A general theorem due to Howe of dual action of a classical group and a certain non-associative algebra on a space of symmetric or alternating tensors is reformulated in a setting of second quantization, and familiar examples in atomic and nuclear physics are discussed. The special case of orthogonal-orthogonal duality is treated in detail. It is shown that, like it was done by Helmers more than half a century ago in the analogous case of symplectic-symplectic duality, one can base a proof of the orthogonal-orthogonal duality theorem and a precise characterization of the relation between the equivalence classes of the dually related irreducible representations on a calculation of characters by combining it with an analysis of the representation of a reflection. Young diagrams for the description of equivalence classes of irreducible representations of orthogonal Lie algebras are introduced. The properties of a reflection of the number non-conserving part in the dual relationship between orthogonal Lie algebras corroborate a picture of an almost perfect symmetry between the partners.

I. INTRODUCTION

This article is a sequel of Ref. 1 with multiple aims. One aim is to relate my results in Ref. 1 and related results in the literature, to a very general duality theorem due to Howe. This theorem was proved in Ref. 2, which I happened to read only after Ref. 1 was published. It has many special cases which were known to and applied by physicists before Ref. 2, but Howe’s theorem places all of this in a nice, unifying picture, as I intend to demonstrate in Sec. III. Howe’s article is written in a professional, mathematical language, which may feel foreign to workers in atomic and nuclear spectroscopy. This may be one reason why his result, while appearing in preprint in 1976, seems virtually unknown to this sector of the physics community. My first task, before entering the discussion of physical applications, is therefore to reformulate Howe’s duality theorem in a, to this readership, hopefully more familiar setting of second quantization. This is done in Sec. II.

The rest of my article is devoted to the special case of orthogonal-orthogonal duality. This case is also a main topic of Ref. 1, where, in particular, I prove an orthogonal-orthogonal duality theorem pertaining to Lie algebras by a calculation of characters similar to one used by Helmers more than half a century ago to obtain an analogous symplectic-symplectic duality theorem. My present investigation springs from the observation that the orthogonal-orthogonal case of Howe’s theorem relates equivalence classes of irreducible representations of a group and a Lie algebra, while mine relates equivalence classes of irreducible representations of two Lie algebras. This makes a difference because the representations of an orthogonal group and its Lie algebra have different reducibilities. The distinguishing feature of the orthogonal groups which gives rise to this difference is the presence of reflections, which form a coset topologically disconnected from the subgroup of rotations. An analysis of the representation of reflections therefore becomes a main theme. The central discussion appears in Sec. V and leads to the conclusion that one can prove the orthogonal-orthogonal special case of Howe’s duality theorem and explicitly describe the relation between the dual equivalence classes by combining my theorem in Ref. 1 with that analysis.

To prepare this discussion I review in Sec. IV essential parts of the representation theory of orthogonal groups and their Lie algebras and introduce generalized Young diagrams to describe the equivalence classes of irreducible representations of orthogonal Lie algebras. The final Sec. VI continues the theme of reflections. I thus define there a reflection of the number non-conserving partner in the dual relationship of Lie algebras, which contributes to a picture of an almost perfect symmetry between this and its number conserving mate.

For precision of the terminology, the word irrep will denote an equivalence class of irreducible representations of a group or Lie algebra or a standard realization within such a
class. Throughout, the base field is understood to be the field $\mathbb{C}$ of complex numbers.

II. HOWE’S DUALITY THEOREM

Consider a system of different kinds of particles. Some of them may be kinds of boson and some of them kinds of fermions. They share a 1-particle state space $V$ with a basis $(|p\rangle, p = 1, \ldots, d)$. A particle of kind $\tau$ is created from the vacuum in the state $|p\rangle$ by the operator $a^\dagger_{p\tau}$. These operators and a set of corresponding annihilation operators $a_{p\tau}$ obey the usual commutation relations

$$[a^\dagger_{p\tau}, a^\dagger_{q\nu}] = [a_{p\tau}, a_{q\nu}] = 0, \quad [a_{p\tau}, a^\dagger_{q\nu}] = \delta_{p\tau, q\nu},$$

where $[\cdot, \cdot]$ denotes the anticommutator $\{\cdot, \cdot\}$ when both $\tau$ and $\nu$ are kinds of fermions and otherwise the commutator $[\cdot, \cdot]$. Any change of the basis $(|p\rangle, p = 1, \ldots, d)$ is required to preserve the commutation relations (1). Despite the notation, $a^\dagger_{p\tau}$ and $a_{p\tau}$ are not assumed Hermitian conjugates. No Hermitian inner product is defined, indeed, on the state space. I call the span $A$ of the set of operators $a^\dagger_{p\tau}$ and $a_{p\tau}$ the space of field operators.

A classical group $G$ is supposed to act on $V$. The classical groups are: the general linear group $GL(d)$, the orthogonal group $O(d)$, and the symplectic group $Sp(d)$. Each of them is a group of linear transformations $g$ of $V$. The matrix elements of $g \in G$ in the basis $(|p\rangle)$ are denoted by $\langle p|g|q\rangle$. There is an induced representation of $G$ on the associative algebra $\mathfrak{A}$ generated by the field operators. This representation is such that $g \in G$ acts distributively on any product of elements $x \in \mathfrak{A}$, so the action of $g$ on $x$ may be written conveniently as a formal similarity map $x \mapsto gxg^{-1}$. In particular $1 \in \mathfrak{A}$ is $G$ invariant. The action of $g$ on field operators preserves the commutation relations (1). Dependent on the kind $\tau$ of particle it may be either cogredient,

$$ga^\dagger_{p\tau}g^{-1} = \sum_q a^\dagger_{q\tau}\langle q|g|p\rangle, \quad ga_{p\tau}g^{-1} = \sum_q \langle p|g^{-1}|q\rangle a_{q\tau},$$

or contragredient,

$$ga^\dagger_{p\tau}g^{-1} = \sum_q \langle p|g^{-1}|q\rangle a^\dagger_{q\tau}, \quad ga_{p\tau}g^{-1} = \sum_q a_{q\tau}\langle q|g|p\rangle.\quad (3)$$

The set of particle kinds with cogredient $G$ action is denoted by $K$ and the set with contragredient $G$ action by $K^*$. Both sets are finite.

The general linear group $GL(d)$ consists of all invertible linear transformation of $V$, while the subgroups $O(d)$ and $Sp(d)$ are defined by the conservation of a non-singular, bilinear form $b$ on $V$,

$$\sum_{rs}\langle b|rs\rangle\langle r|g|p\rangle\langle s|g|q\rangle = \langle b|pq\rangle \quad \forall g \in G.\quad (4)$$

The bilinear form $b$ is symmetric and skew symmetric, respectively, in the cases of $O(d)$ and $Sp(d)$. It follows that in the case of $Sp(d)$, the dimension $d$ is even. A dual bilinear form $b^*$ is defined by

$$\sum_r\langle b|pr\rangle\langle qr|b^*\rangle = \delta_{pq}.\quad (5)$$

It has the same symmetry as $b$ and satisfies

$$\sum_{rs}\langle p|g|r\rangle\langle q|g|s\rangle\langle rs|b^*\rangle = \langle pq|b^*\rangle \quad \forall g \in G.\quad (6)$$

A dual basis $(|p^*\rangle, p = 1, \ldots, d)$ for $V$ may be defined by

$$\langle b|pq^*\rangle = \delta_{pq}.$$
If $G$ acts crogrediently in the basis $|p\rangle, p = 1, \ldots, d$ it acts contragrediently in the basis $|p^*\rangle, p = 1, \ldots, d$ and vice versa. For $G = O(d)$ or $Sp(d)$ there is therefore no need of a distinction between co- and contragredient action, so one can set $K^* = \emptyset$.

The product $|\cdot,\cdot|$ can be extended to the set

$$\mathfrak{h} = \text{span} \{ ab \mid a, b \in A \},$$

(8)

which, by the commutation relations (11), includes the numbers. I thus set $|ab,cd| = |ab,cd|$ when either both $a$ and $b$ or both $c$ and $d$ are boson field operators or both of them are fermion field operators, and $|ab,cd| = \{ ab, cd \}$ when both $ab$ and $cd$ are products of one boson field operator and one fermion field operator. One can check that this defines $|\cdot,\cdot|$ unambiguously as a bilinear product on $\mathfrak{h}$ and that $\mathfrak{h}$ is closed under the action of $|\cdot,\cdot|$.

In particular $|h_1, h_2| = [h_1, h_2] = 0$ when any one of $h_1, h_2 \in \mathfrak{h}$ is a number. The algebra $(\mathfrak{h}, |\cdot,\cdot|)$ is almost a Lie algebra. In Ref. 2, Howe calls the subalgebra $(\mathfrak{h}, |\cdot,\cdot|)$ to be defined in a moment a graded Lie algebra, referring to a grading modulo 2 where numbers and products of two boson field operators or two fermion field operators have grade 0 and products of a boson field operator and a fermion field operator have grade 1. In the terminology of Jacobson (9) $(\mathfrak{h}, |\cdot,\cdot|)$ is a weakly closed subset of the associative algebra $\mathfrak{A}$. The set $\mathfrak{h}$ has a subset

$$\mathfrak{h}^{G} = \{ h \in \mathfrak{h} \mid ghg^{-1} = h \ \forall g \in G \}.$$  

(10)

The algebra $(\mathfrak{h}^{G}, |\cdot,\cdot|)$ is a subalgebra of $(\mathfrak{h}, |\cdot,\cdot|)$ because the map $x \mapsto gxg^{-1}$ preserves relations expressed by the product $|\cdot,\cdot|$.

Finally, the Fock space $\Phi$ is defined by

$$\Phi = \mathfrak{A}|\rangle = \mathfrak{A}^{\dagger}|\rangle$$

(11)

in terms of the vacuum state $|\rangle$. Here $\mathfrak{A}^{\dagger}$ denotes the subalgebra of $\mathfrak{A}$ generated by 1 and the creation operators. An action of $G$ on $\Phi$ is defined by taking literally the formal similarity map $x \mapsto gxg^{-1}$ and assuming that the vacuum is $G$ invariant,

$$g|\rangle = |\rangle \ \forall g \in G.$$  

(12)

Clearly, $\Phi$ and $\mathfrak{A}^{\dagger}$ are isomorphic as vector spaces, so the action of $\mathfrak{A}$ on $\Phi$ can be seen equivalently as an action on $\mathfrak{A}^{\dagger}$. So one avoids introducing the space $\Phi$. This point of view is taken in Ref. 2.

Given these preliminaries, Howe’s duality theorem (Theorem 8 of Ref. 2) can be formulated as follows.

**Theorem 1.** (Howe). The Fock space $\Phi$ has a decomposition

$$\Phi = \bigoplus_{\lambda} X_{\lambda} \otimes \Psi_{\lambda},$$

(13)

where the group $G$ and the algebra $(\mathfrak{h}^{G}, |\cdot,\cdot|)$ act so on $\Phi$ that each space $X_{\lambda}$ carries an irreducible representation of $G$, and each space $\Psi_{\lambda}$ carries an irreducible representation of $(\mathfrak{h}^{G}, |\cdot,\cdot|)$. For $\lambda \neq \mu$, the representations of $G$ on $X_{\lambda}$ and $X_{\mu}$ are inequivalent, and the representations of $(\mathfrak{h}^{G}, |\cdot,\cdot|)$ on $\Psi_{\lambda}$ and $\Psi_{\mu}$ are inequivalent. The spaces $X_{\lambda}$ have finite dimensions.
Remark. In the formulation of the theorem in Ref. 2, \( \mathfrak{h}^G \) is defined as the set of elements in \( (\mathfrak{h},|\cdot,\cdot|,\cdot|) \) which commute with every elements of an image of \( G \)'s Lie algebra. There are cases where this set is larger than the set (10); see my discussion in Ref. 1 of the case of fermions and \( G = O(2) \). The formulation above corresponds to the proof in Ref. 2, which is based on the so-called double commutant theorem and invariant theory.

Howe’s theorem is seen to establish a 1–1 relation between the irreps carried by \( X_\lambda \) and \( \Psi_\lambda \), but it does not specify this relation. More specific results in this respect appear in a later extensive treatise by Howe and also elsewhere in the literature. This is discussed in Sec. III. The structure of \( \mathfrak{h}^G \) is determined in Ref. 2 for each of the three classical groups \( G \). The result can be summarized as follows.

\[ \mathfrak{h}^{\text{GL}(d)} \] is spanned by the operators

\[ \sum_p |a_{pr^*}^\dagger, a_{pv}|, \quad (\tau, v) \in K \times K \cup K^* \times K^*, \]

\[ \sum_p |a_{pr^*}^\dagger, a_{pv}|, \quad (\tau, v) \in K \times K^*. \]

(14)

\[ \mathfrak{h}^{O(d)} \] and \( \mathfrak{h}^{\text{Sp}(d)} \) are spanned by the operators

\[ \sum_p |a_{pr^*}^\dagger, a_{pv}|, \quad \sum_{pq} |a_{pr^*}^\dagger, a_{qv^*}| \langle pq|b^* \rangle, \quad \sum_{pq} \langle b|pq \rangle |a_{pr^*}^\dagger, a_{pv}|, \quad (\tau, v) \in K \times K. \]

(15)

The operators (14) and (15) with products of two creation or two annihilation operators are not linearly independent. By the symmetry of \( |\cdot,\cdot| \), possibly combined with that of \( b \), they are either symmetric or skew symmetric in \( \tau \) and \( v \). In particular \( \tau = v \) is prohibited in the case of skew symmetry.

III. EXAMPLES IN PHYSICS

Several special cases of Theorem II were known in physics before Ref. 2. They concern systems which are either purely bosonic or purely fermionic so that \( \mathfrak{h}^G \) is a Lie algebra. Also, they do not involve contragredient actions of \( \text{GL}(d) \), so from now on, I set \( K^* = \emptyset \) and define \( k = |K| \).

A. \( \text{GL}(d) \)-\( \text{GL}(k) \) duality

The relations which result in the case \( G = \text{GL}(d) \) were noticed very early in the history of quantum mechanics. When \( K^* = \emptyset \), the Lie algebra \( \mathfrak{h}^{\text{GL}(d)} \) is spanned by the operators

\[ \sum_p |a_{pr^*}^\dagger, a_{pv}| = 2 \sum_p |a_{pr}^\dagger a_{pv} = \delta_{\tau v} d, \quad (\tau, v) \in K \times K, \]

(16)

for boson and fermions. The numeric term in Eq. (16) makes no difference with respect to the decomposition of \( \Phi \), so \( \mathfrak{h}^{\text{GL}(d)} \) may be replaced equivalently by the Lie algebra spanned by the operators

\[ \sum_p |a_{pr^*}^\dagger, a_{pv}|, \quad (\tau, v) \in K \times K. \]

(17)

This is the Lie algebra of the representation of \( \text{GL}(k) \) on \( \Phi \) induced by the group of invertible linear transformations acting on the index \( \tau \) of \( a_{pr^*}^\dagger \). Irreps of \( \text{GL}(k) \) stay irreducible upon restriction to the special linear group \( \text{SL}(k) \) of transformations with determinant 1 because the transformations in \( \text{GL}(k) \) deviate from those in \( \text{SL}(k) \) only by numeric factors. Similarly irreps of \( \text{GL}(k) \)'s Lie algebra \( \mathfrak{g}l(k) \) stay irreducible upon restriction to \( \text{SL}(k) \)'s
Lie algebra \( \mathfrak{sl}(k) \) because the transformations deviate only by numeric terms. Finally \( \mathfrak{sl}(k) \) irreps exponentiate to \( \text{SL}(k) \) irreps because \( \text{SL}(k) \) is simply connected. By combination of these facts it follows that, in the statement of the dual relation, \( \mathfrak{h}^{\text{GL}(d)} \) may be replaced equivalently by the \( \text{GL}(k) \) group of transformations acting on \( \tau \), so the relation may be characterized as a \( \text{GL}(d)-\text{GL}(k) \) duality.

Because the operators \( (17) \) conserve the number \( n \) of particles, the sum \( (13) \) splits into parts \( \Phi_n \) with definite \( n \). A state in \( \Phi_n \) is described by a wave function

\[
\phi(p_1, \ldots, p_n, \tau_1, \ldots, \tau_n)
\]

which satisfies

\[
\phi = S \phi
\]

with

\[
S = \frac{1}{n!} \sum_{s \in S(n)} s_p s_\tau \quad \text{and} \quad S = \frac{1}{n!} \sum_{s \in S(n)} (\text{sgn } s) s_p s_\tau
\]

for bosons and fermions, respectively. Here \( S(n) \) denotes the group of permutations of \( n \) elements, and \( s_p \) and \( s_\tau \) the permutation \( s \) applied to the arguments \( p_i \) and \( \tau_i \) of \( \phi \), respectively. The classical example is the system of \( n \) electrons in an atomic shell with principal and azimuthal quantum numbers \( N \) and \( l \). Its wave functions can be written

\[
\phi(m_{l1}, \ldots, m_{ln}, m_{s1}, \ldots, m_{sn})
\]

in terms of the spatial and spin magnetic quantum numbers \( m_l \) and \( m_s \) of one electron. The groups \( \text{GL}(d) \) and \( \text{GL}(k) \) act on the spatial and spin variables, respectively, of a single electron, so \( d = 2l + 1 \) and \( k = 2 \). The wave function \( (21) \) can be expanded on products

\[
\chi_{\mu_1 \nu_1}^\lambda (m_{l1}, \ldots, m_{ln}) \psi_{\mu_2 \nu_2}^\nu (m_{s1}, \ldots, m_{sn}),
\]

where \( \lambda_l \) and \( \lambda_s \) are \( n \)-cell Young diagrams. For a given \( \lambda_l \), the functions \( \chi_{\mu_1 \nu_1}^\lambda \) carry the corresponding irrep of \( \text{GL}(d) \otimes S(n) \) with \( \text{GL}(d) \) and \( S(n) \) acting on the indices \( \mu_l \) and \( \nu_l \) respectively, and similarly \( \psi_{\mu_2 \nu_2}^\nu \) in terms of \( \text{GL}(k) \). (Here, use is made of a well-known relation between symmetry and \( \text{GL}(d) \) irrep sometimes called the Schur or Schur-Weyl duality. That group theory has quantum mechanical applications was seen and communicated by Weyl almost immediately following the birth of quantum mechanics in 1925.)

The matrices of \( S(n) \) irreps can be chosen real orthogonal. For a given Young diagram \( \lambda \), let \( \hat{\lambda} \) denote the conjugate diagram, obtained by reflection in the bisector of the upper left corner. The \( \lambda \) and \( \hat{\lambda} \) irreps have equal dimensions, and their carrier spaces have bases \( (\nu) \) and \( (\nu') \) such that, in an obvious notation,

\[
\langle \nu | s | \nu' \rangle_{\lambda} = \text{sgn } s \langle \hat{\nu} | s | \hat{\nu'} \rangle_{\hat{\lambda}} \quad \forall s \in S(n).
\]

It then follows by the orthogonality relations of unitary matrix elements of irreps of finite groups (sometimes called the Schur orthogonality relations) that when the anti-symmetrizer \( S \) is applied to \( \phi \), only the terms with \( \lambda_l = \hat{\lambda}_s \) survive in the expansion on products \( (22) \) and that these terms combine to sums

\[
\sum_{\nu} \chi_{\mu_1 \nu}^\lambda (m_{l1}, \ldots, m_{ln}) \psi_{\mu_2 \nu}^\nu (m_{s1}, \ldots, m_{sn}).
\]

We have thus arrived in this special case at another proof of the duality theorem, which can obviously be generalized to any pair of dimensions \( d \) and \( k \). We have even got a precise relation between the connected irreps of \( \text{GL}(d) \) and \( \text{GL}(k) \): Their Young diagrams are
mutually conjugate. Moreover, because \( \lambda \) and \( \tilde{\lambda} \) can have no more than \( d \) and \( k \) rows, respectively (see Sec. IV.A), they can have no more than \( k \) and \( d \) columns, respectively. Conversely, the sum (2) can be constructed for every such pair of conjugate Young diagrams \( \lambda \) and \( \tilde{\lambda} \), so all of them occur.

Evidently, an even simpler argument gives an analogous result in the bosonic case. There, the GL\((d)\) and GL\((k)\) irreps have the same Young diagram whose depth does not exceed \( \min(d,k) \). In physics, one usually considers the subgroups \( U(d) \) and \( U(k) \) of unitary transformations rather than the full general linear groups GL\((d)\) and GL\((k)\), and the GL\((d)\)–GL\((k)\) duality is called a \( U(d) \)–\( U(k) \) duality. As the restriction to \( U(d) \) does not break the irreducibility of GL\((d)\) irreps, this makes no difference with respect to the decomposition of \( \Phi_n \).

### B. Sp\((d)\)–Sp\((2k)\) duality

It can be checked that \( h^{O(d)} \) has the structure of \( \mathfrak{sp}(2k) \) for boson systems and \( \mathfrak{o}(2k) \) for fermion systems and vice versa for \( h^{\mathfrak{sp}(d)} \), where \( \mathfrak{o}(2k) \) and \( \mathfrak{sp}(2k) \) are the Lie algebras of O\((2k)\) and Sp\((2k)\). For fermion systems, \( \Phi \) has finite dimension. So has then the irreps of \( \mathfrak{sp}(2k) \) or \( \mathfrak{o}(2k) \) in Eq. (13). In particular, because the group Sp\((2k)\) is simply connected, the \( \mathfrak{sp}(2k) \) irreps exponentiate to Sp\((2k)\) irreps and the Sp\((d)\)–Sp\((2k)\) duality is equivalent to an Sp\((d)\)–Sp\((2k)\) duality. This special case of Theorem I was noticed and proved by Helmers in 1961. His background was a line of research initiated in 1943, when Racah introduced the concept of seniority in an analysis of the n-electron system of Sec. III.A. Racah observed that the states in \( \Phi \) can be arranged in sequences of states of \( v \), \( v+2 \), \( v+4 \), . . . electrons such that the successor of a state in the sequence results when a pair of electrons with total spatial angular momentum \( L = 0 \) is added to its predecessor. The number \( v \) of electrons in the first state of the sequence he called the seniority of the sequence.

In 1949, Racah interpreted this result in terms of group theory, noticing that the Clebsch-Gordan coefficient \( \langle lm|lm'\rangle(00) \) is the matrix element of a symmetric bilinear form on the space \( V \) of spatial 1-electron states. This bilinear form defines an orthogonal group O\((d)\), and the seniority is a function of the irrep of this group. Racah's analysis is, in fact, closely related to Weyl's construction of O\((d)\) irreps (see Sec. IV.B). Racah points out in the same article that one can alternatively define a bilinear form in terms of the Clebsch-Gordan coefficient \( \langle jmjm'\rangle(00) \), where \( j \) is the half-integral quantum number of total, spatial plus spin, angular momentum of the electron, and \( m \) its associated magnetic quantum number. This Clebsch-Gordan coefficient is skew symmetric in \( m \) and \( m' \), and an analysis in terms of a symplectic group Sp\((d)\) ensues. Working in the basis of 1-electron states defined by \( j \) and \( m \) instead of \( m_l \) and \( m_s \) is known as \( jj \) coupling as opposed to \( LS \) coupling.

This analysis was subsequently adapted to the nuclear shell model. There, the spin variable may be supplemented by the variable of isospin, giving rise to more complicated structures. Adopting the \( jj \) coupling scheme, Helmers cast this entire analysis into a framework of second quantization and then proved, by a calculation of characters, the general Sp\((d)\)–Sp\((2k)\) duality theorem covering the cases \( k = 1 \) (electrons, only neutrons, only protons) and \( k = 2 \) (neutrons and protons) as well as any greater number of kinds of fermions. Helmers's proof provides a precise and somewhat peculiar rule in terms of Young diagrams for the association of the connected Sp\((d)\) and Sp\((2k)\) irreps: The Sp\((d)\) Young diagram and a reflected and rotated copy of the Sp\((2k)\) Young diagram fill a rectangle of depth \( d/2 \) and width \( k \) without overlap. (See the analogous orthogonal-orthogonal diagram below.) This rule is given independently by Howe in Ref. 6.

For \( k = 1 \) the group Sp\((2k)\) was identified by Kerman simultaneously with and independently of Helmers. More precisely, Kerman identified the unitary \( \mathfrak{su}(2) \) subalgebra of the \( \mathfrak{so}(2) \) Lie algebra isomorphic to the Lie algebra \( \mathfrak{sp}(2) \) of Sp\((2k)\). This \( \mathfrak{su}(2) \) Lie algebra is known as the quasispin algebra. The 1–1 relation between quasispin and \( jj \) seniority is at the core of an extensive analysis of the nuclear \( k = 1 \) system by Talmi and his coworkers. Also independently of Helmers, Flowers and Szpikowski identified in 1964 the \( \mathfrak{sp}(4) \) Lie al-
gebra pertaining to the case \( k = 2 \) as an \( \mathfrak{o}(5) \) Lie algebra. It is well known that \( \mathfrak{sp}(4) \) and \( \mathfrak{o}(5) \) are isomorphic. In these works, equivalent expressions for the eigenvalues of a certain “pairing” interaction are shown to result whether expressed by \( \mathfrak{sp}(d) \) or \( \mathfrak{sp}(2k) \) quantum numbers upon a suitable association of these quantum numbers. There is no proof that the associated irreps select the same subspace of \( \Phi \), nor that the representation of \( \mathfrak{sp}(d) \oplus \mathfrak{sp}(2k) \) on this subspace is irreducible.

C. \( O(d) \)-\( \mathfrak{o}(2k) \) duality

Helmers anticipates in Ref. 8 that results similar to those obtained there hold when the single-fermion angular momentum quantum number \( l \) is integral so that the Clebsch-Gordan coefficient \( \langle lm|lm' \rangle (00) \) provides a symmetric bilinear form, but the road to such analogous results would turn out fairly long. Closely following their proposal of the \( \mathfrak{o}(5) \) Lie algebra, Flowers and Szpikowski proposed an \( \mathfrak{o}(8) \) Lie algebra of “quasi-spin in \( LS \) coupling” which is recognized as \( \mathfrak{h}^{O(d)} \) in the case when \( V \) is the space of spatial angular momentum states as in Sec. III A and \( k \) equals 4 corresponding to the 4-dimensional space of the nucleonic spin and isospin. Like in the case of \( \mathfrak{o}(5) \) they show that equivalent expressions for the eigenvalues of a pairing interaction result whether expressed by \( \mathfrak{o}(d) \) or \( \mathfrak{o}(8) \) quantum numbers upon a suitable association of these quantum numbers, but there is no proof of duality in the sense of Theorem 11.

Later work in nuclear physics has been based on this suggestion, but not before the work of Rowe, Repka, and Carvalho to appear in 2011 was, to my knowledge, the relation between the dual irreps of \( O(d) \) and \( \mathfrak{o}(2k) \) characterized precisely14 (The case \( k = 4 \) was handled by Rowe and Carvalho in 2007, 20.) In particular, unlike the case of the \( \mathfrak{sp}(d) \)-\( \mathfrak{sp}(2k) \) duality, Howe does not in Ref. 6 provide such a precise characterization in the \( O(d)-\mathfrak{o}(2k) \) case. The argument in Ref. 14 is based on the identification of a state in \( \Phi \) that has highest weight simultaneously with respect to \( \mathfrak{o}(d) \), \( \mathfrak{s}(d) \), \( \mathfrak{s}(k) \), and \( \mathfrak{o}(2k) \). (See Eq. (53).) The \( \mathfrak{gl}(d) \)-\( \mathfrak{gl}(k) \) duality relation then provides a relation between the \( O(d) \) and \( \mathfrak{o}(2k) \) irreps. In 2019 I used Helmers’s method to obtain a result that is closely related to but slightly different from those of Rowe in Ref. 2 and Rowe, Repka, and Carvalho in Ref. 14 by establishing a relation between irreps of the Lie algebras \( \mathfrak{o}(d) \) and \( \mathfrak{o}(2k) \) rather than between such of the group \( O(d) \) and the Lie algebra \( \mathfrak{o}(2k) \). This makes a difference because, consistently with the non-connectedness of \( O(d) \), representations of \( O(d) \) and \( \mathfrak{o}(d) \) have different reducibilities. The relation between my result and those of Refs. 2 and 14 is the topic of Sec. V.

D. \( O(d)-\mathfrak{sp}(2k) \) and \( \mathfrak{sp}(d)-\mathfrak{o}(2k) \) dualities

In the cases of the \( O(d)-\mathfrak{sp}(2k) \) and \( \mathfrak{sp}(d)-\mathfrak{o}(2k) \) dualities, the highest weight of an \( \mathfrak{h}^G \) irrep in the decomposition 13 can be expressed by that of the \( G \) irrep by the method of Rowe, Repka, and Carvalho14. A Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{h}^G \) is spanned by the subset

\[
2 \sum_{p} a^\dagger_{pr} a_{pv} + d, \quad \tau \geq v,
\]

\[
2 \sum_{pq} (b_{pq}) a_{pr} a_{qv}, \quad (25)
\]

of the set 16. The members of its Cartan subalgebra whose eigenvalues on the highest-weight vector defined by \( \mathfrak{b} \) give in any finite-dimensional irreducible representation the row lengths of its Young diagram (compare Sec. VC) are

\[
-\sum_{p} a^\dagger_{pr} a_{pr} - d/2, \quad \tau = 1, \ldots, k.
\]

By the discussion in Sec. III A the number of rows in the \( G \) irrep’s Young diagram cannot exceed \( \min(d,k) \). Let \( \lambda_p, p = 1, \ldots, k, \) denote their lengths in the order from top to bottom.
with trailing zeros if their number is less than \( k \). The highest weight of the \( \mathfrak{g}^G \) irrep is then given in terms of the eigenvalues \( w_\tau \) of the operators (26) by

\[
w_\tau = -\lambda_{k+1-\tau} - d/2.
\] (27)

With every \( w_\tau \) negative, this cannot be a linear combination of fundamental weights with non-negative integral coefficients unless \( \mathfrak{g}^G \simeq \mathfrak{o}(2) \), so in any other case the \( \mathfrak{g}^G \) irrep has infinite dimension.\(^4\)

(The Lie algebra \( \mathfrak{o}(2) \) is 1-dimensional and thus has a continuum of 1-dimensional irreps. Since \( k = 1 \), we are dealing with a single kind of bosons. For the skew symmetric bilinear form \( b \) pertaining to \( \text{Sp}(d) \), the operators \(^{15}\) with products of two creation or two annihilation operators are then absent, whence \( \mathfrak{o}(2) \) conserves the number \( n \) of bosons. In fact \( \mathfrak{o}(2) \) is spanned in its present realization by the single remaining operator \(^{15}\). The representation of \( \text{GL}(d) \) on \( \Phi_n \) is easily seen to belong to the irrep with a 1-row Young diagram of width \( n \). The states in \( \Phi_n \) are also seen to be traceless with respect to \( b \) (compare Sec. IV B), so the restriction to \( \text{Sp}(d) \), whence \( \lambda_1 = n \). The eigenvalue (27) of the single operator (26) is then tautological. The decomposition (13) is therefore, in this pathological case, just the splitting of \( \Phi \) into its subspaces \( \Phi_n \).)

Rowe, Repka, and Carvalho derive in Ref. 14 the relation (27) for \( G = O(d) \). While their proof rests entirely on properties of the Lie algebra \( \mathfrak{sp}(2k) \), they describe their result as an \( O(d)–\text{Sp}(k, \mathbb{R}) \) duality, where \( \text{Sp}(k) \) in their notation is \( \text{Sp}(2k) \) in mine. At the level of Lie algebras the restriction to the reals is irrelevant. One can chose, however, in \( A \) a basis of coordinates and momenta

\[
X_{pr} = c(a^+_p + a_{pr}), \quad P_{pr} = \frac{i}{2c}(a^+_p - a_{pr}),
\] (28)

with \( c \) a numeric constant. These operators obey the Heisenberg commutation relations

\[
[X_{pr}, P_{qv}] = i \delta_{p,qu},
\] (29)

and \( \mathfrak{sp}(2k) \) is spanned by the operators

\[
i \sum_p X_{pr}X_{pv}, \quad i \sum_p P_{pr}P_{pv}, \quad i \sum_p \{X_{pr}, P_{pv}\}.
\] (30)

A real linear combination \( M \) of the operators (30) induce by the commutation map \( x \mapsto [M, x] \) an infinitesimal linear canonical transformation among the “collective” coordinates and momenta

\[
\sum_p X_{pr}, \quad \sum_p P_{pr},
\] (31)

and in classical mechanics the linear canonical transformations form the group \( \text{Sp}(2k, \mathbb{R}) \). This has a double covering group, the so-called metaplectic group \( \text{Mp}(2k) \), faithfully represented on a boson Fock space.\(^{22}\) and the authors of Ref. 14 suggest that the \( \mathfrak{sp}(2k, \mathbb{R}) \) irrep with the highest weight (27) may in general exponentiate to an irrep of \( \text{Mp}(2k) \). An overlapping group of authors has proposed a model of nuclear collective motion based on the Lie algebra \( \mathfrak{sp}(6) \), corresponding to \( k = 3 \), where \( p \) essentially labels the nucleons and \( \tau \) the three spatial dimensions.\(^{23}\) The bosons are quanta of oscillation in the conventional harmonic oscillator potential well of the nuclear shell model. In the literature on this model, \( \mathfrak{sp}(6) \) is called \( \text{Sp}(3, \mathbb{R}) \).

IV. FINITE-DIMENSIONAL REPRESENTATIONS OF \( O(d) \) AND \( \mathfrak{o}(d) \)

In this section, I sketch some elements of the representation theory of the orthogonal groups and their Lie algebras.\(^{45,24}\) At the end, I introduce convenient Young diagrams for the description of finite-dimensional \( \mathfrak{o}(d) \) irreps.
A. \textit{sl}(d) highest-weight vectors on a tensor space

Already met are the functions
\begin{equation}
\chi_{\lambda}^{\mu \nu}(p_1, \ldots, p_n),
\end{equation}
which carry irreducible representations of $GL(d) \otimes S(n)$ and span the space $T_n(d)$ of functions (or tensors)
\begin{equation}
\chi(p_1, \ldots, p_n).
\end{equation}
The action of $g \in GL(d)$ on $\chi$ is by
\begin{equation}
(g \chi)(p_1, \ldots, p_n) = \sum_{(q_1, \ldots, q_n)} \left( \prod_i \langle p_i | g | q_i \rangle \right) \chi(q_1, \ldots, q_n),
\end{equation}
and the action of $s \in S(n)$ by permutation of the arguments $p_i$. Here, $n \geq 1$. One can include $n = 0$ by defining an empty tuple () invariant to a group $S(0) = \{1\}$ and assigning the value 1 to an empty product. Then $T_0(d)$ is 1-dimensional and carries the 1 representation of $GL(d)$ \( \otimes S(0) \cong GL(d) \). The corresponding Young diagram $\lambda$ is the empty diagram. For fixed $\lambda$ and $\nu$ the functions (32) form a basis for an irreducible $GL(d)$ module. Every equivalent module results from this one by the action of a member of the $S(n)$ group algebra. Fixing $\lambda$ and $\nu$ amounts to projecting $T_n(d)$ by one of the primitive idempotents associated with $\lambda$ within the $S(n)$ group algebra. This idempotent can be chosen as the Young symmetrizer
\begin{equation}
Y_{\lambda} = c_{\lambda} \sum_{st} (\text{sgn } s) st
\end{equation}
corresponding to the tableau where the numbers $1, \ldots, n$ are placed in reading order in the cells of the Young diagram $\lambda$. In the sum in Eq. (35), the permutation $s$ runs over all products of permutations of the columns of the tableau, $t$ runs over all products of permutations of the rows of the tableau, and $c_{\lambda} \neq 0$ is a numeric factor ensuring $Y_{\lambda}^2 = Y_{\lambda}$. Consider the function
\begin{equation}
\chi_{\lambda}(p_1, \ldots, p_n) = \prod_i \delta_{p_i, \rho(i)},
\end{equation}
where $\rho(i)$ is the ordinal number of the row in the tableau that contains the number $i$. This product vanishes if $\lambda$ has more than $d$ rows. Otherwise, when $Y_{\lambda}$ acts on $\chi_{\lambda}$, every $t$ in the sum (35) fixes $\chi_{\lambda}$, and every $s$ gives a different product of Kronecker deltas, so $\chi_{\lambda}^{\lambda} = Y_{\lambda} \chi_{\lambda} \neq 0$. The entire irreducible $GL(d)$ module is then generated from $\chi_{\lambda}^{\lambda}$ by the $GL(d)$ group algebra.

The space of linear transformations of $V$ is spanned by the transformations $e_{pq}$ with matrix elements
\begin{equation}
\langle r | e_{pq} | s \rangle = \delta_{rp, sq}.
\end{equation}
The action of $x \in gl(d)$ on $\chi \in T_n(d)$ is by
\begin{equation}
(x \chi)(p_1, \ldots, p_n) = \sum_{(q_1, \ldots, q_n)} \left( \sum_i \langle p_i | x | q_i \rangle \right) \chi(q_1, \ldots, q_n)
\end{equation}
with 0 assigned to an empty sum. For $x = e_{pq}$ and $\chi = \chi_{\lambda}$ this gives a sum of terms where one Kronecker delta $\delta_{p\nu}$ in the product (36) is replaced by $\delta_{p\nu}$. For $p = q$ this changes nothing and because $Y_{\lambda}$ commutes with the action (38), one gets
\begin{equation}
e_{pp} \chi_{\lambda}^{\lambda} = \lambda_p ^{\lambda} \lambda_{\lambda}^{\lambda}.
\end{equation}
where \( \lambda_p, p = 1, \ldots, d \), are the, possibly vanishing, row lengths of the Young diagram \( \lambda \) ordered from top to bottom. For \( p < q \) the factor \( \delta_{p,q} \) becomes identical to a factor \( \delta_{p,p} \) where, after a permutation \( t \) in the sum (35), the number \( j \) is situated vertically above \( i \) in the Young tableau. The term is then killed by the summation over the permutations \( s \), so one gets

\[
e_{pq} \chi_{hw}^\lambda = 0. \tag{40}\]

For \( d \geq 2 \) the transformations \( e_{pp} - e_{p+1,p+1}, p = 1, \ldots, d - 1, \) and \( e_{pq}, p < q \), span a Borel subalgebra \( b \) of \( \mathfrak{so}(d) \subset \mathfrak{gl}(d) \). Its derived Lie algebra \( b' = [b, b] \) is spanned by the second set of transformations, so \( b' \chi_{hw}^\lambda = 0 \) by Eq. (10). Thus \( \chi_{hw}^\lambda \) is a highest-weight function for the representation of \( \mathfrak{sl}(d) \) on \( T_n(d) \). For \( d = 1 \), we have \( \mathfrak{sl}(d) = \mathfrak{sl}(1) = \{0\} \), which is a particularly trivial case of an Abelian Lie algebra. Other cases met below include the 1-dimensional Lie algebra \( \mathfrak{o}(2) \). The irreps of Abelian Lie algebras are 1-dimensional. In the present case the 1-dimensional space \( T_n(1) \) carries the representation \( 0 \rightarrow 0 \) of \( \mathfrak{sl}(1) \). An Abelian Lie algebra is a Cartan and a Borel subalgebra of itself, and any irreducible representation is, being, since 1-dimensional, isomorphic to a linear form, a weight relative to this Cartan subalgebra and indeed the only weight of the representation itself may be considered a highest weight, and any vector in its carrier space a highest-weight vector. In particular the function \( \chi_{hw}^\lambda \) with \( \lambda \) the 1-row, \( n \)-cell Young diagram may be considered a highest-weight function for the representation of \( \mathfrak{sl}(1) \) on \( T_n(1) \).

### B. Irreducible representations of \( \mathfrak{O}(d) \) and \( \mathfrak{o}(d) \) on a tensor space

Let \( m \) be some non-negative integer. Weyl shows that every irreducible module over \( \mathfrak{O}(d) \subset \mathfrak{gl}(d) \) in \( T_n(d) \) is isomorphic for some \( n \leq m \) to a module in the space \( T_n(d) \) of functions \( \chi \in T_n(d) \) that are traceless in the sense

\[
\sum_{p \neq j} \langle b|p_i p_j \rangle \chi(p_1, \ldots, p_n) = 0 \quad \forall i, j, i \neq j. \tag{41}\]

The module in \( T_n^0(d) \) is embedded in an irreducible \( \mathfrak{gl}(d) \) module in \( T_n(d) \), and \( \mathfrak{O}(d) \) modules embedded in this manner in inequivalent \( \mathfrak{gl}(d) \) modules are inequivalent. The Young diagram of the \( \mathfrak{gl}(d) \) irrep may then be assigned to the \( \mathfrak{O}(d) \) irrep. A \( \mathfrak{gl}(d) \) irrep can contain an \( \mathfrak{O}(d) \) irrep in this way if and only if the sum of depths of any two different columns of its Young diagram do not exceed \( d \). The generating function \( \chi_{\lambda_{hw}}^\lambda \) of an \( \mathfrak{gl}(d) \) module defined in Sec. [IV.A] is seen to be traceless when the bilinear form \( b \) is chosen in the form

\[
\langle b|pq \rangle = \delta_{p+q,d+1}, \tag{42}\]

so it then also generates the embedded \( \mathfrak{O}(d) \) module. Each allowed Young diagram has a partner, which I call its complementary Young diagram. Complementary Young diagrams are identical except for the depths \( \lambda_1 \) and \( \lambda'_1 \) of their first columns, which obey \( \lambda_1 + \lambda'_1 = d \). The matrices of a pair of \( \mathfrak{O}(d) \) irreps with different, complementary Young diagrams can be chosen to coincide for rotations \( g \in \mathfrak{O}(d) \), that is, \( \det g = 1 \), and differ by a factor \(-1\) for reflections, that is, \( \det g = -1 \). These representations therefore become identical upon restriction to the subgroup \( \mathfrak{SO}(d) \) of rotations, and this representation can be shown to be irreducible. (The irreducibility follows from that of the derived representation of its Lie algebra \( \mathfrak{o}(d) \), which can be inferred from the general theory of finite-dimensional representations of semisimple Lie algebras mentioned in Sec. [IV.C]). The \( \mathfrak{O}(d) \) coset of reflections is generated by its element

\[
r = \begin{cases} 
-e_{(d+1)/2,(d+1)/2} + \sum_{p \neq (d+1)/2} e_{pp}, & \text{odd } d, \\
e_{d/2,d/2+1} + e_{d/2+1,d/2} + \sum_{p \neq d/2,d/2+1} e_{pp}, & \text{even } d.
\end{cases} \tag{43}\]
It is seen that \( r \chi^\lambda_{\text{hw}} = \chi^\lambda_{\text{hw}} \) for \( \tilde{\lambda}_1 < d/2 \) and \( r \chi^\lambda_{\text{hw}} = -\chi^\lambda_{\text{hw}} \) for \( \tilde{\lambda}_1 > d/2 \). An \( O(d) \) irrep with a self-complementary Young diagram breaks into two \( \text{SO}(d) \) irreps connected by the reflections. Self-complementary Young diagrams only occur for even \( d \).

The Lie algebra \( \mathfrak{o}(d) \) is that of the maximal connected subgroup \( \text{SO}(d) \) of \( O(d) \), so the \( \mathfrak{o}(d) \) irreps occurring on \( \bigoplus_n T_n(d) \) are those of \( \text{SO}(d) \). Let transformations \( \hat{e}_{pq} \) be given by

\[
\hat{e}_{pq} = e_{pq} - e_{q^*p^*} \tag{44}
\]

in terms of the dual basis defined by Eq. (7). Explicitly, \( p^* = d + 1 - p \). Also, assume for the moment that \( d \geq 2 \). The transformations \( \hat{e}_{pq}, p \leq q \), then span a Borel subalgebra of \( \mathfrak{o}(d) \). Because these transformations belong to the \( \mathfrak{sl}(d) \) Borel subalgebra defined in Sec. IV A, the \( \mathfrak{sl}(d) \) highest-weight function \( \chi^\lambda_{\text{hw}} \) is also an \( \mathfrak{o}(d) \) highest-weight function. The transformations \( \hat{e}_{pp}, p = 1, \ldots, [d/2] \), form a basis for the Cartan subalgebra of this Borel subalgebra. If \( \lambda \) and \( \lambda' \) are complementary and \( \tilde{\lambda}_1 < \tilde{\lambda}'_1 \), these transformations are seen from Eq. (39) to have on \( \chi^\lambda_{\text{hw}} \) and \( \chi^{\lambda'}_{\text{hw}} \), the same set of eigenvalues by the action (35), equal to the lengths of the first \( [d/2] \) rows of \( \lambda \). If \( \lambda \) is self-complementary, let \( \lambda' \) be the Young diagram obtained from \( \lambda \) by moving row number \( d/2 \) one step down (thus violating the rule that \( \lambda_p, p = 1, \ldots, d \), should be a non-increasing sequence). Because every transformation \( \chi \) in \( \mathfrak{o}(d) \) satisfies \( (d/2)!d/2+1 = 0 \), the function \( \chi^\lambda_{\text{hw}} \) is also an \( \mathfrak{o}(d) \) highest-weight function. These \( \mathfrak{o}(d) \) highest-weight functions belong to a common irreducible \( \mathfrak{o}(d) \) module because \( \chi^\lambda_{\text{hw}} = r \chi^\lambda_{\text{hw}} \), and this \( \mathfrak{o}(d) \) module must be the one generated by \( \chi^\lambda_{\text{hw}} \). Thus \( \chi^\lambda_{\text{hw}} \) and \( \chi^{\lambda'}_{\text{hw}} \) generate the two irreducible \( \mathfrak{o}(d) \) modules branching out from the \( \mathfrak{o}(d) \) module at the restriction to \( \text{SO}(d) \). The eigenvalues of \( \hat{e}_{pp}, p = 1, \ldots, d/2 \), on \( \chi^{\lambda'}_{\text{hw}} \) by the action (35) are the same as on \( \chi^\lambda_{\text{hw}} \) except for a change of sign of the eigenvalue of \( \hat{e}_{d/2,d/2} \). In summary, for \( d \geq 2 \) the eigenvalues \( w_p \) of \( \hat{e}_{pp}, p = 1, \ldots, [d/2] \), acting by (35) on a highest-weight function of an \( \mathfrak{o}(d) \) module in \( \bigoplus_n T_n(d) \) are integral and satisfy

\[
\begin{align*}
  w_1 &\geq w_2 \geq \cdots \geq \frac{w_{(d-1)/2}}{2} \geq 0, & \text{odd } d, \\
  w_1 &\geq w_2 \geq \cdots \geq \frac{w_{d/2}}{2} \geq 0, & \text{even } d, 
\end{align*}
\tag{45}
\]

and for every set of integers \( w_p \) which obey these rules there is an embedding \( \mathfrak{o}(d) \) module whose Young diagram has row lengths \( \lambda_p = |w_p| \) for \( p \leq d/2 \) and \( \lambda_p = 0 \) for \( p > d/2 \) as well as one with the complementary Young diagram, if different.

Turning to the case \( d = 1 \), one has \( T_n^0(1) = T_n(1) \) for \( n = 0 \) or \( 1 \) and \( T_n^0(1) = 0 \) for \( n \geq 2 \). The representations of \( \mathfrak{o}(1) = \mathfrak{sl}(1) = \{0\} \) on \( T_0^0(1) \) and \( T_0^1(1) \) are the same as those of \( \mathfrak{sl}(1) \). Consistently with the complementarity of the 0-cell and 1-cell Young diagrams they are equivalent. Both functions \( \chi^\lambda_{\text{hw}} \) defined by these two Young diagrams are highest-weight functions. The sequence \( w_p, p = 1, \ldots, [d/2] \), is empty, as is the sequence \( \lambda_p, p = 1, \ldots, [d/2] \), of row lengths of the empty Young diagram. In this way the case \( d = 1 \) conforms to the general rule.

C. Spin representations and \( \mathfrak{o}(d) \) Young diagrams.

Not every finite-dimensional \( \mathfrak{o}(d) \) irrep occurs in \( \bigoplus_n T_n(d) \). There is a general theory which determines every finite-dimensional irrep of a semisimple Lie algebra. Among the Lie algebras \( \mathfrak{o}(d) \) this excludes \( \mathfrak{o}(2) \), which is not semisimple. Otherwise, in terms of the eigenvectors \( w_p \) of the transformations \( \hat{e}_{pp}, p = 1, \ldots, [d/2] \), on the highest-weight vector determined by the Borel subalgebra spanned by the transformations \( \hat{e}_{pp}, p \leq q \), the result is that an \( \mathfrak{o}(d) \) irrep has finite dimension if and only if either all \( w_p \) are integral or all \( w_p \) are half-integral and the rule (44) is obeyed. Incidentally, this is also fulfilled for every \( \mathfrak{o}(2) \) irrep met below. It may be considered conformance to the general rule that the trivial Lie algebra \( \mathfrak{o}(1) = \{0\} \) has only the trivial irrep \( 0 \mapsto 0 \), which may formally be assigned the empty sequence \( w_p, p = 1, \ldots, [d/2] \) with \( d = 1 \).

Besides the \( \mathfrak{o}(d) \) irreps which occur in \( \bigoplus_n T_n(d) \) there is thus a whole set with half-integral \( w_p \), the so-called spin irreps. It is convenient to define Young diagrams to describe
the entire set of finite-dimensional $\mathfrak{o}(d)$ irreps. In view of the close relation between the set of $w_p$, and the row lengths of the Young diagram of an $\mathfrak{o}(d)$ module embedding an $\mathfrak{o}(d)$ module in $\bigoplus_n T_n(d)$ when such one exists, it is natural to let the $\mathfrak{o}(d)$ Young diagrams have rows of lengths $w_p$. Unlike the $\mathfrak{o}(d)$ Young diagrams, the depth of an $\mathfrak{o}(d)$ Young diagram then does not exceed $d/2$. To describe the negative $w_{d/2}$ which can occur for even $d$, one must include rows of negative length. For example the $\mathfrak{o}(6)$ irrep $(w_1, w_2, w_3) = (5, 3, -2)$ may be described by the diagram:

(46)

(To distinguish, for $d = 2$, positive and negative $w_1$ from one another one must mark somehow the edge whence the row extends.) Spin irreps may be described by the inclusion of a column of width $1/2$, for example:

(47)

for the irreps $(w_1, w_2, w_3) = (9/2, 7/2, 3/2)$ and $(w_1, w_2, w_3) = (9/2, 7/2, -3/2)$, respectively.

V. $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ AND $\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ DUALITIES

After all these preparations, I finally get to my task. In 2019, I proved Theorem 2 below by a calculation of characters similar to that of Helmers in Ref. 3. It refers to Young diagrams of irreps of orthogonal Lie algebras as defined in Sec. IV C with $\lambda$ and $w$ describing irreps of $\mathfrak{o}(d)$ and $\mathfrak{o}(2k)$, respectively. Their row lengths are denoted by $\lambda_p$ and $w_\tau$. Irreps of $\mathfrak{o}(d)$ with Young diagrams $\lambda$ that are identical except for opposite, non-zero, values of $\lambda_{d/2}$ (which only occurs for even $d$), and irreps of $\mathfrak{o}(2k)$ with Young diagrams that are identical except for opposite, non-zero, values of $w_k$, are paired so that the Young diagram with positive length of this row represents the pair. With this convention, the theorem reads as follows.

**Theorem 2.** (Neergård). The fermion Fock space $\Phi$ has the decomposition

$$\Phi = \bigoplus X_\lambda \otimes \Psi_w,$$

(48)

where $\mathfrak{o}(d)$ and $\mathfrak{o}(2k)$ act on $\Phi$ that $X_\lambda$ and $\Psi_w$ carry the irreps or pairs of irreps of $\mathfrak{o}(d)$ and $\mathfrak{o}(2k)$ with Young diagrams $\lambda$ and $w$, and the sum runs over all pairs of $\lambda$ and $w$ which fill a $d/2 \times k$ frame without overlap:

(49)

In the illustration, $d = 13$, $k = 4$, $\lambda = (4, 3, 3, 2, 1, 0)$, and $w = (11/2, 7/2, 5/2, 3/2)$, and $w$ represents the pair of this irrep and $w = (11/2, 7/2, 5/2, -3/2)$. The Young diagram $w$ is reflected and rotated so that its rows appear vertically from the right to the left. I deliberately chose an example with an odd $d$ to illustrate that the $\mathfrak{o}(2k)$ irreps are spin
representations when \( d \) is odd such as in the single-\( l \) shell systems of Secs. III B and III C. Because \( k \) is integral, the \( \mathfrak{o}(d) \) irrep is always a non-spin representation, as it should be because \( \mathfrak{o}(d) \) is the Lie algebra of an \( \mathbf{O}(d) \) group acting on \( \Phi \). For this reason, each irreducible \( \mathfrak{o}(d) \) module in \( \Phi \) is also known to have an embedding \( \mathbf{O}(d) \) module.

Theorem 2 is seen to be closely parallel to Helmers’s \( \mathbf{Sp}(d) \)–\( \mathbf{Sp}(2k) \) (or \( \mathfrak{sp}(d) \)–\( \mathfrak{sp}(2k) \)) duality theorem. Both are symmetric in the spaces of dimensions \( d \) and \( 2k \), and the rules for the association of diagrams are identical. It may be noticed that the \( \mathfrak{o}(d) \) irrep is always associated with a pair of \( \mathfrak{o}(2k) \) irreps and vice versa. Thus, if the boundary in Fig. 49 between \( \lambda \) and \( w \) hits the bottom edge of the frame, two \( \mathfrak{o}(d) \) irreps correspond to one \( \mathfrak{o}(2k) \) irrep. If it hits the left edge, it is opposite. The first case actually provides a partial proof of the \( \mathfrak{o}(d) \)–\( \mathfrak{o}(2k) \) case of Theorem 1 and also specifies the relation between the \( \mathbf{O}(d) \) and \( \mathfrak{o}(2k) \) irreps in this case. Indeed, if the border between \( \lambda \) and \( w \) hits the bottom edge of the frame, the \( \mathfrak{o}(d) \) Young diagram is self-complementary, so it represents two different \( \mathfrak{o}(d) \) irreps embedded in an \( \mathbf{O}(d) \) irrep determined by the \( \mathfrak{o}(2k) \) irrep. Let \( \Phi_\psi \) denote the subspace of \( \Phi \) selected by a vector \( \psi \in \Psi_w \). By Theorem 2, \( \Phi_\psi \) is composed of two irreducible \( \mathfrak{o}(d) \) modules belonging to diﬀerent irreps. Each \( \mathfrak{o}(d) \) module is embedded in an \( \mathbf{O}(d) \) module which also contains an \( \mathfrak{o}(d) \) module belonging to the other irrep, and since \( \mathbf{O}(d) \) and \( \mathfrak{o}(2k) \) commute acting on \( \Phi \), each entire \( \mathbf{O}(d) \) module lies within \( \Phi_\psi \). But because each \( \mathfrak{o}(d) \) irrep appears just once in \( \Phi_\psi \), the \( \mathbf{O}(d) \) modules must then coincide.

If the border between \( \lambda \) and \( w \) hits the left edge of the frame, as happens in Fig. 49, and as it always does when \( d \) is odd, the correspondence between the irreps of \( \mathfrak{o}(2k) \) and \( \mathbf{O}(d) \) is less unique. Two \( \mathfrak{o}(2k) \) irreps correspond to the same \( \mathfrak{o}(d) \) irrep, and each corresponding module over \( \mathfrak{o}(d) \) in \( \Phi \) carries an extension to an \( \mathbf{O}(d) \) module. But the \( \mathbf{O}(d) \) representation may belong to any one of two irreps, so based on Theorem 2 alone the range of possible \( \mathbf{O}(d) \) irreps cannot be narrowed further than to those two. Theorem 1 tells us that the \( \mathbf{O}(d) \) irreps corresponding to the two \( \mathfrak{o}(2k) \) irreps must be diﬀerent, leaving still two alternatives for the precise association.

Rowe, Repka, and Carvalho derive the relation

\[
w_\tau = d/2 - \lambda_{k+1-\tau},
\]

(50)

where \( \lambda_\tau \) are the column depths of the \( \mathbf{O}(d) \) Young diagram and \( w_\tau \) the row lengths of the \( \mathfrak{o}(2k) \) Young diagram. Somewhat imprecisely, the Lie algebra \( \mathfrak{o}(2k) \) is called \( \text{SO}(2k) \) in Ref. 14.) This can be illustrated as follows.

Here, \( d = 13, k = 4, \lambda = (4, 3, 3, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0), \) and \( w = (11/2, 7/2, 5/2, -3/2) \) and \( w \) is reflected and rotated as in Fig. 49. The Young diagrams \( \lambda \) and \( w \) fill the \( d/2 \times k \) frame without overlap provided a negative \( w_k \) is understood to cancel a part of \( \lambda \) which extrudes the frame. Theorem 5 of Ref. 14 may be formulated as the statement that every pair of \( \mathbf{O}(d) \) and \( \mathfrak{o}(2k) \) irreps whose Young diagrams combine in this manner, and only those, occur exactly once in the decomposition of \( \Phi \). As detailed below, I could not follow completely the argument in Ref. 14. I shall obtain the rule (50) by analyzing the action on \( \Phi \) of the reflection \( \tau \) given by Eq. (49).

I must prove that when the boundary in Fig. 49 between \( \lambda \) and \( w \) hits the left edge of
the frame, as it happens in the illustration, the pair of $O(d)$ Young diagrams

![Young Diagram](image)

(52)

correspond to the pair of $\sigma(2k)$ Young diagrams

![Young Diagram](image)

(53)
in this order. To this end consider a self-non-complementary $O(d)$ Young diagram $\lambda$. Analogous to the function $\chi_\lambda^{hw}$ defined in Sec. IV A one can define a state

$$\phi_\lambda^{hw} = \left( \prod_i a^\dagger_{\rho(i),\kappa(i)} \right) |\rangle,$$

(54)

where $\kappa(i)$ is the ordinal number of the column containing the number $i$ in any fixed tableau assigned to $\lambda$. (Such states are also considered in Ref. [14]) By copying the reasoning in Sec. IV B one finds that $\phi_\lambda^{hw}$ is a highest-weight state of an $\sigma(2k)$ module belonging to the $\sigma(2k)$ irrep contained in the $O(d)$ irrep described by $\lambda$. It is easily calculated that $r\phi_\lambda^{hw} = \phi_\lambda^{hw}$ for $\tilde{\lambda}_1 < d/2$ and $r\phi_\lambda^{hw} = -\phi_\lambda^{hw}$ for $\tilde{\lambda}_1 > d/2$. Thus $\phi_\lambda^{hw}$ generates an $O(d)$ module with Young diagram $\lambda$. I could not identify in the argument in Ref. [14] the step equivalent to this analysis of the action of $r$.

Now let the definitions of $a^\dagger_{pr}$ and $a_{pr}$ be extended to negative $\tau$ by

$$a^\dagger_{pr} = a_{p^-,\tau}, \quad -k \leq \tau \leq k, \tau \neq 0,$$

(55)

which gives the commutation relations

$$\{a^\dagger_{pr}, a_{qv}\} = \delta_{pr,qv},$$

(56)

valid for every pair of $\tau$ and $v$ in the range of $\tau$ in Eq. (55). It follows that when this range is ordered from $-k$ to $k$ with $0$ omitted, the operators

$$f_{\tau v} = \frac{1}{2} \sum_p [a^\dagger_{pr}, a_{pv}],$$

(57)

obey the same commutation relations in terms of ordinal numbers as do the transformations $e_{pq}$ defined by Eq. (44) in terms of the ordering of the range of $p$ from $1$ to $d$. In particular the operators $f_{\tau v}, \tau \leq v$, span a Borel subalgebra $b$ of $\sigma(2k)$. The derived Lie algebra $b'$ of $b$ is spanned by the operators $f_{\tau v}, \tau < v$. There are two kinds of these operators

$$\sum_p a_{pr} a^\dagger_{pv}, \quad \sum_p a_{p^+,\tau} a_{pv}, \quad \tau > v > 0.$$

(58)

Acting on $\phi_\lambda^{hw}$, the operators of the first kind attempt to move fermions in states $|pr\rangle$ into states with the same $p$ and lower $\tau$, which is impossible because these states are already occupied. The operators of the second kind attempt to annihilate pairs of fermions in pairs of states $|pr\rangle$ and $|qv\rangle$ with $\tau \neq v$ and $p + q = d + 1$, which is also impossible because $\tilde{\lambda}_r + \tilde{\lambda}_v \leq d$ for any pair of different $\tau$ and $v$. In conclusion, $b'\phi_\lambda^{hw} = 0$, and $\phi_\lambda^{hw}$ is also
an \(\mathfrak{o}(2k)\) highest-weight state. One can now calculate the eigenvalues on \(\phi_{hw}^\lambda\) of the basic operators

\[
f_{-\tau,-\tau} = \frac{1}{2} \sum_p [a_{p,-\tau}^\dagger, a_{p,-\tau}] = \frac{d}{2} - \sum_p a_{p\tau}^\dagger a_{p\tau}, \quad \tau > 0,
\]

of the Cartan subalgebra and arrive at the relation (59). It follows that \(\phi_{hw}^\lambda\) generates an irreducible module over \(O(d) \otimes \mathfrak{o}(2k)\) whose \(\lambda\) and \(w\) are combined according to Eq. (59). Since \(\phi_{hw}^\lambda\) exists for every such pair of \(\lambda\) and \(w\), and, by Theorem 2, each \(\mathfrak{o}(2k)\) irrep with a self-complementary Young diagram appears exactly once in combination with a given \(\mathfrak{o}(d)\) irrep, it follows that this \(\mathfrak{o}(d)\) module must be embedded in an \(O(d)\) module belonging to the irrep given by Eq. (59). The entire \(O(d) - \mathfrak{o}(2k)\) case of Theorem 1 and the rule (59) are thus seen to follow from Theorem 2 in combination with the analysis above of the action of a reflection.

When the \(O(d)\) Young diagram \(\lambda\) is self-complementary, and \(\lambda'\) is the Young diagram which, in the analysis of Sec. IV B, produces the involutive automorphism of a reflection. This confirms that the \(\mathfrak{o}(d)\) modules generated by \(\phi_{hw}^\lambda\) and \(\phi_{hw}^{\lambda'}\) combine to an \(O(d)\) module.

VI. REFLECTION OF \(\mathfrak{o}(2k)\)

To see which transformation connects the states \(\phi_{hw}^\lambda\) and \(\phi_{hw}^{\lambda'}\) corresponding to the diagrams \(\lambda\) and \(\lambda'\) in Fig. (52), consider the linear transformation of \(\mathfrak{A}\) which acts distributively on products and is generated in terms of a formal similarity map \(x \mapsto \sigma x \sigma^{-1}\) by

\[
\sigma a_{p1}^\dagger \sigma^{-1} = a_{p1}, \quad \sigma a_{p1} \sigma^{-1} = a_{p1}^\dagger, \quad \sigma a_{p\tau}^\dagger \sigma^{-1} = a_{p\tau}, \quad \sigma a_{p\tau} \sigma^{-1} = a_{p\tau}, \quad \tau > 1.
\]

This preserves the commutation relations (1), and because \((p^*)^* = p\), one gets \(\sigma^2 a \sigma^{-2} = a\) for every \(a \in A\), so \(x \mapsto \sigma x \sigma^{-1}\) is an involution of \(\mathfrak{A}\), that is, equal to its inverse map. Setting

\[
\sigma |\rangle = \left( \prod_p a_{p1}^\dagger \right) |\rangle
\]

and taking literally the formal similarity map, one obtains

\[
\sigma^2 |\rangle = \left( \prod_p a_{p1} \right) \left( \prod_p a_{p1}^\dagger \right) |\rangle = |\rangle,
\]

so \(\sigma\) is an involution of \(\Phi\). It is easily verified that \(x \mapsto \sigma x \sigma^{-1}\) also preserves the span of the operators \(f_{\tau 0}\), given by Eq. (57), which is \(\mathfrak{o}(2k)\). Because it preserves commutation relations within \(\mathfrak{A}\), it preserves, in particular, the commutation relations in \(\mathfrak{o}(2k)\), so it provides an involutionary automorphism of \(\mathfrak{o}(2k)\), which may be called a reflection of \(\mathfrak{o}(2k)\).

The transformation \(a \mapsto r a a^{-1}\) maps \(a_{d+1/2,\tau}^\dagger\) to \(-a_{d+1/2,\tau}\) when \(d\) is odd, and \(a_{d/2,\tau}\) and \(a_{d/2+1,\tau}\) to one another when \(d\) is even, and does not change any other \(a_{p\tau}\), and similarly for the annihilation operators. It follows that the transformations \(a \mapsto r a a^{-1}\) and \(a \mapsto \sigma a \sigma^{-1}\) commute for every \(a \in \mathfrak{A}\). Further,

\[
\sigma r |\rangle = r |\rangle = \left( \prod_p a_{p1} \right) |\rangle, \quad r \sigma |\rangle = r \left( \prod_p a_{p1}^\dagger \right) |\rangle = -\left( \prod_p a_{p1} \right) |\rangle,
\]

(63)
so \( \sigma v = -rv \). Because, for even \( d \), one gets

\[
\sigma \left( \prod_{p=1}^{d/2} a_{p1}^+ \right) |L\rangle = \left( \prod_{p=1}^{d/2} a_{p-1}^+ \right) \left( \prod_{p=1}^{d/2} a_{p1}^+ \right) |L\rangle = (-)^{d/2} \left( \prod_{p=1}^{d/2} a_{p1}^+ \right) |L\rangle,
\]

the highest-weight state \( \phi_{\lambda \omega}^\lambda \) corresponding to a self-complementary \( O(d) \) Young diagram \( \lambda \) is an eigenstate of \( \sigma \) with eigenvalue \((-1)^{d/2}\). By \( \phi_{\lambda \omega}^\mu = \pm r \phi_{\lambda \omega}^\lambda \) and \( \sigma v = -rv \), it follows that the corresponding state \( \phi_{\lambda \omega}^\mu \) with a negative eigenvalue of \( \tilde{e}_{d/2,d/2} \) is an eigenstate of \( \sigma \) with eigenvalue \((-1)^{d/2+1}\). The eigenvalue of \( \sigma \) thus distinguishes the two \( \sigma(d) \) irreps associated with a common \( \sigma(2k) \) irrep from one another in the same way as the eigenvalue of \( r \) distinguishes the two \( \sigma(2k) \) irreps associated with a common \( \sigma(d) \) irrep from one another.

As \( r \) connects the two former, and \( \sigma \) the two latter, a symmetry between the actions of \( r \) and \( \sigma \) with respect to \( \sigma(d) \) and \( \sigma(2k) \) is revealed. The transformation \( \sigma \) may seem to be closely similar to the transformation with this symbol employed by Weyl in his analysis of the restriction from \( O(d) \) to \( SO(d) \).

The similarity of \( \sigma \) to a reflection is displayed even more clearly when one looks at the linear map \( a \mapsto [x,a] \) with \( x \in \sigma(2k) \) and \( a \in A \). It preserves each subspace \( A_p \) of \( A \) spanned for a fixed \( p \) by the operators \( a_{p-1}^+ \), \(-k \leq \tau \leq k, \tau \neq 0 \), and its matrix elements in the basis of these operators does not depend on \( p \). The same holds for the map \( a \mapsto \sigma a \sigma^{-1} \). In a sense, one could thus view our system as a system of \( d \) fermion fields living in a common space \( U \) isomorphic to every \( A_p \). The action of \( \sigma \) on \( A_p \) results in an interchange of the basic operators \( a_{p1}^+ \) and \( a_{p-1}^+ \), which is seen to correspond to the action of \( r \) on \( V \) according to Eq. (43), then \( d \) is even. The system of a single kind of fermion field living in \( U \) in this sense is the \( d = 1 \) case of the general system. For \( d = 1 \), the Fock space \( \Phi \) is isomorphic to the \( 2^k \)-dimensional spinor space, which carries a faithful representation of the double covering group \( Pin(2k) \) of \( O(2k) \).

Because \( \sigma(2k) \) is the Lie algebra of \( Pin(2k) \), the symmetry of Theorem 2 with respect to \( \sigma(d) \) and \( \sigma(2k) \) suggests the existence of a \( \sigma(d) \)-\( Pin(2k) \) duality analogous to the \( O(d) \)-\( \sigma(2k) \) duality. Settling this matter would require an analysis, which I shall not pursue, of the action relative to the said realization of \( Pin(2k) \) of \( \sigma \) and the present realization of \( \sigma(2k) \). For \( k = 1 \), the transformation \( \sigma \) is similar to a particle-hole conjugation. Contrary to a claim in Ref. 1, it is different, however, from the particle-hole conjugation \( \gamma \) of Refs. 27, 29, which obeys \( \gamma^2 \sigma \gamma^{-2} = -\sigma \) for \( a \in A \) and only applies for even \( d \).

In Ref. 1 I employed a particular instance of \( \sigma \). It is instructive to review this example on the background of the present, general definition. The system considered is the atomic shell of Sec. IIIA. The variables \( p \) and \( \tau \) are the magnetic quantum numbers \( m_l \) and \( m_s \), and \( \sigma \) swaps emptiness and occupation of 1-electron states with \( m_s = -1/2 \). The corresponding map \( x \mapsto \sigma x \sigma^{-1} \) is shown in Ref. 1 to transform the total spin \( S \) into a “spin quasi-spin” \( Q \). The Lie algebra \( \sigma(2k) = \sigma(4) \) is spanned by the components of \( S \) and \( Q \), the components of each of them span an \( \sigma(3) \) Lie algebra, and these \( \sigma(3) \) Lie algebras commute. I call them \( \sigma(3)_{S} \) and \( \sigma(3)_{Q} \), and the row lengths of their 1-row Young diagrams \( S \) and \( Q \). (The former is the usual quantum number of total spin. The analogon of \( \sigma(3)_{S} \) for arbitrary \( k \) is the \( sl(k) \) subalgebra of the number conserving \( gl(k) \) subalgebra of \( \sigma(2k) \). Only for \( k = 2 \) does a commuting and non-Abelian subalgebra exist. For \( k = 1 \) one has \( sl(k) = sl(1) = \{0\} \), and \( \sigma(2k) = \sigma(2) \) is 1-dimensional, and for \( k \geq 3 \) the Lie algebra \( \sigma(2k) \) is simple.)

One gets

\[
Q_0 = \sigma \left( \sum_p m_{sp} \right) \sigma^{-1} = \frac{1}{2} (n-d),
\]

where \( n \) is the number of electrons. Two other members of a basis for \( \sigma(3)_{Q} \) raise or lower \( n \) by \( 2 \) units. The row lengths of the Young diagram of an \( \sigma(4) \) irrep are \( w_{1,3} = Q \pm S \). This sheds light on Racah’s original definition of seniority mentioned in Sec. IIIA. The only operators in \( \sigma(4) \) which change the number of electrons belong to \( \sigma(3)_{Q} \), so the sequence of states with constant seniority \( v \) according to Racah’s definition has constant \( Q \). The
leading state has $\frac{1}{2}(v - d) = \frac{1}{2}(n - d) = Q_0 = -Q$, or $v = d - 2Q$, so $v$ and $Q$ are actually equivalent quantum numbers. One further gets $v = d - w_1 - w_2$, which is the area of the $O(d)$ Young diagram $\lambda$ in Fig. (51). At the time of writing Ref. 1, I was unaware of this and only saw that $v$ generally differs from the area of the $O(d)$ diagram $\lambda$ in Fig. (49). This seemed to make this case different from others such as that of the $\text{Sp}(d)$–$\text{Sp}(4)$ duality pertaining to the systems of neutrons of protons in a nuclear $j$ shell, where an appropriately defined seniority equals the area of the Young diagram of the irrep of the number conserving group. In fact, in the atomic system, Racah’s seniority $v$ is also the depth of the 1-column Young diagram of the $\text{Sp}(d)$ group arising in $jj$ coupling, so $Q$ equals Kerman’s quasispin\textsuperscript{17} as well.

VII. CONCLUDING REMARKS

The most important result of this study, in the view of the author, is the demonstration in Sec. V that Helmers’s method of calculation of characters\textsuperscript{3} provides a proof of the $O(d)$–$\sigma(2k)$ special case of Theorem 1 and an explicit association of the participating irreps of $O(d)$ and $\sigma(2k)$ when combined with an analysis of the representation of a reflection. It is an open question whether this method could be adapted to the boson case, where Weyl’s character formula\textsuperscript{31–36} is not available due to the infinite dimensions of the irreps of the number non-conserving Lie algebras. Also the Young diagrams introduced in Sec. IV C to describe irreps of an orthogonal Lie algebra appear to be new in the literature. The properties of the reflection $\sigma$ defined in Sec. VI further corroborates the picture of an almost perfect symmetry between $\sigma(d)$ and $\sigma(2k)$ in their dual relationship already emerging from my study in Ref. 1.

ACKNOWLEDGMENTS

I am indebted to Roger Howe for providing me with copies of some of his publications, which I could not access otherwise, including Ref. 6.

\textsuperscript{1}K. Neergård, J. Math. Phys. 60, 081707 (2019).
\textsuperscript{2}R. Howe, Trans. Am. Math. Soc. 313, 539 (1989 (preprint 1976)).
\textsuperscript{3}K. Helmers, Nucl. Phys. 23, 594 (1961).
\textsuperscript{4}N. Jacobson, Lie Algebras (Interscience Publishers, New York, USA, 1962).
\textsuperscript{5}H. Weyl, The Classical Groups. Their Invariants and Representations (Princeton University Press, Princeton, USA, 1939).
\textsuperscript{6}R. Howe, Isr. Math. Conf. Proc. 8, 1 (1995).
\textsuperscript{7}A. Young, Proc. London Math. Soc. s1-33, 97 (1900).
\textsuperscript{8}A. Young, Proc. London Math. Soc. s1-34, 361 (1902).
\textsuperscript{9}I. Schur, Dissertation (Berlin, Germany, 1901).
\textsuperscript{10}H. Weyl, Gruppentheorie und Quantenmechanik (Hirzel, Leipzig, Germany, 1928).
\textsuperscript{11}F. G. Frobenius, Sitzungsber. Preuß. Akad. (1905), 406.
\textsuperscript{12}D. J. Rowe, J. Repka, and M. J. Carvalho, J. Math. Phys. 52, 013507 (2011).
\textsuperscript{13}G. Racah, Phys. Rev. 63, 367 (1943).
\textsuperscript{14}G. Racah, Phys. Rev. 76, 1352 (1949).
\textsuperscript{15}A. Kerman, J. Math. Phys. 60, 081707 (2019).
\textsuperscript{16}I. Talmi, Simple Models of Complex Nuclei (Harwood Academic Publishers, Chur, Switzerland, 1993).
\textsuperscript{17}B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. 84, 193 (1964).
\textsuperscript{18}B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. 84, 673 (1964).
\textsuperscript{19}B. J. Rowe and M. J. Carvalho, J. Phys. A: Math. Theor. 40, 471 (2007).
\textsuperscript{20}D. Shale, Trans. Am. Math. Soc. 103, 149 (1962).
\textsuperscript{21}G. Rosensteel and D. J. Rowe, Phys. Rev. Lett. 38, 10 (1977).
\textsuperscript{22}R. Goodman and N. R. Wallach, Representations and Invariants of the Classical groups (Cambridge University Press, Cambridge, United Kingdom, 1998).
\textsuperscript{23}E. Cartan, Bull. Soc. Math. France 41, 53 (1913).
\textsuperscript{24}R. Brauer and H. Weyl, Am. J. Math. 57, 425 (1935).
27 E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, United Kingdom, 1935).

28 G. Racah, Phys. Rev. **62**, 438 (1942).

29 J. S. Bell, Nucl. Phys. **12**, 117 (1959).

30 K. Neergård, Phys. Rev. C **91**, 144313 (2015).

31 H. Weyl, Math. Z. **23**, 271 (1925).

32 H. Weyl, Math. Z. **24**, 338 (1926).

33 H. Weyl, Math. Z. **24**, 377 (1926).

34 H. Freudenthal, Indag. Math. **16**, 369 (1954).

35 H. Freudenthal, Indag. Math. **16**, 487 (1954).

36 H. Freudenthal, Indag. Math. **18**, 511 (1956).