Lattice Model with Nearest-Neighbor and Next-Nearest-Neighbor Interactions for Gradient Elasticity

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Abstract

Lattice models for the second-order strain-gradient models of elasticity theory are discussed. To combine the advantageous properties of two classes of second-gradient models, we suggest a new lattice model that can be considered as a discrete microstructural basis for gradient continuum models. It was proved that two classes of the second-gradient models (with positive and negative sign in front the gradient) can have a general lattice model as a microstructural basis. To obtain the second-gradient continuum models we consider a lattice model with the nearest-neighbor and next-nearest-neighbor interactions with two different coupling constants. The suggested lattice model gives unified description of the second-gradient models with positive and negative signs of the strain gradient terms. The sign in front the gradient is determined by the relation of the coupling constants of the nearest-neighbor and next-nearest-neighbor interactions.

PACS: 62.20.Dc; 61.50.Ah

1 Introduction

Elastic deformations of materials can be described by a microscopic approach based on lattice equations \[1, 2\] and by a macroscopic approach based on continuum equations \[3, 4\]. Continuum equations for elasticity can be considered as a limit of lattice dynamics, where the length scales of infinitesimal continuum elements are much greater than that of inter-particle distances in the lattice \[5\]. The theory of nonlocal continuum mechanics was formally initiated by the papers of Eringen \[6\]. Nonlocal elasticity theory is based on the assumption that the forces between material points are a long-range type, thus reflecting the long-range character of inter-atomic forces \[7\]. Nearest-neighbor and next-nearest-neighbor interactions for lattice particles are important type of interactions for materials with non-local properties. Non-local continuum mechanics can be considered by two different approaches \[8\]: the gradient elasticity theory (weak non-locality) and the
integral elasticity theory (strong non-locality). This article focuses on gradient models of non-local elasticity suggested in \[9, 10, 11, 12, 13\]. Mindlin \[9, 10, 11\] presented a theory of elasticity for materials with microstructure, where it was proposed to distinguish quantities of the microscale and the macroscale to take into account a weak non-locality. Eringen \[12\] also formulated a theory of nonlocal elasticity, where the integrals are replaced by gradients. Aifantis \[13\] suggested to extend the linear elastic constitutive relations with the Laplacian of the strain. Usually distinguish two following classes of gradient models with different signs of the strain gradient terms.

The first class is formed by Laplacian-based gradient elasticity models that are described by the linear constitutive relations

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} + l^2 C_{ijkl} \Delta \varepsilon_{kl},
\]

where \(\varepsilon_{ij}\) is the strain, \(\sigma_{ij}\) is the stress, \(C_{ijkl}\) is the stiffness tensor, \(\Delta\) is the Laplace operator and \(l\) is the scale parameter. For \(l^2 = 0\), we have the classical case of the linear elastic constitutive relations that is called the Hooke’s law. The main motivation for using the gradient elasticity of the form (1) has been the description of dispersive wave propagation through heterogeneous media. For the second-gradient models that are defined by (1), it is found that the model becomes unstable for a limited number of wave lengths, while in dynamics, instabilities are encountered for all shorter wave lengths. The corresponding equation for the displacements is unstable for wave numbers \(k > 1/l^2\).

The second class of Laplacian-based gradient elasticity models is described by the linear elastic constitutive relations of the form

\[
\sigma_{ij} = C_{ijkl} \varepsilon_{kl} - l^2 C_{ijkl} \Delta \varepsilon_{kl}.
\]

This equation has the format of equation (1) although the sign of the higher-order term tends to be negative, not positive as in equation (1). Note that whereas the strain gradients with positive sign are destabilizing, the strain gradients with negative sign are stable. The strain gradients in equation (2) are equivalent to those derived from the positive-definite deformation energy density, and therefore the strain gradients in equation (2) are stable. The opposite sign of the strain gradient term in equation (1) makes this term destabilizing. Instabilities manifest themselves in dynamics by an unbounded growth of the response in time without external work. Instabilities are also related to loss of uniqueness in static boundary value problems.

The lattice models are very important in the elasticity theory \[1, 2\]. At the same time, it is formed the opinion that the continuum models described by equation (2) cannot be obtained from lattice models. It is usually assumed that the second class of the gradient models does not have a direct relationship with discrete microstructure and lattice models. This opinion is based on the properties of the Taylor series that is used in homogenization procedure. This problem is described in Section 2 of this paper. In next sections, we propose a lattice model that allows us to remove the lack of the second class of models.
Moreover the suggested type of interaction allows us to have united approach to describe lattice models of the strain-gradient elasticity of two classes. The suggested lattice models give unified description of the second-gradient models with positive and negative signs of the strain gradient terms. A feature of suggested approach is the existence of an operation that transforms \([14, 15, 5]\) (see also \([16, 17]\)) the set of equations for coupled individual particles of lattice into the equation of non-local continuum.

2 Homogenization procedure and Taylor series approach

In order to keep this paper self-contained, we briefly reproduce a derivation of the continuum equation for the gradient elasticity by homogenization approach \([11, 39, 40]\).

In this section, the strain gradient models \([1]\) are derived by means of homogenization of the displacement field of a discrete model. In the lattice model, the particles are replaced by individual masses. The interactions of particle are modeled by springs that connected the point masses. For simplicity, it is assumed that we have one-dimensional lattice, where all particles have the same spring stiffness \(K\), the particle mass \(M\) and the inter-particle distance \(d\).

The gradient elasticity models \([1]\) have been derived from the continualization of the response of a lattice. To illustrate this approach, we will consider the one-dimensional system of particles and springs. All particles have mass \(M\) and all springs have spring stiffness \(K\). The equation of motion of the particle \(n\) is

\[
MD^2 u_n(t) = K \cdot (u_{n+1} - 2u_n(t) + u_{n-1}(t)) + F(n),
\]

where \(M\) and \(K\) are the particle mass and the spring stiffness, respectively, both of which are assumed to be uniform.

In the homogenization procedure, it is assumed the continuous displacement \(u(x,t)\) equals to the lattice displacement \(u_n(t)\) at particle \(n\) by \(u_n(t) = u(nd, t)\), where the particle spacing is denoted as \(d\). The displacement at the neighboring particles is found by means of a Taylor series as

\[
u(x \pm d, t) = u(x, t) \pm d D^1_x u(x) + \frac{d^2}{2} D^2_x u(x) \pm \frac{d^3}{6} D^3_x u(x) + \frac{d^4}{24} D^4_x u(x) + O(d^5).
\]

Next, the displacements of the lattice medium \(u_{n \pm 1}(t)\) are expressed in terms of the continuous displacement. These terms are substituted into equation (3). After division by the cross-section area of the medium \(A\) and the inter-particle distance \(d\) it is found that

\[
\rho D^2_t u(x, t) = E \left( D^2_x u(x, t) + \frac{d^2}{12} D^4_x u(x, t) \right) + f(x)
\]
with the mass density $\rho = M/Ad$, the Young’s modulus $E = Kd/A$, and $f(x) = F(x)/Ad$. Note that all odd derivatives of $u$ have cancelled and signs in front of second-order and fourth-order derivatives coincide. This equation can be written as

$$D_t^2 u(x,t) = C_e^2 D_x^2 u(x,t) + \frac{d^2 C_e^2}{12} D_x^4 u(x,t) + \frac{1}{\rho} f(x),$$

where $C_e = \sqrt{E/\rho}$ is the elastic bar velocity.

When the kinematic relation $\varepsilon = D_x u$ is used, and the equation of motion of the continuum is expresses as

$$\rho D_t^2 u(x,t) = D_x^1 \sigma(x,t) + f(x),$$

the constitutive relation can be retrieved as

$$\sigma = E \left( \varepsilon + \frac{d^2}{12} D_x^2 \varepsilon \right).$$

The positive sign in the relation (8) follows directly from the positive sign of $d^4$-term in the Taylor series (4).

The second-gradient term is preceded by a positive sign, where $d = l \sqrt{12}$. This procedure illustrates the close relation between the discrete microstructure and the gradient non-local continuum with positive sign in (1) and (8).

The homogenization procedure as shown above uniquely leads to a second-order strain gradient term that is preceded by a positive sign. The second-gradient model with negative sign

$$\sigma = E \left( \varepsilon - l^2 D_x^2 \varepsilon \right)$$

cannot be derived from a microstructure of lattice particles by this homogenization procedure.

### 3 Equations of motion for lattice particles

In this paper, we shall use a simplest model to describe the lattice vibration, where all particles are displaced in one direction. We also assume that the displacement of particle from its equilibrium position is determined by a scalar rather than a vector. This model allows us to describe the main properties of a vibrating lattice by using simple equations.

Let us consider a one-dimensional lattice system of interacting particles that are described by the equations of motion

$$M \frac{d^2 u_n(t)}{dt^2} = g_2 \sum_{m=-\infty}^{+\infty} K_2(n,m) \ u_m(t) + g_4 \sum_{m=-\infty}^{+\infty} K_4(n,m) \ u_m(t) + F(n),$$

(10)
where $M$ is the mass of particle, $u_n(t)$ are displacements from the equilibrium, $g_2$ and $g_4$ are coupling constants. The terms $F(n)$ characterize an interaction of the particles with the external on-site force. Let us note some properties of the coefficients $K_s(n, m)$. If we assume the lattice to be displaced as a whole: $u_n(t) = u = \text{const}$, then the internal lattice state cannot be changed in case of absence of external forces $F(n) = 0$. As a result, equations (10) give

$$
\sum_{m=-\infty}^{+\infty} K_s(n, m) = \sum_{m=-\infty}^{+\infty} K_s(m, n) = 0, \quad (s = 2; 4) \quad (11)
$$

for all $n$. This requirement is well-known and it follows from the conservation of total momentum in the lattice [2].

It seems that two terms of (10) with $K_2(n, m)$ and $K_4(n, m)$ can be combined into one without loss of generality. These terms are presented separately to use two different interaction constant $g_2$ and $g_4$ for two type of interactions such as the nearest-neighbor and next-nearest-neighbor interactions. The advantage of such representation is manifested in Section 5. Using two different coupling constants allows us to derive the constitutive relation (1) and (2) with positive and negative depending on the relative values of the coupling constants $g_2$ and $g_4$.

For an unbounded homogeneous lattice, due to its homogeneity the matrix $K_s(n, m)$ has the form $K_s(n, m) = K_s(n - m)$ where $s = 2; 4$. In a simple lattice each particle is an inversion center, and then we have

$$
K_s(n - m) = K_s(m - n) = K_s(|n - m|).
$$

Using the condition (11), we can represent equations (10) in the form

$$
M \frac{d^2u_n(t)}{dt^2} = g_2 \sum_{m=-\infty}^{+\infty} K_2(n - m) \left(u_n - u_m\right) + g_4 \sum_{m=-\infty}^{+\infty} K_4(n - m) \left(u_n - u_m\right) + F(n). \quad (12)
$$

These equations of motion take into account the translation invariance condition of a lattice structure with respect to its displacement as a whole. In equation (12) the interaction terms are translation invariant. The non-invariant interaction terms lead to the divergences in the continuous limit [5].

In this paper we consider the lattice model (12) with the nearest-neighbor and next-nearest-neighbor interactions only. Then the first term $K_2(n - m)$ describes the nearest-neighbor interaction by

$$
K_2(n - m) = -\left(\delta_{n - m, 1} + \delta_{m - n, 1}\right) \quad (13)
$$

with the coupling constant $g_2$. The second term $K_4(n - m)$ describes the next-nearest-neighbor interaction by

$$
K_4(n - m) = -\left(\delta_{n - m, 2} + \delta_{m - n, 2}\right), \quad (14)
$$
and $g_4$ is the coupling constant of this type of interaction.

We can give a discrete mass-spring system that corresponds to the suggested lattice model (12). In Figure 1, we present the mass-spring system with the nearest-neighbor and next-nearest-neighbor interactions.

![Figure 1: Discrete mass-spring system with stiffness coefficients $k_2 = g_2$ and $k_4 = g_4$, the mass $M$ and the distance $d$ that correspond to the lattice model (12) with the nearest-neighbor and next-nearest-neighbor interactions.](image)

4 Map of lattice in the continuum

In this paper to derive a continuum equation from the lattice model we use the approach suggested in [14, 15] instead of the homogenization approach described in Section 2.

In this section, we define the map operation [14, 15, 5] that transforms the equations of motion for $u_n(t)$ of lattice model into continuum equation for a scalar field $u(x, t)$ that describes displacement.

In order to obtain a continuum equation for a lattice equation, we assume that $u_n(t)$ are Fourier coefficients of some function $\hat{u}(k, t)$. We define the field $\hat{u}(k, t)$ on $[-K_0/2, K_0/2]$ by the equation

$$\hat{u}(k, t) = \sum_{n=-\infty}^{+\infty} u_n(t) e^{-ikx_n} = \mathcal{F}_\Delta\{u_n(t)\}, \quad (15)$$

where $x_n = nd$ and $d = 2\pi/K_0$ is distance between equilibrium positions of the lattice particles. The inverse Fourier series transform is defined by

$$u_n(t) = \frac{1}{K_0} \int_{-K_0/2}^{+K_0/2} dk \, \hat{u}(k, t) e^{ikx_n} = \mathcal{F}_\Delta^{-1}\{\hat{u}(k, t)\}. \quad (16)$$
Equations (15) and (16) are the basis for the Fourier transform, which is obtained by transforming from lattice variable to a continuum one in the limit $d \to 0$ ($K_0 \to \infty$). The Fourier transform can be derived from (15) and (16) in the limit as $d \to 0$. We replace the lattice function

$$u_n(t) = \frac{2\pi}{K_0}u(x_n, t)$$

with continuous $u(x, t)$ while letting

$$x_n = nd = \frac{2\pi n}{K_0} \to x.$$ 

Then change ($d \to 0$ or $K_0 \to \infty$) the sum to an integral, and equations (15) and (16) become

$$\tilde{u}(k, t) = \int_{-\infty}^{+\infty} dx \ e^{-ikx}u(x, t) = \mathcal{F}\{u(x, t)\},$$

(17)

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \ e^{ikx}\tilde{u}(k, t) = \mathcal{F}^{-1}\{\tilde{u}(k, t)\}.$$  

(18)

We assume that

$$\tilde{u}(k, t) = \text{Limit} \hat{u}(k, t),$$

where Limit denotes the passage to the limit $d \to 0$ ($K_0 \to \infty$). We note that $\tilde{u}(k, t)$ is a Fourier transform of the field $u(x, t)$, and $\hat{u}(k, t)$ is a Fourier series transform of $u_n(t)$, where we can use $u_n(t) = (2\pi/K_0)u(nd, t)$. The function $\hat{u}(k, t)$ can be derived from $\tilde{u}(k, t)$ in the limit $d \to 0$.

As a result, we define the map from a lattice model into a continuum model by the following operation [14, 15]:

1. The Fourier series transform:

$$\mathcal{F}_\Delta: \ u_n(t) \to \mathcal{F}_\Delta\{u_n(t)\} = \hat{u}(k, t).$$

(19)

2. The passage to the limit $d \to 0$:

$$\text{Limit}: \ \hat{u}(k, t) \to \text{Limit}\{\hat{u}(k, t)\} = \tilde{u}(k, t).$$

(20)

3. The inverse Fourier transform:

$$\mathcal{F}^{-1}: \ \tilde{u}(k, t) \to \mathcal{F}^{-1}\{\tilde{u}(k, t)\} = u(x, t).$$

(21)

Diagrammatically this set of operations for transformation of the displacement can be represented by Figure 2.

![Figure 2: Diagrams of sets of operations for displacement.](image)
We performed similar transformations for differential equations to map the lattice equation into an equation for the elastic continuum. We can represent these sets of transformations of the differential equations in the form of the diagrams presented by Figure 3.

Figure 3: Diagrams of sets of operations for differential equations.

The combination of these operations $F^{-1}$ Limit $F_{\Delta}$ allows us to realize a map of lattice models of interacting particles to models of elastic continuum.

5 Lattice with nearest-neighbor and next-nearest-neighbor interactions

Let us consider a lattice with the nearest-neighbor and next-nearest-neighbor interactions. The nearest-neighbor interaction is described by first terms on the right-hand side of (12) with (13) in the form

$$\sum_{m=-\infty}^{+\infty} K_2(n-m) \left( u_n(t) - u_m(t) \right) = u_{n+1}(t) - 2u_n(t) + u_{n-1}(t).$$

(22)
The next-nearest-neighbor interaction in (12) with (14) is described by the term
\[
\sum_{m=-\infty}^{+\infty} K_4(n-m) \left( u_n(t) - u_m(t) \right) = u_{n+2}(t) - 2u_n(t) + u_{n-2}(t).
\] (23)

Here \( g_2 \) and \( g_4 \) are coupling constants of the nearest-neighbor and next-nearest-neighbor interactions respectively. In general, we have two different coupling constants.

The corresponding continuum equation can be obtained in the limit \( d \to 0 \) by the method suggested in [14, 15, 15]. We have the following statement regarding the suggested lattice model.

**Proposition.** In the continuous limit \( (d \to 0) \) the lattice equations of motion
\[
M \frac{d^2 u_n}{dt^2} = g_2 \cdot \left( u_{n+1} - 2u_n + u_{n-1} \right) + g_4 \cdot \left( u_{n+2} - 2u_n + u_{n-2} \right) + F(n)
\] (24)
are transformed by the combination \( F^{-1} \text{Limit } F_\Delta \) of the operations \([19,21]\) into the continuum equation:
\[
\frac{\partial^2 u(x,t)}{\partial t^2} = G_2 \frac{\partial^2 u(x,t)}{\partial x^2} + G_4 \frac{\partial^4 u(x,t)}{\partial x^4} + \frac{1}{\rho} f(x),
\] (25)
where
\[
G_2 = \frac{(g_2 + 4g_4) d^2}{M}, \quad G_4 = \frac{(g_2 + 16g_4) d^4}{12M}
\] (26)
are finite parameters, and \( f(x) = F(x)/(Ad) \) is the force density, \( \rho = M/(Ad) \) is the mass density.

**Proof.** To derive the equation for the field \( \hat{u}(k,t) \), we multiply equation (24) by \( \exp(-iknd) \), and summing over \( n \) from \(-\infty \) to \(+\infty \). Then
\[
\sum_{n=-\infty}^{+\infty} e^{-iknd} \frac{d^2 u_n}{dt^2} = g_2 \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} \left( u_{n+1} - 2u_n + u_{n-1} \right) +
\]
\[
+ g_4 \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} \left( u_{n+2} - 2u_n + u_{n-2} \right) + \sum_{n=-\infty}^{+\infty} e^{-iknd} F(n).
\] (27)
The first and second terms on the right-hand side of (27) are
\[
g_2 \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} \left( u_{n+1} - 2u_n + u_{n-1} \right) + g_4 \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} \left( u_{n+2} - 2u_n + u_{n-2} \right) =
\]
\[
= g_2 \cdot \left( \sum_{n=-\infty}^{+\infty} e^{-iknd} u_{n+1} - 2 \sum_{n=-\infty}^{+\infty} e^{-iknd} u_n + \sum_{n=-\infty}^{+\infty} e^{-iknd} u_{n-1} \right) +
\]
\[ +g_4 \cdot \left( \sum_{n=\infty}^{-\infty} e^{-iknd} u_{n+2} - 2 \sum_{n=\infty}^{\infty} e^{-iknd} u_n + \sum_{n=\infty}^{\infty} e^{-iknd} u_{n-2} \right) = 
\]
\[ = g_2 \cdot \left( e^{ikd} \sum_{m=\infty}^{\infty} e^{-ikmd} u_m - 2 \sum_{n=\infty}^{\infty} e^{-iknd} u_n + \sum_{j=\infty}^{\infty} e^{-ikjd} u_j \right) + 
\]
\[ = g_4 \cdot \left( e^{2ikd} \sum_{m=\infty}^{\infty} e^{-ikmd} u_m - 2 \sum_{n=\infty}^{\infty} e^{-iknd} u_n + \sum_{j=\infty}^{\infty} e^{-ikjd} u_j \right). \tag{28} \]

Using the definition of \( \hat{u}(k,t) \), equation (28) gives
\[ g_2 \cdot \left( e^{ikd} \hat{u}(k,t) - 2 \hat{u}(k,t) + e^{-ikd} \hat{u}(k,t) \right) + 
\]
\[ + g_4 \cdot \left( e^{2ikd} \hat{u}(k,t) - 2 \hat{u}(k,t) + e^{-2ikd} \hat{u}(k,t) \right) = 
\]
\[ = g_2 \cdot \left( e^{ikd} + e^{-ikd} - 2 \right) \hat{u}(k,t) + g_4 \cdot \left( e^{2ikd} + e^{-2ikd} - 2 \right) \hat{u}(k,t) = 
\]
\[ = 2 \left( g_2 \cdot (\cos (kd) - 1) + g_4 \cdot (\cos (2kd) - 1) \right) \hat{u}(k,t) = 
\]
\[ = 2 \left( -2 g_2 \cdot \sin^2 \left( \frac{kd}{2} \right) - 8 g_4 \cdot \left( \sin^2 \left( \frac{kd}{2} \right) - \sin^4 \left( \frac{kd}{2} \right) \right) \right) \hat{u}(k,t) = 
\]
\[ = -4 (g_2 + 4g_4) \cdot \sin^2 \left( \frac{kd}{2} \right) \hat{u}(k,t) + 16 g_4 \cdot \sin^4 \left( \frac{kd}{2} \right) \hat{u}(k,t). \tag{29} \]

Substitution of (29) into (27) gives
\[ M \frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = -4 (g_2 + 4g_4) \cdot \sin^2 \left( \frac{kd}{2} \right) \hat{u}(k,t) + 16 g_4 \cdot \sin^4 \left( \frac{kd}{2} \right) \hat{u}(k,t) + \mathcal{F}_\Delta \{ F(n) \}. \tag{30} \]

Using the asymptotic behavior of the sine in the form
\[ \sin \left( \frac{kd}{2} \right) = \frac{kd}{2} - \frac{1}{6} \left( \frac{kd}{2} \right)^3 + O((kd)^5), \tag{31} \]
we have
\[ \sin^2 \left( \frac{kd}{2} \right) = \frac{(kd)^2}{4} - 2 \frac{1}{6} \frac{kd}{2} \frac{(kd)^3}{8} + O((kd)^5) = \frac{(kd)^2}{4} - \frac{(kd)^4}{48} + O((kd)^5), \tag{32} \]
\[ \sin^4 \left( \frac{kd}{2} \right) = \frac{(kd)^4}{16} + O((kd)^5). \tag{33} \]
As a result, we can use the representation
\[-4 (g_2 + 4g_4) \cdot \sin^2 \left(\frac{kd}{2}\right) + 16 g_4 \cdot \sin^4 \left(\frac{kd}{2}\right) =
\]
\[= -4 (g_2 + 4g_4) \cdot \frac{(kd)^2}{4} + 4 (g_2 + 4g_4) \cdot \frac{(kd)^4}{48} + 16 g_4 \cdot \frac{(kd)^4}{16} =
\]
\[= -(g_2 + 4g_4) \cdot (kd)^2 + \frac{1}{12} (g_2 + 16g_4) \cdot (kd)^4. \tag{34}\]

Using the finite parameter \(C_e^2 = Kd^2/M\), the transition to the limit \(d \to 0\) in equation (30) gives
\[
\frac{\partial^2 \tilde{u}(k, t)}{\partial t^2} = -G_2 k^2 \tilde{u}(k, t) + G_4 k^4 \tilde{u}(k, t) + \mathcal{F}\{F(x)\}, \tag{35}\]

where
\[
\rho = \frac{M}{Ad}, \quad E = \frac{K d}{A}, \quad G_2 = \frac{E}{\rho} = \frac{(g_2 + 4g_4) d^2}{M}, \quad G_4 = \frac{(g_2 + 16g_4) d^4}{12M}. \tag{36}\]

The inverse Fourier transform \(\mathcal{F}^{-1}\) of (35) has the form
\[
\frac{\partial^2 \mathcal{F}^{-1}\{\tilde{u}(k, t)\}}{\partial t^2} = -G_2 \mathcal{F}^{-1}\{k^2 \tilde{u}(k, t)\} + G_4 \mathcal{F}^{-1}\{k^4 \tilde{u}(k, t)\} + \frac{1}{\rho} f(x).
\]

Then we can use the relation \(\mathcal{F}^{-1}\{\tilde{u}(k, t)\} = u(x, t)\) and the connection between the derivatives and its Fourier transforms
\[
\mathcal{F}^{-1}\{k^2 \tilde{u}(k, t)\} = -\frac{\partial^2 u(x, t)}{\partial x^2}, \quad \mathcal{F}^{-1}\{k^4 \tilde{u}(k, t)\} = +\frac{\partial^4 u(x, t)}{\partial x^4}. \tag{37}\]

As a result, we obtain the continuum equation (25). This ends the proof.

The correspondent principle and relations (36) gives
\[g_2 + 4g_4 = K. \tag{38}\]

It is easy to see that the sign in front the gradient is determined by the value of the coupling constant \(g_4\) for next-nearest-neighbor interaction. If we have the inequalities \(g_4 < 0\) and \(|g_4| > (1/12) K\), then the sign of the constant \(G_4\) will be negative.

If we use the kinematic relation \(\varepsilon(x, t) = \partial u(x, t)/\partial x\), and the equation of motion of the continuum in the form
\[
\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{1}{\rho} \frac{\partial \sigma(x, t)}{\partial x} + \frac{1}{\rho} f(x), \tag{39}\]
then the constitutive relation is represented as

\[
\sigma(x, t) = E \left( \frac{(g_2 + 4g_4)}{EM} \varepsilon(x, t) + \frac{(g_2 + 16g_4)}{12EM} \frac{\partial^2 \varepsilon(x, t)}{\partial x^2} \right)
\]

(40)

with the mass density \( \rho = M/Ad \), and the Young’s modulus \( E = Kd/A \). Using (38), we rewrite relation (40) in the from

\[
\sigma(x, t) = E \left( \varepsilon(x, t) + \frac{(K + 12g_4)}{12K} \frac{\partial^2 \varepsilon(x, t)}{\partial x^2} \right).
\]

(41)

The second-gradient term is preceded by the sign that is defined by \( \text{sgn}(K + 12g_4) \). The scale parameter \( l^2 \) of the gradient elasticity is connected with the coupling constants of the lattice by the equation

\[
l^2 = \frac{|K + 12g_4| d^2}{12K}.
\]

(42)

Equations (38) and (42) give the close relation between the discrete microstructure of lattice and the gradient non-local continuum.

The sign in front the gradient in (41) is determined by the relation of the coupling constants for nearest-neighbor and next-nearest-neighbor interactions. We can list all possible cases:

1. If the coupling constants of the next-nearest-neighbor interaction is

\[
- \frac{1}{4} g_2 < g_4 < -\frac{1}{16} g_2,
\]

then we have the stress-strain constitutive relation with negative sign

\[
\sigma(x, t) = E \left( \varepsilon(x, t) - l^2 \frac{\partial^2 \varepsilon(x, t)}{\partial x^2} \right).
\]

(44)

2. If we have the inequality

\[
g_4 > -\frac{1}{16} g_2,
\]

then the sign in the constitutive relation is positive

\[
\sigma(x, t) = E \left( \varepsilon(x, t) + l^2 \frac{\partial^2 \varepsilon(x, t)}{\partial x^2} \right).
\]

(46)

3. If we have \( g_4 < -(1/4) g_2 \), then the constant \( G_2 \) will be negative and the sign of \( G_4 \) is the same as one of \( g_4 \).

4. If we have \( g_4 = -(1/4) g_2 \), then the constant \( G_2 \) and \( K \) are equal to zero and the sign of \( G_4 \) is the same as one of \( g_4 \).
If we have the condition

$$g_4 = -\frac{1}{16} g_2,$$

then the constant $G_4$ is equal to zero and we have the well-known Hooke’s law without weak nonlocality

$$\sigma(x, t) = E \varepsilon(x, t).$$

As a result we can state that the suggested lattice model gives us an unified approach to the second-gradient elastic continuum models with positive and negative signs of the strain gradient terms. The close relation between the discrete microstructure of lattice and the gradient non-local continuum is given by equations (38) and (42).

The proposed lattice model with the nearest-neighbor and next-nearest-neighbor interactions uniquely leads to second-order strain gradient terms that are preceded by the positive and negative signs. The lattice models with positive value of coupling constant $g_4$ of lattice model leads to the continuum equation with the positive sign in front of the parameter $l^2$. This continuum equation is unstable for wave numbers $k > 1/l^2$. The instability leads to an unbounded growth of the response in time without external work. The negative value of coupling constant $g_4$ of lattice model can lead to stiffness coefficient of the next-nearest-neighbor interaction with non-convex elastic energy potentials in the discrete mass-spring system. At the same time, the correspondent continuum equation with the negative sign in front of the strain gradient are equivalent to those derived from the positive-definite deformation energy density, and therefore these continuum models are stable. We should note that continuum limits for discrete and lattice systems can be considered without convexity hypotheses on discrete energy densities \[18\]. In addition there exist metamaterials with negative stiffness. In the plastic deformation range one can observe a decreasing part of the displacement-force curve, where stiffness along this part of the curve is negative. Viscoelastic materials, nanofilms and molecular chains containing a negative-stiffness phase, anomalies in stiffness and damping have been observed experimentally \[19\] \[20\] \[21\] \[22\]. One of the negative stiffness sources can be obtained from phase transforming materials in the vicinity of their phase transition. A theoretical description of the underlying mechanism from a microscopic viewpoint has been suggested in \[23\]. Drugan \[24\] demonstrated that elastic composite materials having a negative stiffness phase can be stable. A stability analysis for elastic composites with non-positive-definite phase was made by Kochmann and Drugan in \[25\]. Lee and Goverdovskiy \[26\] demonstrated that negative-stiffness elements can be used to improve vibration isolation systems. Elastic composite materials with negative stiffness inclusions can have extreme overall stiffness and mechanical damping \[27\] \[28\].
6 Conclusion

In this paper lattice model with the nearest-neighbor and next-nearest-neighbor interactions for strain-gradient elasticity of continuum is suggested. This model can be considered as a microstructural basis of description for two types of gradient models with positive and negative signs.

We can formulate the main results of this paper in the following short form: (a) The method, suggested in [14, 15] and using the Fourier series and integral transformations, is expanded to elasticity theory; (b) This method is applied to modified lattice model with two coupling constants; (c) The gradient elasticity models with negative sign is derived from this lattice model that have not be done before. Let us give some details about the suggested lattice model with two coupling constants and the proposed method that uses the Fourier transformations.

In the lattice models of gradient elasticity we suggest to use two different coupling constants \( g_2 \) and \( g_4 \). It allows us to derive the constitutive relation that contains these constants. The usual method that based on the direct Taylor expansions (1) allows us to derive the constitutive relation \( \mathcal{N} \) with positive sign only. This traditional approach corresponds to the case \( g_4 = 0 \), where the next-nearest-neighbor interactions are not taken into account. It cannot give the elastic constitutive relations \( \mathcal{P} \) with minus sign. Only incorporation of two types of interactions with two different constants for the nearest-neighbor and next-nearest-neighbor interactions may result in the continuum limit to the gradient elasticity theory with positive and negative signs.

The method based on the Fourier series and integral transformations is represented by Figure 3. This method has been suggested in the papers [14, 15], and it was applied in [16, 17]. Unfortunately this method is not applied to models of gradient elasticity. Moreover there is a common opinion that the gradient models with a negative sign in the constitutive relation \( \mathcal{P} \) cannot be obtained from lattice models. In this paper, the Fourier method is expanded to lattice model with two coupling constants, which is represented by Figure 1. It allows us to prove that the gradient continuum model with negative sign can be obtained from microscopic (lattice) model. This is one of the main results of this paper. We also suggest the conditions (43) for coupling constants, when the gradient continuum model with negative sign can be obtained by the lattice approach.

Let us note some possible extensions of the suggested lattice approach. Using the second-, third, ... nearest-neighbor interactions in the lattice in addition to the nearest-neighbor interaction it can be easily generalized for the case of the high-order gradient elasticity. For this generalization, we can use the terms \( K_{2j}(n, m) = -\left(\delta_{n-m, j} + \delta_{m-n, j}\right) \), where \( j = 1, 2, 3, ... \), that describe an interaction of the \( n \)-particle with two particles with numbers \( n \pm j \). Using models of lattice with long-range interactions, we can get continuum equations with the fractional derivatives [29] of non-integer orders by the methods suggested in [14, 15]. Therefore the suggested lattice model can be extended to get equations for elastic continuum with power-law non-locality [30, 31, 32, 33, 34, 35].
We also assume that the proposed lattice approach can be generalized for lattice with fractal dispersion law \cite{35,36,37}. It allows us to get a possible microscopic basis for gradient elasticity of fractal materials \cite{38} that can be described by different tools (see for example \cite{41,42}). If we consider lattice equation (10), (12), (24) with nonlinear site force \( F(u_n(t)) \) instead of \( F(n) \), then we can get differential equations with nonlinear term \( f(u(x,t)) \) for a continuum in the continuous limit. It is well-known that the localized soliton solutions for lattice models are very important for nonlinear theories \cite{43,44}. Solitons, its interactions, and correspondent discrete lattice models can be considered for different type of physical systems. For example, equations for an electromagnetic line with the second-order couplings can be used to describe the bound solitons with oscillating tails in this line \cite{45}. The approach suggested in this paper can be generalized in order to consider continuum models that correspond to discrete model of a dynamical lattice with the on-site nonlinearity and both nearest-neighbor and next-nearest-neighbor interactions between lattice sites \cite{46,47,48}. We also assume that the unbounded lattice models suggested in this paper can be extended to describe bounded lattices and correspondent continuum models analogous to electromagnetic line described in \cite{45}. We also can assume that suggested approach can be useful for nonlinear deformable-body dynamics \cite{49} for the case of weak non-locality.

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