Existence of $C^k$-Invariant Foliations for Lorenz-Type Maps

Daniel Smania, José Vidarte

Received: 18 February 2016 / Revised: 31 May 2016 / Published online: 22 July 2016
© Springer Science+Business Media New York 2016

Abstract Under conditions similar to those in Shashkov and Shil’nikov (Differ Uravn 30(4):586–595, 732, 1994) we show that a $C^{k+1}$ Lorenz-type map $T$ has a $C^k$ codimension one foliation which is invariant under the action of $T$. This allows us to associate $T$ to a $C^k$ one-dimensional transformation.

Keywords Geometric Lorenz flow · Lorenz-type map · One-dimensional Lorenz-like map · Foliation · Fixed point

1 Introduction

The geometric Lorenz model is an important class of dynamical systems, which was initially studied by Guckenheimer and Williams [11, 13, 31] and Afraimovič et al. [1]. Their aim was to introduce a simple model which has a dynamics similar to the now famous Lorenz system

$$
\dot{x}, \dot{y}, \dot{z} = (10(y-x), 28x - y - xz, -\frac{8}{3}z + xy),
$$

introduced by Lorenz [16]. Lorenz numerically found that in this flow most solutions tend to a certain attracting set, the so-called Lorenz attractor or “strange” attractor, one of the most important early examples of a “chaotic” dynamical system. In particular Lorenz noticed that the Lorenz attractor has sensitive dependence on initial conditions (the so-
called butterfly effect), that is, no matter how close two solutions start, they may have a quite different behaviour in the future. The geometric Lorenz model also has a “strange” attractor with sensitive dependence on initial conditions. The Lorenz and geometric Lorenz attractors also motivated a search for extensions of the notions of hyperbolicity and indeed they are the most representative examples of singular-hyperbolic attractors, a notion that generalizes uniform hyperbolicity for flows in three-dimensional manifolds. They are also the main motivation behind extensions of theses notions to higher dimensions, as the so-called sectional-hyperbolic attractors. For more information on the significance of the Lorenz and geometric Lorenz attractors see Guckenheimer and Holmes [12], Viana [29] and Araújo and Pacifico [4].

Given a $C^{k+1}$ geometric Lorenz flow $X$ on $\mathbb{R}^{n+2}$, by definition there exists a $C^{k+1}$ Poincaré map $T_X: D^* \to D$ associated to it, often called a “Lorenz-type map” [1]. In Shashkov and Shil’nikov [28] it is shown that if a $C^2$ Lorenz-type map $T_X$ satisfies certain conditions, then there exists a $C^1$ codimension one foliation which is invariant under $T_X$. It allows us to associate $T_X$ to a $C^1$ one-dimensional Lorenz like map $f_X: [a, b] \setminus \{c\} \to [a, b]$. This association is the so-called reduction transformation $\mathcal{R}$, so we have $\mathcal{R}T_X = f_X$. So we can study the dynamics of the geometric Lorenz flow (and $T_X$) using the dynamics of such $C^1$ one-dimensional map.

Since the most deep results in one-dimensional dynamics (as the phase-parameter relations in Jakobson’s Theorem [15] and renormalization theory) relies on the study of sufficiently smooth families of one-dimensional maps, the use of these methods and results to the study of geometric Lorenz flows depends on the study the smoothness of the reduction transformation $\mathcal{R}$. That was the main motivation of this work.

There are already impressive results using this approach. For instance Rovella [26] proved that the contracting Lorenz attractor (the so-called Rovella attractor) persists in a measure theoretical sense. More precisely there exists an one-parameter transversal family of $C^3$ vector fields passing through to flow with a Rovella attractor, where on a positive Lebesgue set of parameters the corresponding flow has a transitive non-hyperbolic attractor. This is done using the fact that the one-dimensional Lorenz-like map $f: [a, b] \setminus \{c\} \to [a, b]$, obtained taking the quotient of the dynamics with respect to the invariant stable foliation, is a map of of class $C^k$, $k \geq 3$. There are a few other works where methods of $C^k$ one-dimensional dynamics are applied, as for instance in Morales et al. [21], Robinson [25], Rychlik [27], Araújo and Varandas [6], Araújo and Pacifico [4], Araújo et al. [5], Araújo and Melbourne [3].

We show that if a $C^{k+1}$ Lorenz type map $T$ satisfies certain conditions (see Assumption 2.2), then there exists a $C^k$ codimension one foliation which is invariant under the action of $T$. This generalizes the main result of Shashkov and Shil’nikov [28] for an arbitrary degree of smoothness. It also extends results in Rychlik [27, Corollary 4.2], Rovella [26, Proposition, p. 241] and Morales et al. [21, Lemma 2]. Our main Theorem allows us to introduce new smooth coordinates $\{(x, \eta)\}$ in the domain of $T$ in such way that the map $T$ has the form $T(x, \eta) = (\overline{F}(x, \eta), \overline{G}(\eta))$ (see Afraimovich and Pesin [2, p. 178]); where $\overline{F}$ and $\overline{G}$ are $C^k$ functions, so $T$ can be associated to a $C^k$ one-dimensional transformation $\overline{G}: [a, b] \setminus \{c\} \to [a, b]$.

We hope that the main result of this work will be useful to the study of geometric Lorenz flows since it will allows us to apply to this setting results on one-dimensional dynamics as those obtained in Martens and de Melo [17], Martens and Winckler [18, 19] and Brandão [8] and Brandão et al. [22,23] (Fig. 1).
2 Statement of the Main Result

Let $\mathbb{R}^{n+1} := \mathbb{R}^n \times \mathbb{R}$ be a $(n+1)$-Euclidean space. From now on, the symbol $\| \cdot \|$ denotes a norm in $\mathbb{R}^n$, if applied to a vector or for the corresponding matrix norm if applied to a matrix. We also use the notation

$$\| \cdot \|_D = \sup_{(x,y) \in D^*} \| \cdot \|$$

for norms of matrices and vector functions on $D^*$.

Define

$$D := \{(x, y) \in \mathbb{R}^{n+1} : \|x\| \leq 1, |y| \leq 1\},$$
$$D_+ := \{(x, y) \in D : y > 0\},$$
$$D_- := \{(x, y) \in D : y < 0\},$$
$$D_0 := \{(x, y) \in D : y = 0\},$$
$$D^* := D_- \cup D_+ = D \setminus D_0.$$  \hspace{1cm} (1)

Notice that the sets $D_+$ and $D_-$ are separated by the hyperplane $D_0$.

Let us consider the map $T : D^* \to D$ given by

$$T(x, y) = (F(x, y), G(x, y)) = (\overline{x}, \overline{y}),$$

where the vector function $F$ and the scalar function $G$ are differentiable on $D^*$ and $\partial_y G(x, y)$ is non-vanishing on $D^*$.

**Definition 2.1** We define the following functions:

$$A(x, y) := \partial_x F(x, y)(\partial_y G(x, y))^{-1},$$
Here $A(x, y)$ is a $n \times n$ matrix, $B(x, y)$ is a $n$-column vector and $C(x, y)$ is a $n$-row vector.

**Assumption 2.2** We assume the following conditions hold on $T$:

\[(L_1)\] The functions $F$ and $G$ have the forms

\[
F(x, y) = \begin{cases} 
  x_+^* + |y|\alpha[B_+^* + \varphi_+(x, y)], & y > 0, \\
  x_-^* + |y|\alpha[B_-^* + \varphi_-(x, y)], & y < 0,
\end{cases}
\]

\[
G(x, y) = \begin{cases} 
  y_+^* + |y|\alpha[A_+^* + \psi_+(x, y)], & y > 0, \\
  y_-^* + |y|\alpha[A_-^* + \psi_-(x, y)], & y < 0,
\end{cases}
\]

in a neighborhood of $D_0$, where $A_+^*, A_-^*$ are nonzero constants; $\alpha$ represents a strictly positive constant and the functions $\varphi_{\pm}, \psi_{\pm}$ are of class $C^{k+1}$. The derivatives of $\varphi_{\pm}$ and $\psi_{\pm}$ are uniformly bounded with respect to $x$ and satisfy the estimates:

\[
\left\| \frac{\partial^{l+m} \varphi_{\pm}(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m}, \quad \left\| \frac{\partial^{l+m} \psi_{\pm}(x, y)}{\partial x^l \partial y^m} \right\| \leq K |y|^{\gamma-m},
\]

where $\gamma > k - 1$, $K$ is a positive constant, $l = 0, 1, \ldots, k + 1$ and $m = 0, 1, \ldots, k + 1$.

\[(L_2)\] The following inequality holds:

\[
1 - \|A\|_D > 2\sqrt{\|B\|_D \|C\|_D}.
\]

\[(L_3)\] The following relations hold:

(a)\[
\frac{(2!)^2 (\|A\|_D + \|C\|_D \|B\|_D) \max_{m+n=1} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n \}}{(\|\partial \gamma G\|_D)^{-1} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D} \right)^2} < 1.
\]

(b) for $k \geq 2$

\[
\frac{(2k!)^2 (\|A\|_D + \|C\|_D \|B\|_D) \max_{m+n=k} \{(\|A\|_D + \|B\|_D)^m (\|C\|_D + 1)^n \}}{(\|\partial \gamma G\|_D)^{-k} \left(1 + \|A\|_D + \sqrt{(1 - \|A\|_D)^2 - 4\|B\|_D \|C\|_D} \right)^2} < 1,
\]

and

\[
\|\partial \gamma G\|_D \geq \frac{1}{4} \quad \text{or} \quad \|\partial \gamma F\|_D \geq \frac{1}{4}.
\]

The following sets will be the domains of several maps that will appear along this work.

\[
D_x := \{ x \in \mathbb{R}^n \text{ for which there exists a } y \in \mathbb{R} \text{ with } (x, y) \in D \}.
\]

Given a map $h : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we define its graph as

\[
\text{graph}(h) := \{ (x, h(x)) : x \in U \}.
\]
Definition 2.3 A family of functions $\mathcal{F}_D = \{h(x)\}$ is called a foliation of $D$ with $C^m$ leaves $(m \geq 0)$ given by the graphs of functions $y = h(x)$ if the following three conditions are satisfied:

(a) The domain $\text{Dom}(h(x))$ of every function $h(x) \in \mathcal{F}_D$ is an open and connected set in $D_x$ and its graph lies entirely in $D$;
(b) For every point $(x_0, y_0) \in D$ there is a unique function $h(x) \in \mathcal{F}_D$ such that $x_0 \in \text{Dom}(h(x))$ and $y_0 = h(x_0)$. This function will be denoted by $h(x; x_0, y_0)$;
(c) For every point $(x_0, y_0) \in D$ the function $x \mapsto h(x; x_0, y_0)$ is of class $C^m$.

The graphs of the functions $h(x)$ are called the leaves of $\mathcal{F}_D$ and the leaf that contains $(x_0, y_0) \in D$ will be denoted by $\mathcal{F}(x_0, y_0)$ (Fig. 2).

Definition 2.4 A foliation $\mathcal{F}_D$ is called $C^r$-foliation ($r \geq 0$) if the function

$$(x; x_0, y_0) \mapsto h(x; x_0, y_0)$$

is of class $C^r$.

Definition 2.5 A foliation $\mathcal{F}_D$ is called $T$-invariant if

(a) the hyperplane $D_0 \in \mathcal{F}_D$;
(b) for each leaf $\mathcal{F}(x_0, y_0) \in \mathcal{F}_D$, with $\mathcal{F}(x_0, y_0) \neq D_0$, there is $\mathcal{F}(x_0, y_0) \in \mathcal{F}$ such that $T(\mathcal{F}(x_0, y_0)) \subset \mathcal{F}(x_0, y_0)$.

Remark 2.6 Suppose that $\nu : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is a $C^k$ function completely integrable, that is, there exists a solution for the initial value problem for the differential equation

$$\nabla y(x) = \nu(x, y(x)), \quad y(x_0) = y_0.$$  \hspace{1cm} (3)

for all $(x_0, y_0) \in D$, where $y : U(x_0) \subset D \rightarrow [-1, 1]$ and $U(x_0)$ is a neighborhood of $x_0$. Then, by using Frobenious-Dieudonné Theorem [10, Theorem 10.9.5] we have that

$$\mathcal{F}_D := \{\text{graph}(y) : y \text{ satisfying}(3)\},$$

determines a foliation, that is, the leaves are the graphs of the solutions of the differential equation defined by the function $\nu : D \rightarrow \mathbb{R}^n$. 

Fig. 2 Geometric interpretation of a $T$-invariant foliation
We are ready to state our main result.

**Theorem 2.7 (Main Theorem)** Suppose that the map $T$ satisfies Assumption 2.2. Then, there is a codimension one $T$-invariant $C^k$ foliation $\mathcal{F}_D$ with $C^{k+1}$ leaves.

**Corollary 2.8** Suppose that the map $T$ satisfies Assumption 2.2. Then, there exists a change of variable $\chi : D \to D$ such that $T$ can be associate with a skew-product $\bar{T} : D^* \to D$ of class $C^k$ such that the diagram

$$
\begin{array}{ccc}
D^* & \xrightarrow{T} & D \\
\downarrow \chi & & \downarrow \chi \\
D^* & \xrightarrow{\bar{T}} & D
\end{array}
$$

commutes, that is, $\chi \circ T = \bar{T} \circ \chi$ on $D^*$.

**Proof** The details can be found in [2, p. 178].



3 Overview of the Proof of Main Theorem 2.7

The principal aim of this section is to sketch the proof of our main theorem.

3.1 The Big Picture

Bearing in mind the Remark 2.6 and following the ideas of Robinson [24] and Belickii [7], the foliation $\mathcal{F}_D$ of Theorem 2.7 will be obtained as the integral surfaces of a $C^k$ completely integrable function $\nu : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, which will be a fixed point of an appropriate graph transform $\Gamma$. Next, we give a brief outline of the idea behind the graph transform $\Gamma$, which is also illustrated in Fig. 3.

Our goal is to find a $C^k$ integrable function $\nu^* : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, so that for every integral surface $h$ its graph is invariant under $T(x, y) = (F(x, y), G(x, y)) := (\bar{x}, \bar{y})$. This means that

$$
F(x, h(x)) = \bar{x},
G(x, h(x)) = \bar{h}(\bar{x}),
$$

where $\bar{h}$ is an integral surface of $\nu^*$. To find $\nu^*$ we take any completely integrable function $\nu : D \to \mathbb{R}^n$ and seek a completely integrable function $\nu : D \to \mathbb{R}^n$ so that

$$
F(x, h(x)) = \bar{x},
G(x, h(x)) = \bar{h}(\bar{x}),
$$

where $h$ is an integral surface of $\nu$ and $\bar{h}$ is an integral surface of $\bar{\nu}$.

If such a function exists, we define the graph transform of $\nu$ via $\Gamma(\nu) := \nu$ and note that the desired function $\nu^*$ is a fixed point of the graph transform so that $\Gamma(\nu^*) = \nu^*$. It is not difficult to see that

$$
\Gamma(\bar{\nu})(x, y) = \begin{cases} 
\bar{\nu} \circ T(\bar{x}, \bar{y}) \frac{\partial G(x, \bar{y})}{\partial \bar{y}} - \frac{\partial F(x, \bar{y})}{\partial \bar{y}} , & y \neq 0, \\
0, & y = 0.
\end{cases}
$$
Notice that, in view of Definition 2.1, we can rewrite the operator $\Gamma$ in the following way:

$$\Gamma(v)(x, y) = \begin{cases} 
\left(\nu \circ TA - C \right) \left(1 - \nu \circ TB\right)(x, y), & y \neq 0, \\
0, & y = 0.
\end{cases}$$

It is not difficult to show that the graph transform $\Gamma$ is well defined on a complete sub-space $A_L$ of the continuous function from $D$ to $\mathbb{R}^n$, and that $\Gamma$ has a fixed point $\nu^*$ (see Theorem 3.4). Our goal in this work is to show that the fixed point $\nu^*$ is a completely integrable $C^k$ function. Then, by using the Idea 1 we have that the graphs of the integral surfaces give the foliation $\mathcal{F}_D$ of Theorem 2.7.

### 3.2 The Operator $\Gamma$

In this section we give a rigorous definition and state some properties of the operator $\Gamma$ described informally in the last subsection. We begin by introducing the following definition.

**Definition 3.1** Let $L \geq 0$. We denote by $\mathcal{A}_L$ the set of all functions $\nu : D \to \mathbb{R}^n$ which satisfy the following conditions:

(a) $\nu$ is continuous on $D$;

(b) $\|\nu\| \leq L$;

(c) $\nu(x, 0) = 0$, if $\|x\| \leq 1$.

**Remark 3.2** Since $\mathbb{R}^n$ is a complete normed space, it is not difficult to show that $\mathcal{A}_L$ is a complete metric space with the norm of the supremum.

Now we are ready to define the most important operator of our work. This operator is denoted by $\Gamma$ and is defined as in [28, Eq. 6] by
Definition 3.3

\[ \Gamma: \mathcal{A}_L \rightarrow \Gamma(\mathcal{A}_L) \]

where the function \( \Gamma(\nu): D \rightarrow \mathbb{R}^n \) is given by

\[ \Gamma(\nu)(x, y) = \begin{cases} \frac{(\tau y, A - C)}{(1 - \nu y B)}(x, y), & y \neq 0, \\ 0, & y = 0, \end{cases} \]

with the functions \( A, B \) and \( C \) as in Definition 2.1.

Next, we list a few basic properties of the operator \( \Gamma \). Details may be found in [28, Lemma 1] or [30, Proposition 3.17].

**Proposition 3.4** There is a constant \( L \geq 0 \) such that

(a) \( \Gamma(\mathcal{A}_L) \subset \mathcal{A}_L \).

(b) The operator \( \Gamma: \mathcal{A}_L \rightarrow \mathcal{A}_L \) is a contraction.

(c) The operator \( \Gamma: \mathcal{A}_L \rightarrow \mathcal{A}_L \) has a unique fixed point \( \nu^* \). Moreover \( \nu^* \) is a completely integrable function.

(d) The operator \( \Gamma \) takes completely integrable functions into completely integrable functions. Moreover, if \( \mathcal{F}_D \) and \( \mathcal{F}_D \) are foliations defined by the completely integrable functions \( \nu \) and \( \Gamma(\nu) \) respectively; then \( T \) takes every leaf \( \mathcal{F}(x_0, y_0) \in \mathcal{F}_D, \mathcal{F}(x_0, y_0) \neq D_0 \), into a part of the leaf \( \mathcal{F}_T(x_0, y_0) \in \mathcal{F}_D \), that is, \( T(\mathcal{F}(x_0, y_0)) \subset \mathcal{F}_T(x_0, y_0) \).

**Remark 3.5** It is known from [28, Eq. 81] and [30, Eq. 3.47] that \( L \) can be taken as

\[ L = \frac{-(1 - \| A \|) + \sqrt{\| A \| - 1)^2 - 4\| B \| \| C \|}}{2\| B \|}. \]

Let us state the main proposition of this article.

**Proposition 3.6** Let \( L \geq 0 \) be as in Proposition 3.4. Then, the attracting fixed point \( \nu^* \) of the operator \( \Gamma \) is a function of class \( C^k \).

The proof of this proposition will be given using the following propositions which will be proven in the next sections.

**Proposition 3.7** If \( \mu \in \mathcal{A}_L \) is a \( C^k \) function. Then, the following statements hold:

(a) \( \lim_{(a, b) \rightarrow (x, 0)} D^i(\Gamma(\mu))(a, b) = 0 \), for all \( 1 \leq i \leq k \) and \((x, 0) \in D_0 \).

(b) The function \( \Gamma(\mu) \in \mathcal{A}_L \) is of class \( C^k \) and \( D^i(\Gamma(\mu))(x, 0) = 0 \), for all \( 1 \leq i \leq k \) and \((x, 0) \in D_0 \).

**Proposition 3.8** If \( \nu \in \mathcal{A}_L \) is a \( C^k \) function and \( D^i\nu(x, 0) = 0 \), for all \( 0 \leq i \leq k \) and \((x, 0) \in D_0 \). Then, the following limit exists

\[ \lim_{n \rightarrow \infty} (\Gamma^n(\nu), D(\Gamma^n(\nu)), \ldots, D^k(\Gamma^n(\nu))) = (\nu^*, A_1, A_2, \ldots, A_k), \]

where \( A_1, A_2, \ldots, A_k \) are continuous functions.

**Proof of Proposition 3.6** Let \( \nu \in \mathcal{A}_L \) be a \( C^k \) function. By Proposition 3.6 we have that \( \nu := \Gamma(\nu) \) is a \( C^k \) function and that \( D^k\nu(x, 0) = 0 \). From Proposition 3.8, we have that

\[ \lim_{n \rightarrow \infty} (\Gamma^n(\nu), D(\Gamma^n(\nu)), \ldots, D^k(\Gamma^n(\nu))) = (\nu^*, A_1, A_2, \ldots, A_k). \]
Hence, interchanging the order between the limit and the differentiation in the expression above, which is permissible by [10, Theorem 8.6.3], we obtain
\[ D_j(\lim_{\gamma \to \infty} \Gamma^\gamma(\bar{v})) = A_j, \]
for \(0 \leq j \leq k\). Thus, since \(v^*\) is a global attracting fixed point of \(\Gamma\), it follows that \(D_j(v^*) = A_j\), for \(0 \leq j \leq k\). Therefore, since \(A_j\) is a continuous function, it follows that the function \(v^*\) is of class \(C^k\), which concludes the proof of our main proposition. \(\square\)

Now we are ready to prove our main Theorem 2.7.

**Proof of Theorem 2.7** By Proposition 3.4(c) we have that the attracting fixed point \(v^*\) of the operator \(\Gamma\) is integrable and by Proposition 3.6 we get that \(v^*\) is of class \(C^k\). Thus, the function \(v^*\) defines a foliation \(\mathcal{F}_D\) of class \(C^k\) and by Proposition 3.4(d) it follows that the foliation \(\mathcal{F}_D\) is \(T\)-invariant, which finishes the proof of our main result. \(\square\)

4 Proof of Proposition 3.7

The proof is somewhat lengthy, so we divide it into two parts. In the first part we will establish a formula for the \(k\)th order derivatives of the function \(\Gamma(v)\) at points \((x, y)\), where the \(y\)-component stay away from zero. In the second part we estimate the norms of the \(i\)th derivatives of the functions \(A(x, y), B(x, y)\) and \(C(x, y)\), at points \((x, y)\) around of a neighborhood of \(D_0\).

4.1 Part 1: Formula for Derivatives

Before that, we give some definitions which will be useful in order to find suitable formulas. From now on, \(L(E_1, \ldots, E_k; G)\) denotes the space of continuous \(k\)-multilinear maps of \(E_1, \ldots, E_r\) to \(G\). If \(E_i = E, i \leq k\), this space is denoted by \(L^k(E, F)\). Moreover, \(L^k_s(E; F)\) denotes the subspace of symmetric elements of \(L^k(E, F)\).

**Definition 4.1** (Symmetrizing operator) The Symmetrizing operator \(\text{Sym}^k\) is defined by
\[
\text{Sym}^k: L^k(E; F) \longrightarrow L^k(E; F) \quad A \longmapsto \text{Sym}^k(A) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma A,
\]

where \((\sigma A)(e_1, \ldots, e_k) = A(e_{\sigma(1)}, \ldots, e_{\sigma(k)})\) and \(S_k\) is the group of permutations on \(k\) elements.

**Remark 4.2** The symmetrizing operator \(\text{Sym}^k\) satisfies the following properties:
(a) \(\text{Sym}^k(L^k(E; F)) = L^k_s(E; F)\),
(b) \((\text{Sym}^k)^2 = \text{Sym}^k\),
(c) \(\|\text{Sym}^k\| \leq 1\).

**Definition 4.3** Assume \(B \in L(F_1 \times F_2; G)\). We define the bilinear map
\[
\phi^{(i,(k-i))} : L_i(E; F_1) \times L^{(k-i)}(E; F_2) \longrightarrow L^k(E; G)
\]
by
\[
[\phi^{(i,(k-i))}(A_1, A_2)](e_1, \ldots, e_k) = B(A_1(e_1, \ldots, e_i), A_2(e_{i+1}, \ldots, e_k)).
\]
Definition 4.4 Let $\overline{v}_i : U \to L^i(E; F_1)$ and $\overline{v}_{(k-i)} : U \to L^{(k-i)}(E; F_2)$, we define
\[\phi^{i,(k-i)}(\overline{v}_i, \overline{v}_{(k-i)}) : U \mapsto L^k(E; G) \quad p \mapsto \phi^{i,(k-i)}(\overline{v}_i(p), \overline{v}_{(k-i)}(p)). \tag{7}\]

Definition 4.5 For every tuple $(q, r_1, r_2, \ldots, r_q)$, where $q > 1$, and $r_1 + \cdots + r_q = k$, we define the following continuous multilinear map
\[\phi^{q,r_1,\ldots,r_q} : L^q(F; G) \times L^{r_1}(E; F) \times \cdots \times L^{r_q}(E; F) \to L^k(E; G) \tag{8}\]
where
\[\phi^{q,r_1,\ldots,r_q}(\overline{v}_q, \overline{v}_{r_1}, \ldots, \overline{v}_{r_q}) : E \times \cdots \times E \to G\text{ k-times}\]
is defined as
\[\phi^{q,r_1,\ldots,r_q}(\overline{v}_q, \overline{v}_{r_1}, \ldots, \overline{v}_{r_q}) (e_1, \ldots, e_k) = \overline{v}_q(\overline{v}_{r_1}(e_1, \ldots, e_{j_1}), \ldots, \overline{v}_{r_q}(e_{(j_1 + j_2 + \cdots + j_{(q-1)} + 1)}, \ldots, e_{(j_1 + j_2 + \cdots + j_q)})). \tag{9}\]

Definition 4.6 Let $U \subset E$ such that $\overline{v}_r : U \to L^r(E; F)$, $1 \leq i \leq q$ and $f : V \subset U \to U$ are functions. Then, define
\[\phi^{q,r_1,\ldots,r_q} \ast ((\overline{v}_q \circ f) \times \overline{v}_{r_1} \times \cdots \times \overline{v}_{r_q}) : U \to L^k(E; G) \tag{10}\]
by
\[u \mapsto \phi^{q,r_1,\ldots,r_q}((\overline{v}_q \circ f(u)), \overline{v}_{r_1}(u), \ldots, \overline{v}_{r_q}(u)), \]
where $\phi^{q,r_1,\ldots,r_q}((\overline{v}_q \circ f(u)), \overline{v}_{r_1}(u), \ldots, \overline{v}_{r_q}(u))$ is as in Definition 4.5.

Next, we define generalizations of the $k$th derivative of the composition of two functions.

Definition 4.7 Let $k_1 \geq k_3 \geq k_2 \geq 1$ be integers such that we have the functions $\overline{v}_q : V \subset F \to L^q(F, G)$, for $k_2 \leq q \leq k_3$ and that $f : U \to V$ is a function of class $C^{k_1-k_2+1}$, where $D^i f : U \to L^i(E, F)$, $0 \leq i \leq k_1 - k_2 + 1$ are the derivatives of $f$. Then, we define the function
\[\mathcal{D}C^{(k_1,k_2,k_3)}((\overline{v}_{k_2}, \ldots, \overline{v}_{k_3}), f) : U \to L^{k_1}(E, F)\]
given by
\[\mathcal{D}C^{(k_1,k_2,k_3)}((\overline{v}_{k_2}, \ldots, \overline{v}_{k_3}), f)(p) := \text{Sym}^{k_1} \left( \sum_{n=k_2}^{k_3} \sum_{r_1+\cdots+r_n=k_1} \frac{k_1!}{r_1! \cdots r_n!} \phi^{(n,r_1,\ldots,r_n)} \ast ((\overline{v}_n \circ f) \times D^{r_1}f \times \cdots \times D^{r_n}f)(p) \right), \tag{11}\]
where $\phi^{(n,r_1,\ldots,r_n)} \ast ((\overline{v}_n \circ f) \times \overline{v}_{r_1} \times \cdots \times \overline{v}_{r_n})$ is as in Definition 4.6, and when $k_1 = k_3 = 0$ and $k_2 = 1$, we define the function
\[\mathcal{D}C^{(0,1,0)}(\overline{v}_0, f) : U \to F\]
given by
\[\mathcal{D}C^{(0,1,0)}(\overline{v}_0, f)(p) := (\overline{v}_0 \circ f)(p). \tag{12}\]
Furthermore, if $\nu : V \subset F \to G$ and $f : U \subset E \to V \subset F$ are $C^{k_3}$, then the following notation will be adopted.

$$ DC^{(k_1,k_2,k_3)}(\nu, f) := DC^{(k_1,k_2,k_3)}(D^{k_2}(\nu), \ldots, D^{k_3}(\nu), f). $$ (13)

**Remark 4.8** If $\nu : D \to \mathbb{R}^n$ and $T : D^* \to D$ are functions $C^k$, then:

(i) Applying the chain rule to the function $\nu \circ T$ it is possible to show that

$$ DC^{(k,1,k)}(\nu, T) := D^k(\nu \circ T). $$

Therefore, we conclude that the function in Definitions 4.7 is a generalization of the $k$th derivative of the composition of two functions.

(ii) By Eq. (13) and the symmetry of the function $D^k\nu$, we obtain

$$ DC^{(k,k,k)}(\nu, T) := k!(D^k\nu) \circ T DT \ldots DT. $$ (14)

Next, let define the generalization of the $k$th derivative of the product map $(f \circ g)$ with $h$.

**Definition 4.9** Assume $B \in L(F_1 \times F_2; G)$ and that $k_1 \geq k_3 \geq 1$; $k_2 \geq 0$ are integers such that $f : U \subset E \to V \subset F$ and $B : U \to F_2$ are functions of class $C^{k_3}$ and $C^{k_1-k_2}$ respectively, where $D^i B : U \to L^i(E, F_2)$, for $0 \leq i \leq k_1 - k_2$ are the derivatives of the function $B$, moreover consider the functions $\nu_i : V \subset F \to L^i(F, F_1)$, $0 \leq i \leq k_3$. Then, we define the map

$$ DC^P^{(k_1,k_2,k_3)}(f, (\nu_0, \nu_1, \ldots, \nu_{k_3}), B) : V \to L^{k_1}(E; G), $$

given by

$$ DC^P^{(k_1,k_2,k_3)}(f, (\nu_0, \nu_1, \ldots, \nu_{k_3}), B)(p) := Sym^{k_1} \left( \sum_{n=k_2}^{k_3} \binom{k_1}{n} \phi^{(n,k_1-n)} \left( DC^P^{(n,1,n)}((\nu_1, \ldots, \nu_n), f), D^{k_1-n}B \right) \right)(p), $$ (15)

where $\phi^{(n,k_1-n)}$ is as in Definition 4.4. Moreover, whenever $\nu : U \subset E \to F$ is a function of class $C^{k_3}$, we use the notation

$$ DC^P^{(k_1,k_2,k_3)}(\nu, f, B) := DC^P^{(k_1,k_2,k_3)}(f, (\nu, D(\nu), \ldots, D^{k_3}(\nu)), B). $$ (16)

**Remark 4.10** Let $F_1$ and $F_2$ be the space of the $n$-columns and $n$-rows respectively. We define the multilinear map $B : F_1 \times F_2 \to \mathbb{R}$ given by $B(A, B) = A \times B$, where $A \times B$ is the usual product of matrices. Assume that $\nu : D \to \mathbb{R}^n$, $T : D^* \to D$ and $B : D^* \to F_1$ are $C^k$ functions. Then, by using Leibniz and chain rule applied to the functions $((\nu \circ T).B)$ and $(\nu \circ T)$, respectively and in view of Eq.(4.8) and Definition 4.9 it is easy to see that

$$ DC^P^{(k,0,k)}(\nu, T, B) := D^k((\nu \circ T)B). $$ (17)

This shows that the function in Definition 4.9 generalizes the $k$th derivative of the product of the map $(\nu \circ T)$ with $B$. This fact will be useful later.

Next, we define generalizations of the $k$th derivative of the map $(1 - \nu \circ TB)^{-1}$, where $B$ is a function as in Definition 2.1 and $\nu \in A_L$ is a function of class $C^k$. 

 Springer
Definition 4.11 Let \((q, r_1, \ldots, r_q, r)\) be a tuple with \(q \geq 1\), \(r_1 + \cdots + r_q = k\) and \(r = \max\{r_1, \ldots, r_q\}\) such that \(\overline{v}_i : V \subset F \to L^i_0(F, F_1), 0 \leq i \leq r\) are functions and that \(T : U \subset E \to V \subset F\) and \(B : U \to F_2\) are functions of class \(C^k\). Then, we define the map

\[
\prod_{(r_1, \ldots, r_q, r)} (\overline{v}_0, \ldots, \overline{v}_{r_q}, T, B) : U \to L^k(E, G)
\]
given by

\[
\prod_{(r_1, \ldots, r_q, r)} (\overline{v}_0, \ldots, \overline{v}_{r_q}, T, B) := \mathcal{DCP}(r_1, 0, \cdots, r_1)(T, (\overline{v}_0, \ldots, \overline{v}_{r_1}), B) \times \cdots \times \mathcal{DCP}(r_q, 0, \cdots, r_q)((\overline{v}_0, \ldots, \overline{v}_{r_q}), T, B),
\]

where \(\mathcal{DCP}(r_i, 0, r_i)(T, (\overline{v}_0, \ldots, \overline{v}_{r_i}), B)\), \(1 \leq i \leq q\) is as in Definition 4.9.

Definition 4.12 Under the notations of Definition 4.11, suppose that \(k_1 \geq k_3 \geq k_2 \geq 1\) are integers. Then, we define the map

\[
\mathcal{DICP}(k_1, k_2, k_3)(T, (\overline{v}_0, \ldots, \overline{v}_{(k_1 - k_2) + 1}), B) : U \to L^{k_1}(E, G)
\]
given by

\[
\mathcal{DICP}(k_1, k_2, k_3)((\overline{v}_0, \ldots, \overline{v}_{(k_1 - k_2) + 1}), T, B) := \text{Sym}^{k_3} \left( \sum_{q=0}^{k_3} \sum_{r_1 + \cdots + r_q = k_1} \frac{k!(-1)^q q!}{r_1! \cdots r_q!} (1 - v_0 \circ TB)^{(q+1)} \prod_{(r_1, \ldots, r_q, r)} (\overline{v}_0, \ldots, \overline{v}_{r_q}, T, B) \right),
\]

where \(\prod_{(r_1, \ldots, r_q, r)} (\overline{v}_0, \ldots, \overline{v}_{r_q}, T, B)\) is as in Definition 4.11. Furthermore, if \(\overline{v} : U \subset F \to F_1\) is a \(C^{k_1-k_2+1}\) map, then we use the following notation

\[
\mathcal{DICP}(k_1, k_2, k_3)(\overline{v}, T, B) := \mathcal{DICP}(k_1, k_2, k_3)((\overline{v}, D(\overline{v}), \ldots, D^{k_1-k_2+1}(\overline{v})), T, B).
\]

Remark 4.13 Let \(v \in \mathcal{A}_L, B, T\) be maps as in Definition 3.1, Definition 2.1 and Definition 2 respectively such that \(v\) and \(B\) are \(C^k\). Define \(I : \mathbb{R} - \{0\} \to \mathbb{R}\) by \(I(x) = 1/x\). Since, the function \((1 - \overline{v} \circ TB) : D^x \to \mathbb{R}\) is nonzero, it follows from chain rule applied to \(I \circ (1 - \overline{v} \circ TB)\) and Definition 4.12 that

\[
D^k(1 - \overline{v} \circ TB)^{-1} = \text{Sym}^{k} \circ \sum_{q=1}^{k} \sum_{r_1 + \cdots + r_q = k} k!q! D^r_1(\overline{v} \circ TB) \cdots D^r_q(\overline{v} \circ TB) (1 - v \circ TB)^{(q+1)} \prod_{(r_1, \ldots, r_q, r)} (\overline{v}_0, \ldots, \overline{v}_{r_q}, T, B).
\]

This shows that the function in Definition 4.12 generalizes the \(k\)th derivative of the map \((1 - v \circ B)^{-1}\). This fact will be useful later.

Now, using the last definitions we found a formula for the \(k\)th derivative of the function \(\Gamma(v)\) at the points \((x, y)\), with \(y \neq 0\) (see Lemma 4.18). This generalizes the formulas given in [28, Eq. 11] and [21, Eq. 41], and it will be quite important to prove our main Proposition 3.6. We start by noticing the following simple but very useful lemma.
Lemma 4.14  Under Definitions 2.1, 3.1 and 3.3. Assume that \( \nu \in A_L \) is a \( C^k \) function. Then, for \( y \neq 0 \) the following formulas hold:

\[
D(\Gamma(\nu))(x, y) = (\nu \circ TA - C)D(1 - \nu \circ TB)^{-1}(x, y)
+ D(\nu \circ TA - C)(1 - \nu \circ TB)^{-1}(x, y)
:= (U_1^k(\nu, T, A, B, C) + U_2^k(\nu, T, A, B, C))(x, y);
\]

for \( k \geq 2 \)

\[
D^k(\Gamma(\nu))(x, y) = \text{Sym}^k \circ \nu \circ TA - CD^k(1 - \nu \circ TB)^{-1}(x, y)
+ \text{Sym}^k \circ D^k(\nu \circ TA - C)(1 - \nu \circ TB)^{-1}(x, y)
+ \text{Sym}^k \circ \sum_{q=1}^{k-1} \binom{k}{q} D^q(\nu \circ TA - C)D^{k-q}(1 - \nu \circ TB)^{-1}(x, y).
\]

\[
:= \text{Sym}^k \circ \left(U_1^k(\nu, T, A, B, C) + U_2^k(\nu, T, A, B, C)\right)(x, y)
+ \text{Sym}^k \circ \left(U_3^k(\nu, T, A, B, C)\right)(x, y).
\]

Proof  This is a direct consequence of Leibnitz’s rule.

Lemma 4.15  Under Definitions 2.1 and 3.1. Assume that \( \nu \in A_L \) is a \( C^k \) function and that \( U_1^k(\nu, T, A, B, C): D^* \to L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \) is as in Lemma 4.14. Then the following formulas hold:

\[
U_1^k(\nu, T, A, B, C) = (\nu \circ TA - C)(1 - \nu \circ TB)^{-2}(\nu \circ TDB + D\nu \circ TDT.B),
\]

for \( k \geq 2 \)

\[
U_1^k(\nu, T, A, B, C) = (\nu \circ TA - C)k!(1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(DC^{(k,k,k)}(\nu, T)B\right)
+ (\nu \circ TA - C)k!(1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(DC^{(k,1,(k-1))}(\nu, T)\right)
+ (\nu \circ TA - C)k!(1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(DC^{(k,0,(k-1))}(\nu, T, B)\right)
+ (\nu \circ TA - C)DICP^{(k,2,2)}(\nu, T, B),
\]

where \( DC^{(k_1,k_2,k_3)}(\nu, T), DC^{(k_1,k_2,k_3)}(\nu, T, B) \) and \( DICP^{(k_1,k_2,k_3)}(\nu, T, B) \) are as in Definitions 4.7, 4.9 and 4.12, respectively.

Proof  We do the proof for the formula (25). The proof of the formula (24) is straightforward. From assumption and Remark 4.13 it follows that

\[
U_1^k(\nu, T, A, B, C) = (\nu \circ TA - C)D^k(1 - \nu \circ TB)^{-1}.
\]

\[
:= (\nu \circ TA - C)I_1^k.
\]
We observe that, for $k \geq 2$, by the chain rule,

$$I_1^k = k!(1 - \nabla \circ TB)^{-2} \text{Sym}^k \circ \left(D^k(\nabla \circ TB)\right)$$

$$+ \text{Sym}^k \circ \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \cdot \ldots \cdot r_q!}(1 - \nabla \circ TB)^{-(q+1)} D^{r_1}(\nabla \circ TB) \cdot \ldots \cdot D^{r_q}(\nabla \circ TB).$$

$$:= k!(1 - \nabla \circ TB)^{-2} \text{Sym}^k \circ \left(I_2^k\right) + \text{Sym}^k \circ \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \cdot \ldots \cdot r_q!}(1 - \nabla \circ TB)^{-(q+1)} I_2^k.$$

(27)

Now applying Leibniz’s rule to the function $\nabla \circ TB$, it follows that

$$I_2^k = \text{Sym}^k \circ \left(D^k(\nabla \circ T)B\right) + \text{Sym}^k \circ \left(\sum_{q=0}^{k-1} \binom{k}{q} D^q(\nabla \circ T).D^{k-q}(B)\right).$$

(28)

$$:= \text{Sym}^k \circ \left(I_{2,1}^k B\right) + \text{Sym}^k \circ \left(\sum_{q=0}^{k-1} \binom{k}{q} I_{2,2}^k\right).$$

Moreover, by using the chain rule to the functions $I_{2,1}^k$ and $I_{2,2}^k$ respectively, we get

$$I_{2,1}^k = \text{Sym}^k \circ \left(k!(D^k \nabla) \circ T \underbrace{DT \ldots DT}_{k\text{-times}}\right)$$

$$+ \text{Sym}^k \circ \left(\sum_{q=1}^{k-1} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \cdot \ldots \cdot r_q!}(D^q \nabla) \circ T.D^{r_1} T \cdot \ldots \cdot D^{r_q} T\right),$$

(29)

and

$$I_{2,2}^k = \text{Sym}^q \circ \left(\sum_{n=1}^{q} \sum_{r_1+\ldots+r_n=q} \frac{q!}{r_1! \cdot \ldots \cdot r_n!}(D^n \nu) \circ T.D^{r_1} T \cdot \ldots \cdot D^{r_n} T\right).D^{k-q}(B).$$

(30)

Therefore, by replacing (30) and (29) into (28), and using that $\text{Sym}^k \circ \text{Sym}^k = \text{Sym}^k$, we get

$$I_2^k = \text{Sym}^k \left(k!(D^k \nabla) \circ T \underbrace{DT \ldots DT}_{k\text{-times}} B\right)$$

$$+ \text{Sym}^k \circ \left(\sum_{q=1}^{k-1} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \cdot \ldots \cdot r_q!}(D^q \nabla) \circ F.D^{r_1} T \cdot \ldots \cdot D^{r_q} T B\right)$$

$$+ \text{Sym}^k \circ \sum_{q=0}^{k-1} \binom{k}{q} \text{Sym}^q \left(\sum_{n=1}^{q} \sum_{r_1+\ldots+r_n=q} \frac{q!(D^n \nu) \circ T.D^{r_1} T \cdot \ldots \cdot D^{r_n} T}{r_1! \cdot \ldots \cdot r_n!}.D^{k-q}(B)\right).$$

(31)

Hence, on account of Definitions 4.7 and 4.9 we have

$$I_2^k := \text{Sym}^k \left(\text{DC}^{k,(1,k)}(\nabla, T) B\right) + \text{DC}^{k,(1,(k-1))}(\nabla, T, B) + \text{DC}^{(k,0,(k-1))}(\nabla, T, B).$$

(32)
By similar computation as above, in view of Definitions 4.7 and 4.9 we reach that
\[ I^k_3 := \mathcal{DCP}^{(r_1,0,r_1)}(\nu, T, B) \ldots \mathcal{DCP}^{(r_q,0,r_q)}(\nu, T, B). \] (33)
Hence, by using Definition 4.9, Eq. (33) becomes
\[ I^k_3 = \prod_{(r_1,\ldots,r_q,r)} (\nu, T, B). \] (34)

Whence, on account of Definition 4.12, we get
\[ \text{Sym}^k \circ \left( \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \ldots r_q!} (1 - \nu \circ T B)^{-q} \right) := \mathcal{DICP}^{(k,2,k)}(\nu, T, B). \] (35)

Thus, by replacing (35) and (32) into (27), we get
\[ I^k_1 = \text{Sym}^k \circ \left( \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \ldots r_q!} (1 - \nu \circ T B)^{-q} \right) \]
\[ \quad + \text{Sym}^k \circ \left( \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \ldots r_q!} (1 - \nu \circ T B)^{-q} \right) \]
\[ \quad + \text{Sym}^k \circ \left( \sum_{q=2}^{k} \sum_{r_1+\ldots+r_q=k} \frac{k!}{r_1! \ldots r_q!} (1 - \nu \circ T B)^{-q} \right) \]
\[ \quad + \mathcal{DICP}^{(k,2,k)}(\nu, T, B). \] (36)

Therefore, by replacing (36) into (26), and on account of $\text{Sym}^k \circ \text{Sym}^k = \text{Sym}^k$, it follows formula (25).

**Lemma 4.16** Under Definitions 2.1 and 3.1. Assume that $\nu \in A_L$ is a $C^k$ function and that $U^k_3(\nu, T, A, B, C) : D^* \to L^k(\mathbb{R}^{n+1}, \mathbb{R}^{n})$ is as in Lemma 4.14.

Then, the following formulas hold:
\[ U^k_3(\nu, T, A, B, C) = ((D\nu \circ T) DTA + \nu \circ TDA - D(C)) (1 - \nu \circ T B)^{-1}, \] (37)
for $k \geq 2$

\[ U^k_3(\nu, T, A, B, C) = \text{Sym}^k \left( \mathcal{DCP}^{(k,k,k)}(\nu, T) A \right) (1 - \nu \circ T B)^{-1} \]
\[ \quad + \text{Sym}^k \left( \mathcal{DCP}^{(k,1,k-1)}(\nu, T) A \right) (1 - \nu \circ T B)^{-2} \]
\[ \quad + \text{Sym}^k \left( \mathcal{DCP}^{(k,0,k-1)}(\nu, T) A \right) (1 - \nu \circ T B)^{-2} \]
\[ \quad - \text{Sym}^k \left( D^k(C) \right) (1 - \nu \circ T B)^{-2}, \] (38)
where $\mathcal{DCP}^{(k_1,k_2,k_3)}(\nu, T)$, $\mathcal{DCP}^{(k_1,k_2,k_3)}(\nu, T, B)$, $\mathcal{DICP}^{(k_1,k_2,k_3)}(\nu, T, B)$ are as in Definitions 4.7, 4.9 and 4.12, respectively.

**Proof** The proof is quite similar to the development of Eq. (27).

**Lemma 4.17** Under Definitions 2.1 and 3.1. Assume that $\nu \in A_L$ is a $C^k$ function and that $U^k_3(\nu, T, A, B, C) : D^* \to L^k(\mathbb{R}^{n+1}, \mathbb{R}^{n})$ is as in Lemma 4.14. Then the following formula holds:
\[ U^k_3(\nu, T, A, B, C) \]
\[ = \text{Sym}^k \left( \sum_{q=1}^{k-1} \binom{k}{q} \phi^{(q,k-q)} \left( (\mathcal{DCP}^{(q,0,q)}(\nu, T) A - D^q(C), \mathcal{DICP}^{(k-q,1,k-q)}(\nu, T, B) \right) \right), \] (39)
where $\mathcal{D}\mathcal{C}\mathcal{P}^{(k_1,k_2,k_3)}(\nu, T, B)$, $\phi^{(q,k-q)}$ and $\mathcal{D}\mathcal{I}\mathcal{C}\mathcal{P}^{(k_1,k_2,k_3)}(\nu, T, B)$ are as in Definitions 4.9, 4.6 and 4.12, respectively.

**Proof** The proof is similar to the proof of Lemma 4.15. For more details, see the proof of the formulas $I_1^k$ (Eq. (27)) and $I_2^k$ (Eq. (28)).

As a direct consequence of Lemmas 4.14–4.17 we obtain a formula for the $k$th derivative of the function $\Gamma(\nu)$ at point $(x, y)$, for $y \neq 0$.

**Lemma 4.18** Under Definitions 2.1, 3.1 and 3.3. Assume that $\nu \in A_L$ is a $C^k$ function that $y \neq 0$, then the following formulas hold:

$$
D(\Gamma(\nu))(x, y) = \left((\nu \circ TA - C)(1 - \nu \circ TB)^{-2}(\nu \circ TDB + D\nu \circ TDTB) + (\nu \circ TDA + \nu \circ TDA - DC)(1 - \nu \circ TB)^{-1}\right)(x, y).
$$

for $k \geq 2$

$$
D^k(\Gamma(\nu))(x, y) = \left((\nu \circ TA - C)(1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(D\mathcal{C}^{(k,k,k)}(\nu, T)B\right) + (\nu \circ TA - C)(1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(D\mathcal{C}^{(k,1,k-1)}(\nu, T)\right) + (\nu \circ TA - C)\mathcal{D}\mathcal{I}\mathcal{C}\mathcal{P}^{(k,2,k)}(\nu, T, B) + (1 - \nu \circ TB)^{-1}\text{Sym}^k \circ \left(D\mathcal{C}^{(k,k,k)}(\nu, T)A\right) + (1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(D\mathcal{C}^{(k,1,k-1)}(\nu, T)A + D\mathcal{C}\mathcal{P}^{(k,0,k-1)}(\nu, T, A)\right) - (1 - \nu \circ TB)^{-2}\text{Sym}^k \circ \left(D^k(C)\right) + U_3^k(\nu, T, A, B, C).\right)
$$

where $D\mathcal{C}^{(k_1,k_2,k_3)}(\nu, T)$, $D\mathcal{C}^{(k_1,k_2,k_3)}(\nu, T, B)$, $D\mathcal{I}\mathcal{C}\mathcal{P}^{(k_1,k_2,k_3)}(\nu, T, B)$, $\phi^{(q,k-q)}$ are as in Definitions 4.7, 4.9, 4.12 and 4.6, respectively and $U_3^k(\nu, T, A, B, C)$ is as in Lemma 4.17.

### 4.2 Part 2: The Norm of the $i$th Derivative

In this sub-section we estimate the norms of the $i$th derivative of the functions $A(x, y)$, $B(x, y)$ and $C(x, y)$ around a neighborhood of $D_0$. We start by the following simple but useful lemma.

**Lemma 4.19** Let

$$
d(x, y) = \left\{
\begin{array}{ll}
\alpha(A^+_{y} + \partial_x \psi_{+}(x, y)) + |y|\partial_y \psi_{+}(x, y), & y > 0, \\
\alpha(A^{-}_{y} + \partial_x \psi_{-}(x, y)) + |y|\partial_y \psi_{-}(x, y), & y < 0,
\end{array}
\right.
$$

and

$$
\rho(x, y) = \left\{
\begin{array}{ll}
\frac{1}{\min_{0 \leq j \leq i} |(\alpha A^+_{y} + \psi_{+}(x, y) + y\partial_y \psi_{+}(x, y))|^{i+1}}, & y > 0, \\
\frac{1}{\min_{0 \leq j \leq i} |(\alpha A^-_{y} + \psi_{-}(x, y) + y\partial_y \psi_{-}(x, y))|^{i+1}}, & y < 0,
\end{array}
\right.
$$

then, $d$ and $\rho$ are defined in a neighborhood $\bar{U}$ of $D_0$ and there exists a constant $C \geq 0$ such that the following estimate holds:

$$
\|D^i(d(x, y)^{-1})\| \leq C\rho(x, y)|y|^{i}, \quad \text{for all} \quad (x, y) \in \bar{U}.
$$
Moreover, the limit
\[
\lim_{(x,y) \to (a,0^\pm)} \rho(x, y)
\]
exists, for all \((a, 0) \in D_0\).

**Proof** Since \(A_k^+ \neq 0\), then estimate (44) is a direct consequence of Example 4.13 and norm properties. From Assumption 2.2(L1), it follows that the existence of \(\lim_{(x,y) \to (a,0^\pm)} \rho(x, y)\) and this finishes the proof of lemma. \(\square\)

As a consequence of Lemma 4.19 and Leibnitz rule we get:

**Corollary 4.20** Let \(0 \leq i \leq k\) be an integer. Assume \(A, B\) and \(C\) are as in Definition 2.1 and \(\rho\) as in Lemma 4.19. Then, there is a constant \(C \geq 0\) such that the following inequalities hold
\[
\|D^i A(x, y)\| \leq C\rho(x, y)|y|^{\gamma-i+1},
\]
\[
\|D^i C(x, y)\| \leq C\rho(x, y)|y|^{\gamma-i+1},
\]
\[
\|D^i B(x, y)\| \leq C\rho(x, y)|y|^{\gamma-i},
\]
in a neighborhood of \(D_0\).

**Corollary 4.21** Assume \(T(x, y) = (F(x, y), G(x, y))\) is a map that satisfies the Assumption 2.2(L1). Then, the following relation holds:
\[
\|D^k T(a, b)\| \leq \text{const}\|b\|^{a-k},
\]
in a neighborhood of \(D_0\), where \(\text{const}\) denotes a positive constant.

**Proof** The proof is a direct consequence of Assumption 2.2(Eq. (2.2)) and Leibnitz rule. \(\square\)

**Lemma 4.22** Let \(A, B\) and \(C\) be as in Definition 2.1. Assume that \(\overline{v} \in A_L\) is a \(C^k\) functions and that \(U^k_1(\overline{v}, T, A, B, C)\), \(U^k_2(\overline{v}, T, A, B, C)\) and \(U^k_3(\overline{v}, T, A, B, C)\) are as in Lemmas 4.15, 4.16 and 4.17, respectively. Then
\[
\lim_{(a,b) \to (x,0)} U^k_1(\overline{v}, T, A, B, C)(a, b) = 0,
\]
\[
\lim_{(a,b) \to (x,0)} U^k_2(\overline{v}, T, A, B, C)(a, b) = 0,
\]
\[
\lim_{(a,b) \to (x,0)} U^k_3(\overline{v}, T, A, B, C)(a, b) = 0,
\]
for every \((x, 0) \in D_0\).

**Proof** The result is easy to prove for \(k = 1\). We prove the result for the case \(k \geq 2\).

By Lemma 4.15\((k \geq 2)\), Definition 3.1 and Remark 4.2(iii) we have
\[
\|U^k_1(\overline{v}, T, A, B, C)\| \leq \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|k!\text{DICP}^{(k,k,k)}(\overline{v}, T)(a, b)\| \|B\| + \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|k!\text{DICP}^{(k,1,(k-1))}(\overline{v}, T)(a, b)\| + \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|\text{DICP}^{(k,0,(k-1))}(\overline{v}, T, B)(a, b)\| + \frac{(L\|A(a, b)\| + \|C(a, b)\|)}{(1 - L\|B\|)^2} \|\text{DICP}^{(k,k,k)}(\overline{v}, T, B)(a, b)\|.
\]

\(\square\)
To estimate the first expression of (53), from (14) and norm properties we have
\[ \| D^k \Psi (\nu, T)(a, b) \| \leq \| k! (D^k \nu) \circ T \| \| D^1 (a, b) \| ^k, \]  
(54)
and since \( \nu \) is of class \( C^k \), by using of the Corollary 4.21 we get
\[ \| D^k \Psi (\nu, T)(a, b) \| \leq \text{const} | b | ^{\gamma - k}. \]  
(55)
Whence, in view of Corollary 4.20 we have
\[ \frac{(L \| A(a, b) \| + \| C(a, b) \|)}{(1 - L \| B \| )^2} \| k! D^k \Psi (\nu, T)(a, b) \| \leq \frac{| b |^{\alpha - k + \gamma + 1} (1 - L \| B \| )^2 \text{const}}{(1 - L \| B \| )^2}. \]  
(56)
By similar arguments one can estimates remaining expressions of (53) to obtain
\[ \frac{(L \| A(a, b) \| + \| C(a, b) \|)}{(1 - L \| B \| )^2} \| k! D^k \Psi (\nu, T)(a, b) \| \leq \frac{| b |^{\gamma + 1} (1 - L \| B \| )^2 \text{const}}{(1 - L \| B \| )^2}. \]  
(57)
\[ \frac{(L \| A(a, b) \| + \| C(a, b) \|)}{(1 - L \| D \| )^2} \| D^k \Psi \| \leq \frac{| b |^{\gamma + 1} (1 - L \| D \| )^2 \text{const}}{(1 - L \| D \| )^2}. \]  
(58)
\[ \frac{(L \| A(a, b) \| + \| C(a, b) \|)}{(1 - L \| D \| )^2} \| D^k \Psi \| \leq \frac{| b |^{\alpha - k + \gamma + 1} \text{const}}{(1 - L \| D \| )^2}. \]  
(59)
Therefore, combining the four estimates (59), (58), (57) and (56) with (53) we obtain
\[ \| U^k_1 (\nu, T, A, B, C)(a, b) \| \leq \text{const} | b |^{\alpha - k + \gamma + 1}. \]  
(60)
Hence, by fact that \( \gamma > k - 1 \) and \( \alpha > 0 \) (see Assumption 2.2(L1)) we see that
\[ \lim_{(a, b) \to (x, 0)} \| U^k_1 (\nu, T, A, B, C)(a, b) \| = 0. \]
Repeating the same procedure which deduce estimate (50) , we get estimates (51) and (52).
Thus, we conclude the proof of corollary. \( \Box \)

Proof of Proposition 3.7  
This is a direct consequence of Lemma 4.22. \( \Box \)

5 Proof of Proposition 3.8

The proof of Proposition 3.8 was influenced by the ideas contained in the articles [24, p. 313] and [28, Eq. 3]. The proof is quite long and technical, so we divide it into two steps. Before that, we give the following definition.

Definition 5.1 We define the set \( D_i \) of all the continuous functions \( v_i : D \to \mathbb{L}^1_1 (\mathbb{R}^{n+1}, \mathbb{R}^n) \) such that \( v_i (x, 0) = 0 \), for all \( (x, 0) \in D_0 \), that is
\[ D_i := \{ v_i : D \to \mathbb{L}^1_1 (\mathbb{R}^{n+1}, \mathbb{R}^n) : v_i (x, 0) = 0, \ \text{for all} \ \ (x, 0) \in D_0; \ \ v_i \ \text{is continuous} \}, \]
for every \( 1 \leq i \leq k \), and \( D_0 := \mathbb{A}_L \).

The proof of Proposition 3.8 is somewhat lengthy, so we divide it into two parts. In the first part: we show the existence of functions \( \Psi^i : D_0 \times D_1 \times \cdots \times D_i \to D_i \), so that \( D^i (\Gamma (v_i)) = \Psi^i (\nu_0, D (\nu_0), \ldots, D^i (\nu_0)) \), for all \( 0 \leq i \leq k \). In the second part: we show that the function \( N_i : D_0 \times D_1 \times \cdots \times D_i \to D_0 \times D_1 \times \cdots \times D_i \) given by
\[ N_i (v_0, \nu_1, \ldots, \nu_i) = (\Gamma (v_0), \Psi^1 (\nu_0, \nu_1), \ldots, \Psi^i (\nu_0, \nu_1, \ldots, \nu_i)) \]  
has a global attracting fixed point \( (A_0, A_1, \ldots, A_i) \), for all \( 0 \leq i \leq k \).
5.1 Part 1: Defining the Functions $\Psi^i$

We start by defining a generalization of the function $U^k_i$ (see Corollary 4.15).

**Definition 5.2** Let $1 \leq i \leq k$ be an integer. Let $D_0$ and $D_i$ be sets as in Definition 5.1. We define the function $U^k_i : D_0 \times D_1 \times \cdots \times D_i \to D_i$ given by

$$U^k_i(\nu_0, \nu_1) = (\nu_0 \circ T A - C)(1 - \nu_0 \circ T B)^{-2} (\nu_0 \circ TDB + \nu_1 \circ T DTB),$$

for $i \geq 2$

$$U^k_i(\nu_0, \nu_1, \ldots, \nu_i) = \frac{(\nu_0 \circ T A - C)i!}{(1 - \nu_0 \circ T B)^2} \text{Sym}^i \circ D C^{(j,i,i)}(\nu_i, T) B$$

$$+ \frac{(\nu_0 \circ T A - C)i!}{(1 - \nu_0 \circ T B)^2} \text{Sym}^i \circ D C^{(j,1,(i-1))}(\nu_1, \ldots, \nu_{i-1}, T)$$

$$+ \frac{(\nu_0 \circ T A - C)i!}{(1 - \nu_0 \circ T B)^2} \text{Sym}^i \circ D C P^{(i,0,(i-1))}(\nu_0, \nu_1, \ldots, \nu_{i-1}, T, B)$$

$$+ (\nu_0 \circ T A - C) D I C P^{(i,2,i)}(\nu_0, \nu_1, \ldots, \nu_{i-1}, T, B),$$

where $D C^{(k_1,k_2,k_3)}(\nu, T)$, $D C P^{(k_1,k_2,k_3)}(\nu, T, B)$ and $D I C P^{(k_1,k_2,k_3)}(\nu, T, B)$ are as in Definitions 4.7, 4.9 and 4.12, respectively.

Next, we define a generalization of the function $U^k_2$ (see Corollary 4.16).

**Definition 5.3** Let $1 \leq i \leq k$ be an integer. Let $D_0$, $D_i$ be sets as in Definition 5.1. We define the function $U^k_2 : D_0 \times D_1 \times \cdots \times D_i \to D_i$ given by

$$U^k_2(\nu_0, \nu_1, T) = (\nu_0 \circ T DTA + \nu_0 \circ TDA - DC)(1 - \nu_0 \circ T B)^{-1},$$

for $i \geq 2$

$$(U^k_2)(\nu_0, \nu_1, \ldots, \nu_i) = (1 - \nu_0 \circ T B)^{-1} \text{Sym}^i \circ \left(D C^{(i,i,i)}(\nu_i, T) A \right)$$

$$+ (1 - \nu_0 \circ T B)^{-2} \text{Sym}^i \circ \left(D C^{(i,1,i-1)}(\nu_1, \ldots, \nu_{i-1}, T) A - D^i(C) \right)$$

$$+ (1 - \nu_0 \circ T B)^{-2} \text{Sym}^i \circ \left(D C P^{(i,0,i-1)}(\nu_0, \nu_1, \ldots, \nu_{i-1}, T, A) \right)$$

where $D C^{(k_1,k_2,k_3)}(\nu, T)$ and $D C P^{(k_1,k_2,k_3)}(\nu, T, B)$ are as in Definitions 4.7 and 4.9, respectively.

Next, we define a generalization of the function $U^k_3$ (see Corollary 4.17).

**Definition 5.4** Let $2 \leq i \leq k$ be an integer. Let $D_0$, $D_i$ be the sets as in Definition 5.1. We define the function $U^k_3 : D_0 \times D_1 \times \cdots \times D_i \to D_i$ given by

$$(U^k_3)(\nu_0, \ldots, \nu_i) = \text{Sym}^i \circ \sum_{q=1}^{i-1} \binom{i}{q} \phi^{(q,i-q)}(D C P^{(q,0,q)}(\nu_0, \ldots, \nu_q, T, A)$$

$$- D^i C, D I C P^{(i-1,i-1,q)}(\nu_0, \ldots, \nu_{i-q-1}, T, B)),$$

where $D C P^{(k_1,k_2,k_3)}(\nu, T, B)$, $D I C P^{(k_1,k_2,k_3)}(\nu, T, B)$, $\phi^{(q,i-q)}$ are as in Definitions 4.9, 4.12 and 4.6, respectively.
Next, we define a generalization of the function $D^k \Gamma(v)$ (see Lemma 4.18).

**Definition 5.5** Let $1 \leq i \leq k$ be an integer. Let $D_0$, $D_i$ be sets as in Definition 5.1, and assume the functions $U^i_1$, $U^i_2$ and $U^i_3$ as in Definitions 5.2, 5.3 and 5.4, respectively. We define the function $\Psi^i : D_0 \times D_1 \times \cdots \times D_i \rightarrow D_i$ given by

$$\Psi^i(\overline{v}_0, \overline{v}_1, \overline{v}_2) = (U^i_1 + U^i_2)(\overline{v}_0, \overline{v}_1),$$

(66)

and for $i \geq 2$

$$\Psi^i(\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_i) = (U^i_1 + U^i_2 + U^i_3)(\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_i).$$

(67)

**Remark 5.6** The functions $\Psi^i$ for the cases $i = 1$ and $i = 2$ were established in [28, Eq. 13] and [21, Eq. 42] respectively.

**Proposition 5.7** Let $1 \leq i \leq k$ be an integer. Then, the function $\Psi^i$ given in Definition 5.5 is well-defined. Moreover, if $\overline{v}_0 \in A_L$ is of class $C^i$, then

$$\Psi^i(\overline{v}_0, D\overline{v}_0, \ldots, D^i\overline{v}_0) = D^i \Gamma(\overline{v}_0).$$

(68)

**Proof** To prove that the function $\Psi^i$ is well-defined, it suffices to show that

$$\Psi^i(\overline{v}_0, \ldots, \overline{v}_i) \in D_i, \quad \text{for all } \overline{v}_j \in D_j, 0 \leq j \leq 1.$$  

(69)

That is, by Definition 5.1 we must show that

(a) $\Psi^i(\overline{v}_0, \ldots, \overline{v}_i)$ is continuous on $D$ and

(b) $\Psi^i(\overline{v}_0, \ldots, \overline{v}_i)(x, 0) = 0$, for every $x \in \mathbb{R}^n$, $\|x\| \leq 1$,

for all $\overline{v}_j \in D_j, 0 \leq j \leq 1$. Indeed, by Definition 5.5 we have that $\Psi^i(\overline{v}_0, \ldots, \overline{v}_i)$ is continuous on $D^*$, so it remains to show the continuity of $\Psi^i(\overline{v}_0, \ldots, \overline{v}_i)$ at the points $(x, 0) \in D_0$. Analysis similar to that the proof of Proposition 3.7 shows that

$$\lim_{(a,b) \rightarrow (x,0)} \Psi^i(\overline{v}_0, \ldots, \overline{v}_i)(a, b) = 0,$$

(70)

for all $(x, 0) \in D_0$. Therefore, if we define

$$W^i(x, y) = \begin{cases} \Psi^i(\overline{v}_0, \ldots, \overline{v}_i)(x, y), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

then, we get a continuous extension of $\Psi^i(\overline{v}_0, \ldots, \overline{v}_i)$ on $D$, which completes the proof of (5.1), so $\Psi^i(\overline{v}_0, \ldots, \overline{v}_i) \in D_i$ and therefore $\Psi^i$ is well-defined. The equality in Eq. (68) follows from Definition 5.5 and Lemma 4.18. This concludes the proof.

\[\square\]

### 5.2 Part 2: The Function

$\tilde{N}_i$ In this sub-section we show the following proposition.

**Proposition 5.8** Let $1 \leq i \leq k$ be a integer. Let $\Psi^j$, $1 \leq j \leq i$ be functions as in Definition 5.5. Then the function

$$\tilde{N}_i : D_0 \times D_1 \times \cdots \times D_i \rightarrow D_0 \times D_1 \times \cdots \times D_i$$

(71)

given by

$$\tilde{N}_i(\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_i) = (\Gamma(\overline{v}_0), \Psi^1(\overline{v}_0, \overline{v}_1), \ldots, \Psi^i(\overline{v}_0, \overline{v}_1, \ldots, \overline{v}_i)),$$

(72)

have a global attracting fixed point $(A_0, A_1, \ldots, A_i)$.
5.2.1 Preliminaries

Before proving Proposition 5.8, we state two theorems which will be useful in the sequel.

**Theorem 5.9** (Fiber Contraction Theorem [14]) Let \((X, d_X)\) and \((Y, d_Y)\) be two complete metric spaces, and let \(\Upsilon : X \times Y \to X \times Y\) be a map of the form
\[
\Upsilon(x, y) = (\Gamma(x), \Psi(x, y)).
\]
Assume that

(a) \(\Gamma\) has an attracting fixed point \(x_\infty\), that is,
\[
\Gamma(x_\infty) = x_\infty, \quad \lim_{n \to \infty} \Gamma^n(x) = x_\infty, \quad \text{for all } x \in X;
\]

(b) the family of functions \(\Psi^y : X \to Y\) given by \(\Psi^y(x) = \Psi(x, y)\) depends on \(y\) continuously; that is, if \(x_n \to x\) as \(n \to \infty\), then \(\Psi^y(x_n) \to \Psi^y(x)\) as \(n \to \infty\).

(c) for every \(x \in X\) the map \(\Psi_x := \Psi(x, \cdot) : Y \to Y\) defined by \(\Psi_x(y) := \Psi(x, y)\) is a \(\lambda\)-contraction, with \(\lambda < 1\). This mean that
\[
dY(\Psi_x(y_1), \Psi_x(y_2)) \leq \lambda dY(y_1, y_2),
\]
for all \(x \in X\) and \(y_1, y_2 \in Y\).

Then, if \(y_\infty\) denotes the unique fixed point of \(\Psi_{x_\infty}\), the point \((x_\infty, y_\infty) \in X \times Y\) is an attracting fixed point of \(\Upsilon\), that is,
\[
\lim_{n \to \infty} \Upsilon^n(x, y) = (x_\infty, y_\infty).
\]

**Theorem 5.10** (Perron–Frobenius Theorem for positive matrices [20]) Let \(A = [a_{i, j}]_{n \times n}\) be a real \(n \times n\) positive matrix: \(a_{i, j} > 0\), for \(1 \leq i, j \leq n\), then

(a) \(A\) has a positive simple eigenvalue \(r\) which is equal to the spectral radius of \(A\).

(b) There exists an eigenvector \(x\) with all the coordinates positive, such that \(Ax = rx\).

(c) The eigenvector is the unique vector defined by
\[
Ap = rp, \quad p > 0, \quad \text{and } \|p\|_1 = 1, \quad \text{where } \|p\|_2 = \sum_{i=1}^{n} |p_i|,
\]
and, except for positive multiples of \(p\), there are no other nonnegative eigenvector for \(A\), regardless of the eigenvalue.

(d) An estimate of \(r\) is given by inequalities:
\[
\min_{i} \sum_{j} a_{ij} \leq r \leq \max_{i} \sum_{j} a_{ij}.
\]

Now, we give some elementary properties of multilinear maps. Let us start by fixing some notation. The set \([1, \ldots, n]\) will be denoted by \([n]\). If \(E := \mathbb{R}^n\) and \(F := \mathbb{R}\), then \(\mathcal{F}([k], \{E, F\})\) denotes the set of all the functions \(f : [k] \to \{E, F\}\) (Notice that the cardinality of \(\mathcal{F}([k], \{E, F\})\) is \(2^k\)). Finally \(\pi_F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) and \(\pi_E : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}\) denote the projections of \(\mathbb{R}^{n+1}\) on \(E\) along \(F\) and of \(\mathbb{R}^{n+1}\) on \(F\) along \(E\) respectively.

**Definition 5.11** Assume that \(f \in \mathcal{F}([k], \{E, F\})\) and that \(\heartsuit = E\) or \(\heartsuit = F\). Then, define \(g_{f, \heartsuit} : [n+1] \to \{E, F\}\) by
\[
g_{f, \heartsuit}(i) = \begin{cases} f(i), \text{ if } i \in [n], \\ \heartsuit, \text{ if } i = n + 1. \end{cases}
\]
By \( \Omega([n+1], [\nu]) \) we denote the set of all functions \( g_{f_D} \).

The symbol \( A \cup B \) shall be reserved to denote the union of two disjoint sets.

**Lemma 5.12** The following statement holds:

(a) \( \Omega([n+1], \{E\}) \cup \Omega([n+1], \{F\}) = \mathcal{F}([n+1], \{E, F\}) \).

**Proof** The proof follows immediately from Definition 5.11. \( \square \)

Recall that \( L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \) denotes the space of all the \( k \)-linear maps from \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^n \).

**Definition 5.13** Assume that \( f \in \mathcal{F}([k], \{E, F\}) \). Then, the set of all \( k \)-linear maps \( b \) such that

(a) \( b(\pi_{g(1)}(x_1), \pi_{g(2)}(x_2), \ldots \pi_{g(k)}(x_k)) = 0 \), for every \( g \in \mathcal{F}([k], \{E, F\}), g \neq f \) and for each \( k \)-tuple \((x_1, \ldots, x_k) \in \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} \).

will be denoted by

\[
L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n).
\]

**Lemma 5.14** We have the following properties:

(a) If \( b \in L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \), then

\[
b(x_1, x_2, \ldots, x_k) = \sum_{f \in \mathcal{F}([k], \{E, F\})} b(\pi_{f(1)}(x_1), \ldots, \pi_{f(k)}(x_k)),
\]

for every \((x_1, \ldots, x_k) \in \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} \).

The function \((x_1, x_2, \ldots, x_k) \rightarrow b(\pi_{f(1)}(x_1), \ldots, \pi_{f(k)}(x_k))\) will be denoted by \( b_f \).

(b) If \( f \in \mathcal{F}([k], \{E, F\}) \) and \( b \in L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \), then

\[
b(x_1, x_2, \ldots, x_k) = b(\pi_{f(1)}(x_1), \pi_{f(2)}(x_2), \ldots, \pi_{f(k)}(x_k)),
\]

for every \((x_1, \ldots, x_k) \in \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} \).

(c) \( L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \) can be decomposed into a direct sum of \( 2^k \) \( k \)-linear maps that is,

\[
L^k(\mathbb{R}^{n+1}, \mathbb{R}^n) = \bigoplus_{f \in \mathcal{F}([k], \{E, F\})} L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n),
\]

where \( L_f^k(\mathbb{R}^{n+1}, \mathbb{R}^n) \) as in Definition 5.13.

**Proof** The proof follows from Lemma 5.12 and Eq. (5.13). \( \square \)

5.2.2 Proof of Proposition 5.8

In order to prove Proposition 5.8 we state and prove the following proposition.

**Proposition 5.15** Under the notation of Definitions 5.1 and 5.5. Let \( 1 \leq i \leq k \) be an integer and fix a point \((\nu_0, \ldots, \nu_{i-1}) \in \mathcal{D}_0 \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_{i-1} \). Then, the space \( \mathcal{D}_i \) can be endowed with a norm \(|\cdot|_{\mathcal{D}_i}\) equivalent to the original norm \(|\cdot|_{\mathcal{D}}\) such that the function

\[
\Psi^i(\nu_0, \ldots, \nu_{i-1}, \bullet) : \mathcal{D}_i \rightarrow \mathcal{D}_i
\]

is a contraction with constant of contraction independent of the point \((\nu_0, \nu_1, \ldots, \nu_{i-1})\). 

\( \square \) Springer
The proof of Proposition 5.15 will be given after some lemmas. We set,
\[
\widehat{DT}(x, y) := \begin{bmatrix} A(x, y) & B(x, y) \\ C(x, y) & 1 \end{bmatrix}_{(n+1) \times (n+1)},
\]
where the functions \(A(x, y), B(x, y)\) and \(C(x, y)\) are as in Definition 2.1.

**Lemma 5.16** Let \(M^i : L^i(\mathbb{R}^{n+1}, \mathbb{R}^n) \to L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)\) be a map defined by
\[
M^i(b)(x_1, \ldots, x_i) = b(\widehat{DT} x_1, \ldots, \widehat{DT} x_i).
\]
Then, the space \(L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)\) can be endowed with a norm \(\|\cdot\|_i\) equivalent to \(\|\cdot\|\) such that
\[
\frac{|M^i(b)|_i}{|b|_i} \leq \max_{m,n \in \mathbb{N}} \{(||A||_D + ||B||_D)^m(\|C\|_D + 1)^n\}.
\]

**Proof** Through of the proof, we deal with the case that \(\|B\|_D\) is nonzero, the other case is similar. We will endow \(L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)\) with a new norm \(\|\cdot\|_i\) in the following way: letting
\[
c_{g,f} := ||\pi_g(1)\widehat{DT}\pi_f(1)|| \cdots ||\pi_g(i)\widehat{DT}\pi_f(i)||,
\]
where \(g\) and \(f \in \mathcal{F}([i], \{E, F\})\) while
\[
\pi_g(j)\widehat{DT}\pi_f(j) := \begin{cases} A, & \text{if } g(j) = E \text{ and } f(j) = E, \\
B, & \text{if } g(j) = E \text{ and } f(j) = F, \\
C, & \text{if } g(j) = F \text{ and } f(j) = E, \\
1, & \text{if } g(j) = F \text{ and } f(j) = F.
\end{cases}
\]
Next up, consider the matrix
\[
\Delta := [c_{g,f}]_{2^i \times 2^i},
\]
and notice that since, by assumption \(\|A\|_D, \|B\|_D\) and \(\|C\|_D\) are nonzero, then, in view of (75) and (76) it follows that \(c_{g,f} > 0\). Thus, the matrix \(\Delta\) is positive, and by Perron-Frobenius Theorem 5.10 applied to matrix \(\Delta\), we get

(a) the matrix \(\Delta\) has a positive eigenvalue \(\lambda\),
(b) the matrix \(\Delta\) has an eigenvector \(V\) with entries \(k_f\) such that
\[
\sum_{f \in \mathcal{F}([i], \{E, F\})} k_f = 1,
\]
(c) an estimate of \(\lambda\) is given by inequalities
\[
\min_g \sum_f c_{g,f} \leq \lambda \leq \max_g \sum_f c_{g,f}.
\]
Let \(b \in L^i(\mathbb{R}^{n+1}, \mathbb{R}^n), b \neq 0\), in view of Lemma 5.14(a) we can write
\[
b = \sum_{f \in \mathcal{F}([i], \{E, F\})} b_f,
\]
where \(b_f\) is as in Definition 5.13. Thus, we can define the norm \(\|\cdot\|_i\) on \(L^i(\mathbb{R}^{n+1}, \mathbb{R}^n)\) by
\[
|b|_i := \sum_{f \in \mathcal{F}([i], \{E, F\})} k_f \|b_f\|.
\]
We now will prove that

\[ \text{Indeed, by definition one has } M_i^i(b) \text{ is an } i\text{-linear map, and on account of Lemma 5.14(c) and (80) we have} \]

\[ |M_i^i(b)_i| := \sum_{f \in \mathcal{F}(|i|, \{E, F\})} k_f |M_i^i(b)_f|, \]  

where \( M_i^i(b)_f \) is as in Definition 5.13. But, by using Lemma 5.14(b) we have

\[ M_i^i(b)(x_1, \ldots, x_i) = M_i^i(b)(\pi_{f(1)}(x_1), \ldots, \pi_{f(i)}(x_i)) \]  

and by assumption (5.16) we have

\[ M_i^i(b)(x_1, \ldots, x_i) = b(\hat{D}T x_1, \ldots, \hat{D}T x_i). \]

Thus, combining (84) and (83) we get

\[ M_i^i(b)_f(x_1, \ldots, x_i) = b(\hat{D}T \pi_{f(1)} x_1, \ldots, \hat{D}T \pi_{f(i)} x_i). \]

Furthermore, using Lemma 5.14(a) we can write

\[ b(\hat{D}T \pi_{f(1)}, \ldots, \hat{D}T \pi_{f(i)}) = \sum_{g \in \mathcal{F}(|i|, \{E, F\})} b_g(\pi_{g(1)} \hat{D}T \pi_{f(1)}, \ldots, \pi_{g(i)} \hat{D}T \pi_{f(i)}). \]  

Therefore, it follows from (86) and (85), that

\[ M_i^i(b)_f(x_1, \ldots, x_i) = \sum_{g \in \mathcal{F}(|i|, \{E, F\})} b_g(\pi_{g(1)} \hat{D}T \pi_{f(1)}, \ldots, \pi_{g(i)} \hat{D}T \pi_{f(i)}(x_i)). \]  

Hence, on account of (82) we get

\[ |M_i^i(b)_i| \leq \sum_{f \in \mathcal{F}(|i|, \{E, F\})} k_f \sum_{g \in \mathcal{F}(|i|, \{E, F\})} ||b_g(\pi_{g(1)} \hat{D}T \pi_{f(1)}, \ldots, \pi_{g(i)} \hat{D}T \pi_{f(i)})||. \]  

Since \( b_g \) is \( i \)-linear map, we have

\[ ||b_g(\pi_{g(1)} \hat{D}T \pi_{f(1)}, \ldots, \pi_{g(i)} \hat{D}T \pi_{f(i)})|| \leq ||\pi_{g(1)} \hat{D}T \pi_{f(1)}|| \cdots ||\pi_{g(i)} \hat{D}T \pi_{f(i)}|| . ||b_g|| \]

\[ := c_{g, f}. \]

Consequently, Eq. (88) becomes

\[ |M_i^i(b)_i| \leq \sum_{g \in \mathcal{F}(|i|, \{E, F\})} ||b_g|| \sum_{f \in \mathcal{F}(|i|, \{E, F\})} c_{g, f} k_f. \]  

Notice that, since \( V = [k_f]_{f \in \mathcal{F}(|i|, \{E, F\})} \) is an eigenvector of matrix \( \Delta \), we have

\[ \Delta[k_f]_{f \in \mathcal{F}(|i|, \{E, F\})} = \lambda[k_f]_{f \in \mathcal{F}(|i|, \{E, F\})}, \]

where \( \Delta = [c_{g, f}] \) while \( g \) and \( f \in \mathcal{F}(|i|, \{E, F\}) \).

Hence, if we fix \( g \in \mathcal{F}(|i|, \{E, F\}) \) it is easily seen that

\[ \sum_{f \in \mathcal{F}(|i|, \{E, F\})} c_{g, f} k_f = \lambda k_g. \]
Thus, by replacing (91) into (89) we get
\[
|M^i(b)|_i \leq \lambda \sum_{g \in \mathcal{F}([i], \{E, F\})} ||b_g||_k.
\] (92)

Moreover, by definition we can write
\[
|b|_i = \sum_{g \in \mathcal{F}([i], \{E, F\})} ||b_g||_k.
\] (93)

Therefore, from (93) and (92) one obtains
\[
|M^i(b)|_i \leq \lambda |b|_i.
\] (94)

Through the rest of the proof let us denote by \(\#(S)\) the cardinality of \(S\).

**Claim 5.17** Let \(f\) and \(g \in \mathcal{F}([i], \{E, F\})\) such that \(\#(g^{-1}(E)) = m\) and \(\#(g^{-1}(F)) = n\). Then the following equality holds:
\[
\sum_{f \in \mathcal{F}([i], \{E, F\})} c_{g, f} = (||A||_D + ||B||_D)^m (||C||_D + 1)^n,
\] (95)

where \(c_{g, f}\) as in Eq. (75).

Proof of the Claim. Since \(\#(g^{-1}(E)) = m\) and \(\#(g^{-1}(F)) = n\), then one can consider \(g^{-1}(E) := \{a_1, a_2, \ldots, a_m\}\) and \(g^{-1}(F) := \{b_1, b_2, \ldots, b_n\}\). Thus, by definition we have
\[
\pi_{g(a_i)} \hat{D} \pi_{f(a_i)} := \begin{cases} A, & \text{if } f(a_i) = E, \\ B, & \text{if } f(a_i) = F, \end{cases}
\] (96)

and
\[
\pi_{g(b_i)} \hat{D} \pi_{f(b_i)} := \begin{cases} C, & \text{if } f(b_i) = E, \\ 1, & \text{if } f(b_i) = F. \end{cases}
\] (97)

Now, consider integers \(s, t\) with \(0 \leq s \leq m\), \(0 \leq t \leq n\) and take \(f \in \mathcal{F}([i], \{E, F\})\) such that
\[
\#(g^{-}(E) \cap f^{-1}(E)) = s \quad \text{and} \quad \#(g^{-}(F) \cap f^{-1}(F)) = t.
\]

Then, since \(c_{g, f} := ||\pi_{g(1)} \hat{D} \pi_{f(1)}|| \ldots \ldots ||\pi_{g(i)} \hat{D} \pi_{f(i)}||\), it follows from (97) and (96) that
\[
c_{g, f} = ||A||_D^s ||B||_D^{m-s} ||C||_D^t 1^{n-t}.
\] (98)

In addition, since \(\#g^{-1}(E) = m\) and \(\#g^{-1}(F) = n\), it is not difficult to see that the cardinality of the sets
\[
\mathcal{F}_{g,s}([i], \{E, F\}) := \{f \in \mathcal{F}([i], \{E, F\}) : \text{card}(g^{-}(E) \cap f^{-1}(E)) = s\}
\] (99)

and
\[
\mathcal{F}_{g,t}([i], \{E, F\}) := \{f \in \mathcal{F}([i], \{E, F\}) : \text{card}(g^{-}(F) \cap f^{-1}(F)) = t\}
\] (100)

are \(\binom{m}{s}\) and \(\binom{n}{t}\), respectively. Thus, from (100) and (99), on account of Rule of Product [9, p. 13] we deduce that the cardinality of
\[
\mathcal{F}_{g, st}([i], \{E, F\}) := \mathcal{F}_{g,s}([i], \{E, F\}) \cap \mathcal{F}_{g,t}([i], \{E, F\})
\] (101)
Proposition 5.18
Proof of Proposition 5.18
norm on $D$
From Definition 5.5 one can deduce that

\[
\mathcal{F}([i], \{E, F\}) = \sum_{0 \leq s \leq m} \mathcal{F}_{g,s,t}([i], \{E, F\}),
\]

Whence, on account of (98) and Binomial Theorem we get the following chain of equalities

\[
\sum_{f \in \mathcal{F}([i], \{E, F\})} c_{g,f} = \sum_{s=0}^{m} \sum_{t=0}^{n} \left( \sum_{f \in \mathcal{F}_{g,s,t}([i], \{E, F\})} c_{g,f} \right)
= \sum_{s=0}^{m} \sum_{t=0}^{n} \binom{m}{s} \binom{n}{t} ||A||_{D}^{s} ||B||_{D}^{m-s} ||C||_{D}^{1} t^{n-t}
= (||A||_{D} + ||B||_{D}^{m} ||C||_{D} + 1)^{n}.
\]

Thus Claim 5.17 is proved.
Finally, from (78) and Claim 5.17 we conclude that

\[
\frac{|M^{i}(b)|_{i}}{|b|_{i}} \leq \max_{m,n \in \mathbb{N}} \left\{ (||A||_{D} + ||B||_{D})^{m} (||C||_{D} + 1)^{n} \right\},
\]

(102)
for all $b \in L^{i}(\mathbb{R}^{n+1}, \mathbb{R}^{n})$, which finishes the proof of the lemma.

Now, we are going to prove Proposition 5.15 mentioned in the beginning of the sub-section, which we recall here. Before that, it is important to recall that

$\mathcal{D}_{i} := \{v_{i} : D \rightarrow L^{i}(\mathbb{R}^{n+1}, \mathbb{R}^{n}) : v_{i}(x, 0) = 0; v_{i} \text{ is continuous} \}$.

Proposition 5.18 Under the notation of Definitions 5.1 and 5.5. Let $1 \leq i \leq k$ be an integer and fix a point $(\overline{v}_{0}, \ldots, \overline{v}_{i-1}) \in \mathcal{D}_{0} \times \mathcal{D}_{1} \times \cdots \times \mathcal{D}_{i-1}$. Then, the space $\mathcal{D}_{i}$ can be endowed with a norm $|.|_{i,D}$ equivalent to the original norm $|.|_{D}$ so that the function

$\Psi^{i}(\overline{v}_{0}, \ldots, \overline{v}_{i-1}, \cdot) : \mathcal{D}_{i} \rightarrow \mathcal{D}_{i}$

it is a contraction with constant of contraction independent of the point $(\overline{v}_{0}, \overline{v}_{1}, \ldots, \overline{v}_{i-1})$.

Proof of Proposition 5.18 Let $v_{i} \in \mathcal{D}_{i}$. We define its norm to be

\[
|v_{i}|_{i,D} := \sup\{|v_{i}(x, y)|_{i} : (x, y) \in D\},
\]

(103)
where $|.|_{i}$ is the norm on $L^{i}(\mathbb{R}^{n+1}, \mathbb{R}^{n})$ as in Lemma 5.16. It is easy to check that $|.|_{i,D}$ is a norm on $\mathcal{D}_{i}$ equivalent to $|.|_{D}$.

Let $v^{1}_{i} = \Psi^{i}(\overline{v}_{0}, \overline{v}_{1}, \ldots, \overline{v}_{i-1}, \mu^{1})$ and $v^{2}_{i} = \Psi^{i}(\overline{v}_{0}, \overline{v}_{1}, \ldots, \overline{v}_{i-1}, \mu^{2})$, where $\mu^{1}, \mu^{2} \in \mathcal{D}_{i}$.

From Definition 5.5 one can deduce that

\[
v^{1}_{i} - v^{2}_{i} = (\overline{v}_{0} \circ T A C - C) i!(1 - \overline{v}_{0} \circ T B)^{-2} DC^{(i, i, i)}((\mu^{1} - \mu^{2}), T) B + (1 - \overline{v}_{0} \circ B)^{-1} DC^{(i, i, i)}((\mu^{1} - \mu^{2}), T) A.
\]

(104)

Recall that, by Eq. (14) we have

\[
DC^{(i, i, i)}((\mu^{1} - \mu^{2}), T)(x, y) := i! \partial_{y} G(x, y)(\mu^{1} - \mu^{2}) \circ T(x, y) \overrightarrow{DT}(x, y) \ldots \overrightarrow{DT}(x, y),
\]

(105)
Moreover, by using Assumption 2.2\((\nu \to \infty)\) we get
\[
\begin{align*}
|v_i^1 - v_i^2|_{i,D} & \leq \left| (\mu^1_i - \mu^2_i) \right|_{i,D} \frac{(L||A||_D + ||C||_D)||B||_D}{||\partial_y G(x, y)||^{-i}(1 - L||B||_D)^2} (i!)^2 \Lambda(i) \\
+ \left| (\mu^1_i - \mu^2_i) \right|_{i,D} \frac{||A||_D(1 - L||B||_D)}{||\partial_y G(x, y)||^{-i}(1 - L||B||_D)^2} (i!)^2 \Lambda(i) \\
= (i!)^2 \left| (\mu^1_i - \mu^2_i) \right|_{i,D} \frac{||A||_D + ||C||_D||B||_D}{||\partial_y G(x, y)||^{-i}(1 - L||B||_D)^2} \Lambda(i),
\end{align*}
\]
where
\[
\Lambda(i) := \max_{m, n \in \mathbb{N}} \{(||A||_D + ||B||_D)^m (||C||_D + 1)^n\}.
\]
But, from Eq. (5) we have \(2||B||_D L := 1 - ||A||_D - \sqrt{(1 - ||A||_D)^2 - 4||B||_D||C||_D} \). Hence, Eq. (106) becomes
\[
|v_i^1 - v_i^2|_{i,D} \leq \left| (\mu^1_i - \mu^2_i) \right|_{i,D} \Theta(i),
\]
where
\[
\Theta(i) := \frac{(||A||_D + ||C||_D||B||_D) \max_{m, n \in \mathbb{N}} \{(||A||_D + ||B||_D)^m (||C||_D + 1)^n\}}{(2i)!^{-2}||\partial_y G||^{-i} \left( 1 + ||A||_D + \sqrt{(1 - ||A||_D)^2 - 4||B||_D||C||_D} \right)^2}.
\]
Moreover, by using Assumption 2.2\((L_3)\) one can see that
\[
\Theta(i) < 1, \quad 1 \leq i \leq k.
\]
Therefore, on account of Eq. (107) one obtains that the function
\[
\Psi^i(\nu_0, \nu_1, \ldots, \nu_{i-1}, \bullet) : \mathcal{D}_i \to \mathcal{D}_i
\]
is a contraction independent of the point \((\nu_0, \nu_1, \ldots, \nu_{i-1})\), which finishes the proof. \(\square\)

Before stating the proof of the following lemma, it is convenient to introduce some useful notations. Consider the following norm-spaces \(X_1, \ldots, X_n\) with norm \(||\cdot||_i\), for \(0 \leq i \leq k\) respectively and let \(X := X_1 \times \cdots \times X_n\). Then the norm of the space \(X\) will be denoted by \(||\cdot||_X\) and defined by \(||\cdot||_X := \max\{||\cdot||_i : 1 \leq i \leq n\}||\).

**Lemma 5.19** Under Definition 5.5. Let \(0 \leq i \leq k\) be an integer. Suppose that the sets \(\mathcal{D}_j\), for \(1 \leq j \leq i\) are endowed with the norm \(||\cdot||_j,D\) from Proposition 5.15 and the set \(X_i := \mathcal{D}_0 \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_i\) is endowed with the norm \(||\cdot||_{X_i}\). Then, the family of maps \(\Psi^i(\nu_0, \nu_1, \ldots, \nu_{i-1}) : \mathcal{D}_i \to \mathcal{D}_i\) given by \(\Psi^i(\nu_0, v_1, v_2, \ldots, v_{i-1}) = \Psi^i(v_0, v_1, \ldots, v_{i-1}, \nu_i)\) depends on \(\nu_i\) continuously in the following sense: if \((v_0^0, v_1^0, \ldots, v_{i-1}^0, \nu_i^0) \to (v_0, v_1, \ldots, v_{i-1}, \nu_i)\) as \(n \to \infty\) in the space \(X_{i-1}\), then \(\Psi^i(v_0^0, v_1^0, \ldots, v_{i-1}^0, \nu_i) \to \Psi^i(v_0, v_1, \ldots, v_{i-1}, \nu_i)\) in the space \(\mathcal{D}_i\) for any fixed \(\nu_i \in \mathcal{D}_i\).

**Proof** The proof follows from Definitions 5.5, 5.4, 5.3 and 5.2. \(\square\)

We are going to prove Proposition 5.15, which we recall here.

**Proposition 5.20** Assume the notation of Lemma 5.19. Then, the function
\[
\bar{N}_i : X_i \to X_i
\]

\[\text{ Springer} \]
defined by
\[ \tilde{N}_i(\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_i) = (\Gamma(\vec{v}_0), \Psi^1(\vec{v}_0, \vec{v}_1), \ldots, \Psi^i(\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_i)) \]
has a global attracting fixed point \((A_0, A_1, \ldots, A_i)\).

Proof We proceed by induction on \(i\). Suppose that the statement holds for \(j\) with \(0 \leq j < i\). We wish to show that statement holds for \(i\). To do this, we prove that the map \(\tilde{N}_i = (\tilde{N}_{i-1}, \Psi^i) : X_{i-1} \times Y \to X \times Y\), where \(X_{i-1} = \mathcal{D}_0 \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_{i-1}\) and \(Y = \mathcal{D}_i\), satisfies the three conditions of the Fiber Contraction Theorem 5.9. Indeed,

(a) By inductive hypothesis the function \(\tilde{N}_{i-1} : X_{i-1} \to X_{i-1}\) has a global attracting fixed point \((A_0, \ldots, A_{i-1}) \in X_{i-1}\).
(b) By using Theorem 5.15 applied to \((A_0, \ldots, A_{i-1})\), we have that
\[ \Psi^i(A_0, \ldots, A_{i-1}, \bullet) : \mathcal{D}_i \to \mathcal{D}_i \]
is a contraction. Then by the Banach fixed-point theorem \(\Psi^i(A_0, \ldots, A_{i-1}, \bullet)\) has an attracting fixed point \(A_i\).
(c) It follows from Lemma 5.19 that \(\Psi^i(\cdot, A_i) : X \to Y\) is continuous.

Therefore, from (a), (b), and (c), we deduce that \(\tilde{N}_i : X_i \times X_i\) satisfies the three conditions of Theorem 5.9. Thus, we conclude that there exists a global attracting fixed point \((A_0, A_1, \ldots, A_i)\) to the function \(\tilde{N}_i\), which completes the proof. \(\square\)

Now we are ready to prove the Proposition 3.8, which we recall here.

**Proposition 5.21** If \(\vec{v} \in A_L\) is a \(C^k\) function and \(D^i\vec{v}(x, 0) = 0, 0 \leq i \leq k\) and \((x, 0) \in D_0\). Then the following limit exists
\[ \lim_{n \to \infty} (\Gamma^n(\vec{v}), D(\Gamma^n(\vec{v})), \ldots, D^k(\Gamma^n(\vec{v}))) = (v^*, A_1, A_2, \ldots, A_k), \]
where \(A_1, A_2, \ldots, A_k\) are continuous functions.

**Proof of Proposition 3.8** Let \(\vec{v} \in A_L\) be a \(C^k\) function such that \(D^i\vec{v}(x, 0) = 0\), for all \(0 \leq i \leq k\) and \((x, 0) \in D_0\). By induction, it follows that
\[ \tilde{N}_i^n(\vec{v}, D\vec{v}, \ldots, D^i\vec{v}) = (\Gamma^n(\vec{v}), D(\Gamma^n(\vec{v})), \ldots, D^i(\Gamma^n(\vec{v}))). \]

Hence, on account of Proposition 5.15 one obtains
\[ \lim_{n \to \infty} (\Gamma^n(\vec{v}), D(\Gamma^n(\vec{v})), \ldots, D^k(\Gamma^n(\vec{v}))) = (v^*, A_1, A_2, \ldots, A_k), \]
where \(A_j \in \mathcal{D}_j\), for all \(1 \leq j \leq k\), which concludes the proof. \(\square\)

**Acknowledgements** This work is based on the Ph.D. Thesis of the second author. J. V. was partially supported by FAPESP 2009/17153-9. D.S. was partially supported by CNPq 305537/2012-1. The authors thank the referee for constructive and helpful comments and suggestions that improved this work. Furthermore, the authors thank Nancy Chachapoyas, Luis Mello, Leandro Gomes and Pouya Mehdipour for several useful comments.

**References**

1. Afraımović, V.S., Bykov, V.V., Sil’nikov, L.P.: The origin and structure of the Lorenz attractor. Dokl. Akad. Nauk SSSR 234(2), 336–339 (1977)
2. V.S. Afraımovich and Ya.B. Pesin: Dimension of Lorenz type attractors. In: Mathematical Physics Reviews. Soviet Sci. Rev. Sect. C Math. Phys. Rev., vol. 6, pp. 169–241. Harwood Academic Publishing, Chur (1987)

3. Araújo, V., Melbourne, I.: Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor. Preprint (2015). arXiv:1504.04316

4. Araújo, V., Pinto, M.J.: Three-Dimensional Flows, vol. 53 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer, Heidelberg (2010)

5. Araújo, V., Melbourne, I., Varandas, P.: Rapid mixing for the Lorenz attractor and statistical limit laws for their time-1 maps. Commun. Math. Phys. 340(3), 901–938 (2015)

6. Araújo, V., Varandas, P.: Robust exponential decay of correlations for singular-flows. Commun. Math. Phys. 311(1), 215–246 (2012)

7. Belicki˘ı, G.R.: Functional equations, and conjugacy of local diffeomorphisms of finite smoothness class. Funkcional. Anal. i Priložen. 7(4), 17–28 (1973)

8. Brandão, P.: On the structure of lorenz maps. Preprint (2014). arXiv:1402.2862

9. Cohen, D.I.A.: Basic Techniques of Combinatorial Theory. Wiley, New York (1978)

10. Dieudonné, J.: Foundations of Modern Analysis. Academic Press, New York (1969). Enlarged and corrected printing: Pure Appl. Math. 10-I

11. Guckenheimer, J.: A strange, strange attractor, in the Hopf bifurcation and its applications. Appl. Math. Ser. 19, 368–381 (1976)

12. Guckenheimer, J., Holmes, P.: Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. In: Applied Mathematical Sciences, vol. 42. Springer, New York (1983)

13. Guckenheimer, J., Williams, R.F.: Structural stability of Lorenz attractors. Inst. Hautes Études Sci. Publ. Math. 50, 59–72 (1979)

14. Hirsch, M.W., Pugh, C.C.: Stable manifolds for hyperbolic sets. Bull. Am. Math. Soc. 75, 149–152 (1969)

15. Jakobson, M.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. 81, 39–88 (1981)

16. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. 20, 130–141 (1963)

17. Martens, M., de Melo, W.: Universal models for Lorenz maps. Ergodic Theory Dyn. Syst. 21, 833–860 (2001)

18. Martens, M., Winckler, B.: On the hyperbolicity of Lorenz renormalization. Commun. Math. Phys. 325(1), 185–257 (2013)

19. Martens, M., Winckler, B: Physical measures for infinitely renormalizable Lorenz maps. Preprint (2014). arXiv:1412.8041

20. Meyer, C.: Matrix Analysis and Applied Linear Algebra. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2000)

21. Morales, C.A., Pacífico, M.J., Pujals, E.R.: Strange attractors across the boundary of hyperbolic systems. Commun. Math. Phys. 211(3), 527–558 (2000)

22. Palis, J., Brandão, P., Pinheiro, V.: On the finiteness of attractors for one-dimensional maps with discontinuities. Preprint (2016). arXiv:1401.0232

23. Palis, J., Brandão, P., Pinheiro, V.: On the finiteness of attractors for piecewise $C^2$ maps of the interval. Preprint (2016). arXiv:1506.00276

24. Robinson, C.: Differentiability of the stable foliation for the model Lorenz equations. In: Dynamical Systems and Turbulence, Warwick 1980 (Coventry, 1979/1980), vol. 898 of Lecture Notes in Mathematics, pp. 302–315. Springer, Berlin (1981)

25. Robinson, C.: Transitivity and invariant measures for the geometric model of the Lorenz equations. Ergodic Theory Dyn. Syst. 4(4), 605–611 (1984)

26. Rovella, A.: The dynamics of perturbations of the contracting Lorenz attractor. Bol. Soc. Brasil. Mat. (N.S.) 24(2), 233–259 (1993)

27. Rychlik, M.R.: Lorenz attractors through Šil’nikov-type bifurcation. I. Ergodic Theory Dyn. Syst. 10(4), 793–821 (1990)

28. Shil’nikov, L.P.: On the existence of a smooth invariant foliation in Lorenz-type mappings. Differ. Uravn. 30(4):586–595, 732 (1994)

29. Viana, M.: What’s new on Lorenz strange attractors? Math. Intell. 22(3), 6–19 (2000)

30. Vidarte, J.: Smooth perturbation of Lorenz-Like flow. PhD Thesis, ICMC-USP (2014). http://www.teses.usp.br/teses/disponiveis/55/55135/tde-15072014-155326/en.php

31. Williams, R.F.: The structure of Lorenz attractors. Inst. Hautes Études Sci. Publ. Math. 50, 73–99 (1979)