Teleportation of displaced Fock states: Fidelity and their teleported photon number distributions

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Abstract. We consider the teleportation of displaced Fock states which are highly non-classical states of the quantized electromagnetic field which have a set of remarkable quantum properties that include the peculiar oscillations of their photon number distributions. We use the transfer operator formalism to show that the quantum teleportation of a DFS renders a finite superposition of orthonormal DFS’s and find its explicit mathematical expression in terms of the compression parameter of the correlated EPR states of the quantum channel. The expression for a teleported Fock state is also derived as a particular case of DFS’s teleportation. We finally apply these results to study the fidelity of the teleportation of DFS’s and the teleportation of their photon number statistics.

1. Introduction
Quantum teleportation enables us to transfer an unknown quantum state from a local station to a remote location with the assistance of a shared entangled pair of quantum states - the kind of state first used by Einstein, Podolsky and Rosen in the now called “EPR paradox” - and additionally transmitting an ordinary message via a classical channel of communication between the two parties. At the remote station one in principle obtains an exact replica of the unknown input quantum state at the cost of the destruction of such input state. Initially formulated by Bennett et al. within the context of dichotomic quantum variables [1] or quantum bits, e.g spin-1/2 systems or polarized photons, the protocol was soon extended by Vaidman [2] to quantum systems of continuous variables and applied by Braunstein and Kimble [3] to the bosonic modes of the quantized electro–magnetic field. Based on this scheme Furusawa et al. [4] experimentally demonstrated the quantum teleportation of optical coherent states using squeezed–state entanglement.

In this work we represent the teleportation process by the operator of Hofmann et al. [5]:

\[ T^q_\gamma(\beta) = \sqrt{1 - q^2} \sum_{k=0}^{\infty} q^k \hat{D}(g\beta) |k\rangle \langle k| \hat{D}(-\beta), \]

where \( q = \tanh |r| \), \( r \) is the (complex) compression parameter of the auxiliary EPR state composed by a pair of entangled squeezed states, \( g \) is a gain parameter adjusted by the receiver in order to improve the fidelity of the teleportation [8], and \( \beta \) a complex number associated with a homodyne measurement done by the sender, whose result is communicated using a classical channel in order to define a final displacement operation that the receiver applies to complete
the teleportation process. In particular, we apply this operator on the special class of states known as “displaced Fock states” (DFS’s), or “displaced number states” (DNS’s) which because of their striking physics properties [6] are indeed of a great interest for quantum optics and teleportation.

2. Teleportation of displaced Fock states
A DFS $|\alpha, n\rangle$ is obtained by applying the unitary displacement operator $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$ to a Fock state $|n\rangle$ of the harmonic oscillator, i.e. $|\alpha, n\rangle = \hat{D}(\alpha) |n\rangle$. This amounts to adding a non–zero complex amplitude $\alpha$ to the field amplitude of the Fock state $|n\rangle$, resulting in a new quantum state $|\alpha, n\rangle$.

Applying the teleportation operator $\hat{T}_q^g(\beta)$ to a displaced Fock state $|\alpha, n\rangle$ we have [7]:

$$\hat{T}_q^g(\beta) |\alpha, n\rangle = \sqrt{1 - \frac{q^2}{\pi}} \sum_{k=0}^{\infty} q^k \hat{D}(g \beta) |k\rangle \langle k| \hat{D}(\beta) \hat{D}(\alpha) |n\rangle$$

(2)

$$= \sqrt{1 - \frac{q^2}{\pi} n!} e^{\frac{\alpha^* - \alpha^* \beta}{2}} \sum_{k=0}^{\infty} q^k \hat{D}(g \beta) |k\rangle \langle k| \hat{D}(\alpha - \beta) \hat{a}^\dagger^n |0\rangle .$$

(3)

If we now make for simplicity $\xi = \alpha - \beta$ and insert the identity operator $I = \hat{D}(-\xi) \hat{D}(\xi)$ we get

$$\hat{T}_q^g(\beta) |\alpha, n\rangle = \sqrt{1 - \frac{q^2}{\pi} n!} e^{\frac{\alpha^* - \alpha^* \beta}{2}} \hat{D}(g \beta) \sum_{\ell=0}^{\infty} \binom{n}{\ell} (-\xi^*)^n \sum_{k=0}^{\infty} q^k \langle k| \hat{a}^\dagger \ell |\xi\rangle |k\rangle \text{.}$$

(4)

which after expanding the binomial operator and reordering gives,

$$\hat{T}_q^g(\beta) |\alpha, n\rangle = \sqrt{1 - \frac{q^2}{\pi} n!} e^{\frac{\alpha^* - \alpha^* \beta}{2}} \hat{D}(g \beta) \sum_{\ell=0}^{\infty} \binom{n}{\ell} (-\xi^*)^n \sum_{k=0}^{\infty} q^k \langle k| \hat{a}^\dagger \ell |\xi\rangle |k\rangle \text{.}$$

(5)

Using now the relations $\langle k| \hat{a}^\dagger \ell |k - \ell\rangle = \sqrt{\frac{k!}{(k-\ell)!}} |k - \ell\rangle$ and $|k\rangle = \sqrt{\frac{(k-\ell)!}{k!}} \hat{a}^\dagger \ell |k - \ell\rangle$ we get

$$\hat{T}_q^g(\beta) |\alpha, n\rangle = \sqrt{1 - \frac{q^2}{\pi} n!} e^{\frac{\alpha^* - \alpha^* \beta}{2}} \hat{D}(g \beta) \sum_{\ell=0}^{\infty} \binom{n}{\ell} (-\xi^*)^n \sum_{k=0}^{\infty} q^k \langle k - \ell| \xi\rangle |k - \ell\rangle \text{,}$$

(6)

that may be simplified using the well–known expression $\langle k - \ell|\xi\rangle = e^{-\frac{q^2}{2}} \frac{\xi^{k-\ell}}{\sqrt{(k-\ell)!}}$, so that the expression of the teleported displaced number state can be finally written

$$\hat{T}_q^g(\beta) |\alpha, n\rangle = \sqrt{1 - \frac{q^2}{\pi} n!} e^{(1-gq)\frac{\alpha^* - \alpha^* \beta}{2}} e^{-(1-q^2)\frac{|\alpha - \beta|^2}{2}} \hat{D}(\chi) (q \hat{a}^\dagger + \phi^*)^n |0\rangle \text{,}$$

(7)

where $\chi$ and $\phi$ are complex numbers given by

$$\chi \equiv q \alpha + (g - q) \beta ,$$

$$\phi \equiv (q^2 - 1)(\alpha - \beta) \text{.}$$

(9)
2.1. Some particular cases

Three interesting examples are the teleportation of the coherent state $|\alpha, 0\rangle$, the displaced single photon state $|\alpha, 1\rangle$, and the displaced two photon state $|\alpha, 2\rangle$:

$$\hat{T}^g_q(\beta)|\alpha, 0\rangle = \sqrt{\frac{1 - q^2}{\pi}} e^{(1-qg)\alpha^* \alpha} e^{-\left(1-q^2\right)\frac{\left|\alpha-\beta\right|^2}{2}} \hat{D}\left(\chi\right)|0\rangle$$

(10)

$$\hat{T}^g_q(\beta)|\alpha, 1\rangle = \sqrt{\frac{1 - q^2}{\pi}} e^{(1-qg)\alpha^* \alpha} e^{-\left(1-q^2\right)\frac{\left|\alpha-\beta\right|^2}{2}} \hat{D}\left(\chi\right)\left(\phi^*|0\rangle + q|1\rangle\right)$$

(11)

$$\hat{T}^g_q(\beta)|\alpha, 2\rangle = \sqrt{\frac{1 - q^2}{2\pi}} e^{(1-qg)\alpha^* \alpha} e^{-\left(1-q^2\right)\frac{\left|\alpha-\beta\right|^2}{2}} \hat{D}\left(\chi\right)\left(\hat{\phi}^*|0\rangle + 2q\phi^*|1\rangle + \sqrt{2}q^2|2\rangle\right).$$

(12)

Eq. (10) is just the known expression of a teleported coherent state, which itself is a coherent state $|\alpha\rangle$. Eq. (11) is a superposition of a coherent state component with a displaced one photon component which in the special case $\alpha = 0$ reduces to the known formula of a teleported single photon state $|\alpha\rangle$. In eq. (12) we have again a superposition of orthonormal DFS’s, this time with displaced components of 0, 1 and 2 photons. In general, if we expand the binomial operator $(q\hat{a}^\dagger + \phi^*)^n$ in the r.h.s. of eq. (8) we get a finite superposition of orthonormal DFS’s with a common displacement $\chi = g\alpha + (g-q)\beta$, ranging from the coherent state $|\chi, 0\rangle$ to the displaced $n$–photon state $|\chi, n\rangle$.

For the particular case $\alpha = 0$ eq. (8) give us the expression for the teleportation of a Fock state $|n\rangle$ namely

$$\hat{T}^g_q(\beta)|n\rangle = \sqrt{\frac{1 - q^2}{\pi n!}} e^{-\left(1-q^2\right)|\chi|^2/2} \hat{D}(\gamma)\left(q\hat{a}^\dagger + \theta^*\right)^n|0\rangle,$$

(13)

where $\gamma$ and $\theta$ are complex numbers given by

$$\gamma \equiv (g-q)\beta,$$

$$\theta \equiv (1-q^2)\beta.$$  

(14)

This result is again a finite superposition of orthonormal DFS’s, this time with a common displacement parameter $\gamma \equiv (g-q)\beta$.

2.2. Probability distribution for $\beta$

Let’s recall first that in order to normalize the teleported output states of eqs. (8) and (13) we must divide in each case by its norm $\sqrt{P(\beta)}$ where $P(\beta) = \langle \psi_{\text{out}}(\beta) | \psi_{\text{out}}(\beta) \rangle$ is the probability of measuring $\beta$ during the teleportation process [5]. It can be easily shown using eq. (8) that the probability distribution for $\beta$ in the case of DFS’s teleportation is given by the expression

$$P_{\alpha,n}(\beta) = \frac{1 - q^2}{\pi n!} \exp\left[-(1-q^2)|\alpha-\beta|^2\right] \sum_{k=0}^{n} \frac{n!}{k^k} (1-q^2)^{2(n-k)} |\alpha-\beta|^2 (n-k)$$

(15)

Note that this probability and that which corresponds to Fock states ($\alpha = 0$) are related by a simple displacement $\alpha$ in the $\beta$–plane.
Figure 1. Probability distribution for $\beta$ in the teleportation of the DFS’s (a) $|10 + 10i, 0\rangle$ and (b) $|10 + 10i, 3\rangle$. In both cases we have a compression parameter $q = 0.7$. The projection onto the plane Im$\beta = 10$ (imaginary part of $\alpha$) is shown at the bottom of each surface ((c) and (d)).

Some particular cases of eq. (15) are

$$P_{\alpha,0}(\beta) = \frac{1 - q^2}{\pi} \exp\left(-(1 - q^2)|\alpha - \beta|^2\right)$$  \hspace{1cm} (16)

$$P_{\alpha,1}(\beta) = \frac{1 - q^2}{\pi} \exp\left(-(1 - q^2)|\alpha - \beta|^2\right) \left(|\phi|^2 + q^2\right)$$  \hspace{1cm} (17)

$$P_{\alpha,2}(\beta) = \frac{1 - q^2}{2\pi} \exp\left(-(1 - q^2)|\alpha - \beta|^2\right) \left(|\phi|^4 + 4q^2|\phi|^2 + 2q^4\right)$$  \hspace{1cm} (18)

$$P_{\alpha,3}(\beta) = \frac{1 - q^2}{6\pi} \exp\left(-(1 - q^2)|\alpha - \beta|^2\right) \left(|\phi|^6 + 9q^2|\phi|^4 + 18q^4|\phi|^2 + 6q^6\right)$$  \hspace{1cm} (19)

where the complex number $\phi$ is given in eq. (9). To get the probabilities for the corresponding Fock states we only have to substitute $\alpha = 0$ (and further replace the constant $\phi$ by the complex number $\theta$ given in eq. (14)). In fig. 1 we show the probability distributions $P_{\alpha,0}(\beta)$ and $P_{\alpha,3}(\beta)$ for $\alpha = 10 + 10i$ and $q = 0.7$ (note that this function does not depend on the gain parameter $g$). The first case corresponds to a coherent state, the probability surface draws a poissonian distribution centered at $\alpha = 10 + 10i$. In the second case the probability surface shows a ring of maximum probability with a dip in its center, which is the characteristic shape of these surfaces when $n > 0$. The corresponding surfaces for the Fock states $|0\rangle$ and $|3\rangle$ are identical to those in fig. 1 but centered at $\beta = 0$.

2.3. Coefficients of the DFS’s expansion of the teleported state
In the case of unit gain ($g = 1$) and when the teleported DFS or Fock state is properly normalized, it can be easily verified that the limit $q \to 1$ ($r \to \infty$), which corresponds to perfect entanglement,
Figure 2. Magnitude of the components (squared modulus of the coefficients) of the expansion in DFS’s of the teleported states (a) $\hat{T}_q^g(\beta)|1-2i,1\rangle$, (b) $\hat{T}_q^g(\beta)|1-2i,2\rangle$, (c) $\hat{T}_q^g(\beta)|1-2i,3\rangle$ and (d) $\hat{T}_q^g(\beta)|1-2i,4\rangle$ as a function of the compression parameter $q$ returns exactly the input state as should be expected. We also have that for both DFS’s and Fock states, as the compression parameter $q$ (which ranges between 0 and 1) approaches 0 the magnitude of the coherent component predominates over the rest of the superposition terms, while as $q$ approaches 1 the magnitude of the $n$–photon component approaches 1 while all other components vanish. In fig.2 we plot the dependence on the compression parameter $q$ of the components’ magnitude of the normalized DFS’s expansion of the teleported states $\hat{T}_q^g(\beta)|1-2i,n\rangle$ with photon numbers $n = 1, 2, 3, 4$. Note that these magnitudes do not depend on $g$ and are given by the squared modulus of the corresponding coefficient in the superposition.

3. Fidelity of the teleportation of DFS’s

The standard criterion for the quality of quantum teleportation is called the fidelity and is measured by the overlap between the two states i.e.

$$F(\beta) = |\langle \psi_{\text{in}} | \psi_{\text{out}} \rangle|^2.$$  \hspace{1cm} (20)

The fidelity quantifies the resemblance between the input state $|\psi_{\text{in}}\rangle$ and the output state $|\psi_{\text{out}}\rangle$ generated at the remote station. When perfect teleportation is achieved this two states are identical so that $F = 1$. In terms of the teleportation operator and in the special case of DFS’s teleportation, the fidelity for a single teleportation process is given by,

$$F_{\alpha,n}(\beta) = |\langle \alpha, n | T_q^g(\beta) | \alpha, n \rangle|^2.$$  \hspace{1cm} (21)

Based on the result of eq. (8) we can find a general formula for the fidelity of the teleportation of a DFS.
distribution that particular teleportation event is exactly one, no matter the value of measurement result for $\beta$. The distribution for the case of DFS’s is given by eq. (15). Eqs. (22) and (15) show that when the measurement result for $\beta$ is zero, we evaluate the matrix elements larger values of $\beta$ and for $q = 0.1$ and $q = 1$. The fidelity depends on the measurement result for $\beta$, which is different for each teleportation event and whose probability distribution for the case of DFS’s is given by eq. (15). Eqs. (22) and (15) show that when the measurement result for $\beta$ coincides with the input displacement $\alpha$, the normalized fidelity for that particular teleportation event is exactly one, no matter the value of $q$. In our example of fig. 3 such an event has zero probability of occurrence, but this is not the case when we take larger values of $q$.

4. Teleported photon number distributions for DFS’s

The probability of obtaining $m$ photons when measuring the photon number probability density of a DFS is given by

$$p_{\alpha,n}(m) = |\langle m | \alpha, n \rangle|^2 = |\langle m | \hat{D} (\alpha) | n \rangle|^2.$$ (27)

As shown in [6], we evaluate the matrix elements $\langle m | \hat{D} (\alpha) | n \rangle$ for $m \geq n$ to obtain the probability distribution $p_{\alpha,n}(m)$, e.g.
Figure 3. (a) Fidelity and (b) probability surfaces associated with the teleportation of the displaced Fock state $|1 + i, 2\rangle$ with compression parameter $q = 0.1$ and unit gain $g = 1$. The projection onto the plane $\text{Im} \beta = 10$ (imaginary part of $\alpha$) is shown at the bottom of each surface ((c) and (d))

$$p_{\alpha,0}(m) = \frac{1}{m!} e^{-|\alpha|^2} |\alpha|^{2m}$$

(28)

$$p_{\alpha,1}(m) = \frac{1}{m!} e^{-|\alpha|^2} |\alpha|^{2(m-1)} (|\alpha|^2 - m)^2.$$  

(29)

The first one of these two distributions is the well–known Poisson distribution of the coherent state $|\alpha\rangle$, and the second one when plotted as a function of the number of photons shows two oscillation lobes with a single node in between.

We showed above that the teleported state of given DFS is a superposition of orthonormal displaced number states $|\chi, n\rangle$, we can therefore obtain the photon statistics of a teleported DFS by evaluating the matrix element $\langle m | \hat{D}(\chi) | n \rangle$ for each term of the superposition. This calculations leads [7] to:

$$p^{\text{tel}}_{\alpha,n}(m) = \frac{1 - q^2}{\pi n! m!} e^{-(1-q^2)|\alpha-\beta|^2} e^{-|\chi|^2}$$

$$\times \left| \sum_{k=0}^{n} \binom{n}{k} \phi^{n-k} q^k \chi^{m-k} \left( \sum_{j=0}^{k} (-1)^{k-j} \frac{k! m! |\chi|^{2(k-j)}}{j! (k-j)! (m-j)!} \right)^2 \right|^2$$

(30)

where the complex numbers $\chi$ and $\phi$ are given in Eq. (9). A set of particular results for $n = 0$, 1 and 2 are
Figure 4. Teleported photon probability distributions for the displaced Fock states $|\alpha, n\rangle$ with $\alpha = 7$ and (a) $n = 0$, (b) $n = 1$, (c) $n = 2$ and (d) $n = 3$. In each case we have $\beta = 6.5$, unit gain $g = 1$ and three different values of the compression parameter, namely $q = 0.4$, 0.7 and 0.9. The dotted lines correspond to the non teleported photon number distributions.

\begin{align}
  p_{\alpha,0}^{tel}(m) &= \frac{1 - q^2}{\pi m!} e^{-(1-q^2)|\alpha - \beta|^2} e^{-|\chi|^2} |\chi|^{2m} \\
  p_{\alpha,1}^{tel}(m) &= \frac{1 - q^2}{\pi m!} e^{-(1-q^2)|\alpha - \beta|^2} e^{-|\chi|^2} q\chi^{m-1}(|\chi|^2 - m) + \phi^* \chi^m \\
  p_{\alpha,2}^{tel}(m) &= \frac{1 - q^2}{2 \pi m!} e^{-(1-q^2)|\alpha - \beta|^2} e^{-|\chi|^2} \\
  &\times q^2 \chi^{m-2}\left(|\chi|^4 - 2m|\chi|^2 + m(m - 1)\right) + 2q\phi^* \chi^{m-1}(|\chi|^2 - m) + \phi^2 \chi^m
\end{align}

In order to get the corresponding normalized expressions we must divide by eq. (15) in the general case of eq. (30) and by eqs. (16), (17) and (18) for the particular cases of eqs. (31), (32) and (33). The normalized probability functions $p_{\alpha,0}^{tel}(m)$, $p_{\alpha,1}^{tel}(m)$, $p_{\alpha,2}^{tel}(m)$ and $p_{\alpha,3}^{tel}(m)$ appear plotted in fig. 4 for $\beta = 6.5$, unit gain $g = 1$ and for three different values of the squeezing parameter namely $q = 0.4$, 0.7 and 0.9. For comparison, the photon distribution oscillations of the non–teleported DFS’s are also plotted (in dotted lines).

From fig. 4 we see that as $q$ approaches 1 the characteristics oscillations of the input photon number distribution tend to be perfectly transferred to the teleported state. On the other hand, as $q$ decreases the oscillations attenuate, in fact in the limit $q \to 0$ the photon statistics becomes poissonian. This is in perfect agreement with the fact that as $q \to 0$ the dominant term in the DFS’s superposition of eq. (8) is that corresponding to the coherent component, whose coefficient tends to 1 while the rest of the superposition coefficients vanish.
5. Conclusions
In this work we have studied the teleportation of the displaced Fock states of the quantized electro–magnetic field. The teleported states were found by applying the Hofmann et al. teleportation operator that follows a standard protocol of teleportation of continuous variable quantum systems that is parametrized by two parameters, namely its gain $g$ and the compression parameter $q$ of the auxiliary EPR pair of squeezed states. The teleportation of a DFS $|\alpha, n\rangle$ was shown to render a linear superposition of $(n+1)$ DFS’s, all with a common displacement, beginning with a coherent state and ending with a $n$–photons displaced Fock state. We have also derived the teleportation of one of the most peculiar properties of the DFS’s namely their photon distribution oscillations. The teleported state of a Fock state can be obtained from the teleportation of a DFS by simply considering zero displacement. Setting $g = 1$ and taking the limit when the parameter $q$ approaches 1 of the properly normalized linear superposition state that results from teleporting a given DFS, one recovers the initial DFS. This means that in such limit the fidelity is $F(\beta) = 1$ no matter which result of $\beta$ the sender obtains in its quantum homodyne measurement. In such limit the common displacement $\chi$ of the superposition terms of the teleported state tends to the input state displacement $\alpha$, and the coefficient of the term that contains $n$ photons in that superposition state tends to 1 while the rest of the coefficient vanish.

The fidelity of the teleportation process depends on the measured complex displacement $\beta$, which is different for each teleportation event and whose probability distribution is given by the distribution $P(\beta)$. When the measured value of $\beta$ coincides with the displacement $\alpha$ of the input DFS, the fidelity for that particular teleportation event is exactly 1. For low values of $q$ such an event has very low or zero probability of occurrence but this is not the case for larger values of $q$.

As the compression parameter $q \rightarrow 1$ the singular oscillations of the photon number distribution of the input DFS are perfectly transferred to the teleported state. On the other hand, as $q \rightarrow 0$ the photon distribution statistics of the teleported state evolves to a poissonian distribution. This observation is in perfect agreement with the fact that as $q \rightarrow 0$ the dominant term of the superposition of the teleported DFS is the first one i.e. the coherent state term.

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