Algebraic theories in homotopy theory

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1. Introduction

It is well known in homotopy theory that given a loop space $X$ one can always find a simplicial group $G$ weakly equivalent to $X$, such that the weak equivalence can be realized by maps preserving multiplication. It is also known that loop spaces are not the only class of spaces for which a statement of this kind holds; for example, any $A_\infty$-space is weakly equivalent to a simplicial monoid and every Eilenberg-Mac Lane space $K(G, n)$ with $G$ an abelian group is equivalent to a simplicial abelian monoid. Results like this suggest that there might be some general principle comparing homotopy structures on a space to algebraic structures. Our aim in this paper is to show that there is, in fact, such a principle. To make this precise we need a few definitions.

Definition 1.1. An algebraic theory $\mathbf{T}$ is a small category with objects $T_0, T_1, \ldots$ together with, for each $n$, an expression of $T_n$ as the categorical product in $\mathbf{T}$ of $n$ copies of the object $T_1$. In particular $T_0$ is the terminal object in $\mathbf{T}$. We assume that it is also the initial object.

Given an algebraic theory $\mathbf{T}$, a strict $\mathbf{T}$-algebra $A$ is a product-preserving functor $A: \mathbf{T} \to \text{Spaces}$.

We will denote by $\text{Alg}_\mathbf{T}$ the category of all strict $\mathbf{T}$-algebras with natural transformations of functors as morphisms. A strict $\mathbf{T}$-algebra structure on a space $Y$ is a strict $\mathbf{T}$-algebra $A$ together with an isomorphism $Y \cong A(T_1)$.

Algebraic theories appear naturally in the study of algebraic structures. For example, let $\text{Gr}$ be the category of groups and for $n \geq 0$ let $F_n$ denote the free group generated by the set $\{1, \ldots, n\}$ ($F_0$ is the trivial group). Define $\text{T}_{\text{Gr}}^{\text{op}}$ to be the full subcategory of $\text{Gr}$ with objects $F_0, F_1, \ldots$. Its opposite category $\text{T}_{\text{Gr}}$ is then an algebraic theory. To see this, observe that $n$ inclusions $\{1\} \hookrightarrow \{1, \ldots, n\}$ induce inclusions of groups $F_1 \to F_n$ which express $F_n$ as a coproduct in $\text{T}_{\text{Gr}}^{\text{op}}$ of $n$ copies of $F_1$; it follows that in the opposite category $\text{T}_{\text{Gr}}$ the object $F_n$ is the product of $n$ copies of $F_1$. Suppose that $G$ is an
arbitrary group. We can define a functor

\[ A_G : \mathcal{T}_{\text{Gr}} \to \text{Spaces}, \quad F_n \mapsto \text{Hom}_{\text{Gr}}(F_n, G). \]

It is clear that \( A_G \) is product-preserving, and so \( A_G \) is a strict \( \mathcal{T}_{\text{Gr}} \)-algebra. One can check that the converse is also true: any strict \( \mathcal{T}_{\text{Gr}} \)-algebra \( A \) defines a group structure on the space \( A(F_1) \). This is not surprising, since (by Yoneda’s lemma) the maps \( F_n \to F_1 \) in \( \mathcal{T}_{\text{Gr}} \) correspond exactly to all of the ways of taking \( n \) elements in a group and combining them with the available operations to obtain a single element of the group. The composition in \( \mathcal{T}_{\text{Gr}} \) gives identities between composites of these multivariable operations. A set which possesses such operations satisfying the appropriate identities is exactly a group.

Lawvere [12] showed that strict algebras can be used in this way to describe a wide class of algebraic structures, including, besides groups, monoids, nilpotent and solvable groups of any fixed class, rings, Lie algebras etc. As the example above suggests, the existence of free objects is essential in order to get such a description.

The language of algebraic theories proved to be equally convenient for describing various homotopy invariant structures on spaces. However, in order to allow for homotopy input one needs to relax the definition of a strict algebra. Suppose that \( \mathcal{T} \) is an algebraic theory with objects \( T_n, n \geq 0 \). The expression of \( T_n \) as a product of \( n \) copies of \( T_1 \) gives projection maps

\[ p_k^n : T_n \to T_1, \quad 1 \leq k \leq n. \]

**Definition 1.2.** Suppose that \( \mathcal{T} \) is an algebraic theory. A functor \( X : \mathcal{T} \to \text{Spaces} \) is said to be a homotopy \( \mathcal{T} \)-algebra if \( X \) preserves products up to weak equivalence, i.e., if \( X(T_0) \) is weakly contractible and for each \( n \geq 1 \) the product map

\[ \prod_{k=1}^{n} X(p_k^n) : X(T_n) \to X(T_1)^n \]

is a weak equivalence.

A homotopy \( \mathcal{T} \)-algebra structure on a space \( Y \) is a homotopy \( \mathcal{T} \)-algebra \( X \) together with a weak equivalence \( X(1) \simeq Y \). We can now state our main result which implies that it is always possible to pass from a homotopy \( \mathcal{T} \)-algebra structure on a space \( Y \) to a strict \( \mathcal{T} \)-algebra structure on a space weakly equivalent to \( Y \).

**Theorem 1.3.** Let \( \mathcal{T} \) be an algebraic theory. For any homotopy \( \mathcal{T} \)-algebra \( X \) there exists a weak equivalence \( X \simeq LX \) such that \( LX \) is a strict \( \mathcal{T} \)-algebra.
We will actually prove a somewhat stronger statement (6.4) expressing the relationship of homotopy and strict $T$-algebras as a Quillen equivalence of model categories. In particular, the weak equivalence in the theorem above respects the homotopy $T$-algebra structures on both objects involved.

Theorem 1.3 gives a rigidifying result for homotopy algebras, but the following corollary shows that it is also of consequence for strict algebras.

**Corollary 1.4.** Let $F: \text{Spaces} \to \text{Spaces}$ be a functor preserving weak equivalences and preserving products up to weak equivalence. If $Y$ is a space with a strict $T$-algebra structure for some algebraic theory $T$ then $F(Y)$ is weakly equivalent to a space with a strict $T$-algebra structure.

Indeed, the assumptions on the functor $F$ imply that for any strict $T$-algebra $A$ the composition $F \circ A: T \to \text{Spaces}$ is a homotopy $T$-algebra. Therefore, the statement follows immediately from Theorem 1.3.

Examples of functors for which Corollary 1.4 holds include localization functors [6] and Bousfield-Kan completion functors [5].

**Note 1.5.** Although we define an algebraic theory $T$ as a discrete category (1.1) all statements of this paper remain valid also if we assume that $T$ is a simplicial category (and thus strict, and homotopy $T$-algebras are simplicial functors). The proofs in this case require at most minor changes.

**Relationship to previous results.** The notion of a homotopy algebra is inspired by $\Gamma$-spaces of Segal [13]. We note however that the indexing category $\Gamma^{\text{op}}$ which Segal uses is not an algebraic theory, but falls into a more general class of semi-theories:

**Definition 1.6.** A semi-theory $C$ is a small category with objects $C_0, C_1, \ldots$ and such that for every $n \geq 1$ there is a fixed set of morphisms

$$p_k^n \in \text{Hom}_C(C_n, C_1), \quad 1 \leq k \leq n.$$  

A functor $X: C \to \text{Spaces}$ is a homotopy (resp. strict) $C$-algebra if $X(C_0)$ is weakly contractible (resp. $X(C_0) = *$) and for $n \geq 1$ the product map

$$\prod_{k=1}^n X(p_k^n): X(C_n) \to X(C_1)^n$$

is a weak equivalence (resp. an isomorphism).

Segal proved that giving a homotopy $\Gamma^{\text{op}}$-algebra $X$ amounts to providing the space $X(C_1)$ with a structure of an infinite loop space up to group completion. Other examples of applications of homotopy algebras over semi-theories include characterization of $n$-fold loop spaces [4] and generalized Eilenberg-Mac Lane spaces [1].
In the study of strict algebras the passage from algebraic theories to semi-theories brings nothing new. In fact, for any semi-theory $C$ one can find an algebraic theory $\overline{C}$ such that the categories of strict $C$- and $\overline{C}$-algebras are isomorphic. A result of this kind holds also for homotopy algebras, but in that case the construction of an algebraic theory associated to a semi-theory is more complicated [2]. It can be shown however, that many interesting homotopy structures on spaces can be described directly as homotopy algebras over an algebraic theory. For example, loop spaces can be characterized as homotopy algebras over the theory $T_{Gr}$ which we have already mentioned. Also, $A_{\infty}$-spaces and generalized Eilenberg-Mac Lane spaces can be viewed as homotopy algebras over algebraic theories $T_M$ and $T_{AbM}$ such that the corresponding strict algebras describe respectively monoids and abelian monoids. In each of these cases Theorem 1.3 recovers the results mentioned at the beginning of this paper.

**Organization of the paper.** In Section 2 we state some basic properties of algebraic theories and strict algebras. Then, in Section 3, we recall two standard model category structures defined on a category of functors with values in $\text{Spaces}$. Section 4 contains some remarks on function complexes in model categories. In Section 5 we put a model category structure on the category of strict algebras. Also, we describe a model category expressing the homotopy theory of homotopy $T$-algebras. Finally, in Section 6, Theorem 1.3 is restated in the language of model categories and proved in that form.

**Notation 1.7.** (i) This paper is written simplicially: by $\text{Spaces}$ we denote the category of simplicial sets. Consequently, by ‘space’ we always mean a simplicial set.

(ii) We use extensively the language of model categories of Quillen. Our main references for this subject are [10] and [9].

(iii) Given a category $M$ we will denote by $sM$ the category of simplicial objects in $M$, that is, the category of functors $\Delta^{op} \to M$. If $M$ is a model category then by the model category structure on $sM$ we will always understand Reedy model category structure [10, Ch. 16], where weak equivalences are objectwise weak equivalences while fibrations and cofibrations are defined using matching and latching objects.

(iv) If $M$ is a simplicial model category then so is $sM$. In this case we have the geometric realization functor

$$|-|: sM \to M$$

defined by the coequalizer diagram:
\[ \bigotimes_{\phi: n \to m} X_m \otimes \Delta[n] \longrightarrow \bigotimes_{n} X_n \otimes \Delta[n] \longrightarrow |X_\bullet| \]

for \( X_\bullet \in sM \) (see [9, VII.3]). If \( M = \text{Spaces} \) and so \( sM \) is the category of bisimplicial sets then \(|X_\bullet|\) is just the diagonal of \( X_\bullet \).

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2. Algebraic theories

We start with a brief review of algebraic theories and their strict algebras. For a detailed exposition we refer to [3, §3].

Let \( T \) be an algebraic theory and \( \text{Alg}_T \) the category of strict \( T \)-algebras. We have the forgetful functor
\[ U_T: \text{Alg}_T \to \text{Spaces}, \quad U_T(A) := A(T_1). \]
It is in fact a half of an adjoint pair:

**Proposition 2.1** ([14, 2.3]). The functor \( U_T \) has a left adjoint \( F_T: \text{Spaces} \to \text{Alg}_T \). If \( Y \in \text{Spaces} \) then
\[ F_T(Y)(T_1) = \prod_{n \geq 0} \text{Hom}_T(T_n, T_1) \times Y^n / \sim \]
where the identifications come from the set operations present in any algebraic theory.

The functor \( F_T \) will be called the free \( T \)-algebra functor.

Let \( \text{Spaces}^T \) be the category of all simplicial functors \( T \to \text{Spaces} \). We will often identify the category \( \text{Alg}_T \) with a full subcategory of \( \text{Spaces}^T \).

Using this identification we get:

**Proposition 2.2.** The category \( \text{Alg}_T \) is complete and the limits are computed objectwise.

**Proof.** All limits in \( \text{Spaces}^T \) exist and are computed objectwise, so it is enough to notice that a limit of product-preserving functors also preserves products. \( \Box \)

Let \( J_T: \text{Alg}_T \to \text{Spaces}^T \) denote the embedding of categories. **Proposition 2.2** immediately implies:
Corollary 2.3. The functor $J_T$ preserves limits.

The following fact shows that $\text{Alg}_T$ is a reflective subcategory of $\text{Spaces}_T$.

Proposition 2.4. There exists a functor $K_T: \text{Spaces}_T \to \text{Alg}_T$ left adjoint to $J_T$.

Proof. We use the adjoint functor theorem [11, Thm. 2, p. 117]. By (2.2) and (2.3) it is enough to check the solution set condition.

Let $f: X \to A$ be a morphism in $\text{Spaces}_T$ such that $A \in \text{Alg}_T$. For $n \geq 0$ let

$$f_n: X(T_n) \to A(T_n)$$

denote the restriction of $f$ to $T_n$. By (2.1) the map $f_1$ has a left adjoint

$$g: F_T(X(T_1)) \to A.$$

Define $M_f$ to be the image of $g$:

$$M_f(T_n) := \text{im}(F_T(X(T_1)))(T_n)\overset{g_n}{\to} A(T_n)).$$

Since $g$ is a map of strict $T$-algebras we have $g_n = (g_1)^n$ and it follows that $M_f$ is also a strict algebra. We claim that there exists a morphism $f: X \to M_f$ such that the following diagram commutes:

\[
\begin{array}{ccc}
M_f & \xrightarrow{f} & A \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{f} & A \\
\end{array}
\]

Indeed, it is enough to show that for $n \geq 1$ the image of $f_n$ is contained in $M_f(T_n)$. For $n = 1$ this follows directly from the definition of $M_f$. For $n > 1$, since both $A$ and $M_f$ are strict $T$-algebras, we have

$$\text{im}(f_n) \subseteq \text{im}(f_1)^n$$

and

$$\text{im}(f_1)^n \subseteq (M_f(T_1))^n \cong M_f(T_n).$$

Therefore $\text{im}(f_n) \subseteq M_f(T_n)$ as claimed.

Let $\lambda$ denote the cardinality of the set of simplices of the space $\coprod_n X(T_n) \times \text{Hom}_T(T_n, T_1)$. Since $F_T(X(T_1))$ maps onto $M_f$, from the description of the functor $F_T$ (2.1) we get that the cardinality of the set of simplices of $M_f(T_1)$ cannot exceed $\lambda$. Hence the solution set for $X$ can be chosen to consist of the representatives of isomorphism classes of these strict $T$-algebras $B$ for which the set of simplices of $B(T_1)$ has cardinality not greater than $\lambda$.

\[\square\]

The category $\text{Spaces}_T$ is cocomplete; therefore (2.4) implies:

Corollary 2.5. The category $\text{Alg}_T$ is cocomplete.
Colimits in $\text{Spaces}^T$ are computed objectwise but this is not true in general in $\text{Alg}_T$. However one has the following:

**Proposition 2.6 ([3, 3.4.2]).** Filtered colimits in $\text{Alg}_T$ are computed objectwise. In particular the inclusion functor $J_T: \text{Alg}_T \to \text{Spaces}^T$ preserves filtered colimits.

In Section 6 we will need the following observation which is an easy consequence of the adjointness of $F_T$ and $U_T$.

**Lemma 2.7.** For $1 \leq i \leq m$ let $[n_i] = \{1, \ldots, n_i\}$ be a discrete simplicial set and let

$$\kappa: \prod_{i=1}^m F_T([n_i]) \to F_T\left(\prod_{i=1}^m [n_i]\right)$$

be the map in $\text{Spaces}^T$ induced by inclusions $[n_i] \to \coprod_{i=1}^m [n_i]$. Then $K_T(\kappa)$ is an isomorphism in $\text{Alg}_T$.

Observe that if $A$ is a strict algebra then $K_T A \cong A$. Therefore we get:

**Corollary 2.8.** If $[n_i]$ are simplicial sets as above, $1 \leq i \leq m$, then

$$K_T\left(\prod_{i=1}^m F_T([n_i])\right) \cong F_T\left(\prod_{i=1}^m [n_i]\right).$$

**Remark 2.9.** For $[n] = \{1, \ldots, n\}$ we will denote the functor $F_T([n])$ by $F_n$. Notice that $F_n$ can be described as a functor corepresented by $T_n \in T$:

$$F_n(T_m) = \text{Hom}_T(T_n, T_m).$$

Indeed, for every strict $T$-algebra $A \in \text{Alg}_T$ we have

$$\text{Hom}_{\text{Alg}_T}(\text{Hom}_T(T_n, -), A) \cong A(T_n) \cong A(T_1)^n \cong \text{Hom}_{\text{Spaces}}([n], A(T_1))$$

where the first isomorphism comes from Yoneda’s lemma [11, p.61]. But $F_T$ is left adjoint to the forgetful functor $U_T$ (2.1), and so we have an isomorphism $\text{Hom}_{\text{Spaces}}([n], A(T_1)) \cong \text{Hom}_{\text{Alg}_T}(F_n, A)$. It follows that $\text{Hom}_T(T_n, -)$ must be isomorphic to $F_n$.

### 3. Model category structures on categories of diagrams

We recall here two standard model category structures defined on a category of diagrams of spaces. The model categories describing the homotopy theories of strict and homotopy $T$-algebras ($\S$5) will be derived from $\text{Spaces}^T_{\text{fib}}$. The properties of $\text{Spaces}^T_{\text{cof}}$ on the other hand will allow us to avoid the trouble of working with homotopy function complexes ($\S$4) as shown in (5.8).
Let \( C \) be a small category and let \( \text{Spaces}^C \) denote the category of all functors \( C \to \text{Spaces} \).

**Notation 3.1.** Let \( \text{Spaces}^C_{\text{fib}} \) and \( \text{Spaces}^C_{\text{cof}} \) denote the category \( \text{Spaces}^C \) together with a choice of three classes of morphisms:

- **Spaces}^C_{\text{fib}}**
  - weak equivalences := objectwise weak equivalences
  - fibrations := objectwise fibrations
  - cofibrations := morphisms with the left lifting property with respect to all fibrations which are weak equivalences

- **Spaces}^C_{\text{cof}}**
  - weak equivalences := objectwise weak equivalences
  - cofibrations := objectwise cofibrations
  - fibrations := morphisms with the right lifting property with respect to all cofibrations which are weak equivalences

**Theorem 3.2 ([9, IX 1.4, VIII 2.4]).** Both \( \text{Spaces}^C_{\text{fib}} \) and \( \text{Spaces}^C_{\text{cof}} \) are simplicial model categories. In each case the simplicial structure is given by

\[
(X \otimes K)(c) = X(c) \times K
\]

for any \( X \in \text{Spaces}^C \), \( c \in C \) and a simplicial set \( K \).

Directly from (3.1) one gets that every object of \( \text{Spaces}^C_{\text{cof}} \) is cofibrant. In the remainder of this section we describe a canonical construction of a cofibrant replacement of a diagram of spaces with respect to \( \text{Spaces}^C_{\text{fib}} \) model category structure.

For a category \( C \) as above let \( C^{\text{disc}} \) denote the category with the same objects as \( C \) and with no nonidentity morphisms. The following is readily verified.

**Proposition 3.3.** The forgetful functor \( U: \text{Spaces}^C \to \text{Spaces}^{C^{\text{disc}}} \) has a left adjoint \( F: \text{Spaces}^{C^{\text{disc}}} \to \text{Spaces}^C \) given by

\[
F(X) := \bigsqcup_{c \in C} F_c \otimes X(c)
\]

where \( F_c \in \text{Spaces}^C \) is a functor such that \( F_c(d) := \text{Hom}_C(c, d) \).

Let \( \eta: Y \to UFY \) and \( \varepsilon: FU X \to X \) denote the unit and the counit of this adjunction. As for any pair of adjoint functors the composition

\[
FU: \text{Spaces}^C \to \text{Spaces}^C
\]
defines a cotriple (comonad) [11, p.135] with the structure maps
\[ \varepsilon: FX \to X \quad \text{and} \quad \delta: FX \to (FU)^2X \]
where \( \delta = F\eta U \).

**Definition 3.4.** For \( X \in \text{Spaces}_C \) the standard simplicial resolution of \( X \) is a simplicial object \( FU \cdot X \in s\text{Spaces}_C \) which in the dimension \( k \) consists of a diagram \( FU_k X := (FU)^{k+1}X \). Face and degeneracy operators of \( FU \cdot X \) are given by
\[
(FU_k X \xrightarrow{d_i} FU_{k-1}X) := \left( (FU)^{k+1}X \xleftarrow{(FU)^{k}\varepsilon} (FU)^kX \right)
\]
and
\[
(FU_k X \xrightarrow{s_i} FU_{k+1}X) := \left( (FU)^{k+1}X \xrightarrow{(FU)^{k+1}\delta} (FU)^{k+2}X \right).
\]

If we regard \( X \) as a constant simplicial object we can define a simplicial map \( \phi: FU \cdot X \to X \)
\[
(FU_k X \xrightarrow{\phi_k} X) := \left( (FU)^{k+1}X \xrightarrow{\varepsilon} X \right).
\]
Let \( |\phi|: |FU \cdot X| \to X \) be the geometric realization of \( \phi \) (1.7) taken with respect to the simplicial structure as in (3.2).

**Proposition 3.5.** The map \(|\phi|\) is a weak equivalence.

**Proof.** If \( Y \) is a simplicial object in \( \text{Spaces}_C \) then its realization can be computed objectwise: \( |Y|((c)) = |Y(c)| \), where the space on the left-hand side is the realization (diagonal) of the bisimplicial set \( Y(c) \). Therefore, it suffices to show that \(|\phi^c|: |FU \cdot X(c)| \to |X(c)|\) is a weak equivalence of spaces for all \( c \in C \). One can check that the realization of the map \( \phi^c: X(c) \to FX(c) \)
\[
(X(c) \xrightarrow{\phi^c} FU_k X(c)) := \left( X(c) \xrightarrow{\eta^{k+1}} (FU)^{k+1}X(c) \right)
\]
is a homotopy inverse for \(|\phi^c|\). \( \square \)

We claim that \(|FU \cdot X|\) is a cofibrant replacement for \( X \). In view of (3.5) it remains to show that \(|FU \cdot X|\) is a cofibrant object of \( \text{Spaces}_{\text{fib}}^C \).

**Proposition 3.6.** For every \( X \in \text{Spaces}_C \) the resolution \( FU \cdot X \) is a Reedy cofibrant object in \( s\text{Spaces}_{\text{fib}}^C \).

**Proof.** Let \( \Delta^\text{op}_+ \) denote the subcategory of \( \Delta^\text{op} \) generated by all degeneracy maps \( s_i \) and positive face maps \( d_j, j > 0 \). The category \( \Delta^\text{op}_+ \) is a Reedy category [10, 16.1.2] with the direct subcategory of \( \Delta^\text{op}_+ \) generated by degeneracy maps and the inverse subcategory generated by positive face maps. For a model category \( M \) define \( \text{s} M \) to be the category of functors \( \Delta^\text{op}_+ \to M \) with the Reedy model category structure.
The embedding of categories $\Delta^+_\op \hookrightarrow \Delta^\op$ defines a functor $r: sM \to s_*M$. Since all degeneracy maps of $\Delta^\op$ are contained in $\Delta^+_\op$ we get that $X_* \in sM$ is cofibrant if and only if $r(X)$ is cofibrant in $s_*M$. Take $M = \text{Spaces}_\fib^C$. To prove the statement of the proposition it is then enough to show that $r(FU_*X)$ is a cofibrant object in $s_*\text{Spaces}_\fib^C$.

Since the category $\text{Spaces}^\text{Cdisc}$ is isomorphic to the product $\prod_{c \in C} \text{Spaces}$ it is a model category with weak equivalences, fibrations and cofibrations defined to be objectwise weak equivalences, fibrations and cofibrations. The adjoint pair of functors $(F, U)$ becomes then a Quillen pair between $\text{Spaces}^\text{Cdisc}$ and $\text{Spaces}_\fib^C$. It follows that the induced functors

$$F: s_*\text{Spaces}^\text{Cdisc} \rightleftarrows s_*\text{Spaces}_\fib^C: U$$

also form a Quillen pair with respect to Reedy model category structures [10, 16.11.1]. From Definition 3.4 we see that the object $r(FU_*X)$ is in the image of the functor $F$, hence it suffices to show that every object of $s_*\text{Spaces}^\text{Cdisc}$ is cofibrant. This is however an immediate consequence of the fact that every object of $s_*\text{Spaces}$ is Reedy cofibrant. The proof of this last statement is the same as the proof that every object is cofibrant in the Reedy model category structure on $s\text{Spaces}$ – the category of bisimplicial sets (see [10, 16.7.8]). \(\square\)

Geometric realization of a Reedy cofibrant object is cofibrant [9, VII 3.6], hence (3.6) implies

**Corollary 3.7.** The diagram $|FU_*X|$ is a cofibrant object of $\text{Spaces}_\fib^C$.

### 4. Function complexes

In Section 5 we will introduce a model category for homotopy $T$-algebras. In preparation for that we recall here some properties of function complexes in model categories.

Let $M$ be a simplicial model category and let $\text{HoM}$ denote its homotopy category. In [7] Dwyer and Kan showed that for any $X, Y \in M$ the set of morphisms $\text{Hom}_{\text{HoM}}(X, Y)$ can be replaced by a richer structure of a homotopy function complex, that is a simplicial set $\text{RMap}_M(X, Y)$ such that $\pi_0\text{RMap}_M(X, Y) \cong \text{Hom}_{\text{HoM}}(X, Y)$. Moreover the following holds:

(i) $\text{RMap}_M(X, Y)$ preserves weak equivalences: if $X \simeq X'$ and $Y \simeq Y'$ then $\text{RMap}_M(X, Y) \simeq \text{RMap}_M(X', Y')$.

(ii) The homotopy type of $\text{RMap}_M(X, Y)$ depends only on the class of weak equivalences of $M$: if $M'$ is a model category with the same underlying category as $M$ and with the same class of weak equivalences then $\text{RMap}_M(X, Y) \cong \text{RMap}_{M'}(X, Y)$ for all $X, Y$ in $M$. 


(iii) A morphism \( f: X \to X' \) is a weak equivalence if and only if the induced map \( f_*: \text{RMap}_M(X', Y) \to \text{RMap}_M(X, Y) \) is a weak equivalence of simplicial sets for all \( Y \in M \).

(iv) If \( M \) is a simplicial model category, \( X \) is cofibrant and \( Y \) is fibrant then \( \text{RMap}_M(X, Y) \) is weakly equivalent to the simplicial set \( \text{Map}_M(X, Y) \) where \( \text{Map}_M(X, Y)_k = \text{Hom}_M(X \otimes \Delta[k], Y) \). We will call \( \text{Map}_M(X, Y) \) the simplicial function complex of \( X \) and \( Y \).

Note 4.1. As a consequence of (ii) we do not need to distinguish between homotopy function complexes taken with respect to \( \text{Spaces}^C_{\text{fib}} \) and \( \text{Spaces}^C_{\text{cof}} \) model category structures. Hence from now on \( \text{RMap}(-, -) \) will stand for a homotopy function complex in any of these model categories. Similarly, since simplicial function complexes (see (iv)) are defined using only the simplicial structure of a model category, they are the same in \( \text{Spaces}^C_{\text{fib}} \) and \( \text{Spaces}^C_{\text{cof}} \). We will denote them by \( \text{Map}(-, -) \).

In Section 6 we will refer to the following property of simplicial function complexes.

**Proposition 4.2.** Let \( M \) be a simplicial model category and let \( Y \in M \) be a fibrant object. Assume that \( f: X_* \to X'_* \) is a map of Reedy cofibrant objects in \( sM \) such that

\[
\Phi^*_{X'}: \text{Map}_M(|X'_*|, Y) \to \text{Map}_M(|X_*|, Y)
\]

is a weak equivalence for all \( n \geq 0 \). Then the geometric realization of the map \( f \) induces a weak equivalence

\[
|f|*: \text{Map}_M(|X'_*|, Y) \xrightarrow{\sim} \text{Map}_M(|X_*|, Y).
\]

**Proof.** We have a commutative diagram

\[
\begin{array}{ccc}
\text{Map}_M(|X'_*|, Y) & \xrightarrow{|f|^*} & \text{Map}_M(|X_*|, Y) \\
\Phi^*_{X'} \downarrow & & \Phi^*_{X} \downarrow \\
\text{Map}_M(\text{hocolim}_{\Delta^{op}} X'_*, Y) & \xrightarrow{f^*} & \text{Map}_M(\text{hocolim}_{\Delta^{op}} X_* , Y) \\
\Psi_{X'} \downarrow \simeq & & \Psi_{X} \downarrow \simeq \\
\text{holim}_{\Delta} \text{Map}_M(X'_*, Y) & \xrightarrow{f^*} & \text{holim}_{\Delta} \text{Map}_M(X_* , Y)
\end{array}
\]

The map \( \Phi^*_{X} \) is induced by the Bousfield-Kan map \( \Phi_X: \text{hocolim}_{\Delta^{op}} X_* \to |X_*| \) [10, 19.6.3] (and similarly for \( \Phi^*_{X'} \)), while \( \Psi_{X} \) and \( \Psi_{X'} \) are the isomorphisms of simplicial sets described in [10, 19.1.12]. Since \( X_* \) and \( X'_* \) are Reedy cofibrant
it follows from [10, 19.6.4] that \( \Phi_X \) and \( \Phi_X' \) are weak equivalences and so are the maps they induce on simplicial function complexes. Fibrancy of \( Y \) implies on the other hand, that \( \text{Map}(\cdot, Y) \) is always a Kan complex. Therefore, by our assumption on \( f \) the bottom map \( f^* \) is a weak equivalence (see [10, 19.4.3]), and hence so is the top map. \( \square \)

**Note 4.3.** Suppose than in Proposition 4.2 we have \( M = \text{Spaces}^C_{\text{cof}} \) (3.1). Then the assumption that \( X_\bullet \) and \( X'_\bullet \) are Reedy cofibrant is always satisfied. Indeed, since cofibrations in \( \text{Spaces}^C_{\text{cof}} \) are defined objectwise, \( X_\bullet \in \text{sSpaces}^C_{\text{cof}} \) is Reedy cofibrant if and only if \( X_\bullet(c) \) is a cofibrant bisimplicial set for all \( c \in C \). This last condition however always holds since all bisimplicial sets are cofibrant in the Reedy model category structure on \( \text{sSpaces} \) [10, 16.7.8].

---

5. Model category for homotopy \( T \)-algebras

Let \( T \) be an algebraic theory. Recall that by (2.4) there is an adjoint pair of functors

\[
K_T: \text{Spaces}^T \leftrightarrow \text{Alg}_T: J_T
\]

where \( J_T \) is the inclusion onto a subcategory. This adjunction can be used to put a model category structure on \( \text{Alg}_T \):

**Theorem 5.1** ([14, 3.1]). *The category \( \text{Alg}_T \) is a model category with weak equivalences and fibrations defined as objectwise weak equivalences and fibrations. Then the adjoint pair \( (K_T, J_T) \) becomes a Quillen pair between \( \text{Spaces}^T_{\text{fib}} \) and \( \text{Alg}_T \).*

Our main goal in this section is to construct a model category \( \text{LSpaces}^T \) which reflects the homotopy theory of homotopy \( T \)-algebras.

Recall (2.9) that for each \( n \geq 0 \) there is a functor \( F_n \in \text{Spaces}^T \) given by \( F_n(T_m) := \text{Hom}_T(T_n, T_m) \) (in fact, since \( T_m \cong T_1^m \) we get that \( F_n \in \text{Alg}_T \)). For \( n \geq 1 \) the projections \( p^*_n \) induce maps \( p_n: \coprod F_1 \to F_n \). We define also \( p_0: \coprod F_1 \to F_0 \) to be the unique map from the diagram of empty spaces to \( F_0 \) (alternatively, the morphism \( p_n \) can be described as the map induced by inclusions of sets \( [1] \hookrightarrow [n] \) as in Lemma 2.7). Let \( S := \{ p_0, p_1, \ldots \} \).

**Definition 5.2.** An object \( Z \in \text{Spaces}^T_{\text{fib}} \) is \( S \)-local if it is fibrant and if for each \( n \geq 0 \) the map of homotopy function complexes

\[
p_n^*: \text{RMap}(F_n, Z) \to \text{RMap}(\coprod F_1, Z)
\]

is a weak equivalence.
A morphism \( f: X \to Y \) in \( \text{Spaces}^T_{\text{fib}} \) is an \textit{\( S \)-local equivalence} if the induced map
\[
f^*: \text{RMap}(Y, Z) \to \text{RMap}(X, Z)
\]
is a weak equivalence for every \( S \)-local object \( Z \).

\textit{Note 5.3.} Since both \( \coprod_n F_1 \) and \( F_n \) are cofibrant and \( Z \) is fibrant in \( \text{Spaces}^T_{\text{fib}} \) the map \( p_n^* \) in the definition of \( S \)-local objects above can be replaced by the map of simplicial function complexes (§4(iv)):
\[
p_n^*: \text{Map}(F_n, Z) \to \text{Map}\left( \coprod_n F_1, Z \right).
\]

\textbf{Proposition 5.4.} Let \( \text{LSpaces}^T \) denote the category \( \text{Spaces}^T \) with three distinguished classes of morphisms:

\begin{itemize}
\item weak equivalences := \( S \)-local equivalences
\item cofibrations := cofibrations in \( \text{Spaces}^T_{\text{fib}} \)
\item fibrations := maps with the right lifting property with respect to all cofibrations which are weak equivalences
\end{itemize}

Then \( \text{LSpaces}^T \) is a simplicial model category with the same simplicial structure as \( \text{Spaces}^T_{\text{fib}} \) (3.2).

\textit{Proof.} This is a consequence of a general result [10, 4.1.1] which proves the existence of left Bousfield localizations for a broad class of model categories. The category \( \text{Spaces}^T_{\text{fib}} \) satisfies the assumptions of that theorem by [10, 4.1.4] and [10, 4.1.5] and the model category structure on \( \text{LSpaces}^T \) is obtained by localizing \( \text{Spaces}^T_{\text{fib}} \) with respect to the set \( S \).

Within the model category \( \text{LSpaces}^T \), homotopy \( T \)-algebras can be characterized as follows:

\textbf{Proposition 5.5.} An object \( Z \in \text{LSpaces}^T \) is fibrant if and only if it is a homotopy \( T \)-algebra, fibrant as an object of \( \text{Spaces}^T_{\text{fib}} \).

\textit{Proof.} By [10, 3.5.1] fibrant objects of \( \text{LSpaces}^T \) are exactly the \( S \)-local objects. Therefore, for any fibrant \( Z \in \text{LSpaces}^T \) the maps \( p_n^* \) as in (5.3) are weak equivalences. But for every \( n \geq 0 \) we have \( \text{Map}(\coprod_n F_1, Z) \cong \prod_n Z(T_1) \) and \( \text{Map}(F_n, Z) \cong Z(T_n) \). It follows that \( Z \) is a homotopy \( T \)-algebra. The proof of the other implication is similar.

The next proposition is a consequence of (5.5) and [10, 3.3.12].
Proposition 5.6. If \( Z, Z' \in \text{Spaces}_T^\mathbb{T} \) are homotopy \( T \)-algebras and \( f: Z \to Z' \) is an \( S \)-local weak equivalence then \( f \) is a weak equivalence in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \) (i.e. an objectwise weak equivalence).

Proposition 5.5 says that the homotopy category of \( \text{LSpaces}_T^\mathbb{T} \) is equivalent to the category of homotopy \( T \)-algebras with inverted \( S \)-local equivalences. From (5.6) we see however that it amounts to inverting objectwise weak equivalences. As a consequence we get:

Corollary 5.7. The homotopy theory of homotopy \( T \)-algebras (with objectwise weak equivalences) is equivalent to the homotopy category of \( \text{LSpaces}_T^\mathbb{T} \).

The definition of \( S \)-local equivalences we gave above (5.2) involves maps defined on homotopy function complexes \( \text{RMap}(\cdot, Z) \). In practice it is more convenient to work with simplicial function complexes \( \text{Map}(\cdot, Z) \). Since we assume that \( Z \) is fibrant in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \) we get \( \text{RMap}(X, Z) \simeq \text{Map}(X, Z) \) whenever \( X \in \text{Spaces}_{\mathbb{T}}^\mathbb{T} \) is a cofibrant object. However, the property that \( f: X \to X' \) is an \( S \)-local equivalence can be expressed in terms of simplicial function complexes even when \( X \) or \( X' \) is not cofibrant.

Proposition 5.8. A map \( f: X \to X' \) is an \( S \)-local equivalence if and only if for any homotopy \( T \)-algebra \( \bar{Z} \) fibrant in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \), the induced map

\[
 f^\ast: \text{Map}(X', \bar{Z}) \to \text{Map}(X, \bar{Z})
\]

is a weak equivalence of simplicial sets.

Proof. Assume that \( f \) is an \( S \)-local equivalence. Since fibrant objects of \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \) are also fibrant in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \), the map \( f \) induces a weak equivalence on \( \text{RMap}(\cdot, Z) \) for any \( Z \) as above. Moreover, all objects of \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \) are cofibrant; hence we have \( \text{RMap}(Y, \bar{Z}) \simeq \text{Map}(Y, \bar{Z}) \) for any \( Y \in \text{Spaces}_{\mathbb{T}}^\mathbb{T} \). It follows that the map \( f^\ast: \text{Map}(X', \bar{Z}) \to \text{Map}(X, \bar{Z}) \) must be a weak equivalence.

Conversely, assume that \( f \) induces a weak equivalence on \( \text{Map}(\cdot, \bar{Z}) \) for all homotopy \( T \)-algebras \( \bar{Z} \), fibrant in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \). Let \( Z \) be any homotopy \( T \)-algebra and let \( Z \stackrel{\sim}{\to} \bar{Z} \) denote a fibrant replacement of \( Z \) in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \). Then \( \text{RMap}(\cdot, Z) \simeq \text{RMap}(\cdot, \bar{Z}) \simeq \text{Map}(\cdot, \bar{Z}) \), and so \( f^\ast: \text{RMap}(X', Z) \to \text{RMap}(X, Z) \) is a weak equivalence. Therefore \( f \) is an \( S \)-local equivalence. \( \square \)

6. Strict and homotopy \( T \)-algebras

We begin by proving some properties of the adjunction \( (J_T, K_T) \) (see (2.4)). Let \( A_\bullet \) be a simplicial object in \( \text{Alg}_T \). Abusing notation, by \( |A_\bullet| \), we will always denote the geometric realization of \( A_\bullet \) with respect to the simplicial structure in \( \text{Spaces}_{\mathbb{T}}^\mathbb{T} \) (3.2), that is, the geometric realization of \( J_TA_\bullet \in \text{Spaces}_{\mathbb{T}}^\mathbb{T} \).
Lemma 6.1. If $A_\bullet \in s\text{Alg}_T$ then $|A_\bullet| \in \text{Alg}_T$.

Proof. We need to show that $|A_\bullet|(T_0) \cong *$ and that for $n > 0$ the projection maps $p^n_0: T_n \to T_1$ induce isomorphisms $|A_\bullet|(T_n) \cong (|A_\bullet|(T_1))^n$. As we have already noted in the proof of (3.5) the realization of $A_\bullet$ can be computed objectwise:

$$|A_\bullet|(T_n) \cong |A_\bullet|(T_n)$$

where $|A_\bullet|(T_n)$ is the diagonal of the bisimplicial set $A_\bullet(T_n)$. Since $A_m(T_0) = *$ for all $m$, we have $|A_\bullet|(T_0) = *$. Moreover, since $A_\bullet \in s\text{Alg}_T$, the projection maps induce isomorphisms of bisimplicial sets $A_\bullet(T_n) \cong (A_\bullet(T_1))^n$ for $n > 0$. Since the diagonal of a bisimplicial set commutes with products we get $|A_\bullet|(T_n) \cong |A_\bullet|(T_1)|^n$, and it follows that $|A_\bullet|$ is a strict $T$-algebra. \hfill \Box

Lemma 6.2. If $X_\bullet$ is a simplicial object in $\text{Spaces}_\text{fib}^T$ then $K_T|X_\bullet| \cong |K_TX_\bullet|$.

Proof. By (6.1), $|K_TX_\bullet|$ is an object of $\text{Alg}_T$. Let $\eta_\bullet: X_\bullet \to K_TX_\bullet$ ($= J_TK_TX_\bullet$) be the unit of the adjunction $(K_T, J_T)$. By properties of adjunction there exists a map $\theta: K_T|X_\bullet| \to |K_TX_\bullet|$ such that the following diagram commutes:

$$\begin{array}{ccc}
|X_\bullet| & \xrightarrow{\eta_\bullet} & |K_TX_\bullet| \\
\downarrow{\eta|X_\bullet|} & & \downarrow{\eta|K_TX_\bullet|} \\
K_T|X_\bullet| & \xrightarrow{\theta} & |K_TX_\bullet|
\end{array}$$

The map $\eta|X_\bullet|$ above is the unit of adjunction for $|X_\bullet|$. It is enough to prove that $|\eta_\bullet|_*: \text{Hom}(|K_TX_\bullet|, A) \to \text{Hom}(|X_\bullet|, A)$ is an isomorphism for all $A \in \text{Alg}_T$. Indeed, by properties of adjunction $\eta|X_\bullet|_*: \text{Hom}(K_T|X_\bullet|, A) \to \text{Hom}(|X_\bullet|, A)$ is an isomorphism for any $A$, so it will follow that the map $\theta^*: \text{Hom}(|K_TX_\bullet|, A) \to \text{Hom}(|K_TX_\bullet|, A)$ must be an isomorphism for all $A \in \text{Alg}_T$, and hence $\theta$ is an isomorphism. For a cosimplicial space $Y_\bullet$ let $\text{Tot}(Y_\bullet)$ denote the realization of $Y_\bullet$ [9, VIII.1 p.390]. We have an isomorphism

$$\text{Hom}(|X_\bullet|, A) \cong \text{Tot}(\text{Hom}(X_\bullet, A))_0.$$ 

Therefore, it suffices to show that the realization of the map of cosimplicial spaces $\eta_*^\bullet: \text{Hom}(K_TX_\bullet, A) \to \text{Hom}(X_\bullet, A)$ is an isomorphism. But this follows from the fact that $\eta_n: \text{Hom}(K_TX_n, A) \to \text{Hom}(X_n, A)$ is an isomorphism for any strict $T$-algebra $A$ and for any $n \geq 0$. \hfill \Box

Next, recall that the adjunction $(K_T, J_T)$ is a Quillen pair between $\text{Spaces}_\text{fib}^T$ and $\text{Alg}_T$ (5.1). The following fact shows that this property does not change if we replace the model category structure on $\text{Spaces}_\text{fib}^T$ by the $S$-local structure.
Proposition 6.3. The adjoint functors $(K_T, J_T)$ form a Quillen pair between the categories $\text{LSpaces}_T$ and $\text{Alg}_T$.

Proof. Let $p_n: \coprod_n F_1 \to F_n$ be the map given in Section 5. The model category $\text{LSpaces}_T$ is obtained by localizing $\text{Spaces}_T^{\text{fib}}$ with respect to all maps $p_n$. Thus the above statement follows from the observation that by (2.7) $K_T(p_n)$ is an isomorphism in $\text{Alg}_T$ and from [10, 3.4.20]. □

Since $\text{LSpaces}_T$ can serve as a model category for the homotopy theory of homotopy $T$-algebras (5.7), our main Theorem 1.3 can be now restated as follows.

Theorem 6.4. The Quillen pair of functors

$$K_T: \text{LSpaces}_T \rightleftarrows \text{Alg}_T: J_T$$

is a Quillen equivalence.

This is in turn a consequence of the following:

Lemma 6.5. Let $\eta_X: X \to K_T(X)$ (= $J_TK_T(X)$) denote the unit of the adjunction $(K_T, J_T)$. Then for every cofibrant object $X \in \text{LSpaces}_T$ the map $\eta_X$ is an $S$-local equivalence.

Proof of Theorem 6.4 assuming the lemma. We need to show that if $X$ is a cofibrant object in $\text{LSpaces}_T$, $A$ is fibrant in $\text{Alg}_T$ and $f: X \to A$ is a map in $\text{LSpaces}_T$, then $f$ is an $S$-local equivalence if and only if its adjoint $f^\flat: K_TX \to A$ is a weak equivalence in $\text{Alg}_T$ (that is, an objectwise weak equivalence). Let $f$ be a map as above. We have a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & K_T X \\
\downarrow{f} & & \downarrow{f^\flat} \\
A & & \\
\end{array}$$

where $\eta_X$ is an $S$-local equivalence by Lemma 6.5. If we assume that $f$ is an $S$-local equivalence, then so must be $f^\flat$. But $f^\flat$ is a map of strict $T$-algebras, so (5.6) implies that it is an objectwise weak equivalence. Conversely, if $f^\flat$ is a weak equivalence in $\text{Alg}_T$, then it is also an $S$-local equivalence, and $f = f \circ \eta_X$ is an $S$-local equivalence as well. □

Proof of Lemma 6.5. By (5.8) it suffices to show that for every homotopy $T$-algebra $Z$ fibrant in $\text{Spaces}_T^{\text{cof}}$ and for every cofibrant $X \in \text{Spaces}_T^{\text{fib}}$ the map $\eta_X: X \to K_T X$ induces a weak equivalence of simplicial function complexes

$$\eta_X^*: \text{Map}(K_T X, Z) \to \text{Map}(X, Z).$$
The proof is split into a few steps.

1) $X = \prod_{i=1}^m F_{n_i}$. Recall that by (2.8) we have $K_T(\prod_{i=1}^m F_{n_i}) \simeq F_{\Sigma n_i}$. For any $Z \in \text{Spaces}_{\text{cof}}^T$, 

$$\text{Map} \left( \prod_{i=1}^m F_{n_i}, Z \right) \cong \prod_{i=1}^m \text{Map}(F_{n_i}, Z) \cong \prod_{i=1}^m Z(T_{n_i})$$

and 

$$\text{Map}(F_{\Sigma n_i}, Z) \cong Z(T_{\Sigma n_i}).$$

Moreover, if $Z$ is a homotopy $T$-algebra then $Z(T_{\Sigma n_i}) \simeq Z(T_1)^{\Sigma n_i} \simeq \prod_{i=1}^m Z(T_{n_i})$. One can check that the above weak equivalence is in fact induced by the map $\eta_X$ (notice that in this case $\eta_X = \kappa$ where $\kappa$ is the map as in (2.7)).

2) $X = \coprod_{i \in I} F_{n_i}$ - possibly infinite disjoint union of free strict $T$-algebras. Let $P_I$ denote the category of all finite subsets of $I$ with inclusions of sets as morphisms. Define a functor 

$$\tilde{X} : P_I \to \text{LSpaces}^T, \quad \tilde{X}(A) := \coprod_{i \in A} F_{n_i}.$$ 

Then colim $\tilde{X} = X$ and colim $J_T K_T \tilde{X} = K_T X$. The second equality follows from the fact that $J_T$ commutes with filtered colimits (2.6) and $K_T$ as a left adjoint functor preserves all colimits. The map $\eta_X$ is then a colimit of maps $\eta_{\tilde{X}(A)} : \tilde{X}(A) \to K_T \tilde{X}(A)$. For every $A \in P_I$ the diagram $\tilde{X}(A)$ is of the form in step 1, so that the map induced by $\eta_{\tilde{X}(A)}$ on simplicial function complexes $\text{Map}(-, Z)$ is a weak equivalence. It follows that $\text{hocolim} \eta_{\tilde{X}}$ also induces a weak equivalence 

$$\text{hocolim} \eta_{\tilde{X}} : \text{Map} (\text{hocolim} J_T K_T \tilde{X}, Z) \xrightarrow{\simeq} \text{Map} (\text{hocolim} \tilde{X}, Z).$$

Therefore it is enough to show that colim $\tilde{X} \simeq \text{hocolim} \tilde{X}$ and colim $J_T K_T \tilde{X} \simeq \text{hocolim} J_T K_T \tilde{X}$. By the definition of homotopy colimits [10, 19.1.2] and since the simplicial structure on $\text{Spaces}_{\text{cof}}^T$ is defined objectwise, 

$$\text{(hocolim} \tilde{X})(T_n) \cong \text{hocolim} \tilde{X}(T_n).$$

But $P_I$ is a filtered category and by [5, 3.5, p.331] ordinary and homotopy colimits over filtered categories coincide, so that $\text{(hocolim} \tilde{X})(T_n) \cong \text{(colim} \tilde{X})(T_n)$. Hence $\text{hocolim} \tilde{X} \cong \text{colim} \tilde{X}$. Similarly $\text{hocolim} J_T K_T \tilde{X} \cong \text{colim} J_T K_T \tilde{X}$.

3) $X = \coprod_{i \in I} F_{n_i} \otimes K_i$ ($K_i$ - simplicial set). Let $X_\bullet$ denote a simplicial object in $\text{Spaces}_{\text{cof}}^T$ such that $X_k := \coprod_{i \in I} \coprod_{\sigma \in (K_i)_k} F_{n_i}$. Then [9, VII 3.7] $|X_\bullet| \cong X$. Also by (6.2) we see that $|K_T X_\bullet| \cong K_T X$. Since for all $k \geq 0$ the diagram $X_k$ is of the form considered in step 2, $\eta_X$ must be an $S$-local equivalence. Therefore, by (4.2) and (4.3) the map $\eta_X$ also is an $S$-local equivalence.
4) $X = |FU_*Y|$ (see 3.4) where $Y$ is any object of $\text{Spaces}_{\text{cof}}^T$. Since $FU_kY$ is of the form in step 3 for any $k \geq 0$, we can use (6.2) and (4.2) as above to show that $\eta_X$ must be an $S$-local equivalence.

5) Let $X$ be an arbitrary diagram cofibrant in $\text{Spaces}_{\text{fib}}^T$. Recall (3.5) that we have a weak equivalence $|\varphi|: |FU_*X| \to X$. Consider the commutative diagram

\[
\begin{array}{ccc}
|FU_*X| & \xrightarrow{|\varphi|} & X \\
\eta_{FU_*X} \downarrow & \sim_S & \downarrow \eta_X \\
K_T|FU_*X| & \xrightarrow{K_T|\varphi|} & K_TX
\end{array}
\]

The map $\eta_{FU_*X}$ is an $S$-local equivalence by step 4, so that it suffices to show that $K_T|\varphi|$ is an $S$-local equivalence. The functor $K_T: \text{LSpaces}^T \to \text{Alg}_T$ is a left adjoint in a Quillen pair, and so it preserves all acyclic cofibrations between cofibrant objects. Therefore, by K. Brown’s lemma [8, 9.9] it preserves all weak equivalences between cofibrant objects. Since $|\varphi|$ is a weak equivalence in $\text{LSpaces}^T$ and both $X$ and $|FU_*X|$ are cofibrant we get that $K_T|\varphi|$ is a weak equivalence.

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References

[1] B. Badzioch, Recognition principle for generalized Eilenberg-Mac Lane spaces, in Cohomological Methods in Homotopy Theory, Progr. in Math. 196, 21–26, Birkhäuser, Basel, 2001.

[2] B. Badzioch, From $\Gamma$-spaces to algebraic theories, preprint.

[3] F. Borceux, Handbook of Categorical Algebra, Vol. 2, Encyclopedia of Mathematics and its Applications 51, Cambridge Univ. Press, Cambridge, 1994.

[4] A. K. Bousfield, The simplicial homotopy theory of iterated loop spaces, manuscrupt, 1992.

[5] A. K. Bousfield and D. M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Math. 304, Springer-Verlag, New York, 1972.

[6] E. Dror Farjoun, Cellular Spaces, Null Spaces and Homotopy Localization, Lecture Notes in Math. 1622, Springer-Verlag, New York, 1996.

[7] W. G. Dwyer and D. M. Kan, Function complexes in homotopical algebra, Topology 19 (1980), 427–440.

[8] W. G. Dwyer and J. Spaliński, Homotopy theories and model categories, in Handbook of Algebraic Topology, 73–126, North-Holland, Amsterdam, 1995.

[9] P. G. Goerss and J. F. Jardine, Simplicial homotopy theory, Progr. in Math. 174, Birkhäuser Verlag, Basel, 1999.

[10] P. S. Hirschhorn, Localization of model categories, preprint (version: Oct. 10, 2001), available on http://math.mit.edu/~psh.

[11] S. Mac Lane, Categories for the Working Mathematician, Grad. Texts in Math. 5, Springer-Verlag, New York, 1971.
[12] F. W. Lawvere, Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. U.S.A.* 50 (1963), 869–872.

[13] G. Segal, Categories and cohomology theories, *Topology* 13 (1974), 293–312.

[14] S. Schwede, Stable homotopy of algebraic theories, *Topology* 40 (2001), 1–41.

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