EXACT STRING BACKGROUNDS FROM WZW MODELS
BASED ON NON-SEMISIMPLE GROUPS

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ABSTRACT

We formulate WZW models based on a centrally extended version of the Euclidean group in \(d\)-dimensions. We obtain string backgrounds corresponding to conformal \(\sigma\)-models in \(D = d^2\) space-time dimensions with exact central charge \(c = d^2\) and \(d(d - 1)/2\) null Killing vectors. By identifying the corresponding conformal field theory we show that the one loop results coincide with the exact ones up to a shifting of a parameter.

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1. Introduction

Recently some attention was given to the construction of string backgrounds from WZW models based on non-semi-simple groups. These provide new classes of exactly solvable models and they seem to correspond to plane wave-type with null Killing vectors solutions to string theory [1]. Writing the WZW action corresponding to a non-semi-simple group is not completely straightforward since the quadratic form one usually constructs from the algebra structure constants, namely \( \Omega_{AB} = f_{AC}^D f_{BD}^C \), is not invertible. Nappi and Witten in [2] showed how to circumvent this problem for the particular case of the Euclidean group in two dimensions \( E_2 \) by considering instead \( E_2^c \), i.e. a centrally extended version of it. The corresponding 4-dimensional string background, with exact central charge \( c = 4 \), is of the plane wave-type and is exact to all orders in perturbation theory [2].

The representation theory (and a complete bosonization) for \( E_2^c \) was worked out [3] and string backgrounds in \( D = 3 \) were constructed by gauging various 1-dimensional subgroups of \( E_2^c \) [3] [4]. It has also been shown that the background of [2] corresponds to a larger class of 1-loop solutions to string theory with a null Killing vector which are also exact [5].

In this article we generalize the work of [2] to a larger class of WZW models based on a central extension of the Euclidean group in d-dimensions \( E_d \), which we will accordingly denote by \( E_d^c \). The formulation as WZW models and the underlying current algebra symmetry makes them exactly solvable and one can in principle compute their spectrum using current algebra techniques.

This paper is organized as follows: In Section 2 we work out explicitly the case of the WZW model based on \( E_3^c \). The corresponding \( D = d^2 \)-dimensional conformal \( \sigma \)-model possesses \( d(d - 1)/2 \) null Killing vectors. In Section 3 we identify the corresponding conformal field theory (CFT) (with central charge \( c = d^2 \)) and show that the 1-loop results for the non-linear \( \sigma \)-model corresponding to the WZW action of the previous section are in fact exact to all orders (up to a shifting of a parameter). This fact can be traced back to the existence of \( d(d - 1)/2 \) null Killing vectors as it will be discussed. For simplicity some of the explicit computations of this section are given only for the case of \( E_3^c \) but the final conclusions are true for the general case as well. Section 4 contains discussion and concluding remarks.
2. WZW action based on $E_d^c$

The Euclidean group in $d$-dimensions has $d(d+1)/2$ generators $\{J_{ij}, P_i\}$, $i = 1, 2 \ldots d$. The commutation rules of the corresponding algebra are

$$[J_{ij}, J_{kl}] = i\delta_{[i[k} J_{j]l]} \ , \quad [J_{ij}, P_k] = i\delta_{[i[k} P_{j]} \ , \quad [P_i, P_j] = 0 \ . \quad (2.1)$$

Obviously (2.1) is not a semi-simple algebra because it contains the abelian ideal $\{P_i\}$ and can be thought as the semi-direct sum of the algebra for the $SO(d)$ group and the group of translations in $d$-dimensions, i.e. $so(d) \oplus_s T_d$. If one attempts to write down a WZW action using the quadratic form corresponding to the Killing metric, i.e. $\Omega_{AB} = f_{AC}^D f_{BD}^C$, one discovers that this is reduced to the WZW model for the $d(d-1)/2$-dimensional group $SO(d)$. One can centrally extend (2.1) to $E_d^c$ by adding the set of generators $\{T_{ij}\}$. The additional set of commutation rules which preserve the Jacobi identities are

$$[J_{ij}, T_{kl}] = i\delta_{[i[k} T_{j]l]} \ , \quad [P_i, P_j] = iT_{ij} \quad (2.2)$$

and where the central extension operators $T_{ij}$ commute with everything else. The quadratic form necessary to write down the corresponding WZW action should satisfy the following criteria a) $\Omega_{AB} = \Omega_{BA}$, b) $f_{AB}^D \Omega_{CD} + f_{AC}^D \Omega_{BD} = 0$ and c) it should be non-degenerate, i.e. the inverse matrix $\Omega^{AB}$ obeying $\Omega^{AB} \Omega_{BC} = \delta^A_C$ should exist. The first and the second properties ensure the existence of the quadratic and the Wess-Zumino term in the WZW action and the third one gives a way to properly lower and raise indices. For $E_d^c$ the unique solution that satisfies all of the above criteria is

$$\begin{pmatrix} P_j \\ J_{kl} \\ T_{kl} \end{pmatrix} = \begin{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & \delta_{[i[k} \delta_{j]l]} & \delta_{[i[k} \delta_{j]l]} \\ 0 & \delta_{[i[k} \delta_{j]l]} & 0 \end{pmatrix} \Omega_{AB} ^{-1} \begin{pmatrix} P_i \\ J_{ij} \\ T_{ij} \end{pmatrix} \ . \quad (2.3)$$
The group element $g$ contains $d^2$ parameters and can be parametrized as follows (where summation over repeated indices is implied)

$$g = e^{i\alpha \cdot P} e^{iv \cdot T} h_u , \quad a \cdot P = a_i P_i , \quad v \cdot T = \frac{1}{2} v_{ij} T_{ij} , \quad (2.4)$$

where $v_{ij} = -v_{ji}$ and $h_u$ is an $SO(d)$ group element parametrized in terms of $d(d-1)/2$ parameters $u_{ij}$. Using the formula

$$de^H = \int_0^1 dx \ e^{xH} dHe^{(1-x)H}$$

one can compute the following

$$g^{-1} dg = i(da \cdot P' - \frac{1}{2} da_i a_j T_{ij}') + idv \cdot T' + \frac{1}{2} (dh_u h_u^{-1})_{ij} J_{ij}' , \quad (2.5)$$

where the generators $T'_A = h_u^{-1} T_A h_u$ satisfy the same commutation relations (2.1) as the $T_A$’s. Similarly we compute

$$dgg^{-1} = i(da \cdot P + \frac{1}{2} da_i a_j T_{ij}) + ie^{i\alpha \cdot P} dv \cdot T e^{-i\alpha \cdot P} + e^{i\alpha \cdot P} e^{i\alpha \cdot \nu} dh_u h_u^{-1} e^{-i\alpha \cdot \nu} e^{-i\alpha \cdot P} . \quad (2.6)$$

In section 3 we will need an explicit expression for $dgg^{-1}$. One way to obtain such an expression is to note that $dgg^{-1} = -gdg^{-1}$ and then use (2.5) with $(a_i \rightarrow -a_i, \nu_i \rightarrow -\nu_i, h_u \rightarrow h_u^{-1})$ and $(P_i \rightarrow P'_i, T_{ij} \rightarrow T'_{ij})$ with the additional contribution of the terms $a \cdot dP'$ and $v \cdot dT'$ (these terms contribute when derivatives with respect to the parameters $u_{ij}$ of $h_u$ are taken). The resulting expression is

$$dgg^{-1} = i(da \cdot P + \frac{1}{2} da_i a_j T_{ij}) + idv \cdot T + \frac{1}{2} (dh_u h_u^{-1})_{ij} J_{ij} + a_i (dh_u h_u^{-1})_{ij} P_j$$

$$+ (dh_u h_u)_{ij} \left( \frac{1}{2} a_i a_k + \nu_{ik} \right) T_{jk} . \quad (2.7)$$

The WZW action is defined as (we omit an overall scale factor)

$$S(g) = S_2(g) + S_3(g)$$

$$= -\frac{1}{2\pi} \int_{\Sigma} \ d^2 z \ T(g^{-1} \partial gg^{-1} \bar{\partial} g \Omega) + \frac{1}{6\pi} \int_B \ Tr(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \wedge \Omega) , \quad (2.8)$$

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where $\Omega_{AB} = \text{Tr}(T_A T_B \Omega)$ and $\Sigma = \partial B$. Using (2.5), (2.3) and the fact that $h_u^{-1} \Omega h_u = \Omega$ one easily computes that

$$S_2(g) = S_2(h_u) + \frac{1}{4\pi} \int_{\Sigma} d^2 z \left[ 2 \partial a_i \partial a_i + i(\partial h_u h_u^{-1})_{ij}(\partial a_i a_j - \partial v_{ij}) + i(\partial a_i a_j - \partial v_{ij})(\partial h_u h_u^{-1})_{ij} \right].$$

The computation of $S_3(g)$ is more involved. We will give some of the steps

$$S_3(g) = S_3(h_u) - \frac{1}{6\pi} \int_B \text{Tr}(da \cdot P' \wedge da \cdot P' \wedge (dh_u h_u^{-1})') \Omega)
- \frac{1}{6\pi} \int_B \text{Tr}[(da \cdot P' \wedge (dh_u h_u^{-1})') + (dh_u h_u^{-1})' \wedge da \cdot P') \wedge da \cdot P' \Omega]
+ \frac{i}{12\pi} \int_B \text{Tr}[(dv_{ij} - da_i a_j) T'_{ij} \wedge (dh_u h_u^{-1})' + (dh_u h_u^{-1})' \wedge (dv_{ij} - da_i a_j) T'_{ij} \wedge (dh_u h_u^{-1})']
+ \frac{i}{12\pi} \int_B \text{Tr}[(dh_u h_u^{-1})' \wedge (dh_u h_u^{-1})' \wedge (dv_{ij} - da_i a_j) T'_{ij} \Omega]
= S_3(h_u) + \frac{1}{4\pi} \int_B i(dh_u h_u^{-1})_{ij} \wedge da_i \wedge da_j + (dh_u h_u^{-1})_{il} \wedge (dv_{jl} - 1/2 da_j a_l)
= S_3(h_u) + \frac{i}{4\pi} \int_B d[(dh_u h_u^{-1})_{ij} \wedge (dv_{ij} - da_i a_j)]
= S_3(h_u) + \frac{i}{4\pi} \int_{\Sigma} d^2 z \left[ (\partial h_u h_u^{-1})_{ij}(\partial v_{ij} - \partial a_i a_j) - (\partial v_{ij} - \partial a_i a_j)(\partial h_u h_u^{-1})_{ij} \right],$$

(2.10)

where we used the definition $(dh_u h_u^{-1})' = \frac{1}{2}(dh_u h_u^{-1})_{ij} J'_{ij}$. Combining (2.9) with (2.10) we obtain the final form for the action

$$S(g) = S(h_u) + \frac{1}{2\pi} \int_{\Sigma} d^2 z \left[ \partial a_i \partial a_i + i(\partial a_i a_j - \partial v_{ij})(\partial h_u h_u^{-1})_{ij} \right],$$

(2.11)

where $S(h_u)$ is the WZW action for $SO(d)$ at level $k$. From the action (2.11) one can easily read off the corresponding metric and antisymmetric tensor fields. For the purposes of Section 3 let us note that for $E_3^c$ (2.11) can we rewritten as (In this case any antisymmetric matrix $A_{ij}$ can be parametrized in terms of a vector $A_i$, i.e. $A_{ij} = \epsilon_{ijk} A_k$)

$$S(g) = S(h_u) + \frac{1}{2\pi} \int_{\Sigma} d^2 z \left[ \partial a_i \partial a_i + (2\partial v_k - \epsilon_{ijk} \partial a_j) R^k_{\mu} \partial u^\mu \right],$$

(2.12)
where \( dh_u h_u^{-1} = i J_k R^k \mu d u^\mu \), \( i = 1, 2, 3 \), \( \mu = 1, 2, 3 \). Finally, let us note that if \( J_{ij} \to i \epsilon_{ij} J, T_{ij} \to -i \epsilon_{ij} T, u_{ij} \to -\epsilon_{ij} u, a_i \to -i a_i \), \( i = 1, 2 \) and \( k = -b \) one obtains for \( E_2^c \) the results of [2].

3. CFT corresponding to the WZW model for \( E_d^c \)

In this section we identify the CFT corresponding to the WZW action for \( E_d^c \). The OPE for the current algebra according to (2.1), (2.3) are (we concentrate on the holomorphic part only)

\[
\begin{align*}
J_{ij} J_{kl} &\sim \frac{i \delta_{[ik} J_{j]l]}{z-w} + \frac{k \delta_{[ik} \delta_{j]l]}{(z-w)^2} \\
J_{ij} P_k &\sim \frac{i \delta_{[ik} P_{j]}}{z-w}, \quad P_i P_j \sim \frac{i T_{ij}}{z-w} + \frac{\delta_{ij}}{(z-w)^2} \\
J_{ij} T_{kl} &\sim \frac{i \delta_{[ik} T_{j]l]}{z-w} + \frac{\delta_{[ik} \delta_{j]l]}{(z-w)^2}, \quad T_{ij} T_{kl} \sim 0 .
\end{align*}
\]

The corresponding stress energy tensor

\[
T = \frac{1}{2} : (P_i P_i + J_{ij} T_{ij} - \frac{1}{2} (2d - 3 + k) T_{ij} T_{ij}) : \tag{3.2}
\]

satisfies the Virasoro algebra with central charge \( c = d^2 \) and corresponds to a solution of the Master equation of [3]. With respect to (3.2) all currents are primary fields with conformal dimension one. In the rest of this section we will show that the metric corresponding to the WZW action (2.11) can also be obtained via the operator method [7][8]. For simplicity of the presentation we will concentrate on the case of \( E_3^c \). It is convenient to express the zero modes of the holomorphic currents in (3.1) as first order differential operators. From (2.6) rewritten for \( E_3^c \) we compute the following matrix defined as \( d g g^{-1} = i d X^M E_M^A T_A \), where \( X^M = \{ a_i, v_i, u_\mu \} \)

\[
E_M^A = \begin{pmatrix} P_j & J_j & T_j \\ a_i & \delta_{ij} & 0 & -\frac{1}{2} \epsilon_{ijk} a_k \\ \epsilon_{jkl} R^k_\mu a_l & R^i_\mu & \frac{1}{2} (a_j a_k - a_l a_i \delta_{jk} - 2 \epsilon_{jlk} v_l) R^k_\mu \\ 0 & 0 & \delta_{ij} \end{pmatrix} \tag{3.3}
\]
Its inverse is given by

\[
E_A^M = J_i \begin{pmatrix}
    a_j & u_\mu & v_j \\
    \delta_{ij} & 0 & \frac{1}{2} \epsilon_{ijk} a_k \\
    \epsilon_{ijk} a_k & R^\mu_i & \epsilon_{ijk} v_k \\
    0 & 0 & \delta_{ij}
\end{pmatrix}.
\]  

(3.4)

The first order differential operators defined as \( J_A = i E_A^M \partial_M \) satisfy the commutation relations (2.1), (2.2). Their explicit expressions are

\[
J_{J_i} = i \epsilon_{ijk} (a_k \partial a_j + v_k \partial v_j) + i R^\mu_i \partial u_\mu,
\]

\[
J_{P_i} = i \partial_i + \frac{i}{2} \epsilon_{ijk} a_k \partial v_j, \quad J_{T_i} = i \partial v_i.
\]

(3.5)

The metric and the dilaton can be deduced by comparing [7][8]

\[
HT = (L_0 + \bar{L}_0)T
\]

with

\[
HT = -\frac{1}{\sqrt{G}} \partial_M \sqrt{G} e^\Phi G^{MN} \partial_N T,
\]

where \( H \) is the Hamiltonian of the corresponding CFT, \( L_0 \) and \( \bar{L}_0 \) are the zero modes of the holomorphic and antiholomorphic stress energy tensors and \( T \) denotes tachyonic states of the theory annihilated by the positive modes of the holomorphic and antiholomorphic currents. One can show that the physical condition for closed strings \( (L_0 - \bar{L}_0)T = 0 \) is obeyed and therefore one need only consider the action of \( L_0 \) on \( T \). In this way one obtains a constant dilaton and the inverse metric

\[
G^{MN} = a_i \begin{pmatrix}
    a_j & u_\nu & v_j \\
    \delta_{ij} & 0 & -\frac{1}{2} \epsilon_{ijk} a_k \\
    0 & 0 & R^\nu_j \\
    \frac{1}{2} \epsilon_{ijk} a_k & R^\nu_j & (\frac{1}{4} a \cdot a - k - 3) \delta_{ij} - \frac{1}{4} a_i a_j
\end{pmatrix},
\]

(3.6)
which upon inverting gives the metric corresponding to the action \( (2.12) \), but with \( k \rightarrow k + 3 \). Therefore our result \( (2.12) \) is exact up to the forementioned shifting of \( k \). For the general case of \( E_d^c \) the result for the metric one obtains with the use of the operator method coincides with the one corresponding to the action \( (2.11) \) up to a shifting \( k \rightarrow k + 2d - 3 \) in \( (2.11) \).

4. Discussion and concluding remarks

The space-time corresponding to \( (2.11) \) is a \( d^2 \)-dimensional one with \( d(d - 1)/2 \) null Killing vectors equal in number to the independent components of the matrix \( (v_{ij}) \). The facts that the only result of quantum effects is to shift \( k \) in \( (2.11) \) and that the central charge equals the number of independent fields \( (c = d^2) \) can be traced back to the existence of \( d(d - 1)/2 \) null Killing vectors. Their role in \( (2.11) \) is that of Langrange multipliers which ‘freeze’ out fluctuations of the \( SO(d) \) currents unless sources are introduced for these currents and this is the reason why all renormalization effects (shifting of \( k \)) occur only in the part of \( (2.11) \) corresponding to the WZW model for the group \( SO(d) \). It is worth understanding this point from the beta functions point of view along the lines of \[3\].

The action \( (2.11) \) has the following ‘obvious’ global symmetries

\[
h_u \rightarrow h_u \Lambda, \quad \{ h_u \rightarrow S h_u, v \rightarrow S v S^T, a_i \rightarrow S a_i \}, \quad v \rightarrow v + N
\]

where \( \Lambda, S \) are constant group elements of \( SO(d) \) and \( N \) is a constant antisymmetric matrix. The total number of constant parameters is \( 3d(d - 1)/2 \). It is very likely that one can obtain the background corresponding to \( (2.11) \) by appropriate \( O(2d(d - 1)/2, 2d(d - 1)) \) transformations performed on the background corresponding to a flat space-time with zero antisymmetric tensor in \( D = d^2 \) space-time dimensions (The fact that \( c = d^2 \) is very suggestive).\footnote{For the \( E_2^c \) case this was shown in \[3\] \[3\]. The same is true for all models one obtains by gauging 1-dimensional subgroups of \( E_2^c \).} One can also use similar transformations to generate new string backgrounds.
The signature of the space-time is determined by the sign of the eigenvalues of the quadratic form (2.3). One easily finds that there are \( d(d+1)/2 \) positive and \( d(d-1)/2 \) negative ones. One may consider gauged versions of the WZW model corresponding to \( E_d^c \) as it was done for the case of \( E_2^c \) in [3] [4]. For instance one could gauge the subgroup corresponding to \( E_{d-1}^c \) (for \( d = 2 \) this was done in [4]). That will give a model in \( D = 2d-1 \) space-time dimensions with \( (d-1) \) time-like coordinates. Clearly only for \( d = 2 \) one obtains a space-time with 1-time coordinate in both the \( E_d^c \) and the \( E_d^c/E_{d-1}^c \) cases.

Finally, we believe that it is worth considering WZW models based on other semi-simple groups (in particular those with 1-time coordinate) since they correspond to solvable models with exact conformal invariance. They seem promising candidates for exact CFT theories corresponding to plane wave-type with null Killing vectors solutions to string theory.

**Note added**

After we finished this paper we received ref. [9] where the problem of constructing WZW models based on non-semi-simple groups was also considered.
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