Mass Spectrum for Black Holes in Generic 2-D Dilaton Gravity

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Abstract

Two arguments for the quantization of entropy for black holes in generic 2-D dilaton gravity are summarized. The first argument is based on reduced quantization of the only physical observables in the theory, namely the black hole mass and its conjugate momentum, the Killing time separation. The second one uses the exact physical mass eigenstates for Euclidean black holes found via Dirac quantization. Both methods give the same spectrum: the black hole entropy must be quantized $S = 2\pi n/G$. 

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Progress in the theory of quantum black holes and their thermodynamics essentially depends on the resolution of the problem of Bekenstein-Hawking entropy \[1\]: the explanation of the statistical mechanical origin of this geometric quantity. While the first law of thermodynamics for black holes seems to be a very general and model independent concept \[2\], the statistical mechanical confirmation of the corresponding entropy exists only for a limited set of examples, such as extremal, string-inspired supersymmetric black holes \[3\], and black holes in 2+1 dimensions \[4\]. The universality of black hole thermodynamics \[2\], however, suggests that the explanation of the entropy and energy spectra of black holes might not be found in terms of specific models. Instead one should perhaps search for quantum mechanical mechanisms of their origin in the simplest (vacuum or eternal black hole) cases. One such mechanism has recently been proposed on rather general and heuristic grounds by Bekenstein and Mukhanov \[5\] who argued that the area \(A(M)\) of a black hole with mass \(M\) should be quantized according to the rule:

\[
A(M) = \alpha n
\]  

(1)

where the coefficient \(\alpha\) was determined by requiring the degeneracy of states:

\[
g(M) = \exp S(M) = \exp(A(M)/4)
\]  

(2)

be an integer. A spectrum of this form has also been obtained in a variety of other contexts \[6–8\]. In this talk we will summarize recent arguments \[9\] that suggest a similar quantization condition on the entropy \(S(M)\) of black holes with mass \(M\) in generic 2-D dilaton gravity, namely:

\[
S(M) = 2\pi n/G,
\]  

(3)

but based on different quantum mechanical grounds. Since generic dilaton gravity contains as a special case spherically symmetric gravity, this quantization condition should, if valid, also apply to Euclidean black holes in Einstein gravity.

The classical action for generic dilaton gravity in 2 spacetime dimensions is \[10\]

\[
I = \frac{1}{2G} \int dt dx \sqrt{-g} \left( \eta R(g) + \frac{V(\eta)}{l^2} \right)
\]  

(4)

where \(G\) is a (dimensionless) gravitational constant, and \(l\) is a fundamental constant of dimension length. The most general solution to the field equations in the generic theory up to spacetime diffeomorphisms can be written \[11,12\]:

\[
ds^2 = -(j(\eta) - 2GlM)dt^2 + \frac{1}{(j(\eta) - 2GlM)} dx^2
\]

\[
\eta = x/l
\]  

(5)

where \(j(\eta) = \int_0^\eta d\tilde{\eta}V(\tilde{\eta})\). These solutions all have a Killing vector, whose norm can be written in coordinate invariant form as

\[
|k|^2 = -l^2 |\nabla \eta|^2 = (2GlM - j(\eta))
\]  

(6)
As long as \( j(\eta) \) rises monotonically from zero at \( \eta = 0 \), the above solutions describe Schwarzschild-like black holes with ADM mass \( M \). Note that in the above parametrization the metric is not asymptotically flat, nor is the Killing vector normalized to one at spatial infinity. Without changing the essential features of the following arguments, one can define a new physical, asymptotically flat metric by the conformal reparametrization \( \tilde{g}_{\mu \nu} = g_{\mu \nu} / j(\eta) \). A complete discussion of the necessary conditions for the existence of black hole solutions in the generic theory, and the corresponding thermodynamics, can be found in [12]. The crucial observation for the present purposes is that event horizons are surfaces \( \eta = \eta_{\pm} \) constant for which \( |k|^2 = 0 \). The entropy of the corresponding black hole is proportional to the value of the dilaton at the horizon, namely

\[
S = \frac{2\pi}{G} \eta_-
\]  

(7)

The Hamiltonian formulation for the geometrical theory starts with a decomposition of the metric:

\[
ds^2 = e^{2\rho} \left( -u^2 dt^2 + (dx - vdt)^2 \right)
\]  

(8)

The boundary conditions relevant to black hole thermodynamics require placing the black hole in a box of fixed radius (the value of the dilaton at the boundary is fixed). Moreover we restrict the slices on the interior of the box to end at the bifurcation point of an eternal black hole (i.e. the point at which the Killing vector vanishes), so that we can analytically continue the solutions to Euclidean time without obstruction. As shown in [13] for spherically symmetric gravity, and in [9] for the generic theory, the resulting Hamiltonian is:

\[
H_c = \int_{\sigma^-}^{\sigma^+} dx \left( -\tilde{u} M' + \tilde{v} \mathcal{P} \right) + H_+ - H_-
\]  

(9)

where \( \tilde{u} \) and \( \tilde{v} \) are Lagrange multipliers and \( \Pi_\rho, \Pi_\eta \) are the momenta conjugate to \( \rho, \eta \), respectively. In the above \( \mathcal{P} = \Pi'_\rho - \Pi_\rho \Pi' - \Pi_\eta \Pi' \) is the generator of spatial diffeomorphisms, and the Hamiltonian constraint has been rewritten as the spatial gradient of the mass observable \( \mathcal{M} \) [13,14]:

\[
\mathcal{M} := \frac{1}{2G} \left( e^{-2\rho} \left( G^2 \pi_\rho^2 - (\eta')^2 \right) + \frac{j(\eta)}{l^2} \right)
\]  

(10)

The surface terms \( H_\pm \) are required to make the variations of the Hamiltonian well defined. For the given boundary conditions one finds that [16,17]

\[
H_+ = \sqrt{-\frac{\tilde{g}^+ \tilde{M}(\eta_+)}{lG}} \left( 1 - \sqrt{1 - \frac{2GMl}{\tilde{M}(\eta_+)} \frac{j(\eta_+)}{l^2}} \right)
\]  

(11)

is the quasilocal energy of the black hole in the box, whereas \( H_- = \frac{N_0}{2\pi} S(\mathcal{M}) \) is proportional to the thermodynamic entropy of the black hole, expressed as a function of the mass observable. This generalizes the recent results [13] and [18] for spherically symmetric gravity and string inspired gravity, respectively.

The Hamiltonian analysis of generic dilaton gravity has been elucidated in numerous papers [11,12]. The only diffeomorphism invariant observable is the constant mode, \( M \), of
the mass observable, $M$. Its conjugate variable, $P$, is invariant only under diffeomorphisms that vanish on the boundaries. $P$ has a geometrical interpretation as the Killing time separation between the two ends of the spatial slice under consideration \([15,12,11,14]\). On the constraint surface the Hamiltonian is a function only of the mass $M$ and the action of reduced theory \([15,9]\) reads:

$$S[M, P] = \int dt(P\dot{M} - H(M))$$

(12)

where again, $P$ is the time separation of the two ends of the spatial slice and $M$ is the ADM mass of the black hole. They obviously play the role of angle-action variables, and their naive quantization without knowledge of the global structure of their phase space will be as misleading as the same attempt for the harmonic oscillator rewritten in terms of its angle-action variables. Obviously, a correct quantization (if any) should be based on variables that are derived from the above angle-action ones via some canonical transformation and give rise at the quantum level to a well defined Hilbert space with normalizable states, etc. The recovery of these variables is certainly not a unique procedure, and a leap of faith is needed to justify a specific choice by relying on the universality arguments of the black hole thermodynamics. One such procedure is as follows.

Consider a canonical transformation to new variables

$$X = \sqrt{B(M)/\pi} \cos(2\pi P T(M))$$

(13)

$$\Pi = \sqrt{B(M)/\pi} \sin(2\pi P T(M)).$$

(14)

With arbitrary functions $B(M)$ and $T(M)$ it looks chosen ad hoc, but gets justified in part by noting that it is canonical if and only if the following equation holds

$$\delta B = \frac{\delta M}{T(M)}.$$ 

(15)

This equation can be interpreted as a first law of black hole thermodynamics provided we identify $B(M)$ with Bekenstein-Hawking entropy and $T(M)$ with the temperature of the corresponding black hole of mass $M$:

$$B(M) = S(M)$$

(16)

We have dropped the arbitrary constant and will henceforth assume that the entropy is defined to vanish when $M = 0$. Since $X^2 + \Pi^2 = S(M)/\pi$, finding eigenstates of the mass operator reduces to the eigenvalue problem for a simple harmonic oscillator. Normalizability of the resulting wave function (zero boundary conditions at the infinity of the $X$-configuration space) requires that the eigenvalues of $X^2 + \Pi^2$ be $2n + 1$, where $n$ is a positive integer. This in turn yields a discrete spectrum for the entropy:

$$S(M) = 2\pi(n + 1/2)/G$$

(17)

1While this manuscript was being prepared we were made aware of a paper by Louko and Matela in which a similar canonical transformation was considered.
as claimed.

Remarkably Dirac quantization of the above system in the Euclidean sector, for the same boundary conditions, gives the same spectrum. As shown in [16], by adapting a canonical transformation first used by Cangemi et al [19] for string inspired dilaton gravity, it is possible to find exact, physical eigenstates of the mass observable in the functional Schrödinger representation. These states take the form:

\[ \Psi(\rho^a, \theta_-) = \exp \left( \frac{i}{\hbar} \int_{\sigma_-}^{\sigma_+} dx \, \omega(\rho^2)(\rho^0 \rho^1' + \rho^1 \rho^0') \right) \exp \left( \frac{i}{\hbar} \eta_- \theta_- \right) \]

where \( \rho^a(x), \{a = 0, 1\} \) are new phase space variables (defined on a spatial or “radial” coordinate \( \sigma_- < x < \sigma_+ \)), conjugate to the spatial components of the zweibein fields of the metric in the connection representation of 2D dilaton gravity [19] : \( p_a = e_a \). Moreover, \( \rho^2 = -(\rho^0)^2 + (\rho^1)^2 = |k|^2 \). The specific form of the functional \( \omega(\rho^2) \) is not relevant for the present discussion, except that it vanishes at the bifurcation point given in terms of these variables by \( \rho^a(\sigma_-) = 0 \). What is crucial, however, is the form of \( \theta_- \), which is the variable conjugate to \( \eta_- \), the value of the dilaton at the horizon. As shown in [16] in the Euclidean sector

\[ \theta_- = \arctan(p_0/p_1)|_{\sigma_-} \]

It is therefore periodic and singlevaluedness of the wavefunction requires \( \eta_- = n \). Given the general expression for the entropy Eq.(7), this is precisely the same quantization condition (up to an irrelevant shift) as obtained via reduced quantization above. The positivity of \( n \) does not come from the quantum mechanics in this case. Instead it must be imposed because \( \eta = 0 \) corresponds to a singularity in the theory, where the effective gravitational constant in the theory vanishes\(^\text{2}\).

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\(^{2}\)Alternatively, the physical, asymptotically flat metric has a curvature singularity.
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