THE BURGERS EQUATIONS AND THE BORN RULE

DIMITER PRODANOV

Abstract. The present work demonstrates the connections between the Burgers, diffusion, and Schrödinger’s equations. The starting point is a formulation of the stochastic mechanics, which is modelled along the lines of the scale relativity theory. The resulting statistical description obeys the Fokker-Planck equation. This paper further demonstrates the connection between the two approaches, embodied by the study of the Burgers equation, which from this perspective appears as a stochastic geodesic equation. The main result of the article is the transparent derivation of the Born rule from the starting point of a complex stochastic process, based on a complex Fokker-Planck formalism.

keywords: Burgers equation; Schroedinger equation; diffusion; stochastic mechanics; scale relativity

1. Introduction

The present paper reveals deep connections between the Burgers equation and the Born’s rule in quantum mechanics. The Born rule assigns a probability to any possible outcome of a quantum measurement. It asserts that the probability, associated with an experimental outcome is equal to the squared modulus of the wave function [9, Ch. 2]. The Born rule completes the Copenhagen interpretation of quantum mechanics. On the other hand, it leaves open the question on how these probabilities are to be interpreted. Some authors managed to derive the rule using the machinery of Hilbert spaces (i.e. the Gleason theorem [16]), while others resorted to operational approaches [38, 25]. However, using such abstractions, it may be difficult to distinguish epistemological from ontological aspects of the underlying physics. The Born’s rule was also implicitly demonstrated in the scope of scale relativity theory [29], which was the inspiration of the present work.

The Burgers’ equation was initially formulated by Bateman while modelling the weakly viscous liquid motion [19]. The equation can be derived from the full Navier-Stocks equations of fluid dynamics under some simplifying assumptions. The equation was later studied extensively by Burgers as a cartoon model of turbulence as an attempt for a simplified mean field theory of turbulence [2]. The Burgers’ equation reads

$$\partial_t a + a \partial_x a - \nu \partial_{xx} a = 0 \quad (1)$$

In the initial applications, it was assumed that the viscosity parameter is real, however the equation can be analytically continued for complex values of \( \nu \). The present paper uses all three manifestations of the viscosity coefficient – positive, negative and imaginary – hence the plural form in its title.

There is abundant mathematical literature about the Burgers equation. The one-dimensional solutions are surveyed in [1], while the similarity solutions have been investigated in [31]. At present, the number of applications of the Burgers’ equation is very diverse. The equation has numerous applications in modelling of a
wide variety of physical processes. It has been used to model physical systems, such as surface perturbations, acoustic waves, electromagnetic waves, density waves, or traffic (see for example [18]). The stochastic representation of the Burgers’ equation can be traced back to the seminal works of Busnello et al. [3, 4].

On the other hand, the Burgers’ equation can be introduced from the drastically different perspective. It represents the equation of the drift function of a Brownian diffusion, considered as a stochastic process. This can be rigorously demonstrated using the apparatus of Itô calculus. Recent literature employing this perspective includes [7, 12], however, the focus there was on the inviscid Burgers equation. Einik and Drivas show that the entropy solutions of Burgers have Markov processes of Lagrangian trajectories backward in time for which the Burgers velocity is a backward martingale. A more general approach for the viscous Navier-Stokes equations was introduced by Constantin and Iyer who derived a probabilistic representation of the deterministic 3-dimensional Navier-Stokes equations [7, 8].

The Burgers’ equation also arises in the theory of the Langevin equation as the resulting equation for the velocity of a particle subject to random forces. It is in this latter aspect that the equation can be linked to the theory of quantum mechanics and the Schrödinger equation. The connection arises in two ways – by Nelson’s stochastic mechanics and by Nottale’s scale relativity theory. The two theories are complementary to each other. In scale relativity any material particle is assumed to follow fractal geodesic path, while the stochastic mechanics assumes that particles follow a conservative Brownian motion [5]. Such Brownian motion can be used the model fractal trajectories in terms of their stochastic Markov process embedding [34].

It is noteworthy that the Burgers equation can be linearised exactly by means of the Cole-Hopf transformation, which brings the non-linear equation into one-to-one correspondence with the diffusion equation. From this perspective, the findings of the above works are not surprising. It is also remarkable that, if the viscosity coefficient becomes imaginary, the Cole-Hopf transformation produces the Schrödinger equation. This is not only a formal similarity as the complex viscosity coefficient can be endowed with a precise meaning as will be demonstrated in the present work.

What is not widely recognized is that the Cole-Hopf transformation can be extended into multiple dimensions. The objective of the present paper is to use such non-homogenous Cole-Hopf transform in order to investigate some aspects of the Burgers’ equation in view of multidimensional linear equations, notably the Schrödinger’s equation. This is achieved in the general manner using the techniques of the Geometric algebra and Itô calculus.

Moreover, assuming reversibility of the diffusion, a complex structure can be imposed over the process, which naturally leads to the Schrödinger equation and its conjugate as exact linearizations of the corresponding Burgers equations with imaginary viscosity. This may seem as an arbitrary choice, however it is not so, since introducing such complex structure leads naturally to the Born’s rule for the interpretation of the squared modulus of the Schrödinger’s ψ function as the probability density, appearing in the Fokker-Planck equation of the initial pair of diffusion processes.

As applied to quantum mechanics, it is demonstrated here that the Born’s rule is not arbitrary but stems from the microscopic reversibility in a stochastic sense.
To the best of this author’s knowledge, such derivation has not been demonstrated in literature.

Derivation of the results presented in this manuscript is facilitated by the language of the Geometric Algebra, which allows for a straightforward verification in computer algebra systems [37].

2. Stochastic mechanics

The equations of stochastic mechanics were formulated initially by Fényes [13] and Weizel [39] and later developed by Nelson [26] towards a comprehensive interpretation of quantum mechanics. The stochastic mechanics draws on the formal similarities between the classical statistical mechanics and the Schrödinger equation. In the treatment of the stochastic mechanics, quantum phenomena are described in terms of Brownian motions instead of wave functions.

The main equation of motion is in fact the Langevin equation employing a Wiener driving process, which can be handled by the apparatus of Itō calculus.

Consider the stochastic integral equation with continuous drift and diffusion coefficients

\[ X_t - x_0 = \int_0^t a(X, t) \, dt + \int_0^t b(X, t) \, dW_t \]

where \( a(X, t) \) and \( b(X, t) \) are smooth functions of the variables and \( dW_t \) is an increment of a Wiener process \( dW_t \sim N(0, dt) \) adapted to the past filtration \( F_{t>0} \) — i.e. starting from the initial state \( t = 0 \) and \( x_0 \) is the deterministic initial condition.

The requirement of a filtration is essential for the development of the stochastic calculus. This means that, in a way, one is recording the response of the system at a pre-defined but infinite sequence of intervals in the past. The differential form of this equation is the Langevin equation

\[ dX_t^+ = a(X, t) \, dt + b(X, t) \, dW_t \]

called also stochastic differential equation in the mathematical literature. The superscript indicates adaptation to the past filtration. In most of the derivations in the subsequent sections the dependencies of the \( a \) and \( b \) parameters will be assumed but not denoted explicitly.

The drift and the diffusion fields (e.g. coefficient) can be calculated in the following way. Following Nelson the forward and backward drift, respectively diffusion coefficients, can be identified as the ensemble-averaged velocities [26, 17]. Therefore, one can define a pair of differential operators (e.g. directional derivatives) in the mean sense [27]:

**Definition 1** (Mean velocities).

\[ D_t^+ X := \lim_{dt \to 0} \mathbb{E} \left( \frac{X_{t+dt} - X_t}{dt} \bigg| X_t = x \right), \quad D_t^- X := \lim_{dt \to 0} \mathbb{E} \left( \frac{X_t - X_{t-dt}}{dt} \bigg| X_t = x \right) \]

Defined in this way, it is not necessary to resort to the techniques of non-standard analysis as initially explored by Nelson. In this way,

\[ a = D_t^+ X_t^+ \]

In a similar way, the diffusion coefficient can be rigorously interpreted as the expectation of the fractional velocity [30, 34]:

\[ |b| = \lim_{dt \to 0} \mathbb{E} \left( \frac{\left| X_{t+dt} - X_t \right|}{\sqrt{dt}} \bigg| X_t = x \right) \]
The evolution of the probability density of the stochastic process can be computed from the forward Fokker-Planck equation \[ \frac{\partial_t \rho}{\partial t} + \frac{\partial_x (a \rho)}{\partial x} - \frac{1}{2} \frac{\partial_{xx} (b^2 \rho)}{\partial x} = 0 \] (2) which can be recognized as a conservation law

\[ \frac{\partial_t \rho}{\partial t} + \frac{\partial_x j}{\partial x} = 0 \]

for the probability current \( j := a \rho - \frac{1}{2} \frac{\partial_x b^2 \rho}{\partial x} \). One can define also a backwards process in the sense of the integral equation

\[ x_T - x_t = \int_t^T \hat{a}(X, t) dt + \int_t^T \hat{b}(X, t) d\hat{W}_t \]

which is adapted to the future filtration \( \mathcal{F}_{t<T} \) – i.e. starting from the final state – and \( x_T \) is the deterministic final condition. The backwards diffusion process leads to the anticipative (i.e. anti-Itô) stochastic integrals. It should be noted that the anticipative stochastic integrals are, in a sense, dual to the more common Itô integrals. The differential form can be written in a similar way as

\[ dX_t^- = \hat{a}(X, t) dt + \hat{b}(X, t) d\hat{W}_t \]

Then, in a similar way

\[ D^- X_t^- = \hat{a} \]

Note that in general, \( a \) and \( \hat{a} \) are different velocity fields!

Another result will be important for the subsequent presentation. According to Föllmer [14]:

**Proposition 1.** Suppose that \( b = \hat{b} \) and

\[ E \left( \int_0^T a(X, t)^2 dt \right) < \infty \]

Then the backwards diffusion process has the same density \( \rho \).

Under this condition, the backwards process has the Fokker-Planck equation

\[ \frac{\partial_t \rho}{\partial t} + \frac{\partial_x (\hat{a} \rho)}{\partial x} + \frac{1}{2} \frac{\partial_{xx} (b^2 \rho)}{\partial x} = 0 \] (3)

### 2.1. Velocity fields.

Given this background, the Nelson’s osmotic velocity can be defined from

\[ a - \hat{a} = b^2 \partial_x \log b^2 \rho + \phi(t) \]

where \( \phi(t) \) is an arbitrary \( C^1 \) function of time as

\[ u := \frac{1}{2} (a - \hat{a}) = \frac{b^2}{2} \partial_x \log b^2 \rho \]

and the current velocity as

\[ v := \frac{1}{2} (a + \hat{a}) \]

so that a continuity equation holds for the density

\[ \frac{\partial_t \rho}{\partial t} + \frac{\partial_x (v \rho)}{\partial x} = 0 \]

In order to derive the Schrödinger equation, Nelson’s theory posits a special form of the acceleration without further physical argumentation [27]. This can be considered as a drawback of the original theory.
3. Scale relativity

The nature of the random Wiener process described in the previous section could look mysterious and contrived. This is not so. An intuitive rationale is given by the Scale relativity theory of L. Nottale. The main tenet of the scale relativity theory is that there is no preferred scale of description of the physical reality. Therefore, a physical phenomenon must be described simultaneously at all admissible scales. This lead Nottale to postulating some kind of fractal character of the underlying mathematical variety (i.e. a pseudo-manifold) describing the observables. The theory of such varieties is still underdeveloped, therefore Nottale’s argument should be taken only as analogy. Nottale further posits that the fractal driving process can be approximated in stochastic sense using a Markov process. While Nottale presents a heuristic argument and claims that the prescription of a Wiener process may be generalized he does not proceed to rigorously develop the argument. A rigorous treatment supporting this claim was presented in [34].

On the other hand, the stochastic mechanics fixes from the start the Wiener process as the driving noise. The question of why the Wiener process takes central stage must be addressed further. The answer to this question can be given more easily by an approach inspired by Nottale and is partially given by the argument presented by Gillespie [15]. The original formulation in [15] contains an explicit assumption of existence of the second moment of the distribution, which amounts to assuming Hölder continuity of order $1/2$ as demonstrated in [34].

The scale relativistic approach results in corrections of the Hamiltonian mechanics that arise due to the non-differentiability of paths. Nottale introduces a complex operator differential operator, that he calls the scale derivative. The velocity in scale relativity is not interpreted as a mathematical derivative but as finite difference. Therefore, from mathematical point of view, the fundamental quantities should be treated as asymptotics. The non-differentiability leads to introduction of two velocity fields: $v^+$ for the forward and $v^-$ for the backward velocity. This double field can be embedded in a complex space. Following Nottale [28], the pair of velocity fields is represented by a single complex-valued vector field as

$$v = V - iU$$

with components given by $V := \frac{1}{2} (v^+ + v^-), U := \frac{1}{2} (v^+ - v^-)$ where $V$ is interpreted as the "classical" velocity and $U$ is a new quasi-velocity quantity (i.e. the osmotic velocity in the terminology of Nelson). Such representation will be called complex lifting. Under this lifting Nottale introduces a complex material derivative, which is a pseudo-differential operator acting on scalar functions as

$$D F = \partial_t F + v \cdot \nabla F - i\sigma^2 \nabla^2 F$$

where $\sigma$ is a constant, quantifying the effect of changing the resolution scale. Using this tool, Nottale gives a heuristic derivation of the Schrödinger equation from the classical Newtonian equation of dynamics.

On the other hand, a different embedding choice is also possible

$$u = V + iU$$

resulting in the complex-conjugated differential operator

$$D^* F = \partial_t F + u \cdot \nabla F + i\sigma^2 \nabla^2 F$$
4. The Burgers equation as a stochastic geodesic equation for the velocity field

The use of the Wiener process entails the application of the fundamental Itô Lemma for the forward (i.e. adapted to the past, plus sign) or the backward processes (i.e. adapted to the future, minus sign), respectively. In differential notation it reads

\[ dF(t, X_t) = \partial_t F dt + dX_t^+ \partial_x F + \frac{1}{2} \left[ dX_t^+, dX_t^+ \right] \partial_{xx} F \] (4)

\[ dF(t, X_t) = \partial_t F dt + dX_t^- \partial_x F - \frac{1}{2} \left[ dX_t^-, dX_t^- \right] \partial_{xx} F \] (5)

where, \([dX_t^+, dX_t^+] = b^2 dt \) is the quadratic variation of the process (see for example [30, Ch. 4]).

**Remark 1.** In its essence, the Itô Lemma is just the generalized Taylor development in the \( t \) variable using the algebraical substitution rules \( dt^2 \to 0, \ dW^2 \to dt, \ dW dt \to 0 \).

The term geodesic will be interpreted as a solution of a stochastic variational problem [41, 40]. A brief treatment is given in Appendix. A.1. The stochastic variational problem reads

\[ \delta \int_0^T \left( (D_t^+ X_t^+)^2 - b^2 \right) d\tau = 0 \]

which implies \( \mathbb{E} \, da = 0 \). By application of Itô’s Lemma the forward geodesic equation can be obtained as:

\[ \partial_t a + a \partial_x a + \frac{b^2}{2} \partial_{xx} a = 0 \] (6)

This can be recognized as a Burgers equation with negative kinematic viscosity for the drift field [2].

The backward geodesic equation follows from variational problem for the anticipative process

\[ \delta \int_0^T \left( (D_t^- X_t^-)^2 + b^2 \right) d\tau = 0 \to \mathbb{E} \, d\hat{a} = 0 \]

By an application of the Itô’s lemma for the anticipative process one obtains

\[ \partial_t \hat{a} + \hat{a} \partial_x \hat{a} - \frac{b^2}{2} \partial_{xx} \hat{a} = 0 \] (7)

This can be recognized as a Burgers’ equation with positive kinematic viscosity for the drift field.

5. The real-valued ColeHopf transform

Normalization \( b = 1 \) will be assumed further in the section to simplify presentation. The Burgers equation can be linearized by the ColeHopf transformation [21, 6]. This mapping transforms the non-linear Burgers equation into the linear heat conduction equation in the following way. Let

\[ u = \partial_x \log a \]
Substitution into Eq. 6 leads to
\[ \frac{1}{2u^2} (u \partial_{xxx} u + 2u \partial_{tx} u - \partial_x u \partial_{xx} u - 2 \partial_t u \partial_x u) = 0 \]
This can be recognized as a total spatial derivative
\[ \partial_x u \left( \partial_t u + \frac{1}{2} \partial_{xx} u \right) = 0 \]
Therefore, the transformed equation is equivalent to a solution of the diffusion equation in reversed time
\[ \partial_t u + \frac{1}{2} \partial_{xx} u = 0 \]
wherever \( u \neq 0 \).

It should be noted that if instead of the forward development (i.e. prediction) one takes the backward development (i.e. retrodiction) the usual form of the Burgers equation is recovered:
\[ \partial_t \hat{a} + \hat{a} \partial_x \hat{a} - \frac{1}{2} \partial_{xx} \hat{a} = 0 \]
This corresponds to the anticipative Wiener process, which is subject to the anticipative Itô calculus \([7, 11]\).

6. The Complex Material Derivatives

For simplicity of the discussion, the section focuses on the one-dimensional case.

6.1. Complex embedding. Consider two real-valued Brownian motions
\[
\begin{align*}
    dX_t &:= adt + bdW_t \\
    d\hat{X}_t &:= \hat{a}dt + b\hat{W}_t
\end{align*}
\]
The drift, resp. diffusion coefficients can be further embedded in a complex space as proposed by Pavon [32]. Such embedding is an isomorphism:
\[
\begin{align*}
    a \otimes \hat{a} &\mapsto \mathcal{V} := v - iu \\
    dX_t \otimes d\hat{X}_t &\mapsto d\mathcal{X} = \frac{1}{2} \left( dX_t + d\hat{X}_t - \frac{1}{2} (dX_t - d\hat{X}_t) \right)
\end{align*}
\]
\[d\mathcal{X} = (v - iu) dt + \frac{b}{2} dW_t + \frac{\hat{b}}{2} d\hat{W}_t - i \left( \frac{b}{2} dW_t + \frac{\hat{b}}{2} d\hat{W}_t \right) =
\]
\[
\mathcal{V} dt + \frac{1 - i}{2} b dW_t + \frac{1 + i}{2} \hat{b} d\hat{W}_t = \mathcal{V} dt + \sqrt{-i\sigma} \left( \frac{bdW_t + i\hat{b} d\hat{W}_t}{2\sigma} \right)
\]
where
\[
\sigma = \sqrt{\frac{b^2 + \hat{b}^2}{2}}
\]
Therefore, purely algebraically, we can designate a new complex stochastic variable
\[
dZ_t := \frac{bdW_t + i\hat{b} d\hat{W}_t}{\sqrt{2\sigma}}
\]
so that in differential form
\[
d\mathcal{X} = \mathcal{V} dt + \sqrt{-i\sigma} dZ_t \quad (8)
\]
So far the complex variable $Z_t$ is not completely specified. As an additional postulate, we assume independence of the processes. We further form the double filtration

**Definition 2** (Double filtration). Consider the interval $[0, T]$ and define the double filtration

$$F^2_t := \mathbb{F}_{t>0} \times \mathbb{F}_{t<T}$$

where the future filtration is constrained as

$$[t_1, t_2] \in \mathbb{F}_{t>0} \iff [T - t_2, T - t_1] \in \mathbb{F}_{t<T}$$

The variable $dZ_t$ is adapted to the double filtration and retains the martingale properties according to the Levy Characterization of Brownian motion. Notably, $E dZ_t = 0$. In addition,

$$dZ^2_t = \frac{\hat{b}^2 - b^2}{\hat{b}^2 + b^2} dt$$

Moreover,

$$dZ_t dZ^*_t = \frac{\hat{b}^2 dW^2_t + \hat{b}^2 d\hat{W}^2_t}{\hat{b}^2 + b^2} = dt$$

Therefore, a complex quadratic variation process can be introduced as a lift

$$[dZ_t, dZ_t] := \frac{1}{2} (dZ_t - i dZ^*_t)^2$$

and extended by linearity so that

$$[dZ_t, dZ_t] = -i dZ_t dZ^*_t$$

We further specialize the argument by assuming that $\hat{b} = b(T - t)$ for the stopping time $T$. Then, since $b$ is constant, we immediately obtain $dZ^2_t = 0$.

6.2. **The complex Itô-Nottale Lemma.** The next derivations follow the technique introduced by Pavon [33]. Adding and subtracting equations 4 and 5 gives

$$2dF = 2\partial_t F dt + (dX^+_t + dX^-_t) \partial_x F + \frac{1}{2} [dX^+_t, dX^+_t] \partial_{xx} F - \frac{1}{2} [dX^-_t, dX^-_t] \partial_{xx} F = (dX^+_t + dX^-_t) \partial_x F = 2\partial_t F dt + 2v dt \partial_x F$$

and

$$0 = (dX^+_t - dX^-_t) \partial_x F + \frac{1}{2} [dX^+_t, dX^+_t] \partial_{xx} F + \frac{1}{2} [dX^-_t, dX^-_t] \partial_{xx} F = 2udt \partial_x F + b^2 dt \partial_{xx} F$$

Therefore, in components one can write

$$dF = \left( \partial_t F + \sqrt{-ib} \partial_x F \right) dt + \sqrt{-ib} \partial_x F dZ_t$$

(9)

Therefore, a complex lifted Itô-Nottale differential can be introduced in exactly the same way

$$dF := \partial_t F dt + dX \partial_x F + \frac{1}{2} [dX, dX] \partial_{xx} F$$

(10)

with quadratic variation $[dX, dX] = -i dZ_t dZ^*_t = -ib^2 dt$. 
It should be noted that the complex embedding is not unique. An alternative complex embedding is given by
\[ a \otimes \hat{a} \mapsto U := v + iu \]
\[ dX_t \otimes d\hat{X}_t \mapsto d\mathcal{X} = \frac{1}{2} (X_{t+dt} + X_{t-dt}) + i \frac{1}{2} (X_{t+dt} - X_{t-dt}) \]
with the quadratic variation is \([d\mathcal{X},d\mathcal{X}] = -ib^2 dt\). In this case, the quadratic variation can be defined as
\[ [dZ_t,dZ_t]^* = \frac{1}{2}(dZ_t^* + idZ_t)^2 \]
implying, \([d\mathcal{X},d\mathcal{X}] = idZ_t dZ_t^* = ib^2 dt\).

The same application as above gives the Itô equation for the drift
\[ dG = \left( \partial_t G + U \partial_x G + \frac{ib^2}{2} \partial_{xx} G \right) dt + \sqrt{-ib} \partial_x G dZ_t \quad (11) \]
Moreover,
\[ dG^* = \left( \partial_t G^* + U^* \partial_x G^* - \frac{ib^2}{2} \partial_{xx} G^* \right) dt + \sqrt{-ib} \partial_x G^* dZ_t^* \quad (12) \]
which, can be recognized as the forward Itô equation. Therefore, the equations are dual by complex conjugation.

7. **The Complex Cole-Hopf Transform**

The stochastic geodesic equation can be introduced in the complex setting as well. In this case, the geodesic equation reads
\[ \mathbb{E} d\mathcal{V} = 0 \]
and can be derived from the variational problem
\[ \delta \int_0^T (D_\tau \mathcal{X})^2 d\tau = 0 \]
Note that in this case no regularization of the drift is necessary. This is so because for a constant diffusion coefficient the quadratic variation vanishes: \(dZ_t^2 = 0\).

In the complex case, starting from the generalized Itô differential, the complex velocity field becomes
\[ d\mathcal{V} = \left( \partial_t \mathcal{V} + \mathcal{V} \partial_x \mathcal{V} - \frac{ib^2}{2} \partial_{xx} \mathcal{V} \right) dt + \sqrt{-ib} \partial_x \mathcal{V} dZ_t \]
Therefore, the geodesic equation reads
\[ \partial_t \mathcal{V} + \mathcal{V} \partial_x \mathcal{V} - \frac{ib^2}{2} \partial_{xx} \mathcal{V} = 0 \]
which can be recognized as a generalized Burgers’ equation with imaginary kinematic viscosity coefficient. Applying the complex ColeHopf transformation as \(22\)
\[ \mathcal{V} = -i \partial_x \log U, \quad -\pi < \arg U < \pi \]
and specializing to \(b = 1\) leads to the equation
\[ -\partial_x \frac{1}{U} \left( i \partial_t U + \frac{1}{2} \partial_{xx} U \right) = 0 \]
which can be recognized as a gradient. The last equation is equivalent to the solution of the free Schrödinger equation. On the other hand, the diffusion part is simply

\[-\sqrt{i} (\partial_{xx} \log U) dZ_t = -\sqrt{i} \left( \partial_x \frac{1}{U} \partial_x U \right)\]

since \(-i\sqrt{-i} = -\sqrt{i}\).

Remark 2. The above calculations can be reproduced using the computer algebra system Maxima [35].

Having demonstrated the solution technique, it is instructive to investigate multidimensional generalizations of the Burgers equation and the Cole-Hopf transform.

8. Geometric algebra

In the following section we use the convention of denoting the scalars with Greek letters, the vectors with lowercase Latin letters and multivectors or blades with capital Latin letters. The Euclidean geometric algebra \(G^3(\mathbb{R})\) is generated by the set of 3 orthonormal basis vectors \(E = \{e_1, e_2, e_3\}\) for which the so-called geometric product is defined with properties

\[e_1e_1 = e_2e_2 = e_3e_3 = 1\]

\[e_i e_j = -e_j e_i, \quad i \neq j\]  

(13) \hspace{1cm} (14)

An overview of the topic can be found, for example in the book [23]. The geometric product of two vectors can be decomposed into a symmetrical scalar product and an antisymmetrical wedge product:

\[ab = a \ast b + a \wedge b, \quad a \ast b = b \ast a, \quad a \wedge b = -b \wedge a\]

The scalar product is defined simply as the scalar part of the geometric product between multivectors:

\[A \ast B = \langle AB \rangle_0\]

where the notation \(\langle \rangle_k\) the part of the multivector sum of grade \(k\). Furthermore, the wedge product is extended for blades (products of basis vectors) as

\[a \wedge A_k = \frac{1}{2} (a A_k + (-1)^k A_k a)\]

The Hestenes contraction is a symmetrical operation defined for multivectors of grades \(r\) and \(l\), respectively, as

\[A_r \cdot B_l := \sum_{|r-s|>0} \langle AB \rangle_{|r-s|}\]

while for scalars \(\alpha \cdot A = 0\). Therefore, for vectors

\[a \cdot b = a \ast b\]

It is noteworthy also that for a vector and a blade [20 Ch. 1, p.3]

\[a \cdot A_k = \frac{1}{2} (a A_k - (-1)^k A_k a)\]

which extends the geometric product decomposition also to the products of vectors and blades:

\[aA = a \cdot A + a \wedge A\]
It should be noted that unlike the scalar product the contraction operation is not associative in the general case.

In the most general setting the geometric algebra is a subset of the Clifford algebra $\mathbb{C}_{p,q}$. What is remarkable is the Clifford algebra embedding theorem, which states that the Euclidean geometric algebra is isomorphic to the even part of the Space-Time Algebra $\mathbb{C}_{1,3}^+$:

$$\mathbb{G}^3 \cong \mathbb{C}_{1,3}^+$$

Therefore, all statements concerning Euclidean vectors can be translated into statements about space-time bivectors and vice versa. This allows for an immediate generalization of the theorems of vector analysis using similar notation.

8.1. **Geometric calculus.** The geometric derivative subsumes divergence, curl and gradient operations of the vector calculus. Introduction on the topic can be found in [24]. It is defined in the simplest way as

$$\nabla f = e^j \partial_{x_j} f$$

where $e^j$ are the components of the dual or reciprocal basis, such that

$$x = x_i e^j = x^j e_i$$

for an arbitrary vector $x$. For the dual basis $e^i \ast x = x_i$ since $e^i e_j = e^i \ast e_j = \delta_{i,j}$, where the last symbol is the Kronecker’s symbol.

The geometric derivative is co-ordinate independent. Moreover, it splits into a grade-lowering and grade-increasing parts

$$\nabla f = \nabla \cdot f + \nabla \wedge f$$

The dot represents the Hestenes contraction operation (see discussion in [10]).

8.2. **The stretched gradient operator.**

**Definition 3.** The stretched gradient operator $\mathcal{C}(\nabla)$ is the linear operator acting on the gradient by anisotropically scaling the reciprocal basis vectors along the vector $c$

$$\mathcal{C} : e^k \mapsto c_k e^k$$

So that

$$\mathcal{C}(\nabla) = (c \cdot e^k) e^k \partial_{x_k}$$

(15)

Then it is clear that the stretched gradient commutes with the gradient and the time derivative

$$\partial_t \mathcal{C}(\nabla) = \mathcal{C}(\nabla) \partial_t$$

and

$$\nabla^2 \mathcal{C}(\nabla) = \mathcal{C}(\nabla) \nabla^2$$

under the assumption that $c$ is spatially constant. Also, in components

$$(\mathcal{C}(\nabla) F) \cdot \nabla = c_i \partial_{x_i} F \partial_{x_i}$$

for a scalar function $F$. Furthermore, for a spatially constant scaling $c$

$$\mathcal{C}(\nabla) \mathcal{C}(\nabla) = c_i^2 \partial_{x_i}^2$$

**Proposition 2.**

$$\mathcal{C}(\nabla) F \cdot \nabla = (c \cdot e^i) \partial_{x_i} F \partial_{x_i} = \partial_{x_i} F (c \cdot e^i) \partial_{x_i} = \nabla F \cdot \mathcal{C}(\nabla)$$

for a scalar function $F$ and spatially constant $c$. 
On the other hand, the following identity holds true

**Proposition 3.**

\[(\mathcal{C}(\nabla) F \cdot \nabla) \mathcal{C}(\nabla) F = \mathcal{C}(\nabla) F \cdot (\nabla \mathcal{C}(\nabla) F) = \frac{1}{2} \nabla (\mathcal{C}(\nabla) F)^2\]

for a scalar function \(F\).

9. The vectorized Cole-Hopf transform

Consider the complex, stochastic Itô-Nottale process

\[dX = \mathcal{V} dt + dZ_t\]  \hspace{1cm} (16)

where now \(dZ_t = e_i dZ^i_t\) is also Clifford vector-valued. Using the apparatus of Geometric algebra, the complex Itô differential of Eq. (9) generalizes to

\[dF = (\partial_t F + (\mathcal{V} \cdot \nabla) F - \frac{ib^2}{2} \nabla^2 F) dt + \sqrt{-ib} (dZ_t \cdot \nabla) F\]  \hspace{1cm} (17)

in \(\mathbb{C}\ell_{p,q}\) over the complex numbers \(\mathbb{C}\) for a smooth function \(F\). A sketch of the proof is provided in the remark below.

**Remark 3.** The restrictions on the validity of the formula above are the assumptions that the diffusion coefficient must be a scalar (i.e. homogeneity of space) and the co-ordinate processes \(dZ^i_t\) are uncorrelated. In the multidimensional case, the Taylor development of \(F\) in the direction of the process \(dX\) is

\[dF = \partial_t F dt + (dX \cdot \nabla) F + \frac{1}{2} (dX \cdot \nabla)^2 F + o\left(dX^2 + dt + dX dt\right)\]

On the other hand, in matrix notation

\[(dX \cdot \nabla)^2 F = \mathbf{d}X \cdot \mathbf{H}(F) \cdot \mathbf{d}X^T\]

where \(\mathbf{H}(F) = \{h_{ij} := \partial_{x_i} \partial_{x_j} F\}\) is the Hessian matrix, which is the usual statement of the Multidimensional Itô lemma \[30\]. Evaluating for the Wiener process \(dx^i = dW^i_t\) and using the algebraical rules \(dW^i_t dW^j_t \rightarrow 0\) (independence), \(dW^i_t dt \rightarrow 0\) and \(dW^i_t dW^j_t \rightarrow b^2 dt\) we obtain

\[\mathbf{d}X \cdot \mathbf{H}(F) \cdot \mathbf{d}X^T = b^2 dt \nabla^2 F\]

in Geometric Algebra language.

**Theorem 1** (Cole-Hopf linearization). The complex geodesic equation

\[\partial_t \mathcal{V} + (\mathcal{V} \cdot \nabla) \mathcal{V} - \frac{ib^2}{2} \nabla^2 \mathcal{V} = -\nabla U\]

is linearised by the Cole-Hopf transform

\[\mathcal{V} = -ib^2 \nabla \log F\]

into the Schrödinger-type equation

\[ib^2 \partial_t F = -\frac{b^4}{2} \nabla^2 F + UF\]  \hspace{1cm} (18)
**Proof.** Under so-identified assumptions, the drift equation becomes
\[
d\mathcal{V} = \left( \partial_t \mathcal{V} + (\mathcal{V} \cdot \nabla) \mathcal{V} - \frac{ib^2}{2} \nabla^2 \mathcal{V} \right) dt + \sqrt{-ib} \left( dZ_t \cdot \nabla \right) \mathcal{V}
\]
Therefore, the geodesic equation reads
\[
\partial_t \mathcal{V} + (\mathcal{V} \cdot \nabla) \mathcal{V} - \frac{ib^2}{2} \nabla^2 \mathcal{V} = 0
\]
Under the separate assumption of irrotational flow \( \nabla \times \mathcal{V} = 0 \), a generalized, inhomogeneous Cole-Hopf transform can be introduced by analogy with the scalar case as
\[
\mathcal{V} = C(\nabla) \log F = \frac{C(\nabla) F}{F}
\]
using the stretched gradient. If \( F = 0 \) then trivially \( dF = 0 \). Without loss of generality, assume \( F > 0 \). Under the above substitution using Props. 2 and 3
\[
\partial_t C(\nabla) \log F + (C(\nabla) \log F \cdot \nabla) C(\nabla) \log F - \frac{ib^2}{2} \nabla^2 C(\nabla) \log F = C(\nabla) \partial_t \log F + \frac{1}{2} \nabla (C(\nabla) \log F)^2 - \frac{ib^2}{2} C(\nabla) \nabla^2 \log F = 0
\]
On the other hand,
\[
\nabla^2 \log F = \frac{\nabla^2 F}{F} - \frac{\left( \nabla F \right)^2}{F^2}
\]
Therefore,
\[
C(\nabla) \nabla^2 \log F = C(\nabla) \frac{\nabla^2 F}{F} - C(\nabla) \frac{\left( \nabla F \right)^2}{F^2} = C(\nabla) \frac{\nabla^2 F}{F} - C(\nabla) \left( \nabla \log F \right)^2
\]
Therefore,
\[
\frac{1}{2} \nabla \left( C(\nabla) \log F \right)^2 - \frac{ib^2}{2} C(\nabla) \nabla^2 \log F = \frac{1}{2} \nabla \left( C(\nabla) \log F \right)^2 - \frac{ib^2}{2} C(\nabla) \frac{\nabla^2 F}{F} + \frac{ib^2}{2} C(\nabla) \left( \nabla \log F \right)^2
\]
Finally, we obtain
\[
C(\nabla) \left( \partial_t \log F - \frac{ib^2}{2} \frac{\nabla^2 F}{F} \right) + \frac{1}{2} \left( \nabla \left( C(\nabla) \log F \right)^2 + ib^2 C(\nabla) \left( \nabla \log F \right)^2 \right) = 0
\]
Therefore, for an exact linearisation, the following equation must hold
\[
\frac{1}{2} \left( \nabla \left( C(\nabla) \log F \right)^2 + ib^2 C(\nabla) \left( \nabla \log F \right)^2 \right) = 0
\]
Therefore, one obtains an algebraic system of equations for the coefficients of the stretched gradient in function of the diffusion constant. Let \( \log F = u \), so that in components the equation reads
\[
u = \log F, \quad e^i \left( \partial_x c_i \left( \partial_x u \right)^2 + ib^2 c_i \partial_x \left( \partial_x u \right)^2 \right) = e^i \left( c_i^2 + ib^2 c_i \right) \partial_x \left( \partial_x u \right)^2 = 0
\]
Therefore, the scaling is homogenous so the coefficient can be relabeled as \( c_i \equiv \lambda \) and
\[
\lambda^2 + ib^2 \lambda = 0
\]
Therefore, $\lambda = -ib^2$. Direct calculation verifies the identity:

$$
\nabla (C(\nabla) \log F)^2 + i b^2 C(\nabla) (\nabla \log F)^2 = (-ib^2)^2 (\nabla (\nabla \log F))^2 + i b^2 (-ib^2) (\nabla (\nabla \log F))^2
$$

\[= ((-ib^2)^2 + ib^2(-ib^2)) \nabla (\nabla \log F)^2 = b^4 \left((-i)^2 - i^2\right) \nabla (\nabla \log F)^2 = 0\]

for a real constant scalar $b$. Therefore, exact linearisation is possible and

$$
C(\nabla) \left( \partial_t \log F - \frac{i b^2 \nabla^2 F}{2} \right) = -ib^2 \nabla \left( \partial_t \log F - \frac{i b^2 \nabla^2 F}{2} \right) = \nabla \left( \frac{1}{F} \left(-ib^2 \partial_t F - \frac{b^4}{2} \nabla^2 F\right) \right) = 0
$$

Nothing in the present derivation depends on the properties of the right-hand side (RHS) of the equation. The left-hand side can be equated to a potential gradient from the RHS $-\nabla U$, representing physically a central force. Therefore,

$$
\nabla \left( \frac{1}{F} \left(-ib^2 \partial_t F - \frac{b^4}{2} \nabla^2 F\right) \right) = -\nabla U
$$

So that

$$
ib^2 \partial_t F = -\frac{b^4}{2} \nabla^2 F + UF
$$

and we recognize the form of the Schrödinger equation. \hfill \Box

**Corollary 1.** In the geodesic setting, the drift equation reads

$$
d\mathcal{V} = \sqrt{-ib} (dZ_t \cdot \nabla) \mathcal{V}
$$

Under so-identified Cole-Hopf transform the diffusion term transforms as

$$
\sqrt{-ib} dZ_t C(\nabla) \log F = -i \sqrt{-ib^3} (dZ_t \cdot \nabla) \nabla \log F = -i \sqrt{-ib^3} \nabla (dZ_t \cdot (\nabla \log F))
$$

9.1. **The Conjugated Shrödinger Equation.** The conjugated Shrödinger equation can be derived in the same way.

**Theorem 2.** The complex geodesic equation

$$
\partial_t \mathcal{U} + (\mathcal{U} \cdot \nabla) \mathcal{V} + \frac{ib^2}{2} \nabla^2 \mathcal{U} = -\nabla \mathcal{U}
$$

can be linearised into

$$
ib^2 \partial_t G - \frac{b^4}{2} \nabla^2 G + UG = 0 \quad (19)
$$

where

$$
G = F^*, \quad \mathcal{U} = ib^2 \nabla \log G
$$

**Proof.** Starting from the Itô formula

$$
dF = \left( \partial_t F + (\mathcal{U} \cdot \nabla) F + \frac{ib^2}{2} \nabla^2 F \right) dt + \sqrt{ib} (dZ_t \cdot \nabla) F
$$

The drift equation becomes

$$
d\mathcal{U} = \left( \partial_t \mathcal{U} + (\mathcal{U} \cdot \nabla) \mathcal{V} + \frac{ib^2}{2} \nabla^2 \mathcal{U} \right) dt + \sqrt{ib} (dZ_t \cdot \nabla) \mathcal{U}
$$

Therefore, the geodesic equation reads

$$
\partial_t \mathcal{U} + (\mathcal{U} \cdot \nabla) \mathcal{U} + \frac{ib^2}{2} \nabla^2 \mathcal{U} = 0
$$
Therefore, using the procedure as above we obtain the linearisation condition
\[ \lambda^2 - ib^2 \lambda = 0, \quad c_i = \lambda \]
for the transform
\[ U = ib^2 \nabla \log G \]
Finally,
\[ \nabla \left( \frac{1}{G} \left( ib^2 \partial_t G - \frac{b^4}{2} \nabla^2 G \right) \right) = - \nabla U \]
and
\[ ib^2 \partial_t G - \frac{b^4}{2} \nabla^2 G + UG = 0 \]
Therefore, we can identify \( G = F^* \). \( \square \)

10. **The Complex Fokker-Planck equation implies the Born rule**

The key to the subsequent derivation is the use the Schrödinger equation and its conjugate on equal grounds. A complex Fokker-Planck equation can be introduced based on the reversibility of the process.

**Theorem 3** (Complex Fokker-Planck equation). *The pair of Fokker-Planck equations for the real-valued processes*
\[ \partial_t \rho + \nabla (a \rho) - \frac{1}{2} \nabla^2 (b^2 \rho) = 0 \] \[ \partial_t \rho + \nabla (\hat{a} \rho) + \frac{1}{2} \nabla^2 (b^2 \rho) = 0 \] *implies the complex Fokker-Planck equation and its conjugate*
\[ \partial_t \rho + \nabla \cdot (\mathcal{V} \rho) + \frac{i}{2} \nabla^2 (b^2 \rho) = 0 \] \[ \partial_t \rho + \nabla \cdot (\mathcal{U} \rho) - \frac{i}{2} \nabla^2 (b^2 \rho) = 0, \quad \mathcal{U} = \mathcal{V}^* \]
*for the Itô-Nottale process.*

**Proof.** Starting from the Fokker-Planck equations for the forward and backward processes
\[ \partial_t \rho + \nabla (a \rho) - \frac{1}{2} \nabla^2 (b^2 \rho) = 0 \] \[ \partial_t \rho + \nabla (\hat{a} \rho) + \frac{1}{2} \nabla^2 (b^2 \rho) = 0 \]
and taking sums and differences we obtain
\[ 2\partial_t \rho + \nabla ((a + \hat{a}) \rho) = 2\partial_t \rho + 2\nabla (v \rho) = 0 \] \[ \nabla ((a - \hat{a}) \rho) - \nabla^2 (b^2 \rho) = 2\nabla (u \rho) - \nabla^2 (b^2 \rho) = 0 \]
Therefore, we can formulate a pair of Fokker-Planck equations for the complex velocity and its conjugate as
\[ \partial_t \rho + \nabla \cdot (\mathcal{V} \rho) + \frac{i}{2} \nabla^2 (b^2 \rho) = 0 \] \[ \partial_t \rho + \nabla \cdot (\mathcal{U} \rho) - \frac{i}{2} \nabla^2 (b^2 \rho) = 0, \quad \mathcal{U} = \mathcal{V}^* \]
Finally, the Born rule can be derived as simple consequence of the complex Fokker-Planck equations.

**Theorem 4 (Born’s rule).** Suppose that the above complex Fokker-Planck equations hold. Then

\[
\rho = FF^* 
\]

where \( F \) and \( F^* \) are solutions of the Schrödinger-type equations \( 18 \) and \( 19 \). Moreover,

\[
F = \sqrt{\rho e^{-iS}}
\]

for an analytic phase function \( S(r,t) \).

**Proof.** We use the same notation as in the proof above. Subtracting the two equations leads to

\[
\nabla (\sqrt{\nabla \log FF^* h} - \sqrt{\nabla \log FF^* h^*}) = 0
\]

As shown in the previous section, the Cole-Hopf transform can be specialized to

\[
\nabla (\nabla \log FF^* h) = 0
\]

up to an arbitrary analytic function of time \( h(t) \). Then it follows that

\[
\nabla (\nabla \log FF^* h) = 0
\]

Therefore,

\[
\nabla \log \left( \frac{FF^* h(t)h^*(t)}{\rho} \right) = 0
\]

Since, in general, \( \rho \) is a function of the position it follows that

\[
\nabla \log \left( \frac{FF^* h(t)h^*(t)}{\rho} \right) = 0
\]

must hold. Therefore, in general,

\[
\rho = FF^* f(t)
\]

where, \( f(t) \) is a smooth function of time. Since, \( h \) is arbitrary but analytic function, we can choose

\[
h(t) = e^{ig(t)}
\]

where \( g(t) \) is another smooth function. Therefore,

\[
FF^* = \rho
\]

which is the statement of the Born rule!

Therefore, one can write \( F \) in the form

\[
F = \sqrt{\rho e^{-iS}} = \psi
\]

for an analytic phase function \( S \). Consider, on the other hand, the case where

\[
\log \left( \frac{FF^* h(t)h^*(t)}{\rho} \right) = \log q(t) \Rightarrow FF^* h(t)h^*(t) = q(t)\rho
\]

for a given positive and continuous function \( q(t) \). By integration over 3D space

\[
\int_{\Omega} FF^* h(t)h^*(t)d\omega = \int_{\Omega} q(t)\rho d\omega \Rightarrow q(t) = h(t)h^*(t) \int_{\Omega} FF^* d\omega
\]
Therefore, we can transform $F$ as

$$F' = F \frac{h(t)}{\sqrt{q(t)}}$$

so that the normalization

$$\int_{\Omega} F' F'^* d\omega = 1$$

holds. Therefore, also in this case $F'$ can be interpreted in agreement with the Born’s rule.

11. Concluding Remarks

This work was motivated in part by the premise that inherently non-linear phenomena need development of novel mathematical tools for their description. The second motivation of the present work was to investigate the potential of stochastic methods to represent quantum-mechanical and convection-diffusive systems.

The augmented, in terms of white noise, Newtonian dynamic leads to the stochastic geodesic equation for the drift velocity, which can be recognized as the Burgers equation. If in addition one assumes also path-wise reversibility, this leads to a stochastic description in terms of a pair of Burgers equations. This pair can be put into correspondence with a Schrödinger equation and its conjugate for a wave-function by means of the vectorized Cole-Hopf transform. The use of the Fokker-Planck equation together with the Cole-Hopf transform leads to the Born rule for the wave function.

The complex structure, and hence, the Schrödinger equation can be considered as an ingenious and economical description of the studied phenomena, however such complex structure is not necessarily fundamental. This line of reasoning strongly points out towards the universal but epistemological (!) character of the Schrödinger equation and its unitary evolution. The Born’s rule stems from the time-reversibility of the modelled diffusion processes and does not need to be postulated separately. This corresponds with the time-reversibility of the classical physics kinematics.

Moreover, nowhere in these developments have we assumed anything particular about a "scale" of observations. Therefore, one can reasonably argue that quantum-like phenomena are not confined only to the nanoscale, but in fact can be observed as emerging phenomena on any scale of study.

Appendix A. Appendices

A.1. The Stochastic Variation Problem. The study of stochastic Lagrangian variational principles has been motivated initially by quantum mechanics and optimal control problems. This section gives only sketch for the treatment of the problem. The reader is directed to [41, 40, 32] for more details. In the simplest form this is the minimization of the regularized functional assuming a constant diffusion coefficient $b$.

Definition 4. Consider the interval $I = [a, b]$. A partition of $I$ is a set of $n$ numbers $\mathcal{P}_n[I] := (a < x_1 \ldots x_{n-1} < b)$.
Definition 5. Let $\alpha \in \{0, 1\}$. Define

$$S_\alpha(X|t_0, T) := \lim_{N \to \infty} \mathbb{E} \left( \left( \mathcal{P}_N \right) \sum_{t=t_0}^{t=T} \frac{1}{2} \left( \frac{\Delta X_k}{\Delta t_k} \right)^2 - \sigma \left( \alpha - \frac{1}{2} \right) b^2 \right| X_k = x(\alpha t_k + (1 - \alpha) t_{k+1})$$

for the sequence of partitions $\mathcal{P}_N \subset \mathcal{P}_{N+1} \in \mathbb{F}_\alpha$ and $\sigma$ denoting the sign of the argument, where $\mathbb{F}_0 = \mathbb{F}_{t > 0}$ and $\mathbb{F}_1 = \mathbb{F}_{t < T}$.

On the first place, suppose that $\alpha = 1$ and $N$ is finite. The expectation operator and the finite summation commute so

$$\mathbb{E} \left( \left( \mathcal{P}_N \right) \sum_{t=t_0}^{t=T} \frac{1}{2} \left( \frac{\Delta X_k}{\Delta t_k} \right)^2 - \sigma \left( \alpha - \frac{1}{2} \right) b^2 \right| X_k = x(\alpha t_k + (1 - \alpha) t_{k+1}) =$$

$$= \left( \mathcal{P}_N \right) \sum_{t=t_0}^{t=T} \frac{1}{2} \mathbb{E} \left( \left( \frac{\Delta X_k}{\Delta t_k} \right)^2 - \sigma \left( \alpha - \frac{1}{2} \right) b^2 \right| X_k = x(\alpha t_k + (1 - \alpha) t_{k+1})$$

Then the increments can be interpreted as Itô integrals so that by the Itô isometry since finite summation and integration commute

$$\mathbb{E} \left( \left( \frac{1}{2 \Delta t_k} (\Delta X_k)^2 - \frac{1}{2} b^2 \right| X_k = x(t_k) \right) =$$

$$= \frac{1}{2 \Delta t_k} \left( \int_{t_k}^{t_{k+1}} \text{ad}s \right)^2 + \frac{1}{\Delta t_k} \left( \int_{t_k}^{t_{k+1}} \text{ad}s \right) \mathbb{E} \left( \int_{t_k}^{t_{k+1}} b dw \right) + \frac{1}{2 \Delta t_k} \mathbb{E} \left( \int_{t_k}^{t_{k+1}} b dw \right)^2 - \frac{1}{2} b^2 =$$

by the Middle Value Theorem, where we use the Itô isometry

$$\mathbb{E} \left( \int_{t_k}^{t_{k+1}} b dw \right)^2 = \mathbb{E} \left( \int_{t_k}^{t_{k+1}} b^2 dt \right)$$

Therefore, $S_\alpha(t_0, T)$ is minimal if the drift vanishes on $\mathcal{P}_N$. Suppose that $X_t$ is varied by a small smooth function $\lambda \phi(t, x)$, where the smallness is controlled by $\lambda$, then the Itô lemma should be applied so that $\mathbb{E}(d\delta X_t | F) = 0$ on the difference process $\delta X_t = \lambda \phi(t, x) dt + bdW_t$. Therefore,

$$\mathbb{E}(d\phi | F) = \lambda dt \left( \partial_t \phi + \phi \partial_x \phi + \frac{b^2}{2} \partial_{xx} \phi \right) = 0 \quad (24)$$

should hold. The same calculation can be performed for $\alpha = 0$ if the Itô integral is replaced by the anticipative Itô integral. In this case, $\sigma = -1$ and the integration is reversed

$$\mathbb{E} \left( \left( \frac{1}{2 \Delta t_k} (\Delta X_k)^2 + \frac{1}{2} b^2 \right| X_k = x(t_{k+1}) \right) =$$

$$= \frac{1}{2 \Delta t_k} \left( \int_{t_{k+1}}^{t_k} \text{ad}s \right)^2 + \frac{1}{\Delta t_k} \left( \int_{t_{k+1}}^{t_k} \text{ad}s \right) \mathbb{E} \left( \int_{t_{k+1}}^{t_k} b dw \right) + \frac{1}{2 \Delta t_k} \mathbb{E} \left( \int_{t_{k+1}}^{t_k} b dw \right)^2 + \frac{1}{2} b^2 =$$

$$= \frac{1}{2 \Delta t_k} \left( \int_{t_{k+1}}^{t_k} \text{ad}s \right)^2 + \frac{1}{2 \Delta t_k} \int_{t_{k+1}}^{t_k} b^2 ds + \frac{1}{2} b^2 = a(t^*) \int_{t_k}^{t_{k+1}} \text{ad}s, \quad t^* \in (t_k, t_{k+1})$$
In this case also the backward Itô formula applies as
\[ \mathbb{E}(d\phi|\mathcal{F}) = \lambda dt \left( \partial_t \phi + \phi \partial_x \phi - \frac{b^2}{2} \partial_{xx} \phi \right) = 0 \] (25)

**Remark 4.** The treatment of Pavon [32] uses the symmetrized functional \( S = S_0 + S_1 \) together with a constraint on the anti-symmetrized functional \( S_0 - S_1 \) in the present notation.

The complex geodesic principle is achieved in a more parsimonious way by defining the quantity

**Definition 6.** Define
\[ \mathcal{L}(X|\tau_0, T) := \lim_{N \to \infty} \mathbb{E} \left( (\mathcal{P}_N) \sum_{t=t_0}^{t=T} \frac{1}{2} \left( \Delta \mathcal{X}_{t_k} \right)^2 \mid \mathcal{X}_k = X(t_k) \right) \]
for the sequence of partitions \( \mathcal{P}_N \subset \mathcal{P}_{N+1} \in \mathbb{F}^2 \) and the process
\[ X_t = \int_0^t V ds + \int_0^t \sqrt{-i} b dZ_s \]

Then, in a similar way, we take complex Itô integrals
\[ \mathbb{E} \left( \frac{1}{2\Delta t_k} \left( \Delta \mathcal{X}_{t_k} \right)^2 \mid \mathcal{X}_k = x(t_{k+1}) \right) = \]
\[ \frac{1}{2\Delta t_k} \mathbb{E} \left( \int_{t_{k+1}}^{t_k} V ds \right)^2 + \frac{\sqrt{-i} b}{\Delta t_k} \left( \int_{t_{k+1}}^{t_k} V ds \right) \mathbb{E} \left( \int_{t_{k+1}}^{t_k} dZ_s \right) + \frac{\sqrt{-i} b^2}{2\Delta t_k} \mathbb{E} \left( \int_{t_{k+1}}^{t_k} dZ_s \right)^2 = \]
\[ \frac{1}{2\Delta t_k} \left( \int_{t_{k+1}}^{t_k} V ds \right)^2 = V(\tau) \int_{t_{k+1}}^{t_k} V ds, \quad \tau \in (t_k, t_{k+1}) \]

where we use the result
\[ \mathbb{E} \left( \int_{t_{k+1}}^{t_k} dZ \right)^2 = \mathbb{E} \Delta Z_{t_k}^2 = \mathbb{E} Z_{t_{k+1}}^2 + \mathbb{E} Z_{t_k}^2 + 2\mathbb{E} Z_{t_k} Z_{t_{k+1}} = \mathbb{E} Z_{t_k}^2 + \mathbb{E} Z_{t_{k+1}}^2 = 0 \]
since
\[ \mathbb{E} Z_{t_k}^2 = \mathbb{E} \left( W_{t_k}^2 - \hat{W}_{t_k}^2 + iW_{t_k} \hat{W}_{t_k} \right) = \mathbb{E} \left( W_{t_k}^2 - \hat{W}_{t_k}^2 \right) + i\mathbb{E} \left( W_{t_k} \hat{W}_{t_k} \right) = 0 \]
by the independence of the Brownian motions. Therefore, the expectation of complex drift variation \( \delta X_t = \lambda \psi dt + \sqrt{-ib} dZ_t \) must vanish as well so that
\[ \mathbb{E}(d\psi|\mathcal{F}) = \lambda dt \left( \partial_t \psi + \psi \partial_x \psi - \frac{b^2}{2} \partial_{xx} \psi \right) = 0 \]

Therefore, in 3 dimensions it is immediately generalized to
\[ \mathbb{E}(d\psi|\mathcal{F}) = \lambda dt \left( \partial_t \psi + (\psi \cdot \nabla) \psi - \frac{b^2}{2} \nabla^2 \psi \right) = 0 \]

Moreover, using the notation of mean derivatives
\[ \mathcal{L}(X|0, T) = \int_0^T (D_t X)^2 dt \]
and

\[ S_{0,1}(X(0,T)) = \int_0^T (D_t^+ X^2 \mp b^2) \, dt \]

by the Fubini’s theorem. Therefore, the complex variational problem is homeomorphic to the pair of real-valued variational problems, as expected.

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*Environment, Health and Safety Department, IMEC, Leuven, Belgium, MMSIP, IICT, Bulgarian Academy of Sciences, Bulgaria*