CONTRACTING CONVEX HYPERSURFACES BY
FUNCTIONS OF THE MEAN CURVATURE

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Abstract. This paper concerns the evolution of a closed convex hypersurface in \( \mathbb{R}^{n+1} \), in direction of its inner unit normal vector, where the speed is given by a smooth function depending only on the mean curvature, and satisfies some further restrictions, without requiring homogeneity. It is shown that the flow exists on a finite maximal interval, convexity is preserved and the hypersurfaces shrink down to a single point as the final time is approached. This generalises the corresponding result of Schulze \[20\] for the positive power mean curvature flow to a much larger possible class of flows by the functions depending only on the mean curvature.

1. Introduction and main result

Let \( M^n \) be a smooth, compact oriented manifold of dimension \( n(\geq 2) \) without boundary, and \( X_0 : M^n \rightarrow \mathbb{R}^{n+1} \) a smooth immersion of \( M^n \) into the euclidean space. Consider a one-parameter family of smooth immersions: \( X_t : M^n \rightarrow \mathbb{R}^{n+1} \), evolving according to

\[
\begin{aligned}
\frac{\partial}{\partial t} X (p, t) &= -\Phi(H)(p, t) \cdot \nu(p, t), \quad p \in M^n, \\
X(\cdot, 0) &= X_0(\cdot),
\end{aligned}
\]

where \( \nu(p, t) \) is the outer unit normal to \( M_t \) at \( X(p, t) \) in tangent space \( TM^{n+1} \), \( \Phi \) is a smooth supplementary function defined on an open subset in \( \mathbb{R} \) and satisfying \( \Phi' > 0 \), and \( H(p, t) \) the trace of Weingarten map \( W_\nu(p, t) = -W_\nu(p, t) \) on tangent space \( TM^n \) induced by \( X_t \). Throughout the paper, we will call such a flow \( \Phi(H) \)-flow.

For \( \Phi(H) = H \), we obtain the well-known mean curvature flow, Huisken \[11\] showed that in the Euclidean space \( \mathbb{R}^{n+1} \) any closed convex hypersurface \( M_0 \) evolving by mean curvature flow contracts to a point in finite time, becoming spherical in shape as the limit is approached. In \[12\], he extended this result to compact hypersurfaces in general Riemannian manifolds with suitable bounds on curvature. For \( \Phi(H) = -\frac{1}{H} \), we get the inverse mean curvature flow, which was studied in Euclidean space and hyperbolic space \[12, 8\] and other Riemannian spaces, in particular, Huisken and Ilmanen \[13\] used it to prove the

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Penrose inequality for asymptotically flat 3-manifolds. For \( \Phi(H) = H^3 \), this flow becomes the power mean curvature flow, which has been considered by Schulze in [20] for \( M_0 \) of strictly positive mean curvature hypersurface in the Euclidean space, he proved that the \( H^3 \)-flow has a unique, smooth solution on a finite time interval \([0, T]\) and \( M_t \) converges to a point as \( t \to T \) if \( M_0 \) is strictly convex for \( 0 < \beta < 1 \) or \( M_0 \) is weakly convex for \( \beta \geq 1 \). Here weakly convex and strictly convex, resp., are defined as all the eigenvalues of Weingarten map being positive and nonnegative, resp. But some counterexamples show that in general the evolving hypersurfaces may not become spherical in shape as the limit is approached. In the previous paper [3], the author, together with Li and Wu, extended Schulzes results to \( h \)-convex hypersurfaces in the hyperbolic space, and showed that if the initial hypersurface has mean curvature bounded below, the positive power mean curvature flow has a unique, smooth solution on a finite time interval, and converges to a point if the initial hypersurface is strictly \( h \)-convex for the case that \( 0 < \beta < 1 \), or weakly convex for \( \beta \geq 1 \). Moreover, for the \( H^3 \)-flow case with \( \beta \geq 1 \), it has been found that if the initial hypersurface \( M_0 \) has the ratio of largest to smallest principal curvatures close enough to 1 at every point, then the evolving hypersurfaces contract to a round point: This was first shown in the Euclidean space setting by Schulze [21], then in the hyperbolic space setting by the author, Li and Wu [6].

A feature of the results mentioned above is that the speeds for these flows all depend only on the mean curvature \( H \). However, for other flows with the speeds given by arbitrary functions \( \Phi \) depending only on \( H \), the understanding of convergence is far less complete, except in some specific settings such as closed convex surface expanding (for example, see [22, Smoczyk]). There are many difficulties in understanding such flows with arbitrary speed function \( \Phi \): The first difficulty is to choose “nice” speeds depending only on \( H \) which guarantee that we have suitable inequalities on curvatures along such flows which furthermore ensure that local convexity of initial data are preserved. The second difficulty stems from the greatly increased complexity caused by the presence of the arbitrary function \( \Phi \); for instance, the application of the maximum principle to the evolution equations for geometric quantities either fail or become more subtle, and deriving the sufficient regularity results of solutions for such flows become potentially more complicated than that for the usual geometric flow.

The present paper considers a wide class of such flows with the “nice” speeds \( \Phi(H) \), which satisfy the following additional conditions:

**Assumptions 1.1.** Let \( \Phi : (0, +\infty) \to \mathbb{R} \) is a smooth function such that for all \( x \in (0, +\infty) \) we have

\[
\Phi > 0, \Phi' > 0, \Phi'' \geq \frac{-2\Phi'}{x} \quad \text{and} \quad \Phi\Phi''x + \Phi\Phi' - (\Phi')^2 x \geq 0.
\]

Then we will show the additional technical conditions on the \( \Phi \) which determine that the general picture of behaviour established in the positive power
mean curvature flow by [20] Schulze, remain valid for the \( \Phi \)-flow case. For convenience, we define a function \( g(x) = \frac{1}{\Phi(x)} \) on \((0, +\infty)\), and set \( G(x) = \int_0^x g(s)ds \). The main result achieved can be exactly stated by the following theorem.

**Theorem 1.2.** Assume that \( X_0 : M^n \to \mathbb{R}^{n+1} \) be a smooth convex immersion and that the smooth function \( \Phi : (0, +\infty) \to \mathbb{R} \) is strictly increasing. Then there exists a unique, smooth solution to the flow \( (1.1) \) on a finite maximal time interval \([0, T)\). Furthermore, if the function \( \Phi(x) \) for \( x > 0 \) satisfies the assumptions \( 1.1 \) then \( T \) is between \( G\left(\frac{1}{H_{\text{max}}(0)}\right) \) and \( nG\left(\frac{1}{H_{\text{min}}(0)}\right) \). In particular, in the following two cases that

i) \( M_0 \) is strictly convex for \( \frac{\Phi''}{\Phi'} < 0 \),

ii) \( M_0 \) is weakly convex for \( \Phi'' \geq 0 \) and \( \Phi' \geq \frac{\Phi}{\Phi'} \),

then the hypersurfaces \( M_t \) are strictly convex for all \( t > 0 \) and they contract to a point in \( \mathbb{R}^{n+1} \) as \( t \) is approached to \( T \).

**Remark 1.3.**

1. The positivity on the first order derivative of the function \( \Phi \) is essential to ensure short-time existence like the \( H^0 \)-flow case in [20].

2. In order to drop “bad” terms in the evolution equation for geometric quantities and then apply the maximum principle to show monotonicity of curvature, the more assumptions on \( \Phi \), which is similar as that of [22, Theorem 1], are required. However our hypotheses of \( \Phi \) differ from those in [22] in one important respect: No extra assumption \( \left(\frac{\Phi'' \Phi'}{\Phi'}\right) \leq 0 \) on \( \Phi \) is required, due to focusing on different problems. This shows that our assumptions \( 1.1 \) are more weaker than those in [22, Theorem 1].

There exist many examples of natural flows with the speeds \( \Phi \) satisfying the assumptions \( 1.1 \) which not covered by previous results, for example,

**Example 1.4.**  
(i) \( \Phi(x) = \beta_1 x^\beta_2 + \beta_3 \) defined on all \( x \in (0, +\infty) \), such that the constants \( \beta_i > 0 \), \( i = 1, 2 \), and \( \beta_3 \geq 0 \). Obviously, these include \( \Phi(H) = H \) (the mean curvature) and \( \Phi(H) = H^{\beta} \) (the positive powers of the mean curvature). In particular, the case that \( \beta_2 = 1 \) and \( \beta_3 = 0 \), i.e. \( \Phi(x) = \beta_1 x \) defined on all \( x \in (0, +\infty) \), corresponds the case ii) in Theorem \( 1.3 \). And the case \( \beta_2 > 1 \), in this situation where \( \Phi(x) \) defined on \( \left(\frac{\beta_3}{\beta_1 (\beta_2 - 1)}\right)^{\frac{1}{\beta_2}}, +\infty \) also corresponds the case ii) in Theorem \( 1.3 \).

(ii) \( \Phi(x) = \beta_1 \sinh^{\beta_2}(x) + \beta_3 \):

1) on all \( x \in (0, +\infty) \) and for the constants \( \beta_i > 0 \), \( i = 1, 3 \), and \( \beta_2 \geq 1 \).
In particular, the case that \( \beta_2 > 1 \) and \( \beta_3 = 0 \), i.e. \( \Phi(x) = \beta_1 \sinh^{\beta_2}(x) \) defined on all \( x \in (0, +\infty) \), corresponds the case \( \text{ii} \) in Theorem 1.2.

(iii) \( \Phi(x) = \beta_1 e^{\beta_2 x} + \beta_3 \) on all \( x \in (0, +\infty) \) and for the constants \( \beta_i > 0 \), \( i = 2, 3 \), and \( \beta_1 \geq 1 \).

Remark 1.5.  
(i) All of the above examples can be used in Theorem 1.2. Note that of these, relatively few are covered by the previously results, for example, for the flows with the speed functions \( \Phi(x) = \beta_1 \sinh^{\beta_2}(x) + \beta_3 \) the understanding of behaviors is far less developed.

(ii) Furthermore, observe that the main result of Theorem 1.2 does not require any homogeneity condition, as in [1, 2], et al. Nevertheless, our results are a significant extension of those in [20] in this direction.

(iii) As mentioned in [22] Smoczyk, one can easily check that the function \( \Phi(x) = \ln x \) satisfies almost all conditions of assumptions 1.1, only except in the latest condition such that \( \Phi \Phi''x + \Phi \Phi' - (\Phi')^2 x < 0 \).

We use mainly the methods used in [20] to prove the above theorem, but with technical tricks for choosing the right functions \( \Phi \) to get estimates, which take control of the complications due to the presence of the arbitrary function \( \Phi \). The organization of the paper is as follows: Section 2 introduces the notation for the paper and summarize preliminary results employed in the rest of the paper. Section 3 contains details of short-time existence and uniqueness of solutions and the evolution equations of some geometric quantities, this requires only minor modifications of the power mean curvature flow case due to the more general \( \Phi(H) \)-flow case. Section 4 shows the lower and above bounds on the maximal time, and establishes the higher-order regularity which give rise to the long time existence for solutions of the flow (1.1). Using these, section 5 deduces that solutions of the flow (1.1) remain convex as long as it exists and proves that these hypersurfaces shrink down to a single point in \( \mathbb{R}^{n+1} \) as the final time is approached.

2. Notation and preliminary results

From now on, use the same notation as in [2, 11, 20] in local coordinates \( \{x^i\} \), \( 1 \leq i \leq n \), near \( p \in M^n \) and \( \{y^\alpha\} \), \( 0 \leq \alpha, \beta \leq n \), near \( F(p) \in \mathbb{R}^{n+1} \). Denote by a bar all quantities on \( \mathbb{R}^{n+1} \), for example by \( \bar{g} = \{\bar{g}_{\alpha\beta}\} \) the metric, by \( \bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\} \) the inverse of the metric, by \( \bar{y} = \{\bar{y}^\alpha\} \) coordinates, by \( \bar{\nabla} \) the covariant derivative, by \( \bar{\Delta} \) the rough Laplacian, and by \( \bar{R} = \{\bar{R}_{\alpha\beta\gamma\delta}\} \) the Riemann curvature tensor. Components are sometimes taken with respect to the tangent vector fields \( \partial_\alpha (= \frac{\partial}{\partial x^\alpha}) \) associated with a local coordinate \( \{y^\alpha\} \) and sometimes with respect to a moving orthonormal frame \( e_\alpha \), where \( \bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta} \). The corresponding geometric quantities on \( M^n \) will be denoted.
by $g$ the induced metric, by $g^{-1}, \nabla, \Delta, R, \partial_i$ and $e_i$. Then further important quantities are the second fundamental form $A(p) = \{h_{ij}\}$ and the Weingarten map $\mathcal{W} = \{g^{ik}h_{kj}\} = \{h_i^j\}$ as a symmetric operator and a self-adjoint operator respectively. The eigenvalues $\lambda_1(p) \leq \cdots \leq \lambda_n(p)$ of $\mathcal{W}$ are called the principal curvatures of $X(M^n)$ at $X(p)$. The mean curvature is given by

$$H := \text{tr}_g \mathcal{W} = h_i^i = \sum_{i=1}^n \lambda_i,$$

the total curvature by

$$|A|^2 := \text{tr}_g (\mathcal{W}^2) = h_i^j h_i^j = \sum_{i=1}^n \lambda_i^2,$$

and Gauß-Kronecker curvature by

$$K := \det(\mathcal{W}) = \det\{h_{ij}\} = \prod_{i=1}^n \lambda_i.$$

More generally, the mixed mean curvatures $E_{r}, 1 \leq r \leq n$, are given by the elementary symmetric functions of the $\lambda_i$

$$E_r(\lambda) = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r} = \frac{1}{r!} \sum_{i_1, \ldots, i_r} \lambda_{i_1} \cdots \lambda_{i_r}, \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n,$$

and their quotients are

$$Q_r(\lambda) = \frac{E_r(\lambda)}{E_{r-1}(\lambda)}, \quad \text{for } \lambda \in \Gamma_{r-1},$$

where $E_0 \equiv 1$, and $E_i \equiv 0$, if $r > n$, $\Gamma_r := \{ \lambda \in \mathbb{R}^n | E_i > 0, i = 1, \ldots, r \}$. Denote the sum of all terms in $E_r(\lambda)$ not containing the factor $\lambda_i$ by $E_{r,i}(\lambda)$. Then the following identities for $E_r$ and the properties on the quotients $Q_r$ were proved by Huisken and Sinestrari in [14].

**Lemma 2.1.** For any $r \in \{1, \ldots, n\}$, $i \in \{1, \ldots, n\}$, and $\lambda \in \mathbb{R}^n$,

$$\frac{\partial E_{r+1}}{\partial \lambda_i}(\lambda) = E_{r,i}(\lambda),$$

$$E_{r+1}(\lambda) = E_{r+1,i}(\lambda) + \lambda_i E_{r,i}(\lambda),$$

$$\sum_{i=1}^n E_{r,i}(\lambda) = (n-r) E_r(\lambda),$$

$$\sum_{i=1}^n \lambda_i E_{r,i}(\lambda) = (r+1) E_{r+1}(\lambda),$$

$$\sum_{i=1}^n \lambda_i^2 E_{r,i}(\lambda) = E_1(\lambda) E_{r+1}(\lambda) - (r+2) E_{r+2}(\lambda).$$
Lemma 2.2. i) $Q_{r+1}$ is concave on $\Gamma_r$ for $r \in \{0, \ldots, n-1\}$,  
ii) $\frac{dQ_r}{d\lambda}(\lambda) > 0$ on $\Gamma_r$ for $i \in \{1, \ldots, n-1\}$ and $r \in \{2, \ldots, n-1\}$.

In graphical coordinates, one can adopt a local graph representation for a convex hypersurface given by a height function $u$. For future reference, it is useful to recall here some basic formulae in a graphical representation. First we see that 

$$X(p, t) = (x(p, t), u(x(p, t), t)).$$

So the metric and its inverse are given by

$$g_{ij} = \delta_{ij} + D_i u D_j u, \quad g^{ij} = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2},$$

where $D_i$ denote the derivatives with respect to these local coordinates, respectively. The outward unit normal vector of $M_t$ can be expressed as 

$$(2.1) \quad \nu = \frac{1}{|\xi|} (Du, 1)$$

with

$$(2.2) \quad |\xi| = \sqrt{1 + |Du|^2}.$$ 

The second fundamental form can be expressed as

$$h_{ij} = \frac{D_{ij} u}{(1 + |Du|^2)^{1/2}}$$

and

$$h^i_j = \left( \delta^{ik} - \frac{D^i u D^k u}{1 + |Du|^2} \right) \frac{D_{kj} u}{(1 + |Du|^2)^{1/2}}.$$ 

Then we obtain

$$(2.3) \quad H = g^{ij} h_{ij} = \left( \delta^{ij} - \frac{u^i u^j}{1 + |Du|^2} \right) \frac{u_{ij}}{\sqrt{1 + |Du|^2}}$$

In addition, the Christoffel symbols have the expression:

$$(2.4) \quad \Gamma^k_{ij} = \left( \delta^{kl} - \frac{D^k u D^l u}{1 + |Du|^2} \right) D_{ij} u D_l u.$$ 

3. Short time existence and evolution equations

This section first consider short time existence for the initial value problem (1.1). In order to obtain these results, it suffices to demand that $\Phi'$ is strictly positive.

Theorem 3.1. Assume that $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion and that the smooth function $\Phi : [0, +\infty) \to \mathbb{R}$ is strictly monotone increasing. Then there exists a unique smooth solution $X_t$ of problem (1.1), defined on some time interval $[0, T)$, with $T > 0$. 
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Proof. In fact, if \( f \) is any symmetric function of the curvatures \( \lambda_i, i \in \{1, \ldots, n\} \), it is well known (see e.g. Theorem 3.1 of [15]) that a flow of the form

\[
\frac{\partial}{\partial t} X(p, t) = -f(p, t) \cdot \nu(p, t)
\]

is parabolic on a given hypersurface with the condition \( \frac{\partial f}{\partial \lambda_i} > 0 \) for all \( i \) holds everywhere. Then, given any initial immersion \( X_0 \) satisfying the parabolicity assumption, standard techniques ensure the local existence and uniqueness of a solution to (1.1) with initial value \( X_0 \). In our case \( f = \Phi(H) \) and the condition reads

\[
\frac{\partial \Phi(H)}{\partial \lambda_i} = \Phi' \frac{\partial H}{\partial \lambda_i} = \Phi' > 0,
\]

which is satisfied the condition of Theorem 3.1 of [15]. □

By a direct calculation as in [11], or [1], the following evolution equations of geometric quantities under the flow (1.1) can be easily obtained.

**Theorem 3.2.** On any solution \( M_t \) of (1.1) the following hold:

\[
\begin{align*}
\partial_t g_{ij} &= -2\Phi(H) h_{ij}, \\
\partial_t \nu &= \nabla \Phi(H), \\
\partial_t (d\mu_t) &= -H \Phi d\mu_t,
\end{align*}
\]

\[
\begin{align*}
\partial_t h_{ij} &= \Delta_{g} h_{ij} + \Phi'' \nabla_i H \nabla_j H - (\Phi' H + \Phi) h_{ij}^k h_{kj} + |A|^2 \Phi' h_{ij}, \\
\partial_t h_{i}^j &= \Delta_{g} h_{i}^j + \Phi'' \nabla_i H \nabla_j H - (\Phi' H - \Phi) h_{i}^k h_{kj}, \\
\partial_t H &= \Delta_{g} H + \Phi'' |\nabla H|^2 + |A|^2 \Phi, \\
\partial_t \langle X, \nu \rangle &= \Delta_{g} \langle X, \nu \rangle + |A|^2 \Phi' \langle X, \nu \rangle - (\Phi' H + \Phi).
\end{align*}
\]

Furthermore, the quotients \( Q_r(\lambda) \) satisfy the following evolution equation which is an extension of [20, Lemma 2.4] to hypersurfaces of (1.1) in \( \mathbb{R}^{n+1} \):

**Lemma 3.3.** Suppose \( \Phi \) satisfies that \( \Phi'' \geq 0 \) and \( \Phi' H \geq 0 \). Let \( X : M^n \times [0, T) \to \mathbb{R}^{n+1} \) be a \( \Phi(H) \)-flow with

\[
E_{r-1}(p, t) > 0, \quad E_r(p, t) \geq 0 \quad \text{for all } (p, t) \in M^n \times [0, T).
\]

Then

\[
\partial_t Q_r \geq \Phi' \Delta Q_r + \left[ \Phi' |A|^2 - r(\Phi' H - \Phi) Q_r \right] Q_r.
\]

**Proof.** As in [20], from the evolving equation (3.3) of \( h_i^j \), using

\[
\partial_t Q_r = \frac{\partial Q_r}{\partial h_i^j} \left( \partial_t h_i^j \right) \quad \text{and} \quad \Delta Q_r = \frac{\partial Q_r}{\partial h_i^j} \Delta h_i^j + \frac{\partial^2 Q_r}{\partial h_i^j \partial h_p^k} \nabla^k h_i^j \nabla^k h_p^j
\]
it is easy to calculate the derivative of $Q_r$:

$$\partial_t Q_r = \Phi' \Delta Q_r - \Phi' \frac{\partial^2 Q_r}{\partial h_i^2} - \Phi'' \frac{\partial Q_r}{\partial h_i} \nabla_i H \nabla_j H$$

Choosing a frame $\{e_i\}$ which diagonalises $W$, the fifth and sixth term appearing here can be simplified using the following simple calculation with the aid of Lemma 2.1:

$$\frac{\partial Q_r}{\partial h_i} = \sum_{i=1}^n \frac{\partial Q_r}{\partial \lambda_i} \lambda_i^2 = \frac{1}{E_{r-1}} \left( E_{r-1} \sum_{i=1}^n E_{r-1,i} \lambda_i^2 - E_r \sum_{i=1}^n E_{r-2,i} \lambda_i^2 \right)$$

and

$$\frac{\partial Q_r}{\partial h_i} = \sum_{i=1}^n \frac{\partial Q_r}{\partial \lambda_i} \lambda_i = \frac{1}{E_{r-1}} \left( E_{r-1} \sum_{i=1}^n E_{r-1,i} \lambda_i - E_r \sum_{i=1}^n E_{r-2,i} \lambda_i \right) = Q_r.$$
and
\[ \nabla_k b^i_j = -b^p_i \left( \nabla_k h^q_p \right) b^q_j \]
which implies
\[ \Delta b^i_j = -b^p_i \left( \Delta h^q_p \right) b^q_j + 2 \nabla^k b^i_l \nabla_k b^l_j. \]
Together with equation (3.9), this gives the equality.

**Case 1.** For \( \Phi'' > 0 \), the inequality follows immediately.

**Case 2.** For \(- \frac{2b^i_i}{H} \leq \Phi'' < 0 \), the two gradient terms on the right hand side of the equality in Lemma 3.4 have the desired sign, we have to work a bit more. Note that
\[ -\Phi'' \left( b^p_i \nabla_p H \right) \left( \nabla^q H b^q_j \right) = - \frac{\partial^2 \Phi}{\partial h^m_i \partial h^q_p} \left( b^p_i \nabla_p h^m_i \nabla_q h^q_j \right) b^q_j. \]

As in [20, Lemma 2.5], note that \( H(\lambda) = Q_n(\theta) \), where the \( \theta_i \) are the principle radii, i.e., \( \theta_i = \frac{1}{\nabla_i} \). For general functions \( f, g \) satisfying \( f(h^i_j) = 1/g(b^i_j) \) one can compute that
\[ \frac{\partial f}{\partial h^m_i \partial h^q_p} = \frac{\partial^2 \Phi}{\partial h^m_i \partial h^q_p} b^p_i \nabla_p h^m_i \nabla_q h^q_j \]
\[ = \frac{\partial^2 \Phi}{\partial h^m_i \partial h^q_p} \left( b^p_i \nabla_p h^m_i \nabla_q h^q_j \right) b^q_j. \]

By the chain rule
\[ \frac{\partial \Phi}{\partial h^m_i} (\lambda) = \Phi' \frac{\partial H}{\partial h^m_i} = \Phi' \delta^m_i, \]
and
\[ \frac{\partial^2 \Phi}{\partial h^m_i \partial h^q_p} = \Phi'' \delta^m_i \delta^q_p + \Phi' \frac{\partial^2 H}{\partial h^m_i \partial h^q_p}. \]

From (3.9) (with \( f = H \)) and (3.10), it follows
\[ -\frac{\partial^2 H}{\partial h^m_i \partial h^q_p} b^p_i \nabla_p h^m_i \nabla_q h^q_j b^q_j = - \frac{2}{H} b^p_i \nabla_p H \nabla^q H b^q_j + H^2 \frac{\partial^2 Q_n}{\partial h^m_i \partial h^q_p} b^p_i \nabla_p h^m_i \nabla_q h^q_j b^q_j \]
\[ + 2 \nabla^k b^i_l \nabla_k h^q_j b^q_j, \]
where the Codazzi equation has been used. Now by identities (3.11), (3.8) and (3.10) one can write (3.7) as
\[ \partial_t b^i_j = \Delta^i_j b^i_j - 2\Phi' \nabla_k b^i_k h^q_j \nabla^q h^q_j - \Phi'' \left( b^q_i \nabla_p H \right) \left( \nabla^q H b^q_j \right) - \frac{2\Phi'}{H} b^i_l \nabla_p H \nabla^q H b^q_j \]
\[ + \Phi' \frac{\partial^2 Q_n}{\partial h^m_i \partial h^q_p} b^p_i \nabla_p H \nabla^q H b^q_j + 2 \Phi' \nabla_k b^i_k h^q_j \nabla^q h^q_j + (\Phi' H - \Phi) \delta^i_j - \Phi' \left| A \right|^2 b^i_j \]
\[ = \Delta^i_j b^i_j - \left( \Phi'' + \frac{2\Phi'}{H} \right) b^p_i \nabla_p H \nabla^q H b^q_j + \Phi' H^2 \frac{\partial^2 Q_n}{\partial h^m_i \partial h^q_p} b^p_i \nabla_p h^m_i \nabla_q h^q_j b^q_j \]
\[ + (\Phi' H - \Phi) \delta^i_j - \Phi' \left| A \right|^2 b^i_j. \]
Using the concavity of \( Q_n(\theta) \) and the assumption \(-\frac{2\Phi'}{\pi} \leq \Phi'' \leq 0\), it follows that
\[
\partial_t b^j_i \leq \Delta \phi b^j_i + (\Phi'H - \Phi)\delta^j_i - \Phi|A|^2 b^j_i.
\]

4. The long time existence

The third section has shown that the equation (1.1) has a (unique) smooth solution on a short time if the initial hypersurface in \( \mathbb{R}^{n+1} \) is convex. This section considers the long time behavior of (1.1) and establishes the existence of a solution on a finite maximal interval.

As a first step the maximum principle applied to the evolution equation of \( H \) guarantee that the minimum \( H_{\text{min}} \) of \( H \) is increasing under the flow (1.1) which ensures the uniform parabolicity of our equation.

**Proposition 4.1.** Under the assumptions of Main Theorem 1.2,
\[
H_{\text{min}}(t) \geq \frac{1}{G^{-1}\left(G\left(\frac{1}{H_{\text{min}}(0)}\right) - \frac{2}{n}\right)}
\]

which gives an upper bound on the maximal existence time \( T \): \[
T \leq nG\left(\frac{1}{H_{\text{min}}(0)}\right).
\]

**Proof.** A direct calculation using \(|A|^2 \geq \frac{1}{n}H^2\) and the evolution equation (3.4) of \( H \) gives
\[
\partial_t H_{\text{min}} \geq \frac{1}{n}\Phi(H_{\text{min}})H_{\text{min}}^2.
\]

Now let \( \phi \) be the solution of the ODE
\[
\begin{cases}
\frac{d\phi}{dt} = \frac{1}{n}\Phi(\phi)\phi^2,
\phi(0) = H_{\text{min}}(0),
\end{cases}
\]

then by the maximum principle
\[
H \geq \phi \quad \text{on} \quad 0 \leq t \leq T.
\]

On the other hand \( \phi \) is explicitly given by
\[
\phi(t) = \frac{1}{G^{-1}\left(G\left(\frac{1}{H_{\text{min}}(0)}\right) - \frac{2}{n}\right)},
\]

which implies
\[
H_{\text{min}}(t) \geq \frac{1}{G^{-1}\left(G\left(\frac{1}{H_{\text{min}}(0)}\right) - \frac{2}{n}\right)}.
\]
Thus,

\[ H_{\text{min}}(t) \to \infty \quad \text{as} \quad G\left(\frac{1}{H_{\text{min}}(0)}\right) - \frac{t}{n} \to 0^+, \]

which proves Proposition 4.1. \(\square\)

**Theorem 4.2.** Let \([0, T)\) be the maximal existence interval of the flow \((1.1)\) \(M_t\) with \(\Phi' > 0\) on \([\delta_0, +\infty)\). Then

\[ T \leq nG\left(\frac{1}{H_{\text{min}}(0)}\right). \]

Moreover, \(\max_{M_t} |A|^2 \to +\infty\) as \(t \to T\).

**Proof.** The estimates on the maximal time \(T\) of existence can be easily derived from Proposition 4.1. To complete the proof of the theorem, assume that \(|A|^2\) remains bounded on the interval \([0, T)\), and derive a contradiction. Then the evolution equation \((1.1)\) implies that

\[ |X(p, \sigma) - X(p, \tau)| \leq \int_{\sigma}^{\tau} \Phi(H)(p, t) \, dt \]

for \(0 \leq \tau \leq \sigma < T\). Since \(H\) is bounded from the bound for \(|A|^2\) and the \(\Phi\) is a smooth increasing function, \(X(\cdot, t)\) tends to a unique continuous limit \(X(\cdot, T)\) as \(t \to T\). For any \(t \in [0, T)\), this implies the uniform \(C^2\)-estimates for these hypersurfaces. In order to conclude that \(X(\cdot, T)\) represents a hypersurface \(M_T\), next under this assumption and in view of the evolution equation \((3.1)\) the induced metric \(g\) remains comparable to a fixed smooth metric \(\tilde{g}\) on \(M^n\):

\[ \left| \frac{\partial}{\partial t} \left( \frac{g(u, u)}{\tilde{g}(u, u)} \right) \right| = \left| \frac{\partial g(u, u)}{\partial t} \frac{g(u, u)}{\tilde{g}(u, u)} \right| \leq 2|\Phi(H)||A|\frac{g(u, u)}{\tilde{g}(u, u)}, \]

for any non-zero vector \(u \in TM^n\), so that ratio of lengths is controlled above and below by exponential functions of time, and hence since the time interval is bounded, there exists a positive constant \(C\) such that

\[ \frac{1}{C}\tilde{g} \leq g \leq C\tilde{g}. \]

Then the metrics \(g(t)\) for all different times are equivalent, and they converge as \(t \to T\) uniformly to a positive definite metric tensor \(g(T)\) which is continuous and also equivalent by following Hamilton's ideas in [3].

For \(\alpha > 0\) the uniform \(C^{2,\alpha}\)-estimates can be obtained for these hypersurfaces as follows: For \(-\frac{28c'}{n} \leq \Phi'' < 0\), the speed \(\Phi(H)\) is concave in \(H\) and in this case with the uniform \(C^{2,\alpha}\)-bounds are known in general for operators with concave (see [18], Theorem 2, Chapter 5.5, or also see [16]). For \(\Phi'' \geq 0\), \(M_t\) can be locally reparameterized as graphs given by a height function \(u\). From \((1.1)\) and \((2.1)\), a short computation yields that height function \(u\) satisfies the following parabolic PDE

\[ \partial_t u = \Phi(H)|\xi|, \]
where the mean curvature $H$ and the outward normal vector length $|\xi|$ are given by the expressions (2.3) and (2.2), respectively. The function $\Phi(H)$ in the coordinate system under consideration is a function of $D^2u$ and $Du$. Since $H(\cdot,t)$ is larger than $H_{\min}(0)$ and bounded above by our assumption on $|A|^2$, this implies that $\frac{\Phi(H)}{H}$ is also uniformly H"older continuous functions in space and time. Using this, we can write equation (4.1) as a linear, strictly parabolic partial differential equation

(4.2) \[ \partial_t u = a_{ij} D_i D_j u, \]

with coefficients given by $a_{ij} = g^{ij} \frac{\Phi(H)}{H}$, in $C^\alpha$ in space and time. The interior Schauder estimates by the general theory of Krylov and Safonov \[16\], \[18\] lead to $C^{2,\alpha}$-estimates. In both cases, i.e. $-\Phi'' \leq \Phi'' < 0$, and $\Phi'' > 0$, such a property implies all the higher order estimates by using standard linearization and bootstrap techniques (see \[16\], \[18\]). It is enough to imply bounds on all derivatives of $X$. Therefore the hypersurfaces $M_t$ converge to a smooth limit hypersurface $M_T$. Finally, applying the local existence result with initial data $X(\cdot,t)$, the solution can be continued to a later times, contradicting the maximality of $T$. This completes the proof of Theorem 5.1. □

**Example 4.3.** For the evolution of a sphere $S_0$ with a radius $R_0$ and the origin point of $\mathbb{R}^{n+1}$ its center under the flow (1.1). Since in the sphere case our flow preserves the symmetry, the equation (1.1) reduces to the following ODE for the radius of the spheres

\[
\begin{cases}
\frac{dR(t)}{dt} = -\Phi \left( \frac{n}{R(t)} \right), \\
R(0) = R_0.
\end{cases}
\]

A straightforward analysis for the existence of solution of the above ODE implies that the evolving spheres $S_t$ with radii $R(t)$ contract to the center of the $S_0$ satisfying

\[ R(t) = nG^{-1} \left( G \left( \frac{R(0)}{n} \right) - \frac{t}{n} \right), \]

on a finite maximal existence time $[0,T)$, where $T$ is given by

\[ T = n \left( G \left( \frac{R(0)}{n} \right) \right). \]

5. Preserving convexity

With the notations of Theorem 1.2, this section shall show that convex hypersurface remains so under the $\Phi(H)$-flow.
To show that convexity of $M_t$ is preserved, next consider the evolution of $\lambda_{\min} := \min_{M_t} \lambda_i$ as in Chap. 3 of [10]. In order to do so, define a smooth approximation $A$ to $\max(x_1, \ldots, x_n)$ as follows: for $\delta > 0$ let

$$A_2(x_1, x_2) = \frac{x_1 + x_2}{2} + \sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \delta^2},$$

$$A_{n+1}(x_1, \ldots, x_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} A_2(x_i, A_n(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), \ n \geq 2.$$

The approximation has the following properties, for a proof see ([10], Lemma 3.3).

**Lemma 5.1.** For $n \geq 2$ and $\delta > 0$,

i) $A_n(x_1, \ldots, x_n)$ is smooth, monotonically increasing and convex,

ii) $\max\{x_1, \ldots, x_n\} \leq A_n(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\} + (n-1)\delta$,

iii) $\frac{\partial A_n(x_1, \ldots, x_n)}{\partial x_i} \leq 1$,

iv) $A_n(x_1, \ldots, x_n) - (n-1)\delta \leq \sum_{i=1}^{n} \frac{\partial A_n(x_1, \ldots, x_n)}{\partial x_i} x_i \leq A_n(x_1, \ldots, x_n),$

v) $\sum_{i=1}^{n} \frac{\partial A_n(x_1, \ldots, x_n)}{\partial x_i} = 1$.

Schulze in [20] proved that the minimal principal curvatures of the hypersurfaces under the $H^2$-flow is increasing by applying the properties of $A_0$, which is also valid for the $\Phi(H)$-flow.

**Lemma 5.2.** For $\Phi' > 0$, $\Phi'' \geq -\frac{2\Phi'}{H}$ let $M_t$ be a solution of the $\Phi(H)$-flow (1.1). Suppose the initial hypersurface $M_0$ is strictly convex. Then all $M_t$ are also strictly convex and $\lambda_{\min}(t)$ is monotonically increasing for $t > 0$.

**Proof.** Firstly, note that Proposition 4.1 ensures that Proposition 4.1 ensures that $H$ preserved positivity in time.

**Case 1.** For $\Phi'' \geq 0$, using a frame which diagonalises $W$, consider the evolution of $\lambda_{\min}(t)$ in the evolution equation (3.3) of $W$. Then

$$\partial_t \lambda_{\min}(p, t) \geq \Phi' \Delta \lambda_{\min}(p, t) - (\Phi' H - \Phi) \lambda^2_{\min}(p, t) + |A|^2 \Phi' \lambda_{\min}(p, t)$$

$$= \Phi' \Delta \lambda_{\min}(p, t) + \Phi \lambda^2_{\min}(p, t) + \Phi' \left[|A|^2 (\lambda_{\min}(p, t)) - H \lambda^2_{\min}(p, t)\right].$$

The part in the square brackets is nonnegative by the estimate $|A|^2 \geq H \lambda_{\min}$. Then the maximum principle shows the desired result.

**Case 2.** For $-\frac{2\Phi'}{H} \leq \Phi'' < 0$, observe that the gradient term has the wrong sign, we have to work a little bit more as in [20]. For a fixed $\delta > 0$ now choose a smooth approximation $\lambda_i' := A_n(\theta_1, \ldots, \theta_n)$, to $\max(\theta_1, \ldots, \theta_n)$, as defined in (5.1), where the $\theta_i$ are the eigenvalues of $b_i'$, i.e. $\theta_i = 1/\lambda_i$. By the chain rule

$$\partial_t \lambda_{\min} = \frac{\partial \lambda_{\min}}{\partial t}$$

and

$$\Delta \lambda_{\min} = \frac{\partial^2 \lambda_{\min}}{\partial t^2} + \frac{\partial^2 \lambda_{\min}}{\partial t \partial (\theta_i)} \nabla^\top b_i' \nabla b_i'.$$
grouping the two identities and applying Lemma principle radii evolution $\mathcal{A}$ satisfies the following evolution inequality:

$$\partial_t \mathcal{A} \leq \Phi' \Delta \mathcal{A} - \Phi' \frac{\partial^2 \mathcal{A}}{\partial b_i^s \partial b_i^t} \nabla v_i^s \nabla v_i^t + (\Phi' H - \Phi) \text{tr} \left( \frac{\partial \mathcal{A}}{\partial b_i^t} \right) - \Phi' |\mathcal{A}|^2 \frac{\partial \mathcal{A}}{\partial b_i^t} b_i^t.$$ 

The various terms on the right hand side of this inequality can be easily estimated: First, in view of Lemma 5.1 (i) convexity of $\mathcal{A}$ implies convexity of $\mathcal{A}$, then the second term can be estimated by

$$-\Phi' \frac{\partial^2 \mathcal{A}}{\partial b_i^s \partial b_i^t} \nabla v_i^s \nabla v_i^t \leq 0.$$ 

Using Lemma 5.1 (iv), the third term can be estimated by $(\Phi' H - \Phi)$.

Lemma 5.1 (iv) implies that the next term can be estimated by

$$-\Phi' |\mathcal{A}|^2 (\mathcal{A} - (n - 1) \delta).$$

The following estimate is obtained:

$$(5.3) \quad \partial_t \mathcal{A} \leq \Phi' \Delta \mathcal{A} + (\Phi' H - \Phi) - \Phi' |\mathcal{A}|^2 (\mathcal{A} - (n - 1) \delta).$$

**Case 2.1.** for $\Phi' H - \Phi \leq 0$, at a point $(p, t)$ with $\mathcal{A} - (n - 1) \delta > 0$, this estimate (5.3) gives the following estimate

$$\partial_t \mathcal{A} \leq \Phi' \Delta \mathcal{A},$$

which gives a contradiction if $\mathcal{A}$ attains a first maximum larger than $(n - 1) \delta$.

The limit as $\delta$ is approached to 0 then implies the conclusion of the Lemma.

**Case 2.2.** for $\Phi' H - \Phi > 0$, at a point $(p, t)$ with $\mathcal{A} - (n - 1) \frac{H \Phi'}{\mathcal{A}} \delta > 0$,

$$\partial_t \mathcal{A} \leq \Phi' \Delta \mathcal{A} + (\Phi' H - \Phi) - \Phi' |\mathcal{A}|^2 (\mathcal{A} - (n - 1) \delta)
\leq \Phi' \Delta \mathcal{A} + \left( \Phi' H - \Phi |\mathcal{A}|^2 \right) + |\mathcal{A}|^2 \frac{\Phi}{H} - \Phi
= \Phi' \Delta \mathcal{A} + \mathcal{A} \left( \Phi' - \frac{\Phi}{H} \right) \left( \frac{H}{\mathcal{A}} - |\mathcal{A}|^2 \right),$$

since

$$\Phi' H - \Phi > 0$$

and

$$\frac{H}{\mathcal{A}} \leq \frac{H}{\theta_{max}} = H \lambda_{min} \leq |\mathcal{A}|^2,$$

this gives

$$\partial_t \mathcal{A} \leq \Phi' \Delta \mathcal{A},$$

which gives a contradiction if $\mathcal{A}$ attains a first maximum larger than $(n - 1) \frac{H \Phi'}{\mathcal{A}} \delta$. The limit as $\delta$ is approached to 0 then implies the conclusion of the Lemma. $\square$
Corollary 5.3. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a $\Phi(H)$-flow of strictly convex hypersurfaces. Then

$$|A|(p,t) \leq H(p,t) \leq G^{-1}\left( G\left(\frac{1}{H_{\max}(0)}\right) - t \right).$$

Proof. Lemma 5.2 implies that if $M_0$ is strictly convex, under the flow (1.1), $M_t$ is strictly convex as long as it exists, then $|A| \leq H$, which implies that from the evolution equation (3.4) of $H$

$$\partial_t H_{\max} \leq \Phi H_{\max}^2.$$

Now let $\phi$ be the solution of the ODE

$$\begin{cases}
\frac{d\phi}{dt} = \Phi(\phi)\phi^2, \\
\phi(0) = H_{\max}(0),
\end{cases}$$

then by the maximum principle

$$H \leq \phi \quad \text{on} \quad 0 \leq t \leq T.$$

On the other hand $\phi$ is explicitly given by

$$\phi(t) = G^{-1}\left( G\left(\frac{1}{H_{\max}(0)}\right) - t \right).$$

Thus, this gives the desired estimate. $\square$

Corollary 5.4. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a $\Phi(H)$-flow of weakly convex hypersurfaces. Then $M_t$ is weakly convex for all $t \in [0,T)$ and $T_{\max} \geq G\left(\frac{1}{H_{\max}(0)}\right)$.

Proof. The initial surface $M_0$ can be smoothly approximated by strictly convex hypersurfaces $M_{i0}$, for example choosing the mean curvature flow. Let these hypersurfaces move by $\Phi(H)$-flow, which by Lemma 5.2 remain strictly convex. By Theorem 4.2 and Corollary 5.3 we have a uniform lower bound $T_{\max} \geq G\left(\frac{1}{H_{\max}(0)}\right)$. Using the uniform $C^{2,\alpha}$-estimates from the proof of Theorem 4.2 one can extract a convergent subsequence of strictly convex flows which implies the original flow also had to be convex. $\square$

In the case that $\Phi'' > 0$, $\Phi' \geq \frac{\Phi}{H_{\max}} \geq 0$, the following Proposition shows that weakly convex hypersurfaces immediately become strictly convex along the $\Phi(H)$-flow in $\mathbb{R}^{n+1}$ by using Lemma 3.3.

Proposition 5.5. For $\Phi'' > 0$, $\Phi' \geq \frac{\Phi}{H_{\max}} \geq 0$, let $M_t$ be a solution of the $\Phi(H)$-flow in $\mathbb{R}^{n+1}$. Suppose the initial hypersurface $M_0$ is a weakly convex hypersurface with $H_{\min}(0) > 0$. Then $M_t$ is strictly convex for all $t \in [0,T)$.
Proof. Since $H(t) \geq H_{\text{min}}(0) > 0$ for all all $[0, T)$ along the $\Phi(H)$-flow, $Q_2$ is well-defined and Corollary 5.3 implies that $M_t$ is weakly convex. Then an immediate consequence is

$$Q_2 = \frac{|H|^2 - |A|^2}{2H} \geq 0.$$  

For $t \in [0, \varepsilon], \varepsilon < T$, the bounds on $|A|^2$ implies the bounds on $Q_2$ and $\Phi'$ which implies

$$\left[\Phi'|A|^2 - r(\Phi'H - \Phi)Q_2\right]Q_2 \leq C$$

on this interval. An application of Lemma 3.4 for $\omega := e^{Ct}Q_2$ shows the following estimate:

$$\partial_t \omega \geq \Phi' \Delta \omega.$$  

Suppose that there exists $(p_0, t_0) \in M^n \times (0, \varepsilon)$ with $Q_2(p_0, t_0) = 0$, then also $\omega(p_0, t_0) = 0$. The Harnack’s inequality in the parabolic case (see i.e. [18]) applied to the above equation shows that $\omega \equiv 0$ for all $t \in (0, t_0)$, i.e. $Q_2 \equiv 0$, which is in contradiction to the existence of strictly convex points on $M_t$, and so $Q_2 > 0$ on $M^n \times (0, T)$. An iterative application of this yields that $Q_r > 0$ on $M^n \times (0, T)$. This concludes the Proposition. □

The following step want to show that the flow exists as long as it bounds a non-vanishing volume. In order to achieve this, using a trick of Tso [23] for the Gauß curvature flow, see also [1], [4] and [19], study the evolution under (1.1) of the following function

$$Z_t = \frac{\Phi(H)}{(X, \nu)} - \epsilon.$$  

Here $\epsilon$ is a constant to be chosen later.

**Corollary 5.6.** For $t \in [0, T)$ and any constant $\epsilon$,

$$\partial_t Z = \frac{2\Phi'}{(X, \nu)} + \frac{2\Phi'}{(X, \nu) - \epsilon} (\nabla Z, \nabla \Phi) + Z^2 \left[\frac{\Phi'H}{\Phi} + 1 - \epsilon \frac{\Phi'|A|^2}{\Phi}\right].$$

Proof. From (5.4), (5.3) and (5.5), it follows

$$\partial_t Z = \frac{1}{(X, \nu) - \epsilon} \Phi' \Delta \Phi(H) + \Phi' \Phi \left(|A|^2\right)$$

(5.5)

$$= \frac{\Phi'(H)}{(X, \nu) - \epsilon} \left(\Delta \Phi(H) + |A|^2 \Phi(H) - (\Phi' H + \Phi)\right).$$

Another computation leads to

$$\Phi' \Delta = \frac{\Phi' \Delta \Phi(H)}{(X, \nu) - \epsilon} - \frac{\Phi'|A|^2}{(X, \nu) - \epsilon} - 2 \frac{\Phi'}{(X, \nu) - \epsilon} (\nabla Z, \nabla \Phi).$$

(5.6)

Using (5.6), we can simplify (5.5) as the desired evolution equation for the function $Z$ easily. □
Now we apply the maximum principle to get an upper bound for \( Z \) as long as the evolving hypersurface bounds a non-vanishing volume.

**Theorem 5.7.** Let \( M_t \) be a solution of the \( \Phi(H) \)-flow in \( \mathbb{R}^{n+1} \), where the speed \( \Phi(H) \) satisfies the conditions \( \mathbb{I} \). Suppose the initial hypersurface \( M_0 \) is a convex hypersurface, \( \delta > 0 \), \( q_0 \in \mathbb{R}^{n+1} \) and \( B_\delta(q_0) \subset \Omega_t \) for all \( t \in [0, \tau) \), which boundary is \( M_\ast \). Then

\[
H(p, t) \leq C(M_0, \delta, n) \quad \text{for all} \quad (p, t) \in M^n \times [0, \tau).
\]

**Proof.** Without loss of generality, we take the point \( q_0 \) as the origin of \( \mathbb{R}^{n+1} \) such that \( X \) is the position vector field. Since it is proved previously that \( M_t \) is convex along the flow \( \mathbb{I} \), there exists a constant \( \epsilon > 0 \) in the definition \( 5.4 \) of \( Z \), \( \epsilon = \epsilon(\delta) \), such that the support function \( \langle X, \nu \rangle \) satisfies

\[
\langle X, \nu \rangle \geq 2\epsilon \implies \langle X, \nu \rangle - \epsilon \geq \epsilon > 0.
\]

Combining this, convexity of \( M_t \) implies that \( Z \geq 0 \) and

\[
\left| A \right|^2 \geq \frac{1}{n}H^2.
\]

From Corollary \( \mathbb{I} \), the following inequality can be obtained:

\[
\begin{align*}
\partial_t Z &\leq \Phi' \Delta Z + \frac{2\Phi'}{\langle X, \nu \rangle} \langle \nabla Z, \nabla \Phi \rangle + Z^2 \left[ \left( \frac{\Phi'H}{\Phi} + 1 \right) - \frac{\Phi'H^2}{n\Phi} \right] \\
&= \Phi' \Delta Z + \frac{2\Phi'}{\langle X, \nu \rangle} \langle \nabla Z, \nabla \Phi \rangle + Z^2 \frac{\Phi'H}{n\Phi} \left[ \left( \frac{\Phi}{H} + 1 \right) - \frac{H}{n} \right].
\end{align*}
\]

Notice that

\[
\left( \frac{\Phi}{H} + 1 \right)' = -\frac{\Phi'\Phi'' + \Phi'H - (\Phi')^2 H}{(H\Phi')^2}.
\]

From the assumptions \( \mathbb{I} \) it follows that

\[
\left( \frac{\Phi}{H} + 1 \right)' \leq 0,
\]

which implies the following estimate

\[
\left( \frac{\Phi}{H} + 1 \right) \leq \left( \frac{\Phi(H_{\min}(0))}{H_{\min}(0)\Phi'(H_{\min}(0))} + 1 \right) := D.
\]

Therefore, in view of the assumptions \( \mathbb{I} \) and the above estimate we bound the \( \partial_t Z \) as follows

\[
\partial_t Z \leq \Phi' \Delta Z + \frac{2\Phi'}{\langle X, \nu \rangle} \langle \nabla Z, \nabla \Phi \rangle + Z^2 \frac{\Phi'H}{n\Phi} \left( D - \frac{H}{n} \right).
\]

Assume that in \( (p_0, t_0) \), \( Z \) attains a big maximum \( C \gg 0 \) for the first time. Then

\[
\Phi(H)(p_0, t_0) \geq C(\langle X, \nu \rangle - \epsilon)(p_0, t_0) \geq \epsilon C,
\]
which gives a contradiction if
\[ C \geq \max_{p \in M^n} \left\{ Z(p, 0), \frac{1}{\epsilon} \Phi \left( \frac{nD}{\epsilon} \right) \right\}. \]

□

**Proof of Theorem 1.2.** Theorem 4.2 and Theorem 5.7 ensure that the 191
\( \Phi(H) \)-flow exists as long as it bounds a non-vanishing domain. Lemma 4.1 and Proposition 5.5 show that all hypersurfaces are strictly convex for \( t \in [0, \tau) \), thus \( \lim_{t \to T} \lambda_{mn}(t) = \delta > 0 \). Now by adapting Schulze’s approach in the case 191
\( H^β \)-flow in [20], one can complete the remainder of the proof for convergence 191
to a single point as the final time is approached. □

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