Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map

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Abstract

In this article we seek stability estimates in the inverse problem of determining the potential or the velocity in a wave equation in an anisotropic medium from measured Neumann boundary observations. This information is enclosed in the dynamical Dirichlet-to-Neumann map associated to the wave equation. We prove in dimension $n \geq 2$ that the knowledge of the Dirichlet-to-Neumann map for the wave equation uniquely determines the electric potential and we prove Hölder-type stability in determining the potential. We prove similar results for the determination of velocities close to 1.

Keywords: Stability estimates, Hyperbolic inverse problem, Dirichlet-to-Neumann map.

Contents

1 Introduction 2
1.1 Weak solutions of the wave equation ................. 5

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1 Introduction

In this paper, we are interested in the following inverse boundary value problem: on a Riemannian manifold with boundary, determine the potential or the velocity — i.e. the conformal factor within a conformal class of metrics — in a wave equation from the vibrations measured at the boundary. Let \((M, g)\) be a compact Riemannian manifold with boundary \(\partial M\). All manifolds will be assumed smooth (which means \(C^\infty\)) and oriented. We denote by \(\Delta_g\) the Laplace-Beltrami operator associated to the metric \(g\). In local coordinates, \(g(x) = \sum_{j,k=1}^{n} g_{jk}(x) dx_j \otimes dx_k\), \(\Delta_g\) is given by

\[
\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{\det g} g^{jk} \frac{\partial}{\partial x_k} \right),
\]

(1.1)

Here \((g^{jk})\) is the inverse of the metric \(g\) and \(\det g = \det (g_{jk})\). Let us consider the following initial boundary value problem for the wave equation with bounded (real
valued) electric potential \( q \in L^\infty(M) \)

\[
\begin{align*}
(\partial^2_t - \Delta_g + q(x)) u &= 0, \quad \text{in } (0, T) \times M, \\
u(0, \cdot) &= 0, \quad \partial_t u(0, \cdot) = 0 \quad \text{in } M, \\
u = f, \quad \text{on } (0, T) \times \partial M, \\
\end{align*}
\]  

(1.2)

where \( f \in H^1((0, T) \times \partial M) \). Denote by \( \nu = \nu(x) \) the outer normal to \( \partial M \) at \( x \in \partial M \), normalized so that \( \sum_{j,k=1}^n g^{jk} \nu_j \nu_k = 1 \). We may define the dynamical Dirichlet-to-Neumann map \( \Lambda_{g, q} \) by

\[
\Lambda_{g, q} f = \sum_{j,k=1}^n \nu_j g^{jk} \frac{\partial u}{\partial x_k} \big|_{(0, T) \times \partial M}.
\]  

(1.3)

It is clear that one cannot hope to uniquely determine the metric \( g = (g_{jk}) \) from the knowledge of the Dirichlet-to-Neumann map \( \Lambda_{g, q} \). As was noted in [39], the Dirichlet-to-Neumann map is invariant under a gauge transformation of the metric \( g \). Namely, given a diffeomorphism \( \Psi : M \to M \) such that \( \Psi|_{\partial M} = \text{Id} \) one has \( \Lambda_{\Psi^* g, q} = \Lambda_{g, q} \) where \( \Psi^* g \) denotes the pullback of the metric \( g \) under \( \Psi \). The inverse problem should therefore be formulated modulo the natural gauge invariance. Nevertheless, when the problem is restricted to a conformal class of metrics, there is no such gauge invariance and the inverse problem now takes the form: knowing \( \Lambda_{c, g, q} \), can one determine the conformal factor \( c \) and the potential \( q \)?

Belishev and Kurylev gave an affirmative answer in [5] to the general problem of finding a smooth metric from the Dirichlet-to-Neumann map \( \Lambda_{g, q} \). Their approach is based on the boundary control method introduced by Belishev [4] and uses in an essential way an unique continuation property. Unfortunately it seems unlikely that this method would provide stability estimates even under geometric and topological restrictions. Their method also solves the problem of recovering \( g \) through boundary spectral data. The boundary control method gave rise to several refinements of the results of [5]: one can cite for instance [30], [29] and [1].

In this paper, the inverse problem under consideration is whether the knowledge of the Dirichlet-to-Neumann map \( \Lambda_{g, q} \) on the boundary uniquely determines the electric potential \( q \) (with a fixed metric \( g \)) and whether the knowledge of the Dirichlet-to-Neumann map \( \Lambda_g = \Lambda_{g,0} \) uniquely determines the conformal factor of the metric \( g \) within a conformal class. From the physical viewpoint, our inverse problem consists in determining the properties (e.g. a dispersion term) of an inhomogeneous medium by probing it with disturbances generated on the boundary.
The data are responses of the medium to these disturbances which are measured on the boundary, and the goal is to recover the potential \( q(x) \) and the velocity \( c(x) \) which describes the property of the medium. Here we assume that the medium is quiet initially, and \( f \) is a disturbance which is used to probe the medium. Roughly speaking, the data is \( \partial_\nu u \) measured on the boundary for different choices of \( f \).

In the Euclidian case (\( g = e \)) Rakesh and Symes \([35],[34]\) used complex geometrical optics solutions concentrating near lines with any direction \( \omega \in S^{n-1} \) to prove that \( \Lambda_{e,q} \) determines \( q(x) \) uniquely in the wave equation. In \([35]\), \( \Lambda_{e,q} \) gives equivalent information to the responses on the whole boundary for all the possible input disturbances. Ramm and Sjöstrand \([36]\) extended the results in \([35]\) to the case of a potential \( q \) depending both on space \( x \) and time \( t \). Isakov \([25]\) considered the simultaneous determination of a potential and a damping coefficient. A key ingredient in the existing results, is the construction of complex geometric optics solutions of the wave equation in the Euclidian case, concentrated along a line, and the relationship between the hyperbolic Dirichlet-to-Neumann map and the X-ray transform plays a crucial role.

Regarding stability estimates, Sun \([42]\) established in the Euclidean case stability estimates for potentials from the Dirichlet-to-Neumann map. In \([39]\) and \([41]\) Stefanov and Uhlmann considered the inverse problem of determining a Riemannian metric on a Riemannian manifold with boundary from the hyperbolic Dirichlet-to-Neumann map associated to solutions of the wave equation \((\partial^2_t - \Delta_g)u = 0\). A Hölder type of conditional stability estimate was proven in \([39]\) for metrics close enough to the Euclidean metric in \( C^k, k \geq 1 \) or for generic simple metrics in \([41]\).

Uniqueness properties for local Dirichlet-to-Neumann maps associated with the wave equation are rather well understood (e.g., Belishev \([4]\), Katchlov, Kurylev and Lassas \([29]\), Kurylev and Lassas \([30]\)) but stability for such operators is far from being apprehended. For instance, one may refer to Isakov and Sun \([27]\) where a local Dirichlet-to-Neumann map yields a stability result in determining a coefficient in a subdomain. As for results involving a finite number of data in the Dirichlet-to-Neumann map, see Cheng and Nakamura \([14]\), Rakesh \([34]\). There are quite a few works on Dirichlet-to-Neumann maps, so our references are far from being complete: see also Cardoso and Mendoza \([13]\), Cheng and Yamamoto \([15]\), Eskin \([18]-[19]-[20]\), Hech and Wang \([23]\), Rachele \([33]\), Uhlmann \([43]\) as related papers.

The main goal of this paper is to study the stability of the inverse problem for the dynamical anisotropic wave equation. The approach that we develop is a dynamical approach. Our inverse problem corresponds to a formulation with bound-
ary measurements at infinitely many frequencies. On the other hand, the main methodology for formulations of inverse problems involving a measurement at a fixed frequency, is based on $L^2$-weighted inequalities called Carleman estimates. For such applications of Carleman inequalities to inverse problems we refer for instance to Bellassoued [6], Isakov [25]. Most papers treat the determination of spatially varying functions by a single measurement. As for observability inequalities by means of Carleman estimates, see [8], [9], [10].

Our proof is inspired by techniques used by Stefanov and Uhlmann [41], and Dos Santos Ferreira, Kenig, Salo and Uhlmann [17]. In the last reference, an uniqueness theorem for an inverse problem for an elliptic equation is proved following ideas which in turn go back to the work of Calderón [12]. The heuristic underlying idea is that one can (at least formally) translate techniques used in solving the elliptic equation

$$\partial_t^2 + \Delta_g$$

(which is the prototype of equations studied in [17]) to the case of the wave equation

$$\partial_t^2 - \Delta_g$$

by changing $t$ into $it$. Our problem turns out to be somehow easier because we don’t need to construct complex geometrical solutions, but can rely on classical WKB solutions.

1.1 Weak solutions of the wave equation

Let $(\mathcal{M}, g)$ be a (smooth) compact Riemannian manifold with boundary of dimension $n \geq 2$. We refer to [28] for the differential calculus of tensor fields on Riemannian manifolds. If we fix local coordinates $x = [x_1, \ldots, x_n]$ and let $\left[\partial/\partial x_1, \ldots, \partial/\partial x_n\right]$ denote the corresponding tangent vector fields, the inner product and the norm on the tangent space $T_x\mathcal{M}$ are given by

$$g(X, Y) = \langle X, Y \rangle_g = \sum_{j,k=1}^n g_{jk} \alpha_j \beta_k,$$

$$|X|_g = \langle X, X \rangle_g^{1/2}, \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}.$$
for all vector fields $X$ on $\mathcal{M}$. In local coordinates, we have

$$\nabla_g f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \quad (1.4)$$

The metric tensor $g$ induces the Riemannian volume $dv^n_g = (\det g)^{1/2} dx_1 \wedge \cdots \wedge dx_n$. We denote by $L^2(\mathcal{M})$ the completion of $C^\infty(\mathcal{M})$ with respect to the usual inner product

$$\langle f_1, f_2 \rangle = \int_{\mathcal{M}} f_1(x) f_2(x) dv^n_g, \quad f_1, f_2 \in C^\infty(\mathcal{M}).$$

The Sobolev space $H^1(\mathcal{M})$ is the completion of $C^\infty(\mathcal{M})$ with respect to the norm

$$\|f\|_{H^1(\mathcal{M})}^2 = \|f\|_{L^2(\mathcal{M})}^2 + \|\nabla f\|_{L^2(\mathcal{M})}^2.$$

The normal derivative is given by

$$\partial_\nu u := \nabla_g u \cdot \nu = \sum_{j,k=1}^{n} g^{jk} \nu_j \frac{\partial u}{\partial x_k} \quad (1.5)$$

where $\nu$ is the unit outward vector field to $\partial \mathcal{M}$. Moreover, using covariant derivatives (see [22]), it is possible to define coordinate invariant norms in $H^k(\mathcal{M})$, $k \geq 0$.

Let us consider the following initial boundary value problem for the wave equation

$$\begin{cases}
(\partial_t^2 - \Delta_g + q(x)) v(t, x) = F(t, x) & \text{in } (0, T) \times \mathcal{M}, \\
v(0, x) = 0, \quad \partial_t v(0, x) = 0 & \text{in } \mathcal{M}, \\
v(t, x) = 0 & \text{on } (0, T) \times \partial \mathcal{M}.
\end{cases} \quad (1.6)$$

The following result is well known (see [24]).

**Lemma 1.1** Let $T > 0$ and $q \in L^\infty(\mathcal{M})$, suppose that $F \in \mathcal{H}$, with $\mathcal{H} = L^1(0, T; L^2(\mathcal{M}))$. The unique solution $v$ of (1.6) satisfies

$$v \in C^1(0, T; L^2(\mathcal{M})) \cap C(0, T; H^1_0(\mathcal{M}))$$

and the mapping $F \mapsto \partial_\nu v$ is linear and continuous from $\mathcal{H}$ to $L^2((0, T) \times \partial \mathcal{M})$. Furthermore, there is a constant $C > 0$ such that

$$\|\partial_t v(t, \cdot)\|_{L^2(\mathcal{M})} + \|\nabla v(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \|F\|_{L^1(0, T; L^2(\mathcal{M}))}, \quad (1.7)$$

$$\|\partial_\nu v\|_{L^2((0, T) \times \partial \mathcal{M})} \leq C \|F\|_{\mathcal{H}}. \quad (1.8)$$
A proof of the following lemma may be found for instance in [31].

**Lemma 1.2** Let \( f \in H^1((0, T) \times \partial M) \) be a function such that \( f(0, x) = 0 \) for all \( x \in \partial M \). There exists an unique solution

\[
u \in C^1(0, T; L^2(M)) \cap C(0, T; H^1(M)) \tag{1.9}
\]
to the problem (1.2). Furthermore, the map \( f \mapsto \partial_\nu u \) is linear and continuous from \( H^1((0, T) \times \partial M) \) into \( L^2((0, T) \times \partial M) \).

Therefore the Dirichlet-to-Neumann map \( \Lambda_{g,q} \) defined by (1.3) is continuous. We denote by \( \|\Lambda_{g,q}\| \) its norm in \( \mathcal{L}(H^1((0, T) \times \partial M); L^2((0, T) \times \partial M)) \). Our last remark concerns the fact that when \( q \) is real valued, the Dirichlet-to-Neumann map is self-adjoint; more precisely, we have

\[
\Lambda^*_{g,q} = \Lambda_{g,\bar{q}}.
\]

This simple fact will be proven in section 2. We denote

\[
\Lambda_g = \Lambda_{g,0}
\]

the Dirichlet-to-Neumann map when there is no potential in the wave equation.

### 1.2 Statement of the main results

In this section we state the main stability results. Let us begin by introducing an admissible class of manifolds for which we can prove uniqueness and stability results in our inverse problem. For this we need the notion of simple manifolds [41].

**Definition 1** We say that the Riemannian manifold \((M, g)\) (or more shortly that the metric \(g\)) is simple, if \(\partial M\) is strictly convex with respect to \(g\), and for any \(x \in M\), the exponential map \(\exp_x : \exp_x^{-1}(M) \rightarrow M\) is a diffeomorphism.

Note that if \((M, g)\) is simple, one can extend \((M, g)\) into another simple manifold \(M_1\) such that \(M \subseteq M_1\).

Let us now introduce the admissible set of potentials \(q\) and the admissible set of conformal factors \(c\). Let \(M_0 > 0\), \(k \geq 1\) and \(\varepsilon > 0\) be given. Set

\[
\mathcal{Q}(M_0) = \left\{ q \in H^1(M), \|q\|_{H^1(M)} \leq M_0 \right\}, \tag{1.10}
\]

and

\[
\mathcal{C}(M_0, k, \varepsilon) = \left\{ c \in C^\infty(M), c > 0 \text{ in } \overline{M}, \|1 - c\|_{C^1(M)} \leq \varepsilon, \|c\|_{C^k(M)} \leq M_0 \right\}. \tag{1.11}
\]

The main results of this paper can be stated as follows.
Theorem 1  Let \((\mathcal{M}, g)\) be a simple Riemannian compact manifold with boundary of dimension \(n \geq 2\), let \(T > \text{Diam}_{g}(\mathcal{M})\), there exist constants \(C > 0\) and \(\kappa_1 \in (0, 1)\) such that for any real valued potentials \(q_1, q_2 \in \mathcal{Q}(M_0)\) such that \(q_1 = q_2\) on the boundary \(\partial \mathcal{M}\), we have
\[
\|q_1 - q_2\|_{L^2(\mathcal{M})} \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{\kappa_1}
\]  
(1.12)
where \(C\) depends on \(\mathcal{M}, T, M_0, n,\) and \(s\).

As a corollary of Theorem 1 we obtain the following uniqueness result.

Corollary 1  Let \((\mathcal{M}, g)\) be a simple Riemannian compact manifold with boundary of dimension \(n \geq 2\), let \(T > \text{Diam}_{g}(\mathcal{M})\), let \(q_1, q_2 \in \mathcal{Q}(M_0)\) be real valued potentials such that \(q_1 = q_2\) on \(\partial \mathcal{M}\). Then \(\Lambda_{g, q_1} = \Lambda_{g, q_2}\) implies \(q_1 = q_2\) everywhere in \(\mathcal{M}\).

Theorem 2  Let \((\mathcal{M}, g)\) be a simple Riemannian compact manifold with boundary of dimension \(n \geq 2\), let \(T > \text{Diam}_{g}(\mathcal{M})\), there exist \(k \geq 1, \varepsilon > 0, 0 < \kappa_2 < 1\) and \(C > 0\) such that for any \(c \in \mathcal{C}(M_0, k, \varepsilon)\) with \(c = 1\) near the boundary \(\partial \mathcal{M}\), the following estimate holds true
\[
\|1 - c\|_{L^2(\mathcal{M})} \leq C \|\Lambda_{g} - \Lambda_{cg}\|^{\kappa_2}
\]  
(1.13)
where \(C\) depends on \((\mathcal{M}, g), M_0, n, \varepsilon, k\) and \(s\).

As a corollary of Theorem 2 we obtain the following uniqueness result.

Corollary 2  Let \((\mathcal{M}, g)\) be a simple Riemannian compact manifold with boundary of dimension \(n \geq 2\), let \(T > \text{Diam}_{g}(\mathcal{M})\), there exist \(k \geq 1, \varepsilon > 0,\) such that for any \(c \in \mathcal{C}(M_0, k, \varepsilon)\) with \(c = 1\) near the boundary \(\partial \mathcal{M}\), we have \(\Lambda_{cg} = \Lambda_{g}\) implies \(c = 1\) everywhere in \(\mathcal{M}\).

1.3 Spectral inverse problem

For \(q \in \mathcal{Q}(M_0)\) and \(q \geq 0\), we denote by \(A_q\) the unbounded operator \(A_q = -\Delta_g + q\) with domain \(\mathcal{D}(A_q) = H^1(\mathcal{M}) \cap H^2(\mathcal{M})\).

The spectrum of \(A_q\) consists of a sequence of eigenvalues, counted according to their multiplicities:

\[
0 \leq \lambda_{1,q} \leq \lambda_{2,q} \leq \ldots \leq \lambda_{k,q} \leq \ldots
\]

with \(\lim_{k \to \infty} \lambda_{k,q} = \infty\). The corresponding eigenfunctions are denoted by \((\phi_{k,q})\).

We may assume that this sequence forms an orthonormal basis of \(L^2(\mathcal{M})\).
In the sequel $C$ denotes a generic positive constant depending only on $M$ and $M_0$ ($M_0$ is given by (1.10)). Since $\phi_{k,q}$ is the solution of the following boundary value problem

\[
\begin{aligned}
\left\{
\begin{array}{ll}
(−Δ_g + q)\phi = \lambda_{k,q} & \text{ in } M \\
\phi = 0, & \text{ on } \partial M,
\end{array}
\right.
\end{aligned}
\]

classical $H^2(M)$ a priori estimates imply

\[
\|\phi_{k,q}\|_{H^\sigma(M)} \leq C_{\lambda_{k,q}} \|\phi_{k,q}\|_{L^2(M)} \leq C_{\lambda_{k,q}} \sigma/2, \quad \sigma = 0, 1, 2.
\]

(1.14)

Therefore

\[
\left\|\partial_\nu \phi_{k,q}\right\|_{H^{1/2}(\partial M)} \leq C_{\lambda_{k,q}}.
\]

On the other hand, by Weyl's asymptotics, there exists a positive constant $C \geq 1$ such that

\[
C^{-1} k^{2/n} \leq \lambda_{k,q} \leq C k^{2/n}.
\]

(1.15)

Here $C$ can be chosen uniformly with respect to $q$ provided $0 \leq q(x) \leq M$ for $x \in M$. Therefore we have

\[
\left\|\partial_\nu \phi_{k,q}\right\|_{H^{1/2}(\partial M)} \leq C k^{2/n}.
\]

We fix $r$ such that $n/2 + 1 < r \leq n + 1$ and it follows that

\[
\left( k^{-2r/n} \left\|\partial_\nu \phi_{k,q}\right\|_{H^{1/2}(\partial M)} \right) \in \ell^1.
\]

We recall that $\ell^1$ is the Banach space of real-valued sequences such that the corresponding series is absolutely convergent. This space is equipped with its natural norm.

Let $\omega = (\omega_k)$ be the sequence given by $\omega_k = k^{-2r/n}$ for each $k \geq 1$. We introduce the following Banach spaces

\[
\ell^1_\omega \left( H^{1/2}(\partial M) \right) = \left\{ h = (h_k)_k; \ h_k \in H^{1/2}(\partial M), \ k \geq 1, \text{ and } \left( \omega_k \| h_k \|_{H^{1/2}(\partial M)} \right)_k \in \ell^1 \right\}.
\]

and

\[
\ell^1_\omega (C) = \left\{ y = (y_k)_k; \ y_k \in C, \ k \geq 1, \text{ and } \left( \omega_k |y_k| \right)_k \in \ell^1 \right\}.
\]

The natural norms on those spaces are

\[
\|h\|_{\ell^1_\omega \left( H^{1/2}(\partial M) \right)} = \sum_{k \geq 1} \omega_k \| h_k \|_{H^{1/2}(\partial M)}
\]

9
and
\[ \| y \|_{\ell^1(C)} = \sum_{k \geq 1} \omega_k |y_k|. \]

We will apply Theorem 1 to prove the following result.

**Theorem 3** Let \((\mathcal{M}, g)\) be a simple Riemannian compact manifold with boundary of dimension \(n \geq 2\). There exist \(C > 0\) and \(\kappa_3 \in (0, 1)\) such that the following estimate holds
\[ \| q_1 - q_2 \|_{L^2(\mathcal{M})} \leq C \epsilon^{\kappa_3} \tag{1.16} \]
for any non-negative \(q_1, q_2 \in \mathcal{Q}(M_0)\) which are equal on the boundary \(\partial \mathcal{M}\), where
\[ \epsilon = |\lambda_{q_1} - \lambda_{q_2}|_{\ell^1(C)} + \| \partial_\nu \phi_{q_1} - \partial_\nu \phi_{q_2} \|_{\ell^1(H^{1/2}(\partial \mathcal{M}))} \]
is assumed to be small and \(\partial_\nu \phi_{q_j} = (\partial_\nu \phi_{k,q_j})_{k^j}, j = 1, 2\).

Theorem 3 is an extension of a result in [16] which is itself a variant of a theorem in [2]-[7]. To the best of our knowledge, [2] is the first result in the literature concerned with stability estimates for multidimensional inverse spectral problems.

The outline of the paper is as follows. In section 2 and 3 we collect some of the formulas needed in the paper. In section 4 we construct special geometrical optics solutions to the wave equation. In section 5 and 6, we establish stability estimates for related integrals over geodesics crossing \(\mathcal{M}\) and prove our main results. In section 7 we prove Theorem 3.

## 2 Preliminaries

In this section we collect formulas needed in the rest of this paper. We denote by \(\text{div} X\) the divergence of a vector field \(X \in H^1(T\mathcal{M})\) on \(\mathcal{M}\), i.e. in local coordinates,
\[ \text{div} X = \frac{1}{\sqrt{\text{det} g}} \sum_{i=1}^n \partial_i \left( \sqrt{\text{det} g} \alpha_i \right), \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}. \tag{2.1} \]
If \(X \in H^1(T\mathcal{M})\) we have the divergence formula
\[ \int_{\mathcal{M}} \text{div} X \ dv_g^n = \int_{\partial \mathcal{M}} \langle X, \nu \rangle \ d\sigma_g^{n-1} \tag{2.2} \]
and for $f \in H^1(M)$ Green’s formula reads
\[
\int_M \text{div} X f \, dv^n_g = - \int_M \langle X, \nabla_g f \rangle_g \, dv^n_g + \int_{\partial M} \langle X, \nu \rangle \, f \, d\sigma_{g}^{n-1}. \tag{2.3}
\]

Then if $f \in H^1(M)$ and $w \in H^2(M)$, the following identity holds
\[
\int_M \Delta_g w f \, dv^n_g = - \int_M \langle \nabla_g w, \nabla_g f \rangle_g \, dv^n_g + \int_{\partial M} \partial_\nu w f \, d\sigma_{g}^{n-1}. \tag{2.4}
\]

Let $f_1, f_2 \in H^1((0, T) \times \partial M)$, we denote by $u_1$, respectively by $u_2$, the solutions to (1.2) with potential $q$ and Dirichlet datum $f_1$, respectively $\bar{q}$ and Dirichlet datum $f_2$. By Green’s formula, we have
\[
\int_{\partial M} \Lambda_{g,q} f_1 f_2 d\sigma_{g}^{n-1} = \int_M \Delta_g u_1 u_2 \, dv^n_g + \int_M \langle \nabla_g u_1, \nabla_g u_2 \rangle_g \, dv^n_g
\]
\[
= \int_M u_1 \Delta_g u_2 \, dv^n_g + \int_M \langle \nabla_g u_1, \nabla_g u_2 \rangle_g \, dv^n_g
\]
\[
= \int_{\partial M} f_1 \Lambda_{g,\bar{q}} f_2 d\sigma_{g}^{n-1}.
\]

This shows that
\[
\Lambda_{g,q}^* = \Lambda_{g,\bar{q}}.
\]

In particular, this implies that $\Lambda_{g,q}$ is selfadjoint when $q$ is real-valued (and therefore $\Lambda_g$). From now on, we will suppose the potential to be real-valued.

For $x \in M$ and $\theta \in T_x M$ we denote by $\gamma_{x,\theta}$ the unique geodesic starting at the point $x$ in the direction $\theta$. We denote
\[
SM = \left\{ (x, \theta) \in T M; |\theta|_g = 1 \right\},
\]
\[
S^* M = \left\{ (x, p) \in T^* M; |p|_g = 1 \right\}
\]
the sphere bundle and co-sphere bundle of $M$. The exponential map $\exp_x : T_x M \rightarrow M$ is given by
\[
\exp_x(v) = \gamma_{x,v}(|v|_g v) = \gamma_{x,v}(rv), \quad r = |v|_g. \tag{2.5}
\]

A compact Riemannian manifold $(M, g)$ with boundary is a convex non-trapping manifold, if it satisfies two conditions:
(a) the boundary $\partial M$ is strictly convex, i.e. the second fundamental form of the boundary is positive definite at every boundary point,

(b) for every point $x \in M$ and every vector $\theta \in T_xM$, $\theta \neq 0$, the maximal geodesic $\gamma_{x,\theta}(t)$ satisfying the initial conditions

$$\gamma_{x,\theta}(0) = x \text{ and } \dot{\gamma}_{x,\theta}(0) = \theta$$

is defined on a finite segment $[\tau_-(x,\theta), \tau_+(x,\theta)]$. We recall that a geodesic $\gamma : [a, b] \rightarrow M$ is maximal if it cannot be extended to a segment $[a - \varepsilon_1, b + \varepsilon_2]$, where $\varepsilon_i \geq 0$ and $\varepsilon_1 + \varepsilon_2 > 0$.

The second condition is equivalent to all geodesics having finite length in $M$. An important subclass of convex non-trapping manifold are simple manifolds. Recall that a compact Riemannian manifold $(M, g)$ which is simple satisfies the following properties

(a) the boundary is strictly convex,

(b) there are no conjugate points on any geodesic.

A simple $n$-dimensional Riemannian manifold is diffeomorphic to a closed ball in $\mathbb{R}^n$, and any pair of points on the manifold can be joined by an unique minimizing geodesic.

In the rest of this article, $C$ will be a generic constant which might change from one line to another, but which only depends on the quantities allowed in the statement of the theorems (namely the quantities involved in the sets $\mathcal{Z}, \mathcal{C}$, the manifold $(M, g)$, the dimension $n$, the final time $T$ and the Hölder exponents $\kappa_j$).

3 The geodesical ray transform

We introduce the submanifolds of inner and outer vectors of $SM$

$$\partial_{\pm} SM = \{(x, \theta) \in SM, x \in \partial M, \pm \langle \theta, \nu(x) \rangle < 0\} \quad (3.1)$$

where $\nu$ is the unit outer normal to the boundary. Note that $\partial_+ SM$ and $\partial_- SM$ are compact manifolds with the same boundary $S(\partial M)$, and $\partial SM = \partial_+ SM \cup \partial_- SM$. For $(x, \theta) \in \partial_+ SM$, we denote by $\gamma_{x,\theta} : [0, \tau_+(x,\theta)] \rightarrow M$ the maximal geodesic satisfying the initial conditions $\gamma_{x,\theta}(0) = x$ and $\dot{\gamma}_{x,\theta}(0) = \theta$. Let $C^\infty(\partial_+ SM)$ be the space of smooth functions on the manifold $\partial_+ SM$. The
ray transform (also called geodesic X-ray transform) on a convex non-trapping manifold \( \mathcal{M} \) is the linear operator

\[
\mathcal{I} : C^\infty(\mathcal{M}) \rightarrow C^\infty(\partial_+ SM)
\]  

(3.2)
defined by

\[
\mathcal{I} f(x, \theta) = \int_0^{\tau_+(x, \theta)} f(\gamma_{x, \theta}(t)) dt.
\]  

(3.3)

The right-hand side of (3.3) is a smooth function on \( \partial_+ SM \) because the integration bound \( \tau_+(x, \theta) \) is a smooth function on \( \partial_+ SM \), see Lemma 4.1.1 of [38]. The ray transform on a convex non-trapping manifold \( \mathcal{M} \) can be extended to a bounded operator

\[
\mathcal{I} : H^k(\mathcal{M}) \rightarrow H^k(\partial_+ SM)
\]  

(3.4)

for every integer \( k \geq 1 \), see Theorem 4.2.1 of [38].

The Riemannian scalar product on \( T_x SM \) induces a volume form on \( SM \) denoted by \( d\omega_x(\theta) \) and given by

\[
d\omega_x(\theta) = \sum_{k=1}^{n} (-1)^k \theta^k d\theta^1 \wedge \cdots \wedge \hat{d}\theta^k \wedge \cdots \wedge d\theta^n.
\]

We introduce the volume form \( dv^{2n-1}_g \) on the manifold \( SM \)

\[
dv^{2n-1}_g(x, \theta) = \left| d\omega_x(\theta) \wedge dv^n_g \right|
\]

where \( dv^n_g \) is the Riemannian volume form on \( \mathcal{M} \). By Liouville’s theorem, the form \( dv^{2n-1}_g \) is preserved by the geodesic flow. The corresponding volume form on the boundary \( \partial SM = \{(x, \theta) \in SM, x \in \partial \mathcal{M}\} \) is given by

\[
d\sigma^{2n-2}_g = \left| d\omega_x(\theta) \wedge d\sigma^{n-1}_g \right|
\]

where \( d\sigma^{n-1}_g \) is the volume form of \( \partial \mathcal{M} \).

Let \( L^2_\mu(\partial_+ SM) \) be the space of real valued square integrable functions with respect to the measure \( \mu(x, \theta) d\sigma^{2n-2}_g \) with density \( \mu(x, \theta) = |\langle \theta, \nu(x) \rangle| \). This Hilbert space is endowed with the scalar product given by

\[
\langle u, v \rangle_{L^2_\mu(\partial_+ SM)} = \int_{\partial_+ SM} u(x, \theta) v(x, \theta) \mu(x, \theta) d\sigma^{2n-2}_g.
\]  

(3.5)
The ray transform $I$ is a bounded operator from $L^2(M)$ into $L^2_{\mu}(\partial_+ S M)$ and its adjoint $I^* : L^2_{\mu}(\partial_+ S M) \rightarrow L^2(M)$ is given by

$$I^* \psi(x) = \int_{S_x M} \psi^*(x, \theta) \, d\omega_x(\theta)$$

(3.6)

where $\psi^*$ is the extension of the function $\psi$ from $\partial_+ S M$ to $S M$ constant on every orbit of the geodesic flow, i.e.

$$\psi^*(x, \theta) = \psi(\gamma_{x, \theta}(\tau(x, \theta))).$$

Let $(M, g)$ be a simple metric, we assume, as we may, that $(M, g)$ extends smoothly into a simple manifold such that $M_1 \supseteq M$. Then there exist $C_1 > 0, C_2 > 0$ such that

$$C_1 \|f\|_{L^2(M)} \leq \|I^* I(f)\|_{H^1(M_1)} \leq C_2 \|f\|_{L^2(M)}$$

(3.7)

for any $f \in L^2(M)$, see Theorem 3 in [40]. If $V$ is an open set of the simple Riemannian manifold $(M_1, g)$, the normal operator $I^* I$ is an elliptic pseudodifferential operator of order $-1$ on $V$ whose principal symbol is a multiple of $|\xi|_g$ (see [32, 40]). Therefore there exists a constant $C_k > 0$ such that for all $f \in H^k(V)$ compactly supported in $V$

$$\|I^* I(f)\|_{H^{k+1}(M_1)} \leq C_k \|f\|_{H^k(V)}.$$ (3.8)

4 Geometrical optics solutions

We will now construct geometrical optics solutions of the wave equation. We extend the manifold $(M, g)$ into a simple manifold $M_2 \supseteq M$ and consider a simple manifold $(M_1, g)$ such that $M_2 \supseteq M_1$. The potentials $q_1, q_2$ may also be extended to $M_2$ and their $H^1(M_1)$ norms may be bounded by $M_0$. Since $q_1$ and $q_2$ coincide on the boundary, their extension outside $M$ can be taken the same so that $q_1 = q_2$ in $M_2 \setminus M_1$.

Let us assume for a moment that there exist a function $\psi \in C^2(M)$ which satisfies the eikonal equation

$$|\nabla_g \psi|^2_g = \sum_{i,j=1}^n g^{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} = 1, \quad \forall x \in M_2$$

(4.1)

and a function $a \in H^1(\mathbb{R}, H^2(M))$ which solves the transport equation

$$\frac{\partial a}{\partial t} + \sum_{j,k=1}^n g^{jk} \frac{\partial \psi}{\partial x_j} \frac{\partial a}{\partial x_k} + \frac{1}{2} (\Delta_g \psi)a = 0, \quad \forall t \in \mathbb{R}, x \in M$$

(4.2)
with initial or final data
\[ a(t, x) = 0, \quad \forall x \in M, \quad \text{and} \quad t \leq 0, \text{ or } t \geq T. \quad (4.3) \]

We also introduce the norm \( \| \cdot \|_* \) given by
\[ \| a \|_* = \| a \|_{H^1(0, T; H^2(M))} + \| a \|_{H^3(0, T; L^2(M))}. \quad (4.4) \]

**Lemma 4.1** Let \( q \in L^\infty(M) \), for any \( \lambda > 0 \), the equation
\[ (\partial^2_t - \Delta_g + q(x))u = 0, \quad \text{in} \quad M_T := (0, T) \times M, \]
\[ u(\kappa, x) = \partial_t u(\kappa, x) = 0, \quad \kappa = 0, \text{ or } T \]

has a solution of the form
\[ u(t, x) = a(t, x)e^{i\lambda(\psi(x)-t)} + v_\lambda(t, x), \quad (4.5) \]
such that
\[ u \in C^1(0, T; L^2(M)) \cap C(0, T; H^1(M)), \quad (4.6) \]
and where \( v_\lambda(t, x) \) satisfies
\[ v_\lambda(t, x) = 0, \quad \forall (t, x) \in (0, T) \times \partial M, \]
\[ v_\lambda(\kappa, x) = 0, \quad \partial_t v_\lambda(\kappa, x) = 0 \quad x \in M, \quad \kappa = 0 \text{ or } T \]

and
\[ \lambda \| v_\lambda(t, \cdot) \|_{L^2(M)} + \| \partial_t v_\lambda(t, \cdot) \|_{L^2(M)} + \| \nabla v_\lambda(t, \cdot) \|_{L^2(M)} \leq C \| a \|_* . \quad (4.7) \]

The constant \( C \) depends only on \( T \) and \( M \) (that is \( C \) does not depend on \( a \) and \( \lambda \)).

**Proof.** We set
\[ k(t, x) = - (\partial^2_t - \Delta_g + q) \left( a(t, x)e^{i\lambda(\psi-t)} \right), \quad (t, x) \in (0, T) \times M. \quad (4.8) \]

To prove our Lemma it would be enough to show that if \( v \) solves
\[ \begin{aligned}
(\partial^2_t - \Delta_g + q) v(t, x) &= k(t, x) \quad \text{in} \quad (0, T) \times M, \\
v(\kappa, x) &= 0, \quad \partial_t v(\kappa, x) = 0 \quad \text{in} \quad M, \tau = 0, \text{ or } T \\
v(t, x) &= 0 \quad \text{on} \quad (0, T) \times \partial M,
\end{aligned} \quad (4.9) \]
then the estimates (4.7) holds. We shall prove the estimate for \( \kappa = 0 \), and the \( \kappa = T \) case may be handled in a similar way. We have

\[
-k(t, x) = e^{i\lambda(\psi(x)-t)} \left( \partial_t^2 - \Delta_g + q(x) \right) (a(t, x))  \\
+2i\lambda e^{i\lambda(\psi(x)-t)} \left( \partial_t a + \sum_{j,k=1}^n g^{jk} \frac{\partial \psi}{\partial x_j} \frac{\partial a}{\partial x_k} + \frac{a}{2} \Delta_g \psi \right)  \\
+\lambda^2 a(t, x) e^{i\lambda(\psi(x)-t)} \left( 1 - \sum_{j,k=1}^n g^{jk} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_k} \right). \tag{4.10}
\]

Taking into account (4.1) and (4.2), the right-hand side of (4.10) becomes

\[
k(t, x) = -e^{i\lambda(\psi(x)-t)} \left( \partial_t^2 - \Delta_g + q \right) (a(t, x)) \equiv -e^{i\lambda(\psi(x)-t)} k_0(t, x). \tag{4.11}
\]

where \( k_0 \in H^1(0, T; L^2(M)) \) and satisfies

\[
\|k_0\|_{L^2((0,T) \times M)} + \|\partial_t k_0\|_{L^2((0,T) \times M)} \leq C \|a\|_*.
\]

Since the coefficient \( q \) does not depend on \( t \), the function

\[
w_\lambda(t, x) = \int_0^t v_\lambda(s, x) \, ds
\]

solves the mixed hyperbolic problem (4.9) with right-hand side

\[
k_1(t, x) = \int_0^t k(s, x) ds = \frac{1}{i\lambda} \int_0^t k_0(s, x) \partial_s \left( e^{i\lambda(\psi-s)} \right) ds.
\]

Integrating by parts with respect to \( s \), we conclude that

\[
\|k_1\|_{L^2((0,T) \times M)} \leq \frac{C}{\lambda} \|a\|_*.
\]

By Lemma 1.1 we find

\[
v_\lambda \in C^1(0, T; L^2(M)) \cap C(0, T; H^1_0(M)) \tag{4.12}
\]

and

\[
\|v_\lambda(t, \cdot)\|_{L^2(M)} = \|\partial_t w_\lambda(t, \cdot)\|_{L^2(M)} \leq \frac{C}{\lambda} \|a\|_*.
\tag{4.13}
\]

Since \( \|k\|_{L^2((0,T) \times M)} \leq C \|a\|_* \), using again the energy estimates for the problem (4.9), we obtain

\[
\|\partial_t v_\lambda(t, \cdot)\|_{L^2(M)} + \|\nabla v_\lambda(t, \cdot)\|_{L^2(M)} \leq C \|a\|_*.
\tag{4.14}
\]

The proof is complete. \( \square \)
Remark 1 In the construction of geometrical optics solutions, it is not necessary to assume that the potential is time independent. In the case where the potential $q$ is also time dependent, one can proceed along the following lines. With the same notations, $w_{\lambda}$ satisfies the equation

$$(\partial_t^2 - \Delta_g + q)w_{\lambda} = k_1 + \int_0^t (q(t, x) - q(s, x))v_{\lambda}(s, x) \, ds.$$ 

If one uses Lemma 1.1 on the interval $[0, \tau]$ one gets

$$\|\partial_t w_{\lambda}(\tau, \cdot)\|_{L^2(M)} + \|\nabla_g w_{\lambda}(\tau, \cdot)\|_{L^2(M)} \leq \frac{C}{\lambda} \|a\|_* + C\sqrt{T}\|q\|_{L^\infty} \int_0^\tau \|\partial_t w(s, \cdot)\|_{L^2(M)} \, ds$$

and Gronwall’s inequality allows to conclude

$$\|v_{\lambda}(\tau, \cdot)\|_{L^2(M)} = \|\partial_t w_{\lambda}(\tau, \cdot)\|_{L^2(M)} \leq \frac{C}{\lambda} \|a\|_* \left(1 + T \exp \left(CT^{3/2}\|q\|_{L^\infty}\right) \right).$$

We now proceed to construct a phase function $\psi$ solution to the eikonal equation (4.1) and an amplitude function $a$ solution to the transport equation (4.2).

Let $y \in \partial M_1$. Denote points of $M_1$ by $(r, \theta)$ where $(r, \theta)$ are polar normal coordinates in $M_1$ with center $y$. That is $x = \exp_y(r\theta)$ where $r > 0$ and

$$\theta \in S_y M_1 = \{\xi \in T_y M_1, \|\xi\|_g = 1\}.$$ 

In these coordinates (which depend on the choice of $y$) the metric takes the form

$$\bar{g}(r, \theta) = dr^2 + g_0(r, \theta)$$

where $g_0(r, \theta)$ is a smooth positive definite metric on $S_y M_1$. For any function $u$ compactly supported in $M$, we set for $r > 0$ and $\theta \in S_y M_1$

$$\tilde{u}(r, \theta) = u(\exp_y(r\theta))$$

where we have extended $u$ by 0 outside $M$. To solve the eikonal equation (4.1) it is enough to take

$$\psi(x) = d_g(x, y). \quad (4.15)$$

Then by the simplicity assumption, since $y \in M_2 \backslash \overline{M}$, we have $\psi \in C^\infty(M)$ and

$$\tilde{\psi}(r, \theta) = r = d_g(x, y). \quad (4.16)$$
We now proceed to the transport equation (4.2). Recall that if \( f(r) \) is any function of the geodesic distance \( r \), then

\[
\Delta_{\tilde{g}} f(r) = f''(r) + \frac{\alpha^{-1}}{2} \frac{\partial \alpha}{\partial r} f'(r)
\]  

(4.17)

where \( \alpha = \alpha(r, \theta) \) denotes the square of the volume element in geodesic polar coordinates. The transport equation (4.2) becomes

\[
\frac{\partial \tilde{a}}{\partial t} + \frac{\partial \tilde{\psi}}{\partial r} \frac{\partial \tilde{a}}{\partial r} + \frac{1}{4} \tilde{a} \alpha^{-1} \frac{\partial \alpha}{\partial r} \frac{\partial \tilde{\psi}}{\partial r} = 0.
\]  

(4.18)

Thus \( \tilde{a} \) satisfy

\[
\frac{\partial \tilde{a}}{\partial t} + \frac{\partial \tilde{a}}{\partial r} + \frac{1}{4} \tilde{a} \alpha^{-1} \frac{\partial \alpha}{\partial r} = 0.
\]  

(4.19)

Let \( \phi \in \mathcal{C}_0^\infty(\mathbb{R}) \) and \( b \in H^2(\partial_+ S\mathcal{M}) \), we choose \( \tilde{a} \) of the form

\[
\tilde{a}(t, r, \theta) = \alpha^{-1/4} \phi(t - r) b(y, \theta).
\]  

(4.20)

A simple calculation shows that

\[
\frac{\partial \tilde{a}}{\partial t}(t, r, \theta) = \alpha^{-1/4} \phi'(t - r) b(y, \theta).
\]  

(4.21)

and

\[
\frac{\partial \tilde{a}}{\partial r}(t, r, \theta) = \frac{1}{4} \alpha^{-5/4} \frac{\partial \alpha}{\partial r} \phi(t - r) b(y, \theta) - \alpha^{-1/4} \phi'(t - r) b(y, \theta).
\]  

(4.22)

Finally, (4.22) and (4.21) yield

\[
\frac{\partial \tilde{a}}{\partial t}(t, r, \theta) + \frac{\partial \tilde{a}}{\partial r}(t, r, \theta) = -\frac{1}{4} \alpha^{-1} \tilde{a}(t, r, \theta) \frac{\partial \alpha}{\partial r}.
\]  

(4.23)

If we assume that \( \text{supp} \phi \subset (0, \varepsilon_0) \), with \( \varepsilon_0 > 0 \) small enough so that

\[
T > \text{Diam}_g(\mathcal{M}) + 4\varepsilon_0,
\]  

(4.24)

then for any \( x = \exp_y(r\theta) \in \mathcal{M} \), it is easy to see that \( \tilde{a}(t, r, \theta) = 0 \) if \( t \leq 0 \) and \( t \geq T \).

**Remark 2** If \( T > \text{Diam}_g(\mathcal{M}) + 4\varepsilon_0 \) and \( c_g \) is \( \varepsilon \)-close to \( g \), then we also have \( T > \text{Diam}_{c_g}(\mathcal{M}) + 3\varepsilon_0 \).

18
5 Stability estimate for the electric potential

In this section, we complete the proof of Theorem 1. We are going to use the geometrical optics solutions constructed in the previous section; this will provide information on the geodesic ray transform of the difference of electric potentials.

5.1 Preliminary estimates

The main purpose of this section is to present a preliminary estimate, which relates the difference of the potentials to the Dirichlet-to-Neumann map. As before, we let \( q_1, q_2 \in \mathcal{D}(M_0) \) be real valued potentials. We set

\[
q = (q_1 - q_2) \in H^1_0(M).
\]

Recall that we have extended \( q_1, q_2 \) as \( H^1(M_2) \) in such a way that \( q = 0 \) on \( M_2 \setminus M \).

**Lemma 5.1** There exists \( C > 0 \) such that for any \( a_1, a_2 \in H^1(\mathbb{R}, H^2(M)) \) satisfying the transport equation (4.2) with initial data (4.3) the following estimate holds true:

\[
\left| \int_0^T \int_M q(x) a_1(t, x) \bar{a}_2(t, x) \, dv^n_g \, dt \right| \leq C \left( \lambda^{-1} + \lambda^3 \right) \left\| \Lambda_g, q_1 - \Lambda_g, q_2 \right\| \left\| a_1 \right\| \left\| a_2 \right\| \quad (5.1)
\]

for any sufficiently large \( \lambda > 0 \).

**Proof.** First, if \( a_2 \) satisfies (4.2), (4.3) and \( \lambda \) is sufficiently large, Lemma 4.1 guarantees the existence of the geometrical optics solutions \( u_2 \)

\[
u_2(t, x) = a_2(t, x) e^{i \lambda (\psi(x) - t)} + v_{2,\lambda}(t, x),
\]

to the equation with the electric potential \( q_2 \)

\[
\left( \partial_t^2 - \Delta_g + q_2(x) \right) u(t, x) = 0 \quad \text{in } (0, T) \times M,
\]

\[
u(0, \cdot) = 0, \quad \partial_t u(0, \cdot) = 0 \quad \text{in } M
\]

where \( v_{2,\lambda} \) satisfies

\[
\lambda \left\| v_{2,\lambda}(t, \cdot) \right\|_{L^2(M)} + \left\| \nabla v_{2,\lambda}(t, \cdot) \right\|_{L^2(M)} \leq C \left\| a_2 \right\|
\]

\[
v_{2,\lambda}(t, x) = 0, \quad \forall (t, x) \in (0, T) \times \partial M,
\]

(5.3)
and 
\[ u_2 \in C^1(0, T; L^2(\mathcal{M})) \cap C(0, T; H^1(\mathcal{M})). \]

Let us denote by \( f_\lambda \) the function
\[ f_\lambda(t, x) = a_2(t, x)e^{i\lambda(\psi(x) - t)}, \quad t \in (0, T), \quad x \in \partial \mathcal{M}. \]

Let \( v \) denote the solution of the following initial boundary value problem
\[
\begin{cases}
(\partial_t^2 - \Delta_g + q_1) v = 0, & (t, x) \in (0, T) \times \mathcal{M}, \\
v(0, x) = 0, & \partial_t v(0, x) = 0, \quad x \in \mathcal{M}, \\
v(t, x) = u_2(t, x) := f_\lambda(t, x), & (t, x) \in (0, T) \times \partial \mathcal{M}.
\end{cases}
\tag{5.4}
\]

Taking \( w = v - u_2 \), one gets
\[
\begin{cases}
(\partial_t^2 - \Delta_g + q_1(x)) w(t, x) = q(x) u_2(t, x), & (t, x) \in (0, T) \times \mathcal{M}, \\
w(0, x) = 0, & \partial_t w(0, x) = 0, \quad x \in \mathcal{M}, \\
w(t, x) = 0, & (t, x) \in (0, T) \times \partial \mathcal{M}.
\end{cases}
\]

Since \( q(x) u_2 \in L^1(0, T; L^2(\mathcal{M})) \) by Lemma 1.1, we deduce that
\[ w \in C^1(0, T; L^2(\mathcal{M})) \cap C(0, T; H^1(\mathcal{M})). \]

Therefore, we have constructed a particular solution \( u_1 \in C^1(0, T; L^2(\mathcal{M})) \cap C(0, T; H^1(\mathcal{M})) \) to the backward wave equation
\[
(\partial_t^2 - \Delta_g + q_1(x)) u_1(t, x) = 0, \quad (t, x) \in (0, T) \times \mathcal{M},
\]
\[ u_1(T, x) = 0, \quad \partial_t u_1(T, x) = 0 \quad x \in \mathcal{M}, \]

having the form
\[ u_1(t, x) = a_1(t, x)e^{i\lambda(\psi(x) - t)} + v_{1, \lambda}(t, x), \tag{5.5} \]

corresponding to the electric potential \( q_1 \), with
\[ \lambda \| v_{1, \lambda}(t, \cdot) \|_{L^2(\mathcal{M})} + \| \nabla v_{1, \lambda}(t, \cdot) \|_{L^2(\mathcal{M})} \leq C \| a_1 \|_\ast. \tag{5.6} \]

Integrating by parts and using Green’s formula (2.4), we find
\[
\int_0^T \int_{\mathcal{M}} (\partial_t^2 - \Delta_g + q_1(x)) w \overline{u_1} \, dv_g^n \, dt
\]
\[ = \int_0^T \int_{\mathcal{M}} q(x) u_2 \overline{u_1} \, dv_g^n \, dt = - \int_0^T \int_{\partial \mathcal{M}} \partial_n w \overline{u_1} \, d\sigma_g^{n-1} \, dt. \tag{5.7} \]
Combining (5.7) with (5.4), we deduce

\[
\int_0^T \int_M qu_2 \overline{u}_1 dv^n \, dt = -\int_0^T \int_{\partial M} (\Lambda_{g, q_1} - \Lambda_{g, q_2}) (f_\lambda) g_{\lambda} \, d\sigma_{g}^{n-1} \, dt \quad (5.8)
\]

where

\[
g_{\lambda}(t, x) = a_1(t, x) e^{i\lambda(\psi(x) - t)}, \quad (t, x) \in (0, T) \times \partial M.
\]

It follows from (5.8), (5.5) and (5.2) that

\[
\int_0^T \int_M q(x) a_2(t, x) \overline{u}_1(t, x) dv^n \, dt =
\]

\[
\quad -\int_0^T \int_{\partial M} (\Lambda_{g, q_1} - \Lambda_{g, q_2}) (f_\lambda)(t, x) \overline{g}_{\lambda}(t, x) \, d\sigma_{g}^{n-1} \, dt
\]

\[
\quad -\int_0^T \int_M q(x) a_2(t, x) \overline{u}_1(t, x) e^{i\lambda(\psi - t)} \, dv^n \, dt
\]

\[
\quad -\int_0^T \int_M q(x) v_2,\lambda(t, x) \overline{u}_1(t, x) e^{-i\lambda(\psi - t)} \, dv^n \, dt
\]

\[
\quad -\int_0^T \int_M q(x) v_2,\lambda(t, x) \overline{u}_1(t, x) \, dv^n \, dt. \quad (5.9)
\]

In view of (5.6) and (5.3), we have

\[
\left| \int_0^T \int_M q(x) a_2(t, x) e^{i\lambda(\psi - t)} \, dv^n \, dt \right| \leq C \int_0^T \|a_2(t, \cdot)\|_{L^2(M)} \|v_{1,\lambda}(t, \cdot)\|_{L^2(M)} \, dt \\
\leq C \lambda^{-1} \|a_2\|_\ast \|a_1\|_\ast. \quad (5.10)
\]

Similarly, we deduce

\[
\left| \int_0^T \int_M q(x) \overline{u}_1(t, x) v_{2,\lambda}(t, x) e^{-i\lambda(\psi - t)} \, dv^n \, dt \right| \leq C \lambda^{-1} \|a_1\|_\ast \|a_2\|_\ast.
\]

Moreover we have

\[
\left| \int_0^T \int_M q(x) v_{2,\lambda}(t, x) \overline{u}_1(t, x) \, dv^n \, dt \right| \leq C \lambda^{-2} \|a_1\|_\ast \|a_2\|_\ast.
\]

On the other hand, by the trace theorem, we find

\[
\left| \int_0^T \int_{\partial M} (\Lambda_{g, q_1} - \Lambda_{g, q_2}) (f_\lambda) g_{\lambda} \, d\sigma_{g}^{n-1} \, dt \right|
\]

\[
\leq \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|_{H^1((0, T) \times \partial M)} \|f_\lambda\|_{L^2((0, T) \times \partial M)} \|g_{\lambda}\|_{L^2((0, T) \times \partial M)}
\]

\[
\leq C \lambda^3 \|a_1\|_\ast \|a_2\|_\ast \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|. \quad (5.11)
\]
Inequality (5.1) follows easily from (5.9), (5.10), (5.11), (5.1) and (5.1). This completes the proof of the Lemma.

□

Lemma 5.2 There exist \( C > 0, \beta \in (0, 1) \) such that for any \( b \in H^2(\partial_+ SM_1) \), the following estimate

\[
\left| \int_{S_y M_1} \int_0^{\tau_+(y, \theta)} \bar{q}(s, \theta)b(y, \theta)\mu(y, \theta) \, ds \, d\omega_y(\theta) \right| \\
\leq C \| \Lambda_{g, q_1} - \Lambda_{g, q_2} \| \| b(y, \cdot) \|_{H^2(S_y^+ M_1)} \tag{5.12}
\]

holds for any \( y \in \partial M_1 \).

Here we used the notation

\[ S_y^+ M_1 = \{ \theta \in S_y M_1 : \langle \nu, \theta \rangle < 0 \} \].

We recall that \( \mu \) denotes the density \( -\langle \theta, \nu(y) \rangle g \).

Proof. Following (4.20), we take two solutions to (4.2) and (4.3) of the form

\[
\bar{a}_1(t, r, \theta) = \alpha^{-1/4} \phi(t - r)b(y, \theta), \\
\bar{a}_2(t, r, \theta) = \alpha^{-1/4} \phi(t - r)\mu(y, \theta).
\]

Now we change variables in (5.1), \( x = \exp_y(r\theta), r > 0 \) and \( \theta \in S_y M_1 \), we have

\[
\int_0^T \int_M q(x)a_1(t, x)a_2(t, x) \, dv_g^n \, dt \\
= \int_0^T \int_{S_y M_1} \int_0^{\tau_+(y, \theta)} \bar{q}(r, \theta)\bar{a}_1(t, r, \theta)\bar{a}_2(t, r, \theta) \alpha^{1/2} \, dr \, d\omega_y(\theta) \, dt \\
= \int_0^T \int_{S_y M_1} \int_0^{\tau_+(y, \theta)} \bar{q}(r, \theta)\phi^2(t - r)b(y, \theta)\mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt.
\]

By virtue of Lemma 5.1, we conclude that

\[
\left| \int_0^\infty \int_{S_y M_1} \int_0^{\tau_+(y, \theta)} \bar{q}(r, \theta)\phi^2(t - r)b(y, \theta)\mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \right| \\
\leq C \left( \lambda^{-1} + \lambda^3 \right) \| \Lambda_{g, q_1} - \Lambda_{g, q_2} \| \| \phi \|_{H^2(\mathbb{R})} \| b(y, \cdot) \|_{H^2(S_y^+ M_1)} \tag{5.13}
\]
Since \( \phi(t) = 0 \) for \( t \leq 0 \) or \( t \geq T \), we get

\[
\int_0^\infty \int_{S_yM_1} \int_0^{\tau(y,\theta)} \bar{q}(r,\theta) \phi^2(t-r) b(y,\theta) \mu(y,\theta) \, dr \, d\omega_y(\theta) \, dt
= \left( \int_{-\infty}^\infty \phi^2(t) \, dt \right) \times \int_{S_yM_1} \int_0^{\tau(y,\theta)} \bar{q}(r,\theta) b(y,\theta) \mu(y,\theta) \, dr \, d\omega_y(\theta).
\] (5.14)

Combining (5.13) and (5.14), it follows that

\[
\left| \int_{S_yM_1} \int_0^{\tau(y,\theta)} \bar{q}(s,\theta) b(y,\theta) \mu(y,\theta) \, ds \, d\omega_y(\theta) \right|
\leq C \left( \frac{1}{\lambda} + \lambda^3 \| \Lambda_{g,q_1} - \Lambda_{g,q_2} \| \right) \| b(y,\cdot) \|_{H^2(S_y^+,M_1)}. \]

Finally, minimizing in \( \lambda \) we obtain

\[
\left| \int_{S_yM_1} \int_0^{\tau(y,\theta)} \bar{q}(s,\theta) b(y,\theta) \mu(y,\theta) \, ds \, d\omega_y(\theta) \right|
\leq C \| \Lambda_{g,q_1} - \Lambda_{g,q_2} \|^\beta \| b(y,\cdot) \|_{H^2(S_y^+,M_1)}. \]

This completes the proof of the lemma. \( \square \)

### 5.2 End of the proof of the stability estimate

Let us now complete the proof of the stability estimate in Theorem 1. Using Lemma 5.2 for any \( y \in \partial M_1 \) and \( b \in H^2(\partial_+SM) \) we have

\[
\left| \int_{S_yM_1} I(q)(y,\theta) b(y,\theta) \mu(y,\theta) \, d\omega_y(\theta) \right|
\leq C \| \Lambda_{g,q_1} - \Lambda_{g,q_2} \|^\beta \| b(y,\cdot) \|_{H^2(S_y^+,M_1)}. \]

Integrating with respect to \( y \in \partial M_1 \) we obtain

\[
\left| \int_{\partial_+SM_1} I(q)(y,\theta) b(y,\theta) \langle \theta, \nu(y) \rangle \, d\sigma_g^{2n-2}(y,\theta) \right|
\leq C \| \Lambda_{g,q_1} - \Lambda_{g,q_2} \|^\beta \| b \|_{H^2(\partial_+SM_1)}. \] (5.15)
Now we choose
\[ b(y, \theta) = \mathcal{I} (\mathcal{I}^* \mathcal{I}(q))(y, \theta) . \]

Taking into account (3.8) and (3.4), we obtain
\[ \| \mathcal{I}^* \mathcal{I}(q) \|_{L^2(M_1)}^2 \leq C \| \Lambda_{g, q_1} - \Lambda_{g, q_2} \|_\beta \| q \|_{H^1(M)} . \]

By interpolation, it follows that
\[ \| \mathcal{I}^* \mathcal{I}(q) \|_{H^1(M_1)}^2 \leq C \| \mathcal{I}^* \mathcal{I}(q) \|_{H^2(M_1)} \| q \|_{H^1(M)} \]
\[ \leq C \| \mathcal{I}^* \mathcal{I}(q) \|_{L^2(M_1)} \| q \|_{H^1(M)} \]
\[ \leq C \| \Lambda_{g, q_1} - \Lambda_{g, q_2} \|_\beta/2 . \] (5.16)

Using (3.7), we deduce that
\[ \| q \|_{L^2(M)}^2 \leq C \| \Lambda_{g, q_1} - \Lambda_{g, q_2} \|_\beta/2 . \]

This completes the proof of Theorem 1.

**Remark 3** In the proof of Theorem 1 we have used the time independence of the potential at two stages:

1. In the construction of the remainder term \( \psi(t, x) \) in the proof of Lemma 4.1. But as was noted in Remark 2 this restriction may be bypassed.

2. In equation (5.14) to get rid of the \( \phi^2(t-r) \) term and obtain the ray transform. The adaptation to the time dependent case does not seem to be straightforward.

We leave the case of a time dependent potential as an open problem.

### 6 Stability estimate for the conformal factor

We shall use the following notations. Let \( c \in \mathcal{C}(M_0, k, \varepsilon) \), we denote
\[ \psi_0(x) = 1 - c(x), \quad \psi_1(x) = c^{\alpha/2}(x) - 1, \quad \psi_2(x) = c^{\alpha/2-1}(x) - 1, \]
\[ \phi(x) = \psi_1(x) - \psi_2(x) = c^{\alpha/2-1}(x) (c(x) - 1) . \] (6.1)

Then the following holds
\[ \| \psi_j \|_{C^1(M)} \leq C \| \psi_0 \|_{C^1(M)} , \quad j = 1, 2. \] (6.2)
\[ C^{-1} \| \psi_0 \|_{L^2(M)} \leq \| \phi \|_{L^2(M)} \leq C \| \psi_0 \|_{L^2(M)} . \] (6.3)

The first step in our analysis is the following result.
Lemma 6.1 Let $c \in C^\infty(M)$ be such that $c = 1$ near the boundary $\partial M$. Let $u_1, u_2$ solve the following problem in $(0, T) \times M$ with some $T > 0$

$$\begin{cases}
(\partial_t^2 - \Delta_g)u_1 = 0, & \text{in } (0, T) \times M, \\
u_1(0, \cdot) = \partial_t u(0, \cdot) = 0 & \text{in } M, \\
u_1 = f_1, & \text{on } (0, T) \times \partial M,
\end{cases}(6.4)$$

$$\begin{cases}
(\partial_t^2 - \Delta_{cg})u_2 = 0, & \text{in } (0, T) \times M, \\
u_2(T, \cdot) = \partial_t u(T, \cdot) = 0 & \text{in } M, \\
u_2 = f_2, & \text{on } (0, T) \times \partial M,
\end{cases}(6.5)$$

where $f_k \in H^1((0, T) \times \partial M), k = 1, 2$. Then the following identity

$$\int_0^T \int_{\partial M} (\Lambda_g - \Lambda_{cg}) f_1 \overline{f_2} \, d\sigma_{g}^{n-1} dt = \int_0^T \int_M \varphi_1(x) \partial_t u_1 \partial_t \overline{u_2} \, dv_g^n dt$$

$$- \int_0^T \int_M \varphi_2(x) \langle \nabla_g u_1(t, x), \nabla_g \overline{u_2}(t, x) \rangle \, dv_g^n dt \quad (6.6)$$

holds true for any $f_j \in H^1((0, T) \times \partial M), j = 1, 2$.

**Proof.** We multiply both hand sides of the first equation (6.4) by $\overline{u_2}$; integrating by parts in time and using Green’s formula (2.3)-(2.4) in $(M, g)$, we obtain

$$0 = \int_0^T \int_M (\partial_t^2 u_1 - \Lambda_g u_1) \overline{u_2} \, dv_g^n dt$$

$$= -\int_0^T \int_M \partial_t u_1 \partial_t \overline{u_2} \, dv_g^n dt + \int_0^T \int_M \varphi_1(x) \partial_t u_1 \partial_t \overline{u_2} \, dv_g^n dt$$

$$+ \int_0^T \int_M \sum_{j,k=1}^n (cg)^{jk} \left( \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u_2}}{\partial x_k} \right) \, dv_g^n dt$$

$$- \int_0^T \int_M \varphi_2(x) \left( \sum_{j,k=1}^n g^{jk} \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u_2}}{\partial x_k} \right) \, dv_g^n dt - \int_0^T \int_{\partial M} \partial_{\nu_1} u_1 \overline{f_2} \, d\sigma_{g}^{n-1} dt.$$
Another integration by parts in time and an application of Green’s formula in 
\((\mathcal{M}, cg)\) yield

\[
0 = \int_0^T \int_{\mathcal{M}} u_1 \left( \partial_t^2 \overline{u}_2 - \Delta_{cg} \overline{u}_2 \right) \, dv_{cg} \, dt
\]

\[
+ \int_0^T \int_{\mathcal{M}} \varrho_1(x) \partial_t u_1 \partial_t \overline{u}_2 \, dv_g \, dt
- \int_0^T \int_{\mathcal{M}} \varrho_2(x) \left( \sum_{j,k=1}^n g^{jk} \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u}_2}{\partial x_k} \right) \, dv_g \, dt
\]

\[
+ \int_0^T \int_{\partial \mathcal{M}} \partial_n \overline{u}_2 f_1 \, d\sigma_{g}^{n-1} \, dt
- \int_0^T \int_{\partial \mathcal{M}} \partial_n u_1 f_2 \, d\sigma_{g}^{n-1} \, dt.
\]  

(6.7)

Taking into account the facts that \(\Lambda_{cg}\) is self-adjoint, that \(c = 1\) on \(\partial \mathcal{M}\) and \((\partial_t^2 \overline{u}_2 - \Delta_{cg} \overline{u}_2) = 0\) in \((0, T) \times \mathcal{M}\), it follows that

\[
\int_0^T \int_{\partial \mathcal{M}} (\Lambda_g - \Lambda_{cg}) f_1 \overline{f}_2 \, d\sigma_g^{n-1} \, dt =
\]

\[
\int_0^T \int_{\mathcal{M}} \varrho_1(x) \partial_t u_1 \partial_t \overline{u}_2 \, dv_g \, dt
- \int_0^T \int_{\mathcal{M}} \varrho_2(x) \left( \sum_{j,k=1}^n g^{jk} \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u}_2}{\partial x_k} \right) \, dv_g \, dt.
\]

(6.8)

This completes the proof of the Lemma.

\[\square\]

### 6.1 Modified geometrical optics solutions

As in the case of potentials, we extend the manifold \((\mathcal{M}, g)\) into a simple manifold \(\mathcal{M}_2 \supset \mathcal{M}\) so that \(\mathcal{M}_2 \supset \mathcal{M}_1 \supset \mathcal{M}\) with \((\mathcal{M}_1, g)\) simple. We extend the conformal factor \(c\) by 1 outside the manifold \(\mathcal{M}\); its \(C^k(\mathcal{M}_1)\) norms may also be bounded by \(M_0\). Let \(\psi_1, \psi_2\) be two phase functions solving the eikonal equation with respect respectively to the metrics \(g\) and \(cg\).

\[
|\nabla_g \psi_1|^2_g = \sum_{j,k=1}^n g^{jk} \frac{\partial \psi_1}{\partial x_j} \frac{\partial \psi_1}{\partial x_k} = 1, \quad |\nabla_{cg} \psi_2|^2_{cg} = \sum_{j,k=1}^n cg^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial \psi_2}{\partial x_k} = 1.
\]

(6.9)

Let \(a_2\) solve the transport equation in \(\mathbb{R} \times \mathcal{M}\) with respect the metric \(g\) (as given in section 4)

\[
\frac{\partial a_2}{\partial t} + \sum_{j,k=1}^n g^{jk} \frac{\partial \psi_1}{\partial x_j} \frac{\partial a_2}{\partial x_k} + \frac{a_2^2}{2} \Delta_g \psi_1 = 0.
\]

(6.10)
Let $a_3$ solve the following transport equation in $\mathbb{R} \times \mathcal{M}$ with respect to the metric $c_g$

$$\frac{\partial a_3}{\partial t} + \sum_{j,k=1}^{n} c_{g}^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial a_3}{\partial x_k} + \frac{a_3}{2} \Delta_{c_g} \psi_2 = -\frac{1}{2i} a_2(t, x)(1 - e^{-1}) e^{i\lambda(\psi_1 - \psi_2)} $$

$$\equiv a_2(t, x) \varphi_0(x, \lambda), \quad (6.11)$$

and be such that

$$\|a_3\|_* \leq C \varepsilon \lambda^2 \|a_2\|_* . \quad (6.12)$$

Let us explain the construction of a solution $a_3$ satisfying (6.11) and (6.12). To solve the transport equation (6.11) and (6.12) it is enough to take, in geodesic polar coordinates $(r, \theta)$ (with respect to the metric $c_g$)

$$\tilde{a}_3(t, r, \theta; \lambda) = \alpha_{c_g}^{-1/4}(r, \theta) \int_{0}^{r} \alpha_{c_g}(s, \theta) \tilde{a}_2(s + r - t, s, \theta) \varphi_0(s, \theta, \lambda) \, ds, \quad (6.13)$$

where $\alpha_{c_g}(r, \theta)$ denotes the square of the volume element in geodesic polar coordinates with respect to the metric $c_g$. Using that $\|\varphi_0(\cdot, \lambda)\|_{C^2(M)} \leq C \varepsilon \lambda^2$ and (6.13) we obtain (6.12).

**Lemma 6.2** Let $c \in \mathcal{C}(M_0, k, \varepsilon)$ be such that $c = 1$ near the boundary $\partial \mathcal{M}$. Then the equation

$$(\partial^2_t - \Delta_{c_g}) u = 0, \quad \text{in} \quad (0, T) \times \mathcal{M}, \quad u(0, x) = \partial_t u(0, x) = 0 \quad (6.14)$$

has a solution of the form

$$u_2(t, x) = \frac{1}{\lambda} a_2(t, x) e^{i\lambda(\psi_1(x) - t)} + a_3(t, x; \lambda) e^{i\lambda(\psi_2(x) - t)} + v_{2, \lambda}(t, x) \quad (6.15)$$

when $\lambda$ is large enough, which satisfies

$$\lambda \|v_{2, \lambda}(t, \cdot)\|_{L^2(M)} + \|\nabla g v_{2, \lambda}(t, \cdot)\|_{L^2(M)} + \|\partial_t v_{2, \lambda}(t, \cdot)\|_{L^2(M)} \leq C (\varepsilon \lambda^2 + \lambda^{-1}) \|a_2\|_*. \quad (6.16)$$

The constant $C$ depends only on $T$ and $\mathcal{M}$ (that is $C$ does not depend on $a$, $\lambda$ and $\varepsilon$).

**Proof.** We set

$$k(t, x) = - (\partial^2_t - \Delta_{c_g}) \left( \frac{1}{\lambda} a_2(t, x) e^{i\lambda(\psi_1 - t)} + a_3(t, x, \lambda) e^{i\lambda(\psi_2 - t)} \right). \quad (6.17)$$
To prove our Lemma it would be enough to show that if \( v \) solves
\[
(\partial^2_t - \Delta_{cg}) v = k(t, x)
\] (6.18)
with initial and boundary conditions
\[
v(0, x) = \partial_t v(0, x) = 0, \text{ in } \mathcal{M}, \quad \text{and } \quad v(t, x) = 0 \text{ on } (0, T) \times \partial \mathcal{M}
\] (6.19)
then the estimates (6.16) holds. We have
\[
-k(t, x) = \frac{1}{\lambda} e^{i\lambda(\psi_1 - t)} (\partial^2_t - \Delta_{cg}) a_2
+ 2ie^{i\lambda(\psi_1 - t)} \left( \partial_t a_2 + \sum_{j,k=1}^n c g^j k \frac{\partial \psi_1}{\partial x_j} \frac{\partial a_2}{\partial x_k} + \frac{a_2}{2} \Delta_{cg} \psi_1 \right)
+ \lambda a_2 e^{i\lambda(\psi_1 - t)} \left( 1 - c^{-1} \sum_{j,k=1}^n g^j k \frac{\partial \psi_1}{\partial x_j} \frac{\partial \psi_1}{\partial x_k} \right)
+ e^{i\lambda(\psi_2 - t)} (\partial^2_t - \Delta_{cg}) a_3
+ 2i\lambda e^{i\lambda(\psi_2 - t)} \left( \partial_t a_3 + \sum_{j,k=1}^n c g^j k \frac{\partial \psi_2}{\partial x_j} \frac{\partial a_3}{\partial x_k} + \frac{a_3}{2} \Delta_{cg} \psi_2 \right)
+ \lambda^2 a_3 e^{i\lambda(\psi_2 - t)} \left( 1 - \sum_{j,k=1}^n c g^j k \frac{\partial \psi_2}{\partial x_j} \frac{\partial \psi_2}{\partial x_k} \right).
\] (6.20)
Taking into account (6.9) and (6.10), the right-hand side of (6.20) becomes
\[
-k(t, x) = \frac{1}{\lambda} e^{i\lambda(\psi_1 - t)} (\partial^2_t - \Delta_{cg}) a_2
+ 2ie^{i\lambda(\psi_1 - t)} \left( (c^{-1} - 1) (\nabla_g \psi_1, \nabla_g a_2)_g + \frac{1}{2} a_2 (\Delta_{cg} \psi_1 - \Delta_g \psi_1) \right)
+ 2i\lambda e^{i\lambda(\psi_2 - t)} \left( \partial_t a_3 + \sum_{j,k=1}^n c g^j k \frac{\partial \psi_2}{\partial x_j} \frac{\partial a_3}{\partial x_k} + \frac{a_3}{2} \Delta_{cg} \psi_2 \right)
+ \frac{a_2}{2i} e^{i\lambda(\psi_1 - \psi_2)} (1 - c^{-1})
+ e^{i\lambda(\psi_2 - t)} (\partial^2_t - \Delta_{cg}) a_3.
\] (6.21)
By (6.11) we get
\[ -k(t, x) = \frac{1}{\lambda} e^{i \lambda (\psi_1 - t)} \left( \partial_t^2 - \Delta_{cg} \right) a_2 + 2i e^{i \lambda (\psi_1 - t)} \left( (e^{-1} - 1) \langle \nabla g \psi_1, \nabla_g a_2 \rangle_g + \frac{1}{2} a_2 (\Delta_{cg} \psi_1 - \Delta_g \psi_1) \right) \\
+ e^{i \lambda (\psi_2 - t)} \left( \partial_t^2 - \Delta_{cg} \right) a_3 \]
\[ \equiv \frac{1}{\lambda} e^{i \lambda (\psi_1 - t)} k_0 + e^{i \lambda (\psi_1 - \lambda t)} k_1 + e^{i \lambda (\psi_2 - t)} k_2. \]  
(6.22)

Since \( k_j \in L^1(0, T; L^2(\mathcal{M})) \), by Lemma 1.1 we deduce that
\[ v_\lambda \in C^1(0, T; L^2(\mathcal{M})) \cap C(0, T; H^1_0(\mathcal{M})) \]  
(6.23)

and
\[
\|v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} \\
\leq \frac{C}{\lambda} \left\{ \int_\mathbb{R} \left( \frac{1}{\lambda} \| k_0(s, \cdot) \|_{L^2(\mathcal{M})} + \| k_1(s, \cdot) \|_{L^2(\mathcal{M})} + \| k_2(s, \cdot) \|_{L^2(\mathcal{M})} \right) ds \right\} \\
+ \int_\mathbb{R} \left( \frac{1}{\lambda} \| \partial_t k_0(s, \cdot) \|_{L^2(\mathcal{M})} + \| \partial_t k_1(s, \cdot) \|_{L^2(\mathcal{M})} + \| \partial_t k_2(s, \cdot) \|_{L^2(\mathcal{M})} \right) ds \right\} \\
\leq \frac{C}{\lambda} \left( \frac{1}{\lambda} \| a_2 \|_* + \epsilon \| a_2 \|_* + \epsilon \lambda^2 \| a_2 \|_* \right) \\
\leq C \left( \epsilon \lambda + \frac{1}{\lambda^2} \right) \| a_2 \|_* . \]  
(6.24)

Moreover, we have
\[
\|k\|_{L^2((0, T) \times \mathcal{M})} \leq C \left( \frac{1}{\lambda} \| a_2 \|_* + \epsilon \| a_2 \|_* + \epsilon \lambda^2 \| a_2 \|_* \right) \]  
(6.25)

and by using again the energy estimates for the problem (6.18)-(6.19), we obtain
\[
\| \nabla v_\lambda(t, \cdot) \|_{L^2(\mathcal{M})} + \| \partial_t v_\lambda(t, \cdot) \|_{L^2(\mathcal{M})} \leq C \left( \epsilon \lambda^2 + \frac{1}{\lambda} \right) \| a_2 \|_* . \]  
(6.26)

This ends the proof of Lemma 6.2 \( \square \)

**Lemma 6.3** There exists \( C > 0 \) such that for any \( a_1, a_2 \in H^1(\mathbb{R}, H^2(\mathcal{M})) \) satisfying the transport equation (6.10) with (4.3) the following estimate holds true
\[
\left| \int_0^T \int_{\mathcal{M}} g(x)(a_1 \tilde{a}_2)(t, x) \, dv_g^0 \, dt \right| \leq \|g_0\|_{C(\mathcal{M})} \left( \lambda^{-1} + \epsilon \lambda^3 \right) \| a_1 \|_* \| a_2 \|_* \\
+ \lambda^3 \| a_1 \|_* \| a_2 \|_* \| \Lambda_g - \Lambda_{cg} \| \]  
(6.27)

for any sufficiently large \( \lambda \).
Thus, we have from (6.12), (6.16) and (4.4) the following identity
\[
\mathcal{J}_1(\lambda, \varepsilon) = \int_0^T \mathcal{J}_1(\lambda, \varepsilon)(t, x) \, dv_g^n \, dt
\]
where \(v_{2, \lambda}\) satisfies (6.16) and \(a_3\) satisfies (6.12). Thanks to Lemma 4.1 let \(u_1\) be a solution to the \((\partial_t^2 - \Delta_g)u = 0\) of the form
\[
u_{1, \lambda} = u_1(t, x) = a_1(t, x) e^{i\lambda(\psi_1 - t)} + v_{1, \lambda}(t, x),
\]
where \(v_{1, \lambda}\) satisfies (4.4). Then \(\partial_t \nu_2(t, x)\) is given by
\[
\partial_t \nu_2(t, x) = \frac{1}{\lambda} \partial_t \nu_2(t, x) e^{-i\lambda(\psi_1 - t)} + i \nu_2(t, x) e^{-i\lambda(\psi_1 - t)}
+ \partial_t \nu_3(t, x; \lambda) e^{-i\lambda(\psi_2 - t)} + i \nu_3(t, x, \lambda) e^{-i\lambda(\psi_2 - t)} + \partial_t \nu_{2, \lambda}(t, x).
\]
Let us compute the first term in the right hand side of (6.6). We have
\[
\int_0^T \int_M q_1 \partial_t u_1 \partial_t \nu_2 \, dv_g^n \, dt = \lambda \int_0^T \int_M q_1 a_1 \nu_2 \, dv_g^n \, dt
+ \frac{1}{\lambda} \int_0^T \int_M q_1 (\partial_t a_1 \partial_t \nu_2) \, dv_g^n \, dt - i \int_0^T \int_M q_1 a_1 \partial_t \nu_2 \, dv_g^n \, dt
+ i \int_0^T \int_M q_1 \partial_t \nu_2 \partial_t v_{1, \lambda} e^{-i\lambda(\psi_1 - t)} \, dv_g^n \, dt + i \int_0^T \int_M q_1 (\partial_t a_1 \nu_2) \, dv_g^n \, dt
- i \lambda \int_0^T \int_M q_1 \partial_t \nu_3 \partial_t a_1 e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt + \int_0^T \int_M q_1 \partial_t v_{1, \lambda} \partial_t \nu_3 e^{-i\lambda(\psi_2 - t)} \, dv_g^n \, dt
+ i \lambda \int_0^T \int_M q_1 \nu_3 \partial_t a_1 e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt + \lambda^2 \int_0^T \int_M q_1 a_1 \nu_3 e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt
+ i \lambda \int_0^T \int_M q_1 v_{1, \lambda} \nu_3 e^{-i\lambda(\psi_2 - t)} \, dv_g^n \, dt + \int_0^T \int_M q_1 \partial_t v_{1, \lambda} \partial_t \nu_3 e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt
- i \lambda \int_0^T \int_M q_1 a_1 \partial_t \nu_3 \partial_t \nu_{2, \lambda} e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt + \int_0^T \int_M q_1 \partial_t v_{1, \lambda} \partial_t \nu_{2, \lambda} e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt.
\]
Thus, we have from (6.12), (6.16) and (4.4) the following identity
\[
\int_0^T \int_M q_1 \partial_t u_1 \partial_t \nu_2 \, dv_g^n \, dt = \lambda \int_0^T \int_M q_1 (a_1 \nu_2)(t, x) \, dv_g^n \, dt + \mathcal{J}_1(\lambda, \varepsilon),
\]
where

$$|J_1(\lambda, \varepsilon)| \leq \|\theta_0\|_{C(M)} (1 + \varepsilon \lambda^4) \|a_2\|_* \|a_1\|_*.$$  \hfill (6.31)

On the other hand, we have

$$\nabla_g u_1 = \nabla_g a_1 e^{i\lambda(\psi_1 - t)} + i\lambda \nabla_g \psi_1 a_1 e^{i\lambda(\psi_1 - t)} + \nabla_g v_{1,\lambda}$$

$$\nabla_g \Omega_2 = \frac{1}{\lambda} \nabla_g a_2 e^{-i\lambda(\psi_2 - t)} - i\lambda a_2 \nabla_g a_1 e^{-i\lambda(\psi_1 - t)}$$

$$- i\lambda a_3 \nabla_g \psi_2 e^{-i\lambda(\psi_2 - t)} - \nabla_g a_2 e^{-i\lambda(\psi_2 - t)} + \nabla_g v_{2,\lambda}.$$  \hfill (6.32)

and the second term in the right side of (6.6) becomes

$$\int_0^T \int_M \varphi_2(x) \langle \nabla_g u_1(t, x), \nabla_g \Omega_2(t, x) \rangle_g \, dv^n_g \, dt$$

$$= \lambda \int_0^T \int_M \varphi_2(x) (a_1 \Omega_2)(t, x) \, dv^n_g \, dt + J_2(\lambda, \varepsilon) + J_3(\lambda, \varepsilon)$$  \hfill (6.33)

with

$$J_2(\lambda, \varepsilon) = + \frac{1}{\lambda} \int_0^T \int_M \varphi_2 \langle \nabla_g a_1, \nabla_g \Omega_2 \rangle_g \, dv^n_g \, dt$$

$$- i \int_0^T \int_M \varphi_2 \langle \nabla_g a_1, \nabla_g \psi_1(x) \rangle_g \, dv^n_g \, dt$$

$$+ i \int_0^T \int_M \varphi_2 a_1 \langle \nabla_g \Omega_2, \nabla_g \psi_1 \rangle_g \, dv^n_g \, dt$$

$$+ \frac{1}{\lambda} \int_0^T \int_M \varphi_2 e^{-i\lambda(\psi_2 - t)} \langle \nabla_g \Omega_2, \nabla_g v_{1,\lambda} \rangle_g \, dv^n_g \, dt$$

$$- i \int_0^T \int_M \varphi_2 \Omega_2 e^{-i\lambda(\psi_2 - t)} \langle \nabla_g v_{1,\lambda}, \nabla_g \psi_1 \rangle_g \, dv^n_g \, dt$$

and

$$J_3(\lambda, \varepsilon) = - i\lambda \int_0^T \int_M \varphi_2 a_3 e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g a_1, \nabla_g \psi_2 \rangle_g \, dv^n_g \, dt$$

$$+ \int_0^T \int_M \varphi_2 e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g a_1, \nabla_g \Omega_3 \rangle_g \, dv^n_g \, dt$$

$$+ \int_0^T \int_M \varphi_2 e^{i\lambda(\psi_1 - t)} \langle \nabla_g a_1, \nabla_g \Omega_2 \rangle_g \, dv^n_g \, dt$$

$$+ \lambda^2 \int_0^T \int_M \varphi_2 a_1 \Omega_3 e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g \psi_1, \nabla_g \psi_2 \rangle_g \, dv^n_g \, dt.$$
\[ \begin{align*}
+ i\lambda \int_{0}^{T} \int_{\mathcal{M}} \varrho_2 a_1 e^{i\lambda(\psi_1 - \psi_2)} \left( \nabla_g \varpi_3, \nabla_g \psi_1 \right)_{\mathcal{G}} dv^g_g dt \\
+ i\lambda \int_{0}^{T} \int_{\mathcal{M}} \varrho_2 a_1 e^{i\lambda(\psi_1 - \psi_2)} \left( \nabla_g \psi_1, \nabla_g \varpi_2 \right)_{\mathcal{G}} dv^g_g dt \\
- i\lambda \int_{0}^{T} \int_{\mathcal{M}} \varrho_2 a_1 e^{-i\lambda(\psi_1 - \psi_2)} \left( \nabla_g \varpi_1, \nabla_g \psi_2 \right)_{\mathcal{G}} dv^g_g dt \\
+ \int_{0}^{T} \int_{\mathcal{M}} \varrho_2 \left( \nabla_g \varpi_1, \nabla_g \varpi_2 \right)_{\mathcal{G}} dv^g_g dt \\
+ \int_{0}^{T} \int_{\mathcal{M}} \varrho_2 \left( \nabla_g \varpi_1, \nabla_g \varpi_2 \right)_{\mathcal{G}} dv^g_g dt.
\end{align*} \]

From (6.12), (6.16) and (4.4), we have
\[ |J_2(\lambda, \varepsilon)| + |J_3(\lambda, \varepsilon)| \leq \| \varrho_0 \|_{C(\mathcal{M})} \left( 1 + \varepsilon \lambda^4 \right) \| a_2 \|_{*} \| a_1 \|_{*}. \] (6.34)

Taking into account (6.6), (6.30) and (6.33), we deduce that
\[ \int_{0}^{T} \int_{\partial \mathcal{M}} (\Lambda_g - \Lambda_{cg}) J_{12} T_{12} d\sigma_{\mathcal{S}}^{n-1} dt = \lambda \int_{0}^{T} \int_{\mathcal{M}} \varrho(y)(a_1 \varpi_2)(t, x) dv^g_y dt + J_1(\lambda, \varepsilon) + J_2(\lambda, \varepsilon) + J_3(\lambda, \varepsilon). \] (6.35)

In view of (6.34) and (6.31), we obtain
\[ \left| \int_{0}^{T} \int_{\mathcal{M}} \varrho(y)(a_1 \varpi_2)(t, x) dv^g_y dt \right| \leq \| \varrho_0 \|_{C(\mathcal{M})} \left( \lambda^{-1} + \varepsilon \lambda^3 \right) \| a_1 \|_{*} \| a_2 \|_{*} \\
+ \lambda^3 \| a_1 \|_{*} \| a_2 \|_{*} \| \Lambda_g - \Lambda_{cg} \|. \] (6.36)

This completes the proof. \[ \square \]

### 6.2 Stability estimate of the geodesic ray transform

**Lemma 6.4** Let \( M_0 > 0 \). There exist \( C > 0 \) and \( \beta_j > 0, j = 1, 2, 3 \), such that for any \( b \in H^2(\partial_+, S M_1) \) the following estimate
\[ \left| \int_{\partial_+ S M_1} I(\vartheta) (y, \vartheta) b(y, \vartheta) (\theta, \nu(y)) \ d\sigma^g_{\mathcal{S}} \right| \]
\[ \leq \left( \left( \lambda^{-\beta_1} + \varepsilon \lambda^{\beta_2} \right) \| \varrho_0 \|_{C(\mathcal{M})} + \lambda^{\beta_3} \| \Lambda_g - \Lambda_{cg} \| \right) \| b \|_{H^2(\partial_+ S M_1)}. \] (6.37)

holds for any \( \lambda \) large.
Proof. Following (4.20), we take two solutions of the form
\[
\bar{a}_1(t, r, \theta) = \alpha^{-1/4} \phi(t - r) b(y, \theta),
\]
\[
\bar{a}_2(t, r, \theta) = \alpha^{-1/4} \phi(t - r) \mu(y, \theta).
\]

Now we change variable in (6.27), \( x = \exp_y(r\theta), r > 0 \) and \( \theta \in S_yM_1 \). Then
\[
\int_0^T \int_\mathcal{M} \rho(x) a_1(t, x) a_2(t, x) \, dv^n \, dt
\]
\[
= \int_0^T \int_{S_yM_1 \times 0} \bar{\rho}(r, \theta) \bar{a}_1(t, r, \theta) \bar{a}_2(t, r, \theta) \alpha^{1/2} \, dr \, d\omega_y(\theta) \, dt
\]
\[
= \int_0^T \int_{S_yM_1} \int_0^{\tau_+(y, \theta)} \bar{\rho}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt.
\]

We conclude that
\[
\left| \int_0^\infty \int_{S_yM_1} \int_0^{\tau_+(y, \theta)} \bar{\rho}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \right|
\]
\[
\leq C \left( \lambda^{-1} + \varepsilon \lambda^3 \| q_0 \|_{C(M)} + \lambda^3 \| A_{g, q_1} - A_{g, q_2} \| - \right)
\]
\[
\times \| \phi \|_{H^2(\mathbb{R})} \| b(y, \cdot) \|_{H^2(S^+_yM_1)}. \quad (6.38)
\]

Since \( \phi(t) = 0 \) for \( t \leq 0 \) or \( t \geq T \), we get
\[
\int_0^\infty \int_{S_yM_1} \int_0^{\tau_+(y, \theta)} \bar{\rho}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt
\]
\[
= \left( \int_{-\infty}^\infty \phi^2(t) \, dt \right) \times \int_{S_yM_1} \int_0^{\tau_+(y, \theta)} \bar{\rho}(r, \theta) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta). \quad (6.39)
\]

Combining (6.39) and (6.38), it follows that
\[
\left| \int_{S_yM_1} \int_0^{\tau_+(y, \theta)} \bar{\rho}(s, \theta) b(y, \theta) \mu(y, \theta) \, ds \, d\omega_y(\theta) \right|
\]
\[
\leq C \left( \lambda^{-1} + \varepsilon \lambda^3 \| q_0 \|_{C(M)} + \lambda^3 \| A_{g, q_1} - A_{g, q_2} \| \right) \| b(y, \cdot) \|_{H^2(S^+_yM_1)}. \]

Integrating with respect to \( y \in \partial M_1 \) we obtain
\[
\left| \int_{\partial_sS^+_yM_1} \mathcal{I}(\varrho)(y, \theta) b(y, \theta) \langle \theta, \nu(y) \rangle \, d\sigma_g^{2n-2}(y, \theta) \right|
\]
\[
\leq \left( \lambda^{-\beta_1} + \varepsilon \lambda^{\beta_2} \right) \| q_0 \|_{C^1(M)} + \lambda^{\beta_3} \| A_{g} - A_{g, q_2} \| \} \| b \|_{H^2(\partial_sS^+_M)} \quad (6.40)
\]

33
This completes the proof of the lemma. □

6.3 End of the proof of Theorem 2

This subsection is devoted to the end of the proof of Theorem 2. We need the following known result (see [41] proposition 4.1).

Lemma 6.5 Let \( c \in C^\infty(\mathcal{M}) \) be such that \( \|1 - c\|_{C(\mathcal{M})} \leq \varepsilon \). Then there exists \( C > 0 \) such that

\[
\|d_g - d_{cg}\|_{C(\partial \mathcal{M} \times \partial \mathcal{M})} \leq C \|\Lambda_g - \Lambda_{cg}\|^\mu,
\]

(6.41)

with some \( 0 < \mu < 1 \) depending only on the dimension \( n \).

From this lemma, we can derive the following estimate

Corollary 3 There exists a constant \( C > 0 \) such that the following estimate holds true

\[
\|I^* \mathcal{I}(\varrho)\|_{H^{3/2}(\mathcal{M}_1)} \leq C \left( \|\varrho\|_{C^1(\mathcal{M})}^2 + \|\Lambda_g - \Lambda_{cg}\|^\mu \right)
\]

(6.42)

with some \( 0 < \mu < 1 \) depending on \( n \) only.

Proof. Linearizing near \( g \), we get, as in [21]

\[
d_g(x, y) - d_{cg}(x, y) = \frac{1}{2} \mathcal{I}(\varrho)(x, y) + \mathcal{R}(\varrho)(x, y), \quad \forall x, y \in \partial \mathcal{M},
\]

where, with some abuse of notation, \( \mathcal{I}(\varrho)(x, y) \) stands for \( \mathcal{I}(\varrho)(x, \theta) \) with \( \theta = \exp_x^{-1}(y)/|\exp_x^{-1}(y)| \). The remainder term \( \mathcal{R}(\varrho)(x, y) \) is nonlinear and satisfies the estimate (see [21])

\[
|\mathcal{R}(\varrho)(x, y)| \leq C d_g(x, y) \|\varrho\|_{C^1(\mathcal{M})}^2, \quad \forall x, y \in \partial \mathcal{M},
\]

with \( C > 0 \) uniform in \( c \) if \( 0 < \varepsilon \ll 1 \). By Lemma 6.5 we have

\[
|\mathcal{I}(\varrho)(x, y)| \leq C \left( d_g(x, y) \|\varrho\|_{C^1(\mathcal{M})}^2 + \|\Lambda_g - \Lambda_{cg}\|^\mu \right), \quad \forall x, y \in \partial \mathcal{M}.
\]

Apply \( I^* \) to both sides, and use the estimate \( \|I^* (f)\|_{L^\infty(\mathcal{M}_1)} \leq C \|f\|_{L^\infty(\mathcal{M}_1)} \) to get

\[
\|I^* \mathcal{I}(\varrho)\|_{L^\infty(\mathcal{M}_1)} \leq C \left( \|\varrho\|_{C^1(\mathcal{M})}^2 + \|\Lambda_g - \Lambda_{cg}\|^\mu \right).
\]

(6.43)

Since \( \varrho \) vanishes outside \( \mathcal{M} \) with all derivatives and \( I^* \mathcal{I} \) is a pseudodifferential operator of order \(-1\), we have

\[
\|I^* \mathcal{I}(\varrho)\|_{H^{m+1}(\mathcal{M}_1)} \leq C_m \|\varrho\|_{H^m(\mathcal{M})}
\]
for all integers $m$. Using interpolation, we get
\[
\|I^*I(q)\|_{H^2(M_1)} \leq C \|I^*I(q)\|_{L^2(M_1)}^{2/3} \|I^*I(q)\|_{H^0(M_1)}^{1/3} \\
\leq C \|I^*I(q)\|_{L^\infty(M_1)}^{2/3}.
\]
Therefore, (6.44) and (6.43) imply
\[
\|I^*I(q)\|_{H^2(M_1)}^{3/2} \leq C \left( \|q\|_{C^1(M)}^2 + \|\mathcal{A}_g - \mathcal{A}_{cg}\| \right) \|I^*I(q)\|_{H^2(M_1)}.
\]
This completes the proof. \[\square\]

Let us now prove Theorem 2. We choose
\[
b(y, \theta) = I(I^*I(q))(y, \theta)
\]
and obtain
\[
\|I^*I(q)\|_{L^2(M_1)}^2 \\
\leq C \left( (\lambda^{-\beta_1} + \varepsilon\lambda^{-\beta_2}) \|q_0\|_{C^1(M)} + \lambda^{\beta_3} \|\mathcal{A}_g - \mathcal{A}_{cg}\| \right) \|I^*I(q)\|_{H^2(M_1)}.
\]
By interpolation we have
\[
\|I^*I(q)\|_{H^1(M_1)}^2 \\
\leq C \|I^*I(q)\|_{L^2(M_1)} \|I^*I(q)\|_{H^2(M_1)} \\
\leq C \left( (\lambda^{-\beta_1} + \varepsilon\lambda^{-\beta_2}) \|q_0\|_{C^1(M)} + \lambda^{\beta_3} \|\mathcal{A}_g - \mathcal{A}_{cg}\| \right)^{1/2} \|I^*I(q)\|_{H^2(M_1)}^{3/2}.
\]
Using (6.42) we obtain
\[
\|q\|_{L^2(M)}^2 \leq C \left( (\lambda^{-\beta_1'} + \varepsilon\lambda^{-\beta_2'}) \|q_0\|_{C^1(M)}^{5/2} + \lambda^{\beta_3'} \|\mathcal{A}_g - \mathcal{A}_{cg}\|^{\mu} \right).
\]
Since
\[
\|q_0\|_{C^1(M)} \leq C \|q_0\|_{H^{n/2+\varepsilon}(M)} \leq C \|q_0\|_{L^2(M)}^{4/5} \|q_0\|_{H^\ast(M)}^{1/5} \leq C \|q_0\|_{L^2(M)}^{4/5}
\]
we obtain
\[
\|q\|_{L^2(M)}^2 \leq C \left( (\lambda^{-\beta_1'} + \varepsilon\lambda^{-\beta_2'}) \|q_0\|_{L^2(M)}^2 + \lambda^{\beta_3'} \|\mathcal{A}_g - \mathcal{A}_{cg}\|^{\mu} \right).
\]
Minimising \((\lambda^{-\beta_1'} + \varepsilon\lambda^{-\beta_2'})\) with respect to $\lambda > 0$, we get
\[
C' \|q_0\|_{L^2(M)}^2 \leq \|q\|_{L^2(M)}^2 \leq C \left( \varepsilon^{-\gamma} \|q_0\|_{L^2(M)}^2 + C \varepsilon \|\mathcal{A}_g - \mathcal{A}_{cg}\|^{\mu} \right).
\]
for $\varepsilon > 0$ small enough we conclude and obtain (1.13).
7 Proof of Theorem 3

Let \( q \in L^\infty(\mathcal{M}) \), we first define the elliptic Dirichlet-to-Neumann map; let \( \sigma(A_q) = \{ \lambda_{k,q} \} \) be the spectrum of \( A_q \) and \( \rho(A_q) = \mathbb{C} \setminus \sigma(A_q) \) be the resolvent set of \( A_q \). From well known results (e.g., [31]), for any \( z \in \rho(A_q) \) and \( h \in H^{3/2}(\partial \mathcal{M}) \), the nonhomogeneous boundary value problem

\[
\begin{cases}
(-\Delta_g + q)u = zu, & \text{in } \mathcal{M} \\
u = h, & \text{on } \partial \mathcal{M}
\end{cases}
\]

has an unique solution in \( u_{q,h} \in H^2(\mathcal{M}) \) and the Dirichlet-to-Neumann map

\[ \Pi_{g,q}(z) : f \to \partial_\nu u_{q,h}|_{\partial \mathcal{M}} \]

defines a bounded operator from \( H^{3/2}(\partial \mathcal{M}) \) to \( H^{1/2}(\partial \mathcal{M}) \). We fix \( T > \text{Diam}_g \mathcal{M} \) and consider the following function space

\[ H_1 = \left\{ f \in H^{2n+4}(0,T;H^{3/2}(\partial \mathcal{M})); \partial_j^s f(0,\cdot) = 0, \ 0 \leq j \leq 2n+3 \right\}, \]

and the operator

\[
\mathcal{R}_{g,q} f = \sum_{k \geq 1} \frac{1}{\lambda_{k,q}} \int_0^t \sin \frac{\lambda_{k,q}(t-s)}{\lambda_{k,q}} \langle -\partial_s^{2(n+2)} f(\cdot, s), \partial_\nu \phi_{k,q} \rangle \, ds,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2(\partial \mathcal{M}) \)-scalar product. Then \( \mathcal{R}_{g,q} \) defines a bounded operator from \( H_1 \) to \( H_2 = L^2(0,T;H^s(\partial \mathcal{M})) \).

We will need in the sequel the following three lemmas. Their proof can be found in [2] or can be deduced easily from the results in this reference (see also [16]). We fix \( 0 \leq s < \frac{1}{2} \).

**Lemma 7.1** Let \( q \in L^\infty(\mathcal{M}) \). Then for any \( m > \frac{n}{2} \), \( h \in H^{3/2}(\partial \mathcal{M}) \) and \( z \in \rho(A_q) \), we have

\[
\frac{\partial^m}{\partial z^m} \Pi_{g,q}(z) h = -m! \sum_{k \geq 1} \frac{1}{(\lambda_{k,q} - z)^{m+1}} (h, \partial_\nu \phi_{k,q}) \partial_\nu \phi_{k,q}.
\]

**Lemma 7.2** Let \( N \) be a non negative integer and let \( q_1, q_2 \in L^\infty(\mathcal{M}) \) satisfy \( 0 \leq q_1, q_2 \leq M_0 \) for some positive constant \( M \). Then there exists a positive constant \( C \), depending only on \( M \) and \( M_0 \), such that

\[
\left\| \frac{\partial^j}{\partial z^j} \left[ \Pi_{g,q_1}(z) - \Pi_{g,q_2}(z) \right] \right\|_s \leq \frac{C}{|z|^{p+\frac{3}{2}}}, \quad z \leq 0 \quad \text{and} \quad 0 \leq j \leq N,
\]

where \( \| \cdot \|_s \) denotes the norm in \( L(H^{3/2}(\partial \mathcal{M});H^s(\partial \mathcal{M})) \).
Lemma 7.3 For each \( f \in \mathcal{H}_1 \), we have

\[
\Lambda_{g,q} f = \sum_{j=0}^{n+1} \left[ \frac{\partial^j}{\partial z_j} \Pi_{g,q}(z) \right]_{z=0} (-\partial_t^2 f) + \mathcal{R}_{g,q} f,
\]

(7.1)

where \( \Lambda_{g,q} \) is the restriction of \( \Lambda_{g,q} \) to \( \mathcal{H}_1 \).

First, we remark that for \( q \in L^\infty(M) \) and \( f \in \mathcal{H}_1 \) the problem (1.2) has an unique solution

\[
u \in L^2(0,T;H^2(M)) \cap H^2(0,T;L^2(M)) .
\]

(7.2)

Moreover \( \Lambda_{g,q} \) is a linear and continuous map from \( \mathcal{H}_1 \) into \( \mathcal{H}_2 \). Indeed, for \( f \in \mathcal{H}_1 \), let \( v \) solve the problem

\[
\begin{cases}
(\partial_t^2 - \Delta_g) v = 0, & \text{in } (0,T) \times M, \\
v(0,\cdot) = 0, \quad \partial_t v(0,\cdot) = 0 & \text{in } M, \\
v = f, & \text{on } (0,T) \times \partial M. 
\end{cases}
\]

(7.3)

Then

\[
v \in L^2(0,T;H^2(M)) \cap H^2(0,T;L^2(M)).
\]

Furthermore

\[
\|v\|_{L^2(0,T;H^2(M))} \leq C \|f\|_{\mathcal{H}_1} .
\]

(7.4)

Estimate (7.4) is essentially known, but we give the proof for the readers’ convenience. Let \( v_1 = \partial_t^2 v \). Then, by hyperbolic estimates, we have

\[
\|v_1\|_{L^2(0,T;L^2(M))} \leq C \|f\|_{\mathcal{H}_1} .
\]

On the other hand, since \( \Delta_g v = v_1 \), by the elliptic regularity, we get

\[
\|v\|_{L^2(0,T;H^2(M))} \leq C \left( \|v_1\|_{L^2(0,T;L^2(M))} + \|f\|_{\mathcal{H}_1} \right) .
\]

Thus, we get

\[
\left\| \frac{\partial v}{\partial \nu} \right\|_{H^2} \leq C \|f\|_{\mathcal{H}_1} .
\]

Now, for \( q \in L^\infty(M) \), let \( w \) solve

\[
\begin{cases}
(\partial_t^2 - \Delta_g + q(x)) w = -q(x)v, & \text{in } (0,T) \times M, \\
w(0,\cdot) = 0, \quad \partial_t w(0,\cdot) = 0 & \text{in } M, \\
w = 0, & \text{on } (0,T) \times \partial M,
\end{cases}
\]

(7.5)
we apply Lemma 7.4 to $\partial_t w$, we can prove
\[
\|w\|_{L^2(0,T;H^2(\mathcal{M}))} \leq C \|qv\|_{H^1(0,T;L^2(\mathcal{M}))} \leq C \|f\|_{\mathcal{H}_1}.
\]
Thus, for $u = v + w$, we have
\[
\begin{cases}
(\partial_t^2 - \Delta_g + q(x)) u = 0, & \text{in } (0, T) \times \mathcal{M}, \\
u(0, \cdot) = 0, & \partial_t u(0, \cdot) = 0 \quad \text{in } \mathcal{M}, \\
u = f, & \text{on } (0, T) \times \partial \mathcal{M},
\end{cases}
\tag{7.6}
\]
where
\[
u \in L^2(0, T; H^2(\mathcal{M})) \cap H^2(0, T; L^2(\mathcal{M})),
\]
and
\[
\|\Lambda_{g,q}^s(f)\|_{\mathcal{H}_2} \leq C \|f\|_{\mathcal{H}_1}. \tag{7.7}
\]
We shall denote by $\|\Lambda_{g,q}^s\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)}$ the operator norm of $\Lambda_{g,q}^s$.

**Lemma 7.4** Let $(\mathcal{M}, g)$ be a simple Riemannian compact manifold with boundary of dimension $n \geq 2$, let $T > \text{Diam}_g(\mathcal{M})$, there exist constants $C > 0$ and $\kappa \in (0, 1)$ such that for any real valued potentials $q_1, q_2 \in \mathcal{D}(M_0)$ such that $q_1 = q_2$ on the boundary $\partial \mathcal{M}$, we have
\[
\|q_1 - q_2\|_{L^2(\mathcal{M})} \leq C \|\Lambda_{g,q_1}^s - \Lambda_{g,q_2}^s\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \tag{7.8}
\]
where $C$ depends on $\mathcal{M}, T, M_0, n,$ and $s$.

**Proof**. As in (5.11), we have
\[
\begin{align*}
\left| \int_0^T \int_{\partial \mathcal{M}} \left( \Lambda_{g,q_1}^s - \Lambda_{g,q_2}^s \right) (f_\lambda) g_\lambda \, d\sigma_g^{n-1} \, dt \right| \\
\leq \|\Lambda_{g,q_1}^s - \Lambda_{g,q_2}^s\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \|f_\lambda\|_{\mathcal{H}_1} \|g_\lambda\|_{L^2((0,T) \times \partial \mathcal{M})} \\
\leq C \lambda^{2n+5} \|a_1\|_{**} \|a_2\|_{**} \|\Lambda_{g,q_1}^s - \Lambda_{g,q_2}^s\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \tag{7.9}
\end{align*}
\]
Where
\[
\|a\|_{**} = \|a\|_{H^{2n+4}(0,T;H^2(\mathcal{M}))}.
\]
Thus, we can complete the proof of (7.8) in the same way as in section 5.2. \hfill \Box

We set $P(z) = (\Pi_{g,q_1}(z) - \Pi_{g,q_2}(z))$, from Taylor’s formula, we deduce for $1 \leq j \leq n$, and $z \leq 0$
\[
P^{(j)}(0) = \sum_{p=j}^{n} \frac{(-z)^{p-j}}{(p-j)!} P^{(p)}(z) + \int_0^z \frac{(-\tau)^{n-j}}{(n-j)!} P^{(n+1)}(\tau) \, d\tau. \tag{7.10}
\]
Lemma 7.5  There exist $C > 0$ and $\mu_1 \in (0, 1)$ such that the following estimate

$$\left\| P^{(n+1)}(z) \right\|_s \leq C e^{\mu_1}$$

(7.11)

holds true for any $z \leq 0$. Here $C$ is a positive constant depending on $M_0$, $\mathcal{M}$ and $\left\| \cdot \right\|_s$ denotes the norm in $\mathcal{L}(H^{3/2}(\partial\mathcal{M}); H^s(\partial\mathcal{M}))$.

**Proof.** Let $h \in H^{3/2}(\partial\mathcal{M})$. It follows from Lemma 7.1

$$P^{(n+1)}(z)h = -(n+1)! \sum_{k \geq 1} \frac{1}{(\lambda_{k,q} - z)^{n+2}} \langle h, \partial_{\nu} \phi_{k,q_1} \rangle \partial_{\nu} \phi_{k,q_1}$$

$$+ (n+1)! \sum_{k \geq 1} \frac{1}{(\lambda_{k,q_2} - z)^{n+2}} \langle h, \partial_{\nu} \phi_{k,q_2} \rangle \partial_{\nu} \phi_{k,q_2}.$$ 

We split $P^{(n+1)}(z)h$ into three terms $P^{(n+1)}(z)h = I_1(z)h + I_2(z)h + I_3(z)h$, where

$$I_1(z)h = -(n+1)! \sum_{k \geq 1} \frac{1}{(\lambda_{k,q_1} - z)^{n+2}} - \frac{1}{(\lambda_{k,q_2} - z)^{n+2}} \langle h, \partial_{\nu} \phi_{k,q_1} \rangle \partial_{\nu} \phi_{k,q_1}$$

$$I_2(z)h = -(n+1)! \sum_{k \geq 1} \frac{1}{(\lambda_{k,q_2} - z)^{n+2}} \langle h, \partial_{\nu} \phi_{k,q_1} - \partial_{\nu} \phi_{k,q_2} \rangle \partial_{\nu} \phi_{k,q_1}$$

$$I_3(z)h = -(n+1)! \sum_{k \geq 1} \frac{1}{(\lambda_{k,q_2} - z)^{n+2}} \langle h, \partial_{\nu} \phi_{k,q_2} \rangle [\partial_{\nu} \phi_{k,q_1} - \partial_{\nu} \phi_{k,q_2}].$$

For $I_1(z)h$, we have

$$\left\| I_1(z)h \right\|_{H^{1/2}(\partial\mathcal{M})} \leq (n+1)! \left\| h \right\|_{L^2(\partial\mathcal{M})}$$

$$\times \sum_{k \geq 1} \left| \frac{1}{(\lambda_{k,q_1} - z)^{n+2}} - \frac{1}{(\lambda_{k,q_2} - z)^{n+2}} \right| \left\| \partial_{\nu} \phi_{k,q_2} \right\|_{H^{1/2}(\partial\mathcal{M})}^2. \quad (7.12)$$

On the other hand, noting that $z \leq 0$, $\lambda_{k,q_j} \geq 0$, $j = 1, 2$, we see that

$$\left| \frac{1}{(\lambda_{k,q_1} - z)^{n+2}} - \frac{1}{(\lambda_{k,q_2} - z)^{n+2}} \right| \leq C \max \left( \frac{1}{\lambda_{k,q_1}^{n+3}}, \frac{1}{\lambda_{k,q_2}^{n+3}} \right) |\lambda_{k,q_1} - \lambda_{k,q_2}|$$

$$\leq C \frac{2^{(n+3)}}{k^{2(n+3)}} |\lambda_{k,q_1} - \lambda_{k,q_2}|,$$

where we have used estimate (1.15). On the other hand, since (see (1.14) and (1.15))

$$\left\| \partial_{\nu} \phi_{k,q_2} \right\|_{H^{1/2}(\partial\mathcal{M})}^2 \leq C k^\frac{2}{n},$$

39
we obtain

\[ \|I_1(z)h\|_{H^{1/2}(\partial M)} \leq C \|h\|_{L^2(\partial M)} \frac{1}{k} \sum_{k \geq 1} \frac{\lambda_{k,q_1}}{\lambda_{k,q_2}} |\lambda_{k,q_1} - \lambda_{k,q_2}| \leq C \epsilon \|h\|_{L^2(\partial M)}. \quad (7.13) \]

For \( I_2(z)h \), we have

\[ \|I_2(z)h\|_{H^{1/2}(\partial M)} \leq C \|h\|_{L^2(\partial M)} \sum_{k \geq 1} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2} - z)^{n+2}} \|\partial_{\nu} (\phi_{k,q_1} - \phi_{k,q_2})\|_{L^2(\partial M)}. \quad (7.14) \]

Then Lemma 7.5 yields

\[ \sum_{k \geq 1} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2} - z)^{n+2}} \|\partial_{\nu} (\phi_{k,q_1} - \phi_{k,q_2})\|_{L^2(\partial M)} \leq C \sum_{k \geq 1} \frac{\lambda_{k,q_1}(\lambda_{k,q_1} + \lambda_{k,q_2})}{(\lambda_{k,q_2} - z)^{n+2}} \leq C \epsilon. \quad (7.15) \]

Therefore, we find

\[ \|I_2(z)h\|_{H^{1/2}(\partial M)} \leq C \|h\|_{L^2(\partial M)} \epsilon. \quad (7.16) \]

For \( I_3(z)h \), we have

\[ \|I_3(z)h\|_{H^{1/2}(\partial M)} \leq C \|h\|_{L^2(\partial M)} \sum_{k \geq 1} \frac{1}{\lambda_{k,q_2}^{n+1}} \|\partial_{\nu} (\phi_{k,q_1} - \phi_{k,q_2})\|_{H^{1/2}(\partial M)} \]

\[ \leq C \|h\|_{L^2(\partial M)} \sum_{k \geq 1} \frac{1}{k^{2(n+1)/n}} \|\partial_{\nu} (\phi_{k,q_1} - \phi_{k,q_2})\|_{H^{1/2}(\partial M)} \]

\[ \leq C \|h\|_{L^2(\partial M)} \sum_{k \geq 1} \frac{1}{k^{2(n-1)/n}} \|\partial_{\nu} (\phi_{k,q_1} - \phi_{k,q_2})\|_{H^{1/2}(\partial M)}. \]

Therefore

\[ \|I_3(z)h\|_{H^{1/2}(\partial M)} \leq C \epsilon \|h\|_{L^2(\partial M)}. \quad (7.17) \]

The conclusion follows then from a combination of (7.17), (7.16) and (7.13). \( \square \)

**Proof of Theorem 3** From (7.10) and Lemma 7.2 we obtain

\[ \|P^{(j)}(0)\|_{s} \leq C \left( |z|^{-j-\frac{1+2s}{4}} + |z|^{n-j+1} e^{\mu_1} \right), \]
and then
\[ \left\| P^{(j)}(0) \right\|_s \leq C \left( |z|^{\frac{n-2s}{4}} + |z|^{n+1} e^{\mu_1} \right), \quad \text{if } |z| \geq 1. \]

In particular
\[ \left\| P^{(j)}(0) \right\|_s \leq C \min_{\rho \geq 1} \left( \rho^{\frac{n-2s}{4}} + \rho^{n+1} e^{\mu_1} \right) = C e^{\mu_2} \quad (7.18) \]

where \( \mu_2 \in (0, 1) \). Let \( R_{g,q} \) be defined as in Lemma 7.5. We can proceed as in the proof of Lemma 7.5 to prove
\[ \left\| R_{g,q_1} - R_{g,q_2} \right\|_{L(H_1,H_2)} \leq C e^{\mu_3}. \quad (7.19) \]

From identity (7.1), estimates (7.19) and (7.18), we deduce
\[ \left\| \Lambda^g_{\varepsilon,q_1} - \Lambda^g_{\varepsilon,q_2} \right\|_{L(H_1,H_2)} \leq C e^{\mu_4}, \]

provided that \( \epsilon \) is sufficiently small. To finish, we only need to remark that the traces of the geometrical optics solutions constructed in section 4 in fact satisfy
\[ u_1|_{\partial M}, u_2|_{\partial M} \in H_1 \]
so that in the proof of Theorem 1, the right-hand side of (1.12) may be replaced by
\[ \left\| \Lambda^g_{\varepsilon,q_1} - \Lambda^g_{\varepsilon,q_2} \right\|_{L(H_1,H_2)}. \]

This completes the proof of Theorem 3. \( \square \)

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