Quasi-cliques in inhomogeneous random graphs

Kay Bogerd

k.m.bogerd@tue.nl

Eindhoven University of Technology

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Abstract

Given a graph $G$ and a constant $\gamma \in [0,1]$, let $\omega(\gamma)(G)$ be the largest integer $r$ such that there exists an $r$-vertex subgraph of $G$ containing at least $\gamma \binom{r}{2}$ edges. It was recently shown that $\omega(\gamma)(G)$ is highly concentrated when $G$ is an Erdős-Rényi random graph [Balister, Bollobás, Sahasrabudhe, Veremyev, 2019]. This paper provides a simple method to extend that result to a setting of inhomogeneous random graphs, showing that $\omega(\gamma)(G)$ remains concentrated on a small range of values even if $G$ is an inhomogeneous random graph. Furthermore, we give an explicit expression for $\omega(\gamma)(G)$ and show that it depends primarily on the largest edge probability of the graph $G$.

1 Introduction

Let $G = (V, E)$ be a simple graph, with vertex set $V$ and edge set $E$. Given a subset of vertices $S \subseteq V$, let $G[S]$ denote the subgraph of $G$ induced by $S$. That is, $G[S]$ is a graph with vertex set $S$ and edge set $\{(i, j) : i, j \in S\} \cap E$. A clique is a subset of vertices $C \subseteq V$ such that $G[C]$ is a complete graph, meaning that all vertices in $G[C]$ are connected by an edge. Cliques are an important concept in graph theory, and are often used as a model for community structure [3, 17, 21]. In particular, the problem of finding the largest clique or largest community in a given graph has received much interest [9, 10].

However, for many practical applications the definition of a clique can be too restrictive. Often a few missing edges within a community are fine, as long as the community remains sufficiently well connected. To this end, several relaxations have been proposed for the definition of a clique [25]. One of the most successful of these is known as the $\gamma$-quasi-clique, where $\gamma$ is a parameter [1]. For $\gamma \in [0,1]$, a $\gamma$-quasi-clique is a subset of vertices $S \subseteq V$ such that $G[S]$ contains at least $\gamma \binom{|S|}{2}$ edges. That is, a $\gamma$-quasi-clique is a subset of vertices such that a fraction $\gamma$ of all possible edges between them is present.

Just as for cliques, one would like to know the size of the largest quasi-clique in a given graph [2, 8, 26]. However, it comes as no surprise that finding the largest quasi-clique is a computationally hard problem [23, 24], similar to the problem of finding the largest clique [12, 13, 15]. To circumvent this difficulty, a common approach has been to study the related problem of determining the size of the largest clique or quasi-clique in random graphs. For cliques this approach has been very fruitful, and it turns out that the size of the largest clique is highly concentrated in a variety of random graph models. The first results of this type were obtained for Erdős-Rényi random graphs [6, 18, 19, 20], and later similar results were obtained for random geometric graphs [22], and inhomogeneous random graphs [2, 14].
Recently, the size of the largest quasi-clique was also studied in an Erdős-Rényi random graph, where it was shown that the largest quasi-clique is again highly concentrated \cite{4}. The aim of this paper is to extend that result to the setting of inhomogeneous random graphs. In particular, we formalize a heuristic presented in \cite{3}, and show how this (together with the result from \cite{4}) can be applied to show that the largest quasi-clique remains concentrated on a narrow range of values even in an inhomogeneous random graph.

2 Model and results

We are interested in understanding the behavior of the largest quasi-clique in an inhomogeneous random graph. To this end, define the $\gamma$-quasi-clique number $\omega_\gamma(G)$ of a graph $G$ as the size of the largest subset of vertices $S \subseteq V$ such that the induced subgraph $G[S]$ contains at least $\gamma \binom{|S|}{2}$ edges, where $\gamma \in [0, 1]$ is a parameter. Note that $\omega_0(G)$ is the familiar clique number of $G$, usually denoted simply by $\omega(G)$.

In this paper, we study the behavior of $\omega_\gamma(G)$ when $G$ is distributed according to the random graph model $G(n, \kappa)$. This model has two parameters: the number of vertices $n$, and a symmetric measurable function called a *kernel* $\kappa : [0, 1]^2 \to (0, 1)$. Below we introduce the key concepts of this model, for a more detailed overview we refer the reader to Lovász’s book \cite{16}. An element of $G(n, \kappa)$ is a simple graph $G = (V, E)$ that has $n \in \mathbb{N}$ vertices with vertex set $V = [n] := \{1, \ldots, n\}$, and a random edge set $E$. Each vertex $i \in V$ is assigned a *weight* $W_i$, which is simply a uniform variable on $[0, 1]$, that is $W_i \sim \text{Unif}(0, 1)$. Conditionally on these weights, the presence of an edge between two vertices $i, j \in V$, with $i \neq j$, is modeled by independent Bernoulli random variables with success probability

$$p_{ij} := \mathbb{P}((i, j) \in E \mid (W_k)_{k \in V}) = \kappa(W_i, W_j).$$

(1)

The kernel $\kappa(\cdot, \cdot)$ and the vertex weights $W_i$ are both not allowed to depend on the graph size $n$, and therefore the edge probabilities $p_{ij}$ are independent of $n$. This means that the graphs we consider are necessarily dense and have a number of edges that is quadratic in the graph size.

This brings us to the main result of this paper, which is to show that the $\gamma$-quasi-clique number $\omega_\gamma(G)$ of a graph $G \sim G(n, \kappa)$ is concentrated on a small range of values. Furthermore, this result shows that the size of the largest quasi-clique depends primarily on the densest part of the graph, where the edge probabilities are close to their maximum value. This is made precise by the following result.

**Theorem 1.** Let $\kappa(\cdot, \cdot)$ be a kernel that is continuous and attains its maximum value at the point $(c, c)$ for some $c \in [0, 1]$, and let $p_{\text{max}} := \kappa(c, c)$. Given $p_{\text{max}} < \gamma \leq 1$, define

$$\omega_n^\gamma := \frac{2\log(n)}{D(\gamma, p_{\text{max}})},$$

(2)

where $D(\gamma, p)$ is the Kullback-Leibler divergence between the Bernoulli distributions $\text{Bern}(\gamma)$ and $\text{Bern}(p)$, given by

$$D(\gamma, p) := \begin{cases} \gamma \log \left( \frac{\gamma}{p} \right) + (1 - \gamma) \log \left( \frac{1 - \gamma}{1 - p} \right) & \text{if } \gamma < 1, \\ \log \left( \frac{1}{p} \right) & \text{if } \gamma = 1. \end{cases}$$

(3)

Then, for every $\varepsilon > 0$,

$$\mathbb{P} \left( \omega_n^\gamma(G) \in \left[ (1 - \varepsilon)\omega_n^\gamma, (1 + \varepsilon)\omega_n^\gamma \right] \right) \to 1, \quad \text{as } n \to \infty.$$
To display the applicability of the above result, we show that it can be applied to many well-known random graph models. The simplest example is probably the Erdős-Rényi random graph, which is obtained by setting the kernel $\kappa(x, y)$ to a constant independent of $x$ and $y$. Another commonly used example are the so-called rank-1 random graphs, where $\kappa(x, y) = \varphi(x)\varphi(y)$ for some function $\varphi$. Often the function $\varphi(\cdot)$ is the inverse cumulative distribution function of some distribution $X$, so that $\varphi(W_i)$ can be interpreted as a sample from that distribution. This results in a model similar to that considered in [\[2\]]. The final model that satisfies the conditions in Theorem 1 is the stochastic block model [14], also called the planted partition model in computer science. This model is obtained when the kernel $\kappa(\cdot, \cdot)$ is only allowed to take on finitely many different values.

Note that Theorem 1 gives the first-order behavior of $\omega_n^\gamma$ from (2). More precise results are known for the clique and quasi-clique number in an Erdős-Rényi random graph [2,19], or for the clique number in rank-1 random graphs [5]. Specifically, in those cases the quasi-clique number and clique number are concentrated on two consecutive integers. Therefore, it might be reasonable to expect that it is likewise possible to show such a two-point concentration result in the more general model we consider in this paper. However, this would require a significantly more detailed analyses. The main difficulty here is that the higher order terms of $\omega_n^\gamma$ will likely depend in a complex way on the whole kernel $\kappa(\cdot, \cdot)$ and not just on the maximum value $\kappa(c,c)$. This was also observed for rank-1 random graphs in [5], where several examples are explicitly computed. Thus, the method we use in the proof of Theorem 1 will likely not be precise enough to characterize the higher order terms of $\omega_n^\gamma$ and a different approach would be needed for this.

We end this paper with the proof of Theorem 1. This proof is based on the ideas presented in [2, Section 3.1] combined with the results in [4] and [13].

Proof of Theorem 1 Below we consider the upper and lower bound of (4) separately. Furthermore, we will use the following standard asymptotic notation: given deterministic sequences $a_n$ and $b_n$, we write $a_n = o(b_n)$ when $a_n/b_n \to 0$, and we say that a sequence of events holds with high probability if it holds with probability tending to 1. When limits are unspecified they are taken as the number of vertices $n$ tends to $\infty$.

Upper bound: We first define a coupling between the random graph $G(n; \kappa)$ and the Erdős-Rényi random graph $G(n;\text{p}_{\text{max}})$, where we recall that $\text{p}_{\text{max}} = \kappa(c,c)$ is the maximum edge probability. For $i \neq j \in [n]$, let $U_{ij} \sim \text{Unif}(0,1)$ be independent uniform random variables on $[0,1]$. Conditionally on these uniform random variables and the weights $W_i$, with $i \in [n]$, define

\begin{align}
G &= (V, E), \quad \text{with } V = [n], \quad \text{and } E = \{(i,j) : U_{ij} \leq \kappa(W_i, W_j)\}, \\
G' &= (V', E'), \quad \text{with } V' = [n], \quad \text{and } E' = \{(i,j) : U_{ij} \leq \kappa(c,c)\}. 
\end{align}

It can easily be seen that $G$ is an inhomogeneous random graph, that is $G \sim G(n, \kappa)$. Similarly, $G' \sim G(n,\text{p}_{\text{max}})$ is distributed as an Erdős-Rényi random graph with edge probability $\text{p}_{\text{max}} = \kappa(c,c)$.

Because the edge probabilities satisfy $p_{ij} = \kappa(W_i, W_j) \leq \text{p}_{\text{max}}$ almost surely, for all $i \neq j \in [n]$, the coupling in (5) shows that $\omega_n^\gamma(G) \leq \omega_n^\gamma(G')$ almost surely. Furthermore, by [4, Theorem 1] if $\gamma < 1$ or [13, Theorem 6] if $\gamma = 1$, it follows that

\begin{align}
\omega_n^\gamma(G') \leq \frac{2}{D(\gamma, \text{p}_{\text{max}})} \left( \log(n) - \log \log(n) + \log(eD(\gamma, \text{p}_{\text{max}})/2) \right) + 1 + \varepsilon,
\end{align}

with high probability.
Combining the above, we obtain
\[
\omega(G) \leq \omega(G')
\]
\[
\leq \frac{2}{D(\gamma, p_{\max})} \left( \log(n) - \log \log(n) + \log(eD(\gamma, p_{max})/2) \right) + 1 + \varepsilon
\]
\[
\leq (1 + \varepsilon)\frac{2\log(n)}{D(\gamma, p_{\max})} = (1 + \varepsilon)\omega_n^\gamma,
\]
with high probability. This shows that \( P(\omega(G) \leq (1 + \varepsilon)\omega_n^\gamma) \to 1 \), completing the proof for the upper bound of \( \gamma \).

**Lower bound:** Let \( \delta_n = 1/\log(n) \) and define \( S_n := \{ i \in V : W_i \in [c - \delta_n, c + \delta_n] \} \) to be the subset of vertices that have vertex weight \( W_i \) close to \( c \), where we recall that \( c \) is such that the kernel \( \kappa( \cdot, \cdot ) \) attains its maximal value at the point \((c, c)\). Note that the set \( S_n \) is random and by Hoeffding’s inequality (see \[7, Theorem 2.8\]), for any \( t > 0 \), we have
\[
P(|S_n| \geq E[|S_n|] - t) \leq \exp\left(-2t^2/n\right) \to 0,
\]
where \( E[|S_n|] = nP(W \in [c - \delta_n, c + \delta_n]) = n^{1-o(1)} \) by definition of \( \delta_n \). Furthermore, define \( p_n := \inf_{(x, y) \in [c - \delta_n, c + \delta_n]} \kappa(x, y) \) and observe that \( p_n \to p_{\max} \) by continuity of the kernel, and thus \( D(\gamma, p_n) \to D(\gamma, p_{\max}) \). Using this, together with \( \omega(G) \) and \( t \) fixed, we obtain
\[
(1 - \varepsilon)\frac{2\log(n)}{D(\gamma, p_n)} \leq (1 - \varepsilon/2)\frac{2\log(E[|S_n|] - t)}{D(\gamma, p_{\max})}
\]
\[
\leq (1 - \varepsilon/3)\frac{2\log(|S_n|)}{D(\gamma, p_{\max})}
\]
\[
\leq (1 - \varepsilon/4)\frac{2\log(|S_n|)}{D(\gamma, p_n)},
\]
with high probability.

Similarly to the coupling in \([\delta]\), conditionally on the uniform random variables \( U_{ij} \), for \( i \neq j \in [n] \), and the vertex weights \( W_i \), for \( i \in [n] \), define
\[
G'' = (V'', E''), \quad \text{with} \quad V'' = [n], \quad \text{and} \quad E'' = \{(i, j) : U_{ij} \leq p_n\}.
\]
Note that the graph \( G'' \) is distributed as the Erdős-Rényi random graph \( G(n, p_n) \) with edge probability \( p_n \).

Given a graph \( G \), recall that \( G[S_n] \) denotes the subgraph induced by the vertices in \( S_n \). Because the kernel is continuous around the point \((c, c)\), there exists an \( n \) large enough such that \( \delta_n \) is small enough to ensure that the edge probabilities satisfy \( p_{ij} \geq p_n \) almost surely, for all \( i \neq j \in S_n \) (note that, if the kernel is continuous everywhere then this holds for every \( n \)). Hence, the coupling in \([\delta]\) shows that \( \omega(G) \geq \omega(G[S_n]) \geq \omega(G''[S_n]) \) almost surely, provided \( n \) is large enough. Combining this with \([6] \) and \([4] \) Theorem 1 if \( \gamma < 1 \) or \([10] \) Theorem 6 if \( \gamma = 1 \), we obtain
\[
\omega(G) \geq \omega(G'[S_n]) \geq \omega(G''[S_n])
\]
\[
\geq \frac{2}{D(\gamma, p_n)} \left( \log(|S_n|) - \log \log(|S_n|) + \log(eD(\gamma, p_n)/2) \right) - \varepsilon
\]
\[
\geq (1 - \varepsilon/4)\frac{2\log(|S_n|)}{D(\gamma, p_n)} \geq (1 - \varepsilon)\frac{2\log(n)}{D(\gamma, p_{\max})} = (1 - \varepsilon)\omega_n^\gamma,
\]
with high probability. This shows that \( P(\omega(G) \geq (1 - \varepsilon)\omega_n^\gamma) \to 1 \), completing the proof for the lower bound of \( \gamma \).
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