Pirogov-Sinai Theory With New Contours For Symmetric Models

N.N. Ganikhodjaev\textsuperscript{1,2}, U.A. Rozikov\textsuperscript{1,3}

\textsuperscript{1}Institute of Math. and Infor. Technol., 29, F.Hodjaev str., 100125, Tashkent, Uzbekistan.
\textsuperscript{2}International Islamic University Malaysia, P.O. Box 141, 25710, Kuantan, Malaysia.
\textsuperscript{3} School of Math. Sci. GC University, Lahore, Pakistan.
E-mail: nasirgani@yandex.ru, rozikovu@yandex.ru

Abstract: The contour argument was introduced by Peierls for two dimensional Ising model. Peierls benefited from the particular symmetries of the Ising model. For non-symmetric models the argument was developed by Pirogov and Sinai. It is very general and rather difficult. Intuitively clear that the Peierls argument does work for any symmetric model. But contours defined in Pirogov-Sinai theory do not work if one wants to use Peierls argument for more general symmetric models. We give a new definition of contour which allows relatively easier prove the main result of the Pirogov-Sinai theory for symmetric models. Namely, our contours allow us to apply the classical Peierls argument (with contour removal operation).

Mathematics Subject Classifications (2000). 82B05, 82B20 (primary); 60K35, 05C05 (secondary).

Keywords: Configuration, lattice model, contour, Gibbs measure.

1 Introduction

In many systems of interest, low temperature Gibbs measures are concentrated on configurations which are basically a single configuration plus a small fraction of small ”fluctuations”, also called ”defects”. The boundaries of these ”fluctuations”, define the contours.

The contour argument was pioneered by Peierls in 1936 [8] to demonstrate that the two dimensional Ising model does exhibit phase coexistence at low temperature. The original argument benefited from the particular symmetries of the Ising model. The adaptation of the method to the treatment of non-symmetric models is not trivial, and was developed by Pirogov and Sinai [9], [13] (see also [1]-[7],[15]). A particularly enlightening alternative version of the argument was put forward by Zahradnik [14].

In the Pirogov-Sinai (PS) theory configurations can be described by contours which satisfy Peierls condition. This theory provides tools for a very detailed knowledge of the structure of Gibbs measures in a region in the relevant parameters space (see e.g. [13]). The PS theory is a low temperature expansion which enables to control the entropic fluctuations from the ground states, its natural setup being the lattice systems. But the theory is not limited to such cases and it has been applied to a great variety of situations, covering various types of phase transitions. (see e.g. [3] for details).

The main object of the theory is a family of contours defining a configuration. In the original PS theory the ensemble of contours has more complicated form. In particular, they do not have the ”contour-removal operation” (even for symmetric models) introduced by Peierls.
This paper presents a new definition of the contour on $\mathbb{Z}^d$. Contours defined here more convenient to prove the main theorem of the PS theory for symmetric models. They allow us to use classical Peierls argument (with the contour-removal operation). Such contours for models on the Cayley tree were defined in [10]-[12].

The paper is organized as follows. In section 2 we give all necessary definitions and check the Peierls condition. Section 3 devoted to definition and properties of new contours. In section 4 by the classical Peierls argument we show the existence of $s$ different (where $s$ is the number of ground states) Gibbs measures.

2 Definitions and Peierls condition

2.1. Configuration space and the model. We consider the $d$-dimensional ($d \geq 2$) cubic lattice $\mathbb{Z}^d$. The distance $d(x, y)$, $x, y \in \mathbb{Z}^d$ is defined by

$$d(x, y) = \max_{1 \leq i \leq d} |x_i - y_i|.$$  

For $A \subseteq \mathbb{Z}^d$ a spin configuration $\sigma_A$ on $A$ is defined as a function $x \in A \rightarrow \sigma_A(x) \in \Phi = \{1, 2, \ldots, q\}$; the set of all configurations coincides with $\Omega_A = \Phi^A$. We denote $\Omega = \Omega_{\mathbb{Z}^d}$ and $\sigma = \sigma_{\mathbb{Z}^d}$. Also we define a periodic configuration as a configuration $\sigma \in \Omega$ which is invariant under a subgroup of shifts $\mathbb{Z}^d_s \subset \mathbb{Z}^d$ of finite index. A configuration that is invariant with respect to all shifts is called translational-invariant.

The energy of the configuration $\sigma \in \Omega$ is given by the formal Hamiltonian

$$H(\sigma) = \sum_{A \subseteq \mathbb{Z}^d, \text{diam}(A) \leq r} I(\sigma_A)$$  \hspace{1cm} (2.1)

where $r \in N = \{1, 2, \ldots\}$, $\text{diam}(A) = \max_{x, y \in A} d(x, y)$, $I(\sigma_A) : \Omega_A \rightarrow R$ is a given translational-invariant potential.

Denote by $M_r$ the set of all cubes of linear size $r$.

For $A \subset \mathbb{Z}^d$ with $\text{diam}(A) \leq r$ denote

$$n(A) = |\{b \in M_r : A \subset b\}|,$$

where $|B|$ stands for the number of elements of a set $B$.

The Hamiltonian (2.1) can be rewritten as

$$H(\sigma) = \sum_{b \in M_r} U(\sigma_b),$$  \hspace{1cm} (2.2)

where $U(\sigma_b) = \sum_{A \subset b} \frac{I(\sigma_A)}{m(A)}$.

For a finite domain $D \subset \mathbb{Z}^d$ with the boundary condition $\varphi_{D^c}$ given on its complement $D^c = \mathbb{Z}^d \setminus D$, the conditional Hamiltonian is

$$H(\sigma_D | \varphi_{D^c}) = \sum_{b \in M_r, b \cap D \neq \emptyset} U(\sigma_b),$$  \hspace{1cm} (2.3)
where
\[ \sigma_b(x) = \begin{cases} \sigma(x) & \text{if } x \in b \cap D \\ \varphi(x) & \text{if } x \in b \cap D^c. \end{cases} \]

2.2. The ground state. A ground state of (2.2) is a configuration \( \varphi \) in \( Z^d \) whose energy cannot be lowered by changing \( \varphi \) in some local region. We assume that (2.2) has a finite number of translation-periodic (i.e. invariant under the action of some subgroup of \( Z^d \) of finite index) ground states. By a standard trick of partitioning the lattice into disjoint cubes \( Q(x) \) centered at \( x \in pZ^d \) with an appropriate \( p \) and enlarging the spin space from \( \Phi \) to \( \Phi^Q \) one can transform the model above into a model on \( pZ^d \) with only translation-invariant or non periodic ground states. Such a transformation was considered in [6]. Hence, without loss of generality, we assume translation-invariance instead of translational-periodic and we permute the spin so that the set of ground states of the model be \( GS = GS(H) = \{ \sigma^{(i)}, i = 1, 2, ..., s \}, 1 \leq s \leq q \) with \( \sigma^{(i)}(x) = i \) for any \( x \in Z^d \).

2.3. Gibbs measure. We consider a standard sigma-algebra \( B \) of subsets of \( \Omega \) generated by cylinder subsets; all probability measures are considered on \( (\Omega, B) \). A probability measure \( \mu \) is called a Gibbs measure (with Hamiltonian \( H \)) if it satisfies the DLR equation: \( \forall \) finite \( \Lambda \subset Z^d \) and \( \sigma_{\Lambda} \in \Omega_{\Lambda}: \)
\[ \mu\left(\{ \sigma \in \Omega : \sigma|_{\Lambda} = \sigma_{\Lambda}\}\right) = \int_{\Omega} \mu(d\omega)\nu_\varphi^\Lambda(\sigma_{\Lambda}), \] (2.4)
where \( \nu_\varphi^\Lambda \) is the conditional probability:
\[ \nu_\varphi^\Lambda(\sigma_{\Lambda}) = \frac{1}{Z_{\Lambda,\varphi}} \exp \left( -\beta H \left( \sigma_{\Lambda} | \varphi_{\Lambda^c} \right) \right). \] (2.5)

Here \( \beta = \frac{1}{kT}, \) \( T > 0 \)— temperature and \( Z_{\Lambda,\varphi} \) stands for the partition function in \( \Lambda \), with the boundary condition \( \varphi \):
\[ Z_{\Lambda,\varphi} = \sum_{\tilde{\sigma}_{\Lambda} \in \Omega_{\Lambda}} \exp \left( -\beta H \left( \tilde{\sigma}_{\Lambda} | \varphi_{\Lambda^c} \right) \right). \] (2.6)

2.4. The Peierls condition.
Denote by \( U \) the collection of all possible values of \( U(\sigma_b) \) for any configuration \( \sigma_b, b \in M_r \).

Since \( r < +\infty \) we have \( |U| < +\infty \). Put \( U^{\min} = \min\{ U : U \in U \} \) and
\[ \lambda_0 = \min \left\{ U : U \in U \setminus \{ U \in U : U = U^{\min} \} \right\} - U^{\min}. \] (2.7)

The important assumptions of this paper (see subsection 2.2) are the following:
Assumption A1. The set of all ground states is \( GS = \{ \sigma^{(i)}, i = 1, 2, ..., s \}, 1 \leq s \leq q \).
Assumption A2. \( \lambda_0 > 0 \) i.e. \( U \) has at least two distinct elements.

Let \( P_s \) be the group of permutations on \( \{1, ..., s\} \). For \( g \in P_s, \ g = (g_1, ..., g_s) \) and \( \sigma \in \Omega \) define \( g\sigma \in \Omega \) by
\[ g\sigma(x) = \begin{cases} g_{\sigma(x)}, & \text{if } \sigma(x) \in \{1, ..., s\} \\ \sigma(x), & \text{if } \sigma(x) \in \{s + 1, ..., q\}. \end{cases} \]

Assumption A3. Hamiltonian (2.1) is symmetric i.e. \( H(g\sigma) = H(\sigma) \) for any \( g \in P_s \) and \( \forall \sigma \in \Omega \).
Remark. If a configuration $\sigma$ satisfies
\[ U(\sigma_b) = U_{\text{min}} \quad \forall b \in M_r \quad (2.8) \]
then it is a ground state. Moreover for Hamiltonians on $Z^d$ it is well known that a configuration is a ground state if and only if the condition (2.8) is satisfied (see e.g. [13]).

The relative Hamiltonian is defined by
\[ H(\sigma, \varphi) = \sum_{b \in M_r} (U(\sigma_b) - U(\varphi_b)). \]

**Definition 2.1.** Let $GS$ be the set of all ground states of the relative Hamiltonian $H$. A cube $b \in M_r$ is said to be an improper cube of the configuration $\sigma$ if $\sigma_b \neq \varphi_b$ for any $\varphi \in GS$. The union of the improper cubes of a configuration $\sigma$ is called the boundary of the configuration and denoted by $\partial(\sigma)$.

**Definition 2.2.** The relative Hamiltonian $H$ with the set of ground states $GS$ satisfies the Peierls condition if for any $\varphi \in GS$ and any configuration $\sigma$ coinciding almost everywhere with $\varphi$ (i.e. $|\{x \in Z^d : \sigma(x) \neq \varphi(x)\}| < \infty$)
\[ H(\sigma, \varphi) \geq \lambda |\partial(\sigma)|, \]
where $\lambda$ is a positive constant which does not depend on $\sigma$, and $|\partial(\sigma)|$ is the number of cubes in $\partial(\sigma)$.

**Proposition 2.3.** If assumptions A1-A2 are satisfied then the Peierls condition holds.

Proof. Suppose $\sigma$ coincides almost everywhere with a ground state $\varphi \in GS$ then we have $U(\sigma_b) - U_{\text{min}} \geq \lambda_0$ for any $b \in \partial(\sigma)$ since $\varphi$ is a ground state. Thus
\[ H(\sigma, \varphi) = \sum_{b \in M_r} (U(\sigma_b) - U(\varphi_b)) = \sum_{b \in \partial(\sigma)} (U(\sigma_b) - U_{\text{min}}) \geq \lambda_0 |\partial(\sigma)|. \]
Therefore, the Peierls condition is satisfied for $\lambda = \lambda_0$. The proposition is proved.

3 Contours

Before giving our new contours let us recall the definition of the contour defined in original Pirogov-Sinai theory (see [13]).

**Definition 3.1.** Pair $\Gamma = (M, \sigma(M))$, (where $M = \text{supp} \sigma(M)$ is a finite connected subset of $Z^d$), is called contour of the configuration $\sigma$, if $M$ is a component (maximal connected set) of the boundary $\partial(\sigma)$.

Now we define our contours which are more convenient to use for the symmetric models.

Let $\Lambda \subset Z^d$ be a finite set. Let $\sigma^{(i)}_{\Lambda \setminus c} \equiv i$, $i = 1, ..., s$ be a constant configuration outside of $\Lambda$. For each $i$ we extend the configuration $\sigma_{\Lambda}$ inside $\Lambda$ to the entire lattice by the $i$th constant configuration and denote it by $\sigma^{(i)}_{\Lambda}$. The set of such configurations we denote by $\Omega^{(i)}_{\Lambda}$. 

4
For a given configuration $\sigma^{(i)}_\Lambda \in \Omega^{(i)}_\Lambda$ denote $V^{(j)}_\Lambda \equiv V^{(j)}_\Lambda(\sigma^{(i)}_\Lambda) = \{ t \in \Lambda : \sigma^{(i)}_\Lambda(t) = j \}, j = 1, \ldots, q, j \neq i$. Let $G_{\Lambda,j} = (V^{(j)}_\Lambda, L^{(j)}_\Lambda)$ be a graph such that
\[
L^{(j)}_\Lambda = \{(x,y) \in V^{(j)}_\Lambda \times V^{(j)}_\Lambda : d(x,y) = 1\}, \quad j = 1, \ldots, q, j \neq i.
\]
It is clear, that for a fixed $\Lambda$ the graph $G_{\Lambda,j}$ contains a finite number ($= m$) of maximal connected subgraphs $G_{\Lambda,j,p}$ i.e.
\[
G_{\Lambda,j} = \{G_{\Lambda,j,1}, \ldots, G_{\Lambda,j,m}\}, \quad G_{\Lambda,j,p} = (V^{(j)}_{\Lambda,p}, L^{(j)}_{\Lambda,p}), \quad p = 1, \ldots, m; j \neq i.
\]
Here $V^{(j)}_{\Lambda,p}$ and $L^{(j)}_{\Lambda,p}$ are the set of vertexes and edges of $G^{A\cdot j}_{\Lambda}$, respectively.

The (finite) graph $G_{\Lambda,j,p}, j = 1, \ldots, q, j \neq i, p = 1, \ldots, m$ is called subcontour of the configuration $\sigma^{(i)}_\Lambda$.

The set $V^{(j)}_{\Lambda,p}, j = 1, \ldots, q, j \neq i, p = 1, \ldots, m$ is called interior of $G_{\Lambda,j,p}$, and is denoted by $\text{Int}G_{\Lambda,j,p}$. Note that the configuration $\sigma^{(i)}_\Lambda$ takes the same value $j$ at all points of the connected component $G_{\Lambda,j,p}$. This value is called mark of the subcontour.

For any two subcontours $T_1, T_2$ the distance $\text{dist}(T_1, T_2)$ is defined by
\[
\text{dist}(T_1, T_2) = \min_{x \in \text{Int}T_1, y \in \text{Int}T_2} d(x,y),
\]
where $d(x,y)$ is the distance between $x, y \in Z^d$ (see section 2.1).

**Definition 3.2.** The subcontours $T_1, T_2$ are called adjacent if $\text{dist}(T_1, T_2) \leq r$. A set of subcontours $A$ is called connected if for any two subcontours $T_1, T_2 \in A$ there is a collection of subcontours $T_1 = \tilde{T}_1, \tilde{T}_2, ..., \tilde{T}_l = T_2$ in $A$ such that for each $i = 1, \ldots, l - 1$ the subcontours $\tilde{T}_i$ and $\tilde{T}_{i+1}$ are adjacent.

**Definition 3.3.** Any maximal connected set (component) of subcontours (with given marks) is called contour of the configuration $\sigma^{(i)}_\Lambda$.

For contour $\gamma = \{T_p\}$ denote $\text{Int}\gamma = \cup_p \text{Int}T_p$.

Remarks. Our definition of a contour is different from the Definition 3.1. Indeed: (i) our contour can be non connected subgraph of $Z^d$, but the contours in original PS theory are connected; (ii) By our definition for any two contours $\gamma, \gamma'$ we have $\text{dist}(\gamma, \gamma') > r$. Thus our contours do not interact. This means that for any $\sigma \in \Omega$ there is no a cube $b \in \partial(\sigma)$ with $b \cap \gamma \neq \emptyset$ and $b \cap \gamma' \neq \emptyset$. Such property allows as to use a contour-removal operation. This operation is similar to the one in ordinary Peierls argument [5]: Given a family of contours defining a configuration $\sigma \in \Omega^{(i)}_\Lambda$, the family obtained by omitting one of them is also the family of contours of a (different) configuration in $\Omega^{(i)}_\Lambda$. There is an algorithm of the contour-removal operation to obtain a new configuration as follows. Take the configuration $\sigma$ and change all the spins in the interior of $\gamma$ (which must be removed) to value $i$. This makes $\gamma$ disappear, but leaves intact the other contours. Contours defined in the Definition 3.1 may interact. Therefore the Peierls argument is not directly applicable in that approach.
In the sequel of the paper by contour we mean a contour defined by Definition 3.3. For a given (sub)contour $\gamma$ denote

$$\text{imp} \gamma = \{ b \in \partial : b \cap \gamma \neq \emptyset \}, \quad |\gamma| = |\text{imp} \gamma|.$$ 

By the construction we have $\text{imp} \gamma \cap \text{imp} \gamma' = \emptyset$ for any contours $\gamma \neq \gamma'$.

For a given graph $G$ denote by $V(G)$ the set of its vertices.

Let us define a graph structure on $M_r$ as follows. Two cubes $b, b' \in M_r$ are connected by an edge if $b \cap b' \neq \emptyset$. Denote this graph by $G(M_r)$. Here the vertices of this graph are elements (cubes) of $M_r$. Note that the graph $G(M_r)$ is a locally finite i.e. there is $k = k(d, r) < +\infty$ such that any vertex of $G(M_r)$ has $k$ nearest neighbors. Thus Lemma 1.2 of [2] can be reformulated as follows

**Lemma 3.4.** Let $\tilde{N}_{n,G}(x)$ be the number of connected subgraphs $G' \subset G(M_r)$ with $x \in V(G')$ and $|V(G')| = n$. Then

$$\tilde{N}_{n,G}(x) \leq (ek)^n.$$ 

For $x \in \mathbb{Z}^d$ we will write $x \in \gamma$ if $x \in \text{Int} \gamma$.

Denote $N_n(x) = |\{ \gamma : x \in \gamma, |\gamma| = n \}|$, where as before $|\gamma| = |\text{imp} \gamma|$.

**Lemma 3.5.** $N_n(x) \leq \frac{1}{2}(4ek)^n$.

**Proof.** Consider $\text{imp} \gamma$ as a subgraph of $G(M_r)$. In general $\text{imp} \gamma$ may be non connected subgraph of the graph $G(M_r)$. Denote by $K_\gamma$ the minimal connected subgraph of $G(M_r)$, which contains the contour $\gamma$. It is easy to see that

$$|V(K_\gamma)| \leq 2|\text{imp} \gamma| = 2|\gamma|. \quad (3.1)$$

Using the estimation (3.1) and Lemma 3.4 we obtain

$$N_n(x) \leq \binom{2n}{n} \tilde{N}_{2n,G}(x) \leq 2^{2n-1}(ek)^n = \frac{1}{2}(4ek)^n.$$ 

The lemma is proved.

### 4 Non-uniqueness of Gibbs measure

For $A \subset \mathbb{Z}^d$ denote

$$C(A) = \{ b \in M_r : b \cap A \neq \emptyset \}.$$ 

For $\sigma_\Lambda \in \Omega^{(i)}_\Lambda$ the conditional Hamiltonian (2.3) has the form

$$H^{(i)}(\sigma_\Lambda) \equiv H(\sigma_\Lambda | \sigma_\Lambda^c = i) = \sum_{b \in M_r : b \cap \Lambda \neq \emptyset} U(\sigma_\Lambda, b) =$$

$$\sum_{b \in \partial(\sigma_\Lambda)} (U(\sigma_\Lambda, b) - U^{\min}) + |C(\Lambda)| U^{\min}, \quad (4.1)$$

where $\sigma_{\Lambda, b}(x) = \sigma_\Lambda(x)$ if $x \in \Lambda \cap b$ and $\sigma_{\Lambda, b}(x) = i$ if $x \in \Lambda^c \cap b$. 

6
The Gibbs measure on the space $\Omega^{(i)}_{\Lambda}$ with boundary condition $\sigma^{(i)}$ is defined as

$$
\mu^{(i)}_{\Lambda, \beta}(\sigma_{\Lambda}) = Z^{(i)}_{\Lambda, \beta} \exp(-\beta H^{(i)}(\sigma_{\Lambda})),
$$

(4.2)

where $Z^{(i)}_{\Lambda, \beta}$ is the normalizing factor.

Let us consider a sequence of sets on $\mathbb{Z}^d$

$$
V_1 \subset V_2 \subset ... \subset V_n \subset ..., \ \cup V_n = \mathbb{Z}^d,
$$

and $s$ sequences of boundary conditions outside these sets:

$$
\sigma^{(i)}_n \equiv i, n = 1, 2, ..., i = 1, ..., s.
$$

By very similar argument of proof of the lemma 9.2 in [7] one can prove that each of $s$ sequences of measures $\{\mu^{(i)}_{n, \beta}, n = 1, 2, ..., \}$, $i = 1, ..., s$ contains a convergent subsequence.

We denote the corresponding limits by $\mu^{(i)}_{\beta}, i = 1, ..., s$. Our purpose is to show that for a sufficiently large $\beta$ these measures are different.

**Lemma 4.1.** Suppose assumptions A1, A2 are satisfied. Let $\gamma$ be a fixed contour and $p_i(\gamma) = \mu^{(i)}_{\beta}(\sigma_n \in \Omega_{\gamma_n} : \gamma \in \partial(\sigma_n))$. Then

$$
p_i(\gamma) \leq \exp\{-\beta \lambda_0 |\gamma|\},
$$

(4.3)

where $\lambda_0$ is defined by formula (2.7).

**Proof.** Put $\Omega_{\gamma} = \{\sigma_n \in \Omega^{(i)}_{\gamma_n} : \gamma \subset \partial(\sigma_n)\}$, $\Omega^{0}_{\gamma} = \{\sigma_n : \gamma \cap \partial = \emptyset\}$ and define a (contour-removal) map $\chi_{\gamma} : \Omega_{\gamma} \rightarrow \Omega^{0}_{\gamma}$ by

$$
\chi_{\gamma}(\sigma_n)(x) = \begin{cases} 
  i & \text{if } x \in \text{Int}\gamma \\
  \sigma_n(x) & \text{if } x \notin \text{Int}\gamma.
\end{cases}
$$

When $\gamma$ is fixed then the configuration on $\text{Int}\gamma$ also fixed. Therefore the map $\chi_{\gamma}$ is one-to-one map. For any $\sigma_n \in \Omega^{(i)}_{\gamma_n}$ we have

$$
|\partial(\sigma_n)| = |\partial(\chi_{\gamma}(\sigma_n))| + |\gamma|.
$$

Consequently, using (4.1) one finds

$$
p_i(\gamma) = \frac{\sum_{\sigma_n \in \Omega_{\gamma}} \exp\{-\beta \sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min})\}}{\sum_{\sigma_n \in \Omega_{\gamma}} \exp\{-\beta \sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min})\}} \leq \frac{\sum_{\sigma_n \in \Omega_{\gamma}} \exp\{-\beta \sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min})\}}{\sum_{\sigma_n \in \Omega_{\gamma}} \exp\{-\beta \sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min})\}} = \frac{\sum_{\sigma_n \in \Omega_{\gamma}} \exp\{-\beta \sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min})\}}{\sum_{\sigma_n \in \Omega_{\gamma}} \exp\{-\beta \sum_{b \in \partial(\chi_{\gamma}(\sigma_n))} (U(\chi_{\gamma}(\sigma_{n,b})) - U^{\min})\}}.
$$

(4.4)

Since $\sigma_{n,b} = \chi_{\gamma}(\sigma_{n,b})$, for any $b \in \partial(\sigma_n) \setminus \text{imp}_{\gamma}$ we have

$$
\sum_{b \in \partial(\sigma_n)} (U(\sigma_{n,b}) - U^{\min}) = S_1 + S_2,
$$

(4.5)
where $S_1 = \sum_{b \in \partial(x,\sigma_n)} (U(\sigma_{n,b}) - U_{\text{min}})$; $S_2 = \sum_{b \in \text{imp}} (U(\sigma_{n,b}) - U_{\text{min}})$.

By our construction $\gamma$ is a contour of $\partial(\sigma_n)$ iff $\sigma_n(x) = i$ for any $x \in \mathbb{Z}^d \setminus \text{Int}\gamma$ with $d(x,\text{Int}\gamma) < r$. Consequently, $\text{imp}\gamma$ does not depend on $\sigma_n \in \Omega_{\gamma}$. By assumptions A1-A2 we have $U(\sigma_{n,b}) - U_{\text{min}} \geq \lambda_0 > 0$, for any $b \in \text{imp}\gamma$.

Hence

$$S_2 = \sum_{b \in \text{imp}\gamma} (U(\sigma_{n,b}) - U_{\text{min}}) \geq \lambda_0 |\gamma|, \text{ for any } \sigma_n \in \Omega_{\gamma}. \quad (4.6)$$

Thus from (4.4)-(4.6) one gets (4.3). The lemma is proved.

Now using Lemmas 3.5 and 4.1 by very similar argument of [11] one can prove the following

**Lemma 4.2.** If assumptions A1-A3 are satisfied then for fixed $x \in \Lambda$ uniformly by $\Lambda$ the following relation holds

$$\mu_{\beta}^{(i)}(\sigma_{\Lambda} : \sigma_{\Lambda}(x) = j) \to 0, j \neq i \text{ as } \beta \to \infty.$$ 

This lemma implies the main result, i.e.

**Theorem 4.3.** If A1-A3 are satisfied then for all sufficiently large $\beta$ there are at least $s$ (=number of ground states) Gibbs measures for the Hamiltonian (2.2) on $\mathbb{Z}^d$.

**Acknowledgments.** The work supported by the SAGA Fund P77c of the Ministry of Science, Technology and Innovation (MOSTI) through the Academy of Sciences Malaysia. RUA thanks MOSTI and IIUM, for support and hospitality (in July-August 2007).

**References**

1. Biskup, M., Borgs, C., Chayes, J. T., Kotecký, R.: Partition function zeros at first-order phase transitions: Pirogov-Sinai theory. J. Stat. Phys. **116**, 97-155 (2004)

2. Borgs, C.: Statistical physics expansion methods in combinatorics and computer science, CBMS Lecture Series, Memphis 2003 (in preparation).

3. Bovier, A., Merola, I., Presutti, E., Zahradník, M.: On the Gibbs phase rule in the Pirogov-Sinai regime. J. Stat. Phys. **114**, 1235-1267 (2004)

4. Ganikhodjaev, N., Pah, C. H.: Phase diagrams of multicomponent lattice models. Theor. Math. Phys. **149**, 244-251 (2006).

5. Fernández, R.: Contour ensembles and the description of Gibbsian probability distributions at low temperature. www.univ-rouen.fr/LMRS/persopage/Fernandez, 1998.

6. Lebowitz, J. L., Mazel, A. E.: On the uniqueness of Gibbs states in the Pirogov-Sinai theory. Commun. Math. Phys. **189**, 311-321 (1997)

7. Minlos, R.A.: Introduction to mathematical statistical physics, University lecture series, v.19, AMS, 2000.

8. Peierls, R.: On Ising model of ferro magnetism. Proc. Cambridge Phil. Soc. **32**, 477-481 (1936).

9. Pirogov, S.A., Sinai,Ya. G.: Phase diagrams of classical lattice systems.I, II. Theor. Math. Phys. **25**, 1185-1192 (1975); **26**, 39-49 (1976)
10. Rozikov, U.A.: An example of one-dimensional phase transition. Siber. Adv. Math. 16, 121-125 (2006)

11. Rozikov, U.A.: On $q$-component models on Cayley tree: contour method. Lett. Math. Phys. 71, 27-38 (2005)

12. Rozikov, U. A.: A constructive description of ground states and Gibbs measures for Ising model with two-step interactions on Cayley tree. J. Stat. Phys. 122, 217-235 (2006)

13. Sinai, Ya.G.: Theory of phase transitions: Rigorous Results, Oxford: Pergamon, 1982.

14. Zahradnik, M.: An alternate version of Pirogov-Sinai theory. Commun. Math. Phys. 93, 559-581 (1984)

15. Zahradnik, M.: A short course on the Pirogov-Sinai theory. Rendiconti Math. Serie VII. 18, 411-486 (1998)