Mirror symmetry, mixed motives and $\zeta(3)$

Minhyong Kim$^1$ and Wenzhe Yang$^2$

Mathematical Institute
University of Oxford
Radcliffe Observatory Quarter
Woodstock Road
Oxford, OX2 6GG, UK

Abstract

In this paper, we formulate a natural conjecture on the location of mixed Tate motives in Voevodsky’s triangulated category of motives over $\mathbb{Q}$. This conjecture is almost certainly well-known to practitioners, nevertheless motivation for making it explicit comes from the phenomenon of mirror symmetry in the theory of Calabi-Yau threefolds. Given a one-parameter mirror pair $(M, W)$ of Calabi-Yau threefolds, the prepotential of the complexified Kähler moduli space of $M$ (which has flat coordinate $t$) admits an expansion that has a constant term $-Y_{000}/6$ and a cubic term $-Y_{111} t^3/6$. Here ‘one-parameter’ means that the deformation space of $W$ is one-dimensional. The number $Y_{111}$ is a non-zero integer, while in all examples where $Y_{000}$ has been computed, it is always of the form

$$-3 \chi(M) \zeta(3)/(2 \pi i)^3 + r, \ r \in \mathbb{Q}$$

where $\chi(M)$ is the Euler characteristic of $M$. This paper uses mirror symmetry to deduce that the dual of the limit mixed Hodge structure at the large complex structure limit of $W$ has a direct summand that is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$ whose period in $\mathbb{C}/(2\pi i)^3 \mathbb{Q}$ is the coset of $-(2 \pi i)^3 Y_{000}/(3 Y_{111})$. If further the deformation space of $W$ forms part of a one-parameter algebraic family defined over $\mathbb{Q}$, the computations in this paper provide evidence for the motivic conjecture as well as a motivic interpretation of the occurrence of $\zeta(3)$ in mirror symmetry.

$^1$minhyong.kim@maths.ox.ac.uk
$^2$wenzhe.yang@maths.ox.ac.uk
# Contents

0 Introduction \hspace{1cm} 3

1 Mixed Motives

1.1 Voevodsky’s Mixed Motives \hspace{1cm} 7

1.2 Mixed Tate Motives \hspace{1cm} 8

2 Limit Mixed Hodge Structure

2.1 Gauss-Manin Connection \hspace{1cm} 10

2.2 Canonical Extension \hspace{1cm} 11

2.3 Limit Mixed Hodge Structure \hspace{1cm} 13

2.4 Limit Mixed Hodge Complex \hspace{1cm} 15

2.5 Schmid’s Construction of Limit MHS \hspace{1cm} 17

3 Limit Mixed Motive

3.1 A Naive Construction of Étale Motivic Sheaves \hspace{1cm} 19

3.2 Motivic Nearby Cycle Functor \hspace{1cm} 21

4 Computation of Limit MHS

4.1 Integral Periods of \(\Omega\) \hspace{1cm} 23

4.2 Canonical Periods of \(\Omega\) \hspace{1cm} 25

4.3 Mirror Symmetry \hspace{1cm} 26

4.4 Weight Filtration \hspace{1cm} 28

4.5 Limit Hodge Filtration \hspace{1cm} 32

4.6 Splitness of Limit MHS \hspace{1cm} 33

5 The Motivic Nature of \(\zeta(3)\)

A Mixed Hodge Structure \hspace{1cm} 37

40
A.1 Definition of Mixed Hodge Structure ............................................. 40
A.2 Extensions of MHS ................................................................. 41

B Pure Motives ............................................................................ 43
B.1 Algebraic Cycles ................................................................. 43
B.2 Weil Cohomology Theory ..................................................... 45
B.3 Pure Motives ................................................................. 47
B.4 The Conjectured Abelian Category of Mixed Motives ............... 49

C Mixed Hodge Complex ............................................................. 50
0. Introduction

Over 50 years of research on motives has continuously enriched all areas of number theory and algebraic geometry with unifying themes as well as deep formulas. Nevertheless, Grothendieck’s original vision, supplemented by Beilinson and Deligne [10, 19], whereby an abelian category of motives over $\mathbb{Q}$ is supposed to provide building blocks for arithmetic varieties, has yet to be realised. The best approximation to this idea so far is the construction of a triangulated category $\text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$ of motives [48, 49] that has nearly all the properties conjectured for the derived category of the desired abelian category. That is, $\text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$ is expected to carry a natural motivic $t$-structure, whose heart should be Grothendieck’s abelian category, however construction of such a motivic $t$-structure appears inaccessible for the time being. As far as an abelian category is concerned, the best constructed is that of mixed Tate motives $\text{TM}_{\mathbb{Q}}$, the one whose semi-simplification consists of only the simplest possible pure motives $\mathbb{Q}(n), n \in \mathbb{Z}$. That is to say, the full triangulated subcategory $\text{DTM}_{\mathbb{Q}}$ of $\text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$ generated by $\mathbb{Q}(n), n \in \mathbb{Z}$ does have a motivic $t$-structure whose heart is by definition the abelian category $\text{TM}_{\mathbb{Q}}$. But in fact, even the way in which mixed Tate motives sit inside $\text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$ remains rather mysterious.

Here is a natural conjecture in this regard, which could be regarded as a generalised Hodge conjecture concerning the Hodge realisation functor

$$\mathcal{R} : \text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q}) \to D^b(\text{MHS}_{\mathbb{Q}})$$

to the derived category of mixed Hodge structures.

**Conjecture GH** Suppose a mixed Hodge-Tate structure $P$ occurs as a direct summand of $H^q(\mathcal{R}(\mathcal{N})), q \in \mathbb{Z}$, where $\mathcal{N}$ is an object of $\text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$. Then

$$P \simeq \mathcal{R}(\mathcal{M})$$

where $\mathcal{M}$ is a mixed Tate motive.

Here a mixed Hodge-Tate structure is a mixed Hodge structure whose semi-simplification is a direct sum of $\mathbb{Q}(n), n \in \mathbb{Z}$. Of course this conjecture is likely to be inaccessible in general using current day technologies, nevertheless we might be concerned with the possibility of testing it numerically in some sense. In this paper, we will focus on the following period-theoretic version:

**Conjecture GHP** Suppose a mixed Hodge-Tate structure $P$ occurs as a direct summand of $H^q(\mathcal{R}(\mathcal{N})), q \in \mathbb{Z}$, where $\mathcal{N}$ is an object of $\text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q})$. Suppose further that it is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n), n \geq 3$, then the class of $P$ in

$$\mathbb{C}/(2 \pi i)^n \mathbb{Q} \simeq \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(n))$$

(1.18) is the coset of a rational multiple of $\zeta(n)$.

3
As we will explain later, Conjecture GHP follows from Conjecture GH and the known computations of periods of mixed Tate motives [20]. The main purpose of this paper is to construct examples of Conjecture GHP from limit mixed motives at large complex structure limit in mirror symmetry, a theory we now briefly review. Roughly speaking, mirror symmetry is a conjecture that there exist mirror pairs \((M, W)\) of Calabi-Yau threefolds such that the complexified Kähler moduli space of \(M\) is isomorphic to an open subset of the complex moduli space of \(W\), which is a neighbourhood of a special boundary point known as the large complex structure limit [17, 33].

The complexified Kähler moduli space of \(M\), denoted by \(\mathcal{M}_K(M)\), is essentially the space whose points represent Kähler structures on \(M\), while the complex moduli space of \(W\), denoted by \(\mathcal{M}_C(W)\), is the space whose points represent the complex structures of \(W\), both of which have natural analytic structures [17, 33]. The isomorphism between \(\mathcal{M}_K(M)\) and the special open subset of \(\mathcal{M}_C(W)\) is called mirror map, which is constructed by identifying certain functions on \(\mathcal{M}_K(M)\) with those on \(\mathcal{M}_C(W)\) [14, 17, 33]. Let’s now give a brief description of \(\mathcal{M}_K(M)\) based on [33]. Recall that the Hodge diamond of a Calabi-Yau threefold is of the form

\[
\begin{array}{cccc}
1 & & & \\
& 0 & 0 & \\
& h^{11} & 0 & \\
& h^{21} & h^{21} & 1 \\
& 0 & h^{11} & 0 & \\
& & & \\
& & & \\
1 & & & 
\end{array}
\]

where \(h^{11} := \dim H^{1,1}\) and \(h^{21} := \dim H^{2,1}\), in particular we have \(H^1 = H^5 = 0\). Define \(H^{1,1}(M, \mathbb{R})\) as

\[
H^{1,1}(M, \mathbb{R}) := H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})
\]

i.e. it consists of elements of \(H^2(M, \mathbb{R})\) that can be represented by real closed forms of type \((1, 1)\). The Kähler cone of \(M\) is defined as,

\[
\mathcal{K}_M := \{\omega \in H^{1,1}(M, \mathbb{R}) \mid \omega \text{ can be represented by a Kähler form of } M\}
\]

which is an open subset of \(H^{1,1}(M, \mathbb{R})\) [33]. The complexified Kähler moduli space of \(M\) is defined as [33],

\[
\mathcal{M}_K(M) := (H^2(M, \mathbb{R}) + i \mathcal{K}_M) / H^2(M, \mathbb{Z})
\]

In this paper we will only consider one-parameter mirror pairs \((M, W)\) of Calabi-Yau threefolds, i.e.

\[
\dim H^{1,1}(M) = \dim H^{2,1}(W) = 1
\]

in which case, the Kähler cone \(\mathcal{K}_M\) is the open ray \(\mathbb{R}_{>0}\), hence \(\mathcal{M}_K(M)\) has a very simple description [33],

\[
\mathcal{M}_K(M) = (\mathbb{R} + i \mathbb{R}_{>0}) / \mathbb{Z} = \mathcal{H} / \mathbb{Z}
\]
where $\mathcal{H}$ is the upper half plane of $\mathbb{C}$. Now let $e$ be a basis of $H^2(M, \mathbb{Z})$ (modulo torsion) that lies in the Kähler cone $\mathcal{K}_M$, then every point of $\mathcal{M}_K(M)$ is represented by $t e, t \in \mathcal{H}$, while $et$ is equivalent to $e(t+1)$. Conventionally $t$ is called the flat coordinate of $\mathcal{M}_K(M)$ by physicists [14, 33]. In mirror symmetry, the prepotential $\mathcal{F}$ on the Kähler side admits an expansion of the form [14, 15]

$$\mathcal{F} = -\frac{1}{6} Y_{111} t^3 - \frac{1}{2} Y_{011} t^2 - \frac{1}{2} Y_{001} t - \frac{1}{6} Y_{000} + \mathcal{F}^{np}$$

(0.6)

where $\mathcal{F}^{np}$ is the non-perturbative instanton correction. The term $\mathcal{F}^{np}$ is invariant under the operation $t \to t + 1$ and is exponentially small when $t \to i \infty$, i.e. it admits a series expansion in $\exp 2\pi i t$ with no constant term [14]. The coefficient $Y_{111}$ is the topological intersection number [14, 15]

$$Y_{111} = \int_M e \wedge e \wedge e$$

(0.7)

which is a nonzero integer. Usually, $e$ is chosen to be the cycle represented by a divisor of $M$, in which case $Y_{111}$ is the intersection number of this divisor intersecting with itself twice. The coefficients $Y_{011}$ and $Y_{001}$ could be shown to be rational numbers by mirror symmetry. The coefficient $Y_{000}$ is certainly the most mysterious one and in all examples of mirror pairs where it has been computed, it is of the form [15]:

$$Y_{000} = -3 \chi(M) \frac{\zeta(3)}{(2 \pi i)^3} + r$$

(0.8)

where $\chi(M)$ is the Euler characteristic of $M$ and $r$ is a rational number. Mathematical physicists speculate that equation 0.8 is valid in general for an arbitrary mirror pair but currently there isn’t a proof available. The occurrence of $\zeta(3)$ in $Y_{000}$ is highly interesting, and one might ask if it has an arithmetic origin when combined with the geometry of the mirror threefold $W$.

When the deformation space of $W$ forms part of a one-parameter algebraic family defined over $\mathbb{Q}$, which will be a standard assumption in this paper, we will establish a direct relation between 0.8 and Conjecture GHP. The objects of interest in this paper are limit mixed motives in the sense of Ayoub [3]. They are supposed to be the motivic lifts of the complexes in $D^b(MHS_{\mathbb{Q}})$ that compute the limit mixed Hodge structures, whose construction precisely follows from those of Steenbrink [59] and Schmid [54]. It is natural that the Hodge realisation of the limit mixed motive should be the complex that computes the limit mixed Hodge structure. Unfortunately, the proof of this statement seems to be quite technical and not available yet in literatures, therefore it is stated as a separate conjecture (Conjecture 3.4).

Assuming mirror symmetry and Conjecture 3.4, the main conclusion of this paper is that the existing computations of $Y_{000}$ 0.8 provide highly interesting examples in favour of Conjecture GHP. The key observation is that mirror symmetry forces the limit mixed Hodge structure at the large complex structure limit (obtained from the third cohomology of $W$)
to split (in the abelian category $\text{MHS}_\mathbb{Q}$) into

$$\mathbb{Q}(-1) \oplus \mathbb{Q}(-2) \oplus M$$

(0.9)

where $M$ is a two-dimensional mixed Hodge structure that is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$ (Theorem 4.10).

**Theorem 0.1.** Assuming mirror symmetry conjecture (which will be made precise in section 4.3), the dual object $M^\vee$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$ whose image in $\mathbb{C}/(2\pi i)^3\mathbb{Q}$ is the coset of $-(2\pi i)^3 Y_{000}/(3 Y_{111})$.

In all examples of mirror pairs where $Y_{000}$ has been computed, it is of the form [0.8] therefore the coset of $-(2\pi i)^3 Y_{000}/(3 Y_{111})$ in $\mathbb{C}/(2\pi i)^3\mathbb{Q}$ is the coset of a rational multiple of $\zeta(3)$. So combined with Conjecture 3.4, the limit mixed motives constructed at large complex structure limit provide affirmative cases of Conjecture GHP. On the other hand Conjecture GHP also sheds light on the motivic nature of $\zeta(3)$ in $Y_{000}$. That is, from the converse point of view, if we assume conjectures GHP and 3.4 instead, the computations in this paper prove that $Y_{000}$ must be of the form

$$Y_{000} = \frac{r_1}{(2\pi i)^3} \zeta(3) + r_2, \ r_1, r_2 \in \mathbb{Q}$$

(0.10)

Thus what we have observed is an approximate equivalence between Conjecture GHP and formula [0.8] in mirror symmetry.

The structure of this paper is as follows:

Section 1 contains a very brief introduction to Voevodsky’s category of mixed motives and mixed Tate motives.

Section 2 contains a short review of Gauss-Manin connection, canonical extensions and the construction of limit mixed Hodge structure by Steenbrink and Schmid.

Section 3 talks about the construction of limit mixed motive by Ayoub’s motivic nearby cycle functor, after which Conjecture 3.4 will be formulated.

Section 4 contains detailed computations of limit mixed Hodge structures at large complex structure limit, which is shown to split in a highly interesting way.

Section 5 contains explicit computations which show that the image of $M^\vee$ in $\mathbb{C}/(2\pi i)^3\mathbb{Q}$ is the coset of $-(2\pi i)^3 Y_{000}/(3 Y_{111})$, thus connects the last link in our construction.

The appendix contains elementary materials of mixed Hodge structures and pure motives, which are prerequisites for this paper.
1. Mixed Motives

This section contains a brief introduction to Voevodsky’s triangulated category of mixed motives and the abelian category of mixed Tate motives over $\mathbb{Q}$. However it is not meant to be complete, hence necessary references are given for further reading. On the other hand Appendix $\textbf{A}$ contains an introduction to mixed Hodge structures, while Appendix $\textbf{B}$ contains an introduction to pure motives, both of which are prerequisites for this section. Therefore the readers who are not familiar with them are strongly recommended to consult the appendix or other references before read this section.

1.1. Voevodsky’s Mixed Motives

Let $k$ be a field that admits resolution of singularities and $\Lambda$ be a commutative ring with unit, Voevodsky’s category of mixed motives, denoted by $\text{DM}(k, \Lambda)$, is a rigid tensor triangulated category $\left[49, 62\right]$. The ring $\Lambda$ is called the coefficient ring, while in this paper we are mostly interested in the case where it is $\mathbb{Q}$. The category $\text{DM}(k, \mathbb{Q})$ has nearly all the expected properties of the derived category of the conjectured abelian category of mixed motives defined over $k$ $\left[48\right]$. In this section, we will only list some properties of $\text{DM}(k, \Lambda)$ and use them as a blackbox, while leave the construction of $\text{DM}(k, \Lambda)$ and the proofs to these properties to the excellent references $\left[49, 62\right]$. Meanwhile the first section of $\left[2\right]$ is also very helpful.

1. The category $\text{DM}(k, \Lambda)$ is a rigid tensor triangulated category that contains pure Tate motives $\Lambda(n)$, $n \in \mathbb{Z}$. $\Lambda(0)$ is a unit object of it and $\Lambda(-1)$ is the dual of $\Lambda(1)$,

$$\Lambda(-1) = \text{Hom}(\Lambda(1), \Lambda(0))$$

where $\text{Hom}$ is the internal Hom operator. The Tate object $\Lambda(n)$ satisfies

$$\Lambda(n) = \begin{cases} 
\Lambda(1)^{\otimes n} & \text{if } n \geq 0 \\
\Lambda(-1)^{\otimes n} & \text{if } n < 0 
\end{cases}$$

(1.2)

For an object $\mathcal{N}$ of $\text{DM}(k, \Lambda)$, its Tate twist $\mathcal{N}(n)$ is defined as $\mathcal{N} \otimes \Lambda(n)$, while its dual $\mathcal{N}^{\vee}$ is defined as $\text{Hom}(\mathcal{N}, \Lambda(0))$.

2. There exists a contravariant functor from the category of non-singular projective varieties over $k$ to the category $\text{DM}(k, \Lambda)$

$$M_{gm} : \text{SmProj}/k^{op} \rightarrow \text{DM}(k, \Lambda)$$

(1.3)

which sends a non-singular projective variety $X$ to a constructible object of $\text{DM}(k, \Lambda)$ such that fibered product in $\text{SmProj}/k$ is sent to tensor product in $\text{DM}(k, \Lambda)$, i.e.

$$M_{gm}(X \times_k Y) = M_{gm}(X) \otimes M_{gm}(Y)$$

(1.4)
The definition of constructibility could be found in [2], and the full triangulated subcategory of $DM(k, \Lambda)$ consists of constructible objects will be denoted by $DM_{gm}(k, \Lambda)$, which is the smallest full pseudoabelian triangulated subcategory of $DM(k, \Lambda)$ that contains the image of $M_{gm}$ and is also closed under Tate twists.

3. When the field $k$ admits an embedding into $\mathbb{C}$, say $\sigma : k \to \mathbb{C}$, there exists a Hodge realisation functor, $R_{\sigma} : DM_{gm}(k, \mathbb{Q}) \to D^b(MHS_\mathbb{Q})$ (1.5) such that for every non-singular projective variety $X$, $R_{\sigma}(M_{gm}(X))$ is a complex in $D^b(MHS_\mathbb{Q})$ whose cohomology computes the singular cohomology of $X(\mathbb{C})$ with the natural rational MHS [38, 39, 53]. In this paper we are mostly interested in the case where $k = \mathbb{Q}$, since there is only one embedding of $\mathbb{Q}$ into $\mathbb{C}$, let’s denote the Hodge realisation functor by $R$ for simplicity.

4. The composition of $R_{\sigma}$ with the forgetful functor from $D^b(MHS_\mathbb{Q})$ to the derived category of rational vector spaces $D^b(Vec_\mathbb{Q})$ is (up to an equivalence) the Betti realisation functor $R_{Betti}$ [4], $R_{Betti} : DM_{gm}(k, \mathbb{Q}) \to D^b(Vec_\mathbb{Q})$ (1.6)

1.2. Mixed Tate Motives

We now briefly talk about the abelian category of mixed Tate motives $TM_{\mathbb{Q}}$, while the readers are referred to the paper [46] for details. Let $K_i(k)$ be the $i$-th algebraic $K$-group of the field $k$ [27], there exists a family of Adams operators $\{\psi^l\}_{l \geq 1}$ which act on $K_i(k)$ as group homomorphisms [27]. These Adams operators induce linear maps on the rational vector space $K_i(k) \otimes_\mathbb{Z} \mathbb{Q}$, which induce a decomposition

$$K_i(k) \otimes_\mathbb{Z} \mathbb{Q} = \bigoplus_{j \geq 0} K_i(k)^{(j)}$$

(1.7)

where the eigenspace $K_i(k)^{(j)}$ is defined as,

$$K_i(k)^{(j)} := \{x \in K_i(k) \otimes_\mathbb{Z} \mathbb{Q} : \psi^l(x) = l^j x, \forall l \geq 1\}$$

(1.8)

The strong version of Beilinson and Soulé’s vanishing conjecture claims that [46],

**Conjecture BS** $K_{2q-p}(k)^{(q)} = 0$ if $p \leq 0$ and $q > 0$.

When the field $k$ is $\mathbb{Q}$, Conjecture BS has been proved [24]. Let $DTM_{\mathbb{Q}}$ be the full triangulated subcategory of $DM_{gm}(\mathbb{Q}, \mathbb{Q})$ generated by Tate objects $\mathbb{Q}(n), n \in \mathbb{Z}$, from [46] there exists a motivic $t$-structure on $DTM_{\mathbb{Q}}$ whose heart is defined to be the category of mixed Tate motives $TM_{\mathbb{Q}}$. For two objects $A$ and $B$ of $TM_{\mathbb{Q}}$, an extension of $B$ by $A$ is a short exact sequence,

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

(1.9)
Morphisms between two extensions are given by commutative diagrams of the form,

\[ \begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A' \\
\end{array} \]
\[ \begin{array}{ccc}
& \\ & E \\
& \downarrow \\
& E' \\
\downarrow & & \downarrow \\
& B' \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
0 & \longrightarrow & B \\
\end{array} \]  
(1.10)

The extension \([1.9]\) is said to split if it is isomorphic to the trivial extension,

\[ 0 \longrightarrow A \overset{i}{\longrightarrow} A \oplus B \overset{j}{\longrightarrow} B \longrightarrow 0 \]  
(1.11)

where \(i\) is the natural inclusion and \(j\) is the natural projection. The set of isomorphism classes of extensions of \(B\) by \(A\), denoted by \(\text{Ext}_1^{TM}(B, A)\), has a group structure that is induced by Baer summation, whose zero element is the trivial extension \([1.11]\). The extensions of \(\mathbb{Q}(0)\) by \(\mathbb{Q}(n), n \geq 3\) could be described explicitly by Corollary 4.3 of [46].

**Lemma 1.1.** There exists an isomorphism \(\tau_{n,1}\)

\[ \tau_{n,1} : \text{Ext}_1^{TM}(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow K_{2n-1}(\mathbb{Q})^{(n)}, n \geq 2 \]  
(1.12)

The rank of the higher algebraic \(K\)-group \(K_{2n-1}(\mathbb{Q})\) is well-known \([29]\),

\[ \text{rank } K_{2n-1}(\mathbb{Q}) = \begin{cases} 
0 & \text{if } n = 2k, k \geq 1 \\
1 & \text{if } n = 2k + 1, k \geq 1 
\end{cases} \]  
(1.13)

which yields,

\[ K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} 
0 & \text{if } n = 2k, k \geq 1 \\
\mathbb{Q} & \text{if } n = 2k + 1, k \geq 1 
\end{cases} \]  
(1.14)

As \(K_{2n-1}(\mathbb{Q})^{(n)}\) is a linear subspace of \(K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}\), this immediately implies that

\[ K_{2n-1}(\mathbb{Q})^{(n)} = 0, \text{ if } n = 2k, k \geq 1 \]  
(1.15)

When \(n = 2k + 1, k \geq 1\), as a linear subspace of \(K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}\), \(K_{2n-1}(\mathbb{Q})^{(n)}\) is either 0 or \(\mathbb{Q}\). From \([20, 24, 37]\), there exists a nontrivial extension of \(\mathbb{Q}(0)\) by \(\mathbb{Q}(n)\) when \(n = 2k + 1, k \geq 1\), hence \(\text{Ext}_1^{TM}(\mathbb{Q}(0), \mathbb{Q}(n))\) is nonzero,

\[ \text{Ext}_1^{TM}(\mathbb{Q}(0), \mathbb{Q}(n)) \simeq K_{2n-1}(\mathbb{Q})^{(n)} = \mathbb{Q}, \text{ if } n = 2k + 1, k \geq 1 \]  
(1.16)

The restriction of the Hodge realisation functor \(\mathcal{R}\) to \(TM_{\mathbb{Q}}\) is a functor whose image essentially lies in the full abelian subcategory \(MHT_{\mathbb{Q}}\) that consists of mixed Hodge-Tate structures, i.e. those mixed Hodge structures whose semi-simplifications are direct sums of Tate objects \(\mathbb{Q}(n)\).

\[ \mathcal{R} : TM_{\mathbb{Q}} \rightarrow MHT_{\mathbb{Q}} \]  
(1.17)

From \([24]\), the restriction of \(\mathcal{R}\) to \(TM_{\mathbb{Q}}\) is exact and full-faithful, hence it induces an injective homomorphism from \(\text{Ext}_1^{TM}(\mathbb{Q}(0), \mathbb{Q}(n))\) to \(\text{Ext}_1^{MHT}(\mathbb{Q}(0), \mathbb{Q}(n))\). Since \(MHT_{\mathbb{Q}}\) is a full abelian subcategory of \(MHS_{\mathbb{Q}}\), the isomorphism \([A.13]\) immediately implies,

\[ \text{Ext}_1^{MHT}(\mathbb{Q}(0), \mathbb{Q}(n)) = \text{Ext}_1^{MHS}(\mathbb{Q}(0), \mathbb{Q}(n)) \simeq C/(2 \pi i)^n \mathbb{Q} \]  
(1.18)
Lemma 1.2. When \( n \geq 3 \), the image of \( \text{Ext}^1_{TM_q}(\mathbb{Q}(0), \mathbb{Q}(n)) \) in \( \mathbb{C}/(2 \pi i)^n \mathbb{Q} \) under Hodge realisation is the subgroup of \( \mathbb{C}/(2 \pi i)^n \mathbb{Q} \) consists of elements which are the cosets of rational multiples of \( \zeta(n) \).

Proof. When \( n = 2k, k \geq 2 \), \( \zeta(n) \) is a rational multiple of \( (2 \pi i)^n \), therefore the coset of \( \zeta(n) \) in \( \mathbb{C}/(2 \pi i)^n \mathbb{Q} \) is 0, hence this lemma is a direct result of Lemma 1.1.

When \( n = 2k + 1, k \geq 1 \), from [20, 24, 37], there exists a mixed Tate motive that is a nontrivial extension of \( \mathbb{Q}(0) \) by \( \mathbb{Q}(n) \), whose Hodge realisation in \( \text{Ext}^1_{\text{MHT}_q}(\mathbb{Q}(0), \mathbb{Q}(n)) \) is the coset of a nonzero rational multiple of \( \zeta(n) \). As the coset of \( \zeta(n) \) in \( \mathbb{C}/(2 \pi i)^n \mathbb{Q} \) is nonzero, this lemma is a direct result of Lemma 1.1. \qed

Remark 1.3. Notice that Conjecture GHP follows immediately from Conjecture GH and the computations in this section.

2. Limit Mixed Hodge Structure

In this section, we will review Steenbrink’s and Schmid’s constructions of limit mixed Hodge structures on a singular fiber of an algebraic fibration [54, 59]. Let \( X \) be a quasi-projective variety over \( \mathbb{Q} \) which admits a fibration over a smooth algebraic curve \( S \) over \( \mathbb{Q} \) by a projective morphism \( \pi_\mathbb{Q} \),

\[
\pi_\mathbb{Q} : X \to S
\]  

whose only singular fiber \( Y \) is over a rational point \( 0 \in S \),

\[
Y := \pi_\mathbb{Q}^{-1}(0)
\]

Let’s denote the dimension of the fibers of \( \pi_\mathbb{Q} \) by \( n \). We will assume the singular fiber \( Y \) is reduced with nonsingular components crossing normally (check the semi-stable reduction theorem in Chapter II of [45]). For simplicity now define

\[
X^* := X \setminus Y, \quad S^* := S \setminus \{0\}
\]

By abuse of notation, the restriction of \( \pi_\mathbb{Q} \) to \( X^* \) will also be denoted by \( \pi_\mathbb{Q} \),

\[
\pi_\mathbb{Q} : X^* \to S^*
\]

which is a smooth fibration between smooth varieties. If extend the field to \( \mathbb{C} \), we obtain a smooth fibration \( \pi_\mathbb{C} \) between smooth varieties over \( \mathbb{C} \),

\[
\pi_\mathbb{C} : X^*_\mathbb{C} \to S^*_\mathbb{C}
\]

where \( X^*_\mathbb{C} := X^* \times_{\text{Spec} \mathbb{Q}} \text{Spec} \mathbb{C} \), etc. The analytification of \( \pi_\mathbb{C} \) is a smooth fibration between complex manifolds [61],

\[
\pi^\text{an}_\mathbb{C} : X^*_{\mathbb{C}^\text{an}} \to S^*_{\mathbb{C}^\text{an}}
\]
Since 0 is a smooth point of $S$, the local ring $\mathcal{O}_{S,0}$ is a discrete valuation ring (DVR) \[61\]. Let $\varphi$ be a uniformizer of $\mathcal{O}_{S,0}$, then there exists an open affine neighbourhood $U$ of 0 such that $\varphi$ defines a coordinate on $U^\text{an}_C$. Replace $S$ by $U$ if necessary we will assume $S$ is affine and $\varphi$ defines a coordinate on $S^\text{an}_C$. Now choose a small neighbourhood $\Delta$ of 0 in $S^\text{an}_C$,

$$
\Delta = \{ \varphi \in \mathbb{C} : |\varphi| < \epsilon, \ 0 < \epsilon \leq 1 \}
$$

(2.7)

**Remark 2.1.** In this paper, the choices of a uniformizer $\varphi$, a universal cover $\tilde{\Delta}^* \text{ of } \Delta^*$ and a multi-valued holomorphic function $\log \varphi$ on $\Delta^*$ (which induces a holomorphic function on $\tilde{\Delta}^*$) will be fixed.

The restriction of $\pi^\text{an}_C$ to $\mathcal{X} := (\pi^\text{an}_C)^{-1}(\Delta)$ will be denoted by,

$$
\pi_\Delta : \mathcal{X} \to \Delta
$$

(2.8)

whose only singular fiber $\mathcal{X}_0 := \pi^{-1}_\Delta(0)$ is the analytification of $Y_C$. The restriction of $\pi_\Delta$ to $\mathcal{X}^* := \mathcal{X} \setminus \mathcal{X}_0$ will be denoted by $\pi_{\Delta^*}$,

$$
\pi_{\Delta^*} : \mathcal{X}^* \to \Delta^*
$$

(2.9)

### 2.1. Gauss-Manin Connection

The relative de Rham cohomology sheaf of $\pi^\text{an}_Q$\[2.4\] is defined to be \[1\]

$$
\mathcal{V}_Q := \mathbb{R}^q \pi_{Q,*}(\Omega_{\mathcal{X}^*/S^*})
$$

(2.10)

where $\Omega_{\mathcal{X}^*/S^*}$ is the complex of sheaves of relative differentials \[1,44\]. The complex $\Omega_{\mathcal{X}^*/S^*}$ is naively filtered by the following complexes,

$$
\Omega_{\mathcal{X}^*/S^*}^* : 0 \to \mathcal{O}_{\mathcal{X}^*/S^*} \xrightarrow{d} \Omega_{\mathcal{X}^*/S^*}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathcal{X}^*/S^*}^n \to 0
$$

(2.11)

Since $\pi_Q$ is a smooth projective fibration between smooth varieties, $\mathcal{V}_Q$ is a locally free sheaf over $S^* \[34,44\]$. The fiber of $\mathcal{V}_Q$ over a closed point $\varphi \in S^*$ is the $q$-th algebraic de Rham cohomology $\mathbb{H}^q(X_{\varphi}, \Omega_{X_{\varphi}}^*)$ of $X_{\varphi}$, where $X_{\varphi}$ is considered as a variety defined over the residue field $\kappa(\varphi) \[44,59\]$. The complex $\Omega_{\mathcal{X}^*/S^*}$ is naively filtered by the following complexes,

$$
F^p \Omega_{\mathcal{X}^*/S^*}^* : 0 \to \cdots \to 0 \to \Omega_{\mathcal{X}^*/S^*}^p \xrightarrow{d} \Omega_{\mathcal{X}^*/S^*}^{p+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathcal{X}^*/S^*}^n \to 0
$$

(2.12)

which induces a locally free subsheaf filtration $\mathcal{F}_Q^p$ of $\mathcal{V}_Q$,

$$
\mathcal{F}_Q^p := \text{Im} \left( \mathbb{R}^q \pi_{Q,*}(F^p \Omega_{\mathcal{X}^*/S^*}) \to \mathbb{R}^q \pi_{Q,*}(\Omega_{\mathcal{X}^*/S^*}) \right)
$$

(2.13)

The complexification of the complex \[2.11\] is,

$$
\Omega_{\mathcal{X}^*/S^C}^* : 0 \to \mathcal{O}_{\mathcal{X}^*/S^C} \xrightarrow{d} \Omega_{\mathcal{X}^*/S^C}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{\mathcal{X}^*/S^C}^n \to 0
$$

(2.14)
which is naively filtered by the complexification of the complex \(2.12\)
\[
F^p \Omega^*_X/S^*_C : 0 \to \cdots \to 0 \to \Omega^p_{X/S^*_C} \to \Omega^{p+1}_{X/S^*_C} \to \cdots \to \Omega^n_{X/S^*_C} \to 0 \tag{2.15}
\]
The sheaf \(\mathcal{V}_C\) and its subsheaf filtration \(\mathcal{F}^p_C\) are defined to be,
\[
\mathcal{V}_C := \mathbb{R}^q \pi_{C,*} \left( \Omega^*_X/S^*_C \right), \quad \mathcal{F}^p_C := \mathbb{R}^q \pi_{C,*} \left( F^p \Omega^*_X/S^*_C \right)
\]
which are just the complexifications of \(\mathcal{V}_Q\) and \(\mathcal{F}_Q^p\) respectively. The analytification of the complex \(2.14\) is,
\[
\Omega^*_X/\Delta^* : 0 \to \mathcal{O}_{X/\Delta^*/S^*_C} \to \Omega^1_{X/\Delta^*/S^*_C} \to \cdots \to\]
\[
\Omega^n_{X/\Delta^*/S^*_C} \to 0 \tag{2.17}
\]
which has a naive filtration induced by the analytification of the complex \(2.15\)
\[
F^p \Omega^*_X/\Delta^*/S^*_C : 0 \to \cdots \to 0 \to \Omega^p_{X/\Delta^*/S^*_C} \to \Omega^{p+1}_{X/\Delta^*/S^*_C} \to \cdots \to \Omega^n_{X/\Delta^*/S^*_C} \to 0 \tag{2.18}
\]
The sheaf \(\mathcal{V}^\text{an}_C\) and its subsheaf filtration \(\mathcal{F}^p\text{an}_C\) are defined to be,
\[
\mathcal{V}^\text{an}_C := \mathbb{R}^q \pi_{C,*} \left( \Omega^*_X/\Delta^*/S^*_C \right), \quad \mathcal{F}^p\text{an}_C := \mathbb{R}^q \pi_{C,*} \left( F^p \Omega^*_X/\Delta^*/S^*_C \right)
\]
which are just the analytifications of \(\mathcal{V}_C\) and \(\mathcal{F}_C^p\) respectively. We now introduce the Gauss-Manin connection and we will follow the papers \([44, 63]\) closely, the results in which will be cited freely. The Gauss-Manin connection \(\nabla_Q\) of \(\mathcal{V}_Q\) is an integrable algebraic connection,
\[
\nabla_Q : \mathcal{V}_Q \to \Omega^1_{S^*/Q} \otimes_{\mathcal{O}_{S^*}} \mathcal{V}_Q \tag{2.20}
\]
that satisfies Griffiths transversality,
\[
\nabla_Q(\mathcal{F}_Q^p) \subset \Omega^1_{S^*/Q} \otimes_{\mathcal{O}_{S^*}} \mathcal{F}_Q^{p-1} \tag{2.21}
\]
The connection \(\nabla_Q\) could be canonically extended to the Gauss-Manin connection \(\nabla_C\) of \(\mathcal{V}_C\) by complexification and to the Gauss-Manin connection \(\nabla^\text{an}_C\) of \(\mathcal{V}^\text{an}_C\) by analytification. Over the punctured disc \(\Delta^*\), \(\pi_{\Delta^*}\) defines a local system \(\mathcal{F}_Q^p\text{an}\),
\[
V_Z := \mathbb{R}^q \pi_{\Delta^*,*} \mathbb{Z} \tag{2.22}
\]
whose fiber over a point \(\varphi \in \Delta^*\) is the singular cohomology group \(H^q(\mathcal{X}_\varphi, \mathbb{Z})\) (modulo torsions) \([54, 63]\).

**Remark 2.2.** In this paper torsions of singular homology and cohomology groups are irrelevant, and they could be ignored completely.

The dual of \(V_Z\), denoted by \(V_Z^\vee\), is a local system over \(\Delta^*\) whose fiber over \(\varphi \in \Delta^*\) is the singular homology group \(H_q(\mathcal{X}_\varphi, \mathbb{Z})\) (modulo torsions). Similarly let \(V_Q\) (\(V_C\)) be the local system whose fiber at \(\varphi\) is \(H^q(\mathcal{X}_\varphi, \mathbb{Q})\) (\(H^q(\mathcal{X}_\varphi, \mathbb{C})\)),
\[
V_Q := \mathbb{R}^q \pi_{\Delta^*,*} \mathbb{Q}, \quad V_C := \mathbb{R}^q \pi_{\Delta^*,*} \mathbb{C} \tag{2.23}
\]
which are just \[63\],
\[
V_Q = V_Z \otimes \mathbb{Q}, \quad V_C = V_Z \otimes \mathbb{C}
\]  
(2.24)
The local system \(V_Z\) defines a locally free sheaf \(\mathcal{V}\) over \(\Delta^*\),
\[
\mathcal{V} := V_Z \otimes \mathcal{O}_{\Delta^*}
\]  
(2.25)
with dual \(\mathcal{V}^{'}.\) For every \(\varphi \in \Delta^*\), \(\mathcal{X}_\varphi\) is a projective complex manifold, hence \(H^q(\mathcal{X}_\varphi, \mathbb{C})\) has a Hodge decomposition \[54\],
\[
H^q(\mathcal{X}_\varphi, \mathbb{C}) = \bigoplus_{0 \leq k \leq q} H^{k,q-k}(\mathcal{X}_\varphi)
\]  
(2.26)
which induces a Hodge filtration of \(H^q(\mathcal{X}_\varphi, \mathbb{C})\) through,
\[
F^p_\varphi := \bigoplus_{k \geq p} H^{k,q-k}(\mathcal{X}_\varphi)
\]  
(2.27)
The complex vector space \(F^p_\varphi\) varies holomorphically with respect to \(\varphi\), the union of which forms a holomorphic vector bundle over \(\Delta^*\) whose sheaf of sections is a locally free sheaf \(\mathcal{F}^p\) that induces a subsheaf filtration of \(\mathcal{V}\) \[54\]. From the standard comparison isomorphism, there exist the following canonical isomorphisms \[60\],
\[
I_B : \mathcal{V}^{an}_{\Delta^*} \sim \rightarrow \mathcal{V}
\]
\[
I_B : \mathcal{F}^{p,an}_{\Delta^*} \sim \rightarrow \mathcal{F}^p
\]  
(2.28)
On the sheaf \(\mathcal{V}\), the Gauss-Manin connection \(\nabla\) is the unique connection such that the local sections of \(V_C\) are flat \[54\],
\[
\nabla : \mathcal{V} \rightarrow \Omega^1_{\Delta^*} \otimes \mathcal{O}_{\Delta^*}, \mathcal{V}
\]  
(2.29)
and it also satisfies Griffiths transversality \[54, 63\],
\[
\nabla \mathcal{F}^p \subset \Omega^1_{\Delta^*} \otimes \mathcal{O}_{\Delta^*}, \mathcal{F}^{p-1}
\]  
(2.30)
In fact, the comparison isomorphism \(I_B\) \[2.28\] sends the connection \(\nabla^{an}_{\Delta^*}\) (restricted to \(\mathcal{V}^{an}_{\Delta^*}\) to the connection \(\nabla\) \[21, 22, 36, 52\].

### 2.2. Canonical Extension

Let \(\Omega^*_{X/S}(\log Y)\) be the complex of sheaves of relative algebraic forms over \(S\) with at worst logarithmic poles along the normal crossing divisor \(Y\), similarly it has a naive filtration \(\mathcal{F}^p\Omega^*_{X/S}(\log Y)\) defined similarly as before \[2.12, 59\]. On the curve \(S\), the sheaf \(\tilde{\mathcal{V}}_Q\) and its filtration \(\tilde{\mathcal{F}}^p_\tilde{Q}\) are defined to be
\[
\tilde{\mathcal{V}}_Q := \mathbb{R}^q \pi_{Q,*}(\Omega^*_{X/S}(\log Y))
\]
\[
\tilde{\mathcal{F}}^p_\tilde{Q} := \mathbb{R}^q \pi_{Q,*}(F^p \Omega^*_{X/S}(\log Y))
\]  
(2.31)
which are canonical extensions of $\mathcal{V}$ and $\mathcal{F}_Q^p$. The Gauss-Manin connection $\nabla_Q$ of $\mathcal{V}$ could be canonically extended to a connection $\tilde{\nabla}_Q$ of $\tilde{\mathcal{V}}$ which has a logarithmic pole along the rational point 0 with a nilpotent residue, on the other hand this property also determines the extension of $\mathcal{V}$ and $\mathcal{F}_Q^p$ uniquely \[21, 54\]. The extensions of $\mathcal{V}_C$, $\mathcal{V}_an$, $\mathcal{F}_C^p$ and $\mathcal{F}_C^{p,an}$ could be obtained as the complexifications and analytifications of $\tilde{\mathcal{V}}_Q$ and $\tilde{\mathcal{F}}_Q^p$, which will be denoted by $\tilde{\mathcal{V}}_C$, $\tilde{\mathcal{V}}_an$, $\tilde{\mathcal{F}}_C^p$ and $\tilde{\mathcal{F}}_C^{p,an}$ respectively. Over the disc $\Delta$, $\tilde{\mathcal{V}}_C^{an}$ and $\tilde{\mathcal{F}}_C^{p,an}$ could also be described as $\mathcal{V}_an$ and $\mathcal{F}_C^{p}$.

Under the isomorphism 2.28, $\tilde{\mathcal{V}}_C^{an}|_\Delta$ and $\tilde{\mathcal{F}}_C^{p,an}|_\Delta$ induce extensions of $\mathcal{V}$ and $\mathcal{F}_p$ that will be denoted by $\tilde{V}$ and $\tilde{F}_p$ respectively, which could also be constructed explicitly by the method of Deligne’s canonical extension $\mathcal{21, 36, 54}$. The fiber $\tilde{\mathcal{V}}_{Q|0}$ is a rational vector space that could be described as \[59\],

$$
\tilde{\mathcal{V}}_{Q|0} := (\tilde{\mathcal{V}}_{Q})_0 \otimes_{\mathcal{O}_{S,0}} \mathcal{O}_{S,0}/\mathfrak{m}_{S,0} = H^q(Y, \Omega^*_{X/S}(\log Y)|_Y)
$$

(2.33)
i.e. the hypercohomology of the complex of sheaves obtained from the restriction of $\Omega^*_{X/S}(\log Y)$ to $Y$. By Serre’s GAGA, there are canonical isomorphisms,

$$
\tilde{\mathcal{V}}_C^{an}|_{\mathcal{0}} = \tilde{\mathcal{V}}_{C|\mathcal{0}} = \tilde{\mathcal{V}}_{Q|0} \otimes_{\mathcal{Q}} \mathbb{C}
$$

(2.34)
The fiber $\tilde{\mathcal{V}}_C^{an}|_{\mathcal{0}}$ could also be described as \[52, 59\],

$$
\tilde{\mathcal{V}}_C^{an}|_{\mathcal{0}} := \mathbb{H}^q(\mathcal{X}_0, \Omega^*_{X/\Delta}(\log \mathcal{X}_0)|_{\mathcal{X}_0})
$$

(2.35)
For a closed point $\varphi$ of $S^*$, the fiber $\tilde{\mathcal{V}}_{Q|\varphi}$ is a vector space over $\kappa(\varphi)$, which is \[44, 52\],

$$
\tilde{\mathcal{V}}_{Q|\varphi} = \mathbb{H}^q(Y, \Omega^*_{X/S}(\log Y)|_{X_\varphi}) = H^q_{dR}(X_\varphi)
$$

(2.36)
where $X_\varphi$ is considered as a variety defined over the residue field $\kappa(\varphi)$ of $\varphi$. While in the analytic case, for a point $\varphi$ of $\Delta^*$, the fiber $\tilde{\mathcal{V}}_C^{an}|_{\varphi}$ is \[52, 59\],

$$
\tilde{\mathcal{V}}_C^{an}|_{\varphi} = \mathbb{H}^q(X_\varphi, \Omega^*_{X/\Delta}(\log \mathcal{X}_0)|_{X_\varphi}) = H^q_{dR}(X_\varphi^{an}) \simeq H^q(X_\varphi, \mathbb{C})
$$

(2.37)
where we have used comparison isomorphism $I_B$ 2.28 in the last isomorphism.

The local system $V_C$ over $\Delta^*$ is uniquely determined by a representation of the fundamental group $\pi_1(\Delta^*, \varphi_0)$ \[63\],

$$
\Psi : \pi_1(\Delta^*, \varphi_0) \to \text{GL}(H^q(X_{\varphi_0}, \mathbb{C}))
$$

(2.38)
which gives the monodromy of $V_C$ around 0 \[54\]. The fundamental group $\pi_1(\Delta^*, \varphi_0)$ is isomorphic to $\mathbb{Z}$, and let’s choose a generator $T$, then this representation is determined by the action of $T$ on $H^q(X_{\varphi_0}, \mathbb{C})$. The action of $T$ on $H^q(X_{\varphi_0}, \mathbb{C})$ could be extended to
an automorphism of the sheaf $\tilde{\mathcal{V}} (\tilde{\mathcal{V}}^\text{an}_{\Delta})$, which induces an automorphism $T_0$ of the fiber $\tilde{\mathcal{V}}|_0 (\tilde{\mathcal{V}}^\text{an}|_0)$, for details see Proposition 11.2 of [52]. On the other hand, there exists an endomorphism $N_0$ of $\tilde{\mathcal{V}}^\text{an}|_0$ which is defined as follows [59]. Let $\text{Res}_0$ be the residue map from $\Omega^1_S(\log 0)$ to $\mathbb{C}$ defined to be,

$$\text{Res}_0 (g(t) \, dt/t) := g(0) \quad (2.39)$$

Then the following homomorphism from the germ $(\tilde{\mathcal{V}}^\text{an})_0$ to the fiber $\tilde{\mathcal{V}}^\text{an}|_0$ vanishes on the ideal $(\tilde{\mathcal{V}}^\text{an})_0 \otimes_{\mathcal{O}_{S,0}} m_{S,0}$,

$$(\text{Res}_0 \otimes (\otimes_{\mathcal{O}_{S,0}} \mathcal{O}_{S,0}/m_{S,0})) \circ \nabla^\text{an}_C : (\tilde{\mathcal{V}}^\text{an})_0 \to \tilde{\mathcal{V}}^\text{an}|_0 \quad (2.40)$$

hence it induces an endomorphism $N_0$ of $\tilde{\mathcal{V}}^\text{an}|_0$, which is called the residue of $\nabla^\text{an}_C$ at 0. From Theorem II 3.11 of [22] or Corollary 11.17 of [52] we have,

$$T_0 = \exp(-2 \pi i N_0) \quad (2.41)$$

Corollary 11.19 of [52] tells us that all the eigenvalues of $N_0$ are integers, therefore all the eigenvalues of $T_0$ are 1, which immediately implies that $T_0$ is unipotent.

### 2.3. Limit Mixed Hodge Structure

We now briefly talk about Steenbrink’s construction of limit mixed Hodge structures, and the readers are referred to [40, 59] for more complete treatments. After we fix a universal cover of $\Delta^*$, the nearby cycle sheaf $R\Psi_{\pi_{\Delta}}(\Lambda)$, where $\Lambda$ is $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$, could be constructed, which is a complex of sheaves over the singular fiber $\mathcal{X}_0$ [40, 59]. Steenbrink constructs the following data in [59],

1. A representative of $R\Psi_{\pi_{\Delta}}\mathbb{Z}$ in the derived category $D^+(\mathcal{X}_0, \mathbb{Z})$.

2. A representative of $(R\Psi_{\pi_{\Delta}}\mathbb{Q}, W_*)$ in the filtered derived category $D^+F(\mathcal{X}_0, \mathbb{Q})$, where $W_*$ is an increasing filtration of $R\Psi_{\pi_{\Delta}}\mathbb{Q}$ and

$$R\Psi_{\pi_{\Delta}}\mathbb{Q} \simeq R\Psi_{\pi_{\Delta}}\mathbb{Z} \otimes \mathbb{Q}, \text{ in } D^+(\mathcal{X}_0, \mathbb{Q}) \quad (2.42)$$

3. A representative of $(R\Psi_{\pi_{\Delta}}\mathbb{C}, W_*, F^*)$ in the bifiltered derived category $D^+F_2(\mathcal{X}_0, \mathbb{C})$ where $W_*$ is an increasing filtration of $R\Psi_{\pi_{\Delta}}\mathbb{C}$ and $F^*$ is a decreasing filtration of $R\Psi_{\pi_{\Delta}}\mathbb{C}$ such that,

$$(R\Psi_{\pi_{\Delta}}\mathbb{C}, W_*) \simeq (R\Psi_{\pi_{\Delta}}\mathbb{Q} \otimes \mathbb{C}, W_*) \text{ in } D^+F(\mathcal{X}_0, \mathbb{C}) \quad (2.43)$$

These constructions depend on the choice of $\varphi$ and $\log \varphi$ on $\Delta^*$ [40, 59], both of which have been fixed in this paper.
Theorem 2.3. The data

\[(R\Psi_{\pi_\Delta}Z, (R\Psi_{\pi_\Delta}Q, W_*), (R\Psi_{\pi_\Delta}C, W_*, F^*))\] (2.44)

forms a cohomological mixed Hodge complex of sheaves in the sense of Deligne [23].

Proof. Chapter 11 of [52].

Let’s denote the triangulated category of $\mathbb{Z}$-mixed Hodge complexes by $D^+_\text{MHS}_\mathbb{Z}$, where $*$ is boundedness condition, e.g. $*$ could be $\emptyset, +, - \text{ or } b$. From Proposition 8.1.7 of [23], the mixed Hodge complex of sheaves (2.44) induces an object of $D^+_\text{MHS}_\mathbb{Z}$,

\[(R\Gamma(R\Psi_{\pi_\Delta}Z), R\Gamma(R\Psi_{\pi_\Delta}Q, W_*), R\Gamma(R\Psi_{\pi_\Delta}C, W_*, F^*))\] (2.45)

From [11], for every $q \in \mathbb{Z}$ there exists a cohomological functor $H^q$ from $D^+_\text{MHS}_\mathbb{Z}$ to the abelian category $\text{MHS}_\mathbb{Z}$,

\[H^q: D^+_\text{MHS}_\mathbb{Z} \to \text{MHS}_\mathbb{Z}\] (2.46)

which sends the mixed Hodge complex (2.45) to the MHS, (2.47)

The MHS (2.47) is also expressed as,

\[(H^q(X_0, R\Psi_{\pi_\Delta}Z), (H^q(X_0, R\Psi_{\pi_\Delta}Q), W_*), (H^q(X_0, R\Psi_{\pi_\Delta}C), W_*, F^*))\] (2.48)

Steenbrink proves the following important proposition in [59],

Proposition 2.4. There exists a quasi-isomorphism between the complex of sheaves $R\Psi_{\pi_\Delta}C$ and $\Omega^*_{X/\Delta}(\log X_0)|_{X_0}$ in the derived category $D^+(\mathcal{X}_0, \mathbb{C})$, which depends on the choice of $\varphi$ and $\log \varphi$.

Proof. Chapter 11 of [52].

Hence we have,

\[\tilde{\mathcal{V}}^\text{an}_{\mathbb{C}}|_0 = H^q(X_0, \Omega^*_{X/\Delta}(\log X_0)|_{X_0}) \simeq H^q(X_0, R\Psi_{\pi_\Delta}(\mathbb{C}))\] (2.49)

therefore the MHS (2.47) is called limit mixed Hodge structure [52]. Steenbrink constructs an endomorphism $\nu_N$ of $R\Psi_{\pi_\Delta}C$ in [59], which induces the endomorphism $N_0$ after taking hypercohomology. The endomorphism $\nu_N$ also induces a morphism

\[\nu_N: R\Psi_{\pi_\Delta}Q \to R\Psi_{\pi_\Delta}Q(-1)\] (2.50)

where $(-1)$ is Tate twist in $D^+(\mathcal{X}_0, \mathbb{Q})$. After taking hypercohomology, the morphism (2.50) induces the following important homomorphism,

\[N_0: \mathbb{H}^n(X_0, R\Psi_{\pi_\Delta}(\mathbb{Q})) \to \mathbb{H}^n(X_0, R\Psi_{\pi_\Delta}(\mathbb{Q}))(-1)\] (2.51)

which actually determines the weight filtration $W_*$ in (2.48) uniquely [40, 59, 60]. From [50, 60], the Hodge filtration $F^*$ in (2.48) is also the filtration on $\tilde{\mathcal{V}}^\text{an}_{\mathbb{C}}|_0$ given by the fiber $\tilde{\mathcal{F}}^p\text{an}_{\mathbb{C}}|_0$,

\[F^p = \tilde{\mathcal{F}}^p\text{an}_{\mathbb{C}}|_0 = \tilde{\mathcal{F}}^p|_0 = \tilde{\mathcal{F}}^p|_0 \otimes \mathbb{Q} \mathbb{C}\] (2.52)
2.4. Limit Mixed Hodge Complex

The \( \mathbb{Z} \)-mixed Hodge complex \( 2.45 \) naturally induces a \( \mathbb{Q} \)-mixed Hodge complex \( Y^\bullet \) by forgetting its integral structure,

\[
Y^\bullet := (R\Gamma(R\Psi_{\pi\Delta} \mathbb{Q}), R\Gamma(R\Psi_{\pi\Delta} \mathbb{Q}, W_*), R\Gamma(R\Psi_{\pi\Delta} \mathbb{C}, W_*), F^\bullet) \tag{2.53}
\]

whose hypercohomology is the underlying rational MHS of \( 2.48 \).

\[
\mathcal{H}^q(Y^\bullet) = \left( \mathcal{H}^q(X_0, R\Psi_{\pi\Delta} \mathbb{Q}), \mathcal{H}^q(X_0, R\Psi_{\pi\Delta} \mathbb{Q}, W_*), \mathcal{H}^q(X_0, R\Psi_{\pi\Delta} \mathbb{C}, W_*), F^\bullet \right) \tag{2.54}
\]

Now we need the following lemma from [35].

**Lemma 2.5.** \( \mathcal{H}^q(X_0, R\Psi_{\pi\Delta} \Lambda) \) is isomorphic to \( H^q(X_\varphi, \Lambda) \) when \( \Lambda \) is \( \mathbb{Z}, \mathbb{Q} \) or \( \mathbb{C} \).

Therefore \( \mathcal{H}^q(X_0, R\Psi_{\pi\Delta} \mathbb{Q}) \) is 0 when \( q < 0 \) or \( q > 2 \dim X_\varphi \), which immediately implies that \( \mathcal{H}^q(Y^\bullet) \) is 0 in this case, thus \( Y^\bullet \) is essentially an object of \( D_{\text{MHS}}^b(\mathbb{Q}) \). From Theorem 3.4 of [11], the natural functor \( D^b(\text{MHS}_{\mathbb{Q}}) \hookrightarrow D^b_{\text{MHS}}(\mathbb{Q}) \) is an equivalence of categories, through which \( Y^\bullet \) determines a complex \( Z^\bullet \) of \( D^b(\text{MHS}_{\mathbb{Q}}) \) such that,

\[
H^q(Z^\bullet) = H^q(Y^\bullet), \quad \forall q \in \mathbb{Z} \tag{2.55}
\]

In next section, we will construct a mixed motive of \( \text{DM}_{\text{gm}}(\mathbb{Q}, \mathbb{Q}) \) whose Hodge realisation is conjectured to be isomorphic to \( Z^\bullet \).

2.5. Schmid’s Construction of Limit MHS

Before talk about Schmid’s construction of limit MHS [54], we need to introduce Deligne’s canonical extension of \( V \) [21, 22, 36, 63]. From assumptions on the fibration \( \pi_\mathbb{Q} 2.1 \), the operator \( \Psi(T) \) is unipotent, now let’s define \( N \) to be \( [33] \),

\[
N := \log \Psi(T) \tag{2.56}
\]

**Remark 2.6.** Some literatures define \( N \) to be \(- \log \Psi(T)/(2 \pi i)\), which is in accordance with [2.41]. However the definition [2.56] will make the computations of limit MHS simpler.

Let \( \varphi_0 \) be an arbitrary point of \( \Delta^* \). Every element \( \xi \) of \( H^q(X_{\varphi_0}, \mathbb{C}) \) could be extended to a multi-valued section \( \xi(\varphi) \) of the local system \( V_\mathbb{C} \), and define the single-valued section \( \widehat{\xi}(\varphi) \) of \( V \) to be [21, 22, 36, 63],

\[
\widehat{\xi}(\varphi) := \exp \left( -\frac{\log \varphi}{2 \pi i} N \right) \xi(\varphi) \tag{2.57}
\]

Suppose \( \{\sigma^a\} \) form a basis of \( H^q(X_{\varphi_0}, \mathbb{C}) \), then the sections \( \{\widehat{\sigma}(\varphi)\} \) will form a frame of \( V \) that induces a trivialisation of it over \( \Delta^* \). This trivialisation naturally induces an extension of \( V \) to a locally free sheaf \( \widehat{V} \) over \( \Delta \) that is called Deligne’s canonical extension, in which the sheaf \( F^p \) is extended to a locally free sheaf \( \widehat{F}^p \) over \( \Delta \) [21, 22, 36, 54, 63].
Proposition 2.7. The isomorphisms in (2.28) could be extended to the following isomorphisms 
\[ I_B : \tilde{\mathcal{V}}_{\mathcal{C}}^{an} \mid_{\Delta} \sim \tilde{\mathcal{V}}, \quad I_B : \tilde{\mathcal{F}}_{\mathcal{C}}^{p,an} \mid_{\Delta} \sim \tilde{\mathcal{F}}^p \] 
which induce isomorphisms between their fibers over 0,
\[ I_B : \tilde{\mathcal{V}}_{\mathcal{C}}^{an} \mid_0 \sim \tilde{\mathcal{V}} \mid_0, \quad I_B : \tilde{\mathcal{F}}_{\mathcal{C}}^{p,an} \mid_0 \sim \tilde{\mathcal{F}}^p \mid_0 \]

Proof. See [21, 22, 55, 59, 60].

In Deligne’s canonical extension, the section \( \tilde{\xi}(\varphi) \) of \( \mathcal{V} \) is extended to a section \( \tilde{\xi}(\varphi) \) of \( \tilde{\mathcal{V}} \), in particular the frame \( \{ \tilde{\sigma}^a(\varphi) \} \) of \( \tilde{\mathcal{V}} \), thus \( \{ \tilde{\sigma}^a(0) \} \) forms a basis of the fiber \( \tilde{\mathcal{V}} \mid_0 \). Therefore in Deligne’s canonical extension we have an isomorphism,
\[ \rho_{\varphi_0} : H^q(\mathcal{X}_{\varphi_0}, \mathbb{C}) \rightarrow \tilde{\mathcal{V}} \mid_0, \quad \rho_{\varphi_0}(\xi) = \tilde{\xi}(0) \]
through which the lattice \( H^q(\mathcal{X}_{\varphi_0}, \mathbb{Z}) \) induces a lattice structure \( \tilde{\mathcal{V}} \mid_0, \mathbb{Z} \) on \( \tilde{\mathcal{V}} \mid_0 \) [21, 36, 54],
\[ \tilde{\mathcal{V}} \mid_0, \mathbb{Z} := \rho_{\varphi_0}(H^q(\mathcal{X}_{\varphi_0}, \mathbb{Z})) \]

Similarly \( H^q(\mathcal{X}_{\varphi_0}, \mathbb{Q}) \) induces a rational structure on \( \tilde{\mathcal{V}} \mid_0 \) which satisfies [36, 52, 54],
\[ \tilde{\mathcal{V}} \mid_0, \mathbb{Q} = \tilde{\mathcal{V}} \mid_0, \mathbb{Z} \otimes \mathbb{Z} \mathbb{Q} \]
The action of \( T \) through the representation (2.38) is unipotent, hence it defines a weight filtration \( W_* \) on \( \tilde{\mathcal{V}} \mid_{0,\mathbb{Q}} \), while the fiber \( \tilde{\mathcal{F}}^p \mid_0 \) defines a Hodge filtration \( F^* \) on \( \tilde{\mathcal{V}} \mid_0 \).

Theorem 2.8. The following data form a MHS,
\[ (\tilde{\mathcal{V}} \mid_0, \mathbb{Z}, (\tilde{\mathcal{V}} \mid_0, \mathbb{Q}, W_*), (\tilde{\mathcal{V}} \mid_0, W_*, F^*)) \]

Proof. See [54].

Steenbrink has proved the compatibility between his construction and Schmid’s construction.

Proposition 2.9. The underlying rational MHS of (2.63)
\[ (\tilde{\mathcal{V}} \mid_0, \mathbb{Q}, (\tilde{\mathcal{V}} \mid_0, W_*), (\tilde{\mathcal{V}} \mid_0, W_*, F^*)) \]
is isomorphic to the limit MHS \( H^q(\mathcal{Y}^*) \) (2.54)

Proof. See [59, 60].

18
3. Limit Mixed Motive

This section is devoted to the construction of the limit mixed motive by Ayoub’s motivic nearby cycle functor [3], but first let’s briefly review the construction of the category of étale motivic sheaves, the details of which are left to the paper [9].

3.1. A Naive Construction of Étale Motivic Sheaves

Let Λ be a commutative ring which will be the coefficients ring in the construction of étale motivic sheaves, and in this paper we are mostly interested in the case when Λ is \( \mathbb{Q} \). In order to satisfy some technical assumptions, all schemes in this section are assumed to be separated, Noetherian and of finite Krull dimension. For a base scheme \( U \), the category of étale motivic sheaves with coefficients ring Λ will be denoted by \( D^A(U, \Lambda) \). Here we will follow [9] and give an incorrect naive construction of a category \( D^A(U, \Lambda) \), which nonetheless catches some essences of \( D^A(U, \Lambda) \) and suffices for this paper.

Let \( \text{Sm}/U \) be the category of smooth \( U \)-schemes endowed with étale topology [50] and let \( \text{Sh}_{\text{ét}}(\text{Sm}/U; \Lambda) \) be the abelian category of étale sheaves on \( \text{Sm}/U \) that take values in the abelian category of \( \Lambda \)-modules. A smooth \( U \)-scheme \( Z \) defines a presheaf through

\[
V \in \text{Sm}/U \to \Lambda \otimes \text{Hom}_U(V, Z)
\]

where \( \Lambda \otimes \text{Hom}_U(V, Z) \) is the \( \Lambda \)-module generated by the set \( \text{Hom}_U(V, Z) \), and the sheaf associated to this presheaf will be denoted by \( \Lambda_{\text{ét}}(X) \). In this way we find a Yoneda functor,

\[
\Lambda_{\text{ét}} : \text{Sm}/U \to \text{Sh}_{\text{ét}}(\text{Sm}/U; \Lambda)
\]

which could be considered as the first-step linearisation of the category \( \text{Sm}/S \).

Lemma 3.1. The étale sheaf \( \Lambda_{\text{ét}}(U) \) associated to the identity morphism of \( U \) is the constant étale sheaf on \( \text{Sm}/U \).

Proof. For the smooth \( U \)-scheme \( U \) given by identity morphism, \( \text{Hom}_U(V, U) \) consists of only one element for every scheme \( V \) of \( \text{Sm}/U \), i.e. the structure morphism \( V \to U \). So we have,

\[
\Lambda_{\text{ét}}(U)(V) = \Lambda \otimes \text{Hom}_U(V, U) \simeq \Lambda
\]

It is easy to check that the restriction homomorphism is the identity homomorphism of \( \Lambda \), hence the sheaf \( \Lambda_{\text{ét}}(U) \) is the constant sheaf on \( \text{Sm}/U \). \( \square \)

The next step in the construction of the category of étale motivic sheaves is to take \( \mathbb{A}^1 \)-localisation [9]. Suppose \( D(\text{Sh}_{\text{ét}}(\text{Sm}/U; \Lambda)) \) is the derived category of \( \text{Sh}_{\text{ét}}(\text{Sm}/U; \Lambda) \), and
let $\mathcal{T}_{A^1}$ be the smallest full triangulated subcategory of $D(\text{Sh}_{\text{et}}(\text{Sm}/U; \Lambda))$ that is closed under arbitrary direct sums and also contains all the complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow \Lambda_{\text{et}}(A^1_U \times_U V) \longrightarrow \Lambda_{\text{et}}(V) \longrightarrow 0 \longrightarrow \cdots$$

(3.4)

where $V$ is a smooth $U$-scheme and the morphism from $\Lambda_{\text{et}}(A^1_U \times_U V)$ to $\Lambda_{\text{et}}(V)$ is induced by the projection $A^1_U \times_U V \to V$. Define $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$ to be the Verdier quotient [9],

$$\mathbf{D}A^{\text{et,eff}}(U, \Lambda) := D(\text{Sh}_{\text{et}}(\text{Sm}/U; \Lambda))/\mathcal{T}_{A^1}$$

(3.5)

whose objects are the same as that of $D(\text{Sh}_{\text{et}}(\text{Sm}/U; \Lambda))$, hence by abuse of notations, the objects of $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$ will be denoted by the same symbols as that of $D(\text{Sh}_{\text{et}}(\text{Sm}/U; \Lambda))$.

As the name implies, objects of $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$ will be called effective $U$-motives. The effect of Verdier quotient is that morphisms of $D(\text{Sh}_{\text{et}}(\text{Sm}/U; \Lambda))$ whose cones lie in $\mathcal{T}_{A^1}$ get inverted in $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$. For example, the cone of the morphism

$$\Lambda_{\text{et}}(A^1_U \times_U V) \to \Lambda_{\text{et}}(V)$$

(3.6)

is in $\mathcal{T}_{A^1}$, hence it becomes an isomorphism in $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$, i.e. in $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$, the object $\Lambda_{\text{et}}(A^1_U \times_U V)$ is isomorphic to the object $\Lambda_{\text{et}}(V)$.

**Definition 3.2.** Let $\mathbf{D}A^{\text{et,eff}}_{\text{ct}}(U, \Lambda)$ be the smallest full triangulated subcategory of the category $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$ that contains all the objects of the form $\Lambda_{\text{et}}(Z)$ with $Z \in \text{Sm}/U$ of finite presentation and is also closed under taking direct summand. Objects of $\mathbf{D}A^{\text{et,eff}}_{\text{ct}}(U, \Lambda)$ will be called constructible $U$-motives.

The last step in the construction is stabilisation, and we will follow Ayoub and only give a naive stabilisation in this paper [9]. The injection $\infty_U \hookrightarrow \mathbb{P}^1_U$ induces a morphism,

$$\Lambda_{\text{et}}(\infty_U) \to \Lambda_{\text{et}}(\mathbb{P}^1_U)$$

(3.7)

whose cokernel in $\text{Sh}_{\text{et}}(\text{Sm}/U; \Lambda)$ will be denoted by $\Lambda_{\text{et}}(\mathbb{P}^1_U, \infty_U)$. The motive in $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$ defined by $\Lambda_{\text{et}}(\mathbb{P}^1_U, \infty_U)$ will be called Lefschetz motive,

$$L := \Lambda_{\text{et}}(\mathbb{P}^1_U, \infty_U)$$

(3.8)

The naive stabilisation is to invert the Lefschetz motive $L$,

$$\mathbf{D}A^{\text{et,naive}}(U, \Lambda) := \mathbf{D}A^{\text{et,eff}}(U, \Lambda)[L^{-1}]$$

(3.9)

More precisely, objects of $\mathbf{D}A^{\text{et,naive}}(U, \Lambda)$ are formal pairs $(M, m)$ where $M$ is an object of $\mathbf{D}A^{\text{et,eff}}(U, \Lambda)$ and $m$ is an integer. Morphisms between two objects $(M, m)$ and $(N, n)$ are given by

$$\lim_{r \geq -\min(m, n)} \text{Hom}_{\mathbf{D}A^{\text{et,eff}}(U, \Lambda)}(M \otimes L^{r+m}, N \otimes L^{r+n})$$

(3.10)
This naive stabilisation has the merit of being very straightforward, but suffers many technical problems, e.g. $DA_{\text{et,naive}}(U, \Lambda)$ is not even a triangulated category. However it still catches some of the essences of the category of étale motivic sheaves $DA_{\text{et}}(U, \Lambda)$. More precisely, let $DA_{\text{et,naive}}(U, \Lambda)$ be the full subcategory of $DA_{\text{et,naive}}(U, \Lambda)$ that consists of objects $(M, m)$ such that $M$ is an object of $DA_{\text{et,eff}}(U, \Lambda)$. Under some technical assumptions, which are all satisfied when $U$ is a quasiprojective variety over a field of characteristic 0 and $\Lambda$ is $\mathbb{Q}$, the category $DA_{\text{et,naive}}(U, \Lambda)$ is equivalent to the full triangulated subcategory $DA_{\text{et}}(U, \Lambda)$ of $DA_{\text{et}}(U, \Lambda)$ that consists of constructible objects, which is certainly the most important subcategory of $DA_{\text{et}}(U, \Lambda)$ [9]. Since in this paper we will only be concerned with constructible objects of $DA_{\text{et}}(U, \Lambda)$, this naive stabilisation will suffice for this paper. The defect-free construction of $DA_{\text{et}}(U, \Lambda)$ will not be talked here and is left to the paper [9].

**Remark 3.3.** From the construction of $DA_{\text{et}}(U, \Lambda)$, there exists a covariant functor from $Sm/U$ to $DA_{\text{et}}(U, \Lambda)$, so the construction in this section is covariant. However in Section 1.1, the functor which attaches a mixed motive to a variety is contravariant as we are interested in cohomology theory instead of homology theory, while the difference is just a dual operation [47].

The category $DA_{\text{et}}(U, \Lambda)$ satisfies Grothendieck’s six operations formalism [3, 9], but here we will only mention one such operation. Given a morphism $g : U \to V$, there exists a pushforward functor $g_*$,

$$g_* : \text{Sh}_{\text{et}}(Sm/U; \Lambda) \to \text{Sh}_{\text{et}}(Sm/V; \Lambda)$$

(3.11)

which sends an étale sheaf $G$ of $\text{Sh}_{\text{et}}(Sm/U; \Lambda)$ to $g_* G$ of $\text{Sh}_{\text{et}}(Sm/V; \Lambda)$ such that,

$$g_* G(W) := G(W \times_V U), \ W \in Sm/V$$

(3.12)

The functor $g_*$ could be derived and it induces a functor $Rg_* [9]$,

$$Rg_* : DA_{\text{et,eff}}(U, \Lambda) \to DA_{\text{et,eff}}(V, \Lambda)$$

(3.13)

The functor $g_*$ could be extended to the $L$-spectra, which could again be derived and yields a functor $Rg_* [3, 9]$,

$$Rg_* : DA_{\text{et}}(U, \Lambda) \to DA_{\text{et}}(V, \Lambda)$$

(3.14)

From [3], the functor $Rg_*$ sends constructible objects of $DA_{\text{et}}(U, \Lambda)$ to constructible objects of $DA_{\text{et}}(V, \Lambda)$.

### 3.2. Motivic Nearby Cycle Functor

The category $DA_{\text{et}}(U, \Lambda)$ satisfies the nearby cycle formalism, which realises to the classical nearby cycle functors [2, 8]. Since 0 is a smooth point of $S$, the local ring $\mathcal{O}_{S,0}$ is a discrete valuation ring, and the affine scheme Spec $\mathcal{O}_{S,0}$ admits an injection into $S$ [61],

$$\text{Spec } \mathcal{O}_{S,0} \hookrightarrow S$$

(3.15)
Let the henselisation of $\mathcal{O}_{S,0}$ be $\mathcal{O}^h_{S,0}$, then there is an injective local ring homomorphism from $\mathcal{O}_{S,0}$ to $\mathcal{O}^h_{S,0}$ that induces a morphism \[3.16\],

$$
\text{Spec } \mathcal{O}^h_{S,0} \rightarrow \text{Spec } \mathcal{O}_{S,0}
$$

The affine scheme Spec $\mathcal{O}^h_{S,0}$ consists of two points: a generic point $\eta$ and a closed point $s$ with residue field is $\mathbb{Q}$. For simplicity, let’s denote Spec $\mathcal{O}^h_{S,0}$ by $B$, and we have a henselian trait $(B,s,\eta)$, \[3.17\]

$$
\eta \longrightarrow B \longleftarrow s
$$

The composition of \[3.15\] and \[3.16\] is a morphism $i : B \rightarrow S$ and let $f : X_B \rightarrow B$ be the pull-back of $\pi_\mathbb{Q}$ along $i$, \[3.18\]

$$
\begin{array}{ccc}
X_B & \longrightarrow & X \\
\downarrow f & & \downarrow \pi_\mathbb{Q} \\
B & \longrightarrow & S
\end{array}
$$

then the pull-backs of $f$ along $\eta \rightarrow B$ and $s \rightarrow B$ form a commutative diagram, \[3.19\]

$$
\begin{array}{ccc}
X_\eta & \longrightarrow & X_B & \longleftarrow & X_s \\
\downarrow f_\eta & & \downarrow f & & \downarrow f_s \\
\eta & \longrightarrow & B & \longleftarrow & s
\end{array}
$$

where $X_s$ is just $Y$. There exists a motivic nearby cycle functor $R\Psi_f$, \[3.20\]

$$
R\Psi_f : \mathcal{D}A^{\text{ét}}(X_\eta, \mathbb{Q}) \rightarrow \mathcal{D}A^{\text{ét}}(X_s, \mathbb{Q})
$$

whose construction is left to [3, 8]. From Theorem 10.9 of [8], the functor $R\Psi_f$ sends constructible objects of $\mathcal{D}A^{\text{ét}}(X_\eta, \mathbb{Q})$ to constructible objects of $\mathcal{D}A^{\text{ét}}(X_s, \mathbb{Q})$. From Lemma 3.1, the identity morphism of $X_\eta$ induces the constant étale sheaf $\mathbb{Q}_{\text{ét}}(X_\eta)$ on $\text{Sm}/X_\eta$ which defines a constructible motive of $\mathcal{D}A^{\text{ét}}(X_\eta, \Lambda)$ that is also denoted by $\mathbb{Q}_{\text{ét}}(X_\eta)$. Therefore $R\Psi_f(\mathbb{Q}_{\text{ét}}(X_\eta))$ is a constructible motive of $\mathcal{D}A^{\text{ét}}(X_s, \mathbb{Q})$, which is called nearby motivic sheaf by Ayoub. The nearby motivic sheaf $R\Psi_f(\mathbb{Q}_{\text{ét}}(X_\eta))$ realises to the classical nearby cycle sheaves by Théorème 4.9 of [4] in the Betti realisation case and by Théorème 10.11 of [8] in the $\ell$-adic realisation case, also see Section 1.2 of [7]. The structure morphism $f_s$ in \[3.19\] induces a functor $R(f_s)_* \ [3]$

$$
R(f_s)_* : \mathcal{D}A^{\text{ét}}(X_s, \mathbb{Q}) \rightarrow \mathcal{D}A^{\text{ét}}(\mathbb{Q}, \mathbb{Q})
$$

which sends constructible objects of $\mathcal{D}A^{\text{ét}}(X_s, \mathbb{Q})$ to constructible objects of $\mathcal{D}A^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$. Now define the limit mixed motive $\mathcal{Z}$ to be \[3.22\]

$$
\mathcal{Z} := R(f_s)_* \circ R\Psi_f(\mathbb{Q}_{\text{ét}}(X_\eta))
$$

which will be a constructible object of $\mathcal{D}A^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$. From Theorem 4.4 of [9], the category $\mathcal{D}A^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$ is equivalent to $\mathcal{D}M^{\text{ét}}(\mathbb{Q}, \mathbb{Q})$, a fact that is also discussed in Section 4.3 of [9].
From Theorem 14.30 of [49], $\mathrm{DM}^{st}(\mathbb{Q}, \mathbb{Q})$ is equivalent to the dual of $\mathrm{DM}(\mathbb{Q}, \mathbb{Q})$, therefore the dual $\mathcal{Z}^\vee$ of $\mathcal{Z}$ could be considered as a constructible object of $\mathrm{DM}(\mathbb{Q}, \mathbb{Q})$, i.e. an object of $\mathrm{DM}_{gm}(\mathbb{Q}, \mathbb{Q})$. The motive $R\Psi_f(\mathcal{Q}_{et}(X_\eta))$ realises to the classical nearby cycle sheaf for the Betti realisation by Théorème 4.9 of [4] (when $X_B$ is the base change of a finite type $\mathbb{Q}[\varphi]$-scheme) and for the $\ell$-adic realisation by Théorème 10.11 of [8]. Therefore the realisations of $\mathcal{Z}^\vee$ compute the cohomologies of classical nearby cycle sheaves (in Betti case or $\ell$-adic case) and Ayoub conjectures that more is true.

**Conjecture 3.4.** The Hodge realisation of $\mathcal{Z}^\vee$ in 3.22 is isomorphic to the complex $\mathcal{Z}^\bullet$ in 2.52, i.e. the Hodge realisation of $\mathcal{Z}^\vee$ computes the limit MHS in 2.54 (up to an isomorphism).

However currently it is not available in literature and the proof of it, even though could be considered as ‘routine’, can be technically very hard.

### 4. Computation of Limit MHS

In the rest of this paper, we will only consider the fibration $\pi_Q$ such that the analytification of its complexification realises a deformation of the mirror threefold $W$ that comes from a one-parameter mirror pair $(M, W)$. We will also assume that 0 is the large complex structure limit, the meaning of which will be explained later. From now on, let the restriction of $\pi_{\mathbb{C}}$ to $\mathcal{W} := (\pi_{\mathbb{C}})^{-1}(\Delta)$ be,

$$\pi_\Delta : \mathcal{W} \to \Delta$$

whose only singular fiber is $\mathcal{W}_0 := \pi_\Delta^{-1}(0)$, which is the analytification of $Y_{\mathbb{C}}$. Let’s denote the restriction of $\pi_\Delta$ to $\mathcal{W}^* := \mathcal{W} \setminus \mathcal{W}_0$ by $\pi_\Delta^*$,

$$\pi_\Delta^* : \mathcal{W}^* \to \Delta^*$$

A smooth fiber $\mathcal{W}_\varphi$ of this fibration is a Calabi-Yau threefold with Hodge diamond of the form,

$$
\begin{array}{cccc}
1 & & & \\
& 0 & 0 & \\
& 0 & h^{11} & 0 \\
1 & 1 & 1 & 1 \\
& 0 & h^{11} & 0 \\
& & & \\
& & & 1 \\
\end{array}
$$

(4.3)

where $h^{11} = \dim H^{11}(\mathcal{W}_\varphi)$ is a positive integer.

**Definition 4.1.** For a one-parameter mirror pair $(M, W)$ of Calabi-Yau threefolds, the deformation of $W$ is said to be rationally defined if there exists a fibration of the form $\pi_{\mathbb{C}}$ such that the associated fibration $\pi_{\mathbb{C}}$ realises a deformation of $W$. 

23
Now let's denote the mixed Hodge complex at large complex structure limit by $Y^\bullet_{\text{MS}}$ and it determines a complex $Z^\bullet_{\text{MS}}$ of $D^b(\text{MHS}_\mathbb{Q})$ through the equivalence $D^b(\text{MHS}_\mathbb{Q}) \cong D^b_{\text{MHS}}$. From last section, there exists a limit mixed motive $Z^\bullet_{\text{MS}}$ constructed at the large complex structure limit, whose Hodge realisation is conjectured to be isomorphic to $Z^\bullet_{\text{MS}}$. Now we will compute the limit MHS at large complex structure limit and the following two easy propositions deal with the cases where $q \neq 3$.

**Proposition 4.2.**

$$H^q(Z^\bullet_{\text{MS}}) = 0, \quad \text{when } q \neq 0, 2, 3, 4, 6 \quad (4.4)$$

**Proof.** The Hodge diamond of a smooth fiber $\mathcal{W}_\varphi$ tells us,

$$H^q(\mathcal{W}_\varphi, \mathbb{Q}) = 0, \quad \text{when } q \neq 0, 2, 3, 4, 6 \quad (4.5)$$

Then this proposition is an immediate result of Lemma 2.5. $\square$

**Proposition 4.3.** $H^q(Z^\bullet_{\text{MS}})$ is a Hodge-Tate object when $q = 0, 2, 4, 6$,

$$H^0(Z^\bullet_{\text{MS}}) = \mathbb{Q}(0), \quad H^2(Z^\bullet_{\text{MS}}) = \mathbb{Q}(-1)^{h^{11}}, \quad H^4(Z^\bullet_{\text{MS}}) = \mathbb{Q}(-2)^{h^{11}}, \quad H^6(Z^\bullet_{\text{MS}}) = \mathbb{Q}(-3) \quad (4.6)$$

**Proof.** When $q = 0$, the Hodge diamond tells us,

$$H^0(\mathcal{W}_\varphi, \mathbb{Q}) = \mathbb{Q} \quad (4.7)$$

which immediately implies that the pure Hodge structure on $H^0(\mathcal{W}_\varphi, \mathbb{Z})$ is isomorphic to $\mathbb{Q}(0)$. Hence the subsheaf filtration $\mathcal{F}^p$ of $\mathcal{V} := R^0 \pi_{\Delta^*}^!, \mathbb{Z} \otimes \mathcal{O}_{\Delta^*}$ becomes,

$$\mathcal{F}^0 = \mathcal{V}, \quad \mathcal{F}^1 = 0 \quad (4.8)$$

which shows the limit Hodge filtration on $\widetilde{\mathcal{V}}|_0$ is,

$$F^0 = \widetilde{\mathcal{V}}|_0, \quad F^1 = 0 \quad (4.9)$$

The limit Hodge filtration imposes very strong restriction on the possible limit MHS and in fact the only possibility is,

$$H^0(Z^\bullet_{\text{MS}}) = \mathbb{Q}(0) \quad (4.10)$$

When $q = 2$, the Hodge diamond tells us,

$$H^{2,0}(\mathcal{W}_\varphi, \mathbb{Q}) = H^{0,2}(\mathcal{W}_\varphi, \mathbb{Q}) = 0 \quad (4.11)$$

thus the pure Hodge structure on $H^2(\mathcal{W}_\varphi, \mathbb{Z})$ is isomorphic to $\mathbb{Q}(-1)^{h^{11}}$. The subsheaf filtration $\mathcal{F}^p$ of $\mathcal{V} := R^2 \pi_{\Delta^*}^!, \mathbb{Z} \otimes \mathcal{O}_{\Delta^*}$ becomes,

$$\mathcal{F}^0 = \mathcal{F}^1 = \mathcal{V}, \quad \mathcal{F}^2 = 0 \quad (4.12)$$
and the limit Hodge filtration on $\tilde{V}|_0$ is,
\[ F^0 = F^1 = \tilde{V}|_0, \quad F^2 = 0 \] (4.13)

This limit Hodge filtration again imposes very strong restriction on the possible limit MHS and the only possibility is,
\[ H^2(\mathbb{Z}_{\text{MS}}^*) = \mathbb{Q}(-1)^{h^{11}} \] (4.14)

Similarly we have,
\[ H^4(\mathbb{Z}_{\text{MS}}^*) = \mathbb{Q}(-2)^{h^{11}}, \quad H^6(\mathbb{Z}_{\text{MS}}^*) = \mathbb{Q}(-3) \] (4.15)

The rest of this section is devoted to the computation of the limit MHS $H^3(\mathbb{Z}_{\text{MS}}^*)$, which depends on the mirror symmetry conjecture. In mirror symmetry, the holomorphic three form $\Omega$ is a non-vanishing section of the bundle $F^3_{\text{an}}$ of the mirror threefold. In this paper we are only interested in the case where the deformation of the mirror threefold is rationally defined, hence we require $\Omega$ is rationally defined, i.e. it is a nonvanishing section of $F^3_{\mathbb{Q}}$. We further assume $\Omega$ has logarithmic poles along the smooth component $s$ of the normal crossing divisor $Y$, so that it could be extended to a global section of $F^3_{\mathbb{Q}}$ whose value at the large complex structure limit is nonzero. Now we will briefly review mirror symmetry.

### 4.1. Integral Periods of $\Omega$

Choose a point $\varphi_0 \in \Delta^*$, Poincaré duality implies the existence of a unimodular skew symmetric pairing on $H_3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$, which allows us to choose a symplectic basis $\{A_0, A_1, B_0, B_1\}$ that satisfies the following pairing relations [14, 15, 33],
\[ A_a \cdot A_b = 0, \quad B_a \cdot B_b = 0, \quad A_a \cdot B_b = \delta_{ab} \] (4.16)

Suppose its dual is $\{\alpha^0, \alpha^1, \beta^0, \beta^1\}$, which forms a basis of $H^3(\mathcal{W}_{\varphi_0}, \mathbb{Z})$. The only nonvanishing pairings are,
\[ \alpha^a(A_b) = \delta_b^a, \quad \beta^a(B_b) = \delta_b^a \] (4.17)

**Remark 4.4.** All our treatments will be modulo torsions of homology or cohomology groups.

For simplicity, we will also denote $B_0, B_1$ by $A_2, A_3$ and $\beta^0, \beta^1$ by $\alpha^2, \alpha^3$ respectively. In a simply connected local neighbourhood of $\varphi_0$, $A^a(\varphi)$ could be extended to a local section $A_3(\varphi)$ of the local system $V_{\mathbb{Z}}^\vee$ and $\alpha^b$ could be extended to a local section $\alpha^b(\varphi)$ of the local system $V_{\mathbb{Z}}$, while the basis $\{\alpha^b(\varphi)\}_{b=0}^3$ of $H^3(\mathcal{W}_{\varphi}, \mathbb{Z})$ is dual to the basis $\{A^a(\varphi)\}_{a=0}^3$ of $H_3(\mathcal{W}_{\varphi}, \mathbb{Z})$ [63]. Since the unimodular skew symmetric pairing is preserved by this extension, $\{A^a(\varphi)\}_{a=0}^3$ is actually a symplectic basis of $H_3(\mathcal{W}_{\varphi}, \mathbb{Z})$. The integral periods of $\Omega(\varphi)$ with respect to the basis $\{A^a(\varphi)\}_{a=0}^3$ are defined to be,
\[ z_a(\varphi) = \int_{A_a(\varphi)} \Omega(\varphi), \quad \mathcal{G}_b(\varphi) = \int_{B_b(\varphi)} \Omega(\varphi) \] (4.18)
which are holomorphic functions \[14, 15, 33\]. Now form the column vector,

\[ \Pi(\varphi) := (\mathcal{G}_0(\varphi), \mathcal{G}_1(\varphi), z_0(\varphi), z_1(\varphi))^t \] (4.19)

where \(^t\) means transpose. Under the comparison isomorphism \(I_B\), \(\Omega(\varphi)\) has an expansion with respect to the basis \(\{\alpha^a(\varphi)\}_{a=0}^3\),

\[ I_B(\Omega(\varphi)) = z_0(\varphi) \alpha^0(\varphi) + z_1(\varphi) \alpha^1(\varphi) + \mathcal{G}_0(\varphi) \beta^0(\varphi) + \mathcal{G}_1(\varphi) \beta^1(\varphi) \] (4.20)

However the extension of \(A_a\) to the punctured disc \(\Delta^*\) is generally multi-valued and this multi-valuedness is called monodromy, which is described by a representation \(\Phi\) of the fundamental group of \(\pi_1(\Delta^*, \varphi_0)\) [63],

\[ \Phi : \pi_1(\Delta^*, \varphi_0) \rightarrow \text{Aut}(H_3(W_{\varphi_0}, \mathbb{Z})) \] (4.21)

The fundamental group \(\pi_1(\Delta^*, \varphi_0)\) is isomorphic to \(\mathbb{Z}\) and let \(T\) be a generator, then the representation \(\Phi\) is uniquely determined by \(\Phi(T)\). Since unimodular pairing is preserved by extension, with respect to the basis \(\{A_a\}_{a=0}^3\), the image of the representation \(\Phi\) lies in \(\text{Sp}(4, \mathbb{Z})\) [14, 15, 17]. The monodromy of the extension of \(\alpha^a\) is described by the dual representation \(\Psi\) of \(\Phi\) [63],

\[ \Psi : \pi_1(\Delta^*, \varphi_0) \rightarrow \text{Aut}(H^3(W_{\varphi_0}, \mathbb{Z})) \] (4.22)

The nilpotent operator \(N\) is defined in 2.56.

**Definition 4.5.** The point 0 is a large complex structure limit if the monodromy around it is maximally unipotent [33], i.e.

\[ N^3 \neq 0, \ N^4 = 0 \] (4.23)

In the rest of this paper, 0 will be assumed to be the large complex structure limit of the complex moduli space of \(W\).

### 4.2. Canonical Periods of \(\Omega\)

From Griffiths transversality, the integral periods \(z_a(\varphi)\) and \(\mathcal{G}_b(\varphi)\) are solutions of Picard-Fuchs equation,

\[ \mathcal{L}\Pi_a = 0 \] (4.24)

where \(\mathcal{L}\) is a Picard-Fuchs operator of the form [15, 17, 33],

\[ \mathcal{L} = R_4(\varphi) \vartheta^4 + R_3(\varphi) \vartheta^3 + R_2(\varphi) \vartheta^2 + R_1(\varphi) \vartheta + R_0(\varphi) , \text{ with } \vartheta = \varphi \frac{d}{d\varphi} \] (4.25)
which has regular singularities at 0. Since the monodromy around 0 is maximally unipotent, the solution space of (4.24) has a distinguished basis consists of four linearly independent solutions of the form, 

\[ \varpi_0 = f_0 \]
\[ \varpi_1 = \frac{1}{2 \pi i} (f_0 \log \varphi + f_1) \]
\[ \varpi_2 = \frac{1}{(2 \pi i)^2} (f_0 \log^2 \varphi + 2 f_1 \log \varphi + f_2) \]
\[ \varpi_3 = \frac{1}{(2 \pi i)^3} (f_0 \log^3 \varphi + 3 f_1 \log^2 \varphi + 3 f_2 \log \varphi + f_3) \]

(4.26)

where the choice of the multi-valued holomorphic function \( \log \varphi \) has been fixed and \( \{f_j\}_{j=0}^3 \) are power series of \( \varphi \) that converge in \( \Delta \). If we impose the following conditions,

\[ f_0(0) = 1, \quad f_1(0) = f_2(0) = f_3(0) = 0 \]

(4.27)

the four solutions (4.26) are uniquely determined and they are called canonical periods of \( \Omega \) conventionally [12]. Now form the column vector \( \varpi \),

\[ \varpi := (\varpi_0, \varpi_1, \varpi_2, \varpi_3)^t \]

(4.28)

The integral periods \( \{\Pi_a\}_{a=0}^3 \) form another basis of the solution space of (4.24) therefore there exists a matrix \( S \) of \( GL(4, \mathbb{C}) \) such that,

\[ \Pi_a = \sum_{b=0}^{3} S_{ab} \varpi_b \]

(4.29)

The expansion (4.20) now becomes,

\[ I_B(\Omega(\varphi)) = \sum_{a=0}^{3} \alpha^a(\varphi) \Pi_a(\varphi) = \sum_{a,b} \alpha^a(\varphi) S_{ab} \varpi_b(\varphi) \]

(4.30)

Define the canonical basis \( \{\gamma^a\}_{a=0}^3 \) of \( H^3(W_{\varphi_0}, \mathbb{C}) \) to be,

\[ \gamma^a = \sum_{b=0}^{3} \alpha^b S_{ba} \]

(4.31)

The expansion (4.30) becomes,

\[ I_B(\Omega(\varphi)) = \sum_{a=0}^{3} \gamma^a(\varphi) \varpi_a(\varphi) \]

(4.32)

where \( \gamma^a(\varphi) \) is the extension of \( \gamma^a \). Let \( \{C_a\}_{a=0}^3 \) be the dual of \( \{\gamma^a\}_{a=0}^3 \), it forms a basis of \( H_3(W_{\varphi_0}, \mathbb{C}) \), and the canonical period \( \varpi_a \) is the integration of \( \Omega(\varphi) \) over \( C_a(\varphi) \),

\[ \varpi_a(\varphi) = \int_{C_a(\varphi)} \Omega(\varphi) \]

(4.33)

27
where $C_a(\varphi)$ is the extension of $C_a$. The operator $T$ acts on the basis $\{C_a\}_{a=0}^3$.

$$\Phi(T) C_a = \sum_b (T_{\text{Can}})_{ba} C_b$$  \hspace{1cm} (4.34)

where $(T_{\text{Can}})_{ba}$ is the matrix of $\Phi(T)$ with respect to $\{C_a\}_{a=0}^3$. Through it, $T$ acts on $\varpi_a$ in the way,

$$\varpi_a(\varphi_0) = \int_{C_a} \Omega(\varphi_0) \rightarrow \sum_b (T_{\text{Can}})_{ba} \int_{C_b} \Omega(\varphi_0) = \sum_b (T_{\text{Can}})_{ba} \varpi_b(\varphi_0)$$  \hspace{1cm} (4.35)

which is just the monodromy of $\varpi_a$. The monodromy of $\varpi_a$ is induced by its analytical continuation around 0, i.e. $\log \varphi \rightarrow \log \varphi + 2\pi i$, so $(T_{\text{Can}})$ could be easily found to be,

$$T_{\text{Can}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (4.36)

While $T$ acts on $\{\gamma^a\}_{a=0}^3$ through the dual representation $\Psi$,

$$\Psi(T) \gamma^a = \sum_{b=0}^3 (T^\vee_{\text{Can}})_{ba} \gamma^b$$  \hspace{1cm} (4.37)

where $T^\vee_{\text{Can}}$ is the matrix representation of $\Psi(T)$ with respect to the basis $\{\gamma^a\}_{a=0}^3$,

$$T^\vee_{\text{Can}} = ((T_{\text{Can}})^t)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$  \hspace{1cm} (4.38)

4.3. Mirror Symmetry

The matrix $S$ is very important in this paper, the computation of which needs mirror symmetry, which we now briefly review. From the integral periods $(z_0(\varphi), z_1(\varphi))$ cannot both vanish and locally they form a projective coordinate of $\mathcal{M}_C(W)$, in terms of which $G_a(z)$ is a homogeneous function of degree one. The three-form $\Omega$ could also be expressed as

$$I_B(\Omega(z)) = z_0 \alpha^0(z) + z_1 \alpha^1(z) + G_0(z) \beta^0(z) + G_1(z) \beta^1(z)$$  \hspace{1cm} (4.39)

Griffiths transversality implies,

$$\int_W \Omega(z) \wedge \frac{\partial \Omega(z)}{\partial z_a} = 0$$  \hspace{1cm} (4.40)
which yields the following relation

\[ G_a(z) = \frac{\partial G(z)}{\partial z_a}, \quad \text{where} \quad G(z) := \frac{1}{2} \sum_b z_b G_b(z) \quad (4.41) \]

where \( G \) is called prepotential and it is a homogeneous function of degree two. The Yukawa coupling \( \kappa_{abc} \) satisfies,

\[ \kappa_{abc} = \int_W \Omega(z) \wedge \frac{\partial^3 \Omega(z)}{\partial z_a \partial z_b \partial z_c} = \frac{\partial^3 G(z)}{\partial z_a \partial z_b \partial z_c} \quad (4.42) \]

In all examples of mirror pairs, there exists an integral symplectic basis \((A_0, A_1, B_0, B_1)\) of \( H^3(W_{\phi_0}, \mathbb{Z}) \) such that

\[ z_i(\varphi) = \lambda \varpi_i(\varphi), \quad i = 0, 1 \quad (4.43) \]

where \( \lambda \) is a nonzero constant. Let’s denote the quotient \( \varpi_1/\varpi_0 \) by \( t' \),

\[ t' = \frac{z_1}{z_0} = \frac{\varpi_1}{\varpi_0} = \frac{1}{2\pi i} \log \varphi + \frac{f_1(\varphi)}{f_0(\varphi)} \quad (4.44) \]

which transforms under monodromy in the way,

\[ t' \to t' + 1 \quad (4.45) \]

**Definition 4.6.** The **mirror map** is induced by the identification of the flat coordinate \( t \) of the complexified Kähler moduli space \( \mathcal{M}_K(M) \) with the function \( \varpi_1/\varpi_0 \) on the complex moduli space \( \mathcal{M}_C(W) \), i.e.

\[ t \equiv t' \quad (4.46) \]

On the complex side, the period vector \( \Pi \) is homogeneous of degree 1, and after a rescaling it becomes

\[ \Pi = (G_0/z_0, G_1/z_0, 1, z_1/z_0)^t \quad (4.47) \]

which, in terms of the affine coordinate \( t' \), could be expressed as,

\[ \Pi = (2 G - t' \frac{\partial G}{\partial t'}, \frac{\partial G}{\partial t'}, 1, t')^t \quad (4.48) \]

On the Kähler side, the mirror period vector \( \Pi \) is defined to be \([14, 15] \),

\[ \Pi = (\mathcal{F}_0, \mathcal{F}_1, 1, t)^t, \quad \text{with} \quad \mathcal{F}_0 = 2 \mathcal{F} - t \frac{\partial \mathcal{F}}{\partial t}, \quad \mathcal{F}_1 = \frac{\partial \mathcal{F}}{\partial t} \quad (4.49) \]

Since the prepotential \( \mathcal{F} \) admits an expansion of the form \([14, 15] \), \( \Pi \) could be expressed as,

\[ \Pi = \left( \begin{array}{c} \frac{1}{6} Y_{111} t^3 - \frac{1}{2} Y_{011} t - \frac{1}{2} Y_{001} + 2 \mathcal{F}^{np}(t) - t \frac{d \mathcal{F}^{np}(t)}{dt} \\ -\frac{1}{2} Y_{111} t^2 - Y_{011} t - \frac{1}{2} Y_{001} + \frac{d \mathcal{F}^{np}(t)}{dt} \\ 1 \\ t \end{array} \right) \quad (4.50) \]
The formulation of mirror symmetry conjecture in this paper follows from \([14, 15]\).

**Mirror Symmetry Conjecture** The mirror map induces an isomorphism between \(\mathcal{M}_K(M)\) and a neighbourhood of the large complex structure limit in \(\mathcal{M}_C(W)\), under which the integral period vector \(\Pi\) of \(\mathcal{M}_C(W)\) is identified with the mirror period vector of the mirror map.

**Remark 4.7.** On the complex side, the integral periods \(\varphi\) and the prepotential \(G\) depend on the choice of an integral symplectic basis \(\{A_0, A_1, B_0, B_1\}\) of \(H_3(W,\mathbb{Z})\), while on the Kähler side there is no natural integral symplectic structure. Therefore the identification of \(\varphi\) with \(\Pi\) under mirror map transfers the integral symplectic structure of the complex side to the Kähler side \([15]\). Therefore the values of coefficients \(Y_{011}, Y_{001}\) and \(Y_{000}\) depend on such an integral symplectic structure. For a different choice of integral symplectic basis, it could be shown that the coefficients change in the following way \([14]\),

\[
Y_{011} \rightarrow Y_{011} + n, \quad n \in \mathbb{Z} \\
Y_{001} \rightarrow Y_{001} + k, \quad k \in \mathbb{Z} \\
Y_{000} \rightarrow Y_{000} + r', \quad r' \in \mathbb{Q}
\]  

(4.51)

**Remark 4.8.** We will also want the chosen point \(\varphi_0\) to be close enough to the large complex structure limit such that it lies in this neighbourhood which is isomorphic to \(\mathcal{M}_K(M)\).

Therefore in turn mirror symmetry will also identify the prepotential \(G\) on the complex side with the prepotential \(F\) on the Kähler side. Hence from now on, we will identify \(t'\) with \(t\) and ignore their differences. Now the monodromy of the mirror period \(\Pi_a\) is induced by the operation \(t \rightarrow t + 1\). Let the monodromy matrix of \(\Pi_a\) be \(T_K\), i.e. \(\Pi_a\) transforms in the following way under monodromy,

\[
\Pi_a \rightarrow \sum_{b=0}^{3} (T_K)_{ba} \Pi_b
\]  

(4.52)

Since \(F^{np}(t)\) admits a series expansion in \(\exp 2\pi i t\), hence \(dF^{np}(t)/dt\) is also invariant under \(t \rightarrow t + 1\), the form of \(\Pi_a\) in \([15]\) immediately tells us what is \(T_K\) \([12]\),

\[
T_K = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
\frac{1}{6} Y_{111} - Y_{001} & -\frac{1}{2} Y_{111} - Y_{011} & 1 & 1 \\
\frac{1}{2} Y_{111} - Y_{011} & -Y_{111} & 0 & 1
\end{pmatrix}
\]  

(4.53)

The monodromy of the integral periods \(\Pi\) could be found from the action of the generator \(T\) of \(\pi_1(\Delta^*, \varphi_0)\) on the basis \(\{A_a\}_{a=0}^3\) through the representation \(\Phi\) \([42]\),

\[
\Phi(T).A_a = \sum_{b=0}^{3} A_b (T_C)_{ba}
\]  

(4.54)
where \( T_C \) is the matrix representation of \( \Phi(T) \) with respect to the basis \( \{A_a\}_{a=0}^3 \), which is integral symplectic.

\[
\Pi_a(\varphi_0) = \int_{A_a} \Omega(\varphi_0) \rightarrow \sum_b (T_C)_{ba} \int_{A_b} \Omega(\varphi_0) = \sum_b (T_C)_{ba} \Pi_b(\varphi_0)
\]

The identification of \( \Pi \) with \( \bigoplus \) under mirror map shows that the monodromy matrix of \( \Pi \) equals that of \( \bigoplus \) \[12, 14\]

\[
T_C = T_K
\]

Therefore \( T_K \) is also an integral symplectic matrix which immediately yields the following well-known fact in mirror symmetry.

**Corollary 4.9.** The numbers \( 2Y_{011} \) and \( 6Y_{001} \) are both integers, a priori \( Y_{011} \) and \( Y_{001} \) are rational numbers.

To compute the matrix \( S \) \([4.29]\), we need to look at what happens at the large complex structure limit. The condition \([4.27]\) shows \[12\],

\[
t = \frac{1}{2\pi i} \log \varphi + O(\varphi)
\]

therefore the large complex structure limit corresponds to \( t \to i\infty \). Under this limit, the leading parts of \( \Pi \) and \( \varpi \) are,

\[
\Pi \sim \begin{pmatrix}
\frac{1}{6}Y_{111}t^3 - \frac{1}{2}Y_{001}t - \frac{1}{3}Y_{000} \\
-\frac{1}{2}Y_{111}t^2 - Y_{011}t - \frac{1}{2}Y_{001} \\
1 \\
t
\end{pmatrix}, \quad \varpi \sim \begin{pmatrix}
1 \\
t \\
t^2 \\
t^3
\end{pmatrix}
\]

Since \( \Pi \) is identified with \( \bigoplus \), equation \([4.29]\) becomes,

\[
\Pi_a = \sum_{b=0}^3 S_{ab} \varpi_b
\]

Compare the leading parts of \( \Pi \) and \( \varpi \) in \([4.58]\). \( S \) could be easily evaluated \[12\],

\[
S = \lambda \begin{pmatrix}
-\frac{1}{3}Y_{000} & -\frac{1}{2}Y_{001} & 0 & \frac{1}{6}Y_{111} \\
-\frac{1}{2}Y_{001} & -Y_{011} & -\frac{1}{2}Y_{111} & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

where \( \lambda \) is the constant in equation \([4.43]\). Finally we are ready to compute \( H^3(Z_{\text{MS}}^*) \) \([2.54]\) by methods in \([33, 36, 54]\).
4.4. Weight Filtration

The monodromy operator $N$ induces a weight filtration on the rational vector space $H^3(W_{\varphi_0}, \mathbb{Q})$, which is mapped to the weight filtration $W_*$ on $\tilde{V}_{0,\mathbb{Q}}$ under the isomorphism $\cong$. To compute the weight filtration on $H^3(W_{\varphi_0}, \mathbb{Q})$ induced by $N$, it is more convenient to choose a new basis \( \{\beta^a\}_{a=0}^3 \) of $H^3(W_{\varphi_0}, \mathbb{Q})$ defined as follows. From section 4.3, $Y_{111}$, $Y_{011}$ and $Y_{001}$ are all rational numbers, and now let $S_1$ be,

\[
S_1 = \begin{pmatrix}
0 & \frac{1}{2} Y_{001} & 0 & -Y_{111} \\
-\frac{1}{2} Y_{001} & Y_{011} & -Y_{111} & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

(4.61)

whose determinant $Y_{111}^2$ is a positive integer, therefore $S_1$ lies in $GL(4, \mathbb{Q})$. Now define the new basis \( \{\beta^a\}_{a=0}^3 \) to be,

\[
\beta^a = \sum_{b=0}^3 (S_1)_{ba} \alpha^b
\]

(4.62)

while we have,

\[
\gamma^a = \sum_{b=0}^3 (S_2)_{ba} \beta^b
\]

(4.63)

where the matrix $S_2$ is,

\[
S_2 = \lambda \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
Y_{000} & 0 & 0 & -\frac{1}{6}
\end{pmatrix}
\]

(4.64)

Since \( \{\alpha^a\}_{a=0}^3 \) is the dual of \( \{A_a\}_{a=0}^3 \), the matrix of the action of $T$ on \( \{\alpha^a\}_{a=0}^3 \) is \((T_K)^t)^{-1}\), which equals \((T_C)^t)^{-1}\) from mirror symmetry. From equations 2.56 and 4.62, the operator $N$ acts on \( \{\beta^a\}_{a=0}^3 \) in a simple way,

\[
N \beta^0 = \beta^1, N \beta^1 = \beta^2, N \beta^2 = \beta^3, N \beta^3 = 0
\]

(4.65)

The weight filtration on $H^3(W_{\varphi_0}, \mathbb{Q})$ induced by $N$ could be computed inductively. First $W_{-1}$ and $W_6$ are,

\[
W_{-1} H^3(W_{\varphi_0}, \mathbb{Q}) = 0 \\
W_6 H^3(W_{\varphi_0}, \mathbb{Q}) = H^3(W_{\varphi_0}, \mathbb{Q})
\]

(4.66)

Let $W_0$ and $W_5$ be,

\[
W_0 H^3(W_{\varphi_0}, \mathbb{Q}) = \text{im } N^3 = \mathbb{Q} \beta^3 \\
W_5 H^3(W_{\varphi_0}, \mathbb{Q}) = \text{ker } N^3 = \mathbb{Q} \beta^1 + \mathbb{Q} \beta^2 + \mathbb{Q} \beta^3
\]

(4.67)
Now form the quotient space $W_5/W_0$,
\[ W_5 H^3(W_{\varphi_0}, \mathbb{Q})/W_0 H^3(W_{\varphi_0}, \mathbb{Q}) \simeq \mathbb{Q} \beta^1 + \mathbb{Q} \beta^2 \] (4.68)
As $N^2$ induces a zero map on this quotient space, we have
\[ W_1 H^3(W_{\varphi_0}, \mathbb{Q}) = W_0 H^3(W_{\varphi_0}, \mathbb{Q}) \]
\[ W_4 H^3(W_{\varphi_0}, \mathbb{Q}) = W_5 H^3(W_{\varphi_0}, \mathbb{Q}) \] (4.69)
Then form the quotient space $W_4/W_1$,
\[ W_4 H^3(W_{\varphi_0}, \mathbb{Q})/W_1 H^3(W_{\varphi_0}, \mathbb{Q}) \simeq \mathbb{Q} \beta^1 + \mathbb{Q} \beta^2 \] (4.70)
Since $N$ induces a map on this quotient space that sends $\beta^1$ to $\beta^2$ and $\beta^2$ to 0, we have
\[ W_2 H^3(W_{\varphi_0}, \mathbb{Q}) = W_3 H^3(W_{\varphi_0}, \mathbb{Q}) = \mathbb{Q} \beta^2 + \mathbb{Q} \beta^3 \] (4.71)
In this way we have found the weight filtration on $H^3(W_{\varphi_0}, \mathbb{Q})$ induced by $N$. The isomorphism $\rho_{\varphi_0} \text{[2.60]}$ sends the basis $\{\beta^a\}_{a=0}^3$ of $H^3(W_{\varphi_0}, \mathbb{Q})$ to the basis $\{\tilde{\beta}^a(0)\}_{a=0}^3$ of $\tilde{V}|_{0,\mathbb{Q}}$ and we have
\[ \tilde{\beta}^a(0) = \sum_{b=0}^3 (S_1)_{ba} \tilde{\alpha}^b(0) \] (4.72)
\[ \tilde{\gamma}^a(0) = \sum_{b=0}^3 (S_2)_{ba} \tilde{\beta}^b(0) \]
The isomorphism $\rho_{\varphi_0} \text{[2.60]}$ maps the weight filtration $W_4 H^3(W_{\varphi_0}, \mathbb{Q})$ to the weight filtration $W_4(\tilde{V}|_{0,\mathbb{Q}}) \text{[2.63]}$, hence we have
\[ W_4(\tilde{V}|_{0,\mathbb{Q}}) = W_4(\tilde{V}|_{0,\mathbb{Q}}) = \mathbb{Q} \tilde{\beta}^3(0) \]
\[ W_5(\tilde{V}|_{0,\mathbb{Q}}) = W_5(\tilde{V}|_{0,\mathbb{Q}}) = \mathbb{Q} \tilde{\beta}^2(0) + \mathbb{Q} \tilde{\beta}^3(0) \] (4.73)
\[ W_6(\tilde{V}|_{0,\mathbb{Q}}) = \mathbb{Q} \tilde{\beta}^0(0) + \mathbb{Q} \tilde{\beta}^1(0) + \mathbb{Q} \tilde{\beta}^2(0) + \mathbb{Q} \tilde{\beta}^3(0) \]

### 4.5. Limit Hodge Filtration

The fiber $\tilde{\mathcal{F}}^p|_0$ induces a decreasing filtration on $\tilde{V}|_0$, whose complexification is mapped to the limit Hodge filtration on $\tilde{V}|_0$ under comparison isomorphism $\text{[36, 52, 54, 59, 60]}$,
\[ F^p(\tilde{V}|_0) = I_B(F^p(\tilde{\mathcal{V}}|_0)) = I_B(\tilde{\mathcal{F}}^p|_0) \otimes_{\mathbb{Q}} \mathbb{C} \] (4.74)
where we have used the following canonical isomorphisms $\text{[61]}$,
\[ F^p(\tilde{\mathcal{V}}|_0) = F^p(\tilde{\mathcal{V}}|_0) = F^p(\tilde{\mathcal{V}}|_0) \otimes_{\mathbb{Q}} \mathbb{C} = \tilde{\mathcal{F}}^p|_0 \otimes_{\mathbb{Q}} \mathbb{C} \] (4.75)
From the assumption that the three-form $\Omega$ has logarithmic poles along the smooth components of the singular fiber $Y$, it could be extended to a global section of $\tilde{F}_Q^3$, so we find
\[
F^3(\tilde{\mathcal{V}}_Q|_0) \supset \mathcal{Q} \Omega|_0
\] (4.76)

Shrink the curve $S$ to an open subset if necessary, its tangent sheaf has a section of the form,
\[
\vartheta := \varphi \frac{d}{d\varphi}
\] (4.77)

The Gauss-Manin connection $\nabla_Q$ of $V_Q$ could be canonically extended to a connection $\tilde{\nabla}_Q$ of $\tilde{V}_Q$ that has a logarithmic pole along the point $0$. From Griffiths transversality, $\tilde{\nabla}_{Q,0} \Omega$ is a section of $\tilde{F}_Q^2$, hence we find
\[
F^2(\tilde{\mathcal{V}}_Q|_0) \supset \mathcal{Q} \Omega|_0 + \mathcal{Q} (\tilde{\nabla}_{Q,0} \Omega)|_0
\] (4.78)

Similarly $\tilde{\nabla}_{Q,0} \Omega$ is a section of $\tilde{F}_Q^2$ and $\tilde{\nabla}_{Q,0} \Omega$ is a section of $\tilde{F}_Q^0$, which yields
\[
F^1(\tilde{\mathcal{V}}_Q|_0) \supset \mathcal{Q} \Omega|_0 + \mathcal{Q} (\tilde{\nabla}_{Q,0} \Omega)|_0 + \mathcal{Q} (\tilde{\nabla}_{Q,0}^2 \Omega)|_0
\] (4.79)

The restriction of $\Omega$ to $\Delta$ is mapped to a section of $\tilde{V}$ by comparison isomorphism \[2.59\] and we now compute the value of $I_B(\Omega|_{\Delta})$ at $0$. With respect to the regularised frame $\{\tilde{\gamma}^a(\varphi)\}_{a=0}^3$ of $V$, $I_B(\Omega|_{\Delta^*})$ has an expansion,
\[
I_B(\Omega|_{\Delta^*})|_{\varphi} = \sum_{a=0}^3 \gamma^a(\varphi) \varpi_a(\varphi)
\]
\[
= \sum_{a,b,c} \gamma^a(\varphi) \left( \exp\left(\frac{-\log \varphi}{2 \pi i} N\right) \right)_{ab} \left( \exp\left(\frac{\log \varphi}{2 \pi i} N\right) \right)_{bc} \varpi_c(\varphi)
\]
\[
= \sum_{a,b} \tilde{\gamma}^a(\varphi) \left( \exp\left(\frac{\log \varphi}{2 \pi i} N\right) \right)_{ab} \varpi_b(\varphi)
\] (4.80)

where $\left( \exp\left(\frac{-\log \varphi}{2 \pi i} N\right) \right)_{ab}$ is the matrix of the operator $\exp\left(\frac{-\log \varphi}{2 \pi i} N\right)$ with respect to the basis $\{\gamma^a\}_{a=0}^3$. From this expansion we find that,
\[
I_B(\Omega|_{\Delta^*})|_0 = \sum_{a,b} \lim_{\varphi \to 0} \tilde{\gamma}^a(\varphi) \left( \exp\left(\frac{\log \varphi}{2 \pi i} N\right) \right)_{ab} \varpi_b(\varphi) = \tilde{\gamma}^0(0)
\] (4.81)

To compute $I_B(\tilde{\nabla}_{Q,0}^p \Omega|_{\Delta^*})|_0$, $p > 0$, we will need the following equation from \[17\],
\[
I_B(\tilde{\nabla}_{Q,0}^p \Omega|_{\Delta^*}) = \sum_{a=0}^3 \gamma^a(\varphi) \int_{C_a(\varphi)} \nabla_{Q,0}^p \Omega|_{\Delta^*} = \sum_{a=0}^3 \gamma^a(\varphi) \varpi^j_a(\varphi)
\] (4.82)
from which we have,

\[
I_B(\tilde{\nabla}_{\tilde{Q},\varrho}^1 \Omega_{|\Delta})|_0 = \sum_{a,b} \lim_{\varphi \to 0} \tilde{\gamma}^a(\varphi)(\exp(\log(2\pi i) N))_{ab} \varphi_b(\varphi) = \frac{1}{(2\pi i)} \tilde{\gamma}^1(0)
\]

\[
I_B(\tilde{\nabla}_{\tilde{Q},\varrho}^2 \Omega_{|\Delta})|_0 = \sum_{a,b} \lim_{\varphi \to 0} \tilde{\gamma}^a(\varphi)(\exp(\log(2\pi i) N))_{ab} \varphi^2_b(\varphi) = \frac{2}{(2\pi i)^2} \tilde{\gamma}^2(0)
\]

\[
I_B(\tilde{\nabla}_{\tilde{Q},\varrho}^3 \Omega_{|\Delta})|_0 = \sum_{a,b} \lim_{\varphi \to 0} \tilde{\gamma}^a(\varphi)(\exp(\log(2\pi i) N))_{ab} \varphi^3_b(\varphi) = \frac{6}{(2\pi i)^3} \tilde{\gamma}^3(0)
\]

Therefore we find that \(\{I_B(\tilde{\nabla}_{\tilde{Q},\varrho}^p \Omega_{|\Delta})|_0\}_{p=0}^3\) are linearly independent, which immediately shows,

\[
I_B(F^3(\tilde{\gamma}_{\varrho}|_0)) = \tilde{Q} \tilde{\gamma}^0(0)
\]

\[
I_B(F^2(\tilde{\gamma}_{\varrho}|_0)) = \tilde{Q} \tilde{\gamma}^0(0) + \tilde{Q} \frac{1}{2\pi i} \tilde{\gamma}^1(0)
\]

\[
I_B(F^1(\tilde{\gamma}_{\varrho}|_0)) = \tilde{Q} \tilde{\gamma}^0(0) + \tilde{Q} \frac{1}{2\pi i} \tilde{\gamma}^1(0) + \tilde{Q} \frac{1}{(2\pi i)^2} \tilde{\gamma}^2(0)
\]

\[
I_B(F^0(\tilde{\gamma}_{\varrho}|_0)) = \tilde{Q} \tilde{\gamma}^0(0) + \tilde{Q} \frac{1}{2\pi i} \tilde{\gamma}^1(0) + \tilde{Q} \frac{1}{(2\pi i)^2} \tilde{\gamma}^2(0) + \tilde{Q} \frac{1}{(2\pi i)^3} \tilde{\gamma}^3(0)
\]

From 4.74, the limit Hodge filtration on \(\tilde{\mathcal{V}}|_0\) is the complexification of 4.84.

4.6. Splitness of Limit MHS

Let \(\{x^j\}_{j=0}^3\) be a new basis of \(\tilde{\mathcal{V}}|_0\) given by,

\[
x^j := (2\pi i)^{3-j} \tilde{\beta}^j(0), \quad j = 0, 1, 2, 3
\]

with respect to which, \(\{\tilde{\gamma}^a(0)\}_{a=0}^3\) are expressed as,

\[
\tilde{\gamma}^0(0) = \frac{\lambda}{(2\pi i)^3} x^0 + \frac{\lambda Y_{000}}{3 Y_{111}} x^3
\]

\[
\tilde{\gamma}^1(0) = -\frac{\lambda}{(2\pi i)^2} x^1
\]

\[
\tilde{\gamma}^2(0) = \frac{1}{2} \frac{\lambda}{2\pi i} x^2
\]

\[
\tilde{\gamma}^3(0) = -\frac{\lambda}{6} x^3
\]
The rational vector space \( \tilde{\mathcal{V}}|_{0,\mathbb{Q}} \) is spanned by \( \{(2\pi i)^j x^j\}_{j=0}^3 \), and its weight filtration \( W_*(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) \) now becomes,

\[
\begin{align*}
W_0(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) &= W_1(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \mathbb{Q} x^3 \\
W_2(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) &= W_3(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \mathbb{Q} \frac{1}{(2\pi i)} x^2 + \mathbb{Q} x^3 \\
W_4(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) &= W_5(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) = \mathbb{Q} \frac{1}{(2\pi i)^2} x^1 + \mathbb{Q} \frac{1}{(2\pi i)} x^2 + \mathbb{Q} x^3 \\
W_6(\tilde{\mathcal{V}}|_{0,\mathbb{Q}}) &= \mathbb{Q} \frac{1}{(2\pi i)^3} x^0 + \mathbb{Q} \frac{1}{(2\pi i)^2} x^1 + \mathbb{Q} \frac{1}{(2\pi i)} x^2 + \mathbb{Q} x^3
\end{align*}
\]

(4.87)

The Hodge filtration \( F^*(\tilde{\mathcal{V}}|_0) \) now becomes,

\[
\begin{align*}
F^3(\tilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3} \mathbb{Q}\text{span}\{x^0 + \frac{(2\pi i)^3 Y_{000}}{3 Y_{111}} x^3\} \otimes\mathbb{Q} \mathbb{C} \\
F^2(\tilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3} \mathbb{Q}\text{span}\{x^0 + \frac{(2\pi i)^3 Y_{000}}{3 Y_{111}} x^3, x^1\} \otimes\mathbb{Q} \mathbb{C} \\
F^1(\tilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3} \mathbb{Q}\text{span}\{x^0 + \frac{(2\pi i)^3 Y_{000}}{3 Y_{111}} x^3, x^1, x^2\} \otimes\mathbb{Q} \mathbb{C} \\
F^0(\tilde{\mathcal{V}}|_0) &= \frac{\lambda}{(2\pi i)^3} \mathbb{Q}\text{span}\{x^0 + \frac{(2\pi i)^3 Y_{000}}{3 Y_{111}} x^3, x^1, x^2, x^3\} \otimes\mathbb{Q} \mathbb{C}
\end{align*}
\]

(4.88)

The key observation is the following theorem.

**Theorem 4.10.** Assuming mirror symmetry conjecture, the rational limit MHS \( H^3(\mathbb{Z}_\text{MS}) \) at large complex structure limit splits into

\[
H^3(\mathbb{Z}_\text{MS}) \simeq \mathbb{Q}(-1) \oplus \mathbb{Q}(-2) \oplus \mathbb{M}
\]

(4.89)

where \( \mathbb{M} \) is a two-dimensional MHS with rational vector space

\[
\mathbb{M}_\mathbb{Q} = \mathbb{Q} \frac{1}{(2\pi i)^3} x^0 + \mathbb{Q} x^3
\]

(4.90)

Its weight filtration \( W_* \mathbb{M} \) is,

\[
\begin{align*}
W_{-1} \mathbb{M} &= W_{-2} \mathbb{M} = \cdots = 0 \\
W_0 \mathbb{M} &= W_1 \mathbb{M} = \cdots = W_5 \mathbb{M} = \mathbb{Q} x^3 \\
W_6 \mathbb{M} &= W_7 \mathbb{M} = \cdots = \mathbb{Q} \frac{1}{(2\pi i)^3} x^0 + \mathbb{Q} x^3
\end{align*}
\]

(4.91)
and its Hodge filtration $F^\ast \mathcal{M}$ is,
\[
\begin{align*}
F^4 \mathcal{M} &= F^5 \mathcal{M} = \cdots = 0 \\
F^3 \mathcal{M} &= F^2 \mathcal{M} = F^1 \mathcal{M} = \lambda \left( \frac{2 \pi i}{3} \right)^3 \mathbb{Q} \left( x^0 + \frac{(2 \pi i)^3 Y_{000}}{3 Y_{111}} x^3 \right) \otimes \mathbb{Q} \mathbb{C} \\
F^0 \mathcal{M} &= \lambda \left( \frac{2 \pi i}{3} \right)^3 \left( \mathbb{Q} \left( x^0 + \frac{(2 \pi i)^3 Y_{000}}{3 Y_{111}} x^3 \right) + \mathbb{Q} x^3 \right) \otimes \mathbb{Q} \mathbb{C} \\
F^{-1} \mathcal{M} &= F^{-2} \mathcal{M} = \cdots = F^0 \mathcal{M}
\end{align*}
\]

(4.92)

**Proof.** This is straightforward from the weight filtration $W^\ast (\tilde{\mathcal{V}}|_0, \mathbb{Q})$ 4.87 and Hodge filtration $F^\ast (\tilde{\mathcal{V}}|_0)$ 4.88.

Since $W_{-1} \mathcal{M}$ is 0, $\text{Gr}_W^0 \mathcal{M}$ equals $W_0 \mathcal{M}$. The Hodge filtration $F^\ast \mathcal{M}$ induces a pure Hodge structure of weight 0 on $W_0 \mathcal{M}$, which is isomorphic to $\mathbb{Q}(0)$. The inclusion $W_0 \mathcal{M} \subset \mathcal{M}$ induces an injective morphism from $\mathbb{Q}(0)$ to $\mathcal{M}$, the quotient of which is the pure Hodge structure $\text{Gr}_W^0 \mathcal{M}$ that is isomorphic to $\mathbb{Q}(-3)$. Therefore we have found a short exact sequence in $\text{MHS}_\mathbb{Q}$,
\[
0 \longrightarrow \mathbb{Q}(0) \longrightarrow \mathcal{M} \longrightarrow \mathbb{Q}(-3) \longrightarrow 0
\]

(4.93)

which shows $\mathcal{M}$ is an extension of $\mathbb{Q}(-3)$ by $\mathbb{Q}(0)$. Therefore we have shown that for every $q \in \mathbb{Z}$, $H^q(\mathcal{Z}_\text{MS}^\ast)$ is a mixed Hodge-Tate object and it is nonzero for only finitely many $q$, hence the equivalence in Proposition C.5 immediately implies that $\mathcal{Z}_\text{MS}^\ast$ is essentially an object of $D^b(\text{MHT}_\mathbb{Q})$.

5. **The Motivic Nature of $\zeta(3)$**

In the abelian category $\text{MHS}_\mathbb{Q}$, the dual of an object $\mathcal{H}$ is defined to be \[16, 52\]
\[
\mathcal{H}^\vee := \text{Hom}_{\text{MHS}_\mathbb{Q}}(\mathcal{H}, \mathbb{Q}(0))
\]

(5.1)

The dual operation is exact \[52\], i.e. it sends a short exact sequence to a short exact sequence, therefore the dual of 4.93 is a short exact sequence
\[
0 \longrightarrow \mathbb{Q}(3) \longrightarrow \mathcal{M}^\vee \longrightarrow \mathbb{Q}(0) \longrightarrow 0
\]

(5.2)

**Theorem 0.1** Assuming mirror symmetry conjecture, the dual object $\mathcal{M}^\vee$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$ whose image in $\mathbb{C}/(2 \pi i)^3 \mathbb{Q}$ is the coset of $-(2 \pi i)^3 Y_{000}/(3 Y_{111})$.

**Proof.** The short exact sequence 5.2 immediately shows $\mathcal{M}^\vee$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(3)$. We now compute the object $\mathcal{M}^\vee$ explicitly from Definition A.3. Let $\{x_j\}_{j=0}^3$ be the basis of $(\tilde{\mathcal{V}}|_0) \mathcal{V}$ that is dual to $\{x^j\}_{j=0}^3$
\[
x_j(x^k) = \delta^k_j
\]

(5.3)
The rational vector space of $M^\vee$ is the subspace of $(\tilde{V}_{\{0\},\mathbb{Q}})^\vee$ spanned by $\{(2\pi i)^3 x_0, x_3\}$,

$$(M^\vee)_\mathbb{Q} := \mathbb{Q}(2\pi i)^3 x_0 + \mathbb{Q} x_3$$  \hspace{1cm} (5.4)

The weight filtration $W_\cdot M^\vee$ is,

$W_l M^\vee := \{\phi : \phi(W_r M) \subset W_{r+l} \mathbb{Q}(0)\}$  \hspace{1cm} (5.5)

so we find that,

$W_{-7} M^\vee = W_{-8} M^\vee = \cdots = 0$

$W_{-6} M^\vee = \cdots = W_{-1} M^\vee = \mathbb{Q}(2\pi i)^3 x_0$  \hspace{1cm} (5.6)

$W_0 M^\vee = W_1 M^\vee = \cdots = \mathbb{Q}(2\pi i)^3 x_0 + \mathbb{Q} x_3$

The Hodge filtration $F_\cdot M^\vee$ is,

$F^p M^\vee := \{\phi : \phi(F^r M) \subset F^{r+p} \mathbb{Q}(0)\}$  \hspace{1cm} (5.7)

so we find that,

$F^1 M^\vee = F^2 M^\vee = \cdots = 0$

$F^0 M^\vee = F^{-1} M^\vee = F^{-2} M^\vee = (2\pi i)^3 \mathbb{Q}(-\frac{(2\pi i)^3 Y_{000}}{3 Y_{111}} x_0 + x_3) \otimes \mathbb{Q} \mathbb{C}$  \hspace{1cm} (5.8)

$F^{-3} M^\vee = F^{-4} M^\vee = \cdots = (2\pi i)^3 \left(\mathbb{Q}(-\frac{(2\pi i)^3 Y_{000}}{3 Y_{111}} x_0 + x_3) + \mathbb{Q} x_0\right) \otimes \mathbb{Q} \mathbb{C}$

The results in Appendix [A.2] immediately show that the image of $M^\vee$ in $\mathbb{C}/(2\pi i)^3 \mathbb{Q}$ is the coset of $-(2\pi i)^3 Y_{000}/(3 Y_{111})$.

In all examples of mirror pairs where $Y_{000}$ has been computed, it is always of the form 0.8, hence the image of $M^\vee$ in $\mathbb{C}/(2\pi i)^3 \mathbb{Q}$ is the coset of a rational multiple of $\zeta(3)$.

**Remark 5.1.** This is compatible with Remark [4.7]. For a different choice of an integral symplectic basis of $H_3(W_{x_0}, \mathbb{Z})$, even though $Y_{000}$ is changed to $Y_{000} + r'$, $r' \in \mathbb{Q}$, the coset of $-(2\pi i)^3 Y_{000}/(3 Y_{111})$ in $\mathbb{C}/(2\pi i)^3 \mathbb{Q}$ does not change.

Assuming Conjecture [3.4] and mirror symmetry conjecture, the limit mixed motives $Z_{MS}$ constructed at large complex structure limit of mirror pairs $(M, W)$ such that,

1. The deformation of complex structures of $W$ is rationally defined.
2. The coefficient $Y_{000}$ in the prepotential $F$ of $M$ has been computed.
fulfil compelling examples of Conjecture GHP.

**Remark 5.2.** If we assume mirror symmetry conjecture, conjectures GHP and \([3,4]\) from the beginning, the computations in this paper show that for a one parameter mirror pair \((M,W)\) such that the deformation of \(W\) is rationally defined, \(Y_{000}\) must be of the form

\[
Y_{000} = \frac{r_1}{(2\pi i)^3} \zeta(3) + r_2, \quad r_1, r_2 \in \mathbb{Q} \tag{5.10}
\]

which provides a motivic interpretation of the occurrence of \(\zeta(3)\) in \(Y_{000}\).

**Acknowledgments**

It is a great pleasure to acknowledge many communications with Joseph Ayoub, who generously corrected and clarified many confusions about nearby cycle functors and mixed motives. Further thanks go to Francis Brown, Annette Huber-Klawitter and Marc Levine for very helpful answers to queries about mixed Tate motives. We are also grateful to Noriko Yui for a reading of the draft. W.Y. is very grateful to the Mathoverflow community, especially Mikhail Bondarko, who helped him to learn about Hodge structures and mixed motives. W.Y. is also very grateful for many discussions with Philip Candelas, Xenia de la Ossa and Noriko Yui on the arithmetic of Calabi-Yau manifolds. W.Y. wishes to acknowledge support from the Oxford-Palmer Graduate Scholarship and the generosity of Dr. Peter Palmer and Merton College. M.K. was supported in part by the EPSRC grant ‘Symmetries and Correspondences’, EP/M024830/1.
A. Mixed Hodge Structure

In this section we give a very brief introduction to mixed Hodge structure (MHS), while the readers are referred to [16, 52] for more systematic and complete treatments. Throughout this section, the ring $R$ will be either $\mathbb{Z}$ or $\mathbb{Q}$ for simplicity.

A.1. Definition of Mixed Hodge Structure

An (pure) $R$-Hodge structure $H$ of weight $l \in \mathbb{Z}$ consists of the following data [52],

1. An $R$-module $H_R$ of finite rank.

2. A decreasing filtration $F^*H$ of the complex vector space $H_\mathbb{C} := H_R \otimes_R \mathbb{C}$ such that $H_\mathbb{C}$ admits a decomposition,

$$H_\mathbb{C} = \oplus_{p+q=l} H^{p,q}$$  \hspace{1cm} (A.1)

where $H^{p,q} := F^p \cap \overline{F^q}$. Here the complex conjugation is defined with respect to the real structure $H_R = H_R \otimes_R \mathbb{R}$ of $H_\mathbb{C}$. The definition immediately implies that [52],

$$F^k = \oplus_{p \geq k} H^{p,l-p}$$  \hspace{1cm} (A.2)

The simplest example of Hodge structure is Hodge-Tate object $R(n), n \in \mathbb{Z}$ of weight $-2n$,

**Definition A.1.** The Hodge-Tate object $R(n)$ has $R$ module as,

$$(2\pi i)^n R \subset \mathbb{C}$$  \hspace{1cm} (A.3)

and its Hodge filtration is determined by the following decomposition,

$$R(n)^{-n,-n} = (2\pi i)^n R \otimes_R \mathbb{C}$$  \hspace{1cm} (A.4)

An $R$-mixed Hodge structure (MHS) consists of the following data,

1. An $R$-module $H_R$ of finite rank.

2. An increasing weight filtration $W_*$ of $H_Q := H_R \otimes_R \mathbb{Q}$.

3. A decreasing Hodge filtration $F^*$ of $H_C := H_R \otimes_R \mathbb{C}$.

such that the Hodge filtration $F^*$ induces a pure Hodge structure of weight $l$ on every graded piece $G_{l_i}^{W} W$ [52],

$$G_{l_i}^{W} W := W_i / W_{i-1}$$  \hspace{1cm} (A.5)

Morphisms between two $R$-MHS are defined as linear maps that are compatible with both weight filtrations and Hodge filtrations [16, 52].
Definition A.2. For two $R$-MHS $A$ and $B$, a morphism of weight $2m$ from $A$ to $B$ is a homomorphism $\phi$ from $A_R$ to $B_R$ that satisfies,

$$\phi(W_l A) \subset W_{l+2m}B \forall l$$
$$\phi(F^p A) \subset F^{p+m}B, \forall p$$

(A.6)

The category of $R$-MHS will be denoted by $\text{MHS}_R$. Inside the category $\text{MHS}_R$, the internal Hom operation is defined as [16, 52].

Definition A.3. For two $R$-MHS $A$ and $B$, there exists an $R$-MHS $\text{Hom}(A, B)$ whose $R$-module is,

$$\text{Hom}(A, B)_R := \text{Hom}(A_R, B_R)$$

(A.7)

Its weight filtration and Hodge filtration are,

$$W_l(\text{Hom}(A, B)) = \{\phi : \phi(W_r A) \subset W_{r+l}B, \forall r\}$$
$$F^p(\text{Hom}(A, B)) = \{\phi : \phi(F^r A) \subset F^{r+p}B, \forall r\}$$

(A.8)

In fact $\text{MHS}_R$ is a rigid tensor abelian category [52].

A.2. Extensions of MHS

An extension of $B$ by $A$ in $\text{MHS}_R$ is a short exact sequence,

$$0 \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow 0$$

(A.9)

Morphisms between two extensions are given by commutative diagrams of the form,

$$0 \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow 0$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$0 \longrightarrow A' \longrightarrow H' \longrightarrow B' \longrightarrow 0$$

(A.10)

The extension [A.9] is said to split if it is isomorphic to the trivial extension which is defined as [16, 52],

$$0 \longrightarrow A \overset{i}{\longrightarrow} A \oplus B \overset{j}{\longrightarrow} B \longrightarrow 0$$

(A.11)

where $i$ is the natural inclusion and $j$ is the natural projection.

Definition A.4. The abelian category of mixed Hodge-Tate structures $\text{MHT}_R$ is defined as the smallest full abelian subcategory of $\text{MHS}_R$ that contains Hodge-Tate objects $R(n), n \in \mathbb{Z}$ and is also closed under extension.

The set of isomorphism classes of extensions of $B$ by $A$, denoted by $\text{Ext}^1_{\text{MHS}_R}(B, A)$, has a group structure imposed by Baer summation, whose zero object is the trivial extension [A.11].
Two $R$-MHS $A$ and $B$ are said to be separated if the highest weight of $A$ is lower than the lowest weight of $B$, in which case the extension is also said to be separated. When $A$ and $B$ are separated, there is a canonical and functorial description of the group $\text{Ext}^1_{\text{MHS}_R}(B, A)$,

$$\text{Ext}^1_{\text{MHS}_R}(B, A) = \text{Hom}(B, A)_R \otimes_R \mathbb{C}/(F^0 \text{Hom}(B, A) + \text{Hom}(B, A)_R)$$  \hfill (A.12)

In particular, we have,

**Lemma A.5.** When $n \geq 1$, $Q(n)$ and $Q(0)$ are separated and

$$\text{Ext}^1_{\text{MHS}_Q}(Q(0), Q(n)) = \mathbb{C}/(2 \pi i)^n Q$$  \hfill (A.13)

*Proof.* The weight of $Q(n)$ is $-2n$ and the weight of $Q(0)$ is $0$, so they form a separated pair. The rational vector spaces of $Q(0)$ and $Q(n)$ are respectively,

$$Q(0) : Q \subset \mathbb{C}$$

$$Q(n) : (2 \pi i)^n Q \subset \mathbb{C}$$  \hfill (A.14)

From the definition of internal Hom we have,

$$F^0 \text{Hom}(B, A) = 0$$  \hfill (A.15)

Now we choose an isomorphism,

$$\text{Hom}(Q(0), Q(n)) \otimes_Q \mathbb{C} \simeq \mathbb{C}$$  \hfill (A.16)

such that,

$$\text{Hom}(Q(0), Q(n))_Q \simeq (2 \pi i)^n Q$$  \hfill (A.17)

Now immediately implies A.13.

□

When $n \geq 1$, for every element $\bar{s}$ of $\mathbb{C}/(2 \pi i)^n Q$, we want to construct an extension $H$ of $Q(0)$ by $Q(n)$ such that the image of $H$ under $\text{A.13}$ is $\bar{s}$. The complex vector space $\mathbb{C}^2$ has a natural basis $\{e_j\}_{j=1}^2$ where,

$$e_1 = (1, 0), \ e_2 = (0, 1)$$  \hfill (A.18)

Let the rational vector space of $H$ be,

$$H_Q := Q(2 \pi i)^n e_1 + Q e_2 \subset \mathbb{C}^2$$  \hfill (A.19)

Choose its weight filtration to be,

$$W_{-2n-1} H = W_{-2n-2} H = \cdots = 0$$

$$W_{-2n} = \cdots = W_{-1} = Q(2 \pi i)^n e_1$$

$$W_0 H = W_1 H = \cdots = H_Q$$  \hfill (A.20)
Let $s$ be an arbitrary complex number whose coset is $\tilde{s}$, then choose the Hodge filtration of $H_C$ to be,

$$
F^1 = F^2 = \cdots = 0 \\
F^0 = \cdots = F^{-(n-1)} = \mathbb{C} (se_1 + e_2) \\
F^{-n} = F^{-n-1} = \cdots = \mathbb{C}^2
$$

(A.21)

Now it is easy to see that $H$ admits a short exact sequence,

$$
0 \longrightarrow \mathbb{Q}(n) \longrightarrow H \longrightarrow \mathbb{Q}(0) \longrightarrow 0
$$

(A.22)

where the morphism from $\mathbb{Q}(n)$ to $H$ is the natural inclusion. So $H$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$. From the proof of [A.12] in [16, 52], $H$ is sent to $\tilde{s}$ by the equation [A.13]. The construction of $H$ immediately implies that the isomorphism class of this extension does not depend on the choice of $s$.

## B. Pure Motives

This section contains the construction of pure motives in case the readers need it. However it is not meant to be complete and necessary references will be given for further reading. First, we need to give a brief introduction to algebraic cycles and Weil cohomology theory.

### B.1. Algebraic Cycles

An excellent reference to algebraic cycles is the book [28], which is strongly recommended to the readers. We will follow the notations in [51]. Let $\text{SmProj}/k$ be the category of non-singular projective varieties over a field $k$, it is a symmetric monoidal category with product given by fiber product of varieties and symmetry given by the canonical isomorphism

$$
X \times_k Y \rightarrow Y \times_k X
$$

(B.1)

A prime cycle $Z$ of a non-singular projective variety $X$ is an irreducible algebraic subvariety, whose codimension is defined as $\dim X - \dim Z$. The set of all prime cycles of dimension $r$ (codimension $r$) generates a free abelian group that will be denoted by $C_r(X)$ ($C^r(X)$), and element of $C_r(X)$ ($C^r(X)$) will be called algebraic cycles of dimension $r$ (codimension $r$). Two prime cycles $Z_1$ and $Z_2$ are said to intersect with each other properly if

$$
\text{codim}(Z_1 \cap Z_2) = \text{codim}(Z_1) + \text{codim}(Z_2)
$$

(B.2)

where $Z_1 \cap Z_2$ means the intersection of the underlying point set of $\{Z_i\}_{i=1}^2$.

**Remark B.1.** An irreducible closed subset of $X$ has a natural algebraic variety structure induced from that of $X$ [61]. From now on, we will use this property implicitly.
If two prime cycles $Z_1$ and $Z_2$ intersect with each other properly, the intersection product $Z_1 \cdot Z_2$ is defined as,

$$Z_1 \cdot Z_2 = \sum_T m(T; Z_1 \cdot Z_2) T$$  \hspace{1cm} (B.3)

where the sum is over all irreducible components of $Z_1 \cap Z_2$ and $m(T; Z_1 \cdot Z_2)$ is Serre’s intersection multiplicity formula \[28\]. Extend the definition by linearity, intersection product could be defined for algebraic cycles $Z = \sum_j m_j Z_j$ and $W = \sum_l n_l W_l$ when $Z_j$ and $W_l$ intersect properly for all $j$ and $l$. Therefore there is a partially defined intersection product on algebraic cycles,

$$C^r(X) \times C^s(Y) \rightarrow C^{r+s}(Y)$$

If $f : X \rightarrow Y$ is a morphism between two non-singular projective varieties $X$ and $Y$, the pushforward homomorphism $f_*$ is defined by its action on a prime cycle $Z$ of $X$,

$$f_*(Z) := \begin{cases} 0 & \text{if } \dim f(Z) < \dim Z \\ [k(Z) : k(f(Z))] \cdot f(Z) & \text{if } \dim f(Z) = \dim Z \end{cases}$$  \hspace{1cm} (B.4)

where $k(Z)$ ($k(f(Z))$) is the function field of $Z$ ($f(Z)$) and $[k(Z) : k(f(Z))]$ is the degree of field extension \[61\]. Now we want to define the pullback homomorphism $f^*$. For a prime cycle $W$ of $Y$, the first attempt is to naively try \[28\],

$$f^*(W) := \sum_{T \subset f^{-1}(Z)} \ell_{\mathcal{O}_{X,T}}(\mathcal{O}_{f^{-1}(Z),T}) \cdot T$$  \hspace{1cm} (B.5)

where the sum is over irreducible components of $f^{-1}(Z)$ and $\ell_{\mathcal{O}_{X,T}}(\mathcal{O}_{f^{-1}(Z),T})$ is the length of $\mathcal{O}_{f^{-1}(Z),T}$ in $\mathcal{O}_{X,T}$. However this definition is only partially defined and in general $f^*(W)$ does not make sense \[28\]. The solution to the above problems is to find an equivalence relation $\sim$ on algebraic cycles such that,

1. For two arbitrary cycles $Z_1$ and $Z_2$, there exists a cycle $Z'_1$ in the equivalence class of $Z_1$ such that $Z'_1$ intersects $Z_2$ properly and the equivalence class of $Z'_1 \cdot Z_2$ is independent of the choice of $Z'_1$.

2. For an arbitrary cycle $W$ of $Y$, there exists a cycle $W'$ in the equivalence class of $W$ such that $f^*(W')$ is well defined by equation \[B.5\] and its equivalence class is independent of the choice of $W'$.

Equivalence relation that satisfies these properties will be called adequate equivalence relation. For an adequate equivalence relation $\sim$, let $C_*^r(X)$ be the quotient group $C^r(X)/\sim$, then we have a well defined intersection product,

$$C^r(X)_\sim \times C^s(Y)_\sim \rightarrow C^{r+s}(Y)_\sim$$  \hspace{1cm} (B.6)
and well defined pushforward and pullback homomorphisms,
\[ f_* : C_{r, \sim}(X) \to C_{r, \sim}(Y), \quad f^* : C_{\sim}^r(Y) \to C_{\sim}^r(X) \] (B.7)

The set of adequate equivalence relations could be ordered in a way such that \( \sim_1 \) is said to be finer than \( \sim_2 \) if for every cycle \( Z, Z \sim_1 0 \) implies \( Z \sim_2 0 \). The two most important adequate equivalence relations are rational equivalence and numerical equivalence \[51\]. In fact rational equivalence is the finest adequate equivalence relation and numerical equivalence is the coarsest adequate equivalence relation \[51\].

**B.2. Weil Cohomology Theory**

The note \[18\] is a very good reference to Weil cohomology theory, which our treatment will follow. Let \( \text{Gr}^{\geq 0} \text{Vec}_K \) be the rigid tensor abelian category of finite dimensional graded vector spaces over a field \( K \) of \( \text{char } K = 0 \) \[25\]. Every object \( V \) of \( \text{Gr}^{\geq 0} \text{Vec}_K \) has a decomposition according to the degree of its element,

\[ V = \oplus_{r \geq 0} V_r \] (B.8)

where \( V_r \) consists of elements homogeneous of degree \( r \). Tensor product of \( \text{Gr}^{\geq 0} \text{Vec}_K \) will be denoted by \( \otimes_K \). The category \( \text{Gr}^{\geq 0} \text{Vec}_K \) admits a graded symmetry defined as,

\[ v \otimes_K w \to (-1)^{\deg v \deg w} w \otimes_K v \] (B.9)

when \( v \) and \( w \) are both homogeneous elements. On the other hand, the category \( \text{SmProj}/k \) also admits product and symmetry operation \[B.1\]. A Weil cohomology theory is a symmetric monoidal functor

\[ H^* : \text{SmProj}/k^{\text{op}} \to \text{Gr}^{\geq 0} \text{Vec}_K \] (B.10)

that satisfies a list of axioms \[18\]. We will not give all the axioms here, but instead leave them to \[18\]. Among these axioms is the existence of a cycle map \( \text{cl} \),

\[ \text{cl} : C^*_{\text{rat}}(X)_\mathbb{Q} \to H^*(X) \] (B.11)

which doubles the degree and sends intersection product of cycles to cup product of cohomology classes. Here for an adequate equivalence relation \( \sim \), \( C^*_{\sim}(X)_\mathbb{Q} \) is defined as,

\[ C^*_{\sim}(X)_\mathbb{Q} := C^*_{\sim}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \] (B.12)

A Weil cohomology theory \( H^* \) yields an adequate equivalence relation \( \sim_{H^*} \) defined as \[51\],

\[ Z \sim_{H^*} 0 \iff \text{cl}(Z) = 0 \] (B.13)

However, it is not clear whether \( \sim_{H^*} \) is the same or not for different Weil cohomology theories. We now give three classical examples of Weil cohomology theories. Let \( X \) be a non-singular projective variety of dimension \( n \) over \( k \).
1. If \( \sigma : k \to \mathbb{C} \) is an embedding of \( k \) into \( \mathbb{C} \), the \( \mathbb{C} \)-valued points of \( X_\sigma(\mathbb{C}) \), form a projective complex manifold. The Betti cohomology \( H^*_B,\sigma(X) \) is defined as the singular cohomology of \( X_\sigma(\mathbb{C}) \) with coefficient \( \mathbb{Q} \),

\[
H^*_B,\sigma(X) := H^*(X_\sigma(\mathbb{C}), \mathbb{Q}) \tag{B.14}
\]

Since \( X_\sigma(\mathbb{C}) \) is projective, there exists a Hodge decomposition for every \( m \),

\[
H^m_B,\sigma(X) \otimes \mathbb{Q} \mathbb{C} = \oplus_{p+q=m} H^{p,q}(X_\sigma(\mathbb{C})) \tag{B.15}
\]

which induces a decreasing filtration of \( H^m_B,\sigma(X) \otimes \mathbb{Q} \mathbb{C} \),

\[
F^l(H^m_B,\sigma(X) \otimes \mathbb{Q} \mathbb{C}) := \oplus_{p \geq l} H^{p,m-p}(X_\sigma(\mathbb{C})) \tag{B.16}
\]

2. If \( \text{char} \ k = 0 \), take \( K \) to be \( k \). Let \( \Omega^*_X/k \) be the complex of sheaves of algebraic forms on \( X \),

\[
\Omega^*_X/k : 0 \to \mathcal{O}_{X/k} \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots \to \Omega^n_{X/k} \to 0 \tag{B.17}
\]

The algebraic de Rham cohomology of \( X \), denoted by \( H^\text{dR}(X) \), is the hypercohomology of the complex \( \Omega^*_X/k \),

\[
H^\text{dR}(X) := \mathbb{H}^*(X, \Omega^*_X/k) \tag{B.18}
\]

which is a vector space over \( k \). The note \cite{58} is a very good reference to the computations of hypercohomology. The complex \( \Omega^*_X/k \) admits a naive filtration \( F^p \Omega^*_X/k \),

\[
F^p \Omega^*_X/k : 0 \to 0 \to 0 \to \cdots \to 0 \to \Omega^p_{X/k} \to \cdots \to \Omega^n_{X/k} \to 0 \tag{B.19}
\]

which induces a decreasing filtration of \( H^\text{dR}(X) \) for each \( m \),

\[
F^p H^m_\text{dR}(X) := \text{Im} \left( \mathbb{H}^m(X, F^p \Omega^*_X/k) \to \mathbb{H}^m(X, \Omega^*_X/k) \right) \tag{B.20}
\]

3. If \( \ell \) is a prime number and \( \ell \neq \text{char} \ k \), let \( K \) be \( \mathbb{Q}_\ell \). The étale cohomology of \( X \) is defined as \cite{50},

\[
H^\text{ét}(X)_{\ell} := \lim_{\to \ell, k} H^\text{ét}(X \times k^\text{sep}, \mathbb{Z}/\ell^n) \otimes_{\mathbb{Z}/\ell} \mathbb{Q}_\ell \tag{B.21}
\]

which is a finite dimensional continuous representation of the Galois group \( \text{Gal}(k^\text{sep}/k) \).

The three classical examples above are not totally independent from each other, more precisely there exist canonical comparison isomorphisms between them,

1. The standard comparison isomorphism between Betti cohomology and de Rham cohomology,

\[
I_\sigma : H^m_B,\sigma(X) \otimes \mathbb{Q} \cong H^m_\text{dR}(X) \otimes_{k,\sigma} \mathbb{C} \tag{B.22}
\]

under which \( F^p(H^m_B,\sigma(X) \otimes \mathbb{Q} \mathbb{C}) \) is sent to \( F^p(H^m_\text{dR}(X)) \otimes_{k,\sigma} \mathbb{C} \)

2. The standard comparison isomorphism between Betti cohomology and étale cohomology,

\[
I_{\ell,\sigma} : H^m_B,\sigma(X) \otimes \mathbb{Q}_\ell \cong H^m_\text{ét}(X)_{\ell} \tag{B.23}
\]

which depends on the choice of an extension of \( \sigma \) to \( \bar{\sigma} : k^\text{sep} \to \mathbb{C} \).

46
B.3. Pure Motives

The three examples of classical Weil cohomology theories we give above behave as if they all arise from an algebraically defined cohomology theory over $\mathbb{Q}$, however this is known to be false [51]. Grothendieck’s idea to explain this phenomenon is that there exists a universal cohomology theory in the sense that all Weil cohomology theories are the realisations of it. More precisely there exists a rigid tensor abelian category $\mathbb{M}_{\text{hom}}$ over $\mathbb{Q}$ and a functor $M_{gm}$,

$$M_{gm} : \text{SmProj}/k^{\text{op}} \to \mathbb{M}_{\text{hom}}$$

such that for every Weil cohomology theory $H^*$, there exists a functor $H^*_m$ that makes the following diagram commutes,

$$\text{SmProj}/k^{\text{op}} \xrightarrow{M_{gm}} \mathbb{M}_{\text{hom}} \xrightarrow{H^*} \mathbb{M}_{\text{hom}} \xrightarrow{H^*_m} \text{Gr}^{\geq 0} \text{Vec}_K$$

Now we construct the category of motives $\mathbb{M}_{\sim}$ when $\sim$ is rational equivalence or numerical equivalence [51]. For two non-singular projective varieties $X$ and $Y$, the group of correspondences from $X$ to $Y$ of degree $r$ is defined as,

$$\text{Corr}^r(X, Y) := C^{\dim X + r}(X \times Y)$$

(B.25)

Correspondences could be composed [56],

$$\text{Corr}^r(X, Y) \times \text{Corr}^s(Y, Z) \to \text{Corr}^{r+s}(X, Z)$$

(B.26)

which is defined as,

$$g \times h \to h \circ g := (p_{13})_*( (p_{12})^* g \cdot (p_{23})^* h)$$

(B.27)

where $p_{ij}$ is the corresponding projection morphism. For a morphism $f : Y \to X$, its graph $\Gamma_f$ in $X \times Y$ is an algebraic variety that is isomorphic to $Y$ by projection morphism, therefore $\Gamma_f$ is an element of $\text{Corr}^0(X, Y)$ [61]. On the other hand, a correspondence of $\text{Corr}^0(X, Y)$ could be seen as a multi-valued morphism from $Y$ to $X$. A correspondence $\gamma$ induces a homomorphism from $H^*(X)$ to $H^*(Y)$ by,

$$\gamma_* : x \to p_{2,*} (p_1^* x \cup \text{cl}(\gamma))$$

(B.28)

where $p_1$ ($p_2$) is the projection morphism from $X \times Y$ to $X$ ($Y$). The homomorphism $(\Gamma_f)_*$ induced by $\Gamma_f$ could be seen as the pullback homomorphism $f^*$. The category $\mathbb{M}_{\sim}$ could be constructed by the following three steps [51],
1. Construct a category whose objects are formal symbols,
\[
\{ M_{gm}(X) : X \in \text{SmProj}/k \}
\]  
(B.29)

Let the morphisms between two objects be,
\[
\text{Hom}(M_{gm}(X), M_{gm}(Y)) := \text{Corr}_{\sim}^0(X, Y)_Q
\]  
(B.30)

where
\[
\text{Corr}_{\sim}^r(X, Y) = \text{Corr}^r(X, Y)/\sim, \text{Corr}_{\sim}^r(X, Y)_Q = \text{Corr}_{\sim}^r(X, Y) \otimes \mathbb{Z} \mathbb{Q}
\]  
(B.31)

This category could be seen as the linearisation of \text{SmProj}/k^{op}.

2. Take the pseudo-abelianisation of the category constructed in step 1 and denote this new category by \( M_{\sim}^{\text{eff}} \). The objects of \( M_{\sim}^{\text{eff}} \) are formally,
\[
\{(M_{gm}(X), e) : X \in \text{SmProj}/k \text{ and } e \in \text{Corr}_{\sim}^0(X, X)_Q, e^2 = e \}
\]  
(B.32)

Let the morphisms between two objects be,
\[
\text{Hom}((M_{gm}(X), e), (M_{gm}(Y), f)) := f \circ \text{Corr}_{\sim}^0(X, Y)_Q \circ e
\]  
(B.33)

If denote the graph of the identity morphism of \( \mathbb{P}^1 \) by \( \Delta_{\mathbb{P}^1} \), the object \( (M_{gm}(\mathbb{P}^1), \Delta_{\mathbb{P}^1}) \) has a decomposition \( [51] \),
\[
(M_{gm}(\mathbb{P}^1), \Delta_{\mathbb{P}^1}) = M_{gm}^0(\mathbb{P}^1) \oplus M_{gm}^2(\mathbb{P}^1)
\]  
(B.34)

The component \( M_{gm}^0(\mathbb{P}^1) \) will be called \( \mathbb{Q}(0) \) and the component \( M_{gm}^2(\mathbb{P}^1) \) will be called \( \mathbb{Q}(-1) \).

3. \( M_{\sim} \) is constructed from \( M_{\sim}^{\text{eff}} \) by inverting the object \( \mathbb{Q}(-1) \). The objects of \( M_{\sim} \) are formally,
\[
\{(M_{gm}(X), e, m) : X \in \text{SmProj}/k, e \in \text{Corr}_{\sim}^0(X, X)_Q, e^2 = e, \text{ and } m \in \mathbb{Z} \}
\]  
(B.35)

Let the morphisms between two objects be,
\[
\text{Hom}((M_{gm}(X), e, m), (M_{gm}(Y), f, n)) := f \circ \text{Corr}_{\sim}^{n-m}(X, Y)_Q \circ e
\]  
(B.36)

The category \( M_{\sim}^{\text{eff}} \) is isomorphic to the full subcategory of \( M_{\sim} \) whose objects are of the form \( (M_{gm}(X), e, 0) \).

The Hom set between two objects of \( M_{\sim} \) is a rational vector space whose dimension is finite when \( \sim \) is numerical equivalence. Direct sum in \( M_{\sim} \) is essentially defined by \( [56] \),
\[
(M_{gm}(X), e, m) \oplus (M_{gm}(Y), f, m) := (M_{gm}(X \amalg Y), e \oplus f, m)
\]  
(B.37)
Tensor product in $\text{M}_\sim$ is essentially defined by,

$$(M_{gm}(X), e, m) \otimes (M_{gm}(Y), f, n) := (M_{gm}(X \times Y), e \times f, m + n) \quad (B.38)$$

The object $\mathbb{Q}(0)$ could be shown to be a unit $[51, 56]$. Dual operation in $\text{M}_\sim$ is defined as,

$$(M_{gm}(X), e, m)^\vee := (M_{gm}(X), e^t, \dim X - m) \quad (B.39)$$

where $e^t$ means the transpose of $e$. From the construction of $\text{M}_\sim$, there is a functor,

$$M_{gm} : \text{SmProj}/k^{op} \to \text{M}_\sim \quad (B.40)$$

which sends $X$ to $(M_{gm}(X), \Delta_X, 0)$ and $f : Y \to X$ to $\Gamma f$. It is straightforward to see that every Weil cohomology theory $H^\ast$ automatically factors through $\text{M}_\text{rat}$,

$$\begin{array}{ccc}
\text{SmProj}/k^{op} & \xrightarrow{M_{gm}} & \text{M}_\text{rat} \\
\downarrow^{H^\ast} & & \downarrow^{H^\ast_{\text{rat}}} \\
\text{Gr}^{\geq 0} \text{Vec}_K & \downarrow &
\end{array}$$

however the category $\text{M}_\text{rat}$ is not abelian $[51, 56]$. On the other hand the category $\text{M}_\text{num}$ has been proved to be abelian and semi-simple $[41, 51]$, but it is not known whether an arbitrary Weil cohomology theory $H^\ast$ will factor through it or not. We now explain why. For an algebraic cycle $\gamma$ such that $\gamma \sim_{\text{num}} 0$, it induces a zero morphism in $\text{M}_\text{num}$. In order for $H^\ast$ to factor through $\text{M}_\text{num}$, the induced homomorphism $\gamma_\ast$ $[B.28]$ at cohomology level needs to be zero. This is not known currently, but it is conjectured to be true by Grothendieck,

**Conjecture D** For an algebraic cycle $\gamma$ that is numerically equivalent to 0, its cohomology class $\text{cl}(\gamma)$ is zero for every Weil cohomology theory.

This conjecture also implies that the homological equivalence relation $\sim_{H^\ast}$ defined by Weil cohomology theory $H^\ast$ is the same as numerical equivalence. There are also several other important conjectures about $\text{M}_\text{num}$, together with this one, they are called the standard conjectures $[47]$.  

### B.4. The Conjectured Abelian Category of Mixed Motives

The theory of pure motives could be seen as the universal Weil cohomology theory for non-singular projective varieties over $k$, so one might wonder what is the universal (Bloch-Ogus) cohomology theory for arbitrary varieties over $k$. Beilinson conjectured that there exists a rigid tensor abelian category of mixed motives $\text{MM}_k$ that has similar properties to that of $\text{MHS}_Q$ which forms the universal Bloch-Ogus cohomology theory for arbitrary varieties over $k$ $[48]$. Here we list several expected properties of the conjectured category $\text{MM}_k$. 

49
1. $\text{MM}_k$ is a rigid tensor abelian category whose Hom sets are vector spaces over $\mathbb{Q}$. It contains Tate objects $\mathbb{Q}(n), n \in \mathbb{Z}$ that satisfy,

$$\mathbb{Q}(m) \otimes \mathbb{Q}(n) = \mathbb{Q}(m + n)$$

(B.41)

while $\mathbb{Q}(0)$ is a unit object.

2. There exists a contravariant functor $M_{gm}$ from the category of varieties over $k$ to the derived category of $\text{MM}_k$,

$$M_{gm} : \text{Var}/k^{op} \rightarrow D^b(\text{MM}_k)$$

(B.42)

3. The full subcategory of $\text{MM}_k$ consists of semi-simple objects is equivalent to the category of pure motives.

4. For every object $\mathcal{M}$ of $\text{MM}_k$, there exists a finite weight filtration $W_*(\mathcal{M})$ such that all the graded pieces $\text{Gr}_W^i(\mathcal{M})$ are pure motives.

5. If $\sigma : k \rightarrow \mathbb{C}$ is an embedding, there exists a Hodge realisation functor,

$$\mathcal{R}_\sigma : \text{MM}_k \rightarrow \text{MHS}_\mathbb{Q}$$

(B.43)

which is compatible with all the structures of $\text{MM}_k$ and $\text{MHS}_\mathbb{Q}$. For every variety $X$ over $k$, $\mathcal{R}_\sigma(H^q(M_{gm}(X)))$ is the $q$-th Betti cohomology $H^q_B(X)$ together with the (natural) rational MHS on it [23].

6. The abelian category of mixed Tate motives $\text{TM}_k$ is the smallest full abelian subcategory of $\text{MM}_k$ that contains Tate objects $\mathbb{Q}(n), n \in \mathbb{Z}$ and is also closed under extension.

The construction of an abelian category $\text{MM}_k$ that possesses all the expected properties is still beyond reach. However, now there are several constructions of triangulated tensor categories that satisfy nearly all the expected properties of the derived category of $\text{MM}_k$, except those properties that need these triangulated categories to be realised as the derived category of an abelian category, like $t$-structure. One notable example is Voevodsky’s construction of $\text{DM}(k, \mathbb{Q})$ [49, 62].

C. Mixed Hodge Complex

In this section, we talk about a well-known result C.5. As we could not find a precise reference, we will give a routine proof based on the paper [11]. Let $D^b_{\text{MHS}_\mathbb{Q}}$ be the bounded derived category of rational mixed Hodge complexes, whose construction is left to the paper [11].
Definition C.1. Let $D^b_{\text{MHT}_Q}$ be the full subcategory of $D^b_{\text{MHS}_Q}$ that consists of $\mathbb{Q}$-mixed Hodge complexes whose cohomologies are mixed Hodge-Tate objects,

$$D^b_{\text{MHT}_Q} := \{ F^\bullet \in D^b_{\text{MHS}_Q} : H^q(F^\bullet) \in \text{MHT}_Q, \forall q \in \mathbb{Z} \} \quad (C.1)$$

The restriction of the functor $H^*$ to $D^b_{\text{MHT}_Q}$ is

$$H^* : D^b_{\text{MHT}_Q} \to \text{MHT}_Q \quad (C.2)$$

Now we want to prove that $D^b_{\text{MHT}_Q}$ is actually a full triangulated subcategory of $D^b_{\text{MHS}_Q}$, but first we need the following lemma.

Lemma C.2. The category $\text{MHT}_Q$ of mixed Hodge-Tate objects is closed under taking subquotient object.

Proof. Suppose $B$ is an object of $\text{MHT}_Q$ and $A$ is a subobject of $B$ in the bigger category $\text{MHS}_Q$, i.e.

$$A \subset B \quad (C.3)$$

we want to prove that $A$ is also an object of $\text{MHT}_Q$. As $B$ is a mixed Hodge-Tate object, it has a finite filtration by objects $\{B_i\}_{i=0}^N$ of $\text{MHT}_Q$,

$$0 = B_0 \subset B_1 \subset \cdots \subset B_N = B \quad (C.4)$$

such that the quotients are pure Hodge-Tate objects,

$$B_i/B_{i+1} \simeq \mathbb{Q}(n_i), \ n_i \in \mathbb{Z} \quad (C.5)$$

Let $j$ be the integer such that $A \subset B_j$, but $A \not\subset B_{j-1}$, then there exists a nonzero surjective morphism $f_j$ from $A$ to $B_j/B_{j-1} \simeq \mathbb{Q}(n_j)$ which induces a short exact sequence in $\text{MHS}_Q$,

$$0 \longrightarrow \ker f_j \longrightarrow A \overset{f_j}{\longrightarrow} \mathbb{Q}(n_j) \longrightarrow 0 \quad (C.6)$$

To show $A$ is an object of $\text{MHT}_Q$, we only need to show $\ker f_j$ is an object of $\text{MHT}_Q$, which is done by an easy induction on the dimension of $A$. The subquotient case follows easily from the above proof and the fact that $\text{MHT}_Q$ is an abelian category. \qed

Proposition C.3. $D^b_{\text{MHT}_Q}$ is a triangulated subcategory of $D^b_{\text{MHS}_Q}$.

Proof. For a morphism $f : A^\bullet \to B^\bullet$ in $D^b_{\text{MHS}_Q}$, its distinguished triangle is of the form [11],

$$A^\bullet \to B^\bullet \to \text{Cone} f \to A^\bullet[1] \quad (C.7)$$

which gives a long exact sequence in $\text{MHS}_Q$,

$$\cdots \to H^q(A^\bullet) \to H^q(B^\bullet) \to H^q(\text{Cone} f) \to H^{q+1}(A^\bullet) \to \cdots \quad (C.8)$$
If both $A\bullet$ and $B\bullet$ are objects of $D^b_{\text{MHT}_Q}$, then for every $q \in \mathbb{Z}$ we have,
\[
\mathcal{H}^q(B\bullet) \in \text{MHT}_Q, \quad \mathcal{H}^{q+1}(A\bullet) \in \text{MHT}_Q \tag{C.9}
\]
Lemma [C.2] immediately implies that $\mathcal{H}^q(\text{Cone } f)$ is an object of $\text{MHT}_Q$, hence $\text{Cone } f$ is an object of $D^b_{\text{MHT}_Q}$. Therefore the category $D^b_{\text{MHT}_Q}$ is a full triangulated subcategory of $D^b_{\text{MHS}_Q}$.

\[\square\]

For an object $A\bullet$ of $D^b_{\text{MHT}_Q}$, the truncated complex $\tau_{\leq i}(A\bullet)$ is also an object of $D^b_{\text{MHT}_Q}$. From Lemma 3.5 of [11] we have,

**Proposition C.4.** The functor $H^\bullet$ in [C.2] is the cohomological functor of a non-degenerated $t$-structure on $D^b_{\text{MHT}_Q}$, which is essentially the restriction of the $t$-structure of $D^b_{\text{MHS}_Q}$. The inclusion $\text{MHT}_Q \hookrightarrow D^b_{\text{MHT}_Q}$ is an equivalence between $\text{MHT}_Q$ and the heart of this $t$-structure on $D^b_{\text{MHT}_Q}$.

In section 3 of [11], Beilinson proves that for $A\bullet, B\bullet \in \text{MHS}_Q$,
\[
\text{Hom}^i_{D^b(\text{MHS}_Q)}(A\bullet, B\bullet) = \text{Hom}^i_{D^b_{\text{MHS}_Q}}(A\bullet, B\bullet) \tag{C.10}
\]
from which he concludes that the natural functor $D^b(\text{MHS}_Q) \rightarrow D^b_{\text{MHS}_Q}$ is an equivalence of categories. Since $\text{MHT}_Q$ is a full subcategory of $\text{MHS}_Q$, for $A\bullet, B\bullet \in \text{MHT}_Q$ we immediately have,
\[
\text{Hom}^i_{D^b(\text{MHT}_Q)}(A\bullet, B\bullet) = \text{Hom}^i_{D^b_{\text{MHT}_Q}}(A\bullet, B\bullet) \tag{C.11}
\]
The same proof as in [11] shows,

**Proposition C.5.** The functor $D^b(\text{MHT}_Q) \rightarrow D^b_{\text{MHT}_Q}$ is an equivalence of categories.

**Remark C.6.** This result is certainly well known, but we could not find a precise reference, therefore we provide a routine proof here. This equivalence is compatible with the equivalence $D^b(\text{MHS}_Q) \rightarrow D^b_{\text{MHS}_Q}$ and is essentially induced by the latter one.
References

[1] A. Altman, S. Kleiman. Introduction to Grothendieck duality theory. Springer, N.Y. 2009.

[2] J. Ayoub, Motives and Algebraic Cycles: a Selection of Conjectures and Open Questions. Preprint. http://user.math.uzh.ch/ayoub/PDF-Files/Article-for-Steven.pdf

[3] J. Ayoub, Les Six Opérations de Grothendieck et le Formalisme des Cycles Évanescents dans le Monde Motivique, I. Astérisque, Vol. 314 (2008). Société Mathématique de France. http://user.math.uzh.ch/ayoub/PDF-Files/THES.pdf

[4] J. Ayoub, Note sur les Opérations de Grothendieck et la Réalisation de Betti. Journal de l’Institut Mathématique de Jussieu 9, Issue 02 (2010), 225-263. http://user.math.uzh.ch/ayoub/PDF-Files/Realisation-Betti.pdf

[5] J. Ayoub, The motivic nearby cycles and the conservation conjecture. Algebraic Cycles and Motives, Vol 1, 3-54, London Math. Soc. Lecture Note Ser., 343 (2007), Cambridge Univ. Press. http://user.math.uzh.ch/ayoub/PDF-Files/Leiden.pdf

[6] J. Ayoub, L’Algèbre de Hopf et le Groupe de Galois Motiviques d’un Corps de Caractéristique Nulle, II. Journal für die reine und angewandte Mathematik (Crelles Journal). Volume 2014, Issue 693, Pages 151-226. http://user.math.uzh.ch/ayoub/PDF-Files/GaloisMotivic-2.pdf

[7] J. Ayoub, F. Ivorra and J. Sebag, Motives of Rigid Analytic Tubes and Nearby Motivic Sheaves, http://user.math.uzh.ch/ayoub/PDF-Files/Motive-of-Tube.pdf

[8] J. Ayoub, La Réalisation Étale et les Opérations de Grothendieck. Annales scientifiques de l’Ecole normale supérieure 47, fascicule 1 (2014), 1-145. http://www.math.uiuc.edu/K-theory/0989/Realisation-Etale.pdf

[9] J. Ayoub, A Guide to (Étale) Motivic Sheaves. Proceedings of the ICM 2014, http://user.math.uzh.ch/ayoub/PDF-Files/ICM2014.pdf

[10] A. A. Beilinson, Height pairing between algebraic cycles. K-theory, arithmetic and geometry (Moscow, 1984–1986), 1–25, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[11] A. A. Beilinson, Notes on Absolute Hodge Cohomology. Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part I, Proceeding of a Summer Research Conference held June 12-18, 1983. American Mathematical Society, Volume 55.
[12] V. Braun, P. Candelas and X. de la Ossa, Two One-Parameter Special Geometries, https://arxiv.org/pdf/1512.08367.pdf.

[13] R. Bryant and P. Griffiths, Some Observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle. Progress in Mathematics book series (PM, volume 36).

[14] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. R. Morrison, Mirror Symmetry for Two Parameter Models–I. Nucl. Phys. B 416 (1994), 481–562. Preprint, https://arxiv.org/pdf/hep-th/9308083.pdf.

[15] P. Candelas, X. C. de la Ossa, P. Green and L. Parkes, A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory, Nuclear Physics B359 (1991) 21-74.

[16] J. Carlson, Extensions of Mixed Hodge Structures, Journées de Géometrie Algébrique d’Angers 1979.

[17] D. Cox and S. Katz, Mirror Symmetry and Algebraic Geometry, American Mathematical Society.

[18] A. J. de Jong, Weil Cohomology Theories, http://math.columbia.edu/~dejong/seminar/note_on_...

[19] P. Deligne, Letter to C. Soulé (1985).

[20] P. Deligne, Le Groupe Fondamental de la Droite Projective Moins Trois Points. in: Galois groups over QQ. MSRI publications. 16 (Springer-Verlag 1989) pp. 72-297.

[21] P. Deligne, Local Behavior of Hodge Structures at Infinity. in Mirror Symmetry II. (ed. B. Greene and S. T. Yau, AMS and International Press, 1997) pp. 683-699.

[22] P. Deligne: Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[23] P. Deligne, Théorie de Hodge, III, Publ. Math. IHES 44, (1974), 5-77.

[24] P. Deligne and A. B. Goncharov, Groupes Fondamentaux Motiviques de Tate Mixte. Annales Scientifiques de l’École Normale Supérieure. 38 1 (2005) pp. 1-56.

[25] P. Deligne and J. S. Milne. Tannakian Categories. http://www.jmilne.org/math/xnotes/tc.html

[26] F. Elzein and J. Snoussi, Local Systems and Constructible Sheaves. Springer.

[27] E. Friedlander, An Introduction to K-theory. http://users.ictp.it/~pub_off/lectures/lns023/P...

[28] W. Fulton, Intersection Theory. Springer.
[29] D. R. Grayson, On the $K$-theory of Fields. Contemporary Mathematics, Volume 83 (1989)

[30] P. Griffiths, Periods of Integrals on Algebraic Manifolds, I. (Construction and Properties of the Modular Varieties). American Journal of Mathematics Vol. 90, No. 2 (Apr., 1968), pp. 568-626.

[31] P. Griffiths, Periods of Integrals on Algebraic Manifolds, II: (Local Study of the Period Mapping). American Journal of Mathematics Vol. 90, No. 3 (Jul., 1968), pp. 805-865.

[32] P. Griffiths, Periods of integrals on algebraic manifolds, III (Some global differential-geometric properties of the period mapping). Publications Mathématiques de l’IHÉS (1970), Volume: 38, page 125-180 ISSN: 0073-8301.

[33] M. Gross, D. Huybrechts and D. Joyce, Calabi-Yau Manifolds and Related Geometries. Springer.

[34] A. Grothendieck, On the de Rham Cohomology of Algebraic Varieties, Publ. Math. IHES, 29 (1966) pp. 351-359.

[35] SGA 7, Groupes de monodromie en géométrie algébrique, I, dirigé par A. Grothendieck, Lecture Notes in Math. 288, Springer-Verlag, 1972.

[36] R. Hain, Periods of Limit Mixed Hodge Structures. CDM 2002: Current Developments in Mathematics in Honor of Wilfried Schmid & George Lusztig. Ed. D Jerison, G Lustig, B Mazur, T Mrowka, W Schmid, R Stanley, and ST Yau. Somerville, MA: International Press, 2003. 113–133.

[37] R. Hain and M. Matsumoto, Tannakian Fundamental Groups Associated to Galois Groups. Galois Groups and Fundamental Groups. Ed. L Schneps. Cambridge: Cambridge Univ. Press, 2003. 183–216.

[38] A. Huber, Realization of Voevodskys Motives, J. Algebraic Geom. 9 (2000), no. 4.

[39] A. Huber, Corrigendum to: Realization of Voevodskys motives, J. Algebraic Geom. 13 (2004), no. 1.

[40] L. Illusie, Autour du Théorème de Monodromie Locale, dans Périodes p-adiques, Séminaire de Bures, 1988, Astérisque 223, 1994, 9–57.

[41] U. Jannsen. Motives, numerical equivalence, and semisimplicity. Invent. Math. 107 (1992), 447-452.

[42] N.M. Katz, Nilpotent Connections and the Monodromy Theorem. Applications of a Result of Turrittin. Publ. Math. IHES, 39 (1971) pp. 175–232.
[43] N. M. Katz, The Regularity Theorem in Algebraic Geometry, Proc. Internat. Congress Mathematicians (Nice, 1970), 1, Gauthier-Villars (1971) pp.437–443

[44] N. M. Katz and T. Oda, On the Differentiation of de Rham Cohomology Classes with respect to Parameters, J. Math. Kyoto Univ. Volume 8, Number 2 (1968), 199-213.

[45] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat. Toroidal embeddings I, Lecture notes in mathematics (Springer-Verlag); 339, 1973

[46] M. Levine, Tate motives and the vanishing conjectures for algebraic K-theory. Algebraic K-Theory and Algebraic Topology, ed. P.G. Goerss and J.F. Jardine, NATO ASI Series, Series C, Vol. 407 (1993) 167–188.

[47] M. Levine, Six Lectures on Motives. https://www.uni-due.de/~bm0032/publ/ICTPMotives.pdf

[48] M. Levine, Mixed Motives. An expository article in the Handbook of K-theory, vol 1, E.M. Friedlander, D.R. Grayson, eds., 429–522. Springer Verlag, 2005.

[49] C. Mazza, V. Voevodsky and C. Weibel, Lecture Notes on Motivic Cohomology, American Mathematical Society and Clay Mathematical Institute.

[50] J. S. Milne, Étale Cohomology. Princeton University Press.

[51] J. S. Milne, Motives – Grothendieck’s Dream. http://www.jmilne.org/math/xnotes/MOT.pdf

[52] C. Peters and J. Steenbrink, Mixed Hodge Structures, Springer

[53] J. Riou, Realizations Functors, https://www.math.u-psud.fr/~riou/doc/realizations.pdf

[54] W. Schmid, Variation of Hodge Structure: The Singularities of the Period Mapping. Inventiones math. 22, 211–319 (1973).

[55] C. Schnell, Canonical Extensions of Local Systems, arXiv:0710.2869, https://arxiv.org/abs/0710.2869.

[56] A. J. Scholl, Classical Motives. In: Motives, Seattle 1991, ed. U. Jannsen, S. Kleiman, J-P. Serre. Proc Symp. Pure Math 55 (1994), part 1, 163–187.

[57] The Stacks Project, Tag 0BSK, 10.150. Henselization and strict henselization. https://stacks.math.columbia.edu/tag/0BSK

[58] The Stacks Project, Tag 01FP, 20.26. Čech Cohomology of Complexes. https://stacks.math.columbia.edu/tag/01FP

[59] J. Steenbrink, Limits of Hodge Structures, Inventiones mathematicae 31 (1976): 229-258.
[60] J. Steenbrink and S. Zucker, Variation of Mixed Hodge Structure, I, Inventiones Mathematicae (1985), Volume: 80, page 489-542.

[61] R. Vakil, Foundations of Algebraic Geometry. [http://math.stanford.edu/~vakil/216blog/](http://math.stanford.edu/~vakil/216blog/)

[62] V. Voevodsky, A. Suslin, and E. Friedlander, Cycles, Transfers and Motivic Homology Theories, Annals of Mathematics Studies, vol. 143.

[63] F. E. Zein and J. Snoussi, Local Systems and Constructible Sheaves. Springer.