POISSON STRUCTURES AND BIRATIONAL MORPHISMS ASSOCIATED WITH BUNDLES ON ELLIPTIC CURVES

A. POLISHCHUK

Let $X$ be a complex elliptic curve. In this paper we define a natural Poisson structure on the moduli spaces of stable triples $(E_1, E_2, \Phi)$ where $E_1, E_2$ are algebraic vector bundles on $X$ of fixed ranks $(r_1, r_2)$ and degrees $(d_1, d_2)$, $\Phi : E_2 \to E_1$ is a homomorphism. Such triples are considered up to an isomorphism and the stability condition depends on a real parameter $\tau$. These moduli spaces were introduced by S. Bradlow and O. Garcia-Prada [5]. Our Poisson structure induces a Poisson structure on similar moduli spaces with fixed determinants of $E_1$ and $E_2$. For $E_2 = \mathcal{O}$ and some values of parameters $(r_1, r_2, d_1, d_2, \tau)$ the latter moduli spaces are just the projective spaces. In particular, one of these moduli spaces is $\mathbb{P} \text{Ext}^1(E, \mathcal{O})$, where $E$ is a fixed stable bundle. The corresponding Poisson structures on $\mathbb{P} \text{Ext}^1(E, \mathcal{O})$ were first defined and studied by B. Feigin and A. Odesskii [7]. Moreover, they constructed a flat family of quadratic algebras (Sklyanin algebras) $Q_{d,r}(x)$ where $d = \deg E$, $r = \text{rk} E$, parametrized by $x \in X$ such that $Q_{d,r}(0)$ is the symmetric algebra in $d$ variables and the quadratic Poisson bracket on the symmetric algebra associated with this deformation induces the above Poisson structure on $\mathbb{P}^{d-1}$. The algebra $Q_{d,r}(x)$ is defined as the associative algebra over $\mathbb{C}$ with $d$ generators $t_i$, $i \in \mathbb{Z}/d\mathbb{Z}$ and defining relations

$$
\sum_{n \in \mathbb{Z}/d\mathbb{Z}} \frac{\theta_{j-i+(r-1)n}(0)}{\theta_{j-i-n}(-x)\theta_{rn}(x)} t_{r(j-n)} t_{r(i+n)} = 0 \quad (0.1)
$$

for $i, j \in \mathbb{Z}/d\mathbb{Z}$, where $\theta_m$, $m \in \mathbb{Z}/d\mathbb{Z}$ are certain theta-functions of level $d$ on $X$ (see [3]).

For some other values of parameters we get as moduli space the projective space $\mathbb{P} H^0(E)$ where $E$ is a stable bundle on $X$. When the parameter $\tau$ changes continuously the corresponding moduli space doesn’t change except when $\tau$ passes a finite number of rational values, in which case the moduli space undergoes birational transformations.
(flips) compatible with Poisson structures. In particular, if we start with a stable bundle $E$ of rank $r$ and degree $d > r$ such that $d$ is relatively prime to $r + 1$, then we get a sequence of Poisson birational morphisms connecting $\mathbb{P} \text{Ext}^1(E, \mathcal{O})$ and $\mathbb{P} H^0(E')$ where $E'$ is the unique stable bundle of rank $r + 1$ with $\det E' \simeq \det E$.

On the other hand, using Fourier-Mukai transform one constructs an action of a central extension of $\text{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$ on $D^b(X)$, the derived category of coherent sheaves on $X$ (see [9]). Using this action we construct for every stable bundle $E$ of rank $r$ and degree $d$ an isomorphism (compatible with Poisson structures) between the spaces $\mathbb{P} H^0(E)$ and $\mathbb{P} \text{Ext}^1(E', \mathcal{O})$ where $E'$ is certain stable bundle of degree $d$ and rank $r'$ satisfying the congruence relation $r \cdot r' \equiv -1 \mod (d)$. This reflects the fact noticed in [10] that the corresponding Sklyanin algebras $Q_{d,r}(x)$ and $Q_{d,r'}(x)$ are isomorphic for any $x \in X$.

It turns out that the above birational and regular isomorphisms of projective spaces fit together in the following way. For every $d > 0$ we can consider the disjoint union of $(d - 1)$-dimensional projective spaces indexed by the set of residues $R_d \subset \mathbb{Z}/d\mathbb{Z}$ consisting of $r$ such that both $r$ and $r + 1$ are relatively prime to $d$. Namely, for every $r$ ($0 < r < d$) the corresponding projective space is $\mathbb{P} \text{Ext}^1(E, \mathcal{O})$ where $E$ is stable bundle of rank $r$ and degree $d$. Then the above birational and regular isomorphisms of projective spaces generate the birational action of $S_3$ (the group of permutations in 3 letters) on this disjoint union. The corresponding action of $S_3$ on the set of connected components $R_d$ is generated by operators $r \mapsto r^{-1}$ and $r \mapsto -r - 1$.

Finally, we show how to generalize our Poisson brackets to the similar moduli stacks for other reductive groups. Namely, we fix the following data: a reductive group $G$, its finite-dimensional representation $V$ and a symmetric $g$-invariant tensor $t \in (S^2 g)^g$ where $g$ is the Lie algebra of $G$. These data should satisfy the following condition: the operator $t_* : S^2 V \to S^2 V$ induced by $t$ should be zero. Then we consider the moduli stack of pairs $(P, s)$ where $P$ is a principal $G$-bundle on $X$, $s \in V(P)$ is a section of the associated vector bundle. Given a trivialization of $\omega_X$ we construct a canonical Poisson structure on this moduli stack. In the case when $G = \text{GL}_{r_1} \times \text{GL}_{r_2}$, $V$ is the space of $r_1 \times r_2$-matrices, there is a natural choice of the tensor $t$ such that the above condition is satisfied and we recover our Poisson structure on moduli of triples. The simplest case involving other groups than $\text{GL}$ is
the case $G = \text{GSp}(V)$ where $V$ is the symplectic vector space, $\text{GSp}(V)$ is the group of automorphisms preserving the symplectic form up to a non-zero constant. In this case there is a canonical choice of $t$, so we get a Poisson structure on the moduli stack of pairs $(E,s)$ where $E$ is a vector bundle equipped with a symplectic form $E \times E \to L$ (where $L$ is a line bundle), $s$ is a section of $E$.

1. Stable triples

Let us recall the definition of stable triples from [5]. Let $T = (E_1, E_2, \Phi)$ be a triple consisting of two vector bundles $E_1$ and $E_2$ on $X$ and a homomorphism $\Phi : E_2 \to E_1$. For a real parameter $\sigma$ the $\sigma$-degree of $T$ is defined as follows:

$$\deg_\sigma(T) = \deg(E_1) + \deg(E_2) + \sigma \cdot \text{rk}(E_2).$$

Now the $\sigma$-slope of $T$ is defined by the formula

$$\mu_\sigma(T) = \frac{\deg_\sigma(T)}{\text{rk}(E_1) + \text{rk}(E_2)}.$$  

Note that if $L$ is a line bundle then we can define a tensor of a triple $T$ with $L$ naturally, so that one has $\mu_\sigma(T \otimes L) = \mu_\sigma(T) + \deg L$.

The triple $T$ is called $\sigma$-stable if for every non-zero proper subtriple $T' \subset T$ one has $\mu_\sigma(T') < \mu_\sigma(T)$. Sometimes it is convinient to introduce another stability parameter $\tau = \mu_\sigma(T)$.

The category of triples $T = (E_1, E_2, \Phi)$ is equivalent to the category of extensions

$$0 \to p^* E_1 \to F \to p^* E_2(2) \to 0$$

on $X \times \mathbb{P}^1$ where $p : X \times \mathbb{P}^1 \to X$ is the projection. Indeed, the space of such extensions is $\text{Ext}^1_{X \times \mathbb{P}^1}(p^* E_2(2), p^* E_1) \simeq \text{Hom}_X(E_2, E_1)$. This extension has a unique $\text{SL}_2$-equivariant structure and as shown in [5] the $\sigma$-stability condition on $T$ is equivalent to the $\text{SL}_2$-equivariant stability of $F$ with respect to some polarization on $X \times \mathbb{P}^1$ depending on $\sigma$.

Let us denote by $\mathcal{M}_\sigma = \mathcal{M}_\sigma(d_1, d_2, r_1, r_2)$ the moduli space of $\sigma$-stable triples $T = (E_1, E_2, \Phi)$ on $X$ with $\deg(E_i) = d_i$, $\text{rk} E_i = r_i$. When using another stability parameter $\tau$ we will denote the same moduli space by $\mathcal{M}_\tau$. This moduli space can be constructed using geometric invariant theory as in [5].

We claim that in the case of elliptic curve all these moduli spaces are smooth.

**Lemma 1.1.** The moduli space $\mathcal{M}_\sigma$ is smooth.
Proof. According to [5] we have to show that $H^2(X \times \mathbb{P}^1, \text{End} F)^{\text{SL}_2} = 0$ for the $\text{SL}_2$-equivariant vector bundle $F$ associated with a $\sigma$-stable triple. Consider the exact sequence

$$0 \to K \to \text{End} F \to \text{Hom}(p^* E_1, p^* E_2(2)) \to 0.$$ 

Since the direct image $Rp_*$ of the last term will have no $\text{SL}_2$-invariant part we have $H^*(X \times \mathbb{P}^1, \text{End} F)^{\text{SL}_2} \simeq H^*(X \times \mathbb{P}^1, K)$. Now $K$ sits in the exact triangle

$$\text{Hom}(p^* E_2(2), p^* E_1)) \to K \to \text{End} p^* E_1 \oplus \text{End} p^* E_2 \to \text{Hom}(p^* E_2(2), p^* E_1)[1] \to \ldots$$

It follows that equivariant direct image of $K$ with respect to the projection $p$ is quasi-isomorphic to the complex

$$C^\cdot : \text{End} E_1 \oplus \text{End} E_2 \xrightarrow{d} \text{Hom}(E_2, E_1) \tag{1.2}$$

concentrated in degrees 0 and 1, where $d(A, B) = A\Phi - \Phi B$. We have the exact sequence of cohomologies

$$H^1(X, \text{End} E_1 \oplus \text{End} E_2) \to H^1(X, \text{Hom}(E_2, E_1)) \to H^2(X, C^\cdot) \to 0.$$

Now by Serre duality we have $H^1(X, \text{Hom}(E_2, E_1))^* \simeq H^0(X, \text{Hom}(E_1, E_2))^*$.

According to Lemma 4.4 of [5] this space is zero unless $\Phi$ is an isomorphism. In the latter case the first arrow in the above exact sequence is surjective, so in either case we get $H^2(X \times \mathbb{P}^1, K) = H^2(X, C^\cdot) = 0$. \qed

The proof of this lemma also shows that the tangent space to $M_\sigma$ at a triple $T$ is identified with the hypercohomology space $H^1(X, C^\cdot)$ where $C^\cdot$ is the complex (1.2). This can also be shown directly considering infinitesimal deformations of the first order for triples.

2. Poisson structure

Let us fix a trivialization $\omega_X \simeq \mathcal{O}_X$ of the canonical bundle of $X$. Then we can define a Poisson structure on the moduli space of triples $M_\sigma$. As we have seen above the tangent space to $M_\sigma$ at a triple $T$ is identified with $H^1(X, C)$ where $C$ is the complex (1.2). By Serre duality the cotangent space is isomorphic to $H^0(X, C^\cdot)^* \simeq H^1(X, (C^\cdot)^*) = C^*[-1]$, where the complex $C^*[-1] = ((C^1)^* \xrightarrow{-d^*} (C^0)^*)$ is concentrated in degrees 0 and 1. Using the natural autoduality of $\text{End} E_i$ the complex $C^*[-1]$ can be identified with

$$\text{Hom}(E_1, E_2) \xrightarrow{-d^*} \text{End} E_1 \oplus \text{End} E_2$$
where \(-d^*(\Psi) = (-\Phi \Psi, \Psi \Phi)\). Now let us consider the morphism of complexes \(\phi : C^*[−1] → C\) with components \(\phi_1 = 0\) and
\[
\phi_0 : \text{Hom}(E_1, E_2) → \text{End}E_1 ⊕ \text{End}E_2 : \Psi → (\Phi \Psi, \Psi \Phi).
\]
(2.1)

Since \(d ∘ \phi_0 = 0\), we have indeed the morphism of complexes. Therefore, we can take the induced map on hypercohomologies
\[
H_T = \phi_* : H^1(X, C^*[−1]) → H^1(X, C).
\]

Note that we get a map from the cotangent space to the tangent space of \(M_σ\) at \(T\). This construction easily globalizes to give a morphism \(H\) from the cotangent bundle to the tangent bundle of \(M_σ\).

**Theorem 2.1.** \(H\) defines a Poisson structure on \(M_σ\).

**Proof.** Let us check that \(H^* = −H\). First of all we claim that \(\phi^*[−1] = \phi\) in the homotopy category of complexes. Indeed, by definition \(\phi^*[−1]\) has components \((\phi^*[−1])_0 = 0\) and
\[
(\phi^*[−1])_1 : \text{End}E_1 ⊕ \text{End}E_2 → \text{Hom}(E_2, E_1) : (A, B) → A\Phi + \Phi B.
\]

Now let us consider the map
\[
h : \text{End}E_1 ⊕ \text{End}E_2 → \text{End}E_1 ⊕ \text{End}E_2 : (A, B) → (−A, B).
\]

Immediate check shows that \(h\) provides a homotopy from \(\phi^*[−1]\) to \(\phi\). Now the skew-commutativity of \(H\) follows immediately from the skew-commutativity of the natural pairing \(H^1(C) ⊗ H^1(C’) → H^2(C ⊗ C’)\) that comes from the minus sign in the commutativity constraint for the tensor product of complexes.

The Jacobi identity will be proven in section 3 in more general situation.

We can interpret the Poisson bivector \(H\) in terms of \(SL_2\)-equivariant bundles on \(X × \mathbb{P}^1\) as follows. Let \(F\) be the extension \([5]\) associated with a triple \(T\). Then the tangent space to \(M_σ\) at \(T\) is identified with \(H^1(X × \mathbb{P}^1, \text{End}F)^{SL_2}\), hence by Serre duality the cotangent space is identified with \(H^1(X × \mathbb{P}^1, \text{End}(F)(−2))\). Now we claim that \(H_T\) is induced by some canonical morphism
\[
α : \text{End}F(−2) → \text{End}F
\]
on \(X × \mathbb{P}^1\). Namely, let \(\text{End}(F, p^*E_1)\) be the kernel of the natural projection \(\text{End}F → \text{Hom}(p^*E_1, p^*E_2(2))\). The dual morphism to the embedding gives a morphism \(\text{End}F → \text{End}(F, p^*E_1)^*\). Thus, to construct \(α\) it is sufficient to construct a morphism
\[
\tilde{α} : \text{End}(F, p^*E_1)^*(−2) → \text{End}(F, p^*E_1).
\]
Now the bundle \( \text{End}(F, p^* E_1) \) sits in the following exact triple
\[
0 \to \text{Hom}(p^* E_2(2), p^* E_1) \to \text{End}(F, p^* E_1) \to \text{End}(p^* E_1) \oplus \text{End}(p^* E_2) \to 0.
\]
We have the morphism to the last term of this triple
\[
p^* \phi_0 : \text{Hom}(p^* E_1, p^* E_2) \to \text{End}(p^* E_1) \oplus \text{End}(p^* E_2)
\]
where \( \phi_0 \) is defined in \([2,1]\). It is easy to check that \( p^* \phi_0 \) lifts uniquely to a morphism
\[
\text{Hom}(p^* E_1, p^* E_2) \to \text{End}(F, p^* E_1).
\]
Now we define \( \tilde{\alpha} \) to be the composition of the latter morphism with the natural projection \( \text{End}(F, p^* E_1)^*(-2) \to \text{Hom}(p^* E_1, p^* E_2) \).

One has the natural morphism \( \det : \mathcal{M}_\sigma \to \text{Pic}(X)^2 \) associating to a triple \((E_1, E_2, \Phi)\) the pair of line bundles \((\det E_1, \det E_2)\). We claim that \( \det \) is a Casimir morphism, i.e. preimage of any local function downstairs is a Casimir function upstairs (that is a function having zero Poisson bracket with any other function). Indeed, the cotangent map to \( \det \) is just the natural embedding
\[
i : H^0(X, \mathcal{O}_X)^2 \to H^1(X, C^*)
\]
which factors through \( H^0(X, C^1) = H^0(X, \text{End} E_1) \oplus H^0(X, \text{End} E_2) \). On the other hand, \( H \) factors through the map \( H^1(X, C^*) \to H^1(X, C^0) \), hence the image of \( i \) is killed by \( H \). In particular, the Poisson bracket on \( \mathcal{M}_\sigma \) induces Poisson brackets on the fibers of the morphism \( \det \). These fibers can be identified with moduli spaces \( \mathcal{M}_\sigma(L_1, L_2, r_1, r_2) \) of triples with fixed determinants \( \det E_i \simeq L_i \). Tensoring with a fixed line bundle \( L \) gives a Poisson isomorphism \( \mathcal{M}_\sigma(L_1, L_2, r_1, r_2) \simeq \mathcal{M}_\sigma(L_1 \otimes L^{\otimes r_1}, L_2 \otimes L^{\otimes r_2}, r_1, r_2) \). An automorphism \( \phi : X \to X \) induces an isomorphism of moduli spaces \( \mathcal{M}_\sigma(L_1, L_2, r_1, r_2) \to \mathcal{M}_\sigma(\phi^* L_1, \phi^* L_2, r_1, r_2) \) compatible with Poisson structures.

3. Special cases

For any bundle \( E_1 \) and a subbundle \( E_2 \subset E_1 \) let us denote by \( \text{End}(E_1, E_2) \) the sheaf of local homomorphisms of \( E_1 \) preserving \( E_2 \). In other words, this is the kernel of the natural projection \( \text{End}(E_1) \to \text{Hom}(E_2, E_1/E_2) \).

**Lemma 3.1.** Let \( T = (E_1, E_2, \Phi) \) be a \( \sigma \)-stable triple such that \( \Phi : E_2 \to E_1 \) is an embedding of \( E_2 \) as a subbundle. Then the tangent space to \( \mathcal{M}_\sigma \) at \( T \) can be identified with \( H^1(X, \text{End}(E_1, E_2)) \).

**Proof.** The natural embedding \( \text{End}(E_1, E_2) \to \text{End}(E_1) \oplus \text{End}(E_2) \) induces the quasi-isomorphism \( \text{End}(E_1, E_2) \to C^* \). Hence the assertion. 
\( \square \)
Under the identification of this lemma our Poisson structure at the triple $T$ for which $\Phi$ is an embedding of a subbundle can be described as follows. From the exact triple
\[
0 \to \text{Hom}(E_1/E_2, E_2) \to \text{End}E_1 \to \text{End}(E_1, E_2)^* \to 0
\]
we get a boundary homomorphism
\[
H^0(X, \text{End}(E_1, E_2)^*) \to H^1(X, \text{Hom}(E_1/E_2, E_2)).
\]
Composing it with the natural morphism
\[
H^1(X, \text{Hom}(E_1/E_2, E_2)) \to H^1(X, \text{End}(E_1, E_2))
\]
we get a morphism
\[
H^1(X, \text{End}(E_1, E_2))^* \simeq H^0(X, \text{End}(E_1, E_2)^*) \to H^1(X, \text{End}(E_1, E_2)),
\]
which coincides with $H_T$ under identification of Lemma 3.1. Equivalently, we may start with the natural morphism
\[
H^0(X, \text{End}(E_1, E_2)^*) \to H^0(X, \text{Hom}(E_2, E_1/E_2))
\]
and compose it with the boundary homomorphism
\[
H^0(X, \text{Hom}(E_2, E_1/E_2)) \to H^1(X, \text{End}(E_1, E_2))
\]
coming from the exact triple
\[
0 \to \text{End}(E_1, E_2) \to \text{End}E_1 \to \text{Hom}(E_2, E_1/E_2) \to 0.
\]
\hspace{1cm} (3.1)

The equivalence of this description with the previous one follows immediately from the commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(E_1/E_2, E_2) & \longrightarrow & \text{End}(E_1, E_2) \\
\downarrow \text{id} & & \downarrow \\
\text{Hom}(E_1/E_2, E_2) & \longrightarrow & \text{End}E_1
\end{array}
\]
\[
\begin{array}{ccc}
\text{End}E_1 & \longrightarrow & \text{End}(E_1, E_2)^* \\
\downarrow & & \downarrow \text{id} \\
\text{End}(E_1, E_2)^* & \longrightarrow & \text{Hom}(E_2, E_1/E_2)
\end{array}
\]
\hspace{1cm} (3.2)

It is easy to check using the above descriptions that the restriction of our Poisson bracket to the space of triples for which $\Phi$ is an embedding of a subbundle is a particular case of the Poisson bracket on moduli of principal bundles over parabolic subgroups defined by Feigin and Odesskii in [7].
Lemma 3.2. Keep the assumptions of Lemma 3.1. Then we have an exact triple

$$0 \to \text{End} E_2 \oplus \text{End}(E_1/E_2) \to \ker H_T \to \text{End} E_1/\text{End}(E_1, E_2) \to 0.$$ 

If $\Phi(E_2)$ is a direct summand of $E_1$ then $H$ vanishes at $T$. Otherwise, the dimension of $\ker H_T$ is minimal (and equals to 2) if and only if the bundles $E_1$, $E_2$ and $E_1/E_2$ are simple.

Proof. Considering the second of the above descriptions of $H_T$ we see immediately that there is an exact sequence

$$0 \to \text{End} E_2 \oplus \text{End}(E_1/E_2) \to \ker H_T \to \ker(H^0(X, \text{Hom}(E_2, E_1/E_2)) \to H^1(\text{End}(E_1, E_2)).$$

(3.3)

Using (3.1) the last term can be identified with $\text{End} E_1/\text{End}(E_1, E_2)$. Moreover, the last arrow in (3.3) is surjective since we have a natural map $\text{End} E_1 \to \ker H_T$ coming from the morphism $\text{End} E_1 \to \text{End}(E_1, E_2)^*$, and the composition of this map with the last arrow of (3.3) is just the canonical projection to $\text{End} E_1/\text{End}(E_1, E_2)$.

If $\Phi(E_2)$ is a direct summand in $E_1$ then the boundary homomorphism used in the definition of $H_T$ is zero, hence $H_T = 0$. Otherwise, $\dim \ker H_T = 2$ if and only if $E_2$ and $E_1/E_2$ are simple and all endomorphisms of $E_1$ preserve $\Phi(E_2)$. We claim this can happen only when $E_1$ is also simple. Indeed, let $A : E_1 \to E_1$ be any endomorphism. By assumption $A$ preserves $\Phi(E_2)$. Adding a constant to $A$ we may assume that $A|_{\Phi(E_2)} = 0$. Then it induces a map $E_1/E_2 \to E_2$. However, $\sigma$-stability of our triple implies by Lemma 4.4 of [5] that $\text{Hom}(E_1, E_2) = 0$ since $\Phi$ is not an isomorphism in our situation. It follows that $\text{Hom}(E_1/E_2, E_2) = 0$, hence, $A = 0$. Thus, $\text{End} E_1 = \mathbb{C}$. 

We are mainly interested in the case when $E_2 = \mathcal{O}_X$, det $E_1$ is fixed. In terms of parameter $\tau$ the stability condition on $\Phi : \mathcal{O}_X \to E_1$ is that $\mu(E'_1) < \tau$ for every proper non-zero subbundle $E'_1 \subset E_1$ and $\mu(E_1/E'_1) > \tau$ for every proper subbundle $E'_1 \subset E_1$ such that $\Phi \in H^0(X, E'_1)$.

Now let $E$ be a stable bundle on $X$ of degree $d$ and rank $r$ (in particular, $\gcd(d, r) = 1$). Set $\tau = \mu(E) = \frac{d}{r}$ and consider the moduli space $\mathcal{M}_r(\det E, \mathcal{O}_X, r + 1, 1)$. It is easy to see that the stability condition on such a triple $\Phi : E_2 = \mathcal{O}_X \to E_1$ is equivalent to the condition that $\Phi$ is nowhere vanishing section and the quotient $E_1/\Phi(\mathcal{O}_X)$ is a stable bundle. Moreover, since there exists a unique stable bundle of rank $r$ and determinant $\det E$ it follows that $E_1/\Phi(\mathcal{O}_X) \simeq E$. Thus, we can identify the moduli space of such triples with the projective space $\mathbb{P} \text{Ext}^1(E, \mathcal{O}_X)$. If $\gcd(d, r + 1) = 1$ then generic extension of $E$ by
\[ \mathcal{O}_X \text{ is stable. Hence, according to Lemma } 3.2 \text{ in this case the Poisson bracket on } \mathbb{P} \text{ Ext}^1(E, \mathcal{O}_X) \text{ is symplectic at general point.} \]

Let \( t_x : X \to X \) be the translation by \( x \in X. \) Then by functoriality we have a natural Poisson isomorphism

\[ \mathbb{P} \text{ Ext}^1(E, \mathcal{O}_X) \cong \mathbb{P} \text{ Ext}^1(t_x^*E, \mathcal{O}_X). \]

Note that as \( x \) varies \( t_x^*E \) runs through all stable bundles of given degree \( d \) and rank \( r. \) Let \( K \subseteq X \) be the finite subgroup of order \( d^2 \) consisting of \( x \) such that \( t_x^* \det E \cong \det E. \) Then for \( x \in K \) one has \( t_x^*E \cong E, \) therefore, \( K \) acts on \( \mathbb{P} \text{ Ext}^1(E, \mathcal{O}_X) \) by linear transformations preserving the Poisson structure.

Another special moduli space associated with a fixed stable bundle \( E \) is \( \mathcal{M}_r(\det E, \mathcal{O}_X, \ker E, 1) \) where \( \tau = \mu(E) \). Then the condition on a triple just means that \( E_1 \) is stable, hence isomorphic to \( E, \) and \( \Phi \) is an arbitrary non-zero section. Therefore, this moduli space can be identified with \( \mathbb{P} \text{ Ext}^1(E, \mathcal{O}_X) \) by linear transformations preserving the Poisson structure.

The latter space sits in the exact sequence

\[ 0 \to H^0(X, E(-D)/\mathcal{O}_X) \to H^1(X, \text{End}\,(E, \mathcal{O}_X(D))) \to H^1(X, \text{End}E) \cong H^1(X, \mathcal{O}_X). \]

It follows that (3.4) induces a map \( T_0^*[s] \to H^0(X, E(-D)/\mathcal{O}_X). \) Furthermore, it is easy to check that its composition with the boundary homomorphism \( H^0(X, E(-D)/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \) is zero, hence, we get a map

\[ T_0^*[s] \to H^0(X, E(-D))/H^0(X, \mathcal{O}_X). \]

Now the latter space is naturally embedded into \( T_0^*[s] \) and the composition with this embedding gives our Poisson structure at \([s] \in \mathbb{P} H^0(X, E).\]

**Lemma 3.3.** Let \( H_0^*[s] : T^*[s] \to T_0^*[s] \) be the above Poisson structure on \( \mathbb{P} H^0(X, E). \) If \( \text{rk } E = 1 \) then \( H_0^*[s] = 0. \) Otherwise, one has an exact sequence

\[ 0 \to H^1(X, \text{ad}(E/\mathcal{O}_X(D))) \to \text{coker } H^0[\text{ad}(E/\mathcal{O}_X(D))] \to H^0(D, E|_D) \to 0 \]

where \( D \) is the zero divisor of \( s. \)
Proof. Let us denote by \( V \subset T_{[s]} \) the subspace \( H^0(X, E(-D)) / H^0(X, \mathcal{O}_X) \). Since the image of \( H_{[s]} \) is contained in \( V \) we have the exact sequence
\[
0 \to \text{coker}(T_{[s]} \to V) \to \text{coker} H_{[s]} \to T_{[s]} / V \to 0.
\]
Since \( E(-D) \) is stable of positive slope, it follows that \( H^1(X, E(-D)) = 0 \), hence we have an isomorphism
\[
T_{[s]} / V \cong H^0(X, E) / H^0(X, E(-D)) \cong H^0(D, E|_D).
\]
Now the assertion follows easily from the exact sequence
\[
T_{[s]} \to H^1(X, \text{End}(E, \mathcal{O}(D))) \to H^1(X, \text{End}(E/\mathcal{O}_X)) \oplus H^1(X, \mathcal{O}_X) \to 0.
\]

More generally, we can consider the moduli space \( \mathcal{M}_\tau(L, \mathcal{O}_X, r, 1) \) where \( L \) is a fixed line bundle of degree \( d \), \( \tau = \frac{4}{r} + \epsilon \) where \( \epsilon > 0 \) is sufficiently small. Then we get the moduli space of pairs \( s : \mathcal{O}_X \to E \) where \( E \) is a semistable bundle with determinant \( L \), \( \text{rk} \ E = r \), \( s \) is a section which doesn’t belong to any destabilizing subbundle of \( E \). We have a Casimir morphism from this moduli space to the moduli stack of semistable bundles, so the fibers inherit the Poisson structure. In particular, if we take the semistable bundle \( E = (E_0)^{\oplus k} \) where \( E_0 \) is a stable bundle, then the corresponding fiber is identified with the Grassmannian \( G(k, H^0(E_0)) \) of \( k \)-dimensional subspaces in \( H^0(E_0) \), so we get some family of Poisson structures on the Grassmannians.

4. Fourier transforms

Let \( m : X \times X \to X \) be the group law on \( X \), \( x_0 \in X \) be the neutral element. Let
\[
\mathcal{P} = m^* \mathcal{O}_X(x_0) \otimes p_1^* \mathcal{O}_X(-x_0) \otimes p_2^* \mathcal{O}_X(-x_0)
\]
be the Poincaré line bundle on \( X \times X \) inducing an isomorphism of \( X \) with the dual elliptic curve. We denote by \( \mathcal{F} \) the corresponding Fourier-Mukai transform which is an autoequivalence of the the derived category \( D^b(X) \) of coherent sheaves on \( X \) given by
\[
\mathcal{F}(E) = Rp_{2*}(Lp_1^* E^L \otimes \mathcal{P}).
\]
One has \( \mathcal{F} \circ \mathcal{F} \simeq (- \text{id}_X)^* [-1] \) (see [9]). It is easy to see that for every \( E \in D^b(X) \) one has \( \text{rk} \mathcal{F}(E) = \deg E, \deg \mathcal{F}(E) = - \text{rk} E \). It follows that if \( T : D^b(X) \to D^b(X) \) is a composition of some sequence of Fourier transforms and tensorings with line bundles, then the vector \( v(T(E)) = (\text{rk} T(E), \deg T(E)) \) is obtained from the vector \( v(E) = (\text{rk} E, \deg E) \) by applying some matrix \( A \in \text{SL}_2(\mathbb{Z}) \). Furthermore, one can lift the natural action of \( \text{SL}_2(\mathbb{Z}) \) on vectors \( (\text{rk}, \deg) \) to the action on
of a central extension of $SL_2(\mathbb{Z})$ by $\mathbb{Z}$ on $D^b(X)$. More precisely, we can consider the standard presentation of $SL_2(\mathbb{Z})$ by generators $S = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and $R = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)$ subject to relations

$$S^2 = (RS)^3, \quad S^4 = 1.$$  

Then the central extension in question is the group $\tilde{SL}_2(\mathbb{Z})$ generated by $S$ and $R$ with the only relation $S^2 = (RS)^3$. The action of this group on $D^b(X)$ is the following: $S$ acts as the Fourier-Mukai transform while $R$ acts as tensoring with $\mathcal{O}_X(x_0)$ (see [3]).

We will consider the action of $SL_2(\mathbb{Z})$ on morphisms of stable bundles. For this it will be useful to know the orbits of $SL_2(\mathbb{Z})$ on pairs of primitive vectors in $\mathbb{Z}^2$. First of all, for a pair of vectors $v_1 = (r_1, d_1)$, $v_2 = (r_2, d_2)$ such that $gcd(r_i, d_i) = 1$ for $i = 1, 2$ the determinant $\det(v_1, v_2) \in \mathbb{Z}$ is invariant of $SL_2(\mathbb{Z})$. We consider only pairs $v_1, v_2$ with $\det(v_1, v_2) \not= 0$. For such pairs there is a second $SL_2(\mathbb{Z})$-invariant $\alpha(v_1, v_2) \in (\mathbb{Z}/\det(v_1, v_2))^*$ defined from the condition

$$v_1 \equiv \alpha(v_1, v_2)v_2 \mod \det(v_1, v_2)\mathbb{Z}^2$$

It is easy to see that the $SL_2(\mathbb{Z})$-orbit of such $(v_1, v_2)$ consists of all pairs with the same det and $\alpha$.

Henceforward, we restrict ourselves to considering stable bundles on $X$ with determinant isomorphic to $\mathcal{O}_X(nx_0)$ for some $n$. The reason is that the group $\tilde{SL}_2(\mathbb{Z})$ preserves the set $S_{x_0}$ of objects of the form $E[k]$ where $k \in \mathbb{Z}$, $E$ is either a stable bundle with determinant $\mathcal{O}_X(nx_0)$ for some $n$, or $\mathcal{O}_{x_0}$. An element of $S_{x_0}$ is determined by its degree and rank uniquely up to a shift. It follows that the group $\tilde{SL}_2(\mathbb{Z})$ acts transitively on $S_{x_0}$. Furthermore, an element $T \in \tilde{SL}_2(\mathbb{Z})$ is completely determined by its action on a pair of elements of $S_{x_0}$ which are not isomorphic up to shift.

The first immediate consequence of the action of $\tilde{SL}_2(\mathbb{Z})$ is that for stable bundles $E_1$ and $E_2$ such that $\det E_1 \simeq \det E_2 \simeq \mathcal{O}_X(dx_0)$ and $\text{rk} E_1 \equiv \text{rk} E_2 \mod (d)$ there is a canonical isomorphism

$$\mathbb{P} \text{Ext}^1(E_1, \mathcal{O}_X) \simeq \mathbb{P} \text{Ext}^1(E_2, \mathcal{O}_X).$$

Indeed, under these conditions there is a unique element $T \in \tilde{SL}_2(\mathbb{Z})$ such that $T(\mathcal{O}_X) \simeq \mathcal{O}_X$ and $T(E_1) \simeq E_2$. Considering the action of $T$ on morphisms from $E_1$ to $\mathcal{O}_X[1]$ we get the above isomorphism.

Now for every stable bundle $E$ with $\det E \simeq \mathcal{O}_X(dx_0)$ where $d > 1$ we can find an element $T \in \tilde{SL}_2(\mathbb{Z})$ such that $T(E) \simeq \mathcal{O}_X$. Then $T(\mathcal{O}_X) = E'[n]$ for some stable bundle $E'$ and some $n \in \mathbb{Z}$. Since
Hom(\(O_X, E\)) \neq 0 we should have Hom(\(T(O_X), T(E)\)) \neq 0, hence \(n = 0\) or \(-1\). Consider first the case \(n = -1\). Then one has det \(E' \simeq O_X(dx_0)\) and

\[
 r \cdot r' \equiv -1 \mod (d)
\]

where \(r = \text{rk } E, r' = \text{rk } E'\) (this is deduced comparing invariants \(\alpha\) for the pair of vectors \((v(O_X), v(E))\) and its image under \(T\)). Conversely, for every \(E'\) satisfying these conditions there exists a unique element \(T \in \text{SL}_2(\mathbb{Z})\) sending \(E\) to \(O_X\) and \(O_X\) to \(E'[-1]\). The transformation \(T\) induces an isomorphism

\[
T_* : \mathbb{P}H^0(E) \xrightarrow{\cong} \mathbb{P} \text{Ext}^1(E', O_X).
\]  

(4.1)

(an isomorphism of this kind with \(r = 1, r' = d - 1\) was constructed in [8] by a different method.) Note that in the previous section we identified both sides of the isomorphism \([\mathbb{L}]\) with some special moduli spaces of pairs, in particular, they carry natural Poisson structures.

**Proposition 4.1.** The isomorphism \(T_*\) is compatible with Poisson structures.

**Proof.** Let \(s : O_X \to E\) be a non-zero section, \(O_X \to \tilde{E} \to E'\) be the corresponding extension of \(E'\) by \(O_X\) with class \(T_*(s) \in \text{Ext}^1(E', O_X)\). It suffices to prove that \(T_*\) preserves Poisson structure over a non-empty open subset, hence we can assume that \(E/O_X\) has no torsion.

By Serre duality the cotangent space \(T_{[s]}^* \mathbb{P}H^0(E)\) can be identified with

\[
\ker(\text{Ext}^1(E, O_X) \to H^1(O_X)) \simeq \text{Ext}^1(E/O_X, O_X)/C \cdot e
\]

where \(e \in \text{Ext}^1(E/O_X, O_X)\) is the class of the extension \(O_X \to E \to E/O_X\). Under this identification the Poisson bracket on \(\mathbb{P}H^0(E)\) at the point \([s]\) is induced by the natural morphism

\[
H_{[s]} : \text{Ext}^1(E/O_X, O_X)/C \cdot e \to T_{[s]}^* \mathbb{P}H^0(E)
\]

which comes from the identification of \(T_{[s]}^* \mathbb{P}H^0(E)\) with \(\ker(H^1(\text{End}(E, O_X)) \to H^1(O)^2)\) and from the natural morphism

\[
\text{Ext}^1(E/O_X, O_X) = H^1(\text{Hom}(E/O_X, O_X)) \to H^1(\text{End}(E, O_X)).
\]

In other words, we have a morphism from a neighborhood of \([e]\) in the space of extensions \(\mathbb{P} \text{Ext}^1(E/O_X, O_X)\) to \(\mathbb{P}H^0(E)\) (since in the neighborhood of \([e]\) such an extension is necessarily isomorphic to \(E\)), and \(H_{[s]}\) is just the tangent map to this morphism at the point \([e]\).

Similarly, the Poisson bracket on \(\mathbb{P} \text{Ext}^1(E', O_X)\) at the point \([T_*, s]\) can be identified with the tangent map

\[
\text{Hom}(\tilde{E}, E')/C \cdot f \to T_{[T_*(s)]} \mathbb{P} \text{Ext}^1(E', O_X)
\]
to the local morphism from $\mathbb{P} \text{Hom}(\tilde{E}, E')$ to $\mathbb{P} \text{Ext}^1(E', \mathcal{O}_X)$ at the point $[f]$ where $f : \tilde{E} \to E'$ is the canonical morphism. Here we use the natural identification of $\text{Hom}(\tilde{E}, E')/\mathbb{C} \simeq \ker(H^0(E') \to H^1(\mathcal{O}_X))$ with the cotangent space to $\mathbb{P} \text{Ext}^1(E', \mathcal{O}_X)$ at $[T_s(s)]$. Now we have the following commutative square of local morphisms in the neighborhood of points induced by $s$:

\[
\begin{array}{ccc}
\mathbb{P} \text{Ext}^1(E/\mathcal{O}_X, \mathcal{O}_X) & \longrightarrow & \mathbb{P} H^0(E) \\
\downarrow T_* & & \downarrow T_* \\
\mathbb{P} \text{Hom}(\tilde{E}, E') & \longrightarrow & \mathbb{P} \text{Ext}^1(E', \mathcal{O}_X)
\end{array}
\] (4.2)

Considering the corresponding commutative square of tangent maps we get the compatibility of $T_*$ with Poisson brackets.

For some other choice of autoequivalence $T : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ sending $E$ to $\mathcal{O}_X$ one has $T(\mathcal{O}_X) \simeq (E'')^*$ for a stable bundle $E''$ of degree $\deg E'' = \deg E = d$ and rank $r''$ satisfying the congruence

$$r'' \cdot r \equiv 1 \pmod{d}$$

where $r = \text{rk } E$. In this case we get an isomorphism

$$\mathbb{P} H^0(E) \simeq \mathbb{P} H^0(E'').$$

We claim that it is also compatible with the natural Poisson structures on both sides. Indeed, this is proven exactly as in the above proposition using the following commutative diagram of local morphisms:

\[
\begin{array}{ccc}
\mathbb{P} \text{Ext}^1(E/\mathcal{O}_X, \mathcal{O}_X) & \longrightarrow & \mathbb{P} H^0(E) \\
\downarrow T_* & & \downarrow T_* \\
\mathbb{P} \text{Hom}(E'', E''/\mathcal{O}_X) & \longrightarrow & \mathbb{P} H^0(E'')
\end{array}
\] (4.3)

Combining these isomorphisms we also get Poisson isomorphisms between $\mathbb{P} \text{Ext}^1(E_1, \mathcal{O}_X)$ and $\mathbb{P} \text{Ext}^1(E_2, \mathcal{O}_X)$ for stable bundles $E_1$ and $E_2$ of the same degree $d$ and ranks $r_1$ and $r_2$ satisfying $r_1 r_2 \equiv 1 \pmod{d}$.

We denote by $\mathcal{M}(d, r)$ the projective space $\mathbb{P} \text{Ext}^1(E, \mathcal{O}_X)$ where $E$ is a stable bundle with determinant $\mathcal{O}_X(dx_0)$ and rank $r$ (in particular, $\gcd(d, r) = 1$), considered as a Poisson variety. Then the above results show that $\mathcal{M}(d, r)$ depends only on $d$ and on the residue $r \in (\mathbb{Z}/d\mathbb{Z})^*$, furthermore, we have

$$\mathcal{M}(d, r) \simeq \mathcal{M}(d, r^{-1}).$$ (4.4)
Also for every stable bundle $E$ of degree $d > 0$ and rank $r$ we have an isomorphism of Poisson varieties

$$\mathbb{P}H^0(E) \simeq \mathcal{M}(d, -r^{-1}).$$

The Poisson isomorphism (4.4) is the classical limit of the following isomorphism of Sklyanin algebras

$$Q_{d,r}(x) \simeq Q_{d,r'}(x)$$

for every $\tau \in X$, where $rr' \equiv 1 \mod (d)$ (cf. [6]). To see this isomorphism let us make the substitutions $i = r'i', j = r''i', n = r'(n' + i' - j')$ in the defining relation (0.1) of $Q_{d,r}(x)$. Then using the relation $\theta_{-i}(-x) = a \cdot b^i \cdot \theta_i(x)$ (where $a$ and $b$ are some non-zero constants, $b^d = 1$) we can rewrite the quadratic relations in the form

$$\sum_{n' \in \mathbb{Z}/d\mathbb{Z}} \frac{\theta_{j'-v'-(r'-1)n'}(0)}{\theta_{t'+n'}(x)\theta_{j'-v'-(r'-1)n'}(-x)} t_{j'-n'}t_{i'+n'} = 0. \quad (4.5)$$

Now it is obvious that the map $t_i \mapsto t_{r'i'}$ defines an isomorphism from $Q_{d,r}(x)$ to $Q_{d,r'}(x)$ as required.

5. Birational transformations

If one changes the stability parameter $\tau$ the moduli spaces $\mathcal{M}_\tau(L_1, L_2, r_1, r_2)$ undergo some birational transformations, see [10], [4]. Clearly, these birational transformations are compatible with the Poisson structures on their domain of definition. In particular, considering moduli of pairs $\mathcal{O}_X \to E$ where $\deg E = d$, $\rk E = r$ are such that $\gcd(r, d) = 1$ and $\gcd(r + 1, d) = 1$ we get a Poisson birational transformation from $\mathcal{M}(d, r)$ to $\mathcal{M}(d, -(r + 1)^{-1}) \simeq \mathbb{P}H^0(E)$ where $E$ is a stable bundle of degree $d$ and rank $r + 1$. Let us denote by $R_d \subset \mathbb{Z}/d\mathbb{Z}$ the set of residues $r$ such that $r \in (\mathbb{Z}/d\mathbb{Z})^*$ and $r + 1 \in (\mathbb{Z}/d\mathbb{Z})^*$. The map $\phi : r \mapsto -(r + 1)^{-1}$ preserves $R_d$ and satisfies $\phi^3 = \text{id}$. On the other hand, the involution $\beta : R_d \to R_d : r \mapsto r^{-1}$ also preserves $R_d$ and we have $\phi \beta = \beta \phi^{-1}$. It follows that $\beta$ and $\phi$ generate the action of the symmetric group $S_3$ on $R_d$.

Recall that in the previous section we defined an isomorphism $\mathcal{M}(d, r) \to \mathcal{M}(d, r^{-1}) = \mathcal{M}(d, \beta(r))$.

**Theorem 5.1.** The birational morphisms $\mathcal{M}(d, r) \to \mathcal{M}(d, \phi(r))$ and $\mathcal{M}(d, r) \to \mathcal{M}(d, \beta(r))$ defined above extend to an action of $S_3$ on $\sqcup_{r \in R_d} \mathcal{M}(d, r)$.

**Proof.** For every residue $r \in R_d$ let us denote by $E_r$ a stable bundle with determinant $\mathcal{O}_X(dx_0)$ and rank $\rk E \equiv r \mod (d)$ such that $0 < \rk E < d$. 

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Let us check the relation $\phi^3 = \text{id}$. For this we have to show that the corresponding composition of birational transformations

$$\mathcal{M}(d, r) \to \mathcal{M}(d, \phi(r)) \to \mathcal{M}(d, \phi^2(r)) \to \mathcal{M}(d, r)$$

is the identity. By definition the first arrow is the composition of the map that associates to a generic morphism $f : E_r \to \mathcal{O}_X[1]$ the corresponding morphism $\text{Cone}(f)[−1] : \mathcal{O}_X \to E_{r+1}$ (considered up to constant) with the autoequivalence $T_r \in \widetilde{\text{SL}_2}(\mathbb{Z})$ which sends $E_{r+1}$ to $\mathcal{O}_X[1]$ and $\mathcal{O}_X$ to $E_{\phi(r)}$. It follows that the above triple composition amounts to applying the construction $\text{Cone}(\cdot)[−1]$ thrice (note that in our situation this construction is functorial) and applying the functor $T_{\phi^2(r)}T_{\phi(r)}T_r$. The triple composition of $\text{Cone}(\cdot)[−1]$ is isomorphic to the shift $\text{id}[−2]$, while $T_{\phi^2(r)}T_{\phi(r)}T_r = \text{id}[2]$, hence the assertion.

It remains to check the relation between our birational transformations corresponding to the relation $\phi \beta = \beta \phi$. This amounts to checking the following relation between contravariant functors from $\mathcal{D}b(\mathcal{X})$ to itself:

$$DT_{\beta \phi(r)}DU_{\phi(r)}T_r = U_r$$

where $D : \mathcal{D}b(\mathcal{X})^{\text{op}} \to \mathcal{D}b(\mathcal{X})$ is the duality functor $E \mapsto R\text{Hom}(E, \mathcal{O}_X)$, $U_r \in \widetilde{\text{SL}_2}(\mathbb{Z})$ is the unique element sending $E_r$ to $\mathcal{O}_X$ and $\mathcal{O}_X$ to $E_{\phi(r)}^\ast$. Note that for every $T \in \widetilde{\text{SL}_2}(\mathbb{Z})$ we have $DTD \in \widetilde{\text{SL}_2}(\mathbb{Z})$ (this follows from the compatibility between the Fourier-Mukai transform and duality, cf. [9]), moreover the corresponding involution on $\text{SL}_2(\mathbb{Z})$ is just the conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using this one can check the above identity up to shift. Finally, since both parts send $E_r$ to $\mathcal{O}_X$ the identity follows. \qed

It follows from the above theorem that for every $\sigma \in S_3$ and $r \in R_d$ such that $\sigma(r) = r$ we get a birational automorphism $f_{\sigma}$ of $\mathcal{M}(d, r)$. Since $\beta$ acts as an isomorphism on our moduli spaces there are essentially two different cases to consider: the residue $r \in R_d$ is fixed $\phi$ or by $\phi \beta$. The fixed points of $\phi$ are residues $r$ satisfying

$$r^2 + r + 1 \equiv 0 \mod (d).$$

In this case we get a Poisson birational automorphism $f_\phi$ of $\mathcal{M}(d, r)$ such that $f_\phi^3 = \text{id}$. The map $\phi \beta$ has the only fixed point $r = -2 \mod (d)$ (provided that $d$ is odd), so we get a Poisson birational involution $f_{\phi \beta}$ of $\mathcal{M}(d, d - 2)$.

Let us describe these birational automorphisms more explicitly. Consider first the case when $r^2 + r + 1 \equiv 0 \mod (d)$, i.e. $\phi(r) = r$. Using the notation of the proof of the above theorem we have $T_r(E_{r+1}) = \mathcal{O}_X[1]$,
$T_r(\mathcal{O}_X) = E_r$, and $T_r(E_r) = E_{r+1}[-1]$. Now let us consider the closed subvariety of

$$Z \subset \mathbb{P} \text{Hom}(E_{r+1}, E_r) \times \mathbb{P} \text{Ext}^1(E_r, \mathcal{O}_X) \times \mathbb{P} \text{Hom}(\mathcal{O}_X, E_{r+1})$$

consisting of triples of lines ([v_1], [v_2], [v_3]) such that all three pairwise compositions $v_2 \circ v_1$, $v_3 \circ v_2$ and $v_1 \circ v_3$ are zeroes. It is easy to see that any of three projections of $Z$ to the projective spaces are birational. In fact, general point of $Z$ corresponds to the exact triangle

$$\mathcal{O}_X \xrightarrow{v_3} E_{r+1} \xrightarrow{v_2} E_r \xrightarrow{v_1} \mathcal{O}_X[1].$$

In particular, $Z$ is birational to $\mathcal{M}(d, r) = \mathbb{P} \text{Ext}^1(E_r, \mathcal{O}_X)$. On the other hand, the functor $T_r$ gives rise to the automorphism of $Z$ given by

$$\Phi : (v_1, v_2, v_3) \mapsto (T_r(v_2)[-1], T_r(v_3), T_r(v_1)[-1]).$$

Now $\Phi$ induces our birational automorphism of $\mathcal{M}(d, r)$ with cube equal to the identity. In the simplest non-trivial case $d = 7$, $r = 2$ the functor we use has form $T_r : A \mapsto \mathcal{F}(A(-2x_0))(3x_0)[1]$. In this case one can describe $Z$ as the double blow-up of $\mathcal{M}(7, 2)$. For this it is more convenient to use the isomorphism of $\mathcal{M}(7, 2)$ with $\mathbb{P} H^0(E_3)$. Then we have a natural embedding $S^2 C \to \mathbb{P} H^0(E_3) : D \mapsto H^0(E_3(-D))$ where $D$ is an effective divisor of degree 2 on $X$. We also define the 4-dimensional variety $V \subset \mathbb{P} H^0(E_3)$ containing $S^2 C$ as the union of chords of $S^2 C \subset \mathbb{P} H^0(E_3)$ connecting $D_1$ and $D_2$ in $S^2 C$ such that $D_1 \cap D_2 \neq \emptyset$. Then our variety $Z$ is obtained by first blowing-up of $\mathbb{P} H^0(E_3)$ along $S^2 C$, and then blowing up the proper transform of $V$. The automorphism $\Phi$ cyclically permutes the following three divisors on $Z$: two exceptional divisors and the proper transform of the chord variety of $S^2 C$ (which is a hypersurface in $\mathbb{P} H^0(E_3)$).

In the case $r \equiv -2 \mod (d)$ (where $d$ is odd) the birational autoequivalence of $\mathcal{M}(d, d-2)$ is described as follows. Again using the notation from the proof of Theorem [7] we have $U_{d-1}(E_{d-1}) = \mathcal{O}_X$, $U_{d-1}(\mathcal{O}_X) = E_{d-1}^*$ and $U_{d-1}(E_{d-2}) = E_{d-2}^*[-1]$. Now let us consider the subvariety

$$Y \subset \mathbb{P} \text{Hom}(E_{d-1}, E_{d-2}) \times \mathbb{P} \text{Ext}^1(E_{d-2}, \mathcal{O}_X)$$

consisting of pairs ([v_1], [v_2]) such that the composition $v_2 v_1 \in \text{Ext}^1(E_{d-1}, \mathcal{O}_X)$ is zero. Then both projections of $Y$ to projective spaces are birational. On the other hand, the functor $U_{d-1}$ induces an involution of $Y$ sending $(v_1, v_2)$ to $(U_{d-1}(v_2)^*[-1], U_{d-1}(v_1)^*[-1])$, hence our birational involution of $\mathcal{M}(d, d-2) = \mathbb{P} \text{Ext}^1(E_{d-2}, \mathcal{O}_X)$. 


6. Generalization

In this section we consider a generalization of the main construction of section 2 to the case of principal bundles with other structural groups than GL. For this note that the datum of a triple \((E_1, E_2, \Phi)\) with \(\text{rk } E_i = r_i\) for \(i = 1, 2\) is equivalent to that of a pair \((P, s)\) where \(P\) is a principal bundle with structure group \(\text{GL}_{r_1} \times \text{GL}_{r_2}\) and a section \(s \in V(P)\) of the vector bundle \(V(P)\) associated with the natural representation of \(\text{GL}_{r_1} \times \text{GL}_{r_2}\) on the space of \(r_1 \times r_2\)-matrices.

To generalize this let us consider a general reductive group \(G\) and its representation \(V\). Then one can consider the moduli stack \(\mathcal{M}_{G,V}\) of pairs \((P, s)\) where \(P\) is a principal \(G\)-bundle, \(s \in H^0(X, V(P))\) is a global section of the corresponding vector bundle associated to \(V\) and \(P\). Let \(\mathfrak{g}\) be the Lie algebra of \(G\). Assume that we are given a symmetric invariant tensor \(t \in S^2(\mathfrak{g})^g\). Then \(t\) induces a morphism of \(\mathfrak{g}\)-modules \(t^* : S^2(V) \to S^2(V)\) as follows. Let \(t = \sum_i x_i \otimes y_i\), then

\[t^*(v \otimes v) = \sum (x_i \cdot v) \otimes (y_i \cdot v)\]

where \(\cdot\) denotes the action of \(\mathfrak{g}\) on \(V\).

Assume that \(t^* = 0\). Then fixing a trivialization \(\omega_X \simeq \mathcal{O}_X\) we can construct a Poisson bracket on the smooth locus of \(\mathcal{M}\) as follows. The tangent space to \(\mathcal{M}\) at a point \((P, s)\) can be identified with the hypercohomology space \(H^1(X, C^*)\) where \(C^*\) is the complex \(\mathfrak{g}(P) \xrightarrow{d} V(P)\) concentrated in degrees 0 and 1, where \(\mathfrak{g}(P)\) is the vector bundle associated with \(P\) and the adjoint representation, the map \(d\) is induced by the Lie action of \(\mathfrak{g}(P)\) on \(V(P)\): \(d(A) = A \cdot s\). Hence, the cotangent space can be identified with \(H^1(C^*[-1])\). Now we can construct the morphism of complexes \(\phi : C^*[−1] \to C\) as before setting \(\phi_1 = 0\) and \(\phi_0 : V^*(P) \to \mathfrak{g}(P)\) to be the composition of \(d^*\) with the map \(\mathfrak{g}^*(P) \to \mathfrak{g}(P)\) induced by \(t\). We claim that our condition on \(t\) and \(V\) implies that \(d \circ \phi_0 = 0\) and that the obtained morphism \(H\) from the cotangent space of \(\mathcal{M}\) to the tangent space is skew-symmetric. Indeed, essentially we have to check that for every \(v \in V\) the following composition is zero:

\[V^* \xrightarrow{d^*_v} \mathfrak{g}^* \xrightarrow{t} \mathfrak{g} \xrightarrow{d_\mathfrak{g}} V\]

where \(d_\mathfrak{g}(A) = A \cdot v\). This is equivalent to the condition \(t^*(v \otimes v) = 0\). Now the skew-symmetry follows as before: the homotopy between \(\phi\) and \(\phi^*[−1]\) is constructed using the map \(\mathfrak{g}^*(P) \to \mathfrak{g}(P)\) induced by \(t\).

**Theorem 6.1.** The above construction defines a Poisson bracket on the smooth locus of \(\mathcal{M}\).
Proof. We have to check the Jacoby identity for our bracket. We will use the approach similar to that of [2], [3]. The Jacoby identity can be rewritten in terms of the morphism $H : T^*_M \rightarrow T_M$ as follows:

$$H(\omega_1) \cdot \langle H(\omega_2), \omega_3 \rangle - \langle [H(\omega_1), H(\omega_2)], \omega_3 \rangle + cp(1, 2, 3) = 0$$

(6.1)

where $\omega_i \in T^*_M$ are local 1-forms on $\mathcal{M}$, $[\cdot, \cdot]$ is the commutator of vector fields, $cp(1, 2, 3)$ indicates terms obtained by cyclic permutation of 1, 2 and 3 from the first two terms.

Working over an affine étale open $U \rightarrow \mathcal{M}$ we can represent every 1-form $\omega \in T^*_M(U)$ by a Cech cocycle $(\phi_{ij}, \psi_i)$ for some open covering $\{U_i\}$ of $U \times X$, where $\phi_{ij} \in \Gamma(U_i \cap U_j, V^*(P))$, $\psi_i \in \Gamma(U_i, g^*(P))$ are such that $-d^*\phi_{ij} = \psi_j - \psi_i$ over $U_i \cap U_j$, $P$ is the universal $G$-bundle on $\mathcal{M}$. Similarly, every vector field $v \in T_M(U)$ can be represented by a Cech cocycle $(\alpha_{ij}, \nu_i)$, where $\alpha_{ij} \in \Gamma(U_{ij}, g(P))$, $\nu_i \in \Gamma(U_i, V(P))$ are such that $d\alpha_{ij} = \nu_j - \nu_i$. In terms of these representatives the pairing between $T_M^*$ and $T_M$ takes form

$$\langle (\alpha_{ij}, \nu_i), (\phi_{ij}, \psi_i) \rangle = \text{Tr}\langle (\alpha_{ij}, \psi_j) + (\phi_{ij}, \nu_i) \rangle,$$

(6.2)

where $\text{Tr} : H^1(U \times X, \mathcal{O}_{U \times X}) \rightarrow H^0(U, \mathcal{O}_U)$ is the morphism induced by the trivialization of $\omega_X$.

The map $H$ sends a 1-form $(\phi_{ij}, \psi_i)$ to the vector field represented by the cocycle $(t \circ d^*\phi_{ij}, 0)$ where $t$ is considered as a map $g^* \rightarrow g$. Since $d^*\phi_{ij} = \psi_i - \psi_j$ we have

$$H(\phi_{ij}, \psi_i) = (0, d \circ t(\psi_i)) \mod (\text{im}(\delta))$$

where $\delta$ is the differential in the Cech complex of $C$. Note that since $d \circ t \circ d^* = 0$, we have $d \circ t(\psi_i) = d \circ t(\psi_j)$ over $U_i \cap U_j$, hence we obtain the global section $d \circ t(\psi_i) \in \Gamma(U \times X, V(P))$. It follows that

$$\langle H(\phi_{ij}, \psi_i), (\phi'_{ij}, \psi'_i) \rangle = \text{Tr}(\langle \phi'_{ij}, d \circ t(\psi_i) \rangle) = \text{Tr}(\langle \psi_i' - \psi_j', t(\psi_i) \rangle).$$

(6.3)

Let us consider the relative Atiyah extension for the universal bundle $P$:

$$0 \rightarrow g(P) \rightarrow \mathcal{A}(P) \rightarrow p^*T_M \rightarrow 0,$$

where $p : \mathcal{M} \times X \rightarrow \mathcal{M}$ is the projection, $\mathcal{A}(P)$ is the bundle of relative infinitesimal symmetries of $P$. Then for sufficiently fine covering $\{U_i\}$ a Cech cocycle $(\alpha_{ij}, \nu_i)$ representing a local vector field on $\mathcal{M}$ can be written as follows: $\alpha_{ij} = D_j - D_i$, $\nu_i = D_i(s)$ where $D_i \in \Gamma(U_i, \mathcal{A}(P))$, $s \in V(P)$ is the universal section, the symbol of $D_i$ is equal to the restriction of a given vector field to $U_i$. In particular, for a vector field represented by a cocycle $(0, \nu)$ where $\nu \in \Gamma(U \times X, V(P))$ we have $\nu = D(s)$ for some $D \in \Gamma(U \times X, \mathcal{A}(P))$. 

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After these remarks we can start proving (1.1). Let us denote by $(\phi_{ij}^h, \psi_i^h)$ Cech cocycles representing 1-forms $\omega_h$ for $h = 1, 2, 3$. Let $D^h \in \Gamma(U \times X, \mathcal{A}(P))$ be infinitesimal symmetries corresponding to $H(\omega_h)$ so that $D^h(s) = d \circ t(\psi_i) = t(\psi_i)(s)$ over $U_i$. Then we have

$$H(\omega_1) \cdot \langle H(\omega_2), \omega_3 \rangle = D^1 \cdot \text{Tr}(\langle \psi_i^3 - \psi_j^3, t(\psi_i^2) \rangle) = \text{Tr}(\langle (D^1(\psi_i^3 - \psi_j^3), t(\psi_i^2)) + (\psi_i^3 - \psi_j^3, D^1(t(\psi_i^2))) \rangle).$$

On the other hand, it is easy to compute the commutator:

$$[H(\omega_1), H(\omega_2)] = (0, D^1D^2(s) - D^2D^1(s)) = (0, D^1(t(\psi_i^2)(s)) - D^2(t(\psi_i^1)(s))).$$

Hence, we have

$$\langle [H(\omega_1), H(\omega_2)], \omega_3 \rangle = \text{Tr}(\phi_{ij}^3, D^1(t(\psi_i^2)(s)) - D^2(t(\psi_i^1)(s))) =$$

$$\text{Tr}(\phi_{ij}^3, D^1(t(\psi_i^2))(s) - D^2(t(\psi_i^1))(s)) + t(\psi_i^2)(D^1(s)) - t(\psi_i^1)(D^2(s))) =$$

$$\text{Tr}(\psi_i^3 - \psi_j^3, D^1(t(\psi_i^2)) - D^2(t(\psi_i^1)) - [t(\psi_i^1), t(\psi_i^2)])].$$

It follows that

$$H(\omega_1) \cdot \langle H(\omega_2), \omega_3 \rangle - \langle [H(\omega_1), H(\omega_2)], \omega_3 \rangle =$$

$$\text{Tr}(\langle (D^1(\psi_i^3 - \psi_j^3), t(\psi_i^2)) + \psi_i^3 - \psi_j^3, t(D^1(\psi_i^2)) + t(\psi_i^2) \rangle) =$$

$$\text{Tr}(\langle (D^1(\psi_i^3 - \psi_j^3), t(\psi_i^2)) + \psi_i^3 - \psi_j^3, t(\psi_i^2) \rangle + (\psi_i^3 - \psi_j^3, [t(\psi_i^1), t(\psi_i^2)]) \rangle).$$

Since this is equal to $D^1 \cdot \text{Tr}(\phi_{ij}^3, dt(\psi_i^2)) - \text{Tr}(\phi_{ij}^3, [D^1, D^2](s))$ which is skew-symmetric in $i, j$, we can also skew-symmetrize in $i, j$ the expression obtained above. Then after adding terms obtained by cyclic permutation of 1, 2 and 3 we obtain the trace of the following expression

$$\langle \psi_i^3 - \psi_j^3, [t(\psi_i^1), t(\psi_i^2)] \rangle + \langle [t(\psi_i^1), t(\psi_i^2)] \rangle + cp(1, 2, 3).$$

Up to a coboundary this equals to

$$\langle \psi_i^3 - \psi_j^3, \psi_i^2 \rangle_t - \langle \psi_i^3 - \psi_j^3, \psi_i^2 \rangle_t + cp(1, 2, 3),$$

where we denote $\langle x, y, z \rangle_t = \langle x, t(x), t(y) \rangle$. From the fact that $t \in (S^2 g)^*$ one can deduce easily that $\langle \cdot, \cdot, \cdot \rangle_t$ is $g$-invariant and skew-symmetric. This implies the following identity:

$$\langle \psi_i^3, \psi_j^1, \psi_j^2 \rangle_t - \langle \psi_i^3, \psi_j^1, \psi_j^2 \rangle_t + cp(1, 2, 3) = \langle \psi_i^1, \psi_j^1, \psi_j^2 - \psi_j^3 \rangle_t = - [d^*\phi_{ij}^1, d^*\phi_{ij}^2, d^*\phi_{ij}^3].$$

It remains to notice that $\langle d^*\phi_1, d^*\phi_2, d^*\phi_3 \rangle_t = 0$ for any $\phi_1, \phi_2, \phi_3 \in V^*(P)$. Indeed, we have to show that for any triple of elements $\varphi_1, \varphi_2, \varphi_3 \in V^*$ and any $v \in V$ one has

$$\langle d^*_v \varphi_1, d^*_v \varphi_2, d^*_v \varphi_3 \rangle_t = 0$$
where $d^*_v \varphi_h \in \mathfrak{g}^*$ is defined by $d^*_v \varphi_h(x) = \varphi_h(x \cdot v)$, $h = 1, 2, 3$. Let $t = \sum_i x_i \otimes y_i$. Then

$$\langle d^*_v \varphi_1, d^*_v \varphi_2, d^*_v \varphi_3 \rangle_t = \langle d^*_v \varphi_1, \sum_{i,j} [d^*_v \varphi_2(y_i) x_i, d^*_v \varphi_3(y_j) x_j] \rangle =$$

$$\sum_{i,j} \varphi_1([x_i, x_j] \cdot v) \varphi_2(y_i \cdot v) \varphi_3(y_j \cdot v).$$

Now

$$\sum_{i,j} [x_i, x_j] \cdot v \otimes y_i \cdot v \otimes y_j \cdot v = \sum_{i,j} x_i x_j v \otimes y_i v \otimes y_j v - \sum_{i,j} x_j x_i v \otimes y_i v \otimes y_j v = 0$$

since $\sum_i x_i v \otimes y_i v = 0$. \hfill \Box

Notice that the condition $t_* = 0$ is usually not satisfied when $\mathfrak{g}$ is simple. However, for example if $S^2(V)$ is irreducible and if $\mathfrak{g}$ is simple then we necessarily have $t_* = \lambda \cdot \text{id}$ for some scalar $\lambda$. It follows that we can replace $G$ by its product $G \times \mathbb{G}_m$ with one-dimensional torus, $t$ by its sum with the appropriate multiple of the square of the generator of $\text{Lie}(\mathbb{G}_m)$, so that for the new tensor $t'$ and the same representation $V$ (on which $\mathbb{G}_m$ acts via identity character) the condition $t'_* = 0$ will be satisfied. In the case $G = \text{GL}_{r_1} \times \text{GL}_{r_2}$ and $V = \text{Mat}(r_1, r_2)$ the tensor $t$ is equal to $(t_1, -t_2)$ where $t_1 = \sum E_{ij} \otimes E_{ji}$ is the standard symmetric invariant tensor for $\mathfrak{gl}_{r_1}$, $t_2$ is the similar tensor for $\mathfrak{gl}_{r_2}$. An interesting case is $G = \text{GSp}_{2r}$, the group of invertible matrices preserving the symplectic form up to a scalar. In this case we can take $V$ to be the standard representation of $G$ of rank $2r$, then $S^2(V)$ is isomorphic to the adjoint representation of $\text{Sp}_{2r}$ on which $\text{GSp}_{2r}$ acts naturally, and one can easily find a non-zero invariant tensor $t \in (S^2 \mathfrak{g})^0$ with $t_* = 0$ (in fact, such $t$ is unique up to a constant). The corresponding moduli stack is the stack of the following data: a vector bundle $E$ together with a symplectic form

$$E \otimes E \rightarrow L$$

inducing an isomorphism $E \simeq E^* \otimes L$, and a section $s : \mathcal{O}_X \rightarrow E$. In particular, taking $E$ to be a fixed bundle with a symplectic form as above, we can consider sometimes the appropriate quotient space of $\mathbb{P}H^0(E)$ by the group of GSp-automorphisms of $E$ as a Poisson substack in the above stack. More precisely, we can define the instability condition for such pairs $(E, s)$ depending on a parameter $\tau$: the only difference with the case of GL is that one should consider totally isotropic subbundles of $E$. Then for $\tau = \mu(E) + \epsilon$ we have the Casimir morphism from such moduli space to the stack of semistable
GSp-bundles, hence its fibers inherit the Poisson structure. For example, if $E_0$ is a stable bundle of degree 2 then there is a natural GSp$_4$-structure on $E = E_0 \oplus E_0$ such that both summands are totally isotropic. Then the $\tau$-stability condition (with $\tau = \mu(E_0) + \epsilon$) allows only sections $s \in H^0(E) = H^0(E_0) \oplus H^0(E_0)$ with non-zero projections to both summands. Hence, the space of such sections up to the action of symplectic automorphisms of $E$ is $S^2 \mathbb{P} H^0(E)$, so we get a Poisson structure on the latter variety.

References

[1] A. Bertram, Stable pairs and stable parabolic pairs, J. Algebraic Geom. 3 (1994), 703–724.
[2] F. Bottacin, Poisson structures on moduli spaces of sheaves over Poisson surfaces, Invent. Math. 121 (1995), 421–436.
[3] F. Bottacin, Symplectic geometry on moduli spaces of stable pairs. Ann. Sci. Ecole Norm. Sup. (4) 28 (1995), 391–433.
[4] S. Bradlow, G. Daskalopoulos, R. Wentworth, Birational equivalences of vortex moduli, Topology 35 (1996), 731–748.
[5] S. Bradlow, O. Garcia-Prada, Stable triples, equivariant bundles and dimensional reduction, Math. Ann. 304 (1996), 225–252.
[6] B. Feigin, A. Odesskii, Sklyanin’s elliptic algebras, Functional analysis and its applications, 23 (1989), 207–214.
[7] B. Feigin, A. Odesskii, Vector bundles on elliptic curve and Sklyanin algebras. Preprint.
[8] R. Friedman, J. Morgan, E. Witten, Vector bundles over elliptic fibrations. Preprint.
[9] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. 81 (1981), 153–175.
[10] M. Thaddeus, Stable pairs, linear systems and the Verlinde formula. Invent. Math. 117 (1994), 317–353.