Nonparametric estimation of a regression function using the gamma kernel method in ergodic processes *

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Abstract. In this paper we consider the nonparametric estimation of density and regression functions with non-negative support using a gamma kernel procedure introduced by Chen ([11]). Strong uniform consistency and asymptotic normality of the corresponding estimators are established under a general ergodic assumption on the data generation process. Our results generalize those of Shi and Song ([37]), obtained in the classic i.i.d. framework, and the works of Bouezmarni and Rombouts [3, 5] and Gospodinov and Hirukawa [20] for mixing time series.

Keywords. Ergodic processes, gamma kernel estimation, regression function, strong uniform consistency, central limit theorem.

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1 Introduction

As it is well known, a major drawback of the standard kernel method for nonparametric curve estimation concerns the presence of the so-called bounded effects. Bounded effects occur when the support of the underlying variables is a subinterval of the real line and the estimates are based on a symmetric kernel, leading to an increase of the bias near the boundary of the support. Since the pioneering works of Gasser and Müller (17), Rice (34), Schuster (36) and Gasser et al. (18), several approaches to overcome this problem have been investigated (for an overview of the main correction techniques, the reader is referred to Simonoff (38), Karunamuni and Alberts (25) and Dai and Sperlich (13)). Among the existing proposals, the boundary kernel method has shown to be one of the most popular. The general idea behind this method is to modify the kernel’s form near the endpoints of the support, either by using adaptive kernels in the boundary region and a fixed symmetric kernel in the interior or by considering asymmetric kernels, whose shape and scale parameters change in accordance with the position of the target point, allowing to adjust the local smoothness of the estimate in a natural way.

Asymmetric kernels, namely beta and gamma kernels, were introduced by Brown and Chen (7) and Chen (10, 11) to estimate densities supported in $[0, 1]$ and $[0, +\infty[$, respectively. Apart from having the same support as the curve under consideration, they present other appealing features such as achieving the optimal convergence rate for the mean integrated square error of classical kernels and showing good finite sample performance.

Regarding gamma kernels, which are the goal of our study, Chen’s proposal and its refinements remained a topic of interest for researchers (c.f. Geenens and Wang (19) for an review on the subject and Malec and Schienle (32) and
Gamma kernel regression estimator for ergodic data

Hirukawa and Sakuda ([23]) for a recent simulation studies), although other types of asymmetric kernels have been suggested in the last decades (e.g. the inverse gaussian and the reciprocal inverse gaussian kernels of Scaillet ([35])). However, as pointed out by Koul and Song ([26]), most existing results are devoted to density estimation and address essentially asymptotic bias, variance and mean square error derivations in the i.i.d. setting. In the last few years, there has been increasing attention given to consistency and limiting distributions of both density and regression estimators in this context (c.f. Bouezmarni and Rombouts ([4]), Bouezmarni et al. ([2]), Shi and Song ([37]), Koul and Song ([26])) as well as their natural extensions in stationary time series context (c.f., for instance, Bouezmarni and Rombouts ([3],[5]), Markovich ([33]), in the class of mixing processes, and Chaubey et al. ([9]) in a larger class).

The last authors, arguing that the traditional mixing hypotheses imposed on the observation process are not satisfied in many cases (several examples of ergodic and non-mixing processes may also be found in Bouzébda and Didi ([6])), worked under the general dependence condition of ergodicity introduced by Gyorfi ([21]) and considered by Delecroix et al. ([14]), Delecroix and Rosa ([15]), Yakowitz et al. ([11]) and Laïb and Ould-Saïd ([29]). In fact, this condition gained a renewed interest after the paper of Laïb and Louani ([27]), giving rise to some new consistency results with convergence rate for nonparametric curve estimation (cf. Laïb and Louani ([27],[28]), Chaouch and Khardani ([8]), Bouzébda and Didi ([6]), Benziadi et al. ([1]), Ling and Liu ([30]), Ling et al. ([31])).

Following the works of Chaubey et al. ([9]) and Shi and Song ([37]), we prove, in the present paper, the uniform consistency and the asymptotic normality of density and regression estimators based on the gamma kernel proposed by Chen ([11]), in the framework of discrete time ergodic processes. With respect to the central limit theorem, we remark that, under mild conditions imposed on the
Gamma kernel regression estimator for ergodic data

bandwidth, the convergence rates and the asymptotic variances obtained in our work agree with those of Shi and Song (37) considering the i.i.d. setup.

The paper is organized as follows: section 2 introduces the estimators as well as the general notations and assumptions on the observation process; section 3 provides the main convergence results and a few commentaries concerning their hypotheses; the proofs of the propositions and some auxiliary lemmas are presented in section 4.

2 Assumptions and notations

Let \( \{(X_t, Y_t), t \in \mathbb{Z}\} \) be a \((\mathbb{R}^+)^2\)-valued stochastic process on the probability space \((\Omega, \mathcal{A}, P)\) which is assumed to be strictly stationary and ergodic, with absolutely continuous margin distributions. The density function of \( X_t \) will be denoted by \( f \).

For each \( x \in \mathbb{R}^+ \) such that \( f(x) > 0 \), \( R(x) = E(\Phi(Y_1)/X_1 = x) \) stands for the conditional expectation of \( \Phi(Y_1) \) given \( X_1 = x \), where \( \Phi \) is a known measurable function of \( \mathbb{R}^+ \) into \( \mathbb{R} \) such that \( E(|\Phi(Y_1)|) < +\infty \).

Based on a sample \( \{(X_t, Y_t)\}_{t=1}^n \), our goal is to study some asymptotic properties of the following estimator of \( R(x) \)

\[
R_n(x) = \frac{\sum_{t=1}^n \Phi(Y_t) K_{\alpha(n,x),\beta(n)}(X_t)}{\sum_{t=1}^n K_{\alpha(n,x),\beta(n)}(X_t)},
\]

\( K_{\alpha(n,x),\beta(n)} \) being the density function of the gamma distribution with shape and scale parameters \( \alpha(n,x) = \frac{x}{h_n} + 1 \) and \( \beta(n) = h_n \), respectively given by

\[
K_{\alpha(n,x),\beta(n)}(y) = \frac{1}{\Gamma(\alpha(n,x)) \beta(n)^{\alpha(n,x)}} y^{\alpha(n,x)-1} e^{-\frac{y}{\beta(n)}} 1_{[0,\infty]}(y), \quad y \in \mathbb{R}.
\]

As usual, \( \{h_n\}_{n \in \mathbb{N}} \) is the bandwidth sequence, i.e. \( h_n \in \mathbb{R}^+, n \in \mathbb{N}, \lim_{n \to +\infty} h_n = 0, \)

\[ 
\]
and we adopt the convention that $\frac{y}{0} = 0$, for all $y \in \mathbb{R}$.

In the sequel, we will consider the $\sigma$-fields

$$
F_t = \sigma \{(X_s, Y_s), s \leq t\}, \quad G_t = \sigma \{X_{t+1}, (X_s, Y_s), s \leq t\}, \quad t \in \mathbb{Z},
$$

and we will denote by $C_0(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$ tending to zero at infinity equipped with the sup-norm, $\|\cdot\|_0$, and by $\|\cdot\|_2$ the norm in $L^2(\Omega, \mathcal{A}, P)$.

For easy reference, the general assumptions needed to derive the announced results are gathered thereafter.

(H1) For all $t \in \mathbb{Z}$, the conditional density of $X_t$ given $F_{t-1}$, $f_{F_{t-1}}$, exists; moreover, $f_{F_{t-1}} \in C_0(\mathbb{R})$ and $f \in C_0(\mathbb{R})$.

(H2) $\frac{1}{n} \sum_{t=1}^{n} f_{F_{t-1}} - f \overset{a.s.}{\underset{n \to +\infty}{\to}} 0$.

(H3) $R$ is a continuous and bounded function on $\mathbb{R}_0^+$.

(H4) $E(\Phi(Y_t)/G_{t-1}) = E(\Phi(Y_t)/X_t) = R(X_t), \quad t \in \mathbb{Z}$.

3 Main results

3.1 Strong uniform consistency of $D_n$ and $R_n$

In order to establish the uniform convergence of $R_n$ on $\Delta = [a, b], \ a, b \in \mathbb{R}^+, \ a < b$, we need a preliminary result concerning the behaviour of the gamma kernel estimator of $f$, i.e.

$$(3.1) \quad D_n(x) = \frac{1}{n} \sum_{t=1}^{n} K_{\alpha_n(x), \beta_n}(X_t).$$

**Theorem 3.1.** If conditions (H1) and (H2) are satisfied and the sequence $(h_n)_{n \in \mathbb{N}}$ is such that

$$
\lim_{n \to +\infty} \frac{n h_n}{\log n} = +\infty,
$$

is such that

$$
\lim_{n \to +\infty} \frac{n h_n}{\log n} = +\infty,
$$

then $D_n(x)$ is a strong uniform consistent estimator of $f(x)$, i.e.

$$
\lim_{n \to +\infty} \sup_{x \in [a, b]} \left| D_n(x) - f(x) \right| = 0.
$$

Moreover, if $f_{F_{t-1}} \in C_0(\mathbb{R})$ and $f \in C_0(\mathbb{R})$, then

$$
\lim_{n \to +\infty} \frac{1}{n} \sum_{t=1}^{n} f_{F_{t-1}} - f = 0,
$$

and

$$
\lim_{n \to +\infty} \sup_{x \in [a, b]} \left| D_n(x) - f(x) \right| = 0.
$$

Finally, if $R$ is a continuous and bounded function on $\mathbb{R}_0^+$, then

$$
\lim_{n \to +\infty} E(\Phi(Y_t)/G_{t-1}) = E(\Phi(Y_t)/X_t) = R(X_t), \quad t \in \mathbb{Z}.
$$
Gamma kernel regression estimator for ergodic data

\[
\sup_{x \in \Delta} |D_n(x) - f(x)| \xrightarrow{a.s.} n \to +\infty 0.
\]

**Theorem 3.2.** In addition to conditions (H1) to (H4), suppose that

\[
E\left(\left|\Phi(Y_1)\right|^{\tau+1}\right) < +\infty, \text{ for some } \tau > 0.
\]

If \( I = \inf_{x \in \Delta} f(x) > 0 \) and the sequence \((h_n)_{n \in \mathbb{N}}\) verifies

\[
n\sqrt{h_n} \uparrow +\infty, \quad \exists \theta \in 0, +\frac{\tau}{\tau+1} \left\{ \lim_{n \to +\infty} \frac{n^\theta h_n}{\log n} = +\infty, \right.\]

we have

\[
\sup_{x \in \Delta} |R_n(x) - R(x)| \xrightarrow{a.s.} n \to +\infty 0.
\]

### 3.2 Asymptotic normality of \(D_n\) and \(R_n\)

Let us begin by presenting some additional assumptions.

(H5) \( E(\Phi^{\zeta+2}(Y_1)) \) exists, for some \( \zeta > 0 \).

(H6) (i) \( E(\Phi(Y_t)^2/G_{t-1}) = E(\Phi(Y_t)^2/X_t) = W_2(X_t), \quad t \in \mathbb{Z}. \)

(ii) \( W_{\zeta+2}(y) = E\left(\left|\Phi(Y_1)^{\zeta+2}/X_1 = y\right|\right), \quad y \in \mathbb{R}_0^+, \) is a bounded function.

(iii) \( \sigma^2(y) = V(\Phi(Y_1)/X_1 = y), \quad y \in \mathbb{R}_0^+, \) is a continuous function.

(H7) The second order derivatives of \( f \) and \( R \) are continuous and bounded on \( \mathbb{R}_0^+ \).

(H8) \( \sup_{y \in \mathbb{R}_0^+} \left\| \sum_{l=1}^{n} f^{l-1}(y) - nf(y) \right\|_2^2 = O(n). \)

We are now in position to state the central limit theorems concerning the gamma kernel estimators of \( f \) and \( R \).
Gamma kernel regression estimator for ergodic data

**Theorem 3.3.** Let $x \in \mathbb{R}_0^+$ be such that $f(x) > 0$. In addition to (H1), (H2), (H7) and (H8), suppose that the sequence \( \left( \frac{1}{n} \sum_{t=1}^{n} \left( f^{T_{t-1}} \right)^2 \right)_{n \in \mathbb{N}} \) converges in $C_0(\mathbb{R})$. If

a) \( \lim_{n \to +\infty} n \sqrt{h_n} = +\infty \) and \( \lim_{n \to +\infty} n \sqrt{h_n^5} = 0 \), then

\[
\sqrt{n \sqrt{h_n}} (D_n(x) - f(x)) \xrightarrow{D} n \to +\infty \mathcal{N} \left( 0, \frac{f(x)}{2\sqrt{\pi x}} \right), \quad x > 0;
\]

b) \( \lim_{n \to +\infty} n h_n = +\infty \) and \( \lim_{n \to +\infty} n h_n^3 = 0 \), then

\[
\sqrt{n h_n} (D_n(0) - f(0)) \xrightarrow{D} n \to +\infty \mathcal{N} \left( 0, \frac{f(0)}{2} \right).
\]

**Theorem 3.4.** Let $x \in \mathbb{R}_0^+$ be such that $f(x) > 0$ and suppose that (H1) to (H8) hold. If

a) \( \lim_{n \to +\infty} n \sqrt{h_n} = +\infty \) and \( \lim_{n \to +\infty} n \sqrt{h_n^5} = 0 \), then

\[
\sqrt{n \sqrt{h_n}} (R_n(x) - R(x)) \xrightarrow{D} n \to +\infty \mathcal{N} \left( 0, \frac{\sigma^2(x)}{2\sqrt{\pi x f(x)}} \right), \quad x > 0;
\]

b) \( \lim_{n \to +\infty} n h_n = +\infty \) and \( \lim_{n \to +\infty} n h_n^3 = 0 \), then

\[
\sqrt{n h_n} (R_n(0) - R(0)) \xrightarrow{D} n \to +\infty \mathcal{N} \left( 0, \frac{\sigma^2(0)}{2f(0)} \right).
\]

The general conditions (H1) to (H8) as well as the hypotheses of Theorems 3.2, 3.3 and 3.4 will be discussed in the next section.

### 3.3 Comments on the assumptions

We remark that assumption (H2) as well as the hypothesis concerning the convergence of \( \left( \frac{1}{n} \sum_{t=1}^{n} \left( f^{T_{t-1}} \right)^2 \right)_{n \in \mathbb{N}} \) rely on the ergodic character of the data and became quite common in the general framework of ergodicity considered in the present paper (c.f. Delecroix et al. [14]), Delecroix and Rosa [15], Laïb and...
Ould-Saïd ([29]) and, more recently, condition (A2) (iii) of Laïb and Louani ([27], [28]), condition (A2) of Chaubey et al. ([9]), conditions (C1), (C2), (N1), (N5) of Bouzebda and Didi ([6]), condition (A3) (iii) of Ling and Liu ([30]), condition (A2) 3. of Ling et al. ([31]).

Assumptions (H4) and (H6) (i) are Markov-type conditions similar to the ones considered by Laïb and Louani ([27], [28]) (c.f. (A3) (i), (ii)), Chaouch and Khardani ([8]) (c.f. (A4), p. 69) and Chaubey et al. ([9]) (c.f. (A4) (i), (ii), (A5) (ii), p. 977). To derive the asymptotic distribution of $R_n$ we use the combination of regularity conditions concerning the density, the regression function and higher conditional moments (namely (H6) (ii), (iii) and (H7)) taken from Shi and Song ([37]) (c.f. (A2), (A3) and (A4), p. 3492) and Chaubey et al. ([9]) (c.f. (A5) (ii) and (A6), p. 977).

(H8) is implied by the dependence condition considered by Chaubey et al. ([9]) (c.f. (A7), p. 977), which was introduced Wu ([40]) (c.f. Lemma 3, p. 13) as an alternative to the usual mixing conditions. It is satisfied by several linear and nonlinear time series, as shown by the authors. The interested reader is also referred to Huang et al. ([24]) for a more detailed discussion on the so-called predictive dependence measures related to this hypothesis.

The conditions required on the bandwidth in Theorems 3.2 and 3.3 correspond to those of Shi and Song (c.f.([37]), theorems 3.2 and 3.4, p. 3493 and 3494, respectively). In spite of being more restrictive than the previous ones, our hypotheses are classical in dependence settings such as mixing. In order to assure the condition imposed on $(h_n)_{n \in \mathbb{N}}$ in Theorem 3.2 we may take, for instance, $h_n = n^{-\alpha}$, with $\alpha < \theta$. As for Theorems 3.3 and 3.4 a possible choice is $h_n = n^{-\alpha}$, with $\frac{2}{3} < \alpha < 2$ in a) and $\frac{1}{3} < \alpha < \frac{1}{2}$ in b).
4 Appendix

Firstly let us introduce some further notations and present two essential equalities that will be needed for the proofs.

For $x \geq 0$, we write $R_n(x) = \frac{N_n(x)}{D_n(x)}$, where
$$N_n(x) = \frac{1}{n} \sum_{t=1}^{n} \Phi(Y_t) K_{\alpha(n,x),\beta(n)}(X_t)$$
and $D_n$ is defined by (3.1). Furthermore, consider
$$\overline{N}_n(x) = \frac{1}{n} \sum_{t=1}^{n} E\left( \Phi(Y_t) K_{\alpha(n,x),\beta(n)}(X_t) / F_{t-1} \right)$$
and
$$\overline{D}_n(x) = \frac{1}{n} \sum_{t=1}^{n} E\left( K_{\alpha(n,x),\beta(n)}(X_t) / F_{t-1} \right).$$

Observe that, under hypotheses (H4) and (H6) (i), a routine argument and the properties of conditional expectation lead to

(i) $E\left( (\Phi(Y_t) - R(X_t))^2 K_{\alpha(n,x),\beta(n)}(X_t) / F_{t-1} \right) = E\left( E\left( (\Phi(Y_t) - R(X_t))^2 / G_{t-1} \right) K_{\alpha(n,x),\beta(n)}(X_t) / F_{t-1} \right)$

(ii) $E\left( (\Phi(Y_t) - R(X_t))^2 K_{\alpha(n,x),\beta(n)}(X_t) / F_{t-1} \right) = E\left( (\Phi(Y_t) - R(X_t))^2 / G_{t-1} \right) K_{\alpha(n,x),\beta(n)}^2(X_t) / F_{t-1}$

We can now present the proofs of the referred theorems.

Let us mention that all the constants appearing hereafter will be denoted generically by $C$.  

9
4.1 Proofs of main results

4.1.1 Proof of Theorem 3.2

We have

\[
\sup_{x \in \Delta} |R_n(x) - R(x)| \leq \left( \inf_{x \in \Delta} |D_n(x)| \right)^{-1} \left\{ \sup_{x \in \Delta} |N_n(x) - R(x)f(x)| + \right.
\]
\[
\left. + \sup_{x \in \Delta} |R(x)| \sup_{x \in \Delta} |D_n(x) - f(x)| \right\}.
\]

Since

\[
\inf_{x \in \Delta} |D_n(x)| \geq \inf_{x \in \Delta} f(x) - \sup_{x \in \Delta} |D_n(x) - f(x)|
\]
and \( I = \inf_{x \in \Delta} f(x) > 0 \), it suffices to prove, by Theorem 3.1, that

\[
(4.2) \quad \sup_{x \in \Delta} |N_n(x) - R(x)f(x)| \xrightarrow{a.s.} 0.
\]

To this end, we remark that

\[
\sup_{x \in \Delta} |N_n(x) - R(x)f(x)| \leq A_n + B_n,
\]

with

\[
A_n = \sup_{x \in \Delta} |N_n(x) - \overline{N}_n(x)| \quad \text{and} \quad B_n = \sup_{x \in \Delta} |\overline{N}_n(x) - R(x)f(x)|.
\]

But

\[
B_n \leq \sup_{x \in \Delta} \left| \int_0^{+\infty} R(y) K_{\alpha(n),\beta(n)}(y) \left( \frac{1}{n} \sum_{t=1}^{n} f_{T_{t-1}}(y) - f(y) \right) dy \right| +
\]
\[
+ \sup_{x \in \Delta} \left| \int_0^{+\infty} R(y) K_{\alpha(n),\beta(n)}(y) f(y) dy - R(x)f(x) \right|
\]

and then

\[
B_n \leq \left\| \frac{1}{n} \sum_{t=1}^{n} f_{T_{t-1}} - f \right\|_{0, y \in \mathbb{R}_+^n} \sup_{y \in \mathbb{R}_+^n} |R(y)| +
\]
\[
+ \sup_{x \in \Delta} \left| \int_0^{+\infty} R(y) K_{\alpha(n),\beta(n)}(y) f(y) dy - R(x)f(x) \right|.
\]
Gamma kernel regression estimator for ergodic data

By (H2), the first term of the last sum tends a.s. to zero. On the other hand, the uniform continuity of $Rf$ on $\Delta$ and Lemma 4.2 assure the convergence to zero of the second term.

In what concerns $A_n$, it is bounded by $A_n^+ + A_n^-$, with $A_n^\pm = \sup_{x \in \Delta} \left| \frac{1}{n} \sum_{t=1}^n Z_{t,n}^\pm(x) \right|$, where

\begin{equation}
Z_{t,n}^\pm(x) = \frac{1}{n} \left\{ \Phi^\pm(Y_t) K_{\alpha(n,x),\beta(n)}(X_t) - E \left( \Phi^\pm(Y_t) K_{\alpha(n,x),\beta(n)}(X_t)/F_{t-1} \right) \right\}
\end{equation}

and

\begin{equation}
\Phi^+(Y_t) = \Phi(Y_t) \mathbf{1}_{\{\Phi(Y_t) \geq M_t\}}, \quad \Phi^-(Y_t) = \Phi(Y_t) - \Phi^+(Y_t),
\end{equation}

with $M_t = tk$, $k = \frac{1-a}{2}$, $t \in \mathbb{N}$.

Hence, by Lemma 4.5, the a.s. convergence of $(A_n)_{n \in \mathbb{N}}$ to zero reduces to showing that

$A_n^- \xrightarrow{n \to +\infty} 0$.

With this purpose, let us consider $\delta_n = n^{-\lambda}$, $\lambda > \frac{3}{2}$, and

$\nu_n = \begin{cases} \frac{b-a}{\delta_n}, & \frac{b-a}{\delta_n} \in \mathbb{N} \\ \left\lfloor \frac{b-a}{\delta_n} \right\rfloor + 1, & \frac{b-a}{\delta_n} \notin \mathbb{N}, \end{cases}$

where $[u]$ denotes the integer part of the real number $u$.

Partitioning $\Delta$ into the intervals

$\Delta_{j,n} = [a + (j - 1)\delta_n, a + j\delta_n], \; j \in \{1, \ldots, \nu_n - 1\}$, $\Delta_{\nu_n,n} = [a + (\nu_n - 1)\delta_n, b]$,

we may write

$A_n^- = \max_{1 \leq j \leq \nu_n} \sup_{x \in \Delta_{j,n}} \left| \frac{1}{n} \sum_{t=1}^n Z_{t,n}^-(x) \right| \leq A_{1,n}^- + A_{2,n}^-,$

with

$A_{1,n}^- = \max_{1 \leq j \leq \nu_n} \sup_{x \in \Delta_{j,n}} \left| \frac{1}{n} \sum_{t=1}^n (Z_{t,n}^-(x) - Z_{t,n}^-(x_j,n)) \right|$ and $A_{2,n}^- = \max_{1 \leq j \leq \nu_n} \left| \frac{1}{n} \sum_{t=1}^n Z_{t,n}^-(x_{j,n}) \right|,$
\( x_{j,n} \) being an arbitrary point in \( \Delta_{j,n} \), \( j = 1, \ldots, \nu_n \).

As for \( A_{-1,n} \), note that, for sufficiently large \( n \) and \( j \in \{1, \ldots, \nu_n\} \),
\[
\frac{1}{n} \sum_{t=1}^{n} \left( Z_{t,n}(x) - Z_{t,n}(x_{j,n}) \right) \leq \\
\leq \frac{1}{n} \sum_{t=1}^{n} \left| \Phi(Y_t) \right| \left| K_{\alpha(n,x),\beta(n)}(X_t) - K_{\alpha(n,x_{j,n}),\beta(n)}(X_t) \right| + \\
+ \frac{1}{n} \sum_{t=1}^{n} \left( \left| \Phi(Y_t) \right| \left| K_{\alpha(n,x),\beta(n)}(X_t) - K_{\alpha(n,x_{j,n}),\beta(n)}(X_t) \right| / F_{t-1} \right) \\
\leq \frac{C}{n} \sum_{t=1}^{n} M_t \frac{|x-x_{j,n}|}{h_n},
\]
by the gamma kernel properties (c.f. Lemma 4.4).

Applying the ergodic theorem, we have
\[
\frac{1}{n} \sum_{t=1}^{n} \left( E\left( \left| \Phi(Y_t) \right| \right) + E\left( \left| \Phi(Y_t) \right| / F_{t-1} \right) \right) \quad \overset{a.s.}{\longrightarrow} \quad 2 E \left( \left| \Phi(Y_1) \right| \right).
\]

Thus, \( A_{-1,n} = O \left( \frac{\delta_n}{\sqrt{h_n}} \right) \) a.s. and, taking into account the choice of \( \delta_n \),
\[
A_{-1,n} \overset{a.s.}{\longrightarrow} 0.
\]

Now we study the behaviour of \( A_{-2,n} \). In order to apply Azuma’s inequality, we must find an upper bound for \( |Z_{t,n}(x_{j,n})| \), \( t \in \{1, \ldots, n\} \), \( j \in \{1, \ldots, \nu_n\} \).

Using the fact that, for \( x > 0 \),
\[
K_{\alpha(n,x),\beta(n)}(X_t) \leq \frac{C}{\sqrt{x h_n}}
\]
(c.f. Shi and Song (37), p. 3505, (5.20)), we get
\[
\forall t \in \{1, \ldots, n\}, \forall x \in \Delta, \quad |Z_{t,n}(x)| \leq 2 |\Phi(Y_t)| K_{\alpha(n,x),\beta(n)}(X_t) \leq C \frac{M_t}{\sqrt{h_n}}.
\]
Consequently,
\[
\forall \varepsilon > 0, \quad P\left(A_{-2,n} > \varepsilon\right) \leq \sum_{j=1}^{\nu_n} P\left( \left| \sum_{t=1}^{n} Z_{t,n}(x_{j,n}) \right| > n\varepsilon \right) = O\left( \nu_n \exp\left( - \frac{C \min h_n M_t}{n^2} \right) \right).
\]
The condition \( \lim_{n \to +\infty} \frac{n^\theta h_n}{\log n} = +\infty \) yields the a.s. convergence of \( A_{2,n}^- \) to zero, as \( n \to +\infty \), via the Borel-Cantelli lemma.

### 4.1.2 Proof of Theorem 3.1

The proof is performed over the same steps that Theorem 3.2 by taking \( \Phi = 1 \) (and thus \( R = 1 \)) and considering the same partition of \( \Delta \). In this case, we obtain

\[
\left| A_{1,n}^- \right| = O \left( \frac{\delta_n}{\sqrt{h_n}} \right) \text{ a.s. and } P \left( A_{2,n}^- > \varepsilon \right) = O \left( \nu_n \exp \left( -Cnh_n \right) \right), \quad \varepsilon > 0.
\]

### 4.1.3 Proof of Theorem 3.4

Let us decompose

\[
R_n(x) - R(x) = \frac{N_{1,n}(x) + N_{2,n}(x)}{D_n(x)}, \quad x \geq 0,
\]

where

\[
N_{1,n}(x) = (N_n(x) - R(x)D_n(x)) - (\overline{N}_n(x) - R(x)\overline{D}_n(x)),
\]

\[
N_{2,n}(x) = \overline{N}_n(x) - R(x)\overline{D}_n(x).
\]

The following notations will be used hereafter. For \( t \in \{1, \ldots, n\} \),

\[
U_{t,n}(x) = V_{t,n}(x) - E(V_{t,n}(x)/\mathcal{F}_{t-1}), \quad x \geq 0,
\]

and

\[
V_{t,n}(x) = \begin{cases} 
\frac{\sqrt{n}}{\sqrt{\pi}} (\Phi(Y_t) - R(x)) K_{\alpha(n),\beta(n)}(X_t), & x > 0 \\
\frac{\sqrt{n}}{\sqrt{\pi}} (\Phi(Y_t) - R(0)) K_{\alpha(n),\beta(n)}(X_t), & x = 0.
\end{cases}
\]

a) Consider \( x > 0 \).

Since \( D_n(x) \xrightarrow{P} f(x) \) and \( \sqrt{n} \sqrt{h_n} N_{2,n}(x) \xrightarrow{P} 0 \), by Lemmas 4.6 a) and 4.7 a), respectively, we only need to show that

\[
\sqrt{n} \sqrt{h_n} N_{1,n}(x) \xrightarrow{L} N \left( 0, \frac{\sigma^2(x) f(x)}{2 \sqrt{\pi x}} \right).
\]
Gamma kernel regression estimator for ergodic data

The fact that \( \sqrt{n} \sqrt{h_n} N_{1,n}(x) = \sum_{t=1}^{n} U_{t,n}(x) \) and, for each \( n \in \mathbb{N} \), \( (U_{t,n}(x))_{t \in \{1, \ldots, n\}} \) is a martingale difference with respect to the filtration \( (\mathcal{F}_{t-1})_{t \in \{1, \ldots, n\}} \) allow us to apply the central limit theorem for discrete time martingales. So, according to Hall and Heyde (([22]), p. 58), we must prove that

\[
(4.9) \quad \sum_{t=1}^{n} E \left( U_{t,n}^2(x) / \mathcal{F}_{t-1} \right) \xrightarrow{P \ n \to \infty} \frac{\sigma^2(x) f(x)}{2 \sqrt{\pi x}},
\]

\[
(4.10) \quad \forall \varepsilon > 0, \quad n E \left( U_{t,n}^2(x) \mathbb{1}_{\{|U_{t,n}(x)| > \varepsilon\}} \right) = o(1).
\]

As (4.10) is established in Lemma 4.8 a), the proof is reduced to checking (4.9).

Based on the equality

\[
E \left( U_{t,n}^2(x) / \mathcal{F}_{t-1} \right) = E \left( V_{t,n}^2(x) / \mathcal{F}_{t-1} \right) - E^2 \left( V_{t,n}(x) / \mathcal{F}_{t-1} \right), \quad t = 1, \ldots, n,
\]

it suffices to show that

\[
(4.11) \quad \sum_{t=1}^{n} E \left( V_{t,n}^2(x) / \mathcal{F}_{t-1} \right) \xrightarrow{P \ n \to \infty} \frac{\sigma^2(x) f(x)}{2 \sqrt{\pi x}},
\]

\[
(4.12) \quad \sum_{t=1}^{n} E^2 \left( V_{t,n}(x) / \mathcal{F}_{t-1} \right) \xrightarrow{P \ n \to \infty} 0.
\]

Let us begin by pointing out that

\[
E \left( V_{t,n}^2(x) / \mathcal{F}_{t-1} \right) = \frac{\sqrt{h_n}}{n} E \left( (\Phi(Y_t) - R(X_t))^2 K^2_{\alpha(n,x),\beta(n)}(X_t) / \mathcal{F}_{t-1} \right) + \frac{\sqrt{h_n}}{n} E \left( (R(X_t) - R(x))^2 K^2_{\alpha(n,x),\beta(n)}(X_t) / \mathcal{F}_{t-1} \right),
\]

since \( E(\Phi(Y_t) - R(X_t) / \mathcal{G}_{t-1}) = 0 \) (c.f. (i), p. 9).

Hence, by (ii) (c.f. p. 9), \( \sum_{t=1}^{n} E \left( V_{t,n}(x) / \mathcal{F}_{t-1} \right) \) is equal to

\[
\frac{\sqrt{h_n}}{n} \sum_{t=1}^{n} E \left( K^2_{\alpha(n,x),\beta(n)}(X_t) \left( \sigma^2(X_t) + (R(X_t) - R(x))^2 \right) / \mathcal{F}_{t-1} \right) = L_{1,n}(x) + L_{2,n}(x),
\]

\[
E \left( U_{t,n}^2(x) / \mathcal{F}_{t-1} \right) \xrightarrow{P \ n \to \infty} \frac{\sigma^2(x) f(x)}{2 \sqrt{\pi x}},
\]

\[
\forall \varepsilon > 0, \quad n E \left( U_{t,n}^2(x) \mathbb{1}_{\{|U_{t,n}(x)| > \varepsilon\}} \right) = o(1).
\]

As (4.10) is established in Lemma 4.8 a), the proof is reduced to checking (4.9).

Based on the equality

\[
E \left( U_{t,n}^2(x) / \mathcal{F}_{t-1} \right) = E \left( V_{t,n}^2(x) / \mathcal{F}_{t-1} \right) - E^2 \left( V_{t,n}(x) / \mathcal{F}_{t-1} \right), \quad t = 1, \ldots, n,
\]

it suffices to show that

\[
(4.11) \quad \sum_{t=1}^{n} E \left( V_{t,n}^2(x) / \mathcal{F}_{t-1} \right) \xrightarrow{P \ n \to \infty} \frac{\sigma^2(x) f(x)}{2 \sqrt{\pi x}},
\]

\[
(4.12) \quad \sum_{t=1}^{n} E^2 \left( V_{t,n}(x) / \mathcal{F}_{t-1} \right) \xrightarrow{P \ n \to \infty} 0.
\]

Let us begin by pointing out that

\[
E \left( V_{t,n}^2(x) / \mathcal{F}_{t-1} \right) = \frac{\sqrt{h_n}}{n} E \left( (\Phi(Y_t) - R(X_t))^2 K^2_{\alpha(n,x),\beta(n)}(X_t) / \mathcal{F}_{t-1} \right) + \frac{\sqrt{h_n}}{n} E \left( (R(X_t) - R(x))^2 K^2_{\alpha(n,x),\beta(n)}(X_t) / \mathcal{F}_{t-1} \right),
\]

since \( E(\Phi(Y_t) - R(X_t) / \mathcal{G}_{t-1}) = 0 \) (c.f. (i), p. 9).

Hence, by (ii) (c.f. p. 9), \( \sum_{t=1}^{n} E \left( V_{t,n}(x) / \mathcal{F}_{t-1} \right) \) is equal to

\[
\frac{\sqrt{h_n}}{n} \sum_{t=1}^{n} E \left( K^2_{\alpha(n,x),\beta(n)}(X_t) \left( \sigma^2(X_t) + (R(X_t) - R(x))^2 \right) / \mathcal{F}_{t-1} \right) = L_{1,n}(x) + L_{2,n}(x),
\]
Gamma kernel regression estimator for ergodic data

with

\[ L_{1,n}(x) = \sqrt{h_n} \int_{0}^{+\infty} K_{\alpha,\beta}^2(y) \left( \sigma^2(y) + (R(y) - R(x))^2 \right) \left( \frac{1}{n} \sum_{t=1}^{n} f_{\mathcal{F}_{t-1}}(y) - f(y) \right) dy, \]

\[ L_{2,n}(x) = \sqrt{h_n} \int_{0}^{+\infty} K_{\alpha,\beta}^2(y) \left( \sigma^2(y) + (R(y) - R(x))^2 \right) f(y) dy. \]

Next, note that \(|L_{1,n}(x)|\) is bounded by

\[ \left| \frac{1}{n} \sum_{t=1}^{n} f_{\mathcal{F}_{t-1}} - f \right| \sqrt{h_n} B(2, n, x) E \left( \sigma^2(G_{2,n,x}) + (R(G_{2,n,x}) - R(x))^2 \right) \]

and we may rewrite \(L_{2,n}(x)\) in the form

\[ L_{2,n}(x) = \sqrt{h_n} B(2, n, x) E \left( \left( \sigma^2(G_{2,n,x}) + (R(G_{2,n,x}) - R(x))^2 \right) f(G_{2,n,x}) \right). \]

From (H6)(iii), Lemmas 4.2 and 4.1 b), it follows that \(\sqrt{h_n} \rightarrow \frac{1}{2 \sqrt{\pi x}}\) as \(n \rightarrow +\infty\)

and

\[ \lim_{n \rightarrow +\infty} E \left( \left( \sigma^2(G_{2,n,x}) + (R(G_{2,n,x}) - R(x))^2 \right) f(G_{2,n,x}) \right) = \sigma^2(x) f(x), \]

which concludes the proof of (4.11), by hypothesis (H2).

In order to obtain (4.12), we shall prove the convergence in mean to zero of the same sequence. As

\[ \sum_{t=1}^{n} E^2 \left( \frac{V_{t,n}(x)}{\mathcal{F}_{t-1}} \right) \leq \sum_{t=1}^{n} \frac{h_n}{n} E \left( \left| R(X_t) - R(x) \right|^2 K_{\alpha,\beta}^2(X_t) / \mathcal{F}_{t-1} \right), \]

we have

\[ E \left( \sqrt{h_n} \int_{0}^{+\infty} (R(y) - R(x))^2 K_{\alpha,\beta}^2(y) f(y) dy \right) = \sqrt{h_n} B(2, n, x) E \left( \left| R(G_{2,n,x}) - R(x) \right|^2 f(G_{2,n,x}) \right) \]

Once again, the announced result is implied by Lemmas 4.2 and 4.1 b).

b) For \(x = 0\), the proof follows the same lines as the previous one by replacing \(\sqrt{h_n}\) by \(h_n\), noting that \(B(2, n, 0) = \frac{1}{2 h_n}\) and applying Lemma 4.2.
4.1.4 Proof of Theorem 3.3

Let us consider the decomposition

\[ D_n(x) - D(x) = D_{1,n}(x) + D_{2,n}(x), \quad x \geq 0, \]

where

\[ D_{1,n}(x) = D_n(x) - \overline{D}_n(x), \quad D_{2,n}(x) = \overline{D}_n(x) - f(x). \]  \hfill (4.13)

Again, our strategy is to prove the asymptotic normality of \( \left( \sqrt{n} \sqrt{h_n} D_{1,n}(x) \right) \) and that \( \sqrt{n} \sqrt{h_n} D_{2,n}(x) \overset{P}{\longrightarrow} 0. \)

In view of Remark 4.2, we proceed with the study of the sequence involving \( D_{1,n}(x) \).

As before, \( U_{t,n}(x) = V_{t,n}(x) - E(V_{t,n}(x)/F_{t-1}), \quad x \geq 0, \) considering

\[ V_{t,n}(x) = \left\{ \begin{array}{ll}
\frac{4 \sqrt{n}}{n} K_{\alpha(n),\beta(n)}(X_t), & x > 0 \\
\frac{4 \sqrt{n}}{n} K_{\alpha(n,0),\beta(n)}(X_t), & x = 0.
\end{array} \right. \] \hfill (4.14)

a) Let \( x > 0 \).

To apply the central limit theorem for discrete time martingales, we write \( \sqrt{n} \sqrt{h_n} D_{1,n}(x) = \sum_{t=1}^{n} U_{t,n}(x) \) and, invoking Remark 4.3, we restrict ourselves to showing that

\[ \sum_{t=1}^{n} E\left( U_{t,n}^2(x)/F_{t-1} \right) \overset{P}{\longrightarrow} \frac{f(x)}{2 \sqrt{\pi x}}. \]

Once more, this result follows from

\[ \sum_{t=1}^{n} E\left( V_{t,n}^2(x)/F_{t-1} \right) \overset{P}{\longrightarrow} \frac{f(x)}{2 \sqrt{\pi x}} \quad \text{and} \quad \sum_{t=1}^{n} E^2(V_{t,n}(x)/F_{t-1}) \overset{P}{\longrightarrow} 0. \]

As in Theorem 3.3, we then write \( \sum_{t=1}^{n} E\left( V_{t,n}^2(x)/F_{t-1} \right) = D_{1,n}^a(x) + D_{1,n}^b(x) \), with

\[ D_{1,n}^a(x) = \sqrt{h_n} \int_0^{+\infty} K_{\alpha(n),\beta(n)}^2(y) \left( \frac{1}{n} \sum_{t=1}^{n} f_{F_{t-1}}(y) - f(y) \right) dy, \]
Gamma kernel regression estimator for ergodic data

\[ D^b_{1,n}(x) = \sqrt{h_n} \int_0^{+\infty} K_{\alpha(n,x),\beta(n)}(y) f(y) dy. \]

Noting that

\[ |D^a_{1,n}(x)| \leq \sqrt{h_n} \left\| \frac{1}{n} \sum_{t=1}^n f^{\mathcal{T}_{t-1}} - f \right\| \sup_{y \in \mathbb{R}_0^+} K_{\alpha(n,x),\beta(n)}(y) = o(1) \quad a.s., \]

by (4.5), and

\[ D^b_{1,n}(x) = \sqrt{h_n} B(2,n,x) E\left( f\left( G(2,n,x) \right) \right) \xrightarrow{n \to +\infty} \frac{f(x)}{2 \sqrt{\pi x}}, \]

we have

\[ D^a_{1,n}(x) + D^b_{1,n}(x) \xrightarrow{n \to +\infty} \frac{f(x)}{2 \sqrt{\pi x}} \quad a.s.. \]

In order to study the behaviour of \( \left( \sum_{t=1}^n E^2 \left( V_{t,n}(x)/\mathcal{F}_{t-1} \right) \right) \) \( n \in \mathbb{N} \), observe that the Cauchy-Schwarz inequality leads to

\[ \sum_{t=1}^n E^2 \left( V_{t,n}(x)/\mathcal{F}_{t-1} \right) \leq \frac{\sqrt{h_n}}{n} \sum_{t=1}^n \int_0^{+\infty} K_{\alpha(n,x),\beta(n)}(y) \left( f^{\mathcal{T}_{t-1}}(y) \right)^2 dy \]

\[ = \sqrt{h_n} \int_0^{+\infty} K_{\alpha(n,x),\beta(n)}(y) \left( \frac{1}{n} \sum_{t=1}^n \left( f^{\mathcal{T}_{t-1}}(y) \right)^2 \right) dy. \]

Therefore, denoting by \( f^* \in C_0(\mathbb{R}) \) the limit of \( \left( \frac{1}{n} \sum_{t=1}^n \left( f^{\mathcal{T}_{t-1}} \right)^2 \right) \) \( n \in \mathbb{N} \), the last term is bounded above by

\[ \sqrt{h_n} \left\| \frac{1}{n} \sum_{t=1}^n \left( f^{\mathcal{T}_{t-1}} \right)^2 - f^* \right\|_0 + \sqrt{h_n} \left| E \left( f^* \left( G(1,n,x) \right) \right) \right|, \]

which tends to zero, as \( n \to +\infty \), by Lemmas 4.2 and 4.1 b).

b) For \( x = 0 \), the proof is straightforward by making the adequate substitutions.

4.2 Auxiliary results

Consider \( p, n \) and \( x \) arbitrary numbers in \( [0, +\infty[ \), \( \mathbb{N} \) and \( \mathbb{R}_0^+ \), respectively. For convenience, \( G_p \equiv G(p,n,x) \) will denote, in the sequel, a real random variable following the gamma distribution with shape parameter \( \frac{p x}{h_n} + 1 \) and scale parameter \( \frac{h_n}{p} \).
Lemma 4.1. Let $\varphi$ be a real function defined on $\mathbb{R}_0^+$ and $T$ a real random variable with density function $g$. If $p \in [1, +\infty]$ and $E(\varphi(T)K_{\alpha,n,x}^p(T))$ exists, we have:

a) $E(\varphi(T)K_{\alpha,n,x}^p(T)) = B(p,n,x)E(\varphi(G_p)g(G_p))$, where

$$B(p,n,x) = \frac{\Gamma\left(\frac{p}{n} + 1\right)}{\Gamma\left(\frac{p}{n} + 1\right) p \frac{p}{n} + 1 \frac{h_n}{1}};$$

b) for $x > 0$, $\lim_{n \to +\infty} h_n^{\frac{p-1}{2}} B(p,n,x) = \frac{1}{\sqrt{p} \sqrt{\frac{2}{\pi x}}}.$

Proof. The proof of a) is trivial. As for b), the proof is performed over the same steps of Chen (c.f. ([11]), p. 474, (3.2) and (3.3)), noting that

$$\forall p \in [1, +\infty[, \forall x > 0, \quad B(p,n,x) = \frac{S\left(\frac{z}{n}\right)}{S\left(\frac{z}{n}\right) \left(\sqrt{2\pi x/n}\right)^{p-1} \sqrt{2}}$$

with $S(z) = \frac{\sqrt{2\pi} e^{-z} z^{z+\frac{1}{2}}}{\Gamma(z+1)}$, $z \geq 0$, and taking into account the properties of $S$. □

Remark 4.1. Notice that, for $x = 0$, the result corresponding to b) is obvious since $h_n^{p-1} B(p,n,0) = \frac{1}{p}$.

Lemma 4.2. If $p > 0$, we have $\lim_{n \to +\infty} E(\ell(G_{(p,n,x)})) = \ell(x)$, for every real function $\ell$ defined on $\mathbb{R}_0^+$, continuous and bounded. The convergence is uniform in every interval in which $\ell$ is uniformly continuous.

Proof. Please see Chaubey et al. ([9], p. 975) and Feller ([16], p. 219). □

Lemma 4.3. (Shi and Song ([37], p. 3491, 3501) Under assumption $(H7)$ we have, for all $x \geq 0$,

a) $E(K_{\alpha(n,x),\beta(n)}(X_1)) = f(x) + \frac{2f'(x)+xf''(x)}{2} h_n + o(h_n);$

b) $E(R(X_1)K_{\alpha(n,x),\beta(n)}(X_1)) = R(x)E(K_{\alpha(n,x),\beta(n)}(X_1)) =$

$$= \left[R'(x) f(x) + \frac{2}{2} R''(x) f(x) + x R'(x) f'(x)\right] h_n + o(h_n).$$ □
Lemma 4.4. The gamma kernel has the following property

\[ \exists C > 0 : \forall y \in \mathbb{R}^+, \forall x, u \in \Delta, \left| K_{\alpha(n,x),\beta(n)}(y) - K_{\alpha(n,u),\beta(n)}(y) \right| \leq \frac{C|x-u|}{\sqrt{h_n^3}}, \]

for sufficiently large \( n \).

Proof. Consider \( n \in \mathbb{N}, y > 0 \) and \( x, u \in \Delta \) arbitrarily fixed, with \( x > u \).

Using a similar argument as Shi and Song (([37]), p. 2506), we make a Taylor expansion of \( K_{\alpha(n,x),\beta(n)}(y) \) at \( x = u \) up to the first order:

\[
K_{\alpha(n,x),\beta(n)}(y) - K_{\alpha(n,u),\beta(n)}(y) = \frac{x-u}{h_n^2 \Gamma(\frac{x}{h_n}+1)} \left( \frac{y}{h_n} \right)^{\frac{x}{h_n}} e^{-\frac{y}{h_n}} \left[ \log \left( \frac{y}{h_n} \right) - \Psi \left( \frac{x}{h_n} + 1 \right) \right],
\]

where \( \Psi \) is the digamma function and \( \tilde{x} \in [u, x[ \).

Now, by Stirling’s formula, \( \Psi \) properties and some algebraic manipulations, the second member of the previous equality takes the form:

\[
\frac{x-u}{\sqrt{2 \pi h_n^3}} \exp \left\{ \frac{x}{h_n} \left( 1 - \frac{y}{x} + \log \left( \frac{y}{x} \right) \right) \right\} \left[ \log \left( \frac{y}{h_n} \right) + \log \left( \frac{x}{h_n} \right) - \Psi \left( \frac{x}{h_n} + 1 \right) \right] (1+o(1)).
\]

Therefore, \( \left| K_{\alpha(n,x),\beta(n)}(y) - K_{\alpha(n,u),\beta(n)}(y) \right| \) is bounded by

\[
\frac{|x-u|}{\sqrt{2 \pi h_n^3}} \exp \left\{ \frac{x}{h_n} \left( 1 - \frac{y}{x} + \log \left( \frac{y}{x} \right) \right) \right\} \left[ \left| \log \left( \frac{y}{h_n} \right) \right| + \frac{2h_n}{x} \right] (1+o(1)).
\]

Taking into account that \( 1 - \frac{y}{x} + \log \left( \frac{y}{x} \right) \leq 0 \) and \( \tilde{x} > a \), we can see that

\[
\frac{x}{h_n} \exp \left\{ \frac{x}{h_n} \left( 1 - \frac{y}{x} + \log \left( \frac{y}{x} \right) \right) \right\} < \frac{2h_n}{x}.
\]

On the other hand, for sufficiently large \( n \),

\[
\left| \log \left( \frac{y}{x} \right) \right| \exp \left\{ \frac{x}{h_n} \left( 1 - \frac{y}{x} + \log \left( \frac{y}{x} \right) \right) \right\} < e^{s_n(y)},
\]

where

\[
s_n(y) = \begin{cases} \left( \frac{h_n}{\alpha} + 1 \right) \log \left( 1 + \frac{h_n}{\alpha} \right) - 1, & y \geq 1 \\ \left( \frac{\alpha}{h_n} - 1 \right) \log \left( 1 - \frac{h_n}{\alpha} \right) + 1, & 0 < y < 1. \end{cases}
\]
Gamma kernel regression estimator for ergodic data

The convergence of the sequence \((h_n)_{n \in \mathbb{N}}\) to zero leads to the desired result. □

For the next lemmas recall the notations introduced in subsection 4.1, namely, (4.3), (4.4), (4.7) and (4.13).

Lemma 4.5. If \(E\left(\left|\Phi(Y_1)\right|^\tau+1\right) < +\infty\), for some \(\tau > 0\), and the sequence \((h_n)_{n \in \mathbb{N}}\) satisfies

\[ n \sqrt{h_n} \uparrow + \infty, \quad \exists \theta \in \mathbb{R} \setminus 0, \tau > 0, \lim_{n \to +\infty} \frac{n^\theta h_n}{\log n} = +\infty, \]

then

\[ \sup_{x \in \Delta} \left| \frac{1}{n} \sum_{t=1}^{n} Z^+_t(x) \right| \xrightarrow{\text{a.s.}} 0. \]

Proof. Noting that, for \(x > 0\), \(K_{\alpha(n,x),\beta(n)}(X_t)\) is bounded by \(C / \sqrt{x h_n}\) (c.f. (4.5)), we may write

\[ \sup_{x \in \Delta} \left| \frac{1}{n} \sum_{t=1}^{n} Z^+_t(x) \right| \leq \frac{C}{n \sqrt{a h_n}} \sum_{t=1}^{n} \left( |\Phi^+(Y_t)| + E\left(\left|\Phi^+(Y_t)\right| / F_{t-1}\right) \right). \]

As \(n \sqrt{h_n} \uparrow + \infty\), it suffices to prove, by Kronecker’s lemma, that

\[ \sum_{t=1}^{+\infty} \frac{1}{t \sqrt{h_t}} \left( |\Phi^+(Y_t)| + E\left(\left|\Phi^+(Y_t)\right| / F_{t-1}\right) \right) < +\infty, \text{ a.s.} \]

To this aim, define

\[ W_n = \sum_{t=1}^{n} \frac{1}{t \sqrt{h_t}} \left( |\Phi^+(Y_t)| + E\left(\left|\Phi^+(Y_t)\right| / F_{t-1}\right) \right), \quad n \in \mathbb{N}. \]

The sequence \((W_n)_{n \in \mathbb{N}}\) satisfies the conditions of Van Ryzin’s lemma (c.f. Van Ryzin [39], p. 1765), in particular, \(E(W_{n+1}/F_n) = W_n + W'_n\), with

\[ W'_n = \frac{2}{(n+1) \sqrt{h_{n+1}}} E\left(\left|\Phi^+(Y_{n+1})\right| / F_n\right). \]

Consequently, to prove (4.15), it is enough to show that \(\sum_{n=1}^{+\infty} E(W'_n) < +\infty\).

Applying Hölder and Markov inequalities, with \(p = \tau + 1\) and \(q = \frac{\tau+1}{\tau}\), we get

\[ E\left(\left|\Phi^+(Y_n)\right|ight) \leq E\left(\left|\Phi(Y_n)\right|^p\right)^{\frac{1}{p}} \left( E\left(\left|\Phi(Y_n)\right|^p\right)\right)^{\frac{1}{q}} m_n^{-\frac{q}{p}} = E\left(\left|\Phi(Y_n)\right|^\tau\right) n^{-\tau k} \]

20
Gamma kernel regression estimator for ergodic data

which implies that
\[ \sum_{n=1}^{\infty} E(W'_n) \leq C \sum_{n=1}^{\infty} \frac{1}{n^{1+k} \sqrt{n}}. \]
The fact that \( \lim_{n \to +\infty} \frac{n^k h_n}{\log n} = +\infty \) and the definition of \( k \) assure the announced result. \( \square \)

Lemma 4.6. Assume that (H1) and (H2) are fulfilled. If

a) \( \lim_{n \to +\infty} n \sqrt{h_n} = +\infty, \) then \( D_n(x) \xrightarrow{P} f(x), \ x > 0; \)
b) \( \lim_{n \to +\infty} n h_n = +\infty, \) then \( D_n(0) \xrightarrow{P} f(0). \)

Proof. Assumptions (H1) and (H2) assure that \( (D_n(x))_{n \in \mathbb{N}} \) converges in probability to \( f(x), \) for all \( x \geq 0. \) The result follows then if we prove that
\[ \lim_{n \to +\infty} E((D_n(x) - \overline{D}_n(x))^2) = 0, \ x \geq 0. \]

To this end, we write
\[ E((D_n(x) - \overline{D}_n(x))^2) = \frac{1}{n} \sum_{t=1}^{n} E\left(\left(K_{\alpha(n,x),\beta(n)}(X_t) - E\left(K_{\alpha(n,x),\beta(n)}(X_t)/F_{t-1}\right)\right)^2\right) \]
\[ = \frac{1}{n^2} \sum_{t=1}^{n} \left(E\left(K_{\alpha(n,x),\beta(n)}^2(X_t) - E^2\left(K_{\alpha(n,x),\beta(n)}(X_t)/F_{t-1}\right)\right)\right) \]
\[ \leq \frac{2}{n^2} \sum_{t=1}^{n} E\left(K_{\alpha(n,x),\beta(n)}^2(X_t)\right) = \frac{2}{n} \int_{0}^{+\infty} K_{\alpha(n,x),\beta(n)}^2(y) f(y) dy \]
\[ = \frac{2}{n} B(2, n, x) E(f(G_{2,n,x})). \]

Hence, using Lemmas 4.1 and 4.2, we get

a) if \( x > 0, \) \( E((D_n(x) - \overline{D}_n(x))^2) \leq \frac{2}{n \sqrt{h_n}} \left(\frac{1}{\sqrt{x \pi}} + o(1)\right) (f(x) + o(1)), \)
b) if \( x = 0, \) \( E((D_n(0) - \overline{D}_n(0))^2) \leq \frac{1}{nh_n} (f(0) + o(1)), \)

achieving the proof of the lemma. \( \square \)
Lemma 4.7. Suppose (H3) and (H8) are verified. If

\[
a) \quad \lim_{n \to +\infty} n \sqrt{h_n} = +\infty \quad \text{and} \quad \lim_{n \to +\infty} n \sqrt{h_n^5} = 0, \quad \text{then} \quad \sqrt{n \sqrt{h_n}} N_{2,n}(x) \xrightarrow{P} 0, \quad x > 0;
\]

\[
b) \quad \lim_{n \to +\infty} n h_n = +\infty \quad \text{and} \quad \lim_{n \to +\infty} n h_n^3 = 0, \quad \text{then} \quad \sqrt{n h_n} N_{2,n}(0) \xrightarrow{P} 0.
\]

Proof. Let us begin by decomposing \( N_{2,n}(x) \):

\[
N_{2,n}(x) = \overline{N}_n(x) - R(x)\overline{D}_n(x) = N_{2,n}^a(x) + N_{2,n}^b(x),
\]

with

\[
N_{2,n}^a(x) = \overline{N}_n(x) - R(x)\overline{D}_n(x) - (E(N_n(x)) - R(x)E(D_n(x))),
\]

(4.16)

\[
N_{2,n}^b(x) = E(N_n(x)) - R(x)E(D_n(x)).
\]

(4.17)

For all \( x \geq 0 \),

\[
N_{2,n}^a(x) = \frac{1}{n} \left[ \sum_{t=1}^{n} E\left( (R(X_t) - R(x)) K_{\alpha(n),\beta(n)}(X_t)/F_{t-1} \right) - n E\left( (R(X_t) - R(x)) K_{\alpha(n),\beta(n)}(X_t) \right) \right]
\]

\[
= \frac{1}{n} \int_{0}^{+\infty} K_{\alpha(n),\beta(n)}(y) (R(y) - R(x)) \left( \sum_{t=1}^{n} f_{F_{t-1}}(y) - n f(y) \right) dy.
\]

Setting \( H_n(y) = \sum_{t=1}^{n} f_{F_{t-1}}(y) - n f(y) \), we are led to

\[
|N_{2,n}^a(x)| \leq 2 \sup_{u \in \mathbb{R}_0^+} |R(u)| \frac{1}{n} \int_{0}^{+\infty} K_{\alpha(n),\beta(n)}(y) |H_n(y)| dy.
\]

Therefore, Cauchy-Schwarz inequality and Fubini-Tonelli theorem yield

\[
E \left( |N_{2,n}^a(x)|^2 \right) \leq \frac{C}{n^2} \int_{0}^{+\infty} K_{\alpha(n),\beta(n)}(y) E \left( H_n^2(y) \right) dy
\]

\[
\leq \frac{C}{n^2} \sup_{y \in \mathbb{R}_0^+} \|H_n(y)\|_2^2 = O \left( \frac{1}{n} \right)
\]

22
Gamma kernel regression estimator for ergodic data

since \( \|H_n(y)\|^2_2 = O(n) \), by hypothesis (H8).

a) Consider \( x > 0 \). Applying Markov’s inequality, we obtain

\[
\forall \varepsilon > 0, \quad P \left( \sqrt{n} \sqrt{h_n} |N_{2,n}^a(x)| > \varepsilon \right) = O \left( \sqrt{h_n} \right),
\]

and thus \( \sqrt{n} \sqrt{h_n} N_{2,n}^a(x) \xrightarrow{P} 0 \) as \( n \to +\infty \).

On the other hand, from Lemma 4.3 b), we have

\[
\sqrt{n} \sqrt{h_n} N_{2,n}^b(x) = \sqrt{n} \sqrt{h_n^3} b(x) + \sqrt{n} \sqrt{h_n} o(h_n),
\]

with

\[
b(x) = R'(x) f(x) + \frac{x}{2} R''(x) f(x) + x R'(x) f'(x),
\]

which concludes the proof of a).

b) For \( x = 0 \), a similar reasoning conduces to

\[
\forall \varepsilon > 0, \quad P \left( \sqrt{n} h_n |N_{2,n}^a(0)| > \varepsilon \right) = O(1)
\]

and

\[
\sqrt{n} h_n N_{2,n}^b(0) = \sqrt{n} h_n^3 R'(0) f(0) + \sqrt{n} h_n o(h_n),
\]

completing the proof of the lemma.

\[\square\]

**Remark 4.2.** In a similar way, the convergence of \( \left( \sqrt{n} \sqrt{h_n} D_{2,n}(x) \right) \) in probability to zero, needed in the proof of Theorem 3.4 (c.f. (4.13)), is based on the decomposition

\[
D_{2,n}(x) = D_{2,n}^a(x) + D_{2,n}^b(x),
\]

with

\[
D_{2,n}^a(x) = \overline{D}_n(x) - E(D_n(x)) \quad \text{and} \quad D_{2,n}^b(x) = E(D_n(x)) - f(x).
\]

In this case, we simply get

\[
D_{2,n}^a(x) = \frac{1}{n} \int_0^{+\infty} K_{a(n,x),\beta(n)}(y) \left( \sum_{t=1}^{n} H_n(y) \right) dy
\]

23
Gamma kernel regression estimator for ergodic data

which leads again to the equality $E\left(|D_{2,n}^2(x)|^2\right) = O\left(\frac{1}{n}\right)$, under (H8).

As for the bias term, it suffices to apply Lemma 4.3 a), giving

$$\sqrt{n} \sqrt{h_n} D_{2,n}^b(x) = \sqrt{n} \sqrt{h_n^b} \left(f'(x) + \frac{zf''(x)}{2}\right) + \sqrt{n} \sqrt{h_n} o(h_n).$$

**Lemma 4.8.** Suppose that conditions (H5) and (H6)(ii) hold. If $(h_n)_n \in \mathbb{N}$ is such that

a) \(\lim_{n \to +\infty} n \sqrt{h_n} = +\infty\), then

$$\forall \varepsilon > 0, \quad n E\left(U_{t,n}^2(x) \mathbb{I}\{|U_{t,n}(x)| > \varepsilon\}\right) = o(1), \quad x > 0.$$

b) \(\lim_{n \to +\infty} n h_n = +\infty\), then

$$\forall \varepsilon > 0, \quad n E\left(U_{t,n}^2(0) \mathbb{I}\{|U_{t,n}(0)| > \varepsilon\}\right) = o(1).$$

**Proof.** Let $x \geq 0$ and $\varepsilon > 0$. By corollary 9.5.2 of Chow and Teicher (\cite{12}, p. 131), we have

$$n E\left(U_{t,n}^2(x) \mathbb{I}\{|U_{t,n}(x)| > \varepsilon\}\right) \leq 4n E\left(V_{t,n}^2(x) \mathbb{I}\{|V_{t,n}(x)| > \varepsilon\}\right).$$

Applying once again the inequalities of Hölder and Markov, with $p = \frac{q+1}{q}$ and $q = \frac{q+1}{q} + 1$, the right-hand side of the last inequality is bounded by

$$4n \left\{E\left(|V_{t,n}(x)|^{2p}\right)\right\}^\frac{1}{p}\left\{P\left(|V_{t,n}(x)| > \frac{\varepsilon}{4}\right)\right\}^\frac{1}{q} \leq 4n E\left(|V_{t,n}(x)|^{2p}\right) \left(\frac{\varepsilon}{4}\right)^\frac{2p}{q}.$$

a) Consider $x > 0$. Jensen’s inequality allows us to write

$$n E\left(|V_{t,n}(x)|^{2p}\right) \leq \frac{h_n^p}{n} 2^{p-1} E\left(\left(|\Phi(Y_t)|^{2p} + |R(x)|^{2p}\right) K_{\alpha(n,x),\beta(n)}(X_t)\right)$$

$$= \frac{h_n^p}{n} 2^{p-1} E\left(\left(E\left(|\Phi(Y_t)|^{2p}/X_t\right) + |R(x)|^{2p}\right) K_{\alpha(n,x),\beta(n)}(X_t)\right)$$

$$\leq \frac{h_n^p}{n} 2^{p-1} \left(\sup_{y \in \mathbb{R}^d} W_{2p}(y) + |R(x)|^{2p}\right) B(2p, n, x) E\left(f(G_{2p,n,x})\right).$$

Finally, hypothesis (H6)(ii), the fact that \(\lim_{n \to +\infty} n \sqrt{h_n} = +\infty\) and the results
Gamma kernel regression estimator for ergodic data

\[
h_n^{p-1/2} B(2p, n, x) \xrightarrow{n \to +\infty} \frac{1}{\sqrt{2p}} \left( \frac{1}{\sqrt{2\pi}} \right)^{2p}, \quad \text{and} \quad E\left(f\left(G(2p,n,x)\right)\right) \xrightarrow{n \to +\infty} f(x),
\]

stated in Lemmas 4.1 b) and 4.2, complete the proof of a).

b) For \( x = 0 \), the proof is similar to the previous one. It suffices to replace \( \sqrt{h_n} \) by \( h_n \), to use the fact that \( B(2, n, 0) = \frac{1}{2h_n} \) and to apply Lemma 4.2. \( \square \)

**Remark 4.3.** Analogously, the proof of the condition corresponding to (4.10) in Theorem 3.3 relies essentially on the inequalities

\[
n E\left(|V_{t,n}(x)|^{2p}\right) \leq C \frac{h_n^{p}}{n^{p}} B(2p, n, x) E\left(f\left(G(2p,n,x)\right)\right), \quad x > 0,
\]

\[
n E\left(|V_{t,n}(0)|^{2p}\right) \leq C \frac{h_n^{p}}{n^{p}} B(2p, n, 0) E\left(f\left(G(2p,n,0)\right)\right)
\]

and uses the same arguments as before.

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