Hydraulic Fracture

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Abstract

We consider a variation of Griffith’s analysis of rupture, one which simulates the process of hydrofracturing, where fluid forced into a crack raises the fluid pressure until the crack begins to grow. Unlike that of Griffith, in this analysis fluid pressure drops as a hydrofracture grows. We find that growth of the fracture depends on the ratio of the compliances of the fluid and the fracture, a non-dimensional parameter called $\alpha_0$ here. The pressure needed to initiate a hydrofracture is found to be the same as that derived by Griffith. Once a fracture initiates, for relatively low values of the model parameter $\alpha_0$ ($\alpha_0 \leq 0.2$) the fracture advances spontaneously to a new radius which depends on the value of $\alpha_0$. For $\alpha_0 \leq 0.2$ further fluid injection is required to increase the fracture radius after spontaneous growth stops. For the case where $\alpha_0 > 0.2$ each increment of fracture growth requires injection of more fluid. For the extreme case where $\alpha_0 = 0$ our results are the same as Griffith’s, i.e., a fracture once initiated grows without limit.

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\textsuperscript{3}Joseph B. Walsh died on 30 August 2017 at the age of 86 in Adamsville, R.I. Please read more about Joe Walsh’s life here: Scholz, C. H., Goldsby, D. L., Bernabé, Y., and Evans, B., (2018), Joseph B. Walsh (1930–2017), Eos, 99, https://doi.org/10.1029/2018EO093999. Published on 6 March 2018.
1 Introduction

Griffith [Griffith 1921, 1924], in his fundamental study of fracture, chose to analyze the stress intensity and strain energy associated with an empty crack loaded by normal stress which originate at a distance far away. Here we consider a variation of Griffith’s analysis, one which simulates hydrofracturing, where fluid forced into a crack raises the fluid pressure (and lowers the effective normal stress on the crack walls) until the conditions are energetically favorable for growth of the fracture. Mathematical analysis of the two problems is different because applied stress is held constant or nearly so in Griffith’s analysis whereas in the analysis here fluid pressure drops as a hydrofracture grows. The work done by the constant applied load in Griffith’s analysis causes the fracture, once initiated, to grow without limit. We find that the criterion for fracture initiation in our analysis is the same as that in Griffith’s. On the other hand, we find that although Griffith’s criterion is necessary for initiation, it is not a sufficient condition for fracture growth in all cases; i.e., though growth is possible at Griffith’s value of the internal fluid pressure, it might not necessarily proceed without additional stimulus.

We model a hydrofracture by the circular crack with radius $c$ seen in Figure 1. In the model, before the hydrofracturing procedure begins, the pressure and temperature of the hydrofracturing fluid are assumed to be in equilibrium with the surrounding country rock. The pressures $p$ in the analysis are pressures relative to the initial equilibrium values and so pressure $p$ is zero when fluid injection begins.

The country rock surrounding the fracture is assumed to be an isotropic, elastic, and impermeable infinite medium. The volume $V_f$ of fluid includes all fluid in contact with the fracture; that is, the fluid in the fracture and in the drill hole between the packers. We don’t include time as a parameter and changes in pressure are felt immediately throughout the fluid volume $V_f$. Time, of course, is an important parameter and we discuss the effect of it on our results in a later section.

2 Analysis

2.1 Preliminaries

As discussed above, for the purposes of analysis we consider the initial pressure of the system to be zero. To evaluate the effect of injecting fluid into the cavity we follow Eshelby’s [Eshelby 1957] technique for calculating the effect of a transformational strain upon an inhomogeneity embedded in an infinite matrix. First, we imagine removing the fluid unit from the fracture cavity in the country rock and adding a differential volume $dV_f$ to it. Then, a differential

![Figure 1: Fracture in a borehole. In the analysis we consider the cavity to include both the fracture, assumed to be a penny shaped crack having radius c, and the drill hole between the packers. Total volume is $V_f$. Fluid pressure $p$ is assumed to be uniform throughout the cavity and changes in $p$ occur instantaneously.](image)
pressure $d p_i$ is applied to the unit to return the volume to the original cavity size. The pressure $d p_i$ needed is:

$$
d p_i = \frac{d V_i}{\beta_f V_f},$$

(1)

where $\beta_f$ is the compressibility of the fluid and the volume $V_f$ of fluid is defined above. Once the fluid is compressed under pressure $d p_i$, it is returned to the cavity, the cavity walls remaining stress-free. To equalize stress at the interface, pressure $d p$ is applied to the both the fluid and to the cavity wall while contact is maintained between fluid and the surrounding medium. The decrease $d V_f$ in the volume of the fluid is:

$$
d V_f = \beta_f V_f d p,$$

(2)

where decrease in volume $d V_f$ is positive.

The increase $d V_c$ in the volume $V_c$ of the cavity can be found from (10), (11), and (12) for the conditions here where fracture radius $c_0$ is constant: i.e.,

$$
d V_c = \frac{16}{9} \frac{(1 - \nu^2)}{(1 - 2\nu)^2} \beta_s c_0 d p,$$

(3)

where $\beta_s$ is the compressibility, $\nu$ is the Poisson’s ratio of the elastic matrix. To maintain contact, $d V_f$ must equal $-d V_c$; and for equilibrium at the interface, pressure $d p_i - d p$ from (1) and (2) acting on the fluid must equal pressure $d p$ from (3) acting on the cavity wall. We find from (1), (2), and (3) with these constraints that the increase $d p$ in pressure resulting from an injection of the differential fluid volume $d V_i$ is given by:

$$
d p = \left( \frac{d V_i}{\beta_f V_f} \right) \frac{1}{1 + \alpha_0},
$$

(4)

where

$$
\alpha_0 = \frac{16(1 - \nu^2)}{9(1 - 2\nu)^2} \left( \frac{\beta_s}{\beta_f} \right) c_0^3 V_f,$$

(5)

### 2.2 Griffith Analysis

Consider the penny-shaped crack in an infinite, impermeable, elastic medium in Figure [1]. As discussed in the Introduction, compressible fluid forced into the crack under increasing pressure $p$ via the borehole eventually causes the crack having radius $c_0$ to grow. Griffith reasoned that initiation would occur when conditions were energetically favorable. The components of energy of interest in this problem are the work $d w_p$ done by the loading mechanism, here fluid pressure; the change $d w_c$ in the elastic energy stored in the stressed body; and the energy $d w_s$ needed to create new surface exposed as the crack advances. Griffith’s criterion for the initiation of growth is simply:

$$
d w_s + d w_p + d w_c = 0.
$$

(6)

The relationship between surface energy and surface area is generally taken to be a simple proportionality, i.e.,

$$
d w_s = -2\pi\gamma c dc,
$$

(7)

where $\gamma$ is the surface energy per unit area. Work $d w_p$ done by the fluid pressure $p_f$ acting on the increase $d V_c$ in the volume of the cavity as the crack advances is

$$
d w_p = pd V_c.
$$

(8)

Introducing the expression for $d V_c$ from (3) into (8), we find:

$$
pd V_c = \frac{16}{9} \frac{(1 - \nu^2)}{(1 - 2\nu)^2} \beta_s c_0^3 pd p.
$$

(9)
The change $dw_c$ in elastic strain energy is the result of increase $dc$ in crack radius and decrease $dp$ in fluid pressure; it is conveniently written in differential form as

$$dw_c = \frac{\partial w_c}{\partial c} dc + \frac{\partial w_c}{\partial p} dp. \quad (10)$$

The expression for $w_c$ is [Sack 1946]:

$$w_c = \frac{8(1 - \nu^2)}{9(1 - 2\nu)} \beta_s c^3 p^2. \quad (11)$$

The decrease (partial derivative $(\partial w_c/\partial p) dp$) in elastic strain energy in the body arising from the decrease $-dp$ in fluid pressure is found from (11) to be:

$$\left( \frac{\partial w_c}{\partial p} \right) dp = -\frac{16(1 - \nu^2)}{9(1 - 2\nu)} \beta_s c_0^3 p_0 dp. \quad (12)$$

Combining (6)-(10), and (12), we find that Griffith’s criterion becomes

$$\frac{\partial w_c}{\partial c} \geq 2\pi \gamma c_0. \quad (13)$$

Equation (13) is Griffith’s fracture initiation criterion in compact form. The partial differential term expresses the increase in elastic strain energy arising from a differential increase $dc$ in crack radius carried out with pressure held constant. As established by Griffith, this must be at least equal to the energy required to create new surface. Introducing (11) into (13) and rearranging gives Griffith’s familiar expression for the pressure $p_G$ at which fracture growth is possible:

$$p_G = \left[ \frac{3\pi (1 - 2\nu)}{4(1 - \nu^2)} \left( \frac{\gamma}{\beta_s c_0} \right) \right]^{1/2}. \quad (14)$$

### 2.3 Fracture Growth

[Griffith 1921, 1924] in his analysis considered the balance between elastic and dissipative energies for a crack loaded by stress applied at a remote distance. A consequence of this mode of loading is that once cracking is initiated, the crack grows without limit because the initiating stress remains virtually constant. Hydrofracturing differs not only because the stressed region is limited to a volume of the order of the crack dimensions, but also because fluid pressure decreases as the hydrofracture grows. We now ask the question: Should we expect a hydraulic fracture to grow without limit after the fracture is initiated as Griffith cracks do?

We assume that the speed at which the fracture grows, the viscosity of the hydrofracturing fluid, and any other parameters that involve time are such that in our analysis fluid always completely fills the cavity and that the pressure is uniform throughout the fluid. Note that the mathematical model is the one we have been using in the previous two sections to analyze pressurizing the system and derive the criterion for fracture; we now consider behavior as the crack radius grows from $c_0$ to some arbitrary value $c$.

Let us assume for now that growth occurs without further injection of fluid after initiation at pressure $p_G$. For fracture growth, Griffith’s criterion must be met at each increment of growth. The rate of change in strain energy $w_c$ when the fracture has grown to arbitrary radius $c$ is found from (10) and (11) to be:

$$\left( \frac{\partial w_c}{\partial c} \right) = \frac{8(1 - \nu^2)}{3(1 - 2\nu)} \beta_s c^2 p^2, \quad (15)$$

and the rate of change at initiation, when $c = c_0$, is:

$$\left( \frac{\partial w_c}{\partial c} \right)_0 = \frac{8(1 - \nu^2)}{3(1 - 2\nu)} \beta_s c_0^3 p_0^2. \quad (16)$$
Non-dimensionalizing the criterion by taking the ratio of (15) and (16), we find:

\[
\left( \frac{\partial w_c}{\partial c} \right) / \left( \frac{\partial w_c}{\partial c} \right)_0 = \left( \frac{c}{c_0} \right)^2 \left( \frac{p}{p_G} \right)^2.
\]  

(17)

To find the pressure \( p/p_G \) when the fracture has grown to \( c/c_0 \), we first integrate (4) at constant fracture radius \( c_0 \) to find the injected volume \( V_G \) when the fracture is initiated (i.e., when \( p = p_G \)):

\[
p_G = \frac{V_G}{\beta_f V_f} \left( \frac{1}{1 + \alpha_0} \right).
\]

(18)

The pressure \( p \) when the fracture radius has increased to \( c \), with \( V_G \) remaining constant, is:

\[
p = \frac{V_G}{\beta_f V_f} \left( \frac{1}{1 + \alpha_0 (c/c_0)^3} \right).
\]

(19)

Following the procedure leading to (17), we find

\[
\frac{p}{p_G} = \frac{1 + \alpha_0}{1 + \alpha_0 (c/c_0)^3},
\]

(20)

and (17) becomes:

\[
\left( \frac{\partial w_c}{\partial c} \right) / \left( \frac{\partial w_c}{\partial c} \right)_0 = \left[ \left( \frac{c}{c_0} \right) \frac{1 + \alpha_0}{1 + \alpha_0 (c/c_0)^3} \right]^2.
\]

(21)

Repeating these steps for the surface energy gives:

\[
\left( \frac{\partial w_s}{\partial c} \right) / \left( \frac{\partial w_s}{\partial c} \right)_0 = \left( \frac{c}{c_0} \right).
\]

(22)

The two energy functions in (21) and (22) are plotted in Figure 2. Note in the figure that crack growth is spontaneous for systems having relatively low values of \( \alpha_0 \), but the fracture stops after running a limited distance. Solving (21) and (22) shows that spontaneous growth occurs for \( \alpha_0 \) in the range:

\[
0 \leq \alpha_0 \leq 0.2.
\]

(23)

In the limit as \( \alpha_0 \to 0 \) (i.e., the fracturing fluid is very compressible as for a gas), we revert to Griffith’s model where the load remains approximately constant and fracture growth is unlimited. Note in (20) that loading pressure \( p \) for this extreme case remains constant at Griffith’s value \( p_G \).

Crack growth for systems where \( \alpha_0 > 0.2 \) and for fracture growth beyond the region of spontaneous growth in systems where \( \alpha_0 < 0.2 \) is made possible by injecting fluid into the cavity to raise the pressure. The rate at which fluid must be injected to maintain fracture growth is found by first considering the changes in fluid pressure during growth shown in Figure 3. The fluid pressure \( p_f \) required for continued growth is found by equating (17) and (18) giving:

\[
p_f/p_G = (c/c_0)^{-1/2}.
\]

(24)

Pressure \( p_f/p_G \) in (24) is represented graphically by the heavy black line in Figure 3. The thin lines in the figure give the pressure from (20) in the cavity if the fracture were to grow with no fluid injection.

Consider now the growth of the fracture after injection has brought fluid pressure \( p \) to the Griffith value \( p_G \) in (14). A differential increase \( dc \) in fracture radius with no injection decreases fluid pressure by \( dp/p_G \) in Figure 3, that is, from (20),

\[
\frac{dp}{p_G} = \frac{d}{dc} \left( \frac{1 + \alpha_0}{1 + \alpha_0 (c/c_0)^3} \right) dc.
\]

(25)
Figure 2: The thin lines in the figure are plots of the elastic strain energy release rate $dw_c/dc$ and the heavy black line is the rate $dw_s/dc$ of fracture energy dissipation; both are plotted in the dimensionless form in (21) and (22) as functions of the parameter $\alpha_0$ and crack radius $c/c_0$. For fracture growth, the elastic energy release rate must be greater than the rate at which fracture energy is dissipated, that is the thin lines must be above the heavy black one. We see in the figure that growth occurs spontaneously only over a restricted range of values of $\alpha_0$; that is, as shown in the text, for $\alpha_0 < 0.2$. In cases where $\alpha_0 > 0.2$ and cases where $\alpha_0 < 0.2$ but beyond the range of spontaneous growth, fracture size can be increased only by injecting fluid. Graphically, for any combination of crack length $c/c_0$ and parameter $\alpha_0$ where a line falls below the heavy black line, the crack is stable and additional internal fluid pressure must be increased to make the crack grow.
Figure 3: The thin lines are plots of non-dimensional fluid pressure $p/p_G$ after the fracture has grown to some radius $c/c_0$ without additional fluid being injected given by (20). The heavy black line is the non-dimensional fluid pressure $p_f/p_G$ needed for continued growth as given by (24). For any combination of crack length $c/c_0$ and parameter $\alpha_0$ where a line falls below the heavy black line, the crack is stable and additional internal fluid pressure must be increased to make the crack grow. Lines for $\alpha_0$ in the range $(0 \leq \alpha_0 \leq 0.2)$ lie partially above the heavy black line, indicating that growth in this region is spontaneous, as in Figure 2. In Griffith’s analysis, load is held constant ($p/p_G = 1$) and spontaneous growth is unabated. This condition is represented by the horizontal line (abscissa $p/p_G = 1$) where the line $\alpha_0 = 0$ would plot.

Increasing fracture radius causes a differential change $dp_f/p_G$ in the non-dimensional fluid pressure required for continued growth. We find from (24) that the change $dp_f/p_G$ is:

$$\frac{dp_f}{p_G} = \frac{d}{dc} (c/c_0)^{-1/2} dc. \quad (26)$$

For continued growth of the fracture, fluid pressure $p_f$ in (24) must be maintained, i.e., the rate of pressure change $(dp_f - dp)/p_G$ must be compensated for by injecting fluid into the crack. We see from (4), that the rate $dV_i$ at which fluid must be injected for a fracture having radius $c/c_0$ is given by the expression:

$$dV_i = \beta_f V_f (p_f - p) \left[ 1 + \alpha_0 (c/c_0)^3 \right], \quad (27)$$
or, in non-dimensional form:

$$\frac{dV_i}{V_G} = \frac{d}{dp} \left[ \frac{p_f - p}{p_G} \right] \left[ 1 + \alpha_0 (c/c_0)^3 \right], \quad (28)$$

where $V_G$ is the volume of fluid necessary to force growth of the initial crack ($c = c_0$) given by:

$$V_G = \beta_f V_f p_G. \quad (29)$$

Combining (24)–(28), one finds that the rate $(dV_i/dp)_f$ at which fluid must be injected to maintain growth per unit change in fluid pressure is given by the expression:

$$\left[ \frac{d (V_i/V_G)}{d (p/p_G)} \right]_f = \frac{1 - 5\alpha_0 (c/c_0)^3}{1 + \alpha_0}. \quad (30)$$
Discussion

In our analysis, we find that the pressure required to initiate hydraulic fracturing is the same as that derived by Griffith [1921, 1924]. The subsequent growth of the crack depends on the parameter $\alpha_0$ (see (4) and (5)), where $\alpha_0$ is the ratio of the compliance of the cavity in the country rock surrounding the fracture relative to the compliance of the fracturing fluid itself. The two phases, the cavity and the fluid, include the entire pressurized volume, i.e., both the fracture and the borehole between the packers. We assume, without loss of generality, that the system is in equilibrium and fluid pressure is zero when fluid injection is initiated.

The parameter $\alpha_0$ arises in the first step in the analysis, namely our derivation of the increase $dp$ in pressure in the cavity resulting from injecting a differential volume $dV_f$ at ‘zero’ (i.e. datum) pressure into the cavity. We follow Eshelby’s [Eshelby, 1957] technique for calculating the changes in stress and strain arising from an inhomogeneity undergoing a transformational strain; the procedure is outlined in the text leading to the final expression (4).

One can see in equation (5) definition of the ratio of the compliances of the fluid and cavity phases. The compliance of the fluid component is simply $\beta_f V_f$ where $\beta_f$ is the compressibility of the fluid and $V_f$ is the total volume of the fluid. The fluid is under the same hydrostatic pressure everywhere (for the equilibrium conditions we assume) and so the compliance $\beta_f V_f$ is exact.

On the other hand, the term defining the compliance of the cavity in (5) is exact only for a penny-shaped crack, whereas the deformation of the cavity under pressure includes that of the borehole and the packers. This simplification is acceptable, in our opinion. In the first place, deformation of the hydrofracture is likely to dominate the overall deformation of the cavity because it is the most compliant component – the relatively rigid borehole/packer system contributes to the volume $V_f$ but has relatively little influence on compliance. Also, we are primarily concerned with changes in crack radius, which are not coupled to the borehole/packer unit. And finally, the goal of this analysis is only to point out the important role of $\alpha_0$ in the mechanics of hydrofracturing, and a simple model like ours is not only adequate, but also desirable.

To calculate the fluid pressure $p_G$ required to initiate fracture, we follow Griffith’s [Griffith, 1921, 1924] procedure,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Figure 4: We show in Figure 3 the increase in fluid pressure $p_f$ needed to maintain growth once the fracture has been initiated at $p_G$. Here we plot the volumetric rate at which fluid must be injected per unit change in fracture pressure to maintain growth as given by (4) and (5). Note that the injection volume rate depends strongly on $\alpha_0$. The heavy black line denotes the reference level where no fluid is injected after fracture initiation.}
\end{figure}
with one additional step. Griffith, in his analysis, increased the stress acting on a crack until the work done under constant load as the fracture advanced just equaled the energy needed to create new fracture surface. In our model, increasing the fracture size increases the volume of the cavity causing the fluid pressure to drop. Analysis of the components of work performed during a differential increase in fracture radius (equations (6) through (14)) shows that pressure $p_f$ required to initiate fracture is Griffith’s value $p_G$ even though pressure decreases as the fracture starts to grow.

Although fracture pressure $p_G$ is a necessary condition for growth of the hydrofracture, we find in the section Fracture Growth that it is sufficient over only a range of values of $\alpha_0$. We analyze the advance of the fracture by stipulating that the Griffith criterion must be met at each stage of growth. Our results are summarized in Figures 2, 3, and 4. In Figure 2, we see Griffith’s criterion in graphical form. Griffith postulated that the strain energy release rate must be equal to or greater than the rate at which surface energy is dissipated. We see that this condition is met at $c/c_0 = 1.0$ for any value of $\alpha_0$. Note, however that, as the fracture grows, Griffith’s criterion is met only over a range of values of $\alpha_0$ ($0 \leq \alpha_0 \leq 0.2$). Over this range the fracture radius increases spontaneously and then stops. In cases where $\alpha_0 > 0.2$ or where $\alpha_0 < 0.2$ but beyond the range of spontaneous growth, fracture size can be increased only by increasing energy by increasing fluid pressure.

In Figure 3 we have plotted the increase in fluid pressure needed to increase fracture radius. The heavy black line defines the pressure $p_f$ needed for growth and the thin lines give the pressure $p$ in the fracture when fluid volume remains constant after initiation. We see in the figure that fluid pressure is always sufficient for growth for systems where $\alpha_0$ is in the range ($0 \leq \alpha_0 \leq 0.2$). This, of course is expected from Figure 2 where fractures are seen to grow spontaneously when $\alpha_0$ is in this range. Fluid must be injected for growth when $\alpha_0 > 0.2$ or to activate fractures which have stopped after growing spontaneously.

The rate $(dV_i/dp)_f$ at which fluid must be injected per unit change in pressure to maintain fracture growth is calculated (see [27]) and plotted in Figure 4. Note in the figure that the rate is uniform and positive while the initial crack ($c = c_0$) is pressurized. The rate changes immediately when fluid pressure reaches Griffith fracture pressure, $p_G$ and the fracture begins to grow; the change depending on the value of $\alpha_0$ for the system. We see in the figure for systems where $\alpha_0$ is in the range $0 \leq \alpha_0 \leq 0.2$ that the rate $(dV_i/dp)_f$ remains positive, indicating that fluid is returning to the well head $(dV_i < 0)$ while pressure is decreasing $(dp_f < 0)$. In those regions in the figure where negative values of $(dV_i/dp)_f$ are found, fluid pressure is decreasing while the fracture is growing, even though fluid is being injected into the fracture.

One parameter that we have ignored in the analysis is time. We have assumed that fluid completely fills the fracture during deformation and fluid pressure is the same throughout the cavity. This assumption is questionable for operations involving viscous fluids, fast growing fractures, fractures with very small apertures, etc. In an attempt to get an idea of how much time might affect our results, we analyzed an extreme case where the fluid and fracture properties are such that the fluid remains in the initial cavity while the hydrofracture advances.

The model is that in Figure 1 except that the load is provided by fluid pressure applied over only the area of the initial defect ($c/c_0 \leq 1.0$), and the fracture remains ‘dry’ as it advances. The analysis follows the same steps as those described above, except that elastic strain energy is derived following standard procedure from the stress concentration factor found by Sneddon [Tada, et al. 2000]. Details of the calculation are superfluous in a study like the one here; suffice to say, all steps in the procedure outlined in the text leading to Figures 2, 3, and 4 are the same, requiring only a redefinition of radius $c/c_0$. We find that the elastic strain energy release rate drops precipitously once Griffith’s fracture initiation pressure $p_G$ is reached, and fractures do not advance spontaneously for any values of $\alpha_0$. Further, the pressure increase, or the fluid volume injected, required to make the fracture grow is much more than needed for the ‘wet’ case shown in Figures 2, 3, and 4. These results suggest that it is energetically favorable for the fracture to grow at the rate prescribed by the analysis of the ‘wet’ case described in the text where equilibrium conditions prevail. To carry the calculation further requires analyzing the interrelationship between the flow of fluid and the rate at which the crack advances, an interesting problem but one that is far beyond the scope of a preliminary study such as the one here.
4 ACKNOWLEDGMENTS

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Note from coauthor Stephen Brown: I worked with Joe Walsh off and on throughout my entire career. It was a pleasure to have worked closely with him from 2013-2015 during the last few years of his life on this paper considering the physics of hydraulic fracturing. Unfortunately, it was never published. I am pleased to be able to now provide this work to the science and engineering community as an open access reference for educational use. This paper is likely Joe’s final research contribution. We are all grateful for him. Rest in peace, Joe.

Please read more about Joe Walsh’s life here: Scholz, C. H., Goldsby, D. L., Bernabé, Y., and Evans, B., (2018), Joseph B. Walsh (1930–2017), Eos, 99, https://doi.org/10.1029/2018EO093999. Published on 6 March 2018.

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