Covariant one-loop quantum gravity and Higgs inflation

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The quantisation of scalar field theory and Einstein gravity is investigated using a fully covariant background field formalism, including Vilkovisky-DeWitt corrections. The one-loop divergences, which are relevant for the consistency of the low-energy effective theory, differ substantially from non-covariant calculations. The results are applied to the effective action of Higgs inflation, with a non-minimal gravity coupling parameter $\xi$, where the use of a formalism which is independent of field redefinitions is particularly important. Consistency of the one-loop effective action requires the cut-off $\Lambda < M_p/\sqrt{\xi}$ in the small field and $\Lambda < M_p$ in the large field limit.

I. INTRODUCTION

Quantum gravity is potentially an important ingredient in descriptions of the very early-universe. Of necessity, most early universe applications of quantum gravity are done within the context of an effective field theory, introducing a cut-off scale below which the theory is described with some degree of accuracy by general relativity coupled to the matter sector [1, 2].

At energies below the cut-off scale, we are free to calculate the quantities of interest, such as the effective action, including corrections to the desired order of accuracy. Having relaxed the concept of renormalisability, we still have to address some important technical questions. One issue which provides the focus of this paper is how to remove any dependence of the effective action on the choice of field variables, in other words how to establish covariance on field space. This is closely linked to obtaining an effective action which is independent of the choice of gauge-fixing. This paper will utilise the Vilkovisky-DeWitt method [3, 4] for retaining full covariance, focusing on the one loop effective action for the gravity-scalar system.

Since the existence of the cut-off is an important property in the effective theory, it seems sensible to employ a covariant cut-off regularisation scheme. The Schwinger method is consistent with gauge invariance and general covariance and is well-suited to evaluation of the one loop effective action [5–7]. The divergences become finite, cut-off dependent terms which are important for analysing the self consistency of the theory. These terms are also very closely related to the ones needed in the search for renormalisation group fixed points, which may provide special classes of ultraviolet-complete effective theories if the asymptotic-safety philosophy is adopted [8].

The first non-trivial type of divergence is the quadratic divergence proportional to $\Lambda^2$. The quadratic divergences appear in many places, for example as corrections to the effective Planck mass. This gives the first example of a result which potentially depends on the gauge fixing terms, and the covariant approach gives a new result for the running of the Planck mass,

$$M_p^2(\Lambda) = M_p^2(0) + \frac{\Lambda^2}{24\pi^2}N,$$

where $N$ is the number of scalar fields. The mass $M_p(0)$ can be regarded as the cosmological Planck mass whilst $M_p(\Lambda)$ is the effective mass for small-scale quantum fluctuations. The covariant approach gives a running part to the mass which has has the opposite sign to the one found in previous non-covariant calculations [9, 10].

The second part of this paper considers the importance of quantum corrections to the cosmological model of Higgs inflation [11–16]. In Higgs inflation, the Standard Model of particle physics is coupled to gravity with the Einstein term and a non-minimal Higgs coupling $\xi R|H|^2$. This non-minimal coupling term can be removed by a conformal re-scaling of the metric, the original metric defining the Jordan frame and the new metric defining the Einstein frame. The Jordan frame has the simpler action, but Einstein frame is convenient for descriptions of inflation.

Quantum corrections to Higgs inflation were first considered with reference to the running of the Higgs self-coupling [17–21]. This affects the shape of the Higgs potential in the inflationary regime, and through this predictions of the large-scale structure of the universe. Since the running of the Higgs self-coupling is sensitive to low energy physics, there seemed to be an interesting new link between particle physics and cosmology. However, there where early indications, based on power counting in Feynman diagrams, that the quantum corrections to the model became
problematic if the cut-off is at an energy scale around $M_p/\xi$, which would be below the inflationary scale in the Einstein frame \[17, 22, 23\]. The consistency seemed at first to be restored if the quantum theory was done in the Einstein frame \[24\], but then problems arose with the Goldstone boson sector even in the Einstein frame \[22, 26\]. Subsequently, background field techniques have indicated that the limit $M_p/\xi$ applies in the small field regime but becomes $M_p$ in the large field regime relevant to inflation \[27, 30\].

In order to be able to obtain identical results in both the Jordan and Einstein frames it seems desirable to employ a quantum technology which is covariant under field definitions, hence the relevance of the Vilkovisky-DeWitt methodology (as has been suggested in \[29\]). Furthermore, since ghost loops can affect the degree of divergence in simple power counting arguments, a full one-loop analysis will be attempted in Section III.

The literature on one-loop gravity is substantial. Some of the early work on gravity-scalar systems was done by DeWitt \[31, 32\] and t’Hooft and Veltman \[33\]. Reviews and further references can be found in \[5, 7, 34\]. Results which use Vilkovisky-DeWitt corrections include gauge couplings and the Higgs mass corrections in \[35\] and scalar kinetic and mass terms in \[36\]. The gauge-fixing functionals used here are based on one first used by DeWitt, and supplemented with scalar terms by t’Hooft and Veltman (not to be confused with their electromagnetic gauge). In this paper, Feynman gauge refers to gauge-fixing with gauge parameter $\alpha = 1$ and Landau gauge to gauge fixing with $\alpha \to 0$. The metric conventions used in this paper follow Misner, Thorne and Wheeler \[37\] and the units are ones in which $c = \hbar = 1$.

II. ONE-LOOP CALCULATIONS

In the Vilkovisky-DeWitt formalism, the one-loop contribution to the effective action consists of a contribution from the original fields $\varphi^i$ and a contribution from ghosts $c_i$. The field contribution is given in condensed notation by

$$i\frac{1}{2} \lim_{\alpha \to 0} \ln \det \left( \nabla_i \nabla_j S[\varphi] + \frac{1}{2\alpha} K^i_{\phantom{i}j}[\varphi] K^j_{\phantom{j}i}[\varphi] \right),$$

(2)

where the indices $i, j$ represent both coordinate and internal components. The innovation of Vilkovisky and DeWitt was to put the second functional derivatives into covariant form,

$$\nabla_i \nabla_j S = S_{ij} - \Gamma^k_{ij} S[k,],$$

(3)

where the connection coefficients $\Gamma^i_{jk}$ will be determined by the Levy-Civita connection for the metric on the space of fields. The connection ensures that the result is covariant under field redefinitions. It can be disregarded when the background field is on-shell, i.e. $S_{ij} = 0$, but any off-shell application of the effective action has to include it.

The second term in (2) is a gauge-fixing term for a gauge-fixing functional $\chi_r = K^r_{\phantom{r}i} \delta \varphi^i$, which uses the generator of gauge transformations $K^r_{\phantom{r}i}$. Other gauge-fixing terms can be used, without changing the form of the effective action, provided the field-space connection is suitably modified. However, the one-loop effective action obtained from the Landau gauge in (2) is identical to the one-loop effective action which is fully independent of the gauge-fixing term.

A. Quadratic action

The first task is to evaluate the functional derivatives appearing the one-loop determinant. The basic fields under consideration are the spacetime metric $g_{\mu\nu}$, with Ricci scalar $R$, and a set of scalar fields $\phi^i$. The scalar fields will take values in an internal manifold, with an internal metric $G_{ij}$ and potential $V$. The number of spacetime dimensions is $m$ and the internal space dimension is $N$. The Lagrangian densities for gravity $L_g$, scalar $L_s$ and gauge fixing $L_\chi$ are,

$$L_g = \frac{1}{2\kappa^2} R |g|^{1/2},$$

(4)

$$L_s = -\frac{1}{2} G_{ij}(\phi) g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j |g|^{1/2} - V(\phi) |g|^{1/2},$$

(5)

$$L_\chi = -\alpha^{-1} \chi^\mu \partial_\mu \chi,$$

(6)

where $\partial_\mu$ denotes an ordinary spatial derivative. These are expanded about a background field configuration, so that $L = L_0 + L_1 + \ldots + J^\mu_{\phantom{\mu},\mu}$, where $L_n$ is $n$’th order in perturbations and $J^\mu_{\phantom{\mu},\mu}$ is chosen so that all the terms are first order in derivatives.

We can expand the metric in a simple linear fashion about a background $g_{\mu\nu}$, making the replacement $g_{\mu\nu} \to g_{\mu\nu} + 2\kappa \gamma_{\mu\nu}$. Expansion of the Einstein action to quadratic order has been described in many places \[3, 2, 34\], and the
results are combined in Appendix A. The scalar field is best expanded using the covariant background field method \[32, 33\] where \( \eta^i \) is the tangent vector to the (internal space) geodesic joining the background field \( \varphi^i \) to \( \phi^i \). The covariant background field approach removes the need to include the connection terms for the scalars in Eq. (2), and instead we include them in the derivatives of \( \eta^i \),

\[
D_\mu \eta^i = \nabla_\mu \eta^i + \partial_\mu \psi^b \Gamma^i_{jk} \eta^j.
\]  

(7)

The curvature tensor on the internal space will be denoted by \( R_{ijkl} \). The quadratic order Lagrangian densities are given explicitly in Appendix A.

The scalar field is best expanded using the covariant background field method \[32, 33\]. The connection coefficients can be evaluated using standard formulae for the Levy-Civita connection.

The background field equation in this case is the Einstein field equation \( G_{\mu\nu} = \kappa^2 T_{\mu\nu} = 0 \). The full set of Vilkovisky-DeWitt corrections, including this one, is given in Eq. (A4).

When the various contributions are combined it makes sense to write the result in a form which is suited to the operator methods which will be used to evaluate the effective action. We can combine the metric and scalar variations under the diffeomorphism symmetry.

The final ingredient in (2) is the set of field-space connection terms and background field equations. These have been evaluated on flat spacetime backgrounds by Mackay and Toms \[36\], but they generalise to curved backgrounds very simply. The connection coefficients can be evaluated using standard formulae for the Levy-Civita connection.

The effective gauge potential mixes the gravity and scalar sectors. It is also advantageous to rewrite the terms involving a single spacetime derivative in terms of an effective gauge potential \( A_\mu \), and then

\[
\Phi_a = \left( \begin{array}{c} \gamma_{\mu\nu} \\ \eta_i \end{array} \right).
\]  

(11)

The effective action also has the ghost contribution, obtained from the gauge variation of the gauge-fixing functional \( \chi^\mu \) under the diffeomorphism symmetry.

The effective gauge potential mixes the gravity and scalar sectors, \( A_{\alpha \mu} \), and

\[
A_{\alpha}^{\cdot b} = \left( \begin{array}{c} 0 \\ A_{\alpha}^{\cdot \mu\nu} \end{array} \right),
\]  

(15)

where \( A_{\alpha}^{\cdot \mu\nu} = \kappa g_{\alpha}^{\cdot \mu}(\partial_\nu \phi^i) \) can be read off from Eqs. (A2) and (A3).
B. Regularisation and divergences

The proper-time cut-off regularisation scheme devised by Schwinger proves a fully gauge-invariant means of regularising the one-loop effective action, and gives an explicit representation for the divergent terms at each order in the cut-off. The method is defined in terms of the heat kernel $K(x, x', \tau)$ of an elliptic operator $\Delta$.

$$K(x, x', \tau) = \sum_n u_n(x) u_n^*(x') e^{-\lambda_n \tau}, \quad (16)$$

where $u_n(x)$ are the normalised eigenfunctions of $\Delta$ with eigenvalues $\lambda_n$. In order to use the proper-time method it is necessary to find an analytic continuation of the spacetime from the metric signature $-+++$ to an elliptic signature $+++$.

The definition of $\ln \det \Delta$ with proper-time cut-off $\tau$ is provided by

$$\ln \det \Delta = - \int d^m x |g|^{1/2} \int_\tau^\infty \frac{d\tau'}{\tau'} \Tr K(x, x, \tau'), \quad (17)$$

where $\Tr$ is over the internal indices. For a second order operator, the behaviour of the heat kernel for small $\tau$ is determined by an asymptotic expansion,

$$K(x, x, \tau) \sim (4\pi \tau)^{-m/2} \sum_{r=0}^\infty E_r(\Delta, x) \tau^r. \quad (18)$$

The trace of each coefficient $E_r(x)$ for a covariant operator is given by a local expression invariant under the gauge symmetries.

The small $\tau$ expansion can be used to isolate the divergent parts of $\ln \det \Delta$. Dimensionally, $\tau = 1/\Lambda^2$, where $\Lambda$ is an energy scale. The divergent part of $\ln \det \Delta$ expressed in terms of $\Lambda$ is then

$$\ln \det \Delta|_{\text{div}} \sim - \frac{1}{(4\pi)^m/2} \int d^m x |g|^{1/2} \left( \sum_{r=0}^{m-1} \frac{2}{m-2r} \Tr E_r(\Delta, x) \Lambda^{m-2r} + \Tr E_{m/2} \ln \Lambda \right). \quad (19)$$

The divergent part of the one-loop effective action $\Gamma_{\text{div}}$ is has contributions from the fields with operator $\Delta_f$ and the ghosts with operator $\Delta_g$,

$$\Gamma_{\text{div}} = \lim_{\alpha \to 0} \left\{ -\frac{1}{2} \ln \det \Delta_f|_{\text{div}} + \ln \det \Delta_g|_{\text{div}} \right\}. \quad (20)$$

Analytic continuation back to the metric signature $-+++$ has introduced a factor `$i$' from the spacetime volume integration. In flat spacetime, the cut-off $\Lambda$ becomes the usual energy-momentum cut-off, but in curved space-time there may be a need to re-scale $\Lambda$ to a physical cut-off relevant to an observer’s frame of reference.

Explicit expressions for the traces $\Tr E_n(x)$ are known for certain operators. The relevant operator for the Einstein-scalar system can be read off from the Lagrangian density. There are limited results on this type of non-minimal operator, but recent progress has been made on $\Tr E_1$ and $\Tr E_2$. The general result is known for $\Tr E_1$, and is given in Appendix B. Partial results for $\Tr E_2$ are given in Appendix C.

In four dimensions, the $\Tr E_1$ terms are the coefficients of quadratic divergences. From this point on the value of $m$ will be fixed at $m = 4$ and the reduced Planck mass $M_p = 1/\kappa$. The results of Appendix B translate into the quadratic divergences $\Gamma_{\text{quad}}$,

$$\Gamma_{\text{quad}} = \frac{1}{32\pi^2 M_p^2} \int d^4 x |g|^{1/2} \left\{ -\frac{N + 22}{3} M_p^2 R + \frac{1}{2} N \partial_\mu \phi^i \partial^\mu \phi_i + 2(N + 6) V + M_p^2 R_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - M_p^2 V_{ij} \right\} \quad (21)$$

Recall that we started from an effective theory in which the cut-off takes a finite value, representing some high-energy physics which has been removed from the problem, therefore $\Gamma_{\text{quad}}$ is a finite contribution to the effective action.

The coefficient of the curvature term has a part depending on $N$, which comes from the scalar sector. However, the magnitude and even the sign of this contribution is different from previous scalar calculations because of the Vilkovisky-DeWitt corrections to the effective action. The new result differs by factors of $R + \kappa^2 T$, which vanish on-shell but which affect the renormalisation in a critical way. If we interpret the original action as an effective theory, then the bare mass $M_p$ is the value obtained from integrating out unknown physical effects from energy scales.
above $\Lambda$, therefore the bare mass depends on $\Lambda$. The Planck mass relevant for cosmology is taken from the coefficient of the Ricci scalar in the effective action, which we call $M_p^2(0)$, giving the scaling relation,

$$M_p^2(\Lambda) = M_p^2(0) + \frac{1}{24\pi^2} \Lambda^2 (N + 22).$$

(22)

The scalar kinetic and potential terms can be partly absorbed by field and mass renormalisations, but for $R_{ijkl} \neq 0$ the effective action also contains terms which have to be included in the effective field theory. The self-consistency of the one-loop action depends on these new terms being small, hence there is a requirement that $\Lambda \ll M_p(\Lambda)$.

Current technology is not able to deliver the general expression for $\text{Tr} E_2$ for the full Einstein-scalar operator, so the discussion of the logarithmic divergences will be restricted to constant backgrounds. $\partial_{\mu} \varphi^i = 0$. The logarithmic divergences from Eq. (19) are

$$\Gamma_{\log} = \frac{1}{32\pi^2} \ln \Lambda \int d^4x |g|^{1/2} \left\{ \frac{1}{2} V_{ij} V^{ij} - 2M_p^{-2} V \varphi^i \\
- \frac{3}{2} M_p^{-2} V \varphi^i + (2N + 12) M_p^{-4} V^2 \\
+ \frac{1}{3} R V_{\varphi^i} - \frac{2N + 24}{3} M_p^{-2} RV \\
+ \frac{N + 212}{180} R_{\mu\rho\sigma}^2 - \frac{N + 122}{180} R_{\mu\nu}^2 + \frac{2N + 25}{36} R^2 \right\}.$$  

(23)

The effective action also contains terms which have to be included in the effective field theory. The self-consistency of the one-loop action depends on these new terms being small, hence there is a requirement that $\Lambda \ll M_p(\Lambda)$.

Contributions of the running of the coupling constants in the Standard Model of particle physics suggest that the effective Higgs self-coupling may become negative at large values of the Higgs field [41]. This de-stabilising effect may be overcome by higher order terms in the Higgs potential which arise from Higgs-gravity interactions. These terms are undetermined in the effective theory, but may have some definite coefficients if the large-scale behaviour is determined by a (Wilsonian style) renormalisation group fixed point.

### III. HIGGS INFLATION

The Lagrangian density for Higgs inflation in the Jordan frame contains the standard model Higgs terms and a non-minimal coupling between the metric $g_{\mu\nu}$ and the Higgs doublet field $\mathcal{H}$. The gravity-scalar contributions are

$$\mathcal{L}[g]^{-1/2} = \frac{1}{2\kappa^2} \hat{R} - \xi \hat{R} \mathcal{H}^\dagger \mathcal{H} - \hat{g}^{\mu\nu}(D_\mu \mathcal{H})^\dagger (D_\nu \mathcal{H}) - \hat{V}(\mathcal{H}).$$  

(24)

The Lagrangian density in the Einstein frame is obtained by the conformal transformation

$$\hat{g}_{\mu\nu} = f g_{\mu\nu},$$

(25)

where

$$f = (1 + 2\kappa^2 \xi \mathcal{H}^\dagger \mathcal{H})^{-1}.$$  

(26)

Total derivatives terms have to be removed by integrating by parts, and then

$$\mathcal{L}[g]^{-1/2} = \frac{1}{2\kappa^2} R - f (D_\mu \mathcal{H})^\dagger (D^\mu \mathcal{H}) - \frac{3}{2} \xi^2 \kappa^2 f^2 \hat{g}_{\mu\nu} \hat{g}^{\nu\lambda} \partial_\mu (\mathcal{H}^\dagger \mathcal{H}) - f^2 \hat{V}(\mathcal{H}),$$

(27)

where indices are raised using the Einstein metric. The scalar part of the action is a non-linear sigma model similar to [5] when written in terms of the real components $\mathcal{H}^\dagger = (\phi^1 + i\phi^2, \phi^3 + i\phi^4)/\sqrt{2}$, with $V = f^2 \hat{V}$ and

$$G_{ij} = f \delta_{ij} + 6f^2 \xi^2 \kappa^2 \delta_{ik} \delta_{jl} \phi^k \phi^l.$$  

(28)
When evaluated at the background field \( \varphi^i \), the metric can be expressed in terms of a canonically normalised massive boson and transverse Goldstone mode directions, 

\[
G_{ij} = f(\varphi) \delta_{ij}^\varphi + \chi_i \chi_j
\]  
(29)

where \( \varphi^2 = \delta_{ij} \varphi^i \varphi^j \), \( f(\varphi) = (1 + \xi \kappa^2 \varphi^2)^{-1} \) and

\[
\chi = \int d\varphi f(\varphi) [1 + (6\xi + 1) \xi \kappa^2 \varphi^2]^{1/2}
\]  
(30)

The Ricci curvature and potential derivative terms for the metric are

\[
R_{\chi\chi} = -\frac{3}{2} f^{-1} f_{,\chi\chi} - \frac{3}{4} f^{-2} f_{,\chi}^2,
\]  
(31)

\[
R_{ij}^\chi = -\left( \frac{1}{2} f_{,\chi\chi} + \frac{3}{4} f^{-1} f_{,\chi}^2 \right) \delta_{ij}^\chi,
\]  
(32)

\[
V_{,i}^\chi = (f^2 \dot{V})_{,\chi\chi} + \frac{3}{2} f^{-1} f_{,\chi} (f^2 \dot{V})_{,\chi},
\]  
(33)

\[
V_{,ij} V^{ij} = (f^2 \dot{V})_{,\chi\chi} + \frac{3}{4} f^{-2} f_{,\chi} (f^2 \dot{V})_{,\chi}.
\]  
(34)

When \( \xi \) is large, the derivative of the potential is small for large values of the canonical field \( \chi \) rendering this regime suitable for inflation.

The Jordan and Einstein frames are related by a field redefinition, and the effective action calculated using the covariant approach on field space will be the same, in both frames apart from the definition of the cut-off \( \Lambda \). In the proper-time regularisation scheme, the relation between the proper-times and the cosmological times changes due to the conformal transformation between the frames, hence the cut-off in the Jordan frame \( \hat{\Lambda} \) and the Einstein frame \( \Lambda \) are related by

\[
\hat{\Lambda} = f^{-1/2} \Lambda.
\]  
(35)

The quadratic divergences for the Einstein-scalar sectors of the theory in the Einstein frame are given by the earlier result (21). Consider the kinetic terms in the Lagrangian density of the Higgs boson \( \chi \),

\[
-\frac{1}{2} G_{\chi\chi} \partial_{\mu} \chi \partial^{\mu} \chi.
\]  
(36)

The background value \( G_{\chi\chi} = 1 \), and the quadratic divergences contribute a correction,

\[
G_{\chi\chi,\text{quad}} = -\frac{1}{4\pi^2} \frac{\Lambda^2}{M_p^2} - \frac{1}{16\pi^2} \Lambda^2 R_{\chi\chi}
\]  
(37)

It is worth recalling that the \( R_{\chi\chi} \) term originates from the background mass term \( R_{i\chi j \chi} \), where \( i \) and \( j \) are in the transverse, or Goldstone boson directions, therefore this term is a Goldstone boson loop contribution. In the large and small \( \varphi \) limits the curvature becomes

\[
R_{\chi\chi} = \begin{cases} 
-\frac{3\xi}{M_p^2} & \varphi \ll M_p/\sqrt{\xi} \\
5/6 M_p^2 & \varphi \gg M_p/\sqrt{\xi}
\end{cases}
\]  
(38)

The self-consistency of the effective theory requires that \( |G_{\chi\chi,\text{quad}}| < 1 \), setting an upper limit on the cut-off scale of \( \Lambda < M_p/\sqrt{\xi} \) for small \( \varphi \) and \( \Lambda < M_p \) for large \( \varphi \). The small \( \varphi \) limit is slightly weaker than the result obtained by power counting [22], but the large \( \varphi \) limit is identical.

The regime of interest for Higgs inflation is the large field limit, where the potential has the standard model form in the Jordan frame. In this regime we can ignore the Higgs mass terms and use the potential \( \hat{V} = \lambda |H|^4 \). In the Einstein frame, the potential of the background field \( \varphi \) becomes

\[
V(\varphi) = \frac{1}{4} \lambda f(\varphi)^2 \varphi^4 \sim \frac{1}{4} \lambda \frac{\xi}{\kappa^2} M_p^4 \varphi^4 \sim \frac{1}{2} \lambda \frac{\xi}{\kappa^2} M_p^6 \varphi^2 + \ldots
\]  
(39)
The quadratic divergence contributes a term $V_{\text{quad}}$ (in the $\xi \gg 1$ limit), which can be read off from the effective action Eq. (21). The $V$ term is absorbed by a renormalisation of the quartic coupling, leaving the term

$$V_{\text{quad}} = \frac{1}{32\pi^2} \Lambda^2 V_{ii}^i \sim -\frac{3}{16\pi^2} \frac{\lambda A^2 M^4}{\xi^3 \varphi^2} + \ldots$$  (40)

The leading term in the potential is therefore unaffected by these particular quantum corrections. However, the slope of the potential is important for the observational predictions of inflation, and keeping the contribution from the quadratic divergences small requires a cut-off scale $\Lambda < M_p$. The coefficient of $\varphi^{-2}$ has to be regarded as a new parameter in the effective theory with an undetermined value, independent of the quartic Higgs coupling. The slope of the potential is therefore decoupled from the low energy physics of the Higgs boson. This agrees with the conclusion reached in [27], which was based on examination of the $\chi^6$ vertex. We can be confident that the new one-loop result is frame independent and fully gauge invariant because the calculation has been done using the formalism which is independent of the choice of field variables.

The logarithmic divergences in Eq. (23) do not change the conclusions of the one-loop calculations in any substantial way. For example, consider the term

$$\frac{1}{32\pi^2} V_{ij} V_{\text{ij}} \ln \Lambda \sim \frac{288 \lambda^2 M^8}{\pi^2 \xi^6 \varphi^4} \ln \Lambda$$  (41)

This is smaller than the original term in the potential by a factor of order $\xi^{-2}$.

### IV. CONCLUSIONS

The one-loop effective action for an effective theory of gravity coupled to scalar field has been analysed using an approach which is covariant under field redefinitions and independent of gauge-fixing terms, leading to new results on the running of the coupling constants in the effective theory. When applied to Higgs inflation, the consistency bounds on the cut-off scale are broadly in line with recent results obtained using power counting in Feynman diagrams.

The new results open up the possibility of analysing the full Higgs potential with one-loop quantum gravity contributions. There are limitations set by the necessity for introducing undetermined new terms in the Higgs potential and the undetermined cut-off scale. However, (Wilsonian-style) renormalisation group flows can give further information on the existence of possible fixed point theories [42, 43], which may be important if no new physics intervenes at very large energies. These renormalisation group should be addressed using a covariant formalism, and this can be done using heat kernel coefficients similar to the ones used here.

The new results should also be taken into account when analysing the effects of renormalisation group flows on inflation. The scale dependence of physical parameters can affect the cosmological predictions. For example, the effective Planck mass for large scale cosmology can be different from the effective Planck mass for quantum fluctuations. Furthermore, fixed-point theories are interesting candidates for cosmological models [44–46].

Finally, the one-loop results obtained here can be improved in a number of ways. The logarithmic terms can be extended to analyse the $(\partial \varphi)^4$ and $R^2(\partial \varphi)^2$ terms, although the calculation would be very demanding. It should also be possible to obtain results for the full covariant one-loop effective action, and not just the divergent parts, on space-time backgrounds of interest to early-universe cosmology.

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### Appendix A: Quadratic actions

This appendix contains the quadratic actions obtained after the expansions described in the main body of the paper. Most of these results are standard, however most accounts omit the Vilkovisky-DeWitt corrections. The various contributions to the quadratic action are gravity $\mathcal{L}_g$, scalar $\mathcal{L}_s$, gauge-fixing $\mathcal{L}_\chi$, Vilkovisky-DeWitt corrections $\mathcal{L}_v$. 
and ghosts $\mathcal{L}_c$ given by

$$
\mathcal{L}_{2g} |g|^{-1/2} = -\frac{1}{2} g^{\mu
u}(\rho\sigma) \gamma_{\mu\nu;\alpha} \gamma_{\rho\sigma;\alpha} + \frac{1}{2} P^{\alpha\beta}(\mu\nu)(\rho\sigma) \gamma_{\mu
u;\alpha} \gamma_{\rho\sigma;\beta} + \frac{1}{2} \left( 2 R_{\mu\nu\rho\sigma} - g^{\mu\nu} R^{\rho\sigma} - g^{\rho\sigma} R_{\mu\nu} + 2 g^{\mu\rho} R^{\nu\sigma} - g^{(\mu\nu)}(\rho\sigma) \right) \gamma_{\mu
u;\rho\sigma},
$$

(A1)

$$
\mathcal{L}_{2g} |g|^{-1/2} = -\frac{1}{2} D_\mu \eta^i D^\mu \eta_i + \frac{1}{2} \partial_\mu \varphi^i \partial^\mu \varphi_i R_{i,klj} \eta^k \eta^l - \frac{1}{2} V_{ij} \eta^i \eta^j
$$

$$
- \kappa^2 g^{(\mu\nu)}(\rho\sigma) g_{\alpha\beta} \gamma_{\mu\nu;\gamma} \left( T^{\alpha\beta} - \partial^\alpha \varphi^i \partial^\beta \varphi_i - \kappa g^{(\mu\nu)} \gamma_{\mu\nu;\gamma} V_i \eta^j \right)
$$

$$
+ \kappa g^{(\mu\nu)}(\rho\sigma) \left( \eta_{\rho\sigma;\nu} \partial_\mu \varphi^i \partial^\mu \varphi^j \eta_i + \kappa g^{(\mu\nu)} \gamma_{\rho\sigma;\nu} \eta_i \right),
$$

(A2)

$$
\mathcal{L}_\chi |g|^{-1/2} = -\frac{1}{2} \alpha^{-1} P^{\alpha\beta}(\mu\nu)(\rho\sigma) \gamma_{\mu\nu;\alpha} \gamma_{\rho\sigma;\beta} - \alpha^{-1} \kappa^2 \partial_\mu \varphi^i \partial^\mu \varphi^j \eta_i \eta_j
$$

$$
- \alpha^{-1} \kappa g^{(\mu\nu)}(\rho\sigma) \left( \eta_{\rho\sigma;\nu} \partial_\mu \varphi^i \partial^\mu \varphi^j \eta_i + \kappa g^{(\mu\nu)} \gamma_{\rho\sigma;\nu} \eta_i \right),
$$

(A3)

$$
\mathcal{L}_v |g|^{-1/2} = -\kappa^2 g^{(\mu\nu)}(\rho\sigma) g_{\alpha\beta} \gamma_{\mu\nu;\alpha} \gamma_{\rho\sigma;\beta} \left( C^{\alpha\beta} - \kappa^2 T^{\alpha\beta} \right) + \frac{1}{2} (m-2)^{-1} g^{(\mu\nu)}(\rho\sigma) \gamma_{\mu
u;\rho\sigma} (G - \kappa^2 T)
$$

$$
- \frac{1}{2} \kappa g^{\mu\nu} \gamma_{\mu\nu} \left( \partial^\mu \varphi^\nu - V^{\mu\nu} \right) \eta_i + \frac{1}{2} (m-2)^{-1} (G - \kappa^2 T) \eta_i \eta_i,
$$

(A4)

$$
\mathcal{L}_c |g|^{-1/2} = -\frac{1}{2} \gamma_{\mu\nu;\alpha} \epsilon_{\mu\nu;\alpha} + \frac{1}{2} R^{\mu\nu} c_{\mu} c_{\nu} + 2 \kappa^2 \partial^\mu \varphi^i \partial^\nu \varphi_i c_{\mu} c_{\nu}.
$$

(A5)

where the tensor $P$ is

$$
P^{\alpha\beta}(\mu\nu)(\rho\sigma) = 2 g_{\gamma\delta} g^{(\alpha\gamma)}(\rho\sigma) g^{(\beta\delta)}(\rho\sigma).
$$

(A6)

The gravity and gauge-fixing contributions involving $P$ combine with coefficient

$$
\zeta = \alpha^{-1} - 1
$$

(A7)

This has lead to most work on quantum gravity making use of the simple case $\alpha = 1$, sometimes called Feynman or Feynman-DeWitt gauge. However, the effective action generally depends on the gauge parameter $\alpha$, unless the covariant approach is adopted in which case the correct result is equivalent to the Landau gauge $\alpha \to 0$ limit.

Some terms in the scalar part of the Lagrangian density have been simplified by introducing the scalar stress-energy tensor,

$$
T_{\mu\nu} = \partial_\mu \varphi^i \partial_\nu \varphi_i - g_{\mu\nu} \left( \frac{1}{2} \partial_\alpha \varphi^i \partial^\alpha \varphi_i + V \right)
$$

(A8)

This highlights a cancellation between the scalar part and the Vilkovisky-DeWitt corrections, which actually goes further and includes the cancellation of some Einstein tensor terms.

**Appendix B: The $E_1$ heat kernel coefficient**

The field operator for the Einstein-scalar system is

$$
\Delta_f = -\mathcal{D}_\alpha \mathcal{D}^\alpha - \zeta P^{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta - A_\alpha \mathcal{D}^\alpha - D^\alpha A_\alpha + \mathcal{M}^2,
$$

(B1)

The non-minimal part of the operator is governed by the tensor $P^{\alpha\beta}$ in (A6). This tensor has two useful properties, firstly the symbol $\sigma(P) = P^{\alpha\beta} k_\alpha k_\beta / k^2$ satisfies the identity $\sigma(P)^2 = \sigma(P)$ and secondly $\mathcal{D}_\alpha P^{\beta\gamma} = 0$. Note that combining the gauge potential with the covariant derivative $\mathcal{D}_\mu$ would violate the latter property because $(\mathcal{D}_\alpha + A_\alpha) P^{\beta\gamma} \neq 0$.

Heat kernel coefficients for non-minimal operators of this kind have been investigated recently [30], although the results have to be extended slightly to include the explicit gauge potential terms. In cases where $\mathcal{P} \mathcal{A} = 0$, the traced heat kernel $Tr E_1$ is a sum of invariants,

$$
Tr E_1 (\Delta_f) = a_1 R Tr(I) + a_2 Tr M^2 + a_3 Tr P M^2 + a_4 R Tr P + a_5 R Tr P^2
$$

$$
+ a_6 t_{\mu\nu\rho\sigma} Tr \mathcal{P}^{\mu\nu} \mathcal{P}^{\rho\sigma} F_{\alpha\beta} + a_7 Tr A_\mu A^\mu + a_8 t_{\mu\nu\rho\sigma} Tr \mathcal{P}^{\mu\nu} A^\rho A^\sigma,
$$

(B2)

where $\mathcal{P} = g_{\mu\nu} \mathcal{P}^{\mu\nu}$, $t_{\mu\nu\rho\sigma} = 3 g_{(\mu\nu} g_{\rho\sigma)}/m (m+2)$ and the coefficients are given in table [I].
TABLE I. The coefficients of the invariants in the traced heat kernel coefficient \( \text{Tr} \, E_1(\Delta) \) for operators with non-minimal term \(-\zeta \mathcal{P}^\mu{}\nu \mathcal{D}_\mu \mathcal{D}_\nu \) in spacetime dimension \( m \), where \( u = (1 + \zeta)^{-m/2} \).

| Term | Expression |
|------|------------|
| \( a_1 \) | \( \frac{1}{6} \) |
| \( a_2 \) | \(-1\) |
| \( a_3 \) | \( -\frac{u - 1}{m} \) |
| \( a_4 \) | \( \frac{(m + 2)(m \zeta - 2m + 4\zeta + 10) (u - 1)}{12(m - 2)m(m - 1)} + \frac{\zeta \left(-9m + m^2 + 2\right)}{6(m - 2)m(m - 1)} \) |
| \( a_5 \) | \( \frac{(8 + m^2\zeta + 4\zeta)(u - 1)}{4(m - 2)(m - 1)m^2} + \frac{\zeta}{(m - 2)m(m - 1)} \) |
| \( a_6 \) | \( \frac{2(4 + 2\zeta + m\zeta)(u - 1)}{m - 2} + \frac{4m\zeta}{m - 2} \) |
| \( a_7 \) | \(-1\) |
| \( a_8 \) | \( \frac{4(u - 1)}{\zeta} + 2m \) |

In the case of interest, the curvature \( F_{\alpha\beta} \) is the curvature of the spin 2 tetrad connection, which is used in the derivatives of the metric fluctuations. Since the only terms which mix the spin 2 and scalar sectors are the terms involving \( A_\mu \), a simplification can be made by splitting the result into a part depending on a spin 2 operator \( \Delta^{(2)} \), a part depending on a scalar operator \( \Delta^{(0)} \), and the cross terms, rewriting \( \text{Tr} \, E_1 \) as

\[
\text{Tr} \, E_1(\Delta f) = \text{Tr} \, E_1(\Delta^{(2)}) + \text{Tr} \, E_1(\Delta^{(0)}) + a_7 \text{Tr} \, A_\mu A_\mu + a_8 t_{\mu\nu,\rho} \text{Tr} \, \mathcal{P}^{\mu\nu} A^\rho A^\sigma.
\]  

(B3)

The mass terms in the spin 2 and scalar sectors can be read off from the Lagrangian densities \( (A1-A4) \),

\[
\mathcal{M}^2_{\mu\nu,\rho\sigma} = g_{(\mu\nu)\alpha\beta} \left\{ 2\kappa^2 g^{\alpha\xi} \partial^\beta \phi^i \partial^\sigma \phi^i - 2R^{\alpha\rho\beta\sigma} \\
- \frac{1}{2} g^{\alpha\beta} \left( \partial^{\rho} \phi^i \partial^\sigma \phi^i - R^{\rho\sigma} \right) - \frac{1}{2} g^{\mu\sigma} \left( \partial^{\alpha} \phi^i \partial^\beta \phi^i - R^{\alpha\beta} \right) \right\} \\
- \frac{1}{m - 2} \delta^{(\rho \mu \nu \sigma)} (G - \kappa^2 T),
\]

\[
\mathcal{M}^2_{i,j} = 2(1 + \zeta) \kappa^2 \partial_{\mu} \phi^i \partial^{\mu} \phi^j - \mathcal{R}_{ik}{}^j \partial^{\mu} \phi^k \partial_{\mu} \phi^j \\
+ V_{ij} - (m - 2)^{-1} (G - \kappa^2 T) \delta^j_i,
\]  

(B4)

(B5)

Some of the Vilkovisky-DeWitt corrections are evident in the \( G - \kappa^2 T \) terms, but some have cancelled with other terms. The mass terms can be inserted into the results for \( \text{Tr} \, E_1 \) with non-minimal spin 2 operators found in \( [39] \). Totalling all this together in \( m = 4 \) dimensions and taking the Landau gauge \( \zeta \to \infty \) limit gives

\[
\lim_{\zeta \to \infty} \text{Tr} \, E_1(\Delta f) = -\frac{N + 12}{3} R + 2(N + 6)\kappa^2 V \\
+ \frac{N - 4}{2} \kappa^2 \partial_{\mu} \phi^i \partial^{\mu} \phi^i + \mathcal{R}_{ij} \partial_{\mu} \phi^i \partial^{\mu} \phi^j - V_{i,j}.
\]  

(B6)

For comparison, the Feynman gauge result retaining the Vilkovisky-DeWitt corrections is

\[
\text{Tr} \, E_1(\Delta f)_{\zeta=0} = -\frac{N + 13}{3} R + 2(N + 10)\kappa^2 V
\]
The effect of changing the gauge fixing is quite small, but a large change comes about if we take the non-covariant Feynman gauge result for the operator \( \Delta^{\text{nc}} \), defined by leaving out Vilkovisky-DeWitt corrections,

\[
\text{Tr} E_1(\Delta^{\text{nc}})|_{\zeta=0} = \frac{N - 26}{6} R + 20\kappa^2 V - 2\kappa^2 \partial_\mu \varphi^i \partial^\mu \varphi^i - V_{i}^{i}.
\]

The difference between \( \text{B7} \) and \( \text{B8} \) is proportional to the field equation \( G - \kappa^2 T \) as expected.

The covariant result \( \text{B6} \) is combined with the contribution from the ghost operator \( \Delta^3 \), then the \( E_2 \) coefficient can be expressed in terms of the results for the spin 2 and scalar parts,

\[
\text{Tr} E_2(\Delta_f) = \text{Tr} E_2(\Delta^{(2)}) + \text{Tr} E_2(\Delta^{(0)}) + c_1 \text{Tr} M^2_{A} + c_2 \text{Tr} P M^4_{A},
\]

where the coefficients are given in table II. The mass terms in the spin 2 and scalar sectors are given in \( \text{B31} \) and \( \text{B35} \), and the cross terms which define \( M^2_{A} \) when \( \partial_\mu \phi^i = 0 \) are

\[
M^2_{\mu \nu} = -\frac{1}{2} \kappa g_{\mu \nu} V^{ij}, \quad M^2_{i \rho \sigma} = \frac{1}{2} \kappa g_{\rho \sigma} V_{i}^{i}.
\]

The tensor \( P^{\alpha \beta} \) is given in \( \text{A0} \). The cross terms \( \text{C2} \) and \( \text{C3} \) can be regarded in Feynman diagram language as scalar-scalar-graviton interaction vertices with one scalar from the background. The Vilkovisky-DeWitt corrections have flipped the signs of these terms, but this does not have any affect on the \( E_2 \) heat kernel coefficient.

### TABLE II. The coefficients of the invariants in the traced heat kernel coefficient \( \text{Tr} E_2(\Delta) \) for operators in flat spacetime with non-minimal term \( -\zeta P^{\alpha \nu} D_{\mu} D_\nu \) in spacetime dimension \( m \), where \( u = (1 + \zeta)^{-m/2} \)

| Term Expression | \( \zeta = 0 \) | \( \zeta \to \infty \) |
|------------------|-----------------|-----------------|
| \( c_1 \)       | \( \frac{1}{2} \) | \( \frac{1}{2} \) |
| \( c_2 \)       | \( -\frac{2(1 + \zeta)(u - 1)}{\zeta m(m - 2)} \) | \( \frac{1}{m - 2} \) |

The \( E_2 \) coefficients for spin 2 were evaluated in \( \text{B39} \). Taking the space-time dimensions \( m = 4 \) and the Landau-gauge limit \( \zeta \to \infty \) gives

\[
\lim_{\zeta \to \infty} \text{Tr} E_2(\Delta_f) = \frac{1}{2} V_{ij} V^{ij} - \frac{3}{2} \kappa^2 V_{i}^{i} V^{i} - 2\kappa^2 V V_{i}^{i} + (2N + 12) \kappa^4 V^2
\]

\[
- \frac{2N + 24}{3} \kappa^2 RV + \frac{1}{3} RV_{i}^{i} + \frac{190 + N}{180} R_{\mu \nu \rho \sigma}^2 + \frac{50 - N}{180} R_{\mu \nu}^4 + \frac{41 + 2N}{36} R^2.
\]

\( \text{C4} \)
For comparison, the non-covariant result obtained using Feynman gauge $\zeta = 0$ would be
\[
\text{Tr} E_2(\Delta_f)|_{\zeta=0} = \frac{1}{2} \sum_{i,j} V_{ij} V^{ij} - \kappa^2 V_3 V^3 - 2\kappa^2 V_\perp V^\perp + (2N + 20)\kappa^4 V^2
- \frac{2N + 26}{3} \kappa^2 R V + \frac{1}{3} R V^3 + \frac{190 + N}{180} R_{\mu\nu\rho\sigma}^2 - \frac{190 + N}{180} R_{\mu\nu}^2 + \frac{41 + 2N}{36} R^2.
\]
(C5)

The Feynman-gauge curvature terms are consistent with previous work, e.g. Ref. [3]. The ghost contribution is
\[
\text{Tr} E_2(\Delta_g) = -\frac{11}{180} R_{\mu\nu\rho\sigma}^2 + \frac{43}{90} R_{\mu\nu}^2 + \frac{2}{9} R^2.
\]
(C6)

These combine to give the logarithmic divergences.
[45] M. Hindmarsh, D. Litim, and C. Rahmede, JCAP 1107, 019 (2011), arXiv:1101.5401 [gr-qc].
[46] Z.-Z. Xianyu and H.-J. He, (2014), arXiv:1407.6993 [astro-ph.CO].