NEW DIRECT PROOFS OF VARIATIONAL PRINCIPLES FOR $T$-ENTROPY, SPECTRAL RADIUS OF WEIGHTED SHIFT OPERATORS, AND ENTROPY STATISTIC THEOREM

V. I. Bakhtin (bakhtin@tut.by)$^\dagger$, A. V. Lebedev (lebedev@bsu.by)$^\ddagger$

$^\dagger$Belarusian State University / John Paul II Catholic University of Lublin, Poland; $^\ddagger$Belarusian State University / University of Bialystok, Poland

New direct proofs of variational principles for $t$-entropy, spectral radius of weighted shift operators, and entropy statistic theorem are given. The equivalence of these statements is obtained.

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As is known $t$-entropy serves as a key ingredient in evaluation of spectral radii of weighted shift and transfer operators by means of the corresponding variational principles (see, for example, [1], [2], [3] and the sources cited therein). These variational principles in fact show that the spectral potential (i.e., logarithm of the spectral radius) is the Legendre transform of $t$-entropy (see (5) below). One of the principal elements in all the proofs of these variational principles is the so called ‘entropy statistic theorem’, which proof in turn is far from being elementary (see, for example, [1], [3]). As is shown in [1] Propositions 8.4, 8.5, 8.6, $t$-entropy itself is concave and upper semicontinuous with respect to the $\ast$-weak topology (the proof here again is nonelementary). On this base by Fenchel—Legendre—Moreau duality theorem one can establish the corresponding variational principle for $t$-entropy, namely that $t$-entropy is the Legendre dual to spectral potential (though this fact about $t$-entropy has not been emphasized in the mentioned sources). To summarize the aforementioned argument, the next chain of statements has been proven: ‘entropy statistic theorem’ $\Rightarrow$ ‘variational principle for spectral potential’ $\Rightarrow$ ‘variational principle for $t$-entropy’, where each step is rather nontrivial. In this article we give new direct proofs for the inverted chain: ‘variational principle for $t$-entropy’ $\Rightarrow$ ‘variational principle for spectral potential’ $\Rightarrow$ ‘entropy statistic theorem’. In particular, it means that these statements are equivalent: ‘variational principle for $t$-entropy’ $\Leftrightarrow$ ‘variational principle for spectral potential’ $\Leftrightarrow$ ‘entropy statistic theorem’.

The proofs presented are shorter, more transparent and give new insight into the nature of the objects under study (from our point of view). As a byproduct we automatically obtain concaveness and upper semicontinuity of $t$-entropy (Remark 8) and also write out explicitly an approximating sequence in the variational principle for $t$-entropy (see (17)).

1 The objects of study. Variational principle for spectral potential

Let $(X, \mathcal{A})$ be a measurable space with a positive measure $m$ and $\alpha : X \to X$ be a measurable mapping. This mapping generates a linear shift operator $A$ that maps any
measurable function $f$ to $Af = f \circ \alpha$. We will assume that this operator is bounded in the space of integrable functions $L^1(X, m)$. The boundedness condition for $A$ is equivalent to existence of a positive constant $C$ such that for every measurable set $G \in \mathcal{A}$ the inequality $m(\alpha^{-1}(G)) \leq Cm(G)$ takes place, and in this case one has $\|A\| \leq C$. For every function $\varphi \in L^\infty(X, m)$ we define a weighted shift operator $A_\varphi$ in $L^1(X, m)$ acting according to the formula

$$[A_\varphi f](x) = e^{\varphi(x)} f(\alpha(x)).$$

Let $\lambda(\varphi)$ be the logarithm of its spectral radius:

$$\lambda(\varphi) := \lim_{n \to \infty} \frac{1}{n} \ln \|A^n_\varphi\|, \quad \varphi \in L^\infty(X, m).$$

The functional $\lambda(\varphi)$ will be called the spectral potential of operator $A$.

Recall that a linear functional on $L^\infty(X, m)$ is called positive if its values on nonnegative functions are nonnegative, and it is normalized if its value on the unit function is equal to one. Let $M(X, m)$ be the set of all positive normalized linear functionals on $L^\infty(X, m)$. Clearly, $M(X, m)$ can be identified with the set of all finitely additive probability measures on $\sigma$-algebra $\mathcal{A}$ that are absolutely continuous with respect to $m$ (i.e., they take zero values on the sets of zero measure $m$). Therefore henceforth elements of $M(X, m)$ will be called measures.

A measure $\mu \in M(X, m)$ is called $\alpha$-invariant if $\mu[f \circ \alpha] = \mu[f]$ for all functions $f \in L^\infty(X, m)$. This is equivalent to the property $\mu(\alpha^{-1}(G)) = \mu(G)$ for every measurable set $G$. The family of all $\alpha$-invariant measures from $M(X, m)$ will be denoted by $M_\alpha(X, m)$.

By a measurable partition of unity on $X$ here we mean any finite set $D = \{g_1, \ldots, g_k\}$ consisting of nonnegative functions $g_i \in L^\infty(X, m)$ satisfying the equality $g_1 + \cdots + g_k = 1$. For any invariant measure $\mu \in M_\alpha(X, m)$ its $t$-entropy $\tau(\mu)$ is given by the formulae

$$\tau(\mu) = \inf_{n,D} \frac{\tau_n(\mu, D)}{n},$$

$$\tau_n(\mu, D) = \sup_{\|f\|=1} \sum_{g \in D} \mu[g] \ln \frac{\int_X g \cdot |f \circ \alpha^n| \, dm}{\mu[g]}. \quad (4)$$

The infimum in (3) is taken over all natural numbers $n$ and measurable partitions of unity $D$ on $X$ and the supremum in (4) is taken over all functions $f \in L^1(X, m)$ with unit norm. If for a certain function $g \in D$ we have $\mu[g] = 0$ then it is assumed that the corresponding summand in (4) is zero regardless of the fraction under logarithm. And if $\mu[g] > 0$ and the expression under logarithm takes zero value then it is assumed that $\mu[g] \ln 0 = -\infty$.

It is known (see [2], [3]) that spectral potential and $t$-entropy satisfy the so-called variational principle for spectral potential given by the formula

$$\lambda(\varphi) = \max_{\mu \in M_\alpha(X, m)} \left( \tau(\mu) + \mu[\varphi] \right), \quad \varphi \in L^\infty(X, m).$$

We will give a new proof for this formula in Theorem 10. It will be a simple corollary of the dual result (variational principle for $t$-entropy); namely its new direct proof is the starting point of the next section.
2 New proofs of variational principles for $t$-entropy and spectral potential

Theorem 1 (variational principle for $t$-entropy) The following equality takes place:
\[
\tau(\mu) = \inf_{\varphi \in L^\infty(X,m)} (\lambda(\varphi) - \mu[\varphi]), \quad \mu \in M_\alpha(X,m).
\] (6)

Theorem 2 For each linear functional $\mu$ on the space $L^\infty(X,m)$ that does not belong to $M_\alpha(X,m)$ the following equality takes place:
\[
\inf_{\varphi \in L^\infty(X,m)} (\lambda(\varphi) - \mu[\varphi]) = -\infty.
\] (7)

These two theorems show that it is natural to put
\[
\tau(\mu) = -\infty \quad \text{for all} \quad \mu \in L^\infty(X,m)^* \setminus M_\alpha(X,m).
\] (8)

Then formulae (6) and (7) are united into one:
\[
\tau(\mu) = \inf_{\varphi \in L^\infty(X,m)} (\lambda(\varphi) - \mu[\varphi]), \quad \mu \in L^\infty(X,m)^*.
\] (9)

We start the proof of Theorem 1 with the next lemma presenting, in particular, ‘Young’ inequality (inequality (11)) for the objects considered. Here we take Young in quotation marks since the classical Young inequality (serving as a component of Fenchel—Legendre duality construction) exploits convexity and Legendre dual objects, while up to now and so also in the proof of Lemma 3 we do not use them. In fact inequality (11) will become the classical Young inequality on derivation of Theorem 1 (cf. Remark 9).

Lemma 3 For any $\varphi \in L^\infty(X,m), \mu \in M_\alpha(X,m), n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a measurable partition of unity $D$ such that
\[
\varepsilon + \frac{\ln \|A^\varphi_n\|}{n} \geq \mu[\varphi] + \frac{\tau_n(\mu, D)}{n}
\] (10)
and thus
\[
\lambda(\varphi) \geq \mu[\varphi] + \tau(\mu).
\] (11)

Proof. The proof, in fact, can be extracted from the reasoning in [3] and we give it for completeness of presentation.

Note first that by arbitrariness of $\varepsilon > 0$ inequality (11) follows from (10) by taking infimum with respect to $n, D$.

So it is enough to verify (10).

Let us introduce the notation
\[
S_n\varphi = \varphi + \varphi \circ \alpha + \cdots + \varphi \circ \alpha^{n-1}.
\]
Then (11) implies the equality
\[
A^\varphi_n f = e^{S_n\varphi} f \circ \alpha^n.
\]
For arbitrary numbers $n \in \mathbb{N}$ and $\varepsilon > 0$ we choose a fine measurable partition of unity $D$ such that on the support of each function $g \in D$ the essential oscillation of function $S_n\varphi$ does not exceed $\varepsilon$. This $D$ is in fact the desired partition.
Thus, where \(d\nu\). Let \(c\). Passing here to supremum with respect to functions \(f\) with unit norm and taking into account (4), one obtains the inequality

\[
\varepsilon + \ln \left| e^{S_{n}\varphi} \right| \geq \ln \sum_{g \in D} e^{S_{n}\varphi(g)} \int_{X} |f \circ \alpha^n| \, dm
\]

\[
\geq \sum_{g \in D} \mu[g] \ln \frac{e^{S_{n}\varphi(g)} \int_{X} |f \circ \alpha^n| \, dm}{\mu[g]}
\]

\[
\geq \mu[S_{n}\varphi] + \sum_{g \in D} \mu[g] \ln \int_{X} |f \circ \alpha^n| \, dm / \mu[g].
\]

Passing here to supremum with respect to functions \(f\) with unit norm and taking into account (4), one obtains the inequality

\[
\varepsilon + \ln \|A_n\| \geq \mu[S_{n}\varphi] + \tau_n(\mu, D) = n\mu[\varphi] + \tau_n(\mu, D),
\]

which implies (10). □

Inequality (11) implies that the left hand part in (6) does not exceed its right hand part.

To finish the proof of Theorem 1 we need two more lemmas.

In the next lemma notation \(\lambda(\varphi, A)\) has the same meaning as \(\lambda(\varphi)\) and \(\lambda(n\varphi, A^n)\) denotes logarithm of the spectral radius of operator \(e^{n\varphi}A^n\).

**Lemma 4** The following inequality takes place:

\[
n\lambda(\varphi, A) \leq \lambda(n\varphi, A^n), \quad n \in \mathbb{N}.
\]

**Proof.** Note that for any natural \(k\) one has

\[
\exp\{S_{nk}\varphi\} = \exp\left\{\sum_{i=0}^{n-1} \sum_{j=0}^{k-1} \varphi \circ \alpha^{i+j}\right\} = \prod_{i=0}^{n-1} \exp\left\{\sum_{j=0}^{k-1} \varphi \circ \alpha^{i+j}\right\}.
\]

Let \(c = \text{ess sup} |\varphi|\). Exploiting Hölder inequality in the form

\[
\int_{X} \psi_1 \cdots \psi_n \, d\nu \leq \prod_{i=1}^{n} \left(\int_{X} |\psi_i|^n \, d\nu\right)^{1/n},
\]

where \(d\nu = |f \circ \alpha^{n(k+1)}| \, dm\), one obtains

\[
e^{-nc} \|A_n^{(k+1)}f\| = e^{-nc} \int_{X} e^{S_{n(k+1)}\varphi} |f \circ \alpha^{n(k+1)}| \, dm \leq \int_{X} e^{S_{nk}\varphi} |f \circ \alpha^{n(k+1)}| \, dm
\]

\[
\leq \prod_{i=0}^{n-1} \left(\int_{X} \exp\left\{\sum_{j=0}^{k-1} (n\varphi \circ \alpha^{nj}) \circ \alpha^i\right\} |(f \circ \alpha^{-i}) \circ \alpha^{nk} \circ \alpha^i| \, dm\right)^{1/n}
\]

\[
= \prod_{i=0}^{n-1} \|A_i(e^{n\varphi}A^n)^{k}(A^{-i}f)\|^{1/n} \leq \|A\|^n \|e^{n\varphi}A^n\| \|f\|.
\]

Thus,

\[
-nc + \ln \|A_n^{(k+1)}\| \leq n \ln \|A\| + \ln \|e^{n\varphi}A^n\|.
\]
Dividing the latter inequality by \(k\) and turning \(k \to \infty\) one gets (12). □

Let us fix a measure \(\mu \in M_\alpha(X,m)\), natural number \(n\) and measurable partition of unity \(D\) on \(X\). For these objects there exists a sequence of functions \(f_k \in L^1(X,m)\) with unit norm on which the supremum in (4) is attained. One may choose a subsequence \(f_{k_i}\) of this sequence such that the following limits do exist simultaneously:

\[
\lim_{i \to \infty} \int_X g |f_{k_i} \circ \alpha^n| \, dm =: C_n(\mu, g, D), \quad g \in D.
\] (13)

Then by construction one has

\[
\tau_n(\mu, D) = \sum_{g \in D, \mu[g] > 0} \frac{\mu[g] \ln C_n(\mu, g, D)}{\mu[g]}.
\] (14)

Lemma 5  If \(\tau_n(\mu, D) > -\infty\) then

\[
\sup_{\|f\|=1} \sum_{g \in D, \mu[g] > 0} \frac{\mu[g]}{\mu} \int_X g |f \circ \alpha^n| \, dm = 1.
\] (15)

Proof. Finiteness of \(\tau_n(\mu, D)\) and (14) imply that \(C_n(\mu, g, D) > 0\) whenever \(\mu[g] > 0\). For each \(f \in L^1(X,m)\) with \(\|f\| = 1\) let us consider the function

\[
\eta(t) = \sum_{g \in D, \mu[g] > 0} \frac{\mu[g] \ln \left(1 - t\right) C_n(\mu, g, D) + t \int_X g |f \circ \alpha^n| \, dm}{\mu[g]}.
\]

By definition of the numbers \(C_n(\mu, g, D)\) this function attains its maximal value equal to \(\tau_n(\mu, D)\) at \(t = 0\). Therefore its derivative at \(t = 0\)

\[
\left. \frac{d\eta(t)}{dt} \right|_{t=0} = \sum_{g \in D, \mu[g] > 0} \frac{\mu[g] \int_X g |f \circ \alpha^n| \, dm - C_n(\mu, g, D)}{C_n(\mu, g, D)} = \sum_{g \in D, \mu[g] > 0} \frac{\mu[g] \int_X g |f \circ \alpha^n| \, dm}{C_n(\mu, g, D)} - 1
\]

is nonpositive and so the left hand part in (15) does not exceed its right hand part.

The equality in (15) is attained on the sequence of functions \(f_{k_i}\) from (13). □

Now we can finish the proof of Theorem 1.

Let us fix an arbitrary measure \(\mu \in M_\alpha(X,m)\), natural number \(n\) and measurable partition of unity \(D\) on \(X\).

Suppose at first that there exists a function \(g \in D\) satisfying the inequality \(\mu[g] > 0\) and equality \(\int_X g |f \circ \alpha^n| \, dm = 0\) for all \(f \in L^1(X,m)\). Then by definition one has \(\tau_n(\mu, D) = -\infty\) and therefore \(\tau(\mu) = -\infty\). Thus in this case equality (6) takes the form

\[
-\infty = \inf_{\varphi \in L^\infty(X,m)} \left(\lambda(\varphi) - \mu[\varphi]\right).
\] (16)

Let us verify it.

Consider the family of functions \(\varphi_t = tg/n\) where \(t \in \mathbb{R}\). Inequalities \(0 \leq g \leq 1\) and the Lagrange theorem imply that

\[
e^{n\varphi_t} = e^{tg} \leq 1 + e^{tg}.
\]
Therefore for each function $f \in L^1(X, m)$ with unit norm one has
\[
\|e^{n\varphi_t}A^n f\| = \int_X e^{n\varphi_t} |f \circ \alpha^n| \, dm \leq \int_X (1 + e^t g) |f \circ \alpha^n| \, dm = \int_X |f \circ \alpha^n| \, dm \leq \|A^n\|.
\]
Thus $\|e^{n\varphi_t}A^n\| \leq \|A^n\|$. Applying Lemma 4 we obtain the following estimate
\[
n\lambda(\varphi_t) = n\lambda(\varphi_t, A) \leq \lambda(n\varphi_t, A^n) \leq \ln \|e^{n\varphi_t}A^n\| \leq \ln \|A^n\|.
\]
On the other hand,
\[
\mu[\varphi] = \mu[tg/n] = t\mu[g]/n \to +\infty \quad \text{as} \quad t \to +\infty.
\]
And therefore $\lambda(\varphi_t) - \mu[\varphi_t] \to -\infty$ when $t \to +\infty$. So equality (16) is verified.

It remains to consider the situation when for each function $g \in D$ satisfying the condition $\mu[g] > 0$ there exists a function $f_g \in L^1(X, m)$ such that $\int_X g |f_g \circ \alpha^n| \, dm > 0$. Taking the function $f := \sum g |f_g|$ one obtains that
\[
\int_X g |f \circ \alpha^n| \, dm > 0 \quad \text{as soon as} \quad \mu[g] > 0.
\]
Therefore $\tau_n(\mu, D) > -\infty$. Note also that finiteness of $\tau_n(\mu, D)$ along with (14) implies that the condition $\mu[g] > 0$ automatically implies the inequality $C_n(\mu, g, D) > 0$.

Now let us define the family of functions
\[
\varphi_\varepsilon := \frac{1}{n} \ln \left\{ \sum_{\mu[g] > 0} \frac{\mu[g]}{C_n(\mu, g, D)} g + \sum_{\mu[g] = 0} \varepsilon g \right\}, \quad \varepsilon > 0. \tag{17}
\]
For any integrable function $f$ with unit norm one has
\[
\|e^{n\varphi_\varepsilon}A^n f\| = \int_X e^{n\varphi_\varepsilon} |f \circ \alpha^n| \, dm
\]
\[
= \int_X \sum_{\mu[g] > 0} \frac{\mu[g]}{C_n(\mu, g, D)} g |f \circ \alpha^n| \, dm + \int_X \sum_{\mu[g] = 0} \varepsilon g |f \circ \alpha^n| \, dm
\]
\[
= \sum_{\mu[g] > 0} \mu[g] \int_X g |f \circ \alpha^n| \, dm + \frac{1}{C_n(\mu, g, D)} + \varepsilon \int_X g |f \circ \alpha^n| \, dm \leq 1 + \varepsilon \|A^n\|
\]
(where in the final inequality we exploited Lemma 5). This along with Lemma 4 implies the estimate
\[
n\lambda(\varphi_\varepsilon) \leq \lambda(n\varphi_\varepsilon, A^n) \leq \ln \|e^{n\varphi_\varepsilon}A^n\| \leq \ln(1 + \varepsilon \|A^n\|) \leq \varepsilon \|A^n\|. \tag{18}
\]
On the other hand applying concaveness of logarithm and (14) one obtains
\[
\mu[n\varphi_\varepsilon] = \mu \left[ \ln \left\{ \sum_{\mu[g] > 0} \frac{\mu[g]}{C_n(\mu, g, D)} g + \sum_{\mu[g] = 0} \varepsilon g \right\} \right]
\geq \mu \left[ \sum_{\mu[g] > 0} g \ln \frac{\mu[g]}{C_n(\mu, g, D)} + \sum_{\mu[g] = 0} g \ln \varepsilon \right] = -\tau_n(\mu, D). \tag{19}
\]
Combining (19) and (18) we get

$$\frac{\tau_n(\mu, D)}{n} \geq -\mu[\varphi_\varepsilon] \geq -\mu[\varphi_{\varepsilon}] + \left(\lambda(\varphi_{\varepsilon}) - \frac{\varepsilon\|A^n\|}{n}\right)$$

and therefore

$$\frac{\tau_n(\mu, D)}{n} + \varepsilon\|A^n\| \geq \lambda(\varphi_{\varepsilon}) - \mu[\varphi_{\varepsilon}] \geq \inf_{\varphi \in L^\infty(X, m)} (\lambda(\varphi) - \mu[\varphi]) .$$

This inequality along with arbitrariness of $\varepsilon$, $n$, $D$ and definition (3) of $\tau(\mu)$ implies the inequality

$$\tau(\mu) \geq \inf_{\varphi \in L^\infty(X, m)} (\lambda(\varphi) - \mu[\varphi]) .$$

Together with Young inequality (11) this proves (6).

Now let us prove Theorem 2. For this purpose we need two more lemmas describing the principal properties of the functional $\lambda(\varphi)$.

**Lemma 6**  Spectral potential $\lambda(\varphi)$ possesses the following properties:

a) if $\varphi \geq \psi$, then $\lambda(\varphi) \geq \lambda(\psi)$ (monotonicity);

b) $\lambda(\varphi + t) = \lambda(\varphi) + t$ for all $t \in \mathbb{R}$ (additive homogeneity);

c) $|\lambda(\varphi) - \lambda(\psi)| \leq \text{ess sup } |\varphi - \psi|$ (Lipschitz condition);

d) $\lambda((1-t)\varphi + t\psi) \leq (1-t)\lambda(\varphi) + t\lambda(\psi)$ for $t \in [0, 1]$ (convexity);

e) $\lambda(\varphi + \psi \circ \alpha) = \lambda(\varphi + \psi)$ (strong $\alpha$-invariance).

This lemma is proven in [1], [4].

**Lemma 7**  If a linear functional $\mu$ on $L^\infty(X, m)$ possesses the property

$$\inf_{\varphi \in L^\infty(X, m)} (\lambda(\varphi) - \mu[\varphi]) > -\infty , \quad (20)$$

then $\mu \in M_\alpha(X, m)$. In particular, this is true for every subgradient of the function $\lambda(\varphi)$.

**Proof.** Observe first that any functional $\mu$ satisfying (20) is necessarily positive. Indeed, if $\mu[\varphi] < 0$ for some function $\varphi \geq 0$ then by monotonicity of $\lambda(\varphi)$ (Lemma 6 a) one has $\lambda(-t\varphi) \leq \lambda(0)$ and thus

$$\lambda(-t\varphi) - \mu[-t\varphi] \leq \lambda(0) + t\mu[\varphi] \to -\infty \quad \text{as } t \to +\infty ,$$

which contradicts (20).

Note also that this functional should have norm equal to 1. Indeed, by additive homogeneity of $\lambda(\varphi)$ (Lemma 6 b) we have $\lambda(t) = \lambda(0) + t$ and therefore

$$\lambda(t) - \mu[t] = \lambda(0) + t(1 - \mu[1]) .$$

This function is bounded from below for all $t \in \mathbb{R}$ only if $\mu[1] = 1$. Which by positivity of $\mu$ means that $\|\mu\| = 1$.

Observe now that strong $\alpha$-invariance of $\lambda(\varphi)$ (Lemma 6 e) implies the equality $\lambda(t\varphi \circ \alpha - t\varphi) = \lambda(0)$. Thus,

$$\lambda(t\varphi \circ \alpha - t\varphi) - \mu[t\varphi \circ \alpha - t\varphi] = \lambda(0) - t(\mu[\varphi \circ \alpha] - \mu[\varphi]) .$$
This function is bounded from below for all \( t \in \mathbb{R} \) only if \( \mu[\varphi \circ \alpha] = \mu[\varphi] \) which means \( \alpha \)-invariance of \( \mu \).

Thus condition (20) can be satisfied only when \( \mu \in M_\alpha(X, m). \)

Finally, we observe that any subgradient \( \mu \) of \( \lambda(\varphi) \) satisfies (20). Indeed, let \( \mu \) be a subgradient of \( \lambda(\varphi) \) at a point \( \varphi_0 \). Then

\[
\lambda(\varphi) - \lambda(\varphi_0) \geq \mu[\varphi - \varphi_0] \quad \text{for all } \varphi \in L^\infty(X, m),
\]

that is

\[
\lambda(\varphi) - \mu[\varphi] \geq \lambda(\varphi_0) - \mu[\varphi_0] \quad \text{for all } \varphi \in L^\infty(X, m),
\]

which implies (20).

\[\square\]

Clearly, Theorem 2 is a straightforward corollary of Lemma 7.

**Remark 8** By Lemma 6 the function \( \lambda(\varphi) \) is convex and continuous. Theorems 1 and 2 in essence state that the functional \( -\tau(\mu) \) is the Legendre transform of \( \lambda(\varphi) \). This automatically implies that \( t \)-entropy \( \tau(\mu) \) is concave and upper semicontinuous (in the *-weak topology) on the dual space to \( L^\infty(X, m) \). In [3] concaveness and upper semicontinuity of \( t \)-entropy were proven independently and in an essentially more complicated way.

**Remark 9** Now we see that inequality (11) is indeed the classical Young inequality for the objects considered.

Finally we observe that variational principle (5) can be easily derived from Theorem 1 and Lemmas 6, 7.

**Theorem 10** (variational principle for spectral potential) For each function \( \varphi \in L^\infty(X, m) \) the following equality takes place:

\[
\lambda(\varphi) = \max_{\mu \in M_\alpha(X, m)} \left( \tau(\mu) + \mu[\varphi] \right).
\]

**Proof.** By Lemma 6 the function \( \lambda(\varphi) \) is convex and continuous. Thus at each point \( \varphi_0 \) there exists at least one subgradient \( \mu \) for \( \lambda(\varphi) \). By Lemma 7 this subgradient belongs to \( M_\alpha(X, m) \). By Theorem 1 and definition of a subgradient we have

\[
\tau(\mu) = \inf_{\varphi \in L^\infty(X, m)} (\lambda(\varphi) - \mu[\varphi]) = \lambda(\varphi_0) - \mu[\varphi_0].
\]

Therefore \( \lambda(\varphi_0) = \tau(\mu) + \mu(\varphi_0) \). Combining this equality with Young inequality (11) one obtains (21). \[\square\]

### 3 Variational principles in the spaces of bounded measurable functions

In the above part of the article we considered the spectral potential \( \lambda(\varphi) \) as a function on the space \( L^\infty(X, m) \) and \( t \)-entropy \( \tau(\mu) \) as a function on the dual space \( L^\infty(X, m)^* \). But in fact the spectral potential \( \lambda(\varphi) \) can readily be defined in the same way (i.e., by formula (2)) on the space of bounded measurable functions \( B(X, \mathfrak{A}) \) (i.e., not on the factor space \( L^\infty(X, m) \) of \( B(X, \mathfrak{A}) \)) and \( \tau(\mu) \) can be defined by the same formulae on the dual...
space $B(X, \mathcal{A})^*$. In this case all the above presented results and their proofs remain in force. In particular, in the situation mentioned equality (9) takes form

$$\tau(\mu) = \inf_{\varphi \in B(X, \mathcal{A})} (\lambda(\varphi) - \mu[\varphi]), \quad \mu \in B(X, \mathcal{A})^*, \quad (22)$$

and variational principle (21) looks as

$$\lambda(\varphi) = \max_{\mu \in M_\alpha(X, \mathcal{A})} (\tau(\mu) + \mu[\varphi]), \quad \varphi \in B(X, \mathcal{A}), \quad (23)$$

where $M_\alpha(X, \mathcal{A})$ denotes the set of all $\alpha$-invariant positive normalized linear functionals on $B(X, \mathcal{A})$.

Since the natural factor mapping from $B(X, \mathcal{A})$ onto $L^\infty(X, m)$ is linear and bounded it induces a continuous embedding of $L^\infty(X, m)^*$ in $B(X, \mathcal{A})^*$. Therefore from the formal point of view equality (22) is stronger than (9), and equality (23) is weaker than (21) (here we bear in mind Young inequality (11) which, of course, is also true for $\varphi \in B(X, \mathcal{A})$ and $\mu \in M_\alpha(X, \mathcal{A})$).

The next lemma shows that in fact all the corresponding variational principles are equivalent.

**Lemma 11** If $\mu \in B(X, \mathcal{A})^* \setminus L^\infty(X, m)^*$ then

$$\tau(\mu) = -\infty \quad \text{and} \quad \inf_{\varphi \in B(X, \mathcal{A})} (\lambda(\varphi) - \mu[\varphi]) = -\infty. \quad (24)$$

**Proof.** The assumption $\mu \in B(X, \mathcal{A})^* \setminus L^\infty(X, m)^*$ implies that there exists a function $\varphi \in B(X, \mathcal{A})$ which is equal to zero $m$-almost everywhere and therewith $\mu[\varphi] \neq 0$. For this function we have

$$\inf_{t \in \mathbb{R}} (\lambda(t\varphi) - \mu[t\varphi]) = \inf_{t \in \mathbb{R}} (\lambda(0) - t\mu[\varphi]) = -\infty$$

and therefore

$$\inf_{\varphi \in B(X, \mathcal{A})} (\lambda(\varphi) - \mu[\varphi]) = -\infty.$$

Thus to finish the proof it suffice to verify that $\tau(\mu) = -\infty$ for the mentioned $\mu$.

If $\mu \in B(X, \mathcal{A})^* \setminus M_\alpha(X, \mathcal{A})$ then by definition $\tau(\mu) = -\infty$ (by analogy with (8)). And if $\mu \in M_\alpha(X, \mathcal{A}) \setminus L^\infty(X, m)^*$ then there exists a measurable set $G \in \mathcal{A}$ such that $m(G) = 0$ and therewith $\mu(G) > 0$. Let us take a partition of unity $D = \{g, 1 - g\}$, where $g$ is the characteristic function of this set $G$. In view of (11) one has $\tau_n(\mu, D) = -\infty$ and therefore $\tau(\mu) = -\infty$. □

### 4 A new proof of entropy statistic theorem

As it was noted, ‘entropy statistic theorem’ served as a key instrument in the initial proof of variational principle for spectral potential in [3]. It appears that its proof can also be essentially simplified by means of the variational principle for $t$-entropy. In particular, this means that we obtain an equivalence:

‘variational principle for $t$-entropy’ $\Leftrightarrow$ ‘variational principle for spectral potential’ $\Leftrightarrow$ ‘entropy statistic theorem’.
It will be convenient for us to present a variant of ‘entropy statistic theorem’ in the spaces $B(X, \mathfrak{A})$ and $B(X, \mathfrak{A})^*$.

Let us take an arbitrary point $x \in X$ and a natural number $n$. The corresponding empirical measure $\delta_{x,n}$ is concentrated at points $x$, $\alpha(x)$, $\ldots$, $\alpha^{n-1}(x)$ and its value at each point is equal to $1/n$. For any function $f : X \to \mathbb{R}$ its integral with respect to this measure equals to the ‘empirical mean’

$$\delta_{x,n}[f] = \frac{f(x) + f(\alpha(x)) + \cdots + f(\alpha^{n-1}(x))}{n} = \frac{1}{n} S_n f(x).$$

Clearly, $\delta_{x,n}$ can be considered as a linear functional on the space $B(X, \mathfrak{A})$.

Let $\mu$ be a linear functional on $B(X, \mathfrak{A})$ and $O(\mu)$ be its certain neighborhood in the *-weak topology. By $X_n(O(\mu))$ we denote the set

$$X_n(O(\mu)) := \{ x \in X \mid \delta_{x,n} \in O(\mu) \}.$$

‘Entropy statistic theorem’ is formulated as follows.

**Theorem 12** For any linear functional $\mu$ on $B(X, \mathfrak{A})$ and any number $\varepsilon > 0$ there exists a neighborhood $O(\mu)$ in the *-weak topology and a large number $C(\varepsilon, \mu)$ such that for all natural numbers $n$ and functions $f \in L^1(X, m)$ the next estimate holds:

$$\int_{X_n(O(\mu))} |f \circ \alpha^n| \, dm \leq C(\varepsilon, \mu) e^{n(\tau(\mu) + \varepsilon)} \int_X |f| \, dm. \tag{25}$$

If $\tau(\mu) = -\infty$ then the number $\tau(\mu) + \varepsilon$ in (25) should be replaced by $-1/\varepsilon$.

**Proof.** By the variational principle for $t$-entropy in the form of (22) one can take a function $\varphi \in B(X, \mathfrak{A})$ such that

$$\lambda(\varphi) - \mu[\varphi] < \tau(\mu) + \varepsilon/2$$

(or $\lambda(\varphi) - \mu[\varphi] < -1/\varepsilon - \varepsilon/2$ in the case when $\tau(\mu) = -\infty$). Let us set

$$O(\mu) := \{ \delta \in B(X, \mathfrak{A})^* \mid \lambda(\varphi) - \delta[\varphi] < \tau(\mu) + \varepsilon/2 \}.$$

Then for each point $x \in X_n(O(\mu))$ the following relations are true

$$S_n \varphi(x) = n \delta_{x,n}[\varphi] > n(\lambda(\varphi) - \tau(\mu) - \varepsilon/2).$$

Take a constant $C(\varepsilon, \mu)$ so large that

$$\|A_\varphi^n\| \leq C(\varepsilon, \mu) e^{n(\lambda(\varphi) + \varepsilon/2)}, \quad n \in \mathbb{N}.$$ 

Now (25) follows from the calculation

$$C(\varepsilon, \mu) e^{n(\lambda(\varphi) + \varepsilon/2)} \|f\| \geq \|A_\varphi^n f\| \geq \int_X e^{S_n \varphi} |f \circ \alpha^n| \, dm \geq \int_{X_n(O(\mu))} e^{S_n \varphi} |f \circ \alpha^n| \, dm \geq e^{n(\lambda(\varphi) - \tau(\mu) - \varepsilon/2)} \int_{X_n(O(\mu))} |f \circ \alpha^n| \, dm. \quad \square$$

10
References

[1] A.B. Antonevich, V. I. Bakhtin, A. V. Lebedev. On $t$-entropy and variational principle for the spectral radii of transfer and weighted shift operators. *Ergodic Theory Dynam. Systems* **31** (2011), 995–1045.

[2] A.B. Antonevich, V. I. Bakhtin, A. V. Lebedev. A road to the spectral radius of transfer operators. *Contemporary Mathematics* **567** (2012), 17–51.

[3] V. I. Bakhtin. On $t$-entropy and variational principle for the spectral radius of weighted shift operators. *Ergodic Theory Dynam. Systems* **30** (2010), 1331–1342.

[4] V. I. Bakhtin. Positive processes. *Ergodic Theory Dynam. Systems* **27** (2007), 639–670.