A new kind of the solution of degenerate parabolic equation with unbounded convection term

Abstract: A new kind of entropy solution of Cauchy problem of the strong degenerate parabolic equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} \left( a(u, x, t) \frac{\partial u}{\partial x_i} \right) + \operatorname{div}(E(x, t)u),
\]

is introduced. If \( u_0 \in L^\infty(\mathbb{R}^N) \), \( E = \{ E_i \} \in (L^2(\Omega_T))^N \) and \( \operatorname{div} E \in L^2(\Omega_T) \), by a modified regularization method and choosing the suitable test functions, the BV estimates are got, the existence of the entropy solution is obtained. At last, by Kruzkov bi-variables method, the stability of the solutions is obtained.

Keywords: Cauchy problem, Degenerate parabolic equation, Entropy solution, Unbounded convection term

MSC: 35k10, 35k15, 35K55

1 Introduction

Consider the equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} \left( a(u, x, t) \frac{\partial u}{\partial x_i} \right) + \operatorname{div}(E(x, t)u), \quad (x, t) \in \Omega \times (0, T).
\]

where \( \Omega \) is a open domain in \( \mathbb{R}^N \). Assuming that \( 0 < \alpha \leq a(x, t, s) \leq \beta \), some applicative models related to equation (1) were studied in [12]. If \( \Omega \) is bounded, Boccardo L., Orsina L. and Porretta A. in [2] defined that:

A measurable function \( u \in L^\infty(\Omega) \cap L^2(0, T; H^1_0(\Omega)) \) is a weak solution of equation (1) in the sense of that

\[
< u_t, \varphi > + \iint_{\Omega} a(u, x, t) \nabla u \cdot \nabla \varphi dx dt = - \iint_{\Omega} u \nabla \varphi dx dt,
\]

for every \( \varphi \in L^2(0, T; H^1_0(\Omega)) \), where \( < \ldots > \) denotes the duality product between \( L^2(0, T; H^1_0(\Omega)) \) and \( L^2(0, T; H^{-1}(\Omega)) \). Clearly, if \( a(x, t, s) \equiv 0 \), then (1) becomes the type of conservation law equation, and it is well known that in this case, if one defines the weak solution as (2), then the uniqueness of the solutions is not true.

Also, Boccardo L., Orsina L. and Porretta A. in [2] had introduced the following unbounded entropy solution. A measurable function \( u \in L^\infty(0, T; L^1(\Omega)) \) is an entropy solution of equation (1) if \( T_k(u) \in L^2(0, T; H^1_0(\Omega)) \) for every \( k > 0 \) and \( u \) satisfies

*Corresponding Author: Huashui Zhan: School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, Fujian Province, P.R. China, E-mail: 201211007@xmut.edu.cn
\[ \int_{\Omega} \Theta_k(u - \varphi)(t)dx - \varphi_s, T_k(u - \varphi) > + \int_{0}^{t} \int_{\Omega} a(u, x, s)\nabla T_k(u - \varphi)dxds \leq - \int_{0}^{t} \int_{\Omega} u E \nabla T_k(u - \varphi)dxds + \int_{\Omega} \Theta_k(u_0 - \varphi(0))(t)dx. \]

for almost every \( t \in (0, T) \), for every \( \varphi \in L^2(0; T; H^-1(\Omega)) \cap L^\infty(\Omega) \) such that \( \varphi_t \in L^2(0, T; H^-1(\Omega)) + L^1(\Omega) \), where \( \Theta_k(s) = \int_{0}^{s} T_k(r)dr \). If we check the proof of the theorems in [2], we have found that the condition \( 0 < \alpha < \beta \) acts an important role. If this condition is weakened to \( 0 \leq \alpha \), to get the same conclusions seems difficult. By the way, though the authors did not discussed the uniqueness of the solutions in [2], we believe that the uniqueness of the solutions defined in the sense of inequality (3) is true, and we may study this problem in the future.

In our paper, we will consider Cauchy problem

\[ u_t - \text{div}(a(x, t)\nabla u) = \text{div}(u E), \quad (x, t) \in QT = \mathbb{R}^N \times (0, T), \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \]

If the linear term \( \text{div}(u E) \) is replaced by a nonlinear term \( \text{div}(b(u)) \), equation (4) becomes

\[ u_t - \text{div}(a(x, t)\nabla u) = \text{div}(b(u)), \quad (x, t) \in QT = \mathbb{R}^N \times (0, T). \]

If \( a(x, t, s) \geq 0 \), equation (4) (also equation (6)) now is a degenerate parabolic equation, and generally only has a weak solution. The paper [18] by A. I. Vol’pert and S.I. Hudjaev was the first to be devoted to the solvability of equation (6), the papers [3–6, 8, 14, 15, 18, 20, 23–25] et al. continue to study its posedness problem.

Comparing equation (6) with equation (4), the main obstacle is the unboundedness of \( E \). The unboundedness of \( E \) makes the estimating method used in [3, 8, 14, 23, 25] et al. not effective. To overcome these difficulties, we put forward a new definition of BV-entropy solution for problem (4)–(5), and by modifying the classical parabolic regularizing method, we are able to get the BV estimated formulas. The method used in our paper is completely different from that used in [18, 20, 24], [3, 8, 14, 23, 25] et al. But we use some inspiring techniques in [21]. To the end, some restrictions in \( E \) are added.

### 2 Definition and main results

Let \( S_\eta(s) = \int_{0}^{s} h_\eta(t)dt \) for small \( \eta > 0 \), where \( h_\eta(s) = \frac{2}{\sqrt{\eta}}(1 - \frac{|s|}{\eta})^+. \) Obviously \( h_\eta(s) \in C(\mathbb{R}) \), and

\[ h_\eta(s) \geq 0, \quad |sh_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1; \lim_{\eta \to 0} S_\eta(s) = \text{sgn} s, \lim_{\eta \to 0} S_\eta'(s) = 0. \]

**Definition 2.1.** A function \( u \) is said to be a weak nonnegative solution of the Cauchy problem (4)–(5), if

1. \( u \in BV(Q_T) \cap L^\infty(Q_T) \), there are functions \( g^i \in L^2(0, T; L^1_{loc}(\mathbb{R}^N)), i = 1, 2, \cdots, N \) such that \( \forall \varphi(x, t) \in C_0(Q_T), \)

\[ \iint_{Q_T} g^i(x, t)\varphi(x, t)dxdt = \iint_{Q_T} \tilde{r}(u, x, t)\frac{\partial u}{\partial x^i}\varphi(x, t)dxdt, \]

where \( r = \sqrt{a} \), and

\[ \tilde{r}(u, x, t) = \int_{0}^{1} \sqrt{a(su^+ + (1-s)u^-), x, t} ds. \]

2. For any \( \varphi \in C^2_0(Q_T), \varphi \geq 0, k \in \mathbb{R}, \eta > 0, u \) satisfies

\[ \iint_{Q_T} \{ I_\eta(u - k)\varphi_t - E_1 I_\eta(u - k)\varphi_{x_i} + A_\eta(u, x, t, k)\Delta \varphi + \int_{k}^{u} a(x, s, t, k)S_\eta(s - k)ds\varphi_{x_i} \]
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\[-S'_n(u - k) \mid \nabla \int_0^u \sqrt{a(s, x, t)} ds \mid^2 \varphi + \int_k^u S'_n(s - k) ds E_{i\chi_i} \psi dx dt \geq 0. \quad (8)\]

(3)

\[
\lim_{t \to 0} \int_{B_R} |u(x, t) - u_0(x)| \, dx = 0, \forall R > 0,
\]

where the pairs of equal indices imply a summation from 1 up to \( N \), and

\[
A_n(u, x, t, k) = \int_k^u a(s, x, t) S_n(s - k) ds, \quad I_n(u - k) = \int_0^{u-k} S_n(s) ds.
\]

Clearly if \( u \) is the a solution in Definition 2.1, then \( u \) is the entropy solution defined in [18].

The main results of the paper are the following theorems.

Theorem 2.2. Suppose that \( A''(s, x, t) = \frac{\partial^2}{\partial s^2} A(s, x, t), a_{x_i}(s, x, t) = \frac{\partial a(s, x, t)}{\partial x_i}, A(s, x, t) = A'(s, x, t) = \frac{\partial A(s, x, t)}{\partial s} \geq 0; u_0(x) \in L^\infty(\mathbb{R}^N), u_0(x) \in C(\mathbb{R}^N); E = \{E_i\} \in (L^2(Q_T))^N, \text{div} E \in L^2(Q_T). \) If there is a constant \( \delta > 0 \) such that

\[
a(u, x, t) - \delta \sum_{s=1}^{N+1} (a_{x_i})^2 \geq 0,
\]

then problem (4)–(5) has a generalized solution in the sense of Definition 2.1, where \( x_{N+1} = t \).

Theorem 2.3. Let \( u, v \) be solutions of problem (4)–(5) with initial values \( u_0(x), v_0(x) \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) respectively. Suppose that \( A(s, x, t) \) satisfies the conditions as in Theorem 2.2, and

\[
E \cdot x = E_i(x, t)x_i \geq 0.
\]

Then

\[
\int_{\mathbb{R}^N} |u(x, t) - v(x, t)| \omega_\lambda(x) dx \leq c \int_{\mathbb{R}^N} |u_0 - v_0| \omega_\lambda(x) dx,
\]

where \( c, \lambda \) are positive constants and

\[
\omega_\lambda(x) = \exp(-\lambda \sqrt{1 + |x|^2}).
\]

Corollary 2.4. The solution of problem (4)–(5) is unique.

To explain the reasonableness of Definition 2.1, suppose that equation (4) has a classical solution \( u \). Let \( \varphi \in C^2_0(Q_T), \varphi \geq 0, k \in \mathbb{R}, \eta > 0 \). Multiplying (4) by \( \varphi S_\eta(u - k) \) and integrating over \( Q_T \), we have

\[
\iint_{Q_T} \frac{\partial u}{\partial t} \varphi S_\eta(u - k) dx dt = \iint_{Q_T} \frac{\partial}{\partial x_i} \left( a(u, x, t) \frac{\partial u}{\partial x_i} \right) \varphi S_\eta(u - k) dx dt
\]

(14)

Then

\[
\iint_{Q_T} \frac{\partial u}{\partial t} \varphi S_\eta(u - k) dx dt = \iint_{Q_T} \frac{\partial I_\eta(u - k)}{\partial t} \varphi dx dt = - \iint_{Q_T} I_\eta(u - k) \frac{\partial \varphi}{\partial t} dx dt.
\]

(15)

Then

\[
\iint_{Q_T} \frac{\partial}{\partial x_i} \left( a(u, x, t) \frac{\partial u}{\partial x_i} \right) \varphi S_\eta(u - k) dx dt
\]

(16)
where
\[
\int_0^T \int_{\mathbb{R}^N} u(t, x, t) \frac{\partial u}{\partial x_i} S_\eta(u - k) \varphi_{x_i} dx dt = \int_0^T \int_{\mathbb{R}^N} \frac{\partial A_\eta(u, x, t, k)}{\partial x_j} \varphi_{x_j} dx dt
\]
\[= \int_0^T \int_{\mathbb{R}^N} \left[ \frac{\partial (A_\eta(u, x, t, k) \varphi_{x_i})}{\partial x_j} - A_\eta(u, x, t, k) \Delta \varphi \right] dx dt - \int_0^T \int_{\mathbb{R}^N} \left[ \int_k^u a_{x_j}(s, x, t) S_\eta(s - k) ds \right] \varphi_{x_i} dx dt
\]
\[= - \int_0^T \int_{\mathbb{R}^N} A_\eta(u, x, t, k) \Delta \varphi dx dt - \int_0^T \int_{\mathbb{R}^N} \left[ \int_k^u a_{x_j}(s, x, t) S_\eta(s - k) ds \right] \varphi_{x_i} dx dt. \tag{17}
\]

So
\[
\int_{Q_T} \frac{\partial}{\partial x_i} \left( u(x, t) \frac{\partial u}{\partial x_i} \right) \varphi S_\eta(u - k) dx dt
\]
\[= - \int_{Q_T} S_\eta'(u - k) a(u, x, t) \frac{\partial u}{\partial x_i} \varphi dx dt + \int_{Q_T} A_\eta(u, x, t, k) \Delta \varphi dx dt
\]
\[+ \int_0^T \int_{\mathbb{R}^N} \left[ \int_k^u a_{x_j}(s, x, t) S_\eta(s - k) ds \right] \varphi_{x_i} dx dt
\]
\[= - \int_{Q_T} S_\eta'(u - k) \sum_{i=1}^N |g^i|^2 \varphi dx dt + \int_{Q_T} A_\eta(u, x, t, k) \Delta \varphi dx dt
\]
\[+ \int_0^T \int_{\mathbb{R}^N} \left[ \int_k^u a_{x_j}(s, x, t) S_\eta(s - k) ds \right] \varphi_{x_i} dx dt. \tag{18}
\]

where \(g^i = \sqrt{\eta} \frac{\partial u}{\partial x_i}\).

\[
\int_{Q_T} \text{div}(u E_\eta) S_\eta(u - k) \varphi dx dt = \int_{Q_T} \left( \frac{\partial u}{\partial x_i} E_i + u \frac{\partial E_i}{\partial x_i} \right) S_\eta(u - k) \varphi dx dt
\]
\[= \int_{Q_T} \frac{\partial I_\eta(u - k)}{\partial x_i} \varphi_{x_i} dx dt + \int_{Q_T} u \frac{\partial E_i}{\partial x_i} S_\eta(u - k) \varphi dx dt
\]
\[= - \int_{Q_T} I_\eta(u - k) E_i \varphi_{x_i} dx dt - \int_{Q_T} I_\eta(u - k) E_i \varphi_{x_i} dx dt + \int_{Q_T} u \frac{\partial E_i}{\partial x_i} S_\eta(u - k) \varphi dx dt
\]
\[= - \int_{Q_T} I_\eta(u - k) E_i \varphi_{x_i} dx dt + \int_{Q_T} \int_k^u s S_\eta'(s - k) ds E_i \varphi_{x_i} dx dt. \tag{19}
\]

By (14)–(19), if equation (4) has a classical solution \(u\), then
\[
\int_{Q_T} \left[ I_\eta(u - k) \varphi_t - E_i I_\eta(u - k) \varphi_{x_i} + A(u, x, t, k) \Delta \varphi \right] dx dt
\]
\[+ \int_0^T \int_{\mathbb{R}^N} \left[ \int_k^u a_{x_j}(s, x, t) S_\eta(s - k) ds \right] \varphi_{x_i} dx dt
\]
- \iint_{Q_T} \left[ \frac{S'(u-k)}{\partial_x} a(u,x,t) \frac{\partial u}{\partial x} \varphi - \int_k^u s S'(s-k) dE_{x,i} \varphi \right] dx dt = 0. \quad (20)

Clearly

\iint_{Q_T} \left[ I_\eta(u-k) \varphi_t - E_i I_\eta(u-k) \varphi_{x,i} + A_\eta(u,x,t,k) \Delta \varphi + \int_k^u s S'_\eta(s-k) dE_{x,i} \varphi \right] dx dt

+ \iint_{Q_T} \left[ \int_k^u a_{x,i} (s,x,t) s(s-k) d\varphi_{x,i} dx dt \geq 0. \right. \quad (21)

Let \eta \to 0 in this inequality. We have

\iint_{Q_T} \left[ |u-k| \varphi_t - E_i |u-k| \varphi_{x,i} + \left[ A(u,x,t) - A(k,x,t) \right] \text{sgn}(u-k) \varphi_{x,i} + k \text{sgn}(u-k) E_{i,x,i} \varphi \right] dx dt

+ \iint_{Q_T} \left[ \int_k^u a_{x,i} (s,x,t) s(s-k) d\varphi_{x,i} dx dt \geq 0. \quad (8') \right.

Clearly, if one defines the weak solutions \( u_1, u_2, \) and \( u_3 \) of equation (4) (similarly, also equation (6)) in the sense of formulas (8), (21) and (8') respectively, then \( u_1 \) is also a solution in the senses of inequality (21) and (8'), \( u_2 \) is also the solution in the sense of inequality (8').

If equation (6) is weakly degenerate, Ref. [11, 15, 17, 25] adopted to define the weak solution in sense (8'). In this case, the term \(-S'_\eta(u-k) \sum_{j=1}^N g^j |^2 \varphi \) in (20) seems redudant, and should be drawn away. But, if equation (6) is strongly degenerate, the references [3, 4, 8, 14, 23, 25] tell us that the term \(-S'_\eta(u-k) \sum_{j=1}^N g^j |^2 \varphi \) implies very important information of the uniqueness, it can not be drawn away.

Also, we note that the classical solution \( u \) induces an integral equality (20), while the weak solution formula defined as (8) is an inequality, this is due to that the following weak convergence property.

**Lemma 2.5.** Assume that \( U \subset \mathbb{R}^N \) is an open bounded set and as \( k \to \infty \),

\[ f_k \rightharpoonup f \text{ weakly in } L^q(U), \quad 1 \leq q < \infty, \]

then

\[ \lim_{k \to \infty} \inf \| f_k \|_{L^q(U)}^q \geq \| f \|_{L^q(U)}^q. \] \quad (22)

Generally, inequality (22) can not be an equality. In what follows, one can see that this is why we can only define the weak solution as (8) instead of (20).

**Remark 2.6.** Consider the equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x}, \quad (x,t) \in \mathbb{R} \times (0, T). \] \quad (23)

Voïp'ert A.I. and Hudjaev S.I. in [21] defined that: \( u \in BV(Q_T) \cap L^\infty(Q_T) \) is said to be a weak solution of (23), if

\[ \frac{\partial A(u)}{\partial x} \in L^1_{loc}(Q_T), \quad \text{and for any } 0 \leq \varphi \in C^\infty_0(Q_T), \text{ any } k \in \mathbb{R}, \]

\[ \iint_{Q_T} \text{sgn}(u-k) \left[ (u-k) \frac{\varphi}{\partial t} - \frac{\partial A(u)}{\partial x} \frac{\partial \varphi}{\partial x} \right] dx dt - \iint_{Q_T} \text{sgn}(u-k) \left[ (B(u)-B(k)) \frac{\partial \varphi}{\partial x} \right] dx dt \geq 0. \] \quad (24)

We know that only under the condition \( \frac{\partial A(u)}{\partial x} \in L^\infty(Q_T) \cap BV(Q_T) \) the uniqueness of the solutions in the sense (23) is true. So, an essential improvement of our paper (also [3, 4, 8, 14, 23, 25]) is to get the uniqueness of the solutions in the sense (8) without any bounded restrictions in \( a \).
3 The regularized problem and the proof of Theorem 2.2

Suppose that \( A(s, x, t), u_0(x) \) are smooth as in the assumption of Theorem 2.2 and \( u_0(x) \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \), \( E = \{ E_i \} \in (L^2(\mathbb{T}))^N \), \( \text{div} E \in L^2(\mathbb{T}) \). For any given large positive numbers \( K \), let us introduce the following modified regularized equation.

\[
\frac{\partial u}{\partial t} = \text{div}(a(u, x, t) \nabla u) + \frac{1}{K} \Delta u + \text{div}(u \delta_x * T_K E), \text{ in } \mathbb{T} = \mathbb{R}^N \times (0, T),
\]

\( u(x, 0) = u_0_K(x) \),

where \( \delta_x \) is the mollifier as usual, i.e. let \( y = (x, t) = (x_1, \cdots, x_N, t) \), and

\[
\delta(y) = \begin{cases} \frac{1}{\varepsilon^{N+1}} e^{\frac{-|y|^2}{\varepsilon^2}} & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1, \end{cases}
\]

where

\[
C = \int_{B_1(0)} e^{\frac{1}{2|y|^2}} dy.
\]

For any given \( \varepsilon > 0 \), let

\[
\delta_{\varepsilon}(y) = \frac{1}{\varepsilon^{N+1}} \delta\left(\frac{y}{\varepsilon}\right).
\]

Here, we choose \( \varepsilon = \frac{1}{K} \) especially, and

\[
\delta_{\varepsilon} * T_K(E) = (\delta_{\varepsilon}(E_1) * T_K(E_1), \delta_{\varepsilon}(E_2) * T_K(E_2), \cdots, \delta_{\varepsilon}(E_N) * T_K(E_N)),
\]

\[
T_K(s) = \min\{K, \max\{-K, s\}\}.
\]

Moreover, we suppose that \( \text{supp} u_0 \subset B_K = \{ x : |x| < K \} \), and it satisfies

\[
\lim_{K \to \infty} \|u_0_K - u_0\|_{L^2(\mathbb{R}^N)} = 0, \|u_0\|_{L^\infty} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}.
\]

(27)

It is well-known that there is a classical solutions \( u_K \in C^{2,1}(\mathbb{T}) \) of (25)–(26). By this fact and using the maximum principle, we have

\[
\|u_K\|_{L^\infty} \leq \|u_0\|_{L^\infty}.
\]

(28)

Let \( \text{grad} u_K = (u_{x_1}, u_{x_2}, \cdots, u_{x_N}, u_{x_N+1}) \) and \( u_{x_N+1} = u_t \). For simplicity, we denote \( u_K \) as \( u \) in the following calculation. Let us derivation on \( x_s, s = 1, 2, \cdots, N, N + 1 \) in (25). Then multiplying with \( u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi \) on the two sides, \( 0 \leq \varphi \in C_0^\infty(\mathbb{T}) \), and integrating over \( B_K \), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^N} I_h(|\text{grad} u|) \varphi dx - \frac{1}{K} \int_{\mathbb{R}^N} (\Delta u_{x_s}) u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi dx - \int_{\mathbb{R}^N} \text{div}(a(x, t, u) \nabla u, u_{x_s}) u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi dx
\]

\[
- \int_{\mathbb{R}^N} \text{div}(a(x, t, u) u, \nabla u, u_{x_s}) u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi dx
\]

\[
- \int_{\mathbb{R}^N} \nabla u_{x_s} \cdot \delta_{\varepsilon} * T_K(E) u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi dx - \int_{\mathbb{R}^N} \text{div}(u \frac{\partial \delta_{\varepsilon} * T_K(E)}{\partial x_s}) u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi dx = 0.
\]

(29)

For the last term of the left hand side in (29),

\[
\int_{\mathbb{R}^N} \text{div}[u \frac{\partial \delta_{\varepsilon} * T_K(E)}{\partial x_s}] u_{x_s} \frac{S_h(|\text{grad} u|)}{|\text{grad} u|} \varphi dx
\]
For the other terms of the left hand side in (29), integrating by part, we have

$$
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^N} I_n(\|\nabla u\|) \varphi \, dx + \frac{1}{\mathbb{K}} \int_{\mathbb{R}^N} u x_i u x_j u x_k \frac{\partial^2 I_n(\|\nabla u\|)}{\partial x_i \partial x_j} \varphi \, dx + \frac{1}{\mathbb{K}} \int_{\mathbb{R}^N} S_n(\|\nabla u\|) u x_i \varphi \, dx \\
+ \int_{\mathbb{R}^N} a(u, x, t) u x_i u x_j u x_k \frac{\partial^2 I_n(\|\nabla u\|)}{\partial x_i \partial x_j} \varphi \, dx \\
+ \int_{\mathbb{R}^N} a(u, x, t) S_n(\|\nabla u\|) u x_i \varphi \, dx + \int_{\mathbb{R}^N} a'(u, x, t) u x_i I_n(\|\nabla u\|) \varphi \, dx \\
- \int_{\mathbb{R}^N} \frac{\partial}{\partial x_i} a'(u, x, t) u x_i (\|\nabla u\| S_n(\|\nabla u\|) - I_n(\|\nabla u\|)) \varphi \, dx \\
- \int_{\mathbb{R}^N} a'(u, x, t) \Delta u (\|\nabla u\| S_n(\|\nabla u\|) - I_n(\|\nabla u\|)) \varphi \, dx \\
+ \int_{\mathbb{R}^N} a_{x_i} (u, x, t) u x_i \frac{\partial^2 I_n(\|\nabla u\|)}{\partial x_i \partial x_j} u x_j \varphi \, dx + \int_{\mathbb{R}^N} a_{x_i} (u, x, t) u x_i \varphi \, dx + S_n(\|\nabla u\|) \, dx \\
- \sum_{i=1}^{N} \int_{\mathbb{R}^N} u x_i \frac{\partial (\delta_x * T_K(E_i))}{\partial x_i} + u x_s \frac{\partial^2 (\delta_x * T_K(E_i))}{\partial x_i \partial x_j} \varphi \, dx = 0.
\end{align*}
$$

If we notice that $\varepsilon = \frac{1}{\mathbb{K}}$, then

$$
\frac{\partial (\delta_x * T_K(E_i))}{\partial x_i} = - \int_{\{y:|x-y|<\mathbb{K}\}} \frac{2K^2(x_i - s_i)}{|(x-y)^2|} \frac{K^{N+1}}{A} \frac{1}{e^{(K(x-y))^{p-1}}} T_K(E_i(s)) \, dy,
$$

where $x = (x_1, \ldots, x_N, t)$ as before.

Moreover, it is well known that

$$
\frac{1}{|x-y|^2} e^{(x-y)^2-1} \leq \varepsilon^2.
$$

so, by the facts of that $|x-y| < 1, |T_K(E_i)| \leq K$,

$$
|\frac{\partial (\delta_x * T_K(E_i))}{\partial x_i}| \leq c \int_{\{y:|x-y|<\mathbb{K}\}} \frac{1}{|x-y|^2} \frac{K^{N+3}}{A} e^{(K(x-y))^{p-1}} \, dy \leq c K^{N+3}.
$$

Thus, if we choose that

$$
\varphi(x) = \frac{1}{K^{N+4}} \varphi_1(x), \varphi_1 \in C_0^\infty (\mathbb{R}^N),
$$

$$
\int_{\mathbb{R}^N} u x_i \frac{\partial (\delta_x * T_K(E_i))}{\partial x_i} S_n(\|\nabla u\|) \varphi \, dx \leq \frac{c}{K} \int_{\mathbb{R}^N} \|\nabla u\| \varphi \varphi_1 \, dx.
$$

Similarly, we are able to show that

$$
|\frac{\partial^2 (\delta_x * T_K(E_i))}{\partial x_i \partial x_j}| \leq c K^{N+4},
$$

then

$$
\int_{\mathbb{R}^N} u \frac{\partial^2 (\delta_x * T_K(E_i))}{\partial x_i \partial x_j} S_n(\|\nabla u\|) \varphi \, dx \leq c \int_{\mathbb{R}^N} \varphi \varphi_1 \, dx.
$$
At the same time, if we set

\[
\begin{pmatrix}
    v^i_1 \\
v^i_2 \\
    \vdots \\
v^i_{N+1}
\end{pmatrix} = 
\begin{pmatrix}
    q^{11} & q^{12} & \cdots & q^{1N+1} \\
    q^{11} & q^{12} & \cdots & q^{1N+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    q^{N+11} & q^{N+12} & \cdots & q^{N+1N+1}
\end{pmatrix} 
\begin{pmatrix}
    u_{x_1x_i} \\
u_{x_2x_i} \\
    \vdots \\
u_{x_{N+1}x_i}
\end{pmatrix}
\]

where \((q^{sp})\) is the square root of \(\left(\frac{\partial^2 I_n}{\partial \xi_s \partial \xi_p}\right)\). Then

\[
\sum_{i=1}^{N} |d_x u_{x_i} \frac{\partial^2 I_n}{\partial \xi_s \partial \xi_p} u_{x_s x_i}| = \left| \sum_{i=1}^{N} (a_{x_1} u_{x_1}, a_{x_2} u_{x_2}, \ldots, a_{x_{N+1}} u_{x_{N+1}})(q^{sp}) \right|
\]

\[
= \sum_{i=1}^{N} |a_{x_i} u_{x_i} q^{sp} v^i_p| \leq \sum_{i=1}^{N} \delta \sum_{s,p=1}^{N+1} (a_{x_i} v^i_p)^2 + \frac{1}{4\delta} \sum_{s,p=1}^{N+1} (q^{sp} u_{x_i})^2.
\]

By the assumption

\[
a(u, x, t) - \delta \sum_{i=1}^{N+1} (a_{x_i})^2 \geq 0,
\]

then

\[
\int_{\mathbb{R}^N} a(u, x, t) u_{x_1} u_{x_s} \frac{\partial^2 I_n}{\partial \xi_s \partial \xi_p} u_{x_s x_i} \varphi dx = \int_{\mathbb{R}^N} a_{x_i} u_{x_1} u_{x_s} \frac{\partial^2 I_n}{\partial \xi_s \partial \xi_p} u_{x_s x_i} \varphi dx - \int_{\mathbb{R}^N} a(u, x, t) u_{x_1} \frac{\partial^2 I_n}{\partial \xi_s \partial \xi_p} u_{x_s x_i} \varphi dx
\]

\[
\geq -\frac{1}{4\delta} \int_{\mathbb{R}^N} \sum_{s,p=1}^{N+1} \sum_{i=1}^{N} (q^{sp} u_{x_i})^2 \geq -c \int_{\mathbb{R}^N} |\varphi u_{x_i}|^2.
\]

(33)

\[
\frac{1}{K} \int_{\mathbb{R}^N} S_n(|\varphi u_{x_i}|) u_{x_s} u_{x_s} \varphi x_i dx = -\frac{1}{K^{N+4}} \int I_n(|\varphi u_{x_i}|) \Delta \varphi_1 dx,
\]

(34)

\[
\int_{\mathbb{R}^N} a(u, x, t) \frac{S_n(|\varphi u_{x_i}|)}{|\varphi u_{x_i}|} u_{x_s} u_{x_s} \varphi x_i dx + \int_{\mathbb{R}^N} a'(u, x, t) u_{x_1} I_n(|\varphi u_{x_i}|) \varphi x_i dx
\]

\[
= -\frac{1}{K^{N+4}} \int a(u, x, t) I_n(|\varphi u_{x_i}|) \Delta \varphi_1 dx.
\]

(35)

\[
|\varphi u_{x_i}| S_n(|\varphi u_{x_i}|) - I_n(|\varphi u_{x_i}|) = \int_{0}^{\text{th}_\eta(t)} \varphi x_i d\tau \rightarrow 0, \text{ as } \eta \rightarrow 0,
\]

(36)

By a process of limit, one can assume that

\[
\varphi_1 = \omega_\lambda(x) = \exp(-\lambda \sqrt{1 + |x|^2}),
\]

where \(\lambda\) is a positive constant. Then

\[
\omega_{\lambda x_i} = \omega_\lambda \frac{-\lambda x_i}{\sqrt{1 + |x|^2}}. \quad |\nabla \omega_\lambda| \leq c_\lambda \omega_\lambda, \quad |\Delta \omega_\lambda| \leq c_\lambda \omega_\lambda.
\]

(37)

Let \(\eta \rightarrow 0\) in (30). By (31)–(37), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^N} |\varphi u_{x_i}| dx \leq c_1 + c_2 \int_{\mathbb{R}^N} |\varphi u_{x_i}| dx.
\]
equivalently,
\[
\int_{\mathbb{R}^N} |\nabla u| |\omega_\lambda| \, dx \leq c_1 + c_2 \int_0^T ds \int_{\mathbb{R}^N} |\nabla u| |\omega_\lambda| \, dx,
\]
by Gronwall Lemma, if we return to denote \( u \) as \( u_K \), we have
\[
\int_{\mathbb{R}^N} |\nabla u_K| \, dx \leq c(T, \lambda, \|u_0\|_{L^\infty}). \tag{38}
\]
By (38), from (25), it is easy to show that
\[
\int_{Q_T} (a(u_K, x, t) + \frac{1}{K}) |\nabla u_K| \leq c(T, \lambda, \|u_0\|_{L^\infty}). \tag{39}
\]
By (38), (39) and Kolomogrov's Theorem, there exists a subsequence \( \{u_{K_n}\} \) of the family \( \{u_K\} \) of solutions of regularized problem (25)–(26), which converges strongly in \( L^1(Q_T) \cap L^\infty(Q_T) \) and \( u_{K_n} \to u \) a.e. on \( Q_T \).

\textbf{Proof of Theorem 2.2.} We now prove that \( u \) is a generalized solution of problem (4)–(5). From (39), we have
\[
\frac{\partial}{\partial x_i} \int_0^u a(s, x, t) \, ds \to \frac{\partial}{\partial x_i} \int_0^u a(s, x, t) \, ds \text{ weakly in } L^2_{loc}(\mathbb{R}^N \times (0, T)), i = 1, 2, \cdots, N.
\]
This implies
\[
\frac{\partial}{\partial x_i} \int_0^u a(s, x, t) \, ds \in L^2_{loc}(\mathbb{R}^N \times (0, T)), i = 1, 2, \cdots, N.
\]
Thus \( u \) satisfies (1) in Definition 2.1.

Let \( \varphi \in C^2_0(Q_T), \varphi \geq 0, k \in \mathbb{R}, \eta > 0. \) Multiplying equation (25) by \( \varphi S_\eta(u_K - k) \) and integrating over \( Q_T \), as we have got (20), we obtain
\[
\int_{Q_T} \left[ I_\eta(u_K - k) \varphi_t - E_i I_\eta(u_K - k) \varphi_{x_i} + A(u_K, x, t, k) \Delta \varphi \right] \, dx \, dt
\]
\[
- \frac{1}{K} \int_{Q_T} S'_\eta(u_K - k)(u_K - k) \varphi_{x_i} u_{x_i} \, dx \, dt - \frac{1}{K} \int_{Q_T} S_\eta(u_K - k)(u_K - k) \Delta \varphi \, dx \, dt
\]
\[
+ \frac{1}{K} \int_{Q_T} S'_\eta(u_K - k)(u_K - k) \varphi \, |\nabla u_K|^2 \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} a_{x_i}(s, x, t) S_\eta(s - k) \, dx \, dt
\]
\[
- \int_{Q_T} \left[ S'_\eta(u_K - k) a(u_K, x, t, k) \frac{\partial u_K}{\partial x_i} \frac{\partial u_K}{\partial x_i} \varphi - \int_k^u S'_\eta(s - k) ds E_i x_i \varphi \right] \, dx \, dt. \tag{40}
\]
By Lemma 2.5,
\[
\liminf_{K \to \infty} \int_{Q_T} S'_\eta(u_K - k) a(u_K, x, t, k) \frac{\partial u_K}{\partial x_i} \frac{\partial u_K}{\partial x_i} \varphi \, dx \, dt \geq \int_{Q_T} S'_\eta(u_K - k) |\nabla \int_0^u \sqrt{a(s, x, t)} \, ds|^2 \varphi \, dx \, dt. \tag{41}
\]
Noticing that \( E = \{E_i\} \in (L^2(Q_T))^N \) and \( \text{div} E \in L^2(Q_T) \), let \( K \to \infty \) in (40), we get (8).

The proof of (9) is similar to that in [19] et.al, we omit the details here. \( \square \)
4 The Proof of Theorem 2.3

Let \( \Gamma_u \) be the set of all jump points of \( u \in BV(Q_T) \), \( v \) the normal of \( \Gamma_u \) at \( X = (x, t) \), \( u^+(X) \) and \( u^-(X) \) the approximate limits of \( u \) at \( X \in \Gamma_u \) with respect to \( (v, Y - X) > 0 \) and \( (v, Y - X) < 0 \) respectively. For continuous function \( p(u, x, t) \) and \( u \in BV(Q_T) \), define

\[
\tilde{\rho}(u, x, t) = \int_0^1 p(\tau u^+ + (1 - \tau)u^-, x, t) d\tau, \quad \tilde{u} = \frac{1}{2}(u^+ + u^-),
\]

which is called the composite mean value of \( p \) and \( u \). For a given \( t \), we denote \( \Gamma_u^t \), \( H^t, (v_1^t, \ldots, v_N^t) \) and \( u_\pm^t \) as all jump points of \( u(\cdot, t) \), Hausdorff measure of \( \Gamma_u^t \), the unit normal vector of \( \Gamma_u^t \), and the asymptotic limit of \( u(\cdot, t) \) respectively. By [17], if \( f(s) \in C^1(R), u \in BV(Q_T) \), then \( f(u) \in BV(Q_T) \) and

\[
\frac{\partial f(u)}{\partial x_i} = \chi_i^f(u) \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \ldots, N.
\]

**Lemma 4.1.** Let \( u \) be a solution of problem (4)–(5). Then

\[
a(s, x, t) = 0, \quad s \in I(u^+(x, t), u^-(x, t)) \quad a.e. \text{ on } \Gamma_u,
\]

where \( I(\alpha, \beta) \) denote the closed interval with endpoints \( \alpha \) and \( \beta \).

**Proof.** Denote

\[
\Gamma_1 = \{ (x, t) \in \Gamma_u, v_1(x, t) = \cdots = v_N(x, t) = 0 \}
\]

\[
\Gamma_2 = \{ (x, t) \in \Gamma_u, v_1^2(x, t) = \cdots = v_N^2(x, t) > 0 \}.
\]

First prove \( a(s, x, t) = 0, \quad s \in I(u^+(x, t), u^-(x, t)) \quad a.e. \text{ on } \Gamma_1 \). Since any measurable subset of \( \Gamma_1 \) can be expressed as the union of a Borel set and a set of measure zero, it suffices to prove

\[
a(s, x, t) = 0, \quad s \in I(u^+(x, t), u^-(x, t)) \quad a.e. \text{ on } U \subset \Gamma_1,
\]

where \( U \) is a Borel subset of \( \Gamma_1 \). We may suppose \( \overline{U} \) is compact. By Lemma 3.7.8 in [22], for any bounded function \( f(x, t) \), which is measurable with respect to measure \( \frac{\partial u}{\partial x_i} \), we have

\[
\int_{U^t} f(x, t) \frac{\partial u}{\partial x_i} = \int_0^T dt \int_{U^t} f(x, t) \frac{\partial u}{\partial x_i}.
\]

(45) is equivalent to

\[
\int_{U^t} f(x, t)(u^+(x, t) - u^-(x, t)) v_i dH = \int_0^T dt \int_{U^t} f(x, t)(u_+^t(x, t) - u_-^t(x, t)) v_i dH^t.
\]

The definition of \( \Gamma_1 \) implies that the left hand side vanishes, so we have

\[
\int_0^T dt \int_{U^t} f(x, t)(u_+^t(x, t) - u_-^t(x, t)) v_i dH^t = 0.
\]
Choose \( f(x,t) = \chi_U(x,t) \text{sgn}(u_+^i(x,t) - u_-^i(x,t)) \text{sgn}v_j^i \), where \( \chi_U(x,t) \) denote the characteristic function of \( U \), sum up for \( i \) from 1 up to \( N \). Then we obtain

\[
\int_G dt \int_{U^t} (u_+^i(x,t) - u_-^i(x,t))(|v_1^i| + \cdots + |v_N^i|) dH^t = 0.
\]

(46)

where \( G \) is the projection of \( U \) on the \( t \)-axis. Equality (46) implies for almost all \( t \in G \),

\[
\int_{U^t} (u_+^i(x,t) - u_-^i(x,t))(|v_1^i| + \cdots + |v_N^i|) dH^t = 0
\]

and hence for almost all \( t \in G \),

\[
v_1^i = \cdots = v_N^i = 0.
\]

\( H^t \)-almost everywhere on \( U^t \), which is impossible unless \( \text{mes}G = 0 \).

For any \( \alpha, \beta \) with \( 0 < \alpha < \beta < T \), we choose \( \gamma_j(t) \in C^\infty_0(0, T) \) such that

\[
0 \leq \gamma_j(t) \leq 1, \quad \lim_{\gamma \to \infty} \gamma_j(t) = \chi_{[\alpha, \beta]}(t), \quad \forall t \in [0, T].
\]

By [24], we can choose \( \phi_n \in C^\infty_0(Q_T) \) such that

\[
|\phi_n(x,t)| \leq 1, \quad \lim_{\phi \to \infty} \phi_n = \chi_U \text{ in } L^1(Q_T, |\frac{\partial u}{\partial t}|).
\]

Now from the definition of BV-function, we have

\[
\int_{Q_T} \phi_n(x,t)\gamma_j(t) \frac{\partial u}{\partial t} + \int_{Q_T} E_i u \frac{\partial}{\partial x_i} \phi_n(x,t)\gamma_j(t) dx dt
\]

\[
= \int_{Q_T} A(u,x,t) \Delta \phi_n(x,t)\gamma_j(t) dx dt - \int_{Q_T} \int_0^u a_{x_i}(s,x,t) ds \phi_n(x,t)\gamma_j(t) dx dt.
\]

Letting \( j \to \infty \) leads to

\[
\int_{Q_T} \phi_n(x,t)\chi_{[\alpha, \beta]}(t) \frac{\partial u}{\partial t} + \int_{Q_T} E_i u \frac{\partial}{\partial x_i} \phi_n(x,t)\chi_{[\alpha, \beta]}(t) dx dt
\]

\[
= \int_{Q_T} A(u,x,t) \Delta \phi_n(x,t)\chi_{[\alpha, \beta]}(t) dx dt - \int_{Q_T} \int_0^u a_{x_i}(s,x,t) ds \phi_n(x,t)\chi_{[\alpha, \beta]}(t) dx dt.
\]

Clearly, this equality also holds if \( [\alpha, \beta] \) is replaced by \( (\alpha, \beta) \) and hence it holds even if \( [\alpha, \beta] \) is replaced by any open set \( I \) with \( T \subset (0, T) \). Since \( G \) is a Borel set, by approximation we may conclude that

\[
\int_{Q_T} \phi_n(x,t)\chi_G(t) \frac{\partial u}{\partial t} = \int_{Q_T} A(u,x,t) \Delta \phi_n(x,t)\chi_G(t) dx dt - \int_{Q_T} E_i u \frac{\partial}{\partial x_i} \phi_n(x,t)\chi_G(t) dx dt.
\]

Since \( \text{mes}G = 0 \), the three terms on the right hand vanish and

\[
\int_{Q_T} \phi_n(x,t)\chi_G(t) \frac{\partial u}{\partial t} = 0.
\]

Letting \( n \to \infty \) gives

\[
\int_{U} \frac{\partial u}{\partial t} = \int_{Q_T} \chi_U(x,t)\chi_G \frac{\partial u}{\partial t} = 0.
\]
Hence
\[ \int_U (u^+(x,t) - u^-(x,t))v_i dH = 0, \]  
(47)
which implies \( H(U) = 0 \) and \( H(\Gamma_1) = 0 \) by the arbitrariness of \( U \).

Next we prove \( H(\Gamma_2) = 0 \). Let \( U \) be any Borel subset of \( \Gamma_2 \) which is compact in \( Q_T \). Since \( U \) is a set of \( N + 1 \)-dimensional measure zero and \( \nabla A(u, x, t) \in L^2_{loc}(Q_T) \), we have
\[ \iiint_U \frac{\partial}{\partial x_i} A(u, x, t) dxdtdt = 0, \quad i = 1, \cdots, N, \]
and hence
\[ \int_U (A(u^+(x,t),x,t) - A(u^-(x,t),x,t))v_i dH = 0, \quad i = 1, \cdots, N. \]

Form this it follows by the definition of \( \Gamma_2 \) that
\[ \int_{u^+(x,t)} a(s, x, t) ds = 0, \quad a.e. on \Gamma_2. \]  
(48)
Thus the lemma is proved.

Proof of Theorem 2.3. Let \( u, v \) be two generalized solutions of equation (4) with initial values
\[ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \]  
(49)
By Definition 2.1, we have for any \( \psi \in C^2_0(Q_T) \), \( \varphi \geq 0, \ k, l \in R \),
\[ \iint_{Q_T} \{ I_\eta(u - k)\psi_t - E_\xi(x, t)I_\eta(u - k)\psi_{x_j} + A_\eta(u, x, t, k)\Delta \psi + \int_k^u a_{x_i}(s, x, t)S_\eta(s - k)ds \psi_{x_i} \}
\]
\[-S_\eta'(u - k) | \nabla \int_0^u \sqrt{a(s, x, t)} ds |^2 \psi - \int_0^u sS_\eta'(s - k)ds E_{\xi}(x, t) \psi dxdtdt \geq 0, \]  
(50)
\[ \iint_{Q_T} \{ I_\eta(v - l)\psi_t - E_\xi(y, t)I_\eta(v - l)\psi_{y_i} + A_\eta(v, y, t, l)\Delta \psi + \int_l^v a_{y_i}(y, x, t)S_\eta(s - l)ds \psi_{y_i} \}
\]
\[-S_\eta'(v - l) | \nabla \int_0^v \sqrt{a(s, y, t)} ds |^2 \psi - \int_0^v sS_\eta'(s - l)ds E_{\xi}(y, t) \psi dydt \tau \geq 0. \]  
(51)
Let \( \psi(x, t, y, \tau) \geq 0, \ \psi \in C^2(Q_T \times Q_T), \ \text{supp} \psi(\cdot, \cdot, \cdot, \cdot) \subset Q_T \) if \( (t, y) \in Q_T, \ \text{supp} \psi(x, t, \cdot, \cdot) \subset Q_T \). We choose \( k = v(y, t), \ l = u(x, t), \ \varphi = \psi(x, t, y, \tau) \) in (50) (51) and integrate over \( Q_T \), to get
\[ \iint_{Q_T} \iint_{Q_T} \{ I_\eta(u - v)(\psi_t + \psi_t) - (E_\xi(x, t)\psi_{x_j} + E_\xi(y, t)\psi_{y_i})I_\eta(u - v) + A_\eta(u, x, t, v)\Delta \psi + A_\eta(v, y, t, u)\Delta \psi \]
\[+ \int_u^u a_{x_i}(s, x, t)S_\eta(s - v)ds \psi_{x_i} + \int_u^v a_{y_i}(s, x, t)S_\eta(s - u)ds \psi_{y_i} \]
\[-S_\eta'(u - v)(| \nabla \int_0^u \sqrt{a(s, x, t)} ds |^2 + | \nabla \int_0^v \sqrt{a(s, y, t)} ds |^2 \psi \]
\[-(E_{iX_i} - E_{iY_i}) \int_{0}^{u} sS'_\theta(s - v)ds \psi \| dxdt \| dyd\tau \geq 0. \hspace{1cm} (52)\]

Let \(\psi(x, t, y, \tau) = \phi(x, t)j_h(x - y, t - \tau)\). Where \(\phi(x, t) \geq 0, \phi(x, t) \in C_0^\infty(Q_T)\), and

\[j_h(x - y, t - \tau) = \omega_h(t - \tau)\prod_{i=1}^{N} \omega_h(x_i - y_i), \quad \omega_h(s) = \frac{1}{h} \omega(s)h\]

\(\omega(s) \in C_0^\infty(R), \omega(s) \geq 0, \omega(s) = 0 \text{ if } |s| > 1, \int_{-\infty}^{\infty} \omega(s)ds = 1.\)

it is clear of that

\[\frac{\partial j_h}{\partial \tau} + \frac{\partial j_h}{\partial \tau} = 0, \quad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = 0. \hspace{1cm} (53)\]

\[\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial t} = \frac{\partial \phi}{\partial t}j_h, \quad \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i}j_h. \hspace{1cm} (54)\]

Since \(E_i \in L^2(Q_T)\) and \(\psi \in C_0^\infty(Q_T \times Q_T)\), by the control convergent theorem, we have

\[\lim_{n \rightarrow 0} \int_{Q_T} \int_{Q_T} [E_i(x, t)\psi_{x_i} + E_i(y, \tau)\psi_{y_i}] I_n(u - v)dxdt \| dy \| d\tau = \int_{Q_T} E_i(x, t)[u - v]dxdt \| dy \| d\tau. \hspace{1cm} (55)\]

Let \(h \rightarrow 0\) in the above equality. We have

\[\lim_{h \rightarrow 0} \int_{Q_T} \int_{Q_T} [E_i(x, t)\psi_{x_i} + E_i(y, \tau)\psi_{y_i}][u - v]dxdt \| dy \| d\tau = \int_{Q_T} E_i(x, t)[u - v]\phi_{x_i}dxdt. \hspace{1cm} (56)\]

At the same time, by the fact of that \(\lim_{n \rightarrow 0} sS'_\theta(s) = 0\), and by the test function \(\psi = \phi(x, t)j_h(x - y, t - \tau)\),

\[\lim_{h \rightarrow 0} \lim_{n \rightarrow 0} \int_{Q_T} \int_{Q_T} \left[ (E_iX_i - E_iY_i) \int_{0}^{u} sS'_\theta(s - v)ds \psi \| dxdt \| dy \| d\tau \right]. \hspace{1cm} (57)\]

For the other terms in (52), i.e.

\[\int_{Q_T} \int_{Q_T} [A_h(u, x, t, v)\Delta_x \psi + A_h(v, y, \tau, u)\Delta_y \psi + \int_{0}^{u} a_{X_i}(s, x, t)S_h(s - k)ds \psi_{x_i} + \int_{0}^{v} a_{Y_i}(s, y, \tau)S_h(s - k)ds \psi_{y_i}]

\[-S'_\theta(u - v)(| \nabla \int_{0}^{u} \sqrt{a(s, x, t)}ds |^2 + | \nabla \int_{0}^{v} \sqrt{a(s, y, \tau)}ds |^2)\psi \| dxdt \| dy \| d\tau. \hspace{1cm} (58)\]
we can deal with it as [23], and letting $\eta \to 0, h \to 0$ in (52), we can get
\[
\int_Q \left[ u(x,t) - v(x,t)\phi_t - |u - v|E_i\phi_{x_i} + \text{sgn}(u - v)(A(u,x,t) - A(v,x,t))\Delta \phi \right. \\
+ \left. \int_{\nu} a_x(s,x,t)\text{sgn}(s-v)ds\phi_{x_i} + \int_{\nu} a_x(s,x,t)\text{sgn}(s-u)ds\phi_{x_i} \right] \geq 0.
\] (59)

Let
\[
\eta(t) = \int_{\tau-t}^{\tau-t} \alpha(\sigma)d\sigma, \quad \epsilon < \min\{\tau, T-s\},
\]
where $\alpha(\epsilon)$ is the kernel of mollifier with $\alpha(\epsilon) = 0$ for $t \notin (-\epsilon, \epsilon)$.

By approximation, we can replace $\phi$ in (59) by $\phi(x,t) = \omega(x)\eta(t)$, where $\omega(x)$ is the function of (13), $\eta(t) \in C^1(0, T)$. Using the estimates
\[
|\nabla \omega | \leq C_\omega \omega(x), \quad |\Delta \omega | \leq C_\omega \omega(x),
\]
by the assumption of that (11), we obtain from (59)
\[
\int_{\mathbb{R}^N} |u(x,t) - v(x,t)| \omega(x)dx \\
\leq \int_{\mathbb{R}^N} |u(x, \tau) - v(x, \tau)| \omega(x)dx + c \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x, t) - v(x, t)| \omega(x)dxdt
\]
By Gronwall Lemma
\[
\int_{\mathbb{R}^N} |u(x, \tau) - v(x, \tau)| \omega(x)dx \leq c \int_{\mathbb{R}^N} |u(x, \tau) - v(x, \tau)| \omega(x)dx.
\]
Letting $\tau \to 0$, the proof of Theorem 2.3 is complete.

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