On various types of transitivity for the problem of ordering

T M Ledeneva, N A Kaplieva
Department of Applied Mathematics, Informatics and Mechanics, Voronezh State University, Voronezh, Russia
E-mail: ledeneva-tm@yandex.ru, kaplieva@amm.vsu.ru

Abstract. The article reviews the problem of ranking of a set of alternatives based on the expert matrix of pairwise comparisons. We proposed an algorithm for processing this matrix and its theoretical justification, in which the transitivity property based on the use of triangular norms and conorms plays a key role. The decomposition theorem allows us to proceed to the α-cut of the fuzzy order relation and construct Hasse diagrams for them. For a fixed value α, the Hasse diagram is a graph of the partially ordered set of alternatives. The article also gives the analysis of the influence of various types of transitivity on the ranking of alternatives.

1. Introduction
One of the most common methods of obtaining expert information is the procedure of pairwise comparison of alternatives. The expert is consistently presented with pairs of alternatives. In each pair it is suggested to indicate a better alternative or to determine their equivalence. Experience shows that the expert is not always consistent in his preferences. Some researchers note that in about 30% of comparisons the transitivity of preferences is violated [1], which does not allow us to obtain a solution to the problem. In this case, a special procedure is needed to process the information received from the expert, whose idea is to move to a binary relation that has all the necessary properties to obtain a partial or linear order on a set of alternatives. The purpose of the article is to substantiate this procedure and to study the influence of various types of transitivity on the ranking of alternatives.

2. Materials and methods
2.1. Basic definitions
Definition 1. The fuzzy relation $R$ on a set $X$ is a fuzzy subset of the set $X \times X = X^2$ with the membership function $\mu_R : X \times X \to [0, 1]$, which estimates the degree of execution of the relation $xRy$.

We shall denote the family of fuzzy relations on $X$ by $F(X^2)$. In the matrix form, the fuzzy relation $R$ on the finite set $X$ ($|X| = n$) is given by the matrix $R = (r_{ij})_{n \times n}$, where $r_{ij} = \mu_R(x_i, x_j)$. In the graphical form, the fuzzy relation $R \in F(X^2)$ can be specified in the form of the graph $G_R$ with a set of vertices $X$, each arc $(x_i, x_j)$ of which is assigned a weight – the value of the membership function $\mu_R(x_i, x_j)$. 

**Definition 2.** Suppose $R_1, R_2 \in F(X^2)$. It is said that the fuzzy relation $R_1$ is included in the fuzzy relation $R_2$ ($R_1 \subseteq R_2$), if $\forall (x, y) \in X^2 (\mu_{R_1}(x, y) \leq \mu_{R_2}(x, y))$. If a strict inequality holds for all pairs $(x, y)$, then it is said to be a strict inclusion.

**Definition 3.** $\alpha$-cut of the fuzzy relation $R \in F(X^2)$ is an ordinary relation of the form

$$R_\alpha = \{(x, y) \in X^2 \mid \mu_R(x, y) \geq \alpha\}, \quad \alpha \in (0, 1].$$

The following theorem is of great importance for applications.

**The (decomposition) theorem** [2]. Let $R \subseteq F(X^2)$ be a fuzzy relation on $X$, then we have formula

$$R = \max_\alpha \{\alpha \cdot R_\alpha\},$$

where $\alpha \in (0, 1]$, $R_\alpha$ - $\alpha$-cut, and the sequence of $\alpha$-cuts is ordered by the inclusion relation.

Triangular $T$-norms and $S$-conorms [3] are used to determine the operations of fuzzy relation intersection and union, respectively.

**Definition 4.** A triangular norm ($t$-norm) is a binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$ we have the following properties:

a) $T(x, y) = T(y, x)$ (commutativity),

b) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),

c) $T(x, y) \leq T(x, z)$, if $y \leq z$ (monotonicity),

d) $T(1, x) = x$ (neutral element 1).

**Definition 5.** A triangular conorm ($t$-conorm) is a binary operation $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$ the commutativity, associativity, monotonicity properties and property $S(x, 0) = x$ are fulfilled.

**Definition 6.** The pairs of triangular norms and conorms $T(x, y)$ and $S(x, y)$ are called dual ones if for them the equalities (de Morgan laws)

$$S(x, y) = N(T(N(x), N(y))), \quad T(x, y) = N(S(N(x), N(y))),$$

where $N(x)$ is a strong negation function, are satisfied for all $(x, y) \in [0, 1]^2$.

Examples of dual triangular norms and conorms are presented in Table 1.

| $T(x, y)$ | $S(x, y)$ |
|-----------|-----------|
| $T_M(x, y) = \min(x, y)$ | $S_M(x, y) = \max(x, y)$ |
| $T_P(x, y) = xy$ | $S_P(x, y) = x + y - xy$ |
| $T_L(x, y) = \max(0, x + y - 1)$ | $S_L(x, y) = \min(1, x + y)$ |
| $T_D(x, y) = \{0, \min(x, y), \text{ otherwise}\}$ | $S_D(x, y) = \{1, \max(x, y), \text{ otherwise}\}$ |
| $T_\alpha(x, y) = \frac{xy}{\alpha + (1 - \alpha)(x + y - xy)}, \alpha > 0$ | $S_\beta(x, y) = \frac{(\beta - 1)xy + x + y}{1 + \beta xy}, \beta > -1$ |

**Definition 7.** If for two $t$-norms $T_1$ and $T_2$ we have an inequality $T_1(x, y) \leq T_2(x, y)$ for all $(x, y) \in [0, 1]^2$, then $T_1$ is said to be weaker than $T_2$ (or $T_2$ is stronger than $T_1$).
Between the four basic $t$-norms we have these strict inequalities

$$T_D < T_L < T_P < T_M.$$  

The duality changes the order: if, for some $t$-norms $T_1$ and $T_2$ we have $T_1(x, y) \leq T_2(x, y)$, and if $S_1$ and $S_2$ are the dual $t$-conorms of $T_1$ and $T_2$, respectively, then we get $S_1(x, y) \geq S_2(x, y)$. Thus,

$$S_D > S_L > S_P > S_M.$$  

The operation of intersection $R_1 \cap T_R$ of fuzzy relations $R_1, R_2 \in F(X^2)$ is determined using the triangular norm $T(x, y)$ in the form

$$\mu_{R_1 \cap T_R}(x, y) = T(\mu_{R_1}(x, y), \mu_{R_2}(x, y)).$$

Accordingly, the union $R_1 \cup R_2$ is determined by the membership function

$$\mu_{R_1 \cup R_2}(x, y) = S(\mu_{R_1}(x, y), \mu_{R_2}(x, y)).$$

Composition is related to special operations on fuzzy relations.

**Definition 8.** Let $R_1 \in F(X \times Z)$, $R_2 \in F(Z \times Y)$.

$(max-T)$-composition is a relation $R_1 \circ_T R_2$ with the membership function

$$\mu_{(max-T)(R_1, R_2)}(x, y) = \max_{z \in Z} T(\mu_{R_1}(x, z), \mu_{R_2}(z, y)).$$

$(min-S)$-composition is defined by the relation $R_1 \bullet_S R_2$ and has the membership function

$$\mu_{(min-S)(R_1, R_2)}(x, y) = \min_{z \in Z} S(\mu_{R_1}(x, z), \mu_{R_2}(z, y)).$$

We have the following assertions.

**Assertion 1.** Let $R \in F(X^2)$ and $T_1(x, y), T_2(x, y)$ be a pair of triangular norms, such that $T_1$ is weaker than $T_2$, then $R \overset{T_1}{\circ} R \subseteq R \overset{T_2}{\circ} R$.

**Assertion 2.** Let $R \in F(X^2)$ and $S_1(x, y), S_2(x, y)$ be a pair of triangular conorms, such that $S_1$ is weaker than $S_2$, then $R \overset{S_1}{\bullet} R \subseteq R \overset{S_2}{\bullet} R$.

From these assertions it follows that the above orderings of the basic triangular norms and conorms induce the following inclusions systems for $(max-T)$-compositions and $(min-S)$-compositions:

$$R \overset{L}{\circ} R \subseteq R \overset{P}{\circ} R \subseteq R \overset{D}{\circ} R \subseteq R \overset{M}{\circ} R,$$

$$R \overset{L}{\bullet} R \supseteq R \overset{P}{\bullet} R \supseteq R \overset{D}{\bullet} R \supseteq R \overset{M}{\bullet} R.$$  

The properties of the introduced compositions and their interrelations are determined in [4], in particular:

a) $(max-T)$ and $(min-S)$-compositions are associative,

b) let $R_1 \in F(X \times Z)$, $R_2, R_3 \in F(Z \times Y)$, $T$ and $S$ be arbitrary triangular $t$-norms and $t$-conorms, respectively, then the $(max-T)$-composition is distributive with respect to the $S_M$-union $\cup$, i. e.

$$R_1 \overset{T}{\circ}(R_2 \cup R_3) = (R_1 \overset{T}{\circ} R_2) \cup (R_1 \overset{T}{\circ} R_3),$$
and \((\min-S)\)-composition is distributive with respect to \(T_M\)-intersection \(\cap\), i. e.

\[
R_1 S (R_2 \cap R_3) = (R_1 S R_2) \cap (R_1 S R_3);
\]

c) let \(R_1 \in F(X \times Z), R_2 \in F(Z \times Y)\) and \(T, S\) be a pair of dual norms, then

\[
\mu_{(\max-T)}(R_1, R_2)(x, y) = \mu_{(\min-S)}(R_1, R_2)(x, y);
\]

i. e.

\[
\mathcal{R}_1^T \circ \mathcal{R}_2 = R_1 S R_2.
\]

We note that property c) allows us to speak of the duality of compositions \(R_1 S R_2\) and \(R_1 T R_2\) for the given dual pair \((T, S)\).

2.2. Properties of fuzzy binary relations

For fuzzy relations, the properties of reflexivity, antireflexivity, symmetry, and antisymmetry are introduced in the same way as in the crisp case [2].

The following assertions are proved.

**Assertion 3.** If \(R_1, R_2 \in F(X^2)\) is a reflexive relation, then \(R_1 T R_2\) is also a reflexive relation.

**Assertion 4.** If \(R_1, R_2 \in F(X^2)\) is an antireflexive relation, then \(R_1 S R_2\) is also the antireflexive relation.

The relation of transitivity is richer in content than in a crisp case.

**Definition 9.** The fuzzy relation \(R \in F(X^2)\) is called \((\max-T)\)-transitive if \(R T R \subseteq R\).

**Definition 10.** The fuzzy relation \(R \in F(X^2)\) is called \((\min-S)\)-transitive if \(R \subseteq R S R\).

It is known [2] that if \(R\) is a \((\max-min)\)-transitive relation, then \(\mathcal{R}\) has the property of \((\min-max)\)-transitivity. However, there is a more general

**Assertion 5.** Let \((T, S)\) be a pair of dual norm and conorm, and the fuzzy relation \(R \in F(X^2)\) has the property of \((\max-T)\)-transitivity, then the relation \(\mathcal{R}\) has the property of \((\min-S)\)-transitivity.

The proof of this assertion follows from property c.

It follows from assertion 1 that if \(T_1(x, y), T_2(x, y)\) is a pair of triangular norms, such that \(T_1\) is weaker than \(T_2\), then \((\max-T_2)\)-transitivity follows from \((\max-T_1)\)-transitivity. Similarly, from assertion 2 we obtain that, if \(S_1\) is weaker than \(S_2\), then \((\min-S_2)\)-transitivity of the relation \(R\) implies \((\min-S_1)\)-transitivity.

The transformation of a nontransitive relation into a transitive relation is carried out by means of a transitive closure operation.

**Definition 11.** The transitive closure \(\widehat{R}\), in which the degree of the relation is defined recursively by the operation of a \((\max-T)\)-composition, is called the \(T\)-transitive closure (we shall denote \(\widehat{R}[T]\) to emphasize the dependence on the \(T\)-norm).

**Definition 12.** The transitive closure \(\widehat{R}\), in which the degree of the relation is determined using the \((\min-S)\)-composition, is called the \(S\)-transitive closure (notation \(\widehat{R}[S]\)).

**Assertion 6.** The operation of \((\max-T)\)-transitive closure keeps the properties of reflexivity and symmetry.
Assertion 7. The operation of (min-S)-transitive closure keeps the properties of antireflexivity and antisymmetry.

Thus, if the given relation is reflexive and symmetric, but not transitive, then the transitive closure keeping reflexivity and symmetry adds a (max-T)-transitivity of a certain type that is associated with the corresponding triangular norm. Similarly, if the relation has antireflexivity and antisymmetry properties, the transitive closure adds (min-S)-transitivity to these properties.

The combination of different properties of fuzzy relations generates their types. So, a reflexive, antisymmetric and transitive relation is called an order. If, instead of reflexivity, antireflexivity is required, then we obtain a strict order.

We note an important point. According to the decomposition theorem, a fuzzy relation is represented by a system of ordinary binary relations. It is important that the properties of the initial fuzzy relation \( R \) induce the properties of the corresponding ordinary relations \( \alpha \)-cuts \( R_\alpha \) [4]. We can see that \( \alpha \)-cuts of fuzzy ordering relations are ordinary ordering relations. If the fuzzy relation \( R \) has some type of transitivity, then the ordinary relations \( R_\alpha \) are transitive in the usual sense, and, consequently, the corresponding graphs have no circuits [5]. On the other hand, if the graph does not have circuits, then we can obtain a decomposition into levels and construct an ordinal function for it. This decomposition has the following properties:

a) at the top level, there are vertices that do not have incoming arcs,

b) from a certain vertex of a fixed level, the arcs lead only to the vertices of the lower levels,

c) the vertices on one level are not adjacent,

d) the vertices with no incoming arcs correspond to the pendant vertices.

Definition 13 ([2]). An ordinal function is a mapping \( \rho : X \to \{1, \ldots, K\} \) such that if the vertex \( x_i \in X \) is at the level \( N_k \) in the decomposition into levels of a graph corresponding to a given ordering relation, then \( \rho(x_i) = k \), where \( k \in \{1, \ldots, K\} \), \( K \) is the number of levels.

We can see that for a given vertex \( x_i \), the value of the ordinal function \( \rho(x_i) \) is the length of the maximal path from the source vertex of the graph to a given vertex, where the vertex without the incoming arcs is understood as the source vertex, and the path length is understood as the number of arcs that this path consists of. If we remove transitively closing arcs in the decomposition of the graph, then we obtain the Hasse diagram [5], which is a graph of a partially ordered set.

3. Results and discussion

3.1. Algorithm for ranking alternatives

Let us consider the following problem. Let there be given a set of alternatives \( X = \{x_1, \ldots, x_n\} \), which must be ordered by preference. As a result of the expert survey, the matrix of pairwise comparisons \( R = (r_{ij})_{n \times n} \) is formed, the element \( r_{ij} \in [0, 1] \) of which is the degree of preference for the alternative \( x_i \) in comparison with the alternative \( x_j \). If the alternatives are incomparable, then the corresponding elements in the matrix \( R \) are equal to 0. If \( x_i \) and \( x_j \) are equivalent, then \( r_{ij} = r_{ji} \). Thus, the relation \( R \) is reflexive and asymmetric. We need to order the alternatives according to their preferences.

To solve this problem, we propose the following algorithm:

**S1.** Proceed from the matrix of pairwise comparisons \( R \) to the matrix \( R' = (r'_{ij})_{n \times n} \) by the rule

\[
r'_{ij} = \begin{cases} 
  r_{ij} - r_{ji}, & \text{if } r_{ij} \geq r_{ji}, \\
  0, & \text{if } r_{ij} < r_{ji}.
\end{cases}
\]

We note that this relation \( R' \) is antireflexive and has a perfect antisymmetry, since \( r_{ij} \cdot r_{ji} = 0 \).

**S2.** Choose the appropriate type of (min-S)-composition and proceed to the transitive relation \( P = R'_{[S]} \) with the help of transitive closure while retaining the properties of antireflexivity and antisymmetry.
S3. By the decomposition theorem, we introduce a fuzzy relation \( P \) through ordinary strict-order relations \( P_\alpha \).

S4. With given value \( \alpha \) for the relation \( P_\alpha \), construct a graph, decompose it into levels, and select the Hasse diagram in the resulting hierarchy.

S5. Form the desired ordering on the basis of the Hasse diagram.

3.2. Results of the computing experiment

In order to illustrate the effect of different types of transitivity on the hierarchical structure being formed and the Hasse diagram, we considered examples in which the number of alternatives was not more than 15 (it is known that the dimension of the pairwise comparison method is limited by this number). Let us highlight one of the examples. The results of the rest of the examples fully confirm the conclusions and recommendations made below.

In the procedure of pairwise comparisons, the expert was presented with 7 alternatives. As a result of estimating the degree of preference in each pair of alternatives, a matrix of pairwise comparisons with elements from \([0, 1]\) was generated

\[
\begin{pmatrix}
1.0 & 0.7 & 0.6 & 0.3 & 0.9 & 0.8 & 0.5 \\
0.4 & 1.0 & 0.1 & 0.2 & 0.7 & 0.9 & 0.1 \\
0.0 & 0.0 & 1.0 & 0.5 & 0.6 & 0.8 & 0.2 \\
0.0 & 0.1 & 0.0 & 1.0 & 0.8 & 0.4 & 0.2 \\
0.2 & 0.0 & 0.3 & 0.5 & 1.0 & 0.8 & 0.0 \\
0.7 & 0.7 & 0.1 & 0.0 & 0.9 & 1.0 & 0.1 \\
1.0 & 1.0 & 0.9 & 0.9 & 0.0 & 0.4 & 1.0 \\
\end{pmatrix}
\]

This matrix was used in the above algorithm as the initial information. We considered various types of (\( \min-S \))-transitivity for \( S \)-conorms from table 1 (\( \beta = 0.5, 5, 50, 100 \)). As a result of the algorithm operation for various values of the parameter \( \alpha \), a hierarchical structure and the corresponding Hasse diagram were formed (fig. 1). All hierarchies can be divided into two classes: main and derived, which were obtained by the evolution of the main structure: some links disappear when passing from level to level. The Hasse diagrams coincide for the given main and corresponding derived structures.

![Figure 1. Example of hierarchical structures (main and derived).](image)

Analysis of the results of the computing experiment is the basic for the following conclusions:

1. As the value \( \alpha \) increases, the ordering of alternatives reduces, in fact, to their classification, since alternatives are grouped at one or two levels of the hierarchy.
2. If in the process of constructing hierarchical structure we select only those values which are represented in the matrix of the initial fuzzy relation as $\alpha$, then in the case of $(\min-S_M)$-transitivity, each value $\alpha$ has its own main structure, and hence the Hasse diagram. Evolution of structures does not take place. In the case of other types of transitivity, some types of main structures can evolve, and the Hasse diagram is preserved throughout the evolutionary stage. Evolution is related to the fact that as the decomposition parameter $\alpha$ increases, some links indicating immediate preference become irrelevant.

3. $(\min-S_M)$-transitivity allows us to get good results, but its "action" is limited, since the distribution of vertices (alternatives) over levels is possible only with $\alpha \leq 0.5$ (fig. 2). On the other hand, $(\min-S_L)$-transitivity allows us to order alternatives even when $\alpha$ is close to 1.

4. $(\min-S_P)$-transitivity and $(\min-S_D)$-transitivity cause a more detailed evolution of the structure – the Hasse diagrams are repeated but with a different number of links (fig. 3).

5. $(\min-S_D)$-transitivity for large values of the parameter (50, 100) generates a hierarchy throughout the segment of change of the level $\alpha$, while the rest do it only on values which are close to the values in the source fuzzy relation.

6. For a fixed vertex, the ordering of the values of the ordinal function corresponds to the ordering of the corresponding triangular conorms that generate different types of transitivity (fig. 4).

---

**Figure 2.** Positioning vertices at hierarchy levels with $(\min-S_M)$-transitivity.
Figure 3. Positioning vertices at hierarchy levels with (min-$S_P$)-transitivity.

Figure 4. The ordinal function of a fixed vertex for various types of transitivity.
4. Conclusion
A feature of the proposed algorithm is that it is possible to obtain different ranking of alternatives of a given set depending on the level $\alpha$ values. Since $\alpha$ is selected from the values of the membership function of the initial fuzzy relation, it can be considered as a threshold value of the preference power, which is fixed in the problem. The delicacy of the algorithm is the functional representation of the composition operation. The computing experiment showed a significant dependence of the Hasse diagram structure on the type of composition used in the transitive closure operation.

The research on this problem is the prospect of studying the proposed method.

References
[1] Saaty T L 1994 Fundamentals of Decision Making and Priority Theory (Pittsburgh, PA: RWS publications, 4922 Ellsworth Ave)
[2] and A K 1982 Introduction to the theory of fuzzy sets: trans. from French (Moscow: Radio and Communication)
[3] Klement E P, Mesiar R and Pap E 2004 Fuzzy Set and Systems 143 5–26
[4] Orlovsky S 1981 Problems of decision-making with fuzzy source information (Moscow: Science)
[5] Anderson J A Discrete Mathematics with Combinatorics (New Jersey 07458: Prentice Hall, Upper Saddle River)