THE RAMANUJAN PROPERTY FOR SIMPLICIAL COMPLEXES

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Abstract. Let $G$ be a topological group acting on a simplicial complex $X$ satisfying some mild assumptions. For example, consider a $k$-regular tree and its automorphism group, or more generally, a regular affine Bruhat-Tits building and its automorphism group. We define and study various types of high-dimensional spectra of quotients of $X$ by subgroups of $G$. These spectra include the spectrum of many natural operators associated with the quotients, e.g. the high-dimensional Laplacians.

We prove a theorem in the spirit of the Alon-Boppana Theorem, leading to a notion of Ramanujan quotients of $X$. Ramanujan $k$-regular graphs and Ramanujan complexes in the sense of Lubotzky, Samuels and Vishne are Ramanujan in dimension 0 according to our definition (for $X, G$ suitably chosen).

We give a criterion for a quotient of $X$ to be Ramanujan which is phrased in terms of representations of $G$, and use it, together with deep results about automorphic representations, to show that affine buildings of inner forms of $\text{GL}_n$ over local fields of positive characteristic admit infinitely many quotients which are Ramanujan in all dimensions. The Ramanujan (in dimension 0) complexes constructed by Lubotzky, Samuels and Vishne arise as a special case of our construction. Our construction also gives rise to Ramanujan graphs which are apparently new.

Other applications are also discussed. For example, we show that there are non-isomorphic simplicial complexes which are isospectral in all dimensions.

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Date: July 8, 2016.

2010 Mathematics Subject Classification. 05E18, 11F70, 22D10, 22D25, 46L05.

Key words and phrases. simplicial complex, Ramanujan complex, Ramanujan graph, idempotented $\ast$-algebra, spectrum, affine building, $\ell$-group, reductive group, automorphic form, Ramanujan–Petersson conjecture, Jacquet–Langlands correspondence.

This research was supported by an ERC grant #226135, the Lady Davis Fellowship Trust, and the UBC Mathematics Department.
Introduction

Let $X$ be a connected $k$-regular graph. The spectrum of $X$ is defined to be the spectrum of its adjacency matrix. It is well-known that the spectrum affects many combinatorial properties of $X$ such as behavior of random walks, expansion and mixing properties, and the chromatic number; see the survey [40], for instance.

The graph $X$ is called Ramanujan if all eigenvalues of its adjacency matrix, excluding $k$ and $-k$, lie in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. Let us explain the motivation behind this definition: Denote by $\lambda(X)$ the maximal absolute value of an eigenvalue of the adjacency matrix, excluding $k$ and $-k$. Informally, the smaller $\lambda(X)$ is, the better the aforementioned combinatorial properties of the graph are. However, the Alon–Boppana Theorem [64] states that for any $\varepsilon > 0$, there are only finitely many non-isomorphic $k$-regular graphs with $\lambda(X) < 2\sqrt{k-1} - \varepsilon$. Ramanujan graphs can therefore be thought of as having the smallest possible spectrum one can expect of an infinite family of graphs. In addition, the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ is the spectrum of the $k$-regular tree (Th. 3]), which is the universal cover of any $k$-regular graph, so Ramanujan graphs can be regarded as finite approximations of the infinite $k$-regular tree.

The spectral properties of Ramanujan graphs make them into supreme expander graphs. In addition, they have good mixing properties and large chromatic number, provided they are not bipartite. Some known constructions have large girth as well. Constructing infinite families of non-isomorphic $k$-regular Ramanujan graphs is considered difficult. The first such families were introduced by Lubotzky, Phillips and Sarnak [15] and independently by Margulis [58], assuming $k - 1$ is prime. Morgenstern [62] has extended this to the case $k - 1$ is a prime power. These works rely on deep results of Deligne [19] and Drinfeld [21] concerning the Ramanujan–Petersson conjecture for $\text{GL}_2$. The existence of infinitely many $k$-regular bipartite Ramanujan graphs for arbitrary $k$ was later shown by Marcus, Spielman and Srivastava [56] using different methods; the non-bipartite case remains open.

A high-dimensional generalization of Ramanujan graphs, called Ramanujan complexes, was suggested by Cartwright, Solé and Žuk [10], and later refined by Lubotzky, Samuels and Vishne [51] (see [35] for another generalization of Ramanujan graphs). These complexes are quotients of the affine Bruhat–Tits building of $\text{PGL}_d(F)$, denoted $B_d(F)$, where $F$ is a non-archimedean local field. The building $B_d(F)$ is a contractible simplicial complex of dimension $d - 1$. The spectrum of a quotient of $B_d(F)$, i.e. a simplicial complex whose universal cover is $B_d(F)$, consists of the common spectrum of a certain family of $d - 1$ linear operators associated with the quotient, called the Hecke operators. According to Lubotzky, Samuels and Vishne [51], a quotient of $B_d(F)$ is Ramanujan if its spectrum, which is a subset of $\mathbb{C}^{d-1}$, is contained in the spectrum of the universal cover $B_d(F)$ together with a certain family of $d$ points in $\mathbb{C}^{d-1}$, called the trivial spectrum.

Li [44 Thm. 4.1] proved a theorem in the spirit of the Alon–Boppana Theorem for quotients of $B_d(F)$: If $\{X_n\}_{n \in \mathbb{N}}$ is a family of such quotients satisfying a mild assumption, then the closure of the union of the spectra of $\{X_n\}_{n}$ (in $\mathbb{C}^{d-1}$) contains the spectrum of the universal cover $B_d(F)$. Ramanujan complexes can therefore be thought of as having the smallest possible spectrum that can be expected of an infinite family, or as spectral approximations of the universal cover $B_d(F)$, similarly to Ramanujan graphs. When $d = 2$, the complex $B_d(F)$ is a regular tree, and Ramanujan complexes are just Ramanujan graphs in the previous sense.
The existence of infinite families of Ramanujan complexes was shown by Lubotzky, Samuels and Vishne in \[51\] (see also \[50\]), using Lafforgue’s proof of the Ramanujan–Petersson conjecture for \(\text{GL}_d\) in positive characteristic \[41\]. \text{Li} \[43\] has independently obtained very similar results using a special case of the conjecture established by Laumon, Rapoport and Stuhler \[43\] Th. 14.12 (the notion of Ramanujan complexes used in \[43\] is slightly weaker than the one used in \[51\], but the constructions of \[43\] are in fact Ramanujan in the sense of \[51\]). As in the case of graphs, Ramanujan complexes enjoy various good combinatorial properties: They have high chromatic number \[23, \S6\], good mixing properties \[23, \S4\], they satisfy Gromov’s geometric expansion property \[26\] (see also \[30, \S7\]), and the constructions of \[51\] have high girth in addition \[48\].

The Ramanujan property of quotients of \(\mathcal{B}_d(F)\) is measured with respect to the spectrum of the Hecke operators. In a certain sense, to be made precise in Example \[4.8\] below, these operators capture all spectral information in dimension 0. Therefore, we regard the spectrum of Lubotzky, Samuels and Vishne as the 0-dimensional spectrum. However, one can associate other operators with a simplicial complex such that their spectrum affect combinatorial properties. For example, this is the case for the high-dimensional Laplacians; see for instance \[69, 27, 28, 68\]. Other examples are adjacency operators between various types of facets. These operators are high-dimensional in nature and so their spectrum is a priori not determined by the spectrum of the Hecke operators. The purpose of this work is to treat these and other high-dimensional operators, and to construct examples of complexes which are Ramanujan relative to such operators.

In more detail, let \(\mathcal{X}\) be a simplicial complex and let \(G\) be a group of automorphisms of \(\mathcal{X}\) satisfying certain mild assumptions. For example, one can take \(\mathcal{X}\) to be a \(k\)-regular tree \(\mathcal{T}_k\) and \(G = \text{Aut}(\mathcal{T}_k)\), or \(\mathcal{X} = \mathcal{B}_d(F)\) and \(G = \text{PGL}_d(F)\). Even more generally, \(\mathcal{X}\) can be an affine Bruhat-Tits building (see \[31\]), and \(G\) can be a group of automorphisms acting on \(\mathcal{X}\) in a sufficiently transitive manner. With every quotient of \(\mathcal{X}\) by subgroups of \(G\), called \(G\)-quotient for brevity, we will associate various types of spectra. Among them is the (non-oriented) \(i\)-dimensional spectrum. When \(\mathcal{X} = \mathcal{B}_d(F)\) and \(G = \text{PGL}_d(F)\), or when \(\mathcal{X} = \mathcal{T}_k\) and \(G = \text{Aut}(\mathcal{T}_k)\), our 0-dimensional spectrum coincides with the spectra of quotients of regular graphs and quotients of \(\mathcal{B}_d(F)\) discussed earlier.

We prove a theorem in the spirit of the Alon–Boppana Theorem, generalizing \text{Li}’s aforementioned theorem, stating that if \(\{X_n\}_{n \in \mathbb{N}}\) is a family of \(G\)-quotients of \(\mathcal{X}\) satisfying a mild assumption, then the closure of \(\bigcup_{n \in \mathbb{N}} \text{Spec}(X_n)\) contains \(\text{Spec}(\mathcal{X})\) (Theorem \[5.1\]). This in turn leads to a notion of \text{Ramanujan G-quotients of } \mathcal{X\text{.}}\) In analogy with Ramanujan graphs and Ramanujan complexes, these complexes have the smallest possible spectrum one can expect of an infinite family of \(G\)-quotients of \(\mathcal{X}\). Alternatively, they can be regarded as spectral approximations of the covering complex \(\mathcal{X}\). When \(\mathcal{X}\) is a \(k\)-regular tree (resp. \(\mathcal{B}_d(F)\)), the quotients of \(\mathcal{X}\) which are \text{Ramanujan in dimension } 0\text{ are precisely the Ramanujan graphs (resp. Ramanujan complexes).}

We proceed with establishing a criteria for a quotient of \(\mathcal{X}\) to be Ramanujan. We show that if \(\Gamma \leq G\) is a subgroup such that \(\Gamma \backslash \mathcal{X}\) is a simplicial complex and \(\mathcal{X} \to \Gamma \backslash \mathcal{X}\) is a cover map, then the Ramanujan property of \(\Gamma \backslash \mathcal{X}\) is equivalent to a certain condition on the \(G\)-representation \(L^2(\Gamma \backslash G)\) (Theorem \[6.22\]). This generalizes a similar criterion in \[51\] for the case \(\mathcal{X} = \mathcal{B}_d(F)\). More generally, we show that there is a one-to-one correspondence between the spectrum \(\Gamma \backslash \mathcal{X}\) and

\footnote{The proof in \[51\] assumed the global Jacquet–Langlands correspondence for \(\text{GL}_n\) in positive characteristic that was established later in \[9\].}
a certain class of $G$-subrepresentations of $L^2(\Gamma \backslash G)$ (Theorem 6.21). In case $\mathcal{X}$ is the affine Bruhat-Tits building of an almost simple algebraic group $G$ over $F$, $G = G(F)$, and $\Gamma$ is an arithmetic cocompact lattice in $G$, we restate our criterion in terms of automorphic representations of $G$ (Theorem 7.4).

Finally, we apply our automorphic criterion together with Lafforgue’s proof of the Ramanujan–Petersson conjecture for $\text{GL}_d$ in positive characteristic [11] and the establishment of the global Jacquet–Langlands correspondence for $\text{GL}_d$ in positive characteristic by Badulescu and Roche [5], to give new examples of Ramanujan complexes. Specifically, let $F$ be a non-archimdean local field of positive characteristic, let $D$ be a central division $F$-algebra, let $G = \text{PGL}_d(D) = \text{GL}_d(D)/F^\times$ and let $\mathcal{B}_d(D)$ be the affine Bruhat-Tits building of $G$. Then $\mathcal{B}_d(D)$ admits infinitely many $G$-quotients which are Ramanujan in all dimensions (Theorem 7.22). (In fact, these quotients are completely Ramanujan.) For example, the spectrum of the high dimensional Laplacians of these complexes is contained in the union of the spectrum of the high dimensional Laplacians of the universal cover $\mathcal{B}_d(D)$ together with the trivial spectrum. When $D = F$, our Ramanujan complexes are the Ramanujan complexes constructed in [51]. Thus, the Ramanujan complexes of Lubotzky, Samuels and Vishne [51], which are Ramanujan in dimension 0 in our setting, are in fact Ramanujan on all dimensions. When $d = 2$, our construction gives rise to Ramanujan graphs, which seem to be new when $D \neq F$.

The machinery that we introduce has other applications: When combined with results from [53], we obtain examples of non-isomorphic quotients of $\mathcal{B}_d(F)$ which are isospectral in all dimensions, and more generally, completely isospectral (Example 6.33). In addition, using the classification of irreducible representations of $\text{GL}_d(F)$ (resp. $\text{Aut}(\mathcal{T}_k)$), we show that if $\mathcal{X} = \mathcal{B}_3(F)$ (resp. $\mathcal{X} = \mathcal{T}_k$), then a quotient of $\mathcal{X}$ is Ramanujan in dimension 0 if and only if it is Ramanujan in all dimensions (Propositions 6.30 and 6.31). In the case $\mathcal{X} = \mathcal{B}_3(F)$, this agrees with the related result [36, Th. 2]. In addition, the results of Marcus, Spielman and Srivastava on the existence of $k$-regular Ramanujan graphs [50], and their generalizations by Hall, Puder and Sawin [31] can be translated into representation-theoretic results about the automorphism group of a $k$-regular tree, which resemble the Ramanujan–Petersson conjecture for $\text{GL}_d$ (Corollary 6.28 and the comment after Proposition 6.30). Finally, we show that in certain cases, the Ramanujan property is unaffected by replacing the group $G \subseteq \text{Aut}(\mathcal{X})$ with a commensurable subgroup, even though it affects the way the spectrum is defined (Theorem 6.30).

We now give some brief details about how we define the spectrum of quotients of $\mathcal{X}$. For a simplicial complex $X$ and $i \geq 0$, let $\Omega_i^+(X)$ denote the vector space of $\mathbb{C}$-valued functions of finite support on the $i$-dimensional cells in $X$, and let $\Omega_i^-(X)$ denote the space of $i$-dimensional forms on $X$ with finite support (see [31]). Let $F$ denote $\Omega_i^+$ or $\Omega_i^-$ for some fixed $i$; in general, $F$ can be taken to be a semi-elementary functor from the category of $G$-quotients of $\mathcal{X}$ to the category of pre-Hilbert spaces (Definition 4.11). Every such $F$ gives rise to a certain kind of spectrum that can be associated with $G$-quotients of $\mathcal{X}$, called the $F$-spectrum $\text{Spec}(\mathcal{X})$. A $G$-quotient of $\mathcal{X}$ which is Ramanujan with respect to this spectrum is called $F$-Ramanujan [50]. The non-oriented (resp. oriented) $i$-dimensional spectrum is obtained by taking $F = \Omega_i^+$ (resp. $F = \Omega_i^-$), and the $G$-quotients of $\mathcal{X}$ which are $\Omega_i^+$-Ramanujan are called Ramanujan in dimension $i$.

The action of $G$ on $\mathcal{X}$ induces an action on $F\mathcal{X}$. Let $A$ denote the algebra of $G$-equivariant linear operators on $F\mathcal{X}$. It turns out that elements of $A$ act naturally on $F(\Gamma \backslash \mathcal{X})$ for every subgroup $\Gamma \leq G$. The linear operators that our $F$-spectrum takes into consideration are the elements of $A$. For example, when $F = \Omega_i^+$, the $i$-dimensional Laplacian can be regarded as an element of $A$, and when $F = \Omega_i^-$,
the algebra $A$ contains many natural adjacency operators between $i$-dimensional cells. In addition, for $\mathcal{X} = B_d(F)$ and $F = \Omega_0^+$, the Hecke operators live in $A$, and in fact generate it. We note that one can replace $A$ with a subalgebra of interest.

Let $X$ be a $G$-quotient of $\mathcal{X}$. Naively, one can define the $F$-spectrum of $\mathcal{X}$ as the common spectrum of the operators in $A$ on $FX$. This works well if $A$ is commutative (e.g. when $\mathcal{X} = B_d(F)$ and $F = \Omega_0^+$), but this definition is unsatisfactory in general. Instead, we define the $F$-spectrum of $X$ as the irreducible $A$-submodules of $FX$. For such a definition to work, some complementary theory has to be developed. For example, the collection of irreducible $A$-modules has to be given a topology (because our generalization of Li’s aforementioned theorem uses “closure”), and one has to define a notion of a continuous spectrum — irreducible submodules of $FX$ suffice when $X$ is finite, but this does not work in the infinite case, e.g. for $X = \mathcal{X}$. These and other technicalities are discussed in Chapter 2. They are resolved by introducing a canonical involution on $A$ and invoking the spectral theory of *-algebras ([67], for instance).

We remark that in this general setting, some fundamental facts become difficult to prove. For example, it is reasonable to expect that if $B$ is a subalgebra of $A$, then the spectrum of $FX$ with respect to $A$ determines the spectrum of $FX$ with respect to $B$. This is easy to show when $X$ is finite, but the general case (Theorem 2.35) requires significant work. Another example of a fundamental fact with an involved proof is Theorem 2.31 which is used in the proof of our generalization of Li’s Theorem.

The paper is organized as follows: Chapter 1 is preliminary and recalls Ramanujan complexes as defined in [51]. Chapter 2 concerns with developing a spectral theory for idempotented *-algebras. This chapter is long and technical, but the results and the definitions introduced there are fundamental for the following chapters. Chapter 3 recalls simplicial complexes and certain facts about ℓ-groups acting on them. In Chapter 4 we introduce our notion of spectrum, give some examples, and discuss issues such as dependencies between different types of spectra. In Chapter 5 we prove an generalization of Li’s Theorem (reminiscent of the Alon-Boppana Theorem) and characterize the trivial spectrum, which leads to the definition of Ramanujan quotients of $\mathcal{X}$. Chapter 6 gives a representation-theoretic criterion for a quotient of $\mathcal{X}$ to be Ramanujan. Consequences of this criterion are also discussed. Finally, in Chapter 7 we focus on the case where $\mathcal{X}$ is the affine Bruhat-Tits building of an almost simple algebraic group $G$ over a non-archimedean local field. We relate the Ramanujan property with properties of certain automorphic representations of $G$, and use it to show that the affine building of $\text{PGL}_d(D)$ admits infinitely many quotients which are Ramanujan in all dimensions.

Acknowledgements. We owe a debt of gratitude to Alex Lubotzky for presenting us with the theory of Ramanujan graphs and suggesting this research project. We are also in debt to Lior Silberman for many beneficial discussions. We further thank Anne-Marie Aubert, Alexandru Ioan Badulescu, David Kazhdan, Laurent Lafforgue and Dipendra Prasad for short-yet-crucial correspondences, all concerning Chapter 7. In addition, Amitay Kamber has given us several useful suggestions, for which are grateful. Finally, we thank the participants of the Ramanujan Complexes Seminar that took place at the Hebrew University in the winter of 2013.

Notation

All vector spaces and algebras are over $\mathbb{C}$. The complex conjugate of $z \in \mathbb{C}$ is denoted $\overline{z}$. Algebras are always associative but not necessarily unital.
Inner products of pre-Hilbert and Hilbert spaces will be denoted by triangular brackets \( \langle \cdot, \cdot \rangle \) with the convention that the left component is \( \mathbb{C} \)-linear. The unit sphere of a normed space \( V \) is denoted by \( S^1(V) \). The completion of a pre-Hilbert \( V \) is denoted by \( \overline{V} \). If \( T : V \to V' \) is a linear operator between pre-Hilbert spaces, a dual of \( T \) is an operator \( T^* : V' \to V \) satisfying \( \langle Tv, v' \rangle = \langle v, T^*v' \rangle \) for all \( v \in V \), \( v' \in V' \). The dual is unique if it exists, and it is guaranteed to exist when \( V \) and \( V' \) are Hilbert spaces and \( T \) is bounded.

For a set \( X \), we let \( \ell^2(X) \) denote the set of functions \( \varphi : X \to \mathbb{C} \) with finite support. We endow \( \ell^2(X) \) with the inner product \( \langle \varphi, \psi \rangle = \sum_{x \in X} \varphi(x) \cdot \overline{\psi(x)} \). This makes \( \ell^2(X) \) into a pre-Hilbert space. Its completion is the Hilbert space of square-summable functions on \( X \), denoted \( \ell^2(X) \). The vector space \( \ell^2(X) \) admits a standard basis \( \{e_x\}_{x \in X} \) defined by

\[
e_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.
\]

If \( Y \) is another set and \( f : X \to Y \) is any function, then we define \( f_* : \ell^2(X) \to \ell^2(Y) \) by \( (f_* \varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x) \) for all \( \varphi \in \ell^2(X) \), \( y \in Y \). In particular, we have

\[
f_* e_x = e_{f(x)} \quad \forall x \in X.
\]

This makes \( X \mapsto \ell^2(X) \) into a covariant functor from the category of sets to the category of pre-Hilbert spaces (non-continuous morphisms are allowed).

If \( X \) is an \( \ell \)-space, i.e. a totally disconnected locally compact Hausdorff topological space, we let \( C^\infty_c(X) \) denote the vector space of compactly supported locally constant functions \( \varphi : X \to \mathbb{C} \). If \( X \) is equipped with a Borel measure \( \mu \), which will always be obvious from the context, we make \( C^\infty_c(X) \) into a pre-Hilbert space by setting

\[
\langle \varphi, \psi \rangle = \int_X \varphi \cdot \overline{\psi} \, d\mu.
\]

The completion of \( C^\infty_c(X) \) is clearly \( L^2(X) \), the space of square-integrable functions on \( X \) (considered up to equivalence).

We write \( F \subseteq_Z Z \) to denote that \( F \) is a finite subset of \( Z \). If \( G \) is an \( \ell \)-group, i.e. a totally disconnected locally compact topological group, we write \( K \subseteq_{c.o.} G \) to denote that \( K \) is a compact open subgroup of \( G \). It is well-known that the compact open subgroups of an \( \ell \)-group form a basis of neighborhoods at the identity.

1. Ramanujan Complexes

This preliminary chapter recalls Ramanujan complexes as defined by Lubotzky Samuels and Vishne [51], basing on the work of Cartwright, Solé and Žuk [16].

Let \( F \) be a non-archimedean local field with additive valuation \( \nu \), let \( \mathcal{O} \) be the integer ring of \( F \), and let \( \pi \in \mathcal{O} \) be a uniformizer, i.e. a generator of the maximal ideal of \( \mathcal{O} \). We assume that \( \nu(\pi) = 1 \). Fix an integer \( d \geq 2 \) and let

\[
G = \text{PGL}_d(F) = \text{GL}_d(F)/F^\times,
\]

\[
K = \text{PGL}_d(\mathcal{O}) := \text{im}(\text{GL}_d(\mathcal{O}) \to \text{PGL}_d(F)) .
\]

For \( g \in \text{GL}_d(F) \), denote by \( \overline{g} \) the image of \( g \) in \( G \). The topology of \( F \) induces a topology on \( G \), making it into an \( \ell \)-group. The subgroup \( K \) is compact and open in \( G \), and the function

\[
\begin{align*}
    e & : G \to \mathbb{Z}/d\mathbb{Z} \\
    \overline{g} & \mapsto \nu(\det g) + d\mathbb{Z}
\end{align*}
\]
is well-defined and satisfies \( c(K) = 0 \).

The affine Bruhat-Tits building of \( G \), denoted \( \mathcal{B}_d(F) \), is a \((d-1)\)-dimensional simplicial complex constructed as follows: The vertices of \( \mathcal{B}_d(F) \) are the right cosets \( G/K \). They admit a \( d \)-coloring \( C_0 : G/K \to \mathbb{Z}/d\mathbb{Z} \) given by

\[
C_0(gK) = c(g) \quad \forall g \in G .
\]

To define the edges of \( \mathcal{B}_d(F) \), let

\[
g_1 = \begin{bmatrix} \pi & 1 \\ & \ddots \\ & & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} \pi & & \\ & \ddots & \\ & & \pi \end{bmatrix}, \quad \ldots, \quad g_{d-1} = \begin{bmatrix} \pi \\ \vdots \\ & & 1 \end{bmatrix} \in \text{GL}_d(F)
\]

The double cosets \( \{K\Gamma K\}_{i=1}^{d-1} \) are distinct since \( c(K\Gamma K) = i + d\mathbb{Z} \). Two vertices \( gK, g'K \in G/K \) are adjacent if

\[
g^{-1}g' \in K \cup K\Gamma K \cup K\Gamma K \cup \ldots K\Gamma K .
\]

The \( i \)-dimensional cells of \( \mathcal{B}_d(F) \) are the \((i+1)\)-cliques, namely, they are sets \( \{h_0K, \ldots, h_{i+1}K\} \subseteq G/K \) consisting of pairwise adjacent vertices. The resulting complex is indeed a pure \((d-1)\)-dimensional contractible simplicial complex, which carries additional structure making it into an affine building; see [11 §6.9] for further details. (See also [34] for the definition of the affine building of a general reductive \( p \)-adic Lie group, and [2] for an explicit description of buildings of classical groups.) Note that every \((d-1)\)-simplex in \( \mathcal{B}_d(F) \) consists of \( d \) vertices of different colors.

There is an evident left action of \( G \) on \( \mathcal{B}_d(F) \) which respects the simplicial structure. Let \( \Gamma \leq G \) be a discrete subgroup. Under mild assumptions (see Corollary 3.10 below), one can form the quotient complex \( \Gamma \backslash \mathcal{B}_d(F) \): Its vertices are the double cosets \( \Gamma \backslash G/K \), and its \( i \)-dimensional cells are obtained by projecting the \( i \)-dimensional cells of \( \mathcal{B}_d(F) \) pointwise into \( \Gamma \backslash G/K \). Since \( \mathcal{B}_d(F) \) is contractible (when viewed as a topological space), it is the universal cover of \( \Gamma \backslash \mathcal{B}_d(F) \). Furthermore, when \( \Gamma \) is cocompact in \( G \), the simplicial complex \( \Gamma \backslash \mathcal{B}_d(F) \) is finite.

**Example 1.1.** The complex \( \mathcal{B}_d(F) \) is a \( q+1 \) regular tree, where \( q \) is the size of the residue field of \((F, \nu)\). The quotients \( \Gamma \backslash \mathcal{B}_d(F) \) are therefore \((q+1)\)-regular graphs.

The coloring \( C_0 \) of the vertices of \( \mathcal{B}_d(F) \) does not descend to \( X := \Gamma \backslash \mathcal{B}_d(F) \) in general. However, we can define a color function on the directed edges of \( X \) by

\[
C_1(\Gamma gK, \Gamma g'K) := c(g^{-1}g') \in \mathbb{Z}/d\mathbb{Z}
\]

It can be checked that \( C_1 \) is well-defined when \( \Gamma \backslash \mathcal{B}_d(F) \) is a simplicial complex. Since \( g^{-1}g' \in \bigcup_{i=1}^{d-1} K\Gamma K \) whenever \((gK, g'K) \) is an edge of \( \mathcal{B}_d(F) \), we have

\[
C_1(\Gamma gK, \Gamma g'K) \in \{1 + d\mathbb{Z}, \ldots, (d-1) + d\mathbb{Z}\} .
\]

Write \( X_{\text{vert}} = \Gamma \backslash G/K \) and define \( a_1, \ldots, a_{d-1} : L^2(X_{\text{vert}}) \to L^2(X_{\text{vert}}) \) by

\[
(a_i\varphi)x = \sum_{y \in X_{\text{vert}}} \varphi y \quad \forall \varphi \in L^2(X_{\text{vert}}), \; x \in X_{\text{vert}} .
\]

The operators \( a_1, \ldots, a_{d-1} \) are called the colored adjacency operators or Hecke operators of \( X \). It turns out that \( a_1, \ldots, a_{d-1} \) commute with each other and \( a_i^* = a_{d-i} \) for all \( i \). We may therefore consider the common spectrum

\[
\text{Spec}(a_1, \ldots, a_{d-1}) \subseteq \mathbb{C}^{d-1} .
\]

This set is called the spectrum of \( X \) and denoted \( \text{Spec}_0(X) \). The reason for the subscript 0 will become clear later in the text (specifically, in Example 4.9).
Theorem 1.3 (Li). Let \( \{X_n\}_{n \in \mathbb{N}} \) be a family of quotients of \( B_d(F) \). Let \( r_n \) be the maximal integer \( r \) for which \( X_n \) contains a copy of a ball of radius \( r \) in \( B_d(F) \). If \( \limsup r_n = \infty \), then the closure of \( \bigcup_{n \in \mathbb{N}} \text{Spec}_0(X_n) \) in \( \mathbb{C}^{d-1} \) contains \( \text{Spec}_0(B_d(F)) \).

The set \( \text{Spec}_0(B_d(F)) \) was determined in [51] Th. 2.11. In addition to the bound of Theorem 1.3, the spectrum of a finite quotient of \( B_d(F) \) always contains at least one of \( d \) special points in \( \mathbb{C}^{d-1} \) called the trivial eigenvalues; we refer the reader to [51] [2.3] or Example 5.13 below for their description.

The following definition was suggested by Lubotzky, Samuels and Vishne [51], following Carwright, Solé and Žuk [16].

Definition 1.4 ([51] Df. 1.1]). The complex \( \Gamma \backslash B_d(F) \) is called Ramanujan if \( \text{Spec}_0(\Gamma \backslash B_d(F)) \) is contained in the union of \( \text{Spec}_0(B_d(F)) \) with the set of trivial eigenvalues.

By the previous discussion, Ramanujan complexes can be regarded as quotients of \( B_d(F) \) whose spectrum is as small as one dares to hope, or alternatively, as (finite) spectral estimations of their universal cover \( B_d(F) \). Constructions of Ramanujan complexes were given by Lubotzky, Samuels and Vishne in [51] and [50].

Example 1.5. When \( d = 2 \), we have \( \text{Spec}_0(B_2(F)) = [-2\sqrt{7}, 2\sqrt{7}] \), and the trivial eigenvalues are \(-q-1 \) and \( q+1 \) (where \( q = |O/\pi O| \)). Therefore, a quotient \( X = \Gamma \backslash B_2 \) is Ramanujan if \( \text{Spec}(X) \subseteq [-2\sqrt{7}, 2\sqrt{7}] \cup \{-q-1, q+1\} \). This agrees with the usual definition of Ramanujan \((q+1)\)-regular graphs.

The present work extends the ideas of Lubotzky, Samuels, Vishne [51] and Li [44] to more general simplicial complexes, and to operators different than the Hecke operators, e.g. the high-dimensional Laplacians. Loosely speaking, we will show how to define a Ramanujan property with respect to any prescribed family of associated operators, and show that there are simplicial complexes which are Ramanujan with respect to any such family. For example, the Ramanujan complexes of [51] will be shown to have this property. After we give our general definition of the Ramanujan property in [56] we will address the Ramanujan complexes of Definition 1.4 as \( G \)-quotients of \( B_d(F) \) which are Ramanujan in dimension 0.

2. Idempotented \( \ast \)-Algebras

As a preparation for the next chapters, this chapter develops a spectral theory for idempotented \( \ast \)-algebras. This is an attempt to adapt the spectral theory of \( C^* \)-algebras (see [20], [83] Ch. 14]) to the idempotented algebras often used in the theory of \( p \)-adic Lie groups (see [13]). Of course, this is included in the general theory of \( \ast \)-algebras (67, for instance), but the idempotented case is more tame in nature.
The reader will find many similarities with the theory of C*-algebras. However, the absence of a topology causes certain differences that are pointed throughout.

2A. A Motivating Example. Recall that if $V$ is a Hilbert space and $T : V \to V$ is a bounded linear operator, then the spectrum of $T$, denoted $\text{Spec}(T)$, is the set of elements $\lambda \in \mathbb{C}$ for which $T - \lambda$ is not invertible.

Suppose $T$ is normal, namely $TT^* = T^*T$. Then $V$ can be viewed as a module over the free commutative algebra $A = \mathbb{C}[X, X^*]$ where $X, X^*$ act as $T, T^*$ respectively. The algebra $A$ carries an involution $*$ taking $X$ to $X^*$ and extending the complex conjugation on $\mathbb{C}$. We clearly have

$$\langle au, v \rangle = \langle u, a^*v \rangle \quad \forall a \in A, u, v \in V. \tag{2.1}$$

This is an example of a unitary representation of $(A, *)$. We are interested in extracting the datum of $\text{Spec}(T)$ from the action of $A$ on $V$.

In case $V$ is finite dimensional, this can be done as follows: Since $A$ is commutative and affine over $\mathbb{C}$, all irreducible $A$-modules are 1-dimensional and are of the form $V_{\lambda, \mu} := A/(X - \lambda, X^* - \mu)$ for $\lambda, \mu \in \mathbb{C}$ uniquely determined. There is essentially one way to make $V_{\lambda, \mu}$ into a Hilbert space. However, it is only when $\lambda = \overline{\mu}$ that $V_{\lambda, \mu}$ becomes unitary (i.e. (2.1) is satisfied). Thus, the irreducible unitary representations of $A$ are $\{V_{\lambda} := V_{\lambda, \overline{\lambda}}\}_{\lambda \in \mathbb{C}}$.

Observe that for all $v \in V_{\lambda}$, we have $Xv = \lambda v$ and $X^*v = \overline{\lambda} v$. Furthermore, if $\lambda$ is an eigenvalue of $T$ and $v \in V$ is a corresponding eigenvector, then $Tv = \lambda v$ and $T^*v = \overline{\lambda} v$, because

$$\langle (T^* - \overline{\lambda})v, (T^* - \overline{\lambda})v \rangle = \langle v, (T - \lambda)(T^* - \overline{\lambda})v \rangle = \langle v, (T^* - \overline{\lambda})(T - \lambda)v \rangle = \langle (T - \lambda)v, (T - \lambda)v \rangle. \tag{2.1}$$

Therefore, $\lambda$ is an eigenvalue of $T$ if and only if $V_{\lambda}$ is isomorphic to an $A$-submodule of $V$. Since $V$ is finite dimensional, $\text{Spec}(T)$ consists entirely of eigenvalues. We thus get a one-to-one correspondence

$$\text{Spec}(T) \longleftrightarrow \begin{cases} \text{irreducible submodules} \\ \text{of } V, \text{ up to isomorphism} \end{cases}$$

given by $\lambda \longleftrightarrow V_{\lambda}$.

When $V$ is infinite dimensional, $\text{Spec}(T)$ need not consist of eigenvalues, and a different approach has to be taken; it will be described in 2B below.

The shifting from $T$ to the algebra $A$ allowed us to “forget” the special operator $T$ among the elements of algebra $A$. This idea will be elaborated throughout this chapter, and ultimately manifest in the way we shall define spectrum of simplicial complexes. In the following chapters, we will encounter algebras of operators containing many elements of interest, but without a canonical set of generators. Moreover, the algebras will be non-commutative. Thus, rather than studying the spectrum of individual elements of the algebra, we shall consider irreducible submodules, which can be roughly thought of as the common spectrum of all operators in the algebra.

Before proceeding onward, we recall the following well-known characterization of the spectrum of normal linear operators $T : V \to V$.

**Proposition 2.1.** Assume $T : V \to V$ is a normal operator on a Hilbert space. Then $\lambda \in \text{Spec}(T)$ if and only if $T - \lambda$ is not bounded from below, i.e. for all $\varepsilon > 0$, there is a unit vector $v \in S^1(V)$ such that $\| (T - \lambda)v \| < \varepsilon$. 

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Proof. We only verify the nontrivial direction. Suppose \( \lambda \in \text{Spec}(T) \). Replacing \( T \) with \( T - \lambda \), we may assume \( \lambda = 0 \), so we need to show that \( T \) is unbounded from below. This is clear if \( \ker T \neq 0 \), so we may assume \( T \) is injective. Since \( \|Tv\| = \|T^*v\| \) for all \( v \in V \), \( \ker T^* = 0 \) as well. Thus, if \( v \in (TV)^+ \), then \( 0 = (TV,v) = (V,T^*v) \), and hence \( v = 0 \). Therefore, \( TV \) is dense in \( V \).

Assume by contradiction that \( T \) is bounded from below, namely, there is \( c > 0 \) such that \( c\|x\| \leq \|Tx\| \). We claim that \( TV = V \). Indeed, let \( y \in TV \). Since \( TV \) is dense in \( V \), there is a sequence \( \{x_n\}_{n \in \mathbb{N}} \subseteq V \) such that \( Tx_n \to y \). Since \( \|x_n - x_m\| \leq c^{-1}\|Tx_n - Tx_m\| \), the sequence \( \{x_n\} \) is a Cauchy sequence, and its limit \( x \) satisfies \( Tx = y \), as required. Now, since \( T \) is injective and surjective, it has an inverse \( T^{-1} : V \to V \), and since \( T \) is bounded from below, \( T^{-1} \) is bounded. But this means \( 0 = \lambda \notin \text{Spec}(V) \), a contradiction. \( \square \)

Let \( \{T_1, \ldots, T_n\} \) be a family of operators on \( V \) such that \( T_1, \ldots, T_n, T_1^*, \ldots, T_n^* \) commute. In analogy with Proposition 2.1, the common spectrum of \( \{T_1, \ldots, T_n\} \), denoted

\[
\text{Spec}(T_1, \ldots, T_n),
\]

is defined to be the set of tuples \( (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) such that for all \( \varepsilon > 0 \), there is \( v \in S^1(V) \) with \( \|T_i v - \lambda_i v\| < \varepsilon \) for all \( i \). In fact, since the definition makes sense for any set of operators, we will also use it for arbitrary families of operators.

2B. Idempotented \( * \)-Algebras. The term algebra always refers to an associative \( \mathbb{C} \)-algebra which is not necessarily unital. In this context, the term module also includes non-unital modules. For a left \( A \)-module \( V \) and \( a \in A \), denote by \( a|V \) the linear operator \( [v \mapsto av] \in \text{End}_\mathbb{C}(V) \).

An involution on an algebra \( A \) is a map \( * : A \to A \) such that \( a^{**} = a \), \( (a+b)^* = a^* + b^* \), \( (ab)^* = b^*a^* \) and \( (aa)^* = \overline{a}a^* \) for all \( a, b \in A \) and \( \alpha \in \mathbb{C} \). A \( * \)-algebra is an algebra \( A \) equipped with an involution, which is always denoted \( * \).

A \( * \)-algebra \( A \) is idempotented if for every finite set \( F \subseteq A \) there exists an idempotent \( e \in A \) such that \( e = e^* \) and \( eae = a \) for all \( a \in F \). The set of idempotents \( e \in A \) with \( e^* = e \) is denoted by \( I(A) \).

Example 2.2. Any unital \( * \)-algebra is idempotented (take \( e = 1 \)).

Let \( A \) be an idempotented \( * \)-algebra. A left \( A \)-module \( V \) is said to be smooth if \( AV = V \), or equivalently, if for all \( v \in V \), there exists \( e \in I(A) \) with \( ev = v \). The smooth left \( A \)-modules together with \( A \)-homomorphisms form an abelian category denoted by \( A \)-Mod. The irreducible objects of \( A \)-Mod are the nonzero \( A \)-modules \( V \) for which \( Av = V \) for all \( 0 \neq v \in V \).

Example 2.3. Let \( A \) be a unital involutory algebra. Then the smooth \( A \)-modules are just the unital \( A \)-modules (i.e. modules \( V \) for which \( 1_A \) acts as \( id_V \)).

Example 2.4. Every \( A \)-module \( V \) contains a unique maximal smooth submodule, namely, \( V_{\text{sm}} := AV \).

Another example of an idempotented \( * \)-algebra is the Hecke algebra of an \( \ell \)-group, which is usually not unital. This case is discussed in 6B below.

2C. Unitary Representations. Let \( A \) be an idempotented \( * \)-algebra. A unitary representation of \( A \) is a Hilbert space \( V \) equipped with a left \( A \)-module structure such that

\[
\text{(U1) } (au,v) = (u,a^*v) \text{ for all } a \in A \text{ and } u,v \in V, \\
\text{(U2) } V_{\text{sm}} = AV \text{ is dense in } V, \text{ and } \\
\text{(U3) for all } a \in A, \text{ the operator } a|V : V \to V \text{ is bounded.}
\]
We say that $V$ is irreducible if it is does not have a closed $A$-submodule. Let $\text{Rep}^u(A)$ denote the category whose objects are unitary representations of $A$ and whose morphisms are continuous $A$-module homomorphisms. We further let $\text{Irr}^u(A)$ denote the class of irreducible unitary representations of $A$.

Let $V_1, V_2 \in \text{Rep}^u(A)$. We denote by $\text{Hom}^u_A(V_1, V_2)$ the continuous $A$-homomorphisms from $V_1$ and $V_2$. Likewise, $\text{End}^0_A(V_1)$ denotes the continuous endomorphisms of $V_1$. Homomorphism which preserve the inner product are called unitary. We write $V_1 \leq V_2$ if there is a unitary injective $A$-homomorphism from $V_1$ to $V_2$. The image of $V_1$ in $V_2$ is easily seen to be closed, and hence $V_2 = V_1 \oplus V_2^\perp$.

**Remark 2.5.** If $V \in \text{Rep}^u(A)$ and $V_1$ is a closed $A$-submodule, then $V_1 \in \text{Rep}^u(A)$. Conditions (U1) and (U3) are clear. To see (U2), let $P$ be the orthogonal projection onto $V_1$. Then $P$ is an $A$-homomorphism. Since $AV$ is dense in $V$, the space $P(AV) = AP(V) = AV_1$ is dense in $P(V) = V_1$, as required.

Let $\{V_i\}_{i \in I} \subseteq \text{Rep}^u(A)$. The direct sum $\bigoplus_i V_i$ admits an obvious inner-product making it into a pre-Hilbert space. The completion of $\bigoplus_i V_i$ is denoted $\bigoplus_i V_i$. If for all $a \in A$, we have $\sup_i \|a|_{V_i}\| < \infty$, then the diagonal action of $A$ on $\bigoplus_i V_i$ extends to $\bigoplus_i V_i$ and we may regard $\bigoplus_i V_i$ as a unitary representation of $A$. We denote this by writing $\bigoplus_i V_i \in \text{Rep}^u(A)$. When $I$ is finite, we always have $\bigoplus_i V_i = \bigoplus_i V_i \in \text{Rep}^u(A)$.

We now recall several well-known facts about unitary representations.

**Theorem 2.6** (Schur’s Lemma). Let $V \in \text{Irr}^u(A)$. Then $\text{End}^0_A(V) = \mathbb{C} \text{id}_V$.

**Proof.** This is similar to the proof of [20, Pr. 2.3.1]. Alternatively, see [67, Th. 9.6.1].

**Corollary 2.7.** If $A$ is commutative, then all irreducible unitary representations of $A$ are 1-dimensional.

**Proof.** Let $V \in \text{Irr}^u(A)$. Since $A$ is commutative, the map $a \mapsto a|_V : A \to \text{End}_\mathbb{C}(V)$ takes values in $\text{End}^0_A(V)$, which equals $\mathbb{C} \text{id}_V$ by Schur’s Lemma. This means that $AV$ is 1-dimensional for all $0 \neq v \in V$, hence $\dim V = 1$.

**Proposition 2.8.** Let $V, V' \in \text{Rep}^u(A)$. If there exists a continuous $A$-module isomorphism $f : V \to V'$, then there exists a unitary isomorphism $g : V \to V'$. (This holds even without assuming $V$ and $V'$ satisfy condition (U2).)

**Proof.** This is similar to the argument given in [20, §2.2.2]. (Briefly, let $f = U|f|$ be the polar decomposition of $f$, where $|f|$ is the positive square root of $f^* f$. One can show that $g = U$ is the required isomorphism.)

**2D.** States. In analogy with the theory of $C^*$-algebras and locally compact groups, we now define states. Throughout, $A$ is an idempotented $*$-algebra. We make $\mathbb{I}(A) := \{e \in A | e^2 = e, e^* = e\}$ into a directed set by setting $e \leq e'$ when $e' e e' = e$. In addition, let $A^\vee = \text{Hom}_\mathbb{C}(A, \mathbb{C})$. We shall make repeated use of the fact that $\|e|_V\| \leq 1$ for all $V \in \text{Rep}^u(A)$ and $e \in \mathbb{I}(A)$, which holds since $e|_V$ is an orthogonal projection.

**Lemma 2.9.** For all $V \in \text{Rep}^u(A)$ and $v \in V$, the net $\{e v\}_{e \in \mathbb{I}(A)}$ converges to $v$.

---

2 Some texts use the term topologically irreducible.
Proof. Let \( \varepsilon > 0 \). We need to find \( e \in \mathbb{I}(A) \) such that \( \|v - e'v\| \leq \varepsilon \) for all \( e' \geq e \). Since \( AV \) is dense in \( V \), there is a unit vector \( u \in S^1(AV) \) with \( \|u - v\| < \frac{\varepsilon}{2} \).

Choose \( e \in \mathbb{I}(A) \) with \( eu = u \). Then for all \( e' \geq e \), we have \( e'u = e'eu = eu = u \).

Thus, \( \|u - e'v\| = \|e'(u - v)\| \leq \|e'v\|\|v - u\| \leq \|v - u\| = \frac{\varepsilon}{2} \).

Therefore, for all \( e' \geq e \), we have \( \|v - e'v\| \leq \|v - u\| + \|u - e'v\| < \varepsilon \), as required.

With some work, the following theorem and its corollaries can be derived from results in [67] [9.4], which treat the more complicated situation of general \( * \)-algebras.

However, we found it easier and more comprehensive to include proofs here than explaining how to derive everything from [67] [9.4]. (See also [20] [2.4] for the special case of \( C^* \)-algebras.)

**Theorem 2.10.** Let \( V \in \text{Rep}^1(A) \) and \( v \in S^1(V) \). Define \( \varphi = \varphi_{V,v} \in A' \) by

\[ \varphi_{V,v}(a) = \langle av, v \rangle. \]

Then \( \varphi \) satisfies:

(S1) \( \varphi(a^*a) \in \mathbb{R}_{\geq 0} \) for all \( a \in A \).

(S2) For all \( b \in A \), there is \( r \in \mathbb{R} \) such that \( \varphi(a^*b^*ba) \leq r \varphi(a^*a) \) for all \( a \in A \).

(S3) \( \sup \{ \varphi(e) | e \in \mathbb{I}(A) \} = 1 \).

Conversely, for any \( \psi \in A' \) satisfying conditions (S1)–(S3), there exist \( V, v \) as above with \( \psi = \varphi_{V,v} \) and \( V = \overline{Av} \). The pair \( (V, v) \) is unique up to unitary isomorphism preserving the vector \( v \).

Proof. Unfolding the definition of \( \varphi \), we see that (S1) merely means \( \langle av, av \rangle \in \mathbb{R}_{\geq 0} \) for all \( a \in A \).

(S2) means that \( \|av\| \leq r^{1/2} \|a\| \) for all \( a \in A \), and (S3) means that \( \sup \|ev\|^2 = 1 \).

The conditions now follow from the fact that \( (\cdot, \cdot) \) is an inner product, \( b|v|^2 \) is continuous (take \( r = \|b|v|^2 \) and Lemma 2.29 respectively).

Assume \( \psi \in A' \) satisfies conditions (S1)–(S3). Let \( L = \{ a \in A : \psi(a^*a) = 0 \} \).

Condition (S2) implies that \( L \) is a left ideal. Define \( \langle \cdot, \cdot \rangle : A/L \times A/L \to \mathbb{C} \) by \( \langle x + L, y + L \rangle = \psi(y^*x) \).

Condition (S1) and the definition of \( L \) imply that \( \langle a + L, a + L \rangle > 0 \) for all \( a \in A - L \). Thus, \( (\cdot, \cdot) \) is an inner product on \( A/L \) which clearly satisfies \( \langle ax, y \rangle = \langle x, a^*y \rangle \).

Denote the induced norm on \( A/L \) by \( \| \cdot \| \).

Condition (S2) implies that every \( b \in A \) admits an \( r \in \mathbb{R} \) such that \( \|b|v|^2 \leq r \|a\|^2 \), hence \( b|A/L \) is continuous. Let \( V \) denote the completion of \( A/L \) with respect to its norm.

The left action of \( A \) on \( A/L \) extends to \( V \), making \( V \) into a unitary representation of \( A \) (note that \( AV \supseteq A/L \), which is dense in \( V \)).

We now construct \( v \in S^1(V) \) with \( \psi = \varphi_{V,v} \). For \( e \in \mathbb{I}(A) \), let \( x_e = e + L \). Then \( e \leq e' \) implies \( \langle x_e, x_e' - x_e \rangle = \langle e' - e \rangle = 0 \), hence \( \|x_e\|^2 = \|x_e' - x_e\|^2 \).

We claim that \( \{x_e\}_{e \in \mathbb{I}(A)} \) is a Cauchy net in \( V \). Indeed, by (S3), for all \( \varepsilon > 0 \), there is \( \varepsilon_0 \in \mathbb{I}(A) \) such that \( \|x_{\varepsilon_0}\|^2 = \psi(\varepsilon_0) > 1 - \varepsilon \), and for all \( e \geq \varepsilon_0 \), we have \( \|x_e - x_{\varepsilon_0}\|^2 = \|x_{\varepsilon_0}\|^2 - \|x_e\|^2 < 1 - (1 - \varepsilon) = \varepsilon \), as required.

Similarly, one shows that \( \lim_{e \in \mathbb{I}(A)} \|x_e\|^2 = 1 \).

Let \( v = \lim_{e \in \mathbb{I}(A)} x_e \). Then \( \|v\| = 1 \) and for all \( e \in \mathbb{I}(A) \), we have \( ev = \lim_{e \geq e} x_e v = x_e v = x_e \).

Let \( a \in A \). Then there exists \( e \in \mathbb{I}(A) \) such that \( eae = a \).

Now, \( \langle av, v \rangle = \langle eaev, v \rangle = \langle av, ev \rangle = \langle x_ev, x_e \rangle = \psi(e^*ae) = \psi(e) = \psi(a) \).

This means that \( \varphi_{V,v} = \psi \), as required.

To finish, observe that if \( \psi = \varphi_{V',v'} \), then the morphism taking \( a + L \in A/L \) to \( av' \in V' \) is a well-defined \( A \)-homomorphism preserving the inner product and taking \( x_e \) to \( e'v' \).

Thus, it extends to a unitary isomorphism \( V \to V' \) taking \( v \) to \( v' \) by Lemma 2.29.

**Corollary 2.11.** Let \( V, V' \in \text{Rep}^1(A) \), and let \( v \in S^1(V) \), \( v' \in S^1(V') \). If \( \varphi_{V,v} = \varphi_{V',v'} \), then \( \overline{Av} \cong \overline{Av'} \) as unitary representations.

**Corollary 2.12.** Let \( V, V' \in \text{Rep}^1(A) \) and let \( v \in S^1(V) \), \( v' \in S^1(V') \) be vectors such that \( V = \overline{Av} \) and \( V' = \overline{Av'} \).

Assume that \( t < 0 \) such that \( t \varphi_{V,v}(a^*a) \leq \leq \).
\( \varphi_{V',v}(a^*a) \) for all \( a \in A \). Then there is a continuous \( A \)-module homomorphism \( f : V' \to V \) taking \( v' \) to \( v \).

**Proof.** Let \( L = \{ a \in A : \varphi_{V',v}(a^*a) = 0 \} \) and \( L' = \{ a \in A : \varphi_{V',v}(a^*a) = 0 \} \). Condition (S1) and the assumption \( t\varphi_{V',v}(a^*a) \leq \varphi_{V',v}(a^*a) \) imply that \( L \supseteq L' \). By the proof of Theorem 2.10, the map \( (x + L, y + L) \mapsto \varphi_{V',v}(y^*x) \) is an inner product on \( A/L \) and we have a unitary isomorphism \( \Phi : A/L \to V \) taking \( \lim_{e \in E(A)} (e + L) \) to \( v \). Similarly, we have a unitary isomorphism \( \Phi' : A/L' \to V' \).

Consider the \( A \)-homomorphism \( \Phi : A/L' \to A/L \) given by \( \Phi(a + L') = a + L \). Since \( t\varphi_{V',v}(a^*a) \leq \varphi_{V',v}(a^*a) \), we have \( \|a + L\|^2 \leq t^{-1}\|a + L'\|^2 \) for all \( a \in A \).

This means \( \Phi \) is continuous and hence it extends uniquely to an \( A \)-homomorphism \( A/L' \to A/L \), which clearly takes \( \lim_{e \in E(A)} (e + L') \) to \( \lim_{e \in E(A)} (e + L) \). Now let \( f = \Phi \Phi^{-1} : V' \to V \). Then \( f \) is continuous and takes \( v \) to \( v' \), as required. \( \square \)

A linear functional \( \psi \in A^* \) satisfying conditions (S1)–(S3) of Theorem 2.2 is called a state of \( A \). The function \( \varphi_{V,v} \) is called the state associated with \( V \) and \( v \). Denote by \( E(A) \) the set of states of \( A \). It is easy to see that \( E(A) \) is convex. (Indeed, if \( 0 \leq t \leq 1 \), then \( t\varphi_{V,v} + (1 - t)\varphi_{V',v} = \varphi_{V \oplus V', t/2I_{V \oplus V}, (1 - t)/2I_{V'}} \).

**Remark 2.13.** (i) Concerning condition (S1), any \( \varphi \in A^* \) is completely determined by its values on the set \( \{ a^*a | a \in A \} \). Indeed, for all \( a \in A \), there is \( e \in I(A) \) with \( eae = a \) and \( a + a^* = (e + e^*)a - a^*a - e^*e \). Since for all \( a \in A \), we have \( a = \frac{1}{2}(a + a^*) - i\frac{1}{2}i(a + ia)^* \), it follows that \( \{ a^*a | a \in A \} \) spans \( A \) as a \( \mathbb{C} \)-vector space, hence our claim.

(ii) Give \( A^* \) the topology of pointwise convergence. Then \( E(A) \) is not necessarily closed; the problem lies in condition (S2). It can be shown that this is indeed the case when \( A = \mathbb{C}[X, X^*] \) as in \( \mathbb{2}A \). In contrast, for \( C^* \)-algebras, the set \( E(A) \) is closed \( \mathbb{2}A \) \( \S 3.4 \).

A state \( \psi \) is called pure if it is an extremal point of \( E(A) \). That is, for all \( \psi, \psi' \in E(A) \) and \( 0 < t < 1 \) satisfying \( \psi = t\psi + (1 - t)\psi' \), we have \( \psi = \psi' = \varphi \).

**Proposition 2.14.** Let \( V \in \text{Rep}^A(A) \) and let \( v \in S^1(V) \) be such that \( V = \overline{vA} \). Then \( \varphi_{V,v} \) is pure if and only if \( v \in \overline{IA} \).

**Proof.** This follows from \( \mathbb{67} \) Th. 9.6.4. We give here a full proof for the sake of completeness.

Assume \( V \) is reducible. Then there are nonzero closed \( A \)-submodules \( U \) and \( U' \) such that \( V = U \perp U' \). Let \( P \) and \( P' \) be the orthogonal projections onto \( U \) and \( U' \) respectively. If \( Pv = 0 \), then \( U = \overline{P(Av)} = \overline{P(Av)} = \overline{A(Pv)} = 0 \), so \( Pv = 0 \), and likewise \( P'v \neq 0 \). Let \( t = \|Pv\|^2 \) and \( u = t^{1/2}Pv \), and define \( t' \) and \( u' \) similarly. Then \( t, t' \in \mathbb{R}_{>0} \) and \( t + t' = \|v\|^2 = 1 \). Now, for all \( a \in A \), we have \( \langle av, v \rangle = \langle P(v, P A(v)) + (P'v, P'A(v)) = \langle aPv, P'v \rangle + \langle aP'v, P'v \rangle = t \langle au, u \rangle + t' \langle au', u' \rangle \), hence \( \varphi_{V,v} = t\varphi_{V,u} + (1 - t)\varphi_{V,u'} \). We claim that \( \varphi_{V,v} \neq \varphi_{V,u} \), and therefore \( \varphi_{V,v} \) is not pure. Indeed, since \( \overline{Av} = V \), for all \( \varepsilon > 0 \), there is \( a \in A \) such that \( \|av - u'\varepsilon\| < \varepsilon \). Now, \( \|au\| = \|P(av) - Pu'\| \leq \|P(av - u')\| < \varepsilon \) and hence, \( \|au\| < \varepsilon \). On the other hand,

\[
\langle av, v \rangle \geq (1 - t)\langle au', u' \rangle - t\langle au, u \rangle \geq (1 - t)\langle av, u' \rangle - te
\]

\[
\geq (1 - t)(\langle u', u' \rangle - \|av - u'\|)^{1/2} - t\varepsilon
\]

\[
\geq (1 - t)\|av - u'\| \cdot \|u'\| - t\varepsilon
\]

\[
\geq (1 - t) - (1 - t)\varepsilon = (1 - t) - \varepsilon.
\]

Taking \( t < \frac{1}{2} \), we get \( \langle av, v \rangle \neq \langle au, u \rangle \).
Assume $V$ is irreducible. Let $\psi, \psi' \in E(A)$ and $0 < t < 1$ be such that $\varphi = t\psi + (1-t)\psi'$, and write $\psi = \varphi_{U,v}$ with $\overline{A}u = U$. Condition (S1) implies that $t\psi(a^*) \leq \varphi(a^*)$ for all $a \in A$. Thus, by Corollary 2.12, there is a continuous homomorphism $f : V \to U$ taking $v$ to $u$. By Schur's Lemma, $f^* f : V \to V$ equals $\lambda$ for some $\lambda \in \mathbb{C}$, and it is easy to see that $\lambda \in \mathbb{R}_{>0}$ (notice that $f^* f \neq 0$ since $\langle f^* f v, v \rangle = \|u\|^2 = 1$). Now, for all $a \in A$, $\psi(a) = \langle au, u \rangle = \langle af, v \rangle = \langle f^* f a v, v \rangle = \lambda \langle av, v \rangle = \lambda \varphi_{V,v}(a)$. Condition (S3) implies $\lambda = 1$, so $\psi = \varphi_{V,v}$.

Likewise, $\psi' = \varphi_{V,v}$.

$\square$

2E. Weak Containment. Let $A$ be an idempotented $*$-algebra, and let $A' = \text{Hom}_C(A, C)$. We give $A'$ the topology of point-wise convergence (i.e. the topology induced from the product topology on $\mathbb{C}^A$). Recall from Theorem 2.10 that for $V \in \text{Rep}^a(A)$ and $v \in V$, we define $\varphi_{V,v} \in A'$ by $\varphi_{V,v}(a) = \langle av, v \rangle$. We write $F \subseteq_f A$ to denote that $F$ is a finite subset of $A$. For every such $F$, it is convenient to consider the seminorm $\| \cdot \| : A' \to \mathbb{R}_{\geq 0}$ given by

$$\| \varphi \|_F = \max_{a \in F} | \varphi(a) | .$$

Notice that a net $\{ \varphi_n \}_{n \in I}$ in $A'$ converges to $\varphi$ if and only if $\lim_{n} \| \varphi_n - \varphi \|_F = 0$ for all $F \subseteq_f A$.

Lemma 2.15. Let $V \in \text{Irr}^a(A)$ and $V' \in \text{Rep}^a(A)$. The following conditions are equivalent:

(a) For all $v \in S^1(V)$, $\varepsilon > 0$ and $F \subseteq_f A$, there exists $v' \in S^1(V')$ such that $\| \varphi_{V,v} - \varphi_{V',v'} \|_F < \varepsilon$.

(b) There exists $v \in S^1(V)$ such that for all $\varepsilon > 0$ and $F \subseteq_f A$, there is $v' \in V'$ such that $\| \varphi_{V,v} - \varphi_{V',v'} \|_F < \varepsilon$.

In part (a), we can always take $v' \in S^1(AV')$. Furthermore, if $ev = v$ for some $e \in \text{Irr}(A)$, then $v'$ can be chosen to be in $S^1(\varepsilon V')$.

Proof. (a)$\implies$(b) is clear, so we only prove (b)$\implies$(a).

Let $v_1 \in S^1(V)$, $\varepsilon > 0$ and $F_1 \subseteq_f A$ be given. Since $V$ is irreducible, for all $\delta > 0$, there is $b \in A$ such that

$$\| bv - v_1 \| < \delta \quad \text{ and } \quad \| bv \| = 1 .$$

Choose $e \in \text{Irr}(A)$ such that $ebe = b$ and let $F := \{ b^*ab : a \in F_1 \cup \{ e \} \}$. By assumption, for all $\delta' > 0$, there exists $v' \in V'$ such that $\| \varphi_{V,v} - \varphi_{V',v'} \|_F < \delta'$. Since $\varphi_{V,bv}(a) = \varphi_{V,v}(b^*ab)$ and $\varphi_{V',bv'}(a) = \varphi_{V',v'}(b^*ab)$, this is equivalent to

$$\| \varphi_{V,bv} - \varphi_{V',bv'} \|_{F_1 \cup \{ e \}} < \delta' .$$

In particular, $| 1 - \| bv' \|^2 | = | \varphi_{V,bv}(c) - \varphi_{V',bv'}(c) | < \delta' $ (since $ebe = b$ and $\| bv \| = 1$). Thus, for $\delta' \leq 1$, we have $0 < \| bv' \|^2 < 2$.

Define $v'_1 := \| bv' \|^{-1} bv'$ and let $a \in F_1$. Then

$$| \varphi_{V',bv'}(a) - \varphi_{V',v'_1}(a) | = | \varphi_{V',bv'}(a) - \| bv' \|^2 \varphi_{V',bv'}(a) |$$

$$= | 1 - \| bv' \|^2 | \cdot | \varphi_{V',bv'}(a) | \leq \delta' | \langle abv', bv' \rangle | \leq \delta' \| a \|_{V'} \| bv' \|^2$$

$$\leq 2\delta' \| a \|_{V'} .$$

In addition,

$$| \varphi_{V,v_1}(a) - \varphi_{V,bv}(a) | = | \langle av_1, v_1 \rangle - \langle abv, bv \rangle |$$

$$\leq | \langle av_1, v_1 - bv \rangle | + | \langle av_1 - abv, bv \rangle |$$

$$\leq \| a \|_{V'} \| v_1 \| \cdot \| v_1 - bv \| + \| a \|_{V'} \cdot \| v_1 - bv \| \cdot \| bv \|$$

$$\leq 2\delta \| a \|_{V'} .$$

(2.2)
Thus,
\[
|\varphi_{V,v}(a) - \varphi_{V',v'}(a)| \leq |\varphi_{V,v}(a) - \varphi_{V,bv}(a)| + |\varphi_{V,bv}(a) - \varphi_{V',v'}(a)| \\
+ |\varphi_{V',bv}(a) - \varphi_{V',v'}(a)| \\
< 2\delta |a|_V + \delta' + 2\delta'|a|_{V'} .
\]

Since \(a \in F_1\) and \(F_1\) is finite, we can choose \(\delta\) and \(\delta'\) small enough in advance to have \(|\varphi_{V,v}(a) - \varphi_{V',v'}(a)| < \varepsilon\) for all \(a \in F_1\). This proves (a).

To finish, notice that the vector \(v'_1\) lies \(AV\). In addition, if \(e_1v_1 = v_1\) for some \(e_1 \in I(A)\), then we can apply the previous argument with \(v = v_1\) and \(b = e = e_1\). We then get \(e_1v'_1 = \|bv'\|^{-1}ev' = \|bv'\|^{-1}bv' = v'_1\), so \(v'_1 \in S^1(e_1V')\).

Let \(V \in \text{Irr}^n(A)\) and let \(V' \in \text{Rep}^n(A)\). We say that \(V\) is weakly contained in \(V'\) and write
\(V \prec V'\)
if the equivalent conditions of Lemma 2.15 are satisfied. For example, if \(V \leq V'\), then \(V \prec V'\). The converse is false in general.

The next proposition relates weak containment with common spectrum of linear operators (see 2A for the definition).

**Lemma 2.16.** Let \(V \in \text{Irr}^n(A)\), \(V' \in \text{Rep}^n(A)\) and \(e \in I(A)\). Assume \(V \prec V'\) and let \(F \subseteq_f A\). Then for all \(v \in S^1(V)\) (resp. \(v \in S^1(eV)\)) and \(\varepsilon > 0\), there is \(v' \in S^1(V')\) (resp. \(v' \in S^1(eV')\)) such that \(||av|| - ||av'|| < \varepsilon\) for all \(a \in F\). In particular, \(||a||_V \leq ||a||_{V'}\) for all \(a \in A\).

**Proof.** Observe that \(\varphi_{V,v}(a^*a) = ||av||^2\). The first assertion now follows from Lemma 2.15 and the continuity of the real function \(x \mapsto x^{1/2}\). The last assertion follows from the first assertion because \(||a||_V = \sup\{||av|| : v \in S^1(V)\}\) and likewise for \(||a||_{V'}\).

**Proposition 2.17.** Let \(V \in \text{Irr}^n(A)\), \(V' \in \text{Rep}^n(A)\), and let \(a_1, \ldots, a_n \in A\).

(i) If \(V \prec V'\), then \(\text{Spec}(a_1|_V, \ldots, a_n|_V) \subseteq \text{Spec}(a_1|_{V'}, \ldots, a_n|_{V'})\).

(ii) Suppose \(A\) is generated by \(\{a_1, \ldots, a_n\}\) (as a unital \(\mathbb{C}\)-algebra if \(A\) has a unity) and \(\text{dim} V = 1\). Write \(a_i|_V = \lambda_i\text{id}_V\) for \(\lambda_i \in \mathbb{C}\). Then
\[(\lambda_1, \ldots, \lambda_n) \in \text{Spec}(a_1|_{V'}, \ldots, a_n|_{V'}) \iff V \prec V'.\]

**Proof.** (i) Let \((\lambda_1, \ldots, \lambda_i) \in \text{Spec}(a_1|_V, \ldots, a_n|_V)\) and let \(\varepsilon > 0\). Then there is \(v \in S^1(V)\) such that \(||(a_i - \lambda_i)v|| < \frac{\varepsilon}{2}\) for all \(i\). Since \(AV\) is dense in \(V\), we may choose \(v\) to be in \(AV\). Choose \(e \in I(A)\) with \(ev = v\). Then \((a_i - \lambda_i\varepsilon)v = (a_i - \lambda_i)v\). By Lemma 2.16 there is \(v' \in S^1(eV')\) such that \(||(a_i - \lambda_i\varepsilon)v'|| < ||(a_i - \lambda_i\varepsilon)v|| + \frac{\varepsilon}{2} < \varepsilon\) for all \(i\). This holds for all \(\varepsilon\), so \((\lambda_1, \ldots, \lambda_n) \in \text{Spec}(a_1|_{V'}, \ldots, a_n|_{V'})\).

(ii) Since \(A\) is idempotent, there is an idempotent \(e \in A\) such that \(eae = a_i\) for all \(i\). This idempotent is a necessarily a unity of \(A\). Using this fact, we may replace \(a_i\) with \(a_i - \lambda_i\), and hence assume \(\lambda_i = 0\) for all \(i\).

The direction \((\Rightarrow)\) follows from (i), so we turn to prove \((\Leftarrow)\). Let \(M\) be the free non-commutative monoid on \(n\) letters \(x_1, \ldots, x_n\). Let \(v \in S^1(V)\), \(\varepsilon > 0\) and \(b_1, \ldots, b_n \in A\), where \(b_i = \sum_{w \in M} a_{iw}a_{w}(1, \ldots, a_n)\) for some \(\{a_{iw}\}_{i,w} \in \mathbb{C}\), all but finitely many are zero. We need to show that there exists \(v' \in S^1(V')\) such that \(||b_iv,v) - (b_iv',v')|| < \varepsilon\) for all \(i\). Indeed, since \((0, \ldots, 0) \in \text{Spec}(a_1|_{V'}, \ldots, a_n|_{V'})\), for all \(\delta > 0\), there is \(v' \in S^1(V')\) such that \(||a_i|| < r_i\delta\) where \(r_i = ||a_i|||_V\). Notice
that \( b_1 v = \alpha_{11} v \) since \( a_1 v = \cdots = a_n v = 0 \). We now have

\[
|\langle b_1 v', v' \rangle - \langle b_1 v, v \rangle| = \left| \sum_{w \neq 1} \alpha_{iuous} w(a_1, \ldots, a_n) v, v' \right| - \langle \alpha_{11} v, v \rangle
\]

\[
= \left| \sum_{w \neq 1} \alpha_{iuous} w(a_1, \ldots, a_n) v' \right|
\]

\[
\leq \sum_{w \neq 1} |\alpha_{iuous}| \langle w(r_1, \ldots, r_n) \rangle.
\]

Choosing \( \delta \) small enough in advance, we get \( |\langle b_1 v, v \rangle - \langle b_1 v', v' \rangle| < \varepsilon \) for all \( i \).

Example 2.18. Applying Proposition 2.17 in the setting of 2A with \( a_1 = X \) and \( a_2 = X^* \) implies that \( (\lambda, X) \in \text{Spec}(T, T^*) \iff V_\lambda \prec V. \) Since \( T \) is normal, the former condition is equivalent to \( \lambda \in \text{Spec}(T) \). In particular, we get a one-to-one correspondence

\[
\text{Spec}(T) \longleftrightarrow \left\{ U \in \text{Irr}^n(A) \text{ with } U \prec V, \text{ up to isomorphism} \right\}
\]

We have therefore shown that the spectrum of \( T \) can be recovered from the action of \( A \) on \( V \).

Example 2.19. Generalizing the previous example, let \( A \) be a commutative unital \( * \)-algebra generated by \( \{a_1, \ldots, a_n\} \). Since \( A \) is commutative, every \( V \in \text{Irr}^n(A) \) is 1-dimensional (Corollary 2.7) and hence operators on \( V \) can be viewed as elements of \( C \). We associate with every such \( V \) a vector \( \lambda_V = (a_1|_V, \ldots, a_n|_V) \in C^n \). By Proposition 2.17, for every \( V' \in \text{Rep}^n(A) \), we have a one-to-one correspondence

\[
\text{Spec}(a_1|_V, \ldots, a_n|_V) \longleftrightarrow \left\{ V \in \text{Irr}^n(A) \text{ with } V \prec V', \text{ up to isomorphism} \right\}
\]

induced by \( \lambda_V \leftrightarrow V \).

2F. The Unitary Dual. With Examples 2.18 and 2.19 in mind, we define a notion of spectrum for unitary representations of \( A \).

For \( V \in \text{Irr}^n(A) \), let \([V]\) denote the unitary isomorphism class of \( V \), and let

\[
\widehat{A} = \left\{ [V] \mid V \in \text{Irr}^n(A) \right\}.
\]

The set \( \widehat{A} \) is called the unitary dual of \( A \). It will be the space where points of the spectrum will live. For \( V' \in \text{Rep}^n(A) \), we define the \( A \)-spectrum of \( V' \) to be

\[
\text{Spec}_A(V') = \left\{ [V] \in \widehat{A} \mid V \prec V' \right\}.
\]

Example 2.20. In the setting of 2A, we have an isomorphism \( \widehat{A} \cong C \) via \([V]\) \leftrightarrow \( \lambda \), and under this isomorphism, \( \text{Spec}_A(V) \) coincides with \( \text{Spec}(T) \) (Example 2.18).

We now introduce a topology on \( \widehat{A} \): Let \( V \in \text{Irr}^n(A), v \in S^1(V), \varepsilon > 0 \) and \( F \subseteq I A \). We define

\[
N_{V,v,\varepsilon,F} \subseteq \widehat{A}
\]

3 In the theory of \( C^* \)-algebras, the unitary dual is sometimes called the spectrum of \( A \), in which case the spectrum defined above is called the support of the representation. Similar terminology is used in algebraic geometry. However, since our goal is to demonstrate how the support generalizes the operator spectrum, we preferred to use the term “spectrum”.
to be the set of all isomorphism classes \([U] \in \hat{A}\) for which there is \(u \in U\) such that \(\|\varphi_{V,v} - \varphi_{U,u}\|_F < \varepsilon\). Note that \(u\) is not required to be a unit vector\(^4\). The possible sets \(\{N_{V,v,\varepsilon,F}\}\) form a subsbasis for a topology on \(\hat{A}\).

**Remark 2.21.** The common way to define a topology on the unitary dual of a $$\ast$$-algebra is by pulling back the hull-kernel topology on primitive $$\ast$$-ideals of \(A\) along the map \([V] \mapsto \{a \in A : a|_V = 0\}\). For \(C^*\)-algebras, this gives the topology we have just defined (see [20], §3.1) and Proposition 2.20 below. However, these topologies are different in general, e.g., consider the case \(A = \mathbb{C}[X,X^*]\) as in 2A. Furthermore, Propositions 2.22 and 2.20 below would fail if we were to give \(A\) the pullback of the hull-kernel topology.

It is convenient to call a subset \(S \subseteq \hat{A}\) bounded if \(\sup_{[V] \in S} \|a|_V\| < \infty\) for all \(a \in A\), or equivalently, if \(\bigoplus_{[V] \in S} V \in \text{Rep}^u(A)\) (see 2C).

**Proposition 2.22.** Assume \(A\) is unital, commutative, and \(a_1, \ldots, a_t \in A\) are elements such that \(\{a_1, \ldots, a_t, a_1^*, \ldots, a_t^*\}\) generate \(A\) as a unital algebra. For \(V \in \text{Ir}^u(A)\), denote by \(\lambda_V\) the unique common eigenvalue of \(\{a_1, \ldots, a_t\}\) on \(V\). Then the map

\[
[V] \mapsto \lambda_V : \hat{A} \to \mathbb{C}^t
\]

is a topological embedding. In addition, for all \(V' \in \text{Rep}^u(A)\), we have

\[
\text{Spec}(a_1|_{V'}, \ldots, a_t|_{V'}) = \{\lambda_{V'} | [V] \in \text{Spec}_A(V')\}.
\]

Finally, a subset \(S \subseteq \hat{A}\) is bounded if and only if its image in \(\mathbb{C}^t\) is bounded.

**Proof.** For all \(V' \in \text{Rep}^u(A)\), we have \((\lambda_1, \ldots, \lambda_t) \in \text{Spec}(a_1|_{V'}, \ldots, a_t|_{V'})\) if and only if \((\lambda_1, \ldots, \lambda_t, \bar{\lambda}_1, \ldots, \bar{\lambda}_t) \in \text{Spec}(a_1|_{V'}, \ldots, a_t|_{V'}, a_1^*|_{V'}, \ldots, a_t^*|_{V'})\). Indeed, this easily follows from the fact that \(\|(a_i - \lambda_i)v\| = \|(a_i^* - \bar{\lambda}_i)v\|\) for all \(v \in V'\) when \(a_i a_i^* = a_i a_i^*\) (see 2A). Since the map \((\lambda_1, \ldots, \lambda_t) \mapsto (\lambda_1, \ldots, \lambda_t, \bar{\lambda}_1, \ldots, \bar{\lambda}_t) : \mathbb{C}^t \to \mathbb{C}^{2t}\) is a topological embedding, we may replace \(a_1, \ldots, a_t\) with \(\{a_1, \ldots, a_t, a_1^*, \ldots, a_t^*\}\), and assume that \(\{a_1, \ldots, a_t\}\) generates \(A\) as a unital algebra.

Denote by \(\theta\) the map \([V] \mapsto \lambda_V\). We have seen in Example 2.19 that \(\theta\) is injective and that \(\text{Spec}(a_1|_{V'}, \ldots, a_t|_{V'}) = \{\lambda_{V'} | [V] \in \text{Spec}_A(V')\}\) for all \(V' \in \text{Rep}^u(A)\). It is therefore left to show that \(\theta\) is a topological embedding.

Let us check what is \(N_{V,v,\varepsilon,F}\): Write \(F = \{b_1, \ldots, b_m\}\) and choose polynomials \(f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_t]\) such that \(b_i = f_i(a_1, \ldots, a_t)\). Since \(V\) is 1-dimensional, for all \(v \in S^1(V)\),

\[
(b_i v, v) = f_i(\lambda_V)\|v\|^2 = f_i(\lambda_V).
\]

Likewise, if \([U] \in \hat{A}\), then \((au, u) = f_i(\lambda_U)\|u\|^2\) for all \(u \in U\). Writing \(r = \|u\|^2\), it follows that

\[
N_{V,v,\varepsilon,F} = \bigcup_{r \in \mathbb{R}_{>0}} \left\{ [U] \in \hat{A} : |f_i(\lambda_V) - r f_i(\lambda_U)| < \varepsilon \quad \text{for all } i \right\}.
\]

Therefore,

\[
N_{V,v,\varepsilon,F} = \theta^{-1}(M_{f_1, \ldots, f_m})
\]

where

\[
M_{f_1, \ldots, f_m} := \bigcup_{r \in \mathbb{R}_{>0}} \{ \mu \in \mathbb{C}^n : |f_i(\lambda_V) - r f_i(\mu)| < \varepsilon \quad \text{for all } i \}.
\]

The sets \(M_{f_1, \ldots, f_m}\) are open in \(\mathbb{C}^n\), and moreover, they form an open basis of neighborhoods of \(\lambda_V\). (Indeed, take \(m = n + 1\) and let \(f_1 = x_i\) (\(i \leq n\)) and \(f_{n+1} = 1\); the details are left to the reader.) Thus, \(\theta : \hat{A} \to \mathbb{C}^n\) is a topological embedding.

The final assertion is easy and is left to the reader. \(\square\)

\(^4\) There is some room for variation here. Requiring \(u\) to be a unit vector affects parts (i) and (ii) of Proposition 2.20(ii), but this makes little difference for many $$\ast$$-algebras by Remark 2.27.
Remark 2.23. In the setting of Proposition 2.22 the image of $\hat{A}$ in $\mathbb{C}^t$ can be described as follows: Let $\phi: \mathbb{C}[x_1, \ldots, x_t, y_1, \ldots, y_t] \to A$ be the algebra homomorphism sending $x_i$ to $a_i$ and $y_i$ to $a_i^*$, and let $X$ be the affine scheme $\text{Spec} A$. Then $\phi$ induces a closed embedding (of schemes) $\phi^*: X \to \mathbb{A}^t$, which in turn induces an injection $X(\mathbb{C}) \to \mathbb{C}^t$. We thus view $X(\mathbb{C})$ as a subset of $\mathbb{C}^t$. The image of $\hat{A}$ in $\mathbb{C}^t$ is $\{(\lambda_1, \ldots, \lambda_t) : (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_t) \in X(\mathbb{C})\}$. The technical and straightforward proof is left to the reader.

The following example assumes basic knowledge of quivers and their representations. See [2] for all relevant details.

Example 2.24. Let $A$ be the path algebra of the quiver

$$Q: \begin{array}{c}
\begin{array}{c}
0 \overset{a_{01}}{\longrightarrow} 1
\end{array}
\end{array}$$

That is, $A$ is the free $\mathbb{C}$-algebra generated by $e_0, e_1, a_{01}, a_{10}$ subject to the relations $e_0^2 = e_0$, $e_1^2 = e_1$, $e_0a_{01}e_1 = a_{01}$ and $e_1a_{10}e_0 = a_{10}$. We define $*: A \to A$ to be the only involution satisfying $e_0^* = e_0$, $e_1^* = e_1$, and $a_{01}^* = a_{10}$. It is easy to see that $A$ is a unital $*$-algebra ($e_0 + e_1$ is the unity).

Recall that $A$-modules are in correspondence with representations of $Q$ via sending $V \in A$-Mod to the vector space diagram

$$e_0 V \overset{a_{01}|_V}{\longrightarrow} e_1 V$$

Any $V \in \text{Irr}^u(A)$ can be shown to be isomorphic to one of $\{V_r\}_{r \in \mathbb{R}_{>0}}$, $U_0$, $U_1$ given by the following diagrams, respectively:

The inner products on $V_r = \mathbb{C} \oplus \mathbb{C}$, $U_0 = \mathbb{C} \oplus 0$ and $U_1 = 0 \oplus \mathbb{C}$ are all given by $(\alpha, \beta') = \alpha \beta + (a \beta')$. Using arguments similar to those in the proof of Proposition 2.22 one can show that $\hat{A}$ homeomorphic to two copies of $\mathbb{R}_{>0}$ glued along $\mathbb{R}_{>0}$. The homeomorphism sends $[V_r]$ to $r \in \mathbb{R}_{>0}$ and $U_0$ and $U_1$ to the two points lying over $0$. The details are left to the reader.

This shows that $\hat{A}$ is not $T_1$ in general.

Remark 2.25. The unitary dual of a unital $C^*$-algebra is always quasi-compact [20 Pr. 3.1.8]. However, as follows from Proposition 2.22 or Example 2.24 this is not necessarily the case for unital $*$-algebras.

We finally record the following useful facts about the topology of $\hat{A}$.

Proposition 2.26. Let $V \in \text{Irr}^u(A)$, $v \in S^1(V)$ and let $S \subseteq \hat{A}$. Then:

(i) Then the sets $\{N_{V,v,F}\}_{\varepsilon,F}$ ($\varepsilon > 0$, $F \subseteq f A$) are a basis of open neighborhoods of $[V]$ in $\hat{A}$.

(ii) $[V] \in S$ if and only if for all $\varepsilon > 0$ and $F \subseteq f A$, there exist $[U] \in S$ and $u \in U$ such that $\|\phi_{V,v} - \phi_{U,u}\|_F < \varepsilon$.

---

5 Here is an ad-hoc sketch of proof: Since $e_0 + e_1 = 1_A$, at least one of $e_0V$, $e_1V$ is nonzero. If $e_0V \neq 0$, then $e_0V$ is an irreducible unitary representation of $e_0Ae_0$ (Lemma 2.21 below), and since $e_0Ae_0$ is commutative, $e_0V$ must be 1-dimensional (Corollary 2.21). Likewise, if $e_1V \neq 0$, then dim $e_1V = 1$. The cases $e_0V = 0$ and $e_1V = 0$ lead to $U_1$ and $U_0$, respectively. In all other cases, we may assume $e_0V = e_1V = \mathbb{C}$ and view $a_{10}|_V$, $a_{01}|_V$ as elements of $\mathbb{C}$. If $z = a_{10}|_V$, then $a_{01}|_V = \frac{z}{2}$ because $a_{10}^* = a_{01}$. One can show that the isomorphism type of $V$ determined by $s := z\sqrt{2}$. If $s = 0$, then $V \cong U_0 \oplus U_1$. Otherwise, $V \cong V_r$ for $r = \sqrt{2}$. 

(iii) When $S$ is bounded, $[V] \in \mathcal{S}$ if and only if for all $\epsilon > 0$ and $F \subseteq_f A$, there exist $[U] \in S$ and $u \in S^1(U)$ such that $\|\varphi_{U,u} - \varphi_{U,u}\|_F < \epsilon$.

Proof. (i) Since the sets $\{N_{V,v,e,F}\}_{e,F}$ already form a filter base, it is enough to prove that if $[V] \in N_{U,u,A,G}$ then there are $\epsilon > 0$ and $F \subseteq_f A$ such that $N_{V,v,e,F} \subseteq N_{U,u,A,G}$.

Suppose $[V] \in N_{U,u,A,G}$. Then there is $v' \in V$ such that $\|\varphi_{V,v'} - \varphi_{U,u}\|_G < \delta$. If $v' = 0$, replace it with a vector of sufficiently small magnitude. Fix some $\delta_0 < \delta$ such that $\|\varphi_{V,v'} - \varphi_{U,u}\|_G < \delta_0$, and let $\epsilon, \epsilon' > 0$, to be chosen later. Since $\mathcal{S} = V$, there is $b \in A$ such that $\|bv - v'\| < \epsilon'$ and $\|be\| = ||v'||$ (here we need $v' \neq 0$). Let $F = \{b^*a|a \in G\}$ and let $[W] \in N_{V,v,e,F}$. Then there is $w \in W$ such that $\|\varphi_{V,v} - \varphi_{W,w}\|_F < \epsilon$. As in the proof of Lemma 2.15 the latter implies $\|\varphi_{V,v'} - \varphi_{W,w}\|_F < \epsilon$, and by virtue of (2.2), $|\varphi_{V,v'} - \varphi_{W,w}|(a) - \varphi_{V,v}(a)\| \leq 2\epsilon\|a\|_V\|v'||$ for all $a \in G$. Thus, for all $a \in G$,

$$\|\varphi_{U,u}(a) - \varphi_{W,w}(a)\| \leq |\varphi_{U,u}(a) - \varphi_{V,v'}(a)| + |\varphi_{V,v'}(a) - \varphi_{W,w}(a)|$$

$$+ |\varphi_{W,w}(a) - \varphi_{V,v}(a)| < \delta_0 + 2\epsilon\|a\|_V\|v'|| + \epsilon.$$  

Choose $\epsilon$ and $\epsilon'$ such that $\delta_0 + 2\epsilon\|a\|_V\|v'|| + \epsilon < \delta$ for all $a \in G$. Then we have shown that $N_{V,v,e,F} \subseteq N_{U,u,A,G}$.

(ii) The condition means that $N_{V,v,e,F} \cap S \neq \emptyset$ for all $\epsilon > 0$ and $F \subseteq_f A$, and this is equivalent to $[V] \in \mathcal{S}$ by (i).

(iii) Suppose $[V] \in \mathcal{S}$ and let $F_1 \subseteq_f A$ and $\epsilon > 0$. Since $S$ is bounded, there is $M \in \mathbb{R}$ such that $\|a\|_F < M$ for all $a \in F_1$, $[U] \in S$. Applying the proof of Lemma 2.15 with $M$ in place of $\|a\|_F$ and $v_1 = v$, we obtain $F \subseteq_f A$ and $\delta_0 > 0$ such that for all $[U] \in S$ and $u \in U$ with $\|\varphi_{U,u} - \varphi_{V,v}\|_F < \delta_0$, we have $\|\varphi_{V,v} - \varphi_{U,u}\|_F < \epsilon$. The existence of $U$ and $u$ with $\|\varphi_{U,u} - \varphi_{V,v}\|_F < \delta$ follows from (ii), so this proves one direction. The other direction is immediate from (ii). \qed

Remark 2.27. Proposition 2.26(iii) holds for any $S$ if $\hat{A}$ is locally bounded, i.e. if it admits a basis of bounded open sets. This can be shown to hold when $A$ is a $C^*$-algebra, or when $A$ is the Hecke algebra of an $\ell$-group (see [33]).

2G. Properties of The Spectrum. We now prove several properties of the $A$-spectrum.

Proposition 2.28. Let $V' \in \text{Rep}^n(A)$. Then $\text{Spec}_A(V')$ is bounded and closed.

Proof. The set $\text{Spec}_A(V')$ is bounded by Lemma 2.14. Assume $[V] \in \text{Rep}^n(A)$ and let $v \in S^1(V)$. By Proposition 2.26(iii), for all $\epsilon > 0$ and $F \subseteq_f A$, there is $[U] \in \text{Spec}_A(V')$ and $u \in S^1(U)$ satisfying $\|\varphi_{V,v} - \varphi_{U,u}\|_F < \frac{\epsilon}{2}$. Since $U \prec V'$, there is $v' \in S^1(V')$ such that $\|\varphi_{U,u} - \varphi_{V,v'}\|_F < \frac{\epsilon}{2}$, so $\|\varphi_{V,v} - \varphi_{V,v'}\|_F \leq \|\varphi_{U,u} - \varphi_{V,v'}\|_F + \|\varphi_{U,u} - \varphi_{U,u}\|_F < \epsilon$. As this holds for all $F$ and $\epsilon$, it follows that $V \prec V'$.

Recall that $V' = \text{Hom}_{C}(A, C)$ is endowed with the topology of point-wise convergence. Subsets of $V'$ are given the induced topology. For $V' \in \text{Rep}^n(A)$, let $E(A; V')$ denote the closed convex hull of $\{\varphi_{V,v'}|v' \in S^1(V')\}$ in $V'$. In addition, let $P(A; V')$ be the set of extremal points of $E(A; V')$. Recall that $E(A)$ is the set of all states of $A$. We also let $P(A)$ denote the set of pure states of $A$ (see [21]).

Theorem 2.29. Let $V' \in \text{Rep}^n(A)$. Then:

(i) $E(A; V')$ is compact in $V'$. 

Proof. ...
(ii) $E(A; V')$ is the closed convex hull of $P(A; V')$ and it is contained in $E(A)$.

(iii) $P(A; V') = P(A) \cap E(A; V') = \{ \varphi_{U,u} \mid [U] \in \text{Spec}_A(V'), \, u \in S^1(U) \}.$

Proof. (i) $E(A; V')$ is closed in $A'$ and hence in $C(A)$ (with the product topology), so it is enough to show that it is contained in a compact subset of $C(A)$. Observe that for all $v' \in S^1(V')$, we have $|\varphi_{V,v'}(a)| = |\langle av', v' \rangle| \leq \|a\|_{V'}$. Thus, $E(A; V') \subseteq \{ \varphi \in C(A) : \|\varphi\| \leq \|a\|_{V'} \text{ for all } a \in A \}$. The right hand side is isomorphic to $\prod_{a \in A} \{ \varphi : |\varphi| \leq \|a\|_{V'} \},$ which is compact by Tychonoff’s Theorem.

(ii) The first assertion follows from (i) and the Krein–Milman Theorem ([54, p. 139], for instance). For the second assertion, observe that for all $a, b \in A$, $v' \in S^1(V')$, we have $\varphi_{V,v'}(a^* b a) \leq \varphi_{V,v'}(a') \|b\|_{V'}^2$. From this it is easy to see that elements in the closed convex hull of $\{ \varphi_{V,v'} \mid v' \in S^1(V') \}$ satisfy conditions (S1)–(S3) of Theorem 2.11 and hence lie in $E(A)$.

(iii) The definition of weak containment implies that $\varphi_{U,u} \in E(A; V')$ for all $[U] \in \text{Spec}_A(V')$ and $u \in S^1(U)$. Together with Proposition 2.11 this implies $P(A; V') \supseteq P(A) \cap E(A; V') \supseteq \{ \varphi_{U,u} \mid [U] \in \text{Spec}_A(V'), \, u \in S^1(U) \}$, so it remains to prove that $P(A; V') \subseteq \{ \varphi_{U,u} \mid [U] \in \text{Spec}_A(V'), \, u \in S^1(U) \}$.

Let $\varphi \in P(A; V')$. By (ii), we can write $\varphi = \varphi_{U,u}$ with $U = U_{\overline{a}}$ and $u \in S^1(U)$. Since $E(A; V')$ is the closed convex hull of $\{ \varphi_{V,v'} \mid v' \in S^1(V') \}$ and it is compact, Milman’s converse to the Krein–Milman Theorem ([54, p. 139], for instance) implies that $\varphi_{U,u} \in \{ \varphi_{V,v'} \mid v' \in S^1(V') \}$ (because $\varphi_{U,u}$ is extremal in $E(A; V')$). This easily implies $\varphi_{U,u} \in \{ \varphi_{V,v'} \mid v' \in S^1(V') \}$ for all $a \in A$, and since $U = U_{\overline{a}}$, it follows that $E(A; U) \subseteq E(A; V')$.

Suppose $U$ is reducible. Then $\varphi_{U,u}$ is not pure (Proposition 2.1), and hence there are $U_1, U_2 \in \text{Irr}(A)$, $u_1 \in S^1(U_1)$, $u_2 \in S^1(U_2)$ and $0 < t < 1$ such that $\varphi_{U,u} = t\varphi_{U_1,u_1} + (1 - t)\varphi_{U_2,u_2}$ and $\varphi_{U,u} \neq \varphi_{U_1,u_1}$. By Corollary 2.12, $U_1, U_2 \subseteq U$, and therefore $\varphi_{U_1,u_1}, \varphi_{U_2,u_2} \in E(A; U) \subseteq E(A; V')$. This contradicts our assumption that $\varphi_{U,u}$ is extremal in $E(A; V')$, so $U$ must be irreducible. Since $\varphi_{U,u} \in \{ \varphi_{V,v'} \mid v' \in S^1(V') \}$, this means $U \prec V'$, as required. \[\square\]

Corollary 2.30. Let $0 \neq V' \in \text{Rep}^0(A)$ and let $a \in A$. There is $[V] \in \text{Spec}_A(V')$ such that $\|a\|_V = \|a\|_{V'}$. In particular, $\text{Spec}_A(V') \neq \emptyset$.

Proof. Observe that for all $v' \in S^1(V')$ we have $\varphi_{V,v'}(a^* a) = \|av'\|^2$. Since $\|av'\| = \sup\{\|av'\| \mid v' \in S^1(V') \},$ we have

\[
\psi(a^* a) \leq \|av'\|^2 \quad \forall \psi \in E(V'; A)
\]

Let $S = \{ \psi \in E(A; V') : \psi(a^* a) = \|a\|^2 \}$. We first claim that $S \neq \emptyset$. Since $V' \neq \emptyset$, there is a sequence $\{v'_n \}_{n \in \mathbb{N}} \subseteq S^1(V')$ such that $\lim_{n \to \infty} \|av'_n\| = \|a\|$. Since $E(A; V')$ is compact, $\{ \varphi_{V,v'_n} \}$ has a subsequence converging to some $\psi \in E(A; V')$, which clearly lies in $S$. Now, $S$ is compact since it is closed in $E(A; V')$, so the Krein–Milman Theorem implies that it has an extremal point $\psi$. Using (2.20), it is easy to see that $\psi$ is also extremal in $E(A; V')$, so by Theorem 2.20(iii), there are $[V] \in \text{Spec}_A(V')$ and $v \in S^1(V)$ such that $\psi = \varphi_{V,v}$. Since $\|av\|^2 = \psi(a^* a) = \|a\|^2$, we have $\|av\| \geq \|a\|_{V'}$. On the other hand, $\|av\| \leq \|a\|_{V'}$ by Lemma 2.16. \[\square\]

Theorem 2.31. Let $\{V_i \}_{i \in I}$ be a family of unitary representations of $A$ such that $\bigoplus V_i \in \text{Rep}^0(A)$. Then $\text{Spec}_A(\bigoplus V_i) = \bigcup_i \text{Spec}_A(V_i)$.

Proof. It is clear that $\text{Spec}_A(V_i) \subseteq \text{Spec}_A(\bigoplus V_i)$ for all $i$. Since $\text{Spec}_A(\bigoplus V_i)$ is closed (Proposition 2.28), it is enough to show that $\text{Spec}_A(\bigoplus V_i) \subseteq \bigcup_i \text{Spec}_A(V_i)$.

\[\square\]

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6 The set $E(A)$ is convex, but since it is not closed in general, $\{ \varphi_{V,v'} \}_{v' \in S^1(V')} \subseteq E(A)$ does not immediately imply $E(V'; A) \subseteq E(A)$. 

Assume \([V] \in \text{Spec}_A(\bigoplus_i V_i)\) and let \(v \in S^1(V)\). Then \(\varphi_{V,v} \in P(A; \bigoplus_i V_i)\) (Theorem 2.24(iii)). It is easy to see that \(E(A; \bigoplus_i V_i)\) is the closed convex hull of \(\bigcup E(A; V_i)\), hence by Theorem 2.24(ii), it is the closed convex hull of \(\bigcup P(A; V_i)\). Since \(E(A; \bigoplus_i V_i)\) is compact, Krein’s Converse to the Krein–Milman Theorem [54, p. 139] implies that \(P(A; V) \subseteq \bigcup P(A; V_i)\), hence \(\varphi_{V,v} \in \bigcup P(A; V_i)\). By Proposition 2.26(iii) and Theorem 2.29(iii), this means \([V] \in \bigcup \text{Spec}_A(V_i)\), as required. \(\square\)

**2H. Subalgebras.** Let \(A\) be an idempotented *-algebra. By an *idempotented *-subalgebra of \(A\) we mean a subalgebra \(B \subseteq A\) such that \(B^* = B\) and \((B, \ast|_B)\) is an idempotented *-algebra.

**Example 2.32.** Let \(a_1, \ldots, a_t \in A\). Choose an idempotent \(e \in \mathbb{I}(A)\) such that \(ea_i e = a_i\) for all \(1 \leq i \leq t\). Then the subalgebra \(B\) generated by the elements \(e, a_1, \ldots, a_t\) is an idempotented *-subalgebra of \(A\).

Henceforth, \(B\) is a fixed idempotented *-subalgebra of \(A\). Observe that if \(V' \in \text{Rep}^n(A)\), then \(\overline{BV'} \in \text{Rep}^n(B)\). The purpose of this section is to show that \(\text{Spec}_A(V')\) determines \(\text{Spec}_B(\overline{BV'})\). Informally, this means that the \(A\)-spectrum holds at least as much information as the \(B\)-spectrum.

We start by introducing notation. For any \(V \in \text{Irr}^n(A)\), let \(\mathcal{R}_{A/B}(V) := \text{Spec}_B(\overline{BV}) \subseteq \hat{B}\). In general, \(\mathcal{R}_{A/B}(V)\) is not a singleton, so \(\mathcal{R}_{A/B}\) does not induce a function from \(\hat{A}\) to \(\hat{B}\). However, \(\mathcal{R}_{A/B}\) induces a map \(\mathcal{R}_{A/B} : P(\hat{A}) \to P(\hat{B})\) given by

\[
\mathcal{R}_{A/B}(S) = \bigcup_{[V] \in S} \mathcal{R}_{A/B}(V).
\]

This map has properties resembling continuity and closeness in the following sense.

**Proposition 2.33.** Suppose \(S \subseteq \hat{A}\) is bounded (i.e. \(\bigoplus_{[V] \in S} V \in \text{Rep}^n(A)\)). Then \(\mathcal{R}_{A/B}(\overline{S}) = \mathcal{R}_{A/B}(S)\).

**Proof.** Since \(\bigoplus_{[V] \in S} V \in \text{Rep}^n(A)\), we have \(\overline{S} = \text{Spec}_A(\bigoplus_{[V] \in S} V)\) and \(\mathcal{R}_{A/B}(\overline{S}) = \bigcup_{[V] \in S} \text{Spec}_B(\overline{BV})\) (Theorem 2.31). Since \(B(\bigoplus_{[V] \in S} V) = \bigoplus_{[V] \in S} \overline{BV}\), this means \(\mathcal{R}_{A/B}(\overline{S}) = \text{Spec}_B(\bigoplus_{[V] \in S} \overline{BV}) = \mathcal{R}_{A/B}(S)\). \(\square\)

**Example 2.34.** The assumption \(\bigoplus_{[V] \in S} V \in \text{Rep}^n(A)\) in Proposition 2.33 is essential. Let \(M\) be the multiplicative submonoid of \(\mathbb{C}[X]\) spanned by \(\{X - a \mid a \in \mathbb{C} \setminus \mathbb{Q}\}\), and let \(A\) be the localization \(\mathbb{C}[X] \cdot M^{-1}\) with the unique involution satisfying \(X^* = X\). Elaborating the proof of Proposition 2.24, one sees that \(\hat{A} \cong \mathbb{Q}\) as topological spaces, where the topology on \(\mathbb{Q}\) is induced from \(\mathbb{R}\). The isomorphism is given by sending \(\lambda \in \mathbb{Q}\) to the class \([V_\lambda] \in \hat{A}\) satisfying \(X|_{V_\lambda} = \lambda \text{id}_{V_\lambda}\). Let \(B = \mathbb{C}[X]\) and identify \(\hat{B}\) with \(\mathbb{R}\) similarly. Then \(\mathcal{R}_{A/B}\) turns out to induce a function \(\hat{A} \to \hat{B}\) (Corollary 2.27, this holds whenever \(A\) and \(B\) are commutative and \(A = AB\), which corresponds to the inclusion \(\mathbb{Q} \to \mathbb{R}\)). This map is clearly not closed, e.g. the set \([1, 2] \cap \mathbb{Q}\) is closed in \(\mathbb{Q}\) but not in \(\mathbb{R}\). Translating this to representations, we see that for \(S = \{[V_\lambda] : \lambda \in [1, 2] \cap \mathbb{Q}\}\), we have \(\mathcal{R}_{A/B}(\overline{S}) \subset \mathcal{R}_{A/B}(S)\).

Indeed, \([V_\lambda] : \lambda \in [1, 2] \cap \mathbb{Q}\) is not bounded since \(A\) does not act continuously on \(\bigoplus_{\lambda \in [1, 2] \cap \mathbb{Q}} V_\lambda\) (consider the action of \((X - \sqrt{3})^{-1} \in A\), for instance).

We believe that the inclusion \(\mathcal{R}_B(\overline{S}) \subset \mathcal{R}_B(S)\), which may be regarded as a form of continuity, should also fail when \(S\) is not bounded.

The following theorem explains how \(\text{Spec}_A(V')\) determines \(\text{Spec}_B(\overline{BV'})\).
Corollary 2.37. Suppose \( |U| \in \mathcal{R}_{A/B}(\text{Spec}(A(V'))) \). Then there is irreducible \( V < V' \) such that \( U \leq BV \).

Proof. For any irreducible \( V < V' \) we have \( \text{Spec}(BV) = \mathcal{R}_{A/B}(\text{Spec}(A(V'))) \).

Moreover, for any irreducible \( U < BV \), there exists irreducible \( V < V' \) such that \( U \leq BV \).

Next, let \( U \in \text{Irr}^n(B) \) be such that \( U \leq BV \). Fix \( u \in S^1(BU) \), and choose \( e \in \mathbb{I}(B) \) with \( eu = u \). By Lemma 2.13, \( u \in S^1(eV') \). Choose \( v \in S^1(eV') \), and therefore \( u \leq BV \). Fix some \( u \in S^1(BU) \), and choose \( e \in \mathbb{I}(B) \) with \( eu = u \). By Lemma 2.13, \( u \in S^1(eV') \). Choose \( v \in S^1(eV') \), and therefore \( u \leq BV \).

Moreover, for any irreducible \( U \in \mathbb{I}(B) \), there is \( \phi \in \mathbb{R}_{A/B}(\text{Spec}(A(V'))) \) implies \( \phi_{U,v} \in \{ \phi_{V',v'} \}_{v' \in V'} \). Choose a net \( \{ \phi_{V',v'} \}_{v' \in V'} \) converging to \( \phi \). Since \( \phi \in \mathbb{P}(A(V')) \), we get \( \phi = \phi' \). Now, by the Krein-Milman Theorem [13], there exists an extremal point \( \phi \in S \). We claim that \( \phi \in \mathbb{P}(A(V')) \). Indeed, if \( \psi = (1-t)\phi' + t\phi'' \) for \( \phi', \phi'' \in \mathbb{P}(A(V')) \) and \( 0 < t < 1 \), then \( \phi_{U,v} = \psi_{B,v} = t\phi'_{B,v} + (1-t)\phi''_{B,v} \). Since \( \phi_{U,v} \) is pure (Proposition 2.24), we must have \( \phi'_{B,v} = \phi''_{B,v} = \phi_{U,v} \), which means \( \psi' = \psi'' \in S \), and since \( \phi \) is extremal in \( S \), we get \( \phi = \phi' = \phi'' \).

If \( v \in BV \), then Corollary 2.11 implies that \( U = AU \leq V \) and we are done. We therefore need to show that \( v \in BV \).

Write \( v = v_1 + v_2 \) with \( v_1 \in BV \) and \( v_2 \in (BV)^{\perp} \). Then \( Bu_2 = 0 \), and hence \( \phi_{V',v_2} = \phi_{V',v_2} = \phi_{V',v_2} = \phi_{U,v} \). Thus, for all \( e \in \mathbb{I}(B) \), we have \( ||v_1|| = ||v_2|| = ||v_2||^2 = ||v_2||^2 \). Since \( 1 = ||v_2||^2 = ||v_1||^2 + ||v_2||^2 \), we get \( ||v_2|| = 0 \), and therefore \( v = v_1 \in BV \). \( \square \)

Remark 2.36. In general, it is not true that for all \( U \in \text{Irr}^n(B) \), there is \( V \in \text{Irr}^n(A) \) such that \( U \leq BV \). Indeed, take \( A = \mathbb{C}[X,X^*]^* \) and \( B = \mathbb{C}[X,X^*]^* \). Using Proposition 2.22 and Remark 2.23, we identify \( \hat{A} \) with \( \mathbb{C} \) (take \( a_1 = X \)), and \( \hat{B} \) with \( \mathbb{R} \) (take \( a_1 = X^* \)). The map \( \mathcal{R}_{A/B} \) turns out to induce a function \( \hat{A} \to \hat{B} \), which is not onto, meaning that there are \( U \in \text{Irr}^n(B) \) that are not weakly contained in \( BV \). Indeed, take \( A = \mathbb{C}[X,X^*]^* \) with \( X^* = X \). Then, \( A \) has no unitary representations at all (Corollary 2.7, Corollary 2.39), and \( B \) has many.

In contrast, when \( A \) and \( B \) are \( C^* \)-algebras, every irreducible unitary representation of \( B \) is contained in an irreducible unitary representation of \( A \) [20, Pr. 2.10.2].

Let \( \{a_1,\ldots,a_l\} \) be a family of elements in \( A \) such that \( a_1a_j = a_ja_1 \) and \( a_ia_j = a_ja_i \) for all \( i,j \). The following corollary implies that for all \( V' \in \text{Rep}^n(A) \), one can recover the common spectrum \( \text{Spec}(a_1,V',\ldots,a_l,V') \) from \( \text{Spec}(A(V')) \).

Corollary 2.37. Let \( V' \in \text{Rep}^n(A) \), and let \( a_1,\ldots,a_l \in A \) be elements generating a commutative \( * \)-subalgebra of \( A \). Choose \( e \in \mathbb{I}(A) \) such that \( ea_1e = a_i \) for all \( 1 \leq i \leq l \) (e.g., \( e = 1 \) if \( A \) is unital). Then

\[
\text{Spec}(\text{Rep}^n(A,V')) = \bigcup_{\{V\} \in \text{Spec}(A(V'))} \text{Spec}(a_1,V',\ldots,a_l,V').
\]

When \( V' \neq V \), we further have \( \text{Spec}(\text{Rep}^n(A,V')) = \text{Spec}(a_1,V',a_1,V') \cup \{0,\ldots,0\} \).
Proof. Let $B = \mathbb{C}[e,a_1,\ldots,a_t,a_1^*,\ldots,a_t^*]$. Then $\overline{BW} = BW = eW$ for any $W \in \text{Rep}^0(A)$. The first assertion now follows from Proposition 2.22 and Theorem 2.35. The second assertion holds since $B \cdot (eV')^\perp = 0$. Indeed, for all $v \in (eV')^\perp = (BV')^\perp$, we have $(Be, V') = \langle v, BV' \rangle = 0$, so $Be = 0$. □

21. Corner Subalgebras. Let $A$ be an idempotented involutory algebra. An idempotented $*$-subalgebra $B \subseteq A$ is called a corner subalgebra, or just a corner, if $B = BAB$. Equivalently, $B$ is a corner subalgebra of $A$ if $B = \bigcup_{e \in \mathbb{I}(B)} eAe$.

Example 2.38. (i) Let $e \in \mathbb{I}(A)$. Then $B = eAe$ is a corner subalgebra of $A$. When $A$ is unital, $A$ has a Pierce decomposition $A \cong \bigoplus_{(e \in \mathbb{I}(B))} eAe$ if $\mathbb{I}(B)$ is a corner of $B$.

Proof. Let $\mathbb{I}(B)$ be a corner subalgebra of $A$, let $V \in \text{Rep}^0(A)$, and let $v \in BV$. Choose $e \in \mathbb{I}(B)$ such that $v = ev$. Then $\varphi_{V,v}(a) = \varphi_{BV,v}(ea)$ for all $a \in A$. In particular, $\varphi_{BV,v}$ determines $\varphi_{V,v}$ and vice versa.

Proof. \[ \varphi_{V,v}(a) = \langle av, v \rangle = \langle ava, e \rangle = \langle eava, e \rangle = \varphi_{BV,v}(eaea). \]

Lemma 2.39. Let $V \in \text{Rep}^0(A)$ and let $W$ be a subspace of $V$. Then $\overline{AW} = \overline{AW}$.

Proof. Since $AW \subseteq AW$, we have $\overline{AW} \subseteq \overline{AW}$, and since $aW \subseteq AW$ for all $a \in A$, we have $\overline{AW} = \sum_{a \in A} aW \subseteq \overline{AW}$, and hence $\overline{AW} = \overline{AW}$.

Lemma 2.40. Let $B$ be a corner subalgebra of $A$, let $V \in \text{Rep}^0(A)$, and let $v \in BV$. Choose $e \in \mathbb{I}(B)$ such that $v = ev$. Then $\varphi_{V,v}(a) = \varphi_{BV,v}(ea)$ for all $a \in A$.

Proof. \[ \varphi_{V,v}(a) = \langle av, v \rangle = \langle ava, e \rangle = \langle eava, e \rangle = \varphi_{BV,v}(eaea). \]

Lemma 2.41. Let $B$ be a corner subalgebra of $A$, and let $V, V' \in \text{Rep}^0(A)$.

(i) If $V$ is irreducible and $BV \neq 0$, then $\overline{BV} \in \text{Irr}^0(B)$.

(ii) If $V', V''$ are irreducible and $BV', BV'' \neq 0$, then $V \cong V' \iff \overline{BV} \cong \overline{BV'}$.

(iii) If $V$ is irreducible and $BV \neq 0$, then $V \not\cong V' \iff \overline{BV} \not\cong \overline{BV'}$.

Proof. (i) Let $0 \neq v \in \overline{BV}$. By Lemma 2.19, $v \in \overline{BV}$. Since $V$ is irreducible, we have $\overline{AV} = V$ and hence $\overline{AV} = BABV = BABV \cong \overline{AV} = AV$.

(ii) We only show the nontrivial direction. Let $f : BV \to \overline{BV}$ be a unitary isomorphism, let $v \in \mathbb{S}(BV)$ and let $v' = fv \in BV'$. Then $\varphi_{BV,v} = \varphi_{BV',v'}$ and hence $\varphi_{V,v} = \varphi_{V',v'}$ (Lemma 2.40). By Corollary 2.11, $V \cong V'$.

(iii) Suppose $V \not\cong V'$. Let $v \in \mathbb{S}(BV)$. Choose $e \in \mathbb{I}(B)$ such that $ev = v$. By Lemma 2.15, for all $\varepsilon > 0$ and $F \subseteq B$, there is $v' \in \overline{V'}$ with $\|\varphi_{V,v} - \varphi_{V',v'}\| F < \varepsilon$, hence $\overline{BV} \not\cong \overline{BV'}$. Conversely, assume $\overline{BV} \not\cong \overline{BV'}$ and let $v \in \mathbb{S}(BV)$. Then there is a net $\{v'_\alpha\}_{\alpha \in I} \subseteq BV'$ such that $\lim_{\alpha \to \infty} \varphi_{BV,v'_\alpha} = \varphi_{BV,v}$ in $B^\ast$. By Lemma 2.40 this implies $\lim_{\alpha \to \infty} \varphi_{V,v'_\alpha} = \varphi_{V,v}$, so $V \not\cong V'$.

Theorem 2.42. Let $B$ be a corner subalgebra of $A$. Denote by $\hat{A}(B)$ the set of classes $[V] \in \hat{A}$ such that $BV \neq 0$. Then $\hat{A}(B)$ is open in $\hat{A}$, and the function $[V] \mapsto [BV] : \hat{A}(B) \to \hat{B}$ is a topological embedding.

Proof. Let $[W] \in \hat{A}(B)$ and choose $u \in \mathbb{S}(BW)$ and $e \in \mathbb{I}(B)$ with $eu = u$. We claim that $N_{W,u,0.5,0.5}(e)$ (see 2.19) is contained in $\hat{A}(B)$. Let $[V] \in N_{W,u,0.5,0.5}(e)$. Then there is $v \in V$ such that $|\langle v, v \rangle - \langle u, u \rangle| < 0.5$. Since $\langle u, u \rangle = 1$, this implies $|\langle v, v \rangle| > 0.5$. In particular, $ev \neq 0$, and hence $BV \neq 0$. This proves the first statement of the theorem.
To proceed, let $S$ denote the image of the map $[V] \mapsto [\tilde{B}] : \tilde{A}(B) \to \tilde{B}$. Fix some $[W] \in \tilde{A}(B)$, define $u$ and $e$ as above, and write $U = \tilde{B}W$. By Proposition 2.22(i) and the previous paragraph, the sets $\{N_{U,w,e,f}\}_{0 < c < 0.5, e \in F \subseteq A}$ form a basis of open neighborhoods of $[W]$ in $\tilde{A}(B)$, and the sets $\{N_{U,w,e,F}\}_{0 < c < 0.5, e \in F \subseteq B}$ form a basis of open neighborhoods of $[U]$ in $\tilde{B}$. Note that if $F \subseteq A$, then then $N_{U,w,e,F} = N_{W,u,e,F}$. Indeed, if $[V] \in N_{W,u,e,F}$, then there is $v \in V$ with $\|\varphi_{V,V} - \varphi_{W,u}\| < \varepsilon$. Since $\varphi_{V,V}(a) = \varphi_{V}(eae)$ and $\varphi_{W,u}(a) = \varphi_{W,u}(eae)$ (cf. Lemma 2.40), we get $\|\varphi_{V,V} - \varphi_{W,u}\| \leq \varepsilon$, i.e. $[V] \in N_{U,w,e,F}$. The other direction is similar. Likewise, $N_{U,w,e,F} = N_{U,w,e,F}$ when $F \subseteq B$. Therefore, it is enough to show that for all and $0 < \varepsilon < \frac{1}{2}$ and $e \in F \subseteq e$, the map $[V] \mapsto [\tilde{B}]$ maps $N_{U,w,e,F}$ bijectively onto $N_{U,w,e,F} \cap S$.

Suppose that $[V] \in N_{U,w,e,F}$. Then there is $v \in V$ with $\|\varphi_{V,v} - \varphi_{W,u}\| < \varepsilon$. Since $F \subseteq e\varepsilon$, we have $\varphi_{V,v} = \varphi_{V}(eae)$, and hence $\|\varphi_{V,V} - \varphi_{W,u}\| < \varepsilon$, meaning that $[\tilde{B}] \in N_{U,w,e,F}$. Reversing the argument shows that any $[V] \in \tilde{A}(B)$ with $[\tilde{B}] \in N_{U,w,e,F}$ lies in $N_{U,w,e,F}$. Finally, the map $[V] \mapsto [\tilde{B}]$ is injective by Lemma 2.41(ii).

**Remark 2.43.** It is not true in general that any unitary representation of a corner subalgebra $B$ is of the form $\tilde{B}V$ for $V \in \text{Ir}^n(A)$. For example, take $A$ as in Example 2.22 and let $B = e_0B_0$. Then $B$ is the free unital algebra generated by $Y := a_0, a_10$ (with $Y^* = Y$), and by Remark 2.23 there is a homeomorphism $[U] \mapsto Y|_U : \tilde{B} \to \mathbb{R}$. However, only $U$-s corresponding to $\mathbb{R}_{\geq 0}$ are of the form $\tilde{B}V$ for $V \in \text{Ir}^n(A)$. This follows directly by checking the list of irreducible representations given in Example 2.22. Also, since $Y = a_0, a_10 = a_1^*a_10$, we must have $Y|_{\tilde{B}V} \in \mathbb{R}_{\geq 0}$.

The following theorem is a strengthening of Theorem 2.35 in case $B$ is a corner.

**Theorem 2.44.** Let $B$ be a corner subalgebra of $A$, and let $V' \in \text{Rep}^n(A)$.

(i) $\text{Spec}_B([\tilde{B}]) = \{[\tilde{B}] \mid [V] \in \text{Spec}_A(V') \cap \tilde{A}(B)\}$.

(ii) $\text{Spec}_A(V') \cap \tilde{A}(B) = \{[V] \in \tilde{A} : [\tilde{B}] \in \text{Spec}_B([\tilde{B}])\}$.

**Proof.** (i) That $\text{Spec}_B([\tilde{B}]) \supseteq \{[\tilde{B}] \mid [V] \in \text{Spec}_A(V') \cap \tilde{A}(B)\}$ follows from parts (i) and (ii) of Lemma 2.41. Conversely, if $[U] \in \text{Spec}_B([\tilde{B}])$, then by Theorem 2.35 there exists $V \in \text{Ir}^n(A)$ such that $U \leq [\tilde{B}]$ and $V \in \text{Spec}_A(V')$. Since $\tilde{B}V$ is irreducible (Lemma 2.41(i)), we have $[U] \leq [\tilde{B}]$.

(ii) This follows from Lemma 2.41(ii). □

A family of corner subalgebras $\{B_a\}_{a \in I}$ of $A$ is called full if $\sum_a A = A$. In this case, it is easy to see that for every $A$-module $V$ with $AV \neq 0$, there is $\alpha \in I$ such that $B_aV \neq 0$.

**Example 2.45.** Assume $A$ is unital and $\{e_1, \ldots, e_t\}$ is a family of orthogonal idempotents in $[A]$ such that $\sum_i e_i = 1_A$. Then the family $\{e_iAe_i\}_{i=1}^{t}$ is full. Moreover, if $\{e_\alpha\}_{\alpha \in \Omega} \subseteq [A]$, then the family $\{e_\alpha Ae_\alpha\}_{\alpha \in \Omega}$ is full if and only if $\sum_{\alpha} A = A$. In this case, $\{e_\alpha\}_{\alpha \in \Omega}$ is called a full family of idempotents in $A$.

**Example 2.46.** Suppose there is an idempotent involutory algebra $A'$ such that $A = M_{n}(A')$ and the involution on $A$ is given by $((a_{ij})_{i,j})^* = (a_{ij}^*)_{i,j}$. Let $B'$ denote the $*$-subalgebra of $A$ consisting of matrices of the form

$$\begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & \ddots & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0
\end{pmatrix} \subseteq M_{n}(A') .$$

Then $B$ is a corner subalgebra of $A$ and the family $\{B\}$ is full, i.e. $A = ABA$. In fact, in this case, the map $V \mapsto [\tilde{B}] : \text{Rep}^n(A) \to \text{Rep}^n(B)$ is an equivalence of categories.
resembling a Morita equivalence (see [12], §17–18). A mutual inverse of the functor $V \mapsto BV$ can be constructed as follows: Identify $B$ with $A'$. For any $U \in \text{Rep}^n(B)$, view $U^n$ as column vectors and give it the obvious left $M_n(A')$-module structure.

Define an inner product on $U^n$ as column vectors and give it the obvious left $M_n(A')$-module structure. A morphism is continuous if its underlying $A$-module structure is admissible. Furthermore, it is enough to have this for just one $U \in \text{Rep}^n(B)$.

Let $U \mapsto U^n : \text{Rep}^n(B) \to \text{Rep}^n(A)$ be a mutual inverse of $V \mapsto BV$. The details are left to the reader.

**Corollary 2.47.** Let $\{B_\alpha\}_{\alpha \in I}$ be a full family of corner subalgebras, and let $V' \in \text{Rep}^n(A)$. Then

$$\text{Spec}_A(V') = \{[V] \in A' \mid [\overline{B_\alpha V}] \in \text{Spec}_{B_\alpha}(\overline{B_\alpha V}) \text{ for some } \alpha \in I\}.$$  

**Proof.** This follows from Theorem 2.44 and the fact that for all $V \in \text{Irr}^n(A)$, there is $\alpha \in I$ such that $B_\alpha V \neq 0$. □

Corollary 2.47 and Theorem 2.44 imply for any full family of corner subalgebras $\{B_\alpha\}_{\alpha \in I}$ and $V \in \text{Rep}^n(A)$, the set $\text{Spec}_A(V)$ determines the sets $\{\text{Spec}_{B_\alpha}(\overline{B_\alpha V})\}_{\alpha \in I}$ and vice versa. As a consequence $\{\text{Spec}_{B_\alpha}(\overline{B_\alpha V})\}_{\alpha \in I}$ determines $\text{Spec}_B(V)$ for any idempotented $*$-subalgebra $B'$ of $A$ (Theorem 2.35).

2J. Pre-Unitary Representations. Let $A$ be an idempotented $*$-algebra. Some of the representations that we will encounter later in the text will not be unitary but rather pre-unitary.

A pre-unitary representation of $A$ is a pre-Hilbert space $V$ endowed with a left $A$-module structure such that

(U1) $\langle au, v \rangle = \langle u, a^*v \rangle$ for all $a \in A$ and $u, v \in V$,

(U2) $V$ is smooth (i.e. $V = AV$),

(U3) for all $a \in A$, the operator $a|_V : V \to V$ is bounded.

We denote by $\text{Rep}^n_u(A)$ the category of pre-unitary representations of $A$. The morphisms are continuous $A$-homomorphisms. A pre-unitary representation is said to be irreducible if its underlying $A$-module is irreducible.

If $V$ is a pre-unitary representation, then the action of $A$ extends to the completion $\overline{V}$, which then becomes a unitary representation. Conversely, if $U$ is a unitary representation, then $U_{sm} = AU$ is a pre-unitary representation. We always have $\overline{U_{sm}} = U$, but in general $\overline{(U)_{sm}}$ may be larger than $V$. (However, we shall see in 2K that $\overline{(U)_{sm}} = V$ when $V$ is admissible.) The maps $V \mapsto \overline{V}$ and $U \mapsto U_{sm}$ are in fact functorial, taking continuous $A$-homomorphisms to continuous $A$-homomorphisms.

We extend the spectral theory of the previous sections to pre-unitary representations by defining

$$\text{Spec}_A(V') := \text{Spec}_A(\overline{V'}) \quad \forall \ V' \in \text{Rep}^n_u(A).$$

This means that $[V] \in \hat{A}$ lies in $\text{Spec}_A(V')$ if and only if $V \prec \overline{V'}$. By Lemma 2.16 this is equivalent to:

- For all $v \in S^1(V)$, $\varepsilon > 0$ and $F \subseteq_f A$, there is $v' \in S^1(V')$ such that $\|\varphi_{V,v} - \varphi_{V',v'}\|_F < \varepsilon$ for all $a \in F$.

Furthermore, it is enough to have this for just one $v \in S^1(V)$, and the assumption $v' \in S^1(V')$ can be dropped.

2K. Admissible Representations. We finish this chapter with recalling admissible representations.

Let $A$ be an idempotented $*$-algebra. A unitary (resp. pre-unitary) representation $V$ of $A$ is called admissible if $\dim eV < \infty$ for any $e \in I(A)$. 
Example 2.48. If $A$ is unital, then the admissible representations are precisely the finite-dimensional representations. For such representations, the terms unitary and pre-unitary coincide.

The following theorem summarizes the most important features of admissible representations.

**Theorem 2.49.** Assume $V' \in \text{Rep}^a(A)$ is admissible. Then:

(i) $V'$ is completely reducible, namely, there are irreducible unitary representations $\{V_i\}_{i \in I}$ such that $V \cong \bigoplus_{i \in I} V_i$.  

(ii) If $V \in \text{Irr}^a(A)$, then there are finitely many indices $i \in I$ (possibly none) for which $V \cong V_i$.  

(iii) For all $V \in \text{Irr}^a(A)$, we have $V \prec V' \iff V \leq V'$. In particular, $\text{Spec}_A(V') = \{[V_i] | i \in I\}$.

**Proof.** This is a well-known. We recall the proof for the sake of completeness.

(i) We first claim that if $V' \neq 0$, then $V'$ contains an irreducible submodule. There is $e \in \mathbb{I}(A)$ such that $eV' \neq 0$. Since $V'$ is admissible, $\dim eV' < \infty$. This easily implies that there is $U \in \text{Irr}^a(eAe)$ such that $U \leq eV'$. By Theorem 2.44(ii), there is $V \in \text{Irr}^a(A)$ such that $U = eV$. Let $u \in S^1(U)$. We view $u$ as a vector of both $V$ and $V'$. By Lemma 2.49, $\varphi_{V,u} = \varphi_{V',u}$ and hence $V \leq V'$. (Corollary 2.11).

Now, using Zorn’s Lemma, let $\{V_i\}_{i \in I}$ be a maximal collection of pairwise orthogonal irreducible subrepresentations of $V$. Let $W = \bigoplus_{i \in I} V_i$. Then $W$ is admissible, and hence, if it is nonzero, it admits an irreducible subrepresentation. Since this contradicts the maximality of $\{V_i\}_{i \in I}$, we must have $W = 0$, so $V = \bigoplus_{i \in I} V_i$.

(ii) Choose $e \in \mathbb{I}(A)$ such that $eV \neq 0$. Then $\dim \bigoplus eV_i = \dim eV' < \infty$, and hence $V_i \cong V$ is possible for only finitely many $i$-s.

(iii) Assume $V \prec V'$ and choose $e \in \mathbb{I}(A)$ such that $eV \neq 0$. Let $v \in S^1(eV)$. By Lemma 2.11, there is a net $\{v'_\alpha\}_{\alpha \in J} \subseteq S^1(eV')$ such that $\lim_{\alpha} \varphi_{V',v'_\alpha} = \varphi_{V,v}$. Since $\dim eV' < \infty$, the unit sphere $S^1(eV')$ is compact, and hence there is a subnet $\{v'_\alpha\}_{\alpha \in J'}$ converging to $v' \in S^1(V')$. This means $\lim \varphi_{V,v'_\alpha} = \varphi_{V',v'}$ and hence $\varphi_{V,v} = \varphi_{V',v'}$. By Corollary 2.11, $V \leq V'$.

Let $V' \in \text{Rep}^a(A)$ be admissible, let $[V] \in \hat{A}$, and write $V' = \bigoplus_{i \in I} V_i$ as in Theorem 2.49. The number of $i$-s with $V_i \cong V$ is called the multiplicity of $[V]$ in $V'$ and is denoted $\text{mult}_{V'}[V]$. It can be defined alternatively by

$$\text{mult}_{V'}[V] = \dim_{\mathbb{C}} \text{Hom}^a_{\mathbb{C}}(V,V')$$

(cf. Theorem 2.6), so it is independent of the decomposition $V' = \bigoplus_{i \in I} V_i$. This allows us to consider $\text{Spec}_A(V')$ as a multiset. For our purposes, a multiset consists of a set $X$ together with a map $\text{mult}_X : X \to \mathbb{N} \cup \{0\}$ assigning each $x \in X$ its multiplicity; morphisms of multisets are defined in the obvious way. When $V' \in \text{Rep}^a(A)$ is admissible, we let

$$m\text{-Spec}_A(V')$$

denote the multiset obtained from $\text{Spec}_A(V')$ by setting the multiplicity of $[V]$ to be $\text{mult}_{V'}[V]$.

**Theorem 2.50.** Let $\text{Rep}^{a,\text{ad}}(A)$ (resp. $\text{Rep}^{p,\text{ad}}(A)$) denote the category of admissible unitary (resp. pre-unitary) representations of $A$. Then $\text{Rep}^{a,\text{ad}}(A)$ and $\text{Rep}^{p,\text{ad}}(A)$ are equivalent; the equivalence is given by the functors $U \mapsto U_{\text{sm}}$.

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7 It is likely that the idea of “spectrum with multiplicity” can be extended beyond admissible representations by considering projection valued measures on $\hat{A}$, rather than multisets; developing the necessary theory is beyond the scope of this work.
We need to check that \( U \in \text{Rep}_{\text{ad}}^{n}(A) \) is irreducible if and only \( U_{\text{sm}} \) is irreducible (as an \( A \)-module).

**Proof.** We need to check that \( U \mapsto U_{\text{sm}} : \text{Rep}_{\text{ad}}^{n}(A) \to \text{Rep}_{\text{ad}}^{n}(A) \) and \( V \mapsto \bar{V} : \text{Rep}_{\text{ad}}^{n}(A) \to \text{Rep}_{\text{ad}}^{n}(A) \). In particular, \( U \in \text{Rep}_{\text{ad}}^{n}(A) \) is irreducible if and only \( U_{\text{sm}} \) is irreducible (as an \( A \)-module).

The only detail that is not straightforward is showing that \( (\bar{V})_{\text{sm}} = V \) for all \( V \in \text{Rep}_{\text{ad}}^{n}(A) \). We prove this by showing that if \( V^\perp \) is the orthogonal complement of \( V \) in \( V' := (\bar{V})_{\text{sm}} \), then \( V \oplus V^\perp = V' \). This implies \( \bar{V} = \bar{V} \oplus \bar{V}^\perp \), forcing \( V^\perp = 0 \) and \( V = V' = (\bar{V})_{\text{sm}} \).

Let \( v \in V' \). There is \( e \in I(A) \) such that \( ev = v \). Since \( V \) is admissible, \( \dim eV < \infty \), in which case it is easy to check that \( V' = eV \oplus (eV)^\perp \). Therefore, there are unique \( v_1 \in eV \) and \( v_2 \in (eV)^\perp \) with \( v = v_1 + v_2 \). Since \( v = ev = ev_1 + ev_2 \) and \( (ev_2, evV) = (v_2, eeV) = 0 \), we must have \( v_2 = ev_2 \). This means, \( \langle v_2, V \rangle = \langle ev_2, V \rangle = \langle v_2, evV \rangle = 0 \), so \( v_2 \in V^\perp \). Therefore, \( v = v_1 + v_2 \in V + V^\perp \), as required.

\( \square \)

### 3. Simplicial Complexes

This chapter recalls simplicial complexes. We also discuss topological groups acting on simplicial complexes.

#### 3A. Simplicial Complexes

Recall that a *simplicial complex* consists of a nonempty set \( X \) of finite sets such that subsets of sets in \( X \) are also in \( X \). A partially ordered set \( (Y, \leq) \) which is isomorphic to \( (X, \subseteq) \) for some simplicial complex \( X \) will also be called a simplicial complex. This holds when:

1. for all \( y \in Y \), the partially ordered set \( (\{y' \in Y : y' \leq y\}, \leq) \) is isomorphic to \( (P(\{1, \ldots, n\}), \subseteq) \) for some \( n \in \mathbb{N} \) and
2. for all \( y, y' \in Y \), the set \( \{y, y'\} \) has an infimum relative to \( \leq \).

The elements of a simplicial complex \( X \) are called cells. The complex \( X \) is *locally finite* if every cell in \( X \) is contained in finitely many cells. We let \( X^{(i)} \) denote the sets in \( X \) of cardinality \( i + 1 \). Elements of \( X^{(i)} \) are called \( i \)-dimensional cells, or just \( i \)-cells. The *vertex set* of \( X \) is defined to be \( X_{\text{vert}} := \bigcup_{x \in X} x \). By abuse of notation, we sometimes refer to elements of \( X^{(0)} \) as vertices. A morphism of simplicial complexes \( f : X \to Y \) consists of a function \( f : X_{\text{vert}} \to Y_{\text{vert}} \) such that for all \( i \) and \( x \in X^{(i)} \), we have \( f(x) := \{f(v) : v \in x\} \in Y^{(i)} \). The induced maps \( X^{(i)} \to Y^{(i)} \) and \( X \to Y \) are also denoted \( f \).

The dimension of \( X \), denoted \( \dim X \), is the maximal \( i \) such that \( X^{(i)} \neq \emptyset \). Cells of dimension \( 0 \) are points and \( 1 \)-cells are line segments. If \( x \) and \( x' \) are distinct cells in \( X \), then the *combinatorial distance* of \( x \) from \( x' \), denoted \( d(x, x') \), is the minimal \( t \in \mathbb{N} \) such that there exists a sequence of cells \( y_1, \ldots, y_t \) with \( x \leq y_1 \subseteq y_t \leq x' \) and \( y_i \cap y_{i+1} \neq \emptyset \) for all \( 1 \leq i < t \). We further set \( d(x, x) = 0 \). (This agrees with the combinatorial distance in graphs.) The *ball* of radius \( n \) around \( x \), \( B_X(x, n) \), consists of the cells in \( X \) of distance \( n \) or less from \( x \). When \( X \) is locally finite, all the balls \( B_X(x, n) \) \( (x \neq 0) \) are finite. We say that \( X \) is *connected* if \( X = \bigcup_{n\geq 0} B_X(x, n) \) for some (and hence all) \( x \in X - \{0\} \), or equivalently, if \( d(x, x') < \infty \) for all \( x, x' \in X \).

Every simplicial complex \( X \) can be regarded as a topological space by replacing its abstract simplices with topological simplices, glued in the obvious way. If \( X \) and \( Y \) are connected simplicial complexes, then a morphism \( f : X \to Y \) is a *cover map* if it is a cover map when \( X \) and \( Y \) are regarded as topological spaces. This is equivalent to saying that \( f : X_{\text{vert}} \to Y_{\text{vert}} \) is surjective and induces a bijection \( f : \{x \in X : v \in x\} \to \{y \in Y : f(v) \in y\} \) for all \( v \in X_{\text{vert}} \). In this case, the *deck transformations* of \( f : X \to Y \) are the automorphisms \( h \) of \( X \) satisfying \( f \circ h = f \).

We let \( \hat{X} \) denote \( X - \{0\} \).
Throughout, all simplicial complexes are assumed to be connected and locally finite.

3. Orientation. Let $X$ be a simplicial complex. An oriented cell in $X$ consists of a pair $(x, J)$ where $x \in X$ and $J$ is a full ordering of the vertices of $x$. Two orders on $x$ are equivalent if one can be obtained from the other by an even permutation. We denote by $[x, J]$ the equivalence class of $(x, J)$ and call it an oriented cell. Oriented cells will usually be denoted by the letters $x, y, \ldots$. Note that when $x \in X^{(i)}$ and $i > 0$, there are exactly two possible orientations on $x$. In this case, we write $[x, J]^\text{op}$ for $x$ endowed with the orientation different from the one induced by $J$.

When $i \leq 0$, there is only one possible orientation on $x$, and we set $[x, J]^\text{op} = [x, J]$ for convenience. In addition, for $\{v_0, \ldots, v_i\} \in X^{(i)}$, let

$$[v_0v_1 \ldots v_i] = \{v_0, \ldots, v_i\}, v_0 < v_1 < \cdots < v_i.$$ 

We let $X_{\text{ori}}$ (resp. $X_{\text{ori}}^{(i)}$) denote the set of oriented cells in $X$ (resp. $X^{(i)}$).

We now recall the construction of the $i$-dimensional Laplacian of $X$ (which will also be considered when $X$ is infinite): Recall that the space of $i$-dimensional forms (or just $i$-forms) of finite support on $X$ is defined by

$$\Omega_i^-(X) = \left\{ \varphi \in \ell^2(X_{\text{ori}}^{(i)}) : \varphi(x^\text{op}) = -\varphi(x) \text{ for all } x \in X_{\text{ori}}^{(i)} \right\} i > 0$$
$$\Omega_i^+(X) = \left\{ \varphi \in \ell^2(X_{\text{ori}}^{(i)}) : \varphi(x^\text{op}) = \varphi(x) \text{ for all } x \in X_{\text{ori}}^{(i)} \right\} i = 0$$

We make $\Omega_i^-(X)$ into a pre-Hilbert space by setting

$$\langle \varphi, \psi \rangle = 2 \cdot \sum_{x \in X^{(i)}} \varphi_x \cdot \overline{\psi_x} \quad \forall \varphi, \psi \in \Omega_i^-(X).$$

In the summation over $X^{(i)}$, we pick an arbitrary orientation for $x$. Since both $\varphi$ and $\psi$ reverse sign when the orientation is changed, the expression $\overline{\varphi_x} \cdot \psi_x$ is well-defined. When $i > 0$, the inner product agrees with the usual inner product on $\ell^2(X_{\text{ori}}^{(i)})$.

Recall that the boundary map $\partial_{i+1} : \Omega_{i+1}^-(X) \to \Omega_i^-(X)$ and the coboundary map $\delta_i : \Omega_i^-(X) \to \Omega_{i+1}^-(X)$ are defined by

$$(\partial_{i+1}\varphi)[v_0 \ldots v_i] = \sum_{\{v, v_0, \ldots, v_i\} \in X^{(i+1)}} \varphi[vv_0 \ldots v_i]$$

$$(\delta_i\varphi)[v_0 \ldots v_{i+1}] = \sum_{j=0}^{i+1} (-1)^j \varphi[v_0 \ldots v_j \ldots v_{i+1}]$$

($\hat{e}_j$ means omitting the $j$-th entry). It is easy to check that $\delta_i^* = \partial_{i+1}$. The upper, lower, and total $i$-dimensional Laplacians are defined by

$$\Delta^+_i = \partial_{i+1}\delta_i, \quad \Delta^-_i = \delta_{i-1}\partial_i, \quad \Delta_i = \Delta^+_i + \Delta^-_i$$

respectively (with the convention that $\Delta^-_0 = 0$). All three Laplacians take $\Omega_i^-(X)$ into itself.

Finally, for later usage, we also define

$$\Omega_i^\pm(X) = \ell^2(X_{\text{ori}}^{(i)}),$$

and

$$\Omega_i^+ = \left\{ \varphi \in \ell^2(X_{\text{ori}}^{(i)}) : \varphi(x^\text{op}) = \varphi(x) \text{ for all } x \in X_{\text{ori}}^{(i)} \right\} i > 0$$
$$\Omega_i^- = \left\{ \varphi \in \ell^2(X_{\text{ori}}^{(i)}) : \varphi(x^\text{op}) = -\varphi(x) \text{ for all } x \in X_{\text{ori}}^{(i)} \right\} i = 0$$

We give $\Omega_i^+(X)$ the inner product induced from $\Omega_i^\pm(X)$. The space $\Omega_i^+(X)$ is sometimes called the space of $i$-dimensional anti-forms of finite support on $X$. 
Notice that when \( i > 0 \), we have \( \Omega^\pm_i(X) = \Omega^+_i(X) \oplus \Omega^-_i(X) \) as pre-Hilbert spaces, whereas \( \Omega^+_0(X) = \Omega^-_0(X) \cong \Omega^-_0(X) \). In addition, for \( i > 0 \), the pre-Hilbert space \( \Omega^+_i(X) \) is naturally isomorphic to \( \tilde{F}(X^{(i)}) \); the isomorphism is given by linearly extending \( e_{[x,L]} + e_{[x,L]^*} \mapsto \sqrt{S}_x \) \((x \in X^{(i)})\). We will therefore often identify \( \Omega^+_i(X) \) with \( \tilde{F}(X^{(i)}) \), usually without comment.

3C. Groups Acting on Simplicial Complexes. Let \( G \) be an \( \ell \)-group, i.e. a totally disconnected locally compact Hausdorff topological group. By a \( G \)-complex, we mean a (locally finite, connected) simplicial complex \( X \) on which \( G \) acts faithfully via automorphisms and such that for all \( x \in X \), the stabilizer \( \text{Stab}_G(x) \) is a compact open subgroup of \( G \). The latter is equivalent to \( \text{Stab}_G(v) \) being a compact open subgroup of \( G \) for all \( v \in X_{\text{virt}} \).

Example 3.1. The complex \( B_d(F) \) of Chapter 8 is a \( G \)-complex for \( G = \text{PGL}_{d}(F) \).

Example 3.2. Let \( X \) be a \( k \)-regular tree and let \( G = \text{Aut}(X) \). We give \( X \) the discrete topology and \( G \) the topology of pointwise convergence. It is easy to check that \( G \) is an \( \ell \)-group and \( X \) is a \( G \)-complex.

Generalizing Example 3.2, we have:

Proposition 3.3. Let \( X \) be a simplicial complex. Give \( X \) the discrete topology and \( G := \text{Aut}(X) \) the topology of pointwise convergence. Then:

(i) \( G \) is an \( \ell \)-group and \( X \) is a \( G \)-complex. Consequently, \( X \) is an \( H \)-complex for any closed subgroup \( H \) of \( G \).

(ii) If \( X \) is an \( H \)-complex for an \( \ell \)-group \( H \), then the map \( H \to \text{Aut}(X) = G \) is a closed embedding.

Proof. (i) Let \( v \in X_{\text{virt}} \). The set \( \text{Stab}_G(v) \) is clearly open, so we need to show it is compact. Let \( B_n = B_\chi(v, n) \) and let \( K_n \) be the group of simplicial automorphisms of \( B_n \) that fix \( v \). Give \( K_n \) the topology of point-wise convergence. Since automorphisms preserve combinatorial distance, we have well-defined maps \( g \mapsto g|_{K_{n-1}} : K_n \to K_{n-1} \) \((n \in \mathbb{N})\), and since \( X \) is connected, we have \( \varprojlim K_n \cong \text{Stab}_G(v) \) as topological groups. By assumption, \( X \) is locally finite and hence the groups \( K_n \) are discrete and finite. It therefore follows from Tychonoff’s Theorem that \( \text{Stab}_G(v) \) is compact.

(ii) Observe first that the map \( H \to G \) is continuous. Indeed, the collection \( \{ \text{Stab}_G(v) \mid v \in X_{\text{virt}} \} \) is a subbasis for the open neighborhoods of \( 1_G \), and \( H \cap \text{Stab}_G(v) = \text{Stab}_H(v) \) is open for all \( v \subseteq X_{\text{virt}} \). Fix some \( x \in X \). Then \( \text{Stab}_H(x) \) is compact, hence the inclusion map \( \text{Stab}_H(x) \to \text{Stab}_G(x) \) is closed. This map is also continuous and injective, so it is a topological embedding. Since \( H \) and \( G \) are disjoint unions of cosets of the open subgroups \( \text{Stab}_H(x) \) and \( \text{Stab}_G(x) \) respectively, the map \( H \to G \) is a topological embedding. Finally, \( H \) is closed in \( G \) since the complement \( G - H \) is a union of translations of the open sets \( \text{Stab}_G(x) \) and \( \text{Stab}_H(x) \). \( \square \)

We say that a \( G \)-complex \( X \) is almost transitive or almost \( G \)-transitive when \( G \backslash X \) is finite. The \( G \)-complexes of Examples 3.1 and 3.2 are almost transitive.

Example 3.4. Let \( G \) be an almost simple algebraic group over a non-archimedean local field \( F \), let \( Z \) be the center of \( G \), and let \( G = G(F)/Z(F) \). Let \( B \) be the affine Bruhat-Tits building of \( G \) (see \( [80], [14] \); a more elementary treatment in the case \( G \) is classical can be found in \( [2] \)). The building \( B \) is a pure simplicial complex carrying

\[ \text{the Ramanujan property for simplicial complexes} \]
a faithful left $G$-action such that stabilizers of chambers are compact open and $G$ acts transitively on chambers. Therefore, $B$ is an almost transitive $G$-complex. We further note that the vertices of $B$ have a canonical coloring, which is preserved by $G$ when $G$ is simply-connected. For example, when $G = \text{PGL}_d$, this recovers the construction of $B_d(F)$ with the coloring $C_0$ in Chapter 1. When $G = \text{SL}_d$, we also have $B = B_d(F)$, but in this case $G = \text{im}(\text{SL}_d(F) \to \text{PGL}_d(F))$ and it preserves the coloring $C_0$ (the group $G$ may be smaller than the group of color-preserving automorphisms).

3D. Quotients of Simplicial Complexes. We now recall several facts about quotients of simplicial complexes. The results to follow seem to be known, but we could not find explicit proofs in the literature. We have therefore included the proofs. Throughout, $X$ is an $\ell$-group and $X$ is a $G$-complex. Recall that if $x \in X$ and $x = \{v_1, \ldots, v_\ell\}$ with $v_1, \ldots, v_\ell \in X_{\text{vert}}$, then $gx = \{gv_1, \ldots, gv_\ell\}$.

Let $\Gamma$ be a subgroup of $G$. The partial order on $X$ induces a partial order on $\Gamma \backslash X$ given by $\Gamma x \leq \Gamma y \iff \gamma x \leq y$ for some $\gamma \in \Gamma$. However, $\Gamma \backslash X$ is not a simplicial complex in general, and even when this is the case, the projection map $x \mapsto \Gamma x : X \to \Gamma \backslash X$ may not be a cover map. When both conditions hold, we call $\Gamma \backslash X$ a $G$-quotient of $X$ and write $\Gamma \leq_X G$.

More generally, a complex $X$ is a $G$-quotient of $X$ if there is a cover map $f : X \to X$ such that the group of deck transformations of $f$, call it $\Gamma$, is contained in $G$. In this case, there is a unique isomorphism $\Gamma \backslash X \to X$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\Gamma \backslash X & \xrightarrow{\Gamma \backslash f} & X \\
\end{array}
\]

commutes, so $\Gamma \leq_X G$. We shall see below (Proposition 3.9) that when $\Gamma \leq_X G$, the group of deck transformations of $X \to \Gamma \backslash X$ is $\Gamma$, hence both definitions agree.

Example 3.5. (i) Let $X$ be a $k$-regular tree and let $G = \text{Aut}(X)$. Then any $k$-regular graph (without multiple edges and loops) is a $G$-quotient of $X$.

(ii) Let $X$ be a cyclic graph on 4 vertices, let $Z/4Z$ act on $X$ by cyclic rotations, and let $G = 2Z/4Z$. Then $\Gamma \backslash X$ is not a simplicial complex. The problem is technical and lies in the fact that our definition of simplicial complexes does not allow multiple edges (or cells). Indeed, $\Gamma \backslash X$ consists of two vertices connected by two edges, so it is not a simplicial complex. However, when $G = \text{Aut}(X)$ (which is isomorphic to the dihedral group of order 8), and $X$ is regarded as a topological space, the map $X \to \Gamma \backslash X$ is not a cover map of topological spaces, and the problem of giving $\Gamma \backslash X$ a decent simplicial structure is inherent.

Example 3.6. Let $G = \text{PGL}_d(F)$ and $X = B_d(F)$ be as in Chapter 1 and let $N = \text{im}(\text{SL}_d(F) \to \text{PGL}_d(F))$. It is easy to check that $N^d(F) = N \backslash B_d(F)$ is a simplicial complex consisting of a single simplex of dimension $d - 1$. However, the map $B_d(F) \to N \backslash B_d(F)$ is not a cover map (e.g. by Proposition 3.9 below).

Proposition 3.7. Let $\Gamma \leq G$. Then $\Gamma \backslash X$ is a simplicial complex if and only if

1. $\{\Gamma u_1, \ldots, \Gamma u_\ell\} = \{\Gamma v_1, \ldots, \Gamma v_\ell\}$ implies $\Gamma \{u_1, \ldots, u_\ell\} = \Gamma \{v_1, \ldots, v_\ell\}$ for all $\{u_1, \ldots, u_\ell\}, \{v_1, \ldots, v_\ell\} \in X_{\text{vert}}$.

In this case, $\dim(\Gamma \backslash X) = \dim X$ and the map $x \mapsto \Gamma x : X \to \Gamma \backslash X$ is a morphism of simplicial complexes.

Proof. Assume (C1) holds. Define $Y$ to be the simplicial complex consisting of subsets $\{\Gamma v_1, \ldots, \Gamma v_\ell\} \subseteq \Gamma X_{\text{vert}}$ such that $\{v_1, \ldots, v_\ell\} \in X$. There is a morphism
of partially ordered sets \( \Phi : \Gamma \times X \to Y \) given by \( \Phi(\{v_1, \ldots, v_i\}) = \{v_1, \ldots, v_i\} \). It is clear that \( \Phi \) is onto, and \( \Phi \) is injective by (C1). Thus, \( \Phi \) is an isomorphism and \( \Gamma \times X \) is a simplicial complex.

Suppose now that \( \Gamma \times X \) is a simplicial complex. We shall use the following notion of height in partially ordered sets: Let \((Y, \leq)\) be a partially ordered set and let \( y \in Y \). The *height* of \( y \) in \( Y \), denoted \( h(y) = h_Y(y) \), is the maximal \( n \in \mathbb{N} \cup \{0, \infty\} \) such that there exists a sequence \( y_0 < y_1 < \cdots < y_n = y \) in \( Y \). Automorphisms of \((Y, \leq)\) are easily seen to preserve height. Using this, it is easy to see that for all \( x \in X \), we have \( h_X(x) = h_{\Gamma \times X}(\Gamma x) \). Simplicial complexes \( Y \) satisfy the Ramanujan property when this is satisfied.

Conversely, suppose \( \Gamma \times X \to Y \) is a simplicial complex. We shall use the following notion of height in partially ordered sets: Let \((Y, \leq)\) be a partially ordered set and let \( y \in Y \). The *height* of \( y \) in \( Y \), denoted \( h(y) = h_Y(y) \), is the maximal \( n \in \mathbb{N} \cup \{0, \infty\} \) such that there exists a sequence \( y_0 < y_1 < \cdots < y_n = y \) in \( Y \). Automorphisms of \((Y, \leq)\) are easily seen to preserve height. Using this, it is easy to see that for all \( x \in X \), we have \( h_X(x) = h_{\Gamma \times X}(\Gamma x) \). Simplicial complexes \( Y \) satisfy the Ramanujan property when this is satisfied.

Proof. We have \( h(\Gamma x) = h(x) \). Since \( \Gamma \times X \to Y \) is a cover map, the quotient map \( \Gamma \times X \to Y \) is a cover map if and only if \( \Gamma \times X \to Y \) is a cover map.

**Corollary 3.8.** Assume \( \Gamma \times X \) is a simplicial complex. Then for all \( \gamma, \gamma' \in \Gamma \) and \( v \in \mathcal{X}_{\text{virt}} \), \( \gamma v = \gamma' v \). In particular, if \( \gamma x = x \) for \( x \in X \), then \( \gamma \) fixes the vertices of \( x \).

Proof. We have \( \{\gamma v, \gamma' v\} = \{v\} \), so by (C1), \( \{\gamma v, \gamma' v\} = \{v\} \). Thus, there is \( \gamma' \in \Gamma \) such that \( \gamma'' \gamma v = \gamma'' \gamma' v = v \), which implies \( \gamma v = \gamma' v \). The last assertion follows by taking \( \gamma' = 1 \).

**Proposition 3.9.** Provided \( \Gamma \times X \) is a simplicial complex, the quotient map \( \mathcal{X} \to \Gamma \times \mathcal{X} \) is a cover map if and only if

\[(\text{C2}) \quad \Gamma \cap \text{Stab}_G(v) = \{1_G\} \quad \text{for all} \quad v \in \mathcal{X}_{\text{virt}}.
\]

In this case, \( \Gamma \) acts freely on \( \mathcal{X} \) (i.e. \( \text{Stab}_G(x) = \{1_G\} \quad \text{for all} \quad x \in \mathcal{X} \)), \( \Gamma \) is discrete in \( G \), and the group of deck transformations of \( \mathcal{X} \to \Gamma \times \mathcal{X} \) is \( \Gamma \).

Proof. Assume that (C2) is satisfied. We need to check that for all \( v \in X_{\text{virt}} \), the map \( x \mapsto \Gamma x \) induces a bijection \( \{x \in X : v \in x \} \to \{y \in \Gamma \times X : \Gamma y \leq y \} \). The surjectivity is clear, so it is left to show that whenever \( x, y \in X \) are such that \( x \leq y \) and \( \Gamma x \leq \Gamma y \), we have \( x = y \). Write \( x = \gamma y \) with \( \gamma \in \Gamma \). Then \( \gamma v \in x \), hence \( \{v, \gamma v\} \in \mathcal{X} \). By Corollary 3.8, \( \gamma \in \text{Stab}_G(v) \), so by (C2), \( \gamma = 1_G \) and \( x = y \).

Conversely, suppose \( \mathcal{X} \to \Gamma \times \mathcal{X} \) is a cover map and let \( v \in X_{\text{virt}} \) and \( \gamma \in \Gamma \cap \text{Stab}_G(v) \). Let \( u \in X_{\text{virt}} \) be a neighbor of \( v \), i.e. \( \{u, v\} \in X \) and \( u \neq v \). Then \( \gamma u \) is a neighbor of \( \gamma v \). Since \( \{\gamma u, \gamma v\} = \{\gamma v, \gamma' v\} \), condition (C1) of Proposition 3.7 implies that \( \Gamma \{u, v\} = \Gamma \{\gamma u, v\} \), and since \( x \mapsto \Gamma x \) is a cover map, this means \( \{u, v\} = \{\gamma u, v\} \), hence \( \gamma u = u \) (because \( u \neq v \)). We have therefore showed that \( \gamma \) fixes all the neighbors of \( v \). Proceeding by induction, we see that \( \gamma \) fixes all vertices connected to \( v \). Since \( \mathcal{X} \) is connected and \( G \) acts faithfully, this means \( \gamma = 1_G \), as required.

Suppose henceforth that \( \mathcal{X} \to \Gamma \times \mathcal{X} \) is a cover map. That \( \Gamma \) acts freely on \( \mathcal{X} \) follows from Corollary 3.8 and (C2), and the discreteness of \( \Gamma \) is immediate from (C2) (since \( \text{Stab}_G(v) \) is open in \( G \)). Let \( h \in \text{Aut}(X) \) be a deck transformation of \( x \mapsto \Gamma x \). Then \( \Gamma h x = \Gamma x \) for all \( x \in X \). This means that for all \( x \in X \), there is \( \gamma_x \in \Gamma \) such that \( h x = \gamma_x x \), or equivalently \( s_x := \gamma_x^{-1} h \in \text{Stab}_{\text{Aut}(X)}(x) \). Therefore, for all \( x \in X \), we can write \( h = \gamma_x s_x \) with \( \gamma_x \in \Gamma \) and \( s_x \in \text{Stab}_{\text{Aut}(X)}(x) \), and since \( \Gamma \cap \text{Stab}_{\text{Aut}(X)}(x) = \{1\} \) (because \( \Gamma \) acts freely on \( \mathcal{X} \)), this decomposition...
is unique. We claim that $\gamma_x = \gamma_v$ for all $x \in X$. Indeed, we have $\gamma_x s_x = \gamma_v s_v$ and hence $\{\gamma_x s_x^{-1}, v\} = \{s_x^{-1}, v\} \subseteq x \in X$, so $\gamma_x s_x^{-1} = v$ by Corollary 3.8 and by (C2), $\gamma_x = \gamma_v$. This means that $\gamma_x = \gamma_v$ whenever $x \cap y \neq \emptyset$.

Since $X$ is connected, we have $h = \gamma s$ with $\gamma \in \Gamma$ and $s$ stabilizing every cell in $X$, so $s = 1$ and $h \in \Gamma$.

The following corollary gives an elegant necessary and sufficient condition for having $\Gamma \leq_X G$.

**Corollary 3.10.** $\Gamma \leq_X G \iff d(v, \gamma v) > 2$ for all $1 \neq \gamma \in \Gamma$ and $v \in X_{\text{vert}}$.

**Proof.** Suppose $\Gamma \leq_X G$ and assume $d(v, \gamma v) \leq 2$ for some $v \in X_{\text{vert}}$ and $1 \neq \gamma \in \Gamma$. Then there exists $u \in X_{\text{vert}}$ such that $\{u, v\}, \{u, \gamma v\} \in \mathcal{X}$. By (C1), there is $\gamma' \in \Gamma$ such that $\{\gamma' u, \gamma' v\} = \{u, \gamma v\}$, and this implies $\{\gamma' v, \gamma v\}, \{\gamma' u, u\} \in \mathcal{X}$. By Corollary 3.8, $\gamma' u = u$ and $\gamma' v = \gamma v$, so by (C2) we get $\gamma' = \gamma = 1$.

To see the converse, we verify conditions (C1) and (C2) above. Condition (C2) is straightforward, so we only show (C1). Suppose $\{u_1, \ldots, u_t\}, \{v_1, \ldots, v_s\} \in \mathcal{X}$ satisfy $\{u_1, \ldots, u_t\} = \{\Gamma v_1, \ldots, \Gamma v_s\}$. We may assume that $t, s > 0$ and there is $\gamma \in \Gamma$ such that $\gamma u_1 = v_1$. Now, for all $i$, $d(\gamma u_i, v_1) = d(u_i, u_1) \leq 1$. On the other hand, there is $\gamma' \in \Gamma$ and $j$ such that $\gamma' u_j = v_j$. Thus, $d(\gamma' u_j, v_1) \leq 1$. It follows that $d(\gamma'^{-1} \gamma u_i, u_1) = d(\gamma'(u_i, \gamma u_1) \leq d(\gamma' u_j, v_1) = d(v_1, u_1) = 2$, so by assumption, $\gamma'^{-1} = 1$, meaning that $\gamma u_i = \gamma' u_j \in \{u_1, \ldots, v_s\}$. Therefore, $\gamma \{u_1, \ldots, u_t\} = \{v_1, \ldots, v_s\}$, and likewise, we have $\gamma'^{-1} \{v_1, \ldots, v_s\} \subseteq \{u_1, \ldots, u_t\}$. Thus, $\Gamma \{u_1, \ldots, u_t\} = \Gamma \{v_1, \ldots, v_s\}$.

**Corollary 3.11.** If $\Gamma \leq_X G$, then $\Gamma' \leq_X G$ for all $\Gamma' \leq \Gamma$, and $g^{-1} \Gamma g \leq_X G$ for all $g \in G$.

**Proof.** Use Corollary 3.10.

**Corollary 3.12.** Let $\Gamma$ be a discrete subgroup of $G$ such that $\Gamma \backslash X$ is finite. Then there is a finite subset $S \subseteq \Gamma - \{1\}$ such that any normal subgroup $\Gamma' \leq \Gamma$ with $\Gamma' \cap S = \emptyset$ satisfies $\Gamma' \leq_X G$.

**Proof.** Let $v_1, \ldots, v_s$ be representatives for the $\Gamma$-orbits in $X_{\text{vert}}$. For all $1 \leq i \leq n$, let $S_i = \{\gamma \in \Gamma : d(v_i, \gamma v_i) \leq 2\}$. We claim that $S_i$ is finite. Indeed, $B_X(v_i, 2)$ is finite, and hence $K_i := \{g \in G : d(v_i, g v_i) \leq 2\}$ is a finite union of right cosets of $\text{Stab}_G(v_i)$, so it is compact. Therefore, $S_i := \Gamma \cap K_i$ is discrete and compact, hence finite. Take $S = \bigcup_{i=1}^n S_i - \{1\}$ and suppose $\Gamma' \leq \Gamma$ and $\Gamma' \cap S = \emptyset$. We use Corollary 3.10 to show that $\Gamma' \leq_X G$. Let $v \in X_{\text{vert}}$ and $\gamma' \in \Gamma'$ be such that $d(v, \gamma' v) \leq 2$. There is $\gamma \in \Gamma$ and $1 \leq i \leq n$ such that $v = \gamma v_i$. Thus, $d(v_i, \gamma^{-1} \gamma' v) \geq d(\gamma v_i, \gamma' v) = d(v, v') \leq 2$. Since $\Gamma' \leq \Gamma$, this means $\gamma^{-1} \gamma' \gamma \in \Gamma' \cap S$, which is impossible unless $\gamma^{-1} \gamma' = 1$, so $\gamma' = 1$.

**Proposition 3.13.** Let $\Gamma \leq_X G$. Then $\Gamma \backslash X$ is finite if and only if $X$ is almost $G$-transitive and $\Gamma$ is cocompact in $G$ (i.e. $\Gamma \backslash G$ is compact). In this case, $G$ is unimodular.

**Proof.** Assume $\Gamma \backslash X$ is finite. Then $X$ is clearly almost $G$-transitive. Let $x_1, \ldots, x_n$ be representatives for the $\Gamma$-orbits in $X$ and let $K_i = \text{Stab}_G(x_i)$. Then each $K_i$ is compact and open in $G$, and $\Gamma \backslash X$ can be identified with $\bigsqcup_{i=1}^n \Gamma \backslash G/K_i$ via sending $\Gamma x_i$ to $\Gamma x_i$. Since $\Gamma \backslash X$ is finite, the set $\Gamma \backslash G/K_i$ is finite, so $\Gamma \backslash G$ is the (topological) disjoint union of finitely many epimorphic images of $K_i$, which is compact. Thus, $\Gamma \backslash G$ is compact.

For the converse, identify $\Gamma \backslash X$ with $\bigsqcup_{i=1}^n \Gamma \backslash G/K_i$ as above. Then $\Gamma \backslash G/K_i$ is a compact discrete topological space, hence finite.
Finally, any locally compact group admitting a discrete cocompact subgroup is unimodular by [61, Lm. 1] (for instance), and \( \Gamma \) is discrete by Proposition 3.8.

**Remark 3.14.** A \( G \)-complex \( X \) may not have finite \( G \)-quotients even when it is almost transitive. For example, take \( X = B_d(F) \) and \( G = \text{PGL}_d(O) \) as in Chapter I and let \( B \) be the image of the group of invertible \( d \times d \) upper-triangular matrices in \( G \). The Iwasawa decomposition \( G = BK \) with \( K = \text{PGL}_d(O) \) implies that \( B \) acts transitively on \( B_d(F)^{(0)} = G/K \) and hence almost transitively on \( B_d(F) \). However, \( B \) is not unimodular, so \( B_d(F) \) cannot have finite \( B \)-quotients by Proposition 3.13.

**Remark 3.15.** It is possible that \( \Gamma \backslash X \) can be understood as a simplicial complex with multiple faces even when \( (C1) \) or \( (C2) \) do not hold, e.g. consider Example 3.3(ii). More generally, \( \Gamma \backslash X \) should make sense as a simplicial complex with multiple cells when \( \Gamma \cap \text{Stab}_C(x) = 1 \) for all \( x \in X \). Indeed, this guarantees that no cell in \( X \) is glued to itself, and it can be shown that in this case \( X_{\text{top}} \to \Gamma \backslash X_{\text{top}} \) is a cover map of topological spaces, where \( X_{\text{top}} \) denotes the topological realization of \( X \). The condition \( \Gamma \cap \text{Stab}_C(x) = 1 \) holds in particular when \( \Gamma \) is a torsion-free discrete subgroup, since for any compact open \( K \leq G \), the group \( \Gamma \cap K \) is compact and discrete, hence finite. The torsion-free condition was used in [51], for instance.

Of course, this requires a notion of complexes with multiple faces which is ample enough. However, we do not know of such a notion. For example, regular cell complexes (in the sense [11 Df. A.22]) of are not general enough since they do not allow loops (consider the quotient of a cyclic graph on \( n \) vertices by the cyclic group of order \( n \)), and CW-complexes are too wild as there are very mild assumptions on the gluing maps. It seems likely that a decent definition exists, but this issue is out of the scope of this work. We also comment that with easy modifications, the theory in the following chapters extends to other types of cell complexes which are not simplicial.

Suppose that \( \Gamma \backslash X \) is a simplicial complex. We finish this chapter by showing that one can canonically identify \((\Gamma \backslash X^{(i)})\) with \((\Gamma \backslash (X^{(i)}_\text{ord}))\) with \((\Gamma \backslash (X^{(i)}_\text{ord}))_\text{ori}\), and \((\Gamma \backslash (X^{(i)}_\text{ord}))_\text{ori}\) with \((\Gamma \backslash (X^{(i)}_\text{ord}))_\text{ord}\), where \( X^{(i)}_\text{ord} \) denotes the ordered \( i \)-dimensional cells (see 3.15). The parenthesis in the previous expressions will be dropped henceforth.

**Proposition 3.16.** Let \( \Gamma \leq X \). The following maps are isomorphisms.

(i) \( \Gamma \{v_0, \ldots, v_i\} \to \Gamma \{v_0, \ldots, v_i\} : \Gamma \backslash (X^{(i)}_\text{ori}) \to (\Gamma \backslash X^{(i)}) \)

(ii) \( \Gamma \{v_0, \ldots, v_i\} \to \Gamma \{v_0, \ldots, v_i\} : \Gamma \backslash (X^{(i)}_\text{ord}) \to (\Gamma \backslash X^{(i)})_\text{ord} \)

(iii) \( \Gamma \{v_0, \ldots, v_i\} \to \Gamma \{v_0, \ldots, v_i\} : \Gamma \backslash (X^{(i)}_\text{ord}) \to (\Gamma \backslash (X^{(i)}_\text{ord}))_\text{ord} \)

**Proof.**

(i) This is immediate from condition (C1) and the definition of \( \Gamma \backslash X \).

(ii) It is clear that the map is well-defined and onto, so we only need to show injectivity. If \([\Gamma v_0, \ldots, \Gamma v_i] = [\Gamma u_0, \ldots, \Gamma u_i] \), then \( \{\Gamma v_0, \ldots, \Gamma v_i\} = \{\Gamma u_0, \ldots, \Gamma u_i\} \), so by Proposition 3.7, \( \Gamma \{v_0, \ldots, v_i\} = \Gamma \{u_0, \ldots, u_i\} \). This means that there is \( \gamma \in \Gamma \) and a permutation \( \sigma \in S_{0 \ldots i} \) such that \( \gamma v_r = u_{\sigma r} \) for all \( 0 \leq r \leq i \). In addition, since \([\Gamma v_0, \ldots, \Gamma v_i] = [\Gamma u_0, \ldots, \Gamma u_i] \), there is an even permutation \( \tau \in S_{0 \ldots i} \) and elements \( \gamma_0, \ldots, \gamma_i \in \Gamma \) such that \( \gamma r v_r = u_{\tau r} \) for all \( 0 \leq r \leq i \). Corollary 3.8 now implies that \( \gamma v_r = \gamma_r v_r \) for all \( r \), so \( \sigma = \tau \) and \( \gamma v_0 \ldots v_i = u_0 \ldots u_i \), as required.

(iii) This is similar to (ii). \( \square \)

4. Spectrum in Simplicial Complexes

When studying spectral properties of a simplicial complex, one needs to decide what operators associated with the complex are inspected. For example, in the case of graphs, one usually takes the vertex adjacency operator or the 0-dimensional...
Bultimately generalize the spectrum of graphs and quotients of operators. This will lead to the definition of high dimensional spectrum, which will primarily concern simplicial complexes, there are many relevant operators. However, in the case of graphs, the adjacency operator is defined for all locally finite graphs. In fact, the vertex and edge adjacency operators are defined for all locally finite graphs. The next example is a naive generalization of the vertex and edge adjacency operators to higher dimensions. Example 4.1. Let $X$ be a $k$-regular graph. The vertex adjacency operator of $X$ is the operator $a_0 : \Omega^+_0(X) \to \Omega^+_0(X)$ given by
\[(a_0 \varphi) u = \sum_v \varphi u \quad \forall \varphi \in \Omega^+_0(X), \, u \in X^{(0)},\]
where the sum is taken over all $v \in X^{(0)}$ connected by an edge to $u$. Likewise, the edge adjacency operator of $X$ is the operator $a_1 : \Omega^+_1(X) \to \Omega^+_1(X)$ given by
\[(a_1 \psi) x = \sum_y \psi y \quad \forall \psi \in \Omega^+_1(X), \, x \in X^{(1)},\]
where the sum is taken over all edges $y \in X^{(1)}$ sharing exactly one vertex with $x$. In fact, the vertex and edge adjacency operators are defined for all locally finite graphs.

The spectrum of $a_0$ affects many combinatorial properties of the graph $X$, particularly its Cheeger constant and chromatic number; see the survey [47] for further information. The spectrum of $a_1$ is known to be almost equivalent with the spectrum of $a_0$; we shall recover this later in Example 4.27.

The next example is a naive generalization of the vertex and edge adjacency operators to higher dimensions.

Example 4.2. Let $X$ be any simplicial complex and assume $i, j$ satisfy $0 \leq i < j \leq 2i + 1$. Define $a_{i,j} : \Omega^+_i(X) \to \Omega^+_j(X)$ by
\[(a_{i,j} \varphi) x = \sum_{y \in X^{(i)}} \varphi y \quad \forall \varphi \in \Omega^+_i(X), \, x \in X^{(i)}.\]
That is, the evaluation of $a_{i,j} \varphi$ at an $i$-cell $x$ sums the values of $\varphi$ on $i$-cells $y$ whose union with $x$ is a $j$-cell. In the notation of Example 4.1, we have $a_0 = a_{0,1}$ and $a_1 = a_{1,2}$. However, on the level of edges, we also have $a_{1,3}$. In general, $a_{i,i+1}, \ldots, a_{i,2i+1}$ do not commute.

Example 4.3. Let $X$ be any simplicial complex and let $i \geq 0$. The upper, lower and full Laplacians $\Delta^+_i, \Delta^-_i, \Delta_i$ (see 3B) take $\Omega^+_i(X)$ into itself. Recall from the introduction that $\text{Spec}(\Delta_i)$ affects combinatorial properties of $X$ (see [69], [28], [27] and related works). We also note that when $X$ is a $k$-regular graph, $\Delta_0 = k - a_0$, so $\Delta_0$ and $a_0$ are spectrally equivalent.

If a simplicial complex $X$ admits further structure, e.g. a coloring of the vertices or the edges, then one may also consider operators taking this data into an account. The complex $X = B_d(F)$ of Chapter 1 gives rise to such examples.
Example 4.4. Let $\mathcal{X} = \mathcal{B}_d(F)$ and $G = \operatorname{PGL}_d(F)$ be as in Chapter 1. Recall that with every $G$-quotient $X := \Gamma \backslash \mathcal{B}_d(F)$, we have associated $d - 1$ operators $a_1, \ldots, a_{d-1}: L^2(X^{(0)}) \to L^2(X^{(1)})$ which correspond to the $(d - 1)$-coloring $C_1$ of the directed edges. The spectrum of these operators affects various combinatorial properties of the quotient $X$; see [23], [25]. Notice that $\Omega_0^+ (X)$ is dense in $L^2(X^{(0)})$ (equality holds when $X$ is finite), so the common spectrum of $(a_1, \ldots, a_{d-1})$ on $L^2(X^{(0)})$ is the same as their spectrum on $\Omega_0^+ (X)$.

When $\Gamma$ is contained in $\ker(c: \operatorname{PGL}_d(F) \to \mathbb{Z}/d\mathbb{Z})$ (see Chapter 1), the $d$-coloring $C_0: \mathcal{B}_d(F)^{(0)} \to \mathbb{Z}/d\mathbb{Z}$ descends to $X$, giving rise to operators on $L^2(X^{(0)})$ or $\Omega_0^+ (X)$ which take $C_0$ into account, e.g. restricting a function on $X^{(0)}$ to vertices of a particular color. Such operators were used in [23].

4B. Associated Operators. We now introduce a notion of operators associated with a simplicial complex that includes the examples of [14]. Until we give the definition below, we shall informally address such operators as associated operators.

Let $\operatorname{Sim}$ denote the class of all finite-dimensional locally finite connected simplicial complexes. We first observe that an associated operator $a$ is in fact a family of operators $\{a_X\}_{X \in \mathbb{C}}$ ranging on some class of simplicial complexes $\mathbb{C} \subseteq \operatorname{Sim}$. This class can be all simplicial complexes (Examples 4.2 and 4.3), or a restricted family of quotients of some fixed universal cover (Examples 4.1 and 4.4).

The domain and range of an operator $a_X$ in the family $\{a_X\}_{X \in \mathbb{C}}$ is a pre-Hilbert space varying with $X$; denote it by $FX$. Then $F$ is an assignment from $\mathbb{C}$ to $\phi \mathbb{H}$, the class of pre-Hilbert spaces. In the examples of [14], the assignment $F$ was $\Omega_1^+$ or $\Omega_1^-$. To conclude, the datum of an associated operator consists of a family of linear operators $\{a_X: FX \to FX\}_{X \in \mathbb{C}}$ where $\mathbb{C}$ is a subclass of $\operatorname{Sim}$ and $F: \operatorname{Sim} \to \phi \mathbb{H}$ is an assignment. Of course, this is still too general, and some conditions should be imposed.

We proceed by observing that $\operatorname{Sim}$ and $\phi \mathbb{H}$ can be made into categories; the morphisms of $\operatorname{Sim}$ are morphisms of simplicial complexes [3A] and the morphisms of $\mathbb{H}$ are $C$-linear maps (not-necessarily continuous). Actually, it is usually more convenient to consider the subcategory of $\operatorname{Sim}$ whose objects are those of $\operatorname{Sim}$ and whose morphisms are cover maps. We denote it by

\[ \mathbb{Cov}. \]

Now, the assignments $X \mapsto \Omega_1^+ (X)$, $X \mapsto \Omega_1^- (X)$ and $X \mapsto \Omega_0^+ (X)$ [3B] can be extended to functors $\operatorname{Sim} \to \phi \mathbb{H}$: If $f: X \to Y$ is a simplicial map, then we define $f_*: \Omega_1^+ (X) \to \Omega_1^+ (Y)$ by

\[ (f_* \varphi)y = \sum_{x \in f^{-1}(y)} \varphi(x) \quad \forall \varphi \in \Omega_1^+ (X), \ y \in Y^{(i)}. \]

This makes $X \mapsto \Omega_1^+ (X)$ into a covariant functor. Abusing the notation, we also define $f_*: \Omega_1^- (X) \to \Omega_1^- (Y)$ and $f_*: \Omega_0^+ (X) \to \Omega_0^+ (Y)$ in the same way. This agrees with the notation section. Notice that $f_*$ is continuous if and only if the size of the fibers of $f: X^{(i)} \to Y^{(i)}$ is uniformly bounded, so in order to include infinite simplicial complexes, we must allow non-continuous morphisms in $\phi \mathbb{H}$.

View $\Omega_1^+$ and $\Omega_1^-$ as functors from $\mathbb{Cov}$ to $\phi \mathbb{H}$. It straightforward to check that $a_{ij} = \{a_{ij,X}\}_{X \in \operatorname{Sim}}$ of Example 4.4 is a natural transformation from $\Omega_1^+$ to itself, and $\Delta_i = \{\Delta_i,X\}_{X \in \operatorname{Sim}}$ of Example 4.3 is a natural transformation from $\Omega_1^-$ to itself. That is, for any cover map $f: X \to Y$ in $\mathbb{Cov}$, the following diagrams

\[ \text{commute.} \]
commute:
\[
\begin{array}{ccc}
\Omega^+_1(X) & \xrightarrow{a_{1,X}} & \Omega^+_1(X) \\
\downarrow f_* & & \downarrow f_* \\
\Omega^+_1(Y) & \xrightarrow{a_{1,Y}} & \Omega^+_1(Y)
\end{array}
\quad
\begin{array}{ccc}
\Omega^-_1(X) & \xrightarrow{a_{1,X}} & \Omega^-_1(X) \\
\downarrow f_* & & \downarrow f_* \\
\Omega^-_1(Y) & \xrightarrow{a_{1,Y}} & \Omega^-_1(Y)
\end{array}
\]

This suggests that a general associated operator \( \{a_X : FX \to FX\}_{X \in \mathcal{C}} \) should be a natural transformation, in which case \( \mathcal{C} \) has to be a category and \( F \) has to be a functor. For reasons to become clear later, we also require \( a \) to have a dual, that is, a natural transformation \( \{a_X^\dagger\}_{X \in \mathcal{C}} : F \to F \) such that \( a_X^\dagger \) is a dual of \( a_X \). (Notice that \( a_X \) may not have a dual since \( FX \) is not a Hilbert space. Also, even when \( a_X^\dagger \) exists for all \( X \), it can happen that \( \{a_X^\dagger\}_{X \in \mathcal{C}} \) is not a natural transformation from \( F \) to itself.)

We finally conclude with the definition:

**Definition 4.5.** Let \( \mathcal{C} \) be a subcategory of \( \mathbf{Sim} \) and \( F : \mathcal{C} \to \mathcal{PHil} \) a covariant functor. An associated operator of \((\mathcal{C}, F)\), or just a \((\mathcal{C}, F)\)-operator, is a natural transformation \( a : F \to F \) that admits a dual.

Interesting examples usually arise when \( \mathcal{C} \) is a subcategory of \( \mathbf{Cov} \). Of particular interest is the category of quotients of a \( G \)-complex \( X \), which we now define. It will be our main example, and will play a major role in the sequel.

**Definition 4.6.** Let \( X \) be a \( G \)-complex \((\mathcal{C}, G)\). We define the subcategory
\[
\mathcal{C} = \mathcal{C}(G, X) \subseteq \mathbf{Cov}
\]
as follows: The objects of \( \mathcal{C} \) are \( \{\Gamma \setminus X | \Gamma \leq X, G\} \) (see [41]), where \( \Gamma \setminus X \) is identified with \( X \). The morphisms of \( \mathcal{C} \) are given as follows:

- For all \( \Gamma \leq X, G \), set \( \text{Hom}_\mathcal{C}(X, \Gamma \setminus X) = \{p \circ f | f \in G\} \), where \( p \) is the quotient map \( x \mapsto \Gamma x : X \to \Gamma \setminus X \).
- For all \( 1 \neq \Gamma \leq X, G \), set \( \text{Hom}_\mathcal{C}(\Gamma \setminus X, \Gamma \setminus X) = \{p \circ f | f \in G\} \) where \( p \) is the quotient map \( \Gamma' x \mapsto \Gamma x : \Gamma \setminus X \to \Gamma \setminus X \).
- All other Hom-sets are empty.

In particular, \( \text{End}_\mathcal{C}(X) = G \).

**Example 4.7.** In Example [44], take \( \mathcal{C} = \mathcal{C}(G, X) \) where \( X \) is a \( k \)-regular tree and \( G = \text{Aut}(X) \). The objects of \( \mathcal{C} \) are \( k \)-regular graphs, and the operator \( a_0 = \{a_{0,X}\}_{X \in \mathcal{C}} \) (resp. \( a_1 = \{a_{1,X}\}_{X \in \mathcal{C}} \)) is associated with \( (\mathcal{C}, \Omega_0^+) \) (resp. \( (\mathcal{C}, \Omega_1^+) \)).

In Example [45] take \( \mathcal{C} = \mathcal{C}(\text{PGL}_k(F), \text{B}_k(F)) \) (see Chapter 1). The operators \( a_1, \ldots, a_{d-1} \) are associated with \( (\mathcal{C}, \Omega_0^+), (\mathcal{C}, \Omega_1^+) \).

We finish with introducing another functor from \( \mathbf{Sim} \) to \( \mathcal{PHil} \) that may be considered in applications: For \( X \in \mathbf{Sim} \), let \( \text{Flag}(X) \) denote the set of maximal flags in \( X \), namely, the set of maximal chains of cells in \( X \). If \( X \) is pure of dimension \( d \), then \( \text{Flag}(X) \) is canonically isomorphic to the set of pairs \( (x, \leq) \) such that \( x \in X^{(d)} \) and \( \leq \) is a full ordering of the vertices of \( x \). Let
\[
\Omega_{\text{Flag}}(X) = \tilde{\ell}^2(\text{Flag}(X)) .
\]
We make \( \Omega_{\text{Flag}} \) into a (covariant) functor from \( \mathbf{Sim} \) to \( \mathcal{PHil} \) in the same way we made \( \Omega_+^i, \Omega_-^i, \Omega^\pm_0 \) into functors.
4C. **Spectrum.** Let $\mathcal{C}$ be a subcategory of $\mathbf{Sim}$ and let $F : \mathcal{C} \to \mathbf{pHil}$ be a functor. Denote by $A(\mathcal{C}, F)$ be the collection of all $(\mathcal{C}, F)$-operators. We make $A(\mathcal{C}, F)$ into a unital “*-algebra” (this is a priori not a set) by defining

$$
\{a_X\} + \{a'_X\} = \{a_X + a'_X\},
\{a_X\} : \{a'_X\} = \{a_X \circ a'_X\},
\{a_X\}^* = \{a_X^*\},
\alpha \{a_X\} = \{\alpha a_X\}
$$

for all $\{a_X\}, \{a'_X\} \in A(\mathcal{C}, F)$ and $\alpha \in \mathbb{C}$. The collection $A(\mathcal{C}, F)$ is a set when $\mathcal{C}$ is skeletally small, which we tacitly assume throughout. An idempotent *-subalgebra of $A(\mathcal{C}, F)$ is called an algebra of $(\mathcal{C}, F)$-operators.

Let $A$ be an algebra of $(\mathcal{C}, F)$-operators. We can regard $FX$ as a left $A$-module by setting $a \cdot v = a_X v$ for all $v \in FX$ and $a = \{a_X\} \in A$. We clearly have

$$
\langle au, v \rangle = \langle u, a^* v \rangle \quad \forall u, v \in FX, a \in A(\mathcal{C}, F).
$$

However, a priori, $a|_F X$ is not continuous. When this holds for all $a \in A$, we say that $A$ acts continuously on $FX$. (For example, this is the case when $FX$ is finite dimension.)

To simplify the discussion, assume henceforth that

1. every $(\mathcal{C}, F)$-operator acts continuously on $FX$ for all $X \in \mathcal{C}$, and
2. $FX$ is finite-dimensional when $X$ is finite.

In this case, for all $X \in \mathcal{C}$, the smooth left $A$-module $AFX = A \cdot FX$ can be regarded as a pre-unitary representation of $A$ \([2J]\), and this representation is admissible \([2K]\) when $X$ is finite. (If the identity transformation $id : F \to F$ is in $A$, then $AFX = FX$.) We may therefore apply the spectral theory developed in Chapter \([2]\) to $AFX$ and $A$. In particular, we define the $A$-spectrum of $X$ to be

$$
\text{Spec}_A(X) := \text{Spec}_A(AFX),
$$

and when $X$ is finite, we also define the $A$-spectrum multiset

$$
\text{m-Spec}_A(X) := \text{m-Spec}_A(AFX).
$$

When $A = A(\mathcal{C}, F)$, we will also address these (multi-)sets as the $(\mathcal{C}, F)$-spectrum, or just $F$-spectrum when $\mathcal{C}$ is clear from the context. We then write $\text{Spec}_{\mathcal{C}, F}$ or $\text{Spec}_F$ instead of $\text{Spec}_{A(\mathcal{C}, F)}$. Recall from \([2I]\), \([2G]\) and \([2K]\) that $\text{Spec}_A(X)$ is a closed subset of the unitary dual of $A$, denoted $\widehat{A}$, and, when defined, $\text{m-Spec}_A(X)$ a multiset of elements in $\widehat{A}$.

We stress that the spectrum of any $a \in A$ on $AFX$ can be recovered from $\text{Spec}_A(X)$ as follows from Corollary \([2.57]\). We further note that if $B$ is an algebra of $(\mathcal{C}, F)$-operators contained in $A$, then $\text{Spec}_B(X)$ can be recovered from $\text{Spec}_A(X)$ (Theorem \([2.55]\)). In particular, we can recover $\text{Spec}_A(X)$ from $\text{Spec}_{\mathcal{C}, F}(X)$. Similar claims hold for the spectrum multiset when $X$ is finite.

When condition (1) (resp. (2)) above does not hold, one case still define $\text{Spec}_A(X)$ (resp. $\text{m-Spec}_A(X)$) when $A$ acts continuously on $FX$ (resp. $AFX$ is an admissible $A$-module).

**Example 4.8.** Let $\mathcal{X}$ be a $k$-regular tree, let $G = \text{Aut}(\mathcal{X})$, $\mathcal{C} = \mathcal{C}(\mathcal{X}, G)$ (Definition \([4.6]\)), and write $A_i = A(\mathcal{C}, \Omega_i^+)$ for $i = 0, 1$. We shall see in Example \([4.20]\) below that

$$
A_0 = \mathbb{C}[a_0] \quad \text{and} \quad A_1 = \mathbb{C}[a_1]
$$

where $a_0$ and $a_1$ are the vertex and edge adjacency operators of Example \([4.1]\). By Proposition \([2.22]\) we can identify $\widetilde{A}_0$ with a subset of $\mathbb{C}$ (this set is $\mathbb{R}$, in fact) such
that for all \( V \in \text{Rep}^b(A_0) \), the set \( \text{Spec}_{A_0}(V) \) corresponds to \( \text{Spec}(a_0|_V) \). Therefore, for all \( X \in \mathcal{C} \), the datum of \( \text{Spec}_{A_0}(X) \) is equivalent to the spectrum of the vertex adjacency operator of \( X \), so the \( A_0 \)-spectrum is essentially the same as the usual spectrum of \( k \)-regular graphs. Likewise, \( \text{Spec}_{A_0}(X) \) is equivalent to the spectrum of the edge adjacency operator of \( X \).

**Example 4.9.** Let \( \mathcal{X} = B_d(F) \) and \( G = \text{PGL}_d(F) \) be as in Chapter \( \ref{chp:functors} \) and let \( \mathcal{C} = \mathcal{C}(B_d(F), G) \). We shall see below (Example\( \ref{ex:functors} \)) that \( A_0 : \mathcal{C}(\mathcal{C}, \Omega^+_i) \) is in fact the free commutative unital algebra generated by the natural transformations \( a_1, \ldots, a_{d-1} \) (cf. Chapter \( \ref{chp:functors} \) Example \( \ref{ex:spectra} \)). By Proposition \( \ref{prop:functors} \), we see that \( A_0 \) can be embedded in \( \mathbb{C}^d \) in such a way that \( \text{Spec}_{A_0}(\Gamma \backslash B_d(F)) \) corresponds to the common spectrum of \( (a_1, \ldots, a_{d-1}) \) for all \( \Gamma \leq B_d(F) \) \( G \). Thus, \( \text{Spec}_{A_0}(\Gamma \backslash B_d(F)) \) is equivalent to \( \text{Spec}_{0}(\Gamma \backslash B_d(F)) \) in the sense of Chapter \( \ref{chp:functors} \).

**Remark 4.10.** Let \( F, F' : \mathcal{C} \rightarrow \text{pHil} \) be functors. Then \( \mathcal{A}(\mathcal{C}, F) \) embeds as a \( * \)-subalgebra of \( \mathcal{A}(\mathcal{C}, F \oplus F') \) via \( a \mapsto a \oplus 0 := \{ a_X \oplus 0_{F'X} \} \}_{X \in \mathcal{C}} \). As a result, we can view any algebra \( A \) of \( (\mathcal{C}, F) \)-operators as an algebra of \( (\mathcal{C}, F \oplus F') \)-operators. This transition does not affect the definition of \( \text{Spec}_A(X) \) (when they exist), since \( A(FX \oplus F'X) = AFX \oplus 0 \).

Let \( \mathcal{C} \subseteq \text{Sim} \) and \( i \geq 0 \). We denote the spectrum taken with respect to \( \mathcal{A}(\mathcal{C}, \Omega^+_i), \mathcal{A}(\mathcal{C}, \Omega^-_i), \mathcal{A}(\mathcal{C}, \Omega^\pm_i) \) and \( \mathcal{A}(\mathcal{C}, \Omega^\text{Flag}_i) \) by

\[
\text{Spec}_{\mathcal{C}, \mathcal{C}}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0}, \text{ Spec}_{\mathcal{C}, \mathcal{C}^0},
\]

respectively. We will drop \( \mathcal{C} \) when it is obvious from the context. We call \( \mathcal{C}_i \) (resp. \( \mathcal{C}_{-i} \), \( \mathcal{C}_{-i} \)) the non-oriented (resp. oriented, full) \( i \)-dimensional spectrum, and we call \( \text{Spec}_{\text{Flag}} \) the flag spectrum. Notice that \( \text{Spec}_{\mathcal{C}^0} \) and \( \text{Spec}_{\mathcal{C}^0} \) are in fact identical (since \( \Omega^+_0 = \Omega^-_0 = \Omega^0 \)), so we simply write \( \text{Spec}_{\mathcal{C}} \).

**4D. Elementary Functors.** Let \( G \) be an \( \ell \)-group, let \( \mathcal{X} \) be an almost transitive \( G \)-complex \( \mathcal{X} \), and let \( \mathcal{C} = \mathcal{C}(G, \mathcal{X}) \) (Definition \( \ref{def:functors} \)). In this section we study in detail the algebra of all \( (\mathcal{C}, F) \)-operators when \( F : \mathcal{C} \rightarrow \text{pHil} \) satisfies certain assumptions satisfied by \( \Omega^+_i, \Omega^-_i, \Omega^\pm_i \) and \( \Omega^\text{Flag}_i \).

**Definition 4.11.** A functor \( F : \mathcal{C}(G, \mathcal{X}) \rightarrow \text{pHil} \) is elementary if there exists a covariant functor \( S : \mathcal{C}(G, \mathcal{X}) \rightarrow \text{Set} \) such that

- (E1) There is a unitary natural isomorphism \( \overline{F} \circ S \cong F \).
- (E2) For all \( x \in SX \), the group \( \text{Stab}_G(x) \) is compact open in \( G \) (the action of \( G \) on \( SX \) is via \( S \)). Furthermore, \( \text{Stab}_G(x) \) is contained in the stabilizer of a nonempty cell in \( \mathcal{X} \).
- (E3) For all \( \Gamma \leq H G \), the map \( \Gamma \backslash SX \rightarrow S(\Gamma \backslash \mathcal{X}) \) given \( \Gamma x \mapsto (\text{Spr}_x) x \) (notation as in Definition \( \ref{def:flag} \)) is an isomorphism.
- (E4) \( G \backslash SX \) is finite.

The functor \( F \) is called semi-elementary if there is another functor \( F' : \mathcal{C}(\mathcal{X}, G) \rightarrow \text{pHil} \) such that \( F \oplus F' \) is elementary.

**Example 4.12.** The functors \( \Omega^\pm_i, \Omega^+_i \) and \( \Omega^\text{Flag}_i \) are elementary. Indeed, take \( S \) to be \( X \mapsto X^{(i)} \), \( X \mapsto X^{(i)} \) and \( X \mapsto \text{Flag}(X) \), respectively; condition (E3) follows from Proposition \( \ref{prop:flag} \). The functor \( \Omega^0_i \) is also elementary.

For \( i > 0 \), the functor \( \Omega^-_i \) is semi-elementary since \( \Omega^\pm_i = \Omega^+_i \oplus \Omega^-_i \). More generally, any functor \( F : \mathcal{C}(G, \mathcal{X}) \rightarrow \text{pHil} \) which is an orthogonal summand of a finite direct sum of functors of the form \( \Omega^+_i, \Omega^-_i, \Omega^\text{Flag}_i \) is semi-elementary.

**Remark 4.13.** The second part of condition (E2) will not be used until \( \ref{chp:functors} \). Furthermore, if the topological realization of \( \mathcal{X} \) is a complete metric CAT(0)-space on
which $G$ acts by isometries, then any compact subgroup of $G$ is contained in the stabilizer of a cell in $X$. This follows from the Bruhat–Tits Fixed Point Theorem (Th. 11.23); notice that orbits of compact groups are compact and hence bounded). When $X$ is an affine building, e.g. when $X = B_d(F)$ as in Chapter [3] the complex $X$ can be realized as a complete metric CAT(0)-space [1, Th. 11.16(b)].

For the next proposition, recall that if $\Gamma$ is a group and $V$ is a complex representation of $\Gamma$, then the space of coinvariants $V_\Gamma$ is $V/V(\Gamma)$, where $V(\Gamma) := \text{span}\{v - \gamma v | v \in V, \gamma \in \Gamma\}$. Also notice that if $F : \mathcal{C}(G, X) \to \text{phil}$, then $G$ acts on $FX$ via $F$. Recall from the notation section that for a set $X$, we let $\{e_x\}_{x \in X}$ denote the standard basis of $\ell^2(X)$.

**Proposition 4.14.** (i) Let $S : \mathcal{C}(G, X) \to \text{Set}$ be a functor satisfying condition (E3) of Definition [4.11]. Define $S_1 : \text{Sim} \to \text{Set}$ by

$$S_1(\Gamma \backslash X) = \Gamma \backslash SX,$$

$$S_1(p_\Gamma \circ g) = [x \mapsto \Gamma g x : SX \to \Gamma \backslash SX],$$

$$S_1(p_\Gamma \circ r) = [\Gamma' x \mapsto [x \mapsto \Gamma' \backslash SX \to \Gamma' \backslash SX]$$

for all $\Gamma' \leq \Gamma \leq H, g \in G$ (notation as in Definition [4.6]). Then $S \cong S_1$.

(ii) Let $F : \mathcal{C}(G, X) \to \text{phil}$ be a semi-elementary functor viewed as a functor into $\text{vec}$, the category of vector spaces over $\mathbb{C}$. Write $V = FX$ and define $F_1 : \mathcal{C}(G, X) \to \text{vec}$ by

$$F_1(\Gamma \backslash X) = V_\Gamma,$$

$$F_1(p_\Gamma \circ g) = [v \mapsto gv + V(\Gamma) : V \to V_\Gamma],$$

$$F_1(p_\Gamma \circ r) = [v + V(\Gamma') \mapsto v + V(\Gamma) : V_\Gamma \to V]$$

for all $\Gamma' \leq \Gamma \leq H, g \in G$ (notation as in Definition [4.6]). Then $F \cong F_1$.

Proof. (i) For all $\Gamma \leq H, G$, let $u_{\Gamma \backslash X} : S_1(\Gamma \backslash X) = \Gamma \backslash SX \to S(\Gamma \backslash X)$ be the map $\Gamma x \mapsto (S_{p_\Gamma})x$ (this is well-defined since $p_\Gamma \circ \gamma = p_\Gamma$ for all $\gamma \in \Gamma$). Then $u_{\Gamma \backslash X}$ is an isomorphism by (E3), and it is routine to check that $u = \{u_X\}_{X \in \mathcal{C}} : S_1 \to S$ is a natural transformation.

(ii) For every $X = \Gamma \backslash X \in \mathcal{C}(G, X)$, define $u_{F, X} : F_1X = (FX)_\Gamma \to FX$ by $u_{F, X}(\varphi + (FX)(\Gamma)) = (Fp_\Gamma)\varphi$. This is well-defined since for all $\varphi \in FX$ and $\gamma \in \Gamma$, we have $(Fp_\Gamma)(\varphi - \gamma \varphi) = (Fp_\Gamma)(\varphi - (p_\Gamma \circ \gamma)\varphi) = (Fp_\Gamma)\varphi - (Fp_\Gamma)\varphi = 0$. It is routine to check that $u_F = \{u_{F, X}\}_{X \in \mathcal{C}} : F_1 \to F$ is a natural transformation. It is left to show that $u_F$ is an isomorphism.

Suppose first that $F$ is elementary and write $F = \tilde{\ell} \circ S$ as in Definition [4.11]. By (i), we may assume that $S = S_1$, and hence $F_1(\Gamma \backslash X) = \tilde{\ell}(\Gamma \backslash SX)_\Gamma, F(\Gamma \backslash X) = \tilde{\ell}(\Gamma \backslash SX)$, and $u_{F, \Gamma \backslash X}$ is given by sending $e_x + (FX)(\Gamma)$ to $(S_{p_\Gamma})_\gamma(e_x) = e_{\gamma x}$ for all $x \in SX$ (see the notation section). Since $(FX)(\Gamma) = \text{span}\{e_x - \gamma e_x | x \in SX, \gamma \in \Gamma\}$, this means $u_{F, \Gamma \backslash X}$ is an isomorphism. For general $F$, choose $F'$ such that $F \oplus F'$ is elementary. Then $u_{F \oplus F'}$ is an isomorphism by what we have shown. Since $u_{F \oplus F'} = u_F \oplus u_F$, it follows that $u_F$ is an isomorphism as well. 

**Theorem 4.15.** Let $X$ be an almost transitive $G$-complex, let $\mathcal{C} = \mathcal{C}(G, X)$, let $F : \mathcal{C} \to \text{phil}$ be semi-elementary, and write $A = A(\mathcal{C}, F)$. Then:

(i) The map $\{a_X\}_{X \in \mathcal{C}} \mapsto a_X : A \to \text{End}_G(FX)$ is an isomorphism of unital $*$-algebras, where the involution on $\text{End}_G(FX)$ is given by taking the dual with respect to the inner product on $FX$.

(ii) For every $a \in A$, there is $M = M(a) \in \mathbb{R}_{\geq 0}$ such that $\|a|FX\| \leq M$ for all $X \in \mathcal{C}$. 


When (ii) holds, we say that $A$ acts uniformly continuously on $\mathcal{C}$. We shall need several lemmas for the proof.

**Lemma 4.16.** Let $r_1, \ldots, r_n \in \mathbb{R}$. Then $(r_1 + \cdots + r_n)^2 \leq n(r_1^2 + \cdots + r_n^2)$.

**Proof.** This is well-known. The function $x \mapsto x^2 : \mathbb{R} \to \mathbb{R}^2$ is convex, so by Jensen’s Inequality, $(\frac{r_1}{n} + \cdots + \frac{r_n}{n})^2 \leq \frac{1}{n} + \cdots + \frac{1}{n}$. Now multiply by $n^2$.

**Lemma 4.17.** Let $X$ and $Y$ be sets, let $a : \hat{\ell}^2(X) \to \hat{\ell}^2(Y)$ be a linear operator, and let $\{\alpha_{xy}\}_{x \in X, y \in Y} \subseteq \mathbb{C}$ be the unique complex numbers satisfying $\alpha_{xy} = \sum_y \alpha_{xy} e_y$. Assume that there are $M \in \mathbb{N}$ and $N \in \mathbb{R}$ such that

- (B1) for all $x \in X$, $\|\alpha_{xy}\|^2 = \sum_y |\alpha_{xy}|^2 \leq N$ and
- (B2) for all $y \in Y$, $\# \{x \in X : \alpha_{xy} \neq 0\} \leq M$.

Then $\|a\| \leq \sqrt{MN}$ and $a$ admits a dual $a^* : \hat{\ell}^2(Y) \to \hat{\ell}^2(X)$.

**Proof.** Define $a^* : \hat{\ell}^2(Y) \to \hat{\ell}^2(X)$ by $a^* e_y = \sum_x \alpha_{xy} e_x$. This is well-defined by (B2). It is easy to check that $\langle a^* e_x, e_y \rangle = \alpha_{xy} = \langle e_x, a e_y \rangle$, so $a^*$ is indeed a dual of $a$. Let $\varphi = \sum_x \beta_x e_x \in \ell^2(X)$. Then

$$a \varphi = \sum_x \beta_x \sum_y \alpha_{xy} e_y = \sum_y \left( \sum_x \beta_x \alpha_{xy} \right) e_y.$$

Using Lemma 4.16 (B1) and (B2), we have

$$\|a \varphi\|^2 = \sum_y \left( \sum_x |\beta_x \alpha_{xy}|^2 \right) \leq \sum_y \left( \sum_x |\beta_x \alpha_{xy}| \right)^2 \leq \sum_y M \sum_x |\beta_x \alpha_{xy}|^2 = M \sum_x |\beta_x|^2 \sum_y |\alpha_{xy}|^2 \leq MN \|\varphi\|^2.$$

This means that $\|a\| \leq \sqrt{MN}$.

**Lemma 4.18.** Let $F_1, F_2 : \mathcal{C} \to \bf{phil}$ be two functors. If the conclusion of Theorem 4.15(i) (resp. Theorem 4.15(ii)) holds for $F_1 \oplus F_2$, then it also holds for $F_1$ and $F_2$.

**Proof.** It is enough to prove this for $F_1$. For every $X \in \mathcal{C}$, let $e_X$ denote the orthogonal projection of $(F_1 \oplus F_2) \mathcal{X} = F_1 X \oplus F_2 X$ onto the summand $F_1 X$. Then $e := \{e_X\}_{X \in \mathcal{C}}$ is a self-dual natural transformation from $F_1 \oplus F_2$ to itself, and hence lives in $A(\mathcal{C}, F_1 \oplus F_2)$. It is easy to see that the map

$$A(\mathcal{C}, F_1) \to e A(\mathcal{C}, F_1 \oplus F_2) e$$

$$a_X \in \mathcal{C} \mapsto (a_X \oplus 0_{F_2 X}) X \in \mathcal{C}$$

is an isomorphism of $*$-algebras. Since $\|a_X\| = \|a_X \oplus 0_{F_2 X}\|$, we see that if $A(F_1 \oplus F_2, \mathcal{C})$ acts uniformly continuously on $\mathcal{C}$, then so does $A(F_1, \mathcal{C})$.

Suppose now that $F_1 \oplus F_2$ satisfies the conclusion of Theorem 4.15(i) and let $a_X \in \text{End}_{\mathcal{G}}(F_1 X)$. Then $a_X \oplus 0_{F_2 X} \in \text{End}_{\mathcal{G}}(F_1 X \oplus F_2 X)$, so by assumption, there is unique $b = \{b_X\}_{X \in \mathcal{C}} \in A(\mathcal{C}, F_1 \oplus F_2)$ with $b_X = a_X \oplus 0_{F_2 X}$. Since we also have $(e b) e_X = a_X \oplus 0_{F_2 X}$, we must have $b = e b e$. Thus, there is unique $c = \{c_X\}_{X \in \mathcal{C}} \in A(\mathcal{C}, F)$ such that $b_X = c_X \oplus 0_{F_2 X}$ for all $X \in \mathcal{C}$. In particular, $c_X = a_X$, and $c$ is unique by the uniqueness of $b$.

Let us describe several cases where Lemma 4.18 can be applied.

**Example 4.19.** (i) By definition, $\Omega^\pm_i = \Omega^+_i \oplus \Omega^-_i$ for $i > 0$.

(ii) One can embed $\Omega^+_i$ as an orthogonal summand of $\Omega^{flag}_i$. Observe first that any maximal flag $f = (\emptyset = x_{-1} \subsetneq x_0 \subsetneq x_1 \subsetneq \cdots \subsetneq x_d)$ in $X \in \mathcal{C}$ induces
a partial order on $x_i$, namely, for $u, v \in x_i$, set $u < v$ if and only if there is $0 \leq j < i$ such that $u \in x_j$ and $v \notin x_j$. Denote this order by $\leq$ and write $\zeta(f) = [x_i, \leq] \in X^{(i)}$. Also, for all $x \in X^{(i)}$, let $\eta(x) = \{f \in \text{Flag}(X) : x = \zeta(f)\}$. Notice that $\eta(x)$ is finite since $X$ is locally finite. Now, the map $j_X : \Omega^1(X) \rightarrow \Omega^1_{\text{Flag}}(X)$ sending $e_\alpha$ to $|\eta(\alpha)|^{-1/2} \sum_{f \in \eta(\alpha)} e_f$ is a unitary injection, and $j = \{j_X\}_{X \in \mathcal{F}}$ is a natural transformation from $\Omega^1_{\mathbb{H}}$ to $\Omega^1_{\text{Flag}}$. The image of $j_X$ is indeed an orthogonal summand — its orthogonal complement is $FX := \{\varphi \in \Omega^1_{\text{Flag}}(X) : \sum_{f \in \eta(\alpha)} \varphi(f) = 0\}$ for all $x \in X^{(i)}$. Thus, $\Omega^1_{\text{Flag}} \cong \Omega^1_{\mathbb{H}} \oplus F$.

**Proof of Theorem 1.1.3.** By Lemma 1.1.8 it is enough to prove the theorem when $F$ is elementary. Write $F = \tilde{\ell}^2 \circ S$ where $S : \mathbb{H} \rightarrow \text{Set}$ satisfies conditions (E1)–(E4). By Proposition 1.1.4(i), we may assume $S(\Gamma \setminus \mathcal{X}) = \Gamma \setminus \mathcal{X}$ for all $\Gamma \leq_X G$.

(i) Let $a = \{a_X\} \in A$ and let $pr : \mathcal{X} \rightarrow \Gamma \setminus \mathcal{X}$ be the quotient map. Then $a_{\Gamma \setminus \mathcal{X}} \circ (S pr)_* = (S pr)_* \circ a_X$. The map $S pr$ is the quotient map $\mathcal{X} \rightarrow \Gamma \setminus \mathcal{X}$, and hence $(S pr)_*$ is surjective. This means $a_X$ determines $a_{\Gamma \setminus \mathcal{X}}$ for all $\Gamma \leq_X G$, so the map $\{a_X\} \mapsto a_X$ is injective.

Suppose now that we are given $a_X \in \text{End}_C(F \mathcal{X}) = \text{End}_C(\tilde{\ell}^2(\mathcal{X}))$. For all $\Gamma \leq_X G$, we define $a_{\Gamma \setminus \mathcal{X}} : S(\Gamma \setminus \mathcal{X}) \rightarrow S(\Gamma \setminus \mathcal{X})$ as follows: If $\Gamma x \in \mathcal{X}$ and $ae_x = \sum_{g \in \mathcal{X}} a_{gX} e_g$ (in $\tilde{\ell}^2(\mathcal{X})$), then set

$$a_{\Gamma \setminus \mathcal{X}} e_T = \sum_{g \in \mathcal{X}} a_{gX} e_g,$$

and extend $a_{\Gamma \setminus \mathcal{X}}$ linearly to $\tilde{\ell}^2(\Gamma \setminus \mathcal{X})$. This is well-defined because if we replace $x$ with $\gamma x$ for $\gamma \in \Gamma$, then $ae_x = a_{\gamma X} e_\gamma = \gamma \sum_{y} a_{gY} e_g = \sum_{y} a_{gY} e_y$, and we get $ae_{\gamma X} = \sum_{y} a_{\gamma Y} e_y = \sum_{y} a_{gY} e_y$. It is routine to check that $\{a_X\}_{X \in \mathcal{F}}$ is a natural transformation from $F$ to itself.

Suppose now that $a_X \in \text{End}_C(F \mathcal{X})$ admits a dual $b_X$. We claim that $(a_{\Gamma \setminus \mathcal{X}}^*) = b_{\Gamma \setminus \mathcal{X}}$, and hence $a := \{a_X\}_{X \in \mathcal{F}}$ has a dual. For all $x \in \mathcal{X}$ write $ae_x = \sum_{y \in \mathcal{X}} a_{yX} e_y$ with $\{a_{xy}\}_{x, y \in \mathcal{C}}$. Then $b_X e_y = \sum_{x} a_{xy}^* e_x$. Since $a_X e_x = a_{gX} e_e = ga_X e_x$, for all $g \in G$, $x, y \in \mathcal{X}$, we have $a_{gyx} = a_{x_y (g^{-1})y}$. Now, by (4.4), for all $z, w \in \mathcal{X}$ and $\Gamma \leq_X G$, we have $\langle a_{\Gamma \setminus \mathcal{X}} e_T e_T, e_T \rangle = \langle \sum_{y} a_{yX} e_y, e_{\Gamma w} \rangle = \sum_{y \in \mathcal{G}} a_{yX} e_{z_{\Gamma y}} = \sum_{y \in \mathcal{G}} a_{yX} e_{z_{\Gamma y}} = \langle e_T, \sum_{x} a_{xX} e_T \rangle = \langle e_T, b_{\Gamma \setminus \mathcal{X}} e_T \rangle$, so $(a_{\Gamma \setminus \mathcal{X}}^*) = b_{\Gamma \setminus \mathcal{X}}$.

The previous paragraphs imply that $\{a_X\} \mapsto a_X : A \rightarrow \text{End}_C(F \mathcal{X})$ is injective and surjective, provided that any $a_X \in \text{End}_C(F \mathcal{X})$ admits a dual. We shall verify the latter in the proof of (ii).

(ii) Let $a \in \text{End}_C(F \mathcal{X})$ and define $\{a_X\}_{X \in \mathcal{F}}$ as above. We will apply Lemma 1.1.7 to $a_{\Gamma \setminus \mathcal{X}}$ with constants $N, M$ which are independent of $\Gamma$, thus showing that $a_X$ has a dual and $\|a_{\Gamma \setminus \mathcal{X}}\|$ can be bounded uniformly in $\Gamma$.

Let $\{z_1, \ldots, z_s\}$ be representatives for the $G$-orbits in $\mathcal{X}$ (there are finitely many by (E4)) and write $ae_{z_j} = \sum_{j=1}^{s_j} \alpha_{jX} e_{z_j}$ where $\{z_j\}_{j \in \mathcal{X}}$. Then for all $g \in G$, we have $ae_{gX} = a_{gX} e_g = ga_{gX} e_x = \sum_{j} \alpha_{jX} e_{z_j}$. Thus, by (4.4),

$$a_{\Gamma \setminus \mathcal{X}} e_{TgX} = \sum_{k=1}^{s_j} \alpha_{jX} e_{TgX}.$$

The cosets $\{TgX\}_{j=1}^{s_j}$ may coincide, hence there is a partition $\pi = \pi(g, j)$ of the set $\{1, \ldots, s_j\}$ such that

$$\|a_{\Gamma \setminus \mathcal{X}} e_{TgX}\|^2 = \sum_{s \in \pi} \left| \sum_{k \in S} \alpha_{jX} \right|^2.$$
The number of partitions of \{1, \ldots, s\} is finite, as well as the number of possible \(j\)-s, hence condition (B1) of Lemma 4.17 holds for \(a_{\Gamma \setminus \chi}\) with \(N\) which is independent of \(\Gamma\).

To show (B2), we need to find \(M \in \mathbb{N}\), independent of \(\Gamma\), such that for all \(\Gamma y \in \Gamma \setminus \chi\), there are at most \(M\) orbits \(\Gamma x \in \Gamma \setminus \chi\) with \(\Gamma y \in \text{supp}(a_{\Gamma \setminus \chi}e_{\Gamma x})\). Let \(K_j = \text{Stab}_{\Gamma}(x_j)\), and let \(I_j\) be a set of representatives for the double cosets \(\Gamma \setminus \Gamma / K_j\). Then every \(\Gamma x \in \Gamma \setminus \chi\) can be written as \(\Gamma g x_j\) for unique \(g\) and \(j\). In this case, \(\text{supp}(a_{\Gamma \setminus \chi}e_{\Gamma x}) \subseteq \{\Gamma g z_{jk} | 1 \leq k \leq s_j\}\). Therefore, it is enough to show that for every \(1 \leq j \leq r\) and \(1 \leq k \leq s_j\), there is a bound on \(#\{g \in I_j : \Gamma y = \Gamma g z_{jk}\}\) which is independent of \(y\) and \(\Gamma\). We may assume that \(y = h z_{jk}\) for some \(h \in G\), since otherwise the quantity in question is 0. Let \(L = \text{Stab}_{\Gamma}(z_{jk})\) and put \(K = K_j\). Then \(\Gamma y = \Gamma g z_{jk}\) implies that there is \(\gamma \in \Gamma\) such that \(\gamma h z_{jk} = g z_{jk}\), hence \(h^{-1} \gamma^{-1} g \in L\), or \(\gamma^{-1} g \in hL\). This implies that \(\Gamma g K \cap hL \neq \emptyset\). By (E2), \(L\) and \(K\) are compact and open in \(G\), so the index \([L : K \cap L]\) is finite. Since \(\Gamma g K \cap hL\) is a union of right \((K \cap L)\)-cosets, no more than \([L : K \cap L]\) double cosets \(\Gamma g K\) can intersect \(hL\) non-trivially, which proves that \(#\{g \in I_j : \Gamma y = \Gamma g z_{jk}\}\) \(\leq [L : K \cap L]\). This completes the proof.

Example 4.20. Let \(\chi\) be a \(k\)-regular tree, let \(G = \text{Aut}(\chi)\), and let \(\mathcal{C} = \mathcal{C}(G, \chi)\). We now use Theorem 4.15 to show that \(A_0 := A(\mathcal{C}, \Omega^+_0)\) is freely generated by \(a_0\), the vertex adjacency operator, and \(A_1 := A(\mathcal{C}, \Omega^+_1)\) is freely generated by \(a_1\), the edge adjacency operator. As mentioned in Example 4.18, this implies that:

(i) \(\text{Spec}_{0,0}(\Gamma \setminus \chi)\), the 0-dimensional spectrum of \(\Gamma \setminus \chi\), is equivalent to the spectrum of the vertex adjacency operator of \(\Gamma \setminus \chi\).

(ii) \(\text{Spec}_{1,0}(\Gamma \setminus \chi)\), the non-oriented 1-dimensional spectrum of \(\Gamma \setminus \chi\), is equivalent to the spectrum of the edge adjacency operator of \(\Gamma \setminus \chi\).

(These assertions are false for general \(\chi\) and \(G\).)

Using Theorem 4.15 we identify \(A_0 = \text{End}_{\Gamma}(\Omega^+_0(\chi))\). Fix a vertex \(x \in \chi^{(0)}\), let \(b \in \text{End}_{\Gamma}(\Omega^+_0(\chi)) = A_0\), and write \(b e_x = \sum_{y \in \chi^{(0)}} a_y e_y\). Set \(R_b = \{y \in \chi^{(0)} | d(x, y) = n\}\). Then \(K_x := \text{Stab}_{\Gamma}(x)\) acts transitively on \(R_b\) for all \(n \geq 0\), and for all \(g \in K_x\), we have

\[
\sum_y a_y e_y = b e_x = b(g e_x) = g(b e_x) = g \sum_y a_y e_y = \sum_y a_y e_{gy}.
\]

Therefore, \(a_{gy} = a_y\), and the coefficients \(a_y\) are identical on the set \(R_b\). We may therefore write \(b e_x = \sum_{b \geq 0} \sum_{y \in R_b} a_n e_y\). Let \(m = m(b) \geq 0\) be the least integer such that \(a_n = 0\). We prove by induction on \(m\) that \(b\) is a polynomial in \(a_0\).

Indeed, if \(m = 0\), then \(b e_x = 0\), hence for all \(g \in G\), we have \(b e_{gy} = g(b e_x) = 0\), so \(b = 0\). If \(m > 0\), then \(b' := b - a_m a_0^{m-1}\) satisfies \(m(b') < m(b)\), and by induction, \(b'\) is a polynomial in \(a_0\). Finally, it is easy to show that \(m(a_0^n) = n + 1\), hence \(\{e_x, a_0 e_x, a_0^2 e_x, \ldots\}\) are linearly independent in \(\Omega^+_0(\chi)\), which means \(\{1, a_0, a_0^2, \ldots\}\) are linearly independent in \(A_0\). Thus, \(A_0\) is freely generated by \(a_0\).

Similarly, \(A_1\) is freely generated by \(a_1\).

Example 4.21. The previous example can be generalized to buildings (definitions can be found in [1]): Let \(B\) be a locally finite building with Coxeter system \((W, S)\) and write \(d := \dim B = |S| - 1\). There is a \(W\)-valued distance function \(\delta\) taking two chambers of \(B\) to an element of \(W\) and satisfying certain axioms (see [1] Def. 5.1)). Let \(G\) be a subgroup of Aut(B) such that \(G\) is closed under pointwise convergence, \(\delta(gx, gy) = \delta(x, y)\) for all \(g \in G, x, y \in B^{(d)}\), and \(G\) acts transitively on pairs \((x, y) \in B^{(d)} \times B^{(d)}\) with \(\delta(x, y) = w\) for all \(w \in W\). Then \(B\) is an almost transitive
G-complex (Proposition 4.22(i)). For every \( w \in W \), define \( a_w : \Omega^+_d(B) \to \Omega^+_d(B) \) by
\[
(a_w \varphi)x = \sum_{y \in B \mid \delta(x,y) = w} \varphi y \quad \forall \varphi \in \Omega^+_d(B), \ x \in B^{(d)}.
\]
It is easy to check that \( a_w \in \text{End}_d(\Omega^+_d(B)) \), and hence \( a_w \) can be viewed as an element of \( A_d := \mathcal{A}(\mathcal{C}(G,B), \Omega^+_d) \). Furthermore, using the transitivity property of \( G \), the properties of \( \delta \), and induction on the length of elements in \( W \), one can show that the elements \( \{a_w\}_{w \in W} \) form a basis of \( A_d \).

We finally note that if \( G \) is a simply-connected almost simple algebraic group over a local non-archimedean field \( F \), then its affine Bruhat-Tits building \( B \) satisfies the above assumptions with \( G = \text{im}(G(F) \to \text{Aut}(B)) \) (cf. Example 4.3). For example, when \( G = \text{SL}_d \), we get \( B = B_d(F) \) as in Chapter 1 and \( G = \text{im}(\text{SL}_d(F) \to \text{PGL}_d(F)) \) (the group \( \text{PGL}_d(F) \) does not preserve \( \delta \)-distance).

We finish this section with a proposition relating the \( A \)-spectrum of two \( G \)-quotients of \( \mathcal{X} \).

**Proposition 4.22.** Let \( \Gamma' \leq \Gamma \leq \mathcal{X} G \) with \([\Gamma : \Gamma'] < \infty \), let \( F : \mathcal{C} \to \mathcal{PHil} \) be semi-elementary, and let \( A \) be an algebra of \((\mathcal{C}, F)\)-operators. Then \( \text{Spec}_A(\Gamma' \setminus \mathcal{X}) \subseteq \text{Spec}_A(\Gamma' \setminus \mathcal{X}) \). When \( \Gamma' \setminus \mathcal{X} \) is finite, we also have \( \text{m-Spec}_A(\Gamma' \setminus \mathcal{X}) \subseteq \text{m-Spec}_A(\Gamma' \setminus \mathcal{X}) \) (as multisets).

**Proof.** Choose a functor \( F' : \mathcal{C} \to \mathcal{PHil} \) such that \( F \oplus F' \) is elementary. We may view \( A \) as an algebra of \( A(\mathcal{C}, F \oplus F') \)-operators (Remark 4.10), so assume henceforth that \( F \) is elementary. Write \( F = \ell^2 \circ S \) where \( S \) is as in Definition 4.11. By Proposition 4.21(i), we may also assume \( S(\Delta \setminus \mathcal{X}) = \Delta \setminus \mathcal{X} \) for all \( \Delta \leq \mathcal{X} G \), and that \( \text{Spec}_A(\Gamma' \setminus \mathcal{X}) \) is the quotient map \( \Gamma' \to \Gamma \). The fibers of \( \text{Spec}_A(\Gamma' \setminus \mathcal{X}) \) have size at most \([\Gamma' : \Gamma] < \infty \), and hence the map \( \text{Spec}_A(\Gamma' \setminus \mathcal{X}) \) is continuous. Therefore, this map extends to a continuous map on the completions \( P = \overline{P_{\ell^2}} : \ell^2(\Gamma' \setminus \mathcal{X}) \to \ell^2(\Gamma' \setminus \mathcal{X}) \), which is surjective (because \( P_{\ell^2} = \sigma_{\ell^2} \) for all \( x \in \mathcal{X} \)). Since \( A \) acts continuously on \( \ell^2(\Gamma' \setminus \mathcal{X}) \) and \( \ell^2(\Gamma' \setminus \mathcal{X}) \) (Theorem 4.13(ii)), the action of \( A \) extends to the relevant completions, and \( P \) is an \( A \)-homomorphism. The map \( P \) restricts to a continuous isomorphism of \( A \)-modules (\( \ker P \)) to \( \ell^2(\Gamma' \setminus \mathcal{X}) \). By Proposition 2.8, there exists a unitary isomorphism of \( A \)-modules (\( \ker P \)) to \( \ell^2(\Gamma' \setminus \mathcal{X}) \). This means that \( \text{Spec}_A(\Gamma' \setminus \mathcal{X}) = \text{Spec}_A(\mathcal{A}(\ker P)) \subseteq \text{Spec}_A(\mathcal{A}(\ell^2(\Gamma' \setminus \mathcal{X}))) = \text{Spec}_A(\Gamma' \setminus \mathcal{X}) \). The statement about the multiset spectrum is proved similarly. \( \square \)

The rest of this chapter discusses dependencies between different kinds of spectra, equivalence of oriented and non-oriented spectra (in some cases), and the fact that the spectra of a complex may not be determined from its isomorphism class. The reader can skip this without loss of continuity.

**4E. Dependencies Between Spectra.** Fix a skeletally small subcategory \( \mathcal{C} \subseteq \text{Sim} \) and let \( F, F' : \mathcal{C} \to \mathcal{PHil} \) be two functors (e.g. \( F = \Omega^+_d \) and \( F' = \Omega^+_d \)). For brevity, we write
\[
A_F = A(\mathcal{C}, F), \quad \text{Spec}_F(X) = \text{Spec}_A(\mathcal{C}, F)(X),
\]
and likewise for \( F' \).

The spectra of \( X \in \mathcal{C} \) with respect to \( F \) and \( F' \) are usually not independent in the sense that the presence of certain points from \( \hat{A}_F \) in \( \text{Spec}_F(X) \) may imply the presence of certain points from \( \hat{A}_{F'} \) in \( \text{Spec}_{F'}(X) \). In fact, in some cases, \( \text{Spec}_F(X) \) determines \( \text{Spec}_{F'}(X) \) completely. Such dependencies can sometimes be explained by analyzing \( F \oplus F' \).
Write $A_{F\oplus F'} := A(\mathcal{C}, F \oplus F')$. For all $X \in \mathcal{C}$, let $e_F, X$ and $e_{F'}, X$ be the orthogonal projections of $FX \oplus F'X$ onto $FX$ and $F'X$, respectively. It is easy to see that $e_F = \{e_F, X\}_{X \in \mathcal{C}}$, and $e_{F'} = \{e_{F'}, X\}_{X \in \mathcal{C}}$ live in $A_{F\oplus F'}$, and moreover, that $e_F$ and $e_{F'}$ are idempotents, $e_F + e_{F'} = 1$, $e_F = e_F$ and $e_{F'} = e_{F'}$. There are obvious isomorphisms

$$A_{F} \cong e_F A_{F \oplus F'} e_F,$$

$$A_{F'} \cong e_{F'} A_{F \oplus F'} e_{F'},$$

$$FX \cong e_F (FX \oplus F'X),$$

$$F'X \cong e_{F'} (FX \oplus F'X),$$

which are compatible with the relevant module structures for all $X \in \mathcal{C}$. Provided $A_{F\oplus F'}$ acts continuously on $(F \oplus F')X$, Theorem 2.44 and Corollary 2.47 imply that the datum of $\text{Spec}_{F\oplus F'}(X)$ is equivalent to the data of $\text{Spec}_F(X)$ and $\text{Spec}_{F'}(X)$. However, $\text{Spec}_{F\oplus F'}(X)$ holds this data in a more efficient way which avoids certain dependencies. More formally:

**Proposition 4.23.** In the previous setting, let $\tilde{A}_{F\oplus F'}^{(A_F)} = \{V \in \tilde{A}_{F\oplus F'} \mid e_F V \neq 0\}$ and $\tilde{A}_{F\oplus F'}^{(A_{F'})} = \{V \in \tilde{A}_{F\oplus F'} \mid e_{F'} V \neq 0\}$, and identify the open subset $\tilde{A}_{F\oplus F'}^{(A_F)}$ (resp. $\tilde{A}_{F\oplus F'}^{(A_{F'})}$) with a subspace of $\tilde{A}_F$ (resp. $\tilde{A}_{F'}$) via $[V] \mapsto [e_F V]$ (resp. $[V] \mapsto [e_{F'} V]$) using Theorem 2.43. Then $\tilde{A}_{F\oplus F'} = \tilde{A}_{F\oplus F'}^{(A_F)} \cup \tilde{A}_{F\oplus F'}^{(A_{F'})}$ and under the previous identifications, for all $X \in \mathcal{C}$ such that $\text{Spec}_{F\oplus F'}(X)$ is defined, we have

$$\text{Spec}_F(X) = \text{Spec}_{F\oplus F'}(X) \cap \tilde{A}_{F\oplus F'}^{(A_F)},$$

$$\text{Spec}_{F'}(X) = \text{Spec}_{F\oplus F'}(X) \cap \tilde{A}_{F\oplus F'}^{(A_{F'})}.$$

In particular:

(i) When defined, the $F \oplus F'$-spectrum determines the $F$-spectrum and $F'$-spectrum, and vice versa.

(ii) For any $\eta \in \tilde{A}_{F\oplus F'}^{(A_F)} \cap \tilde{A}_{F\oplus F'}^{(A_{F'})}$, we have $\eta \in \text{Spec}_F(X) \iff \eta \in \text{Spec}_{F'}(X)$ whenever $\text{Spec}_{F\oplus F'}(X)$ is defined. More precisely, if $\eta = [V]$ for $V \in \text{Irr}(A_{F\oplus F'})$, then $[e_F V] \in \text{Spec}_F(X) \iff [e_{F'} V] \in \text{Spec}_{F'}(X)$.

(iii) If $\tilde{A}_{F\oplus F'}^{(A_F)} = \tilde{A}_{F\oplus F'}^{(A_{F'})}$, then the $F$-spectrum determines the $F'$-spectrum whenever the $F \oplus F'$-spectrum is defined.

**Proof.** Notice that $e_F + e_{F'} = 1$ in $A_{F\oplus F'}$, hence $\{e_F, e_{F'}\}$ is a full family of idempotents in the sense of Example 2.45. Thus, everything follows from Theorem 2.44 and Corollary 2.47. We have $\tilde{A}_{F\oplus F'} = \tilde{A}_{F\oplus F'}^{(A_F)} \cup \tilde{A}_{F\oplus F'}^{(A_{F'})}$ by the comment preceding Example 2.45. \qed

**Remark 4.24.** When defined, the $F \oplus F'$-spectrum actually determines the $B$-spectrum for any algebra $B$ of $\mathcal{C}$, $F \oplus F'$-operators, by Theorem 2.45. Proposition 4.23 makes this process more concrete in the cases $B = A_{F}$ and $B = A_{F'}$.

**Example 4.25.** Example 4.19 and Proposition 4.23 imply that for all $i \geq 0$, the flag spectrum determines the full $i$-dimensional spectrum, which in turn determines the non-oriented and the oriented $i$-dimensional spectra, provided all are defined. Also, the non-oriented and the oriented $i$-dimensional spectra determine the full $i$-dimensional spectrum, when defined.

**Example 4.26.** For all $n \in \mathbb{N}$, the $F$-spectrum determines the $F^n$-spectrum and vice versa. More generally, if $G : \mathcal{C} \to \text{Ph11}$ is a functor such that $G$ is a summand of $F^n$ for some $n$, then, when defined, the $F$-spectrum determines the $G$-spectrum.

**Example 4.27.** Let $X$ be a $k$-regular tree, let $G = \text{Aut}(X)$, and let $\mathcal{C} = \mathcal{C}(G, X)$ (Definition 4.1). For $i = 0, 1$, write $A_i = A(\mathcal{C}, \Omega_i^+)$, and recall from Example 4.20.
dependencies can be explained using Proposition 4.23; they essentially follow from other higher dimensional associated operators were shown in [36] and [28]. These to the spectrum defined in Chapter 1 by Example 4.9) and the spectrum of some $\lambda(X)$ by

$$(a_{01}, x \varphi) x = \sum_{x \varphi y \in X^{(0)}} \varphi y \quad \forall x \in X^{(0)}, \varphi \in \Omega^+_1(X)$$

$$(a_{10}, x \psi) y = \sum_{y \ni x \in X^{(0)}} \psi x \quad \forall y \in X^{(1)}, \psi \in \Omega^+_0(X)$$

We may view $a_{01}, X$ and $a_{10}, X$ as operators on $\Omega^+_0(X) = \Omega^+_0(X) \oplus \Omega^+_1(X)$ by setting them to be 0 on $\Omega^+_0(X)$ and $\Omega^+_1(X)$, respectively. It is easy to check that $a_{01}, a_{10} \in A_{0,1}$ and $a_{01} = a_{10}$. Arguing as in Example 4.20 one can show that $A_{0,1}$ is isomorphic to the path algebra $A$ of the quiver

$$0 \xrightarrow{a_{01}} a_{10} \xrightarrow{1}$$

discussed in Example 2.24 (the elements $e_0, e_1, a_{01}, a_{10} \in A_{0,1}$ correspond to the elements with the same name in $A$).

The unitary dual of $A_{0,1}$ is described in Example 2.24 as the gluing of two copies of $\mathbb{R}_{\geq 0}$ along $\mathbb{R}_{>0}$. Since $A_0 = e_0 A_{0,1} e_0$ and $A_1 = e_1 A_{0,1} e_1$, the sets $\hat{A}^{(A_0)}_{0,1}$ and $\hat{A}^{(A_1)}_{0,1}$ correspond to the two copies of $\mathbb{R}_{\geq 0}$, and so $\hat{A}^{(A_0)}_{0,1} \cap \hat{A}^{(A_1)}_{0,1}$ corresponds to $\mathbb{R}_{>0}$, which, in the notation of Example 2.24, $\mathbb{R}_{>0}$ corresponds to the irreducible representations $[V_{\mathbb{R}}] | r \in \mathbb{R}_{>0}$. Therefore, Proposition 4.23 implies that for any $k$-regular graph $X \in \mathscr{C}$ and $r \in \mathbb{R}_{>0}$, we have $[e_0 V_{\mathbb{R}}] \in \text{Spec}_0(X)$ if and only if $[e_1 V_{\mathbb{R}}] \in \text{Spec}_1(X)$.

Recall from Example 4.8 that for $i = 0, 1$ we can identify $\hat{A}_i$ with a subset of $\mathbb{C}$ such that $\text{Spec}_0(X)$ corresponds to $\text{Spec}(a_{i, X})$ for all $X \in \mathscr{C}$. It is easy to check that

$$(4.2) \quad a_{01} a_{10} = a_0 + k e_0 \quad \text{and} \quad a_{10} a_{01} = a_1 + 2 e_1$$

in $A_{0,1}$. Using this, one sees that for all $r \in \mathbb{R}_{>0}$, we have $	ext{Spec}(a_0 |_{r V_{\mathbb{R}}}) = r^2 - k$ and $\text{Spec}(a_1 |_{r V_{\mathbb{R}}}) = r^2 - 2$. Since $\text{Spec}_{\Omega^+_i}(X)$ is always defined (Theorem 4.15 ii)), this means that for all $\lambda \in \mathbb{R}$, we have

$$(4.3) \quad \lambda \in \text{Spec}(a_{i, X}) - \{-k\} \iff \lambda + k - 2 \in \text{Spec}(a_{1, X}) - \{-2\}.$$ 

This is a well-known dependency between the spectrum of the vertex and edge adjacency operators. An alternative way to exhibit this dependency by noting that $a_{10}, X a_{01}, X$ and $a_{01}, X a_{10}, X$ always have the same spectrum except maybe the multiplicity of 0, and then applying (4.2).

**Example 4.28.** Let $G = \text{PGL}_d(F)$ and $\mathcal{X} = \mathcal{B}_d(F)$ be as in Chapter 1. In the case $d = 3$, dependencies between the $\text{Spec}_{\mathcal{C}(G, \mathcal{X})}((\Gamma) \mathcal{B}_d(F))$ (which is equivalent to the spectrum defined in Chapter 1 by Example 4.9) and the spectrum of some other higher dimensional associated operators were shown in [30] and [28]. These dependencies can be explained using Proposition 4.23; they essentially follow from the analysis carried in [30] and [28] and Theorem 0.21 below.

The previous ideas can be extended to arbitrary families of functors $\{ F_\alpha : \mathcal{C} \to \mathfrak{ph}_d \}_{\alpha \in I}$ as follows: Let $F = \bigoplus_{\alpha \in I} F_\alpha$. When $I$ is finite, the $F$-spectrum determines the $F_\alpha$-spectrum for every $\alpha \in I$ and vice versa. However, for infinite $I$, one
has to consider

\[ A_1(F_a) = A(\mathcal{C}, \{F_\alpha\}) := \lim_{\rightarrow} \{A_{\bigoplus_{x \in S} F_x}\}_{S \subseteq I} . \]

Here, \( S \) ranges over the finite subsets of \( I \), and for \( S \subseteq T \subseteq I \), \( A_{\bigoplus_{x \in S} F_x} \) is embedded as a (non-unital) subalgebra of \( A_{\bigoplus_{x \in T} F_x} \) in the obvious way. The algebra \( A_1(F_a) \) is a non-unital idempotent \(*\)-subalgebra of \( A_F = A(\mathcal{C}, F) \), and we define \( \text{Spec}(F_a)(X) := \text{Spec}_{A(F_a)}(FX) \). If \( e_{a,X} \) denotes the orthogonal projection from \( FX \) onto \( F_a X \), then \( \{e_{a,X}\}_{X \in \mathcal{C}} \) form a full system of idempotents in \( A_1(F_a) \) (see Example 4.19). This in turn induces a homeomorphism with the relevant module structures. The resulting spectra can be considered as a variation of the non-oriented (resp. oriented, full) spectrum, seeing all dimensions at once. We shall not investigate such examples in this work, although they are relevant to studying maps relating cells of different dimensions such as the boundary and coboundary maps \( \partial \), which live in \( A_1(\Omega_i^+_{\varepsilon}) \), or adjacency operators between cells of different dimensions, which live in \( A_1(\Omega_i^0) \).

4F. More Dependencies Between Spectra. This section continues discussing dependencies between spectra, but here we take a more explicit approach, focusing on the case \( \mathcal{C} = \mathcal{C}(G, \mathcal{X}) \) where \( \mathcal{X} \) is a fixed almost transitive \( G \)-complex [30, Definition 17].

Observe first that if \( F, F' : \mathcal{C} \to \text{pHil} \) are functors and \( u : F \to F' \) is a unitary natural isomorphism, then \( u \) induces an isomorphisms

\[ a \mapsto uau^{-1} : A_F \to A_{F'} \]

\[ \varphi \mapsto u_X \varphi : FX \to F'X \quad (X \in \mathcal{C}(G, \mathcal{X})) \]

where the second isomorphism is compatible with the relevant module structures. This in turn induces a homeomorphism \( \hat{u} : \hat{A}_F \to \hat{A}_{F'} \) under which \( \text{Spec}_{F'}(X) \) corresponds to \( \text{Spec}_{F'}(X) \) for all \( X \in \mathcal{C} \). We conclude that \( u \) induces an equivalence between the \( F \)-spectrum and the \( F' \)-spectrum of \( G \)-quotients of \( \mathcal{X} \). In particular, when \( F = F' \), the map \( \hat{u} \) is an automorphism of \( \hat{A}_F \), and the spectrum of any \( G \)-quotient of \( \mathcal{X} \) is stable under \( \hat{u} \).

We say that the action of \( G \) on \( \mathcal{X} \) preserves orientation if for all \( x \in \mathcal{X}_\text{ori} \) with \( \dim x > 0 \), \( x \) and \( x^{op} \) are not in the same \( G \)-orbit. Equivalently, this means that whenever \( g \in G, x \in \mathcal{X} \) satisfy \( gx = x \), the element \( g \) induces an even permutation on the vertices of \( x \). We say that the action of \( G \) on \( \mathcal{X} \) preserves order if for any \( x \in \mathcal{X} \) and \( g \in \text{Stab}_G(x) \), we have \( gx = x \) for any vertex \( v \in x \).

Proposition 4.29. In the previous setting:

(i) If \( G \) preserves orientation, then for all \( i > 0 \), there exists a (non-canonical) unitary natural isomorphism \( \Omega_i^+ \cong \Omega_i^- \). In particular, the non-oriented, oriented, and full \( i \)-dimensional spectra determine each other.

(ii) If \( \mathcal{X} \) is pure of dimension \( d \) and \( G \) preserves ordering, then there is a (non-canonical) unitary natural isomorphism \( \theta^{d+1}_i : \Omega_i^0 \cong \Omega_i^\text{Flag} \). In particular, the non-oriented \( d \)-dimensional spectrum determines the flag spectrum, and hence also the non-oriented, oriented, and full spectra in all dimensions \( 0 \leq i \leq d \).
Proposition 4.32. Proof. (i) Choose representatives \(\{x_\alpha\}_{\alpha \in I}\) for the \(G\)-orbits in \(X^{(i)}\) and choose an arbitrary orientation \(x_0 \in X^{(i)}(\alpha)\) for \(x_\alpha\). For all \(\Gamma \leq X\), define \(u_{\Gamma \setminus X} : \Omega^+_\Gamma(\Gamma \setminus X) \to \Omega^+_\Gamma(\Gamma \setminus X)\) by linearly extending \(e_{\Gamma \setminus X}^\alpha + e_{\Gamma \setminus X}^\gamma = e_{\Gamma \setminus X}^\alpha\). This is well-defined since if \(\gamma g x_\alpha = g' x_{\alpha'}\) (\(\gamma, g, g' \in G\)), then \(\alpha' = \alpha\), and \(g'^{-1} \gamma g \in \text{Stab}_G(x_\alpha)\). Since \(G\) preserves orientation, \(\text{Stab}_G(x_\alpha) = \text{Stab}_G(x_{\alpha'})\), and hence \(\gamma g x_\alpha = g' x_{\alpha'}\). It is straightforward to check that \(u = \{e_X\}_{X \in \mathcal{E}}\) is a unitary natural isomorphism from \(\Omega^+_\Gamma\) to \(\Omega^-\Gamma\). All other assertions follow from the discussion before the proposition and Proposition 4.23(i), since \(\Omega^+ = \Omega^+\Gamma \oplus \Omega^-\Gamma\).

(ii) Choose representatives \(\{x_\alpha\}_{\alpha \in I}\) for the \(G\)-orbits in \(X^{(d)}\). Since \(X\) is pure of dimension \(d\), for every \(\alpha \in I\), there are \((d + 1)\)-maximal flags \(\langle x_0, \ldots, x_d \rangle\) with \(x_d = x_\alpha\), call them \(\{f_{\alpha,j}\}_{j=1}^{(d+1)!}\). Now, for all \(\Gamma \setminus X \in \mathcal{E}\), define \(u_{\Gamma \setminus X} : \Omega^{\text{Flag}}(\Gamma \setminus X) \to \Omega^{\text{Flag}}(\Gamma \setminus X)^{(d+1)!}\) by sending \(e_{\Gamma \setminus X} f_{\alpha,j}\) to the image of \(e_{\Gamma \setminus X} x_{\alpha}\) in the \(j\)-th copy of \(\Omega^{\text{Flag}}(\Gamma \setminus X)\). This is well-defined since if \(\gamma g f_{\alpha,j} = g' f_{\beta,k}\) (\(\gamma, g, g' \in G\)), then \(g'^{-1} \gamma g x_\alpha = x_{\beta}\), and so \(\alpha = \beta\) and \(g'^{-1} \gamma g \in \text{Stab}_G(x_\alpha)\). To see that \(j = k\), notice that \(G\) is order preserving and hence \(g'^{-1} \gamma g\) fixes the vertices of \(x_\alpha\). Thus, \(f_{\alpha,j} = g'^{-1} \gamma g f_{\alpha,j} = f_{\beta,k}\), so \(j = k\). It is easy to check that \(u = \{e_X\}_{X \in \mathcal{E}}\) is a unitary natural isomorphism from \(\Omega^{\text{Flag}}\) to \((\Omega^+_\Gamma)^{(d+1)!}\). The other assertions follow from Proposition 4.23(i) and Example 4.19. 

Example 4.30. Let \(G = \text{PGL}_d(F)\) and \(X = B_d(F)\) be as in Chapter 1. Then \(G\) preserves orientation if and only if \(d\) is odd. Indeed, recall that there is a color function \(C_0 : B_d(F)^{(0)} = G/K \to \mathbb{Z}/d\mathbb{Z}\) given by \(C_0(\mathbf{f}K) = \nu_F(\det g) + d\mathbb{Z}\) for all \(g \in \text{GL}_d(F)\). Since adjacent vertices have different colors, \(C_0\) is injective on any cell \(x = \{v_0, \ldots, v_i\} \in B_d(F)\). Suppose now that \(g \in \text{GL}_d(F)\) satisfies \(g \alpha = x\) and \(\nu_F(\det g) + d\mathbb{Z}\). Then \(C_0(\mathbf{f}K v_0) = C_0(v_0) + r\) for all \(0 \leq s \leq i\), and hence \(\mathbf{f}^*\) fixes \(v_0, \ldots, v_i\). When \(d\) is odd, the orbits that \(\mathbf{f}\) induces on \(\{v_0, \ldots, v_d\}\) must have odd sizes, hence \(\mathbf{f}\) induces an even permutation on \(\{v_0, \ldots, v_i\}\). On the other hand, if \(d\) is even, we can take \(x\) to be a \((d-1)\)-cell and \(g\) to be a matrix of determinant \(1\). The permutation \(\mathbf{f}\) induces on \(\{v_0, \ldots, v_{d-1}\}\) is a cycle of size \(d\), which is odd. (The existence of \(x\) and \(g\) as above is easy to see.)

Example 4.31. Let \(X = B_d(F)\) be as in Chapter 1 and let \(G = \text{im}(\text{SL}_d(F) \to \text{PGL}_d(F))\). Then \(G\) preserves order because it preserves the coloring \(C_0\) of the vertices (see Chapter 1). More generally, if \(G\) is a simply-connected almost simple algebraic group over \(F\) with affine building \(B\) and \(G = \text{im}(G(F) \to \text{Aut}(B))\) (cf. Example 3.4), then \(G\) preserves order because it preserves the type of the vertices.

4G. A Note On Isomorphic Complexes. Let \(\mathcal{C} \subseteq \mathcal{S}\) be a skeletal small subcategory, let \(F : \mathcal{C} \to \mathcal{PHIL}\) be a functor, and let \(A\) be an algebra of \((\mathcal{C}, F)\)-operators (e.g. \(A(\mathcal{C}, F)\)). The reader should take notice that if \(X, X' \in \mathcal{C}\) are isomorphic simplicial complexes, then \(\text{Spec}_A(X)\) and \(\text{Spec}_A(X')\) may still differ, because \(X\) and \(X'\) need not be isomorphic in \(\mathcal{C}\). However, when \(\mathcal{C}\) is a full subcategory of \(\mathcal{S}\), and \(F\) sends simplicial isomorphisms to continuous isomorphisms (e.g. if \(F \in \{\Omega^+_\Gamma, \Omega^-\Gamma, \Omega^{\text{Flag}}\}\)), isomorphic simplicial complexes have the same \(A\)-spectrum (cf. Proposition 2.2).

Consider now the case \(\mathcal{C} = \mathcal{C}(G, X)\) and \(A = A(\mathcal{C}, F)\) where \(X\) is a \(G\)-complex. The category \(\mathcal{C}\) is usually not full in \(\mathcal{S}\), but under certain assumptions, one can show that isomorphic complexes in \(\mathcal{C}\) have the same \(A\)-spectrum.

Proposition 4.32. Assume \(X\) is simply-connected when realized as a topological space and \(G = \text{Aut}(X)\). Let \(F : \mathcal{S} \to \mathcal{PHIL}\) be a functor such that:

1. \(F \Gamma_T : F(\Gamma \setminus X) \to F(\Gamma \setminus X)\) is onto for all \(\Gamma \leq X\) \((\text{notation as in Definition } 4.10)\).
is determined by the isomorphism class of the quotient of $A(\mathscr{C}, F)$-spectrum.

Proof. Let $\Gamma, \Gamma' \leq_X G$, let $X = \Gamma \backslash X$, $X' = \Gamma' \backslash X$, and let $u : X \to X'$ be an isomorphism. It is enough to prove that $Fu : FX \to FX'$ is a homomorphism of $A(\mathscr{C}, F)$-modules, since then, by (2) and Proposition 4.32, $FX \cong FX'$ as unitary representations of $A(\mathscr{C}, F)$. Let $\gamma \in A(\mathscr{C}, F)$. Since $X$ is simply-connected, there is $g \in \text{Aut}(X) = G$ such that $p_{\mathcal{T}} \circ g = u \circ p_{\mathcal{T}}$. In particular, $u \circ p_{\mathcal{T}}$ is a morphism in $\mathscr{C}$, and hence $a_X \circ F(u \circ p_{\mathcal{T}}) = F(u \circ p_{\mathcal{T}}) \circ a_X$. Now, let $\psi \in FX$. By (1), there is $\xi \in FX$ such that $(Fp_{\mathcal{T}})\xi = \psi$. Thus,

$$(Fu)(a_X\psi) = (Fu)(a_X(Fp_{\mathcal{T}})\xi) = (Fu)((Fp_{\mathcal{T}})(a_X\xi)) = (F(u \circ p_{\mathcal{T}}))(a_X\xi) = a_X'((F(u \circ p_{\mathcal{T}}))\xi) = a_X'((Fu)((Fp_{\mathcal{T}})\xi)) = a_X'((Fu)\psi),$$

as required. \qed

Remark 4.33. The conditions of Proposition 4.32 are not satisfied when $X = \mathcal{B}_d(F)$ and $G = \text{PGL}_d(F)$ (see Chapter 1), because $\text{Aut}(X)$ is strictly larger than $G$. More precisely, when $d > 2$, $G$ is of index 2 in $\text{Aut}(\mathcal{B}_d(F))$. A representative for the non-trivial $G$-coset in $\text{Aut}(\mathcal{B}_d(F))$ is the automorphism $\tau$ given by $\tau(gK) = (g^{-1})^T K$, where $K = \text{PGL}_d(O)$ and $T$ denotes matrix transposition. A simple modification of the proof Proposition 4.32 shows that every isomorphism class of $\text{Sim}$ with a representative in $\mathscr{C}(G, \mathcal{B}_d(F))$ gives rise to two possible spectra\footnote{Interestingly, this mild issue seems to be unaddressed in all papers we have seen.} (which may coincide in special cases). In more detail, let $\Gamma \leq_X G$. Using Corollary 3.10, it easy to check that $\tau^{-1}\Gamma \tau \leq_X G$, and we have $\Gamma \backslash \mathcal{B}_d(F) \cong \tau^{-1}\Gamma \backslash \mathcal{B}_d(F)$ via $\Gamma v \mapsto \tau^{-1}\Gamma \tau(v)$. The spectra of $\Gamma \backslash \mathcal{B}_d(F)$ and $\tau^{-1}\Gamma \backslash \mathcal{B}_d(F)$ are the two spectra associated with the isomorphism class of $\Gamma \backslash \mathcal{B}_d(F)$. If $S_1 := \text{Spec}_0(\Gamma \backslash \mathcal{B}_d(F))$ and $S_2 := \text{Spec}_0(\tau^{-1}\Gamma \backslash \mathcal{B}_d(F))$ are realized as subsets of $\mathbb{C}^{d-1}$ as in Example 4.30, then $S_1 = \{z_d^{-1}, \ldots, z_1\} \cup \{z_1, \ldots, z_d^{-1}\} \subset S_2$. The technical proof is omitted.

When $d = 2$, one can show by arguing as in Example 4.24 that $A_0(G, \mathcal{B}_d(F)) = \mathbb{C}[a_0]$ and $A_1(G, \mathcal{B}_d(F)) = \mathbb{C}[a_1]$, so the oriented 0- and 1-dimensional spectrum are in fact determined by the isomorphism class of the quotient of $\mathcal{B}_2(F)$.

Remark 4.34. One way to overcome the problem that the spectrum of a $G$-quotient of $X$ is not determined by its isomorphism class is by specifying the cover map. Indeed, for every $G$-quotient $Y$ and a cover map $f : X \to Y$, there exists unique $\Gamma \leq_X G$ and isomorphism $s : \Gamma \backslash X \to Y$ such that $s(\Gamma v) = f(v)$ for all $v \in X_{\text{vert}}$ (cf. [51]). We may therefore define the spectrum of $(Y, f)$ to be the spectrum of $\Gamma \backslash X$.

5. Optimal Spectrum

Recall that a connected $k$-regular graph $X$ is called Ramanujan if the spectrum of its adjacency operator, denoted $\text{Spec}_0(X)$, is contained in $[-2\sqrt{k-1}, 2\sqrt{k-1}] \cup \{\pm k\}$. As explained in the introduction, the definition is motivated by the Alon-Boppana Theorem ([54], see also [55, Pr. 4.2]), stating that for any $\varepsilon > 0$, only finitely many isomorphism classes of $k$-regular graphs $X$ satisfy the tighter bound $\text{Spec}_0(X) \subseteq [-2\sqrt{k-1} + \varepsilon, 2\sqrt{k-1} - \varepsilon] \cup \{\pm k\}$. Ramanujan graphs can therefore be considered as having optimal, or smallest possible, spectrum. Alternatively, Ramanujan graphs can be regarded as spectral approximations of their universal cover, the $k$-regular tree $\mathcal{T}_k$, since $\text{Spec}_0(\mathcal{T}_k) = [-2\sqrt{k-1}, 2\sqrt{k-1}]$ (see [76] p. 252, Apx. 3; this result seems to go back as far as [58, Th. 3]).
The Alon-Boppana Theorem has been extended from \(k\)-regular graphs to further simplicial structures, particular for non-regular graphs \([23]\) and quotients of the building \(B_\delta(F)\) of Chapter 4 (14, Th. 4.1), quoted as Theorem 5.2 above. The general flavor is always that the spectrum of the universal cover is the smallest possible in a certain sense which depends on the exact statement.

We remind the reader that reason for wanting spectrum “as small as possible” is because “small” spectrum usually manifests in good combinatorial properties; see 47 (for graphs) and 23, 69, 28, 27 (for simplicial complexes), for instance.

In this chapter, we prove a theorem in the spirit of the Alon-Boppana Theorem for \(G\)-quotients of an almost transitive \(G\)-complex \(X\) (see 3C). Our theory applies to spectrum taken with respect to any semi-elementary functor \(F\) (see 11). In particular, it applies to the high dimensional spectra introduced at the end of 4C. When \(G = \text{PGL}_d(F)\) and \(X = B_\delta(F)\) as in Chapter 4 and one considers the 0-dimensional spectrum, our theorem is just Li’s Theorem (Theorem 1.3 above). The proof relies on results from Chapter 2 the main novelty is treating spectrum points corresponding to irreducible representations of dimension greater than 1.

With a notion of an optimal \(F\)-spectrum at hand, we proceed with identifying the trivial \(F\)-spectrum of \(G\)-quotients of \(X\). This leads to a notion of \(G\)-quotients of \(X\) which are Ramanujan with respect to \(F\). Taking \(F\) to be \(\Omega^+\) (see 3B), we arrive at the notion \(G\)-quotients of \(X\) which are Ramanujan in dimension \(i\). In case \(X = T_k\) and \(G = \text{Aut}(T_k)\) as above, or \(X = B_\delta(F)\) and \(G = \text{PGL}_d(F)\) as in Chapter 4 Ramanujan in dimension 0 coincides with the usual meaning of Ramanujan in the literature (see 47, 51 for instance).

5A. Lower Bounds on The Spectrum. Throughout, \(X\) is an almost transitive \(G\)-complex \(\{X\}\) and \(\mathcal{E} = \mathcal{E}(G, X)\) (Definition 4.6).

**Theorem 5.1.** Let \(F : \mathcal{E} \to \mathcal{P}hil\) be a semi-elementary functor (e.g. \(\Omega^+, \Omega^-, \Omega^\pm\) or \(\Omega_{\text{Flag}}\); cf. 22), and let \(A\) be an algebra of \((\mathcal{E}, F)\)-operators (cf. 14). Let \(\{\Gamma_\alpha\}_{\alpha \in I}\) be a family of subgroups of \(G\) such that \(\Gamma_\alpha \leq_K G\) for all \(\alpha \in I\), and one of the following conditions, which are equivalent, is satisfied:

1. For every compact \(C \subseteq G\) with \(1 \in C\), there exist \(\alpha \in I\) and \(g \in G\) such that such that \(C \cap g^{-1} \Gamma_\alpha g = 1\).

2. For every \(n \in \mathbb{N}\), there exists \(\alpha \in I\) and \(g \in G\) such that quotient map \(\nabla : X \to \Gamma_\alpha \backslash X\) is injective on the combinatorial ball \(B_X(v, n)\) (see 3A).

Then \[\bigcup_{\alpha} \text{Spec}_A(\Gamma_\alpha \backslash X) \supseteq \text{Spec}_A(X)\].

We first prove the following lemma.

**Lemma 5.2.** Suppose \(\{\Gamma_\alpha\}_{\alpha \in I}\) satisfies condition (1) of Theorem 5.1, and let \(S : \mathcal{E} \to \mathcal{Set}\) be a functor satisfying conditions (E1)–(E4) of Definition 4.11. Then for any finite subset \(T \subseteq SX\), there exist \(\alpha \in I\) and \(g \in G\) such that \(S_{\Gamma_\alpha} : SX \to S(\Gamma_\alpha \backslash X)\) is injective on \(gT\).

**Proof.** We may assume \(T \neq \emptyset\). Let \(C = \bigcup_{x,y \in T} \{g \in G : gx = y\}\). Since \(T\) is finite and stabilizers of elements in \(SX\) are compact, \(C\) is compact, and 1 \(\in C\) since \(T \neq \emptyset\). Therefore, there is \(\alpha \in I\) and \(g \in G\) such that \(g^{-1} \Gamma_\alpha g \cap C = 1\). Now, if \(x, y \in T\) and \(\gamma \in \Gamma_\alpha\) satisfy \(\gamma x = y\), then \(g^{-1} \gamma g \in g^{-1} \Gamma_\alpha g \cap C = 1\), so \(gx = gy\).

This proves that the quotient map \(SX \to \Gamma_\alpha \backslash SX\) is injective on \(gT\). \(\square\)

**Proof of Theorem 5.1** We first prove the equivalence of (1) and (2). If (1) holds, then (2) follows by applying Lemma 5.2 with \(S : \mathcal{Sim} \to \mathcal{Set}\) being the forgetful functor and \(T = B_X(v, n)\). Suppose now that (2) holds, and let \(v_1, \ldots, v_t\) be
representatives for \( G \setminus \mathcal{X}_{\text{ct}} \) (which is finite since \( \mathcal{X} \) is almost \( G \)-transitive). Let \( C \subseteq G \) be compact with \( 1 \in C \). Since \( G \) acts continuously on \( \mathcal{X}_{\text{ct}} \) (when given the discrete topology), \( C \cdot v_i \) is finite for all \( i \). Choose \( n \in \mathbb{N} \) such that \( C \cdot v_i \subseteq B_X(v_i, n) \) for all \( 1 \leq i \leq t \). There exists \( a \in I \) and \( u \in \mathcal{X}_{\text{ct}} \) such that the quotient map \( \mathcal{X} \to \Gamma_a \setminus \mathcal{X} \) is injective on \( B_X(u, n) \). By construction, there is \( g \in G \) and \( 1 \leq i \leq t \) such that \( u = gv_i \). Now, if \( h \in g^{-1} \Gamma_a \cap C \), then \( \Gamma_a g v_i = \Gamma_a g h v_i \) and \( g v_i, g h v_i \in g B_X(v_i, n) = B_X(u, n) \), hence \( g v_i = g h v_i \). It follows that \( h \in \text{Stab}_C(v_i) \cap g^{-1} \Gamma_a g \), so \( h = 1 \) by Proposition \ref{prop:Stab} and Corollary \ref{cor:Stab}

We now prove the main statement. Let \( F' : \mathcal{C} \to \text{phil} \) be a functor such that \( F \oplus F' \) is elementary. By Remark \ref{remark:elementary}, we may view \( A \) as an algebra of \( (F \oplus F', \mathcal{C}) \)-operators. We may therefore assume that \( F \) is elementary, and write \( F = \tilde{f}_2 \circ S \) where \( S \) satisfies conditions (E1)-(E4) in Definition \ref{def:elementary}.

Let \( [V] \in \text{Spec}_A(\mathcal{X}) \). We need to show that \([V]\) belongs to \( \bigcup_{\alpha} \text{Spec}_A(\mathcal{X}_\alpha) \). Corollary \ref{cor:Spec}(ii) implies that \( A \) acts continuously on \( \bigoplus_{\alpha \in I} F(\Gamma_a \setminus \mathcal{X}) \). Therefore, by Theorem \ref{thm:Spec},

\[
\text{Spec}_A\left( \bigoplus_{\alpha \in I} AF(\Gamma_a \setminus \mathcal{X}) \right) = \bigcup_{\alpha} \text{Spec}_A(AF(\Gamma_a \setminus \mathcal{X})) ,
\]

so we only need to show that \( V < \bigoplus_{\alpha \in I} AF(\Gamma_a \setminus \mathcal{X}) \). Let \( v \in S^1(V) \), \( \varepsilon > 0 \) and \( Y \subseteq Y \). Then there is \( \psi \in AFX = AF(\mathcal{X}) \) such that \( |\langle av, v \rangle - \langle av, \psi \rangle| < \varepsilon \) for all \( a \in Y \). Let \( T = \left( \bigcup_{v \in Y} \text{supp}(a v) \right) \cup \text{supp}(\psi) \). By Lemma \ref{lem:Star} there is \( \alpha \in I \) such that \( \text{Star}_\alpha \setminus T \) is injective. Write \( p = pr_\alpha \). Then \(( Fp)(a v) = (a Fp)(v) \). Therefore, \( |\langle av, v \rangle - a(\langle Fp)v \rangle| = |\langle av, v \rangle - \langle av, \psi \rangle| < \varepsilon \) for all \( a \in Y \). Viewing \((Fp)v\) as an element of \( \bigoplus_{\alpha} AF(\Gamma_a \setminus \mathcal{X}) \), we see that \( V < \bigoplus_{\alpha} F(\Gamma_a \setminus \mathcal{X}) \), as required.

\begin{example}
Let \( \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \ldots \) be subgroups of \( G \) satisfying \( \bigcap_n \Gamma_n = 1 \) and \( \Gamma_1 \leq_X G \). Then Theorem \ref{thm:Spec} applies to \( \{\Gamma_n\}_{n \in \mathbb{N}} \). Indeed, by Corollary \ref{cor:Spec}, \( \Gamma_n \leq_X G \) for all \( n \in \mathbb{N} \), and \( \Gamma_1 \) is discrete by Proposition \ref{prop:discrete}. We now show condition (1): Since \( \Gamma_1 \) is discrete and \( C \) is an \( \ell \)-group, there is \( K \leq \text{c.o.} G \) such that \( K \cap \Gamma_1 = 1 \). Let \( C \subseteq G \) be a compact set with \( 1_C \in C \). We may assume \( C \subseteq KT_1 \) without loss of generality. Since \( K \cap \Gamma_1 = 1 \), every \( e \in C \) can be written as \( k \gamma \) for unique \( k \in K \) and \( \gamma \in \Gamma \). In addition, the compactness of \( C \) implies that there are \( \gamma_1, \ldots, \gamma_r \in \Gamma - \{1\} \) such that \( C = K \subseteq \gamma_1 K \cup \cdots \cup \gamma_r K \). Choosing \( n \) such that \( \gamma_1, \ldots, \gamma_r \not\in \Gamma_n \), we get \( \Gamma_n \cap C \subseteq \Gamma_1 \cap K = 1 \).

\begin{corollary}
Let \( G, X, \mathcal{C}, F \) and \( \{\Gamma_n\}_{n \in \mathbb{N}} \) be as in Theorem \ref{thm:Spec} and suppose \( a_i, a_j \in A(\mathcal{C}, F) \) satisfy \( a_i a_j = a_j a_i \) and \( a_i a_j^* = a_j^* a_i \) for all \( i, j \). Write \( V_n = F(\Gamma_n \setminus X) \) and \( V = FX \). Then

\[
\bigcup_{\alpha} \text{Spec}(a_1 | V_\alpha, \ldots, a_t | V_\alpha) \supseteq \text{Spec}(a_1 | V, \ldots, a_t | V) .
\]

\end{corollary}

\begin{proof}
Let \( A \) be the unital \(*\)-subalgebra of \( A(\mathcal{C}, F) \) generated by \( a_1, \ldots, a_t \). Then \( A \) is commutative. The corollary therefore follows from Proposition \ref{prop:Spec} and Theorem \ref{thm:Spec}.
\end{proof}

When \( \mathcal{X} = B_\varepsilon(F), G = \text{PGL}_d(F) \) and \( F = \Omega^d_\varepsilon \), Corollary \ref{cor:Spec} is a result of Li \cite[Th. 4.1]{Li} (quoted as Theorem \ref{thm:Li} above); this follows from Example \ref{example:Li}

The argument in the proof of Theorem \ref{thm:Spec} can be applied in a broader setting. For example, one can similarly prove:
Theorem 5.5. Let $\mathcal{C} \subseteq \text{Sim}$ be a subcategory, let $F \in \{\Omega^+_i, \Omega^-_i, \Omega^+_\text{Flag}, \Omega^-_\text{Flag} \mid i \in \mathbb{N} \cup \{0\}\}$, and let $A$ be an algebra of $(\mathcal{C}, F)$-operators. Let $\{X_\alpha\}_{\alpha \in I} \subseteq \mathcal{C}$, $X \in \mathcal{C}$ and assume that:

1. $A$ acts continuously on $FX$ and $\bigoplus_{\alpha} FX_\alpha$.
2. For any finite set $T \subseteq X_{\text{vert}}$, there exists $\alpha \in I$ and a morphism $p : X \to X_\alpha$ in $\mathcal{C}$ such that $p|_T$ is injective.

Then $\bigcup_{\alpha} \text{Spec}_A(X_\alpha) \supseteq \text{Spec}_A(X)$.

5B. Trivial Spectrum. Let $\mathcal{X}$ be an almost transitive $G$-complex, let $\mathcal{C} = \mathcal{C}(G, \mathcal{X})$, let $F : \mathcal{C} \to \phi\text{hil}$ be semi-elementary (Definition 4.11), and let $A$ be an algebra of $(\mathcal{C}, F)$-operators. It of interest to construct an infinite family of non-isomorphic $G$-quotients $\{\Gamma_n \setminus \mathcal{X}\}_{n \in \mathbb{N}}$ such that $\bigcup_{n} \text{Spec}_A(\Gamma_n \setminus \mathcal{X})$ is as small as possible. Theorem 5.5 implies that under mild assumptions, this set must contain at least $\text{Spec}_A(X)$. However, if we consider finite $G$-quotients, then this is not the only limitation. The A-spectrum of a finite $G$-quotient must contain at least some points from a family called the trivial A-spectrum, which we now describe.

Let $N \leq G$ be a finite-index open subgroup, and let $\Gamma \leq X G$ be such that $\Gamma \leq N$. Then $(F\mathcal{X})_N = \{\varphi - g\varphi \mid \varphi \in FX, g \in N\}$ is an $A$-submodule of $FX$, and hence $(F\mathcal{X})_N = FX/\langle FX\rangle(N)$ has a structure of a left $A$-module.

Lemma 5.6. There is an inner product on $A(F\mathcal{X})_N = (AF\mathcal{X})_N$ such that $A(F\mathcal{X})_N$ becomes a unitary representation of $A$. The unitary isomorphism class of $A(F\mathcal{X})_N$ is independent of the inner product.

Proof. Choose $F'$ such that $F \oplus F'$ is elementary. By Remark 4.11 we may replace $F$ with $F \oplus F'$ and assume $F$ is elementary. Write $F = \ell^2 \circ S$ as in Definition 4.11. By Theorem 4.14(i), we may identify $A$ with an involutive subalgebra of $\text{End}_G(FX)$. The proof of this theorem implies that for any $H \leq G$, one can endow $\ell^2(H \setminus SX)$ with a left $A$-module structure given by $a \cdot e_H = \sum_{y \in SX} \alpha_y e_H y$ where $\{\alpha_y\}_y \subseteq C$ are determined by $a \cdot e = \sum \alpha_y e_y$. Moreover, $\ell^2(H \setminus SX)$ is a pre-unitary representation of $A$. Now, as in the proof of Proposition 4.14(ii), there is an isomorphism $(F\mathcal{X})_N \cong \ell^2(N \setminus SX)$ given by linearly extending $e_x + (FX)(N) \mapsto e_{Nx}$, and it is straightforward to check that this isomorphism is a homomorphism of $A$-modules. This shows the existence of an inner product on $A(F\mathcal{X})_N$ such that $A(F\mathcal{X})_N \in \text{Rep}^u(A)$. Since $[G : N]$ and $G \setminus SX$ are finite, $\dim(F\mathcal{X})_N = \dim \ell^2(N \setminus SX) < \infty$, and hence $A(F\mathcal{X})_N \in \text{Rep}^u(A)$. The unitary isomorphism class of $A(F\mathcal{X})_N$ is uniquely determined by Proposition 2.8.

Write $\Sigma_{A,N} = \text{Spec}_A(A(F\mathcal{X})_N)$.

Proposition 5.7. Let $\Gamma \setminus \mathcal{X}$ be a finite $G$-quotient of $\mathcal{X}$ such that $\Gamma \leq N$. Then $\Sigma_{A,N} \subseteq \text{Spec}_A(\Gamma \setminus \mathcal{X})$. Furthermore, if $\Gamma' \leq N$ is open of finite index, then $\Sigma_{A,N} \subseteq \Sigma_{A,N'}$.

Proof. Identify $F(\Gamma \setminus \mathcal{X})$ with $(F\mathcal{X})_\Gamma$ using Proposition 4.14(ii), and consider the surjective $A$-homomorphism $P : A(F\mathcal{X})_\Gamma \to A(F\mathcal{X})_N$ given by $P(\varphi + (F\mathcal{X})(\Gamma)) = \varphi + (F\mathcal{X})(N)$. Since $\Gamma \setminus \mathcal{X}$ is finite and $F$ is semi-elementary, $A(F\mathcal{X})_\Gamma$ is finite dimensional, and hence $\ker(P)^\perp \cong A(F\mathcal{X})_N$ as $A$-modules. By Proposition 2.8, $\ker(P)^\perp \cong A(F\mathcal{X})_N$ as unitary representations. Since $A \Gamma \setminus \mathcal{X} \cong \ker(P \oplus (\ker P)^\perp$, it follows that $\Sigma_{A,N} = \text{Spec}_A((\ker P)^\perp) \subseteq \text{Spec}_A(A(F(\Gamma \setminus \mathcal{X})) = \text{Spec}_A(\Gamma \setminus \mathcal{X})$.

The second part of the proposition is shown similarly.

We shall prove a partial converse to Proposition 5.7 in Chapter 6 (Proposition 6.34).
For any finite $G$-quotient $\Gamma \backslash \mathcal{X}$, we have $\Sigma_{A,G} \subseteq \text{Spec}_A(\Gamma \backslash \mathcal{X})$.

Proposition 5.7 motivates the following definition.

**Definition 5.9.** The trivial $A$-spectrum is the set $\Sigma_A = \bigcup_N \Sigma_{A,N}$, where $N$ ranges over all finite index open subgroups of $G$.

**Remark 5.10.** Assume $a_1, \ldots, a_t \in A(\mathcal{G}, F)$ satisfy $a_ia_j = a_ja_i$ and $a_ia_j = a_ja_i$ for all $i, j$. Then one can set $\Sigma_{a_1, \ldots, a_t} = \Sigma(at_0 | A(\mathcal{F})_{X})$ and define the trivial common spectrum of $(a_1, \ldots, a_t)$ to be $\Sigma_{a_1, \ldots, a_t} := \bigcap_N \Sigma_{a_1, \ldots, a_t,N}$. The unital $\ast$-subalgebra $A$ of $A(\mathcal{G}, F)$ spanned by $a_1, \ldots, a_t$ is commutative, and if one identifies the $A$-spectrum with the common spectrum of $a_1, \ldots, a_t$ as in Proposition 2.23, then $\Sigma_A$ corresponds to $\Sigma_{a_1, \ldots, a_t}$.

**Remark 5.11.** We actually did not use the fact that $N$ is open anywhere. However, restricting to open subgroups makes no difference, since for any finite-index subgroup $N \leq G$, we have $(FX)_N = (FX_N)$, where $N$ is the closure of $N$ in $G$. It is enough to check this when $F$ is elementary, in which case the claim easily boils down to showing that $N'\mathcal{X} = N\mathcal{X}$, where $F \cong \mathcal{F} \circ S$ as in Definition 4.11.

It is well-known that $\overline{\mathcal{N}} = \bigcap_U N_U$ where $U$ ranges over all neighborhoods of $1_G$. Since stabilizers of elements of $\mathcal{X}$ are open, every $x \in \mathcal{X}$ is stabilized by some neighborhood $U$ of $1_G$, and thus $N'\mathcal{X} = \overline{N}\mathcal{X}$.

**Example 5.12.** Let $\mathcal{X}$ be a $k$-regular tree, let $G = \text{Aut}(\mathcal{X})$, let $\mathcal{G} = \mathcal{G}(G, \mathcal{X})$, and let $a_0 \in A_0 := A(\mathcal{G}, \Omega^+_0)$ be the vertex adjacency operator. Using Example 4.8, we identify the 0-dimensional spectrum with the spectrum of $a_0$. It is well-known (79) or (69) that $G$ has only one proper finite-index subgroup, denoted $H$, which is of index 2 and consists of the elements preserving the canonical 2-coloring of $\mathcal{X}$. The trivial spectrum $\Sigma_{a_0}$ is therefore the spectrum of $a_0$ on $\Omega^+_0(X)_H \cong \hat{\mathbb{Z}}(H, \mathcal{X}(0))$.

Let $u$ and $v$ be representatives for the $H$-orbits of $\mathcal{X}(0)$. Then $\{e_{uH}, e_{vH}\}$ is a basis of $\hat{\mathbb{Z}}(H, \mathcal{X}(0))$ and the action of $a_0$ with respect to this basis is easily seen to be given by $\begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$ (one can regard $H \backslash \mathcal{X}$ as a graph with two vertices connected by $k$ edges). Thus, $\Sigma_{a_0} = \text{Spec}(a_0|_{H\backslash \mathcal{X}}) = \{ \pm k \}$, and we have recovered the usual definition of trivial spectrum for $k$-regular graphs.

If one identifies the 1-dimensional oriented spectrum with the spectrum of the edge adjacency operator $a_1 \in A_1 := A(\mathcal{G}, \Omega^+_1)$ as in Example 4.13, then one similarly finds that $\Sigma_{A_1} = \{ 2k - 2 \}$.

**Example 5.13.** Let $G = \text{PGL}_d(F)$ and $\mathcal{X} = \mathcal{B}_d(F)$ be as in Chapter 1 and let $\mathcal{G} = \mathcal{G}(G, \mathcal{X})$. By Example 4.9, $A_0 := A(\mathcal{G}, \Omega^+_0) = C[a_1, \ldots, a_{d-1}]$, and hence we can identify the 0-dimensional spectrum with the common spectrum of $a_1, \ldots, a_{d-1} \in A_0$. It is well-known that any normal subgroup of $\text{GL}_d(F)$ either contains $\text{SL}_d(F)$ or is contained in the center of $\text{GL}_d(F)$. Since all finite index subgroups contain a normal finite index subgroup, any finite index subgroup of $G = \text{PGL}_d(F)$ contains $N := \text{im}(\text{SL}_d(F) \to \text{PGL}_d(F))$. The trivial spectrum $\Sigma_{A_0}$ is therefore the common spectrum of $a_0, \ldots, a_{d-1}$ on $\Omega^+_0(\mathcal{B}_d(F))_N \cong \ell^2(N, \mathcal{B}_d(F)(0)) \cong \ell^2(\mathcal{B}_d(F)(0))$.

By Example 3.4, $\mathcal{N} \backslash \mathcal{B}_d(F)$ is a simplex of dimension $d - 1$. In fact, the vertex coloring $C_0 : \mathcal{B}_d(F) \to \mathbb{Z}/d\mathbb{Z}$ descends to $\mathcal{N} \backslash \mathcal{B}_d(F)$, so we can write $\mathcal{N} \backslash \mathcal{B}_d(F)_{\text{vert}} = \{v_0, \ldots, v_{d-1}\}$ with $C_0(v_0) = i + d\mathbb{Z}$. Let $e_i := e_{v_i} \in \ell^2(\mathcal{N} \backslash \mathcal{B}_d(F)(0)) \cong \Omega^+_0(\mathcal{N} \backslash \mathcal{B}_d(F))$. An easy computation shows that $a_0 e_j = \alpha_i e_{j+i}$, where $\alpha_i$ is the number of vertices in $\mathcal{B}_d(F)$ with color $\{j + i\} + d\mathbb{Z}$ which are adjacent to some vertex of color $j + d\mathbb{Z}$ (this is independent of the vertex and $j$ since $G$ acts transitively on $\mathcal{B}_d(F)(0)$). Let $q$ denote the cardinality of the residue field of $(F, \nu)$. It is well-known that $\alpha_i$ equals $\binom{i}{d-1}$, the number of $\mathbb{F}_q$-subspaces of dimension $i$ in
\( \mathbb{F}_q^n \) (see the description of \( \mathcal{B}_d(F) \) through \( \mathcal{O} \)-lattices in [51, §2.1], for instance). Let \( \zeta \in \mathbb{C} \) be a primitive \( d \)-th root of unity. Then
\[
\{ \varphi_k := \zeta^0 e_0 + \zeta^k e_1 + \zeta^{2k} e_2 + \cdots + \zeta^{(d-1)k} e_{d-1} \mid 0 \leq k < d \}
\]
is a basis of \( \Omega^0_+((N \setminus \mathcal{B}_d(F)) \) consisting of common eigenvectors of \( (a_1, \ldots, a_{d-1}) \). The multi-eigenvalue corresponding to \( \varphi_k \) is
\[
\left( \left( \begin{array}{c} d \\ q \end{array} \right) \zeta^{-k}, \left( \begin{array}{c} d \\ q \end{array} \right) \zeta^{-2k}, \cdots, \left( \begin{array}{c} d \\ q \end{array} \right) \zeta^{-(d-1)k} \right).
\]
Replacing \( \zeta \) with \( \zeta^{-1} \), we get
\[
\mathcal{T}_{A_0} = \left\{ \left( \left( \begin{array}{c} d \\ q \end{array} \right) \zeta^k, \left( \begin{array}{c} d \\ q \end{array} \right) \zeta^{2k}, \cdots, \left( \begin{array}{c} d \\ q \end{array} \right) \zeta^{(d-1)k} \right) \mid 0 \leq k < d \right\}.
\]
Notice that for \( d = 2 \), the complex \( \mathcal{B}_2(F) \) is a \( (q + 1) \)-regular tree, and \( \mathcal{T}_{A_0} = \{ -q - 1, q + 1 \} \), so this agrees with Example 5.12.

According to Lubotzky, Samuels and Vishne [51, §8], (see the description of \( \mathcal{T}_{A_0} \)), and in Example 5.12, we saw that the trivial spectrum is \( A_0 \)-Ramanujan in the classical sense. In fact, we will see in 6F that
\[
\{ \varphi_k \mid \varphi_k \in \mathcal{T}_{A_0} \}
\]
is a basis of \( \Omega^0_+((N \setminus \mathcal{B}_d(F)) \) consisting of common eigenvectors of \( (a_1, \ldots, a_{d-1}) \).

\[ \mathcal{T}_{A_0} \]

\[ \Gamma \setminus \mathcal{X} \]
is an \( \Omega^0_+ \)-Ramanujan in dimension \( 0 \) \( (\text{cf. Example 5.13}) \). Thus, we identify the 0-dimensional spectrum with the spectrum of \( a_0 \) (cf. Example 4.8).

It is well-known that \( \text{Spec}_{A_0}(\mathcal{X}) = \{ -2\sqrt{k - 1}, 2\sqrt{k - 1} \} \) (76, p. 252, Apx. 3), for instance, and in Example 5.12, we saw that the trivial spectrum is \( \mathcal{T}_{A_0} = \{ \pm k \} \). Thus, \( \text{Spec}_{A_0}(\mathcal{X}) \cup \mathcal{T}_{A_0} = \{ -2\sqrt{k - 1}, 2\sqrt{k - 1} \} \cup \{ \pm k \} \), and it follows that a \( k \)-regular graph is Ramanujan in dimension 0 (or \( \Omega^0_+ \)-Ramanujan) if and only if it is Ramanujan in the classical sense. In fact, we will see in 6F that \( k \)-regular graphs are Ramanujan in dimension 0 if and only if they are completely Ramanujan.

(\text{ii}) Let \( \Gamma = \text{PGL}_d(F) \) and \( \mathcal{X} = \mathcal{B}_d(F) \) be as in Chapter 11. We identify the 0-dimensional spectrum with the common spectrum of \( a_1, \ldots, a_{d-1} \in A_0 := A(\mathcal{O}(G, X), \Omega^0_+) \) as explained in Example 4.9. According Lubotzky, Samuels and Vishne [51, Def. 2.6], a \( G \)-quotient \( \Gamma \setminus \mathcal{B}_d(F) \) is called Ramanujan if \( \text{Spec}_{A_0}(\Gamma \setminus \mathcal{B}_d(F)) \) is contained in the union of \( \text{Spec}_{A_0}(\mathcal{B}_d(F)) \) and \( \mathcal{T}_{A_0} \) (cf. Example 5.13). Thus,
$\Gamma \backslash B_2(F)$ is Ramanujan in dimension 0 if and only if it is Ramanujan in the sense of [51].

We shall see in Chapter 4 that the Ramanujan quotients constructed in [51] are in fact completely Ramanujan. We will also see in [61] that when $d = 3$, a $G$-quotient $\Gamma \backslash B_2(F)$ is Ramanujan in dimension 0 if and only if it is flag Ramanujan. The following proposition shows that A-Ramanujan property behaves well as $A$ and $\Gamma$ vary.

**Proposition 5.16.** Let $F, F' : \mathcal{C} \rightarrow \mathfrak{hil}$ be semi-elementary functors, let $A$ be an algebra of $(\mathcal{C}, F)$-operators, and let $\Gamma' \leq \Gamma \leq \mathcal{X}$. Then:

(i) $\Gamma' \backslash \mathcal{X}$ is $F \otimes F'$-Ramanujan if and only if $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan and $F'$-Ramanujan.

(ii) If $\Gamma' \backslash \mathcal{X}$ is $A$-Ramanujan, then $\Gamma \backslash \mathcal{X}$ is $B$-Ramanujan for any idempotented $*$-subalgebra $B \subseteq A$.

(iii) If $[\Gamma : \Gamma'] < \infty$ and $\Gamma' \backslash \mathcal{X}$ is $A$-Ramanujan, then $\Gamma \backslash \mathcal{X}$ is $A$-Ramanujan.

**Proof.** (i) We use the notation of Proposition 4.23 and the discussion preceding it. Everything follows from this proposition if we show that $\Sigma_{A \oplus B'} = \Sigma_{A'} \cup \Sigma_{A''}$. Let $N \leq \Gamma$ be open of finite index. We have $e_F((F \otimes F')\mathcal{X})_N = (F'\mathcal{X})_N$ and $e_F((F \otimes F')\mathcal{X})_N = (F\mathcal{X})_N$, so, as in the proof of Proposition 4.23 Corollary 2.47 implies that $\Sigma_{A \oplus B'} \subseteq \Sigma_{A'} \cup \Sigma_{A''}$. Taking union over all possible $N$-s, we get $\Sigma_{A \oplus B'} = \Sigma_{A} \cup \Sigma_{A''}$.

(ii) This follows from the definition and from Theorem 2.45.

(iii) This is immediate from Proposition 4.22. □

**Remark 5.17.** At this level of generality, finite $A$-Ramanujan $G$-quotients are not guaranteed to exist; this follows implicitly from [10]. Instead, one may consider $A$-Ramanujan covers: Let $\Gamma' \leq \Gamma \leq \mathcal{X}$ be subgroups such that $\Gamma' \backslash \mathcal{X}$ and $\Gamma \backslash \mathcal{X}$ are finite, and let $p = p_{\Gamma', \Gamma}$ denote the cover map $\Gamma'x \mapsto \Gamma x : \Gamma' \backslash \mathcal{X} \rightarrow \Gamma \backslash \mathcal{X}$. Then, as in the proof of Proposition 4.22, $Fp : F(\Gamma' \backslash \mathcal{X}) \rightarrow F(\Gamma \backslash \mathcal{X})$ is continuous and surjective, and $F(\Gamma' \backslash \mathcal{X}) \cong \ker Fp \oplus F(\Gamma \backslash \mathcal{X})$ as unitary representations of $A$.

In particular, Spec$_A(\Gamma' \backslash \mathcal{X}) \subseteq$ Spec$_A(\Gamma \backslash \mathcal{X})$. We call $\Gamma' \backslash \mathcal{X}$ an $A$-Ramanujan cover of $\Gamma \backslash \mathcal{X}$ if all the $A$-spectrum points of $\Gamma' \backslash \mathcal{X}$ that do not come from $\Gamma \backslash \mathcal{X}$ are in Spec$_A(\mathcal{X})$, namely, if

$$\text{Spec}_A(\ker Fp_{\Gamma', \Gamma}) \subseteq \text{Spec}_A(\mathcal{X}).$$

One can likewise define covers which are $F$-Ramanujan, Ramanujan in dimension $i$, flag Ramanujan, and completely Ramanujan.

It is more reasonable to believe that under certain assumptions, all finite $G$-quotients of $\mathcal{X}$ will have an $A$-Ramanujan cover. Indeed, Marcus, Spielman and Srivastava [56] showed that any bipartite $k$-regular graph admits a $\mathbb{C}[a_0]$-Ramanujan 2-cover ($a_0$ is the vertex adjacency operator; cf. Example 4.13). This was extended to covers of any prescribed rank by Hall, Puder and Sawin [31].

6. Representation Theory

Let $G$ be a unimodular $\ell$-group, let $\mathcal{X}$ be an almost transitive $G$-complex [61], let $F : \mathcal{C}(G, \mathcal{X}) \rightarrow \mathfrak{hil}$ be an elementary functor [10], and let $\Gamma \leq \mathcal{X}$. $G$. In this
chapter, we give a criterion to determine when $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan [5C] which is phrased in terms of the unitary $G$-representation $L^2(\Gamma \backslash G)$.  

Note that by Proposition 3.13, $G$ is unimodular whenever $\mathcal{X}$ has a finite $G$-quotient. Therefore, the assumption that $G$ is unimodular is not unreasonable.

Throughout, $G$ is a unimodular $\ell$-group (i.e. a totally disconnected locally compact group) and $\mu$ is a Haar measure on $G$. A (left) $G$-module is a $\mathbb{C}$-vector space on which $G$ acts (on the left) via $\mathbb{C}$-linear automorphisms. We write $K \leq_{c.o.} G$ to denote that $K$ is a compact open subgroup of $G$. It is well-known that the collection $\{ K : K \leq_{c.o.} G \}$ is basis of neighborhoods of $1_G$. Many computational facts recalled in this chapter can be easily verified using the following identities, which hold whenever the integrals make sense:

$$
\int_{x \in G} \psi(gx) \, d\mu = \int_{x \in G} \psi(xg) = \int_{x \in G} \psi(x^{-1}) \, d\mu = \int_{x \in G} \psi(x) \, d\mu
$$

The characteristic function of $S \subseteq G$ is denoted by $1_S$.

6A. **Unitary and Pre-Unitary Representations.** As usual, a unitary representation of $G$ is a Hilbert space $V$ carrying a $G$-module structure such that the action $G \times V \rightarrow V$ is continuous and $\langle gu, gv \rangle = \langle u, v \rangle$ for all $u, v \in V$, $g \in G$. The representation $V$ is irreducible if $V$ does not contain closed $G$-submodules other than 0 and $V$. The category of unitary representations of $G$ with continuous $G$-homomorphisms is denoted Rep$^u(G)$, and the class of irreducible representations is denoted Irr$^u(G)$.

**Example 6.1.** The right regular representation of $G$ is $L^2(G)$ endowed with the left $G$-action given by

$$(g \varphi)x = \varphi(xg) \quad \forall g, x \in G, \varphi \in L^2(G).$$

The left regular representation of $G$ is $L^2(G)$ endowed with the left $G$-action given by $(g \varphi)x = \varphi(g^{-1}x)$.

To avoid ambiguity between the right and left regular representations, we shall denote them by $L^2(1 \backslash G)$ and $L^2(G/1)$, respectively.

**Example 6.2.** Let $\Gamma$ be a discrete subgroup of $G$. It is well-known that there exists a unique measure $\mu_{\Gamma \backslash G}$ on $\Gamma \backslash G$ such that for all $\varphi \in C^\infty_c(\Gamma \backslash G)$, one has

$$
\int_{x \in \Gamma \backslash G} \varphi(x) \, d\mu = \int_{x \in \Gamma \backslash G} \hat{\varphi}(\Gamma x) \, d\mu_{\Gamma \backslash G},
$$

where $\hat{\varphi} \in C^\infty_c(\Gamma \backslash G)$ is defined by $\hat{\varphi}(\Gamma x) = \sum_{\gamma \in \Gamma} \varphi(\gamma x)$. We make $L^2(\Gamma \backslash G)$ into a unitary representation by setting $(g \varphi)x = \varphi(xg)$ for all $g, x \in G$ and $\varphi \in L^2(\Gamma \backslash G)$.

Let $K \leq G$. For any $G$-module $V$, let

$$
V^K := \{ v \in V : kv = v \text{ for all } k \in K \}.
$$

Recall that $V$ is smooth if $V = \bigcup_{K \leq_{c.o.} G} V^K$. This is equivalent to saying that the action $G \times V \rightarrow V$ is continuous when $V$ is endowed with the discrete topology.

A pre-unitary representation of $G$ is a pre-Hilbert space $V$ carrying a $G$-module structure such that $V$ is smooth and $\langle gu, gv \rangle = \langle u, v \rangle$ for all $u, v \in V$, $g \in G$. The category of pre-unitary representations with continuous $G$-homomorphisms is denoted by Rep$^{pu}(G)$. A pre-unitary representation is irreducible if it is irreducible as a $G$-module and the class of irreducible pre-unitary representation is denoted Irr$^{pu}(G)$.

Every unitary representation $U \in \text{Rep}^u(G)$ contains a maximal pre-unitary subrepresentation $U_{sm} := \bigcup_{K \leq_{c.o.} G} U^K$. It is well-known that $U_{sm}$ is dense in $U$. Conversely, if $V$ is a pre-unitary representation, then the action of $G$ extends uniquely to
a continuous action on the completion $\overline{V}$, which becomes a unitary representation. We always have $(\overline{V}_{\text{sm}}) = V$, but in general $(\overline{V})_{\text{sm}}$ may be larger than $V$.

Finally, recall that a $G$-module $V$ is admissible if $\dim V^K < \infty$ for all $K \leq c.o. \ G$.

**Example 6.3.** Let $\Gamma$ be a cocompact discrete subgroup of $G$. Then $L^2(\Gamma \backslash G)$ is admissible. Indeed, for all $K \leq c.o. \ G$, we have $L^2(\Gamma \backslash G)^K \cong L^2(\Gamma \backslash G/K)$ and $\Gamma \backslash G/K$ is a compact discrete topological space, so it is finite.

6B. The Hecke Algebra. With every unimodular $\ell$-group $G$ one can associate an idempotented $*$-algebra, $\mathcal{H}(G)$, called the Hecke algebra of $G$. It is constructed as follows: As a vector space, $\mathcal{H}(G)$ is $C^\infty(G)$, the set of compactly supported locally constant functions from $G$ to $\mathbb{C}$. The multiplication in $G$ is the convolution product given by

$$(\psi \ast \varphi)g = \int_{x \in G} \psi(x) \cdot \varphi(x^{-1}g) \, d\mu \quad \forall \psi, \varphi \in \mathcal{H}(G), \ g \in G.$$ 

Note that $\mathcal{H}(G)$ is not unital unless $G$ is discrete. The Hecke algebra admits an involution $\ast : \mathcal{H}(G) \to \mathcal{H}(G)$ given by $\psi^*(g) = \overline{\psi(g^{-1})} \quad \forall \psi \in \mathcal{H}(G), \ g \in G$.

For $K \leq c.o. \ G$, we define

$$e_K := \mu(K)^{-1}I_K \in \mathcal{H}(G),$$

where $I_K$ denotes the characteristic function of $K$. It is easy to check that $e_K$ is an idempotent and $e_K^* = e_K$. The idempotent $e_K$ also has the following properties:

(i) $\psi \in \mathcal{H}(G)$ is left $K$-invariant if and only if $\psi = e_K \ast \psi$.

(ii) $\psi \in \mathcal{H}(G)$ is right $K$-invariant if and only if $\psi = \psi \ast e_K$.

Since the compact open subgroups form a basis of neighborhoods of $1_G$, every function $\psi \in \mathcal{H}(G)$ is bi-$K$-invariant for some $K \leq c.o. \ G$. Thus, $\mathcal{H}(G)$ is an idempotented $*$-algebra [2B]. The subalgebra

$$\mathcal{H}_K(G) := e_K \mathcal{H}(G)e_K = \{ \psi \in \mathcal{H}(G) \mid \psi(kgk') = \psi(g) \text{ for all } k, k' \in K \text{ and } g \in G \}$$

is a unitary $*$-subalgebra of $\mathcal{H}(G)$ called the Hecke algebra of $(G, K)$.

**Remark 6.4.** The convolution product $\varphi \ast \psi$ is in fact defined under the milder assumption that one of $\varphi, \psi$ is compactly supported and the other is measurable.

The algebra $\mathcal{H}(G)$ admits another involution given by

$$\psi^\#(g) = \overline{\psi(g^{-1})} \quad \forall \psi \in \mathcal{H}(G), \ g \in G.$$ 

This involution is in fact $\mathbb{C}$-linear, hence it is not an involution in the sense of [2B].

6C. Representations of $G$ vs. Representations of $\mathcal{H}(G)$. Keep the notation of [2B]. Every $V \in \text{Rep}^\text{u}(G)$ can be made into a unitary representation of $\mathcal{H}(G)$ by defining

$$(6.1) \quad \psi \cdot v := \int_{x \in G} \psi(x) \cdot xv \, d\mu \quad \forall \psi \in \mathcal{H}(G), \ v \in V.$$ 

The integral is defined because the function $x \mapsto \psi(x) \cdot xv$ is continuous and compactly supported. That $V$ is indeed a unitary representation of $\mathcal{H}(G)$ follows by computation.

Conversely, every unitary representation of $\mathcal{H}(G)$ can be made into a unitary representation of $G$ as follows: For $g \in G$ and $\psi \in \mathcal{H}(G)$, define $g \ast \psi \in \mathcal{H}(G)$ by $(g \ast \psi)x = \psi(g^{-1}x)$. Now, for $v \in V$, let

$$(6.2) \quad g \cdot v := \lim \{ (g \ast e_K)v \}_{K \leq c.o. \ G}.$$
Here, \( \{ e_K \}_{K \leq c.o. G} \) is viewed as a subnet of \( \mathcal{I}((\mathcal{H}(G)) \) (see 2D). That \( V \) is indeed a unitary representation of \( G \) is routine.

It is also worth noting that for all \( K \leq c.o. G \) and \( v \in V \), one has \( e_K v = v \) if and only if \( v \in V^K \).

We summarize the previous paragraphs in the following well-known proposition.

**Proposition 6.5.** The maps \( \text{Rep}^u(G) \to \text{Rep}^u(\mathcal{H}(G)) \) and \( \text{Rep}^u(\mathcal{H}(G)) \to \text{Rep}^u(G) \) described above are inverses of each other, and they induce an isomorphism of categories (morphisms are mapped to themselves).

Likewise, there is an isomorphism between \( \text{Rep}^{pu}(G) \) and \( \text{Rep}^{pu}(\mathcal{H}(G)) \), and an isomorphism between the category of smooth \( G \)-modules and the category of smooth \( \mathcal{H}(G) \)-modules. (The isomorphisms are defined using the same formulas used in the unitary case. When \( V \) is smooth, the integral in (6.1) is just a finite sum, and the net in (6.2) is eventually constant.)

The previous discussion implies that some of the results of Chapter 2 also apply to unitary and pre-unitary representations of \( G \) (by taking \( A = \mathcal{H}(G) \)). In the sequel, we will freely apply these results to representations of \( G \).

**Example 6.6.** Consider the right regular representation \( L^2(\Gamma \setminus G) \) of Example 6.1. The induced left \( \mathcal{H}(G) \)-module structure on \( L^2(\Gamma \setminus G) \) is given by

\[
\varphi \cdot \psi = \int_{x \in G} \varphi(x) \cdot (x \cdot \psi) \, d\mu .
\]

Applying both sides to \( g \in G \) yields

\[
(\varphi \cdot \psi)g = \left( \int_{x \in G} \varphi(x) \cdot (x \cdot \psi) \, d\mu \right) g = \int_{x \in G} \varphi(x) \cdot (x \cdot \psi) \, d\mu = \int_{x \in G} \varphi(x) \cdot (x \cdot \psi(gx)) \, d\mu = \int_{x \in G} \psi(gx) \cdot \varphi^\#(x^{-1}) \, d\mu = \int_{x \in G} \psi(x) \cdot \varphi^\#(x^{-1} g) = (\psi \cdot \varphi^\#)(g)
\]

Thus, the action of \( \mathcal{H}(G) \) on \( L^2(\Gamma \setminus G) \) is given by

\[
\varphi \cdot \psi = \varphi \ast \varphi^\#.
\]

Likewise, the induced action of \( \mathcal{H}(G) \) on \( L^2(\Gamma \setminus G) \) (see Example 6.2), is also given by \( \varphi \cdot \psi = \psi \ast \varphi^\# \) (when we view \( \psi \in L^2(\Gamma \setminus G) \) as a function on \( \Gamma \)).

**Example 6.7.** Consider \( L^2(G/1) \), the left regular representation of \( G \) (Example 6.1). Then the action of \( \mathcal{H}(G) \) on \( L^2(G/1) \) is given by

\[
\varphi \cdot \psi = \varphi \ast \psi .
\]

The computation is similar to Example 6.6 and is left to the reader.

**Remark 6.8.** Let \( V \) be a unitary representation of \( G \). We can define \( V_{sm} \) by considering \( V \) as a \( G \)-module (6A) and by considering \( V \) as an \( \mathcal{H}(G) \)-module (2B). However, there is no ambiguity because \( \mathcal{H}(G)V = \bigcup_{K \leq c.o. G} e_K V = \bigcup_{K \leq c.o. G} V^K \).

Likewise, \( V \) is admissible as a representation of \( G \) if and only if it is admissible as a representation of \( \mathcal{H}(G) \) because \( e_K V = V^K \) for all \( K \leq c.o. G \).

**6D. Elementary Functors Revised.** Let \( \mathcal{X} \) be an almost transitive \( G \)-complex \( \mathcal{X} \), let \( \mathcal{C} = \mathcal{C}(G, \mathcal{X}) \) (Definition 4.6), and let \( F : \mathcal{C} \to \text{hil} \) be elementary (e.g. \( \Omega^\mathcal{C}_+, \Omega^\mathcal{C}_-, \Omega^\text{Flag} \), see 1D). We further write \( F = \ell^2 \circ S \) where \( S \) is as in Definition 4.11. In this section, we describe \( A(\mathcal{C}, F) \) and \( F \mathcal{X} (\mathcal{X} \in \mathcal{C}) \) in terms of \( \mathcal{H}(G) \). For brevity, write

\[
\mathcal{H} = \mathcal{H}(G)
\]

We shall freely identify \( S(\Gamma \setminus \mathcal{X}) \) with \( \Gamma \setminus S \mathcal{X} \) using Proposition 4.14(i).
The Ramanujan Property for Simplicial Complexes

We set some general notation: Fix representatives \( x_1, \ldots, x_t \) for the \( G \)-orbits in \( S\mathcal{X} \), and for all \( 1 \leq n \leq t \) let

\[ K_n = \text{Stab}_G(x_n) \quad \text{and} \quad e_n = e_{K_n} \in \mathcal{H} \]

\( (K_n \leq \text{c.o.} \ G \) by condition (E2) in Definition 4.11). Let \( \Gamma \leq \mathcal{X} \ G \). The canonical measure \( \mu_{\Gamma \ \cap} \) on \( \Gamma \ \cap \ G \) (see Example 6.2) induces a measure \( \mu_{\Gamma \ \cap} / \Gamma / K_n \) on the discrete topological space \( \Gamma / G / K_n \) by pushing forward.

**Lemma 6.9.** For all \( g, h \in G \), we have (i) \( \eta_{\Gamma \ \cap} / G / K_n(Th) = \sum_{\alpha \in \Gamma} \eta_{K_n}(\gamma h) \) and (ii) \( \mu_{\Gamma \ \cap} / G / K_n(\{gK_n\}) = \mu_{\Gamma \ \cap} / G / (\Gamma \ \cap) gK_n = \mu(K_n) \).

**Proof.** (i) If \( h \notin \Gamma gK_n \), then \( \eta_{\Gamma \ \cap} / G / K_n(Th) = 0 = \sum_{\gamma \in \Gamma} \eta_{K_n}(\gamma h) \), so assume \( h \in \Gamma gK_n \). By condition (E2) in Definition 4.11 \( K_n \) is contained in the stabilizer of a cell \( y \in \mathcal{X} \). Clearly \( gK_n g^{-1} \subseteq \text{Stab}_G(gy) \), so by Proposition 3.9 \( \Gamma \ \cap gK_n g^{-1} = 1 \). This means that there are unique \( \gamma \in \Gamma \) and \( k \in K_n \) such that \( h = \gamma gk \) (if \( \gamma gk = \gamma' g'k' \), then \( \gamma^{-1} \gamma = gk^{-1} k' g^{-1} \in \Gamma \ \cap gK_n g^{-1} = 1 \), so \( \gamma' = \gamma \) and \( k = k' \)). It follows that \( \eta_{\Gamma \ \cap} / G / K_n(Th) = 1 = \sum_{\gamma \in \Gamma} \eta_{K_n}(\gamma h) \).

(ii) In the notation of Example 6.2 part (i) implies that \( \eta_{\Gamma \ \cap} / G / K_n(\Gamma \ \cap) \). Thus, by the definition of the measure on \( \Gamma / G \), we have

\[ \mu_{\Gamma \ \cap} / G / (\{gK_n\}) = \int_{\Gamma / G} \eta_{\Gamma \ \cap} / G / K_n d\mu_{\Gamma / G} = \int_G \eta_{K_n} d\mu = \mu(K_n) \]  

That \( \mu_{\Gamma \ \cap} / G / K_n(\{gK_n\}) = \mu_{\Gamma \ \cap} / G / (\Gamma \ \cap) gK_n \) is immediate. \( \square \)

Recall from the notation section that \( C_c^\infty(\Gamma \ \cap G / K_n) \) denotes the set of locally constant compactly supported functions from \( \Gamma \ \cap G / K_n \) to \( C \). This a subspace of \( L^2(\Gamma \ \cap G / K_n) \) and equality holds if and only if \( \Gamma \ \cap G / K_n \) is finite. Also notice that \( C_c^\infty(\Gamma \ \cap G / K_n) = \overline{L}^2(\Gamma \ \cap G / K_n) \) as \( \mathbb{C} \)-vectors spaces, but these spaces may differ as pre-Hilbert spaces, because

\[ \langle \varphi, \psi \rangle_{\Gamma \ \cap G / K_n} = \mu(K_n) \langle \varphi, \psi \rangle_{\overline{L}^2(\Gamma \ \cap G / K_n)} \]

thanks to Lemma 6.9. In the sequel, we shall view \( C_c^\infty(\Gamma \ \cap G / K_n) \) as a subspace of \( C_c^\infty(\Gamma / G) \) in obvious way.

Let \( \Gamma' \leq \Gamma \). Then the map \( \rho : \Gamma \ \cap G / K_n \rightarrow \Gamma \ \cap G / K_n \) given by \( \rho(\Gamma' gK_n) = \Gamma gK_n \) induces a map \( \rho^* : \overline{L}^2(\Gamma / G / K_n) \rightarrow \overline{L}^2(\Gamma / G / K_n) \). We shall view \( \rho^* \) as a map from \( C_c^\infty(\Gamma' / G / K_n) \) to \( C_c^\infty(\Gamma / G / K_n) \), and denote it by \( P_{\Gamma', \Gamma}^{(n)} \).

Explicitly, \( P_{\Gamma', \Gamma}^{(n)} \eta_{\Gamma \ \cap} / G / K_n = \eta_{\Gamma' \ \cap} / G / K_n \). Furthermore, by Lemma 6.9(i), when viewing \( C_c^\infty(\Gamma / G / K_n) \) as a subspace of \( C_c^\infty(\Gamma / G) \), we have

\[ (P_{\Gamma', \Gamma}^{(n)} \varphi) = \sum_{\gamma \in \Gamma} \varphi(\gamma g) \quad \forall \varphi \in C_c^\infty(\Gamma / G / K_n), g \in G \].

We can now phrase the main result of this section:

**Theorem 6.10.** Define an algebra

\[ B = \left[ \begin{array}{cccc} e_1 \mathcal{H} e_1 & \ldots & e_1 \mathcal{H} e_t \\ \vdots & \ddots & \vdots \\ e_t \mathcal{H} e_1 & \ldots & e_t \mathcal{H} e_t \end{array} \right] \subseteq M_t(\mathcal{H}) \]

and an involution \(* : B \rightarrow B\) by \( (\varphi_{ij})_{i,j}^* = (\varphi_{ji}^*)_i j \). For every \( \Gamma \leq \mathcal{X} \ G \), we make \( \bigoplus_{i=1}^t C_c^\infty(\Gamma / G / K_n) \) into a left \( B \)-module by setting

\[ (\varphi_{ij}) \cdot (\psi_{ij})_{i=1}^j = \left( \sum_{j=1}^t \psi_{ij} \varphi_{ij}^\#_{ij} \right)_{i=1}^t \]

as an algebra.
(see [23] for the definition of #). Then there is an isomorphism of algebras with involutions
\[ A(\mathcal{C}, F) \cong B \]
and isomorphisms of pre-Hilbert spaces
\[ F(\Gamma \setminus X) \cong \bigoplus_{n=1}^{t} C_{c}^{\infty}(\Gamma \setminus G/K_n) \quad \forall \Gamma \leq X G \]
which are compatible with the relevant \( A(\mathcal{C}, F) \)-module and \( B \)-module structures.

The proof is somewhat technical. We first prove the following two lemmas.

**Lemma 6.11.** Define a functor \( F_1 : \mathcal{C}(G, X) \to \text{phil} \) as follows (notation as in Definition [1.7]): For all \( \Gamma \leq \Gamma' \leq \Gamma \leq_X G \) and \( g \in G \), let
- \( F_1(\Gamma \setminus X) = \bigoplus_{n=1}^{t} C_{c}^{\infty}(\Gamma \setminus G/K_n) \),
- \( F_1(p_{\Gamma} \circ g) = \bigoplus_{n=1}^{t} P_{\Gamma}^{n}(g) C_{c}^{\infty}(G/K_n) \), where \( g \) acts on \( C_{c}^{\infty}(G/K_n) \) via the left regular representation \( \Gamma(G) \) (Example [6.7]).
- \( F_1(p_{\Gamma'} \circ g) = \bigoplus_{n=1}^{t} P_{\Gamma'}^{n}(g) \)

Then there is a unitary natural isomorphism \( F \cong F_1 \).

**Proof.** Note that \( F(\Gamma \setminus X) = \ell^{2}(\Gamma \setminus X) \), so we need to construct a natural isomorphism between \( \ell^{2}(\Gamma \setminus X) \) and \( \bigoplus_{n=1}^{t} C_{c}^{\infty}(\Gamma \setminus G/K_n) \). There is an isomorphism \( \Gamma \setminus X \cong \bigcup_{n=1}^{t} \Gamma \setminus G/K_n \) given by sending \( \Gamma g K_n \) to \( \Gamma g x_n \). This induces a unitary isomorphism \( \Phi : \ell^{2}(\Gamma \setminus X) \to \bigoplus_{n=1}^{t} C_{c}^{\infty}(\Gamma \setminus G/K_n) \). Next, by [6.3], we have a unitary isomorphism
\[ \Psi := \bigoplus_{n=1}^{t} \mu(K_n)^{-1/2} \text{id}_{\ell^{2}(\Gamma \setminus G/K_n)} : \bigoplus_{n=1}^{t} \ell^{2}(\Gamma \setminus G/K_n) \to \bigoplus_{n=1}^{t} C_{c}^{\infty}(\Gamma \setminus G/K_n). \]

Now, \( \Psi \circ \Phi : \ell^{2}(\Gamma \setminus X) \to \bigoplus_{n=1}^{t} C_{c}^{\infty}(\Gamma \setminus G/K_n) \) is a unitary isomorphism. That \( \Psi \circ \Phi \) is natural is routine; use [6.3]. \( \square \)

**Lemma 6.12.** Let \( R \) be a ring, possibly non-unital, and let \( e, f \in R \) be idempotents. Consider \( R e \) and \( R f \) as left \( R \)-modules. Then for every \( \phi \in \text{Hom}_R(Re, Rf) \), there exists unique \( a \in eRf \) such that \( \phi(x) = xa \) for all \( x \in Re \). Conversely, \( [x \mapsto xa] \in \text{Hom}_R(Re, Rf) \) for all \( a \in eRf \).

**Proof.** The last assertion is easy, and the uniqueness of \( a \) holds because \( \phi(e) = ea = a \). To see the existence, take \( a = \phi(e) \), and note that for all \( r \in Re \), we have \( \phi(r) = \phi(re) = r \cdot \phi(e) = ra \). \( \square \)

**Proof of Theorem 6.10.** By Lemma 6.11 we may assume \( F = F_1 \). By Theorem 4.15, we have
\[ A(\mathcal{C}, F) \cong \text{End}_G(FX) = \text{End}_G \left( \bigoplus_{n=1}^{t} C_{c}^{\infty}(G/K_n) \right). \]

Under this isomorphism, the involution on \( A(\mathcal{C}, F) \) coincides with taking the adjoint operator with respect to the inner product of \( \bigoplus_{n=1}^{t} C_{c}^{\infty}(G/K_n) \).

View \( C_{c}^{\infty}(G/K_n) \) as a subspace of \( C_{c}^{\infty}(G) = \mathcal{H} \). Then \( C_{c}^{\infty}(G/K_n) = \mathcal{H} e_n \), and it is a pre-unitary \( G \)-subrepresentation of \( \Gamma(G) \). We may therefore view \( \mathcal{H} e_n \) as a pre-unitary representation of \( \Gamma(G) \); by Example 6.7, the (left) action of \( \mathcal{H} \) on \( \mathcal{H} e_n \) is given by \( \varphi \cdot \psi = \varphi \cdot \psi (\varphi \in \mathcal{H}, \psi \in \mathcal{H} e_n) \). We now have
\[ \text{End}_G \left( \bigoplus_{n=1}^{t} C_{c}^{\infty}(G/K_n) \right) = \text{End}_\mathcal{H} \left( \bigoplus_{n=1}^{t} \mathcal{H} e_n \right). \]
Viewing $\bigoplus_{n=1}^t \mathcal{H} e_n$ as column vectors, this gives

$$A(\mathcal{F}, F) \cong \begin{bmatrix} \text{Hom}(\mathcal{H} e_1, \mathcal{H} e_1) & \cdots & \text{Hom}(\mathcal{H} e_1, \mathcal{H} e_1) \\ \vdots & \ddots & \vdots \\ \text{Hom}(\mathcal{H} e_t, \mathcal{H} e_t) & \cdots & \text{Hom}(\mathcal{H} e_t, \mathcal{H} e_t) \end{bmatrix}.$$  

By Lemma 6.12, we can identify $\text{Hom}(\mathcal{H} e_n, \mathcal{H} e_m)$ with $e_m \mathcal{H} e_n$ by letting $\varphi \in e_m \mathcal{H} e_n$ act on $\mathcal{H} e_n$ from the left via $\varphi \cdot \psi = \varphi \ast \varphi^\#$ (note that $(e_m \mathcal{H} e_n)^\# = e_n \mathcal{H} e_m$). This identification turns composition into convolution product and so it gives rise to an isomorphism of (non-involutary) algebras

$$A(\mathcal{F}, F) \cong \begin{bmatrix} e_1 \mathcal{H} e_1 & \cdots & e_1 \mathcal{H} e_t \\ \vdots & \ddots & \vdots \\ e_t \mathcal{H} e_1 & \cdots & e_t \mathcal{H} e_t \end{bmatrix} = B.$$  

From the way we defined this isomorphism, it is immediate that the action of $B$ on

$$F \mathcal{X} = \bigoplus_{n=1}^t C_c^\infty(G/K_n)$$

via the isomorphism $A(\mathcal{F}, F) \cong B$ is given by $(\varphi_{ij}) \cdot (\psi_j)_{t=1}^t = (\sum_{j=1}^t \psi_j \ast \varphi^\#)_{t=1}^t$.

Next, we need to check that the action of $B$ on $F(\Gamma \backslash \mathcal{X}) = \bigoplus_{n=1}^t C_c^\infty(\Gamma \backslash G/K_n)$ via the isomorphism $A(\mathcal{F}, F) \cong B$ is given by $(\varphi_{ij}) \cdot (\psi_j)_{t=1}^t = (\sum_{j=1}^t \psi_j \ast \varphi^\#)_{t=1}^t$.

By Theorem 6.13(i), it is enough to show that for all $(\varphi_{ij}) \in B$, the collection $\{a_X\} \mathcal{X} \in \mathcal{X}$ given by $a_{X \mathcal{X}} = \{(\psi_j) \mapsto (\sum_{j=1}^t \psi_j \ast \varphi^\#)_{t=1}^t : \bigoplus_{n=1}^t C_c^\infty(\Gamma \backslash G/K_n) \rightarrow \bigoplus_{n=1}^t C_c^\infty(\Gamma \backslash G/K_n)\}$ is a natural transformation from $F$ to itself. In fact, the proof of Theorem 6.13(ii) shows that this follows from the weaker condition $F_{\Gamma \backslash \mathcal{X}} \circ a_{X \mathcal{X}} = a_{X \mathcal{X}} \circ F_{\Gamma \backslash \mathcal{X}}$. Working coordinate-wise, this boils down to showing that for all $1 \leq i, j \leq t$, $\varphi \in \mathcal{H} e_j$ and $\psi \in C_c^\infty(G/K_j)$, we have $P_{1, \psi}^{(i)}(\varphi \ast \varphi^\#) = (P_{1, \gamma}^{(i)} \psi) \ast \varphi^\#$.

Using (6.4), for all $g \in G$, we have

$$P_{1, \psi}^{(i)}(\varphi \ast \varphi^\#)g = \sum_{\gamma \in \Gamma} (\psi \ast \varphi^\#)(\gamma g) = \sum_{\gamma \in \Gamma} \int_{x \in G} \psi(x) \cdot \varphi^\#(x^{-1}g) \, d\mu$$

$$= \int_{x \in G} \psi(x) \cdot \varphi^\#(x^{-1}g) \, d\mu = \int_{x \in G} \psi(x) \cdot \varphi^\#(x^{-1}g) \, d\mu$$

$$= \int_{x \in G} (P_{1, \gamma}^{(i)} \psi)(x) \cdot \varphi^\#(x^{-1}g) \, d\mu = ((P_{1, \gamma}^{(i)} \psi) \ast \varphi^\#)g.$$  

The sum and the integral can be exchanged because the function $(\gamma, x) \mapsto \varphi^\#(\gamma x) \cdot \psi(x^{-1}g) : \Gamma \times G \rightarrow \mathbb{C}$ is compactly supported.

It is left to show that the isomorphism $A(\mathcal{F}, F) \cong B$ is an isomorphism of algebras with involution. This amounts to checking that $\bigoplus_{n=1}^t C_c^\infty(G/K_n)$ is a unitary representation of $B$, and after unfolding the definitions, this boils down to showing that for all $\psi \in \mathcal{H} e_n$, $\psi' \in \mathcal{H} e_m$ and $\varphi \in e_m \mathcal{H} e_n$, we have $\langle \psi \ast \varphi^\#, \psi' \rangle_{L^2(G)} = \langle \psi, \psi' \ast (\varphi^\#)^\ast \rangle_{L^2(G)}$. Indeed,

$$\langle \psi \ast \varphi^\#, \psi' \rangle' = \int_{x \in G} (\psi \ast \varphi^\#)(y^{-1}x) \cdot \overline{\psi'}(x) \, d\mu = \int_{y \in G} \int_{x \in G} \psi(y) \cdot \varphi^\#(y^{-1}x) \cdot \overline{\psi'}(x) \, d\mu \, d\mu$$

$$= \int_{y \in G} \int_{x \in G} \psi(y) \cdot \overline{\psi'}(x) \cdot (\varphi^\#)(x^{-1}y) \, d\mu \, d\mu$$

$$= \int_{y \in G} \psi(y) \cdot (\psi' \ast (\varphi^\#)^\ast)(y) \, d\mu = \langle \psi, \psi' \ast (\varphi^\#)^\ast \rangle.$$


as required.

Example 6.13. Suppose $F = \Omega_{+}^{\tau}$. Then $F = \hat{\ell}^{2} \circ S$ where $S : \mathcal{C} \to \text{phil}$ is given by $SX = \chi^{(i)}$. Taking $x_{1}, \ldots, x_{t}$ to be representatives for the $G$-orbits in $\chi^{(i)}$ and setting $K_{n} = \text{Stab}_{G}(x_{n})$ and $e_{n} = e_{K_{n}}$, Theorem 6.10 gives an alternative, more explicit, description of $A(\mathcal{C}, \Omega_{+}^{\tau})$.

Likewise, when $F = \Omega_{+}^{\tau}$, we can take representatives $x_{1}, \ldots, x_{s}$ for the $G$-orbits in $\chi^{(i)}$ and get an alternative description of $A(\mathcal{C}, \Omega_{+}^{\tau})$.

Example 6.14. Let $G = \text{PGL}_{d}(F)$, $K = \text{PGL}_{d}(O)$ and $X = B_{d}(F)$ be as in Chapter 1 and take $F = \Omega_{+}^{\tau}$. Since $B_{d}(F) = G/K$, the group $G$ acts transitively on $B_{d}(F)$.

Taking $x_{1} := K$ as a representative, we have $K_{1} = \text{Stab}_{G}(x_{1}) = K$.

We may therefore identify $A(\mathcal{C}, \Omega_{+}^{\tau})$ with $\mathcal{H}_{K}(G) = \mathcal{H}_{G} e_{K}$ and $\Omega_{\tau}^{+} \mathcal{X}(G/K)$ with $C_{c}^{\infty}(\Gamma \backslash G/K)$ for all $\Gamma \leq_{\mathcal{X}} G$. Under this identification, the action of $\mathcal{H}_{K}(G)$ on $C_{c}^{\infty}(\Gamma \backslash G/K)$ is given by $\varphi \cdot \psi = \psi * \varphi^{\#} (\varphi \in \mathcal{H}_{K}(G), \psi \in C_{c}^{\infty}(\Gamma \backslash G/K))$. Moreover, if we choose the Haar measure $\mu$ on $G$ such that $\mu(K) = 1$, then the isomorphism $\Omega_{\tau}^{+} \mathcal{X}(G/K) \cong C_{c}^{\infty}(\Gamma \backslash G/K)$ is the identity map.

Define $\overline{g}_{1}, \ldots, \overline{g}_{d-1} \in G$ as in Chapter 1. It is well known that $\mathcal{H}_{K}(G)$ is a commutative unital algebra freely generated by the operators $1_{K} F_{K}, \ldots, 1_{K} F_{1} K$, called the Hecke operators (see [55] Ch. V, for instance). An easy computation shows that under the isomorphism, $\Omega_{\tau}^{+} \mathcal{X}(G/K) \cong C_{c}^{\infty}(\Gamma \backslash G/K)$, the operator of $1_{K} F_{K}$ corresponds to the operator $a_{1}$ of Chapter 1. It follows that $A(\mathcal{C}, \Omega_{+}^{\tau})$ is a commutative unital algebra generated by $a_{1}, \ldots, a_{d-1}$.

Remark 6.15. One can extend the analysis of this section to semi-elementary functors $F : \mathcal{C} \to \text{phil}$ as follows: Choose a functor $F' : \mathcal{C} \to \text{hil}$ such that $F \oplus F'$ is elementary and write $F \oplus F' \cong \hat{\ell}^{2} \circ S$ with $S$ as in Definition 4.11. Take representatives $x_{1}, \ldots, x_{t}$ for the $G$-orbits in $S X$ and define $e_{1}, \ldots, e_{t} \in \mathcal{H}$ and the algebra $B$ as in Theorem 6.10. Define $e_{F} = \{ e_{F, X} \}_{X \in \mathcal{C}} \in A(\mathcal{C}, F \oplus F')$ as in [14], namely, $e_{F, X}$ is the orthogonal projection $F X \oplus F' X \rightarrow F X$, and let $\hat{e}_{F}$ be the corresponding idempotent in $B$.

Then the discussion in [14] and Theorem 6.10 imply that there is an isomorphism of involutary unital algebras $A(\mathcal{C}, F) \cong \hat{e}_{F} B \hat{e}_{F}$ and isomorphisms of pre-Hilbert spaces $F(\Gamma \backslash X) \cong \hat{e}_{F} \cdot \prod_{n=1}^{d-1} C_{c}^{\infty}(\Gamma \backslash G/K_{n}) \Gamma \leq_{\mathcal{X}} G$ which are compatible with the actions of $A(\mathcal{C}, F)$ and $\hat{e}_{F} B \hat{e}_{F}$. (Here, $\hat{e}_{F} B \hat{e}_{F}$ acts on $\hat{e}_{F} \cdot \prod_{n=1}^{d-1} C_{c}^{\infty}(\Gamma \backslash G/K_{n})$ via the action of $B$ on $\prod_{n=1}^{d-1} C_{c}^{\infty}(\Gamma \backslash G/K_{n})$ described in Theorem 6.10.)

6E. Weak Containment. In order to apply Theorem 6.10, we need to recall the notion of weak containment for unitary representations of $G$, and relate it to weak containment of unitary representations of $\mathcal{H}(G)$ as defined in [22].

Let $V \in \text{Irr}^{u}(G)$ and $V' \in \text{Rep}^{u}(G)$. Recall that $V$ is weakly contained in $V'$, denoted $V \prec V'$, if for all $v \in S^{1}(V)$, $\varepsilon > 0$ and compact $C \subseteq G$, there exists $v' \in S^{1}(V')$ such that

$$|(gv, v) - (gv', v')| < \varepsilon \quad \forall g \in C.$$ 

In fact, it is easy to see that $v'$ can be taken to be in $S^{1}(V'_{\text{sm}})$, or any prescribed dense subset of $S^{1}(V')$.

The representation $V$ is called tempered if it weakly contained in $L^{2}(1 \backslash G)$, the right regular representation of $G$ (Example 6.1).

Remark 6.16. Let $G$ be a reductive algebraic group over a non-archimedean local field $F$. When $G = G(F)$, there are other definitions of tempereness in the literature. These definitions agree with our definition by [55] §2.4, for instance.
Proposition 6.17. Let $V \in \text{Irr}^u(G)$ and $V' \in \text{Rep}^u(G)$. Then $V \prec V'$ as unitary representations of $G \iff V \prec V'$ as unitary representations of $\mathcal{H}(G)$.

Proof. Write $\mathcal{H} = \mathcal{H}(G)$. Assume $V \prec V'$ as representations of $G$, and let $v \in S^1(V_{sm})$, $\varepsilon > 0$ and $F \subset \mathcal{H}$ a finite set. Let $C = \bigcup_{\varphi \in F} \text{supp}(\varphi)$. Then $C$ is compact and open. For all $\varphi \in \mathcal{H}$, let $\|\varphi\|_1 := \int_{x \in G} |\varphi(x)|^2 \mu$, and let $M = \max\{\|\varphi\|_1 : \varphi \in F\}$.

By assumption, there is $v' \in S^1(V_{sm})$ such that $|\langle gv, v \rangle - \langle gv', v' \rangle| < \varepsilon M^{-1}$ for all $g \in C$. Since $v \in V_{sm}$ and $v' \in V'_{sm}$, there is $K \leq_{co} G$ such that $v \in V^K$ and $v' \in V'^K$. We can choose $K$ small enough such that each $\varphi \in F$ can be written as a finite sum $\sum_j \alpha_j 1_{g_j K}$ with $\{g_j\}_j \subseteq C$ and such that $\{g_j K\}_j$ are disjoint. A straightforward computation shows that $\varphi v = \mu(K) \sum_j \alpha_j g_j v$ and $\varphi v' = \mu(K) \sum_j \alpha_j g_j v'$. Thus,

$$\left| \langle \varphi v, v \rangle - \langle \varphi v', v' \rangle \right| = \left| \mu(K) \sum_j \alpha_j (\langle g_j v, v \rangle - \langle g_j v', v' \rangle) \right| \leq \mu(K) \sum_j |\alpha_j| \varepsilon M^{-1} = \|\varphi\|_1 \varepsilon M^{-1} \leq \varepsilon.$$

This shows that $V \prec V'$ as representations of $\mathcal{H}$.

Conversely, assume $V \prec V'$ as representations of $\mathcal{H}$, and let $v \in S^1(V)$, $\varepsilon > 0$ and $C \subseteq G$ be compact. Suppose first that $v \in V_{sm}$. Then there is $K \leq_{co} G$ such that $v \in V^K = e_K V$. Write $C K = \bigcup_{i=1}^n g_i K$ with $g_1, \ldots, g_n \in G$, and let $\varphi_i = \mu(K)^{-1} 1_{g_i K} \in \mathcal{H}$. By Lemma 2.15, there is $v' \in e_K V'$ such that $|\langle \varphi_i v, v \rangle - \langle \varphi_i v', v' \rangle| < \varepsilon$ for all $i$. For all $g \in C$, there are $x \in K$ and $i$ such that $g = g_i x$. Since $v \in V^K$ and $v' \in V'^K$, we have $\varphi_i v = g_i v = g_i x v = g v$ and $\varphi_i v' = g_i v' = g_i x v' = g v'$, hence $|\langle g v, v \rangle - \langle g v', v' \rangle| < \varepsilon$ for all $g \in C$.

When $v \notin V_{sm}$, we argue as follows: For all $v_1 \in S^1(V)$ and $g \in G$, we have

$$|\langle g v, v \rangle - \langle g v_1, v_1 \rangle| = |\langle g v - v_1, v - v_1 \rangle| \leq \|g v - v_1\| \cdot \|v - v_1\| = 2\|v - v_1\|.$$

Take $v_1 \in S^1(V_{sm})$ close enough to $v$ to have $|\langle g v, v \rangle - \langle g v_1, v_1 \rangle| < \frac{\varepsilon}{2}$. By the previous paragraph, there is $v' \in S^1(V')$ such that $|\langle g v_1, v_1 \rangle - \langle g v', v' \rangle| < \frac{\varepsilon}{2}$ for all $g \in C$. Then $|\langle g v, v \rangle - \langle g v', v' \rangle| < \varepsilon$ for all $g \in C$. \qed

Proposition 6.18. Let $H$ be a normal compact subgroup of $G$ and let $U \in \text{Irr}^u(G/H)$, $V \in \text{Rep}^u(G)$. Then $U \prec V^H$ as representations of $G/H$ if and only if $U \prec V$ as representations of $G$.

Proof. We choose the Haar measure on $G/H$ to be the push-forward of the Haar measure on $G$. This allows us to view $\mathcal{H}(G/H)$ as a *-subalgebra of $\mathcal{H}(G)$). It is easy to check that $\mathcal{H}(G/H)$ is a corner (and also an ideal) of $\mathcal{H}(G)$, and for all $W \in \text{Rep}^u(G/H)$, we have $\mathcal{H}(G/H)^{-1} W \cong W^H$ as unitary representations of $\mathcal{H}(G/H)$). The Proposition therefore follows from Proposition 6.17 and Lemma 2.14(iii). \qed

6F. A Criterion for Being Ramanujan. Let $\mathcal{X}$ be an almost transitive $G$-complex, let $\mathcal{G} = \mathcal{G}(G, \mathcal{X})$ (Definition 3.6), and let $F : \mathcal{G} \to \mathfrak{hil}$ be an elementary functor (Definition 1.11). We assume that $G$ is unimodular, which is automatically the case if $\mathcal{X}$ has finite $G$-quotients, by Proposition 5.39. In this section, we phrase and prove a representation-theoretic criterion for $G$-quotients of $\mathcal{X}$ to be $F$-Ramanujan [68]. In particular, we get criteria for being $\text{Ramanujan in dimension } i$ and completely Ramanujan. When $G = \text{PGL}_d(F)$, $\mathcal{X} = B_d(F)$ and $F = \Omega_0^G$, we recover the criterion given in [51] Prp. 1.5]
Write $\mathcal{H} = \mathcal{H}(G)$. We shall freely view unitary representations of $G$ as unitary representations of $\mathcal{H}$ and vice versa. The relation of weak containment is not affected by these transitions thanks to Proposition 6.17.

**Lemma 6.19.** Let $V \in \text{Irr}^u(G)$ (resp. $V \in \text{Irr}^p(G)$). The following conditions are equivalent.

(a) The image of $G$ in the unitary group of $V$ is finite.
(b) There is an open subgroup of finite index $N \leq G$ such that $V = V^N$.
(c) There is an open subgroup of finite index $N \leq G$ such that $V \leq \hat{L}^2(N\backslash G)$.

(Here, $G$ acts on $\hat{L}^2(N\backslash G)$ via $(g\varphi)x = \varphi(xg)$.)

In this case, $V$ is finite dimensional.

**Proof.** The implications, (a)$\Rightarrow$(b), (c)$\Rightarrow$(a), and the final assertion are straightforward, so we only prove (b)$\Rightarrow$(c). Fix some $0 \neq u \in S^1(V)$. A continuous embedding of $V$ in $\hat{L}^2(N\backslash G)$ is given by and sending $v \in V$ to $[Ng \mapsto (gv,u)] \in \hat{L}^2(N\backslash G)$. That is this is indeed a nonzero $G$-homomorphism is routine. Since $V$ is irreducible, this homomorphism is injective, and since $\hat{L}^2(N\backslash G)$ is finite-dimensional, it is also continuous. Now, by Proposition 2.8 (applied with $A = \mathcal{H}$), there is a unitary embedding $V \to \hat{L}^2(N\backslash G)$. □

An irreducible representation satisfying the equivalent conditions of Lemma 6.19 is said to have finite action. In general, there may be irreducible finite dimensional representations without finite action; consider the case $G = \mathbb{Z}$ with the discrete topology. However, in certain cases, finite dimension and finite action are equivalent.

**Lemma 6.20.** Let $\Gamma \leq G$ be cocompact discrete subgroup. Then for all $V \in \text{Irr}^u(G)$ with $V \prec \hat{L}^2(\Gamma\backslash G)$, we have $\dim V < \infty$ if and only if $V$ has finite action.

**Proof.** By Example 6.3, $\hat{L}^2(\Gamma\backslash G)$ is admissible, so $V \leq \hat{L}^2(\Gamma\backslash G)$ by Theorem 2.19 (iii). Suppose $V$ is finite dimensional. Since $V_{sm}$ is dense in $V$ (Lemma 2.9), we must have $V = V_{sm}$, so $V$ is smooth. Let $v_1, \ldots, v_t$ be a basis of $V$, and choose $K_1, \ldots, K_t \leq \text{c.o.}$ with $v_i \in V^{K_i}$. Then $V = V^K$ for $K = \bigcap_{i=1}^t K_n$. Let $H$ be the smallest normal subgroup of $G$ containing $K$. Then $H$ is open and $V^H = V$. Since $\Gamma$ is cocompact, $\Gamma\backslash G/K$ is finite, and hence so is $\Gamma\backslash G/H = \Gamma H\backslash G$. It follows that $V \subseteq \hat{L}^2(\Gamma H\backslash G)$ (viewed as a subspace of $\hat{L}^2(\Gamma\backslash G)$). The image of $G$ in the unitary group of $\hat{L}^2(\Gamma H\backslash G)$ is clearly finite, so $V$ has finite action. □

**Theorem 6.21.** Write $F = \hat{L}^2 \circ S$ where $S$ is as in Definition 4.11. Let $x_1, x_2, \ldots, x_t$ be representatives of the $G$-orbits in $S\mathcal{X}$, and let $K_n = \text{Stab}_G(x_n)$ ($1 \leq n \leq t$). For any subset $T$ of the unitary dual $\hat{\mathcal{H}}$, define

$$T^{(K_1, \ldots, K_t)} = \{ [V] \in T : V^{K_1} + \cdots + V^{K_t} \neq 0 \}.$$ 

Then there exists an additive functor

$$\mathcal{F} : \text{Rep}^u(\mathcal{H}) \to \text{Rep}^u(\mathcal{A}(\mathcal{H}, F))$$

with the following properties:

(i) $\mathcal{F}$ induces a topological embedding $[V] \mapsto [FV] : \hat{\mathcal{H}}^{(K_1, \ldots, K_t)} \to \hat{\mathcal{A}}(\mathcal{H}, F)$, denoted $\hat{F}$.

(ii) For all $[V] \in \hat{\mathcal{H}}^{(K_1, \ldots, K_t)}$ and $V' \in \text{Rep}^u(\mathcal{H})$, we have $V \prec V' \iff FV \prec FV'$.

(iii) $\mathcal{F}(\hat{L}^2(\Gamma\backslash G)) = F(\mathcal{H}\backslash \mathcal{X})$ for all $\Gamma \leq \mathcal{H} G$.

(iv) $\hat{F}(\text{Spec}_{\mathcal{H}}(\hat{L}^2(\Gamma\backslash G)^{(K_1, \ldots, K_t)})) = \text{Spec}_{\mathcal{A}(\mathcal{H}, F)}(\mathcal{A}\backslash \mathcal{X})$ for all $\Gamma \leq \mathcal{H} G$. Furthermore, when $\Gamma\backslash \mathcal{X}$ is finite, or equivalently, when $\Gamma$ is cocompact in $G$, the
map \( \hat{F} \) induces an isomorphism of multisets \( \text{m-Spec}_{\mathcal{F}}(L^2(\Gamma \backslash G))^{(K_1, \ldots, K_r)} \cong \text{m-Spec}_{A(\mathcal{E}, F)}(\Gamma \backslash \mathcal{X}) \).

(v) Let \( [V] \in \hat{\mathcal{H}}^{(K_1, \ldots, K_r)} \). Then \( \hat{F}[V] \) is in \( \mathfrak{T}_{A(\mathcal{E}, F)} \), the trivial \( A(\mathcal{E}, F) \)-
spectrum (7.4), if and only if \( V \) has finite action. More precisely, we have
\[
\hat{F}(\text{Spec}_{\mathcal{F}}(\hat{F}(N \backslash G))^{(K_1, \ldots, K_r)}) = \mathfrak{T}_{A(\mathcal{E}, F), N},
\]
for any open finite-index subgroup \( N \leq G \).

Proof. Define an involution \( \ast : M_t(\mathcal{H}) \to M_t(\mathcal{H}) \) by \( (\varphi_V)^* = (\varphi_V)^{\text{op}} \). Then \( M_t(\mathcal{H}) \)

is an idempotented involutory algebra. As explained in Example 2.41 there is

an equivalence of categories \( \text{Rep}^u(\mathcal{H}) \sim \text{Rep}^u(M_t(\mathcal{H})) \). Explicitly, the \( M_t(\mathcal{H}) \)-representation corresponding to \( V \in \text{Rep}^u(\mathcal{H}) \) is viewed (viewed as columns vectors)

endowed with the standard left \( M_t(\mathcal{H}) \)-action. In the other direction, the equivalence

is given by \( U \mapsto U^{-1} \mathcal{U} \) where
\[
\mathcal{H}^{-1} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \end{bmatrix} \subseteq M_t(\mathcal{H})
\]

and \( \mathcal{H}^{-1} \) is identified with \( \mathcal{H} \) in the obvious manner. There is a similar equivalence between the corresponding categories of pre-unitary representations.

We identify \( A(\mathcal{E}, F) \) with the algebra \( B \) defined in Theorem 6.10. Observe that
\[
B = eM_t(\mathcal{H})e
\]

where \( e \in M_t(\mathcal{H}) \) is the diagonal matrix \( \text{diag}(e_{K_1}, \ldots, e_{K_r}) \). Define the functor \( F : \text{Rep}^u(\mathcal{H}) \to \text{Rep}^u(B) \) by
\[
F(V) := e \cdot V^t = \bigoplus_{n=1}^t e_{K_n} V = \bigoplus_{n=1}^t V^{K_n}.
\]

(the functor \( F \) acts on morphisms is the obvious way). We similarly define \( F : \text{Rep}^u(\mathcal{H}) \to \text{Rep}^u(B) \). It is easy to see that there is a natural isomorphism
\[
F(V) \cong F(V^t)
\]

for all \( V \in \text{Rep}^u(\mathcal{H}) \). Observe also that for all unitary or pre-unitary \( V \), we have
\[
(6.5) \quad F(V) = eV^t \neq 0 \iff V^{K_1} + \cdots + V^{K_r} \neq 0.
\]

We now prove properties (i)-(v).

(i) Let \( [V] \in \hat{\mathcal{H}}^{(K_1, \ldots, K_r)} \). Then \( V^t \in \text{Irr}^u(M_t(\mathcal{H})) \) by Example 2.46 and by

Lemma 2.41(i) and 6.3. \( F(V) = eV^t \in \text{Irr}^u(B) \). Furthermore, by Theorem 2.42

the maps \( [U] \mapsto \hat{\mathcal{H}}^U : M_t(\mathcal{H}^\circ) \to \hat{\mathcal{H}} \) and \( [U] \mapsto [eU] : (M_t(\mathcal{H}^\circ)(B)) \to \hat{\mathcal{H}} \)

are topological embeddings. Since \( [U] \mapsto [\hat{\mathcal{H}}^U] \) and \( [U] \mapsto [eU] : (M_t(\mathcal{H}^\circ)(B)) \to \hat{\mathcal{H}} \)

the inverse of \( (K_1, \ldots, K_r) \) in \( M_t(\mathcal{H}^\circ) \)

under \( [V] \mapsto [V^t] \) is \( (M_t(\mathcal{H}^\circ)(B)) \), it follows that the map \( [V] \mapsto [eV^t] = [F(V)] : \hat{\mathcal{H}} \to \hat{\mathcal{H}} \)

is bijective with \( B \) is a topological embedding.

(ii) By Lemma 2.41(iii), if \( V \in \text{Irr}^u(\mathcal{H}) \) satisfies \( V^{K_1} + \cdots + V^{K_r} \neq 0 \) and \( V^t \in \text{Rep}^u(\mathcal{H}) \), then \( V \prec V^t \iff V^t \prec V^{\Pi} \) (as representations of \( M_t(\mathcal{H}) \)) \( \iff F(V) = eV^t \prec eV^{\Pi} = F(V^t) \).

(iii) We identify \( F(\Gamma \backslash X) \) with \( \bigoplus_{n=1}^\infty C^\infty(\Gamma \backslash G/K_n) \) as in Theorem 6.10. By

Example 6.6 we have \( F(C^\infty(\Gamma \backslash G)) = \bigoplus_{n=1}^\infty C^\infty(\Gamma \backslash G/K_n) \) as pre-unitary representations of \( B \), so \( F(\ell^2(\Gamma \backslash G)) = F(C^\infty(\Gamma \backslash G)) \cong F(C^\infty(\Gamma \backslash G)) \cong F(\Gamma \backslash X) \).

(iv) This follows from (ii) and (iii). That \( \Gamma \backslash X \) is finite if and only if \( \Gamma \) is compact in \( G \) and \( \Gamma \) does not follow from Proposition 5.13.

(v) Recall from 2.79 that \( \mathfrak{T}_{A(\mathcal{E}, F), N} \) is defined to be \( \text{Spec}_{A(\mathcal{E}, F)}((F \chi)_N) \), where \( (F \chi)_N \) is the space of \( N \)-coinvariants for \( F \chi \). By (ii), it is enough to prove that \( F(\ell^2(N \backslash G)) \cong (F \chi)_N \) as pre-unitary representations of \( B = A(\mathcal{E}, F) \). In fact, by

Lemma 5.6, it is enough to show that \( F(\ell^2(N \backslash G)) \cong (F \chi)_N \) as \( B \)-modules.
Let $C^\infty(N\backslash G)$ be the space of functions $G \to \mathbb{C}$ which are $N$-invariant on the left. We view $C^\infty(N\backslash G)$ as a smooth $G$-module by setting $(g \cdot \varphi)x = \varphi(gx)$. Then $F(N\backslash G) \cong C^\infty(N\backslash G)$ as $G$-modules via $e_{N^g} \mapsto 1_{N^g}$. As in Example 6.10, the corresponding left $\mathcal{H}$-module structure of $C^\infty(N\backslash G)$ is given by $\varphi \cdot \psi = \psi \star \varphi^\#$. It follows that $F(C^\infty(N\backslash G)) = \bigoplus_{n=1}^\infty C^\infty(N\backslash G/K_n)$, where the action of $B$ on the left hand side is given by $(\varphi_1)_i \cdot (\psi_1)_j = (\sum_j \psi_j \star \varphi^\#_{ij}_1)$. By Theorem 6.22, the action of $B$ on $F \mathcal{X} = \bigoplus_{n=1}^\infty C_\infty(G/K_n)$ is also given by $(\varphi_1)_i \cdot (\psi_1)_j = (\sum_j \psi_j \star \varphi^\#_{ij}_1)$, and this action induces the action of $B$ on $(F \mathcal{X})_N = \bigoplus_{n=1}^\infty C_\infty(G/K_n)_N$ (notice that $G$ acts on $C_\infty(G/K_n)_N$ via $(g\varphi)x = \varphi(g^{-1}x)$).

Define $\Phi_n : C_\infty(G/K_n) \to C^\infty(N\backslash G/K_n)$ by $(\Phi_n \varphi)x = \int_{y \in N} \varphi(yx) d\mu$. Since $\Phi_n(g\varphi) = \Phi_n(\varphi)$ for all $g \in N$, this induces a map $C_\infty(G/K_n)_N \to C^\infty(N\backslash G/K_n)_N$, which we also denote by $\Phi_n$. We claim that $\Phi := \bigoplus_{n=1}^\infty \Phi_n : \bigoplus_{n=1}^\infty C_\infty(G/K_n)_N \to \bigoplus_{n=1}^\infty C^\infty(N\backslash G/K_n)$ is an isomorphism of $B$-modules, which will complete the proof. Working component-wise, this amounts to showing that $\Phi_n$ is bijective for all $n$, and for all $n$, $m$, $\varphi \in e_{K_i} \mathcal{H} e_{K_i}$ and $\psi \in C_\infty(G/K_i)$, we have $\Phi_m(\varphi \star \varphi^\#) = (\Phi_n(\psi) \star \varphi^\#)$.

It easy to see that $\Phi_n(1_{GK})$ is a nonzero multiple of $1_{N\sigma K}$, so $\Phi_n$ is surjective. The injectivity follows since $C_\infty(G/K_i)_N \cong \mathcal{P}(N\backslash G/K_i)$ as vector spaces, and hence $\dim C_\infty(G/K_i)_N = [N\backslash G/K_i] = \dim C^\infty(N\backslash G/K_i)$. Finally, for all $g \in G$, we have

$$\Phi_m(\varphi \star \varphi^\#)g = \int_{x \in N} (\psi \star \varphi^\#)(yx) d\mu = \int_{x \in G} \int_{y \in N} \psi(yx) \cdot \varphi^\#(yx^{-1}g) d\mu,$$

so $\Phi_m(\varphi \star \varphi^\#) = (\Phi_n(\psi) \star \varphi^\#)$.

As a corollary, we get the following criterion for checking the $F$-Ramanujan and completely Ramanujan properties.

**Theorem 6.22.** Keep the notation of Theorem 6.21 and let $\Gamma \leq_X G$. Then:

(i) $\Gamma \mathcal{X}$ is $F$-Ramanujan if and only if every irreducible unitary representation $V \prec L^2(\Gamma \backslash G)$ satisfying $V^K_1 + \cdots + V^K_1 \neq 0$ is tempered (i.e. $V \prec L^2(1\backslash G)$) or has finite action.

(ii) $\Gamma \mathcal{X}$ is completely Ramanujan if and only if every irreducible unitary representation $V \prec L^2(\Gamma \backslash G)$ is tempered or has finite action.

When $\Gamma \mathcal{X}$ is finite, one can replace “finite action” with “finite dimension” and “$V \prec L^2(\Gamma \backslash G)$” with “$V \prec L^2(1\backslash G)$”.

**Proof.** (i) The complex $\Gamma \mathcal{X}$ is $F$-Ramanujan precisely when $\text{Spec}_{A(\mathcal{X},F)}(\Gamma \mathcal{X}) \subseteq \text{Spec}_{A(\mathcal{X},F)}(\mathcal{X}) \cup \Sigma_{A(\mathcal{X},F)}$. Taking inverse images relative to $\Gamma$ in Theorem 6.21 yields the equivalence.

(ii) Suppose $\Gamma \mathcal{X}$ is completely Ramanujan. For any $K \leq_{c.o.} G$ such that $K$ stabilizes some nonempty cell of $\mathcal{X}$, define a functor $S : \mathcal{C} \to \text{Set}$ by setting $S(\Gamma \mathcal{X}) = \Gamma G/K$, $S(p \circ g) = [xK \mapsto \Gamma gxK]$, $S(p \circ r) = [\Gamma xK \mapsto \Gamma xK]$ (notation as in Definition 4.4), and let $F = \mathcal{P} \circ S$. It is easy to check that $S$ satisfies conditions (E1)–(E4) of Definition 4.11 and hence $F$ is elementary. Furthermore, $S\mathcal{X} = G/K$ consists of a single $G$-orbit. Taking $x_1 = 1_G K \in S\mathcal{X}$, and $K_1 = \text{Stab}_G(x_1) = K$ in part (i), we get that any irreducible unitary representation $V \prec L^2(\Gamma \backslash G)$ with $V^K \neq 0$ is tempered or has finite action. Since $K$ can be taken to be any sufficiently small
compact open subgroup of $G$, it follows that any irreducible unitary representation $V \prec L^2(\Gamma \setminus G)$ is tempered or has finite action. The other direction is immediate from (i).

The final assertion follows from Lemma 6.20 and Theorem 2.49 (iii) (cf. Proposition 6.13).

Rem 6.23. It sometimes convenient to consider groups acting non-faithfully on $\mathcal{X}$, e.g., a non-adjoint almost simple algebraic group over a local non-archimedean field acting on its affine building (Example 3.4). Theorem 6.22 can be adjusted to

Let $\mathcal{X}$ be a lattice in $\tilde{G}$ such that $\Gamma := \text{im}(\tilde{\Gamma} \to G) \leq \mathcal{X}$. In the notation of Theorem 6.21, write $\tilde{K}_n = \text{Stab}_G(x_n)$. Then $\Gamma \setminus \mathcal{X} = \tilde{\Gamma} \setminus \mathcal{X}$ is $F$-Ramanujan if and only if any $V \prec L^2(\tilde{\Gamma} \setminus \tilde{G})$ with $V^{\mathcal{K}_1} + \cdots + V^{\mathcal{K}_t} \neq 0$ is tempered or has finite action, and $\Gamma \setminus \mathcal{X}$ is completely $\mathcal{K}$-Ramanujan if and only if any $V \prec L^2(\mathcal{X} \setminus \tilde{G})$ with $V^H \neq 0$ is tempered or has finite action. Indeed, notice that $L^2(\tilde{\Gamma} \setminus \tilde{G})^H \cong L^2(\Gamma \setminus G)$ as representations of $G = \tilde{G}/H$, so the previous statements can be translated to the statements of Theorem 6.22 by taking $H$-invariants, thanks to Prop 6.18.

Rem 6.24. The functor $F = \tilde{\ell}^2 \circ S$ used in the proof of Theorem 6.22 (ii) has no a priori combinatorial interpretation. However, the proof can be done using functors defined by combinatorial means. For example, consider the family of functors $(S_{n,m} : \text{Sim} \to \text{Set})_{n,m \in \mathbb{N}}$ defined by letting $S_{n,m} = \{(v_1, \ldots, v_n) \in \mathcal{X}_{\text{vert}} : d(v_i, v_j) \leq m \text{ for all } i, j\}$ ($S_{n,m}$ acts on morphisms in the obvious way). It is easy to see that the collection $\{\text{Stab}_G(x) \mid x \in S_{n,m} \mathcal{X}, n, m \in \mathbb{N}\}$ is a basis of neighborhoods of $1_G$, so we can use $F_{n,m} := \tilde{\ell}^2 \circ S_{n,m}$ in the proof.

Ex 6.25. Let $x_1, \ldots, x_t$ be representatives for the $G$-orbits in $\mathcal{X}^{(i)}$, and let $K_n = \text{Stab}_G(x_n)$ (1 ≤ $n$ ≤ $t$). Then by Theorem 6.22 (i), applied with $F = \Omega^+_i = \tilde{\ell}^2 \circ [\mathcal{X} \to \mathcal{X}^{(i)}]$, a $G$-quotient $\Gamma \setminus \mathcal{X}$ is $\mathcal{K}$-Ramanujan if and only if any irreducible unitary representation $V \prec L^2(\Gamma \setminus G)$ with $V^{\mathcal{K}_1} + \cdots + V^{\mathcal{K}_t} \neq 0$ is tempered or has finite action.

Likewise, letting $x_1, \ldots, x_s$ be a set of representatives for the $G$-orbits in $\mathcal{X}^{(i)}$ and setting $L_n = \text{Stab}_G(x_n)$, we get that $\Gamma \setminus \mathcal{X}$ is $\mathcal{K}$-Ramanujan if and only if any irreducible unitary representation $V \prec L^2(\Gamma \setminus G)$ with $V^L_1 + \cdots + V^L_t \neq 0$ is tempered or has finite action. In this case, the spectrum of the $i$-dimensional Laplacian (31) of $\Gamma \setminus \mathcal{X}$ is contained in the union of the spectrum of the $i$-dimensional Laplacian of $\mathcal{X}$ and the trivial spectrum $\Sigma_\Delta$, (cf. Remark 5.10).

Ex 6.26. Assume $\mathcal{X}$ is a $k$-regular tree and let $G = \text{Aut}(\mathcal{X})$. Choose a vertex $v \in \mathcal{X}^{(0)}$ and write $K = \text{Stab}_G(v)$. Then $\mathcal{X}^{(0)} = Gv$, and, as in Example 6.25, we get that a finite graph $\Gamma \setminus \mathcal{X}$ is $\mathcal{K}$-Ramanujan in dimension 0 if and only if any irreducible dimensional irreducible subrepresentation $V \prec L^2(\Gamma \setminus G)$ with $V^K \neq 0$ is tempered. (Recall from Example 5.15(i) that being $\mathcal{K}$-Ramanujan in dimension 0 is just being a $\mathcal{K}$-regular graph in the classical sense.)

Ex 6.27. Let $G = \text{PGL}_d(F)$, $K = \text{PGL}_d(O)$ and $\mathcal{X} = B_d(F)$ be as in Chapter 11 There is only one $G$-orbit in $B_d(F)^{(0)} = G/K$, represented by $x_1 := K$, and we have $\text{Stab}_G(x_1) = K$. As in Example 6.25 a finite $G$-quotient $\Gamma \setminus B_d(F)$ is $\mathcal{K}$-Ramanujan in dimension 0 if and only if any irreducible dimensional irreducible subrepresentation $V \prec L^2(\Gamma \setminus G)$ with $V^K \neq 0$ is tempered. This statement is 51 Pr. 1.5 (recall from Example 5.15(ii) that being $\mathcal{K}$-Ramanujan in dimension 0 is equivalent to being $\mathcal{K}$-Ramanujan in the sense of 511).
Consider now the case of $G$-quotients which are flag Ramanujan, i.e. $\Omega_{\text{Flag}}$-Ramanujan. Notice that $\Omega_{\text{Flag}} = \hat{\ell} \circ \text{Flag}$. Since $B_d(F)$ is pure, maximal flags in $B_d(F)$ correspond to chambers equipped with a full order on its vertices. Thus, the stabilizer of any maximal flag in $B_d(F)$ is just the pointwise stabilizer of some chamber $x \in B_d(F)^{(d-1)}$. The group $I_x := \bigcap_{v \in x} \text{Stab}_G(v)$ is called an Iwahori subgroup of $G$. For example, the Iwahori group corresponding to the fundamental chamber $x_0 := \{K, K \cap g_1^{-1}K, \ldots, K \cap g_{d-1}^{-1}K\}$ (notation as in Chapter 1) is $I_0 := K \cap \bigcap g_i^{-1}K g_i^{-1} \cap \cdots \cap g_{d-1}^{-1}K g_{d-1}^{-1}$, and an easy computation shows that $I_0$ is the image of

$$I := \begin{bmatrix} O^\times & \pi O & \ldots & \pi O \\ O & O^\times & \ldots & \ldots \\ \vdots & \ddots & \ddots & \pi O \\ O & \ldots & O & O^\times \end{bmatrix} \subseteq \text{GL}_d(O)$$

in $G$. Since $G$ acts transitively on $B_d(F)^{(d-1)}$, any Iwahori group $I_x$ is conjugate to $I_0$, and hence any $[V] \in \mathcal{H}$ with $V^{I_0} \neq 0$ also satisfies $V^{I_0} \neq 0$. By Theorem 6.22(i), this means that a finite quotient $\Gamma \backslash B_d(F)$ is flag Ramanujan if and only if any irreducible infinite-dimensional subrepresentation $V \leq L^2(\Gamma \backslash G)$ with $V^{I_0} \neq 0$ is tempered. In this case, $\Gamma \backslash B_d(F)$ is also Ramanujan in all dimensions (Proposition 5.16(i)), Example 4.19).

6G. Consequences. We now give some consequences of Theorems 6.21 and 6.22.

Let $T_k$ denote the $k$-regular tree and let $G_k = \text{Aut}(T_k)$. In the sequel, we shall freely consider $k$-regular graphs $X$ (with no double edges or loops) as $G_k$-quotients $\Gamma \backslash T_k$. (The implicit choice of the cover map $T_k \to X$ determining $\Gamma$ will not affect the discussion.) Recall from Example 3.13(i) that a $k$-regular graph is Ramanujan in the classical sense if and only if it is Ramanujan in dimension 0, or $\Omega_1$-Ramanujan. We shall henceforth use Ramanujan in dimension 0 to avoid ambiguity.

Marcus, Spielman and Srivastava [58] proved that every bipartite finite $k$-regular graph $X$ admits an 2-cover $X' \to X$ which is Ramanujan in dimension 0 (cf. Remark 5.14). That is, the eigenvalues of the vertex adjacency operator of $X'$ which do not arise from $X$ lie in the interval $[-2\sqrt{k-1},2\sqrt{k-1}]$. This was extended covers of any prescribed rank by Hall, Puder and Sawin [31]. By applying Theorem 6.21 with $X = T_k$, $G = G_k$ and $F = \Omega_1^{2k}$, these results can be restated in a representation-theoretic manner:

**Corollary 6.28.** Let $H_k$ be the index-2 subgroup of $G_k$ consisting of automorphisms preserving the canonical 2-coloring of $X^{(0)}$. Let $K$ be the stabilizer of a vertex of $X'$, and let $\Gamma \leq H_k$ be a cocompact lattice. Then for any $r \in \mathbb{N}$, there exists a sublattice $\Gamma' \leq \Gamma$ of index $r$ such that every irreducible unitary subrepresentation $V$ of the orthogonal complement of $L^2(\Gamma \backslash G_k)$ in $L^2(\Gamma' \backslash G_k)$ satisfying $V^{K'} \neq 0$ is tempered.

We do not know of a representation-theoretic proof of this result.

**Remark 6.29.** Corollary 6.28 suggests the following definition: Let $G$ be an $\ell$-group, let $K_1, \ldots, K_\ell \leq G$, and let $\Gamma$ be a lattice in $G$. Call a subgroup $\Gamma' \leq \Gamma$ a $(K_1, \ldots, K_\ell)$-Ramanujan subgroup if $[\Gamma : \Gamma'] < \infty$ and every irreducible unitary representation $V$ with $V^{K_1} \oplus \cdots \oplus V^{K_\ell} \neq 0$ that is weakly contained in the orthogonal complement of $L^2(\Gamma \backslash G)$ in $L^2(\Gamma' \backslash G)$ is tempered. If $X$ is an almost transitive $G$-complex, $F : \mathcal{C}(G,X) \to \mathcal{H}$ is an elementary functor, $\Gamma \leq \chi G$, and $K_1, \ldots, K_\ell$ are as in Theorem 6.21, then $\Gamma'$ is a $(K_1, \ldots, K_\ell)$-Ramanujan subgroup of $\Gamma$ if and only if $\Gamma' \backslash X \to \Gamma \backslash X$ is an $F$-Ramanujan cover in the sense of Remark 5.14.
It is also reasonable to call \( \Gamma' \leq \Gamma \) a completely Ramanujan subgroup if \([\Gamma : \Gamma'] < \infty\) and any irreducible unitary representation that is weakly contained in the orthogonal complement of \(L^2(\Gamma \backslash G)\) in \(L^2(\Gamma' \backslash G)\) is tempered.

The following results concern equivalences of different types of the Ramanujan property. They follow from representation-theoretic properties of the relevant groups. We expect similar results should hold whenever \( \mathcal{X} \) is an affine building of dimension 1 or 2.

**Proposition 6.30.** A \( k \)-regular graph is Ramanujan in dimension 0 if and only if it is completely Ramanujan [64]. Likewise, a finite covering of \( k \)-regular graphs \( \mathcal{X}' \to \mathcal{X} \) is Ramanujan in dimension 0 if and only if it is completely Ramanujan in the sense of Remark 5.17.

**Proof.** Let \( \Gamma \leq \mathcal{X} \). If \( \Gamma \backslash \mathcal{X} \) is completely Ramanujan, then it is Ramanujan in dimension 0 by definition. Assume the converse, and let \( K \) be the stabilizer of some vertex in \( \mathcal{X} \). By Example 6.26, every irreducible \( V \prec L^2(\Gamma \backslash G) \) with \( V^K \neq 0 \) is tempered of has finite action. Suppose now that \( V \prec L^2(\Gamma \backslash G) \) is irreducible with \( V^K = 0 \). By Olshanski’s classification of the irreducible representations of \( G_k \) [64], \( V \) is either special or supercuspidal, and in both cases it is tempered. (In more detail, in both cases, for all \( u, v \in V \), the matrix coefficient \( \psi_{u,v} := [g \mapsto \langle g^{-1}u, v \rangle] \) is in \( L^2(G) \), and hence \( V \subseteq L^2(G) \); see [99, Pr. 9.6], for instance.) By Theorem 6.22(ii), this means \( \Gamma \backslash \mathcal{X} \) is completely Ramanujan. The assertion about covers is shown similarly. \( \square \)

Proposition 6.30 allows us to slightly strengthen Corollary 6.28 by dropping the assumption \( V^K \neq 0 \) in the end.

**Proposition 6.31.** Let \( G = \text{PGL}_d(F) \) and \( \mathcal{X} = \mathcal{B}_d(F) \) be as in Chapter 7 and suppose \( d \in \{2, 3\} \). Let \( \Gamma \leq \mathcal{X} \). Then \( \Gamma \backslash \mathcal{X} \) is Ramanujan in dimension 0 if and only if \( \Gamma \backslash \mathcal{X} \) is flag Ramanujan (cf. [56, Example 5.17(ii)]).

**Proof.** Being flag Ramanujan implies being Ramanujan in all dimensions (Proposition 6.19(i), Example 6.19), so we need to show the converse. Suppose \( \Gamma \backslash \mathcal{B}_d(F) \) is Ramanujan in dimension 0. Then by Example 6.27 (see there for notation), any irreducible \( V \prec L^2(\Gamma \backslash G) \) with \( V^K \neq 0 \) is tempered or has finite action, and we need to show that the same holds under the milder assumption \( V^{I_0} \neq 0 \). It is therefore enough to show that any irreducible unitary representation \( V \) of \( \text{GL}_d(F) \) with \( V^I \neq 0 \) and \( V^{GL_d(O)} = 0 \) is tempered. However, when \( d \in \{2, 3\} \) this follows from the classification of irreducible representations of \( \text{GL}_d(F) \). Specifically, when \( d = 2 \), \( V \) is necessarily a Steinberg representation, and hence tempered. The case \( d = 3 \) is analyzed in [36] §2.2, Table 2. (A description of the unitary irreducible representations of \( \text{GL}_d(F) \) with a nonzero \( I \)-invariant vector can be given using [17, Pr. 2.6] together with the classification of all unitary irreducible representations of \( \text{GL}_d(F) \) by Tadić [77]. See [81] for the classification of the smooth irreducible representations of \( \text{GL}_d(F) \).) \( \square \)

**Remark 6.32.** In [36] Th. 2], the authors show that the \( \Omega^+_d \)-Ramanujan property of quotients of \( \mathcal{B}_d(F) \) can be described using adjacency operators of 1-cells or adjacency operators of 2-cells. This agrees with Proposition 6.31.

The proof of Proposition 6.31 breaks when \( d \geq 4 \) since then \( \text{GL}_d(F) \) has non-tempered irreducible unitary representations \( V \) with \( V^I \neq 0 \) and \( V^K = 0 \); see [77] or [76] below.

We now give an application to isospectral complexes.
Example 6.33. Let $G = \text{PGL}_d(F)$ and $\mathcal{X} = \mathcal{B}_d(F)$ and assume $d \neq 6$. In [53] (see also [52]), Lubotzky, Samuels and Vishne construct arbitrarily large families of non-isomorphic $G$-quotients of $\mathcal{X}$ which are isospectral in the sense that their $0$-dimensional spectrum is the same. They further note that the high-dimensional Laplacians of these complexes also have the same spectrum. Their example is based on constructing non-commensurable cocompact lattices $\Gamma_1, \Gamma_2, \ldots, \Gamma_m \leq G$ such that $L^2(\Gamma_i \backslash G) \cong \ldots \cong L^2(\Gamma_m \backslash G)$ as representations of $G$. In this case, Theorem 6.21 implies that the quotients $\{\Gamma_i \backslash \mathcal{X}\}_{i=1}^m$ have the same $F$-spectrum for any elementary (and hence semi-elementary) functor $F : \mathcal{C}(G, \mathcal{X}) \to \text{phil}$, that is, the $G$-quotients $\{\Gamma_i \backslash \mathcal{X}\}_{i=1}^m$ are completely isospectral. For example, it follows that the spectrum of any high-dimensional adjacency operator is the same on $\{\Gamma_i \backslash \mathcal{X}\}_{i=1}^m$.

Nevertheless, let $\mathfrak{T}$ be such that $L \mathfrak{T}$ is bipartite (i.e. $\mathfrak{T}$ is abelian and $\mathfrak{T}^2 \mathfrak{T}$ contains nonzero $\mathfrak{T}$-invariant vectors (because $\mathfrak{T}^2 \mathfrak{T}$ is of full rank)); this case, every $\mathfrak{T}$ contains a copy of $\mathfrak{C}$.

Let $\mathfrak{C}$ be a normal finite-index subgroup of $G$. Recall from [53] that $\mathfrak{T}_{A,N} = \text{Spec}_{\mathfrak{A}}(\mathfrak{F}(\mathfrak{C}) \mathfrak{N})$.

(i) If $N$ contains $K_b$ for some $n$, then $\mathfrak{T} \subseteq N \iff \text{Spec}_{\mathfrak{A}}(\mathfrak{F} \mathfrak{X} \mathfrak{N}) \subseteq \text{Spec}_{\mathfrak{A}}(\mathfrak{T} \mathfrak{X} \mathfrak{N})$ (as multisets). In this case, every $[V] \in \mathfrak{T}_{A,N}$ has the same multiplicity in $\text{Spec}_{\mathfrak{A}}(\mathfrak{F} \mathfrak{X} \mathfrak{N})$ and $\text{Spec}_{\mathfrak{A}}(\mathfrak{T} \mathfrak{X} \mathfrak{N})$.

(ii) If $\mathfrak{T}_{A,N} = \bigcup_{n=1}^k \mathfrak{T}_{A,NK_n}$.

(iii) If $N$ contains $K_b$ for some $n$, then $\mathfrak{T} \subseteq N \iff \mathfrak{T}_{A,N} \subseteq \text{Spec}_{\mathfrak{A}}(\mathfrak{T} \mathfrak{X} \mathfrak{N})$. In this case, every $[V] \in \mathfrak{T}_{A,N}$ has multiplicity $1$ in $\text{Spec}_{\mathfrak{A}}(\mathfrak{T} \mathfrak{X} \mathfrak{N})$.

Proof. (i) The direction $(\iff)$ is shown exactly as in the proof of Proposition 5.7 so we turn to prove the other direction. Without loss of generality, $K_1 \subseteq N$. By Theorem 6.21(iv), $\hat{\mathfrak{T}}$ induces an isomorphism of multisets $\text{m-Spec}(L^2(\Gamma) \mathfrak{G}) \to \text{m-Spec}(\mathfrak{T} \mathfrak{X} \mathfrak{N})$ (notice that $L^2(\Gamma) \mathfrak{G}$ is admissible by Proposition 5.13 and Example 6.33). Since $\text{m-Spec}(\mathfrak{F}(\mathfrak{X} \mathfrak{N})) \subseteq \text{m-Spec}(\mathfrak{T} \mathfrak{X} \mathfrak{N})$ and $\mathfrak{F}(C^\infty(N \mathfrak{G})) = (\mathfrak{X} \mathfrak{N})$, since every irreducible subrepresentation of $C^\infty(N \mathfrak{G})$ contains nonzero $K_1$-invariant vectors (because $K_1 \subseteq N$), the representation $L^2(\Gamma) \mathfrak{G}$ contains a copy of $C^\infty(N \mathfrak{G})$. Since $N \leq G$, we have $C^\infty(N \mathfrak{G})^N = C^\infty(N \mathfrak{G})$ and $L^2(\Gamma) \mathfrak{G}^N = L^2(\mathfrak{G} \mathfrak{N})$. Dimension considerations now imply that $N = \Gamma N$ and that that every irreducible subrepresentation of $C^\infty(N \mathfrak{G})$ has the same multiplicity in $C^\infty(N \mathfrak{G})$ and $L^2(\Gamma) \mathfrak{G}$. The latter gives the assertion about the multiplicity of elements of $\mathfrak{T}_{A,N}$, thanks to Theorem 6.21(v).

(ii) Since $G/N$ is abelian and $N \leq G$, we have $NK_n \leq G$ for all $1 \leq n \leq t$. This easily implies that $C^\infty(N \mathfrak{G})^{K_n} = C^\infty(NK_n \mathfrak{G})$, and hence $\text{Spec}(\mathfrak{F}(\mathfrak{G})^{K_1,\ldots,K_t}) = \bigcup_{n=1}^t \text{Spec}(\mathfrak{F}(NK_n \mathfrak{G}))$. The assertion now follows from Theorem 6.21(v).

(iii) Since $G/N$ is abelian, every irreducible subrepresentation of $C^\infty(G/N)$ occurs with multiplicity $1$. Invoking this fact in the proof of (i) gives all claims. □

Example 6.35. (i) Taking $\mathcal{X} = \mathcal{T}_k$, $G = G_k$, $F = \Omega_0^+$ and $N = H_k$ (notation as in Corollary 6.25 in Proposition 6.34) we recover the well-known fact that a $k$-regular connected graph $X$ is bipartite (i.e. $X \cong \Gamma \mathcal{X}$ for $\Gamma \leq X H_k$) if and only if $-k$ is an eigenvalue of its adjacency matrix $a_{0,\mathcal{X}}$ (cf. Example 5.12). We also see that the eigenvalues $k$ and $-k$ can occur with multiplicity at most $1$ in $\text{Spec}(a_{0,\mathcal{X}})$.

(ii) Let $\mathcal{X} = \mathcal{B}_d(F)$, $G = \text{PGL}_d(F)$ and $K = \text{PGL}_d(O)$ by as in Chapter 1. We observed in Example 5.13 that the smallest finite-index subgroup of $G$ is
$N := \text{im}(\text{SL}_d(F) ightarrow \text{PGL}_d(F))$. Since $N \leq G$ and $G/N$ is abelian, Proposition 6.34 implies that the trivial 0-dimensional spectrum $\Sigma_{A(\mathcal{E}(G),\Omega^*_G)}$ coincides with $\Sigma_{A(\mathcal{E}(G),\Omega^*_G), NK}$, and for any $NK \leq N' \leq G$ and cocompact $\Gamma \leq X$ $G$, we have $\Sigma_{A(\mathcal{E}(G),\Omega^*_G), N'} \subseteq \text{Spec}_{0}(\Gamma \backslash \mathcal{X}) \iff \Gamma \leq N'$.

6H. Finite Index Subgroups. Let $\mathcal{X}$ be an almost transitive $G$-complex with $G$ unimodular, let $F : \mathcal{E}(G, \mathcal{X}) \rightarrow \text{pHil}$ be an elementary functor with $F \cong \ell^2 \circ S$ as in Definition 4.11, and let $H$ be a finite-index open subgroup of $G$. Then $\mathcal{X}$ is an almost transitive $H$-complex, and for any $\Gamma \leq X$ $H$, we can consider $\Gamma \backslash \mathcal{X}$ both as a $G$-quotient and as an $H$-quotient of $\mathcal{X}$, giving rise to two possible notions of $F$-Ramanujan-ness for $\Gamma \backslash \mathcal{X}$. In this section, we use Theorem 6.22 to show that, in some cases, these two notions are equivalent.

**Theorem 6.36.** Let $\Gamma \leq X$ $H$.

(i) If $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan as an $H$-quotient of $\mathcal{X}$, then it is $F$-Ramanujan as a $G$-quotient of $\mathcal{X}$.

(ii) The converse of (i) holds if there are representatives $x_1, \ldots, x_t$ for the $G$-orbits in $\mathcal{X}$ such that $\text{Stab}_G(x_n) \subseteq H$ for all $1 \leq n \leq t$.

(iii) $\Gamma \backslash \mathcal{X}$ is completely Ramanujan as an $H$-quotient of $\mathcal{X}$ if and only if $\Gamma \backslash \mathcal{X}$ is completely Ramanujan as a $G$-quotient of $\mathcal{X}$.

For the proof, observe that the Haar measure $\mu$ of $G$ restricts to a Haar measure on $H$. We can extend any function $\varphi \in C_c^\infty(H)$ to a function in $C_c^\infty(G)$ by setting $\varphi$ to be 0 on $G - H$. This allows us to view $\mathcal{H}(H)$ as a $*$-subalgebra of $\mathcal{H}(G)$. If $V$ is a unitary representation of $G$, then the $\mathcal{H}(H)$-module structure of $V$ obtained by considering $V$ as a representation of $H$ coincides with the $\mathcal{H}(G)$-module structure obtained by restricting the $\mathcal{H}(G)$-module structure of $V$ to $\mathcal{H}(H)$.

**Proof.** (i) Write $\mathcal{E}_G = \mathcal{E}(G, \mathcal{X})$ and $\mathcal{E}_H = \mathcal{E}(H, \mathcal{X})$ (Definition 4.10). Since $\mathcal{E}_H \subseteq \mathcal{E}_G$, we have a homomorphism of unital $*$-algebras $\Phi : A(\mathcal{E}_G, F) \rightarrow A(\mathcal{E}_H, F)$ given by $\Phi(a \chi_{x \in \mathcal{E}_G}) = (a \chi_{x \in \mathcal{E}_H})$. The map $\Phi$ injective by Theorem 4.13(i), so we may view $A(\mathcal{E}_G, F)$ as an algebra of $(\mathcal{E}_H, F)$-operators. Now, if $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan as an $H$-quotient, i.e. $A(\mathcal{E}_H, F)$-Ramanujan, then it is $A(\mathcal{E}_G, F)$-Ramanujan by Proposition 6.15(ii), and hence $F$-Ramanujan as a $G$-quotient.

(ii) Write $K_n := \text{Stab}_G(x_n) \subseteq H$. Suppose that $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan as a $G$-quotient. Then by Theorem 6.22(i), any irreducible $V \prec L^2(\Gamma \backslash G)$ with $V^{K_1} + \cdots + V^{K_t} \neq 0$ is tempered or has finite action.

Let $U$ be an irreducible unitary representation of $H$ such that $U \prec L^2(\Gamma \backslash H)$ and $U^{K_1} + \cdots + U^{K_t} \neq 0$. We need to show that $U$ is tempered or has finite action. Since $L^2(\Gamma \backslash H)$ is an $H$-subrepresentation of $L^2(\Gamma \backslash G)$, we have $U \prec L^2(\Gamma \backslash G)$. By Theorem 2.31 and the comment preceding our proof, there is an irreducible unitary representation $V$ of $G$ such that $V \prec L^2(\Gamma \backslash G)$ and $U \leq V$. Since $V^{K_1} + \cdots + V^{K_t} \supseteq U^{K_1} + \cdots + U^{K_t} \neq 0$, either $V$ is tempered, or $V$ has finite action.

When the latter holds, $U$ clearly has finite action. On the other hand, if $V$ is tempered, then $V \prec L^2(1 \backslash G)$, and hence $U \prec L^2(1 \backslash G)$ (as representations of $H$). Since $L^2(1 \backslash G) \cong L^2(1 \backslash H)[G,H]$ as representations of $H$, Theorem 2.31 implies that $U \prec L^2(1 \backslash H)$, so $U$ is tempered, as required.

(iii) One direction follows from (i). The proof of the other direction is similar to the proof of (ii) — simply drop the condition $U^{K_1} + \cdots + U^{K_t} \neq 0$. □

**Example 6.37.** Let $T_k$ be a $k$-regular tree, and let $G = \text{Aut}(T_k)$. Choose a coloring $C_0 : T_k^{(0)} \rightarrow \{0,1\}$ such that each edge in $T_k$ consists of vertices of different colors, and let $H$ be the index 2 subgroup of $G$ preserving the coloring $C_0$. Then $H$-quotients of $T_k$ can be regarded as bipartite $k$-regular graphs with a $\{0,1\}$-coloring.
This coloring gives rise to a richer spectral theory since one can consider the spectra of associated operators [112] taking colors into account. However, despite the extra associated operators, an $H$-quotient of $T_κ$ is completely Ramanujan if and only if it is Ramanujan in dimension 0 as a $G$-quotient (i.e. Ramanujan as a $k$-regular graph in the classical sense). This follows from Corollary 6.30 and Theorems 6.36(iii).

**Example 6.38.** Let $G = \text{PGL}_d(F)$ and $\mathcal{X} = B_d(F)$ be as in Chapter 1. Let $H$ be the index $d$ subgroup of $G$ preserving the vertex coloring $C_0 : B_d(F)^{(0)} \to \mathbb{Z}/d\mathbb{Z}$ (explicitly, $H = \ker(c : G \to \mathbb{Z}/d\mathbb{Z})$ where $c$ is as in Chapter 1). The $H$-quotients of $B_d(F)^{(0)}$ inherit the coloring $C_0$ and hence admit a richer spectral theory. However, since $G$-stabilizers of maximal flags in $B_d(F)$ are contained in $H$, Theorem 6.36(ii) implies that an $H$-quotient of $B_d(F)$ is flag Ramanujan as an $H$-quotient if and only if it is flag Ramanujan as a $G$-quotient.

Furthermore, we mentioned in Remark 4.33 that when $d > 2$, the group $G$ is of index 2 in $\tilde{G} := \text{Aut}(B_d(F))$. It is easy to check that the pointwise $\tilde{G}$-stabilizer of any maximal flag in $B_d(F)$ still lies in $H$, and hence an $H$-quotient of $B_d(F)$ is flag Ramanujan as $H$-quotient if and only if it is flag Ramanujan as an $\tilde{G}$-quotient. This remains true if we replace $H$ with any subgroup between $H$ and $\tilde{G}$.

We also observed in Example 6.13 that the smallest open finite-index subgroup of $G$ is $G_0 := \text{im}(\text{SL}_d(F) \to \text{PGL}_d(F))$. (In fact, $G_0 = H$ if $d$ is coprime to $q(q-1)$, where $q$ is the cardinality of the residue field of $F$.) By Theorem 6.36(iii), for any $H'$ between $G_0$ and $\tilde{G}$ and any $H'$-quotient $\Gamma\backslash B_d(F)$, the condition that $\Gamma\backslash B_d(F)$ is a completely Ramanujan $H'$-quotient is independent of the choice of $H'$, so long as $\Gamma \subseteq H'$.

### 7. Automorphic Representations

Let $k$ be a global field, let $G$ be a simple algebraic group over $k$, let $\nu$ be a non-archimedean place of $k$, and let $F$ be the completion of $k$ at $\nu$. Let $B$ be the affine Bruhat-Tits building of $G := G(F)$. Then $G$ is an $\ell$-group and $B$ is a $G$-complex [64]. Let $\Gamma$ be a congruence subgroup of $G(k)$ such that $\Gamma \leq_B G$ [31]. In this final chapter, we relate certain properties of automorphic representations of $G$ with the condition that the $G$-quotient $\Gamma\backslash B$ is Ramanujan [54]. This is used, together with deep results about automorphic representations, to give examples of infinite families of $G$-quotients which are completely Ramanujan when the group $G$ is an inner form of $\text{PGL}_n$ and char $k > 0$.

We note that such ideas were applied, sometimes implicitly, in [44], [53], [92], [61], [44] and [11] to construct infinite families of Ramanujan regular graphs, Ramanujan complexes (in the sense of Chapter 1), and Ramanujan biregular graphs (see [11]). All of these constructions rely on powerful results about automorphic representations, e.g. the proof of the Ramanujan–Petersson conjecture for $\text{GL}_n$ in positive characteristic by Lafforgue [11] (see also [21], [19]).

Continuing this approach, we Lafforgue’s work together with recent results about the Jacquet–Langlands correspondence ([15] and related works) to show that for any central division algebra $D$ over $F$, the affine Bruhat–Tits building of $\text{PGL}_d(D)$ has infinitely many $\text{PGL}_d(D)$-quotients which are completely Ramanujan [54]. When $D = F$, our construction gives the Ramanujan complexes constructed by Lubotzky, Samuels and Vishne in [61]. Thus, the Ramanujan complexes of [61], which are Ramanujan in dimension 0 (Example 6.13(ii)), are in fact completely Ramanujan.

We alert the reader that in sections 7.1 and 7.2 it is essential that the base field $k$ has positive characteristic.
7A. Adeles. Throughout, \( k \) denotes a global field and \( \mathcal{V} \) is the set of places of \( k \). For the sake of simplicity, we shall assume that \( k \) has positive characteristic, and hence \( \mathcal{V} \) consists entirely of non-archimedean places. All results in this chapter excluding those in [71] and [72] hold when \( k \) is a number field after suitable modifications.

For \( \nu \in \mathcal{V} \), let \( k_\nu \) denote the completion of \( k \) at \( \nu \). We also let \( \nu \) stand for the additive valuation \( \nu : k_\nu \to \mathbb{Z} \cup \{ \infty \} \). The integer ring of \( k_\nu \) is denoted \( \mathcal{O}_\nu \) and we fix a generator \( \pi_\nu \) of the maximal ideal of \( \mathcal{O}_\nu \). For every \( a \in k_\nu \), let

\[
|a|_\nu = q_\nu^{-\nu(a)},
\]

where \( q_\nu := |\mathcal{O}_\nu/\pi_\nu \mathcal{O}_\nu| \).

Let \( S \subseteq \mathcal{V} \) be finite set of places. We denote by \( \mathbb{A}^S \) the ring of adeles over \( k \) away from \( S \), that is,

\[
\mathbb{A}^S = \prod_{\nu \in \mathcal{V} \setminus S} k_\nu := \left\{ (a_\nu)_{\nu} \in \prod_{\nu \in \mathcal{V} \setminus S} k_\nu : a_\nu \in \mathcal{O}_\nu \text{ for almost all } \nu \right\}.
\]

We also write \( \mathbb{A} = \mathbb{A}^0 \). The field \( k \) is a subring of \( \mathbb{A}^S \) via the diagonal embedding, and each of the fields \( k_\nu \) \((\nu \in \mathcal{V} \setminus S)\) embeds as summand of \( \mathbb{A}^S \). We endow \( \prod_{\nu \in \mathcal{V} \setminus S} \mathcal{O}_\nu \) with the product topology, and topologize \( \mathbb{A}^S \) by viewing it as a disjoint union of (additive) cosets of \( \prod_{\nu \in \mathcal{V}} \mathcal{O}_\nu \). The ring \( \mathbb{A}^S \) is therefore a locally compact topological ring. By the product formula, \( k \) is discrete in \( \mathbb{A} \).

Let \( \mathbb{G} \) be an algebraic group over \( k \) with a fixed closed embedding \( j : \mathbb{G} \to \mathrm{GL}_n \). If \( R \) is a commutative domain whose fraction field \( F \) contains \( k \), we set

\[
\mathbb{G}(R) = j(\mathbb{G}(F)) \cap \mathrm{GL}_n(R).
\]

When \( R \) is a topological ring, we embed \( \mathrm{GL}_n(R) \) in \( \mathrm{SL}_{n+1}(R) \) via \((a_{ij}) \mapsto (a_{ij}) \oplus (\det(a_{ij}))^{-1})\) and give \( \mathbb{G}(R) \) the topology induced from \( \mathrm{SL}_{n+1}(R) \subseteq M_{n+1}(R) \cong R^{(n+1)^2} \). This makes \( \mathbb{G}(R) \) into a topological group, which is an \( \ell \)-group if \( R \) is totally disconnected and locally compact. In particular, \( \mathbb{G}(k_v) \) and \( \mathbb{G}(\mathbb{A}) \) are \( \ell \)-groups. We note that the (topological) group \( \mathbb{G}(R) \) is independent of the embedding \( j \) when \( R \) contains \( k \).

For \( \nu \in \mathcal{V} \) and \( I \subseteq \mathcal{O}_\nu \), the \( I \)-congruence subgroup of \( \mathbb{G}(k_\nu) \) is

\[
\mathbb{G}(\mathcal{O}_\nu, I) := \ker \left( \mathbb{G}(\mathcal{O}_\nu) \xrightarrow{j} \mathrm{GL}_n(\mathcal{O}_\nu) \to \mathrm{GL}_n(\mathcal{O}_\nu/I) \right).
\]

The subgroups \( \{ \mathbb{G}(\mathcal{O}_\nu, \pi_\nu^n \mathcal{O}_\nu) \}_{n \geq 0} \) form a basis of compact open neighborhoods at the identity. The group

\[
\mathbb{G}(\mathbb{A}) = \prod_{\nu \in \mathcal{V}} \mathbb{G}(k_\nu) := \left\{ (g_\nu)_{\nu} \in \prod_{\nu \in \mathcal{V}} \mathbb{G}(k_\nu) : g_\nu \in \mathbb{G}(\mathcal{O}_\nu) \text{ for almost all } \nu \right\}
\]

has \( \prod_{\nu \in \mathcal{V}} \mathbb{G}(\mathcal{O}_\nu) \) as a compact open subgroup. Thus, the collection

\[
\left\{ \prod_{\nu \in \mathcal{V}} \mathbb{G}(\mathcal{O}_\nu, \pi_\nu^{n_\nu} \mathcal{O}_\nu) : n_\nu \in \mathbb{N} \cup \{0\} \text{ and } n_\nu = 0 \text{ for almost all } \nu \right\}
\]

is basis of compact open neighborhoods at the identity in \( \mathbb{G}(\mathbb{A}) \). Since \( k \) is discrete in \( \mathbb{A} \), the group \( \mathbb{G}(k) \) is a discrete subgroup of \( \mathbb{G}(\mathbb{A}) \).

When \( \mathbb{G} \) is reductive, the groups \( \mathbb{G}(\mathbb{A}) \) and \( \mathbb{G}(k_\nu) \) \((\nu \in \mathcal{V})\) are unimodular, and the quotient \( \mathbb{G}(k) \backslash \mathbb{G}(\mathbb{A}) \) is compact precisely when \( \mathbb{G} \) is \( k \)-anisotropic; see [70] Th. 5.5 and [32] Cr. 2.2.7.
7B. Smooth Representations Revised. Let $G$ be an $\ell$-group and let $P$ be a closed subgroup of $G$. As usual, given a smooth $P$-module $V$ (see [A]), we let
\[ \text{Ind}_{P}^{G}(V) \]
denote the vector space of locally constant functions $\varphi : G \to V$ satisfying $\varphi(pg) = pg\varphi(g)$ for all $p \in P$, $g \in G$. We make $\text{Ind}_{P}^{G}(V)$ into a left $G$-module by defining $(g \cdot \varphi)h = \varphi(hg)$ for all $g, h \in G$. The $G$-module $\text{Ind}_{P}^{G}(V)$ is smooth when $P \backslash G$ is compact, which will always be the case in the sequel.

Throughout, a character of $G$ means a continuous group homomorphism $\xi : G \to \mathbb{C}^{\times}$. Since $G$ is an $\ell$-group, its characters are locally constant. The character $\xi$ is unitary if $|\xi(g)| = 1$ for all $g \in G$. If $V$ is a smooth $G$-module, we let $\xi V$ denote the vector space $V$ endowed with the $G$-action $(g, v) \mapsto \xi(g)gv$. The $G$-module $\xi V$ is clearly smooth. Characters are in correspondence with 1-dimensional smooth representations of $G$, considered up to isomorphism. In the sequel, we shall freely interchange between characters and their corresponding representations.

We proceed by recalling several facts about admissible representations that will be used in the sequel, often without comment. Note in particular that if $G$ is a reductive algebraic group over $k_{0}$, then all irreducible smooth (resp. unitary) representations of $G(k_{0})$ are admissible ([12], for instance). Here, as usual, a smooth representation $V$ of $G$ is said to be irreducible if it has no nonzero proper $G$-submodules. The class of irreducible smooth $G$-representations is denoted $\text{Irr}_{\text{sm}}(G)$.

**Theorem 7.1** (Schur’s Lemma). Let $V$ be an admissible irreducible smooth representation of $G$. Then $\text{End}_{G}(V) = \mathbb{C} \cdot \text{id}_{V}$.

**Proof.** Let $\psi \in \text{End}_{G}(V)$. Since $V$ is smooth, there is $K \leq k_{0}$. $G$ such that $V^{K} \neq 0$, and since $V$ is admissible, $\dim V^{K} < \infty$. Clearly $\psi(V^{K}) = V^{K}$, hence there exists $\lambda \in \mathbb{C}$ and $0 \neq v \in V^{K}$ such that $v \in \ker(\psi - \lambda)$. Since $\ker(\psi - \lambda)$ is a $G$-submodule of $V$ and $V$ is irreducible, we must have $\ker(\psi - \lambda) = V$, so $\psi = \lambda \text{id}_{V}$. □

Let $Z$ be the center of $G$ and let $V$ be an admissible irreducible smooth representation of $G$. By Schur’s Lemma, there is a character $\chi_{V} : Z \to \mathbb{C}^{\times}$, called the central character of $V$, satisfying $gv = \chi_{V}(g)v$ for all $g \in Z$, $v \in V$.

Let $V$ be a smooth $G$-module. The group $G$ acts on $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ via $(g\phi)v = \phi(g^{-1}v)$. Let $V^{*}$ denote the smooth part of $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and let
\[ \check{V} \]
denote the $G$-module obtained from $V^{*}$ by twisting the $\mathbb{C}$-vector-space structure via the complex conjugate. The representation $\check{V}$ is called the contragradient representation of $V$. It is admissible if and only if $V$ is admissible. If $U$ is another smooth representation and $f : U \to V$ is $G$-equivariant, then we define $\hat{f} : \check{V} \to \check{U}$ by $\hat{f}(\varphi) = \varphi \circ f$. This makes $V \mapsto \check{V}$ into a contravariant functor which restricts to a duality on the full subcategory of admissible representation of $G$.

**Proposition 7.2.** Let $V$ be an irreducible admissible smooth representation. Then, up to scaling, there exists at most one inner product $\langle \ , \ \rangle : V \times V \to \mathbb{C}$ such that $\langle gu, gv \rangle = \langle u, v \rangle$ for all $g \in G$ and $u, v \in V$.

**Proof.** Both $V$ and $\check{V}$ are admissible and irreducible, hence any inner product $\langle \ , \ \rangle : V \times V \to \mathbb{C}$ induces an isomorphism $V \to \check{V}$ mapping $v \in V$ to $[u \mapsto \langle v, u \rangle]$. Schur’s Lemma implies that any two such isomorphisms $V \to \check{V}$ must be the same up to scaling. (It should be noted that we cannot use Proposition 2.8 with $A = \mathcal{A}(G)$ because there is no assumption of continuity.) □
An admissible irreducible smooth $G$-representation $V$ admitting an inner product as in Proposition 7.2 is called unitarizable. In this case, we can view $V$ as a pre-unitary representation, well-defined up to isomorphism.

A pre-unitary representation of $G$ is called irreducible if it is irreducible as a smooth representation. The class of irreducible pre-unitary representations of $G$ is denoted $\text{Irr}^{su}(G)$. By Theorem 2.39 (applied with $A = \mathscr{H}(G)$, cf. 6C), an admissible pre-unitary representation $V$ is irreducible if and only if its completion $\overline{V}$ is irreducible as a unitary representation.

Let $\mathcal{V}$ be an irreducible admissible pre-unitary representation of $G$. For any $u, v \in \mathcal{V}$, define $\varphi_{u, v} : G \to \mathbb{R}$ by

$$\varphi_{u, v}(g) = |\langle gu, v \rangle|.$$ 

Since the central character $\chi$ is unitary, $\varphi_{u, v}$ is $Z$-invariant and hence we may view it as a function on $G/Z$. Recall that $V$ is called supercuspidal if $\varphi_{u, v}$ is compactly supported as function on $G/Z$ for all $u, v \in \mathcal{V}$, and $V$ is called square-integrable if $\varphi_{u, v} \in L^2(G/Z)$ for all $u, v \in \mathcal{V}$. When $G = Z$, all irreducible pre-unitary representations are supercuspidal.

Supercuspidal representations are clearly square-integrable. When $Z = 1$ or $G$ is the group of $k_{\nu}$-points of a reductive algebraic group $G$ over $k_{\nu}$, the square-integrable representations of $G$ are tempered; see [59, Pr. 9.6] for the case $Z = 1$ and [64, 2.4] or [82, p. 265] for the other case.

7C. Local Factors. Let $G$ be a reductive algebraic group over $k$, and write $K_{\nu} = G(O_{\nu})$ (see 7A). The facts stated here are proved in [25].

For every $\nu \in \mathcal{V}$, choose an irreducible smooth representation $V_{\nu} \in \text{Irr}^{sm}(G(k_{\nu}))$. As noted in 7B, the representations $\{V_{\nu}\}_{\nu \in \mathcal{V}}$ are admissible. Suppose that for almost all $\nu$ we have $V_{\nu}^{K_{\nu}} \neq 0$. When this holds, choose a nonzero vector $v_{\nu} \in V_{\nu}^{K_{\nu}}$, and otherwise, choose an arbitrary nonzero vector $v_{\nu} \in V_{\nu}$. Let

$$\bigotimes_{\nu \in \mathcal{V}} V_{\nu} = \lim_{\nu \in \mathcal{V}} \bigotimes_{\nu \in \mathcal{V}} V_{\nu}$$

where $S$ ranges over the finite subsets of $\mathcal{V}$, and for $S \subseteq S'$, we embed $\bigotimes_{\nu \in S} V_{\nu}$ in $\bigotimes_{\nu \in S'} V_{\nu}$ via $\bigotimes_{\nu \in S} v_{\nu} \mapsto \bigotimes_{\nu \in S'} v_{\nu} \otimes \bigotimes_{\nu \in S' \setminus S} v_{\nu}$. It turns out that $V_{K_{\nu}}$ is 1-dimensional for almost all $\nu$, and hence $\bigotimes_{\nu \in \mathcal{V}} V_{\nu}$ is independent of the vectors $v_{\nu} \in V_{\nu}$, up to isomorphism.

The space $\bigotimes_{\nu \in \mathcal{V}} V_{\nu}$ carries an obvious $G(\mathbb{A})$-action making it into an admissible smooth representation of $G(\mathbb{A})$. The representation $\bigotimes_{\nu \in \mathcal{V}} V_{\nu}$ is irreducible, and conversely, every $V \in \text{Irr}^{sm}(G(\mathbb{A}))$ can be factored as $V = \bigotimes_{\nu \in \mathcal{V}} V_{\nu}$, where the factors $\{V_{\nu}\}_{\nu \in \mathcal{V}}$, called the local factors of $V$, are unique up to isomorphism.

A necessary and sufficient condition for $U \in \text{Irr}^{sm}(G(k_{\nu}))$ to be isomorphic to the $\nu$-local-factor of $V \in \text{Irr}^{sm}(G(\mathbb{A}))$ is to have $U \subseteq V$ when $V$ is viewed as a $G(k_{\nu})$-module. Furthermore, if we are given for all $\nu \in \mathcal{V}$ a compact open subgroup $L_{\nu} \leq G(k_{\nu})$ such that $L_{\nu} = K_{\nu}$ for almost all $\nu \in \mathcal{V}$, then $V = \bigotimes_{\nu \in \mathcal{V}} V_{\nu}$ has a nonzero $\prod_{\nu \in \mathcal{V}} L_{\nu}$-invariant vector if and only if each factor $V_{\nu}$ has a nonzero $L_{\nu}$-invariant vector.

Everything stated above remains correct if one replaces smooth representations with pre-unitary representations. One should then chose the vectors $\{v_{\nu}\}_{\nu \in \mathcal{V}}$ to be unit vectors.

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11 Some texts use the term absolutely cuspidal.

12 We believe this statement is correct for a general $\ell$-group $G$. 
7D. Automorphic Representations. Let $G$ be a reductive algebraic group over $k$ and let $Z$ be the center of $G$. Fix a smooth unitary character

$$\omega : Z(\mathbb{A})/\mathbb{A}(k) \to \mathbb{C}^\times.$$ 

We let $L^2_c(G(k)\backslash G(\mathbb{A}))$ denote the space of measurable functions $\varphi : G(k)\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying $\varphi(ag) = \omega(a)\varphi(g)$ for all $a \in Z(\mathbb{A})/\mathbb{A}(k)$ and whose absolute value $|\varphi|$ is square-integrable when viewed as a function on $Z(\mathbb{A})G(k)\backslash G(\mathbb{A})$. The space $L^2_c(G(k)\backslash G(\mathbb{A}))$ is a Hilbert space with respect to the inner product

$$\langle \varphi, \psi \rangle = \int_{x \in Z(\mathbb{A})G(k)\backslash G(\mathbb{A})} \overline{\varphi(x)} \cdot \psi(x) \, d\mu_{Z(\mathbb{A})G(k)\backslash G(\mathbb{A})}.$$ 

Here, $\mu_{Z(\mathbb{A})G(k)\backslash G(\mathbb{A})}$ is a fixed right $G(\mathbb{A})$-invariant measure on $Z(\mathbb{A})G(k)\backslash G(\mathbb{A})$, and $\overline{\varphi(x)} \cdot \psi(x)$ means $\varphi(x') \cdot \overline{\psi(x')}$ where $x = Z(\mathbb{A})x'$ (this expression is independent of the choice of the representative $x'$). Note that $\overline{\varphi(x)} \cdot \overline{\psi(x)}$ is right $Z(\mathbb{A})G(k)$-invariant and can hence be regarded as a function on $Z(\mathbb{A})G(k)\backslash G(\mathbb{A})$.

We let $G(\mathbb{A})$ act on $L^2_c(G(k)\backslash G(\mathbb{A}))$ via $(\varphi x) = \varphi(xy)$, making $L^2_c(G(k)\backslash G(\mathbb{A}))$ into a unitary representation. Recall that an automorphic representation of $G$ with central character $\omega$ is an irreducible pre-unitary representation $V \in \text{Irr}^{pu}(G(\mathbb{A}))$ such that

$$V \prec L^2_c(G(k)\backslash G(\mathbb{A})).$$ 

The representation $V$ is said to be in the discrete spectrum or just discrete if $V \leq L^2_c(G(k)\backslash G(\mathbb{A}))$. In this case, $V$ is admissible and can therefore be written as a product of local factors $\otimes'_\nu V_\nu$ as in [7C].

When $Z$ is finite (as an algebraic group), the group $Z(\mathbb{A})$ is profinite, and in particular compact. Viewing $L^2(G(k)\backslash G(\mathbb{A}))$ as a representation of $Z(\mathbb{A})$ and applying the Peter-Weyl Theorem shows that

$$L^2(G(k)\backslash G(\mathbb{A})) = \bigoplus_{\omega} L^2_c(G(k)\backslash G(\mathbb{A})),$$

where $\omega$ ranges over the unitary characters of $Z(\mathbb{A})/\mathbb{A}(k)$ (one has to choose the Haar measure of $Z(\mathbb{A})/\mathbb{A}(k)$ to have total measure 1 and the right $G(\mathbb{A})$-invariant measures on $G(\mathbb{A})$ and $Z(\mathbb{A})G(k)\backslash G(\mathbb{A})$ such that

$$\int_{x \in Z(\mathbb{A})G(k)\backslash G(\mathbb{A})} \varphi(xy) \, d\mu_{Z(\mathbb{A})/\mathbb{A}(k)} \int_{y \in Z(\mathbb{A})/\mathbb{A}(k)} \varphi(xy) \, d\mu_{Z(\mathbb{A})/\mathbb{A}(k)}$$

for all $\varphi \in C_c^{\infty}(G(k)\backslash G(\mathbb{A}))$. We may therefore define automorphic representations of $G$ as irreducible pre-unitary representations $V \in \text{Irr}^{pu}(G(\mathbb{A}))$ such that $V \prec L^2(G(k)\backslash G(\mathbb{A}))$.

Assume henceforth that $Z$ is finite and $G(k)\backslash G(\mathbb{A})$ is compact. This holds when $G$ is semisimple and $k$-anisotropic. Then $L^2(G(k)\backslash G(\mathbb{A}))$ is admissible (Example [5.3]), and hence any automorphic representation is in the discrete spectrum (Theorem 2.49). We now determine under what conditions a pre-unitary representation of $G(k_\nu)$ is a local factor of a discrete automorphic representation.

Fix $\eta \in V$. We shall view $G(\mathbb{A})$ as $G(k_\eta) \times G(\mathbb{A}^{(\eta)})$ (notation as in [7A]). Choose $K_\eta \leq_c G(k_\eta)$ and $K^\eta \leq_c G(\mathbb{A}^{(\eta)})$, and let $K := K_\eta \times K^\eta \leq_c G(\mathbb{A})$. The double coset space

$$G(k)\backslash G(\mathbb{A})/(G(k_\eta) \times K^\eta)$$

is compact and discrete, hence finite. Let $(1, g_1), \ldots, (1, g_t) \in G(k_\eta) \times G(\mathbb{A}^{(\eta)})$ be representatives for the double cosets. For each $1 \leq i \leq t$, define

$$\Gamma_i = G(k) \cap (G(k_\eta) \times g_i K^\eta g_i^{-1})$$
and view \( \Gamma_i \) as a subgroup of \( G(k_\eta) \). It is a standard fact that there is an isomorphism of topological (right) \( G(k_\eta) \)-spaces

\[
\bigsqcup_{i=1}^{t} \Gamma_i \backslash G(k_\eta) \to G(k) \backslash G(A)/(1 \times K^n)
\]

given by sending \( \Gamma_i g \) to \( G(k)(g, g_i)(1 \times K^n) \). In particular, \( \Gamma_i \) is a cocompact lattice in \( G(k_\eta) \) for all \( 1 \leq i \leq t \).

**Proposition 7.3.** In the previous setting, the following holds:

(i) Let \( U \) be an irreducible pre-unitary subrepresentation of \( L^2(\Gamma_i \backslash G(k_\eta)) \) such that \( U^K_{\eta} \neq 0 \). Then \( U \) is the \( \eta \)-local-factor of an automorphic representation \( V \) of \( G \) with \( V^K \neq 0 \).

(ii) Conversely, if \( V = \bigotimes \nu V_\nu \) is an automorphic representation of \( G \) such that \( V^K \neq 0 \), then there is \( 1 \leq i \leq t \) such that \( V_\eta \) is isomorphic to a subrepresentation of \( L^2(\Gamma_i \backslash G(k_\eta)) \) and \( V^K_{\eta} \neq 0 \).

**Proof.** (i) By the previous discussion, we may view \( U \) as a \( G(k_\eta) \)-submodule of \( L^2(G(k) \backslash G(A)/(1 \times K^n)) \). Since \( L^2(G(k) \backslash G(A)/(1 \times K^n)) \) decomposes as a direct sum of irreducible unitary representations of \( G(A) \), the projection of \( U \) onto one of those representations, call it \( V_1 \), must be nonzero, so \( U \) must be a local factor of \( V := (V_1)^{\otimes m} \). Since \( V^K \) contains a copy of \( U^K_{\eta} \subseteq L^2(G(k) \backslash G(A)/K) \), we have \( V^K \neq 0 \).

(ii) Let \( 0 \neq \varphi \in V^K \). Since \( V \) is in the discrete spectrum, we may view \( \varphi \) as a function in \( L^2(G(k) \backslash G(A)/(1 \times K^n)) \), which is isomorphic to \( \bigoplus_{i=1}^{t} L^2(\Gamma_i \backslash G(k_\eta)) \) as \( G(k_\eta) \)-modules. Let \( U_1 \) be the \( G(k_\eta) \)-module generated by \( \varphi \). Then there is \( i \) such that the projection of \( U_1 \) onto \( L^2(\Gamma_i \backslash G(k_\eta)) \) is nonzero. Let \( U \) be an irreducible smooth \( G(k_\eta) \)-submodule of this image (it exists because \( L^2(\Gamma_i \backslash G(k_\eta)) \) is admissible and hence completely reducible, cf. Theorem 2.49). Then we must have \( U = V_\eta \). That \( V^K_{\eta} \neq 0 \) follows from \( V^K \neq 0 \); see (C). Suppose now that \( G \) is almost simple, let \( G = G(k_\eta)/Z(k_\eta) \) and write

\[
\Gamma_i = \text{im}(\Gamma_i \to G).
\]

Then \( G \) acts faithfully on the affine Bruhat-Tits building \( B \) of \( G(k_\eta) \), making it into an almost transitive \( G \)-complex (Example 3.3). Recall from 3.1 that for \( \Gamma \leq G \), we write \( \Gamma \leq B \) to denote that \( \Gamma \backslash B \) is a simplicial complex and that the quotient map \( B \to \Gamma \backslash B \) is a cover map. Combining Proposition 7.3 with Theorem 6.22 and Remark 6.23 we get:

**Theorem 7.4.** Let \( \mathcal{C}(G, B) \to \mathfrak{phil} \) be an elementary functor (e.g. \( \Omega^1_c, \Omega^1_{\text{flag}} \) or \( \Omega^1_{\text{flag}} \)). Write \( F \equiv \mathcal{F} \circ S \) as in Definition 4.14, let \( x_1, \ldots, x_s \) be representatives for the \( G \)-orbits in \( SB \), and let \( L_j = \text{Stab}_{G(k_\eta)}(x_j) \times K^n \) (\( 1 \leq j \leq s \)). Assume that for any automorphic representation \( V = \bigotimes \nu V_\nu \) of \( G \) with \( V^{L_1} \oplus \ldots V^{L_s} \neq 0 \) (resp. \( V^Z(k_\eta) \times K^n \neq 0 \)), the local factor \( V_\eta \) is tempered or finite-dimensional. Then \( \Gamma_i \backslash B \) is \( F \)-Ramanujan (resp. completely Ramanujan) for every \( 1 \leq i \leq t \) such that \( \Gamma_i \leq B \). The converse holds when \( \Gamma_i \leq B \) for all \( 1 \leq i \leq t \).

**Remark 7.5.** (i) Fix one of the representatives \( g_i \) and choose some \( \rho \in V - \{ \eta \} \) and \( K^{(\eta, \rho)} \leq \text{c.o.} \ G(A^{(\eta, \rho)}) \). For every \( n \geq 0 \), let \( K^n(\eta) = G(\O_\rho, \pi_\rho \O_\rho) \times K^{(\eta, \rho)} \), \( \Gamma_i(\eta) = G(k) \cap (G(k_\eta) \times g_i K^n(\eta) g_i^{-1}) \) and \( \Gamma_i(\eta) = \text{im}(\Gamma_i(\eta) \to G) \). Then \( \{ \Gamma_i(\eta) \}_{n \geq 0} \) is a decreasing family of normal subgroups of \( \Gamma_i(0) \) with trivial intersection. Since \( \Gamma_i(0) \backslash B \) is finite, Corollary 3.12 implies that there is \( n_0 \in \mathbb{N} \) such that \( \Gamma_i(\eta) \leq B \). In particular, we have \( \Gamma_i \leq G \) for \( \Gamma = G(k) \cap (G(k_\eta) \times g_i K^n g_i^{-1}) \).
whenever $K^n \subseteq K^n(n_0)$. Notice, however, that replacing $K^n$ with $K^n(n_0)$ may increase the number of double cosets in $G(k) \backslash G(A)/(G(k_0) \times K^n)$ and $n_0$ depends a priori on the representative $g_i$.

(ii) When $G$ is simply-connected and $k_0$-isotropic, strong approximation ([57, 71]) implies that $G(k) \backslash G(A)/(G(k_0) \times K^n)$ consists of a single double coset for every $K^n$. Thus, by (i), there is $K^n \leq_{c.o.} G(k_0)$ such that $\Gamma_0 = \Gamma_1 \leq_{c.o.} G(k_0)$ whenever $K^n \subseteq K^n$, in which case the last statement of Theorem 7.4 can be applied.

Given information about the automorphic spectrum of $G$, one can apply Theorem 7.4 to show existence of Ramanujan $G$-quotients of $B$. We mention here several places where such ideas were applied in the literature, sometimes implicitly or in an equivalent formulation:

- Lubotzky, Phillips and Sarnak [45], and independently Margulis [58], constructed infinite families of Ramanujan $(p + 1)$-regular graphs for every prime $p$ using results of Eichler [22] and Igusa [34] about modular forms. (See also Deligne’s proof of the Ramanujan–Petersson conjecture for modular forms [19].) In our setting, this corresponds to taking $k = \mathbb{Q}$ and $G$ to be an inner form of $\text{PGL}_2$ which splits over $k_0$.
- Morgenstern [62] used Drinfeld’s proof of the Ramanujan–Petersson conjecture for $\text{GL}_2$ when $\text{char} k > 0$ [21] to construct infinite families of Ramanujan $(q + 1)$-regular graphs for every prime power $q$. Again, the corresponding group $G$ is an inner form of $\text{PGL}_2$.
- Lubotzky, Samuels and Vishne [51] applied Lafforgue’s proof of the Ramanujan–Petersson conjecture for $\text{GL}_d$ when $\text{char} k > 0$ [11] to construct infinite families of Ramanujan complexes (in the sense of Chapter 1). The corresponding group $G$ is an inner form of $\text{PGL}_n$ which splits over $k_0$.
- Li [41] independently gave similar constructions of Ramanujan complexes, using results of Laumon, Rapoport and Stuhler, who proved a special case of the Ramanujan–Petersson conjecture for anisotropic inner forms of $\text{GL}_n$ [13, Th. 14.12]. (In fact, in [41] it is only shown that the complexes are $\mathbb{C}[a_i, a_i^*]$-Ramanujan for all $0 < i < d$ (notation as in Chapter 1 and [50] cf. Proposition 2.22). However, they are in fact Ramanujan in the sense of Chapter 1 by [51, Pr. 1.5] or Example 6.27)
- Ballantine and Ciubotaru [11] constructed infinite families of Ramanujan $(q + 1, q^3 + 1)$-biregular graphs for every prime power $q$. The corresponding group $G$ is an inner form of $\text{SU}(3)$, and they use the classification of the automorphic spectrum of $G$ due to Rogawski [73].

We hope our work will facilitate further results of this kind.

Let $D$ be a central division algebra over $k_0$. The rest of this chapter concerns with showing that when $\text{char} k > 0$, the affine building of $\text{PGL}_d(D)$ admits infinitely many non-isomorphic $\text{PGL}_d(D)$-quotients which are completely Ramanujan. The proof uses Theorem 7.4 to transfer the problem to a question about automorphic representations, and then applies Lafforgue’s work on the Ramanujan conjecture for $\text{GL}_n$ [11], together with the Jacquet–Langlands correspondence in positive characteristic, established by Badulescu and Roche [9].

7E. The Affine Building of $\text{PGL}_d(D)$. Fix $\nu \in \mathcal{V}$ and let $F = k_{\nu}$. Given a central simple $F$-algebra $A$, let $\deg A$ denote the degree of $A$, and let $\text{Nrd}_{A/F} : A \to F$ denote the reduced norm map; see [10] §1 for the relevant definitions and further details. There is a reductive algebraic group $\text{GL}_{n, A}$ over $F$, unique up to isomorphism, such that for every commutative $F$-algebra $R$, the groups $\text{GL}_{n, A}(R)$ and $\text{GL}_{n, A}(A \otimes_F R)$ are naturally isomorphic. The topology on $F$ induces a topology
on $GL_n(A) = GL_{n,A}(F)$, making it into an $\ell$-group $\mathfrak{G}_A$. We further let $PGL_{n,A} = GL_{n,A}/Z$, where $Z \cong \text{G}_m,F$ is the center of $GL_{n,A}$.

Let $D$ be a finite dimensional central division $F$-algebra of degree $r$. By [72, §12], the additive valuation $\nu : F \to \mathbb{Z} \cup \{\infty\}$ extends uniquely to an additive valuation $\nu_D : D \to \mathbb{R} \cup \{\infty\}$ given by:

$$\nu_D(x) = r^{-1}\nu(\text{Nrd}_{D/F}(x)).$$

Since the residue field of $F$ is finite, $\text{im}(\nu_D) = \frac{1}{r}\mathbb{Z}$ and the residue division ring of $(D, \nu_D)$ is the Galois field of cardinality $q^r$ [72, Th. 14.3]. We fix an element $\pi_D \in D$ with $\nu_D(\pi_D) = \frac{1}{r}$ and write

$$\mathcal{O}_D = \{x \in D : \nu_D(x) \geq 0\}.$$

The topology on $D = GL_{d,D}(F)$ coincides with the topology induced by $\nu_D$ and the topology on $M_d(D) = GL_{d,D}(F)$ coincides with the topology induced from $M_d(D) \cong D^{d^2}$.

The affine Bruhat-Tits building of $PGL_d(D) := GL_d(D)/F^\times$, denoted $B_d(D)$, is a simplicial complex of dimension $(d - 1)$. It is constructed exactly as the affine Bruhat-Tits buildings of $PGL_d(F)$ described in Chapter 11 with the following modifications: Take $G = PGL_d(D)$, let $K$ be the subgroup generated by the images of $GL_d(\mathcal{O}_D)$ and

$$\begin{bmatrix}
\pi_D \\
\vdots \\
\pi_D
\end{bmatrix}
$$

in $G$, and replace $\pi = \pi_v$ with $\pi_D$. See [2, §3] for further details and an alternative construction. The group $PGL_d(D)$ acts on $B_d(D)$ on the left via its action on $B_d(D)^{(0)} = PGL_d(D)/K$, making $B_d(D)$ into an almost transitive $PGL_d(D)$-complex. The building $B_2(D)$ is a $(q^r + 1)$-regular tree.

One can define a vertex coloring $C_0$ and a directed-edge coloring $C_1$ on $B_d(D)$ as in Chapter 11 using the map $c : PGL_d(D) \to \mathbb{Z}/d\mathbb{Z}$ given by

$$c(gF^\times) = \nu(\text{Nrd}_{M_d(D)/F}(g)) + d\mathbb{Z}.$$

One can then define the operators $a_1, \ldots, a_{d-1}$ of Chapter 11 for $PGL_d(D)$-quotients of $B_d(D)$. It can be shown using the building axioms that $a_1, \ldots, a_{d-1}$ still commute among themselves and that $a_i^* = a_{d-i}$. Furthermore, it seems correct that $a_1, \ldots, a_{d-1}$ generate $A(\mathfrak{g}(PGL_d(D), B_d(D)), \Omega^+_{d/2})$, and hence their common spectrum is equivalent to the 0-dimensional spectrum (cf. Example 4.8). We will not need this fact, however.

7F. Representations of $GL_d(D)$. Fix $\nu \in \mathcal{V}$, let $F = k_{\nu}$, and let $D$ be a finite dimensional central division $F$-algebra of degree $r$. This section recalls various facts about representations of $GL_d(D)$.

Let $n_1, \ldots, n_t \in \mathbb{N}$ and let $n = n_1 + \cdots + n_t$. Denote by $P_{(n_1, \ldots, n_t)}$ and $M_{(n_1, \ldots, n_t)}$ the closed algebraic subgroups of $GL_{n,D}$ consisting of block matrices of the form

$$\begin{bmatrix}
* & \cdots & * \\
\vdots & \ddots & \vdots \\
* & \cdots & *
\end{bmatrix}$$

and

$$\begin{bmatrix}
* \\
\vdots \\
*\end{bmatrix},$$

where the $i$-th $*$ on the diagonal stands for an $n_i \times n_i$ matrix. The group $P_{(n_1, \ldots, n_t)}$ is a standard parabolic subgroup of $GL_{n,D}$ and $M_{(n_1, \ldots, n_t)}$ is its standard Levi factor. We write $P_{(n_1, \ldots, n_t)} = P_{(n_1, \ldots, n_t)}(F)$ and $M_{(n_1, \ldots, n_t)} = M_{(n_1, \ldots, n_t)}(F)$. Let
\(\delta_{(n_1, \ldots, n_t)}\) denote the unimodular character of \(P_{(n_1, \ldots, n_t)}\). Given smooth representations \(V_i \in \text{Rep}^{sm}(\GL_{n_i}(D))\) \((1 \leq i \leq t)\), let

\[
V_1 \times V_2 \times \cdots \times V_t = \text{Ind}^{\GL_n(D)}_{\GL_{n_1}(1, \ldots, n_t)}\left(\delta^{1/2}_{(n_1, \ldots, n_t)}(V_1 \otimes V_2 \otimes \cdots \otimes V_t)\right)
\]

Here, \(V_1 \otimes V_2 \otimes \cdots \otimes V_t\) is viewed as an \(M_{(n_1, \ldots, n_t)}\)-module, which is in turn viewed as a \(P_{(n_1, \ldots, n_t)}\)-module via the homomorphism \(P_{(n_1, \ldots, n_t)} \to M_{(n_1, \ldots, n_t)}\) removing the blocks above the diagonal.

The operation \(\times\) is associative up to a natural isomorphism [34 Pr. 1.1(b)] but not commutative in general. However, when \(V_1, \ldots, V_t\) are of finite length, semisimplification of \(V_1 \times \cdots \times V_t\) does not depend on the order of terms [34 Th. 1.9]. In particular, if \(V_1 \times \cdots \times V_t\) is irreducible, then \(V_1 \times \cdots \times V_t \cong V_{\sigma_1} \times \cdots \times V_{\sigma_t}\) for any permutation \(\sigma\) on \(\{1, \ldots, t\}\). When \(V_1, \ldots, V_t\) are admissible, there is a canonical-up-to-scaling isomorphism \((V_1 \times \cdots \times V_t)^{\ast} \cong V_1 \times \cdots \times V_t\) (this is similar to [13 Pr. 2.25c]). If \(V_1, \ldots, V_t\) are pre-unitary and one identifies \(V_i\) with \(\tilde{V}_i\) using the inner product on \(V_i\) (cf. the proof of Theorem [74]), then the previous isomorphism gives rise to an inner product on \(V_1 \times \cdots \times V_t\), making it into a pre-unitary representation.

**Theorem 7.6.** For all \(1 \leq i \leq t\), let \(V_i \in \text{Irr}^{pu}(\GL_{n_i}(D))\) be square-integrable. Then \(V_1 \times \cdots \times V_t\) is an irreducible pre-unitary tempered representation of \(\GL_n(D)\), where \(n = n_1 + \cdots + n_t\). Any irreducible pre-unitary tempered representation of \(\GL_n(D)\) is obtained in this manner with \((V_1, n_1), \ldots, (V_t, n_t)\) uniquely determined up to isomorphism and reordering.

**Proof.** By [32 Pr. III.4.1], every irreducible tempered pre-unitary representation of \(\GL_n(D)\) is a subquotient of \(V_1 \times \cdots \times V_t\), where \(V_i \in \text{Irr}^{sm}(\GL_{n_i}(D))\) is square-integrable, and \((V_1, n_1), \ldots, (V_t, n_t)\) are uniquely determined up to isomorphism and reordering. It is therefore enough to show that \(V_1 \times \cdots \times V_t\) is irreducible. This is shown in [18 Th. B.2.d] when \(F = 0\) and in [6 Th. 1.1] when \(\text{char } F > 0\). (In fact, \(V_1 \times \cdots \times V_t\) is irreducible under the milder assumption that each \(V_i\) is pre-unitary; see [74 and 4] for the cases \(\text{char } F = 0\) and \(\text{char } F \neq 0\), respectively.) \(\square\)

Let \(\xi : \GL_n(D) \to \mathbb{C}^\times\) be defined by \(\xi_n(g) = |\text{Nrd}_{\GL_n(D)/F}(g)|^\nu\). For \(V \in \text{Rep}^{sm}(\GL_{n_i}(D))\) and \(\alpha \in \mathbb{R}\), we write

\[\nu^{\alpha}V = \xi_{n}^\alpha V.\]

Given \(V_i \in \text{Rep}^{sm}(\GL_{n_i}(D))\) \((1 \leq i \leq t)\) with \(n = \sum n_i\), there is an obvious canonical isomorphism

\[\nu^{\alpha_1}(V_1) \times \cdots \times \nu^{\alpha_t}(V_t) \cong \nu^{\alpha_1\nu^{\alpha_2}}(V_1 \times \cdots \times \nu^{\alpha_t}(V_t))\]

**Theorem 7.7.** For all \(1 \leq i \leq t\), let \(V_i \in \text{Irr}^{pu}(\GL_{n_i}(D))\) be tempered, and let \(\alpha_1, \ldots, \alpha_t \in \mathbb{R}\) satisfy \(\alpha_1 > \cdots > \alpha_t\). Then \(\nu^{\alpha_1}V_1 \times \cdots \times \nu^{\alpha_t}V_t\) has a unique irreducible quotient, called the Langlands quotient. When unitarizable, the Langlands quotient of \(\nu^{\alpha_1}V_1 \times \cdots \times \nu^{\alpha_t}V_t\) is not tempered if \(t > 1\).

**Proof.** The first part follows from [75 Thm. 4.1(1)]. The second part follows from the uniqueness claim in [75 Thm. 4.1(2)]. \(\square\)

For the sake of simplicity, we henceforth specialize to the case \(D = F\). The results to follow extend to general \(D\) after some modifications; see [78 and 4].

Let \(V \in \text{Irr}^{pu}(\GL_{n}(F))\) be tempered, and let \(s \in \mathbb{N}\). By Theorem [77] the representation \(\nu^{\frac{1}{2s}}V \times \nu^{\frac{1}{2s}}V \times \cdots \times \nu^{\frac{1}{2s}}V\) has a unique irreducible quotient, which we denote by

\[u(V, s)\]

We will occasionally use the following alternative characterization of \(u(V, s)\):
Proposition 7.8. Let $V \in \text{Irr}^{\text{ps}}(\text{GL}_r(F))$ be tempered and assume $u(V,s)$ is unitarizable. Then

$$I_s(V) := \text{Ind}_{P_{(r,...,r)}}^{\text{GL}_r(F)} (V \otimes \cdots \otimes V)_{\text{s times}}.$$

has a unique irreducible subrepresentation which is isomorphic to $u(V,s)$. 

Proof. Since $u(V,s)$ is unitarizable, $u(V,s) \cong u(V,s)^*$. It is therefore enough to show that $I_s(V) \cong (\nu^{\frac{1-s}{2}} V \times \cdots \times \nu^{\frac{1-s}{2}} V)^*$. Indeed,

$$I_s(V) = \text{Ind}_{P_{(r,...,r)}}^{\text{GL}_r(F)} (V \otimes \cdots \otimes V)$$

$$= \text{Ind}_{P_{(r,...,r)}}^{\text{GL}_r(F)} (\nu^{\frac{1-s}{2}} V \times \cdots \times \nu^{\frac{1-s}{2}} V)$$

$$= \text{Ind}_{P_{(r,...,r)}}^{\text{GL}_r(F)} (\delta_{(r,...,r)}^{1/2} (\nu^{\frac{1-s}{2}} V \times \cdots \times \nu^{\frac{1-s}{2}} V))$$

$$= \nu^{\frac{1-s}{2}} V \times \nu^{\frac{1-s}{2}} V \times \cdots \times \nu^{\frac{1-s}{2}} V$$

$$\cong (\nu^{\frac{1-s}{2}} V)^* \times (\nu^{\frac{1-s}{2}} V)^* \times \cdots \times (\nu^{\frac{1-s}{2}} V)^*$$

$$\cong (\nu^{\frac{1-s}{2}} V \times \cdots \times \nu^{\frac{1-s}{2}} V)^*.$$

We used the fact that $V \cong \bar{V}$ in the last isomorphism. □

Example 7.9 (cf. [34, Ex. 3.2]). Let $\chi : \text{GL}_1(F) = F^\times \rightarrow \mathbb{C}^\times$ be a unitary character. Then $u(\chi_i, s) \in \text{Irr}^{\text{ps}}(\text{GL}_r(F))$ is the character $\chi \circ \text{Nrd}_{M_i(F) / F}$. To see this, observe that the function $g \mapsto \chi(\text{Nrd}_{M_i(F) / F}(g))$ is in $I_s(\chi)$, so the character $\chi \circ \text{Nrd}_{M_i(F) / F}$ must be the unique irreducible $\text{GL}_n(F)$-submodule of $I_s(\chi)$.

Theorem 7.10. If $V \in \text{Rep}^{\text{ps}}(\text{GL}_r(F))$ is square-integrable, then $u(V,s)$ is unitarizable.

Proof. This follows from [77, Th. D]. □

Theorem 7.11. Let $r, s \in \mathbb{N}$, let $n = rs$, and let $V$ be a pre-unitary supercuspidal representation of $\text{GL}_n(F)$. Then:

(i) $I_s(V)$ of Proposition 7.8 has a unique irreducible quotient, denoted $T(V,s)$.

(ii) $T(V,s)$ is unitarizable and square-integrable. Moreover, any square-integrable representation of $\text{GL}_n(F)$ is obtained in this manner, with $V$ and $s$ uniquely determined up to isomorphism.

Proof. See [34, Prp. 2.10] for (i) and [34, §9.3, Thm. 6.1] for (ii). □

Let $K := \text{GL}_n(O_r)$. An irreducible smooth $\text{GL}_n(F)$-module $V$ is called unramified if $V^K \neq 0$. In this case, $\dim V^K = 1$. (This holds because $V^K$ is an irreducible representation of $\mathcal{H}_K(\text{GL}_n(F))$, which is well-known to be commutative.) For example, a character $\chi$ of $\text{GL}_1(F) = F^\times$ is unramified when $O_r^\times \subseteq \ker \chi$.

Suppose we are given $V_i \in \text{Irr}^{\text{ps}}(\text{GL}_{n_i}(F))$ ($1 \leq i \leq t$) with $n = \sum_i n_i$. If each $V_i$ is unramified, then $V_1 \times \cdots \times V_t$ has a unique irreducible unramified subquotient, and conversely, if such a subquotient exists, then $V_1, \ldots, V_t$ must all be unramified. This follows easily from the Iwasawa decomposition $P_{(n_1,\ldots,n_t)} K = \text{GL}_n(F)$.

Proposition 7.12. Let $V \in \text{Irr}^{\text{ps}}(\text{GL}_r(F))$ be tempered and unramified. Then:

(i) There are unramified unitary characters $\chi_1, \ldots, \chi_r : \text{GL}_1(F) \rightarrow \mathbb{C}^\times$ such that $V \cong \chi_1 \times \cdots \times \chi_r$.

(ii) $u(V,s)$ is unramified for all $s \geq 1$.

Proof. (i) By Theorem 7.6 we may assume $V$ is square-integrable. By Theorem 7.11 we can write $V = T(U,t)$ for $t \in \mathbb{N}$ and $U$ a supercuspidal representation of $\text{GL}_{r/t}(F)$. Then $U$ is unramified. In this case, [17, Pr. 2.6] implies that there
are characters $\psi_1, \ldots, \psi_{r/t} : \GL_1(F) \to \mathbb{C}^\times$ such that $U$ is a subrepresentation of $\psi_1 \cdots \psi_{r/t}$. Since $U$ is supercuspidal, we must have $r/t = 1$ [84, Pr. 1.10], so $U$ is an unramified character of $\GL_1(F)$. Now, by Example 7.20, $u(U)$ is the unique unramified subquotient of $I_r(U)$, so $u(U, t) \cong T(U, t)$. Since $u(U, t)$ is non-tempered when $t > 1$ (Theorem 7.47) and $T(U, t)$ is square-integrable (Theorem 7.14), we must have $t = 1$, hence $V = U$ is an unramified character of $\GL_1(F)$.

(ii) Write $V = \chi_1 \times \cdots \times \chi_r$ as in (i). Then $u(V, s)$ is the unique irreducible subrepresentation of

\[
(\nu_1^{-s} \chi_1 \times \cdots \times \nu_r^{-s} \chi_r) \times \cdots \times (\nu_1^{-s} \chi_1 \times \cdots \times \nu_r^{-s} \chi_r).\]

We claim that this subrepresentation is isomorphic to $u(\chi_1, s) \times \cdots \times u(\chi_r, s)$, which is irreducible by [77, Th. A] or [84, Th. 4.2]). By Example 7.3 u(\chi, n) is unramified for all $i$, so this would show that $u(V, s)$ is unramified.

Suppose first that $\chi_1 = \cdots = \chi_r = \chi$. It is convenient to introduce $Z(\chi, s) := \nu^{-s} u(\chi, s)$. Alternatively, $Z(\chi, s)$ is the unique irreducible subrepresentation of $\chi \times \nu \chi \times \cdots \times \nu^{s-1} \chi$. For $U \in \Irr^{n=1}(\GL_n(F))$, we write $U^{\times r}$ to abbreviate $U \times \cdots \times U$ ($r$ times). We claim that

1. $Z(\chi, s) \times Z(\chi, s+1) \cong Z(\chi, s+1) \times Z(\chi, s)$

2. $Z(\chi, s+1)^{\times r}$ is isomorphic to a subrepresentation of $Z(\chi, s)^{\times r} \times (\nu^s \chi)^{\times r}$.

To prove (1), it enough to show that $Z(\chi, s) \times Z(\chi, s + 1)$ is irreducible, and this follows from [84, Th. 4.2] in [84]. $Z(\chi, s)$ is denoted by $(a)$ (where $a$ is the segment $\{\chi, \nu \chi, \ldots, \nu^{s-1} \chi\}$). We prove (2) by induction on $r$. The case $r = 1$ is immediate from the characterization of $Z(\chi, s)$ as the unique irreducible subrepresentation of $\chi \times \nu \chi \times \cdots \times \nu^{s-1} \chi$. For $r > 1$, the induction hypothesis implies that $Z(\chi, s)^{\times r} \times (\nu^s \chi)^{\times r}$ contains a copy of $Z(\chi, s) \times Z(\chi, s+1)^{\times (r-1)} \times \nu^s \chi$. By (1), the latter is isomorphic to $Z(\chi, s+1)^{\times (r-1)} \times Z(\chi, s) \times \nu^s \chi$, which contains a copy of $Z(\chi, s+1)^{\times (r-1)} \times Z(\chi, s+1) = Z(\chi, s+1)^{\times r}$.

Now, applying (2) repeatedly, we see that $\chi^{\times r} \times (\nu^{s-1} \chi)^{\times r} \times \cdots \times (\nu^{s-1} \chi)^{\times r}$ contains a copy of $Z(\chi, s)^{\times r}$. This implies that $u(\chi, s)^{\times r} = (\nu^{-s} Z(\chi, s))^{\times r} \cong (\nu^{-s} Z(\chi, s))^{\times r}$ is isomorphic to a subrepresentation of $u(\nu^{-s} \chi, s)^{\times r} \times \cdots \times (\nu^{-s} \chi, s)^{\times r}$ as required.

Suppose now that $\chi_1, \ldots, \chi_r$ are arbitrary, and write $\{\rho_1, \ldots, \rho_t\} = \{\chi_1, \ldots, \chi_r\}$ where $\rho_1, \ldots, \rho_t$ are distinct. Let $r_i$ denote the number of indices $j$ with $\rho_i = \chi_j$. Since $\rho_1, \ldots, \rho_t$ are unitary, [84 Pr. 1.11] implies that $\rho_i \times \rho_j$ is irreducible for all $i \neq j$, hence $\rho_i \times \rho_j \cong \rho_j \times \rho_i$. Using this, we can rearrange the terms in

\[
(\nu_1^{-s} \chi_1 \times \cdots \times \nu_r^{-s} \chi_r) \times \cdots \times (\nu_1^{-s} \chi_1 \times \cdots \times \nu_r^{-s} \chi_r),
\]

to get

\[
((\nu_1^{-s} \rho_1)^{\times r_1} \times \cdots \times (\nu_1^{-s} \rho_1)^{\times r_1}) \times \cdots \times ((\nu_r^{-s} \rho_t)^{\times r_t} \times \cdots \times (\nu_r^{-s} \rho_t)^{\times r_t}).
\]

By the previous paragraphs, this representation contains a copy of $u(\rho_1, s)^{\times r_1} \times \cdots \times u(\rho_t, s)^{\times r_t} \cong u(\chi_1, s) \times \cdots \times u(\chi_r, s)$, so we are done. \qed

7G. The Jacquet-Langlands Correspondence. Let $\nu$, $F$, $D$ be as in [77] and let $r = \deg D$. For all $n \in \mathbb{N}$, write

\[
G_n = \GL_{nr}(F) \quad \text{and} \quad G'_n = \GL_n(D).
\]

We choose Haar measures $\mu_{G_n}$, $\mu_{G'_n}$ for $G_n$, $G'_n$ respectively. Two elements $g \in G_n$, $g' \in G'_n$ are said to be in correspondence, denoted $g \leftrightarrow g'$, if they have the same reduced characteristic polynomial (see [72, 9a]). The element $g$ (resp. $g'$) is regular.
if the roots of its reduced characteristic polynomial in an algebraic closure of $F$ are
distinct. Denote by $\hat{G}_n$ (resp. $\hat{G}'_n$) the set of regular elements in $G_n$ (resp. $G'_n$). An
element $g \in G_n$ is called $D$-compatible if there is $g' \in G'_n$ such that $g \leftrightarrow g'$. This
is equivalent to saying that the degrees of the irreducible factors of the reduced
characteristic polynomial of $g$ are divisible by $r$ \cite[LM. 2.1]{7}.

Let $V$ be an irreducible smooth representation of $G'_n$. Harish-Chandra \cite{33}
showed that there exists a unique function $\psi_V : \hat{G}'_n \to \mathbb{C}$ such that $\psi_V$ is lo-
cally constant, stable under conjugation, and for any $\varphi \in \mathcal{H}(G'_n)$ supported on $G'_n$
one has
\[
\int_{g' \in \hat{G}'_n} \psi_V(g') \cdot \varphi(g') \, d\mu_{G'_n}.
\]
(In fact, this holds for any $\varphi \in \mathcal{H}(G'_n)$; see \cite[§2.5]{3}.) Notice that $\psi_V$ is independent
of the Haar measure $\mu_{G'_n}$. The function $\psi_V$ is called the function character of $V$.
It determines $V$ up to isomorphism. A representation $V \in \text{Ir}^{\text{sm}}(G'_n)$ is called
$D$-compatible if $\psi_V$ does not vanish on the set of regular $D$-compatible elements.

Let $\hat{G}^{(D)}_n$ denote the isomorphism classes of $D$-compatible irreducible pre-unitary
representations of $G_n$ and let $\hat{G}_n$ denote the isomorphism classes of all irreducible
pre-unitary representations of $G'_n$. To avoid cumbersome statements, we will occasion-
ally identify irreducible representations with their isomorphism classes.

**Theorem 7.13.** There exists a unique map $LJ : \mathcal{G}^n \to \hat{G}_n$, called local Jacquet–
Langlands correspondence, with the property that for all $V \in \mathcal{G}^{(D)}_n$, there is $\varepsilon =
\varepsilon(V) \in \{\pm 1\}$ such that
\[
\psi_V(g) = \varepsilon \cdot \psi_{LJ(V)}(g')
\]
for all $g \in G_n$, $g' \in G'_n$ with $g \leftrightarrow g'$. The map $LJ$ has the following additional
properties:

(i) $LJ$ restricts to a bijection between the square-integrable representations
of $G_n$ and the square integrable representations of $G'_n$ (considered up to
isomorphism). In particular, all square-integrable representations of $G_n$
are $D$-compatible.

(ii) Let $n_1, \ldots, n_t \in \mathbb{N}$ and let $V_i \in \text{Rep}^{\text{sm}}(\text{GL}_{n_i}(F))$ ($1 \leq i \leq t$). Assume that
$\sum_i n_i = nr$ and $V := V_1 \times \cdots \times V_t$ is irreducible. If $r \mid n_i$ and $V_i \in \mathcal{G}^{(D)}_n$
for all $i$, then $V$ is $D$-compatible and
\[
LJ(V) = LJ(V_1) \times \cdots \times LJ(V_t)
\]
Otherwise, $V$ is not $D$-compatible.

(iii) If $[V] \in \mathcal{G}^{(D)}_n$ is tempered, then so does $LJ(V)$.

**Proof.** The existence of $LJ$ was established in \cite{8} and \cite[Cor. 4.5]{3} in the cases
char $F = 0$ and char $F > 0$, respectively.

(i) See \cite[Theorem 3.2]{13} and \cite[Cor. 4.5]{10} (char $F > 0$).

(ii) This follows from the formula for the function character of an induced rep-
resentation in \cite[Th. 3]{31} (see also the comment following that theorem). See also
\cite[Pr. 3.4]{7}.

(iii) This follows from (i), (ii) and Theorem 7.6 \hfill $\square$

**Remark 7.14.** (i) The map $LJ$ is neither injective nor surjective in general \cite[Rm. 3.2]{7}
and Proposition 7.15 below). When $D = F$, the map $LJ$ is just the identity map.

(ii) The Jacquet-Langlands correspondence is often defined to be the inverse of
$LJ$, denoted $JL$.
Proposition 7.15. Let $\chi: F^* \to \mathbb{C}^*$ be a unitary character. Then $\text{LJ}_\nu(u(\chi, r)) = \text{LJ}_\nu(T(\chi, r)) \cong \chi \circ \text{Nrd}_{D/F}$. Furthermore, any $V \in \tilde{G}_1^{(D)}$ with $\text{LJ}_\nu(V) \cong \chi \circ \text{Nrd}_{D/F}$ is isomorphic to $u(\chi, r)$ or $T(\chi, r)$. (See [7] for the definition $u(\chi, r)$ and $T(\chi, r)$.)

Proof. Recall from Example 7.9 that $u(\chi, r) = \chi \circ \text{Nrd}_{D/F}$. It is easy to check that $\psi_{\chi \circ \text{Nrd}_{D/F}} = \chi \circ \text{Nrd}_{D/F}|_{\tilde{G}_1}$ and $\psi_{\chi \circ \text{Nrd}_{\nu(r/F)}|F} = \chi \circ \text{Nrd}_{\nu(F/F)}|_{\tilde{G}_1}$, hence $\text{LJ}_\nu(\chi \circ \text{Nrd}_{\nu(F/F)}|F) = \chi \circ \text{Nrd}_{D/F}$ (if $g \leftrightarrow g'$, then $\text{Nrd}_{\nu(F/F)}(g) = \text{Nrd}_{D/F}(g')$). By [3] Th. 5.2, this means that $\text{LJ}_\nu(T(\chi, r)) = \chi \circ \text{Nrd}_{D/F}$. (In more detail, the map $\text{LJ}_\nu$ is compatible with the Zelevinsky–Aubert involution, which takes $u(\chi, r)$ to $T(\chi, r)$ and leaves the supercuspidal representation $\chi \circ \text{Nrd}_{D/F}$ fixed.)

Suppose now that $\text{LJ}_\nu(V) = \chi \circ \text{Nrd}_{D/F}$. By Tadic's classification of unitary representations of $\text{GL}_r(F)$ [7], we can write

$$V = u(V_1, s_1) \times \cdots \times u(V_\ell, s_\ell)$$

$$\times (\nu^\alpha u(U_1, t_1) \times \nu^{-\alpha} u(U_1, t_1)) \times \cdots \times (\nu^\alpha u(U_m, t_m) \times \nu^{-\alpha} u(U_m, t_m))$$

where $V_1, \ldots, V_\ell, U_1, \ldots, U_m$ are square-integrable and $\alpha_1, \ldots, \alpha_m \in (0, \frac{1}{2})$. By Theorem 7.13(ii), we must have $\ell = 1$ and $m = 0$, so $V = u(V_1, s_1)$. By [3] Th. 5.2(3)], $V$ is not $D$-compatible unless $s_1 = 1$ or $s_1 = r$. In the first case, $V = V_1$ is square integrable and hence $V \cong T(\chi, r)$ by Theorem 7.13(i). In the second case, $V_1$ is a character $\xi: \text{GL}_1(F) \to \mathbb{C}^*$ and $V = u(\xi, r) = \xi \circ \text{Nrd}_{\nu(F/F)}$. This means that $\xi \circ \text{Nrd}_{\nu(F/F)}$ and $\chi \circ \text{Nrd}_{\nu(F/F)}$ coincide on $\tilde{G}_1$, so $\xi = \chi$. $\square$

Let $E$ be a division algebra of degree $r$ over $k$. For every $\nu \in \mathcal{V}$, write $E_\nu = E \otimes_k {k}_\nu$. Then $E_\nu \cong M_{m_\nu}(D_\nu)$ where $D_\nu$ is a central division $k_\nu$-algebra. Let $T$ denote the set of places $\nu \in \mathcal{V}$ at which $E$ ramifies, i.e. $m_\nu < r$. It is well-known that $T$ is finite [72, Thm. 32.1].

Let $G = \text{GL}_{r,k}$ and let $G' = \text{GL}_{1,E}$. Notice that for $\nu \not\in T$, we have $G'({k}_\nu) \cong \text{GL}_{r}({k}_\nu) \cong G({k}_\nu)$. A discrete automorphic representation $V = \bigotimes_\nu V_\nu$ of $G$ is said to be $E$-compatible if each local factor $V_\nu$ is $D_\nu$-compatible.

Theorem 7.16. In the previous setting, there exists a unique map $\text{LJ}$ from the $E$-compatible discrete automorphic representations of $G$ to the discrete automorphic representations of $G'$ with the property that the $\nu$-local-component of $\text{LJ}(\bigotimes_\nu V_\nu)$ is $\text{LJ}_\nu(V_\nu)$ for all $\nu \in \mathcal{V}$. The map $\text{LJ}$ is bijective.

Proof. See [3] Th. 18.1] (char $k = 0$) and [5] Th. 3.2] (char $k > 0$). $\square$

7H. Cuspidal and Residual Representations. Let $G$, $Z$ and $\omega$ be as in [7]

Discrete automorphic representations of $G$ are classified as cuspidal or residual. A discrete automorphic representation $V \leq \text{L}_2^\omega(G({k})) \setminus G({k}))$ is cuspidal if it consists of functions $\varphi \in \text{L}_2^\omega(G({k})) \setminus G({k}))$ satisfying

$$\int_{n \in N({k}) \setminus N(\omega)} \varphi(n g) d\mu_{N({k}) \setminus N(\omega)} = 0$$

for any $g \in G({k})$ and $N$, where $N$ ranges over the unipotent radicals of the parabolic subgroups of $G$. Otherwise, $V$ is called residual.

From now and until the end of [72] we assume that char $k > 0$. Under this assumption, Lafforgue proved the following result, known as the Ramanujan–Petersson conjecture for $\text{GL}_d$.

Theorem 7.17 (Lafforgue; [11] Th. VI.10]). Let $V = \bigotimes_\nu V_\nu$ be a cuspidal automorphic representation of $\text{GL}_d$. Then $V_\nu$ is tempered for all $\nu \in \mathcal{V}$.

When $G = \text{GL}_d$ with $d$ prime, the residual spectrum consists of 1-dimensional representations. For general $d$, the residual spectrum of $\text{GL}_d$ can be described as
follows: Suppose $d = rs$ and let $P_{(r,...,r)}$ and $M_{(r,...,r)}$ be as in [7]. Let $U$ be a cuspidal automorphic representation of $GL_r$ and, in analogy with Proposition 7.18 let

$$I_s(U) = \text{Ind}_{P_{(r,...,r)}}^{GL(r)}(U \otimes \cdots \otimes U)$$

(the induction is not normalized and $I_s(U)$ has no a priori structure of a pre-unitary representation).

**Theorem 7.18** (Mœglin, Waldspurger: [59]). For every cuspidal automorphic representation $U$ of $GL_r$, the representation $I_s(U)$ has a unique irreducible subrepresentation, denoted $u(U,s)$. The representation $u(U,s)$ is unitarizable. Furthermore, every residual automorphic representation of $GL_d$ is isomorphic to $u(U,s)$ for unique $1 < s | d$ and $U$ as above.

**Corollary 7.19.** Let $V$ be a residual automorphic representation of $GL_d$ and write $V = u(U,s)$ as in Theorem 7.18. Let $\{U_{\nu}\}_{\nu \in \mathcal{V}}$ be the local factors of $U$. Then $V \cong \bigotimes'_{\nu \in \mathcal{V}} u(U_{\nu},s)$. In particular, the local factors of $V$ are not tempered.

**Proof.** By Theorem 7.18 the local factors of $U$ are tempered, and hence $u(U_{\nu},s)$ is well-defined and non-tempered (Theorem 7.7). Since $U_{\nu}$ is unramified for almost all $\nu$, the representation $u(U_{\nu},s)$ is unramified for almost all $\nu$ (Proposition 7.12(ii)). Thus, the restricted product $\bigotimes'_{\nu \in \mathcal{V}} u(U_{\nu},s)$ is well-defined.

Since $I_s(U)$ has a unique irreducible submodule, it is enough to give a nonzero homomorphism from $\bigotimes'_{\nu \in \mathcal{V}} u(U_{\nu},s)$ to $I_s(U)$. Recall from Proposition 7.7 that $u(U_{\nu},s)$ is a subrepresentation of $I_s(U_{\nu}) = \text{Ind}_{P_{(r,...,r)}}^{GL_d(k)}(U_{\nu} \otimes \cdots \otimes U_{\nu})$ (s times). Choose a unit vector $u_{\nu} \in U_{\nu}$ which is $GL_r(O_{\nu})$-invariant if $U_{\nu}$ is unramified. When $U_{\nu}$ is unramified, $I_s(U_{\nu})$ contains a unique function $f_{\nu} : GL_d(k_{\nu}) \to U_{\nu} \otimes \cdots \otimes U_{\nu}$ with $f_{\nu}(GL_d(O_{\nu})) = u_{\nu} \otimes \cdots \otimes u_{\nu}$. The function $f_{\nu}$ is the only $GL_d(O_{\nu})$-invariant element in $I_s(U_{\nu})$ up to scaling, hence $f_{\nu} \in u(U_{\nu},s)$. If $U_{\nu}$ is ramified, then choose an arbitrary nonzero $f_{\nu} \in u(U_{\nu},s)$. We use the vectors $\{u_{\nu}\}$ to form $\bigotimes'_{\nu \in \mathcal{V}} U_{\nu} = U$ and the functions $\{f_{\nu}\}$ to form $\bigotimes'_{\nu \in \mathcal{V}} u(U_{\nu},s)$. We further identify $\bigotimes'_{\nu \in \mathcal{V}} u(U_{\nu},s)$ with $U \otimes \cdots \otimes U$ (as representations of $P_{(r,...,r)}(\mathbb{A})$) in the obvious way, using the vectors $\{u_{\nu} \otimes \cdots \otimes u_{\nu}\}$ when forming the restricted product. Now, define $\Phi : \bigotimes'_{\nu \in \mathcal{V}} u(U_{\nu},s) \to I_s(U)$ by sending $\bigotimes'_{\nu \in \mathcal{V}} h_{\nu}$ (with $h_{\nu} = f_{\nu}$ for almost all $\nu$) to the function $h \in I_s(U)$ given by $h((g_{\nu})_{\nu \in \mathcal{V}}) = \bigotimes'_{\nu \in \mathcal{V}} f_{\nu}(g_{\nu})$. It is straightforward to check that $\Phi$ is a well-defined nonzero $GL_d(\mathbb{A})$-homomorphism. \□

**Corollary 7.20.** Let $V = \bigotimes'_{\nu \in \mathcal{V}} V_{\nu}$ be a discrete automorphic representation of $GL_d$ and let $\theta \in V$. Then $V$ is cuspidal if and only if $V_{\theta} \in \text{Irr}^{\text{ram}}(GL_d(k_{\theta}))$ is tempered.

We finally record the following well-known fact.

**Proposition 7.21.** Let $V = \bigotimes'_{\nu \in \mathcal{V}} V_{\nu}$ be a discrete automorphic representation of $GL_d$. If one of the local factors $V_{\nu}$ is finite dimensional, then $V$ is 1-dimensional.

**Proof.** We first observe that $SL_d(k_{\nu})$ acts trivially on $V_{\nu}$. Indeed, since $V_{\nu}$ is smooth and finite-dimensional, the group $L = 1 + \pi_{\nu} M_d(O_{\nu})$ acts trivially on $V_{\nu}$ for $r$ sufficiently large. The normal subgroup generated by $L$ is easily seen to contain $SL_d(k_{\nu})$ (look on elementary matrices), hence our claim.

We view $V_{\nu}$ as a $GL_d(k_{\nu})$-subrepresentation of $V$. Since $V$ is irreducible as a $GL_d(\mathbb{A})$-module, it is generated by $V_{\nu}$, and hence $SL_d(k_{\nu})$ acts trivially on $V$ (because $SL_d(k_{\nu}) \leq GL_d(k_{\nu})$). The functions in $V$ are therefore $GL_d(k)$-invariant on the left and $SL_d(k_{\nu})$-invariant on the right. Since $SL_d(k_{\nu}) \leq GL_d(k)$ and $V$ consists of locally constant functions (it is smooth), the functions in $V$ are $GL_d(k)SL_d(k_{\nu})$-invariant on the left. By strong approximation (for $SL_d$), the group $GL_d(k)SL_d(k_{\nu})$
Example 7.23. For every $\Gamma$ as in the theorem, the spectrum of the Laplacian $\Delta$ is contained in the union of the spectrum of the $i$-dimensional Laplacian of $\mathcal{B}_d(D)$ with the trivial spectrum $\mathcal{T}_\Delta$, (apply Proposition 5.16 with $A = A(\mathcal{O}_E(D), \mathcal{B}_d(D)), \Omega_\Delta^+$ and $B = \mathbb{C}[\Delta]$; see also Proposition 2.22). This also holds for the adjacency operators considered in Example 4.2. Therefore, when $d = 2$, the quotient $\Gamma \backslash \mathcal{B}_2(D)$ is a Ramanujan $(q^*_i + 1)$-regular graph.

We shall need the following lemmas for the proof.

Lemma 7.24. The group $k_\theta \mathcal{O}_E^\times$ is normal in $E_\theta^\times$ and $E_\theta^\times / k_\theta \mathcal{O}_E^\times \cong \mathbb{Z} / rd\mathbb{Z}$. 

71. Ramanujan Complexes. Let $F$ be a local non-archimedean field of positive characteristic, let $D$ be a central division $F$-algebra and let $d > 1$. We finally prove that $\mathcal{B}_d(D)$, the affine Bruhat-Tits building of $\mathcal{PGL}_d(D)$, admits infinitely many $\mathcal{PGL}_d(F)$-quotients which are completely Ramanujan. The Ramanujan complexes constructed in [51], which are $\mathcal{PGL}_d(F)$-quotients of $\mathcal{B}_d(F)$ that are Ramanujan in dimension 0 (Example 5.15(ii)), will arise as special cases of our construction. In particular, they are completely Ramanujan.

Let the global field $k$ and $\eta \in \mathcal{V}$ be chosen such that $F = k_\eta$. Write $r = \deg D$, choose a central division $k$-algebra $E$ of degree $dr$, and define $E_\nu$, $D_\nu$, $m_\nu$ and $T$ as in (7C) We assume that:

- $E_\eta \cong M_d(D)$, i.e. $D_\eta \cong D$ and $m_\eta = d$.
- There is $\theta \in T$ such that $E_\theta$ is a division ring, i.e. $m_\theta = 1$.

Existence of a suitable $E$ for any prescribed $D$ and $d$ follows from the Albert–Brauer–Hasse–Noether Theorem ([72, Rem. 32.12(ii)], for instance). We further write $G = \mathcal{GL}_{dr,k}$, $G' = \mathcal{GL}_{d,E}$ and $H' = \mathcal{PGL}_{d,E}$. Since $E$ is a division ring, $H'$ is $k$-anisotropic. Note also that

- $H'(k_\eta) = \mathcal{PGL}_d(D)$
- $H'(k_\theta) = E_\theta^\times / k_\theta^\times$
- $H'(k_\nu) = \mathcal{PGL}_{dr}(k_\nu)$ when $\nu \notin T$

Finally, let

$$K_\theta = \text{im} \left( O_{E_\theta}^\times \to H'(k_\theta) \right) = k_\theta^\times O_{E_\theta}^\times / k_\theta^\times$$

(see [71] for the definition of $O_{E_\theta}$). We shall view the $k$-adeles $\mathbb{A}$ as $k_\eta \times \mathbb{A}^{(n)}$ and $\mathbb{A}^{(n)}$ as $k_\theta \times \mathbb{A}^{(n,\theta)}$ (notation as in [7A]).

We are now ready to state our main result. When $D = F$, it is just Theorem 1.2 of [51] with the difference that we show complete Ramanujan-ness whereas [51] shows Ramanujan-ness in dimension 0. (Also, in [51], it is assumed that $m_\nu = 1$ for all $\nu \in T$, but this assumption can be dropped thanks to [5].)

**Theorem 7.22.** Let $K^0$ be a compact open subgroup of $H'(\mathbb{A}^{(n)})$ and let $\Gamma = H'(k) \cap (H'(k_\eta) \times K^0)$ be viewed as a cocompact lattice in $H'(k_\eta) = \mathcal{PGL}_d(D)$ (cf. (7D)). Assume that either

1. $K^0$ contains $K_\theta \times 1$ (view $H'(\mathbb{A}^{(n)})$ as $H'(k_\eta) \times H'(\mathbb{A}^{(n,\theta)})$), or
2. $D = F$ and $d$ is prime,

and $\Gamma \leq \mathcal{B}_d(D) \mathcal{PGL}_d(D)$ (cf. [72]). Then $\Gamma \backslash \mathcal{B}_d(D)$ is completely Ramanujan.

**Example 7.23.** For every $\Gamma$ as in the theorem, the spectrum of the $i$-dimensional Laplacian $\Delta_i$ of $\mathcal{B}_d(D)$ is contained in the union of the spectrum of the $i$-dimensional Laplacian of $\mathcal{B}_d(D)$ with the trivial spectrum $\mathcal{T}_\Delta$, (apply Proposition 5.16 with $A = A(\mathcal{O}_E(D), \mathcal{B}_d(D)), \Omega_\Delta^+$ and $B = \mathbb{C}[\Delta]$; see also Proposition 2.22). This also holds for the adjacency operators considered in Example 4.2. Therefore, when $d = 2$, the quotient $\Gamma \backslash \mathcal{B}_2(D)$ is a Ramanujan $(q^*_i + 1)$-regular graph.

We shall need the following lemmas for the proof.

**Lemma 7.24.** The group $k_\theta^\times O_{E_\theta}^\times$ is normal in $E_\theta^\times$ and $E_\theta^\times / k_\theta^\times O_{E_\theta}^\times \cong \mathbb{Z} / rd\mathbb{Z}$. 


Proof. Define \( \Phi : E_\theta^\times \to \frac{1}{\nu} \mathbb{Z}/\mathbb{Z} \) by \( \Phi(g) = \nu E_\theta(g) + \mathbb{Z} \). Then \( \Phi \) is onto \( \mathbb{Z}/\nu \mathbb{Z} \) and its kernel is easily seen to be \( k_\Theta^\times \mathbb{O}_{E_\theta}^\times \). \( \square \)

**Lemma 7.25.** Let \( V \) be an irreducible pre-unitary representation of \( \text{PGL}_d(D) \). Then \( V \) is tempered as a representation of \( \text{PGL}_d(D) \) if and only if it is tempered as a representation of \( \text{GL}_d(D) \).

**Proof.** Use the equivalent conditions for temperedness in \( [65], \text{§2.4} \). \( \square \)

**Proof of Theorem 7.22.** Notice that \( \Gamma \) is just \( \prod_i \) in the setting of \( [73] \) when \( (1, q_i) \) represents the trivial double coset \( H'(k_i) \cdot (H'(k_\eta) \times K') \). Thus, by Theorem 7.22 it is enough to show that for every automorphic representation \( V' = \otimes' V'_\nu \) of \( H' \) with \( V'^{\times} \not\cong 0 \), the local component \( V'_\nu \) is tempered or finite-dimensional. View \( V' \) as an automorphic representation of \( G' \) with trivial central character. By Lemma 7.25 it is enough to show that \( V'_\eta \) is tempered or finite-dimensional as a representation of \( G'(k_\eta) = \text{GL}_d(D) \). By Theorem 7.16 there is a discrete automorphic representation \( V = \otimes' V'_\nu \) of \( G \) such that \( L_{I,\nu}(V'_\nu) = V'_\nu \) for all \( \nu \in \mathcal{V} \).

Suppose first that \( D = F \) and \( d = 1 \). Then \( \eta \not\in T \) and hence \( V'_\eta = V_\eta \). If \( V \) is cuspidal, then \( V_\eta \) is tempered by Theorem 7.17. Otherwise, \( V \) is 1-dimensional \( [71] \), so \( V_\eta \) is finite-dimensional.

Suppose now that \( K' \) contains \( K \times 1 \). Then \( V'^{K'} \not\cong 0 \), or rather \( V'^{K'_\eta} \otimes_{K'} \mathbb{O}^\times_{E_\theta} \not\cong 0 \). By Lemma 7.24 we may view \( V'_\nu \) as an irreducible unitary representation of \( E_\theta^\times / k_\theta^\times \mathbb{O}^\times_{E_\theta} \), which is finite and abelian, hence \( \dim V'_\nu = 1 \). Since every element of \( E_\theta^\times \) with reduced norm 1 is a commutator \( [63] \), there is a unitary character \( \chi : k_\theta^\times \to \mathbb{C}^\times \) such that \( V'_\nu \cong \chi \circ \text{Nrd}_{E_\theta / k_\theta} \). Since \( L_{I,\nu}(V'_\nu) = V'_\nu \cong \chi \circ \text{Nrd}_{E_\theta / k_\theta} \) (cf. Example 7.9), Proposition 7.18 implies that \( V_\eta \cong \chi(\nu, \text{rd}) = \chi \circ \text{Nrd}_{M_\nu(k_\eta) / k_\eta} \) (cf. Example 7.9). or \( V_\eta \cong \bar{T}(\chi, \text{rd}) \) (cf. Theorem 7.11).

If \( V_\eta \cong \chi \circ \text{Nrd}_{M_\nu(k_\eta) / k_\eta} \), then \( V \) is 1-dimensional (Proposition 7.21). In particular, \( V_\eta \) is 1-dimensional, and as explained previously, we can write \( V_\eta = \xi \circ \text{Nrd}_{M_\nu(F) / F} \) for a unitary character \( \xi : F^\times \to \mathbb{C}^\times \). Thus, by Proposition 7.15 \( V'_\eta = L_{I,\nu}(V'_\nu) = \xi \circ \text{Nrd}_{M_\nu(D) / D} \) and \( \dim V'_\eta = 1 \).

If \( V_\eta \cong \bar{T}(\chi, \text{rd}) \), then \( V_\eta \) is tempered (Theorem 7.11(ii)), and hence so is \( V_\eta \) (Corollary 7.20). Now, by Theorem 7.13(ii), \( V'_\eta = L_{I,\nu}(V'_\nu) \) is also tempered. This completes the proof. \( \square \)

**Remark 7.26.** (i) As explained in Remark 7.5(i), there exists \( K_\theta^\eta \subseteq_{\text{c.o.}} \text{H}'(k(\eta)) \) containing \( K_\theta \times 1 \) such that \( \Gamma \) of Theorem 7.22 satisfies \( \Gamma \leq_{\text{c.o.}} \text{PGL}_d(D) \) whenever \( K_\eta \subseteq K_\theta^\eta \).

(ii) One can choose the implicit closed embedding \( j : \text{H}' \to \text{GL}_{m,k} \) such that \( \text{H}'(O_\theta) = K_\theta^\eta \); see \( [51], \text{§5} \). Assuming this, let

\[
R = \{ x \in k : \nu(x) \geq 0 \text{ for all } \nu \in \mathcal{V} \setminus \{\eta\} \},
\]

and choose non-negative integers \( \{n_\nu\}_{\nu \in \mathcal{V} \setminus \{\eta\}} \) such that \( n_\nu = 0 \) almost always and \( n_\eta = 0 \). Taking \( K_\eta = \prod_{\nu \neq \eta} \text{H}'(O_\nu, n_\nu^0 \mathbb{O}_\nu) \) and writing \( I = \prod_{\nu \neq \eta} n_\nu^0 R \), the lattice \( \Gamma = \text{H}'(k) \cap \text{H}'(k_\eta) \times K_\eta^\times \) is just the congruence subgroup

\[
\text{H}'(R, I) = \ker(\text{H}'(R) \to \text{GL}_m(R) \to \text{GL}_m(R/I)).
\]

By Remark 7.3(i) or Corollary 7.12, for every \( \nu \in \mathcal{V} \setminus \{\eta\} \), there is \( n_\nu \) such that \( \Gamma \leq_{\text{c.o.}} \text{PGL}_d(D) \) whenever \( n_\nu \geq n_\nu \).

(iii) When \( D = F \), Lubotzky, Samuels and Vishne \( [50] \) gave an explicit description of some of the quotients \( \Gamma \backslash B_d(F) \) as Cayley complexes of certain finite groups. It is interesting to find similar descriptions for \( \Gamma \backslash B_d(D) \).
Remark 7.27. Let $\text{SL}_{d,D} = \ker(\text{Nrd} : \text{GL}_{d,D} \to \text{G}_{m,F})$ and let $H$ be the image of $\text{SL}_{d,D}(F)$ in $\text{PGL}_{d}(D)$. If the lattice $\Gamma$ of Theorem 7.22 is contained in $H$, then $\Gamma \backslash \mathcal{B}_{d}(D)$ is completely Ramanujan as an $H$-quotient of $\mathcal{B}_{d}(D)$. This follows from Theorem 6.30 (ii).

7J. A Remark about The Spectrum. While we have shown the existence of infinitely many completely Ramanujan quotients of $\mathcal{B}_{d}(D)$, we did not compute the spectrum of even a single associated operator! As most combinatorial applications require explicit bounds on spectra of operators of interest, we now comment about how such bounds might be obtained.

We retain the general setting of $\mathbb{H}$, $G$ is an $\ell$-group, $\mathcal{X}$ is an almost transitive $G$-complex, $\mathcal{C} = \mathcal{C}(G, \mathcal{X})$, and $F : \mathcal{C} \to \text{phil}$ is an elementary functor (semi-elementary functors can be treated using Remark 4.10). Proposition 2.22 allows us to shift the discussion from $(\mathcal{C},\text{F})$-operators to algebras of $(\mathcal{C},\text{F})$-operators [4C].

Given such an algebra $A_0$, embed it in the algebra $B$ of Theorem 6.10 and let $\mathcal{F}$ and $\hat{\mathcal{F}}$ be defined as in Theorem 6.21. If a classification of the irreducible unitary representations of $G$ is known, then one can compute the $A_0$-spectrum of $\mathcal{X}$ by finding $\text{Spec}_{A_0} \hat{\mathcal{F}}(\mathcal{V})$ for every tempered $\mathcal{V}$ in the domain of $\mathcal{F}$ and taking the union. The trivial $A_0$-spectrum can be computed similarly by letting $\mathcal{V}$ range over the irreducible unitary subrepresentations of $A(\mathcal{X})_N$ (see [5B]), where $N$ ranges over the finite-index open subgroups of $G$.

When $G = \text{GL}_{d}(D)$, a classification of the irreducible unitary representations is known; see [77] (for $D = F$) and [78], [74], [3]. Thanks to [17, Pr. 2.6], the classification simplifies significantly if one is only interested in representations admitting a nonzero vector fixed under an Iwahori subgroup. The approach we sketched was successfully applied in [30] and [28] with $G = \text{PGL}_{d}(F)$, in [24] with $G = \text{PGSp}_{4}(F)$, and also in [51, Th. 2.11, §2.4], to some extent. (In these sources, the functor $\mathcal{F}$ is implicit and turns out to take $\mathcal{V}$ to $\mathcal{V}^L$ where $L$ is a certain parahoric subgroup of $G$.) As can be seen from [30], [28], [24], one must engage in lengthy case-by-case analysis, and the number of cases increases with the split-rank of $G$. It is therefore interesting to look for more economical approaches.

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“So math. Very algebra. Many logic. Much variable. Wow.”

– Doge