NON-COMMUTATIVE WIDTH AND GOPAKUMAR-VAFA INVARIANTS

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Abstract. We show that the non-commutative widths for flopping curves on smooth 3-folds introduced by Donovan-Wemyss are described by Katz’s genus zero Gopakumar-Vafa invariants.

1. Introduction

1.1. Result. Let $X$ be a smooth quasi-projective complex 3-fold and $f: X \to Y$ a birational flopping contraction which contracts a single rational curve $\mathbb{P}^1 \cong C \subset X$ to a point $p \in Y$. In the paper [DW], Donovan-Wemyss introduced a new invariant associated to $f$, the contraction algebra $A_{\text{con}}$, given by the universal non-commutative deformation algebra of the curve $C$ in $X$. The algebra $A_{\text{con}}$ is finite dimensional, and it is commutative if and only if $C$ is not a $(1,-3)$-curve. Furthermore if $A_{\text{con}}$ is commutative, the dimension of $A_{\text{con}}$ coincides with Reid’s width $[\text{Rei}]$ of $C$. Based on this observation, Donovan-Wemyss defined the following generalizations of Reid’s width

$$\text{wid}(C) := \dim_{\mathbb{C}} A_{\text{con}}, \quad \text{cwid}(C) := \dim_{\mathbb{C}} A_{\text{con}}^{ab},$$

which they called \textit{non-commutative width} and \textit{commutative width} respectively.

On the other hand, Katz [Kat08] defined genus zero Gopakumar-Vafa (GV) invariants as virtual numbers of one dimensional stable sheaves on $X$. For $j \geq 1$, the genus zero GV invariant $n_j \in \mathbb{Z}_{\geq 0}$ of curve class $j[C]$ on $X$ is shown in [Kat08] to coincide with the multiplicity of the Hilbert scheme of $X$ at some subscheme $C^{(j)} \subset X$ with curve class $j[C]$. The purpose of this short note is to describe Donovan-Wemyss’s widths in terms of Katz’s genus zero GV invariants. The main result is as follows:

Theorem 1.1. We have the following formulas

$$\text{wid}(C) = \sum_{j=1}^{l} j^2 \cdot n_j, \quad \text{cwid}(C) = n_1.$$  

Here $l$ is the scheme theoretic length of $f^{-1}(p)$ at $C$. 


Here we remark that the identity of $c\text{wid}(C)$ is almost obvious from the definitions, and the identity of $\text{wid}(C)$ is more interesting. The result of Theorem 1.1 indicates that one can study non-commutative widths without using non-commutative algebras. Conversely, one may compute genus zero GV invariants by computing contraction algebras. The proof of Theorem 1.1 is an easy application of the main result of [DW], combined with some deformation argument. By [DW], the algebra $A_{\text{con}}$ defines the non-commutative twist functor, describing Bridgeland-Chen’s flop-flop autoequivalence of $\text{D}^b\text{Coh}(X)$. On the other hand, after taking the completion at $p$, the morphism $f$ deforms to flopping contractions of disjoint $(-1, -1)$-curves, such that the number of $(-1, -1)$-curves with curve class $j[C]$ coincides with $n_j$. Now the flop-flop autoequivalence deforms along the deformation of $f$, hence the non-commutative twist functor also deforms: the resulting deformation is a composition of Seidel-Thomas’s spherical twists along $(-1, -1)$-curves. We then relate the Hilbert polynomial of a cohomology sheaf of the kernel object of the non-commutative twist functor with that of the above composition of the spherical twists, and obtain the desired identity of $\text{wid}(C)$.

1.2. Examples and a Remark. Here we describe some examples of Theorem 1.1.

Example 1.2. In Theorem 1.1 we have $l = 1$ if and only if $C$ is either a $(-1, -1)$ or a $(0, -2)$-curve. In this case, we have $\text{wid}(C) = c\text{wid}(C)$, and it coincides with Reid’s width (cf. [DW, Example 3.12]). On the other hand, the genus zero GV invariant $n_1$ also coincides with Reid’s width as indicated in [BKL01, Section 1].

Example 1.3. Suppose that $Y = \text{Spec } R_k$, where $R_k$ is defined by

$$R_k = \mathbb{C}[u, v, x, y]/(u^2 + v^2y = x(x^2 + y^{2k+1})).$$

There is a flopping contraction $f : X \to Y$ with $l = 2$. The contraction algebra $A_{\text{con}}$ is computed in [DW] Example 3.14]

$$A_{\text{con}} \cong \mathbb{C}(x, y)/(xy = -yx, x^2 = y^{2k+1})$$

$$A_{\text{con}}^{ab} \cong \mathbb{C}[x, y]/(xy = 0, x^2 = y^{2k+1}).$$

It follows that

$$\text{wid}(C) = 3(2k + 1), \quad c\text{wid}(C) = 2k + 3.$$ 

The result of Theorem 1.1 indicates that $n_1 = 2k + 3$ and $n_2 = k$.

We also have the following remark:

\footnote{Wemyss pointed out to the author that the non-commutative widths are commutative things, as they are computed using some Ext-groups on commutative algebras. See [DW, Remark 5.2].}
Remark 1.4. We have $n_j \geq 1$ for $1 \leq j \leq l$. So Theorem 1.1 implies that

$$\text{wid}(C) \geq \sum_{j=1}^{l} j^2.$$  

The above lower bound is better than the lower bound in [DW, Remark 3.17].

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2. Preliminary

2.1. 3-fold flopping contractions. Let $X$ be a smooth quasi-projective complex 3-fold. By definition, a flopping contraction is a birational morphism

$$f : X \to Y$$

which is isomorphic in codimension one, $Y$ has only Gorenstein singularities and the relative Picard number of $f$ equals to one. In what follows, we always assume that the exceptional locus $C$ of $f$ is isomorphic to $\mathbb{P}^1$, and set

$$p := f(C) \in Y.$$  

We say that $C \subset X$ is $(a, b)$ curve if $N_{C/X}$ is isomorphic to $\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$. It is well-known that $(a, b)$ is either one of the following:

$$(a, b) = (-1, -1), \ (0, -2), \ (1, -3).$$

We denote by $l$ the length of $\mathcal{O}_{f^{-1}(p)}$ at the generic point of $C$, where $f^{-1}(p)$ is the scheme theoretic fiber of $f$ at $p$. Then we have

$$l \in \{1, 2, 3, 4, 5, 6\}$$

and $l = 1$ if and only if $C$ is not a $(1, -3)$-curve (cf. [KM92, Section 1]). Moreover if $l = 1$, then we have

$$\tilde{\mathcal{O}}_{Y, p} \cong \mathbb{C} [x, y, z, w] / (x^2 + y^2 + z^2 + w^{2k})$$

for some $k \in \mathbb{Z}_{\geq 1}$. The number $k$ is called width of $C$ in [Rei].
2.2. **Contraction algebras.** In the setting of Subsection 2.1, we set $R = \widehat{O}_{Y,p}$, and take the following completion of $(2)$

$$\hat{f} : \widehat{X} := X \times_Y \text{Spec } R \to \hat{Y} := \text{Spec } R.$$ 

Then there is a line bundle $\mathcal{L}$ on $\widehat{X}$ such that $\text{deg}(\mathcal{L}|_C) = 1$. We define the vector bundle $\mathcal{N}$ on $\widehat{X}$ to be the extension

$$0 \to \mathcal{L}^{-1} \to \mathcal{N} \to \mathcal{O}_{\widehat{X}} \to 0$$

given by the minimum generators of $H^1(\widehat{X}, \mathcal{L}^{-1})$. We set $U := O_{\widehat{X}} \oplus N$, $N := \hat{f}_* N$ and

$$A := \text{End}_{\widehat{X}}(U) \cong \text{End}_R (R \oplus N).$$

By Van den Bergh [dB04, Section 3.2.8], we have a derived equivalence

$$R \text{Hom}_{\widehat{X}}(U, -) : D^b \text{Coh}(\widehat{X}) \sim \to D^b \text{mod } A$$

whose inverse is given by $- \otimes_A U$. Here mod $A$ is the category of finitely generated right $A$-modules.

**Definition 2.1.** ([DW, Definition 2.11]) The contraction algebra $A_{\text{con}}$ is defined to be $A/I_{\text{con}}$, where $I_{\text{con}}$ is the two sided ideal of $A$ consisting of morphisms $R \oplus N \to R \oplus N$ factoring through a member of $\text{add}(R)$. Here $\text{add}(R)$ is the set of summands of finite sums of $R$.

By [DW] Proposition 2.12, the algebra $A_{\text{con}}$ is finite dimensional.

**Remark 2.2.** The algebra $A_{\text{con}}$ is commutative if and only if $C$ is not a $(1, -3)$-curve (cf. [DW, Theorem 3.15]). In this case, $A_{\text{con}}$ is isomorphic to $\mathbb{C}[t]/(t^k)$, where $k$ is the width of $C$ which appears in (3). See [DW] Example 3.12.

The contraction algebra $A_{\text{con}}$ coincides with the universal algebra which represents the non-commutative deformation functor of $O_C(-1)$

$$\text{Def}_{O_C(-1)} : \text{Art}_1 \to \text{Sets}.$$ 

Here $\text{Art}_1$ is the category of finite dimensional $\mathbb{C}$-algebras $\Gamma$ with some additional conditions, and the functor (6) assigns each $\Gamma$ to the set of isomorphism classes of flat deformation of $O_C(-1)$ to $\text{Coh}(O_X \otimes_{\mathbb{C}} \Gamma)$. We refer [DW] Section 2] for details of the functor (6). Since $A_{\text{con}}$ represents the functor (6), there is the universal non-commutative deformation of $O_C(-1)$

$$\mathcal{E} \in \text{Coh}(O_X \otimes_{\mathbb{C}} A_{\text{con}}).$$ 

Let $A_{\text{con}}^{ab}$ be the abelization of $A_{\text{con}}$. The algebra $A_{\text{con}}^{ab}$ is a commutative Artinian local $\mathbb{C}$-algebra, which represents the commutative deformation functor

$$\text{cDef}_{O_C(-1)} : \text{cArt}_1 \to \text{Sets}.$$
Here $\text{cArt}_1$ is the category of commutative Artinian local $\mathbb{C}$-algebras, and the functor $\mathcal{F}$ is the restriction of the functor $\mathcal{G}$ to $\text{cArt}_1$. We refer [DW, Section 3] for details of the above representabilities.

2.3. **Flop equivalences.** The contraction algebra $A_{\text{con}}$ plays an important role in describing Bridgeland-Chen’s flop-flop autoequivalence. Let us consider the flop diagram of (2)

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^\dagger \\
\downarrow f & & \downarrow f^\dagger \\
Y & &
\end{array}
\]

By [Bri02] and [Che02], we have the derived equivalence

\[
\Phi_{X \rightarrow X^\dagger}^\mathcal{O}_{X \times Y \times X^\dagger} : D^b \text{Coh}(X) \xrightarrow{\sim} D^b \text{Coh}(X^\dagger).
\]

Here we use the notation in Appendix A for the Fourier-Mukai functors. Composing (10) twice, we obtain the autoequivalence

\[
\Phi_{X^\dagger \rightarrow X}^\mathcal{O}_{X \times Y \times X^\dagger} \circ \Phi_{X \rightarrow X^\dagger}^\mathcal{O}_{X \times Y \times X^\dagger} : D^b \text{Coh}(X) \xrightarrow{\sim} D^b \text{Coh}(X).
\]

The result of [DW, Proposition 7.18] shows that (11) has an inverse isomorphic to the non-commutative twist functor $T_E$ associated to the universal object (7). Namely $T_E$ is the autoequivalence of $D^b \text{Coh}(X)$ which fits into the distinguished triangle

\[
R \text{Hom}(\mathcal{E}, F) \overset{L}{\otimes}_{A_{\text{con}}} \mathcal{E} \rightarrow F \rightarrow T_E(F)
\]

for any $F \in D^b \text{Coh}(X)$. If $C$ is a $(-1, -1)$-curve, the functor $T_E$ coincides with Seidel-Thomas twist [ST01] along $\mathcal{O}_C(-1)$. If $C$ is a $(0, -2)$-curve, then $T_E$ coincides with the author’s generalized twist [Tod07]². The kernel object of the equivalence $T_E$ is given by

\[
\text{Cone} \left( R \text{Hom}_A(A_{\text{con}}, A) \overset{L}{\otimes}_{A_{\text{con}} \otimes A} (\mathcal{U}' \boxtimes \mathcal{U}) \rightarrow \mathcal{O}_{\Delta_X} \right).
\]

Here $\Delta_X \subset X \times X$ is the diagonal (cf. [DW, Lemma 6.16]).

**Lemma 2.3.** The object $R \text{Hom}_A(A_{\text{con}}, A) \overset{L}{\otimes}_{A_{\text{con}} \otimes A} (\mathcal{U}' \boxtimes \mathcal{U})$ is isomorphic to $\mathcal{F}[-2]$ for $\mathcal{F} \in \text{Coh}(X \times X)$ satisfying the following: there is a filtration

\[
0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{\dim A_{\text{con}}} = \mathcal{F}
\]

such that each subquotient $\mathcal{F}_j/\mathcal{F}_{j-1}$ is isomorphic to $\mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)$.

²In [Tod07], it was stated that $T_E$ is isomorphic to (11), but it was wrong: the correct statement is $T_E$ is an inverse of (11). We explain details in Appendix B.
Proof. By the definition of Art1 in [DW, Definition 2.1], there is a $\mathbb{C}$-algebra homomorphism $A_{\text{con}} \to \mathbb{C}$ such that its kernel $n \subset A_{\text{con}}$ is nilpotent. The ideal $n \subset A_{\text{con}}$ is two-sided, and $A_{\text{con}}/n$ is a one-dimensional $A^{\text{op}} \otimes A$-module. We have the filtration of $A^{\text{op}} \otimes A$-modules

$$0 = n^m \subset n^{m-1} \subset \cdots \subset n \subset A_{\text{con}}$$

for some $m > 0$ such that each subquotient $n^i/n^{i+1}$ is an $A_{\text{con}}/n$-module. Since $A_{\text{con}}/n = \mathbb{C}$, the object $n^i/n^{i+1}$ is a finite direct sum of $A_{\text{con}}/n$. Therefore it is enough to show that

$$(13) \quad R\text{Hom}_{A}(A_{\text{con}}/n, A) \overset{L}{\otimes}_{A^{\text{op}} \otimes A} (U^\vee \boxtimes U)$$

$$\cong \mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)[-2].$$

Let $S \in \text{mod } A$ be the object given by $S := R\text{Hom}_{\hat{X}}(U, \mathcal{O}_C(-1))$. Note that we have $\text{dim}_\mathbb{C} S = 1$. The object $S$ is the unique simple $A_{\text{con}}$-module (cf. [DW, Section 2.3]), hence $A_{\text{con}}/n$ viewed as a right $A_{\text{con}}$-module is isomorphic to $S$. On the other hand, the vector bundle $U^\vee \boxtimes U$ on $\hat{X} \times \hat{X}$ is a tilting vector bundle. Hence we have a derived equivalence

$${\mathcal{R}}\text{Hom}_{\hat{X} \times \hat{X}}(U^\vee \boxtimes U, -) : D^b(\text{Coh}(\hat{X} \times \hat{X})) \xrightarrow{\sim} D^b(\text{mod}(A^{\text{op}} \otimes A))$$

with inverse given by $- \otimes_{A^{\text{op}} \otimes A} (U^\vee \boxtimes U)$. Let $\mathbb{D}$ be the dualizing functor $R\text{Hom}_{\hat{X}}(-, \mathcal{O}_{\hat{X}})$ on $D^b(\text{Coh}(\hat{X}))$. We have $\mathbb{D}(\mathcal{O}_C(-1)) \cong \mathcal{O}_C(-1)[-2]$, and

$${\mathcal{R}}\text{Hom}_{\hat{X} \times \hat{X}}(U^\vee \boxtimes U, \mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)[-2])$$

$$\cong {\mathcal{R}}\text{Hom}_{\hat{X} \times \hat{X}}(\mathbb{D}(U) \boxtimes U, \mathbb{D}(\mathcal{O}_C(-1)) \boxtimes \mathcal{O}_C(-1))$$

$$\cong {\mathcal{R}}\text{Hom}_{\hat{X}}(\mathcal{O}_C(-1) \boxtimes \mathcal{O}_C(-1)) \otimes_{\mathcal{O}_C} {\mathcal{R}}\text{Hom}_{\hat{X}}(\mathcal{U}, \mathcal{O}_C(-1))$$

$$\cong {\mathcal{R}}\text{Hom}_A(S, A) \otimes_{\mathcal{O}_C} S$$

$$\cong {\mathcal{R}}\text{Hom}_A(A_{\text{con}}/n, A).$$

Therefore we obtain the desired isomorphism (13). □

2.4. Genus zero Gopakumar-Vafa invariants. The genus zero GV invariants defined in [Kat08] count one dimensional stable sheaves $F$ on Calabi-Yau 3-folds satisfying $\chi(F) = 1$. In the setting of Subsection 2.1, the variety $X$ may not be Calabi-Yau, but so in a neighborhood of $C$. Since $C$ is rigid in $X$, we can define the genus zero GV invariant with curve class $j[C]$ on $X$ as well. Indeed in [Kat08], the genus zero GV invariants of $X$ are shown to coincide with the multiplicities of the Hilbert scheme of $X$ at some subschemes supported on $C$. Let $p \in H \subset Y$ be a general hypersurface, and $\overline{T} \subset X$ its proper
transformation. Then we have $C \subset \overline{H}$. Let $I \subset \mathcal{O}_{\overline{H}}$ be the ideal sheaf of $C$. For $j \geq 1$, we have the subscheme $C^{(j)} \subset X$ given by
\[ O_{C^{(j)}} = (\mathcal{O}_{\overline{H}}/I)^j/Q \]
where $Q$ is the maximum zero dimensional subsheaf of $\mathcal{O}_{\overline{H}}/I$. Let $I \subset \mathcal{O}_{\overline{H}}$ be the ideal sheaf of $C$. For $j \geq 1$, we have the subscheme $C^{(j)} \subset X$ given by
\[ O_{C^{(j)}} = (\mathcal{O}_{\overline{H}}/I)^j/Q \]
where $Q$ is the maximum zero dimensional subsheaf of $\mathcal{O}_{\overline{H}}/I$.

**Definition 2.4.** For $1 \leq j \leq l$, we define $n_j \in \mathbb{Z}_{\geq 1}$ to be
\[ n_j := \dim C \mathcal{O}_{\text{Hilb}(X),C^{(j)}}. \]
By convention, we define $n_j = 0$ for $j > l$.

Since $\mathcal{O}_{\text{Hilb}(X),C^{(j)}}$ is a finitely generated Artinian $\mathbb{C}$-algebra, the number $n_j$ is well-defined. If $l = 1$, the number $n_1$ equals to the width $k$ in (3) as indicated in [BKL01, Section 1]. In general, Katz [Kat08] shows that $n_j$ coincides with the genus zero GV invariant of $X$ with curve class $j[C]$. The number $n_j$ also appears in the context of deformations in the following way. By [BKL01, Section 2.1], there exists a flat deformation of (4) $X \rightarrow Y$ where $T$ is a Zariski open neighborhood of $0 \in \mathbb{A}^1$ such that $g_0 : X_0 \rightarrow Y_0$ is isomorphic to $\hat{f}$ in (1), and $g_t : X_t \rightarrow Y_t$ for $t \in T \setminus \{0\}$ is a flopping contraction whose exceptional locus is a disjoint union of $(-1,-1)$-curves. Here $X_t, Y_t$ are the fibers of $X \rightarrow T, Y \rightarrow T$ at $t \in T$ respectively. Then the number $n_j$ coincides with the number of $g_t$-exceptional $(-1,-1)$-curves $C' \subset X_t$ for $t \neq 0$ whose curve class equals to $j[C]$, i.e. for any line bundle $L$ on $X$, we have
\[ \deg(L|_{C'}) = j \deg(L|_C) \]
where we regard $C$ as a curve on the central fiber of $X \rightarrow T$. In what follows, we write the exceptional locus of $g_t$ for $t \neq 0$ as
\[ C_{j,k} \subset X_t, \ 1 \leq j \leq l, \ 1 \leq k \leq n_j \]
where $C_{j,k}$ is a $(-1,-1)$-curve with curve class $j[C]$.

**3. Proof of Theorem 1.1**

**Proof.** The identity $\text{cwid}(C) = n_1$ is almost obvious from the definitions of both sides. Indeed since $A_{\text{con}}^{ab}$ represents the commutative deformation functor (5), the scheme $\text{Spec} A_{\text{con}}^{ab}$ is the component of the moduli scheme of one dimensional stable sheaves on $X$ containing
$\mathcal{O}_C(-1)$. By tensoring the line bundle $\mathcal{L}$ in Subsection 2.2, the scheme $\text{Spec} \ A^{ab}_{\text{con}}$ is isomorphic to the component of the moduli scheme of stable sheaves containing $\mathcal{O}_C$, which defines the invariant $n_1$. By the proof of [Kat08, Proposition 3.3], the degree of the virtual fundamental cycle of $\text{Spec} \ A^{ab}_{\text{con}}$ coincides with the dimension of $A^{ab}_{\text{con}}$. Therefore $\text{cwid}(C) = n_1$ holds.

We show the identity of $\text{wid}(C)$. The morphism $g$ in (14) is a flopping contraction, and the argument of [Che02, Section 6] shows that $g$ admits a flop

\[ \xymatrix{ \mathcal{X} \ar@/^/[rr]^\psi \ar@/_/[dr]_g & & \mathcal{X}^\dagger \ar@/_/[dl]_{g^\dagger} } \]

such that we have the derived equivalence

\[ \Phi_{\mathcal{X} \to \mathcal{X}^\dagger}^{\mathcal{O}_{\mathcal{X} \times Y}, \mathcal{X}^\dagger} : D^b \text{Coh}(\mathcal{X}) \sim D^b \text{Coh}(\mathcal{X}^\dagger). \]

By composing the above equivalence twice, we obtain the autoequivalence

\[ (16) \quad \Phi_{\mathcal{X}^\dagger \to \mathcal{X}}^{\mathcal{O}_{\mathcal{X} \times Y}, \mathcal{X}^\dagger} \circ \Phi_{\mathcal{X} \to \mathcal{X}^\dagger}^{\mathcal{O}_{\mathcal{X} \times Y}, \mathcal{X}^\dagger} : D^b \text{Coh}(\mathcal{X}) \sim D^b \text{Coh}(\mathcal{X}). \]

Let $\Psi$ be an inverse of the equivalence (16), and

\[ \mathcal{P} \in D^b \text{Coh}(\mathcal{X} \times \mathcal{T} \mathcal{X}) \]

the kernel object of $\Psi$. By [Che02, Lemma 6.1], for each $t \in T$, we have the commutative diagram

\[ \xymatrix{ D^b \text{Coh}(\mathcal{X}) \ar[r]^\Psi & D^b \text{Coh}(\mathcal{X}) \ar[d]^{L_{i^t}} \ar[d]_{L_{i^t}} \ar[r]^{\Phi_\mathcal{X}^t \mathcal{X}^\dagger} & D^b \text{Coh}(\mathcal{X}_t) \ar[d]_{L_{j^t}} \ar[r]^{\Psi_t} & D^b \text{Coh}(\mathcal{X}_t) \ar[d]_{L_{j^t}} \ar[r]^{\Phi_\mathcal{X}^t \mathcal{X}^\dagger} & D^b \text{Coh}(\mathcal{X}_t). \}

Here $i^t : \mathcal{X}_t \hookrightarrow \mathcal{X}$ is the inclusion, and $\Psi_t$ is the Fourier-Mukai functor with kernel $\mathcal{P}_t := L_{i^t} \mathcal{P}$, where $j^t$ is the inclusion

\[ j^t := (i^t \times i^t) : \mathcal{X}_t \times \mathcal{X}_t \hookrightarrow \mathcal{X} \times \mathcal{T} \mathcal{X}. \]

The functor $\Psi_t$ is an equivalence, and it has an inverse given by the composition (cf. [Che02, Corollary 4.5])

\[ (17) \quad \Phi_{\mathcal{X}_t \to \mathcal{X}_t}^{\mathcal{O}_{\mathcal{X}_t \times \mathcal{Y}_t}, \mathcal{X}_t^\dagger} \circ \Phi_{\mathcal{X}_t \to \mathcal{X}_t}^{\mathcal{O}_{\mathcal{X}_t \times \mathcal{Y}_t}, \mathcal{X}_t^\dagger} : D^b \text{Coh}(\mathcal{X}_t) \sim D^b \text{Coh}(\mathcal{X}_t). \]

Therefore by [DW, Proposition 7.18], the equivalence $\Psi_0$ is isomorphic to the non-commutative twist functor $T_\mathcal{E}$ in (12). By the uniqueness of Fourier-Mukai kernels in Lemma A.1 below, we have

\[ (18) \quad \mathcal{P}_0 \cong \text{Cone} \left( \mathcal{F}_0[-2] \to \mathcal{O}_{\Delta_{x_0}} \right). \]

Here $\mathcal{F}_0$ is a sheaf $\mathcal{F}$ on $X \times X$ given in Lemma 2.3 restricted to $\tilde{X} \times \tilde{X}$. 
For $t \neq 0$, the birational map $\mathcal{X}_t \dashrightarrow \mathcal{X}^t$ is the composition of flops at $(-1, -1)$-curves $C_{j,k}$ for $1 \leq j \leq l$, $1 \leq k \leq n_j$. Hence the equivalence $\Psi_t$ for $t \neq 0$ is isomorphic to the compositions of all the spherical twists along $\mathcal{O}_{C_{j,k}}(-1)$ for $1 \leq j \leq l$, $1 \leq k \leq n_j$. Therefore using Lemma A.1 again, we have

\begin{equation}
(19) \quad \mathcal{P}_t \cong \text{Cone} \left( \mathcal{F}_t[-2] \rightarrow \mathcal{O}_{\Delta \mathcal{X}_t} \right)
\end{equation}

where $\mathcal{F}_t$ is a sheaf on $\mathcal{X}_t \times \mathcal{X}_t$ defined by

\begin{equation}
(20) \quad \mathcal{F}_t := \bigoplus_{j=1}^l \bigoplus_{k=1}^{n_j} \mathcal{O}_{C_{j,k}}(-1) \boxtimes \mathcal{O}_{C_{j,k}}(-1).
\end{equation}

**Lemma 3.1.** We have $\mathcal{H}^i(\mathcal{P}) = 0$ for $i \neq 0, 1$.

**Proof.** For any $t \in T$, we have the distinguished triangle

$$\mathcal{P} \rightarrow \mathcal{P} \rightarrow j_{tt}^* \mathcal{P}_t.$$ 

By (18) and (19), we have $\mathcal{H}^i(\mathcal{P}_t) = 0$ for any $t \in T$ and $i \neq 0, 1$. By taking the long exact sequence of cohomologies of the above triangle, we obtain $j_{tt}^* \mathcal{H}^i(\mathcal{P}) = 0$ for any $t \in T$ and $i \neq 0, 1$. Therefore we have $\mathcal{H}^i(\mathcal{P}) = 0$ for $i \neq 0, 1$.

**Lemma 3.2.** We have $\mathcal{H}^0(\mathcal{P}) \cong \mathcal{O}_{\Delta \mathcal{X}}$ and $\mathcal{H}^1(\mathcal{P})$ is flat over $T$. Furthermore we have $j_{tt}^* \mathcal{H}^1(\mathcal{P}) \cong \mathcal{F}_t$ for any $t \in T$.

**Proof.** By Lemma 3.1 we have the distinguished triangle in $D^b \text{Coh}(\mathcal{X} \times_T \mathcal{X})$

$$\mathcal{H}^0(\mathcal{P}) \rightarrow \mathcal{P} \rightarrow \mathcal{H}^1(\mathcal{P})[-1].$$

Applying $Lj_{tt}^*$, we obtain the distinguished triangle in $D^b \text{Coh}(\mathcal{X}_t \times \mathcal{X}_t)$

$$Lj_{tt}^* \mathcal{H}^0(\mathcal{P}) \rightarrow \mathcal{P}_t \rightarrow Lj_{tt}^* \mathcal{H}^1(\mathcal{P})[-1].$$

By taking the long exact sequence of cohomologies, we have

$$Lj_{tt}^* \mathcal{H}^0(\mathcal{P}) \cong j_{tt}^* \mathcal{H}^0(\mathcal{P}), \quad \mathcal{F}_t \cong j_{tt}^* \mathcal{H}^1(\mathcal{P})$$

and the exact sequence

\begin{equation}
(21) \quad 0 \rightarrow j_{tt}^* \mathcal{H}^0(\mathcal{P}) \rightarrow \mathcal{O}_{\Delta \mathcal{X}_t} \rightarrow \mathcal{H}^{-1}(Lj_{tt}^* \mathcal{H}^1(\mathcal{P})) \rightarrow 0.
\end{equation}

Below we denote by $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times_T \mathcal{X}$, $\Delta_t: \mathcal{X}_t \rightarrow \mathcal{X}_t \times \mathcal{X}_t$ the diagonal morphisms, and $\mathcal{C}$ the exceptional locus of $g: \mathcal{X} \rightarrow \mathcal{Y}$. The isomorphism $\mathcal{F}_t \cong j_{tt}^* \mathcal{H}^1(\mathcal{P})$ implies that $\mathcal{H}^1(\mathcal{P})$ is supported on $\mathcal{C} \times \mathcal{C}$, hence $\mathcal{H}^{-1}(Lj_{tt}^* \mathcal{H}^1(\mathcal{P}))$ is supported on $\mathcal{C}_t \times \mathcal{C}_t$. The exact sequence (21) also implies that $\mathcal{H}^{-1}(Lj_{tt}^* \mathcal{H}^1(\mathcal{P}))$ is supported on $\Delta_{\mathcal{X}_t}$, hence on $\Delta_{\mathcal{X}_t} \cap (\mathcal{C}_t \times \mathcal{C}_t) = \Delta_{\mathcal{X}_t}(\mathcal{C}_t)$. It follows that $\mathcal{H}^0(\mathcal{P})$ is written as $\Delta_{\mathcal{X}_t}$ for a rank one torsion free sheaf $\mathcal{I}$ on $\mathcal{X}_t$, and the exact sequence (21) is given by $\Delta_{tt}$ of the exact sequence of the following form

\begin{equation}
(22) \quad 0 \rightarrow i_{tt}^* \mathcal{I} \rightarrow \mathcal{O}_{\Delta_{\mathcal{X}_t}} \rightarrow \mathcal{O}_{\mathcal{C}_{tt}} \rightarrow 0
\end{equation}
for some subscheme $C'_t \subset X$ supported on $C_t$. Also by the generic flatness, there is a non-empty Zariski open subset $U \subset T$ such that $H^{-1}(Lj'_t^*H^1(P)) = 0$ for all $t \in U$. This implies that $C'_t = \emptyset$ for all $t \in U$, hence $\mathcal{I}$ is isomorphic to $\mathcal{O}_X$ away from $C_t$ for $t \in T \setminus U$. By taking the double dual of $\mathcal{I}$, we obtain the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_{C'} \to 0$$

where $C'$ is supported on $C_t$ for $t \in T \setminus U$. If $C' \neq \emptyset$, then $j'_t^*H^0(\mathcal{P}) \cong \Delta_t^*i_t^*\mathcal{I}$ contains the non-zero sheaf $\Delta_t^*i_t^*H^1(O_{C'})$ for some $t \in T \setminus U$ supported on $C_t$, which contradicts to (21). Therefore $C' = \emptyset$ and $H^0(\mathcal{P}) \cong O_{\Delta_X}$ holds.

Now in the sequence (22), we have $i'_t^*\mathcal{I} \cong O_{X_t}$ for any $t \in T$, hence $C'_t = \emptyset$ as $C'_t$ has codimension bigger than or equal to two in $X_t$. This implies that $H^{-1}(Lj'_t^*H^1(P)) = 0$ for any $t \in T$, hence $H^1(P)$ is flat over $T$. □

By the above lemma, the sheaf $\mathcal{F}_t$ for $t \neq 0$ is a flat deformation of $\mathcal{F}_0$. Since they have compact supports, $\mathcal{F}_0$ and $\mathcal{F}_t$ have the same Hilbert polynomials. It follows that, for a $g$-ample line bundle $\mathcal{L}$ on $X$ with $d := \deg(\mathcal{L}|_C) > 0$, we have the equality

$$(23) \quad \chi(\mathcal{F}_0 \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \chi(\mathcal{F}_t \otimes (\mathcal{L} \boxtimes \mathcal{L})).$$

By Lemma 2.3 and the Riemann-Roch theorem, we have

$$\chi(\mathcal{F}_0 \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \dim C \cdot \chi(O_C(-1) \otimes \mathcal{L})^2 = \dim C \cdot d^2.$$

By the definition of $\mathcal{F}_t$ for $t \neq 0$ in (20), we have

$$\chi(\mathcal{F}_t \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \sum_{j=1}^t \sum_{k=1}^{n_j} \chi(O_{C_{j,k}}(-1) \otimes \mathcal{L})^2 = \sum_{j=1}^t j^2 \cdot n_j \cdot d^2.$$

Here we have used the relation (15) for $C' = C_{j,k}$. Since $d > 0$, the equality (23) implies the desired equality for $\text{wid}(C)$. □

APPENDIX A. Uniqueness of Fourier-Mukai kernels

Let $Y$ be a quasi-projective complex variety, or a spectrum of a completion of a finitely generated $\mathbb{C}$-algebra at some maximum ideal. Suppose that $f_i : X_i \to Y$ are projective morphisms for $i = 1, 2$, and $X_i$ are regular schemes. Given an object

$$\mathcal{P} \in D^b \text{Coh}(X_1 \times X_2)$$

supported on $X_1 \times_Y X_2$, we have the Fourier-Mukai functor

$$\Phi^\mathcal{P}_{X_1 \to X_2} : D^b \text{Coh}(X_1) \to D^b \text{Coh}(X_2)$$
defined by

$$
\Phi_P^{X_1 \to X_2}(-) := \mathbb{R}p_{2*}(Lp_1^*(-) \otimes P)
$$

where $p_i: X_1 \times X_2 \to X_i$ is the projection. The above functor preserves coherence since $p_2|\text{supp}(P)$ is projective. For another regular scheme $X_3$, a projective morphism $f_3: X_3 \to Y$ and an object $Q \in D^b\text{Coh}(X_2 \times X_3)$ supported on $X_2 \times_Y X_3$, we have

$$
\Phi^Q_{X_2 \to X_3} \circ \Phi^P_{X_1 \to X_2} \simeq \Phi^{Q\circ P}_{X_1 \to X_3}
$$

where $Q \circ P$ is defined by (cf. [Che02, Proposition 2.3])

$$
Q \circ P := \mathbb{R}p_{13*}(p_{12}^*P \otimes p_{23}^*Q).
$$

Here $p_{ij}: X_1 \times X_2 \times X_3 \to X_i \times X_j$ is the projection.

If $Y = \text{Spec } \mathbb{C}$ and $\Phi^P_{X_1 \to X_2}$ is an equivalence, then Orlov [Orl97] showed that the kernel object $P$ is unique up to an isomorphism, i.e. $\Phi^P_{X_1 \to X_2} \simeq \Phi^Q_{X_1 \to X_2}$ implies $P \simeq Q$. It should be well-known that the same claim holds without $Y = \text{Spec } \mathbb{C}$ assumption, but as the author cannot find a reference we include a proof here.

**Lemma A.1.** For $P, Q \in D^b\text{Coh}(X_1 \times X_2)$ supported on $X_1 \times_Y X_2$, suppose that the following conditions hold:

- We have an isomorphism of functors $\Phi^P_{X_1 \to X_2} \simeq \Phi^Q_{X_1 \to X_2}$.
- The functors $\Phi^P_{X_1 \to X_2}$, $\Phi^Q_{X_1 \to X_2}$ are equivalences.

Then we have $P \simeq Q$.

**Proof.** Let $Q^*$ be the object of $D^b\text{Coh}(X_1 \times X_2)$ given by

$$
Q^* := \mathbb{R}\text{Hom}_{X_1 \times X_2}(Q, O_{X_1 \times X_2}) \otimes p_1^!*\omega_{X_1}[\dim X_1].
$$

By the Grothendieck duality, the functor $\Phi^{Q^*}_{X_2 \to X_1}$ is the right adjoint of $\Phi^Q_{X_2 \to X_1}$, hence an inverse of it. We have

$$
\Phi^{Q^*}_{X_2 \to X_1} \circ \Phi^P_{X_1 \to X_2} \simeq \Phi^{Q^*\circ P}_{X_1 \to X_1}
$$

and it is isomorphic to the identity functor. Then $\Phi^{Q^*\circ P}_{X_1 \to X_1}$ sends $O_x$ to $O_x$ for any $x \in X_1$, and $O_{X_1}$ to $O_{X_1}$. Applying the argument of [Huy06 Corollary 5.23], it follows that $Q^* \circ P \simeq O_{\Delta X_1}$. Similarly we have $Q \circ Q^* \simeq O_{\Delta X_2}$. We obtain

$$
P \simeq O_{\Delta X_2} \circ P \simeq Q \circ Q^* \circ P \simeq Q \circ O_{\Delta X_1} \simeq Q
$$

as desired. \qed
Appendix B. Correction on flop-flop autoequivalence

In this occasion, I would correct a wrong statement in [Tod07, Section 3] on the description of flop-flop autoequivalence. Let us consider the equivalence

\[ \Phi_{O_{X \times Y}^{\times 1}} \circ \Phi_{O_{X \times Y}^{\times 1}} : D^b \text{Coh}(X) \rightarrow D^b \text{Coh}(X) \]  

associated to the flop diagram (9). In [Tod07, Theorem 3.1], it was stated that if \( C \) is either a \((-1, -1)\)-curve or a \((0, -2)\)-curve, then the functor (24) is isomorphic to the generalized (commutative) twist functor \( T_{\bar{E}} \). However this turns out to be wrong: the correct statement is that the equivalence (24) is an inverse of \( T_{\bar{E}} \). Indeed the statement in [Tod07, Section 3] that the equivalence \[ \Phi_{O_{X \times Y}^{\times 1}} : D^b \text{Coh}(X) \rightarrow D^b \text{Coh}(X^{\dagger}) \]  

takes \( O_C(-1)[1] \) to \( O_{C^{\dagger}}(-1) \) was wrong: it should be corrected that \( (25) \) takes \( O_C(-1) \) to \( O_{C^{\dagger}}(-1)[1] \). Then replacing \( T_{\bar{E}} \) with \( T_{\bar{E}}^{-1} \) in the proof of [Tod07, Theorem 3.1], we obtain the statement that (11) is isomorphic to \( T_{\bar{E}}^{-1} \).

We explain why the above statement in [Tod07, Section 3] was wrong. In [Tod07, Section 3], I referred [Tod08, Ver 1, Lemma 5.1], which in turn referred [Bri02, (4.8)] that the equivalence (25) induces the equivalence

\[ \text{Per}(X/Y) \rightarrow \text{Per}(X^{\dagger}/Y). \]  

(Here we have used the fact that the equivalence (25) coincides with the equivalence \( \Phi \) given in [Bri02, Section 4] by [Che02].) However (25) was not correct: it should be corrected that (25) induces the equivalence

\[ \text{Per}(X/Y) \rightarrow \text{Per}(X^{\dagger}/Y). \]  

Indeed let \( C_X \subset \text{Coh}(X) \) be the category of sheaves \( F \) with \( Rf_*F = 0 \). Then [Bri02, (4.5)] shows that (25) takes \( C_X \) to \( C_{X^{\dagger}}[1] \). On the other hand, as \( p\text{Per}(X/Y) \) is the gluing of \( \text{Coh}(Y) \) and \( C_X[-p] \) (not \( C_X[p] \)) by the definition, the equivalence (25) should reduce the perversity one. After correcting (26) as (27), the argument of [Tod08, Ver 1, Lemma 5.1] shows that (25) takes \( O_C(-1) \) to \( O_{C^{\dagger}}(-1)[1] \).

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3In the notation of [Bri02, (4.8)], the equivalence \( p\text{Per}(W/X) \cong p+1\text{Per}(Y/X) \) should be corrected as \( p\text{Per}(W/X) \cong p-1\text{Per}(Y/X) \).
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