UNIFORM STABILIZATION OF 1-D SCHRÖDINGER EQUATION WITH INTERNAL DIFFERENCE-TYPE CONTROL

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Abstract. In this paper, we consider the stabilization problem of 1-D Schrödinger equation with internal difference-type control. Different from the other existing approaches of controller design, we introduce a new approach of controller design so called the parameterization controller. At first, we rewrite the system with internal difference-type control as a cascaded system of a transport equation and Schrödinger equation; Further, to stabilize the system under consideration, we construct a target system that has exponential stability. By selecting the solution of nonlocal and singular initial value problem as parameter function and defining a bounded linear transformation, we show that the transformation maps the closed-loop system to the target system; Finally, we prove that the transformation is bounded inverse. Hence the closed-loop system is equivalent to the target system.

1. Introduction. Time delay often occurs in many practical control systems especially in distributed parameter control systems [20, 21]. It is well known that time delay even if any small delay may cause periodic oscillations and destabilize the systems [8, 4, 5, 6]. Moreover, in actual systems, small time delay causes catastrophic behavior [16] and makes many control laws not apply to the partial differential control systems, although these control laws are valid in the absence of time delay. The detailed results please see the references [4, 5, 6]. Therefore it is necessary to redesign the stabilizing controller for systems with time delay.

In recent years, time delay problems have been extensively studied, especially the stabilization of systems with time delay becomes a hot topic in control field.

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Due to the counterexamples in the references \cite{4, 5, 6}, Xu et al. \cite{22} firstly considered the stabilization of a 1-d wave equation with difference-type control under the collocated boundary feedback and proved so called "$\frac{1}{2}$-rule" of stability for the boundary difference-type control. Later, Nicaise et al. \cite{14, 15} discussed the condition of the stability of the abstract second-order evolution equations with distributed time delay. And under the "$\frac{1}{2}$-rule" assumption, the authors proved the exponential stability of systems. Different from the "$\frac{1}{2}$-rule" verification, Shang and Xu \cite{17} started to design a new dynamic feedback controller for an Euler-Bernoulli system with input delay in the boundary. Such a new feedback control removes the $\frac{1}{2}$ restriction of parameters. Slight later, Shang and Xu \cite{18} extended the dynamic feedback controller design to more complicated control form for the Euler-Bernoulli beam. However, the main difficulty lies in the stability analysis of the closed-loop system, this is because the multiplier techniques and spectral analysis approach are failed to apply. To overcome the difficulty in stability analysis, based on the system feedback equivalence point of view, Feng, Xu and Chen \cite{7} introduced a new design approach of the feedback control for an Euler-Bernoulli beam with internal pure time-delay control.

At the same time, the investigation of stabilization problem for Schrödinger equations has been developing. For example, Machtyngier and Zuazua \cite{12} considered the stabilization of the Schrödinger equation by multiplier techniques; Liu and Wang \cite{11} applied the backsteping method to study the stabilization problem of an anti-stable Schrödinger equation by boundary feedback with only displacement observation; Cui, Liu and Xu \cite{3} studied the stabilization of Schrödinger equation with constrained boundary control. For the time-delay problem, Guo and Yang \cite{9} considered output feedback stabilization of 1-d Schrödinger equation by the boundary observation with pure time delay; Nicaise and Rebiai \cite{13} obtained the exponential stabilization of a Schrödinger equation with boundary or internal difference-type control. Note that in papers mentioned above, the authors mainly applied the collocated feedback to stabilize the systems. Different from the collocated feedback, based on the "partial state" predictor, Cui, Han and Xu \cite{2} designed a dynamical feedback control to stabilize the 1-d Schrödinger equation with boundary difference-type control \cite{19}. Recently, Chen, Xie and Xu \cite{1} applied the approach comes from Feng et al.\cite{7} to the multi-dimensional Schrödinger equation with interior pure delay control and obtained the exponential stabilization. Moreover, Li, Chen and Xie \cite{10} used this method to consider the stabilization problem for the Schrödinger equation with an input time delay, and obtained the exponential stability with arbitrary decay rate of the closed-loop system. However, in the above literatures, the authors did not discuss whether the approach comes from Feng et al.\cite{7} apply the case of difference-type control or not.

Motivated by this idea, in this paper, we shall extend the approach in Feng et al.\cite{7} to the case of difference-type control for a 1-d Schrödinger equation. Let us consider the 1-D Schrödinger equation with interior difference-type control, whose behaviour is governed by the partial differential equation

\[
\begin{align*}
& w_t(x, t) = -i w_{xx}(x, t) + \alpha u(x, t) + \beta u(x, t - \tau), \quad x \in (0, 1), \quad t > 0, \\
& w(0, t) = w_x(1, t) = 0, \\
& w(x, 0) = w_0(x), \quad x \in [0, 1], \\
& u(x, t - \tau) = h(x, t), \quad t \in [0, \tau],
\end{align*}
\]  \hspace{1cm} (1)

where $w(x, t)$ is the complex-valued state of the system, $i$ is the pure imaginary
unit, $\alpha, \beta \in \mathbb{R}$ and $|\alpha| + |\beta| \neq 0$; $u(x, t)$ is the control input function defined on $(0, 1) \times [-\tau, \infty)$; $w_0(x)$ is the initial state of system and $h(x, t)$ is the controller memory.

This is an extensive model of Schrödinger equation with the distributed controls and input delays. Note that, if $\alpha = 0$ and $\beta = 1$, Li, Chen and Xie [10] studied the exponential stability of the system (1) through establishing the equivalence relationship between the original closed-loop system and the target one. In this paper, we shall extend this approach to the case of difference-type control for a 1-d Schrödinger equation.

Here we set $$v(x, s, t) = u(x, t + s - \tau), \quad x \in [0, 1], \quad s \in [0, \tau], \quad t \geq 0.$$ By a simple computation, the system (1) is equivalent to a coupled system

$$w_t(x, t) = -iw_{xx}(x, t) + \alpha v(x, \tau, t) + \beta v(x, 0, t), \quad x \in (0, 1), \quad t > 0,$$

$$v_t(x, s, t) = v_s(x, s, t), \quad x \in (0, 1), \quad s \in (0, \tau), \quad t > 0,$$

$$v(x, \tau, t) = u(x, t), \quad x \in [0, 1], \quad t > 0,$$

$$w(0, t) = w_x(1, t) = 0, \quad t > 0,$$

$$w(x, 0) = w_0(x), \quad x \in [0, 1],$$

$$v(x, s, 0) = u(x, s - \tau) = h(x, s), \quad x \in [0, 1], \quad s \in [0, \tau].$$

In this paper, we shall design a feedback controller such that the closed-loop system of (2) is exponentially stable.

The rest of this paper is organized as follows. In section 2, we design a parameterization state feedback controller of the system (2) and construct an exponentially stable target system. Furthermore we select suitable kernel functions and construct a bounded linear transformation such that $K$ maps the solution of system (2) to a solution of the target system. In section 3, to prove the bounded invertibility of $K$, we construct a bounded linear transformation by different parameterizing controller that maps the target system to the closed-loop system under selecting the kernel functions. In section 4, we consider the solvability of kernel functions. Finally in section 5 we give a conclusion.

2. Design of feedback controller of (2). In this section, we shall design the state feedback control of the system (2). Suppose that $(w(x, t), v(x, s, t))$ is a solution of (2), we take the control $u(x, t)$ of the form

$$u(x, t) = \int_0^\tau \int_0^1 p(x, \tau - r, y)v(y, r, t)dy\, dr + \int_0^1 \eta(x, \tau, y)w(y, t)dy, \quad t \geq 0,$$

where $p(x, s, y)$ and $\eta(x, s, y)$ are undetermined parameter functions, which are called the parameterization controller. Then the closed-loop system corresponding to (2) is

$$w_t(x, t) = -iw_{xx}(x, t) + \alpha v(x, \tau, t) + \beta v(x, 0, t), \quad x \in (0, 1), \quad t > 0,$$

$$v_t(x, s, t) = v_s(x, s, t), \quad x \in (0, 1), \quad s \in (0, \tau), \quad t > 0,$$

$$v(x, \tau, t) = \int_0^\tau \int_0^1 p(x, \tau - r, y)v(y, r, t)dy\, dr + \int_0^1 \eta(x, \tau, y)w(y, t)dy, \quad x \in [0, 1],$$

$$w(0, t) = w_x(1, t) = 0, \quad t > 0,$$

$$w(x, 0) = w_0(x), \quad v(x, s, 0) = h(x, s), \quad x \in [0, 1], \quad s \in [0, \tau].$$

In what follows, we shall select suitable kernel functions $p(x, s, y)$ and $\eta(x, s, y)$ such that the solution of (4) is exponentially stable at arbitrary rate $k$. For the aim,
we take the state space $\mathcal{H} = L^2[0, 1] \times L^2([0, 1] \times [0, \tau])$ equipped with the norm

$$
\| (f, g) \|_{\mathcal{H}} = \left( \int_0^1 |f(x)|^2 dx + \int_0^\tau \int_0^1 |g(x, s)|^2 dx ds \right)^{\frac{1}{2}}, \quad \forall (f, g) \in \mathcal{H}.
$$

It is easily to show that $\mathcal{H}$ is a Hilbert space.

### 2.1. Construction of target system.

In this subsection, we construct a target system which decays exponentially at rate $k > 0$.

Note that, in the absence of time delay and $u(t) = -kw(x, t)$, $k > 0$, the system

$$
\begin{cases}
    w_t(x, t) = -iw_{xx}(x, t) - kw(x, t), \quad x \in (0, 1), \quad t > 0, \\
    w(0, t) = w_x(1, t) = 0, \\
    w(x, 0) = w_0(x), \quad x \in [0, 1]
\end{cases}
$$

is exponentially stable and decays exponentially at rate $k$ in space $L^2[0, 1]$.

Now let us consider the first order partial differential equation

$$
\begin{cases}
    z_t(x, s, t) = z_s(x, s, t), \quad x \in [0, 1], \quad s \in (0, \tau), \quad t > 0, \\
    z(x, \tau, t) = 0, \\
    z(x, s, 0) = z_0(x, s), \quad x \in [0, 1], \quad s \in [0, \tau].
\end{cases}
$$

Obviously

$$
z(x, s, t) = \begin{cases}
    z_0(x, t + s), \quad t + s \in (0, \tau), \\
    0, \quad t + s \geq \tau
\end{cases}
$$

is a solution of (5). So we can construct the following coupled system

$$
\begin{cases}
    w_t(x, t) = -iw_{xx}(x, t) - kw(x, t) + \beta z(x, 0, t), \quad x \in (0, 1), \quad t > 0, \\
    z_t(x, s, t) = z_s(x, s, t), \quad x \in [0, 1], \quad s \in [0, \tau], \quad t > 0, \\
    z(x, \tau, t) = 0, \quad x \in [0, 1], \\
    w(0, t) = w_x(1, t) = 0, \\
    w(x, 0) = w_0(x), \quad x \in [0, 1], \\
    z(x, s, 0) = z_0(x, s), \quad x \in [0, 1], \quad s \in [0, \tau].
\end{cases}
$$

This can be regarded as a target system. It is easy to know that the following assertion holds true.

**Theorem 2.1.** For any $(w_0, z_0) \in \mathcal{H}$, the solution of system (7) decays exponentially at rate $k$ in the sense of norm in $\mathcal{H}$.

### 2.2. Selecting kernel functions and transformation.

In this subsection, we shall select the suitable kernel functions $p(x, s, y)$ and $\eta(x, s, y)$ and construct a transformation that maps the system (4) to the system (7).

**Theorem 2.2.** Suppose that $\eta(x, s, y)$ is a solution of the following partial differential equation

$$
\begin{cases}
    \eta_t(x, s, y) = -i\eta_{yy}(x, s, y), \quad x, y \in (0, 1), \quad s \in (0, \tau), \\
    \eta(x, s, 0) = \eta_y(x, s, 1) = 0, \\
    \beta \eta(x, 0, y) + \alpha \eta(x, \tau, y) = -k\delta(y - x),
\end{cases}
$$

where $\delta(\cdot)$ is impulse function satisfying $\int_0^1 \delta(y-x)f(y)dy = \begin{cases}
    1, & y = x, \\
    0, & y \neq x
\end{cases}$ for any test function $f(y)$.
Set \( p(x, s, y) = \beta \eta(x, s, y), \quad q(x, \tau - s, y) = \alpha \eta(x, s, y), \) where \( x, y \in [0, 1], \) \( s \in [0, \tau]. \) And define a linear transformation \( K \) on \( H \) by

\[
\begin{bmatrix}
  w(x) \\
  z(x, s)
\end{bmatrix} = K
\begin{bmatrix}
  w(x) \\
  v(x, s)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  v(x, s) - \int^s_0 \int^1_0 p(x, s - r, y)v(y, r)dydr \\
  + \int^s_0 \int^1_0 q(x, r - s, y)v(y, r)dydr - \int^1_0 \eta(x, s, y)w(y)dy
\end{bmatrix}.
\] (9)

Then the transformation \( K \) maps the solution of system (4) to a solution of system (7).

**Proof.** The solvability of (8) will be given in section 4. Here we only prove the transformation \( K \) maps the solution of system (4) to a solution of system (7).

Let \( (w(x, t), v(x, s, t)) \) be a solution of system (4). Set

\[
(w(x, t), z(x, s, t))^T = K(w(x, t), v(x, s, t))^T.
\]

We shall prove that \( (w(x, t), z(x, s, t)) \) is a solution to (7).

We begin with checking \( z(x, s, t) \). From the definition of \( K \) in (9), we know

\[
z(x, s, t) = v(x, s, t) - \int^s_0 \int^1_0 p(x, s - r, y)v(y, r, t)dydr \\
+ \int^s_0 \int^1_0 q(x, r - s, y)v(y, r, t)dydr - \int^1_0 \eta(x, s, y)w(y, t)dy.
\]

Differentiating \( z(x, s, t) \) with respect to \( s \) and \( t \) respectively, we have

\[
z_s(x, s, t) = v_s(x, s, t) - \int^s_0 \int^1_0 p(x, 0, y)v(y, s, t)dy - \int^1_0 q(x, 0, y)v(y, s, t)dy \\
- \int^s_0 \int^1_0 p_s(x, s - r, y)v(y, r, t)dydr \\
- \int^s_0 \int^1_0 q_s(x, r - s, y)v(y, r, t)dydr - \int^1_0 \eta_s(x, s, y)w(y, t)dy,
\]

and

\[
z_t(x, s, t) = v_t(x, s, t) - \int^s_0 \int^1_0 p(x, s - r, y)v_t(y, r, t)dydr \\
+ \int^s_0 \int^1_0 q(x, r - s, y)v_t(y, r, t)dydr - \int^1_0 \eta(x, s, y)w_t(y, t)dy
\]

\[
= v_s(x, s, t) - \int^s_0 \int^1_0 p(x, s - r, y)v_t(y, r, t)dydr \\
+ \int^s_0 \int^1_0 q(x, r - s, y)v_t(y, r, t)dydr \\
- \int^1_0 \eta(x, s, y)[-iw_{\eta y}(y, t) + \alpha v(y, t) + \beta v(y, 0, t)]dy
\]

\[
= v_s(x, s, t) - \int^1_0 p(x, 0, y)v(y, s, t)dy + \int^1_0 p(x, s, y)v(y, 0, t)dy \\
- \int^s_0 \int^1_0 p_s(x, s - r, y)v(y, r, t)dydr + \int^1_0 q(x, \tau - s, y)v(y, \tau, t)dy
\]
Moreover, in fact, we only need to check the differential equation.

\[ \begin{align*}
- \int_0^1 q(x, 0, y)v(y, s, t)dy - & \int_s^\tau \int_0^1 g(x, r - s, y)v(y, r, t)dy \ dr \\
+ & \int_0^1 \int_0^1 w_y(1, t) - i\eta(x, s, 0)w_y(0, t) - i\eta_y(x, s, 1)w(1, t) \\
+ & i\eta_y(x, s, 0)w(0, t) + i \int_0^1 \eta_{yy}(x, s, y)w(y, t)dy \\
- & \alpha \int_0^1 \eta(x, s, y)v(y, \tau, t)dy - \beta \int_0^1 \eta(x, s, y)v(y, 0, t)dy \\
(\text{using the equality of } z_s(x, s, t) \text{ and boundary condition of } w) \\
= & \ z_s(x, s, t) + \int_0^1 [\eta_s(x, s, y) + i\eta_{yy}(x, s, y)]w(y, t)dy \\
+ & \int_0^1 [p(x, s, y) - \beta\eta(x, s, y)]v(y, 0, t)dy \\
+ & \int_0^1 [q(x, \tau - s, y) - \alpha\eta(x, s, y)]v(y, \tau, t)dy \\
- & i\eta(x, s, 0)w_y(0, t) - i\eta_y(x, s, 1)w(1, t) \\
(\text{using the differential equation in (8) and its boundary conditions}) \\
= & \ z_s(x, s, t).
\end{align*} \]

Moreover

\[ \begin{align*}
z(x, \tau, t) &= v(x, \tau, t) - \int_0^\tau \int_0^1 p(x, \tau - r, y)v(y, r, t)dydr - \int_0^1 \eta(x, \tau, y)w(y, t)dy, \\
(\text{using the condition about } v(x, \tau, t) \text{ in (4)}) \\
&= 0.
\end{align*} \]

So \( z(x, s, t) \) satisfies the differential equation and the boundary condition in (7).

Next we check \( w \) satisfies the differential equation in (7) and the boundary condition. In fact, we only need to check the differential equation.

Since \( z(x, \tau, t) = 0 \) and

\[ \begin{align*}
z(x, 0, t) &= v(x, 0, t) + \int_0^\tau \int_0^1 q(x, r, y)v(y, r, t)dydr - \int_0^1 \eta(x, 0, y)w(y, t)dy, \\
\text{so} \\
\beta z(x, 0, t) &= \beta z(x, 0, t) + \alpha z(x, \tau, t) \\
&= \beta \left[ v(x, 0, t) + \int_0^\tau \int_0^1 q(x, r, y)v(y, r, t)dydr - \int_0^1 \eta(x, 0, y)w(y, t)dy \right] \\
&\quad + \alpha \left[ v(x, \tau, t) - \int_0^\tau \int_0^1 p(x, \tau - r, y)v(y, r, t)dydr - \int_0^1 \eta(x, \tau, y)w(y, t)dy \right] \\
&= \beta v(x, 0, t) + \alpha v(x, \tau, t) - \int_0^1 [\beta \eta(x, 0, y) + \alpha \eta(x, \tau, y)]w(y, t)dy \\
&\quad + \int_0^\tau \int_0^1 [\beta q(x, r, y) - \alpha p(x, \tau - r, y)]v(y, r, t)dydr \\
&= \beta v(x, 0, t) + \alpha v(x, \tau, t) - \int_0^1 [\beta \eta(x, 0, y) + \alpha \eta(x, \tau, y)]w(y, t)dy,
\end{align*} \]
where we have used the equality $\beta q(x, r, y) - \alpha p(x, \tau - r, y) = \beta \alpha \eta(x, \tau - r, y) - \alpha \beta \eta(x, \tau - r, y) = 0$. Further, we get the equality

$$\beta z(x, 0, t) + \int_0^1 [\beta \eta(x, 0, y) + \alpha \eta(x, \tau, y)]w(y, t)dy = \beta v(x, 0, t) + \alpha v(x, \tau, t).$$

Therefore, from the differential equation in (4) we get

$$w_t(x, t) = -iw_{xx}(x, t) + \alpha v(x, \tau, t) + \beta v(x, 0, t)$$

$$= -iw_{xx}(x, t) + \beta z(x, 0, t) + \int_0^1 [\beta \eta(x, 0, y) + \alpha \eta(x, \tau, y)]w(y, t)dy$$

(being the initial value condition in (8))

$$= -iw_{xx}(x, t) - kw(x, t) + \beta z(x, 0, t).$$

So $w(x, t)$ satisfies the differential equation and the boundary conditions in (7).

Finally we need to determine the initial value condition in (7). Since

$$z(x, s, t) = v(x, s, t) - \int_0^s \int_0^1 p(x, s - r, y)v(y, r, t)dydr$$

$$+ \int_0^t \int_0^1 q(x, r - s, y)v(y, r, t)dydr - \int_0^1 \eta(x, s, y)w(y, t)dy,$$

taking $t = 0$ leads to

$$z_0(x, s) = h(x, s) - \int_0^s \int_0^1 p(x, s - r, y)h(y, r)dydr$$

$$+ \int_0^s \int_0^1 q(x, r - s, y)h(y, r)dydr - \int_0^1 \eta(x, s, y)w_0(y)dy.$$ That is $(w_0(x), z_0(x, s))^T = \mathbb{K}(w_0(x), h(x, s))$. The proof is completed.

The following theorem shows that $\mathbb{K}$ is a bounded linear operator on $\mathcal{H}$.

**Theorem 2.3.** Let the linear operator $\mathbb{K}$ be defined as (9) and $\eta(x, s, y)$ be a solution to (8), then the following assertions are true:

1. For any $f \in L^2[0, 1]$, there exists a positive $M_1$ such that

$$\int_0^1 \int_0^1 \left( \int_0^1 [\eta(x, s, y)f(y)dy] \right)^2 dxds \leq M_1^2 \int_0^1 |f(y)|^2dy.$$

2. For any $v \in L^2([0, 1] \times [0, \tau])$, there exists a positive $M_2$ such that

$$\int_0^\tau \int_0^1 \left( \int_0^s \int_0^1 \eta(x, s - r, y)v(y, r)dydr \right)^2 dxds \leq M_2^2 \int_0^\tau \int_0^1 |v(y, r)|^2dydr.$$

**Proof.** By the solvability of (8) in section 4, we obtain

$$\eta(x, s, y) = -k \sum_{n=0}^\infty c_n(s)\phi_n(x)\phi_n(y),$$

where

$$c_n(s) = \frac{e^{\mu_n s}}{\alpha e^{\nu_n \tau} + \beta}, \quad \mu_n = \frac{(2n + 1)^2}{4} \pi^2 i, \quad n \geq 0$$

and $\{\phi_n(x), n \geq 0\}$ is an orthonormal basis for $L^2[0, 1]$. 

(1) For any \( f(x) \in L^2[0,1] \), since \( \{ \phi_n(x) \} \) is an orthonormal basis for \( L^2[0,1] \), it holds that
\[
f(x) = \sum_{n=0}^{\infty} a_n(f) \phi_n(x), \quad a_n(f) = (f, \phi_n(x))_{L^2} = \int_0^1 f(x) \overline{\phi_n(x)} dx
\]
and
\[
\int_0^1 |f(x)|^2 dx = \sum_{n=0}^{\infty} |a_n(f)|^2.
\]
So we have
\[
\int_0^1 \eta(x, s, y)f(y)dy = \int_0^1 -k \sum_{n=0}^{\infty} c_n(s) \phi_n(x) \phi_n(y)f(y)dy
\]
\[
= -k \sum_{n=0}^{\infty} c_n(s) \phi_n(x) \int_0^1 f(y)\phi_n(y)dy = -k \sum_{n=0}^{\infty} c_n(s)a_n(f)\phi_n(x),
\]
and hence
\[
\int_0^1 \left| \int_0^1 \eta(x, s, y)f(y)dy \right|^2 dx = k^2 \sum_{n=0}^{\infty} |c_n(s)|^2 |a_n(f)|^2.
\]
Since \( c_n(s) = \frac{ae^{\mu n} - \beta}{ae^{\mu n} + \beta} \) and \( \inf_{n \geq 0} |ae^{\mu n} + \beta| = \delta > 0 \), from the expression of \( c_n(s) \) we can get \( \int_0^\tau |c_n(s)|^2 ds \leq \frac{\tau}{\delta^2} \). Thus
\[
\int_0^\tau \int_0^1 \left| \int_0^1 \eta(x, s, y)f(y)dy \right|^2 dx ds = \int_0^\tau k^2 \sum_{n=0}^{\infty} |c_n(s)|^2 |a_n(f)|^2 ds
\]
\[
= k^2 \sum_{n=0}^{\infty} |a_n(f)|^2 \int_0^\tau |c_n(s)|^2 ds \leq k^2 \frac{T^2}{\delta^2} \sum_{n=0}^{\infty} |a_n(f)|^2 = M^2 \int_0^1 |f(y)|^2 dy.
\]

(2) For any \( v(y, r) \in L^2([0,1] \times [0,\tau]) \)
\[
v(y, r) = \sum_{n=0}^{\infty} b_n(r) \phi_n(y), \quad b_n(r) = \int_0^1 v(y, r)\phi_n(y)dy,
\]
then it holds that
\[
\int_0^\tau \int_0^1 |v(y, r)|^2 dydr = \int_0^\tau \int_0^1 \left| \sum_{n=0}^{\infty} b_n(r) \phi_n(y) \right|^2 dydr = \sum_{n=0}^{\infty} \int_0^\tau |b_n(r)|^2 dr.
\]
A simple calculation gives
\[
-\beta \int_0^s \int_0^1 \eta(x, s - r, y)v(y, r)dydr + \alpha \int_0^\tau \int_0^1 \eta(x, \tau - (r - s), y)v(y, r)dydr
\]
\[
= \beta k \int_0^s \int_0^1 \sum_{n=0}^{\infty} c_n(s - r) \phi_n(x) \phi_n(y)v(y, r)dydr
\]
\[
- \alpha k \int_0^\tau \int_0^1 \sum_{n=0}^{\infty} c_n(\tau - r + s) \phi_n(x) \phi_n(y)v(y, r)dydr
\]
\[
= \sum_{n=0}^{\infty} \phi_n(x) \left( \beta k \int_0^s c_n(s - r) b_n(r) dr - \alpha k \int_0^\tau c_n(\tau - r + s) b_n(r) dr \right).
\]
In this section we shall prove that the instability assertion.

It seems that we do not assert the stability of the system (4) from it. In general, we have

where \(\tilde{K}\). That is

\[
\int_0^T \int_0^1 \left| -\beta \int_0^s \int_0^1 \eta(x, s - r, y)v(y, r)dydr \\
+\alpha \int_s^T \int_0^1 \eta(x, \tau - (r - s), y)v(y, r)dydr \right|^2 dxds \\
= \int_0^T \sum_{n=1}^{\infty} \left| \beta k \int_0^T c_n(s - r)b_n(r)dr - \alpha k \int_s^T c_n(\tau - r + s)b_n(r)dr \right|^2 ds \\
\leq k^2 \int_0^T \sum_{n=1}^{\infty} \left( \int_0^s \beta^2 |c_n(s - r)|^2 dr \cdot \int_0^T |b_n(r)|^2 dr \\
+ \int_s^T \alpha^2 |c_n(\tau - r + s)|^2 dr \cdot \int_0^T |b_n(r)|^2 dr \right) ds \\
\leq k^2 \int_0^T \sum_{n=1}^{\infty} \left( \int_0^s \beta^2 |c_n(s - r)|^2 dr + \int_s^T \alpha^2 |c_n(\tau - r + s)|^2 dr \right) \\
\left( \int_0^T |b_n(r)|^2 dr + \int_0^T |b_n(r)|^2 dr \right) ds \\
\leq k^2 \int_0^T \left( \sum_{n=1}^{\infty} (\alpha^2 + \beta^2) \int_0^T |c_n(r)|^2 dr \cdot \int_0^T |b_n(r)|^2 dr \right) ds \\
\leq k^2 \frac{(\alpha^2 + \beta^2)\tau^2}{\delta^2} \sum_{n=1}^{\infty} \int_0^T |b_n(r)|^2 dr = k^2 \frac{(\alpha^2 + \beta^2)\tau^2}{\delta^2} \int_0^T \int_0^1 |v(y, r)|^2 dydr \\
= M_2^2 \int_0^T \int_0^1 |v(y, r)|^2 dydr,
\]

where \(M_2^2 = k^2 \frac{(\alpha^2 + \beta^2)\tau^2}{\delta^2}\). That is

\[
\int_0^T \int_0^1 \left| -\beta \int_0^s \int_0^1 \eta(x, s - r, y)v(y, r)dydr \\
+\alpha \int_s^T \int_0^1 \eta(x, \tau - (r - s), y)v(y, r)dydr \right|^2 dxds \leq M_2^2 \int_0^T \int_0^1 |v(y, r)|^2 dydr.
\]

The desired result follows.

3. Stability analysis of (4). In the previous section, we proved the boundedness of \(K\). It seems that we do not assert the stability of the system (4) from it. In this section we shall prove that \(K\) is bounded inverse and hence give the stability assertion.

Similar to the previous section, to find out \(K^{-1}\), we first define the following linear transformation \(\mathcal{T}\) on \(\mathcal{H}\),

\[
\mathcal{T} \begin{bmatrix} w(x) \\ z(x, s) \end{bmatrix} = \begin{bmatrix} w(x) \\ v(x, s) \end{bmatrix} \begin{bmatrix} w(x) \\ z(x, s) \end{bmatrix} \\
= w(x) + \int_s^T \int_0^1 \eta(x, \tau - (r - s), y)v(y, r)dydr \\
+ \int_0^s \int_0^1 \eta(x, s - r, y)v(y, r)dydr - \int_0^1 \tilde{\eta}(x, s, y)w(y)dy,
\]

where \(\tilde{p}(x, s, y), \tilde{q}(x, s, y), \tilde{\eta}(x, s, y)\) are determined by the following equation
\[
\begin{align*}
\tilde{\eta}(x, s, y) &= -i\tilde{\eta}_{yy}(x, s, y) - k\tilde{\eta}(x, s, y), \quad x, y \in (0, 1), \quad s \in (0, \tau), \\
\tilde{\eta}(x, s, 0) &= \tilde{\eta}_y(x, s, 1) = 0, \\
\beta \tilde{\eta}(x, 0, y) + \alpha \tilde{\eta}(x, \tau, y) &= k\delta(y - x), \\
\beta \tilde{\eta}(x, s, y) &= \beta \tilde{\eta}(x, s, y), \\
\beta \tilde{\eta}(x, s, y) &= \alpha \tilde{\eta}(x, \tau - s, y)
\end{align*}
\]
whose solvability is given in the next section.

To verify \( T \) is the bounded inverse operator of \( K \), herein we shall adopt the different approach to prove it.

**Theorem 3.1.** Let \( T \) be defined as (10). Then \( T \) maps the solution of system (7) to a solution of system (2) with control \( u(x, t) \)
\[
u(x, t) = -\int_0^\tau \int_0^1 p(x, \tau - r, y)z(y, r, t)dy dr - \int_0^1 \tilde{\eta}(x, \tau, y)w(y, t)dy. \tag{12}\]

**Proof.** Suppose that \( (w(x, t), z(x, s, t)) \) is the solution of system (7). According to the definition of transformation \( T \), it holds that
\[
v(x, s, t) = z(x, s, t) - \int_s^\tau \int_0^1 \tilde{p}(x, s - r, y)z(y, r, t)dy dr \\
+ \int_s^\tau \int_0^1 \tilde{q}(x, r - s, y)z(y, r, t)dy dr - \int_0^1 \tilde{\eta}(x, s, y)w(y, t)dy. \tag{13}\]
We shall verify that \( (w(x, t), v(x, s, t)) \) satisfies the equation (2).
Differentiating (13) with respect to \( s \) and \( t \) leads to
\[
v_s(x, s, t) = z_s(x, s, t) - \int_0^1 \tilde{p}(x, 0, y)z(y, s, t)dy \\
- \int_s^\tau \int_0^1 \tilde{p}(x, s - r, y)z(y, r, t)dy dr - \int_0^1 \tilde{q}(x, 0, y)z(y, s, t)dy \\
- \int_s^\tau \int_0^1 \tilde{q}(x, r - s, y)z(y, r, t)dy dr - \int_0^1 \tilde{\eta}(x, s, y)w(y, t)dy \tag{14}\]
and
\[
v_t(x, s, t) = z_t(x, s, t) - \int_s^\tau \int_0^1 \tilde{p}(x, s - r, y)z(y, r, t)dy dr \\
+ \int_s^\tau \int_0^1 \tilde{q}(x, r - s, y)z(y, r, t)dy dr \\
+ \int_0^1 \tilde{\eta}(x, s, y)[-iw_{yy}(y, t) - kw(y, t) + \beta z(y, 0, t)]dy \\
(\text{using the differential equation in (7)}) \\
- \int_0^1 \tilde{\eta}(x, s, y)[-iw_{yy}(y, t) - kw(y, t) + \beta z(y, 0, t)]dy \\
(\text{Integrating by parts}) \\
= z_s(x, s, t) - \int_0^1 \tilde{p}(x, 0, y)z(y, s, t)dy + \int_0^1 \tilde{p}(x, s, y)z(y, 0, t)dy \\
- \int_0^s \int_0^1 \tilde{p}(x, s - r, y)z(y, r, t)dy dr + \int_0^1 \tilde{q}(x, \tau - s, y)z(y, \tau, t)dy.
\]
In addition, we have

\[ -\int_0^1 \tilde{q}(x,0,y)z(y,s,t)dy - \int_s^\tau \int_0^1 \tilde{q}_r(x,r-s,y)z(y,r,t)dydr \]

\[ + i\tilde{\eta}(x,1,0)w_y(1,t) - i\tilde{\eta}(x,0,0)w_y(0,t) - i\tilde{\eta}_y(x,s,1)w(1,t) \]

\[ + i\tilde{\eta}_y(x,s,0)w(0,t) + i \int_0^1 \tilde{\eta}_{yy}(x,s,y)w(y,t)dy \]

\[ + k \int_0^1 \tilde{\eta}(x,s,y)w(y,t)dy - \beta \int_0^1 \tilde{\eta}(x,s,y)z(y,0,t)dy \]

(using the equality (14) and the boundary conditions in (7))

\[ = v_s(x,s,t) + \int_0^1 [\tilde{p}(x,s,y) - \beta \tilde{\eta}(x,s,y)]z(y,0,t)dy \]

\[ + \int_0^1 [\tilde{\eta}_s(x,s,y) + i\tilde{\eta}_{yy}(x,s,y) + k\tilde{\eta}(x,s,y)]w(y,t)dy \]

\[ - i\tilde{\eta}(x,s,0)w_y(0,t) - i\tilde{\eta}_y(x,s,1)w(1,t) \]

(using the differential equation and boundary conditions in (11))

\[ = v_s(x,s,t). \]

In addition,

\[ v(x,\tau,t) \]

\[ = z(x,\tau,t) - \int_0^\tau \int_0^1 \tilde{p}(x,\tau-r,y)z(y,r,t)dydr - \int_0^1 \tilde{\eta}(x,\tau,y)w(y,t)dy \]

\[ = - \int_0^\tau \int_0^1 \tilde{p}(x,\tau-r,y)z(y,r,t)dydr - \int_0^1 \tilde{\eta}(x,\tau,y)w(y,t)dy \]

\[ = u(x,t). \] (15)

Therefore \( v(x,s,t) \) satisfies the differential equation and boundary condition in (2).

Next we check that \( w(x,t) \) also satisfies the equation in (2). Since

\[ v(x,0,t) = z(x,0,t) + \int_0^\tau \int_0^1 \tilde{q}(x,r,y)z(y,r,t)dydr - \int_0^1 \tilde{\eta}(x,0,y)w(y,t)dy, \]

we have

\[ w_t(x,t) + iw_{xx}(x,t) - [\alpha v(x,\tau,t) + \beta v(x,0,t)] \]

\[ = w_t(x,t) + iw_{xx}(x,t) - \alpha v(x,\tau,t) - \beta z(x,0,t) \]

\[ - \beta \int_0^\tau \int_0^1 \tilde{q}(x,r,y)z(y,r,t)dydr + \beta \int_0^1 \tilde{\eta}(x,0,y)w(y,t)dy \]

(using the expression of \( v(x,\tau,t) \) in (15))

\[ = w_t(x,t) + iw_{xx}(x,t) - \beta z(x,0,t) + \alpha \int_0^\tau \int_0^1 \tilde{p}(x,\tau-r,y)z(y,r,t)dydr \]

\[ + \alpha \int_0^1 \tilde{\eta}(x,\tau,y)w(y,t)dy - \beta \int_0^\tau \int_0^1 \tilde{q}(x,r,y)z(y,r,t)dydr \]

\[ + \beta \int_0^1 \tilde{\eta}(x,0,y)w(y,t)dy \]

\[ = w_t(x,t) + iw_{xx}(x,t) - \beta z(x,0,t) \]

\[ - \int_0^\tau \int_0^1 [\beta \tilde{q}(x,r,y) - \alpha \tilde{p}(x,\tau-r,y)]z(y,r,t)dydr \]
The following statements are true:

**Theorem 3.2.**

Let

\[ H \]

and decays exponentially at rate \( k \) given as (3). Then the solution of the closed-loop system (4) is exponentially stable.

\[ TK = I \] and \( KT = I \). The result of corollary is obviously verified.

**Theorem 3.3.**

For any \((w_0(x), h(x, s)) \in H^1[0, 1] \times H^1([0, 1] \times [0, \tau])\), from the final steps of Theorem 2.2 and 3.1 we know that

\[ \mathbb{K}(w_0, h)^T = (w_0, z_0)^T , \quad \mathbb{T}(w_0, z_0)^T = (w_0, h)^T . \]

Therefore, \( \mathbb{T} \) is a bounded linear operator on \( \mathcal{H} \).

**Theorem 3.2.** Let \( \mathbb{T} \) be defined as (10) and \( \eta(x, s, y) \) be a solution to (11). Then the following statements are true:

(1) For any \( g \in L^2([0, 1]) \), there exists a positive \( M_3 \) such that

\[ \int_0^\tau \int_0^1 \left| \int_0^1 \eta(x, s, y)g(y)dy \right|^2 \, dxds \leq M_3^2 \int_0^1 |f(y)|^2 \, dy; \]

(2) For any \( x \in L^2([0, 1] \times [0, \pi]) \), there exists a positive \( M_4 \) such that

\[ \int_0^\tau \int_0^1 \left| -\beta \int_0^s \int_0^1 \eta(x, s, r, y)z(y, r)dydr + \alpha \int_s^\tau \int_0^1 \eta(x, \tau - (r - s), y)z(y, r)dydr \right|^2 \, dxds \leq M_4^2 \int_0^\tau \int_0^1 |z(y, r)|^2 \, dydr . \]

The proof of Theorem 3.2 is entirely similar to one of Theorem 2.3, so we shall omit it.

According to Theorems 2.2 and 3.1, we have the following corollary.

**Corollary 1.** Let \( \mathbb{K} \) and \( \mathbb{T} \) be defined as (9) and (10) respectively. Then \( \mathbb{T} = \mathbb{K}^{-1} \).

For any \((w_0(x), h(x, s)) \in H^1[0, 1] \times H^1([0, 1] \times [0, \tau])\), from the final steps of Theorem 2.2 and 3.1 we know that

\[ \mathbb{K}(w_0, h)^T = (w_0, z_0)^T , \quad \mathbb{T}(w_0, z_0)^T = (w_0, h)^T . \]

Therefore, \( \mathbb{T} \) is a bounded linear operator on \( \mathcal{H} \).

**Theorem 3.3.** Let \( p \) and \( \eta \) satisfy the conditions in Theorem 2.2, and let \( u(x, t) \) be given as (3). Then the solution of the closed-loop system (4) is exponentially stable and decays exponentially at rate \( k \) in the sense of norm in \( \mathcal{H} \).
Proof. Assume that \((w(x,t), v(x, s, t))\) and \((w(x,t), z(x, s, t))\) are solutions to (2) and (7) respectively. According to Theorem 3.2, we have
\[
\|(w(x,t), v(x, s, t))\|_\mathcal{H} \leq \|T\|\|(w(x,t), z(x, s, t))\|_\mathcal{H}.
\]
Applying Theorem 2.1, the desired result follows. \(\square\)

4. Solvability of (8) and (11). In this section, we consider the solvability of the nonlocal and singular initial value problems (8) and (11).

Firstly we introduce a lemma which is a result of second order differential operator.

Lemma 4.1. Let the differential operator \(L\) in \(L^2(0,1)\) be defined as
\[
(Lf)(x) = -if''(x), \quad D(L) = \{ f(x) \in H^2(0,1)| f(0) = f'(1) = 0 \}.
\]
Then \(L\) is a skew-adjoint operator with compact resolvent in \(L^2(0,1)\). The eigenvalues of \(L\) are \(\mu_n = \frac{(2n+1)^2 \pi^2}{4}i, \quad n \in \mathbb{N}\), and the eigenvector \(\{\phi_n(x), n \in \mathbb{N}\}\) corresponding to \(\{\mu_n, n \in \mathbb{N}\}\) forms an orthonormal basis for \(L^2(0,1)\).

Theorem 4.2. Suppose that \(\inf|\beta + \alpha e^{\mu_n s}| > 0\), \(\phi_n(x)\) and \(\mu_n\) are defined as Lemma 4.1 and satisfy the eigenvalue equation
\[
\begin{align*}
-\alpha \phi_n''(x) &= \mu_n \phi_n(x), \quad x \in (0,1), \\
\phi_n(x) &= \phi_n'(x) = 0.
\end{align*}
\]
Then the partial differential equation (8)
\[
\begin{align*}
\eta_n(x, s, y) &= -i \eta_{yb}(x, s, y), \quad x, y \in (0,1), \quad s \in (0, \tau), \\
\eta(x, s, 0) &= \eta_{y}(x, s, 1) = 0, \\
\beta \eta(x, 0, y) + \alpha \eta(x, \tau, y) &= -k \delta(y-x)
\end{align*}
\]
has a unique solution
\[
\eta(x, s, y) = -k \sum_{n=0}^{\infty} \frac{e^{\mu_n s}}{\beta + \alpha e^{\mu_n \tau}} \phi_n(x) \phi_n(y).
\]
Proof. Let
\[
\eta_n(x, s) = \int_0^1 \eta(x, s, y) \phi_n(y)dy.
\]
Differentiating \(\eta_n(x, s)\) with respect to \(s\) and using the equation (8), we have
\[
\frac{\partial \eta_n(x, s)}{\partial s} = \int_0^1 \eta(x, s, y) \phi_n(y)dy = -i \int_0^1 \eta_{yb}(x, s, y) \phi_n(y)dy
\]
\[
= -i \int_0^1 \eta(x, s, y) \phi_n''(y)dy = \mu_n \eta_n(x, s).
\]
Further it holds that
\[
\eta_n(x, s) = \eta_n(x, 0) e^{\mu_n s}.
\]
In addition, the condition \(\beta \eta(x, 0, y) + \alpha \eta(x, \tau, y) = -k \delta(y-x)\) yields
\[
\int_0^1 \beta \eta(x, 0, y) \phi_n(y)dy + \int_0^1 \alpha \eta(x, \tau, y) \phi_n(y)dy = \int_0^1 -k \delta(y-x) \phi_n(y)dy,
\]
that is \(\beta \eta_n(x, 0) + \alpha \eta_n(x, \tau) = -k \phi_n(x)\), or equivalently
\[
\beta \eta_n(x, 0) + \alpha \eta_n(x, 0) e^{\mu_n \tau} = -k \phi_n(x).
\]
Thus we get
\[ \eta_n(x, 0) = \frac{-k\phi_n(x)}{\beta + \alpha e^{\mu_n\tau}} \]
and hence
\[ \eta(x, s, y) = -k \sum_{n=0}^{\infty} \frac{e^{\mu_n s}}{\beta + \alpha e^{\mu_n\tau}} \phi_n(x)\phi_n(y). \]
The desired result is obtained. \( \square \)

**Theorem 4.3.** Suppose that \( \inf |\beta + \alpha e^{(\mu_n-k)\tau}| > 0 \), \( \phi_n(x) \) and \( \mu_n \) are defined as before and also satisfy the boundary eigenvalue problem (16). Then the partial differential equation (11)
\[
\begin{align*}
\tilde{\eta}_n(x, s, y) &= -i\tilde{\eta}_{yy}(x, s, y) - k\tilde{\eta}(x, s, y), \quad x, y \in (0, 1), \quad s \in (0, \tau), \quad k > 0, \\
\tilde{\eta}(x, s, 0) &= \tilde{\eta}_y(x, s, 1) = 0, \\
\beta\tilde{\eta}(x, 0, y) + \alpha\tilde{\eta}(x, \tau, y) &= k\delta(y-x)
\end{align*}
\]
has a unique solution
\[ \tilde{\eta}(x, s, y) = k \sum_{n=0}^{\infty} \frac{e^{(\mu_n-k)s}}{\beta + \alpha e^{(\mu_n-k)\tau}} \phi_n(x)\phi_n(y). \]

**Proof.** Let
\[ \tilde{\eta}_n(x, s) = \int_0^1 \tilde{\eta}(x, s, y)\phi_n(y)dy. \]
Differentiating \( \eta_n(x, s) \) with respect to \( s \) and using the equation (11), we have
\[
\frac{\partial \tilde{\eta}_n(x, s)}{\partial s} = \int_0^1 \tilde{\eta}_s(x, s, y)\phi_n(y)dy = \int_0^1 [-i\tilde{\eta}_{yy}(x, s, y) - k\tilde{\eta}(x, s, y)]\phi_n(y)dy
\]
\[ = -i \int_0^1 \tilde{\eta}(x, s, y)\phi_n''(y)dy - k \int_0^1 \tilde{\eta}(x, s, y)\phi_n(y)dy = (\mu_n - k)\tilde{\eta}_n(x, s). \]
Thus
\[ \tilde{\eta}_n(x, s) = \tilde{\eta}_n(x, 0)e^{(\mu_n-k)s}. \]
From \( \beta\tilde{\eta}(x, 0, y) + \alpha\tilde{\eta}(x, \tau, y) = k\delta(y-x) \), we can get \( \beta\tilde{\eta}(x, 0, y) + \alpha\tilde{\eta}(x, \tau, y) = k\phi_n(x) \) and
\[ \tilde{\eta}_n(x, 0) = \frac{k\phi_n(x)}{\beta + \alpha e^{(\mu_n-k)\tau}}. \]
Therefore, it holds that
\[ \tilde{\eta}(x, s, y) = k \sum_{n=0}^{\infty} \frac{e^{(\mu_n-k)s}}{\beta + \alpha e^{(\mu_n-k)\tau}} \phi_n(x)\phi_n(y). \]
This ends the proof. \( \square \)

5. **Numerical simulations.** In this section, we shall give some numerical simulations to demonstrate the effectiveness of system proposed with difference-type control and the stability of the closed-loop system (7). All numerical results were made with Matlab R2018b by adopting the central difference method in space (with a fixed space step \( dx = 0.025 \)) and the implicit Euler method in time (with a fixed time step \( dt = 0.005 \)). In the whole process of computation, we use the initial values of system (7) \( w(x; 0) = 5\cos(3x) \), \( z_0(s, 0) = \frac{1}{\sqrt{2\pi}}e^{-x^2} \) and take the delay time \( \tau = 0.5 \) and the feedback gain \( k = 2 \).
Firstly, we shall give the dynamic behaviour of system (1) in absence of control which is a free system, whose dynamic behaviour is shown in Figure 1.

![Figure 1](image1.png)  
(a) The real part of \( w(x,t) \)  
(b) The imaginary part of \( w(x,t) \)

**Figure 1.** The dynamic behaviour of system (1) for \( \alpha = \beta = 0 \)

![Figure 2](image2.png)  
(a) The real part of \( w(x,t) \)  
(b) The imaginary part of \( w(x,t) \)

**Figure 2.** The dynamic behaviour of system (1) for \( \alpha = 1, \beta = 0 \) under \( U(t) = -kw(x,t) \)

Under the feedback control law \( U(t) = -kw(x,t) \), Figure 2-3 shows that the dynamic behaviour of the closed-loop system (1) with \( \alpha = 1, \beta = 0 \) and \( \alpha = 2, \beta = 1 \) respectively. Figure 2 shows that, the solution to the closed-loop system (1) can decay to zero. But Figure 3 shows that, under the same feedback control, the solution to the closed-loop system (1) with difference-type control can not decay to zero. From Figure 2-3, we can see that, the feedback control law \( U(t) = -kw(x,t) \) fails to stabilize the system (1). Thus it is necessary to design a new parameterization controller for stabilizing the system (1).

When \( \alpha = 2, \beta = 1 \), Figure 4 shows that the dynamic behaviour of the system (1) under the control (3). From Figure 4, we see that the solution to the closed-loop system (1) can decay faster to zero at rate \( k = 2 \). In fact, \( k \) is the exponential decay rate of the closed-loop system and any desired decay rate can obtained by choosing a proper \( k \).
6. Conclusion. In this paper, we studied the stabilization problem of 1-D Schrödinger equation with internal difference-type control. Different from the existing dynamic feedback controller design, we adopt design of parameterization controller, which is a state feedback controller. By constructing a suitable target system and selecting parameter functions, we define a bounded invertible linear transformation and establish the equivalence between the system under consideration and the target system. In this process, the key issue is the selection of parameter functions. For the different target system, there is different selection of parameter functions. Usually, the parameter function satisfies the partial differential equation with certain singular initial value condition. In this paper we mainly use the solutions of two partial differential equations with nonlocal and singular initial value as parameter functions. Such a process forms a new approach of controller design.

The advantage of such an approach is to overcome the difficulty in stability analysis of the closed-loop system. This is because stability of the target system is known and the closed-loop system is equivalent to the target system. In addition, the approach of controller design can be applied to high-dimensional systems. But, it is a key problem we need to solve for obtaining the expression of parameter
equations and proving the boundedness and invertibility of linear transformations. Therefore, in the future, we shall calculate such high-dimensional system in detail.

However we shall note that, in order to prove the invertibility and boundedness of the transformation, we have to use the expression of solution of partial differential equation with nonlocal and singular initial value problem. For some complicated models or choosing more complicated target systems, it is difficult to obtain the expression of their solutions. So how to use the differential equation of parameter function to prove the boundedness of transformation is a key issue. Moreover, it is also a worthy studying problem for extending the design of parameterization controller to the more complicated difference-type control including the distributed term and high-dimensional system.

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