THE UNIVERSAL PERTURBATIVE QUANTUM 3-MANIFOLD
INVARIANT, ROZANSKY-WITTEN INVARIANTS, AND THE
GENERALIZED CASSON INVARIANT

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Abstract

Let $Z^{LMO}$ be the 3-manifold invariant of [7]. It is shown that $Z^{LMO}(M) = 1$, if the first
Betti number of $M$, $b_1(M)$, is greater than 3. If $b_1(M) = 3$, then $Z^{LMO}(M)$ is completely
determined by the cohomology ring of $M$. A relation of $Z^{LMO}$ with the Rozansky-Witten
invariants $Z^R_X[M]$ is established at a physical level of rigour. We show that $Z^R_X[M]$ satisfies
appropriate connected sum properties suggesting that the generalized Casson invariant ought to
be computable from the LMO invariant.

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1 Introduction

In [W], E. Witten explained the Jones polynomial using physics. In doing so, he introduced mathematicians to the partition function $Z_{G,k}^{CS}(M,L)$ of the topological quantum field theory associated to the Chern Simons action, for a Lie group $G$ and coloured link $L \subset M$. Its physical definition is given by a Feynman path integral over the infinite dimensional space of connections.

In general, one expects that topological field theories defined using the path integral, or perturbative versions of these, can be given a rigorous definition through surgery formulae, just as is the case for other quantum invariants, such as the Reshetikhin-Turaev invariants, $Z_{G,k}^{RT}$, or the more recent universal invariant $Z_{LMO}^{M}$, of T. Le, J. Murakami, and T. Ohtsuki and the Arhus invariant, $Z_{A}$, [2]. Invariants have also been given through integral formulae, as is the case for the Kontsevich integral $Z_{K}^{K}$ (see [2] [3]), the Bott-Taubes invariant $Z_{BT}^{BT}$, [3], [3], and the invariant $Z_{BC}^{BC}$ of Bott and Cattaneo [4], [5]. (It is conjectured that $Z_{K}^{K} = Z_{BT}^{BT}$.)

Our intent in this paper is to study the invariant $Z_{LMO}^{LMO}(M)$. $Z_{LMO}^{LMO}(M)$ lies in $A(\theta)$, the vector space of Feynman diagrams modulo anti-symmetry and $IHX$ relations. This vector space is not well-understood, except in low degrees. (See Vogel, [3], for an attempt to understand the structure of $A(\theta)$.)

Quantum invariants (or perturbative versions of these) are a rich source of data for the study of knots, links, and 3-manifolds. Nevertheless, their relationship to classical topology remains obscure, hampering their use in problem-solving. A notable exception is the Alexander polynomial of a knot, which, through its interpretation as the Conway polynomial (together with the solution of the Conway weight system on uni-trivalent graphs [3]), gives a computation of the one loop part of the Kontsevich integral. Another recent advance has been the computation of the Milnor invariants from the Kontsevich integral [3].

For 3-manifolds, one has the result that the degree one term of the LMO invariant, $Z_{LMO}^{1}$, is the Casson-Walker-Lescop invariant [2], [3]. However, beyond this, the topological significance of $Z_{LMO}^{LMO}$ remains a mystery. For example, it is not even known whether or not the degree two term of $Z_{LMO}^{LMO}$ vanishes in the simply connected case (which of course would be implied by a positive solution of the Poincaré conjecture, since $Z_{LMO}^{LMO}(S^{3}) = 1$).

One possible programme for attempting to tie the quantum invariants to homotopy data is through generalization of the Casson invariant to groups other than $SU(2)$, e.g., $SU(N)$. Recent advances on the mathematical side [2], for $SU(3)$, as well as on the physical side, by Rozansky and Witten [2], may make this programme tractable. The purpose of this paper is to give a conjectural relationship between the generalized Casson invariants and $Z_{LMO}^{LMO}$ and some partial evidence for its veracity. We consider this conjecture to be an explicit form of the basic philosophical viewpoint of [2], who believe their invariants are of finite type.

Indeed, we may summarize the underlying ideas of [2] as follows. On the one hand a comparison of the Rozansky-Witten invariants to the perturbative Chern-Simons theory indicates that,

\[\text{By coloured link, one means that to each link component, one associates a representation of } G\]

\[\text{The connections are on an underlying principle } G\text{-bundle lying over } M.\]
for \( b_1(M) = 0 \), they both arise from one universal theory. The difference between the two rests in the choice of weight system (in [?] a rigorous mathematical weight system \( W_X^{RW} \) is given). On the other hand, (and perhaps the deepest part of the theory) equivalence between certain physical theories allows one to identify the Rozansky-Witten invariants for a particular choice of hyper-Kähler manifold \( X_G \) with a regularized Euler characteristic [?] of the moduli space of flat \( G \)-connections on the 3-manifold \( M \), \( \chi_G(M) \). The equivalence comes from the work of Seiberg and Witten on 3-dimensional theories (and is the analogue of their work in four dimensions). The twist of the first theory yields the gauge theoretic model of the Euler characteristic while the twist of the second is the Rozansky-Witten model. The equivalence of the physical theories suggests the equivalence of their twisted topological versions. Thus, from the physics side, one expects that

\[
Z_{X_G}^{RW}(M) = \chi_G(M). \tag{1.1}
\]

One important consequence of (??) is its potential use in computing \( \chi_G(M) \).

On the mathematical side there are, currently, a number of candidates for a universal perturbative quantum invariant. These include the LMO invariant, \( Z^{LMO} \), the Århus invariant, \( Z^\Lambda \), and the invariant of Bott and Cattaneo, \( Z^{BC} \). It has been suggested [?] that although the LMO invariant agrees with the Århus invariant, for rational homology spheres, that nevertheless \( Z^{LMO} \) is more directly related to the Rozansky-Witten sigma model theory than to the Chern-Simons gauge theory.

For \( b_1(M) > 0 \), (except for \( b_1(M) = 1 \) and \( \text{Tor} \, H_1(M, \mathbb{Z}) \neq 0 \)) the first author and collaborators, [?] [?], have calculated \( Z^{LMO} \) from classical data. Here, we perform analogous computations in the Rozansky-Witten theory and observe, for \( b_1(M) > 0 \), that these results agree. Specifically, we show, for \( b_1(M) > 0 \), and under the conditions mentioned above,\(^5\) that at the physical level of rigour,

\[
W_X^{RW}\left(Z^{LMO}_{n}(M)\right) = Z_X^{RW}(M), \tag{1.2}
\]

where \( X \) denotes a hyper-Kähler manifold of dimension \( 4n \).

One might naively conjecture that (??) holds for all 3-manifolds. However, for \( b_1(M) = 0 \), which is the case of most interest, numerous considerations, including connected sum formulae and normalization conditions, indicate that the equality (??) should be modified. We introduce invariants \( \lambda_X^k(M) \) for all \( k \) and \( X \) which are computed from the Rozansky-Witten theory, and the formulae now suggest\(^6\) that

\[
|H_1(M, \mathbb{Z})|^{n-k} W_X^{RW}\left(Z^{LMO}_k(M)\right) = \lambda_X^k(M). \tag{1.3}
\]

We now propose, that on correcting for the trivial connection, one should replace (??) with the equality

\[
\lambda_X^k(M) = \lambda_G(M), \tag{1.4}
\]

\(^5\)The computations that we make for the Rozansky-Witten theory suggest that it is to be expected that the results of [?] hold even when the manifold has torsion in \( H_1(M, \mathbb{Z}) \).

\(^6\)This corresponds to the fact that \( Z^{RT}_{G,k}/Z^{RT}_{G,k}(S) \) and \( \sum_n |H_1|^{-n} Z^{LMO}_n \) are both multiplicative.
where $\Lambda_G(M)$ is the, still to be mathematically defined, G-Casson invariant, and where $\text{rank}(G) = n$. In this way, we obtain the purely mathematical

Conjecture:

$$W_{X_G}^{RW} \left( Z_{LMO}^{G}(M) \right) = \lambda_G(M).$$

(N.b., for $SU(2)$ this equality holds by the computation of $Z_{LMO}$ combined with those on the physics side [?].)

The equalities above are certainly suggestive. On the one hand, $Z_{LMO}(M)$ satisfies axioms$^7$ of topological quantum field theory (TQFT), see [?], as is the case for the Chern-Simons theory $Z^{RT}_{G,k}$. On the other hand, $Z_{X}^{RW}$ is given as a topological sigma-model. As explained in [?], the actions of the Chern-Simons theory, and the Rozansky-Witten theory are formally analogous (see section 4 below).

We begin this paper with a computation, which originally appeared in [?], of $Z_{LMO}(M)$ for manifolds whose first Betti number, $b_1(M)$, is greater than or equal to 3. Subsequently, computations for $b_1(M) = 2$, [?], and $b_1(M) = 1$, [?], followed. These computations were inspired by the work of T. Le, [?], who showed that the invariant $Z_{LMO}(M)$, restricted to homology spheres, is the universal finite type invariant$^8$ in the sense of Ohtsuki [?].

Specifically, in section 3, we will give a proof of parts (i) and (ii) of the following result (part (iii) was proven in [?] and part (iv) in [?]). Let $\lambda_M$ denote the Lescop invariant of $M$, see [?].

Theorem 1.

(i) Suppose $b_1(M) > 3$. Then $Z_{LMO}(M) = 1$.

(ii) There are non-zero $\gamma_n \in A_n(\emptyset)$, such that if $b_1(M) = 3$, then $Z_{LMO}(M) = \Sigma_n \lambda^n_M \gamma_n$.

(iii) [?] There are non-zero $\mathcal{H}_n \in A_n(\emptyset)$, such that if $b_1(M) = 2$, then $Z_{LMO}(M) = \Sigma_n \lambda^n_M \mathcal{H}_n$.

(iv) [?] For $H_1(M) = \mathbb{Z}$, $Z_{LMO}(M)$ determines and is determined by $A(M)$, the Alexander Polynomial of $M$.

Remark. In fact, though not observed in [?], but as suggested from the combinatorics of the physical approach, one can show using equality (2) in [?] that $\gamma_n = \pm \mathcal{H}_n$.

Sections ??-?? of this paper concern heuristic results, reminiscent of theorem 1 and the hypothetical equality $W_{X}^{RW}(Z_{nLMO}(M)) = Z_{X}^{RW}(M)$. Specifically, we give a heuristic proof, (i.e. at the physical level of rigour) of the following:

Heuristic Theorem 2.

(i) Suppose $b_1(M) > 3$. Then $Z_{X}^{RW}(M) = 0$.

(ii) There are constants $c_X$, such that if $b_1(M) = 3$, then $Z_{X}^{RW}(M) = c_X \lambda^n_M$.

(iii) There are constants $c'_X$, such that if $b_1(M) = 2$, then $Z_{X}^{RW}(M) = c'_X \lambda^n_M$.

(iv) For $b_1(M) = 1$, $Z_{X}^{RW}(M)$ is determined by Reidemeister Torsion.

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$^7$Actually, the TQFT axioms hold for certain truncations of $Z_{LMO}$.

$^8$See [?] for an expository account of the theory of finite type invariants.
Actually, the equality $W^X_R(W^X_R(Z_{LMO}(M))) = Z^X_R(M)$ suggests that $c_X = W^X_R(\gamma_n)$ and $c'_X = W^X_R(\mathcal{H}_n)$. Our calculations indicate that this is so and furthermore, show that $c_X = c'_X$.

The final sections, ??-??, are devoted to deriving the properties of the $\lambda_X$ invariants that are required to motivate (??) and our conjecture (??).

Let us conclude this introduction with a few remarks on our proof of heuristic theorem 2. While, for some purposes, the perturbative Feynman diagram expansion may be useful, e.g., for obtaining weight systems, our approach is essentially non-perturbative. In general the path integral formalism may have uses beyond giving us a definition of invariants. One can define theories via path integrals and after passing to the perturbation theory completely forgo the path integral formulation. This leads to an interesting set of combinatorial problems, having to do with the type of diagrams to be considered, as well as their frequency. On the other hand, it may happen that the path integral can be performed in a, more or less, elementary manner. In this case the combinatorial issues are by-passed, and in addition one obtains nicely re-summed formulae. An example of such a situation is the derivation of the Verlinde formula [?] for the dimension of the space of holomorphic sections of the $k$'th tensor power of the determinant line bundle over the space of flat connections on a Riemann surface. Similarly, for the Rozansky-Witten invariants, we will see that it is better to ‘perform’ the path integral directly rather than to expand out first.

The main thrust of our physical computations is then to avoid working directly with diagrams. However, in order to make the relationship with [?] somewhat more transparent we will, on occasion, explain certain phenomena at the diagrammatic level.

2 The Invariant $Z^{LMO}(M)$.

The invariant $Z^{LMO}(M)$ is computed in general from the Kontsevich integral (denoted here by $Z^K(L)$) of any framed link $L: \coprod_{j=1}^l S^1 \to \mathbb{R}^3$, such that surgery on $L$, denoted by $S^3(L)$, produces $M$. $Z^K(L)$ lies in $A(\coprod_{j=1}^l S^1)$.

Before stating the result, we recall (see, e.g., [?], [?], [?], [?]) that $A(X)$ denotes the graded-completed Q-vector space of Feynman diagrams $X \cup \Gamma$ on the compact 1-manifold $X$. The space $A(X)$ is graded by the degree, where the degree of a diagram is half the number of vertices of $\Gamma$. Using the notation of [HM], we let $\coprod_{j=1}^l I_j$ denote the disjoint union of $l$ copies of the interval, and we set $A(l) = A(\coprod_{j=1}^l I_j)$. $A(l)$ is a Hopf algebra, and one has that $A(1) = A(S^1)$. Moreover, any embedding $I \to X$ gives rise to a well defined action of $A(1)$ on $A(X)$. In particular, $A(1)^{\otimes l}$ acts on $A(l)$ and on $A(\coprod_{j=1}^l S^1)$.

For $1 \leq i, j \leq l$, we let $\xi_{ij}$ be the degree 1 diagram $X \cup \Gamma$, with $X = \coprod_{j=1}^l I_j$, where $\Gamma$ is a chord with vertices on the $i$-th and $j$-th components ($i$ may be equal to $j$). We set $\xi_{12} = [\xi_{12}, \xi_{23}]$, where $[a, b] = ab - ba$ denotes the Lie bracket of $a$ and $b$. ($\xi_{123}$ is represented by the diagram $X \cup \Gamma$, with $X = \coprod_{j=1}^3 I_j$, where $\Gamma$ is the $Y$-graph of degree 2 with one vertex on each component of $X$.)

In [?], maps $i_n: A_{n+i}(\coprod_{j=1}^l S^1) \to A_i(\emptyset)$ were defined for $i \leq n$. (We set $i_n = 0$ otherwise.) We denote by $p_n: A(l) \to A(\coprod_{j=1}^l S^1)$ the quotient mapping. We set $\gamma_0 = 1$, and $\gamma_n = i_n(p_3(\frac{\xi_{123}}{2n})) \in A_n(\emptyset)$. Note that $A_1(\emptyset)$ is 1-dimensional and $A_1(\emptyset)^{\otimes n}$ is a direct summand
of $A_n(\emptyset)$. Moreover, it is easily seen from the definition of $i_n$ that the image of $\gamma_n$ in $A_1(\emptyset) \otimes \mathbb{R}$ is nonzero. Hence $\gamma_n$ is nonzero.

For a set $A$, we set $|A|$ to be the cardinality of $A$, if this is finite, and 0 otherwise. For a 3-manifold $M$ with $b_1(M) = 3$, we define $\lambda_M = |\text{Tor}(H_1(M))| \cdot \frac{|H^3(M)|}{|H^1(M) \otimes \mathbb{R}|^2}$, where $i: H^1(M) \otimes \mathbb{R} \to H^3(M)$ is given by the cup product $a \otimes b \otimes c \mapsto a \cup b \cup c$. This is Lescop’s invariant, for $b_1(M) = 3$, see [?] section 5.3.

Theorem 1.

(i) Suppose $b_1(M) > 3$. Then $Z_{\text{LMO}}(M) = 1$.

(ii) Suppose $b_1(M) = 3$. Then $Z_{\text{LMO}}(M) = \Sigma_n \lambda_M \gamma_n$.

The theorem will be proven in the next section. We first recall here how $Z_{\text{LMO}}(M)$ is defined. One puts $\nu = Z^K(U)$, where $U$ is the trivial knot with framing zero. Then $Z_{\text{LMO}}^n(M)$ is the degree $n$ part of the expression

\[
\frac{i_n(Z^K(L)\nu^{\otimes l})}{(i_n(\nu^2 \exp(\frac{\xi_{11}}{2})))^b_+ (i_n(\nu^2 \exp(\frac{-\xi_{11}}{2})))^b_-}
\]

considered to lie in $A_{\leq n}(\emptyset)$, where $b_+, b_-$ denote the number of positive and negative eigenvalues of the linking matrix of $L$. (It was shown in [?] that the expressions in the denominators are invertible.)

Remark. For later use we note that for $k \leq n$, in [?] the degree $k$ part of the above expression (3.3) was denoted by $\Omega_n(M)^{(k)}$. Thus in particular, $Z_{\text{LMO}}^n(M) = \Omega_n(M)^{(n)}$.

In the proof of the theorem, we will need to make use of certain facts.

1) $i_n$ satisfies the property that it vanishes on diagrams which have fewer than $2n$ vertices on some component.

2) Let $\sigma$ be a string link whose closure is $L$. Then $Z^K(L) = p_i(Z^K(\sigma)\nu_l)$ (see [2]). Here $Z^K(\sigma)$ lies in $A(l)$ and $\nu_l \in A(l)$ is obtained from $\nu = \nu_1$ by the operator which takes a diagram on the interval to the sum of all lifts of vertices to each of the $l$ intervals. It is known that $\nu$ and hence $\nu_l$ is a sum of diagrams each of which has each component of $\Gamma$ non-simply connected (see [?]).

3) Let $\sigma$ be an $l$-component string link. Then $Z^K(\sigma) = \exp(\xi^t + \xi^h)$, where $\xi^t$ is a linear combination of diagrams for which $\Gamma$ is a tree, and $\xi^h$ is a linear combination of diagrams for which $\Gamma$ is connected, but not simply connected. If we denote by $A'(X)$ the quotient of $A(X)$ obtained by setting to zero all diagrams for which some component of $\Gamma$ is not simply connected, and denote by $Z'(\sigma)$, the image in $A'(l)$ of $Z^K(\sigma)$, then it was shown in [?] that the Milnor invariants of $\sigma$ determine, and are determined by $Z'(\sigma) = \exp(\xi^t)$. We will need the fact that if the linking numbers and framings are zero, then $\xi^t$ has degree $\geq 2$, and moreover, the coefficient of $\xi_{123}$ is the Milnor invariant $\mu_{123}$. (See [?]).
3 Proof of Theorem 1.

The theorem will be proven progressively, starting from the case where $M$ is obtained via surgery on an algebraically split link $L$ (i.e., one with vanishing linking numbers) having 3 components all of which are zero-framed. In this case, $H_1(M) = \mathbb{Z}^3$, so that $\text{Tor}(H_1(M)) = 0$. Moreover, using the Poincaré dual interpretation of cup product, one easily checks from the definition of the Milnor invariant $\mu = \overline{\mu}_{123}(L)$ in terms of intersections of Seifert surfaces (which can be completed to surfaces in $M$, that $|\mu| = \frac{H_1(M)}{|H_1(M)|}$. It follows that $\lambda_M = \mu^2$ in this case.

The theorem in this case is an immediate consequence of the observation that the only term contributing to $Z_n^{LMO}(M)$ is $p_3(\overline{\mu}_{123}^{2n})$. To see this let $\sigma$ be a zero framed string link whose closure is $L$. Then by 3) above and [2] (since the linking and framings are zero and $\mu = \overline{\mu}_{123}(L)$), $Z^K(\sigma) = \exp(\mu \xi_{123} + \xi')$, where $\xi'$ is a linear combination of diagrams, all of which consist of diagrams for which $\Gamma$ is either not simply connected, or is a tree of degree $\geq 3$. Note that in each case, such a diagram has a ratio of external vertices (the univalent vertices of $\Gamma$) to internal vertices which is < 3, whereas this ratio for $\xi_{123}$ is 3. It follows that every term of $Z^K(L)\mu^{\otimes l} = p_3(Z^K(\sigma)\mu)^{\otimes l}$, which has at least $2n$ vertices on every component, must have at least $\frac{6n}{3} = 2n$ internal vertices. Hence such a term has degree at least $\frac{6n+2n}{2} = 4n$, and has degree precisely $4n$ if and only if that term is $p_3(\overline{\mu}_{123}^{2n})$.

Now suppose that $M = S^3(L)$, where $L$ is an algebraically split link having $l$ components, and such that $L$ contains a 3-component sublink $L_0$ which is zero-framed. We set $L_1 = L \setminus L_0$.

We first suppose that $L_0$ and $L_1$ are separated by a 2-sphere, so that $M$ is a connected sum. Recall from ([3], 5.1), that if $M$ is a connected sum of $M'$ and $M''$ such that $b_1(M') > 0$, then one has the formula $Z_n^{LMO}(M) = Z_n^{LMO}(M')|H_1(M'')|^n$. Setting $M' = S^3(L_0)$ and $M'' = S^3(L_1)$, then this formula shows that the result for $M$ is implied by the result for $M'$ shown earlier. (This includes the vanishing result if $b_1(M) > 3$, since in this case $b_1(M') > 0$, and hence $[H_1(M'')] = 0$.)

If $L_0$ and $L_1$ are not separated by a 2-sphere, i.e., $L \neq L_0 \coprod L_1$, the result still follows, since one has that $i_\ast(Z^K(L)\nu^{\otimes l}) = i_\ast(Z^K(L_0 \coprod L_1)\nu^{\otimes l})$. To see this, let $\sigma$ be a string link, whose closure is $L$, such that the first 3 components, $\sigma_0$, close up to give $L_0$. Set $\sigma_1 = \sigma \setminus \sigma_0$. Let $\sigma_0 \times \sigma_1$ denote the juxtaposition of $\sigma_0$ and $\sigma_1$. We wish to compare $Z^K(\sigma)$ and $Z^K(\sigma_0 \times \sigma_1)$. One has that $Z^K(\sigma_0) = \exp(\xi_0)$, $Z^K(\sigma_1) = \exp(\xi_1)$, and hence that $Z^K(\sigma_0 \times \sigma_1) = \exp(\xi_0 + \xi_1)$. Moreover, $Z^K(\sigma) = \exp(\xi_0 + \xi_1)$, where $\xi'$ is a sum of diagrams for which $\Gamma$ is connected, has degree $\geq 2$ (since $L$ is algebraically split), and has a vertex on $\sigma_0$ and on $\sigma_1$. Since $\xi_0$ is also of degree $\geq 2$, it follows that every term of $(Z^K(L) - Z^K(L_0 \coprod L_1))\nu^{\otimes l} = p_1((Z^K(\sigma) - Z^K(\sigma_0 \times \sigma_1))\nu)\nu^{\otimes l}$ is a sum of terms which satisfy that each component of $\Gamma$, with a vertex on one of the 3 components of $L_0$, has degree at least 2 and that some such component must also have a vertex lying on $L_1$. It follows that any such term, having at least $2n$ external vertices on each component of $L$, must have more than $2n$ internal vertices, and hence that such a term is in the kernel of $i_\ast$ (since it is of degree $> nl + n$).

Now suppose that $M = S^3(L)$, where $L$ is arbitrary. Let $B$ be the linking matrix of $L$. It is well known that $B$ becomes diagonalizable after taking the direct sum with a certain diagonal matrix $D$ having non-zero determinant. Let $L'$ denote a link whose linking matrix is $D$. Then if $M''$ denotes $S^3(L \coprod L')$, the theorem holds for $M''$, since $L \coprod L'$ is equivalent to an algebraically
split link $L''$, via handle sliding (so that $M'' = S^3(L'')$). Then the theorem holds also for $M$, using the formula $Z_n^{LO}(M''') = Z_n^{LO}(M)|H_1(S^3(L'))|^n$ (since $|H_1(S^3(L')) = |det(B)| \neq 0$).

4 Review of Rozansky-Witten Theory.

The theory whose partition function is believed to yield the G-Casson invariant is a twisted version of $N = 4$ super-Yang-Mills theory in 3-dimensions [2], [3]. Seiberg and Witten [2] have given a solution of the physical theory with $G = SU(2)$ in the coulomb branch. The coulomb branch of a theory corresponds to an analysis at a particular (low) energy scale. This solution has, as its moduli space, the reduced $SU(2)$ 2-Monopole moduli space, that is the Atiyah-Hitchin space $X_{AH}$. Since the topological theory should not depend on which scale we are looking at, we can twist the low energy theory of Seiberg and Witten and in this way we are led to equating the $SU(2)$ Casson invariant with a particular path integral over the space of maps from a 3-manifold to $X_{AH}$.

Rather more generally it is believed that the moduli space for for the physical theory with group $G$ is some monopole moduli space. For example for $SU(n)$ it is believed to be the reduced $SU(2)$ n-monopole moduli space. These moduli spaces are all hyper-Kähler. We denote those hyper-Kähler manifolds that arise as the moduli space of the coulomb branch of the $G$ physical theory by $X_G$. From this point of view the $G$ Casson invariant is then equated with a particular path integral over the space of maps from a 3-manifold to some hyper-Kähler $X_G$. The path integral in question, $Z_{X_G}^{RW}[M]$, was described and analysed in [2]. Given some subtleties that we will address later, one expects that $Z_{X_G}^{RW}[M]$ and $\lambda_G(M)$ if not equal are very closely related. (The exact statement was given in the introduction (??).)

4.1 The Rozansky-Witten Model

Rozansky and Witten [?] defined a path integral, and so invariants for a 3-manifold, for any hyper-Kähler $X$. This section is devoted to describing the objects that go into defining that path integral.

Let $\phi$ be a map from the 3-manifold $M$ to a hyper-Kähler manifold $X$. In local coordinates on $X$, the map is denoted $\phi^i$, $i = 1, \ldots, 4n$. We write $\phi \in \text{Map}(M, X)$. Denote by $T_\phi \text{Map}(M, X)$ the tangent space of $\text{Map}(M, X)$ at $\phi$. One may identify $T_\phi \text{Map}(M, X)$ with $\Omega^0(M, \phi^* (TX))$. Since one has, $TX \equiv R T^{(1,0)}X \equiv V$, for the tangent bundle of $X$ (see the review in Appendix ??). We define $\eta$ to be a Grassman variable$^{10}$ on $\Omega^0(M, \phi^* (V))$, that is $\eta \in \Lambda^1(\Omega^0(M, \phi^* (V))^*)$ which is, in local coordinates, denoted by $\eta^I(x)$, $I = 1, \ldots, 2n$. Let $\chi$ be a Grassman variable on $\Omega^1(M, \phi^* (V))$, that is it is an element of $\Lambda^1(\Omega^1(M, \phi^* (V))^*)$ which, in local coordinates, we denote by $\chi^I$.

We define a Lagrangian (density on $M$) $L = L_1 + L_2$.

$$L_1 = \frac{1}{2} g_{IJ}(\phi) \phi^I \phi^J = \epsilon_{IJ}(\phi) \phi^I \phi^J \nabla \eta^J$$

$$L_2 = \frac{1}{3} \left( \epsilon_{IJK} \phi^I \nabla \chi^J + \frac{1}{3} \Omega_{IJKL}(\phi) \phi^I \phi^J \phi^K \eta^L \right).$$

$^{9}$ $\phi^i$ is the composite of $\phi$, restricted to the inverse image of the coordinate neighborhood, with the $i$-th coordinate function. Thus $\phi^i$ is not a function defined on all of $M$, but only on some open set.

$^{10}$ The definition of what it means to be a Grassman variable on a vector space is explained in Appendix ??
The covariant derivative $\nabla$ is defined with the pullback of the Levi-Civita connection on $V$,

$$\nabla^I_J = d\delta^I_J + (d\phi^i)\Gamma^I_{IJ},$$  

and $\ast$ is the Hodge star operator on $M$ thought of as a Riemannian manifold. The two Lagrangians are separately invariant under a pair of BRST transformations. One does not need to pick a complex structure to exhibit these, however that level of generality is not required and we pick now a complex structure on $X$ so that the $\phi^I$ are local holomorphic coordinates with respect to this complex structure. In this complex structure we can pick a basis, $Q, \overline{Q}$ for the BRST charges which act by

$$\overline{Q}\phi^I = 0, \quad \overline{Q}\phi^J = T^I_J \eta^I,$$

$$\overline{Q}\eta^I = 0, \quad \overline{Q}\chi^I = -d\phi^I,$$

and

$$Q\phi^I = \eta^I, \quad Q\phi^J = 0,$$

$$Q\eta^I = 0, \quad Q\chi^I = T^I_J d\phi^J - \Gamma^I_{JK} \eta^J \chi^K.$$

These BRST operators satisfy the algebra

$$Q^2 = 0, \{Q, \overline{Q}\} = 0, \overline{Q}^2 = 0.$$  

The BRST invariant sigma model action is

$$S = \int_M (L_1 + L_2).$$  

We note that $L_1$ is both $Q$ and $\overline{Q}$ exact. Indeed one has

$$L_1 = \langle d\phi, d\phi \rangle + \langle \chi, T\nabla\eta \rangle$$

$$= -\overline{Q}\langle \chi, d\phi \rangle$$

$$= \overline{Q} Q \left( \frac{1}{2} \epsilon_{IJK} \chi^I \ast \chi^J \right),$$  

where the inner product for $X \in \Omega^1(M, T^{(1,0)}X)$ and $Y \in \Omega^1(M, T^{(0,1)}X)$, is defined to be

$$\langle X, Y \rangle = g_{IJ} X^I \ast Y^J.$$  

In order to write $L_1$ we needed to pick a metric on $M$. However, as $L_1$ is BRST exact, nothing depends on the choice made (see [?] section 2 for how this is established) and, ultimately, this explains why this theory produces a 3-manifold invariant.

Now we have a gauge theory interpretation of the sigma model action (??) as a gauge fixed action. Firstly, $L_2$, is BRST invariant (and metric independent). However, it is not BRST exact. So we may consider it to be the initial gauge invariant Lagrangian that needs to be augmented with a gauge fixing term, in order to arrive at a well defined theory. The gauge fixing term should be BRST exact and we see that $L_1$ fits the bill. In section ?? we will, for $b_1(M) = 1$, take this point of view and gauge fix an invariant Lagrangian. In fact what one finds is a theory that looks a great deal like the Chern-Simons theory of Witten. This suggestive analogy will
be taken up again when we make a more comprehensive comparison with Chern-Simons theory below.

The action (??) at first sight defines quite a complicated theory. However, as it is a topological theory, one may expect rather drastic simplifications. This is, indeed, the case.

There are various arguments that are available (see [?] and [?]) that establish that one may as well instead consider the Lagrangians

\[
L_1 \rightarrow L_1 = \frac{1}{2} g_{ij}(\phi_0) d\phi_{\perp}^i \star d\phi_{\perp}^j + \epsilon_{IJ}(\phi_0) \chi^I \star d\eta_{\perp}^J
- \Omega_{IJKL}(\phi_0) T_{IJ} \chi^I \eta_0^L \phi_{\perp}^M \star d\phi_{\perp}^K
\]

\[
L_2 \rightarrow L_2 = \frac{1}{2} \left( \epsilon_{IJ}(\phi_0) \chi^I d\chi^J + \frac{1}{3} \Omega_{IJKL}(\phi_0) \chi^I \chi^J \chi^K \eta_0^L \right).
\]

The notation in these formulae is as follows. Set \( \phi_{\perp} \in \Omega^0(M, \phi_0^*(TX)) \), where the \( \phi_0^* \) are the constant maps and the \( \phi_{\perp}^i \) are required to be orthogonal to the \( \phi^0_i \), that is \( f_M * \phi_{\perp}^i = 0 \). The \( \eta^I \) are also expanded as, \( \eta^I = \eta_0^I + \eta_{\perp}^I \), where the \( \eta_0^I \) are harmonic 0-forms with coefficients in the fibre \( V_{\phi_0} \) of the \( \text{Sp}(n) \) bundle \( V \rightarrow X \) and the \( \eta_{\perp}^I \) are orthogonal to these. Though not indicated in the formulae we will, below, also decompose the \( \chi^I \) fields in a similar fashion, \( \chi^I = \chi_0^I + \chi_{\perp}^I \) where the \( \chi_0^I \) are harmonic 1-forms with coefficients in the fibre \( V_{\phi_0} \) and the \( \chi_{\perp}^I \) are orthogonal to these in the obvious way.

The theory that we will analyze in the following sections is the one defined in terms of \( L_1 + L_2 \). This theory is rather simple to get a handle on, as we will see.

4.2 Path Integral Properties

Before proceeding we should mention that we will normalize the bosonic part of the path integral measure as done in [?]. This means that on occasion certain factors of \( 2\pi \) will make an appearance and those can always be traced back to our choice of normalization. Somewhat more involved is the question of sign of the path integral. Different approaches to fixing this have been explained in [?] and [?], and we will take the signs to be as given in those references. The question of framing in the path integral is not adressed here. The issues involved are spelt out in [?].

4.3 Relationship with Chern-Simons Theory

In this section we review the relationship between Chern-Simons theory and the Rozansky-Witten model. Though this relationship has already been explained in [?], we include it here so that we may refer back to it as we go along.

Recall that the Chern-Simons action is

\[
L_{CS} = \text{Tr} \left( A^a A^a + \frac{2}{3} A^a A^b A^c \right)
\]

where the trace is understood to be normalized so as to agree with the standard inner product on the Lie algebra. We compare this with (??). Notice that there is almost a direct match if we make the following substitutions

\[
A^a \rightarrow \chi^I
\]

\[
\text{Tr} T_a T_b \rightarrow \epsilon_{IJ}
\]

\[
f_{abc} \rightarrow \Omega_{IJKL}(\phi_0) \eta_0^L
\]
