Reverse and improved inequalities for operator monotone functions

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1. Introduction. Consider a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). An operator \(T\) is said to be positive (denoted by \(T \geq 0\)) if \(\langle Tx, x \rangle \geq 0\) for all \(x \in H\) and also an operator \(T\) is said to be strictly positive (denoted by \(T > 0\)) if \(T\) is positive and invertible. A real valued continuous function \(f(t)\) on \([0, \infty)\) is said to be operator monotone if \(f(A) \geq f(B)\) holds for any \(A \geq B \geq 0\), which is defined as \(A - B \geq 0\).

In 1934, K. Löwner [6] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144–145]:

**Theorem 1.** A function \(f : [0, \infty) \to \mathbb{R}\) is operator monotone in \([0, \infty)\) if and only if it has the representation

\[
(1.1) \quad f(t) = f(0) + bt + \int_{0}^{\infty} \frac{ts}{t+s} \, dm(s)
\]
where $b \geq 0$ and $m$ is a positive measure on $[0, \infty)$ such that

$$\int_0^\infty \frac{s}{1 + s} dm(s) < \infty.$$ 

We recall the important fact proved by Löwner and Heinz which states that the power function $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, see [5].

Let $f : (0, \infty) \to (0, \infty)$ be a continuous function. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [3] or [7].

Consider the family of functions defined on $(0, \infty)$ by

$$f_p(t) := \frac{p - 1}{p} \left( \frac{tp - 1}{tp - 1 - 1} \right)$$

if $p \in [-1, 2] \setminus \{0, 1\}$ and

$$f_0(t) := \frac{t}{1 - t} \ln t,$$

$$f_1(t) := \frac{t - 1}{\ln t} \quad \text{(logarithmic mean)}.$$ 

We also have the functions of interest:

$$f_{-1}(t) = \frac{2t}{1 + t} \quad \text{(harmonic mean)}, \quad f_{1/2}(t) = \sqrt{t} \quad \text{(geometric mean)}.$$ 

In [2], the authors showed that $f_p$ is operator monotone for $1 \leq p \leq 2$. In the same category, we observe that the function

$$g_p(t) := \frac{t - 1}{tp - 1}$$

is an operator monotone function for $p \in (0, 1]$, see [3].

It is well known that the logarithmic function $\ln$ is operator monotone and in [3], the author proved that the functions

$$f(t) = t (1 + t) \ln \left( 1 + \frac{1}{t} \right), \quad g(t) = \frac{1}{(1 + t) \ln (1 + \frac{1}{t})}$$

on $(0, \infty)$ are also operator monotone.

Let $A$ and $B$ be strictly positive operators on a Hilbert space $H$ such that $B - A \geq m1_H > 0$. In 2015, T. Furuta [4] obtained the following result for any non-constant operator monotone function $f$ on $[0, \infty)$:

$$f(B) - f(A) \geq f(\|A\| + m) - f(\|A\|) \geq f(\|B\|) - f(\|B\| - m) > 0.$$
If $B > A > 0$, then
\[ f(B) - f(A) \geq f\left(\|A\| + \frac{1}{\|(B - A)^{-1}\|}\right) - f(\|A\|) \]
\[ \geq f(\|B\|) - f\left(\|B\| - \frac{1}{\|(B - A)^{-1}\|}\right) > 0. \tag{1.3} \]

The inequality between the first and third term in (1.3) was obtained earlier by H. Zuo and G. Duan in [9].

By taking $f(t) = t^r$, $r \in (0, 1]$ in (1.3), Furuta obtained the following refinement of the celebrated Löwner–Heinz inequality
\[ B^r - A^r \geq \left(\|A\| + \frac{1}{\|(B - A)^{-1}\|}\right)^r - \|A\|^r \]
\[ \geq \|B\|^r - \left(\|B\| - \frac{1}{\|(B - A)^{-1}\|}\right)^r > 0 \]
provided $B > A > 0$.

With the same assumptions for $A$ and $B$, we have the logarithmic inequality [4]:
\[ \ln B - \ln A \geq \ln \left(\|A\| + \frac{1}{\|(B - A)^{-1}\|}\right) - \ln (\|A\|) \]
\[ \geq \ln (\|B\|) - \ln \left(\|B\| - \frac{1}{\|(B - A)^{-1}\|}\right) > 0. \tag{1.5} \]

Notice that the inequalities between the first and third terms in (1.4) and (1.5) were obtained earlier by M. S. Moslehian and H. Najafi in [8].

Motivated by the above results, we show in this paper that if $f : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ and there exist positive numbers $d > c > 0$ such that the condition $d1_H \geq B - A \geq c1_H > 0$ is satisfied, then
\[ d \frac{f(c) - f(0)}{c}1_H \geq f(B) - f(A) \geq c \frac{f(d + \|A\|) - f(\|A\|)}{d}1_H \geq 0. \tag{1.6} \]

Some examples of interest, including a refinement and a reverse of the Löwner–Heinz inequality, are also provided.

2. Main Results. We have:

**Theorem 2.** Assume that $f : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ given by representation (1.1). Let $A \geq 0$ and assume that there exist positive numbers $d > c > 0$ such that
\[ d1_H \geq B - A \geq c1_H > 0. \tag{2.1} \]
Then
\[
d \left( \frac{f(c) - f(0)}{c} - b \right)_{1_H} \geq f(B) - f(A) - b(B - A)
\]
(2.2)
\[
\geq c \left( \frac{f(d + \|A\|) - f(\|A\|)}{d} - b \right)_{1_H} \geq 0.
\]

**Proof.** Since the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator monotone in \([0, \infty)\),
then \( f \) can be written as in the equation (1.1) and for \( A, B \geq 0 \) we have
the representation
\[
f(B) - f(A)
\]
(2.3)
\[
= b(B - A) + \int_0^{\infty} s \left[ B(B + s1_H)^{-1} - A(A + s1_H)^{-1} \right] dm(s).
\]

Observe that for \( s > 0 \),
\[
B(B + s1_H)^{-1} - A(A + s1_H)^{-1}
\]
\[
= (B + s1_H - s1_H)(B + s1_H)^{-1} - (A + s1_H - s1_H)(A + s1_H)^{-1}
\]
\[
= (B + s1_H)(B + s1_H)^{-1} - s1_H(B + s1_H)^{-1} - (A + s1_H)(A + s1_H)^{-1} + s1_H(A + s1_H)^{-1}
\]
\[
= 1_H - s1_H(B + s1_H)^{-1} - 1_H + s1_H(A + s1_H)^{-1}
\]
\[
= s \left[ (A + s1_H)^{-1} - (B + s1_H)^{-1} \right].
\]

Therefore, (2.3) becomes (see also [4])
\[
f(B) - f(A)
\]
(2.4)
\[
= b(B - A) + \int_0^{\infty} s^2 \left[ (A + s1_H)^{-1} - (B + s1_H)^{-1} \right] dm(s).
\]

The function \( g(t) = -t^{-1} \) is operator monotone on \((0, \infty)\), operator Gâteaux
differentiable and the Gâteaux derivative is given by
\[
\nabla g_T(S) := \lim_{t \to 0} \left[ \frac{g(T + tS) - g(T)}{t} \right] = T^{-1}ST^{-1}
\]
(2.5)
for \( T, S > 0 \).

Consider the continuous function \( g \) defined on an interval \( I \) for which the
corresponding operator function is Gâteaux differentiable and for selfadjoint
operators \( C, D \) with spectra in \( I \) we consider the auxiliary function defined
on \([0, 1]\) by
\[
g_{C, D}(t) = g \left( (1 - t)C + tD \right), \; t \in [0, 1].
\]
If \( g_{C, D} \) is Gâteaux differentiable on the segment
\[
[C, D] := \{(1 - t)C + tD, \; t \in [0, 1]\},
\]
then, by the properties of the Bochner integral, we have
\[
(2.6) \quad g(D) - g(C) = \int_0^1 \frac{d}{dt} (g_{C,D}(t)) \, dt = \int_0^1 \nabla g_{(1-t)C+tD} \, (D - C) \, dt.
\]

If we write this equality for the function \( g(t) = -t^{-1} \) and \( C, D > 0 \), then we get the representation
\[
(2.7) \quad C^{-1} - D^{-1} = \int_0^1 ((1-t) \, C + tD)^{-1} \, (D - C) \, ((1-t) \, C + tD)^{-1} \, dt.
\]

Now, if we replace in (2.7): \( C = A + s_1H \) and \( D = B + s_1H \) for \( s > 0 \), then we get
\[
(2.8) \quad (A + s_1H)^{-1} - (B + s_1H)^{-1} = \int_0^1 ((1-t) \, A + tB + s_1H)^{-1} \, (B - A) \, ((1-t) \, A + tB + s_1H)^{-1} \, dt.
\]

By the representation (2.4), we derive the following identity of interest
\[
(2.9) \quad f(B) - f(A) = b \, (B - A) + \int_0^\infty s^2 \left[ \int_0^1 ((1-t) \, A + tB + s_1H)^{-1} \times (B - A) \, ((1-t) \, A + tB + s_1H)^{-1} \, dt \right] \, dm(s)
\]
for \( A, B \geq 0 \).

From the representation (2.9) we get
\[
\frac{f(x) - f(0) - bx}{x} = \int_0^\infty s^2 \left( \int_0^1 (tx + s)^{-1} \, x \, (tx + s)^{-1} \, dt \right) \, dm(s)
\]
for \( B = x_1H, A = 0 \), which for \( x > 0 \) gives
\[
(2.10) \quad \frac{f(x) - f(0)}{x} - b = \int_0^\infty s^2 \left( \int_0^1 (tx + s)^{-2} \, dt \right) \, dm(s).
\]

Since \( 0 < c_1H \leq B - A \leq d_1H \), we have
\[
c \, ((1-t) \, A + tB + s_1H)^{-2} \leq ((1-t) \, A + tB + s_1H)^{-1} \, (B - A) \, ((1-t) \, A + tB + s_1H)^{-1} \leq d \, ((1-t) \, A + tB + s_1H)^{-2}
\]
for \( t \in [0,1], s > 0 \) and by (2.9), we get
\[
(2.11) \quad c \int_0^\infty s^2 \left( \int_0^1 ((1-t) \, A + tB + s_1H)^{-2} \, dt \right) \, dm(s) \leq f(B) - f(A) - b \, (B - A) \leq d \int_0^\infty s^2 \left( \int_0^1 ((1-t) \, A + tB + s_1H)^{-2} \, dt \right) \, dm(s).
\]
Observe that for \( t \in [0, 1] \) and \( s > 0 \) we have
\[
(1 - t) A + tB + s1_H = A + t(B - A) + s1_H \\
\geq 0 + tc1_H + s1_H = (tc + s)1_H.
\]
This implies that
\[
((1 - t) A + tB + s1_H)^{-1} \leq (tc + s)^{-1}1_H.
\]
Therefore
\[
\int_{0}^{\infty} s^2 \left( \int_{0}^{1} ((1 - t) A + tB + s1_H)^{-2} dt \right) dm(s) \\
\leq \int_{0}^{\infty} s^2 \left( \int_{0}^{1} (tc + s)^{-2} dt \right) dm(s) 1_H \\
= \left( \frac{f(c) - f(0)}{c} - b \right)1_H \text{ (by (2.10))}
\]
and by (2.11), we get
\[
(2.12) \quad f(B) - f(A) - b(B - A) \leq d \left( \frac{f(c) - f(0)}{c} - b \right)1_H.
\]
We also have
\[
(1 - t) A + tB + s1_H = A + t(B - A) + s1_H \leq A + td1_H + s1_H \\
= (1 - t) A + t(d1_H + A) + s1_H.
\]
Since \( A \leq \|A\|1_H \), then
\[
(1 - t) A + t(d1_H + A) + s1_H \leq ((1 - t) \|A\| + t(d + \|A\|) + s)1_H,
\]
which implies that
\[
(1 - t) A + tB + s1_H \leq ((1 - t) \|A\| + t(d + \|A\|) + s)1_H
\]
for \( t \in [0, 1] \) and \( s > 0 \).

This implies that
\[
((1 - t) A + tB + s1_H)^{-1} \geq ((1 - t) \|A\| + t(d + \|A\|) + s)^{-1}1_H
\]
and
\[
((1 - t) A + tB + s1_H)^{-2} \geq ((1 - t) \|A\| + t(d + \|A\|) + s)^{-2}1_H
\]
for \( t \in [0, 1] \) and \( s > 0 \).

Therefore
\[
\int_{0}^{\infty} s^2 \left( \int_{0}^{1} ((1 - t) A + tB + s1_H)^{-2} dt \right) dm(s) \\
\geq \int_{0}^{\infty} s^2 \left( \int_{0}^{1} ((1 - t) \|A\| + t(d + \|A\|) + s)^{-2} dt \right) dm(s) 1_H (\geq 0)
\]
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\[ \frac{1}{d} \int_{0}^{\infty} s^2 \left( \int_{0}^{1} ((1 - t) \|A\| + t (d + \|A\|) + s)^{-1} (d + \|A\| - \|A\|) \right. \]
\[ \quad \times \left. ((1 - t) \|A\| + t (d + \|A\|) + s)^{-1} dt \right) dm (s) 1_H \]
\[ = \frac{1}{d} \left[ (f (d + \|A\|) - f (\|A\|) - bd) \right] 1_H \]
(by identity (2.9) for \( d + \|A\| \) and \( \|A\| \))
\[ = \left( \frac{f (d + \|A\|) - f (\|A\|)}{d} - b \right) 1_H \geq 0. \]

By (2.11), we get
\[ f (B) - f (A) - b (B - A) \]
\[ \geq c \int_{0}^{\infty} s^2 \left( \int_{0}^{1} ((1 - t) A + t B + s 1_H)^{-2} dt \right) dm (s) 1_H \]
\[ \geq c \left( \frac{f (d + \|A\|) - f (\|A\|)}{d} - b \right) 1_H \geq 0. \]

The inequalities (2.12) and (2.13) imply (2.2). \( \square \)

From the first inequality in (2.2) we get
\[ \frac{df (c)}{c} - f (0) \]
\[ \geq \frac{d f (c) - f (0)}{c} 1_H - b [d 1_H - (B - A)] \geq f (B) - f (A) \]
and since \( d 1_H - (B - A) \geq 0 \) and \( b \geq 0 \),
\[ \frac{df (c)}{c} - f (0) \]
\[ \geq d \frac{f (c) - f (0)}{c} 1_H - b [d 1_H - (B - A)]. \]

From the second inequality in (2.2) we have
\[ f (B) - f (A) \geq b [(B - A) - c] + c \frac{f (d + \|A\|) - f (\|A\|)}{d} 1_H \]
\[ \geq c \frac{f (d + \|A\|) - f (\|A\|)}{d} 1_H \geq 0 \]
since \( b [(B - A) - c 1_H] \geq 0. \)

Therefore we have the following result which does not contain the value \( b \):

**Corollary 1.** Assume that \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator monotone on \([0, \infty)\), \( A \geq 0 \) and that there exist positive numbers \( d > c > 0 \) such that the condition (2.1) is satisfied. Then
\[ \frac{df (c)}{c} - f (0) \]
\[ \geq \frac{d f (c) - f (0)}{c} 1_H \geq f (B) - f (A) \geq c \frac{f (d + \|A\|) - f (\|A\|)}{d} 1_H \geq 0. \]

**Remark 1.** If we take \( f (t) = t^r, r \in (0, 1] \), in (2.14), then we get
\[ \frac{dc^{-1}}{c} 1_H \geq B^r - A^r \geq c \frac{(d + \|A\|)^r - \|A\|^r}{d} 1_H \geq 0, \]
provided that the condition (2.1) is satisfied and $A \geq 0$.

Let $\varepsilon > 0$. Consider the function $f_\varepsilon : [0, \infty) \to \mathbb{R}$, $f_\varepsilon(t) = \ln(\varepsilon + t)$. This function is operator monotone on $[0, \infty)$ and by the second inequality in (2.14), we get

$$
\ln (B + \varepsilon 1_H) - \ln (A + \varepsilon 1_H) \geq c \frac{\ln (d + \|A\| + \varepsilon) - \ln (\|A\| + \varepsilon)}{d} 1_H > 0.
$$

By taking the limit over $\varepsilon \to 0^+$ in (2.16), we get

$$
\ln (B) - \ln (A) \geq \frac{\ln (d + \|A\|) - \ln (\|A\|)}{d} 1_H > 0
$$

for $d 1_H \geq B - A \geq c 1_H > 0$ and $A > 0$.

It is well known that if $P \geq 0$, then

$$
|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle
$$

for all $x, y \in H$.

Therefore, if $T > 0$, then

$$
0 \leq \langle x, x \rangle^2 = \langle T^{-1}Tx, x \rangle^2 = \langle Tx, T^{-1}x \rangle^2
\leq \langle Tx, x \rangle \langle TT^{-1}x, T^{-1}x \rangle = \langle Tx, x \rangle \langle x, T^{-1}x \rangle
$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$
1 \leq \langle Tx, x \rangle \langle x, T^{-1}x \rangle \leq \langle Tx, x \rangle \sup_{\|x\|=1} \langle x, T^{-1}x \rangle = \langle Tx, x \rangle \|T^{-1}\|,
$$

which implies the following operator inequality

$$
(2.18) \quad \frac{1}{\|T^{-1}\|} 1_H \leq T.
$$

**Corollary 2.** Assume that $f : [0, \infty) \to \mathbb{R}$ is operator monotone on $[0, \infty)$ and $B > A \geq 0$, then

$$
\|B - A\| \| (B - A)^{-1} \| \left[ f \left( \| (B - A)^{-1} \|^{-1} \right) - f(0) \right] 1_H
\geq f(B) - f(A)
\geq \frac{f(\|B - A\| + \|A\|) - f(\|A\|)}{\| (B - A)^{-1} \| \|B - A\|} 1_H
\geq \frac{f(\|B\|) - f(\|A\|)}{\| (B - A)^{-1} \| \|B - A\|} 1_H \geq 0.
$$

**Proof.** Since $B - A > 0$, by (2.18) we get

$$
\frac{1}{\| (B - A)^{-1} \|} 1_H \leq B - A \leq \|B - A\| 1_H.
$$
So, if we write the inequality (2.14) for 
\[ c = \frac{1}{\|B - A\|}, \]
and \( d = \|B - A\| \), then we get

\[
\|B - A\| \frac{1}{\|B - A\|} \left[ f \left( \frac{1}{\|B - A\|} \right) - f (0) \right] \geq f (B) - f (A)
\]

(2.20)

\[
\geq f \left( \frac{\|B - A\| + \|A\|}{\|B - A\|} \right) - f (\|A\|) \geq 0.
\]

Also, we have \( \|B - A\| + \|A\| \geq \|B\| \) and since \( f \) is nondecreasing, then

\[
f (\|B - A\| + \|A\|) - f (\|A\|) \geq f (\|B\|) - f (\|A\|) \geq 0.
\]

(2.21)

By (2.20) and (2.21) we derive (2.19).

**Remark 2.** By making use of a similar argument as in Remark 1, we can also derive the logarithmic inequality

\[
\ln (B) - \ln (A) \geq \frac{\ln (\|B - A\| + \|A\|) - \ln (\|A\|)}{\|B - A\|} \geq \frac{\ln (\|B\|) - \ln (\|A\|)}{\|B - A\|} > 0
\]

for \( A > 0 \) and \( B - A > 0 \).

**3. Some Examples.** Assume that \( B > A \geq 0 \) and \( r \in (0, 1] \). Then by (2.19) we have, for the operator monotone function \( f(t) = t^r \) on \( [0, \infty) \), the following refinement and reverse of L"owner–Heinz inequality

\[
\|B - A\| \frac{1}{\|B - A\|} \left[ (1 - r) \ln (\|B - A\|) - r \ln \|A\| \right] \geq \|B\| \ln \|B\| - \|A\| \ln \|A\| \geq 0.
\]

(3.1)

Consider the function

\[
f_0 (t) := \begin{cases} \frac{t}{1 - t} \ln t & \text{for } t > 0, \\ 0 & \text{for } t = 0, \end{cases}
\]

which is operator monotone on \([0, \infty)\). By (2.19), we then have

\[
\frac{\|B - A\|}{\|B - A\|} \ln \|B - A\| \geq B (1_H - B)^{-1} \ln B - A (1_H - A)^{-1} \ln A
\]

(3.2)

\[
\geq \frac{\|B\| \ln \|B\| - \frac{\|A\|}{\|B - A\|} \ln \|A\|}{\|B - A\|} \geq 0
\]
for $B > A > 0$ and $\|A\|, \|B\|, \|(B - A)^{-1}\| \neq 1$.

The function $f(t) = \ln(t + 1)$ is also operator monotone on $[0, \infty)$, so by (2.19) we have

$$\|B - A\| \|(B - A)^{-1}\| \ln \left( \|(B - A)^{-1}\|^{-1} + 1 \right)_{1_H} \geq \ln (B + 1_H) - \ln (A + 1_H)$$

$$\geq \frac{\ln (\|B - A\| + \|A\| + 1) - \ln (\|A\| + 1)}{\|(B - A)^{-1}\| \|B - A\|}$$

$$\geq \frac{\ln (\|B\| + 1) - \ln (\|A\| + 1)}{\|(B - A)^{-1}\| \|B - A\|} 1_H > 0$$

(3.3)

for $B > A \geq 0$.

Consider the function $f_{-1}(t) = \frac{2t}{1+t}$, $t \in [0, \infty)$, which is operator monotone, then by (2.19) we derive

$$\|B - A\| \|(B - A)^{-1}\|^{-1} 1_H \geq B (1_H + B)^{-1} - A (1_H + A)^{-1}$$

$$\geq \frac{\|B\| - \|A\|}{\|(B - A)^{-1}\| \|B - A\| (1 + \|B\|) (1 + \|A\|)} 1_H > 0$$

(3.4)

for $B > A \geq 0$.

The interested reader may state other similar inequalities by employing the operator monotone functions presented in Introduction. We omit the details.

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