The Ultrarelativistic Kerr-Geometry

and

its Energy-Momentum Tensor

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Abstract

The ultrarelativistic limit of the Schwarzschild and the Kerr-geometry together with their respective energy-momentum tensors is derived. The approach is based on tensor-distributions making use of the underlying Kerr-Schild structure, which remains stable under the ultrarelativistic boost.

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**Introduction**

In a classical work [1] Aichelburg and Sexl obtained the ultrarelativistic limit of the Schwarzschild geometry. However, they noticed that the straightforward limit of the metric tensor itself does not exist even in the sense of distributions. They cured this problem by introducing a coordinate transformation whose singular behavior compensated that of the boost, thereby obtaining a sensible limit. Moreover, they were able to calculate the energy-momentum tensor, which provided a physical interpretation of the limit geometry as being generated by a massless ”point”-particle. Inspired by recent interest in the quantum-mechanical scattering of ultrarelativistic particles [2, 3] their work was generalised by applying essentially the techniques described above to the Kerr-Newman spacetime-family [4]. Although the formalism produced finite results, especially the energy-momentum tensor of the rotating case lacks a clear physical interpretation.

The present work advocates a different route of attack to the problem, which mainly relies on the theory of tensor-distributions and the Kerr-Schild type of all the geometries under consideration. Starting from the recently derived energy-momentum tensors of the Schwarzschild and the Kerr-spacetime [5, 6], we calculate their ultrarelativistic limits. Solving the Einstein-equations leads to the
corresponding ultrarelativistic geometries, which are again of Kerr-Schild type. More precisely, they turn out to be pp-waves [7].

We have organised our work in the following way: In the first part we rederive the ultrarelativistic Schwarzschild-geometry, focusing mainly on the “regularisation-dependence”. The second part uses the Kerr-Schild structure to obtain the energy momentum-tensor of Schwarzschild and Kerr. Finally, we derive the ultrarelativistic Kerr-metric by solving the Einstein equations.

1) Regularisation-dependence and the ultrarelativistic Schwarzschild geometry

In order to illustrate our approach, it is useful to briefly review the method of Aichelburg and Sexl [1] for boosting the Schwarzschild-geometry from a slightly different point of view. Let us start our investigation from the Schwarzschild geometry in Kerr–Schild form [8],

\[ g_{ab} = \eta_{ab} + f k_a k_b, \]  

with \( r = \sqrt{x^i x^i} \) and \( f(r) = 2m/r \). \( \eta_{ab} \) denotes the flat Minkowski metric of the decomposition and \( k^a = (1, x^i/r) \) the principal null direction (with respect to
The existence of the flat background metric $\eta_{ab}$ provides us with a natural notion of boosts as its associated isometries. From a more physical point of view, the metric $g_{ab}$ approaches asymptotically $\eta_{ab}$, which allows us to choose a boosted asymptotical observer rather than a static one. Having in mind to apply a boost to (1) it is useful to rewrite $m r$ with respect to a general Lorentz-frame associated with $\eta_{ab}$

$$m^2 r^2 = (P \cdot x)^2 + m^2 x^2, \quad P^a = m(1, 0, 0, 0).$$

In order to obtain a sensible ultrarelativistic limit for $P^a$ one has to ensure that its square vanishes in the same way as the boost velocity increases [1], thereby turning $P^a$ into a null vector $p^a$. Thus the naive limit of (1) is given by

$$g_{ab} = \eta_{ab} + 8 \frac{\vartheta(px)}{px} p_a p_b, \quad (2)$$

where $\vartheta$ denotes the step-function. We note that the boosted geometry again is of Kerr–Schild type, the flat part remaining unchanged under Lorentz-transformations. However, the profile function $\vartheta(px)/px$ is not locally integrable and thus a meaningless quantity even in the sense of distributions. Aichelburg and Sexl circumvented this problem by introducing an $m$-dependent coordinate-transformation together with the boost and performing the distributional limit afterwards. How-
ever, this amounts to a continuation of \( \psi(px)/px \) to the whole of test-function space, whose general form is given by

\[
\left( \frac{\psi(px)}{px} \right)_f \varphi := \int_{\mathbb{R}^4} pdx \, \bar{p} dx \, d^2\tilde{x} \, \frac{\psi(px)}{px} \left[ \varphi(px) - \psi(e^{f(\tilde{x},\bar{p}x)} - px)\varphi(0) \right]
\]

(3)

where \( \bar{p}x \) and \( \tilde{x} \) denote the remaining conjugate null and spacelike coordinates, i.e. \( \bar{p}^2 = 0, \bar{p}p = -1, \bar{p}\tilde{\partial} = \bar{p}\tilde{\partial} = 0. \) (3) coincides with the original profile function everywhere except for the plane \( px = 0, \) more precisely for all test functions supported in \( \mathbb{R}^4 - \{px = 0\}. \) It is important to note that the continuation (3) generally contains an arbitrary function \( f, \) which may depend on the remaining coordinates too. The explicit \( f \)-dependence is most easily displayed by employing the following identity

\[
\left[ \frac{\psi(px)}{px} \right]_f = \left[ \frac{\psi(px)}{px} \right]_{f=0} - \delta(px) f(\tilde{x}, \bar{p}x).
\]

Restricting \( f(\bar{p}x, \tilde{x}) \) to be independent of \( \bar{p}x \) allows us to perform the distributionally well-defined coordinate-transformation \( 4 \)

\[
\hat{x} = x + 4p \, \psi(px) \log px
\]
leaving us with an Minkowskian geometry everywhere except for the null plane $px = 0$. The final form of the ultrarelativistic metric is thus given by

$$g_{ab} = \eta_{ab} - 8\delta(px)f(\tilde{x})p_ap_b.$$ (4)

The result (4) illustrates that the most general regularisation introduces an arbitrary function $f(\tilde{x})$ depending on the coordinates $\tilde{x}$ which contributes only on the plane of the shock wave. It should be noted that the function $f$ cannot be determined in this approach, without any additional information. The situation is quite analogous to second quantized field theory, where ambiguities are a necessary consequence of regularising the infinite result. As in quantum field theory, a physical normalisation condition has to be imposed on the theory in order not to lose predictibility. In the next chapter, we will use the recently calculated energy-momentum tensor as basis for a different approach.

2) Energy-momentum Tensor of the ultrarelativistic Schwarzschild and Kerr-geometry

Our proposal to attack the problem will again rely on the Kerr-Schild structure, i.e. the flat part of the metric, which allows to define the notion to the concept of
a boost. This time however, we will focus on the energy-momentum tensor and show that its ultrarelativistic limit exists unambiguously, which allows us to solve the Einstein-equations for the metric directly. We will do this in the following for Schwarzschild as well as for Kerr. As pointed out in a previous work [5, 6] the decomposition enables us to calculate the energy-momentum tensor of the whole Kerr-Newman spacetime-family. With respect to Kerr-Schild coordinates one finds

\begin{align*}
\text{Schwarzschild: } T^a_{\; b} &= -m\delta^{(3)}(x)(\partial_t)^a(dt)_b \\
\text{Kerr: } T^a_{\; b} &= \frac{m\delta(z)}{8\pi} \left\{ \frac{2}{a} \left[ \frac{a^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \delta(\rho - a) \right] (dt)^a(\partial_t)_b \\
&+ \left[ (\partial_t)^a(e_\phi)_b - (e_\phi)^a(dt)_b \right] \left[ \frac{2 \rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \frac{\pi}{a} \delta(\rho - a) \right] \\
&+ \frac{2}{a} \left[ - \frac{\rho^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} + 2\delta(\rho - a) \right] (e_\phi)^a(e_\phi)_b \\
&- \frac{2 \vartheta(a - \rho)}{a \sqrt{a^2 - \rho^2}}(e_\rho)^a(e_\rho)_b \right\} ,
\end{align*}

(5)

where $e_\rho$ and $e_\phi$ are the polar basis vectors of the spacelike 2-plane, which is orthogonal to the boost-plane. Changing now to an arbitrary Lorentz-frame asso-
associated with $\eta_{ab}$, (5) becomes

Schwarzschild: $T^a_b = \delta(Qx)\delta^{(2)}(\tilde{x}) \, P^a P_b$,

Kerr: $T^a_b = \frac{\delta(Qx)}{8\pi} \left\{ \frac{2}{a} \left( - \left[ \frac{a^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] + \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} + \delta(\rho - a) \right) P^a P_b 
+ m(P^a(e_\phi)_b + (e_\phi)^a P_b) \left( 2 \left[ \frac{\rho \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] - \frac{\pi}{a} \delta(\rho - a) \right) 
+ \frac{2m^2}{a} \left( - \left[ \frac{\rho^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] - \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} + 2\delta(\rho - a) \right) (e_\phi)^a (e_\phi)_b 
- \frac{2m^2}{a} \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} (e_\rho)^a (e_\rho)_b \right\} ,$

where $Q^a$ denotes the spacelike vector spanning together with its timelike counterpart $P^a$ the 2-plane of the boost. The calculation of the ultrarelativistic limit $m \to 0$ of the energy-momentum tensor now merely reduces to replacing $P^a$ and $Q^a$ by their null limit $p^a$.

Schwarzschild: $T^a_b = \delta(px)\delta^{(2)}(\tilde{x}) \, p^a p_b$,

Kerr: $T^a_b = \frac{\delta(px)}{8\pi} \left\{ \frac{2}{a} \left( - \left[ \frac{a^2 \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] + \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} + \delta(\rho - a) \right) p^a p_b 
\right\} ,$

Note however, that in the Kerr-case we have restricted ourselves to boosts along the symmetry axis. The result for the Schwarzschild case coincides with the one obtained in [1]. Taking into account that the limit geometries are not only of Kerr-Schild type but moreover pp-waves, the integration of the Einstein-equations reduces to solving a 2-dimensional Poisson-equation, which in the case
of Schwarzschild is readily integrated to give the profile function

\[ f(px, \tilde{x}) = -8\delta(px) \log \rho. \]  

(6)

Let us emphasize that during the whole computation we neither encountered any singularities, nor had to perform any kind of "subtraction" procedure. This is due to the fact that the ultrarelativistic limit of the energy-momentum tensor is a perfectly well-defined quantity. The singularities encountered in the usual approach are mainly due to the fact that the ultrarelativistic boost changes the boundary conditions for the metric, which is no longer asymptotically flat. The situation is very similar to the one encountered in calculating the Green-function of the Laplace equation in an arbitrary number, say \( n \), of dimensions. Throwing away the constant part of the solution by imposing natural boundary conditions, renders the limit \( n \to 2 \) singular, since the logarithm does not vanish at infinity. However, the concept of the delta function remains same in any number of dimensions.

3) Ultrarelativistic Kerr-geometry

Having now calculated the energy-momentum tensor of the ultrarelativistic Kerr-geometry, the corresponding metric is obtained by solving the Einstein equations.
As pointed out in the previous paragraph, due to the pp-wave character of the geometry this reduces to a 2-dimensional Poisson-equation for the profile function $f$. Splitting $f(px, \tilde{x})$ into $f(\tilde{x})\delta(px)$ leaves us with

$$\tilde{\Delta}f = 4\left(\frac{a \vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \frac{1}{a} \delta(\rho - a) - \frac{1}{a} \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right). \quad (7)$$

Integration of this equation is most easily achieved by dividing the range of $\rho$ into two regions $0 < \rho < a$ and $a < \rho < \infty$, which in distributional language amounts to a corresponding condition on the support of the test-functions. In the first region (7) becomes

$$\tilde{\Delta}f_1 = 4\left(\frac{a}{\sqrt{a^2 - \rho^2}} - \frac{1}{a} \frac{1}{\sqrt{a^2 - \rho^2}} \right),$$

which integrates to

$$f_1 = 8 \log\left(\frac{\rho}{a + \sqrt{a^2 - \rho^2}}\right) + \frac{4}{a} \sqrt{a^2 - \rho^2} + C \log\left(\frac{\rho}{a}\right) + D,$$

where $C, D$ denote arbitrary integration constants. In the second region (7) is turned into the Laplace equation

$$\tilde{\Delta}f_2 = 0 \quad \text{with the general solution} \quad f_2 = C' \log\left(\frac{\rho}{a}\right) + D'.$$
where \( a \) is included for later convenience. Gluing together both pieces employing step functions, i.e. \( f = \vartheta(a - \rho)f_1 + \vartheta(\rho - a)f_2 \), yields

\[
\tilde{\Delta} f = 4 \left[ \frac{a\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] - \frac{4}{a} \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} + 2\pi \delta^{(2)}(\tilde{x})(8 + C) \\
+ \frac{1}{a} \delta(\rho - a)(C' - C - 4) + (D' - D) \partial_i (e_i^\dagger \delta(\rho - a)),
\]

where we made use of the identities

\[
\tilde{\Delta} \left( \vartheta(a - \rho) \frac{1}{a} \sqrt{a^2 - \rho^2} \right) = - \frac{1}{a} \frac{\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} - \left[ \frac{a\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] + \frac{1}{a} \delta(\rho - a),
\]

\[
\tilde{\Delta} \left( \vartheta(a - \rho) \log \left( \frac{\rho}{a + \sqrt{a^2 - \rho^2}} \right) \right) = \left[ \frac{a\vartheta(a - \rho)}{\sqrt{a^2 - \rho^2}} \right] + 2\pi \delta^{(2)}(\tilde{x}) - \frac{1}{a} \delta(\rho - a),
\]

\[
\tilde{\Delta} \left( \vartheta(a - \rho) \log \frac{\rho}{a} \right) = 2\pi \delta^{(2)}(\tilde{x}) - \frac{1}{a} \delta(\rho - a),
\]

\[
\tilde{\Delta} \left( \vartheta(\rho - a) \log \frac{\rho}{a} \right) = \frac{1}{a} \delta(\rho - a).
\]

Imposing \((\ref{5})\) fixes the constants to be \( C = -8, D = D', C' = -8 \), which produces the ultrarelativistic profile function

\[
f = -8 \log \rho + \vartheta(a - \rho) \left( 8 \log \left( \frac{\rho}{a + \sqrt{a^2 - \rho^2}} \right) + \frac{4}{a} \sqrt{a^2 - \rho^2} \right),
\]

\[(8)\]

where the value of the remaining constant \( D \) has been fixed to \(-8 \log a\) in order to produce \((\ref{3})\) in the limit \( a \to 0 \). Comparing \((8)\) with the result obtained in...
[4], we have found a closed form of the profile function, which is a well-defined distribution. This is mainly due to the fact that no singularities were encountered during the calculation and therefore no subtraction ambiguities arose either. Moreover, the energy-momentum tensors of geometries, boosted and unboosted, are in direct relation.

**Conclusion**

In this paper we proposed a method for obtaining ultrarelativistic versions of the Kerr and the Schwarzschild-geometry. Our approach is mainly based on the energy-momentum tensor, whose ultrarelativistic form is readily obtained. Finally, solving the Einstein equations produces a well-defined ultrarelativistic geometry.

The main ingredient for this calculation is provided by the Kerr-Schild decomposition of the original geometries, since it allows the calculation of the energy-momentum tensors as well as the definition of the boost. We believe that the main advantage of the presented approach lies in its ”regularisation”-independence, since all the quantities involved are well-defined distributions. An interesting further generalisation will cover the scattering properties of our solution and an
extension to the charged case.

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