Mediated Entanglement And Correlations In A Star Network Of Interacting Spins

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We investigate analytically a star network of spins, in which all spins interact exclusively with a central spin through Heisenberg XX couplings of equal strength. We find that the central spin correlates and entangles the other spins at zero temperature to a degree that depends on the total number of spins. Surprisingly, the entanglement depends on the evenness or oddness of this number and some correlations are substantial even for an infinite collection of spins. We show how symmetric multi-party states for optimal sharing and splitting of entanglement can be obtained in this system using a magnetic field.

For a long time spin correlations in 1D chains and higher dimensional lattices of interacting spins have been a subject of extensive interest. Recently, the same systems have been studied from the point of view of truly quantum correlations or entanglement. However, lattices of various dimensions are not the only physical systems whose fabrication is possible with current technology. In particular, with the advent of quantum computation, various technologies have evolved which can make any member of an array of qubits (systems isomorphic to spin-1/2) controllably interact with any other member. It thus becomes possible to visualize structures of interacting spins which do not fall into the category of lattices in various dimensions. One very simple structure that one can imagine, is a spin-star, as opposed to the extensively studied spin-chains. In such a spin star, there is a preferred spin, which we call the central spin which interacts with all the other spins. All the non-central spins (which we will call the outer spins), on the other hand, do not directly interact among themselves. The structure is clearly depicted in Fig.1. 0 depicts the central spin. The spins 1 – 5 interact only with the central spin and not with each other. The architecture is analogous to the star distribution networks used in communications. To our knowledge, not just entanglement and correlations, but also the statistical mechanics of such a structure remains unexplored.

The star configuration with couplings of equal strength has many symmetry properties due to its invariance under the exchange of any two outer spins and we solve it exactly in the case of an Heisenberg XX interaction. The XX model was intensively investigated for spin chains by Lieb, Schultz and Mattis, as opposed to the extensively studied spin-chains. In such a spin star, there is a preferred spin, which we call the central spin which interacts with all the other spins. All the non-central spins (which we will call the outer spins), on the other hand, do not directly interact among themselves. The structure is clearly depicted in Fig.1. 0 depicts the central spin. The spins 1 – 5 interact only with the central spin and not with each other. The architecture is analogous to the star distribution networks used in communications. To our knowledge, not just entanglement and correlations, but also the statistical mechanics of such a structure remains unexplored.

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The Hamiltonian which describes our system is given by

$$H = \mathcal{J} \left( \sum_{\text{outer}} \sigma_{ix} + \sum_{\text{outer}} \sigma_{iy} \right)$$

where the summation over 'outer' refers to the outer spins, \(\sigma_{ix}\) and \(\sigma_{iy}\) denote the \(\sigma_x\) and \(\sigma_y\) Pauli operators for the ith outer spin and \(\sigma_{ox}\) and \(\sigma_{oy}\) denote the \(\sigma_x\) and \(\sigma_y\) Pauli operators for the central spin. It can be shown that \(J_x = (1/2) \sum_{\text{outer}} \sigma_{ix}, J_y = (1/2) \sum_{\text{outer}} \sigma_{iy}\) and \(J_z = (1/2) \sum_{\text{ring}} \sigma_{ix}\) obey the standard angular momentum commutation relations (we have taken \(\hbar = 1\)). This implies that the outer spins collectively behave as a single spin with spin operator \(\mathbf{J} = iJ_x + jJ_y + kJ_z\). It can be

FIG. 1. This figure depicts the star configuration of spins. The spin labelled 0 is the central spin, which interacts by the XX Heisenberg interaction with spins 1 - 5 around it.
shown that \( J^2 = J_x^2 + J_y^2 + J_z^2 \) commutes with \( H \). It will also help to define the total angular momentum operator \( F = (1/2)\sigma_0 + J \), where \( \sigma_0 = i\sigma_{0x} + i\sigma_{0y} + k\sigma_{0z} \) and it can be shown that the \( z \)-component, \( F_z \) obeys \([H, F_z] = 0\), \([J_x, F_z] = 0\). Therefore simultaneous eigenstates of \( H \), \( J^2 \) and \( F_z \) can be constructed.

It is convenient to recast the Hamiltonian in Eq.3 using the raising and lowering operators \( \sigma_\pm = (\sigma_x \pm i\sigma_y) \) and \( J_\pm = (1/2)\sum_{\text{outer}} \sigma_\pm \) as

\[
H = J (\sigma_0 + J_- + \sigma_0 - J_+) \quad (2)
\]

The above Hamiltonian thus represents a resonant interaction between a spin-(1/2) and a higher spin (with operator \( J \)) system. Inspired by the above form of \( H \) (in particular, its similarities with the Jaynes-Cummings model of quantum optics \([13]\)), we conjecture the following state as the general form of its eigenstates

\[
\frac{1}{\sqrt{2}} \left( (0) | j, m \rangle \pm |1) | j, m - 1 \rangle \right) \quad (3)
\]

where the first ket in each term denotes the central spin \((|0\rangle \text{ and } |1\rangle \text{ stand for the } |\pm 1/2 \rangle \text{ and } |\mp 1/2 \rangle \text{ spin states of the central spin})\), and the second ket is an eigenstate of \( J^2 \). \( j \) is the quantum number associated with eigenstates of \( J^2 \) (eigenvalue of \( J^2 \) is \( j(j+1) \)), and \( m \) is the quantum number for \( J_z \). The way \(|j, m\rangle\) is paired with central spin state \(|0\rangle \text{ and } |j, m - 1 \rangle \text{ with } |1\rangle \) in Eq.3 means that the state is an eigenstate of \( F_z \) with eigenvalue \( m - 1/2 \). Eq.3 is valid for \( m = j \) to \( m = -j + 1 \). There are also two additional states where only one of the terms exist: \(|1 \rangle \ |j, j \rangle \), because \(|0 \rangle \ |j, j + 1 \rangle \) does not exist, and similarly \(|0 \rangle \ |j, -j \rangle \).

We now seek to prove that states of the form in Eq.3 are indeed eigenstates by applying the Hamiltonian in Eq.3 to them. Applying \( H \) gives

\[
\begin{align*}
J(1/\sqrt{2}) \left( (1) J_- | j, m \rangle \pm |0 \rangle J_+ | j, m - 1 \rangle \right) \\
= J(1/\sqrt{2}) \left( |1 \rangle \sqrt{(j + m)(j + m - 1)} | j, m - 1 \rangle \right. \\
\pm |0 \rangle \sqrt{(j - (m - 1))(j + (m - 1) + 1)} | j, m \rangle \\
= \pm J \sqrt{(j + m)(j - m + 1)} \\
\times (1/\sqrt{2}) \left( (0) | j, m \rangle \pm |1 \rangle | j, m - 1 \rangle \right)
\end{align*}
\]

where the standard relations for \( J_\pm | j, m \rangle \) have been used in the second step. This confirms that states in Eq.3 are eigenstates, with energy eigenvalues

\[
E = \pm J \sqrt{(j + m)(j - m + 1)} \quad (4)
\]

Next we prove that these states account for all possible eigenstates, so that we have found a complete eigen-basis for \( H \).

The total number of eigenstates of the form Eq.3 is \( 4j \) for a given value of \( j \) (as \( m \) runs from \( j \) to \(-j + 1 \) and there are two states for each \( m \) due to the \( \pm \) in Eq.3). In addition, there are the two states where one of the terms in Eq.3 was missing. In total therefore, for a given value of \( j \), there are \( 4j + 2 \) possible eigenstates. If we can now enumerate the possible values of \( j \) (which is the label for the total angular momentum of the outer spins), we can find out the number of eigenstates we have been able to account for.

Suppose there are \( N \) outer spins. Let us first consider the different ways of generating different \( m \) values. Then there are \( N C_0 = 1 \) ways of having all these spins aligned in the same direction, giving \( m = N/2 \). If all the outer spins but one are aligned, then \( m = (N - 1)/2 - 1/2 \), and there will be \( N C_1 \) ways of getting this value of \( m \). For the general case of \( N-r \) spins aligned and \( r \) spins anti-aligned to them, then \( m = (N - r)/2 - r/2 \) and there are \( N C_r \) ways of getting this value of \( m \).

We now have to group the above \( m \) values under suitable \( j \) values. Consider the case where \( m = N/2 \). As there is only one arrangement of spins giving rise to this, there should be only one value of \( j = N/2 \) to account for this. In other words, if there was more than one allowed \( j = N/2 \) value for the collection of spins, there would be more than one \( m = N/2 \), which we know, is not the case. Next consider \( m = (N - 2)/2 \). There are \( N C_1 \) such ways of obtaining this value of \( m \). One of these ways will arise from when \( j = N/2 - 1 \), which means there must be \( N C_1 - N C_0 \) of \( j = (N - 2)/2 \) to account for the remaining ways of getting this value of \( m \). This procedure for determining possible \( j \) continues until all \( m \) are accounted for. In general, there are \( N C_r - N C_{r-1} \) ways of obtaining \( j = (N - 2r)/2 \), with allowed values of \( r \) ranging from \( r = 0 \) to \( r = N/2 \) if \( N \) is even, or \( r = (N - 1)/2 \) if \( N \) is odd.

Given these values and \( j \), and that each value of \( j \) can produce \( 4j + 2 \) eigenstates, then a summation expressing the total number of states we have accounted for is

\[
\sum_{r=0}^{x} (N C_r - N C_{r-1}) \times \left( 4 \left[ (N - 2r)\frac{1}{2} \right] + 2 \right) \quad (5)
\]

where \( x = N/2 \) if \( N \) is even, or \((N-1)/2 \) if \( N \) is odd. The result of the summation is determined by observing that successive terms in summation cancel out some of the preceding terms. The result is independent of whether \( N \) is even or odd, and is \( 2^{N+1} \). This means the total number of eigenstates of the type we identified in Eq.3 is \( 2^{N+1} \), which is equal to the dimension of the Hilbert space of all the \( N \) outer spins plus the central spin. Therefore the states in Eq.3 (when combined with states of the type \(|1 \rangle \langle j, j \rangle \text{ and } |0 \rangle \langle j, -j \rangle \)) describe all the eigenstates of \( H \).

The ground state is the state with the lowest energy. To identify it we will first have to assume a sign of \( J \), which we take to be positive. All our results about entanglement and correlations will be exactly the same for negative \( J \). In Eq.3 the lowest energy is obtained when \( j \) has its maximum possible value and \( m \) has its minimum
possible value. For the case \( N \) odd, the lowest energy is when \( m = \frac{1}{2} \) i.e. the eigenstate

\[
\left(1/\sqrt{2}\right) \left( |0\rangle \, |N/2, 1/2\rangle - |1\rangle \, |N/2, -1/2\rangle \right)
\]  

(6)

and if \( N \) is even, then in fact the ground state is degenerate because there are two states with the lowest possible energy, when \( m = 0 \) or \( m = 1 \). These are

\[
\frac{1}{\sqrt{2}} \left( |0\rangle \, |N/2, 0, 0\rangle - |1\rangle \, |N/2, -2\rangle \right)
\]

\[
\frac{1}{\sqrt{2}} \left( |0\rangle \, |N/2, 1, 1\rangle - |1\rangle \, |N/2, 0, 0\rangle \right)
\]

(7)

The reason for the above difference between \( N \) even and \( N \) odd is that the two cases lead to an integral and half integral value of \( j \) respectively. When \( j \) is half integral, \( m = \pm 1/2 \) is allowed and gives an unique ground state. For \( j \) integral, the \( 0, -1 \) and \( 1, 0 \) form two distinct \( j, m \) pairs to combine with the central spin-1/2 particle to give two degenerate ground states.

To compute entanglement and correlations, it is useful to have expressions in terms of the states of the individual outer spins for these ground states. Let \( |0\rangle \) and \( |1\rangle \) stand for the \(-1/2\) and \(1/2\) spin states of any outer spin. For \( N \) odd, the state \( |N/2, 1/2\rangle \) is an equal superposition of all states with \((N + 1)/2\) ones and \((N - 1)/2\) zeros with no relative phase between them. The state \( |N/2, -1/2\rangle \) is the same type of state with \((N - 1)/2\) ones and \((N + 1)/2\) zeros. For example, for \( N = 3 \),

\[
\frac{1}{2} \left( |011\rangle + |101\rangle + |110\rangle \right)
\]

\[
\frac{1}{2} \left( |100\rangle + |010\rangle + |001\rangle \right)
\]

(8)

There are similar expressions for the ground state for \( N \) even. The \( |N/2, 0\rangle \) state is an equal superposition of all states with an equal number of zeros and ones, with no relative phase between the superposed states. \(|N/2, \pm 1\rangle \) is the same type of state with \(N/2\pm 1\) ones and rest zeros.

Given the ground state, we are able to calculate the entanglement between any two outer spins in this state (i.e. at zero temperature). The symmetry of the problem implies that the entanglement will be the same between any two outer spins. Since \( H \) and \( F_z \) commute, the form of the reduced density matrix for any two spins is

\[
\begin{pmatrix}
  v & 0 & 0 & 0 \\
  0 & w & z & 0 \\
  0 & z & x & 0 \\
  0 & 0 & 0 & y
\end{pmatrix}
\]

in the standard basis \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle \} \). For such density matrices, a measure of entanglement called the concurrence is given by

\[
C = 2 \max\{ |z| - \sqrt{vw}, 0 \}
\]

For \( N \) odd, the concurrence comes out as

\[
C = 2 \max\{ \frac{1}{2N}, 0 \} = \frac{1}{N}
\]

For the case of \( N \) even, where there are two ground states, a similar procedure is followed, except that the reduced density matrix is now described as an equal mixture of the two states. This gives the concurrence as

\[
C = 2 \max\{ \frac{1}{2N} - \frac{1}{2N(N-1)}, 0 \} = \frac{1}{N} - \frac{1}{N(N-1)}
\]

Thus the entanglement goes to zero as \( N \to \infty \), which is expected, as the entanglement is mediated by the central spin. The total entangling capability (and thereby mediating capability) of the central spin is divided among a larger and larger number of outer spins as \( N \) becomes larger. However, on going from an even \( N \) to an odd \( N + 1 \) number of outer spins, the concurrence rises from \( 1/\sqrt{N - 1}/(N^2 - N) \) to \( 1/(N + 1) \). Thus on increasing \( N \) there are curious oscillations in concurrence of amplitude \( 2/(N(N - 1)(N + 1)) \). These oscillations, due to the different expressions for concurrence in the cases of even and odd \( N \), is due to mixedness of the zero temperature density matrix for even \( N \). On application of a magnetic field in the \(+Z\) direction, the state \( |0\rangle \, |N/2, 0\rangle - |1\rangle \, |N/2, -1\rangle \) becomes the non-degenerate ground state for even \( N \) and the oscillations in entanglement disappear.

There are also interesting results to be noted for correlation functions in the star network. The \( \langle \sigma_{1z} \sigma_{2z} \rangle \) correlations follow the same pattern as the entanglement, but

\[
\langle \sigma_{1z} \sigma_{2z} \rangle = \frac{1}{2} + \frac{1}{2N} \quad \text{for odd } N
\]

\[
\langle \sigma_{1z} \sigma_{2z} \rangle = \frac{1}{2} + \frac{1}{2N} - \frac{1}{2N(N-1)} \quad \text{for even } N.
\]

What is particularly surprising above, is the non-vanishing nature of the correlations in the large \( N \) limit. The solitary central spin is capable to imposing spin order in the \( X \) direction (so that \( \langle \sigma_{1x} \sigma_{2x} \rangle = 1/2 \)), even when there are an infinite number of outer spins to order. The same result holds for \( \langle \sigma_{1y} \sigma_{2y} \rangle \). Thus our system provides an effective way of imposing order simultaneously in the \( X \) and \( Y \) directions for an infinite collection of spins. This result also highlights a crucial difference between entanglement and correlations: while a finite dimensional quantum system cannot be individually entangled to each member of an infinite collection of systems, it can indeed be correlated individually to each of them.

We now show that the application of a magnetic field allows us to change the ground state to

\[
|\alpha\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \, |N/2, -N/2 + 1\rangle - |1\rangle \, |N/2, -N/2\rangle \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( |0\rangle \{000\ldots1\} - |1\rangle \{000\ldots0\} \right)
\]

(9)
where \{000\ldots1\} is a normalized state that is an equal superposition of all states with only one \ket{1} with no relative phase between the superposed components. This state has the special significance that the concurrence between the central spin and each of the outer spins is $1/\sqrt{N}$, which is the maximum consistent with symmetric splitting of the total entanglement of one qubit with a collection of $N$ qubits among the $N$ qubits [16]. To our knowledge, this is the first identification of the canonical state $|\alpha\rangle$ as the ground state of an interacting spin system. Once the ground state $|\alpha\rangle$ is generated, if the central spin is measured and found to be in the state $|0\rangle$, the rest of the spins are projected onto the state \{000\ldots1\}. The central spin can now be removed to make the state dynamically steady (except for decoherence and spontaneous decay effects). This state has the property that the concurrence between any two spins is $2/N$, which is the maximum possible entanglement in a collection of $N$ spins in which all pairs of spins are equally entangled [17].

To show that it is possible to make $|\alpha\rangle$ the ground state, we consider the energy eigenvalues in a uniform magnetic field $B$ in the $+Z$ direction (in which the eigenstates remain unchanged)

$$E = \pm J \sqrt{(j+m)(j-m+1)} + (m-\frac{1}{2})B$$

When $B$ becomes so high that the second term in $E$ dominates, the relative ordering of the energy levels will be determined purely by $m$, with the ground state being state with $m = \frac{N}{2}$, i.e. $|\beta\rangle = |0\rangle |000\ldots0\rangle$. It is straightforward to show that $|\beta\rangle$ has an energy lower than $|\alpha\rangle$ for $B > J\sqrt{N}$. Before $|\beta\rangle$ becomes the ground state, there is a range of $B$ in which $|\alpha\rangle$ is the ground state. To prove this, we will have to show that the energy of $|\alpha\rangle$ will be less than that of all other states in a certain range. The value of $B$ for which $|\alpha\rangle$ has the same energy as a general state described by $j$ and $m$ is given by

$$B = \frac{J \left( \sqrt{N} - \sqrt{(j+m)(j-m+1)} \right)}{\frac{\sqrt{N}}{2} - 1} - m$$

$B$ attains its largest value, when $j$ is a maximum and $m$ has it’s most negative value. This happens when $j = \frac{N}{2}$ and $m = -\frac{N}{2} + 2$ for which $B = J\sqrt{N} \left( \sqrt{2(1-1/N)} - 1 \right) < J\sqrt{N}$. Therefore, for a magnetic field with a range of $J\sqrt{N} \left( \sqrt{2(1-1/N)} - 1 \right) < B < J\sqrt{N}$, the state $|\alpha\rangle$ is the ground state.

In this letter we have introduced and solved a spin-star, an architecture of interacting spins which cannot be classified as a lattice in any dimension. It is physically realizable in various arrays of qubits designed for quantum computation. Testing for the entanglement and correlations predicted here would serve as a benchmark test for the functioning of arrays of qubits. If spin-spin interactions can be extended to long distances, the spin-star could be used for distribution of entanglement through the multiparty states for optimal symmetric sharing and splitting of entanglement. Exploration of the full statistical mechanics of a spin-star, would be interesting future work.

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