Computing quadratic function fields with high 3-rank via cubic field tabulation

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Abstract
We present recent results on the computation of quadratic function fields with high 3-rank. Using a generalization of a method of Belabas on cubic field tabulation and a theorem of Hasse, we compute quadratic function fields with 3-rank \( \geq 1 \), of imaginary or unusual discriminant \( D \), for a fixed \(|D| = q^{\deg(D)}\). We present numerical data for quadratic function fields over \( \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \) and \( \mathbb{F}_{13} \) with \( \deg(D) \leq 11 \). Our algorithm produces quadratic function fields of minimal genus for any given 3-rank. Our numerical data mostly agrees with the Friedman-Washington heuristics for quadratic function fields over the finite field \( \mathbb{F}_q \) where \( q \equiv -1 \pmod{3} \). The data for quadratic function fields over the finite field \( \mathbb{F}_q \) where \( q \equiv 1 \pmod{3} \) does not agree closely with Friedman-Washington, but does agree more closely with some recent conjectures of Malle.

1 Introduction
Let \( D \) be a square-free non-constant polynomial in \( \mathbb{F}_q[t] \) and \( \text{Cl}(D) \) the ideal class group of the quadratic function field \( \mathbb{F}_q(t, \sqrt{D}) \). For any prime \( p \), the number
$r_p(D)$, which denotes the number of cyclic factors in the $p$-Sylow subgroup of $\text{Cl}(D)$, is called the $p$-rank of the group $\text{Cl}(D)$. In short, we say that the quadratic function field $\mathbb{F}_q(t, \sqrt{D})$ has $p$-rank $r$ if $\text{Cl}(D)$ has $p$-rank equal to $r$.

In this paper, we describe an algorithm for finding all quadratic function fields of bounded genus with three rank greater than zero. Our algorithm is based on our previous work ([Rozenhart 08], [Rozenhart 09] and [Rozenhart 10]) for tabulating cubic function fields of bounded discriminant, and is inspired by Belabas’ algorithm [Belabas 04] for tabulating quadratic number fields of bounded discriminant with three rank greater than zero. Like Belabas’ algorithm, our algorithm makes use of a theorem of Hasse [Hasse 30] relating the 3-rank of a quadratic function field of discriminant $D$ to the number of triples of conjugate cubic function fields with discriminant $D$. This theorem is also used in [Jacobson 10], although it is used in the “reverse” direction, in the sense that information on the ideal class group of quadratic function fields is used to generate cubic function fields. Our approach is the opposite: we tabulate cubic function fields and use this data to generate information on 3-ranks of quadratic fields.

Our 3-rank algorithm, like Belabas’ [Belabas 04], is exhaustive in the sense that all quadratic function fields with positive 3-rank and fixed discriminant size $|D|$ are produced by the method. As a consequence, the resulting algorithm also produces minimum discriminant sizes for a given positive 3-rank value. This is in contrast to [Bauer 08], where in general minimum discriminant sizes are not produced for a given 3-rank value. Our goal is to obtain these minimal discriminant sizes for a given positive 3-rank value $r$, and not necessarily to produce record 3-rank values. For results along these lines, see [Bauer 08] or [Berger 10].

As our method is exhaustive, we can generate data on the distribution of 3-rank values for bounded $|D|$. Cohen and Lenstra [Cohen 84a, Cohen 84b] gave heuristics on the behavior of class groups of quadratic number fields. For example, they provided heuristic estimates for the probability that the $p$-rank of a given class group is equal to $r$ for a given prime $p$ and non-negative integer $r$. None of these heuristics are proved, but there is a large amount of numerical evidence supporting their validity, as seen, for example, in [Jacobson 06, te Riele 03], among others. These heuristics imply that the ideal class group of a quadratic number field is expected to have low $p$-rank for any prime $p$. As such, there is a large body of literature devoted to the construction of families of quadratic number fields of large $p$-rank, with 3-ranks of particular interest.

The function field analogue of the Cohen-Lenstra heuristics, the Friedman-Washington heuristics [Friedman 89], attempt to explain statistical observations about divisor class groups of quadratic function fields. Other previous results attempting to numerically verify the Friedman-Washington heuristics include computations of class groups for small genus over small base fields by Feng and Sun [Feng 90] and computations of class groups of real quadratic function fields of genus 1 over large base fields by Friesen [Friesen 00]. To the knowledge of the authors, the computational data contained herein is the most extensive since the work of Feng and Sun [Feng 90] and Friesen [Friesen 00].
We were able to generate examples of minimal genus and 3-rank as large as four for quadratic function fields over $\mathbb{F}_q$ for $q = 5, 7, 11, 13$. As expected, we did not find fields with higher 3-rank than any known examples, but we did find numerous examples of fields with 3-rank as high as four and smaller genus than any others known. In addition, the data we generated yields evidence for the validity of the Friedman-Washington heuristic for $q = 5, 11$. The data for $q = 7, 13$ does not agree closely with Friedman-Washington, but instead with some recent conjectures of Malle [Malle 08, Malle 09]. This discrepancy is due to the presence of cube roots of unity in the base field.

This paper is organized as follows. After a brief review of some preliminaries from the theory of algebraic function fields and cubic function field tabulation in Section 2, we proceed with some background material on quadratic function fields and the Friedman-Washington heuristics in Section 3. We give a brief discussion of the algorithm in Section 4. The 3-rank data generated, and a comparison to the Friedman-Washington and other conjectures are presented in Section 5. Finally, we make some concluding remarks in Section 6.

2 Function Field Preliminaries and Cubic Function Field Tabulation

For a general introduction to algebraic function fields, we refer the reader to Rosen [Rosen 02] or Stichtenoth [Stichtenoth 09]. Let $\mathbb{F}_q$ be a finite field of characteristic at least 5, and set $\mathbb{F}_q^* = \mathbb{F}_q\backslash\{0\}$. Denote by $\mathbb{F}_q[t]$ and $\mathbb{F}_q(t)$ the ring of polynomials and the field of rational functions in the variable $t$ over $\mathbb{F}_q$, respectively. For any non-zero $H \in \mathbb{F}_q[t]$ of degree $n = \deg(H)$, we let $|H| = q^n = q^{\deg(H)}$, and denote by $\text{sgn}(H)$ the leading coefficient of $H$. For $H = 0$, we set $|H| = 0$. This absolute value extends in the obvious way to $\mathbb{F}_q(t)$. Note that in contrast to the absolute value on the rationals, the absolute value on $\mathbb{F}_q(t)$ is non-Archimedean.

We now give a brief summary of the cubic function field tabulation results needed for this paper. These results can be found in [Rozenhart 09, Rozenhart 10].

2.1 Binary Forms and Reduction

A binary quadratic form over $\mathbb{F}_q[t]$ is a homogeneous quadratic polynomial in two variables with coefficients in $\mathbb{F}_q[t]$. We denote the binary quadratic form $H(x, y) = Px^2 + Qxy + Ry^2$ by $H = (P, Q, R)$. The discriminant of $H$ is the polynomial $D(H) = Q^2 - 4PR \in \mathbb{F}_q[t]$. A polynomial $F$ in $\mathbb{F}_q[t]$ is said to be imaginary if $\deg(F)$ is odd, unusual if $\deg(F)$ is even and $\text{sgn}(F)$ is a non-square in $\mathbb{F}_q^*$, and real if $\deg(F)$ is even and $\text{sgn}(F)$ is a square in $\mathbb{F}_q^*$. Correspondingly, a binary quadratic form is said to be imaginary, unusual or real according to whether its discriminant is imaginary, unusual or real.

A binary cubic form over $\mathbb{F}_q[t]$ is a homogeneous cubic polynomial in two variables with coefficients in $\mathbb{F}_q[t]$. We denote the binary cubic form $f(x, y) =$
\[ ax^3 + bx^2y + cxy^2 + dy^3 \] by \( f = (a, b, c, d) \). The discriminant of \( f = (a, b, c, d) \) is the polynomial

\[ D(f) = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 \in \mathbb{F}_q[t] . \]

We assume throughout that binary cubic forms \( f = (a, b, c, d) \) are primitive, i.e. \( \gcd(a, b, c, d) = 1 \).

Let \( F \) be a binary quadratic or cubic form over \( \mathbb{F}_q[t] \). If \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) is a \( 2 \times 2 \) matrix with entries in \( \mathbb{F}_q[t] \), then the action of \( M \) on \( F \) is defined by \( F \circ M = f(\alpha x + \beta y, \gamma x + \delta y) \). We obtain an equivalence relation from this action by restricting to matrices \( M \in GL_2(\mathbb{F}_q[t]) \), the group of \( 2 \times 2 \) matrices over \( \mathbb{F}_q[t] \) whose determinant lies in \( \mathbb{F}_q^* \). That is, two binary quadratic or cubic forms \( F \) and \( G \) over \( \mathbb{F}_q[t] \) are said to be equivalent if

\[ F(\alpha x + \beta y, \gamma x + \delta y) = G(x, y) \]

for some \( \alpha, \beta, \gamma, \delta \in \mathbb{F}_q[t] \) with \( \alpha \delta - \beta \gamma \in \mathbb{F}_q^* \). Up to some power of \( \det(M) \), equivalent binary forms have the same discriminant. Furthermore, the action of the group \( GL_2(\mathbb{F}_q[t]) \) on binary forms over \( \mathbb{F}_q[t] \) preserves irreducibility and over \( \mathbb{F}_q(t) \) and primitivity.

As in the case of integral binary cubic forms, any binary cubic form \( f = (a, b, c, d) \) over \( \mathbb{F}_q[t] \) is closely associated with its Hessian

\[ H_f(x, y) = -\frac{1}{4} \begin{vmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{vmatrix} = (P, Q, R) , \]

where \( P = b^2 - 3ac, Q = bc - 9ad, \) and \( R = c^2 - 3bd \). Note that \( H_f \) is a binary quadratic form over \( \mathbb{F}_q[t] \). The Hessian has a number of useful properties, which are easily verified by direct computation:

\[ H_{f \circ M} = (\det(M))^2 (H_f \circ M) \quad \text{for any} \quad M \in GL_2(\mathbb{F}_q[t]), \quad \text{and} \]

\[ D(H_f) = -3D(f) . \]

We now briefly summarize the reduction theory for binary quadratic and cubic forms. Fix a primitive root \( h \) of \( \mathbb{F}_q^* \). As in Artin [Artin 24], for any binary quadratic form \( f \), the discriminant \( D \) of \( f \) is endowed with the normalization \( \text{sgn}(D) = 1 \) or \( \text{sgn}(D) = h \), where 1 or \( h \) is chosen depending on whether or not \( \text{sgn}(D) \) is a square in \( \mathbb{F}_q^* \). We can impose this restriction since the discriminant of a function field is only unique up to square factors in \( \mathbb{F}_q^* \). We define the set \( S = \{ h^i : 0 \leq i \leq (q - 3)/2 \} \). Then \( a \in S \) if and only if \( -a \notin S \). In particular, note that \( S \) is non-empty, since \( 1 \in S \).

**Definition 2.1**

1. Let \( f = (P, Q, R) \) be an imaginary or unusual binary quadratic form of discriminant \( D \). Then \( f \) is reduced if
(a) $|Q| < |P|$, and either $Q = 0$ or $\text{sgn}(Q) \in S$;
(b) Either $|P| < |R|$ and $\text{sgn}(P) \in \{1, h\}$, or $|P| = |R|$ and $\text{sgn}(P) = 1$;
(c) If $|P| = |R|$ and $\text{sgn}(P) = 1$, then $f$ is lexicographically smallest among the $q + 1$ binary quadratic forms in its equivalence class satisfying conditions (a) and (b) above.

2. Let $f = (a, b, c, d)$ be a binary cubic form with imaginary or unusual Hessian $H_f = (P, Q, R)$ of discriminant $-3D$. Then $f$ is reduced if

(a) $\text{sgn}(a) \in S$, and if $Q = 0$, then $\text{sgn}(d) \in S$.
(b) $H_f$ is reduced, and in addition, if $|P| = |R|$, then $f$ is lexicographically smallest among all binary cubic forms in its equivalence class with Hessian $H_f$.

Note that there are rare occasions where two equivalent binary cubic forms can have the same reduced unusual Hessian $H_f = (P, Q, R)$; see Theorem 4.21 of [Rozenhart 09]. The proof of the following theorem can be found in [Rozenhart 09].

**Theorem 2.2**

1. Every equivalence class of imaginary or unusual binary quadratic forms contains a unique reduced representative, and there are only finitely many reduced imaginary or unusual binary quadratic forms of any given discriminant.

2. Every equivalence class of binary cubic forms with imaginary or unusual Hessian contains a unique reduced representative, and there are only finitely many reduced binary cubic forms of any given discriminant with imaginary or unusual Hessian.

### 2.2 The Davenport-Heilbronn Theorem

We now describe the Davenport-Heilbronn set $\mathcal{U}$ for function fields and state the function field version of the Davenport-Heilbronn theorem. For brevity, we let $[f]$ denote the equivalence class of any primitive binary cubic form $f$ over $\mathbb{F}_q[t]$. Fix any irreducible polynomial $p \in \mathbb{F}_q[t]$. We define $\mathcal{V}_p$ to be the set of all equivalence classes $[f]$ of binary cubic forms such that $p^2 \nmid D(f)$. In other words, if $D(f) = i^2\Delta$ where $\Delta$ is square-free, then $f \in \mathcal{V}_p$ if and only if $p \nmid i$. Hence, $f \in \bigcap_p \mathcal{V}_p$ if and only if $D(f)$ is square-free.

Now let $\mathcal{U}_p$ be the set of equivalence classes $[f]$ of binary cubic forms over $\mathbb{F}_q[t]$ such that

- either $[f] \in \mathcal{V}_p$, or
- $f(x, y) \equiv \lambda(\delta x - \gamma y)^3 \pmod{p}$ for some $\lambda, \gamma, \delta \in \mathbb{F}_q[t]/(p)$ with $\lambda$ non-zero, $x, y \in \mathbb{F}_q[t]/(p)$ not both zero, and in addition, $f(\gamma, \delta) \not\equiv 0 \pmod{p^2}$. 


Finally, we set $V = \bigcap_p V_p$ and $U = \bigcap_p U_p$; the latter is the set under consideration in the Davenport-Heilbronn theorem for function fields. For the purpose of computing 3-ranks, we are only interested in whether $[f] \in V \subset U$. This amounts to testing whether or not a polynomial in $\mathbb{F}_q[t]$ is square-free, which is straightforward. The Davenport-Heilbronn theorem is stated for completeness. A more general version of the theorem has been proved by Taniguchi [Taniguchi 06].

**Theorem 2.3** Let $q$ be a prime power with $\gcd(q, 6) = 1$. Then there exists a discriminant-preserving bijection between $\mathbb{F}_q(t)$-isomorphism classes of cubic function fields and equivalence classes of binary cubic forms over $\mathbb{F}_q[t]$ belonging to $U$.

The main tool for obtaining 3-rank data is a modified algorithm for tabulating cubic function fields, described in [Rozenhart 08], [Rozenhart 09], and [Rozenhart 10], in conjunction with a generalization of a theorem of Hasse that describes a connection between quadratic and cubic fields. Hasse's Theorem states that for a given quadratic number field of fixed square-free discriminant whose ideal class group has 3-rank $r$, there are $(3^r - 1)/2$ non-isomorphic cubic fields of the same discriminant. A function field analogue of Hasse's result appears in [Jacobson 10] (see Theorem 3.2 below) and is used in conjunction with the tabulation algorithm to obtain 3-rank information.

Other necessary tools for accomplishing our task are the Davenport-Heilbronn theorem for function fields, and the reduction theory of binary forms over $\mathbb{F}_q[t]$ just described. In order to compute the 3-rank of a quadratic function field of square-free discriminant $D$, we need only count the number of $\mathbb{F}_q(t)$-isomorphism classes of cubic function fields of that same discriminant and positive unit rank, see Theorem 3.2. In turn, to list all $\mathbb{F}_q(t)$-isomorphism classes of cubic function fields for fixed $D$, it suffices to enumerate the unique reduced representatives of all equivalence classes of binary cubic forms of discriminant $D$ for all $D \in \mathbb{F}_q[t]$. Bounds on the coefficients of such a reduced form (as given in Algorithm 4.1) show that there are only finitely many candidates for any reduced binary cubic form of a fixed discriminant (see also part 2 of Theorem 2.2). These bounds can then be employed in nested loops to test whether each form found belongs to $U$ and has square-free discriminant (lies in $V$). A more precise description of the algorithm is given in Section 4.

3 Quadratic Function Fields and 3-rank

We now give some relevant terminology regarding quadratic fields and 3-ranks, along with some discussion of the Friedman-Washington heuristics.

The Friedman-Washington heuristic is entirely analogous to the Cohen–Lenstra heuristic. Loosely speaking, Friedman and Washington predict that given a fixed finite field $\mathbb{F}_q$ and an abelian $p$-group $H$, where $p$ is an odd prime that does not divide $\text{char}(\mathbb{F}_q)$, $H$ occurs as the $p$-Sylow part of the divisor class.
group of a quadratic function field over $\mathbb{F}_q$ with frequency inversely proportional to $|\text{Aut}(H)|$. The precise statement is given below.

**Conjecture 3.1 [Friedman 89]** Let $p$ be an odd prime, where $p$ does not divide $\text{char}(\mathbb{F}_q)$, and $g_K$ the genus of the quadratic extension $K$. Then a finite abelian group $H$ of $p$-power order appears as the $p$-Sylow part $\text{Cl}_p$ of the class group of a quadratic extension $K$ of the rational function field over a finite field $\mathbb{F}_q(t)$ with a frequency inversely proportional to the number of automorphisms of $H$. That is,

$$
\lim_{g \to \infty} \left( \sum_{G \leq g} \frac{1}{\sum_{G \leq g}} \right) = |\text{Aut}(H)|^{-1} \prod_{j=1}^{\infty} (1 - p^{-j}).
$$

(1)

For fixed and small values of $q$, the heuristic is still a conjecture; Achter [Achter 06, Achter 08] proved a weaker version of this assertion where $q \to \infty$ inside $\lim_{g \to \infty}$. A newer, related result, due to Ellenberg, Venkatesh and Westerland [Ellenberg 09], is that the upper and lower densities of imaginary quadratic extensions of $\mathbb{F}_q(t)$ for which the $p$-part of the class group is isomorphic to any given finite abelian $p$-group converges to the right-hand side of equation (3.1), as $q \to \infty$ with $q \not\equiv 1 \pmod{p}$. A different probability distribution, one consistent with Malle’s conjectures [Malle 08, Malle 09], is obtained for $q \equiv 1 \pmod{p}$ due to the presence of $p$-th roots of unity (see [Garton 10]). We also discuss the presence of roots of unity in the base field in Section 5. In subsequent sections, we will compare our numerical results to the Friedman-Washington result to see how close our computations come to this heuristic.

Based on this heuristic, the probability that the $p$-rank of an ideal class group of an imaginary quadratic function field is equal to $r$, as given in Cohen and Lenstra [Cohen 84b] for number fields and in Lee [Lee 04] for function fields, is given by

$$
p^{-r^2} \eta_{\infty}(p) \prod_{k=1}^{r} (1 - p^{-k})^{-2},
$$

(2)

where $\eta_{\infty}(p) = \prod_{k \geq 1} (1 - p^{-k})$. For $p = 3$ and the values of $r = 0, 1, 2, 3$ and 4, we obtain the approximate probabilities 0.56128, 0.42009, 0.019692, 0.0008739 and 4.0964 $\times 10^{-8}$ respectively.

The key connection between quadratic and cubic function fields that allows us to adapt the tabulation algorithm to finding 3-ranks of quadratic function fields is a modified version of theorem of Hasse [Hasse 30] for the function field setting. It gives a precise formula for the number of isomorphism classes of cubic function fields for a fixed square-free discriminant $D$ in terms of the 3-rank of the quadratic field with discriminant $D$. 


Theorem 3.2 (Jacobson et al. [Jacobson 10]) If \( D \) is a square-free polynomial in \( \mathbb{F}_q[t] \) and \( K = \mathbb{F}_q(t, \sqrt{D}) \), then the number of isomorphism classes of cubic function fields of discriminant \( D \) and unit rank at least one is

\[
\frac{3^{r_3(D)} - 1}{2},
\]

where \( r_3(D) \) is the 3-rank of the ideal class group of the quadratic function field \( K \).

Theorem 3.2 gives an explicit formula relating the 3-rank of a quadratic function field of square-free discriminant \( D \) and the number of isomorphism classes of cubic function fields of the same discriminant \( D \) with unit rank at least one. In fact, a much more precise statement can be given in certain cases. First, the notion of a dual hyperelliptic function field is needed, along with definitions of escalatory and non-escalatory. Let \( n \) be any non-square in \( \mathbb{F}_q^* \).

Let \( D \in \mathbb{F}_q[t] \) be an unusual discriminant, and set \( D' = nD \), so that \( D' \) is a real discriminant. Then \( K = \mathbb{F}_q(t, y) \) with \( y^2 = D \) and \( K' = \mathbb{F}_q(t, y') \) with \( (y')^2 = D' \) are said to be dual hyperelliptic (quadratic) function fields.

Let \( l \) be an odd prime dividing \( q+1 \), \( K/\mathbb{F}_q(t) \) an unusual quadratic function field and \( K'/\mathbb{F}_q(t) \) its dual real quadratic function field. If \( r \) and \( r' \) denote the \( l \)-rank of the ideal class group of \( K/\mathbb{F}_q(t) \) and \( K'/\mathbb{F}_q(t) \), respectively, then \( r = r' \) or \( r = r' + 1 \). In the latter case, the regulator of \( K'/\mathbb{F}_q(t) \) is a multiple of \( l \) (see Lee [Lee 07]). The cases \( r = r' + 1 \) and \( r = r' \) are referred to as escalatory and non-escalatory, respectively.

Denote by \((e_1, f_1; \ldots; e_r, f_r)\) the signature of the place at infinity of \( \mathbb{F}_q(t) \) in a finite extension \( L \) of \( \mathbb{F}_q(t) \), so \( e_i \) is the ramification index and \( f_i \) the residue degree of the \( i \)-th infinite place of \( L \) for \( 1 \leq i \leq r \). The precise count of the number of cubic function fields of discriminant \( D \) and a given signature can now be stated in terms of the 3-rank of the quadratic function field of discriminant \( D \). In the cases of interest, we have an exact count of the number of isomorphism classes of cubic function fields having given signature. In the case \( q \equiv 1 \pmod{3} \), we have less information in the sense that it is unknown how many fields there are with only one infinite place. The complete statement and proof appears in [Jacobson 10].

Theorem 3.3 Let \( r \) be the 3-rank of the quadratic function field of discriminant \( D \). If \( D \) is imaginary, then there are exactly \( (3^r - 1)/2 \) isomorphism classes of cubic function fields of discriminant \( D \), all of which have signature \((1,1;2,1)\).

- For \( q \equiv -1 \pmod{3} \):
  - Suppose \( D \) is real.
    * In the escalatory case, \( (3^r - 1)/2 \) isomorphism classes of cubic function fields have signature \((1,1;1,1;1,1)\) and \( 3^r \) isomorphism classes of cubic function fields have signature \((1,3)\). No other signatures occur.
In the non-escalatory case, all \((3^r - 1)/2\) isomorphism classes of cubic function fields of discriminant \(D\) have signature \((1, 1; 1; 1, 1)\). No other signatures occur.

- For \(q \equiv 1 \pmod{3}\):
  - Suppose \(D\) is real. Then there are exactly \((3^r - 1)/2\) isomorphism classes of cubic function fields of discriminant \(D\) with signature \((1, 1; 1; 1, 1)\). There are also fields of discriminant \(D\) with signature \((1, 3)\) or \((3, 1)\), but there is as yet no exact formula for the number of such fields.
  - Suppose \(D\) is unusual. Then there are exactly \((3^r - 1)/2\) isomorphism classes of cubic function fields of discriminant \(D\), all of which have signature \((1, 1; 1, 2)\).

We note that no exact count by signature is known for \(q \equiv 1 \pmod{3}\) and \(D\) real.

4 The Algorithm and its Complexity

We now briefly describe the 3-rank algorithm for discriminants \(D\) such that \(-3D\) is imaginary or unusual. The basic algorithm follows from the algorithm for tabulating cubic function fields from [Rozenhart 08, Rozenhart 09, Rozenhart 10], except instead of outputting minimal polynomials \(f(x, 1)\), we simply increment a counter for each square-free discriminant found. The counter and corresponding discriminant values are then output. Since \(-3D\) is imaginary or unusual, the value of the counter for each discriminant is a number of the form \((3^r - 1)/2\), where \(r\) is the 3-rank of the quadratic function field of discriminant \(D\). Discriminants and the number of cubic fields with that discriminant are stored in a hash table, and output to a file once the main for loops are exited. The basic algorithm for generating fields with positive 3-rank can now be described. We loop over each coefficient of a binary cubic form satisfying the bounds given in [Rozenhart 09], Section 4.5. For each binary cubic form \(f\) encountered in the loop, we test whether or not \(f\) is reduced, lies in \(U\), whether \(-3D\) is imaginary or unusual, and whether \(|D| \leq X\). If this is the case, the number of forms found with discriminant \(D\) is incremented by one in our hash table. A modified version of the algorithm, with various improvements, appears in Algorithm 4.1.

In practice and in this paper, a modified version of the algorithm is used that includes various speed-ups, including “shortening” the loop on \(|d|\) by determining whether \(D\) has odd or even degree given some degree value of \(d\). These improvements led to speed-ups of up to a factor of 6 over the original algorithm in some cases. Details can be found in [Rozenhart 09, Rozenhart 10]. Some
Algorithm 4.1 3-rank algorithm for computation of quadratic fields where $-3D$ is imaginary (resp. unusual)

**Input:** A prime power $q$ not divisible by 2 or 3, a primitive root $h$ of $\mathbb{F}_q$, the set $S = \{1, h, h^2, \ldots, h^{(q-3)/2}\}$, and a positive integer $X$.

**Output:** A table where each entry is a discriminant $D$ and a number of the form $(3^r-1)/2$ with $-3D$ imaginary (resp. unusual), $\text{sgn}(-3D) \in \{1, h\}$ (resp. $\text{sgn}(-3D) = h$), and $|D| \leq X$. The positive integer $r$ is the 3-rank of the quadratic function field $\mathbb{F}_q(t, \sqrt{D})$.

1. **for** $|a| \leq X^{1/4}$ **AND** $\text{sgn}(a) \in S$ **do**
2.  **for** $|b| \leq X^{1/4}$ **do**
3.   **for** $|c| \leq X^{1/2}/|b|$ **do**
4.     **m1** := $2(\deg(b) + \deg(c))$;
5.     **m2** := $\deg(a) + 3\deg(b)$
6.   **for** $i = 0$ to $\log_q(X)/2 - \deg(a)$ **do**
7.     **m3** := $\deg(a) + \deg(b) + \deg(c) + i$;
8.     **m4** := $3\deg(b) + i$
9.     **m5** := $2(\deg(a) + i)$
10.    **m** := $\max\{m1, m2, m3, m4, m5\}$
11. **if** ($m$ is not taken on by a unique term among the $m_i$) **OR** ($m$ is taken on by a unique term AND $m$ is odd (resp. even) AND $q^m \leq X$) **then**
12.     **Compute** $P := b^2 - 3ac$;
13.     **Compute** $t1 := bc$;
14.     **Compute** $t2 := c^2$
15. **for** $|d| = q^i$ **do**
16.     **Set** $f := (a, b, c, d)$;
17.     **Compute** $Q := t1 - 9ad$;
18.     **Compute** $R := t2 - 3bd$;
19.     **Compute** $-3D = -3D(f) = Q^2 - 4PR$;
20. **if** $-3D$ is imaginary (resp. unusual) **AND** $|D| \leq X$ **AND** $f$ is reduced **AND** $[f] \in \mathcal{V}$ **then**
21.     **Increment** counter for discriminant $D$ by one in the hash table;
extra routines are necessary for the unusual case, and are the same as those required for the tabulation algorithm in the unusual case [Rozenhart 09].

The asymptotic complexity of the algorithm for generating fields with positive 3-rank is the same as for the tabulation algorithm for cubic function fields, namely $O(q^4 X^{1+\epsilon})$ field operations [Rozenhart 10] for $-3D$ imaginary and $O(q^5 X^{1+\epsilon})$ field operations for $-3D$ unusual, which is roughly linear in $X$ if $q$ is small. The main difference between the two algorithms is that instead of testing if a form lies in $U$, we test if it lies in $V$ by testing whether or not the cubic form has square-free discriminant. This requires only one gcd computation. The 3-rank program does require the storage of discriminant and 3-rank data in a hash table, with the contents of the table displayed at the end of the algorithm.

In Belabas [Belabas 04], a number of modifications to the basic algorithm for computing the 3-rank of a quadratic number field were suggested and implemented. We give a brief summary of these modifications here, and explain why we refrained from making similar changes to our program, but the cost of this is negligible compared to the rest of the algorithm.

First, we did not use Belabas’ “cluster” approach. This approach, where one loosens the conditions for a form to be reduced and looks for a large number of forms in a given interval, finally proceeding with class group computations on the clusters found, was not used as we sought to avoid a large number of direct class group computations, except for verification of a small sample of examples. This does however warrant further investigation.

The other main variants of Belabas’ 3-rank program are dedicated to speeding up the square-free test for integers. As square-free testing for polynomials is straightforward and efficient, this aspect of Belabas’ work was not explored.

5 Numerical Results for Quadratic Function Fields with $-3D$ Imaginary or Unusual

Table 1 presents the results of our computations for the 3-rank counts of quadratic function fields for $q = 5, 7, 11, 13$ for various imaginary and unusual discriminants using Algorithm 4.1. We implemented our counting algorithm using the C++ programming language coupled with the number theory library NTL [Shoup 10]. The lists of quadratic fields and their (positive) 3-rank values were computed on a 2.95 GHz Pentium 4 machine running Linux with 4 GB of RAM and 180 MB cache, with the exception of the degree 11 discriminants over $\mathbb{F}_5$ and the degree 9 discriminants over $\mathbb{F}_7$, for which a machine with 3.6 GHz Pentium 4 processor running Linux with 6 GB of RAM was used. Each table entry consists of the base field size $q$, the degree bound on the discriminant $D$ and the corresponding genus $g$, the total number of square-free discriminants of that degree, the 3-rank of $\mathbb{F}_q(t, \sqrt{D})$, the total number of fields found with that 3-rank, and the total elapsed time to find all quadratic function fields with the given degree and 3-rank at least 1.
Table 1: 3-ranks of quadratic function fields over $\mathbb{F}_q$

| $q$ | deg($D$), $g$ | # of $D$ | 3-rank | Total | Total elapsed time |
|-----|--------------|----------|--------|-------|-------------------|
| 5   | 3, 1         | 200      | 1      | 80    | 0.03 seconds      |
|     | 4, 1         | 500      | 1      | 200   | 0.88 seconds      |
|     | 5, 2         | 5000     | 1      | 1600  | 1.08 seconds      |
|     |              |          | 2      | 10    |                   |
| 6   | 2            | 12500    | 1      | 4780  | 27.91 seconds     |
|     |              |          | 2      | 100   |                   |
| 7   | 3            | 125000   | 1      | 40680 | 30.01 seconds     |
|     |              |          | 2      | 1180  |                   |
| 8   | 3            | 312500   | 1      | 115460| 23 minutes, 55.16 sec |
|     |              |          | 2      | 2205  |                   |
| 9   | 4            | 3125000  | 1      | 1297160| 22 minutes, 25.94 sec |
|     |              |          | 2      | 51300 |                   |
|     |              |          | 3      | 40    |                   |
| 10  | 4            | 7812500  | 1      | 3240440| 5 hours,          |
|     |              |          | 2      | 128160| 12 min, 0.42 sec  |
|     |              |          | 3      | 100   |                   |
| 11  | 5            | 78125000 | 1      | 31731960| 8 hours,         |
|     |              |          | 2      | 1167200| 52 min, 39.63 sec |
|     |              |          | 3      | 1880  |                   |
| 7   | 3, 1         | 588      | 1      | 196   | 0.18 seconds      |
|     |              |          | 2      | 14    |                   |
| 4, 1|              | 2058     | 1      | 672   | 17.43 seconds     |
|     |              |          | 2      | 42    |                   |
| 5   | 2            | 28812    | 1      | 8400  | 18.87 seconds     |
|     |              |          | 2      | 588   |                   |
|     |              |          | 3      | 63    |                   |
| 6   | 2            | 100842   | 1      | 31052 | 9 minutes, 12.27 sec |
|     |              |          | 2      | 3115  |                   |
|     |              |          | 3      | 63    |                   |
| 7, 3|              | 1411788  | 1      | 433188| 16 minutes, 55.07 sec |
|     |              |          | 2      | 42924 |                   |
|     |              |          | 3      | 840   |                   |
| 8, 3|              | 4941258  | 1      | 1514646| 10 hours,         |
|     |              |          | 2      | 142632| 26 min, 38.81 sec |
|     |              |          | 3      | 2310  |                   |
| 9, 4|              | 69177612 | 1      | 22087352| 23 hours,       |
|     |              |          | 2      | 2675890| 51 min, 11.96 sec |
|     |              |          | 3      | 90874 |                   |
|     |              |          | 4      | 588   |                   |
| 11  | 3, 1         | 2420     | 1      | 1100  | 2.58 seconds      |
|     |              |          | 2      | 2970  |                   |
| 5, 2|              | 292820   | 1      | 110000| 21 min, 31.46 sec |
|     |              |          | 2      | 2970  |                   |

Continued on next page
Table 1 – continued from previous page

| $q$ | deg($D$), $g$ | # of $D$ | 3-rank | Total      | Total elapsed time            |
|-----|---------------|----------|--------|------------|-------------------------------|
| 7, 3| 35431220      | 1        | 14187140 | 1 day, 2 hours, 28 minutes |
|     |               | 2        | 506220  | 1 day, 2 hours, 28 minutes |
|     |               | 3        | 660     | 1 day, 2 hours, 28 minutes |
| 13  | 3, 1          | 4056     | 1352    | 7.03 sec   |
|     |               | 2        | 130     | 7.03 sec   |
| 5, 2| 685464        | 1        | 209664  | 1 hour, 11 min, 28.69 sec   |
|     |               | 2        | 20046   | 1 hour, 11 min, 28.69 sec   |
|     |               | 3        | 312     | 1 hour, 11 min, 28.69 sec   |
| 7, 3| 115843416     | 1        | 36389184| 6 days, 1 hour, 16 minutes  |
|     |               | 2        | 4009824 | 6 days, 1 hour, 16 minutes  |
|     |               | 3        | 108108  | 6 days, 1 hour, 16 minutes  |
|     |               | 4        | 494     | 6 days, 1 hour, 16 minutes  |

We were able to produce examples of escalatory and non-escalatory cases in $F_5$; this was ascertained by computing the class groups of these particular quadratic function fields and their corresponding dual discriminants. For example, the real quadratic function field of discriminant $D = t^{10} + 2t^9 + t^8 + 4t^7 + 2t^6 + 3t^5 + 3t^4 + 4t^3 + 3t^2 + t$ has 3-rank 2, but its unusual dual field of discriminant $-3D$ has 3-rank 3. The escalatory case did not occur for the field $F_7$ in our computations, since $D$ and $-3D$ are both unusual in this case. The discovery of escalatory cases in $F_5$ would provide an easy way to obtain a higher 3-rank for free via the dual field, so the existence of such examples is important.

For each of the finite fields specified above, $h = 2$ was chosen as a primitive root for $F_q$ with the exception of $F_7$, where $h = 3$ was chosen. This completely determines the set $S$ specified previously. These sets were \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4, 5, 8\} and \{1, 2, 3, 4, 6, 8\} for $F_5$, $F_7$, $F_{11}$ and $F_{13}$ respectively.

We note that our algorithm in both the imaginary and the unusual case is particularly successful at finding quadratic function fields of high 3-rank and small genus because our method is exhaustive. This means that any examples of a given 3-rank value found by our algorithm are minimal, in the sense that any quadratic field with the same 3-rank must have genus at least as large as the examples found by our algorithm. For example, our algorithm beats the Diaz y Diaz method, ([Diaz y Diaz 78]; adapted to quadratic function fields in [Bauer 08]), in the sense that fields of 3-rank equal to 3 were found for genus 4 fields over $F_5$ using our algorithm. The minimal genus yielded by the Diaz y Diaz method in [Bauer 08] for quadratic function fields over $F_5$ with 3-rank equal to 3 is $g = 5$. For $F_{11}$ and $F_{13}$ and 3-rank values 3 and 4, we found examples of smaller genus than those given in [Bauer 08]. The minimal genus values were the same for $F_7$ and 3-rank 4 for both methods.

An explicit comparison of our method to Diaz y Diaz’s algorithm is given in Tables 2 and 3. The third column denotes the minimal genus found with Diaz y Diaz’ method yielding the corresponding 3-rank specified in column 2.
The fourth column denotes the minimal genus found with our method for the same 3-rank. The fifth column gives an example discriminant of minimal degree with given 3-rank found by our method. In Table 3, the unusual discriminant is given, rather than the dual real discriminant produced by the algorithm. These tables indicate a genuine improvement over previous methods in this regard with respect to finding minimal genera with high 3-rank.

Since our algorithm is exhaustive (as we count the number of cubic fields for a given degree and hence all possible square-free discriminants of that degree with a given positive 3-rank), we compared our data to the Friedman-Washington heuristic, in an effort to provide numerical evidence in support of the heuristic’s validity. A comparison of our 3-rank data for imaginary and unusual discriminants for the field $F_5$ to the expected value given by the Friedman-Washington results is given in Figure 1. We compared our data to equation (2) in the imaginary and unusual cases for $p = 3$. The solid line denotes the value of equation (2) for $r = 1$ and the dotted line denotes the proportion of 3-rank one values found up to a given bound for the degree found by our algorithm. As seen in Figure 1, our data mostly agrees with the probabilities predicted by Friedman and Washington. The data for $F_{11}$ (figure omitted) does not agree as closely, but this is likely because computations were not carried far enough to obtain a sufficient sample size.

We also compared our $F_7$ data to the Friedman-Washington probability. The data is a poor fit to the probabilities given by Friedman and Washington. The
data for 3-rank one fields differs from the expected Friedman-Washington probabilities by more than 50% in some cases. However, as noted in Malle [Malle 08], the Cohen-Lenstra-Martinet heuristics for \( p \)-ranks may fail when primitive \( p \)-th roots of unity lie in the base field, which is the case for \( \mathbb{F}_7 \). Achter’s result [Achter 06], while weaker than Friedman and Washington’s original conjecture, defines a function \( \alpha(g, r) \) expressing the same probability that does take roots of unity into account. In order to account for this discrepancy for number fields, Malle [Malle 09] proposed alternative conjectural formulas to cover the case where primitive \( p \)-th roots of unity lie in the base field, provided primitive \( p^2 \)-th roots of unity that are themselves not also \( p \)-th roots of unity do not lie in the base field. The modified formula for the probability that a quadratic function field has \( p \)-rank equal to \( r \) in this case is

\[
p^{-(r^2 + r)/2}(\eta_{\infty}(p)/\eta_{\infty}(p^2)) \prod_{k=1}^r (1 - p^{-k})^{-1},
\]

where again \( \eta_{\infty}(p) = \prod_{k \geq 1} (1 - p^{-k}) \). For \( p = 3 \) and the values of \( r = 0, 1, 2, 3 \) and 4, we obtain the approximate probabilities 0.64032, 0.31950, 0.03994, 1.5361 \times 10^{-3} and 1.9261 \times 10^{-5} respectively.

For \( p = 3 \), we compared our data for \( \mathbb{F}_7 \) to Malle’s formula and Achter’s \( \alpha(g, r) \) function in Figure 2. As seen in Figure 2, our data agrees more closely with the probabilities predicted by Malle. In addition, our data is a good match
to the values of Achter’s $\alpha(g, r)$ function. The value of $\alpha(g, 1)$ for $g \leq 3$ appears in [Achter 06]. Our data agrees more closely, and in one case exactly, with this formula as compared to the original Friedman-Washington heuristic. It is also interesting to note that Achter’s function $\alpha$ converges to Malle’s formula as $g \to \infty$, giving additional evidence for the validity of a version of Malle’s conjecture for function fields. This result is Proposition 3.1 of Malle [Malle 09].

6 Conclusion

This paper presented a method for finding quadratic function fields of high 3-rank of discriminant $D$ with $-3D$ imaginary or unusual. Our computations were carried out up to degree bounds $\deg(D) \leq 11$ for $q = 5$, and various smaller bounds for $q = 7, 11, 13$.

It would be interesting to modify the algorithm to the case where $-3D$ is real. As noted in [Rozenhart 09, Rozenhart 10], this is currently an open problem. In this case it is unclear how to single out efficiently a unique reduced representative in each equivalence class of binary cubic forms, and in fact there are generally exponentially many reduced forms in a given equivalence class. We also note that all our counts by 3-rank (column 5 of Table 1) are divisible by $q$, and the total discriminant counts (column 3 of Table 1) are divisible by $q^2(q - 1)$. This phenomenon is the subject of future investigation.
Computing $p$-ranks for quadratic function fields where $p \neq 3$ via a similar indirect technique would also be of interest. Unfortunately, a special connection between certain function fields and $p$-rank values of lower degree function fields, as given by Hasse in the case of cubic fields and $p = 3$ has not been explored for higher degree function fields or other values of $p$. Other techniques for computing $p$-ranks of quadratic function fields are currently being investigated.

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