How large can $P(G, L) - P(G, k)$ be for $k$-assignments $L$

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Abstract

Let $G = (V, E)$ be a simple graph with $n$ vertices and $m$ edges, $P(G, k)$ be the chromatic polynomial of $G$ and $P(G, L)$ be the number of $L$-colorings of $G$ for any $k$-assignment $L$. In this article, we show that if $k \geq m - 1 \geq 3$, then $P(G, L) - P(G, k) \geq 2c^3 n^{-5} \sum_{uv \in E} |L(u) \setminus L(v)|$, where $c \geq \frac{(m-1)(m-3)}{8}$, and in particular, if $G$ is $K_3$-free, then $c \geq \left(\frac{m-2}{2}\right) + 2\sqrt{m} - 3$. Consequently, $P(G, L) \geq P(G, k)$ whenever $k \geq m - 1$.

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1 Introduction

Let $\mathbb{N}$ be the set of positive integers. For any graph $G$, let $V(G)$ and $E(G)$ be the vertex set and edge set of $G$ respectively. For any $k \in \mathbb{N}$, let $[k] := \{1, 2, \ldots, k\}$, and a proper $k$-coloring of $G$ is a map $f : V(G) \rightarrow [k]$ such that $f(u) \neq f(v)$ for each pair of adjacent vertices $u$ and $v$ in $G$. Let $P(G, k)$ denote the number of proper $k$-colorings of $G$. Introduced by Birkhoff [1] in 1912, $P(G, k)$ is called the chromatic polynomial of $G$. More details on $P(G, k)$ can be found in [1, 2, 3, 4, 8, 10, 11].

The notion of list-coloring was introduced independently by Vizing [13] and by Erdős, Rubin and Taylor [6]. A map $L : V(G) \rightarrow 2^{\mathbb{N}}$ is called an assignment of $G$. For any $k \in \mathbb{N}$, a $k$-assignment of $G$ is an assignment $L$ of $G$ with $|L(v)| = k$ for all $v \in V(G)$. Given any

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assignment $L$ of $G$, an $L$-coloring of $G$ is a map $f : V(G) \rightarrow \mathbb{N}$ with the property that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(u) \neq f(v)$ for each pair of adjacent vertices $u$ and $v$ in $G$. Let $P(G, L)$ denote the number of $L$-colorings of $G$. For any $k \in \mathbb{N}$, let $P_l(G, k)$ be the minimum value of $P(G, L)$ among all $k$-assignments $L$ of $G$. We call $P_l(G, k)$ the list-color function of $G$. More details on $P_l(G, k)$ can be found in [12].

It is known that $P(G, k)$ is a polynomial of degree $|V(G)|$ (see Theorem [3]). However, due to Donner [5], $P_l(G, k)$ is in general not a polynomial of $k$. By the definitions of $P(G, k)$ and $P_l(G, k)$, $P_l(G, k) \leq P(G, k)$ holds for every $k \in \mathbb{N}$. Clearly, $P_l(G, k) = P(G, k)$ does not hold for some graphs $G$ and some numbers $k \in \mathbb{N}$. For example, $P(G, 2) \geq 2$ for each bipartite graph $G$, but $P_l(K_{s,s}, 2) = 0$ for each $s \geq 2$. It is not difficult to verify that $P(G, k) = P_l(G, k)$ holds for any chordal graph $G$ and $k \in \mathbb{N}$ (see [9]). For any simple graph $G$, Donner [5] showed that $P(G, k) = P_l(G, k)$ holds when $k$ is sufficiently large, answering a problem proposed by Kostochka and Sidorenko [9], and Thomassen [12] proved that $P(G, k) = P_l(G, k)$ when $k > |V(G)|^{10}$. In 2017, Wang, Qian and Yan [14] significantly improved this result by showing that $P(G, k) = P_l(G, k)$ holds for any $k \in \mathbb{N}$ with $k \geq \frac{(m-1)}{\ln(1+\sqrt{2})} \approx 1.135(m-1)$, where $m$ is the number of edges in $G$.

In this article, we will establish a lower bound of $P(G, L) - P(G, k)$ for an arbitrary $k$-assignment $L$ with $k \geq m - 1$. It tells how large the gap between $P(G, L)$ and $P(G, k)$ can be whenever $L(u) \neq L(v)$ for some pair of adjacent vertices $u$ and $v$. It follows directly that $P_l(G, k) = P(G, k)$ holds whenever $k \geq m - 1$.

Let $c_4(G)$ be the minimum integer $r$ such that each edge in $G$ is contained in at most $r$ 4-cycles of $G$. Clearly, $c_4(G) = 0$ when the girth of $G$ is at least 5.

**Theorem 1.** Let $G = (V, E)$ be a simple graph with $n$ vertices and $m$ ($\geq 4$) edges. Then, for any $k$-assignment $L$ of $G$ with $k \geq m - 1$,

$$P(G, L) - P(G, k) \geq c \cdot \frac{2k^{n-5}}{3} \sum_{uv \in E} |L(u) \setminus L(v)|,$$

where $c \geq \frac{(m-1)(m-3)}{8}$, and particularly, when $G$ is $K_3$-free, $c \geq \binom{m-1}{2} - c_4(G) \geq \binom{m-2}{2} + 2\sqrt{m} - 3$.

Note that for any chordal graph $G$, if $L$ is a $k$-assignment of $G$ with $L(u) \neq L(v)$ for some edge $uv$ in $G$, then $P(G, L) > P(G, k)$. Thus, the following conclusion follows from Theorem 1 directly.

**Corollary 2.** Let $G$ be any simple graph with $m$ edges and let $k \in \mathbb{N}$ with $k \geq m - 1$. For any $k$-assignment $L$ of $G$, $P(G, L) > P(G, k)$ if and only if $L(u) \neq L(v)$ for some edge $uv$ in $G$. 


In Section 2, we show that if \(-m\) deleting one of any two parallel edges joining the same pair of vertices. Thus, \(G\) is a simple graph with \(n\) vertices and \(m\) edges and \(\eta\) be a fixed bijection from \(E\) to \([m]\). A broken cycle of \(G\) (with respect to \(\eta\)) is a path \(B = v_1v_2\cdots v_r\) of \(G\), where \(r \geq 3\), such that \(v_1v_r \in E\) and \(\eta(v_1v_r) < \eta(v_iv_{i+1})\) for each \(i = 1, 2, \cdots, r-1\). Let \(\mathcal{B}(G)\) be the collection of edge sets \(E(B)\) over all broken cycles \(B\) of \(G\), and let \(\mathcal{NB}(G)\) be the set of \(A \subseteq E\) with \(E_0 \not\subseteq A\) for each \(E_0 \in \mathcal{B}(G)\). Obviously, for each \(A \in \mathcal{NB}(G)\), the spanning subgraph \((V, A)\) has no cycles, implying that \(0 \leq |A| \leq n - 1\). For each \(i\) with \(0 \leq i \leq n - 1\), let \(\mathcal{NB}_i(G)\) be the set of \(A \in \mathcal{NB}(G)\) with \(|A| = i\).

For any \(e \in E\) and \(1 \leq i \leq n - 1\), let \(\mathcal{NB}_i(G, e)\) be the set of \(A \in \mathcal{NB}_i(G)\) with \(e \in A\). Note that \(|\mathcal{NB}_i(G, e)|\) depends on \(\eta\) although \(\eta\) is not included in the notation. For example, if \(G\) is \(K_3\), then \(|\mathcal{NB}_2(G, e)|\) is either 1 or 2. Let \(Q_\eta(G, e, x)\) denote the polynomial defined below:

\[
Q_\eta(G, e, x) := \sum_{1 \leq i \leq n-1} \left|\mathcal{NB}_i(G, e)\right| x^{n-i} = \sum_{2 \leq i \leq n-1} \left|\mathcal{NB}_i(G, e)\right| x^{n-i}.
\]  

(2)

For any edge \(e \in E\), let \(G/e\) denote the simple graph obtained from \(G\) by contracting \(e\) and deleting one of any two parallel edges joining the same pair of vertices. Thus, \(|E(G/e)| = m - 1 - t|, where \(t\) is the number of 3-cycles in \(G\) containing \(e\).

In Section 2, we show that if \(x \geq m - 1 \geq 2\) and \(n \geq 4\), \(Q_\eta(G, e, x) \geq \frac{2|\mathcal{NB}_2(G/e)|}{3} x^{n-4}\) holds. Then, in Section 3 we find a lower bound of \(|\mathcal{NB}_2(G/e)|\) in terms of \(m\). Finally, in Section 4 we show that for any \(k\)-assignment \(L\) of \(G\), \(P(G, L) - P(G, k)\) is bounded below by \(\frac{1}{k} \sum_{e=uv \in E} (|L(u) \setminus L(v)| Q_\eta(G, e, k))\). Theorem 1 then follows immediately.

2 A lower bound of \(Q_\eta(G, e, x)\)

In this section, we always assume that \(G = (V, E)\) is a simple graph with \(n\) vertices and \(m\) edges and \(\eta\) is a fixed bijection from \(E\) to \([m]\). Due to Whitney [15], the coefficients of \(P(G, x)\) can be expressed in terms of the sizes of \(\mathcal{NB}_i(G)\)’s.

Theorem 3 [15]. \(P(G, x)\) can be expressed as \(P(G, x) = \sum_{i=0}^{n-1} (-1)^i |\mathcal{NB}_i(G)| x^{n-i}\).

In this section, we shall find a lower bound of \(Q_\eta(G, e, x)\) for any edge \(e\) under the condition \(x \geq m - 1\). By the definition of \(\mathcal{NB}_i(G, e)\), we first have the following relation between \(|\mathcal{NB}_i(G, e)|\) and \(|\mathcal{NB}_{i+1}(G, e)|\).

Lemma 4. For any \(e \in E\) and any \(i\) with \(1 \leq i \leq n - 2\), \(|\mathcal{NB}_{i+1}(G, e)| \leq (m-i)|\mathcal{NB}_i(G, e)|\).
Proof. When \( i \geq m \), the inequality is trivial, as both sides are 0. Now assume that \( 1 \leq i \leq m - 1 \). Lemma 4 then follows directly from the following facts:

(i). for each \( A \in \mathcal{B}_{i+1}(G, e) \) and each \( e' \in A \setminus \{e\} \), \( A \setminus \{e'\} \in \mathcal{B}_i(G, e) \); and

(ii). for each \( A' \in \mathcal{B}_i(G, e) \), there are at most \( m - i \) edges \( e' \) in \( E \setminus A' \) with \( A' \cup \{e'\} \in \mathcal{B}_{i+1}(G, e) \). \( \square \)

We can now apply Lemma 4 to find a lower bound of \( Q_\eta(G, e, x) \).

**Theorem 5.** Assume that \( n \geq 3 \). For any edge \( e \) in \( G \) and \( x \geq 0 \),

\[
Q_\eta(G, e, x) \geq \sum_{1 \leq i \leq n - 1 \atop i \text{ odd}} \frac{|\mathcal{B}_i(G, e)|}{i} (x - m + i)x^{n-i-1}.
\]

(3)

In particular, if \( n \) is even, then,

\[
Q_\eta(G, e, x) \geq \sum_{1 \leq i \leq n - 1 \atop i \text{ odd}} \frac{|\mathcal{B}_i(G, e)|}{i} (x - m + i)x^{n-i-1} + \frac{|\mathcal{B}_{n-1}(G, e)|}{n-1} x.
\]

(4)

**Proof.** By Lemma 4, for any \( i \) with \( 1 \leq i \leq n - 1 \), as \( x \geq 0 \),

\[
\frac{|\mathcal{B}_i(G, e)|}{i} x^{n-i} - |\mathcal{B}_{i+1}(G, e)| x^{n-i-1} \geq \frac{|\mathcal{B}_i(G, e)|}{i} x^{n-i} - \frac{(m - i)|\mathcal{B}_i(G, e)|}{i} x^{n-i-1} = \frac{|\mathcal{B}_i(G, e)|}{i} (x - m + i)x^{n-i-1}.
\]

(5)

By the definition of \( Q_\eta(G, e, x) \), the result follows from (5). \( \square \)

Although \( |\mathcal{B}_i(G, e)| \) depends on \( \eta \), it is bounded below by \( |\mathcal{B}_{i-1}(G/e)| \) which is determined only by graph \( G/e \) and \( i \).

**Lemma 6.** For any edge \( e \) in \( G \) and \( 1 \leq i \leq n - 1 \), \( |\mathcal{B}_i(G, e)| \geq |\mathcal{B}_{i-1}(G/e)| \).

**Proof.** Let \( \eta|_{E(G/e)} \) be the fixed bijection on the edge set of \( G/e \). It suffices to show that \( A \cup \{e\} \in \mathcal{B}_i(G, e) \) for each \( A \in \mathcal{B}_{i-1}(G/e) \).

Suppose that \( A \cup \{e\} \notin \mathcal{B}_i(G, e) \). Then, there exists \( B \in \mathcal{B}(G) \) with \( B \subseteq A \cup \{e\} \). As \( A \in \mathcal{B}_{i-1}(G/e) \), \( B \nsubseteq A \), which implies that \( e \in B \) and \( B \setminus \{e\} \subseteq A \). However, \( B \in \mathcal{B}(G) \) implies that \( B \setminus \{e\} \in \mathcal{B}(G/e) \), a contradiction to the assumption that \( A \in \mathcal{B}_{i-1}(G/e) \).

Hence Lemma 6 follows. \( \square \)
Combining Theorem 5 and Lemma 6 we have a lower bound of $Q_\eta(G, e, x)$ in terms of $|\mathcal{NB}_2(G/e)|$ and $x$.

**Corollary 7.** For any edge $e$ in $G$ and any real number $x$ with $x \geq m - 1$, if $n = 4$, then $Q_\eta(G, e, x) \geq \frac{|\mathcal{NB}_2(G/e)|}{3} x$; and if $n \geq 5$, then

$$Q_\eta(G, e, x) \geq \sum_{3 \leq i \leq n - 1} \frac{(i - 1) \cdot |\mathcal{NB}_i(G, e)|}{i} x^{n-i-1} \geq \frac{2|\mathcal{NB}_2(G/e)|}{3} x^{n-4}. \quad (6)$$

## 3 Lower bounds of $|\mathcal{NB}_2(G/e)|$

In this section, we still assume that $G = (V, E)$ is a simple graph with $|V| = n$ and $|E| = m$, and we shall find a lower bound of $|\mathcal{NB}_2(G/e)|$ in terms of $m$ for an arbitrary edge $e$ in $G$. Given any simple graph $H$, by the definition of $|\mathcal{NB}_2(H)|$ or Corollary 2.3.1 in [4], $|\mathcal{NB}_2(H)|$ has the following expression:

$$|\mathcal{NB}_2(H)| = \left( \frac{|E(H)|}{2} \right) - \Delta(H), \quad (7)$$

where $\Delta(H)$ is the number of 3-cycles in $H$.

First consider the special case that $G$ is $K_3$-free. Recall that $c_4(G)$ is the minimum integer $r$ such that each edge $e$ in $G$ is contained in at most $r$ 4-cycles of $G$. For any $u \in V$, let $N_G(u)$ denote the set of vertices in $G$ adjacent to $u$, and let $d_G(u) = |N_G(u)|$.

**Lemma 8.** For any edge $e$ in $G$, if $G$ is $K_3$-free and $m \geq 3$, then

$$|\mathcal{NB}_2(G/e)| \geq \left( \frac{m-1}{2} \right) - c_4(G) \geq \left( \frac{m-2}{2} \right) + 2\sqrt{m} - 3. \quad (8)$$

**Proof.** As $G$ is $K_3$-free, then $G/e$ has exactly $m - 1$ edges and at most $c_4(G)$ 3-cycles. Thus, applying (7) implies that $|\mathcal{NB}_2(G/e)| \geq \left( \frac{m-1}{2} \right) - c_4(G)$ for any edge $e \in E$.

Since $\left( \frac{m-1}{2} \right) - \left( \frac{m-2}{2} \right) - 2\sqrt{m} + 3 = (\sqrt{m} - 1)^2$, it remains to show that $c_4(G) \leq (\sqrt{m} - 1)^2$. It suffices to show that for each edge $e'$ in $G$, the number of 4-cycles in $G$ containing $e'$, denoted by $c_4(e')$, is at most $(\sqrt{m} - 1)^2$. Let $e' = uv \in E$, $N'(u) := N_G(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_p\}$ and $N'(v) := N_G(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_q\}$. As $G$ is $K_3$-free, $N'(u) \cap N'(v) = \emptyset$. If $p = 0$ or $q = 0$, then $c_4(e') = 0 < (\sqrt{m} - 1)^2$. Now, assume $p \geq 1$ and $q \geq 1$. Clearly, $c_4(e')$ is equal to...
Lemma 11. If implying that \( p = c \) the size of the edge set \( \triangle \) By applying Theorem 9, we can find an upper bound of \(|c|\). Now we are going to find a lower bound of \( |c| \). We shall apply the following theorem obtained by Fisher in [7].

Proof. Let \( w \) be a vertex in \( H - w \) be the subgraph of \( H \) induced by \( V(H) \setminus \{w\} \). Then \( |E(H)| \leq |E(H)| - s \) and \( |E(H - w)| = |E(H)| - s \). Then,

\[
\triangle(H) = |E(H_0)| + \triangle(H - w) \\
\leq |E(H)| - s + \frac{1}{6}(|E(H)| - s) \left( \sqrt{8(|E(H)| - s)} + 1 - 3 \right) \\
= \frac{|E(H)| - s \left( 3 + \sqrt{8(|E(H)| - s)} + 1 \right)}{6},
\]

where the second last expression follows from Theorem 9. As \( t \geq s \), the lemma holds. \( \square \)

Lemma 10. For any simple graph \( H \), if the maximum degree of \( H \) is at least \( t \), then

\[
\triangle(H) \leq \frac{|E(H)| - t}{6} \left( 3 + \sqrt{8(|E(H)| - t)} + 1 \right).
\]

Proof. Let \( w \) be a vertex in \( H \) with \( d_H(w) = s \geq t \). Let \( H_0 \) be the subgraph of \( H \) induced by \( N_H(w) \), and let \( H - w \) be the subgraph of \( H \) induced by \( V(H) \setminus \{w\} \). Then \( |E(H_0)| \leq |E(H)| - s \) and \( |E(H - w)| = |E(H)| - s \). Then,

\[
\triangle(H) = |E(H_0)| + \triangle(H - w) \\
\leq |E(H)| - s + \frac{1}{6}(|E(H)| - s) \left( \sqrt{8(|E(H)| - s)} + 1 - 3 \right) \\
= \frac{|E(H)| - s \left( 3 + \sqrt{8(|E(H)| - s)} + 1 \right)}{6},
\]

where the second last expression follows from Theorem 9. As \( s \geq t \), the lemma holds. \( \square \)

Lemma 11. If \( m \geq 4 \), then for any edge \( e \) in \( G \), \(|\mathcal{B}_2(G/e)| \geq \frac{(m-1)(m-3)}{8} \).

Proof. Let \( e \) be any edge in \( G \) and let \( t \) be the number of 3-cycles in \( G \) containing \( e \). Then \( m \geq 2t + 1 \) and \(|E(G/e)| = m - t - 1 \). By (11) and Theorem 9 \(|\mathcal{B}_2(G/e)| \geq g(t, m) \), where

\[
g(t, m) = \left( \frac{m - t - 1}{2} \right) - \frac{(m - t - 1)}{6} \left( \sqrt{8(m - t - 1) + 1 - 3} \right)
\]
Note that \( f(x) := \frac{1}{2}x^2 - \frac{2}{9}\sqrt{8x + 1} \) is strictly increasing for \( x \geq 1 \). Since \( g(t, m) = f(m-1-t) \), it is routine to verify that when \( m \geq 4 \),

\[
g(0, m) > g(1, m) = \frac{(m-2)^2}{2} - \frac{m-2}{6}\sqrt{8m-15} > \frac{(m-1)(m-3)}{8}. \quad (14)
\]

It remains to consider the case \( t \geq 2 \). Note that \( |E(G/e)| = m - t - 1 \) and the vertex in \( G/e \) produced after contracting \( e \) is of degree at least \( t \). By (7) and Lemma 10,

\[
|\mathcal{F}_2(G/e)| \geq \binom{m-t-1}{2} - \frac{2(m-2t-1)}{6} \left( 3 + \sqrt{8(m-2t-1)+1} \right). \quad (15)
\]

Then, by (15),

\[
|\mathcal{F}_2(G/e)| - \frac{(m-1)(m-3)}{8} \geq \frac{m-2t-1}{24} \left( 9m - 6t - 27 - 4\sqrt{8(m-2t-1)+1} \right) \geq 0 \quad (16)
\]

when \( t \geq 2 \) and \( m \geq 2t + 1 \). Hence Lemma 11 holds. 

By Corollary 7 and Lemmas 8 and 11 the following conclusion holds.

**Theorem 12.** For any edge \( e \) in \( G \) and any real number \( x \) with \( x \geq m - 1 \geq 3 \), \( Q_\eta(G, e, x) \geq \frac{2}{3}x^{n-4} \) holds, where \( c \geq \frac{(m-1)(m-3)}{8} \), and in particular, if \( G \) is \( K_3 \)-free, then \( c \geq \binom{m-1}{2} - c_4(G) \geq \binom{m-2}{2} + 2\sqrt{m-3} \).

## 4 Proving Theorem 1

In this section, we always assume that \( G = (V, E) \) is a simple graph with \( n \) vertices and \( m \) edges, \( \eta \) is a fixed bijection from \( E \) to \([m]\), and \( L \) is a \( k \)-assignment of \( G \), where \( k \geq 2 \).

For any integer \( i \) with \( 0 \leq i \leq n-1 \), let \( \mathcal{F}_i(G) \) be the set of spanning forests \( F = (V, A) \) of \( G \) with \( A \in \mathcal{F}_i(G) \). Clearly, each \( F \in \mathcal{F}_i(G) \) has exactly \( n-i \) components. We can represent \( F \) by the set \( \{T_1, T_2, \cdots, T_{n-i}\} \) if \( T_1, T_2, \cdots, T_{n-i} \) are the components of \( F \).

For any subgraph \( H \) of \( G \), define \( \beta(H) = \left| \bigcap_{v \in V(H)} L(v) \right| \). It can be proved by applying the
inclusion-exclusion principle that

\[ P(G, L) = \sum_{i=0}^{n-1} (-1)^i \sum_{\{T_1, \ldots, T_{n-i}\} \in \mathcal{N} \Phi_i(G)} \prod_{j=1}^{n-i} \beta(T_j). \quad (17) \]

By Theorem 3 and (17), we have

\[ P(G, L) - P(G, k) = \sum_{i=1}^{n-1} (-1)^i \sum_{\{T_1, \ldots, T_{n-i}\} \in \mathcal{N} \Phi_i(G)} \left( \prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \right). \quad (18) \]

For any edge \( e = uv \) in \( G \), let \( \alpha(e) = |L(u) \setminus L(v)| \). For any \( F = \{T_1, \ldots, T_{n-i}\} \in \mathcal{N} \Phi_i(G) \), a lower bound for \( \prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \) was obtained in [14], as stated below.

**Lemma 13 ([14]).** For any \( i \) with \( 0 \leq i \leq n - 1 \) and \( F = \{T_1, \ldots, T_{n-i}\} \in \mathcal{N} \Phi_i(G) \),

\[ \prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \geq -k^{n-i-1} \sum_{e \in E(F)} \alpha(e). \quad (19) \]

We are now going to establish an upper bound for \( \prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \). We first introduce the following result.

**Lemma 14.** Let \( d_1, d_2, \ldots, d_r \) be any non-negative real numbers, and \( q_1, q_2, \ldots, q_r \) be any positive real numbers, where \( r \geq 1 \). If \( x \geq \max_{1 \leq i \leq r} d_i \), then

\[ (x - d_1)(x - d_2) \cdots (x - d_r) \leq x^r - \frac{x^{r-1}}{q_1 + \cdots + q_r} \sum_{i=1}^{r} q_i d_i. \quad (20) \]

**Proof.** Assume that \( d_1 \geq d_2 \geq \cdots \geq d_r \). It is trivial to verify that \( d_1 \geq \frac{1}{q_1 + \cdots + q_r} \sum_{i=1}^{r} q_i d_i \). As \( 0 \leq (x - d_i) \leq x \) for all \( 2 \leq i \leq r \), the result follows immediately. \( \square \)

**Lemma 15.** For \( 1 \leq i \leq n - 1 \) and \( F = \{T_1, \ldots, T_{n-i}\} \in \mathcal{N} \Phi_i(G) \),

\[ \prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \leq -\frac{k^{n-i-1}}{i} \sum_{e \in E(F)} \alpha(e). \quad (21) \]
Proof. For each $T_j$ with $E(T_j) \neq \emptyset$, we have

$$\beta(T_j) \leq k - \max_{e \in E(T_j)} \alpha(e) \leq k - \frac{1}{|E(T_j)|} \sum_{e \in E(T_j)} \alpha(e). \quad (22)$$

Assume that $E(T_j) \neq \emptyset$ for each $j$ with $1 \leq j \leq s$ while $E(T_j) = \emptyset$ for each $j$ with $s + 1 \leq j \leq n - i$. As $|E(T_1)| + \cdots + |E(T_s)| = i$ and $k \geq \alpha(e)$ for each edge $e$ in $G$, by Lemma 14

$$\prod_{j=1}^{s} \beta(T_j) \leq \prod_{j=1}^{s} \left( k - \frac{1}{|E(T_j)|} \sum_{e \in E(T_j)} \alpha(e) \right) \leq k^s - \frac{k^{s-1}}{|E(T_1)| + \cdots + |E(T_s)|} \sum_{j=1}^{s} \sum_{e \in E(T_j)} \alpha(e)$$

$$= k^s - \frac{k^{s-1}}{i} \sum_{e \in E(F)} \alpha(e). \quad (23)$$

As $\beta(T_j) = k$ for each $j$ with $s + 1 \leq j \leq n - i$, (21) follows. □

Recall that $Q_{\eta}(G, e, k)$ is a function defined in (2) and $\mathcal{A}_i(G, e)$ is the set of $A \in \mathcal{A}_i(G)$ with $e \in A$. We are now going to find a lower bound of $P(G, L) - P(G, k)$ in terms of $\alpha(e)$’s and $Q_{\eta}(G, e, k)$’s.

**Lemma 16.** $P(G, L) - P(G, k) \geq \frac{1}{k} \sum_{e \in E} (\alpha(e)Q_{\eta}(G, e, k)).$

Proof. By (18) and applying Lemma 13 for even $i$’s and Lemma 15 for odd $i$’s,

$$P(G, L) - P(G, k) = \sum_{i=1}^{n-1} (-1)^i \sum_{\{T_1, \ldots, T_{n-i}\} \in \mathcal{A}_{i}(G)} \left( \prod_{j=1}^{n-i} \beta(T_j) - k^{n-i} \right)$$

$$\geq \sum_{\begin{subarray}{c} 1 \leq i \leq n-1 \cr i \text{ odd} \end{subarray}} \frac{k^{n-i-1}}{i} \sum_{T \in \mathcal{A}_i(G)} \sum_{e \in E(T)} \alpha(e) - \sum_{\begin{subarray}{c} 2 \leq i \leq n-1 \cr i \text{ even} \end{subarray}} k^{n-i-1} \sum_{F \in \mathcal{A}_i(G)} \sum_{e \in E(F)} \alpha(e)$$

$$= \sum_{\begin{subarray}{c} 1 \leq i \leq n-1 \cr i \text{ odd} \end{subarray}} \frac{k^{n-i-1}}{i} \sum_{e \in E} \alpha(e)|\mathcal{A}_i(G, e)| - \sum_{\begin{subarray}{c} 2 \leq i \leq n-1 \cr i \text{ even} \end{subarray}} k^{n-i-1} \sum_{e \in E} \alpha(e)|\mathcal{A}_i(G, e)|$$

$$= \sum_{e \in E} \alpha(e) \left( \sum_{\begin{subarray}{c} 1 \leq i \leq n-1 \cr i \text{ odd} \end{subarray}} \frac{|\mathcal{A}_i(G, e)|}{i} k^{n-i-1} - \sum_{\begin{subarray}{c} 2 \leq i \leq n-1 \cr i \text{ even} \end{subarray}} |\mathcal{A}_i(G, e)| k^{n-i-1} \right)$$

$$= \frac{1}{k} \sum_{e \in E} \alpha(e)Q_{\eta}(G, e, k). \quad (24)$$
We are now going to prove Theorem 7.

**Proof of Theorem 7.** As $m \geq 4$, we have $n \geq 4$. For each edge $e$ of $G$, $G/e$ has at least 2 edges, implying that $|\mathcal{B}_2(G/e)| > 0$. As $k \geq m - 1$, by Theorem 12 and Lemma 16,

$$P(G, L) - P(G, k) \geq \frac{1}{k} \sum_{e \in E} (\alpha(e)Q_{\eta}(G, e, k)) \geq \frac{2ck^{n-5}}{3} \sum_{e \in E} \alpha(e),$$

(25)

where $c \geq \frac{(m-1)(m-3)}{8}$, and if $G$ is $K_3$-free, then $c \geq \binom{m-1}{2} - c_4(G) \geq \binom{m-2}{2} + 2\sqrt{m} - 3$. □

We guess that there exists a constant $c > 0$ such that $P_l(G, k) = P(G, k)$ holds whenever $k \geq cn$ or $k \geq c\Delta(G)$, where $\Delta(G)$ is the maximum degree of $G$.

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