The Simple Connectedness of Tame Algebras with Separating Almost Cyclic Coherent Auslander–Reiten Components

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Abstract
We study the simple connectedness of the class of finite-dimensional algebras over an algebraically closed field for which the Auslander–Reiten quiver admits a separating family of almost cyclic coherent components. We show that a tame algebra in this class is simply connected if and only if its first Hochschild cohomology space vanishes.

Keywords Simply connected algebra · Hochschild cohomology · Auslander–Reiten quiver · Tame algebra · Generalized multicoil algebra

Mathematics Subject Classification (2010) Primary 16G70 · Secondary 16G20

1 Introduction and the Main Results

Throughout the paper $k$ will denote a fixed algebraically closed field. By an algebra is meant an associative finite-dimensional $k$-algebra with an identity, which we shall assume (without loss of generality) to be basic. Then such an algebra has a presentation $A \cong kQ_A/I$, where $Q_A = (Q_0, Q_1)$ is the ordinary quiver of $A$ with the set of vertices $Q_0$ and the set of arrows $Q_1$ and $I$ is an admissible ideal in the path algebra $kQ_A$ of $Q_A$. If the quiver $Q_A$ has no oriented cycles, the algebra $A$ is said to be triangular. For an algebra $A$, we denote by $\text{mod} A$ the category of finitely generated right $A$-modules, and by $\text{ind} A$ a full subcategory of $\text{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable modules. We shall denote by $\text{rad} A$ the Jacobson radical of $\text{mod} A$, and by $\text{rad}^\infty A$ the intersection of all powers $\text{rad}^i A$, $i \geq 1$, of $\text{rad} A$. Moreover, we denote by $\Gamma_A$ the Auslander–Reiten quiver of $A$, and by $\tau_A$ and $\tau_A^-$ the Auslander–Reiten translations.
$D \text{Tr}$ and $\text{Tr} \, D$, respectively. We will not distinguish between a module in $\text{ind} \, A$ and the vertex of $\Gamma_A$ corresponding to it. Following [45], a family $\mathcal{C}$ of components is said to be *generalized standard* if $\text{rad}^\infty_\mathcal{C}(X, Y) = 0$ for all modules $X$ and $Y$ in $\mathcal{C}$. We note that different components in a generalized standard family $\mathcal{C}$ are orthogonal, and all but finitely many $\tau_A$-orbits in $\mathcal{C}$ are $\tau_A$-periodic (see [45, (2.3)]). We refer to [37] for the structure and homological properties of arbitrary generalized standard Auslander–Reiten components of algebras.

Following Assem and Skowroński [7], a triangular algebra $A$ is called *simply connected* if, for any presentation $A \cong kQ_A/I$ of $A$ as a bound quiver algebra, the fundamental group $\pi_1(Q_A, I)$ of $(Q_A, I)$ is trivial (see Section 2). The importance of these algebras follows from the fact that often we may reduce (using techniques of Galois coverings) the study of the module category of an algebra to that for the corresponding simply connected algebras. Let us note that to prove that an algebra is simply connected seems to be a difficult problem, because one has to check that various fundamental groups are trivial. Therefore, it is worth looking for a simpler characterization of simple connectedness. In [44, Problem 1] Skowroński has asked, whether it is true that a tame triangular algebra $A$ is simply connected if and only if the first Hochschild cohomology space $H^1(A)$ of $A$ vanishes. This equivalence is true for representation-finite algebras [3, Proposition 3.7] (see also [12] for the general case), for tilted algebras (see [5] for the tame case and [25] for the general case), for quasitilted algebras (see [3] for the tame case and [26] for the general case), for piecewise hereditary algebras of type any quiver [25], and for weakly shod algebras [4].

A prominent role in the representation theory of algebras is played by the algebras with separating families of Auslander–Reiten components. A concept of a separating family of tubes has been introduced by Ringel in [40, 41] who proved that they occur in the tube system of components is said to be *separating* (see Section 2). The importance of these algebras follows from the fact that often we may reduce (using techniques of Galois coverings) the study of the module category of an algebra to that for the corresponding simply connected algebras. Let us note that to prove that an algebra is simply connected seems to be a difficult problem, because one has to check that various fundamental groups are trivial. Therefore, it is worth looking for a simpler characterization of simple connectedness. In [44, Problem 1] Skowroński has asked, whether it is true that a tame triangular algebra $A$ is simply connected if and only if the first Hochschild cohomology space $H^1(A)$ of $A$ vanishes. This equivalence is true for representation-finite algebras [3, Proposition 3.7] (see also [12] for the general case), for tilted algebras (see [5] for the tame case and [25] for the general case), for quasitilted algebras (see [3] for the tame case and [26] for the general case), for piecewise hereditary algebras of type any quiver [25], and for weakly shod algebras [4].

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Simply Connected Algebras

Let $A$ be a tame algebra with a separating family of almost cyclic coherent components in $\Gamma_A$. Then $A$ is simply connected if and only if $H^1(A) = 0$.

It has been proved in [33, Theorem A] that the Auslander–Reiten quiver $\Gamma_A$ of an algebra $A$ admits a separating family $\mathcal{C}_A$ of almost cyclic coherent components if and only if $A$ is a generalized multicoil enlargement of a finite product of concealed canonical algebras $C_1, \ldots, C_m$ by an iterated application of admissible algebra operations of types (ad 1)–(ad 5) and their duals. These algebras are called generalized multicoil algebras (see Section 3 for details). Note that for such an algebra $A$, we have that $A$ is triangular, gl. dim $A \leq 3$, and pd$_A M \leq 2$ or id$_A M \leq 2$ for any module $M$ in $\text{ind } A$ (see [33, Corollary B and Theorem E]). Moreover, let $\Gamma_A = \mathcal{P}_A \cup \mathcal{C}_A \cup \mathcal{Q}_A$ be the induced decomposition of $\Gamma_A$. Then, by [33, Theorem C], there are uniquely determined quotient algebras $A^{(i)} = A_1^{(i)} \times \cdots \times A_m^{(i)}$ and $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of $A$ which are the quasitilted algebras of canonical type such that $\mathcal{P}_A = \mathcal{P}^{(i)}_A$ and $\mathcal{Q}_A = \mathcal{Q}^{(r)}_A$.

Let $A$ be a generalized multicoil algebra obtained from a concealed canonical algebra $C = C_1 \times \cdots \times C_m$ and $C = A_0, A_1, \ldots, A_n = A$ be an admissible sequence for $A$ (see Section 3). In order to formulate the next result we need one more definition. Namely, if the sectional paths occurring in the definitions of the operations (ad 4), (fad 4), (ad 4$\ast$) (fad 4$\ast$) come from a component or two components of the same connected algebra $A_i$, $i \in \{0, \ldots, n - 1\}$, then we will say that $\Gamma_{A_{i+1}}$ contains an exceptional configuration of modules.

The following theorem is the second main result of the paper.

**Theorem 1.2** Let $A$ be a generalized multicoil algebra obtained from a family $C_1, \ldots, C_m$ of simply connected concealed canonical algebras. Assume moreover that $\Gamma_A$ does not contain exceptional configurations of modules. Then there are quotient algebras $A^{(i)} = A_1^{(i)} \times \cdots \times A_m^{(i)}$ and $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of $A$ such that the following statements are equivalent:

(i) $A$ is simply connected.

(ii) $A_i^{(i)}$ and $A_i^{(r)}$ are simply connected, for any $i \in \{1, \ldots, m\}$.

(iii) $H^1(A) = 0$.

(iv) $H^1(A_i^{(i)}) = 0$ and $H^1(A_i^{(r)}) = 0$, for any $i \in \{1, \ldots, m\}$.

(v) $A$ is strongly simply connected.

This paper is organized as follows. In Section 2 we recall some concepts and facts from representation theory, which are necessary for further considerations. Section 3 is devoted to
describing some properties of almost cyclic coherent components of the Auslander–Reiten quivers of algebras, applied in the proofs of the preliminary results and the main theorems. In Section 4 we present and prove several results applied in the proof of the first main result of the paper. Sections 5 and 6 are devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively. The aim of the final Section 7 is to present examples illustrating the main results of the paper.

For basic background on the representation theory of algebras we refer to the books [6, 41–43], for more information on simply connected algebras we refer to the survey article [2], and for more details on algebras with separating families of Auslander–Reiten components and their representation theory to the survey article [35].

2 Preliminaries

2.1 Let $A$ be an algebra and $A \cong kQ_A/I$ be a presentation of $A$ as a bound quiver algebra. Then the algebra $A = kQ_A/I$ can equivalently be considered as a $k$-linear category, of which the object class $A_0$ is the set of points of $Q_A$, and the set of morphisms $A(x, y)$ from $x$ to $y$ is the quotient of the $k$-vector space $kQ_A(x, y)$ of all formal linear combinations of paths in $Q_A$ from $x$ to $y$ by the subspace $I(x, y) = kQ_A(x, y) \cap I$ (see [11]). A full subcategory $B$ of $A$ is called convex (in $A$) if any path in $A$ with source and target in $B$ lies entirely in $B$. For each vertex $v$ of $Q_A$ we denote by $S_v$ the corresponding simple $A$-module, and by $P_v$ (respectively, $I_v$) the projective cover (respectively, the injective envelope) of $S_v$.

2.2 One-point Extensions and Coextensions Frequently an algebra $A$ can be obtained from another algebra $B$ by a sequence of one-point extensions and one-point coextensions. Recall that the one-point extension of an algebra $B$ by a $B$-module $M$ is the matrix algebra

$$B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $B[M]$ contains $Q_B$ as a convex subquiver and there is an additional (extension) point which is a source. $B[M]$-modules are usually identified with triples $(V, X, \varphi)$, where $V$ is a $k$-vector space, $X$ a $B$-module and $\varphi : V \to \text{Hom}_B(M, X)$ a $k$-linear map. A $B[M]$-linear map $(V, X, \varphi) \to (V', X', \varphi')$ is then identified with a pair $(f, g)$, where $f : V \to V'$ is $k$-linear, $g : X \to X'$ is $B$-linear and $\varphi' f = \text{Hom}_B(M, g) \varphi$. One defines dually the one-point coextension $[M]B$ of $B$ by $M$ (see [41]).

2.3 Tameness and Wildness Let $A$ be an algebra and $K[x]$ the polynomial algebra in one variable $x$. Following [17], the algebra $A$ is said to be tame if, for any positive integer $d$, there exists a finite number of $K[x] - A$-bimodules $M_i$, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isoclasses of indecomposable $A$-modules of dimension $d$ are of the form $K[x]/(x - \lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \ldots, n_d\}$. Recall that, following [17], the algebra $A$ is wild if there is a $k(x, y)$-$A$-bimodule $M$, free of finite rank as left $k(x, y)$-module, and the functor $- \otimes_{k(x, y)} M : \text{mod } k(x, y) \to \text{mod } A$ preserves the indecomposability of modules and sends nonisomorphic modules to nonisomorphic modules. From Drozd’s Tame and Wild Theorem [17] the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras and the second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional algebras over $k$. 
Hence, a classification of the finite dimensional modules is only feasible for tame algebras. It has been shown by Crawley-Boevey [16] that, if \( A \) is a tame algebra, then, for any positive integer \( d \geq 1 \), all but finitely many isomorphism classes of indecomposable \( A \)-modules of dimension \( d \) are invariant on the action of \( \tau_A \), and hence, by a result due to Hoshino [23], lie in stable tubes of rank one in \( \Gamma_A \).

### 2.4 Hochschild Cohomology of Algebras

Let \( A \) be an algebra. Denote by \( C^\bullet A \) the Hochschild complex \( C^\bullet = (C^i, d^i)_{i \in \mathbb{Z}} \) defined as follows: \( C^i = 0 \), \( d^i = 0 \) for \( i < 0 \), \( C^0 = kA_A \), \( C^i = \text{Hom}_k(A^{\otimes i}, A) \) for \( i > 0 \), where \( A^{\otimes i} \) denotes the \( i \)-fold tensor product over \( k \) of \( A \) with itself, \( d^0 : A \to \text{Hom}_k(A, A) \) with \( (d^0x)(a) = ax - xa \) for \( x, a \in A \), \( d^1 : C^i \to C^{i+1} \) with

\[
(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i a_{i+1})
\]

for \( f \in C^i \) and \( a_1, a_2, \ldots, a_{i+1} \in A \). Then \( H^1(A) = H^1(C^\bullet A) \) is called the \( i \)-th Hochschild cohomology space of \( A \) (see [14, Chapter IX]). Recall that the first Hochschild cohomology space \( H^1(A) \) of an algebra \( A \) is isomorphic to the space \( \text{Der}(A, A)/\text{Der}^0(A, A) \) of outer derivations of \( A \), where \( \text{Der}(A, A) = \{ \delta \in \text{Hom}_k(A, A) \mid \delta(ab) = a\delta(b) + \delta(a)b, \text{ for } a, b \in A \} \) is the space of \( k \)-linear derivations of \( A \) and \( \text{Der}^0(A, A) \) is the subspace \( \{ \delta_x \in \text{Hom}_k(A, A) \mid \delta_x(a) = ax - xa, \text{ for } a \in A \} \) of inner derivations of \( A \).

### 2.5 Concealed Canonical Algebras

An important role in our considerations will be played by certain tilts of canonical algebras introduced by Ringel [41]. Let \( p_1, p_2, \ldots, p_t \) be a sequence of positive integers with \( t \geq 2 \), \( 1 \leq p_1 \leq p_2 \leq \ldots \leq p_t \), and \( p_1 \geq 2 \) if \( t \geq 3 \). Denote by \( \Delta(p_1, \ldots, p_t) \) the quiver of the form

\[
\begin{align*}
\alpha_1 & \leftarrow \cdots \leftarrow \alpha_{p_1} \\
\alpha_{p_1} & \leftarrow \alpha_{p_1 + 1} & \alpha_{p_1 + 1} & \leftarrow \alpha_{p_1 + 2} & \alpha_{p_1 + 2} & \leftarrow \alpha_{p_1 + 3} & \alpha_{p_1 + 3} & \leftarrow \alpha_{p_1 + 4} & \cdots \\
& \quad \cdots \\
\alpha_{p_2} & \leftarrow \alpha_{p_2 + 1} & \alpha_{p_2 + 1} & \leftarrow \alpha_{p_2 + 2} & \alpha_{p_2 + 2} & \leftarrow \alpha_{p_2 + 3} & \alpha_{p_2 + 3} & \leftarrow \alpha_{p_2 + 4} & \cdots \\
& \quad \cdots \\
\alpha_{p_t} & \leftarrow \alpha_{p_t + 1} & \alpha_{p_t + 1} & \leftarrow \alpha_{p_t + 2} & \alpha_{p_t + 2} & \leftarrow \alpha_{p_t + 3} & \alpha_{p_t + 3} & \leftarrow \alpha_{p_t + 4} & \cdots \\
& \quad \cdots
\end{align*}
\]

For \( t \geq 3 \), consider a \( (t + 1) \)-tuple of pairwise different elements of \( \mathbb{P}_1(k) = k \cup \{ \infty \} \), normalized such that \( \lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1 \), and the admissible ideal \( I(\lambda_1, \lambda_2, \ldots, \lambda_t) \) in the path algebra \( k\Delta(p_1, \ldots, p_t) \) of \( \Delta(p_1, \ldots, p_t) \) generated by the elements

\[
a_{ip_1} \cdots a_{i2} a_{i1} + a_{ip_2} \cdots a_{i2} a_{i1} + \lambda_i a_{ip_1} + a_{i2} a_{i1}, \quad 3 \leq i \leq t.
\]

Then the bound quiver algebra \( \Lambda(p, \lambda) = k\Delta(p_1, \ldots, p_t)/I(\lambda_1, \lambda_2, \ldots, \lambda_t) \) is said to be the canonical algebra of type \( p = (p_1, \ldots, p_t) \). Moreover, for \( t = 2 \), the path algebra \( \Lambda(p) = k\Delta(p_1, p_2) \) is said to be the canonical algebra of type \( p = (p_1, p_2) \). It has been proved in [41, Theorem 3.7] that if \( \Lambda \) is a canonical algebra of type \( (p_1, \ldots, p_t) \) then \( \Gamma_\Lambda = \mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda \) for a \( \mathbb{P}_1(k) \)-family \( \mathcal{T}^\Lambda \) of stable tubes of tubular type \( (p_1, \ldots, p_t) \), separating \( \mathcal{P}^\Lambda \) from \( \mathcal{Q}^\Lambda \). Following [27], a connected algebra \( C \) is called a concealed canonical algebra of type \( (p_1, \ldots, p_t) \) if \( C \) is the endomorphism algebra \( \text{End}_\Lambda(T) \), for...
some canonical algebra $\Lambda$ of type $(p_1, \ldots, p_l)$ and a tilting $\Lambda$-module $T$ whose indecomposable direct summands belong to $\mathcal{P}^\Lambda$. Then the images of modules from $T^\Lambda$ via the functor $\text{Hom}_\Lambda(T, -)$ form a separating family $\mathcal{T}^C$ of stable tubes of $\Gamma_C$, and in particular we have a decomposition $\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C$. It has been proved by Lenzing and de la Peña [28, Theorem 1.1] that the class of (connected) concealed canonical algebras coincides with the class of all connected algebras with a separating family of stable tubes. It is also known that the class of concealed canonical algebras of type $(p_1, p_2)$ coincides with the class of hereditary algebras of Euclidean types $\tilde{\mathbb{A}}_m, m \geq 1$ (see [22]). Recall also that the canonical algebras of types $(2, 2, 2, 2), (3, 3, 3), (2, 4, 4)$ and $(2, 3, 6)$ are called the tubular canonical algebras, and an algebra which is tilting-cotilting equivalent to a tubular canonical algebra is called a tubular algebra (see [18, 21, 41]).

2.6 Simple Connectedness Let $(Q, I)$ be a connected bound quiver. A relation $\varrho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is minimal if $m \geq 2$ and, for any nonempty proper subset $J \subset \{1, \ldots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. We denote by $\alpha^{-1}$ the formal inverse of an arrow $\alpha \in Q_1$. A walk in $Q$ from $x$ to $y$ is a formal composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_l^{\varepsilon_l}$ (where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$ for all $i$) with source $x$ and target $y$. We denote by $e_x$ the trivial path at $x$. Let $\sim$ be the homotopy relation on $(Q, I)$, that is, the smallest equivalence relation on the set of all walks in $Q$ such that:

(a) If $\alpha : x \to y$ is an arrow, then $\alpha^{-1} \sim e_y$ and $\alpha \alpha^{-1} \sim e_x$.
(b) If $\varrho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all $i, j$.
(c) If $u \sim v$, then $uw' \sim vw'$ whenever these compositions make sense.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes $\tilde{u}$ of closed walks $u$ starting and ending at $u$ has a group structure defined by the operation $\tilde{u} \cdot \tilde{v} = \tilde{u}\tilde{v}$. Since $Q$ is connected, $\pi_1(Q, I, x)$ does not depend on the choice of $x$. We denote it by $\pi_1(Q, I)$ and call it the fundamental group of $(Q, I)$.

Let $A \cong kQ_A/I$ be a presentation of a triangular algebra $A$ as a bound quiver algebra. The fundamental group $\pi_1(Q_A, I)$ depends essentially on $I$, so it is not an invariant of $A$. A triangular algebra $A$ is called simply connected if, for any presentation $A \cong kQ_A/I$ of $A$ as a bound quiver algebra, the fundamental group $\pi_1(Q_A, I)$ of $(Q_A, I)$ is trivial [7].

Example 2.7 Let $A = kQ/I$ be the bound quiver algebra given by the quiver $Q$ of the form

$$
\begin{array}{cccc}
5 & \lambda & 1 & \alpha \\
\lambda & 1 & \beta & 2 \\
\beta & 2 & \delta & 4 \\
\gamma & 3 & & \\
\end{array}
$$

and $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $\gamma \beta, \delta \alpha - a \delta \beta, a \lambda$, where $a \in k \setminus \{0\}$. Then $\pi_1(Q, I)$ is trivial. Moreover, the triangular algebra $A$ is simply connected. Indeed, any choice of a basis of $\text{rad}_A/\text{rad}_A^2$ will lead to at least one minimal relation with target 1 and source $i \in \{3, 4\}$ or with target 5 and source 2.

Remark 2.8 It is known, for example, that the following important classes of algebras are simply connected: the iterated tilted algebras of Dynkin type (see [1, Proposition 3.5]), the iterated tilted algebras of Euclidean types $\tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_p, n \geq 4$, $p = 6, 7, 8$, the tubular algebras (see [7, Corollary 1.4]), and the $pg$-critical algebras (see [38, Corollary 3.3]).
3 Almost Cyclic Coherent Auslander–Reiten components

3.1 Generalized Multicoil Algebras
It has been proved in [32, Theorem A] that a connected component $\Gamma$ of an Auslander–Reiten quiver $\Gamma_A$ of an algebra $A$ is almost cyclic and coherent if and only if $\Gamma$ is a generalized multicoil, that is, can be obtained, as a translation quiver, from a finite family of stable tubes by a sequence of operations called admissible. We recall briefly the generalized multicoil enlargements of algebras from [33, Section 3].

Given a generalized standard component $\Gamma$ of $\Gamma_A$, and an indecomposable module $X$ in $\Gamma$, the support $S(X)$ of the functor $\text{Hom}_A(X, -)|_{\Gamma}$ is the $k$-linear category defined as follows [9]. Let $\mathcal{H}_X$ denote the full subcategory of $\Gamma$ consisting of the indecomposable modules $M$ in $\Gamma$ such that $\text{Hom}_A(X, M) \neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \rightarrow N$ (with $M, N$ in $\mathcal{H}_X$) such that $\text{Hom}_A(X, f) = 0$. We define $S(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the $k$-linear category $S(X)$ with its quiver.

Recall that a module $X$ in mod $A$ is called a brick if $\text{End}_A(X) \cong k$.

Let $A$ be an algebra and $\Gamma$ be a family of generalized standard infinite components of $\Gamma_A$. For an indecomposable brick $X$ in $\Gamma$, called the pivot, five admissible operations are defined, depending on the shape of the support $S(X)$ of the functor $\text{Hom}_A(X, -)|_{\Gamma}$. These admissible operations yield in each case a modified algebra $A'$ such that the modified translation quiver $\Gamma'$ is a family of generalized standard infinite components of the Auslander–Reiten quiver $\Gamma_A'$ of $A'$ (see [32, Section 2] or [35, Section 4] for the figures illustrating the modified translation quiver $\Gamma'$).

(ad 1) Assume $S(X)$ consists of an infinite sectional path starting at $X$:

$$X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

Let $t \geq 1$ be a positive integer, $D$ be the full $t \times t$ lower triangular matrix algebra, and $Y_1, \ldots, Y_t$ denote the indecomposable injective $D$-modules with $Y = Y_1$ the unique indecomposable projective-injective $D$-module. We set $A' = (A \times D)[X \oplus Y]$. In this case, $\Gamma'$ is obtained by inserting in $\Gamma$ the rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, [\frac{1}{1}])$ for $i \geq 0$, $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$. If $t = 0$ we set $A' = A[X]$ and the rectangle reduces to the sectional path consisting of the modules $X'_i, i \geq 0$.

(ad 2) Suppose that $S(X)$ admits two sectional paths starting at $X$, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

where $t \geq 1$. In particular, $X$ is necessarily injective. We set $A' = A[X]$. In this case, $\Gamma'$ is obtained by inserting in $\Gamma$ the rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, [\frac{1}{1}])$ for $i \geq 1, 1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$.

(ad 3) Assume $S(X)$ is the mesh-category of two parallel sectional paths:

$$Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_t$$

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{t-1} \rightarrow X_t \rightarrow \cdots$$

with the upper sectional path finite and $t \geq 2$. In particular, $X_{t-1}$ is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of $\Gamma$ obtained by deleting the arrows $Y_i \rightarrow \tau^{-1}_A Y_{i-1}$. We assume that the union $\overline{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the modules $\tau^{-1}_A Y_{i-1}, 2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\hat{\Gamma}$ and a cofinite full translation subquiver $\Gamma^*$, containing the pivot $X$. We set $A' = A[X]$. In this
case, Γ′ is obtained from Γ* by inserting the rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, \begin{bmatrix} 1 & 1 \end{bmatrix})$ for $i \geq 1, 1 \leq j \leq i$, and $X_i' = (k, X_i, 1)$ for $i \geq 0$.

(ad 4) Suppose that $S(X)$ consists of an infinite sectional path, starting at $X$ $X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ and $Y = Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_t$ with $t \geq 1$, is a finite sectional path in $\Gamma_A$. Let $r$ be a positive integer. Moreover, we consider the translation quiver $\Gamma$ of $\Gamma$ obtained by deleting the arrows $Y_i \rightarrow \tau_{A^{-1}} Y_{i-1}$. We assume that the union $\hat{\Gamma}$ of connected components of $\Gamma$ containing the vertices $\tau_{A^{-1}} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\Gamma$ is a disjoint union of $\hat{\Gamma}$ and a cofinite full translation subquiver $\Gamma^*$, containing the pivot $X$. For $r = 0$ we set $A' = A[X \oplus Y]$.

In this case, $\Gamma'$ is obtained from $\Gamma^*$ by inserting the rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, \begin{bmatrix} 1 \end{bmatrix})$ for $i \geq 0, 1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$ for $i \geq 0$.

For $r \geq 1$, let $G$ be the full $r \times r$ lower triangular matrix algebra, $U_{1,r+1}, U_{2,r+1}, \ldots, U_{r,r+1}$ denote the indecomposable projective $G$-modules, $U_{r,1}, U_{r,2}, \ldots, U_{r,r}$ denote the indecomposable injective $G$-modules, with $U_{r,1}$ the unique indecomposable projective-injective $G$-module. We define the matrix algebra $A' = \begin{bmatrix} A & 0 \ 0 & k \ 0 & 0 \ \vdots & \vdots & \ddots & \vdots \ Y & k & \cdots & 0 \ X \oplus Y & k & \cdots & k \ k & k & \cdots & k \ \end{bmatrix}$ with $r+2$ columns and rows. In this case, $\Gamma'$ is obtained from $\Gamma^*$ by inserting the following modules $U_{sl} = \begin{cases} (k, Y_{1,s-1}, 1) & \text{for } s = 1, 1 \leq l \leq t, \\ (k, U_{s-1,l-1}, 1) & \text{for } 2 \leq s \leq r, 1 \leq l < t+s, \\ (k, 0, 0) & \text{for } 2 \leq s \leq r, l = t+s, \end{cases}$ $Z_{ij} = (k, X_i \oplus U_{ij}, \begin{bmatrix} 1 \end{bmatrix})$ for $i \geq 0$ and $1 \leq j \leq t + r$.

and $X_i' = (k, X_i, 1)$ for $i \geq 0$. In the above formulas $U_{sl}$ is treated as a module over the algebra $A_s = A_{s-1}[U_{s-1,1}]$, where $A_0 = A$ and $U_{01} = Y$ (in other words $A_s$ is an algebra consisting of matrices obtained from the matrices belonging to $A'$ by choosing the first $s+1$ rows and columns).

We note that the quiver $Q_{A'}$ of $A'$ is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r + 1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

For the definition of the next admissible operation we need also the finite versions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), which we denote by (fad 1), (fad 2), (fad 3) and (fad 4), respectively. In order to obtain these operations we replace all infinite sectional paths of the form $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ (in the definitions of (ad 1), (ad 2), (ad 3), (ad 4)) by the finite sectional paths of the form $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_s$. For the operation (fad 1) $s \geq 0$, for (fad 2) and (fad 4) $s \geq 1$, and for (fad 3) $s \geq t - 1$. In all above operations $X_s$ is injective (see the figures for (fad 1)–(fad 4) in [32, Section 2] or [35, Section 4]).

(ad 5) We define the modified algebra $A'$ of $A$ to be the iteration of the extensions described in the definitions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), and their finite versions corresponding to the operations (fad 1), (fad 2), (fad 3) and (fad 4). In this case, $\Gamma'$ is obtained in the following three steps: first we are doing on $\Gamma$ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly zero) of the operation.
(fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective modules have a common cofinite (infinite) sectional subpath. By an (ad 5)-pivot we mean an indecomposable brick $X$ from the last (ad 4) operation used in the whole process of creating (ad 5).

Moreover, together with each of the admissible operations (ad 1)–(ad 5), we consider its dual, denoted by (ad 1$^*$)–(ad 5$^*$). These dual operations are also called admissible. Following [32], a connected translation quiver $\Gamma$ is said to be a generalized multicoil if $\Gamma$ can be obtained from a finite family $T_1, T_2, \ldots, T_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 1$^*$), (ad 2), (ad 2$^*$), (ad 3), (ad 3$^*$), (ad 4), (ad 4$^*$), (ad 5) or (ad 5$^*$). If $s = 1$, such a translation quiver $\Gamma$ is said to be a generalized coil. The admissible operations of types (ad 1)–(ad 3), (ad 1$^*$)–(ad 3$^*$) have been introduced in [8–10], and the admissible operations (ad 4) and (ad 4$^*$) for $r = 0$ in [30].

Finally, let $C$ be a (not necessarily connected) concealed canonical algebra and $\mathcal{T}^C$ a separating family of stable tubes of $\Gamma_C$. Following [33] we say that an algebra $A$ is a generalized multicoil enlargement of $C$ using modules from $\mathcal{T}^C$ if there exists a sequence of algebras $C = A_0, A_1, \ldots, A_n = A$ such that $A_{i+1}$ is obtained from $A_i$ by an admissible operation of one of the types (ad 1)–(ad 5), (ad 1$^*$)–(ad 5$^*$) performed either on stable tubes of $\mathcal{T}^{A_i}$, or on generalized multicoils obtained from stable tubes of $\mathcal{T}^{A_i}$ by means of operations done so far. The sequence $C = A_0, A_1, \ldots, A_n = A$ is then called an admissible sequence for $A$. Observe that this definition extends the concept of a coil enlargement of a concealed canonical algebra introduced in [10]. We note that a generalized multicoil enlargement $A$ of $C$ invoking only admissible operations of type (ad 1) (respectively, of type (ad 1$^*$)) is a tubular extension (respectively, tubular coextension) of $C$ in the sense of [41]. An algebra $A$ is said to be a generalized multicoil algebra if $A$ is a connected generalized multicoil enlargement of a product $C$ of connected concealed canonical algebras.

**Proposition 3.2** [33, Proposition 3.7] Let $C$ be a concealed canonical algebra, $\mathcal{T}^C$ a separating family of stable tubes of $\Gamma_C$, and $A$ a generalized multicoil enlargement of $C$ using modules from $\mathcal{T}^C$. Then $\Gamma_A$ admits a generalized standard family $\mathcal{G}^A$ of generalized multicoils obtained from the family $\mathcal{T}^C$ of stable tubes by a sequence of admissible operations corresponding to the admissible operations leading from $C$ to $A$.

The following theorem, proved in [33, Theorem A], will be crucial for our further considerations.

**Theorem 3.3** Let $A$ be an algebra. The following statements are equivalent:
(i) $\Gamma_A$ admits a separating family of almost cyclic coherent components.
(ii) $A$ is a generalized multicoil enlargement of a concealed canonical algebra $C$.

**Remark 3.4** The concealed canonical algebra $C$ is called the core of $A$ and the number $m$ of connected summands of $C$ is a numerical invariant of $A$. We note that $m$ can be arbitrarily large, even if $A$ is connected. Let us also note that the class of algebras with generalized standard almost cyclic coherent Auslander–Reiten components is large (see [34, Proposition 2.9] and the following comments).

We note that the class of tubular extensions (respectively, tubular coextensions) of concealed canonical algebras coincides with the class of algebras having a separating family of ray tubes (respectively, coray tubes) in their Auslander–Reiten quiver (see [27, 29]). Moreover, these algebras are quasitilted algebras of canonical type.
We recall also the following theorem on the structure of the module category of an algebra with a separating family of almost cyclic coherent Auslander–Reiten components proved in [33, Theorems C and F].

**Theorem 3.5** Let $A$ be an algebra with a separating family $\mathcal{C}^A$ of almost cyclic coherent components in $\Gamma_A$, and $\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A$ the associated decomposition of $\Gamma_A$. Then the following statements hold.

(i) There is a unique full convex subcategory $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ of $A$ which is a tubular coextension of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$ such that $\Gamma_{A^{(l)}} = \mathcal{P}^{A^{(l)}} \cup \mathcal{T}^{A^{(l)}} \cup \mathcal{Q}^{A^{(l)}}$ for a separating family $\mathcal{T}^{A^{(l)}}$ of coray tubes in $\Gamma_{A^{(l)}}$, $\mathcal{P}^A = \mathcal{P}^{A^{(l)}}$, and $A$ is obtained from $A^{(l)}$ by a sequence of admissible operations of types (ad 1)–(ad 5) using modules from $\mathcal{T}^{A^{(l)}}$.

(ii) There is a unique full convex subcategory $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of $A$ which is a tubular extension of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$ such that $\Gamma_{A^{(r)}} = \mathcal{P}^{A^{(r)}} \cup \mathcal{T}^{A^{(r)}} \cup \mathcal{Q}^{A^{(r)}}$ for a separating family $\mathcal{T}^{A^{(r)}}$ of ray tubes in $\Gamma_{A^{(r)}}$, $\mathcal{Q}^A = \mathcal{Q}^{A^{(r)}}$, and $A$ is obtained from $A^{(r)}$ by a sequence of admissible operations of types (ad 1*)–(ad 5*) using modules from $\mathcal{T}^{A^{(r)}}$.

(iii) $A$ is tame if and only if $A^{(l)}$ and $A^{(r)}$ are tame.

In the above notation, the algebras $A^{(l)}$ and $A^{(r)}$ are called the left and right quasitilted algebras of $A$. Moreover, the algebras $A^{(l)}$ and $A^{(r)}$ are tame if and only if $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean type or tubular algebras.

Recall that an algebra $A$ is **strongly simply connected** if every convex subcategory of $A$ is simply connected (see [44]). Clearly, if $A$ is strongly simply connected then $A$ is simply connected. We need the following result proved in [31, Theorem 1.1].

**Theorem 3.6** Let $A$ be an algebra with a separating family of almost cyclic coherent components in $\Gamma_A$ without exceptional configurations of modules. Then there are quotient algebras $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ and $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of $A$ such that the following statements are equivalent:

(i) $A$ is strongly simply connected.

(ii) $A_i^{(l)}$ and $A_i^{(r)}$ are strongly simply connected, for any $i \in \{1, \ldots, m\}$.

### 4 Preliminary Results

#### 4.1 Branch Extensions and Coextensions

Let $A$ be an algebra and $A \cong kQ_A/I$ be a presentation of $A$ as a bound quiver algebra. For a given vertex $v$ in $Q_A$, we denote by $v^\rightarrow$ (respectively, by $v^\leftarrow$) the set of all arrows of the quiver $Q_A$ starting at $v$ (respectively, terminating at $v$). Let now $K$ be a branch at a vertex $v \in Q_A$ and $E \in \text{mod } A$. Recall that the branch extension $A[E, K]$ by the branch $K$ [41, (4.4)] is constructed in the following way: to the one-point extension $A[E]$ with extension vertex $w$ (that is, rad $P_w = E$) we add the branch $K$ by identifying the vertices $v$ and $w$. If $E_1, \ldots, E_n \in \text{mod } A$ and $K_1, \ldots, K_n$ is a set of branches, then the branch extension $A[E_i, K_i]_{i=1}^n$ is defined inductively as: $A[E_i, K_i]_{i=1}^n = (A[E_i, K_i]_{i=1}^{n-1})[E_n, K_n]$. The concept of branch coextension is defined dually.
Lemma 4.2 Let $A$ be a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$. Moreover, let $C = A_0, \ldots, A_p = A^{(l)}$, $A_{p+1}, \ldots, A_n = A$ be an admissible sequence for $A$, $j \geq p$, $X \in \text{ind} A_j$ be an (ad 2) or (ad 3)-pivot, and $A_{j+1}$ be the modified algebra of $A_j$. If $v$ is the corresponding extension point then there is a unique vertex $u \in A^{(l)} \setminus A^{(r)}$ that satisfies:

(i) Each $\alpha \in v^-$ is the starting point of a nonzero path $\omega_\alpha \in A(v, u)$.

(ii) There are at least two different arrows in $v^-$. Moreover, if $\alpha, \beta \in v^-$, and $\alpha \neq \beta$, then $\omega_\alpha - \omega_\beta \in I$.

Proof We know from [33, Section 4] that $A^{(l)}$ is a unique maximal convex branch coextension of $C = C_1 \times \cdots \times C_m$ inside $A$, that is, $A^{(l)} = B_1^{(l)} \times \cdots \times B_m^{(l)}$, where $B_i^{(l)}$ is a unique maximal convex branch coextension of $C_i$ inside $A$, $i \in \{1, \ldots, m\}$. More precisely, $B_i^{(l)} = \bigcap_{i=1}^n [K_j, E_j]C_i$, where $K_1, \ldots, K_n$ are branches, $i \in \{1, \ldots, m\}$. Then there exists $s \in \{1, \ldots, m\}$ such that $u \in B_s^{(l)}$ and $A_{j+1} = A_j[X]$. If $X$ is an (ad 2)-pivot (respectively, (ad 3)-pivot), then in the sequence of earlier admissible operations, there is an operation of type (ad 1*) or (ad 5*) which contains an operation (fad 1*) which gives rise to the pivot $X$ of (ad 2) (respectively, to the pivot $X$ of (ad 3)) and to the modules $Y_1, \ldots, Y_t$ in the support of $\text{Hom}_A(X, -)$ restricted to the generalized multicoil containing $X$ - see definition of (ad 3)). The operations done after must not affect the support of $\text{Hom}_A(X, -)$ restricted to the generalized multicoil containing $X$. Note that in general, in the sequence of earlier admissible operations, there can be an operation of type (ad 5) which contains an operation (fad 4) which gives rise to the pivot $X$ of (ad 2) (respectively, to the pivot $X$ of (ad 3)) but from Lemma [33, Lemma 3.10] this case can be reduced to (ad 5*) which contains an operation (fad 1*).

Let $X$ be an (ad 2)-pivot, $A_{j+1} = A_j[X]$, and $u, u_1, \ldots, u_t$ (where $X = I_u, Y_i = I_{u_i}$ for $i \in \{1, \ldots, t\}$ - see definition of (ad 2)) be the points in the quiver $Q_{A_j}$ of $A_j$ corresponding to the new indecomposable injective $A_j$-modules obtained after performing the above admissible operation (ad 1*) or the operation (fad 1*). Then $u, u_1, \ldots, u_t \in A^{(l)}$. Since $X = \text{rad} P_v$, there must be a nonzero path from $v$ to each vertex $u$ which is a predecessor of $u$. Hence, each $\alpha \in v^-$ is the starting arrow of a nonzero path from $v$ to $u$, and there are at least two arrows in $v^-$, namely: one from $v$ to $u_t$ and one from $v$ to a point in $\text{Supp} X_1$, where $X_1$ is the immediate successor of $X$ on the infinite sectional path in $S(X)$ (see definition of (ad 2)). Moreover, since $P_v(u) = X(u) = k$, all paths from $v$ to $u$ are congruent modulo $I_{j+1}$. The bound quiver $Q_{A_{j+1}}$ of $A_{j+1}$ is of the form

\[
\text{Supp } X_1 \quad \begin{array}{c}
\text{Q}_{A_{j+1}} \\
\text{Q}_{A_j}
\end{array}
\]

\[
\begin{array}{c}
u \\
\downarrow \\
u_1 \quad \cdots \quad u_t \\
\uparrow \\
u
\end{array}
\]

where $A_{j+1}(v, u)$ is one-dimensional. From the proofs of [33, Theorems A and C], we have $u \in A^{(l)} \setminus A^{(r)}$, $v \in A^{(r)} \setminus A^{(l)}$, and $u_1, \ldots, u_t \in A^{(l)} \cap A^{(r)}$.

Let now $X$ be an (ad 3)-pivot, $A_{j+1} = A_j[X]$, and assume that we had $r$ consecutive admissible operations of types (ad 1*) or (fad 1*), the first of which had $X_t$ as a pivot, and
these admissible operations built up a branch $K$ in $A_j$ with points $u, u_1, \ldots, u_t$ in $Q_{A_j}$, so that $X_{t-1}$ and $Y_t$ are the indecomposable injective $A_j$-modules corresponding respectively to $u$ and $u_1$, and both $Y_t$ and $I_{A_j}^{-1}Y_1$ are coray modules in the generalized multicoil containing the (ad 3)-pivot $X$ (where $X, X_{t-1}, X_t, Y_1$ and $Y_t$ are as in the definition of (ad 3)). Then $u, u_1 \in A^{(l)}$ and $X$ is the indecomposable $A_j$-module given by: $X(w) = 0$ if $w < u_1$, $X(w) = k$ if $u_1 < w$, and $X(w) = X_{t-1}(w)$ in any other case. Since $X = \rad P_v$, there must be a nonzero path from $v$ to each vertex $w$ which is a predecessor of $u$, but those which are predecessors of $u_1$. Hence, each $\alpha \in v^{-}$ is the starting arrow of a nonzero path from $v$ to $u$, and there are at least two arrows in $v^{-}$, namely: one from $v$ to $u_1$ and one from $v$ to a point in $\Supp X_t$, where $X_t$ is the immediate successor of $X_{t-1}$ on the infinite sectional path in $S(X)$ (see definition of (ad 3)). Moreover, since $P_v(u) = X_{t-1}(u) = k$, all paths from $v$ to $u$ are congruent modulo $I_{j+1}$. The bound quiver $Q_{A_{j+1}}$ of $A_{j+1}$ is of the form

$$Q_{A_j}$$

$$\begin{align*}
&\begin{array}{c}
\text{Supp } X_t \\
\text{rest of } K
\end{array}
\end{align*}$$

where $A_{j+1}(v, u)$ is one-dimensional. Again, from the proofs of [33, Theorems A and C], we have $u \in A^{(l)} \setminus A^{(r)}$, $v \in A^{(r)} \setminus A^{(l)}$, $u_1$ and the vertices of the branch $K$ belong to $A^{(l)} \cap A^{(r)}$. 

**Lemma 4.3** Let $A$ be a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$. Moreover, let $C = A_0, \ldots, A_p = A^{(l)}, A_{p+1}, \ldots, A_n = A$ be an admissible sequence for $A$, $j \geq p$, $X \in \ind A_j$ be an (ad 1)-pivot, $A_{j+1}$ be the modified algebra of $A_j$, and $v$ be the corresponding extension point. Then the following statements hold.

(i) If there is a vertex $u \in A^{(l)} \setminus A^{(r)}$ such that each $\alpha \in v^{-}$ is the starting point of a nonzero path $\omega_\alpha \in A(v, u)$, then:

(a) The vertex $u$ is unique.

(b) There are at least two different arrows in $v^{-}$.

(c) If $\alpha, \beta \in v^{-}$, and $\alpha \neq \beta$, then $\omega_\alpha - \omega_\beta \in I$.

(ii) If $X|_{C_i} = 0$ for any $i \in \{1, \ldots, m\}$, then $X$ is uniserial.

**Proof** Since $X$ is an (ad 1)-pivot, the support $S(X)$ consists of an infinite sectional path $X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ starting at $X$. Let $t \geq 1$ be a positive integer, $D$ be the full $t \times t$ lower triangular matrix algebra, and $Y_1, \ldots, Y_t$ be the indecomposable injective $D$-modules with $Y_1$ the unique indecomposable projective-injective $D$-module (see definition of (ad 1)).

(i) Again, we know from [33, Section 4] that $A^{(l)}$ is a unique maximal convex branch coextension of $C = C_1 \times \cdots \times C_m$ inside $A$, that is, $A^{(l)} = B^{(l)}_1 \times \cdots \times B^{(l)}_m$, where $B^{(l)}_i$ is a unique maximal convex branch coextension of $C_i$ inside $A$, $i \in \{1, \ldots, m\}$. More precisely, $B^{(l)}_i = I_{K_{i1}}[K_{1i}, E_j]C_i$, where $K_{1i}, \ldots, K_{ti}$ are branches, $i \in \{1, \ldots, m\}$. Assume that there is a vertex $u \in A^{(l)} \setminus A^{(r)}$ such that each $\alpha \in v^{-}$ is the starting point of a
nonzero path \( \omega_\alpha \in A(v, u) \). Then there exists \( s \in \{1, \ldots, m\} \) such that \( u \in B_s^{(l)} \). Moreover, \( A_{j+1} = (A_j \times D)[X \oplus Y_1] \) and the bound quiver \( Q_{A_{j+1}}|_{\text{Supp } X} \) is of the form

![Diagram of a quiver](image)

where \( v_1, \ldots, v_t \) are the points in the quiver \( Q_{A_{j+1}} \) of \( A_{j+1} \) corresponding to the new indecomposable projective \( A_{j+1} \)-modules. Then \( A_{j+1} \) is the extension of \( B_s^{(l)} \) at \( X \) by the extension branch \( K \) consisting of the points \( v, v_1, \ldots, v_t \), that is, we have \( A_{j+1} = A_j[X, K] \). Since \( u \) does not belong to \( A^{(r)} \) and for any \( \alpha \in v \rightarrow \) it is the starting point of a nonzero path \( \omega_\alpha \in A(v, u) \), we get that \( u \) is the coextension point of the admissible operation \((\text{ad } 2^*)\) or \((\text{ad } 3^*)\). By \([10, \text{Lemma } 3.1]\) the admissible operations \((\text{ad } 2^*)\) and \((\text{ad } 3^*)\) commute with \((\text{ad } 1)\), so we can apply \((\text{ad } 2^*)\) after \((\text{ad } 1)\) (respectively, \((\text{ad } 3^*)\) after \((\text{ad } 1)\)).

Using now \([10, \text{Lemma } 3.3]\) (respectively, \([10, \text{Lemma } 3.4]\)), we are able to replace \((\text{ad } 1)\) followed by \((\text{ad } 2^*)\) (respectively, \((\text{ad } 1)\) followed by \((\text{ad } 3^*)\)) by an operation of type \((\text{ad } 1^*)\) followed by an operation of type \((\text{ad } 2)\) (respectively, \((\text{ad } 1^*)\) followed by an operation of type \((\text{ad } 3)\)). Therefore, the statements (a), (b) and (c) follow from \(\text{Lemma } 4.2\).

(ii) A case by case inspection (which admissible operation gives rise to the \((\text{ad } 1)\)-pivot \( X \)) shows that \( X \) is either simple module or the support of \( X \) is a linearly ordered quiver of type \( \Lambda_t \).

**Lemma 4.4** Let \( A \) be a generalized multicoil enlargement of a concealed canonical algebra \( C = C_1 \times \cdots \times C_m \). Moreover, let \( C = A_0, \ldots, A_p = A^{(l)}, A_{p+1}, \ldots, A_n = A \) be an admissible sequence for \( A, j \geq p, X \in \text{ind } A \) be an \((\text{ad } 4)\) or \((\text{ad } 5)\)-pivot, \( A_{j+1} \) be the modified algebra of \( A_j \), and \( v \) be the corresponding extension point. If there is a vertex \( u \in A^{(l)} \setminus A^{(r)} \) such that for pairwise different arrows \( \alpha_1, \ldots, \alpha_q \in v \rightarrow, q \geq 2 \) there are paths \( \omega_{\alpha_1}, \ldots, \omega_{\alpha_q} \in A(v, u) \), then for arbitrary \( f, g \in \{1, \ldots, q\}, f \neq g \), one of the following cases holds:

1. At least one of \( \omega_{\alpha_f}, \omega_{\alpha_g} \) is zero path.
2. The paths \( \omega_{\alpha_f}, \omega_{\alpha_g} \) are nonzero and \( \omega_{\alpha_f} - \omega_{\alpha_g} \in 1 \).

**Proof** It follows from \([33, \text{Section } 4]\) that \( A^{(l)} \) is a unique maximal convex branch coextension of \( C = C_1 \times \cdots \times C_m \) inside \( A \), that is, \( A^{(l)} = B_1^{(l)} \times \cdots \times B_m^{(l)} \), where \( B_i^{(l)} \) is a unique maximal convex branch coextension of \( C_i \) inside \( A, i \in \{1, \ldots, m\} \). More precisely, \( B_i^{(l)} = j=1^i[K_j, E_j]C_i \), where \( K_1, \ldots, K_i \) are branches, \( i \in \{1, \ldots, m\} \). Assume that there is a vertex \( u \in A^{(l)} \setminus A^{(r)} \) such that for pairwise different arrows \( \alpha_1, \ldots, \alpha_q \in v \rightarrow, q \geq 2 \), there are paths \( \omega_{\alpha_1}, \ldots, \omega_{\alpha_q} \in A(v, u) \). Then there exists \( s \in \{1, \ldots, m\} \) such that \( u \in B_s^{(l)} \).

Let \( X \) be an \((\text{ad } 4)\)-pivot and \( Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_t \) with \( t \geq 1 \), be a finite sectional path in \( \Gamma_{A_j} \) (as in the definition of \((\text{ad } 4)\)). Note that this finite sectional path is the linearly oriented quiver of type \( \Lambda_t \) and its support algebra \( \Lambda \) (given by the vertices corresponding to the simple composition factors of the modules \( Y_1, Y_2, \ldots, Y_t \)) is a tilted algebra of the path.
algebra $D$ of the linearly oriented quiver of type $A_r$. From [41, (4.4)(2)] we know that $\Lambda$ is a bound quiver algebra given by a branch in $x$, where $x$ corresponds to the unique projective-injective $D$-module. Let $\Gamma$ be a generalized multicoil of $\Gamma_{A_{j+1}}$ obtained by applying the admissible operation (ad 4), where $X$ is the vertex contained in the generalized multicoil $\Omega_1$, and $Y_1$ is the starting vertex of a finite sectional path contained in the generalized multicoil $\Omega_1$ or $\Omega_2$. So, $\Gamma$ is obtained from $\Omega_1$ or from the disjoint union of two generalized multicoils $\Omega_1$, $\Omega_2$ by the corresponding translation quiver admissible operations. In general, $\Omega_1$ and $\Omega_2$ are components of the same connected algebra or two connected algebras. Hence, we get two cases. In the first case $X, Y_1 \in \Omega_1$ or $X \in \Omega_1, Y_1 \in \Omega_2$ and $\Omega_1, \Omega_2$ are two components of the same connected algebra. In the second case $X \in \Omega_1, Y_1 \in \Omega_2$ and $\Omega_1, \Omega_2$ are two components of two connected algebras. Therefore, the bound quiver $QA_{j+1}$ of $A_{j+1}$ in the first case is of the form

$$QA_{j}$$

for $r = 0$ and

$$QA_{j}$$

for $r \geq 1$, where the index $r$ is as in the definition of (ad 4), $v$ is the extension point of $A_{j}[X], w$ is the extension point of $A_{j}[Y_1], w_1, \ldots, w_d$ belong to the branch in $w$ generated by the support of $Y_1 \oplus \cdots \oplus Y_r$, and $\alpha \beta_1 \cdots \beta_h = 0$ for some $h \in \{1, \ldots, d+1\}$. In the second case the bound quiver $QA_{j+1}$ of $A_{j+1}$ is of the form

$$QA_{j}$$

for $r = 0$ and

$$QA_{j}$$
for \( r \geq 1 \), where the index \( r \) is as in the definition of \((ad 4)\), \( v \) is the extension point of \( A_j[X] \), \( w \) is the extension point of \( A_j[Y_1] \), \( w_1, \ldots, w_d \) belong to the branch in \( w \) generated by the support of \( Y_1 \oplus \cdots \oplus Y_t, \alpha \beta_1 \cdots \beta_h = 0 \) for some \( h \in \{1, \ldots, d + 1\} \), and \( y \) is the coextension point of \( A_j \) such that \( y \in A(l) \setminus A(r) \). More precisely, \( y \in B_s(l) \), where \( s' \in \{1, \ldots, m\} \) and \( s' \neq s \). Moreover in both cases, we have \( P_v(u) = X(u) = k \) or \( P_v(u) = X(u) = 0 \), and hence all nonzero paths from \( v \) to \( u \) are congruent modulo \( I_{j+1} \). So, \( A_{j+1}(v, u) \) is at most one-dimensional. We note that in the first case, the definition of \((ad 4)\) (see the shape of the bound quiver \( QA_{j+1} \) of \( A_{j+1} \)) implies that if the paths \( \alpha \beta g, \alpha \beta g \in A_{j+1}(v, u) \) are nonzero and \( \alpha \beta g - \alpha \beta g \in I \), then there is also a zero path \( \alpha \beta h \in A_{j+1}(v, u) \) for some \( h \in \{1, \ldots, q\} \), \( h \neq f \neq g \).

Let \( X \) be an \((ad 5)\)-pivot and \( \Gamma \) be a generalized multcoil of \( \Gamma_{A_{j+1}} \) obtained by applying this admissible operation with pivot \( X \). Then \( \Gamma \) is obtained from the disjoint union of the finite family of generalized multicoils \( \Omega_1, \Omega_2, \ldots, \Omega_e \) by the corresponding translation quiver admissible operations, \( 1 \leq e \leq l \), where \( l \) is the number of stable tubes of \( \Gamma_C \) used in the whole process of creating \( \Gamma \). Since in the definition of admissible operation \((ad 5)\) we use the finite versions \((fad 1)\)–\((fad 4)\) of the admissible operations \((ad 1)\)–\((ad 4)\) and the admissible operation \((ad 4)\), we conclude that the required statement follows from the above considerations.

**Remark 4.5** Let \( A \) be a generalized multcoil enlargement of a concealed canonical algebra \( C \). We know from Theorems 3.3 and 3.5 that \( A \) can be obtained from \( A(l) \) by a sequence of admissible operations of types \((ad 1)\)–\((ad 5)\) or \( A \) can be obtained from \( A(r) \) by a sequence of admissible operations of types \((ad 1^*)\)–\((ad 5^*)\). We note that all presented above lemmas can be formulated and proved for dual operations \((ad 1^*)\)–\((ad 5^*)\) in a similar way.

### 4.6 The Separating Vertex

Let \( A \) be a triangular algebra. Recall that a vertex \( v \) of \( QA \) is called *separating* if the radical of \( P_v \) is a direct sum of pairwise nonisomorphic indecomposable modules whose supports are contained in different connected components of the subquiver \( Q(v) \) of \( QA \) obtained by deleting all those vertices \( u \) of \( QA \) being the source of a path with target \( v \) (including the trivial path from \( v \) to \( v \)).

We have the following lemma which follows from the proof of [44, Proposition 2.3] (see also [2, Lemma 2.3]).

**Lemma 4.7** Let \( A \) be a triangular algebra and assume that \( A = B[X] \), where \( v \) is the extension vertex and \( X = \text{rad}_A P_v \). If \( B \) is simply connected and \( v \) is separating, then \( A \) is simply connected.

Let \( D \) be the same as in the definition of \((ad 1)\), that is, the full \( t \times t \) lower triangular matrix algebra. Denote by \( Y_1, \ldots, Y_t \) the indecomposable injective \( D \)-modules with \( Y = Y_1 \) the unique indecomposable projective-injective \( D \)-module.

**Lemma 4.8** Let \( A \) be a triangular algebra and assume that \( A = (B \times D)[X \oplus Y] \), where \( v \) is the extension vertex and \( X \oplus Y = \text{rad}_A P_v \). If \( B \) is simply connected and \( v \) is separating, then \( A \) is simply connected.

**Proof** Since the module \( P_v \) is a sink in the full subcategory of \( \text{ind} \ A \) consisting of projectives, the vertex \( v \) is a source in \( QA \). Moreover, \( A = (B \times D)[X \oplus Y] \), where \( X \) is the indecomposable direct summand of \( \text{rad}_A P_v \) that belongs to \( \text{mod} \ B \) and \( Y \) is a directing
module (that is, an indecomposable module which does not lie on a cycle in \( \text{ind} \ A \)) such that \( \text{rad}_A \ P_0 = X \oplus Y \). Therefore, the proof follows from the proof of [44, Proposition 2.3] (see also the proof of Lemma 2.3 in [2]).

4.9 The Pointed Bound Quiver

In order to carry out the construction of the free product of two fundamental groups of bound quivers, and in analogy with algebraic topology where pointed spaces are considered, one can define a \textit{pointed bound quiver} \((Q, I, x)\), that is, a bound quiver \((Q, I)\) together with a distinguished vertex \(x\) (see [13, Section 3]). Given two pointed bound quivers \(Q' = (Q', I', x')\) and \(Q'' = (Q'', I'', x'')\), we can assume, without loss of generality, that \(Q'_0 \cap Q''_0 = Q'_1 \cap Q''_1 = \emptyset\). We define the quiver \(Q = Q' \sqcup Q''\) as follows: \(Q_0 = Q'_0 \cup Q''_0\) in which we identify \(x'\) and \(x''\) to a single new vertex \(x\), and \(Q_1 = Q'_1 \cup Q''_1\). Then, \(Q'\) and \(Q''\) are identified to two full convex subquivers of \(Q\), so walks on \(Q'\) or \(Q''\) can be considered as walks on \(Q\). Thus, \(I'\) and \(I''\) generate two-sided ideals of \(kQ\) which we denote again by \(I'\) and \(I''\). We define \(I\) to be the ideal \(I' + I''\) of \(kQ\). It follows from this definition that the minimal relations of \(I\) are precisely the minimal relations of \(I'\) together with the minimal relations of \(I''\) give the minimal relations needed to determine the homotopy relation on \((Q, I)\). Moreover, we can consider an element \(\bar{w} \in \pi_1(Q', I', x')\) as an element \(\bar{w} \in \pi_1(Q, I, x)\) (we denote by \(\bar{w}\) the homotopy class of a walk \(w\)). Conversely, any (reduced) walk \(w\) in \(Q\) has a decomposition \(w = w'_1w'_2w'_3\ldots w'_nw''_n\), where \(w'_i\) and \(w''_i\) are walks in \(Q'\) and \(Q''\) for \(i \in \{1, \ldots, n\}\), respectively. Moreover, this decomposition is unique, up to reduced walk, and compatible with the homotopy relations involved. This leads us to the following proposition.

**Proposition 4.10** [13, Proposition 3.1] With the notations above we have:

(i) \((Q, I, x)\) is the coproduct of \((Q', I', x')\) and \((Q'', I'', x'')\) in the category of pointed bound quivers.

(ii) \(\pi_1(Q, I, x) \cong \pi_1(Q', I', x') \sqcup \pi_1(Q'', I'', x'').\)

5 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1 and recall the relevant facts.

We know from Theorem 3.3 that the Auslander–Reiten quiver \(\Gamma_A\) of \(A\) admits a separating family of almost cyclic coherent components if and only if \(A\) is a generalized multicoil enlargement of a concealed canonical algebra \(C\). Let \(C = C_1 \times C_2 \times \cdots \times C_l \times C_{l+1} \times \cdots \times C_m\) be a decomposition of \(C\) into product of connected algebras such that \(C_1, C_2, \ldots, C_l\) are of type \((p_1, p_2)\) and \(C_{l+1}, C_{l+2}, \ldots, C_m\) are of type \((p_1, \ldots, p_l)\) with \(t \geq 3\). Following [36], by \(h_i\) we denote the number of all stable tubes of rank one from \(\Gamma_{C_i}\), with \(1 \leq i \leq l\), used in the whole process of creating \(A\) from \(C\), and \(h_i = 0\), if \(i + 1 \leq i \leq l\). Moreover, let

\[
e_i = \begin{cases} 
0 & \text{if } C_i \text{ is of type } (p_1, \ldots, p_l) \text{ with } t \geq 3 \\
1 & \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1, p_2 \geq 2 \\
2 & \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1 = 1, p_2 \geq 2 \\
3 & \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1 = p_2 = 1,
\end{cases}
\]

for \(i \in \{1, \ldots, m\}\). We define also \(f_{C_i} = \max(e_i - h_i, 0)\), for \(i \in \{1, \ldots, m\}\) and set \(f_A = \sum_{i=1}^m f_{C_i} = \sum_{i=1}^l f_{C_i}\). Note that we can apply the operations \((\text{ad} 4)\), \((\text{fad} 4)\), \((\text{ad} 4^*)\), \((\text{fad} 4^*)\) in two ways. The first way is when the sectional paths occurring in the definitions of these operations come from a component or two components of the same connected algebra. The second one is, when these sectional paths come from two components of two
connected algebras. By $d_A$ we denote the number of all operations $(\text{ad} \ 4), (\text{fad} \ 4), (\text{ad} \ 4^*)$ or $(\text{fad} \ 4^*)$ which are of the first type, used in the whole process of creating $A$ from $C$.

The Hochschild cohomology of a connected generalized multicoil algebra $A$ has been described in [36, Theorem 1.1] using the numerical invariants of $A$ ($f_A$, $d_A$ and the others), depending on the types of admissible operations $(\text{ad} \ 1)–(\text{ad} \ 5)$ and their duals, leading from a product $C$ of concealed canonical algebras to $A$. Here, we will only need information about the first Hochschild cohomology of $A$, namely from [36, Theorem 1.1(iii)] we have:

**Theorem 5.1** Let $A$ be a connected generalized multicoil algebra. Then $\dim_k H^1(A) = d_A + f_A$.

We are now able to complete the proof of Theorem 1.1.

Since $A$ is tame, we may restrict to the generalized multicoil enlargements of tame concealed algebras. Namely, we have the following consequence of Theorem 3.3 and [33, Theorem F]: $A$ is tame and $\Gamma_A$ admits a separating family of almost cyclic coherent components if and only if $A$ is a tame generalized multicoil enlargement of a finite family $C_1, \ldots, C_m$ of tame concealed algebras (concealed canonical algebras of Euclidean type).

We first show the necessity. Suppose that $A$ is simply connected. We must show that the first Hochschild cohomology $H^1(A)$ vanishes. Assume to the contrary that $H^1(A) \neq 0$. Then by Theorem 5.1, $d_A + f_A \neq 0$. If $d_A \neq 0$, then it follows from the proof of Lemma 4.4 (and its dual version) that $A$ is not simply connected, a contradiction. Therefore, we may assume that $d_A = 0$ and $f_A \neq 0$. Since $f_A = \sum_{i=1}^l \max\{e_i - h_i, 0\} \neq 0$, we get that $\max\{e_j - h_j, 0\} \neq 0$ for some $j \in \{1, \ldots, l\}$. Note that, from Lemmas 4.2, 4.3, 4.4 and their proofs (and also from their dual versions - see Remark 4.5), we know how the bound quiver algebra changes after applying a given admissible operation. We have three cases to consider:

1. Assume that the algebra $C_j$ is of type $(p_1, p_2)$ with $p_1, p_2 \geq 2$. Then $e_j = 1$ and $h_j = 0$. The bound quiver algebra $A = kQ/I$ is given by the quiver $Q$ which can be visualized as follows:

![Diagram of a quiver algebra](https://via.placeholder.com/150)

where $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $\varepsilon_1 \alpha_1, \alpha_2 \gamma_1, \varepsilon_1 \gamma_1 - \varepsilon_2 \gamma_2, \beta_2 \xi, \alpha_{p_1 - 1} \omega, \delta \alpha_p, \sigma_1 \beta_{p_2 - 1}, \sigma_2 \sigma_3 \psi$, elements from parts $A, B, D$ of $Q$, and elements from $C_i$. Therefore, $\pi_1(Q, I)$ is not trivial and so $A$ is not simply connected. More
precisely, it follows from Proposition 4.10 that $\pi_1(Q, I) = \mathbb{Z} \sqcup \pi_1(A) \sqcup \pi_1(B) \sqcup \pi_1(D) \sqcup \pi_1(C_i)$.

(2) Assume that the algebra $C_j$ is of type $(p_1, p_2)$ with $p_1 = 1$, $p_2 \geq 2$. Then $e_j = 2$, $h_j = 0$ or $h_j = 1$ and we have two subcases to consider. If $e_j = 2$ and $h_j = 0$, then the bound quiver algebra $A = kQ/I$ is given by the quiver $Q$ which can be visualized as follows:

where $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $\gamma\beta_{p_2}$, $\beta_{p_2-1}\omega$, $\sigma_1\beta_1$, $\sigma_2\sigma_3\varphi$, elements from parts $A, B$ of $Q$, and elements from $C_i$. Therefore, $\pi_1(Q, I)$ is not simply connected. More precisely, it follows from Proposition 4.10 that $\pi_1(Q, I) = \mathbb{Z} \sqcup \pi_1(A) \sqcup \pi_1(B) \sqcup \pi_1(C_i)$. If $e_j = 2$ and $h_j = 1$, then the bound quiver algebra $A = kQ/J$ is given by the quiver $Q$ which can be visualized as in the previous subcase with the ideal $J$ of $kQ$ generated by the elements $\gamma\alpha_1 - a\gamma\beta_{p_2} + \beta_2\beta_1, \beta_{p_2-1}\omega, \sigma_1\beta_1$, $\sigma_2\sigma_3\varphi$, elements from parts $A, B$ of $Q$, and elements from $C_i$, where $a \in k \setminus \{0\}$. Note that in general, we can apply to a stable tube $T$ of one of the following admissible operations: (ad 1), (ad 4), (ad 5) or their dual versions (with an infinite sectional path belonging to $T$). Since $h_j = 1$, we applied (in the above visualization) an admissible operation from the set $S = \{(\text{ad} 1), (\text{ad} 4), (\text{ad} 5)\}$ to the algebra $C_j$ with pivot the regular $C_j$-module corresponding to the indecomposable representation of the form

lying in a stable tube of rank 1 in $\Gamma_{C_j}$ (see [42, XIII.2.4(c)]), where $a \in k \setminus \{0\}$. More precisely, if we apply (ad 1) with parameter $t = 0$, then we have to remove the arrow $\varepsilon$ and the part $B$. Observe also that $A$ is not simply connected, because $A$ is isomorphic to the algebra $A' = kQ/J'$, where the ideal $J'$ of $kQ$ is generated by the elements of $J \setminus \{\gamma\alpha_1 - a\gamma\beta_{p_2} + \beta_2\beta_1\} \cup \{\gamma\alpha_1\}$ and $\pi_1(Q, J')$ is not trivial. Again, it follows from Proposition 4.10 that $\pi_1(Q, J') = \mathbb{Z} \sqcup \pi_1(A) \sqcup \pi_1(B) \sqcup \pi_1(C_i)$. If we apply an admissible operation from the set $S^* = \{(\text{ad} 1^*), (\text{ad} 4^*), (\text{ad} 5^*)\}$ to the algebra $C_j$, the proof follows by dual arguments.

(3) Assume that the algebra $C_j$ is of type $(p_1, p_2)$ with $p_1 = p_2 = 1$. Then $e_j = 3$, $h_j = 0$, $h_j = 1$ or $h_j = 2$ and we have three subcases to consider. Note that in this case all stable tubes in $\Gamma_{C_j}$ have ranks equal to 1. Now, if $e_j = 3$ and $h_j = 0$, then $j = l = 1$ and the path algebra $A = kQ$ is given by the Kronecker quiver $Q$: $\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ$. Therefore,
\[ \pi_1(Q) \cong \mathbb{Z} \text{ and so } A \text{ is not simply connected. If } e_j = 3 \text{ and } h_j = 1, \text{ then the bound quiver algebra } A = kQ/J \text{ is given by the quiver } Q \text{ which can be visualized as follows:} \\
\begin{array}{cccccc}
\circ & \xleftarrow{\alpha} & \circ & \xleftarrow{\gamma} & \circ & \xrightarrow{\epsilon} \text{ part } A \\
\circ & \xrightarrow{\beta} & \circ & \xrightarrow{\lambda} \end{array}
\]

with the ideal } J \text{ in the path algebra } kQ \text{ of } Q \text{ over } k \text{ generated by the element } \gamma\alpha - a\gamma\beta \text{ and elements from part } A \text{ (the rest of } Q), \text{ where } a \in k \setminus \{0\}. \text{ Since } h_j = 1, \text{ we applied (in the above visualization) an admissible operation from the set } S \text{ to the algebra } C_j \text{ with pivot the regular } C_j\text{-module corresponding to the indecomposable representation of the form } k \xleftarrow{a} \frac{1}{a} k \text{ lying in a stable tube of rank } 1 \text{ in } \Gamma_{C_j} \text{ (see [42, XIII.2.4(c)]}, \text{ where } a \in k \setminus \{0\}. \text{ More precisely, if we apply (ad 1) with parameter } t = 0, \text{ then we have to remove the arrow } \epsilon \text{ and the part } A. \text{ Observe also that } A \text{ is not simply connected, because } A \text{ is isomorphic to the algebra } A' = kQ/J', \text{ where the ideal } J' \text{ of } kQ \text{ is generated by the elements of } J \setminus \{\gamma\alpha - a\gamma\beta\} \cup \{\gamma\alpha\} \text{ and } \pi_1(Q, J') \text{ is not trivial. Again, it follows from Proposition 4.10 that } \pi_1(Q, J') = \mathbb{Z} \sqcup \pi_1(A). \text{ Moreover, if we apply an admissible operation from the set } S^* \text{ to the algebra } C_j, \text{ the proof follows by dual arguments. If } e_j = 3 \text{ and } h_j = 2, \text{ then the bound quiver algebra } A = kQ/L \text{ is given by the quiver } Q \text{ which can be visualized as follows:} \\
\begin{array}{cccccc}
\text{part } A & \xrightarrow{\lambda} & \circ & \xleftarrow{\delta} & \circ & \xleftarrow{\gamma} & \circ & \xrightarrow{\epsilon} \text{ part } B \\
\end{array}
\]

with the ideal } L \text{ of } kQ \text{ generated by the elements } \gamma\alpha - a\gamma\beta, a\alpha\delta - b\beta\gamma, \gamma\alpha\delta \text{ and elements from parts } A, B \text{ of } Q, \text{ where } a, b \in k \setminus \{0\} \text{ and } a \neq b. \text{ Since } h_j = 2, \text{ we applied (in the above visualization) one admissible operation from the set } S \text{ and one from the set } S^* \text{ to the algebra } C_j \text{ with pivots the regular } C_j\text{-modules corresponding to the indecomposable representations of the form} \\
k \xleftarrow{a} \frac{1}{a} k \text{ and } k \xleftarrow{b} := k
\]

lying in different stable tubes of rank } 1 \text{ in } \Gamma_{C_j} \text{ (see [42, XIII.2.4(c)]}, \text{ where } a, b \in k \setminus \{0\} \text{ and } a \neq b. \text{ More precisely, if we apply (ad 1) (respectively, (ad 1*)) with parameter } t = 0, \text{ then we have to remove the arrow } \epsilon \text{ and the part } B \text{ (respectively, the arrow } \lambda \text{ and the part } A). \text{ Observe also that } A \text{ is not simply connected, because } A \text{ is isomorphic to the algebra } A' = kQ/L', \text{ where the ideal } L' \text{ of } kQ \text{ is generated by the elements of } L \setminus \{\gamma\alpha - a\gamma\beta, a\alpha\delta - b\beta\gamma\} \cup \{\gamma\alpha, a\delta\} \text{ and } \pi_1(Q, L') \text{ is not trivial. Again, it follows from Proposition 4.10 that } \pi_1(Q, L') = \mathbb{Z} \sqcup \pi_1(A) \sqcup \pi_1(B). \text{ In a similar way, one can show all the cases of applying two admissible operations from the set } S \cup S^* \text{ to any two stable tubes of rank one from the Auslander–Reiten quiver of the Kronecker algebra.} \]

We now show the sufficiency. We note from Theorem 3.5 that there is a unique full convex subcategory } A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)} \text{ of } A \text{ which is a tubular coextension of the product } C_1 \times \cdots \times C_m = C \text{ of a family } C_1, \ldots, C_m \text{ of tame concealed algebras (see remarks immediately after } Theorem 5.1) \text{ such that } A \text{ is obtained from } A^{(l)} \text{ by a sequence of admissible operations of types (ad 1)} \text{–(ad 5). We shall prove our claim by induction on the number of admissible operations leading from } A^{(l)} \text{ to the algebra } A. \text{ Note that we can apply an admissible operation (ad 2), (ad 3), (ad 4) or (ad 5) if the number of all successors of the module } Y_i \text{ (which occurs in the definitions of the above admissible operations) is finite for each } 1 \leq i \leq t. \text{ Indeed, if this is not the case, then the family of generalized multicoils obtained after applying such admissible operation is not sincere, and then it is not}
operation. Since applying again Lemma 4.7, we get that the finite versions (fad 1)–(fad 4) of the admissible operations (ad 1)–(ad 4) and the admissible operation leading from $A_k$ to $A$ is of type (ad 1), (ad 2) or (ad 3), then $A_k$ is a connected algebra.

If $X$ is an (ad 1)-pivot, then $A = A_k[X]$ or $A = (A_k \times D)[X \oplus Y]$, where $\text{rad}_A P_v = X$ or $\text{rad}_A P_v = X \oplus Y$ respectively, $D$ is the full $t \times t$ lower triangular matrix algebra over $k$ for some $t \geq 1$, and $Y$ is the unique indecomposable projective-injective $D$-module (see definition of (ad 1)). Applying Lemma 4.7 or Lemma 4.8 respectively, we conclude that $A$ is simply connected.

If $X$ is an (ad 2)-pivot or (ad 3)-pivot, then $A = A_k[X]$, where $\text{rad}_A P_v = X$. Applying Lemma 4.7, we conclude that $A$ is simply connected.

Let $X$ be an (ad 4)-pivot and $Y = Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_t$ with $t \geq 1$ be a finite sectional path in $\Gamma_{A_k}$. Then, for $r = 0$, $A = A_k[X \oplus Y]$, and for $r \geq 1$,

$$A = \begin{bmatrix} A_k & 0 & 0 & \ldots & 0 \\ Y & k & 0 & \ldots & 0 \\ Y & k & k & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y & k & k & \ldots & k \\ X \oplus Y & k & k & \ldots & k \end{bmatrix}$$

with $r + 2$ columns and rows (see definition of (ad 4)). We note that $Y_i$ is directing $A$-module for each $1 \leq i \leq t$. Indeed, since $H^1(A) = 0$, we get $d_A = 0$, and so $A_k$ is not connected.

Now, if $r = 0$, then $A = A_0[X \oplus Y]$ and $\text{rad}_A P_v = X \oplus Y$. Then it follows from Lemma 4.7 that $A$ is simply connected.

If $r \geq 1$, then observe that the modified algebra $A$ of $A_k$ can be obtained by applying $r + 1$ one-point extensions in the following way: $A_k^{(0)} = A_k[U_0]$, $A_k^{(1)} = A_k^{(0)}[U_{11}]$, $A_k^{(2)} = A_k^{(1)}[U_{21}]$, $\ldots$, $A_k^{(r-1)} = A_k^{(r-2)}[U_{r-1,1}]$ and finally $A = A_k^{(r)} = A_k^{(r-1)}[X \oplus U_{r1}]$, where $U_0 = Y$, $U_{j1}$ is a projective $A_k^{(j-1)}$-module such that $\text{rad}_{A_k^{(j-1)}} U_{j1} = U_{j-1,1}$, for $r \geq 1, 1 \leq j \leq r$. We denote by $v_j$ the extension vertex of $A_k^{(j-1)}$, for $1 \leq j \leq r$. Since the vertex $v_1$ of $Q_{A_k^{(0)}}$ is separating and $\text{rad}_{A_k^{(0)}} P_{v_1} = U_{01}$, applying Lemma 4.7, we conclude that the algebra $A_k^{(0)}$ is simply connected. Further, since the vertex $v_2$ of $Q_{A_k^{(1)}}$ is separating, $\text{rad}_{A_k^{(1)}} P_{v_2} = U_{11}$, and $A_k^{(0)}$ is simply connected, it follows from Lemma 4.7 that $A_k^{(1)}$ is simply connected. Iterating a finite number of times the same arguments, we get that $A_k^{(r-1)}$ is simply connected. Finally, since the vertex $v$ of $Q_A$ is separating and $\text{rad}_A P_v = X \oplus U_{r1}$, applying again Lemma 4.7, we get that $A$ is simply connected.

Let $X$ be an (ad 5)-pivot. Since in the definition of admissible operation (ad 5) we use the finite versions (fad 1)–(fad 4) of the admissible operations (ad 1)–(ad 4) and the admis-
sible operation (ad 4), we conclude that the required statement follows from the above considerations.

This finishes the proof of Theorem 1.1.

6 Proof of Theorem 1.2

Let $A$ be a generalized multicoil algebra. Then $A$ is a connected generalized multicoil enlargement of a concealed canonical algebra $C$. Let $C = C_1 \times C_2 \times \cdots \times C_l \times C_{l+1} \times \cdots \times C_m$ be a decomposition of $C$ into product of connected algebras such that $C_1, C_2, \ldots, C_l$ are of type $(p_1, p_2)$ and $C_{l+1}, C_{l+2}, \ldots, C_m$ are of type $(p_1, \ldots, p_t)$ with $t \geq 3$. Since $C_i, i \in \{1, \ldots, m\}$, are simply connected, we get $l = 0$. Moreover, by the assumption, the sectional paths occurring in the definitions of the operations (ad 4), (fad 4), (ad 4*), (fad 4*) come from two components of two connected algebras. Applying Theorems 3.3 and 3.5 we infer that there exists a unique factor algebra $A(l) = A(l)_1 \times \cdots \times A(l)_m$ of $A$ which is a tubular coextension of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$, and a unique factor algebra $A(r) = A(r)_1 \times \cdots \times A(r)_m$ of $A$ which is a tubular extension of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$. Since $A(l)$ and $A(r)$ are quasitilted algebras (of canonical types), the equivalence (ii) and (iv) follows from [26, Theorem 1]. Clearly, (v) implies (i).

We now show that (i) implies (iii). Since all algebras $C_1, \ldots, C_m$ are of type $(p_1, \ldots, p_t)$ with $t \geq 3 (l = 0)$, we get $f_A = 0$. Assume to the contrary that $H_1(A(l)) \neq 0$. Then, by Theorem 5.1, $d_A + f_A \neq 0$. Therefore, $d_A \neq 0$ and it follows from the proof of Lemma 4.4 (and its dual version) that $A$ is not simply connected, a contradiction with (i).

We show that (iii) implies (iv). Assume to the contrary that there exists $i \in \{1, \ldots, m\}$ such that $H_1(A(l)_i) \neq 0$ or $H_1(A(r)_i) \neq 0$. Without loss of generality, we may assume that $H_1(A(l)_i) \neq 0$ for some $i \in \{1, \ldots, m\}$. Since $A(l)_i$ is a tubular coextension of a concealed canonical algebra $C_i$, we have that $A(l)_i$ is a generalized multicoil enlargement of $C_i$, and so, by Theorem 5.1, $\dim_k H_1(A(l)_i) = d_{A(l)_i} + f_{A(l)_i}$. Moreover, by our assumption on $C_i$, we have $f_{A(l)_i} = 0$. Hence $d_{A(l)_i} \neq 0$. Since $d_A \geq d_{A(l)_i}$, we get a contradiction with (iii).

In order to finish the proof we will show that (iv) implies (v). Assume that $H_1(A(l)_i) = 0$ and $H_1(A(r)_i) = 0$, for any $i \in \{1, \ldots, m\}$. We know that for each $i \in \{1, \ldots, m\}$, $A(l)_i$ (respectively, $A(r)_i$) is a tubular coextension (respectively, extension) of a concealed canonical algebra $C_i$ of type $(p_1, \ldots, p_t)$, $t \geq 3$ and $H_1(C_i) = 0$, by [20, Theorem 2.4]. Then $H_1(B) = 0$ for every full convex subcategory $B$ of $A(l)_i$ (respectively, $A(r)_i$). Therefore, it follows from [44, Theorem 4.1] that $A(l)_i$ and $A(r)_i$ are strongly simply connected, for any $i \in \{1, \ldots, m\}$. Moreover, by our assumption on $A$, the Auslander–Reiten quiver $\Gamma_A$ does not contain exceptional configurations of modules. Applying now Theorems 3.3 and 3.6 we infer that $A$ is strongly simply connected.

7 Examples

We start this section with the following remark.

\[ \text{Simply Connected Algebras} \]
Remark 7.1 We can apply Theorem 1.1 to important classes of algebras. For example, to the cycle-finite algebras with separating families of almost cyclic coherent Auslander–Reiten components. Indeed, it is known (see [8]) that every cycle-finite algebra is tame.

Example 7.2 Let $A = kQ/I$ be the bound quiver algebra given by the quiver $Q$ of the form

and $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $\alpha\beta$, $\gamma\delta$, $\eta\varepsilon$, $\kappa\lambda\varrho$, $\xi\kappa\lambda$, $\nu\alpha$. Then $A$ is a generalized multicoil enlargement of a concealed canonical algebra $C$, where $C$ is the hereditary algebra of Euclidean type $\tilde{D}_6$ given by the vertices 1, 2, ..., 7. Indeed, consider the dimension-vectors

$$\begin{align*}
a_1 &= 0 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix},
a_2 &= 0 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix},
a_3 &= 0 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix},
a_4 &= 0 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix},
a_5 &= 0 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix},
a_6 &= 0 \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}$$

We apply $(\text{ad } 1^*)$ with pivot the simple regular $C$-module with vector $a_1$, and with parameter $t = 0$. The modified algebra $B_1$ is given by the quiver with vertices 1, 2, ..., 8 bound by $\alpha\beta = 0$. Now, we apply $(\text{ad } 1^*)$ with pivot the indecomposable $B_1$-module with vector $a_2$, and with parameter $t = 2$. The modified algebra $B_2$ is given by the quiver with vertices 1, 2, ..., 11 bound by $\alpha\beta = 0$. Next, we apply $(\text{ad } 1^*)$ with pivot the indecomposable $B_2$-module with vector $a_3$, and with parameter $t = 3$. The modified algebra $B_3$ is given by the quiver with vertices 1, 2, ..., 15 bound by $\alpha\beta = 0$, $\gamma\delta = 0$. In the next step we apply $(\text{ad } 1^*)$ with pivot the indecomposable $B_3$-module with vector $a_4$, and with parameter $t = 0$. The modified algebra $B_4$ is given by the quiver with vertices 1, 2, ..., 16 bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$. Next, we apply the admissible operation $(\text{ad } 5)$ in two steps. The first step: we apply the operation $(\text{fad } 3)$ with pivot the indecomposable $B_4$-module with vector $a_5$, and with parameters $t = 3$, $s = 2$. The modified algebra $B_5$ is given by the quiver with vertices 1, 2, ..., 17 bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$. The second step: we apply the operation $(\text{ad } 4)$ with pivot the indecomposable $B_5$-module with vector $a_6$, and
with a finite sectional path consisting of the indecomposable $B_5$-modules with dimension-vectors

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0011 & 0011 & 0011 & 0011 & 00 \\
1 & 0 & 0 & 0 & 0
\end{array}
\]

and with parameter $r = 4$. The modified algebra is equal to $A$.

Then the left quasitilted algebra $A^{(l)}$ of $A$ is the convex subcategory of $A$ being the bound quiver algebra $kQ^{(l)}/I^{(l)}$, where $Q^{(l)}$ is a full subquiver of $Q$ given by the vertices $1, 2, \ldots, 16$ and $I^{(l)} = kQ^{(l)} \cap I$ is the ideal in $kQ^{(l)}$. The right quasitilted algebra $A^{(r)}$ of $A$ is the convex subcategory of $A$ being the bound quiver algebra $kQ^{(r)}/I^{(r)}$, where $Q^{(r)}$ is a full subquiver of $Q$ given by the vertices $1, 2, \ldots, 7, 14, 15, \ldots, 18$ and $I^{(r)} = kQ^{(r)} \cap I$ is the ideal in $kQ^{(r)}$. Note that $A^{(l)}$ and $A^{(r)}$ are tame.

It follows from Theorems 3.3, 3.5(iii) and the above construction that the Auslander–Reiten quiver $\Gamma_A$ of the tame algebra $A = kQ/I$ admits a separating family of almost cyclic coherent components. Further, $\pi_1(Q, I) \cong \mathbb{Z}$ and hence $A$ is not simply connected. Moreover, by Theorem 5.1, the first Hochschild cohomology space $H^1(A) \cong k (d_A = 1, f_A = 0)$. We also note that, since $A^{(l)}$ and $A^{(r)}$ are tame tilted algebras of Euclidean type $\mathbb{D}_4$ such that $H^1(A^{(l)}) = 0$ and $H^1(A^{(r)}) = 0$, it follows from [5, Theorem] that $A^{(l)}$ and $A^{(r)}$ are simply connected (and even strongly simply connected from [5, Corollary]). We refer to [33, Example 4.1] (see also [35, Example 9.13]) for a more extensive example of the tame algebra with a separating family of almost cyclic coherent components which is not simply connected. Finally, we also mention that $A$ is a generalized multicoil algebra such that $\Gamma_A$ contains the exceptional configurations of modules.

Example 7.3 We borrow the following example from [31]. Let $A = kQ/I$ be the bound quiver algebra given by the quiver $Q$ of the form

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
1 & & & & & \\
\downarrow & \alpha & \downarrow & \beta & \downarrow & \\
2 & & & & & \\
\downarrow & & & & & \\
3 & 8 & 7 & & & \\
\downarrow & \gamma & \downarrow & \delta & \downarrow & \\
9 & & & & & \\
\downarrow & & & & & \\
4 & 6 & & & & \\
\downarrow & & & & & \\
5 & & & & & \\
\downarrow & & & & & \\
& & & & & \\
\downarrow & \varphi_1 & \downarrow & \varphi_3 & \downarrow & \\
20 & & & & & \\
\downarrow & & & & & \\
24 & & & & & \\
\downarrow & & & & & \\
31 & & & & & \\
\downarrow & & & & & \\
16 & & & & & \\
\downarrow & & & & & \\
13 & & & & & \\
\downarrow & \lambda & \downarrow & \kappa & \downarrow & \\
12 & & & & & \\
\downarrow & & & & & \\
17 & & & & & \\
\downarrow & & & & & \\
14 & & & & & \\
\downarrow & \mu & \downarrow & \omega_1 & \downarrow & \\
15 & & & & & \\
\downarrow & & & & & \\
33 & & & & & \\
\downarrow & & & & & \\
30 & & & & & \\
\downarrow & & & & & \\
29 & & & & & \\
\downarrow & \pi_2 & \downarrow & \pi_1 & \downarrow & \\
27 & & & & & \\
\downarrow & & & & & \\
21 & & & & & \\
\downarrow & & & & & \\
26 & & & & & \\
\downarrow & & & & & \\
19 & & & & & \\
\downarrow & & & & & \\
18 & & & & & \\
\downarrow & & & & & \\
11 & & & & & \\
\downarrow & & & & & \\
10 & & & & & \\
\downarrow & \eta & \downarrow & \xi_2 & \downarrow & \\
15 & & & & & \\
\downarrow & & & & & \\
6 & & & & & \\
\downarrow & & & & & \\
\end{array}
\]

and $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $\alpha \beta, \gamma \delta, \eta \epsilon, \kappa \lambda \varphi_1, \varphi_2 \psi, \varphi_4 \psi, \xi_3 \omega_1, \zeta_1 \varphi_1, \zeta_1 \psi_2, \zeta_2 \xi_3 \xi_2 \xi_1 - \zeta_1 \psi, \pi_1 \xi_2, \pi_1 \omega_1 - \pi_2 \omega_2, \mu \kappa \lambda, \nu \xi_1$. Then $A$ is a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times C_2$, where $C_1$ is the hereditary algebra of Euclidean type $\mathbb{D}_6$ given by the vertices $1, 2, \ldots, 7$, and $C_2$ is the hereditary algebra of Euclidean type $\mathbb{D}_5$ given by the vertices $20, 21, \ldots, 24$. Indeed,
we apply (ad 1*) to \( C_1 \) with pivot the simple regular \( C_1 \)-module \( S_4 \), and with parameter \( t = 0 \). The modified algebra \( B_1 \) is given by the quiver with the vertices 1, 2, \ldots, 8 bound by \( \alpha \beta = 0 \). Next, we apply (ad 1*) to \( B_1 \) with pivot the indecomposable injective \( B_1 \)-module \( I_8 \), and with parameter \( t = 2 \). The modified algebra \( B_2 \) is given by the quiver with the vertices 1, 2, \ldots, 11 bound by \( \alpha \beta = 0 \). Now, we apply (ad 1*) to \( B_2 \) with pivot the indecomposable \( B_2 \)-module \( \tau_{B_2} S_{10} \), and with parameter \( t = 3 \). The modified algebra \( B_3 \) is given by the quiver with the vertices 1, 2, \ldots, 15 bound by \( \alpha \beta = 0, \gamma \delta = 0 \). Next, we apply (ad 1*) to \( B_3 \) with pivot the simple \( B_3 \)-module \( S_{14} \), and with parameter \( t = 0 \). The modified algebra \( B_4 \) is given by the quiver with the vertices 1, 2, \ldots, 16 bound by \( \alpha \beta = 0, \gamma \delta = 0, \eta \epsilon = 0 \). In the next step we apply (ad 1*) to \( C_2 \) with pivot the simple regular \( C_2 \)-module \( S_{22} \), and with parameter \( t = 3 \). The modified algebra \( B_5 \) is given by the quiver with the vertices 20, 21, \ldots, 28 bound by \( \varphi_3 \psi = 0, \varphi_4 \psi = 0 \). Now, we apply (ad 1*) to \( B_5 \) with pivot the simple \( B_5 \)-module \( S_{27} \), and with parameter \( t = 2 \). The modified algebra \( B_6 \) is given by the quiver with the vertices 20, 21, \ldots, 31 bound by \( \varphi_3 \psi = 0, \varphi_4 \psi = 0, \xi_3 \omega_1 = 0 \). Next, we apply (ad 2) to \( B_6 \) with pivot the indecomposable injective \( B_6 \)-module \( I_{25} \), and with parameter \( t = 3 \). The modified algebra \( B_7 \) is given by the quiver with the vertices 20, 21, \ldots, 32 bound by \( \varphi_3 \psi = 0, \varphi_4 \psi = 0, \xi_3 \omega_1 = 0, \xi_1 \varphi_1 = 0, \xi_1 \varphi_2 = 0, \xi_2 \xi_3 \xi_2 \xi_1 = \xi_1 \psi \). Now, we apply (ad 3) to \( B_7 \) with pivot the indecomposable \( B_7 \)-module \( \tau_{B_7} S_{30} \), and with parameter \( t = 2 \). The modified algebra \( B_8 \) is given by the quiver with the vertices 20, 21, \ldots, 33 bound by \( \varphi_3 \psi = 0, \varphi_4 \psi = 0, \xi_3 \omega_1 = 0, \xi_1 \varphi_1 = 0, \xi_1 \varphi_2 = 0, \xi_2 \xi_3 \xi_2 \xi_1 = \xi_1 \psi, \pi_1 \xi_2 = 0, \pi_1 \omega_1 = \pi_2 \omega_2 \). Finally, we apply (ad 5) to \( B_4 \times B_8 \) in two steps. The first step: we apply (fad 3) with pivot the indecomposable \( B_4 \)-module \( \tau_{B_4} S_{14} \), and with parameters \( t = 3, s = 2 \). The modified algebra \( B_9 \) is given by the quiver with the vertices 1, 2, \ldots, 17 bound by \( \alpha \beta = 0, \gamma \delta = 0, \eta \epsilon = 0, \kappa \lambda \varphi = 0 \). The second step: we apply (ad 4) with pivot the simple \( B_8 \)-module \( S_{26} \), and with the finite sectional path \( I_{16} \rightarrow \tau_{B_9} S_{15} \rightarrow I_{14} \rightarrow S_{17} \) consisting of the indecomposable \( B_9 \)-modules, and with parameters \( t = 4, r = 1 \). The modified algebra is equal to \( A \).

Then the left quasi\-tilted algebra \( A(l) \) of \( A \) is the convex subcategory of \( A \) being the product \( A(l) = A^{(l)}_1 \times A^{(l)}_2 \), where \( A^{(l)}_1 = kQ^{(l)}_1 / I^{(l)}_1 \) is the branch extension of the tame concealed algebra \( C_1 \), \( Q^{(l)} \) is a full subquiver of \( Q \) given by the vertices 1, 2, \ldots, 16 and \( I^{(l)}_1 = kQ^{(l)}_1 \cap I \) is the ideal in \( kQ^{(l)}_1 , A^{(l)}_2 = kQ^{(l)}_2 / I^{(l)}_2 \) is the branch coextension of the tame concealed algebra \( C_2 , Q^{(l)}_2 \) is a full subquiver of \( Q \) given by the vertices 20, 21, \ldots, 31 and \( I^{(l)}_2 = kQ^{(l)}_2 \cap I \) is the ideal in \( kQ^{(l)}_2 \). The right quasi\-tilted algebra \( A^{(r)} \) of \( A \) is the convex subcategory of \( A \) being the product \( A^{(r)} = A^{(r)}_1 \times A^{(r)}_2 \), where \( A^{(r)}_1 = C_1 , A^{(r)}_2 = kQ^{(r)}_2 / I^{(r)}_2 \) is the branch extension of the tame concealed algebra \( C_2 , Q^{(r)}_2 \) is a full subquiver of \( Q \) given by the vertices 14, 15, \ldots, 24, 26, 27, 28, 30, 31, 32, 33 and \( I^{(r)}_2 = kQ^{(r)}_2 \cap I \) is the ideal in \( kQ^{(r)}_2 \). Note that \( A^{(l)}_1 , A^{(r)}_1 , A^{(l)}_2 , A^{(r)}_2 \) are tame.

It follows from Theorems 3.3, 3.5(iii) and the above construction that \( A \) is tame and \( \Gamma_A \) admits a separating family of almost cyclic coherent components. Moreover, by Theorem 5.1, the first Hochschild cohomology space \( H^1(\bar{A}) = 0 \) (\( d_A = 0, f_A = 0 \)). Then, a direct application of Theorem 1.1 shows that the algebra \( A \) is simply connected. In fact, it follows from [31, Theorem 1.2] that \( A \) is strongly simply connected. We also note that, since \( A^{(l)}_1 , A^{(l)}_2 , A^{(r)}_1 \) and \( A^{(r)}_2 \) are tame tilted algebras of Euclidean type \( \tilde{D} \) such that \( H^1(A^{(l)}_1) = 0, H^1(A^{(l)}_2) = 0, H^1(A^{(r)}_1) = 0 \) and \( H^1(A^{(r)}_2) = 0 \) it follows from [5, Theorem] that \( A^{(l)}_1 , A^{(l)}_2 , A^{(r)}_1 \) and \( A^{(r)}_2 \) are simply connected (and even strongly simply connected from [5, Corollary]). Finally, we mention that \( C_1 , C_2 \) are simply connected, \( A \) is a generalized multicoil.
algebra, $\Gamma_A$ does not contain exceptional configurations of modules, and so this example illustrates also Theorem 1.2.

**Example 7.4** Let $A = kQ/I$ be the bound quiver algebra given by the quiver $Q$ of the form

$$
\begin{array}{cccccccc}
16 & \overset{\lambda}{\leftarrow} & 1 & \overset{\alpha}{\leftarrow} & 2 & \overset{\beta}{\leftarrow} & 3 & \overset{e}{\leftarrow} & 11 & \overset{\kappa}{\leftarrow} & 12 \\
13 & \overset{\phi}{\leftarrow} & 14 & \overset{\psi}{\leftarrow} & 15 \\
5 & \overset{\delta}{\leftarrow} & 4 & \overset{\gamma}{\leftarrow} & 10 & \overset{\eta}{\leftarrow} & 9 & \overset{\pi}{\leftarrow} & 17
\end{array}
$$

and $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $a\gamma\beta\alpha\lambda - \delta\phi\lambda$, $\gamma\varepsilon$, $b\pi\omega\nu\mu - \pi\eta\xi$, $\zeta\mu$, $\varphi\psi\kappa$, where $a, b \in k \setminus \{0\}$. Then $A$ is a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times C_2$, where $C_1$ is the hereditary algebra of Euclidean type $\tilde{A}_4$ given by the vertices $1, 2, \ldots, 5$, and $C_2$ is the hereditary algebra of Euclidean type $\tilde{A}_4$ given by the vertices $6, 7, \ldots, 10$. Indeed, we apply $(ad 1^*)$ to $C_1$ with pivot the simple regular $C_1$-module $S_3$, and with parameter $t = 2$. The modified algebra $B_1$ is given by the quiver with the vertices $1, 2, \ldots, 5, 11, 12, 13$ bound by $\gamma\varepsilon = 0$. Next, we apply $(ad 4)$ to $B_1 \times C_2$ with pivot the simple regular $C_2$-module $S_7$ and with the finite sectional path $I_{12} \rightarrow S_{13}$ consisting of the indecomposable $B_1$-modules, and with parameters $t = 2, r = 1$. The modified algebra $B_2$ is given by the quiver with the vertices $1, 2, \ldots, 15$ bound by $\gamma\varepsilon = 0$, $\zeta\mu = 0$, $\varphi\psi\kappa = 0$. Now, we apply $(ad 1^*)$ with parameter $t = 0$ to the algebra $B_2$ with pivot the regular $C_1$-module corresponding to the indecomposable representation of the form

$$
\begin{array}{c}
k \leftarrow a \kappa \leftarrow 1 \\
1 \overset{1}{\leftarrow} \\
k \leftarrow 1 \\
k \leftarrow 1
\end{array}
$$

lying in a stable tube of rank 1 in $\Gamma_{C_1}$ (see [42, XIII.2.4(c)]). The modified algebra $B_3$ is given by the quiver with the vertices $1, 2, \ldots, 16$ bound by $\gamma\varepsilon = 0$, $\zeta\mu = 0$, $\varphi\psi\kappa = 0$, $a\gamma\beta\alpha\lambda = \delta\phi\lambda$, where $a \in k \setminus \{0\}$. Finally, we apply $(ad 1)$ with parameter $t = 0$ to the algebra $B_3$ with pivot the regular $C_2$-module corresponding to the indecomposable representation of the form

$$
\begin{array}{c}
k \leftarrow b \kappa \leftarrow 1 \\
1 \overset{1}{\leftarrow} \\
k \leftarrow 1 \\
k \leftarrow 1
\end{array}
$$

lying in a stable tube of rank 1 in $\Gamma_{C_2}$ (see [42, XIII.2.4(c)]). The modified algebra is then equal to $A$.

Then the left quasitilted algebra $A^{(l)}$ of $A$ is the convex subcategory of $A$ being the product $A^{(l)} = A_1^{(l)} \times A_2^{(l)}$, where $A_1^{(l)} = kQ_1^{(l)}/I_1^{(l)}$ is the branch coextension of $C_1$, $Q_1^{(l)}$ is a full subquiver of $Q$ given by the vertices $1, 2, \ldots, 5, 11, 12, 13, 16$ and $I_1^{(l)} = kQ_1^{(l)} \cap I$ is the ideal in $kQ_1^{(l)}$, $A_2^{(l)} = C_2$. The right quasitilted algebra $A^{(r)}$ of $A$ is the convex subcategory of $A$ being the product $A^{(r)} = A_1^{(r)} \times A_2^{(r)}$, where $A_1^{(r)} = C_1$, $A_2^{(r)} = kQ_2^{(r)}/I_2^{(r)}$ is the branch extension of $C_2$, $Q_2^{(r)}$ is a full subquiver of $Q$ given by the vertices
6, 7, . . . , 10, 12, 13, 14, 15, 17 and $I_2^{(r)} = kQ_2^{(r)} \cap I$ is the ideal in $kQ_2^{(r)}$. Note that $A_1^{(l)}$, $A_2^{(l)}$, $A_1^{(r)}$ and $A_2^{(r)}$ are tame.

It follows from Theorems 3.3, 3.5(iii) and the above construction that $A$ is tame and $\Gamma_A$ admits a separating family of almost cyclic coherent components. Moreover, we have $h_1 = 1$, $e_1 = 1$, $h_2 = 1$, $e_2 = 1$, $f_{C_1} = 0$, $f_{C_2} = 0$, $f_A = f_{C_1} + f_{C_2} = 0$, and $d_A = 0$. Therefore, by Theorem 5.1, the first Hochschild cohomology space $H^1(A) = 0$.

Then, a direct application of Theorem 1.1 shows that the algebra $A$ is simply connected. We note that, by [19, Proposition 1.6], $H^1(A_1^{(l)}) \cong k$, $H^1(A_1^{(r)}) \cong k$. Since $A_1^{(l)}$ and $A_2^{(r)}$ are generalized multicoil algebras, we get by Theorem 5.1 that $H^1(A_1^{(l)}) = 0$, $H^1(A_2^{(r)}) = 0$.

We also mention that $A_1^{(r)} = C_1$, $A_2^{(l)} = C_2$ are not simply connected, $A_1^{(l)}$, $A_2^{(r)}$ are simply connected, by [3, Theorem A], and so $A$ is not strongly simply connected. Moreover, by the above construction we know that $A$ is a generalized multicoil algebra, such that $\Gamma_A$ does not contain exceptional configurations of modules. Therefore, this example shows that simple connectedness assumption imposed on the considered concealed canonical algebras is essential for the validity of Theorem 1.2.

We end this section with an example of a wild generalized multicoil algebra, illustrating Theorem 1.2.

**Example 7.5** Let $A = kQ/I$ be the bound quiver algebra given by the quiver $Q$ of the form

![Quiver Diagram](attachment:image.png)

and $I$ the ideal in the path algebra $kQ$ of $Q$ over $k$ generated by the elements $\alpha_1\alpha_2\alpha_3\alpha_4 + \beta_1\beta_2\beta_3\beta_4 + \gamma_1\gamma_2$, $\alpha_1\beta_1$, $\alpha_2\alpha_1$, $\xi_1\alpha_4$, $\epsilon_1\alpha_2$, $\epsilon_1\delta_1 - \epsilon_2\delta_2$, $\alpha_3\beta_1$, $\xi_1\rho_1 - \xi_2\rho_2$, $\nu_1\gamma_2$, $\gamma_1\theta_2$, $\nu_1\theta_2 - \nu_2\theta_1$, $\varphi_1\psi_1$, $\varphi_2\psi_1$, $\eta_1\varphi_2$, $\eta_1\varphi_3$, $\psi_2\kappa_1$, $\eta_2\kappa_2 - \eta_1\psi_1\kappa_1$, $\omega_2\kappa_2$, $\omega_1\sigma_3\sigma_2$. Then $A$ is a generalized multicoil algebra. Indeed, $A$ is a generalized multicoil enlargement of a canonical algebra $C = C_1 \times C_2$, where $C_1$ is the tubular canonical algebra of type $(2, 4, 4)$ given by the vertices $0, 1, \ldots, 8$ bound by $\alpha_1\alpha_2\alpha_3\alpha_4 + \beta_1\beta_2\beta_3\beta_4 + \gamma_1\gamma_2 = 0$, and $C_2$ is the canonical algebra of Euclidean type $\mathbb{D}_4$ given by the vertices $23, 24, \ldots, 27$. It is known that $\Gamma_{C_1}$ admits an infinite family $T^{C_1}_\lambda$, $\lambda \in \mathbb{P}(k)$, of pairwise orthogonal stable tubes, having a stable tube, say $T^{C_1}_1$, of rank 4 with the mouth formed by the modules $S_5 = \tau_{C_1}S_6$, $S_6 = \tau_{C_1}S_7$, $S_7 = \tau_{C_1}E$, $E = \tau_{C_1}S_5$, where $E$ is the unique indecomposable $C_1$-module with the dimension vector $\dim E = 11111111110000$, and a unique stable tube, say $T^{C_1}_2$, of rank 2 with
the mouth formed by the modules $S_1 = \tau C_1 F$, $F = \tau C_1 S_1$, where $F$ is the unique indecomposable $C_1$-module with the dimension vector $\dim F = 11111$ (see [41, (3.7)]). Moreover, $\Gamma C_2$ admits an infinite family $T^2_\mu$, $\mu \in \mathbb{P}(k)$, of pairwise orthogonal stable tubes, having a stable tube, say $T^2_1$, of rank 2 with the mouth formed by the modules $S_{25} = \tau C_2 G$, $G = \tau C_2 S_25$, where $G$ is the unique indecomposable $C_2$-module with the dimension vector $\dim G = 11111$. We have the following sequence of the modified algebras. First, we apply (ad 1*) to $C_1$ with pivot the simple regular $C_1$-module $S_7$, and with parameter $t = 2$. The modified algebra $B_1$ is given by the quiver with the vertices $0, 1, \ldots, 11$ bound by $\alpha_1 \delta_1 = 0$. Next, we apply (ad 1*) to $B_1$ with pivot the simple $B_1$-module $S_6$, and with parameter $t = 3$. The modified algebra $B_2$ is given by the quiver with the vertices $0, 1, \ldots, 15$ bound by $\alpha_1 \delta_1 = 0, \alpha_2 \sigma_1 = 0$. Now, we apply (ad 1) to $B_2$ with pivot the simple $B_2$-module $S_5$, and with parameter $t = 1$. The modified algebra $B_3$ is given by the quiver with the vertices $0, 1, \ldots, 17$ bound by $\alpha_1 \delta_1 = 0, \alpha_2 \sigma_1 = 0, \xi_1 \alpha_4 = 0$. Next, we apply (ad 3) to $B_3$ with pivot the indecomposable $B_3$-module $\tau B_1 I_{10}$, and with parameter $t = 2$. The modified algebra $B_4$ is given by the quiver with the vertices $0, 1, \ldots, 18$ bound by $\alpha_1 \delta_1 = 0, \alpha_2 \sigma_1 = 0, \xi_1 \alpha_4 = 0, \xi_2 \alpha_5 = 0, \xi_1 \gamma_1 = 0, \delta_1 \gamma_1 = 0$. Further, we apply (ad 2*) to $B_4$ with pivot the indecomposable projective $B_4$-module $P_{16}$, and with parameter $t = 1$. The modified algebra $B_5$ is given by the quiver with the vertices $0, 1, \ldots, 19$ bound by $\alpha_1 \delta_1 = 0, \alpha_2 \sigma_1 = 0, \xi_1 \alpha_4 = 0, \xi_1 \gamma_1 = 0, \xi_2 \gamma_2 = 0, \xi_2 \rho_2 = 0$. Now, we apply (ad 1) to $B_5$ with pivot the simple regular $B_5$-module $S_1$, and with parameter $t = 1$. The modified algebra $B_6$ is given by the quiver with the vertices $0, 1, \ldots, 21$ bound by $\alpha_1 \delta_1 = 0, \alpha_2 \sigma_1 = 0, \xi_1 \alpha_4 = 0, \xi_2 \gamma_2 = 0, \xi_2 \rho_2 = 0, \gamma_1 \theta_2 = 0, \gamma_1 \theta_2 = 0$. Now, we apply (ad 1*) to $C_2$ with pivot the simple regular $C_2$-module $S_{25}$, and with parameter $t = 2$. The modified algebra $B_7$ is given by the quiver with the vertices $23, 24, \ldots, 30$ bound by $\varphi_1 \psi_1 = 0, \varphi_4 \psi_1 = 0$. Next, we apply (ad 1) to $B_8$ with pivot the indecomposable $B_8$-module $\tau B_8 S_{29}$, and with parameter $t = 1$. The modified algebra $B_9$ is given by the quiver with the vertices $23, 24, \ldots, 32$ bound by $\varphi_1 \psi_1 = 0, \varphi_3 \psi_1 = 0, \eta_1 \varphi_2 = 0, \eta_1 \varphi_3 = 0$. Now, we apply (ad 2*) to $B_9$ with pivot the indecomposable projective $B_9$-module $P_{31}$, and with parameter $t = 1$. The modified algebra $B_{10}$ is given by the quiver with the vertices $23, 24, \ldots, 33$ bound by $\varphi_1 \psi_1 = 0, \varphi_4 \psi_1 = 0, \eta_1 \varphi_2 = 0, \eta_1 \varphi_3 = 0, \varphi_2 \kappa_1 = 0, \eta_2 \kappa_2 = \eta_1 \psi_1 \kappa_1$. Next, we apply (ad 4) to $B_7 \times B_{10}$ with pivot the simple $B_{10}$-module $S_{32}$, and with the finite sectional path $I_{13} \rightarrow I_{14} \rightarrow I_{15}$ consisting of the indecomposable $B_7$-modules, and with parameters $t = 3, r = 4$. The modified algebra is then equal to $A$.

Then the left quasitilted algebra $A^{(l)}$ of $A$ is the convex subcategory of $A$ being the product $A^{(l)} = A_1^{(l)} \times A_2^{(l)}$, where $A_1^{(l)} = k Q^{(l)}_1 / I^{(l)}_1$ is the branch coextension of $C_1$, $Q^{(l)}_1$ is a full subquiver of $Q$ given by the vertices $0, 1, \ldots, 15, 17, 19, 21, 22$ and $I^{(l)}_1 = k Q^{(l)}_1 \cap I$ is the ideal in $k Q^{(l)}_1$, $A_2^{(l)} = k Q^{(l)}_2 / I^{(l)}_2$ is the branch coextension of $C_2$, $Q^{(l)}_2$ is a full subquiver of $Q$ given by the vertices $23, 24, \ldots, 30, 33$ and $I^{(l)}_2 = k Q^{(l)}_2 \cap I$ is the ideal in $k Q^{(l)}_2$. The right quasitilted algebra $A^{(r)}$ of $A$ is the convex subcategory of $A$ being the product $A^{(r)} = A_1^{(r)} \times A_2^{(r)}$, where $A_1^{(r)} = k Q^{(r)}_1 / I^{(r)}_1$ is the branch extension of $C_1$, $Q^{(r)}_1$ is a full subquiver of $Q$ given by the vertices $0, 1, \ldots, 8, 10, 11, 16, 17, 18, 20, 21$ and $I^{(r)}_1 = k Q^{(r)}_1 \cap I$ is the ideal in $k Q^{(r)}_1$, $A_2^{(r)} = k Q^{(r)}_2 / I^{(r)}_2$ is the branch extension of $C_2$, $Q^{(r)}_2$ is a
full subquiver of $Q$ given by the vertices $13, 14, 15, 23, 24, \ldots, 27, 31, 32, 34, 35, \ldots, 38$ and $I_2^{(r)} = kQ_2^{(r)} \cap I$ is the ideal in $kQ_2^{(r)}$. Then, $A_l^{(l)}$ and $A_r^{(r)}$ are the quasitilted algebras of wild types $(4, 4, 13), (4, 4, 9)$, respectively. Moreover, $A_2^{(l)}$ and $A_2^{(r)}$ are tame.

It follows from [7, Corollary 1.4] that $C_1$ is simply connected. Moreover, $C_2$ is also simply connected. By the above construction we know that $A$ is a generalized multicoil algebra obtained from $C_1, C_2$ and $\Gamma_A$ does not contain exceptional configurations of modules. Further, by Theorem 5.1, the first Hochschild cohomology space $H^1(A) = 0 (d_A = 0, f_A = 0)$ and $H^1(A_l^{(l)}) = 0, H^1(A_2^{(l)}) = 0, H^1(A_r^{(r)}) = 0, H^1(A_2^{(r)}) = 0$. Then, a direct application of Theorem 1.2 shows that the algebras $A_l^{(l)}, A_r^{(l)}, A_l^{(r)}, A_2^{(r)}$ and $A$ are simply connected.

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