Energy-Momentum Conservation and
Holographic S-Matrix

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We investigate the consequence of the energy-momentum conservation law for the holographic S-matrix from AdS/CFT correspondence. It is shown that the conservation law is not a natural consequence of conformal invariance in the large $N$ limit. We predict a new singularity for the four point correlation function of a marginal operator. Only the two point scattering amplitude is explicitly calculated, and the result agrees with what is expected.
1. Introduction and Intuitive Observation

Although there exist several nonperturbative formulations of M theory and string theory in a flat spacetime, it has been hard to do a quantitative calculation in the right regime where the formulation is supposed to be valid. This difficulty has to do with the unusual physics which is required of a holographic theory [1]. The most recent proposal is an explicit ansatz for the S-matrix in the flat space limit using a convolution of conformal correlators in the boundary conformal field theory [2,3], for a related discussion, see [4]. A few puzzling aspects are already pointed out within this context in [5].

The ansatz of [2,3] involves taking a peculiar high energy limit along with a large N limit. For instance, the type IIB string theory defined on $R^{10}$ is encoded in the super Yang-Mills theory on $S^3 \times R$ in the large N limit. Of course the whole theory may not exist in the large N limit when the Yang-Mills coupling constant is held fixed. The conjecture of [2,3] rather asserts that a certain subsector must exist. If one possesses infinite calculational power to calculate all the relevant correlation functions in SYM, the IIB string S-matrix can be constructed nonperturbatively.

Of course for the time-being we are not yet that powerful. On the contrary, the Maldacena conjecture [6] has been used to make predictions about the SYM theory. Although we do not know much about the nonperturbative S-matrix, a few principles are certainly applicable. Traditionally, very general principles such as Lorentz invariance, unitarity and analyticity constitute strong constraints on the S-matrix. We expect that these constraints transform into the ones on a subset of correlation functions in SYM via the holographic S-matrix ansatz. Our purpose in this paper is to point out that the simplest consequence of Lorentz invariance, the energy-momentum conservation law, is not a bona fide consequence of conformal invariance. Notice that the isometry of the anti-de Sitter space is the conformal group. Taking the large radius limit, the conformal group contracts to the Poincare group. Rather surprisingly, already at the level of the 4 point amplitude, implementation of energy-momentum conservation requires the existence of a new type of singularity in the 4 point correlation functions in the large N limit. This singularity, to our knowledge, is not dictated by conformal symmetry.

In order to extract information about physical process happening in the center of the anti-de Sitter space, well-focused wave packets must be prepared. A precise ansatz for an incoming particle or an outgoing particle is given in [2]. We will focus on massless particles, for we will work with the type IIB string theory, and the only stable states are
those of supergraviton. Denote the creation operator of an incoming particle by $\alpha_{\omega e^{-}}$, where $\omega = RE$ is the dimensionless energy, $R$ is the radius of $AdS_5$, and $e$ is a unit 4 vector. This particle carries a momentum $\omega e$ tangent to $AdS_5$ in its center. The state is smeared over $S^5$, or it carries a zero momentum in the internal space. Similarly, denote the annihilation operator of an outgoing particle by $\alpha_{\omega e^{+}}$. For a scalar particle, the ansatz of \cite{2} is

$$\alpha_{\omega e^{-}} = \omega^{-3/2} \int dtd\Omega \exp\left\{ -\frac{\omega}{2} \left[ (t + \pi/2)^2 + |x + e|^2 \right] - i\omega(t + \pi/2) \right\} \mathcal{O}(t, x),$$

$$\alpha_{\omega e^{+}} = \omega^{-3/2} \int dtd\Omega \exp\left\{ -\frac{\omega}{2} \left[ (t - \pi/2)^2 + |x - e|^2 \right] + i\omega(t - \pi/2) \right\} \mathcal{O}(t, x),$$

where $\mathcal{O}$ is the appropriate operator corresponding to the scalar field \cite{7,8,9}. For the dilaton, it is proportional to $\text{tr} F^2$, for the axion, it is proportional to $\text{tr} F \wedge F$. We assumed that $\mathcal{O}$ is properly normalized, so some $\omega$ independent numerical factors in \cite{2} were dropped. The integral $\int d\Omega$ is over the unit $S^3$ which is parameterized by $x$.

The above ansatz clearly indicates that the incoming (outgoing) particle originates (ends up) at time $-\pi/2$ ($\pi/2$). In the large $R$ limit, the proper time goes to $\pm \infty$. The Gaussian factor helps to focus the beam in the direction $e$. A S-matrix element is given by

$$S = \lim_{N \to \infty} \Phi^{-1} \langle \prod_i \alpha_{\omega e^{-}} \prod_j \alpha_{\omega e^{+}} \rangle,$$

where $\Phi$ is a normalization factor. For a fixed string coupling constant, due to the relation $R^4 = 4\pi g_s N\alpha'^2$, the large $R$ limit is achieved by taking the large $N$ limit.

In a conformal field theory, both the two point functions and three point functions are fixed up to a numerical coefficient by conformal symmetry. One would expect that the calculation of the two point amplitudes and three point amplitudes using eqs. (1.1), (1.2) is a simple matter. Actually, as shown in the next section, an exact form of three amplitude can be obtained only after some tedious calculation. In this section we will be content with a qualitative examination of these amplitudes.

The geometry $S^3 \times R$ is conformal to $R^4$, so correlation functions on $S^3 \times R$ can be obtained from those on $R^4$ using the conformal transformation. For instance, the Euclidean distance between the two points $x$ and $y$, $r^2 = |x - y|^2$ is mapped to

$$\exp(\tau_x + \tau_y)(\cosh(\tau_x - \tau_y) - x \cdot y),$$

where...
where we parametrize $R^4$ by the radial coordinates $r = \exp \tau$ and the unit sphere $S^3$. $x$ and $y$ are unit 4 vectors. Now $\tau$ and $x$ parametrize $S^3 \times R$.

In a correlation function, the extra factors such as $\exp(\tau_1 + \tau_2)$ are removed by a conformal factor. If $F(r_{ij}^2)$ is a correlation function on $R^4$, then the corresponding correlation function on $S^3 \times R$ is obtained by simply replacing $r_{ij}^2$ with $\cosh(\tau_i - \tau_j) - x_i \cdot x_j$. To obtain the correlation function on a Minkowskian $S^3 \times R$, we wick-rotate $\tau$ to $i\tau$, and add a term $i\epsilon$:

$$r_{ij}^2 = \cos(t_i - t_j) - x_i \cdot x_j + i\epsilon,$$

where the $i\epsilon$ prescription is introduced to ensure causality in the boundary conformal theory. To see this, assume $t_i - t_j$ small, and the angle $\phi_{ij}$ between $x_i$ and $x_j$ small, we obtain

$$-\frac{1}{2}(t_i - t_j)^2 + \frac{1}{2}\phi_{ij}^2 + i\epsilon,$$

we see that if one uses $r_{ij}^{-2}$ as the propagator, the signal will propagate along the future light-cone for a positive energy mode of the form $\exp(-i\omega t)$.

The scaling dimension of an operator corresponding to a massless scalar field is $\Delta = 4$. The two point function is therefore

$$\langle O(t_1, x_1)O(t_2, x_2) \rangle = (\cos(t_1 - t_2) - x_1 \cdot x_2 + i\epsilon)^{-4}$$

up to a normalization constant.

The two point scattering amplitude is obtained using (1.2). Without taking the large $N$ and high energy limit, there is no energy-momentum conservation, since as shown in the appendix of [2], the incoming and outgoing waves have finite width in both energy and momentum. The width of $\omega$ is proportional to $\sqrt{\omega}$. Using $\omega = RE$, the width of $E$ is proportional to $\sqrt{E/R}$ and goes to zero in the large $R$ limit. So energy-momentum conservation has to be recovered in this limit. We shall show in the next section that the energy conservation is always guaranteed in this limit.

However, the momentum conservation rather imposes strong constraints on the behavior of conformal correlators in the large $N$ limit. As we will see in the next section, the convolution of (1.2) using ansatz (1.1) is rather subtle. To obtain the exact numerical answer, one cannot simply replace the Gaussian distribution of (1.1) by a simpler one, say a delta function. To see the momentum conservation, though, we will do this in this section.
In the case of two point amplitude, replacing the Gaussian wave packets by delta functions, we obtain
\[(1 - e_1 \cdot e_2)^{-4}.
\] (1.7)
Together with a factor depending on \(\omega_i\), we will have a null result if \(e_1 \cdot e_2 \neq 1\). The above expression is singular if \(e_1 \cdot e_2 = 1\) or equivalently \(e_1 = e_2\). Thus we hope that a more careful calculation will result a delta function \(\delta^3(e_1 - e_2)\) for the two point amplitude. Here we define the delta function by
\[\int \delta^3(e_1 - e_2) d\Omega_2 = 1.
\]
Together with the energy conservation \(\omega_1 = \omega_2\), or \(E_1 = E_2\), this implies the momentum conservation \(E_1 e_1 = E_2 e_2\). The conservation is due to the fact that the two point conformal correlation function is singular if one point sits on the future light-cone of the other point.

The three point correlation function of operator \(O\) is fixed by conformal invariance. Again up to a constant, it is
\[
\langle O(t_1, x_1)O(t_2, x_2)O(t_3, x_3) \rangle = (\cos(\pi - t_2) - x_1 \cdot x_2 + i\epsilon)^{-2} \\
(\cos(t_2 - t_3) - x_2 \cdot x_3 + i\epsilon)^{-2}(\cos(t_1 - t_3) - x_1 \cdot x_3 + i\epsilon)^{-2}.
\] (1.8)
Consider the case where two states are incoming, one is outgoing. The Gaussian wave packets force \(t_1, t_2\) to center around \(-\pi/2\), and \(t_3\) to center around \(\pi/2\). If we simply replace the Gaussian wave packets by delta functions, the above expression becomes
\[(1 - e_1 \cdot e_2)^{-2}(1 - e_2 \cdot e_3)^{-2}(1 - e_1 \cdot e_3)^{-2}.
\] (1.9)
This function is singular whenever \(e_i = e_j\) for \(i \neq j\). It is most singular when all \(e_i\) are equal. Thus we expect that a more careful treatment will lead to delta functions forcing all \(e_i\) to be equal. Indeed this is the consequence of momentum conservation for a three point amplitude involving only massless particles: That all momenta of outlegs must be collinear. This is most easily seen in the following physical way. If, say, the two incoming states are not collinear, then one can go to the center of mass frame in which the end product of the scattering can never be a single massless particle.

Mathematically, the above result, as in the two point amplitude case, is a consequence of causality in the boundary theory: Whenever two points are separated by a null geodesics, then the correlation function becomes singular. For the three point function, it is most singular when they lie on the same light-cone.
This raises a puzzle already at the level of the three point amplitudes. Imagine that there are stable massive particles. In this case the three point correlation is still given by a formula similar to (1.8) if one replaces $2$ by $(\Delta_i + \Delta_j - \Delta_k)/2$, if the particles carry zero momentum in the internal space $S^5$. Again the correlation function becomes singular when $e_i$ and $e_j$ are equal, provided $\Delta_i + \Delta_j > \Delta_k$. However this has nothing to do with the momentum conservation for massive particles. It is therefore quite interesting to note that there are no stable massive stringy states in the type IIB theory. Thus this puzzle is avoided. A related feature is that the conformal dimension of a stringy state is divergent in the large $N$ limit \cite{7,8}, thus although the convolution (1.2) exists for finite $N$, its large $N$ limit does not exist.

It is more interesting to see what happens to the four point correlation function when the momentum conservation is imposed. The four point correlation function of operator $O$ of scaling dimension 4 can not be fixed by conformal symmetry alone. Up to a scaling factor, it is a function of two independent cross-ratios. Define the cross-ratios

$$a = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2}, \quad b = \frac{r_{12}^2 r_{34}^2}{r_{23}^2 r_{14}^2},$$

(1.10)

the correlation function can be written as, for instance

$$\langle O(t_1, x_1) \ldots O(t_4, x_4) = F_4(r_{ij}^2) = \prod_{i<j\leq 4} r_{ij}^{-8/3} f(a, b),$$

(1.11)

where $f$ is a undetermined function of $a$ and $b$. All $r_{ij}^2$ are as given in (1.4).

Unlike in some 2D conformal field theories, $f(a, b)$ is not constrained in SYM as far as we know, since no other nontrivial symmetries extending conformal invariance have been discovered by far. The scaling factor in (1.11) is singular whenever two points are separated by a null geodesics. $f(a, b)$ can be singular too in this case, since one of $a$ and $b$ vanishes or becomes infinity. Unlike for the two and three point amplitudes, energy-momentum conservation in general does not require two points being on the same light-cone. To see the general consequence, we follow the same strategy as the above to replace $r_{ij}^2$ by their “on-shell” value:

$$r_{12}^2 = 1 - e_1 \cdot e_2 = 2 \sin^2(\phi_{12}/2), \quad r_{34}^2 = 2 \sin^2(\phi_{34}/2),$$

$$r_{13}^2 = -1 + e_3 \cdot e_4 = -2 \sin^2(\phi_{13}/2), \quad r_{24}^2 = -2 \sin^2(\phi_{24}/2),$$

$$r_{23}^2 = -2 \sin^2(\phi_{23}/2), \quad r_{14}^2 = -2 \sin^2(\phi_{14}/2),$$

(1.12)
where we assumed that particles 1 and 2 are incoming, and particles 3 and 4 are outgoing; \( \phi_{ij} \) is the angle between momentum \( k_i \) and momentum \( k_j \).

Now use the energy conservation \( E_1 + E_2 = E_3 + E_4 \) and the momentum conservation law \( E_1 e_1 + E_2 e_2 = E_3 e_3 + E_4 e_4 \), we derive

\[
s = 4E_1E_2 \sin^2(\phi_{12}/2) = 4E_3E_4 \sin^2(\phi_{34}/2), \tag{1.13}
\]

where \( s \) is one of the Mandelstam variables. Similarly

\[
t = -4E_1E_3 \sin^2(\phi_{13}/2) = -4E_2E_4 \sin^2(\phi_{24}/2), \tag{1.14}
\]

\[
u = -4E_1E_4 \sin^2(\phi_{14}/2) = -4E_2E_3 \sin^2(\phi_{23}/2).
\]

We see that those “on-shell” distances in \((1.12)\) are simply related to the Mandelstam variables. However, \( r_{ij}^2 \) also depends on individual energies. It appears that the only way to eliminate the dependence on energies is to use the two cross-ratios

\[
a = \frac{s^2}{t^2}, \quad b = \frac{s^2}{u^2}. \tag{1.15}
\]

Now since Mandelstam variables satisfy relation \( s + t + u = 0 \), we obtain

\[
\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = 1. \tag{1.16}
\]

In other words, the above relation is a consequence and energy-momentum conservation.

Our experience with the two and three point amplitudes tells us that in order for the 4 point scattering amplitude to obey energy-momentum conservation, the four point correlation function must be singular when \((1.16)\) is satisfied. The correlation is also singular whenever one of \( r_{ij}^2 \) vanishes. As already explained, \( r_{ij}^2 = 0 \) has nothing to do with energy-momentum conservation, the singularity of \( f(a,b) \) at \( 1/\sqrt{a} + 1/\sqrt{b} = 1 \) must be severe than the singularity of the correlation at \( r_{ij}^2 = 0 \).

The new singularity we observed above is not dictated by conformal invariance at all. Also, this singularity is demanded only in the large N limit, since for finite N there is no momentum conservation in the anti-de Sitter space. As we will explain in detail in the next section, \( f(a,b) \) must be singular if

\[
4ab = (ab - a - b)^2, \tag{1.17}
\]

and \((1.16)\) is only one of the four solutions

\[
\frac{1}{\sqrt{a}} \pm \frac{1}{\sqrt{b}} = \pm 1 \tag{1.18}
\]

to the above equation.
2. Explicit Calculations

We will see that an explicit calculation based on the ansatz of [2] is quite difficult. We will be able to obtain a closed form for the two point amplitude, we will not be able to complete the calculation of the three point amplitude. However, we will show that energy-momentum conservation is ensured in the large $R$ limit. We discuss the calculation of the four point amplitude. A lot of work is left for future.

First, we want to simplify the calculation of a general amplitude a bit. Shifting $t \rightarrow t \mp \pi/2$, for an incoming particle or an outgoing one, one can always ignore the quadratic term in $t$ in ansatz (1.1), if one remembers that the integral over $t$ will always pick up the most important contribution around $t \sim 0$. To see this, consider the integral

$$
\int dt e^{-\frac{\omega}{2} t^2 + i \omega t} f(t, \ldots),
$$

(2.1)

where the dots denote other variables. $f(t)$ is a periodic function of $t$ of period $2\pi$. Express $f(t, \ldots)$ as

$$
f(t, \ldots) = \int d\omega \tilde{f}(\omega, \ldots) e^{-i \omega t},
$$

then

$$
\int dt e^{-\frac{\omega}{2} t^2 + i \omega t} f(t, \ldots) = \sqrt{\frac{2\pi}{\omega}} \int d\omega' e^{-\frac{(\omega - \omega')^2}{\omega}} \tilde{f}(\omega', \ldots).
$$

(2.2)

The integral over $\omega'$ centers around $\omega$ with width $\delta \omega' \sim \sqrt{\omega}$, compared with the principle value $\omega$, this deviation tends to zero in the large $\omega$ limit. Thus the above integral is approximately

$$
2\pi \tilde{f}(\omega, \ldots) = \int dt e^{i \omega t} f(t, \ldots).
$$

(2.3)

Compared with this approximate value, the deviation is about

$$
\delta \omega \frac{d \tilde{f}(\omega, \ldots)}{d\omega},
$$

(2.4)

so it can be ignored if

$$
\tilde{f}'(\omega, \ldots) / \tilde{f}(\omega, \ldots) \ll 1/\sqrt{\omega}.
$$

This condition is generally satisfied. Alternatively, as we already saw, the function $f(t, \ldots)$ has infinitely many poles. Integral over $t$ will pick up these poles, the Gaussian factor $\exp(-(\omega/2)t^2)$ helps to suppress all these poles except the one closest to zero. In this case, the other factor $\exp(i \omega t)$ is more important than the Gaussian factor, since it oscillates fast around $t \sim 1/\sqrt{\omega}$. 


To demonstrate the above result, let us see how the energy conservation is derived in the large $\omega$ limit. For a general scattering amplitude, we have

$$
\int \prod_i dt_i e^{-\frac{1}{2} \sum i \omega_i t_i^2 + \frac{1}{2} \sum i a_i \omega_i t_i f(t_i, \ldots)},
$$

(2.5)

where $a_i = 1$ for an outgoing state, and it is $-1$ for an incoming state. Use a new set of times: $t_1, t_i = \tau_i + t_1, i = 2, \ldots, n$. The function $f(t_i, \ldots)$ is a function of $\tau_i$ only, since the correlation function in SYM is invariant under a time translation. Performing the integral over $t_1$ first, we obtain the integrand for $\tau_i$

$$
\sqrt{\frac{2\pi}{\omega}} e^{-\frac{i}{\omega} (\sum a_i \omega_i)^2 - \frac{1}{2} (\sum \omega_i \tau_i)^2 - \frac{1}{2} \sum \omega_i \tau_i^2 + i \sum a_i \omega_i \tau_i f(\tau_i, \ldots)},
$$

(2.6)

where $\omega = \sum \omega_i$, and we omitted a term

$$
-\frac{i}{\omega} (\sum a_i \omega_i)(\sum \omega_i \tau_i)
$$

in the exponential. This is because the first exponential together with the prefactor gives rise to a delta function in the large $R$ limit

$$
\frac{2\pi}{R} \delta(\sum a_i E_i),
$$

(2.7)

which is just the energy conservation law. This factor can be obtained without including the Gaussian factors. The remaining Gaussian factor in (2.6) is positive definite in $\tau_i$, and as we argued before, can be ignored so long if we pick up poles closest to zero in $\tau_i$.

The integral over variables on $S^3$ is much more complicated, and since the Gaussian factor $\exp(-\omega_i/2)|x_i \pm e_i|^2$ is the only nontrivial factor in the convolution, one has to treat the convolution carefully. In the following, we will examine the two, three and four amplitudes separately. Before doing that, let us remark that this Gaussian factor can be replaced by

$$
e^{-\frac{\omega_i}{2} |x_i \pm e_i|^2} = \int \frac{d^4 k_i}{(2\pi \omega_i)^2} e^{-\frac{k_i^2}{2\omega_i} + i k_i (e_i \pm x_i)}
= \int \frac{d^4 k_i}{(2\pi)^2} e^{-\frac{k_i^2}{2} + i \sqrt{\omega_i} k_i \cdot (e_i \pm x_i)},
$$

(2.8)
2.1. Two Point Amplitude

As shown above, the integration over the “center of times” results in a factor

\[
\frac{2\pi}{R} \delta(E_1 - E_2).
\] (2.9)

The remaining part is

\[
\Phi^{-1} N_2(R E_1)^{-3} \int \frac{d^4 k_1}{(2\pi)^2} \frac{d^4 k_2}{(2\pi)^2} e^{-k_1^2/k_2^2/2 + i\sqrt{\omega_1} \sum k_i \cdot e_i} F_2(k_i),
\] (2.10)

with

\[
F_2(k_i) = \int dtd\Omega_1 d\Omega_2 e^{-i\omega_1 t - i\sqrt{\omega_1} \sum a_i k_i \cdot x_i (\cos t - x_1 \cdot x_2 - i\epsilon)^{-4}},
\] (2.11)

where in eq. (2.10) we introduced a normalization factor \(N_2\) depending on a normalization of operator \(O\).

The integral over \(t\) in (2.11) can be performed first. It picks up a pole at \(t = \phi_{12} - i\epsilon\), where \(x_1 \cdot x_2 = \cos \phi_{12}\). Other poles are suppressed by a Gaussian factor we have omitted.

The leading contribution is

\[
F_2(k_i) = \frac{\pi}{3} \omega_1^3 \int d\Omega e^{-i\phi_{12} - i\sqrt{\omega_1} \sum a_i k_i \cdot x_i} \frac{1}{\sin^4(\phi_{12} - i\epsilon)},
\] (2.12)

other terms are suppressed by powers of \(\omega_1\). Performing the integration over \(\Omega_2\) first, we have

\[
F_2 = \frac{8\pi^3}{3} \omega_1^4 \int d\Omega_1 e^{i\sqrt{\omega_1} (k_2 - k_1) \cdot x_1} = \frac{32\pi^4}{3} \omega_1^4 \frac{\pi J_1(\sqrt{\omega_1}|k_1 - k_2|)}{\sqrt{\omega_1}|k_1 - k_2|}.
\] (2.13)

We will see momentarily that the fact that the above result is a function of only \(k_1 - k_2\) ensures the momentum conservation in the large \(R\) limit.

Plugging back the above result into (2.10), ignoring the prefactor in (2.10) for the moment, we have, after changing variables \(k_1 - k_2 = \kappa_2 / 2\), \(k_1 + k_2 = \kappa_1 / 2\)

\[
\frac{2^5\pi}{3} \omega_1^4 \int d^4\kappa_1 e^{-\kappa_1^2 + i\sqrt{\omega_1} \kappa_1 \cdot (e_1 - e_2)} \int \frac{d^4\kappa_2}{2\sqrt{\omega_1} |\kappa_2|} e^{-\kappa_2^2 + i\sqrt{\omega_1} \kappa_2 \cdot (e_1 + e_2)} J_1(2\sqrt{\omega_1} |\kappa_2|).
\] (2.14)

The integral over \(\kappa_1\) is separated from that over \(\kappa_2\). The first integral results in

\[
\pi^2 e^{-\omega_1 |e_1 - e_2|^2}
\]

which in the large \(\omega\) limit tends to

\[
\pi^2 \left( \frac{\pi}{\omega_1} \right)^{3/2} \delta^3(e_1 - e_2).
\] (2.15)
Due to the delta function, the second integral in (2.14) is simplified. The integral over \( \kappa \) can be separated into the radial part and the angular part, and the latter can be easily performed. In the end, we obtain

\[
\frac{2^8 \pi^5}{3} (\pi \omega_1)^{3/2} \delta^3(e_1 - e_2) \int dk J_1^2(2\sqrt{\omega_1}k)e^{-k^2}. \tag{2.16}
\]

In the large \( \omega_1 \) limit, the Bessel function \( J_1 \) can be replaced by its asymptotic form, namely

\[
\int_0^\infty dk J_1^2(2\sqrt{\omega_1}k)e^{-k^2} \to \frac{1}{\pi \sqrt{\omega_1}} \int dk \cos^2(2\sqrt{\omega_1}k)e^{-k^2},
\]

and since for a large \( \omega_1 \), the \( \cos \) factor can be replaced by its average value 1/2, the value of the above integral is \((1/4)(1/\sqrt{\pi \omega_1})\). We have checked this result by using a formula for the integral in (2.16) involving the Bessel function. Substitute this into (2.16), we find

\[
\frac{(2\pi)^6}{3} \omega_1 \delta^3(e_1 - e_2). \tag{2.17}
\]

Together with the prefactor in (2.10) and the delta function in (2.9), the end result is

\[
\langle \omega_1 e_1 - \omega_2 e_2 \rangle = \Phi^{-1} N_2 \frac{(2\pi)^7}{3R^3} E_1^{-2} \delta(E_1 - E_2) \delta^3(e_1 - e_2). \tag{2.18}
\]

Using the identity

\[
\delta^4(E_1 e_1 - E_2 e_2) = E_1^{-3} \delta(E_1 - E_2) \delta^3(e_1 - e_2)
\]

it follows that

\[
\langle \omega_1 e_1 - \omega_2 e_2 \rangle = \Phi^{-1} N_2 \frac{(2\pi)^7}{3R^3} E_1 \delta^4(E_1 e_1 - E_2 e_2). \tag{2.19}
\]

Conservation of momentum for the two point amplitude also ensures conservation of energy, so we have obtained the right delta function, as expected by our intuitive argument in the last section. It remains to check whether the kinetic factor is also right. Note that the first normalization factor in (2.19) depends on the overlap of the two wave functions \([2]\), so it can depend on the energy. The other normalization factor, \( N_2 \), does not depend on energy, although it can be a function of \( R \) and \( g_s \).

The wave function of \([2]\) takes the form

\[
F(t, x) = e^{-i\omega(t \cdot e \cdot x)} - \frac{1}{2} (x_1^2 + (t \cdot e)^2)
\]

\[
\tag{2.20}
\]
near the center of AdS space. The Gaussian factor can be ignored so long if the spacetime region has a scale smaller than $1/\sqrt{\omega}$. The proper scale is $R/\sqrt{\omega} \sim \sqrt{R}$. So in the large $R$ limit the Gaussian factor can be ignored and we have a plan wave. Since the creation operator is defined by

$$\alpha = \int dV \hat{\phi} \partial_t F(t, x) + \ldots,$$

so close to the center of the AdS space we have, roughly

$$\hat{\phi}(t, x) = \int \frac{d^4k}{E} (\alpha^+(k)e^{-iE+ik\cdot x} + \alpha(k)e^{iE-k\cdot x}),$$

where all the coordinates are the proper ones, unlike the ones in (2.20). Thus we expect that the scattering amplitude, up to a numerical factor, must be

$$\langle \alpha_\omega \omega_1 \omega_2 \rangle = R^5 E_1 \delta^4(E_1 e_1 - E_2 e_2),$$

where the volume factor $R^5$ comes from the internal space $S^5$, since the particles have zero momenta in the internal space. The kinetic factor is precisely the same as in (2.19). Therefore, it appears that the normalization $\Phi$ is order 1, and $N_2 \sim R^8 \sim N^2 g_s$.

2.2. Three Point Amplitude

Consider the three point amplitude with two incoming particles. It is more convenient to integrate out $t_3$ first, with $t_1 = \tau_1 + t_3$, $t_2 = \tau_2 + t_3$. As before a delta function ensuring energy conservation results. The amplitude is

$$A_3 = N_3 \frac{2\pi}{R} \delta(E_1 + E_2 - E_3) \prod \omega_i^{-3/2} \int \prod d^4k_i (2\pi)^2 e^{-1/2 \sum k_i^2 + \sum \sqrt{\omega_i}k_i e_i F_3(k_i)},$$

with

$$F_3(k_i) = \int d\tau_1 d\tau_2 \prod i \omega_i e^{-i\omega_i \tau_i + i\sqrt{\omega_i}k_i \cdot x_i}$$

$$(\cos(\tau_1 - \tau_2) - x_1 \cdot x_2 + i\epsilon)^{-2}(\cos \tau_1 - x_1 \cdot x_3 - i\epsilon)^{-2}(\cos \tau_2 - x_2 \cdot x_3 - i\epsilon)^{-2},$$

where we have reflected $x_3 \rightarrow -x_3$. Denote $x_i \cdot x_j$ by $\cos \phi_{ij}$.

To perform the integral over $\tau_i$ first, we use the following formula

$$(\cos \tau - \cos \phi \pm i\epsilon)^{-2} = -\frac{1}{\sin^2(\phi \pm i\epsilon)} \int_{-\infty}^{\infty} d\omega |\omega| e^{i\omega \tau \pm i|\omega|(\phi \pm i\epsilon)},$$
which is derived from
\[(\cos \tau - \cos \phi \pm i \epsilon)^{-1} = \mp \frac{i}{\sin(\phi \pm i \epsilon)} \int d\omega e^{i \omega \tau \pm i |\omega|(\phi \pm i \epsilon)} \] (2.26)
by taking derivative with respect to \(\phi\) once. With the Fourier transform (2.25), the integral over \(\tau_i\) in (2.24) is readily performed, with the result
\[F_3(k_i) = -\int \prod_i d\Omega_i \frac{\exp(i\sqrt{\omega_i} k_i \cdot x_i)}{\sin^2(\phi_{12} + i \epsilon) \sin^2(\phi_{13} - i \epsilon) \sin^2(\phi_{23} - i \epsilon)} \int d\omega |\omega(\omega_1 - \omega)(\omega_2 + \omega)| e^{i |\phi_{12} - i|\omega_1 - \omega|\phi_{13} - i|\omega_2 + \omega|\phi_{23}|}. \]

We do not know how to carry out the calculation of the above integral. One thing is certain, though, that due to the singular behavior of the integrand, the integral is peaked around \(\phi_{12} = \phi_{13} = \phi_{23} = 0\). We thus expect that \(F_3\) will be a function of \(\sum \sqrt{\omega_i} k_i\) only. We now argue that this ensures the momentum conservation in the large \(R\) limit. Introduce new vectors
\[l_1 = \frac{1}{\sqrt{2\omega_3}} \sum \sqrt{\omega_i} k_i, \]
\[l_2 = -\sqrt{\omega_2/\omega_3} k_1 + \sqrt{\omega_1} k_2 \]
\[l_3 = \sqrt{\omega_1/(2\omega_3)} k_1 + \sqrt{\omega_2/(2\omega_3)} k_2 - 1/\sqrt{2} k_3. \]

With the above relations, it is easy to see that \(\sum k_i^2 = \sum l_i^2\), using the fact \(\omega_3 = \omega_1 + \omega_2\). Now \(F_3\) is a function of \(l_1\) only. We can perform the integral in (2.23) over \(l_2\) and \(l_3\) first. We note in particular that in the exponential in (2.23), the \(l_3\) dependent part is
\[-\frac{1}{2} l_3^2 + i \frac{l_3}{\sqrt{2}\omega_3} (\sum a_i \omega_i e_i). \]

It is seen that the integral over \(l_3\) results in a delta function ensuring momentum conservation, in the large \(R\) limit.

After some calculation, we obtain
\[A_3 = N_3 \frac{2^{11/2} \pi^{5/2} E_3^{3/2}}{E_1^3 E_2^3} \delta(\sum a_i E_i) \delta^4(\sum a_i E_i e_i) \int \frac{d^4l_1}{(2\pi)^2} e^{-1/2 l_1^2 + i \sqrt{2\omega_1} l_1 \cdot e_1} F_3(l_1). \]

(2.30)
2.3. Four Point Amplitude

We have much less to say about the four point amplitude. Although we trust that the condition for energy-momentum conservation derived in the previous section is necessary, we are not able to prove that it is also sufficient. As in the two and three point amplitude cases, we can always write the four point amplitude as

\[ A_4 = N_4 \int \prod_{i=1}^{4} \frac{d^4 k_i}{(2\pi)^2} e^{-\frac{1}{2} \sum k_i^2 + i \sum \sqrt{\omega_i k_i \cdot e_i} F_4(k_i)}, \]  

(2.31)

where \( F_4 \) is given by a similar formula as (2.24). We omitted a factor conserving energy.

To see whether the momentum conservation is true in the large \( R \) limit, as in the previous subsection, we introduce a set of new vectors

\[ l_i = \sum_j \Omega_{ij} k_j, \]  

(2.32)

with \( \{\Omega_{ij}\} \) being an orthogonal matrix. We can choose \( \Omega_{1i} = a_i \sqrt{\omega_i / \omega} \), where \( \omega = \sum \omega_i \).

If in the large \( R \) limit, \( F_4(l_i) \) is less dependent on \( l_1 \) than on other \( l_i \), the integral over \( l_1 \) in (2.31) can be performed first, thus resulting in a delta function associated with the momentum conservation.

It can be shown that the condition that \( 1/\sqrt{a} + 1/\sqrt{b} = 1 \) is a singularity of \( f(a, b) \) is a necessary one, where \( f(a, b) \) is a function introduced in (1.11). It is far from clear whether it is also the sufficient condition for conserving momentum.

Finally, we want to show that if \( 1/\sqrt{a} + 1/\sqrt{b} = 1 \) is a singularity of \( f(a, b) \), then both \( 1/\sqrt{a} - 1/\sqrt{b} = 1 \) and \( 1/\sqrt{a} - 1/\sqrt{b} = -1 \) are also singularities of \( f(a, b) \). \( f(a, b) \) is a symmetric function of \( a \) and \( b \). To see this, we go to the Euclidean space. Exchange point 1 with point 2, \( a \) and \( b \) are exchanged. Exchange point 2 and point 3 we are led to \( a \rightarrow 1/a, b \rightarrow b/a \), thus

\[ F(1/a, b/a) = F(a, b) = F(b, a). \]  

(2.33)

Now, \( 1/\sqrt{a} + 1/\sqrt{b} = 1 \) is the same as \( \sqrt{a} - \sqrt{a/b} = 1 \). This means that \( f(1/a, b/a) \) is singular when this relation is satisfied. Renaming the variables, we conclude that \( f(a, b) \) is singular when \( 1/\sqrt{a} - 1/\sqrt{b} = 1 \). Exchanging \( a \) with \( b \), we deduce that \( f(a, b) \) is singular when \( 1/\sqrt{a} - 1/\sqrt{b} = -1 \).
3. Conclusion

We have only scratched the surface of the problem of investigating the consequence of Lorentz invariance for the holographic S-matrix and associated correlation functions in the large N limit. For instance, more constraints can be derived from the requirement that $A_4$ is a function of only $s$ and $t$, apart from a kinetic factor. Even more interesting, is the consequence of causality in the flat space limit. We leave these problems for future investigations.

Our main result in this paper is the identification of a new singularity in the four point amplitude, in the large N limit. This means that the dominant contribution to the scattering amplitude comes from around this “saddle point”. This reminds us the problem of sensitive initial conditions raised in [3]. It is observed there that if the two beams aimed at the center of the AdS are emitted from the boundary with a time difference greater than $1/R$, then the beams will miss each other. In the large N limit, this time difference can be arbitrarily small. It appears that a kind of sharp saddle point may help to understand this puzzle. Presumably this is a consequence of locality in the bulk space. It remains to see whether bulk locality together boundary conformal invariance guarantee bulk Poincare invariance in the large N limit.

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References

[1] G. ’t Hooft, “Dimensional reduction in quantum gravity”, hep-th/9310026; L. Susskind, “The world as a hologram”, hep-th/9409089.
[2] J. Polchinski, “S-Matrices from AdS Spacetime”, hep-th/9901076.
[3] L. Susskind, “Holography in the Flat Space Limit”, hep-th/9901079.
[4] V. Balasubramanian, S. B. Giddings and A. Lawrence, “What Do CFTs Tell Us About Anti-de Sitter Spacetimes”, hep-th/9902052; S. B. Giddings, “The boundary S-matrix and the AdS to CFT dictionary”, hep-th/9903048.
[5] J. Polchinski, L. Susskind and N. Toumbas, “Negative Energy, Superluminosity and Holography”, hep-th/9903228.
[6] J. M. Maldacena, “The Large N Limit of Superconformal Field Theories and Supergravity”, hep-th/9711200.
[7] E. Witten, “Anti De Sitter Space And Holography”, hep-th/9802150.
[8] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge Theory Correlators from Non-Critical String Theory”, hep-th/9802109.
[9] V. Balasubramanian, P. Kraus and A. Lawrence, “Bulk vs. Boundary Dynamics in Anti-de Sitter Spacetime”, hep-th/9805171.