Diagram technique for the heat kernel of the covariant Laplace operator

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Abstract

We present a diagram technique used to calculate the Seeley–DeWitt coefficients for a covariant Laplace operator. We use the combinatorial properties of the coefficients to construct a matrix formalism and derive a formula for an arbitrary coefficient.

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1 Introduction

Research motivations. The Fock proper-time method described in [1] was first used to work with Green’s functions but then found application in the field of gauge invariance [2] and quantum gravitation [3–5]. This approach is currently a rather convenient tool in mathematical physics [12] and especially theoretical physics, where it is used in loop computations and in finding divergences and renormalization. As an example, we mention the computations of two and three loops for the Yang–Mills theory in the background field method [17, 18]. Computing the coefficients of the heat kernel expansion in the power series in the proper time (Seeley–DeWitt coefficients) is a separate labor-consuming problem. In the case of a second-order differential operator, answers have been obtained up to the power $\tau^{5-\frac{d}{2}}$, where $d$ is the space dimension [6–11]. The most general survey of this field can be found in [19].

Our main subject here is the covariant Laplace operator in a space with a flat metric. In the framework of the general theory, we consider its heat kernel, whose coefficients in the power series expansion in the proper time satisfy relations (47). These differential equations admit solutions (14) and (25), but such a form of the solutions is inconvenient when calculating the trace is required. Our main goal here is to study the combinatorial properties of the Seeley–DeWitt coefficients based on the developed diagram technique. This paper generalizes and mathematically justifies the formalism proposed in [21].

The covariant perturbation theory [13–15], where the expansion of the operator exponential is also used, is closest to our proposed approach. But the authors of [13–15] were not interested in the combinatorics in higher orders of the perturbation theory. The higher orders of the heat kernel expansion were obtained in [16] when determining the restrictions on the form of the gauge field strength. We do not impose such restrictions here.

Results of the paper. This paper consists of two basic parts. The main result in the first part is the development of a diagram technique for working with the heat kernel of the covariant Laplace operator. In Definitions 2–4, we introduce the basic notions of this diagram technique. We also prove a theorem on differentiating a diagram, which plays the key role in computations. Further, as an example, we obtain the first coefficients arising in calculations with the determinant in the Yang–Mills theory [20, 22]. In the second part of the paper, we derive the formula for an arbitrary Seeley–DeWitt coefficient $a_n(x, x)$ by using the matrices and operators acting on matrices. Our main result is contained in formulas (51), (53) and (60). In conclusion, we calculate the coefficient $a_3(x, x)$ of the power $\tau^{3-d/2}$.

2 Heat kernel

The fundamental solution for the operator $A_0 = -\partial^\mu \partial_\mu$ satisfies the equation

$$A_0 G_0(x, y) = \delta(x - y).$$

(1)

To obtain $G_0(x, y)$ by the heat kernel method [1], we must determine the function $K_0(x, y; \tau)$ that solves the problem

$$\left( \frac{\partial}{\partial \tau} + A_0 \right) K_0(x, y; \tau) = 0, \quad K_0(x, y; 0) = \delta(x - y).$$

(2)
Then
\[ G_0(x, y) = \int_0^\infty d\tau K_0(x, y; \tau). \] (3)

If we consider the Laplace operator \( A_1 = A_0 + V(x) \) with a sufficiently good potential, then solving the problem
\[ \left( \frac{\partial}{\partial \tau} + A_1 \right) K_1(x, y; \tau) = 0, \quad K_1(x, y; 0) = \delta(x - y), \] (4)
we obtain the corresponding fundamental solution
\[ G_1(x, y) = \int_0^\infty d\tau K_1(x, y; \tau). \] (5)

By the general theory, for a certain class of potentials, we can write the formula for the logarithm of the determinant
\[ \ln \det \frac{A_1}{A_0} = -\int_0^\infty \frac{d\tau}{\tau} \text{tr} \left( e^{-A_1\tau} - e^{-A_0\tau} \right), \] (6)
where the integral operators \( e^{-A_1\tau} \) and \( e^{-A_0\tau} \) are respectively associated with \( K_1(x, y; \tau) \) and \( K_0(x, y; \tau) \).

3 Classical solution

Let an \( N \)-dimensional vector \( f(x) \) and an \( N \times N \) matrix \( B_\mu(x) \) with smooth coefficients be given. We calculate the operator \( A_1 \) as
\[ A_1 f(x) = -D_{x\nu}D_{x\mu}f(x), \] (7)
where
\[ D_{x\nu}f(x) = (\partial_{x\nu} + B_\mu(x)) f(x). \] (8)

With regard to problem statement (4), we seek the solution in the form
\[ K_1(x, y; \tau) = K_0(x, y; \tau) \sum_{n=0}^\infty \tau^na_n(x, y), \] (9)
where the function
\[ K_0(x, y; \tau) = \frac{1}{(4\pi\tau)^d/2} \exp \left( -\frac{|x - y|^2}{4\tau} \right) \] (10)
is a solution of problem (2). We then have the following assertion.

**Proposition 1.** Function (9) is a formal solution of problem (2) if
\[ (x - y)^\mu D_{x\nu}a_0(x, y) = 0, \]
\[ ((n + 1) + (x - y)^\mu D_{x\nu}) a_{n+1}(x, y) = -A_1a_n(x, y), \quad n \geq 0. \] (11)
Let the arguments $x$ and $y$ belong to $\mathbb{R}^d$. Then the operator $x^\mu \partial_\mu$ is the degree operator. Hence, if we assume that the solution can be expanded in a Taylor series in a neighborhood of a point, then each monomial of the form
\[ x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \alpha_j \in \mathbb{N} \cup \{0\}, \quad \forall j \in \{1, \ldots, d\}, \]
is subject to the condition $\alpha_1 + \ldots + \alpha_d = 0$. This immediately implies the alternative that either $\alpha_j = 0$ for all $j \in \{1, \ldots, d\}$ or there exists a number $j \in 1, \ldots, d$ such that $\alpha_j < 0$. A solution that can be expanded in a Taylor series at all points of the considered domain is said to be classical.

We introduce the operator of the P-ordered exponential for a fixed field $-B_\mu(x)$ and a contour $\gamma$ that is the segment connecting the initial and final points.

**Definition 1.** The P-exponential is defined by the formula
\[ \Phi(x, y) = 1 + \sum_{n=1}^{\infty} (-1)^n \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{dz_{\nu_1}}{ds_1} \cdots \frac{dz_{\nu_n}}{ds_n} B_{\nu_1}(z_1) \cdots B_{\nu_n}(z_n), \quad (12) \]
where the points $z_1, \ldots, z_{n-1}$ are located on the contour in the indexed order, the parameterizations $z_j(\cdot)$ are maps of the interval $[0, 1]$ into a part of the interval from $y$ to $z_{j-1}$ for $j = 2, \ldots, n$ and from $y$ to $x$ for $j = 1$.

The P-ordered exponential has the following basic properties:

- We have the equality
  \[ \Phi(x, x) = 1. \quad (13) \]
- Function $(12)$ is a solution of the integral equation
  \[ \Phi(x, y) = 1 - \int_0^1 ds \frac{dz^\nu(s)}{ds} B_\nu(z(s)) \Phi(z(s), y). \quad (14) \]
- The inverse operator has the form (see the proof in the appendix)
  \[ \Phi^{-1}(x, y) = \Phi(y, x). \quad (15) \]

**Proposition 2.** The function $\Phi(x, y)$ is a classical solution in $\mathbb{R}^d$ of the problem
\[ (x - y)^\mu D_\mu a_0(x, y) = 0, \quad a_0(x, y)|_{x=y} = 1. \quad (16) \]

**Proof:** It suffices to note that by choosing the parameterization in the form $z^\mu(s) = (1-s)y^\mu + sx^\mu$ and integrating by parts, we can represent the derivative as
\[ \partial_\nu(\Phi(x, y)) = -B_\mu(x)\Phi(x, y) - \int_0^1 dz^\nu(s) s H_{\mu\nu}(z(s), y), \quad (17) \]
where the operator $H_{\mu\nu}$ antisymmetric in the indices is given by
\[ H_{\mu\nu}(z, y) = \partial_\nu(B_\nu(z)\Phi(z, y)) - \partial_\nu(B_\nu(z)\Phi(z, y)). \quad (18) \]
Proposition 3. If the three points \( x, y \) and \( z \) are collinear, then
\[
\Phi(x, y) = \Phi(x, z)\Phi(z, y).
\] (19)

Proof: Because of the preceding properties and the vector collinearity condition \((x - y)^\mu = \text{const} \cdot (x - z)^\mu\), the left- and right-hand sides of (19) solve the problem
\[
(x - y)^\mu(\partial_{x^\mu} + B_\mu(x))\Psi(x, y) = 0, \quad \Psi(x, y)|_{x=y} = 1,
\] (20)
which proves the proposition. ■

To solve the other differential equations in (47), it suffices to understand how the solution of the following model problem is constructed. Let a sufficiently smooth and well-decreasing function \( V(x, y) \) be given and its expansion in a Taylor series around a point \( y \) have the form
\[
V(x, y) = \sum_{k=0}^{\infty} \frac{(x - y)^{\nu_1\ldots\nu_k}}{k!} V_{\nu_1\ldots\nu_k}(y),
\]
where we write \((x - y)^{\nu_1\ldots\nu_n} := (x - y)\nu_1 \cdot \ldots \cdot (x - y)\nu_n\) for convenience. Then we have the following assertion.

Proposition 4. The classical solution in \( \mathbb{R}^d \) of the equation
\[
((n + 1) + (x - y)^\mu D_{x^\mu}) W_{n+1}(x, y) = V(x, y)
\] (21)
has the form
\[
W_{n+1}(x, y) = \int_0^1 ds s^n \Phi(x, z(s))V(z(s), y).
\] (22)

Proof: It suffices to make the change
\[
W_{n+1}(x, y) = e^{-(n+1)\pi_\mu \ln(x-y)^\mu} \Phi(x, y)\widetilde{W}_{n+1}(x, y),
\] (23)
where
\[
\pi_\mu \ln (x - y)^\mu = \sum_{k=1}^{d} \pi_k \ln (x - y)^k, \quad \pi \in \mathbb{R}^d: \sum_{i=1}^{d} \pi_i = 1.
\]
We can then write the solution as
\[
\widetilde{W}_{n+1}(x, y) = \text{const}(y) + \int_0^1 ds s^{-1} e^{-(n+1)\pi_\mu \ln(z(s)-y)^\mu} \Phi^{-1}(z(s), y)V(z(s), y).
\] (24)
Solution (22) follows from the condition that the solution is bounded, Proposition 2 and the expression
\[
e^{(n+1)\pi_\mu \ln(z(s)-y)^\mu} = s^{n+1}e^{(n+1)\pi_\mu \ln (x-y)^\mu}.
\]

Corollary 1. It follows from Propositions 1–3 that the recursive system of classical solutions of differential equations (47) with the initial condition has the form
\[
a_{n+1}(x, y) = -\int_0^1 ds s^n \Phi(x, z(s))(A_1a_n(z(s), y)).
\] (25)
4 Differentiation formulas for the P-exponential

The formulas for differentiating the ordered exponential with the first and second arguments play a key role in constructing the diagram technique. In the case of covariant differentiation with respect to the first argument, a similar derivation can be found in [23]. We can prove the following assertion.

**Proposition 5.** Let $x, y \in \mathbb{R}^d$ then we have the equality

$$D_{x^\mu} \Phi(x, y) = \int_0^1 ds \, s \, \frac{d}{ds} \Phi(x, z) F_{\nu\mu}(z) \Phi(z, y),$$

(26)

where

$$F_{\nu\mu}(x) = \partial_{x^\nu} B_\mu(x) - \partial_{x^\mu} B_\nu(x) + [B_\nu(x), B_\mu(x)]$$

(27)

and $s$ is the parameterization parameter, $z_\nu(s) = (1 - s)y_\nu + sx_\nu$.

**Proof:** If we introduce the field

$$K_\mu(y) = -\int_0^1 dz^\nu(s) \, sH_{\mu\nu}(y, z(s))$$

and take relations (17) and (18) into account, then we obviously obtain

$$D_{x^\nu} \Phi(x, y) = K_\mu(x), \quad (x - y)^\nu K_\nu(z) \equiv 0.$$  

(28)

We can also show that

$$K_\mu(x) = -\int_0^1 ds \, s(x - y)^\nu (F_{\mu\nu} \Phi(z, y) + B_\nu(z)K_\mu(z)).$$

(29)

Further, directly substituting (26) in (29) and taking property (14) into account, we see that the formula to be proved holds.

The formula for differentiating the ordered exponential with respect to the second argument can be derived using (26). Indeed, if we differentiate the equality $\Phi^{-1}(x, y) \Phi(x, y) = 1$ and take basic properties (13), (15) and (19) into account, then we obtain the following assertion.

**Proposition 6.** Let $x, y \in \mathbb{R}^d$, then we have the equality

$$\partial_{x^\nu} \Phi(y, x) = \Phi(y, x)B_\mu(x) - \int_0^1 ds \, s \, \frac{d}{ds} \Phi(y, z) F_{\nu\mu}(z) \Phi(z, x).$$

(30)

Formula (30) is transformed after the changes $x \leftrightarrow y$ and $s \rightarrow 1 - s$, and we obtain the following assertion.

**Proposition 7.** Let $x, y \in \mathbb{R}^d$, then we have the equality

$$\partial_{y^\nu} \Phi(x, y) = \Phi(x, y)B_\mu(y) + \int_0^1 ds \, (1 - s) \, \frac{d}{ds} \Phi(x, z) F_{\nu\mu}(z) \Phi(z, y).$$

(31)
5  Diagram technique

5.1  Motivation

We note that the diagram technique presented here is not related to Feynman diagrams but is mainly used to obtain compact expressions and conveniently calculate and analyze the properties.

As already noted, a key role in constructing the diagram technique is played by formulas (26), (30) and (31). Indeed, we note that under the covariant differentiation, the function $\Phi(x, y)$ becomes an integral of the product of ordered exponentials and the field strength. Therefore, continuing the differentiation, we obtain only ordered exponentials and derivatives of the field strength. A more detailed analysis confirms this assumption and allows obtaining a method for controlling the coefficients arising under the action of the differentiation operators.

5.2  Basic definitions

The diagram technique consists of several basic elements.

**Definition 2.** A function $\Phi(x, y)$ is associated with a line with the arguments $x \rightarrow y$.

$$\Phi(x, y) = x \rightarrow y.$$ 

**Definition 3.** A function between ordered exponentials is associated with a circle depending on the following parameters:

1. a set of Greek indices $\mu_1 \ldots \mu_n$ associated with

$$\nabla_{\mu_1} \cdots \nabla_{\mu_{n-1}}(d(z - y)^\rho F_{\rho\mu_n}(z)),
$$

where all operators act on the variable $z$ and $\nabla_{x^\nu} \cdot = \partial_{x^\nu} \cdot + [B_\mu(x), \cdot]$, 

2. the parameterization parameter $s^k$ raised to an appropriate power, and 

3. a parameter $(y \rightarrow x)$ representing the integral over the straight line from $y$ to $x$ with the parameterization $z_\nu = (1 - s)y_\nu + sx_\nu$.

For example, formula (26) becomes

$$\int_0^1 ds \frac{dz^\nu}{dz} s \Phi(x, z) F_{\nu\mu}(z) \Phi(z, y) = x \rightarrow y $$

where the two lines correspond to the functions $\Phi(x, z)$ and $\Phi(z, y)$ and the circle with the parameters $\mu, (y \rightarrow x)$, and $s^1$ corresponds to the integral from $x$ to $y$ of the form $dz_\nu(s)F_{\nu\mu}(z(s))$ with weight $s$. In the integration, the circle runs through the abovementioned interval.
Definition 4. Let \( z_\nu = (1 - s)y_\nu + sx_\nu \). Then a construction of the form

\[
\int_0^1 ds \ s^n \ x \z(s) \ z(s) \quad (\forall \text{ diagram})
\]

corresponds to the diagram

\[
x \xrightarrow{s^{n+1}(y \to x)} y \quad (\forall \text{ diagram}).
\]

This definition is based on formula (25).

5.3 Main example

To understand the structure of the diagram technique, it is expedient to calculate the first iteration of system (25). In the zeroth order, the solution of system (47) is determined by the formula \( a_0(x, y) = \Phi(x, y) \) in Definition 1 (see formula (12)). Further, we apply the Laplace operator and the integration operator.

First, by formula (26), we have

\[
D_x \mu \Phi(x, y) = \int_0^1 ds \ s^n \Phi(x, z) F_{\nu \mu}(z) \Phi(z, y),
\]

или

\[
D_x \mu x \rightarrow y = x \mu s_1^n(y \to x) y.
\]

Second, it follows from relations (26) and (31) that the second covariant derivative has the form

\[
D_{x^\nu} D_{x^\rho} \Phi(x, y) = \int_y^x dz^\nu(s_1) s_1 \int_z^x dz^\rho(s_2) s_2 \Phi(x, z') F_{\nu \mu}(z') \Phi(z', z) F_{\nu \mu}(z) \Phi(z, y) +
\]

\[
+ \int_y^x dz^\nu(s_1) s_1^2 \int_z^x dz^\rho(s_2) (1 - s_2) \Phi(x, z') F_{\nu \mu}(z') \Phi(z', z) F_{\nu \mu}(z) \Phi(z, y) +
\]

\[
+ \int_y^x dz^\nu(s_1) s_1^2 \Phi(x, z) F_{\nu \mu}(z) \int_y^z dz^\rho(s_2) s_2 \Phi(z, z') F_{\nu \mu}(z') \Phi(z', y) +
\]

\[
+ \int_y^x s_1 \Phi(x, z) (s_1 \partial_{s_1} (dz^\nu(s_1) F_{\nu \mu}(z))) \Phi(z, y) + \int_y^x dz^\nu(s_1) s_1^2 \Phi(x, z) B^\mu(z) F_{\nu \mu}(z) \Phi(z, y) -
\]

\[
- \int_y^x dz^\nu(s_1) s_1^2 \Phi(x, z) F_{\nu \mu}(z) B^\mu(z) \Phi(z, y),
\]

which in the diagram language is equivalent to
where the first three terms in (33) correspond to the first three diagrams and the last three terms form the action of the operator $\nabla_\mu$ on the field strength and are depicted by the fourth diagram. Third, after integration, we obtain

$$\int_0^1 ds \, x'(s) (D_\nu D_\mu x \rightarrow y)|_{x=z'(s)} =$$

where we use the standard parameterization $z'_\mu(s) = (1-s)y_\mu + sx_\mu$ and Definition 4.

### 5.4 Theorem on the differentiation of diagrams

The main problem in this section is to learn how the covariant derivative can be applied to an arbitrary diagram, to simplify the diagram technique, and to eliminate unnecessary parameters.

**Proposition 8.** Let two continuous functions $f(x)$ and $g(x)$ and the parameterization

$$z_\mu(s) = (1-s)y_\mu + sy'_\mu, \quad x_\mu(t) = (1-t)z_\mu + ty'_\mu \quad (34)$$

be given. Then

$$\int_y^y ds \int_z^y dx_\mu s^n t^p f(z) g(x) = \int_y^y dx'_\mu \int_y^x dx'_\rho \int_z^y dx'_\rho (s')^n (t')^p \left( \frac{1-s'}{1-t'} \right)^p f(z') g(x'), \quad (35)$$

where

$$x'_\mu(t') = (1-t')y_\mu + t'y'_\mu, \quad z'_\mu(s') = (1-s')y_\mu + s'x'_\mu. \quad (36)$$

**Proof:** It suffices to change

$$t \rightarrow t' = (1-s)t + s, \quad (37)$$
then to use the Fubini theorem

\[(s, t') \in [0, 1] \times [s, 1] \rightarrow (s, t') \in [0, t'] \times [0, 1], \tag{38}\]

and to make another change \(s \rightarrow s/t'.\)

**Proposition 9.** We have the equality

\[
\begin{align*}
\rho s_1^n (z_3 \rightarrow z_1) + \mu s_3^n (y \rightarrow z_1) + x 
\rho (1 - s_2)(z_3 \rightarrow z_1) \mu s_3^{n+1}(y \rightarrow z_1) 
\rho s_2^{n+1}(y \rightarrow z_1) = x 
\nu s_1^{k+1}(y \rightarrow x) 
\mu s_3^n(y \rightarrow z_2)
\end{align*}
\]

**Proof:** This assertion can be proved in several stages. Proposition 8 on the permutation of integrals implies that

\[
\rho s_1^n (z_3 \rightarrow z_1) + \mu s_3^n (y \rightarrow z_1) + x 
\rho (1 - s_2)(z_3 \rightarrow z_1) \mu s_3^{n+1}(y \rightarrow z_1) 
\rho s_2^{n+1}(y \rightarrow z_1) = x 
\nu s_1^{k+1}(y \rightarrow x) \mu s_3^n(y \rightarrow z_2)
\]

Proposition 8 similarly implies the result for the second diagram

\[
\rho (1 - s_2)(z_3 \rightarrow z_1) + \mu s_3^{n+1}(y \rightarrow z_2) 
\rho s_2^{n+2}(y \rightarrow z_1) = x 
\nu s_1^{k+1}(y \rightarrow x) \mu s_3^{n+1}(1 - s_3)(y \rightarrow z_2)
\]

The parameters satisfy the relations

\[
s_2^{n+1} s_3^n \left( \frac{1 - s_3}{1 - s_2 s_3} \right) + s_2^{n+1} s_3^{n+1} - s_2^{n+2} s_3^{n+1} \left( \frac{1 - s_3}{1 - s_2 s_3} \right) = s_2^{n+1} s_3^n. \tag{39}\]

\[\blacksquare\]
Proposition 10. We have the equality

\[ y + \rho s_1(x \rightarrow z_3) s_1^{k+1}(y \rightarrow x) + \mu s_1^n(y \rightarrow z_1) \mu s_3^n(y \rightarrow z_1) = x + \rho(1 - s_3)(z_3 \rightarrow z_1) + \rho s_2^{n+1}(y \rightarrow z_1) = \]

Proposition 11. We have the equality

\[ y + \nu s_1^k(x \rightarrow z_3) s_1^{k+1}(y \rightarrow x) s_1^n(y \rightarrow z_1) = x + \nu(1 - s_3)(z_3 \rightarrow z_1) + \nu s_2^{n+1}(y \rightarrow z_1) = \]

Corollary 2. It follows from Propositions 9–11 that we can neglect the third parameter in Definition 3 because the integrals are always taken from the point \( y \) to the nearest point on the left.

Corollary 3. The formula for the first order becomes

\[
\int_0^1 ds \left. z'(s) (D_x \rho D_{x'} x \rightarrow y) \right|_{x = z(s)} = 2 x + \left. s_1^{k+1}(x \rightarrow z_3) \mu s_1^2 \right. \mu s_2^2 + \left. s_1^n(y \rightarrow z_2) \right. y + \]

Summarizing, we formulate a theorem on the differentiation of diagrams.

Theorem 1. Let there be an arbitrary diagram with \( i \) lines, \( k \) circles, and \( j \) crosses and the operation \( D_\rho \) be applied to the diagram. Let \( t \) be a new parameterization. Then the following three cases are possible:

1. If the derivative acts on a line, then the line is replaced with a circle with two lines on the sides and with the parameters \( \rho \) and \( t \) raised to the power equal to the power of the second parameter on the vertex to the right incremented by 1 (if the line is the rightmost, then the power is equal to 1). In this case, the power of each weight to the left of the differentiated line increases by 1. As a result, we have \( i \) diagrams.
2. If the derivative acts on a circle, then the power of its second parameter increases by 1, and the index \( \rho \) is attached on the left to the others. In this case, the power of each second parameter of the vertices to the left increases by 1. As a result, we have \( k \) diagrams.

3. If the derivative acts on a cross, then the diagram vanishes.

We therefore have \( i + k \) diagrams after differentiation.

For example,

\[
D_{x^\mu} x = s^1 \quad \nu_1 \ldots \nu_p s^2 \quad y =
\]

\[
= x \quad s^1 \quad \mu^{k+1} \quad s^n_1 \quad \nu_1 \ldots \nu_p s^2 \quad y + \\
+ x \quad s^{k+1} \quad \mu^{n+1} \quad s^n_1 \quad \nu_1 \ldots \nu_p s^2 \quad y + \\
+ x \quad s^{k+1} \quad \nu_1 \ldots \nu_p s^{n+1} \quad \mu \quad y + \\
+ x \quad s^{k+1} \quad \nu_1 \ldots \nu_p s^{n+1} \quad \mu \quad y.
\]

**Corollary 4.** The differentiation operation splits into the operation \((\nabla)\) acting on each circle and the operation of addition of a new circle (complying with the rule of increase in the powers of the parameterization parameters).

### 5.5 First order

The first order can be written as

\[
2 x \quad s^1 \quad \mu s^2_1 \quad \mu s^2_1 \quad y + \\
+ x \quad s^1 \quad \mu s^2_1 \quad y.
\]

To determine the contribution obtained by calculating the trace, we must take \( y = x \). Hence, we must determine the zeroth-order term of the Taylor series expansion. It is easy to see that the first term of the diagram starts from the quadratic term because each circle with a single index contains a linear contribution. In turn, the second term gives the result \( \nabla^\mu ((x - y)^\nu F_{\nu\mu})|_{x=y} \), which is obviously equal to zero because the field strength is antisymmetric.

### 5.6 Second order

To calculate the second order, it suffices to apply the covariant derivative twice to the first order and integrate. Further, we must set \( y = x \) and choose the contributions with nonzero traces. We must take into account that, first, each circle must contain at least two Greek indices because the contribution for \( y = x \) is otherwise zero; second, a circle with two equal Greek indices contributes zero; third, a diagram with one circle and with more than two Greek indices contributes zero. With these remarks taken into account, the formula becomes
For $y = x$, all ordered exponentials become the unit. Hence, using the relation

$$ (\nabla_\rho ((x - y)\nu F_{\nu\mu}(x)))|_{x = y} = F_{\rho\mu}(x), \quad (40) $$

we can write the answer as

$$ \frac{1}{12} F_{\rho\mu}(x) F^{\rho\mu}(x). \quad (41) $$

6 Operators acting on matrices

6.1 Motivation

The rules of the diagram technique described in the preceding section easily allow determining
the coefficients. As an example, we obtained standard results (41) for the first and second
orders. But the procedure is recursive. In this section, we present a formalism that allows
deriving the final series for the heat kernel of the covariant Laplace operator for $y = x$.

6.2 Matrices

It was previously shown that the diagram vertices have two parameters and each diagram can
therefore be associated with a matrix with two rows. The number of columns in such a matrix
is equal to the total number of circles and crosses in the diagram.

The matrices have the following basic properties:

1. The matrices have two rows and $n$ columns preserving the order, where $n$ is the total
   number of vertices in the diagram.

2. The elements in the first row can be of two types: either $1$, if the column corresponds to
   a cross in the diagram, or a set of Greek indices, if the column corresponds to a circle.

3. The second row contains powers of the second parameters of the vertices.

4. The unit $1$ denotes an element that does not contain columns.

We thus obtain matrices of the form $1, M^{2\times 1}, M^{2\times 2}, \ldots$. We note that in each order of the heat
kernel, there are finitely many matrices of a finite size. Such matrices can be added only if
they have the same elements and the same size. It is clear that the basis in this case is at most
countable because the first and the second row can contain only elements from a countable set.

6.3 Definitions of the operators

According to the rules listed above, any diagram can be written as a matrix, and it is hence
convenient to introduce an operator calculating the diagram at the point $y = x$. 
Definition 5. An operator $\Upsilon$ associates each set of Greek indices with a structure corresponding to it by Definition 3, where the differential $d(z - y)$ is replaced with $z - y$, the point $z = y = x$ is chosen, and the product of elements of the upper row is then divided by the product of elements of the lower row.

For example,

$$\Upsilon \mathbf{1} = 1, \quad \Upsilon \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) = \frac{1}{6} (\nabla_z ((z - y)^p F_{\rho\mu}(z))) |_{z=y=x}. \quad (42)$$

It was previously noted that the covariant derivative of the diagram reduces either to the operator of addition of a circle with an index or to the operator of index addition (complying with the rule of variation in the weight powers). The integration operator also reduces to the operator of cross addition. It is easy to see that we can introduce the following similar operators acting on matrices:

Definition 6. The operator of additional multiplication

$$B^\mu \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right) = \left( \begin{array}{c} \mu \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 + 1 \\ k_2 \\ \vdots \\ k_n \end{array} \right) + \cdots + \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 + 1 \\ k_2 + 1 \\ \vdots \\ k_n + 1 \end{array} \right). \quad (43)$$

Definition 7. The operator of column addition

$$A^\mu \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right) = \left( \begin{array}{c} \mu \\ \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 + 1 \\ k_2 \\ \vdots \\ k_n \end{array} \right) + \cdots + \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \mu \\ \nu_n \\ k_1 + 1 \\ k_2 + 1 \\ \vdots \\ k_n + 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 + 1 \\ k_2 + 1 \\ \vdots \\ k_n + 1 \end{array} \right). \quad (44)$$

Definition 8. The integration operator

$$S_{l/1}^\mu \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right) = \left( \begin{array}{c} \mu/1 \\ \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \\ l \\ k_1 \\ k_2 \\ \vdots \\ k_n \end{array} \right). \quad (45)$$

Using these definitions, we can write heat kernel (9) for the covariant Laplace operator in the form

$$K_1(x, x; \tau) = \Upsilon \left( 1 + \sum_{n=1}^{\infty} \tau^n \prod_{k=1}^{n} S_k^{1}(A^\mu + B^\mu)(A^{\mu_k} + B^{\mu_k}) \right) \mathbf{1}. \quad (46)$$

6.4 Commutators of the operators

The main idea is to use the commutators of the operators defined above to move the operators of additional multiplication and of column addition to the right in (46) and thus avoid the recursive procedure. The following assertion can easily be verified.

Proposition 12. We have the equalities

$$A^\mu S_k^1 = S_{k+1}^1 A^\mu + S_k^1 S_{k+1}^{\mu}, \quad B^\mu S_k^1 = S_{k+1}^1 B^\mu, \quad A^\mu S_k^{\nu_1 \ldots \nu_n} = S_{k+1}^{\nu_1 \ldots \nu_n} A^\mu + S_k^{\mu} S_{k+1}^{\nu_1 \ldots \nu_n}, \quad B^\mu S_k^{\nu_1 \ldots \nu_n} = S_{k+1}^{\nu_1 \ldots \nu_n} B^\mu + S_k^{\nu_1 \ldots \nu_n}. \quad (47)$$

This assertion gives only the relations used in the further proofs.
6.5 Properties of the integration operators

Relations (12) show that the commutators of the operators \( A \) and \( B \) with the operators \( S \) again become \( S \) operators. The whole construction for the heat kernel is therefore reducible to a sum of combinations of integration operators. Hence, we must introduce several structures to simplify the further description.

First, we introduce the following operator series. Let \( I_n = \{ n, \ldots, 1 \} \) and let \( \mu_{I_n} = \mu_n \ldots \mu_1 \) be a multi-index. Then

\[
\Sigma^\mu_{I_n} = \sum_\sigma \sum_{\nu_{i=1}^k} \mu_{I_n \setminus \nu_i} \nu_i \mu_{I_n \cap \nu_i} \nu_i \mu_{I_n \setminus \nu_i} \mu_{I_n \cap \nu_i} \nu_i \mu_{I_n \setminus \nu_i},
\]

where the sums are taken over partitions with the properties

- \( \sigma_j \subset I_n \) for all \( j \in \{1, \ldots, k\} \),
- \( \sigma_i \cap \sigma_j = \emptyset \) for \( i \neq j \),
- \( \cup_{i=1}^k \sigma_i = I_n \), and
- \( \sigma_s(i) > \sigma_s(j) \) for any \( s \in \{1, \ldots, k\} \) and any \( i, j \in \{1, \ldots, \sharp \sigma_s\} \) such that \( i > j \).

Here, the operator \( \sharp \) counts the number of elements in a set. The sum in formula (48) is finite, and there are hence no problems related to convergence.

If we act by such an operator on \( 1 \) (this does not impose any restrictions, because any matrix can be obtained from the unit by a certain set of integration operators) and then apply the operator \( \Upsilon \), then we consequently obtain a sum containing the field strengths and numerical coefficients. To obtain the exact result, we must take several assertions into account.

**Proposition 13.** We have the equalities

\[
\nabla_{\mu_{I_n}} ((x - y)^\rho F_{\rho\nu}(x)) = (x - y)^\rho \nabla_{\mu_{I_n}} F_{\rho\nu}(x) + \sum_{k=1}^n \nabla_{\mu_{I_n \setminus \nu}} F_{\mu_k\nu}(x),
\]

\[
\nabla_{\mu_{I_n}} ((x - y)^\rho F_{\rho\nu}(x)) \bigg|_{x=y} = \sum_{k=1}^n \nabla_{\mu_{I_n \setminus \nu}} F_{\mu_k\nu}(x).
\]

**Proposition 14.** We have the equalities

\[
\widehat{S}^\mu_{I_n\nu} = \Upsilon S^\mu_{I_n\nu}, \quad \Pi = \frac{1}{l} \sum_{k=1}^n \nabla_{\mu_{I_n \setminus \nu}} F_{\mu_k\nu}(x);
\]

where we introduce the additional notation \( \widehat{S}^\mu_{I_n\nu} \) for convenience.

**Proposition 15.** For any sets of indices \( I \) and \( J \) and positive integers \( l \) and \( k \), we have the relation

\[
\Upsilon(S^\mu_{I_k} S^\mu_{I_j}) = (\Upsilon S^\mu_{I_k}) (\Upsilon S^\mu_{I_j}) = \frac{1}{l \cdot k} \left( \sum_{(i_1, \ldots, i_l) \in \sharp \nu_{I_k}} \nabla_{\mu_{I_k}} F_{\mu_{i_1\nu}}(x) \right) \left( \sum_{(j_1, \ldots, j_l) \in \sharp \nu_{I_j}} \nabla_{\mu_{I_j}} F_{\mu_{j_1\nu}}(x) \right) = \widehat{S}^\mu_{I_k} \widehat{S}^\mu_{I_j}.
\]
The proof of this assertion follows from Propositions 13 and 14. It is clear that this property can be generalized to arbitrarily many integration operators. As a result, we can state that the structure $\Upsilon \Sigma \mathbb{1}$ is completely known and can be expressed in terms of the field strength and its derivatives according to the following assertion.

**Proposition 16.** We have the equality

$$
\Upsilon \Sigma^\mu_1 = \sum_{\sigma} \sum_{k=1}^{\mu} S_k^1 \frac{(A^k_\mu + B^k_\mu)(A^k_\mu + B^k_\mu)}{ slick \sum_{\sigma} (A + B)^\mu_1 \sigma_1 \Sigma_{l+k} S_{l+k}^1 (A + B)^\mu_1 \sigma_1,}
$$

where the summation complies with the rules described above.

**Remark:** Proposition 15 also holds in the case where the integration operators $S$ are replaced with the operators $\Sigma$. This assertion can also be generalized to the case of arbitrarily many factors.

### 6.6 Series for the heat kernel

Formula (46) is a Taylor series in powers of the proper time. A separate Seeley–DeWitt coefficient has the form

$$
a_n(x, x) = \Upsilon \left( \prod_{k=1}^{n} S_k^1 (A^k_\mu + B^k_\mu)(A^k_\mu + B^k_\mu) \right) \mathbb{1.}
$$

To write the coefficient in terms of the field strength and its derivatives, we need the following assertion.

**Proposition 17.** Let $I$ be a set of indices with $|I| = n$ and $l$ be a positive integer. Then we have the operator equality

$$(A + B)^\mu_1 S_l^1 = \sum_{k=0}^{n} \sum_{\sigma_k} \Sigma_{l+k}^1 (A + B)^\mu_1 \sigma_k, \quad (55)$$

where the sums are taken over all possible subsets $\sigma_k \subset I$ satisfying the conditions $|\sigma_k| = k$ and $\sigma_k(i) > \sigma_k(j)$ for all $i, j \in \{1, \ldots, k\}$ such that $i > j$.

**Proof:** We verify formula (55) by induction. For $n = 1$, everything is obvious:

$$(A + B)^\mu_1 S_l^1 = S_{l+1}^1 (A + B)^\mu_1 + S_{l+1}^1 \sigma_k. \quad (56)$$

We suppose that (55) is satisfied for $n = j$ and prove it for $n = j + 1$:

$$(A + B)^\mu_1 (A + B)^\mu_1 S_l^1 = \sum_{k=0}^{j} \sum_{\sigma_k} (A + B)^\mu_1 \sigma_k S_{l+k}^1 (A + B)^\mu_1 \sigma_k = \sum_{k=0}^{j} \sum_{\sigma_k} \left( \Sigma_{l+k}^1 S_{l+k}^1 (A + B)^\mu_1 \sigma_k + \Sigma_{l+k+1}^1 S_{l+k+1}^1 (A + B)^\mu_1 \sigma_k \right). \quad (57)$$

It remains to redefine the indices, and this implies (55). The proof of the proposition is complete.

It is more convenient to substitute formula (55) in (54) in several steps.
Proposition 18. Let $I = I_{2n}$. Then

$$
\prod_{k=1}^{n} S_k^1(A^{\mu_{2k}} + B^{\mu_{2k}})(A^{\mu_{2k-1}} + B^{\mu_{2k-1}}) =
$$

$$
= \sum_{p_1=0}^{2} \sum_{p_2=0}^{4-p_1} \ldots \sum_{p_{n-1}=0}^{2(n-1)-\sum_{k=1}^{n-2} p_k} \sum_{\sigma} S_{n+1-p_1}^{1,\sum_{i=1}^{p_1}} S_{n+1-p_1}^{1,\sum_{i=1}^{p_2}} S_{n+2-p_1-p_2}^{1,\sum_{i=1}^{p_3}} \ldots
$$

$$
\ldots S_{2n-1-\sum_{j=1}^{n} p_{j-1}}^{1,\sum_{i=1}^{p_n}} (A + B)^{\sum_{i=1}^{n-1} \sigma_{p_{i}}},
$$

where the sums are taken over all possible subsets $\sigma_{p_{k}}$ satisfying the conditions $\sigma_{p_{k}} = p_{k}$, $\sigma_{p_{k}} \subset (I_{2n} \setminus I_{2(n-k)}) \setminus (\cup_{j=1}^{k-1} \sigma_{p_{j}})$, and $\sigma_{p_{k}}(i) > \sigma_{p_{k}}(j)$ for all $i, j \in \{1, \ldots, p_{k}\}$ such that $i > j$.

To prove (58), we apply formula (55) several times.

Proposition 19. Let $I$ be a set of indices. Then

$$
\Upsilon(A + B)^{\mu_{i}} 1 = \Upsilon_0^{\mu_{i}} 1 = \tilde{\Sigma}^{\mu_{i}}.
$$

This assertion follows from the definition of operators. At the last stage, we must act by operator (58) on the unit and then multiply from the right by the operator $\Upsilon$. We can formulate the final theorem summarizing the obtained results.

Theorem 2. Let $\mu_{2k-1} = \mu_{2k}$ for all $k \in \{1, \ldots, n\}$. Then

$$
a_n(x, x) = \sum_{p_1=0}^{2} \sum_{p_2=0}^{4-p_1} \ldots \sum_{p_{n-1}=0}^{2(n-1)-\sum_{k=1}^{n-2} p_k} \sum_{\sigma} \frac{\tilde{S}_{n+1-p_1}^{\mu_{\sigma_{p_1}}} \tilde{S}_{n+1-p_1}^{\mu_{\sigma_{p_2}}} \tilde{S}_{n+2-p_1-p_2}^{\mu_{\sigma_{p_3}}} \ldots}{n + 1 - p_1 n + 2 - p_1 - p_2 \ldots} \frac{\tilde{S}_{2n-1-\sum_{j=1}^{n} p_{j-1}}^{\mu_{\sigma_{p_n}}} \tilde{S}_0^{\mu_i \cup \sum_{i=1}^{n-1} \sigma_{p_{i}}}}{2n - 1 - \sum_{j=1}^{n} p_{j-1} n},
$$

where the summation over $\sigma$ complies with the rules described in Proposition 18.

6.7 Third order

As an example, we calculate the results in the third order, which were first obtained in [7]. For this, we must use formulas (51), (53), and (60) adapted to this particular case:

$$
\frac{2 \cdot 4 - p_1 \cdot 5 - p_2 - p_1}{3}.
$$

To simplify the calculations, we note that

$$
(x - y)^{\nu} F_{\nu \mu}(x)|_{y=x} = 0, \quad F_{\mu \mu}(x) = 0,
$$

and the operator $\tilde{S}_l^{\mu \nu \tau}$ is symmetric in the indices $\mu$ and $\nu$. Taking the last remarks into account, we see that a nonzero contribution is given by the terms with $p_1 = p_2 = 0$ and $p_1 = 0$, $p_2 = 3$, and formula (61) hence becomes

$$
a_3(x, x) = \frac{1}{60} \tilde{S}_0^{\mu \nu \tau \nu} + \frac{1}{12} \left( \tilde{S}_2^{\mu \nu \tau \nu} \tilde{S}_0^{\tau \nu} + \tilde{S}_2^{\mu \nu \tau \nu} \tilde{S}_0^{\mu \nu} \right).
$$
The formulas (48) and (51) implies that
\[
\tilde{\Sigma}_0^{\mu \tau \nu} = \frac{4}{3} \nabla_\mu F_{\tau \nu}(x) \nabla_\mu F_{\tau \nu}(x) + \frac{2}{3} \nabla_\mu F_{\mu \nu}(x) \nabla_\tau F_{\tau \nu}(x) + \\
+ F_{\tau \nu}(x) \nabla_\mu F_{\tau \nu}(x) + \nabla_\mu \nabla_\mu F_{\tau \nu}(x) F_{\tau \nu}(x) - 2 F_{\tau \mu}(x) F_{\mu \nu}(x) F_{\nu \tau}(x)
\] (64)

and
\[
\tilde{\Sigma}_2^{\mu \tau \nu} + \tilde{\Sigma}_0^{\mu \tau \nu} = - \frac{1}{15} \nabla_\mu F_{\mu \nu}(x) \nabla_\tau F_{\tau \nu}(x).
\] (65)

After expressions (64) and (65) are substituted in (63), the final formula hence becomes
\[
a_3(x, x) = \frac{1}{45} \nabla_\mu F_{\tau \nu}(x) \nabla_\mu F_{\tau \nu}(x) + \frac{1}{180} \nabla_\mu F_{\mu \nu}(x) \nabla_\tau F_{\tau \nu}(x) + \\
+ \frac{1}{60} F_{\tau \nu}(x) \nabla_\mu \nabla_\mu F_{\tau \nu}(x) + \frac{1}{60} \nabla_\mu \nabla_\mu F_{\tau \nu}(x) F_{\tau \nu}(x) - \frac{1}{30} F_{\tau \mu}(x) F_{\mu \nu}(x) F_{\nu \tau}(x);
\] (66)

which gives the standard answer after taking the trace and integrating.

## 7 Appendix: Inverse P-exponential

Let \(x_1 \geq \ldots \geq x_k\) be the indicator function of the set
\[
\{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 \geq \ldots \geq x_k\}
\]
for \(k > 1\) and unit for \(k = 1\). Let the function \(\mathcal{N}\) be given by the formula
\[
\mathcal{N}(x_1, \ldots, x_k) = \sum_{n=1}^{k} \sum_{k_1 + \ldots + k_n = k} (-1)^n x_{1} \geq \ldots \geq x_{k} \cdot \ldots \cdot x_{k_1 + \ldots + k_n - 1 + 1} \geq \ldots \geq x_{k_n}.
\] (67)

The following lemmas can be proved by induction.

**Lemma 1:** Let \(c_n(x_1, \ldots, x_n) = x_1 \geq \ldots \geq x_n\) for \(n \geq 1\) and \(c_0 = 1\), and for \(n \geq 1\), let
\[
d_n(x_1, \ldots, x_n) = -c_n(x_1, \ldots, x_n) - \sum_{k=1}^{n-1} d_{n-k}(x_1, \ldots, x_{n-k}) c_k(x_{n-k+1}, \ldots, x_n).
\] (68)

Then \(d_n(x_1, \ldots, x_n) = \mathcal{N}(x_1, \ldots, x_n)\) for \(n \geq 1\).

**Lemma 2:** We have the equality \(\mathcal{N}(x_1, \ldots, x_n) = (-1)^n x_{1 \leq \ldots \leq n}\).

**Proof:** For \(k = 1\), the formula is obvious. We suppose that it holds for \(n = k - 1\). Then for \(n = k\),
\[
\mathcal{N}(x_1, \ldots, x_n) = - \left( \sum_{k=1}^{n} (-1)^{n-k} x_{1 \leq \ldots \leq n-k} x_{n-k+1} \geq \ldots \geq x_n \right).
\]

But
\[
x_{y_1 \leq \ldots \leq y_{n-1}} x_{y_n} - x_{y_1 \leq \ldots \leq y_{n-2}} x_{y_{n-1} \geq y_n} = x_{y_1 \leq \ldots \leq y_n} - x_{y_1 \leq \ldots \leq y_{n-2}} x_{y_{n-2} \geq \ldots \geq y_n},
\]
and for \(2 \leq j \leq n - 1\),
\[
x_{y_1 \leq \ldots \leq y_{j-1}} x_{y_{j} \geq \ldots \geq y_{n}} - x_{y_1 \leq \ldots \leq y_{j-1}} x_{y_{j-1} \geq \ldots \geq y_{n}} = x_{y_1 \leq \ldots \leq y_{j-1}} x_{y_{j-1} \geq \ldots \geq y_{n}}.
\]
which completes the proof. ■
To determine the inverse operator, we can regard (12) as a series in the background field. Indeed, after the transformation $B_\mu(y) \rightarrow pB_\mu(y)$, where $p$ is a certain variable, the operator can be expanded in a series

$$
\Phi(x, y) = 1 + \sum_{n=1}^{\infty} p^n a_n, \quad a_n = a_n(B, x, y).
$$

(69)

The ansatz for the inverse operator has a similar form:

$$
\Phi^{-1}(x, y) = 1 + \sum_{n=1}^{\infty} p^n b_n, \quad b_n = b_n(B, x, y).
$$

(70)

The relation $\Phi^{-1}(x, y)\Phi(x, y) = 1$ implies the system of recurrence relations

$$
b_n = -a_n - \sum_{k=1}^{n-1} b_{n-k}a_k, \quad n \geq 1,
$$

(71)

whence we take Definition 1 and Lemmas 1 and 2 into account and obtain $\Phi^{-1}(x, y) = \Phi(y, x)$.

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References

[1] V. Fock, *Die eigenzeit in der klassichen und in der quantenmechanick*, Phys. Z. Sowjetunion 12, 404–425 (1937)

[2] J. Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. 82, 664–679 (1951)

[3] B. S. DeWitt, *Quantum theory of gravity: I. The canonical theory*, Phys. Rev. 160, 1113–1148 (1967)

[4] B. S. DeWitt, *Quantum theory of gravity: II. The manifestly covariant theory*, Phys. Rev. 162, 1195–1239 (1967)

[5] B. S. DeWitt, *Quantum theory of gravity: III. Applications of the covariant theory*, Phys. Rev. 162, 1239–1256 (1967)

[6] H. P. McKeans, I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. 1, 43–69 (1967)

[7] P. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Diff. Geom. 10, 601–618 (1975)
[8] P. Amsterdamski, A. L. Berkin, D. J. O’Connor, b8 ‘Hamidew’ coefficient for a scalar field, Class. Quant. Grav. 6, 1981–1991 (1989)

[9] I. G. Avramidi, The covariant technique for the calculation of the heat kernel asymptotic expansion, Phys. Lett. B 238, 92–97 (1990)

[10] I. G. Avramidi, A covariant technique for the calculation of the one-loop effective action, Nucl. Phys. B 355, 712–754 (1991)

[11] A. E. M. van de Ven, Index-free heat kernel coefficients, Class. Quant. Grav. 15, 2311–2344 (1998)

[12] B. Iochum, C. Levy, D. Vassilevich, Spectral action beyond the weak-field approximation, Commun. Math. Phys. 316, 595–613 (2012)

[13] A. O. Barvinsky, G. A. Vilkovisky, Beyond the Schwinger–DeWitt technique: Converting loops into trees and in–in currents, Nucl. Phys. B 282, 163–188 (1987)

[14] A. O. Barvinsky, G. A. Vilkovisky, Covariant perturbation theory (II): Second order in the curvature. General algorithms, Nucl. Phys. B 333, 471–511 (1990)

[15] A. O. Barvinsky, G. A. Vilkovisky, Covariant perturbation theory (III): Spectral representations of the third-order form factors, Nucl. Phys. B, 333, 512–524 (1990)

[16] I. G. Avramidi, Heat kernel on homogeneous bundles over symmetric spaces, Commun. Math. Phys. 288, 963–1006 (2009)

[17] I. Jack, H. Osbornm Two-loop background field calculations for arbitrary background fields, Nucl. Phys. B 207, 474–504 (1982)

[18] J. P. Bornsen, A. E. M. van de Ven, Three-loop Yang–Mills β-function via the covariant background field method, Nucl. Phys. B 657, 257–303 (2003)

[19] D. V. Vassilevich, Heat kernel expansion: User’s manual, Phys. Rep. 388, 279–360 (2003)

[20] L. D. Faddeev, Mass in Quantum Yang-Mills Theory: Comment on a Clay Millenium problem, arXiv:0911.1013 [math-ph] (2009)

[21] A. V. Ivanov, About renormalization of the Yang–Mills theory and the approach to calculation of the heat kernel, EPJ Web of Conferences 158, 07004 (2017)

[22] S. E. Derkachev, A. V. Ivanov, L. D. Faddeev, Renormalization scenario for the quantum Yang–Mills theory in four-dimensional space–time, Theoret. and Math. Phys. 192:2, 1134–1140 (2017)

[23] Graham M. Shore, Symmetry restoration and the background field method in gauge theories, Annals of physics 137, 262-305 (1981)