Acylindrical action on the hyperplanes of a CAT(0) cube complex

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Abstract

We prove that if a group acts essentially and acylindrically on the hyperplanes of a finite-dimensional CAT(0) cube complex then it is either acylindrically hyperbolic or virtually cyclic. An action on a CAT(0) cube complex is acylindrical on the hyperplanes if the intersection of the stabilisers of two hyperplanes which are sufficiently far away from each other has its cardinality uniformly bounded.

1 Introduction

A group $G$ is acylindrically hyperbolic [Osi13] if it admits an action on a Gromov-hyperbolic space $X$ which is non-elementary (i.e., with a infinite limit set) and acylindrical, i.e., for every $d \geq 0$, there exist some $R, N \geq 0$ such that, for all $x, y \in X$,

$$d(x, y) \geq R \Rightarrow \# \{g \in G \mid d(x, gx), d(y, gy) \leq d\} \leq N.$$ 

This definition, due to Bowditch (see [Bow06]), essentially generalises the acylindrical actions on trees introduced by Sela [Sel97]. In this context, the acylindricity reduces to the weak acylindricity: there exist some constants $R, N \geq 0$ such that, for all $x, y \in X$,

$$d(x, y) \geq R \Rightarrow |\text{stab}(x) \cap \text{stab}(y)| \leq N.$$ 

Generalising [Mar15, Theorem A], we proved in [Gen16a, Theorem 8.33] that a similar equivalence holds for (hyperbolic) CAT(0) cube complexes, usually considered as generalised trees in higher dimensions.

Theorem 1. [Gen16a] An action on a hyperbolic CAT(0) cube complex is acylindrical if and only if it is weakly acylindrical.

As a consequence, a group acting weakly acylindrically on a hyperbolic CAT(0) cube complex is acylindrically hyperbolic (or virtually cyclic). Recently, Chatterji and Martin improved this corollary:

Theorem 2. [CM16] A group acting essentially, without fixed point at infinity and (non-uniformly) weakly acylindrically on a finite-dimensional irreducible CAT(0) cube complex is either acylindrically hyperbolic or virtually cyclic.

Recall that an action on a CAT(0) cube complex is essential whenever the orbit of any vertex does not lie in the neighborhood of some hyperplane, and that a CAT(0) cube complex is irreducible if it does not split as a cartesian product of two unbounded factors. As suggested by the work of Caprace and Sageev [CS11], irreducible CAT(0) cube complexes may be thought of as cube complexes with a hyperbolic behaviour, justifying the previous result.
In fact, we gave in [Gen16a] another equivalent condition to the acylindricity of an action on a hyperbolic CAT(0) cube complex. Let us say that a group acts \textit{acylindrically on the hyperplanes} of a CAT(0) cube complex if there exist $R, N \geq 1$ such that, for every hyperplanes $J_1, J_2$ separated by at least $R$ other hyperplanes, $|\text{stab}(J_1) \cap \text{stab}(J_2)| \leq N$.

**Theorem 3.** [Gen16a] An action on a hyperbolic CAT(0) cube complex is acylindrical if and only if it is acylindrical on the hyperplanes.

In general, being acylindrical on the hyperplanes is stronger than being weakly acylindrical (see Lemma 27), but this hypothesis allows us to simplify the assumptions of Theorem 2.

**Theorem 4.** Let $G$ be a group acting essentially on a finite-dimensional CAT(0) cube complex. If the action is acylindrical on the hyperplanes, then $G$ is either acylindrically hyperbolic or virtually cyclic.

This is the main result of this paper. For convenience, we introduce the following terminology. An action is

- \textit{non-uniformly acylindrical on the hyperplanes} if there exists some constant $R \geq 1$ such that, for every hyperplanes $J_1, J_2$ separated by at least $R$ other hyperplanes, $\text{stab}(J_1) \cap \text{stab}(J_2)$ is finite;
- \textit{non-uniformly weakly acylindrical} if there exists $R \geq 0$ such that, for all $x, y \in X$, $d(x, y) \geq R$ implies $|\text{stab}(x) \cap \text{stab}(y)| < +\infty$;
- \textit{non-uniformly acylindrical} if, for every $d \geq 0$, there exists some constant $R \geq 0$ such that, for all $x, y \in X$,
  $$d(x, y) \geq R \Rightarrow \# \{g \in G \mid d(x, gx), d(y, gy) \leq d\} < +\infty.$$  

In fact, Theorem 2 is the first step of our proof of Theorem 4. We will give two different alternative proofs of this result, which we think to be of independent interest. As a consequence of Theorem 4, we deduce (see Section 6 for precise definitions):

**Proposition 5.** If a finitely-generated group contains a finitely-generated codimension-one subgroup which has uniformly finite height and which satisfies the bounded packing property, then $G$ is either acylindrically hyperbolic or virtually cyclic.

**Corollary 6.** Let $G$ be a finitely-generated group acting essentially on a uniformly locally finite CAT(0) cube complex $X$. If $X$ contains a hyperplane whose stabiliser is finitely-generated and has uniformly finite height then $G$ is either acylindrically hyperbolic or virtually cyclic.

The paper is organised as follows. In Section 2, we begin by giving some preliminaries on CAT(0) cube complexes, needed in the sequel. Then, we study WPD contracting isometries in Section 3, and give the first proof of Theorem 2. It is worth noticing that the main result of this section produces also an alternative proof of [CM16, Theorem 1.1]; see Remark 21. The second proof of Theorem 2 is given in Section 4, thanks to a non-uniformly acylindrical action on the contact graph. Finally, Section 5 is dedicated to the proof of Theorem 4 and its consequences on codimension-one subgroups are studied in Section 6.
2 Preliminaries

A cube complex is a CW complex constructed by gluing together cubes of arbitrary (finite) dimension by isometries along their faces. Furthermore, it is nonpositively curved if the link of any of its vertices is a simplicial flag complex (i.e., \(n + 1\) vertices span a \(n\)-simplex if and only if they are pairwise adjacent), and CAT(0) if it is nonpositively curved and simply-connected. See [BH99, page 111] for more information.

A fundamental feature of cube complexes is the notion of hyperplane. Let \(X\) be a nonpositively curved cube complex. Formally, a hyperplane \(J\) is an equivalence class of edges, where two edges \(e\) and \(f\) are equivalent whenever there exists a sequence of edges \(e = e_0, e_1, \ldots, e_{n-1}, e_n = f\) where \(e_i\) and \(e_{i+1}\) are parallel sides of some square in \(X\). Notice that a hyperplane is uniquely determined by one of its edges, so if \(e \in J\) we say that \(J\) is the hyperplane dual to \(e\). Geometrically, a hyperplane \(J\) is rather thought of as the union of the midcubes transverse to the edges belonging to \(J\).

The neighborhood \(N(J)\) of a hyperplane \(J\) is the smallest subcomplex of \(X\) containing \(J\), i.e., the union of the cubes intersecting \(J\). In the following, \(\partial N(J)\) will denote the union of the cubes of \(X\) contained in \(N(J)\) but not intersecting \(J\), and \(X \setminus J = (X \setminus N(J)) \cup \partial N(J)\). Notice that \(N(J)\) and \(X \setminus J\) are subcomplexes of \(X\).

**Theorem 7.** [Sag95, Theorem 4.10] Let \(X\) be a CAT(0) cube complex and \(J\) a hyperplane. Then \(X \setminus J\) has exactly two connected components.

The two connected components of \(X \setminus J\) will be referred to as the halfspaces associated to the hyperplane \(J\).

**Distances \(\ell_p\).** There exist several natural metrics on a CAT(0) cube complex. For example, for any \(p \in (0, +\infty)\), the \(\ell_p\)-norm defined on each cube can be extended to a distance defined on the whole complex, the \(\ell_p\)-metric. Usually, the \(\ell_1\)-metric is referred to as the combinatorial distance and the \(\ell_2\)-metric as the CAT(0) distance. Indeed, a CAT(0) cube complex endowed with its CAT(0) distance turns out to be a CAT(0) space [Lea13, Theorem C.9], and the combinatorial distance between two vertices corresponds to the graph metric associated to the 1-skeleton \(X^{(1)}\). In particular, combinatorial geodesics are edge-paths of minimal length, and a subcomplex is combinatorially convex if it contains any combinatorial geodesic between two of its points.

In fact, the combinatorial metric and the hyperplanes are strongly linked together: the combinatorial distance between two vertices corresponds exactly to the number of hyperplanes separating them [Hag08, Theorem 2.7], and

**Theorem 8.** [Hag08, Corollary 2.16] Let \(X\) be a CAT(0) cube complex and \(J\) a hyperplane. The two components of \(X \setminus J\) are combinatorially convex, as well as the components of \(\partial N(J)\).
The $\ell_\infty$-metric, denoted by $d_\infty$, is also of particular interest. Alternatively, given a CAT(0) cube complex $X$, the distance $d_\infty$ between two vertices corresponds to the distance associated to the graph obtained from $X^{(1)}$ by adding an edge between two vertices whenever they belong to a common cube. Nevertheless, the distance we obtain stays strongly related to the combinatorial structure of $X$:

**Proposition 9.** [BvdV91, Corollary 2.5] Let $X$ be a CAT(0) cube complex and $x, y \in X$ two vertices. Then $d_\infty(x, y)$ is the maximal number of pairwise disjoint hyperplanes separating $x$ and $y$.

**Combinatorial projection** In CAT(0) spaces, and so in particular in CAT(0) cube complexes with respect to the CAT(0) distance, the existence of a well-defined projection onto a given convex subspace provides a useful tool. Similarly, with respect to the combinatorial distance, it is possible to introduce a combinatorial projection onto a combinatorially convex subcomplex, defined by the following result.

**Proposition 10.** [Gen15, Lemma 1.2.3] Let $X$ be a CAT(0) cube complex, $C \subset X$ be a combinatorially convex subspace and $x \in X \setminus C$ be a vertex. Then there exists a unique vertex $y \in C$ minimizing the distance to $x$. Moreover, for any vertex of $C$, there exists a combinatorial geodesic from it to $x$ passing through $y$.

The following proposition will be especially useful:

**Proposition 11.** [Gen16b, Proposition 2.6] Let $X$ be a CAT(0) cube complex, $C$ a combinatorially convex subspace, $p : X \to C$ the combinatorial projection onto $C$ and $x, y \in X$ two vertices. The hyperplanes separating $p(x)$ and $p(y)$ are precisely the hyperplanes separating $x$ and $y$ which intersect $C$.

**Definition 12.** Let $L \geq 0$. Two hyperplanes are $L$-well-separated if any collection of hyperplanes intersecting both our two hyperplanes, and which does not contain any facing triple (i.e., three hyperplanes such that each one does not separate the two others), has cardinality at most $L$. Moreover, they are

- **well-separated** if they are $L$-well-separated for some $L \geq 0$;
- **strongly separated** if they are 0-well-separated.

Now, it is possible to characterize the well-separation thanks to the combinatorial projection:

**Proposition 13.** Let $J, H$ be two hyperplanes and let $p : X \to N(J)$ denote the combinatorial projections onto $N(J)$. Then $J$ and $H$ are $L$-well-separated if and only if $p(N(H))$ has diameter at most $L$.

**Proof.** Suppose first that $J$ and $H$ are $L$-well-separated. Let $x, y \in N(H)$ be two vertices and let $\mathcal{H}$ denote the set of the hyperplanes separating them. According to Proposition 11, any hyperplanes separating $p(x)$ and $p(y)$ separates $x$ and $y$. Therefore, $\mathcal{H}$ defines a family of hyperplanes intersecting both $J$ and $H$ which does not contain any facing triple, hence

$$d(x, y) = \# \mathcal{H} \leq L.$$ 

We have proved that $p(N(H))$ has diameter at most $L$.

Conversely, suppose that $p(N(H))$ has diameter at most $L$, and let $\mathcal{H}$ be a finite family of hyperplanes intersecting both $J$ and $H$ which does not contain any facing triple. If $x, y \in N(H)$ are two vertices separated by each hyperplane of $\mathcal{H}$, it follows from Proposition 11 that each hyperplane of $\mathcal{H}$ separates $p(x)$ and $p(y)$, hence
We have proved that $J$ and $H$ are $L$-well-separated.

**Combinatorial isometries of CAT(0) cube complexes.** Let $X$ be a CAT(0) cube complex and $g \in \text{Isom}(X)$ an isometry. As a consequence of [Hag07], we know that exactly one of the following possibilities must happen:

- $g$ is *elliptic*, i.e., $g$ stabilises a cube of $X$;
- $g$ is *loxodromic*, i.e., there exists a bi-infinite combinatorial geodesic on which $(g)$ acts by translations.

Naturally, if $g$ is loxodromic, we call an *axis* of $g$ a bi-infinite combinatorial geodesic $\gamma$ on which $(g)$ acts by translations. We denote by $\mathcal{H}(\gamma)$ the set of the hyperplanes intersecting $\gamma$.

We will say that an isometry is *quasiconvex* if it admits a quasiconvex combinatorial axis. Recall from [Gen16b, Proposition 3.3] that:

**Proposition 14.** [Gen16b] A bi-infinite combinatorial geodesic $\gamma$ is quasiconvex if and only if the join of hyperplanes in $\mathcal{H}(\gamma)$ are uniformly thin.

A join of hyperplanes $(\mathcal{H}, \mathcal{V})$ is the data of two collections of hyperplanes which do not contain any facing triple so that any hyperplane of $\mathcal{H}$ is transverse to any hyperplane of $\mathcal{V}$. It is $C$-thin if $\min(\#\mathcal{H}, \#\mathcal{V}) \leq C$.

## 3 WPD contracting isometries

If a group $G$ acts on a metric space $(S, d)$, and if $g \in G$, we say that $g$ is *WPD* if, for every $d \geq 0$ and $x \in S$, there exists some $m \geq 1$ such that

$$\{ h \in G \mid d(x, hx), d(g^m x, hg^m x) \leq d \}$$

is finite. In [Osi13, Theorem 1.2], Osin proves that a group is acylindrically hyperbolic if and only if it is not virtually cyclic and it acts on a Gromov-hyperbolic space with a WPD loxodromic isometry. This characterization was generalised in [BBF14, Theorem H] by Bestvina, Bromberg and Fujiwara as:

**Theorem 15.** [BBF14] If a group acts on a metric space with a contracting isometry, then it is either virtually cyclic or acylindrically hyperbolic.

Recall that, given a metric space $X$, an isometry $g \in X$ is *contracting* if

- $g$ is *loxodromic*, i.e., there exists $x_0 \in X$ such that $n \mapsto g^n \cdot x_0$ defines a quasi-isometric embedding $\mathbb{Z} \to X$;
- if $C_g = \{ g^n \cdot x_0 \mid n \in \mathbb{Z} \}$, then the diameter of the nearest-point projection of any ball disjoint from $C_g$ onto $C_g$ is uniformly bounded.

For instance, any loxodromic isometry of a Gromov-hyperbolic space is contracting. In [Gen16b, Theorem 3.13], we characterized contracting isometries of CAT(0) cube complexes. In particular,

**Theorem 16.** [Gen16b] An isometry of a CAT(0) cube complex is contracting if and only if it skewers a pair of well-separated hyperplanes.
Now, notice that a hyperplane separating $J_1^+, J_2^+$ delimited by $J_1, J_2$ respectively such that $g^n J_1^+ \subseteq J_2^+$ for some $n \geq 1$.

The main result of this section is:

**Theorem 17.** Let $G$ be a group acting on a CAT(0) cube complex. Then $g \in G$ is a WPD contracting isometry if and only if it skewers a pair $(J_1, J_2)$ of well-separated hyperplanes such that the intersection $\text{stab}(J_1) \cap \text{stab}(J_2)$ is finite.

To prove this theorem, the following result will be needed.

**Proposition 18.** If a group $G$ acts on a CAT(0) cube complex $X$ with a quasiconvex WPD element $g \in G$, then, for every $n \geq 1$, $g^n$ is a WPD element as well.

We begin with a probably well-known lemma; we include a proof here because no reference could be found.

**Lemma 19.** Let $G$ be a group acting on a metric space $(S, d)$ and $g \in G$. Then $g$ is WPD if and only if there exists some $x \in S$ such that, for every $d \geq 0$, there exists some $m \geq 1$ such that \{ $h \in G \mid d(x, hx), d(g^m x, hg^m x) \leq d$ \} is finite.

**Proof.** The implication is clear. Conversely, fix some $d \geq 0$ and some $y \in S$. By assumption, there exist $x \in S$ and $m \geq 1$ such that

$$\{ h \in G \mid d(x, hx), d(g^m x, hg^m x) \leq 2d(x, y) + d \}$$

is finite. Noticing that, for every $h \in G$,

$$d(x, hx) \leq d(x, y) + d(y, hy) + d(hy, hx) = 2d(x, y) + d(y, hy)$$

and similarly

$$d(g^m x, hg^m x) \leq d(g^m x, g^m y) + d(g^m y, hg^m y) + d(hg^m y, hg^m x) \leq 2d(x, y) + d(g^m y, hg^m y),$$

we deduce that \{ $h \in G \mid d(y, hy), d(g^m y, hg^m y) \leq d$ \} is finite. Therefore, $g$ is WPD. \( \Box \)

**Lemma 20.** Let $X$ be a CAT(0) cube complex and $\gamma$ be combinatorial geodesic between two vertices $x, y$ such that every join of hyperplanes in $\mathcal{H}(\gamma)$ are $C$-thin. If $g \in \text{Isom}(X)$ satisfies $d(x, gx), d(y, gy) \leq d$, then $d(z, gz) \leq C + 6d$ for every $z \in \gamma$.

**Proof.** First, fix two combinatorial geodesics $[x, gx]$ and $[y, gy]$, and denote by

- $\mathcal{H}_1$ the set of the hyperplanes separating $\{gx, z\}$ and $\{gz, y\}$;
- $\mathcal{H}_2$ the set of the hyperplanes separating $\{x, gz\}$ and $\{z, gy\}$;
- $\mathcal{H}_3$ the set of the hyperplanes separating $\{x, gx\}$ and $\{z, gz\}$.

Now, notice that a hyperplane separating $x$ and $z$ either belongs to $\mathcal{H}_2$ or $\mathcal{H}_3$, or separates $x$ and $gx$ or $y$ and $gy$. Because $d(x, gx), d(y, gy) \leq d$, we deduce that

$$|d(x, z) - \#\mathcal{H}_2 - \#\mathcal{H}_3| \leq 2d.$$

Similarly, a hyperplane separating $gx$ and $gz$ either belongs to $\mathcal{H}_1$ or $\mathcal{H}_3$, or separates $x$ and $gx$ or $y$ and $gy$, hence

$$|d(gx, gz) - \#\mathcal{H}_1 - \#\mathcal{H}_3| \leq 2d.$$

Consequently,

$$|\#\mathcal{H}_1 - \#\mathcal{H}_2| = |\#\mathcal{H}_1 + \#\mathcal{H}_3 - d(gx, gz) + d(x, z) - \#\mathcal{H}_2 - \#\mathcal{H}_3| \leq |d(gx, gz) - \#\mathcal{H}_1 - \#\mathcal{H}_3| + |d(x, z) - \#\mathcal{H}_2 - \#\mathcal{H}_3| \leq 2d + 2d = 4d.$$
Then, since a hyperplane separating \( z \) and \( gz \) either belongs to \( \mathcal{H}_1 \) or \( \mathcal{H}_2 \), or separates \( x \) and \( gx \) or \( y \) and \( gy \), we deduce that

\[
d(z, gz) \leq \#\mathcal{H}_1 + \#\mathcal{H}_2 + 2d \leq \min(\#\mathcal{H}_1, \#\mathcal{H}_2) + 6d.
\]

To conclude, it is sufficient to notice that any hyperplane of \( \mathcal{H}_1 \) is transverse to any hyperplane of \( \mathcal{H}_2 \), so that \((\mathcal{H}_1, \mathcal{H}_2)\) defines a join of hyperplanes in \( \mathcal{H}(\gamma) \), which has to be \( C \)-thin, ie., \( \min(\#\mathcal{H}_1, \#\mathcal{H}_2) \leq C \). \(\square\)

**Proof of Proposition 18** Let \( \gamma \) be a quasiconvex combinatorial axis for \( g \); according to Proposition 14, we know that there exists some \( C \geq 1 \) such that any join of hyperplanes in \( \mathcal{H}(\gamma) \) is \( C \)-thin. We fix some vertex \( x \in \gamma \) and some \( d \geq 0 \). Because \( g \) is WPD, there exists some \( m \geq 1 \) such that

\[
\{h \in G \mid d(x, hx), d(g^m x, hg^m x) \leq C + 6d\}
\]
is finite. Now, let \( h \in G \) satisfy \( d(x, hx), d(g^m x, hg^m x) \leq d \). Because \( g^m x \) is a vertex of \( \gamma \) between \( x \) and \( g^m x \), it follows from the previous lemma that \( d(g^m x, hg^m x) \leq C + 6d \). We conclude that

\[
\{h \in G \mid d(x, hx), d(g^m x, hg^m x) \leq d\}
\]
is finite, so that \( g^n \) is WPD according to Lemma 19. \(\square\)

**Proof of Theorem 17** First, suppose that \( g \) skewers a pair \( (J_1, J_2) \) of well-separated hyperplanes such that the intersection \( \text{stab}(J_1) \cap \text{stab}(J_2) \) is finite. We already know that \( g \) is contracting thanks to Theorem 16.

If \( J_1^+, J_2^+ \) are halfspaces delimited by \( J_1, J_2 \) respectively such that \( g^n J_1^+ \subsetneq J_2^+ \) for some \( n \geq 1 \), notice that \( H = \text{stab}(J_1) \cap \text{stab}(g^n J_1) \) is finite. Indeed, because there exist only finitely-many hyperplanes separating \( J_1 \) and \( g^n J_1 \), \( H \) contains a finite-index subgroup \( H_0 \) stabilising each of these hyperplanes; since \( J_2 \) separates \( J_1 \) and \( g^n J_1 \), we deduce that \( H_0 \) is a subgroup of \( \text{stab}(J_1) \cap \text{stab}(J_2) \), which is finite. A fortiori, \( H \) must be finite.

Therefore, if we fix a combinatorial axis \( \gamma \) of \( g \), there exists a hyperplane \( J \in \mathcal{H}(\gamma) \) and \( n \geq 1 \) such that \( J \) and \( g^n J \) are disjoint and \( \text{stab}(J) \cap \text{stab}(g^n J) \) is finite. Fix some vertex \( x \in \gamma \cap N(J) \) and some \( d \geq 0 \). If we set

\[
F = \{h \in G \mid d(x, hx), d(g^{n+2d} x, hg^{n+2d} x) \leq d\},
\]
according to Lemma 19, it is sufficient to prove that \( F \) is finite to conclude that \( g \) is a WPD element of \( G \).

Let \( \mathcal{W} = \{g^k J \mid 0 \leq k \leq n + 2d\} \). We claim that for all but at most \( 2d \) elements of \( F \), an element of \( \mathcal{W} \) is sent to a hyperplane separating \( x \) and \( g^{n+2d} x \); for convenience, let \( \mathcal{H}(x, g^{n+2d} x) \) denote the set of the hyperplanes separating \( x \) and \( g^{n+2d} x \). Let \( f \in F \) and \( H \in \mathcal{W} \). Because \( H \) separates \( x \) and \( g^{n+2d} x \), necessarily \( fH \) separates \( fx \) and \( fg^{n+2d} x \). Now, if \( fH \) does not separate \( x \) and \( g^{n+2d} x \), necessarily \( fH \) must separate either \( x \) and \( fx \) or \( g^{n+2d} x \) and \( fg^{n+2d} x \). On the other hand, \( d(x, hx), d(g^{n+2d} x, hg^{n+2d} x) \leq d \), so we know that there can exist at most \( 2d \) such hyperplanes.

Thus, if \( \mathcal{L} \) denotes the set of functions \( (S \subset \mathcal{W}) \to \mathcal{H}(x, g^{n+2d} x) \), where \( S \) has co cardinality at most \( 2d \) in \( \mathcal{W} \), any element of \( F \) induces an element of \( \mathcal{L} \). If \( F \) is infinite, since \( \mathcal{L} \) is finite, there exist infinitely many pairwise distinct elements \( g_{01}, g_{12}, \ldots \in F \) inducing the same function of \( \mathcal{L} \). In particular, \( g_{01}^{-1} g_{12} \) stabilise each hyperplane of a subset \( S \subset \mathcal{W} \) of co cardinality at most \( 2d \). We deduce that there exists some \( 0 \leq k \leq 2d \) such that

\[
\text{stab}(g^k J) \cap \text{stab}(g^{k+n} J) = (\text{stab}(J) \cap \text{stab}(g^n J))^k.
\]
is infinite, which contradicts our assumption.

Conversely, suppose that \( g \) is a WPD contracting isometry. According to Theorem \[16\] \( g \) skewers a pair of well-separated hyperplanes, so there exist some hyperplane \( J \) and some constant \( n \geq 1 \) such that \( J \) and \( g^nJ \) are \( L \)-well-separated. Let \( C \) denote the combinatorial projection of \( N(g^nJ) \) onto \( N(J) \). According to Proposition \[13\] \( C \) has diameter at most \( L \).

Now, fix some vertex \( z \in C \). Because \( g^n \) is WPD as well, since any contracting contracting isometry is quasi convex (see for example \[Gen16b\] Lemma 2.20) so that Proposition \[18\] applies, there exists some \( m \geq 1 \) such that
\[
\{ h \in G \mid d(z, hx), d(g^{nm}z, hg^{mn}z) \leq L \}
\]
is finite. First, we want to prove that \( H = \bigcap_{i=0}^{m+1} \text{stab}(g^{ni}J) \) is finite. Because \( H \) stabilises \( J \) and \( g^nJ \), a fortiori \( H \) stabilises \( C \), whose diameter is at most \( L \); similarly, because \( H \) stabilises \( g^{nm}J \) and \( g^{n(m+1)}J \), \( H \) must stabilise \( g^{nm}C \), which is the combinatorial projection of \( N(g^{n(m+1)}J) \) onto \( N(g^{nm}J) \), and so has diameter at most \( L \). Therefore, \( z \in C \) and \( g^{nm}z \in g^{nm}C \) implies
\[
d(z, hx) \leq L \text{ and } d(g^{nm}z, hg^{mn}z) \leq L
\]
for every \( h \in H \). We conclude that \( H \) is finite. On the other hand, because there exist only finitely many hyperplanes separating \( J \) and \( g^{n(m+1)}J \), we know that \( H \) is a finite-index subgroup of \( \text{stab}(J) \cap \text{stab}(g^{n(m+1)}J) \). Consequently, we have proved that \( g \) skewers the pair \((J, g^{n(m+1)}J)\) of well-separated hyperplanes where \( \text{stab}(J) \cap \text{stab}(g^{n(m+1)}J) \) is finite.

**Remark 21.** Theorem \[17\] provides also an alternative proof of \[CMI6\] Theorem 1.1], which states that if a group \( G \) acts essentially without fixed point at infinity on an irreducible finite dimensional CAT(0) cube complex such that there exist two hyperplanes whose stabilisers intersect along a finite subgroup, then \( G \) must be acylindrically hyperbolic or virtually cyclic. The beginning of the argument remains unchanged: finding two strongly separated hyperplanes \( J_1, J_2 \) such that \( \text{stab}(J_1) \cap \text{stab}(J_2) \) is finite. Next, instead of constructing an über-contraction, we deduce from \[CS11\] Double Skewering Lemma] that there exists an isometry \( g \in G \) skewering the pair \((J_1, J_2)\). According to Theorem \[17\] \( g \) turns out to be a WPD contracting isometry, so that \( G \) must be acylindrically hyperbolic or virtually cyclic as a consequence of Theorem \[15\].

**First proof of Theorem \[\].** According to \[CS11\] Proposition 5.1, \( X \) contains a pair \((J_1, J_2)\) of strongly separated hyperplanes. Then, it follows from \[CS11\] Double Skewering Lemma] that there exists an element \( g \in G \) skewering this pair. This proves that there exist a hyperplane \( J \) and an integer \( n \geq 1 \) such that \( J \) and \( g^{kn}J \) are strongly separated for every \( k \geq 1 \). If we prove that \( \text{stab}(J) \cap \text{stab}(g^{kn}J) \) is finite for some \( k \geq 1 \), then we will be able to conclude that \( g \) is a WPD contracting isometry according to Theorem \[17\] so that the conclusion will follow from Theorem \[15\].

Since the combinatorial projection of \( N(J) \) onto \( N(g^{kn}J) \) is a vertex according to Proposition \[13\] and conversely, we deduce that \( \text{stab}(J) \cap \text{stab}(g^{kn}J) \) fixes two vertices \( x \in N(J) \) and \( x_k \in N(g^{kn}J) \). If we choose \( k \) sufficiently large so that the distance between \( x \) and \( x_k \) turns out to be sufficiently large, we deduce from the non-uniform weak acylindricity of the action that \( \text{stab}(J) \cap \text{stab}(g^{kn}J) \) is finite. \[\]
4 Action on the contact graph

In [Hag14], Hagen associated to any CAT(0) cube complex \( X \) a hyperbolic graph, namely the contact graph \( \Gamma X \). This is the graph whose vertices are the hyperplanes of \( X \) and whose edges link two hyperplanes \( J_1, J_2 \) whenever \( N(J_1) \cap N(J_2) \neq \emptyset \). In [BHS14, Corollary 14.5], Behrstock, Hagen and Sisto proved that, if a group acts geometrically on a CAT(0) cube complex, which admits an invariant factor system, then the induced action on the contact graph is acylindrical. The question of whether this action is acylindrical without additional assumption on the cube complex remains open. The main result of this section suggests a positive answer. It is worth noticing that, although we are not able to deduce a complete acylindricity of the action on the contact graph, no assumption is made on the cube complex and the action of our group is not supposed to be geometric but only to satisfy some weak acylindrical condition.

**Theorem 22.** Let \( G \) be a group acting on a CAT(0) cube complex \( X \). If \( G \acts \Gamma X \) is non-uniformly weakly acylindrical, the induced action \( G \acts \Gamma X \) is non-uniformly acylindrical. In particular, any loxodromic isometry of \( G \) turns out to be WPD.

Given two hyperplanes \( J \) and \( H \), let \( \Delta(J,H) \) denote the maximal length of a chain of pairwise strongly separated hyperplanes \( V_1, \ldots, V_n \) separating \( J \) and \( H \), i.e., for every \( 2 \leq i \leq n-1 \) the hyperplane \( V_i \) separates \( V_{i-1} \) and \( V_{i+1} \). Our first result will be a direct consequence of Lemma 25 and Lemma 26 proved below.

**Proposition 23.** For every hyperplanes \( J \) and \( H \), we have

\[
\Delta(J, H) \leq d_{\Gamma X}(J, H) \leq 5\Delta(J, H).
\]

To prove our lemmas, the following result proved in [Hag12, Chapter 3] will be needed.

**Lemma 24.** Let \( V_1, \ldots, V_n \) be the successive vertices of a geodesic in \( \Gamma X \) and let \( H_1, \ldots, H_m \) denote the hyperplanes separating \( V_1 \) and \( V_n \) in \( X \).

(i) For every \( 2 \leq i \leq n-1 \), there exists some \( 1 \leq j \leq m \) such that \( d_{\Gamma X}(V_i, H_j) \leq 1 \).

(ii) For every \( 1 \leq j \leq m \), there exists some \( 2 \leq i \leq n-1 \) such that \( d_{\Gamma X}(V_i, H_j) \leq 1 \).

**Lemma 25.** If \( J, H \) are two hyperplanes satisfying \( d_{\Gamma X}(J, H) \geq 5n \), there exist at least \( n \) pairwise strongly separated hyperplanes separating \( J \) and \( H \) in \( X \).

**Proof.** Let \( J = V_0, V_1, \ldots, V_{r-1}, V_r = H \) be a geodesic in \( \Gamma X \) between \( J \) and \( H \). According to Lemma 24 for every \( 1 \leq k \leq r-1 \), there exists a hyperplane \( S_k \) separating \( J \) and \( H \) such that \( d_{\Gamma X}(V_k, S_k) \leq 1 \). For every \( 1 \leq k \leq (r-1)/5 \) and every \( 1 \leq j \leq (r-1)/5-k \), we have

\[
d_{\Gamma X}(S_{5k}, S_{5(k+j)}) \geq d_{\Gamma X}(V_{5k}, V_{5(k+j)}) - d_{\Gamma X}(V_{5k}, S_{5k}) - d_{\Gamma X}(V_{5(k+j)}, S_{5(k+j)})
\]

\[
\geq 5j - 1 - 1 \geq 3
\]

Therefore, \( S_{5k} \) and \( S_{5(k+j)} \) are strongly separated.

**Lemma 26.** Let \( J \) and \( H \) be two hyperplanes. If they are separated in \( X \) by \( n \) pairwise strongly separated hyperplanes \( V_1, \ldots, V_n \) such that \( V_i \) separates \( V_{i-1} \) and \( V_{i+1} \) in \( X \) for all \( 2 \leq i \leq n-1 \), then \( d_{\Gamma X}(J, H) \geq n \).

**Proof.** Let \( J = S_0, S_1, \ldots, S_{r-1}, S_r = H \) be a geodesic in \( \Gamma X \) between \( J \) and \( H \). According to Lemma 24 for every \( 1 \leq k \leq n \), there exists some \( 1 \leq n_k \leq r-1 \) such that \( d_{\Gamma X}(V_k, S_{n_k}) \leq 1 \). Notice that, for every \( 1 \leq i < j \leq n \), because \( V_i \) and \( V_j \) are strongly separated, necessarily \( n_i \neq n_j \). Let \( \varphi \) be a permutation so that the sequence \( (n_{\varphi(k)}) \) be increasing. We have
where we used the inequality \( d_{\Gamma X}(V_{\varphi(k)}, V_{\varphi(k+1)}) \geq 3 \), which precisely means that \( V_{\varphi(k)} \) and \( V_{\varphi(k+1)} \) are strongly separated. This completes the proof.

**Proof of Theorem 22** Let \( R_0 \) be the constant given by the non-uniform weak acylindricity of the action \( G \curvearrowright X \). Let \( \epsilon > 0 \) and \( R \geq 5(R_0 + 4(\epsilon + 4\delta) + 6) \). Now, fix two hyperplanes \( J \) and \( H \) satisfying \( d_{\Gamma X}(J, H) \geq R \) and let

\[
F = \{ g \in G \mid d_{\Gamma X}(J, gJ), d_{\Gamma X}(H, gh) \leq \epsilon \}.
\]

According to Proposition 23, there exist \( m \geq R_0 + 4(\epsilon + 4\delta) + 6 \) pairwise strongly separated hyperplanes \( V_1, \ldots, V_m \) separating \( J \) and \( H \) and such that \( V_i \) separates \( V_{i-1} \) and \( V_{i+1} \) for every \( 2 \leq i \leq m - 1 \). In particular, there exist \( 1 \leq p < r < s < q \leq m \) such that

\[
\begin{cases}
|p - q|, |q - s| \geq \epsilon + 2 + 8\delta \\
r, |m - s| > \epsilon \\
|p - q| \geq R_0
\end{cases}
\]

We claim that, for every \( g \in F \) and every hyperplane \( W \) separating \( V_p \) and \( V_q \), \( gW \) separates \( V_r \) and \( V_s \).

First notice that \( d_{\Gamma X}(W, gW) \leq \epsilon + 8\delta + 2 \), where \( \delta \) is the hyperbolicity constant of \( \Gamma X \). Indeed, if \( S_0 = J, S_1, \ldots, S_{r-1}, S_r = H \) is a geodesic in \( \Gamma X \) between \( J \) and \( H \), according to Proposition 23, there exists \( 1 \leq j \leq r - 1 \) such that \( d_{\Gamma X}(W, S_j) \leq 1 \). On the other hand, because \( d_{\Gamma X} \) is \( 8\delta \)-convex [CDP90 Corollaire 5.3] and \( d_{\Gamma X}(J, gJ), d_{\Gamma X}(H, gh) \leq 1 \) by our hypotheses, we have \( d_{\Gamma X}(S_j, gS_j) \leq \epsilon + 8\delta \). Therefore,

\[
d_{\Gamma X}(W, gW) \leq d_{\Gamma X}(W, S_j) + d_{\Gamma X}(S_j, gS_j) + d_{\Gamma X}(gS_j, gW) \leq \epsilon + 8\delta + 2.
\]

For convenience, for every \( 1 \leq i \leq m \), let \( V_i^+ \) (resp. \( V_i^- \)) denote the half-space delimited by \( V_i \) containing \( H \) (resp. \( J \)). If \( gW \subset V_{r+1}^- \), then \( V_{r+1}, \ldots, V_p \) separate \( W \) and \( gW \), hence \( d_{\Gamma X}(W, gW) > |r - p| \geq \epsilon + 2 + 8\delta \) according to Proposition 23, which contradicts what we have noticed. Therefore, \( gW \subset V_r^+ \); otherwise, \( gW \) must intersect \( V_{r+1}^+ \) and \( V_r^- \), and a fortiori \( V_{r+1} \) and \( V_r \), but this is impossible since they are strongly separated. Thus, \( gW \subset V_r^+ \). Similarly, we prove that \( gW \subset V_s^- \). Finally, \( gW \subset V_r^+ \cap V_s^- \).

Now, if \( gW \) does not separate \( V_r \) and \( V_s \), then \( gW^+ \subset V_r^+ \cap V_s^- \) for some half-space \( W^+ \) delimited by \( W \). In particular, because \( W^+ \) contains \( J \) or \( H \), we deduce that either \( V_1, \ldots, V_r \) separate \( J \) and \( gJ \) or \( V_s, \ldots, V_m \) separate \( H \) and \( gh \). Thus, it follows from Proposition 23 that either \( d_{\Gamma X}(J, gJ) \geq r > \epsilon \) or \( d_{\Gamma X}(H, gh) \geq |m - s| > \epsilon \), which is a contradiction. Therefore, \( gW \) separates \( V_r \) and \( V_s \).

Let \( \mathcal{H}(a, b) \) denote the set of the hyperplanes separating \( V_a \) and \( V_b \), and let \( \mathcal{L} \) denote the set of functions \( \mathcal{H}(p, q) \to \mathcal{H}(r, s) \). We have proved that any element of \( F \) induces a function of \( \mathcal{L} \). If \( F \) is infinite, there exist \( g_0, g_1, g_2, \ldots \in F \) inducing the same function of \( \mathcal{L} \). In particular, \( g_0^{-1}g_1, g_0^{-1}g_2, \ldots \) belong to \( I := \bigcap_{W \in \mathcal{H}(p, q)} \text{stab}(W) \). On the other hand, any element of \( I \) stabilizes \( V_{p+1} \) and \( V_{q-1} \), and a fortiori the combinatorial projections.
of $N(V_{p+1})$ onto $N(V_{q-1})$ and of $N(V_{q-1})$ onto $N(V_{p+1})$. So, by applying Proposition 13 we find two vertices $x \in N(V_{p+1})$ and $y \in N(V_{q-1})$ fixed by $I$. Now, $x$ and $y$ are separated by $V_{p+2}, \ldots, V_{q-2}$, hence $d(x, y) \geq |p - q| + 4 > R_0$. The non-uniform weak acylindrical hyperbolicity of the action $\hat{G} \acts X$ implies that $I$ must be finite, a contradiction. Therefore, $F$ is necessarily finite. We conclude that the induced action $G \acts \Gamma X$ is non-uniform acylindrical.

Second proof of Theorem 4 According to [CS11, Proposition 5.1], $X$ contains a pair $(J_1, J_2)$ of strongly separated hyperplanes. Then, it follows from [CS11, Double Skewering Lemma] that there exists an element $g \in G$ skewering this pair. On the other hand, we know that an isometry which skewers a pair of strongly separated hyperplanes induces a loxodromic isometry on the contact graph $\Gamma X$. This is essentially a consequence of Proposition 23; otherwise see [Hag12, Theorem 6.1.1]. Furthermore, $g$ is WPD according to Theorem 22. Therefore, we have proved that $G$ acts on a hyperbolic space with a WPD isometry. The conclusion follows.

5 Proof of the main theorem

We begin by noticing that an non-uniformly acylindrical action on the hyperplanes is non-uniformly weakly acylindrical, so that Theorem 2 applies under the former hypothesis.

Lemma 27. Let $G$ be a group acting on a finite-dimensional CAT(0) cube complex $X$. If the action is non-uniformly acylindrical on the hyperplanes, then it is non-uniformly weakly acylindrical.

Proof. Let $R \geq 1$ be such that, for any hyperplanes $J_1$ and $J_2$ separated by at least $R$ hyperplanes, $\text{stab}(J_1) \cap \text{stab}(J_2)$ is finite. Without loss of generality, we may suppose that $R \geq \dim(X)$. Now, let $L$ be the Ramsey number $\text{Ram}(R + 2)$; by definition, if you color the vertices of the complete graph containing at least $L$ vertices, so that two adjacent vertices have different colors, then there exists a monochromatic set with at least $R + 2$ vertices. If $x, y \in X$ are two vertices satisfying $d(x, y) \geq L$, then $\text{stab}(x) \cap \text{stab}(y)$ contains a finite-index subgroup $H$ which stabilises each hyperplane separating $x$ and $y$. On the other hand, because $d(x, y) \geq L$ and that $X$ does not contain $R + 2$ pairwise transverse hyperplanes, we know that there exist at least $R + 2$ pairwise disjoint hyperplanes separating $x$ and $y$. Therefore, $H$ stabilises two hyperplanes which are separated by at least $R$ other hyperplanes. We conclude that $H$, and a fortiori $\text{stab}(x) \cap \text{stab}(y)$, is finite.

Proof of Theorem 4 According to [CS11, Proposition 2.6], it is possible to write $\mathcal{H} = \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_r$ such that $X$ is isomorphic to the cartesian product of the restriction quotients $X(\mathcal{H}_1) \times \cdots \times X(\mathcal{H}_r)$, where each factor is irreducible, and $G$ contains a finite-index subgroup $\hat{G}$ such that this decomposition is $\hat{G}$-invariant. Notice that, for each $1 \leq i \leq r$, the induced action $\hat{G} \acts X(\mathcal{H}_i)$ is again essential and acylindrical on the hyperplanes, because these properties are preserved under taking a restriction quotient (see [CS11, Proposition 3.2] for essential actions). Therefore, $G$ contains a finite-index subgroup $\hat{G}$ acting essentially and acylindrically on the hyperplanes on a finite-dimensional irreducible CAT(0) cube complex $Y$.

If $\hat{G} \acts Y$ has no fixed point at infinity, then Theorem 2 implies that $\hat{G}$, and a fortiori $G$, is acylindrically hyperbolic or virtually cyclic. Otherwise, [CF16, Proposition 2.26] implies that two cases may happen: either $\hat{G}$ has a finite orbit in the Roller boundary of $Y$, or $\hat{G}$ contains a finite-index subgroup $\hat{G}$ and $Y$ admits a $\hat{G}$-invariant restriction quotient $Z$ such that the induced action $\hat{G} \acts Z$ has no fixed point at infinity. In
the latter case, we can apply Theorem \[2\] to deduce that \(\hat{G}\), and a fortiori \(G\), is either acylindrically hyperbolic or virtually cyclic.

From now on, suppose that the action \(\hat{G} \curvearrowright Y\) has a finite orbit in the Roller boundary of \(Y\). In particular, \(\hat{G}\) contains a finite-index subgroup \(\tilde{G}\) fixing an ultrafilter \(\alpha\) in the Roller boundary of \(Y\). It follows from [CPI16 Theorem B.1] that \(\tilde{G}\) contains a normal subgroup \(F\), which is locally elliptic in the sense that any finitely-generated subgroup of \(F\) fixes a point of \(Y\), such that the quotient \(\tilde{G}/F\) is a finitely-generated free abelian group.

**Claim 28.** There exists a constant \(K\) such that any elliptic subgroup \(H\) of \(\hat{G}\) has cardinality at most \(K\).

Because the action \(\hat{G} \curvearrowright Y\) is acylindrical on the hyperplanes, there exist two constants \(L, N\) such that, for any hyperplane \(J_1\) and \(J_2\) of \(Y\) separated by at least \(L\) hyperplanes, the intersection \(\text{stab}(J_1) \cap \text{stab}(J_2)\) has cardinality at most \(N\). Let \(x \in Y\) be a point fixed by \(H\). Let us denote

\[
U(x, \alpha) = \{ J \text{ hyperplane } | \ x(J) \neq \alpha(J) \},
\]

where \(x\) is thought of as a principal ultrafilter. Alternatively, \(U(x, \alpha)\) can be interpreted as the set of the hyperplanes intersecting a combinatorial ray starting from \(x\) and pointing to \(\alpha\). Because \(H\) fixes \(x\) and \(\alpha\), \(U(x, \alpha)\) is \(H\)-invariant. If \(\mathcal{H}_i\) denotes the set of the hyperplanes \(J\) of \(U(x, \alpha)\) satisfying \(d_\infty(x, N(J)) = i\), then

\[
U(x, \alpha) = \mathcal{H}_0 \sqcup \mathcal{H}_1 \sqcup \mathcal{H}_2 \sqcup \cdots.
\]

Notice that, because \(H\) fixes \(x\) and stabilises \(U(x, \alpha)\), each \(\mathcal{H}_i\) is \(H\)-invariant. On the other hand, according to [Gen16b Claim 4.9], for each \(i \geq 0\), \(\mathcal{H}_i\) is a collection of pairwise transverse hyperplanes, hence \(\# \mathcal{H}_i \leq \dim(X)\). Consequently, if \(H\) has cardinality at least \(N + 2 \cdot (\dim(X)!)^2\), then there exist at least \(N + 2\) pairwise distinct elements \(h_0, \ldots, h_{N+1} \in H\) which induce the same permutation on \(\mathcal{H}_0\) and \(\mathcal{H}_{L+1}\); in particular, \(h_0^{-1}h_1, \ldots, h_0^{-1}h_{N+1}\) define \(N + 1\) pairwise distinct elements fixing the hyperplanes of \(\mathcal{H}_0\) and \(\mathcal{H}_{L+1}\). But, according to [Gen16b Claim 4.10], there exist two hyperplanes \(J_1 \in \mathcal{H}_0\) and \(J_2 \in \mathcal{H}_{L+1}\) separated by at least \(L\) hyperplanes, so we have proved that \(\text{stab}(J_1) \cap \text{stab}(J_2)\) has cardinality at least \(N + 1\), contradicting the acylindricity on the hyperplanes. Therefore, \(H\) has cardinality at most \(N + 2 \cdot (\dim(X)!)^2\), proving our claim.

Suppose now by contradiction that the subgroup \(F < \hat{G}\) is infinite. In particular, \(F\) contains an infinite countable subgroup \(C = \{ g_1, g_2, \ldots \}\). For every \(i \geq 1\), let \(C_i\) denote the subgroup \(\langle g_1, \ldots, g_i \rangle\). We have

\[
C_1 \subset C_2 \subset C_3 \subset \cdots \subset \bigcup_{i \geq 1} C_i = C.
\]

Because \(F\) is locally elliptic, each \(C_i\) has to be elliptic and the previous claim implies that its cardinality is bounded above by a constant which does not depend on \(i\). Therefore, the sequence \(C_1 \subset C_2 \subset \cdots\) must be eventually constant, and in particular \(C\) is necessarily finite, a contradiction. We conclude that \(F\) is finite.

So we know that there exists an exact sequence \(1 \to F \to \tilde{G} \to \mathbb{Z}^k \to 1\) for some \(k \geq 0\), where \(F\) is finite. If \(k \leq 1\), then \(\tilde{G}\) is either finite or virtually infinite cyclic. From now on, suppose that \(k \geq 2\). Let \(a, b \in \tilde{G}\) be the lifts of two independent elements of \(\mathbb{Z}^k\). In particular, for every \(p \geq 1\), the commutator \([a, b^p]\) belongs to the subgroup \(F\); because \(F\) is finite, we deduce that there exist two integers \(n \neq m\) such that \([a, b^m] = [a, b^n]\), which implies \([a, b^{m-n}] = 1\). Let \(A\) denote the subgroup of \(\tilde{G}\) generated by \(a\) and \(b^{m-n}\); this is a free abelian subgroup of rank two.
We know from Fact 28 that the induced action $A \curvearrowright Y$ has no global fixed point, so, by a result proved by Sageev (see [CFI16, Proposition B.8]), there exist a hyperplane $J$ of $Y$ and an element $g \in A$ such that $gJ^+ \subsetneq J^+$ for some half-space $J^+$ delimited by $J$. In particular, for every $n \geq 1$, the hyperplanes $J$ and $g^nJ$ are separated by $n - 1$ hyperplanes, namely $gJ, g^2J, \ldots, g^nJ$. Therefore, because the action $A \curvearrowright Y$ is also acylindrical on the hyperplanes, for some sufficiently large $n \geq 1$, the intersection $\text{stab}_A(J) \cap \text{stab}_A(g^nJ)$ is finite, and a fortiori trivial since $A$ is torsion-free. Thus, because $A$ is abelian,

$$\text{stab}_A(J) = \text{stab}_A(J) \cap \text{stab}_A(g^nJ) = \text{stab}_A(J) \cap \text{stab}_A(g^nJ) = \{1\}.$$ 

It follows from [Sag95, § 3.3] that $\{1\}$ is a codimension-one subgroup of $A$ (see the next section for a precise definition), or equivalently, that $A$ has at least two ends. We get a contradiction, since we know that $A$ is a free abelian group of rank two.

We conclude that $G$ is necessarily virtually cyclic. 

**Remark 29.** In the statement of Theorem 4, acylindrical on the hyperplanes cannot be replaced with non-uniformly acylindrical on the hyperplanes. Indeed, it is a consequence of the construction given in the proof of [Ser03, Theorem I.6.15] that any countable locally finite group acts essentially on a CAT(0) cube complex, transitive on the hyperplanes. Explicitly, let $G$ be a countable locally finite group. Because it is countable, it can be written as a union of finitely-generated subgroups $G_1 \subset G_2 \subset \cdots$, which are finite since $G$ is locally finite; for a specific example, you may consider

$$
\mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \cdots \subset \bigoplus \limits_{i \geq 1} \mathbb{Z}_2.
$$

Now, let $T$ be the graph whose vertices are the cosets $gG_n$, where $g \in G$ and $n \geq 1$, and whose edges link two cosets $gG_n$ and $hG_{n+1}$ if their images in $G/G_n$ coincide. This is a tree, the action is essential and non-uniformly acylindrical on the hyperplanes. Notice that the action is non-uniformly weakly acylindrical (but not weakly acylindrical). This does not contradict Theorem 2 because $G$ fixes the point at infinity corresponding to the geodesic ray $(G_1, G_2, G_3, \ldots)$.

**Question 30.** Let $G$ be a group acting essentially on a finite-dimensional CAT(0) cube complex. If the action is non-uniformly acylindrical on the hyperplanes, is $G$ either acylindrically hyperbolic or (locally finite)-by-cyclic?

### 6 An application to codimension-one subgroups

Given a finitely-generated group $G$ and a subgroup $H \leq G$, we can define the relative number of ends $e(G, H)$ as the number of ends of the quotient of a Cayley graph of $G$ (with respect to a finite generating set) by the action of $H$ by left-multiplication; this definition does not depend on the choice of the generating set. See [Sco77] for more information. If $e(G, H) \geq 2$, we say that $H$ is a codimension-one subgroup. The main result of [Sag95] states that to any codimension-one subgroup is associated an essential action of $G$ on a CAT(0) cube complex, transitive on the hyperplanes, so that the hyperplane-stabilisers contain a conjugate of our codimension-one subgroup as a finite-index subgroup. Unfortunately, this complex may be infinite-dimensional. Nevertheless, the CAT(0) cube complex we construct turns out to be finite-dimensional if our codimension-one subgroup $H \leq G$ is finitely-generated and satisfies the bounded packing property: fixing a Cayley graph of $G$ (with respect to a finite generating set), for every $D \geq 1$, there is a number $N$ so that, for any collection of $N$ distinct cosets of $H$ in $G$, at least two are separated by a distance at least $D$. Although not stated explicitly, this idea goes back to [Sag97]; see [HW08] for more information. Therefore,
Theorem 31. Let $G$ be a finitely-generated group and $H$ a finitely-generated codimension-one subgroup. Suppose that $H$ satisfies the bounded packing property. Then $G$ acts essentially on a finite-dimensional CAT(0) cube complex, transitively on the hyperplanes, so that its hyperplane stabilisers contain a conjugate of $H$ as a finite-index subgroup.

The combination of the previous result with Theorem 31 essentially proves Proposition 32. First, recall a subgroup $H \leq G$ has (uniformly) finite height if there exists some $n \geq 1$ such that, for every collection of $n$ distinct cosets $g_1 H, \ldots, g_n H$, the cardinality of the intersection $\bigcap_{i=1}^n g_i H g_i^{-1}$ is (uniformly) finite.

Proof of Proposition 32. It follows from Theorem 31 that $G$ acts essentially on a finite-dimensional CAT(0) cube complex $X$, transitively on the hyperplanes, so that its hyperplane-stabilisers contain a conjugate of $H$ as a finite-index subgroup. Since there is only one orbit of hyperplanes, there exists some $M \geq 0$ such that any hyperplane-stabiliser contains a conjugate of $H$ as subgroup of index at most $M$. Then, since $H$ has uniformly finite height, we can fix some $n, N \geq 1$ such that, for every collection of $n$ distinct cosets $g_1 H, \ldots, g_n H$, the intersection $\bigcap_{i=1}^n g_i H g_i^{-1}$ has cardinality at most $N$.

Let $J_1, J_2$ be two hyperplanes separated by at least $n \cdot \dim(X)$ other hyperplanes. Let $\mathcal{H}$ be the set of the hyperplanes separating $J_1$ and $J_2$. For $i \geq 0$, let $\mathcal{H}_i$ denote the set of the hyperplanes $J$ of $\mathcal{H}$ such that $d_{\infty}(N(J), N(J_i)) = i$, i.e., the maximal number of pairwise disjoint hyperplanes separating $J_1$ and $J$ is $i$; it is not difficult to notice that the hyperplanes of $\mathcal{H}_i$ are pairwise transverse, hence $\# \mathcal{H}_i \leq \dim(X)$ (see for instance the proof of [Gen16b, Claim 4.9]). In particular, because $J_1$ and $J_2$ are separated by at least $n \cdot \dim(X)$ hyperplanes, we know that $\mathcal{H}_0, \ldots, \mathcal{H}_{n-1}$ are non-empty. Moreover, $K = \text{stab}(J_1) \cap \text{stab}(J_2)$ contains a subgroup $K_0$ of index at most $(\dim(X))!^n$ which stabilises each hyperplane of $\mathcal{H}_0, \ldots, \mathcal{H}_{n-1}$. Therefore, $K_0$ is contained in the intersection $I$ of the stabilisers of $n$ distinct hyperplanes. On the other hand, according to the group theoretical lemma below, $I$ contains the intersection of $n$ conjugates of $H$, corresponding to $n$ distinct cosets, as a subgroup of index at most $M^n!$. We conclude that $\# K_0 \leq \# I \leq M^n! \cdot N$, and finally that $\# K \leq (\dim(X))!^n \cdot \# K_0 \leq M^n! \cdot (\dim(X))!^n \cdot N$.

We have proved that, for any pair of hyperplanes separated by at least $n \cdot \dim(X)$ other hyperplanes, the intersection of their stabilisers has cardinality at most $M^n! \cdot (\dim(X))!^n \cdot N$. Thus, the action $G \acts X$ is acylindrical on the hyperplanes. It follows from Theorem 31 that $G$ is either acylindrically hyperbolic or virtually cyclic.

Lemma 32. Let $G$ be a group and $G_1, H_1, \ldots, G_k, H_k \leq G$ a collection of subgroups. Suppose that, for every $1 \leq i \leq k$, $H_i$ is a subgroup of $G_i$ of index at most $n_i$. Then $\bigcap_{i=1}^k H_i$ has index at most $(n_1 \cdots n_k)!$ in $\bigcap_{i=1}^k G_i$.

Proof. Let $\bigcap_{i=1}^k G_i$ act by diagonal left multiplication on $G_1/H_1 \times \cdots \times G_k/H_k$. This defines a homomorphism

$$\varphi : \bigcap_{i=1}^k G_i \to \text{Sym}(G_1/H_1 \times \cdots \times G_k/H_k).$$

We clearly have $\ker(\varphi) \subset \bigcap_{i=1}^k H_i$, hence

$$\left| \bigcap_{i=1}^k G_i/\bigcap_{i=1}^k H_i \right| \leq \left| \bigcap_{i=1}^k G_i/\ker(\varphi) \right| \leq \# \text{Sym}(G_1/H_1 \times \cdots \times G_k/H_k) = (n_1 \cdots n_k)!.$$
This proves our lemma.

\textbf{Proof of Corollary}\ref{corollary}. Let $J$ be a hyperplane whose stabiliser is finitely-generated and has uniformly finite height. Because the action is essential, $H = \text{stab}(J)$ is a codimension-one subgroup according to \cite[§ 3.3]{Sag95}, and we know that $H$ satisfies the bounded packing property according to \cite[Theorem 3.2]{HW08}. Thus, the conclusion follows directly from Proposition \ref{proposition}.

\textbf{Question 33.} If the subgroup of Proposition \ref{proposition} does not satisfy the bounded packing property, does the conclusion still hold?

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