GROWTH IN LINEAR ALGEBRAIC GROUPS AND PERMUTATION GROUPS: TOWARDS A UNIFIED PERSPECTIVE

Abstract. By now, we have a product theorem in every finite simple group $G$ of Lie type, with the strength of the bound depending only in the rank of $G$. Such theorems have numerous consequences: bounds on the diameters of Cayley graphs, spectral gaps, and so forth. For the alternating group $\text{Alt}_n$, we have a quasipolylogarithmic diameter bound (Helfgott-Seress 2014), but it does not rest on a product theorem.

We shall revisit the proof of the bound for $\text{Alt}_n$, bringing it closer to the proof for linear algebraic groups, and making some common themes clearer. As a result, we will show how to prove a product theorem for $\text{Alt}_n$ – not of full strength, as that would be impossible, but strong enough to imply the diameter bound.

1. Introduction

My personal route in the subject started with the following result.

**Theorem 1.1 (Product theorem [Hel08]).** Let $G = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})$, $p$ a prime. Let $A \subset G$ generate $G$. Then either

$$|A \cdot A \cdot A| \geq |A|^{1+\delta}$$

or

$$A^k = G,$$

where $\delta > 0$ and $k \in \mathbb{Z}^+$ are absolute constants.

Here $|S|$ is the number of elements of a set $S$, and $A^k$ denotes $\{a_1 \ldots a_k : a_i \in A\}$. (We also write $AB$ for $\{ab : a \in A, b \in B\}$ and $A^{-1}$ for $\{a^{-1} : a \in A\}$.)

Theorem 1.1 gives us an immediate corollary on the diameter of any Cayley graph $\Gamma(G, A)$ of $G$. The **diameter** of a graph is the maximal distance $d(v_1, v_2)$ over all pairs of vertices $v_1, v_2$ of a graph $\Gamma$; in turn, the distance $d(v_1, v_2)$ between two vertices is the length of the shortest path between them, where the length of a path is defined as its number of edges. In the particular case of a (directed) Cayley graph $\Gamma(G, A)$, the diameter equals the least $\ell$ such that every element $g \in G$ can be expressed as a product of elements of $A$ of length $\leq \ell$.

**Corollary 1.2.** Let $G = \text{SL}_2(\mathbb{Z}/p\mathbb{Z})$, $p$ a prime. Let $S \subset G$ generate $G$. Then the diameter of the Cayley graph $\Gamma(G, S)$ is at most

$$diameter \leq (\log |G|)^C,$$

where $C$ is an absolute constant.

**Proof.** Apply Theorem 1.1 to $A = S$, $A = S^3$, $A = S^9$, etc. \hfill $\square$

The product theorem has other applications, notably to spectral gaps and expander graphs ([BG08], [BGS10], [BGS11]). It has been generalized several times.$^1$

$^1$ It was soon determined that $k = 3$ ([Gow08], [BNP08]).
times, to the point where now it is known to hold for all finite simple groups of Lie type and bounded rank \cite{BG11}, \cite{PS16}. When we say bounded rank, we mean that the constants $\delta$ and $C$ in these generalizations of Thm. 1.1 and Cor. 1.2 depend on the rank of the group $G$.

Babai’s conjecture states that the bound (1.1) holds any finite, simple, non-abelian $G$, and any set of generators $S$ of $G$, with $C$ an absolute constant (i.e., one constant valid for all $G$). By the classification of finite simple groups (henceforth: CFSG), every finite, simple, non-abelian group $G$ is either (a) a simple group of Lie type, or (b) an alternating group $\text{Alt}(n)$, or (c) one of a finite number of sporadic groups. Being finite in number, the sporadic groups are irrelevant for the purposes of the asymptotic bound (1.1). It remains, then, to consider whether Babai’s conjecture is true for $\text{Alt}(n)$, and for simple, finite groups of Lie type whose rank goes to infinity.

Part of the problem is that, in either of these two cases, the natural generalization of Thm. 1.1 is false: counterexamples due to Pyber and Spiga \cite{PPSS12}, \cite{Spi12} show that $\delta$ has to depend on the rank of $G$, or on the index $n$ in $\text{Alt}(n)$, at least if there are no additional conditions. Nevertheless, Babai’s conjecture is still believed to be true.

Some of the ideas leading to Thm. 1.1 and its generalizations were useful in the proof of the following result, even though the overall argument looked rather different.

**Theorem 1.3 \cite{HS14}**. Let $G = \text{Alt}(n)$ or $G = \text{Sym}(n)$. Then,

\[
\text{diam } G \leq e^{C(\log \log |G|)^4 \log \log \log |G|},
\]

where $C$ is an absolute constant.

Here we write $\text{diam } G$ for the “worst-case diameter”

\[
\text{diam } G = \max_{A \subseteq G : \langle A \rangle} \text{diam}(\Gamma(G, A)),
\]

i.e., the same sort of quantity that we bounded in Cor. 1.2.

Theorem 1.3 is not as strong as Babai’s conjecture for $\text{Alt}(n)$ or $\text{Sym}(n)$, since the quantity on the right of (1.2) is larger than $(\log |G|)^C$. The proof of Thm. 1.3 did not go through the proof of an analogue of a product theorem; it used another kind of inductive process.

One of our main aims in what follows is to give a different proof of Theorem 1.3. Some of its elements are essentially the same as in the original proof, sometimes in improved or simplified versions. Others are more closely inspired by the tools developed for the case of groups of Lie type.

Theorem 1.3 – or rather a marginally weaker version thereof (Thm. 6.1), with an additional factor of $\log \log \log |G|$ in the exponent – will follow as a direct consequence of the following product theorem, which is new. It is, naturally, weaker than a literal analogue of Thm. 1.1 since such as analogue would be false, by the counterexamples we mentioned.

**Theorem 1.4.** Let $A \subseteq \text{Sym}(n)$ be such that $A = A^{-1}$, $e \in A$, and $\langle A \rangle$ is 3-transitive. There are absolute constants $C, c > 0$ such that the following holds. Assume that $|A| \geq n^{C(\log n)^2}$. Then either

\[
|A^{nC}| \geq |A|^{1 + \frac{\log \log |A|}{\log \log n} + \frac{\log \log |A|}{\log \log n}^2 - \frac{\log \log n}{\log \log n}}.
\]
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or

\[(1.4) \quad \text{diam}(\Gamma((A), A)) \leq n^C \text{diam}(G),\]

where \(G\) is a transitive group on \(m \leq n\) elements such that either (a) \(m \leq e^{-1/10}n\) or (b) \(G \approx \text{Alt}(m), \text{Sym}(m)\).

Our general objective will be to make the proof for \(\text{Alt}(n)\) and \(\text{Sym}(n)\) not just simpler but closer to that for groups of Lie type of bounded rank. Part of the motivation is that the next natural aim is to study in depth groups of Lie type of unbounded rank, which combine features of both kinds of groups.

**Overall idea.** To prove the growth of sets \(A\) in a group \(G\), we study the actions of a group \(G\). First of all, every group acts on itself, by left and right multiplication, and by conjugation. The study of these actions is always useful; it gives us lemmas valid for every group. Then there are the actions that exist for a given kind of group.

A linear algebraic group acts by linear transformations on affine space. It then makes sense to see how the action of the group affects varieties, and what this tells us about sets of elements in the group.

In the case of the symmetric group \(\text{Sym}(\Omega)\), \(|\Omega| = n\), we have no such nicely geometric action. What we do have is an action on a set \(\Omega\), that, while completely unstructured, is very small compared to the group. This fact allows us to use short random walks to obtain elements whose action on \(\Omega\) and low powers follows an almost uniform distribution.

It is then unsurprising that the strategies for linear algebraic groups and symmetric groups diverge: the actions that characterize the two kinds differ. Nevertheless, it is possible to unify the strategies to some extent. We shall see that the role played by generic elements – in the sense of algebraic geometry – in the study of growth in linear algebraic groups is roughly analogous to the role played in permutation groups by random elements – in the sense of being produced by random walks.

**Further perspectives.** A “purer” product theorem would state that either (1.3) holds or, say, \(A^\log n = G\). The switch to diameters in conclusion (1.4) is not just somewhat ungainly; it also slows down the recursion. If (1.4) were replaced by \(A^{\log n} = G\), we would then obtain an exponent of 3 instead of 4 in Theorem 1.3. Such a “purer” result is not contradicted by the existing counterexamples, and so remains a plausible goal.

Yet another worthwhile goal would be to remove the dependence on the Classification of Finite Simple Groups (CFSG). The proof here uses the structure theorem in [Cam81]/[Lie84], which relies on CFSG. The proof in [HS14] also depended on CFSG, for essentially the same reason: it used [BS92] Thm. 1.4, which uses [Cam81]/[Lie84].

Incidentally, there is a flaw in [BS92] Thm. 1.4 (proof and statement), as L. Pyber pointed out to the author. We fix it in [H] with input from Pyber; the amended statement is in Prop. 4.15. The bound in [HS14] is not affected when we replace [BS92] Thm. 1.4 by Prop. 4.15 in the proof of [HS14].

**Notation.** We write actions on the right, i.e., if \(G\) acts on \(X\), and \(g \in G, x \in X\), we write \(x^g\) for the element to which \(g\) sends \(x\).

As is usual, we write \(f(x) = O(g(x))\) to mean that there exists a constant \(C > 0\) (called an implied constant) such that \(|f(x)| \leq Cg(x)\) for all large enough \(x\). We also write \(f(x) \ll g(x)\) to mean that \(f(x) = O(g(x))\), and \(f(x) \gg g(x)\) to...
mean, for \( g \) taking positive values, that there is a constant \( c > 0 \) (called, again, an implied constant) such that \( f(x) \geq cg(x) \) for all large enough \( x \). When we write \( O^*(c) \), \( c \) a non-negative real, we simply mean a quantity whose absolute value is at most \( c \).

Given \( h \in G \), we write \( C(h) \) for the centralizer \( \{ g \in G : gh = hg \} \) of \( h \). Given \( H \leq G \), we write \( C(H) \) for the centralizer \( \{ g \in G : gh = hg \forall h \in H \} \) of \( H \).

As should be clear by now, and as is standard, we write \( \text{Alt}(\Omega) \) for the alternating group on a set \( \Omega \), and \( \text{Alt}(m) \) for the abstract group isomorphic to \( \text{Alt}(\Omega) \) for any set \( \Omega \) with \( n \) elements. We define \([n] = \{1, 2, \ldots, n\} \).

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2. Toolbox

2.1. Special sets. In the proofs of growth for groups of Lie type, some of the main tools are statements on intersections with varieties. A typical statement is of the following kind.

**Lemma 2.1.** Let \( G = \text{SL}_2(K) \), \( K \) a finite field. Let \( A \subset G \) be a set of generators of \( G \) with \( A = A^{-1} \). Let \( V \) be a one-dimensional irreducible subvariety of \( \text{SL}_2 \). Then, for every \( \delta > 0 \), either \( |A^3| \geq |A|^{1+\delta} \) holds, or the intersection of \( A \) with \( V \) has

\[
\ll |A|^{|V|_{\text{dim SL}_2}}+O(\delta) \leq |A|^{1/3+O(\delta)}
\]

elements. The implied constants depend only on the degree of \( V \).

Special statements of this kind were proved and used in [He08] and [He11], and have been central to the main strategy since then. They were fully generalized in [PS16]. As it happens, Larsen and Pink, in the course of their work on finite subgroups of linear groups, had proven results of the same kind – for subgroups \( H \), instead of sets \( A \), but for all simple linear groups \( G \). Their procedure was adapted in [BGT11] to give essentially the same general result as in [PS16].

(Incidentally, the main purpose of Larsen and Pink was to prove without CFSG a series of statements that follows from CFSG. For this purpose, they developed tools that were, in some sense, both concrete and general. It was these features that let the tools be generalized later to sets, as opposed to subgroups. This is not the only time that preexistent work on doing without CFSG has proved fruitful in this context; we will see another instance when we examine random walks and permutation groups.)

There is an obvious difficulty in adapting such work to the study of permutation groups: in \( \text{Sym}(n) \), there seems to be no natural concept of a “variety”, let alone its degree and its dimension.

The approach we will follow here is to strip to the proof of a statement such as Lemma 2.1 to its barest bones, so that the main idea becomes a statement about an abstract group. We will later be able to see how to apply it to obtain a useful result on permutation groups.

The proof of Lemma 2.1 goes as follows. First, we show that, for generic \( g_1, g_2 \in \text{SL}_2(K) \), the map \( \phi : V \times V \times V \to G \) given by

\[
\phi(v_0, v_1, v_2) = v_0 \cdot g_1 V g_1^{-1} \cdot g_2 V g_2^{-1}
\]
is almost-injective, in the sense that the preimage of a generic point of the (closure of) the image is zero-dimensional. "A generic point" here means "a point outside a subvariety of positive codimension". Similarly, "for $g_1$, $g_2$ generic" means that the pairs $(g_1, g_2)$ for which the map $\phi$ is not almost-injective lie in a variety of positive codimension $W$ in $\text{SL}_2 \times \text{SL}_2$. Now, because $A$ generates $G$, a general statement on escape of subvarieties shows that there exists a pair $(g_1, g_2) \in A^k \times A^k$ outside $W$, where $k$ is a constant depending only on the degree of $V$. ("Escape of subvarieties" was an argument known before [Hel08]. The statement in [EMO05] Prop. 3.2] is over $\mathbb{C}$, but the argument of the proof there is valid over an arbitrary field; see, e.g., [Hel11] Prop. 4.1.)

Then we examine the image of $(A \cap V) \times (A \cap V) \times (A \cap V)$ under $\phi$. If $\phi$ is injective, then the image has exactly the same size as the domain, namely, $|A \cap V|^3$. In general, for $\phi$ almost-injective, the image will have size $|A \cap V|^3$. At the same time, the image is contained in $A^{1+2k+1+2k+1+2k} = A^{8k+3}$. Hence

$$|A \cap V| \leq \left|A^{8k+3}\right|^{1/3}.$$ 

Let us prove an extremely simple general statement that expresses the main idea of the statement we have just sketched.

**Lemma 2.2.** Let $G$ be a group. Let $A, B \subset G$ be finite. Then

$$|AB^{-1}| \geq \frac{|A|^2}{|AA^{-1} \cap BB^{-1}|}.$$ 

In particular, if $AA^{-1} \cap BB^{-1} = \{e\}$, then

$$|AB^{-1}| \geq |A||B|.$$ 

The condition $AA^{-1} \cap BB^{-1} = \{e\}$ is fulfilled if, for instance, $A \subset H_1$, $B \subset H_2$, where $H_1$, $H_2$ are subgroups of $G$ with $H_1 \cap H_2 = \{e\}$.

**Proof.** Consider the map $\phi : A \times B \to AB^{-1} \subset G$ defined by

$$(a, b) \mapsto ab^{-1}.$$ 

Clearly, as with any map from $A \times B$ to $G$,

$$\text{(2.1)} \quad |\text{im}(\phi)| \geq \frac{|A \times B|}{\max_{x \in G} |\phi^{-1}(x)|},$$ 

and of course $|AB^{-1}| \geq |\text{im}(\phi)|$.

So, let us bound $\phi^{-1}(x)$. Say $\phi(a, b) = x = \phi(a', b')$. Then

$$a^{-1} a' = b(b')^{-1}. \quad \text{(2.2)}$$ 

In particular, given $a$, $b$, and $b(b')^{-1}$, we can reconstruct $a'$ and $b'$. Moreover, again by (2.2), $b(b')^{-1}$ lies in $AA^{-1} \cap BB^{-1}$. Letting $(a, b)$ be fixed, and letting $(a', b')$ vary among all elements of $\phi^{-1}(x)$, we see that

$$|\phi^{-1}(x)| \leq |AA^{-1} \cap BB^{-1}|.$$ 

By (2.1), we are done. \qed

We can apply the same idea to obtain growth assuming only that an intersection of many sets is empty.
Lemma 2.3. Let $G$ be a group. Let $A_0, A_1, \ldots, A_k \subset G$ be finite. Then there is at least one $0 \leq j \leq k - 1$ such that

$$|A_j A_{j+1}^{-1}| \geq \frac{A^{k+1}}{\left|\bigcap_{j=0}^{k} A_j A_{j+1}^{-1}\right|^{1/k}},$$

where $A$ is the geometric average $(\prod_{j=0}^{k} |A_j|)^{1/(k+1)}$. In particular, if

(2.3) \[ \bigcap_{j=0}^{k} A_j A_{j+1}^{-1} = \{e\}, \]

then

$$|A_j A_{j+1}^{-1}| \geq A^{k+1}.$$

We will typically apply this lemma to sets $A_j$ that are conjugates of each other, and so all of the same size $A$. If $A \subset H$, $A = g_j A g_j^{-1}$ and $\bigcap_{j=0}^{k} g_j H g_j^{-1} = \{e\}$, then condition (2.3) holds.

Proof. Consider the map

$$\phi : A_0 \times A_1 \times \ldots \times A_k \to A_0 A_1^{-1} \times A_1 A_2^{-1} \times \ldots \times A_{k-1} A_k^{-1}$$

given by

$$(a_0, a_1, \ldots, a_k) \mapsto (a_0 a_1^{-1}, a_1 a_2^{-1}, \ldots, a_{k-1} a_k^{-1}).$$

Clearly,

$$\prod_{j=0}^{k-1} |A_j A_{j+1}^{-1}| = |\text{im}(\phi)| \geq \frac{\prod_{j=0}^{k} |A_j|}{\max_{x \in G} |\phi^{-1}(x)|}.$$

Say $\phi(a_0, a_1, \ldots, a_j) = x = \phi(a'_0, a'_1, \ldots, a'_j)$. Then, since $a_j a_{j+1}^{-1} = a'_j \left(a'_j a_{j+1}^{-1}\right)$ for all $0 \leq j < k$, we see that $a_j^{-1} a'_j = a_{j+1}^{-1} a'_j$ for all $0 \leq j < k$. Thus, $(a'_0, a'_1, \ldots, a'_j)$ is determined by $(a_0, a_1, \ldots, a_j)$ and the single element

$$a_0^{-1} a'_0 = a_1^{-1} a'_1 = \ldots = a_k^{-1} a'_k,$$

which lies in $\bigcap_{j=0}^{k} A_j A_{j+1}^{-1}$. We conclude that

$$|\phi^{-1}(x)| \leq \left|\bigcap_{j=0}^{k} A_j A_{j+1}^{-1}\right|.$$

We will later see how to obtain the weak orthogonality condition

(2.4) \[ \bigcap_{j=0}^{k} g_j H g_j^{-1} = \{e\} \]

for some kinds of subgroups of permutation groups.
2.2. Subgroups and quotients. We will need a couple of basic lemmas on subgroups and quotients. As explained in [HS14 §3.1–3.2] and [Hel15 §4.1], they are all easy applications of an orbit-stabilizer principle for sets [Hel15 Lemma 4.1]. We can also prove them by using the pigeonhole principle directly.

For a group and , write for the map taking each to the right coset containing . Thus, for instance, equals the number of distinct cosets intersecting .

**Lemma 2.4** ([Hel11 Lem. 7.2]). Let be a group and a subgroup thereof. Let be a non-empty finite set. Then

\[
|AA^{-1} \cap H| \geq \frac{|A|}{|\pi_{G/H}(A)|} \geq \frac{|A|}{|G:H|}.
\]

**Proof.** By pigeonhole, there is a coset of containing at least elements of . Fix . Then, for each , we obtain a distinct element .

**Lemma 2.5.** Let be a group and a subgroup thereof. Let be a non-empty finite set. Then, for any \(k \geq 1\),

\[
|A^{k+1}| \geq \frac{|A^k \cap H|}{|AA^{-1} \cap H|} \cdot |A|.
\]

In other words, growth in a subgroup implies growth in the group.

**Proof.** It is clear that

\[
|A^{k+1}| \geq \left| \left( A^k \cap H \right) \cdot A \right| \geq \left| (A \cap H)^k \right| \cdot |\pi_{G/H}(A)|.
\]

At the same time, by Lemma 2.4

\[
\left| (AA^{-1} \cap H) \right| \cdot |\pi_{G/H}(A)| \geq |A|.
\]

**Lemma 2.6** ([HS14 Lem. 3.7]). Let be a group, let be subgroups of , and let be a non-empty finite set. Then

\[
|\pi_{K/H}(AA^{-1} \cap K)| \geq \frac{|\pi_{G/H}(A)|}{|\pi_{G/K}(A)|} \geq \frac{|\pi_{G/H}(A)|}{|G:K|}.
\]

In other words: if intersects cosets of in , then intersects at least \(r|G:H|/|G:K| = r|K:H|\) cosets of in . (As usual, all of our cosets are right cosets of a group , etc.) We quote the proof in [HS14 Lem. 3.7].

**Proof.** Since intersects cosets of in , and cosets of in , the pigeonhole principle implies that there exists a coset such that intersects at least \(k = |\pi_{G/H}(A)|/|\pi_{G/K}(A)|\) cosets . Let be elements of in distinct cosets of in . Then \(a_i a_i^{-1} \in AA^{-1} \cap K\) for each \(i = 1, \ldots, k\). Note that \(H a_i a_i^{-1}, \ldots, H a_k a_k^{-1}\) are \(k\) distinct cosets of .

2.3. Graphs and random walks. For us, a graph is a directed graph, that is, a pair \((V, E)\), where \(V\) is a set and \(E\) is a subset of the set of ordered pairs of elements of \(V\). (We allow loops, that is pairs, \((v, v)\).) A multigraph is the same as a graph, but with \(E\) a multiset, i.e., edges may have multiplicity \(>1\).

Given a group \(G\) and a set of generators \(A \subset G\), the Cayley graph \(\Gamma(G, A)\) is defined to be the pair \((G, \{(g, ga) : g \in G, a \in A\})\). It is connected because \(A\) is a set of generators. Given a group \(G\), a set of generators \(A \subset G\) and a set \(X\) on which \(G\) acts, the Schreier graph \(\Gamma(G, A; X)\) is the pair \((X, \{(x, xa^i) : x \in X, a \in A\})\).

We take a random walk on a graph or multigraph \(\Gamma\) by starting at a given vertex \(v_0\) and deciding randomly, at each step, to which neighbor \(w\) of our current location \(v\) to move. (A neighbor of \(v\) is a vertex \(w\) such that \((v, w) \in E\).) We choose \(w\) with uniform probability among the neighbors of \(v\), if \(\Gamma\) is a graph, or with probability proportional to the multiplicity of \(w\), if \(\Gamma\) is a multigraph.

In a lazy random walk, at each step, we first throw a fair coin to decide whether we are going to bother to move at all. (Of course, if we decide to move, and \((v, v)\) is an edge, we might move from \(v\) to itself.) Our random walks will always be lazy, for the sake of eliminating some technicalities.

We say that the \((\ell_\infty, \epsilon)\)-mixing time in a regular, symmetric (multi)graph \(\Gamma = (V, E)\) is at most \(t\) if, for every (lazy) random walk of length \(\geq t\), the probability that it ends at any given vertex lies between \((1 - \epsilon)/|V|\) and \((1 + \epsilon)/|V|\). We will use the fact that (multi)graphs with few vertices have small mixing times.

**Proposition 2.7.** Let \(\Gamma\) be a connected, regular and symmetric multigraph of valency \(d\) and with \(N\) vertices. Then the \((\ell_\infty, \epsilon)\)-mixing time is at most \(N^2d \log(N/\epsilon)\).

**Proof.** This is a well-known fact; see, e.g., the exposition in [Hel15, §6]. The main idea is to study the spectrum of the adjacency operator, meaning the operator \(\mathcal{A}\) taking each function \(f : V \to \mathbb{C}\) to a function \(\mathcal{A}f\) whose value at \(v\) is the average of \(f(w)\) over the neighbors \(w\) of \(v\) in the graph \(\Gamma\). The connectedness of \(\Gamma\) is used to show that, for every non-constant eigenfunction of \(\mathcal{A}\), the corresponding eigenvalue \(\lambda\) cannot be too close to 1; it is at most \(1 - 1/N^2d\). The bound on the mixing time then follows. \(\square\)

In particular, Prop. 2.7 holds when \(\Gamma\) is any Schreier graph \(\Gamma(G, A; \Omega^{(k)})\) of the action of a permutation group \(G \leq \text{Sym}(\Omega)\) on the set \(\Omega^{(k)}\) of \(k\)-tuples of distinct elements of \(\Omega\). The point is that \(N = |\Omega^{(k)}| \leq |\Omega|^k/k!\) is very small compared to \(\text{Sym}(\Omega)\) (which is of course of size \(|\Omega|!\)) for \(k\) bounded. We can make sure that \(A\) is small as well, by the following simple lemma.

**Lemma 2.8.** Let \(A \subset \text{Sym}(n)\). Then there is a subset \(A_0 \subset A \cup A^{-1}\) such that \(\langle A_0 \rangle = \langle A \rangle\), \(|A_0| \leq 4n\) and \(A_0 = A_0^{-1}\).

**Proof.** Choose an element \(g_1 \in A\), and then an element \(g_2 \in A\) such that \(\langle g_1 \rangle \leq \langle g_1, g_2 \rangle\), and then an element \(g_3 \in A\) such that \(\langle g_1, g_2 \rangle \leq \langle g_1, g_2, g_3 \rangle\), ... Since the longest subgroup chain in \(\text{Sym}(n)\) is of length \(\leq 2n - 3\) [Bab86], we must stop in \(r \leq 2n - 3 < 2n\) steps. Let \(A_0 = \{g_1, g_1^{-1}, \ldots, g_r, g_r^{-1}\}\). \(\square\)

The point is that, while we cannot assume we can produce random, uniformly distributed elements of \(G = \langle A \rangle\) as short products in \(A\) (we cannot assume what we are trying to prove, namely, that the diameter is small), we can take short, random products of elements of \(A\), and their action on \(\Omega^{(k)}\) is like that of random,
uniformly distributed elements. This observation was already used in \[BBS04\] to prove the following.

**Lemma 2.9 \([BBS04]\).** Let \(A \subset \text{Sym}(\Omega)\), \(|\Omega| = n\), be such that \(A = A^{-1}\) and \(G = \langle A \rangle\) is 3-transitive. Assume there is a \(g \in A\), \(g \neq e\), with \(|\text{supp}(g)| \leq (1/3 - \epsilon)n\), \(\epsilon > 0\). Then

\[
\text{diam}(\Gamma(G, A)) \ll \epsilon\ n^8(\log n)^c,
\]

where \(c\) is an absolute constant.

**Sketch of proof.** See \[BBS04\] or the exposition in \[Hel15\] \(\S 6.2\). The main idea is as follows. Let \(A_0\) be as in Lemma 2.8 and let \(h \in A_0^m \subset A^m\) be the outcome of a random walk on \(A_0\) of length \(\leq m\), where \(m \geq 4n^3 \log(n/\epsilon')\) and \(\epsilon' = \epsilon/100\) (say).

Then, by Prop. 2.7, for any \((x, y) \in \Omega\), the probability that \(h\) takes \(x\) to \(y\) is almost exactly \(1/n\). In particular, for \(x \in \text{supp}(g)\), the probability that \(x^h \in \text{supp}(g)\) is almost exactly \(|\text{supp}(g)|/n\).

Hence

\[
|\text{supp}(g) \cap \text{supp}(hgh^{-1})| \leq \frac{|\text{supp}(g)|^2}{n} \leq \left(\frac{1}{3} - \epsilon\right) |\text{supp}(g)|.
\]

A quick calculation shows that the commutator \([g, h^{-1}] = g^{-1}hgh^{-1}\) obeys

\[
|\text{sup}(\ [g, h^{-1}]\ )| \leq 3 |\text{supp}(g) \cap \text{supp}(hgh^{-1})|,
\]

and so, for \(g' = [g, h^{-1}]\), \(|\text{supp}(g')| \leq |\text{supp}(g)|^2/n \leq (1 - 3\epsilon)|\text{supp}(g)|\).

We iterate until, after \(O(\log \log n)\), we obtain an element \(h\) of support of size 2 or 3. (Additional care is taken in the process so that our element \(h\) is never trivial. It is, in fact, convenient to take \(m \geq 4n^3 \log(n/\epsilon')\) from the beginning, so that the probability that \(h\) takes a pair \((x, x') \in \Omega^{(2)}\) to a pair \((y, y') \in \Omega^{(2)}\) is almost exactly \(1/|\Omega^{(2)}| = 2/n(n-1)\).) We conjugate \(h\) by elements of \(A_0^m\) to obtain a set \(C\) consisting of all 2-cycles or 3-cycles. It is clear that \(\text{diam}(\Gamma(G, C)) \ll n\). \(\square\)

We shall now use short random walks to construct elements \(g_j\) such that a weak orthogonality condition in the sense of (24) holds for some kinds of sets \(B\). By an orbit of a set \(B \subset \text{Sym}(\Omega)\) we mean a subset of \(\Omega\) of the form \(xB, x \in \Omega\).

**Lemma 2.10.** Let \(A, B \subset \text{Sym}(\Omega), \ |\Omega| = n, e \in B\). Let \(0 < \rho < 1\). Assume that \(\langle A \rangle\) is 2-transitive, and that \(B\) has no orbits of length \(> \rho n\). Then there are \(g_1, g_2, \ldots, g_k \in (A \cup A^{-1} \cup \{e\})^m, k \ll (\log n)/|\log \rho|, m \ll n^6 \log n, \) such that

\[
\bigcap_{j=1}^{k} g_jBg_j^{-1} = \{e\}.
\]

If we required only that \(g \in \langle A \rangle\) (and not \(g_k \in (A \cup A^{-1} \cup \{e\})^m\)), and \(B\) were assumed to be a group, then this Lemma would be the “splitting lemma” in \[Bab82\] \(\S 3\). The fact that the proof can be adapted illustrates what we were saying: short random products act on \(\Omega^{(2)}\) as random elements do. Prop. 5.2 in \[HS14\] is an earlier generalization of Babai’s “splitting Lemma”, based on the same idea.

**Proof.** Let \(g_1, \ldots, g_k\) be the outcome of \(k\) independent random walks of length \(\leq m\) on \(A_0\), where \(m \geq 4n^6 \log(n/\epsilon)\), \(\epsilon > 0\) and \(A_0 \subset A \cup A^{-1}\) is as inLemma 2.8. Then, by Prop. 2.7, for any \((x, y), (x', y') \in \Omega^{(2)}\) and any \(1 \leq j \leq k\), the probability that \(g_j\) takes \((x, y)\) to \((x', y')\) lies between \((1 - \epsilon)/|\Omega^{(2)}|\) and \((1 + \epsilon)/|\Omega^{(2)}|\).
Since $B$ has no orbits of length $>\rho n$, there are at most $\rho |\Omega (2)|$ pairs $(x', y') \in \Omega (2)$ such that $x'$ and $y'$ lie in the same orbit of $H$. Hence, for any $(x, y) \in \Omega (2)$ and any $1 \leq j \leq k$, the probability that $x^{g_j}$ and $y^{g_j}$ lie in the same orbit of $B$ is at most $(1 + \epsilon)^\rho$. Since $g_1, \ldots , g_k$ were chosen independently, it follows that the probability that $x^{g_j}$ and $y^{g_j}$ are in the same orbit for every $1 \leq j \leq k$ is at most $(1 + \epsilon)^\rho k$.

Now, $x^{g_j}$ and $y^{g_j}$ are in the same orbit for every $1 \leq j \leq k$ if and only if $x$ and $y$ are in the same orbit of $B' = \bigcup_{j=1}^k g_j B g_j^{-1}$. The probability that at least two distinct $x, y$ lie in the same orbit of $B'$ is therefore at most

$$n^2((1 + \epsilon)^\rho)^m.$$ 

We let $\epsilon = \rho^{-1/2} - 1$, so that $(1 + \epsilon)^\rho = \rho^{1/2}$. Then, for $k > 2(\log n^2)/|\log \rho|$, $n^2((1 + \epsilon)^\rho)^m < 1$.

In other words, with positive probability, no two distinct $x, y$ lie in the same orbit of $B'$, i.e., $B'$ equals $\{e\}$. Thus, there exist $g_1, \ldots , g_m$ such that $B' = \{e\}$. $\square$

It is a familiar procedure in combinatorics (sometimes called the probabilistic method) to prove that a lion can be found at a random place of the city with positive probability, and to conclude that there must be a lion in the city. What we have done is prove that, after a short random walk, we come across a lion with positive probability (and so there is a lion in the city).

**Corollary 2.11.** Let $A, B \subset \text{Sym}(\Omega)$, $|\Omega| = n$. Let $0 < \rho < 1$. Assume that $\langle A \rangle$ is 2-transitive, and that $BB^{-1}$ has no orbits of length $> \rho n$. Then there is a $g \in (A \cup A^{-1} \cup \{e\})^m$, $m \ll n^6 \log n$, such that

$$|BB^{-1}gBB^{-1}g^{-1}| \geq |B|^{1 + \frac{\log |\rho|}{\log n}}.$$ 

**Proof.** By Lemma 2.10 applied to $BB^{-1}$ rather than $B$, there are $g_1, g_2, \ldots , g_k \in (A \cup A^{-1})^m$, $k \ll (\log n)/|\log \rho|$, $m \ll n^6 \log n$, such that

$$\bigcap_{j=1}^k g_j BB^{-1}g_j^{-1} = \{e\}.$$ 

Hence, by Lemma 2.3 with $A_j = g_j BB^{-1}g_j^{-1}$ (and $g_0 = e$, say), there is a $0 \leq j \leq k - 1$ such that

$$|A_j A_{j+1}^{-1}| \geq |BB^{-1}|^{1 + \frac{1}{2} + \frac{1}{2}}.$$ 

Since $|A_j A_{j+1}^{-1}| = |g_j BB^{-1}g_j^{-1}g_{j+1}BB^{-1}g_{j+1}^{-1}| = |BB^{-1}gBB^{-1}g^{-1}|$ for $g = g_j^{-1}g_{j+1}$, we are done. $\square$

**Corollary 2.12.** Let $A \subset \text{Sym}(\Omega)$, $|\Omega| = n$, with $A = A^{-1}$ and $e \in A$. Let $0 < \rho < 1$. Assume that $\langle A \rangle$ is 2-transitive. Let $\Sigma \subset \Omega$ be such that $(A^3)_{\langle \Sigma \rangle}$ has no orbits of length $> \rho n$. Then either

$$|\Sigma| \geq \frac{|\log \rho|}{3(\log n)^2} \log |A|$$ 

or

$$|A| \geq |A|^{1 + \frac{\log |\rho|}{3 \log n}}.$$ 

We can assume $\rho \leq (n - 1)/n$. Thus $\epsilon = \rho^{-1/2} - 1$ implies $\epsilon \gg 1/n$, and so $\log(n/\epsilon) \ll \log n$. 

---

(2.7) $|\Sigma| \geq \frac{|\log \rho|}{3(\log n)^2} \log |A|$

or

(2.8) $|A| \geq |A|^{1 + \frac{\log |\rho|}{3 \log n}}$
for some $l \ll n^6 \log n$.

Compare to [HST14, Cor. 5.3].

Proof. Let $B = (A^2)_{\Sigma} = (A^{-1})_{\Sigma}$. Since $BB^{-1} \subseteq (A^4)_{\Sigma}$, we see that $BB^{-1}$ has no orbits of length $> pm$. Apply Corollary 2.11. We obtain that

$$|A| \geq |B|^{1 + \frac{\log pl}{m}}$$

for $l = 4m + 2, m \ll n^6 \log n$. At the same time, by Lemma 2.4

$$|B| = |AA^{-1} \cap \text{Sym}(\Omega)_{\Sigma}| \geq \frac{|A|}{|\text{Sym}(\Omega): \text{Sym}(\Omega)_{\Sigma}|} \geq \frac{|A|}{n!e}.$$

Hence, either (2.7) holds, or

$$|B| > \frac{|A|}{\frac{\log pl}{2 \log n}} = |A|^{1 - \frac{\log pl}{2 \log n}},$$

and so

$$|A| \geq |A|^{\left(1 - \frac{\log pl}{2 \log n}\right) \left(1 + \frac{\log pl}{2 \log n}\right)} = |A|^{\left(1 + \frac{\log pl}{2 \log n}\right) \left(1 - \frac{1}{\log n}\right)}.$$

We can assume $1 - 1/ \log n \geq 2/3$, as otherwise (2.8) holds trivially.

2.4. Generating an element of large support. We will need to produce an element of $\text{Sym}(n)$ of very large support (almost all of $\{1, 2, \ldots, n\}$). It is not difficult to carry out this task using short random walks.

Lemma 2.13. Let $g \in \text{Sym}(n)$ have support $\geq \alpha n$, $\alpha > 0$. Let $A \subseteq \text{Sym}(n)$ generate a 2-transitive group. Assume $A = A^{-1}, e \in A$. Then, provided that $n$ is larger than a constant depending only on $\alpha$, there are $\gamma_i \in A^n, 1 \leq i \leq \ell$, where $\ell = O((\log n)/\alpha)$, such that the support of

$$\gamma_1 g \gamma_1^{-1} \cdot \gamma_2 g \gamma_2^{-1} \cdots \gamma_{\ell} g \gamma_{\ell}^{-1}$$

has at least $n - 1$ elements.

Proof. Let $h_1, h_2 \in \text{Sym}(n), m_i = |\text{supp}(h_i)|$. By Prop. 2.7, a random walk of length $r = [4n^5 \log(n^2/\epsilon)]$ gives us an element $\sigma$ of $A'$ sending any given pair of distinct elements $x, y \in \{1, \ldots, n\}$ to any given pair of distinct elements $x', y' \in \{1, \ldots, n\}$ with probability $(1 + O^{*}(\epsilon))/n(n-1)$.

An element $x \in \{1, \ldots, n\}$ can fail to be in the support of $h_1 \sigma h_2 \sigma^{-1}$ only if (a) $x \notin \text{supp}(h_1)$, $x \notin \text{supp}(h_2 \sigma^{-1}$, or (b) $x \in \text{supp}(h_1)$ and $h_2 \sigma^{-1}$ sends $x^{h_1}$ to $x$. For $x$ random, case (a) happens with probability at most $(1 - m_1/n) \cdot (1 + \epsilon)(1 - m_2/n)$. In case (b), $\sigma$ must send $x^{h_1}$ to an element that is not fixed by $h_2$, and, moreover, it must send $x$ to $x^{h_1 \sigma h_2}$. Now, we know that, even given that $\sigma$ sends an element $x_0$ (in this case, $x_0 = x^{h_1}$) to some specific element $y_0$, it will still send any $x \neq x_0$ to any $y \neq y_0$ with almost equal probability. Hence,

$$\text{Prob}(x \notin \text{supp}(h_1 \sigma h_2 \sigma^{-1})) \leq (1 + \epsilon) \left(1 - \frac{m_1}{n}\right) \left(1 - \frac{m_2}{n}\right) + \frac{m_1 m_2}{n} \frac{1}{n-1}$$

We set $\epsilon = 1/n$ and assume $m_1, m_2 < n$. Then we have

$$\text{Prob}(x \notin \text{supp}(h_1 \sigma h_2 \sigma^{-1})) \leq (1 + \epsilon) \left(1 - \frac{m_1}{n}\right) \left(1 - \frac{m_2}{n}\right) + \frac{1}{n}.$$

The expected value of $n - |\text{supp}(h_1 \sigma h_2 \sigma^{-1})|$ is thus at least $(1 + \epsilon)(n - m_1)(1 - m_2/n) + 1/n$. Hence there is a $\sigma \in A'$ such that $n - |\text{supp}(h_1 \sigma h_2 \sigma^{-1})|$ is at least that much.
We apply this first with \( h_1 = h_2 = g \), and obtain a \( \sigma_1 = \sigma \) as above; define \( g_1 = g_1 g \sigma_i^{-1} \). Then we iterate: we let \( h_1 = g_1, h_2 = g \), and obtain a \( \sigma_2 = \sigma \) such that \( g_2 = g_1 g_2 g_2 \sigma_2^{-1} \) has large support; and so forth, with \( h_1 = g_{i-1}, h_2 = g \) at the \( i \)th step. We obtain

\[
1 - \frac{\text{supp}(g_i)}{n} \geq (1 + \epsilon) \left( 1 - \frac{\text{supp}(g)}{n} \right) \left( 1 - \frac{\text{supp}(g_{i-1})}{n} \right) + \frac{1}{n},
\]

where \( g_0 = g \), and so, for \( r = (1 + \epsilon)(1 - \text{supp}(g)/n) \) (which is \( < 1 \)) and \( k \geq 0 \),

\[
1 - \frac{\text{supp}(g_k)}{n} \geq r^k \left( 1 - \frac{\text{supp}(g)}{n} \right) + \frac{1}{(1-r)n} \geq r^k (1- \alpha) + \frac{1}{(1-r)n}.
\]

We let \( k = \lceil \log (n)/((1- \alpha)/r) \rceil \) and obtain

\[\text{supp}(g_k) \geq n - 1 - \frac{1}{(1-r)}.\]

For \( n \geq 2/\alpha \), we have \( r \leq (1 + \alpha/2)(1- \alpha) < 1 - \alpha/2 \), and so \( 1/(1-r) \leq 2/\alpha \) and \( k \ll (\log n)/\alpha \).

We can assume \( \text{supp}(g_k) < n \), as otherwise we are done. Now apply the procedure at the beginning with \( h_1 = h_2 = g_k \). We obtain \( \text{Prob}(x \notin \text{supp}(g_k g_k g_2 \sigma_1^{-1})) < 2/n \), provided that \( n \) is larger than a constant depending only on \( \alpha \). Hence there is a \( \sigma \in A^r \) such that \( \text{supp}(g_k \sigma g_k \sigma_1^{-1}) \geq n - 1 \). Since

\[g_k \sigma g_k \sigma^{-1} = g \cdot \sigma_1 g \sigma_1^{-1} \cdots \sigma_k g \sigma_k^{-1} \cdot \sigma g \sigma^{-1} \cdot (\sigma \sigma_1) g (\sigma \sigma_1)^{-1} \cdots (\sigma \sigma_k) g (\sigma \sigma_k)^{-1},\]

we set \( l = 2k + 2 \) and are done. \( \Box \)

It may be useful to compare Lemma 2.13 to analogous results on random subproducts in the sense of \[\text{BLS97}.\] Such results make weaker assumptions (transitivity instead of double transitivity) and give weaker conclusions (support \( \geq n/2 \) instead of support \( \sim n \); see \[\text{Ser03}, \text{Lemma 2.3.1}, \text{HS14, Lemma 4.3}].

2.5. Stabilizers and stabilizer chains. Let \( A \subset \text{Sym}(\Omega), |\Omega| = n \). Given a subset \( \Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset \Omega \), we write \( A_{(\Sigma)} \) and \( A_{\Sigma} \) for the pointwise and setwise stabilizers, respectively:

\[A_{(\Sigma)} = A_{(\alpha_1, \ldots, \alpha_k)} = \{ g \in A : \alpha_j^g = \alpha_j \; \forall 1 \leq j \leq k \},\]

\[A_{\Sigma} = A_{(\alpha_1, \ldots, \alpha_k)} = \{ g \in A : \Sigma^g = \Sigma \}.
\]

A stabilizer chain is simply a chain of subsets

\[A \supset A_{(\alpha_1)} \supset A_{(\alpha_1, \alpha_2)} \supset \ldots,\]

where \( \alpha_1, \alpha_2, \ldots \in \{1, 2, \ldots, n\} \). Stabilizer chains have been studied starting with Sims [\text{Sim70}] (in the case of \( A \) equal to a subgroup \( H \)). It is useful to find long chains of stabilizers such that the orbits

\[\alpha_j^{A_{(\alpha_1, \ldots, \alpha_{j-1})}}\]

are long.

Why do we want stabilizer chains with long orbits? Here is one reason.

Lemma 2.14. Let \( A \subset \text{Sym}(\Omega), |\Omega| = n \). Let \( \rho \in (0, 1) \). Let \( \Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset \Omega \) be such that, for every \( 1 \leq j \leq k \),

\[
(2.9) \quad |\alpha_j^{A_{(\alpha_1, \ldots, \alpha_{j-1})}}| \geq \rho n.
\]
Then \( A^k \) intersects at least \((pn)^k\) right cosets of \( \text{Sym}(\Omega) \), and the restriction of the setwise stabiliser \((A^{-k}A^k)\Sigma \) to \( \Sigma \) is a subset of \( \text{Sym}(\Sigma) \) with at least \( \rho^k k! \) elements.

This result was shown in the proof of [Pyb93, Lemma 3] for \( A \) a subgroup, and in [HS14, Lemma 3.19] for general \( A \).

**Proof.** First of all, notice that \( A^k \) sends \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) to at least \((pn)^k\) distinct \( k \)-tuples. This is shown as follows. Let \( 1 \leq j \leq k \). Let \( \Delta_j \) denote the orbit \( A^{(\alpha_1, \ldots, \alpha_{j-1})} \). For each \( \delta \in \Delta_j \), choose an element \( g_\delta \in A^{(\alpha_1, \ldots, \alpha_{j-1})} \) sending \( \alpha_j \) to \( \delta \). Let \( S_\delta = \{ g_\delta : \delta \in \Delta_i \} \). Clearly, \( |S_\delta| = |\Delta_i| \) and \( S_\delta \subseteq A \). Now let \((s_1, s_2, \ldots, s_k), (s'_1, s'_2, \ldots, s'_k)\) be two distinct elements of \( S_1 \times \cdots \times S_k \). Then \( s_k \cdots s_2s_1 \) and \( s'_k \cdots s'_2s'_1 \) send \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) to two different \( k \)-tuples: if \( j \) is the least index such that \( s_j \neq s'_j \), then

\[
\alpha_j^{s_k s_{k-1} \cdots s_j} = \alpha_j^{s'_j} \neq \alpha_j^{s'_j} = \alpha_j^{s'_k s_{k-1} \cdots s_j},
\]

and so

\[
\alpha_j^{s_k s_{k-1} \cdots s_j s_{j-1} \cdots s_1} \neq \alpha_j^{s'_k s_{k-1} \cdots s_j s_{j-1} \cdots s_1} = \alpha_j^{s'_k s_{k-1} \cdots s_j s_{j-1} \cdots s_1}.
\]

Hence \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) is sent to at least \( |S_1| \cdots |S_k| \geq (pn)^k \) distinct tuples by the action of \( S_k \cdots S_1 \subseteq A^k \).

In other words, \( A^k \) intersects at \((pn)^k\) cosets \( \text{Sym}(\Omega) \langle \Sigma \rangle g \) of \( \text{Sym}(\Omega) \langle \Sigma \rangle \). By Lemma [2.6]

\[
\pi_\langle \text{Sym}(\Omega) \rangle \langle \text{Sym}(\Omega) \rangle (A^k A^{-k} \cap \text{Sym}(\Omega) \langle \Sigma \rangle) \geq \frac{\pi_{\text{Sym}(\Omega) / \langle \text{Sym}(\Omega) \rangle} (A^k)}{[\text{Sym}(\Omega) : \langle \text{Sym}(\Omega) \rangle \langle \Sigma \rangle]} \geq \frac{(pn)^k}{n(n-1) \cdots (n-k+1)/k!} \geq \rho^k k!.
\]

Now, two elements of \( \text{Sym}(\Omega) \langle \Sigma \rangle \) lie in different cosets of \( \langle \text{Sym}(\Omega) \rangle \langle \Sigma \rangle \) if and only their restrictions to \( \Sigma \) are distinct. Since \( A^k A^{-k} \cap \text{Sym}(\Omega) \langle \Sigma \rangle = (A^k A^{-k}) \langle \Sigma \rangle \), we have shown that the restriction \((A^{-k}A^k)\Sigma \) to \( \Sigma \) is of size at least \( \rho^k k! \).

2.6. **Composition factors.** **Primitive groups.** Let us recall some standard definitions. A composition factor of a group \( G \) is a quotient \( H_{i+1} / H_i \) in a composition series of \( G \), i.e., a series

\[
1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G,
\]

where every quotient is simple. By the Jordan-Hölder theorem, whether or not an abstract group is a composition factor of \( G \) does not depend on the particular composition series of \( G \) being used.

A section of a group \( G \) is a quotient \( H / N \), where \( 1 \leq N \triangleleft H \leq G \). A composition factor is, by definition, a section.

A block system for a permutation group \( G \leq \text{Sym}(\Omega) \) is a partition of \( \Omega \) preserved by \( G \), that is, a partition of \( \Omega \) into blocks (sets) \( B_1, \ldots, B_k \) such that, if \( x, y \in B_i \), \( g \in G \) and \( x^g \in B_j \), then \( y^g \in B_j \). A maximal block system is one that has blocks of size \( > 1 \) and cannot be subdivided into a finer partition with blocks of size \( > 1 \). (In other words, it is a system of minimal non-trivial blocks.) A minimal block system is one that is not the refinement of any block system other than the trivial partition of \( \Omega \) into one set \( \Omega \).

The group \( G \) is primitive if it has no block systems with more than \( 1 \) and fewer than \( |\Omega| \) blocks, i.e., no block systems other than (a) the partition of \( \Omega \) into the
single set $\Omega$ and (b) the partition of $\Omega$ into one-element sets. It follows from the
definitions that a group $G \leq \text{Sym}(\Omega)$ acts as a primitive group on any minimal
block system.

2.7. Tools from the theory of permutation groups. The following result
guarantees the existence of an element of small support in a group under rather
mild conditions. It is essentially due to Wielandt. Thanks are due to L. Pyber
for the reference.

**Lemma 2.15.** For any $\epsilon > 0$, there are $C_1, C_2 \geq 0$ such that the following
holds. Let $n \geq C_1$. Let $G \leq \text{Sym}(\Omega)$, $|\Omega| = n$, be a group containing a section
isomorphic to $\text{Alt}(k)$ for $k \geq C_2 \log n$. Then there is a $g \in G$, $g \neq e$, such that
$|\text{supp}(g)| < \epsilon n$.

This Lemma replaces [HS14, Lem. 3.19], which was based on [BS87, Lem. 3].

**Proof.** Suppose there is no $g \in G$, $g \neq e$, such that $|\text{supp}(g)| < \epsilon n$; we say
$G$ has minimal degree at least $\epsilon n$. We can assume without loss of generality that
$C_2 \log n \leq k \leq 2C_2 \log n$, since, given that $G$ contains a section isomorphic to $\text{Alt}(k)$, it contains a section isomorphic to $\text{Alt}(k')$ for all
$k' \leq k$.

Let $\omega = \min(\epsilon, 0.4)$. Then, by [DM96, Thm. 5.5A], we have $n > \left(\begin{array}{c}k \\ s\end{array}\right)$ for
$s = \lfloor \mu(k + 1) \rfloor$, $\mu = (1 - \omega)^{1/5}$. By Stirling’s formula,

$$\left(\begin{array}{c}k \\ s\end{array}\right) = \frac{k!}{s!(k-s)!} \gg \omega \frac{1}{\sqrt{k}} \left(\frac{1}{\mu(1-\mu)^{1-\mu}}\right)^k$$

for $k$ greater than a constant depending only on $\mu$. This gives a contradiction
with $C_2 \log n \leq k \leq 2C_2 \log n$ for $n \geq C_1$ when $C_1$ and $C_2$ are large enough in
terms of $\epsilon$. (We use the condition $k \leq 2C_2 \log n$ to ensure that the effect of $1/\sqrt{k}$
is negligible.)

We also need a result telling us that a large subset of $\text{Sym}(\Sigma)$ generates a large
symmetric subgroup in a few steps.

**Lemma 2.16.** Let $H \leq \text{Sym}(\Sigma)$, $|\Sigma| = k$. Let $\rho \in (1/2, 1)$. If $|H| \geq \rho^k k!$ and $k$
is larger than a constant depending only on $\rho$, then there exists an orbit $\Delta \subset \Sigma$
of $H$ such that $|\Delta| \geq \rho |\Sigma|$ and $H|\Delta$ is $\text{Alt}(\Delta)$ or $\text{Sym}(\Delta)$.

**Proof.** By [DM96, Thm. 5.2B], which is a somewhat strengthened version of
[Lie83, Lem. 1.1]. We simply need to check that

$$[\text{Sym}(\Sigma) : H] < \min\left(\frac{1}{2} \left(\begin{array}{c}k \\ \lfloor k/2 \rfloor\end{array}\right), \left(\begin{array}{c}k \\ m\end{array}\right)\right)$$

for $m = \lceil \rho k \rceil$. This inequality follows from Stirling’s formula for $k$ larger than a
constant depending on $\rho$. 

2.8. Diameter comparisons: directed and undirected graphs. We wish to
derive a (version of) Theorem 1.3, which is a bound on the diameter of a directed
graph, from Theorem 1.4, which is a statement on sets $A$ satisfying $A = A^{-1}$.
It would be natural to expect such a statement to imply only a bound on the
diameter of an undirected graph. As it happens, the distinction between directed
and undirected graph matters little in this context, thanks to the following result.
Lemma 2.17 ([Bab06], Thm. 1.4). Let $G$ be a finite group and $A$ a set of generators of $G$. Then
\[ \text{diam} \Gamma(G, A) \ll (\log |G|)^3 \cdot (\text{diam} \Gamma(G, A \cup A^{-1}))^2, \]
where the implied constant is absolute.

It is thus enough to prove Theorem 1.3 (and analogous statements) for sets $S$ satisfying $S = S^{-1}$: simply replace $S$ by $S \cup S^{-1}$, and use Lemma 2.17.

3. Finding few generators for a transitive group

Let us be given a set $A \subset \text{Sym}(\Omega)$. When is it the case that there is a small subset $A'$ of $(A \cup A^{-1} \cup \{e\})^\ell$ (say) that generates $\langle A \rangle$, or at least a transitive subgroup of $\langle A \rangle$? Can we put further conditions on the elements of $A'$, such as, for instance, that they all be conjugates of each other?

Our motivation for considering this question is the following. We will find it necessary to do what amounts to bounding the size of the intersection of a slowly growing set $A$ with the centralizer of an element of large support. It would stand to reason that there should be stronger bounds than for the intersection of $A$ with a subgroup without long orbits: being in the centralizer is a more restrictive condition.

Given one of our main tools (Lemma 2.3; see in particular the remark between its proof and its statement), we can quickly reduce this task to the following one: given an element $h$ of large support and a set of generators $A$ of $G$, find $g_1, \ldots, g_k \in (A \cup A^{-1} \cup \{e\})^\ell$ such that
\begin{equation}
C(h) \cap g_1 C(h) g_1^{-1} \cap \ldots \cap g_k C(h) g_k^{-1}
\end{equation}
is equal to $\{e\}$, or at least very small.

It is clearly enough for the group
\begin{equation}
\langle h, g_1 hg_1^{-1}, \ldots, g_k hg_k^{-1} \rangle
\end{equation}
to be transitive: the centralizer $H$ of a transitive subgroup of $\text{Sym}(n)$ is semiregular (that is, no element of $H$ other than $e$ fixes any point), and thus has $\leq n$ elements.

Let us, then, show that there are $g_1, \ldots, g_k \in (A \cup A^{-1} \cup \{e\})^\ell$, $k$ and $\ell$ small, such that the group (3.2) is transitive. We will be able to prove what we want with $k \ll \log \log n$, assuming that $\langle A \rangle$ is 4-transitive.

(As we will later discuss, reaching the bound $k \ll \log n$ is substantially easier. An analogous, but not identical, result with $k \sim \log n$ can be found in [BLS97, Lemma 5.13].)

Our proof will be by iteration, with the iterative step being given by the next proposition. We will be working with partitions into orbits, but will prove the proposition for general partitions. Recall that, given two partitions $\mathcal{P}, \mathcal{Q}$ of a set $\Omega$, the join $\mathcal{P} \lor \mathcal{Q}$ is the finest partition that is coarser (not necessarily strictly so) than both $\mathcal{P}$ and $\mathcal{Q}$. The trivial partition of $\Omega$ is the partition $\{\Omega\}$.

Given a partition $\mathcal{P}$ of $\Omega$ and an element $x \in \Omega$, we define $S_\mathcal{P}(x)$ to be the element of $\mathcal{P}$ containing $x$. Let $s_\mathcal{P}(x) = |S_\mathcal{P}(x)|$.

The total variation distance between two probability measures $\mu_1, \mu_2$ on a finite set $X$ is defined to be
\[ \delta(\mu_1, \mu_2) = \max_{S \subseteq X} |\mu_1(S) - \mu_2(S)|. \]
Thus, for instance, $\mu$ is at total variation distance at most $\epsilon$ from the uniform distribution if $\mu(S) = |S|/|X| + O^*(\epsilon)$ for every $S \subset X$. Suppose this is the case. Then, given any function $f : X \to \mathbb{R}$ with $0 \leq f(x) \leq T$ for every $x \in X$, we can easily estimate the expected value $\mathbb{E}_\mu(f(x))$ of $f(x)$ with respect to $\mu$: clearly,

$$f(x) = \int_0^T 1_{L(f,t)}(x) dt \quad \text{("layer-cake decomposition")},$$

where $L(f,t) = \{x \in X : f(x) \geq t\}$, and so

$$\mathbb{E}_\mu(f(x)) = \int_0^T \text{Prob}_\mu(x \in L(f,t)) dt = \int_0^T \left( \frac{L(f,t)}{|X|} dt + O^*(\epsilon) \right) dt.$$

Applying the same idea to the uniform distribution, without the error term $O^*(\epsilon)$, we obtain that

$$\mathbb{E}_\mu(f(x)) = \frac{1}{|X|} \sum_{x \in X} f(x) + O^*(\epsilon T). \quad (3.3)$$

**Lemma 3.1.** Let $P$ be a partition of a finite set $\Omega$ with $|\Omega| = n$. Let $m \geq 2$. Denote by $\rho \in \text{Sym}(\Omega)$ be taken at random with a distribution such that, for any element $\vec{v}$ of the set $\Omega^{(2)}$ of ordered pairs of distinct elements of $\Omega$, the probability distribution of $\vec{v}$ is at total variation distance $\leq \epsilon$ from the uniform distribution on $\Omega^{(2)}$. Assume that the same is true for the probability distribution of $\vec{v}^{-1}$ as well.

1. With positive probability, the proportion of elements $x$ of $\Omega$ such that $s_{P \vee \rho^0}(x) \geq m$ is $\geq 1 - (1 - \rho)^2 - \epsilon$.
2. Assume that $\epsilon \leq \rho/100$, $n \geq 100$ and $2 \leq m \leq n/2$. Then, with positive probability, the proportion of elements $x$ of $\Omega$ such that $s_{P \vee \rho^0}(x) \geq (1 + \rho/3)m$ is $\geq \rho^2/8$.
3. Assume that $\epsilon \leq \min(\rho/25, \rho/4m)$ and $n \geq 250$. Then, with positive probability, $P \vee P^\rho$ contains at least one set of size at least

$$\min \left( \frac{\rho}{10n}, \frac{\rho - \epsilon}{2m^2} \right).$$

The proof is straightforward, in that we will proceed by taking expected values. We are giving constants simply for concreteness; they have not been optimized. Conclusion (2) is substantially weaker than what we could obtain by means of more complicated variance-based arguments such as those we will use in the proof of Lemma 3.2.

**Proof.** Let $B$ be the set of all $x \in \Omega$ such that $s_P(x) < m$. For each $x \in B$, the probability that $x^g \in B$ is $\leq |B|/n + \epsilon = 1 - \rho + \epsilon$. Hence, the expected value of the number of $x \in B$ such that $x^g \in B$ is $\leq (1 + \epsilon - \rho)|B| = (1 + \epsilon - \rho)(1 - \rho)n \leq (1 - \rho)^2 + \epsilon)n$. It obviously follows that the number of such $x$ is $\leq (1 - \rho)^2 + \epsilon)n$ with positive probability. In other words, conclusion (1) holds.

For each $x \in \Omega$ such that $s_P(x) \geq m$, we choose a subset $Z(x) \subset S_P(x)$ of size $m$, in such a way that, for every set $S$ in $P$, every element of $S$ is contained in exactly $m$ sets $Z(x)$, $x \in S$. (For instance, we may identify each element of $P$ having $m' \geq m$ elements with the set $\mathbb{Z}/m'\mathbb{Z}$, and then let $Z(x) = \{x, x + 1, \ldots, x + m - 1\}$ mod $m'$ for every $x \in \mathbb{Z}/m'\mathbb{Z}$. We can easily see that every element of $\mathbb{Z}/m'\mathbb{Z}$ is then contained in exactly $m'$ sets $Z(x)$.) For every $x \in \Omega$
such that \( s_P(x) < m \), we let \( Z(x) = \emptyset \). We write \( z(x) \) for \( |Z(x)| \); we see that \( z(x) \) can take only the values \( m \) or \( 0 \).

We see immediately that

\[
\sum_{x,x' \in \Omega} |Z(x) \cap Z(x')| = \sum_{x \in \Omega} \sum_{y \in Z(x)} |\{x' \in S_P(x) : y \in Z(x')\}|
\]

(3.4)
\[
= \sum_{x \in \Omega} \sum_{y \in Z(x)} m = \rho m^2 n,
\]
a fact that will be useful later.

We write \( Z_g(x) \) for \( Z(x^{g^{-1}})^g \), and \( z_g(x) \) for \( |Z_g(x)| \). By definition,

\[
Z_g(x) \subset \left(S_P(x^{g^{-1}})^g\right) = S_{P^g}(x).
\]

Clearly,

\[
s_{P \cdot P^g}(x) \geq z(x) + z_g(x) - |Z(x) \cap Z_g(x)|.
\]

(3.5)

For any \( x \),

\[
\mathbb{E}(z_g(x)) = \mathbb{E}\left(|Z(x^{g^{-1}})|\right) = m \cdot \text{Prob}\left(x^{g^{-1}} \notin B\right)
\]

\[
= m \cdot (\rho + O^*(\epsilon)),
\]
since \( B \) is the set of elements \( x \) of \( \Omega \) such that \( Z(x) = \emptyset \), and \( |Z(x)| = m \) for all \( x \in \Omega \setminus B \).

Given \( x \in \Omega \setminus B \) and a \( y \in Z(x) \), we should estimate the probability that \( y \) is an element of \( Z(x) \cap Z_g(x) \), where \( g \) is, as always, taken at random. Evidently, if \( y = x \), then \( y \notin Z_g(x) \) if \( Z_g(x) \) is empty, and \( y \in Z_g(x) \) otherwise. If \( y \neq x \), we have \( y \in Z_g(x) \) if and only if \( g^{-1} \) sends \((x, y)\) to an element of

\[
S = \{(x', y') : x' \in \Omega \setminus B, y' \in Z(x), y' \neq x'\}.
\]
The number of elements of \( S \) is \( |\Omega \setminus B| \cdot (m - 1) \). Hence

\[
\text{Prob}(y \in Z_g(x)) \leq \frac{|\Omega \setminus B| \cdot (m - 1)}{n(n - 1)} + \epsilon \leq \frac{\rho m}{n} + \epsilon.
\]

Therefore,

\[
\mathbb{E}(\{|Z(x) \cap Z_g(x)|\}) \leq \text{Prob}(x \in Z_g(x)) + \sum_{y \in Z(x) \text{ for } y \neq x} \text{Prob}(y \in Z_g(x))
\]

\[
\leq \rho + \epsilon + \sum_{y \in Z(x) \text{ for } y \neq x'} \left(\frac{\rho m}{n} + \epsilon\right) \leq \rho \left(1 + \frac{m^2}{n}\right) + \epsilon m,
\]

and so

\[
\mathbb{E}\left(\frac{1}{\rho m} \sum_{x \in \Omega \setminus B} (z_g(x) - |Z(x) \cap Z_g(x)|)\right)
\]

(3.6)
\[
\geq \frac{1}{|\Omega \setminus B|} \sum_{x \in \Omega \setminus B} \left(m \cdot (\rho - \epsilon) - \rho \left(1 + \frac{m^2}{n}\right) - \epsilon m\right)
\]

\[
= \rho m - \rho \left(1 + \frac{m^2}{n}\right) - 2\epsilon m.
\]
Thus, with positive probability,
\[
\frac{1}{\rho m} \sum_{x \in \Omega \setminus B} (z_g(x) - |Z(x) \cap Z_g(x)|) \geq \rho m - \rho \left(1 + \frac{m^2}{n}\right) - 2\epsilon m.
\]

The contribution of all \(x \in \Omega \setminus B\) such that \(z_g(x) - |Z(x) \cap Z_g(x)| \leq \rho m/3\) is at most \(\rho m/3\). Each one of the other \(x \in \Omega \setminus B\) contributes at most \(m/\rho n\). Hence, the number of all \(x \in \Omega \setminus B\) such that \(z_g(x) - |Z(x) \cap Z_g(x)| > \rho m/3\) is
\[
\geq \frac{\left(\frac{2}{3}m - \left(1 + \frac{m^2}{n}\right)\right) \rho - 2\epsilon m}{m/\rho n}
\]
\[
\geq \left(\frac{2}{3} - \left(\frac{1}{m} + \frac{m}{n}\right)\right) \rho^2 n - 2\epsilon \rho n \geq \left(\frac{1}{6} - \frac{2}{n}\right) \rho^2 n - 2\epsilon \rho n \geq \frac{\rho^2}{8} n,
\]
where we use the assumptions \(2 \leq m \leq n/2\), \(\epsilon \leq \rho/100\), \(n \geq 100\). By \(3.5\), we obtain that conclusion \((2)\) holds. It remains to prove conclusion \((3)\).

Let \(x \in \Omega\). For each \(y \in S_P(x)\), every element of \(Z_g(y)\) lies in \(S_{P_{y'}P_{y}}(y)\) and hence in \(S_{P_{y'}P_{y}}(x)\). Therefore, by inclusion-exclusion, for \(U \subset S_P(x)\) arbitrary,
\[
|S_{P_{y'}P_{y}}(x)| \geq \left| \bigcup_{y \in U} S_{P_{y'}}(y) \right| \geq \sum_{y \in U} z_g(y) - \sum_{y, y' \in U, y \neq y'} |Z_g(y) \cap Z_g(y')|.
\]

By our assumptions on the distribution of \(g\),
\[
\mathbb{E} \left( \sum_{y \in U} z_g(y) \right) = \sum_{y \in U} \mathbb{E} (z_g(y)) = \sum_{y \in U} \mathbb{E} \left( z(yg^{-1}) \right) \geq \sum_{y \in U} (\rho - \epsilon)m = (\rho - \epsilon)m|U|,
\]
and, similarly,
\[
\mathbb{E} \left( \sum_{y \in U} z_g(y) \right) \leq (\rho + \epsilon)m|U|.
\]

We can apply \((3.3)\) to \(X = \Omega^{(2)}\) and \(f(x, x') = |Z(x) \cap Z(x')|\), with the probability distribution on \(X\) given by \((y, y')^{g^{-1}}\), where \((y, y')\) is a given element of \(\Omega^{(2)}\) and \(g\) is taken randomly in the sense we have been using throughout. Then, by \((3.4)\) and the fact that \(0 \leq f(x, x') \leq m\) for all \((x, x') \in X\),
\[
\mathbb{E} \left( |Z_g(y) \cap Z_g(y')| \right) = \frac{\rho m^2 n - mn}{n(n - 1)} + O^*(\epsilon m).
\]

Hence
\[
\mathbb{E} \left( \sum_{y, y' \in U, y \neq y'} |Z_g(y) \cap Z_g(y')| \right) = |U|(|U| - 1) \left( \frac{\rho m^2 n - mn}{n(n - 1)} + O^*(\epsilon m) \right).
\]
\[
\leq \left( \frac{\rho m^2}{n} + \epsilon m \right) \cdot |U|^2.
\]
Therefore, by \((3.7)\),

\[ \mathbb{E}(s_{P \vee P^g}(x)) \geq (\rho - \epsilon)m|U| - \left( \frac{\rho m^2}{n} + \epsilon m \right)|U|^2. \]

In general, the maximum of an expression \(at - bt^2\), \(a, b > 0\), is of course attained when \(t = a/2b\); moreover, since \(a(t_0 - \Delta) - b(t_0 - \Delta)^2 = a^2/4b - b\Delta^2\), we see that \(a(t_0 - \Delta) - b(t_0 - \Delta)^2 \geq a^2/4b - b\). We let \(a = (\rho - \epsilon)m, b = \rho m^2/n + \epsilon m\).

Suppose first that \(t_0 > m\). Then we simply choose any \(x \in \Omega\) with \(|S_P(x)| \geq m\), and choose \(U \subset S_P(x)\) with \(|U| = m\). Since \(t_0 = a/2b > m\), we see that \(am - bm^2 = am(1 - bm/a) > am/2\), and so

\[ \mathbb{E}(s_{P \vee P^g}(x)) \geq \frac{am}{2} = \frac{\rho - \epsilon}{2}m^2. \]

Now suppose that \(t_0 \leq m\). Then there is an \(x \in \Omega\) such that \(|S_P(x)| \geq t_0\). We choose \(U \subset S_P(x)\) with \(|U| = |t_0|\), and obtain that

\[ \mathbb{E}(s_{P \vee P^g}(x)) \geq \frac{a^2}{4b} - b. \]

Clearly \(1/(r_1 + r_2) \geq \min(1/2r_1, 1/2r_2)\) for \(a, b > 0\). Hence

\[
\frac{a^2}{4b} = \frac{a^2/4}{\rho m^2/n + \epsilon m} \geq \frac{a^2}{8} \min \left( \frac{n}{\rho m^2}, \frac{1}{\epsilon m} \right) \\
\geq \frac{(1 - \epsilon/\rho)^2}{8} \min \left( \rho n, \frac{\rho^2}{\epsilon m} \right) \geq \frac{(1 - \epsilon/\rho)^2}{8} \min \left( \rho n, 4\rho m^2 \right),
\]

where we use the assumption \(\epsilon \leq \rho/4m\). Again by \(\epsilon \leq \rho/4m\), we see that \(t_0 = a/2b \leq m\) implies that

\[(1 - \epsilon)\rho m = a \leq 2bm = 2\frac{\rho m^3}{n} + 2\epsilon m^2 \leq 2\frac{\rho m^3}{n} + \frac{\rho m}{2},\]

and so \(4\rho m^2 \geq (1 - 2\epsilon)\rho n\). Therefore

\[
\frac{a^2}{4b} - b \geq \frac{(1 - \epsilon/\rho)^2}{8}(1 - 2\epsilon)\rho m - \left( \frac{\rho m^2}{n} + \epsilon m \right) \\
\geq \frac{(1 - \epsilon/\rho)^2}{8}(1 - 2\epsilon)\rho m - \frac{1 - 2\epsilon}{4}\rho - \epsilon m.
\]

If \(m \geq \rho m/10\), then \(P\) contained sets of size \(\geq \rho m/10\) to begin with, and hence so does \(P \vee P^g\). If \(m < \rho m/10\), we obtain that

\[
\mathbb{E}(s_{P \vee P^g}(x)) \geq \frac{a^2}{4b} - b \geq \left( \frac{(1 - \epsilon/\rho)^2}{8}(1 - 2\epsilon) - \frac{\epsilon}{10} - \frac{1 - 2\epsilon}{4n} \right)\rho m \\
\geq \left( \frac{(24/25)^2}{8} \cdot \frac{23}{25} - \frac{1}{250} - \frac{1}{600} \right)n \geq 0.10031\rho m > \frac{\rho m}{10}
\]

by the assumptions \(\epsilon \leq \rho/25 \leq 1/25\) and \(n \geq 150\). Thus, conclusion \((\ref{eq:3.8})\) holds. \(\Box\)

Simply using Lemma \((\ref{lem:3.7})\) repeatedly, we could give a proof of Prop. \((\ref{prop:3.3})\) with \(k\) in the order of \(\log n\). Our crucial induction step, allowing us \(k \ll \log \log n\), will be provided by the following Lemma. The proof will proceed by variance-based bounds. (In other words, we will be using Chebyshev’s inequality.)
Lemma 3.2. Let $P$ be a partition of a finite set $\Omega$ with $|\Omega| = n$. Let $m \geq 2$. Denote by $\rho$ the proportion of elements $x$ of $\Omega$ such that $s_P(x) \geq m$. Let $g \in \text{Sym}(\Omega)$ be taken at random with a distribution as in Lemma 3.1 with $\epsilon \leq \min(1/1000, \rho m/n)$.

Assume that $\rho \geq 999/1000$ and $1000 \leq m \leq \sqrt{n}/100$. Then, with positive probability,

$$s_{P\cup P_0}(x) \geq \frac{m^2}{2}$$

for more than $n/2$ elements $x$ of $\Omega$.

We will use several estimates in the proof of part (3) of Lemma 3.1. We shall use the same notation as in that proof: $Z(x)$, $z(x)$, $Z_g(x)$ and $z_g(x)$ are the same as there.

Proof. Let $f_g(x) = \sum_{y \in Z(x)} z_g(y)$. By (3.8) and (3.9),

$$(\rho - \epsilon)m \cdot z(x) \leq E(f_g(x)) \leq (\rho + \epsilon)m \cdot z(x).$$

Therefore, writing $E_F = \frac{1}{n} \sum_{x \in \Omega} F(x)$, we see that

$$\tag{3.11} (\rho - \epsilon)pm^2 \leq E(E_{f_g}) \leq (\rho + \epsilon)pm^2,$$

where we take $g$ at random, as always. Let

$$R_g = \frac{1}{n} \sum_{x \in \Omega} \sum_{y, y' \in Z(x)} |Z_g(y) \cap Z_g(y')|.$$

Then, by (3.10),

$$\tag{3.13} E(R_g) \leq \rho \left( \frac{pm^2}{n} + \epsilon m \right) m^2 = \frac{\rho^2 m^4}{n} + \epsilon \rho m^3 \leq \frac{2\rho^2 m^4}{n},$$

where we use the assumption $\epsilon \leq \rho m/n$.

Let us now bound the expected value of $\sum_{x \in \Omega} f_g(x)^2$. Clearly

$$f_g(x)^2 = \sum_{y, y' \in Z(x)} z_g(y)z_g(y')$$

$$\tag{3.14} = \sum_{y \in Z(x)} z_g(y)^2 + \sum_{y, y' \in Z(x)} z_g(y)z_g(y').$$

Now, for $x$ such that $Z(x)$ is non-empty,

$$\tag{3.15} E \left( \sum_{y \in Z(x)} z_g(y)^2 \right) \leq \sum_{y \in Z(x)} (\rho + \epsilon)m^2 = (\rho + \epsilon)m^3$$

and

$$\tag{3.16} E \left( \sum_{y, y' \in Z(x)} z_g(y)z_g(y') \right) \leq \sum_{y, y' \in Z(x)} (\rho^2 + \epsilon)m^2$$

$$= (\rho^2 + \epsilon)m^2 \cdot (m^2 - m).$$
Therefore,
\[
E \left( \frac{1}{n} \sum_{x \in \Omega} f_g(x)^2 \right) \leq (\rho + \epsilon) \rho m^3 + (\rho^2 + \epsilon) \rho(m^4 - m^3) \\
= (\rho - \rho^2) \rho m^3 + (\rho^2 + \epsilon) \rho m^4.
\]

We have just established a bound on the expectation of the variance: for
\[
(3.18) \quad V_f = \frac{1}{n} \sum_{x \in \Omega} f(x)^2 - \left( \frac{1}{n} \sum_{x \in \Omega} f(x) \right)^2
\]
we quickly see, by (3.11) and (3.17), that
\[
(3.19) \quad E(V_f) = E \left( \frac{1}{n} \sum_{x \in \Omega} f_g(x)^2 \right) - \left( \frac{1}{n} \sum_{x \in \Omega} f_g(x) \right)^2
\leq (\rho - \rho^2) \rho m^3 + (\rho^2 + \epsilon) \rho m^4 - ((\rho + \epsilon) \rho m^2)^2
\leq (1 - \rho) \rho^3 m^4 + \epsilon_1 m^4,
\]
where
\[
(3.20) \quad \epsilon_1 = \epsilon \rho + \frac{(1 - \rho) \rho^2}{m}.
\]

We may call $V_{f_g}$ the variance of $f_g$, just as we may call $E_{f_g}$ the expectation of $f_g$.

Now we should give a bound on the variance $V(E_{f_g})$ of $E_{f_g}$. Clearly
\[
(3.21) \quad E \left( E_{f_g}^2 \right) = E \left( \left( \frac{1}{n} \sum_{x \in \Omega} f_g(x) \right)^2 \right) = \frac{1}{n^2} \cdot E \left( \sum_{x,x' \in \Omega} f_g(x)f_g(x') \right).
\]

By the definition of $f_g(x)$ and the fact that, for every $y \in \Omega$ with $Z(y) \neq \emptyset$, $y \in Z(x)$ for exactly $m$ values of $x \in \Omega$,
\[
(3.22) \quad \sum_{x,x' \in \Omega} f_g(x)f_g(x') = \sum_{x,x' \in \Omega} \sum_{y \in Z(x) \cap Z(x')} z_g(y)^2 + \sum_{x,x' \in \Omega} \sum_{y \in Z(x), y' \in Z(x')} z_g(y)z_g(y')
\]
\[
= \sum_{y \in \Omega, Z(y) \neq \emptyset} m^2 z_g(y)^2 + \sum_{y,y' \in \Omega, y \neq y', Z(y), Z(y') \neq \emptyset} m^2 z_g(y)z_g(y').
\]

Much as in (3.15) and (3.16),
\[
E \left( \sum_{y \in \Omega, Z(y) \neq \emptyset} m^2 z_g(y)^2 \right) \leq \sum_{y \in \Omega, Z(y) \neq \emptyset} m^2 \cdot (\rho + \epsilon)m^2 = (\rho + \epsilon) \rho m^4 n
\]
and
\[
\sum_{y,y' \in \Omega, Z(y), Z(y') \neq \emptyset} m^2 z_g(y)z_g(y') \leq \sum_{y,y' \in \Omega, Z(y), Z(y') \neq \emptyset} m^2 \cdot (\rho^2 + \epsilon)m^2
\leq (\rho^2 + \epsilon) \rho^2 m^4 n^2.
\]
Hence
\[ \mathbb{E} \left( E_{f_g}^2 \right) \leq \frac{(\rho + \epsilon) \rho}{n} m^4 + (\rho^2 + \epsilon) \rho^2 m^4, \]
and so, by (3.11),
\[ \forall \left( E_{f_g} \right) \leq \frac{(\rho + \epsilon) \rho}{n} m^4 + ((\rho^2 + \epsilon) - (\rho - \epsilon)^2) \rho^2 m^4 \leq \epsilon_2 m^4, \]
where
\[ \epsilon_2 = (1 + 2\rho) \rho^2 \epsilon + \frac{(\rho + \epsilon) \rho}{n}. \]
Here, of course, \( \forall \left( E_{f_g} \right) = \mathbb{E} \left( E_{f_g}^2 \right) - \mathbb{E} \left( E_{f_g} \right)^2 = \mathbb{E} \left( (E_{f_g} - \mathbb{E} \left( E_{f_g} \right))^2 \right). \)
By (3.11), (3.13), (3.19) and (3.23) and Cauchy-Schwarz, we conclude that
\[ \mathbb{E} \left( E_{f_g}^2 - c_1 V_{f_g} - c_2 \left( E_{f_g} - \mathbb{E} \left( E_{f_g} \right) \right)^2 - c_3 n \cdot R_g \right) \]
\[ \geq \mathbb{E} \left( E_{f_g} \right)^2 - c_1 \mathbb{E} \left( V_{f_g} \right) - c_2 \mathbb{E} \left( E_{f_g} \right) - c_3 n \cdot \mathbb{E} \left( R_g \right) \geq K m^4 \]
for any \( c_1, c_2, c_3 > 0, \) where
\[ K = (\rho - \epsilon)^2 \rho^2 - ((1 - \rho) \rho^3 + \epsilon_1) c_1 - \epsilon_2 c_2 - 2\rho^2 c_3. \]
We will choose \( c_1, c_2, c_3 \) so that \( K \) is positive. Then the probability that
\[ V_{f_g} \leq \frac{E_{f_g}^2}{c_1}, \quad E_{f_g} \geq \sqrt{c_2} \left| E_{f_g} - \mathbb{E} \left( E_{f_g} \right) \right|, \quad n R_g \leq \frac{E_{f_g}^2}{c_3} \]
will be positive. What happens when (3.25) is the case?
1. First of all, \( E_{f_g} \geq \sqrt{c_2} \left| E_{f_g} - \mathbb{E} \left( E_{f_g} \right) \right| \) implies
\[ \frac{\sqrt{c_2}}{\sqrt{c_2} + 1} \mathbb{E} \left( E_{f_g} \right) \leq E_{f_g} \leq \frac{\sqrt{c_2}}{\sqrt{c_2} - 1} \mathbb{E} \left( E_{f_g} \right). \]
2. By Chebyshev’s inequality, if (3.25) is the case, then for any \( \tau > 0, \) the number of \( x \in \Omega \) such that
\[ 1 - \tau \leq \frac{f_g(x)}{E_{f_g}} \leq 1 + \tau \]
does not hold is at most \( (n V_{f_g} / E_{f_g}) / \tau^2 \leq n / c_1 \tau^2. \)
3. By (3.12) and the last inequality in (3.25), for any \( \tau' > 0, \) the number of \( x \in \Omega \) such that
\[ \sum_{y, y' \in Z(x)} |Z_g(y) \cap Z_g(y')| \leq \tau' E_{f_g} \]
does not hold is \( \leq R_g n / \tau' E_{f_g} \leq \frac{E_{f_g}^2}{c_3 \tau' E_{f_g}} = E_{f_g} / c_3 \tau'. \)
Hence, for \( \geq (1 - 1 / c_1 \tau^2) n - E_{f_g} / c_3 \tau' \) values of \( x \in \Omega, \) by (3.7) and (3.11),
\[ |S_{P \vee P'}(x)| \geq f_g(x) - \tau' E_{f_g} \geq (1 - \tau - \tau') E_{f_g} \geq \frac{\sqrt{c_2}}{\sqrt{c_2} + 1} (1 - \tau - \tau') \mathbb{E} \left( E_{f_g} \right) \]
\[ \geq \frac{\sqrt{c_2}}{\sqrt{c_2} + 1} (1 - \tau - \tau')(\rho - \epsilon) \rho m^2. \]
Moreover, by (3.11) and (3.26),
\[ \frac{E_{f_g}}{c_3 \tau'} \leq \frac{\sqrt{c_2}}{\sqrt{c_2} - 1} \frac{E \left( E_{f_g} \right)}{c_3 \tau'} \leq \frac{\sqrt{c_2}}{\sqrt{c_2} - 1} (\rho + \epsilon) \rho m^2. \]
Thus, by the assumption $m \leq \sqrt{n}/100$, we obtain that
\[
\left(1 - \frac{1}{c_1 \tau^2}\right)n - \frac{E_{f_{\rho}}}{c_3 \tau^2} \geq \rho'n
\]
for
\[(3.28)\quad \rho' = 1 - \frac{1}{c_1 \tau^2} - \frac{\sqrt{c_2}}{\sqrt{c_2} - 1} \frac{(\rho + \epsilon)^2}{100^2 c_3 \tau^2}.
\]

It is time to choose the parameters $c_1, c_2, c_3, \tau$ and $\tau'$. We let
\[(3.29)\quad c_1 = \frac{1}{4\delta_1}, \quad \delta_1 = \frac{(1 - \rho)\rho^3 + \epsilon_1}{\rho^4}, \quad c_2 = \frac{1}{4\epsilon_2}, \quad c_3 = \frac{\rho^2}{8},
\]
and $\tau = 1/4, \tau' = 1/12$. Then $K \geq (\rho - \epsilon)^2 - 3/4 > 0$, by our assumptions on $\rho$ and $\epsilon$.

In fact, since we are assuming $\epsilon \leq 1/1000$, $m \geq 1000$, $\rho \geq 999/1000$ and $n \geq (100m)^2 \geq 10^9$,
\[
\epsilon_1 \leq \epsilon_1 + \frac{1 - \rho}{m} \leq 0.001001, \quad \epsilon_2 \leq 3\epsilon + \frac{(1 + \epsilon)}{n} \leq 0.003001.
\]
\[
\delta_1 \leq \rho^{-1} - 1 + \frac{\epsilon_1}{\rho} \leq \frac{1000}{999} - 1 + \frac{0.001001}{(999/1000)^2} \leq 0.002007
\]
by (3.20), (3.24), and (3.29). Hence, by (3.27),
\[
s_{P\lor P}(x) = |S_{P\lor P}(x)| \geq \frac{\sqrt{c_2}}{\sqrt{c_2} + 1} \geq \frac{1}{1 + \sqrt{c_2}} \left(1 - \frac{1}{4} - \frac{1}{12}\right) \frac{998}{1000} \frac{999}{1000} m^2 > \frac{m}{2}
\]
for at least $\rho'n$ elements $x$ of $\Omega$. Moreover, by (3.28),
\[
\rho' \geq 1 - \frac{4\delta_1}{\tau^2} - \frac{1 + \epsilon}{1 - \sqrt{4\delta_2} 100^2 \cdot \frac{\rho^2}{8} \cdot \frac{1}{12}} > 0.86 > \frac{1}{2}
\]
\[\square\]

**Proposition 3.3.** Let $P$ be a partition of a finite set $\Omega$ with $|\Omega| = n$. Assume that at least $\geq \rho n$ elements of $\Omega$, $\rho > 0$, lie in sets in $P$ of size $> 1$. Let $A \subset \text{Sym}(\Omega)$ be a set of generators of a 4-transitive subgroup of $\text{Sym}(n)$. Let $h \in \text{Sym}(\Omega)$ have support of size $n - c$, where $0 \leq c < n$.

Then there are $g_1, \ldots, g_k \in (A \cup A^{-1} \cup \{e\})^v$, $k = O(\log \log n) + O_{\rho,c}(1)$, $v = O(n^{10})$, such that the partition $Q_k$ defined by
\[
Q_0 = P, \quad Q_j = Q_{j-1} \lor Q_{j-1}^{g_j h_j^{-1}} \quad \text{for } 1 \leq j \leq k
\]
is the trivial partition of $\Omega$.

An example given by W. Sawin suggests that 4-transitivity is a necessary assumption.

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4On MathOverflow, in comments to [Hel].
Proof. Let $A_0 \subset A \cup A^{-1}$ be as in Lemma 2.8 and let $g \in A_0^v \subset (A \cup A^{-1})^v$ be the outcome of a random walk of length $v$, where $v = \lceil n^5 \log(n^4/\epsilon) \rceil$ for a given $\epsilon > 0$. Then, by Prop. 2.7 applied to the Schreier graph $\Gamma(G, A_0; \Omega(4))$, given any two elements $\vec{v}_1, \vec{v}_2$ of the set $\Omega(4)$ of quadruples of distinct elements of $\Omega$, the probability that $g$ takes $\vec{v}_1$ to $\vec{v}_2$ lies between $(1 - \epsilon)/|\Omega(4)|$ and $(1 + \epsilon)/|\Omega(4)|$.

A moment’s thought shows that, since the support of $h$ is of size $n - c$, the number of quadruples $(r, s, r', s') \in \Omega(4)$ such that $r^h = r'$ and $s^h = s'$ is at least $(n - c)(n - c - 3)$ and at most $(n - c)(n - c - 1)$. Given any $(x, y, x', y') \in \Omega(4)$, the probability that $ghg^{-1}$ takes $(x, y)$ to $(x', y')$ equals the probability that $g$ takes $(x, y, x', y')$ to a tuple $(r, s, r', s')$ such that $r^h = r'$ and $s^h = s'$, and so lies between $(1 - \epsilon)(n - c)(n - c - 3)/|\Omega(4)|$ and $(1 + \epsilon)(n - c)(n - c - 1)/|\Omega(4)|$.

Therefore, for any $(x, y) \in \Omega(2)$ and any subset $S \subset \Omega(2)$, the probability that $(x, y)^{ghg^{-1}} \in S$ is at least

$$
(1 - \epsilon)\frac{(n - c)(n - c - 3)}{|\Omega(4)|(|S| - 4n)}
$$

(Here $|S| - 4n$ is a lower bound for the number of pairs in $S$ not containing $x$ or $y$.) We bound (3.30) from below by

$$(1 - \epsilon)\frac{(n - c)(n - c - 3)}{(n - 2)(n - 3)} \frac{|S|}{n(n - 1)} - \frac{4n(n - c)(n - c - 3)}{n(n - 1)(n - 2)(n - 3)} \geq \frac{|S|}{|\Omega(2)|} - \epsilon',$$

where

$$\epsilon' = \epsilon + \left(1 - \frac{(n - c)(n - c - 3)}{(n - 2)(n - 3)}\right) + \frac{4n(n - c)(n - c - 3)}{(n - 1)(n - 2)(n - 3)} \leq \epsilon + \frac{c}{n - 3} + \frac{c}{n} + \frac{4n}{(n - 1)(n - 2)} \leq \epsilon + \frac{8 + 3c}{n},$$

by $n \geq 6$. Since we can apply the same bound to the complement of $S$, we conclude that the distribution of $(x, y)^{ghg^{-1}}$ is at total variation distance at most $\epsilon'$ from the uniform distribution. We can apply the same statement to $h^{-1}$ instead of $h$. Hence, we can apply Lemmas 3.1 and 3.2 with $ghg^{-1}$ instead of $g$, and $\epsilon'$ instead of $\epsilon$, provided that their conditions on $\epsilon'$, $\rho$, $m$, and $n$ hold.

If $n$ is bounded by a constant $C$, say, then we can proceed as follows: at every step, we look at an element $L$ of $Q_j$ of maximal length (size), and let $g$ be as above, with $\epsilon = 1/100$, say. (We could even take $v = \lceil n^5 \log(n^2/\epsilon) \rceil$, and look only at the Schreier graph $\Gamma(G, A; \Omega).$) Then any given element of $L$ is sent to any given element of $\Omega \setminus L$ with positive probability, and so, trivially, $L$ becomes larger in $Q_{j+1} = Q_j \cup Q_j^0$ with positive probability, i.e., for at least one $g$. We set $g_{j+1}$ equal to that $g$. After at most $k = C$ steps, we obtain, then, that $Q_k$ consists of a single set, equal to $\Omega$, and so we are done.

We can assume, then, that $n$ is greater than a constant $C$. We start by applying parts (1) and (2) of Lemma 3.1 a bounded number of times so that we get to a state in which either the conditions of Lemma 3.2 are fulfilled or $m > \sqrt{n}/50$. (If the conditions are already fulfilled, then, of course, this initial stage may be skipped.) The initial value $m_0$ of $m$ will be $2$. We let $\rho_0 = \rho$. We let $\epsilon = \rho/4000$; since we can assume that $n \geq 4000(8 + 3c)/\rho$, we obtain from (3.31) that $\epsilon' \leq \rho/4000 + (8 + 3c)/n \leq \rho/2000$. In particular, the condition $\epsilon' \leq \rho/100$ in part (2) of Lemma 3.1 is satisfied.

(1) We begin by applying part (1) of Lemma 3.1 repeatedly, with $m$ held constant. At the $j$th step, Lemma 3.1 guarantees us the existence of a
$g_j \in A^j_0$ such that, for $Q_j = Q_{j-1} \lor Q_{j-1}^{gh^{-1}}$, the proportion $\rho_j$ of elements $x \in \Omega$ for which $s_{Q_j}(x) \geq m$ satisfies

$$1 - \rho_j \leq (1 - \rho_{j-1})^2 + \epsilon' \leq 1 - 2\rho_{j-1} + \frac{\rho_{j-1}^2}{2000} + \rho_{j-1}^2$$

$$\leq 1 - \frac{3\rho_{j-1}}{2} + \frac{\rho_{j-1}^2}{2} = (1 - \rho_j) \left(1 - \frac{\rho_j}{2}\right)$$

provided that $\rho_{j-1} \leq 999/1000$. Thus, letting $j$ be at least about

$$\frac{\log 1000}{\log \left(1 - \frac{\rho_0}{2}\right)}$$

(which is $O(1/\rho_0)$), we obtain a new value $\rho_j$ of $\rho$ such that $\rho_j \geq 999/1000$, say.

(2) If $m \geq 1000$, we stop. Otherwise, we apply part (2) of Lemma 3.1. We are guaranteed the existence of an element $g_{j+1} \in A^j_0$ such that, for $Q_{j+1} = Q_j \lor Q_j^{gh^{-1}}$, the proportion of elements $x \in \Omega$ such that $s_{Q_j}(x) \geq (1 + \rho_j/3)m$ is at least $\rho_j^2/8$. We choose that $g$, let our new values $m_{j+1}$ and $\rho_{j+1}$ of $m$ and $\rho$ be

$$m_{j+1} = \left\lceil \left(1 + \frac{\rho_j}{3}\right) m_j \right\rceil \geq \left(1 + \frac{333}{1000}\right) m_j, \quad \rho_{j+1} = \frac{\rho_j^2}{8} \geq \frac{(999/1000)^2}{8} > \frac{1}{9},$$

and go back to step 1.

It is clear that, after $s = O(1/\rho) + O(\log \max(c, 1))$ steps, we obtain a partition $Q_s$ such that the proportion of elements $x \in \Omega$ satisfying $s_{Q_s}(x) \geq 1000 \cdot \max(c, 1)$ is at least $999/1000$.

Now – and here we are at the heart of the proof of this proposition – we go again through an iterative procedure, only we will now be alternating a bounded number of applications of part (1) of Lemma 3.1 and an application of Lemma 3.2, rather than $O(1/\rho)$ applications of part (1) of Lemma 3.1 and an application of part (2) of Lemma 3.1. Throughout the iteration, $\rho$ stays bounded from below by $1/2$. We let $\epsilon = 1/n$, so that, by (3.31),

$$\epsilon' \leq \epsilon + \frac{8 + 3c}{n} = \frac{9 + 3c}{n},$$

and thus $\epsilon' \leq 1/1000$ and $\epsilon' \leq 500 \max(c, 1)/n \leq \rho m/n$ both hold. The conditions of Lemma 3.2 are thus satisfied for as long as $m \leq \sqrt{n}/100$.

The important part this time is that Lemma 3.2 enables us to take $m_{j+1} = \lceil m_j^2/2 \rceil$, rather than $m_{j+1} = \lceil (1 + \rho/3)m_j \rceil$ as in Lemma 3.1(2). Thanks to this fact, after only $O(\log \log n)$ steps, we obtain a partition $Q_{s'}$ such that the proportion of elements $x \in \Omega$ satisfying $s_{Q_{s'}}(x) > \sqrt{n}/100$ is at least $999/1000$.

We are almost done. We apply part (3) of Lemma 3.1 with $m = \lceil \sqrt{n}/100 \rceil$, $\rho = 999/1000$ and $\epsilon = 1/n$ (and thus $\rho' = 1/n$) as in (3.32). We obtain a partition $Q_{s'+1} = Q_{s'} \lor Q_{s'}^{gh'}$, containing at least one set of size $\geq \min(pm/9, (\rho - \epsilon)m^2/2) > ((998/1000)/20000)n > n/20100$. Tautologically, the proportion $\rho$ of elements of $\Omega$ lying in that set is $> 1/20100$.

We then alternate a bounded number of applications of part (1) of Lemma 3.1 and an application of part (2) of the same Lemma, iterating a bounded number of times, to obtain a partition $Q_{s''}$ such that at least $n/2$ of its elements lie in sets of size $> n/2$. 


Finally, we apply part (1) (with $\epsilon = 1/n$) $O(\log \log n)$ times, and obtain a partition $Q_s'''$ such that the proportion $x$ of elements of $\Omega$ lying in sets in the partition of size $> n/2$ is at least $1 - 2\epsilon' \geq 1 - (18 + 6c)/n$. Since $|\Omega| = n$, there can be at most one set in the partition of length $> n/2$; that is, at least $n - (18 + 6c)$ elements of $\Omega$ lie in one and the same set $S$.

We now proceed as in the case of $n$ bounded, choosing at each step a $g \in A_0$ that increases the size of the orbit $S$ by at least 1. After a bounded number of steps, we obtain that all elements of $\Omega$ lie in the same set of the partition, i.e., the final partition consists of the single set $\Omega$.

We come to the main result of this section.

**Proposition 3.4.** Let $\Omega$ be a finite set of size $|\Omega| = n$. Let $g_0 \in \text{Sym}(\Omega)$ have support of size $\geq \alpha n$, $\alpha > 0$. Let $A \subseteq \text{Sym}(\Omega)$ with $\langle A \rangle$ 4-transitive.

Then there are $\gamma_i \in (A \cup A^{-1} \cup \{e\})^v$, $1 \leq i \leq \ell$, where $\ell = O((\log n)/\alpha)$, and $g_i \in (A \cup A^{-1} \cup \{e\})^v$, $1 \leq i \leq k$, $v = O(n^{10})$, $k = O(\log \log n)$, such that, for

$$h = \gamma_1 g_0 \gamma_1^{-1} \cdot \gamma_2 g_0 \gamma_2^{-1} \cdots \gamma_\ell g_0 \gamma_\ell^{-1},$$

the group

$$\langle h, g_1 h g_1^{-1}, g_2 h g_2^{-1}, \ldots, g_k h g_k^{-1} \rangle$$

is transitive.

**Proof.** Let $\gamma_1, \ldots, \gamma_\ell$ be as in Lemma 2.13 (applied with $g_0$ instead of $g$ and $A \cup A^{-1} \cup \{e\}$ instead of $A$), so that the element $h$ defined in (3.33) has support of size $n - c$ with $c = 0$ or $c = 1$. (If $n$ is less than a constant, we do not need apply Lemma 2.13 we can simply let $h = g_0$, as $c = n - |\text{supp}(h)|$ will be bounded.) Write $h$ as a product of disjoint cycles, and let $P$ be the partition of $\Omega$ given by the cycles.

We can now apply Proposition 3.3 with $\rho = (n - 1)/n$ and $c = 0$ or $c = 1$. It is clear, inductively, that, for $0 \leq j \leq k$, $Q_j$ is finer (not necessarily strictly) than the partition of $P$ given by the orbits of

$$\langle h, g_1 h g_1^{-1}, g_2 h g_2^{-1}, \ldots, g_j h g_j^{-1} \rangle.$$

Since $Q_k$ is the trivial partition, it follows that the group in (3.34) has a single orbit. \qed

## 4. Babai-Seress revisited

The proof of the main result in [BS92] has what looks like a bookkeeping mistake, or rather two mistakes, at the very end ([BS92] p. 242), “Proof of Theorem 1.4”: the right side of the last displayed equation has a factor of $\text{diam}(\text{Alt}(m(G)))$ where it should have a product of squares of several such factors. We will show how to fix the result and its proof.

The intermediate result [BS92] Thm. 2.3], is in fact correct. We will prove it again here (4.1), in part for the sake of clarity, and in part so as to give an improved version (Prop. 4.6). We will be following [BS92] §3–4] quite closely.

### 4.1. Imprimitivity and structure trees.

A tree is a graph without cycles, with one vertex labeled as the root vertex, or root for short. The vertices at level $j$ are those at distance $j$ from the root. The leaves of a tree are the vertices at maximal distance from the root. That maximal distance is called the height $h$ of the tree. A child of a vertex $v$ at level $j$, $0 \leq j < h$, is a vertex at level $j + 1$
connected to \( v \) by an edge. A descendant of \( v \) is a child of \( v \), or a child of a child, etc.

The following definition has its origin in the study of algorithms on permutation groups, and in particular [LMSS]. It provides a convenient way to work with permutation groups that may not be primitive.

**Definition 1.** Let \( G \leq \text{Sym}(\Omega) \) be a transitive permutation group. A structure tree \( T \) for \((G, \Omega)\) is defined as follows. The set of leaves is \( \Omega \). If \( G \) is primitive, then it consists of a root vertex and, for each leaf, an edge connecting the root to the leaf. If \( G \) is not primitive, we choose a maximal block system \( B_1, \ldots, B_k \), define a structure tree for \( G \) as a group acting transitively on \( B_1, \ldots, B_k \), and then draw edges from the vertex corresponding to each \( B_i \) to the elements of \( B_i \) (which then become the leaves).

It is clear that \( G \) has a natural action on \( T \). Given a vertex \( v \) of \( T \), we define the stabilizer \( G_v \leq G \) to be the setwise stabilizer of the block corresponding to \( v \) (or the stabilizer of the element of \( \Omega \) corresponding to \( v \), if \( v \) is a leaf). If \( w \) is a descendant of \( v \), then \( G_w \) is a subgroup of \( G_v \). For \( v \) a vertex that is not a leaf, define \( K_v \) to be the intersection \( \cap_w G_w \), where \( w \) ranges over all children of \( v \). It is clear that \( K_v \) is a normal subgroup of \( G_v \). It is also clear that \( G_v/K_v \) acts primitively on the set of children of \( v \), due to the maximality of the block systems used in Definition 1.

It is easy to see from the definition that \( G \) acts transitively on all vertices of \( T \) at a given level. It follows that the normal core \( N_j = \bigcap_{g \in G} g G_v g^{-1} \) of the stabilizer \( G_v \) of a vertex \( v \) depends only on the level of \( v \). Indeed, it is the intersection \( \cap_w G_w \), where \( w \) ranges over all \( w \) at the same level as \( v \). Moreover, if the level \( j \) of a vertex \( v \) is less than the height of the tree (i.e., \( v \) is not a leaf), then the normal core \( \bigcap_{g \in G} g K_v g^{-1} \) of \( K_v \) equals \( N_j+1 \).

Part of the reason for working with the groups \( G_v \), rather than just with \( N_j \), is that \( G_v \) acts transitively on the block corresponding to \( v \), whereas \( N_j \) may not.

The following lemmas are quoted as “folklore” in [BS92] §3.

**Lemma 4.1.** Let \( H \) be a subgroup of a direct product of simple groups \( M_1 \times M_2 \times \ldots \times M_k \) such that the projection \( \pi_i : H \to M_i \) is surjective for every \( 1 \leq i \leq k \). Then \( H \) is isomorphic to a direct product \( \prod_{i \in I} M_i \), where \( I \subset \{1, 2, \ldots, k\} \).

**Proof.** If the projection \( \phi : H \to M_2 \times M_3 \times \ldots \times M_k \) is injective, we apply the lemma to the image of \( H \) as \( M_2 \times \ldots \times M_k \) and are done, by induction. Suppose \( \phi \) is not injective. Let \( h_1, h_2 \in H \) be distinct elements such that \( \phi(h_1) = \phi(h_2) \). Then \( \phi(h_1^{-1} h_2) = e \), and so \( h = h_1^{-1} h_2 \) lies in \( H \cap M_1 \). Let \( g \in M_1 \) be arbitrary. There is an element \( g' \) of \( H \) mapped to \( g \) by \( \pi_1 \); conjugating \( h \) by it, we obtain \( g' h (g')^{-1} = ghg^{-1} \), which must then lie in \( H \cap M_1 \). Since we can do as much for every \( g \in M_1 \), we conclude that \( H \cap M_1 \) must contain the subgroup of \( M_1 \) generated by all elements of the form \( ghg^{-1} \). Since \( M_1 \) is simple, that subgroup is precisely \( M_1 \), and so \( H \cap M_1 = M_1 \).

Let \( K = H \cap \{e\} \times M_2 \times \ldots \times M_k \). Since \( H \cap M_1 = M_1 \), we know that \( H \sim M_1 \times K \). For each \( 2 \leq i \leq k \), the image \( \pi_i(K) \) is invariant under conjugation by \( \pi_i(H) = M_i \), and thus must be either \( \{e\} \) or \( M_i \). We eliminate all indices \( i \) for which \( \pi_i(K) = \{e\} \), and apply the Lemma inductively to \( K \) as a subgroup of the direct product of the remaining \( M_i \).

**Lemma 4.2.** Let \( H_1 \triangleleft H_2 \leq G \), \( H_2/H_1 \) simple. Let \( N_i = \bigcap_{g \in G} g H_1 g^{-1} \). Then \( N_2/N_1 \) is isomorphic to a direct product of copies of \( H_2/H_1 \).
Proof. We may assume that \( N_2 \neq N_1 \). By the second isomorphism theorem, \( H_1 \cap N_2 \) is a normal subgroup of \( H_2 \cap N_2 = N_2 \), and \( N_2/(H_1 \cap N_2) \) is isomorphic to \( N_2H_1/H_1 \), which is a normal subgroup of \( H_2/H_1 \). Since \( H_2/H_1 \) is simple, that subgroup is either trivial or all of \( H_2/H_1 \). If it were trivial, then \( N_2 \leq H_1 \), and so \( N_2 \leq gH_1g^{-1} \) for every \( g \in G \); it would follow immediately that \( N_2 = N_1 \cap N_2 = N_1 \). Thus, we may assume that \( N_2/(H_1 \cap N_2) \) is isomorphic to \( H_2/H_1 \).

The same argument applied to any conjugate \( gH_1g^{-1}, g \in G \), instead of \( H \) shows that \( N_2/(gH_1g^{-1} \cap N_2) \) is isomorphic to \( H_2/H_1 \). Now, \( N_2/N_1 \) is isomorphic to its image under the natural map \( N_2/N_1 \to (N_2/(gH_1g^{-1} \cap N_2))_{g \in G} \), since \( N_1 = \bigcap_{g \in G}(gH_1g^{-1} \cap N_2) \). We apply Lemma 4.1 and obtain that \( N_2/N_1 \) is isomorphic to a direct product of groups of the form \( N_2/(gH_1g^{-1} \cap N_2) \sim H_2/H_1 \). \[ \square \]

The following is an extremely useful (and by now standard) consequence of the Classification Theorem and the O’Nan-Scott theorem. This is the one way in which the Classification Theorem is needed for the proof of our results.

**Proposition 4.3 (Cam81, Lie84).** Let \( G \leq \text{Sym}(\Omega) \) be a primitive group, where \( |\Omega| = n \). Then either

1. \( |G| \leq n^{O(\log n)} \), or
2. there is a subgroup \( N \trianglelefteq G \), \( [G:N] \leq n \), isomorphic to a direct product \( \text{Alt}(m)^r = \text{Alt}(m) \times \cdots \times \text{Alt}(m) \), where \( r \geq 1 \), \( m \geq 5 \) and \( n = \binom{m}{k} \) for some \( 1 \leq k \leq m - 1 \).

Proof. Just a few words on how the statement follows from [Lie84, Main Thm.]. Case (ii) there asserts that there is a set \( \Delta \subset \Omega \) with \( |\Delta| < 9 \log_2 n \) such that \( G_{(\Delta)} = \{e\} \); since \( [G : G_{(\Delta)}] \leq n^{|\Delta|} \), it follows immediately that conclusion [1] holds.

Assume, then, that we are in case (i) in [Lie84, Main Thm.]; that case gives us a subgroup \( N \) as in (2) here. It also gives us that \( m \geq 2 \), \( [G:N] \leq 2^r \) and \( n = \binom{m}{k} \) for some \( 1 \leq k \leq m - 1 \). Clearly, \( n \geq m^r \).

If \( m \geq \max(2r, 5) \), then \( [G:N] \leq (2r)^r \leq m^r \leq n \), and so we obtain conclusion [2]. If \( m < 2r \), then, since \( r \leq \log_m n \leq \log_2 n \), we see that \( [N] = (m!/2)^r \leq n^{mr} < n^m < n^{\max(2r, 5)} \leq n^{O(\log n)} \), whereas \( [G:N] \leq 2^r \) and \( (2r)^r \leq n^{O(\log n)} \). Hence \( [G] = n^{O(\log n)} \), that is, conclusion [1] holds. \[ \square \]

The motivation for wanting subgroups \( \text{Alt}(m) \) with \( m \geq 5 \) is of course that \( \text{Alt}(m) \) is then simple.

We could do without the following bound\(^5\) in that using a trivial bound in case (2) of Prop. 4.3 would be enough for our purposes; our intermediate results would become somewhat weaker, but our final results (Thms. 4.4 and 6.1) would not be affected. At the same time, there is no reason to avoid the lemma we are about to state. It does use the Classification Theorem, but only in the sense that it uses [Lie84, Main Thm.] (or rather the version with sharp constants in Mar02).

**Lemma 4.4.** (GPRV Thm. 1.3) Let \( G \leq \text{Sym}(\Omega) \) be a primitive group, where \( |\Omega| = n \). Let \( \{e\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{\ell} = G \). Then

\[
\ell \leq \frac{8 \log n}{3 \log 2} - \frac{4}{3}.
\]

The trivial bound assuming Prop. 4.3 would be \( \ell \ll (\log n)^2 \).

\(^5\)Thanks are due to D. Holt for the reference.
Lemma 4.5 ([Bab86]). The length $\ell$ of any subgroup chain $\{e\} = H_0 \leq H_1 \leq \ldots \leq H_\ell = \text{Sym}(n)$ of $\text{Sym}(n)$ is at most $2n - 3$.

The trivial bound would be $\ell \leq (\log |\text{Alt}(n)|)/\log 2 = O(n \log n)$.

We will now prove a version of [BS92] Thm. 2.3;

**Proposition 4.6.** Let $G \leq \text{Sym}(\Omega)$ be transitive, $|\Omega| = n$. Then $G$ has a series of normal subgroups $\{e\} = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_\ell = G$, $H_i \triangleleft G$, with $\{1, \ldots, \ell\}$ being partitioned into two sets, $A, B$, such that these properties hold:

1. each quotient $H_{i+1}/H_i$ is a direct product of at most $2n$ copies of a simple group $M_i$,
2. for each $i \in A$, the group $M_i$ is an alternating group $\text{Alt}(m_i)$ with $m_i \geq 5$,
3. $m = \prod_{i \in A} m_i$ satisfies $m \leq n$,
4. if $G \neq \text{Alt}(\Omega)$ and $G \neq \text{Sym}(\Omega)$, then $m_i \leq n/2$ for every $i \in A$,
5. $\prod_{i \in B} |M_i| = (n/m)^{O(\log(n/m))} \cdot m = n^{O(\log n)}$,
6. $\ell = O(\log n)$.

All implied constants are absolute.

The series $\{e\} = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_\ell = G$ is a refinement of the series $\{e\} = N_1 \triangleleft \ldots \triangleleft N_h = G$ defined by the structure tree as above.

**Proof.** We construct a structure tree as in Def. [1]. We choose a leaf $v$ and denote by $v_0, v_1, \ldots, v_h, v_h = v$ all vertices on the path from the root $v_0$ to $v$. For $0 \leq i \leq h - 1$, let $G_i = G_{v_i}/K_{v_i}$.

Apply Prop. [3] with $G_i$ instead of $G$, and with the set of children of $v_i$ instead of $\Omega$. Write $n_i$ for the number of children of $v_i$. Clearly, $\prod_{0 \leq i \leq h-1} n_i = n$, and so $h \leq (\log n)/\log 2 = O(\log n)$.

If conclusion [1] holds, we simply take a composition series

$$\{e\} = S_{i,0} \triangleleft S_{i,1} \triangleleft \ldots \triangleleft S_{i,\ell_i} = G_i$$

of $G_i$. By Lemma [4] $\ell_i \ll \log n_i$. Write $\tilde{S}_{i,j}$ for the preimage of $S_{i,j}$ under the map $G_{v_i} \to G_i = G_{v_i}/K_{v_i}$. Let

$$H_{i,j} = \bigcap_{g \in G} g\tilde{S}_{i,j}g^{-1}.$$

By Lemma [4] for $1 \leq j \leq \ell_i$, $H_{i,j}/H_{i,j-1}$ is a direct product of copies of the simple group $M_{i,j} = \tilde{S}_{i,j}/\tilde{S}_{i,j-1} = S_{i,j}/S_{i,j-1}$. By Lemma [4] there are at most $2n$ such copies. Evidently,

$$\prod_{1 \leq j \leq \ell_i} |M_{i,j}| = |G_i| = n_i^{O(\log n_i)}.$$

We include every one of the groups $M_{i,j}$ in the set $B$ (to be implicitly redefined as a set of indices at the end of the proof).

If conclusion [2] of Prop. [3] holds, we write $r_i$ for $r$, $m_i$ for $m$ and $k_i$ for $k$, and let

$$\{e\} = S_{i,r_i} \triangleleft S_{i,r_i+1} \triangleleft \ldots \triangleleft S_{i,\ell_i} = G_i/N$$

be a composition series of $G_i/N$. Since $|G_i/N| \leq n_i$, we see that $\ell_i = r_i + O(\log n_i) = O(\log n_i)$. Given that $N \sim \text{Alt}(m_i)^{r_i}$, we can write

$$\{e\} = A_{i,0} \triangleleft A_{i,1} \triangleleft \ldots \triangleleft A_{i,r_i} = N,$$
where $A_{i,j}/A_{i,j-1} \sim \text{Alt}(m_i)$ for $1 \leq j \leq r_i$. This time, we define $\tilde{A}_{i,j}$ to be the preimage of $A_{i,j}$ under the map $G_{v_i} \to G_i = G_{v_i}/K_{v_i}$, and $\tilde{S}_{i,j}$ to be the image of $S_{i,j}$ under the composition $G_{v_i} \to G_i \to G_i/N$.

Let

$$H_{i,j} = \begin{cases} \bigcap_{g \in G} g\tilde{A}_{i,j}g^{-1} & \text{if } 0 \leq j \leq r_i, \\ \bigcap_{g \in G} g\tilde{S}_{i,j}g^{-1} & \text{if } r_i < j \leq \ell_i. \end{cases}$$

We let $M_{i,j} = S_{i,j}/S_{i,j-1}$ for $r_i < j \leq \ell_i$, and $M_{j,1} = A_{i,j}/A_{i,j-1} \sim \text{Alt}(m_i)$ for $1 \leq j \leq r_i$.

We know from Prop. 4.3 that $(m_i/k_i)^{r_i} \leq n_i$, where $1 \leq k_i \leq m_i - 1$. If $r_i \geq 2$, then $m_i \leq \sqrt{n_i}$, and $m_i \leq n_i/2$; if $r_i = 1$ and $k_i \geq 2$, then $m_i(m_i - 1) \leq 2n_i$ and so, since $m_i \geq 5$, $m_i \leq n_i/2$. If $r_i = 1$ and $k_i = 1$, then $m_i = n_i$. In that case, if $G$ is not primitive, then $m_i = n_i \leq n/2$, whereas, if $G$ is primitive, $m_i = n_i = n$ and so $\text{Alt}(n) \leq G$.

For all $1 \leq j \leq \ell_i$, $M_{i,j}$ is simple. By Lemma 4.2, for $1 \leq j \leq \ell_i$, $H_{i,j}/H_{i,j-1}$ is a direct product of copies of the simple group $M_{i,j}$; by Lemma 4.5 there are at most $2n$ such copies. We include $M_{i,j} \sim \text{Alt}(m_i)$ in $A$ for $1 \leq j \leq r_i$ and $M_{i,j}$ in $B$ for $r_i < j \leq \ell_i$.

It is clear – whether conclusion [1] or [2] holds – that, for any $i$ less than the height $h$ of our tree, $H_{i,0} = \bigcap_{g \in G} gK_{v_i}g^{-1} = N_{i+1}$, whereas $H_{i,\ell_i} = \bigcap_{g \in G} gG_{v_i}g^{-1} = N_i$. Hence we can define our subgroups $H_0, H_1, \ldots, H_{\ell}$ to be $H_{h-1,0}, H_{h-1,1}, \ldots, H_{h-1,\ell_{h-1}} = H_{h-2,0}, \ldots, H_{h-2,\ell_{h-2}}, \ldots, H_{1,1} = H_{0,0}, \ldots, H_{0,0}$.

The (trivial) accounting is left to the reader. \hfill \Box

4.2. Reduction of the diameter problem to the case of alternating groups. First, a lemma essentially due to Schreier. The statement is as in [BS92] Lemma 5.1.

**Lemma 4.7** (Schreier). Let $G$ be a finite group. Let $N \vartriangleleft G$. Then

$$\text{diam } G \leq (2 \text{diam}(G/N) + 1) \text{diam}(N) + \text{diam}(G/N) \leq 4 \text{diam}(G/N) \text{diam}(N).$$

**Proof.** Let us be given a set $A = \{g_1, \ldots, g_r\}$ of generators of $G$. Write $d_1$ for $\text{diam}(G/N)$, $d_2$ for $\text{diam}(N)$ and $m$ for $|G/N|$. Then, by the definition of diameter, there are $\sigma_1, \ldots, \sigma_m \in (A \cup A^{-1} \cup \{e\})^{d_1}$ giving us a full set of representatives of $G/N$. As is well-known,

$$S = \{\sigma_ig_j\sigma_k^{-1} : 1 \leq i, k \leq m, 1 \leq j \leq r\} \cap N$$

is a set of generators of $N$ (Schreier generators). Hence, $N = (S \cup S^{-1} \cup \{e\})^{d_2}$, and so

$$G = \{\sigma_1, \ldots, \sigma_m\} \cdot N \subset (A \cup A^{-1} \cup \{e\})^{d_1} \cdot (S \cup S^{-1} \cup \{e\})^{d_2} \subset (A \cup A^{-1} \cup \{e\})^{d_1+2d_2} \cdot (S \cup S^{-1} \cup \{e\})^{d_2}.$$

\hfill \Box

**Corollary 4.8.** Let $G$ be a finite group. Let $\{e\} \vartriangleleft H_1 \vartriangleleft H_2 \vartriangleleft \ldots \vartriangleleft H_{\ell} = G$. Then

$$\text{diam}(G) \leq 4^{\ell-2} \prod_{i=0}^{\ell-1} \text{diam } (H_{i+1}/H_i).$$

**Proof.** Immediate from Lemma 4.7. \hfill \Box
Lemma 4.9. ([BS92] Lemma 5.4) Let $G = T_1 \times T_2 \times \cdots \times T_n$, where the $T_i$ are non-abelian simple groups. Let $\text{diam}(T_i) = d_i$, $d = \max_i d_i$. Then $\text{diam}(G) \ll n^3d^2$.

We will go over the ideas of the proof of the Lemma in a moment, when we improve it in the special case of the alternating group (Lemma 4.13). L. Pyber pointed out to the author that the dependence on $d$ could and should be improved so as to be linear; the quadratic dependence of Lemma 4.9 on $d$ is one of the gaps in the proof of the main result in [BS92]. It is actually enough to improve the dependence on $d$ in the alternating case. The tool we will use is a simple lemma, similar to [BS92] Prop. 5.8.

Lemma 4.10. Let $g \in \text{Alt}(\Omega)$, $g \neq e$, $|\Omega| \geq 4$. Then there is an $h \in \text{Alt}(\Omega)$ such that $[g, h]$ is either a 3-cycle or a product of two disjoint 2-cycles.

Here $[g, h]$ denotes the commutator $g^{-1}h^{-1}gh$.

Proof. Write $g$ as a product of disjoint cycles. If $g$ contains two disjoint 3-cycles $(abc)$, $(def)$, let $h$ equal $(ad)(bc)(ef)$. Then $[g, h] = (af)(bd)$. If $g$ contains two disjoint 2-cycles $(ab)(cd)$, let $h$ be the 3-cycle $(abc)$; then $[g, h]$ will be $(ac)(bd)$. If $g$ contains a $k$-cycle $(abcd\ldots)$, $k \geq 4$, let $h = (abc)$. Then $[g, h] = (adc)$. Finally, if $g$ consists of a single 3-cycle $(abc)$, let $d$ be an element of $\Omega$ different from $a$, $b$ and $c$, and define $h$ to be $(bcd)$. Then $[g, h] = (ad)(bc)$. □

The following lemma is of course extremely familiar.

Lemma 4.11. Let $n \geq 5$. Then every element of $\text{Alt}(\Omega)$, $|\Omega| = n$, can be written as (a) the product of at most $n-1$ 3-cycles, (b) the product of at most $(n+1)/2$ elements of the form $(ab)(cd)$.

Proof. We prove part (a) by induction. If $g \in \text{Alt}(\Omega)$ is not the identity, then there is an $a \in \Omega$ such that $b = a^g$ is distinct from $a$, and another $c \neq a, b$ that is also in the support of $g$. Then, for $g' = g \cdot (bac)$, we see that $a^{g'} = a$ and $\text{supp}(g') \subset \text{supp}(g)$, and so $|\text{supp}(g')| \leq |\text{supp}(g)| - 1$.

We prove part (b) in the same way: if $|\text{supp}(g)| \geq 4$, there are distinct $a, b, c, d$ such that $a^g = b$ and $c^g = d$; then, for $g' = g \cdot (ba)(dc)$, $|\text{supp}(g')| \leq |\text{supp}(g)| - 2$. If $|\text{supp}(g)| = 3$, then $g$ is a 3-cycle $(abc)$, and so, for $b', c'$ not in the support of $g$, $g$ equals the product of $(ac)(b'c')$ and $(bc)(b'c')$. □

The following result is classical, easy and very well-known. According to [KM07], it was first proved in [Mil99].

Lemma 4.12. Let $m \geq 5$. Then every element of $\text{Alt}(m)$ is a commutator, i.e., expressible in the form $[x, y]$, $x, y \in \text{Alt}(m)$.

Now we come to the proof of an improved version of Lemma 4.9 in the case of the alternating group.

Lemma 4.13. Let $G = T_1 \times T_2 \times \cdots \times T_n$, where the $T_i$ are alternating groups $\text{Alt}(m_i)$, $m_i \geq 5$. Let $\text{diam}(T_i) = d_i$, $d = \max_i d_i$, $m = \max_i m_i$. Then $\text{diam}(G) \ll n^3md$.

L. Pyber suggests using [LS01] to prove an analogous improvement on Lemma 4.9 for arbitrary finite simple groups $T_i$. 
Moreover, let $S$ be a set of generators of $G$, and let $A = S \cup S^{-1} \cup \{e\}$. Write $\pi_i : G \to T_i$ for the projection of $G$ to $T_i$.

By the definition of $d$, $\pi_i (A^d) = T_i$ for every $1 \leq i \leq m$. The set $A^{2d+1} \cap \ker(\pi_i)$ must then contain a set of generators of $\ker(\pi_i)$ (namely, Schreier generators). In particular, for any $j \neq i$, $A^{2d+1} \cap \ker(\pi_j)$ contains at least one element $g_{i,j}$ such that $\pi_j (g_{i,j}) \neq e$. By Lemma 4.10 and $\pi_j (A^d) = T_j$, there is an $h \in A^d$ such that $\pi_j ([g, h]) \in \pi_j (A^{6d+2})$ is either a 3-cycle or the product of two disjoint 2-cycles. Hence, conjugating $[g, h]$ by all elements of $A^d$, we obtain either all 3-cycles in $T_j$ or all products of disjoint 2-cycles in $T_j$. By Lemma 4.11, we can express every element of $T_j$ as a product of (a) at most $m_j - 1$ 3-cycles, (b) at most $(m_j + 1)/2$ 3-cycles. At the same time, $[g, h]$ is in $\ker(\pi_i)$, and so, obviously, are its conjugates. Hence, for $B_i = A^{(8d+2)m_j} \cap \ker(\pi_i)$, we see that $\pi_j (B_i) = T_j$.

Now, say that, for $S, S' \subset \{1, \ldots, n\} \setminus \{j\}$, there are sets $B_S, B_{S'} \subset A^k$ satisfying $\pi_j (B_S) = \pi_j (B_{S'}) = T_j$ as well as $B_S \subset \ker(\pi_i)$ for every $i \in S$ and $B_{S'} \subset \ker(\pi_i)$ for every $i \in S'$. Then $B_{S \cup S'} = \{x, y : x \in B_S, y \in B_{S'}\}$ is a subset of $A^k$ contained in $\ker(\pi_i)$ for every $i \in S \cup S'$. Moreover, by Lemma 4.12, $\pi_j (B_{S \cup S'}) = T_j$.

We apply this procedure repeatedly, first expressing $Z_i = \{1, \ldots, n\} \setminus \{j\}$ as the union of two disjoint sets $S, S'$ of size $\lceil (n-1)/2 \rceil$ and $\lfloor (n-1)/2 \rfloor$, respectively, and then doing a recursion, expressing at each point the set we are given as the union of two disjoint sets of sizes differing by at most 1, until we reach the single-element sets $S = \{i\}, i \neq j$. We obtain a subset $B_{Z_j}$ of $A^{4\lceil \log_2 n \rceil^k} \subset A^{4n^2k}$, where $k = (8d + 2)m_j$, such that $B_{Z_j} \subset \ker(\pi_i)$ for every $i \neq j$, and $\pi_j (B_{Z_j}) = T_j$.

(Here we note that $4^{\lceil \log_2 n \rceil^k} \leq 4^{\log_2 n + 1} \leq 4 \cdot 4^{\log_2 n} = 4n^2$.)

Multiplying the sets $B_{Z_j}$, we obtain that $A^{m_j (8d+2)m} \subset \ker(\pi_i)$ contains all of $G$. □

We will also need a very easy analogue for abelian simple groups.

**Lemma 4.14.** Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ ($n$ times). Then $\text{diam}(G) \leq n \lfloor p/2 \rfloor$.

**Proof.** Let $S$ be a set of generators of $G$, and let $A = S \cup S^{-1} \cup \{e\}$. We can see $G$ both as a group and as a vector space (which we may call $V$) over $\mathbb{Z}/p\mathbb{Z}$. We choose a non-identity element $v$ of $A$.

Trivially, every element of the linear span $\langle v \rangle$ of $v$ can be written in the group as $v^j$ for some $j \in \mathbb{Z}$ with $|j| \leq (p-1)/2$. We project the elements of $A$ to $A$ mod $\langle v \rangle$, and thus reduce the problem to that for the space $V$ mod $\langle v \rangle$ instead of $V = G$. □

We finally come to the fixed (and improved) version of [BS92] Thm. 1.4.

**Proposition 4.15.** Let $G \leq \text{Sym}(\Omega), |\Omega| = n$, be transitive. Then there are $m_1, \ldots, m_k \geq 3$ with $\prod_{i=1}^k m_i \leq n$ such that

$$\text{diam}(G) \leq n^{O(\log n)} \prod_{i=1}^k \text{diam} (\text{Alt} (m_i)).$$

Moreover,

1. for every $1 \leq i \leq k$, $G$ has a composition factor isomorphic to $\text{Alt} (m_i)$,
2. if $G \neq \text{Alt}(\Omega), \text{Sym}(\Omega)$, then $m_i \leq n/2$ for every $1 \leq i \leq k$. 


Proof. Apply Proposition 4.6. By Cor. 4.8,
\[
\text{diam}(G) \leq 4^{O(\log n)} \prod_{i=0}^{\ell-1} \text{diam}(H_{i+1}/H_i).
\]
By Lemma 4.9, Lemmas 4.13–4.14 and Prop. 4.6(1),
\[
\text{diam}(H_{i+1}/H_i) \ll \begin{cases} n^3 m_i \text{diam}(M_i) & \text{if } i \in A, \\
^3 \text{diam}(M_i)^2 & \text{if } i \in B.
\end{cases}
\]
Hence
\[
\text{diam}(G) \leq 4^{O(\log n)} (O(n^3))^{\ell} \prod_{i \in A} (m_i \text{diam}(M_i)) \prod_{i \in B} \text{diam}(M_i)^2.
\]
Trivially, \(\text{diam}(M_i) \leq |M_i|\), and so, by Prop. 4.6(5),
\[
\prod_{i \in B} \text{diam}(M_i) \leq \prod_{i \in B} |M_i| \leq n^{O(\log n)},
\]
whereas, by Prop. 4.6(2),
\[
\prod_{i \in A} \text{diam}(M_i) = \prod_{i \in A} \text{diam}(\text{Alt}(m_i)).
\]
Finally, by Prop. 4.6(3) and (6), \(\prod_{i \in A} m_i \leq n\) and \(\ell \ll \log n\). \(\square\)

5. Main argument

Let us set out to prove our main result (Theorem 1.4). Part of the general strategy will be as in [HS14], but much simplified.

Throughout, \(A \subset \text{Sym}(\Omega)\), \(|\Omega| = n\), with \(A = A^{-1}\), \(e \in A\), and \(\langle A \rangle\) is 3-transitive. We assume that \(\log |A| \geq C(\log n)^3\), where \(C > 0\) is a constant large enough for our later uses.

5.1. Existence of a large prefix. For \(j = 1, 2, \ldots\), we choose distinct elements \(\alpha_1, \alpha_2, \ldots \in \Omega\) such that, for \(j = 1, 2, \ldots\),
\[
(5.1) \quad \left| \alpha_j^{A^j(\alpha_1, \alpha_2, \ldots, \alpha_{j-1})} \right| \geq \rho n,
\]
where we set \(\rho = e^{-1/5} = 0.818\ldots\), say. We stop when \(A^j(\alpha_1, \alpha_2, \ldots, \alpha_k)\) has no orbits of size \(\geq \rho n\). Inequality (5.1) holds for \(1 \leq j \leq k\).

Let \(\Sigma = \{\alpha_1, \ldots, \alpha_{k-1}\}\). By Corollary 2.12 (applied with \(\Sigma \cup \alpha_k\) instead of \(\Sigma\)), either (2.8) holds, and we are done, or
\[
(5.2) \quad k \geq \frac{\log |A|}{15(\log n)^2}.
\]
We can assume henceforth that (5.2) holds.

By Lemma 2.14 the restriction \(A^{8(k-1)}|_{\Sigma}\) of \(\Sigma\) is a subset of \(\text{Sym}(\Sigma)\) with at least \(\rho^{k-1}(k-1)!\) elements. Let \(A' = A^{8(k-1)}|_{\Sigma}\), \(H = \langle A' \rangle\). If, as we may assume, \(k\) is larger than an absolute constant, then, by Lemma 2.16 there exists an orbit \(\Delta \subset \Omega\) of \(H|_{\Sigma}\), such that \(|\Delta| \geq \rho \cdot (k-1)\) and \(H|_{\Delta}\) contains \(\text{Alt}(\Delta)\). Thus, in particular, \(H\) has a section isomorphic to \(\text{Alt}(k-1)\), namely, the quotient defined by restricting either \(H\) or a subgroup of \(H\) of index 2 to \(\Delta\).
5.2. The case of descent. Applying Lemma 2.15 with $\epsilon = 1/8$, we see that $H$ contains an element $g_0 \neq e$ such that $|\text{supp}(g_0)| < n/8$, assuming, as we may, that $n$ is greater than an absolute constant and that $k - 1 \geq C_2 \log n$, where $C_2$ is an absolute constant.

Let $O = \alpha_k^{(A^4)(\Sigma)}$. We know that (5.1) holds for $j = k$, i.e., $|O| \geq \rho m$. Denote by $O'$ the orbit of $H$ containing $O$.

Suppose first that either $|O'| \leq e^{-1/10} n$ or $H|O' \neq \text{Alt}(O')$, $\text{Sym}(O')$. Define $D$ to be the diameter of $H|O'$. By definition, $g_0|O' \in ((A^{8k})_\Sigma|O'|D)$. Hence, there is a $g' \in A^{8kD}$ such that $g'|_{O'} = g|_{O'}$. The support of $g'$ is at most

$$|\Omega \setminus O'| + n/8 \leq (1 - e^{-1/5})n + n/8 \leq 0.307 \cdot n < n/3.$$ 

Hence, by Lemma 2.9 we obtain that

$$\text{diam}(\Gamma((A), A \cup g)) \ll n^8(\log n)^{O(1)},$$

and so

$$\text{diam}(\Gamma((A), A)) \ll 8kDn^8(\log n)^{O(1)} \ll n^{10}D.$$ 

Thus we attain conclusion (1.4) in Theorem 1.4. We call this case the case of descent.

5.3. The case of growth. Assume henceforth that $H|O'$ is either $\text{Alt}(O')$ or $\text{Sym}(O')$ and that $|O'| \geq e^{-1/10} n$. We can then of course assume that $|O'| \geq 6$, and so the action of $H$ on $O'$ is 4-transitive.

Let $B = (A^2)_{(\alpha_1, \alpha_2, \ldots, \alpha_k)}$. Then, by (5.1), every orbit of $BB^{-1}$ is of length $< \rho n = e^{-1/5} n \leq e^{-1/10}|O'|$. We apply Cor. 2.11 with $A'|O'$ instead of $A$, $B|O'$ instead of $B$, $O'$ instead of $O$, and $\rho$, and obtain that there is a $g \in (A')^m \subset A^{8km}$, $m \ll n^6 \log n$, such that

$$|B^2gB^2g^{-1}| \geq |B|^{1+1/10\log n}.$$ 

Since $g$ is in the setwise stabilizer of $\Sigma$, we know that $B^2gB^2g^{-1}$ is a subset of $(A^{16km+8})(\Sigma)$. Therefore, by Lemma 2.4

$$\left|\frac{A^{16km+8}}{n}\right| \geq \frac{|A|^{1+1/10\log_n}}{n}.$$ 

We have obtained what we wanted: growth in a subgroup – namely, the subgroup $\text{Sym}(\Omega)_{(\alpha_1, \ldots, \alpha_k)}$ of $\text{Sym}(\Omega)$. We apply Lemma 2.5 with $\text{Sym}(\Omega)_{(\alpha_1, \ldots, \alpha_k)}$ instead of $H$ and $32km + 16$ instead of $k$, and obtain that

$$|A^{32km+17}| \geq \frac{|B|}{n^{\log n}} |A|.$$ 

Only one thing remains: to ensure that $|B|$ is not negligible compared to $|A|$. By Lemma 2.3

$$|B| = \left|(A^2)_{(\alpha_1, \ldots, \alpha_k)}\right| \geq \frac{|A|}{nk}.$$ 

Thus, if $k \leq (\log_n |A|)/2$, we see that $|B| \geq \sqrt{|A|}$, and so

$$|A^{32km+17}| \geq \frac{1}{n} |A|^{1+1/20\log n} \geq |A|^{1+1/31\log n},$$

say, yielding a strong version of conclusion (1.3). Assume from now on that $k > (\log_n |A|)/2$. 

Since (5.1) holds for \( j = k \), and since we can assume that \( \rho n > 1 \), there is at least one non-trivial element of \((A^4)_\Sigma\). Call it \( g_0 \). If \( \text{supp}(g_0) \leq n/4 \), then, by Lemma 2.9

\[
\text{diam}(\Gamma(\text{Sym}(\Omega), A)) \leq 4 \text{diam}(\Gamma(\text{Sym}(\Omega), (A^4)_\Sigma \cup A)) \ll n^{8} (\log n)^c,
\]

and we are done. Assume, then, that \( \text{supp}(g_0) > n/4 \), and so \( |\text{supp}(g_0) \cap O'| > n/4 - (1 - e^{-1/10})n \geq n/7 \geq |O'|/7 \).

Now we can finish the argument in any of two closely related ways. One way would involve combining Prop. 3.4 with Lemma 2.3 as we said at the beginning of §3. However, we will find it simpler to proceed in a way closer to the procedure explained in [HS14] §1.5. We apply Prop. 3.4 with \( O' \) instead of \( \Omega \) and \((A')_{|O'}\) instead of \( A \). We obtain \( \gamma_i \in (A' \cup (A')^{-1} \cup \{e\})^{n^6}, 1 \leq i \leq \ell', \) where \( \ell = O(7 \log n) = O(\log n) \), and \( g_1, \ldots, g_{k'} \in (A')^\nu, v = O(n^{10}), k' = O(\log n) \), such that, for

\[
h = \gamma_1 g_0 \gamma_1^{-1} \cdot \gamma_2 g_0 \gamma_2^{-1} \cdots \gamma_\ell g_0 \gamma_\ell^{-1} \in (A^{4\ell+2\ell \cdot 8(k-1)n^6}_\Sigma),
\]

the group

\[
\langle h, g_1 h g_1^{-1}, g_2 h g_2^{-1}, \ldots, g_{k'} h g_{k'}^{-1} \rangle
\]

acts transitively on \( O' \). Write \( h_0 = h, h_i = g_i h g_i^{-1} \) for \( 1 \leq i \leq k' \). Since \( h \) fixes \( \Sigma \) pointwise and \( g_i \) fixes \( \Sigma \) setwise, \( h_i \) fixes \( \Sigma \) pointwise for every \( 0 \leq i \leq k' \). Thus, \( h_i \in (A^{4\ell+2\ell \cdot 8(k-1)n^6+2\nu})_\Sigma \). By the same argument, the map

\[
\phi : g \mapsto (gh_0 g^{-1}, gh_1 g^{-1}, \ldots, gh_{k'} g^{-1})
\]

sends \((A')^2 \subset (A^{16k})_\Sigma\) to a subset of the Cartesian product

\[
(A^{32k+4\ell+2\ell \cdot 8(k-1)n^6+2\nu})_\Sigma \times \cdots \times (A^{32k+4\ell+2\ell \cdot 8(k-1)n^6+2\nu})_\Sigma \quad (k' + 1 \text{ times}).
\]

Moreover, two elements \( g, g' \) satisfy \( \phi(g) = \phi(g') \) if and only if \( g^{-1} g' h_i (g^{-1} g')^{-1} \) for every \( 0 \leq i \leq k' \), i.e., if and only if \( g^{-1} g' \) lies in \( C(\langle h_0, h_1, \ldots, h_{k'} \rangle) \).

We know that \( \langle h_0, h_1, \ldots, h_{k'} \rangle \) acts transitively on \( O' \). It is easy to show that an element of the centralizer of a transitive group can have a fixed point if and only if it is the identity. Thus, if two distinct \( g, g' \in ((A')^2)_{\alpha_k} \) satisfy \( \phi(g) = \phi(g') \), then, since \( g(g')^{-1} \) fixes \( \alpha_k \), it must act as the identity on \( O' \). In other words, \( g(g')^{-1} \) is a non-identity element of support of size \( \leq n - |O'| \leq (1 - e^{-1/10})n \leq n/10 \).

We can now apply Lemma 2.9 (Babai-Beals-Seress), and obtain that

\[
\text{diam}(\Gamma(\langle A \rangle, A)) \ll 8kn^{8} (\log n)^4 \ll n^{10}.
\]

Assume, then, that the restriction of \( \phi \) to \((A')^2)_{\alpha_k}\) is injective. Then

\[
\left| (A^{32k+4\ell+2\ell \cdot 8(k-1)n^6+2\nu})_\Sigma \right|^{k'+1} \geq \left| ((A')^2)_{\alpha_k} \right|
\]

and so

\[
\left| (A^N)_\Sigma \right| \geq \left| ((A')^2)_{\alpha_k} \right|^{\frac{1}{k'+1}} \geq \left( \frac{|A'|}{n} \right)^{\frac{1}{k'+1}}
\]
for \( N = 32k + 4\ell + 2\cdot 8(k - 1)n^6 + 2v = O\left(n^{10}\right)\). Since \( \Sigma = \{\alpha_1, \ldots, \alpha_{k-1}\} \) and \(|A'| \geq \rho^{-1}(k-1)!\), where \( \rho = e^{-1/5} \), it follows that

\[
\left|A^{2N}\right|_{(\alpha_1, \ldots, \alpha_k)} \geq \frac{\left(A^N\right)_{(\Sigma)}}{n} \geq \frac{|A'|/n}{n^{1/2}} \geq \frac{(\rho^{k-1}(k-1)!/n)^{1/2}}{n^{1/2}} \geq \frac{(\rho^k k!)}{n^2}.
\]

Since \( B = (A^2)_{(\alpha_1, \ldots, \alpha_k)} \), we know from Lemma 2.5 that

\[
\left|A^{2N+1}\right| \geq \frac{\left(A^{2N}\right)_{(\alpha_1, \ldots, \alpha_k)}}{|B|} \cdot |A|.
\]

Hence either

\[
|A^{2N+1}| \geq \frac{(\rho^k k!)}{n^{2/2}} |A|
\]

or \(|B| \geq (\rho^k k!)^{1/(2k'+2)} / n\). In the latter case, by (5.3),

\[
|A^{2km+17}| \geq \left(\frac{\rho^k k!}{n^{2/2}}\right)^{10\log n} \frac{|A|}{n} \gg \frac{(\rho^k k!)^{20(k'+1)\log n}}{n} |A|.
\]

The amount on the right in (5.4) is clearly greater than that on the right in (5.5), so we can focus on bounding the right side of (5.5) from below.

By \( \rho = e^{-1/5} \), Stirling’s formula, and the assumptions that \( \log |A| \geq C(\log n)^3 \) (or even just \( \log |A| > C(\log n)^2 \)) and \( k > (\log n |A|)/2, \)

\[
\rho^k k! \gg \left(\frac{k}{e^{6/5}}\right)^k \geq \left(\frac{\log |A|}{2e^{6/5} \log n}\right)^{\log |A|^{2 \log n}} \geq (\log |A|) \log |A| \log |A| = |A| \log |A| \log |A|.
\]

Hence, again by \( \log |A| \geq C(\log n)^3 \),

\[
\frac{(\rho^k k!)}{10k' \log n} \geq |A|^{\log \log |A|} \log \log n \geq |A|^{\log \log |A|} \log \log n.
\]

Taking \( N' = \max(2N + 1, 32km + 17) = O\left(n^{10}\right)\), we conclude that

\[
\left|A^{N'}\right| \geq |A|^{1+ \frac{\log \log |A|}{O(\log n)^2 \log \log n}}.
\]

Theorem 1.4 is thus proved.

6. Iteration

We can now prove a marginally weaker version of Theorem 1.3. The reader will notice that the proof we are about to give works for any 3-transitive group \( G \), not just for \( G = \text{Alt}(n) \) and \( G = \text{Sym}(n) \). However, by [Pyb93, Cor. to Thm. A], every 3-transitive and in fact every 2-transitive group \( G \) on \( n \) elements that is not \( \text{Alt}(n) \) or \( \text{Sym}(n) \) has \( \exp(O((\log n)^3)) \) elements. Thus, in such a case, the result we are about to prove would be trivial.

**Theorem 6.1.** Let \( G = \text{Alt}(n) \) or \( \text{Sym}(n) \). Let \( S \) be a set of generators of \( G \). Then

\[
diam \Gamma(G, S) \leq e^{K(\log n)^4(\log \log n)^2},
\]

where \( K \) is an absolute constant.
Since $|\text{Alt}(n)| \geq n!/2 \gg (n/e)^n$, it follows immediately that, for $G = \text{Alt}(n)$ and for $G = \text{Sym}(n)$,
\[\text{diam} \Gamma(G, S) \leq e^{O((\log \log |G|)^4(\log \log \log |G|)^2)},\]
where the implied constant is absolute.

**Proof.** We can assume that $e \in S$. By Lemma 2.17, we can assume that $S = S^{-1}$ as well.

For any $k \geq 1$, if $S^k = S^{k+1}$, then $S^k = S^{k'}$ for every $k' > k$, and so $S^k = \langle S \rangle = G$. So, if $S^k \neq G$, $|S^{k+1}| \geq |S^k| + 1$. Applying this statement for $k = 1, 2, \ldots, m$, we see that, for any $m$, $|S^m| \geq \min(m, |G|)$. Let $A_0 = S^m$
for $m = \lceil \exp (C(\log n)^3) \rceil$, where $C$ is as in the statement of Thm. 1.3, Then, assuming $n$ is larger than a constant, $|A_0| \geq \exp (C(\log n)^3)$. (If $n$ is not larger than a constant, then the theorem we are trying to prove is trivial.)

We apply Theorem 1.4 to $A_0$ instead of $A$. If conclusion (1.3) holds, we stop. If conclusion (1.3) holds, we let $A_1 = A_0^{C^3}$ and apply Theorem 1.4 to $A_1$. We keep on iterating until conclusion (1.4) holds, and then we stop. We thus have $A_0, A_1, \ldots, A_k$, $k \geq 0$, such that $A_{i+1} = A_i^{C_1}$ for $0 \leq i \leq k - 1$,
\[|A_{i+1}| \geq |A_i|^{1+c \frac{\log |A_i|}{(\log n)^2 \log \log n}},\]
(i.e., conclusion (1.3) holds) for $0 \leq i \leq k - 1$, and conclusion (1.4) holds for $A_k$.

Let us bound $k$. Write $r_i = \log |A_i|$. By (6.2),
\[r_{i+1} = \left(1 + c \frac{\log r_i}{(\log n)^2 \log \log n}\right) r_i.\]

We also know that $r_0 \geq 2$ (or really rather more) and $r_k \leq \log |G|$. The number of steps needed for $r_i$ to double is
\[\leq \left\lceil \frac{(\log n)^2 \log \log n}{c \log r_i} \right\rceil \leq \frac{2(\log n)^2 \log \log n}{c \log r_i},\]
where we use the fact that $\lceil y \rceil \leq 2y$ for $y \geq 1$ and we assume, as we may, that $c \leq 1$. We conclude that $k$ is at most $(2/c)(\log n)^2 \log \log n$ times
\[
\sum_{r=2^j}^{\log |G|} \frac{1}{\log r} = \sum_{0 \leq j \leq \log_2 \log |G|} \frac{1}{j \log 2} \ll \log \log |G| \ll \log \log n.
\]
Write this bound in the form
\[k \leq C'(\log n)^2(\log \log n)^2.\]

We see that $A_k \subseteq A_0^{C^3} = A_0^l = S^{lm}$ for $l \leq \exp (CC'(\log n)^3(\log \log n)^2)$ and, as before, $m \leq \exp (C'(\log n)^3)$.

(The author would like to thank L. Pyber profusely for pointing out that, as we have just seen, the presence of $\log \log |A|$ in the exponent in (1.3) means we save a factor of $(\log n)/\log \log n$ in the bound on $k$.)

By conclusion (1.4), which holds for $A_k$
\[\text{diam} \Gamma(G, S) \leq |lm| \cdot \text{diam} \Gamma(G, A_k) \leq lmC \text{diam}(G'),\]
where \( G' \) is a transitive group on \( n' \leq n \) elements such that either (a) \( n' \leq e^{-1/10}n \) or (b) \( G' \simeq \text{Alt}(n') \), \( \text{Sym}(n') \). If \( n' \leq e^{-1/10}n \) and either \( G \simeq \text{Alt}(n') \) or \( G \simeq \text{Sym}(n') \), then
\[
diam(G') \leq \max(\text{diam}(\text{Sym}(n'))), \text{diam}(\text{Alt}(n')) \leq 4 \text{ diam}(\text{Alt}(n'))
\]
by Lemma 4.7. If \( G \simeq \text{Alt}(n'), \text{Sym}(n') \), then, we apply Prop. 4.15 and obtain that
\[
diam(G) \leq (n')^{C''} \log n' \prod_{i=1}^{k} \text{diam}(\text{Alt}(m_i)) \leq e^{C''(\log n)^2} \prod_{i=1}^{k} \text{diam}(\text{Alt}(m_i)),
\]
where \( \prod_{i=1}^{k} m_i \leq n' \leq n \), \( m_i \leq n'/2 \leq n/2 \) for every \( 1 \leq i \leq k \), and \( C'' \) is an absolute constant. Clearly, \( \text{Im}(C, 4, e^{C''(\log n')^2}) \leq e^{C''(\log n)^2} \) for \( C'' \) an absolute constant, provided that (say) \( n \geq e^{3/2} \).

We can assume, as an inductive hypothesis, that Theorem 6.1 is true for \( G_1 = \text{Alt}(n_1) \), \( n_1 \leq e^{-1/10}n \). In other words,
\[
diam(G_1) \leq e^{K(\log n_1)^4(\log \log n_1)^2},
\]
If (6.3) above applies, we let \( n_1 = n' \), and obtain that
\[
diam(\Gamma(G, S)) \leq e^{C''(\log n)^3(\log \log n)^2} e^{K(\log n')^4(\log \log n')^2}
\leq e^{(C''(\log n)^3 + K((\log n) - 1/10)^4)(\log \log n)^2}.
\]
For \( K > (10/3.99) \cdot C'' \) (say) and \( n \) larger than a constant,
\[
C''(\log n)^3 + K \left( (\log n) - \frac{1}{10} \right)^4 \leq K(\log n)^4,
\]
and so Theorem 6.1 is true for \( n \).

If (6.4) applies instead, then
\[
diam(\Gamma(G, S)) \leq e^{C''(\log n)^3(\log \log n)^2} \prod_{i=1}^{k} e^{K(\log m_i)^4(\log \log m_i)^2},
\]
where \( \prod_{i=1}^{k} m_i \leq n \) and \( m_i \leq n'/2 \) for all \( 1 \leq i \leq k \). If \( k = 1 \), we proceed as above, with \( 1/2 \) instead of \( e^{-1/10} \). If \( k > 1 \), then, assuming, as we may, that \( m_1 \geq m_i \) for all \( 2 \leq i \leq k \),
\[
\sum_{i=1}^{k} (\log m_i)^4 = (\log m_1)^4 + \sum_{i=2}^{k} (\log m_i)^4 \leq (\log m_1)^4 + (\sum_{i=2}^{k} \log m_i)^k
\leq (\log m_1)^4 + (\log \prod_{i=2}^{k} m_i)^4 \leq (\log m_1)^4 + (\log n - \log m_1)^4
\leq (\log 2)^4 + (\log n/2)^4,
\]
since \( 2 \leq m_1 \leq n/2 \). Hence, much as above,
\[
diam(\Gamma(G, S)) \leq e^{C''(\log n)^3(\log \log n)^2} e^{K(\log n - \log 2)^4 + (\log 2)^4)(\log \log n)^2}.
\]
For \( K > 1/(3.99 \log 2) \) and \( n \) larger than a constant,
\[
C''(\log n)^3 + K((\log n) - \log 2)^4 + (\log 2)^4 \leq K(\log n)^4,
\]
and so Theorem 6.1 is true for \( n \) in this case as well. \( \square \)
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