Bootstrapping Structural Change Tests

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Abstract

This paper analyses the use of bootstrap methods to test for parameter change in linear models estimated via Two Stage Least Squares (2SLS). Two types of test are considered: one where the null hypothesis is of no change and the alternative hypothesis involves discrete change at $k$ unknown break-points in the sample; and a second test where the null hypothesis is that there is discrete parameter change at $l$ break-points in the sample against an alternative in which the parameters change at $l+1$ break-points. In both cases, we consider inferences based on a sup-Wald-type statistic using either the wild recursive bootstrap or the wild fixed bootstrap. We establish the asymptotic validity of these bootstrap tests under a set of general conditions that allow the errors to exhibit conditional and/or unconditional heteroskedasticity, and report results from a simulation study that indicate the tests yield reliable inferences in the sample sizes often encountered in macroeconomics. The analysis covers the cases where the first-stage estimation of 2SLS involves a model whose parameters are either constant or themselves subject to discrete parameter change. If the errors exhibit unconditional heteroskedasticity and/or the reduced form is unstable then the bootstrap methods are particularly attractive because the limiting distributions of the test statistics are not pivotal.

JEL classification: C12, C13, C15, C22

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1 Introduction

Linear models with endogenous regressors are commonly employed in time series econometric analysis. In many cases, the parameters of these models are assumed constant throughout the sample. However, given the span of many economic time series data sets, this assumption may be questionable and a more appropriate specification may involve parameters that change value during the sample period. Such parameter changes could reflect legislative, institutional or technological changes, shifts in governmental and economic policy, political conflicts, or could be due to large macroeconomic shocks such as the oil shocks experienced over the past decades and the productivity slowdown. It is therefore important to test for parameter - or structural - change. Various tests for structural change have been proposed with one difference between them being in the type of structural change against which the tests are designed to have power. In this paper, we focus on the scenario in which the potential structural change consists of discrete changes in the parameter values at unknown points in the sample, known as break - (or change-) points. Within this framework, two types of hypotheses tests are of natural interest: tests of no parameter change against an alternative of change at a fixed number of break-points, and tests of whether the parameters change at \( \ell \) break-points against an alternative that they change at \( \ell + 1 \) points. These hypotheses tests are of interest in their own right, and also because they can form the basis of a sequential testing strategy for estimating the number of parameter break-points, see Bai and Perron (1998).

Hall, Han, and Boldea (2012) (HHB hereafter) propose various statistics for testing these hypotheses in linear models with endogenous regressors based on Two Stage Least Squares (2SLS). Their tests are the natural extensions of the analogous tests for linear models with exogenous regressors estimated via OLS that are introduced in the seminal paper by Bai and Perron (1998). A critical issue in the implementation of these tests in a 2SLS setting is whether or not the reduced form for the endogenous regressors is stable. If it is then, under certain conditions, HHB’s test statistics converge in distribution to the same distributions as their OLS counterparts and are pivotal, see HHB and Perron and Yamamoto (2014). However, if the reduced form itself is unstable and/or there is unconditional heteroskedasticity, then these limiting distributions no longer apply (HHB), and are, in fact, no longer pivotal (Perron and Yamamoto, 2014). This is a severe drawback as in most cases of interest the reduced form is likely to be unstable. This problem has been circumvented in two ways. HHB suggest a testing strategy based on dividing the sample into sub-samples over which the RF is stable but this is inefficient compared to inferences based on the whole sample, and can be infeasible if the sub-samples are small. Perron and Yamamoto (2015) propose using a variant of Hansen (2000)’s fixed regressor bootstrap to calculate the critical values of the test. Their simulation evidence suggests the use of this bootstrap improves the reliability of inferences but they do not establish the asymptotic validity of the method.

In this paper, we explore the use of bootstrap versions of 2SLS-based tests for parameter change in far greater detail than previous studies. We consider inferences based on two different types of bootstrap versions of the structural change tests, provide formal proofs of their asymptotic validity and report simulations results that

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1 For example, Brady (2008) examines consumption smoothing using by regressing consumption growth on consumer credit, the latter being endogenous because it depends on liquidity constraints. Zhang, Osborn, and Kim (2008), Kleibergen and Mavroeidis (2009), Hall, Han, and Boldea (2012) and Kim, Manopimoke, and Nelson (2014) investigate the New Keynesian Phillips curve, where inflation is driven by expected inflation and marginal costs, both endogenous since they are correlated with inflation surprises. Bunzel and Enders (2010) and Gian and Su (2014) estimate the forward-looking Taylor rule, a model where the Federal fund rate is set based on expected inflation and output, both endogenous as they depend either on forecast errors or on current macroeconomic shocks. All these studies test for structural change in their estimated equations as part of their analysis.

2 Perron and Yamamoto (2015) propose an alternative approach based on OLS.

3 An alternative approach is to estimate the number and location of the breaks via an information criteria, see Hall, Osborn, and Sakkas (2015). However, this approach has the drawback that inferences can be sensitive to the choice of penalty function.
demonstrate that the bootstrap tests provide reliable inferences in the finite sample sizes encountered in practice. More specifically, we consider the case where the right-hand side variables of the equation of interest contains endogenous regressors, contemporaneously exogenous variables, lagged values of both and lagged values of the dependent variable. This equation of interest is part of a system of equations that is completed by the reduced form for the endogenous regressors and equations for the contemporaneously exogenous variables. This system of equations is assumed to follow a Structural Vector Autoregressive (SVAR) model in which the parameters of the mean are subject to discrete shifts at a finite number of break-points in the sample. Both the number and location of the break-points are unknown to the researcher. These break-points define regimes over which the parameters are constant, and it is assumed that the implied reduced form VAR is stable within each such regime. The errors of the VAR are assumed to follow a vector martingale difference sequence that potentially exhibits both conditional and unconditional heteroskedasticity. Given this error structure, we explore methods for inference based on the wild bootstrap proposed by Liu (1988) because it has been found to replicate the conditional and unconditional heteroskedasticity of the errors in other contexts. In particular, we consider two versions of the wild bootstrap: the wild recursive bootstrap (which generates recursively the bootstrap observations) and the wild fixed-regressor bootstrap (which adds the wild bootstrap residuals to the estimated conditional mean, thus keeping all lagged regressors fixed). These bootstraps have been proposed by Gonçalves and Kilian (2004) to test the significance of parameters in autoregressions with (stationary) conditional heteroskedastic errors. Our primary focus is on bootstrap versions of sup-\textit{Wald} -type statistics to test for structural changes in the parameters of the equation of interest (with endogenous variables) estimated by 2SLS, but our validity arguments also extend straightforwardly to analogous sup-\textit{F}-type statistics. While our primary focus is on models where the reduced form for the endogenous regressors is unstable, our results also cover the case where this reduced form is stable. In the latter case, the test statistics have a pivotal limiting distribution under conditions covered by our framework, specialized to errors that are unconditionally homoskedastic. For these situations, the bootstrap methods we propose are expected to provide a superior approximation to finite sample behaviour compared to the limiting distribution because the bootstrap, by its nature, incorporates sample information. Thus bootstrap versions of the tests are attractive in this setting as well.

In the case where there are no endogenous regressors in the equation of interest, our framework reduces to a linear model estimated by Ordinary Least Squares (OLS). For this set-up, Hansen (2000) proposes the wild fixed-design bootstrap to test for structural changes using a sup-\textit{F} statistic. Very recently, Georgiev, Harvey, Leybourne, and Taylor (2018) (GHLT, hereafter) consider Hansen (2000)’s bootstrap for versions of sup-\textit{F} type tests for parameter variation in predictive regressions with exogenous regressors. Both Hansen (2000) and GHLT establish the asymptotic validity of this bootstrap within the settings they consider. There are some similarities and important differences between our framework (specialized to the no endogenous regressor case) and those in Hansen (2000) and GHLT. We adopt similar assumptions about the error process to GHLT and like both Hansen (2000) and GHLT consider fixed regressor bootstrap tests of a null of constant parameters versus an alternative of parameter change. Important differences include: GHLT allow for strongly persistent variables whereas our framework assumes the system is stable within (suitably defined) regimes; our analysis covers tests for additional breaks in the model, the use of the recursive bootstrap and also inferences based on

\footnote{In fact, GHLT demonstrate that Hansen (2000)’s proof of the asymptotic validity of the bootstrap needs an amendment when the predictive regressors are (near-) unit root processes.}
sup-Wald tests. Thus our results for this case complement those of Hansen (2000) and GHLT.5

Although the frameworks are different, Hansen (2000), GHLT and our own study all find their bootstrap versions of the structural change tests work well in finite samples. Interestingly, Chang and Perron (2018) find that bootstrap-based inferences about the location of breaks have similar advantages in finite samples.6 Collectively, our paper and these other recent studies suggest the use of the bootstrap can yield reliable inferences in linear models with multiple break-points in the sample sizes encountered in practice.

An outline of the paper is as follows. Section 2 lays out the model, test statistics and their bootstrap versions. Section 3 details the assumptions and contains theoretical results establishing the asymptotic validity of the bootstrap tests. Section 4 contains simulation results that provide evidence on the finite sample performance of the structural change tests. Section 5 concludes. Appendix A contains all the tables for Section 4, with additional simulations relegated to a Supplementary Appendix. Appendix B contains the proofs, with some background bootstrap methods. Section 4 contains simulation results that provide evidence on the finite sample performance of the structural change tests. Section 5 concludes. Appendix A contains all the tables for Section 4, with additional simulations relegated to a Supplementary Appendix. Appendix B contains the proofs, with some background

Notation: Matrices and vectors are denoted with bold symbols, and scalars are not. Define for a scalar $N$, the generalized vec operator $\text{vect}_{s=1:N}(A_s) = (A'_1, \ldots, A'_N)'$, stacking in order the matrices $A_s$, $(s=1, \ldots, N)$, which have the same number of columns. Let $\text{diag}_{s=1:N}(A_s) = \text{diag}(A_1, \ldots, A_N)$ be the matrix that puts the blocks $A_1, \ldots, A_N$ on the diagonal. If it is clear over which set vect and diag operations are taken, then the subscript $s=1:N$ is dropped on these operators. If $N$ is the number of breaks in a quantity, $T_1, \ldots, T_N$ are the ordered candidate change-points and $T$ the number of time series observations, where $\tau_0 = 0$, $\tau_{N+1} = 1$, and where $\tau_N = (\tau_0, \text{vect}_{s=1:N}(\tau_s)'(\tau_{N+1})')$ is a partition of the time interval $[1, T]$ divided by $T$, such that $[T_{\tau_s}] = T_s, \tau_N$ for $s = 1, \ldots, N + 1$. Define the regimes where parameters are assumed constant as $I_{s, \tau_N} = [T_{s-1} + 1, T_s]$ for $s = 1, \ldots, N$. Below the breaks in the structural equation are denoted by $\tau_N = \lambda_m$, and those in the reduced form by $\tau_N = \pi_h$, where $m$ an $h$ are the number of breaks in each equation. A superscript zero on any quantity refers to the true quantity, which is a fixed number, vector or matrix. For any random vector or matrix $Z$, denote by $||Z||$ the Euclidean norm for vectors, or the Frobenius norm for matrices. Finally, $0_a$ and $0_a \times a$ denote, respectively, an $a \times 1$ vector and a $a \times a$ matrix of zeros, and $1_A$ denotes an indicator function that takes the value one if event $A$ occurs.

2 The model and test statistics with their bootstrap versions

This section is divided into three sub-sections. Section 2.1 outlines the model. Section 2.2 outlines the hypotheses of interest and the test statistics. Section 2.3 presents the bootstrap versions of the test statistics.

2.1 The model

Consider the case where the equation of interest takes the form

$$y_t = \begin{cases} w'_i \ \beta_{i(1)}^{(h)} + u_t, & i = 1, \ldots, m + 1, \ \ t \in I_{s,\lambda^m}, \end{cases}$$

5The wild fixed-regressor bootstrap is also included in the recent simulation study exploring the finite sample properties of inference methods about the location of the break-point in models estimated via OLS reported in Chang and Perron (2018).

6Chang and Perron (2018) report results from a comprehensive simulation study that investigates the finite sample methods of various methods for constructing confidence intervals for the break fractions in linear regression models with exogenous regressors. They consider variants of the intervals based on i.i.d., wild and sieve bootstraps.
where \( \mathbf{w}_t = \text{vect}(x_t, z_{1,t}) \), \( z_{1,t} \) includes the intercept, \( r_t \) and lagged values of \( y_t, x_t \), and \( r_t, \beta^{0}_{(i)} \) are parameters in regime \( i \). The key difference between \( x_t \) and \( r_t \) is that \( x_t \) represents the set of explanatory variables which are correlated with \( u_t \), and \( r_t \) represents the set of explanatory variables that are uncorrelated with \( u_t \). We therefore refer to \( x_t \) as the *endogenous* regressors and \( r_t \) as the *contemporaneously exogenous* regressors.\(^7\) Equation (1) can be re-written as:

\[
y_t = x'_t \beta^0_{(i)} + z'_{1,t} \beta^0_{z_t} + u_t = w'_t \beta^0_{i} + u_t,
\]

where \( \beta^0 = \beta^0_{(i)} \) if \( t \in I_i, \xi^0_i, i = 1, \ldots, m \) and similar notation holds for \( \beta_{x,t} \) and \( \beta_{z,t} \). For simplicity, we refer to (1) as the “structural equation” (SE).

The SE is assumed to be part of a system that is completed by the following equations for \( x_t \) and \( r_t \). The reduced form (RF) equation for the endogenous regressors \( x_t \) is a regression model with \( h \) breaks \((h + 1 \text{ regimes})\), that is:

\[
x'_t = z'_t \Delta^0_{(i)} + v'_t, \quad j = 1, \ldots, h + 1, \quad t \in I_i, \pi^0_i.
\]

The vector \( z_t \) includes the constant, \( r_t \) and lagged values of \( y_t, x_t \) and \( r_t \). It is assumed that the variables in \( z_{1,t} \) are a strict subset of those in \( z_t \) and therefore we write \( z_t = \text{vect}(z_{1,t}, z_{2,t}) \). Equation (2) can also be rewritten as:

\[
x'_t = z'_t \Delta^0_{(i)} + v'_t,
\]

where \( \Delta^0 = \Delta^0_{(i)} \) if \( t \in I_i, \pi^0_i, i = 1, \ldots, h + 1 \). The contemporaneously exogenous variables \( r_t \) are assumed to be generated as follows,

\[
r'_t = z'_{3,t} \Phi^0_{(i)} + \zeta'_{t}, \quad i = 1, \ldots, d + 1, \quad t \in I_i, \omega^0_i,
\]

where \( z_{3,t} \) includes the constant and lagged values of \( r_t, y_t \) and \( x_t \).

Equations (1), (2) and (3) imply \( \tilde{z}_t = \text{vect}(y_t, x_t, r_t) \) evolves over time via a SVAR process whose parameters are subject to discrete shifts at unknown points in the sample. To present the reduced form VAR version of the model, define \( n = \text{dim}(\tilde{z}_t) \) and let \( \tau_N \) denote the partition of the sample such that all three equations have constant parameters within the associated regimes.\(^8\) We can then write equations (1), (2) and (3) as:

\[
\tilde{z}_t = c_{\tilde{z}_s} + \sum_{i=1}^{p} C_{i,s} \tilde{z}_{t-i} + e_t, \quad \begin{bmatrix} \tau_{s-1} T \end{bmatrix} + 1 \leq t \leq \begin{bmatrix} \tau_s T \end{bmatrix}, \quad s = 1, 2, \ldots, N + 1,
\]

where \( e_t = A^{-1}_s e_t, \)

\[
A_s = \begin{bmatrix}
1 & -\beta^0_{r,s} & -\beta^0_{y,s} \\
0 & I_{p_1} & \Delta^0_{r,s} \\
0 & 0 & I_{p_2}
\end{bmatrix}, \quad (5)
\]

\( \beta^0_{r,s} \) denotes the sub-vector of \( \beta^0 \) that contain the coefficients on \( r_t \) in (1) \( (\beta^0_{r,s} \text{ and } \beta^0_{y,s} \text{ are the values of } \beta^0_{r,t}) \)

\(^7\)This terminology is taken from Wooldridge (1994)[p.349] and reflects that fact \( r_t \) may be correlated with \( u_{ts} \) for \( t \neq s. \)

\(^8\)For example, suppose \( m = 1, h = 2 \) and \( d = 1 \) with \( \lambda^0_i = [0, 0.5, 1], \pi^0_i = [0, 0.3, 0.5, 1] \) and \( \omega^0_i = [0, 0.7, 1] \), then \( N = 3 \) and \( \tau_N = [0, 0.3, 0.5, 0.7, 1] \).
and $\beta_0^u$ for $[\tau_{s-1}T] + 1 \leq t \leq [\tau_sT])$: $\Delta_{r,s}^0$ denotes the sub-matrix of $\Delta_s^0$ that contains the coefficients on $r_t$ in (2) ($\Delta_{r,s}^0$ and $\Delta_s^0$ are the values of $\Delta_{r,t}^0$ and $\Delta_t^0$ for $[\tau_{s-1}T] + 1 \leq t \leq [\tau_sT]$), and $\epsilon_t = \text{vect}(u_t, v_t, \zeta_t)$. For ease of notation, we assume the order of the VAR is the same in each regime, but our results easily extend to the case where the order varies by regime.

### 2.2 Testing parameter variation

As stated in the introduction, this paper focuses on the issue of testing for structural change in the SE. Within the model described above, there are two types of test that are of particular interest. The first tests the null hypothesis of no parameter change against the alternative of a fixed number of parameter changes in the sample that is, a test of $H_0 : m = 0$ versus $H_1 : m = k$. The second tests the null of a fixed number of parameter changes against the alternative that there is one more, that is, it tests $H_0 : m = \ell$ versus $H_1 : m = \ell + 1$. We consider appropriate test statistics for each of these scenarios in turn below.

As the tests are based on the Wald principle, calculation of the test statistics here requires 2SLS estimation of the SE under $H_1$. On the first stage, the RF is estimated via least squares methods. If the number and location of the breaks in the RF are known then this estimation is straightforward. However, in general, neither the number or location of the breaks is known and so they must be estimated. For our purposes here, it is important that both $h$ and $\pi_h^0$ are consistently estimated and that $\hat{\pi}_h$, the estimator of $\pi_h^0$, converges sufficiently fast (see Lemma 7 in the Appendix B). These properties can be achieved by estimating the RF either as a system or equation by equation, and using a sequential testing strategy to estimate $h$; see, respectively Qu and Perron (2007) and Bai and Perron (1998). Provided the significance levels of the tests shrink to zero slowly enough, $\hat{h}$ approaches $h$ with probability one as the sample size $T$ grows; e.g. see Bai and Perron (1998)[Proposition 8]. The same consistency result holds if we estimate $h$ via the information criteria; e.g. see Hall, Osborn, and Sakkas (2013). For this reason, in the rest of the theoretical analysis, we treat $h$ as known. However, we explore the potential sensitivity of the finite sample performance of the tests for structural change in the SE to estimation of $h$ in our simulation study. Let $\hat{\Delta}_{(j)}$ be the estimator of $\Delta_{(j)}^{0}$, $\hat{\Delta}_j = \sum_{j=1}^{h+1} \hat{\Delta}_{(j)} 1_{t \in I_j}$, where $I_j = \{[\hat{\pi}_{j-1}T] + 1, [\hat{\pi}_{j-1}T] + 2, \ldots, [\hat{\pi}_jT]\}$, and $\hat{x}_t = \hat{\Delta}_j z_t'$ that is, $\hat{x}_t$ is the predicted value for $x_t$ from the estimated RF.

**Case (i):** $H_0 : m = 0$ versus $H_1 : m = k$

Under $H_1$, the second stage estimation involves estimation via OLS of the model,

$$y_t = w_i'\beta_{(i)} + \text{error}, \quad i = 1, \ldots, k + 1, \quad t \in I_{i,k},$$

(6)

for all possible $k$-partitions $\lambda_k$. Let $\hat{\beta}_{(j)}$ denote the OLS estimator of $\beta_{(j)}$ in (6), $\hat{\beta}_{\lambda_k}$ denote the OLS estimator of $\text{vect}_{i=1:k+1}(\beta_{(j)})$ in (6) (that is, $\hat{\beta}_{\lambda_k}$ is the OLS estimator of $\text{vect}_{i=1:k+1}(\beta_{(j)})$ based on partition $\lambda_k$).\(^9\) To present the sup-Wald test, we define $\hat{R}_k = \hat{R}_k \otimes I_p$ where $\hat{R}_k$ is the $k \times (k + 1)$ matrix whose $(i,j)^{th}$ element, $\hat{R}_k(i,j)$, is given by: $\hat{R}_k(i,i) = 1$, $\hat{R}_k(i,i+1) = -1$, $\hat{R}_k(i,j) = 0$ for $i = 1, 2, \ldots, k$, and $j \neq i$, $j \neq i + 1$. Also let

\(^9\)Strictly, $\beta_{(j)}$ depends on $\lambda_k$ but we have suppressed this to avoid excessive notation.
\( A_{k} = \{ \lambda_k : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon \}. \) With this notation, the test statistic is:

\[
\text{sup-Wald}_T = \sup_{\lambda_k \in A_{k}} \text{Wald}_{T \lambda_k},
\]

\[
\text{Wald}_{T \lambda_k} = T \beta_{\lambda_k}' R_{\lambda_k}^{-1} \left( R_{\lambda_k} V_{\lambda_k} R_{\lambda_k}^{-1} \right)^{-1} R_{\lambda_k} \beta_{\lambda_k},
\]

where:

\[
\hat{V}_{\lambda_k} = \text{diag}_{i=1:k+1} (\hat{V}_i), \quad \hat{V}_i = \hat{Q}_i^{-1} \hat{M}_i \hat{Q}_i^{-1}, \quad \hat{Q}_i = T^{-1} \sum_{t \in I_{i, \lambda_k}} \hat{w}_t \hat{w}_t', \quad \hat{M}_i \stackrel{P}{\rightarrow} \lim_{T \rightarrow \infty} \text{Var} \left( T^{-1/2} \sum_{t \in I_{i, \lambda_k}} \gamma_t^0 \zeta_t (u_t + v_t^i \beta_{x,i}^0) \right),
\]

and \( \beta_{x,i}^0 \) is the true value of \( \beta_{x,i}^0 \) for \( i = 1, 2, \ldots, m + 1 \) under \( H_0 \).

As mentioned in the introduction, our framework assumes the errors are a m.d.s. that potentially exhibits heteroskedasticity, and so the natural choice of \( \hat{M}_i \) is the Eicker-White estimator, see Eicker (1967) and White (1980). This can be constructed using the estimator of \( \beta_{x,i} \) in (6) under either \( H_0 \) or \( H_1 \), where \( \beta_{x,i} \) are the elements of \( \beta_i \) containing the coefficients on \( \hat{x}_t \). For the purposes of the theory presented below, it does not matter which is used because the null hypothesis is assumed to be true. However, the power properties may be sensitive to this choice. In our simulation study reported below, we use the Eicker-White estimator based on \( \hat{\beta}_{x,i} \), the estimator of \( \beta_{x,i} \) under \( H_1 \), that is,

\[
\hat{M}_i = \hat{E} W \left[ \tilde{y}_t' \zeta_t (\hat{u}_t + \hat{v}_t^i \hat{\beta}_{x,i}) ; I_{i, \lambda_k} \right],
\]

where \( \hat{u}_t = y_t - w_t^i \hat{\beta}_i \) for \( t \in I_{i, \lambda_k} \), \( \hat{v}_t = x_t - \Delta_t \zeta_t', \tilde{y}_t = [\Delta_t, \Pi] \) and, for any vector \( \alpha_t \) and \( I \subseteq \{1, 2, \ldots, T\} \), \( \hat{E} W \left[ \alpha_t ; I \right] = T^{-1} \sum_{t \in I} \alpha_t a_t' \).

**Case (ii):** \( H_0 : m = \ell \) versus \( H_1 : m = \ell + 1 \)

Following the same approach used by Bai and Perron (1998) for OLS based inferences, suitable tests statistics can be constructed as follows. The model with \( \ell \) breaks is estimated via a global minimization of the sum of squared residuals associated with the second stage of the 2SLS estimation of the SE. For each of the \( \ell + 1 \) regimes of this estimated model, the sup-Wald statistic for testing no breaks versus one break is calculated. Inference about \( H_0 : m = \ell \) versus \( H_1 : m = \ell + 1 \) is based on the largest of these \( \ell + 1 \) sup-Wald statistics.

More formally, let the estimated SE break fractions for the \( \ell \)-break model be \( \hat{\lambda}_i \) and the associated break points be denoted \( \{ \hat{T}_i \}_{i=1}^{\ell} \) where \( \hat{T}_i = [T \hat{\lambda}_i] \). Let \( \hat{I}_i = I_{i, \lambda_k} \), the set of observations in the \( i \)th regime of the \( \ell \)-break model and partition this set as \( \hat{I}_i = \hat{I}_i^{(1)} (\varpi_i) \cup \hat{I}_i^{(2)} (\varpi_i) \) where \( \hat{I}_i^{(1)} (\varpi_i) = \{ t : [\hat{\lambda}_{i-1} T] + 1, [\hat{\lambda}_{i-1} T] + 2, \ldots, [\varpi_i T] \} \) and \( \hat{I}_i^{(2)} (\varpi_i) = \{ t : [\varpi_i T] + 1, [\varpi_i T] + 2, \ldots, [\hat{\lambda}_i T] \} \). Consider estimation of the model

\[
y_t = \hat{w}_t^i \beta_{(j)} + \text{error}, \quad j = 1, 2 \quad t \in \hat{I}_i^{(j)},
\]

for all possible choices of \( \varpi_i \) (where for notational brevity we suppress the dependence of \( \beta_{(j)} \) on \( i \)). Let \( \hat{\beta}_i (\varpi_i) = \text{vect} (\hat{\beta}_{(1)} (\varpi_i), \hat{\beta}_{(2)} (\varpi_i)) \) be the OLS estimators of \( \text{vect}(\beta_{(1)}, \beta_{(2)}) \) from (11). Also let \( \mathcal{N}(\hat{\lambda}_i) = [\hat{\lambda}_{i-1} + \epsilon, \hat{\lambda}_i - \epsilon] \).
The sup-Wald statistic for testing $H_0 : m = \ell$ versus $H_1 : m = \ell + 1$ is given by

$$\text{sup-Wald}_T(\ell + 1 | \ell) = \max_{i=1,2,\ldots,\ell+1} \left\{ \sup_{w_i \in N(\lambda_i)} T \hat{\delta}(w_i) \right\}$$

(12)

where

$$
V(w_i) = \text{diag} \left( \hat{V}_1(w_i), \hat{V}_2(w_i) \right), \quad \hat{V}_j(w_i) = (\hat{Q}_j(w_i))^{-1} M_j(w_i) (\hat{Q}_j(w_i))^{-1}, \\
\hat{Q}_j(w_i) = T^{-1} \sum_{t \in I_i} \tilde{w}_t \tilde{w}_t', \quad M_j(w_i) = \mathbb{E} \hat{W} \left[ \hat{Y}_t' \hat{z}_t(\tilde{u}_t + \tilde{v}_t \tilde{\beta}_{x,t}) ; \hat{I}_t^{(j)} \right].
$$

2.3 Bootstrap versions of the test statistics

In this section, we introduce the bootstrap analogues of the test statistics presented in the previous section. As noted above, our framework assumes the error vector $\epsilon_t$ to be a m.d.s that potentially exhibits conditional and unconditional heteroskedasticity, and so we use the wild bootstrap proposed by Liu (1988) because it has been found to replicate the conditional and unconditional heteroskedasticity of the errors in other contexts. We consider both the wild recursive (WR) bootstrap and the wild fixed (WF) bootstrap. These procedures differ in their treatment of the right-hand-side variables in the bootstrap samples as described below.

Generation of the bootstrap samples:

Let $z_t^b = \text{vec}(y_t^b, x_t^b, r_t)$ where $y_t^b$ and $x_t^b$ denote the bootstrap values of $y_t$ and $x_t$. Note that because $r_t$ is contemporaneously exogenous its sample value is used in the bootstrap samples. In all cases below, the bootstrap residuals are obtained as $u_t^b = \hat{u}_t \nu_t$ and $v_t^b = \hat{v}_t \nu_t$, where $\hat{u}_t$ and $\hat{v}_t$ are the (non-centered) residuals under the null hypothesis and $\nu_t$ is a random variable that is discussed further in Section 3.

For the WR bootstrap, $(y_t^b)$ and $(x_t^b)$ are generated recursively as follows:

$$
x_t^{b'} = z_t^{b'} \hat{\Delta}_t + v_t^{b'}, \quad (13)$$

$$
y_t^b = x_t^b \hat{\beta}_{x,t} + z_t^{b'} \hat{\beta}_{x,t} + u_t^b, \quad (14)$$

where the vector $z_t^b$ contains a constant, $r_t$ and lags of $y_t^b$, $x_t^b$ and $r_t$; $\hat{\beta}_{x,t}$ and $\hat{\beta}_{x,t}$ are the sample estimates of $\beta_{x,t}^0$ and $\beta_{x,t}^0$ under $H_0$ of the test in question.

For the WF bootstrap, $z_t$ is kept fixed and, following Gonçalves and Kilian (2004), the bootstrap samples are generated as follows:

$$
x_t^{b'} = z_t^{b'} \hat{\Delta}_t + v_t^{b'}, \quad (15)$$

$$
y_t^b = x_t^b \hat{\beta}_{x,t} + z_t^{b'} \hat{\beta}_{x,t} + u_t^b, \quad (16)$$

where again $\hat{\beta}_{x,t}$ and $\hat{\beta}_{x,t}$ are the sample estimates of $\beta_{x,t}^0$ and $\beta_{x,t}^0$ under $H_0$ of the test in question.

Case (i): $H_0 : m = 0$ vs $H_1 : m = k$

First consider the WR bootstrap. 2SLS estimation is implemented in the bootstrap samples as follows. On the

$^{10}$The comment above about the calculation of $M_{j(i)}$ apply equally to $M_j(w_i)$.

$^{11}$Liu (1988) developed the wild bootstrap has been developed in Liu (1988) following suggestions in Wu (1986) and Beran (1986) in the context of static linear regression models with (unconditionally) heteroskedastic errors.
first stage, the following model is estimated via OLS

\[ x_t^b = z_t^b \Delta_j + \text{error}, \quad t \in I_j^*, \quad j = 1, 2, \ldots, h + 1, \]

to obtain \( \hat{\Delta}_j^b = \left\{ \sum_{t \in I_j^*} z_t^b z_t^b \right\}^{-1} \sum_{t \in I_j^*} z_t^b x_t^b \). Define \( \hat{\Delta}_j^b = \sum_{j=1}^{h+1} \sum_{t \in I_j^*} \hat{x}_t^b = z_t^b \Delta_j^b \), and \( \hat{w}_t^b = \text{vect}(\hat{x}_t^b, z_{1,t}) \).

For a given \( k \)-partition \( \lambda_k \), the second stage of the 2SLS in the bootstrap samples involves OLS estimation of

\[ y_t^b = \hat{w}_t^b \hat{\beta}(i) + \text{error}, \quad i = 1, \ldots, k + 1, \quad t \in I_{\lambda_k}, \quad (17) \]

and let \( \hat{\beta}_{\lambda_k}^b \) be the resulting OLS estimator of \( \text{vect}_{i=1:k+1}(\hat{\beta}(i)) \). The WR bootstrap version of the sup-Wald statistic is:

\[ \sup-Wald^b_{\lambda_k} = \sup_{\lambda_k \in \Lambda_k} Wald^b_{\lambda_k}, \quad (18) \]

\[ Wald^b_{\lambda_k} = T \beta_{\lambda_k}^b R_k (R_k \hat{\lambda}_{\lambda_k} R_k)^{-1} R_k \beta_{\lambda_k}^b, \]

where:

\[ \hat{V}_{\lambda_k}^b = \text{diag}_{i=1:k+1}(\hat{V}_{\lambda_k}^b), \quad \hat{V}_{\lambda_k}^b = (\hat{Q}_{\lambda_k}^b)^{-1} \hat{M}_{\lambda_k}^b (\hat{Q}_{\lambda_k}^b)^{-1}, \quad \hat{Q}_{\lambda_k}^b = T^{-1} \sum_{t \in I_{\lambda_k}} \hat{w}_t^b \hat{w}_t^b, \quad (19) \]

\[ \hat{M}_{\lambda_k}^b = \hat{E}\hat{W} \left[ \hat{T}_t^b z_t^b (u_t^b + v_t^b \hat{\beta}_{\lambda_k}^b); I_{\lambda_k} \right], \quad \hat{T}_t^b = (\hat{\Delta}_j^b, I). \quad (20) \]

Now consider the WF bootstrap, for which \( y_t^b \) and \( x_t^b \) are generated via (15)-(16). The first stage of the 2SLS involves LS estimation of

\[ x_t^b = z_t^b \Delta_j + \text{error}, \quad t \in I_j^*, \quad j = 1, 2, \ldots, h + 1, \]

to obtain \( \hat{\Delta}_j^b = \left\{ \sum_{t \in I_j^*} z_t^b z_t^b \right\}^{-1} \sum_{t \in I_j^*} z_t^b x_t^b \). Now re-define \( \hat{\Delta}_j^b = \sum_{j=1}^{h+1} \sum_{t \in I_j^*} \hat{x}_t^b = z_t^b \Delta_j^b \), and \( \hat{w}_t^b = \text{vect}(\hat{x}_t^b, z_{1,t}) \). For a given \( k \)-partitions \( \lambda_k \), the second stage of the 2SLS in the bootstrap samples involves OLS estimation of (17) and let \( \hat{\beta}_{\lambda_k}^b \) be the resulting OLS estimator of \( \text{vect}_{i=1:k+1}(\hat{\beta}(i)) \). The WF bootstrap sup-Wald statistic is defined as in (18) with \( Wald^b_{\lambda_k} \) defined as in (19) only with \( \hat{w}_t^b \) and \( \hat{\Delta}_j^b \) redefined in the way described in this paragraph, and \( \hat{M}_{\lambda_k}^b \) in (21) replaced by \( \hat{M}_{\lambda_k}^b = \hat{E}\hat{W} \left[ \hat{T}_t^b z_t^b (u_t^b + v_t^b \hat{\beta}_{\lambda_k}^b); I_{\lambda_k} \right] \).

Case (ii): \( H_0 : m = \ell \) versus \( H_1 : m = \ell + 1 \)

For each bootstrap the first stage of the 2SLS estimation and the construction of \( \hat{w}_t \) is the same as described under Case (i) above. Let \( I_j^{(i)} \) be defined as in the discussion of Case (ii) in Section 2.2, and consider

\[ y_t^b = \hat{w}_t^b \hat{\beta}(i) + \text{error}, \quad j = 1, 2 \quad t \in I_j^{(i)}, \quad (22) \]

for all possible choices of \( \omega_i \) (where, once again, we suppress the dependence of \( \hat{\beta}(i) \) on \( i \)). Let \( \hat{\beta}^b(\omega_i) = \text{vect}(\hat{\beta}_{(1)}(\omega_i), \hat{\beta}_{(2)}(\omega_i)) \) be the OLS estimators of \( \text{vect}(\hat{\beta}(1), \hat{\beta}(2)) \) from (22). The bootstrap version of
The asymptotic validity of the bootstrap tests

In this section, we establish the asymptotic validity of the bootstrap versions of the test statistics described above. To this end we impose the following conditions.

Assumption 1. If $m > 0$, $T_i^0 = [T^0]\lambda_i$, where $0 < \lambda_i^0 < \ldots < \lambda_m^0 < 1$.

Assumption 2. If $m > 0$, $\beta_{(i+1)}^0 - \beta_{(i)}^0 \neq 0_{p_1 + q_1}$ is a vector of constants for $i = 1, \ldots, m$.

Assumption 3. If $h > 0$, then $T^*_1 = [T^0]\pi_1^0$, where $0 < \pi_1^0 < \ldots < \pi_k^0 < 1$.

Assumption 4. If $h > 0$, $\Delta_{(i+1)}^0 - \Delta_{(j)}^0 \neq 0_{q \times p_1}$ is a matrix of constants for $j = 1, \ldots, h$.

Assumption 5. If $k > 0$, then $0 < \omega_1^0 < \ldots < \omega_k^0 < 1$ and $\Phi_{(i+1)}^0 - \Phi_{(i)}^0 \neq 0_{q \times p_2}$ is a matrix of constants for $i = 1, \ldots, k$.

Assumption 6. The first and second stage estimations in 2SLS are over respectively all partitions of $\pi$ and $\lambda$ such that $T_1 - T_{i-1} > \max(q - 1, \epsilon T)$ for some $\epsilon > 0$ and $\epsilon < \min_i(\lambda_{i+1}^0 - \lambda_i^0)$ and $\epsilon < \min_j(\pi_{j+1}^0 - \pi_j^0)$.

Assumption 7. (i) $p < \infty$; (ii) $|I_n - C_{1,a} - C_{2,a}a^2 - \cdots - C_{q,a}a^q| \neq 0$, for all $s = 1, \ldots, N + 1, \lambda^0$ and all $|a| \leq 1$.

Assumption 8. $\text{rk}(\mathcal{Y}_n^0) = p_1 + q_1$ where $\mathcal{Y}_n^0 = (\Delta^0_l, \Pi)$ and $\Pi' = (I_{q_1}, 0_{q_1 \times (q - q_1)})$.

Assumption 9. The innovations can be written as $e_t = SD_tI_t$, where:

(i) $S$ is a $n \times n$ lower triangular non-stochastic matrix with real-valued diagonal elements $s_{ii} = 1$ and elements below the diagonal equal to $s_{ij}$ (which are also zero for $i > p_1 + 1, j < p_1 + 1$), such that $SS'$ is positive definite;

(ii) $I_t = \text{vect}(t_{u,t}, t_{w,t}, I_{\zeta,t})$ is a $n \times 1$ vector m.d.s. w.r.t to $\mathcal{F}_t = \{I_t, I_{t-1}, \ldots\}$ to which it is adapted, with conditional covariance matrix $\Sigma_{t|t-1} = E(I_t I_t' \mid \mathcal{F}_{t-1}) = \text{diag}(\Sigma_{t|t-1}^{(1)}, \Sigma_{t|t-1}^{(2)})$ and unconditional variance $E(I_t I_t') = I_n$.

(iii) $\sup_t \text{E}[\|I_t\|^{4+\delta}] < \infty$ for some $\delta > 0$.

(iv) $E((I_t I_t') \otimes I_{t-1}) = \rho_1$, for all $i \geq 0$, with $\sup_{i \geq 0} \|\rho_1\| < \infty$.

(v) $E((I_t I_t') \otimes (I_{t+j} I_{t+j}')) = \rho_{i,j}$, for all $i, j \geq 0$ with $\sup_{i,j \geq 0} \|\rho_{i,j}\| < \infty$. 

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Assumption 9'. Let \( \mathbf{n}_t = \text{vec}(l_{u,t}, l_{v,t}) \). Then:

(i) Assumption 9(iv) holds with \( E[(\mathbf{n}_t \mathbf{n}_t') \otimes \mathbf{n}_{t-1}] = \mathbf{0} \) for all \( i \geq 1 \).

(ii) Assumption 9(v) holds with \( E[(\mathbf{n}_t \mathbf{n}_t') \otimes (\mathbf{n}_{t-j} \mathbf{n}_{t-j}')] = \mathbf{0} \) for all \( i, j \geq 1 \) and \( i \neq j \).

(iii) Assumption 9(v) holds with \( E[(\mathbf{n}_t \mathbf{n}_t') \otimes (\mathbf{n}_{t-j} \mathbf{l}_{t-j})] = \mathbf{0} \) for all \( i \geq 1 \) and \( j \geq 0 \).

Assumption 10. (i) \( \nu_t \overset{IID}{\sim} (0, 1) \) independent of the original data generated by (1), (2) and (3); (ii) \( E^b |\nu_t|^{4+\delta^*} = \bar{c} < \infty \), for some \( \delta^* > 0 \), for all \( t \), where \( E^b \) denotes the expectation under the bootstrap measure.

Before presenting our main theoretical results, we discuss certain aspects of the assumptions.

Remark 1. Assumptions 1-5 indicate that the breaks are “fixed” in the sense that the size of the associated shifts in the parameters between regimes is constant and does not change with the sample size.

Remark 2. It follows from Assumption 7 that \( \bar{z}_t \) follows a finite order VAR in (4) that is stable within each regime.

Remark 3. Assumption 8 is the identification condition for estimation of the structural equation parameters; see HHB for further discussion.

Remark 4. From Assumption 9 it follows that \( \mathbf{e}_t \) is a vector m.d.s. relative to \( F_{t-1} \) with time varying conditional and unconditional variance given by \( E(\mathbf{e}_t \mathbf{e}_t' | F_{t-1}) = SD_t \Sigma_t \mathbf{1}_{t-1} D_t S' \) and \( E(\mathbf{e}_t \mathbf{e}_t') = SD_t D_t S' \) respectively. The m.d.s. property implies that all the dynamic structure in the SE and RF for \( \mathbf{x}_t \) is accounted for by the variables in \( z_{1,t} \) and \( z_t \) respectively. As noted by Boswijk, Cavaliere, Rahbek, and Taylor (2016) and GHLT, Assumption 9 allows for \( \mathbf{e}_t \) to exhibit conditional and unconditional heteroskedasticity of unknown and general form that can include single or multiple variance shifts, variances that follow a broken trend or follow a smooth transition model. When \( D_t = D \), the unconditional variance is constant but we may have conditional heteroskedasticity. When \( \Sigma_t = I_n \), the unconditional variance may still be time-varying. Note that Assumption 9 (i)-(ii) imply that \( \mathbf{x}_t \) is endogenous and \( \mathbf{r}_t \) is contemporaneously exogenous in the SE. Assumption 9(iv) allows for leverage effects (the correlation between the conditional variance and \( l_{t-i} \) is nonzero, when \( i \geq 1 \)). Assumption 9(v) allows for (asymmetric) volatility clustering (the conditional variance is correlated with cross-products \( l_{t-i} l_{t-j} \), for \( i, j \geq 1 \)).\(^{12}\)

Remark 5. Assumption 9 is only imposed in the case of the WR bootstrap. Assumption 9(i)-(iii) is needed because the WR bootstrap sets to zero certain covariance terms in the distribution of the bootstrapped parameter estimates given the data. This happens because these moments depend on products of bootstrap errors at different lags and these terms have zero expectation under the bootstrap measure due to the fact that \( \nu_t \) is mean zero and i.i.d. Assumption 9(i) is a restriction on the leverage effects and Assumption 9(ii) is a restriction of the asymmetric effects allowed in volatility clustering. Note that Assumption 9(i) is only needed when we have an intercept in (4).

Assumption 9(iii) arises because the WR design bootstraps the lags of \( y_t \) and \( \mathbf{x}_t \) in (4), but it does not bootstrap \( \mathbf{r}_t \) and its lags. Therefore, certain fourth cross-moments involving both types of quantities are set to zero by the WR bootstrap, leading to the restriction on clustering effects in Assumption 9(iii) (where \( i = j \) is imposed for replicating certain variances, and \( i \neq j \) is imposed for replicating certain covariances in the asymptotic distribution of the parameter estimates).

\(^{12}\)The clustering is asymmetric if \( \rho_{i,j} \neq 0 \) when \( i \neq j \).
Remark 6. There are several choices for the distribution of $\nu_t$, the random variable used in construction of the bootstrap errors: Gonçalves and Kilian (2004) use the standard normal distribution, while Mammen (1993) and Liu (1988) suggested a two-point distribution. In this paper, we report simulation results for Liu (1988)'s two-point distribution, which we found performed the best compared to the other distributions in simulations not reported here. This conclusion is similar to Davidson and Flachaire (2008) and Davidson and MacKinnon (2010).

The following theorems establish the asymptotic validity of the bootstrap versions of the sup-Wald tests.

Theorem 1. If the WF bootstrap is used let Assumptions 1-10 hold and if the WR bootstrap is used let Assumptions 1-10 and Assumption 9 hold. If $y_t$, $x_t$ and $r_t$ are generated by (1), (2) and (3) and $m = 0$ then it follows that

$$\sup_{c \in \mathbb{R}} \left| P^b \left( \sup \text{Wald}_T \leq c \right) - P \left( \sup \text{Wald}_T \leq c \right) \right| \xrightarrow{P} 0$$

as $T \to \infty$, where $P^b$ denotes the probability measure induced by the bootstrap.

Theorem 2. If the WF bootstrap is used let Assumptions 1-10 hold and if the WR bootstrap is used let Assumptions 1-10 and Assumption 9 hold. If $y_t$, $x_t$ and $r_t$ are generated by (1), (2) and (3) and $m = \ell$ then it follows that:

$$\sup_{c \in \mathbb{R}} \left| P^b \left( \sup \text{Wald}_T^\ell \leq c \right) - P \left( \sup \text{Wald}_T^\ell \leq c \right) \right| \xrightarrow{P} 0$$

as $T \to \infty$, where $P^b$ denotes the probability measure induced by the bootstrap.

Remark 7. The proof rests on showing the sample and bootstrap statistics have the same limiting distribution. Although this distribution is known to be non-pivotal if the RF is unstable (see Perron and Yamamoto, 2014), to our knowledge this distribution has not previously been presented in the literature. A formal characterization of this distribution is provided in the Supplementary Appendix.

Remark 8. Theorems 1-2 cover the case where the reduced form is stable and the errors are unconditionally homoskedastic. In this case, the sup-Wald tests are asymptotically pivotal and so the bootstrap is expected to provide a superior approximation to finite sample behaviour compared to the limiting distribution because the bootstrap, by its nature, incorporates sample information. However, a formal proof is left to future research.

Remark 9. HHB also propose testing the hypotheses described above using sup-F tests. While F-tests are designed for use in regression models with homoskedastic errors,\(^{13}\) wild bootstrap versions of the tests can be used as a basis for inference when the errors exhibit heteroskedasticity. In the Supplementary Appendix, we present WR bootstrap and WF bootstrap versions of appropriate sup-F statistics for testing both $H_0 : m = 0$ versus $H_1 : m = k$ and $H_0 : m = \ell$ versus $H_1 : m = \ell + 1$, and show that these bootstrap versions of the sup-F tests are asymptotically valid under the same conditions as their sup-Wald counterparts. Simulation evidence indicated no systematic difference in the finite sample behaviour of the sup-Wald and sup-F tests for a given null and bootstrap method, and so further details about this approach are relegated to the Supplementary Appendix.

Remark 10. In the special case where there are no endogenous regressors in the equation of interest then our framework reduces to one in which a linear regression model is estimated via OLS. For this set-up, the asymptotic validity of wild fixed bootstrap versions of sup-F test for parameter variation (our Case(i) above)\(^{13}\)If the reduced form is stable then the limiting distribution of the sup-F statistics are only pivotal if the errors are homoskedastic.
has been established under different sets of conditions by Hansen (2000) and GHLT. Hansen (2000) considers the case where the marginal distribution of the exogenous regressors changes during the sample. GHLT consider Hansen (2000)'s bootstrap in the context of predictive regressions with strongly persistent exogenous regressors. Our results complement these earlier studies because we provide results for the wild recursive bootstrap and a theoretical justification for tests of \( \ell \) breaks against \( \ell + 1 \) based on bootstrap methods.

4 Simulation results

In this section, we investigate the finite sample performance of the bootstrap versions of the sup-Wald and sup-\( F \) statistics. We consider a number of designs that involve stability or instability in the SE and/or the RF. In all the designs the variable \( x_t \) is endogenous and the SE is estimated by 2SLS. Recalling from above that \( h \) and \( m \) denote the true number of breaks in the RF and SE respectively, the four scenarios we consider are as follows.

- **Scenario: \((h,m)=(0,0)\)**
  
  The DGP is as follows:

  \[
  x_t = \alpha_x + r'_t \delta_{x}^0 + \delta_{x1,t}^0 x_{t-1} + \delta_{y1,t}^0 y_{t-1} + v_t, \quad \text{for } t = 1, \ldots, T, \tag{24}
  \]

  \[
  y_t = \alpha_y + x_t \beta_x^0 + \beta_{r1,t}^0 r_{1,t} + \beta_{y1,t}^0 y_{t-1} + u_t, \quad \text{for } t = 1, \ldots, T, \tag{25}
  \]

  where the parameters of the SE - see equation (25) - are \( \alpha_y = 0.5, \beta_x^0 = 0.5, \beta_{r1}^0 = 0.5, \beta_{y1}^0 = 0.8 \); the parameters of the RF in equation (24) are \( \alpha_x = 0.5, \delta_{x1}^0 = (1.5, 1.5, 1.5, 1.5)' \) a 4 \times 1 parameter vector, \( \delta_{x1}^0 = 0.5, \delta_{y1}^0 = 0.2 \); \( r_t = (r_{1,t}, r_{2,t})' \).

- **Scenario: \((h,m)=(1,0)\)**
  
  The DGP is as follows:

  \[
  x_t = \alpha_{x(1)} + r'_t \delta_{r1(1)}^0 + \delta_{x1(1)}^0 x_{t-1} + \delta_{y1(1)}^0 y_{t-1} + v_t, \quad \text{for } t = 1, \ldots, [T/4], \tag{26}
  \]

  \[
  = \alpha_{x(2)} + r'_t \delta_{r1(2)}^0 + \delta_{x1(2)}^0 x_{t-1} + \delta_{y1(2)}^0 y_{t-1} + v_t, \quad \text{for } t = [T/4] + 1, \ldots, T, \tag{27}
  \]

  \[
  y_t = \alpha_y + x_t \beta_x^0 + \beta_{r1,t}^0 r_{1,t} + \beta_{y1,t}^0 y_{t-1} + u_t, \quad \text{for } t = 1, \ldots, T, \tag{28}
  \]

  where the parameters of the SE - equation (28) - are the same as in scenario \((h,m)=(0,0)\), and the RF parameters - equations (26)-(27) - are: \( \alpha_{x(1)} = 0.1, \alpha_{x(2)} = 0.5, \delta_{r1(1)}^0 = (0.1, 0.1, 0.1, 0.1)' \), \( \delta_{r1(2)}^0 = (1.5, 1.5, 1.5, 1.5)' \), \( \delta_{x1(1)}^0 = 0.1, \delta_{x1(2)}^0 = 0.5, \delta_{y1(1)}^0 = 0.1 \), and \( \delta_{y1(2)}^0 = 0.2 \). In our simulation study, prior to testing the null hypothesis of zero breaks in the SE parameters from (28), we test sequentially for breaks in the RF parameters (assuming for a maximum of 2 breaks) by applying our bootstrap sup-Wald test.

- **Scenario: \((h,m)=(0,1)\)**
  
  The DGP is as follows:

  \[
  x_t = \alpha_x + r'_t \delta_x^0 + \delta_{x1,t}^0 x_{t-1} + \delta_{y1,t}^0 y_{t-1} + v_t, \quad \text{for } t = 1, \ldots, T, \tag{29}
  \]

  \[
  y_t = \alpha_y + x_t \beta_x^0 + \beta_{r1,t}^0 r_{1,t} + \beta_{y1,t}^0 y_{t-1} + u_t, \quad \text{for } t = 1, \ldots, [3T/4], \tag{30}
  \]

  \[
  = \alpha_y + x_t \beta_x^0 + \beta_{r1,t}^0 r_{1,t} + \beta_{y1,t}^0 y_{t-1} + u_t, \quad \text{for } t = [3T/4] + 1, \ldots, T, \tag{31}
  \]
where the parameter values for the RF equation (29) are as in scenario \((h, m) = (0, 0)\), and the parameters on the SE equations (30)-(31) are: \(\alpha_y(1) = 0.5, \alpha_y(2) = -0.5; \beta^0_{x(1)} = 0.5, \beta^0_{x(2)} = -0.5; \beta^0_{r_1(1)} = 0.5, \beta^0_{r_1(2)} = -0.5, \beta^0_{y_1(1)} = 0.8, \text{ and } \beta^0_{y_1(2)} = 0.1.\)

- **Scenario: \((h, m) = (1, 1)\)**

  The DGP is as follows:

  \[
  \begin{align*}
  x_t &= y_t + r_t \gamma_t + \omega_t + \epsilon_t, & & t = 1, \ldots, [T/4], \\
  y_t &= y_t + r_t \gamma_t + \omega_t + \epsilon_t, & & t = [T/4] + 1, \ldots, T,
  \end{align*}
  \]

  where the parameters of the RF equations (32)-(33) are as in scenario \((h, m) = (1, 0)\) and the parameters in the SE equations (34)-(35) are as in \((h, m) = (0, 1)\). In our simulation study, prior to testing the null hypothesis of zero breaks in the SE parameters from (28), we test sequentially for breaks in the RF parameters (assuming for a maximum of 2 breaks) by applying our bootstrap sup-Wald test.

  For the four scenarios above we consider the following choices for \(u_t, v_t\) and \(r_t\):

  - **Case A:** \(u_t\) and \(v_t \overset{IID}{\sim} N(0, 1), \text{ Cov}(u_t, v_t) = 0.5, t = 1, \ldots, T, \) \(r_t \overset{IID}{\sim} N(0, 1, 1).\)

  - **Case B:** \(u_t\) and \(v_t\) are GARCH(1,1) processes i.e. \(u_t\) and \(v_t\) are i.i.d. \(N(0, 1), \text{ Cov}(u_t, v_t) = 0.5, t = 1, \ldots, T, \) \(r_t \overset{IID}{\sim} N(0, 1, 1).\)

  - **Case C:** \(u_t\) and \(v_t \overset{IID}{\sim} N(0, 1), \text{ Cov}(u_t, v_t) = 0.5, t = 1, \ldots, [T/3]; u_t\) and \(v_t \overset{IID}{\sim} N(0, 2), \text{ Cov}(u_t, v_t) = 0.5, t = [T/3] + 1, \ldots, T.\)

  - **Case D:** \(u_t\) and \(v_t\) are as in Case D and \(r_t \overset{IID}{\sim} N(0, 1, 1)\) for \(t = 1, \ldots, [3T/5]\), and \(r_t \overset{IID}{\sim} N(0, 1, 1.5I_4)\) for \(t = [3T/5] + 1, \ldots, T.\)

  In **Case A**, the errors \(u_t\) and \(v_t\) are homoskedastic and the contemporaneous exogenous regressors \(r_t\) are stable.

  In **Case B**, the errors are conditionally heteroskedastic. In **Case C** the errors have a contemporaneous upward shift in the unconditional variance, while in **Case D** there is also an upward shift in the variance of \(r_t\).

  In our simulations we consider the behavior of the bootstrap tests both under their null and alternative hypotheses. For scenarios \((h, m) = (0, 0)\) and \((h, m) = (1, 0)\) we consider the behavior of the sup-Wald test. For scenarios \((h, m) = (0, 1)\) and \((h, m) = (1, 1)\) we consider the performance of the sup-Wald test. In order to assess the power of our bootstrap tests we also consider the case when the null hypotheses are not true and there is an additional break in the SE parameters at \([T/2]\). More exactly we consider in all the four scenarios described above the following:

  \[
  y_t = (\alpha_y(1) + g) + x_t (\beta^0_{x(1)} + g) + (\beta^0_{r_1(1)} + g) r_1 t + (\beta^0_{y_1(1)} + g) y_{t-1} + u_t, \quad t = [T/2] + 1, \ldots, \tilde{T},
  \]

  with \(g\) a constant; \(i = 1\) and \(\tilde{T} = T\) for scenarios \((h, m) = (0, 0)\) and \((h, m) = (1, 0)\), and the equation for \(y_t\) for \(t < [T/2] + 1\) is the same as that given in the two scenarios \((h, m) = (0, 0)\) and \((h, m) = (1, 0)\); \(i = 2\) and \(\tilde{T} = [3T/4]\) for scenarios \((h, m) = (1, 0)\) and \((h, m) = (1, 1)\), and the equation for \(y_t\) for \(t < [T/2] + 1\) and
of the Supplementary Appendix we also present similar
Liu
14
1
1
32
different values of \( g \): \( g = -0.007, -0.009 \) for scenario \((h, m) = (0, 0)\); \( g = -0.05, -0.07 \) for scenario \((h, m) = (1, 0)\);
\( g = 0.3, 0.4 \) for scenario \((h, m) = (0, 1)\); \( g = -0.5, 0.5 \) for scenario \((h, m) = (1, 1)\).

For scenarios \((h, m) = (1, 0)\) and \((h, m) = (1, 1)\) we have tested for the presence of max 2 breaks in the RF
for \( x_t \) (in (26)-(27) and (32)-(33) respectively) prior to testing for breaks in the SE. More exactly we tested the
null hypothesis \( H_0 : h = \ell \) against \( H_1 : h = \ell + 1, \ell = 0, 1 \) using the WR and WF bootstrap sup-Wald for OLS.
If the bootstrap \( p \)-value (given by the fraction of bootstrap statistics more extreme than the sup-Wald based on
the original sample) was larger than 5%, then we imposed the \( \ell \) breaks (assumed under null \( H_0 : h = \ell \) in the
RF and estimated their locations which were subsequently accounted for in the estimation of the SE.

We now describe other features of the calculations before discussing the results. For the WR and the WF
bootstraps the auxiliary distribution (from Assumption 10) is the Rademacher distribution proposed by Liu
(1988) which assigns 0.5 probability to the value \( \nu_t = -1 \) and 0.5 probability to \( \nu_t = 1 \), \( t = 1, \ldots, T \). The same \( \nu_t \)
is used to obtain both the bootstrap residuals \( u_t^b = \hat{u}_t\nu_t \) and \( v_t^b = \hat{v}_t\nu_t \) in order to preserve the contemporaneous
correlation between the error terms. We consider \( T = 120, 240, 480 \) for the sample size and \( B = 399 \) for the
number bootstrap replications. All results are calculated using \( N = 1,000 \) replications.

The reported rejection rates of the WR and WF bootstraps are calculated as: \( N^{-1} \sum_{j=1}^{N} 1_{t_j \geq t_1^{b_{1-\alpha_1,j}}} \), where \( \alpha_1 = 0.10, 0.05, 0.01 \) are the nominal values of the tests; \( t_j \) is the statistic (sup-Wald) computed from the original sample; \( t_1^{b_{1-\alpha_1,j}} \) is \( 1 - \alpha_1 \) quantile of the bootstrap distribution calculated as \((1 - \alpha_1)(B + 1)\) bootstrap order statistic from the sample of bootstrap statistics in simulation \( j = 1, \ldots, N \).

For the WR bootstrap, the bootstrap samples were generated recursively with start-up values for \( y_t^b \) and \( x_t^b \)
being given by the first observations from the sample \((x_t, y_t)\); see Davidson and MacKinnon (1993).

In all settings, the bootstrap samples are generated by imposing the null hypothesis. The value of \( \epsilon \), the
trimming parameter in Assumption 6, is set to 0.15 which is a typical value used in the literature.

We now turn to our results. We present results for the sup-Wald test under both the null and alternative
hypotheses in Tables 1-4 of the paper. In Tables 6.1-6.4 of the Supplementary Appendix we also present similar
results for the sup-F test. The first two columns of these tables give the rejection rates of the tests under the
null hypothesis, while columns 3-6 give the rejection rates of the tests under the alternative hypothesis.\(^{14}\)

From the first two columns of Tables 1-4, it can be seen that the WR bootstrap works better in general
than the WF bootstrap. The latter has large size distortions for scenarios \((h, m) = (0, 0), (h, m) = (0, 1)\) and
\((h, m) = (1, 0)\) whether the errors are conditionally homoskedastic, are conditionally heteroskedastic or have a
break in the unconditional variance. For scenario \((h, m) = (1, 1)\), the WF bootstrap is only slightly undersized
or oversized. Regarding the behavior of the sup-Wald test under the alternative hypothesis, the main conclusion
that emerges from columns 3-6 of Tables 1-4 is that the power is influenced in small sample (\( T = 120 \)) by the
number of breaks in RF and SE, the distribution of the errors \( u_t \) and \( v_t \), the distribution of \( r_t \), as well as the
number of breaks in the variance of the errors and in the variance of \( r_t \). When there is a break in SE, we need a
larger \( g \) in (36) to be able to see an increase in the power of the test, compared with scenarios with no break in SE
\((g = 0.3, 0.4\) for scenario \((h, m) = (0, 1)\), and \( g = -0.5, 0.5 \) for scenario \((h, m) = (1, 1)\), while \( g = -0.007, -0.009
\) for scenario \((h, m) = (0, 0)\) and \( g = -0.05, -0.07 \) for scenario \((h, m) = (1, 0))\). This can be explained by the fact

\(^{14}\)The rejection rates under the alternative are not level-adjusted, but since we have used the same sequence of random numbers
for repetition \( i, i = 1, \ldots, N \), in the experiments under both null and the alternative hypotheses, one can always subtract (or add)
the positive (or negative) size discrepancy (relative to the nominal size) from the rejection rate under the alternative in order to
obtain the level-adjusted power of the test; see Davidson and MacKinnon (1998).
that the second break in the SE is tested over smaller samples than the first break in the SE. Moreover, the power is lower for the smallest sample ($T = 120$) when the error terms have an upward shift in the variance (Case C in Tables 1-4) and the contemporaneous exogenous regressors also have an upward shift in their variance (Case D). However, for $T = 240, 480$ the power increases sharply in all cases.

In Tables 3 and 4 we have sequentially tested for the presence of max 2 breaks in the RF for $x_t$ (in (26)-(27) and (32)-(33) respectively) using the WR/WR sup-Wald for OLS, and the resulting number of RF breaks was imposed in each simulation prior to estimating the RF and SE and computing the test statistics for 2SLS. The fraction of times that 0, 1, 2 breaks were detected in RF (out of 1,000 replications of the scenarios), is given in Tables 6.7-6.8 from Section 6 of the Supplemental Appendix. To assess the impact of the pre-testing in RF (in the first two columns of Tables 3 and 4), we have obtained the rejection frequencies of the bootstrap tests when the number of breaks in the RF is held at the true number; see (the first two columns of) Tables 6.5 and 6.6 from Section 6 of the Supplemental Appendix. To complement our results, we have also considered a break in RF of smaller size than the one mentioned after (26)-(27) by taking $\delta_{r,1}^0 = (1, 1, 1, 1)'$ (and the rest of the parameter values are as mentioned after (26)-(27)); see Tables 6.9 and 6.10 from Section 6 of the Supplemental Appendix.

Looking at the results for the sup-Wald our results suggest that in the smaller samples ($T = 120, 240$) the recursive bootstrap is clearly to be preferred over the fixed regressor bootstrap. In the larger sample ($T = 480$), the case for the WR over the WF is more marginal as the latter yields only slightly oversized tests. This relative ranking of the two methods is intuitive from the perspective of Davidson’s (2016) first “golden rule” of bootstrap, which states: “The bootstrap DGP [...] must belong to the model [...] that represents the null hypothesis.” The fixed regressor bootstraps treat the lagged dependent variables in the RF and SE as fixed across bootstrap samples, and as such do not seem to replicate the true model that represents the null hypothesis. This would seem to point toward a recommendation to use the WR but it is important to note an important caveat to our results: our designs involve models for which both recursive and fixed bootstraps are valid. As discussed in Section 3, the fixed regressor bootstrap is asymptotically valid under weaker conditions than the recursive bootstrap. Therefore, while the recursive bootstrap works best in the settings considered here, there may be other settings of interest in which only the fixed bootstrap is valid and so would obviously be preferred.

5 Concluding remarks

In this paper, we analyse the use of bootstrap methods to test for parameter change in linear models estimated via Two Stage Least Squares (2SLS). Two types of test are considered: one where the null hypothesis is of no change and the alternative hypothesis involves discrete change at $k$ unknown break-points in the sample; and a second test where the null hypothesis is that there is discrete parameter change at $l$ break-points in the sample against an alternative in which the parameters change at $l + 1$ break-points. In both cases, we consider inferences based on a sup-Wald-type statistic using either the wild recursive bootstrap or the wild fixed regressor bootstrap. We establish the asymptotic validity of these bootstrap tests under a set of general conditions that allow the errors to exhibit conditional and/or unconditional heteroskedasticity and the regressors to have breaks in their marginal distributions. While we focus on inferences based on sup-Wald statistics, our arguments are easily extended to establish the asymptotic validity of inferences based on bootstrap versions of the analogous tests based on sup-$F$ statistics; see the Supplementary Appendix.

Our simulation results show that the wild recursive bootstrap is more reliable compared to the wild fixed
repressor bootstrap, yielding sup-Wald-type tests with empirical size equal or close to the nominal size. The gains from using the wild recursive bootstrap are quite clear in the smaller sample sizes, but are more marginal in the largest sample size \((T = 480)\) in our simulation study. This would seem to point toward a recommendation to use the wild recursive bootstrap but it is important to note that the wild fixed bootstrap is asymptotically valid under less restrictive conditions than the wild recursive bootstrap. Thus, while both bootstraps are valid in our simulation design, there may be other circumstances when the recursive bootstrap is invalid and the fixed bootstrap would be preferred. The powers of the bootstrap tests are affected in small sample by the characteristics of the error distribution, but in moderate sample sizes often encountered in macroeconomics, there is a very sharp increase in power.

Our analysis covers the cases where the first-stage estimation of 2SLS involves a model whose parameters are either constant or themselves subject to discrete parameter change. If the errors exhibit unconditional heteroscedasticity and/or the reduced form is unstable then the bootstrap methods are particularly attractive because the limiting distributions are non-pivotal. As a result, critical values have to be simulated on a case-by-case basis. In principle it may be possible to simulate these critical values directly from the limiting distributions presented in our Supplementary Appendix replacing unknown moments and parameters by their sample estimates but this would seem to require knowledge (or an estimate of) the function driving the unconditional heteroskedasticity. In contrast, the bootstrap approach is far more convenient because it involves simulations of the estimated data generation process using the residuals and so does not require knowledge of the form of heteroskedasticity. Furthermore, our results indicate that the bootstrap approach yields reliable inferences in the sample sizes often encountered in macroeconomics.
## A Tables

Table 1: Scenario: \((h,m)=(0,0)\) - rejection probabilities from testing \(H_0 : m = 0\) vs. \(H_1 : m = 1\) with bootstrap sup-Wald test.

|          | WR bootstrap |          | WR bootstrap |          | WR bootstrap |          | WR bootstrap |          |
|----------|---------------|----------|---------------|----------|---------------|----------|---------------|----------|
|          | Size          | Power    | Size          | Power    | Size          | Power    | Size          | Power    |
|          | \(g = 0\)    | \(g = -0.007\) | \(g = 0\)    | \(g = -0.007\) | \(g = 0\)    | \(g = -0.007\) | \(g = 0\)    | \(g = -0.007\) |
| Case A   | T 10% 5% 1% | 10% 5% 1% | T 10% 5% 1% | 10% 5% 1% | T 10% 5% 1% | 10% 5% 1% | T 10% 5% 1% | 10% 5% 1% |
|          | Case B        |          | Case C        |          | Case D        |          |                |          |
|          | 120 11.8 6.1 1.6 | 120 9.3 4 0.8 | 120 9.9 5.6 1.5 | 120 10.7 5.3 1 | 120 10.8 5.09 1.15 | 120 12 5.9 0.7 | 120 9.9 5.1 0.7 | 120 10.4 5.1 0.9 |
|          | 151 8.7 2.4 | 129 6.4 0.9 | 15.7 8.3 1.9 | 13.8 7.3 1.9 | 9.76 5.52 1.04 | 14.4 8.5 1.7 | 15.2 8.4 1.2 | 12.2 6.2 1.7 |
|          | 59.2 48.3 25 | 99.7 99.7 99.6 | 99.8 99.7 99.7 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
|          | 61.1 55.3 31.5 | 99.8 99.7 99.7 | 99.8 99.7 99.7 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
|          | 79.4 70.3 49.1 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
|          | 84.5 75 56.3 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |

**Notes.** The first two columns refer to the case when \(H_0 : m = 0\) is true \((g = 0\) in equation (36)). The next columns refer to the case when we test for \(H_0 : m = 0\), but \(H_1 : m = 1\) is true \((g = -0.007, -0.009\) in equation (36)). Under the null and the alternative hypotheses we impose \(h = 0\) in the RF.
Table 2: Scenario: \((h,m)=(0,1)\) - rejection probabilities from testing \(H_0 : m = 1\) vs. \(H_1 : m = 2\) with bootstrap sup-Wald test.

|       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |
|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|
|       | Size         |       | g=0          |       | Power        |       | g=0.3        |       | Power        |       | g=0.4        |
| Size  | g=0          |       | g=0.3        |       | g=0.4        |       |               |       |               |       |               |
| T     | 10% 5% 1%    |       | 10% 5% 1%    |       | 10% 5% 1%    |       | 10% 5% 1%    |       | 10% 5% 1%    |       | 10% 5% 1%    |
| 120   | 10.7 5 1.2   | 15.5 9.9 5.6 | 54.9 36.9 10.8 | 61.5 45.6 19.8 | 78.1 60.3 24.8 | 82.2 67.8 38.5 |
| 240   | 10.2 4.9 0.5 | 12.5 7.1 3.4 | 99.5 98.9 89.9 | 99.6 98.6 92.2 | 100 100 98.8 | 100 100 99.1 |
| 480   | 8 4.5 1     | 8.6 4.4 0.8 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |

**Case A**

|       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |
|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|
|       |              |       | g=0          |       | g=0.3        |       | g=0.4        |       |               |       |               |
| 120   | 9.7 4.6 1    | 16 10.2 6.5 | 62.3 44.8 16.1 | 67.2 53.9 26.2 | 82.2 67.6 31.1 | 84.1 73.6 44.7 |
| 240   | 10.6 5.2 1.2 | 13.8 8.1 3 | 99.3 97.5 86.2 | 99.1 93 91.5 | 99.9 99.6 96.6 | 100 99.9 98.3 |
| 480   | 8 4.2 0.9    | 8.4 4.8 0.8 | 100 99.8 99.5 | 99.8 99.7 99.5 | 100 100 100 | 100 100 99.9 |

**Case B**

|       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |
|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|
|       |              |       | g=0          |       | g=0.3        |       | g=0.4        |       |               |       |               |
| 120   | 10.5 5.2 0.9 | 16.3 11 5.8 | 26.3 14.8 3.3 | 36.1 21.4 6.6 | 40.1 24.5 7.5 | 51.1 34.6 13 |
| 240   | 11 4.8 0.9   | 13.2 8.3 2.4 | 83.1 68.7 31.4 | 87.2 77.6 47.2 | 98.5 93.4 68.7 | 99 97 80.1 |
| 480   | 10.4 5.6 0.5 | 11.2 6.1 1.2 | 100 99.9 98.4 | 100 99.9 99.2 | 100 100 100 | 100 100 100 |

**Case C**

|       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |       | WR bootstrap |       | WF bootstrap |
|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|-------|--------------|
|       |              |       | g=0          |       | g=0.3        |       | g=0.4        |       |               |       |               |
| 120   | 11.6 5.8 1.5 | 15.3 9.5 5.3 | 39.8 24.1 6.5 | 51.2 33.2 13.3 | 64.8 43.3 14 | 72.5 54.8 24.2 |
| 240   | 11.5 6 1     | 14.9 9.1 2.9 | 98.9 94.6 73 | 98.9 97.1 82.7 | 100 99.8 95.6 | 100 99.9 97.9 |
| 480   | 9.6 4 1.3    | 9.5 5.3 1.5 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |

**Case D**

Notes. The first two columns refer to the case when \(H_0 : m = 1\) is true \((g=0\) in equation \((36)\)). The next columns refer to the case when we test for \(H_0 : m = 1\), but \(H_1 : m = 2\) is true \((g = 0.3, 0.4\) in equation \((36)\)). Under the null and the alternative hypotheses we impose \(h = 0\) in the RF.
Table 3: Scenario: \((h,m)=(1,0)\) - rejection probabilities from testing \(H_0 : m = 0\) vs. \(H_1 : m = 1\) with bootstrap sup-Wald test; number of breaks in the RF was estimated and imposed in each simulation using a sequential strategy based on the WR/WF sup-Wald for OLS

| WR bootstrap | WF bootstrap |
|--------------|--------------|
| Size \(g = 0\) | Size \(g = 0\) |
| Power \(g = -0.05\) | Power \(g = -0.05\) |
| Power \(g = -0.07\) | Power \(g = -0.07\) |

| Case A | T | 10% | 5% | 1% |
|--------|---|-----|----|----|
|        | 120 | 10.2 | 3.7 | 0.9 |
|        | 240 | 10.8 | 5.7 | 0.8 |
|        | 480 | 10.9 | 5.2 | 0.9 |

| Case B | T | 10% | 5% | 1% |
|--------|---|-----|----|----|
|        | 120 | 10.1 | 4.8 | 1.0 |
|        | 240 | 10.4 | 5.2 | 1.2 |
|        | 480 | 11.0 | 5.4 | 0.7 |

| Case C | T | 10% | 5% | 1% |
|--------|---|-----|----|----|
|        | 120 | 9.6  | 4.3 | 0.9 |
|        | 240 | 11.8 | 6.0 | 0.6 |
|        | 480 | 10.8 | 5.9 | 1.1 |

| Case D | T | 10% | 5% | 1% |
|--------|---|-----|----|----|
|        | 120 | 10.2 | 4.8 | 1.0 |
|        | 240 | 10.6 | 5.7 | 0.4 |
|        | 480 | 11.6 | 6.2 | 0.9 |

Notes. The first two columns refer to the case when \(H_0 : m = 0\) is true \((g=0\) in equation \((36)\)). The next columns refer to the case when we test for \(H_0 : m = 0\), but \(H_1 : m = 1\) is true \((g = -0.05, -0.07\) in equation \((36)\)). Prior to testing \(H_0 : m = 0\) vs. \(H_1 : m = 1\) (for all columns above), we tested sequentially for the presence of maximum two breaks in the RF (we used the WR/WF bootstrap sup-Wald for OLS to test \(H_0 : \ell = \ell\) vs. \(H_1 : \ell + 1, \ell = 0, 1\)). If breaks are detected in RF, the number of breaks and the estimated locations are imposed when estimating the SE.
Table 4: Scenario: \((h,m)=(1,1)\) - rejection probabilities from testing \(H_0 : m = 1\) vs. \(H_1 : m = 2\) with bootstrap sup-Wald test; number of breaks in the RF was estimated and imposed in each simulation using a sequential strategy based on the WR/WF sup-Wald for OLS

|           | WR bootstrap |          | WR bootstrap |          | WR bootstrap |          | WR bootstrap |          | WR bootstrap |          | WF bootstrap |
|-----------|--------------|----------|--------------|----------|--------------|----------|--------------|----------|--------------|----------|--------------|
|           | Size         |          | Power        |          | Power        |          | Power        |          | Power        |          | Power        |
|           | \(g = 0\)   |          | \(g = 0.5\)  |          | \(g = -0.5\) |          | \(g = -0.5\) |          | \(g = -0.5\) |          | \(g = -0.5\) |
| T         | 10% 5% 1%    |          | 10% 5% 1%    |          | 10% 5% 1%    |          | 10% 5% 1%    |          | 10% 5% 1%    |          | 10% 5% 1%    |

**Case A**

|       |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 120 | 8.8 | 4.7 | 0.7 | 8.7 | 4.5 | 0.8 | 52  | 40.7| 16  | 57.4| 45.4| 22.7| 85  | 71.9| 32.1| 88.2| 74.8| 37.5|
|       | 240 | 10.4| 5.7 | 0.7 | 10.4| 5.2 | 0.8 | 99.8| 99.4| 97.6| 99.6| 99.4| 97.4| 100 | 100 | 99.8| 100 | 100 | 99.7|
|       | 480 | 9.7 | 4.2 | 0.7 | 10.2| 4.6 | 0.8 | 100 | 100 | 100 | 100 | 99.8| 99.1| 100 | 100 | 100 | 100 | 100 | 100 |

**Case B**

|       |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 120 | 8.9 | 3.7 | 0.9 | 8.7 | 3.4 | 0.9 | 50.1| 40.1| 18.6| 54.8| 45.4| 24.4| 81.8| 70.9| 38.3| 85.5| 73  | 39.9|
|       | 240 | 10.8| 4.7 | 0.8 | 10.6| 5.3 | 0.9 | 98.8| 98.3| 96  | 99.2| 98.7| 95.8| 99.6| 99.5| 98.1| 98  | 99.6| 98.3|
|       | 480 | 9.9 | 4.1 | 0.9 | 10.9| 5.4 | 0.9 | 100 | 100 | 99.8| 100 | 99.8| 99.6| 100 | 100 | 100 | 100 | 100 | 100 |

**Case C**

|       |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 120 | 9.1 | 3.5 | 1   | 9.4 | 4   | 0.4 | 30.7| 17.5| 3.1 | 38.3| 25.9| 8   | 45.1| 25.4| 6.9 | 49.7| 31.7| 8.8 |
|       | 240 | 10.3| 5.2 | 1   | 10.2| 5   | 1   | 98.6| 96.8| 86.2| 99  | 97.7| 88.4| 99.3| 98.5| 86.9| 100 | 99.8| 90.8|
|       | 480 | 11.3| 4.8 | 1   | 12.1| 5.3 | 0.6 | 100 | 100 | 99.9| 100 | 99.2| 99.9| 100 | 100 | 99.7| 100 | 100 | 99.9|

**Case D**

|       |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       | 120 | 10.1| 4.4 | 1.6 | 8.5 | 3.8 | 0.6 | 36.8| 23.4| 6.3 | 42.1| 30.4| 12.4| 69.3| 52.6| 16.4| 76.8| 59.4| 25.4|
|       | 240 | 10.9| 4.9 | 0.8 | 11.8| 5.2 | 0.8 | 99.2| 98.6| 94  | 99.6| 98.9| 94.5| 99.5| 99.4| 98  | 99.9| 99.9| 98.6|
|       | 480 | 10.2| 5.3 | 1.4 | 11  | 5.6 | 1.2 | 100 | 100 | 100  | 100 | 100 | 98.1 | 100 | 100 | 100 | 100 | 100 |

Notes. The first two columns refer to the case when \(H_0 : m = 1\) is true (\(g=0\) in equation (36)). The next columns refer to the case when we test for \(H_0 : m = 1\), but \(H_1 : m = 2\) is true (\(g = -0.5, 0.5\) in equation (36)). Prior to testing \(H_0 : m = 1\) vs \(H_1 : m = 2\) (for all columns above), we tested sequentially for the presence of maximum two breaks in the RF (we used the WR/WF bootstrap sup-Wald for OLS to test \(H_0 : h = \ell\) vs. \(H_1 : \ell + 1, \ell = 0,1\)). If breaks are detected in RF, the number of breaks and the estimated locations are imposed when estimating the SE.
B Appendix: Proof of Theorems

For the purposes of our analysis, it is convenient to write the system in (4) as a VAR(1) model.\textsuperscript{15} To this end, define:

\[
\begin{align*}
\xi_t &\equiv \begin{bmatrix}
\tilde{z}_t \\
\tilde{z}_{t-1} \\
\vdots \\
\tilde{z}_{t-p+1}
\end{bmatrix}_{np \times 1} & \quad F_s &\equiv \begin{bmatrix}
C_{1,s} & C_{2,s} & \cdots & C_{p-1,s} & C_{p,s} \\
I_n & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & I_n & \cdots & 0_{n \times n} & 0_{n \times n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{n \times n} & 0_{n \times n} & \cdots & I_n & 0_{n \times n}
\end{bmatrix}_{np \times np}, \\
\eta_t &\equiv \begin{bmatrix}
e_t \\
o_n \\
\vdots \\
o_n
\end{bmatrix}_{np \times 1}.
\end{align*}
\]

Then equation (4) is the first \( n \) entries of:

\[
\xi_t = \mu_s + F_s \xi_{t-1} + \eta_t, 
\]

where we have suppressed the dependence of \( \xi_t \) and \( \eta_t \) on \( s \) for notational convenience.

From Assumption 9 it follows that \( \eta_t \) is a vector m.d.s. relative to \( \mathcal{F}_{t-1} \) with conditional covariance matrix

\[
E(\eta_t \eta_j') | \mathcal{F}_{t-1} = \begin{cases}
\Omega_{jt}^{-1} \text{, for } t = j, \\
0_{np \times np} \text{ otherwise},
\end{cases}
\]

where

\[
\Omega_{jt}^{-1} \equiv \begin{bmatrix}
A_s^{-1} \Sigma_{jt} A_s^{-1} & 0_{n \times n(p-1)} \\
0_{n \times n(p-1)} & 0_{n(p-1) \times n(p-1)}
\end{bmatrix},
\]

and time-varying unconditional covariance matrix

\[
\Omega_t \equiv E(\eta_t \eta_t') = \begin{bmatrix}
A_s^{-'} \Sigma_t A_s^{-1} & 0_{n \times n(p-1)} \\
0_{n \times n(p-1)} & 0_{n(p-1) \times n(p-1)}
\end{bmatrix},
\]

where

\[
\Sigma_t = SD_t \Sigma_{jt} D_t' S'.
\]

From (37), it follows that within each regime we have, for \( t = [\tau_{s-1} T] + 1, [\tau_{s-1} T] + 2, \ldots, [\tau_s T], \)

\[
\xi_s = F_s^{[\tau_{s-1} T]} \xi_{[\tau_{s-1} T]} + \xi_t + \left( \sum_{l=0}^{[\tau_{s-1} T]-1} F_s^l \right) \mu_s, 
\]

where \( \xi_t = \sum_{l=0}^{[\tau_{s-1} T]-1} F_s^l \eta_{t-l}, \) \( \{ \eta_t \} \) is a m.d.s. sequence, and, from Assumption 7, all the eigenvalues of \( F_s \) have modulus less than one.

The following lemmas are used in proofs; Lemmas 2 and 4-8 are proven in the Supplementary Appendix.

\textsuperscript{15}For example, see Using Hamilton (1994)p.259.
which also contains the asymptotic distributions of the sup Wald test statistics. The rest of the lemmas are proven below.

**Lemma 1.** If \( \{ \vartheta_t, F_t \} \) is a mean-zero sequence of \( L^1 \)-mixingale random variables with constants \( \{ c_t, c_t' \} \) that satisfy
\[
\lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} c_t, \quad \text{and} \quad \sup_{t} E|\vartheta_t|^b < \infty \text{ for some } b > 1, \text{ then } \sup_{s \in [0,1]} |T^{-1} \sum_{t=1}^{T} \vartheta_t| \to 0.
\]

This follows from applying the LLN in Andrews (1988) [Theorem 1], modified to be a uniform LLN in the proof of Lemma A2 of Andrews (1993).

**Lemma 2.** For \( s = 1, \ldots, N + 1 \), where \( N \) is the total number of breaks in the coefficients of the \( \text{VAR}(p) \) representation of \( \tilde{z}_t \), define the following functions: \( F(\tau) = F_s, A(\tau) = A_s, \mu(\tau) = \mu_s, \gamma(\tau) = \gamma_s \) for \( \tau \leq \tau_a \). Also, define the function \( \Sigma(\tau) \) on \( \tau \in [0,1] \) as follows \( \Sigma(0) = 0, \text{ and } \Sigma(\tau) = \Sigma_t \text{ for } \tau \in ((t-1)/T, t/T], \text{ } t = 1, \ldots, T \). Let \( S \) and \( S_{\tau} \) be the selection matrices such that \( z_t = \text{vect}(1, S_{\tau} \xi_1, S_{\tau} \xi_{t-1}) = \text{vect}(1, r_t, S_{\tau} \xi_{t-1}) \), and
\[
Q_{\tau}(\tau) = \begin{bmatrix}
1 & [S_{\tau} \xi_1(\tau)]' & [S \xi_1(\tau)]' \\
S_{\tau} \xi_1(\tau) & S_{\tau} \xi_2(\tau) S_{\tau}' & S_{\tau} (\mu(\tau) Q_1(\tau) + F(\tau) Q_2(\tau)) S_{\tau}' \\
S \xi_1(\tau) & (S_{\tau} (\mu(\tau) Q_1(\tau) + F(\tau) Q_2(\tau)) S_{\tau}')' & S \xi_2(\tau) S_{\tau}'
\end{bmatrix},
\]
where \( Q_1(\tau) = \{ I_{op} - F(\tau) \} \mu(\tau) \) and
\[
Q_2(\tau) = \sum_{i=0}^{\infty} F(\tau)^i \begin{bmatrix}
A(\tau)^{-1} \Sigma(\tau) A(\tau)^{-1}' & 0_{n \times n(p-1)} \\
0_{P \times n(p-1)} & 0_{n \times n} 
\end{bmatrix} (F(\tau)^i)' + Q_1(\tau) Q_1(\tau).
\]

Also, let \( Q(\tau) = \sum_{i=1}^{N} \gamma(\tau) Q(\tau) \gamma(\tau) d\tau \).

Under Assumptions 1-8,
\[
Q(\tau) = T^{-1} \sum_{t \in h, \lambda} Y_t' z_t z_t' Y_t \overset{P}{\to} Q_{\tau}.
\]

**Lemma 3.** If \( (a_1, F_t) \) is a \( 0 \times 1 \) vector of m.d.s. with \( \sup_{t} E|a_{1,t}|^{2+\delta^*} < \infty \text{ for some } \delta^*>0 \) and all elements \( a_{1,1} \) of the vector \( a_{1,1} \), if \( T^{-1} \sum_{t=1}^{T} |E(a_{1,1}'(F_{t-1}) - E(a_{1,1}))| \overset{P}{\to} 0 \) uniformly in \( r \) and if \( T^{-1} \sum_{t=1}^{T} E(a_{1,1}) \rightarrow r I_{p_0} \), uniformly in \( r \), then \( T^{-1/2} \sum_{t=1}^{T} a_{1,1} \rightarrow B(r) \), a \( 0 \times 1 \) vector of independent standard Brownian motions.

Lemma 3 provides sufficient conditions so that Theorem 3 in Brown (1971) is satisfied.

**Lemma 4.** Under Assumption 9,
(i) \( T^{-1} \sum_{t=1}^{T} E(I_{p_1} | F_{t-1}) \overset{P}{\to} r I_{p_1} \) uniformly in \( r \).
(ii) \( T^{-1} \sum_{t=1}^{T} E((I_{p_1}') \otimes I_{p_1} | F_{t-1}) \overset{P}{\to} r I_{p_1} \) uniformly in \( r \), for all \( i \geq 0 \).
(iii) \( T^{-1} \sum_{t=1}^{T} E((I_{p_1}') \otimes (I_{p_1-1} | F_{t-1}) \overset{P}{\to} r I_{p_1} \) uniformly in \( r \), for all \( i,j \geq 0 \).

For the following lemmas and the rest of the proofs, we need additional notation. Define \( \tilde{S}_1 = [I_{p_1} \quad 0_{(p_1+1) \times p_2}] \) and \( \tilde{S}_2 = [0_{p_2 \times (p_1+1)} \quad I_{p_2}] \). Also, define the following vectors of Brownian motions: \( B_{0}(r), \) a \( n \times 1 \) vector with variance \( r I_{p_0} \), \( B_{1}(r), \) a \( n \times 2 \) vector with variance \( r I_{p_1} \) for all \( l \leq 1, B_{2}(r) = \text{vect}(B_{uc}(r), B_{ac}(r)) \) with \( B_{uc}(r) \) of dimension \( p_2 \times 1 \) and \( B_{ac}(r) \) of dimension \( p_1 p_2 \times 1 \), where the variance of \( B_{ac}(r) \) is \( r(\tilde{S}_1 \otimes \tilde{S}_2) \rho_{0,0} (\tilde{S}_1 \otimes \tilde{S}_2)' = r \rho_{\varepsilon,0} \).

The covariances of these processes are: \( \text{Cov}(B_{1}(r_1), B_{1}(r_2)) = \min(r_1, r_2) \rho_{1,1} \) for all \( l, \kappa \geq 1, l \neq \kappa \), and \( \text{Cov}(B_{2}(r_1), B_{2}(r_2)) = \min(r_1, r_2) \rho_{2,2} \delta_1 \delta_2 \rho_{0,1} \rho_{2,2} \delta_1 \delta_2 \rho_{0,0} \delta_1 \delta_2 \), where \( \rho_{0,1} \) and \( \rho_{0,0} \) are given in Assumption 9(v) and (iv) respectively, and \( \rho_{u,0} \) is of dimension \( p_2 \times n \).
Lemma 5. For fixed \(n^*\), under Assumption 9,

\[
T^{-1/2} \sum_{t=1}^{[T\tau]} \text{ve}(l_t, t_{t-1}, \ldots, t_{t-n^*}, u_t, l_{\xi,t}, u_{\xi,t} \otimes l_{\xi,t}) \Rightarrow \text{ve}(B_0(\tau), B_1(\tau), \ldots, B_{n^*}(\tau), B_{\xi}(\tau)),
\]

where if \(t - l < 0\), the rest of the elements of this sum are artificially set to zero.

Now define for \(b = 1, 2\) and any \(n^b \times 1\) vectors \(a, a_{\#} = \text{ve}(a, 0_{n^b(p^b-1)})\), and for any \(n^b \times n^b\) matrices \(A\), let \(A_{\#} = \text{diag}(A, 0_{n^b(p^b-1) \times n^b(p^b-1)})\), except for \(\beta_{x,s,\#}\), which is \(\beta_{x,s,\#} = \text{ve}(0, \beta_{x,s}^0, 0_{p_2+n(p-1)})\) and the subscript \(s\) indicates the value of \(\beta_{x,s}^0\) in the stable regime \(I_s = [[\tau_{s-1}T] + 1, [\tau_s T]].\) If \(m = 0\), then \(\beta_{x,s}^0 = \beta_{x,s}\) and \(\beta_{x,s,\#} = \text{ve}(0, \beta_{x,s}^0, 0_{p_2+n(p-1)})\). Let \(S_u = \text{ve}(1, 0_{n-1}, 0_{n(p-1)})\) and \(S_1 = S_u\) or \(S_1 = \beta_{x,s,\#}\), where the value \(S_1\) takes is clarified in each context where the distinction between the two values is necessary. Let \(S\), defined in Assumption 9, and \(D(\tau)\), the function such that \(D(\tau) = D_t\) for \(\tau \in [\frac{1}{n^1}, \frac{1}{n^1}]\), be partitioned as follows:

\[
S = \begin{pmatrix}
1 & 0_{1 \times p_1} & 0_{1 \times p_2} \\
p_{s_1} & S_{s_1} & 0_{p_1 \times p_2} \\
p_{p_2 \times 1} & 0_{p_2 \times p_1} & S_{p_2}
\end{pmatrix}, \quad D(\tau) = \begin{pmatrix}
d_u(\tau) & 0_{1 \times p_1} & 0_{1 \times p_2} \\
p_{p_1 \times 1} & D_u(\tau) & 0_{p_1 \times p_2} \\
p_{p_2 \times 1} & 0_{p_2 \times p_1} & D_\xi(\tau)
\end{pmatrix}, \quad (40)
\]

where \(s_{p_1}\) is of dimension \(p_1 \times 1\), \(S_1\) and \(D_u(\tau)\) are of dimension \(p_1 \times p_1\), and \(S_{p_2}\) and \(D_\xi(\tau)\) are of dimension \(p_2 \times p_2\). For any interval \([[\tau_{s-1}T] + 1, [\tau_s T]]\) where the coefficients of the VAR representation in (4) are stable, let:

\[
M_1(\tau_{s-1}, \tau_s) = (S_1^sS_{\#}^s) \left( \int_{\tau_{s-1}}^{\tau_s} D_{\#}(\tau) \mathrm{d}B_{\#}(\tau) \right)
\]

\[
M_{2,1}(\tau_{s-1}, \tau_s) = \sum_{i=0}^{\infty} ((S_1^sS_{\#}^s) \otimes (S_sF_s^i)) \left( \int_{\tau_{s-1}}^{\tau_s} D_{\#}(\tau) \mathrm{d}B_{\#}(\tau) \right) \otimes \mu_s
\]

\[
M_{2,2}(\tau_{s-1}, \tau_s) = \sum_{i=0}^{\infty} ((S_1^sS_{\#}^s) \otimes (S_sF_s^{i+1}A_{\#}^{-1}s_{\#})) \int_{\tau_{s-1}}^{\tau_s} (D_{\#}(\tau) \otimes D_{\#}(\tau)) \mathrm{d}B_{\#}(\tau)
\]

\[
M_{2,3}(\tau_{s-1}, \tau_s) = \int_{\tau_{s-1}}^{\tau_s} (d_u(\tau) \otimes D_\xi(\tau)) \mathrm{d}B_{\xi}(\tau)
\]

\[
M_{2,4}(\tau_{s-1}, \tau_s) = \int_{\tau_{s-1}}^{\tau_s} (d_u(\tau) \otimes D_\xi(\tau)) \mathrm{d}B_{\xi}(\tau) + ((\beta_{x,s}^0S_{p_1} \otimes S_{p_2}) \int_{\tau_{s-1}}^{\tau_s} (D_{\#}(\tau) \otimes D_\xi(\tau)) \mathrm{d}B_{\xi}(\tau)
\]

\[
M_{3,1}(\tau_{s-1}, \tau_s) = \sum_{i=0}^{\infty} ((S_1^sS_{\#}^s) \otimes (S_sF_s)) \left( \int_{\tau_{s-1}}^{\tau_s} D_{\#}(\tau) \mathrm{d}B_{\#}(\tau) \right) \otimes \mu_s
\]

\[
M_{3,2}(\tau_{s-1}, \tau_s) = \sum_{i=0}^{\infty} ((S_1^sS_{\#}^s) \otimes (S_sF_s^{A_{\#}^{-1}s_{\#}}S_{\#})) \int_{\tau_{s-1}}^{\tau_s} (D_{\#}(\tau) \otimes D_{\#}(\tau)) \mathrm{d}B_{\#}(\tau)
\]

\[
M(\tau_{s-1}, \tau_s) = \text{ve}(M_1(\tau_{s-1}, \tau_s), M_2(\tau_{s-1}, \tau_s), M_3(\tau_{s-1}, \tau_s))
\]

where \(S_r\) was defined in Lemma 2.

Lemma 6. Let the interval \(I_\tau\) contain \(N_\tau\) breaks from the total set of \(N\) breaks. Then, under Assumptions 1-9,

\[
T^{-1/2} \sum_{t \in I_\tau} z_{t \tau}u_t \Rightarrow \hat{M}_\tau = \begin{cases}
M(\lambda_{s-1}, \tau_s) + \sum_{j=1}^{N_\tau} M(\lambda_{s+j-1}, \tau_{s+j}) + M(\tau_{s+N_\tau}, \lambda_s) & \text{if } N_\tau \geq 2 \\
M(\lambda_{s-1}, \tau_s) + M(\tau_s, \lambda_1) & \text{if } N_\tau = 1 \\
M(\lambda_{s-1}, \lambda_1) & \text{if } N_\tau = 0
\end{cases}
\]

with \(S_1 = S_u\). Similarly, \(T^{-1/2} \sum_{t \in I_\tau} z_t w_t \beta_{x,\#(i)}^{(s)} \Rightarrow \hat{M}_\tau\) but with \(S_1 = \beta_{x,\#(i)} = \text{ve}(0, \beta_{x,\#(i)}, 0_{p_2+n(p-1)})\). If \(m = 0\), then \(S_1 = \beta_{x,\#(i)}\).
Lemma 7. Under Assumptions 1-9,

(i) if \( h > 0 \), then \( T(\hat{\pi}_i - \pi^0_i) = O_p(1), i = 1, \ldots, h + 1; \)

(ii) \( T^{1/2}(\Delta_i - \Delta^0_i) = O_p(1) \) for \( i = 1, \ldots, h + 1; \)

(iii) if \( m > 0 \), then \( T(\hat{\lambda}_i - \lambda^0_i) = O_p(1), i = 1, \ldots, m + 1. \)

Lemma 8. Under Assumption 9, uniformly in \( r \),

(i) \( T^{-1} \sum_{t=1}^{[Tr]} \{ \epsilon_t \epsilon_t' - E(\epsilon_t \epsilon_t') \} \xrightarrow{P} 0, \)

(ii) \( T^{-1} \sum_{t=1}^{[Tr]} \{ \epsilon_t \epsilon_t' \odot \epsilon_{t-1} - E(\epsilon_t \epsilon_t' \odot \epsilon_{t-1}) \} \xrightarrow{P} 0 \) for all \( i \geq 0, \)

(iii) \( T^{-1} \sum_{t=1}^{[Tr]} \{ (\epsilon_t \epsilon_t') - E(\epsilon_t \epsilon_t') \} \xrightarrow{P} 0 \) for all \( i, j \geq 0 \)

(iv) Parts (i)-(iii) hold with \( t \) replaced by \( l, l_{t-1}, l_{t-j} \), replacing \( \epsilon_t, \epsilon_{t-1}, \epsilon_{t-j} \).

Lemma 9. Let \( \hat{Q}^b_{(i)} = T^{-1} \sum_{t \in I_i} \hat{Y}_t^b z_t^b + b^b \). Then, under Assumptions 1-9, \( \hat{Q}^b_{(i)} = Q_i + o_p(1) \), where

\[
Q_i = \int_{\lambda_{i-1}}^{\lambda_i} Y(\tau)Qz(\tau)Y(\tau)d\tau.
\]

Proof of Lemma 9.

For the WF bootstraps, \( z_t^b = z_t \), and therefore Lemma 9 holds by Lemma 2. Consider the WR bootstrap, first for \( I_i = \hat{I}_s \). Define \( \hat{z}_t^s = (y_t^s, x_t^s, r_t)' \), and:

\[
\hat{Q}^b_{(i)} = T^{-1} \sum_{t \in I_i} \hat{Y}_t^b z_t^b \hat{Y}_t = \hat{Y}_s \left[ \begin{array}{ccc}
\Delta_{\tau} & A^b_1 & A^b_2 \\
S_rA^b_1 & S_rB^b_1S_r & S_rB^b_2S_r \\
S_A^b & S_B^bS_r & S_B^bS_r
\end{array} \right] \hat{Y}_s,
\]

where

\[
A^b_1 = T^{-1} \sum_{t \in I_i} \xi^b_t, \quad A^b_2 = T^{-1} \sum_{t \in I_i} \xi^b_{t-1}
\]

and

\[
B^b_1 = T^{-1} \sum_{t \in I_i} \chi^b_t\chi^b_t, \quad B^b_2 = T^{-1} \sum_{t \in I_i} \xi^b_t\xi^b_{t-1}, \quad B_3 = T^{-1} \sum_{t \in I_i} \xi^b_{t-1}\xi^b_{t-1}.
\]

Note that, because \( r_t \) is kept fixed, \( S_rA^b_1 = T^{-1} \sum_{t \in I_i} S_r\xi_t = S_rA_1, \) and \( S_rB^b_1S_r = S_rB_1S_r \), where \( A_1, A_2 \) are the sample counterparts of \( A^b_1, B^b_1 \) defined at the beginning of the proof of Lemma 2. By Lemma 2, the result in Lemma 9 holds automatically for these terms. We now analyze the rest of the terms. To that end, we first derive some preliminary results.

- Preliminary results and bootstrap notation. Note that in any stable subinterval \( \hat{I}_s \),

\[
\hat{z}_t^b = \hat{c}^b_{t,s} + \sum_{i=1}^P \hat{C}^b_{t,s} \hat{z}_{t-i} + e_t^b, \quad [\tau_{s-1}T] + 1 \leq t \leq [\tau_sT], \ s = 1, 2, \ldots, N + 1,
\]

where \( e_t^b = \hat{A}^b - e_t^b \), \( e_t^b = \text{vec}(u_t^b, v_t^b, \zeta_t) \), of size \( n \times 1 \), and the elements of \( \hat{A}_s, \hat{c}^b_{t,s} \) and \( \hat{C}^b_{t,s} \) corresponding to the equation for \( r_t \) are the true parameters, not the estimated ones. Then,

\[
\xi^b_t = \hat{\mu}^b + \hat{F}_s \xi^b_{t-1} + e_t^b
\]

and

\[
\xi^b_t = F_s^{[\tau_{s-1}T]} \xi^b_{[\tau_{s-1}T]} + \left( \sum_{t=0}^{[\tau_{s-1}T]-1} F_s^t \right) \hat{\mu}_s + \sum_{t=0}^{[\tau_{s-1}T]-1} F_s^t \eta^b_{t-1},
\]

where \( \hat{\xi}_t^b = \text{vec}(\hat{z}_t^b, \hat{z}_{t-1}^b, \ldots, \hat{z}_{t-p+1}^b) \), \( \eta^b_t = \hat{A}^{-1}_s e^b_t \), and \( \hat{F}_s, \hat{\mu}_s \) are defined as \( F_s, \mu_s \), but replacing the true
coefficients that are estimated by 2SLS with those estimated counterparts. Also, let \( \hat{\eta}_t = \hat{\epsilon}_t,\# = \hat{A}_{s,\#}^{-1} \hat{\epsilon}_{t,\#} \), where \( \hat{\epsilon}_t = \text{vect}(\hat{u}_t, \hat{v}_t, \zeta_t) \).

We now show two results that we repeatedly need in the proofs: \( T^{-\alpha} \xi_t^b = o_p(1) \) and \( T^{-\alpha} \xi_t^{b\prime} = o_p(1) \) for any \( \alpha > 0 \).

For this purpose, we first show that \( E^b(T^{-\alpha} \eta_t^b) = o_p(1) \) and that \( \text{Var}^b(T^{-\alpha} \eta_t^b) = o_p(1) \). Then, by Markov’s inequality, for any \( C > 0 \), \( P^b(T^{\alpha}\|\eta_t^b - E^b(\eta_t^b)\| \geq C) \leq C^{-2}T^{-2\alpha}\text{Var}^b(\eta_t^b)^{-1} \), so completing the proof.

Let \( \mathcal{I} = \text{vect}(0_{p+1}, t_{p+1}, 0_{p_2+n(p+1)}) \) and \( \mathcal{J} = [\text{diag}(J_{p+1}, J_{p+2}+1)]\# \), where \( t_a \) is a \( a \times 1 \) vector of ones, and \( J_a = t_a t_a\# \). Let \( \nu_t = \text{vect}(\nu_t, t_{p+1}, t_{p+2})\# \). Then \( E^b(\nu_t) = \mathcal{I} \) and \( E^b(\nu_t, \nu_t) = \mathcal{J} \).

Also, let \( g_{t}^b = e_{t,\#}^b = \hat{\epsilon}_{t,\#} \odot \nu_t \), where \( \odot \) is the element-wise multiplication. Then \( g_t^b = \hat{A}_{s,\#}^1 \eta_t^b \), and letting \( \hat{g}_t = \hat{\epsilon}_{t,\#} \), it follows that \( g_t^b = \hat{g}_t \odot \nu_t \). Further, let \( \hat{g}_{t,1} \equiv \text{vect}(\hat{u}_t, \hat{v}_t, 0_{p_2+n(p-1)}) \) and \( \hat{g}_{t,2} \equiv \text{vect}(0_{p+1}, \zeta_t, 0_{n(p-1)}) \). Also, note that \( \hat{g}_t^b = \hat{A}_{s,\#}^{-1}(\hat{g}_t \odot \nu_t) \). Then:

\[
E^b(\eta_t^b) = E^b(\hat{A}_{s,\#}^{-1}(\hat{g}_t \odot \nu_t)) = (\hat{A}_{s,\#}^{-1}g_t^b), (\hat{A}_{s,\#}^{-1}g_t^b, \nu_t)\mathcal{I} = \hat{A}_{s,\#}^{-1}(\hat{g}_t^b, \nu_t)\mathcal{I}
\]

\[
E^b(\eta_t^b \eta_t^{b\prime}) = E^b(\hat{A}_{s,\#}^{-1}(\hat{g}_t \odot \nu_t)(\hat{g}_t \odot \nu_t)\mathcal{J} \hat{A}_{s,\#}^{-1})
\]

\[
= \hat{A}_{s,\#}^{-1}[(g_t^b g_t^{b\prime}) \odot \mathcal{J}] \hat{A}_{s,\#}^{-1} = \hat{A}_{s,\#}^{-1}((\hat{g}_t^b \hat{g}_t^{b\prime}) \odot \mathcal{J}) \hat{A}_{s,\#}^{-1}
\]

\[
\text{Var}^b(\eta_t^b) = E^b(\eta_t^b \eta_t^{b\prime}) - E^b(\eta_t^b)E^b(\eta_t^{b\prime}) = \hat{A}_{s,\#}^{-1}
\]

\[
= \hat{A}_{s,\#}^{-1}((\hat{g}_t^b \hat{g}_t^{b\prime}) \odot \mathcal{J}) \hat{A}_{s,\#}^{-1}
\]

By Lemma 7, Lemma 8 and standard 2SLS theory, \( \hat{g}_{t,1} = \text{vect}(\hat{u}_t, \hat{v}_t, 0_{p_2+n(p-1)}) = O_p(1) \), and \( \hat{A}_{s,\#} = A_{s,\#} + o_p(1) \), therefore \( \hat{A}_{s,\#}^{-1}\hat{g}_{t,1} = O_p(1) \), so \( E^b(T^{-\alpha} \eta_t^b) = o_p(1) \).

Next, we show that \( T^{-\alpha} \xi_t^b = o_p(1) \) by induction. First, recall that \( \xi_0^b = \xi_0 \), and therefore, \( T^{-\alpha} \xi_t^b = T^{-\alpha} \hat{\mu}_b + \hat{F}_t T^{-\alpha} \xi_0^b + T^{-\alpha} \eta_t^b = o_p(1) \) because \( \hat{\mu}_b - \mu_s = o_p(1) \), \( \hat{F}_t - \hat{F}_s = o_p(1) \), and \( T^{-\alpha} \eta_t^b = o_p(1) \). Now let \( T^{-\alpha} \xi_{t-1}^b = o_p(1) \); then for \( t, t - 1 \in I_s \), \( T^{-\alpha} \xi_t^b = T^{-\alpha} \hat{\mu}_b + \hat{F}_t T^{-\alpha} \xi_{t-1}^b + T^{-\alpha} \eta_t^b = o_p(1) + \hat{F}_t o_p(1) + o_p(1) = o_p(1) \).

Therefore, it follows that:

\[
T^{-\alpha} \xi_t^b = o_p(1).
\]

Next, we show that \( T^{-\alpha} \xi_t^b \xi_t^{b\prime} = o_p(1) \), also by mathematical induction. Note that, from the results above,

\[
T^{-\alpha} \xi_t^b \xi_t^{b\prime} = T^{-\alpha}(\hat{\mu}_b + \hat{F}_t \xi_{t-1}^b + \eta_t^b) (\hat{\mu}_b + \hat{F}_t \xi_{t-1}^b + \eta_t^b)^\prime
\]

\[
= T^{-\alpha} \hat{\mu}_b \hat{\mu}_b^\prime + \hat{F}_t (T^{-\alpha} \xi_{t-1}^b \xi_{t-1}^b) \hat{F}_t^\prime + T^{-\alpha} \eta_t^b \eta_t^{b\prime} + T^{-\alpha} \hat{\mu}_b \xi_{t-1}^b \eta_t^b + (T^{-\alpha} \hat{\mu}_b \xi_{t-1}^b \hat{F}_t^\prime)
\]

\[
+ T^{-\alpha} \hat{\mu}_b \eta_t^b + (T^{-\alpha} \hat{\mu}_b \eta_t^b)^\prime + \hat{F}_t T^{-\alpha/2} \xi_{t-1}^b T^{-\alpha/2} \eta_t^b + (\hat{F}_t T^{-\alpha/2} \xi_{t-1}^b T^{-\alpha/2} \eta_t^b)^\prime
\]

\[
= \hat{F}_t (T^{-\alpha} \xi_{t-1}^b \xi_{t-1}^b) \hat{F}_t^\prime + T^{-\alpha} \eta_t^b \eta_t^{b\prime} + o_p(1).
\]

Now consider \( \text{vect}(T^{-\alpha} \eta_t^b \eta_t^{b\prime}) = T^{-\alpha} \eta_t^b \odot \eta_t^{b\prime} \). We have:

\[
\eta_t^b \odot \eta_t^b = (\hat{A}_{s,\#}^{-1} \odot \hat{A}_{s,\#}^{-1})(g_t^b \odot g_t^b) \text{ and } E^b(g_t^b \odot g_t^b)(g_t^b \odot g_t^b)^\prime = E^b((g_t^b g_t^b)^\prime \odot (g_t^b g_t^b)^\prime).
\]
Since \( g_j^i g_j^i \) and \( \nu_j \nu_j \), the typical non-zero elements of \( E^b((g_j^i g_j^i) \otimes (g_j^i g_j^i)) \) are \( O_p(1) \otimes (\nu_j \nu_j) \), with \( j = 0, 1, \ldots, 4 \). By Assumption 10, sup \( E^b(\nu_j) < \infty \), it follows that \( E^b((g_j^i g_j^i) \otimes (g_j^i g_j^i)) = O_p(1) \), which implies that \( T^{\alpha} E^b((\eta_i^b \eta_i^b) \otimes (\eta_i^b \eta_i^b)) = O_p(T^{\alpha}) \). By Markov’s inequality, for any \( C > 0 \), \( P^b(T^{-\alpha} \| \eta_i^b \otimes \eta_i^b \| > C) \leq T^{-2\alpha} C^{-2} E^b(\| \eta_i^b \otimes \eta_i^b \|)^2 \leq T^{-2\alpha} C^{-2} E^b((\eta_i^b \eta_i^b) \otimes (\eta_i^b \eta_i^b))) \), if it follows that \( T^{-\alpha} \eta_i^b \eta_i^b = o_p(1) \).

Using this result in (49), by a similar mathematical induction argument as for \( T^{-\alpha} \xi_i^b = o_p(1) \), it follows that

\[
T^{-\alpha} \xi_i^b \xi_i^b = o_p(1). \tag{50}
\]

Besides (48) and (50), in the proof below we will assume that \( |I_n - \hat{C}_1 a - \hat{C}_2 a^2 - \cdots - \hat{C}_p a^p| \neq 0 \), for all \( s = 1, \ldots, N + 1 \), and all \( |a| \leq 1 \); otherwise the estimated system is not stationary. Then we show in the Supplementary Appendix, Section 2, that \( \sum_{t=0}^{\infty} \| F_t \| < \infty \), and similarly, it can be shown that \( \sum_{t=0}^{\infty} \| \hat{F}_t \| < \infty \).

Moreover, the results in the Supplementary Appendix Section 3 show that \( R_{s,t} = \hat{F}_s - F_s \) is such that

\[
\sum_{t=0}^{\infty} \| R_{s,t} \| = \| F_s - F_s \| = o_p(1), \tag{51}
\]

an argument which will be used repeatedly in the proofs.

**Now consider the case where \( I_t = I_s \) first, and analyze \( A_{t,t}^b \).** From (48),

\[
A_{t,t}^b = T^{-1} \sum_{s \in \bar{I}_t} \xi_t^b = T^{-1} \sum_{s \in \bar{I}_t} \xi_t^b = T^{-1} \sum_{r \in \bar{I}_t} \xi_t^b \tag{52}
\]

Therefore, we now derive the limit of \( A_{t,t}^b \). Note that

\[
\xi_t^b = \mu_s + \hat{F}_s \xi_{t-1} + \eta_t = \hat{F}_s \xi_{t-1} + \xi_t^b = \left( \sum_{r=t}^{\infty} \| R_{s,r} \| \right) \mu_s,
\]

where \( \xi_t^b = \sum_{t=0}^{\infty} \| R_{s,t} \| \). Therefore, \( A_{t,t}^b = \sum_{s=1}^{4} A_{t,t}^b \), where \( \Delta r \) is \( \left[ \{T, \{T - r_{s-1} \} \} \right] \), and

\[
A_{t,1}^b = T^{-1} \sum_{r=1}^{\infty} \| R_{s,r} \| \mu_s, \quad A_{t,2}^b = T^{-1} \sum_{r=1}^{\infty} \| R_{s,r} \| \mu_s, \quad A_{t,3}^b = T^{-1} \sum_{r=1}^{\infty} \| R_{s,r} \| \mu_s.
\]

We show that \( A_{t,1}^b = o_p(1) \). First, we show \( E^b(A_{t,1}^b) = o_p(1) \). Second, we show \( \text{Var}^b(A_{t,1}^b) = o_p(1) \). which by Markov’s inequality implies that \( A_{t,1}^b = o_p(1) \). Consider \( E^b(A_{t,1}^b) \) with \( E^b(\eta_{t-1}^b) = \sum_{t=0}^{\infty} \| R_{s,t} \| \mu_s \).

We have \( \xi_t = \mu_s + \hat{F}_s \xi_{t-1} + \eta_t = \mu_s + \hat{F}_s \xi_{t-1} + \eta_t \). Then,

\[
\hat{\eta}_t = \eta_t + (\mu_s - \hat{\mu}_s) + (F_s - \hat{F}_s) \xi_{t-1}, \tag{53}
\]

\[
\hat{g}_t = A_{s,t} \hat{\eta}_t = A_{s,t} \eta_t + A_{s,t} (\mu_s - \hat{\mu}_s) + A_{s,t} (F_s - \hat{F}_s) \xi_{t-1} \tag{54}
\]

Therefore, \( \eta_t^b = (\eta_t + (\mu_s - \hat{\mu}_s) + (F_s - \hat{F}_s) \xi_{t-1}) \).

Note that \( \mu_s - \hat{\mu}_s = (\hat{c}_{z,s} - c_z) = \text{vec}(d_1, d_1, \hat{0}_{p_z}, \hat{0}_{p_z}) \), where \( d_1, d_1 \) are of dimension 1 and \( p_1 \times 1 \), respectively, and this holds because the rows \( p_2 + 1 : n \) are not estimated since the equation for \( r_t \) is not estimated. Let \( \hat{a}_1, \hat{a}_1, \hat{a}_2, \hat{a}_2, \) be rows 1, \( 2 : p_1 + 1 \) and \( p_1 + 2 : n \) of the matrix \( \hat{A}_t \) respectively. Note that like \( A_t^{-1}, \hat{A}_t^{-1} \) is upper triangular with \( \hat{A}_{p_2} = \hat{0}_{p_2 \times (p_1 + 1), I_p} \) because the equation for \( r_t \) is not estimated, and \( r_t \) is assumed
contemporaneously exogenous. Therefore,

\[
\begin{bmatrix}
\tilde{A}_{s, \#}^{-1} (\mu_s - \tilde{\mu}_s)
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}_{s, \#}^{-1} (\tilde{c}_{\xi, s} - c_{\xi, s}) \\
0_{n(p-1)}
\end{bmatrix} =
\begin{bmatrix}
\tilde{a}_{1,\cdot} (\tilde{c}_{\xi, s} - c_{\xi, s}) \\
\tilde{A}_{p1, \cdot} (\tilde{c}_{\xi, s} - c_{\xi, s}) \\
\tilde{A}_{p2, \cdot} (\tilde{c}_{\xi, s} - c_{\xi, s}) \\
0_{n(p-1)}
\end{bmatrix} =
\begin{bmatrix}
\tilde{a}_{1,\cdot} (\tilde{c}_{\xi, s} - c_{\xi, s}) \\
\tilde{A}_{p1, \cdot} (\tilde{c}_{\xi, s} - c_{\xi, s}) \\
0_{p2} \\
0_{n(p-1)}
\end{bmatrix},
\]

so

\[
(\tilde{A}_{s, \#} (\mu_s - \tilde{\mu}_s)) \circ \mathcal{I} = 0_{np}.
\]  

(56)

By similar arguments, because the \(p_1 + 2 : n\) rows of \(\tilde{F}_s\) are equal to the corresponding rows of \(F_s\), \(\tilde{A}_{s, \#} (\tilde{F}_s - F_s)\) are equal to zero, therefore

\[
(\tilde{A}_{s, \#} (\tilde{F}_s - F_s) \xi_{t-1}) \circ \mathcal{I} = 0_{np}.
\]  

(57)

Using (56)-(57), and letting \(\tilde{t} = t - \tau_{s-1} T\), we have:

\[
E^b(A_{11}^b) = \sum_{i=1}^3 \mathcal{H}_i,
\]

where:

\[
\mathcal{H}_1 = T^{-1} \sum_{t \in I_1} \sum_{l=0}^{t-1} \tilde{F}_s^l \tilde{A}^{-1}_{s, \#} ((\tilde{A}_{s, \#} \eta_{t-l}) \circ \mathcal{I}) = T^{-1} \sum_{t \in I_1} \sum_{l=0}^{\tilde{t}-1} \tilde{F}_s^l \tilde{A}^{-1}_{s, \#} ((\tilde{A}_{s, \#} \eta_{t-l}) \circ \mathcal{I}),
\]

\[
\mathcal{H}_2 = T^{-1} \sum_{t \in I_1} \sum_{l=0}^{t-1} \tilde{F}_s^l \tilde{A}^{-1}_{s, \#} ((\tilde{A}_{s, \#} (\mu_s - \tilde{\mu}_s)) \circ \mathcal{I}) = 0_{np},
\]

\[
\mathcal{H}_3 = T^{-1} \sum_{t \in I_1} \sum_{l=0}^{t-1} \tilde{F}_s^l \tilde{A}^{-1}_{s, \#} ((\tilde{A}_{s, \#} (F_s - \tilde{F}_s) \xi_{t-1}) \circ \mathcal{I}) = 0_{np}.
\]

Since \(\tilde{A}_{s, \#} A_{s, \#}^{-1} = I_{n, \#} + o_p(1)\), it follows that:

\[
\mathcal{H}_1 = T^{-1} \sum_{t \in I_1} \sum_{l=0}^{\tilde{t}-1} \tilde{F}_s^l A_{s, \#}^{-1} (g_{t-l} \circ \mathcal{I}) + T^{-1} \sum_{t \in I_1} \sum_{l=0}^{\tilde{t}-1} R_{s, l} A_{s, \#}^{-1} (g_{t-l} \circ \mathcal{I}) + o_p(1)
\]

\[
= \mathcal{H}_1^{(1)} + \mathcal{H}_1^{(2)} + o_p(1).
\]

From Assumptions 7 and 9, and using \(\sum_{l=0}^{\infty} \|F_s^l\| < \infty \) and \(\sum_{l=0}^{\infty} \|R_{s, l}\| < \infty \), both proven in the Supplementary Appendix Sections 2-3, it can be shown that \(\sum_{l=0}^{\tilde{t}-1} \tilde{F}_s^l A_{s, \#}^{-1} (g_{t-l} \circ \mathcal{I}) \) and \(T^{-1} \sum_{t \in I_1} \sum_{l=0}^{\tilde{t}-1} R_{s, l} A_{s, \#}^{-1} (g_{t-l} \circ \mathcal{I}) \) are \(L^1\)-mixingales satisfying the conditions of Lemma 1, therefore \(\mathcal{H}_1^{(1)} = o_p(1)\) and \(\mathcal{H}_1^{(2)} = o_p(1)\). Hence \(\mathcal{H}_1 = o_p(1)\), so \(E^b(A_{11}^b) = o_p(1)\).

Second, we show that \(Var^b(A_{11}^b) = o_p(1)\). To that end, note that

\[
E^b(\eta_{t-l}^{b, \cdot} \eta_{t-\kappa}^{b, \cdot}) = \tilde{A}_{s, \#}^{-1} E^b((\tilde{g}_{t-l} \circ \nu_{t-l})(\tilde{g}_{t-\kappa} \circ \nu_{t-\kappa}))(\tilde{A}_{s, \#}^{-1})' = \tilde{A}_{s, \#}^{-1} ((\tilde{g}_{t-l} \tilde{g}_{t-\kappa}^l \circ \mathcal{I}))(\tilde{A}_{s, \#}^{-1})',
\]

for \(l \neq \kappa\), \(E^b(\nu_{t-l} \nu_{t-\kappa}^l) = \mathcal{I} = [\text{diag}(0_{p_1+1} \times 0_{p_1+1})]_{\#} = \mathcal{J}_2\). Therefore, exploiting the upper triangular structure of \(\tilde{A}_{s, \#}^{-1}\) with \(p_2 \times p_2\) lower right block equal to \(I_{p_2}\), for \(l \neq \kappa\),

\[
E^b(\eta_{t-l}^{b, \cdot} \eta_{t-\kappa}^{b, \cdot}) = \tilde{A}_{s, \#}^{-1} ((\tilde{g}_{t-l} \tilde{g}_{t-\kappa}^l) \circ \mathcal{J}_2)(\tilde{A}_{s, \#}^{-1})' = ((\tilde{A}_{s, \#}^{-1} \tilde{g}_{t-l}) \circ \mathcal{I}))(\tilde{A}_{s, \#}^{-1} \tilde{g}_{t-\kappa}) \circ \mathcal{I})' = \eta_{t-1,2}^{\cdot, \cdot} \eta_{t-\kappa, 2, \kappa},
\]

where \(\eta_{t,2} = g_{t,2} = [\text{vec}(0_{p_1+1} \times 0_{p_1+1})]_{\#}\). For \(l = \kappa\), \(E^b(\eta_{t-l}^{b, \cdot} \eta_{t-\kappa}^{b, \cdot}) = \tilde{A}_{s, \#}^{-1} ((\tilde{g}_{t-l} \tilde{g}_{t-\kappa}^l) \circ \mathcal{J})(\tilde{A}_{s, \#}^{-1})',\)

where \(E^b(\nu_{t-l} \nu_{t-\kappa}^l) = \mathcal{J}_2\), so \(T^{-1} \sum_{t \in I_1} (\tilde{g}_{t-l} \tilde{g}_{t-\kappa}^l) \circ \mathcal{J} = T^{-1} \sum_{t \in I_1} \tilde{g}_{t-l} \tilde{g}_{t-\kappa}^l + o_p(1)\) by Assumption 9(ii), Lemma 8 followed by standard 2SLS theory. Therefore,

\[
T^{-1} \sum_{t \in I_1} E^b(\eta_{t-l}^{b, \cdot} \eta_{t-\kappa}^{b, \cdot}) = \tilde{A}_{s, \#}^{-1} ((T^{-1} \sum_{t \in I_1} \tilde{g}_{t-l} \tilde{g}_{t-\kappa}^l) (\tilde{A}_{s, \#}^{-1})' +
\]

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\( o_p(1) = T^{-1} \sum_{i \in I} \eta_{t-i} \eta_{t-i} + o_p(1) \). Hence, 

\[
\begin{align*}
\text{Var}^h(A_{1,1}^h) &= T^{-2} \sum_{i \in I} F_i'(\xi_i \xi_i') = V_1 + V_2 + o_p(1) \quad (58) \\
V_1 &= T^{-2} \sum_{i \in I} \sum_{i=0}^{t-1} F'_i \eta_{t-i} \eta_{t-i} (\hat{F}_s') \\
V_2 &= T^{-2} \sum_{i \in I} \sum_{i=0 \neq k}^{t-1} F'_i \eta_{t-i} \eta_{t-i,2} (\hat{F}_s') \\&= B_{1,1}^{(1)} + B_{1,1}^{(2)} + o_p(1) = B_1(\tau_{s-1}, \tau_s) + o_p(1), \quad (59)
\end{align*}
\]

Consider \( V_1 \). We analyze first \( V_1^* \), where

\[
V_1^* = T^{-1} \sum_{i \in I} \sum_{i=0}^{t-1} F'_i \eta_{t-i} \eta_{t-i} (\hat{F}_s') = B_{1,1}^{(1)} + o_p(1) = B_1(\tau_{s-1}, \tau_s) + o_p(1). \quad (60)
\]

Now consider \( V_2 \), where

\[
\begin{align*}
V_2 &= T^{-2} \sum_{i \in I} \sum_{i=0}^{t-1} F'_i \eta_{t-i} \eta_{t-i} (\hat{F}_s') \\
\end{align*}
\]

Similarly to \( V_1^* \), because \( \hat{F}_s' - F_s' = R_{s,l} = o_p(1) \), we can show that \( V_2^{(2)} = o_p(1) \). From (51), \( R_{s,l} \) is such that \( \sum_{i=0}^{t-1} R_{s,l} = \| \hat{F}_s - F_s \| O_p(1) = o_p(1) \). Therefore, by the same arguments as for \( V_2^{(2)} = o_p(1) \), one can show that \( V_2^{(3)} = o_p(1) \), therefore \( TV_2 = B_1(\tau_{s-1}, \tau_s) + o_p(1) \), and \( V_1 = o_p(1) \).

By similar arguments to the analysis of the term \( B_{1,1} \) in the Supplementary Appendix Section 1, proof of Lemma 2, \( TV_2 = o_p(1) \). Substituting \( V_2 = o_p(1) \) and \( V_1 = o_p(1) \) into (58), it follows that \( \text{Var}^h(A_{1,1}^h) = V_1 + V_2 = o_p(1) \), and, by Markov’s inequality, that \( A_{1,1}^h = o^h_p(1) \). It also follows that:

\[
TV \text{Var}^h(A_{1,1}^h) = B_1(\tau_{s-1}, \tau_s) + o_p(1), \quad (61)
\]

a stronger result that we need later in this proof.

Consider now \( A_{1,2}^h, A_{1,3}^h, A_{1,4}^h \). We have \( A_{1,2}^h = T^{-1} \Delta \tau_s T \sum_{i=0}^{\Delta \tau_s} F_i \mu_s + o_p(1) \) and by Assumption 7, it follows that \( A_{1,2}^h = T^{-1} \Delta \tau_s \sum_{i=0}^{\Delta \tau_s} F_i \mu_s + o_p(1) \) and \( \sum_{i=1}^{\infty} \| \hat{F}_s'' \| = O_p(1) \).

\[
\| A_{1,3}^h \| = \| T^{-1} \sum_{i=1}^{\Delta \tau_s} \hat{F}_i \xi_{[\tau_{s-1}, \tau_s]} \| \leq \| \sum_{i=1}^{\infty} \hat{F}_i'' \| T^{-1} \xi_{[\tau_{s-1}, \tau_s]} = O_p(1) o^h_p(1) = o^h_p(1).
\]
Since $\sum_{l=1}^{T} l \hat{F}_s^l = O_p(1)$ and $\mu_s - \mu_s = o_p(1)$, $A_{1,4} = o_p(1)$. Combining these results, we obtain:

$$A_1^h = \Delta \tau_s (I_{n_p} - F_s)^{-1} \mu_s + o_p(1) = \int_{\tau_{s-1}}^{\tau_s} \hat{Q}_s(\tau) d\tau + o_p(1) = A_1 + A_1^h(1);$$

where $A_1 = \int_{\tau_{s-1}}^{\tau_s} \hat{Q}_s(\tau) d\tau + o_p(1)$ from (1.1) in Supplementary Appendix, Section 1, and $A_i, i = 1, 2, 4$ are the sample equivalents of $A_i^h$. From (52), it follows that $A_2^h = A_1 + o_p(1) = A_2 + o_p(1) + \int_{\tau_{s-1}}^{\tau_s} \hat{Q}_s(\tau) d\tau$.

- Next, analyze $B_1^h$. First, note that because $T^{-1} \xi_s^{j} \xi_s^{\prime} = \hat{o}_s(1)$ as shown in the preliminaries of this proof,

$$B_1^h = T^{-1} \sum_{l \in \tilde{I}_s} \xi_l^{j} \xi_l^{\prime} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime,$$

so we analyze instead $B_1^{j}$. Note that $B_1^{j} = \sum_{l=1}^{3} B_{1,l}^{j} + \sum_{j=1}^{3} \left\{ B_{1,3+j}^{j} + B_{1,3+j}^{j} \right\}$, where

$$B_{1,1}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \hat{F}_s^{l} \eta_{l-1} \eta_{l-1}^\prime = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime,$$

$$B_{1,2}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime,$$

$$B_{1,3}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime,$$

$$B_{1,4}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime,$$

$$B_{1,5}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime.$$ 

Let

$$B_{1}(\tau_{s-1}, \tau_s) = \sum_{l=0}^{\infty} F_s^{l} \left( A_s^{-1} \int_{\tau_{s-1}}^{\tau_s} \sum d\tau A_s^{-1} \right) F_s^{l}.$$ 

We show $B_{1}^{j} - B_{1}(\tau_{s-1}, \tau_s) = o_p(1)$ by showing that $E^{j}(B_{1,1}^{j} - B_{1}(\tau_{s-1}, \tau_s)) = o_p(1)$ and $Var^{j}(\text{vec} B_{1,1}^{j}) = o_p(1)$.

$$B_{1,1}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \eta_{l-1}^\prime \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime = \sum_{l=0}^{\infty} F_s^{l} \left( A_s^{-1} \int_{\tau_{s-1}}^{\tau_s} \sum d\tau A_s^{-1} \right) F_s^{l}.$$ 

$$E^{j}(B_{1,1}^{j}) = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \eta_{l-1}^\prime \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \eta_{l-1} \right) \left( \sum_{l=0}^{l-1} \hat{F}_s^{l} \mu_s \right)^\prime = \sum_{l=0}^{\infty} F_s^{l} \left( A_s^{-1} \int_{\tau_{s-1}}^{\tau_s} \sum d\tau A_s^{-1} \right) F_s^{l}.$$ 

where the last equality above follows from (61). We have

$$\text{vec} B_{1,1}^{j} = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \left( \hat{F}_s^{l} \hat{A}_s^{-1} \right) \otimes \left( \hat{F}_s^{l} \hat{A}_s^{-1} \right) \right) \left( \hat{g}_t^{l} \otimes \hat{g}_t^{l} \right) = T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \left( \hat{F}_s^{l} \hat{A}_s^{-1} \right) \otimes \left( \hat{F}_s^{l} \hat{A}_s^{-1} \right) \right) \left( \hat{g}_t^{l} \otimes \hat{g}_t^{l} \right) \left( \hat{g}_t^{l} \otimes \hat{g}_t^{l} \right)$$

$$= T^{-1} \sum_{l \in \tilde{I}_s} \left( \sum_{l=0}^{l-1} \left( \hat{F}_s^{l} \hat{A}_s^{-1} \right) \otimes \left( \hat{F}_s^{l} \hat{A}_s^{-1} \right) \right) \left( \hat{g}_t^{l} \otimes \hat{g}_t^{l} \right) \left( \hat{g}_t^{l} \otimes \hat{g}_t^{l} \right) = \sum_{l=0}^{\infty} F_s^{l} \left( A_s^{-1} \int_{\tau_{s-1}}^{\tau_s} \sum d\tau A_s^{-1} \right) F_s^{l}.$$ 

$$\text{Var}^{j}(\text{vec} B_{1,1}^{j}) = E^{j}(\text{vec} B_{1,1}^{j}) - E^{j}(\text{vec} B_{1,1}^{j}) E^{j}(\text{vec} B_{1,1}^{j})' = E^{j}(\text{vec} B_{1,1}^{j}) - E^{j}(\text{vec} B_{1,1}^{j}) E^{j}(\text{vec} B_{1,1}^{j})' + o_p(1).$$
We need to show that  
\[ E^b(\text{vec}(\hat{B}^b_{\text{I},1}(\text{vec}(\hat{B}^b_{\text{I},1})))) = \text{vec}(\hat{B}_1(\tau_{s-1}, \tau_s)) \text{ vec}(\hat{B}_1(\tau_{s-1}, \tau_s))'. \]

Letting \( \tilde{t} = t^* - [\tau_{s-1} T] \),  
\[
E^b(\text{vec}(\hat{B}^b_{\text{I},1}(\text{vec}(\hat{B}^b_{\text{I},1}))))' = \left[ T^{-1} \sum_{t \in \tilde{t}} \sum_{t', \kappa=0}^{t-1} \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right) (g_{t'-\kappa}^b \otimes \nu_{t'-\kappa}) \right] \\
\times \left[ T^{-1} \sum_{t,t' \in \tilde{t}} \sum_{t', \kappa=0}^{t-1} \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right) (g_{t'-\kappa}^b \otimes \nu_{t'-\kappa}) \right] \\
= T^{-2} \sum_{t,t' \in \tilde{t}} \sum_{t', \kappa=0}^{t-1} \sum_{t', \kappa=0}^{t-1} \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right) G((\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1})' \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}')). \\
\sum_{i=1}^9 O_i,
\]
where \( O_i \) are the terms corresponding to nine cases when \( G \neq O_{(n \tau_p) \times (n \tau_p^2)} \). Case (1) is when \( t - \kappa = t - l, t^* - \kappa^* = t^* - l^* \), \( t - \kappa \neq t^* - \kappa^* \); we show below that \( O_1 = \text{vec}(\hat{B}_1(\tau_{s-1}, \tau_s)) \text{ vec}(\hat{B}_1(\tau_{s-1}, \tau_s))' + o_p(1) \). For brevity, the rest of the cases are defined and analyzed in Supplementary Appendix, Section 4, where we show that
\[
O_i = o_p(1) \text{ for } i = 2, \ldots, 9. \quad (63)
\]

By Assumption 10, \( E^b(\nu_1 \nu_1' \otimes (\nu_{t-1} \nu_{t-1}')) = E^b(\nu_1 \nu_1') E^b(\nu_{t-1} \nu_{t-1}) \) since \( E^b(\nu_1^2 \nu_{t-1}^2) = E^b(\nu_1^2) E^b(\nu_{t-1}^2) = 1 \), \( E^b(\nu_1 \nu_{t-1}) = 0 \) and \( E^b(\nu_{t-1} \nu_{t-1}) = 0 \) (these are elements of \( E^b(\nu_1 \nu_1' \otimes (\nu_{t-1} \nu_{t-1}')) \)). Hence, conditional on the data, we have, by Assumption 10,
\[
\begin{align*}
G &= E^b((g_{t-\kappa}^b \otimes \nu_{t-\kappa})(g_{t-\kappa}^b \otimes \nu_{t-\kappa}'))' \\
&= E^b((\hat{g}_{t-\kappa}^b \otimes \nu_{t-\kappa})(\nu_{t-\kappa} \otimes \nu_{t-\kappa}))' \\
&= E^b((\hat{g}_{t-\kappa}^b \otimes \nu_{t-\kappa})(\nu_{t-\kappa} \otimes \nu_{t-\kappa}))' \\
&= E^b((\hat{g}_{t-\kappa}^b \otimes \nu_{t-\kappa})(\nu_{t-\kappa} \otimes \nu_{t-\kappa}))' \\
&= (\hat{g}_{t-\kappa} \otimes \nu_{t-\kappa})(\nu_{t-\kappa} \otimes \nu_{t-\kappa}),
\end{align*}
\]

hence
\[
O_1 = \left[ T^{-1} \sum_{t \in \tilde{t}} \sum_{t', \kappa=0}^{t-1} \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right) (g_{t'-\kappa}^b \otimes \nu_{t'-\kappa}) \right] \\
\times \left[ T^{-1} \sum_{t,t' \in \tilde{t}} \sum_{t', \kappa=0}^{t-1} \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right) (g_{t'-\kappa}^b \otimes \nu_{t'-\kappa}) \right] \\
= \Delta_{t,t'} T^{-1} \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right) \left( T^{-1} \sum_{t \in \tilde{t}} (g_{t-\kappa} \otimes \nu_{t-\kappa}) \right) \\
\times \left( T^{-1} \sum_{t \in \tilde{t}} (g_{t-\kappa} \otimes \nu_{t-\kappa}) \right) \left( (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \otimes (\hat{F}_{\tilde{s}}^\kappa \hat{A}_{\tilde{s},\#}^{-1}) \right)',
\]

where the last two lines follow because \( g_{t-\kappa} \otimes \nu_{t-\kappa} = \text{vec}(g_{t-\kappa} \nu_{t-\kappa}) \), and
\[
T^{-1} \sum_{t \in \tilde{t}} (g_{t-\kappa} \otimes \nu_{t-\kappa}) = T^{-1} \sum_{t \in \tilde{t}} \text{vec}(g_{t-\kappa} \nu_{t-\kappa}) = \text{plim}_{T \to \infty} T^{-1} \sum_{t \in \tilde{t}} \text{vec}(g_{t-\kappa} \nu_{t-\kappa}) + o_p(1),
\]

which follows by standard 2SLS theory and Lemma 8. So, \( O_1 = \text{vec}(\hat{B}_1(\tau_{s-1}, \tau_s)) \text{ vec}(\hat{B}_1(\tau_{s-1}, \tau_s))' + o_p(1) \).

Therefore, \( \text{Var}(\text{vec}(\hat{B}^b_{\text{I},1})) = o_p(1) \), so by Markov’s inequality,
\[
\hat{B}^b_{\text{I},1} = B_1(\tau_{s-1}, \tau_s) + o_p(1).
\]
Next, because $\mu_s = \mu_s + o_p(1)$, and $F_s = F_s + o_p(1)$, and $\sum_{l=0}^{\infty} \| F_s^{l} - F_s^{l} \| = o_p(1)$ as shown in Supplementary Appendix, Section 3, $B_{1,2}^b = B_{1,2} + o_p(1) = \mathbb{B}_2(\tau_{s-1}, \tau_s) + o_p(1)$, where $\mathbb{B}_2(\tau_{s-1}, \tau_s) = \int_{\tau_{s-1}}^{\tau_s} Q_1(\tau)Q_1(\tau) d\tau$, and $B_{1,2}$ is the sample equivalent of $B_{1,2}^b$ (and in general, $B_{1,i}, i = 1, \ldots, 6$ are the sample equivalents of $B_{1,i}^b$, defined in the proof of Lemma 2 in Supplementary Appendix, Section 1). Also, we have $B_{1,3}^b = B_{1,3} + o_p(1) = o_p(1)$, because, as shown in the preliminaries $T^{-\alpha} \xi b_t \xi b_t' = o_p(1)$, and $F_s$ is exponentially decaying with $l$.

Consider $B_{1,4}^b$.

$$B_{1,4}^b = T^{-1} \sum_{l \in I_s} \left( \sum_{i=0}^{l-1} \xi b_{\tau_{i-1}} \right) \left( \sum_{i=0}^{l-1} \xi b_{\tau_{i-1}} \mu_s \right)' = T^{-1} \sum_{l \in I_s} \left( \sum_{i=0}^{l-1} \xi b_{\tau_{i-1}} (\hat{A}_{s,i}^{-1} g_{\tau_{i-1}} \mu_s) \right)' \left( \sum_{i=0}^{l-1} \xi b_{\tau_{i-1}} \mu_s \right)' .$$

We show $B_{1,4}^b = o_p(1)$. To that end note that

$$E^b(B_{1,4}^b) = T^{-1} \sum_{l \in I_s} \left( \sum_{i=0}^{l-1} F_s^b \eta_{\tau_{i-1}} \right)' \left( \sum_{i=0}^{l-1} F_s^b \mu_s \right)' + o_p(1) = o_p(1),$$

by similar arguments as for its sample equivalent $B_{1,4}$ defined in the proof of Lemma 2. So, $E^b(B_{1,4}^b) = o_p(1)$.

Moreover, by similar arguments as before, it can be shown that $\| \text{vec}\text{Var}^b(B_{1,4}^b) \| = o_p(1)$. Hence, by Markov’s inequality, $B_{1,4}^b = o_p(1)$. Similarly, because $T^{-\alpha} \xi b_{\tau_{s-1}T} = O_p(1)$ for any $\alpha > 0$, we can show that $B_{1,5}^b = o_p(1)$, and $B_{1,6}^b = o_p(1)$. Putting all the results for $B_{1,i}^b$ together, $i = 1, \ldots, 6$ we conclude $B_{1}^b = \mathbb{B}_1(\tau_{s-1}, \tau_s) + o_p(1) = B_1 + o_p(1)$, where $B_1 = \mathbb{B}_1(\tau_{s-1}, \tau_s) + \mathbb{B}_2(\tau_{s-1}, \tau_s) + B_3$ defined in the proof of Lemma 2.

Because $B_3 = B_1^b + o_p(1)$, it follows that $B_3 = B_1 + B_2 + B_3 + o_p(1)$, where the same result was shown to hold for $B_3$ defined in the proof of Lemma 2. Now consider $B_2^b$. Using $\xi b = \hat{\mu}_s + \hat{F}_s \xi b_{\tau_{s-1}} + \eta_{b}^t$, it follows that:

$$B_2^b = \mu_s A_{1,2}^b + \hat{F}_s B_3^b + T^{-1} \sum_{l \in I_s} \xi b_{\tau_{s-1}} \eta_{b}^t + o_p(1).$$

By similar arguments as for some elements of $B_1$, it can be shown that $T^{-1} \sum_{l \in I_s} \xi b_{\tau_{s-1}} \eta_{b}^t = o_p(1)$. Therefore,

$$B_2^b = \int_{\tau_{s-1}}^{\tau_s} \mu(\tau) Q_1(\tau) d\tau + \int_{\tau_{s-1}}^{\tau_s} F(\tau) Q_2(\tau) d\tau + o_p(1).$$

Therefore, for $I_i = \tilde{I}_s$,

$$\hat{Q}_i^b = \int_{\tau_{s-1}}^{\tau_s} \gamma(\tau) Q_2(\tau) d\tau + o_p(1) = Q_i(\tilde{I}) + o_p(1).$$

For other regimes, by similar arguments as in the end of the proof of Lemma 2,

$$\hat{Q}_i^b = \int_{\tau_{s-1}}^{\tau_s} \gamma(\tau) Q_2(\tau) d\tau + o_p(1) = Q_i(\tilde{I}) + o_p(1),$$

concluding the proof.

\[ \Box \]

**Lemma 10.** $T^{-1/2} \sum_{l \in I,} z_l g_l b \xi b_{\tau_{s-1}} \rightarrow \text{in probability uniformly in } \lambda_k$, where $\tilde{M}_k$ is as defined in Lemma 6, and $S_{\tau_{s-1}}^b = S_u$ or $S_{\tau_{s-1}}^b = (\hat{\beta}_{\tau_{s-1}}(1))$. If $m = 0$, then $S_{\tau_{s-1}}^b = S_u$ or $S_{\tau_{s-1}} = \hat{\beta}_{\tau_{s-1}}$.\]

**Lemma 11.** $T^{-1/2} \sum_{l \in I,} z_l g_l b \xi b_{\tau_{s-1}} \rightarrow \text{in probability uniformly in } \lambda_k$.

For the proofs of Lemma 10-11, it suffices to consider $S_{\tau_{s-1}}^b = S_u$ or $S_{\tau_{s-1}}^b = \hat{\beta}_{\tau_{s-1}}$, therefore considering $m = 0$. If $S_{\tau_{s-1}}^b = (\hat{\beta}_{\tau_{s-1}}(1))$, by Lemma 7 followed by standard 2SLS theory, $\hat{\beta}_{\tau_{s-1}} = \beta_{\tau_{s-1}} + O_p(T^{-1/2})$ so $S_{\tau_{s-1}}^b = S_1 + O_p(T^{-1/2})$,
and the results follow in a similar fashion.

Additionally to the notation already defined at the beginning of the proof of Lemma 9, we use the following results and notation, some relevant for both Lemma 10 and 11. Consider the partition $\hat{I}_s$, then for the WR bootstrap we have $\hat{z}_{\hat{i}}^b = \hat{c}_{\hat{i},s} + \sum_{i=1}^p \hat{C}_{i,s} \hat{z}_{\hat{i}-1} + e_{\hat{i}}^b$, and for both WR and WF bootstraps, we have $e_{\hat{i}}^b = \hat{A}_s^{-1} e_{\hat{i}}^b$. We have for the WR bootstrap:

$$\xi_{\hat{i}}^b = \hat{\mu}_s + \hat{F}_s \xi_{\hat{i}-1} + \eta_{\hat{i}}^b = \hat{F}_s^{-\lfloor \tau_{\hat{i}}-T \rfloor} \xi_{\hat{i}-1} + \sum_{l=0}^{T-\lfloor \tau_{\hat{i}}-T \rfloor-1} \hat{F}_s \eta_{\hat{i}-l} + \left( \sum_{l=0}^{T-\lfloor \tau_{\hat{i}}-T \rfloor-1} \hat{F}_s \right) \hat{\mu}_s$$

(64)

except that in (64) when $s = 1$ and we are in the first regime $\hat{I}_1 = [1, \ldots, [\tau_1 T]]$, we have that $\xi_{\hat{i}}^b = \xi_0$, where $\xi_{\hat{i}}^b = \text{vect}_{j=0:(p-1)}(\hat{z}_{\hat{i}-1}^b)$. Let $F_{\hat{i}}^b = \{ \nu_1, \nu_{t-1}, \ldots, \nu_1 \}$. Recall that, by Assumption 10,

$$E^b(\nu_i) = E^b(\nu_i | F_{\hat{i}-1}^b) = \text{vect}(0_{p_{t+1}, \nu_{t+2}, 0_{n(p-1)\times 1}}) = \mathcal{I}$$

(65)

$$E^b(\nu_i | \nu'_j) = E^b(\nu_i | \nu'_j, F_{\hat{i}-1}^b) = (\text{diag}(J_{p_{t+1}, J_{p_2}}))_{\#} = J$$

(66)

$$E^b(\nu_i | \nu'_j, F_{\hat{i}-1}^b) = (\text{diag}(0_{p_{t+1}, J_{p_2}}))_{\#} = J_2$$

(67)

Furthermore, recall that $\xi_{\hat{i}} = \hat{\mu}_s + \hat{F}_s \xi_{\hat{i}-1} + \eta_{\hat{i}}$, and therefore $\eta_{\hat{i}} = \eta_{\hat{i}} + (\hat{\mu}_s - \hat{\mu}_s) + (F_s - \hat{F}_s) \hat{\xi}_{\hat{i}-1}$. By backward substitution of $\xi_{\hat{i}} = \hat{\mu}_s + \hat{F}_s \xi_{\hat{i}-1} + \eta_{\hat{i}}$, we have that: $\xi_{t-1} = F_{\hat{i}-1}^{t-1} \xi_{[\tau_{\hat{i}}-T]} + \sum_{l=1}^{T-2} F_{\hat{i}}^l \eta_{t-l-1} + \left( \sum_{l=1}^{T-2} F_{\hat{i}}^l \right) \hat{\mu}_s$, where $t = \hat{t} - [\tau_{\hat{i}}-T]$. We also have $\eta_{\hat{i}}^b = \hat{A}_s^{-1} \eta_{\hat{i}}^b = \hat{A}_s^{-1} (g_{\hat{i}} \ominus \nu_{\hat{i}})$, where recall that $\ominus$ is the element-wise multiplication. Hence:

$$g_{\hat{i}} = (\hat{A}_s, \hat{\eta}_{\hat{i}}) \ominus \nu_{\hat{i}} + (\hat{A}_s, (\hat{\mu}_s - \hat{\mu}_s) \ominus \nu_{\hat{i}}) + (\hat{A}_s, (F_s - \hat{F}_s) \hat{\xi}_{\hat{i}-1}) \ominus \nu_{\hat{i}}$$

(68)

$$\eta_{\hat{i}}^b = \hat{A}_s^{-1} (\hat{A}_s, \hat{\eta}_{\hat{i}}) \ominus \nu_{\hat{i}} + \hat{A}_s^{-1} ((\hat{A}_s, (\hat{\mu}_s - \hat{\mu}_s) \ominus \nu_{\hat{i}}) + \hat{A}_s^{-1} ((\hat{A}_s, (F_s - \hat{F}_s) \hat{\xi}_{\hat{i}-1}) \ominus \nu_{\hat{i}}$$

(69)

$$\eta_{\hat{i}}^b = \hat{\eta}_{\hat{i}}^{b,A} + \hat{\eta}_{\hat{i}}^{b,B} + \hat{\eta}_{\hat{i}}^{b,C}$$

(70)

where $\|\hat{A}_s\| \leq \|A_s\|_\# + \|A_s - A_s\|_\# = \|A_s\|_\# + o_p(1)$ and hence $\hat{A}_s^{-1} = I_{n,\#} + o_p(1)$. Moreover, $\|\hat{A}_s^{-1}\|_\# \leq \|A_s^{-1}\|_\# + \|A_s^{-1} - A_s^{-1}\|_\# = \|A_s^{-1}\|_\# + o_p(1)$.

Finally, for a vector $o$, we denote $O^{(j_1,j_2)}$ its sub-vector with elements $j_1$ to $j_2$ selected in order, and for a matrix $O$, we denote by $O^{(j_1,j_2; j_1'; j_2')}$ its sub-matrix consisting of rows $j_1$ to $j_2$ and columns $j_1'$ to $j_2'$.  

**Proof of Lemma 10.** As for the proof of Lemma 9, consider the interval $I_i = \hat{I}_s$. Let $S_{\hat{i}}^b = S_u$ or $S_{\hat{i}}^b = \hat{\beta}_{x,u}$. We derive the asymptotic distribution of $T^{-1/2} \sum_{\hat{i} \in \hat{I}_s} z_{\hat{i}}^b g_{\hat{i}}^b S_{\hat{i}}^b$:

$$T^{-1/2} \sum_{\hat{i} \in \hat{I}_s} z_{\hat{i}}^b g_{\hat{i}}^b S_{\hat{i}}^b = \begin{bmatrix}
T^{-1/2} \sum_{\hat{i} \in \hat{I}_s} g_{\hat{i}}^b S_{\hat{i}}^b \\
T^{-1/2} \sum_{\hat{i} \in \hat{I}_s} S_{\hat{i}}^b S_{\hat{i}}^b \\
T^{-1/2} \sum_{\hat{i} \in \hat{I}_s} S_{\hat{i}}^b S_{\hat{i}}^b \\
\end{bmatrix} \begin{bmatrix}
\tilde{\xi}_1^b \\
\tilde{\xi}_2^b \\
\tilde{\xi}_3^b \\
\end{bmatrix}$$

(71)
• Consider first $\mathcal{E}_1^b$. By (69),

\[
\mathcal{E}_1^b = T^{-1/2} \sum_{t \in I_s} g_t^b S_t^b = T^{-1/2} \sum_{t \in I_s} g_t^b A_t S_t^b + T^{-1/2} \sum_{t \in I_s} g_t^b B_t^b S_t^b + T^{-1/2} \sum_{t \in I_s} g_t^b C_t^b S_t^b
\]

\[
= T^{-1/2} \sum_{t \in I_s} S_t^{\prime b}((\hat{A}_{s,#} \eta_t) \circ \nu_t) + T^{-1/2} \sum_{t \in I_s} S_t^{\prime b}((\hat{A}_{s,#}(\mu_t - \hat{\mu}_s)) \circ \nu_t)
\]

\[
+ T^{-1/2} \sum_{t \in I_s} S_t^{\prime b}((\hat{A}_{s,#}(F_s - \hat{F}_s)\xi_{t-1}) \circ \nu_t)
\]

\[
= \mathcal{E}_{1,1}^b + \mathcal{E}_{1,2}^b + \mathcal{E}_{1,3}^b.
\]

Because $\|S_t^b - S_t\| = o_p(1)$, $\|\hat{\mu}_s - \mu_s\| = o_p(1)$, $\|\hat{A}_s - A_s\| = o_p(1)$ and $\|\hat{A}_s^{-1} - A_s^{-1}\| = o_p(1)$ and $\sup_{t \in [1]} \|\hat{F}_t^b - F_t^b\| = o_p(1)$, whenever a $o_p(1)$ term is written with the estimated quantities instead of the true one, the difference is $o_p^b(1)$, so asymptotically negligible. Therefore, we proceed in the rest of the proof by replacing the estimated parameters mentioned above with their true values, and denote the remainder by $o_p^b(1)$.

Using these replacements, one can show that $\mathcal{E}_{1,1}^b = o_p^b(1)$ and $\mathcal{E}_{1,2}^b = o_p^b(1)$. So, $\mathcal{E}_1^b = S_t^b T^{-1/2} \sum_{t \in I_s} ((\hat{A}_{s,#} \eta_t) \circ \nu_t) + o_p^b(1)$. Since $\hat{A}_{s,#} \eta_t = A_{s,#} \hat{A}_{s,#} g_t = g_t = \epsilon_{t,#}$, it follows that

\[
\mathcal{E}_1^b = S_t^b T^{-1/2} \sum_{t \in I_s} (g_t \circ \nu_t) + o_p^b(1).
\]  

(72)

• First, let $S_t = S_{u,t}$. Then $\mathcal{E}_1^b = T^{-1/2} \sum_{t \in I_s} u_t \nu_t + o_p(1) = T^{-1/2} \sum_{t \in I_s} d_{u,t} u_t \nu_t + o_p(1) = \mathcal{E}_{1,1}^b + o_p(1)$, where recall that $d_{u,t} = d_{1,t}$ and $l_{u,t}$ is the first element of $I_t$.

We now derive the limiting distribution of $\mathcal{E}_{1,1}^b$, in two parts: in part (i), we show that Lemma 3 holds for $\tilde{\mathcal{E}}_{1,1}^b = T^{-1/2} \sum_{t \in I_s} u_t \nu_t$, i.e. $\tilde{\mathcal{E}}_{1,1}^b \overset{d}{\rightarrow} \mathcal{B}_0^{(1)}(r)$ in probability, where $\mathcal{B}_0^{(1)}(r)$ is the first element of $\mathcal{B}_0(r)$ defined just before Lemma 6; in part (ii), we show that the condition of Theorem 2.1 of Hansen (1992) holds, that is, the bootstrap unconditional variance of $\mathcal{E}_{1,1}^b$ converges in probability to the unconditional variance of $T^{-1/2} \sum_{t \in I_s} d_{u,t} u_t \nu_t = \mathcal{E}_1$ (or equivalently, to the variance of $\mathcal{B}_0^{(1)}(r)$). Note that here $\mathcal{E}_1 = S_t (T^{-1/2} \sum_{t \in I_s} g_t)$ is the sample equivalent of $\mathcal{E}_1^b$, defined in the proof of Lemma 6 in the Supplementary Appendix, Section 1.

Part (i). First, $\nu_t$ is i.i.d, so conditional on the data, $\mathbb{E}^b(u_t \nu_t | F_{t-1}) = l_{u,t} \mathbb{E}^b(\nu_t | F_{t-1}) = 0$, so $\mathbb{E}^b(u_t \nu_t)$ is a.m.s. Second, for some $C > 0$, $\sup_t \mathbb{E}(\mathbb{E}^b(u_t \nu_t | F_{t-1})^2) = \sup_t \mathbb{E}(l_{u,t}^2) < C$ by Assumption 9(iii) and Assumption 10(ii), so $\sup_t \mathbb{E}(l_{u,t}^2) < C \overset{p}{\rightarrow} C$. Third, by Lemma 8(iv),

\[
\mathbb{E}^b(\tilde{\mathcal{E}}_{1,1}^b)^2 = T^{-1} \sum_{t=1}^{[T]} \mathbb{E}^b(l_{u,t}^2 \nu_t^2) = T^{-1} \sum_{t=1}^{[T]} l_{u,t}^2 - r \overset{p}{\rightarrow} 0.
\]

Forth, because $l_{u,t} \nu_t$ is i.i.d conditional on the data, the conditional and unconditional bootstrap second moments are the same, so $\mathbb{E}^b(\tilde{\mathcal{E}}_{1,1}^b)^2 = 0$. This shows that $\tilde{\mathcal{E}}_{1,1}^b = T^{-1/2} \sum_{t \in I_s} u_t \nu_t \overset{d}{\rightarrow} \mathcal{B}_0^{(1)}(r)$ in probability (uniformly in $r$).

Part (ii). By Assumption 9(ii), $\mathbb{E}(d_{u,t}^2 l_{u,t}^2) = d_{u,t}^2$. Therefore, by Lemma 8(iv), uniformly in $r$,

\[
\mathbb{E}^b(\tilde{\mathcal{E}}_{1,1}^b)^2 - \mathbb{E}(\mathcal{E}_1^2) = T^{-1} \sum_{t=1}^{[T]} [d_{u,t}^2 l_{u,t}^2 - \mathbb{E}(d_{u,t}^2 l_{u,t}^2)] \overset{p}{\rightarrow} 0.
\]

Therefore, by Theorem 2.1 in Hansen (1992), $T^{-1/2} \sum_{t=1}^{[T]} d_{u,t} l_{u,t} \nu_t \overset{d}{\rightarrow} \int_0^r d_{u,t}(\tau) d\mathcal{B}_0^{(1)}(\tau) = \mathbb{M}_1(\tau_{s-1}, \tau_s)$ in probability, where $\mathbb{M}_1(\tau_{s-1}, \tau_s)$ is defined just before Lemma 6. So for $S_t = S_{u,t}$, $\tilde{\mathcal{E}}_{1,1}^b \overset{d}{\rightarrow} \mathbb{M}_1(\tau_{s-1}, \tau_s)$.

Now let $S_t = \beta_{s,#}$ and note that $S_t^b = \beta_{s,#}^0 T^{-1/2} \sum_{t \in I_s} \nu_t$. Recall that by the decomposition of $S$ and a
decomposition of $D_t$ exactly as $D(\tau)$ in (40), we have:

$$g_t \odot \nu_t = (SD_t)_{\#} \odot \nu_t = \text{vec}(d_{u,t}l_{u,t}\nu_t, s_{p,t}d_{u,t}l_{u,t}\nu_t + S_{p,t}D_{v,t}l_{v,t}\nu_t, S_{p,t}l_{\xi,t})_{\#}. \quad (73)$$

Therefore, $E_t^1 = \beta^\nu_t T^{-1/2} \sum_{t \in I_t} v_t \nu_t = (\beta^\nu_{s,t} s_{p,t}) \left( T^{-1/2} \sum_{t \in I_t} d_{u,t}l_{u,t}\nu_t \right) + (\beta^\nu_{s,t} s_{p,t}) \left( T^{-1/2} \sum_{t \in I_t} D_{v,t}l_{v,t}\nu_t \right).$

Because $E(l_{v,t}l_{u,t}) = I_{p,t}$, by similar derivations as for $T^{-1/2} \sum_{t \in I_t} d_{u,t}l_{u,t}\nu_t \overset{d}{=} \int_0^T d_{u,t}(\tau) d\tilde{B}_0^1(\tau)$ in probability, it can be shown that $(\beta^\nu_{s,t} s_{p,t}) \left( T^{-1/2} \sum_{t \in I_t} D_{v,t}l_{v,t}\nu_t \right. \quad (74)$

$$\Rightarrow \left( \beta^\nu_{s,t} s_{p,t} \right) \left( T^{-1/2} \sum_{t \in I_t} D_{v,t}l_{v,t}\nu_t \right)$$

$$= (S_{p,t}) \int_{\tau_{t-1}}^{\tau_t} D(\tau) d\tilde{B}_0^1(\tau) = M_1(\tau_{t-1}, \tau_t),$$

with variance matrix $V_{M_1(\tau_{t-1}, \tau_t)}$ given in the Supplementary Appendix, Section 1, Proof of Lemma 6.

Next, consider $E_t^2$. From (64) we have that: $\xi_{t-1}^b = \mathcal{F}_{s(t)}^{\tau_t} + \sum_{l=0}^{T-1} \mathcal{F}_{s(\tau)}^{\tau_t} \xi_{t-1} + \left( \sum_{l=0}^{T-1} \mathcal{F}_{s(t)}^{\tau_t} \right) \mu_s$. Then, replacing estimated parameters with the true ones and denoting the remainder by $\epsilon_t^b(1)$ for reasons discussed earlier,

$$E_t^b = T^{-1/2} \sum_{t \in I_t} S\xi_{t-1}^b g_t^b S_t$$

$$= T^{-1/2} \left( S_{t-1}^b \xi_{t-1}^b \right) + T^{-1/2} \sum_{t \in I_t} \left( S_{t-1}^b \xi_{t-1}^b \right) \left( S_{t-1}^b \xi_{t-1}^b \right)$$

$$+ T^{-1/2} \sum_{t \in I_t} \left( S_{t-1}^b \xi_{t-1}^b \right) \left( S_{t-1}^b \xi_{t-1}^b \right)$$

$$+ T^{-1/2} \sum_{t \in I_t} \left( S_{t-1}^b \xi_{t-1}^b \right) \left( S_{t-1}^b \xi_{t-1}^b \right)$$

$$+ o_t(1) = E_t^b + \epsilon_{3,1}^b + \epsilon_{3,2}^b + \epsilon_{3,3}^b + \epsilon_{3,4}^b + o_t(1). \quad (75)$$

First, note that by (48), $\|\xi_{t-1}^b\| = O_p(T^{-\alpha})$, and also that $\|g_{t-1}^b \xi_{t-1}^b \| = O_p(T^{-\alpha})$, for any $\alpha > 0$. Therefore, $E_{3,1}^b = o_t(1)$. For the same reason and by the fact that $\|F_t^b\|$ is exponentially decaying with $l$,

$$E_{3,2}^b \leq \|S_t^b\| \sup_{l} \|F_t^b\| \|\xi_{t-1}^b \| = E_t^b + \|S_t^b\| \sup_{l} \|F_t^b\| o_t(1) = o_t(1). \quad (76)$$

Next, note that by similar derivations as for (1.13) in Supplementary Appendix, Section 1, and artificially setting $g_{t-1} = 0, \nu_{t-1} = 0$ for all $t \leq l$ (as in Boswijk, Cavaliere, Rahbek, and Taylor (2016)) we have, for $\bar{n} = [\tau_{t-1} T] + 2$, and $\bar{I}_{\bar{n}} = [\tau_{t-1} T] + 2, [\tau_{T}],$

$$E_{3,4}^b = \sum_{l=1}^{\bar{n} - T - 2} S F_s^b \left( T^{-1/2} \sum_{t \in I_t} (S_t^b g_{t-1}^b) \eta_{t-1}^b \right)$$

$$\equiv E_{3,4}^b (\Delta_{T}, T - 2) - L. \quad (76)$$
We now show that \( L = \phi^b_1(1) \). Let \( S'^*_t \mathbf{g}_t = u_t \). Then,

\[
(S'_t \mathbf{g}_{t, -1}) (g_{t-1} \odot \nu_{t-1}^{-1}) (S'_t \mathbf{g}_{t, -1})' = \nu^2_t \begin{bmatrix}
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\end{bmatrix} \nu^2_t \begin{bmatrix}

\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} & \nu^2_{t-1} \\
\end{bmatrix}
\]

Then, letting

\[
E(\sum_t (S'_t \mathbf{g}_{t, -1}) (g_{t-1} \odot \nu_{t-1}^{-1}) (S'_t \mathbf{g}_{t, -1})) = \sup_t E(u^2_t ((g_{t-1} \mathbf{g}_{t, -1}) \odot \mathcal{J})).
\]

By Assumption 9, the non-zero elements of \( E(u^2_t ((g_{t-1} \mathbf{g}_{t, -1}) \odot \mathcal{J}) \), do not depend on \( t \), and are elements of linear functions \( \rho_{0,t} \), so they are uniformly bounded in \( l \), Therefore, for element \( \mathcal{L}^{(a,b)} \) of the matrix \( L \), and constants \( c > 0, c_1 > 0 \),

\[
\sup_t E(E(\mathcal{L}^{(a,b)})) \leq T^{-1/2} \sum_{i=1}^{\Delta_T - 2} \left( |SF^i_t A_{\nu}^{-1}(a,b)| \right) \sup_{j=1}^{\Delta_T} E(\mathcal{L}^{(a,b)}(g_{n+j}^{-1}(1)) \odot \mathcal{J}) \leq T^{-1/2} \sum_{i=1}^{\Delta_T - 2} \left( |A_{\nu}^{-1}(a,b)| \right) \sup_{j=1}^{\Delta_T} \mathcal{L}^{(a,b)} \rightarrow 0.
\]

Therefore, \( L = \phi^b_1(1) \) for \( S_t = S_n \), and by similar arguments, \( L = \phi^b_1(1) \) for \( S_t = \beta_\mathbf{x} \).

Next, we analyze \( \hat{\mathcal{E}}^{\beta}_{3,4}(\Delta_T, T - 2) \). To that end, let for now \( S_t = S_n \), and note that a crucial term in \( \hat{\mathcal{E}}^{\beta}_{3,4}(\Delta_T, T - 2) \) is \( \mathcal{L}^{(a,b)}_t(l) = T^{-1/2} \sum_{t=1}^{T} u_t \nu_t (g_{t-1} \odot \nu_{t-1}) \) for \( l \geq 1 \). By the structure of \( S \) and \( D_t \) in (40), letting \( \nu_t = \operatorname{vect}(u_t l_{t+1}, u_t l_{t+1}) \),

\[
\mathcal{L}^{(a,b)}_t(l) = T^{-1/2} \sum_{t=1}^{T} u_t \nu_t (g_{t-1} \odot \nu_{t-1}) = T^{-1/2} \sum_{t=1}^{T} d_{u_t l_{t+1} u_t l_{t+1}} S_{\nu} D_{\nu} l_{t+1} l_{t+1}.
\]

Then, let \( \mathcal{E}^b_{l,t} = l_{u_t l_t} \) and \( \mathcal{E}^b_{l,t} = l_{u_t l_t} \odot \operatorname{vect}(u_t l_{t+1}, u_t l_{t+1}) \).

\[
L^{(a,b)}_t(l) = T^{-1/2} \sum_{t=1}^{T} \mathcal{L}^{(a,b)}_t(l) = T^{-1/2} \sum_{t=1}^{T} (d_{u_t} S_{\nu} D_{\nu} l_{t+1} l_{t+1}) (\mathcal{E}^b_{l,t})^{-1}.
\]
generalized inverse. Therefore, in the rest of the analysis, we no longer need to verify this condition, except for deriving the limit of the unconditional bootstrap variance, and proceed to verify the rest of the conditions. First, \( E^{b}(c_{t,l}^{b}) = E_{t,l} \odot \text{vec}(E^{b}(\nu_{t-l} | F_{t-1}^{b}))t_{p_{1}+1}, E^{b}(\nu_{t} | F_{t-1}^{b})p_{2} = 0_{n} \), so \( E_{t,l}^{b} \) is a m.d.s. Second, for \( \phi_{t}^{b} \) denoting a typical element of \( E_{t,l}^{b} \), and \( \phi_{t} \) denoting the corresponding element of \( E_{t,l} \), we have that \( \sup_{t} E(\phi_{t}^{b} | \nu_{t-l} | F_{t-1}^{b} < \infty \) by Assumptions 9(iii) and Assumption 10(ii) for the first \( p_{1}+1 \) elements of \( E_{t,l}^{b} \), or we have that \( \sup_{t} E(\phi_{t}^{b} | \nu_{t} | F_{t-1}^{b} \) sup \( E^{b}(\nu_{t-l} | F_{t-1}^{b} \) < \infty \) by the same assumptions, for the rest of the elements.

Third, to facilitate showing that \( \text{Var}^{b}(B_{l,T,A}^{b}(r) | F_{t-1}^{b}) - \text{Var}^{b}(B_{l,T,A}^{b}(r)) \to 0 \), note that, from (78), \( \text{Var}^{b}(E^{b} | \nu_{l}^{2}+\epsilon^{*}) = \sup_{t} E(\phi_{t}^{b} | \nu_{t}^{2}+\epsilon^{*} \) sup \( E^{b}(\nu_{t-l} | F_{t-1}^{b} \) < \infty \) by the same assumptions, for the rest of the elements.

To verify the last condition in Lemma 3, because \( \nu_{t-l}, \nu_{l}^{2} \) is i.i.d. and \( \sup_{t} E(\phi_{t}^{b} \nu_{l}^{2}+\epsilon^{*} \) < \infty \, by Lemma 1 and Lemma 8(iv), (proceeding element-wise, first conditional on the sample, then unconditionally),

\[
\text{Var}^{b}(B_{l,T,A}^{b}(r) | F_{t-1}^{b}) - \text{Var}^{b}(B_{l,T,A}^{b}(r)) = T^{-1} \sum_{l=1}^{[Tr]} \left[ \begin{array}{ccc}
\nu_{l}^{2}-1 & \nu_{l}^{2} & \nu_{l}^{2} \\
\nu_{l}^{2} & \nu_{l}^{2} & \nu_{l}^{2} \\
\nu_{l}^{2} & \nu_{l}^{2} & \nu_{l}^{2}
\end{array} \right] d_{l}^{n} = 0_{p_{2} \times p_{2}},
\]

Therefore,

\[
B_{l,T,A}^{b}(r) = T^{-1/2} \sum_{l=1}^{[Tr]} c_{l,l}^{b} d_{l}^{n} \Rightarrow B_{l}^{(1:n)}(r),
\]

where \( B_{l} \) was defined just before Lemma 6.

Now let \( S_{l} = \beta_{x} \). Then:

\[
L_{2}^{b}(l) = T^{-1/2} \sum_{l=1}^{[Tr]} \beta_{x}^{l} \nu_{l}(g_{l} \ominus \nu_{l}) = \beta_{x}^{l} S_{l} \left[ l_{u,t} \nu_{l}(g_{l} \ominus \nu_{l}) \right] + T^{-1/2} \sum_{l=1}^{[Tr]} \beta_{x}^{l} S_{l} D_{l}^{u} \nu_{l}(g_{l} \ominus \nu_{l}) = \beta_{x}^{l} S_{l} \left[ l_{u,t} \nu_{l} \right] + \beta_{x}^{l} S_{l} \left[ l_{u,t} \nu_{l} \right] \nu_{l} = l_{2}^{b} + L_{2}^{b}.
\]

The distribution \( L_{2}^{b} \) follows from (81) and part (ii) below. Following similar steps as for \( L_{1}^{b}(l) \) above (where \( S_{l} = S_{u} \), it can be shown that under Assumptions 1-10 and Assumption 9'(iii), which imposes \( E_{u} n_{t-l} \beta_{x}^{l} \) =
where the last line follows by replacing $l_i$ so we proceed with each block $2$, where the relevant bootstrap condition we have to show, by analogy to $(84)$, we have:

$$
\text{block}_n(\mathbf{O}) = \mathbf{O}^{(n(n-1)+1:n,n(n-1)+1:n^{*})},
$$

that is, the operation that selects the $(\kappa, \kappa^*)$ $n \times n$ sub-matrix of the matrix $\mathbf{O}$. Also, for a $\bar{n} \times \bar{n}$ square matrix $\mathbf{O}_1$, define the operation that makes $\mathbf{O}_1$ block diagonal at row $j$ as follows:

$$
\text{block}_{\bar{n}, \kappa^*}(\mathbf{O}_1) = \text{diag}(\mathbf{O}_1^{(1:j-1:j)}, \mathbf{O}_1^{(j+1:n:j+n)}).
$$

Then, by similar arguments to $\mathcal{S}_1 = \mathcal{S}_u$,

$$
\text{vec}(\mathbf{B}_{1,T,A}^b(r), \mathbf{B}_{1,T,B}^b(r)) \overset{db}{\Rightarrow} \mathbf{B}_1^{(1:n(p_1+1))}(r),
$$

where the relevant bootstrap condition we have to show, by analogy to $\mathcal{S}_1 = \mathcal{S}_u$, is that $E^t(\mathbf{B}_{1,T,A}^b(r_1)\mathbf{B}_{1,T,B}^b(r_2)) \overset{db}{\Rightarrow} \mathbf{B}_1^{(1:n)}(r_1)(\mathbf{B}_1^{(n+1:n(p_1+1))}(r_2))^\prime) = o_p(1)$, a condition proven below for $r_1 = r_2 = r$, because when $r_1 \neq r_2$, the proof follows in a similar fashion. Letting $\mathbf{B}_{1,T}^* = \text{vec}(\mathbf{B}_{1,T,A}^b(r_1), \mathbf{B}_{1,T,B}^b(r_1))$, note that

$$
E^t(\mathbf{B}_{1,T,A}^b(r)\mathbf{B}_{1,T,B}^b(r)) = [\text{block}_{1,2}(\text{Var}^b(\mathbf{B}_{1,T}^*))]\text{block}_{1,3}(\text{Var}^b(\mathbf{B}_{1,T}^*)), \ldots, \text{block}_{1,p_1+1}(\text{Var}^b(\mathbf{B}_{1,T}^*))],
$$

so we proceed with each block $2, \ldots, p_1 + 1$, and let $b_i = l_{u,t}u_{v,t}$ for $\kappa = 1, \ldots, p_1$. Then, noting that $l_{u,t}u_{v,t}^t(l_{t-1} \otimes \nu_{t-1})$ is a m.d.s. with respect to $\mathcal{F}_{1,l}$ for $l \geq 1$, we have:

$$
\text{block}_{1,\kappa+1}(\text{Var}^b(\mathbf{B}_{1,T}^*)) = T^{-1} \sum_{t=1}^{[T\tau]} E^t[b_i^2u_i^t(l_{t-1} \otimes \nu_{t-1})(l_{t-1} \otimes \nu_{t-1})]' = T^{-1} \sum_{t=1}^{[T\tau]} \text{blockdiag}_{\rho_{p_1+1}}(b_i(l_{t-1} \otimes \nu_{t-1})) = \text{block}_{1,\kappa+1}(\text{Var}^b(\rho_{1:t})) = \text{block}_{1,\kappa+1}(\rho_{1:t}),
$$

where the last line follows by replacing $l_{u,t}$ by $b_i$ in (80), by Lemma 8(iv) and by Assumption 9(iii).

Part (ii). First, let $\mathcal{S}_1 = \mathcal{S}_u$, and recall that, from (79), we need the distribution of

$$
\mathcal{L}^b(l) = T^{-1/2} \sum_{t=1}^{[T\tau]} u_t^t g_t(l_{t-1} \otimes \nu_{t-1}) = T^{-1/2} \sum_{t=1}^{[T\tau]} (d_{u,t}^b S_{D_{l-1,t}}(\cdot))^b_{l,t}.
$$

By Hansen (1992), Theorem 2.1, because $\|d_{u,t}^b S_{D_{l-1,t}}\|$ is bounded by Assumption 9(ii), and $D(\tau - \frac{\tau}{T}) = D_{l-1}$ when $\tau \in [\frac{T}{T}, \frac{T}{T}]$, we have:

$$
\mathcal{L}^b(l) = \int_{0}^{T} d_u(\tau) S_{D_{l-1,t}}(\tau)^{db} \Rightarrow \int_{0}^{T} d_u(\tau) S_{D_{l-1,t}}(\tau)^{db} \Rightarrow 0,
$$

where the convergence follows because $\text{Var}^b(\mathcal{L}^b(l)) - \mathcal{L}_1(l) \overset{db}{\Rightarrow} 0$, which can be shown by similar arguments to
(80), and using Lemma 8(iii) instead of 8(iv). Similarly, for $S_i = \beta_x$, we have:

\[
\begin{align*}
L^b_{2,2}(l) &= (\beta_x s \nu_l) L^b_{1,1}(l) + (\beta_x \otimes I_{n,\#})(S_{\nu_l} \otimes S_{\#}) T^{-1/2} \sum_t E[(D_{\nu_l} \otimes D_{l-1,\#}) (l_{t-1,\#} \otimes \nu_{t-1})] \\
&\quad \doteq (\beta_x s \nu_l) \int^\tau_0 d u(\tau) S_{\nu_l} \otimes D_{l-1,\#} \, d B^{(1,n)}(\tau) + (\beta_x s \nu_l) \int^\tau_0 [D_{\nu_l} \otimes D_{\#}(\tau)] \, d B^{(n+1,n)}(\tau) \\
&= \mathbb{E}[S_{\#} S_{\#}] \int^\tau_0 (D_{\#}(\tau) \otimes D_{\#}(\tau)) \, d B^{(1,1)}(\tau).
\end{align*}
\]

Next, we derive the distribution of $\mathcal{E}_{3,4}(n^*)$. This follows by similar arguments as above if we can verify that the off-diagonal elements of the bootstrap covariance $\text{Cov}^b(B^b_{-1,T}(r), B^b_{-1,T}(r))$ converge in probability to the counterpart elements of the covariance $\text{Cov}(B^1_{l-1,n}(n^*)^+(r), B^1_{l-1,n}(n^*)^+(r))$ for $l \neq l^*$. We only do so for $\text{block}_1(\text{Cov}(B^b_{1,T}(r), B^b_{1,T}(r)))$; the rest follows by similar reasoning.

\[
\begin{align*}
\text{block}_1(\text{Cov}(B^b_{1,T}(r), B^b_{1,T}(r))) &= T^{-1} \sum_t E[(\nu_t \otimes \nu_{t-1}) \otimes \text{diag}(0_{p_1+1,p_1+1}, J_2)] \\
&\quad + T^{-1} \sum_t E[(\nu_t \otimes \nu_{t-1}) \otimes \text{diag}(0_{p_1+1,p_1+1}, J_2)] \\
&\quad \doteq \text{block}_1(\nu_t \otimes \nu_{t-1}) \\
&\quad \doteq \text{block}_1(\nu_t \otimes \nu_{t-1}).
\end{align*}
\]

because of Assumption 9(ii) we imposed that $E[t^2_{u,l} m_{l-1,l}] = 0_{(p_1+1) \times (p_1+1)}$ for $l > 1$, $l \neq l^*$, and Assumption 9(iii) we imposed that $E[t^2_{u,l} m_{l-1,l}] = 0_{(p_1+1) \times (p_1+1)}$ for $l > 1$, $l \neq l^*$. In the general setting, for $S_i = S_u$ or $S_i = \beta_x$, by analogy we need $E[(n_i \nu_t) \otimes (m_i \nu_{t-1})] = 0_{(p_1+1) \times (p_1+1)}$ for $l > 1$, $l \neq l^*$, and $E[(n_i \nu_t) \otimes (n_i \nu_{t-1})] = 0_{(p_1+1) \times (p_1+1)}$ for $l > 1$, $l \neq l^*$, which are also satisfied by Assumption 9(ii)-(iii).

Using (87)-(88) in the expression for $\mathcal{E}_{3,4}^b(n^*) = \sum_{t=0}^{n^*} \mathbb{E}^b_{3,4} A^{-1}_s \left\{ T^{-1/2} \sum_{t=0}^{n^*} S_{i}^b g_{i} \mu_{t} (g_{t-1,l} \otimes \nu_{t-1,l}) \right\}$, it follows that, for a fixed $n^*$,

\[
\mathcal{E}_{3,4}^b(n^*) \doteq \sum_{t=0}^{n^*} \mathbb{E}^b_{3,4} A^{-1}_s ((S_{i}^b S_{\#}) \otimes S_{\#}) \int^\tau_{t-1} \mu_{t} (g_{t-1,l} \otimes \nu_{t-1,l}) \, d B^{(1,1)}(\tau).
\]

Now as in the proof of Lemma 6, setting $n^* = T^\alpha$ for some $\alpha \in (0, 1)$, and noting that the remainder $\mathcal{E}_{3,4}^b_{\Delta \tau_r T - 2} - \mathcal{E}_{3,4}^b(n^*) = O_p(1)$, it can be shown that:

\[
\mathcal{E}_{3,4}^b_{\Delta \tau_r T - 2} \doteq \sum_{t=0}^{n^*} \mathbb{E}^b_{3,4} A^{-1}_s ((S_{i}^b S_{\#}) \otimes S_{\#}) \int^\tau_{t-1} \mu_{t} (g_{t-1,l} \otimes \nu_{t-1,l}) \, d B^{(1,1)}(\tau) = M_{3,2}(\tau_{s-1,\#}, \tau_s),
\]

where $M_{3,2}(\tau_{s-1,\#}, \tau_s)$ as defined above is also the asymptotic distribution of the sample counterpart of $\mathcal{E}_{3,4}^b_{\Delta \tau_r T - 2}$, that is $\mathcal{E}_{3,4}^b_{\Delta \tau_r T - 2}$, featuring in Supplementary Appendix, Section 1 in (1.16), whose variance exists and is derived right after that equation. Therefore,

\[
\mathcal{E}_{3,4}^b \doteq M_{3,2}(\tau_{s-1,\#}, \tau_s). \tag{89}
\]

Now consider $\mathcal{E}_{3,4}^b = T^{-1/2} \sum_{t=0}^{n^*} (S_{i}^b g_{i}) \left[ \mathbb{S} \left( \sum_{l=0}^{t-2} \sum_{s=1}^{T} \right) \mu_{t} \right]$ in (75). By similar analysis as for $\mathcal{E}_{3,3}^b$, it can be shown that:

\[
\mathcal{E}_{3,3}^b \doteq \sum_{t=0}^{n^*} (S_{i} S_{\#}) \otimes (S_{i} S_{\#}) \int^\tau_{t-1} D_{\#}(\tau) \, d B^{(1,1)}(\tau) = M_{3,1}(\tau_{s-1,\#}, \tau_s). \tag{90}
\]

Also, $\mathcal{E}_{3,3}^b + \mathcal{E}_{3,4}^b$ can be shown to jointly converge to $M_{3,1}(\tau_{s-1,\#}, \tau_s) + M_{3,2}(\tau_{s-1,\#}, \tau_s)$, provided that $\text{Cov}(B^b_{0,T}(r), B^b_{0,T}(r)) - \text{Cov}(B_{0,T}(r), B^1_{l-1,n}(n^*)^+(r)) \approx 0$ uniformly in $r$ for all $l \geq 1$, where $B^b_{0,T} =$
\[ \sum_{t=1}^{T_r} l_t \otimes \nu_t, \text{ and recall that } B_{i,T}(r) = \text{vect}(B_{i,T,A}(r), B_{i,T,B}(r)). \] Consider \( \text{Cov}^b(B_{0,T}^{(1)}(r), B_{i,T}(r)) = (\text{Cov}^b(B_{0,T}(r), B_{i,T}(r)))^{(1,n,1)} \) (so for \( S_t = S_0 \); the proof for the rest of the elements is similar).

\[
\text{Cov}^b(B_{0,T}^{(1)}(r), B_{i,T,A}(r)) = T^{-1/2} \sum_{t=1}^{T_r} (l_{u,t} \nu_t)(T^{-1/2} \sum_{t=1}^{T_r} l_{u,t} \nu_t (l_{t-1} \otimes \nu_{t-1})
\]
\[
= T^{-1} \sum_{t=1}^{T_r} l_{u,t} \nu_t (l_{t-1} \otimes \nu_{t-1}) + T^{-1} \sum_{t,t'=1,t \neq t'} l_{u,t} l_{u,t'} \nu_t \nu_{t'} (l_{t-1} \otimes \nu_{t-1})
\]
\[
= \mathcal{L}_{3,1}^b + \mathcal{L}_{3,2}^b.
\]

Now note that by Lemma 6(iv),

\[
\text{E}^b(\mathcal{L}_{3,1}^b) = T^{-1} \sum_{t=1}^{T_r} l_{u,t} \nu_t (l_{t-1} \otimes \nu_{t-1}) = T^{-1} \sum_{t=1}^{T_r} \begin{bmatrix} l_{u,t} \nu_t(l_{t-1} \otimes \nu_{t-1}) \end{bmatrix} \otimes \text{E}^b \begin{bmatrix} \nu_t l_{t-1} \\ \nu_t \nu_{t-1} \end{bmatrix}
\]
\[
= T^{-1} \sum_{t=1}^{T_r} l_{u,t} \nu_t (l_{t-1} \otimes \nu_{t-1}) \rightarrow \text{r}^b(\rho^{(1,n,1)}_1) \otimes \mathbb{I}.
\]

\[
\text{E}^b(\mathcal{L}_{3,2}^b) = T^{-1} \sum_{t,t'=1,t \neq t'} l_{u,t} l_{u,t'} \nu_t \nu_{t'} (l_{t-1} \otimes \nu_{t-1}) \rightarrow \text{E}^b \begin{bmatrix} \nu_t l_{t-1} \\ \nu_t \nu_{t-1} \end{bmatrix} = 0_n.
\]

Therefore, \( \text{Cov}^b(B_{0,T}^{(1)}(r), B_{i,T,A}(r)) - \text{Cov}(B_{0}^{(1)}(r), B_{i}^{(1,n)}(r)) = o_p(1) \), by the restriction in Assumption 9(i), which ensures that \( \rho^{(1,n,1)}_1 = \rho^{(1,n,1)}_1 \) \( \otimes \mathbb{I} \); also note that in the general definition of \( S_t \), by analogy, we need \( \text{E}^b((n_t n'_t) \otimes n_{t-1}) = 0_{(p+1)^2 \times (p+1)} \) for \( l \geq 1 \), imposed in Assumption 9(i). So,

\[
\mathcal{E}_{3,3}^b + \mathcal{E}_{3,4}^b \rightarrow \mathbb{M}_3(\nu_{t-1}, \nu_t) = \mathbb{M}_3(\nu_{t-1}, \nu_t).
\]

Because we showed that \( \mathcal{E}_{3,1}^b = o_p^b(1) \) and \( \mathcal{E}_{3,2}^b = o_p(1) \), it follows that: \( \mathcal{E}_{3}^b \rightarrow \mathbb{M}_3(\nu_{t-1}, \nu_t) \), and that \( \text{vect}(\mathcal{E}_{3}^b, \mathcal{E}_{3}^b) \rightarrow \text{vect}(\mathbb{M}_3(\nu_{t-1}, \nu_t), \mathbb{M}_3(\nu_{t-1}, \nu_t)) \).

**Now consider** \( \mathcal{E}_{3}^b \). Note that \( \nu_t = \nu_t \nu_t \) is not bootstrapped, and recall that \( \nu_t = \mu_t + \eta_t + F_t \xi_t \). Therefore, replacing again estimated parameters by the true values, because the rest of the terms are \( o_p^b(1) \) (therefore also replacing, as before, \( g_t \) with \( g_t \nu_t \)),

\[
\mathcal{E}_{3}^b = \left\{ T^{-1/2} \sum_{t \in \tilde{I}_t} [S_t'(g_t \nu_t)] S_t \mu_s + T^{-1/2} \sum_{t \in \tilde{I}_t} [S_t'(g_t \nu_t)] S_t F_t \xi_{t-1} + T^{-1/2} \sum_{t \in \tilde{I}_t} [S_t'(g_t \nu_t)] S_t \eta_t \right\} 
\[
= \mathcal{E}_{2,1} + \mathcal{E}_{2,2} + \mathcal{E}_{2,3} + o_p^b(1).
\]

Now consider \( \mathcal{E}_{2,1} \). From (72) and (74), without any restrictions on \( \rho_i, \rho_j \) except those in Assumption 9,

\[
\mathcal{E}_{2,1}^b = S_t \mathcal{E}_{2,1}^b + o_p^b(1) \rightarrow \left\{ \left( S_t'(S_{\#}) \int_{\tau_{t-1}}^{\tau_t} D(\tau) d B_{0,\#}(\tau) \right) S_t \mu_s \right\} \rightarrow \left( S_t'(S_{\#}) \otimes (S_t F_t^2) \right) \left[ \left( \int_{\tau_{t-1}}^{\tau_t} D(\tau) d B_{0,\#}(\tau) \right) \otimes \mu_s \right],
\]

(91)

where the latter is the first term in \( \mathcal{E}_{2,1}(\nu_{t-1}, \nu_t) = \sum_{t=0}^{\infty} (S_t'(S_{\#}) \otimes (S_t F_t^2)) \left[ \left( \int_{\tau_{t-1}}^{\infty} D(\tau) d B_{0,\#}(\tau) \right) \otimes \mu_s \right] \) (the rest of the terms appear from the distribution of \( \mathcal{E}_{2,2} \) as seen below).

Now consider \( \mathcal{E}_{2,2} = T^{-1/2} \sum_{t \in \tilde{I}_t} [S_t'(g_t \nu_t)] S_t \eta_t \). Recall from the arguments above (55) that \( A_{\#}^{-1} \) is also upper triangular with rows \( p_1 + 2 : n \) equal to \( [0_{p_2} \ 0_{p_2 \times p_3} \ I_{p_2}] \). Therefore, it can be shown that \( S_t \eta_t = \)
\[ S_r \Lambda_{s,\#}^{-1} g_t = \zeta_t, \text{ so for } S_t = S_u, \]

\[ E_r^b = T^{-1/2} \sum_{t \in I} [S_t'(g_t \circ \nu_t)] S_r \eta_t^b = T^{-1/2} \sum_{t \in I} u_t \nu_t \zeta_t + o_p(1), \]

where the last equality follows from replacing \( S_t' A_{s,\#} \eta_t^b = S_t' A_{s,\#} ((\hat{A}_s^{-1} \tilde{g}_t) \circ \nu_t) \) with \( S_t'(g_t \circ \nu_t) \), as for \( E_t^b \), since the rest of the terms are of lower order in all the relevant sums. Note that:

\[ E_r^b = T^{-1/2} \sum_{t \in I} (S_{p2} D_{\zeta,t} d_{u,t}) u_t \nu_t l_{\zeta,t}, \]

and consider first \( B^b_{u\zeta,T}(r) = T^{-1/2} \sum_{i=1}^{[Tr]} l_{u,t} l_{\zeta,t} \nu_t \).

**Part (i).** Since \( \nu_t \) is i.i.d., \( E^b(l_{u,t} l_{\zeta,t} \nu_t | F_{t-1}^{\tilde{b}}) = 0 \), for any element \( l_{u,t} l_{\zeta,t} \). Also, for some \( c > 0 \), \( \sup_t E^b \left[ l_{u,t} l_{\zeta,t} | 2^{2+\delta} \right] \right] \leq \sup_t E^b \left[ l_{u,t} l_{\zeta,t} | 2^{2+\delta} \right] \). Because \( \nu_t \) is i.i.d., the conditional and unconditional bootstrap moments are the same, and it remains to verify that \( \text{Var}^b(B^b_{u\zeta,T}(r)) = \text{Var}(B_{u\zeta}(r)) = o_p(1) \), where \( B_{u\zeta}(r) \) was defined just before Lemma 6.

\[ \text{Var}^b(B^b_{u\zeta,T}(r)) = T^{-1} \sum_{i=1}^{[Tr]} (l_{u,t} l_{\zeta,t} l_{U,T}^b) E^b(\nu_t^2) = T^{-1} \sum_{i=1}^{[Tr]} (l_{u,t} l_{\zeta,t} l_{U,T}^b) \overset{p}{\rightarrow} \rho_{u\zeta,0,0} = \text{Var}(B_{u\zeta}(r)), \]

where \( \rho_{u\zeta,0,0} \) was defined before Lemma 5, and the convergence follows by Lemma 5(iv) and Assumption 9.

**Part (ii).** Because \( \text{Var}^b(B^b_{u\zeta,T}(r)) \overset{p}{\rightarrow} \text{Var}(B_{u\zeta}(r)) \), using Lemma 5(iii), it follows by Hansen (1992), Theorem 2.1, that:

\[ E_r^b = T^{-1/2} \sum_{t \in I} (S_{p2} D_{\zeta,t} d_{u,t}) u_t \nu_t l_{\zeta,t} \overset{d}{\rightarrow} S_{p2} \int_{\tau_{s-1}}^{\tau_{s}} d_u(t) D_{\zeta}(t) dB_{\zeta}(t) = M^{(1)}_{2,3}(\tau_{s-1}, \tau_{s}), \]

where \( M^{(1)}_{2,3}(\tau_{s-1}, \tau_{s}) \) was defined right before Lemma 6. Similarly, it can be shown that for \( S_t = \beta_{x,\#} \), without restrictions on \( \rho_{0,0} \) besides those imposed in Assumption 9,

\[ E_r^b = M^{(2)}_{2,3}(\tau_{s-1}, \tau_{s}). \]

Next, consider \( E_r^{b,2} \). By backward substituting \( \xi_{t-1} \),

\[ E_r^{b,2} = S_r F_r T^{-1/2} \sum_{t \in I} [S_t'(g_t \circ \nu_t)]^b \xi_{t-1} = S_r F_r T^{-1/2} [S_t'(g_t \circ \nu_t)]^b [\xi_{\tau_{s-1},T}] + S_r F_r T^{-1/2} \sum_{t \in \tilde{I}} F_\tau^T \xi_{t-1} + 1 \left( \sum_{i=0}^{\tilde{t} - 2} F_\tau^T \xi_{t-1} \right) + S_r T^{-1/2} \sum_{t \in \tilde{I}} [S_t'(g_t \circ \nu_t)] \left( \sum_{i=0}^{\tilde{t} - 2} F_\tau^T \xi_{t-1} \right) \]

\[ = \sum_{t=1}^{\tilde{t}} E_r^{b,2,1}. \]

First, note that \( E_r^{b,2,1} = o_p(1) \) because, as shown before, \( (g_t \circ \nu_t) = O_p(1) \) and \( \xi_{\tau_{s-1},T} = O_p(1) \) from the proof of Lemma 2 in the Supplementary Appendix. Next, because \( \sum_{i=0}^{\tilde{t}} \| F_\tau^T \| \) is bounded, because \( \text{Var}^b(g_t \circ \nu_t) = O_p(1) \) and \( \xi_{\tau_{s-1},T} = O_p(1) \), \( E_r^{b,2,2} = o_p(1) \).

Next, by similar arguments as for \( E_r^{b,3} \), and noting that no restrictions are needed on \( \rho_t, \rho_{tj} \) besides those in
Assumption 9 (because $E_{2,3}^b$ has at the basis the same random process as $E_{1,1}^b$)

$$E_{2,2,3}^b = S_r F_r T^{-1/2} \sum_{t \in I_r^*} E_t \left[ S_t^b (g_t \otimes \nu_t) \right] \left[ \sum_{t=0}^{l-2} F_s^t \mu_s \right] = S_r F_r T^{-1/2} \sum_{t \in I_r^*} E_t \left[ S_t^b (g_t \otimes \nu_t) \right] \left[ \sum_{t=0}^{l-2} F_s^t \mu_s \right] = M_1 (\tau_{s-1}, \tau_s) - \left( \left( S_t^b S^b \right) \otimes (S_r F_r^b) \right) \left[ \int_{t_{r-1}}^{t_r} D^b (\tau) d B_0, # (\tau) \otimes \mu_s \right].$$

Now consider $E_{2,2,4}^b$.

$$E_{2,2,4}^b = S_r T^{-1/2} \sum_{t \in I_r^*} E_t \left[ S_t^b (g_t \otimes \nu_t) \right] \left[ \sum_{t=0}^{l-2} F_s^t \eta_{t-1} \right].$$

By similar arguments as for $E_{3,4}^b$ in (76) and just below it,

$$E_{2,2,4}^b = S_r T^{-1/2} \sum_{t \in I_r^*} E_t \left[ S_t^b (g_t \otimes \nu_t) \right] \left[ \sum_{t=0}^{l-2} F_s^t \eta_{t-1} \right] + \psi_{p}^b (1) = E_{2,2,4}^b (\Delta_r, T - 2) + \psi_{p}^b (1).$$

Next, we analyze $E_{2,2,4}^b (n^*)$, first for a fixed $n^*$. Note that a crucial term in $E_{2,2,4}^b (n^*)$ is $L_{2}^b (I)$ is $T^{-1/2} \sum_{t=1}^{[T_r]} u_t \eta_{t-1}$ for $l \geq 1$, because $\eta_{t-1} = A^* g_{t-1}$. By the structure of $S$ and $D_t$ in (40), recalling that $\nu_t = \text{vec}(\nu_u t_{p,u+1}, \nu_{p+1})$,

$$g_{t-1} = \begin{bmatrix} d_{u,t-l_{u,t-l}} \\ s_{u,t-l_{u,t-l}} + S_{1,u} D_{u,t-l_{u,t-l}} \end{bmatrix} = S_{#} D_{t-1, #} (\nu_{t-1} \otimes \nu_{t-1}).$$

Let $S_1 = S_u$. Then, letting $E_{1,1,5}^b = l_{u,t-l_{t-1}}$ and $E_{2,1,5}^b = l_{u,t-l_{t-1}}$, we have

$$L_{2}^b (I) = T^{-1/2} \sum_{t=1}^{[T_r]} d_{u,t} S_{#} D_{t-1, #} (\nu_{t-1}), \quad \text{for } l \geq 1.$$

**Part (i).** First, consider $B_{1,1,5}^b (r) = T^{-1/2} \sum_{t=1}^{[T_r]} l_{u,t-l_{t-1}}$, for $l \geq 1$. Because $\nu_t$ is i.i.d., it is m.d.s under the bootstrap measure conditional on the data, so by arguments similar to before, $B_{1,1,5}^b (r) \approx B_{1,1}^b (r)$, provided that $\text{Var}^b (B_{1,1,5}^b (r)) \approx \text{Var} (B_{1,1}^b (r))$, which we verify below:

$$\text{Var}^b (B_{1,1,5}^b (r)) = T^{-1} \sum_{t=1}^{[T_r]} l_{u,t-l_{t-1}} E^b (\nu_{t-1}^2) = T^{-1} \sum_{t=1}^{[T_r]} l_{u,t-l_{t-1}} E^b (\nu_{t-1}^2) \approx \text{Var} (B_{1,1}^b (r)) = \text{block}_{1,1} (\rho_{l,l}).$$

The previous to last statement above follows by Lemma 8(iv) without restrictions on the form of $\rho_{l,l}$ besides the ones in Assumption 9. Therefore, $B_{1,1,5}^b (r) \approx B_{1,1}^b (r)$.

**Part (ii).** By Hansen (1992), Theorem 2.1, and Lemma 8(iii), $L_{2}^b (I)$ defined in (96) is such that:

$$L_{2}^b (I) \approx \int_0^r d_u (r) S_{#} D_{#} (\tau) d B_{l, #} (\tau) = ((S_{#} \otimes S_{#}) L_{2}^b (I)) \approx \text{vec} (B_{l,1,5}^b (r), B_{1,1,5}^b (r)) \approx \text{vec} (B_{1,1}^b (r), B_{1,1}^b (r)),$$

which can be shown using Lemma 3 and Lemma 8(iv), because $\text{Cov}^b (B_{l,1,5}^b (r), B_{1,1,5}^b (r)) \approx \text{Cov} (B_{1,1}^b (r), B_{1,1}^b (r)) = \text{block}_{1,1} (\rho_{l,l});$
the latter condition holds because:

\[ \mathbb{E}^b(B^b_{l,T,C}(r)(B^b_{l',T,C}(r))^\prime) = T^{-1} \sum_{t=1}^{[T]} \ell^b_t \ell^b_{t-1} \mathbb{E}^b(\nu_t^2) + T^{-1} \sum_{t=1}^{[T]} \ell^b_u \ell^b_{u,1} \ell^b_{u-1} \ell^b_{u-1} \mathbb{E}^b(\nu_t^2) \]

where the last statement follows by Lemma 8(iv), and under Assumptions 1-10. By analogy, no other restrictions besides Assumptions 1-10 are needed also when \( S_i = \beta_{x, #} \).

Therefore, by Hansen (1992) and Lemma 8(iii), for a fixed \( n^* \),

\[ \mathcal{E}^b_{n^*} = S_r \left( \sum_{i=0}^{n^*} F_s^i A_s^{-1} \left( T^{-1/2} \sum_{t \in I_i} (S'_i g_t^i) g_{t-1} \right) \right) \]

\[ \overset{d}{=} \left( \sum_{i=0}^{n^*} (S'_i S_#^i) \otimes (S_i F_s^{i+1} A_s^{-1} S_#^i) \right) \int_{\tau_1}^{\tau_r} (D_{\#}(\tau) \otimes D_{\#}(\tau)) dB_{1,#}(\tau). \]

Letting as before \( n^* = T^\alpha \), it can be shown that \( \mathcal{E}^b_{n^*}(\Delta \tau_s T - 2) = \mathcal{E}^b_{n^*}(1) \), and therefore

\[ \mathcal{E}^b_{2,2} = \mathcal{E}^b_{n^*}(\Delta \tau_s T - 2) + o_p(1), \quad \mathcal{E}^b_{2,2} = \mathcal{E}^b_{n^*}(1), \quad \text{and} \quad (95) \text{ and } (97) \text{ into } (94), \text{ and then using } (91), \text{ it follows that:} \]

\[ \mathcal{E}^b_{2,2} \overset{d}{=} M_{2,1}(\tau_{s-1}, \tau_s) \]

\[ = \mathcal{E}^b_{2,2} + M_{2,1}(\tau_{s-1}, \tau_s) + M_{2,2}(\tau_{s-1}, \tau_s), \]

because the joint convergence of \( \mathcal{E}^b_{2,3}, \mathcal{E}^b_{2,4} \) can be shown as above under Assumptions 1-10. Because all these terms share the same \( \nu_t \), it can be shown that they also jointly converge with \( \mathcal{E}^b_{2,3} \) and their bootstrap covariance to the covariances of the relevant limits, under Assumptions 1-10.

Therefore, for \( S_i = S_u \),

\[ \mathcal{E}^b = \sum_{i=1}^{3} \mathcal{E}^b_{2,i} \overset{d}{=} M_{2,1}(\tau_{s-1}, \tau_s) + M_{2,2}(\tau_{s-1}, \tau_s) + M_{2,3}(\tau_{s-1}, \tau_s) = M_2(\tau_{s-1}, \tau_s). \]

Similarly, for \( S_i = \beta_{x, #} \), \( \mathcal{E}^b \overset{d}{=} M_2(\tau_{s-1}, \tau_s) \), completing the proof for the distribution of \( \mathcal{E}^b \), which note that we proved only under Assumptions 1-10.

Now note that because \( \mathcal{E}^b \) featured as part of \( \mathcal{E}^b \), their joint convergence was already shown, and recall that it also followed under Assumptions 1-10. It remains to verify the condition:

\[ \text{Cov}^b(\text{vec}(\mathcal{E}^b_{2,1}, \mathcal{E}^b_{2,2})) = \text{Cov}(\text{vec}(M_2(\tau_{s-1}, \tau_s), M_3(\tau_{s-1}, \tau_s))) \overset{p}{\to} 0, \]

because then \( \text{vec}_{i=1:3}(\mathcal{E}^b_{2,i}) \overset{d}{=} \text{vec}_{i=1:3}(M_{2,1}(\tau_{s-1}, \tau_s)) = M_1(\tau_{s-1}, \tau_s). \) This condition follows by similar arguments as before, if we show that (C 1) \( \text{Cov}^b(\mathcal{E}^b_{2,3}, \mathcal{E}^b_{2,4}) \) converges to the joint covariance of their respective limits, and that (C 2) \( \text{Cov}^b(\mathcal{E}^b_{2,2,4}, \mathcal{E}^b_{2,3}) \) converges to the joint covariance of their respective limits. For (C 1), by arguments as before, it suffices to show \( \text{Cov}^b(B_{l',T}(r), B_{u,#}(r)) = \text{Cov}(B_{1,n}(r), B_{u,#}(r)) \overset{d}{=} 0 \) (here, we set \( S_i = S_u \) for all terms and that is why we consider the first \( n \times 1 \) elements of \( B^b_{l}(r) \)); the proofs for the case \( S_i = \beta_{x, #} \) is
similar and is briefly discussed below). Note:

\[
\text{Cov}^b(B^b_{i,T,A}(r), B^b_{u T}(r)) = E^b(T^{-1} \sum_{t=1}^{[T]} (l_{t-1} - l_{t-1}^{T,0}) \otimes \text{vec}(v_t v_{t-1} v_{t+1} v_{t+1}^T)) (l_{t-1} - l_{t-1}^{T,0})
\]

\[
= T^{-1} \sum_{t=1}^{[T]} \left[ \left( \left( \sum_{i=1}^{[T]} (l_{t-1,i} - l_{t-1}^{T,0}) \otimes \text{vec} \left( 0_{(p_1+1) \times p_2} \right) \right) \right) + T^{-1} \sum_{t=1}^{[T]} \mathbb{E} \left[ \text{vec} \left( 0_{(p_1+1) \times p_2} \right) \right] \right]
\]

which shows why we need \( \text{E}(l_{t-1,i}^2 n_t - l_{t-1}^{T,0}) = 0_{(p_1+1) \times p_2} \) imposed in Assumption 9(iii). In the general case, for \( b \) as defined before, by analogy, the condition needed and imposed in Assumption 9(iii) is that for \( l \geq 1 \),

\( \text{E}((n_t n_t^T) \otimes (n_t - l_{t-1}^{T,0})) = 0_{(p_1+1)^2 \times (p_1+1) p_2} \).

For (C 2), notice that from (97),

\[
\mathcal{E}_{2,2,4}^b \overset{d}{\rightarrow} \mathbb{M}_{2,2}(\tau_{s-1}, \tau_s) = \sum_{i=0}^{\infty} (S_i^b \otimes (S_i^b A_i^{-1} S_i^b)) \int_{\tau_{s-1}}^{\tau_s} (D_{i}^{\otimes} \otimes D_{i}^{\otimes}) \, dB_{i+1}^{\otimes}
\]

while from (89),

\[
\mathcal{E}_{3,4}^b \overset{d}{\rightarrow} \sum_{i=0}^{\infty} (S_i^b A_i^{-1} S_i^b) \int_{\tau_{s-1}}^{\tau_s} (D_{i}^{\otimes} \otimes D_{i}^{\otimes}) \, dB_{i+1}^{\otimes} = [\mathbb{P}(\tau_{s-1}, \tau_s), \mathbb{P}(\tau_{s-1}, \tau_s), \ldots, \mathbb{P}(\tau_{s-1}, \tau_s)]
\]

Therefore, they jointly converge. It follows that for \( i = \tilde{i} \),

\[
T^{-1/2} \sum_{t \in I,} z_t^b g_t^b S_t^b = \text{vec}_{i=1:3} (\mathcal{E}_{i}^b) \overset{d}{\rightarrow} \mathbb{M}_{i}^{(\tau_{s-1}, \tau_s)}
\]

Using exactly the same arguments as in the end of the proof of Lemma 6, \( T^{-1/2} \sum_{t \in I,} z_t^b g_t^b S_t^b \overset{d}{\rightarrow} \mathcal{M}_i \) for \( i \neq \tilde{i} \), completing the proof.

\[\square\]

**Proof of Lemma 11.** As for the proof of Lemma 6, consider the interval \( i = \tilde{i} \). Let \( S_{\tilde{i}}^b = S_u \) or \( S_{\tilde{i}}^b = \beta_{u_{\tilde{i}}} \). We need the asymptotic distribution of \( Z_{\tilde{i}}^b = T^{-1/2} \sum_{t \in I,} z_t^b g_t^b S_t^b \).

\[
Z_{\tilde{i}}^b = T^{-1/2} \sum_{t \in I,} z_t^b g_t^b S_t^b = \left[ \begin{array}{c}
T^{-1/2} \sum_{t \in I,} S_t^b g_t^b S_t^b \\
T^{-1/2} \sum_{t \in I,} S_t^b g_t^b S_t^b \\
T^{-1/2} \sum_{t \in I,} S_t^b g_t^b S_t^b
\end{array} \right] = \left[ \begin{array}{c}
F_{1}^b \\
F_{2}^b \\
F_{3}^b
\end{array} \right]
\]

Note that \( F_{1}^b = \mathcal{E}_{1}^b \), and \( F_{2}^b = \mathcal{E}_{2}^b \), defined in (71) and analyzed in the proof of Lemma 10. Also note that, using (100) in the proof of Lemma 10, and replacing as in the proof of Lemma 10, estimated parameters with true values because their difference is asymptotically negligible,

\[
F_{1}^b = S \left( T^{-1/2} \sum_{t \in I,} S_t^b (g_t \circ \nu_t) \right) + o^b_p(1)
\]

\[
F_{2}^b = S \left( T^{-1/2} \sum_{t \in I,} S_t^b (g_t \circ \nu_t) \right) + o^b_p(1)
\]
Since they involve the same underlying random quantity, just scaled differently ($S$ versus $S_r F_s$), the desired distribution for $\mathcal{F}_S^b$ follows directly from the analysis of $\mathcal{E}_{1,2}^b$ in Lemma 10. Careful inspection of the proof of Lemma 10 (focusing on the analysis of $\mathcal{E}_{1}^b$ and $\mathcal{E}_{2}^b$ only) also shows that $\mathcal{Z}_T^b \overset{d}{\rightarrow} M(\tau_{s-1}, \tau_s)$, and indicates that this result holds under Assumptions 1-10, without the need for Assumption 9'. By a similar argument as for the proof of Lemma 6 in the Supplemental Appendix, Section 1, when $t_i \neq \bar{t}_s$, $\mathcal{Z}_T^b \overset{d}{\rightarrow} \bar{M}_i$, completing the proof.

**Proof of Theorem 1.**

We consider only the WR bootstrap; for the WF bootstrap, the results follow in a similar fashion. Let for simplicity $t_i = I_i \lambda_k$. From (8)-(10) and for the Eicker-White estimator $\hat{M}(i)$,

$$Wald_{T \lambda_k} = T \hat{\beta}_{\lambda_k}^\prime R_k^{-1} R_k \hat{\beta}_{\lambda_k}, \quad \hat{V}_{\lambda_k} = \text{diag}_{i=1:k+1}(\hat{Q}_{(i)}^{-1}) \hat{M}_{(i)} \hat{Q}_{(i)}^{-1}$$

From (19)–(21),

$$Wald_{T \lambda_k}^b = T \hat{\beta}_{\lambda_k}^\prime R_k^{-1} R_k \hat{\beta}_{\lambda_k}, \quad \hat{V}_{\lambda_k} = \text{diag}_{i=1:k+1}(\hat{Q}_{(i)}^{-1}) \hat{M}_{(i)} \hat{Q}_{(i)}^{-1}$$

Now consider $\hat{\beta}_{\lambda_k} = \text{vec}(\hat{\beta}_{\lambda_k})$. Let $Q_j = T^{-1} \sum_{t \in I_j} z_t z_t^\prime$. By Lemma 2, $\hat{Q}_{j} \overset{p}{\rightarrow} \int_{\tau_{j - 1}}^{\tau_{j}} Q_{z} d\tau = Q_{z,j}$. Therefore, from the proof of Theorem B 1 in the Supplementary Appendix, Section 1,

$$T^{1/2}(\hat{\beta}_{\lambda_k} - \beta^0) = Q_i^{-1} Y_i^0 (T^{-1/2} \sum_{t \in I_i} z_t u_t + T^{-1/2} \sum_{t \in I_i} z_t v_t^{0}) - T^{-1} \sum_{t \in I_i} z_t^\prime \left\{ \sum_{j=1}^{b+1} l_{j+1}^{0} \right\} + o_p(1).$$

From Lemma 9 and 10, $\hat{Y}_i = Y_i^0 + o_p(1)$. Also, $\hat{Q}_{j} = T^{-1} \sum_{t \in I_j} z_t^\prime z_t^\prime \overset{p}{\rightarrow} Q_{z,j}$ (in probability) by the proof of Lemma 9, therefore:

$$T^{1/2}(\hat{\beta}_{\lambda_k}^0 - \beta^0) = Q_i^{-1} Y_i^0 (T^{-1/2} \sum_{t \in I_i} z_t^b (u_t^b + v_t^{0})^0) - T^{-1} \sum_{t \in I_i} z_t^\prime \left\{ \sum_{j=1}^{b+1} l_{j+1}^{0} \right\} + o_p(1).$$

From Lemma 6 and 10, we have that $T^{-1/2} \sum_{t \in I_i} z_t^v (u_t^b)^0 - T^{-1/2} \sum_{t \in I_i} z_t v_t^{0} = o^-_p(1)$ and $T^{-1/2} \sum_{t \in I_i} z_t^b u_t - T^{-1/2} \sum_{t \in I_i} z_t u_t = o^-_p(1)$. Therefore, from (103)-(104), $T^{1/2}(\hat{\beta}_{\lambda_k}^0 - \beta^0) = o^-_p(1)$.

Because $M_{(i)}$ and $\hat{M}_{(i)}^b$ estimate the same part of the variance of $T^{1/2}(\hat{\beta}_{\lambda_k} - \beta^0)$, and $T^{1/2}(\hat{\beta}_{\lambda_k}^0 - \beta^0)$ respectively, from Lemma 2, 6, 9, and 10, it follows that $\hat{M}_{(i)} - \hat{M}_{(i)} = o^-_p(1)$. Putting these results together, $\sup_{r \in R} |P^b (\text{sup-Wald}_T^b \leq c) - P(\text{sup-Wald}_T \leq c)| \overset{p}{\rightarrow} 0$ as $T \to \infty$.

**Proof of Theorem 2.**

Inspecting the alternative representation of the sup-Wald$_T^b(\ell + 1/\ell)$ in the proof of Theorem B 2 in the Supplemental Appendix, Section 1, and defining the same representation for sup-Wald$_T^b(\ell + 1/\ell)$, the desired result follows using the same steps as in the proof of Theorem 1.
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In the proofs below we refer to the equation numbers from the paper and to the equation numbers from this Supplemental Appendix by (eqnnumber) and respectively (sectionnumber.eqnnumber). Also the tables from this Supplementary Appendix are referred to by sectionnumber.tablenumber, and the tables from the paper are referred to by tablenumber. All the sections referenced below refer to Supplementary Appendix sections.

1 Proofs of Lemmas 2, 4-8 and asymptotic distribution of the sup-Wald tests

**Proof of Lemma 2.**

To begin, consider the case where \(I_i = [\tau_{s-1}T + 1, \tau_s T]\), where \([\tau_{s-1}T + 1, \tau_s T]\) are the intervals for which the coefficients in the VAR(\(p\)) representation of \(\tilde{z}_t\) are stable, and \(s = 1, \ldots, N\), with \(N\) being the total number of breaks in the slope coefficients of the VAR(\(p\)) representation for \(\tilde{z}_t\) following from Assumptions 1-5. Then, letting \(I_i = \tilde{I}_s \sim [\tau_{s-1}T + 1, \tau_s T]\), we have:

\[
\hat{Q}_{z(i)} = T^{-1} \sum_{t \in I_i} z_t z_t' = A + o_p(1),
\]

where

\[
A = T^{-1} \sum_{t \in I_i} \begin{bmatrix}
1 & (S_r \xi_t)' & (S_{\xi_{t-1}})' \\
S_r \xi_t & S_r \xi_t (S_r \xi_t)' & S_r \xi_t (S_{\xi_{t-1}})' \\
S_{\xi_{t-1}} & S_{\xi_{t-1}} (S_r \xi_t)' & S_{\xi_{t-1}} (S_{\xi_{t-1}})'
\end{bmatrix} = \begin{bmatrix}
\Delta \tau_s & A_1' S_r' & A_2' S_r' \\
S_r A_1 & S_r B_1 S_r' & S_r B_2 S_r' \\
S_r A_2 & S_r B_2 S_r' & S_r B_3 S_r'
\end{bmatrix},
\]

where

\[
A_1 = T^{-1} \sum_{t \in I_i} \xi_t, \quad A_2 = T^{-1} \sum_{t \in I_i} \xi_{t-1}, \quad B_1 = T^{-1} \sum_{t \in I_i} \xi_t \xi_t', \quad B_2 = T^{-1} \sum_{t \in I_i} \xi_t \xi_{t-1}', \quad B_3 = T^{-1} \sum_{t \in I_i} \xi_{t-1} \xi_{t-1}'.
\]
Consider $A_1$. We have $A_1 = \sum_{i=1}^{4} A_{1,i}$, where below we denote $\Delta \tau_s T = [\tau_s T] - [\tau_{s-1} T]$, and:

$$
A_{1,1} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \xi_t^i \\
A_{1,2} = T^{-1} \frac{\sum_{t=0}^{\lfloor \tau_s T \rfloor} (\sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l) \mu_s - A_{1,4}}{\sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l} \\
= T^{-1} \frac{1}{\sum_{l=0}^{\lfloor \tau_s T \rfloor} \sum_{t=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l \mu_s - A_{1,4} \\
= T^{-1} \Delta \tau_s T \sum_{t=0}^{\sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l} \mu_s, \\
A_{1,3} = T^{-1} \frac{\sum_{t=0}^{\lfloor \tau_s T \rfloor} (\sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l)}{\sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l \mu_s, \\
A_{1,4} = -T^{-1} \frac{\sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l}{\sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l} \mu_s.
$$

Let $\xi_{t,i}$ be the $i^{th}$ element of $\xi_t$. From Assumptions 7 and 9, $\{\xi_{t,i}, F_t\}$ is a $L^1$-mixingale satisfying the conditions of Lemma 1 and so $A_{1,1} \xrightarrow{p} 0$. From Assumption 7, it follows that $A_{1,2} \xrightarrow{p} A_{1,4}$. From Assumptions 7 and 9, it follows that $\text{Var}[\xi_{t,i}] = O(1)$ and so, again using Assumption 7, it follows that $A_{1,3} = o_p(1)$. From Assumption 7, it follows that $\sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l = O(1)$ and hence $A_{1,4} = o(1)$. Combining these results, we obtain:

$$A_1 = a(\tau_{s-1}, \tau_s) + o_p(1),$$

where

$$a(\tau_{s-1}, \tau_s) = \Delta \tau_s (I_{np} - F_s)^{-1} \mu_s = \int_{\tau_{s-1}}^{\tau_s} Q_1(\tau). \quad (1.1)$$

By similar arguments, we have $A_2 = a(\tau_{s-1}, \tau_s) + o_p(1)$. Now consider $B_1$. Since

$$B_1 = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \left( F_s^{t-\lfloor \tau_s T \rfloor} \xi_{t,i} \right) + T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \xi_t + \left( \sum_{t=0}^{\lfloor \tau_s T \rfloor} \sum_{l=0}^{t-\lfloor \tau_s T \rfloor-1} F_s^l \right) \mu_s,$$

we have $B_1 = \sum_{i=1}^{3} B_{1,i} + \sum_{j=1}^{3} \left( B_{1,3+j} + B_{4,3+j} \right)$, where, setting $\bar{t} = t - [\tau_{s-1} T],$

$$B_{1,1} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \xi_t\xi_t^i = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \left( \sum_{l=0}^{t-1} F_s^l \eta_{t-l} \right) \left( \sum_{l=0}^{t-1} F_s^l \eta_{t-l} \right)'$$

$$B_{1,2} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \left( \sum_{l=0}^{t-1} F_s^l \mu_s \right) \left( \sum_{l=0}^{t-1} F_s^l \mu_s \right)'$$

$$B_{1,3} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} F_s^l \xi_{t,i} \xi_{t,i}^l \left( \sum_{l=0}^{t-1} F_s^l \mu_s \right)'$$

$$B_{1,4} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \xi_t \left( \sum_{l=0}^{t-1} F_s^l \mu_s \right)'$$

$$B_{1,5} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} (\xi_{t,i} \xi_{t,i})^l F_s^l$$

$$B_{1,6} = T^{-1} \sum_{t=0}^{\lfloor \tau_s T \rfloor} \left( \sum_{l=0}^{t-1} F_s^l \mu_s \right) (\sum_{l=0}^{t-1} F_s^l \mu_s)'$$.
Note that $B_{1,1} = \sum_{i=1}^{3} B_{i,1}^{(i)}$ where

$$
B_{1,1}^{(1)} = T^{-1} \sum_{t \in I_t} \sum_{l=0}^{T-1} F_s^l \Omega_{t-l|t-l-1} F_s^l', \\
B_{1,1}^{(2)} = T^{-1} \sum_{t \in I_t} \sum_{l=0}^{T-1} F_s^l \left( \eta_{l-1} \eta_{t-l} - \Omega_{t-l|t-l-1} \right) F_s^l', \\
B_{1,1}^{(3)} = T^{-1} \sum_{t \in I_t} \sum_{l=0, l \neq j}^{T-1} F_s^l \eta_{l-j} F_s^l'.
$$

It can be shown that $B_{1,1}^{(i)} = C_1 + C_2$, where

$$
C_1 = \sum_{l=0}^{(\Delta \tau T)-1} F_s^l \left\{ T^{-1} \sum_{t \in I_t} \Omega_{t-l|t-1} \right\} F_s^l', \\
C_2 = -T^{-1} \sum_{l=1}^{(\Delta \tau T)-1} F_s^l \left( \sum_{j=[l,T]-l}^{l+1} \Omega_{j|j-1} \right) F_s^l'.
$$

Under our Assumptions, it is shown in Section 2 that $C_2 = o_p(1)$. Furthermore, we have:

$$
C_1 \xrightarrow{p} \sum_{l=0}^{\infty} F_s^l \left[ A_{s}^{-1} \int_{\tau_{t-1}}^{\tau_{t-1}} \Sigma(\tau) \, d\tau \, A_{s}^{-1}' \right]_{0_{n \times n(p-1)}}^{0_{n \times n(p-1)}} \times 0_{n(p-1) \times n(p-1)} = B_1(\tau_{s-1}, \tau_s),
$$

where $\Sigma(a) = S\{D(a)\}^2$, and $D(a) = \text{diag}_{i=1:n}(d_i(a))$, with $d_i(a)$ defined in Assumption 9.

Now consider $B_{1,1}^{(2)}$. Recall that the only non-zero elements of $\eta_{l-1} \eta_{t-l} - \Omega_{t-l|t-l-1}$ are in the upper left-hand block and take the form:

$$
A_{s}^{-1} (\epsilon_{t-l} \epsilon_{t-l}^l - \Sigma_{t-l|t-l-1}) A_{s}^{-1}'.
$$

Since each element of the matrix $\epsilon_t \epsilon_t^l - E[\epsilon_t \epsilon_t^l | F_{t-1}]$ is a mean-zero m.d.s., and has uniformly bounded $(2 + \delta)$ moments (Assumption 9), it follows that each element of the matrix $\sum_{l=0}^{T-1} F_s^l \left( \eta_{l-1} \eta_{t-l} - \Omega_{t-l|t-l-1} \right) F_s^l'$ is a $L^1$-mixingale with constants that are uniformly bounded. Therefore, by Lemma 1, it follows that $B_{1,1}^{(2)} \xrightarrow{p} 0$.

Now consider $B_{1,1}^{(3)}$. Under Assumptions 7 and 9, each element of the matrix $\sum_{l=0, l \neq j}^{T-1} F_s^l \eta_{l-j} F_s^l'$ is a $L^1$-mixingale satisfying the conditions of Lemma 1, so $B_{1,1}^{(3)} \xrightarrow{p} 0$. Therefore,

$$
B_{1,1} \xrightarrow{p} B_1(\tau_{s-1}, \tau_s).
$$

From Assumption 7 and Section 2, it follows that:

$$
B_{1,2} \rightarrow \Delta \tau_s (I_{np} - F_s)^{-1} \mu_s \mu_s' (I_{np} - F_s)^{-1}' = \int_{\tau_{s-1}}^{\tau_s} Q_1(\tau) Q_1'(\tau) = B_2(\tau_{s-1}, \tau_s).
$$

Now consider $B_{1,3}$. Under Assumptions 7 and 9, it follows that $\xi_{[\tau_{s-1}, T]} \xi_{[\tau_{s-1}, T]} = o_p(1)$ and so from Assumption 7, $B_{1,3} = o_p(1)$. Next, consider $B_{1,4}$. Under Assumptions 7 and 9, $\xi_t \left( \sum_{l=0}^{T-1} F_s^l \mu_s \right)'$ is a m.d.s. with uniformly bounded $(2 + \delta)$ moments, so $B_{1,4} = o_p(1)$. Next, consider $B_{1,5}$. Under Assumptions 7 and 9, it can be shown that $\xi_t$ is a $L^1$-mixingale satisfying Lemma 1. Hence

$$
B_{1,5} = T^{-1} \sum_{t \in I_t} \sum_{l=0}^{(\Delta \tau T)-1} F_s^l \eta_{l-1} O_p(1) F_s^l' = o_p(1).
$$

Finally, it is shown in Section 2 that under Assumption 7, $B_{1,6} = o_p(1)$.  

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Combining these results, we obtain:
\[ B_1 \overset{p}{\rightarrow} \mathbb{B}(\tau_{s-1}, \tau_s), \]

where
\[ \mathbb{B}(\tau_{s-1}, \tau_s) = \mathbb{B}_1(\tau_{s-1}, \tau_s) + \mathbb{B}_2(\tau_{s-1}, \tau_s). \]  

(1.5)

Next, define \( \tilde{I}_s^- = \{[\tau_{s-1}T] + 2, [\tau_{s-1}T] + 3, \ldots, [\tau_sT]\} \), and note that \( \tilde{I}_s = \tilde{I}_s^- \cup \{[\tau_{s-1}T] + 1\} \). Similarly to \( B_1 \),

\[ B_3 = T^{-1} \sum_{t \in \tilde{I}_s^-} \xi_{t-1} \xi_{t-1} - T^{-1} \sum_{t \in \tilde{I}_s^-} \xi_{t-1} \xi_{t-1} + T^{-1} \xi_{[\tau_{s-1}T] - 1} \xi_{[\tau_{s-1}T] - 1} \]
\[ = T^{-1} \sum_{t \in \tilde{I}_s^-} \xi_t \xi_t - T^{-1} \xi_{[\tau_{s-1}T]} \xi_{[\tau_{s-1}T]} + T^{-1} \xi_{[\tau_{s-1}T] - 1} \xi_{[\tau_{s-1}T] - 1} \]
\[ = \mathbb{B}(\tau_{s-1}, \tau_s) + o_p(1) - T^{-1} \xi_{[\tau_{s-1}T]} \xi_{[\tau_{s-1}T]} + T^{-1} \xi_{[\tau_{s-1}T] - 1} \xi_{[\tau_{s-1}T] - 1}. \]  

(1.6)

We now show that \( \sup_t \mathbb{E} \| \xi_t \xi_t \| < c \) for some \( c > 0 \). By backward substitution of the first regime \( s = 1 \), \( \xi_t = F_s \xi_0 + \left( \sum_{t=0}^{s-1} F_s \right) \mu_s + \xi_t \). Let \( c_t = \left( \sum_{t=0}^{s-1} F_s \right) \mu_s \), and note that \( \| c_t \| < c_0 \) for some \( c_0 > 0 \), by arguments similar to Section 2. Also,

\[ \mathbb{E} \| \xi_t \xi_t \| \leq \mathbb{E} \| F_s \xi_0 \xi_0 \xi_s \| + \| c_t c_t \| + \mathbb{E} \| \xi_t \xi_t \| + 2 \mathbb{E} \| F_s \xi_0 c_t \| + 2 \mathbb{E} \| c_t \xi_t \| + 2 \mathbb{E} \| \xi_t \| F_s \xi_t \| \].

First, \( \sup_t \mathbb{E} \| F_s \xi_0 \xi_0 \xi_s \| \leq \sup_t \| F_s \| \mathbb{E} \| \xi_0 \| < c_1 \) by Assumptions 7 and 9 for some \( c_1 > 0 \). Second, \( \sup_t \| c_t c_t \| < \sup_t \| c_t \| < c_0^2 = c_2 \). Third, \( \sup_t \mathbb{E} \| \eta_t \xi_t \| \leq \sup_t \| \eta_t \|^{2/2} \mathbb{E} \| \eta_t \|^{2/2} < c^* \) for some \( c^* > 0 \) by Assumption 9. Therefore, \( \sup_t \mathbb{E} \| \xi_t \xi_t \| \leq \sup_t \left( \sum_{t=0}^{s-1} \| F_s \| \| F_s \| \sup_t \| \eta_t \| \right) \leq c^* \sup_t \left( \sum_{t=0}^{s-1} \| F_s \| \| F_s \| \right) < c_3 \) for some \( c_3 > 0 \). Similarly, \( \sup_t \left( 2 \mathbb{E} \| F_s \xi_0 c_t \| + 2 \mathbb{E} \| c_t \xi_t \| + 2 \mathbb{E} \| \xi_t \| F_s \xi_t \| \right) < c_4 \) for some \( c_4 > 0 \). Therefore,

\[ \sup_t \mathbb{E} \| \xi_t \xi_t \| < c_1 + c_2 + c_3 + c_4 = c < \infty, \]  

(1.7)

so, by Markov’s inequality, \( T^{-1} \xi_{[\tau_{s-1}T]} \xi_{[\tau_{s-1}T]} = o_p(1) \) and \( T^{-1} \xi_{[\tau_{s-1}T] - 1} \xi_{[\tau_{s-1}T] - 1} = o_p(1) \) for \( s = 1 \). For the other regimes \( s > 1 \), repeated backward substitution yields the same result. Substituting this result into (1.6), it follows that \( B_3 \overset{p}{\rightarrow} \mathbb{B}(\tau_{s-1}, \tau_s) \).

Now consider \( B_2 \):

\[ B_2 = T^{-1} \sum_{t \in \tilde{I}_s^-} \xi_t \xi_{t-1} = T^{-1} \sum_{t \in \tilde{I}_s^-} (\mu_s + F_s \xi_{t-1} + \eta_t) \xi_{t-1} \]
\[ = \mu_s T^{-1} \sum_{t \in \tilde{I}_s^-} \xi_{t-1} + T^{-1} \sum_{t \in \tilde{I}_s^-} \eta_t \xi_{t-1} + F_s \mathbb{B}(\tau_{s-1}, \tau_s) + o_p(1). \]  

(1.8)

First, note that \( \mu_s T^{-1} \sum_{t \in \tilde{I}_s^-} \xi_{t-1} = \mu_s \lambda'_2 = \mu_s \lambda' \tau_{s-1}, \tau_s + o_p(1) \). Next, \( \eta_t \xi_{t-1} \) is a m.d.s. sequence, so each of its element is \( L^1 \)-mixingale with bounded constants as defined in Lemma 1. Also, \( \sup_t \mathbb{E} \| \eta_t \xi_{t-1} \|^{1+\delta/2} < \sup_t (\mathbb{E} \| \eta_t \|^2)^{1+\delta/2} \sup_t (\mathbb{E} \| \eta_t \|^2)^{1+\delta/2} < \infty \), so \( \eta_t \xi_{t-1} \) satisfies Lemma 1 element-wise, so \( T^{-1} \sum_{t \in \tilde{I}_s^-} \eta_t \xi_{t-1} \overset{p}{\rightarrow} 0 \). Substituting these results into (1.8), we obtain:

\[ B_2 \overset{p}{\rightarrow} \mu_s \lambda' \tau_{s-1}, \tau_s + F_s \mathbb{B}(\tau_{s-1}, \tau_s). \]  

(1.9)
Therefore,

\[
A \overset{\rho}{\to} \left[ \begin{array}{ccc}
\Delta \tau_s & \mathcal{A}'(\tau_{s-1}, \tau_s)S' & \mathcal{A}'(\tau_{s-1}, \tau_s)S' \\
S_r \mathcal{A}(\tau_{s-1}, \tau_s) & S_r \mathcal{B}(\tau_{s-1}, \tau_s)S' & S_r (\mathbf{\mu}, \mathcal{A}'(\tau_{s-1}, \tau_s) + F_r \mathcal{B}(\tau_{s-1}, \tau_s))S' \\
S \mathcal{A}(\tau_{s-1}, \tau_s) & S (\mathbf{\mu}, \mathcal{A}'(\tau_{s-1}, \tau_s) + F_r \mathcal{B}(\tau_{s-1}, \tau_s))S' & S \mathcal{B}(\tau_{s-1}, \tau_s)S'
\end{array} \right].
\]

Now consider the case of \( I_i \) containing \( N_i \) breaks from the total set of \( N \) breaks, that is, there is an \( s \) such that \( \tau_{s-1} < \lambda_i - 1 \leq \tau_s \) and \( \tau_{s+N_i} \leq \lambda_i < \tau_{s+N_i+1} \). Then, generalizing the previous results which were for \( I_i = \tilde{I}_i = [[\tau_{s-1}T] + 1, [\tau_sT]] \), we have by similar arguments that: \( \hat{Q}_{z,(i)} \overset{\rho}{\to} \int_{\lambda_i - 1}^{\lambda_i} Q_z(\tau) d\tau \).

By Lemma 7, \( \hat{Y}_i = Y_i^0 + o_p(1) \), therefore

\[
\hat{Q}(i) \overset{\rho}{\to} \int_{\lambda_i - 1}^{\lambda_i} Y'(\tau)Q_z(\tau)Y(\tau) d\tau = Q_i.
\]

\[\square\]

**Proof of Lemma 4.** Let \( \phi_t \) denote the \((a, b)\) element of the matrix \([ E (l_i^j \mid F_{t-1}) - I_n ], [ E ((l_i^j)^{\otimes} l_{i-1} \mid F_{t-1}) - \rho_i ]\) or \([ E ((l_i^j)^{\otimes} l_{i-1,l_{i-1}} \mid F_{t-1}) - \rho_{i,j} ]\). Note that \( \phi_t \) is a m.d.s., so it is a \( L^1 \)-mixingale with constants satisfying Lemma 1. We now show that \( \sup_t E|\phi_t|^b < \infty \), letting \( b = 1 + \delta/4 > 1 \) (for example).

From Jensen’s inequality, \( E|\phi_t|^b \leq E( E(\phi_t|F_{t-1})^b ) \), \( E|\phi_t|^b \leq E|\phi_t|^b \).

(1.10)

Consider \( E( E(\phi_t|F_{t-1}) - E(\phi_t) )^b ) \). We have:

\[
|E(\phi_t|F_{t-1}) - E(\phi_t) |^b \leq 2^b \max \left( |E(\phi_t|F_{t-1})|^b, |E(\phi_t)|^b \right).
\]

From (1.10), \( \sup_t E( |E(\phi_t|F_{t-1})|^b ) \leq \sup_t E( |\phi_t|^b |F_{t-1}) = \sup_t E(|\phi_t|^b \), so \( \sup_t E( |E(\phi_t|F_{t-1}) - E(\phi_t) |^b ) \leq 2^b \sup_t E|\phi_t|^b < \infty \). Therefore, the conditions of Lemma 1 are satisfied, so \( T^{-1} \sum_{t=1}^{[\tau_T]} E(\phi_t|F_{t-1}) - E(\phi_t) \overset{\rho}{\to} 0 \) uniformly in \( r \).

\[\square\]

**Proof of Lemma 5.**

By Assumption 9(i) and Lemma 4, Assumption 2 in Boswijk, Cavaliere, Rahbek, and Taylor (2016) is satisfied; therefore, the proof follows by exactly the same arguments as in the proof of their Lemma 2, page 79, paragraphs 1-2.

\[\square\]

**Proof of Lemma 6.**

We first derive the asymptotic distribution of \( T^{-1/2} \sum_{t \in I_i} z_t u_t \) and \( T^{-1/2} \sum_{t \in I_i} z_t v_t^j \beta_x^j \) for interval \( I_i = \tilde{I}_i = [[\tau_{s-1}T] + 1, \ldots, [\tau_sT]] \) when all the coefficients of the VAR are stable. Let \( g_t = g_t = \text{vect}(\epsilon_t, 0_{n(1-p)}) = \text{vect}(u_t, v_t, \zeta_t, 0_{n(p-1)}) \).

Then \( \eta_t = A_{s}^{-1} g_t \) and \( g_t = A_{s} \eta_t \). Hence, \( T^{-1/2} \sum_{t \in I_i} z_t u_t = T^{-1/2} \sum_{t \in I_i} z_t g_t S_u \) and \( T^{-1/2} \sum_{t \in I_i} z_t v_t^j \beta_x^j = \)
Next, from above, note that
\[ Z_T = T^{-1/2} \sum_{t \in I_s} Z_t g_t^i \beta_{x_s, #}. \]
We now derive the limit of \( Z_T = T^{-1/2} \sum_{t \in I_s} z_t g_t^i S_t \).

\[
Z_T = T^{-1/2} \sum_{t \in I_s} z_t \eta_t^i A_{t, #} S_t = \begin{bmatrix}
T^{-1/2} \sum_{t \in I_s} \eta_t^i A_{t, #} S_t \\
T^{-1/2} \sum_{t \in I_s} S_{t} \xi_t \eta_t^i A_{t, #} S_t \\
T^{-1/2} \sum_{t \in I_s} S_{t} \xi_{t-1} \eta_t^i A_{t, #} S_t
\end{bmatrix} \equiv \begin{bmatrix}
\mathcal{E}_1 \\
\mathcal{E}_2 \\
\mathcal{E}_3
\end{bmatrix}.
\]

Note that \( \eta_t = A_{s, #}^{-1} S_{#} D_{t, #} l_{t, #} \).

- Consider first \( \mathcal{E}_1 \). Notice that \( \mathcal{E}_1 = \mathcal{E}_1^T = T^{-1/2} \sum_{t \in I_s} S_{t} A_{s, #} \eta_t = \text{vec}(\mathcal{E}_1) = S_{s} \xi_t A_{s, #} \tilde{\mathcal{E}}_1 \), where \( \tilde{\mathcal{E}}_1 = T^{-1/2} \sum_{t \in I_s} \eta_t \). Hence, consider \( \tilde{\mathcal{E}}_1 \). By Lemma 5, \( B_0(T) = T^{-1/2} \sum_{t=1}^{T} l_t \Rightarrow B_0 \). By Assumption 9 and Lemma 4, the conditions in Lemmas 1-2 of Bowwijk, Cavaliere, Rahbek, and Taylor (2016) are satisfied, so:

\[
\tilde{\mathcal{E}}_1 = T^{-1/2} \sum_{t \in I_s} \eta_t = A_{s, #}^{-1} S_{#} T^{-1/2} \sum_{t \in I_s} (D_{t, #} l_{t, #}) = A_{s, #}^{-1} S_{#} \left( \int_{t-1}^{T} D_{#} (\tau) dB_{0,T, #}(\tau) \right) \Rightarrow A_{s, #}^{-1} S_{#} \left( \int_{t-1}^{T} D_{#} (\tau) dB_{0,T, #}(\tau) \right).
\]

Using the fact that \( A_{s, #}^{-1} A_{s, #}^{-1} = I_{#}, I_{#} S_{#} = S_{#} \),

\[
\mathcal{E}_1 = S_{s} \xi_t A_{s, #} \tilde{\mathcal{E}}_1 \Rightarrow (S_{s} \xi_t) \left( \int_{t-1}^{T} D_{#} (\tau) dB_{0, #}(\tau) \right) = M_{t}(\tau_{t-1}, \tau_t), \tag{1.11}
\]

with variance

\[
V_{M_{t}(\tau_{t-1}, \tau_t)} = (S_{s} \xi_t) \left( \int_{t-1}^{T} D_{#} (\tau) D_{#} (\tau) d\tau \right) (S_{s} \xi_t)^{\prime}. \tag{1.12}
\]

- Next, consider \( \mathcal{E}_3 \). Note that \( \mathcal{E}_3 = \text{vec}(\mathcal{E}_3) = (S_{s} \xi_t) \otimes S_{t} T^{-1/2} \sum_{t \in I_s} (\eta_t \otimes \xi_t) \otimes \xi_{t-1} \). Hence consider \( \tilde{\mathcal{E}}_3 \), letting \( I = [\tau_{t-1} + 2, \tau_T] \) and recall that \( I_{s} = [\tau_{t-1} + 2, \tau_T] \). Then:

\[
\tilde{\mathcal{E}}_3 = T^{-1/2} \sum_{t \in I_s} \eta_t \otimes \xi_t \otimes \xi_{t-1}
= T^{-1/2} I_{s} [\tau_{t-1} + 2, \tau_T] + T^{-1/2} \sum_{t \in I_s} \eta_t \otimes \left( F_s^t \xi_{t-1} \right)
+ T^{-1/2} \sum_{t \in I_s} \eta_t \otimes \left( \sum_{l=0}^{\tau_T} F_s^l \right) \mu_s
= \sum_{t=0}^{\tau_T} \tilde{\mathcal{E}}_{3,t}.
\]

Note that \( E \| \tilde{\mathcal{E}}_{3, t} \| \leq T^{-1/2} \sup_t (E \| \eta_t \|^2)^{1/2} \sup_t (E \| \xi_t \|^2)^{1/2} < c_{\eta} \sup_t (E \| \xi_t \|^2)^{1/2} \) for some \( c_{\eta} > 0 \) by Assumption 9. Since \( \sup_t (E \| \xi_t \|^2) < c \) by (1.7), \( \tilde{\mathcal{E}}_{3, t} \to 0 \).

Next, from above, \( \xi_{[\tau_{t-1} T]} = O_p(1) \), and

\[
\tilde{\mathcal{E}}_{3,2} = \left( T^{-1/2} \sum_{t \in I_s} (I_{#} \otimes F_{s}^{t-\tau_{t-1} T}) \otimes (\eta_t \otimes I_{#}) \right) \xi_{[\tau_{t-1} T]},
\]

where for some \( c^* > 0 \),

\[
E \| T^{-1/2} \sum_{t \in I_s} (I_{#} \otimes F_{s}^{t-\tau_{t-1} T}) \otimes (\eta_t \otimes I_{#}) \| \leq T^{-1/2} \sum_{t \in I_s} \| I_{#} \otimes F_{s}^{t-\tau_{t-1} T} \| \| E \| \eta_t \| \| I_{#} \| \leq c^* T^{-1/2} \sum_{t=0}^{\tau_{t-1} T-1} \| F_{s}^{t+1} \| \to 0,
\]

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where we used \( \| A \otimes B \| = \| A \| \| B \| \), and the last statement follows by Assumptions 7 and 9 and the derivations in Section 2 below. Therefore, by the Markov inequality, it follows that \( \tilde{\mathcal{E}}_{3,2} = o_p(1) \).

Next, we show that \( \tilde{\mathcal{E}}_{3,4} = \sum_{l=0}^{T-2} [I_{np} \otimes F_s^l] \left( T^{-1/2} \sum_{t \in I_l} [\eta_t \otimes \eta_{t-l-1}] \right) + o_p(1) \equiv \tilde{\mathcal{E}}_{3,4}^{(1)} + o_p(1) \). To that end, let \( \tilde{n} = [\tau_{s-1} T] \).

\[
T^{-1/2} \sum_{t \in I_l} \eta_t \otimes \left[ \sum_{i=0}^{T-l-2} F_s^i \eta_{t-l-1} \right] = T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^0 \eta_{\tilde{n}+1}] + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^0 \eta_{\tilde{n}+2} + F_s^1 \eta_{\tilde{n}+1}] \\
+ T^{-1/2} \eta_{\tilde{n}+4} \otimes [F_s^0 \eta_{\tilde{n}+3} + F_s^1 \eta_{\tilde{n}+2} + F_s^2 \eta_{\tilde{n}+1}] + \ldots \\
+ T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T} \otimes [F_s^0 \eta_{\tilde{n}+\Delta \tau,T-1} + F_s^1 \eta_{\tilde{n}+\Delta \tau,T-2} + \ldots + F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+1}] \\
= (T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^0 \eta_{\tilde{n}+1}]) + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^0 \eta_{\tilde{n}+2} + \ldots] + T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T} \otimes [F_s^0 \eta_{\tilde{n}+\Delta \tau,T-1}] + \\
+ (T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^1 \eta_{\tilde{n}}]) + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^1 \eta_{\tilde{n}+1}] + T^{-1/2} \eta_{\tilde{n}+4} \otimes [F_s^1 \eta_{\tilde{n}+2}] + \ldots + T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T} \otimes [F_s^1 \eta_{\tilde{n}+\Delta \tau,T-2}] \\
- T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^2 \eta_{\tilde{n}}] + \\
+ (T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^2 \eta_{\til{n}-1}]) + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^2 \eta_{\tilde{n}}] + T^{-1/2} \eta_{\tilde{n}+4} \otimes [F_s^2 \eta_{\tilde{n}+1}] + \ldots + T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T} \otimes [F_s^2 \eta_{\tilde{n}+\Delta \tau,T-3}] \\
- T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^2 \eta_{\tilde{n}-1}] + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^2 \eta_{\tilde{n}}] + \ldots + T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T} \otimes [F_s^2 \eta_{\tilde{n}+\Delta \tau,T-2}] \\
+ (T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+3} \otimes \eta_{\tilde{n}+\Delta \tau,T}) + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+4} \otimes \eta_{\tilde{n}+\Delta \tau,T}) + \ldots + T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T} \otimes [F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+1}]) \\
- T^{-1/2} \eta_{\tilde{n}+2} \otimes [F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+3} \otimes \eta_{\tilde{n}+\Delta \tau,T}) + T^{-1/2} \eta_{\tilde{n}+3} \otimes [F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+4} \otimes \eta_{\tilde{n}+\Delta \tau,T}) + \ldots + T^{-1/2} \eta_{\tilde{n}+\Delta \tau,T-1} \otimes [F_s^\Delta \tau,-T+2 \eta_{\tilde{n}+2}]) \\
= \tilde{\mathcal{E}}_{3,4}^{(1)} - T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} [I_{np} \otimes F_s^l] \sum_{j=0}^{l-1} \eta_{\tilde{n}+2+j} \otimes \eta_{\tilde{n}+2+j-(l+1)}. \\
\tag{1.13}
\]

Note that by Assumption 7, for some \( c, c^* > 0 \),

\[
E \| T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} [I_{np} \otimes F_s^l] \sum_{j=0}^{l-1} \eta_{\tilde{n}+2+j} \otimes \eta_{\tilde{n}+2+j-(l+1)} \| \\
\leq T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} \| I_{np} \otimes F_s^l \| \sum_{j=0}^{l-1} E \| \eta_{\tilde{n}+2+j} \otimes \eta_{\tilde{n}+2+j-(l+1)} \| \\
\leq T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} \left\| I_{np} \otimes F_s^l \right\| \sum_{j=0}^{l-1} \left( \| \mathbb{E}(\eta_{\tilde{n}+2+j} \otimes \eta_{\tilde{n}+2+j-(l+1)}) \| \right)^{1/2} \\
\leq T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} \left\| I_{np} \otimes F_s^l \right\| \sum_{j=0}^{l-1} \left\| A_s^{-1} \otimes A_s^{-1} \right\|^{1/2} \left( \mathbb{E}([\mathbf{I}_{\tilde{n}+2+j} \mathbf{I}_{\tilde{n}+2+j}] \otimes [\mathbf{I}_{\tilde{n}+2+j} \mathbf{I}_{\tilde{n}+2+j-(l+1)}]) \right)^{1/2} \\
\leq c T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} \sum_{j=0}^{l-1} \left\| I_{np} \otimes F_s^l \right\| \| \mathbb{E}(\mathbf{I}_{\tilde{n}+2+j} \mathbf{I}_{\tilde{n}+2+j}) \otimes [\mathbf{I}_{\tilde{n}+2+j} \mathbf{I}_{\tilde{n}+2+j-(l+1)}] \right\|^{1/2} \\
\leq c T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} \sum_{j=0}^{l-1} \left\| I_{np} \otimes F_s^l \right\| \sup_j \left( \| \mathbf{I}_{\tilde{n}+2+j} \mathbf{I}_{\tilde{n}+2+j-(l+1)} \| \right)^{1/2} \leq c^* \sum_{l=1}^{\Delta \tau,T-2} l \| F_s^l \| \to 0, \\
\tag{1.14}
\]

where the last statement follows from Assumptions 7 and 9 and arguments similar to the ones in Section 2.

Therefore, by Markov’s inequality, \( T^{-1/2} \sum_{l=1}^{\Delta \tau,T-2} [I_{np} \otimes F_s^l] \sum_{j=0}^{l-1} \eta_{\tilde{n}+2+j} \otimes \eta_{\tilde{n}+2+j-(l-1)} = o_p(1) \), so

\[
\tilde{\mathcal{E}}_{3,4} = \tilde{\mathcal{E}}_{3,4}^{(1)} + o_p(1). 
\]

By similar arguments,

\[
\tilde{\mathcal{E}}_{3,3} = \tilde{\mathcal{E}}_{3,3}^{(1)} + o_p(1),
\]

where \( \tilde{\mathcal{E}}_{3,3}^{(1)} = \sum_{l=0}^{\Delta \tau,T-2} [I_{np} \otimes F_s^l] \left( T^{-1/2} \sum_{t \in I_l} [\eta_t \otimes \mu_s] \right) \).

Putting the results about \( \tilde{\mathcal{E}}_{i,4} \) for \( i = 1, 2, 3, 4 \) together, and letting

\[
\tilde{\mathcal{E}}_{4}(l) = [I_{np} \otimes F_s^l] \left( T^{-1/2} \sum_{t \in I_l} \eta_t \otimes \mu_s \otimes \eta_{t-l-1} \right).
\]

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and $\tilde{E}_4(\Delta T - 2) = \sum_{l=0}^{\Delta T - 2} \tilde{E}_4(l)$, we have:

$$\tilde{E}_4(l) + o_p(1) = \tilde{E}_4(\Delta T - 2) + o_p(1). \quad (1.15)$$

We are interested in the asymptotic distribution of $\tilde{E}_4(\Delta T - 2)$. First, consider the asymptotic distribution of $\tilde{E}_4(l)$ for a given $l$. First, $B_{0,T}(\tau;#) = T^{-1/2} \sum_{t=1}^{[Tr]} l_{#} \Rightarrow B_{0,#}(\tau)$. We also have that: $\eta_l \odot \eta_{l-1} = (A_{x,#}^{-1} \odot A_{x,#}^{-1})(g_t \odot g_{t-1}) = (A_{x,#}^{-1} \odot A_{x,#}^{-1})([S_{#}D_{l,#}] \odot (S_{#}D_{l-1,#}))(l_{#} \odot l_{l-1,#})$. By Lemma 5, for $l \geq 0$, $B_{l+1,T}(\tau;#) = T^{-1/2} \sum_{t=1}^{[Tr]} (l_{#} \odot l_{l-1,#}) \Rightarrow B_{l+1}(r)$. Also, we have that $B_{l+1,T}(\tau;#) = T^{-1/2} \sum_{t=1}^{[Tr]} (l_{#} \odot l_{l-1,#}) \Rightarrow B_{l+1}(r)$, with variance matrix $\rho_{perm}^{\tau,\tau} = \rho_{l+1,l+1,#}P_1P_{l+1}P_2$, where $P_1, P_2$ are permutation matrices defined as in Tracy and Jinadasa (1979), Theorem 7, equation (18); note that this result holds because $l_{#} \odot l_{l-1,#}$ is a permutation of $(l \odot l_{l-1,#})$.

Now let $u_{l,#}^{(n)} = \text{vec}(l_t, l_t \odot l_{l-2}, \ldots, l_t \odot l_{l-1,n})$ and $B_{T}^{(n)}(\tau;#) = T^{-1/2} \sum_{t=1}^{[Tr]} u_{l,#}^{(n)} \Rightarrow B(\tau^{(n)};#) = \text{vec}(B_{t}^{(n)}(r))$, with variance $V_{B^{(n)}}(r) = r\begin{bmatrix} I_n & \rho_{0,1} & \ldots & \rho_{0,n-1} \\ \rho_{1,0} & \rho_{1,1} & \ldots & \rho_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n-1,0} & \rho_{n-1,1} & \ldots & \rho_{n-1,n-1} \end{bmatrix}$.

We have:

$$\tilde{E}_4([\Delta T - 2]) = \tilde{E}_4(n^{*}) + \tilde{E}_4([\Delta T - 2] - \tilde{E}_4(n^{*}))$$

$$= \sum_{l=0}^{\Delta T - 2}[I_{n^#} \odot F_l^#]\left(T^{-1/2} \sum_{t=1}^{[Tr]} (\eta_t \odot \mu_t)\right) + \sum_{l=0}^{\Delta T - 2}[I_{n^#} \odot F_l^#]\left(T^{-1/2} \sum_{t=1}^{[Tr]} (\eta_t \odot \eta_{l-1})\right) + o_p(1)$$

$$= \tilde{E}_{*,1}(n^{*}) + \tilde{E}_{*,2}(n^{*}) + \tilde{E}_{*,3}(n^{*}) + \tilde{E}_{*,4}(n^{*}) + \tilde{E}_{*,5}(n^{*}) + o_p(1)$$

Using the convergence $B_{l,T}(\tau;#) \Rightarrow B_{l}(r)$, $l = 0, 1, \ldots, n^*$, and Theorem 2.1 in Hansen (1992), it follows that, as $T \to \infty$, with $n^* = T^\alpha$ and $\alpha \in (0, 1)$,

$$\tilde{E}_{*,1}(n^{*}) + \tilde{E}_{*,2}(n^{*}) \Rightarrow \sum_{l=0}^{\infty}([A_{l,#}^{-1}S_{#}] \odot F_l^#)\left([\int_{r_{l-1}}^{r} D_{#}(\tau) dB_{0,T}(\tau)\right) \odot \mu_l$$

$$+ \sum_{l=0}^{\infty}([A_{l,#}^{-1}S_{#}] \odot (F_l^#A_{l,#}^{-1}S_{#}))\left([\int_{r_{l-1}}^{r} D_{#}(\tau) dD_{#}(\tau - \frac{l-1}{l})\right) dB_{l+1,T}(\tau)$$

where the variance of the right hand side quantity exists by Assumption 7 and 9. Next, note that under the same
assumptions, for \( n^* = T^\infty \) and \( T \to \infty \),

\[
\| \tilde{\mathcal{E}}_4([\Delta \tau_s T] - 2) - \tilde{\mathcal{E}}_4(n^*) \| \leq \sum_{l=n^*+1}^{[\Delta \tau_s T]-2} \| (A^{-1}_{s,\#} S_\#) \otimes F_l^\prime \left( \left[ \int_{\tau_{l-1}}^{\tau_s} D_\#(\tau) \, dB_{0,\#}(\tau) \right] \otimes \mu_s \right) \|
+ \sum_{l=n^*+1}^{[\Delta \tau_s T]-2} \| (A^{-1}_{s,\#} S_\#) \otimes (F_l^\prime A^{-1}_{s,\#} S_\#) \| \int_{\tau_{l-1}}^{\tau_s} (D_\#(\tau) \otimes D_\#(\tau)) \, dB_{l+1,\#}(\tau) \|
\leq \sum_{l=n^*+1}^{[\Delta \tau_s T]-2} \| F_l^\prime \| \| A^{-1}_{s,\#} S_\# \| O_p(1) + \sum_{l=n^*+1}^{[\Delta \tau_s T]-2} \| F_l^\prime \| \| A^{-1}_{s,\#} S_\# \|^2 O_p(1) \xrightarrow{P} 0.
\]

Therefore,

\[
\tilde{\mathcal{E}}_3 = \tilde{\mathcal{E}}_3([\Delta \tau_s T] - 2) + o_p(1) = \sum_{l=0}^{\infty} \left( (A^{-1}_{s,\#} S_\#) \otimes F_l^\prime \right) \left( \left[ \int_{\tau_{l-1}}^{\tau_s} D_\#(\tau) \, dB_{0,\#}(\tau) \right] \otimes \mu_s \right) + \sum_{l=0}^{\infty} \left( (A^{-1}_{s,\#} S_\#) \otimes (F_l^\prime A^{-1}_{s,\#} S_\#) \right) \int_{\tau_{l-1}}^{\tau_s} (D_\#(\tau) \otimes D_\#(\tau)) \, dB_{l+1,\#}(\tau),
\]

and because \( \tilde{\mathcal{E}}_3 = ((S_l A_{s,\#}) \otimes S) \tilde{\mathcal{E}}_3 \), we have:

\[
\mathcal{E}_3 \Rightarrow \sum_{l=0}^{\infty} \left( (S_l A_{s,\#}) \otimes (S F_l^\prime) \right) \left( \left[ \int_{\tau_{l-1}}^{\tau_s} D_\#(\tau) \, dB_{0,\#}(\tau) \right] \otimes \mu_s \right) + \sum_{l=0}^{\infty} \left( (S_l A_{s,\#}) \otimes (S F_l^\prime A^{-1}_{s,\#} S_\#) \right) \int_{\tau_{l-1}}^{\tau_s} (D_\#(\tau) \otimes D_\#(\tau)) \, dB_{l+1,\#}(\tau) = M_{3,1}(\tau_{s-1}, \tau_s) + M_{3,2}(\tau_{s-1}, \tau_s) = M_{3}(\tau_{s-1}, \tau_s).
\]

Note that \( M_{3}(\tau_{s-1}, \tau_s) \) depends on the parameters in the interval \( \tilde{l}, A_s \), and \( \mu_s \), so when the interval over which \( M_{3}(\cdot) \) is evaluated changes, these coefficients also change. Also note that the variance matrix of \( M_{3}(\tau_{s-1}, \tau_s) \) is

\[
V_{M_{3}(\tau_{s-1}, \tau_s)} = V_{M_{3,1}(\tau_{s-1}, \tau_s)} + V_{M_{3,2}(\tau_{s-1}, \tau_s)} + V_{M_{3,1}(\tau_{s-1}, \tau_s), M_{3,2}(\tau_{s-1}, \tau_s)} + V_{M_{3,1}(\tau_{s-1}, \tau_s), M_{3,2}(\tau_{s-1}, \tau_s), M_{3,3}(\tau_{s-1}, \tau_s)},
\]

where using the fact that the variance of \( dB_{0,\#}(\cdot) \) is \( I_{\#} \) for all \( l \),

\[
V_{M_{3,1}(\tau_{s-1}, \tau_s)} = \sum_{l=0}^{\infty} \left( (S_l S_\#) \otimes (S F_l^\prime) \right) \left( V_0 \otimes \mu_s, \mu_s^\prime \right) \left( (S_l S_\#) \otimes (S F_l^\prime) \right)^\prime
\]

\[
V_0 = \int_{\tau_{l-1}}^{\tau_s} D_\#(\tau) \, dB_{0,\#}(\tau) \, d\tau
\]

\[
V_{M_{3,2}(\tau_{s-1}, \tau_s)} = \sum_{l=0}^{\infty} \left( (S_l S_\#) \otimes (S F_l^\prime A^{-1}_{s,\#} S_\#) \right) V_{l,\#} \left( (S_l S_\#) \otimes (S F_l^\prime A^{-1}_{s,\#} S_\#) \right)^\prime
\]

\[
V_{l,\#} = \int_{\tau_{l-1}}^{\tau_s} (D_\#(\tau) \otimes D_\#(\tau)) \, dB_{l+1,\#}(\tau) \, d\tau
\]

\[
V_{M_{3,1}(\tau_{s-1}, \tau_s), M_{3,2}(\tau_{s-1}, \tau_s)} = \sum_{l=0}^{\infty} \left( (S_l S_\#) \otimes (S F_l^\prime A^{-1}_{s,\#} S_\#) \right) V_{l} \left( I_{l,\#} \otimes \mu_s^\prime \right) \left( (S_l S_\#) \otimes (S F_l^\prime) \right)^\prime
\]

\[
V_{l} = \int_{\tau_{l-1}}^{\tau_s} (D_\#(\tau) \otimes D_\#(\tau)) \rho^{perm}_{l-1,\#} (D_\#(\tau)^\prime \otimes 1) \, d\tau
\]

where \( \rho^{perm}_{l,\#} = P_l \rho_{l,\#} P_{l-1} \), with \( P_l, P_{l-1} \) the previously defined permutations, and the only non-zero block of \( \rho_{l,\#} \) is given by the upper-left block \( \rho_{l,\#} \) in Assumption 9(v); \( \rho^{perm}_{l,\#} \) is a permutation of \( \rho_{l,\#} \) which ensures that \( \rho^{perm}_{l,\#} = E(dB_{l,\#}(\cdot) \, dB_{0,\#}(\cdot)) = E((I_{l,\#} \otimes I_{l-1,\#}) \, dB_{l,\#}(\cdot)) \); and the only non-zero block of \( \rho_{l,\#} \) is given by upper-left block \( \rho_{l} \) in Assumption 9(iv).
Consider $\mathcal{E}_2$. Substituting into the expression for $\mathcal{E}_2$ the expression $\xi_t = \mu_t + F_t \xi_{t-1}$, we have:

$$\mathcal{E}_2 = \text{vec}(\mathcal{E}_2) = ((S_t' \mathbf{A}_{s,#}) \otimes S_r) T^{-1/2} \sum_{t \in I} (\eta_t \otimes \xi_t) + ((S_t' \mathbf{A}_{s,#}) \otimes S_r) T^{-1/2} \sum_{t \in I} (\eta_t \otimes \mu_t) + \text{vec}(\mathcal{E}_2)$$

$$\mathcal{E}_2 = \mathcal{E}_{2,1} + \mathcal{E}_{2,2} + \mathcal{E}_{2,3}.$$

Consider $\mathcal{E}_{2,1}$ and $\mathcal{E}_{2,2}$ first. Using similar arguments as before,

$$\mathcal{E}_{2,1} = \sum_{t=0}^{\infty} (S_t' \mathbf{S}_{#}) (S_r F_t^s) \left( \left[ \int_{\tau_{t-1}}^{T} D_{#} (\tau) d B_{t+1, #} (\tau) \right] \otimes \mu_t \right) = \mathcal{M}_{2,1}(\tau_{t-1}, \tau_t)$$

$$\mathcal{E}_{2,2} = \sum_{t=0}^{\infty} (S_t' \mathbf{S}_{#}) (S_r F_t^{s+1} A_{s,#}^{-1} S_#) \left( \left[ \int_{\tau_{t-1}}^{T} (D_{#} (\tau) \otimes D_{#} (\tau)) d B_{t+1, #} (\tau) \right] \right) = \mathcal{M}_{2,2}(\tau_{t-1}, \tau_t)$$

with variances derived similarly to $V_{\mathcal{M}_{2,1}(\tau_{t-1}, \tau_t)}$ and $V_{\mathcal{M}_{2,2}(\tau_{t-1}, \tau_t)}$ above:

$$V_{\mathcal{M}_{2,1}(\tau_{t-1}, \tau_t)} = \sum_{t=0}^{\infty} ((S_t' \mathbf{S}_{#}) (S_r F_t^s))' V_0 ((S_t' \mathbf{S}_{#}) (S_r F_t^s))'$$

$$V_{\mathcal{M}_{2,2}(\tau_{t-1}, \tau_t)} = \sum_{t=0}^{\infty} ((S_t' \mathbf{S}_{#}) (S_r F_t^{s+1} A_{s,#}^{-1} S_#))' V_{1,1} ((S_t' \mathbf{S}_{#}) (S_r F_t^{s+1} A_{s,#}^{-1} S_#))'$$

and covariance derived similarly to $V_{\mathcal{M}_{2,1}(\tau_{t-1}, \tau_t), \mathcal{M}_{2,2}(\tau_{t-1}, \tau_t)}$ above:

$$V_{\mathcal{M}_{2,1}(\tau_{t-1}, \tau_t), \mathcal{M}_{2,2}(\tau_{t-1}, \tau_t)} = \sum_{t=0}^{\infty} ((S_t' \mathbf{S}_{#}) (S_r F_t^{s+1} A_{s,#}^{-1} S_#))' V_{1} ((S_t' \mathbf{S}_{#}) (S_r F_t^{s+1} A_{s,#}^{-1} S_#))'$$

Consider next $\mathcal{E}_{2,3}$, where $\mathcal{E}_{2,3} = ((S_t' \mathbf{A}_{s,#}) \otimes S_r) T^{-1/2} \sum_{t \in I} (\eta_t \otimes \eta_t) = T^{-1/2} \sum_{t \in I} S_r \eta_t g_t^t S_r$. Note that $\mathbf{A}_s$ given in (5) is upper triangular with ones on the main diagonal. Hence, $\mathbf{A}_s^{-1}$ is also upper triangular and has ones on the main diagonal. Denote by $a_{1,}$, the first row of $\mathbf{A}_s^{-1}$, by $a_{p_1,}$ the subsequent $p_1$ rows, and by $a_{p_2,} = [0_{p_2} \ 0_{p_2 \times p_1} \ I_{p_2}]$ the subsequent $p_2$ rows. Then:

$$S_r \eta_t = S_r A_{s,#}^{-1} g_t = S_r \begin{bmatrix} a_{1,} \ 
\epsilon_t \\
A_{p_1,} \ 
\epsilon_t \\
A_{p_2,} \ 
\epsilon_t \\
0_{n(p-1)} \end{bmatrix} = S_r \begin{bmatrix} a_{1,} \ 
\epsilon_t \\
A_{p_1,} \ 
\epsilon_t \\
\zeta_t \\
0_{n(p-1)} \end{bmatrix} = \zeta_t,$$

and $g_t^t S_r$ is equal to either $u_t$ or $v_t^0 \beta^0$. Let $\mathcal{E}_{2,3} = \mathcal{E}_{2,3}^{(1)} = T^{-1/2} \sum_{t \in I} \zeta_t u_t$ (when $g_t^t S_r = u_t$) and let $\mathcal{E}_{2,3} = \mathcal{E}_{2,3}^{(2)} = T^{-1/2} \sum_{t \in I} \zeta_t v_t^0 \beta^0$ (when $g_t^t S_r = v_t^0 \beta^0$).

Consider first $\mathcal{E}_{2,3}^{(1)}$. To that end, by Assumption 9 we have:

$$D_t = \begin{bmatrix} d_{u,t} & 0'_{p_1} & 0'_{p_2} \\
0_{p_1} & D_{v,t} & 0_{p_1 \times p_2} \\
0_{p_2} & 0'_{p_1 \times p_2} & D_{\zeta,t} \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0'_{p_1} & 0'_{p_2} \\
s_{p_1} & s_{p_1} & 0_{p_1 \times p_2} \\
0_{p_2} & 0'_{p_1 \times p_2} & s_{p_2} \end{bmatrix}, \quad l_t = \begin{bmatrix} l_{u,t} \\
l_{v,t} \\
l_{\zeta,t} \end{bmatrix},$$

where $D_t$ is partitioned exactly as $D(\tau)$ in the notation preceding the statement of Lemma 6, $S$ is partitioned.
exactly the same way in the notation section before Lemma 6, \( l_{u,t} \) is a scalar and \( l_{v,t} \) is a \( p_1 \times 1 \). Therefore,

\[
\begin{bmatrix}
  u_t \\
  v_t \\
  \zeta
\end{bmatrix} = \epsilon_t = SD_l l_t = \begin{bmatrix}
  d_{u,t} & l_{u,t} \\
  s_{p_1} & d_{u,t} + s_{p_1} D_{v,t} l_{v,t} \\
  S_{p_2} & D_{\zeta,t} l_{\zeta,t}
\end{bmatrix}.
\]

(1.29)

Let \( B_{u\zeta,T}(r) = T^{-1/2} \sum_{t=1}^{[Tr]} l_{u,t} l_{\zeta,t} \). By Lemma 5, \( B_{u\zeta,T}(\cdot) \Rightarrow B_{u\zeta}(\cdot) \). By Assumption 9, \( S, D \) are bounded, so by Theorem 2.1 in Hansen (1992), we have:

\[
\mathcal{E}_{2,3}^{(1)} = T^{-1/2} \sum_{t \in I_r} u_t \zeta_t - \sum_{t \in I_r} S_{p_2} (d_{u,t} D_{\zeta,t}) (l_{u,t} l_{\zeta,t}) \Rightarrow S_{p_2} \int_{\tau_{r-1}}^{\tau_r} (d_u(\tau) D_{\zeta}(\tau)) dB_{u\zeta}(\tau)
\]

(1.30)

with variance

\[
V_{M_2, 3, (\tau_{s-1}, \tau_s)}^{(1)} = S_{p_2} \int_{\tau_{r-1}}^{\tau_r} d^2(\tau) D_{\zeta}(\tau) \rho_{u, \zeta, 0, 0} D_{\zeta}(\tau) S_{p_2} d\tau
\]

(1.31)

Consider \( \mathcal{E}_{2,3}^{(2)} \). Similarly to \( \mathcal{E}_{2,3}^{(1)} \), \( B_{e\zeta,T}(r) = T^{-1/2} \sum_{t \in I_r} l_{v,t} \zeta_t \Rightarrow B_{e\zeta}(r) \), where recall that \( B_{e\zeta}(\cdot) \) is a \( p_2 \times 1 \) vector of Brownian motions with variance \( \rho_{e, \zeta, 0, 0} = \mathbb{E} (l_{v,t} l_{v,t}') \) which is just the \((p_1 p_2) \times (p_1 p_2)\) lower-right block of \( \rho_{e, 0, 0} \). Also, by Lemma 5, \( B_{e\zeta,T}(\cdot) = \text{vect}(B_{u\zeta,T}(\cdot), B_{e\zeta,T}(\cdot)) \) also jointly converge to \( B_{\zeta}(\cdot) \), a process defined just before Lemma 5. Therefore,

\[
\mathcal{E}_{2,3}^{(2)} = \beta_{y_{x,s}}'(\cdot) T^{-1/2} \sum_{t \in I_r} (v_t \otimes \zeta_t)
\]

\[
= (\beta_{y_{x,s}}'(\cdot) \otimes 1) (T^{-1/2} \sum_{t \in I_r} (s_{p_1} d_{u,t} l_{u,t}) \otimes (S_{p_2} D_{\zeta,t} l_{\zeta,t}))
\]

\[
+ (\beta_{y_{x,s}}'(\cdot) \otimes 1) (T^{-1/2} \sum_{t \in I_r} (S_{p_2} D_{v,t} l_{v,t}) \otimes (S_{p_2} D_{\zeta,t} l_{\zeta,t}))
\]

\[
\Rightarrow ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2}) \int_{\tau_{r-1}}^{\tau_r} d_u(\tau) D_{\zeta}(\tau) dB_{u\zeta}(\tau) + ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2}) \int_{\tau_{r-1}}^{\tau_r} D_{v}(\tau) D_{\zeta}(\tau) dB_{e\zeta}(\tau))
\]

(1.32)

with variance

\[
V_{M_2, 3, (\tau_{s-1}, \tau_s)}^{(2)} = V^{(1)} + V^{(2)} + V^{(3)} + (V^{(3)})'
\]

(1.33)

\[
V^{(1)} = ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2}) \int_{\tau_{r-1}}^{\tau_r} d_u(\tau) D_{\zeta}(\tau) \rho_{u, \zeta, 0, 0} (d_u(\tau) D_{\zeta}(\tau))' d\tau ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2})'\rho_{u, \zeta, 0, 0} (d_u(\tau) D_{\zeta}(\tau))'
\]

\[
V^{(2)} = ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2}) \int_{\tau_{r-1}}^{\tau_r} D_{v}(\tau) D_{\zeta}(\tau) \rho_{u, \zeta, 0, 0} (D_{v}(\tau) D_{\zeta}(\tau))' d\tau ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2})'\rho_{u, \zeta, 0, 0} (D_{v}(\tau) D_{\zeta}(\tau))'
\]

\[
V^{(3)} = ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2}) \int_{\tau_{r-1}}^{\tau_r} (d_u(\tau) D_{\zeta}(\tau)) \rho_{u, v, \zeta, 0, 0} (D_{v}(\tau) D_{\zeta}(\tau))' d\tau ((\beta_{y_{x,s}}'(\cdot) \otimes S_{p_2})'\rho_{u, v, \zeta, 0, 0} (D_{v}(\tau) D_{\zeta}(\tau))'
\]

where \( \rho_{u, v, \zeta, 0, 0} \) is the upper-right block of \( \rho_{x, 0, 0} \).

In conclusion, we have:

\[
\mathcal{E}_2 \Rightarrow M_{2,1}(\tau_{s-1}, \tau_s) + M_{2,2}(\tau_{s-1}, \tau_s) + M_{2,3}(\tau_{s-1}, \tau_s) = M_{2}(\tau_{s-1}, \tau_s),
\]

where \( M_{2,2}(\tau_{s-1}, \tau_s) = M_{2,2}^{(1)}(\tau_{s-1}, \tau_s) \) when \( g_i^\prime S_1 = u_t \) and \( M_{2,2}(\tau_{s-1}, \tau_s) = M_{2,2}^{(2)}(\tau_{s-1}, \tau_s) \) when \( g_i^\prime S_1 = v_t \beta_{y_{x,s}}'(\cdot) \) with asymptotic variance \( V_{M_2, 2, (\tau_{s-1}, \tau_s)} \) which can be derived by similar arguments as those used for \( V_{M_1, \tau_{s-1}, \tau_s} \).
and \( V_{M_3(T-1,T_s)} \). We have:

\[
V_{M_2(\tau_{s-1},\tau_s)} = V_{M_2,1(\tau_{s-1},\tau_s)} + V_{M_2,2(\tau_{s-1},\tau_s)} + \mathcal{M}_{M_2,1}(\tau_{s-1},\tau_s) + V_{M_2,1(\tau_{s-1},\tau_s), M_2,2(\tau_{s-1},\tau_s)}
\]

(1.34)

\[
+ V'_{M_2,1(\tau_{s-1},\tau_s), M_2,2(\tau_{s-1},\tau_s)} + V_{M_2,1(\tau_{s-1},\tau_s), M_2,2(\tau_{s-1},\tau_s), M_2,3(\tau_{s-1},\tau_s)} + V'_{M_2,1(\tau_{s-1},\tau_s), M_2,3(\tau_{s-1},\tau_s)}
\]

(1.35)

\[
+ V'_{M_2,2(\tau_{s-1},\tau_s), M_2,3(\tau_{s-1},\tau_s)} + V'_{M_2,2(\tau_{s-1},\tau_s), M_2,3(\tau_{s-1},\tau_s), M_2,4(\tau_{s-1},\tau_s)}
\]

(1.36)

where the terms in (1.34) are given in (1.26), (1.27), (1.31) and (1.28) respectively, and \( V_{M_2,1(\tau_{s-1},\tau_s), M_2,3(\tau_{s-1},\tau_s)} \) and \( V_{M_2,2(\tau_{s-1},\tau_s), M_2,3(\tau_{s-1},\tau_s)} \) can be obtained similarly using \( \rho_{\xi,0} \) (the upper block of \( \rho_{\xi,0} \) derived before the proof of Lemma 6) and \( \rho_{\xi,0} \) (the lower block of \( \rho_{\xi,0} \)) respectively for \( i = 1, 2 \).

Now note that \( E_1, E_2, E_3 \) are functions of the same underlying Brownian motions which were shown to jointly converge, therefore they also jointly converge:

\[
Z_T = \left[ \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \right] \Rightarrow \left[ \begin{array}{c} M_1(\tau_{s-1}, \tau_s) \\ M_2(\tau_{s-1}, \tau_s) \\ M_3(\tau_{s-1}, \tau_s) \end{array} \right] \equiv M(\tau_{s-1}, \tau_s).
\]

This completes the proof for \( I_i = \tilde{I}_i = [\lfloor \tau_{s-1}T \rfloor + 1, \lfloor \tau_sT \rfloor] \). Now consider the case of \( I_i \) containing \( N_i \) breaks from the total set of \( N \) breaks, that is, there is an \( s \) such that \( \tau_{s-1} < \lambda_{i-1} \leq \tau_s \) and \( \tau_{s+N_i-1} \leq \lambda_i < \tau_{s+N_i} \). Then, generalizing the previous results which were for \( I_i = \tilde{I}_i = [\lfloor \tau_{s-1}T \rfloor + 1, \lfloor \tau_sT \rfloor] \),

\[
Z_T \Rightarrow \left\{ \begin{array}{ll}
M(\lambda_{i-1}, \tau_s) + \sum_{j=1}^{N_i} M(\tau_{s+j-1}, \tau_{s+j}) + M(\tau_{s+N_i}, \lambda_i) & \text{if } N_i \geq 2 \\
M(\lambda_{i-1}, \tau_s) + M(\tau_s, \lambda_i) & \text{if } N_i = 1 \\
M(\lambda_{i-1}, \lambda_i) & \text{if } N_i = 0.
\end{array} \right.
\]

Proof of Lemma 7.

Because of Assumptions 4 of fixed breaks, this lemma fits the setting of Hall, Han, and Boldea (2012) and Bai and Perron (1998). If the reduced form is stable, then \( h = 0 \), and Lemma 7(ii) follows from Lemma 2 and Lemma 6. Lemma 7(iii) follows from Hall, Han, and Boldea (2012) Theorems 1-2, where their Assumptions 6-11 are automatically satisfied by our assumptions and Lemma 2 as follows: their Assumption 6 is our Assumption 1, their Assumption 7 is our Assumption 6, their Assumption 8 is automatically satisfied by our Assumption 9, their Assumption 9 by our Assumption 8, and their Assumptions 10-11 hold by Lemma 2 and Assumption 7. If \( h > 0 \), Lemma 7(i) is the same as Hall, Han, and Boldea (2012) Assumption 19(i), and is a special case of Proposition 1 in Bai and Perron (1998), where the Bai and Perron (1998) Assumptions 1-3 hold by Lemma 2 and our Assumption 7, the Bai and Perron (1998) Assumption 4 encompasses Assumption 7 as a special case, and the Bai and Perron (1998) Assumption 5 is exactly our Assumption 3. Lemma 7(ii) follows from Lemma 2 and 6. Lemma 7(iii) follows from Hall, Han, and Boldea (2012) Theorem 8(i)-(ii). In particular, their Assumption 6 is the same as our Assumption 1, their Assumption 8 is automatically satisfied by our Assumption 9, their Assumptions 10-11 hold by Lemma 2, their Assumption 17 is our Assumption 3, their Assumption 18 is our Assumption 6, and their Assumption 19(ii) holds by our Assumption 8.
Proof of Lemma 8.
Let $\phi_t$ be the $(a, b)$ element of $\epsilon_t \epsilon_t', (\epsilon_t \epsilon_t') \otimes \epsilon_{t-1}$, or $(\epsilon_t \epsilon_t') \otimes (\epsilon_{t-1} \epsilon_{t-j})$, for $i, j \geq 0$.

$$T^{-1} \sum_{t=1}^{[Tr]} \phi_t = T^{-1} \sum_{t=1}^{[Tr]} (\phi_t - E(\phi_t|F_{t-1})) + T^{-1} \sum_{t=1}^{[Tr]} (E(\phi_t|F_{t-1}) - E(\phi_t)).$$

Note that $(\phi_t - E(\phi_t|F_{t-1}))$ is a m.d.s. so it is a $L^1$-mixingale with uniformly bounded constants. Moreover, by similar arguments as in the proof of Lemma 4, for $b = 1 + \delta/4 > 1$ (for example), $\sup E|\phi_t|^b < \infty$. Therefore, by Lemma 1, $T^{-1} \sum_{t=1}^{[Tr]} (\phi_t - E(\phi_t|F_{t-1})) \overset{p}{\to} 0$ uniformly in $r$. By similar arguments, $T^{-1} \sum_{t=1}^{[Tr]} (E(\phi_t|F_{t-1}) - E(\phi_t)) \overset{p}{\to} 0$ uniformly in $r$, completing the proof. \hfill \Box

**Theorem B 1.** Under Assumption 1-8 and the null hypothesis $k = 0$,

$$Wald_T \lambda_k = T \beta_{\lambda_k}^* R_k^t \left(R_k \hat{\lambda}_k R_k^t\right)^{-1} R_k \beta_{\lambda_k} \Rightarrow N(\lambda_k),$$

where $N(\lambda_k) = [\text{vector} \{\lambda_{i_k} = 1 + (Q_{i_k}^{-1} N_{i_k})\}] R_k^t \left(R_k \text{vector} \{\lambda_{i_k} = 1 + (Q_{i_k}^{-1} N_{i_k})\}, \text{vector} \{Q_{i_k}, V_{i_k}, N_{i_k}\} \right)$ are defined in equations (1) in the paper, and (1.37) and (1.39) in this Appendix.

**Proof of Theorem B 1.** Recall that $\hat{\Delta}_t = \Delta_{(j)}$ if $t \in \{[x_t^{-1} T] + 1, [\tilde{x}_t^{-1} T] \}$ for $j = 1, \ldots, h + 1$, and $\hat{Y}_t = (\Delta_t, \Pi)$. For notation ease, set $I_t = I_{i, \lambda_k}$, and recall that $Wald_T \lambda_k = T \beta_{\lambda_k}^* R_k^t \left(R_k \hat{\lambda}_k R_k^t\right)^{-1} R_k \beta_{\lambda_k}$, where

$$\hat{\lambda}_k = \text{vector} \{I_{i, \lambda_k} = 1 + (V_{i, \lambda_k}^{-1} \beta_{\lambda_k}^*)\} R_k^t \left(R_k \text{vector} \{I_{i, \lambda_k} = 1 + (V_{i, \lambda_k}^{-1} \beta_{\lambda_k}^*)\}, \text{vector} \{\hat{V}_{i, \lambda_k}, \hat{\lambda}_k^2\} \right).$$

By Lemma 2, $\hat{Q}_{(i)} \overset{p}{\to} Q_{(i)}$. Letting $\tilde{M}(\tau) = M(\tau)|_{S_i = S_{i+k}} + M(\tau)|_{S_i = \beta_{0,i+k}}$, and $M(\tau_{i-1}, \tau_i) = \int_{\tau_{i-1}}^{\tau_i} d\tilde{M}(\tau)$, we have:

$$\lim_{T \to \infty} \text{Var} \left(T^{-1/2} \sum_{t \in I_t} \hat{\lambda}_k \hat{Y}_t z_t (u_t + \beta_{0,i,k}^z) \right) = \int_{\lambda_{i-1}}^{\lambda_i} \mathcal{Y}^t(\tau) \text{Var}(d\tilde{M}(\tau)) \mathcal{Y}^t(\tau) = \mathbb{V}_t. \tag{1.37}$$

Now consider $\hat{\beta}_{\lambda_k} = \text{vector}(\hat{\beta}_{i, \lambda_k})$. Under the null hypothesis, $\beta_{0,i,k}^z = \beta_0^z$, and so therefore we write:

$$\hat{u}_t = y_t - \hat{w}_t \beta_0^z = y_t + (x_t - \hat{\delta}_t) \beta_0^z = u_t + \beta_0^z - z_t^\prime \left(\Delta_t - \Delta_0^z\right) \beta_0^z. \tag{1.38}$$

where, by Lemma 7,

$$T^{1/2} (\Delta_t - \Delta_0^z) = \sum_{j=1}^{h+1} 1_t \in I_t \left\{T^{-1} \sum_{t \in I_t} z_t^\prime \right\}^{-1} T^{-1/2} \sum_{t \in I_t} z_t^\prime u_t + o_p(1),$$

where $I_t = \{[x_t^{-1} T] + 1, [\tilde{x}_t^{-1} T] + 2, \ldots, [x_0 T]\}$. From the definition of $\hat{\beta}_{i, \lambda_k}$, it follows that under $H_0$, we have:

$$T^{1/2} (\hat{\beta}_{i, \lambda_k} - \beta_0^z) = \left(T^{-1} \sum_{t \in I_t} \hat{u}_t \hat{w}_t^\prime \right)^{-1} T^{-1/2} \sum_{t \in I_t} \hat{u}_t \hat{u}_t^\prime$$

$$= \left(T^{-1} \sum_{t \in I_t} \hat{u}_t \hat{w}_t^\prime \right)^{-1} \left(T^{-1/2} \sum_{t \in I_t} \hat{w}_t (u_t + \beta_0^z)\right)$$

$$- T^{-1} \sum_{t \in I_t} \hat{w}_t z_t^\prime \left\{\sum_{j=1}^{h+1} 1_t \in I_t \left\{T^{-1} \sum_{t \in I_t} z_t^\prime \right\}^{-1} T^{-1/2} \sum_{t \in I_t} z_t^\prime u_t + o_p(1)\right\}$$

$$= \left(T^{-1} \sum_{t \in I_t} \hat{Y}_t^t z_t^\prime \hat{Y}_t^\prime \right)^{-1} \left(T^{-1/2} \sum_{t \in I_t} \hat{Y}_t^t z_t (u_t + \beta_0^z)\right)$$

$$- T^{-1} \sum_{t \in I_t} \hat{Y}_t^t z_t^\prime \left\{\sum_{j=1}^{h+1} 1_t \in I_t \left\{T^{-1} \sum_{t \in I_t} z_t^\prime \right\}^{-1} T^{-1/2} \sum_{t \in I_t} z_t^\prime u_t + o_p(1)\right\}.$$
where \( I_i = \{ \lfloor \lambda_i-1 \rfloor + 1, \lfloor \lambda_i-1 \rfloor + 2, \ldots, \lfloor \lambda_iT \rfloor \} \). Therefore, letting \( I_i \) contain \( N_i \) true breaks of the VAR representation in (4), letting \( s_0 = \lambda_{i-1}, s_{N_i+1} = \lambda_i \), as well as denoting \( Q_{x,j} = \int_{\pi_j}^{\pi_j} Q_x \),

\[
T^{1/2}(\hat{\beta}_1 - \beta^0) \Rightarrow Q_i^{-1} T^{1/2} \sum_{j=1}^{b+1} 1_{i \in I_j} Q_{x,j} \tilde{M}_j{T_{\pi_j}} E_{x,x} \]

where \( \tilde{M}_j \) is defined as \( \tilde{M}_i \), but with \( \pi_0 \) replacing \( \lambda_{i-1} \) and \( \pi_j \) replacing \( \lambda_i \).

**Theorem B.2.** Under Assumptions 1-8 and the null hypothesis \( H_0 : m = \ell \),

\[
\sup \text{Wald}_{T}(\ell + 1 | \ell) = \max_{i=1, \ldots, \ell + 1} \left\{ \sup_{\pi \in \mathcal{N}(\lambda_i)} W_{i,1}(\pi_i) \{ W_{i,2}(\pi_i) \}^{-1} W_{i,1}(\pi_i) \right\},
\]

where \( W_{i,1}(\pi_i) \) and \( W_{i,2}(\pi_i) \) are defined in (1.43)-(1.44).

**Proof of Theorem B.2.** We begin by deriving an alternative representation of the sup-Wald_{T}(\ell + 1 | \ell). Define \( \tilde{E}_i(\pi) \) to be the \((\hat{T}_i - \hat{T}_{i-1}) \times d_{\beta} \) matrix with \( \beta^0 \)th row given by

\[
\{ \tilde{E}_i(\pi) \}_{\ell} = \tilde{w}_i', \quad \text{for} \quad t = \hat{T}_{i-1} + \ell, \quad \ell = 1, 2, \ldots, [\pi T],
\]

\[
= 0_{d_{\beta}}, \quad \text{for} \quad \ell = [\pi T] + 1, [\pi T] + 2, \ldots, \hat{T}_i.
\]

Then we can re-parameterize the model in (11) as

\[
y_i = \tilde{W}_i \gamma + \tilde{E}_i(\pi_i) \alpha + \text{error} \tag{1.40}
\]

where \( y_i \) is the \((\hat{T}_i - \hat{T}_{i-1}) \times 1 \) vector with \( \beta^0 \) element \( y_{\hat{T}_{i-1}+\ell} \) and \( \tilde{W}_i = \text{vect}_{\hat{T}_{i-1}+1: \hat{T}_i}(\tilde{w}_i) \). If \( \hat{\alpha}(\pi_i) \) denotes the OLS estimator of \( \alpha \) based on (1.40) then it follows by straightforward arguments that \( R_i \tilde{\beta}(\pi_i) = \hat{\alpha}(\pi_i) \).

Using the Frisch-Waugh theorem, we have

\[
R_i \tilde{\beta}(\pi_i) = \hat{\alpha}(\pi_i) = \left\{ \tilde{E}_i(\pi)' M_{\tilde{W}_i} \tilde{E}_i(\pi) \right\}^{-1} \tilde{E}_i(\pi)' M_{\tilde{W}_i} y_i,
\]

where \( M_{\tilde{W}_i} = I_{\hat{T}_i - \hat{T}_{i-1}} - W_i \left( W_i' W_i \right)^{-1} W_i' \).

Let \( E_i(\pi) \) and \( W_i \) be defined analogously to \( \tilde{E}_i(\pi) \) and \( \tilde{W}_i \) only replacing \( \hat{T}_{i-1} \) and \( \hat{T}_i \) by \( T_{0,i} \) and \( T_i \) respectively. Further define \( \tilde{u}_i \) to be the \((T_i - T_{0,i}) \times 1 \) vector with \( \beta^0 \) element \( \hat{u}_{T_{0,i}+\ell} \), where

\[
\tilde{u}_i \equiv y_t - \tilde{w}_i' \beta^0_{(i)} = u_t + (x_t - \hat{x}_t)' \beta^0_{(i)} = u_t + v_t' \beta^0_{(i)} - z_t' \left( \Delta T_i - \Delta T_{0,i} \right) \beta^0_{(i)}.
\]

Under \( H_0 \), from Lemma 7, it follows that \( \hat{T}_i - T_{0,i} = O_p(1) \) for \( i = 1, 2, \ldots, \ell \), so we can treat \( T_{0,i} \) as known for the rest of this proof. Additionally, \( \beta^0_{(i)} \) is constant in interval \([T_{0,i} + 1, T_i] \), so \( R_i \text{vect}(\beta^0_{(i)}, \beta^0_{(i)}) = 0_{2d_{\beta}}, \)

therefore:

\[
T^{1/2} R_i \tilde{\beta}(\pi_i) = T^{1/2} \left\{ E_i(\pi)' M_{\tilde{W}_i} E_i(\pi) \right\}^{-1} E_i(\pi)' M_{\tilde{W}_i} \tilde{u}_i + o_p(1),
\]

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uniformly in \( \varpi \). Therefore,

\[
\sup_{i = 1, 2, \ldots, \ell - 1} \text{Wald}_T(\ell + 1 | \ell) = \max_{i = 1, 2, \ldots, \ell + 1} \left\{ \sup_{\varpi_i \in \mathcal{N}(\lambda_i)} \text{Wald}_T(\varpi_i; \ell) \right\} + o_p(1) \tag{1.41}
\]

where

\[
\text{Wald}_T(\varpi_i; \ell) = T \tilde{u}_i' \text{MW}_i E_i(\varpi) \left\{ E_i(\varpi)' M \text{W}_i E_i(\varpi) \right\}^{-1} \left( R_i \tilde{V}(\varpi_i) R_i \right)^{-1} \times \left\{ E_i(\varpi)' M \text{W}_i E_i(\varpi) \right\}^{-1} E_i(\varpi)' M \text{W}_i \tilde{u}_i, \tag{1.42}
\]

where

\[
\tilde{V}(\varpi_i) = \text{diag} \left( \tilde{V}_1(\varpi_i), \tilde{V}_2(\varpi_i) \right), \quad \tilde{V}_i(\varpi_i) = \{ \tilde{Q}_j(\varpi_i) \}_{j = 1}^{p} \{ \tilde{Q}_j(\varpi_i) \}_{j = 1}^{p}, \quad \tilde{M}_j(\varpi_i) = \lim_{T \to \infty} \text{Var} \left( \frac{T^{-1/2} \sum_{t \in I_i(\varpi)} \tilde{\tau}_t \tilde{M}_j \left( \sum_{t \in I_i(\varpi)} \tilde{\tau}_t \right) \left( u_t + \tilde{v}_t^{0} \beta_{\pi(i)}^{0} \right) \right), \]

and, \( I_i^{(1)}(\varpi_i) = \{ t : [\lambda_{i-1}^0, \lambda_i^0] + 1, [\lambda_{i-1}^0, \lambda_i^0] + 2, \ldots, [\varpi_i, T] \} \) and \( I_i^{(2)}(\varpi_i) = \{ t : [\varpi_i, T] + 1, [\varpi_i, T] + 2, \ldots, [\lambda_i^0, T] \} \).

To derive the limit of the \( \text{Wald}_T(\varpi_i; \ell) \), note that by Lemma 7 and \( \tilde{Y}_i \overset{P}{\to} \tilde{Y}_i^0 \), \( T^{-1/2} \sum_{t = T_{i-1}^0}^{T_i} \tilde{w}_t \tilde{u}_i \Rightarrow \mathcal{N}(\lambda_{i-1}^0, \varpi_i) \), where the definition of \( \mathcal{N}(\cdot, \cdot) \) for this entire section is as in equation (1.39) but with \( \beta_{\pi(i)}^{0} \) replaced by \( \beta_{\pi(i)}^{0} \) when \( \mathcal{N}(\cdot, \cdot) \) is evaluated in (a subset of) the interval \( [\lambda_{i-1}^0, \lambda_i^0] \). Similarly, \( T^{-1/2} \sum_{t = T_{i-1}^0}^{T_i} \tilde{w}_t \tilde{u}_i \Rightarrow \mathcal{N}(\lambda_{i-1}^0, \lambda_i^0) \). Similarly to (1.37), one can derive \( \mathcal{V}_{i, j}(\varpi) = \lim_{T \to \infty} \text{Var} \left( \frac{T^{-1/2} \sum_{t \in I_i(\varpi)} \tilde{\tau}_t \tilde{M}_j \left( \sum_{t \in I_i(\varpi)} \tilde{\tau}_t \right) \left( u_t + \tilde{v}_t^{0} \beta_{\pi(i)}^{0} \right) \right) \).

Using these results, Lemma 2, and letting \( Q(\tau) = Y(\tau) Q_2(\tau) Y(\tau) \), we have:

\[
T^{-1} \tilde{W}_i' \tilde{W}_i \overset{P}{\to} \int_{\lambda_{i-1}^0}^{\lambda_i^0} Q(\tau) d\tau \equiv \mathcal{Q}_i^0
\]

\[
T^{-1} E_i(\varpi_i)' \tilde{W}_i = \tilde{Q}_1(\varpi_i) \overset{P}{\to} \int_{\lambda_{i-1}^0}^{\lambda_i^0 \varpi_i} Q(\tau) d\tau \equiv \mathcal{Q}_{i, 1}(\varpi_i),
\]

\[
\tilde{Q}_2(\varpi_i) \overset{P}{\to} \int_{\lambda_{i-1}^0 \varpi_i}^{\lambda_i^0} Q(\tau) d\tau \equiv \mathcal{Q}_{i, 2}(\varpi_i),
\]

\[
\tilde{M}_1(\varpi_i) \overset{P}{\to} \mathcal{V}_{i, 1}(\varpi_i),
\]

\[
\tilde{M}_2(\varpi_i) \overset{P}{\to} \mathcal{V}_{i, 2}(\varpi_i),
\]

\[
T^{-1/2} E_i(\varpi_i)' \tilde{u}_i \Rightarrow \mathcal{N}(\lambda_{i-1}^0, \varpi_i),
\]

\[
T^{-1/2} \tilde{W}_i' \tilde{u}_i \Rightarrow \mathcal{N}(\lambda_{i-1}^0, \lambda_i^0).
\]

It then follows by the continuous mapping theorem (CMT) that:

\[
\text{Wald}_T(\varpi_i; \ell) \Rightarrow \mathcal{W}_{i, 1}(\varpi_i) \{ \mathcal{W}_{i, 2}(\varpi_i) \}^{-1} \mathcal{W}_{i, 1}(\varpi_i)
\]

where

\[
\mathcal{W}_{i, 1} = \left( \mathcal{Q}_{i, 1}(\varpi_i) \{ \mathcal{Q}_i^0 \}^{-1} \mathcal{Q}_{i, 1}(\varpi_i) \right)^{-1} \left( \mathcal{Q}_{i, 1}(\varpi_i) \{ \mathcal{Q}_i^0 \}^{-1} \mathcal{N}(\lambda_{i-1}^0, \lambda_i^0) \right), \tag{1.43}
\]

\[
\mathcal{W}_{i, 2}(\varpi_i) = R_i \text{diag}_{j = 1, 2} (\mathcal{V}_{i, j}(\varpi_i)) R_i'. \tag{1.44}
\]
Therefore, by the CMT, we have

\[ \sup \text{Wald}_T(\ell + 1 | \ell) = \max_{i=1,2,\ldots,\ell+1} \left\{ \sup_{\omega_t \in \mathcal{N}(\lambda_i)} \mathcal{W}_{i,1}(\omega_i)'(\mathcal{W}_{i,2}(\omega_i))^{-1}\mathcal{W}_{i,1}(\omega_i) \right\}. \]

\[ \square \]

2 Analysis of $C_2$, $B_6$ and $B_{1,2}$ defined in Section 1, Proof of Lemma 2

This section considers $C_2$, $B_6$ and $B_{1,2}$ in more detail, but it also shows that $\sum_{i=0}^{\infty} \|F_s^i\| < \infty$.

Let $\{\gamma_i; i = 1, 2, \ldots, np\}$ be the eigenvalues of $F_s$. We consider four cases relating to the roots of the characteristic equation of VAR in (1): (i) real and distinct; (ii) real and repeated; (iii) complex and distinct; (iv) complex and repeated. (The case of repeated complex and real roots is easily deduced from Case’s (i) and (iv) and so is omitted for brevity.)

Case (i): real and distinct roots

We have, for $\Omega_j = A_s^{-1} \Sigma_j (A_s^{-1})'$ and $\Sigma, \Sigma$ defined in the main paper, at the beginning of its appendix,

\[ \|C_2\| = T^{-1} \left\| \sum_{l=1}^{(\Delta \tau, T)-1} F_s^l \left\{ \sum_{j=\tau, T}\cdots -l+1 \Omega_j \right\} F_s^l' \right\|, \]

\[ \leq T^{-1} \sum_{l=1}^{(\Delta \tau, T)-1} \|F_s^l\| \left\| \sum_{j=\tau, T}\cdots -l+1 \Omega_j \right\| \|F_s^l'\|, \]

\[ \leq T^{-1} \sum_{l=1}^{(\Delta \tau, T)-1} \|F_s^l\|^2 \left\| \sum_{j=\tau, T}\cdots -l+1 \Omega_j \right\|, \]

Given the definition of $\Omega_j$, we have

\[ \|\Omega_j\| = \|A_s^{-1} \Sigma_j (A_s^{-1})'\| \leq \|A_s^{-1}\|^2 \sum_{j=\tau, T}\cdots -l+1 \Sigma_j \]

Under our assumptions, we have $\|A_s^{-1}\|^2 = O(1)$. Since $\sup \|\Sigma_t\| < C_\Sigma$ for some finite positive constant $C_\Sigma$, we have have for $l = 1, 2, \ldots, (\Delta \tau, T) - 1$:

\[ \left\| \sum_{j=\tau, T}\cdots -l+1 \Sigma_j \right\| \leq lC_\Sigma \]

Now consider $\|F_s^l\|$. From Hamilton (1994) (p.259) and Lütkepohl (1993) (p.460), it follows that under the assumptions about the roots,\(^{16}\)

\[ F_s = P^{-1} \Gamma P \]

where $\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{np})$ and $P$ is a real matrix with $P = O(1)$, and hence that

\[ F_s^l = P^{-1} \Gamma^l P. \]

\(^{16}\)The dependence of both $P$ and $\Gamma$ on $s$ is suppressed for ease of notation.
Therefore, we have:

\[ \| F_s^i \| \leq \| P^{-1} \| \| T^i \| \| P \| \leq \| P^{-1} \| \| P \| \sqrt{\mu} \gamma_{\text{max}, r^2} = \sqrt{\mu} \gamma_{\text{max}, r^2}, \quad (2.45) \]

where \( \gamma_{\text{max}, r^2} \) is the positive square root of the maximum eigenvalue of \( T^2 \). Therefore,

\[ \| \mathcal{C}_2 \| \leq T^{-1} \sum_{l=1}^{(\Delta \tau_r T)^{-1}} \sqrt{\mu} \gamma_{\text{max}, \gamma^2} \mathcal{C}_2 \]

Since \( \sum_{i=1}^{\infty} i \gamma^i = \gamma (1 - \gamma)^{-2} = O(1) \), it follows that \( \mathcal{C}_2 = o(1) \).

Now consider \( \mathcal{B}_6 \). We have

\[
\| \mathcal{B}_6 \| = T^{-1} \left\| \sum_{t \in I_t} \left( \sum_{i=0}^{i-1} F_s^i \mu_s \right) \xi_{[\tau_s - t]} F_s^i \right\| \\
\leq T^{-1} \sum_{t \in I_t} \sum_{i=0}^{i-1} \| F_s^i \mu_s \| \| \xi_{[\tau_s - t]} F_s^i \| \\
\leq T^{-1} \| \mu_s \| \| \xi_{[\tau_s - t]} \| \sum_{t \in I_t} \sum_{i=0}^{i-1} \| F_s^i \| \| F_s^i \| .
\]

Using (2.45), we have:

\[
\sum_{t \in I_t} \sum_{i=0}^{i-1} \| F_s^i \| \| F_s^i \| \leq K \sum_{t \in I_t} \sum_{i=0}^{i-1} \gamma^{t+i}
\]

where \( \gamma = \gamma_{\text{max}, \gamma^2} \) and \( K \) is a bounded positive constant. Expanding the sums and using \( 0 < \gamma < 1 \), it can be shown that \( \sum_{t \in I_t} \sum_{i=0}^{i-1} \gamma^{t+i} \leq \sum_{i=1}^{\infty} t \gamma^i = (1 - \gamma)^{-2} = O(1) \) and so

\[ \| \mathcal{B}_6 \| \leq T^{-1} K \| \mu_s \| \| \xi_{[\tau_s - t]} \| (1 - \gamma)^{-2} = o_p(1) \xrightarrow{P} 0. \]

Now consider \( \mathcal{B}_{1.2} \). Recall that

\[ B_{1.2} = T^{-1} \sum_{t \in I_t} \left( \sum_{i=0}^{i-1} F_s^i \mu_s \right) \left( \sum_{i=0}^{i-1} F_s^i \mu_s \right) ', \]

where \( \tilde{t} = t - [\tau_s - t] \).

\[
\sum_{i=0}^{i-1} F_s^i \mu_s = \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s - \sum_{t} F_s^i \mu_s
\]

we have

\[
B_{1.2} = T^{-1} \sum_{t \in I_t} \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right) \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right)' - T^{-1} \sum_{t \in I_t} \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right) \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right)' - T^{-1} \sum_{t \in I_t} \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right) \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right)' + T^{-1} \sum_{t \in I_t} \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right) \left( \sum_{i=0}^{\Delta \tau_r T-1} F_s^i \mu_s \right). \quad (2.46)
\]
The first term on the right-hand side (rhs) of (2.46) converges to \( B_2(\tau_{s-1}, \tau_2) \). Since
\[
\sum_{t \in I_t} \sum_{l=1}^{\Delta \tau \tau_{T-1}} F_{l}^{t} \mu_s = \sum_{l=1}^{\Delta \tau \tau_{T-1}} |F_{l}^{t} \mu_s|
\]
and is \( O(1) \) by same arguments as above, it follows that the second and third terms on the rhs of (2.46) are \( o(1) \). Now consider the fourth term. Noting that
\[
\sum_{l=j}^{\Delta \tau \tau_{T-1}} F_{l}^{j} \mu_s = F_{j}^{j} \mu_s + \sum_{l=j+1}^{\Delta \tau \tau_{T-1}} F_{l}^{j} \mu_s
\]
and so
\[
\left( \sum_{l=j}^{\Delta \tau \tau_{T-1}} F_{l}^{j} \mu_s \right) \left( \sum_{l=j}^{\Delta \tau \tau_{T-1}} F_{s}^{l} \mu_s \right) = F_{j}^{j} \mu_s \mu_s^{'} F_{s}^{j} + \sum_{l=j+1}^{\Delta \tau \tau_{T-1}} F_{l}^{j} \mu_s \left( \sum_{l=j+1}^{\Delta \tau \tau_{T-1}} F_{s}^{l} \mu_s \right) + \sum_{l=j+1}^{\Delta \tau \tau_{T-1}} F_{l}^{j} \mu_s \left( \sum_{l=j+1}^{\Delta \tau \tau_{T-1}} F_{s}^{l} \mu_s \right) \mu_s^{'} F_{s}^{j}.
\]
Using similar arguments to above, it can be shown that the rhs of (2.47) is \( O(1) \) and so the fourth term on the rhs of (2.46) is \( o(1) \). This completes the derivation.

Case (ii) roots are real but not distinct

Using the Jordan decomposition, Magnus & Neudecker (1991, p.17), there is a nonsingular matrix \( P \) such that
\[
F = P^{-1} \Gamma P
\]
where
\[
\Gamma = \text{blockdiag}(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)
\]
and \( \Gamma_j \) is the \( n_j \times n_j \) matrix
\[
\Gamma_j = \begin{bmatrix}
\gamma_j & 1 & 0 & \ldots & 0 \\
0 & \gamma_j & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & \gamma_j
\end{bmatrix},
\]
and \( \{\gamma_j\} \) are the eigenvalues of \( F \).\(^{17}\) As before, we have:
\[
F^t = P^{-1} \Gamma^t P
\]
where from Lütkepohl (1993) (p.460),
\[
\Gamma = \text{blockdiag}(\Gamma_1^t, \Gamma_2^t, \ldots, \Gamma_k^t)
\]
\(^{17}\)An eigenvalue may appear in more than one block, see Lütkepohl (1993) (p.460).
where \( l_j = 0 \) if \( c > l \).

Given the block diagonal structure,

\[
\| R^l \| = \left( \sum_{i=1}^{k} \| R_i^l \| ^2 \right)^{1/2},
\]

and from (2.48),

\[
\| R_i^l \| ^2 = \sum_{m=0}^{n-1} \sum_{c=0}^{m} \left\{ \left( \frac{l}{c} \right) \gamma_{c}^{l-c} \right\} ^2.
\]

Set \( \pi = \max_j \{ n_j \} \). Then:

\[
\| R_i^l \| ^2 \leq \sum_{m=0}^{\pi} \sum_{c=0}^{m} \left\{ \left( \frac{l}{c} \right) \gamma_{c}^{l-c} \right\} ^2,
\]

where \( \gamma^* = \max\{ \tilde{\gamma}_i; i = 1, 2, \ldots, np \} \), and \( \tilde{\gamma}_i = | \gamma_i | \). For \( 0 \leq c \leq \pi \) and \( l = 1, 2, \ldots, \pi \), we have (as \( | \gamma_i | < 1 \))

\[
\left\{ \left( \frac{l}{c} \right) \gamma_{c}^{l-c} \right\} ^2 \leq \left( \frac{\pi}{\pi - l} \right)^2.
\]

and so

\[
\| R^l \| ^2 \leq (\pi + 1)^2 \left( \frac{\pi}{\pi - l} \right)^{-2}.
\]

Therefore, for \( l \geq \pi \), we have

\[
\| R^l \| \leq K_1 \frac{\pi}{\pi - l},
\]

for some finite positive constant \( K_1 \).

We have

\[
\sum_{l=1}^{(\Delta \tau, T)-1} \| F_i^l \|^2 dC_\Sigma = \sum_{l=1}^{\pi} \| F_i^l \|^2 dC_\Sigma + \sum_{l=\pi+1}^{(\Delta \tau, T)-1} \| F_i^l \|^2 dC_\Sigma,
\]

where the first term on the right-hand side is evidently \( O(1) \) as \( \pi \) is finite. So now consider the other term. By similar arguments to Case (i),

\[
\sum_{l=\pi+1}^{(\Delta \tau, T)-1} \| F_i^l \|^2 dC_\Sigma \leq K_3 \sum_{l=\pi+1}^{(\Delta \tau, T)-1} l\| R^l \| \leq K \sum_{l=\pi+1}^{(\Delta \tau, T)-1} \frac{\pi}{\pi - l} = O(1)
\]

(The last quality holds by the following reasoning. We have \( \sum_{j=0}^{\infty} x^j = (1 - x)^{-1} \). Differentiating both sides with respect to \( x \) gives \( \sum_{j=1}^{\infty} jx^{j-1} = (1 - x)^{-2} \) and so \( \sum_{j=1}^{\infty} jx^j = x(1 - x)^{-2} \). The second derivative gives \( \sum_{j=2}^{\infty} j(j-1)x^{j-2} = 2(1 - x)^{-3} \) and so \( \sum_{j=2}^{\infty} j^2x^j = 2x^2(1 - x)^{-3} \) and \( \sum_{j=2}^{\infty} j^2x^j = 2x^2(1 - x)^{-3} - \{ x(1 - x)^{-2} - x \} = O(1) \). Repeated differentiation and this logic yields: \( \sum_{j=\pi}^{\infty} j^3x^j = O(1) \).)
It then follows by similar arguments to Case (ii) that $C_2 = o(1)$. The proof that $B_6 = o_p(1)$ and $B_{1,2} \rightarrow \mathbb{B}(\tau_{s-1}, \tau_s)$ follow via similar arguments and so is omitted for brevity.

**Case 3: distinct complex roots**

Without loss of generality, we consider the case where the first two eigenvalues of $F$ are complex and the remainder are all real. Let $\gamma_1 = \mu + i\nu$ and $\gamma_2 = \mu - i\nu$. Given that the VAR is stationary, we have $r = \sqrt{\mu^2 + \nu^2} < 1$. In this case, we have:

$$F^i = P^{-1} \overrightarrow{T} P$$

where

$$\overrightarrow{T} = \begin{bmatrix} \mu & -\nu & 0 & 0 & \ldots & 0 \\ \nu & \mu & 0 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_3 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ldots & \gamma_{np} \end{bmatrix}$$

Using polar coordinate, we write $\mu = r\cos(\theta)$ and $\nu = r\sin(\theta)$. It then follows that

$$\overrightarrow{T'} = \begin{bmatrix} r\cos(l\theta) & -r\sin(l\theta) & 0 & 0 & \ldots & 0 \\ r\sin(l\theta) & r\cos(l\theta) & 0 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_3^l & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ldots & \gamma_{np}^l \end{bmatrix}$$

Thus we have

$$\|\overrightarrow{T'}\|^2 = 2r^{2l} + \gamma_3^{2l} + \ldots + \gamma_{np}^{2l}.$$ 

We can then apply the same argument as in Case (i) to show $C_2 = o(1)$ provided we replace $\gamma_{\text{max},F}$ by $\max\{r,|\gamma_3|,\ldots,|\gamma_{np}|\}$. The proof that $B_6 = o_p(1)$ and $B_{1,2} \rightarrow \mathbb{B}_2(\tau_{s-1}, \tau_s)$ follow via similar arguments and so is omitted for brevity.

**Case 4: repeated complex roots**

Without loss of generality, we consider the case where the first four eigenvalues of $F$ are complex and repeated, and the remainder are all real. Let $\gamma_1 = \gamma_3 = \mu + i\nu$ and $\gamma_2 = \gamma_4 = \mu - i\nu$. Given that the VAR is stationary, we have $r = \sqrt{\mu^2 + \nu^2} < 1$. In this case, we have:

$$F^i = P^{-1} \overrightarrow{T'} P$$
we have, via repeated back substitution,

\[ T = \begin{bmatrix} \mu & -\nu & 1 & 0 & 0 & 0 & \ldots & 0 \\ \nu & \mu & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \mu & -\nu & 0 & 0 & \ldots & 0 \\ 0 & 0 & \nu & \mu & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \gamma_5 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \gamma_{np} \end{bmatrix}, \]

and

\[ \overline{T} = \begin{bmatrix} r^l a(l) & -r^l b(l) & l r^l a(l - 1) - l r^{l-1} b(l - 1) & 0 & \ldots & 0 \\ r^l b(l) & r^l a(l) & l r^l b(l - 1) - l r^{l-1} a(l - 1) & 0 & 0 & \ldots & 0 \\ 0 & 0 & r^l a(l) & -r^l b(l) & 0 & 0 & \ldots & 0 \\ 0 & 0 & r^l b(l) & r^l a(l) & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \gamma_5 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \gamma_{np} \end{bmatrix}, \]

where \( a(l) = \cos(l\theta) \) and \( b(l) = \sin(l\theta) \). Therefore, we have\(^{18}\)

\[ \|\overline{T}\|^2 = 4r^{2l} + 2l^2r^{2(l-1)}\gamma_5^2 + \ldots + \gamma_{np}^2 \]

and

\[ \|\overline{T}\| = Kl^2 \gamma_s^{(l-1)} \]

where \( \gamma_s = \max\{r, |\gamma_5|, \ldots, |\gamma_{np}|\} \). By a similar argument to Case (ii), it follows that \( C_2 = o(1) \). The proof that \( B_6 = o(p(1)) \) and \( B_{1,2} \to B_2(\tau_s-1, \tau_s) \) follow via similar arguments and so is omitted for brevity.

Inspecting the derivations, it also becomes clear that we showed that \( \sum_{l=0}^{\infty} \| F_{s}^l \| < \infty \).

### 3 Proof of (51)

We have

\[ \sum_{l=0}^{\infty} \| \dot{F}_{s}^l - F_{s}^l \| = \sum_{l=1}^{\infty} \| \dot{F}_{s}^l - F_{s}^l \|. \]

Using

\[ \dot{F}_{s}^l - F_{s}^l = \dot{F}_{s}^{l-1}(\dot{F}_{s} - F_{s}) + (\dot{F}_{s}^{l-1} - F_{s}^{l-1})F_{s}, \]

we have, via repeated back substitution,

\[ \dot{F}_{s}^{l} - F_{s}^{l} = \sum_{j=1}^{l} \dot{F}_{s}^{j-1}(\dot{F}_{s} - F_{s})F_{s}^{j-1}. \]

\(^{18}\)Using \( \cos^2(\theta) + \sin^2(\theta) = 1 \).
Therefore, it follows that
\[ \| \hat{F}_s^\ell - F_s^\ell \| \leq \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| \hat{F}_s - F_s \| \| F_s^{\ell-j} \| = \| \hat{F}_s - F_s \| \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \|, \]
and
\[ \sum_{\ell=0}^{\infty} \| \hat{F}_s^\ell - F_s^\ell \| = \sum_{\ell=1}^{\infty} \| \hat{F}_s^\ell - F_s^\ell \| \leq \| \hat{F}_s - F_s \| \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \|. \]

We now show that \( \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} \| \hat{F}_s^j - F_s^j \| \| F_s^{\ell-j} \| = o_p(1) \) for the cases in Section 2.

**Case (i): real and distinct roots**

From (2.45), it follows that (using \( K \) repeatedly to denote any finite constant)
\[ \| \hat{F}_s^j - F_s^j \| \leq K \hat{\gamma}_s^{j-1}, \]
\[ \| F_s^{\ell-j} \| \leq K \gamma_s^{\ell-j}. \]
where \( \gamma_s \) denotes \( \gamma_{\text{max}, F} \) from Section 2, and \( \hat{\gamma}_s \) is its analogue based on \( \hat{F}_s \). Let \( \gamma_\bullet = \max\{\gamma_s, \hat{\gamma}_s\} \). Then, we have
\[ \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \| \leq K \ell \gamma_\bullet^{\ell-1}, \]
and so
\[ \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \| \leq K \sum_{\ell=1}^{\infty} \ell \gamma_\bullet^{\ell-1} = O_p(1), \]
from Assumption 7.\(^{19}\)

**Case (ii): roots are real but repeated**

Using (2.49), we have for \( \ell \geq \bar{n} \) (and \( \bar{n} \) is defined as in Section 2 Case (ii))
\[ \| \hat{F}_s^j - F_s^j \| \leq K (j-1)^n \gamma_s^{j-1}, \]
\[ \| F_s^{\ell-j} \| \leq K (\ell-j)_n \gamma_s^{\ell-j}. \]
where \( \gamma_s \) is defined as in Section 2 Case (ii) and \( \hat{\gamma}_s \) is defined analogously only replacing \( F_s \) with \( \hat{F}_s \). For ease of presentation, assume the same multiplicity of eigenvalues for \( F_s \) and \( \hat{F}_s \); it is straightforward to modify the argument to allow for different multiplicities. Using \( j \leq \ell \), we have
\[ \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \| \leq K \ell^{2n} \gamma_\bullet^{\ell-1}, \]
and
\[ \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \| \leq K \ell^{2n+1} \gamma_\bullet^{\ell-1}. \]
This gives
\[ \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} \| \hat{F}_s^{j-1} \| \| F_s^{\ell-j} \| \leq K \sum_{\ell=1}^{\infty} \ell^{2n+1} \gamma_\bullet^{\ell-1} = O_p(1), \]
\(^{19}\)Note that \( \hat{F}_s \overset{p}{\to} F_s \) implies \( \hat{\gamma}_s \overset{p}{\to} \gamma_s \) and \( 0 < \gamma_s < 1 \) from Assumption 7.
using the results noted toward the end of Section 2, Case (ii). The proof for the rest of the cases follow by similar reasoning.

4 Proof of (63)

We need to analyze the following cases:

2) \( t - \kappa = t - t^* \), \( t - l = t - l^* \), \( t - \kappa \neq t - l \)
3) \( t - l = t - t^* \), \( t - \kappa = t - l^* \), \( t - \kappa \neq t - l \)
4) \( t - l = t - \kappa = t - l^* \), \( t - \kappa \neq t - l \)
5) \( t - \kappa \neq t - l \neq t^* - t^* \neq t^* - l^* \)
6) \( t - \kappa = t - l = t - t^* \), \( t - \kappa \neq t - l^* \)
7) \( t - \kappa = t - l = t - l^* \), \( t - \kappa \neq t - t^* \)
8) \( t - \kappa = t^* - l^* = t^* - t^* \), \( t - \kappa \neq t - l \)
9) \( t - l = t^* - l^* = t^* - t^* \), \( t - \kappa \neq t - l \)

For cases 2)-9) we show that \( O_i = o_p(1), i = 2, \ldots, 9 \). Recall that the estimation error in \( \hat{g}_t \) is asymptotically negligible and \( \hat{F}_t^l - F_s = \mathcal{R}_{s,t} \), where \( \sum_{i=0}^{\infty} \| R_{s,t} \| = \| \hat{F}_s - F_s \| O_p(1) = o_p(1) \), and \( \mathcal{A}^{-1}_{s,t} = \mathcal{A}^{-1}_{s,t} + O_p(T^{-1}) \). These arguments will be used to replace the estimated values with the true values in all terms \( O_i, i = 2, \ldots, 9 \), because the limits will be the same. Consider case 2).

\[
O_2 = T^{-2} \sum_{t,t^* \in I} \sum_{l,k=0,l\neq k}^{l-1} \left( (\hat{F}_s^l A^{-1}_{s,t}) \otimes (\hat{F}_s^l A^{-1}_{s,t}) \right) \mathcal{G} \left( (\hat{F}_s^{l^*} A^{-1}_{s,t^*}) \otimes (\hat{F}_s^{l^*} A^{-1}_{s,t^*}) \right)
\]

where

\[
\mathcal{G} = E^b(g^b_{t-k} g^b_{t-l}) \otimes E^b(g^b_{t-k} g^b_{t-l}) = ((g_{t-k} \hat{g}_{t-k}^l \otimes (g_{t-k} \hat{g}_{t-k}^l)) \otimes (E^b(\nu_{t-k} \nu_{t-k}) \otimes E^b(\nu_{t-k} \nu_{t-k}))
\]

Note that

\[
\| O_2 \| \leq T^{-2} \sum_{t,t^* \in I} \sum_{l,k=0,l\neq k}^{l-1} \left( (\hat{F}_s^l A^{-1}_{s,t}) \otimes (\hat{F}_s^l A^{-1}_{s,t}) \right) \mathcal{G} \left( (\hat{F}_s^{l^*} A^{-1}_{s,t^*}) \otimes (\hat{F}_s^{l^*} A^{-1}_{s,t^*}) \right)
\]

because \((g_{t-k} g_{t-l}) \subseteq I\) is the upper left block \( \epsilon_{t-k}^l \epsilon_{t-l}^l \), and \( \| (\epsilon_{t-k}^l \epsilon_{t-l}^l) \| \subseteq \sup_{t-k} \| \epsilon_{t-k}^l \| \| \epsilon_{t-l}^l \| = (\sup_t \| \epsilon_t \|)^2 = O(1) \) by Assumption 9.

\( \text{If roots of } \mathcal{F}_s \text{ and } \mathcal{F}_l \text{ have different multiplicities then define } n \text{ to be the max over the two. Recall that } \hat{\gamma}_s, \hat{\gamma}_s \text{ and hence the multiplicities match with probability one as } T \to \infty. \)
For a generic scalar \( N \) and generic matrices \( A_j, j = 1, \ldots, N, \) \( \text{vec}_N(A) = \text{vec}_{p,1,N}(A); \) \( \text{mat}_{N,N}(A) \) a \( N \times N \) block matrix with typical block given by \( A. \) Note that \( \text{vec}_N(A) = \text{mat}_{N,1}(A) \) and \( \text{mat}_{N,N}(a) \) is a \( N \times N \) matrix with typical scalar element \( a. \) Then:

\[
E_b^b((\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \otimes (\nu_{t-\kappa}^b \nu_{t-\kappa}^b)) = \begin{bmatrix}
\text{mat}_{1+p_1,1+p_1}(Y_1) & \text{mat}_{1+p_1,p_2}(Y_2) & \text{mat}_{1+p_1,p_2}(0_{p_2 \times n(p-1)}) \\
(\text{mat}_{p_2,p_2}(J))' & \text{mat}_{p_2,n(p-1)}(0_{p_2 \times n(p-1)}) & \text{mat}_{p_2,n(p-1)}(0_{p_2 \times n(p-1)}) \\
(\text{mat}_{1+p_1,n(p-1)}(0_{n(p-1) \times p_2})') & \text{mat}_{n(p-1),n(p-1)}(0_{n(p-1) \times p_2}) & \text{mat}_{n(p-1),n(p-1)}(0_{n(p-1) \times p_2})
\end{bmatrix}
\]

\[
Y_1 = \begin{bmatrix}
\text{mat}_{1+p_1,1+p_1}(E_b^b(\nu_1^b)) & \text{mat}_{1+p_1,p_2}(0_{1+p_1 \times n(p-1)}) \\
(\text{mat}_{p_2,p_2}(J))' & 0_{p_2 \times n(p-1)} \\
0_{n(p-1) \times p_2} & 0_{n(p-1) \times n(p-1)}
\end{bmatrix}
\]

\[
Y_2 = \begin{bmatrix}
\text{mat}_{1+p_1,1+p_1}(E_b^b(\nu_1^b)) & \text{mat}_{1+p_1,p_2}(0_{1+p_1 \times n(p-1)}) \\
(\text{mat}_{p_2,p_2}(J))' & 0_{p_2 \times n(p-1)} \\
0_{n(p-1) \times p_2} & 0_{n(p-1) \times n(p-1)}
\end{bmatrix}
\]

We can see from above that the only nonzero elements of \( E_b^b(\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \otimes (\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \) are \( E_b^b(\nu_1^b), E_b^b(\nu_1^b) \) and \( 1. \) By similar arguments as for case 2) with the additional assumption that \( E_b^b(\nu_1^b) \leq \bar{c}, \bar{c} > 0, \) it follows that \( O_4 = o_p(1). \)

For case 5) we have

\[
O_5 = T^{-2} \sum_{t,t^* \in I_s} \sum_{\nu_{t-\kappa}^b \otimes (\nu_{t-\kappa}^b), \nu_{t-\kappa}^b \otimes g_{t-\kappa}^b} \nu_{t-\kappa}^b \otimes g_{t-\kappa}^b \nu_{t-\kappa}^b \otimes g_{t-\kappa}^b \nu_{t-\kappa}^b \otimes g_{t-\kappa}^b \\
\mathcal{G} = E_b^b \left( \left( g_{t-\kappa}^b \otimes g_{t-\kappa}^b \right) \left( g_{t-\kappa}^b \otimes g_{t-\kappa}^b \right) \right)
\]

\[
= \left( (\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \otimes (\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \right) \otimes E_b^b \left( (\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \otimes (\nu_{t-\kappa}^b \nu_{t-\kappa}^b) \right);
\]
where

\[
E^b((\nu_{t-k} \nu'_{t-\nu}) \otimes (\nu_{t-k} \nu'_{t-\nu})) = \begin{bmatrix}
\text{mat}_{1+p_1,1+p_1}(0_{np \times np}) & \text{mat}_{1+p_1,p_2}(0_{np \times np}) & \text{mat}_{1+p_1,n(p-1)}(0_{np \times np}) \\
\text{mat}_{1+p_1,p_2}(0_{np \times np}) & \text{mat}_{p_2,p_2}(\mathcal{J}) & \text{mat}_{p_2,n(p-1)}(0_{np \times np}) \\
\text{mat}_{1+p_1,n(p-1)}(0_{np \times np}) & \text{mat}_{p_2,n(p-1)}(0_{np \times np}) & \text{mat}_{n(p-1),n(p-1)}(0_{np \times np})
\end{bmatrix}
\]

which only selects the cross products \((\zeta_{t-k} \zeta'_{t-\nu}) \otimes (\zeta_{t-k} \zeta'_{t-\nu})\). We have

\[
\|O_6\| \leq T^{-2} \sum_{t,t' \in I_t} \sum_{t',\zeta_{t-k} \zeta'_{t-\nu}} \left( \left\| \left(\sum_{l=0}^{\infty} ((F^s_{t} A^{-1}_{s,\#}^t) \otimes (F^s_{t} A^{-1}_{s,\#}^t)) \right) ((g_{t-k} g'_{t-\nu}) \otimes (g_{t-k} g'_{t-\nu})) \otimes \mathcal{J}_3 \right\| + o_p(1) \right)
\]

Consider now case 6). We have:

\[
O_6 = T^{-2} \sum_{t,t' \in I_t} \sum_{t',\zeta_{t-k} \zeta'_{t-\nu}} \left( \left\| \left(\sum_{l=0}^{\infty} ((F^s_{t} A^{-1}_{s,\#}^t) \otimes (F^s_{t} A^{-1}_{s,\#}^t)) \right) ((\hat{F}^s_{t} A^{-1}_{s,\#}^t) \otimes (\hat{F}^s_{t} A^{-1}_{s,\#}^t)) \right\| + o_p(1) \right)
\]

where

\[
\mathcal{G} = E^b((g^b_{t-k} \otimes g^b_{t-k}) (g^b_{t-k} \otimes g^b_{t-\nu}) (g_{t-k} g'_{t-\nu}) \otimes (g_{t-k} g'_{t-\nu})),
\]

with

\[
E^b((\nu_{t-k} \nu'_{t-\nu}) \otimes (\nu_{t-k} \nu'_{t-\nu})) = \begin{bmatrix}
\text{mat}_{1+p_1,1+p_1}(\hat{Y}_1) & \text{mat}_{1+p_1,p_2}(\hat{Y}_2) & \text{mat}_{1+p_1,n(p-1)}(0_{np \times np}) \\
\text{mat}_{1+p_1,p_2}(\hat{Y}_2)' & \text{mat}_{p_2,p_2}(\mathcal{J}_2) & \text{mat}_{p_2,n(p-1)}(0_{np \times np}) \\
\text{mat}_{1+p_1,n(p-1)}(0_{np \times np}) & \text{mat}_{p_2,n(p-1)}(0_{np \times np}) & \text{mat}_{n(p-1),n(p-1)}(0_{np \times np})
\end{bmatrix}
\]

which only has \(E(\nu^2_{t}) < c^*\) (by Assumption 10) and 1 as non-zero elements. Hence,

\[
\|O_6\| \leq T^{-2} \sum_{t,t' \in I_t} \sum_{t',\zeta_{t-k} \zeta'_{t-\nu}} \left( \left\| \left(\sum_{l=0}^{\infty} ((F^s_{t} A^{-1}_{s,\#}^t) \otimes (F^s_{t} A^{-1}_{s,\#}^t)) \right) ((g_{t-k} g'_{t-\nu}) \otimes (g_{t-k} g'_{t-\nu})) \otimes \mathcal{J}_3 \right\| + o_p(1) \right)
\]

which follows by similar arguments as for case 2) and the fact that \(\|\mathcal{J}\|^2 = O(1)\). Similarly, under no additional assumptions, \(O_i = o_p(1), i = 7, 8, 9\).
5 Validity of the WF and WR bootstrap for the sup-\(F\)-statistic

We begin by defining the sup-\(F\) statistics in both the sample and the bootstrap. For ease of reference some equations in the main paper are repeated.

Case (i): \(H_0 : m = 0\) versus \(H_1 : m = k\)

Under \(H_0\), the second stage estimation involves regression via OLS of \(y_t\) on \(\hat{w}_t\) where \(\hat{w}_t = (\hat{a}'_t, z'_{t, \ell}, \ell)'\) using the complete sample. Let \(SSR_0\) denote the residual sum of squares from this estimation. Under \(H_1\), the second stage estimation involves estimation via OLS of the model in equation (6) in the paper, that is,

\[
y_t = \hat{w}'_t \beta(i) + \text{error}, \quad i = 1, ..., k + 1, \quad t \in I_i, k,
\]

for all possible \(k\)-partitions \(\lambda_k\). Let \(SSR_k(\lambda_k; \hat{\beta}_M)\) denote the residual sum of squares associated with this estimation. The sup-\(F\) test statistic is defined as:

\[
\sup-F_T = \sup_{\lambda_k \in \Lambda, k} F_T(\lambda_k).
\]

where

\[
F_T(\lambda_k) = \left( \frac{T - (k + 1)d_{\beta}}{kd_{\beta}} \right) \left( \frac{SSR_0 - SSR_k(\lambda_k; \hat{\beta}_M)}{SSR_k(\lambda_k; \hat{\beta}_M)} \right),
\]

and \(\Lambda_{\epsilon, k} = \{ \lambda_k : |\lambda_{i+1} - \lambda_i| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_k \leq 1 - \epsilon \} \) and \(d_{\beta} = \dim(\beta(i)) = q_1 + p_1\).

The WR bootstrap version of the sup-\(F\) statistic is calculated as follows. Let \(\hat{w}_t^b\) be calculated as described for the WR bootstrap in Section 2.3. For a given \(k\)-partition \(\lambda_k\), the second stage of the 2SLS in the bootstrap samples involves OLS estimation of (17) that is,

\[
y_t^b = \hat{w}_t^b \beta(i) + \text{error}, \quad i = 1, ..., k + 1, \quad t \in I_i, k;
\]

let \(SSR_k(\lambda_k; \hat{\beta}_M^b)\) denote the residual sum of squares associated with this estimation. Let \(SSR_0^b\) be the residual sum of squares associated with estimation of (17) subject to the restriction that \(\beta(i)\) takes the same value in each of the \(k + 1\) regimes. The WR bootstrap version of the sup-\(F\) statistic:

\[
\sup-F_T^b = \sup_{\lambda_k \in \Lambda_{\epsilon, k}} F_T^b(\lambda_k),
\]

where

\[
F_T^b(\lambda_k) = \left( \frac{T - (k + 1)d_{\beta}}{kd_{\beta}} \right) \left( \frac{SSR_0^b - SSR_k^b(\lambda_k; \hat{\beta}_M^b)}{SSR_k^b(\lambda_k; \hat{\beta}_M^b)} \right).
\]

The only difference for the WF bootstrap version of the sup-\(F\) statistic the only difference is that \(\hat{w}_t^b\) is calculated as described for the WF bootstrap in Section 2.3 of the paper.

Case (ii): \(H_0 : m = \ell\) versus \(H_1 : m = k + 1\)

As in Section 2.2, let the estimated break fractions for the \(\ell\)-break model be \(\hat{\lambda}_\ell\) and the associated break points be denoted \(\{\hat{T}_i\}_{i=1}^\ell\) where \(\hat{T}_i = [\hat{T}_{i, \ell}]\). Let \(\hat{I}_i = I_{i, \lambda_k}\), the set of observations in the \(i^{th}\) regime of the \(\ell\)-break model and partition this set as \(\hat{I}_i = \hat{I}_i^{(1)}(\pi_i) \cup \hat{I}_i^{(2)}(\pi_i)\) where \(\hat{I}_i^{(1)}(\pi_i) = \{ t : [\hat{\lambda}_{i-1} T] + 1, [\hat{\lambda}_{i-1} T] + 2, \ldots, [\pi_i T] \} \)
and \( \hat{I}^{(2)}_t(\varpi_i) = \{ t : [\varpi_i]T + 1, [\varpi_i]T + 2, \ldots, [\hat{\lambda}_i]T \} \). Consider estimation of the model in (11) that is,

\[
y_t = \hat{w}_t^{(j)} \beta(j) + \text{error}, \quad j = 1, 2, \quad t \in \hat{I}^{(j)}_t,
\]

for all possible choices of \( \varpi \) (where for notational brevity we suppress the dependence of \( \beta(j) \) on \( t \)). Let \( SSR_i(\varpi_i) \) be the residual sum of squares associated with this estimation, and let \( SSR_i^{(b)} \) be the residual sum of squares associated with estimation of the model subject to the restriction that \( \beta^{(1)} = \beta^{(2)} \). The \( \sup - F \) statistic for the same test is given by

\[
\sup - F_T(\ell + 1 | \ell) = \max_{i=1,2,\ldots,\ell+1} \left\{ \sup_{\varpi_i \in \mathcal{N}(\hat{\lambda}_i)} \left( \frac{SSR_i - SSR_i(\varpi_i)}{SSR_i^{(b)}} \right) \frac{\hat{T}_i - \bar{T}_{i-1} - d_{\beta}}{d_{\beta}} \right\}
\]

(5.54)

where \( \mathcal{N}_i(\hat{\lambda}_i) = [\hat{\lambda}_{i-1} + \epsilon, \hat{\lambda}_i - \epsilon] \).

For each bootstrap the first stage of the 2SLS estimation and the construction of \( \hat{w}_t \) is the same as described above in Case (i). The second stage of the 2SLS involves estimation via OLS of (22) that is,

\[
y^{(b)}_t = \hat{w}_t^{(b)} \beta(j) + \text{error}, \quad j = 1, 2, \quad t \in \hat{I}^{(j)}_t.
\]

for all possible \( \varpi \) (where, once again, we suppress the dependence of \( \beta(j) \) on \( i \)). Let \( SSR_i^{(b)}(\varpi_i) \) be the residual sum of squares associated with this estimation, and \( SSR_i^{(b)} \) be the residual sum of squares associated with estimation of the model subject to the restriction that \( \beta^{(1)} = \beta^{(2)} \). The bootstrap version of \( \sup - F_T(\ell + 1 | \ell) \) is given by

\[
\sup - F_T^{(b)}(\ell + 1 | \ell) = \max_{i=1,2,\ldots,\ell+1} \left\{ \sup_{\varpi_i \in \mathcal{N}(\hat{\lambda}_i)} \left( \frac{SSR_i^{(b)} - SSR_i^{(b)}(\varpi_i)}{SSR_i^{(b)}} \right) \frac{\hat{T}_i - \bar{T}_{i-1} - d_{\beta}}{d_{\beta}} \right\}
\]

(5.55)

To establish the asymptotic validity of the bootstrap versions \( \sup - F_T^{(b)} \) and \( \sup - F_T^{(b)}(\ell + 1 | \ell) \), it is most convenient to work with alternative formulae for the \( F \)-statistics. To illustrate, consider Case (i). From standard LS theory,\(^{21}\) it follows that:

\[
F_{T \lambda_k} = \frac{T}{kd_{\beta}} \hat{\beta}_k \hat{V}_k \hat{R}_k^r (\hat{R}_k \hat{V}_k \hat{R}_k^r)^{-1} \hat{R}_k \hat{\beta}_k
\]

where \( d_{\beta} \) is the dimension of \( \beta, \hat{V}_k = \hat{\sigma}^2(\lambda_k) \text{diag}_{i=1,k+1}(\hat{Q}^{-1}_i) \) and \( \hat{\sigma}^2(\lambda_k) = SSR_k(\lambda_k; \hat{\beta}_k)/(T - (k + 1)d_{\beta}) \).

The asymptotic validity of the bootstrap version of the \( \sup - F \) statistics can then be established using similar arguments to the proofs of Theorems 1-2.

**Theorem B.3.** Under Assumption 1-10 for the WF bootstrap, and Assumptions 1-10 and Assumption 9 for the WR bootstrap, (i) under the null hypothesis \( m = 0 \),

\[
\sup_{c \in \mathbb{R}} \left| \hat{P}^b \left( \sup - F_T^{(b)} \leq c \right) - P \left( \sup - F_T \leq c \right) \right| \xrightarrow{p} 0 \quad \text{as } T \to \infty;
\]

(ii) under the null hypothesis \( m = \ell \),

\[
\sup_{c \in \mathbb{R}} \left| \hat{P}^b \left( \sup - F_T^{(b)}(\ell + 1 | \ell) \leq c \right) - P \left( \sup - F_T(\ell + 1 | \ell) \leq c \right) \right| \xrightarrow{p} 0 \quad \text{as } T \to \infty.
\]

**Proof of Theorem B.3.**

\(^{21}\)For example, see Greene (1994) (p.163). We are grateful to a referee for drawing our attention to this alternative representation of the \( F \)-statistic.
(i) Recall that $Wald_{T \lambda_k} = T \hat{\beta} \lambda_k \hat{R}_k \hat{V} \lambda_k \hat{R}_k^{-1} \hat{R}_k \hat{\beta} \lambda_k$, where $\hat{V} \lambda_k = \text{diag}_{i=1:k+1}(\hat{V}(i))$, where $\hat{V}(i) = Q^{-1}(i) \hat{M}(i) Q^{-1}(i)$. It can be recognized that the version of the $F \lambda_k$ described above only differs from the $Wald_{T \lambda_k}$ in terms of the choice of matrix in the center of the quadratic form. Clearly an analogous representation is available for the bootstrap versions of the test.

By Lemmas 2, 5 and 6 in the paper, it can be shown that the bootstrap equivalent of $\hat{\sigma}^2(\lambda_k)$ is such that its difference with $\hat{\sigma}^2(\lambda_k)$ is $o_p(1)$. For the rest of the quantities, the analysis is similar to the proof of Theorem 1 in the paper.

(ii) Recalling the alternative representation of sup-$Wald_{\ell+1|\ell}$ in the proof of Theorem B 2, the proof follows as for part (i).
6 More simulation evidence

In this section we present further simulation evidence on the WR and WF bootstrap sup-Wald and sup-F using the same DGPs as in the main paper.

We have considered the behavior of the sup-F test under both the null and the alternative hypotheses. From the first two columns of Tables 6.1-6.4 it can be seen that the WR sup-F test works better than the WF sup-F who is in general oversized. Comparing the two tests, the sup-F and the sup-Wald, their WR versions are similar under the null, while the WF sup-Wald is less size distorted than the WF sup-F in general (see column 2 of Tables 1-4 from the paper for the sup-Wald and column 2 of Tables 6.1-6.4 below for the sup-F). Regarding the power, it can be seen from columns 3-6 of Tables 6.1-6.4 that the sup-F is more powerful than the sup-Wald for $T = 120$, but for $T = 240, 480$, the sup-Wald is as powerful or slightly more powerful than the sup-F after adjusting for size (see for example Table 4 of the paper and Table 6.4 below).

In Tables 3 and 4 of the main paper we have tested sequentially for the presence of max 2 breaks in the RF for $x_t$ (in (26)-(27) and (32)-(33) respectively). The fraction of times that 0, 1, 2 breaks were detected in RF (out of 1,000 replications of the scenarios), is given in Tables 6.7-6.8. These tables indicate that for all sample sizes considered, only in about 5 percent of the cases the null $H_0 : h = \ell = 1$ was not rejected and that the null $H_0 : h = \ell = 0$ was rejected all the time in general.

In order to assess the impact of the pre-testing in RF, in Tables 6.5 and 6.6 we have obtained the rejection frequencies of the bootstrap tests when the number of breaks in the RF is held at the true number, $h = 1$, and the estimated location is imposed in the estimation of RF and SE and computing the test statistics sup-Wald and sup-F for 2SLS. Comparing these tables with the first two columns of Tables 3 and 4 from the paper, and Tables 6.3 and 6.4 below we can see that the rejection frequencies are similar. Note that the true number of breaks in RF for Tables 6.5 and 6.6 is the same, $h = 1$, but the DGP for the RF is different since in Table 6.6 (corresponding to scenario $(h, m) = (1, 1)$) the break in the SE results in a break in the mean of $y_{t-1}$, a regressor in the RF. Table 6.5 corresponds to the scenario when there is no break in SE.

In Tables 6.9 and 6.10 we have considered a break in RF of smaller size than the one mentioned after (26)-(27) by taking $\delta_{x_{t(i)}} = (1, 1, 1, 1)'$ (and the rest of the parameters’ values are as mentioned after (26)-(27)). Tables 6.9 and 6.10 present the rejection frequencies for the WR and WF bootstrap sup-Wald under the null hypothesis when we have sequentially tested for the presence of max 2 breaks in the RF for $x_t$ (in (26)-(27) and (32)-(33) respectively) using the WR/WR sup-Wald for OLS, and the resulting number of RF breaks was imposed in each simulation prior to estimating the RF and SE and computing the test statistics for 2SLS (see the first two columns of Tables 6.9 and 6.10). Columns 3-4 of Tables 6.9 and 6.10 present the rejection frequencies when the number of breaks in the RF is held at the true number, $h = 1$ (and the estimated location is taken into account in the estimation of SE). The last two columns report fraction of times that 0, 1, 2 breaks were detected in RF out of 1,000 replications of the scenarios. We notice that for $T = 120$, in 5-9 percent of the cases no break was detected in RF, but nevertheless the rejection frequencies of the bootstrap sup-Wald (columns 1-2 of Tables 6.9 and 6.10) remain close to the case when no pre-testing in RF took place (columns 3-4 of Tables 6.9 and 6.10) with the WR bootstrap performing again better than the WF bootstrap. Note that the true number of breaks in RF for Tables 6.9 and 6.10 is the same, $h = 1$, but the DGP for the RF is different because in Table 6.10 (which corresponds to scenario $(h, m) = (1, 1)$) the break in the SE results in a break in the mean of $y_{t-1}$, a regressor in the RF. In Table 6.9, there is no break in SE.
Table 6.1: Scenario: \((h,m)=(0,0)\) - rejection probabilities from testing \(H_0: m = 0\) vs. \(H_1: m = 1\) with bootstrap sup-\(F\) test.

|       | Case A       |       |       | Case B       |       |       | Case C       |       |       | Case D       |       |       |
|-------|--------------|-------|-------|--------------|-------|-------|--------------|-------|-------|--------------|-------|-------|
|       | WR bootstrap | WF bootstrap | Power | WR bootstrap | WF bootstrap | Power | WR bootstrap | WF bootstrap | Power | WR bootstrap | WF bootstrap | Power |
|       | Size         | Size   |       |              |              |       |              |              |       |              |              |       |
|       | \(g=0\)      | \(g=0\) | \(g=-0.007\) | \(g=-0.007\) | \(g=-0.009\) |       | \(g=-0.009\) |       |       | \(g=-0.009\) |       |       |
| \(T\) | 10\% 5\% 1\% | 10\% 5\% 1\% | 10\% 5\% 1\% | 10\% 5\% 1\% | 10\% 5\% 1\% |       | 10\% 5\% 1\% |       |       | 10\% 5\% 1\% |       |       |
|       | Size         | Power  |       |              |              |       |              |              |       |              |              |       |
|       | \(g=0\)      | \(g=-0.007\) |       | \(g=-0.009\) |       |       | \(g=-0.009\) |       |       |       |       |       |
| 120   | 9.8 5.1 1.4  | 9.5 5.6 1.7  | 99.9 99.9 99.7 | 99.9 99.8 99.7 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 240   | 9.4 4.6 0.5  | 8.8 4.5 0.7  | 99.9 99.9 97  | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 480   | 10.9 5.9 1.5 | 9.4 5.4 1.6  | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 120   | 11.3 6.1 1.7 | 11.5 6.3 2.2 | 99.4 99.4 98.6 | 99.6 99.4 99  | 99.8 99.7 99.6 |       | 99.9 99.9 99.7 |       |       |       |       |       |
| 240   | 10.6 5.9 1   | 9.7 4.9 1   | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 480   | 10.3 6.2 1.9 | 9.4 5.4 1.3 | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 120   | 9.1 5 1     | 9.8 5.4 1   | 99.7 99.5 99.4 | 99.7 99.6 99.4 | 99.9 99.9 99.9 |       | 100 100 100 |       |       |       |       |       |
| 240   | 11.1 5.7 0.9 | 11.2 5.5 0.8 | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 480   | 9.6 4.7 1.1  | 8.7 4.9 0.7 | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 120   | 10.4 5.8 0.9 | 10.6 4.9 0.9 | 100 100 99.7 | 99.9 99.9 99.7 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 240   | 10 4.5 1    | 9.6 5 1.1  | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |
| 480   | 10.5 3.6 0.9 | 8.8 3.6 0.9 | 100 100 100 | 100 100 100 | 100 100 100 |       | 100 100 100 |       |       |       |       |       |

Notes. The first two columns refer to the case when \(H_0: m = 0\) is true (\(g=0\) in equation (36)). The next columns refer to the case when we test for \(H_0: m = 0\), but \(H_1: m = 1\) is true (\(g = -0.007, -0.009\) in equation (36)). Under the null and the alternative hypotheses we impose \(h = 0\) in the RF.
Table 6.2: Scenario: \((h,m)=(0,1)\) - rejection probabilities from testing \(H_0 : m = 1\) vs. \(H_1 : m = 2\) with bootstrap sup-\(F\) test.

| WR bootstrap  | WF bootstrap | WR bootstrap  | WF bootstrap | WR bootstrap  | WF bootstrap |
|---------------|--------------|---------------|--------------|---------------|--------------|
| Size          | Power        | Size          | Power        | Size          | Power        |
| \(g=0\)       | \(g=0\)      | \(g=0.3\)     | \(g=0.3\)    | \(g=0.4\)     | \(g=0.4\)    |
| T 10% 5% 1%   | T 10% 5% 1%  | T 10% 5% 1%   | T 10% 5% 1%  | T 10% 5% 1%  | T 10% 5% 1%  |
| 120 10.8 5.6 1.2 | 14.2 7.7 1.7 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
| 240 9.3 4.7 1.1 | 11.4 5.5 1.4 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
| 480 8.7 4.7 1.5 | 9.8 5.2 1.7 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |

**Case A**

| 120 13.1 6.9 1.9 | 15.3 8.7 2.7 | 98.6 97.5 95.1 | 98.4 97.6 95.9 | 99.4 99.3 98.8 | 99.6 99.3 98.6 |
| 240 11.1 4.9 0.9 | 13 6.2 1.2 | 99.9 99.7 98.9 | 99.9 99.6 98.9 | 100 100 99.8 | 100 99.9 99.8 |
| 480 10 5.1 1.3 | 10.6 5.8 1.2 | 100 99.9 99.8 | 99.9 99.8 99.8 | 100 100 100 | 100 100 100 |

**Case B**

| 120 12.1 7.2 1.1 | 17.6 10.8 2.4 | 96.8 93.7 82.5 | 98.5 96.3 87.1 | 99.9 99.8 99.1 | 100 100 99.4 |
| 240 10.7 4.9 0.7 | 15.5 8.5 2 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
| 480 11.3 6.4 1 | 14.1 8.4 1.9 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |

**Case C**

| 120 11.9 7.1 1.4 | 16.3 10.2 2.9 | 99.8 99.8 99.2 | 99.9 99.8 99.8 | 100 100 100 | 100 100 100 |
| 240 11.5 5.4 1.4 | 17 8.4 2.2 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |
| 480 11.6 6 1.1 | 14.2 7.5 1.7 | 100 100 100 | 100 100 100 | 100 100 100 | 100 100 100 |

**Case D**

Notes. The first two columns refer to the case when \(H_0 : m = 1\) is true (\(g=0\) in equation (36)). The next columns refer to the case when we test for \(H_0 : m = 1\), but \(H_1 : m = 2\) is true (\(g = 0.3, 0.4\) in equation (36)). Under the null and the alternative hypotheses we impose \(h = 0\) in the RF.
Table 6.3: Scenario: \((h,m)=(1,0)\) - rejection probabilities from testing \(H_0: m = 0\) vs. \(H_1: m = 1\) with bootstrap sup-\(F\) test; number of breaks in the RF was estimated and imposed in each simulation using a sequential strategy based on the WR/WF sup-\(F\) for OLS.

| Case | Size | Power | | Size | Power | | Size | Power | | Size | Power |
|------|------|-------|------|------|-------|------|------|-------|------|------|------|-------|------|------|-------|
|      |      |       |      |      |       |      |      |       |      |      |      |       |      |      |       |
| 120  | 7.6  | 3.6   | 0.7  | 11.5 | 5.8   | 0.9  | 72.6 | 67    | 56.1 | 76.4 | 70.7 | 60   | 86.9 | 81.6 | 71.5 | 88.4 |
| 240  | 7.7  | 3.2   | 0.2  | 11.3 | 5     | 0.8  | 97.5 | 96.9  | 93.8 | 99.7 | 97.4 | 94.5 | 99.5 | 99.1 | 98   | 99.7 |
| 480  | 9.7  | 5     | 0.9  | 12.4 | 5.6   | 1.3  | 100  | 99.9  | 99.9 | 100  | 100  | 100  | 100  | 100  | 100  | 100  |
| 120  | 10   | 5.8   | 1.1  | 12.7 | 6.4   | 1.3  | 71.8 | 64.9  | 53.4 | 76.4 | 68.9 | 56.3 | 85.3 | 80.1 | 69.6 | 87.6 |
| 240  | 8.8  | 3.9   | 0.8  | 10.3 | 4.7   | 0.9  | 96.7 | 95.3  | 91.1 | 97.6 | 96.1 | 92.7 | 99.1 | 98.6 | 96.8 | 99.3 |
| 480  | 9.1  | 4.7   | 1.2  | 10.6 | 6     | 1.2  | 100  | 99.9  | 99.9 | 100  | 100  | 100  | 100  | 100  | 100  | 100  |
| 120  | 8.9  | 4.4   | 0.6  | 14.4 | 7.7   | 1.8  | 61.9 | 54.4  | 40.9 | 67.2 | 60.6 | 47.2 | 75.6 | 68.8 | 58.2 | 80.5 |
| 240  | 10.1 | 5.1   | 0.8  | 14.4 | 8     | 1.5  | 93.5 | 91.5  | 86.1 | 94.2 | 93.1 | 89.1 | 97.2 | 96.3 | 93.7 | 99.7 |
| 480  | 8.4  | 4     | 0.6  | 10.9 | 5.6   | 0.9  | 99.8 | 99.6  | 99.4 | 99.9 | 99.8 | 99.5 | 100  | 100  | 100  | 100  |
| 120  | 9.3  | 4.4   | 0.6  | 13.1 | 6.1   | 1.6  | 63.9 | 56.1  | 42.5 | 68.2 | 60.6 | 45.8 | 78.1 | 72.8 | 61.1 | 80.6 |
| 240  | 9.3  | 4.2   | 0.5  | 12.5 | 6.1   | 1.5  | 94   | 92.5  | 88.7 | 94.7 | 93   | 89.8 | 97.9 | 96.9 | 94.3 | 98.2 |
| 480  | 8.5  | 4     | 0.8  | 11.6 | 5.4   | 0.8  | 99.8 | 99.5  | 99.3 | 99.8 | 99.5 | 99.4 | 100  | 100  | 100  | 100  |

Notes. The first two columns refer to the case when \(H_0: m = 0\) is true (\(g=0\) in equation (36)). The next columns refer to the case when we test for \(H_0: m = 0\), but \(H_1: m = 1\) is true (\(g = -0.05, -0.07\) in equation (36)). Prior to testing \(H_0: m = 0\) vs \(H_1: m = 1\) (for all columns above), we tested sequentially for the presence of maximum two breaks in the RF (we used the WR/WF bootstrap sup-\(F\) for OLS to test \(H_0: h = \ell\) vs. \(H_1: \ell + 1, \ell = 0, 1\)). If breaks are detected in RF, the number of breaks and the estimated location are imposed when estimating the SE.
Table 6.4: Scenario: \((h,m) = (1,1)\) - rejection probabilities from testing \(H_0 : m = 1\ vs.\ H_1 : m = 2\) with bootstrap sup-\(F\) test; number of breaks in the RF was estimated and imposed in each simulation using a sequential strategy based on the WR/WF sup-\(F\) for OLS.

| Case | WR bootstrap Size | WF bootstrap Size | WR bootstrap Power \(g = 0.5\) | WF bootstrap Power \(g = 0.5\) | WR bootstrap Power \(g = -0.5\) | WF bootstrap Power \(g = -0.05\) |
|------|------------------|------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| A    |                  |                  |                         |                         |                         |                         |
| T 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% |
| 120 | 14.4 6 0.7 | 14.3 8.6 1.8 | 98.2 98.2 97.7 | 98.4 98.2 98 | 98.9 98.9 98.6 | 99 98.8 98.5 |
| 240 | 9.8 4.8 0.7 | 12.2 6.1 1.5 | 98.7 98.7 98.6 | 98.8 98.8 98.7 | 98.9 98.9 98.6 | 99 98.9 98.6 |
| 480 | 8.2 4.5 0.9 | 9.5 5.4 1.1 | 100 100 99.9 | 100 100 99.8 | 99.9 99.9 99.9 | 99.9 99.9 99.9 |
| B    |                  |                  |                         |                         |                         |                         |
| T 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% |
| 120 | 14.4 5.4 1.4 | 14.4 7.2 1.9 | 96.7 95.8 93.6 | 96.9 96.4 94.8 | 97.4 96.8 94.9 | 97.5 97 95.9 |
| 240 | 9.7 5.1 1.1 | 11.3 6.1 1.2 | 98.2 98.1 97.2 | 98 97.9 97.2 | 98.4 98.3 97.3 | 98.3 98 96.9 |
| 480 | 9.7 5 1.1 | 10.8 5.6 0.9 | 98.7 98.7 98.5 | 98.6 98.6 98.3 | 99.2 99.2 99.2 | 98.8 98.7 98.7 |
| C    |                  |                  |                         |                         |                         |                         |
| T 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% |
| 120 | 9.8 3.7 0.4 | 14.5 7.6 1 | 96.3 94.1 86 | 97.2 95.7 89.8 | 93.1 90.4 78.3 | 94.3 91.8 80.2 |
| 240 | 10.6 5.2 0.13 | 14.2 8.4 1.6 | 97.4 97.4 97.3 | 97.5 97.5 97.4 | 97.1 97.1 97 | 96.7 96.7 96.6 |
| 480 | 10.1 5 0.8 | 12.4 7.5 1.2 | 98.5 98.5 97.9 | 98.6 98.6 98.1 | 97.1 97.1 97 | 97.1 97 96.8 |
| D    |                  |                  |                         |                         |                         |                         |
| T 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% | 10% 5% 1% |
| 120 | 8.6 4.6 0.8 | 13 7.5 1.3 | 98.3 97.8 96.4 | 98.2 97.9 96.8 | 98.5 98.1 96.6 | 99 98.4 96.9 |
| 240 | 10.2 4.6 1.2 | 14.1 7.1 1.4 | 99 98.9 98.6 | 99 98.9 98.5 | 98.5 98.5 98.3 | 98.6 98.6 98.4 |
| 480 | 9.9 5 0.9 | 12.4 6.9 1.6 | 99.7 99.6 99.4 | 99.7 99.6 99.5 | 99.6 99.1 98.8 | 99.5 99.2 98.7 |

Notes. The first two columns refer to the case when \(H_0 : m = 1\) is true \((g = 0\) in equation (36)). The next columns refer to the case when we test for \(H_0 : m = 1\), but \(H_1 : m = 2\) is true \((g = -0.5, 0.5\) in equation (36)). Prior to testing \(H_0 : m = 1\ vs.\ H_1 : m = 2\) (for all columns above), we tested sequentially for the presence of maximum two breaks in the RF (we used the WR/WF bootstrap sup-\(F\) for OLS to test \(H_0 : h = \ell\ vs.\ H_1 : \ell + 1, \ell = 0, 1\)). If breaks are detected in RF, the number of breaks and the estimated location are imposed when estimating the SE.
Table 6.5: Scenario: \((h,m)=(1,0)\) - rejection probabilities from testing \(H_0 : m = 0\) vs. \(H_1 : m = 1\) with bootstrap sup-Wald and sup-F tests; the number of breaks in the RF is held at the true number \((h = 1)\); \(H_0 : m = 0\) is true.

|      | WR bootstrap sup-Wald | WF bootstrap sup-Wald | WR bootstrap sup-F | WF bootstrap sup-F |
|------|-----------------------|-----------------------|--------------------|--------------------|
|      | 10%  5%  1%           | 10%  5%  1%           | 10%  5%  1%        | 10%  5%  1%        |
| \(T\) |                       |                       |                    |                    |
| 120  | 11.0  4.3  0.9        | 14.8  7.0  1.3        | 8.0  4.2  0.9      | 12.1  5.9  1.1     |
| 240  | 10.3  5.5  1.0        | 13.1  5.8  0.9        | 8.6  4.1  0.4      | 11.6  5.9  0.9     |
| 480  | 10.5  4.8  0.5        | 12.3  6.2  1.0        | 10.1  5.3  0.9     | 12.5  6.2  1.2     |
|      |                       |                       |                    |                    |
| \(Case A\) |                       |                       |                    |                    |
| 120  | 9.4  5.2  0.8         | 13.2  7.0  1.6        | 11.4  5.5  1.3     | 11.4  5.5  1.3     |
| 240  | 9.6  4.7  1.3         | 12.2  6.3  1.4        | 9.8  4.7  1.1      | 9.8  4.7  1.1      |
| 480  | 11.5  4.9  0.4        | 12.9  6.4  1.1        | 9.9  5.3  1.5      | 9.9  5.3  1.5      |
|      |                       |                       |                    |                    |
| \(Case B\) |                       |                       |                    |                    |
| 120  | 10.2  4.4  1.0        | 14.8  7.0  1.8        | 10.2  5.0  0.8     | 12.0  5.0  0.8     |
| 240  | 10.8  5.0  0.4        | 15.4  7.4  1.6        | 10.8  5.3  0.6     | 10.8  5.3  0.6     |
| 480  | 9.7  5.6  1.0         | 11.8  6.1  1.3        | 9.1  4.9  1.0      | 9.1  4.9  1.0      |
|      |                       |                       |                    |                    |
| \(Case C\) |                       |                       |                    |                    |
| 120  | 10.6  5.2  1.2        | 14.1  7.0  1.5        | 9.6  4.8  1.0      | 9.6  4.8  1.0      |
| 240  | 10.3  5.9  1.1        | 14.7  7.5  2.0        | 9.7  4.7  0.5      | 9.7  4.7  0.5      |
| 480  | 10.3  5.6  0.8        | 13.3  6.3  1.5        | 9.8  4.4  0.7      | 9.8  4.4  0.7      |

Notes: For both sup-F and sup-Wald bootstrap tests the number of breaks in the RF is held at the true number \((h = 1)\), we estimated the location of the RF break and imposed it when the SE was estimated.
Table 6.6: Scenario: \((h,m)=(1,1)\) - rejection probabilities from testing \(H_0: m = 1\) vs. \(H_1: m = 2\) with bootstrap sup-Wald and sup-\(F\) tests; the number of breaks in the RF is held at the true number \((h = 1)\) \(H_0: m = 1\) is true.

|          | WR bootstrap sup-Wald | WR bootstrap sup-Wald | WR bootstrap sup-\(F\) | WR bootstrap sup-\(F\) |
|----------|-----------------------|-----------------------|------------------------|------------------------|
|          | 10% 5% 1%              | 10% 5% 1%              | 10% 5% 1%              | 10% 5% 1%              |
| **Case A** |                       |                       |                        |                        |
|   120    |  9.3  5.2  0.7        |  8.8  4.5  0.8        |  10.9  5.8  1.1        |  14.8  8.8  2.1        |
|   240    | 10.4  5.6  0.7        | 10.2  5.2  0.9        | 10.5  5.6  1.1        | 12.7  7.6  1.5        |
|   480    |  9.6  4.3  0.7        |  10  4.6  0.8        |  8.7  4.8  1.1        |  10.3  5.6  1.4        |
| **Case B** |                       |                       |                        |                        |
|   120    |  9.4  4.1  0.9        |  8.6  3.2  0.8        | 12.1  5.7  1.5        | 15.1  8  2.4        |
|   240    | 10.4  4.6  0.8        | 10.3  5.2  0.9        | 10.6  6  1.3        | 12.7  6.9  2        |
|   480    | 10.3  4.2  0.8        | 11  5.3  0.7        | 10.4  5.4  1.2        | 11.6  6  1.2        |
| **Case C** |                       |                       |                        |                        |
|   120    |  9.8  3.9  1.1        |  9.4  4  0.3        | 10  4.7  0.6        | 15.1  8  1.6        |
|   240    | 10  5.1  1.2        | 10.2  5  1        | 10.5  5  0.8        | 15.1  8.6  2.7        |
|   480    | 11.2  4.7  1        | 12.1  5.3  0.6        | 10.3  5.9  1        | 12.6  8  1.5        |
| **Case D** |                       |                       |                        |                        |
|   120    | 10.1  4.4  1.6        |  8.5  3.8  0.6        |  9.7  4.8  1.1        | 13.5  7.5  1.7        |
|   240    | 11  5.1  0.9        | 11.8  5.1  0.6        | 11.2  4.5  1.3        | 14.8  8.2  1.6        |
|   480    |  9.7  5.1  1.4        | 10.7  5.2  1.2        | 11.4  5.7  1.2        | 12.9  7.3  1.8        |

**Notes.** For both sup-\(F\) and sup-Wald bootstrap tests the number of breaks in the RF is held at the true number \((h = 1)\), we estimated the location of the RF break and imposed it when the SE was estimated.
Table 6.7: Percentage of times (out of 1,000 replications) when 0, 1 and 2 breaks in RF were detected with sup-Wald and sup-F bootstrap tests before testing the true null hypothesis $H_0 : m = 0$ vs. $H_1 : m = 1$ in Scenario:$(h,m)=(1,0)$.

|          | WR sup-Wald | WF sup-Wald | WR sup-F | WF sup-F |
|----------|-------------|-------------|----------|----------|
|          | % of RF breaks | % of RF breaks | % of RF breaks | % of RF breaks |
| $T$      | 0 1 2       | 0 1 2       | 0 1 2    | 0 1 2  |
| 120      | 0 94.6 5.4 0 98.7 1.3 0 93.3 6.7 0 92.6 7.4 |
| 240      | 0 95.8 4.2 0 95.6 4.4 0 93.6 6.4 0 93.3 6.7 |
| 480      | 0 94.4 5.6 0 93.6 6.4 0 95.1 4.9 0 94.7 5.3 |

Case A

|          | WR sup-Wald | WF sup-Wald | WR sup-F | WF sup-F |
|----------|-------------|-------------|----------|----------|
|          | % of RF breaks | % of RF breaks | % of RF breaks | % of RF breaks |
| 120      | 0.3 93.7 6 0.5 98.1 1.4 0 91.1 8.9 0 91.1 8.9 |
| 240      | 0 95.9 4.1 0 95.5 4.5 0 90.9 9.1 0 90.3 9.7 |
| 480      | 0 94.8 5.2 0 94.8 5.2 0 93.3 6.7 0 93.3 6.7 |

Case B

|          | WR sup-Wald | WF sup-Wald | WR sup-F | WF sup-F |
|----------|-------------|-------------|----------|----------|
|          | % of RF breaks | % of RF breaks | % of RF breaks | % of RF breaks |
| 120      | 0 94 6 0.2 98.4 1.4 0 93.5 6.5 0 93 7 |
| 240      | 0 93.8 6.2 0 95.9 4.1 0 93.5 6.5 0 92.7 7.3 |
| 480      | 0 93.6 6.4 0 93.2 6.8 0 93.9 6.1 0 93.5 6.5 |

Case C

|          | WR sup-Wald | WF sup-Wald | WR sup-F | WF sup-F |
|----------|-------------|-------------|----------|----------|
|          | % of RF breaks | % of RF breaks | % of RF breaks | % of RF breaks |
| 120      | 0 94.6 5.4 0.2 97.4 2.4 0 94.3 5.7 0 94.1 5.9 |
| 240      | 0 95.8 4.2 0 95.8 4.2 0 94.3 5.7 0 94.1 5.9 |
| 480      | 0 93.5 6.5 0 91.5 8.5 0 94.2 5.8 0 94.1 5.9 |

Case D

Notes. In RF we tested $H_0 : h = \ell$ vs $H_1 : h = \ell + 1$, $\ell = 0,1$ with the WR and WF bootstrap sup-Wald and sup-F tests. A(n) (additional) break was detected if the WR or WF bootstrap $p$-value was smaller than 5%.
Table 6.8: Percentage of times (out of 1,000 replications) when 0, 1 and 2 breaks in RF were detected with sup-Wald and sup-F bootstrap tests before testing the true null hypothesis $H_0 : m = 1$ vs. $H_1 : m = 2$ in Scenario $(h,m) = (1,1)$.

|        | WR sup-Wald  | WF sup-Wald  | WR sup-F    | WF sup-F    |
|--------|--------------|--------------|-------------|-------------|
|        | % of RF breaks | % of RF breaks | % of RF breaks | % of RF breaks |
| $T$    | 0 1 2        | 0 1 2        | 0 1 2       | 0 1 2       |
| 120    | 0 94.8 5.2   | 0 98.6 1.4   | 0 94.5 5.5  | 0 93.9 6.1  |
| 240    | 0 95.9 4.1   | 0 95.8 4.2   | 0 94.4 5.6  | 0 93.9 6.1  |
| 480    | 0 96.3 3.7   | 0 96.2 3.8   | 0 94.7 5.3  | 0 94.6 5.4  |
|        |              |              | Case A      |              |
| 120    | 0.5 93.6 5.9 | 0.8 98 1.2   | 0 92.8 7.2  | 0.01 92.89 7.1 |
| 240    | 0.2 95.6 4.2 | 0.2 95 4.8   | 0 92 8     | 0 91.9 8.1  |
| 480    | 0 96.2 3.8   | 0 96.1 3.9   | 0 93.4 6.6  | 0 93.2 6.8  |
|        |              |              | Case B      |              |
| 120    | 0.1 94.2 5.7 | 0.4 98 1.6   | 0 93.4 6.6  | 0 93.6 6.4  |
| 240    | 0 93.7 6.3   | 0 95.8 4.2   | 0 94.2 5.8  | 0 93.8 6.2  |
| 480    | 0 94.1 5.9   | 0 94.6 5.4   | 0 94.4 5.6  | 0 94.2 5.8  |
|        |              |              | Case C      |              |
| 120    | 0 94.4 5.6   | 0.2 97.8 2   | 0 94.1 5.9  | 0 94.2 5.8  |
| 240    | 0 95.7 4.3   | 0 96 4      | 0 94.7 5.3  | 0 94.9 5.1  |
| 480    | 0 95 5       | 0 93.6 6.4   | 0 93.8 6.2  | 0 94.1 5.9  |

Notes. In RF we tested $H_0 : h = \ell$ vs $H_1 : h = \ell + 1, \ell = 0,1$ with the WR and WF bootstrap sup-Wald and sup-F tests. A(n) (additional) break was detected if the WR or WF bootstrap p-value was smaller than 5%.
Table 6.9: Scenario: \((h,m)=(1,0)\) - rejection probabilities from testing \(H_0: m = 0\) vs. \(H_1: m = 1\) with bootstrap sup-Wald test (size of break in RF is smaller than in Table 3 of the paper)

|       | WR bootstrap |     | WF bootstrap |     | WR bootstrap |     | WF bootstrap |     | WR bootstrap breaks detected |     | WF bootstrap breaks detected |
|-------|--------------|-----|--------------|-----|--------------|-----|--------------|-----|-------------------------------|-----|-----------------------------|
|       | RF tested    | 10%| 5% | 1%         | 10%| 5% | 1%         | 10%| 5% | 1%   | 0 | 1 | 2 | 0 | 1 | 2 |
| **T** |              |    |    |            |    |    |            |    |    |      |   |   |   |   |   |   |
| 120   | 9.9          | 4.9| 1.1 | 15.5 | 8.8 | 2.2 | 10.3 | 5.7 | 1   | 15.6 | 7.6 | 2.3 | 5  | 87.2 | 7.8 | 5.5 | 92.4 | 2.1 |
| 240   | 9.5          | 4.6| 0.8 | 13.4 | 6.5 | 1.2 | 9.6  | 4.5 | 0.8 | 12.8 | 7.5 | 1.2 | 0  | 93.2 | 6.8 | 0  | 96.3 | 3.7 |
| 480   | 9.3          | 4.5| 0.8 | 11.6 | 5.7 | 1   | 9.5  | 4.7 | 0.8 | 11.5 | 5.6 | 1.1 | 0  | 95.4 | 4.6 | 0  | 95.7 | 4.3 |
| **Case A** |              |    |    |            |    |    |            |    |    |      |   |   |   |   |   |   |
| 120   | 10.6         | 5.3| 1   | 15   | 8.1 | 2   | 11.6 | 5.9 | 1.1 | 14.6 | 8.3 | 1.9 | 6.4 | 85.3 | 8.3 | 6.8 | 91.5 | 1.7 |
| 240   | 9.7          | 4.8| 0.9 | 11.9 | 6.2 | 1.3 | 9.5  | 4.6 | 1   | 12  | 6.1 | 1.4 | 0.1 | 93  | 6.9 | 0.1 | 95.6 | 4.3 |
| 480   | 9.8          | 5.2| 0.9 | 11.3 | 6.4 | 1.3 | 9.9  | 4.9 | 0.5 | 11.3 | 5.9 | 1.4 | 0  | 95  | 5  | 0  | 94.8 | 5.2 |
| **Case B** |              |    |    |            |    |    |            |    |    |      |   |   |   |   |   |   |
| 120   | 9.7          | 4.8| 1   | 15.6 | 7.8 | 1.7 | 9.5  | 4.7 | 1   | 14.2 | 7.9 | 1.4 | 8  | 84.9 | 7.1 | 9.1 | 88.9 | 2  |
| 240   | 9.9          | 5.4| 0.7 | 15.5 | 10  | 2.5 | 10.7 | 5.5 | 1.2 | 15  | 9.7 | 1.9 | 0  | 94.8 | 5.2 | 0  | 96.3 | 3.7 |
| 480   | 9.4          | 4.9| 1.5 | 11.9 | 6.5 | 2   | 9.8  | 4.4 | 1.2 | 12.5 | 6.4 | 2   | 0  | 92.8 | 7.2 | 0  | 92.3 | 7.7 |
| **Case C** |              |    |    |            |    |    |            |    |    |      |   |   |   |   |   |   |
| 120   | 7.6          | 3.6| 0.7 | 11.3 | 6.2 | 1.4 | 7.6  | 3.7 | 0.4 | 11.6 | 5  | 1.3 | 8.1 | 86.3 | 5.6 | 8.9 | 89.1 | 2  |
| 240   | 8.9          | 4.1| 0.8 | 13.2 | 7.5 | 1.9 | 9.5  | 4.4 | 0.8 | 12.9 | 7.7 | 1.3 | 0  | 93.9 | 6.1 | 0  | 95.6 | 4.4 |
| 480   | 10.3         | 5.5| 1.1 | 13.2 | 7.4 | 1.2 | 10.1 | 5.7 | 1.1 | 13.4 | 7.1 | 1.7 | 0  | 92.9 | 7.1 | 0  | 93.3 | 6.7 |

Notes. All the columns refer to the case when \(H_0: m = 0\) is true. The first two columns correspond to the case when we tested sequentially for a maximum of 2 breaks (using the WR/WF bootstrap sup-Wald for OLS to test \(H_0: h = \ell\) vs \(H_1: h = \ell + 1, \ell = 0, 1\)). The resulting number of RF breaks and their estimated location was imposed in each simulation prior to estimating RF and SE and testing \(H_0: m = 0\). The next two columns refer to the case when under the null and the alternative hypotheses the number of breaks in RF is held at the true number (\(h = 1\)) and the estimated location of the RF break in imposed when the SE was estimated. The last two columns give the percentage of times that the bootstrap tests detected and imposed 0, 1 or 2 breaks when SE was estimated.
Table 6.10: Scenario: $(h,m)=(1,1)$ - rejection probabilities from testing $H_0: m = 1$ vs. $H_1: m = 2$ with bootstrap sup-Wald test (size of break in RF is smaller than in Table 4 of the paper)

|       | WR bootstrap |   | WF bootstrap |   | WR bootstrap |   | WF bootstrap |   | WR bootstrap breaks detected |   | WF bootstrap breaks detected |
|-------|--------------|---|--------------|---|--------------|---|--------------|---|-------------------------------|---|-------------------------------|
|       | T  | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% | 0 | 1 | 2 | 0 | 1 | 2 |
| Case A 120 | 9.4 | 5 | 0.8 | 9.8 | 5.2 | 1 | 9.7 | 4.9 | 0.8 | 9.3 | 4.4 | 0.7 | 7.5 | 85 | 7.5 | 8.2 | 89.5 | 2.3 |
| 240 | 9.1 | 5.3 | 0.8 | 10.5 | 4.7 | 0.6 | 9.7 | 5.5 | 0.8 | 10.3 | 4.6 | 0.5 | 0.5 | 92.6 | 6.9 | 0.7 | 95.9 | 3.4 |
| 480 | 9.7 | 5.5 | 1.1 | 10.8 | 5.6 | 1 | 9.6 | 5.3 | 1 | 10.9 | 5.4 | 0.9 | 0.5 | 95.7 | 3.8 | 0.5 | 96 | 3.5 |
| Case B 120 | 8.3 | 3.7 | 0.6 | 8.1 | 3.6 | 0.9 | 8.4 | 4 | 0.7 | 7.7 | 3.2 | 0.6 | 7.7 | 84 | 8.3 | 7.8 | 90 | 2.2 |
| 240 | 10.1 | 5 | 0.9 | 10.6 | 5.2 | 1.1 | 10.1 | 5.2 | 1 | 10.5 | 5.3 | 1 | 1.3 | 91.5 | 7.2 | 1.5 | 94.4 | 4.1 |
| 480 | 9.5 | 5.2 | 1.1 | 10.6 | 5.7 | 1.5 | 9.1 | 4.9 | 0.9 | 10.3 | 5.4 | 1.2 | 0.8 | 94.5 | 4.7 | 1 | 94.8 | 4.2 |
| Case C 120 | 8.7 | 4.3 | 0.8 | 9.3 | 4.4 | 0.4 | 9.3 | 4.3 | 0.9 | 9 | 4.1 | 0.2 | 8.9 | 84.8 | 6.3 | 10.6 | 87.5 | 1.9 |
| 240 | 10.2 | 5.3 | 1.1 | 10.9 | 4.9 | 0.9 | 10.3 | 5.5 | 1.2 | 10.7 | 5 | 0.9 | 1.2 | 93.1 | 5.7 | 1.5 | 94.9 | 3.6 |
| 480 | 10.6 | 5.4 | 1 | 12.2 | 6 | 1 | 10.4 | 5.2 | 1.1 | 11.9 | 5.6 | 0.8 | 0.4 | 93.6 | 6 | 0.5 | 93.9 | 5.6 |
| Case D 120 | 9.8 | 4.4 | 0.5 | 9.3 | 4.2 | 0.5 | 10 | 4.2 | 0.5 | 8.6 | 3.5 | 0.1 | 9 | 84.2 | 6.8 | 10.3 | 87.4 | 2.3 |
| 240 | 9.8 | 5.4 | 0.8 | 11.1 | 4.5 | 0.4 | 9.9 | 5.5 | 0.8 | 11.1 | 4.5 | 0.4 | 0.4 | 93.7 | 5.9 | 0.5 | 95.1 | 4.4 |
| 480 | 9.7 | 4.9 | 1.4 | 10.6 | 5.2 | 1.4 | 9.8 | 4.7 | 1.5 | 10.6 | 5.1 | 1.3 | 0.1 | 94 | 5.9 | 0.2 | 94.2 | 5.6 |

Notes. All the columns refer to the case when $H_0: m = 1$ is true. The first two columns correspond to the case when we tested sequentially for a maximum of 2 breaks (using the WR/WF bootstrap sup-Wald for OLS to test $H_0: h = \ell$ vs $H_1: h = \ell + 1, \ell = 0, 1$). The resulting number of RF breaks and their estimated location was imposed in each simulation prior to estimating RF and SE and testing $H_0: m = 1$. The next two columns refer to the case when under the null and the alternative hypotheses the number of breaks in RF is held at the true number ($h = 1$) and the estimated location of the RF break in imposed when the SE was estimated. The last two columns give the percentage of times that the bootstrap tests detected and imposed 0, 1 or 2 breaks when SE was estimated.