DIFFUSION PHENOMENA FOR THE WAVE EQUATION WITH SPACE-DEPENDENT DAMPING IN AN EXTERIOR DOMAIN

MOTOTHIRO SOBAJIMA AND YUTA WAKASUGI

Abstract. In this paper, we consider the asymptotic behavior of solutions to the wave equation with space-dependent damping in an exterior domain. We prove that when the damping is effective, the solution is approximated by that of the corresponding heat equation as time tends to infinity. Our proof is based on semigroup estimates for the corresponding heat equation and weighted energy estimates for the damped wave equation. The optimality of the decay rate for solutions is also established.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an exterior domain with smooth boundary. We consider the initial-boundary value problem to the wave equation with space-dependent damping

$$
\begin{align*}
\begin{cases}
u_{tt} - \Delta \nu + a(x)\nu_t & = 0, \\
u(x, t) & = 0, \\
(u, u_t)(x, 0) & = (u_0, u_1)(x),
\end{cases} 
\end{align*}
$$

(1.1)

Here $u = u(x, t)$ is a real-valued unknown function. The coefficient of the damping term $a(x)$ is a radially symmetric function on the whole space $\mathbb{R}^N$ satisfying $a \in C^2(\mathbb{R}^N)$ and

$$a(x) = a_0|x|^{-\alpha} + o(|x|^{-\alpha}) \quad \text{as} \quad |x| \to \infty$$

with some $a_0 > 0$ and $\alpha \in [0, 1)$, and then we may assume $0 \notin \overline{\Omega}$ without loss of generality. The initial data $(u_0, u_1)$ belong to $[H^3(\Omega) \cap H^1_0(\Omega)] \times [H^2(\Omega) \cap H^1_0(\Omega)]$ with the compatibility condition of second order and satisfy $\text{supp} (u_0, u_1) \subset \{x \in \Omega; \, |x| < R_0\}$ with some $R_0 > 0$. Here we recall that for a nonnegative integer $k$, with the assumption $a \in C^{\max(k, 2)}$, the initial data $(u_0, u_1) \in H^{k+1} \times H^k$ satisfy the compatibility condition of order $k$ if $u_p = 0$ on $\partial \Omega$ for $p = 0, 1, \ldots, k$, where $u_p$ are successively defined by $u_p = \Delta u_{p-2} - a(x)u_{p-1}$ $(p = 2, \ldots, k)$. Then, it is known that (1.1) admits a unique solution

$$u \in \bigcap_{i=0}^{k+1} C^i([0, \infty); H^{k+1-i}(\Omega))$$

(see Ikawa [I] Theorem 2).
Also, we consider the initial-boundary value problem to the corresponding heat equation

\[
\begin{cases}
  v_t - a(x)^{-1}\Delta v = 0, & x \in \Omega, t > 0, \\
  v(x, t) = 0, & x \in \partial\Omega, t > 0, \\
  v(x, 0) = v_0(x), & x \in \Omega.
\end{cases}
\]

(1.2)

Our aim is to prove that the asymptotic profile of the solution to (1.1) is given by a solution of (1.2) as time tends to infinity. Namely, the solution of the damped wave equation (1.1) has the diffusion phenomena.

The diffusive structure of the damped wave equation has been studied for a long time. Matsumura [8] proved \(L^p-L^q\) estimates of solutions in the case \(\Omega = \mathbb{R}^N\) and \(a(x) \equiv 1\). On the other hand, Mochizuki [13] considered the case where \(\Omega = \mathbb{R}^N\) and the coefficient \(a = a(x, t)\) satisfies \(0 \leq a(x, t) \leq C(1 + |x|)^{-\alpha}\) with \(\alpha > 1\), and proved that in general the energy of the solution does not decay to zero. Moreover, for some initial data, the solution approaches to a solution to the wave equation without damping in the energy sense. Mochizuki and Nakazawa [14] generalized it to the case of exterior domains with star-shaped complement. Matsuyama [10] further extended it to more general domains by adding the assumption of the positivity of \(a(x, t)\) around \(\partial\Omega\).

On the other hand, when the coefficient \(a\) satisfies \(a(x) \geq C(1 + |x|)^{-\alpha}\) with some \(\alpha \in (0, 1]\), Matsumura [9] and Uesaka [23] showed that the energy of the solution decays to zero. When \(\Omega = \mathbb{R}^N\) and the coefficient \(a(x)\) is radially symmetric and satisfies \((\alpha)\), Todorova and Yordanov [22] introduced a suitable weight function of the form \(t^{-\alpha}e^\varepsilon\), which originates from [21] and [3], and proved an almost optimal energy estimate

\[
\int_{\mathbb{R}^N} (|u_t|^2 + |\nabla u|^2) dx \leq C(1 + t)^{-\frac{N-\alpha}{2N-\alpha} - \frac{1}{2} + \varepsilon} \|(u_0, u_1)\|_{H^1 \times L^2}^2.
\]

After that, Radu, Todorova and Yordanov [18] extended it to estimates for higher order derivatives. Nishihara [15] also established a weighted energy method similar to [22] and obtained decay estimates of the solution to the nonlinear problem

\[
u_{tt} - \Delta u + (1 + |x|^2)^{-\alpha/2}u_t + |u|^{p-1}u = 0.
\]

Based on the energy method of [15], the second author [24] proved the same type estimates as those obtained in [18] and applied them to the diffusion phenomena for the damped wave equation (1.1) when \(\Omega = \mathbb{R}^N\) and \(a(x) = (1 + |x|^2)^{-\alpha/2}\) with \(0 \leq \alpha < 1\).

Ikehata, Todorova and Yordanov [11] considered the damping satisfying \(a(x) \geq \mu (1 + |x|^2)^{-1/2}\), which corresponds to the case \(\alpha = 1\). They proved that the energy of the solution decays as \(O(t^{-\mu})\) if \(1 < \mu < N\) and \(O(t^{-N+\varepsilon})\) with arbitrary \(\varepsilon > 0\) if \(\mu \geq N\), respectively.

Recently, Nishiyama [16] studied the abstract damped wave equation \(u_{tt} + Au + Bu_t = 0\) and proved the diffusion phenomena by using the resolvent argument when the damping term \(B\) is strictly positive. Radu, Todorova and Yordanov [19] considered the equation \(Cu_{tt} + Au + u_t = 0\) and obtained a similar result via the method of diffusion approximation.

In this paper, we prove the diffusion phenomena for the damped wave equation (1.1) in an exterior domain and for more general damping term than [24]. Moreover, we prove the optimality of the decay rate for the solution of the corresponding
parabolic problem (1.2) in a special case, which implies that the asymptotic profile of the solution of the damped wave equation (1.1) is actually given by a solution of the corresponding heat equation (1.2) (see Proposition 2.7). Our main result is the following:

**Theorem 1.1.** Let $u$ be a solution of (1.1) with initial data $(u_0, u_1)$ and let $v$ be a solution of (1.2) with $v_0 = u_0 + a(x)^{-1}u_1$. Then, for any $\varepsilon > 0$, there exists $C > 0$ such that we have

$$\|\sqrt{a(\cdot)}(u(\cdot, t) - v(\cdot, t))\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{N}{2(N-1)}} - \frac{N}{2(N-1)} + \varepsilon \|(u_0, u_1)\|_{H^2 \times H^1(\Omega)}$$

for any $t \geq 1$.

The proof of the above theorem consists of following three parts.

Firstly, in the next section, we investigate the heat semigroup $e^{tL_*}$ generated by the Friedrichs extension $L_*$ of $L = a(x)^{-1}\Delta$. In particular, we prove the semigroup estimate

$$\|\sqrt{a(\cdot)e^{tL_*}} f\|_{L^2} \leq Ct^{-\frac{N}{2(N-1)}} \|a(\cdot)f\|_{L^1}$$

by using Beurling-Deny criteria (see e.g., Ouhabaz [17, Section 2]) and weighted Gagliardo-Nirenberg inequalities. A similar argument can be found in Liskevich and Sobol [6] ($L^p$-analysis of $L$ is studied in Metafune and Spina [11] and Metalfune, Okazawa, Spina and the first author [12].) Moreover, under some additional assumptions, we also have the optimality of the above estimate.

Secondly, in Section 3, we prove the almost sharp higher-order energy estimate of solutions to the damped wave equation (1.1) by using the Todorova-Yordanov-type weight function

$$\Phi_{A,\beta}(x, t) = \exp \left( \beta \frac{A(x)}{1 + t} \right).$$

This has also been proved by Radu, Todorova and Yordanov [18] when $\Omega = \mathbb{R}^N$. In this paper, we give an alternate proof based on the argument of [15] and a Hardy-type inequality.

In the final step, in Section 4, we rewrite the difference of solutions to (1.1) and (1.2) as

$$u(t) - v(t) = -\int_{t/2}^t e^{(t-s)L_*}[a(\cdot)^{-1}u_{ss}(s)] ds$$

$$- e^{\frac{t}{4}L_*}[a(\cdot)^{-1}u_t(t/2)]$$

$$- \int_0^{t/2} \frac{\partial}{\partial s} \left( e^{(t-s)L_*}[a(\cdot)^{-1}u_s(s)] \right) ds.$$

Applying the heat semigroup estimate for $e^{tL_*}$ and the energy estimate for the time-derivatives of $u$, we prove that the each term of the right-hand side decays faster.

For the end of this section, we introduce the notation used throughout this paper. The letter $C$ indicates the generic constant, which may change from line to line. We denote the set of all compactly supported smooth functions in $U \subset \mathbb{R}^N$ as $C_c^\infty(U)$ and the $L^p$ norm by $\| \cdot \|_{L^p}$, that is,

$$\|f\|_{L^p} = \begin{cases} \left( \int_U |f(x)|^p dx \right)^{1/p} & (1 \leq p < \infty), \\ \text{ess sup} |f(x)| & (p = \infty). \end{cases}$$
For a nonnegative integer $k$, $\| \cdot \|_{H^k}$ denotes the Sobolev norm, that is,
\[ \| f \|_{H^k} = \left( \sum_{|\alpha| \leq k} \| \partial^\alpha f \|_{L^2}^2 \right)^{\frac{1}{2}}. \]

For an interval $I$ and a Banach space $X$, we define $C^r(I; X)$ as the space of $r$-times continuously differentiable mapping from $I$ to $X$ with respect to the topology in $X$.

### 2. The Semigroup Generated by $a(x)^{-1}\Delta$

In this section, we study the semigroup generated by the operator
\[ L = a(x)^{-1}\Delta \quad \text{in } \Omega \]
endowed with the Dirichlet boundary condition. Since the coefficient $a$ is positive and satisfies (a0), we may assume that there exists $c_0 \in (0, 1)$ such that
\[ c_0|x|^{-\alpha} \leq a(x) \leq c_0^{-1}|x|^{-\alpha}, \quad x \in \overline{\Omega}. \]

We remark that the results of this section require only (2.1) and we do not need that $a$ is radially symmetric and satisfies (a0). We introduce the weighted $L^p$-spaces
\[ L^p_{d\mu} := \left\{ f \in L^p_{\text{loc}}(\Omega) : \| f \|_{L^p_{d\mu}} := \int_{\Omega} |f(x)|^p a(x) \, dx < \infty \right\}, \quad 1 \leq p < \infty \]
and the bilinear form
\[ a(u, v) := \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx, \]
\[ D(a) := \left\{ u \in C^\infty_c(\overline{\Omega}) : u(x) = 0 \quad \forall x \in \partial \Omega \right\}. \]
in a Hilbert space $L^2_{d\mu}$. Then the form $a$ is closable, and therefore, we denote $a_*$ as a closure of $a$. Then we can see that

**Lemma 2.1.** The bilinear form $a_*$ can be characterized as follows:

(2.2) $D(a_*) = \left\{ u \in L^2_{d\mu} \cap H^1(\Omega) : \int_{\Omega} \frac{\partial u}{\partial x_j} \cdot \varphi \, dx = - \int_{\Omega} u \frac{\partial \varphi}{\partial x_j} \, dx \quad \forall \varphi \in C^\infty_c(\mathbb{R}^N) \right\}$.

(2.3) $a_*(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx.$

**Proof.** We denote $D$ as the function space on the right-hand side of (2.2). It is clear that $D(a) \subset D$. Since $D$ is closed with respect to the norm
\[ \| u \|_D := \left( \| \nabla u \|_{L^2(\Omega)}^2 + \| u \|_{L^2_{d\mu}}^2 \right)^{\frac{1}{2}}, \]
we deduce $D(a_*) \subset D$.

Conversely, let $u \in D$. To prove $u \in D(a_*)$, it suffices to find the sequence $\{u_n\}_n \subset D(a)$ such that
\[ u_n \to u \quad \text{in } \quad L^2_{d\mu} \quad \text{as } n \to \infty, \quad a(u_n - u_m, u_n - u_m) \to 0 \quad \text{as } n, m \to \infty. \]

Set a function $\zeta \in C^\infty(\mathbb{R}; [0, 1])$ as
\[ \zeta(s) = \begin{cases} 1 & \text{if } s < 0, \\ 0 & \text{if } s > 1 \end{cases} \]
and define a family of cut-off functions \( \{ \zeta_n \}_n \subset C_c^\infty(\mathbb{R}^N; [0, 1]) \) as

\[
\zeta_n(x) = \zeta(\log(|x|) - \log R - n).
\]

Then we take \( u_n(x) := \zeta_n(x)u(x) \) for \( n \in \mathbb{N} \). Observe that \( u_n \in H^1_0(\Omega) \), \( u_n \to u \) in \( L^2_{d\mu} \). Moreover, for \( n, m \in \mathbb{N} \) with \( n < m \),

\[
\int_\Omega |\nabla(u_n - u_m)|^2 \, dx = \int_\Omega |\nabla((\zeta_m - \zeta_n)u)|^2 \, dx
\]

\[
\leq 2 \int_\Omega |\zeta_m - \zeta_n|^2 |\nabla u|^2 \, dx + 2 \int_\Omega |\nabla(\zeta_m - \zeta_n)|^2 |u|^2 \, dx.
\]

Noting that \( a \) satisfies (2.1) and then

\[
|\nabla \zeta_n| = \left| \frac{x}{|x|^2} c'(\log(|x|) - \log R - n) \right| \leq \chi_{\{ |x| > R^n \}} |x|^{-1} \leq C \chi_{\{ |x| > R^n \}} a(x)^{\frac{1}{2}},
\]

for some constant \( C > 0 \) (\( \chi_U \) denotes the indicator function of \( U \)), we obtain

\[
\int_\Omega |\nabla(u_n - u_m)|^2 \, dx \to 0
\]

as \( n, m \to \infty \). Since for every \( n \in \mathbb{N} \), \( u_n \) is compactly supported in \( \overline{\Omega} \), \( u_n \) can be also approximated by functions in \( C_c^\infty(\overline{\Omega}) \) in the sense of \( H^1_0(\Omega) \)-topology. This means that \( u \in D(a_s) \). Since (2.1) can be verified by the above approximation, the proof is completed.

\[\square\]

**Lemma 2.2** (The Friedrichs extension). The operator \(-L_\ast \) in \( L^2_{d\mu} \) defined by

\[
D(L_\ast) := \left\{ u \in D(a_s) : \exists f \in L^2_{d\mu} \text{ s.t. } a_s(u, v) = (f, v)_{L^2_{d\mu}} \quad \forall v \in D(a_s) \right\},
\]

\[
-L_\ast u := f
\]

is nonnegative and selfadjoint in \( L^2_{d\mu} \). Therefore \( L_\ast \) generates an analytic semigroup \( e^{tL_\ast} \) on \( L^2_{d\mu} \) and satisfies

\[
\|e^{tL_\ast} f\|_{L^2_{d\mu}} \leq \|f\|_{L^2_{d\mu}}, \quad \|L_\ast e^{tL_\ast} f\|_{L^2_{d\mu}} \leq \frac{1}{t} \|f\|_{L^2_{d\mu}}, \quad \forall f \in L^2_{d\mu}.
\]

Furthermore, \( L_\ast \) is an extension of \( L \) defined on \( C_c^\infty(\overline{\Omega}) \) with Dirichlet boundary condition.

**Proof.** By [20] Theorem X.23 we see that \(-L_\ast \) is nonnegative and selfadjoint in \( L^2_{d\mu} \). Moreover, for every \( u, v \in D(a) \),

\[
-Lu, v\rangle_{L^2_{d\mu}} = \int_\Omega (-Lu)v \, d\mu = \int_\Omega (-\Delta u)v \, dx = a(u, v) = a_s(u, v).
\]

Therefore \( u \in D(L_\ast) \) and \( L_\ast u = Lu \). \[\square\]

**Lemma 2.3.** We have

\[
\{ u \in H^2(\Omega) \cap H^1_0(\Omega) : a(x)^{-\frac{1}{2}} \Delta u \in L^2(\Omega) \} \subset D(L_\ast)
\]

and its inclusion is continuous.
Lemma 2.4. \( \square \) This inequality gives the continuity of the inclusion in the assertion.

It suffices to show respective inequalities for \( \varphi \in D(\alpha) \).

Proof. Applying [17, Theorems 2.7 and 2.13], we respectively obtain the positivity and \( -\text{contractivity of } e^{tL^\ast} \).

Therefore \( u \in D(L_\ast) \) and \( L_\ast u = a(x)^{-\frac{1}{2}}\Delta u \). Moreover, we have

\[
\|u\|_{D(L_\ast)}^2 = \|u\|_{L^2}^2 + \|L_\ast u\|_{L^2}^2 = \|a(\cdot)^{\frac{1}{2}}u\|_{L^2}^2 + \|a(\cdot)^{-\frac{1}{2}}\Delta u\|_{L^2}^2 \\
\leq \|a\|_{L^\infty} \|u\|_{L^2}^2 + \|a(\cdot)^{-\frac{1}{2}}\Delta u\|_{L^2}^2 \\
\leq \|a\|_{L^\infty} \|u\|_{H^2}^2 + \|a(\cdot)^{-\frac{1}{2}}\Delta u\|_{L^2}^2.
\]

This inequality gives the continuity of the inclusion in the assertion. \( \square \)

Lemma 2.4. The semigroup \( \{e^{tL^\ast}\}_{t \geq 0} \) given by Lemma 2.2 is sub-martkovian, that is, \( e^{tL^\ast} \) is positively preserving:

\[ 0 \leq f \in L^2_{d\mu} \implies e^{tL^\ast} f \geq 0 \]

and \( L^\infty \)-contractive:

\[ \|e^{tL^\ast} f\|_{L^\infty} \leq \|f\|_{L^\infty}, \quad \forall f \in L^2_{d\mu} \cap L^\infty(\Omega). \]

Proof. Let \( u \in D(\alpha_u) \) be fixed. We note that Lemma 2.2 implies \( |u|, Pu := (1 \wedge |u|)\text{sign }u \in D(\alpha_u) \). Moreover, by direct computation, we have

\[ a_u(|u|, |u|) = \int_\Omega \nabla |u|^2 dx = \int_\Omega |\nabla u|^2 dx = a_u(u, u) \]

and

\[ a_u(Pu, u - Pu) = \int_\Omega |\nabla u|^2 \chi_{\{|u|<1\}}(1 - \chi_{\{|u|<1\}}) dx = 0. \]

Applying [17] Theorems 2.7 and 2.13, we respectively obtain the positivity and \( L^\infty \)-contractivity of \( e^{tL^\ast} \). \( \square \)

Lemma 2.5 (The embedding \( D(\alpha_u) \hookrightarrow L^{q_u}_{d\mu} \)). If \( N \geq 3 \), then there exists \( C_{N,\alpha} > 0 \) such that

\[ \|u\|_{L^{q_u}_{d\mu}} \leq C_{N,\alpha} a_u(u, u)^{\frac{1}{2}}, \quad \forall u \in D(\alpha_u), \]

where \( q_u := \frac{2(N-\alpha)}{N-2} > 2 \). If \( N = 2 \), then for every \( 2 < q < \infty \), there exists \( C_{2,\alpha,q} > 0 \) such that

\[ \|u\|_{L^{q_u}_{d\mu}} \leq C_{2,\alpha,q} a_u(u, u)^{\frac{1}{2} - \frac{1}{q}} \|u\|_{L^2_{d\mu}}^{\frac{1}{2}}, \quad \forall u \in D(\alpha_u). \]

Proof. It suffices to show respective inequalities for \( u \in D(\alpha) \). First we prove the case \( N \geq 3 \). Let \( u \in D(\alpha) \). Then noting that

\[ q_u = 2^\ast \left(1 - \frac{\alpha}{2}\right) + \alpha, \]
we see from the Hölder inequality that
\[ \|u\|_{L^{\infty}_{\alpha}}^{q} \leq c_0^{-1} \int_{\Omega} |u(x)|^{q} |x|^{-\alpha} \, dx = c_0^{-1} \int_{\Omega} \left( |u(x)|^{2} \right)^{1-\frac{q}{2}} \left( |u(x)|^{2} |x|^{-2} \right)^{\frac{q}{2}} \, dx \]
\[ \leq c_0^{-1} \left( \int_{\Omega} |u(x)|^{2} \, dx \right)^{1-\frac{q}{2}} \left( \int_{\Omega} |u(x)|^{2} |x|^{-2} \, dx \right)^{\frac{q}{2}}. \]

Using Gagliardo-Nirenberg and Hardy inequalities, we have
\[ \|u\|_{L^{q}_{\alpha}}^{q} \leq c_0^{-1} C_{\infty}^{2\alpha} \|\nabla u\|_{L^{2}} \|\nabla u\|_{L^{2}}^{\alpha} = C \|\nabla u\|_{L^{2}}. \]

If \( N = 2 \), then set \( \Phi(x) = |x|^{-\frac{\alpha}{2}}x \) for \( x \in \Omega \). Then we see by change of variables that
\[ \|u\|_{L^{q}_{\alpha}}^{q} \leq c_0^{-1} \int_{\Omega} |u(x)|^{q} |x|^{-\alpha} \, dx = \frac{2}{(2-\alpha)c_0} \int_{\Phi(\Omega)} \left| u(\Phi^{-1}(y)) \right|^{q} \, dy. \]

Gagliardo-Nirenberg inequality for \( v := u \circ \Phi^{-1} \) implies
\[ \|v\|_{L^{q}(\Phi(\Omega))} \leq C \|\nabla v\|_{L^{2}(\Phi(\Omega))}^{1-\frac{\alpha}{2}} \|u\|_{L^{q}(\Phi(\Omega))}^{\frac{\alpha}{2}}. \]

Noting that \( \Phi^{-1}(y) = |y|^{\frac{\alpha}{2-\alpha}} y \), we have
\[ \int_{\Phi(\Omega)} |\nabla v(y)|^{2} \, dy = \int_{\Phi(\Omega)} |D\Phi(y)\nabla u(\Phi^{-1}(y))|^{2} \, dy \]
\[ \leq \left( \frac{2}{2-\alpha} \right)^{2} \int_{\Phi(\Omega)} |\nabla u(\Phi^{-1}(y))|^{2} |y|^{\frac{2\alpha}{2-\alpha}} \, dy \]
\[ = \left( \frac{2}{2-\alpha} \right)^{2} \int_{\Phi(\Omega)} |\nabla u(\Phi^{-1}(y))|^{2} |\Phi^{-1}(y)|^{\alpha} \, dy. \]

Using change of variables again yields
\[ \|\nabla v\|_{L^{2}(\Phi(\Omega))}^{2} \leq \frac{2}{2-\alpha} \int_{\Omega} |\nabla u(x)|^{2} \, dx. \]

Combining the inequalities above, we obtain the desired inequality. \( \square \)

**Proposition 2.6.** Let \( e^{tL_f} \) be given in Lemma 2.2. For every \( f \in L^{2}_{d_{\mu}} \), we have
\[ \|e^{tL_f}\|_{L^{\infty}} \leq C t^{-\frac{N-\alpha}{2\alpha-\alpha}} \|f\|_{L^{2}_{d_{\mu}}}. \] (2.4)

Moreover, for every \( f \in L^{1}_{d_{\mu}} \cap L^{2}_{d_{\mu}} \), we have
\[ \|e^{tL_f}\|_{L^{2}_{d_{\mu}}} \leq C t^{-\frac{N-\alpha}{2\alpha-\alpha}} \|f\|_{L^{1}_{d_{\mu}}} \] (2.5)
and
\[ \|L_{*}e^{tL_f}\|_{L^{2}_{d_{\mu}}} \leq C t^{-\frac{N-\alpha}{2\alpha-\alpha}} \|f\|_{L^{1}_{d_{\mu}}}. \] (2.6)

**Proof.** If \( N \geq 3 \), then by Lemma 2.2 we have for every \( t > 0 \) and \( f \in L^{2}_{d_{\mu}} \),
\[ \|e^{tL_f}\|_{L^{2}_{d_{\mu}}} \leq C_{N,\alpha} \|e^{tL_f} f, e^{tL_f} f, e^{tL_f} f\|_{L^{2}_{d_{\mu}}}^{\frac{1}{2}} \]
\[ = C_{N,\alpha} (-L_{*}e^{tL_f} f, e^{tL_f} f, e^{tL_f} f)_{L^{2}_{d_{\mu}}}^{\frac{1}{2}} \]
\[ \leq C_{N,\alpha} \|L_{*}e^{tL_f} f\|_{L^{2}_{d_{\mu}}} \|e^{tL_f} f\|_{L^{2}_{d_{\mu}}}^{\frac{1}{2}} \]
\[ \leq C_{N,\alpha} t^{-\frac{1}{2}} \|f\|_{L^{2}_{d_{\mu}}}. \]
Therefore noting Lemmas 2.4 and applying Lemma 6.1] with \((\alpha, r) = (\frac{1}{2}, q_*)\), we obtain (2.5). Moreover, for every \(f \in \L^1_{d\mu} \cap \L^2_{d\mu}\) and \(g \in \L^2_{d\mu}\),

\[
\left\|e^{tL_*}f, g\right\|_{\L^2_{d\mu}} = \left|\int_\Omega f(e^{tL_*}g)d\mu\right| \leq \|f\|_{\L^1_{d\mu}} \|e^{tL_*}g\|_{\L^\infty} \leq C t^{-\frac{N-\alpha}{2}} \|f\|_{\L^1_{d\mu}} \|g\|_{\L^2_{d\mu}}.
\]

Therefore (2.5) is proved. The estimate (2.6) is easily proved by noting

\[
\left\|L_*e^{tL_*}f\right\|_{\L^1_{d\mu}} = \left\|L_*e^{\frac{t}{2}L_*} \left(e^{\frac{t}{2}L_*}f\right)\right\|_{\L^2_{d\mu}} \leq \frac{2}{t} \left\|e^{\frac{t}{2}L_*}f\right\|_{\L^2_{d\mu}}
\]

and using (2.6). We can also verify the case of \(N = 2\) by a similar computation. \(\square\)

Finally, we discuss about the optimality of the estimate (2.6).

**Proposition 2.7.** If \(N \geq 2\) and \(a(x) = |x|^{-\alpha}\), then there exist a function \(f \in C_c^\infty(\Omega)\) and constants \(\varepsilon > 0\) and \(t_0 > 0\) such that

\[
\left\|e^{tL_*}f\right\|_{\L^2_{d\mu}} \geq \varepsilon t^{-\frac{N-\alpha}{2}}, \quad t \geq t_0.
\]

In other words, the decay rate \(t^{-\frac{N-\alpha}{2}}\) of \(e^{tL_*}\) is optimal for initial data in \(C_c^\infty(\Omega)\).

**Proof.** First we consider the case \(N \geq 3\). Let \(R\) satisfy \(K \subset B(0, R/2)\). Define the function

\[
G(x, t) = t^{-\frac{N-\alpha}{2}} \left[1 - t^{\frac{N-2}{2}} |x|^{2-N}\right] \exp\left[-\frac{|x|^{2-\alpha}}{(2-\alpha)^2 t}\right], \quad x \in \Omega, t > 0
\]

and \(G_+(x, t) := (G(x, t))^+\). By [5] Lemma 5.5] we can see that

\[
\frac{\partial G}{\partial t} = |x|^{\alpha} \Delta G \quad \text{in} \quad \Omega = \mathbb{R}^N \setminus \{0\},
\]

and

\[
\begin{cases}
G(x, t) \geq 0 & \text{if } |x|^{2-\alpha} \geq t, \\
G(x, t) \leq 0 & \text{if } |x|^{2-\alpha} \leq t.
\end{cases}
\]

If \(g \in \L^2_{d\mu}\) satisfies \(g \geq G_+(x, tR)\) with \(t_R = R^{2-\alpha}\), then since \(e^{tL_*}\) is preserves positivity, we see that for every \(x \in B(0, R) \cap \Omega\),

\[
G(x, t + t_R) - e^{tL_*}g \leq G(x, t + t_R) \leq 0
\]

and then

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left\|\left(G(x, t + t_R) - e^{tL_*}g\right)_+\right\|_{\L^2_{d\mu}}^2 &= \int_\Omega \left(\frac{\partial G}{\partial t}(x, t + t_R) - \frac{d}{dt}e^{tL_*}g\right)G(x, t + t_R) - e^{tL_*}g a(x) \, dx \\
&= \int_{\mathbb{R}^N \setminus B(0, R)} \Delta \left(G(x, t + t_R) - e^{tL_*}g\right) \left(G(x, t + t_R) - e^{tL_*}g\right)_+ \, dx \\
&= -\int_{\mathbb{R}^N \setminus B(0, R)} \nabla \left(G(x, t + t_R) - e^{tL_*}g\right)_+ \, dx \\
&\leq 0.
\end{align*}
\]
Hence we have $e^{tL^*} g \geq G_+(x, t + t_R)$ and then
\[
\|e^{tL^*} g\|_{L^2_{\mu}}^2 \geq \int_{\Omega} G_+(x, t + t_R)^2 a(x) \, dx
\]
\[
= \int_{\mathbb{R}^N \setminus B(0, (t + t_R)^{-\frac{1}{2}})} G_+(x, t + t_R)^2 \, dx
\]
\[
= (t + t_R)^{-\frac{N-\alpha}{2}} \int_{\mathbb{R}^N \setminus B(0,1)} (1 - |y|^2 - 2\exp\left[-\frac{|y|^{2-\alpha}}{(2-\alpha)^2}\right] \, dy
\]
\[
= \tilde{C}(t + t_R)^{-\frac{N-\alpha}{2}}.
\]
To conclude the proof, we choose a cut-off function $\bar{\eta}(x) \in C_0^\infty(\mathbb{R}^N)$ such that
\[
\|(1 - \bar{\eta})g\|_{L^2_{\mu}} \leq 3^{-\frac{N-\alpha}{2\alpha}} C^{-1} \tilde{C},
\]
where $C$ is the constant given by $L^2_{\mu}$ estimate. Then
\[
\|e^{tL^*} (1 - \bar{\eta})g\|_{L^2_{\mu}} \leq Ct^{-\frac{N-\alpha}{2\alpha}} \|(1 - \bar{\eta})g\|_{L^2_{\mu}} \leq \tilde{C}(3t)^{-\frac{N-\alpha}{2\alpha}}
\]
and therefore for every $t \geq t_R$,
\[
\|e^{tL^*} \bar{\eta}g\|_{L^2_{\mu}} \geq \|e^{tL^*} g\|_{L^2_{\mu}} - \|e^{tL^*} (1 - \bar{\eta})g\|_{L^2_{\mu}}
\]
\[
\geq \tilde{C}(t + t_R)^{-\frac{N-\alpha}{2\alpha}} - \tilde{C}(3t)^{-\frac{N-\alpha}{2\alpha}}
\]
\[
\geq \tilde{C} \left(2^{-\frac{N-\alpha}{2\alpha}} - 3^{-\frac{N-\alpha}{2\alpha}} \right) \left(\frac{1}{t_R}\right)^{\frac{N-\alpha}{2\alpha}}.
\]
Since $\bar{\eta}g \in C^\infty(\Omega)$ is compactly supported, the proof is completed if $N \geq 3$.

If $N = 2$, then replacing $G$ with
\[
\tilde{G}(x, t) = t^{-1} \log \left(\frac{|x|^{2-\alpha}}{t}\right) \exp\left[-\frac{|x|^{2-\alpha}}{(2-\alpha)^2t}\right], \quad x \in \Omega, t > 0,
\]
we can verify the same conclusion. Hence the proof is completed. \qed

3. Weighted energy estimates for damped wave equation

In this section, we prove almost sharp estimates for time-derivatives of the solution to the damped wave equation \eqref{damped-wave}.

In this section, we assume the coefficient of the damping $a(x)$ is radially symmetric function on the whole space $\mathbb{R}^N$ and satisfies $a \in C^2(\mathbb{R}^N)$ and
\[
(a_0') \quad a_1(1 + |x|)^{-\alpha} \leq a(x) \leq a_2(1 + |x|)^{-\alpha}
\]
with some $a_1, a_2 > 0$ and $\alpha \in [0, 1)$. Then, by Todorova and Yordanov \cite[Proposition 1.3]{TY99}, there exists a solution to the Poisson equation
\[
\Delta A_0(x) = a(x) \quad \text{in } \mathbb{R}^N
\]
satisfying
\[
(a1) \quad A_1(1 + |x|)^{2-\alpha} \leq A_0(x) \leq A_2(1 + |x|)^{2-\alpha},
\]
\[
(a2) \quad h_a := \lim_{R \to \infty} \sup_{x \in \mathbb{R}^N \setminus B_R} \left(\frac{\left|\nabla A_0(x)^2\right|}{a(x)}\right) < \infty
\]
with some $A_1, A_2 > 0$. 

Lemma 3.1. Let $a$ and $A_0$ satisfy (H0), (H1), (a1) and (a2). Then for every $\varepsilon > 0$ there exists $c_0 \geq 0$ such that $A(x) = A_0(x) + c_0$ satisfies
\[ \frac{[\nabla A(x)]^2}{a(x)A(x)} \leq h_\alpha + \varepsilon \]
for any $x \in \Omega$.

Proof. For any $\varepsilon > 0$, there exists a constant $R > 0$ such that
\[ \frac{[\nabla A_0(x)]^2}{a(x)A_0(x)} \leq h_\alpha + \varepsilon \]
holds for any $|x| > R$. For $x \in \Omega \cap \{|x| \leq R\}$, taking a constant $c_0 > 0$ sufficient large, we have
\[ \frac{|\nabla (A_0(x) + c_0)|^2}{a(x)(A_0(x) + c_0)} \leq \frac{\|\nabla A_0\|_{L^\infty(\Omega \cap \{|x| \leq R\})}^2}{\inf_{x \in \Omega \cap \{|x| \leq R\}} a(x)} c_0^{-1} \leq h_\alpha + \varepsilon. \]
The proof is completed. \hfill \Box

Let us recall the finite speed propagation property of the wave equation (see [2]).

Lemma 3.2 (Finite speed of propagation). Let $u$ be the solution of (1.1) with the initial data $(u_0, u_1)$ satisfying supp $(u_0, u_1) \subset \{x \in \Omega; |x| \leq R_0\}$. Then, one has
\[ \text{supp } u(\cdot, t) \subset \{x \in \Omega; |x| \leq R_0 + t\} \]
and therefore $|x|/(t_0 + t) \leq 1$ for $t_0 \geq R_0, t \geq 0$ and $x \in \text{supp } u(\cdot, t)$.

We define the weight function $\Phi$. For a constant $\beta > 0$ and a function $A(x)$, we put
\[ \Phi_{A, \beta}(x, t) := \exp \left( \beta \frac{A(x)}{1 + t} \right). \]

The main result of this section is the following.

Proposition 3.3. Let $a(x) \in C^2(\mathbb{R}^N)$ be a radially symmetric function satisfying (H0) and let $A(x)$ be a solution to the Poisson equation (1.1) which satisfies (H1), (H2). Then, for any $\varepsilon > 0$, there exists a constant $\beta > 0$ such that the following holds: Let $k \geq 0$ be an integer and assume that $a \in C^{\max(k,2)}(\mathbb{R}^N)$ and the initial data $(u_0, u_1)$ belong to $[H^{k+1} \cap H_0^1(\Omega)] \times [H^k \times H_0^1(\Omega)]$ with the compatibility condition of order $k$ and the support condition supp $(u_0, u_1) \subset \{x \in \Omega; |x| < R_0\}$ for some $R_0 > 0$. Then, there exist constants $t_0 > 0$ and $C = C(N, k, R_0, \varepsilon) > 0$ such that the solution $u$ to (1.1) satisfies
\begin{align}
\int_{\Omega} \Phi_{A, \beta}(x, t) a(x) |\partial_t^k u(x, t)|^2 dx \\
&\leq C(t_0 + t)^{-1/2} \|u_0 - 2^{k+\varepsilon} \|(u_0, u_1)\|^2_{H^{k+1} \times H^k}, \\
\int_{\Omega} \Phi_{A, \beta}(x, t) |\partial_t^k \nabla_x u(x, t)|^2 dx \\
&\leq C(t_0 + t)^{-1/2 \varepsilon} \|(u_0, u_1)\|^2_{H^{k+1} \times H^k}
\end{align}
for $t \geq 0$. 
In particular, as stated in Introduction, if $a(x)$ is a radially symmetric function on the whole space $\mathbb{R}^N$ and satisfies (a0) with some $\alpha \in [0, 1)$, then there exists a solution $A(x)$ to the Poisson equation (3.1) satisfying

$$A(x) = \frac{a_0}{(2-\alpha)(N-\alpha)} |x|^{2-\alpha} + o(|x|^{2-\alpha}),$$

(a3)

$$h_\alpha = \frac{2-\alpha}{N-\alpha}.$$  

This was also proved in [22, Proposition 1.3]. Therefore, from the above proposition, we obtain the following energy estimates for time-derivatives of the solution to (1.1).

**Corollary 3.4.** Let $k \geq 0$ be an integer. In addition to the assumption in Proposition 3.3, we assume that $a(x)$ satisfies (a0). Then for any $\varepsilon > 0$, there exist constants $t_0 > 0$ and $C = C(N, k, R, \varepsilon) > 0$ such that the solution $u$ to (1.1) satisfies

$$\int_\Omega \Phi_{A,\beta}(x, t) |\partial_t^k u(x, t)|^2 dx \leq C(t_0 + t) \frac{a_0}{(1+t)^{N-\alpha}} \|u_0, u_1\|_{H^{k+1} \times H^k}^2,$$

for $t \geq 0$.

In order to prove Proposition 3.3, we use a weighted energy method which originates from Todorova and Yordanov [21], [22] and Ikehata [3]. Our proof is based on the argument of [15] and a Hardy-type inequality Lemma 3.6.

**Lemma 3.5.** We have

$$\partial_t \Phi_{A,\beta} = -\beta A(x) \frac{A(x)}{1+t} \Phi_{A,\beta},$$

$$\nabla \Phi_{A,\beta} = \beta \frac{\nabla A}{1+t} \Phi_{A,\beta},$$

$$\Delta \Phi_{A,\beta} = \beta \frac{\Delta A}{1+t} \Phi_{A,\beta} + \beta \frac{\nabla A}{1+t} \Phi_{A,\beta}.$$  

In particular, we have

$$-\Delta \Phi_{A,\beta} + \frac{|\nabla \Phi_{A,\beta}|^2}{\Phi_{A,\beta}} = -\beta \frac{a(x)}{1+t} \Phi_{A,\beta},$$

$$\frac{|\nabla \Phi_{A,\beta}|^2}{\partial_t \Phi_{A,\beta}} = -\beta \frac{|\nabla A|^2}{A} \Phi_{A,\beta}.$$  

**Proof.** It suffices to observe that

$$\partial_t \Phi_{A,\beta}(x, t) = \exp \left[ \beta \frac{A(x)}{1+t} \right] \left( -\beta \frac{A(x)}{(1+t)^2} \right) = -\beta \frac{A(x)}{(1+t)^2} \Phi_{A,\beta}(x, t),$$

$$\nabla \Phi_{A,\beta}(x, t) = \exp \left[ \beta \frac{A(x)}{1+t} \right] \left( \beta \frac{\nabla A}{1+t} \right) = \beta \frac{\nabla A}{1+t} \Phi_{A,\beta}(x, t),$$

$$\Delta \Phi_{A,\beta}(x, t) = \beta \frac{\Delta A}{1+t} \Phi_{A,\beta}(x, t) + \beta \frac{\nabla A}{1+t} \Phi_{A,\beta}(x, t).$$

Combining the equalities above, we deduce the assertion.
Next, we prove a Hardy-type inequality with the weight function $\Phi_{A,\beta}$, which is essential in our weighted energy method.

**Lemma 3.6.** We have

\begin{equation}
\frac{\beta}{1+t} \int_{\Omega} a(x)|u|^2 \Phi_{A,\beta} \, dx \leq \int_{\Omega} |\nabla u|^2 \Phi_{A,\beta} \, dx.
\end{equation}

**Proof.** Integration by parts gives

\begin{align*}
0 & \leq \int_{\Omega} \Phi_{A,\beta}^{-1} |\nabla (\Phi_{A,\beta} u)|^2 \, dx \\
& = \int_{\Omega} (\Phi_{A,\beta} |\nabla u|^2 + 2(\nabla \Phi_{A,\beta} \cdot \nabla u) u + \frac{|\nabla \Phi_{A,\beta}|^2}{\Phi_{A,\beta}} |u|^2) \, dx \\
& = \int_{\Omega} \Phi_{A,\beta} |\nabla u|^2 \, dx + \int_{\Omega} \left( -\Delta \Phi_{A,\beta} + \frac{|\nabla \Phi_{A,\beta}|^2}{\Phi_{A,\beta}} \right) |u|^2 \, dx.
\end{align*}

By Lemma 3.5 we have

\begin{align*}
-\Delta \Phi_{A,\beta} + \frac{|\nabla \Phi_{A,\beta}|^2}{\Phi_{A,\beta}} &= -\beta \frac{\Delta A(x)}{1+t} \Phi_{A,\beta} = -\beta \frac{a(x)}{1+t} \Phi_{A,\beta}.
\end{align*}

Thus, we obtain (3.3). \hfill \Box

Let us turn to the proof of Proposition 3.3. In what follows, we fix an arbitrary constant $\varepsilon > 0$ and take a constant $c_0 > 0$ so that Lemma 3.1 holds.

Let us define

\begin{align*}
E_1(t; u) &= \int_{\Omega} (|\nabla u|^2 + |u_t|^2) \Phi_{A,\beta} \, dx, \\
E_2(t; u) &= \int_{\Omega} (2uu_t + |u|^2) \Phi_{A,\beta} \, dx, \\
F(t; u) &= \int_{\Omega} \left( a(x) + \frac{A(x)}{(1+t)^2} \right) |u_t|^2 \Phi_{A,\beta} \, dx.
\end{align*}

**Remark 3.1.** In order to prove Proposition 3.3 for $k \in \mathbb{Z}_{\geq 0}$, we assume additional regularity on the initial data:

\begin{equation}
(u_0, u_1) \in [H^{k+2}(\Omega) \cap H^1_0(\Omega)] \times [H^{k+1}(\Omega) \cap H^1_0(\Omega)]
\end{equation}

and hence, the solution $u$ of (1.1) has the regularity $\cap_{j=0}^{k+2} C^j([0, \infty); H^{k+2-j}(\Omega))$, since the initial data satisfies the compatibility condition of order $k$ (see [1, Theorem 2]). After proving the conclusion of Proposition 3.3 for sufficiently regular initial data, we apply the density argument and obtain the same estimate for the initial data belonging to $[H^{k+1}(\Omega) \cap H^1_0(\Omega)] \times [H^k(\Omega) \cap H^1_0(\Omega)]$.

**Lemma 3.7.** Let $\Phi \in C^2(\Omega \times [0, \infty))$ satisfy $\Phi > 0$ and $\partial_t \Phi < 0$ and let $u$ be a solution of (1.1). Then

\begin{align*}
\frac{d}{dt} \left[ \int_{\Omega} (|\nabla u|^2 + |u_t|^2) \Phi \, dx \right] &= \int_{\Omega} (\partial_t \Phi)^{-1} |\partial_t \Phi \nabla u - u_t \nabla \Phi|^2 \, dx \\
& \quad + \int_{\Omega} \left( -2a(x)\Phi + \partial_t \Phi - (\partial_t \Phi)^{-1} |\nabla \Phi|^2 \right) |u_t|^2 \, dx.
\end{align*}
In particular, if we put $\Phi = \Phi_{A,\beta}$, and $\beta < 2(h_a + \varepsilon)^{-1}$, then there exists $\gamma_1(\beta) > 0$ such that

\begin{equation}
(3.4) \hspace{1cm} \frac{d}{dt} E_1(t; u) \leq -\gamma_1(\beta) F(t; u).
\end{equation}

Proof. We have

\[
\frac{d}{dt} \left[ \int_\Omega \left( |\nabla u|^2 + |u_t|^2 \right) \Phi \, dx \right] = \int_\Omega |\nabla u|^2 (\partial_t \Phi) \, dx + 2 \int_\Omega (\nabla u \cdot \nabla u_t) \Phi \, dx + 2 \int_\Omega u_{tt} u_t \Phi \, dx + \int_\Omega |u_t|^2 (\partial_t \Phi) \, dx.
\]

Thus, we have the first assertion. In particular, if $\Phi = \Phi_{A,\beta}$, then

\[ -2a(x)\Phi + \partial_t \Phi - (\partial_t \Phi)^{-1} |\nabla \Phi|^2 \leq - \left( 2a(x) + \beta \frac{A}{(1+t)^2} - \beta \frac{|\nabla A|^2}{A} \right) \Phi_{A,\beta}. \]

By Lemma 3.8, $|\nabla A(x)|^2 / A(x) \leq (h_a + \varepsilon)a(x)$. This and $\beta < 2(h_a + \varepsilon)^{-1}$ lead to $-2a(x) + \beta |\nabla A(x)|^2 / A(x) \leq -2\beta(h_a + \varepsilon)a(x)$. Therefore, there exists some constant $\gamma_1(\beta) > 0$ such that

\[
\int_\Omega \left( -2a(x)\Phi_{A,\beta} + \partial_t \Phi_{A,\beta} - (\partial_t \Phi_{A,\beta})^{-1} |\nabla \Phi_{A,\beta}|^2 \right) |u_t|^2 \, dx \leq -\gamma_1(\beta) F(t; u).
\]

From this and the first assertion, we reach the desired estimate. \hfill \square

Furthermore, by multiplying (3.4) by $(t_0 + t)^m$ with arbitrary $m \geq 0$, we have the following.

**Lemma 3.8.** Let $u$ be a solution of (1.1) and let $m \geq 0$. Then there exists $t_1 \geq 1$ such that for every $t_0 \geq t_1$ and $t > 0$,

\[ \frac{d}{dt} \left[ (t_0 + t)^m E_1(t; u) \right] \leq m(t_0 + t)^{m-1} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} \, dx - \frac{\gamma_1(\beta)}{2} (t_0 + t)^m F(t; u). \]

Proof. Let $t_1$ be a constant determined later and let $t_0 \geq t_1$. Multiplying (3.4) by $(t_0 + t)^m$, we have

\[
\frac{d}{dt} \left[ (t_0 + t)^m E_1(t; u) \right] \leq m(t_0 + t)^{m-1} E_1(t; u) - \gamma_1(\beta)(t_0 + t)^m F(t; u)
\]

\[ \leq m(t_0 + t)^{m-1} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} \, dx + (t_0 + t)^m \left[ m(t_0 + t)^{-1} \int_\Omega |u_t|^2 \Phi_{A,\beta} \, dx - \gamma_1(\beta) F(t; u) \right]. \]
By Lemma 3.2, we have $a(x) \geq a_1(1 + |x|)^{-\alpha} \geq a_1 t^{-\alpha}$ if $t_1 \geq R_0 + 1$. Since $\alpha < 1$, retaking $t_1$ larger so that

$$m(t_0 + t)^{-1} \leq a_1^{-1} m(t_0 + t)^{\alpha - 1} a(x) \leq \frac{\gamma_1(\beta)}{2} a(x),$$

we have the desired estimate. □

**Lemma 3.9.** Let $\Phi \in C^2(\Omega \times [0, \infty))$ satisfy $\Phi > 0$ and $\partial_t \Phi < 0$ and let $u$ be a solution to (1.1). Then, we have

$$\frac{d}{dt} \left[ \int_{\Omega} (2wu_t + a(x)|u|^2) \Phi \, dx \right]$$

$$= 2 \int_{\Omega} uu_t(\partial_t \Phi) \, dx + \int_{\Omega} a(x)|u|^2(\partial_t \Phi) \, dx$$

$$+ 2 \int_{\Omega} |u|^2 \Phi \, dx + 2 \int_{\Omega} u(u_t + a(x)u_t) \Phi \, dx$$

$$= 2 \int_{\Omega} uu_t(\partial_t \Phi) \, dx + \int_{\Omega} a(x)|u|^2(\partial_t \Phi) \, dx + 2 \int_{\Omega} |u|^2 \Phi \, dx + 2 \int_{\Omega} u(\Delta u) \Phi \, dx.$$  

Noting that integration by parts yields

$$\int_{\Omega} u(\Delta u) \Phi \, dx = \int_{\Omega} |\nabla u|^2 \Phi \, dx - \int_{\Omega} u(\nabla u \cdot \nabla \Phi) \, dx$$

$$= - \int_{\Omega} |\nabla u|^2 \Phi \, dx - \frac{1}{2} \int_{\Omega} (\nabla(|u|^2) \cdot \nabla \Phi) \, dx$$

$$= - \int_{\Omega} |\nabla u|^2 \Phi \, dx + \frac{1}{2} \int_{\Omega} (\Delta \Phi)|u|^2 \, dx,$$

we deduce

$$\frac{d}{dt} \left[ \int_{\Omega} (2wu_t + a(x)|u|^2) \Phi \, dx \right] + 2 \int_{\Omega} |\nabla u|^2 \Phi \, dx$$

$$= 2 \int_{\Omega} uu_t(\partial_t \Phi) \, dx + 2 \int_{\Omega} |u|^2 \Phi \, dx + \int_{\Omega} (a(\Delta \Phi) + \Phi \, dx.$$  

This proves the first assertion. If $\Phi = \Phi_{A,\beta}$, then from Lemma 3.3, we obtain

$$a(x)\partial_t \Phi_{A,\beta} + \Delta \Phi_{A,\beta} = -\beta a(x)A(x)\frac{A(x)}{(1 + t)^2} \Phi_{A,\beta} + \beta \frac{A(x)}{1 + t} \Phi_{A,\beta} + \left| \frac{\nabla A(x)}{1 + t} \right|^2 \Phi_{A,\beta}.$$  

By Lemma 3.9, we have $|\nabla A(x)|^2 \leq (h_a + \varepsilon) a(x)A(x)$ and hence,

$$a(x)\partial_t \Phi_{A,\beta} + \Delta \Phi_{A,\beta} \leq \frac{\beta}{1 + t} a(x)\Phi_{A,\beta} - \frac{\beta(1 - \beta(h_a + \varepsilon))}{(1 + t)^2} a(x)A(x)\Phi_{A,\beta}.$$
From this and Lemma 3.6 we conclude
\[
\int_{\Omega} (a(x)\partial_t \Phi_{A,\beta} + \Delta \Phi_{A,\beta}) |u|^2 \, dx \\
\leq \int_{\Omega} |\nabla u|^2 \Phi_{A,\beta} \, dx - \frac{\beta(1 - \beta(h_a + \varepsilon))}{(1 + t)^2} \int_{\Omega} a(x)A(x) |u|^2 \Phi_{A,\beta} \, dx.
\]

On the other hand, the Young inequality and Lemma 3.2 yield
\[
2 \int_{\Omega} uu_t (\partial_t \Phi_{A,\beta}) \, dx \leq \frac{2\beta}{(1 + t)^2} \int_{\Omega} uu_t A \Phi_{A,\beta} \, dx \\
\leq \frac{4\beta}{(1 - \beta(h_a + \varepsilon))(1 + t)^2} \int_{\Omega} a(x)^{-1} A(x) |u_t|^2 \Phi_{A,\beta} \, dx \\
+ \frac{\beta(1 - \beta(h_a + \varepsilon))}{(1 + t)^2} \int_{\Omega} a(x)A(x) |u|^2 \Phi_{A,\beta} \, dx.
\]

Therefore, we have
\[
\frac{d}{dt} E_2(t; u) + \int_{\Omega} |\nabla u|^2 \Phi_{A,\beta} \, dx \\
\leq 2 \int_{\Omega} |u_t|^2 \Phi_{A,\beta} \, dx + \frac{4\beta(R_0 + 1 + t)^\alpha}{c_0(1 - \beta(h_a + \varepsilon))(1 + t)^2} \int_{\Omega} A(x) |u_t|^2 \Phi_{A,\beta} \, dx.
\]

Finally, Lemma 3.1 gives
\[
2 \int_{\Omega} |u|^2 \Phi_{A,\beta} \, dx \leq 2a_1^{-1}(R_0 + 1 + t)^\alpha \int_{\Omega} a(x) |u_t|^2 \Phi_{A,\beta} \, dx
\]
and hence,
\[
\frac{d}{dt} E_2(t; u) + \int_{\Omega} |\nabla u|^2 \Phi_{A,\beta} \, dx \leq \gamma_2(\beta)(R_0 + 1 + t)^\alpha F(t; u)
\]
with some $\gamma_2(\beta) > 0$, which completes the proof. \(\square\)

As in Lemma 3.8 by multiplying (3.5) by $(t_0 + t)^l$ with arbitrary $l \geq 0$, we have the following.

**Lemma 3.10.** Let $l \geq 0$. There exists $t_2 \geq t_1$ such that for every $t_0 \geq t_2$ and $t \geq 0$,
\[
\frac{d}{dt} \left[ (t_0 + t)^l E_2(t; u) \right] + (t_0 + t)^l \int_{\Omega} |\nabla u|^2 \Phi_{A,\beta} \, dx \\
\leq l(1 + \varepsilon)(t_0 + t)^{l-1} \int_{\Omega} a(x) |u|^2 \Phi_{A,\beta} \, dx + 2\gamma_2(\beta)(t_0 + t)^{l+\alpha} F(t; u).
\]

**Proof.** Let $l \geq 0$ and let $t_2 \geq t_1$ be a constant determined later. We assume that $t_0 \geq t_2$. Multiplying (3.5) by $(t_0 + t)^l$, we have
\[
\frac{d}{dt} \left[ (t_0 + t)^l E_2(t; u) \right] + (t_0 + t)^l \int_{\Omega} |\nabla u|^2 \Phi_{A,\beta} \, dx \\
\leq (t_0 + t)^l \left[ l(t_0 + t)^{-1} E_2(t; u) + \gamma_2(\beta)(R_0 + 1 + t)^\alpha F(t; u) \right].
\]
We estimate the right-hand side. From the Schwarz inequality and Lemma 3.2 we obtain
\[ 2l(t_0 + t)^{-1} \int_\Omega u_l \Phi_{A,\beta} \, dx \]
\[ \leq \frac{l^2}{\gamma_2(\beta)}(t_0 + t)^{-2+\alpha} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} \, dx + \gamma_2(\beta)(t_0 + t)^{-\alpha} \int_\Omega a(x)|u_l|^2 \Phi_{A,\beta} \, dx \]
\[ \leq \frac{l^2}{\gamma_2(\beta)}(t_0 + t)^{-2+\alpha} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} \, dx + \gamma_2(\beta)(t_0 + t)^{\alpha} \int_\Omega a(x)|u_l|^2 \Phi_{A,\beta} \, dx \]
Here we have also used Lemma 3.2 and the definition of $F(t;u)$. Taking $t_2 \geq t_1$ such that
\[ \frac{l^2}{\gamma_2(\beta)}t_2^{-1+\alpha} \leq l\beta \varepsilon \]
and hence,
\[ \frac{l^2}{\gamma_2(\beta)}(t_0 + t)^{-2+\alpha} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} \, dx \leq l\beta \varepsilon(t_0 + t)^{-1} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} \, dx. \]
Consequently, we reach
\[ \frac{d}{dt} \left[ (t_0 + t)^l E_2(t;u) \right] + (t_0 + t)^l \int_\Omega |\nabla u|^2 \Phi_{A,\beta} \, dx \]
\[ \leq l(1 + \beta \varepsilon)(t_0 + t)^{l-1} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} \, dx + 2\gamma_2(\beta)(t_0 + t)^l F(t;u). \]
This completes the proof. \qed

In particular, by choosing $l = (h_0 + 4\varepsilon)^{-1}$ and $\beta = (h_0 + 2\varepsilon)^{-1}$, we have the following estimate.

**Lemma 3.11.** For every $t_0 \geq t_2$ and $t \geq 0$, we have
\[ \frac{d}{dt} \left[ (t_0 + t)^{\frac{1}{h_0 + 4\varepsilon}} E_2(t;u) \right] \]
\[ \leq -\frac{\varepsilon}{h_0 + 4\varepsilon}(t_0 + t)^{\frac{1}{h_0 + 4\varepsilon}} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} \, dx + 2\gamma_2(\beta)(t_0 + t)^{\frac{1}{h_0 + 4\varepsilon}} F(t;u). \]

**Proof.** We choose $l = (h_0 + 4\varepsilon)^{-1}$ and $\beta = (h_0 + 2\varepsilon)^{-1}$ in Lemma 3.10. Moreover, Lemma 3.6 implies
\[ l(1 + \beta \varepsilon)(t_0 + t)^{l-1} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} \, dx \leq \frac{h_0 + 3\varepsilon}{h_0 + 4\varepsilon}(t_0 + t)^l \int_\Omega |\nabla u|^2 \Phi_{A,\beta} \, dx, \]
which gives the desired estimate. \qed

From Lemmas 3.8 and 3.11 we obtain Proposition 3.3 when $k = 0$.

**Proof of Proposition 3.3** when $k = 0$. Let $t_0 \geq t_2$ be a constant determined later. Taking $m = (h_0 + 4\varepsilon)^{-1} + \alpha$ in Lemma 3.8, we have
\[ \frac{d}{dt} \left[ (t_0 + t)^{\frac{1}{h_0 + 4\varepsilon}+\alpha} E_1(t;u) \right] \leq \left( \frac{1}{h_0 + 4\varepsilon} + \alpha \right) (t_0 + t)^{\frac{1}{h_0 + 4\varepsilon}+\alpha-1} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} \, dx \]
\[ - \frac{\gamma_1(\beta)}{2}(t_0 + t)^{\frac{1}{h_0 + 4\varepsilon}+\alpha} F(t;u). \]
Set \( \nu = \min\{ \frac{2\alpha}{\gamma_2}, \frac{1}{4\nu_0} \} \) and define
\[
E_0(t; u) = (t_0 + t)^{\frac{1}{\alpha + \nu}} E_1(t; u) + \nu(t_0 + t)^{\frac{1}{\alpha + \nu}} E_2(t; u).
\]
Then, the Schwarz inequality and Lemma 3.2 imply
\[
2\nu|uu_t| \leq \frac{1}{2} a(x)|u|^2 + 2\nu a(x)^{-1}|u_t|^2 \\
\leq \frac{1}{2} a(x)|u|^2 + \frac{\nu a_2}{2} (t_0 + t)^\alpha |u_t|^2 \\
\leq \frac{1}{2} a(x)|u|^2 + \frac{1}{2} (t_0 + t)^\alpha |u_t|^2
\]
and hence, \( E(t; u) \) is equivalent to
\[
(t_0 + t)^{\frac{1}{\alpha + \nu}} \left( |\nabla u|^2 + |u_t|^2 \right) \Phi_{A,\beta} dx + (t_0 + t)^{\frac{1}{\alpha + \nu}} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} dx.
\]
By Lemmas 3.8 and 3.10 with \( l = (h_a + 4\varepsilon)^{-1} \), we have
\[
\frac{d}{dt} E_0(t; u) \\
\leq (t_0 + t)^{\frac{1}{\alpha + \nu}} \left[ \left( \frac{1}{h_a + 4\varepsilon} + \alpha \right) (t_0 + t)^\alpha - \frac{\varepsilon \nu}{h_a + 4\varepsilon} \right] \int_\Omega |\nabla u|^2 \Phi_{A,\beta} dx.
\]
Therefore, there exists \( t_3 \geq t_2 \) such that for every \( t_0 \geq t_3 \) and \( t \geq 0 \),
\[
\frac{d}{dt} E_0(t; u) \\
\leq -\frac{\varepsilon \nu}{2(h_a + 4\varepsilon)} (t_0 + t)^{\frac{1}{\alpha + \nu}} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} dx.
\]
Integrating it over \([0, t]\), we have
\[
(3.6) \quad E_0(t; u) + \frac{\varepsilon \nu}{2(h_a + 4\varepsilon)} \int_0^t (t_0 + s)^{\frac{1}{\alpha + \nu}} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} dx ds \leq E_0(0; u).
\]
In particular, we obtain
\[
(t_0 + t)^{\frac{1}{\alpha + \nu}} \int_\Omega a(x)|u|^2 \Phi_{A,\beta} dx \leq C E_0(0, u)
\]
with some constant \( C > 0 \), which depending on \( t_0, \nu, \varepsilon \). This gives the first assertion of (3.2) when \( k = 0 \).

Next, we prove the second assertion of (3.2). We note that the estimate (3.6) also gives
\[
(3.7) \quad \int_0^t (t_0 + s)^{\frac{1}{\alpha + \nu}} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} dx ds \leq C E_0(0, u).
\]
By choosing \( m = (h_a + 4\varepsilon)^{-1} + 1 \) in Lemma 3.8 we have
\[
\frac{d}{dt} \left[ (t_0 + t)^{\frac{1}{\alpha + \nu} + 1} E_1(t; u) \right] \\
\leq \left( \frac{1}{h_a + 4\varepsilon} + 1 \right) (t_0 + t)^{\frac{1}{\alpha + \nu} + 1} \int_\Omega |\nabla u|^2 \Phi_{A,\beta} dx - \frac{\gamma_1(\beta)}{2} (t_0 + t)^{\frac{1}{\alpha + \nu} + 1} F(t; u).
\]
Integrating it over $[0, t]$ and using (5.7), we deduce
\[
(t_0 + t)^{\frac{1}{a + \alpha} + 1} E_1(t; u) + \int_0^t (t_0 + s)^{\frac{1}{a + \alpha} + 1} F(s; u) ds \leq C E_1(0; u) + C \int_0^t (t_0 + s)^{\frac{1}{a + \alpha}} \int_\Omega |\nabla u|^2 \Phi_{A, \beta} \, dx \, ds \leq C E_0(0; u).
\]
In particular, we obtain
\[
(3.8) \quad (t_0 + t)^{\frac{1}{a + \alpha} + 1} \int_\Omega |\nabla u|^2 \Phi_{A, \beta} \, dx + \int_0^t (t_0 + s)^{\frac{1}{a + \alpha} + 1} F(s, u) ds \leq C E_0(0; u).
\]
This proves the second assertion of (3.2) when $k = 0$. □

Finally, we give a proof of Proposition 3.3 when $k \geq 1$.

**Proof of Proposition 3.3 for $k \geq 1$.** For $k \in \mathbb{Z}_{\geq 0}$, we define
\[
E_k(t; u) = (t_0 + t)^{\frac{1}{a + \alpha} + k} E_1(t; \partial_t^k u) + \nu (t_0 + t)^{\frac{1}{a + \alpha} + 2 k} E_2(t; \partial_t^k u).
\]
By differentiating the equation (1.1) with respect to $t$, we notice that $u_t$ also satisfies the equation (1.1). Therefore, we apply Lemmas 3.3 and 3.10 for $u_t$ with $m = (h_a + 4\varepsilon)^{-1} + 2 + \alpha$ and $l = (h_a + 4\varepsilon)^{-1} + 2$ to obtain
\[
\frac{d}{dt} \left((t_0 + t)^{\frac{1}{a + \alpha} + k + 2} E_1(t; u_t)\right) \leq \left(\frac{1}{h_a + 4\varepsilon} + \alpha + 2\right) (t_0 + t)^{\frac{1}{a + \alpha} + k + 1} \int_\Omega |\nabla u_t|^2 \Phi_{A, \beta} \, dx
\]
\[
- \frac{\gamma_1(\beta)}{2} (t_0 + t)^{\frac{1}{a + \alpha} + k + 2} F(t; u_t)
\]
and
\[
\frac{d}{dt} \left((t_0 + t)^{\frac{1}{a + \alpha} + k + 2} E_2(t; u_t)\right) + (t_0 + t)^{\frac{1}{a + \alpha} + k + 2} \int_\Omega |\nabla u_t|^2 \Phi_{A, \beta} \, dx \leq \left(\frac{1}{h_a + 4\varepsilon} + 2\right) (1 + \beta\varepsilon)(t_0 + t)^{\frac{1}{a + \alpha} + k + 1} \int_\Omega a(x)|u_t|^2 \Phi_{A, \beta} \, dx
\]
\[
+ 2\gamma_2(\beta)(t_0 + t)^{\frac{1}{a + \alpha} + k + 2 + \alpha} F(t; u_t)
\]
Hence, there exists a constant $t_4 \geq t_3$ such that for every $t_0 \geq t_4$ and $t \geq 0$,
\[
\frac{d}{dt} E_1(t; u) + \nu (t_0 + t)^{\frac{1}{a + \alpha} + 2} \int_\Omega |\nabla u_t|^2 \Phi_{A, \beta} \, dx \leq \left(\frac{1}{h_a + 4\varepsilon} + 2\right) (1 + \beta\varepsilon)(t_0 + t)^{\frac{1}{a + \alpha} + 1} \int_\Omega a(x)|u_t|^2 \Phi_{A, \beta} \, dx
\]
\[
\leq \left(\frac{1}{h_a + 4\varepsilon} + 2\right) (1 + \beta\varepsilon)(t_0 + t)^{\frac{1}{a + \alpha} + 1} F(t; u).
\]
Integrating it over \([0, t]\) and \(3.8\) lead to

\[
E_1(t; u) + \int_0^t (t_0 + s)^{\frac{1}{\alpha+\varepsilon}+2} \int_\Omega |\nabla u_t|^2 \Phi_{A, \beta} \, dx \leq C \int_0^t (t_0 + s)^{\frac{1}{\alpha+\varepsilon}+1} F(s; u_t) \, ds \\
\leq C (E_1(0; u) + E_0(0; u)).
\]

In particular, we have the first assertion of \(3.2\) for \(k = 1\), since \(E_1(0; u) + E_0(0; u) \leq C\|(u_0, u_1)\|_{H^2 \times H^1}^2\). Moreover, we obtain

\[
\int_0^t (t_0 + s)^{\frac{1}{\alpha+\varepsilon}+2} \int_\Omega |\nabla u_t|^2 \Phi_{A, \beta} \, dx \leq C\|(u_0, u_1)\|_{H^2 \times H^1}^2.
\]

This estimate together with Lemma \(3.8\) with \(m = (h_a + 4\varepsilon)^{-1} + 3\) imply

\[
(t_0 + t)^{\frac{1}{\alpha+\varepsilon}+3} E_1(t; u_t) + \frac{\gamma_2(\beta)}{2} \int_0^t (t_0 + s)^{\frac{1}{\alpha+\varepsilon}+3} F(s; u_t) \, ds \\
\leq CE_1(0; u_t) + C \int_0^t (t_0 + s)^{\frac{1}{\alpha+\varepsilon}+2} \int_\Omega |\nabla u|^2 \Phi_{A, \beta} \, dx
\]

and hence,

\[
(t_0 + t)^{\frac{1}{\alpha+\varepsilon}+3} \int_\Omega |\nabla u_t|^2 \, dx + \int_0^t (t_0 + s)^{\frac{1}{\alpha+\varepsilon}+3} F(s; u_t) \, ds \leq C\|(u_0, u_1)\|_{H^2 \times H^1}^2,
\]

which gives the second assertion of \(3.2\) for \(k = 1\). We can repeat the same argument inductively and we obtain \(3.2\) for any \(k \in \mathbb{Z}_{\geq 0}\). \(\square\)

4. Diffusion phenomena for the damped wave equation

In this section, we give a proof of Theorem \(1.1\). The argument is similar to that of \cite{24}. We will apply Proposition \(3.3\) with \(k = 2\) and assume that the initial data satisfies

\[
(u_0, u_1) \in [H^3(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]
\]

with the compatibility condition of second order and \(\text{supp}(u_0, u_1) \subset \{x \in \Omega; |x| \leq R_0\}\) with some \(R_0 > 0\). Hence, the corresponding solution \(u\) of \(1.1\) has the regularity \(u \in \cap_{j=0}^3 C^j([0, \infty) ; H^{3-j}(\Omega))\) and \(\text{supp} u(\cdot, t) \subset \{x \in \Omega; |x| \leq R_0 + t\}\).

We rewrite the equation \(1.1\) as the heat equation on \(L_{d \mu}^2\) defined in Section 2:

\[
u_t - a(x)^{-1} \Delta u = -a(x)^{-1} u_{tt}.
\]

Then we remark that

**Lemma 4.1.** Assume that the initial data satisfies \((u_0, u_1) \in [H^3(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]\) with the compatibility condition of second order and \(\text{supp}(u_0, u_1) \subset \{x \in \Omega; |x| \leq R_0\}\). Then for every \(t \geq 0\),

\[
u(x, t) = (e^{t L_{d \mu}} u_0)(x) - \int_0^t e^{(t-s)L_{d \mu}} [a(\cdot)^{-1} u_{tt}(\cdot, s)] \, ds,
\]

where \(L_{d \mu}\) is the Friedrichs extension of \(L = a(x)^{-1} \Delta\) in \(L_{d \mu}^2\) given in Lemma \(3.2\).
Proof. In view of \[7, \text{Theorem 4.3.1}], it suffices to show that
\begin{align}
(4.1) \quad & u \in C^1([0, \infty); L^2_{d\mu}) \cap C([0, \infty); D(L_*)), \\
(4.2) \quad & f = a^{-1}u_t(t) \in C^1([0, \infty); L^2_{d\mu}).
\end{align}

In fact, if the above condition holds, then
\begin{align}
(4.3) \quad & \begin{cases}
  w_t + L_* w = -f, & t \in [0, \infty), \\
  w(0) = u_0 \in D(L_*)
\end{cases}
\end{align}

has a unique classical solution in $C^1([0, \infty); L^2_{d\mu}) \cap C([0, \infty); D(L_*)$ which is represented by
\[
  w(x, t) = (e^{L_*} u_0)(x) - \int_0^t e^{(t-s)L_*} f(s, s) \, ds, \quad t \geq 0.
\]

Since $u$ also satisfies (4.3), it follows from the uniqueness that $u = w$.

Now we prove (4.1) and (4.2). (4.1) is verified by using Lemmas 3.2 and 2.3 with $u \in C^1([0, \infty); L^2(\Omega)) \cap C([0, \infty); H^2 \cap H^1_0(\Omega))$. Finally, we see from the regularity $u \in C^1([0, \infty); L^2(\Omega))$ and Lemma 3.2 that (4.2) is satisfied. This completes the proof. \qed

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By the Leibniz rule
\[
  \frac{\partial}{\partial s} \left( e^{(t-s)L_*} [a(\cdot)^{-1} u_s(s)] \right) = -L_* e^{(t-s)L_*} [a(\cdot)^{-1} u_s(s)] + e^{(t-s)L_*} [a(\cdot)^{-1} u_{ss}(s)]
\]

and the fundamental theorem of calculus in Bochner integral, we have
\[
  \int_0^t e^{(t-s)L_*} [a(\cdot)^{-1} u_{ss}(\cdot, s)] \, ds = \int_0^t \left[ e^{(t-s)L_*} [a(\cdot)^{-1} u_s(s)] \right] ds
\]
\[
  + e^{tL_*} [a(\cdot)^{-1} u_t(\cdot, t/2)] - e^{tL_*} [a(\cdot)^{-1} u_1]
\]
\[
  + \int_0^{t/2} L_* e^{(t-s)L_*} [a(\cdot)^{-1} u_s(\cdot, s)] \, ds.
\]

Combining Lemma 1.1 and the above equality yield
\begin{align}
(4.4) \quad & u(x, t) - e^{tL_*} [u_0 + a(\cdot)^{-1} u_1] = - \int_{t/2}^t e^{(t-s)L_*} [a(\cdot)^{-1} u_{ss}(\cdot, s)] \, ds \\
& \quad - e^{tL_*} [a(\cdot)^{-1} u_t(\cdot, t/2)] \\
& \quad - \int_0^{t/2} L_* e^{(t-s)L_*} [a(\cdot)^{-1} u_s(\cdot, s)] \, ds.
\end{align}

We estimate the each term of the right-hand side by using Propositions 2.6 and 3.3. Hence, we obtain
\begin{align}
(4.5) \quad & \|u(\cdot, t) - e^{tL_*} [u_0 + a(\cdot)^{-1} u_1]\|_{L^2_{d\mu}} \leq J_1 + J_2 + J_3,
\end{align}
where
\[ J_1 = \int_{t/2}^t \left\| e^{(t-s)\mathcal{L}^*}[a(\cdot)^{-1}u_{s\cdot}(\cdot, s)] \right\|_{L_{x\mu}^2} ds, \]
\[ J_2 = \left\| e^{\frac{t}{2}\mathcal{L}^*}[a(\cdot)^{-1}u_{t\cdot}(\cdot, t/2)] \right\|_{L_{x\mu}^2}, \]
\[ J_3 = \int_{t/2}^{t} \left\| L_s e^{(t-s)\mathcal{L}^*}[a(\cdot)^{-1}u_{s\cdot}(\cdot, s)] \right\|_{L_{x\mu}^2} ds. \]

Let us fix \( \varepsilon > 0 \) so that \( \varepsilon < \frac{2-2\alpha}{2-\alpha} \). We first estimate \( J_1 \). By \( a(x) \geq a_1(1 + |x|)^{-\alpha} \) with some \( a_1 > 0 \) and (21), we have
\[ a(x)^{-2} \leq C(1 + |x|)^{2\alpha} \leq \left( \frac{1 + |x|}{t} \right)^{2\alpha/2-\alpha} t^{2\alpha/2-\alpha} \leq t^{2\alpha/2-\alpha} \Phi_{A,\beta}(x, t). \]

We use Lemma 2.2 and then, Proposition 3.3 with \( k = 2 \) to obtain
\[ J_1 \leq \int_{t/2}^t \left\| a(\cdot)^{-1}u_{s\cdot}(\cdot, s) \right\|_{L_{x\mu}^2} ds \]
\[ \leq \int_{t/2}^t s^{2\alpha/2-\alpha} \left\| \sqrt{\Phi_{A,\beta}(\cdot, s)} u_{s\cdot}(\cdot, s) \right\|_{L_{x\mu}^2} ds \]
\[ \leq C t^{2\alpha/2-\alpha} \int_{t/2}^t (t + s) - \frac{N-\alpha}{2+\varepsilon} t^{2\alpha/2-\alpha} \left\| (u_0, u_1) \right\|_{H^2 \times H^1} ds \]
\[ \leq C (1 + t)^{-\frac{N-\alpha}{2+\varepsilon}} t^{2\alpha/2-\alpha} \left\| (u_0, u_1) \right\|_{H^2 \times H^1}. \]

Next, we estimate \( J_2 \). In the same way, we have
\[ J_2 \leq \left\| a(\cdot)^{-1}u_{t\cdot}(\cdot, t/2) \right\|_{L_{x\mu}^2} \]
\[ \leq t^{2\alpha/2-\alpha} \left\| \sqrt{\Phi_{A,\beta}(\cdot, t)} u_{t\cdot}(\cdot, t/2) \right\|_{L_{x\mu}^2} \]
\[ \leq (1 + t)^{-\frac{N-\alpha}{2+\varepsilon}} t^{2\alpha/2-\alpha} \left\| (u_0, u_1) \right\|_{H^2 \times H^1}. \]

Finally, we estimate \( J_3 \). By Proposition 2.6, we have
\[ (4.6) \quad J_3 \leq \int_{t/2}^t (t - s) - \frac{N-\alpha}{2+\varepsilon} \left\| a(\cdot)^{-1}u_{s\cdot}(\cdot, s) \right\|_{L_{x\mu}^1} ds. \]

The Schwarz inequality and Proposition 3.2 lead to
\[ \left\| a(\cdot)^{-1}u_{s\cdot}(\cdot, s) \right\|_{L_{x\mu}^1} \]
\[ = \int_{\Omega} \left| u_{s\cdot}(x, s) \right| dx \]
\[ \leq \left( \int_{\Omega} \Phi_{A,\beta}(x, s) a(x) |u_{s\cdot}(x, s)|^2 dx \right)^{1/2} \left( \int_{\Omega} \Phi_{A,\beta}(x, s)^{-1} a(x)^{-1} dx \right)^{1/2} \]
\[ \leq C (t_0 + s)^{-\frac{N-\alpha}{2+\varepsilon}} t^{2\alpha/2-\alpha} \left\| (u_0, u_1) \right\|_{H^2 \times H^1}. \]
Here we have used
\[
\int_{\Omega} \Phi_{A,\beta}(x,s)^{-1}a(x)^{-1}dx \leq C \int_{\Omega} s^{\frac{N}{2(2-N)}} \Phi_{A,\beta}(x,s)^{-1} \left( \frac{(1+|x|)^{2-\alpha}}{s} \right)^{\frac{N\alpha}{2}} dx \\
\leq C s^{\frac{N-\alpha}{2}}.
\]

Hence, from (4.6) we obtain
\[
J_3 \leq \int_0^{t/2} (t-s)^{-\frac{N-\alpha}{2\alpha} - \frac{2-2\alpha}{2\alpha} + \varepsilon} \|(u_0, u_1)\|_{H^2 \times H^1} \\
\leq C(1+t)^{-\frac{N-\alpha}{2\alpha} - \frac{2-2\alpha}{2\alpha} + \varepsilon} \|(u_0, u_1)\|_{H^2 \times H^1}.
\]

These estimates and (4.5) lead the conclusion
\[
\|u(\cdot, t) - e^{tL_*}[u_0 + a(\cdot)^{-1}u_1]\|_{L^2} \leq C(1+t)^{-\frac{N-\alpha}{2\alpha} - \frac{2-2\alpha}{2\alpha} + \varepsilon} \|(u_0, u_1)\|_{H^2 \times H^1}.
\]

This completes the proof. □

Acknowledgement

The authors are deeply grateful to Professor Mitsuru Sugimoto for his careful reading of the manuscript and helpful comments and suggestions. This work is supported by Grant-in-Aid for JSPS Fellows 15J01600 of Japan Society for the Promotion of Science.

References

[1] M. Ikawa, Mixed problems for hyperbolic equations of second order, J. Math. Soc. Japan 20 (1968), 580–608.
[2] M. Ikawa, Hyperbolic partial differential equations and wave phenomena, American Mathematical Society (2000).
[3] R. Ikehata, Some remarks on the wave equation with potential type damping coefficients, Int. J. Pure Appl. Math. 21 (2005), 19–24.
[4] R. Ikehata, G. Todorova, B. Yordanov, Optimal decay rate of the energy for wave equations with critical potential, J. Math. Soc. Japan 65 (2013), 183–236.
[5] N. Ioku, G. Metafune, M. Sobajima, C. Spina, $L^p$-$L^q$ estimates for homogeneous operators, Commun. Contemp. Math., to appear (doi:10.1142/S0219199715500376).
[6] V. Liskevich, Z. Sobol, Estimates of integral kernels for semigroups associated with second-order elliptic operators with singular coefficients, Potential Anal. 18 (2003), 359–390.
[7] A. Lunardi, “Analytic semigroups and optimal regularity in parabolic problems,” Progress in Nonlinear Differential Equations and their Applications 16, Birkhäuser Verlag, Basel, 1995.
[8] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. 12 (1976), 169–189.
[9] A. Matsumura, Energy decay of solutions of dissipative wave equations, Proc. Japan Acad., Ser. A 53 (1977), 232–236.
[10] T. Matsuyama, Asymptotic behavior of solutions for the wave equation with an effective dissipation around the boundary, J. Math. Anal. Appl. 271 (2002), 467–492.
[11] G. Metafune, C. Spina, A degenerate elliptic operator with unbounded diffusion coefficients, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 25 (2014), 109–140.
[12] G. Metafune, N. Okazawa, M. Sobajima C. Spina, Scale invariant elliptic operators with singular coefficients, J. Evol. Equ., to appear (doi:10.1007/s00028-015-0307-1).
[13] K. Mochizuki, Scattering theory for wave equations with dissipative terms, Publ. Res. Inst. Math. Sci. 12 (1976), 383–390.
[14] K. Mochizuki, H. Nakazawa, Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation, Publ. RIMS, Kyoto Univ. 32 (1996), 401–414.
[15] K. Nishihara, Decay properties for the damped wave equation with space dependent potential and absorbed semilinear term, Comm. Partial Differential Equations 35 (2010), 1402–1418.
[16] H. Nishiyama, Remarks on the asymptotic behavior of the solution to an abstract damped wave equation, arXiv:1505.01794v2.

[17] E.M. Ouhabaz, “Analysis of heat equations on domains,” London Mathematical Society Monographs Series 31, Princeton University Press, Princeton, NJ, 2005.

[18] P. Radu, G. Todorova, B. Yordanov, Higher order energy decay rates for damped wave equations with variable coefficients, Discrete Contin. Dyn. Syst. Ser. S. 2 (2009), 609–629.

[19] P. Radu, G. Todorova, B. Yordanov, The generalized diffusion phenomenon and applications, SIAM J. Math. Anal. 48 (2016), 174–203.

[20] M. Reed, B. Simon, “Methods of modern mathematical physics II, Fourier analysis, self-adjointness,” Academic Press, New York-London, 1975.

[21] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001), 464–489.

[22] G. Todorova, B. Yordanov, Weighted L2-estimates for dissipative wave equations with variable coefficients, J. Differential Equations 246 (2009), 4497–4518.

[23] Y. Uesaka, The total energy decay of solutions for the wave equation with a dissipative term, J. Math. Kyoto Univ. 20 (1979), 57–65.

[24] Y. Wakasugi, On diffusion phenomena for the linear wave equation with space-dependent damping, J. Hyp. Diff. Eq. 11 (2014), 795–819.

E-mail address, M. Sobajima: msobajima1984@gmail.com

(M. Sobajima) Department of Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, 162-8601, Tokyo, Japan

E-mail address, Y. Wakasugi: yuta.wakasugi@math.nagoya-u.ac.jp

(Y. Wakasugi) Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602 Japan