Multipliers and integration operators between conformally invariant spaces

Daniel Girela 1 · Noel Merchán 2

Abstract
In this paper we are concerned with two classes of conformally invariant spaces of analytic functions in the unit disc $D$, the Besov spaces $B^p(1 \leq p < \infty)$ and the $Q_s$ spaces $(0 < s < \infty)$. Our main objective is to characterize for a given pair $(X, Y)$ of spaces in these classes, the space of pointwise multipliers $M(X, Y)$, as well as to study the related questions of obtaining characterizations of those $g$ analytic in $D$ such that the Volterra operator $T_g$ or the companion operator $I_g$ with symbol $g$ is a bounded operator from $X$ into $Y$.

Keywords Möbius invariant spaces · Besov spaces · $Q_s$ spaces · Multipliers · Integration operators · Carleson measures

Mathematics Subject Classification 30H25 · 47B38

1 Introduction

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc of the complex plane $\mathbb{C}$ and let $\mathcal{Hol}(D)$ be the space of all analytic functions in $D$ endowed with the topology of uniform convergence on compact subsets.
If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set
\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,
\]
and
\[
M_\infty(r, f) = \sup_{|z|=r} |f(z)|.
\]
If $0 < p \leq \infty$ the Hardy space $H^p$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that
\[
\|f\|_{H^p} \overset{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty.
\]
We mention [18] for the theory of $H^p$-spaces.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that
\[
\|f\|_{A^p_\alpha} \overset{\text{def}}{=} \left( (\alpha + 1) \int_{\mathbb{D}} (1 - |z|)^\alpha |f(z)|^p dA(z) \right)^{1/p} < \infty.
\]
The unweighted Bergman space $A^p_0$ is simply denoted by $A^p$. Here, $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [19], [36] and [58] for the theory of these spaces.

We let $\text{Aut}(\mathbb{D})$ denote the set of all disc automorphisms, that is, of all one-to-one analytic maps $\varphi$ from $\mathbb{D}$ onto itself. It is well known that $\text{Aut}(\mathbb{D})$ coincides with the set of all Möbius transformations from $\mathbb{D}$ onto itself:
\[
\text{Aut}(\mathbb{D}) = \{ \lambda \varphi_a : |\lambda| = 1, \ a \in \mathbb{D} \},
\]
where $\varphi_a(z) = (a - z)/(1 - \overline{a}z) \ (z \in \mathbb{D})$.

A linear space $X$ of analytic functions in $\mathbb{D}$ is said to be conformally invariant or Möbius invariant if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \text{Aut}(\mathbb{D})$ and, moreover, $X$ is equipped with a semi-norm $\rho$ for which there exists a positive constant $C$ such that
\[
\rho(f \circ \varphi) \leq C \rho(f), \quad \text{whenever} \ f \in X \text{ and } \varphi \in \text{Aut}(\mathbb{D}).
\]
The articles [8] and [44] are fundamental references for the theory of Möbius invariant spaces which have attracted much attention in recent years (see, e.g., [3,16,17,30,47,57,58]).

The Bloch space $\mathcal{B}$ consists of all analytic functions $f$ in $\mathbb{D}$ such that
\[
\rho_\mathcal{B}(f) \overset{\text{def}}{=} \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.
\]
The Schwarz-Pick lemma easily implies that $\rho_\mathcal{B}$ is a conformally invariant seminorm, thus $\mathcal{B}$ is a conformally invariant space. It is also a Banach space with the norm $\| \cdot \|_\mathcal{B}$ defined by $\|f\|_\mathcal{B} = |f(0)| + \rho_\mathcal{B}(f)$. The little Bloch space $\mathcal{B}_0$ is the set of those $f \in \mathcal{B}$ such that $\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0$. Alternatively, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. A classical reference for the theory of Bloch functions is [7]. Rubel and Timoney [44] proved that $\mathcal{B}$ is the largest “reasonable” Möbius invariant space. More precisely, they proved the following result.

**Theorem A** Let $X$ be a Möbius invariant linear space of analytic functions in $\mathbb{D}$ and let $\rho$ be a Möbius invariant seminorm on $X$. If there exists a non-zero decent linear functional $L$ on $X$ which is continuous with respect to $\rho$, then $X \subset \mathcal{B}$ and there exists a constant $A > 0$ such that $\rho_\mathcal{B}(f) \leq A \rho(f)$, for all $f \in X$. 

@article{Springer}
Here, a linear functional $L$ on $X$ is said to be decent if it extends continuously to $\mathcal{H}(\mathbb{D})$.

The space $BMOA$ consists of those functions $f$ in $H^2$ whose boundary values have bounded mean oscillation on the unit circle $\partial \mathbb{D}$ as defined by F. John and L. Nirenberg. There are many characterizations of $BMOA$ functions. Let us mention the following:

If $f \in \mathcal{H}(\mathbb{D})$, then $f \in BMOA$ if and only if $\|f\|_{BMOA} \equiv |f(0)| + \rho_s(f) < \infty$, where

$$\rho_s(f) = \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2}.$$

It is well known that $H^\infty \subset BMOA \subset B$ and that $BMOA$ equipped with the seminorm $\rho_s$ is a Möbius invariant space. The space $VMOA$ consists of those $f \in BMOA$ such that $\lim_{|a| \to 1} \|f \circ \varphi_a - f(a)\|_{H^2} = 0$, it is the closure of the polynomials in the $BMOA$-norm. We mention [28] as a general reference for the space $BMOA$.

Other important Möbius invariant spaces are the Besov spaces and the $Q_s$ spaces.

For $1 < p < \infty$, the analytic Besov space $B^p$ is defined as the set of all functions $f$ analytic in $\mathbb{D}$ such that $f' \in A^{p-2}$. All $B^p$ spaces ($1 < p < \infty$) are conformally invariant with respect to the semi-norm $\rho_{B^p}$ defined by

$$\rho_{B^p}(f) \equiv \|f'\|_{A^{p-2}}$$

(see [8, p. 112] or [16, p. 46]) and Banach spaces with the norm $\| \cdot \|_{B^p}$ defined by $\|f\|_{B^p} = |f(0)| + \rho_{B^p}(f)$. An important and well-studied case is the classical Dirichlet space $B^2$ (often denoted by $D$) of analytic functions whose image has a finite area, counting multiplicities.

The space $B^1$ requires a special definition: it is the space of all analytic functions $f$ in $\mathbb{D}$ for which $f'' \in A^1$. Although the semi-norm $\rho$ defined by $\rho(f) = \|f''\|_{A^1}$ is not conformally invariant, the space itself is. An alternative definition of $B^1$ with a conformally invariant semi-norm is given in [8], where it is also proved that $B^1$ is contained in any Möbius invariant space. A lot of information on Besov spaces can be found in [8,16,17,37,56,58]. Let us recall that

$$VMOA \subsetneq B_0, \quad BMOA \subsetneq B,$$

$$B^1 \subsetneq B^p \subsetneq B^q \subsetneq VMOA \subsetneq BMOA, \quad 1 < p < q < \infty.$$

If $0 \leq s < \infty$, we say that $f \in Q_s$ if $f$ is analytic in $\mathbb{D}$ and

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z, a)^s \, dA(z) < \infty,$$

where $g(z, a) = \log((1 - |az|)/|a - z|)$ is the Green function of $\mathbb{D}$. These spaces were introduced by Aulaskari and Lappan [12] while looking for characterizations of Bloch functions (see [50] for the case $s = 2$). For $s > 1$, $Q_s$ is the Bloch space, $Q_1 = BMOA$, and

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \quad 0 < s_1 < s_2 < 1.$$

It is well known [14,46] that for every $s$ with $0 \leq s < \infty$, a function $f \in \mathcal{H}(\mathbb{D})$ belongs to $Q_s$ if and only if

$$\rho_{Q_s}(f) \equiv \left( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) \right)^{1/2} < \infty.$$

All $Q_s$ spaces ($0 \leq s < \infty$) are conformally invariant with respect to the semi-norm $\rho_{Q_s}$. They are also Banach spaces with the norm $\| \cdot \|_{Q_s}$ defined by $\|f\|_{Q_s} = |f(0)| + \rho_{Q_s}(f)$. We mention [52,53] as excellent references for the theory of $Q_s$-spaces.
Let us recall the following two facts which were first observed in [10].

If \( 0 < p \leq 2 \), then \( B^p \subset Q_s \) for all \( s > 0 \).

If \( 2 < p < \infty \), then \( B^p \subset Q_s \) if and only if \( 1 - \frac{2}{p} < s \).

For \( g \) analytic in \( \mathbb{D} \), the Volterra operator \( T_g \) is defined as follows:
\[
T_g(f)(z) \overset{\text{def}}{=} \int_0^z g'(\xi) f(\xi) \, d\xi, \quad f \in \mathcal{H} \text{ol}(\mathbb{D}), \quad z \in \mathbb{D}.
\]

We define also the companion operator \( I_g \) by
\[
I_g(f)(z) \overset{\text{def}}{=} \int_0^z g(\xi) f'(\xi) \, d\xi, \quad f \in \mathcal{H} \text{ol}(\mathbb{D}), \quad z \in \mathbb{D}.
\]

The integration operators \( T_g \) and \( I_g \) have been studied in a good number of papers. Let us just mention here that Pommerenke [43] proved that \( T_g \) is bounded on \( H^2 \) if and only if \( g \in BMOA \) and that Aleman and Siskakis [4] characterized those \( g \in \mathcal{H} \text{ol}(\mathbb{D}) \) for which \( T_g \) is bounded on \( H^p \) \( (p \geq 1) \), while Aleman and Cima characterized in [1] those \( g \in \mathcal{H} \text{ol}(\mathbb{D}) \) for which \( T_g \) maps \( H^p \) into \( H^q \). Aleman and Siskakis [5] studied the operators \( I_g \) and \( T_g \) acting on Bergman spaces.

For \( g \in \mathcal{H} \text{ol}(\mathbb{D}) \), the multiplication operator \( M_g \) is defined by
\[
M_g(f)(z) \overset{\text{def}}{=} g(z) f(z), \quad f \in \mathcal{H} \text{ol}(\mathbb{D}), \quad z \in \mathbb{D}.
\]

If \( X \) and \( Y \) are two Banach spaces of analytic function in \( \mathbb{D} \) continuously embedded in \( \mathcal{H} \text{ol}(\mathbb{D}) \) and \( g \in \mathcal{H} \text{ol}(\mathbb{D}) \) then \( g \) is said to be a multiplier from \( X \) to \( Y \) if \( M_g(X) \subset Y \). The space of all multipliers from \( X \) to \( Y \) will be denoted by \( M(X, Y) \) and \( M(X) \) will stand for \( M(X, X) \).

Using the closed graph theorem we see that for the three operators \( T_g, I_g, M_g \), we have that if one of them maps \( X \) into \( Y \) then it is continuous from \( X \) to \( Y \). We remark also that
\[
T_g(f) + I_g(f) = M_g(f) - f(0)g(0).
\]

Thus if two of the operators \( T_g, I_g, M_g \) are bounded from \( X \) to \( Y \) so is the third one.

It is well known that if \( X \) is nontrivial then \( M(X) \subset H^\infty \) (see, e.g., [2, Lemma 1.1] or [48, Lemma 1.10]), but \( M(X, Y) \) need not be included in \( H^\infty \) if \( Y \nsubseteq X \). However, when dealing with Möbius invariant spaces we have the following result.

**Proposition 1** Let \( X \) and \( Y \) be two Möbius invariant spaces of analytic functions in \( \mathbb{D} \) equipped with the seminorms \( \rho_X \) and \( \rho_Y \), respectively. Suppose that there exists a non-trivial decent linear functional \( L \) on \( Y \) which is continuous with respect to \( \rho_Y \). Let \( g \in \mathcal{H} \text{ol}(\mathbb{D}) \). Then the following statements hold.

(i) If \( M_g \) is continuous from \( (X, \rho_X) \) into \( (Y, \rho_Y) \), then \( g \in H^\infty \).

(ii) If \( I_g \) is continuous from \( (X, \rho_X) \) into \( (Y, \rho_Y) \), then \( g \in H^\infty \).

Before embarking into the proof of Proposition 1, let us mention that, as usual, throughout the paper we shall be using the convention that \( C = C(p, \alpha, q, \beta, \ldots) \) will denote a positive constant which depends only upon the displayed parameters \( p, \alpha, q, \beta \ldots \) (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions \( E_1, E_2 \) we write \( E_1 \lesssim E_2 \), or \( E_1 \gtrsim E_2 \), if there exists a positive constant \( C \) independent of the arguments such that \( E_1 \leq CE_2 \), respectively \( E_1 \geq CE_2 \). If we have \( E_1 \lesssim E_2 \) and \( E_1 \gtrsim E_2 \) simultaneously then we say that \( E_1 \) and \( E_2 \) are equivalent and we write \( E_1 \asymp E_2 \). Also, if \( 1 < p < \infty \), \( p' \) will stand for its conjugate exponent, that is, \( \frac{1}{p} + \frac{1}{p'} = 1 \).
Proof of Proposition 1  Since \( X \) is conformally invariant, \( \text{Aut}(\mathbb{D}) \subset X \) \([8, \text{p. 114}]\) and
\[
\rho_X(\varphi_a) \asymp 1, \quad a \in \mathbb{D}. \tag{4}
\]
Suppose that \( M_g \) is continuous from \((X, \rho_X)\) into \((Y, \rho_Y)\). Using this, Theorem A, and (4) we obtain
\[
\rho_B(g \varphi_a) \lesssim \rho_Y(g \varphi_a) \lesssim \rho_X(\varphi_a) \lesssim 1, \quad a \in \mathbb{D}.
\]
This implies that
\[
(1 - |a|^2) \left| g'(a)\varphi_a(a) + g(a)\varphi'_a(a) \right| \lesssim 1, \quad a \in \mathbb{D}.
\]
Since \( \varphi(a) = 0 \) and \( \varphi'_a(a) = -(1 - |a|^2)^{-1} \), it follows that
\[
|g(a)| \lesssim 1, \quad a \in \mathbb{D},
\]
that is, \( g \in H^\infty \).

Similarly, if we assume that \( I_g \) is continuous from \((X, \rho_X)\) into \((Y, \rho_Y)\), we obtain
\[
\rho_B(I_g(\varphi_a)) \lesssim 1, \quad a \in \mathbb{D}.
\]
This implies that
\[
(1 - |a|^2) \left| (I_g(\varphi_a))'(a) \right| = (1 - |a|^2)|\varphi'_a(a)||g(a)| = |g(a)| \lesssim 1, \quad a \in \mathbb{D}.
\]

For notational convenience, set
\[
BQ = \{Q_s: 0 \leq s < \infty\} \cup \{B^p: 1 \leq p < \infty\}.
\]

The main purpose of this paper is characterizing, for a given pair of spaces \( X, Y \in BQ \), the functions \( g \in \mathcal{H}(\mathbb{D}) \) such that the operators \( M_g, T_g \) and/or \( I_g \) map \( X \) into \( Y \). When \( X \) and \( Y \) are Besov spaces this question has been extensively studied (see, e.g. [9,26,32,45,49,59]). Thus we shall concentrate ourselves to study these operators when acting between a certain Besov space \( B^p \) and a certain \( Q_s \) space and when acting between \( Q_{s_1} \) and \( Q_{s_2} \) for a certain pair of positive numbers \( s_1, s_2 \).

2 Multipliers and integration operators from Besov spaces into \( Q_s \)-spaces

For \( \alpha > 0 \), the \( \alpha \)-logarithmic Bloch space \( B_{\log, \alpha} \) is the Banach space of those functions \( f \in \mathcal{H}(\mathbb{D}) \) which satisfy
\[
\|f\|_{\log, \alpha} \overset{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \left( \log \frac{2}{1 - |z|^2} \right)^{\alpha} |f'(z)| < \infty. \tag{5}
\]
For simplicity, the space \( B_{\log, 1} \) will be denoted by \( B_{\log} \).

It is clear that \( B_{\log, \alpha} \subset B_0 \), for all \( \alpha > 0 \). Using the characterization of \( VMOA \) in terms of Carleson measures [28, p. 102], it follows easily that
\[
B_{\log, \alpha} \subset VMOA, \quad \text{for all} \quad \alpha > 1/2.
\]
In particular, \( B_{\log} \subset VMOA \).
Brown and Shields [15] showed that $M(B) = B_{\log} \cap H^\infty$. The spaces $M(B^p, B)$ ($1 \leq p < \infty$) were characterized in [25]. Namely, Theorem 1 of [25] asserts that $M(B^1, B) = H^\infty$ and

$$M(B^p, B) = H^\infty \cap B_{\log, 1/p'}, \quad 1 < p < \infty,$$

where $p'$ is the exponent conjugate to $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

In this section we extend these results. In particular, we shall obtain for any pair $(p, s)$ with $2 < p < \infty$ and $0 < s < \infty$ a complete characterization of the space of multipliers $M(B^p, Q_s)$.

Let us start with the case $s \geq 1$ which is the simplest one.

**Theorem 1** Let $g \in \mathcal{H}ol(\mathbb{D})$. Then:

(i) $I_g$ maps $B^1$ into $B$ if and only if $g \in H^\infty$.

(ii) $M_g$ maps $B^1$ into $B$ if and only if $g \in H^\infty$.

(iii) $T_g$ maps $B^1$ into $B$ if and only if $g \in B$.

**Proof** If $I_g(B^1) \subset B$ then, using Proposition 1, it follows that $g \in H^\infty$.

To prove the converse it suffices to recall that $B^1 \subset B$. Indeed, suppose that $g \in H^\infty$ and take $f \in B^1$. Then

$$(1 - |z|^2) \left| (I_g(f))' (z) \right| = (1 - |z|^2) |f' (z)||g(z)| \leq \|f\|_B \|g\|_{H^\infty}.$$

Thus $I_g(f) \in B$.

Hence (i) is proved. Now, (ii) is contained in [25, Theorem 1].

It remains to prove (iii). If $T_g(B^1) \subset B$ then $T_g(1) = g - g(0) \in B$ and, hence $g \in B$.

Conversely, if $g \in B$ and $f \in B^1$ then, using the fact that $B^1 \subset H^\infty$, we obtain

$$(1 - |z|^2) \left| (T_g(f))' (z) \right| = (1 - |z|^2) |g'(z)||f (z)| \leq \|g\|_B \|f\|_{H^\infty}.$$

Thus $T_g(f) \in B$. Hence (iii) is also proved. $\square$

**Theorem 2** Suppose that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and let $g \in \mathcal{H}ol(\mathbb{D})$. Then:

(i) $I_g$ maps $B^p$ into $B$ if and only if $g \in H^\infty$.

(ii) $M_g$ maps $B^p$ into $B$ if and only if $g \in H^\infty \cap B_{\log, 1/p'}$.

(iii) $T_g$ maps $B^p$ into $B$ if and only if $g \in B_{\log, 1/p'}$.

**Proof** If $I_g$ maps $B^p$ into $B$ then Proposition 1 implies that $g \in H^\infty$. Conversely, using that $B^p \subset B$, we see that if $g \in H^\infty$ and $f \in B^p$ then

$$(1 - |z|^2) \left| (I_g(f))' (z) \right| = (1 - |z|^2) |f' (z)||g(z)| \leq \|f\|_B \|g\|_{H^\infty}.$$

Hence, $I_g(f) \in B$. Thus (i) is proved and (ii) reduces to (6). $\square$

Finally, (iii) follows from the following more precise result.
Theorem 3 Suppose that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and let $g \in \mathcal{Hol}(\mathbb{D})$. Then the following conditions are equivalent.

(a) $T_g$ maps $B^p$ into $\mathcal{B}$.
(b) $g \in \mathcal{B}_{\log, 1/p'}$.
(c) $T_g$ maps $B^p$ into $\mathcal{B}_0$.

Proof of Theorem 3 (a)⇒(b) Suppose (a). By the closed graph theorem $T_g$ is a bounded operator from $B^p$ into $\mathcal{B}$, hence

$$ (1 - |z|^2)|g(z)f(z)| \lesssim \|f\|_{B^p}, \quad z \in \mathbb{D}, \quad f \in B^p. \quad (7) $$

For $a \in \mathbb{D}$ with $a \neq 0$, set

$$ f_a(z) = \left( \log \frac{1}{1 - |a|^2} \right)^{-1/p} \log \frac{1}{1 - \overline{a}z}, \quad z \in \mathbb{D}. \quad (8) $$

It is readily seen that $f_a \in B^p$ for all $a$ and that $\|f_a\|_{B^p} \asymp 1$. Using this and taking $f = f_a$ and $z = a$ in (7), we obtain

$$ (1 - |a|^2)|g'(a)| \left( \frac{1}{1 - |a|^2} \right)^{1/p} \lesssim 1, $$

that is, $g \in \mathcal{B}_{\log, 1/p'}$.

(b)⇒(c) Suppose (b) and take $f \in B^p$. It is well known that

$$ |f(z)| = o \left( \left( \log \frac{1}{1 - |z|^2} \right)^{1/p'} \right), \quad \text{as } |z| \to 1, $$

(see, e.g., [37,56]). This and (b) immediately yield that $T_g(f) \in \mathcal{B}_0$.

The implication (c)⇒(a) is trivial. Hence the proof of Theorem 3 is finished and, consequently, Theorem 2 is also proved. \( \square \)

Let us turn now to the case $0 < s \leq 1$. We shall consider first the Volterra operators $T_g$. For $0 < s < \infty$ and $\alpha > 0$ we set

$$ Q_{s, \log, \alpha} = \left\{ f \in \mathcal{Hol}(\mathbb{D}) : \sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|} \right)^{2\alpha} \int_{\mathbb{D}} |f'(z)|^2 (1 - |q_a(z)|^2)^s dA(z) < \infty \right\}. $$

We have the following results.

Theorem 4 Suppose that $0 < s \leq 1$ and let $g \in \mathcal{Hol}(\mathbb{D})$. Then:

(i) $T_g$ maps $B^1$ into $Q_s$ if and only if $g \in Q_s$.
(ii) If $1 < p < \infty$, $0 < s \leq 1$, and $T_g$ maps $B^p$ into $Q_s$, then $g \in Q_{s, \log, 1/p'}$.
(iii) If $1 < p < \infty$, then $T_g$ maps $B^p$ into $Q_1 = \text{BMOA}$ if and only if $g \in Q_{1, \log, 1/p'}$.
(iv) If $2 < p < \infty$, $0 < s < 1$, and $1 - \frac{2}{p} < s$ then $T_g$ maps $B^p$ into $Q_s$ if and only if $g \in Q_{s, \log, 1/p'}$.

Before we get into the proofs of these results we shall introduce some notation and recall some results which will be needed in our work.

If $I \subset \partial \mathbb{D}$ is an interval, $|I|$ will denote the length of $I$. The Carleson square $S(I)$ is defined as $S(I) = \{ re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1 \}$. Also, for $a \in \mathbb{D}$, the Carleson box $S(a)$ is defined by

$$ S(a) = \left\{ z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, \quad \left| \frac{\arg(a \bar{z})}{2\pi} \right| \leq \frac{1 - |a|}{2} \right\}. $$
If $s > 0$ and $\mu$ is a positive Borel measure on $\mathbb{D}$, we shall say that $\mu$ is an $s$-Carleson measure if there exists a positive constant $C$ such that

$$\mu(S(I)) \leq C|I|^s,$$

for any interval $I \subset \partial \mathbb{D}$.

or, equivalently, if there exists $C > 0$ such that

$$\mu(S(a)) \leq C(1 - |a|^s),$$

for all $a \in \mathbb{D}$.

A 1-Carleson measure will be simply called a Carleson measure.

These concepts were generalized in [55] as follows: If $\mu$ is a positive Borel measure in $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$, we say that $\mu$ is an $\alpha$-logarithmic $s$-Carleson measure if there exists a positive constant $C$ such that

$$\frac{\mu(S(I)) \left( \log \frac{2\pi}{|I|} \right) ^\alpha}{|I|^s} \leq C,$$

for any interval $I \subset \partial \mathbb{D}$

or, equivalently, if

$$\sup_{a \in \mathbb{D}} \frac{\mu(S(a)) \left( \log \frac{2}{1 - |a|^2} \right) ^\alpha}{(1 - |a|^2)^s} < \infty.$$

Carleson measures and logarithmic Carleson measures are known to play a basic role in the study of the boundedness of a great number of operators between analytic function spaces. In particular we recall the Carleson embedding theorem for Hardy spaces which asserts that if $0 < p < \infty$ and $\mu$ is a positive Borel measure on $\mathbb{D}$ then $\mu$ is a Carleson measure if and only if the Hardy space $H^p$ is continuously embedded in $L^p(d\mu)$ (see [18, Chapter 9]).

In the next theorem we collect a number of known results which will be needed in our work.

**Theorem B**

(i) If $0 < s \leq 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, then $f \in Q_s$ if and only if the Borel measure $\mu$ on $\mathbb{D}$ defined by

$$d\mu(z) = (1 - |z|^2)^s |f'(z)|^2 dA(z)$$

is an $s$-Carleson measure.

(ii) If $0 \leq \alpha < \infty$, $0 < s < \infty$, and $\mu$ is a positive Borel measure on $\mathbb{D}$ then $\mu$ is an $\alpha$-logarithmic $s$-Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|^2} \right) ^\alpha \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \right) ^s d\mu(z) < \infty.$$

(iii) If $1 < p \leq 2$ then $B^p \subset Q_s$ for all $s > 0$.

(iv) If $2 < p < \infty$ and $1 - \frac{2}{p} < s$, then $B^p \subset Q_s$.

(v) For $s > -1$, we let $\mathcal{D}_s$ be the space of those functions $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_s} \overset{\text{def}}{=} |f(0)| + \left( \int_{\mathbb{D}} (1 - |z|^2)^s |f'(z)|^2 dA(z) \right)^{1/2} < \infty.$$

Suppose that $0 < s < 1$ and $\alpha > 1$, and let $\mu$ be a positive Borel measure on $\mathbb{D}$. If $\mu$ is an $\alpha$-logarithmic $s$ Carleson measure, then $\mu$ is a Carleson measures for $\mathcal{D}_s$, that is, $\mathcal{D}_s$ is continuously embedded in $L^2(d\mu)$. 

\copyright Springer
Let us mention that (i) is due to Aulaskari, Stegenga and Xiao [13], (ii) is due to Zhao [55], (iii) and (iv) were proved by Aulaskari and Csordas in [10], and (v) is due to Pau and Peláez [41, Lemma 1].

Using Theorem B (ii) and the fact that

\[ 1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2}, \]

we see that for a function \( f \in \mathcal{H}_{\text{ol}}(\mathbb{D}) \) we have that \( f \in \mathcal{Q}_{s, \log, \alpha} \) if and only if the measure \( \mu \) defined by \( d\mu(z) = (1 - |z|^2)^s |f'(z)|^2 dA(z) \) is a \( 2\alpha \)-logarithmic \( s \)-Carleson measure.

**Proof of Theorem 4 (i)** Suppose that \( T_g \) maps \( B^1 \) into \( \mathcal{Q}_s \). Since the constant functions belong to \( B^1 \), we have that \( T_g(1) = g - g(0) \in \mathcal{Q}_s \) and, hence, \( g \in \mathcal{Q}_s \).

To prove the converse, suppose that \( g \in \mathcal{Q}_s \). Then the measure \( \mu \) defined by

\[ d\mu(z) = (1 - |z|^2)^s |g'(z)|^2 dA(z) \]

is an \( s \)-Carleson measure. Take now \( f \in B^1 \), then \( f \in H^\infty \) and, hence,

\[ (1 - |z|^2)^s \left| \left( T_g(f) \right)'(z) \right|^2 = (1 - |z|^2)^s |g'(z)|^2 |f(z)|^2 \leq \| f \|_{H^\infty}^2 (1 - |z|^2)^s |g'(z)|^2. \]

Since \( \mu \) is an \( s \)-Carleson measure, it follows readily that the measure \( \nu \) given by \( d\nu(z) = (1 - |z|^2)^s \left| \left( T_g(f) \right)'(z) \right|^2 dA(z) \) is also an \( s \)-Carleson measure and, hence, \( T_g(f) \in \mathcal{Q}_s \).

**Proof of Theorem 4 (ii)** Suppose that \( 0 < s \leq 1, 1 < p < \infty \), and that \( T_g \) maps \( B^p \) into \( \mathcal{Q}_s \). For \( a \in \mathbb{D} \setminus \{0\} \), set

\[ f_a(z) = \left( \log \frac{1}{1 - |a|^2} \right)^{-1/p} \log \frac{1}{1 - a\overline{z}}, \quad z \in \mathbb{D}, \]

as in (8). We have that \( \| f_a \|_{B^p} \asymp 1 \) and it is also clear that

\[ |f_a(z)| \asymp \left( \log \frac{1}{1 - |a|^2} \right)^{1/p'}, \quad z \in S(a). \]

Using these facts, we obtain

\[ \frac{\left( \log \frac{1}{1 - |a|^2} \right)^{2/p'}}{(1 - |a|^2)^s} \int_{S(a)} (1 - |z|^2)^s |g'(z)|^2 dA(z) \]

\[ \asymp \frac{1}{(1 - |a|^2)^s} \int_{S(a)} (1 - |z|^2)^s |g'(z) f_a(z)|^2 dA(z) \]

\[ = \frac{1}{(1 - |a|^2)^s} \int_{S(a)} (1 - |z|^2)^s \left| \left( T_g(f_a) \right)'(z) \right|^2 dA(z). \]

The fact that \( T_g \) is a bounded operator from \( B^p \) into \( \mathcal{Q}_s \), implies that the measures \( (1 - |z|^2)^s \left| \left( T_g(f_a) \right)'(z) \right|^2 dA(z) \) are \( s \)-Carleson measures with constants controlled by \( \| T_g \|^2 \).

Then it follows that the measure \( (1 - |z|^2)^s |g'(z)|^2 dA(z) \) is a \( 2/p' \)-logarithmic \( s \)-Carleson measure and, hence, \( g \in \mathcal{Q}_{s, \log, 1/p'} \).

**Proof of Theorem 4 (iii) and (iv)** In view of (ii) we only have to prove that if \( g \in \mathcal{Q}_{s, \log, 1/p'} \) then \( T_g \) maps \( B^p \) into \( \mathcal{Q}_s \).
Hence, take \( g \in Q_{s, \log, 1/p'} \) and set
\[
K(g) = \sup_{a \in \mathbb{D}} \left( \log \frac{2}{1-|a|} \right)^{2/p'} \int_{\mathbb{D}} |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z),
\]
and take \( f \in B^p \). Set \( F = T_g(f) \), we have to prove that \( F \in Q_s \) or, equivalently, that the measure \( \mu_F \) defined by
\[
d\mu_F(z) = (1 - |z|^2)^s |F'(z)|^2 \, dA(z)
\]
is an \( s \)-Carleson measure. Let \( a \in \mathbb{D} \). Using the well known fact that
\[1 - |a|^2 \simeq |1 - a z|, \quad z \in S(a),\]
we obtain
\[
\frac{1}{(1 - |a|^2)^s} \int_{S(a)} |F'(z)|^2 (1 - |z|^2)^s \, dA(z) \simeq \int_{S(a)} |F'(z)|^2 \frac{(1 - |z|^2)^s (1 - |a|^2)^s}{|1 - a z|^{2s}} \, dA(z)
\]
\[
= \int_{S(a)} |f(z)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z)
\]
\[
\leq 2 \int_{\mathbb{D}} |f(a)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z)
\]
\[
+ 2 \int_{\mathbb{D}} |f(z) - f(a)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z)
\]
\[
= 2T_1(a) + 2T_2(a).
\]
(9)

Using the fact that
\[
|f(a) - f(0)| \lesssim \|f\|_{B^p} \left( \log \frac{2}{1-|a|^2} \right)^{1/p'},
\]
we obtain
\[
T_1(a) \lesssim \|f\|^2_{B^p} \left( \log \frac{2}{1-|a|^2} \right)^{2/p'} \int_{\mathbb{D}} |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) \lesssim K(g) \|f\|^2_{B^p}.
\]
(11)

To estimate \( T_2(a) \) we shall treat separately the cases \( s = 1 \) and \( 0 \leq s < 1 \).

Let us start with the case \( s = 1 \). Then
\[
T_2(a) = \int_{\mathbb{D}} |f(z) - f(a)|^2 |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z).
\]
Making the change of variable \( w = \varphi(z) \) in the last integral, we obtain
\[
T_2(a) = \int_{\mathbb{D}} |(f \circ \varphi_a)(w) - f(a)|^2 |(g \circ \varphi_a)'(w)|^2 (1 - |w|^2) \, dA(w).
\]
Since \( Q_{1, \log, 1/p'} \subset Q_1 = BMOA \), \( g \in BMOA \) and then it follows that, for all \( a \in \mathbb{D} \), \( g \circ \varphi_a \in BMOA \) and \( \rho_s(g \circ \varphi_a) = \rho_s(g) \). This gives that all the measures \( (1 - |w|^2)(g \circ \varphi_a)'(w))^2 \, dA(w) \) \((a \in \mathbb{D})\) are Carleson measures with constants controlled by \( \|g\|^2_{BMOA} \). Then, using the Carleson embedding theorem for \( H^2 \) and the fact that \( B^p \) is continuously embedded in \( BMOA \), it follows that
\[
T_2(a) \lesssim \|g\|^2_{BMOA} \|f \circ \varphi_a - f(a)\|^2_{H^2} \lesssim \|g\|^2_{BMOA} \|f\|^2_{BMOA} \lesssim \|g\|^2_{BMOA} \|f\|^2_{B^p}.
\]
Putting together this, (9), and (11), we see that the measure $\mu_F$ is a Carleson measure. This finishes the proof of part (iii).

To finish the proof of part (iv) we proceed to estimate $T_2(a)$ assuming that $2 < p < \infty$, $0 < s < 1$, and $1 - \frac{2}{p} < s$. Notice that

$$T_2(a) = (1 - |a|^2)^s \int_{\mathbb{D}} \left| \frac{f(z) - f(a)}{(1 - \overline{a} z)^s} \right|^2 |g'(z)|^2 (1 - |z|^2)^s \, dA(z).$$

Since $0 < s < 1$, $2/p' > 1$, and the measure $(1 - |z|^2)^s |g'(z)|^2 \, dA(z)$ is a $2/p'$-logarithmic $s$-Carleson measure, using Theorem $B$ (v), it follows that

$$T_2(a) \lesssim (1 - |a|^2)^s \left( |f(a) - f(0)|^2 + \int_{\mathbb{D}} \left| \frac{f(z) - f(a)}{(1 - \overline{a} z)^s} \right|^2 (1 - |z|^2)^s \, dA(z) \right).$$

The growth estimate (10) and simple computations yield

$$T_2(a) \lesssim \|f\|_{B^p}^2 (1 - |a|^2)^\gamma \left( \log \frac{2}{1 - |a|^2} \right)^{2/p'} + \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^\gamma \, dA(z)$$

$$+ \int_{\mathbb{D}} \frac{|f(z) - f(a)|^2}{|1 - \overline{a} z|^2} (1 - |\varphi_a(z)|^2)^s \, dA(z)$$

$$\lesssim \|f\|_{B^p}^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^\gamma \, dA(z) + \int_{\mathbb{D}} \frac{|f(z) - f(a)|^2}{|1 - \overline{a} z|^2} (1 - |\varphi_a(z)|^2)^s \, dA(z).$$

By Theorem $B$ (iv), our assumptions on $s$ and $p$ imply that $B^p$ is continuously embedded in $Q_s$. Hence, $f \in Q_s$. This implies that

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^\gamma \, dA(z) \leq \|f\|_{Q_s}^2 \lesssim \|f\|_{B^p}^2$$

and that

$$\int_{\mathbb{D}} \frac{|f(z) - f(a)|^2}{|1 - \overline{a} z|^2} (1 - |\varphi_a(z)|^2)^s \, dA(z) \lesssim \|f\|_{Q_s}^2 \lesssim \|f\|_{B^p}^2,$$

by a result proved by Pau and Peláez in [41, pp. 551–552]. Consequently, we have proved that $T_2(a) \lesssim \|f\|_{B^p}^2$. This, together with (9) and (11), shows that $\mu_F$ is an $s$-Carleson measure as desired. Thus the proof is also finished in this case. $\square$

The case when $1 < p \leq 2$ and $0 < s < 1$ remains open. This is so because if we set $\alpha = 2/p'$, then $\alpha \leq 1$ and, hence, $\alpha$ is not in the conditions of Theorem $B$ (v). We can prove the following result.

**Theorem 5** Suppose that $1 < p \leq 2$ and $0 < s < 1$, and let $g \in \mathcal{H}ol(\mathbb{D})$. The following statements hold.

(i) If $T_g$ maps $B^p$ into $Q_s$ then $g \in Q_{s, \log, 1/p'}$.
(ii) If $\alpha > 1/2$ and $g \in Q_{s, \log, \alpha}$ then $T_g$ maps $B^p$ into $Q_s$.

**Proof** (i) follows from part (ii) of Theorem 4.

Let us turn to prove (ii). Suppose that $0 < s < 1$, $\alpha > 1/2$, and $g \in Q_{s, \log, \alpha}$. Set

$$K(g) = \sup_{a \in \mathbb{D}} \left( \log \frac{2}{1 - |a|} \right) \int_{\mathbb{D}} |g'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z),$$
and take \( f \in B^p \). Set \( F = T_g(f) \), we have to prove that \( F \in Q_s \) or, equivalently, that the measure \( \mu_F \) defined by
\[
d\mu_F(z) = (1 - |z|^2)^s |F'(z)|^2 \, dA(z)
\]
is an \( s \)-Carleson measure. Now we argue as in the proof of Theorem 4 (iv). For \( a \in \mathbb{D} \), we obtain
\[
\frac{1}{1 - |a|^2} \int_{S(a)} |F'(z)|^2(1 - |z|^2)^s \, dA(z) \lesssim 2T_1(a) + 2T_2(a),
\]
where \( T_1(a) \) and \( T_2(a) \) are defined as in the proof of Theorem 4. Using (10) and the fact that \( \frac{1}{p'} \leq \frac{1}{2} < \alpha \), we obtain
\[
|f(a) - f(0)| \lesssim \|f\|_{B^p} \left( \log \frac{2}{1 - |a|^2} \right)^\alpha.
\]
This yields
\[
T_1(a) \lesssim \|f\|_{B^p}^2 \left( \log \frac{2}{1 - |a|^2} \right)^{2\alpha} \int_{\mathbb{D}} |g'(z)|^2(1 - |z|^2)^s \, dA(z) \lesssim K(g)\|f\|_{B^p}^2.
\]

To estimate \( T_2(a) \), observe that the measure \( (1 - |z|^2)^s |g'(z)|^2 \, dA(z) \) is a \( 2\alpha \)-logarithmic \( s \)-Carleson measure. Since \( 2\alpha > 1 \), using Lemma 1 of [41], this implies that the measure \( (1 - |z|^2)^s |g'(z)|^2 \, dA(z) \) is a Carleson measure for \( D_s \). Then, arguing as in the proof of Theorem 4 (iv), we obtain \( T_2(a) \lesssim \|f\|_{B^p}^2 \). This, together with (13) and (12), implies that the measure \( \mu_F \) is an \( s \)-Carleson measure. \( \square \)

Regarding the operators \( I_g \) and \( M_g \) we have the following results.

**Theorem 6** Let \( g \in \mathcal{Hol}(\mathbb{D}) \), then:

1. If \( 1 < p \leq 2 \) and \( 0 < s \leq 1 \) then:
   1.1 If \( I_g \) maps \( B^p \) into \( Q_s \) if and only if \( g \in H^\infty \).
   1.2 If \( M_g \) maps \( B^p \) into \( Q_s \), then \( g \in Q_{s, \log, 1/p'} \cap H^\infty \).
   1.3 If \( g \in Q_{s, \log, \alpha} \cap H^\infty \) for some \( \alpha > 1/2 \) then \( M_g \) maps \( B^p \) into \( Q_s \).
2. If \( 2 < p < \infty \) and \( 1 - \frac{2}{p} < s \leq 1 \) then:
   2.1 If \( I_g \) maps \( B^p \) into \( Q_s \) if and only if \( g \in H^\infty \).
   2.2 If \( M_g \) maps \( B^p \) into \( Q_s \) if and only if \( g \in Q_{s, \log, 1/p'} \cap H^\infty \).
3. If \( 2 < p < \infty \) and \( 0 < s \leq 1 - \frac{2}{p} \) then:
   3.1 If \( I_g \) maps \( B^p \) into \( Q_s \) if and only if \( g \equiv 0 \).
   3.2 If \( M_g \) maps \( B^p \) into \( Q_s \) if and only if \( g \equiv 0 \).

**Proof of Parts (1) and (2) of Theorem 6** Using Proposition 1 it follows that if either \( I_g \) or \( M_g \) maps \( B^p \) into \( Q_s \) for any pair \( (s, p) \) with \( 0 < s < \infty \) and \( 1 < p < \infty \) then \( g \in H^\infty \).

Suppose now that \( s \) and \( p \) are in the conditions of (1) or (2) and that \( g \in H^\infty \). Take \( f \in B^p \). We have to prove \( I_g(f) \in Q_s \) or, equivalently, that the measure
\[
(1 - |z|^2)^s |f'(z)|^2 |g(z)|^2 \, dA(z)
\]
is an \( s \)-Carleson measure.
Using (1) and (2), we see that \( B^p \subset Q_s \). Hence \( f \in Q_s \) which is the same as saying that \( (1 - |z|^2)^{s'} |f'(z)|^2 dA(z) \) is an \( s \)-Carleson measure. This and the fact that \( g \in H^\infty \) trivially yield (14). Thus (1a) and (2a) are proved. Then (1b), (1c), and (2b) follow using Proposition 1, the fact that if two of the operators \( T_g, I_g, M_g \) map \( B^p \) into \( Q_s \) so does the third one, Theorem 4, and Theorem 5.

In order to prove Theorem 6 (3), for \( 2 < p < \infty \) we shall consider the function \( F_p \) defined by

\[
F_p(z) = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}2^k/p} z^{2^k}, \quad z \in \mathbb{D}. \tag{15}
\]

Using [10, Corollary 7] or [14, Theorem 6], we see that \( F_p \in B^p \) and \( F_p \notin Q_{1-2/p} \). Hence

\[
F_p \in B^p \setminus Q_s, \quad 0 < s \leq 1 - \frac{2}{p}, \quad 2 < p < \infty. \tag{16}
\]

Let us estimate the integral means \( M_2(r, F_p') \). We have

\[
zF_p'(z) = \sum_{k=1}^{\infty} 2^{k/p'} k^{-1/2} z^{2^k}, \quad z \in \mathbb{D}
\]

and, hence,

\[
M_2(r, F_p')^2 \geq \sum_{k=1}^{\infty} 2^{2k/p'} k^{-1} r^{2^k+1}, \quad 0 < r < 1.
\]

Set \( r_n = 1 - 2^{-n} \) \( (n = 1, 2, \ldots) \). Then

\[
M_2(r_n, F_p')^2 \geq \sum_{k=1}^{\infty} 2^{2k/p'} k^{-1} r_n^{2^k+1}
\]

\[
\geq 2^{2n/p'} n^{-1} r_n^{2n+1} \geq 2^{2n/p'} n^{-1} \times \frac{1}{(1 - r_n)^{2/p'} \log \frac{2}{1-r_n}}, \quad n = 1, 2, \ldots.
\]

This readily yields

\[
M_2(r, F_p')^2 \geq \frac{1}{(1 - r)^{2/p'} \log \frac{2}{1-r}}, \quad 0 < r < 1. \tag{17}
\]

**Proof of part (3) of Theorem 6** Suppose that \( 2 < p < \infty \) and \( 0 < s \leq 1 - \frac{2}{p} \) and \( g \in \mathcal{H}(\mathbb{D}) \) is not identically zero.

Suppose first that either \( I_g \) or \( M_g \) maps \( B^p \) into \( Q_s \). We know that then \( g \in H^\infty \) and then, by Fatou’s theorem and the Riesz uniqueness theorem, we know that \( g \) has a finite non-tangential limit \( g(e^{i\theta}) \) for almost every \( \theta \in [0, 2\pi] \) and that \( g(e^{i\theta}) \neq 0 \) for almost every \( \theta \). Then it follows that there exist \( C > 0, r_0 \in (0, 1) \), and a measurable set \( E \subset [0, 2\pi] \) whose Lebesgue measure \( |E| \) is positive such that

\[
|g(re^{i\theta})| \geq C, \quad \theta \in E, \quad r_0 < r < 1. \tag{18}
\]

Since \( F_p \) is given by a power series with Hadamard gaps, Lemma 6.5 in [60, Vol. 1, p. 203] implies that

\[
\int_E |F_p'(re^{i\theta})|^2 d\theta \asymp M_2(r, F_p')^2, \quad 0 < r < 1. \tag{19}
\]
Using the fact that \( s \leq 1 - \frac{2}{p} \), (18), (19), and (17), we obtain
\[
\int_0^1 (1 - r)^s M_2(r, F_p g)^2 \, dr \geq \int_0^1 (1 - r)^{1 - \frac{2}{p}} M_2(r, F_p g)^2 \, dr
\]
\[
\gtrsim \int_{r_0}^1 (1 - r)^{1 - \frac{2}{p}} \int_E |F_p'(re^{i\theta})|^2 |g(re^{i\theta})|^2 \, d\theta \, dr \gtrsim \int_{r_0}^1 (1 - r)^{1 - \frac{2}{p}} \int_E |F_p'(re^{i\theta})|^2 \, d\theta \, dr
\]
\[
\gtrsim \int_{r_0}^1 (1 - r)^{1 - \frac{2}{p}} M_2(r, F_p')^2 \, dr \gtrsim \int_{r_0}^1 \frac{dr}{(1 - r) \log \frac{1}{1 - r}} = \infty. \tag{20}
\]

If we assume that \( I_g \) maps \( B^p \) into \( Q_s \) then \( I_g(F_p) \in Q_s \) and then, using [11, Proposition 3.1], it follows that
\[
\int_0^1 (1 - r)^s M_2(r, F_p g)^2 \, dr < \infty.
\]
This is in contradiction with (20).

Assume now that \( M_g \) maps \( B^p \) into \( Q_s \). Since 1 and \( F_p \) belong to \( B^p \), we have that \( g \) and \( F_p g \) belong to \( Q_s \) and then, by [11, Proposition 3.1],
\[
\int_0^1 (1 - r)^s M_2(r, g')^2 \, dr < \infty \tag{21}
\]
and
\[
\int_0^1 (1 - r)^s M_2(r, (F_p g')^2) \, dr < \infty. \tag{22}
\]
Notice that \( F_p \in H^\infty \) and then
\[
M_2(r, F_p g') \lesssim M_2(r, g'), \quad 0 < r < 1.
\]
This and (21) imply that
\[
\int_0^1 (1 - r)^s M_2(r, F_p g)^2 \, dr < \infty. \tag{23}
\]
We have arrived to a contradiction because it is clear that (20) and (23) cannot be simultaneously true.

In the other direction we have the following result.

**Theorem 7** Suppose that \( 0 < s < \infty \) and \( 1 \leq p < \infty \) and let \( g \in \mathcal{H}ol(\mathbb{D}) \). Then the following conditions are equivalent

(i) \( M_g \) maps \( Q_s \) into \( B^p \).

(ii) \( g \equiv 0 \).

**Proof** Suppose that \( g \neq 0 \). Choose an increasing sequence \( \{r_n\}_{n=1}^\infty \subset (0, 1) \) with \( \lim\{r_n\} = 1 \) and a sequence \( \{\theta_n\}_{n=1}^\infty \subset [0, 2\pi] \) such that
\[
|g(r_n e^{i\theta_n})| = M_\infty(r_n, g), \quad n = 1, 2, \ldots.
\]
For each \( n \) set
\[
f_n(z) = \log \frac{1}{1 - e^{-i\theta_n} z}, \quad z \in \mathbb{D}.
\]
Notice that \( M(r_1, g) > 0 \) and that the sequence \( \{M(r_n, g)\} \) is increasing. Set

\[
f_n(z) = \log \frac{1}{1 - e^{-i\theta_n} z}, \quad z \in \mathbb{D}, \quad n = 1, 2, \ldots.
\]

We have that \( f_n \in Q_s \) for all \( n \) and

\[
\|f_n\|_{Q_s} \asymp 1.
\]

Assume that \( M_g \) maps \( Q_s \) into \( B^p \). Then, by the closed graph theorem, \( M_g \) is a bounded operator from \( Q_s \) into \( B^p \). Hence the sequence \( \{g f_n\}_{n=1}^\infty \) is a bounded sequence on \( B^p \), that is,

\[
\|g f_n\|_{B^p} \lesssim 1.
\]

Then it follows that

\[
M(r_1, g) \log \frac{1}{1 - r_n} \leq M(r_n, g) \log \frac{1}{1 - r_n} = |g(r_n e^{i\theta_n}) f_n(r_n e^{i\theta_n})| 
\]

\[
\lesssim \|g f_n\|_{B^p} \left( \log \frac{1}{1 - r_n} \right)^{1/p'} \lesssim \left( \log \frac{1}{1 - r_n} \right)^{1/p'}.
\]

This is a contradiction. \( \square \)

## 3 Multipliers and integration operators between \( Q_s \) spaces

As we mentioned above the space of multipliers \( M(B) = M(Q_s) \) \( s > 1 \) was characterized by Brown and Shields in [15]. Ortega and Fàbrega [40] characterized the space \( M(BMOA) = M(Q_1) \). Pau and Peláez [41] and Xiao [54] characterized the spaces \( M(Q_s) \) \( 0 < s < 1 \) closing a conjecture formulated in [51]. Indeed, Theorem 1 of [41] and Theorem 1.2 of [54] assert the following.

**Theorem C** Suppose that \( 0 < s \leq 1 \) and let \( g \) be an analytic function in the unit disc \( \mathbb{D} \). Then:

(i) \( T_g \) maps \( Q_s \) into itself if and only if \( g \in Q_s, \log, 1 \).
(ii) \( I_g \) maps \( Q_s \) into itself if and only if \( g \in H^\infty \).
(iii) \( M_g \) maps \( Q_s \) into itself if and only if \( g \in Q_s, \log, 1 \cap H^\infty \).

We shall prove the following results.

**Theorem 8** Suppose that \( 0 < s_1 \leq s_2 \leq 1 \) and let \( g \in \mathcal{H}ol(\mathbb{D}) \). Then:

(i) \( T_g \) maps \( Q_{s_1} \) into \( Q_{s_2} \) if and only if \( g \in Q_{s_2, \log, 1} \).
(ii) \( I_g \) maps \( Q_{s_1} \) into \( Q_{s_2} \) if and only if \( g \in H^\infty \).
(iii) \( M_g \) maps \( Q_{s_1} \) into \( Q_{s_2} \) if and only if \( g \in Q_{s_2, \log, 1} \cap H^\infty \).

**Theorem 9** Suppose that \( 0 < s_1 < s_2 \leq 1 \) and let \( g \in \mathcal{H}ol(\mathbb{D}) \). Then the following conditions are equivalent:

(i) \( I_g \) maps \( Q_{s_2} \) into \( Q_{s_1} \).
(ii) \( M_g \) maps \( Q_{s_2} \) into \( Q_{s_1} \).
(iii) \( g \equiv 0 \).
Proof For $a \in \mathbb{D}$ we set

$$h_a(z) = \log \frac{2}{1 - a z}, \quad z \in \mathbb{D}.$$  

Then $h_a \in Q_{s_1}$ for all $a \in \mathbb{D}$ and

$$\|h_a\|_{Q_{s_1}} \asymp 1.  \quad (24)$$

- If $T_g$ maps $Q_{s_1}$ into $Q_{s_2}$ then $T_g$ is a bounded operator from $Q_{s_1}$ into $Q_{s_2}$. Using this and (24), it follows that for all $a \in \mathbb{D}$ the measure

$$\int_{S(a)} (1 - |z|^2)^{s_2} |g'(z)|^2 |h_a(z)|^2 \, dA(z)$$

is an $s_2$-Carleson measure and that

$$\int_{S(a)} (1 - |z|^2)^{s_2} |g'(z)|^2 \, dA(z) \lesssim (1 - |a|^2)^{s_2}, \quad a \in \mathbb{D}.  \quad (25)$$

Since

$$|h_a(z)| \asymp \log \frac{2}{1 - |a|^2}, \quad z \in S(a),$$

(25) implies that

$$\left( \log \frac{2}{1 - |a|^2} \right)^2 \int_{S(a)} (1 - |z|^2)^{s_2} |g'(z)|^2 \, dA(z) \lesssim (1 - |a|^2)^{s_2}.$$  

This is the same as saying that the measure $(1 - |z|^2)^{s_2} |g'(z)|^2 \, dA(z)$ is a 2-logarithmic $s_2$-Carleson measure or, equivalently, that $g \in Q_{s_2, \log, 1}$.

If $g \in Q_{s_2, \log, 1}$ then, by Theorem C, $T_g$ maps $Q_{s_2}$ into itself. Since $Q_{s_1} \subset Q_{s_2}$, it follows trivially that $T_g$ maps $Q_{s_1}$ into $Q_{s_2}$. Hence (i) is proved

- Proposition 1 shows that if $I_g$ maps $Q_{s_1}$ into $Q_{s_2}$ then $g \in H^\infty$.

Conversely, suppose that $g \in H^\infty$. In order to prove that $I_g$ maps $Q_{s_1}$ into $Q_{s_2}$, we have to prove that for any $f \in Q_{s_1}$ the measure

$$\int_{S(a)} (1 - |z|^2)^{s_1} |g(z)|^2 |f'(z)|^2 \, dA(z)$$

is an $s_1$-Carleson measure. So, take $f \in Q_{s_1}$. Then

$$\int_{S(a)} (1 - |z|^2)^{s_1} |f'(z)|^2 \, dA(z)$$

is an $s_1$-Carleson measure. Then it follows that

$$\int_{S(a)} (1 - |z|^2)^{s_2} |g(z)|^2 |f'(z)|^2 \, dA(z)$$

$$\leq \|g\|_{H^\infty} (1 - |a|^2)^{s_2-s_1} \int_{S(a)} (1 - |z|^2)^{s_1} |f'(z)|^2 \, dA(z)$$

$$\lesssim (1 - |a|^2)^{s_2}.$$  

This shows that $(1 - |z|^2)^{s_2} |g(z)|^2 |f'(z)|^2 \, dA(z)$ is an $s_2$-Carleson measure as desired, finishing the proof of (ii).

- If $M_g$ maps $Q_{s_1}$ into $Q_{s_2}$ then, Proposition 1, $g \in H^\infty$. Then (i) implies that $I_g$ maps $Q_{s_1}$ into $Q_{s_2}$. Since $M_g(f) = I_g(f) + T_g(f) + f(0) g(0)$, it follows that $T_g$ maps $Q_{s_1}$ into $Q_{s_2}$. Then (i) yields $g \in Q_{s_2, \log, 1}$. Then we have that $g \in Q_{s_2, \log, 1} \cap H^\infty$.

Conversely, if $g \in Q_{s_2, \log, 1} \cap H^\infty$ then (i) and (ii) immediately give that both $T_g$ and $I_g$ map $Q_{s_1}$ into $Q_{s_2}$ and then so does $M_g$.  

 Springer
Some results from [11] will be used to prove Theorem 9. As we have already noticed if $0 < s \leq 1$ and $f \in Q_s$ then $\int_0^1 (1 - r)^s M_2(r, f')^2 \, dr < \infty$. Using ideas from [27], Aulaskari, Girela and Wulan [11, Theorem 3.1] proved that this result is sharp in a very strong sense.

**Theorem D** Suppose that $0 < s \leq 1$ and let $\varphi$ be a positive increasing function defined in $(0, 1)$ such that

$$\int_0^1 (1 - r)^s \varphi(r)^2 \, dr < \infty.$$  

Then there exists a function $f \in Q_s$ given by a power series with Hadamard gaps such that $M_2(r, f') \geq \varphi(r)$ for all $r \in (0, 1)$.

**Proof of Theorem 9** Suppose that $g \not\equiv 0$ and that either $I_g$ or $M_g$ maps $Q_{s_2}$ into $Q_{s_1}$. By Proposition 1, $g \in H^\infty$ and then it follows that there exist $C > 0, r_0 \in (0, 1)$, and a measurable set $E \subset [0, 2\pi]$ whose Lebesgue measure $|E|$ is positive such that $|g(re^{i\theta})| \geq C, \theta \in E, \ r_0 < r < 1$.

- Suppose that $I_g$ maps $Q_{s_2}$ into $Q_{s_1}$. Then we use Theorem D to pick a function $F \in Q_{s_2}$ given by a power series with Hadamard gaps so that

$$M_2(r, F') \geq \frac{1}{(1 - r)^{(1 + s_1)/2}}, \quad 0 < r < 1. \quad (26)$$

Since $I_g(F) \in Q_{s_1}$, 

$$\int_0^1 (1 - r)^{s_1} M_2(r, F')^2 \, dr < \infty. \quad (27)$$

However, using Lemma 6.5 in [60, Vol. 1, p. 203] and (26), it follows that

$$\int_0^1 (1 - r)^{s_1} M_2(r, F')^2 \, dr \geq \int_{r_0}^1 (1 - r)^{s_1} \int_E |F'(re^{i\theta})|^2 |g(re^{i\theta})|^2 \, d\theta \, dr$$

$$\geq \int_{r_0}^1 (1 - r)^{s_1} \int_E |F'(re^{i\theta})|^2 \, d\theta \, dr$$

$$\times \int_{r_0}^1 (1 - r)^{s_1} M_2(r, F')^2 \, dr$$

$$\geq \int_{r_0}^1 (1 - r)^{-1} \, dr$$

$$= \infty.$$  

This is in contradiction with (27).

- Suppose now that $M_g$ maps $Q_{s_2}$ into $Q_{s_1}$. Take $\varepsilon > 0$ with $s_2 - s_1 - \varepsilon > 0$ and use Theorem D to pick a function $H \in Q_{s_2}$ given by a power series with Hadamard gaps so that

$$M_2(r, H') \geq \frac{1}{(1 - r)^{(1 + s_1 + \varepsilon)/2}}, \quad 0 < r < 1. \quad (28)$$

Since $gH \in Q_{s_1}$ we have that

$$\int_0^1 (1 - r)^{s_1} M_2(r, g'H + gH')^2 \, dr < \infty. \quad (29)$$
Using Lemma 6.5 in [60, Vol. 1, p. 203] and (28), we obtain as above that
\[
\int_0^1 (1-r)^{s_1+\varepsilon} M_2(r, H'g)^2 \, dr \gtrsim \int_{r_0}^1 (1-r)^{s_1+\varepsilon} \int_E |H'(re^{i\theta})|^2 \, d\theta \, dr \\
\gtrsim \int_{r_0}^1 (1-r)^{s_1+\varepsilon} M_2(r, H')^2 \, dr \\
= \infty.
\]
(30)

Notice that \(g \in Q_{s_1}\). Using this and the fact that
\[
|H(z)| \lesssim \log \frac{2}{1-|z|}, \quad z \in \mathbb{D},
\]
it follows that
\[
\int_0^1 (1-r)^{s_1+\varepsilon} M_2(r, Hg')^2 \, dr \lesssim \int_0^1 (1-r)^{s_1+\varepsilon} \left( \log \frac{2}{1-r} \right)^2 M_2(r, g')^2 \, dr \\
\lesssim \int_0^1 (1-r)^{s_1+\varepsilon} M_2(r, g') \, dr < \infty.
\]
(31)

We have arrived to a contradiction because (29), (30), and (31) cannot hold simultaneously.

\[\square\]

**Remark 1** The implication (ii) \(\Rightarrow\) (iii) in Theorem 9 was obtained by Pau and Peláez [42, Corollary 4] using the fact that there exists a function \(f \in Q_{s_2}, f \not\equiv 0\), whose sequence of zeros is not a \(Q_{s_1}\)-zero set.

This idea gives also the following:
\[
M(B, Q_s) = \{0\}, \quad 0 < s \leq 1.
\]

Indeed, it is well known that there exists a function \(f \in B, f \not\equiv 0\), whose sequence of zeros does not satisfy the Blaschke condition [7,31]. If \(g \not\equiv 0\) were a multiplier from \(B\) into \(Q_s\) for some \(s \leq 1\) then the sequence of zeros of \(fg\) would satisfy the Blaschke condition.

But this is not true because all the zeros of \(f\) are zeros of \(gf\).

### 4 Some further results

The inner-outer factorization of functions in the Hardy spaces plays an outstanding role in lots of questions. In many cases the outer factor \(O_f\) of \(f\) inherits properties of \(f\). Working in this setting the following concepts arise as natural and quite interesting.

A subspace \(X\) of \(H^1\) is said to have the \(f\)-property (also called the property of division by inner functions) if \(h/I \in X\) whenever \(h \in X\) and \(I\) is an inner function with \(h/I \in H^1\).

Given \(v \in L^\infty(\partial \mathbb{D})\), the Toeplitz operator \(T_v\) associated with the symbol \(v\) is defined by
\[
T_v f(z) = P(vf)(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{v(\xi) f(\xi)}{\xi - z} \, d\xi, \quad f \in H^1, \quad z \in \mathbb{D}.
\]
Here, \(P\) is the Szegö projection.
A subspace $X$ of $H^1$ is said to have the $K$-property if $T_{\psi}(X) \subset X$ for any $\psi \in H^\infty$.

These notions were introduced by Havin in [34]. It was also pointed out in [34] that the $K$-property implies the $f$-property: indeed, if $h \in H^1$, $I$ is inner and $h/I \in H^1$ then $h/I = T_{\psi} h$.

In addition to the Hardy spaces $H^p$ ($1 < p < \infty$) many other spaces such as the Dirichlet space [34,38], several spaces of Dirichlet type including all the Besov spaces $B^p$ ($1 < p < \infty$) [20–22,39], the spaces $BMOA$ and $VMOA$ [35], and the $Q_s$ spaces ($0 < s < 1$) [23] have the $f$-property. The Hardy space $H^1$, $H^\infty$ and $VMOA \cap H^\infty$ are examples of spaces which have the $f$-property but fail to have the $K$-property [35].

The first example of a subspace of $H^1$ not possessing the $f$-property is due to Gurarii [33] who proved that the space of analytic functions in $D$ whose sequence of Taylor coefficients is in $\ell^1$ does not have the $f$-property. Anderson [6] proved that the space $B_0 \cap H^\infty$ does not have the $f$-property. Later on it was proved in [29] that if $1 \leq p < \infty$ then $H^p \cap B$ fails to have the $f$-property also.

Since we have already mentioned the Besov spaces $B^p$ ($1 < p < \infty$) and the $Q_s$ spaces ($0 < s \leq 1$) have the $K$-property (and, also, the $f$-property), it seems natural to investigate whether the spaces of multipliers and the spaces $Q_{s,\log,\alpha}$ that we have considered in our work have also these properties. We shall prove the following results.

**Theorem 10** The spaces of multipliers $M(B^p, Q_s)$ ($0 < s \leq 1$, $1 \leq p < \infty$), $M(Q_{s1}, Q_{s2})$ ($0 < s_1, s_2 \leq 1$), and $M(B^p, B^q)$ ($1 \leq p, q < \infty$) have the $f$-property.

**Theorem 11** For $\alpha > 0$ and $0 < s < 1$ the space $Q_{s,\log,\alpha}$ has the $K$-property.

**Lemma 1** Let $X$ and $Y$ be to Banach spaces of analytic functions which are continuously contained in $H^1$. Suppose that $X$ contains the constants functions and that $Y$ has the $f$-property. Then the space of multipliers $M(X, Y)$ also has the $f$-property.

**Proof** Since $X$ contains the constants functions $M(X, Y) \subset Y \subset H^1$.

Suppose that $F \in M(X, Y)$, $I$ is an inner function, and $F/I \in H^1$. Take $f \in X$. Then $f F \in Y \subset H^1$ and then $f F/I \in H^1$. Since $Y$ has the $f$-property, it follows that $f F/I \in Y$. Thus, we have proved that $F/I \in M(X, Y)$. $\square$

**Theorem 11** will follow from a characterization of the spaces $Q_{s,\log,\alpha}$ in terms of pseudoanalytic continuation. We refer to Dyn’kin’s paper [24] for similar descriptions of classical smoothness spaces, as well as for other important applications of the pseudoanalytic extension method.

Let, $\mathbb{D}_-$ denotes the region $\{z \in \mathbb{C} : |z| > 1\}$, and write $z^\ast \overset{\text{def}}{=} 1/\bar{z}$, $z \in \mathbb{C} \setminus \{0\}$.

We shall need the Cauchy–Riemann operator
\[
\bar{\partial} = \frac{\partial}{\partial z} \overset{\text{def}}{=} \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad z = x + iy.
\]

The following result is an extension of [23, Theorem 1].

**Theorem 12** Suppose that $0 < s < 1$, $\alpha > 0$, and $f \in \cap_{0 < q < \infty} H^q$. Then the following conditions are equivalent.

1. $f \in Q_{s,\log,\alpha}$. 
(ii) \[
\sup_{|a| < 1} \left( \log \frac{2}{1 - |a|} \right)^{2\alpha} \int_{\mathbb{D}} |f'(z)|^2 \left( \frac{1}{|\varphi_a(z)|^2} - 1 \right)^s \, dA(z) < \infty.
\]

(iii) There exists a function \( F \in C^1(\mathbb{D}) \) satisfying
\[
F(z) = O(1), \quad \text{as} \ z \to \infty,
\]
\[
\lim_{r \to 1^+} F(re^{i\theta}) = f(e^{i\theta}), \quad \text{a.e. and in } L^q([-\pi, \pi]) \text{ for all } q \in [1, \infty),
\]
\[
\sup_{|a| < 1} \left( \log \frac{2}{1 - |a|} \right)^{2\alpha} \int_{\mathbb{D}} \|\partial F(z)\|_2^2 \left( |\varphi_a(z)|^2 - 1 \right)^s \, dA(z) < \infty.
\]

Theorem 12 can be proved following the arguments used in the proof of [23, Theorem 1], we omit the details. Once Theorem 12 is established, noticing that if \( \alpha > 0 \) and \( 0 < s < 1 \) then \( Q_s, \log \alpha \subset Q_s \subset BMOA \), Theorem 11 can be proved following the steps in the proof of [23, Theorem 2]. Again, we omit the details.

Acknowledgements We wish to thank the referees for reading carefully the paper and making a number of nice suggestions to improve it.

References

1. Aleman, A., Cima, J.A.: An integral operator on \( H^p \) and Hardy’s inequality. J. Anal. Math. 85, 157–176 (2001)
2. Aleman, A., Duren, P.L., Martín, M.J., Vukotić, D.: Multiplicative isometries and isometric zero-divisors. Can. J. Math. 62(5), 961–974 (2010)
3. Aleman, A., Simbotin, A.: Estimates in Möbius invariant spaces of analytic functions. Complex Var. Theory Appl. 49(7–9), 487–510 (2004)
4. Aleman, A., Siskakis, A.G.: An integral operator on \( H^p \). Complex Variables Theory Appl. 28(2), 149–158 (1995)
5. Aleman, A., Siskakis, A.G.: Integration operators on Bergman spaces. Indiana Univ. Math. J. 46(2), 337–356 (1997)
6. Anderson, J.M.: On division by inner factors. Comment. Math. Helv. 54(2), 309–317 (1979)
7. Anderson, J.M., Clunie, J., Pommerenke, Ch.: On Bloch functions and normal functions. J. Reine Angew. Math. 270, 12–37 (1974)
8. Arazy, J., Fisher, S.D., Peetre, J.: Möbius invariant function spaces. J. Reine Angew. Math. 363, 110–145 (1985)
9. Arcozzi, N., Rochberg, R., Sawyer, E.: Carleson measures for analytic Besov spaces. Rev. Mat. Iberoam. 18(2), 443–510 (2002)
10. Aulaskari, R., Csordas, G.: Besov spaces and the \( Q_{q,0} \) classes. Acta Sci. Math. (Szeged) 60(1–2), 31–48 (1995)
11. Aulaskari, R., Girela, D., Wulan, H.: Taylor coefficients and mean growth of the derivative of \( Q_p \) functions. J. Math. Anal. Appl. 258(2), 415–428 (2001)
12. Aulaskari, R., Lappan, P.: Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, complex analysis and its applications (Harlow), In: Pitman Research Notes in Mathematics, vol. 305. Longman Scientific and Technical, pp. 136–146 (1994)
13. Aulaskari, R., Stegenga, D.A., Xiao, J.: Some subclasses of BMOA and their characterization in terms of Carleson measures. Rocky Mt. J. Math. 26(2), 485–506 (1996)
14. Aulaskari, R., Xiao, J., Zhao, R.: On subspaces and subsets of BMOA and UBC. Analysis 15(2), 101–121 (1995)
15. Brown, L., Shields, A.L.: Multipliers and cyclic vectors in the Bloch space. Michigan Math. J. 38(1), 141–146 (1991)
16. Donaire, J.J., Girela, D., Vukotić, D.: On univalent functions in some Möbius invariant spaces. J. Reine Angew. Math. 553, 43–72 (2002)
17. Donaire, J.J., Girela, D., Vukotić, D.: On the growth and range of functions in Möbius invariant spaces. J. Anal. Math. 112, 237–260 (2010)
18. Duren, P.L.: Theory of \( H^p \) spaces (reprint). Academic Press, New York (2000)
19. Duren, P.L., Schuster, A.P.: Bergman spaces. In: Mathematics Surveys and Monographs, vol. 100. American Mathematical Society, Providence (2004)
20. Dyakonov, K.M.: Factorization of smooth analytic functions via Hilbert-Schmidt operators (in Russian). Algebra i Analiz 8(4), 1–42 (1996) (English translation in St. Petersburg Math. J. 8(4), 543–569 (1997)
21. Dyakonov, K.M.: Equivalent norms on Lipschitz-type spaces of holomorphic functions. Acta. Math. 178, 143–167 (1997)
22. Dyakonov, K.M.: Holomorphic functions and quasiconformal mappings with smooth modulii. Adv. Math. 187, 146–172 (2004)
23. Dyn'kin, E.M.: The pseudoanalytic extension. J. Anal. Math. 60, 45–70 (1993)
24. Girela, D.: Growth of the derivative of bounded analytic functions. Complex Variables Theory Appl. 20(1–4), 221–227 (1992)
25. Girela, D.: Analytic functions of bounded mean oscillation. In: Mekrijärvi, R., Aulaskari, R. (Eds.) Complex Function Spaces. Univ. Joensuu Dept. Math. Rep. Ser. 4. Univ. Joensuu, Joensuu, vol. 2001, pp. 61–170 (1999)
26. Girela, D., González, C., Peláez, J.A.: Multiplication and division by inner functions in the space of Bloch functions. Proc. Am. Math. Soc. 134(5), 1309–1314 (2006)
27. Girela, D., Merchán, N.: A generalized Hilbert operator acting on conformally invariant spaces. Banach J. Math. Anal. 12(2), 374–398 (2018)
28. Girela, D., Nowak, M., Waniurski, P.: On the zeros of Bloch functions. Math. Proc. Camb. Philos. Soc. 129(1), 117–128 (2000)
29. Girela, D., Peláez, J.A.: Carleson measures, multipliers and integration operators for spaces of Dirichlet type. J. Funct. Anal. 241(1), 334–358 (2006)
30. Girela, D., Merchán, N.: A certain class of compression operators that are connected with the divisibility of analytic functions (in Russian). Ukrain. Mat. Z. 24, 692–695 (English translation in Ukrainian Math. J. 24(1973), 559–561) (1972)
31. Girela, D., Peláez, J.A.: Multipliers of Möbius invariant $Q_p$ spaces. Math. Z. 261(3), 545–555 (2009)
32. Girela, D., Peláez, J.A.: On the zeros of functions in Dirichlet-type spaces. Trans. Am. Math. Soc. 363(4), 1981–2002 (2011)
33. Pau, J., Peláez, J.A.: Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces (in Russian). Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222, 3806–3827 (1995) (Issled. po Linein. Oper. i Teor. Funktsii 23, 45–77, 308; English translation in J. Math. Sci. (New York) 87, no. 5, 1997)
34. Timoney, R.M.: Natural function spaces. J. Lond. Math. Soc. (2) 41(1), 78–88 (1990)
35. Vinogradov, S.A.: Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces (in Russian). Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222, 3806–3827 (1995) (Issled. po Linein. Oper. i Teor. Funktsii 23, 45–77, 308; English translation in J. Math. Sci. (New York) 87, no. 5, 1997)
36. Wu, Z.: Carleson measures and multipliers for Dirichlet spaces. J. Funct. Anal. 169(1), 148–163 (1999)
50. Xiao, J.: Carleson measure, atomic decomposition and free interpolation from Bloch space. Ann. Acad. Sci. Fenn. Ser. A I Math. 19, 35–46 (1994)
51. Xiao, J.: The $Q_p$ corona theorem. Pac. J. Math. 194(2), 491–509 (2000)
52. Xiao, J.: Holomorphic $Q$ classes. Lecture Notes in Mathematics, vol. 1767. Springer, New York (2001)
53. Xiao, J.: Geometric $Q$ functions. In: Frontiers in Mathematics. Birkhäuser, Boston (2006)
54. Xiao, J.: The $Q_p$ Carleson measure problem. Adv. Math. 217(5), 2075–2088 (2008)
55. Zhao, R.: On logarithmic Carleson measures. Acta Sci. Math. (Szeged) 69(3–4), 605–618 (2003)
56. Zhu, K.: Analytic Besov spaces. J. Math. Anal. Appl. 157, 318–336 (1991)
57. Zhu, K.: A class of Möbius invariant function spaces. Illinois J. Math. 51(3), 977–1002 (2007)
58. Zhu, K.: Operator Theory in Function Spaces. Marcel Dekker, New York (1990) (reprint: Mathematics Surveys and Monographs, vol. 138. American Mathematical Society, Providence, 2007)
59. Zorboska, N.: Multiplication and Toeplitz operators on the analytic Besov spaces. In: Begehr, H. G. W., Nicolosi, F. (eds.)More Progress in Analysis: Proceedings of the 5th International. Isaac Congress. Catania, Italy, 25–30 July 2005. World Scientific, pp. 387–396 (2009)
60. Zygmund, A.: Trigonometric Series, Vol. I and Vol. II, 2nd edn. Cambridge University Press, Cambridge (1959)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.