**PT-symmetric non-Hermitian Hamiltonian and invariant operator in periodically driven SU(1,1) system**

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We study in this paper the time evolution of PT-symmetric non-Hermitian Hamiltonian consisting of periodically driven SU(1,1) generators. A non-Hermitian invariant operator is adopted to solve the Schrödinger equation, since the time-dependent Hamiltonian is no longer a conserved quantity. We propose a scheme to construct the non-Hermitian invariant with a PT-symmetric but non-unitary transformation operator. The eigenstates of invariant and its complex conjugate form a bi-orthogonal basis to formulate the exact solution. We obtain the non-adiabatic Berry phase, which reduces to the adiabatic one in the slow time-variation limit. A non-unitary time-evolution operator is found analytically. As an consequence of the non-unitarity the ket (|ψ(t)⟩) and bra (⟨ψ(t)|) states are not normalized each other. While the inner product of two states can be evaluated with the help of a metric operator. It is shown explicitly that the model can be realized by a periodically driven oscillator.

I. INTRODUCTION

In 1998, Bender and Boettcher showed that the Hermiticity of Hamiltonian is sufficient but not necessary condition to have real spectrum [1]. The non-Hermitian Hamiltonians can still possess real and positive eigenvalues [1], provided that the parity-time symmetry (PT-symmetry) is maintained instead of Hermiticity. Soon after the originally proposed Hamiltonians [1–3] have been extended to different kinds of PT-symmetric non-Hermitian systems. The predicted properties of PT-symmetric Hamiltonians [3, 4] have been observed at the classical level in a wide variety of laboratory experiments involving superconductivity [5, 6], optics [7–10], microwave cavities [11], atomic diffusion [12], and nuclear magnetic resonance [13].

The study of PT symmetric oscillator has attracted much attention in recent years. In 2005, Cem Yuces studied the exactly solvable generalized PT symmetric harmonic oscillator problem [14]. Joseph Schindler et al. presented in 2011 a simple experimental set-up, which displays all the novel phenomena of PT symmetry for a coupled oscillator pair [15]. Subsequently Carl M. Bender et al. observed PT phase transition in a PT-symmetric two-oscillator model [16], and studied twofold transitions as well [17]. A systematic analysis is provided by Jesús Cuevas et al. for a prototypical nonlinear oscillator with PT-symmetry [18]. The phase transition arises when the number of coupled oscillator-pairs increases from 1 to N, which is sufficient large [19]. A partial PT symmetry is examined by Alireza Beygi et al. for a chain of N coupled harmonic-oscillators [20]. Exact analytical solutions are found by Andreas Fring and Thomas Frith for a two-dimensional time-dependent non-Hermitian describing coupled two harmonic oscillators, which possess infinite dimensional Hilbert space in the broken PT-symmetry regime [21]. And then they provided a time-dependent Dyson map and metric recently [22]. The dynamics of the average displacement of a mechanical oscillator is also investigated in different regimes for the PT-symmetric-like optomechanical system [23].

Quantum harmonic oscillator is exactly solvable and effectively applied to many other systems [24], for example, a cavity mode in cavity quantum electrodynamics [25] or a mode of an LC radio-frequency resonator in circuit quantum electrodynamics [26]. Entanglement dynamics is researched in short- and long-range harmonic oscillators [27].

Since the time dependent Hamiltonian is no longer a conserved quantity it is necessary to construct a PT-symmetric non-Hermitian invariant operator to solve the time dependent system. The Hermitian invariants were proposed by Lewis and Riesenfeld to investigate the dynamics and quantization of time-dependent systems long ago [28, 29]. Explicit Hermitian invariant operators are constructed by Y. Z. Lai et al. for the time-dependent quantum systems consisting of SU(1,1) and SU(2) generators [30]. B. Khantoul et al. propose a scheme to deal with certain time-dependent non-Hermitian Hamiltonian, which involves the use of invariant operators, which are pseudo-Hermitian with respect to the time-dependent metric operator [31]. Topological invariants are also discussed in non-Hermitian systems [32–35].

For the non-Hermitian Hamiltonian the invariant operators have to be PT-symmetric and non-Hermitian. Subsequently PT-symmetric transformations are required to construct the invariant operators instead of the usual unitary transformation in the system described by the Hermitian Hamiltonian [30]. Therefore, the main aim of this paper is to invite an alternative method with PT-symmetric non-Hermitian invariant operators for the time-evolution solution of the non-Hermitian Hamiltonian.

If the time dependence of the Hamiltonian depends on a set of parameters, a cyclic evolution of the Hamiltonian in the parameter space leads to an additional phase that has geometric significance and is known as Berry’s phase [36]. This phase shift, which reveals a gauge structure in quantum mechanics, has attracted both theoretical and experimental interests [37–39]. We obtain the nonadiabatic Berry phase, which reduces...
to the adiabatic one in the slow time-varying limit for the non-Hermitian Hamiltonian.

The paper is organized as follows: in Sec. II we put forward a PT-symmetric non-Hermitian Hamiltonian consisting of periodically driven $SU(1,1)$ generators. The PT-symmetric non-Hermitian invariant operator is constructed by means of PT-symmetric transformation in order to obtain the time evolution of quantum states. We explain bi-orthogonal basis approach and metric operator in Sec. III. Exact solutions of the Schrödinger equations are found along with the nonadiabatic Berry phase, which reduces the adiabatic one in the slow varying limit. The time-evolution operator is presented in Sec. IV. We end with a conclusion and discussion in the last section.

II. PT-SYMMETRIC NON-HERMITIAN HAMILTONIAN AND INVARIANT OPERATOR

In classical mechanics PT transformation is simply the reflections of space and time coordinates. While the PT transformation operator in quantum mechanics can be defined as

$$\hat{\Theta} = \hat{U} K \hat{\Theta}^{-1} = K \hat{U}^\dagger$$

where $\hat{U}$ is the usual unitary operator for the space-time reflection and $K$ denotes the operation of complex conjugate. The position and momentum operators become $\hat{\Theta} \hat{x} \hat{\Theta}^{-1} = -\hat{x}$, $\hat{\Theta} \hat{p} \hat{\Theta}^{-1} = \hat{p}$ under PT transformation, and $\hat{\Theta} \hat{a} \hat{\Theta}^{-1} = -\hat{a}$, $\hat{\Theta} \hat{a}^\dagger \hat{\Theta}^{-1} = -\hat{a}^\dagger$ for the boson operators in harmonic oscillator. The PT transformation operator is antilinear and antunitary. The $SU(1,1)$ generators satisfy the commutation relation

$$[\hat{S}_+ \hat{S}_-, \hat{S}_z] = -2 \hat{S}_z, \quad [\hat{S}_z \hat{S}_+, \hat{S}_z] = \pm \hat{S}_z,$$  \hspace{1cm} (1)

And the $SU(1,1)$ Lie algebra has a realization in terms of boson creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ such that\cite{30, 40}

$$\hat{S}_z = \frac{1}{2} \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad \hat{S}_+ = \frac{1}{2} (\hat{a}^\dagger)^2, \quad \hat{S}_- = \frac{1}{2} (\hat{a})^2,$$  \hspace{1cm} (2)

Under the PT transformation the $SU(1,1)$ generators transform as

$$\hat{\Theta} \hat{S}_\pm \hat{\Theta}^{-1} = \hat{S}_\mp, \quad \hat{\Theta} \hat{S}_z \hat{\Theta}^{-1} = \hat{S}_z, \quad \hat{\Theta} \hat{S}_- \hat{\Theta}^{-1} = -\hat{S}_-.$$  \hspace{1cm} (3)

The commutation relation Eq.(1) is PT invariant.

We consider the PT-symmetric non-Hermitian Hamiltonian written as

$$\hat{H}(t) = \Omega S_z + G \left( \hat{S}_+ e^{i \phi(t)} - \hat{S}_- e^{-i \phi(t)} \right)$$  \hspace{1cm} (4)

in which

$$\phi(t) = \omega t$$

with $\omega$ being the driving frequency and $G$ denotes a coupling parameter. The Hamiltonian is obviously non-Hermitian

$$\hat{H}(t) \neq \hat{H}^\dagger(t) = \Omega S_z - G \left( \hat{S}_+ e^{i \phi(t)} - \hat{S}_- e^{-i \phi(t)} \right),$$

since $\hat{S}_z = \left( \hat{S}_- \right)^\dagger$. As a matter of fact the Hamiltonian $\hat{H}(t)$ describes a periodically driving harmonic-oscillator

$$\hat{H}(t) = \left[ \frac{\Omega}{2} + i \frac{G}{2} \sin \phi(t) \right] \hat{x}^2 + \left[ \frac{\Omega}{4} - \frac{G}{2} \sin \phi(t) \right] \beta^2 - i \frac{G}{2} \cos \phi(t) \left( \hat{x} \hat{p} + \hat{p} \hat{x} \right),$$

in the coordinate and momentum representation of boson operators $\hat{a} = (\hat{x} + i \hat{p}) / \sqrt{2}$ and $\hat{a}^\dagger = (\hat{x} - i \hat{p}) / \sqrt{2}$. The Schrödinger equation is covariant under the PT transformation

$$i \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle,$$

(natural unit $\hbar = 1$) with $|\psi(t)\rangle = \hat{\Theta} |\phi(t)\rangle$. We can solve the time-dependent $SU(1,1)$ system with non-Hermitian Hamiltonian in the formalism of Schrödinger equation.

Since the Hamiltonian is no long a conserved quantity an invariant operator is required to solve the time dependent system\cite{28–30}. For the PT-symmetric Hamiltonian the invariant operator $\hat{I}(t)$, which satisfies the condition

$$i \frac{d\hat{I}(t)}{dt} = \frac{d}{dt} \hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = 0,$$  \hspace{1cm} (5)

should also be invariant under the PT transformation. To this end we can construct $\hat{I}(t)$ from the PT-symmetric $SU(1,1)$-generator $\hat{S}_z$ with a PT-symmetric transformation such that

$$\hat{I}(t) = \hat{R}(t) \hat{S}_z \hat{R}^{-1}(t).$$  \hspace{1cm} (6)

The transformation operator considered as

$$\hat{R}(t) = e^{\eta \left( \hat{S}_+ e^{i \phi(t)} + \hat{S}_- e^{-i \phi(t)} \right)},$$  \hspace{1cm} (7)

$$\hat{R}^{-1}(t) = e^{-\eta \left( \hat{S}_+ e^{i \phi(t)} + \hat{S}_- e^{-i \phi(t)} \right)},$$

(with a real parameter $\eta$ to be determined) is PT symmetric, since

$$\hat{\Theta} \hat{R}(t) \hat{\Theta}^{-1} = \hat{R}(t).$$

However the operator $\hat{R}(t)$ is non-unitary with $\hat{R}^\dagger \neq \hat{R}^{-1}$ different from the ordinary quantum mechanics. Occasionally it is Hermitian $\hat{R}^\dagger(t) = \hat{R}(t)$ in this particular model but is unnecessary in general. The invariant operator $\hat{I}(t)$ is PT-symmetric, however non-Hermitian

$$\hat{I}(t) \neq \hat{I}^\dagger(t) = \hat{R}^{-1}(t) \hat{S}_z \hat{R}(t).$$

With the PT-symmetric transformation operator Eq.(7) we obtain the following relations\cite{41}

$$\hat{R}^{-1}(t) \hat{S}_+ \hat{R}(t) = \hat{S}_+ \cos^2 \left( \frac{\eta}{2} \right) - \hat{S}_- e^{-i \phi(t)} \sin (\eta)$$

$$+ \hat{S}_+ e^{-2i \phi(t)} \sin^2 \left( \frac{\eta}{2} \right),$$

$$\hat{R}^{-1}(t) \hat{S}_- \hat{R}(t) = \hat{S}_- \cos^2 \left( \frac{\eta}{2} \right) + \hat{S}_+ e^{i \phi(t)} \sin (\eta)$$

$$+ \hat{S}_- e^{2i \phi(t)} \sin^2 \left( \frac{\eta}{2} \right),$$

$$\hat{R}^{-1}(t) \hat{S}_0 \hat{R}(t) = \hat{S}_0 \cos (\eta)$$

$$+ \frac{1}{2} \sin (\eta) \left( \hat{S}_+ e^{i \phi(t)} - \hat{S}_- e^{-i \phi(t)} \right).$$  \hspace{1cm} (8)
and
\[ i\hat{R}^{-1}(t) \frac{\partial}{\partial t} \hat{R}(t) = 2\frac{d\phi}{dt} \hat{S}_z \sin^2 \left( \frac{\eta}{2} \right) - \frac{d\phi}{2dt} \sin(\eta) \left( \hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)} \right) \] (9)

By straightforward algebra the invariant operator \( \hat{I}(t) \) is found as [41]
\[ \hat{I}(t) = \cos(\eta) \hat{S}_z + \frac{1}{2} \sin(\eta) \left( \hat{S}_+ e^{i\phi(t)} - \hat{S}_- e^{-i\phi(t)} \right), \] (10)
under the auxiliary condition
\[ G \cos(\eta) = -\frac{1}{2}(\phi + \Omega) \sin(\eta), \] (11)
derived from Eq. (5), Eq. (6) and Eq. (7).

III. EXACT SOLUTION AND NON-ADIABATIC BERRY PHASE

Since the invariant operator \( \hat{I}(t) \) is non-Hermitian the eigenstates of it along are not an orthonormal basis. The bi-orthogonal basis [42–44] are requested respectively for the \( PT \)-symmetric invariant-operator \( \hat{I}(t) \) and its complex conjugate \( \hat{I}^\dagger \) to obtain the exact solution of the Schrödinger equation.

A. Bi-orthogonal basis and metric operator

The eigenstates of \( \hat{S}_z, \hat{S}_\pm |n\rangle = k_n |n\rangle \), are Fock states \( |n\rangle \) with eigenvalues
\[ k_n = \frac{1}{2} \left( n + \frac{1}{2} \right), \]
which are nothing but the eigenvalues of harmonic oscillator. The eigenstates of \( \hat{I}(t) \) are obviously given by
\[ \hat{I}(t) |n\rangle_r = k_n |n\rangle_r, \quad |n\rangle_r = \hat{R}(t) |n\rangle, \]
with the same eigenvalues \( k_n \), which are conserved quantities. The subscript "r" denotes the ket states. Since the \( PT \)-symmetric transformation operator \( \hat{R}(t) \) is non-unitary, the states \( |n\rangle_r \) cannot be normalized i. e. \( \langle n(t) | n(t) \rangle_r \neq 1 \). The complex conjugate operator of invariant \( \hat{I}(t) \) is seen to be
\[ \hat{I}^\dagger = \left[ \hat{R}(t) \hat{S}_z \hat{R}^{-1}(t) \right]^\dagger = \hat{R}^{-1}(t) \hat{S}_z \hat{R}(t), \]
which possesses eigenstates denoted by
\[ \hat{I}^\dagger |n\rangle_r = k_n |n\rangle_r, \quad |n\rangle_l = \hat{R}^{-1}(t) |n\rangle, \]
with real eigenvalues too and the subscript "l" indicates the bra states in the orthogonality condition. The two sets of the ket and bra basis \( |n(t)\rangle_r \) and \( |n(t)\rangle_l \) form a bi-orthogonal basis [42–44] with the orthogonality condition
\[ \langle n(t) | m(t) \rangle_r = \delta_{nm}. \] (12)

The completeness relation is
\[ \sum_n \langle n(t) | n(t) \rangle = \sum_n |n(t)\rangle_r \langle n(t) | n(t) \rangle = 1. \]

According to Refs. [44], we can define new metric operator \( \hat{\chi} \) relating the ket and bra states by
\[ |n(t)\rangle_r = \hat{\chi} |n(t)\rangle_l, \]
in which the metric operator for the present model is
\[ \hat{\chi} = \left( \hat{R}^{-1}(t) \right)^2. \] (13)

With the metric operator orthogonality condition Eq. (12) becomes
\[ \langle n(t) | \hat{\chi} | m(t) \rangle = \delta_{nm} \]
without using the two sets of basis.

B. Solution and LR phase

According to Lewis and Riesenfeld (LR) theory [28, 29], the general solution of the Schrödinger equation is superposition of the eigenstates of invariant operator \( \hat{I}(t) \)
\[ |\psi(t)\rangle = \sum_n C_n e^{i\alpha_n(t)} |n(t)\rangle_r, \] (14)
in which the time-independent coefficient \( C_n \) can be determined by initial condition. Under the \( PT \)-symmetric transformation
\[ |\psi'(t)\rangle = \hat{R}^{-1} |\psi(t)\rangle, \]
the original time-dependent Schrödinger equation \( i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \) becomes
\[ i \frac{\partial}{\partial t} |\psi'(t)\rangle = \hat{H}' |\psi'(t)\rangle, \] (15)
with the new Hamiltonian given by
\[ \hat{H}' = \left( \hat{R}^{-1} \hat{H} \hat{R} - i \hat{R}^{-1} \frac{\partial}{\partial t} \hat{R} \right). \]
The solution of Schrödinger equation Eq. (15) is
\[ |\psi'(t)\rangle = \sum_n C_n e^{i\alpha_n(t)} |n\rangle, \] (16)
seen from Eq. (14). Substituting the \( |\psi'(t)\rangle \) in Eq. (16) into the Schrödinger equation (15) yields the LR phase
\[ \alpha_n(t) = - \int_0^t dt' \langle n| \hat{H}'(t') |n\rangle \]
\[ = \int_0^t dt' \langle n| \left[ i \frac{\partial}{\partial t'} - \hat{H}'(t') \right] |n\rangle \cdot \delta_{nn}. \] (17)
The periodically driven Hamiltonian can be solved exactly by means of the LR method with the help of invariant operator. The first term in LR phase $\alpha_n(t)$ is usually regarded as the non-adiabatic Berry phase. Using Eqs.(8,9) we obtain the LR phase

$$\alpha_n(t) = -k_n \int_0^t [\Omega - 2\Gamma (t')] dt',$$

(18)

where $\Gamma (t)$ is given by

$$\Gamma = (\phi + \Omega) \sin^2 \left( \frac{\eta}{2} \right) + G \sin (\eta).$$

(19)

From the auxiliary equation Eq.(11), the parameter $\eta$ is determined as

$$\sin^2 \left( \frac{\eta}{2} \right) = \frac{1}{2} \pm \frac{\omega + \Omega}{2 \sqrt{(\omega + \Omega)^2 + 4G^2}}.$$  

(20)

C. Berry phase

According to definition, the first term in Eq.(17) gives rise to the Berry phase indicated by $\gamma_n$, which in one period of the driven field $T = 2\pi/\omega$ is evaluated from Eq.(9) as

$$\gamma_n(T) = i \int_0^T \langle n(t) \mid \frac{\partial}{\partial t} \mid n(t) \rangle dt = 2k_n \int \sin^2 \left( \frac{\eta}{2} \right) d\phi.$$  

(21)

Substituting Eq.(20) into Eq.(21) we find the non-adiabatic Berry phase

$$\gamma_n(T) = \pi \left(n + \frac{1}{2}\right) \left(1 \mp \frac{\omega + \Omega}{\sqrt{(\omega + \Omega)^2 + 4G^2}}\right).$$

In the adiabatic approximation that $\dot{\phi} = \omega = 0$ the Berry phase becomes the well known form

$$\gamma_n(T) = \pi \left(n + \frac{1}{2}\right) \left(1 \mp \frac{\Omega}{\sqrt{\Omega^2 + 4G^2}}\right).$$

D. Average energies

The invariant operators $\hat{I}(t)$ and its conjugate $\hat{I}^\dagger(t)$ possess real eigenvalues in the $PT$-symmetric $SU(1, 1)$ system. From Eqs.(17,18,19), one can easily confirm that the transformed Hamiltonian does not depend on time

$$\hat{R}^{-1}(t) \hat{H}(t) \hat{R}(t) = (\Omega - 2\Gamma_0) \hat{S}_z$$

in which

$$\Gamma_0 = \Omega \sin^2 \left( \frac{\eta}{2} \right) + G \sin (\eta)$$

in the adiabatic approximation. Average energies at the eigenstates of invariant operators $\hat{I}(t)$ and its conjugate $\hat{I}^\dagger(t)$ are entirely real values too

$$\langle n(t) \mid \hat{H}(t) \mid n(t) \rangle = \langle n \mid (\Omega - 2\Gamma_0) \hat{S}_z \mid n \rangle = (\Omega - 2\Gamma_0) k_n.$$  

The Hamiltonian $\hat{H}(t)$ what we considered is in an unbroken $PT$-symmetric phase according to Ref.[16].

IV. THE TIME-EVOLUTION OPERATOR

The time-evolution operator can be derived in terms of the eigenstates of invariant operator. Substituting the LR phase Eq.(18) into the general solution of the Schrödinger equation (14) we have

$$|\psi(t)\rangle = \hat{R}(t) \sum_n C_n e^{-i\epsilon(t)k_n} |n\rangle = \hat{R}(t) e^{-i\epsilon(t)\hat{S}_z} \sum_n C_n |n\rangle, $$

(22)

where

$$\epsilon (t) = \int_0^t dt' [\Omega - 2\Gamma (t')] .$$

Assume that the initial state at time $t = 0$ is denoted by

$$|\psi (0)\rangle = \hat{R}(0) \sum_n C_n |n\rangle$$

The state at time $t$ is generated by the time evolution operator such that

$$|\psi (t)\rangle = \hat{U}(t, 0) |\psi (0)\rangle ,$$

where the time evolution operator is

$$\hat{U}(t, 0) = \hat{R}(t) e^{-i\epsilon(t)\hat{S}_z} \hat{R}^{-1}(0).$$

(23)

The time evolution operator is not unitary

$$\hat{U}^{-1}(t, 0) \neq \hat{U}^\dagger(t, 0),$$

since the inverse operator is

$$\hat{U}^{-1}(t, 0) = \hat{R}(0) e^{i\epsilon(t)\hat{S}_z} \hat{R}^{-1}(t),$$

while the complex conjugate reads

$$\hat{U}^\dagger(t, 0) = \hat{R}^{-1}(0) e^{i\epsilon(t)\hat{S}_z} \hat{R}(t).$$

The bra state

$$\langle \psi (t) | = \langle \psi (0) | \hat{U}^\dagger(t, 0),$$

is not normalized with the corresponding ket state

$$\langle \psi (t) | \psi (t) \rangle \neq 1$$

While it can be normalized with the help of metric operator[44]

$$(\psi (t), \psi (t)) \equiv \langle \psi (t) | \hat{\chi} (t) \mid \psi (t) \rangle = 1.$$  

In general the inner product of two states $|\psi (t)\rangle$ and $|\varphi (t)\rangle$ is evaluated as

$$(\varphi (t), \psi (t)) \equiv \langle \varphi (t) | \hat{\chi} (t) \mid \psi (t) \rangle .$$
V. CONCLUSION

The $PT$ symmetry is a more general property of dynamic system, since both Newton’s and Maxwell’s equations are invariant under the $PT$ transformation. If the non-Hermitian Hamiltonian is $PT$ symmetric, the Schrödinger equation is also invariant. We demonstrate an analytical formalism to solve the periodically driven non-Hermitian Hamiltonian consisting of generators of $SU(1,1)$ Lie algebra. A $PT$-symmetric non-Hermitian invariant operator $\hat{I}(t)$ is constructed in terms of $PT$-symmetric but non-unitary transformation-operator $\hat{R}(t)$. As a consequence two sets of basis $|n(t)\rangle$, $|\langle n(t)|\rangle$ respectively for invariant $\hat{I}(t)$ and its complex conjugate operator $\hat{I}^*(t)$ are required to serve as an orthonormal basis. In the considered model with unbroken $PT$-symmetry[16], the invariant operators possess real eigenvalues and the average energies of Hamiltonian are also real. We obtain the LR phase and non-adiabatic Berry phase, which reduces to the adiabatic one in the slowly varying limit. The non-unitary time-evolution operator is formulated explicitly. The ket ($|\psi(t)\rangle$) and bra ($\langle \psi(t)|$) states evolve with time respectively by the evolution operators $\hat{U}(t,0)$ and $\hat{U}^\dagger (0,t)$. The inner product of two states are evaluated with the help of a metric operator $\chi(t)$. This model Hamiltonian can be realized by periodically driven harmonic oscillator.

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[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[2] C. M. Bender, D. C. Brody, H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002).
[3] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[4] A. Mostafazadeh, J. Geom. Methods Mod. Phys. 7, 1191 (2010).
[5] J. Rubinstein, P. Sternberg, and Q. Ma, Phys. Rev. Lett. 99, 167003 (2007).
[6] N. M. Chchelkatchev, A. A. Golubov, T. I. Baturina, and V. M. Vinokur, Phys. Rev. Lett. 109, 150405 (2012).
[7] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
[8] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature Phys. 6, 192 (2010).
[9] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. 106, 213901 (2011).
[10] L. Feng, M. Ayache, J. Huang, Y.-L. Xu, M. H. Lu, Y. F. Chen, Y. Fainman, and A. Scherer, Science 333, 729 (2011).
[11] S. Bittner, B. Dietz, U. Günther, H. L. Harney, M. Miski-Ogлу, A. Richter, and F. Schäfer, Phys. Rev. Lett. 108, 024101 (2012).
[12] K. F. Zhao, M. Schaden, and Z. Wu, Phys. Rev. A 81, 042903 (2010).
[13] C. Zheng, L. Hao, and G. L. Long, Philos. Trans. R. Soc. A 371, 20120053 (2013).
[14] C. Yuce Phys. Lett. A 336, 290 (2005).
[15] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, Phys. Rev. A 84, 040101 (2011).
[16] C. M. Bender, B. K. Berntson, D. Parker, and E. Samuel, Am. J. Phys. 81, 173 (2013).
[17] C. M. Bender, M. Gianfreda, S. K. Özdemir, B. Peng, and L. Yang, Phys. Rev. A 88, 062111 (2013).
[18] J. Cuevas, P. G. Kevrekidis, A. Saxena, and A. Khare, Phys. Rev. A 88, 032108 (2013).
[19] C. M. Bender, M. Gianfreda, S. P. Klevansky, Phys. Rev. A 90, 022114 (2014).
[20] A. Beygi, S. P. Klevansky, C. M. Bender, Phys. Rev. A 91, 062101 (2015).
[21] A. Fring, T. Frith, J Phys A-Math Theor. 51, 265301 (2018).
[22] A. Fring, T. Frith, Mod. Phys. Lett. A 35, 2050041 (2020).
[23] H. Xu, D. G. Lai, Y. B. Qian, B. P. Hou, A. Miranowicz, and F. Nori, Phys. Rev. A 104, 053518 (2021).
[24] P. A. M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, Oxford, 1958).
[25] A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, Jean-Michel Raimond, and S. Haroche, Science 288, 2024 (2000).
[26] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R. S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature (London) 431, 162 (2004).
[27] M. Ghasemi Nezhadaghighi and M. A. Rajabpour, Phys. Rev. B 90, 205438 (2014).
[28] H. R. Lewis, Jr., J. Math. Phys. 9, 1976 (1968).
[29] H. R. Lewis, Jr., and W. B. Ziesenberg, J. Math. Phys. 10, 1458 (1969).
[30] Y. Z. Lai, J. Q. Liang, H. J. W. Müller-Kirsten, J. G. Zhou, Phys. Rev. A 53, 3691 (1996).
[31] B. Khantoul, A. Bounames, M. Maamache, Eur. Phys. J. Plus 132, 258 (2017).
[32] S. D. Liang, G. Y. Huang, Phys. Rev. A 87, 012118 (2013).
[33] S. Yao, Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).
[34] A. Ghatak, T. Das, J. Phys. Condens. Mat. 31, 263001 (2019).
[35] F. Song, S. Yao, Z. Wang, Phys. Rev. Lett. 123, 246801 (2019).
[36] M. V. Berry, Proc. R. Soc. London A 392, 45 (1984).
[37] Y. Zhang, Y. W. Tan, H. L. Stormer, P. Kim, Nature 438, 201 (2005).
[38] H. Watanabe, M. Oshikawa, Phys. Rev. X 8, 021065 (2018).
[39] S. Enomoto, T. Matsuda, Phys. Rev. D 99, 063005 (2019).
[40] J. Q. Liang and L. F. Wei, New Advances in Quantum Physics (Science Press, Beijing, 2020).
[41] Y. Z. Lai, J. Q. Liang, H. J. W. Müller-Kirsten and J. G. Zhou, J. Phys. A: Math. Gen. 29, 1773 (1996).
[42] C. P. Sun, Phys. Scr. 48, 393 (1993).
[43] P. T. Leung, W. M. Suen, C. P. Sun, and K. Young, Phys. Rev. E 57, 6101 (1998).
[44] T. Shi, C. P. Sun, arXiv: 0905.1771 (2009).