M-curves of degree 9 with three nests

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Abstract

The first part of Hilbert’s sixteenth problem deals with the classification of the isotopy types realizable by real plane algebraic curves of a given degree $m$. For $m = 9$ the classification of the $M$-curves is still wide open. Let $C_9$ be an $M$-curve of degree 9 and $O$ be a non-empty oval of $C_9$. If $O$ contains in its interior $\alpha$ ovals that are all empty, we say that $O$ together with these $\alpha$ ovals, forms a nest. The present paper deals with the $M$-curves with three nests. Let $\alpha_i, i = 1, 2, 3$ be the numbers of empty ovals in each nest. We prove that at least one of the $\alpha_i$ is odd. This is a step towards a conjecture of A. Korchagin, claiming that at least two of the $\alpha_i$ should be odd.

1 Introduction

1.1 Real and complex schemes

The first part of Hilbert’s sixteenth problem deals with the classification of the isotopy types realizable by real plane algebraic curves of given degree. Let $A$ be a real algebraic non-singular plane curve of degree $m$. Its complex part $C_A \subset \mathbb{C}P^2$ is a Riemannian surface of genus $g = (m - 1)(m - 2)/2$; its real part $R_A \subset \mathbb{R}P^2$ is a collection of $L \leq g + 1$ circles embedded in $\mathbb{R}P^2$. If $L = g + 1$, we say that $A$ is an $M$-curve. A circle embedded in $\mathbb{R}P^2$ is called oval or pseudo-line depending on whether it realizes the class 0 or 1 of $H_1(\mathbb{R}P^2)$. If $m$ is even, the $L$ components of $R_A$ are ovals; if $m$ is odd, $R_A$ contains exactly one pseudo-line, which will be denoted by $J$. An oval separates $\mathbb{R}P^2$ into a Möbius band and a disc. The latter is called the interior of the oval. An oval of $R_A$ is empty if its interior contains no other oval. One calls exterior oval an oval that is surrounded by no other oval.

Two ovals form an injective pair if one of them lies in the interior of the other one. Let us call the isotopy type of $R_A \subset \mathbb{R}P^2$ the real scheme of $A$;
it will be described with the following notation due to Viro. The symbol \( \langle J \rangle \) stands for a curve consisting in one single pseudo-line; \( \langle n \rangle \) stands for a curve consisting in \( n \) empty ovals. If \( X \) is the symbol for a curve without pseudo-line, \( 1\langle X \rangle \) is the curve obtained by adding a new oval, containing all of the others in its interior. Finally, a curve which is the union of 2 disjoint curves \( \langle A \rangle \) and \( \langle B \rangle \), having the property that none of the ovals of one curve is contained in an oval of the other curve, is denoted by \( \langle A\Pi B \rangle \). The classification of the real schemes which are realizable by \( M \)-curves of a given degree in \( \mathbb{R}P^2 \) is part of Hilbert’s sixteenth problem. This classification is complete up to degree 7. For \( m \geq 8 \), one restricts the study to the case of the \( M \)-curves. The classification is almost complete for \( m = 8 \), and still wide open for \( m = 9 \). A systematic study of the case \( m = 9 \) has been done, the main contribution being due to A. Korchagin. See e.g. [11], [8], [9], [10], [13] for the constructions, and [6], [7], [11], [1], [3], [4], [14], [15] for the restrictions.

Let us briefly recall some facts about complex orientations. The complex conjugation \( \text{conj} \) of \( \mathbb{C}P^2 \) acts on \( \mathbb{C}A \) with \( \mathbb{R}A \) as fixed points sets. Thus, \( \mathbb{C}A \setminus \mathbb{R}A \) is connected, or splits in 2 homeomorphic halves which are exchanged by \( \text{conj} \). In the latter case, we say that \( A \) is dividing. Let us now consider a dividing curve \( A \) of degree \( m \), and assume that \( \mathbb{C}A \) is oriented canonically. We choose a half \( \mathbb{C}A_+ \) of \( \mathbb{C}A \setminus \mathbb{R}A \). The orientation of \( \mathbb{C}A_+ \) induces an orientation on its boundary \( \mathbb{R}A \). This orientation, which is defined up to complete reversion, is called complex orientation of \( A \). One can provide all the injective pairs of \( \mathbb{R}A \) with a sign as follows: such a pair is positive if and only if the orientations of its 2 ovals induce an orientation of the annulus that they bound in \( \mathbb{R}P^2 \). Let \( \Pi_+ \) and \( \Pi_- \) be the numbers of positive and negative injective pairs of \( A \). If \( A \) has odd degree, each oval of \( \mathbb{R}A \) can be provided with a sign: given an oval \( O \) of \( \mathbb{R}A \), consider the Möbius band \( \mathcal{M} \) obtained by cutting away the interior of \( O \) from \( \mathbb{R}P^2 \). The classes \( [O] \) and \( [2J] \) of \( H_1(\mathcal{M}) \) either coincide or are opposite. In the first case, we say that \( O \) is negative; otherwise \( O \) is positive. Let \( \Lambda_+ \) and \( \Lambda_- \) be respectively the numbers of positive and negative ovals of \( \mathbb{R}A \). The complex scheme of \( A \) is obtained by enriching the real scheme with the complex orientation: let e.g. \( A \) have real scheme \( \langle J \Pi I_1(\alpha) \Pi \beta \rangle \). The complex scheme of \( A \) is encoded by \( \langle J \Pi I_\epsilon(\alpha_+ \Pi \alpha_-) \Pi \beta_+ \Pi \beta_- \rangle \) where \( \epsilon \in \{+,-\} \) is the sign of the non-empty oval; \( \alpha_+, \alpha_- \) are the numbers of positive and negative ovals among the \( \alpha \); \( \beta_+, \beta_- \) are the numbers of positive and negative ovals among the \( \beta \) (remember that all signs are defined with respect to the orientation of \( J \)).
Rokhlin-Mishachev formula: If $m = 2k + 1$, then
$$2(\Pi_+ - \Pi_-) + (\Lambda_+ - \Lambda_-) = L - 1 - k(k + 1)$$

Fiedler theorem: Let $L_t = \{L_t, t \in [0, 1]\}$ be a pencil of real lines based in a point $P$ of $\mathbb{R}P^2$. Consider two lines $L_{t_1}$ and $L_{t_2}$ of $L_t$, which are tangent to $\mathbb{R}A$ at two points $P_1$ and $P_2$, such that $P_1$ and $P_2$ are related by a pair of conjugated imaginary arcs in $\mathbb{C}A \cap (\bigcup L_t)$.

Orient $L_{t_1}$ coherently to $\mathbb{R}A$ in $P_1$, and transport this orientation through $L_t$ to $L_{t_2}$. Then this orientation of $L_{t_2}$ is compatible to that of $\mathbb{R}A$ in $P_2$.

1.2 Results

The main result of the present paper is the following

**Theorem 1** Let $C_9$ be an $M$-curve of degree 9 with real scheme $\langle J \Pi 1(\alpha_1) \Pi 1(\alpha_2) \Pi 1(\alpha_3) \Pi \beta \rangle$. At least one of the $\alpha_i, i = 1, 2, 3$ is odd.

This Theorem represents a step towards a conjecture from [8].

**Conjecture 1** Let $C_9$ be an $M$-curve of degree 9 with real scheme $\langle J \Pi 1(\alpha_1) \Pi 1(\alpha_2) \Pi 1(\alpha_3) \Pi \beta \rangle$. At least two of the $\alpha_i, i = 1, 2, 3$ are odd.

Theorem 1 prohibits the 53 real schemes $\langle J \Pi 1(\alpha_1) \Pi 1(\alpha_2) \Pi 1(\alpha_3) \Pi \beta \rangle$ ($\alpha_1 + \alpha_2 + \alpha_3 + \beta = 25, \alpha_1 \leq \alpha_2 \leq \alpha_3$) with $\alpha_1, \alpha_2, \alpha_3$ even. Among them, the 12 ones with $\beta = 1$ had already been excluded by A. Korchagin in [11]. The proof is an improvement of the classical restriction methods. The latters combine Bezout’s theorem with auxiliary lines or conics, the Rokhlin and Rokhlin-Mishachev formulas, and Fiedler’s theorem. We use supplementarily rational cubics and quartics (single curves or pencils of such curves), and Orevkov’s complex orientation formulas for $M$-curves of degree $4d + 1$ with 4 nests [17]. All of the arguments, and hence the statements, are also valuable for pseudo-holomorphic curves.

We prove also a few results on complex orientations and rigid isotopy for the curves $C_9$ with some $\alpha_i$ odd.

2 First properties

2.1 Descriptive lemmas and definitions

Let $C_9$ be an $M$-curve of degree 9. Given an empty oval $X$ of $C_9$, we often will have to consider one point chosen in the interior of $X$. For simplicity,
we shall call this point also $X$. In the following, it will be clear from the context whether we speak of the oval or of the point $X$. We denote the pencil of lines based in $X$ by $\mathcal{F}_X$. Let $[XY]$, and $[XY]'$ be the two segments of line determined by $X$ and $Y$, cutting $\mathcal{J}$ respectively an even and an odd number of times. We say that $[XY]$ is the principal segment determined by $X, Y$. Let $X, Y, Z$ be three ovals of $C_9$. Corresponding three points $X, Y$ and $Z$ determine 4 triangles of $\mathbb{R}P^2$. We will call principal triangle and denote by $\triangle XYZ$ the triangle whose sides are the principal segments $[XY]$, $[YZ]$ and $[ZX]$. Let $C_2$ be a conic passing through 5 points $A, B, C, D, E$, in this ordering. Then, we write $C_2 = ABCDE$. If $F$ lies in the interior of $C_2$, we write $F \in C_2$.

**Definition 1** An ordered group of empty ovals $F_1, \ldots, F_n$ of $C_9$ is said to lie in a convex position if for each triple $F_i, F_j, F_k$, the principal triangle $F_iF_jF_k$ does not contain any other oval of the group and $F_1, \ldots, F_n$ are the successive vertices of $\bigcup F_iF_jF_k$ (the convex hull of the group).

**Definition 2** Let $O$ be a non-empty oval of $C_9$. We will say that $C_9$ has a jump in $O$ if there exist 2 empty ovals $B$ and $C$ inside of $O$, and 2 empty ovals $A$ and $D$ outside of $O$, such that: $A$ lies inside of an oval $O'$ different from $O$, and a line passing through $A$ and $D$ separates $B$ and $C$ in $\text{Int}(O)$ (Figure 1).

Let $C_9$ have real scheme $\langle \mathcal{J} II 1\langle \alpha_1 \rangle II 1\langle \alpha_2 \rangle II 1\langle \alpha_3 \rangle II \beta \rangle$. We shall call nest $O_i$ each configuration $1\langle \alpha_i \rangle$ formed by $O_i$ and its interior ovals. Let $O_1, O_2, O_3$ be the non-empty ovals of $C_9$, and $A_i, i = 1, 2, 3$ be empty ovals
of $O_3$. The lines $A_1A_2$, $A_2A_3$, $A_3A_1$ and the pseudo-line $J$ separate $\mathbb{R}P^2$ in 4 triangles $T_0, T_1, T_2, T_3$ and 3 quadrangles $Q_1, Q_2, Q_3$. Notice that, by Bezout's theorem, $J$ does not cut $T_0$ (Figure 2).

The lemmas 1, 2, 3, 5 hereafter are proven in the article [3].

Lemma 1 If $C_9$ has a jump, then $D$ is exterior.

Lemma 2 Assume $C_9$ has a jump, say in $O_3$. Let $A, B, C, D$ be ovals giving rise to the jump, with $A$ interior to $O_i, i = 1$ or 2. Let $A'$ be any other empty oval in $O_1 \cup O_2$. Then, $A', B, C, D$ also give rise to a jump.

Definition 3 Let $O$ be a non-empty oval of $C_9$ and $S$ be an oval of $C_9$ lying inside of another non-empty oval $O'$. The curve $C_9$ has $n$ jumps in $O$ with repartition $(l_1, \ldots, l_{2n+1})$ if a pencil of lines $F_S$ sweeping out $O$ meets successively $2n + 1$ groups of ovals, which are situated alternatively in, out, $\ldots$, in $\text{Int}(O)$ and have cardinals $l_1, \ldots, l_{2n+1}$.

It follows from Lemma 2 that the number of jumps in $O$ and their repartition does not depend on the choice of $S$. Thus Definition 3 is correct.

Lemma 3 Let $C_9$ have a jump in $O_3$. Let $B, C \in \text{Int}(O_3)$ and $D$ exterior be such that for any $A \in \text{Int}(O_i), i = 1, 2$, the line $AD$ separates $B$ from $C$ in $\text{Int}(O_3)$. Up to permutation of $B, C$, the ovals $A_1, A_2, C, D, B$ lie in convex position.
When we consider a curve $C_9$ with a jump, we actually ignore which of the two segments $[AD]$, $[AD]'$ cuts $O_3$. Figure 3 shows both possibilities.

**Lemma 4** Let $\{i, j, k\} = \{1, 2, 3\}$. All of the lines through 2 ovals interior to $O_i$ cut the same segment of line $[A_jA_k]$ or $[A_jA_k]'$.

**Proof** Assume $i = 1$. Let $A, B, C$ be 3 ovals in $\text{Int}(O_1)$. By Bezout’s theorem with the conic through $A, B, C, A_2, A_3$, the lines $AB, AC, BC$ must all cut the same segment $[A_2A_3]$ or $[A_2A_3]'$. □

**Definition 4** A non-empty oval $O_i$ of $C_9$ is separating if any line $(AA')$ joining two ovals of $\text{Int}(O_i)$ cuts the principal segment $[A_2A_3]$. Otherwise, $O_i$ is non-separating.

**Lemma 5** The curve $C_9$ has at most one jump.

**Lemma 6** If $C_9$ has a jump in $O_i$, then $O_i$ is non-separating.

**Proof** This follows immediately from Lemma 3.

**Definition 5** Let $C_9$ have a jump in $O_3$, and $A$ be any empty oval of $O_1 \cup O_2$. If $O_3$ cuts the principal segment $[AD]$, then $O_3$ is crossing, otherwise $O_3$ is non-crossing.

**Lemma 7** Let $C_9$ have a jump in $O_3$.

1. If $O_3$ is crossing, there are no ovals in $T_3$.
2. If $O_3$ is non-crossing, there are no ovals in $T_0 \cup T_1 \cup T_2$.

**Proof** Let $C_2$ be the conic through $A_1, A_2, B, C, E$. 6
1. Let $E \in T_3$. One has a priori $C_2 = A_1EA_2CB$ or $BA_1CA_2E$. Applying Bezout’s theorem with $C_9$, one gets: $C_2 = A_1EA_2CB$, and the arc $CB$ of $C_2$ lies inside of $O_3$. Thus, $D \notin C_2$, and $E \in A_1A_2CDB$. The conic $A_1A_2CDB$ cuts $C_9$ at 20 points. Contradiction.

2. Let $E \in T_0$, then $C_2 = A_1EA_2BC$ cuts $C_9$ at 20 points. Let $E \in T_1 \cup T_2$. By symmetry, we can suppose that $E \in T_1$. One has a priori $C_2 = A_1A_2ECB$ or $A_1A_2BCE$. By Bezout’s theorem with $C_9$, one must have $C_2 = A_1A_2ECB$, and the arc $CB$ of $C_2$ lies inside of $O_3$. Thus $D \in C_2$ and $E \in A_1A_2CDB$. The conic $A_1A_2CDB$ cuts $C_9$ at 20 points. Contradiction.

\[ \square \]

2.2 Complex orientations

Let $a_i^\pm$ be the numbers of positive and negative interior ovals of the nest $O_i$. Let $A$ be any empty oval of $O_j \cup O_k$. Consider the pencil of lines $\mathcal{F}_A$, sweeping out $O_i$. By Fiedler’s theorem, the empty ovals met by this pencil have alternating orientations. It follows from lemmas 1, 2 and 4 that the ordering of the ovals in the chain is independent from the choice of $A$. There is at most one jump in $O_i$, thus $|a_i^+ - a_i^-| \leq 2$. The equality occurs if and only if $O_i$ has a jump with repartition $l_1, l_2, l_3$, with each $l_n, n = 1, 2, 3$ odd. Let us call \textit{principal ovals} the ovals $O_1, O_2, O_3, A_1, A_2, A_3$. Let us call \textit{base ovals} the empty principal ovals $A_1, A_2, A_3$. Denote by $\epsilon_n, n = 1, 2, 3, 4, 5, 6$, $\epsilon_n \in \pm 1$ the respective contributions of these 6 ovals to $\Lambda_+ - \Lambda_-$. Let $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ be the contributions to $\Lambda_+ - \Lambda_-$ brought respectively by the non-principal ovals of the zones $T_0, Q_1, Q_2, Q_3, T_1, T_2, T_3$.

**Lemma 8** One has:

- $\lambda_0 + \lambda_1 - \lambda_4 = -\frac{1}{2}(\epsilon_3 + \epsilon_6 + \epsilon_2 + \epsilon_5)$,
- $\lambda_0 + \lambda_2 - \lambda_5 = -\frac{1}{2}(\epsilon_3 + \epsilon_6 + \epsilon_1 + \epsilon_4)$,
- $\lambda_0 + \lambda_3 - \lambda_6 = -\frac{1}{2}(\epsilon_2 + \epsilon_5 + \epsilon_1 + \epsilon_4)$,
- $3\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 + \sum \epsilon_i = 0$,
- $2(\lambda_0 - \lambda_4 - \lambda_5 - \lambda_6) = -(\Lambda_+ - \Lambda_-)$.

**Proof** Apply Fiedler’s Theorem to the pencils of lines $\mathcal{F}_{A_1} : A_3 \rightarrow T_0 \cup Q_1 \cup T_1 \rightarrow A_2$, $\mathcal{F}_{A_2} : A_1 \rightarrow T_0 \cup Q_2 \cup T_2 \rightarrow A_3$ and $\mathcal{F}_{A_3} : A_2 \rightarrow T_0 \cup Q_3 \cup T_3 \rightarrow A_1$. The last identity is obtained substracting $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \sum \epsilon_i = \Lambda_+ - \Lambda_-$ from the fourth identity. \( \square \)

**Lemma 9** If $\alpha_i$ is odd, then the oval $O_i$ is non-separating.
Proof Let $O_1$ be separating. By lemma 6, $O_1$ has no jump. Consider
the Fiedler chain formed by the empty ovals in $Int(O_1)$. Let $A_1$ and $A'_1$ be
the 2 extreme ovals of this chain, such that $A'_1 \in T_0$ and $A_1 \in T_1$. Take as
base ovals first the triple $(A_1, A_2, A_3)$ and then the triple $(A'_1, A_2, A_3)$. For
either case, we write the contributions of the triangles, the quadrangles and
the base ovals to $\Lambda_+ - \Lambda_-$. One has: $\epsilon'_i = \epsilon_i$ for $i = 1, 2, 3, 5, 6$;
$\lambda'_i = \lambda_i$ for $i = 1, 2, 3, 5, 6$.

1. $\alpha_1$ even, $A_1$ positive:
   \[\lambda'_0 = \lambda_0 + 1, \lambda'_4 = \lambda_4 + 1, \epsilon_4 = 1, \epsilon'_4 = -1\]
2. $\alpha_1$ even, $A_1$ negative:
   \[\lambda'_0 = \lambda_0 - 1, \lambda'_4 = \lambda_4 - 1, \epsilon_4 = -1, \epsilon'_4 = 1\]
3. $\alpha_1$ odd, $A_1$ positive:
   \[\lambda'_0 = \lambda_0 - 1, \lambda'_4 = \lambda_4 + 1, \epsilon_4 = \epsilon'_4 = 1\]
4. $\alpha_1$ odd, $A_1$ negative:
   \[\lambda'_0 = \lambda_0 + 1, \lambda'_4 = \lambda_4 - 1, \epsilon_4 = \epsilon'_4 = -1\]

Write the fourth identity in Lemma 8 for either choice of the base ovals.
Subtracting the one identity from the other, one gets:
\[3(\lambda_0 - \lambda'_0) - (\lambda_4 - \lambda'_4) - (\epsilon_4 - \epsilon'_4) = 0.\]
The two cases where $\alpha_1$ is odd yield a contradiction. \(\square\)

3 Inequalities

Let $C_9$ be, as in the previous section, an $M$-curve of degree 9 with real scheme
\[\langle J \Pi_1 \alpha_1 \Pi_1 \alpha_2 \Pi_1 \alpha_3 \Pi \beta \rangle.\]
Let us perform a Cremona transformation $cr : (x_0 : x_1 : x_2) \rightarrow (x_1 x_2 : x_0 x_2 : x_0 x_1)$ with base points $A_1, A_2, A_3$.
We shall denote the respective images of the lines $(A_1 A_2), (A_2 A_3), (A_3 A_1)$
by $A_3, A_1, A_2$. For the other points, we use the same notation as before $cr$. The curve $C_9$ is mapped onto a curve $C_{18}$ of degree 18 with 3 singular
points. We shall call main part of $C_{18}$ the piece formed by the images of $J$
and the principal ovals. See Figure 4 where $cr(A_i)$ and $cr(O_i)$ stand for the
images of the ovals $A_i$ and $O_i$.

An oval $A$ of $C_{18}$ will be said to be interior, exterior, positive or negative
if its preimage is. Let $O = cr(J)$. One has $Int(O) = cr(T_0 \cup T_1 \cup T_2 \cup T_3)$,
$Ext(O) = cr(Q_1 \cup Q_2 \cup Q_3)$. The ovals of $Int(O)$ and their preimages will
be called $O$-inner ovals; the ovals of $Ext(O)$ and their preimages will be
called $O$-outer ovals.
Figure 4: The singular curve $C_{18}$

**Definition 6** A base line $A_iA_j$ and a conic $C_2$ are mutually maximal if $C_2$ cuts $A_iA_j$ and $C_2$ cuts each component $cr(O_k)$ and $cr(A_k)$ at 4 points.

**Lemma 10** Let $\{i, j, k\} = \{1, 2, 3\}$. If a base point $A_i$ lies inside of a conic $C_2$, then the two base lines $A_iA_j$ and $A_iA_k$ are maximal with respect to $C_2$.

**Proof** Let $P_1, P_1' = C_2 \cap (A_2A_3)$, $P_2, P_2' = C_2 \cap (A_1A_3)$ and $P_3, P_3' = C_2 \cap (A_1A_2)$. If $A_i$ lies inside of $C_2$, then $C_2$ meets $P_k, P_j, P_k', P_j'$ in this ordering. Each arc joining two consecutive points cuts $cr(O_k), cr(A_k), cr(O_j)$ and $cr(A_j)$.

**Lemma 11** Let $C_2$ be a conic passing through 5 ovals $B_1, \ldots B_5$ of $C_{18}$. Let $A_i, A_j$ be 2 of the base points, lying outside of $C_2$, such that the line $A_iA_j$ cuts $C_2$. If any of the following conditions is verified, then $A_iA_j$ is maximal with respect to $C_2$.

1. Each arc of $C_2 \setminus (C_2 \cap A_iA_j)$ passes through an oval $B_m$ that is exterior to $cr(O_k)$, or cuts $O$.

2. $\text{Int}(C_2) \cup \text{Int}(O)$ is orientable and $A_i, A_j$ lie on different arcs of $O \setminus (O \cap C_2)$,

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We leave the proof to the reader.

**Lemma 12** Let $C_2$ be a conic passing through 5 ovals $B_1, \ldots, B_5$ of $C_{18}$, and having at least 4 intersection points with $\mathcal{O}$. Then one of the three base lines, say $A_1A_2$ is non-maximal with respect to $C_2$, and the points $A_1, A_2$ lie outside of $C_2$.

**Proof** If the three base lines are maximal with respect to $C_2$, then $C_2$ cuts the images of the principal ovals at 24 points, $\mathcal{O}$ at 4 points, and the union $\cup B_i, i = 1, \ldots, 5$ at 10 points. Contradiction. A base line, say $A_1A_2$ is non-maximal; Lemma 10 implies that $A_1, A_2$ lie outside of $C_2$. □

**Lemma 13** Let $C_2 = B_1B_2B_3B_4B_5$ verify: $\text{Int}(\mathcal{O}) \cup \text{Int}(C_2)$ is orientable, $B_1, B_5$ are $\mathcal{O}$-outer ovals, and each arc $B_iB_{i+1}$, $i = 1, \ldots, 4$ of $C_2$ cuts $\mathcal{O}$ an odd number of times. Then the arc $B_5B_1$ of $C_2$ does not cut $\mathcal{O}$. One of the base lines, say $A_1, A_2$ is non-maximal for $C_2$ and the arc $s$ of $\mathcal{O} \setminus (\mathcal{O} \cap C_2)$ containing $A_1, A_2$ has endpoints on two consecutive arcs of $C_2 = B_1B_2B_3B_4B_5$.

**Proof** Lemma 12 implies that a base line, say $A_1A_2$, is non-maximal for $C_2$; by Lemma 11 (2), the points $A_1, A_2$ lie on the same arc $s$ of $\mathcal{O} \setminus (\mathcal{O} \cap C_2)$, that is exterior to $C_2$. Bezout’s theorem applied to $C_{18}$ with the lines $B_lB_m$ implies that the endpoints of $s$ are either on two consecutive arcs of $C_2$, or on the same arc of $C_2$. In the first case, one can assume that the consecutive arcs are $B_1B_2$, $B_4B_3$. Assume that $B_5B_1$ cuts $\mathcal{O}$ and consider the conics $C_2(A_1) = A_1B_3B_4B_5B_1$ and $C_2(A_2) = A_2B_3B_4B_5B_1$. Both of them cut $\mathcal{O}$ at 6 points (Figure 5). One of the base points $A_j$, $j = 2, 1$ lies in the interior of the conic $C_2(A_i), i = 1, 2$. The preimage of $C_2(A_i)$ is a rational cubic $C_3(A_i)$ passing through $A_1, A_2, A_3, B_3, B_4, B_5, B_1$, with double point at $A_i$ (Figure 6). This cubic cuts: each of the ovals $A_i, O_i$ at 4 points, each of the other base ovals $A_k, O_k, A_j, O_j$ at 2 points, the set $\{B_3, B_4, B_5, B_1\}$ at 8 points, and $\mathcal{J}$ at 5 points. Hence in total 29 intersection points with $C_9$. Contradiction. In the second case, one can assume that the endpoints of $s$ are on $B_1B_2, B_2B_3$, or $B_3B_4$. With similar arguments as above, one gets again a contradiction, letting $C_2(A_i), i = 1, 2$ be respectively: $A_1B_2B_3B_4B_5$, $A_1B_4B_5B_1B_2$ and $A_1B_1B_2B_3B_4$. □

**Lemma 14** The curve $C_{18}$ cannot contain a configuration of 6 ovals $B_1, D_3, B_2, D_1, B_3, D_2$ lying in convex position, with $B_i \in \text{Int}(\mathcal{O})$, $D_i \notin \text{Int}(\mathcal{O})$
Figure 5: The arc $B_1B_5$ of $C_2$ cuts $O$

Figure 6:
Figure 7: Convex configuration of 6 ovals of $C_{18}$

**Proof** Let $B_i, D_i$ verify the conditions of the Lemma (Figure 7). The 6 segments $[B_iD_j]$ bounding the convex hull of the 6 points cut each $\mathcal{O}$ once. Consider the 3 conics $B_1D_3D_1B_2, B_3D_2D_3B_1$ and $B_2D_1D_2B_1D_3$. If $C_2$ is one of these 3 conics, let 2 base points lie on the same exterior arc of $\mathcal{O} \setminus (\mathcal{O} \cap C_2)$. By Lemma 13, these points lie in the interior of one of the other 2 conics (Figure 8). Contradiction. □

In the proofs of the next two propositions, we consider conics passing through some empty ovals of $C_{18}$. Several times, we find a conic that is maximal with respect to the 3 base lines. The maximality follows always from Lemma 11 (1): each base line separates on this conic a pair of exterior ovals. Let $L, L'$ be 2 lines and $D$ be a point, we denote by $(L, L', \hat{D})$ the sector $(L, L')$ that does not contain $D$.

**Proposition 1** Let the base ovals $A_1, A_2, A_3$ of $C_9$ be such that $T_0$ contains only exterior ovals of $C_9$. One has $|\lambda_0| \leq 3$. If $\lambda_0 = \pm 3$, then $\sum \epsilon_i = \mp 6$ and $\alpha_1, \alpha_2, \alpha_3$ are all even.

**Proof** Notice that the choice of $A_i$ is unique if $O_i$ is separating, and arbitrary if $O_i$ is non-separating. Perform the cremona transformation $cr$, and denote by $B_i, i = 1, \ldots, n$ the ovals of $C_{18}$ lying in $T_0$. Assume there exist $B_i, B_j, B_k, B_l$ such that $B_l$ lies in the triangle $B_iB_jB_k$ that does not cut the base lines. Consider the pencil of conics $\mathcal{F}_{B_iB_jB_kB_l}$, the conics of this pencil are all maximal with respect to the 3 base lines. Perform $cr^{-1}$, one obtains a pencil of rational quartics passing through $A_1, A_2, A_3, B_i, B_j, B_k, B_l$, the first 3 points being double points. Any quartic of the pencil meets: the
union of the principal ovals at 24 points, the set of ovals \( \{B_i, B_j, B_k, B_l\} \) at 8 points, and \( \mathcal{J} \) at 4 points. Hence in total 36 intersection points with \( C_9 \). The other empty ovals of \( C_9 \) cannot be swept out. Contradiction. Thus, the ovals of \( T_0 \) lie in convex position in this triangle. Consider the maximal pencils of lines \( \mathcal{F}_{B_i} \). Each of them gives rise to a cyclic ordering of all other ovals of \( C_{18} \). Consider two ovals \( B_i, B_j \) that are consecutive for some pencil \( \mathcal{F}_{B_k} \). Then, they are consecutive for any pencil based in another empty oval \( D \). Indeed, assume that \( B_iB_j \) are not consecutive for some pencil \( \mathcal{F}_D \). There exists a conic \( C_2 \) passing through \( B_i, B_j, B_k, D \) and a fifth oval, that is maximal with respect to the 3 base lines and cuts \( \mathcal{O} \) at 4 points, which is impossible. Thus, one may speak of a Fiedler chain of ovals in \( T_O \), without refering to a base point. Assume \(|\lambda_0| \geq 3\), so there are at least 3 distinct Fiedler chains of ovals in \( T_O \). Let \( B_1, B_2, B_3 \) be 3 extreme ovals of 3 such chains, with the same sign. Let \( \{i, j, k\} = \{1, 2, 3\} \), and denote by \( [B_iB_j] \) the segment \( B_iB_j \) contained in \( T_0 \). The pencil of lines \( \mathcal{F}_{B_i} \) sweeping out \([B_jB_k]\) must meet an oval \( D_i \) outside of \( T_O \), hence comes a configuration of 6 ovals. Consider the 3 conics determined by the 3 ovals in \( T_0 \) and 2 of the other ovals. By Bezout’s theorem with \( C_{18} \), these conics are: \( B_1D_3B_2D_1B_3, B_2D_1B_3D_2B_1, B_3D_2B_1D_3B_2 \). To each conic determined by 5 given points, we associate the pentagon having these points as vertices. Choose a line at infinity \( L \) that does not cut any of the pentagons interior to the 3 conics. The points \( B_1, D_3, B_2, D_1, B_3, D_2 \) lie in convex position in the affine plane \( \mathbb{R}P^2 \setminus L \). The hexagon \( \mathcal{H} = B_1D_3B_2D_1B_3D_2 \) gives rise to a natural cyclic ordering of the 6 lines supporting its edges. Let \( Z_k, k \in \{1, \ldots, 6\} \) be the
following facts. Let
rise to a cyclic ordering of all other ovals of
in

\[ B \]
\[ C \]
are consecutive for some pencil
\[ \mid \]
interior pentagons. The points
conics determined by the 3 ovals
\[ B \]
\[ \lambda \]
T
splits into 6 successive groups, that are alternatively inside and outside of
\[ T_0 \]
By Fiedler’s theorem:
\[ \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 = 0 \]
Combining this with the fourth identity in Lemma 8, one gets:
\[ 2\lambda_0 = -\sum \epsilon_i \]
Thus,
\[ |\lambda_0| \leq 3 \]
and if
\[ \lambda_0 = \pm 3 \]
than
\[ \sum \epsilon_i = \mp 6 \]
If
\[ |\lambda_0| = 3 \]
then
\[ C_9 \]
sweeping out the 6 triangles, and the pencils
\[ F_{D_i} \]
\[ i = 1, 2, 3 \]
sweeping out the 4 triangles that do not have
\[ D_i \]
as vertex. The cyclic chain of ovals splits into 6 successive groups, that are alternatively inside and outside of
\[ T_0 \]
Consider 2 ovals
\[ B_i, \]
that are consecutive for some pencil
\[ F_{B_k} \]
Then,
\[ B_iB_j \]
are consecutive for any other pencil
\[ F_D \]
based in another empty oval
\[ D \]
Thus, one may speak of a Fiedler chain of ovals in
\[ cr(T_i) \]
without referring to a base point. Assume that
\[ |\lambda_{l+3}| \geq 3 \]
so there are at least 3 distinct Fiedler chains of ovals in
\[ cr(T_i) \]
Let
\[ B_1, B_2, B_3 \]
be 3 extreme ovals of 3 such chains, with the same sign.
Let \[ \{i, j, k\} = \{1, 2, 3\} \]
and denote by
\[ [B_iB_j] \]
the segment
\[ B_iB_j \]
contained by
\[ c(T_i) \]
The pencil of lines
\[ F_{B_i} \]
sweeping out
\[ [B_jB_k] \]
meet an oval
\[ D_i \notin \{B_1, \ldots, B_n\} \]
hence comes a configuration of 6 ovals. Consider the 3 conics determined by the 3 ovals
\[ B_1, B_2, B_3 \]
and 2 of the other ovals. By Bezout’s theorem with
\[ C_{18} \]
these conics are:
\[ B_1D_3B_2D_1B_3, B_2D_1B_3D_2B_1, B_3D_2B_1D_3B_2 \]
Choose a line at infinity
\[ L \]
that does not cut any of the 3 interior pentagons. The points
\[ B_1, D_3, B_2, D_1, B_3, D_2 \]
lie in convex position in the affine plane
\[ \mathbb{R}P^2 \setminus L \]
The hexagon
\[ H = B_1D_3B_2D_1B_3D_2 \]
gives rise to a natural cyclic ordering of the 6 lines supporting its edges. Let
\[ Z_k, k \in \{1, \ldots, 6\} \]
be the 6 triangles that are supported by triples of consecutive
lines, such that $Z_k$ and $\mathcal{H}$ intersect along an edge. All of the remaining ovals of $C_{18}$ lie in $\cup Z_k$. There is a natural cyclic ordering of the empty ovals of $C_{18}$ given by the pencils of lines $\mathcal{F}_{B_i}, i = 1, 2, 3$ sweeping out the 6 triangles, and the pencils $\mathcal{F}_{D_i}, i = 1, 2, 3$ sweeping out the 4 triangles that do not have $D_i$ as vertex. By Lemma 14, one of the $D_i$, say $D_1$ lies inside of $\mathcal{O}$. Let $T = cr(Q_1) \cup cr(T_1)$ be the triangle $A_1A_2A_3$ containing the $B_j$. If a $D_i, i \in \{2, 3\}$ lies in $T$, then $D_i$ lies in $cr(Q_i)$. Whatever triangle $A_1A_2A_3$ contains $D_i$, either line $B_1D_3, B_1D_2$ must cut $\mathcal{O}$ twice in $T$. Thus, the ovals $D_2$ and $D_3$ lie outside of $\mathcal{O}$. Let $Z_1, Z_2$ be the triangles having $D_1$ as a vertex, they contain only $\mathcal{O}$-inner ovals. If $X$ is an $\mathcal{O}$-inner oval in $Z_3 \cup Z_4 \cup Z_5 \cup Z_6$, then $X \in \{B_1, \ldots, B_n\}$. The pencil of lines $\mathcal{F}_{D_3}$ sweeping out $\mathcal{O}$ meets successively $\mathcal{O}$-inner, $\mathcal{O}$-outer and again $\mathcal{O}$-inner ovals. □

4 M-curves with three nests and a jump

4.1 Pencils of rational cubics

Let $C_9$ be an $M$-curve of degree 9 with three nests and a jump. See Figure 3 where one makes $A_3 = B$. The 21 ovals of $C_9 \setminus \{O_1, O_2, O_3, A_1, A_2, A_3, C\}$ are swept out by the pencil of conics $\mathcal{F}_{A_1A_2A_3C} : A_1A_3 \cup A_2C \to A_1A_2 \cup A_3C \to A_2A_3 \cup A_1C$ if $O_3$ is crossing, and by the pencil of conics $\mathcal{F}_{A_1A_2A_3C} : A_1A_3 \cup A_2C \to A_1A_2 \cup A_3C$ if $O_3$ is non-crossing. In both cases, the 21 ovals are distributed in two Fiedler chains. Denote by $P, Q$ the pair of starting points of the chains, where $P$ is a points of tangency with $O_3$ and $Q$ is a point of tangency with $J$. Denote by $P', Q'$ the endpoints of the chains, where $P'$ is a points of tangency with $O_3$ and $Q'$ is a point of tangency with $J$. The pair of Fiedler chains is then: $\langle P \to P', Q \to Q' \rangle$ or $\langle P \to Q', Q \to P' \rangle$. Perform the cremona transformation $cr$ based in $A_1, A_2, A_3$. The upper and lower part of Figure 9 show either case $O_3$ crossing and $O_3$ non-crossing.

Lemma 15 Let $C_9$ be an $M$-curve of degree 9 with three nests and a jump. All of the ovals in $T_0 \cup T_1 \cup T_2 \cup T_3$ are consecutive for the maximal portion of $\mathcal{F}_{A_1A_2A_3C}$ formed by conics intersecting $J$. If $O_3$ is crossing, they form a Fiedler chain: $\mathcal{O}$-inner ovals $\to P'$. Thus, $\lambda_0 - \lambda_1 = \lambda_5 = 0$ or $-\epsilon_3$. If $O_3$ is non-crossing, they form a Fiedler chain: $P \to \mathcal{O}$-inner ovals. Thus, $\lambda_6 = 0$ or $-\epsilon_3$.

Proof Let $O_3$ be crossing, let $E \in T_0 \cup T_1 \cup T_2$ and $H$ be an oval met after $E$ by the pencil of conics $\mathcal{F}_{A_1A_2A_3C} : A_1A_3 \cup A_2C \to A_1A_2 \cup A_3C \to A_1C \cup A_2A_3$ We shall prove that $H$ must be also in $T_0 \cup T_1 \cup T_2$. Assume
Figure 9:
first that $E \in T_0 \cup T_1$. Perform the cremona transformation $cr(A_1, A_2, A_3)$, and consider the pencil of conics $\mathcal{F}_{A_1ECD}$. The possible positions for the double lines of this pencil are shown in Figure 10. The preimage of $\mathcal{F}_{A_1ECD}$ is the pencil of rational cubics $\mathcal{F}_{A_1A_2A_3ECD}$. This pencil has five singular cubics, whose images in $\mathcal{F}_{A_1ECD}$ are the three double lines and the two conics through $A_2$ respectively $A_3$. If $E \in T_0$, there is only one possible sequence of singular cubics for $\mathcal{F}_{A_1A_2A_3ECD}$ (Figure 11). If $E \in T_1$, there are four possible sequences of singular cubics for $\mathcal{F}_{A_1A_2A_3ECD}$ (see Figures 12, 13, 14, 15). Let $H$ be one of the remaining ovals of $C_{18}$. Let $E \in T_0$, $H$ is swept out in the portion $\mathcal{F}_{A_1ECD} : A_2A_1DCE \to A_1E \cup CD \to A_1EA_3DC \to A_1C \cup ED$. Moreover, if $H$ is met after $E$ by the pencil of lines $\mathcal{F}_C : A_2 \to A_3 \to A_1$, then $H$ lies inside of $\mathcal{O} = cr(\mathcal{J})$. Let $E \in T_1$, $H$ is swept out by $\mathcal{F}_{A_1ECD}$ in the portion:

- $A_2A_1ECD \to A_1E \cup CD \to A_1C \cup ED$ (case 1)
- $A_1DCEA_3 \to A_1E \cup CD \to A_1C \cup ED$ (case 2)
- $A_2A_1ECD \to A_1E \cup CD \to A_1C \cup ED$ (case 3)
- $A_2A_1ECD \to A_1E \cup CD \to A_1EA_3DC \to A_1C \cup ED$ (case 4)

In all cases, if $H$ is met after $E$ by the pencil of lines $\mathcal{F}_C : A_2 \to A_3 \to A_1$, then $H$ lies inside of $\mathcal{O} = cr(\mathcal{J})$. Let $E \in T_2$. Notice that there are four possibilities for the pencil of rational cubics $\mathcal{F}_{A_2A_2A_1CDA_3}$, which are deduced from the pencils $\mathcal{F}_{A_1A_1A_2A_3ECD}, E \in T_1$ by an axial symmetry switching $(A_1, A_3)$ with $(A_2, C)$. The result follows immediately. □
Figure 11: $E \in T_0$
Figure 12: $E \in T_1$ case 1
Figure 13: \( E \in T_1 \), case 2
Figure 14: $E \in T_1$, case 3
Figure 15: $E \in T_1$, case 4
Figure 16: $F \in T_3$
Let $O_3$ be non-crossing and $F \in T_3$. Let $H$ be an oval met before $F$ by the pencil of conics $F_{A_1A_2A_3C} : A_1A_3 \cup A_2C \to A_1A_2 \cup A_3C$. We shall prove that $H$ must also be in $T_3$. Perform the cremona transformation $cr$, and consider the pencil of conics $F_{A_3FCD}$. The double lines of this pencil are shown in Figure 10. The preimage of $F_{A_3FCD}$ is the pencil of rational cubics $F_{A_1A_2A_3A_3FCD}$. This pencil has five singular cubics, whose images in $F_{A_3FCD}$ are the three double lines and the two conics through $A_1$ respectively $A_2$. There is only one possible sequence of singular cubics for $F_{A_1A_1A_2A_3A_3ECD}$ (Figure 16). Let $H$ be one of the remaining ovals of $C_{18}$, $H$ is swept out by $F_{A_3FCD}$ in the portion $A_3A_1FDC \to A_3F \cup CD \to A_3A_2FCD$. Moreover, if $H$ is met before $F$ by the pencil of lines $F_{C} : A_2 \to A_3$, then $H$ lies inside of $O$. □

Lemma 16 Let $C_9$ be an $M$-curve of degree 9 with a jump. One of the three possibilities hereafter arises:

1. $\lambda_0 - \lambda_4 - \lambda_5 - \lambda_6 = 0$ and $\Pi_+ - \Pi_- = 4$
2. $O_3$ is crossing, $\lambda_0 - \lambda_4 - \lambda_5 = -1$, $\epsilon_3 = 1$, $\Pi_+ - \Pi_- = 3$
3. $O_3$ is non-crossing, $\lambda_6 = 1$, $\epsilon_3 = -1$, $\Pi_+ - \Pi_- = 3$

Proof It follows immediately from Lemma 15 and the fact that $\Pi_+ - \Pi_- \leq 4$.

5 Complex orientations again

5.1 Orevkov’s complex orientation formulas

Let $C_m$ be an $M$-curve of degree $m = 4d + 1$, $d \geq 2$. In this subsection, we shall call nest $N$ of depth $n$ a configuration of ovals $(o_1, \ldots, o_n)$ such that $o_i$ lies in the interior of $o_j$ for all pairs $i, j$ with $j > i$. A nest is maximal if it is not a subset of a bigger nest of $C_m$. We assume that there exist 4 maximal nests $N_i, i \in \{1, 2, 3, 4\}$ of $C_m$ verifying: if $F$ is a pencil of conics based in the 4 innermost ovals of the nests, any conic of $F$ intersects the union of the 4 nests and $J$ at least $2m - 2$ points. Let $V_i$ be the outermost oval of the nest $N_i$. We shall call big ovals the ovals that belong to the union of the nests $N_i$, and small ovals the other ovals. For $S, s \in \{+, -\}$, let $\pi^S_s(N_i)$ be the numbers of pairs of ovals $(O, o)$ with signs $(S, s)$ such that $O$ is an oval of $N_i$ and $o$ is an empty oval contained in $Int(O)$. Let $\pi_i = (\pi^+_i - \pi^-_i)(N_i)$, $\pi'_i = (\pi^+_i - \pi^-_i)(N_i)$ Let $\Pi^S_s(N_i)$ be the number of pairs $(O, o)$ with signs
the two extreme interior ovals, such that

\[ O \]

is negative, we say that

\[ S \]

simpler one, writing:

\[ \mu \]

encoded as follows: 1 of the 3 nests

\[ O \]

Let

\[ p_1, \ldots, p_4 \]

be 4 points distributed in the innermost ovals of the 4 nests. If \( N_i, N_j, N_k \) have all depth \( d \), call principal triangle \( p_ip_jp_k \) the triangle \( p_ip_jp_k \) that does not intersect \( J \). The formulas hereafter are proven in [17] (with slightly different notations):

**First complex orientation formula (Orevkov)** Let \( C_m \) be such that the nests \( N_i, i \in \{1, 2, 3, 4\} \) have respective depths \( d, d, d - 1 \), and \( p_1, l \in \{1, 2, 3, 4\} \) lies in the principal triangle determined by the other three points \( p_i, p_j, p_k \), then:

\[ \pi_i + \pi_j + \pi_k + \pi'_l = N_i^2 + N_j^2 + N_k^2 + M_l^2 \]

**Second complex orientation formula (Orevkov)** Let \( C_m \) be such that: the nests \( N_l, l \in \{1, 2, 3, 4\} \) have all depth \( d \), some \( V_i, i \in \{1, 2, 3\} \) coincides with \( V_4 \), but the nests \( N_1', N_2', N_3', N_4' \) are pairwise disjoint, with \( N_i' = N_i \setminus V_i \). Let us denote by \( V \) the oval \( V_i = V_4 \), and by \( T \) be the principal triangle \( p_1p_2p_3 \). Let

\[ \Pi_l = \Pi^+(N_l') - \Pi^-(N_l'), \Pi'_l = \Pi^+(N_l) - \Pi^-(N_l). \]

Let \( \text{Int}^+(V) = \text{Int}(V) \setminus T, \text{Int}^-(V) = \text{Int}(V) \cap T. \) For any big oval \( O \neq V \subset N_l, \) let \( \text{Int}^\pm(O) = \text{Int}(O) \). For \( l \in \{i, 4\} \), denote by \( \tilde{\Pi}_l(N_l') \) the numbers of pairs \( (O, o) \) with signs \( (S, s) \) where \( O \subset N_l \) is big and \( o \subset \text{Int}^S(O) \) is small. Let

\[ \tilde{\Pi}_l = \tilde{\Pi}_l^+(N_l') - \tilde{\Pi}_l^+(N_l), \]

\[ \tilde{\Pi}'_l = \tilde{\Pi}_l^-(N_l') - \tilde{\Pi}_l^-(N_l). \]

If \( p_i, i \in \{1, 2, 3\} \) lies in the principal triangle \( p_ipkp_4 \), then:

\[ \tilde{\Pi}'_i + \Pi_j + \Pi_k + \tilde{\Pi}_4 = Q_i^2 - 2Q_i + P_j^2 - P_j + P_k^2 - P_k + P_4^2 - P_4 + \nu(V), \]

where \( \nu(V) = 0 \) if \( V \) is positive, and 1 if \( V \) is negative.

### 5.2 Proof of the conjecture for the case \( \alpha_1, \alpha_2, \alpha_3 \) even

Let \( C_9 \) be an \( M \)-curve of degree 9 with real scheme \( \langle J \ 1 \ 1 \alpha_1 \ 1 \alpha_2 \ 1 \alpha_3 \ 1 \beta \ \rangle \). The complex scheme of \( C_9 \) is determined by the complex schemes of the 3 nests \( \mathcal{O}_i = 1(\alpha_i), i \in \{1, 2, 3\} \). The complex scheme \( \mathcal{S}_i \) of a nest \( \mathcal{O}_i \) is encoded as follows: 1,\( \langle a_i^+ \Pi a_i^- \rangle \), where \( a_i^+ - a_i^- = \alpha_i, a_i^+ - a_i^- \in \{0, \pm 1, \pm 2\}. \) Let \( \mu_i \in \pm \) be the sign of \( a_i^+ - a_i^- \). We replace the standard encoding by a simpler one, writing: \( \mathcal{S}_i = \nu_i \) if \( a_i^+ - a_i^- = 0, \mathcal{S}_i = (\nu_i, \mu_i, \mu_i) \) if \( a_i^+ - a_i^- = \pm 2 \)

\[ \mathcal{S}_i = (\nu_i, \mu_i) \] if \( a_i^+ - a_i^- = \pm 1. \) Assume \( \mathcal{O}_i \) is separating. Let \( A_i, A'_i \) be the two extreme interior ovals, such that \( A_i \in T_i' \) and \( A'_i \in T_0. \) If \( A'_i \) is negative, we say that \( \mathcal{O}_i \) is \( (\nu_i, u) \), otherwise \( \mathcal{O}_i \) is \( (\nu_i, d) \), where the letters
Lemma 17 Let $C_9$ have some exterior oval $B \in T_i$, $i \in \{0, 1, 2, 3\}$. Then $E_i = 0$, where

\[
\begin{align*}
E_0 &= \pi_1 + \pi_2 + \pi_3 - (N_1 + N_2 + N_3), \\
E_1 &= \pi'_1 + \pi_2 + \pi_3 - (M_1 + N_2 + N_3), \\
E_2 &= \pi_1 + \pi'_2 + \pi_3 - (N_1 + M_2 + N_3), \\
E_3 &= \pi_1 + \pi_2 + \pi'_3 - (N_1 + N_2 + M_3).
\end{align*}
\]

Proof The first formula applies making $N_i$, $i = 1, 2, 3$ and $N_4 = B$. □

Let $O_i$ be separating. Remember that by Lemma 9, $\alpha_i$ must be even. Again, choose base ovals $A_1, A_2, A_3$, and let $A_4$ be a fourth oval, interior to $O_i$, lying in $T_i$. Let $N_i = (A_i, O_i)$, $l = 1, 2, 3$ and $N_4 = (A_4, O_i)$. Let:

\[
F_i = \Pi'_{l} + \Pi_4 - (Q^2_l - 2Q_l + P^2_4 - P_4 + \nu(O_i)), \quad \text{and} \quad G_i = P^2_l - P_1 - \Pi_i.
\]

It is easily seen that $G_i$ depends only on $S_i$, and $F_i$ depends only on $S_i$.

Lemma 18 Let $C_9$ have three nests and separating $O_i$. Then, $F_i = G_j + G_k$.

Proof The second formula applies with the nests $N_i, l = 1, 2, 3, 4$. □

In Figure 17, 18 we computed the terms appearing in Lemmas 17, 18.

Lemma 19 Let $C_9$ be an M-curve of degree 9 with three nests. If the union $T_0 \cup T_1 \cup T_2 \cup T_3$ is empty, then $C_9$ verifies: $S_1, S_2 \in \{(+,-), (-,+)\}$, $S_3 \in \{(+,-,-), (-,+,+)\}$.

Proof One has $\lambda_0 - \lambda_4 - \lambda_5 - \lambda_6 = 0$, $\Lambda_+ - \Lambda_- = 0$, $\Pi_+ - \Pi_- = 4$. □

Theorem 1 Let $C_9$ be an M-curves of degree 9 with real scheme $\langle J \Pi 1(\alpha_1) \Pi 1(\alpha_2) \Pi 1(\alpha_3) \Pi \beta \rangle$. At least one of the $\alpha_i$, $i = 1, 2, 3$ is odd.

Proof By Lemma 16, $C_9$ has no jump, so it can realize 4 complex schemes. The last column $Z$ of Figure 19 contains the indices of the triangles $T_i, i \in$
\[
\begin{array}{cccccc}
S_l & \pi_l & \pi'_l & N_l & M_l & G_l \\
- & 0 & 0 & 0 & 1 & 0 \\
+ & 0 & 0 & 1 & 0 & 1 \\
(-,+)^{+} & 0 & 1 & 0 & 1 & 0 \\
(+,-) & 1 & 0 & 1 & 0 & 0 \\
(-,-) & 0 & -1 & 0 & 1 & 0 \\
(+,+) & -1 & 0 & 1 & 0 & 2 \\
(,+,-) & 0 & 2 & 0 & 1 & 0 \\
(+,-,-) & 2 & 0 & 1 & 0 & -1 \\
(-,-,-) & 0 & -2 & 0 & 1 & 0 \\
(+,+,-) & -2 & 0 & 1 & 0 & 3 \\
\end{array}
\]

Figure 17:

\[
\begin{array}{cc}
S_i & F_i \\
(-,d) & 0 \\
(-,u) & -1 \\
(+,d) & 0 \\
(+,u) & -1 \\
\end{array}
\]

Figure 18:

\{0, 1, 2, 3\} that may contain exterior ovals. In Figure 20, we assume that \(O_i\) is separating and compute the term \(F_i - G_j - G_k\). In Figure 21, we list the a priori possible complex types for \(C_9\) together with the data \(Z\). The first two types contradict to Lemma 19. For the other types, choose each \(A_i, i \in \{1, 2, 3\}\) in such a way that \((A_i, O_i)\) is a positive pair, compute the values of the \(\lambda_l, l \in \{0, \ldots, 6\}\) combining the identities: \(\lambda_0 - \lambda_4 - \lambda_5 - \lambda_6 = -4, \lambda_0 + \lambda_1 - \lambda_4 = 0, \lambda_0 + \lambda_2 - \lambda_5 = 0, \lambda_0 + \lambda_3 - \lambda_6 = 0\). For the last 3 cases, one gets \(\lambda_0 \leq -4\), which contradicts to Proposition 1. For the remaining 2 cases, one has \(\lambda_6 = 4\) or 5, which contradicts to Proposition 2. □

\[
\begin{array}{cccccccc}
S_1 & S_2 & S_3 & E_0 & E_1 & E_2 & E_3 & Z \\
- & - & - & 0 & -1 & -1 & -1 & (0) \\
- & - & + & -1 & -2 & -2 & 0 & (3) \\
- & + & + & -2 & -3 & -1 & -1 & \emptyset \\
+ & + & + & -3 & -2 & -2 & -2 & \emptyset \\
\end{array}
\]

Figure 19:
\[
\begin{array}{cccc}
\bar{S}_1 & S_j & S_k & F_i - G_j - G_k \\
(-, d) & - & - & 0 \\
(-, d) & - & + & -1 \\
(-, d) & + & + & -2 \\
(-, u) & - & - & -1 \\
(-, u) & - & + & -2 \\
(-, u) & + & + & -3 \\
(+, d) & - & - & 0 \\
(+, d) & - & + & -1 \\
(+, d) & + & + & -2 \\
(+, u) & - & - & -1 \\
(+, u) & - & + & -2 \\
(+, u) & + & + & -3
\end{array}
\]

Figure 20:

\[
\begin{array}{cccc}
\bar{S}_1 & \bar{S}_2 & \bar{S}_3 & Z \\
(+, n) & (+, n) & (+, n) & 0 \\
(-, n) & (+, n) & (+, n) & 0 \\
(-, n) & (-, n) & (+, n) & (3) \\
(-, n) & (-, n) & (+, d) & (3) \\
(-, n) & (-, n) & (-, n) & (0) \\
(-, d) & (-, n) & (-, n) & (0) \\
(-, d) & (-, d) & (-, n) & (0) \\
(-, d) & (-, d) & (-, d) & (0)
\end{array}
\]

Figure 21:
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