Strong Approximation of the Anisotropic Random Walk Revisited

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Abstract
We study the path behavior of the anisotropic random walk on the two-dimensional lattice $\mathbb{Z}^2$. Strong approximation of its components with independent oscillating Brownian motions are proved.

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1 Introduction and main results
We consider random walks on the square lattice $\mathbb{Z}^2$ of the plane with possibly unequal symmetric horizontal and vertical step probabilities, so that these probabilities can only depend on the value of the vertical coordinate. In particular, if such a random walk is situated at a site on the horizontal line $y = j \in \mathbb{Z}$, then at the next step it moves with probability $p_j$ to either vertical neighbor, and with probability $1/2 - p_j$ to either horizontal neighbor. A substantial motivation for studying such two-dimensional random walks on anisotropic lattice has originated from transport problems of statistical physics.

More formally, consider the random walk \{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\} on $\mathbb{Z}^2$ with the transition probabilities

$$P(C(N+1) = (k+1,j)|C(N) = (k,j)) = P(C(N+1) = (k-1,j)|C(N) = (k,j)) = \frac{1}{2} - p_j,$$

$$P(C(N+1) = (k,j+1)|C(N) = (k,j)) = P(C(N+1) = (k,j-1)|C(N) = (k,j)) = p_j,$$
for \((k, j) \in \mathbb{Z}^2, N = 0, 1, 2, \ldots\) We assume throughout the paper that \(0 < p_j \leq 1/2\) and \(\min_{j \in \mathbb{Z}} p_j < 1/2\). Unless otherwise stated we assume also that \(C(0) = (0, 0)\). In this paper we will have the following condition
\[
n^{-1} \sum_{j=1}^{n} p_j^{-1} = 2\gamma_1 + o(n^{-\tau}), \quad n^{-1} \sum_{j=1}^{n} p_j^{-1} = 2\gamma_2 + o(n^{-\tau})
\] (1.1)
as \(n \to \infty\) for some constants \(1 < \max(\gamma_1, \gamma_2) < \infty\), and \(1/2 < \tau \leq 1\). We will point out how this condition is different from the previous similar results. The case \(p_j = 1/4, j = 0, \pm 1, \pm 2, \ldots\) corresponds to simple symmetric random walk on the plane. For this case we refer to Erdős and Taylor [18], Dvoretzky and Erdős [17], and Révész [25]. The case \(p_j = 1/2\) for some \(j\) means that the horizontal line \(y = j\) is missing. If all \(p_j = 1/2\), then the random walk takes place on the \(y\) axis, so it is only a one-dimensional random walk, and this case is excluded from the present investigations. The case however when \(p_j = 1/2, j = \pm 1, \pm 2, \ldots\) but \(p_0 = 1/4\) is an interesting one which is the so-called random walk on the two-dimensional comb. In this case \(\gamma_1 = \gamma_2 = 1\). For this model we may refer to Weiss and Havlin [30], Bertacchi and Zucca [2], Bertacchi [1], Csáki et al. [8]. In the comb model the scaling of the horizontal and vertical coordinates are different, namely for the first coordinate the scaling is of order \(N^{1/4}\), so it is a so called sub-diffusion, and can be approximated with an iterated Wiener process while the second coordinate is of order \(N^{1/2}\), and hence it can be approximated with a Wiener process.

In our paper Csáki et al. [10] we considered the case when both coordinates are of order \(N^{1/2}\), hence can be approximated simultaneously with independent Wiener processes. More precisely we investigated the case when in (1.1) we have \(\lambda_1 = \lambda_2 > 1\). In Csáki et al. [10] we proved that

**Theorem A** Under the condition (1.1) with \(\lambda = \lambda_1 = \lambda_2 > 1\), and \(1/2 < \tau \leq 1\), on an appropriate probability space for the random walk

\[
\{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}
\]

one can construct two independent standard Wiener processes \(\{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\}\) so that, as \(N \to \infty\), we have with any \(\varepsilon > 0\)
\[
\left| C_1(N) - W_1\left(\frac{\gamma - 1}{\gamma} N\right) \right| + \left| C_2(N) - W_2\left(\frac{1}{\gamma} N\right) \right| = O(N^{5/8 - \tau/4 + \varepsilon}) \quad a.s.
\] (1.2)

The case \(\lambda_1 = \lambda_2 > 1\) has a considerable complex history, we refer to the interested reader to [10]. Here we just mention a few names; Silver et al. [28], Seshadri et al. [26], Shuler [27], Westcott [29]. Some of the most important contribution to this topic is due to Heyde [19], [20] and Heyde et al. [21]. Let \(\{Y(t), t \geq 0\}\) be a diffusion process on the same probability space as \(\{C_2(n)\}\) whose distribution is defined by
\[
Y(t) = W(A^{-1}(t)), \quad t \geq 0,
\]
where \( \{W(t), t \geq 0\} \) is a standard Brownian motion, (or standard Wiener process) and

\[
A(t) = \int_0^t \sigma^{-2}(W(s)) \, ds
\]

and

\[
\sigma^2(y) = \begin{cases} 
\frac{1}{\gamma_1} & \text{for } y \geq 0, \\
\frac{1}{\gamma_2} & \text{for } y < 0.
\end{cases}
\]

Here \( A^{-1}(\cdot) \) is the inverse of \( A(\cdot) \). The process \( Y(t) \) is called oscillating Brownian motion if \( \gamma_1 \neq \gamma_2 \), that is a diffusion with speed measure \( m(dy) = 2\sigma^{-2}(y)dy \).

**Remark 1.1** Observe that \( A(t) \) above is equal to

\[
A(t) = \gamma_1 \int_0^t I(W(s) \geq 0) \, ds + \gamma_2 \int_0^t I(W(s) < 0) \, ds.
\]  

(1.3)

Let

\[
-1 \sum_{j=1}^k p_j^{-1} = 2\gamma_1 + \varepsilon_k \quad -1 \sum_{j=-k}^{k-1} p_j^{-1} = 2\gamma_2 + \varepsilon_k^*
\]  

(1.4)

then the main result of Heyde et al. [21] is that

**Theorem B** ([21]) Suppose that in (1.4) \( \varepsilon_k \) and \( \varepsilon_k^* \) are \( o(1) \) as \( k \to \infty \). Then

\[
\sup_{0 \leq t \leq N} |N^{-1/2}C_2([Nt]) - Y(t)| \to 0 \quad a.s.
\]

Observe that here \( \gamma_1 \) and \( \gamma_2 \) might be different, and the convergence rates are much less restrictive, but the approximation is only for the second component.

Let's define an arbitrary set \( B \subset \mathbb{Z} \) such that for

\[
p_i = \frac{1}{4} \quad \text{if } \quad i \in B \
\]

and \( p_i = \frac{1}{2} \quad i \in \mathbb{Z} \setminus B \).

(1.5)

Thus we remove from the two-dimensional integer lattice all the horizontal edges which do not belong to the \( i \)-levels in \( B \). In our paper [9] we investigated a simple random walk on the half-plane half-comb (HPHC) structure, which is another interesting special case where we define the set \( B = \{i = 0, 1, 2, \ldots\} \), that is to say, all horizontal lines under the \( x \)-axis are deleted. Our main result there reads as follows.

**Theorem C** ([9]) On an appropriate probability space for the HPHC random walk

\( \{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\} \) with \( p_j = \frac{1}{4}, j = 0, 1, 2, \ldots, p_j = \frac{1}{2}, j = -1, -2, \ldots \)

one can construct two independent standard Wiener processes \( \{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\} \) such that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)

\[
|C_1(N) - W_1(N - A_2^{-1}(N))| + |C_2(N) - W_2((A_2^{-1}(N))| = O(N^{3/8 + \varepsilon}) \quad a.s.,
\]
where \( A_2(t) = 2 \int_0^t I(W_2(s) \geq 0) \, ds + \int_0^t I(W_2(s) < 0) \, ds \).

Clearly in this case \( \lambda_1 = 2 \) and \( \lambda_2 = 1 \), \( \tau \) can be selected to be 1. In our paper Csáki and Földes \([11]\) we considered the case when the set \( B \) is much more general than in Theorem C. Our main result in that paper can be formulated as follows:

**Theorem D**  \([11]\) Let

\[
|B_n| := |B \cap \{-n, n\}| \sim cn
\]

with some constant \( c > 0 \), where \( B \) is defined in \([1.5]\) and \( |B_n| \) stands for the (finite) number of elements in the set \( B_n \). Under the conditions \([1.1]\) with \( \max(\gamma_1, \gamma_2) > 1 \) on an appropriate probability space for the random walk \( \{C(N) = (C_1(N), C_2(N)); \, N = 0, 1, 2, \ldots\} \) one can construct two independent standard Wiener processes \( \{W_1(t); \, t \geq 0\}, \{W_2(t); \, t \geq 0\} \) so that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)

\[
|C_1(N) - W_1 \left(N - A_2^{-1}(N)\right)| + |C_2(N) - W_2 \left(A_2^{-1}(N)\right)| = O(N^{5/8-\gamma/4+\varepsilon}) \quad \text{a.s.} \tag{1.7}
\]

where \( A_2^{-1}(\cdot) \) is the inverse of \( A_2(\cdot) \).

So in Theorem D we have exactly our condition \([1.1]\), but all the \( p_i \)-s has to be 1/2 or 1/4. Our goal in this paper is to get a common generalization of Theorems A, B, C and D, namely we only need condition \([1.1]\) and no other restrictions for the \( p_i \)-s.

**Theorem 1.1** Under the conditions \([1.1]\), and \( 1 \leq \gamma_2 < \gamma_1 \) on an appropriate probability space for the random walk \( \{C(N) = (C_1(N), C_2(N)); \, N = 0, 1, 2, \ldots\} \) one can construct two independent standard Wiener processes \( \{W_1(t); \, t \geq 0\}, \{W_2(t); \, t \geq 0\} \) so that, as \( N \to \infty \), we have with any \( \varepsilon > 0 \)

\[
|C_1(N) - W_1 \left(N - A_2^{-1}(N)\right)| + |C_2(N) - W_2 \left(A_2^{-1}(N)\right)| = O(N^{5/8-\gamma/4+\varepsilon}) \quad \text{a.s.} \tag{1.8}
\]

where \( A_2(t) = \lambda_1 \int_0^t I(W_2(s) \geq 0) \, ds + \lambda_2 \int_0^t I(W_2(s) < 0) \, ds \).

and \( A_2^{-1}(\cdot) \) is the inverse of \( A_2(\cdot) \).

**Remark 1.2** If \( \gamma_1 = \gamma_2 > 1 \) then \( A_2(t) = \gamma_1 t \) and our theorem coincides with Theorem A. So we made the supposition that \( 1 \leq \gamma_2 < \gamma_1 \), instead of \( 1 < \max(\gamma_1, \gamma_2) < \infty \), even though it is not necessary but makes the flow of argument easier.

## 2 Preliminaries

First we are to redefine our random walk \( \{C(N); \, N = 0, 1, 2, \ldots\} \). It will be seen that the process described right below is equivalent to that given in the Introduction (cf. \([2.2]\) below).

To begin with, on a suitable probability space consider two independent simple symmetric (one-dimensional) random walks \( S_1(\cdot) \), and \( S_2(\cdot) \). We may assume that on the same probability space
we have a double array of independent geometric random variables \( \{G^{(j)}_i, i \geq 1,j \in \mathbb{Z}\} \) which are independent from \( S_1(\cdot) \) and \( S_2(\cdot) \), where \( G^{(j)}_i \) has the following geometric distribution

\[
P(G^{(j)}_i = k) = 2p_j(1 - 2p_j)^k, \quad k = 0, 1, 2, \ldots
\]  

(2.1)

We now construct our walk \( C(N) \) as follows. We will take all the horizontal steps consecutively from \( S_1(\cdot) \) and all the vertical steps consecutively from \( S_2(\cdot) \). First we will take some horizontal steps from \( S_1(\cdot) \), then exactly one vertical step from \( S_2(\cdot) \), then again some horizontal steps from \( S_1(\cdot) \) and exactly one vertical step from \( S_2(\cdot) \), and so on. Now we explain how to get the number of horizontal steps on each occasion. Consider our walk starting from the origin proceeding first horizontally \( G^{(0)}_1 \) steps (note that \( G^{(0)}_1 = 0 \) is possible with probability \( 2p_0 \)), after which it takes exactly one vertical step, arriving either to the level \( 1 \) or \( -1 \), where it takes \( G^{(1)}_1 \) or \( G^{(-1)}_1 \) horizontal steps (which might be no steps at all) before proceeding with another vertical step. If this step carries the walk to the level \( j \), then it will take \( G^{(j)}_1 \) horizontal steps, if this is the first visit to level \( j \), it takes \( G^{(j)}_2 \) horizontal steps, if this is its second visit at level \( j \) and so on. In general, if we finished the \( k \)-th vertical step and arrived to the level \( j \) for the \( i \)-th time, then it will take \( G^{(j)}_i \) horizontal steps.

Let now \( H_N, V_N \) be the number of horizontal and vertical steps, respectively from the first \( N \) steps of the just described process. Consequently, \( H_N + V_N = N \), and

\[
\{C(N); N = 0, 1, 2, \ldots\} = \{(C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}
\]

\[
d = \{(S_1(H_N), S_2(V_N)); N = 0, 1, 2, \ldots\},
\]  

(2.2)

where \( d \) stands for equality in distribution.

We will need the following two lemmas from Csáki at al. [10]

**Lemma A** Let \( \{G^{(j)}_i, i = 1,2,\ldots,n_j, j = 1,2,\ldots,K\} \) be independent geometric random variables with distribution

\[
P(G^{(j)}_i = k) = \alpha_j(1 - \alpha_j)^k, \quad k = 0, 1, 2, \ldots,
\]

where \( 0 < \alpha_j \leq 1 \). Put

\[
B_K = \sum_{j=1}^{K} \sum_{i=1}^{n_j} G^{(j)}_i, \quad \sigma^2 = \text{Var}B_K = \sum_{j=1}^{K} \frac{n_j(1 - \alpha_j)}{\alpha_j^2}.
\]

Then, for \( \lambda < -\sigma^2 \log(1 - \alpha_j) \) for each \( j \in [1, K] \), we have

\[
P\left(\left| \sum_{j=1}^{K} \sum_{i=1}^{n_j} \left( G^{(j)}_i - \frac{1 - \alpha_j}{\alpha_j} \right) \right| > \lambda \right) \leq 2 \exp\left(-\frac{\lambda^2}{2\sigma^2} + \sum_{\ell=3}^{K} \frac{\lambda^\ell}{\ell! 2^{\ell} \sum_{j=1}^{j} \alpha_j} \right).
\]  

(2.3)
Lemma B Assume the conditions of Lemma A and put $M = \sum_{j=1}^{K} n_j$. For $M \to \infty$ and $K \to \infty$ assume moreover that

$$K = K(M) = O(M^{1/2+\delta}), \quad \max_{1 \leq j \leq K} n_j = O(M^{1/2+\delta}),$$

(2.4)

for all $\delta > 0$,

$$\frac{1}{\alpha_j} \leq c_1 |j|^{1-\tau}, \quad j = 0, 1, 2, \ldots$$

(2.5)

for some $1/2 < \tau \leq 1$ and $c_1 > 0$,

$$\sum_{j=1}^{K} \frac{1}{\alpha_j} = O(K), \quad \frac{1}{\sigma} \leq \frac{c_2}{M^{1/2}}$$

(2.6)

for some $c_2 > 0$. Then we have as $K, M \to \infty$,

$$\sum_{j=1}^{K} \sum_{i=1}^{n_i} G_{i}^{(j)} = \sum_{j=1}^{K} n_j \frac{1 - \alpha_j}{\alpha_j} + O(M^{3/4-\tau/4+\varepsilon}) \quad a.s.$$  

(2.7)

for some $\varepsilon > 0$.

Remark 2.1 Lemmas A and B in [10] are formulated for $-K \leq j \leq K$, but it is obvious that the present one can be stated and proved word by word as it is in [10].

Let $\{X_i\}$ be a sequence of independent i.i.d. random variable, with $P(X_i = \pm 1) = 1/2$. Let $S(n) = \sum_{i=1}^{n} X_i$. Then $S(n)$ is a simple symmetric walk on the line. Its local time is defined by $\xi(j, n) = \#\{k : 0 < k < n, S(k) = j\}$, $n = 1, 2, \ldots$ for any integer $j$. Define $M_n = \max_{0 \leq k \leq n} |S(k)|$. Then we have the usual law of the iterated logarithm (LIL) and Chung’s LIL [6].

Lemma C

$$\limsup_{n \to \infty} \frac{M_n}{(2n \log \log n)^{1/2}} = 1, \quad \liminf_{n \to \infty} \left(\frac{\log \log n}{n}\right)^{1/2} M_n = \frac{\pi}{\sqrt{8}} \quad a.s.$$ 

The following result is from Heyde [19], (see also in [12] Lemma 5)

Lemma D For the simple symmetric random walk for any $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \sup_{x \in \mathbb{Z}} \frac{|\xi(x+1, n) - \xi(x, n)|}{n^{1/4+\varepsilon}} = 0 \quad a.s.$$  

For the next Lemma see Kesten [22].
Lemma E For the maximal local time

\[ \xi(n) = \sup_{x \in \mathbb{Z}} \xi(x, n) \]

we have

\[ \limsup_{n \to \infty} \frac{\xi(n)}{2n \log \log n}^{1/2} = 1 \quad \text{a.s.} \]

We need a simple lemma about the properties of \( A(t) \) from Csáki and Földes [11].

Lemma F Consider \( A(t) \) defined by \( \text{(1.3)} \) and let \( \alpha(t) = A(t) - t \). Let \( \gamma_1 > \gamma_2 \geq 1 \). Then

- \( A(t) \) and \( \alpha(t) \) are nondecreasing
- \( \gamma_2 t \leq A(t) \leq \gamma_1 t \) and \( \frac{\alpha}{\gamma_1} \leq A^{-1}(t) \leq \frac{\alpha}{\gamma_2} \).

We will need the famous KMT strong invariance principle (cf. Komlós et al. [23]).

Lemma G On an appropriate probability space one can construct \( \{S(n), n = 1, 2, \ldots\} \), a simple symmetric random walk on the line and a standard Wiener process \( \{W(t), t \geq 0\} \) such that as \( n \to \infty \),

\[ S(n) - W(n) = O(\log n) \quad \text{a.s.} \]

The next lemma is a simultaneous strong approximation result of Révész [24].

Lemma H On an appropriate probability space for a simple symmetric random walk \( \{S(n); n = 0, 1, 2, \ldots\} \) with local time \( \{\xi(x, n); x = 0, \pm 1, \pm 2, \ldots; n = 0, 1, 2, \ldots\} \) one can construct a standard Wiener process \( \{W(t); t \geq 0\} \) with local time process \( \{\eta(x, t); x \in \mathbb{R}; t \geq 0\} \) such that, as \( n \to \infty \), we have for any \( \varepsilon > 0 \)

- \( |S(n) - W(n)| = O(n^{1/4+\varepsilon}) \quad \text{a.s.} \)

and

- \( \sup_{x \in \mathbb{Z}} |\xi(x, n) - \eta(x, n)| = O(n^{1/4+\varepsilon}) \quad \text{a.s.,} \)

simultaneously.

Concerning the increments of the Wiener process we quote the following result from Csörgő and Révész (14, page 69).

Lemma I Let \( 0 < a_T \leq T \) be a non-decreasing function of \( T \). Then, as \( T \to \infty \), we have

\[ \sup_{0 \leq t \leq T - a_T} \sup_{s \leq a_T} |W(t + s) - W(t)| = O(a_T(\log(T/a_T) + \log \log T))^{1/2} \quad \text{a.s.} \]

The above statement is also true if \( W(\cdot) \) replaced by the simple symmetric random walk \( S(\cdot) \).
For a simple random walk with local time $\xi(\cdot, \cdot)$, let
\[ \hat{A}(n) = \gamma_1 \sum_{j=0}^{\infty} \xi(j, n) + \gamma_2 \sum_{j=1}^{\infty} \xi(-j, n). \] (2.8)

We will need the following lemma from Csáki and Földes [11]

**Lemma J** On a probability space as in Lemma H
\[ |\hat{A}(n) - A(n)| = O(n^{3/4+\epsilon}) \quad a.s. \]
where $A(.)$ is defined in (1.3). The next lemma is using some ideas from Heyde [19].

**Lemma 2.1** Let $\{S(i)\}_{i=1}^{\infty}$ a simple symmetric random walk with local time $\xi(i, n)$. Let $\beta_j > 0 \quad j = 1, 2, \ldots$ Suppose that
\[ n^{-1} \sum_{j=1}^{n} \beta_j^{-1} = \rho + o(n^{-\tau}), \] (2.9)
as $n \to \infty$ for some constants $0 < \rho < \infty$, and $1/2 < \tau \leq 1$. Then
\[ \sum_{j=1}^{N} \xi(j, N) \frac{1}{\beta_j} = \rho \sum_{j=1}^{\infty} \xi(j, N) + O(N^{5/4-\tau/2+\epsilon}) \quad a.s. \] (2.10)

**Proof:** Introduce the notation:
\[ \frac{1}{j} \sum_{k=1}^{j} \frac{1}{\beta_k} = \kappa_j \quad j = 1, 2, \ldots. \]
\[ \sum_{j} \xi(j, N) \frac{1}{\beta_j} = \sum_{j=1}^{\infty} \xi(j, N)(j\kappa_j - (j-1)\kappa_{j-1}) \]
\[ = \sum_{j=1}^{\infty} j\kappa_j(\xi(j, N) - \xi(j + 1, N)) \]
\[ = \sum_{j=1}^{\infty} j(\kappa_j - \rho)(\xi(j, N) - \xi(j + 1, N)) + \rho \sum_{j=1}^{\infty} j(\xi(j, N) - \xi(j + 1, N)) \]
\[ = \rho \sum_{j=1}^{\infty} \xi(j, N) + \sum_{j=1}^{\infty} j(\kappa_j - \rho)(\xi(j, N) - \xi(j + 1, N)) \]
Observe that from (2.9) we have that
\[ |j(\kappa_j - \rho)| \leq cj^{1-\tau} \]
for some \( c > 0 \). Now applying Lemma C for \( S(\cdot) \), and Lemma D, we get that
\[ \sum_{j=1}^{\infty} j(\kappa_j - \rho)(\xi(j, N) - \xi(j + 1, N)) \]
\[ = O(N^{1/4+\epsilon}) \sum_{j=1}^{\max_{k \leq N} |S(k)|} j^{-\tau} = O(N^{1/4+\epsilon})O(N^{1-\tau/2+\epsilon}) = O(N^{5/4-\tau/2+\epsilon}) \text{ a.s.,} \]
where here and throughout the paper the value of \( \epsilon \) might change from line to line. \( \square \)

3 Proofs

Proof of Theorem 1.1 Recall that \( H_N \) and \( V_N \) are the number of horizontal and vertical steps respectively of the first \( N \) steps of \( \{C(\cdot)\} \). First we would like to approximate \( H_N \) almost surely as \( N \to \infty \).

Consider the sum
\[ G^{(j)}_1 + G^{(j)}_2 + \ldots + G^{(j)}_{\xi_2(j, V_N)} \]
which is the total number of horizontal steps on the level \( j \), where \( \xi_2(\cdot, \cdot) \) is the local time of the walk \( S_2(\cdot) \). This statement is slightly incorrect if \( j \) happens to be the level where the last vertical step (up to the total of \( N \) steps) takes the walk. In this case the last geometric random variable might be truncated. However the error which might occur from this simplification will be part of the \( O(\cdot) \) term. This can be seen as follows. Let
\[ H_N^* = \sum_j \xi_2(j, V_N) \sum_{i=1}^{G^{(j)}_i} G^{(j)}_i, \]
where \( G^{(j)}_i \) has distribution (2.1). Clearly
\[ H_N^* - H_N \leq \max_j G^{(j)}_{\xi_2(j, V_N)}. \]
Here and in the sequel
\[ \sum_j = \sum_{\min_0 \leq k \leq V_N, S_2(k) \leq j} \sum_{\max_0 \leq k \leq V_N, S_2(k)} \]
and

\[
\max_j = \max_{\min_0 \leq k \leq V_N} \text{max}_{0 \leq k \leq \max_0 \leq k \leq V_N} S_2(k)
\]

Note that from (1.1) we have

\[
\frac{1}{\alpha_j} = \frac{1}{2p_j} \leq c_1 |j|^{1-\tau} \quad j = \pm 1, \pm 2, \ldots
\]

\[
P(\max_j G_2^{(j)}(j, V_N) > N^{1/2+\delta}) \leq \sum_j P(G_1^{(j)} > N^{1/2+\delta}) \leq \sum_j (1 - \alpha_j)^{N^{1/2+\delta}} \tag{3.1}
\]

\[
\leq \sum_j \exp(-\alpha_j N^{1/2+\delta}) \leq \sum_j \exp(-cN^{\tau-1}N^{1/2+\delta})
\]

\[
\leq N^{1/2+\delta^*} \exp(-cN^{\tau-1}N^{1/2+\delta}) \leq \exp(-cN^\varepsilon)
\]

with some small \( \varepsilon > 0, \delta > 0, \delta^* > 0 \). In the last line we used that \( 1/2 < \tau \leq 1 \) and Lemma C. Here and in what follows the value of \( c \) can be different from line to line. By Borel Cantelli we have now that for \( N \to \infty \)

\[
H^*_N - H_N \leq N^{1/2+\delta} \tag{3.2}
\]

almost surely for any \( \delta > 0 \).

The next step is to show that \( \sum_j \xi_2(j, V_N) \frac{1}{2p_j} \) is close to \( A(V_N) \).

To see this we apply Lemma 2.1 for the vertical walk \( S_2(\cdot) \) two times (separately for positive and negative \( j \) indices) with \( \beta_j = 2p_j = \alpha_j \) for \( j = \pm 1, \pm 2, \ldots \) and \( \rho = \gamma_1 \) and \( \gamma_2 \) respectively for positive and negative indices and with \( V_N \) instead of \( N \) to conclude, that

\[
\sum_j \frac{\xi_2(j, V_N)}{\alpha_j} = \gamma_1 \sum_{j=1}^{\infty} \xi_2(j, V_N) + \gamma_2 \sum_{j=1}^{\infty} \xi_2(-j, V_N) + \xi_2(0, V_N) \frac{1}{2p_0} + O(N^{5/4-\tau/2+\varepsilon}) \tag{3.3}
\]

\[
= \hat{A}_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}) = A_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}).
\]

where in the last line we used Lemmas E and J and that \( V_n \leq N \).

The rest of proof will be different for \( \gamma_2 > 1 \) and for \( \gamma_2 = 1 \).

Consider first the case \( \gamma_2 > 1 \). In this case we will apply Lemma B twice for

\[
H^*_N = \sum_j \left( G_1^{(j)} + G_2^{(j)} + \ldots + G_\xi^{(j)}_{\xi_2(j, V_N)} \right).
\]
Introduce $V_N(+) = \sum_{j=0}^{\infty} \xi_2(j, V_N)$ and $V_N(-) = \sum_{j=1}^{+\infty} \xi_2(-j, V_N)$, and let $K = \max_{0 \leq k \leq V_N} |S_2(k)|$. Define

$$\sigma_1^2 = \sum_{j=0}^{K} \frac{\xi_2(j, V_N)(1 - \alpha_j)}{\alpha_j^2}, \quad \text{and} \quad \sigma_2^2 = \sum_{j=1}^{K} \frac{\xi_2(-j, V_N)(1 - \alpha_{-j})}{\alpha_{-j}^2}.$$ 

Let $M = V_N(+) - V_N(-)$ for the first and second application respectively and $n_j = \xi_2(j, V_N)$, $\alpha_j = 2p_j$, $j = 0, 1, 2, \ldots$ for the first and $n_j = \xi_2(j, V_N)$, $\alpha_j = 2p_j$, $j = -1, -2, \ldots - K$ indices for the second application.

We need to check all the assumptions of the lemma in both cases. (2.4) follows from Lemma C and Lemma E, and (2.3) follows from (1.1). The first part of (2.6) follows from (2.9) with $\rho$ equal $\gamma_1$ and $\gamma_2$ respectively. It remains to verify the second part of (2.6). Using Lemma 2.1 we have almost surely as $N \to \infty$,

$$\frac{\sigma_1^2}{V_N(+)\sum_{j \geq 0} \xi_2(j, V_N)(1 - 2p_j)} = \frac{1}{V_N(+)}\sum_{j \geq 0} \frac{\xi_2(j, V_N(+))(1 - 2p_j)}{(2p_j)^2}$$

$$\geq \frac{1}{V_N(+)}\sum_{j \geq 0} \frac{\xi_2(j, V_N(+))(1 - 2p_j)}{2p_j} = \frac{1}{V_N(+)}\sum_{j \geq 0} \frac{\xi_2(j, V_N(+))}{2p_j} - 1 \to \gamma_1 - 1 > 0.$$ 

where the last inequality follows from our supposition of $\gamma_1 > 1$. The corresponding argument for $\sigma_2^2$ goes the same way using now that $\gamma_2 > 1$ as well. So we checked all the conditions of Lemma B and we can conclude that we have almost surely, as $N \to \infty$,

$$H_N^+(+) = \sum_{j \geq 0} \left( G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j, V_N)}^{(j)} \right) = \sum_{j \geq 0} \frac{\xi_2(j, V_N)(1 - 2p_j)}{2p_j} + O(N^{3/4 - \tau/4 + \varepsilon})$$

and a similarly

$$H_N^+(-) = \sum_{j < 0} \left( G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j, V_N)}^{(j)} \right) = \sum_{j < 0} \frac{\xi_2(j, V_N)(1 - 2p_j)}{2p_j} + O(N^{3/4 - \tau/4 + \varepsilon})$$

As for the case $\gamma_2 = 1$ we don’t have (3.6), we need a different argument, as follows.

Recall (1.1) and that in this condition $1/2 < \tau \leq 1$. Select $\delta > 0$ such that $\tau = 1/2 + 2\delta$ should hold. We will show that for $N$ big enough

$$H_N^+(-) \leq N^{1 - \tau/2 + \delta} \quad \text{a.s.}$$

To this end observe that for $N$ big enough by Lemma C and Lemma E

$$M_N \leq N^{1/2} \log N \quad \text{a.s. and} \quad \xi(j, V_N) \leq \xi(N) \leq N^{1/2} \log N \quad \text{a.s.}$$

where $M_N = \max_{0 \leq k \leq N} |S_2(k)|$. Introduce the notation $r_N = N^{1/2} \log N$.

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implying that

\[ H^*_N(\cdot) \leq \sum_{j=1}^r \sum_{i=1}^r G_i^{(-j)} \quad \text{a.s.} \]

(3.8)

Let \( N_k = k^k \), \( \lambda_N = N^{1-\tau/2+\delta} \) with some \( \delta > 0 \). Then \( \frac{N_{k+1}}{N_k} \sim e(k+1) \). From Markov inequality and (1.1) we have that

\[
\begin{align*}
\mathbb{P}\left( \sum_{j=1}^{rN_{k+1}} \sum_{i=1}^{rN_{k+1}} G_i^{(-j)} > \lambda_{N_k} \right) & \leq \frac{\mathbb{E}\left( \sum_{j=1}^{rN_{k+1}} \sum_{i=1}^{rN_{k+1}} G_i^{(-j)} \right)}{\lambda_{N_k}} \\
& = \frac{\sum_{j=1}^{rN_{k+1}} \sum_{i=1}^{rN_{k+1}} \frac{1-2p_{j-1}}{2p_{j-1}}}{\lambda_{N_k}} \\
& = \frac{r_{N_{k+1}} O(r_{N_{k+1}})^{1-\tau}}{(N_k)^{1-\tau/2+\delta}} \\
& \sim (e(K+1))^{1-\tau/2+\delta} \frac{1}{(K+1)^{\frac{K+1}{2}}}.
\end{align*}
\]

(3.9)

So we got the \((K+1)\)-th term of a convergent series. By Borel-Cantelli and the monotonicity of \( H^*_N(\cdot) \) we conclude that for \( N_K \leq N \leq N_{K+1} \)

\[ H^*_N(\cdot) \leq H^*_{N_{K+1}}(\cdot) \leq \lambda_{N_K} \leq \lambda_N = N^{1-\tau/2+\delta} \quad \text{a.s.} \]

(3.10)

for \( N \) big enough, proving (3.7). Applying now Lemma 2.1 with \( \rho = 1 \) and \( \beta_j = 2p_{j-1}, j = 1, 2, \ldots \) imply that

\[
\sum_{j<0} \xi_2(j, V_N) \frac{1}{2p_j} = V_N(\cdot) + O(N^{5/4-\tau/2+\varepsilon})
\]

or equivalently

\[
\sum_{j<0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} = O(N^{5/4-\tau/2+\varepsilon})
\]

Consequently

\[
H^*_N(\cdot) = \sum_{j<0} \xi_2(j, V_N) \frac{1-2p_j}{2p_j} + O(N^{5/4-\tau/2+\varepsilon}) + N^{1-\tau/2+\delta}
\]

(3.11)
as we can select \( \delta \) to be arbitrary small. Consequently, we have by (3.5), (3.6) and (3.11) that for \( 1 \leq \lambda_1 < \lambda_2 \)

\[
H_N^* = \sum_j \left( G_1^{(j)} + G_2^{(j)} + \ldots + G_{\xi_2(j,V_N)}^{(j)} \right) = \sum_j \xi_2(j,V_N) \frac{1-2p_j}{2p_j} + O(N^{3/4-\tau/4+\varepsilon}) + O(N^{5/4-\tau/2+\varepsilon}).
\]

\[
= -V_N + \sum_j \xi_2(j,V_N) \frac{1}{2p_j} + O(N^{3/4-\tau/4+\varepsilon}) + O(N^{5/4-\tau/2+\varepsilon}).
\]

\[
= -V_N + \hat{A}_2(V_N) + O(N^{3/4-\tau/4+\varepsilon}) + O(N^{5/4-\tau/2+\varepsilon}).
\]

\[
= -V_N + A_2(V_N) + O(N^{3/4+\varepsilon}) + O(N^{5/4-\tau/2+\varepsilon}) a.s.
\] (3.12)

where we used the definition of \( \hat{A}_2(.) \) and Lemma J.

Clearly, using (3.2) and (3.12)

\[
N = H_N + V_N = H_N^* + V_N + O(N^{1/2+\delta}) = A_2(V_N) + O(N^{5/4-\tau/2+\varepsilon}) a.s.
\]

and

\[
V_N = A_2^{-1}(N) + O(N^{5/4-\tau/2+\varepsilon}) a.s.
\]

**Remark 3.1** In the previous line we used the fact that \( A_2^{-1}(u+v) - A_2^{-1}(u) \leq v \). To see this, first recall from Lemma 3.1 that \( A_2(t), A_2^{-1}(t) \) and \( \alpha(t) = A_2(t) - t \) are all nondecreasing. Then

\[
v = A_2(A_2^{-1}(u+v)) - A_2(A_2^{-1}(u)) = \alpha(A_2^{-1}(u+v)) + A_2^{-1}(u+v) - \alpha(A_2^{-1}(u)) - A_2^{-1}(u) \geq A_2^{-1}(u+v) - A_2^{-1}(u).
\]

So we can conclude, using Lemmas H and I that

\[
C_2(N) = S_2(V_N) = W_2(V_N) + O(N^{1/4+\varepsilon}) = W_2((A_2^{-1}(N)) + O(N^{5/4-\tau/2+\varepsilon}) + O(N^{1/4+\varepsilon})
\]

\[
= W_2((A_2^{-1}(N)) + O(N^{5/8-\tau/4+\varepsilon}) a.s.
\]

\[
C_1(N) = S_1(H_N) = S_1(N-V_N) = W_1(N-V_N) + O(N^{1/4+\varepsilon}) = W_1(N-A_2^{-1}(N)) + O(N^{5/8-\tau/4+\varepsilon}) a.s.
\]

proving our theorem. \( \square \)

**Remark 3.2** We could extend our result for the case \( \gamma_1 = \gamma_2 = 1 \), by using the argument of the case \( \gamma_2 = 1 \) for both of the positive and the negative side. Then we would get the following result:
Under the conditions (1.1), and $\gamma_1 = \gamma_2 = 1$ on an appropriate probability space for the random walk $\{C(N) = (C_1(N), C_2(N)); N = 0, 1, 2, \ldots\}$ one can construct two independent standard Wiener processes $\{W_1(t); t \geq 0\}, \{W_2(t); t \geq 0\}$ so that, as $N \to \infty$, we have with any $\varepsilon > 0$

$$|C_1(N)| + |C_2(N) - W_2(N)| = O(N^{5/8-\tau/4+\varepsilon}) \quad \text{a.s.}$$ (3.13)

Remark 3.3 In our paper ([11] Lemma 4.1) we calculated the density function of $A^{-1}(t)$ and $t - A^{-1}(t)$. Our result was the following:

Suppose that $\gamma_1 > \gamma_2 \geq 1$.

$$P(A^{-1}(t) \in dv) = \frac{t}{\pi v \sqrt{(v\gamma_1 - t)(t - \gamma_2 v)}} dv \quad \text{for} \quad \frac{t}{\gamma_1} < v < \frac{t}{\gamma_2},$$

$$P(t - A^{-1}(t) \in dv) = \frac{t}{\pi(t - v) \sqrt{((\gamma_1 - 1)t - \gamma_1 v)(t(1 - \gamma_2) + \gamma_2 v)}} dv$$

$$\quad \text{for} \quad t \left(1 - \frac{1}{\gamma_2}\right) < v < t \left(1 - \frac{1}{\gamma_1}\right).$$

As in our Theorem 1.1 our random walk $\{C(N) = (C_1(N), C_2(N)\}$ is approximated with the same pair of oscillating Wiener processes as in Theorem D, we get the same consequences as in case of the $p_i$’s were restricted to be 1/2 or 1/4. We proved the following laws of the iterated logarithm ([11]).

Corollary A ([11] Corollary 4.1). Under condition (1.1) with $\gamma_1 > \gamma_2 \geq 1$ the following laws of the iterated logarithm hold.

$$\limsup_{t \to \infty} \frac{W_1(t - A^{-1}(t))}{\sqrt{t \log \log t}} = \limsup_{N \to \infty} \frac{C_1(N)}{\sqrt{N \log \log N}} = \sqrt{2 \left(1 - \frac{1}{\gamma_1}\right)} \quad \text{a.s.,}$$

$$\liminf_{t \to \infty} \frac{W_1(t - A^{-1}(t))}{\sqrt{t \log \log t}} = \liminf_{N \to \infty} \frac{C_1(N)}{\sqrt{N \log \log N}} = -\sqrt{2 \left(1 - \frac{1}{\gamma_1}\right)} \quad \text{a.s.,}$$

$$\limsup_{t \to \infty} \frac{W_2(A^{-1}(t))}{\sqrt{t \log \log t}} = \limsup_{N \to \infty} \frac{C_2(N)}{\sqrt{N \log \log N}} = \sqrt{\frac{2}{\gamma_1}} \quad \text{a.s.,}$$

$$\liminf_{t \to \infty} \frac{W_2(A^{-1}(t))}{\sqrt{t \log \log t}} = \liminf_{N \to \infty} \frac{C_2(N)}{\sqrt{N \log \log N}} = -\sqrt{\frac{2}{\gamma_2}} \quad \text{a.s.}$$
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