FUNCTIONS OF EXPONENTIAL GROWTH IN A HALF-PLANE, SETS OF UNIQUENESS AND THE MÜNTZ–SZÁSZ PROBLEM FOR THE BERGMAN SPACE

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ABSTRACT. We introduce and study some new spaces of holomorphic functions on the right half-plane $\mathbb{R}$. In a previous work, S. Krantz, C. Stoppato and the first named author formulated the Müntz–Szász problem for the Bergman space, that is, the problem to characterize the sets of complex powers $\{\zeta^{\lambda_j-1}\}$ with $\text{Re} \lambda_j > 0$ that form a complete set in the Bergman space $A^2(\Delta)$, where $\Delta = \{ \zeta : |\zeta - 1| < 1 \}$.

In this paper, we construct a space of holomorphic functions on the right half-plane, that we denote by $M^2_\omega(\mathbb{R})$, whose sets of uniqueness $\{\lambda_j\}$ correspond exactly to the sets of powers $\{\zeta^{\lambda_j-1}\}$ that are a complete set in $A^2(\Delta)$.

We show that $M^2_\omega(\mathbb{R})$ is a reproducing kernel Hilbert space and we prove a Paley–Wiener type theorem among several other structural properties.

We introduce a transform $M_\Delta$ modelled on the classical Mellin transform and show that $M^2_\omega(\mathbb{R}) = 2^{-z} \Gamma(1 + z) M_\Delta(A^2(\Delta))$. We determine a sufficient condition on a set $\{\lambda_j\}$ to be a set of uniqueness for $M^2_\omega(\mathbb{R})$, thus providing a sufficient condition for the solution of the Müntz–Szász for the Bergman space.

INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we introduce and begin the analysis of a space of holomorphic functions on the right half-plane. The initial motivation for such a study arose in the work on Bergman spaces of worm domains in $\mathbb{C}^2$ by S. Krantz and the first named author. In collaboration also with C. Stoppato [7] we stated the Müntz–Szász problem for the Bergman space and proved a preliminary result.

We denote by $\Delta$ the disk $\{ \zeta : |\zeta - 1| < 1 \}$, by $dA$ the Lebesgue measure in $\mathbb{C}$ and consider the (unweighted) Bergman space $A^2(\Delta)$. Then the complex powers $\{\zeta^{\lambda_j-1}\}$ with $\text{Re} \lambda_j > 0$ are well defined and in $A^2(\Delta)$. We denote by $\mathcal{R}$ the right half-plane and by $\overline{\mathcal{R}}$ its closure.

Following [7], the Müntz–Szász problem for the Bergman space is the question of characterizing the sequences $\{\lambda_j\}$ in $\mathcal{R}$ such that $\{\zeta^{\lambda_j-1}\}$ is a complete set in $A^2(\Delta)$, that is, span$\{\zeta^{\lambda_j-1}\}$ is dense in $A^2(\Delta)$.

The classical Müntz–Szász theorem concerns with the completeness of a set of powers $\{t^{\lambda_j-1/2}\}$ in $L^2([0, 1])$, where $\text{Re} \lambda_j > 0$. The solution was provided in two papers separate by C. Müntz [12] and by O. Szász [13] where they show that the set $\{t^{\lambda_j-1/2}\}$ is complete $L^2([0, 1])$ if and only if

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if the sequence \( \{ \lambda_j \} \) is a set of uniqueness for the Hardy space of the right half-plane \( H^2(\mathcal{R}) \), that is, if \( f \in H^2(\mathcal{R}) \) and \( f(\lambda_j) = 0 \) for every \( j \), then \( f \) is identically 0.

As in the classical case, in order to study the M"{u}ntz–Szász problem for the Bergman space we wish to transform the question into characterizing the sets of uniqueness for some (Hilbert) space of holomorphic functions. We now outline our paradigm.

For \( f \in A^2(\Delta) \) and \( z \in \mathcal{R} \) we define the Mellin–Bergman transform

\[
M(\lambda) f = \frac{1}{\pi} \int_{\Delta} f(\zeta) \zeta^{-1} dA(\zeta).
\]

The function \( \zeta^{-1} \) is well defined and belongs to \( A^2(\Delta) \). Then a set \( \{ \zeta^{-1} \} \) is complete in \( A^2(\Delta) \) if and only if \( f \in A^2(\Delta) \) and \( M(\lambda_j) f = 0 \) for all \( j \) implies that \( f \) vanishes identically.

Thus, the main task becomes to characterize the space \( M(\Delta) \) and study its sets of uniqueness. To this end we introduce a space of holomorphic functions on \( \mathcal{R} \).

**Definition.** For \( 0 < b < \infty \), denote by \( S_b \) the vertical strip \( \{ z = x + iy : 0 < x < b \} \) and by \( H^2(S_b) \) the classical Hardy space

\[
H^2(S_b) = \{ f \text{ holomorphic in } S_b : \sup_{0 < x < b} \int_{-\infty}^{+\infty} |f(x + iy)|^2 dy < \infty \}.
\]

On \( \mathcal{R} \) consider the Borel measure \( \omega = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \delta_{\frac{n}{2}}(x) \otimes dy \), and the space \( L^2(\mathcal{R}, d\omega) \); explicitly, the norm is given by

\[
\|f\|_{L^2(\mathcal{R}, d\omega)}^2 = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} |f\left(\frac{n}{2} + iy\right)|^2 dy < \infty.
\]

We define \( M^2(\mathcal{R}, d\omega) \) to be the space of holomorphic functions \( f \) on \( \mathcal{R} \) such that:

(I) \( f \in H^2(S_b) \) for every \( 0 < b < \infty \);

(B) \( f \in L^2(\mathcal{R}, d\omega) \).

Observe that condition (I) implies that \( f \) admits a boundary value function in \( L^2(\mathcal{R}) \) and we take such boundary values as the definition of \( f \) on the imaginary axis, which is a positive \( \omega \)-measure set. More formally, we could define \( M^2(\mathcal{R}, d\omega) \) as the closure in \( L^2(\mathcal{R}, d\omega) \) of the functions satisfying (I) that admits continuous extension to the closure of \( \mathcal{R} \).

We also point out that the measure \( \omega \) has been found in a constructive way and that it satisfies a uniqueness property, see Theorem 5 below.

Observing that \( M^2(\mathcal{R}, d\omega) \) is a subspace of \( L^2(\mathcal{R}, d\omega) \), we prove

**Theorem 1.** The space \( M^2(\mathcal{R}, d\omega) \) is a Hilbert space with reproducing kernel and with the unique inner product such that

\[
\|f\|_{M^2(\mathcal{R})}^2 = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} |f\left(\frac{n}{2} + iy\right)|^2 dy < \infty.
\]

Notice that trivially \( H^2(\mathcal{R}) \subset M^2(\mathcal{R}) \) as a closed subspace. Moreover, the following simple facts hold true (see Proposition 1.2):

(i) the function \( \Gamma(1 + \delta z) \in M^2(\mathcal{R}) \) for \( 0 < \delta < 1 \), but \( \Gamma(1 + z) \notin M^2(\mathcal{R}) \);

(ii) there exists \( f \) holomorphic in \( \mathcal{R} \) and satisfying (B), but \( f \notin H^2(S_b) \) if \( b > \frac{1}{2} \).
We study some basic structural properties of \( \mathcal{M}_\omega^2(\mathbb{R}) \) and we begin by proving the following Paley–Wiener-type theorem. We define the Fourier transform of \( \psi \in L^1(\mathbb{R}) \)

\[
(\mathcal{F}\psi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x)e^{-ix\xi} \, dx.
\]

**Theorem 2.** For \( f \in \mathcal{M}_\omega^2(\mathbb{R}) \), let \( f(0 + i\cdot) = f_0 \). Then \( \mathcal{F}f_0 \in L^2(\mathbb{R}, e^{2\xi} \, d\xi) \) and

\[
\|f\|_{\mathcal{M}_\omega^2(\mathbb{R})} = \|\mathcal{F}f_0\|_{L^2(\mathbb{R}, e^{2\xi} \, d\xi)}.
\]

Conversely, if \( \psi \in L^2(\mathbb{R}, e^{2\xi} \, d\xi) \) and for \( z \in \mathbb{R} \) we set

\[
f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\xi)e^{z\xi} \, d\xi,
\]

then \( f \in \mathcal{M}_\omega^2(\mathbb{R}) \), equality (3) holds, and \( \psi = \mathcal{F}f_0 \).

**Theorem 3.** The reproducing kernel for \( \mathcal{M}_\omega^2(\mathbb{R}) \) is given by

\[
K(z, w) = \frac{1}{2\pi} \frac{\Gamma(z + \overline{w})}{\Gamma(z + 1)\Gamma(1 + \overline{w})}.
\]

We are in the position to describe the image of \( A^2(\Delta) \) under the Mellin–Bergman transform \( M_\Delta \). Given any set \( \Omega \subseteq \mathbb{C} \) we denote by \( \text{Hol}(\Omega) \) the holomorphic functions on \( \Omega \).

**Definition.** We define

\[
\mathcal{H} = \{ g \in \text{Hol}(\mathbb{R}) : \frac{\Gamma(1+z)}{2^z} g(z) \in \mathcal{M}_\omega^2(\mathbb{R}) \}
\]

with norm

\[
\|g\|^2_{\mathcal{H}} = \left( \frac{\Gamma(1+z)}{2^z} g \right)^2_{\mathcal{M}_\omega^2(\mathbb{R})} = \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} |g(\frac{n}{2} + iy)|^2 \frac{\left| \Gamma(\frac{n}{2} + 1 + iy) \right|^2}{\Gamma(n+1)} \, dy.
\]

**Theorem 4.** The Mellin–Bergman transform

\[
M_\Delta : A^2(\Delta) \to \mathcal{H}
\]

is a surjective isometry. The space \( \mathcal{H} \) consists of holomorphic functions on \( \mathbb{R} \) that are of exponential type at most \( \pi/2 \) and the polynomials are dense in \( \mathcal{H} \). Moreover, it is a Hilbert space with reproducing kernel

\[
H(z, w) = \frac{1}{2\pi} \frac{\Gamma(z + \overline{w})}{\Gamma(1+z)\Gamma(1+\overline{w})}.
\]

We stress the fact that we prove that

\[
M_\Delta(A^2(\Delta)) = \frac{2^z}{\Gamma(1+z)} \mathcal{M}_\omega^2(\mathbb{R}) =: \mathcal{H}.
\]

It is interesting to notice that the space \( \mathcal{H} \) has already appeared in the literature, in a different context \([8, 9]\). While on one hand \( \mathcal{H} \) may be more natural being the isometric image of \( A^2(\Delta) \) through \( M_\Delta \), the space \( \mathcal{M}_\omega^2(\mathbb{R}) \) turns out to enjoy more manageable properties, since

\[1\]We are grateful to A. Aleman for pointing these references to us.
the measure $\omega$ is translation invariant in $\mathcal{R}$ and there is no analogue of the Paley–Wiener type Theorem\textsuperscript{2} for $\mathcal{H}$. We will collect some properties of $\mathcal{H}$ and further remarks in Section \textsuperscript{3}.

It is worth pointing out that the measure $\omega$ was constructed in a direct way and satisfies the uniqueness property in the next result.

**Theorem 5.** The measure $\omega$ is the unique positive translation invariant Borel measure in $\overline{\mathcal{R}}$ such that, for every $f, g \in A^2(\Delta)$ we have

$$\langle f, g \rangle_{A^2(\Delta)} = \left\langle \frac{\Gamma(z+1)}{2^z} M_\Delta(f), \frac{\Gamma(z+1)}{2^z} M_\Delta(g) \right\rangle_{L^2(\overline{\mathcal{R}}, d\omega)}.$$

Clearly, the sets of uniqueness of $\mathcal{H}$ and $\mathcal{M}_\omega^2(\mathcal{R})$ coincide, and the same holds for the zero-sets. We obtain that the zero-sets for $\mathcal{M}_\omega^2(\mathcal{R})$ are also zero-sets for the functions of exponential type $\pi/2$ and that zero-sets for the functions of exponential type $\tau < \pi/2$ are zero-sets for $\mathcal{M}_\omega^2(\mathcal{R})$, see Proposition \textsuperscript{4.1}. The reverse inclusions hold for the sets of uniqueness of the corresponding spaces.

In order to describe our next result we need to recall some classical definitions. Given a sequence $\{z_j\}$ with $|z_j| \to +\infty$, its exponent of convergence is $\rho_1 = \inf\{\rho > 0 : \sum_{j=1}^{+\infty} 1/|z_j|^\rho < \infty\}$, while the counting function is $n(r) = \#\{z_j : |z_j| \leq r\}$. The upper and lower densities $d^+ = d^+_\{z_j\}$ are then defined as

$$d^+ = \limsup_{r \to +\infty} \frac{n(r)}{r^{\rho_1}}, \quad d^- = \liminf_{r \to +\infty} \frac{n(r)}{r^{\rho_1}}.$$

In order to avoid vanishing of infinite order at finite points, we assume the functions to be regular in $\overline{\mathcal{R}}$ and we denote by $\text{Hol}(\overline{\mathcal{R}})$ such space.

**Theorem 6.** Let $\{z_j\} \subseteq \mathcal{R}$, $1 \leq |z_j| \to +\infty$. The following properties hold.

(i) If $\{z_j\}$ has exponent of convergence 1 and upper density $d^+ < \frac{1}{2}$, then $\{z_j\}$ is a zero-set for $\mathcal{M}_\omega^2(\mathcal{R}) \cap \text{Hol}(\overline{\mathcal{R}})$.

(ii) If $\{z_j\}$ is a zero-set for $\mathcal{M}_\omega^2(\mathcal{R}) \cap \text{Hol}(\overline{\mathcal{R}})$, then

$$\limsup_{R \to +\infty} \frac{1}{\log R} \sum_{|z_j| \leq R} \text{Re} \left(1/z_j\right) \leq \frac{2}{\pi}.$$

**Theorem 7.** A sequence $\{z_j\}$ of points in $\mathcal{R}$ such that $\text{Re} z_j \geq \varepsilon_0$, for some $\varepsilon_0 > 0$ and that violates condition \textsuperscript{[5]}, is a set of uniqueness for $\mathcal{M}_\omega^2(\mathcal{R})$.

As a consequence, if $\{z_j\}$ is a sequence as above, the set of powers $\{\zeta^{z_j-1}\}$ is a complete set in $A^2(\Delta)$.

We point out that a classical result of W. Fuchs\textsuperscript{[4]} shows that there exist sequences $\{z_j\}$ with exponent of convergence 1 and lower density $d^- > \frac{1}{2}$ such that $\{z_j\}$ is a set of uniqueness for $\mathcal{M}_\omega^2(\mathcal{R}) \cap \text{Hol}(\overline{\mathcal{R}})$. We will prove the above theorem and compare it with Fuchs’ result in Section \textsuperscript{5}.

For $1 \leq p < \infty$ we also consider the spaces

$$\mathcal{M}_\omega^p(\mathcal{R}) = \{f \in \text{Hol}(\overline{\mathcal{R}}) : f \in H^p(S_b), \text{ for all } b > 0, \text{ and } f \in L^p(\overline{\mathcal{R}}, d\omega)\}.$$
with norm
\[ \|f\|_{L^p(\mathcal{R},d\omega)}^p = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} |f\left(\frac{n}{2} + iy\right)|^p \, dy < \infty. \]

We recall that \( H^p(S_b) = \{ f \text{ holomorphic in } S_b : \sup_{0 < x < b} \int_{-\infty}^{+\infty} |f(x + iy)|^p \, dy < \infty \} \) and that \( f \in H^p(S_b) \) admits boundary values that are \( p \)-integrable, so that, as in the case \( p = 2 \), integration over the imaginary is well defined.

Finally we prove

**Theorem 8.** The orthogonal projection operator \( P : L^2(\mathcal{R},d\omega) \to \mathcal{M}^2 \), is unbounded as operator
\[ P : L^p(\mathcal{R},d\omega) \cap L^2(\mathcal{R},d\omega) \to \mathcal{M}^p \]
for every \( p \neq 2 \).

To the best of our knowledge, this is the first paper that deals with a mixed Hardy–Bergman type condition that appears in the definition of \( \mathcal{M}_\omega^2(\mathcal{R}) \). We find it remarkable that this space appears naturally in the attempt of solving the Müntz–Szász problem for the Bergman space. In [5] Fuchs studied the Müntz–Szász problem for sets of exponential on the positive half-line. In [14] we study some generalization of \( \mathcal{M}_\omega^2(\mathcal{R}) \), obtained by different choices of the measure \( \omega \).

In order to relate ours to some previous work, we mention that in [3] B. Jacob, J. Partington and S. Pott studied spaces of holomorphic function in \( \mathcal{R} \) whose norm is defined by the condition \( \sup_{\varepsilon > 0} \|f(\varepsilon + \cdot)\|_{L^p(\mathcal{R},d\mu)} < +\infty \), where \( \mu \) is a translation invariant Borel measure on \( \mathcal{R} \). While, on one hand, this class of spaces contains as particular cases the classical Hardy and Bergman spaces, \( \mathcal{M}_\omega^2(\mathcal{R}) \) does not fall in this class. For, the finiteness of the above norm requires the function to be bounded in each half-plane \( \{ \text{Re } z \geq \varepsilon_0 \} \), for \( \varepsilon_0 > 0 \), while both \( \mathcal{M}_\omega^2(\mathcal{R}) \) and \( \mathcal{H} \) contain functions of exponential growth.

We also mention that in [15] A. Sedletskii studies the completeness of sets of exponentials in weighted \( L^p \) spaces on \((0, +\infty)\) in terms of zeros of functions the classical Bergman space on a half-plane.

1. **Basic properties of \( \mathcal{M}_\omega^2(\mathcal{R}) \)**

We begin recalling some well-known facts about Hardy spaces on a strip. For \( 0 \leq a < b < \infty \) we denote by \( S_{(a,b)} \) the vertical strip \( \{ z = x + iy : a < x < b \} \) and by \( S_{[a,b]} \) its closure. As before, we simply write \( S_b \) to denote the strip \( S_{(0,b)} \). The classical Hardy space \( H^2(S_{(a,b)}) \) is
\[ H^2(S_{(a,b)}) = \{ f \text{ holomorphic in } S_{(a,b)} : \sup_{a < x < b} \int_{-\infty}^{+\infty} |f(x + iy)|^2 \, dy < \infty \}. \]

**Theorem 1.1.** (Paley–Wiener)

(i) Let \( F \in H^2(S_b) \). Then \( F_x := F(x + iy) \) admits limit in \( L^2(\mathcal{R}) \) as \( x \to 0^+ \) and \( x \to b^- \), that we denote by \( F_0 \) and \( F_b \), respectively. Moreover, \( e^{ib}\mathcal{F}F_0 \in L^2(\mathcal{R}) \) and for every \( x \in [0,b] \)
\[ \mathcal{F}F_x(\xi) = e^{ix\xi} \mathcal{F}F_0(\xi). \]

(ii) \( H^2(S_b) \) is a Hilbert space with the unique inner product such that
\[ \|F\|^2_{H^2(S_b)} = \|F_0\|^2_{L^2(\mathcal{R})} + \|F_b\|^2_{L^2(\mathcal{R})}. \]
(iii) If $\psi \in L^2(\mathbb{R})$ and $e^{i\xi} \psi \in L^2(\mathbb{R})$, then

$$F(z) = e^{-\frac{1}{2\pi z}} \int_{-\infty}^{\infty} \psi(\xi)e^{iz\xi}d\xi$$

is in $H^2(S_b)$ and $\mathcal{F}F_0 = \psi$.

The proofs of these facts can be found in [13], Theorems I-VII.

Next we prove Theorem 1. From now on, for simplicity of notation, we write $\mathcal{M}^2$ in place of $\mathcal{M}^2_\omega(\mathbb{R})$.

**Proof of Theorem 1.** This is elementary and we include the details for completeness.

Let $\{f_m\}$ be sequence in $\mathcal{M}^2$, Cauchy in the $L^2(\mathcal{R}, d\omega)$-norm. Then $\{f_m(\frac{k}{2} + i\cdot)\}$ is a Cauchy sequence in $L^2(\mathbb{R})$ for every $k$, since

$$\|f_m - f_{m'}\|_{L^2(\mathcal{R}, d\omega)}^2 = \sum_{k=0}^{+\infty} \frac{2^k}{k!} \int_{-\infty}^{+\infty} |f_m(\frac{k}{2} + iy) - f_{m'}(\frac{k}{2} + iy)|^2 dy < \varepsilon$$

implies that for any fixed $n$,\n
$$\|f_m(\frac{k}{2} + i\cdot) - f_{m'}(\frac{k}{2} + i\cdot)\|_{L^2(\mathbb{R})}^2 < \varepsilon'$$

for $m, m' \geq N$, and $0 \leq k \leq n$. By (ii) in the previous theorem, $\{f_m\}$ is a Cauchy sequence in $H^2(S_{\frac{b}{2}})$, so that $\{f_m\}$ converges to $f \in H^2(S_b)$, for any $n$. By analytic continuation $f \in H^2(S_b)$ for all $b > 0$.

Now it is clear that $f_m \to f$ in the $L^2(\mathcal{R}, d\omega)$-norm, that is, $\mathcal{M}^2$ is closed. The fact that it is a Hilbert space with reproducing kernel follows at once, since point evaluations are bounded in $H^2(S_b)$ and that $f \in \mathcal{M}^2$, its $H^2(S_b)$-norms are controlled by a constant (depending on $b$) times the $L^2(\mathcal{R}, d\omega)$-norm. The conclusion about the inner product is now clear. \qed

We take a look at the elementary inclusions between the basic function spaces.

**Proposition 1.2.**

(i) Let $0 < \delta < 1$ and $\varepsilon_0 > 0$. Then $\Gamma(\varepsilon_0 + \delta z) \in \mathcal{M}^2$ if and only if $\delta < 1$.

(ii) Let $G(z) = (z + 1)^{-1} \exp \{ie^{2\pi i z}\}$. Then $G \in L^2(\mathcal{R}, d\omega)$ is holomorphic in $\mathcal{R}$, but $G \notin H^2(S_b)$ if $b > \frac{1}{2}$. On the other hand, $\Gamma(1 + z) \in H^2(S_b)$ for $b > 0$, but $\Gamma(1 + z) \notin \mathcal{M}^2$.

(iii) Let $h$ be a function regular in $\mathcal{R}$ and of exponential type $\tau < \pi/2$ and let $2\tau/\pi < \delta < 1$. Then $F(z) = h(z)\Gamma(1 + \delta z) \in \mathcal{M}^2$.

**Proof.** It is well known that for $c > 0$,

$$\|\Gamma(c + i\cdot)\|_{L^2(\mathbb{R})}^2 = \int_0^{+\infty} e^{-2x} \frac{dx}{x} = 2^{-2c}\Gamma(2c),$$

see Lemma 2.3 in [2] e.g. (and also the discussion in Section [3]). Therefore, for every $b > 0$,

$$\sup_{0 < x < b} \|\Gamma(\varepsilon_0 + \delta x + i\delta \cdot)\|_{L^2(\mathbb{R})}^2 = C_b < +\infty.$$  

Thus, in order to prove $\Gamma(\varepsilon_0 + \delta z) \in \mathcal{M}^2$ it suffices to show that $\Gamma(\varepsilon_0 + \delta z) \in L^2(\mathcal{R}, d\omega)$.
For some $Q > 0$ large enough we have

$$
\|\Gamma(\varepsilon_0 + \delta \cdot)\|_{M^2}^2 = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \|\Gamma(\varepsilon_0 + \delta \frac{n}{2} + i\delta \cdot)\|_{L^2(\mathbb{R})}^2 \\
\leq \frac{1}{\delta} \sum_{n=0}^{+\infty} \frac{2^{(1-\delta)n}}{n!} \Gamma(2\varepsilon_0 + \delta n) \\
\leq C \sum_{n=0}^{+\infty} \frac{2^{(1-\delta)n}}{n!} n^{\varepsilon_0 \delta \cdot} e^{\delta n \log n} \\
\leq C \sum_{n=0}^{+\infty} Q^n e^{(\delta-1)n \log n} \\
< \infty.
$$

Since it is easy to see that the norm $\|\Gamma(\varepsilon_0 + \delta \cdot)\|_{M^2}$ is infinite if $\delta = 1$, this proves (i).

In order to prove (ii), notice that

$$
\begin{align*}
&|\exp \{ie^{2\pi iy}\}| = \exp \{e^{-2\pi y} \cos(2\pi x)\},
\end{align*}
$$

is bounded for $x = \frac{a}{2}, n = 0, 1, 2 \ldots$. This implies that $G(z) = (z + 1)^{-1} \exp \{ie^{2\pi iy}\} \in L^2(\mathbb{R}, d\omega).$ However, $G \not\in H^2(S_b)$ if $\cos(2\pi x) > 0$ for some $x < b$.

Finally, recall the asymptotic of the Gamma function (see [10] e.g.), valid for $|\arg z| \leq \pi - \delta,

(9)

$$
\Gamma(z) = \sqrt{2\pi} e^{(z-\frac{1}{2})\log z - z} \left[1 + O(1/|z|)\right].
$$

Then $\Gamma(\varepsilon_0 + \delta z)$ is regular in $\mathbb{R}$ and for $|z| \geq 1$, $z = x + iy,$

$$
|\Gamma(\varepsilon_0 + \delta z)| \leq C |z|^{-\frac{1}{2}} \exp \{\text{Re}[\varepsilon_0 + \delta z](\log(\varepsilon_0 + \delta z) - 1)\} \\
\leq C |z|^{\varepsilon_0 - \frac{1}{2}} \exp \{\delta x \log |z| - \delta |y| \arctan(|y|/x)\}.
$$

(10)

Let $2\tau/\pi < \delta < 1$ and take $\varepsilon_0 = 1$ (for simplicity). When $0 \leq x \leq b$ we have $\tau < \tau' < \delta \arctan(|y|/x)$ for $|y| \geq N$ sufficiently large. Then (10) gives

$$
|h(z)||\Gamma(1 + \delta z)| \leq e^{\tau|z|} e^{-\tau'|y|} e^{\delta x \log |z|} \leq C
$$

uniformly in the strip $S_b$. In order to bound the $M^2$-norm we observe that $|h(z)| \leq Ce^{\tau(x+|y|)}$ and using (10) we estimate

$$
\int_{-\infty}^{+\infty} e^{2\tau|y|} |\Gamma(1 + \delta \frac{n}{2} + i\delta y)|^2 dy \\
\leq \int_{-\infty}^{+\infty} e^{2\tau|y|} (n + |y|) \exp \{\delta n \log (\frac{n}{2} + |y|) - 2\delta |y| \arctan (2|y|/n)\} dy \\
= \left( \int_{|y| \leq \alpha n} + \int_{|y| > \alpha n} \right) (n + |y|) \exp \{2\tau|y| + \delta n \log (\frac{n}{2} + |y|) - 2\delta |y| \arctan (2|y|/n)\} dy \\
= I + II,
$$

with $\alpha > \frac{1}{2}$ to be fixed. Then it follows at once that

(11)

$$
I \leq Cn^2(1 + \alpha)^{\delta n} e^{2\tau\alpha n} e^{\delta n \log n} \leq Cp^n e^{\delta n \log n},
$$
for some \( p > 0 \). On the other hand, since \( \tau / \delta < \pi / 2 \) we can select \( \alpha \) large enough so that 
\[
\delta \arctan(2\alpha) > \tau.
\]
Then we have
\[
II \leq C 2^{\delta n} \int_{|y| > \alpha n} \big|y\big|^{1+\delta n} \exp \left\{ -2\big(\delta \arctan(2\alpha) - \tau\big) \big|y\big| \right\} dy
\]
\[
\leq C 2^{\delta n} \frac{\Gamma(2+\delta n)}{\left[2\big(\delta \arctan(2\alpha) - \tau\big)\right]^{2+\delta n}}
\]
(12)

for some \( q > 0 \) and where we have applied the simple estimate, valid for \( A,B,y_0 > 0, \)
\[
\int_{y_0}^{+\infty} e^{-Ay} y^B dy = \frac{1}{A^{B+1}} \int_{A y_0}^{+\infty} e^{-t} t^B dt \leq \frac{\Gamma(B+1)}{A^{B+1}}.
\]

Putting (11) and (12) together we then obtain
\[
\|h \Gamma(1+\delta \cdot)\|_{\mathcal{M}^2}^2 \leq C \sum_{n=0}^{+\infty} \frac{2^n}{n!} (p^n + q^n) e^{\delta n \log n}
\]
\[
\leq C \sum_{n=0}^{+\infty} Q^n e^{(\delta-1)n \log n}
\]
\[
< \infty.
\]

This proves (iii).

\[ \square \]

**Proof of Theorem 2** Since \( f \in H^2(S_{\mathbb{R}}) \) for every \( n \), using (5) we have,
\[
\|f\|_{\mathcal{M}^2}^2 = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \|f(\frac{\pi}{2} + i \cdot)\|_{L^2(\mathbb{R})}^2
\]
\[
= \sum_{n=0}^{+\infty} \frac{2^n}{n!} \| \mathcal{F}(f(\frac{\pi}{2} + i \cdot))\|_{L^2(\mathbb{R})}^2
\]
\[
= \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} e^{n \xi} |\mathcal{F}f_0(\xi)|^2 d\xi
\]
(13)

Conversely, let \( \psi \in L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi) \) and \( f \) be defined by (4). Notice that the integral converges absolutely for \( z \in \mathcal{R} \) since
\[
\int_{-\infty}^{+\infty} |\psi(\xi) e^{2\varepsilon \xi}| d\xi \leq \|\psi\|_{L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi)} \left( \int_{-\infty}^{+\infty} e^{2\varepsilon \xi} e^{-2\varepsilon \xi} d\xi \right)^{1/2} < \infty.
\]
Then $f$ is well defined and holomorphic in $\mathcal{R}$. Notice also that $e^{\frac{\pi}{2} \xi} \psi \in L^2(\mathbb{R})$ for $n = 0, 1, 2, \ldots$ so that by (iii) in Theorem [1.1] $f \in H^2(S^\mathbb{R})$ for every $n$, and $\mathcal{F}f_0 = \psi$. Now the same argument as in [3] gives [3]. □

As a consequence of the Paley–Wiener-type theorem we determine the reproducing kernel of $M^2$.

**Proof of Theorem 3.** Let $K_z \in M^2$ be such that $(f, K_z)_{M^2} = f(z)$ for every $z \in \mathcal{R}$ and every $f \in M^2$. Using (8) again we have,

$$f(z) = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \langle f\left(\frac{n}{2} + i \cdot \right), K_z\left(\frac{n}{2} + i \cdot \right) \rangle_{L^2}$$

$$= \sum_{n=0}^{+\infty} \frac{2^n}{n!} \langle \mathcal{F}\left(f\left(\frac{n}{2} + i \cdot \right)\right), \mathcal{F}\left(K_z\left(\frac{n}{2} + i \cdot \right)\right) \rangle_{L^2}$$

$$= \sum_{n=0}^{+\infty} \frac{2^n}{n!} \langle e^{\frac{\pi}{2} \xi} \mathcal{F}f_0, e^{\frac{\pi}{2} \xi} \mathcal{F}K_z\left(\frac{n}{2} + i \cdot \right) \rangle_{L^2}$$

$$= \int_{-\infty}^{+\infty} \mathcal{F}f_0(\xi) \mathcal{F}K_z\left(\frac{n}{2} + i \cdot \right) e^{2\xi} d\xi,$$

where switching the integral with the sum is justified by Theorem 2.

On the other hand,

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\xi} \mathcal{F}f_0(\xi) d\xi,$$

so that

$$\mathcal{F}K_z(0)(\xi) = \frac{1}{\sqrt{2\pi}} e^{-2\xi} e^{\frac{\pi}{2} \xi},$$

and

$$K_z(w) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{w \xi} e^{-2\xi} e^{\frac{\pi}{2} \xi} d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{\xi(w+\bar{\xi})} e^{-2\xi} d\xi$$

$$= \frac{1}{2\pi} \int_{0}^{+\infty} t^{w+\bar{\xi}-1} e^{-t} dt$$

$$= \frac{1}{2\pi} \Gamma(w+\bar{\xi}) \frac{1}{2^{w+\bar{\xi}}}. \quad \square$$

2. Properties of Mellin–Bergman transforms

Next we study the Mellin–Bergman transform $M_{\Delta}$, as defined in [11], when acting on $A^2(\Delta)$. We set $\|f\|_{A^2(\Delta)}^2 = \frac{1}{\pi} \int_{\Delta} |f(\bar{\zeta})|^2 dA(\zeta)$. Observe that $M_{\Delta}f(z) = \langle f, \zeta^{w-1} \rangle_{A^2(\Delta)}$, so that $M_{\Delta}$ is linear and $M_{\Delta}f$ is holomorphic in $\mathcal{R}$.

We need the explicit expression of the inner product of the powers $\zeta^\alpha$ and $\zeta^\beta$ in $A^2(\Delta)$. The next result appears in [7].
Lemma 2.1. Let \( \text{Re} \alpha, \text{Re} \beta > -1 \). Then
\[
\frac{1}{\pi} \iint_{\Delta} \zeta^\alpha \overline{\zeta}^\beta \, dA(\zeta) = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 2)\Gamma(\beta + 2)}.
\]
In particular, \( \zeta^{\alpha-1} \in A^2(\Delta) \) if and only if \( \text{Re} z > 0 \) and in this case
\[
\|\zeta^{\alpha-1}\|^2_{A^2(\Delta)} = \|\zeta^{\beta-1}\|^2_{A^2(\Delta)} = \frac{\Gamma(2 \text{Re} z)}{\Gamma(z + 1)^2}.
\]

The lemma provides us with a few explicit examples.

Example 2.2. By taking \( \alpha = 0 \) and \( \beta = \pi - 1 \), we obtain that \( M_\Delta(1) = 1 \), and similarly, for \( k = 1, 2, \ldots \)
\[
M_\Delta(\zeta^k)(z) = \frac{\Gamma(z + k + 1)}{k! \Gamma(z + 1)} = \frac{1}{k!} (z + 1) \cdots (z + k);
\]
hence a polynomial of degree \( k \) in \( z \).

We now obtain the following property on the growth of functions in \( M_\Delta(A^2(\Delta)) \).

Proposition 2.3. For \( f \in A^2(\Delta) \), \( M_\Delta f \) is holomorphic of exponential type at most \( \pi/2 \) in \( \mathcal{R} \).

Proof. It is clear that \( M_\Delta f \) is holomorphic in \( \mathcal{R} \) since
\[
|\partial_z (f(\zeta)\overline{\zeta}^{\alpha-1})| \leq |f(\zeta)| |\log |\zeta|| |\overline{\zeta}^{\alpha-1}|
\leq |f(\zeta)| |\zeta|^{\text{Re} z - \epsilon - 1} e^{\text{Im} z \pi/2},
\]
which is absolutely integrable if \( \text{Re} z > \epsilon \).

Next,
\[
(14) \quad |M_\Delta f(z)| \leq \|f\|_{A^2(\Delta)} \|\zeta^{\alpha-1}\|_{A^2(\Delta)} \leq C \frac{\Gamma(2 \text{Re} z)^{1/2}}{|\Gamma(z + 1)|}.
\]
A straightforward application of the asymptotics of the Gamma function \( \varGamma \) shows that \( M_\Delta f \) is of exponential type at most \( \pi/2 \). We leave the details to the reader.

We also observe that there exist functions \( f \) such that \( M_\Delta f \) is of type \( \pi/2 - \epsilon \), for every \( \epsilon > 0 \). For, for \( \zeta_0 \in \Delta \), let \( B_{\zeta_0} \) be the reproducing kernel for \( A^2(\Delta) \) at \( z_0 \). Then,
\[
|M_\Delta(B_{\zeta_0})(z)| = |\zeta_0^{\alpha-1}| \geq e^{y \arg \zeta_0}.
\]
The conclusion follows by taking \( \zeta_0 \) with \( \arg \zeta_0 \geq \pi/2 - \epsilon \) and \( z \in \mathcal{R} \), \( z = \delta + iy \).

We now turn to proving Theorem 4, therefore establishing the properties the Mellin–Bergman transforms of \( A^2(\Delta) \)-functions.

We break Theorem 4 into two results, first of which is the following theorem.

As usual, we denote by \( A^2(\mathcal{R}) \) the (unweighted) Bergman space, then the Paley–Wiener theorem for \( A^2(\mathcal{R}) \) shows that the Fourier transform is a surjective isometry between \( A^2(\mathcal{R}) \) and \( L^2((-\infty, 0), d\xi/|\xi|) \).
**Theorem 2.4.** Let \( F \in A^2(\mathcal{R}) \). Then there exists \( \psi \in L^2\left((\infty, 0), d\xi/|\xi|\right) \) such that for \( z \in \mathcal{R} \)

\[
F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{z\xi} \psi(\xi) \, d\xi
\]

and

\[
\|F\|^2_{A^2(\mathcal{R})} = \frac{1}{2} \|\psi\|^2_{L^2((\infty, 0), d\xi/|\xi|)}.
\]

Conversely, if \( \psi \in L^2\left((\infty, 0), d\xi/|\xi|\right) \) and \( F \) is defined by (15) then \( F \in A^2(\mathcal{R}) \) and (16) holds.

A proof of this result can be found in [1] or [3], e.g. This result implies that if \( F \in A^2(\mathcal{R}) \), setting

\[
F_0 = \lim_{x \to 0^+} F^{-1}(e^{x(\cdot)} \psi)
\]

in the sense of tempered distributions, we obtain that \( F \) admits boundary values \( F_0 = F(0 + i \cdot) \) such that \( F F_0 = \psi \).

The next result can be obtained from the well-known property of the Gamma function for \( \Re \lambda > -1 \),

\[
\int_{0}^{+\infty} t^\lambda e^{-t} e^{ixt} \, dt = \frac{\Gamma(\lambda + 1)}{(1 - ix)^{\lambda+1}}.
\]

**Lemma 2.5.** Let \( \Re \lambda > 0 \) and \( h(w) = (1 + w)^{-\lambda-1} \), then \( h \in A^2(\mathcal{R}) \) and

\[
\mathcal{F}h_0(\xi) = \sqrt{2\pi} \frac{|\xi|^\lambda}{\Gamma(1 + \lambda)} e^{\xi} \chi_{\{|\xi|<0\}}.
\]

We recall that the (re-normalized) classical Mellin transform is

\[
M\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \varphi(t) t^{z-1} \, dt,
\]

where \( \varphi \) is a function defined on \((0, +\infty)\).

Next, we reduce the problem to characterize the space \( M\left(L^2\left((0, +\infty), \frac{2\xi}{x} \, d\xi\right)\right) \).

**Theorem 2.6.** There exists a surjective isometry

\[
T : A^2(\Delta) \to L^2\left((0, +\infty), \frac{2\xi}{x} \, d\xi\right)
\]

such that for \( g \in A^2(\Delta) \) and \( z \in \mathcal{R} \)

\[
M_{\Delta}g(z) = -\sqrt{2\pi} \left(\frac{2}{z+1}\right) M(Tg)(z).
\]

**Proof.** For \( w \in \mathcal{R} \) we let \( \phi(w) = 2(w + 1)^{-1} \). Then \( \phi : \mathcal{R} \to \Delta \) is a biholomorphic mapping and \( f \mapsto \frac{1}{\sqrt{\pi}} \phi'(f \circ \phi) =: \tilde{f} \) is a surjective isometry of \( A^2(\Delta) \) onto \( A^2(\mathcal{R}) \). Notice that

\[
(z^2-1)^{-1}(w) = -\frac{2}{\sqrt{\pi}} \left(\frac{1}{w+1}\right)^2.
\]
Then, if \( f \in A^2(\Delta) \), using Lemma 2.5 (with \( z = \lambda \in \mathbb{R} \)) we have

\[
M_{\Delta}f(z) = \frac{1}{\pi} \int_{\Delta} f(\zeta) \zeta^{-1} dA(\zeta) = \langle f, \zeta^{-1} \rangle_{A^2(\Delta)}
\]

\[
= -\frac{2^z}{\sqrt{\pi}} \langle \hat{f}, (w + 1)^{-\zeta^{-1}} \rangle_{A^2(\mathcal{R})}
\]

\[
= -\sqrt{\frac{1}{2 \Gamma(1 + z)}} \mathcal{F}\mathcal{F}_0(\xi) e^{\xi z} \mathcal{F}(\xi e^{\xi \int_{0}^{z} e^{-2\xi} \xi^{-1} d\xi})
\]

\[
= -\sqrt{\frac{1}{2 \Gamma(1 + z)}} M(e^{-t} \mathcal{F}\mathcal{F}_0(-t))(z);
\]

\( M \) being the Mellin transform. Setting for \( t > 0 \), \( Tf(t) = e^{-t} \mathcal{F}\mathcal{F}_0(-t) \) the desired conclusion follows at once. \( \square \)

3. Mapping properties of the Mellin transform

In order to complete the proof of Theorem 4 we need the following result.

**Theorem 3.1.** The mapping

\[
M : L^2((0, +\infty), \frac{e^{2\xi}}{\xi} d\xi) \to \mathcal{M}^2
\]

is a surjective isometry.

The mapping properties of \( M \) as operator between function spaces have been studied in [2]. We begin the proof of theorem with the following

**Lemma 3.2.** The mapping

\[
M : L^2((0, +\infty), \frac{e^{2\xi}}{\xi} d\xi) \to \mathcal{M}^2
\]

is a partial isometry.

**Proof.** We first show that if \( \varphi \in L^2((0, +\infty), \frac{e^{2\xi}}{\xi} d\xi) \) then \( M\varphi \) is well defined and holomorphic in \( \mathcal{R} \). Writing \( z = x + iy \) we have

\[
|M\varphi(z)| \leq \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |\varphi(\xi)| \xi^{x-1} d\xi
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \|\varphi\|_{L^2((0, +\infty), \frac{e^{2\xi}}{\xi} d\xi)} \left( \int_{0}^{+\infty} e^{-2\xi} \xi^{2x-1} d\xi \right)^{1/2}
\]

\[
= \frac{1}{\sqrt{2\pi}} \frac{\Gamma(2x)^{1/2}}{2^x} \|\varphi\|_{L^2((0, +\infty), \frac{e^{2\xi}}{\xi} d\xi)}.
\]
Moreover, therefore, the integral defining $M\varphi$ converges absolutely for every $z \in \mathcal{R}$, and a similar argument shows that $M\varphi$ is also holomorphic in $\mathcal{R}$. Notice that

$$M\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\varphi \circ \exp)(s) e^{zs} \, ds = \mathcal{F}^{-1}((\varphi \circ \exp)e^{x(\cdot)})(y).$$

Moreover,

$$\|M(x + iy)\|_{L^2(\mathbb{R})}^2 = \|\varphi\|_{L^2((0, +\infty), e^{2\xi - 1}d\xi)}^2 \leq C_x \|\varphi\|_{L^2((0, +\infty), \frac{e^{2\xi}}{\xi}d\xi)}^2 ,$$

uniformly in $x \in (0, b]$. Hence, $\varphi \in L^2((0, +\infty), \frac{e^{2\xi}}{\xi}d\xi)$ implies that $M\varphi \in H^2(S_b)$ for every $b > 0$.

Finally, let $\varphi, \psi \in L^2((0, +\infty), \frac{e^{2\xi}}{\xi}d\xi)$ and first assume that they have compact support. Then, we have

$$\langle M\varphi, M\psi \rangle_{L^2} = \frac{1}{2\pi} \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} M\varphi(n \frac{\pi}{2} + iy)\overline{M\psi(n \frac{\pi}{2} + iy)} \, dy$$

$$= \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} \mathcal{F}^{-1}((\varphi \circ \exp)e^{\frac{\pi}{2}y})(y)\overline{\mathcal{F}^{-1}((\psi \circ \exp)e^{\frac{\pi}{2}y})(y)} \, dy$$

$$= \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} \varphi(e^y)\overline{\psi(e^y)} e^{\frac{\pi}{2}y} \, dy$$

$$= \int_{-\infty}^{+\infty} \varphi(t)\overline{\psi(t)} e^{\frac{\pi}{2}y} \, dy$$

$$= \int_{0}^{+\infty} \varphi(t)\overline{\psi(t)} e^{\frac{\pi}{2}y} \, dy.$$

Therefore, $M : L^2((0, +\infty), \frac{e^{2\xi}}{\xi}d\xi) \to M^2$ is a partial isometry, i.e.

$$\langle M\varphi, M\psi \rangle_{M^2} = \langle \varphi, \psi \rangle_{L^2((0, +\infty), \frac{e^{2\xi}}{\xi}d\xi)}$$

for all $\varphi, \psi \in L^2((0, +\infty), \frac{e^{2\xi}}{\xi}d\xi).$ \hfill $\square$

To order to prove Theorem 3.1 we need some density results in $M^2$, that are consequences of the Paley–Wiener-type Theorem 2.

For $1 \leq p < \infty$ we denote by $M^p(\mathcal{E})$ the subspace of $M^p$ of functions that are holomorphic for $\text{Re} \ z > -\varepsilon$ and that are in $H^p(S(-\varepsilon, b))$ for every $b > 0$.

Lemma 3.3. Let $\varepsilon' > 0$ and $\psi \in L^2(\mathbb{R}, e^{2\xi \xi}e^{2\varepsilon \xi}d\xi) \cap L^2(\mathbb{R}, e^{2\varepsilon \xi}d\xi)$. Then $\psi \in L^2(\mathbb{R}, e^{2\varepsilon \xi}e^{2\varepsilon \xi}d\xi)$ for $0 < \varepsilon \leq \varepsilon'$ and let $f$ be defined by (1). If $f(z) = f(z + \varepsilon)$, then $f_\varepsilon \in M^2(\mathcal{E})$ and $f_\varepsilon \to f$ in $M^2$ as $\varepsilon \to 0^+$. Hence, $\bigcap_{\varepsilon > 0} M^2(\mathcal{E})$ is dense in $M^2$. 
Proof. It is clear that $\psi \in L^2(\mathbb{R}, e^{2\varepsilon \xi} e^{2\varepsilon \xi} d\xi) \cap L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi)$ for $0 < \varepsilon < \varepsilon'$. Notice that

$$f_{\varepsilon}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\xi) e^{(z+\varepsilon)\xi} d\xi,$$

it is holomorphic in $\text{Re} \, z > -\varepsilon$, and since $e^{\varepsilon(\cdot)} \psi \in L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi)$, $f_{\varepsilon} \in \mathcal{M}^2$. Moreover, since $f \in H^2(S_b)$ for every $b > 0$, $f_{\varepsilon} \in H^2(S_{(-\varepsilon, b)})$ for every $b > 0$. Hence, $f_{\varepsilon} \in \mathcal{M}^2_{(\varepsilon)}$.

Next, since $F(f(x + \varepsilon + i\cdot))(\xi) = \psi(\xi)e^{(x+\varepsilon)\xi}$, we have

$$\|f - f_{\varepsilon}\|^2_{\mathcal{M}^2} = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \left| \mathcal{F}(f(\frac{x}{2} + i\cdot)) - \mathcal{F}(f(\frac{x}{2} + \varepsilon + i\cdot)) \right|^2_{L^2(\mathbb{R})} = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} e^{n \xi} |(1 - e^{\varepsilon \xi})\psi(\xi)|^2 d\xi = \int_{-\infty}^{+\infty} |(1 - e^{\varepsilon \xi})\psi(\xi)|^2 e^{2\varepsilon \xi} d\xi \to 0,$$

as $\varepsilon \to 0$. Since $L^2(\mathbb{R}, e^{2\varepsilon \xi} e^{2\varepsilon \xi} d\xi) \cap L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi)$ is dense in $L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi)$ the conclusion follows. \qed

**Proposition 3.4.** The subspace $\bigcap_{\varepsilon > 0} \mathcal{M}^2_{(\varepsilon)} \cap \mathcal{M}^1_{(\varepsilon)}$ is dense in $\mathcal{M}^2$.

**Proof.** Let $\psi \in C^\infty_0(\mathbb{R})$ and $f$ be defined by \[\mathbb{M}.\] By the previous result $f \in \mathcal{M}^2_{(\varepsilon)}$. Now, denoting by $\|\psi\|_{W^s(\mathbb{R})}$ the standard Sobolev norm,

$$\|f\|_{L^1(\mathbb{R}, d\omega)} \leq \|(1 + y^2)^{-1/2}\|_{L^2(\mathbb{R}, d\omega)} \|f\|_{L^2(\mathbb{R}, d\omega)} \leq C \left( \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} (1 + y^2)^{-1} (e^{\frac{x}{2} \xi(\psi)}(y))^2 dy \right)^{1/2} \leq C \left( \sum_{n=0}^{+\infty} \frac{2^n}{n!} \|e^{\frac{x}{2} \xi(\psi)}\|_{W^2(\mathbb{R})}^2 \right)^{1/2} \leq C \left( \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} e^{n \xi} (n^4 |\psi(\xi)|^2 + n^2 |\psi'(\xi)|^2 + |\psi''(\xi)|^2) d\xi \right)^{1/2} \leq C \left( \int_{-\infty}^{+\infty} (|\psi(\xi)|^2 + n^2 |\psi'(\xi)|^2 + |\psi''(\xi)|^2) P(d\xi/\xi) (e^{2\varepsilon \xi}) d\xi \right)^{1/2}$$

where $P$ is the polynomial $t^4 + t^2 + 1$. The right hand side above is finite since $\psi \in C^\infty_0$, so that $f \in L^1(\mathbb{R}, d\omega)$. Arguing as before, we see that for every $x \geq -\varepsilon$

$$\|f(x + i\cdot)\|_{L^1(\mathbb{R})} \leq C \|e^{x(\cdot)} \psi\|_{W^2(\mathbb{R})}^2,$$

hence $f \in H^1(S_{(-\varepsilon, b)})$ for every $b > 0$, that is, $f \in \mathcal{M}^2_{(\varepsilon)} \cap \mathcal{M}^1_{(\varepsilon)}$. Since $C^\infty_0$ is dense in $L^2(\mathbb{R}, e^{2\varepsilon \xi} d\xi)$ the conclusion follows. \qed
Proof of Theorem 3.1. We only need to show that $M$ is onto.

It is a well-known fact that if $g \in L^2(\{c\} + i\mathbb{R}) \cap L^1(\{c\} + i\mathbb{R})$ for all $c \in (a, b)$ and $g(x + iy) \to 0$ as $|y| \to +\infty$ uniformly in $x \in (a, b)$, then for $\xi > 0$

\begin{equation}
M_c^{-1}g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(c + it)\xi^{-c-it} \, dt
\end{equation}

is independent of $c \in (a, b)$ and satisfies $MM_c^{-1}g = g$.

For $\varepsilon > 0$ fixed, for $f \in M^2(\varepsilon) \cap M^1(\varepsilon)$, $f(x + iy) \to 0$ as $|y| \to +\infty$ uniformly in $x \in (a, b)$. Therefore, $M_{\alpha}^{-1}f$ is independent of $n = 0, 1, 2, \ldots$ and satisfies $MM_{\alpha}^{-1}f = f$. We set

\begin{equation}
M^{-1}f = M_{\alpha}^{-1}f.
\end{equation}

Having constructed an inverse of $M$ on a dense subspace of $M^2$, if we show that on this subspace

\begin{equation}
\|M^{-1}f\|_{L^2((0, +\infty), \frac{e^{2\xi}}{\xi} \, d\xi)} = \|f\|_{M^2},
\end{equation}

the conclusion will follow.

For $f \in L^2(\{c\} + i\mathbb{R})$ define $\varphi_c(\xi) = \xi^{-c}f(c - i \log \xi)$. Then, setting $L^2_c = L^2((0, +\infty), t^{2c} \, dt)$, we have

\begin{equation}
\|\varphi_c\|_{L^2_c} = \|f\|_{L^2(\{c\} + i\mathbb{R})},
\end{equation}

since

\begin{equation}
\|\varphi_c\|^2_{L^2_c} = \int_0^{+\infty} |f(c - i \log \xi)|^2 \frac{d\xi}{\xi} = \int_{-\infty}^{+\infty} |f(c - it)|^2 \, dt = \|f\|^2_{L^2((0, +\infty), t^{2c} \, dt)}.
\end{equation}

We claim that for such $f$ and $\xi > 0$ we have

\begin{equation}
\xi^cM_c^{-1}f(\xi) = M\varphi_c(c + i \log \xi).
\end{equation}

For,

\begin{align*}
M_c^{-1}f(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(c + it)\xi^{-c-it} \, dt \\
&= \frac{\xi^{-c}}{\sqrt{2\pi}} \int_0^{+\infty} f(c - i \log s)\xi^{i \log s} \frac{ds}{s} \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} s^c \varphi_c(s)\xi^{i \log s} \frac{ds}{s} \\
&= \xi^{-c}M\varphi_c(c + i \log \xi),
\end{align*}

as we claimed.

Lemma 2.3 in [2] shows that

$M : L^2_c \to L^2(\{c\} + i\mathbb{R})$.
is an isometry. Finally, using (23) and (24) we have

\[ \|Mf\|^2_{L^2((0, +\infty), e^{2\xi} d\xi)} = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_0^{+\infty} |M^{-1}f(\xi)|^2 \xi^n d\xi = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_0^{+\infty} |M^{-1}f(\xi)|^2 \xi^n \frac{d\xi}{\xi} \]

\[ = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_0^{+\infty} |M \varphi_{1/2}(n^2 + i \log \xi)|^2 \frac{d\xi}{\xi} = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \int_{-\infty}^{+\infty} |f(n^2 + iy)|^2 dy \]

This completes the proof. \( \Box \)

We conclude this section discussing some property of the space \( \mathcal{H} := \mathcal{M}^2 \).

**Proof of Theorem 4.** We have already proved that \( M_\Delta : A^2(\Delta) \to \mathcal{H} \) is a surjective isometry. Proposition 2.3 shows that elements of \( \mathcal{H} \) are holomorphic functions in \( \mathbb{R} \) of exponential type at most \( \pi/2 \), while Example 2.2 easily implies that the polynomials are dense in \( \mathcal{H} \).

Clearly, the reproducing kernel of \( \mathcal{H} \) is given by

\[ H(z, w) = \frac{2^z}{\Gamma(1+z)} \left( \frac{1}{2\pi} \frac{\Gamma(z+w)}{2^{z+w}} \right) = \frac{\Gamma(z+w)}{2\pi \Gamma(1+z) \Gamma(1+w)}. \]

**Remark 3.5.** It is not difficult to show (see [14]) that \( \mathcal{H} \) contains functions that are of exponential type \( \pi/2 - \varepsilon \) for any \( \varepsilon > 0 \), and whose restriction to the imaginary axis is again of exponential type. This shows that no Paley–Wiener type theorem can hold for \( \mathcal{H} \). Moreover, the measure on \( \mathcal{R} \) that appears in the norm of \( \mathcal{H} \)

\[ d\mu(x + iy) := \sum_{n=0}^{+\infty} \delta_{\frac{n}{2}}(x) \otimes \frac{|\Gamma(n^2 + iy)|^2}{\Gamma(n+1)} dy \]

is not translation invariant, in contrast to the invariance of \( \omega \).

The results of the next section concerning the zero-sets indicate that it is easier to exploit the properties of \( \mathcal{M}^2 \) than the ones of \( \mathcal{H} \).

### 4. Zero-sets

Denote by \( \mathcal{E}_\tau(\mathcal{K}) \) and \( \mathcal{E}_{<\tau}(\mathcal{K}) \) respectively, the space of holomorphic functions on \( \mathcal{K} \) that are of exponential type \( \tau \) and of exponential type less than \( \tau \), respectively. Let \( \mathcal{Z}(\mathcal{K}) \) denote the collection of zero-sets for the space \( \mathcal{K} \).

**Proposition 4.1.** We have the inclusions

\[ \mathcal{Z}(\mathcal{E}_{<\frac{\pi}{2}}(\mathcal{K})) \subseteq \mathcal{Z}(\mathcal{M}^2(\mathcal{K})) \subseteq \mathcal{Z}(\mathcal{E}_{\frac{\pi}{2}}(\mathcal{K})). \]
Proof. From Proposition 2.3 and Theorem 4 we have
\[ \mathcal{M}^2(\mathcal{R}) \cap \text{Hol}(\mathcal{R}) \subseteq \Gamma(1 + \cdot)E_4(\mathcal{R}). \]
The conclusion now follows from the above inclusion and Proposition 1.2 (iii). \(\square\)

In order to prove Theorem 6 we need a couple of preliminary results that may be of independent interest. Recall that the elementary Weierstrass factor \(E(z,p)\) equals \(1 - z\) when \(p = 0\), while \(E(z,p) = (1 - z)e^{z + \cdots + z^p/p}\), for a positive integer \(p\).

**Proposition 4.2.** Let \(\{z_j\}_{j=1,2,\ldots} \subseteq \mathcal{R}\), with \(|z_j| \to +\infty\), \(|z_j| > 1\) having exponent of convergence \(1\) and assume \(d^+ < \infty\). Set \(z_j = -z_j\) and consider the sequence \(\{z_j\}_{j \neq 0}\). Then the infinite product \(\prod_{j \neq 0} E(z/z_j, 1)\) converges to an entire function \(\Pi(z)\) of exponential type at most \(\pi d^+\).

Proof. The sequence \(\{z_j\}_{j \neq 0}\) has exponent of convergence \(1\) so that the product \(\prod_{j \neq 0} E(z/z_j, 1)\) converges to an entire function \(\Pi(z)\). Thus, we only need to prove the statement about the exponential type.

Since \(E(z/z_j, 1)E(-z/z_j, 1) = E(z^2/z_j^2, 0)\) we have
\[
\log |\prod_{j \neq 0} E(z/z_j, 1)| = \log |\prod_{j=1}^{+\infty} E(z^2/z_j^2, 0)|
\leq \sum_{j=1}^{+\infty} \log |E(z^2/z_j^2, 0)|
\leq \sum_{j=1}^{+\infty} \log (1 + |z^2/z_j^2|).
\]
Setting \(|z| = r\) we then have
\[
\log |\Pi(z)| \leq \sum_{j=1}^{+\infty} \log (1 + r^2/|z_j|^2)
\leq \int_1^{+\infty} \log (1 + r^2/t^2) \, dn(t)
\leq \log (1 + r^2/t^2)n(t) \bigg|_1^{+\infty} + 2r^2 \int_1^{+\infty} \frac{n(t)}{t^3(1 + r^2/t^2)} \, dt.
\]
Notice that
\[
\log (1 + r^2/t^2)n(t) \bigg|_1^{+\infty} = \lim_{R \to +\infty} \log (1 + r^2/R^2)n(R) \leq C \lim_{R \to +\infty} (r^2/R^2)R = 0.
\]
Therefore, given \( \varepsilon > 0 \) there exists \( A > 0 \) large enough so that

\[
\log |\Pi(z)| \leq 2r^2 \int_{-\infty}^{\infty} \frac{n(t)}{t^2 + r^2} \, dt = 2r^2 \left( \int_{1}^{A} + \int_{A}^{+\infty} \right) \frac{n(t)}{t^2 + r^2} \, dt
\]

\[
\leq 2r^2 \left( C \int_{1}^{A} \frac{1}{t^2 + r^2} \, dt + (d^+ + \varepsilon) \int_{A}^{+\infty} \frac{1}{t^2 + r^2} \, dt \right)
\]

\[
\leq 2r^2 \left( C \frac{A}{1 + r^2} + \frac{(d^+ + \varepsilon)}{r} \int_{0}^{+\infty} \frac{1}{1 + s^2} \, ds \right)
\]

\[
\leq CA + \pi(d^+ + \varepsilon)r,
\]

as \( r \to +\infty \), and \( \Pi(z) \) is of exponential type at most \( \pi d^+ \).

In order to obtain a necessary condition for a sequence to be a zero-set for \( M^2 \), we recall the classical Carleman formula for the right half-plane.

**Theorem 4.3. (Carleman)** Let \( f \in \text{Hol}(\overline{\mathcal{R}}) \) and let \( \{z_j\} \) its zero-set, with \( r_j \geq 1 \), where \( z_j = r_j e^{i\theta_j} \). Then, for \( R \geq 1 \) we have

\[
\sum_{r_j \leq R} \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j
\]

\[
R = \frac{1}{2\pi} \int_{1}^{R} \left( \frac{1}{y^2} - \frac{1}{R^2} \right) \log |f(iy)f(-iy)| \, dy + \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{i\theta})| \cos \theta \, d\theta + A(R),
\]

where \( A(R) \) is a bounded function of \( R \).

**Proposition 4.4.** Let \( f \in \text{Hol}(\overline{\mathcal{R}}) \) be such that

\[
\text{(i) } \sup_{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}} |f(Re^{i\theta})| \leq c_1 e^{BR\log R} \text{ with } B > 0 \text{ and some } c_1 > 0;
\]

\[
\text{(ii) } |f(iy)| \leq c_2 e^{A|y|} \text{ with } A > 0, \text{ and some } c_2 > 0.
\]

If \( \{z_j\} \) are the zeros of \( f \) with \( |z_j| \geq 1 \), then

\[
\sup_{R > 0} \frac{1}{\log R} \sum_{r_j \leq R} \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j \leq \frac{1}{\pi} (A + 2B).
\]

**Proof.** Denote by \( I(R) + J(R) + A(R) \) the right hand side in (25). Consider \( I(R) \) and set

\[
I_\pm(R) = \frac{1}{2\pi} \int_{1}^{R} \left( \frac{1}{y^2} - \frac{1}{R^2} \right) \log^\pm |f(iy)f(-iy)| \, dy.
\]

Clearly we have \( I_-(R) \leq 0 \leq I_+(R) \). Moreover, since (ii) implies \( \log^+ |f(iy)f(-iy)| \leq 2A|y| + C \), it follows that

\[
0 \leq \frac{1}{\log R} \frac{1}{2\pi} \int_{1}^{R} \left( \frac{1}{y^2} - \frac{1}{R^2} \right) (Ay + C) \, dy,
\]

which tends to \( \frac{A}{\pi} \) as \( R \to +\infty \).

Next we consider \( J(R)/\log R \). We observe that since \( f \) vanishes of finite order in \( \mathcal{R} \) we have

\[
\left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(Re^{i\theta})| \cos \theta \, d\theta \right| \leq C.
\]
Therefore, using (i) we have
\[
\left| \frac{J(R)}{\log R} \right| \leq \frac{C}{\pi R \log R} + \frac{1}{\pi R \log R} \int_{-\frac{\pi}{2}}^{\pi} \log_+ |f(Re^{i\theta})| \cos \theta d\theta 
\]
\[
\leq \frac{C}{\pi R \log R} + \frac{2B}{\pi} ,
\]
which tends to \( \frac{2B}{\pi} \), as \( R \to +\infty \).

Now notice that the left-hand side of (25) is non-negative and increasing in \( R \). Therefore,
\[
0 \leq \lim\sup_{R \to +\infty} \frac{1}{\log R} \left( I_+(R) + I_-(R) + J(R) + A(R) \right) = \frac{A}{\pi} + \frac{2B}{\pi} + \lim\sup_{R \to +\infty} \frac{I_- (R)}{\log R} .
\]
Thus, \( I_- (R)/\log R \) must remain bounded from below it follows that
\[
(27) \quad \lim\sup_{R \to +\infty} \frac{1}{\log R} \sum_{r_j \leq R} \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j \leq \frac{1}{\pi} (A + 2B) . \quad \square
\]

**Proof of Theorem 6.** (i) Using Proposition 4.2 we can construct an entire function \( \Pi \) of exponential type \( \tau < \frac{\pi}{2} \), whose zeros in \( \mathcal{K} \) is exactly the sequence \( \{ z_j \} \). From Proposition 1.2 (iii), there exists \( \delta > 0 \) such that \( \Pi(z) \Gamma(1 + \delta z) \) is in \( M^2(\mathcal{K}) \cap \text{Hol}(\mathcal{K}) \).

(ii) From Theorem 4 it follows that any \( f \in M^2(\mathcal{R}) \cap \text{Hol}(\mathcal{K}) \) satisfies the hypotheses of Proposition 4.4, so that using the asymptotics for the Gamma function (9), conclusion (26) holds for \( f \) with \( A = \varepsilon \), for every \( \varepsilon > 0 \) and \( B = 1 \).

Observe that, for \( 0 \leq \delta < 1 \),
\[
\frac{1}{r_j} - \frac{r_j}{R^2} \geq \frac{\delta}{r_j} \quad \text{if and only if} \quad r_j \leq R \sqrt{1 - \delta} .
\]
Therefore,
\[
\sum_{r_j \leq R} \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j \geq \sum_{r_j \leq R \sqrt{1 - \delta}} \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j 
\]
\[
\geq \delta \sum_{r_j \leq R \sqrt{1 - \delta}} \frac{1}{r_j} \cos \theta_j ,
\]
so that,
\[
\lim\sup_{R \to +\infty} \frac{1}{\log R} \sum_{r_j \leq R} \left( \frac{1}{r_j} - \frac{r_j}{R^2} \right) \cos \theta_j \geq \delta \lim\sup_{R \to +\infty} \frac{1}{\log R} \sum_{r_j \leq R \sqrt{1 - \delta}} \frac{1}{r_j} \cos \theta_j 
\]
\[
= \delta \lim\sup_{R' \to +\infty} \frac{1}{\log R' - \log \sqrt{1 - \delta}} \sum_{r_j \leq R'} \frac{1}{r_j} \cos \theta_j 
\]
\[
= \delta \lim\sup_{R \to +\infty} \frac{1}{\log R} \sum_{r_j \leq R} \frac{1}{r_j} \cos \theta_j .
\]
The conclusion now follows. \( \square \)
Remark 4.5. Since the Hardy space $H^2(\mathcal{R})$ is contained in $\mathcal{M}^2(\mathcal{R})$, so that $\mathcal{Z}(H^2(\mathcal{R})) \subseteq \mathcal{Z}(\mathcal{M}^2(\mathcal{R}))$, it follows that if $\{z_j\}$ is such that $|z_j| \to +\infty$ and
\[ \sum_j \frac{\Re z_j}{1 + |z_j|^2} < \infty, \]
then $\{z_j\}$ is also a zero-set for $\mathcal{M}^2(\mathcal{R}) \cap \text{Hol}(\mathcal{R})$. We now show that:

(a) $\mathcal{Z}(H^2(\mathcal{R})) \subset \mathcal{Z}(\mathcal{M}^2(\mathcal{R}))$;

(b) There exist sequences in $\mathcal{Z}(\mathcal{M}^2(\mathcal{R}))$ (actually, in $\mathcal{Z}(H^2(\mathcal{R}))$) that do not satisfy condition (i) in Theorem 6 hence, this condition is not necessary.

(a) Any sequence $\{z_j\}$ satisfying condition (i) in Theorem 6 contained in a sector $|\arg z_j| \leq \vartheta < \frac{\pi}{2}$ and such that $\sum_j \frac{1}{|z_j|} = +\infty$, is in $\mathcal{Z}(\mathcal{M}^2(\mathcal{R}))$ but not in $\mathcal{Z}(H^2(\mathcal{R}))$. As special cases, we find all sequences of the form $z_j = a e^{i\vartheta}$, $j = 1, 2, \ldots$, with $a > 2$.

(b) Any sequence $\{z_j\}$ contained in a strip $S_0$ with exponent of convergence $\rho > 1$ and such that $\sum_j \Re(1/z_j) < \infty$, is in $\mathcal{Z}(H^2(\mathcal{R}))$ hence in $\mathcal{Z}(\mathcal{M}^2(\mathcal{R}))$ but does not satisfy condition (i) in Theorem 6. As special cases, we find all sequences of the form $z_j = a + i j^\alpha$, $j = 1, 2, \ldots$, with $\frac{1}{2} < \alpha < 1$.

5. Sets of uniqueness and the Müntz–Szász problem for the Bergman space

The Müntz–Szász problem for the Bergman space was formulated by S. Krantz, C. Stoppato and the first author, see [2]. In Theorem 3.1 of the same paper, using an ad hoc method, it is shown that if $\lambda_k = \varepsilon_0 + a_k + i b$, where $\varepsilon_0 > 0$, $b \in \mathcal{R}$ and $0 < a < 1$, the set $\{\lambda_k\}$ is a complete set in $A^2(\Delta)$.

We recall a classical result by Fuchs [1], concerning sets of uniqueness for functions that are of exponential type in a half-plane. It says that if $\lambda_k > 0$, $\lambda_{k+1} - \lambda_k > \delta > 0$ and the sequence $\{\lambda_k\}$ has lower density $d^- > \frac{1}{2}$, then $\{\lambda_k\}$ is a set of uniqueness for the functions that are of exponential type $\frac{1}{2}$ in $\mathcal{R}$. Since the sets of uniqueness satisfy the inverse inclusions of Proposition 4.1, it follows that the set $\{\lambda_k\}$ is a complete set in $A^2(\Delta)$.

In light of Proposition 4.1, it is clear that Theorem 3.1 in [2] follows from Fuchs’ result. Our Theorem 7 applied to a sequence $\{\lambda_k\}$ on the positive half-line gives that if it has lower density greater than $\frac{1}{2}$, regardless of being separated or not, then $\{\lambda_k\}$ is a set of uniqueness for $\mathcal{M}^2$. Thus, while it does not contain Fuchs’ result, it is of much more general nature, since it only assumes a lower bound of a quantity depending on the number of the $\{\lambda_k\}$ in the half-disks $\{|z| \leq R, \Re z > 0\}$ and not on any type of distribution of the $\lambda_k$’s.

Proof of Theorem 7 Let $\{z_j\} \subseteq \{|\Re z| \geq \varepsilon_0\}$ be such that (6) is violated and let $f \in \mathcal{M}^2$ vanish on $\{z_j\}$. If $f \in \text{Hol}(\mathcal{R})$ the result follows at once from Theorem 6 (ii).

For a generic $f \in \mathcal{M}^2$ and $0 < \varepsilon < \varepsilon_0$, we consider the points $w_j = z_j - \varepsilon$ and the function $f_\varepsilon(z) = f(z + \varepsilon)$. Then $f_\varepsilon \in \mathcal{M}^2 \cap \text{Hol}(\mathcal{R})$ and vanishes at the $w_j$’s. If we show that the sequence $\{w_j\}$ violates condition (6), it would follow that $f_\varepsilon$, hence $f$, is identically zero. Notice that $|w_j| = |z_j - \varepsilon| < |z_j|$ so that
\[ \Re \frac{1}{|w_j|} \geq \frac{\Re z_j - \varepsilon}{|z_j|^2} = \Re \left( \frac{1}{z_j} \right) - \frac{\varepsilon}{|z_j|^2}. \]
Therefore,
\[
\sum_{|z_j| \leq R} \Re \left( \frac{1}{z_j} \right) \leq \sum_{|z_j| \leq R} \Re \left( \frac{1}{w_j} \right) + \frac{\varepsilon}{|z_j|^2} \\
\leq \sum_{|w_j| \leq R} \Re \left( \frac{1}{w_j} \right) + \sum_{|z_j| \leq R} \frac{\varepsilon}{|z_j|^2} \\
\leq \sum_{|w_j| \leq R} \Re \left( \frac{1}{w_j} \right) + \frac{\varepsilon}{\varepsilon_0} \sum_{|z_j| \leq R} \Re \left( \frac{1}{z_j} \right).
\]

Hence,
\[
\left( 1 - \frac{\varepsilon}{\varepsilon_0} \right) \limsup_{R \to +\infty} \frac{1}{\log R} \sum_{|z_j| \leq R} \Re \left( \frac{1}{z_j} \right) \leq \liminf_{R \to +\infty} \frac{1}{\log R} \sum_{|w_j| \leq R} \Re \left( \frac{1}{w_j} \right),
\]
for every \( 0 < \varepsilon < \varepsilon_0 \). The conclusion now follows by taking \( \varepsilon > 0 \) sufficiently small. \( \square \)

6. PROOF OF THEOREMS 5 AND 8

We remark that in the next proof in particular we show how the measure \( \omega \) was determined.

Proof of Theorem 5. By Theorems 2.6 and 3.1 it suffices to show that, if the Mellin transform \( M : L^2(\mathbb{R}, \frac{e^{2\xi}}{2} d\xi) \to L^2(\mathbb{R}, d\mu) \) is an isometry and \( \mu \) is translation invariant, then \( \mu = \omega \).

Assume then that \( d\mu = d\nu(x) \otimes dy \). Let \( \eta_{1,\varepsilon}, \eta_{2,\varepsilon'} \) be smooth cut-off functions with support in \([\varepsilon, 1/\varepsilon]\) and \([-1/\varepsilon', 1/\varepsilon']\), respectively, and identically 1 in \([2\varepsilon, 1/(2\varepsilon)]\) and \([-1/(2\varepsilon'), 1/(2\varepsilon')\] \), respectively. Then
\[
\eta_{1,\varepsilon}(x) \eta_{2,\varepsilon'}(y) \, d\nu(x) \otimes dy \to d\mu(x, y)
\]
as Borel measures, as \( \varepsilon, \varepsilon' \to 0^+ \).

Notice that for \( \varphi \in \mathcal{C}_0^\infty(0, +\infty) \), \( M \varphi = \mathcal{F}^{-1}(\varphi \circ \exp) \) is an entire function. Then, for any \( \varphi, \psi \in \mathcal{C}_0^\infty(0, +\infty) \) we have
\[
\langle \varphi, \psi \rangle_{L^2((0, +\infty), \frac{e^{2\xi}}{2} d\xi)}
= \int_{\mathbb{R}} \int_{\mathbb{R}} M \varphi(x + iy) \overline{M \psi(x + iy)} \, d\nu(x) \, dy \\
= \lim_{\varepsilon, \varepsilon' \to 0^+} \int_{\mathbb{R}} \int_{\mathbb{R}} M \varphi(x + iy) \overline{M \psi(x + iy)} \eta_{1,\varepsilon}(x) \, d\nu(x) \, \eta_{2,\varepsilon'}(y) \, dy \\
= \frac{1}{2\pi} \lim_{\varepsilon, \varepsilon' \to 0^+} \int_0^{+\infty} \int_0^{+\infty} \varphi(s) \psi(t) \left( \int_{\mathbb{R}} \int_{\mathbb{R}} e^{s iy t - iy} \eta_{1,\varepsilon}(x) \, d\nu(x) \, \eta_{2,\varepsilon'}(y) \, dy \right) \frac{ds \, dt}{s - t},
\]
where we can interchange the integration order since the integrals converge absolutely. The inner integral in the right hand side above equals, using Fubini’s theorem again,
\[
\frac{1}{2\pi} \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{s iy t - iy} \eta_{2,\varepsilon'}(y) s^x t^y \eta_{1,\varepsilon}(x) \, dy \, d\nu(x) \\
= \frac{1}{\sqrt{2\pi}} \mathcal{F}(\eta_{2,\varepsilon'})(\log(s/t)) \int_0^{+\infty} e^{x \log(st)} \eta_{1,\varepsilon}(x) \, d\nu(x) \\
= \mathcal{F}(\eta_{2,\varepsilon'})(\log(s/t)) \mathcal{F}(\eta_{1,\varepsilon})(i \log(st)),
\]
observing that $\mathcal{F}(\eta, \nu)\nu$ is entire. Plugging this equality into the right hand side of (28) we find that

$$
\langle \varphi, \psi \rangle_{L^2((0, +\infty), \frac{2\pi}{\xi} d\xi)} = \lim_{\varepsilon, \varepsilon' \to 0^+} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(s)\overline{\psi(t)}\mathcal{F}(\eta_2, \varepsilon')(i\log(s/t))\mathcal{F}(\eta_1, \varepsilon\nu)(i\log(st)) \frac{ds dt}{s t}
$$

$$
= \lim_{\varepsilon, \varepsilon' \to 0^+} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(e^y)\overline{\psi(e^v)}\mathcal{F}(\eta_2, \varepsilon')(y - v)\mathcal{F}(\eta_1, \varepsilon\nu)(i(y + v)) dy dv
$$

$$
= \lim_{\varepsilon \to 0^+} \sqrt{2\pi} \int_{-\infty}^{+\infty} \varphi(e^y)\overline{\psi(e^v)}\mathcal{F}(\eta_1, \varepsilon\nu)(2iy) dy
$$

$$
= \lim_{\varepsilon \to 0^+} \sqrt{2\pi} \int_{0}^{+\infty} \varphi(\xi)\overline{\psi(\xi)}\mathcal{F}(\eta_1, \varepsilon\nu)(2i\log \xi) \frac{d\xi}{\xi}.
$$

(29)

Hence, if such a measure $\nu$ exists, for $\xi > 0$ it must satisfy the identity

$$
e^{2\xi} = \sqrt{2\pi} \lim_{\varepsilon \to 0^+} \mathcal{F}(\eta_1, \varepsilon\nu)(2i\log \xi)
$$

$$
= \lim_{\varepsilon \to 0^+} \int_{0}^{+\infty} e^{2x\log \xi} \eta_1, \varepsilon(x) d\nu(x)
$$

$$
= \int_{0}^{+\infty} e^{2x\log \xi} d\nu(x)
$$

(30)

by the monotone convergence theorem. This in particular implies that:

- $\nu(\{0\}) = 1$ (by letting $\xi \to 0^+$) and $\nu([0, +\infty)) = e^2$;
- $x \mapsto \xi^{2x} \in L^1(d\nu)$, for all $\xi > 0$;
- for all $s \in \mathbb{R}$,

$$
\exp\{2e^{\xi} \} = \int_{0}^{+\infty} e^{xs} d\nu(x).
$$

Hence, $\mathcal{F}\nu$ can be extended to the entire function $\mathcal{F}\nu(t + is) = \frac{1}{\sqrt{2\pi}} \exp\{2e^{-\frac{i}{2} t + i\frac{s}{2}}\}$, so that

$$
\nu = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left\{ \exp\{2e^{-\frac{i}{2} (\cdot)}\} \right\}.
$$

(31)

Thus, if $\nu$ exists, it is unique and it is given by (31).

Observe that the function $\exp\{2e^{-\frac{i}{2} x}\}$ is bounded on $\mathbb{R}$, so that its (inverse) Fourier transform is well defined as a tempered distribution. Moreover, it is the restriction to the real line of the entire function

$$
G(z) = \exp\{2e^{-\frac{i}{2} z}\} = \sum_{n=0}^{+\infty} \frac{2^n}{n!} e^{-i\frac{n}{2} z},
$$

and by [11] Theorem 2 we know that $\mathcal{F}^{-1}G$ is a finite Borel measure with support in $[0, +\infty)$, that in fact we can compute explicitly. The series $\sum_{n=0}^{+\infty} \frac{2^n}{n!} e^{-i\frac{n}{2} x}$ converges uniformly on compact subsets of the line to the bounded function $\exp\{2e^{-\frac{i}{2} x}\}$, hence in the sense of tempered
Proof of Theorem 8. For $1 < p < \infty$, the dual of $\mathcal{M}^p$ with respect to the $L^2(\mathcal{R}, d\omega)$-inner product is $\mathcal{M}^{p'}$, with $1/p + 1/p' = 1$. If the projection $P : L^p(\mathcal{R}, d\omega) \rightarrow \mathcal{M}^p$ were bounded, $K_w \in \mathcal{M}^{p'}$, for any $w \in \mathcal{R}$. Thus, since $P$ is self-adjoint, it suffices to show that $K_w \not\in L^p(\mathcal{R}, d\omega)$ for any $p > 2$.

This is a simple application of the asymptotics of the Gamma function. For $w = u + iv \in \mathcal{R}$ fixed

$$
\|K_w\|_{\mathcal{M}^p}^p = \frac{1}{(2\pi)^p} \left\| \frac{\Gamma(\cdot + \overline{w})}{2^{(\cdot + \overline{w})}} \right\|_{\mathcal{M}^p}^p
= \frac{1}{(2\pi)^p} \sum_{n=0}^{+\infty} \frac{2^n(1-\frac{u^2}{2})}{n!} \int_{-\infty}^{+\infty} |\Gamma(\frac{x}{2} + u + iy)|^p dy.
$$

Now, using (9) we see that there exists an absolute constant $C > 0$ such that

$$
|\Gamma(\frac{x}{2} + u + iy)|^p \geq C \exp \left\{ p \left( \frac{n-1}{2} + u \right) \log \left( \left( \frac{n}{2} + u \right)^2 + y^2 \right)^{1/2} - \frac{n}{2} - u - |y| \arctan \left( \frac{|y|}{\frac{n}{2} + u} \right) \right\}
\geq C e^{-p\mu} \exp \left\{ p \left( \frac{n-1}{2} + u \right) \log \left( \frac{n}{2} + u \right) - \frac{n}{2} - \frac{1}{\frac{n}{2}} |y| \right\}
\geq C e^{-p\mu} e^{-p\frac{1}{2}|y|} \exp \left\{ \frac{p(n-1)}{2} \log \left( \frac{n}{2} \right) - \frac{pn}{2} \right\}.
$$

Therefore,

$$
\|K_w\|_{\mathcal{M}^p}^p \geq C_u \sum_{n=0}^{+\infty} \frac{2^n(1-\frac{u^2}{2})}{n!} \exp \left\{ \frac{p(n-1)}{2} \log \left( \frac{n}{2} \right) - \frac{pn}{2} \right\}
\geq C_u \sum_{n=0}^{+\infty} \frac{2^n(1-\frac{u^2}{2}) \log 2}{n!} e^{\frac{p}{2} n \log n},
$$

which clearly diverges when $p > 2$. □

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