Cooling by heating

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We introduce the idea of actually cooling quantum systems by means of incoherent thermal light, hence giving rise to a counter-intuitive mechanism of “cooling by heating”. In this effect, the mere incoherent occupation of a quantum mechanical mode serves as a trigger to enhance the coupling between other modes. This notion of effectively rendering states more coherent by driving with incoherent thermal quantum noise is applied here to the opto-mechanical setting, where this effect occurs most naturally. We discuss two ways of describing this situation, one of them making use of stochastic sampling of Gaussian quantum states with respect to stationary classical stochastic processes. The potential of experimentally demonstrating this counter-intuitive effect in opto-mechanical systems with present technology is sketched.

Cooling in quantum physics is usually achieved in just the same way as it occurs in classical physics or in common everyday situations: One brings a given system into contact with a colder bath. Coherent driving of quantum systems can effectively achieve the same aim, most prominently in instances of laser cooling of ions or in its opto-mechanical variant, cooling mechanical degrees of freedom using the radiation pressure of light. The coherence then serves a purpose of, in a way, rendering the state of the system “more quantum”. In any case, in these situations, the interacting body should first and foremost be cold or coherent.

In this work, we introduce a paradigm in which thermal hot states of light can be used to significantly cool down a quantum system. To be specific, we will focus on an opto-mechanical implementation of this idea: This type of system seems to be an ideal candidate to demonstrate this effect with present technology: it should however be clear that several other natural instances can well be conceived. Intuitively speaking, it is demonstrated that due to the driving with thermal noise, the interaction of other modes can be effectively enhanced, giving rise to a “transistor-like” effect. We flesh out this effect at hand of two approaches following different approximation schemes. The first approach is essentially a weak coupling master equation, while the second approach makes use of stochastic samplings with respect to colored classical stochastic processes, which constitutes an interesting and practical tool to study such quantum optical systems of several modes in its own right.

The observation made here adds to the insight that appears to be appreciated only fairly recently, in that quantum noise does not necessarily only give rise to heating, decoherence, and dissipation, providing in particular a challenge in applications in quantum metrology and in quantum information science. When suitably used, quantum noise can also assist in processes thought to be necessarily of coherent nature, in noise-driven quantum phase transitions, quantum criticality, in entanglement distillation, or in quantum computation. It turns out that thermal noise, when appropriately used, can also assist in cooling. Alas, this counter-intuitive effect is not in contradiction to the laws of thermodynamics, as is plausible when viewing this set-up as a thermal machine or heat engine operating in the quantum regime.

The system under consideration. We consider a system of two optical modes at frequencies $\omega_a$ and $\omega_b$, respectively, that are coupled to a mechanical degree of freedom at frequency $\omega_c$. The Hamiltonian of the entire system is assumed to be well-approximated by $H = H_0 + H_1$, where the free part is given by $H_0 = \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + \hbar \omega_c c^\dagger c$, and the interaction can be cast into the form

$$H_1 = \hbar g(a^\dagger b^\dagger + b a)(c^\dagger + c).$$

It is convenient to move to a rotating interaction picture with respect to $\hbar \omega_0 (a^\dagger a + b^\dagger b)$. The radiation pressure interaction is invariant under this transformation, while $H_0$ simplifies to

$$H_0' = \hbar \Delta a^\dagger a + \hbar \omega_c c^\dagger c,$$

where $\Delta = \omega_a - \omega_b$. For most of what follows, the frequencies are chosen such that $\Delta = \omega_c$, as we will see is the optimal resonance for cooling the mechanical resonator. This can be realized by tuning the mechanical degree of freedom or the cavity mode splitting. In fact, this is exactly the setting proposed in Ref. as a feasible three-mode optoacoustic interaction, in an idea that can be traced back to studies of parametric oscillatory instability in Fabry-Perot interferometers. Similarly, with systems of high-finesse optical cavities coupled to thin semi-transparent membranes, or of double-microdisk whispering-gallery resonators or of opto-mechanical crystals such a situation can be achieved. Surely numerous other architectures are well conceivable.

In addition to this coherent dynamics, the system is assumed to undergo natural damping and decoherence – un-
and master equation for modes \( b \) and \( L \) in mind, the Liouvillian in Eq. (2) can be decomposed as approximation if the back action on mode \( a \) is negligible for small couplings \( g \) and \( \gamma \). Having this picture in mind, the Liouvillian in Eq. (2) can be decomposed as \( \mathcal{L} = \mathcal{L}_{\text{sys}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{bath}} \), where \( \mathcal{L}_{\text{bath}} = \mathcal{L}_b \) and

\[
\mathcal{L}_{\text{sys}} = -\frac{i}{\hbar}[H', \cdot] + \mathcal{L}_a + \mathcal{L}_c, \quad \mathcal{L}_{\text{int}} = -\frac{i}{\hbar}[H_1, \cdot]. \tag{6}
\]

Using projection operators techniques [22], one can derive a master equation for the reduced system \( \rho_{a,c} = \text{tr}_b[\rho] \)

\[
\dot{\rho}_{a,c}(t) = \mathcal{L}_{\text{sys}}\rho_{a,c}(t) + \text{tr}_b \int_0^\infty ds e^{\mathcal{L}_{\text{int}}s}[\mathcal{L}_{\text{int}}\rho_{a,c}(t-s) \otimes \rho_b]. \tag{7}
\]

Here \( \mathcal{L}_{\text{int}} = \mathcal{L}_{\text{sys}} + \mathcal{L}_{\text{bath}} \). Making use of the explicit expression \( \mathcal{L} \) for \( \mathcal{L}_{\text{int}}, \) we have

\[
\dot{\rho}_{a,c}(t) = \mathcal{L}_{\text{sys}}\rho_{a,c}(t) - \frac{\hbar^2}{2} \text{tr}_b[H_1, \int_0^\infty ds e^{\mathcal{L}_{\text{int}}s}[\mathcal{L}_{\text{int}}\rho_{a,c}(t-s) \otimes \rho_b]]. \tag{8}
\]

In what follows, we will make a sequential approximation of the interaction Hamiltonian \( H_1 \) and the damping mechanism. Eq. (7) – up to second order expansion in the coupling \( g \), which constitutes the first approximation step (a) – can also be written as

\[
\dot{\rho}_{a,c}(t) = \mathcal{L}_{\text{sys}}\rho_{a,c}(t) - \frac{\hbar^2}{2} \text{tr}_b[H_1, \int_0^\infty ds e^{\mathcal{L}_{\text{int}}s}[\mathcal{L}_{\text{int}}\rho_{a,c}(t-s) \otimes \rho_b]], \tag{9}
\]

where \( \mathcal{L}_{\text{int}}^+ \) acts only on the Hamiltonian \( H_1 \), corresponding to a “dissipative interaction picture” with respect to \( \mathcal{L}_r \). We start from Eq. (1) and (b) neglect the term proportional to \( a^\dagger a \) because we assume mode \( a \) to be weakly perturbed from its ground state. In contrast, we allow the physical optical bath of mode \( b \) to have an arbitrary temperature and therefore we cannot neglect the term proportional to \( b^\dagger b \). We rewrite the approximated \( H_1 \) as

\[
H_1' = \hbar g(a^\dagger b + b^\dagger a + \delta)(c + c^\dagger), \tag{10}
\]

where the operator \( \delta = b^\dagger b - n_b \) represents the intensity fluctuations of mode \( b \). In order to have vanishing first moments with respect to mode \( b \), the mean force proportional to \( \langle b^\dagger b \rangle \) has been subtracted, which is responsible of merely shifting the resonator equilibrium position. Since \( \omega - \omega_b = \omega_c \), the (c) rotating wave approximation (RWA) of Eq. (9) is

\[
H_1'' = \hbar g(a^\dagger bc + ab^\dagger c^\dagger) + \hbar g\delta(c + c^\dagger). \tag{11}
\]

As will be explained later in more details, the first term of the Hamiltonian is responsible for the cooling of the mechanical resonator, while the second term corresponds to an additional heating noise.

In order to compute the partial trace in Eq. (5), we need the two-time correlation functions of the thermal light in mode \( b \),

\[
\langle be^{\mathcal{L}_{\text{int}}s}b^\dagger \rangle = e^{-\kappa s} n_b, \quad \langle \delta e^{\mathcal{L}_{\text{int}}s} \delta \rangle = e^{-2\kappa s} (n_b^2 + n_b).
\]

The exponential functions in Eqs. (11) determine the time scale of the integral kernel in Eq. (8), which will be of the order of \( \kappa^{-1} \). Within this time scale (d) we can neglect the effect of the mechanical reservoir (\( \gamma \ll \kappa \)), and the action of the map \( e^{\mathcal{L}_{\text{int}}s} \) on the system operators will be

\[
e^{\mathcal{L}_{\text{int}}s}a = e^{-(\kappa + i\Delta)s}a, \quad e^{\mathcal{L}_{\text{int}}s}c = e^{-(\gamma + i\omega_c)s}c \approx e^{-i\omega_c s}c.
\]

We can finally perform the integral in Eq. (8), and since all the odd moments of \( \rho_b \) vanish, the cooling and heating terms in Eq. (11) generate two independent contributions to the master equation, respectively

\[
\mathcal{L}_{\text{cool}} = \frac{g^2}{2\kappa} ((1 + n_b)D_{ac^\dagger} + n_b D_{a^\dagger c}), \tag{12}
\]

\[
\mathcal{L}_{\text{heat}} = \frac{2kg^2(n_b^2 + n_b)}{4\kappa^2 + \omega_c^2} (D_{c^\dagger} + D_c), \tag{13}
\]
where in calculating \( \mathcal{L}_{\text{heat}} \) we (e) kept only the counter-
rotating terms. The effect of \( \mathcal{L}_{\text{heat}} \) is simply a renormaliza-
tion of the mean occupation number of the mechanical bath

\[
n_c \mapsto \tilde{n}_c = n_c + \frac{2\kappa g^2(n_b + n_b)}{\gamma(4\kappa^2 + \omega_c^2)},
\]

always increasing, as expected, the effective temperature of the environment. Denoting with \( \tilde{\rho}_{a,c} \) the corresponding renor-
malized Liouvillian, the master equation can be written as

\[
\dot{\rho}_{a,c} = (\tilde{\mathcal{L}}_{\text{sys}} + \tilde{\mathcal{L}}_{\text{cool}})\rho_{a,c}.
\]

With respect to Eq. (2), Eq. (14) can be numerically solved with
much less computational resources but we have to remind
ourselves that this approach is valid only within the RWA and
for weak coupling: \( \gamma, g \ll \omega_c \). Another advantage of Eq.
(14) is that the corresponding adjoint equations for the number
operators \( n_a = a^\dagger a \) and \( n_c = b^\dagger b \) are closed with respect
to these operators, that is

\[
\hat{n}_a = -2\kappa n_a - \frac{g^2}{\kappa}(n_b + 1) n_a - n_b n_c - n_a n_c, \\
\hat{n}_c = -2\kappa n_c - \frac{g^2}{\kappa}(n_c + 1) n_a - n_b n_c - 2\gamma n_c.
\]

Assuming (f) that the factorization property \( \langle n_a n_c \rangle \approx
\langle n_a \rangle \langle n_c \rangle \) holds – which is essentially a mean-field
approach which is expected to be good in case of small correlations,
or, again as assumed, for small values of \( g \) – we can find analyti-
cal expressions for the steady state expectation values:

\[
\langle n_c \rangle = \frac{\tilde{n}_c - \eta}{2} + \left( \frac{\tilde{n}_c + \eta + 2}{4} - \frac{\kappa n_b n_c}{\gamma} \right)^{1/2}, \\
\langle n_a \rangle = \frac{(\tilde{n}_c - \langle n_c \rangle)\gamma}{\kappa},
\]

where \( \eta = 1 + n_b(1 + \kappa/\gamma) + 2\kappa^2/g^2 \).

**Description 2:** Sampling with respect to colored sta-
tionary classical stochastic processes. In this approach, we start
from the exact dynamics Eq. (2) but treat mode \( b \) as a classi-
cal thermal field and neglect any feed-back from the resonator.
We substitute the bosonic operator with a complex amplitude
\( \beta_t \) giving rise to a semi-classical picture. The pa-
rameter \( \beta_t \) can be described as a classical stochastic process
de ned by the stochastic differential equation (SDE)

\[
d\beta_t = -\kappa \beta_t dt + \sqrt{\kappa n_b}(dW^{(x)} + idW^{(y)}),
\]

with independent Wiener increments \( \text{(10)} \) obeying the Itô
rules \( dW^{(x)}dW^{(b)} = \delta_{a,b} dt, \) \( dW^{(a,b)} dt = 0 \). The dynamics of the remaining modes \( a \) and \( c \) instead, can be efficiently
treated quantum mechanically; this is true, since for every single
realization of the process \( \text{(15)} \), the evolution defines a
Gaussian completely positive map and therefore the corre-
sp onding Gaussian state \( \rho^{(\beta)}_{a,c}(t) = e^{(\beta t)}(\rho_{a,c}) \) can be de-
scribed entirely in terms of first and second moments. The ac-
tual quantum state of the system will in general not be exactly
Gaussian, it can nonetheless be simulated by sampling over
many Gaussian states associated with different realizations of \( \beta_t \). Only the respective weight in the convex combinations
are such that the resulting state can be non-Gaussian. The re-
sulting state \( \rho_{a,c}(t) = E_\rho^{(\beta)}(t) \) will be our semi-classical
description of the system.

It is convenient to introduce a vector of quadratures op-
erators \( u = [x_c, y_c, x_a, y_a] \), where \( x_j = (j + j^\dagger)/\sqrt{2}, \)
\( y_j = i(j^\dagger - j)/\sqrt{2} \) and \( j = a, c \). From Eq. (2), we get a
SDE for the first moments

\[
\frac{d\langle u \rangle_t}{dt} = A_t \langle u \rangle_t + f_t,
\]

where

\[
A_t = \begin{bmatrix}
-\gamma & \omega_c & 0 & 0 \\
-\omega_c & -\gamma & g\beta_t^{(x)} & g\beta_t^{(y)} \\
-g\beta_t^{(y)} & 0 & -\kappa & \Delta \\
g\beta_t^{(x)} & 0 & -\Delta & -\kappa
\end{bmatrix},
\]

\[
\beta_t^{(x)} = (\beta_t^* + \beta_t), \quad \beta_t^{(y)} = (\beta_t^* - \beta_t).
\]

The second moments can be arranged in the matrix \( V_t = \text{Re} \langle uu^\dagger \rangle_t \), satisfying the SDE

\[
\frac{dV_t}{dt} = A_t V_t + V_t A_t^T + D + F_t,
\]

where \( D = \text{diag}[\gamma(2n_c + 1), \gamma(2n_c + 1), \kappa, \kappa] \), and \( F_t = f_t(u)_t + \langle u \rangle_t f_t^T \). The statistical average over many realiza-
tions of \( V_t \) will be an estimator for the second moments of the quantum state \( V(t) = E(V_t) \). In particular, the first two di-
agonal elements give the effective phonon number of the me-
chanical oscillator, since \( \langle n_c \rangle(t) = (V_{11}(t) + V_{22}(t) - 1)/2 \). The three stochastic differential equations (15) can be numerically integrated in sequential order. In our simulations, see Fig. 2 we used the Euler method, for each time step \( dt \) sampling the associated Wiener increments in Eq. (15) with
normal distributions of variance \( \sigma^2 = dt \).
the matrix $F_t$. However, the same process $\beta_t$ is also contained in the matrix $A_t$ and corresponds to a cooling noise, up to the approximations identical to the above “good noise”. The reason is quite evident from Eq. (17), where we observe that the coupling between the hot mechanical oscillator and the cold optical mode is mediated by the thermal fluctuations of $\beta_t$. This opto-mechanical coupling, which would be zero without noise, leads to a sympathetic cooling of the mechanical mode.

Example. We will now discuss the effect of cooling by heating at hand of an example using realistic parameters in an opto-mechanical setting.Fig. 2 shows the effective temperature of the mechanical mode as a function of the number of photons in mode $b$: Here, effective temperature is defined as the temperature $T$ of a Gibbs state

$$\rho_c(T) = \frac{e^{-\hbar \omega_c c^\dagger c/(kT)}}{tr(e^{-\hbar \omega_c c^\dagger c/(kT)})}$$

such that $\langle n_c \rangle = tr(\rho_c(T)c^\dagger c)$. One quite impressively encounters the effect of cooling by heating, for increasing photon number and hence effective temperature of this optical mode. For very large values of the photon number, the “bad noise” eventually becomes dominant, resulting again in a heating up of the mechanical mode. Note that needless to say, the effective temperature of the optical mode $b$ is usually larger than the mechanical one by many orders of magnitude (approximately $10^{13}$K for reasonable parameters).

Putting things upside down, one could also conceive settings similar to the one discussed here as demonstrators of small heat engines [15] operating at the quantum mechanical level, where $b$ takes the role of an “engine” and mode $a$ of a “condenser”. To fully explore these implications for feasibly realizing quantum thermal machines constitutes an exciting perspective. It would also be interesting to fully flesh out the potential for the effect to assist in generating non-classical states [24]. Finally, quite intriguingly, this work may open up ways to think of optically cooling mechanical systems without using lasers at all, but rather with basic, cheap LEDs emitting incoherent light.

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![FIG. 2. Room temperature cooling with parameters reminding of those typical in realistic experiments [3]: $\omega_c = 2 \pi \text{MHz}, \kappa = 0.2 \omega_c$, $g = 0.3 \times 10^{-5} \omega_c$, and $\gamma = 10^{-3} \omega_c$. The black line shows the predictions of the steady state using Description 1, the dots are a result from stochastic sampling using Description 2 (with 100 realizations), which qualitatively coincide well. One clearly finds that an increased population of mode $b$ leads to a significant cooling of the mechanical mode – up to a point when eventually the “bad noise” becomes dominant.](image)

Summary. In this work, we have established the notion of cooling by heating, which means that cooling processes can be assisted by means of incoherent hot thermal light. We focused on an opto-mechanical implementation of this paradigm. We also introduced new theoretical tools to grasp the situation of driving by quantum noise, including sampling techniques over stochastic processes. To experimentally demonstrate this counterintuitive effect should be exciting in its own right.

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**SUPPLEMENTARY MATERIAL**

In this supplementary material, we compare the methods of Description 1 and 2 of the main text with an exact simulation of the master equation (2) in a truncated, finite-dimensional Hilbert space of the three involved modes, \( \mathcal{H} = \mathbb{C}^{N_a} \otimes \mathbb{C}^{N_b} \otimes \mathbb{C}^{N_c} \). The unique stationary state of Eq. (2) can easily be found numerically; a dimension \( d = N_a N_b N_c \) of the total Hilbert space of, say, \( d \lesssim 400 \), is well feasible. This is obviously an essentially exact method for small occupation numbers in each of the three modes, and the error made can easily be estimated. This analysis, see Fig. 3, together with analogous ones in similar regimes, shows that the methods used here are also suitable in the deep quantum regime.

**FIG. 3.** Stochastic simulation – introduced in Description 2 – of the mean number of phonons as a function of time (black line), compared with the exact steady state (blue line) and with the analytical approximation given in Description 1 (red line). Parameters: \( n_b = n_c = 1 \), \( \kappa = 0.1 \omega_c \), \( \gamma = 0.01 \omega_c \), \( g = 0.006 \omega_c \).