WHITTAKER QUANTUM UNIVERSES

H. Rosu and J. Socorro

Instituto de Física de la Universidad de Guanajuato, Apdo Postal E-143, León, Gto, México

Summary. - We show that closed, radiation-filled Friedmann-Robertson-Walker quantum universes of arbitrary factor ordering obey the Whittaker equation. We also present the formal Witten factorization as well as the double Darboux strictly isospectral scheme for the Whittaker equation.

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If in the “time-time” component of Einstein’s equations for Friedmann-Robertson-Walker (FRW) universes \( \dot{a}^2 + k = \frac{8\pi G}{3} \rho a^2 \) one substitutes a canonical momentum \( \pi_a \) conjugate to the scale factor and quantizes the system according to the common procedure \( \pi_a \rightarrow -i(d/da) \), one will obtain the corresponding Wheeler-DeWitt (WDW) equation

\[
\left[ -a^{-p} \frac{d}{da} a^p \frac{d}{da} + \left( \frac{3\pi}{2G} \right)^2 \frac{1}{k^2} \left( ka^2 - \frac{8\pi G}{3} \rho a^4 \right) \right] u(a) = 0.
\] (1)

The parameter \( p \) enters as a consequence of the ambiguity in the ordering of \( a \) and \( d/da \). Amongst the most used orderings, the \( p = 1 \) is the so-called Laplacian one, which has been used for the two-dimensional mini-superspace describing a massive scalar field in a closed universe, i.e., a model obtained through the substitution \( \frac{8\pi G}{3} \rho \rightarrow m^2 \phi^2 \), \( k = 1 \) in Eq. (1) and adding \( \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} \) to the \( a \) derivatives turned into partial ones too, while the dependent variable becomes \( u(a, \phi) \).

On the other hand, the \( p = 2 \) ordering is convenient for the semiclassical approximation \[2\].

\[1\] E-mail: rosu@ifug3.ugto.mx; Fax: 0052-47-187611
\[2\] E-mail: socorro@ifug4.ugto.mx
In the matter sector various contributions may be introduced but here we shall consider only the radiation case, i.e.,

\[
\frac{8\pi G}{3} \rho \rightarrow \frac{8\pi G}{3} \left[ \rho_r \left( \frac{a_0}{a} \right)^4 \right],
\]

(2)

where \( \rho_r \) is the radiation energy density. The most compact form of the WDW equation in the FRW metric can be written down as follows

\[
\left[ -\ddot{\tilde{a}} - p \frac{d}{d\tilde{a}} \tilde{a}^p \frac{d}{d\tilde{a}} + V(\tilde{a}) \right] u(\tilde{a}) = 0,
\]

(3)

where the tilde variables are rescaled and dimensionless ones. Thus, only with the radiation matter sector the WDW “potential” function turns out to be of the form

\[
V(\tilde{a}) \equiv \tilde{a}^2 - \tilde{\beta}^2,
\]

(4)

where \( \tilde{a}^2 = \frac{3\pi}{2c^2 k} a^2 \) is the tilde scale factor of the universe, and \( \tilde{\beta}^2 = \frac{4\pi^2}{k^2} \rho_r a_0^4 \) expresses the radiation effect on the cosmological expansion; \( k \) is considered positive (closed universes).

With the ansatz \( u(\tilde{a}) \equiv g(\tilde{a}) e^{-\tilde{a}^2/2} \), and with the further change of variable \( x = \tilde{a}^2 \), one gets the confluent hypergeometric equation for \( g(\tilde{a}) \) as follows

\[
x \frac{d^2 g}{dx^2} + \left( \frac{1 + p}{2} - x \right) \frac{dg}{dx} + \left( \frac{\tilde{\beta}^2 - 1 - p}{4} \right) g = 0.
\]

(5)

The complete solution of (5) is the superposition of confluent hypergeometric (Kummer) functions \( M \) and \( U \)

\[
g(\tilde{a}) = C_m M\left(\frac{1 + p - \tilde{\beta}^2}{4}, \frac{1 + p}{2}; \tilde{a}\right) + C_u U\left(\frac{1 + p - \tilde{\beta}^2}{4}, \frac{1 + p}{2}; \tilde{a}\right),
\]

where \( C_m \) and \( C_u \) are superposition constants. Thus, mathematically speaking, the problem may be considered as solved up to fixing some boundary conditions. However, for several particular purposes (such as Darboux constructions, see below) one should make another mathematical step, which is to pass to a self-adjoint form of the confluent equation. This has first been done by Whittaker in 1904 [4] by eliminating the first derivative term, i.e., by substituting \( g = |\tilde{a}|^{-\frac{p+1}{2}} \exp(-\tilde{a}/2)y(\tilde{a}) \) in the confluent equation (one can also get a self-adjoint form by multiplying the confluent equation
by the weight function $\tilde{a}^{\frac{p-1}{2}} e^{-\tilde{a}}$. The Whittaker equation reads

$$y''_{\lambda,\mu}(\tilde{a}) - \frac{1}{4} \left( 1 - \frac{\tilde{\beta}^2}{\tilde{a}} - \frac{1 - |(p-1)/2|^2}{\tilde{a}^2} \right) y_{\lambda,\mu}(\tilde{a}) = 0 ,$$

(6)

where $\lambda = (\tilde{\beta}/2)$, $\mu = (p-1)/4$, and the general solution is the linear combination $y_{\lambda,\mu} = C_1 M_{\lambda,\mu} + C_2 W_{\lambda,\mu}$ \[2\]. Notice that $V_{confl} = \frac{1}{4} \left( 1 - \frac{\tilde{\beta}^2}{\tilde{a}} - \frac{1 - |(p-1)/2|^2}{\tilde{a}^2} \right)$ can be thought of as a Schrödinger potential, and the Whittaker equation as a Schrödinger equation at zero energy.

The two Whittaker functions are defined as follows \[3\]

$$M_{\lambda,\mu}(\tilde{a}) = e^{-\tilde{a}/2} \tilde{a}^{\mu+1/2} M\left(\mu - \lambda + \frac{1}{2}, 2\mu + 1; \tilde{a}\right)$$

(7)

and

$$W_{\lambda,\mu}(\tilde{a}) = \frac{\Gamma(2\mu)}{\Gamma(\mu - \lambda + 1/2)} M_{\lambda,-\mu}(\tilde{a}) + \frac{\Gamma(-2\mu)}{\Gamma(-\mu - \lambda + 1/2)} M_{\lambda,\mu}(\tilde{a}) ,$$

(8)

where $M$ is the Kummer function. In Eq. (8) the right-hand side is replaced by its limiting value when $2\mu$ is an integer or zero. As is well-known the problem of boundary conditions in quantum cosmology is more involved than in quantum mechanics. The most used choices are the Hartle-Hawking “no boundary” one \[5\] and Vilenkin’s “tunneling” condition \[5\]. At the level of the Whittaker equation the choice of the boundary conditions depends on what feature one would like to emphasize more. For example, in atomic physics the physically acceptable solution is the $W_{\lambda,\mu}$ function only because it is exponentially declining, though singular at the origin. However, in quantum cosmology the exponentially growing $M_{\lambda,\mu}$ is also an acceptable solution. Suppose we would like to adopt Lemaître’s Primeval Atom paradigm \[5\]. In that case one selects $W_{\lambda,\mu}$ and gives “atomic” meaning to the subscripts, i.e., $\lambda$ should be considered as an effective principal quantum number and $\mu = l + 1/2$, where $l$ stands for an angular quantum number. As a matter of fact, the Rydberg states of the Lemaître atom are the most interesting for the physical case. For this, one should use the asymptotic properties of $W_{\lambda,\mu}$ at large $\lambda$ and $\mu$ \[8\].

Let us address the problem of particular cases. Since both Whittaker functions can be expressed in terms of the Kummer function $M$, there is nothing new for this issue with respect to the confluent equation. In particular for polynomial solutions the constraints are $p \neq -2m - 1$
and the “quantization” conditions $\tilde{\beta}^2 = 4n + p + 1$, where $n$ is the polynomial degree $[3]$. For example, the $p = 1$ factor ordering leads to Laguerre polynomials whereas the $p=2$ factor-ordering provides Hermite polynomials $[3]$. Many other special cases are given in $[3]$.

The Whittaker form of the confluent equation is needed in order to perform the single Darboux (Witten $[10]$) construction and the double Darboux one $[11]$. However, the single Darboux construction might be useless from the physical point of view if the Witten superpotentials are singular. This would also mean singular partner potentials. To see this, suppose we would like to use $M_{\lambda,\mu}$ for the single Darboux construction, which is nothing but a factorization of the Whittaker equation $[12]$. The superpotential is the negative of the logarithmic derivative of the $M_{\lambda,\mu}$. This can be shown to be $\frac{1}{2} - \frac{\mu+\lambda}{a} - \frac{a}{b} \frac{M(a+1,b+1;\tilde{a})}{M(a,b;\tilde{a})}$. One can find zones in the $(a, b)$-plane where $M(a, b; \tilde{a})$ has no zeros (see Fig. 13.1 in $[3]$). However, the singularity in the first term at the branch point $\tilde{a} = 0$ cannot be deleted unless $\mu = -1/2$ for which the function $M_{\lambda,\mu}$ is actually undefined (one can also see that the second confluent parameter should be nought, $b = 0$, and therefore the ratio $a/b$ becomes singular). For all the cases $2\mu = \pm 1, \pm 2, ...$ one must work with the function $W_{\lambda,\mu}$, but again one should know the nodeless regions in the parameter plane. Despite this difficulty, the Witten factorization of the Whittaker equation can be performed formally as an intermediate step for the double Darboux transform. It is accomplished by means of the operators $A_1 = \frac{d}{da} + W(\tilde{a})$ and $A_2 = -\frac{d}{da} + W(\tilde{a})$, where $W$ denotes the superpotential function. The initial Riccati equation of the Witten scheme is $V_{\text{conf}} = W^2 - W'$ and the partner (“fermionic”) equation is $V_{f,\text{conf}} = W^2 + W'$, corresponding to the “fermionic” Whittaker equation for which the factorizing operators are applied in reversed order. On the other hand, the double Darboux construction can be performed with the general Whittaker solution $y_{\lambda,\mu}$ without any constraints, because the singularities introduced by the solutions with zeros do not appear in the final result $[13]$. The one-parameter family of strictly isospectral Whittaker potentials will be

$$V_{\text{iso}}(\tilde{a}; \epsilon) = \frac{1}{4} \left( 1 - \frac{\tilde{\beta}^2}{\tilde{a}} - \frac{1 - [(p - 1)/2]^2}{\tilde{a}^2} \right) - \frac{4y_{\lambda,\mu}y'_{\lambda,\mu}}{J_{\lambda,\mu} + \epsilon} - \frac{2y^4_{\lambda,\mu}}{(J_{\lambda,\mu} + \epsilon)^2},$$

where $J_{\lambda,\mu}(\tilde{a}) \equiv \int_0^{\tilde{a}} y^2_{\lambda,\mu}(z)dz$ and $\epsilon$ is the family parameter, which is a real, positive number.
The derivative $y_{\lambda,\mu}'$ can be written down as follows

$$y_{\lambda,\mu}' = \frac{2\tilde{a} - \tilde{\beta}^2}{4\tilde{a}} y_{\lambda,\mu} - \frac{1}{\tilde{a}} y_{\lambda+1,\mu} + C_1 \frac{5 + p + \tilde{\beta}^2}{4\tilde{a}} M_{\lambda+1,\mu}, \quad (10)$$

where $C_1$ is the first superposition constant in the general Whittaker solution. For the double Darboux construction in the particular case $p = 1$ see [14].

The wavefunctions of the Whittaker strictly isospectral one-parameter modes can be written down as follows [13, 14]

$$y_{\lambda,\mu,iso} \propto \frac{y_{\lambda,\mu}}{\mathcal{J}_{\lambda,\mu} + \epsilon}. \quad (11)$$

We can hint on the following physical picture. Since the strictly isospectral scheme introduces in principle an infinity of Whittaker modes of the common WDW zero energy cosmology, one may try to identify those modes with a class of “radiation-only-containing” cosmic structures. These structures are characterized by the mode numbers $\lambda$ and $\mu$, i.e., by their radiation content and quantum factor ordering, by the set of superposition constants, and by the isospectral family parameter, which is a sort of decoherence parameter. Moreover, following techniques presented in [13], one can introduce multiple-parameter families of strictly isospectral modes, that is more decoherence parameters.

Finally, we remark that the class of Whittaker-WDW equations holds not only for radiation-filled universes. Some other matter sectors can be introduced without changing the confluent character of the equation. However, the radiation-filled case is well suited for resonator-cavity models of the universe at “quantum” gravitational scales [15].

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\[ z \left[ \sqrt{z} \frac{d}{dz} - \mu \frac{1}{\sqrt{z}} + \frac{1}{2} \sqrt{z} \right] \left[ \sqrt{z} \frac{d}{dz} + \frac{1}{2} (2\mu - 1) \frac{1}{\sqrt{z}} - \frac{1}{2} \sqrt{z} \right] \Psi_{\lambda,\mu} - z(\mu - \lambda - \frac{1}{2}) \Psi_{\lambda,\mu} = 0 . \]

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