Holonomy and Skyrmee’s model

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Abstract

In this paper we consider two generalizations of the Skyrmee model. One is a variational problem for maps from a compact 3-manifold to a compact Lie group. The other is a variational problem for flat connections. We describe the path components of the configuration spaces of smooth fields for each of the variational problems. We prove that the invariants separating the path components are well-defined for (not necessarily smooth) fields with finite Skyrmee energy. We prove that for every possible value of these invariants there exists a minimizer of the Skyrmee functional. Throughout the paper we emphasize the importance of holonomy in the Skyrmee model. Some of the results may be useful in other contexts. In particular, we define the holonomy of a distributionally flat $L^2_{loc}$ connection; the local developing maps for such connections need not be continuous.

1 Introduction

In 1961 T. H. R. Skyrme introduced a model to describe self-interacting meson fields, [19], [20], [21]. For a review of the physical and mathematical literature on the Skyrmee model see [3], [6], [9].

The static fields of the original Skyrmee model may be described as maps $u$ from $\mathbb{R}^3$ into the group of unit quaternions, $\text{Sp}(1) \simeq SU(2)$. It is required that these maps satisfy the boundary condition $u(\infty) = 1$ and have finite energy

$$E(u) = \int_{\mathbb{R}^3} \frac{1}{2} |u^{-1} du|^2 + \frac{1}{4} |u^{-1} du \wedge u^{-1} du|^2 \, dx,$$

where

$$|u^{-1} du|^2 = \sum_{j=1}^{3} |u^{-1} \frac{\partial u}{\partial x_j}|^2,$$

$$|u^{-1} du \wedge u^{-1} du|^2 = \sum_{j<k} |\left[ u^{-1} \frac{\partial u}{\partial x_j}, u^{-1} \frac{\partial u}{\partial x_k} \right]|^2,$$

and $|q|^2 = q \cdot \bar{q} = \sum_{j=1}^{4} (q_j)^2$. In addition, the space of smooth maps satisfying the boundary condition splits into infinitely many different components classified by the degree of the map.

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In the very first paper \cite{19}, p.129, Skyrme noted that \( u^{-1}du \) is a flat connection. In terms of the connection \( a = u^{-1}du \) the energy takes the form

\[
E[a] = \int_{\mathbb{R}^3} \frac{1}{2}|a|^2 + \frac{1}{16} ||[a, a]|^2 \, dx.
\]

Some generalizations of this functional have been considered previously, \cite{12, 11}. It is natural to generalize the original setting in several directions. First, one may replace \( \mathbb{R}^3 \) with an arbitrary Riemannian 3-manifold. Second, one may consider maps into arbitrary Lie groups. Third, the model may be described in terms of flat connections on principal \( G \)-bundles.

If the manifold, \( M^3 \) is non-compact, the existence of ground states is a difficult open problem. On the other hand, if \( M^3 \) is compact (and \( G = SU(2) \)), the existence of ground states is much easier and has been established in \cite{9}. The purpose of the present paper is to understand the underlying geometry of the space of maps and/or the space of connections together with the interaction between this geometry and the analytical behavior of the Skyrme fields. All the geometric features are present for closed 3-manifolds. For this reason we restrict our attention to compact Lie groups.

The gauge group is the group of automorphisms of the bundle. It may be identified with \( \Gamma(\Lambda^2\mathcal{M} \otimes \mathfrak{g}) \). Indeed, \( \mathcal{L}(0 \cdot g) = \mathcal{L}(0) = 0 \) implies \( \frac{1}{2}|dbg|^2 + \frac{1}{16} ||dbg, dbg||^2 = 0 \). The energy density \( \mathcal{L}(a) \) is not gauge invariant unless the gauge transformation is \( B \)-covariantly constant \( (dbg = 0) \). Thus, the density \( \mathcal{L}(a) \) is not gauge invariant unless the gauge transformation is \( B \)-covariantly constant \( (dbg = 0) \). The \( B \)-covariantly constant gauge transformations can be identified with the centralizer of the holonomy of \( B \), \cite{3}. Thus, the energy density is invariant under the group exactly when \( B \) has central holonomy. By the holonomy reduction theorem, \cite{10}, there is a subbundle \( Q \) of \( P \) with structure group \( Z \), the center of \( G \), so that \( \text{Ad}_p \cong Q \times Z \mathfrak{g} \). Since \( Z \) acts trivially on the Lie algebra \( \mathfrak{g} \), we have an isomorphism \( Q \times Z \mathfrak{g} \cong M \times \mathfrak{g} \), i.e., \( ([g, X]) \rightarrow ([g], X) \). In other words, the gauge group becomes Maps\( (M, G) \) in our case. The flat connections on \( P \) correspond to the set \( \{ a \in \Gamma(T^*\mathcal{M} \otimes \mathfrak{g}) \mid da + \frac{1}{2}[a, a] + F_B = 0 \} \), where \( F_B \) is the curvature of the reference connection \( B \) regarded as an element of \( \Gamma(\Lambda^2\mathcal{M} \otimes \mathfrak{g}) \). Given a unitary representation \( \alpha : G \rightarrow U(n) \), the associated complex vector bundle \( P \times_a \mathbb{C}^n \) has first Chern class \( -\frac{1}{2\pi i} \text{Trace}(\alpha(F_B)) \). It follows that the curvature of \( B \) is a closed central 2-form with integral periods. Conversely, any closed central 2-form with integral periods arises as the curvature of the central connection on the bundle.

To summarize, we have two distinct variational problems, one for equivalence classes of maps from a closed 3-manifold to a compact Lie group, and one for flat connections \( A = B + a \) modulo \( B \)-covariantly constant gauge transformations.
In this paper we describe the path components of the configuration spaces of smooth fields for each of the above variational problems. We prove that the invariants separating the path components are well-defined for (not necessarily smooth) fields with finite Skyrme energy. We prove that for every possible value of these invariants there exists a minimizer of the Skyrme functional. Throughout the paper we emphasize the importance of holonomy in the Skyrme model. Some of the results may be useful in other contexts. In particular, we define the holonomy of a distributionally flat $L^2_{\text{loc}}$ connection; the local developing maps for such connections need not be continuous.

We now describe the contents of the paper in more detail. In Section 2 below we prove that the homotopy classes of maps from $M$ to $G$, $[M, G]$, are in bijective correspondence with $G/G_0 \times H^3(M, H_3(\tilde{G})) \times H^1(M, H_1(G_0))$. Here $G_0$ is the identity component of $G$, and $\tilde{G}$ is the universal covering group of $G_0$. Since the energy, $E(u)$, is $G$-invariant, the relevant set of classes of maps is $[M, G]/G$. This set is isomorphic to $H^3(M, H_3(\tilde{G})) \times H^1(M, H_1(G_0))$. We also give a concrete analytic description of the corresponding invariants in Sections 2, 3 and 4.

The topological type of a connection modulo $B$-covariantly constant gauge is specified by a Chern-Simons invariant and a holonomy representation. These correspond to $H^3(M, H_3(\tilde{G}))$ and $H^1(M, H_1(G_0))$, respectively. The Chern-Simons invariant is well defined for connections with finite Skyrme energy. In Section 3 we give a definition of holonomy that is valid for distributionally flat $L^2_{\text{loc}}$ connections. In particular, our definition is valid for flat connections with bounded Skyrme energy.

This generalization of holonomy requires a nonlinear version of Poincaré’s lemma. More precisely, in Section 3 we prove that any distributionally flat $L^2_{\text{loc}}$ connection is locally trivial. For more regular connections (namely, $A \in W^{1,3/2}$), this follows from a theorem of K. Uhlenbeck.  

In Section 4 we prove that the invariants are well defined for maps with bounded Skyrme energy. In Section 5 we consider the minimization problems for the Skyrme functionals. In particular, we prove that for any fixed set of invariants there exists a map attaining these values that minimizes the Skyrme energy $E(u)$ in this class. Moreover, we also solve the analogous problem in the space of flat connections.

## 2 Homotopy classes of maps

Let $M$ be a closed 3-manifold. Let $G$ be a compact Lie group, $G_0$ its identity component, and $\tilde{G}$ the universal covering group of $G_0$, with covering projection $p: \tilde{G} \to G_0$. Denote by $[M, G]$ the free homotopy classes of continuous maps from $M$ to $G$.

### 2.1 Algebraic description

**Proposition 1** As sets,

$$[M, G] \cong G/G_0 \times H^3(M; H_3(\tilde{G})) \times H^1(M; H_1(G_0)).$$

**Proof.** Pick a point $x_0 \in M$. There is an isomorphism $\phi_0: [M, G] \to G/G_0 \times [M, G_0]$ given by $\phi_0([u]) = ([u(x_0)], [u(x_0)^{-1} \cdot u])$. Now we will regard $[M, G_0]$ as based homotopy classes.

The homotopy classes of maps into a Lie group form a group under pointwise multiplication. The covering projection, $p: \tilde{G} \to G_0$, induces a homomorphism $p_*: [M, \tilde{G}] \to [M, G_0]$ by composition. The homomorphisms from $\pi_1(M)$ to $\pi_1(G_0)$ form a group under pointwise multiplication. The natural map, $\pi_1: [M, G_0] \to \text{Hom}(\pi_1(M), \pi_1(G_0))$, is a group homomorphism. Consider the sequence

$$1 \to [M, \tilde{G}] \xrightarrow{p_*} [M, G_0] \xrightarrow{\pi_1} \text{Hom}(\pi_1(M), \pi_1(G_0)) \to 1. \quad (1)$$

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If \( p_*(\bar{u}) = 1 \), there exists a homotopy \( H : M \times I \to G_0 \) to the constant map making the following diagram commute:

\[
\begin{array}{ccc}
M & \xrightarrow{\bar{u}} & \hat{G} \\
i_0 & \downarrow{\hat{H}} & \downarrow{p} \\
M \times I & \xrightarrow{H} & G_0
\end{array}
\]

By the homotopy lifting theorem, \( [22] \), there is a homotopy \( \hat{H} \) that makes the extended diagram commute. By the unique lifting theorem, \( [22] \), \( \hat{H} \) is constant; hence, \( p_* \) is injective.

Since \( \hat{G} \) is simply connected, \( \pi_1(\hat{G}) \) is the trivial homomorphism for all \( \bar{u} \in [M, \hat{G}] \). So, the image of \( p_* \) lies in the kernel of \( \pi_1 \). If \( \pi_1(u) = 1 \), there is a map \( \bar{u} \) so that \( p_* \circ \bar{u} = u \), by the lifting theorem, \( [22] \). Thus, the sequence (1) is exact.

Any closed, connected 3-manifold admits a Heegaard splitting, \( [17] \), \( [13] \), and, therefore, a CW decomposition with exactly one 0-cell and one 3-cell. Given a homomorphism, \( \alpha \in \pi_1(M) \), construct a map \( u_\alpha : M \to G_0 \) as follows. On the 0-skeleton, define \( u^{(0)}_\alpha(x_0) = 1 \). Any 1-cell of \( M \) specifies an element of \( \pi_1(M) \). Pick a representative of the corresponding class in \( \pi_1(G_0) \) and define \( u^{(1)}_\alpha \) on the 1-cell via this representative. The attaching map of any 2-cell is trivial in \( \pi_1(M) \), so the composition with \( u^{(1)}_\alpha \) is trivial in \( \pi_1(G_0) \) and thus extends to a map of the disk into \( G_0 \). Define \( u^{(2)}_\alpha \) on the 2-cell via this map. The composition of the attaching map of the 3-cell with \( u^{(2)}_\alpha \) is trivial in \( \pi_2(G_0) \) since \( \pi_2 \) of any Lie group is trivial, \( [4] \). Thus, the composition extends to a map of the 3-disk that may be used to define the map \( u_\alpha = u^{(3)}_\alpha \). By construction, \( \pi_1([u_\alpha]) = \alpha \), i.e., \( \pi_1 \) is surjective. Thus, the sequence (1) is exact.

Consider the sequence

\[
\begin{array}{ccccccc}
M^{(2)} & \xrightarrow{i_2} & M^{(3)} & \xrightarrow{q_2} & M^{(3)}/M^{(2)} & \xrightarrow{\partial_2} & SM^{(2)} & \xrightarrow{-} & SM^{(3)}.
\end{array}
\]  \tag{2}

The space \( M^{(3)}/M^{(2)} \) is homeomorphic to \( D^2/S^2 \). Under this identification, \( \partial_2(x) = \left( f^{(3)}(\frac{x}{|x|}), |x| \right) \) for \( x \in D^3 \), where \( f^{(3)} : S^2 \to M^{(2)} \) is the attaching map for the 3-cell. The sequence (2) is co-exact, \( [22] \). Therefore, it induces the exact sequence

\[
[SM^{(2)}, \hat{G}] \to [M^{(3)}/M^{(2)}, \hat{G}] \to [M, \hat{G}] \to [M^{(2)}, \hat{G}] = 0.
\]

Similarly, the sequence

\[
\begin{array}{ccccccc}
M^{(1)} & \xrightarrow{i_2} & M^{(2)} & \xrightarrow{q_2} & M^{(2)}/M^{(1)} & \xrightarrow{\partial_2} & SM^{(1)} & \xrightarrow{-} & SM^{(2)}.
\end{array}
\]  \tag{3}

\[
\begin{array}{ccccccc}
S(M^{(2)}/M^{(1)}) & \xrightarrow{-} & S^2M^{(1)}
\end{array}
\]

4
induces the exact sequence
\[ [S^2 M^{(1)}, \tilde{G}] \to [S(M^{(2)}/M^{(1)}), \tilde{G}] \to [SM^{(2)}, \tilde{G}] \to [SM^{(1)}, \tilde{G}] = 0. \]
These exact sequences may be spliced together to give
\[ [SM^{(2)}, \tilde{G}] \xrightarrow{\partial^*_2} [M^{(3)}/M^{(2)}, \tilde{G}] \xrightarrow{q^*_2} [M, \tilde{G}] \to 0 \]
Thus,
\[ [M, \tilde{G}] \cong [M^{(3)}/M^{(2)}, \tilde{G}]/\text{Ker } q^*_2 \cong [M^{(3)}/M^{(2)}, \tilde{G}]/\text{Im } \partial^*_2 \]
\[ \cong [M^{(3)}/M^{(2)}, \tilde{G}]/\text{Im } (\partial^*_2 \circ q^*_1). \]
The sequences (2), (3) induce
\[ H_3(M^{(3)}) \xrightarrow{q^*_2} H_3(M^{(3)}/M^{(2)}) \xrightarrow{\partial^*_3} H_2(M^{(2)}) \]
and
\[ H_2(M^{(2)}) \xrightarrow{q^*_2} H_2(M^{(2)}/M^{(1)}) \xrightarrow{\partial^*_2} H_1(M^{(1)}), \]
which give
\[ H_3(M^{(3)}/M^{(2)}) \xrightarrow{q_1^* \circ \partial^*_2} H_2(M^{(2)}/M^{(1)}) \]
and the commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(H_2(M^{(2)}/M^{(1)}), \pi_3(\tilde{G})) & \xrightarrow{q_1^* \circ \partial^*_2} & \text{Hom}(H_3(M^{(3)}/M^{(2)}), \pi_3(\tilde{G})) \\
F_2 & & F_3 \\
[S(M^{(2)}/M^{(1)}), \tilde{G}] & \xrightarrow{\partial^* \circ q^*_1} & [M^{(3)}/M^{(2)}, \tilde{G}] \\
\end{array}
\]
(4)
Here \( F_3([u])[\gamma] = [u \circ r_3^{-1}(\gamma)] \) and \( F_2([u])(\gamma) = [u \circ r_2^{-1} \circ (\partial^*_2)^{-1}(\gamma)] \), where \( \partial^*_2 : H_3(S(M^{(2)}/M^{(1)})) \to H_2(M^{(2)}/M^{(1)}) \) is the suspension isomorphism, and \( r_2 : \pi_3(M^{(3)}/M^{(2)}) \to H_3(M^{(3)}/M^{(2)}) \) and \( r_3 : \pi_3(S(M^{(2)}/M^{(1)})) \to H_3(S(M^{(2)}/M^{(1)})) \) are the Hurewicz isomorphisms. Note, that the vertical arrows in (4) are isomorphisms. Thus,
\[ [M, \tilde{G}] \cong \text{coKer}(\partial^* \circ q^*) = H^3(M; \pi_3(\tilde{G})) \cong H^3(M; H_3(G)). \]
Since the fundamental group of a Lie group is abelian, \( \pi_1(G_0) \cong H_1(G_0; \mathbb{Z}) \). It follows that \( \text{Hom}(\pi_1(M), \pi_1(G_0)) \cong \text{Hom}(H_1(M), H_1(G_0)) \). The universal coefficient theorem, (4), gives
\[ 0 \to \text{Ext}_2^1(H_0(M), H_1(G_0)) \to H^1(M; H_1(G_0)) \to \text{Hom}(H_1(M), H_1(G_0)) \to 0. \]
Since $\text{Ext}^2_\mathbb{Z}(\mathbb{Z},-) = 0$, we have $\text{Hom}(H_1(M), H_1(G_0)) \cong H^1(M; H_1(G_0))$.

Recall that the Skyrme functional is invariant under right multiplication by elements of $G$. We are therefore interested in the classification of maps from $M$ to $G$ up to homotopy and right translation in $G$. The following is a corollary of Proposition 1.

**Corollary 1**

$$[M, G]/G \cong H^3(M; H_3(\tilde{G})) \times H^1(M; H_1(G_0)).$$

### 2.2 Analytical description

We will now develop analytical expressions for the homotopy invariants starting with $H^3(M; H_3(\tilde{G}))$. Since the second homology of any 3-manifold is torsion free, the universal coefficient theorem gives $H^3(M; H_3(\tilde{G})) \cong \text{Hom}_\mathbb{Z}(H_3(M), H_3(\tilde{G}))$. The universal covering group is the direct product of $\mathbb{R}^n$ together with a finite collection of compact, simply connected, simple Lie groups, $\tilde{G} \cong \mathbb{R}^n \times \tilde{G}_1 \times \ldots \times \tilde{G}_N$. (5)

It is known, [2], that $\pi_3$ of any compact simple Lie group is $\mathbb{Z}$. Therefore, $H_3(\tilde{G})$ is free, and, by the universal coefficient theorem, $H^3(\tilde{G}) \cong \text{Hom}_\mathbb{Z}(H^3(\tilde{G}); \mathbb{Z})$.

We are interested in maps from $M$ to $G$, and not every map can be lifted to a map from $M$ to $\tilde{G}$. We would like to identify $H^3(\tilde{G}; \mathbb{Z})$ with a subgroup of $H^3(G; \mathbb{R})$ in order to express the homotopy $H^3$-invariant as the integral over $M$ of the pull-back of a 3-form. The needed identification is the topic of the next lemma.

Let $Z$ be the center of $G_0$ and $Z_0$ be the identity component of $Z$. Let $G^k$ be the connected Lie subgroup of $G_0$ corresponding to the Lie algebra of $\tilde{G}^k$. Let $d^k$ be the degree of the cover $\tilde{G}^k \to G^k$.

**Lemma 1** The covering map, $p : \tilde{G} \to G$, induces a surjective homomorphism

$$p^* : H^3(G_0; \mathbb{R}) \to H^3(\tilde{G}; \mathbb{R}),$$

which restricts to an isomorphism

$$\bigoplus_{k=1}^N H^3(G^k, \frac{1}{d^k} \mathbb{Z}) \to H^3(\tilde{G}; \mathbb{Z}).$$

**Proof.** Consider the diagram:

$$\begin{array}{ccc}
p^{-1}(Z_0) & \longrightarrow & \tilde{G} \\
p & \downarrow & \downarrow q \\
Z_0 & \longrightarrow & G_0 \\
i_Z & \uparrow & \uparrow \\
& & G_0/Z_0
\end{array}$$

(6)

The homomorphism $i_Z : \pi_1(Z_0) \to \pi_1(G_0)$ is injective, [3], so $\pi_1(p^{-1}(Z_0)) = 1$ and $p^{-1}(Z_0) \cong \mathbb{R}^n$. Therefore, the homotopy exact sequence applied to the top row of (6) implies that $\pi_m(\tilde{G}) \cong \pi_m(\tilde{G}/p^{-1}(Z_0))$. 

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Thus, $H_3((G/Z_0)) \cong \pi_3(\tilde{G}/p^{-1}(Z_0)) \cong \pi_3(\tilde{G}) \cong H_3(G)$. Now, $G_0/Z_0$ being semisimple, has finite fundamental group. Define a transfer map $\tau : H_3(G_0/Z_0) \to H_3((G_0/Z_0))$ by sending every simplex to the sum of its lifts. The compositions $\tau \circ q_*$ and $q_* \circ \tau$ are multiplications by $|\pi_1(G_0/Z_0)|$, the order of the group. Thus, $p^*$ is surjective. The transfer map from $H^3(\tilde{G}^k; \mathbb{R})$ to $H^3(G^k; \mathbb{R})$ induces an isomorphism $H^3(G^k; \mathbb{Z}) \cong H^3(\tilde{G}^k; \mathbb{Z})$. The result now follows from the Künneth formula.

For each $\tilde{G}^k$ in the decomposition (3), we have a bi-invariant 3-form on $\tilde{G}^k$ defined by

$$\tilde{\Theta}^k(X_g, Y_g, Z_g) = -\frac{K_{\tilde{G}^k}}{32\pi^2} \text{Tr} (\text{ad}[L_{g^{-1}}^*, X_g, \text{ad}(L_{g^{-1}}^* Z_g)]).$$

The form $\tilde{\Theta}^k$ is closed because the Killing form is biinvariant. Integrating it over any $SU(2)$ in $\tilde{G}^k$, we see that it is non-trivial. The constants $K_{\tilde{G}^k}$ are chosen so that the integral of this 3-form is 1 on any primitive $S^3$ in $\tilde{G}^k$. The value of this constant for every simple, simply connected, compact Lie group is listed in Table 1.

We now summarize how these constants are determined. The first observation is that $\pi_3$ of every simple, simply connected, compact Lie group is generated by a homomorphic image of $Sp(1) = SU(2) = Spin(3)$. To see this, one uses the homotopy exact sequence of the fibrations

$$U(n) \to U(n+1) \to S^{2n+1} \quad \text{and} \quad SU(n) \to U(n) \to S^4$$

for $A_n$, $SO(m) \to SO(m+1) \to S^{m+1}$

for $B_n$ and $D_n$, and

$$Sp(n) \to Sp(n+1) \to S^{4n+3}$$

for $C_n$.

The exceptional groups are treated separately beginning with the inclusions (see [1])

$$\text{Spin}(3) \to \text{Spin}(9) \to F_4 \to \text{Spin}(10) \times_{\mathbb{Z}_4} S^1 \to E_6 \to \text{Spin}(12) \times_{\mathbb{Z}_2} \text{Sp}(1) \to E_7 \to SO(16) \to E_8.$$

Let $\gamma$ denote the generator of $\pi_3(\text{Spin}(3))$. We have already seen that $\pi_3(\text{Spin}(9))$ and $\pi_3(\text{Spin}(10) \times_{\mathbb{Z}_4} S^1)$ are generated by $\gamma$. It follows that $\pi_3(F_4)$ is generated by $\gamma$ as well. Notice that $\text{Spin}(3)$ is naturally included in the Spin(12) factor of Spin(12)×_{Z_2} Sp(1). It follows that $\gamma$ is primitive in $\pi_3(\text{Spin}(12) \times_{Z_2} \text{Sp}(1))$, thus $\pi_3(E_6)$ is generated by $\gamma$. Once again, $\gamma$ generates $\pi_3(SO(16))$, and, hence, generates $\pi_3(E_7)$. Finally, the homotopy exact sequence of the fibration $SO(16) \to E_8 \to E_8/\text{SO}(16)$ shows that $\gamma$ generates $\pi_3(E_8)$ since the perversely named octooctonionic projective plane $E_8/\text{SO}(16)$ is 4-connected, [13], p.361. For the group $G_2$ we use the homotopy exact sequence of the fibration $SU(3) \to G_2 \to S^6$.

A straightforward evaluation of the integral $\int_{SU(2)} \tilde{\Theta}$ shows that $K_{SU(2)} = 1$. For the other groups, one must integrate $\tilde{\Theta}$ over $\gamma$. Let $h : SU(2) \to \tilde{G}^k$ be a homomorphism generating $\pi_3(\tilde{G}^k)$. The constant

| group, $G$ | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $SU(n+1)$ | $\frac{1}{n+1}$ | $\frac{1}{2n-1}$ | $\frac{1}{2n-2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\text{Spin}(2n+1)$ | $\frac{1}{2n-1}$ | $\frac{1}{2n-2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Table 1: Normalizing constants
is then the ratio of the value of the $SU(2)$-Killing form of a non-zero vector in $\mathfrak{su}(2)$ to the value of the $G^k$-Killing form of its image under $h_*$. For the classical groups these ratios may be inferred from \cite{[18]} pp. 197, 199, 201, and 203, for example. For $E_6$, $E_7$, and $E_8$ these ratios may be found in \cite{[1]} p. 87, 77, and 43. We could not find references listing the constants for $F_4$ and $G_2$, so we compute them here.

A nice description of $G_2$ may be found in \cite{[7]}. By definition, $G_2$ is the group of endomorphisms of the purely imaginary octonions preserving the forms $\text{Re}(x^* y)$ and $\text{Re}([x, y]^* z)$. Its Lie algebra may be realized as the space of 7 by 7 matrices of the form

$$
\begin{bmatrix}
0 & -\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 \\
\lambda_2 & 0 & -\mu_3 & -\mu_4 & -\mu_5 & -\mu_6 & -\mu_7 \\
\lambda_3 & \mu_3 & 0 & -\mu_5 & -\lambda_4 & -\lambda_5 & -\lambda_6 \\
\lambda_4 & \mu_4 & \lambda_6 - \mu_5 & 0 & -\nu_5 & -\nu_6 & -\nu_7 \\
\lambda_5 & \mu_5 & \lambda_6 + \mu_4 & \nu_5 & 0 & -\lambda_2 - \nu_7 & \nu_6 - \lambda_3 \\
\lambda_6 & \mu_6 & -\lambda_4 + \mu_7 & \nu_6 & \lambda_2 + \nu_7 & 0 & -\mu_3 - \nu_5 \\
\lambda_7 & \mu_7 & -\lambda_5 - \mu_6 & \nu_7 & \lambda_3 - \nu_6 & \mu_3 + \nu_5 & 0
\end{bmatrix}
$$

and the basis of the Lie algebra is obtained by setting each of the 14 parameters in turn to 1 and the others to 0. The Lie algebra of the $SU(3)$ mentioned above is obtained by setting all of the $\lambda$’s to 0. The homomorphism $h$ induces a homomorphism of Lie algebras. The image of

$$v = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathfrak{su}(2)$$

in the Lie algebra of $G_2$ is obtained by substituting $\nu_5 = 1$ and setting the rest of the parameters to 0. The 14 by 14 matrix for $\text{ad}(h_*(v))$ in the given basis is obtained by direct computation, and the trace of its square is $(-16)$, thus, giving $K_{G_2} = -8/16 = 1/2$.

A matrix representation for $F_4$ would require 52 by 52 matrices. Instead, we use generators and relations. Recall, that the $m$-dimensional Clifford algebra $CL_m$ has $m$ generators, $e_1, \ldots, e_m$, with relations $e_i^2 = -1$, and $e_i e_j = -e_j e_i$ for $i \neq j$. The Lie algebra $\text{spin}(m)$ has $e_i e_j, i < j$, as a basis, together with the usual Lie bracket, $[e_i e_j, e_k e_l] = e_i e_j e_k e_l - e_k e_l e_i e_j$. Note, that the only nonzero brackets are of the form $[e_i e_j, e_k e_l] = 2e_j e_i - [e_i e_j, e_k e_l]$ when $j \neq \ell$. One may use this to recover the constants $K_{\text{Spin}(m)}$. We identify the 9-dimensional spinors with the positive 10-dimensional spinors, i.e.,

$$\Delta_9^+ = \Delta_9 = \{a \in CL_{10}^{\text{even}} \otimes \mathbb{C} | e_{2j-1}e_{2j} \otimes i \cdot a = -a, j = 1, 2, 3, 4, 5\}.$$

Here $CL_{10}^{\text{even}}$ is the subspace of $CL_{10}$ generated by products of even numbers of $e_j$’s. Define $e_j = 1 - e_{2j-1}e_{2j} \otimes i$ and $\omega_j = e_{2j-1} + e_{2j} \otimes i$. Then

$$\{ e_1 e_3 e_5, \quad \omega_1 e_2 e_3 e_5, \quad \omega_1 e_2 e_3 e_5, \quad \omega_1 e_2 e_3 e_5 \}
\{ e_1 e_2 e_3 e_4, \quad e_1 e_2 e_3 e_4, \quad e_1 e_2 e_3 e_4, \quad e_1 e_2 e_3 e_4 \}
\{ e_1 e_2 e_3 e_4, \quad e_1 e_2 e_3 e_4, \quad e_1 e_2 e_3 e_4, \quad e_1 e_2 e_3 e_4 \}\}$$

is a basis for $\Delta_9$. As a vector space, the Lie algebra $f_4$ is $\text{spin}(9) \oplus \Delta_9$. The Lie bracket of $f_4$ restricted to $\text{spin}(9)$ is the usual $\text{spin}(9)$-bracket. The Lie bracket between the $\text{spin}(9)$ and $\Delta_9$ factors is given by $[a, v] = a v$, the usual representation induced from Clifford multiplication. For the remaining brackets see \cite{[1]}. Lemma 6.2. The Lie algebra of the primitive $SU(2)$ in $f_4$ is generated by $e_1 e_2, e_1 e_3$, and $e_2 e_3$. A
direct computation generates the 52 by 52 matrix for the $f_4$ adjoint representation of $e_1 e_2$. The trace of the square of this matrix is $-72$, giving $K_{F_4} = -8/ -72 = 1/9$.

Recall that $G^k$ is the connected subgroup of $G_0$ with universal covering group $\hat{G}^k$. Let $\Theta^k$ be the 3-form on $G$ given by

$$\Theta^k(X_g, Y_g, Z_g) = -\frac{1}{32\pi^2} K_{\hat{G}^k} \text{Tr} \left( \text{ad} \left[ L_{g^{-1}vX_g}, L_{g^{-1}vY_g} \right] \text{ad} \left( L_{g^{-1}vZ_g} \right) \right),$$

where now $X_g, Y_g,$ and $Z_g$ are tangent vectors on $G$, and $L_{g^{-1}v} W_g$ is the orthogonal projection of $L_{g^{-1}vW_g}$ onto the Lie algebra of $G^k$. Note, that since $\hat{G}^k$ is a $d^k$-fold cover of $G^k$, the form $\Theta^k$ is not in general an integral class.

Summarizing, we have the following lemma.

**Lemma 2** The cohomology group, $H^3(\hat{G}; \mathbb{Z})$, is the free abelian group generated by $(\iota \circ p)^* \Theta^k$, where $p : \hat{G}^k \to G_0$ is the covering projection, and $\iota : G_0 \to G$ is the inclusion.

Before we can get numerical expressions for the three dimensional part of the homotopy invariants, we need to consider the second invariant that corresponds to

$$f_3$$

Before we can get numerical expressions for the three dimensional part of the homotopy invariants, we need to consider the second invariant that corresponds to $H^1(M; H_1(G_0))$, see Corollary 1.

Using the identification $H_1(G_0) \cong \pi_1(G_0) \cong \{p^{-1}(1)\} \subset \hat{G}$, a map $u : M \to G$ induces a $\hat{G}$-connection $u^{-1}du$. Given a loop $\gamma : ([0,1], \{0,1\}) \to (M, x_0)$, solve the system of ordinary differential equations, $\frac{dg_\gamma}{d\theta} = u^{-1}du g_\gamma$, $g_0 = 1$ to obtain a path $g_\gamma : [0,1] \to \hat{G}$. The element of $H^1(M; H_1(G_0))$ corresponding to $u$ is $\alpha_u$, where $\alpha_u([\gamma]) = g_\gamma^1$. This is just the holonomy of the connection. We take up the full description of holonomy in Section 3.

Note, that $H^1(M; H_1(G_0))$ is a finitely generated abelian group. Let $\alpha_1, \ldots, \alpha_b$ be its generators, and $r_1 \alpha_1 = \cdots = r_b \alpha_b$ be the relations, where $r_1 | r_2 | \cdots | r_b$. There is no canonical choice for the classes $\alpha_\ell$, but it is not difficult to construct a set given any closed connected 3-manifold. By the proof of Proposition 1, there are maps $v_\ell : M \to G_0$ with $(v_\ell)_* = \alpha_\ell : H_1(M) \to H_1(G_0)$. It is usually possible to construct such maps explicitly.

Given a map $u : M \to G$, one obtains the map $u_0 = u(x_0)^{-1}u$ form $M$ into $G_0$. This map induces a map $u_0 : H_1(M) \to H_1(G_0)$. We write $u_0 = \sum_{\ell=1}^b a_\ell \alpha_\ell$, where $a_\ell = a_\ell(u) \in \mathbb{Z}_{r_\ell}$.

We are in position now to state the analytical description of $[M, G]/G$.

**Proposition 2** Given any element $[u]$ of $[M, G]/G$, define the numbers $a_\ell(u)$ and

$$c^k(u) = \int_M (u (\prod_{1}^{b} v^{a_{\ell}}_{\ell})^{-1})^* \Theta^k.$$

We have $a_\ell(u) \in \mathbb{Z}_{r_\ell}$ and $c^k(u) \in \mathbb{Z}$. Any $(b+N)$-tuple

$$(a_1, a_b, c_1, \ldots, c_N) \in \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_b} \times \mathbb{Z}^N,$$

is obtained from some map $u : M \to G$. Two maps are equivalent if and only if they produce the same $(b+N)$-tuple.
Remark 1 The three dimensional part of this invariant is a torsor, i.e., an affine space modeled on integers with no canonical choice of 0. Since
\[ \int_M (uw)^* \Theta^k = \int_M u^* \Theta^k + \int_M w^* \Theta^k, \]
we could encode the information contained in \( c^k(u) \) as \( \int_M u^* \Theta^k \). However, these numbers are no longer integers, but the differences \( \int_M u^* \Theta^k - \int_M w^* \Theta^k \) always are.

Remark 2 The integrals \( \int_M u^* \Theta^k \) are well defined for maps \( u: M \to G \) with finite Skyrme energy, \( E(u) \), since \( u^{-1}du \in L^2 \) and \( |u^{-1}du, u^{-1}du| \in L^2 \). In Section 4 we will see that all of the numbers \( a_\ell \) and \( c^k(u) \) are well defined for maps with finite Skyrme energy.

3 Connections and holonomy

In the previous section we described the one dimensional component of the homotopy invariant using the solutions of a system of ordinary differential equations. While this is legitimate for smooth maps, it does not directly apply to all maps with finite Skyrme energy. To circumvent this problem, we will give a new definition of holonomy in the spirit of Čech cohomology that is well defined for distributionally flat \( L^2 \)-connections and reduces to the usual definition for smooth connections.

3.1 Function spaces

Every compact Lie group has a faithful unitary representation, [3]. Pick such a representation and identify \( G \) with its image, a subgroup of \( U(N) \subset \mathbb{C}^{N^2} \). Use the corresponding representation to identify the Lie algebra of \( G \), \( g \), with a subalgebra of \( \mathfrak{u}(N) \subset \mathbb{C}^{N^2} \). These identifications allow us to consider \( G \) and \( g \) valued maps as maps into \( \mathbb{C}^{N^2} \).

Given a Riemannian manifold \( \Omega \), denote by \( W^{s,p}(\Omega, \mathbb{C}^m) \) the Sobolev space of \( \mathbb{C}^m \)-valued functions on \( \Omega \). We use these spaces only when \( s \) is nonnegative integer, in which case the norm can be chosen as
\[ \|u\|_{W^{s,p}(\Omega, \mathbb{C}^m)}^p = \int_\Omega \sum_{0 \leq k \leq s} |\nabla^k u|^p d\text{vol}_\Omega. \]
We define \( W^{s,p}(\Omega, G) \) to be the space of maps \( u \in W^{s,p}(\Omega, \mathbb{C}^{N^2}) \) with \( u(x) \in G \) for almost all \( x \in \Omega \). We define \( W^{s,p}(\Omega, g) \) similarly. Notice, that \( W^{s,p}(\Omega, G) \subset L^\infty(\Omega, \mathbb{C}^{N^2}) \), since \( G \) is compact.

Remark 3 The space \( W^{1,p}(\Omega, G) \) forms a group under pointwise multiplication.

We will sometimes shorten the notation to \( W^{s,p} \) and will use the same notation for the corresponding spaces of differential forms on \( \Omega \). Different faithful representations of \( G \) lead to equivalent norms. For compact manifolds, different Riemannian metrics also lead to equivalent norms provided the boundary is sufficiently regular.

3.2 Nonlinear Poincaré Lemma

In this section we prove an important lemma that we use to define the holonomy for Sobolev connections.

Let \( I^m \) denote a unit cube in \( \mathbb{R}^m \).
Lemma 3 Given any $L^2$ $g$-valued 1-form $A$ on $I^m$ such that

$$dA + \frac{1}{2} [A, A] = 0$$

in the sense of distributions, there exists $u \in W^{1,2}(I^m, G)$ such that $u^{-1} \in W^{1,2}(I^m, G)$ and $A = u^{-1} du$. Furthermore, for any two such maps, $u$ and $v$, there exists $g \in G$ so that $u(x) = g \cdot v(x)$, for almost every $x \in I^m$.

Proof. Choose coordinates parallel to the edges of the cube. In coordinates, the $g$-valued 1-form $A$ can be written as $A_k(x) dx^k$, where each $A_i(x)$ is a matrix-valued function. We then have $[A, A] = (A_i A_j - A_j A_i) dx^i \wedge dx^j$. Here and in what follows we use the summation convention.

The following observation is important for our construction. Let $f$ be a scalar function in $L^p(I^m)$, for some $1 \leq p < \infty$. By Fubini’s theorem, for almost all values of $x^1 \in I^1$ the function of $m-1$ variables $f(x^1, \cdot)$ is in $L^p(I^{m-1})$. We will extend $f$ outside of $I^m$ by 0. Denote by $T_\epsilon f$ the mollification of $f$ defined as follows:

$$(T_\epsilon f)(x^1, \ldots, x^m) = \int \zeta_\epsilon(x^1 - y^1) \ldots \zeta_\epsilon(x^m - y^m) f(y^1, \ldots, y^m) \, dm_y,$$

where $\zeta_\epsilon(t) = \epsilon^{-1} \zeta(\epsilon^{-1} t)$ with $\zeta$ a smooth, even, compactly supported bump-function with integral 1. Choose a sequence $\epsilon_k \to 0$ so that

$$\sum_{k=1}^{\infty} \int_{I^m} |T_{\epsilon_k} f - f|^p \, dm < \infty.$$

Then,

$$\int_{I^1} \left( \sum_{k=1}^{\infty} \int_{I^{m-1}} |T_{\epsilon_k} f(x^1, \cdot) - f(x^1, \cdot)|^p \, dx^2 \ldots dx^m \right) \, dx^1 < \infty.$$

This implies that there is a subset $\hat{I}^1$ of full measure in $I^1$ for each point $x^1$ of which $T_{\epsilon_k} f(x^1, \cdot)$ converges to $f(x^1, \cdot)$ in $L^p(I^{m-1})$. If there is a finite number of functions $f$, the set $\hat{I}^1$ can be chosen to accommodate all of them.

We now return to the connection $A \in L^2(I^m, G)$. Using the above observation we translate the coordinates in $I^m$ as follows. For almost every point $x^1_0$ of the interval $I^1$, the restrictions of all $A_i$ to the hyperplane $x^1 = x^1_0$ lie in $L^2(I^{m-1})$ and $(A_i)_{\epsilon_k}(x^1_0, \cdot) = (T_{\epsilon_k} A_i)(x^1_0, \cdot)$ converge to $A_i(x^1_0, \cdot)$ in $L^2(I^{m-1})$. In addition, for almost every such $x^1_0$, the restrictions of all Lie brackets $[A_j, A_l]$ to the hyperplane $x^1 = x^1_0$ lie in $L^1(I^{m-1})$ and $([A_j, A_l])_{\epsilon_k}(x^1_0, \cdot) = (T_{\epsilon_k} [A_j, A_l])(x^1_0, \cdot)$ converge to $[A_j, A_l](x^1_0, \cdot)$ in $L^1(I^{m-1})$. Fix one such point $x^1_0$ inside $I^1$. Similarly (sparring the sequence $\epsilon_k$, if necessary) we may fix values $x^2_0, \ldots, x^m_0$ so that the restrictions of all $A_i$ to the slices $\{x^1_0\} \times \ldots \times \{x^k_0\} \times I^{m-k}$ are in $L^2(I^{m-k})$ and the mollifications $\int \zeta_\epsilon(x^1_0 - y^1) A_i(x^1_0, \ldots, x^{k-1}_0, y^k, \cdot) \, dy^k$ converge to $A_i(x^1_0, \ldots, x^{k-1}_0, x^k, \cdot)$ in $L^2(I^{m-k})$. In addition, the restrictions of all $[A_j, A_l]$ to the slices $\{x^1_0\} \times \ldots \times \{x^k_0\} \times I^{m-k}$ are in $L^1(I^{m-k})$ and the mollifications $\int \zeta_\epsilon(x^1_0 - y^1) [A_j, A_l](x^1_0, \ldots, x^{k-1}_0, y^k, \cdot) \, dy^k$ converge to the bracket $[A_j, A_l](x^1_0, \ldots, x^{k-1}_0, x^k, \cdot)$ in $L^1(I^{m-k})$. Almost every point in $I^m$ satisfies these “slice” conditions. By translating the coordinates we set $x^1_0 = 0$.

After these preliminary remarks we turn to the construction of $u$. We obtain $u(x)$ as $u_\alpha(x)$, where $u_\alpha : \{0\} \times I^k \to G$ is defined inductively, setting $u_0(0) = 1$ and defining successive terms via the system of
Ordinary differential equations

\[
\frac{d}{dt} v(t, x^{n-k}, \ldots, x^n) = v(t, x^{n-k}, \ldots, x^n) A_{n-k-1}(0, t, x^{n-k}, \ldots, x^n), \quad v(0, x^{n-k}, \ldots, x^n) = u_k(x^{n-k}, \ldots, x^n),
\]

(8)

Setting \( u_{k+1}(x^{n-k-1}, x^{n-k}, \ldots, x^n) = v(x^{n-k-1}, x^{n-k}, \ldots, x^n) \). We will prove that system (8) has a unique solution in \( L^2 \) with \( \frac{dv}{dt} \) also in \( L^2 \). It is not hard to see that, for any \( i \geq n-k \), the quantity \( w_i = \partial_i u_{k+1} - u_{k+1} A_i \) formally satisfies the differential equation

\[
\partial_j w_i = \partial_i (\partial_j u_{k+1} - u_{k+1} A_j) + w_i A_j + u_{k+1} (\partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i),
\]
on the slice \( \{0\} \times I^{k+1} \) with \( j = n-k-1 \). Using the defining differential equation for \( u_{k+1} \) and hypothesis (9), one obtains \( \partial_j w_i = w_i A_j \). The initial condition \( w_i(x_j = 0) = 0 \) follows from the induction step. So, formally, \( w_i = 0 \), which implies that \( u_{k+1} \in W^{1,2} \). We now make this argument precise.

The initial value problem

\[
\frac{d}{dt} v(t) = v(t) a(t), \quad v(0) = v_0 \in G,
\]

(9)

with \( a \in L^2(I^1, g) \), has a unique solution \( v \in W^{1,2}(I^1, G) \). To prove this, we mollify \( a \) to get \( a_\varepsilon(t) = (\zeta_\varepsilon * a)(t) \) and solve the smooth system

\[
\frac{d}{dt} v_\varepsilon(t) = v_\varepsilon(t) a_\varepsilon(t), \quad v_\varepsilon(0) = v_0,
\]

(10)

obtaining smooth functions \( v_\varepsilon \) with values in \( G \). Since \( G \) is compact, the functions \( v_\varepsilon \) are uniformly bounded, and they are equicontinuous on \( I^1 \), since

\[
|v_\varepsilon(t_2) - v_\varepsilon(t_1)| = \left| \int_{t_1}^{t_2} \frac{d}{ds} v_\varepsilon(s) a_\varepsilon(s) \, ds \right| \leq C_1 \int_{t_1}^{t_2} |a_\varepsilon(s)| \, ds \\
\leq C_1 |t_2 - t_1| \left( \int_{t_1}^{t_2} |a_\varepsilon(s)|^2 \, ds \right)^{\frac{1}{2}} \leq C_1 \|a\|_{L^2(I^1, g)} |t_2 - t_1| \frac{1}{2}
\]

Hence, there is a sequence \( \varepsilon_k \to 0 \) for which \( v_{\varepsilon_k} \) converges uniformly on \( I^1 \) to a continuous \( G \)-valued function \( v \). Also, \( v_\varepsilon a_\varepsilon \to v a \) in \( L^2 \). This implies that \( \frac{dv_{\varepsilon_k}}{dt} \) converges in \( L^2 \); the limit is the distributional derivative of \( v \). This shows that \( v \in W^{1,2}(I^1; G) \), and that \( v \) is a solution of (9). Note, that for all \( t \) we have

\[
v(t) = v_0 + \int_0^t v(s) a(s) \, ds,
\]

(11)

and this equation is equivalent to (8). If \( w \in W^{1,2}(I^1; G) \) is another solution of (9), then the difference, \( p(t) = v(t) - w(t) \), satisfies \( |p(t)| \leq \int_0^t |p(s)||a(s)| \, ds \), and is therefore identically zero.

This argument establishes the base case of the induction, namely, \( u_1(0, \cdot) \in W^{1,2}(I^1, G) \). To complete the induction, we will prove that \( u_{k+1} A\upharpoonright_{\{0\} \times I^{k+1}} = \partial_i u_{k+1} \) for \( i \geq n-k-1 \) and \( u_{k+1} \in W^{1,2}(\{0\} \times I^{k+1}, G) \) when \( \partial_i u_k = u_k A_i\upharpoonot_{\{0\} \times I^k} \) for \( i \geq n-k \) and \( u_k \in W^{1,2}(\{0\} \times I^k, G) \). We first need to show that the equation

\[
dA\upharpoonot_{\{0\} \times I^{k+1}} + \frac{1}{2} \left[ A\upharpoonot_{\{0\} \times I^{k+1}}, A\upharpoonot_{\{0\} \times I^{k+1}} \right] = 0
\]

(12)

holds in the sense of distributions. We prove this inductively. By the hypothesis of the lemma, we have
\[
\int_{I^n} (A_j \partial_i \eta - A_i \partial_j \eta) \, d^n x = \int_{I^n} [A_i, A_j] \eta \, d^n x
\] (13)

for any smooth scalar function \( \eta \) supported in the interior of the cube. When both \( i \) and \( j \) are larger than 1, equation (13) will also hold for scalar functions of the form

\[
\eta_\epsilon(x^1, \ldots, x^n) = \int \zeta(-x^1)\zeta(y^2 - x^2) \cdots \zeta(y^n - x^n) \phi(y^2, \ldots, y^n) \, d^n y.
\]

Substitute this \( \eta_\epsilon \) in (13), integrate by parts to move all derivatives onto \( \phi \), and change the order of integration to obtain

\[
\int_{I^n} (A_j \partial_i \eta - A_i \partial_j \eta) \, d^n x = \int_{I^n} [A_i, A_j] \eta \, d^n x.
\]

Pass to the limit as \( \epsilon \) goes to 0 along the sequence chosen in the beginning of the proof. The result will be

\[
\int_{I^{n-1}} (A_j(0, \cdot) \partial_i \phi - A_i(0, \cdot) \partial_j \phi) \, d^{n-1} x = \int_{I^{n-1}} [A_i(0, \cdot), A_j(0, \cdot)] \phi \, d^{n-1} x.
\]

Repeating this argument, in a finite number of steps we obtain (12).

We next consider a mollified version of system (8),

\[
\frac{d}{dt} v^\epsilon = v^\epsilon (A_{n-k-1}|_{0} \times I^{k+1}) (t, x^{n-k}, \ldots, x^n),
\]

(14)

Let \( \Phi^\epsilon(t, s; x^{n-k}, \ldots, x^n) \) be the solution of

\[
\frac{d}{dt} \Phi^\epsilon(t, s; \cdot) = \Phi^\epsilon(t, s; \cdot) (A_{n-k-1}|_{0} \times I^{k+1}) (t, \cdot), \quad \Phi^\epsilon(s, s; \cdot) = 1.
\]

In terms of \( \Phi^\epsilon \), the solution of (14) can be written as

\[
v^\epsilon(x^{n-k-1}, \ldots, x^n) = u_k(x^{n-k}, \ldots, x^n) \Phi^\epsilon(x^{n-k-1}, 0; x^{n-k}, \ldots, x^n),
\]

and it is a smooth function of all of the variables. For any \( i \geq n-k \), set \( w^\epsilon_i = \partial_i v^\epsilon - v^\epsilon (A_i|_{0} \times I^{k+1}) \) on \( \{0\} \times I^{k+1} \). Differentiate equation (14) to obtain (with \( j = n-k - 1 \))

\[
\partial_j w^\epsilon_i = \partial_i (\partial_j v^\epsilon - v^\epsilon (A_j|_{0} \times I^{k+1}) + w^\epsilon_i (A_j|_{0} \times I^{k+1})
+ v^\epsilon \left( \partial_i (A_j|_{0} \times I^{k+1}) - \partial_j (A_i|_{0} \times I^{k+1}) \right)
+ (A_i|_{0} \times I^{k+1}) (A_j|_{0} \times I^{k+1}) - (A_j|_{0} \times I^{k+1}) (A_i|_{0} \times I^{k+1}) \right),
\]

on the slice \( \{0\} \times I^{k+1} \). Notice that \( \partial_j v^\epsilon - v^\epsilon (A_j|_{0} \times I^{k+1}) = 0 \) by (14) and

\[
F^\epsilon_{ij} := \partial_j (A_i|_{0} \times I^{k+1}) - \partial_j (A_i|_{0} \times I^{k+1})
+ (A_i|_{0} \times I^{k+1}) (A_j|_{0} \times I^{k+1}) - (A_j|_{0} \times I^{k+1}) (A_i|_{0} \times I^{k+1})
= (A_i|_{0} \times I^{k+1}) (A_j|_{0} \times I^{k+1}) - (A_j|_{0} \times I^{k+1}) (A_i|_{0} \times I^{k+1})
- \left( (A_i A_j - A_j A_i)|_{0} \times I^{k+1} \right),
\] (15)
by equation \((12)\). Also,

\[
\begin{align*}
   w_t^\epsilon (0, x_n^{n-k}, \ldots, x_n) &= \partial_x (u_k)^\epsilon - (u_k)^\epsilon (A^1|_{(0) \times I_{k+1}})^\epsilon \\
   &= \left( \partial_x (u_k)^\epsilon - (u_k)^\epsilon A^1|_{(0) \times I_k} \right)^\epsilon \\
   &+ (u_k A^1|_{(0) \times I_k})^\epsilon - (u_k)^\epsilon (A^1|_{(0) \times I_k})^\epsilon \\
   &+ (u_k)^\epsilon ((A^1|_{(0) \times I_k})^\epsilon - (A^1|_{(0) \times I_k+1})^\epsilon) \\
   &= \left( u_k A^1|_{(0) \times I_k} \right)^\epsilon - (u_k)^\epsilon (A^1|_{(0) \times I_k})^\epsilon \\
   &+ (u_k)^\epsilon ((A^1|_{(0) \times I_k})^\epsilon - (A^1|_{(0) \times I_k+1})^\epsilon) =: w_{t0}^\epsilon,
\end{align*}
\]

by the induction hypothesis. Thus, \(w_t^\epsilon (x_n^{n-k-1}, x_n^{n-k}, \ldots, x_n)\) is the solution of the following initial value problem:

\[
\partial_x w_t^\epsilon = w_t^\epsilon (A_j)^\epsilon + v^\epsilon F_{ij}^\epsilon,
\]

Therefore,

\[
w_t^\epsilon (x^j, \cdot) = w_{t0}^\epsilon \Phi^\epsilon (x^j_0, 0; \cdot) + \int_0^{x^j} (v^\epsilon F_{ij})|_{x^j_0 = 0} \Phi^\epsilon (x^j, s; \cdot) ds.
\]

Notice that \(w_{t0}^\epsilon \to 0\) in \(L^1\{(0) \times I^k\}\), since \(u_k \in W^{1,2}\{(0) \times I^k, G\}\), and \((A^1|_{(0) \times I_k})^\epsilon \to A^1|_{(0) \times I_k}\) in \(L^2\{(0) \times I^k, G\}\), and, due to the choice of the coordinates at the beginning of the argument, \((A^1|_{(0) \times I_k})^\epsilon|_{x^j_0 = 0} \to A^1|_{(0) \times I_k}\) in \(L^2\{(0) \times I^k, G\}\). In addition, \(\Phi^\epsilon\) is smooth and uniformly bounded, so

\[
\| w_{t0}^\epsilon \Phi^\epsilon (x^j, 0; \cdot)\|_{L^1(\{0\} \times I^{k+1})} \overset{\epsilon \to 0}{\to} 0.
\]

Since \((A^1)|_{(0) \times I^{k+1}} \to A^1|_{(0) \times I^{k+1}}\) in \(L^2(\{0\} \times I^{k+1})\), we have \(F_{ij}^\epsilon \to 0\) in \(L^1\) (see equation \((13)\)). Both \(v^\epsilon\) and \(\Phi^\epsilon\) are smooth and uniformly bounded, thus \(w_t^\epsilon\) tends to 0 in \(L^1(\{0\} \times I^{k+1}, C^{0,\gamma})\), i.e.,

\[
\| \partial_x v^\epsilon - v^\epsilon (A^1|_{(0) \times I^{k+1}}) \|_{L^1(\{0\} \times I^{k+1}, C^{0,\gamma})} \to 0.
\]

Because the \(L^1\)-norm of \(v^\epsilon (A^1|_{(0) \times I^{k+1}})\) is uniformly bounded, \((10)\) implies that the \(L^1\)-norm of \(\partial_x v^\epsilon\) is bounded, while equation \((13)\) shows that \(\partial_x v^\epsilon\) is bounded in \(L^1\) as well. Thus, \(v^\epsilon\) is bounded in \(W^{1,1}(I^{k+1}, G)\). Since in our case the embedding \(W^{1,1} \to L^1\) is compact, there exists a sequence \(\epsilon_m \to 0\) so that \(v^\epsilon\) converges strongly in \(L^1\) and almost everywhere in \(I^{k+1}\) to some \(v \in L^1(I^{k+1}, G)\). In addition, we have \(\partial_x v = v A^1(0, \cdot)\) in the sense of distributions for all \(n-k-1 \leq i \leq n\). Hence, \(v \in W^{1,2}(I^{k+1}, G)\).

We still need to check the initial conditions in \((8)\). Taking a further subsequence if necessary, we will have \(v^m(t, \cdot) \to v(t, \cdot)\) in \(L^1(I^k)\) for almost every \(t \in I^1\). Indeed, let \(\epsilon_m\) be such that \(\|v^m - v\|_{L^1(I^{k+1})} \leq 2^{-m}\). Then,

\[
\int_{I^1} (\sum_{m=1}^{\infty} \int_{I^k} |v^m - v| \, dx) \, dt = \sum_{m=1}^{\infty} \int_{I^{k+1}} |v^m - v| \, dx \leq 1,
\]

which proves the convergence almost everywhere. To simplify the notation we will omit the subscript \(m\) on the subsequence in the future. Multiply equation \((14)\) by an arbitrary \(\eta \in L^2(I^k)\) and an arbitrary smooth \(\varphi\) and integrate to obtain

\[
\varphi(t)(v^\epsilon(t), \eta) = \varphi(0)(v^\epsilon(0), \eta) + \int_0^t ((v^\epsilon(s), \eta) \varphi'(s) + (v^\epsilon(s)(A^1)|_{(0) \times I_k} \varphi(s)) ds
\]

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where \((\xi, \eta) = \int_0 \xi \eta d^k x\). Each of the terms on the right hand side converge, as \(\epsilon \to 0\), to the corresponding term with \(v\). Hence, the left side must have a limit as well. For almost all \(t \in I^1\) this limit must be \(\varphi(t)(v(t), \eta)\). As a result, \(v\) is a weakly continuous function from \(I^1\) to \(L^2(I^k)\). It satisfies the differential equation

\[
\frac{d}{dt} (v(t), \eta) = (v(t)) A_j(0, t), \eta
\]

almost everywhere, and \(v(t) \to v(0) = u_k\) weakly in \(L^2(I^k)\). The fact, that \(v\) is the unique solution to equation (\ref{eq:1}), follows from the argument given in the base case. Thus, we have \(du = u A\). Recall, that 

Now assume that \(\tilde{u}^{-1} d\tilde{u} = u^{-1} du\) for some other \(\tilde{u} \in W^{1,2}(I^n, G)\). Since \(\tilde{u} u^{-1}\) belongs to \(W^{1,2}(I^n, G)\), we compute:

\[
d(\tilde{u} u^{-1}) = d\tilde{u} u^{-1} -\tilde{u} u^{-1} d\tilde{u}^{-1} u^{-1} = \tilde{u} (\tilde{u}^{-1} d\tilde{u} - u^{-1} du) u^{-1} = 0.
\]

Hence, \(\tilde{u}(x) = g u(x)\) for some constant \(g \in G\). Now the lemma is proved.

### 3.3 The holonomy representation for \(L^2_{\text{loc}}\) connections

Let \(\Omega\) be an \(n\)-dimensional Riemannian manifold. Let \(P\) be a principal \(G\)-bundle over \(\Omega\). Let \(\{\varphi_\nu : \mathcal{U}_\nu \times G \to P\}_{\nu \in \mathcal{N}}\) be a bundle atlas. The transition functions, \(\psi_{\mu \nu} : \mathcal{U}_\nu \cap \mathcal{U}_\mu \to G\), are specified by \(\varphi_\nu(x, g) = \varphi_\nu(x, \psi_{\mu \nu}(x) \cdot g)\). The associated local sections, \(\sigma_\nu : \mathcal{U}_\nu \to P\), are given by \(\sigma_\nu(x) = \varphi_\nu(x, 1)\).

Let \(A\) be a connection on \(P\). Locally we may express \(A\) via \(A_\nu = \sigma_\nu^* A\). The local connection forms satisfy \(A_\nu = \psi_{\nu^{-1}}^{-1} A_\mu \psi_{\mu \nu} + \psi_{\nu^{-1}} \psi_{\mu \nu}^{-1} d\psi_{\mu \nu}\). We say that \(A\) is an \(L^2_{\text{loc}}\)-connection if all \(A_\nu\) are locally \(L^2\) 1-forms with values in \(\mathfrak{g}\). The curvature of \(A\) is defined as \(F_A = dA + \frac{1}{2} [A, A]\). Locally, we have \((F_A)_\nu = \sigma_\nu^* F_A = d A_\nu + \frac{1}{2} [A_\nu, A_\nu]\), which can be interpreted as a distribution for an \(L^2_{\text{loc}}\)-connection. A connection \(A\) is flat if \(F_A = 0\).

For smooth flat connections, the usual holonomy map is a map from \(\pi_1(\Omega, x_0)\) to \(G\). This map depends on a base point above \(x_0\) in \(P\). Changing the base point in \(P\) will conjugate the holonomy map by an element of \(G\). The equivalence class of the holonomy map up to conjugation is called the holonomy representation. We are about to generalize the notion of holonomy representation to distributionally flat \(L^2_{\text{loc}}\) connections. Note that the value of the local gauge, \(u\), from the nonlinear Poincaré lemma in the previous subsection, is not defined at every point. This is why we do not generalize the holonomy map. To generalize the holonomy representation, we will triangulate the manifold \(\Omega\) and use the edge-path group in place of the fundamental group.

We now review the definition of the edge-path group of a pointed simplicial complex, \cite{B儿子}. Let \(K\) be a simplicial complex. Denote the vertex set of \(K\) by \(K^{(0)}\). Select a vertex \(p_0\) as the base point. In general, vertices will be denoted by letters \(p, q, \ldots\), the oriented edges will be denoted by \(e = [p, q]\) where \(i(e) = p\) is the initial vertex and \(f(e) = q\) is the final vertex of \(e\). The closed star of the vertex \(p\) will be denoted by \(\text{st}(p)\). An edge-path, \(\zeta\), is a non-empty finite sequence of oriented edges \([e_1, \ldots, e_\ell]\) with \(f(e_j) = i(e_{j+1})\). A closed, pointed edge-path is one with \(i(e_1) = f(e_\ell) = p_0\). Two edge-paths, \(\zeta_1\) and \(\zeta_2\), are called simply equivalent if one can be obtained from the other by replacing a single edge, \(e = [p, r]\) by a two-edge path \([p, q][q, r]\) (or vice versa), where \(p, q,\) and \(r\) are (not necessarily distinct) vertices of the same 2-simplex in \(K\). Two edge-paths are equivalent if one can be obtained from the other by a finite sequence of such moves.

The set of equivalence classes of all closed pointed edge-paths forms a group, called the edge-path group, \(E(K, p_0)\). This group is canonically isomorphic to the fundamental group, \(\pi_1([K], p_0)\), \cite{B儿子}, Theorem 3.6.17.

It is known that any smooth manifold admits a unique PL-structure compatible with the smooth structure, \cite{B儿子}. In particular, there exists a simplicial complex, \(K\) whose topological realisation, \([K]\), is homeomorphic to \(\Omega\). Furthermore, we may assume that any nonempty intersection of closed stars of vertices is
piecewise smoothly equivalent to the unit cube in \( \mathbb{R}^n \). Any two such complexes have a common subdivision. Given any open cover of \( \Omega \), subdividing the complex, if necessary, we may assume that the closed star of any vertex is contained in some element of the cover. Fix a complex \( K \), retaining the properties described above, subordinate to a bundle atlas and fix a vertex, \( p_0 \), as the base point. Let \( \phi : E(K, p_0) \to \pi_1([K], p_0) \) be the canonical isomorphism. For each vertex \( p \) choose an index \( \nu(p) \) with \( \text{st}(p) \subset \mathcal{U}_\nu(p) \). Denote \( \psi_{\nu(p)}(\nu(p)) \) by \( \psi_{\nu(p)} \), \( \varphi_{\nu(p)} \) by \( \varphi(p) \), and \( A_{\nu(p)} \) by \( A_p \).

Let us define the holonomy representation of a distributionally flat \( L^2_{\text{loc}} \) connection, \( A \). By the nonlinear Poincaré lemma, there exists \( u_p \in W^{1,2}(\text{st}(p), G) \) so that \( A_p = u_p^{-1} \text{du}_p \). Pick such \( u_p \) for each vertex \( p \). A collection of such functions is called a local developing map. We have,

\[
(u_p \psi_{\nu(p)})^{-1} d(u_p \psi_{\nu(p)}) = \psi_{\nu(p)}^{-1} u_p^{-1} d\psi_{\nu(p)} + \psi_{\nu(p)}^{-1} d\psi_{\nu(p)} = \psi_{\nu(p)}^{-1} A_p \psi_{\nu(p)} + \psi_{\nu(p)}^{-1} d\psi_{\nu(p)} = u_p^{-1} d\text{du}_p.
\]

From the second part of the nonlinear Poincaré lemma it follows, that there exists a \( g_{[p,q]} \in G \) so that

\[
u(p) = g_{[p,q]} u_p \psi_{\nu(p)}.
\]

**Definition 1** The holonomy representation of a flat connection, \( A \), is the conjugacy class of the map \( \rho_A : \pi_1(\Omega, p_0) \to G \) given by

\[
\rho_A(\phi([e_1, \ldots, e_\ell])) = g_{e_1} \cdots g_{e_\ell}.
\]

**Lemma 4** This definition is independent of the representative of the edge-path group, the choice of \( u_p, \varphi(p), \) and the triangulation \( K \).

**Proof.** Consider a simple equivalence of two edge-paths generated by \([p, r] \leftrightarrow [p, q][q, r]\). Observe that \(\psi_{[r,q]} \psi_{[p,q]} = \psi_{[p,r]}\). Since \(p, q, \) and \(r\) lie within a single simplex, the set \(V_{p,q,r} = \text{st}(p) \cap \text{st}(q) \cap \text{st}(r)\) is nonempty. By the definition of \(g_e\),

\[
g_{[p,q]} \cdot \psi_{[p,q]} \cdot \psi_{[p,q]} = u_p |_{V_{p,q,r}} = g_{[p,q]} |_{V_{p,q,r}} \cdot \psi_{[p,q]} \cdot \psi_{[p,q]} = g_{[q,r]} |_{V_{p,q,r}} \cdot \psi_{[q,r]} = g_{[q,r]} \cdot g_{[p,q]} |_{V_{p,q,r}} \cdot \psi_{[p,q]} \cdot \psi_{[p,q]}
\]

Thus, \(g_{[p,q]} \cdot g_{[q,r]} = g_{[p,r]}\) and \(\rho_A \circ \phi\) respects simple equivalence.

By the nonlinear Poincaré lemma, any other choice of local developing map, \(u'_p\), is related to \(u_p\) by \(u'_p = h_p \cdot u_p\), for some constants \(h_p\). The corresponding edge labels are given by \(g'_{[p,q]} = h_p \cdot g_{[p,q]} \cdot h_q^{-1}\). This implies that \(\rho_A\) changes by conjugation by \(h_p\), i.e., \(\rho'_A(\gamma) = h_p \cdot \rho_A(\gamma) \cdot h_p^{-1}\).

Let \(K'\) be a subdivision of \(K\), and let \(\phi' : E(K', p_0) \to \pi_1([K'], [K], p_0)\) be the corresponding isomorphism. Any loop in \(\Omega\) is represented by an edge-path \(\zeta = [e_1, \ldots, e_\ell]\) in \(K\). In \(K'\) the same loop is represented by a subdivision \(\zeta' = [e_{1,1}, \ldots, e_{1,k_1}, \ldots, e_{\ell,k_\ell}]\). An edge, \([p, q]\), of \(\zeta\) is subdivided into a subpath \([p = r_1, r_2], \ldots, [r_{k-1}, r_k = q]\). The complex \(K'\) is subordinate to the same bundle atlas as \(K\). Thus, we may choose \(\nu(r_1) = \ldots = \nu(r_m) = \nu(p)\) and \(\nu(r_{m+1}) = \ldots = \nu(r_k) = \nu(q)\), and also, \(ur_1 = \ldots = ur_m = u_p, ur_{m+1} = \ldots = ur_k = u_q,\) and \(\varphi_{r_1} = \ldots = \varphi_{r_m} = \phi_p, \varphi_{r_{m+1}} = \ldots = \varphi_{r_k} = \phi_q\). This implies \(g'_{[p=r_1,r_2]} \cdot \ldots \cdot g'_{[r_{k-1}, r_k = q]} = 1 \cdot \ldots \cdot 1 \cdot g_{[p,q]} \cdot 1 \cdots 1\). Thus, \(\rho_A\) does not depend on the triangulation.

Let \(\{\varphi'_\nu : \mathcal{U}_\nu \times G \to P\}\) be a second bundle atlas. By taking intersections, if necessary, we assume that it has the same open cover \(\{\mathcal{U}_\nu\}\). Subdividing, if necessary, we may assume that the complex \(K'\) is the same. We may therefore pick the same indexing functions, \(\nu(p)\). There exist transition functions \(\partial_\nu : \mathcal{U}_\nu \to G\)

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where the equivalence is given by $u_P$ with the description of the admissible reference connections. Let $x$ of

This completes the proof of the lemma.

Remark 5 One can prove that for smooth connections our definition of the holonomy representation agrees with the usual one.

4 Configuration spaces and the Skyrme functional

In this section we generalize the original Skyrme model to two separate settings. In the first setting the fields are maps from a 3-manifold to a compact Lie group. In the second setting, the fields are flat connections on a principle bundle over a 3-manifold.

To define the Skyrme functional for maps, fix a Riemannian metric on the base manifold $M$, and fix a faithful unitary representation of the compact Lie group $G$. Define the norm on the Lie algebra $g$ as $|X|^2 = -\frac{1}{8} \text{Tr}(\text{ad}(X)\text{ad}(X))$. With the help of the Riemannian metric, this extends to a norm on Lie algebra valued forms in a standard way. The Skyrme functional for maps is defined to be

$$E(u) = \int_M \frac{1}{2} |u^{-1} du|^2 + \frac{1}{4} |u^{-1} du \wedge u^{-1} du|^2 \, d\text{vol}_M.$$  

The configuration space of Skyrme maps is defined to be

$$\mathcal{S}^G(M) = \{ u \in W^{1,2}(M, G) | E(u) < \infty \} / \sim,$$

where the equivalence is given by $u(\cdot) \sim u(\cdot) g$, for $g \in G$.

To define the Skyrme functional for connections, we need a special type of reference connection. We start with the description of the admissible reference connections. Let $P$ be a principal $G$-bundle over $M$, and let $B$ be a (not necessarily flat) smooth connection on $P$. Given a base point $x_0 \in M$ and a lift $\hat{x}_0 \in P$ of $x_0$, we define a holonomy homomorphism $\varrho_B : \Omega_{x_0} \to G$, where $\Omega_{x_0}$ is the based loop group of $M$, as follows. Given a based loop $\gamma$ in $M$, construct the horizontal lift $\hat{\gamma} : [0, 1] \to P$ with $\hat{\gamma}(0) = \hat{x}_0$. By definition, $\varrho_B(\gamma)$ is the unique element of $G$ satisfying $\hat{\gamma}(1) \cdot \varrho_B(\gamma) = \hat{x}_0$. Note that the horizontal lift $\hat{\gamma}$ may be constructed from any lift $\tilde{\gamma}$ starting at $\hat{x}_0$ as $\hat{\gamma}(t) = \tilde{\gamma}(t) g(t)$, where $g : [0, 1] \to G$ is the solution of the equation $\dot{g}(t) + i\tilde{\gamma}(t) B g(t) = 0$ with $g(0) = 1$.

Remark 5 In this paper we have used three flavors of holonomy: the holonomy homomorphism, the holonomy map, and the holonomy representation. The holonomy homomorphism is defined for arbitrary smooth connections $B$. If the connection $B$ is flat, then the holonomy homomorphism $\varrho_B$ depends only on the homotopy class of $\gamma$, and therefore reduces to the holonomy map, $\varrho_B : \pi_1(M, x_0) \to G$. If the lift of the base point $\hat{x}_0$ is changed, then the holonomy map changes by conjugation. The equivalence class of $\varrho_B$ up to conjugacy is the holonomy representation.
A connection $B$ is called central if $g_B(\Omega_{\text{alg}})$ is contained in the center, $Z_G$, of $G$. A principal bundle is called central if it admits a central connection. Clearly, any bundle with abelian structure group is central. In general, a principal $G$-bundle $P$ is central if and only if $P \times_G (G/Z_G)$ is a trivial bundle. For connected Lie groups, there is an obstruction in $H^2(M; \pi_1(G/Z_G))$ which vanishes if and only if the bundle $P \times_G (G/Z_G)$ is trivial.

Let $P$ be a central bundle with central connection $B$. Any other connection on $P$ may be expressed as $A = B + \text{pr}^*a$, where $a$ is a $\mathfrak{g}$-valued 1-form on $M$. Our generalization of the Skyrme functional uses this description:

$$E[a] = \int_M \frac{1}{2}|a|^2 + \frac{1}{16}|[a, a]|^2 \, d\text{vol}_M.$$ 

The corresponding configuration space of Skyrme potentials is defined to be

$$S^{B,G}[M] = \{a \in L^2(M, \mathfrak{g}) \mid da + \frac{1}{2}[a, a] + F_B = 0, E[a] < \infty\}/\sim,$$

where $a \sim g^{-1}ag$ for $g \in G$.

**Remark 6** The Skyrme functional is not gauge invariant. It is only invariant under constant gauge transformations. This is why the configuration space of Skyrme fields is infinite dimensional.

The transformation $u \mapsto u^{-1}du$ takes the configuration space of Skyrme maps into the configuration space of Skyrme potentials with the trivial reference connection. We may view this as a transformation $\mathcal{D} : S^G(M) \to S^{B,G}[M]$, or as a transformation $\hat{\mathcal{D}} : S^G(M) \to S^{B,G}[M]$, where $\theta$ ($\hat{\theta}$) is the trivial connection on $M \times G$ ($M \times \hat{G}$). We have $E(u) = E[Du] = E[D\hat{u}]$. We will reconsider the map $\mathcal{D}$ at the end of this section.

In Section 2 we described the path components of the space of smooth maps from $M$ to $G$ up to multiplication by $G$ on the right. It turns out that the numerical invariants specifying these components, Proposition 2, are well defined for Sobolev maps. To show this, we combine the description of these invariants given before Proposition 2 with the general definition of the holonomy representation from Section 3. We need to relate the holonomy of a connection on the trivial bundle $M \times G$ to the holonomy of the corresponding connection on the trivial bundle $M \times \hat{G}$. The following lemma addresses a slightly more general situation in which the standard map $G \to G$ is replaced by a Lie group homomorphism $H \to G$.

**Lemma 5** Let $f : H \to G$ be a homomorphism of Lie groups such that $f_* : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. Let $f : M \times H \to M \times G$ be the obvious map. If $A$ is a flat connection on $M \times G$, then $\hat{A} = f_\ast^{-1}f^*A$ is a flat connection on $M \times H$, and the holonomy representations are related by the formula: $\rho_A = f \circ \rho_{\hat{A}}$.

**Proof.** Let $R_k$ denote right multiplication by $k$ in a given Lie group. Let $L_s$ denote left multiplication by an element $s$ of any space with a right group action. First note that $\hat{A}$ is right invariant. Indeed,

$$R_k^*\hat{A} = f_\ast^{-1} R_k^* f^* A = f_\ast^{-1} \hat{f}^* R_{f(h)}^* A = f_\ast^{-1} \hat{f}^* (\text{Ad} f(h)^{-1})_s A = (\text{Ad} h^{-1})_s \hat{A},$$

where we have used the equalities $\hat{f} \circ R_k = R_{f(h)} \circ \hat{f}$ and $(\text{Ad} f(h)) \circ f = f \circ (\text{Ad} h)$. Now notice that $\hat{A}$ satisfies the second condition in the definition of a connection. Indeed,

$$\hat{A}(L_{(x,h)}^* X) = f_\ast^{-1} A(\hat{f}_s L_{(x,h)}^* X) = f_\ast^{-1} A(L_{(x,f(h))}^* f_s X) = f_\ast^{-1} f_s X = X,$$

for any $X$ in $\mathfrak{h}$, since $\hat{f} \circ L_{(x,h)} = L_{(x,f(h))} \circ f$. The connection $\hat{A}$ is flat because $A$ is.
To compute the holonomy representations we use the trivial bundle atlases and a fixed triangulation. Let $p$ be a vertex of the triangulation and $\tilde{\sigma}_p$ denote the local section $\tilde{\sigma}_p : st(p) \to M \times H$ which is simply $\tilde{\sigma}_p(x) = (x, 1)$. Then, $\sigma_p = \tilde{f} \circ \tilde{\sigma}_p$. Let $\tilde{u}_p$ be the associated local developing map for $\tilde{A}$. The following computation shows that $u_p = f \circ \tilde{u}_p$ is a local developing map for $A$.

$$u_p^{-1} du_p = f(\tilde{u}_p^{-1}) f_d \tilde{u}_p = f(\tilde{u}_p^{-1}) f_s \tilde{u}_p, \tilde{u}_p^{-1} d\tilde{u}_p = f(\tilde{u}_p^{-1}) f_s (L_{\tilde{u}_p})_s \hat{\sigma}_p^* f_s^{-1} \hat{f}^* A$$

$$= f(\tilde{u}_p^{-1}) f_s (L_{\tilde{u}_p})_s f_s^{-1} \sigma_p^* A = f(\tilde{u}_p^{-1}) (L_{f(\tilde{u}_p)})_s f_s^{-1} \sigma_p^* A = \sigma_p^* A.$$  

Here we used $f \circ L_{\tilde{u}_p} = L_{f(\tilde{u}_p)} \circ f$. It follows that $g_s = f(\tilde{g}_s)$, and so, $\rho_{\tilde{A}} = f \circ \rho_A$ establishing the lemma.

Returning to the construction of invariants for Sobolev maps, we will apply the previous lemma to the standard map $\tau : G \to G_0 \to G$. When $u$ is an element of $S^G(M)$, the form $A = \theta + \text{pr}^* Du$ is a connection on $M \times G$. Here $\theta$ is the trivial connection on $M \times G$. Notice that $u$ is a local developing map for $A$ associated to the trivial bundle atlas. The corresponding holonomy representation, $\rho_A$, is therefore trivial. Now, for $\tilde{A} = \theta + \text{pr}^* Du = \tau^{-1}_* \hat{A}$, the previous lemma implies $\rho_{\tilde{A}} = \tau \circ \rho_A = 1$. Since $\tau^{-1}(1) \cong \pi_1(G_0) \cong H_1(G_0)$, we have $\rho_{\tilde{A}}(\pi_1(M)) \subset Z_G$. In general, the holonomy is only well defined up to conjugation, but in this case, the holonomy is a well defined map, $\rho_{\tilde{A}} : \pi(1) \to H_1(G_0)$. Since $H_1(G_0)$ is abelian, this must factor through a map from $H_1(M)$ to $H_1(G_0)$. Denote the corresponding element of $H^1(M; H_1(G_0))$ by $\alpha_u$. The constants $a_\ell(u)$ from Proposition 2 are just the coordinates of $\alpha_u$ in the basis $\{\alpha_\ell\}$ of $H^1(M; H_1(G_0))$.

Given any element $a \in H^1(M; H_1(G_0))$, there exists a smooth map $v_a : M \to G_0$ with $(v_a)_* = a$. Fix such a map for each cohomology class. Combining the above discussion with Remark 1 and Remark 2 we see that the integral $\int_M (u(v_a))^{-1})^* \Theta^k$ is well defined for Sobolev maps. We partition the configuration space of maps into the following sectors:

$$S^G_{\alpha_1, \ldots, \alpha_N}(M) = \{u \in S^G(M) | \alpha_u = \alpha, \int_M (u(v_{\alpha_u}))^{-1})^* \Theta^k = c^k\}.$$  

We do not yet know that two maps have the same invariants if and only if they live in the same path component. It is true for smooth maps, and it would be true if we knew that any finite Skyrme energy Sobolev map could be approximated by smooth maps with uniformly bounded Skyrme energy.

We now turn to Skyrme potentials. As we have shown, the holonomy is well defined for $L^2_{loc}$ distributionally flat connections. We first partition $S^{B, G}[M]$ using the holonomy representation as follows:

$$S^{B, G}_p[M] = \{a \in S^{B, G}[M] | \rho_{B+a} = \rho\}.$$  

In the smooth case it is well known that two flat connections are gauge equivalent if and only if they have the same holonomy. To describe the path components of the space of smooth flat connections with fixed holonomy, it is convenient to fix a reference connection in this class. Let $B + b$ be one such connection. Any other connection, $B + a$, is obtained from $B + b$ by a gauge transformation, $B + a = (B + b)^u = B + u^{-1} b u + u^{-1} d_B u$, where $u$ is a map from $M$ to $G$. Conversely, given any such $u$, one obtains a connection $(B + b)^u$ with the same holonomy. The connections with fixed holonomy modulo constant gauge transformation are in one-to-one correspondence with the maps $u : M \to G$ modulo right multiplication by elements of $G$. The path components of the space of equivalence classes of smooth flat connections with fixed holonomy are determined by the one dimensional invariant and the three dimensional invariant of the gauge, $u$, as described above. We will now show that these invariants are well defined for Skyrme potentials with fixed holonomy.

We start by showing that two $L^2_{loc}$ distributionally flat connections have the same holonomy if and only if they are gauge equivalent. We only prove this for connections on central bundles.
Let $\Omega$ be a Riemannian manifold with sufficiently smooth boundary. For central principal $G$-bundles over $\Omega$ we define the gauge group to be $\mathcal{G} = W^{1,2}(\Omega, G)$, see Remark [[[3]]].

**Lemma 6** Two $L^2_{\text{loc}}$ distributionally flat connections on a central bundle are gauge equivalent if and only if they have the same holonomy.

**Proof.** Let $B$ be a smooth central reference connection. Let $A = B + a$ be an $L^2_{\text{loc}}$ distributionally flat connection, and take $g \in \mathcal{G}$. The element $g$ acts on $A$ by $A^g = B + g^{-1}a g + g^{-1}d_B g$. To compute the holonomy representation of each connection, we fix a central bundle atlas and a triangulation. On the star of the vertex $p$ we have $A_p = B_p + a_p$ and $A^g_p = B_p + g^{-1}a_p g + g^{-1}d_B g$. Pick a developing map $u_p$ for $A_p$, i.e., $u_p^{-1}du_p = A_p$. Then $u^g_p = u_p g$ will be a developing map for $A^g_p$. Now, $u_p = g_{[p,q]}u_q \psi_{qp}$, so $u_p \cdot g = g_{[p,q]}u_q \psi_{qp} \cdot g = g_{[p,q]}u_q g \psi_{qp}$. Thus, the holonomy representation is the same.

Now, assume that $A^1 = B + a^1$ and $A^2 = B + a^2$ have the same holonomy representation. We fix a central bundle atlas and a triangulation. Pick local developing maps $u^1_p$ and $u^2_p$, and let $g^1_{[p,q]}$ and $g^2_{[p,q]}$ denote the corresponding edge labels. The holonomy representation is defined up to conjugacy. Multiplying $u^2$ at the base vertex $p_0$ by a constant we may assure that the holonomy maps of $A^1$ and $A^2$ are the same. Choose a maximal tree in the 1-skeleton of the triangulation with root $p_0$. Any vertex is connected to the root by a unique path, $(p_0, p_1, \ldots, p_m)$. We inductively modify the local developing map $u^2$ as follows. Set $\tilde{u}^2_{p_0} = u^2_{p_0}$, and recursively define

$$\tilde{u}^2_{p_{i+1}} = g^1_{[p_{i+1}, p_i]} g^2_{[p_i, p_{i+1}]} u^2_{p_i+1}.$$ 

The required gauge transformation is defined on each star by $g = (u^1_p)^{-1} \tilde{u}^2_p$. On the overlaps corresponding to the edges of the tree, these functions agree by construction. Any edge, $e$, not in the maximal tree belongs to a circuit, say, $[[q_1, q_2], \ldots, [q_{m-1}, q_m], [q_m, q_1]]$, with $q_1 = p_0$ and each edge $[q_i, q_{i+1}]$, except $e$, in the tree. Let $\tilde{g}^e_f$ be the edge labels constructed from $\tilde{u}^2$. When $f$ is in the tree, $\tilde{g}^e_f = g^1_f$ by construction. The product $g^1_{[q_i, q_2]} \ldots g^1_{[q_{m-1}, q_m]} g^1_{[q_m, q_1]}$ is equal to $\tilde{g}^e_f [q_i, q_2] \ldots \tilde{g}^e_f [q_{m-1}, q_m] \tilde{g}^e_f [q_m, q_1]$, because the holonomies agree. Hence, $\tilde{g}^e_f = g^1_f$, and so $\tilde{g}^e_f = g^1_f$ for every edge $f$. Now, $(u^1_p)^{-1} \tilde{u}^2_p = (u^1_p)^{-1} g^1_{[p_{i+1}, p_i]} g^2_{[p_i, p_{i+1}]} u^2_{p_{i+1}} = (u^1_p)^{-1} \tilde{u}^2_p$, for any two neighboring vertices $p$ and $q$. This implies that the function $g = (u^1)^{-1} u^2$ is well defined globally on $\Omega$. It belongs to $W^{1,2}_{\text{loc}}(\Omega, G)$, and it is easy to check that $(A^1)^g = A^2$. The lemma is proved.

Returning to Skyrme potentials on a closed 3-manifold $M$, fix a reference connection $B$ and holonomy $\rho$. Fix $b \in \mathcal{S}^{B,G}_{\rho}$ as a reference field. Given any other field $a \in \mathcal{S}^{B,G}_{\rho}[M]$, by the previous lemma, there exists $u \in W^{1,2}(M, G)$ such that $B + a = (B + b)u$. The one dimensional invariant of $a$ is nothing more then $\alpha_a$ described previously. In this setting we will denote it $\alpha[a]$. Define $u$ to be $u = u^\alpha[a]$. The three dimensional invariants are given by

$$c^k[a] = -\frac{K_{\hat{G}}}{192 \pi^2} \int_M \text{Tr}(\text{ad}(\hat{u}^{-1} du \hat{u}^{-1} du) \text{ad}(\hat{u}^{-1} du)).$$

Here $k$ is the index corresponding to the simple factor $\hat{G}^k$ of the covering group $\hat{G}$, and the constants $K_{\hat{G}^k}$ are those presented in Table [[[4]]] see Section [[[52]]]. The $\overline{\text{Tr}}(\hat{u}^{-1} du)$ means orthogonal projection onto the Lie algebra of $\hat{G}^k$. We include $v^1_\alpha$ to ensure that the invariants are integral in the smooth case.

**Lemma 7** The invariant $c^k[a]$ is well defined for Skyrme potentials with fixed holonomy.

**Proof.** Given a reference potential $b \in \mathcal{S}^{B,G}_{\rho}$, any other potential, $a$, with the same holonomy satisfies $a = u^{-1} bu + u^{-1} du$, for some $u \in W^{1,2}(M, G)$. Since

$$|\text{Tr}(\text{ad}(\hat{u}^{-1} du \hat{u}^{-1} du) \text{ad}(\hat{u}^{-1} du))| \leq c |\text{Tr}(\hat{u}^{-1} du \hat{u}^{-1} du \wedge \hat{u}^{-1} du |),$$

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Remark 7 \textbf{The map} \[ D : S^G_{\rho \in \mathcal{C}^1, \ldots, c^N}(M) \to S^G_{\rho \in \mathcal{C}^1, \ldots, c^N}[M] \] \textbf{is well defined. Indeed, by the very definition} \textbf{D preserves the invariants. To show that} \textbf{Du} \textbf{is distributively flat for any} u \in W^{1,2}(M,G), \textbf{consider in a chart the approximating sequence} (u^{-1})_n d(u), \textbf{where} (\cdot)_n \textbf{is the usual mollification. This sequence converges to} u^{-1}du \textbf{in the sense of distributions. The differential,} \[ d((u^{-1})_n d(u)) = (u^{-1}du u^{-1})_n \wedge (du)_n, \] \textbf{converges to} u^{-1}du \wedge u^{-1}du \textbf{in} L^1, \textbf{hence, in the sense of distributions.}

5 \textbf{Existence of minimizers}

In this section we prove that each sector of the configuration space of Skyrme maps contains a minimizer and each sector of the configuration space of Skyrme potentials contains a minimizer as well. We use the direct method of the calculus of variations, i.e., we prove that for suitable minimizing sequences the topological invariants are preserved and the energy is lower semicontinuous. We begin by analyzing the holonomy. The following lemma establishes convergence of the holonomy and, therefore, one-dimensional invariants.

Let \( \Omega \) \textbf{be an} n\textbf{-dimensional Riemannian manifold. We define a topology on the space of representations} \( \pi_1(\Omega, x_0) \to G \textbf{as follows. We say that} \rho_n \textbf{converges to} \rho, \textbf{if for every finite subset} \{\gamma_1, \ldots, \gamma_k\} \subset \pi_1(\Omega, x_0) \textbf{there exist representatives} \tilde{\rho}_n, \tilde{\rho} \textbf{so that} \tilde{\rho}_n(\gamma_j) \to \tilde{\rho}(\gamma_j) \textbf{in} G \textbf{for every} j. \textbf{It is not hard to see that the}
topology defined by this convergence is Hausdorff when \( \pi_1(\Omega, x_0) \) is finitely generated. We next prove a simple lemma about the holonomy representation.

**Lemma 8** If \( A^{(n)} \) is a sequence of \( L^1_{loc} \) distributionally flat connections that converges weakly to a distributionally flat connection, \( A \), then there exists a subsequence \( A^{(n_k)} \) so that \( \rho_{A^{(n_k)}} \to \rho_A \).

**Proof.** We are in the setting from Section 3. Let \( \psi_{qp} \) be transition functions for the underlying bundle. As in the definition of holonomy, there exist functions \( u_p^{(n)} \) and group elements \( g_{[p,q]}^{(n)} \) so that \( A_p^{(n)} = (u_p^{(n)})^{-1} du_p^{(n)} \) and \( u_p^{(n)} = g_{[p,q]}^{(n)} u_q^{(n)} \psi_{qp} \). Since \( A_p^{(n)} \) are uniformly bounded in \( L^2(st(p)) \), after taking a subsequence we may assume that \( u_p^{(n)} \) converges weakly in \( W^{-1,2} \), strongly in \( L^2 \), and almost everywhere to some \( u_p \). At the same time, since \( G \) is compact, we may also assume that \( g_{[p,q]}^{(n)} \) converges to some \( g_{[p,q]} \). It follows that \( A_p = (u_p)^{-1} du_p \) and \( u_p = g_{[p,q]} u_q \psi_{qp} \). Since we only need to check convergence on a finite set of group elements, this concludes the proof.

In order to address the three-dimensional invariants, we will need a special case of Tartar’s div - curl lemma, see [16]. This lemma will also be used later in the proof of the two main theorems.

**Lemma 9** Let \( M \) be a smooth 3-dimensional Riemannian manifold. Let \( \omega^m_1 \in L^2 \) be a sequence of matrix-valued 1-forms and \( \omega^m_2 \in L^2 \) be a sequence of matrix-valued 2-forms on \( M \). If \( \omega^m_1 \) converges weakly in \( L^2 \) to a form \( \omega_1 \) and \( \omega^m_2 \) converges weakly in \( L^2 \) to a form \( \omega_2 \), and if each sequence \( dw^m_1 \) and \( dw^m_2 \) is precompact in \( W^{-1,2}_{loc}(M) \), then \( \omega^m_1 \wedge \omega^m_2 \) converges to \( \omega_1 \wedge \omega_2 \) in the sense of distributions.

Returning to the settings of Lemma 8, work on a closed 3-manifold \( M \) with a fixed a reference field \( b \in S^B_{\rho,G} \).

**Lemma 10** Given a sequence \( a_n \in S^{B,b,G}_{\rho,\alpha}[M] \) and \( a \in S^{B,b,G}_{\rho,\alpha}[M] \) such that \( a_n \to a \) and \( a_n \wedge a_n \to a \wedge a \) in \( L^2(M) \), there exists a subsequence such that \( c^k[a_n] \to c^k[a] \).

**Proof.** Given any field \( a_n \in S^{B,b,G}_{\rho,\alpha}[M] \), by Lemma 8, there exists a gauge transformation \( u_n \in W^{-1,2}(M,G) \) such that \( u_n = u_n^{-1} bu_n + u_n^{-1} du_n \). Recall that \( \|u_n\|_{L^{\infty}} \) is uniformly bounded, therefore \( du_n \) is uniformly bounded in \( L^2 \). Hence, upon taking a subsequence, we may assume that there exists \( u \in W^{-1,2}(M,G) \) such that \( u_n \to u \) in \( W^{-1,2}(M,G) \), \( u_n \to u \) in \( L^2(M,G) \). Note that this implies \( a = u^{-1} bu + u^{-1} du \) and \( a_n \to a \) in \( L^2(M,G) \). From the definition of the invariant \( c^k \), it is clear that distributional convergence of \( \text{Tr}(u_n^{-1} du_n \wedge u_n^{-1} du_n \wedge u_n^{-1} du_n) \) to \( \text{Tr}(u^{-1} du \wedge u^{-1} du \wedge u^{-1} du) \) implies the convergence of the invariants. As in the proof of Lemma 6, we have

\[
\text{Tr}(u_n^{-1} du_n \wedge u_n^{-1} du_n \wedge u_n^{-1} du_n) = \text{Tr}(a_n \wedge a_n \wedge a_n - 3a_n \wedge a_n \wedge u_n^{-1} bu_n + 3u_n^{-1} b \wedge bu_n \wedge a_n - b \wedge b \wedge a) .
\]

Note that \( du_n = -a_n \wedge a_n - F_{B_n} \) is bounded in \( L^2 \). The Bianchi identity implies that \( d(a_n \wedge a_n) = 0 \). By hypotheses, \( a_n \to a \) and \( a_n \wedge a_n \to a \wedge a \) in \( L^2(M) \). Hence, \( a_n \wedge a_n \wedge a_n \) converges to \( a \wedge a \wedge a \) in the sense of distributions by the div - curl lemma. Since \( u_n^{-1} \) converges strongly to \( u^{-1} \) in \( L^2 \) and (taking a subsequence) \( bu_n \) converges weakly to \( bu \) in \( L^2 \), the product \( u_n^{-1} bu_n \) converges to \( u^{-1} bu \) in the sense of distributions. However, \( \|u_n^{-1} bu_n\|_{L^2} = \|u^{-1} bu\|_{L^2} = \|b\|_{L^2} \), so, upon taking a further subsequence, \( u_n^{-1} bu_n \) converges to \( u^{-1} bu \) weakly in \( L^2 \), and, hence, strongly in \( L^2 \). Since \( u_n \wedge a_n \to a \wedge a \) in \( L^2(M) \), we see that \( \text{Tr}(a_n \wedge a_n \wedge u_n^{-1} bu_n) \) converges to \( \text{Tr}(a \wedge a \wedge u^{-1} bu) \) distributionally. Similarly (on a further subsequence), \( u_n^{-1} b \wedge bu_n \) converges to \( u^{-1} b \wedge bu \) strongly in \( L^2 \) (recall: \( b \wedge b \in L^2 \)). Since \( a_n \to a \), we see
that \(\text{Tr}(u_n^{-1}b \wedge bu_n \wedge a_n)\) converges distributionally to \(\text{Tr}(u^{-1}b \wedge bu \wedge a)\). Transition from \(u\) to \(\mathfrak{u}\) is as in Lemma 2. This proves the lemma.

We are now ready for the existence theorems.

**Theorem 1** Each sector \(S^{B,b,G}_{\rho,\alpha; c_1,\ldots,c_N}[M]\) contains a minimizer of the Skyrme energy.

**Proof.** Let \(a_n \in S^{B,b,G}_{\rho,\alpha; c_1,\ldots,c_N}[M]\) be a minimizing sequence. As in the proof of the previous lemma, \(d\alpha_n = -a_n \wedge a_n - F_B\) is bounded in \(L^2\) and \(d(a_n \wedge a_n) = 0\). After taking a subsequence, we may assume that \(a_n \rightarrow a\) in \(L^2(M)\) and \(a_n \wedge a_n\) converges weakly to some 2-forms in \(L^2(M,g)\). By the \(\text{div} - \text{curl}\) lemma, \(a_n \wedge a_n\) converges to \(a \wedge a\) in the sense of distributions and, hence, weakly in \(L^2\).

By Lemma 8, \(\rho_{B+a} = \rho_{B+a_n} = \rho\). As in the proof of Lemma 10, taking a subsequence, we produce a sequence \(u_n \in W^{1,2}(M,G)\) weakly converging to \(u \in W^{1,2}(M,G)\) such that \(a_n = u_n^{-1}bu_n + u_n^{-1}du_n\), and \(u_n^{-1}du_n \rightarrow u^{-1}du\) in \(L^2(M,g)\). By definition, \(\alpha[a] = \rho_{\tilde{\theta} + \text{pr}^*\tilde{D}u_n}\), where \(u\) satisfies \(a = u^{-1}bu + u^{-1}du\).

Applying Lemma 8 to \(\tilde{\theta} + \text{pr}^*\tilde{D}u_n\) we conclude that \(\alpha = \alpha[a_n] \rightarrow \alpha[a]\), so that \(\alpha[a] = \alpha\). By Lemma 10, \(c^k[a] = c^k\). This completes the proof.

**Corollary 2** Each sector \(S^G_{\alpha; c_1,\ldots,c_N}(M)\) of the Skyrme maps contains a minimizer.

**Proof.** By Remark 6, the map \(\mathcal{D}\) maps \(S^G_{\alpha; c_1,\ldots,c_N}(M)\) into \(S^{\theta,G}_{1,\alpha; c_1,\ldots,c_N}[M]\). It is surjective by Lemma 5. Also, \(\mathcal{D}\) preserves energy, i.e., \(E(u) = E[\mathcal{D}u]\). Any map \(u\) in the inverse image of a minimizer in \(S^{\theta,G}_{1,\alpha; c_1,\ldots,c_N}[M]\) minimizes \(E(\cdot)\). End of proof.

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