ON SINGULAR LAGRANGIANS AFFINE IN VELOCITIES

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ABSTRACT. The properties of Lagrangians affine in velocities are analyzed in a geometric way. These systems are necessarily singular and exhibit, in general, gauge invariance. The analysis of constraint functions and gauge symmetry leads us to a complete classification of such Lagrangians.

1. Introduction

The Dirac’s method of dealing with constrained systems [1-3], developed when looking for a way of quantizing such systems by means of a “canonical quantization”-like procedure, has been shown to be a very efficient method and the geometric approach to the theory of systems defined by singular Lagrangians has deserved a big amount of attention during the past years.

In particular, Lagrangians that are affine in time-derivatives have been analysed [4] in the framework of presymplectic geometry and the geometric theory of time-independent singular Lagrangians [5]. They are important because their Euler-Lagrange equations become systems of first-order differential equations, appearing as constraints, instead of systems of second-order as it would happen with regular Lagrangians. So they will play a relevant rôle in many cases, not only in Physics, where many equations are first-order, as in Dirac equation, but also in other fields as in biology dynamics and Chemistry. On the other side, these systems are singular and then they are giving rise to gauge ambiguities and gauge symmetries.

The geometric study of a particular type of singular Lagrangian, those which are affine in velocities, was carried out in [4] in the framework of autonomous systems, with the aim of studying the inverse problem of Lagrangian mechanics and the theory of non-point symmetries from a new geometric perspective. Almost simultaneously Faddeev and Jackiw developed a method for the quantization of such singular systems which became soon very popular and received much attention from most part of theoretical physicists. This procedure of dealing with such systems is usually referred to as Faddeev–Jackiw method of quantization [6-12].

One of the most important properties of singular Lagrangians is the existence of infinitesimal gauge symmetries which are related to the Second Noether theorem. This,
which is particularly important for field theories, has only sense in the framework of
time-dependent systems, the time playing the rôle of base coordinates in field theory.
A geometric approach to Second Noether Theorem was given in [13], but the geometric
theory of the time-dependent description of such affine in velocities Lagrangian systems
has never been developed, as far as we know, even if it is very important as the only way
of fully-understanding the meaning of Noether’s Second Theorem establishing the relation-
ship between singular Lagrangians and gauge-transformations. Then we feel that a
re-examination of the problem of singular Lagrangian that are first-order in velocities
will be very useful and will allow us a better understanding of the theory.

The two fundamental aspects of these systems described by these first-order in velocities
Lagrangians are the presence of first order equations and constraints, which make possible
this alternative Faddeev–Jackiw method of quantization [6,14]. This method is based on
the reduction theory for the presymplectic form defined by the singular Lagrangian, but
it admits an alternative by means of the addition of the constraints with some Lagrange
multipliers in order to obtain the symplectic extension [11,12,15,16]. Once the symplectic
structure has been got and, therefore, Poisson brackets have been defined, we can make use
of the canonical quantization procedure. In particular, the FJ-method uses the reduction
technique and has been applied in many different fields, running form condensed matter
[17-19] and astrophysics [20], until fluid dynamics [21] and, of course, high-energy physics
[22]. Even the own Schrödinger equation can be derived using this method [23].

It is also to be noted that the FJ-method can be generalized to include also fermionic
degrees of freedom, i.e. non-commutative variables [24] and the corresponding canonical
quantization can also be used in super-symmetric theories [8,25,26] with applications in
super-gravity (see e.g. [27]).

The paper is then organized as follows. In Section 2 we summarize the results of [4]. A
Lagrangian approach to Hamilton equations from a geometric perspective will be given in
Section 3. The framework for the geometric description of time-dependent singular systems
as given in [28] will be indicated in Section 4, where we will also include a recipe obtained
from [13] (see also [28]) for the search of gauge symmetries for singular Lagrangian systems.
The geometric theory of time-dependent Lagrangians which are affine in the velocities will
be developed in Section 5 and the gauge invariance of such systems will be studied in
Section 6. Finally, the theory is illustrated with several examples.

As a matter of notation, tangent and cotangent bundles will be denoted by $\tau_M : T M \to M$ and $\pi_M : T^* M \to M$, respectively. The set of vector fields along a map
$f : M \to N$ (see e.g. [29]), also called $f^*$-derivations in [30], i.e. maps $X : M \to T N$ such that $\tau_N \circ X = f$, will be denoted $\mathcal{X}(f)$. Examples of such kind of vector
fields are $T f \circ Y$ and $Z \circ f$, where $Y \in \mathcal{X}(M)$, $T f : T M \to T N$ is the tangent map
corresponding to $f$, and $Z \in \mathcal{X}(N)$. It has been shown in Pidello and Tulczyjew [30]
that a vector field $X$ along $f$ determines two $f^*$-derivations of scalar forms on $M$: one
of type $i_*$ and degree $-1$, denoted $i_X$, and other of type $d_*$, denoted $d_X$, defined in the
following way: given $\alpha \in \Lambda^p(N)$ and $v_1, \ldots, v_{p-1} \in T_x M$, we define $i_X \alpha \in \Lambda^{p-1}(M)$ by
$\langle i_X \alpha(x), (v_1, \ldots, v_{p-1}) \rangle = \langle \alpha(f(x)), (X(m), f_{sx}(v_1), \ldots, f_{sx}(v_{p-1})) \rangle$, and $d_X \alpha \in \Lambda^p(M)$
is given by $d_X \alpha = d_{iX} \alpha + i_X d\alpha$ [31].
2. GEOMETRIC APPROACH TO TIME-INDEPENDENT LAGRANGIANS WHICH ARE AFFINE IN THE VELOCITIES

In the geometric description of a time-independent Lagrangian system the states are described by points of the tangent bundle $\tau_Q : TQ \to Q$ of the configuration space $Q$, which is assumed to be an $n$-dimensional differentiable manifold. We are interested in the geometric study of systems described by a Lagrangian including only terms up to first order in velocities, namely, with a coordinate expression

$$L(q, v) = m_j(q) v^j - V(q),$$

where summation on repeated indices is understood.

The property of a function being linear in the fibre coordinates (velocities) of the tangent bundle is intrinsic because it is preserved under point transformations. If $\phi \in C^\infty(TQ)$ takes the form $\phi = \phi_i(q) v^i$ in a particular set of local coordinates, then, under the change of coordinates $q^i = q^i(q)$, we have that in order to

$$\phi_i(q) v^i = \phi'_i(q') \left( \frac{\partial q'^i}{\partial q^j} \right) v^j$$

the components $\phi_i(q)$ should transform as

$$\phi_j(q) = \phi'_i(q') \left( \frac{\partial q'^i}{\partial q^j} \right), \quad j = 1, \ldots, n,$$

i.e., the functions $\phi_i(q)$’s should transform as the components of an associated basic 1-form $\alpha \in \bigwedge^1(TQ)$, $\alpha = \phi_i(q) dq^i$. This suggests us that there is thus a one-to-one linear correspondence between basic 1-forms and linear homogeneous functions that we establish next: if $\mu \in \bigwedge^1(Q)$ is a 1-form, then $\tilde{\mu}$ denotes the function $\tilde{\mu} \in C^\infty(TQ)$ defined by $\tilde{\mu} = \iota_t \mu$, where $t \in X(\tau_Q)$ is the identity map $id : TQ \to TQ$ viewed as a vector field along the tangent bundle projection $\tau_Q$; in fibred coordinates, $t = v^i \partial/\partial q^i$. That is, $\tilde{\mu}(q, v) = \langle \mu(q), t(q, v) \rangle$, the contraction making sense because $t(q, v) \in T_q Q$. In coordinates, when $\mu = m_j(q) dq^j$, $\tilde{\mu}(q, v) = m_i(q) v^i$.

The geometric theory for systems described by a Lagrangian $L = \tilde{\mu} - \tau_Q^* V$, with $V \in C^\infty(Q)$, that in coordinates of the bundle induced from those of a chart in $Q$ becomes $L = m_j(q) v^j - V(q)$, has been studied in [4]: the energy $E_L$ and the presymplectic form $\omega_L$ are given by $E_L = \tau_Q^* (V)$, $\theta_L = \tau_Q^* \mu$, and, therefore, $\omega_L = -\tau_Q^* (d\mu)$, which in coordinates reads as follows:

$$\omega_L = \left( \frac{\partial m_i}{\partial q^k} - \frac{\partial m_k}{\partial q^i} \right) dq^i \wedge dq^k.$$

The Hessian matrix $W$ with elements $W_{ij} = \partial^2 L/\partial v^i \partial v^j$ vanishes identically and, therefore, all the $\tau_Q$-vertical vectors, i.e. $\xi^i(q,v) \partial/\partial v^i$, are in the kernel of $\omega_L$.

The search for the other elements in the kernel of $\omega_L$ starts by looking for a local basis $\{Z_a = \{z^i_a\}\}$, of the module of eigenvectors (if any) corresponding to the null eigenvalue of the matrix $A$ with elements given by elements

$$A_{ij} = \frac{\partial m_j}{\partial q^i} - \frac{\partial m_i}{\partial q^j}, \quad i, j = 1, \ldots, n.$$
Then, a basis for the kernel of $\omega_L$ is generated by the vector fields $X_a = z_a^i(q) \partial/\partial q^i$ and $\partial/\partial v^i$, with $a = 1, \ldots, n - n_0$, where $n_0$ denotes the rank of the matrix $A$. The primary constraint submanifold $M$ is then determined by the constraint functions (see e.g. [5]) $\phi_X = X E_L$, with $X \in \text{Ker} \omega_L$, that in the case we are considering are $\phi_a = X_a E_L$, i.e.

$$\phi_a = z_a^i(q) \frac{\partial V}{\partial q^i},$$

(2.4)

because the energy is $\tau_Q$-projectable, $E_L = \tau^* V$, and then $\partial E_L/\partial v^i \equiv 0$.

The general solution for the dynamical equation $i(X) \omega_L = dE_L$ will be given by

$$X = [\eta^i + \lambda^a z_a^i] \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i},$$

(2.5)

with $\eta^i(q)$ being a solution of

$$A_{ij} \eta^j = \frac{\partial V}{\partial q^i},$$

(2.6)

and where $\lambda^a$ and $f^i$ are arbitrary functions on $TQ$.

There is a special class of vector fields $D$ in $TQ$ which are called second order vector fields, hereafter shortened as SODE fields, which are characterized by $S(D) = \Delta$, where $S$ denotes the vertical endomorphism [32-34] and $\Delta$ is the Liouville vector field generating dilations along the fibres. They can also be characterized by $T\tau_Q \circ D = t$.

The constraint functions for a Lagrangian given by (2.1) are basic functions, i.e. holonomic constraints, defining a submanifold $Q'$ of $Q$. Consequently, the secondary constraints functions for the existence of a solution restriction of second order vector field will be $\delta \phi_a$ and are given by linear functions in the velocities. A solution $X$ given by (2.5) can be the restriction of a SODE only in those points of $TQ$ for which

$$A_{ij} v^j = \frac{\partial V}{\partial q^i}.$$  

(2.7)

In these points, the general solution of the dynamical equation is given by

$$X = [v^i + \lambda^a z_a^i] \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i},$$

(2.8)

while the SODE condition corresponds to the choice $\lambda^a = 0$.

The particularly simple case in which $(Q, d\mu)$ is a symplectic manifold, i.e. $\det A \neq 0$, and therefore $Q$ is even dimensional, was also studied in [4], where it was shown that then $\text{Ker} \omega_L$ is made of all $\tau_Q$-vertical vectors and therefore there will be no dynamical constraint function. The globally defined solution of the dynamical equation is

$$X = \eta^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial v^i},$$

with the functions $\eta^i$ uniquely determined by

$$\eta^i = (A^{-1})^i j \frac{\partial V}{\partial q^j}.$$  

(2.9)
The Marsden–Weinstein theory of reduction for the presymplectic system defined by the Lagrangian (2.1), \((TQ, \omega_L = -\tau_Q(d\mu), E_L = \tau_Q(V))\), establishes that the reduced symplectic manifold is \((Q, -d\mu)\) with Hamiltonian function \(V\).

The Hamiltonian formalism for the Lagrangian (2.1) was also studied in [4]: the primary constraint submanifold \(P_1 = \mathcal{F}L(TQ)\) is determined by \(\Phi_j(q, p) = p_j - m_j(q) = 0\), i.e. is the graph of the form \(\mu\). Using the identification of it with the base \(Q\) the pull-back of the canonical 1-form \(\omega_0\) in \(T^*Q\) is \(-d\mu\). The Poisson brackets of the constraint functions are \(\{\Phi_j, \Phi_k\} = \{p_j - m_j(q), p_k - m_k(q)\} = A_{jk}\), and therefore, when \(d\mu\) is symplectic, all the constraints are of the second class. The Hamiltonian function \(H\) is defined on \(P_1\) by the restriction of the function \(\tilde{V} = \pi_Q^*V\). The general theory leads again to the study of the Hamiltonian dynamical system \((Q, -d\mu, V)\), as in the Lagrangian case.

Let now \(L\) be a singular Lagrangian for which all constraint functions \(\phi_a = X_aE_L\), \(a = 1, \ldots, n - n_0 = N\), are holonomic; we consider an extended configuration space \(\mathbb{R}^N \times Q\) and denote by \(pr_2\) the natural projection \(pr_2 : \mathbb{R}^N \times Q \to Q\). We can then introduce a new Lagrangian \(L_1\) in \(T(\mathbb{R}^N \times Q)\) by

\[ L_1 = \tilde{L} + \lambda^a \tilde{\phi}_a , \]

(2.10)

where the tilde stands for the \(Tpr_2\)-pull-back and \(\lambda^a\) are the new additional coordinates (whose corresponding velocities will be represented by \(\zeta^a\)). Taking into account that

\[ \omega_{L_1} = \tilde{\omega}_L, \quad E_{L_1} = \tilde{E}_L - \lambda^a \tilde{\phi}_a , \]

(2.11)

we see that \(\text{Ker} \omega_{L_1}\) is generated by the set of vector fields projecting onto the vector fields \(X_a\) of \(\text{Ker} \omega_L\), plus \(\partial/\partial \lambda^a\) and \(\partial/\partial \zeta^a\). The constraint functions determined by \(\partial/\partial \lambda^a\) are the (pull-back of the) original primary constraint functions \(\phi_a\) and SODE condition leads us to consider the tangent bundle of the new configuration space \(\mathbb{R}^k \times Q'\). The solutions of the dynamics will project under \(pr_2\) onto the solutions of the original problem.

A similar approach can be done when we use \(d\tilde{\phi}_a\) instead of \(\tilde{\phi}_a\) as constraint functions and we replace the original Lagrangian \(L\) for \(L_2 = \tilde{L} + \lambda^a d\tilde{\phi}_a\). In this case \(E_{L_2} = \tilde{E}_L\) and \(\omega_{L_2} = \tilde{\omega}_L + d\tilde{\phi}_a \wedge d\lambda^a\), from which we can see that

\[ X = \xi^i \frac{\partial}{\partial q^i} + \xi^a \frac{\partial}{\partial \lambda^a} + \eta^i \frac{\partial}{\partial v^i} + \eta^a \frac{\partial}{\partial \zeta^a} \]

is in \(\text{Ker} \omega_{L_2}\) if and only if

\[ A\xi - \zeta^a \nabla \phi_a - W\eta = 0, \quad \xi \cdot \nabla \phi^a = 0, \quad W\xi = 0, \]

from which we see that we will obtain as constraint functions (the pull-back of) those obtained directly from \(L\), and the dynamics will correspond, up to the gauge ambiguity in the coordinates \(\lambda^a\), to the dynamics obtained in \(Q'\).

The relation \(\lambda^a d\tilde{\phi}_a = d(\lambda^a \tilde{\phi}_a) - \zeta^a \phi_a\) shows that the Lagrangian \(L_2\) may be replaced by \(L_3 = \tilde{L} - \zeta^a \tilde{\phi}_a\), which is quite similar to \(L_1\) with the change of \(\lambda^a\) for its velocity \(\zeta^a\).

In the more general situation for which \(d\mu\) is singular, the primary constraint functions (2.4) will be holonomic and therefore the previous remarks show us that they will determine
a submanifold \( Q' \subset Q \) characterized by some constraint functions \( \phi_a \) and the corresponding secondary constraint functions for a second order evolution will be \( \tilde{d}\phi_a \). Every such a constraint can be incorporated in a new Lagrangian \( L \) defined in the tangent bundle of a new configuration space \( \tilde{Q} = Q \times \mathbb{R}^{(n-n_0)} \), by

\[
L(q^i, \lambda, v^i, \zeta) = L(q^i, v^i) + \lambda^a \tilde{d}\phi_a(q^i, v^i),
\]

where \((q^i, \lambda^a, v^i, \zeta^a)\) denote the coordinates on the tangent bundle \( T\tilde{Q} \).

The expressions for \( \theta_L \) and \( \omega_L \) are

\[
\theta_L = \tilde{\theta}_L + \lambda^a \tilde{d}\phi_a \implies \omega_L = \tilde{\omega}_L + \tilde{d}\phi_a \wedge d\lambda^a,
\]

and therefore the rank of \( \omega_L \) may be higher than that of \( \omega_L \) and this is the starting point in the Faddeev–Jackiw approach. The energy \( E_L \) is the pull-back of \( \tilde{E}_L \), \( E_L = \tilde{E}_L \).

3. A Lagrangian approach to Hamilton equations

Let \( M \) be the configuration space of a mechanical system and consider \( Q = T^* M \) endowed with its exact canonical symplectic structure \( \omega = -d\theta_0 \), where \( \theta_0 \) is the Liouville 1-form in \( T^* M \) (see e.g. [35]). Then, given a function \( H \in C^\infty(T^* M) \), let us define the linear Lagrangian \( L \in C^\infty(T(T^* M)) \) by

\[
L = \tilde{\theta}_0 - \tau_{T^* M}^* \theta_0 - \tau_{T^* M}^* H,
\]

which in local coordinates is written

\[
L(q, p; v, u) = p_j v^j - H(q, p).
\]

In this case \( \theta_L \in C^\infty(T(T^* M)) \) is given by \( \theta_L = \tau_{T^* M}^* \theta_0 \) and \( E_L = \tau_{T^* M}^* H \). The matrix \( A \) given by (2.3) is now the symplectic matrix

\[
J = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}.
\]

The Kernel of \( \omega_L = -d\theta_L \) is then made up by the \( \tau_{T^* M} \)-vertical vector fields

\[
Y_{f,g} = f^i(q, p; v, u) \frac{\partial}{\partial v^i} + g^i(q, p; v, u) \frac{\partial}{\partial u^i},
\]

and therefore the presymplectic system defined by \( L, (T(T^* M), \omega_L, E_L) \), admits a global dynamics which is not uniquely defined but given by

\[
\Gamma_{f,g} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + f^i(q, p; v, u) \frac{\partial}{\partial v^i} + g^i(q, p; v, u) \frac{\partial}{\partial u^i},
\]

where \( f \) and \( g \) are arbitrary functions. The integral curves of each one of such vector fields are determined by the system of differential equations

\[
\begin{aligned}
\dot{q}^i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial q^i} \\
\dot{v}^i &= f^i(q, p; v, u) \\
\dot{u}^i &= g^i(q, p; v, u)
\end{aligned}
\]
However, only in those points for which
\begin{equation}
\begin{aligned}
  v^i &= \frac{\partial H}{\partial p_i} \\
  u^i &= -\frac{\partial H}{\partial q^i}
\end{aligned}
\end{equation}
the solution can be chosen to be the restriction of a SODE vector field. The preceding equations determine a submanifold $C$ on $T(T^*M)$ and the condition on the restriction of the vector field $\Gamma_{f,g}$ to be tangent to $C$ determines the functions $f^i$ and $g^i$ by means of
\begin{equation}
\begin{aligned}
  f^i &= \Gamma \left( \frac{\partial H}{\partial p_i} \right) \\
  g^i &= -\Gamma \left( \frac{\partial H}{\partial q^i} \right),
\end{aligned}
\end{equation}
obtaining in this way the vector field on $C$
\begin{equation}
\Gamma_C = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} + \Gamma \left( \frac{\partial H}{\partial p_i} \right) \frac{\partial}{\partial v^i} - \Gamma \left( \frac{\partial H}{\partial q^i} \right) \frac{\partial}{\partial u^i}.
\end{equation}

The dimension of $C$ is only twice that of $M$ and then it can be parametrized by $(q^i, p_i)$. The expression of $\Gamma_C$ in these new coordinates is
\begin{equation}
\Gamma_C = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i},
\end{equation}
and therefore, here the Hamilton equations arise as determining the integral curves of the uniquely defined vector field in the submanifold $C$ in which such a SODE solution of the dynamical equation can be found. Then, a curve $\gamma : I \to M$ whose lift to $TM$ lies in $C$ is the solution we were looking for.

4. Geometric description of time-dependent singular system

For the reader convenience we introduce in this Section the notation to be used and a summary of several properties and results of interest for the following sections.

The evolution space of a time-dependent mechanical system whose configuration space is the $n$-dimensional manifold $Q$ is $\mathbb{R} \times TQ$ [36], which is the space of 1-jets of the trivial bundle $\pi : \mathbb{R} \times Q \to \mathbb{R}$, $j^1 \pi = \mathbb{R} \times TQ$; $\mathbb{R}$ is endowed with a volume form $dt \in \Lambda^1(\mathbb{R})$ and represents the Newtonian time. There is one vector field in $\mathbb{R}$, $d/dt \in \mathfrak{X}(\mathbb{R})$, such that $i(d/dt)dt = 1$. The main geometric tools to be used in the geometric description of time-dependent mechanics are those of jet bundle geometry [37,38]. The $k$-jet bundle of $\pi$ is $J^k \pi = \mathbb{R} \times T^k Q$, with $T^k Q$ representing the space of $k$-velocities. In particular, $J^2 \pi = \mathbb{R} \times T^2 Q$ is the space of accelerations and $J^1 \pi = \mathbb{R} \times TQ$, the space of velocities. By convention we will write $J^0 \equiv \mathbb{R} \times Q$. For each pair of indices $k, l$ such that $k > l$, there is a natural projection $\pi_{k,l} : \mathbb{R} \times T^k Q \to \mathbb{R} \times T^l Q$ and we will denote $\pi_k = \pi \circ \pi_{k,0} : \mathbb{R} \times T^k Q \to \mathbb{R}$, the projection of $J^k \pi$ onto $\mathbb{R}$.

If $\sigma \in \text{Sec}(\pi)$, then $j^k \sigma \in \text{Sec}(\pi_k)$ will denote the $k$-jet prolongation of $\sigma$. So, if $\sigma(t) = (t, \sigma^i(t))$, we have $j^k \sigma(t) = (t, \sigma^i(t), d\sigma^i/dt, \ldots, d^k \sigma^i/dt^k)$. We also recall that the differential 1-forms $\theta \in \Lambda^1(J^k \pi)$ such that $(j^k \sigma)^* \theta = 0$, no matter the section $\sigma$, are
called contact 1-forms of $J^k\pi$. They are the constraint 1-forms for the so called Cartan distribution.

The theory of time-dependent Lagrangian systems makes an extensive use of the notion of vector field along a map. In particular, there exist vector fields along $\pi_{k+1,k}$, $T^k$, representing the total derivative operators. They are defined by means of $T^k \circ j^{k+1}\sigma = T(j^k\sigma) \circ d/dt, \forall \sigma \in \text{Sec}(\pi)$. For every $F \in C^\infty(\mathbb{R} \times T^k Q)$, the Lie derivative of $F$ with respect to $T^k$, i.e. the function $d_{T^k}F = i_{T^k}dF \in C^\infty(\mathbb{R} \times T^{k+1} Q)$, represents the total time derivative of $F$, usually written as $dF/dt$ or simply $\dot{F}$. In fibred coordinates $(t, q^i, v^i)$ for $\mathbb{R} \times TQ$ and the corresponding ones, $(t, q^i, v^i, a^i)$, for $\mathbb{R} \times T^2 Q$, we have $T^0 = \partial/\partial t + v^i \partial/\partial q^i \in \mathfrak{X}(\pi_{1,0})$ and $T^1 = \partial/\partial t + v^i \partial/\partial q^i + a^i \partial/\partial v^i \in \mathfrak{X}(\pi_{2,1})$. Obviously $T^0$ and the operator $\tau \in \mathfrak{X}(\tau_Q)$ introduced in Section 2 are related, $T \rho \circ T^0 = \tau \circ \rho_2$, with $\rho$ and $\rho_2$ being the projections onto the second factor of $\mathbb{R} \times Q$ and $\mathbb{R} \times TQ$, respectively.

A vector field $X \in \mathfrak{X}(\mathbb{R} \times Q)$ can be lifted to $\mathbb{R} \times TQ$ giving rise to a unique vector field $X^1 \in \mathfrak{X}(\mathbb{R} \times TQ)$ which is $\pi_{1,0}$-projectable onto $X$ and preserves the set of contact 1-forms of $\mathbb{R} \times TQ$. If the coordinate expression of $X$ is $X = \tau \partial/\partial t + X^i \partial/\partial q^i$, then

$$X^1 = \tau \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + (\dot{X}^i - v^i \dot{\tau}) \frac{\partial}{\partial v^i}.$$  

Vector fields of type $X^1$ are called infinitesimal contact transformations (hereafter ICT); it is worthy to note that 

$$(fX)^1 = (\pi_{1,0}^* f) X^1 + \hat{f} X^V,$$

where $X^V = S(X^1)$. Here $S$ is the vertical endomorphism on $\mathbb{R} \times TQ$ [31], which is a $(1,1)$-tensor field whose expression in the fibred coordinates $(t, q^i, v^i)$ is $S = \partial/\partial v^i \otimes (dq^i - v^i dt)$. Note that the local 1-forms given by $\theta^i = dq^i - v^i dt$ generate the set of contact 1-forms of $\mathbb{R} \times TQ$ and the Cartan distribution is but $\text{Ker} S$.

A similar definition works for the prolongation of vector fields $X = \tau(t, q, v) \partial/\partial t + X^i(t, q, v) \partial/\partial q^i \in \mathfrak{X}(\pi_{1,0})$: the vector field along $\pi_{2,1}$ given by

$$X^1 = \tau(t, q, v) \frac{\partial}{\partial t} + X^i(t, q, v) \frac{\partial}{\partial q^i} + (\dot{X}^i - v^i \dot{\tau}) \frac{\partial}{\partial v^i}$$

is the first prolongation of $X$. Thus $T^1 = (T^0)^1$.

More details about these notions and constructions can be found in [28].

The key object on which the geometric formulation of the dynamics corresponding to a time-dependent Lagrangian $L \in C^\infty(\mathbb{R} \times TQ)$ is based is the Poincaré–Cartan 1-form, defined by

$$\Theta_L = dL \circ S + L \, dt \in \bigwedge^1(\mathbb{R} \times TQ). \quad (4.1)$$

In fibred coordinates $\Theta_L = (\partial L/\partial v^i) \theta^i + L \, dt$. Another important object related with $L$ is the Euler-Lagrange 1-form, which is defined by $\delta L = i_{T^1}d\Theta_L \in \bigwedge^1(\mathbb{R} \times T^2 Q)$, with local coordinate expression $\delta L = L_i \theta^i$, where

$$L_i = \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \right) - \pi_{2,1}^* \left( \frac{\partial L}{\partial q^i} \right)$$

are the variational derivatives of the Lagrangian $L$. 
The dynamical equation to be considered here is

\[ i(\Gamma)d\Theta_L = 0, \]  

(4.2)
and, according to Hamilton’s principle, the condition for the section \( \sigma \in \text{Sec}(\pi) \) to be an extremal of the action is

\[ (j^1\sigma)^* [i(Z)d\Theta_L] = 0, \quad \forall Z \in \mathfrak{X}(\mathbb{R} \times TQ). \]

The problem is to find a vector field \( \Gamma \in \mathfrak{X}(\mathbb{R} \times TQ) \) which is solution of the dynamical equation (4.2) and whose integral curves are the first prolongation \( j^1\sigma \) of sections \( \sigma \in \text{Sec}(\pi) \). In other words, \( \Gamma \) must be a time-dependent SODE, i.e., a vector field \( \Gamma \) such that \( T\pi_1,0 \circ \Gamma = T_0 \). Its integral curves are parametrized by \( t \) and the local expression of \( \Gamma \) is \( \Gamma = \partial/\partial t + v^i \partial/\partial q^i + \Gamma_i \partial/\partial v^i \). Both SODE-type and ICT-type fields belong to \( \text{Ker} S \), but in general a SODE is not an ICT.

The submanifold of \( \mathbb{R} \times TQ \) where the dynamical equation (4.2) possesses such kind of solutions is given by the following theorem [28]:

**Theorem.** Let \( M_L \) denote the coisotropic subbundle

\[ M_L = \{ Z \in T(\mathbb{R} \times TQ) \mid S(Z) \in \mathcal{V}_{1,0}(\text{Ker} d\Theta_L) \} , \]

and \( \mathcal{V}_{1,0}(\cdot) \) means the \( \pi_{1,0} \)-vertical part, which is but the kernel of the differential of the Legendre transformation \( \mathcal{F}L : \mathbb{R} \times TQ \rightarrow \mathbb{R} \times T^*Q \), that is, \( \mathcal{V}_{1,0}(\text{Ker} d\Theta_L) = \text{Ker} T\mathcal{F}L \).

Then, there exist solutions of the dynamical equation which are restrictions of a SODE field in the points, and only in that points, of the set defined by

\[ M = \{ z \in \mathbb{R} \times TQ \mid d\Theta_L(D,Z)(z) = 0, \forall Z \in M_L, \ D \text{ any SODE } \} . \]

When \( L \) is regular, \( M_L = \{ 0 \} \) and \( M = \mathbb{R} \times TQ \) and, consequently, there is no restriction on the motion. But there exist (primary) constraints for a singular Lagrangian which are given by the following conditions

\[ \Phi_Z = d\Theta_L(D,Z) = 0, \quad Z \in M_L. \]  

(4.3)

The functions \( \Phi_Z \) are the primary constraint functions. Obvious conditions for the consistency of the dynamics \( \Gamma \) compatible with the constraints (4.3) are given by

\[ \chi_Z = \Gamma(\Phi_Z)|_M = 0 \]  

(4.4)
which either give rise to the secondary constraints or (partially) fix the dynamics \( \Gamma \). When the process is iterated, we will hopefully arrive at the final constraint submanifold \( M_f \), on which there exist solutions of the dynamical equation which are the restriction onto \( M_f \) of SODE fields tangent to \( M_f \).

Note that here the only ingredient is the singular Lagrangian \( L \) which provides both the (nonholonomic) constraint functions and the dynamics. In this sense this is a problem that does not coincide with the more frequently studied constrained situation in which the
starting point is a given nonholonomic constraint distribution [39]. This latter situation, that is receiving much attention during the last years (see e.g. [40,41]), and is important by its relation with the theory of connections and by its control theoretical applications, is different from the present case of first order singular Lagrangians.

The dynamics obtained by applying the constraint algorithm sketched above may be not unique, a fact which is related with the gauge invariance of the Lagrangian. The appropriate geometric tool to deal with a gauge infinitesimal transformation

\[ \delta q^i = \epsilon \sum_{\alpha=0}^R \frac{d^\alpha g}{dt^\alpha} X^i_\alpha(t, q, v), \quad \delta v^i = \frac{d}{dt}(\delta q^i), \]  

(4.5)

where \( g = g(t) \) is an arbitrary function of the time, is that of a vector field along the bundle map \( \pi_{1,0} \). In fact, let \( \{ X_\alpha = X^i_\alpha(t, q, v) \partial /\partial q^i \mid \alpha = 0, 1, \ldots, R \} \) be a family of \( R + 1 \) \( \pi \)-vertical vector fields along \( \pi_{1,0} \) and \( g(t) \) an arbitrary function in \( \mathbb{R} \). Then, if \( X_g \) is the \( \pi \)-vertical vector field along \( \pi_{1,0} \)

\[ X_g = \epsilon \sum_{\alpha=0}^R \frac{d^\alpha g}{dt^\alpha} X_\alpha, \]  

(4.6)

its first prolongation \( X^1_g \) is the infinitesimal generator of the gauge transformation (4.5).

Such \( X_g \) is said to be a gauge symmetry of the Lagrangian \( L \) if there exists a function \( F_g \in C^\infty(\mathbb{R} \times TQ) \) such that

\[ d_{X^1_g} L = X^1_g L = \frac{dF_g}{dt} \]  

(4.7)

In such case \( \langle \delta L, X_g \rangle + dG_g /dt = 0 \), with \( G_g = F_g - \langle \Theta_L, X_g \rangle \), and conversely, if there exists a function \( G_g \) such that \( \langle \delta L, X_g \rangle + dG_g /dt = 0 \), then \( d_{X^1_g} L = dF_g /dt \) with \( F_g = G_g + \langle \Theta_L, X_g \rangle \). (The contractions \( \langle \delta L, X_g \rangle \) and \( \langle \Theta_L, X_g \rangle \) make sense because of the \( \pi_{2,0} \)- and \( \pi_{1,0} \)-semibasic character of \( \delta L \) and \( \Theta_L \), respectively.)

All this mathematical apparatus can be used to give a geometric version of Noether’s Second Theorem as it can be seen in [13,28]. The theorem essentially establishes that a gauge-invariant Lagrangian is necessarily singular and it satisfies the so called Noether identities

\[ \sum_\alpha (-1)^\alpha \frac{d^\alpha \langle \delta L, X_\alpha \rangle}{dt^\alpha} = 0, \]

i.e.

\[ \sum_{i,\alpha} (-1)^\alpha \frac{d^\alpha \langle L_i X^i_\alpha \rangle}{dt^\alpha} = 0, \]

with \( L_i \) being the variational derivatives of the Lagrangian \( L \). However, a singular Lagrangian needs not to be gauge-invariant and a method for the determination of the gauge symmetry underlying a given singular Lagrangian has been developed in [13] (see also [28]). The method is based on the necessary conditions which are derived from the second Noether’s theorem and it also tells us whether, or not, a given Lagrangian is gauge-invariant. Let \( X_g \) given by (4.6) be the wanted gauge symmetry of \( L \) and choose a vector field \( \tilde{X}_\alpha \in \mathfrak{X}(\mathbb{R} \times TQ) \) such that \( T_{\pi_{1,0} \circ \tilde{X}_\alpha} = X_\alpha \). Assume that

\[ G_g = \sum_{\alpha=0}^{R-1} G_\alpha \frac{d^\alpha g}{dt^\alpha}. \]
It follows from the symmetry condition (4.7) that the vector field $X_R$ belongs to the distribution $M_L - \text{Ker } S$ and the functions $G_\alpha$ satisfy the relations

$$G_R = 0, \quad \dot{G}_\alpha + G_{\alpha - 1} + \langle \delta L, X_\alpha \rangle = 0 \quad (\alpha = 1, \ldots, R), \quad \dot{G}_0 + \langle \delta L, X_0 \rangle = 0,$$

from which it follows that all the functions $G_\alpha$ must be $FL$-projectable and satisfy the recursive relations

$$G_{\alpha - 1} = -d\Theta_L(D, \tilde{X}_\alpha) - D(G_\alpha),$$

with $D$ being any sode; in particular, $G_{R - 1} = \dot{G}_0 + \langle \delta L, X_0 \rangle = 0$ the algorithm enters into a cycle and the solutions appear cyclically repeated. We can take $R = N - 1$ to be the higher order for the derivatives of $g(t)$ and the algorithm ends up.

In the case when in some step there is not any solution, we stop the process and return to make (if possible) a new choice for the solution in a previous step. If there is no solution in any case, we have to conclude that the Lagrangian is not gauge invariant.

The next sections are devoted to show the application of these constructions to the case of a Lagrangian which is linear, or more accurately affine, in the velocities $v^i$.

5. **Geometric theory of time-dependent Lagrangians which are affine in velocities**

From the geometric viewpoint, a time-dependent Lagrangian $L$ which is affine in the velocities arises from a 1-form $\lambda \in \bigwedge^1(\mathbb{R} \times Q)$ in the following way: $L = i_{\mathbf{T}^0} \lambda \in C^\infty(\mathbb{R} \times TQ)$. In fact, if $\lambda = m_i(t, q) dq^i - V(t, q) dt$, then the Lagrangian, to be denoted by $\tilde{\lambda}$, is

$$\tilde{\lambda} = m_i(t, q) v^i - V(t, q). \quad (5.1)$$

Obviously the time-independent case we have dealt with in Sect. 2 is simply a special case of this: given $\mu \in \bigwedge^1(Q)$ and $V \in C^\infty(Q)$ then we consider the 1-form $\lambda = \rho^* \mu - \rho^* V dt \in \bigwedge^1(\mathbb{R} \times Q)$ which yields the ($\rho_2$-pull-back of the) time-independent Lagrangian (2.1).

Coming back to the general case, the basic geometric features for this Lagrangian are:

1. The Poincaré–Cartan 1-form is given by $\Theta_{\tilde{\lambda}} = \pi_{1,0}^* \lambda$.

2. $FL$-projectability means $\pi_{1,0}$-projectability, because

$$FL \lambda = \mu \circ \lambda \circ \pi_{1,0},$$
where \( \mu = \pi_\mathbb{R} \times \text{id}_{T^*Q} \) is the natural projection of \( T^*(\mathbb{R} \times Q) \) onto \( \mathbb{R} \times T^*Q \).

3. \( \mathcal{V}_{1,0}(\text{Ker } d\Theta_\lambda) = \text{Ker } T \pi_{1,0} \), so that \( M_\lambda = T(\mathbb{R} \times TQ) \).

Taking into account all these facts, we find that the primary constraint functions are given by
\[
\Phi_Z = d\Theta_\lambda(D, Z), \quad Z \in T(\mathbb{R} \times TQ), \quad D \text{ any SODE},
\]
i.e. \( \Phi_Z = i(Z)(i_{T^0}d\lambda) \). The primary constraint manifold is described in the following terms:
\[
\mathcal{M} = \{ z \in \mathbb{R} \times TQ \mid \Phi_Z(z) = 0 \} \equiv \{ z \in \mathbb{R} \times TQ \mid i_{T^0}d\lambda(z) = 0 \}.
\]
However, if we recall that there exists a local basis of \( \mathfrak{X}(\mathbb{R} \times TQ) \) is made up from a SODE and vector fields \( Y^1 \) and \( Y^V = S(Y^1) \), with \( Y \in \text{Ker } T \pi \) [28], we see that the only effective constraint functions are those given by
\[
\Phi_Y = d\Theta_\lambda(Y^1, D), \quad Y \in \text{Ker } T \pi.
\]

Then the functions \( \Phi_Y \) are in a one to one correspondence with the elements of \( \text{Ker } T \pi \). They also verify the property
\[
\Phi_{Y_1 + fY_2} = \Phi_{Y_1} + (\pi^*_{1,0}f)\Phi_{Y_2}, \quad f \in C^\infty(\mathbb{R} \times Q).
\]

We will say that \( \Phi_Y \) and \( \Phi_{\overline{Y}} \) are 'linearly dependent' if \( \Phi_{\overline{Y}} = \Phi_{fY} \) for some \( f \in C^\infty(\mathbb{R} \times Q) \) everywhere non-null. This property trivially takes place when \( \overline{Y} = fY \) but in the case when \( \overline{Y} \) and \( Y \) are not dependent it is a property related with the existence of a gauge symmetry of the Lagrangian as we will see later.

In local coordinates \((t, q^i)\) for \( \mathbb{R} \times Q \) and \((t, q^i, v^i)\) for \( \mathbb{R} \times TQ \), respectively, the vector field \( Y \) is written \( Y = Y^i(t, q) \partial/\partial q^i \) and \( d\lambda = \frac{1}{2} A_{ij} dq^i \wedge dq^j - \omega_i dq^i \wedge dt \), where
\[
A_{ij} = \frac{\partial m_j}{\partial q^i} - \frac{\partial m_i}{\partial q^j} \quad \text{and} \quad \omega_i = \frac{\partial V}{\partial q^i} + \frac{\partial m_i}{\partial t}.
\]

Consequently, the primary constraint functions are
\[
\Phi_Y = (A_{ij}v^j - \omega_i) Y^i,
\]
i.e. they are affine in the velocities. A basis for such constraint functions is made up from the following functions
\[
\Phi_i = \Phi_{\partial/\partial q^i} = A_{ij}v^j - \omega_i = 0, \quad i = 1, \ldots, n.
\]

Note that these equations are the Euler-Lagrange equations obtained from the Lagrangian (5.1) and all of them appear in this formalism as constraint equations.

We can deduce from (5.4) that a primary constraint function \( \Phi_Y \) is \( \pi_{1,0} \)-projectable (i.e., holonomic) if and only if the 1-form \( i(Y) d\lambda \) is \( \pi \)-semibasic. In an equivalent way, \( \Phi_Y \) is \( \pi_{1,0} \)-projectable if and only if
\[
Y \in \Omega_\Lambda = [\text{Ker } T \pi]|_{\Lambda} \cap \text{Ker } T \pi.
\]
(The superscript $\perp d\lambda$ means $d\lambda$-orthogonal complement). In local coordinates as above, $\Omega_\lambda$ is spanned by vector fields $Y = Y^i(t, q) \partial/\partial q^i$ such that $A_{ij} Y^i = 0$ and then the corresponding constraint function $\Phi_Y = \omega_i Y^i = 0$ is projectable.

Obviously, the maximum number of linearly independent constraint functions equals the dimension $n$ of $Q$ and the maximum number of the holonomic ones is $p = n - \text{rank } A$, where $A$ is the skew-symmetric matrix of elements $A_{ij}$ (5.6).

As far as the dynamics is concerned, any SODE $\Gamma$ is a solution on $\mathcal{M}$ of the dynamical equation because of (5.2). The consistency conditions (4.4), $\Gamma \Phi = 0$, will generate additional (secondary) constraints and/or fix (maybe partially) the dynamics. More accurately, the consistency conditions for non-projectable primary constraint functions will determine some components of the SODE, while the projectable ones will give rise to secondary constraints that are affine in the velocities, which fix the dynamics. The uniqueness is got when all the constraints so obtained are independent and in the maximum number. The two fundamental cases are.

I) The first fundamental case arises when $\Omega_\lambda = 0$. Then the matrix $A$ is regular and therefore there are no primary holonomic constraints. The $n$ independent constraints (5.8) are non-holonomic and lead to $n$ equations determining in a unique way the dynamics on $\mathcal{M}$, i.e.

$$v^j = (A^{-1})^{jk} \omega_k .$$

Of course, this is only possible when the dimension of $Q$ is even, $n = 2m$.

Let us analyse this “regular” case in geometrical terms. The condition $\Omega_\lambda = 0$ means that the distribution $\text{Ker } d\lambda$ is 1-dimensional, so $d\lambda$ is a contact form on $\mathbb{R} \times Q$ generating an exact contact structure on $\mathbb{R} \times Q$. In fact, every nonzero vector field $Z \in \text{Ker } d\lambda$ satisfies the condition $i(Z) dt \neq 0$, as it follows from the trivial fact that $Z$ also belongs to $[\text{Ker } T \pi] \perp d\lambda$, and if the condition $i(Z) dt = 0$ is fulfilled $Z$ should also be $\pi$-vertical and, consequently, $Z = 0$. Moreover for every pair $Z, Z' \in \text{Ker } d\lambda$ the vector field $Z'' = [i(Z) dt] Z' - [i(Z') dt] Z$ lies in $\Omega_\lambda$, $Z'' \in \Omega_\lambda$, i.e. $Z'' = [i(Z) dt/i(Z') dt] Z$.

On the other hand, no new constraints arise from the consistency conditions (4.4) and the $n$ independent non-holonomic constraints will fix the dynamics on $\mathcal{M}$ in a unique way. Let $\eta \in \text{Ker } d\lambda$ be a vector field such that $i(\eta) dt = 1$. It generates locally $\text{Ker } d\lambda$ and in coordinates turns out to be

$$\eta = \frac{\partial}{\partial t} + (A^{-1})^{ij} \omega_j \frac{\partial}{\partial q^i} .$$

The description (5.3) of $\mathcal{M}$ means that $T^0(z) \in \text{Ker } d\lambda(\pi_{1,0}(z))$, $z \in \mathcal{M}$, and, consequently, we can assert that the ICT $\eta^1$ is, at least on $\mathcal{M}$, a SODE field. It is such that $i(\eta^1) d\Theta_\lambda|_{\mathcal{M}} = 0$, thus the dynamical equation (4.2) is satisfied by $\eta^1$ on $\mathcal{M}$ and the SODE $\Gamma$ compatible with the constraints is the one generated by $\eta^1$.

In summary, when $\Omega_\lambda = 0$, the Lagrangian system on $\mathbb{R} \times T Q$ reduces to the (in general, time-dependent) Hamiltonian system $(\mathbb{R} \times Q, d\lambda, H)$. The manifold $\mathbb{R} \times Q$ is the extended phase-space and the Hamiltonian function $H$ is essentially the energy $E_{\lambda}$, which is a holonomic function: $E_{\lambda} = \pi_{1,0}^* H$; in local coordinates $H = V$.

Both dynamical systems are equivalent in the sense that the integral curves of the dynamics in $\mathbb{R} \times T Q$ are the first prolongation to $\mathbb{R} \times T Q$ of the integral curves of the Hamiltonian dynamics in $\mathbb{R} \times Q$. 

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Using the 2-form $d\lambda$ we can define a Poisson bracket in $\mathbb{R} \times Q$ according to the following construction. Given a 1-form $\alpha \in \bigwedge^1(\mathbb{R} \times Q)$ there exists a vector field $X_\alpha \in \mathfrak{X}(\mathbb{R} \times Q)$ such that $i(X_\alpha) d\lambda = \alpha$ iff $i(Z)\alpha = 0$, $\forall Z \in \text{Ker}\, d\lambda$; obviously such a $X_\alpha$ is not unique, the indeterminacy being $\text{Ker}\, d\lambda$ itself. Although $\alpha$ does not satisfy the condition above one can take the “$d\lambda$-semibasic” part, given by

$$\alpha^\lambda = \alpha - \frac{i(Z)\alpha}{i(Z) dt} dt, \quad Z \in \text{Ker}\, d\lambda,$$

which, in fact, depends only on $\text{Ker}\, d\lambda$ and not on a particular $Z \in \text{Ker}\, d\lambda$, i.e. there exist vector fields $X_\alpha$ such that $i(X_\alpha) d\lambda = \alpha^\lambda$.

In local coordinates, if $\alpha = a^0 dt + a^i dq^i$, then

$$\alpha^\lambda = a^i \left(dq^i - (A^{-1})^{ij} \omega_j dt\right)$$

and

$$X_\alpha = X^0 \eta + (A^{-1})^{jk} \alpha^j \frac{\partial}{\partial q^k},$$

where $X^0$ is an arbitrary function and $\eta$ is given by (5.10); in particular, for $\alpha = df$,

$$d^\lambda f = (df)^\lambda = \frac{\partial f}{\partial q^i} (dq^i - (A^{-1})^{ij} \omega_j dt)$$

and

$$X_f = X_{df} = X^0 \eta + (A^{-1})^{jk} \frac{\partial f}{\partial q^j} \frac{\partial}{\partial q^k}.$$ 

The Poisson bracket $\{f, g\}$ of the functions $f$ and $g$ is then defined by the rule

$$\{f, g\} = d\lambda(X_g, X_f) = d^\lambda g(X_f).$$

In local coordinates,

$$\{f, g\} = (A^{-1})^{jk} \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial q^k},$$

the fundamental Poisson brackets being $\{q^i, q^k\} = (A^{-1})^{jk}$ (see [42]). So the equation of motion derived from the constraint equations (5.8) can be written in the form $\dot{q}^i = \{q^i, q^k\} \omega_k$.

In the autonomous case $(Q, -d\mu)$ is a symplectic manifold and we obtain the Hamiltonian system $(Q, -d\mu, V)$ analysed in detail in Section 2. In particular, the equations of motion read

$$\dot{q}^i = \{q^i, q^k\} \frac{\partial V}{\partial q^k},$$

(see (2.9)).

II) Second case: $\Omega_\lambda \neq 0$. Then $A$ is singular and there will be $r = \text{rank}\, A$ independent non-projectable constraint functions, and then the number of effective projectable constraint functions is not greater than $p = n - r$. These functions generate secondary constraint functions that together with the non-projectable functions will determine, at least partially, the dynamics. The uniqueness of solution for the dynamics depends on whether or not the primary and secondary constraints are independent. The analysis of gauge invariance in the following section will clarify these points.
6. Gauge invariance of first order Lagrangian systems

In this section we will show how to make use of the algorithm of gauge symmetry explained in Section 4. It starts by taking \( G_R = 0 \) and choosing a vector field \( \tilde{X}_R \in (M_\lambda - \text{Ker} S) \cap \text{Ker} T \pi_1 \) in such a way that the function \( G_{R-1} \) given by \( G_{R-1} = d\Theta_\lambda(\tilde{X}_R, D) \) be a \( \pi_{1,0} \)-projectable primary constraint function. Then, we can choose \( \tilde{X}_R = Y^1 \) (that is, \( X_R = T \pi_{1,0} \circ Y^1 = Y \circ \pi_{1,0} \), where \( Y \in \Omega_\lambda \)), so that \( G_{R-1} \) turns out to be \( G_{R-1} = \Phi_Y \) that is a \( \pi_{1,0} \)-projectable function. Moreover, we can see that the 1-form \( \lambda_{R-1} = i(\tilde{X}_{R-1})d\Theta_\lambda - dG_{R-1} \) is \( \pi_{1,0} \)-semibasic, no matter of the choice of \( \tilde{X}_{R-1} \). We can write \( \tilde{X}_{R-1} \) as a sum

\[
\tilde{X}_{R-1} = Y_1^1 + Y_2^Y, \quad Y_1, Y_2 \in \text{Ker} T \pi;
\]

and, consequently,

\[
G_{R-2} = d\Theta_\lambda(Y_1^1, D) - D(\Phi_Y) = \Phi_{Y_1} - \chi_Y
\]

is the difference between a primary constraint function and a secondary one. The algorithm only works if it is possible to choose \( G_{R-2} \) as being a \( \pi_{1,0} \)-projectable function. The iteration of this procedure will give rise to a sequence of functions \( G_{R-k} \) analogous to \( G_{R-2} \), and therefore a gauge symmetry will be obtained in this way if we arrive at a secondary constraint function \( \chi_Y \) that it was a primary one, making possible to have \( G_{R-2} = 0 \).

Thus the starting point to have a gauge symmetry is to know the set of holonomic (primary) constraint functions. If two (or more) of them are linearly dependent it will be possible to choose \( X_R \) in such a way that \( G_{R-1} = 0 \). In the opposite case we need to know whether or not the primary constraint function gives rise to a secondary constraint function which is a primary one.

More specifically, there are the following four possibilities and correspondingly the four types of affine in velocities Lagrangians:

I) \( \Omega_\lambda = 0 \), i.e., \( A \) is regular. In this case there are no holonomic constraints and there is no gauge symmetry at all. The dynamics is uniquely determined by the non-holonomic constraints (see Sect. 5).

II.1) \( \Omega_\lambda \neq 0 \), i.e., \( A \) is singular, and there are two (non trivially) linearly dependent holonomic primary constraint functions \( \Phi_T \) and \( \Phi_Y \), i.e. \( \Phi_T = \Phi_{fY} = (\pi^*_1 f) \Phi_Y \). In this case, \( \nabla f Y \) is non-null and choosing \( \tilde{X}_R \) as \( \tilde{X}_R = (\nabla f Y)^1 \), we will get \( G_{R-1} = 0 \) (i.e., the higher order for the derivatives of the arbitrary function \( g(t) \) is \( R = 0 \)) and come to the gauge symmetry

\[
X_g = g(\nabla f Y) \circ \pi_{1,0}.
\]

II.2) \( \Omega_\lambda \neq 0 \) and there is a holonomic primary constraint function \( \Phi_Y \), \( Y \neq Y \in \Omega_\lambda \), giving rise to a secondary constraint function \( \chi_Y = D(\Phi_Y) \) which is a primary one, namely, there exists a vector field \( \nabla \in \text{Ker} T \pi_1 \) such that \( \chi_Y = \Phi_T = d\Theta_\lambda(\nabla^1, D) \). In this case, choosing \( Y_1 = \nabla \) we find that \( G_{R-2} = d\Theta_\lambda(\nabla^1, D) - D(\Phi_Y) = 0 \). Therefore the algorithm tells us that \( R = 1 \) and the gauge symmetry is

\[
X_g = g(\nabla \circ \pi_{1,0}) + g(Y \circ \pi_{1,0}).
\]
II.3) $\Omega_\lambda \neq 0$ and none of the secondary constraints is primary. $G_{R-2}$ is not holonomic and the algorithm cannot go on, i.e. there is no gauge symmetry. The dynamics is uniquely determined by the full set of constraints (both primary and secondary).

In summary, we will have gauge symmetry when there exist holonomic constraint functions generating secondary constraint functions that generate a free set with the non-projectable constraint functions.

7. Examples

Finally, several examples will be used to illustrate the Lagrangian analysis made in Sections 5 and 6. As a matter of notation we will use subindices instead of upperindices in the coordinates $q$ and the velocities $v$.

**Example 1.** The well-known two-dimensional Lotka–Volterra system can be derived from the following Lagrangian which is affine in the velocities [43]:

$$L = \frac{\ln y}{2x} v_x - \frac{\ln x}{2y} v_y - (a \ln y + b \ln x - x - y),$$

where $a$ and $b$ are positive constants. Considered as a Lagrangian function on $\mathbb{R} \times T^* \mathbb{R}^2_+$ it derives from the 1-form

$$\lambda = \frac{\ln y}{2x} dx - \frac{\ln x}{2y} dy - (a \ln y + b \ln x - x - y) dt \in \bigwedge^1 (\mathbb{R} \times \mathbb{R}^2_+).$$

Therefore,

$$A = \begin{pmatrix} 0 & \frac{1}{xy} \\ \frac{1}{xy} & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} \frac{b}{x} - 1 \\ \frac{x}{a} - 1 \end{pmatrix}.$$ 

The matrix $A$ is regular and then $\Omega_\lambda = 0$. All of the primary constraint functions (5.8) are non-holonomic,

$$\Phi_x = \Phi_{\partial/\partial x} = -\frac{v_y}{xy} - \frac{b}{x} + 1, \quad \Phi_y = \Phi_{\partial/\partial y} = \frac{v_x}{xy} - \frac{a}{y} + 1,$$

and, consequently, the reduced system (5.10) on $\mathbb{R} \times \mathbb{R}^2_+$ is

$$\eta = \frac{\partial}{\partial t} + x(a - y) \frac{\partial}{\partial x} - y(b - x) \frac{\partial}{\partial y}.$$ 

The system of differential equations for its integral curves constitutes the 2-dimensional Lotka–Volterra system

$$\dot{x} = x(a - y), \quad \dot{y} = -y(b - x).$$

This is a Hamiltonian system with a symplectic structure

$$\sigma = \frac{1}{xy} dy \wedge dx,$$
i.e. with defining Poisson bracket \( \{x, y\} = xy \), and Hamiltonian function

\[
H = a \ln y + b \ln x - x - y .
\]

This Lagrangian is of Type I.

**Example 2.** Another regular case is that provided by the Lagrangian studied in [44]

\[
L = \frac{1}{2} \left[ (q_2 + q_3)v_1 - q_1 v_2 + (q_4 - q_1)v_3 - q_3 v_4 \right] - \frac{1}{2}(2q_2q_3 + q_2^2 + q_3^2)
\]

coming from the 1-form \( \lambda \in \wedge^1(\mathbb{R} \times \mathbb{R}^4) \) given by

\[
\lambda = \frac{1}{2} \left[ (q_2 + q_3)q_1 - q_1 q_2 + (q_4 - q_1)q_3 - q_3 q_4 \right] - \frac{1}{2}(2q_2q_3 + q_2^2 + q_3^2) dt ,
\]

from which we obtain

\[
A = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

and

\[
\omega = \begin{pmatrix} 0 \\ q_3 \\ q_2 + q_3 \\ q_4 \end{pmatrix}.
\]

The matrix \( A \) has rank 4 and then, as \( \Omega_\lambda = 0 \), there is not any holonomic constraint. The generating set of primary constraints (5.8) is

\[
\Phi_1 = -v_2 - v_3 = 0, \quad \Phi_2 = v_1 - q_3 = 0, \quad \Phi_3 = v_1 - v_4 - q_2 - q_3 = 0, \quad \Phi_4 = v_3 - q_4 = 0 ,
\]

and the secondary ones, \( \Gamma(\Phi_i) = 0 \), with \( i = 1, \ldots, 4 \), determine a unique dynamics which is the restriction of a SODE on the constraint manifold \( \mathcal{M} \), namely

\[
\Gamma' = \Gamma|_{\mathcal{M}} = \frac{\partial}{\partial t} + q_3 \frac{\partial}{\partial q_1} - q_4 \frac{\partial}{\partial q_2} + q_4 \frac{\partial}{\partial q_3} - q_2 \frac{\partial}{\partial q_4} + v_3 \frac{\partial}{\partial v_1} - v_4 \frac{\partial}{\partial v_2} + v_4 \frac{\partial}{\partial v_3} - v_2 \frac{\partial}{\partial v_4}.
\]

The reduced system on \( \mathbb{R} \times \mathbb{R}^4 \) is

\[
\eta = \frac{\partial}{\partial t} + q_3 \frac{\partial}{\partial q_1} - q_4 \frac{\partial}{\partial q_2} + q_4 \frac{\partial}{\partial q_3} - q_2 \frac{\partial}{\partial q_4} \in \text{Ker } d\lambda
\]

Note that the restriction of \( \Gamma' \) onto \( \mathcal{M} \) coincides with that of \( \eta^1 \).

**Example 3.** Let us now consider the Lagrangian of the type II.1

\[
L = q_1 v_2 + q_2 v_3 + q_2 v_4 - q_2 (q_4 - q_3)
\]

generated by the 1-form

\[
\lambda = q_1 dq_2 + q_2 dq_3 + q_2 dq_4 - q_2(q_4 - q_3) dt \in \wedge^1(\mathbb{R} \times \mathbb{R}^4),
\]
from which we obtain

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\omega = \begin{pmatrix}
0 \\
q_4 - q_3 \\
-q_2 \\
q_2
\end{pmatrix}.
\]

The primary constraint functions (5.8) are

\[
\Phi_1 = v_2, \quad \Phi_2 = -v_1 + v_3 + v_4 - q_4 + q_3, \quad \Phi_3 = -v_2 + q_2, \quad \Phi_4 = -v_2 - q_2.
\]

The distribution \(\Omega_\lambda\) is spanned by the two vector fields \(\partial/\partial q_1 + \partial/\partial q_3\) and \(\partial/\partial q_1 + \partial/\partial q_4\), and yields to two holonomic primary constraint functions

\[
\Phi_1 + \Phi_3 = q_2, \quad \Phi_1 + \Phi_4 = -q_2
\]

that are linearly dependent. The Lagrangian is gauge invariant, and the symmetry vector (6.3) and the associated function \(F_g\) are \((R = 0)\)

\[
X_g = g(t) \left( 2 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_3} + \frac{\partial}{\partial q_4} \right) \quad \text{and} \quad F_g = 2g(t)q_2.
\]

The dynamics \(\Gamma\) is determined by the set of (primary and secondary) constraints \(\Phi_i, \chi_i, i = 1, \ldots, 4:\)

\[
\Gamma = \frac{\partial}{\partial t} + (v_3 + v_4 - q_4 + q_3) \frac{\partial}{\partial q_1} + v_3 \frac{\partial}{\partial q_3} + v_4 \frac{\partial}{\partial q_4} + (C_3 + C_4 - v_4 + v_3) \frac{\partial}{\partial v_1} + C_3 \frac{\partial}{\partial v_3} + C_4 \frac{\partial}{\partial v_4},
\]

where \(C_3\) and \(C_4\) are arbitrary functions.

**Example 4.** The Lagrangian defined by the 1-form

\[
\lambda = (q_2 - q_3) \, dq_1 - q_2 \, dq_3 - (q_2 - q_1) \, dt \in \bigwedge^1 (\mathbb{R} \times \mathbb{R}^3)
\]

is of Type II.2. The matrices \(A\) and \(\omega\) are given, respectively, by

\[
A = \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix}, \quad
\omega = \begin{pmatrix}
-q_3 \\
q_3 \\
q_2 - q_1
\end{pmatrix}.
\]

The set of primary constraints (5.8) is generated by

\[
\Phi_1 = -v_2 + v_3 + q_3, \quad \Phi_2 = v_1 - v_3 - q_3, \quad \Phi_3 = -v_1 + v_2 - q_2 + q_1.
\]

The rank of \(A\) is 2 and therefore the distribution \(\Omega_\lambda\) is 1-dimensional. It is generated by the vector field \(\partial/\partial q_1 + \partial/\partial q_2 + \partial/\partial q_3\) and there is one primary holonomic constraint function, namely, \(\Phi_1 + \Phi_2 + \Phi_3 = q_1 - q_2\), whose corresponding secondary constraint function, \(\chi = v_1 - v_2\), is likewise a primary constraint: \(\chi = \Phi_{\partial/\partial q_1 + \partial/\partial q_2} = \Phi_1 + \Phi_2\). Consequently,
this Lagrangian is gauge invariant: starting from \( X_R = \partial/\partial q_1 + \partial/\partial q_2 + \partial/\partial q_3 \) and choosing \( X_{R-1} = \partial/\partial q_1 + \partial/\partial q_2 \) we obtain the gauge symmetry (6.4)

\[
X_g = g(t) \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right) + \dot{g}(t) \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3} \right),
\]

and the corresponding function \( F_g = g(t)(q_1 - q_3) - \dot{g}(t)q_3 \). The dynamics on \( \mathcal{M} \) is given by

\[
\Gamma = \frac{\partial}{\partial t} + (v_3 + q_3) \frac{\partial}{\partial q_1} + (v_3 + q_3) \frac{\partial}{\partial q_2} + v_3 \frac{\partial}{\partial q_3} + (C + v_3) \frac{\partial}{\partial v_1} + (C + v_3) \frac{\partial}{\partial v_2} + C \frac{\partial}{\partial v_3},
\]

with \( C \) an arbitrary function.

**Example 5.** The 1-form

\[
\lambda = t \, q_2 \, dq_1 - q_1 \, dq_2 - [q_1 - (t + 1)q_2] \, q_3 \, dt \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^3)
\]

gives the time-dependent Lagrangian of the type II.2

\[
\hat{\lambda} = t \, q_2 \, v_1 - q_1 \, v_2 - [q_1 - (t + 1)q_2] \, q_3 .
\]

The matrices \( A \) and \( \omega \) are given, respectively, by

\[
A = \begin{pmatrix} 0 & -(t + 1) & 0 \\ t + 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} q_2 + q_3 \\ -(t + 1)q_3 \\ q_1 - (t + 1)q_2 \end{pmatrix}.
\]

Two constraint functions of those determined by (5.8) are non-holonomic

\[
\Phi_1 = -(t + 1)v_2 - q_2 - q_3, \quad \Phi_2 = (t + 1)v_1 + (t + 1)q_3,
\]

while \( \Phi_3 \) is holonomic,

\[
\Phi_3 = -q_1 + (t + 1)q_2.
\]

It gives rise to a secondary constraint, \( \chi_3 = -v_1 + (t + 1)v_2 + q_2 \), which is a primary one, \( \chi_3 = -\Phi_1 - \Phi_2/(t + 1) \). The constraint function \( \chi_3 \) corresponds to the vector field \(-\partial/\partial q_1 - (t + 1)\partial/\partial q_2 \) and, consequently, \( \hat{\lambda} \) is gauge invariant. The symmetry vector \( X_g \) (6.4) and the function \( F_g \) are \((R = 1)\)

\[
X_g = g(t) \left( -\frac{\partial}{\partial q_1} - \frac{1}{t + 1} \frac{\partial}{\partial q_2} \right) + \dot{g}(t) \frac{\partial}{\partial q_3}, \quad F_g = g(t) \left( q_2 - \frac{t}{t + 1} q_1 \right).
\]

The local expression for the dynamical vector field is

\[
\Gamma = \frac{\partial}{\partial t} - q_3 \frac{\partial}{\partial q_1} - \frac{q_2 + q_3}{t + 1} \frac{\partial}{\partial q_2} + v_3 \frac{\partial}{\partial q_3} - v_3 \frac{\partial}{\partial v_1} - \frac{2v_2 + v_3}{t + 1} \frac{\partial}{\partial v_2} + C \frac{\partial}{\partial v_3},
\]

with \( C \) being an arbitrary function.
Example 6. A Lagrangian of the type II.3 is the one provided by a slight modification of the 1-form \( \lambda \) in Example 4:

\[
\lambda = (q_2 - q_3) dq_1 + q_2 dq_3 + (q_1 - q_2) q_3 dt \in \bigwedge^1 (\mathbb{R} \times \mathbb{R}^3).
\]

In this case,

\[
A = \begin{pmatrix} 0 & -1 & 1 \\
1 & 0 & 1 \\
-1 & -1 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} -q_3 \\
q_3 \\
q_2 - q_1 \end{pmatrix}.
\]

The primary constraint functions given by (5.8) are

\[
\Phi_1 = -v_2 + v_3 + q_3, \quad \Phi_2 = v_1 + v_3 - q_3, \quad \Phi_3 = -v_1 - v_2 + q_1 - q_2.
\]

The distribution \( \Omega_\lambda \) is spanned by the vector field \( \partial/\partial q_1 - \partial/\partial q_2 - \partial/\partial q_3 \). There is a holonomic constraint function, \( \Phi_1 - \Phi_2 - \Phi_3 = 2q_3 - q_1 + q_2 \), giving rise to a secondary one, \( \chi = 2v_3 - v_1 + v_2 \), which is not primary, that is, there is no \( \pi \)-vertical vector field \( Y \) such that \( \chi = \Phi Y \). Consequently, there is no gauge symmetry at all, and the dynamics \( \Gamma \) on the constraint manifold \( \mathcal{M} \) is unique:

\[
\Gamma = \frac{\partial}{\partial t} + q_3 \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right) + v_3 \left( \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} \right).
\]

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REFERENCES

1. Dirac P.A.M., Generalized Hamiltonian dynamics, Canad. J. Math. 2 (1950), 129–148.
2. Bergmann P.G. and Goldberg I., Transformations in phase space and Dirac brackets, Phys. Rev. 98 (1955), 531–38.
3. Sudarshan E.C.G. and Mukunda N., Classical dynamics: a modern perspective, Wiley-Interscience, John Wiley and Sons, New York-London-Sydney, 1974.
4. Cariñena J.F., López C and Rañada M.F., Geometric Lagrangian approach to first order systems and applications, J. Math. Phys. 29 (1988), 1134–42.
5. Cariñena J.F., Theory of Singular Lagrangians, Forts. Phys. 38 (1990), 641–680.
6. Faddeev L. and Jackiw R., Hamiltonian reduction of unconstrained and constrained systems, Phys. Rev. Lett. 60 (1988), 1692–94.
7. Floreanini R., and Jackiw R., Self-Dual Fields as Charge Density Solitons, Phys. Rev. Lett. 59 (1987), 1873–76.
8. Kulshreshta D.S. and Müller-Kirsten H.J.W., Quantization of systems with constraints: the Faddeev–Jackiw method versus Dirac’s method applied to superfields, Phys. Rev. D 43 (1991), 3376–83.
9. Kulshreshta D.S. and Müller-Kirsten H.J.W., Faddeev–Jackiw quantization of selfdual fields, Phys. Rev. D 45 (1992), R 393–97.
10. Floreanini R., Percacci R, and Sezgin E., Infinite-dimensional algebras in Chern-Simons Quantum Mechanics, Phys. Lett. B 261 (1991), 51–56.
11. Barcelos–Neto J. and Wotzasek C., *Symplectic quantization of constrained systems*, Int. J. Modern Phys. A 7 (1992), 1737–47.
12. Barcelos–Neto J. and Wotzasek C., *Faddeev–Jackiw quantization and constraints*, Int. J. Modern Phys. A 7 (1992), 4981–5003.
13. Cariñena J.F., Fernández-Núñez J. and Martínez E., *A geometric approach to Noether’s Second Theorem in time-dependent Lagrangian Mechanics*, Lett. Math. Phys. 23 (1991), 51–63.
14. Jackiw R., *(Constrained) Quantization without tears*, in “Constraint Theory and Quantization Methods” (Colomo F., Lusanna L. and Marmo G., eds.), World Scientific, 1994.
15. Montani H., *Symplectic analysis of constrained systems*, Int. J. Mod. Phys. A 8 (1993), 4319–4337.
16. Montani H. and Wotzasek C., *Faddeev–Jackiw quantization of nonabelian systems*, Mod. Phys. Lett. A 8 (1993), 3387–3396.
17. Foussats A., Greco A. and Zandron O. S., *Dynamical model and path integral formalism for Hubbard operators*, Int. J. Theor. Phys. 38 (1999), 1439–1453.
18. Foussats A., Greco A. and Zandron O. S., *First-order Lagrangians and path-integral quantization in the t-J model*, Ann. Phys. 275 (1999), 238–253.
19. Foussats A. and Greco A., *Large-N expansion based on the Hubbard-operator path integral representation and its application to the t – J model*, Phys. Rev. B 65 (2002), 195107.
20. Garriga J., Montes X., Sasaki M. and Tanaka T., *Canonical quantization of cosmological perturbations in the one-bubble open universe*, Nucl. Phys. B 513 (1998), 343–374.
21. Mendes A.C.R., Oliveira W. and Takakura F.I., *Turbulence as a constrained system*, hep-th/0004169.
22. Blas H. and Pimentel B.M., *The Faddeev–Jackiw approach and the conformal affine sl(2) Toda model coupled to the matter field*, Ann. Phys. 282 (2000), 67–86.
23. Gergely L.A., *On Hamiltonian formulations of the Schrödinger system*, Ann. Phys. 298 (2002), 394–402.
24. Govaerts J., *Hamiltonian reduction of first-order actions*, Int. J. Mod. Phys. A 5 (1990), 3625–3640.
25. Barcelos–Neto J. and Cheb-Terrab E.S., *Faddeev–Jackiw quantization in superspace*, Z. Phys. C 54 (1992), 133–138.
26. Foussats A. and Zandron O. S., *About the supersymmetric extension of the symplectic Faddeev–Jackiw quantization formalism*, J. Phys. A 30 (1997), L 513–517.
27. Foussats A., Repetto, C., Zandron O.P. and Zandron O.S., *Faddeev–Jackiw quantization method in conformal three-dimensional supergravity*, Int. J. Theor. Phys. 36 (1997), 55–65.
28. Cariñena J.F. and Fernández-Núñez J., *Geometric theory of time-dependent singular Lagrangians*, Forts. Phys. 41 (1993), 517–52.
29. Cariñena J.F., López C. and Martínez E., *Sections along a map applied to higher-order Lagrangian Mechanics. Noether’s theorem*, Acta Applicandae Mathematicae 25 (1991), 127–51.
30. Pidello G. and Tulczyjew W. M., *Derivations of Differential Forms on Jet Bundles*, Annali di Mathematica Pura ed Aplicata 147 (1987), 249–65.
31. Cariñena J.F., Fernández-Núñez J. and Martínez E., *Noether's theorem in time-dependent Lagrangian Mechanics*, Rep. Math. Phys. 31 (1992), 189–204.
32. Crampin M., *On the differential geometry of the Euler–Lagrange equations and the inverse problem of Lagrangian dynamics*, J. Phys. A: Math. Gen. 14 (1981), 2567–75.
33. Grifone J., *Structure presque tangente et connexions I*, Ann. Inst. Fourier 22 (1) (1972), 287–334.
34. Grifone J., *Structure presque tangente et connexions II*, Ann. Inst. Fourier 22 (3) (1972), 291–338.
35. Abraham R. and Marsden J.E., *Foundations of Mechanics*, Reading, Ma: Benjamin/Cummings, 1978.
36. de León M. and Rodrigues P.R., *Methods of Differential Geometry in Analytical Mechanics*, North Holland, 1989.
37. Saunders D.J., *The Geometry of Jet Bundles*, LMS 142, Cambridge University Press, 1989.
38. Crampin M., *Jet bundle techniques in Analytical Mechanics*, Quaderni del Consiglio Nazionale delle Ricerche, Gruppo Nazionale di Fisica Matematica, n. 47, 1995.
39. Bloch A.M., Krishnaprasad P.S., Marsden J.E. and Murray R.M., *Nonholonomic mechanical system with symmetry*, Arch. Rational Mech. Anal 135 (1996), 21–99.
40. Sarlet W., Cantrijn F., Saunders D.J., *A geometrical framework for the study of non-holonomic Lagrangian systems*, J. Phys. A:Math. Gen. **28** (1995), 3253–3268.

41. Sarlet W., Cantrijn F., Saunders D.J., *A geometrical framework for the study of non-holonomic Lagrangian systems II*, J. Phys. A:Math. Gen. **29** (1996), 4265–4274.

42. Bergvelt M.J. and de Kerf E.A., *Poisson brackets for Lagrangians Linear in the Velocity*, Lett. Math. Phys. **10** (1985), 13–19.

43. Fernández-Núñez J., *Lagrangian Structure of the Two-Dimensional Lotka-Volterra System*, Int. J. Theor. Phys. **37** (1998), 2457–2462.

44. Farias R., *Conventional Hamiltonian for First Order Differential Systems*, Hadronic J. **8** (1985), 93–99.