Stochastic control problems and HJB equations with excluded parameters of random inputs

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Submitted February 14, 2019. Revised: March 3, 2020

Abstract

This paper introduces a new type of second order stochastic backward Hamilton-Jacobi-Bellman (HJB) equations for optimal stochastic control problems with a currently observable but non-predicable parameter process, in addition to the driving Brownian motion. The main feature of this HJB equation is that it excludes specifications of the parameter process which dynamics can be unspecified or unknown. This allows to reduce the dimension of the state space. The paper considers the case of control dependent diffusion coefficients and fully nonlinear HJB equations under so-called Cordes conditions.

Key words: stochastic optimal control, Hamilton-Jacobi-Bellman equation, backward SPDEs, dimension reduction, fully nonlinear equations, Cordes conditions,

Mathematical Subject Classification (2010): 91G80 93E20, 91G10

Introduction

This paper considers optimal stochastic control problems in the continuous time setting. The theory of these problems is well developed. In the diffusion Markovian setting, the value function is usually represented by a parabolic Hamilton-Jacobi-Bellman (HJB) equation. In non-Markovian control problems, the backward Hamilton-Jacobi-Bellman equations equations have to be replaced by corresponding backward SPDEs; this was first observed by Peng (1992).
Stochastic Partial Differential Equations (SPDEs) are well studied in the literature, including the case of forward and backward equations; see, e.g., Walsh (1986), Alós et al (1999), Chojnowska-Michalik (1987), Rozovsky (1990), Zhou (1992), Pardoux (1993), Bally et al (1994), Chojnowska-Michalik and Goldys (1995), Maslowski (1995), Da Prato and Tubaro (1996), Gyöngy (1998), Mattingly (1999), Duan et al (2003), Caraballo et al (2004), Mohammed et al (2008), Feng and Zhao (2012), and the bibliography therein. In particular, backward SPDEs (BSPDEs) represent versions of the so-called Bismut-Peng equations where the diffusion term is not given a priori but needs to be found; see e.g. Hu and Peng (1991), Peng (1992), Zhou (1992), Dokuchaev (2008, 2018), Du and Tang (2012), Du et al (2013), Hu and Peng (1995), Hu et al (2002), Ma and Yong (1999), and the bibliography therein.

In Bender and Dokuchaev (2016a,b) and Dokuchaev (2017), some special BSPDEs were derived for the value functions of special problems with linear state equations. They represented analogs of Hamilton-Jacobi-Bellman equations for some non-Markovian stochastic optimal control problems associated with pricing of swing options in continuous time. These equations are not exactly differential, since their solutions can be discontinuous in time, and they allow very mild conditions on the underlying driving stochastic processes with unspecified dynamics. More precisely, the method does not have to assume a particular evolution law of the underlying process; the underlying processes do not necessarily satisfy stochastic differential equations of a known kind with a given structure. In particular, the First Order BSPDEs describe the value function even in the situation where the underlying price process cannot be described via a stochastic equation ever described in the literature. The numerical solution requires just to calculate certain conditional expectations of the functions of the process without using its evolution law (see the discussion in Section 4). It can be also noted that these equations are not the same as the first order deterministic HJB equations known in the deterministic optimal control.

The present paper extends these results on the setting with controlled diffusion with observed stochastic parameter with unspecified dynamics. The paper considers a model where the controlled process is described as a stochastic Ito process with coefficients depending on a random currently observable but unpredictable process \( Z(t) \) being independent on the driving Brownian motion. It is shown that the value function satisfies a second order BSPDE being a stochastic analog of the Hamilton-Jacobi-Bellman equation These stochastic equations are not exactly differential, since their solutions can be discontinuous in time, and they allow very mild conditions on the processes \( Z(t) \) with unspecified dynamics. Similarly to the First Order BSPDEs introduced in Bender and Dokuchaev (2016a,b), the presented BSPDEs do not include the parameters of a particular
evolution law of $Z(t)$; these processes $Z(t)$ do not necessarily satisfy stochastic differential equations of a known kind with a given structure. This could be used to reduce the dimension of the equations even for the case when the dynamic of $Z(t)$ is known; see Remark 3.1 below. The paper covers the case of control dependent diffusion coefficients and fully nonlinear HJB equations under so-called Cordes conditions.

The paper is organised as the following. In Section 1 the control problem is described. In Section 2 some background results on weak solutions of Ito equations and related parabolic equations are provided. In Section 3 the main results on existence of optimal controls and backward SPDEs for the value functions are given. Section 4 contains the proofs.

1 Stochastic control problem

We are given an open domain $D \subseteq \mathbb{R}^n$ such that either $D = \mathbb{R}^n$ or $D$ is bounded with $C^{2+\alpha}$-smooth boundary $\partial D$ for some $\alpha > 0$; if $n = 1$, then the condition of smoothness is not required. Let $T > 0$ be given, and let $Q \triangleq D \times (0,T)$.

We are given a standard complete probability space $(\Omega, \mathcal{F}, P)$ and a $n$-dimensional Wiener process $w(t) = (w_1(t), ..., w_n(t))$, $t \geq 0$, with independent components such that $w(0) = 0$.

Let an integer $d > 0$ be given. Let $Z(t), t \geq 0$, be a $d$-dimensional right continuous stochastic process with left limits that is independent on $w$.

Let $\{F^w_t\}_{t \geq 0}$ be the filtration generated by $w$, and let $\{F^Z_t\}_{t \geq 0}$ be the filtration generated by $Z$.

In addition, let $\{G_t\}_{t \geq 0}$ be the filtration generated by $(w, Z)$, and let $\{\bar{G}_t\}_{t \geq 0}$ be the filtration such that $\bar{G}_t$ is the completion of all events $\{A \cap B : A \in F^w_t, B \in F^Z_t\}$.

We assume that $F^w = F^Z_0 = \mathcal{G}_0$, and that this $\sigma$-algebra is the completion of the trivial $\sigma$-algebra $\{\Omega, \emptyset\}$.

We denote by $\omega$ the elements of the set $\Omega = \{\omega\}$.

Let $\Delta \subset \mathbb{R}^m$ be a compact set.

Consider a controlled Ito equations

$$dy(t) = f(y(t), u(y(t), t), Z(t), t)dt + \beta(y(t), u(y(t), t), Z(t), t)dw(t).$$

For $t \geq s$ and a random vector $a$, we denote by $y^{a,s}(t)$, the solution of this equation with the initial condition

$$y(s) = a. \quad (1.1)$$
One selects functions $u(\cdot) : D \times [0, T] \times \Omega \to \Delta$ as controls.

Random vectors $a$ and $y(t)$ take values in $\mathbb{R}^n$.

We assume that the functions $f(x, v, z, t) : \mathbb{R}^n \times \Delta \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^n$ and $\beta(x, v, z, t) : \mathbb{R}^n \times \Delta \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{n \times n}$ are continuous and bounded, together with the derivatives $\frac{\partial f}{\partial x}(x, v, z, t)$ : $\mathbb{R}^n \times \Delta \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^n$ and derivatives $\frac{\partial^2 f}{\partial x_1 \partial x_2}(x, v, z, t) : \mathbb{R}^n \times \Delta \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{n \times n}$ for $i, j = 1, \ldots, n$.

Consider a bounded Borel measurable function $\varphi(x, v, z, t) : \mathbb{R}^n \times \Delta \times \mathbb{R}^d \times \mathbb{R}$ such that the function $\varphi(x, r, z, t) : \Delta \to \mathbb{R}$ is continuous for all $x, z, t$.

For $s \in [0, T)$, let $\mathcal{A}$ be the set of all initial random vectors $a$, such that $a \in D$ a.s., $a$ is independent on $(w, Z)$, and that there exists $\rho \in H^{-1}$ such that $(\rho, \Psi)_{H^0} = \mathbb{E}\Psi(a)$ for any $\Psi \in H^1$. It can be noted that if $\rho \in L_1(D)$ then it is the probability density function of $a$. If $n = 1$, then the set $\mathcal{A}$ includes non-random $a \in D$; in this case, $\rho$ can be associated with the delta-function supported at $a \in D$. In all cases, we say that $\rho \in H^{-1}$ describes the probability distribution of $a$.

For $a \in \mathcal{A}$ and some measurable function $\varphi : \mathbb{R}^n \times \Delta \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$, let $\tau^{a,s} \triangleq \inf\{t : y^a_s(t) \notin D\}$, and let

$$F(a, u(\cdot)) \triangleq \mathbb{E} \int_0^{\tau^{a,s}_{\text{A},T}} \varphi(y^a_s(t), u(t), Z(t), t) dt. \quad (1.2)$$

Consider control problem

Minimize $F(a, u(\cdot))$ over $u(\cdot)$. \hfill (1.3)

Admissible controls

Let us describe admissible controls.

Let $U_0$ be the class of admissible control functions $u(y, t, \omega) : \bar{D} \times [0, T] \times \Omega \to \Delta$ that are measurable and $\mathcal{F}_t^2$-adapted for any $(y, t) \in \bar{D} \times [0, T]$.

Let $C'(\Delta)$ be the space of real valued continuous functions defined on $\Delta$. Let $C'(\Delta)^*$ be its dual space.

Let $\Delta_R = \{u \in C'(\Delta)^* : \langle u, 1 \rangle_{C'(\Delta)^*, C(\Delta)} = 1, \langle u, \nu \rangle \geq 0 \ (\forall \nu \in C(\Delta) : \nu \geq 0)\}$. It can be noted that $u \in \Delta_R$ can be associated with a probability measure on $\Delta$, so $\langle u, \nu \rangle_{C'(\Delta)^*, C(\Delta)}$ would be an averaging of the function $\nu(v)$ over $\Delta$ with resect to this probability measure.

Let $U_R$ be the class of admissible control functions $u(y, t, \omega) : \bar{D} \times [0, T] \times \Omega \to \Delta_R$ that are $\mathcal{F}_t^2$-adapted for any $(y, t) \in \bar{D} \times [0, T]$.
We assume that $U_0 \subset U_R$, meaning that any $v \in \Delta$ is associated with the corresponding Dirac measure.

2 Some background definitions and results

For a Banach space $X$, we denote by $\| \cdot \|_X$ the norm, and we denote by $X^*$ its dual space. We will use notation $\langle a, b \rangle$ for $b(a)$, where $b : X \to \mathbb{R}$ is an element of $X^*$.

For a Hilbert space $X$, we denote by $(\cdot, \cdot)_X$ the scalar product in $X$.

We denote Euclidean norm in $\mathbb{R}^k$ as $| \cdot |$, and $\bar{G}$ denotes the closure of a region $G \subset \mathbb{R}^k$.

We introduce some spaces of real valued functions.

We denote by $W^m_q(D)$ the Sobolev space of functions that belong to $L^q(D)$ together with first $m$ derivatives, $q \geq 1$. In particular,

$$\| u \|_{W^1_2(D)} \overset{\Delta}{=} \left( \| u \|_{L^2(D)}^2 + \sum_{i=1}^n \| \frac{\partial u}{\partial x_i} \|_{L^2(D)}^2 \right)^{1/2}.$$

Let $H^0 \overset{\Delta}{=} L^2(D)$, and let $H^1 \overset{\Delta}{=} W^0_2(D)$ be the closure in the $W^1_2(D)$-norm of the set of all smooth functions $u : D \to \mathbb{R}$ such that $u|_{\partial D} \equiv 0$. Let $H^2 = W^2_2(D) \cap H^1$ be the space equipped with the norm of $W^2_2(D)$. The spaces $H^k$ and $W^k_2(D)$ are called Sobolev spaces, $k = 0, 1, 2$; they are Hilbert spaces, and $H^k$ is a closed subspace of $W^k_2(D)$, $k = 1, 2$. As usual, we assume that $W^0_2(D) = H^0$.

Let $H^{-k} \overset{\Delta}{=} (H^k)^*$ be the dual spaces to the spaces $H^k$, $k = 1, 2$.

If $Y$ is the dual space for a space $X$, we denote the dual pairing by $\langle \cdot, \cdot \rangle_{Y,X}$.

We denote by $\ell_k$ and $\bar{\ell}_k$ the Borel measure and the Lebesgue measure in $\mathbb{R}^k$ respectively, and we denote by $\mathcal{B}_k$ the $\sigma$-algebra of Borel sets in $\mathbb{R}^k$. We denote by $\bar{\mathcal{B}}_k$ the completion of $\mathcal{B}_k$ with respect to the measure $\ell_k$, or the $\sigma$-algebra of Lebesgue sets in $\mathbb{R}^k$.

We denote by $\bar{\mathcal{P}}_Z$ the completion (with respect to the measure $\bar{\ell}_1 \times \mathcal{P}$) of the $\sigma$-algebra of subsets of $[0, T] \times \Omega$, generated by functions that are progressively measurable with respect to $\mathcal{F}^Z_t$.

For $k = -2, -1, 0, 1, 2$, we introduce spaces

$$X^k(s, t) \overset{\Delta}{=} L^2([s, t] \times \Omega, \bar{\mathcal{P}}_Z, \bar{\ell}_1 \times \mathcal{P}; H^k), \quad \tilde{X}^k(s, t) \overset{\Delta}{=} L^2(\Omega, \mathcal{F}^Z_t, \mathcal{P}, L^2([s, t], \bar{\mathcal{B}}_1, \bar{\ell}_1, H^k),$$

$$\mathcal{Z}_t^k \overset{\Delta}{=} L^2(\Omega, \mathcal{F}^Z_t, \mathcal{P}; H^k), \quad C^k(s, t) \overset{\Delta}{=} C([s, t]; \mathcal{Z}_t^k).$$

The spaces $X^k(s, t)$, $\tilde{X}^k(s, t)$, and $\mathcal{Z}_t^k$, are Hilbert spaces.
Further, we introduce spaces

\[ Y^k(s,t) \overset{\Delta}{=} X^k(s,t) \cap C^{k-1}(s,t), \quad \bar{Y}^k(s,t) \overset{\Delta}{=} \bar{X}^k(s,t) \cap C^{k-1}(s,t), \quad k = 1, 2. \]

Note that spaces \( X^k \) and \( Y^k \) include adapted to \((w, Z)\) functions only, and the spaces \( X^k \) and \( Y^k \) include functions only that are not necessarily adapted.

For brevity, we will use the notations \( X^k(0,T), \bar{X}^k(0,T), C^k(0,T), Y^k \overset{\Delta}{=} Y^k(0,T) \), and \( \bar{Y}^k \overset{\Delta}{=} \bar{Y}^k(0,T) \).

The assumption on the regularity of related linear parabolic equations

The control problem \([1.3]\) is formulated for a challenging case where the diffusion coefficients depend on the control. This case is difficult even in the Markovian setting (i.e. where \( Z(t) \equiv 0 \)), because corresponding forward and backward Kolmogorov parabolic equations for distributions are equations in non-divergent form with discontinuous coefficients at higher derivatives; they do not feature sufficient regularity in a case of non-smooth closed loop controls \( u(\cdot, t) \). Their investigation is most complicated because, in general, in the case of discontinuous coefficients, the uniqueness of a solution for nonlinear parabolic or elliptic equations can fail, and there is no a priori estimate for partial derivatives of a solution; see. e.g. Krylov (1987) and the literature therein. On the other hand, a typical optimal control is not expected to be smooth.

There are two main approaches to overcoming these difficulties via relaxation of the requirements for the solution. One approach is to consider the so-called viscosity solutions; see, e.g., Crandall and Lions (1983). Another approach is to accept solutions with measure-valued second derivatives; see e.g. Krylov (1980,1987). In this paper, we will not be using either of these approaches. Instead, we restrict our consideration only by the cases where the backward Kolmogorov equations for controls \( u \in U_R \) features solutions with \( L^2 \)-integrable derivatives. This still can cover some important case. For example, this setting covers the case where the part of the diffusion coefficient depending on the control is restricted in size (Condition \([2.1]\) below).

Let \( b(x, v, z, t) \overset{\Delta}{=} \beta(x, v, z, t)\beta(x, v, z, t)^\top / 2 \). We assume that

\[ \inf_{\xi \in \mathbb{R}^n, (x, t) \in Q, v \in \Delta, z \in \mathbb{R}^d} \frac{\xi^\top b(x, v, z, t)\xi}{|\xi|^2} > 0. \]

We assume that the domains for \( b, f, \) and \( \varphi \) are extended to \( \mathbb{R}^n \times \Delta_R \times \mathbb{R} \) as the following. For \((x, u, z, t) \in \mathbb{R}^n \times \Delta_R \times \mathbb{R}^d \times \mathbb{R}\), we assume that \( f(x, u, z, t) = \langle f(x, ., z, t), u \rangle_{C(\Delta)^*, C(\Delta)} \) and
\[ b(x, u, z, t) = (b(x, z, t), u)_{C(\Delta)\ast, C(\Delta)}. \]

It can be noted that \( u \in \Delta_R \) can be associated with a probability measure on \( \Delta \); therefore, these extensions represent averaging over this measure.

Let us define differential operators

\[ A(x, u, z, t)V = \sum_{i,j=1}^{n} b_{ij}(x, u, z, t) \frac{\partial^2 V}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} f_i(x, u, z, t) \frac{\partial V}{\partial x_i}(x), \quad (2.1) \]

where \( b_{ij}, f_i, x_j \) are components of \( f, x, b \). Since these functions depend on \( Z(t), \) i.e., these operators have random coefficients.

In \( Q = D \times (0, T) \), consider, for an admissible \( u(\cdot) \), a boundary value problem

\[ \frac{\partial V}{\partial t}(x, t) + A(x, u(x, t), Z(t), t)V(x, t) = -\psi(x, t), \quad V|_{x \in \partial D} = 0, \quad V(x, T) = 0. \quad (2.2) \]

**Condition 2.1** Assume that the function \( b \) is such that, for any \( u \in U_R \), the problem (2.2) has a unique solution \( V \in \bar{Y}^2 \) for any \( \psi \in X^0 \). Moreover,

\[ \int_0^T \left( \left\| \frac{\partial V}{\partial t}(\cdot, t) \right\|_{H^0}^2 + \left\| V(\cdot, t) \right\|_{H^2}^2 \right) dt \leq c\|\psi\|_{L^2(Q)}^2 \quad \text{a.s.,} \quad (2.3) \]

where \( c > 0 \) depends on \( D, n, T, f, b \) only.

The following result describes some special cases where Condition 2.1 holds.

**Lemma 2.1** Condition 2.1 holds if at least one of the following conditions is satisfied.

(i) \( b(x, v, z, t) \equiv b(x, z, t) \);

(ii) The matrix \( b \) has the form \( b(x, v, z, t) = \bar{b}(x, z, t) + \hat{b}(x, v, z, t) \), where \( \bar{b}(x, z, t) = \bar{b}(x, z, t)^\top \)

is a continuous bounded matrix such that

\[ C_b \triangleq \inf_{\xi \in \mathbb{R}^n, (x, t) \in Q, z \in \mathbb{R}^d} \frac{\xi^\top \bar{b}(x, z, t) \xi}{|\xi|^2} > 0, \quad (2.4) \]

and where

\[ \sup_{(x, t) \in Q, z \in \Delta, v \in \mathbb{R}^d, i, k=1} \sum_{i,k=1}^{n} \hat{b}_{ik}(x, v, z, t)^2 < \frac{C_b^2}{n}. \]

(iii) The matrix \( b \) has the form \( b(x, v, z, t) = \bar{b}(x, z, t) + \hat{b}(x, v, z, t) \), where \( \bar{b}(x, z, t) = \bar{b}(x, z, t)^\top \) is a continuous bounded matrix such that (2.1) holds. The matrix function \( \hat{b}(x, v, t) \) is symmetric and such that there exists a set \( N \subseteq \{1, \ldots, n\} \) such that

\[ \hat{b}_{ij} \equiv \hat{b}_{ji} \equiv 0 \quad \forall i, j : i \notin N, j \notin N, \]
and there exists a set \( \{\gamma_k\}_{k \in \mathbb{N}} \) such that \( \gamma_k \in (0, 2) \) for all \( k \) and

\[
\left( \sum_{k \in \mu} \frac{1}{2 \gamma_k} \right) \sup_{(x,t) \in Q, \nu \in \Delta, z \in \mathbb{R}^d} \left( \sum_{i \in \mu} \tilde{b}_{ik}(x,v,z,t)^2 + 4 \sum_{i \notin \mu} \tilde{b}_{ik}(x,v,z,t)^2 + \frac{\gamma_k}{2 - \gamma_k} \tilde{b}_{kk}(x,v,z,t)^2 \right) < C_b^2.
\]

The result of Lemma 2.1 was obtained in Dokuchaev (1997); a related result can be found in Dokuchaev (2005).

Conditions (ii) and (iii) represents analogs of the so-called Cordes conditions that ensure regularity of solutions of boundary value problems for second order equations and that are known as Cordes conditions.

**On Cordes conditions: some historical remarks**

It is known that discontinuity of the higher order coefficients for linear parabolic and elliptic equations in non-divergent form causes problems with regularity of the solutions. This makes analysis of corresponding HJB equations difficult; see, e.g. Krylov (1987) and the literature therein. A possible approach is to consider so-called viscosity solutions; see, e.g., Fleming and Soner (1993). These solutions may not have all integrable derivatives. In some cases, it is still possible to have solutions with \( L_2 \)-integrable derivatives; this can be achieved with some restrictions the scale of discontinuities for the higher order coefficients. The original Cordes conditions restricts the scattering of the eigenvalues of the matrix of the coefficients at higher derivatives (see Cordes (1956)). Related conditions from Talenti (1965), Koshelev (1982), Kalita (1989), Landis (1998), on the eigenvalues are also called Cordes type conditions. A closed condition is presented implicitly in the proof of the uniqueness of a weak solution in Gihman and Skorohod (1975), Section 3 of Chapter 3.

Cordes (1956) considered elliptic equations. Landis (1998) considered both elliptic and parabolic equations. Koshelev (1982) considered systems of elliptic equations of divergent type and Hölder property of solutions. Kalita (1989) considered union of divergent and nondevirgent cases.

Conditions from Cordes (1956) are such that they are not necessary satisfied even for constant non-degenerate matrices \( b \), therefore, the condition for \( b = b(x) \) means that the corresponding inequalities are satisfied for all \( x_0 \) for some non-degenerate matrix \( \theta(x_0) \) and \( \tilde{b}(x) = \theta(x_0)^T b(x) \theta(x_0) \), where \( x \) is from \( \varepsilon \)-neighborhood of \( x_0 \) (\( \varepsilon > 0 \) is given). Conditions in Lemma 2.1 ensure also solvability and uniqueness for first boundary value problem for nondivergent parabolic equation with
discontinuous diffusion coefficients. Moreover, conditions in Lemma 2.1 ensure prior estimate required in Condition 2.1 in contrast with the existing literature. Second order SPDEs satisfying Cordes conditions similar to the ones in Lemma 2.1 were considered in Dokuchaev (2005) for forward stochastic SPDEs. Some comparison of different types of Cordes conditions can be found in Dokuchaev (1997,2005).

Some auxiliary operators

It can be seen that Condition 2.1 implies continuity, for any $u \in U_R$, of the following linear operators

$$
\bar{L}(u(\cdot)) : X^0 \to X^2, \quad L(u(\cdot)) : X^0 \to X^2, \quad \mathcal{L}(u(\cdot)) : X^0 \to H^1
$$

defined such that

$$
\bar{V} = \bar{L}(u(\cdot))\psi, \quad V = L(u(\cdot))\psi, \quad V(\cdot,0) = \mathcal{L}(u(\cdot))\psi,
$$

where $V$ is the solution of the problem (2.2), and where $V(\cdot,t) = \mathcal{E}(\bar{V}(\cdot,t)|\mathcal{F}_t^Z)$. (Remind that $Z^k_0 = H^k$). The corresponding adjoint operators $\mathcal{L}(u(\cdot))^* : H^{-1} \to X^0$ are linear and continuous as well.

On weak solutions of Ito equations

We consider solutions of (1.1) for $u \in U_R$. In this case, we assume that $\beta$ is defined as a square-root of the matrix $2b$; to ensure that the choice of the square-root is unique, we can require, for example, that the matrix $\beta$ is positive-definite everywhere.

It can be noted that $b(\cdot, u, \cdot)$ is affine in $u \in U_R$, but this is not necessarily the case for $\beta(\cdot, u, \cdot)$.

We consider weak solutions of (1.1) for $u \in U_R$ as described in the following lemma.

**Lemma 2.2** Let $a \in L^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ be independent on $w$. Then, for any $u \in U_R$, there exists a set

$$
\left\{ (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}), (w(t), \mathcal{F}_t), y^{a,0}(t) \right\},
$$

where $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ is a probability space such that $a \in L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, $(w(t), \mathcal{F}_t)$ is a $n$-dimensional Wiener process on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, $\mathcal{F}_t \subseteq \hat{\mathcal{F}}$ is a filtration of $\sigma$-algebras of events such that $w(t) - w(s)$ do not depend on $(a,Z)$ and on $\mathcal{F}_s$ for $t > s$, and $y^{a,0}(t)$ is the solution of (1.1) for this $w(t)$. 

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This theorem can be found, in particular, in Krylov (1980), Chapter 2; it is formulated therein for non-random \((a, Z(\cdot))\), which is unessential since \((a, Z)\) are independent on \(w\).

The results of Lemma 2.3 below were obtained in Dokuchaev (1997).

**Lemma 2.3** Let \(a \in \mathfrak{A}\), and let \(\rho\) describes the probability distribution of \(a\). Assume that Condition 2.1 is satisfied. Then, for any \(u(\cdot) \in U_R\), equation 1.1 has a unique weak solution, meaning that the solution is univalent with respect to the probability distribution.

In this paper, we consider problem (1.3) with weak solutions of equation (1.1).

**Lemma 2.4** Let \(a \in \mathfrak{A}\), and let \(\rho\) describes the probability distribution of \(a\). Then, for any measurable \(u \in U_R\),

\[
\mathbb{E} \left\{ \int_0^{a_{0,T}} \varphi(y^{a,0}(t), u(t), Z(r), t) dt \bigg| \mathcal{F}_T^2 \right\} = \langle \rho, \bar{V}(\cdot, 0) \rangle_{H^{-1}, H^1} \quad \text{a.s.}
\]

and

\[
F(a, u(\cdot)) = \mathbb{E} \langle \rho, \bar{V}(\cdot, 0) \rangle_{H^{-1}, H^1} = \langle \rho, V(\cdot, 0) \rangle_{H^{-1}, H^1}
\]

where \(\bar{V} = \bar{L}(u(\cdot))\varphi(\cdot, u(\cdot), Z(\cdot), \cdot) \in \bar{Y}^2\), and \(V = L(u(\cdot))\varphi(\cdot, u(\cdot), Z(\cdot), \cdot) \in Y^2\). In addition,

\[
|F(a, u(\cdot))| \leq C\|\rho\|_{H^{-1}} \|\varphi(\cdot, u(\cdot), Z(\cdot), \cdot)\|_{X^0},
\]

where \(C > 0\) is a constant occurring in Condition 2.1.

By Lemma 2.4, the functionals \(F(a, u(\cdot))\) are defined for Borel measurable \(u(\cdot) \in U_R\), and \(F(a, u(\cdot)) = \langle \rho, V(\cdot, 0) \rangle_{H^{-1}, H^1}\), where \(V = L(u(\cdot))\varphi(\cdot, u(\cdot), Z(\cdot), \cdot)\). However, the value \(\langle \rho, V(\cdot, 0), \rho \rangle_{H^{-1}, H^1}\) is defined also for \(u(\cdot) \in U_R\), and it does not depend on the choice of a representative of a class of equivalency; for \(u(\cdot) \in U_R\) there exists a Borel measurable equivalent function \(u(\cdot) \in U_R\).

Respectively, we presume that the functionals \(F(a, u(\cdot))\) are extended on \(u(\cdot) \in U_R\).

**Lemma 2.5** Under the assumptions of Lemma 2.4, the weak solution \(y^{a,0}(t)\) of equation 1.1 with \(s = 0\), considered on the boundary \(D\), has the conditional distribution given \(\mathcal{F}_T^2\) featuring the probability density function \(p = \mathcal{L}(u(\cdot))^* \rho \in L_2(Q) \quad \text{a.s.}\). Moreover, \(p \in X^0\).

**Corollary 2.1** (The Maximum Principle). Assume that conditions of Lemma 2.2 are satisfied and, in addition, that \(\varphi(x, v, z, t) \geq 0\) for a.e. \(x, t\) for all \(v \in \Delta_R\), \(z \in \mathbb{R}^d\), \(\langle \rho, \phi \rangle_{H^{-1}, H^1} \geq 0\) a.s. for all \(\phi \in H^1\) such that \(\phi(x) \geq 0\) a.s., where \(\langle \cdot, \cdot \rangle_{H^{-1}, H^1}\) denote the natural pairing between \(H^{-1}\) and \(H^1\). Then, for any \(u \in U_R\), we have that \(V(x, t, \omega) \geq 0\) and \(p(x, t, \omega) \geq 0\) a.e., where \(V = L(u(\cdot))\varphi + \mathcal{L}(u(\cdot))\Phi\) and \(p = \mathcal{L}(u(\cdot))^* \rho\).
The following lemma provides a strengthened version of the maximum principle.

**Lemma 2.6** Let \( a \in \mathfrak{A} \), and let \( \rho \in H^{-1} \) describing the probability distribution of \( a \) be such that 
\[
\langle \rho, \phi \rangle_{H^{-1},H^1} > 0 \quad \text{for all } \psi \in H^1 \text{ such that mes } \{ x \in D : \psi(x) > 0 \} > 0.
\]
Then, under the assumptions of Lemma 2.5, \( p(x,t,\omega) > 0 \) for a.e. \( x \in D, \ t \in (0,T] \), \( \omega \in \Omega \).

### 3 The main results

Up to the end of this paper, we assume that Condition 2.1 holds.

**Theorem 3.1** Let \( a \in \mathfrak{A} \). Then there exists an optimal solution \( u \in U_R \) for problem (1.3).

**Theorem 3.2** Assume that the set \( \Delta \) is convex. In this case, the following holds.

(i) There exists \( \hat{u}(\cdot) \in U_0 \) such that \( \hat{V} = L(\hat{u}(\cdot))\varphi(\cdot, \hat{u}(\cdot), Z(\cdot), \cdot) \) satisfies the following modification of the Hamilton-Jacobi-Bellman equation
\[
\hat{V}(x,t) = \mathbb{E} \left\{ \int_t^T \inf_{v \in \Delta} \left[ A(x,v,Z(s),s)\hat{V}(x,s) + \varphi(x,v,Z(s),s) \right] ds \right\},
\]
\[
\hat{V}(x,t)|_{x\in\partial D} = 0, \quad \hat{V}(x,T) = 0,
\]
for all \( t \in [0,T] \) for a.e. \( x \in D \) a.s..

(ii) If \( \hat{u} \in U_0 \) is such as described above, then \( F(a,\hat{u}(\cdot)) = \langle \rho, \hat{V}(\cdot,0) \rangle_{H^{-1},H^1} \) and
\[
F(a,\hat{u}(\cdot)) \leq F(a,u(\cdot)) \quad \forall u(\cdot) \in U_0, a \in \mathfrak{A}.
\]

Remark 3.1 Equation (3.1) does not include the parameters of a particular evolution law of \( Z(t) \); these processes \( Z(t) \) do not necessarily satisfy stochastic differential equations or jump-diffusion equation of a known kind with a given structure. This could be used to reduce the dimension of the equations even for the case when the dynamic of \( Z(t) \) is known. Assume, for example, that the controlled process \( y(t) \) is \( n \)-dimensional and that a scalar process \( Z(t) \) is defined as \( Z(t) = CX(t) \), where \( C \in \mathbb{R}^{1 \times N} \), and where \( X(t) \) is a \( N \)-dimensional solution of an Ito equation. Then the traditional parabolic HJB equation in Markovian setting would require the state space \( \mathbb{R}^{n+N} \). On the other hand, the state space for equation (3.1) is \( \mathbb{R}^n \); this gives a significant dimension reduction for large \( N \).
4 Proofs

4.1 Proof of Lemma 2.6

Let $P_0\{\cdot|(a, Z)\}$ be a probability measure that is equivalent to $P\{\cdot|(a, Z)\}$ and such that the process $y^{a,0}(t)$ is a martingale on the conditional probability space given $(a, Z)$; this measure exists by Girsanov Theorem. In this case, for any $\alpha \in D$ and any $\varepsilon > 0$,

$$P_0(\sup_{t \in [0,T]} |y^{a,0}(t) - \alpha| \leq \varepsilon | F^Z_T) > 0 \quad \text{a.s.}$$

This follows from the properties of standard one-dimensional Wiener processes and from the Dambis-Dubins-Schwartz theorem applied to the components of the vector process $y^{a,0}(\cdot)$.

In this case, for any $\alpha \in D$ and any $\varepsilon > 0$,

$$P_0(\sup_{t \in [0,T]} |y^{a,0}(t) - \alpha| \leq \varepsilon | F^Z_T) > 0 \quad \text{a.s.}$$

This implies that

$$P(\sup_{t \in [0,T]} |y^{a,0}(t) - \alpha| \leq \varepsilon | F^Z_T) > 0 \quad \text{a.s.}
\quad (4.1)$$

Further, suppose that there exists a domain $D_0 \subset D$ such that $P(y^{a,0}(T) \in D_0 | F^Z_T) = 0$. Let $D_\varepsilon := \{x \in D_0 : \text{dist}(x, \partial D_0) > \varepsilon\}$. Let $\varepsilon > 0$ be such that $\text{mes} D_\varepsilon > 0$.

By the properties of $\rho$, it follows that $P(a \in D_\varepsilon) > 0$. Clearly, (4.1) implies that

$$P(y^{a,0}(t) \in D_0 \quad \forall t \in [0,T] | F^Z_T, a \in D_\varepsilon) > 0 \quad \text{a.s.}$$

This completes the of Lemma 2.6 $\Box$

4.2 Proof of Theorem 3.1

We denote by $U_R^w$ the set $U_R$ provided with the weak topology of the space being dual to the space $L^1(D \times [0,T] \rightarrow C(\Delta))$. It is known that the set $U_R^w$ is convex, compact, and sequentially compact; see e. g. Varga (1972), Ch.IV.

For $u \in U_R$, the operators $A(\cdot, u(\cdot), \cdot, t) \triangleq A(x, u(x,t), Z(t), t) : \mathcal{Z}_T^2 \rightarrow \mathcal{Z}_T^0$ are continuous, hence the adjoint operators $A^*(\cdot, u(\cdot), \cdot, t) : \mathcal{Z}_T^0 \rightarrow \mathcal{Z}_T^{-2}$ are continuous; here $\mathcal{Z}_T^{-2} = (\mathcal{Z}_T^2)^*$. These operators are such that

$$\langle A^*(\cdot, u(\cdot), \cdot, t) \eta, \psi \rangle_{\mathcal{Z}_T^{-2}, \mathcal{Z}_T^0} = \langle \eta, A(\cdot, u(\cdot), \cdot, t) \psi \rangle_{\mathcal{Z}_T^0}, \quad \psi \in \mathcal{Z}_T^2, \quad \eta \in \mathcal{Z}_T^0, \quad t \in [0,T].$$
Proposition 4.1 Let $\rho \in H^0$ describes the probability distribution of $a$ (in particular, if $\rho \in H^0$ then $\rho$ is the density for $a$). Let $u_i(\cdot) \in U_R$ be such that the derivatives up to the second order for $u_i(x,t,\omega)$ with respect to $x$ are bounded, $\alpha_i \geq 0$, $i = 1, \ldots, N$, $\sum_{i=1}^N \alpha_i = 1$, $N = 1, 2, 3, \ldots$. Let $p_i = L(u_i(\cdot))^* \rho$. Let $\tilde{p}(x,t) = \sum_{i=1}^N \alpha_i p_i(x,t)$. Let us consider control

$$\tilde{u}(x,t) = \tilde{p}(x,t)^{-1} \sum_{i=1}^N \alpha_i p_i(x,t) u_i(x,t).$$

(It can be seen that $\tilde{u} \in U_R$ since $\Delta_R$ is a convex set). Let $p_u = L^*(\tilde{u}(\cdot))\rho$. Then

$$p_u = \tilde{p}, \quad \sum_{i=1}^N \alpha_i F(a, u_i(\cdot)) = F(a, \tilde{u}(\cdot)).$$

**Proof of Proposition 4.1** In this proof, we use that the coefficients $f$ and $b$ defined as $f(x,v,z,t) = \langle f(x,\cdot, z,t), v \rangle_{C(\Delta),C(\Delta)}$ and $b(x,v,z,t) = \langle b(x,\cdot, z,t), v \rangle_{C(\Delta),C(\Delta)}$ are affine with respect to the probability measures $v \in \Delta_R$. It can be noted that, however, that it is not required that the functions $f(x,\cdot, z,t) : \Delta \to \mathbb{R}^n$ and $b(x,\cdot, z,t) : \Delta \to \mathbb{R}^{n\times n}$ are affine with respect to $u \in \Delta$.

Since the function $\varphi(x,v,Z(t),t)$ is affine in $v \in U_R$, we have that

$$\sum_{i=1}^N \alpha_i F(a, u_i(\cdot)) = \mathbb{E} \int_Q dx\, dt \sum_{i=1}^N \alpha_i p_i(x,t) \varphi(x,u_i(x,t),Z(t),t) = \mathbb{E} \int_Q \tilde{p}(x,t) \varphi_i(x,\tilde{u}(x,t),Z(t),t) dx\, dt.$$

Therefore, it suffices to show that $p_u = \tilde{p}$.

Let us assume first that $\rho \in H^0$. Since we selected smooth in $x$ controls $u_i = u_i(x,t,\omega)$, we have that satisfy the forward Kolmogorov equations

$$\frac{\partial p_i}{\partial t}(x,t) = A^*(x,u_i(x,t),Z(t),t) p_i(x,t),$$

$$p_i(x,t,\omega)|_{x \in \partial D} = 0, \quad \tilde{p}_i(x,0,\omega) = \rho(x). \quad (4.2)$$

Here $A^*(\cdot, u_i(\cdot, t), Z(t), t) : H^1 \to H^{-1}$ are the formally operators for the operators $A(\cdot, u_i(\cdot, t), Z(t), t) : H^1 \to H^{-1}$.

The classical theory for these parabolic equations ensures that these equation have unique solutions $p_i \in Y^1$; moreover, we have that $p_i \in Y^2$ as well; see, e.g., Ladyzhenskaya (1985), Sections III.4-III.5).
We sum Kolmogorov’s equations for \( p_i \) (or, more precisely, for \( \alpha_i p_i \)), using the fact that the functions \( b(x, v, Z(t), t) \) and \( f(x, v, Z(t), t) \) are affine in \( u \in U_R \), meaning relations such as

\[
\sum_{i=1}^{N} \alpha_i p_i(x, t)f(x, u_i(x, t), Z(t), t) = (f(x, \cdot, t), \sum_{i=1}^{N} \alpha_i p_i(x, t)u_i(x, t))_{C^\infty(\Delta), C(\Delta)}
\]

\[
= \tilde{p}(x, t)\langle f(x, \cdot, Z(t), t), u(x, t) \rangle_{C^\infty(\Delta), C(\Delta)} = \tilde{p}(x, t)f(x, \tilde{u}(x, t), t).
\]

It gives that

\[
\sum_{i=1}^{N} \alpha_i A^\ast(\cdot, u_i(\cdot), \cdot)p_{u_i}(\cdot, \cdot) = A^\ast(\cdot, \tilde{u}(\cdot), \cdot)p_{\tilde{u}}(\cdot, \cdot);
\]

the equality here is in \( X^{-2} \). From the form of the corresponding forward Kolmogorov equation for the conditional density \( p \) on the conditional probability space given \( Z \), we obtain that \( p_u = \tilde{p} \) if \( \rho \in H^0 \). Since \( H^0 \) is everywhere dense in \( H^{-1} \), it follows that this identity holds also in the case if \( \rho \in H^{-1} \). This completes the proof of Proposition [4.1]. □

Further, let \( J \overset{\Delta}{=} \inf_{U_R} F(a, u(\cdot)) \). Clearly, there exists a sequence of controls \( u_i(\cdot) \in U_R \), \( i = 1, 2, \ldots, \) such that the derivatives up to the second order for \( u(x, t, \omega) \) with respect to \( x \) are bounded, and that \( F(a, u_i(\cdot)) \to J \) as \( i \to +\infty \). Let \( \tilde{p}_i \overset{\Delta}{=} \mathcal{L}(u_i(\cdot))^\ast \rho \). By passing to a weakly converging subsequence, we see that there exists \( \tilde{p} \in X^0 \) such that \( \tilde{p}_i \to \tilde{p} \) as \( i \to +\infty \) weakly in \( X^0 \). By the Mazur Theorem (see, e.g., Yosida (1995), p.173), there exists a sequence of convex combinations \( p_i(\cdot) = \sum_{j=1}^{i} \alpha_j \tilde{p}_j(\cdot), \alpha_j = \alpha_j(i), \alpha_j \geq 0, \sum_{j=1}^{i} \alpha_j = 1 \) such that \( p_i(\cdot) \to \tilde{p}(\cdot) \) in \( X^0 \). By Proposition [4.1], for any \( p_i(x, t) \) there exists \( v_i(\cdot) \in U_R \), such that \( p_i(x, t) = \mathcal{L}^\ast(v_i(\cdot))\rho \) and \( F(a, v_i(\cdot)) = \sum_{j=1}^{i} \alpha_j F(a, u_j(\cdot)) \). In addition, it is easy to see that \( F(a, v_i(\cdot)) \to J \) as \( i \to +\infty \).

The set \( U_R \) is compact in the topology of \( U_R \). Passing to a subsequence, we obtain that there exists \( \hat{u}(\cdot) \) such that \( v_i(\cdot) \to \hat{u}(\cdot) \) as \( i \to +\infty \) in the topology of \( U_R \).

Let us define operators

\[
A_i(\cdot, t) \overset{\Delta}{=} A(x, v_i(x, t), Z(t), t) : \mathcal{Z}_T^2 \to \mathcal{Z}_T^0, \quad \tilde{A}_i(\cdot, t) \overset{\Delta}{=} A^\ast(x, \hat{u}(x, t), Z(t), t) : \mathcal{Z}_T^0 \to \mathcal{Z}_T^{-2}.
\]

These operators are continuous.

By the assumptions on \( u_i \) and \( v_i \), it follows that \( p_i \) belong to \( Y^1 \) and that \( p(t) \) represent conditional densities given \( \mathcal{F}^2_T \) for processes \( y^{a,0}(t) \) being killed on the boundary. Hence they satisfy the forward Kolmogorov equations

\[
p_i(\cdot, t) = \rho + \int_0^t A_i^\ast(\cdot, s)p_i(\cdot, s)ds.
\]

Furthermore,

\[
A_i^\ast p_i - \tilde{A}_i^\ast \tilde{p} = R_{1,i} + R_{2,i},
\]
where
\[ R_{1,i} = A_i^* \hat{p} - \hat{A}^* \hat{p}, \quad R_{2,i} = A_i^* p_i - A_i^* \hat{p}, \]
i.e. \( R_{1,i} = [A_i^* - \hat{A}^*] \hat{p} \) and \( R_{2,i} = A_i^* [p_i - \hat{p}] \).

Clearly, \( \| R_{2,i} \|_{X^{-2}} \to 0 \) as \( i \to +\infty \) and
\[ R_{1,i} \to 0 \text{ weakly in } X^{-2} \text{ as } i \to +\infty. \]

Hence, for any \( t \in [0, T] \),
\[ \int_0^t A_i^*(\cdot, s)p_i(\cdot, s)ds \to \int_0^t \hat{A}^*(\cdot, s)\hat{p}(\cdot, s)ds \text{ weakly in } Z^{-2}_T \text{ as } i \to +\infty. \]

It follows that
\[ \hat{p}(\cdot, t) = \rho + \int_0^t \hat{A}^*(\cdot, s)\hat{p}(\cdot, s)ds. \]

These equalities hold for all \( t \in [0, T] \) in \( Z^{-2}_T \).

Similarly, we obtain that, for any \( \psi \in Z^{-2}_T \),
\[ \langle \hat{p}(\cdot, t), \psi \rangle_{Z^{-2}_T, Z^0_T} = \langle \rho, \psi \rangle_{Z^{-2}_T, Z^0_T} + \int_0^t \langle \hat{A}^*(\cdot, s)\hat{p}(\cdot, s), \psi \rangle_{Z^{-2}_T, Z^0_T}ds \]
\[ = \langle \hat{p}(\cdot, 0), \psi \rangle_{Z^{-2}_T, Z^0_T} + \int_0^t \langle \hat{A}^*(\cdot, s)\hat{p}(\cdot, s), \psi \rangle_{Z^{-2}_T, Z^0_T}ds. \]

Hence
\[ \langle \hat{p}'(\cdot, t), \psi \rangle_{Z^{-2}_T, Z^0_T} = \langle \hat{A}(\cdot, t)\hat{p}(\cdot, t), \psi \rangle_{Z^{-2}_T, Z^0_T} \text{ for a.e. } t. \]

Let us show that \( \hat{p} = \mathcal{L}^*(\bar{u}(\cdot)) \). For this, it suffices to show that
\[ (\xi, \hat{p})_{X^0} = \langle \rho, \bar{V}(\cdot, 0) \rangle_{H^{-1, H}} \text{ for any } \xi \in X^0, \bar{V} = L(\bar{u}(\cdot))\xi. \]

For this \( \bar{V} \), we have that
\[ \langle \rho, \bar{V}(\cdot, 0) \rangle_{H^{-1, H}} = \langle \rho, \bar{V}(\cdot, 0) \rangle_{H^{-1, H}} - \langle \hat{p}(\cdot, T), \bar{V}(\cdot, T) \rangle_{Z^{-2}_T, Z^0_T} \]
\[ = -\int_0^T [\langle \hat{p}'(\cdot, t), \bar{V}'(\cdot, t) \rangle_{Z^0_T} + \langle \hat{p}'(\cdot, t), \bar{V}(\cdot, t) \rangle_{Z^{-2}_T, Z^0_T}]dt \]
\[ = -\int_0^T [\langle -\bar{A}\bar{V}(\cdot, t) - \xi, \hat{p}'(\cdot, t) \rangle_{Z^0_T} + \langle \hat{p}'(\cdot, t), \bar{V}(\cdot, t) \rangle_{Z^{-2}_T, Z^0_T}]dt \]
\[ = (\xi, \hat{p})_{X^0} + \int_0^T [\langle \bar{A}\bar{V}(\cdot, t), \hat{p}(\cdot, t) \rangle_{Z^0_T} - \langle \hat{p}'(\cdot, t), \bar{V}(\cdot, t) \rangle_{Z^{-2}_T, Z^0_T}]dt = (\xi, \bar{p})_{X^0}. \]

Hence \( \hat{p} = \mathcal{L}^*(\bar{u}(\cdot)) \).
Further, we have that
\[
F(a, v_1(\cdot)) - F(a, \hat{u}(\cdot)) = E \int_Q (p_i(x, t) \varphi(x, v_1(x, t), Z(t), t) - \hat{p}(x, t) \varphi(x, \hat{u}(x, t), Z(t), t)) dx \, dt
\]
\[
= E \int_Q (\hat{p}(x, t) \varphi(x, v_1(x, t), Z(t), t) - \hat{p}(x, t) \varphi(x, \hat{u}(x, t), Z(t), t)) dx \, dt
\]
\[
+ E \int_Q (p_i(x, t) \varphi(x, v_1(x, t), Z(t), t) - \hat{p}(x, t) \varphi(x, v_1(x, t), Z(t), t)) dx \, dt \to 0 \quad \text{as} \quad i \to +\infty.
\]
Hence \( F(a, \hat{u}(\cdot)) = J \). This proves the existence of an optimal control \( \hat{u}(\cdot) \in U_R \) for problem \( \text{[11.3]} \) and completes the proof of Theorem \( \text{3.1} \). \hfill \square

### 4.3 Proof of Theorem 3.2

Let \( a \in \mathbb{A} \) be such that \( a \) has the probability density function \( \rho \in H^0 \) such that \( \rho(x) > 0 \) for any \( x \in D \). Let \( \hat{u} \in U_R \) be the optimal control that exists by Lemma \( \text{3.1} \). For a given \( \mu \in U_R \) and \( \varepsilon \in [0, 1] \), we consider a family of controls \( u_\varepsilon = u_\varepsilon, \mu \in U_R \) such that \( u_\varepsilon = (1 - \varepsilon)\hat{u} + \varepsilon\mu(\cdot) \) for all \( \varepsilon \in [0, 1] \). Since \( \Delta_R \) is a convex set, we have that \( u_\varepsilon \in U_R \) (\( \forall \varepsilon, \mu \)).

We denote
\[
A(x, t) \triangleq A(x, \hat{u}(x, t), Z(t), t), \quad A_\varepsilon(x, t) \triangleq A(x, u_\varepsilon(x, t), Z(t), t),
\]
\[
V_\varepsilon(x, t) \triangleq L(u_\varepsilon(\cdot))\varphi(\cdot, u_\varepsilon(\cdot), Z(\cdot), \cdot), \quad V_\varepsilon(x, t) \triangleq L(u_\varepsilon(\cdot))\varphi(\cdot, u_\varepsilon(\cdot), Z(\cdot), \cdot),
\]
\[
\hat{\varphi} \triangleq \varphi(\cdot, \hat{u}(\cdot, t), Z(\cdot), \cdot), \quad \varphi_\varepsilon \triangleq \varphi(\cdot, u_\varepsilon(\cdot, t), Z(\cdot), \cdot), \quad \bar{\varphi}_\varepsilon \triangleq \frac{\partial V_\varepsilon}{\partial t} - \hat{A}V_\varepsilon.
\]
Let \( \Phi(u(\cdot)) \triangleq F(a, u(\cdot)) \).

**Proposition 4.2** For any \( \varepsilon \in [0, 1] \), we have that
\[
\Phi(u_\varepsilon(\cdot)) - \Phi(\hat{u}(\cdot)) = E \int_Q \hat{p}(x, t) ((A_\varepsilon(x, t) - \hat{A}(x, t))V_\varepsilon(x, t) + \varphi(x, u_\varepsilon(x, t), Z(t), t) - \varphi(x, \hat{u}(x, t), Z(t), t)) dx \, dt
\]
\[
+ E \zeta_\varepsilon, \quad (4.3)
\]
where
\[
\zeta_\varepsilon(x, t) \triangleq \int_Q \hat{p}(x, t)(A_\varepsilon(x, t) - \hat{A}(x, t))(\hat{V}_\varepsilon(x, t) - \hat{V}(x, t)) dx \, dt. \tag{4.4}
\]

**Proof of Proposition 4.2** It follows from the definitions that
\[
\Phi(\hat{u}(\cdot)) = (\hat{V}(\cdot, 0), \hat{p}(\cdot, 0))_{Z^q} = \int_0^T (\hat{\varphi}, \hat{p})_{Z^q} dt,
\]
\[
\Phi(u_\varepsilon(\cdot)) = (V_\varepsilon(\cdot, 0), p_\varepsilon(\cdot, 0))_{Z^q} = \int_0^T (\varphi_\varepsilon, p_\varepsilon)_{Z^q} dt.
\]
and
\[ \Phi(u_\varepsilon(\cdot)) = (V_\varepsilon(\cdot,0), \hat{p}(\cdot,0))_{\mathbb{Z}_T^0} = \int_0^T (\bar{\varphi}_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} \, dt = \int_0^T (\varphi_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} \, dt + \int_0^T ((A_\varepsilon - \hat{A})\hat{V}_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} \, dt, \]

since, by the definitions,
\[ \varphi_\varepsilon = -\frac{\partial \hat{V}_\varepsilon}{\partial t} - A_\varepsilon \hat{V}_\varepsilon = \bar{\varphi}_\varepsilon - (A_\varepsilon - \hat{A})\hat{V}_\varepsilon. \]

Using that \( \hat{p} \in X^0 \), we obtain that
\[ \Phi(u_\varepsilon(\cdot)) = \int_0^T (\varphi_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} \, dt + \int_0^T ((A_\varepsilon - \hat{A})\hat{V}_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} \, dt. \]

Hence
\[ \Phi(u_\varepsilon(\cdot)) - \Phi(\hat{u}(\cdot)) = \int_0^T \left\{ (A_\varepsilon \hat{V}, \hat{p})_{\mathbb{Z}_T^0} - (\hat{A}\hat{V}, \hat{p})_{\mathbb{Z}_T^0} + (\varphi_\varepsilon - \hat{\varphi}, \hat{p})_{\mathbb{Z}_T^0} \right\} \, dt + \mathbb{E} \xi_\varepsilon, \]

since
\[ \xi_\varepsilon = \int_0^T \left\{ ((A_\varepsilon - \hat{A})(V_\varepsilon - \hat{V}), \hat{p})_{\mathbb{Z}_T^0} \right\} \, dt \]
\[ = \int_0^T \left\{ (A_\varepsilon V_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} - (\hat{A}V_\varepsilon, \hat{p})_{\mathbb{Z}_T^0} - (A_\varepsilon \hat{V}, \hat{p})_{\mathbb{Z}_T^0} - (\hat{A}\hat{V}, \hat{p})_{\mathbb{Z}_T^0} \right\} \, dt. \]

**Proposition 4.3** There exists a limit
\[
\lim_{\varepsilon \to 0} \frac{\Phi(u_\varepsilon(\cdot)) - \Phi(\hat{u}(\cdot))}{\varepsilon} = \mathbb{E} \int_Q \hat{p}(x,t) \left\{ A(x, \mu(x,t), Z(t), t)\hat{V}(x,t) + \varphi(x, \mu(x,t), Z(t), t) \right. \]
\[ \left. - A(x, \hat{u}(x,t), t)\hat{V}(x,t) - \varphi(x, \hat{u}(x,t), Z(t), t) \right\} dx \, dt. \] (4.5)

**Proof of Proposition 4.3.** By the choice of the a family of controls, the first integral in the right hand size of (4.3) coincides with the right hand side of (4.5) multiplied by \( \varepsilon \), for any \( \varepsilon \in (0, 1] \).

Let \( W_\varepsilon = \hat{V}_\varepsilon - \hat{V} \). The lemma will be proved if we show that
\[ J_\varepsilon = \varepsilon^{-1} \mathbb{E} \int_Q \hat{p}(x,t) \left( A_\varepsilon(x,t) - \hat{A}(x,t) \right) W_\varepsilon(x,t) \, dx \, dt \to 0 \text{ as } \varepsilon \to 0. \]

Let
\[ \tilde{b}_\varepsilon(x,t) = b(x, u_\varepsilon(x,t), Z(t), t) - b(x, \hat{u}(x,t), Z(t), t) = \varepsilon [b(x, \mu(x,t), t) - b(x, \hat{u}(x,t), Z(t), t)], \]
\[ \tilde{f}_\varepsilon(x,t) = f(x, u_\varepsilon(x,t), Z(t), t) - f(x, \hat{u}(x,t), Z(t), t) = \varepsilon [f(x, \mu(x,t) - f(x, \hat{u}(x,t), Z(t), t)], \]
\[ \tilde{\varphi}_\varepsilon(x,t) = \varphi(x, u_\varepsilon(x,t), Z(t), t) - \varphi(x, \hat{u}(x,t), Z(t), t) = \varepsilon [\varphi(x, \mu(x,t) - \varphi(x, \hat{u}(x,t), Z(t), t)]. \]

The second equalities in the above formulae follow from the choice of the a family of controls.
Let

\[ \xi_\varepsilon = -\hat{\varphi}_\varepsilon(x,t) + \sum_{i,j=1}^{n} b_{ij}(x,t) \cdot \partial^2 \hat{V}(x,t) + \partial \hat{V}(x,t) \hat{f}_\varepsilon(x,t). \]

By the choice of \( u_\varepsilon \), it follows that

\[ \mathbb{E} |\xi_\varepsilon| \leq \text{const} \|W_\varepsilon\|_{X^2}. \]

Further, \( W_\varepsilon \) is such that, in \( Q \),

\[ \frac{\partial W_\varepsilon}{\partial t} + A_\varepsilon W_\varepsilon = \xi_\varepsilon, \quad W_\varepsilon|_{x \in \partial D} = 0, \quad W_\varepsilon|_{t=0} = 0, \]

We have that \( |J_\varepsilon| \leq C \|W_\varepsilon\|_{Y^2} \|\hat{p}\|_{L^2(Q)} \leq C \|\xi_\varepsilon\|_{X^0} \|\rho\|_{H^{-1}} \) for a constant \( C > 0 \). Since \( b, f, \lambda \) are bounded functions, we have that

\[ |\xi_\varepsilon(x,t)| \leq \varepsilon C_1 \left( \sum_{i,j=1}^{n} \left| \frac{\partial^2 \hat{V}}{\partial x_i \partial x_j} \right| + \left| \frac{\partial \hat{V}}{\partial x}(x,t) \right| + \sup_{v \in \Delta} |\varphi(x,v,t)| \right), \]

where \( C_1 > 0 \).

From the choice of the controls \( u_\varepsilon(\cdot) \), we obtain that \( \|\xi_\varepsilon\|_{X^0} \leq \varepsilon \|\hat{V}\|_{Y^2} \). This completes the proof of Proposition 4.3.

We are now in the position to prove Theorem 3.2(i).

Let \( \Xi \) be the set of \((x,t)\) that are Lebesgue points of

\[ \hat{p}(x,t)[A(x,\hat{u}(x,t),Z(t),t)\hat{V}(x,t) + \varphi(x,\hat{u}(x,t),Z(t),t)]. \]

Clearly, \( \text{mes}\{(\Theta \times [0,T])\setminus\Xi\} = 0 \). By the continuity \( b(x,v,t), f(x,v,t), \) and \( \varphi(x,v,t) \), by Luzin Theorem from Shilov and Gurevich (2012), p.87, we obtain that for all \( v \in \Delta \) the vectors \((x,t)\in\Xi\) are Lebesgue points of \( \hat{p}(x,t)(A(x,v,t)\hat{V}(x,t) + \varphi(x,v,t)). \)

Further, let us consider \( \mu(\cdot) \in U_R \) such that \( \mu(x,t) = \hat{u}(x,t) \) for \((x,t) \notin B, \mu(x,t) = v \) for \((x,t) \in B \), where \( v \in \Delta \) represent Dirac measure, \( B \subset \Theta \times (0,T) \) are arbitrary domains such that they form a Vitali system of sets properly shrinking in the sense of definition from Shilov and Gurevich (2012) to each point \((x,t)\in\Xi\). Hence

\[ A(x,\hat{u}(x,t),t)\hat{V}(x,t) + \varphi(x,\hat{u}(x,t),Z(t),t) \leq A(x,v,t)\hat{V}(x,t) + \varphi(x,v,Z(t),t) \quad \text{a.e. } \forall v \in \Delta_R. \]

Then the statement of the proof of Theorem 3.2(i) for \( \hat{u} \in U_R \) follows from Proposition 4.3. To complete the proof of Theorem 3.2(i), we need to show that there exists \( \hat{u} \in U_0 \) with the required properties.
Proposition 4.4 There exists $\tilde{u}(\cdot) \in U_0$ such that, for a.e. $x, t$

$$A(x, \tilde{u}(x, t), Z(t), t)\hat{V}(x, t) + \varphi(x, \tilde{u}(x, t), Z(t), t) \leq A(x, v, Z(t), t)\hat{V}(x, t) + \varphi(x, v, Z(t), t) \quad \forall v \in \Delta.$$  

and

$$\hat{V} = L(\tilde{u}(\cdot))\varphi(\cdot, \tilde{u}(\cdot), Z(\cdot), \cdot) = L(\tilde{u}(\cdot))\varphi(\cdot, \tilde{u}(\cdot), Z(\cdot), \cdot), \quad (4.6)$$

where $\tilde{u} \in U_R$ is an optimal control for described in the proof above.

Proof of Proposition 4.4 Let $R \overset{\Delta}{=} D \times \Delta \times [0, T]$, and let $z(x, v, t) \overset{\Delta}{=} A(x, v, t)\hat{V}(x, t) + \varphi(x, v, t)$. Let $R' \subset R$ be such that $\text{mes}\{R \setminus R'\} = 0$, and $R' = \bigcup_{k=1}^{+\infty} R_k$, where $R_k = R_k(\omega)$ are random $\mathcal{F}_T^\omega$-measurable compact sets defined a.s. such that the function $z(x, v, t)$ is continuous on $R_k$. (These $R_k = R_k(\omega)$ exist by the Luzin Theorem from Shilov and Gurevich (2012), p.87. Let

$$S_k \overset{\Delta}{=} \{(x, \tilde{v}, t) \in R_k : z(x, \tilde{v}, t) = \inf_{v \in \Delta} z(x, v, t)\}, \quad S = \bigcup_{k=1}^{+\infty} S_k.$$  

The set $S$ is $\sigma$-compact. By Lemma B from Fleming and Rishel (1975), p.277, there exists a desired function $\tilde{u}(\cdot) \in U_0$ such that $(x, \tilde{u}(x, t), t) \in S$ a.s. for a.e. $x, t$. In particular, this means that

$$A(x, \tilde{u}(x, t), t)\hat{V}(x, t) + \varphi(x, \tilde{u}(x, t), Z(t), t) = A(x, \tilde{u}(x, t), t)\hat{V}(x, t) + \varphi(x, \tilde{u}(x, t), Z(t), t) \quad \text{a.e.}$$

hence (4.6) holds. This completes the proof of Proposition 4.4. \hfill $\square$

The proof of Theorem 3.2 (i) follows from Proposition 4.4.

Let us prove Theorem 3.2 (ii). Let $\tilde{a} \in \mathfrak{A}$, and let $\tilde{a} \in U_0$ be an optimal control for corresponding problem (1.3) that exists according to Theorem 3.2 (i). Let $\hat{V} = L(\tilde{u}(\cdot))\varphi(\cdot, \tilde{u}(\cdot), Z(\cdot), \cdot)$.

Further, let $a \in \mathfrak{A}$ be such that its probability density $\rho \in H^0$. Let $u \in U_R$ be any. It follows from the definitions that

$$A(x, \tilde{u}(x, t), Z(t), t)\hat{V}(x, t) + \varphi(x, \tilde{u}(x, t), Z(t), t) = A(x, u(x, t), Z(t), t)\hat{V}(x, t) + \varphi(x, u(x, t), Z(t), t) + \psi(x, t),$$

where

$$\psi(x, t) \overset{\Delta}{=} A(x, \tilde{u}(x, t), Z(t), t)\hat{V}(x, t) + \varphi(x, \tilde{u}(x, t), Z(t), t) - A(x, a, Z(t), t)\hat{V}(x, t) - \varphi(x, a, Z(t), t).$$

Hence $\hat{V} = L(u(\cdot))[\varphi(\cdot, u(\cdot), \cdot) + \psi]$. By the definition of the operator $L * (u(\cdot))$, we have that

$$(\hat{V}(\cdot, 0), \rho)_{Z^0} = E \int_0^T (\varphi(\cdot, \tilde{u}(\cdot), Z(\cdot), t), p_n(\cdot, t))_{Z^0} dt + \bar{R},$$
where \( p_u \triangleq L(u(\cdot))^* \rho, \)

\[
\bar{R} \triangleq \mathbb{E} \int_0^T (\psi(\cdot, t), p_u(\cdot, t))_{Z_t^0} dt.
\]

On the other hand, it follows from the definitions that

\[
(\hat{V}(\cdot, 0), \rho)_{Z^0_p} = F(a, \tilde{u}(\cdot)), \quad \mathbb{E} \int_0^T (\varphi(\cdot, \tilde{u}(\cdot), t), p_u(\cdot, t))_{Z_t^0} dt = F(a, u(\cdot)).
\]

By the choice of \( p_u, \tilde{u}, \) and \( \hat{V}, \)

\[
p_u(x, t, \omega) \geq 0, \quad \psi(x, t, \omega) \leq 0 \quad \text{a.e.}
\]

Hence \( \bar{R} \leq 0. \) This proves the statement of Theorem 3.2(ii) and completes the proof of Theorem 3.2. □

5 Discussion and further research

Similarly to the first order SPDEs introduced in Bender and Dokuchaev (2016a,b), equation (3.1) is an analog of the HJB equation. In our case, it has some special features. For example, solutions of (3.1) are not necessarily continuous in \( t \) since the filtration \( \mathcal{F}_t^Z \) can be discontinuous for the case where \( Z(t) \) is discontinuous.

The proofs in the present paper are different from the proofs from Bender and Dokuchaev (2016a,b) and from the proofs from Dokuchaev (2017); the present proof since it uses the regularity properties of non-degenerate parabolic equations and does use the time discretisation implemented in the cited papers. Therefore, the proofs in the present are more straightforward. However, it is unlikely that this proof can be extended on the special control problems considered in Bender and Dokuchaev (2016a,b) and Dokuchaev (2017); these problems can be regarded as degenerate problems in domains with boundaries. The regularity of corresponding value functions in these papers was not covered by the existing literature; it was analyzed directly using the time discretisation.

Alternatively to solution of equation (3.1), the value function \( \hat{V} \) can be estimated via Monte-Carlo method combined with the dual pathwise optimization method, similarly to Section 7 in Bender and Dokuchaev (2016a) or Section 4 in Dokuchaev (2017). In this case, equation (3.1) can be useful of calculation of the optimal control as the process where the minimum in Proposition 4.4 is achieved for optimal \( \hat{V}. \)

The present paper considers only the case where the value function has \( L_2 \)-integrable second order derivatives in \( x. \) It could be interesting to extend the results of this this paper on more
general class diffusion coefficients. This may require to consider viscosity solutions of parabolic equations with measure-valued derivatives.

References

[1] E. Alós, J. A. León, D. Nualart. (1999). Stochastic heat equation with random coefficients. *Probability Theory and Related Fields*, **115** (1), 41–94.

[2] V. Bally, I. Gyongy, E. Pardoux. (1994). White noise driven parabolic SPDEs with measurable drift, *Journal of Functional Analysis* **120**, 484–510.

[3] C. Bender and N. Dokuchaev. (2016a). A first-order BSPDE for swing option pricing. *Mathematical Finance* **26** (3), 461–491.

[4] C. Bender and N. Dokuchaev. (2016b). A first-order BSPDE for swing option pricing: Classical solutions. *Mathematical Finance*, online published in 2015, in press.

[5] H.O. Cordes. (1956). Über die erste Randwertaufgabe bei quasilinearan Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen, Math. Ann. 131, 278-312.

[6] T. Caraballo, P. E. Kloeden, B. Schmalfuss. (2004). Exponentially stable stationary solutions for stochastic evolution equations and their perturbation, *Appl. Math. Optim.* **50**, 183–207.

[7] A. Chojnowska-Michalik, B. Goldys. (1995). Existence, uniqueness and invariant measures for stochastic semilinear equations in Hilbert spaces, *Probability Theory and Related Fields*, **102**(3), 331–356.

[8] M. Crandall and P. L. Lions. (1983a). Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc. 277, 1-42

[9] G. Da Prato, L. Tubaro. (1996). Fully nonlinear stochastic partial differential equations, *SIAM Journal on Mathematical Analysis* **27**, No. 1, 40–55.

[10] N.G. Dokuchaev. (1997). Cordes conditions and some alternatives for parabolic equations and discontinuous diffusion. *Differential equations* **33**, N 4, 433-442.

[11] Dokuchaev, N. (2005). Parabolic Ito equations and second fundamental inequality. *Stochastics* **77**, iss. 4., pp. 349-370.
[12] N. Dokuchaev. (2018). On degenerate backward SPDEs in bounded domains under non-local conditions. *Stochastics* 90 (8), 1170-1189.

[13] K. Du, S. Tang. (2012). Strong solution of backward stochastic partial differential equations in $C^2$ domains. *Probability Theory and Related Fields*, 154, 255–285.

[14] K. Du, S. Tang, Q. Zhang. (2013). $W^{m,p}$-solution ($p \geq 2$) of linear degenerate backward stochastic partial differential equations in the whole space. *Journal of Differential Equations* 254 (7), 2877–2904.

[15] K. Du, Q. Zhang. (2013). Semi-linear degenerate backward stochastic partial differential equations and associated forward–backward stochastic differential equations *Stochastic Processes and their Applications* 123 (5), 1616–1637.

[16] J. Duan, K. Lu, B. Schmalfuss. (2003). Invariant manifolds for stochastic partial differential equations, *Ann. Probab.*, 31, 2109–2135.

[17] C. Feng, H. Zhao. (2012). Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding. *Journal of Functional Analysis* 262, 4377–4422.

[18] W.H. Fleming, R.W. Rishel. (1975). Deterministic and Stochastic Optimal Control. New York-Heidelberg-Berlin. Springer-Verlag.

[19] W. Fleming and H. M. Soner. (1993). Controlled Markov Processes and Viscosity Solutions, Springer, Berlin-Heidelberg-New York.

[20] I.I. Gihman and A.V. Skorohod. (1975). *The Theory of Stochastic Processes*. Vol. 2. Springer-Verlag, New York.

[21] I. I. Gikhman, T.M. Mestechkina. (1983). The Cauchy problem for stochastic first-order partial differential equations. Theory of Random Processes 11, 25–28.

[22] I. Gyöngy. (1998). Existence and uniqueness results for semilinear stochastic partial differential equations. *Stochastic Processes and their Applications*, 73 (2), 271–299.

[23] Y. Hu, S. Peng. (1995). Solution of forward-backward stochastic differential equations. *Probability Theory and Related Fields* 103, 273–283.

[24] Y. Hu, J. Ma, J. Yong. (2002). On semi-linear degenerate backward stochastic partial differential equations, *Probab. Theory Related Fields* 123 (3), 381–411.
[25] E.A. Kalita. (1989). Regularity of solutions of Cordes-type elliptic systems of any order. *Doklady Acad. Sci. Ukr. SSR, A*, 5, 12-15.

[26] A.I. Koshelev. (1982). On exact conditions of regularity of solutions of for elliptic systems and Liouville theorem. *Dokl. Akad. Nauk. SSSR*, 265, Iss.6, 1309-1311.

[27] N.V. Krylov. (1980). *Controlled diffusion processes*. New York, USA: Springer.

[28] N.V. Krylov. (1987). *Nonlinear Elliptic and Parabolic Equations of the Second Order*. Springer, Netherlands.

[29] H. Kunita. (1990) *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, Cambridge.

[30] O.A. Ladyzhenskaia. (1985). *The Boundary Value Problems of Mathematical Physics*. New York: Springer-Verlag.

[31] E.M. Landis. (1998). *Second Order equations of elliptic and parabolic Type*, vol. 171, Amer. Math. Soc., Providence, R.I., English transl. in Translations of Math. Monographs.

[32] P.L. Lions. (1982). Generalized solutions of Hamilton-Jacobi equations. Research Notes in Mathematics, Vol. 69, Pitman Advanced Publishing Program, Boston, 317 pp.

[33] Y. Liu, H.Z. Zhao. (2009). Representation of pathwise stationary solutions of stochastic Burgers equations, *Stochastics and Dynamics* 9 (4), 613–634.

[34] J. Ma, J. Yong, J. (1999). On linear, degenerate backward stochastic partial differential equations. *Probability Theory and Related Fields* 113 (2) (1999), 135–170.

[35] B. Maslowski. (1995). Stability of semilinear equations with boundary and pointwise noise, *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* (4), 22, No. 1, 55–93.

[36] J. Mattingly. (1999). Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity, *Comm. Math. Phys.*, 206 (2), 273–288.

[37] S.-E. A. Mohammed, T. Zhang, H. Z. Zhao. (2008). The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations, *Mem. Amer. Math. Soc.* 196 (917), 1-105.

[38] E. Pardoux. (1993). Stochastic partial differential equations, a review, *Bull. Sci. Math.* 117 (1), 29–47.
[39] S. Peng. (1992). Stochastic Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.* 30, 284–304.

[40] B. L. Rozovskii. (1990). *Stochastic Evolution Systems; Linear Theory and Applications to Non-Linear Filtering*, Kluwer Academic Publishers, Dordrecht-Boston-London.

[41] G.E. Shilov and B. L. Gurevich. (2012). Integral, Measure, and Derivative. Dover Books on Mathematics. New York.

[42] G. Talenti. (1965). Sopra una classe di equazioni ellittiche a coefficienti misurabili. *Ann. Math. Pure. Appl.* 69, 285-304.

[43] J. Varga. (1972). Optimal Control for Differential and functional Equations. Academic Press, New York.

[44] J. B. Walsh. (1986). An introduction to stochastic partial differential equations, Lecture Notes in Mathematics, 1180, Springer, New York.

[45] K. Yosida, Functional Analysis (Springer, Berlin Heilderberg New York, 1995.

[46] X. Y. Zhou. (1992). A duality analysis on stochastic partial differential equations, *Journal of Functional Analysis* 103 (2), 275–293.