Distributed Stochastic Variance Reduced Gradient Methods

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Abstract

We study the problem of minimizing the average of $N$ convex functions using $m$ machines with each machine storing locally $n = \frac{N}{m}$ functions. We design a distributed stochastic variance reduced gradient (DSVRG) algorithm which, when the condition number $\kappa$ of the problem is less than $n^{0.9}$ and when the data is assumed to be randomly partitioned onto the machines, simultaneously achieves the optimal parallel runtime, communication and rounds of communication among all first order methods up to constant factors. DSVRG and its extension DPPASVRG also outperform existing distributed algorithms in terms of the rounds of communication as long as $\kappa$ is not too large compared to $n$. For the regime when $n$ is relatively small, we propose a distributed accelerated SVRG algorithm which further reduces the parallel runtime.

1 Introduction

In this paper, we focus on the distributed optimization problem of minimizing a sum of $N$ convex functions in $d$ variables, i.e.,

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{N} (f_1(x) + \cdots + f_N(x))$$

(1)

using $m$ machines with $n = \frac{N}{m}$ functions stored on each machine\textsuperscript{1}. We assume that each of $f_i$ is $L$-smooth (so is $f$) and $f$ is $\mu$-strongly convex, and we define $\kappa = L/\mu$ to be the condition number of $f$.

In the design of distributed algorithms, one often needs to consider more types of resources than in the standard sequential scenario. Apriori, one is only interested in the runtime of algorithm – the time differential between the start and end times of the job. However, it is often not clear how to theoretically count the runtime to reflect the true runtime in practice – unlike sequential algorithms, communications between machines also takes significant time. Moreover, the latency in initiating communication is also non-negligible, and often the bottleneck of the whole system.

Given this state of affairs, we study the distributed optimization problem with three main resources in mind, the parallel runtime, the amount of total communication, and the number of rounds of communication. To facilitate the theoretical study, we use the following simplified model for distributed computation with messages passing that has been used in the literature \cite{1, 2}: Each machine has the same computation speed and we define the parallel runtime as the largest running time of machines, while counting communication as using zero time. The amount of communication will be counted in bits. Moreover, we assume the

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\textsuperscript{1}For simplicity, we assume $N$ is divisible by $m$ so that $n$ is an integer. However, all of our results can be easily generalized to general $N$. We also require each machine to have space for storing $2n$ or more functions in certain circumstances. See Section \textsuperscript{3} for details.
Therefore, when the number of machines \( m \) design an additional data distribution step (which causes small overhead in Section 3) that finds an \( \epsilon \) Shamir [14] prove that for a class of \( \delta \) random partition, the overhead of the data distribution procedure (in terms of bits and rounds of communication, is 0) \( \kappa \) parallel runtime is \( O(\frac{\sqrt{\log N}}{\epsilon}) \) gradient computation, which is faster than DSVRG, especially when \( \epsilon = n^{-10} \) is accurate enough for most machine learning applications since it exceeds the machine precision of real numbers, and typically \( \epsilon = \Theta(\frac{1}{\sqrt{N}}) \ll \frac{1}{n^{0.1}} \) in empirical risk minimization.

Besides being optimal, the distributed SVRG (DSVRG) method in this paper outperforms recently proposed distributed optimization algorithms DISCO [9] and DANE [7] in both rounds and parallel runtime. In particular, when \( \kappa < \min\{n^{1.5}, N\} \), DSVRG uses fewer number of rounds than DISCO and DANE. Using the recent progresses on accelerating first-order methods [12, 13], we propose DPPASVRG which uses smaller number rounds than DiSCO and DANE for any \( \kappa \) and \( n \). (See Table 2 and 1 and Section 5).

We also make the connection to recent lower bounds [14] for the distributed optimization: Arjevani and Shamir [14] prove that for a class of \( \delta \)-related functions and for a class of algorithms, the round complexity achieved by DiSCO is optimal, while our algorithms have better round complexity than DiSCO. When \( \kappa = \Theta(\frac{1}{\sqrt{mn}}) \), our algorithms do fall into the class concerned by [14]. However, we assume more structure (that is the data forms a random partition) on the functions than \( \delta \)-relatedness, and our algorithms heavily exploit this additional structure. Communication lower bounds for distributed optimization under random partition of the data remains an interesting open question.

Finally, in the regime when \( n \) is much smaller than \( \kappa \), to compensate DSVRG and further minimize runtime, we propose distributed accelerated SVRG (DASVRG) method. It uses \( O(\sqrt{\kappa} \log \frac{1}{\epsilon}) \) rounds, and its parallel runtime is \( O((n + \kappa/m) \log(\frac{1}{\epsilon})) \) gradient computation, which is faster than DSVRG, especially when \( \kappa \) is relatively large compared to \( n \).

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2Initial error \( \epsilon_0 \) is considered as a constant so we sometimes dropped it in the discussion below for simplicity
3For example, for the regime of parameters above, under the assumption that originally the data on the machines forms a random partition, the overhead of the data distribution procedure (in terms of bits and rounds of communication, is 0)
4If \( n = 10^{5} \), then \( n^{0.8} = 10^{5} \) which is already much larger than most clusters.
Table 1: In this table, we assume that $\kappa = \Theta(\sqrt{mn})$ for risk minimization problems as done in [9,7,6,8], and only compare against DISCO which has the lowest rounds of communication. The communication costs of all the algorithms are essentially equal to $d$ times the number of rounds, while the first three algorithms need to communicate an additional $O(\log^2(1/\epsilon))$ data points during the data allocation phase. $C$ and $G$ are defined in Table 2.

| Algorithm      | Rounds                          | Runtime                     | Remarks                   |
|----------------|--------------------------------|-----------------------------|---------------------------|
| DSVRG          | $(1 + \sqrt{\frac{m}{n}}) \log \frac{1}{\epsilon}$ | $G(n + \sqrt{mn}) \log \frac{1}{\epsilon}$ | $m < n^{6,8}\epsilon > \frac{1}{n^m}$ |
| DSVRG-2        | $O(1)$                         | $nG$                        |                           |
| DAPPSVRG       | $(1 + \frac{m}{n^{i^2}}) \log(1 + \sqrt{\frac{m}{n}}) \log \frac{1}{\epsilon}$ | $G(n + n^{1.5}m^{0.5}) \log(1 + \sqrt{\frac{m}{n}}) \log \frac{1}{\epsilon}$ |                           |
| DISCO (quad)   | $m^{2.5} \log \frac{1}{\epsilon}$ | $Cm^{2.5} \log \frac{1}{\epsilon}$ | $f_i \sim D$               |

2 Related Work

Recently, there have been several distributed optimization algorithms proposed for problem (1). We list most of them in Table 2. The algorithms proposed in this paper are DSVRG, DSVRG-2, and DASVRG.

COCOA+ [15, 16] is a distributed coordinate optimization algorithm which can be applied to the conjugate dual formulation of (1). In COCOA+, each machine is only updating a subset of dual variables contained in a local problem. Any optimization algorithm can be used as a subroutine in each machine as long as it reduces the optimality gap of the local problem by a constant factor.

Under the statistical assumption that $f_i \sim D$, the DANE [7] and DISCO [9] algorithms require very few rounds and communication in the big data regime of large $n$. As $n$ increases, they require exactly $\log \frac{1}{\epsilon}$ rounds which is independent of the condition number. However, DISCO and DANE have large running times due to the need of solving a linear system each round, which requires a matrix inverse at each iteration. This is infeasible for problems of large dimensionality. As an alternative, Zhang and Xiao [9] suggest solving the linear system with an inexact solution using another optimization algorithm, but this still has large runtime for ill-conditioned problems. Furthermore, DANE only applies to quadratic functions, and DISCO only applies to self-concordant functions with easily computed Hessian.

In practice on a given dataset, DANE and DISCO need to perform a pre-shuffle step before the algorithm is run to mimic the stochastic setting. Similarly, DSVRG and DSVRG-2 also require a data allocation step to mimic sampling with replacement, but do not assume $f_i \sim D$.

The DSVRG and DSVRG-2 algorithms proposed here also require very few rounds of communication, but have much improved runtime. DSVRG requires $O((1 + \frac{m}{n}) \log \frac{1}{\epsilon})$ rounds of communication which is strictly better than DANE, and better than DISCO when $n^{1.5} > \kappa$. DPPASVRG, as an extension of DSVRG, outperforms DiSCO in terms of rounds in all the regimes. Moreover, DSVRG and DSVRG-2 and DPPASVRG only require computing the gradients of $f_i$‘s in each round which needs much less computation. DSVRG-2 further leverages the setting of large $n$. When $n^{1-25} > \kappa$, DSVRG-2 requires a constant number of rounds that is independent of $\kappa$, with running time that is close to the optimal of $O(nG)$. We note that DSVRG requires a one-round data distribution procedure (see Section 3 for the details), which almost has no overhead for $\kappa = \Theta(\sqrt{N})$, and which in general requires a one round shuffling step of the data when $\kappa \lesssim N$, and increase the space used for each machine by a constant factor.

To minimize runtime, we also propose the distributed accelerated SVRG (DASVRG) algorithm. For problems where $\kappa$ is large, DASVRG requires computation of only $(n + \frac{\kappa}{m} + \log N) \log \frac{1}{\epsilon}$ gradients. This algorithm is related to the serial accelerated SVRG algorithm in [17] and largely inspired work by the work of [18] in the non-stochastic setting, but the algorithm is more suitable for distributed computation.

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5The examples in [9] are all of the type $f_i(x) = g(a_i^T x)$ for some scalar function $g$.

6The results reported in the table for DANE and DISCO actually require the assumption $f_i \sim D$, but this is impossible to ensure in practice. In empirical studies, we have observed that shuffling is sufficient.

7For $\kappa \gg N$, there are also ways to apply data allocation steps with reasonable overhead. See Section 3 for details.
In this section, we consider a simple distributed stochastic variance reduced gradient (DSVRG) method that is based on a parallelization of serial SVRG. SVRG method \cite{Johnson2013, Zhang2013} consists of multiple stages in which the algorithm calculates the batch gradient at some reference point and then takes roughly $\kappa$ iterative updates. It’s straightforward to see that the batch gradient part can be completely parallelized while the iterative update part cannot. However, the key observation here is that when $\kappa \ll N$, the runtime for computing the batch gradient dominates that for the iterative updates and therefore parallelizing the former will improve the runtime. Moreover, when $\kappa \ll n$, the parallel runtime for computing batch gradient still dominates that for iterative updates and therefore the failure of parallelizing the iterative updates is of less an issue. Note that $\kappa \ll n$ is already a reasonable regime since as argued in \cite{Arjevani2016, Arjevani2015, Arjevani2017, Jaggi2012}, $\kappa$ is typically set to $\Theta(\sqrt{N})$ and therefore $\kappa \ll n$ when $m \ll n$.

However, there remains one caveat: In order to guarantee the convergence of SVRG, an unbiased stochastic gradient needs to be constructed by sampling over the whole data set. A priori, this requires communication of data points (or the gradients of the data points) between the machines since the machines can only access local data. Moreover, as shown by Arjevani and Shamir \cite{Arjevani2016}, if the functions stored on all the machines don’t have any good structure (for example, random-like structure), then a (large) class of natural algorithms must have a large number of rounds of communication.

We can get around with these lower bounds by two different means: a) In section 3.1, we allow the distributed algorithm to have a one-round data allocation/shuffle step that doesn’t count as communication cost. We observe that this assumption is typically very reasonable since either we anyway need to distribute the data onto the machines or a random shuffle of the data is very efficient. b) In section 3.2, we assume that the data is stored on the machine originally and the partition of the functions are uniformly random. Using this intrinsic randomness of the data, we can show that when $\kappa = \Theta(\sqrt{N})$, a slightly more sophisticated data allocation process only uses $O(1)$ communications of the data point and returns a sequence of i.i.d data points that the machine can access locally.

Finally in Section 3.3, we provide the formal description of the algorithm and its guarantees.

### 3.1 Data Allocation

In this subsection, we consider the situation where a random shuffle step is relatively cheap. We will use a modification of random shuffle to prepare both a partition of the dataset and a sequence of i.i.d data points.
drawn from uniformly from the dataset. The data allocation procedure in Algorithm 1 only costs twice the time and space compared to random shuffle. This also suggests that when the data was moved onto the machines at the first time, this data allocation step should be performed for future use.

**Algorithm 1** Data Allocation: DA($N, m, q$)

**Input:** Index set $[N]$, sample size $q$ and the number of machines $m$.

**Output:** For each $j \in [m]$, a set $S_j \subset [N]$, a multi-set $R_j \subset [N]$ and data $\{f_i | i \in S_j \cup R_j\}$ stored on machine $j$.

1: Randomly partition $[N]$ into $m$ disjoint sets $S_1, \ldots, S_m$ of the same size $n = \frac{N}{m}$.
2: for machine $j = 1$ to $m$ do
3: Sample a multi-set $R_j$ of size $q$ with replacement from $[N]$.
4: Machine $j$ gets $\{f_i | i \in S_j \cup R_j\}$.
5: end for

Note that after the data allocation step, each machine stores at most $q + n$ data points and $R_1 \cup \cdots \cup R_m$ forms a sequence of i.i.d samples of length $qm$. In Algorithm 3, the machines will use the samples in $R_i$'s (in the order of $R_1, \ldots, R_m$) to do the iterative updates in SVRG, and due to the locality of the samples, the communication cost of accessing these samples is essentially $O(d)$ bits per $q$ samples. We will choose $q = n$ typically so that space used by each machine only increases by a factor of 2. When more space is available, we can use choose larger $q$ (see more discussion in Section 3.3).

### 3.2 Data Allocation+ for Random Partitioned Data

In this subsection we assume that the machines have a random partition of the data initially.

**Assumption 1** (random partition). *Initially, machine $j$ stores a subset $S_j$ of functions, where $S_1, \ldots, S_m$ is a uniform random partition of the set $[N] = \{1, 2, \ldots, N\}$ with $|S_j| = N/m = n$.*

Given the assumption above, the machines can have access to a permutation of the functions in a local way – the concatenation of $S_1, \ldots, S_m$ is a random permutation. However, we actually need a sequence of functions that are drawn with replacement. Our key observation to fix this issue is that the random partition already provides enough randomness, and given a random permutation, it is very easy to build a sequence of i.i.d samples of length $Q$ (with replacement). If $Q$ is not too large, then the new sequence is pretty close to the random permutation so that we don’t have to communicate a lot.

**Algorithm 2** Data Allocation+: DA+(N, m, Q)

**Input:** Index set $[N]$, the number of machines $m$. A random partition $S_1, \ldots, S_m$ of $[N]$. The length of target sequence $Q$ (which is assumed to be a multiple of $n$).

**Output:** A sequence $r_1, \ldots, r_Q$ and multi-sets $R_1, \ldots, R_{Q/n} \subset [N]$, with $R_j = \{r_{(j-1)n+1}, \ldots, r_{jn}\}$.

Center samples $r_1, \ldots, r_Q$ and $R_1, \ldots, R_{Q/n}$:

1: concatenate the subsets $S_1, \ldots, S_m$ (in this fixed order) and obtain a random permutation $i_1, \ldots, i_N$ of $N$, where $i_{(j-1)n+1}, \ldots, i_{jn}$ are from the subset $S_j$.
2: for $\ell = 1$ to $Q$ do
3: with probability $1 - (\ell - 1)/N$, let $r_\ell = i_\ell$
4: for any $\ell' < \ell$, with probability $1/N$, let $r_\ell = i_{\ell'}$.
5: end for
6: Let $R_j = \{r_{(j-1)n+1}, \ldots, r_{jn}\}$ for all $1 \leq j \leq Q/n$.

Machines pass data points in one round:

7: Machine $j$ acquires data points in $R_j \setminus S_j$ from the machines that have them.
Lemma 1. Under assumption 4, the sequence $r_1, \ldots, r_Q$ output by Algorithm 2 has the same joint distribution as a sequence of i.i.d uniform samples with replacement from $[N]$. Moreover, machines obtain the set $R_i \cup S_j$ on their local storage in one round with expected communication of at most $Q^2/N$ data points. Moreover, when $Q \leq n$, the bias and round of communication are both 0.

Proof. Conditioned on $i_1, \ldots, i_{\ell-1}$ and $r_1, \ldots, r_\ell-1$, the random variable $i_\ell$ has uniform distribution over $[N] \setminus \{i_1, \ldots, i_{\ell-1}\}$. Therefore by the definition, random variable $r_\ell$ (conditioned on the same event) has uniform distribution over the set $[N]$. Thus we complete the proof of the first part of the lemma.

To analyze the communication, we note that $i_\ell \neq r_\ell$ with probability $(\ell-1)/N$. Only when $i_\ell \neq r_\ell$, a communication of a data point might potentially needed. Therefore, the expected communication is upper bounded by $\sum_{\ell=1}^Q (\ell-1)/N \leq Q^2/N$. \hfill \square

3.3 Algorithms and guarantees

Given the sequence of samples $R_1, \ldots, R_\ell$ that are prepared by Algorithm 2, we can just almost trivially parallelize SVRG by using the samples in $R_\ell$’s to do the iterative updates.

At each stage of SVRG, the machines simply compute the batch gradient in parallel and simulate the iterative update using the sequence of samples $R_1, \ldots, R_\ell$ prepared by one of the data allocation procedures (Algorithm 2 and Algorithm 1). The machines do the iterative updates in order from 1 to $m$. As soon as a machine, say, machine $j$ uses up all of its samples in $R_j$, then it passes the iterate to the next machine. The key point here is that machines should never reuse any samples in $R$ twice. In each stage, there are only $T = O(\kappa)$ iterative updates, and in total at most $\kappa \log(1/\epsilon)$ iterative updates, therefore the length of the sequence only need to be at most $O(\kappa \log(1/\epsilon))$. Moreover, when $\kappa < n$, the iterative updates in each stage can be finished by at most 2 machines and therefore the number of rounds is at most $O(\log(1/\epsilon))$.

We describe formally the outer loop of this algorithm in Algorithm 3 and the inner loop in Algorithm 4.

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**Algorithm 3** Distributed SVRG (DSVRG)

**Input:** An initial solution $\tilde{x}^0 \in \mathbb{R}^d$, data $\{f_i\}_{i=1, \ldots, N}$, the number of machine $m$, the size for pre-sampling $r$, a step length $\eta < \frac{1}{4}$, the number of iterations $T$ in each stage, and the number of stages $K$.

**Output:** $\tilde{x}^K$

1: Option 1: $(R_1, \ldots, R_m) = DA(N, m, q)$ and $D = \{R_1, R_2, \ldots, R_m\}$
2: Option 2: $(R_1, \ldots, R_{TK/n}) = DA_+(N, m, TK)$ and $D = \{R_1, R_2, \ldots, R_{TK/n}\}$
3: $k \leftarrow 1$
4: for $\ell = 0, 1, 2, \ldots, K - 1$ do
5:     Center sends $\tilde{x}^\ell$ to each machine
6:     for machine $j = 1, 2, \ldots, m$ in parallel do
7:         Compute $h_\ell^j = \sum_{i \in S_j} \nabla f_i(\tilde{x}^\ell)$ and send it to center
8:     end for
9:     Center computes $h^\ell = \frac{1}{N} \sum_{j=1}^m h_\ell^j$ and send it to each machine
10:    $(\tilde{x}^{\ell+1}, D, k) \leftarrow SS-SVRG(\tilde{x}^\ell, D, k, \eta, T)$
11: end for

The convergence of Algorithm 3 is established by the following theorem. The proof follows from the analysis in [4] and [3] and is deferred to the Appendix.

**Theorem 1.** Suppose $\eta < \frac{1}{4T}$, using option 1, or using option 2 under assumption 4, assuming additionally $TK \geq qm$ for option 1, Algorithm 3 guarantees that

$$
\mathbb{E} \left[ f(\tilde{x}^K) - f(x^*) \right] \leq 3 \left( \frac{1}{\mu \eta (1 - 4L \eta T)} + \frac{4L \eta (T + 1)}{(1 - 4L \eta T)} \right)^K \left[ f(\tilde{x}^0) - f(x^*) \right].
$$

It’s possible that one machine used up its samples in $r$ and hand the iterate to the next machine, and then the next machine will finish the iterative updates by itself since $\kappa \ll n$. In this regime, we can also further reduce the number of rounds by asking a new machine to do the updates for each new stage.
Since we need \( n = \frac{T}{N} \) data allocation step is negligible. When distribution step every \( N/T \) designed for the regime of \( O(\text{rounds of communication}) \). Moreover, by Lemma 1, the communication of the data allocation step only costs \( \epsilon \) find an \( O(\text{rounds of communication}) \) in Corollary 1, DSVRG needs \( O(\text{rounds of communication}) \) of communication in total.

Suppose the average solution \( \bar{x} \) Output: \( x \)

\begin{algorithm}
\textbf{Algorithm 4} Single-Stage SVRG: SS-SVRG(\( \bar{x}, h, D, k, \eta, T \))
\begin{algorithmic}
\Input A solution \( \bar{x} \in \mathbb{R}^d \), a gradient \( h \), \( m \) multi-sets of indexes \( D = \{D_1, D_2, \ldots, D_m\} \), the index of the active machine \( k \), a step length \( \eta \leq \frac{1}{\mathcal{E}} \), and the number of iteration \( T \).
\Output The average solution \( \bar{x}_T \), the updated multi-sets of indexes \( D = \{D_1, D_2, \ldots, D_m\} \), and the updated index of active machine \( k \)
\begin{algorithmic}
\State \( x_0 = \bar{x} \) and \( x_0 = 0 \)
\For {\( t = 0, 1, 2, \ldots, T - 1 \)}
\State Machine \( k \) samples an instance \( i \) from \( D_k \) and computes \( x_{t+1} = x_t - \eta (\nabla f_i(x_t) - \nabla f_i(\bar{x}) + h) \), \( \bar{x}_{t+1} = \frac{x_{t+1} + t\bar{x}_t}{t+1} \), \( D_k \leftarrow D_k \setminus \{i\} \)
\If {\( D_k = \emptyset \)}
\State \( x_{t+1} \) and \( \bar{x}_{t+1} \) are sent to machine \( k + 1 \)
\State \( k \leftarrow k + 1 \)
\EndIf
\EndFor
\end{algorithmic}
\end{algorithm}

Theorem 1 leads to the following corollary when a specific choice of \( T \) and \( \eta \) in Algorithm 3.

\textbf{Corollary 1.} Suppose \( \eta = \frac{1}{16L} \), \( T = 96\kappa \), and either of the two conditions in Theorem 2 is satisfied in Algorithm 3. Then, Algorithm 3 guarantees that

\[ \mathbb{E} \left[ f(\bar{x}^K) - f(x^*) \right] \leq 3 \left( \frac{8}{9} \right)^K [f(\bar{x}^0) - f(x^*)]. \] (2)

According to Corollary 1, DSVRG can find an \( \epsilon \)-optimal solution for \( 1 \) after \( K = O(\log(1/\epsilon)) \) stages with \( T = O(\kappa) \) iterations in each stage. For option 1, since DSVRG needs one round of communication to compute a batch gradient at each stage and one round after every \( \frac{T}{q} \) inner iterations, it needs \( O(1 + \frac{T}{q}) \log(1/\epsilon) \) rounds of communication in total.

Under option 2, the amount of instances stored in each machine is at most \( 2n \), and with \( T = O(\kappa) \) as in Corollary 1, DSVRG needs \( O(1 + \frac{T}{m}) \log(1/\epsilon) \) rounds of communications to find an \( \epsilon \)-optimal solution. Since we needs \( n \geq \frac{T\kappa}{m} \) to support \( TK \) inner iterations in DSVRG, we require \( O(TK) = O(\kappa \log(1/\epsilon)) \leq O(nm) = O(N) \) to apply DSVRG. The communication cost of the data allocation step is \( O(TK^2/N) = O(\kappa^2 \log^2(1/\epsilon)/N) \) data points. When \( \kappa = N \), the cost could be potentially as large as \( N \) data points. However, this communication can be done in one round and potentially we anyway need to move all the data points onto the machines, which costs \( N \) data points communication. In practice, a shuffle step before the start of the algorithm does not cost much overhead and is also required in the implementations of DISCO and DANE.

Moreover, in the regime when \( \kappa = \Theta(\sqrt{N}) \), DSVRG needs \( O(1 + \frac{\kappa}{m}) \log(1/\epsilon) \) rounds of communication to find an \( \epsilon \)-optimal. Note that \( \kappa/n = \sqrt{m/n} < 1 \) when \( m < n \). Therefore in this case we only need \( O(\log(1/\epsilon)) \) rounds of communication. Moreover, by Lemma 4 the communication of the data allocation step only costs \( O(TK^2/N) = O(\log^2(1/\epsilon)) \) data points of communication. When \( \kappa = 0 \), by Lemma 1 the communication overhead of the data distribution step is even 0.

As a short summary, when \( \kappa < n \) or \( \kappa = O(\sqrt{N}) \), option 2 should be preferred and the overhead of the data allocation step is negligible. When \( n < \kappa \leq N \), the overhead of both DA (with \( \bar{q} \) chosen to be \( n \)) and DA+ is one round of communication with a large amount of bits communicated. Our algorithm is not designed for the regime of \( \kappa \gg N \), although in this case, either option 1 can be used with large \( q \) or data allocation can be performed whenever the sequence of i.i.d samples \( D \) is used up.

At the end of this section, we show that a modified version of the DSVRG algorithm can find an \( \epsilon \)-optimal solution for \( 1 \) with a nearly constant number of rounds when \( n \gg \kappa \).

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9When \( TK > N \) which corresponds to the case when \( \kappa \log(1/\epsilon) \geq N \), one should modify Algorithm 4 by doing the data distribution step every \( N/T \) rounds. We left the details to the readers since we are mainly interested in the regime when \( \kappa \ll N \).
Theorem 2 (DSVRG-2). Suppose $\kappa < \frac{1}{16} n^{1-2\delta}$, where $\delta < \frac{1}{2}$ is a constant. DSVRG using Option 1 with $r = n$, $\eta = \frac{1}{16\delta n^{1-2\delta}}$, $T = n$ and $K = \frac{1}{3\log n} \log \frac{1}{\epsilon}$, ensures that
\[
\mathbb{E}[f(\hat{x}^K) - f(x^*)] \leq \epsilon [f(\hat{x}^0) - f(x^*)],
\]
after $\frac{2\log(1/\epsilon)}{\delta \log n}$ rounds of communications and $\frac{2n\log(1/\epsilon)}{\delta \log n}$ gradient computations.

For most machine learning problems formulated as (1), an accuracy level of $\epsilon = O\left(\frac{1}{N}\right)$ is generally acceptable. This can be obtained in $\frac{2}{3} (1 + \frac{1}{\log n}) \approx \frac{4}{3}$ rounds of communication when $n > m$, which is mostly true in practice. We denote this implementation of DSVRG by DSVRG-2.

4 Distribution Accelerated SVRG

Algorithm 5 Mini-Batch Stochastic Variance Reduced Gradient: MiniBG($\hat{x}, x, h, b$)

Input: reference point $\hat{x}$, current iterate $x$, the gradient at $\hat{x}$, i.e., $h = \nabla f(\hat{x})$, batch size $b$.

Output: $g$

1. Machines publicly sample an index set $S$ of size $b$ without replacement from $[N]$.
2. for machine $j = 1, 2, \ldots, m$ in parallel do
3. Computes $p_j = \sum_{i \in S \cap S} \nabla f_i(x)$ and $\tilde{p}_j = \sum_{i \in S \cap S} \nabla f_i(\hat{x})$ and send them to center.
4. end for
5. Center computes $g = \frac{1}{|S|} \sum_{j=1}^m p_j - \frac{1}{|S|} \sum_{j=1}^m \tilde{p}_j + h$.

Algorithm 6 Distributed Accelerated SVRG (DASVRG)

Input: An initial solution $\hat{x}^0 \in \mathbb{R}^d$, data $\{f_i\}_{i=1}^N$, the number of machine $m$, control parameters $(\tau, \alpha, \eta)$, the number of iterations $T$ in each stage, and the number of stages $K$, mini-batch size $b$.

Output: $\hat{x}^K$

1. DD($N, 0, m$)
2. for $\ell = 0, 1, 2, \ldots, K - 1$ do
3. Center sends $\hat{x}^\ell$ to each machine
4. for machine $j = 1, 2, \ldots, m$ in parallel do
5. Compute $h^\ell_j = \sum_{i \in S_j} \nabla f_i(\hat{x}^\ell)$ and send it to center.
6. end for
7. Center computes $h^\ell = \frac{1}{N} \sum_{j=1}^m h^\ell_j$ and send it to each machine
8. $x_0 = y_0 = z_0 \leftarrow \hat{x}^\ell$
9. for $t = 0, 1, 2, \ldots, T - 1$ do
10. $x_{t+1} = (1 - \tau) y_t + \tau z_t$
11. $y_{t+1} \leftarrow $ MiniBG($\hat{x}^\ell, x_{t+1}, h, b$)
12. $y_{t+1} = x_{t+1} - \eta g_{t+1}$
13. $z_{t+1} = z_t - \alpha g_{t+1}$
14. end for
15. $\hat{x}^{\ell+1} = \frac{1}{T} \sum_{t=1}^T x_t$
16. end for

It is known that the classical gradient descent algorithm can be accelerated using auxiliary solution sequences. A similar acceleration technique has been applied to SVRG by Nitanda [17] to improve its convergence rate. In the accelerated SVRG method in [17], a mini-batch of data instances is sampled in an appropriate size to construct the stochastic gradient. In this section, we propose another variant of accelerated SVRG and its distributed version (DASVRG) using the similar idea.
We present the accelerated SVRG method in Algorithm 5 and Algorithm 6. Similar to DSVRG in the previous section, DASVRG also consists of $K$ stages with $T$ inner iterations conducted in each stage. Since the accelerated SVRG method constructs the stochastic gradient in each inner iteration based on a mini-batch of data whose size can be large, Algorithm 5 is designed to distribute the computation of stochastic gradient of machines in order to reduce the runtime of accelerated SVRG.

Algorithm 5 presents the main scheme of DASVRG. The sequence $z_t$ and $y_t$ are auxiliary solution sequences and $x_t$ is the output solution sequence. However, different from the method in [17] where stochastic gradients are constructed at the auxiliary sequence, our method constructs stochastic gradients at the output sequence. In Algorithm 5, machines publicly sample without replacement an index set $S$ of size $b$ and machine $j$ only compute the gradient of $f_j$ in $S$ which it has access to. Therefore, machine $j$ has to compute $|S_j \cap S|$ gradients so that the runtime of Algorithm 5 is $\max \{|S_j \cap S| \}$ in each inner iteration for each stage. Note that, different from DSVRG, machines in DASVRG have to communicate not only at the beginning but also in each inner iteration in each stage. Therefore, DASVRG requires $K(1 + T) = O(\sqrt{N} \log(1/\epsilon))$ rounds of communication which is more than DSVRG needs. However, according to Proposition 1, the runtime of DASVRG is at most $O((n + \sqrt{\kappa} \log(1/\epsilon)) \log(1/\epsilon))$ of DASVRG when $m$ is large.

5 Distributed Proximal Point Accelerated SVRG

In this section, we use the idea of [13] and [12] to further speed up the DSVRG.

Algorithm 7 Distributed Proximal Point Accelerated SVRG (DPPASVRG)

**Input:** An initial solution $\tilde{x}^0 \in \mathbb{R}^d$, data $\{f_i\}_{i=1,\ldots,N}$, the number of machine $m$, regularization parameter $\lambda$, and accuracy parameter $\delta$

**Output:** $\tilde{x}^K$

1. DA$(N, m, Q)$
2. Initialize $y_0 = x_0$
3. Initialize $\alpha_0 = \sqrt{\frac{\mu}{\mu + \lambda}}$
4. for $\ell = 0, 1, 2, \ldots, K - 1$ do
5. Let $x_t = \text{arg min}_{x \in \mathbb{R}^d} f_\lambda(x; y_t)$ using DSVRG with accuracy $\delta$.
6. Let $\alpha_t^2$ be the solution to $\alpha_t^2 = (1 - \alpha_t)\alpha_{t-1}^2 + \frac{\mu}{\mu + \lambda}$.
7. Update $y_t = x_t + \beta_t(x_t - x_{t-1})$, where $\beta_t = \frac{\alpha_{t-1}(1 - \alpha_{t-1})}{\alpha_{t-1}^2 + \alpha_t}$.
8. end for
Definition 1. Define the proximal point objective

\[ f_\lambda(x; s) = f(x) + \frac{\lambda}{2} \|x - s\|^2. \]

The condition number is \( \kappa(f_\lambda) = \frac{L + \lambda}{\mu + \lambda}. \)

The idea behind DPPASVRG is that for the choice of \( \lambda, f_\lambda \) has condition number \( \kappa(f_\lambda) = O(n) \). Thus DSVRG needs a constant number of rounds to halve the error. The accelerated proximal point algorithm has acceleration as part of the outer loop, so it requires \( O(\sqrt{1 + \frac{\lambda}{\mu}} \log \frac{1}{\epsilon}) \) outer iterations. This translates to total number of rounds \( \sqrt{1 + \frac{\lambda}{\mu}} \log \frac{1}{\epsilon} = O(\sqrt{1 + \kappa/n} \log 1/\epsilon) \).

Theorem 4. The number of rounds of communication of DPPASVRG (Algorithm 7) with \( \mu = \max\left\{ \frac{L}{n-1} - \frac{n}{n-1} \mu, 0 \right\} \) and \( \delta = \left( \frac{\mu}{\mu + \lambda} \right)^2 \) is given by \( \tilde{O}(\sqrt{1 + \frac{2\kappa}{n} \log \frac{2\kappa}{n}}) \) and the parallel runtime is \( \tilde{O}(n\sqrt{1 + \frac{2\kappa}{n} \log \frac{2\kappa}{n}}) \) gradients.

Proof. We first compute the condition number of \( \kappa(f_\lambda) \).

\[ \kappa(f_\lambda) = \frac{L + \lambda}{\mu + \lambda}. \]

Let us choose \( \lambda \) such that \( \kappa(f_\lambda) < n \). We solve to obtain

\[ \frac{L}{n-1} - \frac{n}{n-1} \mu < \lambda, \]

which is satisfied when \( \lambda = \max\left\{ \frac{L}{n-1} - \frac{n}{n-1} \mu, 0 \right\} \).

The number of times DSVRG is called is given by Theorem 3.1 of [12] is \( O(\sqrt{1 + \frac{\lambda}{\mu}} \log \frac{1}{\epsilon}) = O(\sqrt{1 + \frac{\kappa}{n} \log \frac{1}{\epsilon}}) \).

Using Proposition 3.2 of [12], DSVRG needs to find a point \( x_{k+1} \) with accuracy \( \delta \leq \left( \frac{\mu}{\mu + \lambda} \right)^2 \) meaning \( f_\lambda(x_{k+1}; s) - f_\lambda(x_k; s) - f^* \), which requires \( \left( 1 + \frac{\kappa(f_\lambda)}{n} \right) \log \frac{1}{\delta} = O(\log(1 + \frac{\kappa}{n})) \) rounds of communication. Thus the total number of rounds is \( O(\sqrt{1 + \frac{\kappa}{n}} \log \frac{1}{\delta} \log(1 + \frac{\kappa}{n})) \). Each round requires \( n \) gradients, so we arrive at a runtime of \( O(n\sqrt{1 + \frac{\kappa}{n}} \log \frac{1}{\delta} \log(1 + \frac{\kappa}{n})) \) gradients.

\( \square \)

6 Numerical Experiments

In this section, we conduct numerical experiments to compare the DSVRG, DASVRG and DPPASVRG algorithms with the DISCO algorithm by Zhang and Xiao [9] over synthetic data. We consider the problem of regularized logistic regression, \( \min_{x \in \mathbb{R}^d} f(x) = \frac{1}{N} \sum_{i=1}^{N} \log(1 + \exp(-b_{i}A_{i}^T x)) + \frac{\lambda}{2} \|x\|^2, \) where \( A_{i} \in \mathbb{R}^d \) and \( b_{i} \in \{-1, 1\}, \) for \( i = 1, \ldots, N, \) are the data points and their class labels. We generate \( A_{i} \) as rows of a \( N \times d \) random matrix \( A \) following the experimental setup in [19] [20]. According to this data generating process, the Hessian matrix of the objective function of at \( x = 0 \) can be approximated by \( \frac{1}{4} \mathbb{E}[A^T A] + \lambda I. \)

According to [19] [20], the eigenvalues of \( \mathbb{E}[A^T A] \) lie within the interval \( \left[ \frac{1}{1+\omega}, \frac{2}{(1-\omega)^2(1+\omega)} \right] \) so as \( \omega \to 1 \) the condition number \( \kappa \) increases. To demonstrate the effectiveness of these three methods under different settings, we choose \( d = 50, N = 10^5, \omega \in \{0.25, 0.5, 0.75\} \) and \( m \in \{10, 100\}. \) In all experiments, we set \( \lambda = 10^{-4}. \)

For each setting, we approximate \( A \) and \( \mu, \) respectively, by \( \lambda + \) the largest and smallest eigenvalues of the Hessian matrix of \( f(x) \) at \( x = 0. \) In the experiments, we implement DSVRG by choosing \( \eta = \frac{1}{100}, K = 2 \) and \( T = 20\kappa. \) For DPPASVRG, we apply DSVRG with \( \eta = \frac{1}{100}, \) \( K = 2 \) and \( T = 30\kappa \) to solve \( \min_{x} f_\lambda(x; y) \) approximately. For DASVRG, we choose the mini-batch size \( b = 10\sqrt{n} \) and set the parameters \( \alpha = \min\left\{ \frac{\lambda}{12L(N-b)}, \frac{1}{\sqrt{1+\omega}} \right\}, \eta = \frac{1}{100}, \tau = \frac{\eta}{\alpha+\eta} \) and \( T = \max\left\{ \frac{144L(N-b)}{\eta\alpha\beta(N-1)}, 12\sqrt{n} \right\}. \) In the implementation of
DSVRG algorithm, we use Option 1 with $q = \frac{N}{m}$ so that the solution is passed to the next machine after every $\frac{N}{m}$ stochastic gradient steps. In each main iteration of DISCO, an inexact Newton direction is found using a preconditioned distributed conjugate gradient method whose performance depends on a preconditioning.
parameter. We choose this parameter to be $\sqrt{m} \times 0.1$ in all simulated experiments.

The numerical results are presented in Figure 1 and Figure 2. In Figure 1, the horizontal axis presents the number of rounds of communication conducted by algorithm. In Figure 2, the horizontal axis presents the computation cost defined as the exact or equivalent number of inner products of two vectors of size $d$ computed in algorithm. In both figures, the vertical axis represents the optimality gap in logarithmic scale. According to Figure 1, we find that, to obtain the solution of the same optimality gap, DSVRG or DPPASVRG requires similar number of rounds of communication to DISCO and, in some cases, fewer than DISCO. However, according to Figure 2, DISCO will require more parallel computational cost than other method if the communication between machines and centers are faster. To find a good solution with the shortest runtime, one should use DASVRG.

We also conduct the same comparison of these three methods over three real datasets\footnote{http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html}: Covtype ($N = 581012, d = 54$), RCV1 ($N = 20242, d = 47236$), and News20 ($N = 19996, d = 1355191$). For each data, we choose $\lambda = 10^{-5}$ and $m = 10$. The numerical results are showed in Figure 3 and Figure 4 for the comparison over the rounds of communication and computation cost (parallel runtime). A similar conclusion can be derived from these results, that is, DISCO is more efficient in terms of the communications while DSVRG and DASVRG can be more efficient in terms of parallel runtime.

![Figure 3](image1.png) ![Figure 4](image2.png)

**Figure 3:** For all cases, $m = 10, \lambda = 10^{-5}$.

![Figure 4](image3.png)

**Figure 4:** For all cases, $m = 10, \lambda = 10^{-5}$.

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Appendix

The proof of Theorem 7. Under both option 1 and option 2, the sequence of samples in $R_1, \ldots, R_t$ (for option 2 and $t = m$ for option 1) has the same joint distribution as a sequence of samples i.i.d drawn uniformly from the dataset. Therefore Algorithm 3 will behave statistically the same as SVRG when applied to the whole data stored in the a single machine. Then, the conclusion is directly from the convergence rate of SVRG given in Theorem 1 in [4].

The proof of Proposition 7. Since machine $j$ computes $|S_j \cap S| = b_j$ gradients in parallel, we only need to bound the expectation of $\max_j b_j$ from above. Let $X_i^j$ be the indicator variable which equals one if the $i$-th data of in $S_j$ is selected in $S$. Then we have that $b_j = X_1^j + \cdots + X_n^j$ for $j = 1, 2, \ldots, m$.

It is easy to check that random variables $\{X_i^j\}_{1 \leq j \leq m, 1 \leq i \leq n}$ are negatively associated. (For its definition, see [21, 22].) Therefore, the random vectors $(X_1^1, \ldots, X_n^j)_{j=1,2,\ldots,m}$ are also negatively associated. It is also known that negative associated random variables satisfy Bernstein inequality 11.

Therefore we obtain that with probability at least $1 - N^{-5}$,

$$|b_j - \mathbb{E}[b_j]| = \left|X_1^j + \cdots + X_n^j - \frac{b}{m}\right| \leq O(\log n) + O(\sqrt{b/m \log n}).$$

It follows by AM-GM inequality that $b_j \leq \mathbb{E}[b_j] + O(\log n) + (b/m + O(\log n)) = 2b/m + O(\log n)$. Taking union bound over all $j$’s, we obtain that for all $j$, $b_j \leq 2b/m + O(\log n)$.  

We start with a well known result about the variance of the output $g$ by the MiniBG that has been shown by [3, 4, 17]. We provide the proof here only for completeness.

Lemma 2. The output $g$ of MiniBG satisfies that $\mathbb{E}g = \nabla f(x)$ and

$$\mathbb{E}\|g - \mathbb{E}g\|^2 \leq \frac{4L(N-b)}{b(N-1)} (f(x) - f(x^*) + f(\tilde{x}) - f(x^*)),$$

where the expectation is taken over $S$ generated in MiniBG.

Proof. Let $v_i = \nabla f_i(x) - \nabla f_i(\tilde{x}) + h$ for $i \in [N]$ and $g = \frac{1}{S} \sum_{i \in S} v_i$ so that $\mathbb{E}g = \mathbb{E}v_i = \nabla f(x)$. Since $S$ is a set of indexes randomly sampled from $[N]$ without replacement, the following equalities holds for any $x$ according to Theorem B Chapter 7 (page 208) in [24],

$$\mathbb{E}\|g - \mathbb{E}g\|^2 = \mathbb{E}\|g - \nabla f(x)\|^2 = \frac{1}{b} \frac{N-b}{N-1} \mathbb{E}\|v_i - \nabla f(x)\|^2. \tag{5}$$

According to equation (12) in [4], we have

$$\mathbb{E}\|v_i - \nabla f(x)\|^2 \leq 4L (f(x) - f(x^*) + f(\tilde{x}) - f(x^*))$$

which can be applied to the right hand side of (5) to obtain the conclusion of this lemma.

Theorem 3 can be easily obtained from the following proposition.

Proposition 2. Suppose the control parameters $(\tau, \alpha, \eta)$ are chosen as $\alpha = \min\{\frac{b(N-1)}{12L(N-b)}, \frac{1}{\sqrt{b}}\}, \eta = \frac{1}{\tau}$ and $\tau = \frac{n}{\alpha + \eta}$. If $T = \max\{\frac{144L(N-b)}{m\alpha(N-1)}, 12\sqrt{\kappa}\}$, Algorithm 6 guarantees that

$$\mathbb{E}_t [f(\tilde{x}^{t+1}) - f(x^*)] \leq \frac{3}{4} (f(\tilde{x}^t) - f(x^*)),$$

where $\mathbb{E}_t$ is the conditional expectations conditioning on $\tilde{x}^t$.

11 To see a simple proof, modify the proof of Theorem 0.2 in [23] by applying the definition of negative association in the first equation, and then the rest of the proof follows straightforwardly.
Proof. For simplicity, within stage \( \ell \), we define \( \delta_t = f(x_t) - f(x^*) \) for \( t = 0, 1, \ldots, T \) and let \( \delta = \frac{1}{T} \sum_{t=1}^{T} \delta_t \) so that \( f(x_{t+1}) - f(x^*) \leq \delta \) due to the convexity of \( f_i \). Therefore, it suffices to prove that \( E_t \delta \leq \frac{3}{4} \cdot \delta_0 \).

The convexity of \( f \) implies that, for any \( t \),

\[
\delta_{t+1} = f(x_{t+1}) - f(x^*) \leq \nabla f(x_{t+1})^T (x_{t+1} - x^*) - \frac{\alpha}{2} \|g_{t+1}\|^2 + \frac{1}{2} \|z_t - x^*\|^2 - \frac{1}{2} \|z_{t+1} - x^*\|^2 \tag{6}
\]

By the fact that \( x_{t+1} - z_t = \frac{1-\tau}{\tau} (y_t - x_{t+1}) \) and the convexity of \( f \), we have

\[
A_1 = \frac{1}{1-\tau} \nabla f(x_{t+1})^T (y_t - x_{t+1}) - \frac{1}{\tau} (f(y_t) - f(x_{t+1})). \tag{7}
\]

Since \( z_{t+1} - z_t = -\alpha g_{t+1} \), simple algebraic manipulation leads to,

\[
\alpha A_2 = \alpha (\nabla f(x_{t+1}) - g_{t+1})^T (z_t - x^*) + \frac{\alpha}{2} \|g_{t+1}\|^2 + \frac{1}{2} \|z_t - x^*\|^2 - \frac{1}{2} \|z_{t+1} - x^*\|^2 \tag{8}
\]

Let \( E_t \) be the conditional expectations conditioning on \( \{x_s, y_s, z_s\}_{s=0, \ldots, t} \) in stage \( \ell \). According to Lemma 2, we have \( E_t g_{t+1} = \nabla f(x_{t+1}) \). Taking expectation \( E_t \) on both sides of (8) and dividing them by \( \alpha \), we obtain that

\[
E_t A_2 = \frac{\alpha}{2} E_t \|g_{t+1}\|^2 + \frac{1}{2\alpha} \|z_t - x^*\|^2 - \frac{1}{2\alpha} E_t \|z_{t+1} - x^*\|^2. \tag{9}
\]

Because \( f \) is \( L \)-smooth, we have

\[
E_t f(y_{t+1}) \leq f(x_{t+1}) + E_t \langle \nabla f(x_{t+1}), y_{t+1} - x_{t+1} \rangle + \frac{L}{2} E_t \|y_{t+1} - x_{t+1}\|^2 \leq f(x_{t+1}) - \frac{\eta}{2} E_t \|\nabla f(x_{t+1})\|^2 + \frac{L\eta^2}{2} E_t \|g_{t+1}\|^2 \tag{10}
\]

Multiplying both sides of (10) by \( \frac{\alpha}{\eta} \) and adding it to (9) gives

\[
E_t A_2 \leq \frac{\alpha}{\eta} (f(x_{t+1}) - E_t f(y_{t+1})) + \frac{1}{2\alpha} \|z_t - x^*\|^2 - \frac{1}{2\alpha} E_t \|z_{t+1} - x^*\|^2 - \alpha \|\nabla f(x_{t+1})\|^2 + \left( \frac{L\eta + 1}{2} \right) \alpha E_t \|g_{t+1}\|^2. \tag{11}
\]

According to Lemma 2 again, we have

\[
E_t \|g_{t+1}\|^2 = \|\nabla f(x_{t+1})\|^2 + E_t \|g_{t+1} - E g_{t+1}\|^2 \leq \|\nabla f(x_{t+1})\|^2 + \frac{4L(N-b)}{b(N-1)} (\delta_{t+1} + \delta_0) \tag{12}
\]

For simplicity, let \( D = \frac{4L(N-b)}{b(N-1)} \). Applying (12) and \( \eta = \frac{1}{L} \) to (11)

\[
E_t A_2 \leq \frac{\alpha}{\eta} (f(x_{t+1}) - E_t f(y_{t+1})) + \frac{1}{2\alpha} \|z_t - x^*\|^2 - \frac{1}{2\alpha} E_t \|z_{t+1} - x^*\|^2 + \alpha D (\delta_{t+1} + \delta_0). \]

Combining this inequality with (6) and (7) and using the fact that \( \frac{1-\tau}{\tau} = \frac{\alpha}{\eta} \), we have

\[
(1 - \alpha D) \delta_{t+1} \leq \frac{\alpha}{\eta} (f(y_t) - E_t f(y_{t+1})) + \frac{1}{2\alpha} \|z_t - x^*\|^2 - \frac{1}{2\alpha} E_t \|z_{t+1} - x^*\|^2 + \alpha D \delta_0,
\]

15
whose expectation conditioning on $\tilde{x}^t$ is

$$(1 - \alpha D)E_\ell \delta_{t+1} \leq \frac{\alpha}{\eta} (E_\ell f(y_t) - E_\ell f(y_{t+1})) + \frac{1}{2\alpha} E_\ell \|z_t - x^*\|^2 - \frac{1}{2\alpha} E_\ell \|z_{t+1} - x^*\|^2 + \alpha D \delta_0.$$ 

Summing this inequality over $t = 0, 1, \ldots, T - 1$ and dividing by $T$ yields,

$$(1 - \alpha D)E_\ell \tilde{\delta} = \frac{1 - \alpha D}{T} \sum_{t=0}^{T-1} E_\ell \delta_{t+1} \leq \frac{\alpha}{\eta T} (f(y_0) - E_\ell f(y_T)) + \frac{1}{2\alpha T} \|z_0 - x^*\|^2 + \alpha D \delta_0.$$ 

Due to the fact that $x_0 = y_0 = z_0$, the $\mu$-strong convexity of $f$, and $f(y_T) \geq f(x^*)$, the inequality above implies

$$E_\ell \tilde{\delta} \leq \frac{1}{(1 - \alpha D)} \left( \frac{\alpha}{\eta T} + \frac{1}{\mu \alpha T} + \alpha D \right) \delta_0$$

$$\leq \frac{3}{2} \left( \frac{\sqrt{\kappa}}{T} + \frac{1}{\mu \alpha T} + \frac{1}{3} \right) \delta_0$$

$$\leq \frac{3}{2} \left( \frac{1}{12} + \frac{1}{12} + \frac{1}{3} \right) \delta_0 = \frac{3}{4} \delta_0,$$

where the second inequality is because $\alpha = \min \{ \frac{1}{3D}, \frac{1}{\sqrt{\mu L}} \}$ so that $\alpha \leq \frac{1}{3D}$ and $\alpha \leq \frac{1}{\sqrt{\mu L}}$ and the third inequality is because $T = \max \{ \frac{36D}{\mu}, 12\sqrt{\kappa} \}$ so that $\frac{\sqrt{\kappa}}{T} \leq \frac{1}{12}$ and $\frac{1}{\mu \alpha T} \leq \frac{1}{12}$. The conclusion is then implied by $E_\ell \tilde{\delta} \leq \frac{3}{4} \cdot \delta_0$. 

The proof of Theorem. The conclusion is obtained by applying Proposition recursively for $\ell = 0, 1, \ldots, K - 1$. 

□