\). All non-equivalent topological vacuum gauge potentials are classified by topological Chern-Simons number \(n_{CS}\)
\[
n_{CS} = \frac{1}{16\pi^2} \int d^3 x \epsilon^{ijk} \left[ A_i^a \partial_j A_k^a + \frac{1}{3} f^{abc} A_i^a A_j^b A_k^c \right]
\]
\[
≡ \frac{1}{16\pi^2} \int \omega^{(3)}_{CS},
\]
where \(\omega^{(3)}_{CS}\) is a differential Chern-Simons 3-form which is closed on space of vacuum configurations of \(\tilde{A}_\mu\).

The construction of Cho-Duan-Ge gauge invariant Abelian projection provides a novel approach to classification of topological structure of the classical vacuum. It has been shown that the vacuum gauge potential in
where Explicit expressions for the topological charges $g$ and $\mathbf{g}$ are introduced next. We introduce explicit expressions for Killing vector fields. Since each configuration of Killing vector field determines a topological number in addition to the Chern-Simons number. So that, we have one to one correspondence between Killing vector fields and pure gauge potentials. Under the Weyl symmetry transformation which plays an important role in classical and quantum theory.

To study the Weyl structure of vacuum and classical solutions we will use an explicit construction for Killing vector fields. Let us start from the constant vectors in $SU(3)$ color space

$$\hat{\xi}_i = \delta_i^a, \quad (i = 3, 8).$$

To obtain a more general functional form for Killing vectors it is convenient to apply a local $SU(3)$ gauge transformation to the constant vectors $\hat{\xi}_i$

$$\hat{\xi}_i \rightarrow \hat{\xi}_i = U \hat{\xi}_i,$$

where $U$ is an arbitrary matrix group element of $SU(3)$. Notice, the Killing vectors $\hat{m}_3, \hat{m}_8$ are orthogonal to each other.

Let us consider one parameter subgroup of $SU(3)$ gauge transformations acting on $\hat{\xi}_i$ as follows

$$\hat{\xi}_3 \rightarrow \hat{\xi}_3(\delta) = \hat{\xi}_3 \cos \delta + \hat{\xi}_8 \sin \delta,$$

$$\hat{\xi}_8 \rightarrow \hat{\xi}_8(\delta) = \hat{\xi}_3 \sin 2\delta + \hat{\xi}_8 \cos 2\delta,$$

where, the transformation law for the second vector $\hat{\xi}_8$ is determined by the consistence requirement with $d-$product, namely, for any Killing vector $\hat{m}_3$ the second vector $\hat{m}_8$ is defined by $d-$symmetry. So that, the vectors $\hat{\xi}_i$ transform on different representations of the group $SO(2)$.

The most general expression for the Killing vector field $\hat{n}_i(\delta)$ can be obtained from $\hat{m}_i$ by applying the $\delta$-transformation

$$\hat{n}_i(\delta) = \hat{m}_i \cos \delta + \hat{m}_8 \sin \delta,$$

$$\hat{n}_8(\delta) = \hat{m}_3 \sin 2\delta + \hat{m}_8 \cos 2\delta.$$

The Killing vectors $\hat{n}_3, \hat{n}_8$ are not orthogonal to each other in general

$$\hat{n}_i^2 = 1,$$

$$\hat{n}_3^2 \hat{n}_8^2 = \sin(3\delta).$$

Notice, that one has only one independent Killing vector $\hat{n}_3$ since the second vector $\hat{n}_8$ is defined by $d-$symmetry transformation. The orthonormality condition $\hat{n}_3^2 = 1$ and $d-$symmetry imply that the Killing vector $\hat{n}_3$ has exactly six independent degrees of freedom. So that, $\hat{n}_3$ alone parameterizes the whole homogeneous coset space $M^6$ in a consistent manner.

We define a generalized monopole vector field $\hat{C}_\mu(\delta)$ and corresponding vector magnetic field $\hat{H}_{\mu\nu}(\delta)$ by the
same definitions \[14\]. One can verify that the magnetic field strength \( \tilde{H}_{\mu
u}(\delta) \) belongs to the Cartan plane only under a certain condition for the angle \( \delta \)

\[
\tilde{H}_{\mu
u}(\delta_k) = H^i_{\mu
u}(\delta_k)\hat{n}_i(\delta_k),
\]

\[
\sin(3\delta_k) = 0, \quad \delta_k = \frac{\pi k}{3},
\]

(25)

where \( k = (0, 1, \ldots, 5) \). Notice, the expressions for the magnetic potential \( \tilde{C}_\mu^i(\delta_k) \) and field strength \( \tilde{H}_{\mu
u}(\delta_k) \) are the same for different angle values \( \delta_k \). The Abelian dual magnetic potentials \( \tilde{C}_\mu^i(\delta_k) \) are defined for the respective magnetic fields \( H^i_{\mu
u}(\delta_k) \) by

\[
H^i_{\mu
u}(\delta_k) = \partial_\mu \tilde{C}^i_\nu(\delta_k) - \partial_\nu \tilde{C}^i_\mu(\delta_k).
\]

(26)

One has the following relationship for the dual magnetic potentials \( \tilde{C}_\mu^i(\delta_k) \) for different angles \( \delta_k \)

\[
\tilde{C}^3_\mu(\delta_k) = \tilde{C}^3_\mu(\delta_k) \cos \delta_k + \tilde{C}^8_\mu(\delta_k) \sin \delta_k,
\]

\[
\tilde{C}^8_\mu(\delta_k) = \tilde{C}^3_\mu(\delta_k) \sin(2\delta_k) + \tilde{C}^3_\mu(\delta_k) \cos(2\delta_k),
\]

(27)

where \( \tilde{C}^3_\mu, \tilde{C}^8_\mu \) are dual magnetic potentials given at zero angle value, \( \delta = 0 \). It should be stressed, that the dual magnetic potentials in the last equations are defined up to dual magnetic transformation \( U(1) \times U'(1) \). So that, to find explicit expressions for \( \tilde{C}_\mu^i(\delta_k) \) one should start from the given expressions for the magnetic field \( H^i_{\mu
u} \).

Now we can construct a vacuum sextet realizing the representation of the Weyl symmetry group \( Z_6 \)

\[
\tilde{A}_\mu^{vac}(\delta_k) = -\tilde{C}_\mu^i(\delta_k)\hat{n}_i(\delta_k) + \tilde{C}_\mu^i(\delta_k).
\]

(28)

The expressions for the vacuum potential \( \tilde{A}_\mu^{vac}(\delta_k) \) and for the vacuum equation \( \tilde{F}_{\mu
u} = 0 \) are highly non-linear. However, due to Abelian structure of the vacuum gauge potential, one can verify that any linear combination of \( \tilde{A}_\mu^{vac}(\delta_k) \) with coefficients \( c_k \) satisfying the condition \( \sum_k c_k = 1 \) represents a vacuum as well. A Weyl symmetric vacuum is given by the symmetric linear superposition of \( \tilde{A}_\mu^{vac}(\delta_k) \)

\[
\tilde{A}_\mu^{Weyl} = \frac{1}{6} \sum_{k=0,\ldots,5} \tilde{A}_\mu^{vac}(\delta_k).
\]

(29)

One can choose angle values \( \delta_k = (0, \frac{2\pi}{3}, \frac{4\pi}{3}) \) corresponding to \( I, U, V \)-vacuums for corresponding \( SU(2) \) subgroups of \( SU(3) \). This allows to factorize the reflection subgroup from the full Weyl group and define a reduced Weyl symmetric vacuum

\[
\tilde{A}_\mu^{Weyl} = \frac{1}{3}(\tilde{A}_\mu^{vac} + \tilde{A}_\mu^{vac} + \tilde{A}_\mu^{vac}).
\]

(30)

In the next section we will consider a special parametrization for the Killing vector fields and derive the corresponding vacuum gauge potentials. Explicit expressions for \( I, U, V \)-type and Weyl symmetric vacuums are given in Appendix. The Weyl symmetric vacuum is invariant under the basic Weyl permutation group \( Z_3 \) and can be useful in search of new essentially \( SU(3) \) classical solutions.

IV. ANSATZ FOR INSTANTON SOLUTIONS

Our approach to vacuum construction in terms of the Killing vectors on \( SU(3) \) allows to define a more general ansatz for searching possible classical solutions in \( SU(3) \) Yang-Mills theory. In this section we apply a spherically symmetric vacuum ansatz with two topological numbers to study possible non-trivial instanton solutions. Let us consider a standard ’t Hooft ansatz for \( n = 1 \) \( SU(2) \) instanton \([3]\):

\[
\tilde{A}_\mu = f(\rho) U \partial_\mu U^{-1},
\]

\[
U = \frac{x_4 + i\sigma_3 x_i}{\rho},
\]

(31)

where \( x_\mu = (\vec{E}, x_4) \) represent Cartesian coordinates in Euclidean four dimensional space-time, and \( \sigma_i \) are Pauli matrices. The expression for the pure gauge potential \( U \partial_\mu U^{-1} \) in \( (31) \) can be reproduced in our approach using the expression for the \( SU(2) \) vacuum gauge potential \( (18) \) with a Killing vector defined by

\[
\hat{m}_3 = e^{-n\phi_3}e^{-(\pi - \theta)\epsilon_3} = \begin{pmatrix}
\sin \theta \cos(n\phi) \\
\sin \theta \sin(n\phi) \\
-\cos \theta
\end{pmatrix},
\]

(32)

where \( \epsilon_3 = (0,0,1) \) for the case of \( SU(2) \) group.

In the case of \( SU(3) \) gauge theory we define a Killing vector field \( \hat{m}_3 \) by the following gauge transformation

\[
\hat{m}_3 = e^{-n'\phi(-\frac{1}{3}t_3 + \frac{\sqrt{2}}{3}t_8)} e^{\theta t_7} e^{-(n-\frac{1}{2}n')\phi t_3} e^{-\theta t_2} \hat{\xi}_3
\]

(33)

where \( \hat{\xi}_3 = (0,0,1) \) for the case of \( SU(2) \) group.

The parameters \( n, n' \) determine the topological structure of the gauge theory and they are related to instanton and monopole topological charges \([18, 19]\). The vector \( \hat{m}_8 \) can be obtained from
\( \hat{m}_3 \) by using the \( d \)-product

\[
\hat{m}_3 = \sqrt{3} d^{abc} \hat{m}_1 \hat{m}_2 \hat{m}_3 = \begin{pmatrix}
0 \\
\sqrt{\frac{2}{5}} (1 - \cos \theta) \\
0 \\
\frac{2}{\sqrt{3}} \sin \theta \cos \theta' \phi \\
\frac{2}{\sqrt{3}} \sin \theta \sin \theta' \phi \\
\frac{1}{4} (1 + 3 \cos \theta)
\end{pmatrix}.
\] (34)

Explicit expressions for respective \( I, U, V \) and Weyl symmetric vacuum gauge potentials are given in Appendix.

The \( I \)-vacuum is defined by a pure gauge potential corresponding to embedding \( I \)-type \( SU(2) \) subgroup into \( SU(3) \) can be written in a Weyl symmetric form \( (p = 1, 2, 3) \)

\[
\hat{A}_{\mu vac} = -\frac{2}{3} \sum_{p=1,2,3} \left( \hat{C}_{\mu p} \hat{m}_p + \hat{m}_p \times \partial_\mu \hat{m}_p \right),
\] (35)

where \( \hat{C}_{\mu p} \) represent Weyl symmetric combinations of fields \( \hat{C}_{\mu 3,8} \)

\[
\hat{C}_{\mu 1} = \frac{1}{2} (\hat{C}_{\mu 3} + \sqrt{3} \hat{C}_{\mu 8}), \\
\hat{C}_{\mu 2} = \frac{1}{2} (\hat{C}_{\mu 3} - \sqrt{3} \hat{C}_{\mu 8}), \\
\hat{C}_{\mu 3} = \hat{C}_{\mu 3},
\] (36)

and we have similar definitions for the Weyl symmetric combinations \( \hat{m}_p \).

Let us now consider the Weyl symmetric structure of instanton solutions corresponding to \( I, U, V \)-spin \( SU(2) \) subgroups of \( SU(3) \). For \( I \)-spin \( SU(2) \) subgroup one can introduce an instanton ansatz with three trial functions \( f_\rho (\rho) \)

\[
\hat{A}^I_{\mu} = -\frac{2}{3} \sum_{p=1,2,3} f_\rho (\rho) \left( (q_\rho \partial_\rho \gamma + C_\mu p) \hat{m}_p + \hat{m}_p \times \partial_\mu \hat{m}_p \right),
\] (37)

where the number parameters \( q_\rho \) satisfy Weyl symmetry condition \( q_1 + q_2 + q_3 = 0 \). In this section we use simple notations \( q_1 \equiv i, q_2 \equiv u, q_3 \equiv v \).

The (anti-) self-duality equations are

\[
F^a_{\mu \nu} = \pm \frac{1}{2} \sqrt{g} \epsilon_{\mu \nu \rho \sigma} g^{\rho \tau} g^{\sigma \xi} F^a_{\tau \xi}.
\] (38)

One has three independent sets of self-duality equations:

\[
F^a_{\rho \theta} = \frac{2}{\rho \sin \theta} F^a_{\phi \gamma},
\] (39)

\[
F^a_{\rho \phi} = -\frac{2}{\rho \sin \theta} (F^a_{\theta \gamma} - \cos \theta F^a_{\phi \phi}),
\] (40)

\[
F^a_{\rho \gamma} = \frac{2}{\rho \sin \theta} (F^a_{\phi \phi} - \cos \theta F^a_{\phi \gamma}).
\] (41)

The first set of Eqs. (39) is most simple for solving. One has \( F^a_{\rho \theta} = F^a_{\phi \gamma} = 0 \). Notice, all other components of the field strength are non-zero. There are only three independent equations among the remaining ones in (39) corresponding to indices \( a = 1, a = 4 \) and \( a = 6 \). Each equation actually implies two equations since it has one part which does not include the dependence on the angle \( \theta \) and another part which includes terms proportional to \( \cos \theta \). It is convenient to start with equation (39) with index \( a = 4 \) which produces the following two equations:

\[
-f_1' + 5 f_2' + 8 f_3' + \frac{1}{3} \rho \left[ 2 \left( 2(n + 3'n')u f_1' \right) + 2(n + 3'n')v f_2' + 2i f_3'(12n - 8n f_3 + 3n' f_3) \\
+f_1(-12nu + (2n - 3'n')(u + v)f_2' - (4in + 3in') \\
-8nu + 3n'f_3) + f_2(-12nv - (4in + 3in') \\
-8nv + 3n'f_3) \right] = 0,
\] (42)

\[
3(f_1' - f_2') + \frac{2n'}{\rho} \left[ 2uf_1' + 2vf_2' - (i + v)f_2f_3 \\
+2if_3' + f_1(-u + v)f_2 - (i + u)f_3 \right] = 0,
\] (43)

The Eq. (39) with index \( a = 6 \) produces the following two equations:

\[
5f_1' + 5f_2' + 2f_3' + \frac{2n'}{\rho} \left[ -3uf_1' \\
+f_1(6u - 2(u - v)f_2 + (i - u)f_3) \\
+f_2(3v(-2 + f_2) + (-i + v)f_3) \right] = 0,
\] (44)

\[
3f_1' - 3f_2' + \frac{2n'}{3\rho} \left[ -7uf_1' - 7vf_2' \\
+f_1(6u + 2(u + v)f_2 + 5if_3 - uf_3) \\
+2i(-6 + f_3)f_2 + f_2(6v + 5if_3 - v f_3) \right] = 0.
\] (45)

Summing up the Eqs. (43) and (44) one obtains the equation

\[
u'(u f_1 + v f_2 + 2(u + v)f_3)(f_1 + f_2 + 4f_3 - 6) = 0.
\] (46)

Careful analysis shows that the last equation has a non-trivial instanton solution only when \( n' = 0 \).

Taking into account the equation (39) with index \( a = 1 \) and Eq. (44) one can obtain the following relations

\[
f_1(\rho) = f_2(\rho) \equiv f(\rho) \\
f_3' = 5f_2' = 0, \\
\rho f'' = 2n(u + v)(f - 1)(3f - 4).
\] (47)

The solution to the last equation is

\[
f = \frac{4a^2 + \rho^{2n(u + v)}}{3a^2 + \rho^{2n(u + v)}},
\] (48)

where \( a \) is a dimensional integration constant (instanton size). Substitution of the solution into the remaining
self-duality equations fixes the values of the parameters $n = 1$, $u = v = \frac{1}{2}$. Finally, the solution is given by

\begin{align*}
    f_1 &= f_2 \equiv f = \frac{4a^2 + \rho^2}{3a^2 + \rho^2}, \\
    f_3 &\equiv g = \frac{-2a^2 + \rho^2}{3a^2 + \rho^2}.
\end{align*}

(49)

For self-dual instantons with winding number $n = 1, n' = 0$ the solution is given by two functions $f, g$. Up to re-scaling the parameter $a$ this is the known 't Hooft instanton solution.

The $U, V$-type $SU(2)$ embedded instanton solutions can be obtained in a similar manner starting from the constant color vectors

\begin{align*}
    \hat{\xi}^U &= (0, 0, \frac{1}{2}, 0, 0, 0, 0, \sqrt{\frac{3}{2}}), \\
    \hat{\xi}^V &= (0, 0, \frac{1}{2}, 0, 0, 0, 0, -\sqrt{\frac{3}{2}}). 
\end{align*}

(50)

To obtain the instanton solution from the Weyl symmetric vacuum we generalize the ansatz

\[ \bar{A}^{Weyl}_\mu = c_1(\rho)\bar{A}^I_\mu + c_2(\rho)\bar{A}^U_\mu + c_3(\rho)\bar{A}^V_\mu . \]

(51)

In general $\bar{A}^{I, U, V}_\mu$ depend on six functions $f^{I, U, V}, g^{I, U, V}$, different winding numbers $n^{I, U, V}$ and different parameters $q^{I, U, V}_\mu$. It is surprising, this ansatz produces a unique instanton solution which is given by the sum of $I, U, V$–instantons with the same functions $(f, g)$ given in (49) and coefficients $c_1 = c_2 = c_3 = \frac{1}{3}$ (the parameters $q^{I, U, V}_\mu$ are different and determined by winding number $n$ for $I, U, V$–instanton solutions). The winding numbers for $I, U, V$–instantons must be the same, i.e., $n^I = n^U = n^V = +1$.

Notice, one can choose three $SU(2)$ vacuums corresponding to various values of the angle $\delta$. All vacuums are gauge equivalent, and the instanton solution in the Weyl symmetric form is gauge equivalent to $SU(2)$ embedded instanton. For the symmetric set of angles $\delta_k = (0, \frac{2\pi}{3}, \frac{4\pi}{3})$ the Weyl symmetric solution coincides exactly with $I$ type $SU(2)$ instanton. So that, all $I, U, V$ and Weyl symmetric vacuums are the same for instanton solutions due to the condition $n' = 0$.

The $U$-type instanton solution is given by

\begin{align*}
    f_1 &= f_3 \equiv f, \quad f_2 \equiv g, \\
    q_1 &= -q_2 - q_3, \\
    q_2 &= \frac{1}{2n}, \quad q_3 = -\frac{1}{n}, \\
    n &= \pm 1,
\end{align*}

(52)

the $V$-type instanton solution is given by

\begin{align*}
    f_2 &= f_3 \equiv f, \quad f_1 \equiv g, \\
    q_1 &= -q_2 - q_3, \\
    q_2 &= -n, \quad q_3 = \frac{1}{2n}, \\
    n &= \pm 1,
\end{align*}

(53)

where the functions $f, g$ are given by the same functions as in the case of $I$–instanton.

Our conclusion is the following, spherically symmetric instanton solutions are insensitive to the presence of two topological numbers. Even though the Chern-Simons number for each $I, U, V$ vacuum is expressed by the sum

\[ n_{CS} = 2p_i n + \frac{q_k - 3p_i}{2} n', \]

(54)

the self-dual equations admit a unique solution with a constraint $n' = 0$. Our analysis is restricted by spherically symmetric solutions, it is still possible that there might exist non-spherically symmetric, essentially $SU(3)$ instanton which admits a non-zero value for $n'$.

Notice, that the parametrization (53) is not unique. For instance, one can perform a different gauge transformation of the constant vector $\xi_3$

\[ \tilde{m}_3 = e^{-w_2} e^{\beta t_2} e^{\theta t_7} e^{-w_1} e^{\theta t_5} e^{-\beta t_4} \xi_3, \]

(55)

where $w_1, w_2$ are winding numbers corresponding to rotations by angles $\phi$ and $\gamma$. The corresponding Chern-Simons number is the same for $I, U, V$ type vacuum gauge potentials

\[ n_{CS} = w_1 w_2. \]

(56)

The parametrization (55) is more suitable since it does not include the gauge parameters $p_i, q_i$ present in (54). The presence of different expressions for the Chern-Simons number in terms of topological numbers of the homotopy group $\pi_2(M^6)$ is reflection of the fact that the topological numbers $(n, n')$ (or $w_{1,2}$), as well as the Chern-Simons number, are not gauge invariant (contrary to topological Pontryagin number). One should notice, since the third homotopy group $\pi_3(M^6)$ coincides with $\pi_3(SU(3))$ the degenerated vacuum structure rather can not be removed through vacuum tunneling effect. We expect that such a degenerated vacuum structure manifests itself due to generation of monopole condensation in QCD.

V. WEYL REPRESENTATION FOR MONOPOLES

In this Section we consider the Weyl symmetric structure of singular and finite energy monopoles in $SU(3)$ QCD. Our consideration of finite energy monopoles coincides formally with the BPS limit of monopole solutions in Yang-Mills-Higgs theory. However, one should stress, that in QCD one should not introduce the Higgs potential that provides the spontaneous symmetry breaking scale. Our point is that the mass scale parameter in monopole solutions in QCD must represent a free parameter. The Higgs field $\Phi^a$ is treated as a deformation of the Killing vector within the pure QCD theory. A possible mechanism of generation of the kinetic term for the Higgs
field in the Lagrangian can be realized due to quantum dynamics of gluons. Such a mechanism takes place in the Faddeev-Skyrme model [29, 30], where the topological Killing vector field \( \hat{n} \) gains dynamical degrees of freedom in the effective theory of QCD [51].

V.1. Singular monopoles

Let us first consider the Weyl symmetric structure of singular monopole solutions. Singular monopoles can be constructed using the Killing vectors \( \hat{n}_i(\delta) \) defined by Eqs. (28, 35, 37). Under the orthogonality condition \( \hat{n}_3 \cdot \hat{n}_8 = \sin(3 \delta_k) = 0 \) one can calculate vector magnetic field \( \vec{H}^{K} \) and corresponding magnetic fields \( H^{K}_{\mu \nu} \), \( K = (I, U, V) \), for each angle \( \delta_k = \left(0, \frac{2\pi}{3}, \frac{4\pi}{3}\right) \) respectively

\[
\vec{H}^{K}_{\mu \nu} = -\partial_{\mu} \vec{C}^{\nu K} - \partial_{\nu} \vec{C}^{\mu K} + \vec{C}^{\mu K} \times \vec{C}^{\nu K} = H^{K}_{3\mu \nu} \hat{n}_3 + H^{K}_{8\mu \nu} \hat{n}_8, \tag{57}
\]

Explicit expressions for the magnetic fields of \( I, U, V \)-type monopoles are given by the following relationships

\[
H^{I}_{3\mu \nu} = (n - n') \cos \theta \sin \theta (\partial_{\mu} \partial_{\nu} \phi - \partial_{\nu} \partial_{\mu} \phi),
\]
\[
H^{I}_{8\mu \nu} = \sqrt{3} \sin \theta (\partial_{\mu} \partial_{\nu} \phi - \partial_{\nu} \partial_{\mu} \phi), \tag{58}
\]
\[
H^{U}_{3\mu \nu} = (n - n') \cos \theta \sin \theta (\partial_{\mu} \partial_{\nu} \phi - \partial_{\nu} \partial_{\mu} \phi),
\]
\[
H^{U}_{8\mu \nu} = \sqrt{3} \left(-n + n' \cos \theta (\partial_{\nu} \partial_{\mu} \phi - \partial_{\mu} \partial_{\nu} \phi), \right) \tag{59}
\]
\[
H^{V}_{3\mu \nu} = (-n + n') \cos \theta (\partial_{\nu} \partial_{\mu} \phi - \partial_{\mu} \partial_{\nu} \phi),
\]
\[
H^{V}_{8\mu \nu} = \sqrt{3} \left(-n + n' \cos \theta (\partial_{\nu} \partial_{\mu} \phi - \partial_{\mu} \partial_{\nu} \phi) \right). \tag{60}
\]

Dual Abelian magnetic potentials \( \vec{C}^{K}_{\mu} \) can be easily derived using (7). All field strengths \( H^{I, U, V}_{3\mu \nu} \) contain the terms proportional to \( n' \cos \theta \). These terms prevent the fulfillment of equations of motion. To provide the above monopole configurations to be solutions of the equations of motion one can use the freedom in the definition of the restricted potential \( \vec{A}_{\mu} \). Namely, it is enough to define the Abelian gauge potential \( A^{I}_{\mu} \) (“photon”) in Eq. (1) in an appropriate way to cancel the terms proportional to \( n' \cos \theta \) in \( H^{K}_{3\mu \nu} \). For instance, for \( I \)-spin case one can define the Abelian gauge potential as follows \[ A^{I}_{\mu} = -\frac{n'}{4} \cos(2\theta) \partial_{\mu} \phi, \tag{61} \]
and similarly for \( U \)- and \( V \)-type monopole solutions. With this, the configurations \( 58, 60 \) represent exact singular monopole solutions of \( I, U, V \) type. The corresponding monopole charges are defined by

\[
g^{I}_{K} = \int \vec{H}^{K}_{\mu \nu} \cdot \hat{n}^{I}_{K} dS^{\mu \nu}, \tag{62}
\]

It is easy to calculate the monopole charges for the singular monopoles:

\[
g^{I}_{3} = 4\pi(n - n'), \quad g^{I}_{8} = 4\pi(\sqrt{3}/2n'),
\]
\[
g^{U}_{3} = 4\pi(n - n'), \quad g^{U}_{8} = 4\pi(\sqrt{3}/2n),
\]
\[
g^{V}_{3} = 4\pi(-n + n'), \quad g^{V}_{8} = 4\pi(\sqrt{3}/2(n - n')) \tag{63}
\]

For all magnetic charges one has the same expression for the total magnetic charge

\[
g^{tot} = \sqrt{(g^{3})^{2} + (g^{8})^{2}} = \sqrt{n^{2} - nn' + n'^{2}} \tag{64}
\]

For unit monopole charge one has six types of monopoles corresponding to various combinations of \( n, n' \) (three other monopole solutions are obtained by reflections from \( I, U, V \) monopoles).

It should be noticed, that there is a principal difference between structures of the monopole solutions in \( SU(2) \) and \( SU(3) \) Yang-Mills theories. The singlet representation of \( SU(2) \) monopoles with vanishing monopole charge, \( g = 0 \), can be constructed from monopole and anti-monopole solutions. In \( SU(3) \) case the colorless state with a total vanishing monopole charge can be constructed in a different way, from three \( I, U, V \) monopoles. From the Eq. (63) it follows that the total sum of \( I, U, V \) monopole charges \( g^{I}_{K} \) for such a system is zero.

V.2. Finite energy \( SU(3) \) QCD monopoles

A well known finite energy monopole is given by the ’t Hooft-Polyakov monopole solution in \( SU(2) \) Yang-Mills-Higgs theory [52, 30]. Simple generalizations of \( SU(2) \) ’t Hooft-Polyakov monopole to the case of \( SU(3) \) theory were considered in [37, 41]. To find finite energy monopole solutions in \( SU(3) \) Yang-Mills-Higgs theory with color scalar octet one can start with the Killing vector \( \hat{n}_3 \) given by a simple gauge transformation

\[
\hat{n}_3 = e^{-i\sigma_3} e^{-i\sigma_2} \xi_3. \tag{55}
\]

For a non-trivial embedded finite energy monopole solution with two magnetic charges corresponding to Cartan algebra of \( SU(3) \) one should rather apply a parametrization for \( \hat{n}_3 \) similar to one given in [53] which would imply two monopole charges due to presence of two winding numbers \( (n, n') \). Unfortunately, in that case the ansatz for the gauge potential becomes non-spherically symmetric

\[
\Phi^a = \hat{n}_3 F(r, \theta) + \hat{n}_8 G(r, \theta),
\]
\[
\vec{A}_{\mu} = U_{\mu}(r, \theta) \hat{n}_3 + S_{\mu}(r, \theta) \hat{n}_8 + \hat{n}_3 \times \partial_{\mu} \hat{n}_3 P(r, \theta) + \hat{n}_8 \times \partial_{\mu} \hat{n}_8 Q(r, \theta). \tag{66}
\]

The origin of this lies in the non-trivial homotopy group \( \pi_2(SU(3)/U(1) \times U(1)) \). Namely, in the presence of the winding number \( n' \) the singular monopole solution includes the term with the Abelian gauge potential \( A^I_{\mu} \hat{n}_3 \).
This term is incompatible with spherically symmetric ansatz for \( \Phi \) due to the equation of motion for Higgs field

\[
(\bar{D} \bar{D} \Phi)^a = \lambda \Phi^a (\Phi^2 - \eta^2). \tag{67}
\]

The term \( A_i^a \hat{m}_3 \) implies that the l.h.s. of the equation does not belong to the Cartan plane \((\hat{m}_3, \hat{m}_8)\), so that one has to introduce six functions with angle dependence in the ansatz \([54]\).

To study the structure of monopole solutions in \( SU(3) \) theory one can start with I-type Killing vector \( \hat{m}_3 \) defined by the gauge transformation \([55]\). The simplest ansatz is the following

\[
\Phi^a = \hat{m}_3 F(r) + \hat{m}_8 G(r),
\]

\[
\hat{A}_\mu = \hat{m}_3 \times \partial_\mu \hat{m}_3 P(r) + \hat{m}_8 \times \partial_\mu \hat{m}_8 Q(r). \tag{68}
\]

One can treat the functions \( F(r), G(r) \) as length deformations of the Killing vectors \( \hat{m}_i \). Due to the chosen parametrization of \( \hat{m}_3 \) the second Killing vector is just a constant color vector \( \hat{m}_8^a = \delta_8^a \). So, the field \( Q(r) \) can be omitted in the last equation. We consider a standard Yang-Mills-Higgs Lagrangian in Minkowski spacetime with a flat metric \( \eta_{mn} = (- ++ +) \)

\[
\mathcal{L}^{YMH} = -\frac{1}{4} \text{Tr} \bar{F}_{mn} \bar{F}^{mn} - \frac{1}{2} (D_m \Phi)^a (D^n \Phi)^a - \frac{\lambda}{4} (\Phi^2 - \eta^2). \tag{69}
\]

Substituting the ansatz \([68]\) into the equations of motion leads to a system of ordinary differential equations

\[
\begin{align*}
r^2 P'' &= (1 + P)(r^2 F^2 + P(P + 2)), \\
r^2 F'' + 2 r F' &= F(2 P(P + 2) + 2 - \frac{\lambda}{3} r^2 (F^2 + G^2 - v^2)), \\
r G'' + 2 G' &= -\frac{\lambda}{3} G(F^2 + G^2 - v^2). \tag{70}
\end{align*}
\]

Let us consider the solution structure of the equations in the BPS limit, \( \lambda = 0 \). In the case of non BPS limit the solution was obtained numerically in \([41]\). Notice, even though \( \lambda = 0 \) in the BPS limit, one has still an effect of the Higgs potential which implies the asymptotic boundary condition \( \Phi(r = \infty) = v \). With this the solution to \([70]\) reads

\[
\begin{align*}
P(r) &= -1 + \frac{vr}{\sinh(vr)}, \\
F(r) &= \frac{vr \coth(vr) - 1}{r}, \\
G(r) &= C_1. \tag{71}
\end{align*}
\]

As we mentioned above, in QCD one should not introduce any Higgs potential since in the confinement phase the color symmetry is unbroken, so that one has no any pre-fixed scale parameter like the spontaneous symmetry breaking parameter \( v \). An interesting observation is that in the absence of Higgs potential, one has still a class of finite energy monopole solutions parameterized by an arbitrary mass scale parameter \( \mu \)

\[
\begin{align*}
P(r) &= -1 + \frac{\mu r}{\sinh(\mu r)}, \\
F(r) &= \frac{\mu r \coth(\mu r) - 1}{r}, \\
G(r) &= \mu C_1. \tag{72}
\end{align*}
\]

The solution coincides formally with \([71]\) up to the replacement \( v \leftrightarrow \mu \). The introduced mass scale parameter \( \mu \) represents a free parameter which is treated as an intrinsic property of classical monopole solutions in a pure QCD where generation of a mass scale is a result of dynamical symmetry breaking. The integration constant \( C_1 \) describes two different classes of solutions. The zero value of the constant, \( C_1 = 0 \), implies a vanishing function \( G(r) = 0 \). Such a solution corresponds to the trivial embedding \( SU(2) \) monopole. Non-zero value of \( C_1 \) defines I-type embedding of the monopole corresponding to \( I \)-spin subgroup \( SU(2) \). For \( U \) and \( V \) type monopole solutions the respective functions \( G(r) \) become non-constant functions \([41, 42]\). Let us consider a more general class of monopole solutions by performing the gauge transformation \([28]\) for the Killing vectors in the Cartan plane with an arbitrary embedding

\[
\begin{align*}
r^2 F''(r) - (1 + \cos^2 \delta P(r))(P(r)(2 + \cos^2 \delta P(r)) + r^2 W^2(r)) = 0, \\
r^2 W''(r) + 2 r W'(r) - 2 W(r)(1 + \cos^2 \delta P(r))^2 = 0, \\
r Z''(r) + 2 Z'(r) = 0, \tag{73}
\end{align*}
\]

where we have redefined the variables in the following way

\[
\begin{align*}
F(r) &= -\frac{W(r) \cos(2\delta) - 2 Z(r) \sin \delta}{1 - 2 \cos(2\delta)}, \\
G(r) &= -\frac{Z(r) - W(r) \sin \delta}{1 - 2 \cos(2\delta)}. \tag{74}
\end{align*}
\]

For finite energy monopole configurations the equation for the function \( Z(r) \) has a constant solution

\[
Z(r) = \mu C. \tag{75}
\]

The remaining equations in \([73]\) after proper redefinitions

\[
\begin{align*}
P(r) &= \frac{1}{\cos^2 \delta} B(r), \\
W(r) &= \frac{1}{\cos \delta} R(r). \tag{76}
\end{align*}
\]
reduce to equations for the functions $B(r)$, $R(r)$ identical to first two equations in (70). Finally, the solution is given by

$$P(r) = \frac{1}{\cos \delta} \left( \frac{\mu r}{\sin |\mu r|} - 1 \right),$$

$$F(r) = \frac{\cos[2\delta] \sec \delta (\mu r \coth |\mu r| - 1) - 2 Cr \sin \delta}{r(2 \cos(2\delta) - 1)},$$

$$G(r) = \frac{Cr + \tan \delta (1 - \mu r \coth |\mu r|)}{r(2 \cos(2\delta) - 1)}. \quad (77)$$

The Weyl representation for $I, U, V$ type monopoles can be obtained from the last equations by choosing respective values for the angle $\delta_k = (0; \frac{2\pi}{3}; \frac{4\pi}{3})$. One should notice, that all solutions for arbitrary angle $\delta$ are gauge equivalent. So that, the energy density and gauge invariant magnetic flux do not depend on angle $\delta$. The only dependent term on the integration constant $C$ is presented by the gauge invariant term $\Phi^2$. Gauge dependent monopole charges corresponding to the magnetic fields $H_{k,3}^{\mu m} = F_{\mu \nu} n_{3,8}^{k,8}$ include dependence on $\delta_k, \mu$. The solution (77) implies that there are six critical values $\delta_{CR} = \pi + \frac{\pi k}{3}, \ (k = 0, 1, 2, ..., 5), \ at \ which \ solutions \ gain \ infinite \ energy. \ At \ these \ critical \ points \ the \ Killing \ vectors \ \hat{n}_{3,8} \ become \ (anti-) \ parallel \ to \ each \ other. \ This \ implies \ that \ the \ variety \ of \ monopole \ solutions \ is \ divided \ into \ six \ sectors \ which \ are \ separated \ by \ infinite \ energy \ barrier \ and \ can \ not \ be \ connected \ by \ a \ smooth \ rotation \ in \ the \ Cartan \ plane.

VI. DISCUSSION

We have applied the formalism of gauge invariant Abelian projection to study of the Weyl symmetric structure of classical $SU(3)$ QCD vacuum. The topological structure of the vacuum can be described naturally in terms of Killing vector fields, i.e., by the homotopy groups $\pi_{2,3}(SU(3)/U(1) \times U(1))$. Whereas the third homotopy group provides the topological number which is equivalent to the Chern-Simons number, the second homotopy group implies that topologically non-equivalent classical vacuums are described by two topological numbers in general. We have shown that one can construct a classical vacuum for each given Killing vector field. Topologically non-equivalent vacuums in terms of Killing vectors form the Weyl sextet representation. It is an interesting feature of $SU(3)$ gauge theory that it has a non-trivial Weyl symmetric vacuum presented by a symmetric sum of $I, U, V$-type vacuums. Our construction of a more general vacuum can be useful in search of essentially $SU(3)$ instanton and monopole solutions.

The Weyl symmetry in QCD manifests itself in the presence of classical singular and finite energy monopole solutions which form representations of the Weyl group as well. In particular, the lowest dimension representations are given by singlet and sextet representations. This indicates to possible existence of similar structures of the vacuum and monopole condensates in QCD. Indeed, it has been shown that the quantum one-loop effective potential in QCD has a vacuum which is completely Weyl symmetric. Notice, that the orthogonality condition for the Killing vector fields $\hat{n}_i$ provides a necessary minimal energy condition for the classical vacuum. Similarly, the absolute minimum of energy in the quantum one-loop effective potential of QCD is realized when two independent homogeneous background color magnetic fields are orthogonal to each other.

At two-loop level one might have six degenerated vacuums forming Weyl sextet. At quantum level the infinite energy barrier between different Weyl sectors of monopoles becomes finite. We conjecture that two-loop effective potential has a Weyl symmetric vacuum in addition to possible six Weyl degenerated vacuums. To preserve the color symmetry there are two possibilities. First one is that the depth of the Weyl symmetric vacuum is larger than the depth of Weyl degenerated vacuums, so it provides a true vacuum. In the case if the energy of the Weyl sextet vacuums is lower than the energy of the Weyl symmetric vacuum the color symmetry can still remain unbroken due to possible vacuum tunneling which will remove the vacuum degeneracy. This will provide that Weyl symmetry as a part of whole color symmetry remains unbroken giving a possible answer to the problem of origin of color confinement in QCD.

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Appendix: Notations, useful relations

In consideration of instanton solutions in Section IV we use four dimensional cylindrical polar coordinate system

$$x = \rho \cos \frac{\theta}{2} \cos \frac{\gamma + \phi}{2},$$

$$y = \rho \cos \frac{\theta}{2} \sin \frac{\gamma + \phi}{2},$$

$$z = \rho \sin \frac{\theta}{2} \cos \frac{\gamma - \phi}{2},$$

$$t = \rho \sin \frac{\theta}{2} \sin \frac{\gamma - \phi}{2},$$

$$0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi, \ 0 \leq \gamma \leq 4\pi. \quad (78)$$
The corresponding metric $g_{\mu\nu}$ is given by

\begin{align*}
g_{rr} &= 1, \\
g_{\theta\theta} &= g_{\phi\phi} = g_{\gamma\gamma} = \rho^2, \\
g_{\phi\gamma} &= \frac{\rho^2 \cos \theta}{4}.
\end{align*}

(79)

In the analysis of static monopole solutions we use the standard three dimensional spherical coordinate system. In the derivation of Eqn. (33) the following relationship is used

\begin{equation}
\cos \alpha \gamma = 1 - M + M \cos \alpha + \gamma^2 \sin \alpha,
\end{equation}

\begin{equation}
M = \frac{1}{4} \begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 4 & 0 \\ -\sqrt{3} & 0 & 3 \end{pmatrix}.
\end{equation}

Relations for other $SU(3)$ group elements can be found in a similar manner.

Explicit expressions for $I$, $U$, $V$ type vacuum gauge potentials are given by the following equations

\begin{equation}
\vec{A}_{I\text{ vac} \mu} = \begin{pmatrix} \\
\frac{1}{32} (3(9n - 8n)\partial_r \phi + 4(u - 3v)\partial_r \gamma - 2 \cos \theta ((4n + n')\partial_r \phi + 8(2u + v)\partial_r \gamma) - \cos [2\theta] (n'\partial_r \phi + 4(u + v)\partial_r \gamma)) \\
\frac{1}{4} \left( 8 \sin[n' \phi] \partial_r \theta + \cos[n' \phi] (2(2n - n')\partial_r \phi + 4(u - v)\partial_r \gamma + \cos \theta(n'\partial_r \phi + 4(u + v)\partial_r \gamma)) \right) \sin[n' \phi] \sin \theta \\
\frac{1}{4} \left( -8 \cos[n' \phi] \partial_r \theta + \cos[n' \phi] (2(2n - n')\partial_r \phi + 4(u - v)\partial_r \gamma + \cos \theta(n'\partial_r \phi + 4(u + v)\partial_r \gamma)) \right) \cos[n' \phi] \sin \theta \\
\sqrt{3} \left( -3(8n + 11n')\partial_r \phi + 4(5u + v)\partial_r \gamma + 6((4n - 3n')\partial_r \phi - 8v\partial_r \gamma) \cos \theta + 3(n'\partial_r \phi + 4(u + v)\partial_r \gamma) \cos [2\theta] \right) \\
\end{pmatrix}
\end{equation}

\begin{equation}
\vec{A}_{U\text{ vac} \mu} = \begin{pmatrix} \\
\frac{1}{4} \cos \frac{\theta}{2} (\sin [(n - n') \phi] \partial_r \phi + v \cos [(n - n') \phi] \sin \theta \partial_r \gamma) \\
\cos \frac{\theta}{2} (-\cos [(n - n') \phi] \partial_r \phi + v \sin [(n - n') \phi] \sin \theta \partial_r \gamma) \\
\frac{1}{8} ((7n' - 6n)\partial_r \phi - (4u + v)\partial_r \gamma + ((n' - 2n)\partial_r \phi + 4(u + 2v) \cos \theta \partial_r \phi + v \cos [2\theta] \partial_r \gamma) \\
\sin \frac{\theta}{2} (\sin [n \phi] \partial_r \theta + v \cos [n \phi] \sin \theta \partial_r \gamma) \\
\sin \frac{\theta}{2} (-\cos [n \phi] \partial_r \theta + v \sin [n \phi] \sin \theta \partial_r \gamma) \\
\frac{1}{4} \left( 4 \sin[n' \phi] \partial_r \phi - \cos[n' \phi] \sin \theta ((n' - 2n)\partial_r \phi + 2(2u + v)\partial_r \gamma + 2v \cos \theta \partial_r \gamma) \right) \\
\frac{1}{4} \left( -4 \cos[n' \phi] \partial_r \theta - \cos[n' \phi] \sin \theta ((n' - 2n)\partial_r \phi + 2(2u + v)\partial_r \gamma + 2v \cos \theta \partial_r \gamma) \right) \\
\sqrt{3} \left( -3(2n + 3n')\partial_r \phi - (4u + 5v)\partial_r \gamma - 3(\cos \theta ((n' - 2n)\partial_r \phi + 4u\partial_r \gamma) + v \cos [2\theta] \partial_r \gamma) \right) \\
\end{pmatrix}
\end{equation}

(82)
The Weyl symmetric vacuum is written as follows

\begin{equation}
\vec{A}_{\text{vac} \mu}^W = \begin{pmatrix}
\cos \frac{\theta}{2} (\sin[(n-n')\phi] \partial_\mu \theta + u \cos[(n-n')\phi] \sin \theta \partial_\mu \gamma) \\
\cos \frac{\theta}{2} (-\cos[(n-n')\phi] \partial_\mu \theta + u \sin[(n-n')\phi] \sin \theta \partial_\mu \gamma) \\
\frac{1}{8} (7n' - 6n) \partial_\mu \phi + (3u + 4v) \partial_\mu \gamma + ((n' - 2n) \partial_\mu \phi + 4(u - v) \cos \theta \partial_\mu \gamma + u \cos[2\theta] \partial_\mu \gamma)
\end{pmatrix}
\end{equation}

\begin{equation}
\frac{1}{8} \left( (7n' - 6n) \partial_\mu \phi + (3u + 4v) \partial_\mu \gamma + ((n' - 2n) \partial_\mu \phi + 4(u - v) \cos \theta \partial_\mu \gamma + u \cos[2\theta] \partial_\mu \gamma) \right)
\end{equation}

The Weyl symmetric vacuum is written as follows

\begin{equation}
\vec{A}_{\text{vac} \mu}^{W,}\text{sym} = \begin{pmatrix}
\frac{1}{12} \cos \frac{\theta}{2} (12 \sin[(n-n')\phi] \partial_\mu \theta - \cos[(n-n')\phi] \sin \theta (n' \partial_\mu \phi + 4p \partial_\mu \gamma)) \\
-\frac{1}{12} \cos \frac{\theta}{2} (12 \cos[(n-n')\phi] \partial_\mu \theta + \sin[(n-n')\phi] \sin \theta (n' \partial_\mu \phi + 4p \partial_\mu \gamma)) \\
\frac{1}{96} ((83n' - 72n) \partial_\mu \phi - 4q \partial_\mu \gamma + (6n' - 4n) \partial_\mu \phi + 4(\theta - 7p) \partial_\mu \gamma) \cos \theta - \cos[2\theta] (n' \partial_\mu \phi + 4p \partial_\mu \gamma))
\end{pmatrix}
\end{equation}

\begin{equation}
\frac{1}{12} \sin \frac{\theta}{2} (12 \sin[(n-n')\phi] \partial_\mu \theta - \cos[(n-n')\phi] \sin \theta (n' \partial_\mu \phi + 4p \partial_\mu \gamma)) \\
-\frac{1}{12} \sin \frac{\theta}{2} (12 \cos[(n-n')\phi] \partial_\mu \theta + \sin[(n-n')\phi] \sin \theta (n' \partial_\mu \phi + 4p \partial_\mu \gamma)) \\
\sin[(n-n')\partial_\mu \theta + \frac{1}{24} \cos[(n-n') \sin \theta (6(2n' - n') \partial_\mu \phi + 2(p - q) \partial_\mu \gamma + n' \cos \theta \partial_\mu \phi + 4p \cos \theta \partial_\mu \gamma) \\
- \cos[(n-n') \partial_\mu \theta + \frac{1}{24} \sin[(n-n') \sin \theta (6(2n' - n') \partial_\mu \phi + 2(p - q) \partial_\mu \gamma + n' \cos \theta \partial_\mu \phi + 4p \cos \theta \partial_\mu \gamma)
\end{pmatrix}
\end{equation}

\begin{equation}
\frac{\sqrt{3}}{288} (-3(24n + 35n') \partial_\mu \phi + 4(4p - q) \partial_\mu \gamma + 6(12n - 7n') \partial_\mu \phi + 3((2n - n') \partial_\mu \phi + 4(u + v) \partial_\mu \gamma) - 3u \cos[2\theta] \partial_\mu \gamma) \end{equation}

(83)

(84)
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