A NOTE ON THE SPACES OF VARIABLE INTEGRABILITY 
AND SUMMABILITY OF ALMEIDA AND HÄSTÖ

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Abstract. We address an open problem posed recently by Almeida and Hästö. 
They defined the spaces \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) of variable integrability and summability 
and showed that \( \| \cdot \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) is a norm if \( q \geq 1 \) is constant 
almost everywhere or if \( 1/p(x) + 1/q(x) \leq 1 \) for almost every \( x \in \mathbb{R}^n \). Nevertheless, the 
natural conjecture (expressed also by Almeida and Hästö) is that the expression 
is a norm if \( p(x), q(x) \geq 1 \) almost everywhere. We show that \( \| \cdot \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) is 
an or m i f \( 1 \leq q(x) \leq p(x) \) for almost every \( x \in \mathbb{R}^n \). Furthermore, we construct 
an example of \( p(x) \) and \( q(x) \) with \( \min(p(x), q(x)) \geq 1 \) for every \( x \in \mathbb{R}^n \) such 
that the triangle inequality does not hold for \( \| \cdot \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \).

1. Introduction

For the definition of the spaces \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) we closely follow [1]. Spaces of variable 
integrability \( L_{p(\cdot)}(\cdot) \) and variable sequence spaces \( \ell_{q(\cdot)}(\cdot) \) were first considered in 1931 
by Orlicz [5], but the modern development started with the paper [4]. We refer to 
[3] for an excellent overview of the vastly growing literature on the subject.

First of all we recall the definition of the variable Lebesgue spaces \( L_{p(\cdot)}(\Omega) \), where 
\( \Omega \) is a measurable subset of \( \mathbb{R}^n \). A measurable function \( p : \Omega \to (0, \infty] \) is called a 
variable exponent function if it is bounded away from zero. For a set \( A \subset \Omega \) we denote 
\( p^+_A = \operatorname{ess-sup}_{x \in A} p(x) \) and \( p^-_A = \operatorname{ess-inf}_{x \in A} p(x) \); we use the abbreviations 
\( p^+ = p^+_\Omega \) and \( p^- = p^-_\Omega \). The variable exponent Lebesgue space \( L_{p(\cdot)}(\Omega) \) consists of 
all measurable functions \( f \) such that there exists an \( \lambda > 0 \) such that the modular
\[
\varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda) = \int_{\Omega} \varphi_{p(x)} \left( \frac{|f(x)|}{\lambda} \right) \, dx
\]
is finite, where
\[
\varphi_p(t) = \begin{cases} 
  t^p & \text{if } p \in (0, \infty), \\
  0 & \text{if } p = \infty \text{ and } t \leq 1, \\
  \infty & \text{if } p = \infty \text{ and } t > 1.
\end{cases}
\]
This definition is nowadays standard and was also used in \cite{1} Section 2.2 and \cite{3} Definition 3.2.1.

If we define \( \Omega_\infty = \{ x \in \Omega : p(x) = \infty \} \) and \( \Omega_0 = \Omega \setminus \Omega_\infty \), then the Luxemburg norm of a function \( f \in L_{p(\cdot)}(\Omega) \) is given by

\[
\| f \|_{L_{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda) \leq 1 \}
\]

where on

\[
\| f \|_{L_{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega_0} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right. \ \text{and} \ |f(x)| \leq \lambda \ \text{for a.e.} \ x \in \Omega_\infty \right\}.
\]

If \( p(\cdot) \geq 1 \), then it is a norm, but it is always a quasi-norm if at least \( p^- > 0 \); see \cite{4} for details. We denote the class of all measurable functions \( p : \mathbb{R}^n \to (0, \infty] \) such that \( p^- > 0 \) by \( \mathcal{P}(\mathbb{R}^n) \) and the corresponding modular is denoted by \( \varrho_{p(\cdot)} \) instead of \( \varrho_{L_{p(\cdot)}(\mathbb{R}^n)} \).

To define the mixed spaces \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) we have to define another modular. For \( p, q \in \mathcal{P}(\mathbb{R}^n) \) and a sequence \( (f_\nu)_{\nu \in \mathbb{N}_0} \) of \( L_{p(\cdot)}(\mathbb{R}^n) \) functions we define

\[
\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu = 0}^{\infty} \inf \left\{ \lambda_\nu > 0 : \varrho_{p(\cdot)} \left( \frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}} \right) \leq 1 \right\},
\]

where we put \( \lambda^{1/\infty} = 1 \). The (quasi-) norm in the \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) spaces is defined as usual by

\[
\| f_\nu \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} = \inf \{ \mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \leq 1 \}.
\]

This (quasi-)norm was used in \cite{1} to define the spaces of Besov type with variable integrability and summability. Spaces of Triebel-Lizorkin type with variable indices have recently been considered in \cite{2}. The appropriate \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) space is a normed space whenever \( \text{ess-inf}_{x \in \mathbb{R}^n} \min(p(x), q(x)) \geq 1 \). This was the expected result and coincides with the case of constant exponents.

As pointed out in the remark after Theorem 3.8 in \cite{1}, the same question is still open for the \( \ell_{q(\cdot)}(L_{p(\cdot)}) \) spaces.

2. When does \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) define a norm?

In Theorem 3.6 of \cite{1} the authors proved that if the condition \( \frac{1}{p(x)} + \frac{1}{q(x)} \leq 1 \) holds for almost every \( x \in \mathbb{R}^n \), then \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) defines a norm. They also proved in Theorem 3.8 that \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) is a quasi-norm for all \( p, q \in \mathcal{P}(\mathbb{R}^n) \).

Furthermore, the authors of \cite{1} posed a question if the (rather natural) condition \( p(x), q(x) \geq 1 \) for almost every \( x \in \mathbb{R}^n \) ensures that \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) is a norm.

We give (in Theorem 1) a positive answer if \( 1 \leq q(x) \leq p(x) \leq \infty \) almost everywhere on \( \mathbb{R}^n \). Furthermore, in Theorem 2 we construct two functions \( p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) such that \( \text{inf}_{x \in \mathbb{R}^n} \min(p(x), q(x)) \geq 1 \), but the triangle inequality does not hold for \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \).

2.1. Positive results. We summarize in the following theorem all the cases when the expression \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) is known to be a norm. We include the proof of the case discussed already in \cite{1} for the sake of completeness.

**Theorem 1.** Let \( p, q \in \mathcal{P}(\mathbb{R}^n) \) such that either \( p(x) \geq 1 \) and \( q \geq 1 \) is constant almost everywhere or \( 1 \leq q(x) \leq p(x) \leq \infty \) for almost every \( x \in \mathbb{R}^n \), or \( 1/p(x) + 1/q(x) \leq 1 \) for almost every \( x \in \mathbb{R}^n \). Then \( \| : \|_{\ell_{q(\cdot)}(L_{p(\cdot)})} \) defines a norm.
Proof. If \( p(x) \geq 1 \) and \( q \geq 1 \) is constant almost everywhere, then the proof is trivial.

In the remaining cases, we want to show that

\[
\| f_\nu + g_\nu \|_{q(\cdot)}(L_{p(\cdot)}) \leq \| f_\nu \|_{q(\cdot)}(L_{p(\cdot)}) + \| g_\nu \|_{q(\cdot)}(L_{p(\cdot)})
\]

for all sequences of measurable functions \( \{f_\nu\}_{\nu \in \mathbb{N}_0} \) and \( \{g_\nu\}_{\nu \in \mathbb{N}_0} \). Let \( \mu_1 > 0 \) and \( \mu_2 > 0 \) be given with

\[
\varrho_{q(\cdot)}(L_{p(\cdot)}) \left( \frac{f_\nu}{\mu_1} \right) \leq 1 \quad \text{and} \quad \varrho_{q(\cdot)}(L_{p(\cdot)}) \left( \frac{g_\nu}{\mu_2} \right) \leq 1.
\]

We want to show that

\[
\varrho_{q(\cdot)}(L_{p(\cdot)}) \left( \frac{f_\nu + g_\nu}{\mu_1 + \mu_2} \right) \leq 1.
\]

For every \( \varepsilon > 0 \), there exist sequences of positive numbers \( \{\lambda_\nu\}_{\nu \in \mathbb{N}_0} \) and \( \{\Lambda_\nu\}_{\nu \in \mathbb{N}_0} \) such that

\[
\varrho_{p(\cdot)} \left( \frac{f_\nu(x)}{\mu_1 \lambda_\nu^{1/q(x)}} \right) \leq 1 \quad \text{and} \quad \varrho_{p(\cdot)} \left( \frac{g_\nu(x)}{\mu_2 \Lambda_\nu^{1/q(x)}} \right) \leq 1,
\]

together with

\[
\sum_{\nu=0}^{\infty} \lambda_\nu \leq 1 + \varepsilon \quad \text{and} \quad \sum_{\nu=0}^{\infty} \Lambda_\nu \leq 1 + \varepsilon.
\]

We set

\[
A_\nu := \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2}, \quad \text{i.e.} \quad \sum_{\nu=0}^{\infty} A_\nu \leq 1 + \varepsilon.
\]

We shall prove that

\[
\varrho_{p(\cdot)} \left( \frac{f_\nu(x) + g_\nu(x)}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right) \leq 1 \quad \text{for all} \quad \nu \in \mathbb{N}_0.
\]

Let \( \Omega_0 := \{x \in \mathbb{R}^n : p(x) < \infty\} \) and \( \Omega_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\} \). We put for every \( x \in \Omega_0 \)

\[
F_\nu(x) := \left( \frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \right)^{p(x)} \quad \text{and} \quad G_\nu(x) := \left( \frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \right)^{p(x)}.
\]

Then (1) may be reformulated as

\[
\int_{\Omega_0} F_\nu(x) \, dx \leq 1 \quad \text{and} \quad \text{ess-sup}_{x \in \Omega_\infty} \frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \leq 1
\]

and

\[
\int_{\Omega_0} G_\nu(x) \, dx \leq 1 \quad \text{and} \quad \text{ess-sup}_{x \in \Omega_\infty} \frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \leq 1.
\]

Our aim is to prove (2), which reads

\[
\int_{\Omega_0} \left( \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} \, dx \leq 1 \quad \text{and} \quad \text{ess-sup}_{x \in \Omega_\infty} \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \leq 1.
\]
We first prove the second part of (5). First we observe that (3) and (4) imply that
\[ |f_\nu(x)| \leq \mu_1 \lambda_\nu^{1/q(x)} \quad \text{and} \quad |g_\nu(x)| \leq \mu_2 \Lambda_\nu^{1/q(x)} \]
hold for almost every \( x \in \Omega_\infty \). Using \( q(x) \geq 1 \) and Hölder’s inequality in the form
\[ \frac{\mu_1 \lambda_\nu^{1/q(x)} + \mu_2 \Lambda_\nu^{1/q(x)}}{\mu_1 + \mu_2} \leq \left( \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2} \right)^{1/q(x)}, \]
we get
\[ \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \leq 1. \]

If \( q(x) = \infty \), only notational changes are necessary.

Next we prove the first part of (5). Let \( 1 \leq q(x) \leq p(x) < \infty \) for almost all \( x \in \Omega_0 \). Then we use Hölder’s inequality in the form
\[ F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2 \]
\[ \leq (\mu_1 + \mu_2)^{1-1/q(x)} (\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu)^{1/q(x)-1/p(x)} (F_\nu(x) \mu_1 + G_\nu(x) \mu_2)^{1/p(x)}. \]
If \( 1/p(x) + 1/q(x) \leq 1 \) for almost every \( x \in \Omega_0 \), then we replace (6) by
\[ F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2 \]
\[ \leq (\mu_1 + \mu_2)^{1-1/p(x)-1/q(x)} (\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu)^{1/q(x)} (F_\nu(x) \mu_1 + G_\nu(x) \mu_2)^{1/p(x)}. \]
Using (6), we may continue:
\[ \int_{\Omega_0} \left( \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} \, dx \]
\[ = \int_{\Omega_0} \left( \frac{F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2}{\mu_1 + \mu_2} \right)^{p(x)} \left( \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2} \right)^{-\frac{p(x)}{q(x)}} \, dx \]
\[ \leq \int_{\Omega_0} \frac{\mu_1 \lambda_\nu}{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu} \int_{\Omega_0} F_\nu(x) \, dx + \frac{\mu_2 \Lambda_\nu}{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu} \int_{\Omega_0} G_\nu(x) \, dx \leq 1, \]
where we also used (3) and (4). If we start with (7) instead, we proceed in the following way:
\[ \int_{\Omega_0} \left( \frac{|f_\nu(x) + g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)} \right)^{p(x)} \, dx \]
\[ = \int_{\Omega_0} \left( \frac{F_\nu(x)^{1/p(x)} \lambda_\nu^{1/q(x)} \mu_1 + G_\nu(x)^{1/p(x)} \Lambda_\nu^{1/q(x)} \mu_2}{\mu_1 + \mu_2} \right)^{p(x)} \left( \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2} \right)^{-\frac{p(x)}{q(x)}} \, dx \]
\[ \leq \int_{\Omega_0} \frac{F_\nu(x) \mu_1 + G_\nu(x) \mu_2}{\mu_1 + \mu_2} \int_{\Omega_0} F_\nu(x) \, dx + \frac{\mu_2}{\mu_1 + \mu_2} \int_{\Omega_0} G_\nu(x) \, dx \leq 1. \]
In both cases, this finishes the proof of (5). \( \square \)
Remark 1. (i) A simpler proof of Theorem 1 is possible (and was proposed to us by the referee) if $1 \leq q(x) \leq p(x) \leq \infty$. Namely, if $1 \leq q \leq p \leq \infty$, $\lambda > 0$ and $t \geq 0$, then

$$\varphi_p \left( \frac{t}{\lambda^{1/q}} \right) = \varphi_{\frac{p}{q}} \left( \frac{\varphi_q(t)}{\lambda} \right),$$

where we use the convention that $\frac{p}{q} = 1$ if $p = q = \infty$. This allows us to simplify the modular $\varrho_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})}$ to

$$\varrho_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})}(f) = \sum_{\nu=0}^{\infty} \left\| \varphi_{q^{(\cdot)}}(|f|) \right\|_{\frac{p^{(\cdot)}}{q^{(\cdot)}}}.$$

This shows that $\varrho_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})}(f)$ is a composition of only convex functions. Hence, it is a convex modular, and therefore it induces a norm. Unfortunately, we were not able to find such a simplification for the case $1/p(x) + 1/q(x) \leq 1$. The advantage of our proof of Theorem 1 is that it proves both cases in a unified way.

(ii) Let us observe that (5) loses its sense if $p < q = \infty$. This shows why (4) (which was already used in [1] for $q^+ < \infty$) has to be applied with certain care.

(iii) The method of the proof of Theorem 1 can actually be used to show that under the conditions posed on $p^{(\cdot)}$ and $q^{(\cdot)}$ in Theorem 1 $\varrho_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})}$ is a convex modular, which is a stronger result than the norm property.

2.2. Counterexample.

Theorem 2. There exist functions $p, q \in P(\mathbb{R}^n)$ with $\inf_{x \in \mathbb{R}^n} p(x) \geq 1$ and $\inf_{x \in \mathbb{R}^n} q(x) \geq 1$ such that $\left\| \cdot, \ell_{q^{(\cdot)}}(L_{p^{(\cdot)}}) \right\|$ does not satisfy the triangle inequality.

Proof. Let $Q_0, Q_1 \subset \mathbb{R}^n$ be two disjoint unit cubes, let $p(x) := 1$ everywhere on $\mathbb{R}^n$, and put $q(x) := \infty$ for $x \in Q_1$ and $q(x) := 1$ for $x \notin Q_1$. Let $f_1 = \chi_{Q_0}$ and $f_2 = \chi_{Q_1}$. Finally, we put $f = (f_1, f_2, 0, \ldots)$ and $g = (f_2, f_1, 0, \ldots)$.

We calculate for every $L > 0$ fixed

$$\inf \left\{ \lambda_1 > 0 : \varrho_{p^{(\cdot)}} \left( \frac{f_1(x)}{\lambda_1^{1/q(x)} L} \right) \leq 1 \right\} = \inf \left\{ \lambda_1 > 0 : \frac{1}{\lambda_1 L} \leq 1 \right\} = 1/L$$

and

$$\inf \left\{ \lambda_2 > 0 : \varrho_{p^{(\cdot)}} \left( \frac{f_2(x)}{\lambda_2^{1/q(x)} L} \right) \leq 1 \right\} = \inf \left\{ \lambda_2 > 0 : \frac{1}{L} \leq 1 \right\}.$$ 

If $L \geq 1$, then the last expression is equal to zero; otherwise it is equal to $\infty$.

We obtain

$$\left\| f \right\|_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})} = \inf \{ L > 0 : \varrho_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})}(f/L) \leq 1 \} = \inf \{ L > 0 : 1/L + 0 \leq 1 \} = 1,$$

and the same is also true for $\left\| g \right\|_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})}$. It is therefore enough to show that $\left\| f + g \right\|_{\ell_{q^{(\cdot)}}(L_{p^{(\cdot)}})} > 2$. 
Using the calculation

\[
\inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f_1(x) + f_2(x)}{L \cdot \lambda^{1/q(x)}} \right) \leq 1 \right\} = \inf \left\{ \lambda > 0 : \int_{Q_0} \frac{1}{L \cdot \lambda} + \int_{Q_1} \frac{1}{L} \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \frac{1}{L \cdot \lambda} + \frac{1}{L} \leq 1 \right\} = \frac{1}{L - 1},
\]

which holds for every \( L > 1 \) fixed, we get

\[
\|f + g\|_{\varrho_{q(\cdot)}(L_{p(\cdot)})} = \inf \left\{ L > 0 : \varrho_{l_{q(\cdot)}(L_{p(\cdot)})} \left( \frac{f + g}{L} \right) \leq 1 \right\}
\]

\[
= \inf \left\{ L > 0 : 2 \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{f_1(x) + f_2(x)}{L \cdot \lambda^{1/q(x)}} \right) \leq 1 \right\} \leq 1 \right\}
\]

\[
= \inf \left\{ L > 1 : 2 \cdot \frac{1}{L - 1} \leq 1 \right\} = 3.
\]

\( \blacksquare \)

**Remark 2.** Let us observe that \( 1 \leq q(x) \leq p(x) \leq \infty \) holds for \( x \in Q_0 \) and that \( 1/p(x) + 1/q(x) \leq 1 \) is true for \( x \in Q_1 \). It is therefore necessary to interpret the assumptions of Theorem 1 in a correct way, namely, that one of the conditions of Theorem 2 holds for (almost) all \( x \in \mathbb{R}^n \). This is not to be confused with the statement that for (almost) every \( x \in \mathbb{R}^n \) at least one of the conditions is satisfied, which is not sufficient.

**Remark 3.** A similar calculation (which we shall not repeat in detail) shows that one may also put \( q(x) := q_0 \) large enough for \( x \in Q_1 \) to obtain a counterexample. Hence there is nothing special about the infinite value of \( q \), and the same counterexample may be reproduced with uniformly bounded exponents \( p, q \in \mathcal{P}(\mathbb{R}^n) \).

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