The étale open topology over the fraction field of a Henselian local domain

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Abstract
Suppose that $R$ is a local domain with fraction field $K$. If $R$ is Henselian, then the $R$-adic topology over $K$ refines the étale open topology. If $R$ is regular, then the étale open topology over $K$ refines the $R$-adic topology. In particular, the étale open topology over $L((t_1, \ldots, t_n))$ agrees with the $L[[t_1, \ldots, t_n]]$-adic topology for any field $L$ and $n \geq 1$.

Keywords
étale open topology, Henselian rings, topological fields

MSC (2020)
12E30, 14G27

1 | INTRODUCTION

Throughout, $K$ and $L$ are fields, all rings are commutative with unit, the “dimension” of a ring is the Krull dimension, and by convention a “local ring” is not a field. The étale open topology (or $\mathcal{E}_K$-topology) on a $K$-variety $V$ is a topology on the set $V(K)$ of $K$-points introduced in [13]; see Section 1.1 for definitions. The $\mathcal{E}_K$-topology recovers the “natural” topology on $V(K)$ in several cases of interest. For example, when $K$ is algebraically closed, the étale open topology is the Zariski topology on $V(K)$, and when $K$ is $\mathbb{R}$ or $\mathbb{Q}_p$ (or any local field other than $\mathbb{C}$), the étale open topology is the analytic topology on $V(K)$. The étale open topology is primarily of interest when $K$ is a large field in the sense of Pop [18]. In fact, the étale open topology is discrete when $K$ is nonlarge [13, Theorem C]. (See [2, 20] for more information on largeness, including a definition.) The étale open topology has applications to large fields: It yielded a classification of model-theoretically stable large fields, and provided new insights into several known results on large fields [13, Theorem D, 9.2].

If $\mathcal{T}$ is a field topology on $K$, then $\mathcal{T}$ induces a topology on $V(K)$ in a natural way. We call the resulting topology the $\mathcal{T}$-topology on $V$. In cases like $K = \mathbb{R}$ or $K = \mathbb{Q}_p$, the étale open topology is induced by a field topology. On the other hand, this fails when $K$ is algebraically closed. (The Zariski topology on $V \times W$ is not the product topology of the Zariski topologies on $V$ and $W$.) One might hope to characterize when the étale open topology is induced by a field topology. Some results in this direction were obtained in [13, Theorem B and Section 8]. In this paper, we give a new example where the étale open topology is induced by a field topology, namely, when $K$ is the fraction field of a Henselian regular local ring such as $\mathbb{C}[[x, y]]$.
Fact 1.1 describes what we have established so far with Tran concerning the relationship between the étale open and other topologies, in order [13, Theorem A, Proposition 6.1, Proposition 6.14, Theorem B, Theorem 6.15, Theorem B].

**Fact 1.1.** Suppose that $V$ is a $K$-variety and $v$ is a nontrivial valuation on $K$.

1. The $\mathcal{E}_K$-topology on $V(K)$ refines the Zariski topology.
2. If $K$ is separably closed, then the $\mathcal{E}_K$-topology on $V(K)$ agrees with the Zariski topology.
3. If $<\!$ is a field order on $K$, then the $\mathcal{E}_K$-topology on $V(K)$ refines the $<\!$-topology.
4. If $K$ is real closed, then the $\mathcal{E}_K$-topology on $V(K)$ agrees with the order topology.
5. If the Henselization of $(K, \mathcal{E}_K)$ is not separably closed, then the $\mathcal{E}_K$-topology on $V(K)$ refines the valuation topology. (Hence, if the value group of $v$ is not divisible or the residue field of $v$ is not algebraically closed, then the $\mathcal{E}_K$-topology on $V(K)$ refines the valuation topology.)
6. If $K$ is not separably closed and $v$ is Henselian, then the $\mathcal{E}_K$-topology on $V(K)$ agrees with the valuation topology.

If $\mathcal{R}$ is a Henselian valuation ring with fraction field $K$, then Fact 1.1.6 says that the étale open topology over $K$ is the valuation topology, unless $K$ is separably closed or the valuation is trivial. For example, the étale open topology over $L((t))$ is the valuation topology. It is natural to ask whether Fact 1.1.6 generalizes to fraction fields of Henselian local domains such as $L[[t_1, \ldots, t_n]]$. We first need an analogue of the valuation topology. If $\mathcal{R}$ is a local domain with fraction field $K$, then $\{\alpha \mathcal{R} + \beta : \alpha \in K^\times, \beta \in K\}$ is a basis of opens for a nondiscrete Hausdorff field topology on $K$ [21, Theorem 2.2]. We call this the $\mathcal{R}$-adic topology on $K$. This is the coarsest ring topology on $K$ with $R$ open. When $R$ is a nontrivial valuation ring, the $R$-adic topology is the valuation topology. As mentioned above, the $R$-adic topology induces a topology on $V(K)$ for each $K$-variety $V$.

**Theorem 1.2.** Suppose that $\mathcal{R}$ is a local domain with fraction field $K$ and $V$ is a $K$-variety.

1. If $\mathcal{R}$ is Henselian, then the $\mathcal{R}$-adic topology on $V(K)$ refines the $\mathcal{E}_K$-topology.
2. If $\mathcal{R}$ is regular, then the $\mathcal{E}_K$-topology on $V(K)$ refines the $\mathcal{R}$-adic topology.

Hence, the étale open topology over the fraction field $L((t_1, \ldots, t_n))$ of $L[[t_1, \ldots, t_n]]$ agrees with the $L[[t_1, \ldots, t_n]]$-adic topology.

See Section 1.1 for the definitions of regularity and Henselianity. Examples of regular Henselian local rings are the ring $L[[t_1, \ldots, t_n]]$ of formal power series for any field $L$, and the ring $L[t_1, \ldots, t_n]$ of convergent power series for $L$ a local field. Additionally, the ring $L(t_1, \ldots, t_n)_{\text{reg}} \cap L[[t_1, \ldots, t_n]]$ is a regular Henselian local ring for any field $L$, as it is the Henselization of the regular local ring $L[t_1, \ldots, t_n]_{(t_1, \ldots, t_n)}$ (see [23, 07PX, 07PV, 0A1W, 06LN]).

A one-dimensional regular local ring is a discrete valuation ring [6, 11.1], so the one-dimensional case of Theorem 1.2 follows from Fact 1.1.5. If $\mathcal{R}$ is a Noetherian local domain of dimension at least two, then the $\mathcal{R}$-adic topology is not induced by a valuation (see Proposition 2.1), so Theorem 1.2 does not follow from Fact 1.1.

As noted above, the étale open topology is connected to the class of large fields in field theory. Specifically, the étale open topology on $K = \mathcal{A}(K)$ is nondiscrete if and only if $K$ is large [13, Theorem C]. Thus, the first claim of Theorem 1.2 can be seen as a topological refinement of the fact, proven in [19, Theorem 1.1], that fraction fields of Henselian local domains are large.

Both refinements in Theorem 1.2 can be strict. For example, the localization of $L[t_1, \ldots, t_n]$ at the ideal generated by $t_1, \ldots, t_n$ is a regular local ring $R$ with nonlarge fraction field $L(t_1, \ldots, t_n)$, so the étale open topology over $L(t_1, \ldots, t_n)$ is discrete and hence strictly refines the $R$-adic topology. Theorem 1.3 shows that the other refinement may also be strict.

**Theorem 1.3.** Fix a prime $p$. There is a subring $E$ of $\mathbb{Z}_p$ such that:

1. $E$ is a one-dimensional Noetherian Henselian local domain,
2. $\mathbb{Q}_p$ is the fraction field of $E$, and
3. the $E$-adic topology on $\mathbb{Q}_p$ strictly refines the $p$-adic topology.

By Fact 1.1.6, the étale open topology over $\mathbb{Q}_p$ agrees with the $p$-adic topology, hence the $E$-adic topology on $\mathbb{Q}_p$ strictly refines the étale open topology. Therefore, some assumption like regularity is needed in Theorem 1.2.2. In an upcoming
paper with Dittmann, we will show that excellence is another sufficient condition. In particular, we will show that for an excellent Henselian local domain \( R \), the \( R \)-adic topology on the fraction field is the étale open topology. (In this paper, the ring \( E \) of Theorem 1.3 has nonreduced completion and is hence not excellent; see Remark 5.10 below.) More generally, the upcoming paper suggests that the comparison between the \( R \)-adic topology and étale open topology is connected to resolution of singularities over \( R \).

We now discuss an application to definable sets. “Definable” means “first-order definable in the language of rings, possibly with parameters” and a field is Henselian if it admits a nontrivial Henselian valuation. It is a well-known and important fact that definable sets in characteristic zero Henselian fields are well behaved with respect to the valuation. In particular, if \( K \) is a characteristic zero Henselian valued field, then every definable subset of \( K^n \) is a finite union of definable \( \mathcal{E}_K \)-open subsets of Zariski closed subsets of \( K^n \) [24]; note that this applies to \( L(\{(\cdot)\}) \). Definable sets in fraction fields of characteristic zero Henselian local domains need not be well behaved. The following roughly follows the argument in [8, Example 10] for \( \mathbb{Q}(X, Y) \). Suppose \( \text{Char}(L) = 0 \) and \( n \geq 2 \). Jensen and Lenzig [1, Theorem 3.34] showed that \( L((t_1, \ldots, t_n)) \) defines \( \{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n : \alpha_1^2 + \cdots + \alpha_n^2 = 1 \} \). Becker and Lipschitz [3] showed that \( L[[t_1, \ldots, t_n]] \) defines \( \mathbb{N} \), Delon [5] showed that \( L[[t_1, \ldots, t_n]] \) uniformly defines all subsets of \( \mathbb{N} \), hence \( L((t_1, \ldots, t_n)) \) defines the standard model of second-order arithmetic. Nevertheless, the existentially definable sets are still somewhat tame, as observed in [1, 8, 25].

**Corollary 1.4.** Suppose that \( R \) is a Henselian local domain with fraction field \( K \), \( K \) is perfect, and \( X \subseteq K^n \) is existentially definable. Then, there are pairwise disjoint irreducible smooth subvarieties \( V_1, \ldots, V_k \) of \( \mathbb{A}^n \) and \( O_1, \ldots, O_k \) such that each \( O_i \) is a definable \( R \)-adically open subset of \( V_i(K) \) and \( X = O_1 \cup \ldots \cup O_k \).

Corollary 1.4 follows directly from Theorem 1.2.1, and the theorem, proven in [25, Theorem B], that if \( K \) is perfect, then every existentially definable subset of \( K^n \) is a finite union of definable \( \mathcal{E}_K \)-open subsets of Zariski closed subsets of \( K^n \).

### 1.1 Conventions and background

A \( K \)-variety is a separated \( K \)-scheme of finite type, \( n \) is a natural number, \( \mathbb{A}^n \) is \( n \)-dimensional affine space over \( K \), and \( \mathbb{G}_m \) is the scheme-theoretic multiplicative group over \( K \), that is, \( \mathbb{A}^n = \text{Spec} K[x_1, \ldots, x_n] \) and \( \mathbb{G}_m = \text{Spec} K[y, y^{-1}] \). We let \( V(K) \) be the set of \( K \)-points of a \( K \)-variety \( V \). Recall that \( \mathbb{A}^n(K) = \mathbb{K}^n \) and \( \mathbb{G}_m(K) = \mathbb{K}^\times \). We let \( \text{Frac}(R) \) be the fraction field of a domain \( R \).

Suppose that \( R \) is a local ring with maximal ideal \( m \). Then, \( R \) is Henselian if for any \( g \in R[x] \) and \( \alpha \in R \) such that \( g(\alpha) \equiv 0 \pmod{m} \) and \( g'(\alpha) \not\equiv 0 \pmod{m} \), there is \( \alpha^* \in R \) such that \( g(\alpha^*) = 0 \) and \( \alpha^* \equiv \alpha \pmod{m} \). Likewise, \( R \) is regular if \( R \) is Noetherian and \( m \) admits a \( d \)-element generating set, where \( d = \text{dim } R \).

We briefly recall the étale open topology. Suppose that \( V \) is a \( K \)-variety. An étale image in \( V(K) \) is a set of the form \( f(X(K)) \) for an étale morphism \( f : X \to V \) of \( K \)-varieties. The collection of étale images in \( V(K) \) forms a basis for the étale open topology (or \( \mathcal{E}_K \)-topology) on \( V(K) \). Fact 1.5 below gathers some basic facts on the étale open topology from [13]. Parts (1) and (2) are [13, Theorem A, Lemma 5.3], while parts (3) and (4) are immediate consequences of (1) and (2).

**Fact 1.5.** Suppose that \( V \to W \) is a morphism of \( K \)-varieties. Then,

1. the induced map \( V(K) \to W(K) \) is \( \mathcal{E}_K \)-continuous;
2. if \( V \to W \) is étale, then the induced map \( V(K) \to W(K) \) is \( \mathcal{E}_K \)-open;
3. the map \( K \to K, x \mapsto ax + \beta \) is an \( \mathcal{E}_K \)-homeomorphism for any \( a \in K^\times, \beta \in K \);
4. if \( n \) is prime to \( \text{Char}(K) \), then \( \{ \alpha^n : \alpha \in K^\times \} \) is an étale open subset of \( K \).

We will need Fact 1.6 below, proven in [13, Lemma 4.2, Lemma 4.8].

**Fact 1.6.** Let \( \mathcal{F} \) be a Hausdorff field topology on \( K \). If the \( \mathcal{F} \)-topology on each \( K^n = \mathbb{A}^n(K) \) refines the \( \mathcal{E}_K \)-topology, then the \( \mathcal{F} \)-topology on \( V(K) \) refines the \( \mathcal{E}_K \)-topology for any \( K \)-variety \( V \). If the \( \mathcal{E}_K \)-topology on \( K \) refines \( \mathcal{F} \), then the \( \mathcal{E}_K \)-topology on \( V(K) \) refines the \( \mathcal{F} \)-topology for any \( K \)-variety \( V \).

We finally prove a general lemma.
Lemma 1.7. Suppose $R$ is a local domain and $K = \text{Frac}(R)$. The following are equivalent:

1. The $\mathcal{E}_K$-topology on $V(K)$ refines the $R$-adic topology for any $K$-variety $V$.
2. The $\mathcal{E}_K$-topology on $K$ refines the $R$-adic topology.
3. $R$ is an $\mathcal{E}_K$-open subset of $K$.
4. $R$ contains a nonempty $\mathcal{E}_K$-open subset of $K$.

If $K$ is additionally large and perfect, then (1)–(4) is equivalent to the following:

5. $R$ contains an infinite set, which is existentially definable in $K$.
6. There is a morphism $f : V \to \mathbb{A}^1$ of $K$-varieties such that $f(V(K))$ is infinite and contained in $R$.

We will only use the equivalence of (1)–(4) at present. If $K$ is not large, then the étale open topology is discrete and hence trivially refines the $R$-adic topology.

Proof. Fact 1.6 shows that (1)$\iff$(2). It is clear that (2)$\implies$(3). The definition of the $R$-adic topology and Fact 1.5.3 together show that (3)$\implies$(2). The equivalence of (3) and (4) follows by Fact 1.5.3 as $R$ is an additive subgroup of $K$. Suppose that $K$ is large and perfect. By [13, Theorem C], the $\mathcal{E}_K$-topology on $K$ is not discrete, so any nonempty étale image in $K$ is infinite. If (4) holds, then $R$ contains a nonempty étale image, (6) follows immediately and (5) follows as étale images are existentially definable. (6) implies (5) as $f(V(K))$ is existentially definable. Finally, an infinite existentially definable subset of $K$ has nonempty $\mathcal{E}_K$-interior [25, Theorem B], hence (5) implies (4).

\[\square\]

2 \hspace{1em} $R$-ADIC TOPOLOGIES AND V-TOPOLOGIES

Let $\mathcal{T}$ be a Hausdorff nondiscrete field topology on $K$. Then, $\mathcal{T}$ is a $V$-topology if for every neighborhood $U \ni 0$, there is a neighborhood $V \ni 0$ such that $xy \in V$ if $x \in U$ or $y \in U$. A field topology $\mathcal{T}$ is a $V$-topology if and only if $\mathcal{T}$ is induced by a nontrivial valuation or absolute value [7, Theorem B.1].

Proposition 2.1. Suppose that $R$ is an $n$-dimensional Noetherian local domain with fraction field $K$. If $n \geq 2$, then the $R$-adic topology on $K$ is not a $V$-topology. Hence, if $R$ is in addition regular, then the $R$-adic topology is a $V$-topology if and only if $n = 1$. In particular, the $L[[t_1, \ldots, t_n]]$-adic topology on $L((t_1, \ldots, t_n))$ is a $V$-topology if and only if $n = 1$.

A Noetherian local ring has finite Krull dimension [6, 8.2.2], so our use of “$n$” is justified. The proof of Proposition 2.1 requires the following lemmas.

Lemma 2.2. Suppose that $R$ is a local domain, $K$ is the fraction field of $R$, $v, v^* : K^\times \to \mathbb{Z}$ are homomorphisms such that $v, v^*$ are both nonnegative on $R \setminus \{0\}$, and some $\beta \in K^\times$ satisfies $v(\beta) < 0 < v^*(\beta)$. Then, the $R$-adic topology on $K$ is not a $V$-topology.

Before proving Lemma 2.2, we use it to show that the $L[[t, t^*]]$-adic topology on $L((t, t^*))$ is not a $V$-topology. Let $v$ be the $t$-adic valuation and $v^*$ be the $t^*$-adic valuation on $L((t, t^*))$. Set $\beta = t^*/t$. Then, $v, v^*$ are both nonnegative on $L[[t, t^*]]$ and $v(\beta) < 0 < v^*(\beta)$.

Proof (of Lemma 2.2). Suppose otherwise. Then, there is $c \in K^\times$ such that $xy \in cR \Rightarrow x \in R$ or $y \in R$.

For every $n \geq 1$, we have $v(c/\beta^n) < 0$, and if $n$ is sufficiently large, then $v^*(c/\beta^n) = v^*(c) - nv^*(\beta) < 0$. Thus, there is $n$ such that $\beta^n, c/\beta^n \notin R$, but $(\beta^n)(c/\beta^n) \in cR$, a contradiction.

\[\square\]

Fact 2.3 [14, Theorem 144]. Suppose $R$ is a Noetherian domain with dimension $\geq 2$. Then, $R$ contains infinitely many distinct prime ideals of height one.

We now prove Proposition 2.1. We let $R_p$ be the localization of a ring $R$ at a prime ideal $p$. 


Proof. The second claim follows from the first as a one-dimensional regular local ring is a discrete valuation ring (DVR). We prove the first claim. Applying Fact 2.3 fixes height one prime ideals \( p \neq p^* \) in \( R \). Then \( R_p, R_{p^*} \) are one-dimensional Noetherian domains. Let \( v : R \setminus \{0\} \to \mathbb{Z} \) be given by declaring \( v(a) \) to be the length of the \( R_p \)-module \( R_p/aR_p \). Then, \( v \) extends to a map \( K^\times \to \mathbb{Z} \) using \( p^* \). By [10, Definition A.3], \( v, v^* \) are well-defined homomorphisms. Note that \( v, v^* \) are nonnegative on \( R \) as the length is nonnegative. As \( p, p^* \) are distinct height one prime ideals, neither is contained in the other. Fix \( \alpha \in p \setminus p^* \), \( \alpha^* \in p^* \setminus p \), and set \( \beta = \alpha^*/\alpha \).

Note that \( v(\beta) < 0 < v^*(\beta) \). Apply Lemma 2.2.

The valuational approach to \( L((t, t^*)) \) generalizes to the regular case. Suppose that \( R \) is a regular local ring of dimension \( \geq 2 \). Let \( p, p^* \) be distinct height one prime ideals of \( R \). A localization of a regular ring is regular [6, Corollary 19.14] so \( R_p, R_{p^*} \) are one-dimensional regular local rings, hence DVR’s. Note that the induced discrete valuations on \( K \) satisfy Lemma 2.2.

3 | PROOF OF THEOREM 1.2.1

In this section, we suppose that \( R \) is a Henselian local domain with maximal ideal \( m \) and fraction field \( K \). We will need the following variant of Hensel’s lemma.

Fact 3.1. Suppose that \( g \in R[x] \) and \( \alpha \in R \) satisfy \( g'(\alpha) \neq 0 \) and \( g(\alpha) \equiv 0 \pmod{g'(\alpha)^2m} \). Then, there is \( \alpha^* \in R \) such that \( g(\alpha^*) = 0 \) and \( \alpha^* - \alpha \in (g(\alpha)/g'(\alpha))^m \).

We let \( \alpha \in R/m, \bar{p}(y) \in (R/m)[y] \) be the reduction mod \( m \) of \( \alpha \in R, p \in R[y] \), respectively.

Proof. After possibly replacing \( g(x) \) with \( g(x + \alpha) \), we suppose \( \alpha = 0 \). Fix \( c_0, \ldots, c_n \in R \) with \( g(x) = c_0 + c_1x + \cdots + c_nx^n \). Then, \( c_1 = g'(0) \neq 0 \) and \( c_0/c_1^2 = g(\alpha)/g'(\alpha)^2 \in m \). Let \( p \in R[x] \) be given by

\[
p(x) = x + \sum_{i=2}^n c_i c_1^{i-2} \left( \frac{c_0}{c_1^2} \right)^i (x - 1)^i.
\]

As \( c_0/c_1^2 \in m \) we have \( \bar{p}(x) = x \). Hence \( \bar{p}(0) = 0 \) and \( \bar{p}'(0) = 1 \). As \( R \) is Henselian, there is \( \beta \in m \) such that \( p(\beta) = 0 \).

Then,

\[
0 = p(\beta) = \beta + \sum_{i=2}^n c_i c_1^{i-2} \left( \frac{c_0}{c_1^2} \right)^i (\beta - 1)^i = 1 + (\beta - 1) + \sum_{i=2}^n c_i \left( \frac{c_0}{c_1^2} \right)^i (\beta - 1)^i.
\]

Hence,

\[
0 = c_0 p(\beta) = c_0 + c_0 (\beta - 1) + \sum_{i=2}^n c_i \left( \frac{c_0}{c_1} \right)^i (\beta - 1)^i
= c_0 + c_1 \left( \frac{c_0}{c_1} \right)(\beta - 1) + \sum_{i=2}^n c_i \left( \frac{c_0}{c_1} \right)^i (\beta - 1)^i
= \sum_{i=0}^n c_i \left( \frac{c_0}{c_1} \right)^i (\beta - 1)^i.
\]

Set \( \alpha^* = (c_0/c_1)(\beta - 1) \). Then, \( \alpha^* \in (c_0/c_1)R \) as \( \beta \in R \). Note that \( g(\alpha^*) = 0 \).

We proceed with the proof of Theorem 1.2.1. We recall the notion of a standard étale morphism; see [17, Definition 3.5.38] for more details. A morphism \( f : X \to Y \) between affine schemes is called standard étale if the corresponding map in the category of rings is of the form, \( R \to R[x]/(f) \), where \( g, f \in R[x] \) and \( f \) is monic with \( f' \) invertible in \( R[x]/(f) \). Given an affine \( K \)-variety \( V \), a standard étale image in \( V(K) \) is a set of the form \( f(X(K)) \) for a standard étale morphism.
Lemma 4.1. In this section, we suppose that \( f : X \to V \) of the origin and show that 0 lies in the \( f : X \to V \). JOHNSON et al. by Fact 1.1.5. Hence, we suppose that \( \partial f \in \mathcal{O} \) of the origin and show that 0 lies in the \( f : X \to V \). We proceed with the proof of Theorem 1.2.2. If \( R \) is one dimensional, then \( K \) is a discretely valued field; this case follows by Fact 1.1.5. Hence, we suppose that \( \dim R \geq 2 \). By Lemma 1.7, it is enough to produce an \( \mathcal{O}_K \)-neighborhood of 0 contained in \( R \). First, suppose \( \Char(R/m) \neq 2 \). This implies that \( \Char(K) \neq 2 \). Let \( S = \{ b^2 : b \in K^* \} \), \( f : K \to K \) be given by \( f(x) = \)
1 + x^4, and Ω = f^{-1}(S). Note 0 ∈ Ω. By Fact 1.5, S and Ω are \( \mathcal{C}_K \)-open sets. By Lemma 4.2, \( \Omega \subseteq R \cup R^{-1} \). Fix a height one prime ideal \( p \) of \( R \). The localization \( R_p \) is a regular local ring of dimension 1, and hence a discrete valuation ring. Let \( \nu : K^* \to \mathbb{Z} \) be the associated valuation. By Fact 1.1.5, the \( \mathcal{C}_K \)-topology on \( K \) refines the \( \nu \)-topology. Therefore, the valuation ideal \( pR_p = \{ a \in K : \nu(a) > 0 \} \) is \( \mathcal{C}_K \)-open. The intersection \( \Omega \cap pR_p \) is an \( \mathcal{C}_K \)-open neighborhood of 0. We show that \( \Omega \cap pR_p \subseteq R \). Suppose \( a \in \Omega \cap pR_p \). As \( a \in pR_p \) we have \( \nu(a) > 0 \), hence \( \nu(a^{-1}) < 0 \), hence \( a^{-1} \notin pR_p \), so \( a^{-1} \notin R \). As \( a \in \Omega \), we have \( a \in R \).

The case when \( \text{Char}(R/m) = 2 \) is similar, using \( S = \{ b^3 : b \in K^* \} \) and \( f(a) = 1 + a^3 \).

## 5 PROOF OF THEOREM 1.3

Fix a prime \( p \), let \( \nu : K^* \to \mathbb{Z} \) be the valuation ideal. Let \( \nu : Q_p^* \to \mathbb{Z} \) be the \( p \)-adic valuation. Fix a \( (x, 1 + x^4 + 4x^5) \) is a \( \mathcal{C}_K \)-open set. By Lemma 4.2, \( \Omega \subseteq R \cup R^{-1} \). Fix a height one prime ideal \( p \) of \( R \). The localization \( R_p \) is a regular local ring of dimension 1, and hence a discrete valuation ring. Let \( \nu : K^* \to \mathbb{Z} \) be the associated valuation. By Fact 1.1.5, the \( \mathcal{C}_K \)-topology on \( K \) refines the \( \nu \)-topology. Therefore, the valuation ideal \( pR_p = \{ a \in K : \nu(a) > 0 \} \) is \( \mathcal{C}_K \)-open. The intersection \( \Omega \cap pR_p \) is an \( \mathcal{C}_K \)-open neighborhood of 0. We show that \( \Omega \cap pR_p \subseteq R \). Suppose \( a \in \Omega \cap pR_p \). As \( a \in pR_p \) we have \( \nu(a) > 0 \), hence \( \nu(a^{-1}) < 0 \), hence \( a^{-1} \notin pR_p \), so \( a^{-1} \notin R \). As \( a \in \Omega \), we have \( a \in R \).

The case when \( \text{Char}(R/m) = 2 \) is similar, using \( S = \{ b^3 : b \in K^* \} \) and \( f(a) = 1 + a^3 \).

## 5 PROOF OF THEOREM 1.3

Fix a prime \( p \), let \( \nu : K^* \to \mathbb{Z} \) be the valuation ideal. Let \( \nu : Q_p^* \to \mathbb{Z} \) be the \( p \)-adic valuation. Fix a \( (x, 1 + x^4 + 4x^5) \) is a \( \mathcal{C}_K \)-open set. By Lemma 4.2, \( \Omega \subseteq R \cup R^{-1} \). Fix a height one prime ideal \( p \) of \( R \). The localization \( R_p \) is a regular local ring of dimension 1, and hence a discrete valuation ring. Let \( \nu : K^* \to \mathbb{Z} \) be the associated valuation. By Fact 1.1.5, the \( \mathcal{C}_K \)-topology on \( K \) refines the \( \nu \)-topology. Therefore, the valuation ideal \( pR_p = \{ a \in K : \nu(a) > 0 \} \) is \( \mathcal{C}_K \)-open. The intersection \( \Omega \cap pR_p \) is an \( \mathcal{C}_K \)-open neighborhood of 0. We show that \( \Omega \cap pR_p \subseteq R \). Suppose \( a \in \Omega \cap pR_p \). As \( a \in pR_p \) we have \( \nu(a) > 0 \), hence \( \nu(a^{-1}) < 0 \), hence \( a^{-1} \notin pR_p \), so \( a^{-1} \notin R \). As \( a \in \Omega \), we have \( a \in R \).

The case when \( \text{Char}(R/m) = 2 \) is similar, using \( S = \{ b^3 : b \in K^* \} \) and \( f(a) = 1 + a^3 \).

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The case when \( \text{Char}(R/m) = 2 \) is similar, using \( S = \{ b^3 : b \in K^* \} \) and \( f(a) = 1 + a^3 \).
Proposition 5.5. The $E$-adic topology on $\mathbb{Q}_p$ strictly refines the $p$-adic topology.

Proof. We have $E \subseteq \mathbb{Z}_p$, so the $E$-adic topology refines the $p$-adic topology by Lemma 5.4. We show that $E$ is not open in the $p$-adic topology. Suppose $O$ is a nonempty $p$-adically open subset of $\mathbb{Q}_p$. By Lemma 5.1, there is $a \in O$ such that $\partial a \notin \mathbb{Q}_p \setminus \mathbb{Z}_p$. Then, $a \notin E$, hence $O \notin E$.

Proposition 5.6. $E$ is Henselian.

Proof. Given $g \in \mathbb{Q}_p[x]$, $g(x) = a_0 + a_1 x + \cdots + a_d x^d$, we let $\partial g \in \mathbb{Q}_p[x]$ be $\partial(a_0) + \partial(a_1)x + \cdots + \partial(a_d)x^d$. Note that if $g \in E[x]$, then $\partial g \in \mathbb{Z}_p[x]$. As above, we let $m$ be the maximal ideal of $E$.

Fix $g \in E[x]$ and $a \in E$, such that $g(a) \equiv 0 \pmod{m}$ and $g'(a) \not\equiv 0 \pmod{m}$. We will produce $a^* \in E$ such that $g(a^*) = 0$ and $a^* \equiv a \pmod{m}$. As $m = E \cap p\mathbb{Z}_p$, we have $g(a) \equiv 0 \pmod{p\mathbb{Z}_p}$ and $g'(a) \not\equiv 0 \pmod{p\mathbb{Z}_p}$. As $\mathbb{Z}_p$ is Henselian, there is $a^* \in \mathbb{Z}_p$ such that $g(a^*) = 0$ and $a^* \equiv a \pmod{m}$. It suffices to show that $a^* \in E$: as $m = E \cap p\mathbb{Z}_p$, this will yield $a^* \equiv a \pmod{m}$, and hence that $E$ is Henselian. To show that $a^* \in E$, it is enough to show that $\partial a^* \in \mathbb{Z}_p$. We have

$$0 = \partial(0) = \partial(g(a^*)) = (\partial g)(a^*) + g'(a^*)\partial(a^*)$$

hence

$$\partial(a^*) = \frac{(\partial g)(a^*)}{-g'(a^*)}.$$ 

As $g \in E[x]$, we have $\partial g \in \mathbb{Z}_p[x]$, hence $(\partial g)(a^*) \in \mathbb{Z}_p$. As $a^* \equiv a \pmod{p\mathbb{Z}_p}$, we have $g'(a^*) \equiv g'(a) \not\equiv 0 \pmod{p\mathbb{Z}_p}$, and so $g'(a^*) \in \mathbb{Z}_p$. Therefore, $\partial a^* \in \mathbb{Z}_p$.

It remains to show that $E$ is one-dimensional Noetherian. We will use the following fact.

Fact 5.7. Suppose that $R$ is a domain, $R$ is not a field, and there is $n$ such that every ideal in $R$ admits an $n$ element generating set. Then $R$ is one-dimensional Noetherian.

Fact 5.7 is a theorem of Cohen [4, Corollary 1, Theorem 9, and Theorem 10]. To apply Fact 5.7, we will show that every ideal is generated by two elements. We first prove a technical lemma.

Lemma 5.8. Suppose $a, a', a'' \in \mathbb{Q}_p$, $v(a) \leq \min\{v(a'), v(a'')\}$ and $v(\partial(a'/a)) \leq v(\partial(a''/a))$. Then, $a'' \in aE + a'E$.

Proof. We may assume $a'' \neq 0$. Then, $v(a) \leq v(a'') < \infty$, so $a \neq 0$. After replacing $a, a', a''$ by $a/a, a'/a, a''/a$, we may suppose $a = 1$. Then, $0 = v(a) \leq \min\{v(a'), v(a'')\}$, so $a', a'' \in \mathbb{Z}_p$. Additionally, $v(\partial a) \leq v(\partial a')$. This yields $b \in \mathbb{Z}_p$ such that $\partial(a'') = b\partial(a')$. By continuity of multiplication, there is a $p$-adically open neighborhood $U \subseteq \mathbb{Z}_p$ of $b$ such that if $b^* \in U$, then $\partial(a'') - b^*\partial(a') \in \mathbb{Z}_p$. By Lemma 5.1, there is $b^* \in U$ such that $\partial(b^*) \in \mathbb{Z}_p$. Then, $b^* \in E$. Let $c = a'' - b^*a'$. Then, $a'' = c + b^*a'$. We claim that $c \in E$, so that $a'' \in E + a'E$. We have $c \in \mathbb{Z}_p$, as $a', a'', b^* \in \mathbb{Z}_p$. We have

$$\partial(c) = \partial(a'') - \partial(b^*a') = \partial(a'') - b^*\partial(a') - a'\partial(b^*) .$$

Note that $\partial(a'') - b^*\partial(a')$ and $a'\partial(b^*)$ are both in $\mathbb{Z}_p$. Hence, $\partial c \in \mathbb{Z}_p$, so $c \in E$.

Proposition 5.9. $E$ is one-dimensional Noetherian.

Proof. By Fact 5.7, it suffices to show that any ideal $I$ in $E$ has a two element generating set. We may suppose $I \neq \{0\}$. As $E \subseteq \mathbb{Z}_p$, we have $v(a) \geq 0$ for all $a \in I$. Fix $a \in I$ minimizing $v(a)$; note that $a \neq 0$. For any $a^* \in I$, we have

$$v(\partial(a^*/a)) = v\left( \frac{a\partial(a^*) - a^*\partial(a)}{a^2} \right) = v(a\partial(a^*) - a^*\partial(a)) - 2v(a).$$
As $a, a^*, \delta(a), \delta(a^*) \in \mathbb{Z}_p$ we have $v(a\delta(a^*) - a^*\delta(a)) \geq 0$, hence $v(\delta(a^*/a)) \geq -2v(a)$. Therefore, we may select $a' \in I$ minimizing $v(\delta(a'/a))$. We show that $I = aE + a'E$. Fix $a'' \in I$. Then, $v(a) \leq \min\{v(a'), v(a'')\}$ and $v(\delta(a'/a)) \leq v(\delta(a''/a))$. Apply Lemma 5.8.

Remark 5.10. Our ring $E$ is similar to Ferrand and Raynaud’s example of a Noetherian one-dimensional local domain with nonreduced completion [9, Proposition 3.1]. Their example probably satisfies an analogue of Theorem 1.3, with $C\{t\}$ replacing $\mathbb{Q}_p$. Likewise, our $E$ also has nonreduced completion, by Lemma 5.11 below. This implies that $E$ is not excellent [23, 07QT, 07GH, 07QK] and not regular [23, 07NY, 00NP].

Lemma 5.11. The completion of $E$ is $\mathbb{Z}_p[x]/(x^2)$.

Proof. It is enough to produce a ring embedding $\tau: E \to \mathbb{Z}_p[x]/(x^2)$ such that $\tau$ gives a dense topological embedding from the $m$-adic topology on $E$ to the $p$-adic topology on $\mathbb{Z}_p[x]/(x^2)$. Let $\tau: E \to \mathbb{Z}_p[x]/(x^2)$ be $\tau(a) = a + \delta(a)x$. Note that $\tau$ is an injective ring homomorphism. The $p$-adic topology on $\mathbb{Z}_p[x]/(x^2)$ agrees with the product topology given by the natural bijection $\mathbb{Z}_p[x]/(x^2) \to \mathbb{Z}_p^2$. By Lemma 5.1, the image of $\tau$ is dense. The $m$-adic topology on $E$ agrees with the restriction of the $E$-adic topology on $\mathbb{Q}_p$ to $E$. (It suffices to show that for any nonzero ideal $I$ in $E$ we have $m^n \subseteq I$ for some $n$. This holds because $E/I$ is a local Artinian ring, as dim $E = 1$.) Meanwhile, $(\mathbb{Q}_p, \mathbb{Z}_p, \delta)$ is a “lifted diffeovalued field” (see [12, Definition 8.10]) that is “dense” (see [12, Definition 8.12]) by [12, Proposition 8.19] and Lemma 5.1 above. The $E$-adic topology on $\mathbb{Q}_p$ is the “diffeovaluation topology” (see [12, Definition 8.16]). Therefore, [12, Proposition 8.21] shows that the collection of sets of the form $\{\alpha \in E: v(\beta - \alpha) > \gamma$ and $v(\beta^* - \delta \alpha) > \gamma^*\}$ for $\beta, \beta^* \in E$ and $\gamma, \gamma^* \in \mathbb{Z}$ is a basis for the $E$-adic topology on $\mathbb{Q}_p$. This implies that $\tau: E \to \mathbb{Z}_p^2$ is a topological embedding.

Remark 5.12. We discuss our usage of the axiom of choice. Note that $\delta$ is a discontinuous additive homomorphism $\mathbb{Q}_p \to \mathbb{Q}_p$, because $\delta$ is nonzero but vanishes on the dense set $\mathbb{Q} \subseteq \mathbb{Q}_p$. This implies that $\delta$ is not measurable, hence existence of $\delta$ requires a strong application of the axiom of choice; see, for example, [22, Section 2]. One can avoid this. In fact, our argument goes through for $R$ a characteristic zero Henselian DVR, $K = \text{Frac}(R)$, and $\delta: K \to K$ a derivation with dense graph. For example, fix $t \in \mathbb{Q}_p$, transcendental over $\mathbb{Q}$ and let $F$ be the algebraic closure of $\mathbb{Q}(t)$ in $\mathbb{Q}_p$. Then, $F$ is a dense subfield of $\mathbb{Q}_p$ and $F \cap \mathbb{Z}_p$ is a Henselian DVR with fraction field $F$. Let $\delta^* \in \mathbb{Q}$ be the unique $\mathbb{Q}$-linear derivation $\mathbb{Q}(t) \to \mathbb{Q}(t)$ with $\delta^* t = 1$. Then, $\delta^*$ uniquely extends to a $\mathbb{Q}$-linear derivation $\delta: F \to F$ as $F/\mathbb{Q}(t)$ is algebraic [26, Proposition 1.15]. This extension does not require choice. Our example goes through with $\mathbb{Z}_p, \mathbb{Q}_p$ replaced by $F \cap \mathbb{Z}_p, F$, respectively. At the end, one obtains a $p$-adically closed field $F$ of transcendence degree 1 and a subring $E \subseteq F$ satisfying the conclusions of Theorem 1.3, with $\mathbb{Q}_p$ replaced by $F$.

6 | FINAL REMARK

We showed in [13, Theorem B] that the étale open topology over $K$ is induced by a $V$-topology if and only if $K$ is infinite, $t$-Henselian, and not separably closed ($t$-Henselianity is a topological generalization of Henselianity, see [21, section 7]). We have produced examples such as $L((t_1, \ldots, t_p))$ for $n > 1$ where the étale open topology is induced by a field topology that is not a $V$-topology. We do not know a general criterion for when the étale open topology is induced by a field topology.

ACKNOWLEDGMENTS

We would like to thank Marcus Tressl who encouraged us to write this down, as well as two anonymous referees, who caught many errors. WJ was partially supported by the National Natural Science Foundation of China (Grant No. 12101131). JY was partially supported by GeoMod AAPG2019 (ANR-DFG), Geometric and Combinatorial Configurations in Model Theory.

CONFLICT OF INTERESTS STATEMENT

The authors declare no potential conflict of interests.

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ENDNOTES

1When $V$ is $\mathbb{A}^n$, we take the product topology on $\mathbb{K}^n$. When $V \subseteq \mathbb{A}^n$ is an affine variety, we take the subspace topology on $V(\mathbb{K}) \subseteq \mathbb{K}^n$. When $V$ is an arbitrary variety, we cover $V$ by affine open subvarieties and glue the corresponding topologies. See [16, chapter I.10] for details.

2In [12, section 8], the residue characteristic 0 assumption was only used in [12, Proposition 8.9]. Hence, the definitions and results quoted in the proof are unaffected.

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How to cite this article: W. Johnson, E. Walsberg, and J. Ye, The étale open topology over the fraction field of a Henselian local domain, Math. Nachr. 296 (2023), 1928–1937. https://doi.org/10.1002/mana.202100418