Locally convex quasi $C^*$-normed algebras

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Abstract

If $A_0[\| \cdot \|_0]$ is a $C^*$-normed algebra and $\tau$ a locally convex topology on $A_0$ making its multiplication separately continuous, then $\tilde{A}_0[\tau]$ (completion of $A_0[\tau]$) is a locally convex quasi $*$-algebra over $A_0$, but it is not necessarily a locally convex quasi $*$-algebra over the $C^*$-algebra $\tilde{A}_0[\| \cdot \|_0]$ (completion of $A_0[\| \cdot \|_0]$). In this article, stimulated by physical examples, we introduce the notion of a locally convex quasi $C^*$-normed algebra, aiming at the investigation of $\tilde{A}_0[\tau]$; in particular, we study its structure, $*$-representation theory and functional calculus.

1. Introduction

In the present paper we continue the study introduced in [7] and carried over in [13] and [8]. At this stage, it concerns the investigation of the structure of the completion of a $C^*$-normed algebra $A_0[\| \cdot \|_0]$, under a locally convex topology $\tau$ “compatible” to $\| \cdot \|_0$, that makes the multiplication of $A_0$ separately continuous. The case when $A_0[\| \cdot \|_0]$ is a $C^*$-algebra and $\tau$ makes the multiplication jointly continuous was considered in [7, 13], while the analogue case corresponding to separately continuous multiplication was discussed in [8], where the so-called locally convex quasi $C^*$-algebras were introduced. In this work, prompted by examples that one meets in physics, we introduce the notion of locally convex quasi $C^*$-normed algebras, which is wider than that of locally convex quasi $C^*$-algebras, starting with a $C^*$-normed algebra $A_0[\| \cdot \|_0]$ and a locally convex topology $\tau$, “compatible” to $\| \cdot \|_0$, making the multiplication of $A_0$ separately continuous. For example, let $M_0$ be a $C^*$-normed algebra of operators on a Hilbert space $H$, endowed with the operator norm $\| \cdot \|_0$, $D$ a dense subspace of $H$ such that $M_0D \subset D$ and $\tau_{s^*}$ the strong$^*$-topology on $M_0$ defined by $D$. Then, the $C^*$-algebra $\tilde{M}_0[\| \cdot \|_0]$ does not leave $D$ invariant, in general, and so the multiplication $ax$ of $a \in \tilde{M}_0[\tau_{s^*}]$ and $x \in \tilde{M}_0[\| \cdot \|_0]$ is not necessarily well-defined, therefore $\tilde{M}_0[\tau_{s^*}]$ is not a locally convex quasi $C^*$-algebra.

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over the $C^*$-algebra $\widetilde{M}_0[\|\cdot\|]$. Hence, it is meaningful to study not only locally convex quasi $C^*$-algebras, but also locally convex quasi $C^*$-normed algebras.

For locally convex quasi “$C^*$-normed algebras” we obtain analogous results to those in [8] for locally convex quasi “$C^*$-algebras” despite of the lack of completion and of weakening the condition (T$_3$) of [8].

In Section 3 we consider a $C^*$-algebra $A_0[\|\cdot\|]$ with a “regular” locally convex topology $\tau$ and show that every unital pseudo-complete symmetric locally convex $*$-algebra $A[\tau]$ such that $A_0[\|\cdot\|] \subset A[\tau] \subset \widetilde{A}_0[\tau]$ is a $GB^*$-algebra over the unit ball $U(A_0)$ of $A_0[\|\cdot\|]$. The latter algebras have been defined by G.R. Allan [2] and P.G. Dixon [12] and play an essential role in the unbounded $*$-representation theory. In Section 4 we define the notion of locally convex quasi $C^*$-normed algebras and study their general theory, while in Section 5 we investigate the structure of commutative locally convex quasi $C^*$-normed algebras. In the final Section 6 we present locally convex quasi $C^*$-normed algebras of operators and then we study questions on the $*$-representation theory of locally convex quasi $C^*$-normed algebras and functional calculus for the “commutatively quasi-positive” elements of $\widetilde{A}_0[\tau]$.

Topological quasi $*$-algebras were introduced in 1981 by G. Lassner [15, 16], for facing solutions of certain problems in quantum statistics and quantum dynamics. But only later (see [17, p. 90]) the initial definition was reformulated in the right way, having thus included many more interesting examples. Quasi $*$-algebras came in light in 1988 (see [19], as well as [20, 9, 10]), serving as important examples of partial $*$-algebras initiated by J.-P. Antoine and W. Karwowski in [4, 5]. A lot of works have been done on this topic, which can be found in the treatise [3], where the reader will also find a relevant rich literature. Partial $*$-algebras and quasi $*$-algebras keep a very prominent place in the study of unbounded operators, where the latter are the foundation stones for mathematical physics and quantum field theory (see, for instance, [3, 14, 6, 20]).

Our motivation for such studies comes, on the one hand, from the preceding discussion and the promising contribution of the powerful tool that the $C^*$-property offers to such studies and, on the other hand, from the physical examples of locally convex quasi $C^*$-normed algebras in “dynamics of the BCS-Bogolubov model” [16] that will be shortly discussed in Section 7.

2. Preliminaries

Throughout the whole paper we consider complex algebras and we suppose that all topological spaces are Hausdorff. If an algebra $A$ has an identity element, this will be denoted by 1, and an algebra $A$ with identity 1 will be called unital.

Let $A_0[\|\cdot\|]$ be a $C^*$-normed algebra. The symbol $\|\cdot\|_0$ of the $C^*$-norm will also
denote the corresponding topology. Let $\tau$ be a topology on $A_0$ such that $A_0[\tau]$ is a locally convex $*$-algebra. The topologies $\tau$, $\| \cdot \|_0$ on $A_0$ are called compatible, whenever for any Cauchy net $\{x_\alpha\}$ in $A_0[\| \cdot \|_0]$ such that $x_\alpha \to 0$ in $\tau$, $x_\alpha \to 0$ in $\| \cdot \|_0$ [8]. The completion of $A_0$ with respect to $\tau$ will be denoted by $\tilde{A}_0[\tau]$. In the sequel, we shall call a directed family of seminorms that defines a locally convex topology $\tau$, a defining family of seminorms.

A partial $*$-algebra is a vector space $A$ equipped with a vector space involution $*: A \to A : x \mapsto x^*$ and a partial multiplication defined on a set $\Gamma \subset A \times A$ such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ and $\lambda, \mu \in \mathbb{C}$ imply $(x, \lambda y_1 + \mu y_2) \in \Gamma$;

(iii) for every $(x, y) \in \Gamma$, a product $xy \in A$ is defined, such that $xy$ depends linearly on $x$ and $y$ and satisfies the equality $(xy)^* = y^*x^*$.

Given a pair $(x, y) \in \Gamma$, we say that $x$ is a left multiplier of $y$ and $y$ is a right multiplier of $x$.

Quasi $*$-algebras are essential examples of partial $*$-algebras. If $A$ is a vector space and $A_0$ a subspace of $A$, which is also a $*$-algebra, then $A$ is said to be a quasi $*$-algebra over $A_0$ whenever:

(i) The multiplication of $A_0$ is extended on $A$ as follows: The correspondences $A \times A_0 \to A : (a, x) \mapsto ax$ (left multiplication of $x$ by $a$) and $A_0 \times A \to A : (x, a) \mapsto xa$ (right multiplication of $x$ by $a$) are always defined and are bilinear;

(ii) $x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$, for all $x_1, x_2 \in A_0$ and $a \in A$;

(iii) the involution $*$ of $A_0$ is extended on $A$, denoted also by $*$, such that $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$, for all $a \in A_0$ and $a \in A$.

For further information cf. [3]. If $A_0[\tau]$ is a locally convex $*$-algebra, with separately continuous multiplication, its completion $\tilde{A}_0[\tau]$ is a quasi $*$-algebra over $A_0$ with respect to the operations:

- $ax := \lim_{a} x_\alpha a$, (left multiplication), $x \in A_0, a \in \tilde{A}_0[\tau]$,
- $xa := \lim_{a} x a_\alpha$, (right multiplication), $x \in A_0, a \in \tilde{A}_0[\tau]$,

where $\{x_\alpha\}_{a \in \Sigma}$ is a net in $A_0$ such that $a = \tau$-lim $x_\alpha$.

- An involution on $\tilde{A}_0[\tau]$ like in (iii) is the continuous extension of the involution on $A_0$.

A $*$-invariant subspace $A$ of $\tilde{A}_0[\tau]$ containing $A_0$ is called a quasi $*$-subalgebra of $\tilde{A}_0[\tau]$ if $ax, xa$ belong to $A$ for any $x \in A_0, a \in A$. One easily shows that $A$ is a quasi $*$-algebra over $A_0$. Moreover, $A[\tau]$ is a locally convex space that contains $A_0$ as a dense subspace and for every fixed $x \in A_0$, the maps $A[\tau] \to A[\tau]$ with $a \mapsto ax$ and $a \mapsto xa$ are continuous. An algebra of this kind is called locally convex quasi $*$-algebra over $A_0$. 
We denote by $\mathcal{L}^1(D, \mathcal{H})$ the set of all (closable) linear operators $X$ such that $D(X) = D$, $D(X^*) \supseteq D$. The set $\mathcal{L}^1(D, \mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication $\lambda X$, the involution $X \mapsto X^\dagger = X^* \vert D$ and the (weak) partial multiplication $X_1 \vartriangleright X_2 = X_1^\dagger X_2$, defined whenever $X_2$ is a weak right multiplier of $X_1$ (we shall write $X_2 \in R^w(X_1)$ or $X_1 \in L^w(X_2)$), that is, iff $X_2 D \subset D(X_1^\dagger)$ and $X_1^* D \subset D(X_2^*)$. $\mathcal{L}^1(D, \mathcal{H})$ is neither associative nor semiassociative.

**Definition 2.1.** Let $D$ be a dense subspace of a Hilbert space $\mathcal{H}$. A *-representation $\pi$ of $\mathcal{A}[\tau]$ is a linear map from $\mathcal{A}$ into $\mathcal{L}^1(D, \mathcal{H})$ (see beginning of Section 4) with the following properties:

(i) $\pi$ is a $*$-representation of $\mathcal{A}_0$;
(ii) $\pi(a) \vartriangleright = \pi(a^*)$, $\forall a \in \mathcal{A}$;
(iii) $\pi(ax) = \pi(a) \vartriangleright \pi(x)$ and $\pi(xa) = \pi(x) \vartriangleright \pi(a)$, $\forall a \in \mathcal{A}$ and $x \in \mathcal{A}_0$, where $\vartriangleright$ is the (weak) partial multiplication of $\mathcal{L}^1(D, \mathcal{H})$ (ibid.) Having a $*$-representation $\pi$ as before, we write $D(\pi)$ in the place of $D$ and $\mathcal{H}_\pi$ in the place of $\mathcal{H}$. By a $(\tau, \tau_{s^*})$-continuous $*$-representation $\pi$ of $\mathcal{A}[\tau]$, we clearly mean continuity of $\pi$, when $\mathcal{L}^1(D(\tau), \mathcal{H}_\pi)$ carries the locally convex topology $\tau_{s^*}$ (see Section 4).

In what follows, we shall need the concept of a $GB^*$-algebra introduced by G.R. Allan [2] (see also [12]), which we remind here. Let $\mathcal{A}[\tau]$ be a locally convex $*$-algebra with identity 1 and let $B^*$ denote the collection of all closed, bounded, absolutely convex subsets $B$ of $\mathcal{A}[\tau]$ with the properties: $1 \in B$, $B^* = B$ and $B^2 \subset B$. For each $B \in B^*$, the linear span $A[B]$ of $B$ is a normed $*$-algebra under the Minkowski functional $\| \cdot \|_B$ of $B$. When $A[B]$ is complete for each $B \in B^*$, then $\mathcal{A}[\tau]$ is called pseudo-complete. Every unital sequentially complete locally convex $*$-algebra is pseudo-complete [1, Proposition (2.6)]. A unital locally convex $*$-algebra $\mathcal{A}[\tau]$ is called symmetric (resp. algebraically symmetric) if for every $x \in \mathcal{A}$ the element $1 + x^* x$ has an Allan-bounded inverse in $\mathcal{A}$ [2, pp. 91,93] (resp. if $(1 + x^* x)$ has inverse in $\mathcal{A}$). A unital symmetric pseudo-complete locally convex $*$-algebra $\mathcal{A}[\tau]$, such that $B^*$ has a greatest member, say $B_0$, is said to be a $GB^*$-algebra over $B_0$. In this case, $A[B_0]$ is a $C^*$-algebra.

### 3. $C^*$-normed algebras with regular locally convex topology

Let $A_0[\| \cdot \|_0]$ be a $C^*$-normed algebra and $\widetilde{A}_0[\| \cdot \|_0]$ the $C^*$-algebra completion of $A_0[\| \cdot \|_0]$. Consider a locally convex topology $\tau$ on $A_0$ with the following properties:

(T1) $A_0[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication.
(T2) $\tau \leq \| \cdot \|_0$, with $\tau$ and $\| \cdot \|_0$ being compatible.
Then, compatibility of $\tau$, $\| \cdot \|_0$ implies that:

- $A_0[\| \cdot \|_0] \hookrightarrow \tilde{A}_0[\| \cdot \|_0] \hookrightarrow \tilde{A}_0[\tau]$;

- $\tilde{A}_0[\tau]$ is a locally convex quasi $*$-algebra over the $C^*$-normed algebra $A_0[\| \cdot \|_0]$, but it is not necessarily a locally convex quasi $*$-algebra over the $C^*$-algebra $\tilde{A}_0[\| \cdot \|_0]$, since $\tilde{A}_0[\| \cdot \|_0]$ is not a locally convex $*$-algebra under the topology $\tau$.

**Question.** Under which conditions one could have a well-defined multiplication of elements in $\tilde{A}_0[\tau]$ with elements in $\tilde{A}_0[\| \cdot \|_0]$?

We consider the case that the locally convex topology $\tau$ defined by a directed family of seminorms, say $(p_\lambda)_{\lambda \in \Lambda}$, satisfies in addition to the conditions (T) and (T) an extra “good” condition for the $C^*$-norm $\| \cdot \|_0$, called regularity condition, denoted by (R).

That is,

(R) $\forall \lambda \in \Lambda, \exists \lambda' \in \Lambda$ and $\gamma_\lambda > 0 : p_\lambda(xy) \leq \gamma_\lambda \|x\|_0 p_{\lambda'}(y), \forall x, y \in A_0[\| \cdot \|_0]$.

In this regard, we have the following

**Lemma 3.1.** Suppose $A_0[\| \cdot \|_0]$ is a $C^*$-normed algebra and $\tau$ a locally convex topology on $A_0$ satisfying the conditions (T) and the regularity condition (R) for $\| \cdot \|_0$. Let $a$ be an arbitrary element in $\tilde{A}_0[\tau]$ and $y$ an arbitrary element in $\tilde{A}_0[\| \cdot \|_0]$. Then, the left resp. right multiplication of $a$ with $y$ is defined by

$$a \cdot y = \tau - \lim_{\alpha, n} x_\alpha y_n \quad \text{resp.} \quad y \cdot a = \tau - \lim_{\alpha, n} y_n x_\alpha,$$

where $\{x_\alpha\}_{\alpha \in \Sigma}$ is a net in $A_0[\tau]$ converging to $a$, $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in $A_0[\| \cdot \|_0]$ converging to $y$ and $\forall \lambda \in \Lambda, \exists \lambda' \in \Lambda$ and $\gamma_\lambda > 0$:

$$p_\lambda(a \cdot y) \leq \gamma_\lambda \|y\|_0 p_{\lambda'}(a), p_\lambda(y \cdot a) \leq \gamma_\lambda \|y\|_0 p_{\lambda'}(a).$$

Under this multiplication $\tilde{A}_0[\tau]$ is a locally convex quasi $*$-algebra over the $C^*$-algebra $\tilde{A}_0[\| \cdot \|_0]$.

The proof of Lemma 3.1 follows directly from the regularity condition (R). If $A_0[\tau]$ is a locally convex $*$-algebra with jointly continuous multiplication and $\tau \preceq \| \cdot \|_0$, then it satisfies the regular condition (R) for $\| \cdot \|_0$.

**Lemma 3.2.** Let $A_0[\| \cdot \|_0]$ be a $C^*$-normed algebra and $A_0[\tau]$ an $m^*$-convex algebra satisfying conditions (T) and (R). If $(p_\lambda)_{\lambda \in \Lambda}$ is a defining family of $m^*$-seminorms for $\tau$ (i.e., submultiplicative $*$-preserving seminorms) and there is $\lambda_0 \in \Lambda$ such that $p_{\lambda_0}$ is a norm, then $\tau \sim \| \cdot \|_0$, where $\sim$ means equivalence of the respective topologies. In particular, if $A_0[\| \cdot \|]$ is a normed $*$-algebra such that $\| \cdot \| \leq \| \cdot \|_0$ and $\| \cdot \|, \| \cdot \|_0$ are compatible, then $\| \cdot \| \sim \| \cdot \|_0$. 

By (T2) and (R) we have $\mathcal{A}_0[\| \cdot \|_0] \hookrightarrow \mathcal{A}_0[\tau] \hookrightarrow \mathcal{A}_0[\rho_{\lambda_0}]$, which by the basic theory of $C^*$-algebras (see e.g., [18, Proposition 5.3]) implies that $\|x\|_0 \leq \rho_{\lambda_0}(x)$, for all $x \in \mathcal{A}_0$. Hence, $\tau \sim \| \cdot \|_0$.

By Lemma 3.2 there does not exist any normed $*$-algebra containing the $C^*$-algebra $\mathcal{A}_0[\| \cdot \|_0]$ properly and densely.

We now consider whether a $GB^*$-algebra over the unit ball $\mathcal{U}(\mathcal{A}_0[\| \cdot \|_0])$ exists in $\mathcal{A}_0[\tau]$. If $\mathcal{A}_0[\tau]$ has jointly continuous multiplication and $\mathcal{U}(\mathcal{A}_0[\| \cdot \|_0])$ is $\tau$-closed in $\mathcal{A}_0[\tau]$ then $\mathcal{A}_0[\tau]$ is a $GB^*$-algebra over $\mathcal{U}(\mathcal{A}_0[\| \cdot \|_0])$, (cf. [13, Theorem 2.1]).

**Theorem 3.3.** Let $A_0[\| \cdot \|_0]$ be a unital $C^*$-normed algebra and $\mathcal{A}_0[\tau]$ a locally convex $*$-algebra such that $\tau$ satisfies the conditions (T1),(T2), the regularity condition (R) for $\| \cdot \|_0$ and makes the unit ball $\mathcal{U}(\mathcal{A}_0[\| \cdot \|_0])$ $\tau$-closed. Then every algebraically symmetric locally convex $*$-algebra $\mathcal{A}[\tau]$ such that $\mathcal{A}_0[\| \cdot \|_0] \subset \mathcal{A}[\tau] \subset \mathcal{A}_0[\tau]$ is a $GB^*$-algebra over $\mathcal{U}(\mathcal{A}_0[\| \cdot \|_0])$.

**Proof.** The proof can be done in a similar way to that of [7, Theorem 2.2]. Here we give a simpler proof. Without loss of generality we may assume that $\mathcal{A}_0[\| \cdot \|_0]$ is a $C^*$-algebra. Then we have (see, e.g., proof of [7, Lemma 2.1]):

1. $(1 + a^*a)^{-1} \in \mathcal{U}(\mathcal{A}_0)$, $\forall a \in \mathcal{A}$.

Moreover, we show that

2. $\mathcal{U}(\mathcal{A}_0)$ is the largest member in $\mathcal{B}^*(\mathcal{A})$.

It is clear that $\mathcal{U}(\mathcal{A}_0) \in \mathcal{B}^*(\mathcal{A})$. Suppose now that $B$ is an arbitrary element in $\mathcal{B}^*(\mathcal{A})$ and take $a = a^*$ in $B$. Let $\mathcal{C}(a)$ be the maximal commutative $*$-subalgebra of $\mathcal{A}$ containing $a$ and

$$C_1 \equiv (\mathcal{U}(\mathcal{A}_0) \cap \mathcal{C}(a)) \cdot (B \cap \mathcal{C}(a)).$$

Then, clearly $C_1^* = C_1$; by the regular condition (R) $C_1$ is $\tau$-bounded in $\mathcal{C}(a)$, while by the commutativity of $\mathcal{U}(\mathcal{A}_0) \cap \mathcal{C}(a)$ and $B \cap \mathcal{C}(a)$ one has that $C_1^2 \subset C_1$. It is now easily seen that $\overline{C_1^*} \subset B^*(\mathcal{C}(a))$, where $B^*(\mathcal{C}(a)) = \{B \cap \mathcal{C}(a) : B \in B^*(\mathcal{A})\}$. Thus, there is $B_1 \in B^*(\mathcal{A})$ such that $\overline{C_1^*} = B_1 \cap \mathcal{C}(a)$.

Since $\mathcal{C}(a)$ is commutative and pseudo-complete, $B^*(\mathcal{C}(a))$ is directed [1, Theorem (2.10)]. So for each $B \in B^*(\mathcal{A})$ there is $B_1 \in B^*(\mathcal{A})$ such that

$$(B \cup \mathcal{U}(\mathcal{A}_0)) \cap \mathcal{C}(a) \subset B_1 \cap \mathcal{C}(a).$$

Hence $\mathcal{A}_0 \cap \mathcal{C}(a) \subset A[B_1] \cap \mathcal{C}(a)$, where $\mathcal{A}_0 \cap \mathcal{C}(a)$ is a $C^*$-algebra and $A[B_1] \cap \mathcal{C}(a)$ a normed $*$-algebra. An application of Lemma 3.2 gives

$$\|x\|_0 = \|x\|_{B_1}, \forall x \in \mathcal{A}_0 \cap \mathcal{C}(a).$$
Furthermore, it follows from (1) that \( x(I + \frac{1}{n}x^*x)^{-1} \in \mathcal{A}_0 \). Thus,

\[
\|x(I + \frac{1}{n}x^*x)^{-1} - x\|_{B_1} \leq \frac{1}{n}\|xx^*x\|_{B_1}, \quad \forall x \in A[B_1] \cap C(a), n \in \mathbb{N},
\]

which implies that \( \mathcal{A}_0 \cap C(a) = \| \cdot \|_{B_1}\text{-dense in } A[B_1] \cap C(a) \). Therefore, from (3.1) and the fact that \( \mathcal{A}_0 \cap C(a) \) is a \( C^* \)-algebra we get \( \mathcal{A}_0 \cap C(a) = A[B_1] \cap C(a) \). It follows that \( B \cap C(a) \subseteq B_1 \cap C(a) = \mathcal{U}(\mathcal{A}_0) \cap C(a) \), from which we conclude

\[
(3.2) \quad a \in \mathcal{U}(\mathcal{A}_0), \quad \forall a \in B, \quad \text{with } a^* = a.
\]

Now taking an arbitrary \( a \in B \) we clearly have \( a^*a \in B \), hence from (3.2) \( a^*a \in \mathcal{U}(\mathcal{A}_0) \), which gives \( a \in \mathcal{U}(\mathcal{A}_0) \). So, \( B \subset \mathcal{U}(\mathcal{A}_0) \) and the proof of (2) is complete. Now, since \( \mathcal{U}(\mathcal{A}_0) \) is the greatest member in \( B^*(\mathcal{A}) \), we have that \( A[\mathcal{U}(\mathcal{A}_0)] \) coincides with the \( C^* \)-algebra \( \mathcal{A}_0 \), therefore it is complete. So [1, Proposition 2.7] implies that \( \mathcal{A}[\tau] \) is pseudo-complete, hence a \( GB^* \)-algebra over \( \mathcal{U}(\mathcal{A}_0) \).

4. Locally convex quasi \( C^* \)-normed algebras

Let \( \mathcal{A}_0[\| \cdot \|_0] \) be a \( C^* \)-normed algebra and \( \tau \) a locally convex topology on \( \mathcal{A}_0 \) with \( \{p_\lambda\}_{\lambda \in \Lambda} \) a defining family of seminorms. Suppose that \( \tau \) satisfies the properties (T1), (T2). The regularity condition (R), considered in the previous Section 2, for \( \| \cdot \|_0 \), is too strong (see Section 6). So in the present Section we weaken this condition, and we use it together with the conditions (T1), (T2), in order to investigate the locally convex quasi \( * \)-algebra \( \widetilde{\mathcal{A}_0}[\tau] \). The weakened condition (R) will be denoted by (T3) and it will read as follows:

\[
(\text{T3}) \quad \forall \lambda \in \Lambda, \ \exists \lambda' \in \Lambda \text{ and } \gamma_\lambda > 0 : p_\lambda(xy) \leq \gamma_\lambda\|x\|_0p_{\lambda'}(y), \text{ for all } x, y \in \mathcal{A}_0 \text{ with } xy = yx.
\]

Then, we first consider the question stated in Section 3, just before Lemma 3.1, concerning a well-defined multiplication between elements of \( \widetilde{\mathcal{A}_0}[\tau] \) and \( \widetilde{\mathcal{A}_0}[\| \cdot \|_0] \).

If \( \mathcal{A}_0[\| \cdot \|_0] \) is commutative and \( \tau \) satisfaction the conditions (T1) - (T3), then \( \tau \) fulfills clearly the regularity condition (R) for \( \| \cdot \|_0 \), and so by Lemma 3.1, for arbitrary \( a \in \widetilde{\mathcal{A}_0}[\tau] \) and \( y \in \widetilde{\mathcal{A}_0}[\| \cdot \|_0] \) the left and right multiplications \( a \cdot y \) and \( y \cdot a \) are defined, respectively, and \( \widetilde{\mathcal{A}_0}[\tau] \) is a locally convex quasi \( * \)-algebra over the \( C^* \)-algebra \( \widetilde{\mathcal{A}_0}[\| \cdot \|_0] \).

We consider now the afore-mentioned question in the noncommutative case; for this we set the following

Definition 4.1. Let \( a \in \widetilde{\mathcal{A}_0}[\tau] \) and \( y \in \widetilde{\mathcal{A}_0}[\| \cdot \|_0] \). We shall say that \( y \) commutes strongly with \( a \) if there is a net \( \{x_\alpha\}_{\alpha \in \Sigma} \in \mathcal{A}_0[\| \cdot \|_0] \) such that \( x_\alpha \rightarrow_{\tau} a \) and \( x_\alpha y = yx_\alpha \), for every \( \alpha \in \Sigma \).
In the rest of the paper, \( \widetilde{A}_0[|| \cdot ||_0] \), denotes the completion of the \( C^\ast \)-algebra \( \widetilde{A}_0[|| \cdot ||_0] \) with respect to the locally convex topology \( \tau \). As a set it clearly coincides with \( \widetilde{A}_0[\tau] \), but there are cases that we need to distinguish them (see Remark 4.6).

**Remark 4.2.** Let \( a \in \widetilde{A}_0[\tau] \) and \( y \in \widetilde{A}_0[|| \cdot ||_0] \). Whenever \( y \in A_0 \), the multiplications \( ay \) and \( ya \) are always defined by

\[
ay = \lim_{\alpha} x_\alpha y \quad \text{and} \quad ya = \lim_{\alpha} yx_\alpha,
\]

where \( \{ x_\alpha \}_{\alpha \in \Sigma} \) is a net in \( A_0 \) converging to \( a \) with respect to \( \tau \). Hence, we may define the notion \( y \) commutes with \( a \), as usually, i.e., when \( ay = ya \). But, even if \( y \) commutes with \( a \), one has, in general, that \( y \) does not commute strongly with \( a \). Thus, the notion of strong commutativity is clearly stronger than that of commutativity.

**Lemma 4.3.** Let \( A_0[|| \cdot ||_0] \) be a \( C^\ast \)-normed algebra and \( \tau \) a locally convex topology on \( A_0 \) that satisfies the properties \( (T_1) - (T_3) \). Let \( a \in \widetilde{A}_0[\tau] \) and \( y \in \widetilde{A}_0[|| \cdot ||_0] \) be strongly commuting. Then the multiplications \( a \cdot y \) resp. \( y \cdot a \) are defined by

\[
a \cdot y = \tau - \lim_{\alpha} x_\alpha y \quad \text{resp.} \quad y \cdot a = \tau - \lim_{\alpha} yx_\alpha \quad \text{and} \quad a \cdot y = y \cdot a,
\]

where \( \{ x_\alpha \}_{\alpha \in \Sigma} \) is a net in \( \widetilde{A}_0[|| \cdot ||_0] \), \( \tau \)-converging to \( a \) and commutating with \( y \). The preceding multiplications provide an extension of the multiplication of \( A_0 \). Moreover, an analogous condition to \( (T_3) \) holds for the elements \( a, y \), i.e.,

\[
(T'_3) \; \forall \lambda \in \Lambda \exists \lambda' \in \Lambda \quad \text{and} \quad \gamma_\lambda > 0 : p_\lambda(a \cdot y) \leq \gamma_\lambda ||y||_0 p_\lambda(a).
\]

**Proof.** Existence of the \( \tau - \lim_{\alpha} x_\alpha y \) in \( \widetilde{A}_0[\tau] \):

Note that \( \{ x_\alpha y \}_{\alpha \in \Sigma} \) is a \( \tau \)-Cauchy net in \( \widetilde{A}_0[|| \cdot ||_0] \). Indeed, from \( (T_3) \), for every \( \lambda \in \Lambda \), there are \( \lambda' \in \Lambda \) and \( \gamma_\lambda > 0 \) such that

\[
p_\lambda(x_\alpha y - x_\alpha' y) = p_\lambda((x_\alpha - x_\alpha') y) \leq \gamma_\lambda ||y||_0 p_\lambda(x_\alpha - x_\alpha') \longrightarrow 0.
\]

Hence, \( \tau - \lim_{\alpha} x_\alpha y \) exists in \( \widetilde{A}_0[|| \cdot ||_0] \), which, as already noticed, as a set clearly coincides with \( \widetilde{A}_0[\tau] \).

The existence of the \( \tau - \lim_{\alpha} yx_\alpha \) in \( \widetilde{A}_0[|| \cdot ||_0] \) is similarly shown and clearly \( \tau - \lim_{\alpha} yx_\alpha = \tau - \lim_{\alpha} x_\alpha y \).

Independence of \( \tau - \lim_{\alpha} x_\alpha y \) from the choice of the net \( \{ x_\alpha \}_{\alpha \in \Sigma} \):

Let \( \{ x_\beta' \}_{\beta \in \Sigma'} \) be another net in \( A_0 \) such that \( x_\beta' \xrightarrow{\tau} a \) and \( x_\beta'y = yx_\beta' \), for all \( \beta \in \Sigma' \). Then,

\[
x_\alpha - x_\beta' \xrightarrow{\tau} 0 \quad \text{with} \quad (x_\alpha - x_\beta')y = y(x_\alpha - x_\beta') \forall (\alpha, \beta) \in \Sigma \times \Sigma'.
\]
Moreover, by (T$_3$), for every $\lambda \in \Lambda$, there exist $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ such that
\[ p_\lambda ((x_\alpha - x'_\beta)y) \leq \gamma_\lambda \|y\|_0 p_\lambda(x_\alpha - x'_\beta) \to 0; \]
this completes the proof of our claim. Thus, we set
\[ a \cdot y := \tau - \lim x_\alpha y, \text{ resp. } y \cdot a := \tau - \lim y x_\alpha; \]
this clearly implies $a \cdot y = y \cdot a$. Furthermore, using again (T$_3$) we conclude that
\[ \forall \lambda \in \Lambda \exists \lambda' \in \Lambda \text{ and } \gamma_\lambda > 0 : p_\lambda(a \cdot y) \leq \gamma_\lambda \|y\|_0 p_\lambda(a), \forall a \in \widetilde{A}_0[\tau] \text{ and } y \in \widetilde{A}_0[\| \cdot \|_0], \]
and this proves (T$_3'$).

Now, following [8] we define notions of positivity for the elements of $\widetilde{A}_0[\tau]$.

**Definition 4.4.** Let $a \in \widetilde{A}_0[\tau]$. Consider the set
\[ (A_0)_+ := \{ x \in A_0 : x^* = x \text{ and } sp_{A_0}(x) \subseteq [0, \infty) \}, \]
where $sp_{A_0}(x)$ means spectrum of $x$ in $A_0$. Clearly $(A_0)_+$ is contained in the positive cone of the C*-algebra $\widetilde{A}_0[\| \cdot \|_0]$. The element $a$ is called *quasi-positive* if there is a net $\{x_\alpha\}_{\alpha \in \Sigma}$ in $(A_0)_+$ such that $x_\alpha \to a$. In particular, $a$ is called *commutatively quasi-positive* if there is a commuting net $\{x_\alpha\}_{\alpha \in \Sigma}$ in $(A_0)_+$ such that $x_\alpha \to a$.

Denote by $\widetilde{A}_0[\tau]_{q+}$ the set of all quasi-positive elements of $\widetilde{A}_0[\tau]$ and by $\widetilde{A}_0[\tau]_{cq+}$ the set of all commutatively quasi-positive elements of $\widetilde{A}_0[\tau]$.

An easy consequence of Definition 4.4 is the following

**Lemma 4.5.**

\[ (A_0)_+ \cap (\widetilde{A}_0[\| \cdot \|_0]_+) \subseteq \widetilde{A}_0[\tau]_{cq+} \cap \widetilde{A}_0[\| \cdot \|_0]_{q+}. \]

(2) $\widetilde{A}_0[\tau]_{q+}$ is a positive wedge, but it is not necessarily a positive cone. $\widetilde{A}_0[\tau]_{cq+}$ is not even a positive wedge, in general.

**Remark 4.6.** As we have mentioned before, the equality $\widetilde{A}_0[\| \cdot \|_0]_{q+} = \widetilde{A}_0[\tau]$ holds set-theoretically. We consider the following notation:
\[ \widetilde{A}_0[\| \cdot \|_0]_{q+} \equiv \{ a \in \widetilde{A}_0[\tau] : \exists a \text{ net } \{x_\alpha\}_{\alpha \in \Sigma} \text{ in } \widetilde{A}_0[\| \cdot \|_0]_+ : x_\alpha \to a \}, \]
\[ \widetilde{A}_0[\| \cdot \|_0]_{cq+} \equiv \{ a \in \widetilde{A}_0[\tau] : \exists \text{ a commuting net } \{x_\alpha\}_{\alpha \in \Sigma} \text{ in } \widetilde{A}_0[\| \cdot \|_0]_+ : x_\alpha \to a \}. \]
Then,

\[(4.1) \quad \bar{A}_0[\| \cdot \|_0^\sim][\tau]_{q+} = \bar{A}_0[\tau]_{q+}, \quad \text{but} \quad \bar{A}_0[\| \cdot \|_0^\sim][\tau]_{cq+} \supset \neq \bar{A}_0[\tau]_{cq+}, \quad \text{in general.}\]

If \(A_0\) is commutative, then

\[\bar{A}_0[\tau]_{cq+} = \bar{A}_0[\| \cdot \|_0^\sim][\tau]_{cq+} = \bar{A}_0[\| \cdot \|_0^\sim][\tau]_{q+} = \bar{A}_0[\tau]_{q+}.\]

The following Proposition 4.7 plays an important role in the present paper. It is a generalization of Proposition 3.2 in [8], stated for locally convex quasi \(C^*\)-algebras, to the case of locally convex quasi \(C^*\)-normed algebras.

**Proposition 4.7.** Let \(A_0[\| \cdot \|_0]\) be a unital \(C^*\)-normed algebra and \(\tau\) a locally convex topology on \(A_0\) that fulfills the conditions (T1) - (T3). Suppose that the next condition (T4) holds:

\[(T4) \quad \text{The set } \mathcal{U}(\bar{A}_0[\| \cdot \|_0]+) \equiv \{ x \in \bar{A}_0[\| \cdot \|_0]+ : \|x\|_0 \leq 1 \} \text{ is } \tau\text{-closed in } \bar{A}_0[\tau].\]

Then, \(\bar{A}_0[\tau]\) is a locally convex quasi \(*\)-algebra over \(A_0\) with the properties:

1. \(a \in \bar{A}_0[\tau]_{cq+}\) implies that \(1 + a\) is invertible with \((1 + a)^{-1}\) in \(\mathcal{U}(\bar{A}_0[\| \cdot \|_0])_+\).

2. For \(a \in \bar{A}_0[\tau]_{cq+}\) and \(\varepsilon > 0\), the element \(a_\varepsilon := a \cdot (1 + \varepsilon a)^{-1}\) is well-defined, \(a - a_\varepsilon \in \bar{A}_0[\| \cdot \|_0]_{cq+}\) and \(a = \tau - \lim_{\varepsilon \downarrow 0} a_\varepsilon\).

3. \(\bar{A}_0[\tau]_{cq+} \cap (-\bar{A}_0[\tau]_{cq+}) = \{0\}\).

4. Furthermore, suppose that the following condition

\[(T_5) \quad \bar{A}_0[\tau]_{q+} \cap \bar{A}_0[\| \cdot \|_0] = \bar{A}_0[\| \cdot \|_0]+\]

is satisfied. Then, if \(a \in \bar{A}_0[\tau]_{cq+}\) and \(y \in \bar{A}_0[\| \cdot \|_0]_+\) with \(y - a \in \bar{A}_0[\tau]_{q+}\), one has that \(a \in \bar{A}_0[\| \cdot \|_0]+\).

**Proof.** (1) There exists a commuting net \(\{x_\alpha\}_{\alpha \in \Sigma}\) in \((A_0)_+\) with \(x_\alpha \xrightarrow{\tau} a\) and \(x_\alpha x_{\alpha'} = x_{\alpha'}x_\alpha\), for all \(\alpha, \alpha' \in \Sigma\). Using properties of the positive elements in a \(C^*\)-algebra, and condition (T3) we get that for every \(\lambda \in A\) there are \(\lambda' \in A\) and \(\gamma_\lambda > 0\) such that:

\[p_\lambda((1 + x_\alpha)^{-1} - (1 + x_{\alpha'})^{-1}) = p_\lambda((1 + x_\alpha)^{-1}(x_{\alpha'} - x_\alpha)(1 + x_{\alpha'})^{-1}) \leq \gamma_\lambda\|((1 + x_\alpha)^{-1}||_0((1 + x_{\alpha'})^{-1}) a_{\alpha,\alpha'} - x_{\alpha} \xrightarrow{\alpha,\alpha'} 0.\]

Hence \(\{(1 + x_\alpha)^{-1}\}_{\alpha \in \Sigma}\) is a Cauchy net in \(\bar{A}_0[\tau]\) consisting of elements of \(\mathcal{U}(\bar{A}_0[\| \cdot \|_0])_+\), the latter set being \(\tau\)-closed by (T4). Hence, there exists \(y \in \mathcal{U}(\bar{A}_0[\| \cdot \|_0])_+\) such that

\[(4.2) \quad (1 + x_\alpha)^{-1} \xrightarrow{\tau} y.\]

We shall show that \((1 + a)^{-1}\) exists in \(\mathcal{U}(\bar{A}_0[\| \cdot \|_0])_+\) and coincides with \(y\). It is easily seen that, for each index \(\alpha \in \Sigma\), \((1 + x_\alpha)^{-1}\) commutes strongly with \((1 + a)\), so that
\((I + a) \cdot (I + x_a)^{-1}\) is well-defined (Lemma 4.3). Similarly, \((x_a - a) \cdot (I + x_a)^{-1} = I - (I + a) \cdot (I + x_a)^{-1}\) is well-defined, therefore using \((T'_3)\) of Lemma 4.3, we have that for all \(\lambda \in \Lambda\) there are \(\lambda' \in \Lambda\) and \(\gamma_\lambda > 0\) with

\[
p_\lambda(I - (I + a) \cdot (I + x_a)^{-1}) = p_\lambda((x_a - a) \cdot (I + x_a)^{-1}) \leq \gamma_\lambda p_{\lambda'}(x_a - a) \rightarrow 0.
\]

Thus, \((I + a) \cdot (I + x_a)^{-1} \rightarrow I\). By the above,

\[
I + x_a \rightarrow I + a \quad \text{and} \quad (I + x_a)y = y(I + x_a), \forall a \in \Sigma.
\]

Hence, \(y\) commutes strongly with \(I + a\), therefore \((I + a) \cdot y\) is well-defined by Lemma 4.3. Now, since \(x_a \rightarrow a\), we have that

\[
\forall \lambda \in \Lambda \quad \forall \varepsilon > 0, \exists \alpha_0 \in \Sigma : p_\lambda(x_{a'} - a) < \varepsilon, \forall a' \geq \alpha_0.
\]

Using \((T_3), (T'_3)\) of Lemma 4.3, and relations (4.3), (4.2) we obtain

\[
p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + a) \cdot y) \\
\leq p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + x_{a_0})(I + x_a)^{-1}) \\
+ p_\lambda((I + x_{a_0})(I + x_a)^{-1} - (I + x_{a_0})y) + p_\lambda((I + x_{a_0})y - (I + a)y) \\
\leq \gamma_\lambda p_{\lambda'}(a - x_{a_0}) + \gamma_\lambda \|I + x_{a_0}\| p_{\lambda'}((I + x_a)^{-1} - y) + \gamma_\lambda p_{\lambda'}(x_{a_0} - a) \\
< 2\varepsilon + \gamma_\lambda \|I + x_{a_0}\| p_{\lambda'}((I + x_a)^{-1} - y), \quad \forall \varepsilon > 0.
\]

Hence,

\[
0 \leq \lim_{\alpha} p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + a) \cdot y) \leq 2\varepsilon, \quad \forall \varepsilon > 0,
\]

which implies

\[
\lim_{\alpha} p_\lambda((I + a) \cdot (I + x_a)^{-1} - (I + a) \cdot y) = 0.
\]

Consequently,

\[
(4.4) \quad (I + a) \cdot (I + x_a)^{-1} \rightarrow_{\tau} (I + a) \cdot y.
\]

Similarly, \((I + x_a)^{-1} \cdot (I + a) \rightarrow y \cdot (I + a)\). So from (4.3) and (4.4) we conclude that \((I + a) \cdot y = y \cdot (I + a) = 1\), therefore \(y = (I + a)^{-1}\).

(2) By (1), for every \(\varepsilon > 0\), the element \((I + \varepsilon a)^{-1}\) exists in \(\mathcal{U}((A_0[\|\cdot\|_0])_{\tau})\), and commutes strongly with \(a\). Hence (see Lemma 4.3), \(a_\varepsilon := a \cdot (I + \varepsilon a)^{-1}\) is well-defined. Moreover, applying \((T'_3)\) of Lemma 4.3, we have that for all \(\lambda \in \Lambda\), there exist \(\lambda' \in \Lambda\) and \(\gamma_\lambda > 0\) such that

\[
p_\lambda(I - (I + \varepsilon a)^{-1}) = \varepsilon p_\lambda(a \cdot (I + \varepsilon a)^{-1}) \leq \varepsilon \gamma_\lambda\|(I + \varepsilon a)^{-1}\| p_{\lambda'}(a) \leq \varepsilon \gamma_\lambda p_{\lambda'}(a).
\]
Therefore,
\[
\tau - \lim_{\varepsilon \downarrow 0} (1 + \varepsilon a)^{-1} = 1.
\]
On the other hand, since \((1 - (1 + \varepsilon a)^{-1})\) commutes strongly with \(a\) and \(a_\varepsilon = \varepsilon^{-1}(1 - (1 + \varepsilon a)^{-1})\), \(\varepsilon > 0\), we have
\[
(1 - (1 + \varepsilon a)^{-1}) \cdot a = a \cdot (1 - (1 + \varepsilon a)^{-1}) = a - a_\varepsilon \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0] \sim [\tau]_{cq+}.
\]

Using (4.5), (4.6) and same arguments as in (4.4), we get that
\[
\tau - \lim_{\varepsilon \downarrow 0} a_\varepsilon = a.
\]

(3) Let \(a \in \widetilde{\mathcal{A}}_0[\tau]_{cq+} \cap (-\widetilde{\mathcal{A}}_0[\tau]_{cq+})\) and \(\varepsilon > 0\) sufficiently small. By (2) (see also Remark 4.6), we have
\[
\widetilde{\mathcal{A}}_0[\| \cdot \|_0] \sim [\tau]_{cq+} \ni a \cdot (1 + \varepsilon a)^{-1} \underset{\tau}{\longrightarrow} a; \text{ in the same way } -a \cdot (1 - \varepsilon a)^{-1} \underset{\tau}{\longrightarrow} -a.
\]

Now the element
\[
x_\varepsilon \equiv a \cdot (1 + \varepsilon a)^{-1} - (-a) \cdot (1 - \varepsilon a)^{-1} = 2a \cdot (1 + \varepsilon a)^{-1}(1 - \varepsilon a)^{-1}
\]
belongs to \(\widetilde{\mathcal{A}}_0[\| \cdot \|_0]_+\) by (1) and the functional calculus of commutative \(C^*\)-algebras. Similarly,
\[
x_\varepsilon = 2(-a) \cdot (1 - \varepsilon a)^{-1}(1 + \varepsilon a)^{-1} \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0]_+.\text{ Hence,}
\]
\[
x_\varepsilon \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0]_+ \cap (-\widetilde{\mathcal{A}}_0[\| \cdot \|_0]_+) = \{0\}, \text{ so that } a \cdot (1 + \varepsilon a)^{-1} = -a \cdot (1 - \varepsilon a)^{-1}.
\]

Furthermore, by (2),
\[
a = \tau - \lim_{\varepsilon \downarrow 0} a \cdot (1 + \varepsilon a)^{-1} = \tau - \lim_{\varepsilon \downarrow 0} (-a) \cdot (1 - \varepsilon a)^{-1} = -a, \text{ so } a = 0.
\]

(4) Note that \(y - a_\varepsilon = (y - a) + (a - a_\varepsilon) \in \widetilde{\mathcal{A}}_0[\tau]_{q^+}\), since (by (4) and (2) resp.) the elements \(y - a, a - a_\varepsilon\) belong to \(\widetilde{\mathcal{A}}_0[\tau]_{q^+}\) and the latter set is a positive wedge according to Lemma 4.5(2). On the other hand,
\[
a_\varepsilon = a \cdot (1 + \varepsilon a)^{-1} = (1 + \varepsilon a)^{-1} \cdot a = \varepsilon^{-1}(1 - (1 + \varepsilon a)^{-1}) \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0].
\]

Thus, taking under consideration the assumption \((T_5)\) we conclude that
\[
y - a_\varepsilon \in \widetilde{\mathcal{A}}_0[\tau]_{q^+} \cap \widetilde{\mathcal{A}}_0[\| \cdot \|_0] = \widetilde{\mathcal{A}}_0[\| \cdot \|_0]_+,\text{ which clearly gives } \|a_\varepsilon\|_0 \leq \|y\|_0, \text{ for every } \varepsilon > 0. \text{ Applying } (T_4), \text{ we show that } a \in \widetilde{\mathcal{A}}_0[\| \cdot \|_0]_+. \tag*{\square}
\]

**Definition 4.8.** Let \(\mathcal{A}_0[\| \cdot \|_0]\) be a unital \(C^*\)-normed algebra, \(\tau\) a locally convex topology on \(\mathcal{A}_0\) satisfying the conditions \((T_1) - (T_5)\) (for \((T_4),(T_5)\) see the previous proposition). Then,
• A quasi *-subalgebra \( A \) of the locally convex quasi *-algebra \( \widetilde{A}_0[\tau] \) over \( A_0 \) containing \( \widetilde{A}_0[\| \cdot \|_0] \) is said to be a locally convex quasi \( C^* \)-normed algebra over \( A_0 \).

A locally convex quasi \( C^* \)-normed algebra \( A \) over \( A_0 \) is said to be normal if \( a \cdot y \in A \) whenever \( a \in A \) and \( y \in \widetilde{A}_0[\| \cdot \|_0] \) commute strongly.

A locally convex quasi \( C^* \)-normed algebra \( A \) over \( A_0 \) is called a locally convex quasi \( C^* \)-algebra if \( A_0[\| \cdot \|_0] \) is a \( C^* \)-algebra.

Note that the condition (\( T_3 \)) in the present paper is weaker than the condition

\[(T_3) \forall \lambda \in \Lambda, \exists \lambda' \in \Lambda : p_\lambda(xy) \leq \|x\|_0p_\lambda(y), \forall x, y \in A_0 \text{ with } xy = yx\]

in [8]. Nevertheless, results for locally convex quasi \( C^* \)-algebras in [8] are valid in the present paper for the wider class of locally convex \( C^* \)-normed algebras. It follows, by the very definitions, that a locally convex quasi \( C^* \)-algebra is a normal locally convex quasi \( C^* \)-normed algebra. A variety of examples of locally convex quasi \( C^* \)-algebras are given in [8], Sections 3 and 4. Examples of locally convex quasi \( C^* \)-normed algebras are presented in Sections 6 and 7.

An easy consequence of Definition 4.8 and Lemma 4.3 is the following

**Lemma 4.9.** Let \( A_0[\| \cdot \|_0] \) and \( \tau \) be as in Definition 4.8. Then the following hold:

1. \( \widetilde{A}_0[\tau] \) is a normal locally convex quasi \( C^* \)-normed algebra over \( A_0 \).

2. Suppose \( A \) is a commutative locally convex quasi \( C^* \)-normed algebra over \( A_0 \). Then \( A \cdot \widetilde{A}_0[\| \cdot \|_0] \equiv \text{linear span of } \{a \cdot y : a \in A, y \in \widetilde{A}_0[\| \cdot \|_0]\} \) is a commutative locally convex quasi \( C^* \)-algebra over \( \widetilde{A}_0[\| \cdot \|_0] \) under the multiplication \( a \cdot y \) \((a \in A, y \in \widetilde{A}_0[\| \cdot \|_0])\). In particular, if \( A \) is normal, then \( A \) is a commutative locally convex quasi \( C^* \)-algebra over \( \widetilde{A}_0[\| \cdot \|_0] \).

5. **Commutative locally convex quasi \( C^* \)-normed algebras**

In this Section, we discuss briefly some results on the structure of a commutative locally convex quasi \( C^* \)-normed algebra \( A[\tau] \) and on a functional calculus for its quasi-positive elements, that are similar to those in [8, Sections 5 and 6].

Let \( A[\tau] \) be a commutative locally convex quasi \( C^* \)-normed algebra over \( A_0 \) (see Definition 4.8). Then,

\[A_0[\| \cdot \|_0] \subset \widetilde{A}_0[\| \cdot \|_0] \subset A[\tau] \subset A[\tau] \cdot \widetilde{A}_0[\| \cdot \|_0] \subset \widetilde{A}_0[\tau],\]

where \( A_0[\| \cdot \|_0] \) is a commutative unital \( C^* \)-normed algebra and \( A[\tau] \cdot \widetilde{A}_0[\| \cdot \|_0] \) is a commutative locally convex quasi \( C^* \)-algebra over the unital \( C^* \)-algebra \( \widetilde{A}_0[\| \cdot \|_0] \) according to Lemma 4.9(2). Thus, using some results of the Sections 5, 6 in [8] for the latter algebra we obtain information for the structure of \( A[\tau] \).
Let $W$ be a compact Hausdorff space, $C^* = \mathbb{C} \cup \{\infty\}$, and $\mathfrak{F}(W)_+$ a set of $C^*$-valued positive continuous functions on $W$, which take the value $\infty$ on at most a nowhere dense subset $W_0$ of $W$. The set

$$\mathfrak{F}(W) \equiv \{fg_0 + h_0 : f \in \mathfrak{F}(W)_+ \text{ and } g_0, h_0 \in C(W)\},$$

where $C(W)$ is the $C^*$-algebra of all continuous $\mathbb{C}$-valued functions on $W$, is called the set of $C^*$-valued continuous functions on $W$ generated by the wedge $\mathfrak{F}(W)_+$ and the $C^*$-algebra $C(W)$. Using [8, Definition 5.6] and $\mathfrak{F}(W)$ we get the following theorem, which is an application of Theorem 5.8 of [8] for the commutative locally convex quasi $C^*$-algebra $A[\tau] \cdot \tilde{A}_0[\parallel \cdot \parallel_0]$ over the unital commutative $C^*$-algebra $\tilde{A}_0[\parallel \cdot \parallel_0]$, with $A[\tau]_{q^+} \cdot \tilde{A}_0[\parallel \cdot \parallel_0]$, in the place of $\mathcal{M}(A_0, A[\tau]_{q^+})$.

**Theorem 5.1.** There exists a map $\Phi$ from $A[\tau]_{q^+} \cdot \tilde{A}_0[\parallel \cdot \parallel_0]$ onto $\mathfrak{F}(W)$, where $W$ is the compact Hausdorff space corresponding to the Gel’fand space of the unital commutative $C^*$-algebra $\tilde{A}_0[\parallel \cdot \parallel_0]$, such that:

(i) $\Phi(A[\tau]_{q^+}) = \mathfrak{F}(W)_+$ and $\Phi(\lambda a + b) = \lambda \Phi(a) + \Phi(b)$, $\forall a, b \in A[\tau]_{q^+}$, $\lambda \geq 0$;

(ii) $\Phi$ is an isometric $*$-isomorphism from $A[\tau]_{q^+} \cdot \tilde{A}_0[\parallel \cdot \parallel_0]$ onto $C(W)$;

(iii) $\Phi(ax) = \Phi(a)\Phi(x)$, $\Phi((\lambda a + b)x) = (\lambda \Phi(a) + \Phi(b))\Phi(x)$ and $\Phi(a(x_1 + x_2)) = \Phi(a)(\Phi(x_1) + \Phi(x_2))$, $\forall a, b \in A[\tau]_{q^+}$, $x, x_1, x_2 \in A_0$ and $\lambda \geq 0$.

Further we consider a functional calculus for the quasi-positive elements of the commutative locally convex quasi $C^*$-normed algebra $A[\tau]$ over $A_0$. For this, we must extend the multiplication of $A[\tau]$.

Let $a, b \in A[\tau]_{q^+}$. Then (see also [8, Definition 6.1]), $a$ is called left multiplier of $b$ if there are nets $\{x_\alpha\}_{\alpha \in \Sigma}$, $\{y_\beta\}_{\beta \in \Sigma'}$ in $(A_0)_+$ such that $x_\alpha \rightarrow_{\tau} a$, $y_\beta \rightarrow_{\tau} b$ and $x_\alpha y_\beta \rightarrow_{\tau} c$, where the latter means that the double indexed net $\{x_\alpha y_\beta\}_{(\alpha, \beta) \in \Sigma \times \Sigma'}$ converges to $c \in A[\tau]$. Then, we set

$$a \cdot b := c - \lim_{\alpha, \beta} x_\alpha y_\beta,$$

where the multiplication $a \cdot b$ is well defined, in the sense that it is independent of the choice of the nets $\{x_\alpha\}_{\alpha \in \Sigma}$, $\{y_\beta\}_{\beta \in \Sigma'}$, as follows from the proof of Lemma 6.2 in [8] applying arguments of the proof of Proposition 4.7. In the sequel, we simply denote $a \cdot b$ by $ab$. In analogy to Definition 6.3 of [8], if $x, y \in \tilde{A}_0[\parallel \cdot \parallel_0]$ and $a, b \in A[\tau]_{q^+}$ with a left multiplier of $b$, we may define the product of the elements $ax$ and $by$ as follows:

$$(ax)(by) := (ab)xy.$$

The spectrum of an element $a \in A[\tau]_{q^+}$, denoted by $\sigma_{\tilde{A}_0[\parallel \cdot \parallel_0]}(a)$, is defined as in Definition 6.4 of [8].
So using Theorem 5.1, it is shown (cf., for instance, Lemma 6.5 in [8]) that for every \( a \in \mathcal{A}[\tau]_{q^+} \), one has that \( \sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a) \) is a locally compact subset of \( \mathbb{C}^* \) and \( \sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a) \subset \mathbb{R}_+ \cup \{ \infty \} \).

According to the above, and taking into account the comments after Lemma 6.5 in [8] with \( \mathcal{A}_0[\| \cdot \|_0] \) in the place of \( \mathcal{A}_0 \), the next Theorem 5.2 provides a generalization of [8, Theorem 6.6] in the setting of commutative locally convex quasi \( C^* \)-normed algebras. In particular, Theorem 5.2 supplies us with a functional calculus for the quasi-positive elements of the commutative locally convex quasi \( C^* \)-normed algebra \( \mathcal{A}[\tau] \).

**Theorem 5.2.** Let \( a \in \mathcal{A}[\tau]_{q^+} \). Let \( a^n \) be well-defined for some \( n \in \mathbb{N} \). Then there is a unique \(*\)-isomorphism \( f \rightarrow f(a) \) from \( \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \) into \( \mathcal{A}[\tau] \cdot \mathcal{A}_0[\| \cdot \|_0] \) such that:

(i) If \( u_0(\lambda) = 1 \), with \( u_0 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \) and \( \lambda \in \sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a) \), then \( u_0(a) = 1 \).

(ii) If \( u_1(\lambda) = \lambda \) with \( u_1 \in \bigcup_{k=1}^n \mathcal{C}_k(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \) and \( \lambda \in \sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a) \), then \( u_1(a) = a \).

(iii) \( (\lambda_1 f_1 + f_2)(a) = \lambda_1 f_1(a) + f_2(a) \), \( \forall f_1, f_2 \in \mathcal{C}_k(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \) and \( \lambda_1 \in \mathbb{C} \); \( (f_1 f_2)(a) = f_1(a) f_2(a) \), \( \forall f_j \in \mathcal{C}_k(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \), \( j = 1, 2 \), with \( k_1 + k_2 \leq n \).

(iv) Denoting with \( \mathcal{C}_b \) the set of the bounded and continuous functions, the map \( f \rightarrow f(a) \) restricted to \( \mathcal{C}_b(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \) is an isometric \(*\)-isomorphism of the \( C^* \)-algebra \( \mathcal{C}_b(\sigma_{\mathcal{A}_0[\| \cdot \|_0]}(a)) \) on the closed \(*\)-subalgebra of \( \mathcal{A}_0[\| \cdot \|_0] \) generated by \( 1 \) and \( (1 + a)^{-1} \).

Applying Theorem 5.2 and Proposition 4.7 in the proof of [8, Corollary 6.7] we get the following

**Corollary 5.3.** Let \( a \in \mathcal{A}[\tau]_{q^+} \) and \( n \in \mathbb{N} \). Then, there exists unique \( b \) in \( \mathcal{A}[\tau]_{q^+} \cdot \mathcal{A}_0[\| \cdot \|_0] \) such that \( a = b^n \). The unique element \( b \) is called quasi \( n \)-th root of \( a \) and we write \( b = a^{\frac{1}{n}} \).

### 6. Structure of noncommutative locally convex quasi \( C^* \)-normed algebras

Using the notation of [8, Section 4] (see also [3]), let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{D} \) a dense subspace of \( \mathcal{H} \) and \( \mathcal{M}_0[\| \cdot \|_0] \) a unital \( C^* \)-normed algebra on \( \mathcal{H} \), such that

\[
\mathcal{M}_0 \mathcal{D} \subset \mathcal{D}, \quad \text{but} \quad \mathcal{M}_0[\| \cdot \|_0] \mathcal{D} \nsubseteq \mathcal{D}.
\]

Then, the restriction \( \mathcal{M}_0 \mid \mathcal{D} \) of \( \mathcal{M}_0 \) to \( \mathcal{D} \) is an \( O^* \)-algebra on \( \mathcal{D} \), so that an element \( X \) of \( \mathcal{M}_0 \) may be regarded as an element \( X \mid \mathcal{D} \) of \( \mathcal{M}_0[\mathcal{D}] \). Moreover, let

\[
\mathcal{M}_0 \subset \mathcal{M} \subset \mathcal{L}(\mathcal{D}, \mathcal{H}),
\]
where \( \mathcal{M} \) is an \( O^* \)-vector space on \( \mathcal{D} \), that is, a \( * \)-invariant subspace of \( \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \). Denote by \( \mathcal{B}(\mathcal{M}) \) the set of all bounded subsets of \( \mathcal{D}[t_{\mathcal{M}}] \) (\( t_{\mathcal{M}} \) is the graph topology on \( \mathcal{M} \); see [14, p.9]) and by \( \mathcal{B}_f(\mathcal{D}) \) the set of all finite subsets of \( \mathcal{D} \). Then \( \mathcal{B}_f(\mathcal{D}) \subset \mathcal{B}(\mathcal{M}) \) and both of them are admissible in the sense of [8, p. 522].

We recall the topologies \( \tau_{s^*}, \tau_u^*(\mathcal{B}), \tau_u^*(\mathcal{M}) \) defined in [8, pp. 522-523]. More precisely, for an arbitrary admissible subset \( \mathcal{B} \) of \( \mathcal{B}(\mathcal{M}) \), and any \( \mathfrak{M} \in \mathcal{B} \) consider the following seminorm:

\[
p_{\mathfrak{M}}^s(X) := \sup_{\xi \in \mathfrak{M}} \|X\xi\| + \|X\xi\|, \quad X \in \mathcal{M}.
\]

We call the corresponding locally convex topology on \( \mathcal{M} \) induced by the preceding family of seminorms, strongly\(^*\) \( \mathcal{B} \)-uniform topology and denote it by \( \tau_u^*(\mathcal{B}) \). In particular, the strongly\(^*\) \( \mathcal{B}(\mathcal{M}) \)-uniform topology will be simply called strongly\(^*\) \( \mathcal{M} \)-uniform topology and will be denoted by \( \tau_u^*(\mathcal{M}) \). In Schmüdgen’s book [17], this topology is called bounded topology. The strongly\(^*\) \( \mathcal{B}_f(\mathcal{D}) \)-uniform topology is called strong\(^*\)-topology on \( \mathcal{M} \), denoted by \( \tau_{s^*} \). All three topologies are related in the following way:

\[
\tau_{s^*} \leq \tau_u^*(\mathcal{B}) \leq \tau_u^*(\mathcal{M}).
\]

Then, one gets that

\[
(6.1) \quad \mathcal{M}_0[\| \cdot \|_0] \subset \widetilde{\mathcal{M}}_0[\| \cdot \|_0] \subset \widetilde{\mathcal{M}}_0[\tau_u^*(\mathcal{B})] \subset \widetilde{\mathcal{M}}_0[\tau_{s^*}] \subset \mathcal{L}^1(\mathcal{D}, \mathcal{H}).
\]

In this regard, we have now the following

**Proposition 6.1.** Let \( \mathcal{M}_0[\| \cdot \|_0], \mathcal{M} \) be as before. Let \( \mathcal{B} \) be any admissible subset of \( \mathcal{B}(\mathcal{M}) \). Then \( \widetilde{\mathcal{M}}_0[\tau_u^*(\mathcal{B})] \) is a locally convex quasi \( C^* \)-normed algebra over \( \mathcal{M}_0 \), which is contained in \( \mathcal{L}^1(\mathcal{D}, \mathcal{H}) \). In particular, \( \widetilde{\mathcal{M}}_0[\tau_{s^*}] \) is a locally convex quasi \( C^* \)-normed algebra over \( \mathcal{M}_0 \). Furthermore, if \( A \in \widetilde{\mathcal{M}}_0[\tau_u^*(\mathcal{B})] \) and \( Y \in \widetilde{\mathcal{M}}_0[\| \cdot \|_0] \) commute strongly, then \( A \square Y \) is well-defined and

\[
A \square Y = A \cdot Y = Y \cdot A = Y \square A.
\]

**Proof.** It is easily checked that \( \widetilde{\mathcal{M}}_0[\tau_u^*(\mathcal{B})] \) and \( \widetilde{\mathcal{M}}_0[\tau_{s^*}] \) are locally convex quasi \( C^* \)-normed algebras over \( \mathcal{M}_0 \). Suppose now that \( A \in \mathcal{M}_0[\tau_u^*(\mathcal{B})] \) and \( Y \in \mathcal{M}_0[\| \cdot \|_0] \) commute strongly. Then, there is a net \( \{X_\alpha\}_{\alpha \in \Sigma} \) in \( \mathcal{M}_0 \) such that \( X_\alpha Y = Y X_\alpha \), for all \( \alpha \in \Sigma \) and \( A = \tau_u^*(\mathcal{B}) - \lim_{\alpha} X_\alpha \). Since

\[
(A^\dagger \xi)|Y\eta\rangle = \lim_{\alpha} (X_\alpha^\dagger \xi)|Y\eta\rangle = \lim_{\alpha} (\xi|X_\alpha Y\eta\rangle = \lim_{\alpha} (\xi|YX_\alpha \eta\rangle = (\xi|YA\eta\rangle
\]

for all \( \xi, \eta \in \mathcal{D} \), it follows that \( A \square Y \) is well-defined and \( A \square Y = YA \). Furthermore, since

\[
A \cdot Y = \tau_u^*(\mathcal{B}) - \lim_{\alpha} X_\alpha Y = \tau_{s^*} - \lim_{\alpha} X_\alpha Y,
\]
we have

\[(A \cdot Y)\xi = \lim_{\alpha} X_\alpha Y \xi = Y A \xi = (A \square Y)\xi\]

for each \(\xi \in \mathcal{D}\). Hence, \(A \cdot Y = A \square Y\). \(\square\)

**Proposition 6.2.** \(\mathcal{L}^1(\mathcal{D}, \mathcal{H})[\tau_{s^*}]\) is a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{L}^1(\mathcal{D})_b \equiv \{X \in \mathcal{L}^1(\mathcal{D}) : X \in \mathcal{B}(\mathcal{H})\}\).

*Proof.* Indeed, as shown in [3, Section 2.5], \(\mathcal{L}^1(\mathcal{D})_b\), is a \(C^*\)-normed algebra which is \(\tau_{s^*}\) dense in \(\mathcal{L}^1(\mathcal{D}, \mathcal{H})\). Hence, \(\mathcal{L}^1(\mathcal{D}, \mathcal{H})\) is a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{L}^1(\mathcal{D})_b\). \(\square\)

**Remark 6.3.** The following questions arise naturally:

1. What is exactly the \(C^*\)-algebra \(\mathcal{L}^1(\mathcal{D})_b[\| \cdot \|_0]\)? Under what conditions may one have the equality \(\mathcal{L}^1(\mathcal{D})_b[\| \cdot \|_0] = \mathcal{B}(\mathcal{H})\)?

2. Is \(\mathcal{L}^1(\mathcal{D}, \mathcal{H})\) a locally convex quasi \(C^*\)-algebra under the strong* uniform topology \(\tau_{s^*}\)?

More precisely, does the equality \(\mathcal{L}^1(\mathcal{D})_{b^*}[\tau_{s^*}^u] = \mathcal{L}^1(\mathcal{D}, \mathcal{H})\) hold?

We expect the answer to these questions to depend on the properties of the topology \(t_1 \equiv t_{\mathcal{L}^1(\mathcal{D}, \mathcal{H})}\) given on \(\mathcal{D}\) and we conjecture positive answers in the case where \(\mathcal{D} \equiv \mathcal{D}^\infty(T)\), with \(T\) a positive self-adjoint operator in a Hilbert space \(\mathcal{H}\), and \(\| \cdot \|_0\) the operator norm in \(\mathcal{B}(\mathcal{H})\). We leave these questions open.

- In the rest of this Section we consider conditions under which a locally convex quasi \(C^*\)-normed algebra is continuously embedded in a locally convex quasi \(C^*\)-normed algebra of operators.

So let \(\mathcal{A}[\tau]\) be a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{A}_0\) and \(\mathcal{D}\) a dense subspace in a Hilbert space \(\mathcal{H}\). Let \(\pi : \mathcal{A} \longrightarrow \mathcal{L}^1(\mathcal{D}, \mathcal{H})\) be a \(*\)-representation. Then we have the following:

**Lemma 6.4.** Let \(\mathcal{A}[\tau]\) be a locally convex quasi \(C^*\)-normed algebra over \(\mathcal{A}_0\) and \(\pi : \mathcal{A} \longrightarrow \mathcal{L}^1(\mathcal{D}, \mathcal{H})\) a \((\tau, \tau_{s^*}(\mathcal{B}))\)-continuous \(*\)-representation of \(\mathcal{A}\). Then,

1. \(\pi\) is a \(*\)-representation of the \(C^*\)-algebra \(\mathcal{A}_0[\| \cdot \|_0]\);
2. \(\pi(\mathcal{A})[\tau_{s^*}^u(\mathcal{B})]\) resp. \(\pi(\mathcal{A})[\tau_{s^*}]\) are locally convex quasi \(C^*\)-normed algebras over \(\pi(\mathcal{A}_0)\).

*Proof.* (1) Since \(\mathcal{A}_0 \subset \mathcal{A}_0[\| \cdot \|_0] \subset \mathcal{A}\) and \(\pi\) is a \(*\)-representation of \(\mathcal{A}\), it follows that

\[
\pi(ay) = \pi(a) \circ \pi(y), \quad \forall a \in \mathcal{A}_0[\| \cdot \|_0], \forall y \in \mathcal{A}_0.
\] (6.2)
Now we show that
\[(6.3)\]
\[\pi(ab) = \pi(a)\pi(b), \quad \forall a, b \in \widetilde{A}_0[\| \cdot \|_0].\]

Indeed, let \(a, b\) be arbitrary elements of \(\widetilde{A}_0[\| \cdot \|_0]\). Then, there exists a sequence \(\{y_n\}\) in \(A_0\) such that \(b = \| \cdot \|_0 - \lim_{n \to \infty} y_n\). Hence, \(ab = \| \cdot \|_0 - \lim_{n \to \infty} ay_n\).

Moreover, it is easily seen that \(\pi\) is also \((\tau, \tau_{s^*})\)-continuous and so, by (6.2),
\[
\langle \pi(b)\xi|\pi(a^*)\eta \rangle = \lim_{n \to \infty} \langle \pi(y_n)\xi|\pi(a^*)\eta \rangle = \lim_{n \to \infty} \langle \pi(a)\pi(y_n)\xi|\eta \rangle = \lim_{n \to \infty} \langle \pi(ay_n)\xi|\eta \rangle = \langle \pi(ab)\xi|\eta \rangle,
\]
for every \(\xi, \eta \in D\). Thus, (6.3) holds.

For any \(\xi \in D\), we put
\[f(a) = \langle \pi(a)\xi|\xi \rangle, \quad a \in \widetilde{A}_0[\| \cdot \|_0].\]

Then, by (6.3), \(f\) is a positive linear functional on the unital \(C^*\)-algebra \(\widetilde{A}_0[\| \cdot \|_0]\).

Hence, we have
\[\| \pi(a)\xi \|^2 = f(a^*a) \leq f(1)\|a\|_0^2 = \|\xi\|^2\|a\|_0^2\]
for all \(a \in \widetilde{A}_0[\| \cdot \|_0]\), which implies that \(\pi\) is bounded. This completes the proof of (1).

(2) \(\pi(A)\) is a quasi \(*\)-subalgebra of the locally convex quasi \(*\)-algebras \(\pi(\widetilde{A})[\tau^u(B)]\) and \(\pi(\widetilde{A})[\tau_{s^*}]\) over \(\pi(A_0)\). Furthermore, by (1), \(\pi(\widetilde{A}_0[\| \cdot \|_0])\) is a \(C^*\)-algebra and
\[\pi(\widetilde{A}_0)[\| \cdot \|_0] = \pi(\widetilde{A}_0[\| \cdot \|_0]) \subset \pi(A)\]

\[\square\]

**Remark 6.5.** Let \(A[\tau]\) be a locally convex quasi \(C^*\)-normed algebra over \(A_0\), and \(\pi\) a \((\tau, \tau_{s^*}(B))\)-continuous \(*\)-representation of \(A\), where \(B\) is an admissible subset in \(\mathcal{B}(\pi(A))\). Let \(a \in A\) be strongly commuting with \(y \in \widetilde{A}_0[\| \cdot \|_0]\). Then \(\pi(a)\) commutes strongly with \(\pi(y)\). The converse does not necessarily hold. So even if \(A[\tau]\) is normal, the locally convex quasi \(C^*\)-normed algebra \(\pi(A)\) over \(\pi(A_0)\) is not necessarily normal.

We are going now to discuss the faithfulness of a \((\tau, \tau_{s^*})\)-continuous \(*\)-representation of \(A\). For this, we need some facts on sesquilinear forms, for which the reader is referred to [8, p. 544]. We only recall that if
\[S(A_0) := \{\tau\text{-continuous positive invariant sesquilinear forms } \varphi \text{ on } A_0 \times A_0\},\]
we say that the set \(S(A_0)\) is *sufficient*, whenever
\[a \in A \text{ with } \varphi(a, a) = 0, \forall \varphi \in S(A_0), \text{ implies } a = 0,\]
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where $\tilde{\varphi}$ is the extension of $\varphi$ to a $\tau$-continuous positive invariant sesquilinear form on $A \times A$.

From the next results, Theorem 6.6 and Corollary 6.7 can be regarded as generalizations of the analogues of the Gel’fand-Naimark theorem, in the case of locally convex quasi $C^*$-algebras proved in [8, Section 7]. Theorem 6.6 is proved in the same way as [8, Theorem 7.3].

**Theorem 6.6.** Let $A[\tau]$ be a locally convex quasi $C^*$-normed algebra over a unital $C^*$-normed algebra $A_0$. The following statements are equivalent:

(i) There exists a faithful $(\tau, \tau_s^*)$-continuous $*$-representation of $A$.

(ii) The set $S(A_0)$ is sufficient.

**Corollary 6.7.** Suppose $S(A_0)$ is sufficient. Then, the locally convex quasi $C^*$-normed algebra $A[\tau]$ over $A_0$ is continuously embedded in a locally convex quasi $C^*$-normed algebra of operators.

We end this Section with the study of a functional calculus for the commutatively quasi-positive elements (see Definition 4.4) of $A[\tau]$.

Let $A[\tau]$ be a locally convex quasi $C^*$-normed algebra over a unital $C^*$-normed algebra $A_0$. If $a \in A[\tau]_{q^+}$, then by Proposition 4.7(1), the element $(1 + a)^{-1}$ exists and belongs to $U(\bar{\tilde{A}_0}[\| \cdot \|_0])$. Denote by $C^*(a)$ the maximal commutative $C^*$-subalgebra of the $C^*$-algebra $\bar{\tilde{A}_0}[\| \cdot \|_0]$ containing the elements 1 and $(1 + a)^{-1}$.

**Lemma 6.8.** $C^*(a)[\tau]$ is a commutative unital locally convex quasi $C^*$-algebra over $C^*(a)$ and $a \in C^*(a)[\tau]_{q^+}$.

**Proof.** Since $C^*(a)$ is a unital $C^*$-algebra, we have only to check the properties $(T_1)$ – $(T_5)$. We show $(T_1)$; the rest of them, as well as the fact that $a \in C^*(a)[\tau]_{q^+}$ are proved by the same way as in [8, Proposition 7.6 and Corollary 7.7]. From the condition $(T_3)$ for $A_0[\tau]$, we have that for all $\lambda \in \Lambda$, there exist $\lambda' \in \Lambda$ and $\gamma_\lambda > 0$ such that

$$p_\lambda(xy) \leq \gamma_\lambda \|x\|_0 p_{\lambda'}(y), \forall x, y \in C^*(a).$$

So, $C^*(a)[\tau]$ is a locally convex $*$-algebra with separately continuous multiplication. □

By Lemma 6.8 and Theorem 5.2 we can now obtain a functional calculus for the commutatively quasi-positive elements of the noncommutative locally convex quasi $C^*$-normed algebra $A[\tau]$ (see also [8, Theorem 7.8, Corollary 7.9]).

**Theorem 6.9.** Let $A[\tau]$ be an arbitrary locally convex quasi $C^*$-normed algebra over a unital $C^*$-normed algebra $A_0$ and $a \in A[\tau]_{q^+}$. Suppose that $a^n$ is well-defined for some $n \in \mathbb{N}$. Then, there is a unique $*$-isomorphism $f \to f(a)$ from $\bigcup_{k=1}^{n} C_k(\sigma C^*(a)(a))$ into $A[\tau] \cdot C^*(a)$ such that:
(i) If $u_0(\lambda) = 1$, with $u_0 \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ and $\lambda \in \sigma_{C^*(a)}(a)$, then $u_0(a) = 1$.

(ii) If $u_1(\lambda) = \lambda$ with $u_1 \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ and $\lambda \in \sigma_{C^*(a)}(a)$, then $u_1(a) = a$.

(iii) $(\lambda_1 f_1 + f_2)(a) = \lambda_1 f_1(a) + f_2(a)$, $f_1, f_2 \in \bigcup_{k=1}^n C_k(\sigma_{C^*(a)}(a))$ and $\lambda_1 \in \mathbb{C}$;

$$(f_1 f_2)(a) = f_1(a) f_2(a), \forall f_j \in C_{k_j}(\sigma_{C^*(a)}(a)), j = 1, 2, \text{ with } k_1 + k_2 \leq n.$$ 

(iv) The map $f \rightarrow f(a)$ restricted to $C_b(\sigma_{C^*(a)}(a))$ is an isometric $*$-isomorphism of the $C^*$-algebra $C_b(\sigma_{C^*(a)}(a))$ on the $C^*$-algebra $C^*(a)$.

Using Theorem 6.9 and applying Corollary 5.3 for the commutative unital locally convex quasi $C^*$-algebra $C^*(a)[\tau]$, we conclude the following

**Corollary 6.10.** Let $A[\tau]$ and $A_0$ be as in Theorem 6.9. If $a \in A[\tau]_{cq^+}$ and $n \in \mathbb{N}$, there is a unique element $b \in A[\tau]_{cq^+} \cdot C^*(a)$, such that $a = b^n$. The element $b$ is called commutatively quasi $n$th-root of $a$ and is denoted by $a^{\frac{1}{n}}$.

### 7. Applications

Locally convex quasi $C^*$-normed algebras arise, as we have discussed throughout this paper, as completions of a $C^*$-normed algebra with respect to a locally convex topology which satisfies a series of requirements. Completions of this sort actually occur in quantum statistics.

In statistical physics, in fact, one has to deal with systems consisting of a very large number of particles, so large that one usually considers this number to be **infinite**. One begins by considering systems living in a **local region** $V$ ($V$ is, for instance, a bounded region of $\mathbb{R}^3$ for gases or liquids, or a finite subset of the lattice $\mathbb{Z}^3$ for crystals) and requires that the set of local regions is directed, i.e., if $V_1$, $V_2$ are two local regions, then there exists a third local region $V_3$ containing both $V_1$ and $V_2$. The observables on a given bounded region $V$ are supposed to constitute a $C^*$-algebra $A_V$, where all $A_V$’s have the same norm, and so the $*$-algebra $A_0$ of local observables, $A_0 = \bigcup_V A_V$, is a $C^*$-normed algebra. Its uniform completion is, obviously, a $C^*$-algebra (more precisely, a quasi local $C^*$-algebra) that in the original algebraic approach was taken as the observable algebra of the system. As a matter of fact, this $C^*$-algebraic formulation reveals to be insufficient, since for many models there is no way of including in this framework the thermodynamical limit of the local Heisenberg dynamics [6]. Then a possible procedure to follow in order to circumvent this difficulty is to define in $A_0$ a new locally convex topology, $\tau$, called, for obvious reasons, **physical topology**, in such a way that the dynamics in the thermodynamical limit belongs to the completion of $A_0$ with respect to $\tau$. For that purpose, a class of topologies for the $*$-algebra $A_0$ of local observables of a quantum system was proposed by Lassner in [15, 16]. We will sketch in
what follows this construction. Let \( \mathcal{A}_0 \) be a C*-normed algebra to be understood as the algebra of local observables described above; thus we will suppose that \( \mathcal{A}_0 = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda \), where \( \{ \mathcal{A}_\lambda \}_{\lambda \in \Lambda} \) is a family of C*-algebras labeled by a directed set of indices \( \Lambda \).

Assume that, for every \( \alpha \in \Sigma \) (\( \Sigma \) a given set of indices), \( \pi_\alpha \) is a *-representation of \( \mathcal{A}_0 \) on a dense subspace \( \mathcal{D}_\alpha \) of a Hilbert space \( \mathcal{H}_\alpha \), i.e. each \( \pi_\alpha \) is a *-homomorphism of \( \mathcal{A}_0 \) into the partial \( O^* \)-algebra \( \mathcal{L}^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha) \) endowed, for instance, with the topology \( \tau_\alpha(\mathcal{L}^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha)) \). We shall assume that \( \pi_\alpha(x) \mathcal{D}_\alpha \subset \mathcal{D}_\alpha \), for every \( \alpha \in \Sigma \) and \( x \in \mathcal{A}_0 \).

Since every \( \mathcal{A}_\lambda \) is a C*-algebra, each \( \pi_\alpha \) is a bounded and continuous *-representation, i.e. \( \overline{\pi_\alpha(x)} \in \mathcal{B}(\mathcal{H}_\alpha), \| \pi_\alpha(x) \| \leq \| x \|_o \), for every \( x \in \mathcal{A}_0 \). So each \( \pi_\alpha \) can be extended to the C*-algebra \( \tilde{\mathcal{A}}_0[\| \cdot \|_o] \) (we denote the extension by the same symbol). The family is supposed to be faithful, in the sense that if \( x \in \tilde{\mathcal{A}}_0[\| \cdot \|_o], x \neq 0 \), then there exists \( \alpha \in \Sigma \) such that \( \pi_\alpha(x) \neq 0 \). Let us further suppose that \( \mathcal{D}_\alpha = \mathcal{D}^\infty(M_\alpha) = \bigcap_{n \in \mathbb{N}} \mathcal{D}(M_\alpha^n) \), where \( M_\alpha \) is a selfadjoint operator. Without loss of generality we may assume that \( M_\alpha \geq I_\alpha \), with \( I_\alpha \) the identity operator in \( \mathcal{B}(\mathcal{H}_\alpha) \). Under these assumptions, a physical topology \( \tau \) can be defined on \( \mathcal{A}_0 \) by the family of seminorms

\[
p_\alpha'(x) = \| \pi_\alpha(x) f(M_\alpha) \| + \| \pi_\alpha(x^*) f(M_\alpha) \|, \quad x \in \mathcal{A}_0,
\]

where \( \alpha \in \Sigma \) and \( f \) runs over the set \( \mathcal{F} \) of all positive, bounded and continuous functions \( f(t) \) on \( \mathbb{R}^+ \) such that

\[
\sup_{t \in \mathbb{R}^+} t^k f(t) < \infty, \quad \forall \; k = 0, 1, 2, \ldots .
\]

Then, \( \mathcal{A}_0[\tau] \) is a locally convex *-algebra with separately continuous multiplication (i.e. \( (T_1) \) holds). In order to prove that \( \tilde{\mathcal{A}}_0[\tau] \) is a locally convex quasi C*-normed algebra, we need to prove that \( (T_2)-(T_3) \) also hold. As for \( (T_2) \), we have, for every \( \alpha \in \Sigma \),

\[
p_\alpha'(x) = \| \pi_\alpha(x) f(M_\alpha) \| + \| \pi_\alpha(x^*) f(M_\alpha) \| \leq 2 \| f(M_\alpha) \| \| \pi_\alpha(x) \| \leq 2 \| f(M_\alpha) \| \| x \|_o, \quad x \in \mathcal{A}_0.
\]

The compatibility of \( \tau \) with \( \| \cdot \|_o \) follows easily from the closedness of the operators \( f(M_\alpha)^{-1} \) and the faithfulness of the family \( \{ \pi_\alpha \}_{\alpha \in \Sigma} \) of *-representations.

The condition (R) does not hold, in general, but, on the other hand, if \( x, y \in \mathcal{A}_0 \) with \( xy = yx \), we have

\[
p_\alpha'(xy) = \| \pi_\alpha(xy) f(M_\alpha) \| + \| \pi_\alpha((xy)^*) f(M_\alpha) \|
\]

\[
= \| \pi_\alpha(xy) f(M_\alpha) \| + \| \pi_\alpha(x^* y^*) f(M_\alpha) \|
\]

\[
\leq \| \pi_\alpha(x) \| (\| \pi_\alpha(y) f(M_\alpha) \| + \| \pi_\alpha(y^*) f(M_\alpha) \|)
\]

\[
= \| \pi_\alpha(x) \| p_\alpha'(y) \leq \| x \|_o p_\alpha'(y).
\]

Hence \( (T_3) \) holds. As for \( (T_4) \), we begin with noticing that for every \( \alpha \in \Sigma \), \( \pi_\alpha(\mathcal{A}_0) \) is an \( O^* \)-algebra of bounded operators in \( \mathcal{D}_\alpha \). Hence, its closure in \( \mathcal{L}^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau_\alpha^u(\mathcal{L}^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha))] \)
is a locally convex C*-normed algebra of operators, by Proposition 6.1. Moreover, every \( \pi_\alpha \) can be extended by continuity to \( \tilde{A}_0[\| \cdot \|_0] \). The extension, that we denote by the same symbol, takes values in \( L^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau_\alpha(L^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha))] \), since this space is complete. Now, if \( \{ \lambda \} \) is a net in \( U(\tilde{A}_0[\| \cdot \|_0])^\tau \)-converging to \( x \in \tilde{A}_0[\| \cdot \|_0] \), then \( x = x^* \) and \( \pi_\alpha(x^\lambda) \to \pi_\alpha(x) \) in \( L^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha)[\tau_\alpha(L^1(\mathcal{D}_\alpha, \mathcal{H}_\alpha))] \), for every \( \alpha \in \Sigma \). Thus \( \pi_\alpha(x) \geq 0 \) and \( \| \pi_\alpha(x) \| \leq 1 \), for every \( \alpha \in \Sigma \), since the same is true for every \( x^\lambda \). By constructing a faithful representation \( \pi \) by direct sum of the \( \pi_\alpha \)'s, one easily realizes that \( x \geq 0 \) and \( \| x \|_0 \leq 1 \). The inclusion \( \tilde{A}_0[\tau]_q \cap \tilde{A}_0[\| \cdot \|_0] \subset \tilde{A}_0[\| \cdot \|_0]^\tau \) in Condition (T_5) can be proved in similar fashion. The converse inclusion comes from Lemma 4.5. Thus Condition (T_5) holds.

Then we conclude that

**Statement 7.1.** \( A \equiv \tilde{A}_0[\tau] \) is a locally convex quasi C*-normed algebra, which can be understood as the quasi *-algebra of the observables of the physical system.

A more concrete realization of the situation discussed above is obtained for the so-called BCS model. Let \( V \) be a finite region of a \( d \)-dimensional lattice \( \Lambda \) and \( |V| \) the number of points in \( V \). The local C*-algebra \( A_V \) is generated by the Pauli operators \( \sigma_p = (\sigma_p^1, \sigma_p^2, \sigma_p^3) \) and by the unit \( 2 \times 2 \) matrix \( e_p \) at every point \( p \in V \). The \( \sigma_p \)'s are copies of the Pauli matrices localized in \( p \).

If \( V \subset V' \) and \( A_V \in A_{V'} \), then \( A_V \to A_{V'} = A_V \otimes (\bigotimes_{p \in V' \setminus V} e_p) \) defines the natural imbedding of \( A_V \) into \( A_{V'} \).

Let \( \vec{n} = (n_1, n_2, n_3) \) be a unit vector in \( \mathbb{R}^3 \), and put \( \langle \vec{\sigma} \cdot \vec{n} \rangle = n_1 \sigma^1 + n_2 \sigma^2 + n_3 \sigma^3 \). Then, denoting as \( Sp(\vec{\sigma} \cdot \vec{n}) \) the spectrum of \( \vec{\sigma} \cdot \vec{n} \), we have \( Sp(\vec{\sigma} \cdot \vec{n}) = \{1, -1\} \). Let \( |\vec{n}\rangle \in C^2 \) be a unit eigenvector associated with 1.

Let now denote by \( \mathbf{n} := (\vec{n}_p)_{p \in \Lambda} \) an infinite sequence of unit vectors in \( \mathbb{R}^3 \) and \( |\mathbf{n}\rangle = \bigotimes_p |\vec{n}_p\rangle \) the corresponding unit vector in the infinite tensor product \( \mathcal{H}_\infty = \bigotimes_p \mathbb{C}^2 \). We put \( A_0 = \bigcup_V A_V \) and \( D^0_{\mathbf{n}} = A_0|\mathbf{n}\rangle \) and we denote the closure of \( D^0_{\mathbf{n}} \) in \( \mathcal{H}_\infty \) by \( \mathcal{H}_{\mathbf{n}} \). As we saw above, to any sequence \( \mathbf{n} \) of three-vectors there corresponds a state \( |\mathbf{n}\rangle \) of the system. Such a state defines a realization \( \pi_{\mathbf{n}} \) of \( A_0 \) in the Hilbert space \( \mathcal{H}_{\mathbf{n}} \). This representation is faithful, since the norm completion \( A_S \) of \( A_0 \) is a simple C*-algebra.

A special basis for \( \mathcal{H}_{\mathbf{n}} \) is obtained from the *ground state* \( |\mathbf{n}\rangle \) by *flipping* a finite number of spins using the following strategy:

Let \( \vec{n} \) be a unit vector in \( \mathbb{R}^3 \), as above, and \( |\vec{n}\rangle \) the corresponding vector of \( C^2 \). Let us choose two other unit vectors \( \vec{n}_1, \vec{n}_2 \) so that \( (\vec{n}, \vec{n}_1, \vec{n}_2) \) form an orthonormal basis of \( \mathbb{R}^3 \). We put \( \vec{n}_\pm = \frac{1}{2}(\vec{n}_1 \pm i\vec{n}_2) \) and define \( |m, \vec{n}\rangle := (\vec{\sigma} \cdot \vec{n}_\pm)^m |\vec{n}\rangle \) \( (m = 0, 1) \). Then we have

\[
(\vec{\sigma} \cdot \vec{n}_\pm)|m, \vec{n}\rangle = (-1)^m |m, \vec{n}\rangle \quad (m = 0, 1).
\]
Thus, the set \( \{ |m, n\rangle = \otimes_p |m_p, \vec{n}_p\rangle; \; m_p = 0,1, \; \sum_p m_p < \infty \} \) forms an orthonormal basis in \( \mathcal{H}_n \).

In this space we define the unbounded self-adjoint operator \( M_n \) by

(7.1) \[ M_n |m, n\rangle = (1 + \sum_p m_p) |m, n\rangle. \]

\( M_n \) counts the number of the flipped spins in \( |m, n\rangle \) with respect to the ground state \( |n\rangle \). Now we put

\[ D_n = \bigcap_k D(M_n^k). \]

The representation \( \pi_n \) is defined on the basis vectors \( \{|m, n\rangle\} \) by

\[ \pi_n(\sigma^i_p) |m, n\rangle = \sigma^i_p |m_p, \vec{n}_p\rangle \otimes (\prod_{p' \neq p} |m_{p'}, \vec{n}_{p'}\rangle) \quad (i = 1, 2, 3). \]

This definition is then extended in obvious way to the whole space \( \mathcal{H}_n \). It turns out that \( \pi_n \) is a bounded representation of \( \mathcal{A}_0 \) into \( \mathcal{H}_n \). For more details we refer to [20, 11]. Hence, the procedure outlined above applies, showing that a natural framework for discussing the BCS model is, indeed, provided by locally convex quasi C*-normed algebras considered in this paper. We argue that an analysis similar to that of [11] can be carried out also in the present context, so that for suitable finite volume hamiltonians, the thermodynamical limit of the local dynamics can be appropriately defined in \( \tilde{\mathcal{A}}_0[\tau] \).

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