What is Generic Structure of the Three-dimensional Magnetic Reconnection?

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The probability of occurrence of various topological configurations of the three-dimensional reconnection in a random magnetic field is studied. It is found that a specific six-tail spatial configuration should play the dominant role, while all other types of reconnection (in particular, the axially-symmetric fan-like structures) are realized with a much less probability. A characteristic feature of the six-tail configuration is that at the sufficiently large scales it is approximately reduced to the well-known two-dimensional X-type structure; and this explains why the two-dimensional models of reconnection usually work quite well.

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It is commonly recognized that reconnection of the magnetic field lines is of fundamental importance in many branches of plasma physics, ranging from the laboratory installations to various astrophysical objects \cite{1,2,3}. However, this phenomenon was usually studied in the two-dimensional (2D) approximation. The papers on the three-dimensional (3D) reconnection are few in number (e.g., review \cite{4}); and most of them are based on some predetermined types of topology of the magnetic field lines \cite{5,6,7}. As far as we know, the problem of the generic topological configuration(s) of the 3D reconnection was never studied before in a systematic manner, and this is just the aim of the present paper.

A key ingredient of the magnetic reconnection is presence of the so-called null (or singular) points, where all components of the magnetic field \( B \) disappear, which makes this phenomenon possible. Otherwise, it would be prohibited by the theorem of rectification from the theory of differential equations (e.g., Ch. 2 in book \cite{8}): the field lines in the vicinity of a non-singular point should be approximately parallel to each other, which evidently prevents them from reconnection. In the 2D approximation, the null points possess a universal topology of X-type; while much more diverse topological configurations are allowed, in general, in the 3D case.

It was implicitly assumed in many recent papers that the most typical case of the 3D null point is the fan-like structure, i.e., “collision” of two oppositely directed magnetic fluxes with subsequent outflow in the equatorial plane, as illustrated in the left-hand side of Fig.\textsuperscript{1}. Then, the configurations with a finite number of the fan “vanes” (or “tails”) look as particular cases of this generic structure (the right-hand side of the figure). Unfortunately, the probabilities of the above-mentioned configurations were never calculated explicitly, and it will be seen from the subsequent consideration that such an analysis results in the nontrivial conclusions.

Specifically, we shall consider random realizations of the potential magnetic field

\[
\mathbf{B} = -\mathbf{\nabla} \psi, \quad (1)
\]

where the magnetic potential \( \psi \) satisfies the usual Laplace equation:

\[
\Delta \psi = 0. \quad (2)
\]

(The potential field approximation is widely used, for example, in the solar physics and for some kinds of laboratory devices, although it may be less relevant for treating the magnetospheric reconnection.)

Assuming the origin of the spherical coordinate system \((r, \theta, \varphi)\) to be the spot of reconnection, solution of Eq. (2) can be written by the standard way as

\[
\psi(r, \theta, \varphi) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} r^j \psi_{jm}(\theta, \varphi), \quad (3)
\]

where

\[
\psi_{jm}(\theta, \varphi) = P^m_j(\cos \theta) \left[ a_{jm} \cos(m \varphi) + b_{jm} \sin(m \varphi) \right] \quad (4)
\]

are the spherical functions, and \( P^m_j \) are the adjoint Legendre polynomials. (The terms with negative powers of \( r \) in are not taken into account here because we are interested only in the finite solutions.) To avoid dealing with the infinite sum, it is convenient to assume that expression (3) is cut off at some sufficiently large value of \( j \) and, therefore, contains only the finite number of terms \( N \). In other words, \( N \) is the dimensionality of the space of coefficients \( a_{jm} \) and \( b_{jm} \).

Now, if these coefficients are assumed to be random numbers, we get a random realization of the magnetic field \( \mathbf{B} \). It is a separate problem what are the reasonable probability distributions for the coefficients. However, it is important to emphasize that the most of our
Then, the magnetic field components take the form:
\[ \psi^{(1)} = r \left\{ a_{10} \cos \theta - (1 - \cos^2 \theta)^{1/2}[a_{11} \cos \varphi + b_{11} \sin \varphi] \right\}; \]  
(5)
and its substitution into Eq. (1) results in
\[ B_r^{(1)} = - \left\{ a_{10} \cos \theta - (1 - \cos^2 \theta)^{1/2}[a_{11} \cos \varphi + b_{11} \sin \varphi] \right\}. \]  
(6)

Since \( r = 0 \) is assumed to be a null point (i.e., all components of the magnetic field, including \( B_r \), should vanish), we arrive at the requirement:
\[ a_{10} = a_{11} = b_{11} = 0. \]  
(7)

Because of these three constraints, the null point of any kind will be realized only in a subspace of the random expansion coefficients \( a_{jm} \) and \( b_{jm} \) with the dimensionality \( N - 3 \) or less.

Next, at \( j = 2 \), the magnetic potential is written as
\[ \psi^{(2)} = r^2 \left\{ \frac{1}{2} (3 \cos^2 \theta - 1) a_{20} - 3 \sin \theta \cos \left[ a_{21} \cos \varphi + b_{21} \sin \varphi \right] + 3 \sin^2 \theta \left[ a_{22} \cos(2\varphi) + b_{22} \sin(2\varphi) \right] \right\}. \]  
(8)

Since we are interested in structure of the magnetic field lines rather than in absolute values of the field, it is convenient to introduce the normalized coefficients (denoted by a single subscript):
\[ a_m = a_{2m}/a_{20}, \quad b_m = b_{2m}/a_{20}, \quad \text{where} \quad m = 1, 2. \]  
(9)

Then, the magnetic field components take the form:
\[ B_r^{(2)} = -2a_{20}r \left\{ \frac{1}{2} (3 \cos^2 \theta - 1) \right. \]  
\[ \left. - \frac{3}{2} \sin(2\theta) \left[ a_{1} \cos \varphi + b_{1} \sin \varphi \right] + 3 \sin^2 \theta \left[ a_{22} \cos(2\varphi) + b_{22} \sin(2\varphi) \right] \right\}, \]  
(10a)
\[ B_\theta^{(2)} = -3a_{20}r \left\{ \sin(2\theta) \left[ -\frac{1}{2} + a_{2} \cos(2\varphi) + b_{2} \sin(2\varphi) \right] \right. \]  
\[ \left. - \cos(2\theta) \left[ a_{1} \cos \varphi + b_{1} \sin \varphi \right] \right\}, \]  
(10b)
\[ B_\varphi^{(2)} = -3a_{20}r \left\{ 2 \sin \theta \left[ -a_{2} \sin(2\varphi) + b_{2} \cos(2\varphi) \right] \right. \]  
\[ \left. + \cos \theta \left[ a_{1} \sin \varphi - b_{1} \cos \varphi \right] \right\}. \]  
(10c)

Following the standard procedures, the equation of a magnetic field line can be written as
\[ \frac{dr}{B_r/(a_{20}r)} = \frac{r d\theta}{B_\theta/(a_{20}r)} = \frac{r \sin \theta d\varphi}{B_\varphi/(a_{20}r)}. \]  
(11)

Since the quantities \( B_r^{(2)}/(a_{20}r) \), \( B_\theta^{(2)}/(a_{20}r) \), and \( B_\varphi^{(2)}/(a_{20}r) \) do not depend on \( r \), in the limit \( r \to 0 \) we arrive at the conditions specifying the field lines passing immediately through the null point:
\[ B_\theta^{(2)}/(a_{20}r) = 0, \quad B_\varphi^{(2)}/(a_{20}r) = 0. \]  
(12)

Substitution of the detailed expressions (10a), (10c) into Eqs. (12) gives the following set of algebraic equations:
\[ \sin(2\theta^*) \left[ -\frac{1}{2} + a_{2} \cos(2\varphi^*) + b_{2} \sin(2\varphi^*) \right] \]  
\[ - \cos(2\theta^*) \left[ a_{1} \cos \varphi^* + b_{1} \sin \varphi^* \right] = 0, \]  
(13a)
\[ 2 \sin \theta^* \left[ -a_{2} \sin(2\varphi^*) + b_{2} \cos(2\varphi^*) \right] \]  
\[ + \cos \theta^* \left[ a_{1} \sin \varphi^* - b_{1} \cos \varphi^* \right] = 0, \]  
(13b)

where \( \theta^* \) and \( \varphi^* \) are the angles at which the field line enters (or leaves) the null point.

First of all, it can be easily checked that the above set of equations is preserved under the transformation:
\( \theta^* \to \pi - \theta^*, \varphi^* \to \varphi^* + \pi. \) Consequently, the magnetic field lines passing through the null point always appear as the oppositely directed pairs. Therefore, the geometric structures with an odd number of tails (e.g., \( n = 5 \) or \( 7 \) in Fig. 1) cannot exist at all.

Next, let us begin to analyze the particular solutions of Eqs. (13a), (13b). The simplest case evidently takes place at \( a_1 = b_1 = a_2 = b_2 = 0 \) or, in the original designations,
\[ a_{2m} = 0, \quad b_{2m} = 0, \quad \text{where} \quad m = 1, 2. \]  
(14)

Then, these equations are reduced just to the condition
\[ \sin(2\theta^*) = 0. \]  
(15)
which has solutions of the two types:

\[
\theta^* = 0, \pi \text{ and } \theta^* = \pi/2 \text{ (at any } \varphi^*). \tag{16}
\]

This represents a combination of the polar axis and a disk in the equatorial plane, i.e., exactly the fan-like structure depicted in the left-hand side of Fig. 1.

Because of the 4 constraints given by Eqs. (14), this structure seems to be realized in the subspace of coefficients \(a_{jm}\) and \(b_{jm}\), with dimensionality \(N-3-4 = N-7\). However, it should be born in mind that these constraints were formulated for the specific situation when a “spine” of the “fan” was oriented exactly along the polar axis of our coordinate system. In general, such fan-like structure can be rotated in space by three Euler angles, which effectively removes three constraints. So, the dimensionality of the subset of coefficients will be \(N-7+3 = N-4\).

Returning to the general case of arbitrary coefficients \(a_1, b_1, a_2,\) and \(b_2\), it can be naturally assumed that the set of two algebraic equations (13a), (13b) for two unknown variables \(\theta^*\) and \(\varphi^*\) should have a discrete set of solutions, i.e., the number of asymptotic tails in Fig. 1 should be finite. Moreover, as follows from a more careful mathematical analysis, this number is always equal to 6 (except for some special subsets of the coefficients \(a_i, b_i\) with lower dimensionality). To prove this fact, it is convenient to reduce the above-mentioned set of equations to the single equation for the azimuthal angle \(\varphi^*\):

\[
F(\eta(\varphi^*), \zeta(\varphi^*)) = 0, \tag{17}
\]

where \(\eta = \cos \varphi^*, \zeta = \sin \varphi^*,\) and

\[
F(\eta, \zeta) = 4\left[2a_2\eta \zeta - b_2(\eta^2 - \zeta^2)\right](a_1\zeta - b_1\eta)
\times \left[-\frac{1}{2} + a_2(\eta^2 - \zeta^2) + 2b_2\eta\zeta\right]
- \left\{4\left[2a_2\eta \zeta - b_2(\eta^2 - \zeta^2)^2\right] - (a_1\zeta - b_1\eta)^2\right\}
\times (a_1\eta + b_1\zeta). \tag{18}
\]

Then, if the roots \(\varphi^*\) have been found, the corresponding values of the polar angle \(\theta^*\) can be easily restored from one of equations (13a) or (13b).

Since formula (18) represents a quite complex polynomial expression, the simplest approach to resolve our task is just to perform a statistical simulation: let us generate a sufficiently large sequence of random coefficients \(a_1, a_2, b_1, b_2\) (e.g., as a Gaussian distribution with a zero mean) and then analyze behavior of the function \(F(\eta(\varphi^*), \zeta(\varphi^*))\) graphically (Fig. 2). Surprisingly, it was found that the plot of \(F\) intersects the horizontal axis always in 3 points at the interval \(\varphi^* \in [0, \pi]\) (and, consequently, in 6 points at the interval \(\varphi^* \in [0, 2\pi]\)). In fact, a subsequent careful analysis enabled us to get a rigorous mathematical proof of this fact. However, because of the cumbersome formulas, this proof requires a separate paper; so that we prefer to appeal here just to the results of statistical simulation. Besides, it was established that these six solutions of Eqs. (13a), (13b) correspond geometrically to the six “tails” which are mutually orthogonal to each other.

Therefore, we have found that a generic 3D null point, responsible for the magnetic reconnection, should have the specific six-tail structure, i.e., possess 6 asymptotic directions of the magnetic field. This is because it is realized in the subspace of coefficients of the random field with dimensionality \(N-3\), i.e., almost in the entire space allowed for the null point by the constraints (7).

All other configurations (particularly, the widely-discussed fan-type structure or more exotic geometric structures outlined in the old work [20]) should emerge with a much less probability, because they are realized in the subspaces of coefficients with lower dimensionality. It is especially important to emphasize that, since these conclusions are based only on the dimensionality of the relevant subspaces, they should be valid for any nonsingular probability distribution of the random-field coefficients. (So, the particular Gaussian distribution used in the simulations presented in Fig. 2 does not affect the final result.)

It is interesting, of course, to discuss a pattern of the magnetic field lines in the vicinity of the above-mentioned generic configuration. To avoid cumbersome formulas, let us consider the simplest (but completely representative) case \(a_1 = b_1 = b_2 = 0, a_2 \neq 0,\) which corresponds to the situation when the six-tail structure is oriented along the axes of the coordinate system. Then, Eqs. (13a) and (13b) are simplified to

\[
\sin(2\theta^*) \left[ -\frac{1}{2} + a_2 \cos(2\varphi^*) \right] = 0, \tag{19a}
\]

\[
a_2 \sin \theta^* \sin(2\varphi^*) = 0. \tag{19b}
\]

Their solutions are evidently \(\theta^* = 0, \pi\) and \(\varphi^* = \frac{\pi}{2}, \varphi^* = \frac{\pi}{2}\).
0, π/2, π, 3π/2, which correspond just to the six semiaxes of the coordinate system.

Next, omitting the unessential common multiplier a20, expressions (10a)–(10c) for the magnetic field components are reduced to

\[ B_r = -2r \left[ \frac{1}{2} (3 \cos^2 \theta - 1) + 3a_2 \sin^2 \theta \cos(2\varphi) \right], \quad (20a) \]

\[ B_\theta = -3r \sin(2\theta) \left[ -\frac{1}{2} + a_2 \cos(2\varphi) \right], \quad (20b) \]

\[ B_\varphi = 6a_2 r \sin \theta \sin(2\varphi). \quad (20c) \]

As expected, \( B_\theta \) and \( B_\varphi \) vanish immediately at the coordinate axes, while \( B_r \) changes its sign on the opposite sides from the origin.

Substituting expressions (20a)–(20c) into (11) and performing the integration, we can easily find formulas for the magnetic field lines in three coordinate planes. For example, in the \( xy \)-plane (\( \theta = \pi/2 \)) the final result will take the form:

\[ r = C \left( \left| \sin \varphi \right|^{1/3} \left| \cos \varphi \right|^{1/3} \right)^{-1/2}, \quad (21) \]

where \( C \) is an arbitrary constant.

Behaviour of this function has three qualitatively different regimes, depending on the value of coefficient \( a_2 \):

(a) If \( a_2 < -1/6 \) or \( a_2 > 1/6 \), then \( r \rightarrow \infty \) both at \( \varphi \rightarrow 0 \) and \( \varphi \rightarrow \pi/2 \). This evidently corresponds to the field lines of hyperbolic type.

(b) If \(-1/6 < a_2 < 0\), then \( r \rightarrow \infty \) at \( \varphi \rightarrow 0 \) and \( r \rightarrow 0 \) at \( \varphi \rightarrow \pi/2 \). These are the field lines of parabolic type with the parabola axis oriented in \( x \)-direction.

(c) If \( 0 < a_2 < 1/6 \), then \( r \rightarrow 0 \) at \( \varphi \rightarrow 0 \) and \( r \rightarrow \infty \) at \( \varphi \rightarrow \pi/2 \). These are the field lines of parabolic type with the parabola axis oriented in \( y \)-direction.

As regards the field lines in two other coordinate planes, they can be shown to have a hyperbolic structure in this range of \( a_2 \). Just this case is illustrated in Fig. 3.

Furthermore, it can be proved that the same pattern of the field lines can be associated with all other above-mentioned cases merely by interchanging the roles of the various coordinate axes. Using the terminology adopted in the theory of differential equations, we should say that the field lines have the node structure in one of the coordinate planes and the saddle structure in two other planes. Let us mention also that the discussed six-tail arrangement of the magnetic field lines in the vicinity of 3D null point is not actually a new finding: it was occasionally mentioned in the earlier works (e.g., Fig. 1 in paper [21]) but was never assumed to play a dominant role in the 3D magnetic reconnection.

By the way, using the pattern presented in Fig. 3, it can be easily understood why the probability of occurrence of the fan-type structure (left-hand panel in Fig. 4) should be substantially suppressed as compared to the six-tail structure. Let us pay attention to the behavior of field lines in the \( xy \)-plane and assume that initially the value of the parameter \( a_2 \) corresponds to the case (c), as depicted in the left-hand panel of Fig. 4. Next, let us assume that the value of \( a_2 \) gradually decreases and becomes negative, which refers to the case (b). From the geometrical point of view, this process corresponds to a gradual decrease in the curvature of the field lines; and at some instant they become bent in another direction, i.e., the entire pattern remains parabolic but the parabola axis jumps by \( \pi/2 \) (right-hand panel in Fig. 4). The boundary between these two cases is just the fan-type structure (depicted in the central panel), which is realized at \( a_2 = 0 \). In other words, there are infinitely many six-tail configurations of the types (b) and (c) and only one intermediate fan-type configuration. This explains pictorially why the probability of its realization should be very small.

At last, returning to Fig. 3, attention should be drawn to the fact that the six asymptotic directions of the mag-

FIG. 3: Sketch of magnetic field lines in the six-tail configuration. Solid (blue) curves represent the field lines in the horizontal \( xy \)-plane; and dotted (red) curves, in the vertical \( yz \)-plane. Field lines in another vertical plane \( xz \), perpendicular to the plane of the figure, are not shown here; they have the same hyperbolic structure as in the \( yz \)-plane.

FIG. 4: Appearance of the fan-type structure as an intermediate case between the two six-tail configurations. Only field lines in the \( xy \)-plane are drawn here; the field lines in the \( xz \)- and \( yz \)-planes are always of the saddle type.
netic field are rather different from each other. Four of them ($y$, $-y$, $z$, and $-z$) can be called “dominant”, because the most of field lines tend to approach one of these directions when they go away from the null point. (Directions of the arrows are not important here.) On the other hand, two other asymptotic directions ($x$ and $-x$) should be called “recessive”, because the most of field lines tend to depart from them. Therefore, the recessive directions will be “lost” when viewed “from a large distance”, and the entire pattern will look like a classical two-dimensional X-point structure. This fact can explain why 2D models of the magnetic reconnection often work quite well. Let us mention that the above behavior of the field lines is confirmed also by our straightforward numerical simulations of the geometry of random magnetic fields, which will be published elsewhere.

In summary, we calculated the probability of occurrence for various kinds of the 3D null points in random magnetic fields and studied in detail the structure of the magnetic field lines in their vicinity. As a result, it was found that: (1) contrary to the intuitive expectations, the most likely case of the 3D null point, responsible for the magnetic reconnection, is the specific six-tail structure, in which all tails are oriented in the mutually orthogonal directions; (2) all other kinds of the 3D null points (in particular, the intuitively-appealing fan-type structure) are realized with much less probabilities, as schematically summarized in Fig. 1; (3) the six-tail structure possesses one cross-section with the purely parabolic field lines (the node) and two cross-sections with the purely hyperbolic field lines (the saddles); (4) at the sufficiently large distances, the generic six-tail structure is approximately reduced to a quasi-2D configuration with the well-known topology of X-point, which explains why the 2D approach is often a good approximation for the magnetic reconnection. Therefore, it may be conjectured that the specific 3D effects should be important, first of all, in the small-scale magnetic reconnection events (e.g., solar micro- and nano-flares).

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