Nearest Neighbor and Contact Distance Distribution for Binomial Point Process on Spherical Surfaces

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Abstract—This letter characterizes the statistics of the contact distance and the nearest neighbor (NN) distance for binomial point processes (BPP) spatially-distributed on spherical surfaces. We consider a setup of \( n \) concentric spheres, with each sphere \( S_k \) has a radius \( r_k \) and \( N_k \) points that are uniformly distributed on its surface. For that setup, we obtain the cumulative distribution function (CDF) of the distance to the nearest point from two types of observation points: (i) the observation point is not a part of the point process and located on a concentric sphere with a radius \( r_k < r_k \forall k \), which corresponds to the contact distance distribution, and (ii) the observation point belongs to the point process, which corresponds to the nearest-neighbor (NN) distance distribution.

Index Terms—Stochastic geometry, binomial point process, distance distribution.

I. INTRODUCTION

A. Motivation

Cellular coverage has become one of the top needs of the modern society due to its importance in various applications such as healthcare, remote education, industry, and much more. For that reason, it is important to ensure cellular coverage all over the globe including remote areas, rural regions, and many other under-served locations. However, due to the lack of infrastructure, majority of these areas receive bad coverage due to lack of incentive for network operators to invest in these locations. Recent advances in Low Earth Orbit (LEO) satellite communications are providing a promising solution to solve the coverage problem in under-served locations. Recent works, such as [1], [2], have identified technological advances and highlighted open issues in the field of satellite communications. The recent advances have motivated companies such as SpaceX to get permission to build a constellation of 4425 LEO satellites to supply low latency communication. They have a plan for setting 1600 satellites in 1150 km altitude orbits at the first stage [3].

The spatial distribution of the LEO satellite plays an essential role as it strongly affects the performance of the satellite communication systems. In this paper, we propose to model the locations of the satellites using tools from stochastic geometry. Stochastic geometry is one of the mathematical tools that enable us to model different types of wireless networks and analyze their properties [4]. We develop a new traceable approach where we model the locations of the LEO satellites as a BPP on a sphere. The developed framework is essential for studying the performance of the LEO satellite communication system. However, it is first needed to understand the fundamental characteristics of the distances emerging from this point process, which is the main contribution of this paper.

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B. Related work

The knowledge of statistics of distances between various components of the wireless networks is crucial. Relevant literature has mainly focused on Spatial point processes on a 2D plane. As an example, the letter [5] characterizes the derivation of the CDFs of contact and nearest-neighbor distances for a Thomas cluster process (TCP), which is a special case of Poisson cluster process. Even though there were works with point distribution on the sphere in the 1950s, statistical research on point processes on the sphere could date back to the 1970s, such as a study of random sets on the sphere by Milne’s [6]. Statistical methods that are developed for analyzing a distribution of points on a region of the sphere, including modeling and estimating techniques for a specified model, are studied in the recent work [7]. However, statistical methods for point processes on the sphere are still surprisingly underdeveloped since most of the current work deals only with the moments of the distances or derive the exact distribution only for precise system models.

Furthermore, the reference [8] gives expressions for the distributions in a d-dimensional homogeneous Poisson point process (PPP) and which have many implications for large wireless networks of randomly distributed nodes. Considering the factors in [9] such as global coverage, target application scenario and cost of constructing the constellation in design of zodiac, the paper [10] models LEO constellation composed a total of \( N \) satellites are distributed in \( M \) circular orbital planes at heights \( h_m \) for \( m = 1, ..., M \). The geometry of this system model resembles our system in the way of design, but we have concentric spheres instead of orbital planes. Also, they aim to solve the optimization problem using proposed matching algorithms to reduce the number of required ground stations. In our case, we study the distribution of nearest point analysis.

C. Contributions

The main contributions of this work are as follows. First, we model \( n \) concentric spheres with \( N_k \) points uniformly distributed on each sphere \( \forall k \). Then we use concepts from point process theory and stochastic geometry to provide a new tractable model for studying distribution in satellite networks on spheres. In particular, we model the location of points as a spherical BPP to study the distribution of nearest-neighbor distance for two different locations of observation point which are (i) observation point is not a part of the point process and located on the Earth, and (ii) observation point is a part of the point process and located on \( k^{th} \) sphere. The key findings are applied to study satellite network characteristics.
II. SYSTEM MODEL

As stated above, the analysis in this paper is motivated by the recent advances in the area of LEO satellite communication systems. Hence, our objective is to provide a model that captures two kinds of communication links: (i) links between gateways on the earth and LEO satellites, and (ii) inter-satellite links between LEO satellites. For the former, it is important to derive the distribution of the distance between a point on the earth and its nearest LEO satellite. For the latter, and for the sake of satellite backhaul links, it is important to derive the distribution of the distance between a given LEO satellite and its nearest neighbor. Given that current LEO satellite deployment are considering various altitudes with various number of satellites at each altitude, we consider a system composed of \( n \) concentric spheres, denoted by \( S_k \subset \mathbb{R}^3 \), \( \forall k \). On each sphere, \( N_k \) points are uniformly distributed. Each sphere is defined by the altitude \( a_k \) (height of \( k^{th} \) sphere to the surface of Earth) and the radius \( r_k = r_e + a_k \), where \( r_e \) is the radius of the earth. Hence, the considered point process is defined as \( \Phi = \bigcup_{k=1}^n \Phi_k \) on \( \bigcup_{k=1}^n S_k \). We denote its corresponding counting measure by \( N \), such that \( N(A) \) denotes the number of points in \( \Phi \) falling in the region \( A \subseteq \bigcup_{k=1}^n S_k \). For each BPP \( \Phi_k \), fixed number \( N_k \) of points are independently distributed on a sphere \( S_k \) defined as

\[
S_k \triangleq \{(r_k, \varphi, \theta) : r_k = r_e + a_k, 0 \leq \varphi \leq \pi, 0 \leq \theta < 2\pi \},
\]

where the \((r_k, \varphi, \theta)\) represents the spherical coordinate of any point on \( \mathbb{R}^3 \). The nearest neighbor or contact distance (depending on the definition of the observation point) is the distance from the observation point to the nearest point in \( \Phi \) and is given by \( D \). The corresponding distribution \( F_D(d) \triangleq P(D < d) \) is the nearest neighbor or contact distance distribution function.

A. Scenario-1 Description

The observation point is located on the Earth. The corresponding distribution is

\[
F_D(d) \triangleq P(D < d) = 1 - \prod_{k=1}^n P(D_k \geq d),
\]

where \( P(D_k \geq d) = F_{D_k}(d) = 1 - F_{D_k}(d) \) is the complementary cumulative distribution function (CCDF) of the contact distance \( D_k \) from the observation point to the nearest point on \( k^{th} \) sphere.

By definition, we know that if \( d < a_k \) then \( F_{D_k}(d) = 0 \). For \( d > a_k \), \( F_{D_k}(d) \) is the probability that the number of points on a given spherical cap \( A_k \) at height \( h(d, k, 0) \) is greater than zero:

\[
F_{D_k}(d) \triangleq P(D_k < d) = P(N(A_k) > 0).
\]

Hence, the complementary CDF of \( D_k \) can be computed as follows.

\[
P(D_k \geq d) = P(N(A_k) = 0) = [P(z_k < z(d, k, 0))]^{N_k},
\]

where \( z_k = r_k \cos \varphi \), and \( z(d, k, 0) = r_k - h(d, k, 0) \). With Pythagoras’ theorem, we can easily derive the expression of \( h(d, k, 0) \). Assuming that the communication between any point on the earth and an LEO satellite requires a Line-of-Sight (LoS), the maximum distance, \( d_{\text{max}}(k, 0) \), that can be taken from the observation point also forms a spherical cap \( A_{\text{max},k} \) with height \( h_{\text{max}}(k, 0) \). When the number of points in \( A_{\text{max},k} \) is zero, it means that there are no points in \( S_k \) that have an LoS with the observation point. Hence, for that scenario, we assume that \( D_k = \infty \). As a result, the CCDF of \( D_k \) for \( d > d_{\text{max}}(k, 0) \) is

\[
P(D_k \geq d) = P(N(A_{\text{max},k} = 0) = [P(z_k < z_{\text{max}}(k, 0))]^{N_k},
\]

where \( z_{\text{max}}(k, 0) = r_e \). Combining all the conditions which are \( d < a_k \), \( a_k < d < d_{\text{max}}(k, 0) \) and \( d > d_{\text{max}}(k, 0) \), we can derive the complete \( F_{D_k}(d) \) \( \forall k \).

B. Scenario-2 Description

The observation point is located on the \( S_k \), and the point is part of the point process. So, the corresponding distribution is

\[
F_D(d) = 1 - \prod_{k=1}^{i-1} P(D_{k,i} \geq d) [P(D_i \geq d)] \prod_{k=i+1}^{n} P(D_{k,i} \geq d),
\]

where \( D_{k,i} \) is the distance between the observation point and the nearest point on \( S_k \), and \( D_i \) is the distance between the observation point and the nearest point on the same sphere \( S_i \). Here, the complementary CDFs correspond to NN distance distribution for (a) below.
the $i^{th}$ sphere, (b) on the $i^{th}$ sphere and (c) above the $i^{th}$ sphere respectively. Fig. 2 shows the system model for the case (c). As can be seen from the figure, we have two spherical caps formed on $S_k$ with corresponding heights $h(d,k,i)$ and $h_{\text{max}}(k,i)$. We follow the same procedure as Scenario-1 to derive the complete CDF for each case where conditions are $|a_k-a_i| < d$, $|a_k-a_i| < d < d_{\text{max}}(k,i)$ and $d > d_{\text{max}}(k,i)$ for (a) and (c) and $d < d_{\text{max}}(k,i)$ and $d > d_{\text{max}}(k,i)$ for case (b). Also, we get $h(d,k,i)$, $h_{\text{max}}(k,i)$ and $d_{\text{max}}(k,i)$ for each case separately by using Pythagoras’ theorem.

### III. DISTANCE DISTRIBUTION

In this section, we determine the distribution of the nearest distance from a specified observation point for a general BPP.

**Theorem 1** (Scenario-1: Contact distance distribution),

$$F_D(d) \triangleq P(D < d) = 1 - \prod_{k=1}^{n} P(D_k \geq d), \quad (1)$$

where the complementary CDF of $D_k$ is

$$P(D_k \geq d) = \begin{cases} 
1, & d < a_k \\
1 - \frac{1}{2} \arccos \left( 1 - \frac{d^2}{2r_k^2} \right) N_k, & d \in (a_k, d_{\text{max}}(k,0)) \\
1 - \frac{1}{2} \arccos \left( \frac{r_k^2}{2r_k^2} \right)^N_k, & d > d_{\text{max}}(k,0).
\end{cases}$$

**Proof:** See Appendix A.

**Theorem 2** (Scenario-2: NN distance distribution),

$$F_D(d) \triangleq P(D < d) = 1 - \prod_{k=1}^{n} P(D_k \geq d), \quad (2)$$

where the complementary CDF of $D_k$ is

$$P(D_k \geq d) = \begin{cases} 
1, & d < |a_k-a_i| \\
\left[ 1 - \frac{1}{2} \arccos \left( 1 - \frac{d^2}{2r_k^2} \right) \right]^{N_k}, & d \in (|a_k-a_i|, d_{\text{max}}(k,0)) \\
\left[ 1 - \frac{1}{2} \arccos \left( \frac{r_k^2}{2r_k^2} - \frac{d_{\text{max}}^2(k,i)}{2r_k^2} \right) \right]^{N_k}, & d > d_{\text{max}}(k,i).
\end{cases}$$

**A. Numerical Results**

In this section, we provide numerical results for the given scenarios. As shown in Fig.3 and Fig.4, Theorems 1 and 2 are perfectly matched with simulation, which confirms the accuracy of our analysis.

In Fig.3, we plot the CDF of contact distance for three different cases. The red and black graphs have the exact same number of concentric spheres and same altitudes for each sphere. The red graph has more uniformly distributed points on each sphere, which is the reason for more fast convergence.

In Fig.4, each graph shows CDF of nearest-neighbor distance when the observation point is on the choice of $k^{th}$ sphere for $k = 1, ..., 4$. From these results, we notice that CDF is improved by decreasing the attitude and increasing number of points.
IV. Conclusion

In this letter, we presented a stochastic geometry formulation using the spatial point process to model the point distribution in satellite networks. The model is accompanied by the derivation of exact analytical expressions for the CDFs of nearest-neighbor and contact distance distribution for the spherical BPP where known numbers of points are independently distributed on concentric spheres. Simulation results and analysis demonstrate that the method is accurate and effective which can provide some theoretical basis for further researches.

APPENDIX

A. Proof of Theorem 1

The proof of Theorem 1 and 2 follow the same steps.

• If $d < a_k$, we have $P(D_k \geq d) = 1$ $\forall k$.

• If $a_k < d < d_{\max}(k,0)$, then we have the contact distance distribution, $P(D_k \geq d) = P(N(A_k) = 0)$

$\quad = [P(z_k < z(d,k,0))]^{N_k}$

$\quad = [P(r_k \cos \varphi < z(d,k,0))]^{N_k}$

$\quad = [P(\varphi > \frac{\arccos(z(d,k,0))}{r_k}) + P(\varphi < -\frac{\arccos(z(d,k,0))}{r_k})]^{N_k}$

$\quad = [1 - \frac{1}{\pi} \arccos(\frac{z_{\max}(k,0)}{r_k})]^{N_k}$,

where $h(d,k,0) = \frac{d^2-a_k^2}{2r_k}$ and $d_{\max}(k,0) = \sqrt{2r_k a_k + a_k^2}$ are easily derived by Pythagoras’ theorem and $z(d,k,0) = r_k - h(d,k,0)$.

• If $d > d_{\max}(k,0)$, skipping all the intermediate steps get $P(D_k \geq d) = P(N(A_{\max,k}) = 0) = [P(z_k < z_{\max}(k,0))]^{N_k}$

$\quad = [1 - \frac{1}{\pi} \arccos(\frac{z_{\max}(k,0)}{r_k})]^{N_k}$,

where $z_{\max}(k,0) = r_k$ $\forall k$.

This concludes the proof.

B. Proof of Theorem 2

For $S_k$, for $k = 1, \ldots, n, k \neq i$

• If $d < |a_k - a_i|$, we have $P(D_k \geq d) = 1$.

• If $|a_k - a_i| < d < d_{\max}(k,i)$,

$P(D_k \geq d) = P(N(A_k) = 0) = [P(z_k < z(d,k,i))]^{N_k}$

$\quad = [1 - \frac{1}{\pi} \arccos(\frac{z(d,k,i)}{r_k})]^{N_k}$,

where $h(d,k,i) = \frac{d^2-(a_k-a_i)^2}{2r_k}$, $z(d,k,i) = r_k - h(d,k,i)$ and $d_{\max}(k,i) = \sqrt{r_k^2 - r_i^2} + \sqrt{r_i^2 - r_k^2}$.

• If $d > d_{\max}(k,i)$,

$P(D_k \geq d) = P(N(A_{\max,k}) = 0) = [P(z_k < z_{\max}(k,i))]^{N_k}$

$\quad = [1 - \frac{1}{\pi} \arccos(\frac{z_{\max}(k,i)}{r_k})]^{N_k}$,

where $h_{\max}(k,i) = \frac{(r_k+r_i)^2-d_{\max}^2(k,i)}{2r_i}$ and $z_{\max}(k,i) = r_k - h_{\max}(k,i)$.

For $S_k$, $\forall k = i$.

For the remaining point process $P_{\Phi_1}$, then the point process simply becomes a spherical BPP with $N_i-1$ points.

• If $d < d_{\max}(i,i)$, then

$P(D_i \geq d) = P(N(A_i) = 0) = [P(z_i < z(d,i,i))]^{N_i-1}$

$\quad = [1 - \frac{1}{\pi} \arccos(\frac{z(d,i,i)}{r_i})]^{N_i-1}$,

where $h(d,i,i) = \frac{d^2}{2r_i}$, $d_{\max}(i,i) = 2\sqrt{r_i^2 - r_k^2}$ and with $z(d,i,i) = r_i - h(d,i,i)$.

• If $d > d_{\max}(i,i)$, then

$P(D_i \geq d) = P(N(A_{\max,i}) = 0) = [P(z_i < z_{\max}(i,i))]^{N_i-1}$

$\quad = [1 - \frac{1}{\pi} \arccos(\frac{z_{\max}(i,i)}{r_i})]^{N_i-1}$,

where $h_{\max}(i,i) = \frac{2r_i^2}{r_i}$ and $z_{\max}(i,i) = r_i - h_{\max}(i,i)$.

This concludes the proof.

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