Optimal Control and Stability Analysis of Nonlinear Control-Affine Systems

Armen Bagdasaryan
Department of Mathematics, College of Engineering and Technology, American University of the Middle East, Kuwait
E-mail: armen.bagdasaryan@aum.edu.kw

Abstract. In this paper we will discuss the problems of optimal control and stabilization for nonlinear control-affine systems of the form

\[ \dot{x} = A(x) + B(x)u \]

where \( x = x(t), \ x \in \mathbb{R}^n, \ u \in \mathbb{R}, |u| \leq 1 \)

and vector functions \( A(x), B(x) \) are assumed to be smooth in the domain \( D \subset \mathbb{R}^n, \ 0 \in D, A(0) = 0 \). We give an in-depth analysis of optimal control problems for nonlinear dynamical systems and then consider for the above systems the problem of synthesis of continuous control \( u = u(x), u(0) = 0 \), that stabilizes the system at the equilibrium point \( (x, u) = (0, 0) \). The solution to the problem is based on the transformation of the system to canonical form, and on nonlinear stabilization of the system.

1. Introduction: Optimal control and stability of dynamical systems

Optimal control problems play major and important roles in science and engineering. Solutions of optimal control problems, from theoretical point of view, can be obtained using Pontryagin’s principle, and also Hamilton-Jacobi-Bellman equation. However, in most cases, solving optimal control problems analytically is too difficult or impossible due to their complexity and significant nonlinearity of systems. In such cases, iterative or numerical methods are employed [1, 2, 3, 4].

Consider a nonlinear differential system with control

\[ \dot{x} = f(t, x(t), u(t)), \quad t \in \tau = [t_0, t_\tau], \]

where \( x \in \mathbb{R}^n \) is a state in the state space, \( u \in \mathcal{U}(t, x) \subset \mathbb{R}^d \) is a control function, \( f : \mathcal{R} \to \mathbb{R}^n \) is a nonlinear, continuous function with domain \( \mathcal{R} \subseteq \mathbb{R}^n \), and functions \( h_x = x(t) \) and \( h_u = u(t) \), \( t \in [t_0, t_\tau] \), are piecewise smooth and piecewise continuous functions, respectively; and the pair \( (h_x = x(t), h_u = u(t)) \) is called a trajectory-control pair defined on the interval \( \tau \).

The standard optimal control problem for the system (1) is defined as

\[ \mathcal{F} = F(x(t_\tau)) \to \inf, \quad \text{where} \ x \in \mathcal{X}(t) \subset \mathbb{R}^n, \ x(t_0) = x_0, \ x(t_\tau) \in \mathcal{C} \subset \mathbb{R}^n \]

on the set \( \mathcal{H} \) of admissible functions \( h = (h_x, h_u) \), defined as follows: for \( x \in \mathbb{R}^n, \ u \in \mathcal{U}, \ t_\tau > 0 \), we say that \( h = (h_x, h_u) \) is admissible if the trajectory \( x(\cdot, x_0, u) \) of (1) associated with \( u \) and starting at \( x_0 \) is well-defined on \( [t_0, t_\tau] \).
Alternatively, the system (1) can also be represented in the form of differential inclusion
\[ \dot{x} \in V(t,x) = f(t,x,u(t,x)). \]

Nowadays, the real-life optimal control problems (1)-(2) are fully nonlinear and degenerate [5, 6, 7, 8, 9]. Degeneracy results from passive differential connections in the optimal control problem setting. For differential control systems the degeneracy is characterized by linear functional dependencies involving control functions in the problem description. The class of degenerate optimal control problems is quite widespread, and can appear both as an independent problem and as a result of reformulation of the original control problem and transition to an equivalent system. This transformation, being universal, means that the differential control system can be transformed into an equivalent control affine system in a non-unique way.

For example, the following system is equivalent to (1)
\[ \dot{x} = f(t,x,u_0) + \sum_{i=1}^{\ell} \alpha_i (f(t,x,u_i) - f(t,x,u_0)), \quad \ell \leq d \] (3)
where \[ j = 0,1,\ldots,j_\tau, \quad x^k(j+1) = x(t_\tau(j)), \quad x(t_0(j+1)) = x^k(j+1), \] (4)
\[ x^k(0) = x_0, \quad u_i \in U, \quad \sum_{i=1}^{\ell} \alpha_i \leq 1, \quad \alpha_i \geq 0. \]
The system (3)-(4) is a hybrid affine system, where the right-hand side of (3) contains convex linear combinations of the right-hand side of the original system (1) for different values of \( u \in U \).

As a consequence, piecewise (control) affine models has recently become a relevant and powerful tool in the approximation of general smooth nonlinear systems that usually capture many features of general systems, and enable a tractable mathematical analysis. The use of optimal control in the class of linear systems permits a substantial reduction of the computations determining the laws of optimal control. It is also an efficient method for solving nonlinear optimal control problems and nonlinear system stability analysis.

The stability of dynamical systems is one of the most basic problems in system theory. The most complete contribution to the stability analysis of nonlinear dynamical systems was introduced by Lyapunov [10]. Lyapunov’s results, along with the Krasovskii–LaSalle invariance principle [11, 12], provide a powerful framework for analyzing the stability of nonlinear dynamical systems. Lyapunov methods have also been used to obtain stabilizing feedback control functions for nonlinear systems that guarantee closed-loop system stability.

The Lyapunov theory includes two methods, Lyapunov’s indirect method and Lyapunov’s direct method. The Lyapunov’s indirect method states that the dynamical system
\[ \dot{x} = f(x,t), \quad \text{where} \ f(0,t) = 0 \text{ for all} \ t \geq 0, \] (5)
has a locally exponentially stable equilibrium point at the origin, if and only if the real parts of the eigenvalues of the Jacobian matrix of \( f \) at zero are all strictly negative.

Consider the system (5) and define
\[ A(t) = \left. \frac{\partial f(x,t)}{\partial x} \right|_{x=0} \] (6)
to be the Jacobian matrix of \( f(x,t) \) with respect to \( x \), evaluated at the origin.

For each fixed \( t \),
\[ R(x,t) = f(x,t) - A(t)x \]
approaches zero as \( x \) approaches zero. In the case when \( R(x,t) \) does not approach zero uniformly, it is required that the stronger condition to be true
\[ \lim_{\|x\| \to 0} \sup_{t \geq 0} \frac{\|R(x,t)\|}{\|x\|} = 0. \] (7)
If equation (5) holds, then the system
\[ \dot{z} = A(t)z \] (8)
is referred to as the (uniform) linearization of equation (5) about the origin. The stability of the linearization, when it exists, determines the local stability of the original nonlinear equation.

**Theorem 1.1** (Stability by linearization). Consider the dynamical system \( \dot{x} = f(x,t) \), where \( f(0,t) = 0 \) for all \( t \geq 0 \). Assume that
\[
\lim_{\|x\| \to 0} \sup_{t \geq 0} \frac{\|R(x,t)\|}{\|x\|} = 0.
\]
Further, let \( A(\cdot) \) defined in (6) be bounded. If 0 is a uniformly asymptotically stable equilibrium point of (8) then it is a locally uniformly asymptotically stable equilibrium point of (5).

Consequently, in order to prove uniform asymptotic stability of the nonlinear system using the preceding theorem 1.1, one requires uniform asymptotic stability of the linearized system. When the system (5) is time-invariant, then the indirect method says that if the eigenvalues of
\[ A(t) = \frac{\partial f(x)}{\partial x} \bigg|_{x=0} \]
are located in the open left half complex plane, then the origin is asymptotically stable.

The theorem 1.1 proves that global uniform asymptotic stability of the linearization implies local uniform asymptotic stability of the original nonlinear system. The estimates provided by the proof of the theorem can be used to give a bound on the domain of attraction of the origin.

Considering the stability of closed-loop or feedback systems, if one has a constructive method to determine the feedback control, such that (1) has the stability properties one would like it to have, the problem can be solved in a very satisfactory manner, which is the case in linear control theory. In linear control theory \( f \) is assumed to be affine in \( x \) and in \( u \) and one searches for a linear feedback that makes the equilibrium at the origin exponentially stable. This means that there is an \( n \times n \)-matrix \( A \) and an \( n \times d \)-matrix \( B \), such that the system (1) has the form
\[ \dot{x} = Ax + Bu(x), \]
and one wants to determine a \( d \times n \)-matrix \( K \), such that \( u(x) = Kx \) makes the equilibrium point at the origin stable, which holds when \( \text{rank}[A|B] = n \), where \( [A|B] = [B, AB, \ldots, A^{n-1}B] \) is the matrix obtained by writing down the columns of \([B, AB, \ldots, A^{n-1}B]\) consequently.

If the function \( f \) is not affine, one can still try to linearize the system about equilibrium and use results from linear control theory to make the origin a locally exponentially stable equilibrium. The disadvantage of this method is, that its region of attraction is not necessarily very large and although there are some methods to slightly extend it, these methods are far from solving the general problem. The lack of a general analytical method to determine a successful feedback function for nonlinear systems has made exact methods and approaches less popular, which has led to the development of approximate, iterative, or fuzzy control design techniques.

Lyapunov’s direct method is a mathematical extension of the fundamental physical observation, that an energy dissipative system must eventually settle down to an equilibrium point. The method is a generalization of the idea that if there is some “measure of energy”, described in terms of positive definite functions, in a system then one can study the rate of change of the energy of the system to ascertain stability. It states that if there is an energy-like function \( V \) for the dynamical system that is strictly decreasing along its trajectories, then the equilibrium at the origin is asymptotically stable. The function \( V \) is then said to be a Lyapunov function for the system. A Lyapunov function provides via its preimages a lower bound of
the region of attraction of the equilibrium. This bound is non-conservative, it extends to the boundary of the domain of the Lyapunov function. Though the Lyapunov theory of dynamical systems is the most useful general theory for studying the stability of nonlinear systems and the methods it provides are very powerful, they have major drawbacks. The indirect method has purely local nature. In general, it does not give any idea how large the region of attraction might be. It follows from the direct method, that one can extract important information regarding the stability of the equilibrium at the origin if one has a Lyapunov function for the system, but it does not provide any prescription for determining the Lyapunov function $V$.

Consider the autonomous system $\dot{x} = f(x)$, $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and assume that the origin is an exponentially stable equilibrium of the system. The standard method to verify the exponential stability of the origin is to solve the Lyapunov equation, i.e. to find a positive definite matrix $Q \in \mathbb{R}^{n \times n}$ that is a solution to $J^T Q + Q J = -P$, where $J = Df(0)$ is the Jacobian of $f$ at the origin and $P \in \mathbb{R}^{n \times n}$ is an arbitrary positive definite matrix. Then the function $x \mapsto x^T Q x$ is a local Lyapunov function for the system $\dot{x} = f(x)$, i.e. it is a Lyapunov function for the system in some neighborhood of the origin. The size of this neighborhood — region (basin) of attraction — is a priori not known and is, except for linear $f$, in general a poor estimate of the region. This method to compute local Lyapunov functions is constructive because there is an algorithm to solve the Lyapunov equation that succeeds whenever it possesses a solution [13]. The construction of Lyapunov functions for true nonlinear systems is a much harder problem than for linear systems. However, it has been studied intensively in the last decades and there have been numerous proposals of how to construct Lyapunov functions numerically. A complete Lyapunov function, first introduced by Conley [14], is a generalization of a Lyapunov function for compact invariant sets to an object completely characterizing the decomposition of a flow into a chain-recurrent and a gradient-like part, suggested to be referred to as the Fundamental Theorem of Dynamical Systems [15]. Some algorithmic approaches to construct approximations to complete Lyapunov functions for discrete and continuous dynamical systems have been proposed.

Stability properties of control-affine systems can be studied and analyzed following a recently introduced method to extremum seeking control [16, 17, 18, 19, 21] which rests upon the problem of stabilizing a control-affine system by means of output feedback for states in which the output function attains an extreme value. An essential role in this problem is played by approximating Lie brackets of suitably defined vector fields [17, 20, 21, 22]. In its general form, the purpose of extremum seeking control is not restricted to control-affine systems with real-valued outputs, but is aimed at optimizing the output of a general nonlinear control system without any information about the model or the current system state. We refer the reader to [18, 19] for an overview of different approaches and strategies on obtaining extremum seeking control. In this regard, it is worth mentioning that a universal solution to this problem is not known.

Another approach to analyzing the stability of nonlinear systems is based on vector Lyapunov functions [23, 24, 25], which is a generalization of classical Lyapunov function. Vector Lyapunov theory has been developed to weaken the hypothesis of standard Lyapunov theory in order to enlarge the class of Lyapunov functions that can be used for analyzing system stability. The use of vector Lyapunov functions and recently control vector Lyapunov functions offers a more flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. Weakening the hypothesis on the Lyapunov function enlarges the class of Lyapunov functions that can be used for analyzing system stability, e.g. each component of a vector Lyapunov function need not be positive definite with a negative or even negative-semidefinite derivative. Alternatively, the time derivative of the vector Lyapunov function need only satisfy an element-by-element inequality involving a vector field of a certain comparison system. Since in this case the stability properties of the comparison system imply the stability properties of the dynamical system, the use of vector
Lyapunov theory can significantly reduce the complexity (i.e., dimensionality) of the dynamical system being analyzed [26, 27, 28, 29, 30].

Synthesizing Lyapunov functions for nonlinear dynamical systems is very difficult in general problem and often involves solving large, nonconvex optimization problems. To avoid solving hard optimization problems directly, several sample-based Lyapunov function synthesis methods have been proposed, such as: using simulation data to help postulate the region of attraction of continuous-time polynomial systems and finding Lyapunov function candidates for solving bilinear matrix inequalities; applying iterative approach that formulates Lyapunov function candidates for continuous-time nonlinear systems from simulation traces and improves the result by the counterexample trace generated by a falsifier at each iteration, or methods that provide a finite-step termination guarantee for iterative algorithm and show that the algorithm terminates with either a valid Lyapunov function or a certification that Lyapunov functions of a specific form do not exist; or, recently, methods and algorithms based on vector Lyapunov functions.

Hence, development of systematic techniques for estimating the bounds on the regions of attraction of equilibrium points of nonlinear dynamical systems is an important and active area of research and involves searching for the "best" Lyapunov functions.

2. Linear approximation: a classical approach to stabilization

We consider nonlinear control-affine systems of the form

$$\dot{x} = A(x) + B(x)u$$  \hspace{1cm} (9)$$

where

$$x = x(t), \; x \in \mathbb{R}^n, \; u \in \mathbb{R}, \; |u| \leq 1$$

and the vector functions $A(x), B(x)$ are assumed to be smooth, $A(x), B(x) \in C^\infty(D)$, in the domain $D \subset \mathbb{R}^n$, $0 \in D$, $A(0) = 0$, and then tackle the problem of synthesis of continuous control $u = u(x)$, $u(0) = 0$, that stabilizes the system at the equilibrium point $(x, u) = (0, 0)$.

The stabilization problem that we have formulated for the system (9) can be solved by using the linear approximation of (9)

$$\dot{x} = Ax + bu, \quad A = \left. \frac{\partial A(x)}{\partial x} \right|_{x=0}, \quad b = B(0), \quad |u| \leq 1$$  \hspace{1cm} (10)$$

and then solving the stabilization problem for the system (10) at the equilibrium point $x = 0$ by means of methods of linear theory. If for the linear approximation (10) the solution

$$u = c^T x, \quad c \in \mathbb{R}^n$$  \hspace{1cm} (11)$$

does exist, then the equilibrium point $x = 0$ of the system (10) under the control $u$, that is of the system

$$\dot{x} = Gx, \quad G = A + bc^T$$  \hspace{1cm} (12)$$
is asymptotically stable. Therefore, for the system (9) under the same control $u$, that is for the system

$$\dot{x} = A(x) + B(x)c^T x$$  \hspace{1cm} (13)$$

the equilibrium point $x = 0$ is also asymptotically stable, which follows from the observation that linear approximation of the system (13) coincides with the system (12).

In order to take into account the restrictions on the control $u$, one can consider the following bounded control function.
\[ \tilde{u}(x) = \text{rc} \left( c^T x \right) = \begin{cases} c^T x, & \text{if } |c^T x| \leq 1 \\ \text{sign} \left( c^T x \right), & \text{if } |c^T x| > 1. \end{cases} \]  
(14)

Since the control \( u = c^T x \) is continuous at the point \( x = 0 \) and is equal to zero at this point, it follows from (14) that the controls \( \tilde{u}(x) \) and \( u = c^T x \) coincide in some neighborhood of \( x = 0 \).

The set of points
\[ \mathcal{P} = \{ x \in \mathbb{R}^n : |c^T x| < 1 \} \]
(15)
can serve as such a neighborhood.

Hence, the control \( u = \tilde{u}(x) \) satisfying the condition \( |u| \leq 1 \) is a solution to the stabilization problem for the system (9) in a certain neighborhood of the equilibrium point \( x = 0 \).

### 3. System stability and the region of stability

System stability is characterized by analyzing the response of a dynamical system to small perturbations in the system states. Specifically, an equilibrium point of a dynamical system is said to be stable if, for sufficiently small values of initial disturbances, the perturbed motion remains in an arbitrarily prescribed small region of the state space. More precisely, stability is equivalent to continuity of solutions as a function of the system initial conditions over a neighborhood of the equilibrium point uniformly in time. Stability analysis aims to show that a set of initial states of a dynamical system would stay around and potentially converge to an equilibrium point.

It often relies on constructing Lyapunov functions that can find positive invariant sets, certify the stability of equilibrium points, and estimate their region of attraction. However, finding Lyapunov functions for general nonlinear systems is not an easy task and often requires substantial expertise and manual effort.

Consider a nonlinear autonomous dynamical system
\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in I_{x_0}, \]
(16)
where \( x(t) \in D \subseteq \mathbb{R}^n \), \( t \in I_{x_0} \), is the system state vector, \( f : D \rightarrow \mathbb{R}^n \) is nonlinear continuous function on \( D \), and \( I_{x_0} = [0, \tau_{x_0}) \), \( 0 \leq \tau_{x_0} \leq \infty \), and \( D \) is an open set in \( \mathbb{R}^n \) with \( 0 \in D \). Without loss of generality, we assume that \( 0 \) is the interior point of \( D \), \( 0 \in \text{int}(D) \), and is an equilibrium point of the system, i.e., \( f(0) = 0 \). We assume that for every initial condition \( x(0) \in D \) and every \( \tau_{x_0} > 0 \), the nonlinear dynamical system (16) possesses a unique solution \( x : [0, \tau_{x_0}) \rightarrow D \) on the interval \( [0, \tau_{x_0}) \).

A control \( u \) is said to be a stabilizing control if for every initial condition \( x(0) \) from \( D \subseteq \mathbb{R}^n \), for the closed-loop system, it holds that \( x(t) \rightarrow x_0 \) when \( t \rightarrow t_0 \), \( (x_0 = 0 \text{ in (9)}) \).

To study the stability properties of autonomous dynamical systems at the equilibrium points, Lyapunov functions can be used.

**Definition 3.1** (Lyapunov stability [31]). The zero solution \( x(t) = 0 \), i.e. the equilibrium point \( x = 0 \) of the differential system (16) is said to be
- **Lyapunov stable**, if for all \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) > 0 \) such that if \( \|x(0)\| < \delta \) then \( \|x(t)\| < \epsilon, \ t \geq 0 \).
- **asymptotically stable**, if it is Lyapunov stable and there exists \( \delta > 0 \) such that if \( \|x(0)\| < \delta \) then \( \lim_{t \to \infty} \|x(t)\| = 0 \).
- **exponentially stable**, if there exist constants \( \alpha > 0, \beta > 0 \), and \( \delta > 0 \) such that if \( \|x(0)\| < \delta \) then \( \|x(t)\| \leq \alpha e^{-\beta t} \|x(0)\|, \ t \geq 0 \).
Lyapunov stability describes the local behavior of autonomous systems around their points of equilibrium. It also describes the global behavior of autonomous systems, both asymptotic and exponential, if the statements in the definition hold for all $x(0) \in \mathbb{R}^n$.

For an asymptotically stable equilibrium point, the region of attraction is the set of initial states from which the trajectories of the autonomous system converge to the equilibrium.

**Definition 3.2 (Region of attraction).** Let $x = 0$ be the asymptotically stable equilibrium point. The region of attraction of the nonlinear dynamical system (16) is defined as

$$\mathcal{A}_0 = \{x_0 \in D \mid \text{if } x(0) = x_0, \text{ then } \lim_{t \to \infty} x(t) = 0\}.$$ 

The dynamical system (16) generates a continuous flow in the state space, which is a one-parameter family of homeomorphisms that define the behavior of the dynamical system.

For $x \in D$, the map $f(t, x) : T_{x_0} \to D$ defines the trajectory of (16) through $x \in D$. For $t \in T_{x_0}$, the map $f(t, \cdot) : D \to D$ is denoted by $f_t(x_0) = f_t$, and the set of mappings $f : T_{x_0} \times D \to D$ defined by $f_t(x_0) = f(t, x_0)$ for every $x_0 \in D$ give the flow of the dynamical system (16).

**Definition 3.3 (Positive invariant set).** A set $\mathcal{S} \subset D \subseteq \mathbb{R}^n$ is said to be a positive invariant set for the dynamical system (16) if $f_t(\mathcal{S}) \subseteq \mathcal{S}$ for all $t \geq 0$, where $f_t(\mathcal{S}) = \{f_t(x) : x \in \mathcal{S}\}$.

**Theorem 3.4 (System stability [31]).** Consider the nonlinear dynamical system (16) and assume there exists a continuously differentiable function $V(x) : D \to \mathbb{R}$ with derivative along the trajectories of (16) given by $V(x) = V'(x)f(x)$ and with domain $D$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad x \in \mathcal{S} \setminus \{0\}$$

$$V'(x)f(x) \leq 0, \quad x \in \mathcal{S},$$

where $\mathcal{S} \subseteq \mathbb{R}^n$ and $0 \in \text{int}(\mathcal{S})$.

Then the zero solution $x(t) = 0$ of (16), or the equilibrium point $x = 0$, is Lyapunov stable. If, in addition,

$$V'(x)f(x) < 0, \quad x \in \mathcal{S} \setminus \{0\}$$

then the zero solution $x(t) = 0$ of (16), or the equilibrium $x = 0$, is asymptotically stable.

**Remark 3.5.** For an equilibrium point $x_{eq} \neq 0$ of (16), theorem (3.4) holds true with $V(0) = 0$ and $x \in \mathcal{S} \setminus \{0\}$, i.e. $x \neq 0$, is replaced by $V(x_{eq}) = 0$ and $x \neq x_{eq}$.

Any continuously differentiable function $V(\cdot)$ that satisfies (17) is called a Lyapunov function candidate for the nonlinear dynamical system (16). If, additionally, $V(\cdot)$ satisfies (18) or (19), then $V(\cdot)$ is called a Lyapunov function or a valid Lyapunov function.

Now let $\ell = \inf_{x \in D \setminus \mathcal{S}} V(x)$. Then the set defined as

$$\mathcal{S} = \{x \mid V(x) \leq \ell\}$$

is an approximation of the region of attraction and $\mathcal{S} \subset \mathcal{S}$ is the largest sublevel set of $V(x)$ that is contained in $\mathcal{S}$. If $\mathcal{S}$ is a positive invariant set, then the existence of the Lyapunov function $V(x)$ certifies that $\mathcal{S} \subseteq \mathcal{A}_0$, which means that $\mathcal{S}$ itself is an estimate of the region of attraction.

In our approach we find the Lyapunov function for a linear approximation of the affine system under study. In order to get a guaranteed estimate for the attraction region of the state $x = 0$ of the closed-loop system (9)-(14), that is the system

$$\dot{x} = A(x) + B(x)\bar{u}(x),$$

we make use of Lyapunov function $v(x)$ of linear approximation of (20) in the neighborhood of the state $x = 0$. Without loss of generality, we may always assume that this neighborhood is
contained in \( \mathcal{P} \). The linear approximation of the system (20) exists and coincides with (12). The equilibrium point \( x = 0 \) of the system (12) is asymptotically stable, therefore, for any positive definite quadratic form

\[ w(x) = x^TWx, \quad W \in \mathbb{M}_n[\mathbb{R}], \quad W^T = W > 0 \]

one can construct the positive definite quadratic form

\[ v(x) = x^TVx, \quad V \in \mathbb{M}_n[\mathbb{R}], \quad V^T = V > 0 \]

such that

\[ \dot{v}(x)|_{(12)} = -w(x). \]

To find the quadratic form \( v(x) \), we can use the matrix Lyapunov equation

\[ G^TV + VG = -W \]  \hspace{1cm} (21)

with respect to the symmetric matrix \( V \). The equation (21) can be solved by integrating the matrix differential equation

\[ \dot{V}(t) = G^TV(t) + V(t)G + W \]

with the initial condition \( V(0) = 0 \), since \( V = \lim_{t \to +\infty} V(t) \). The following criterion can be used for the process of integration to be considered completed

\[ \|G^TV(t) + V(t)G + W\| < \theta \]

where \( \theta \) is an admissible integration error. Then the quadratic form \( v(x) \) together with the matrix \( V \) will be the required Lyapunov function.

Consider the set of points \( \mathcal{S} \) in the state space \( \mathbb{R}^n \), given by the equation

\[ \dot{v}(x)|_{(20)} = 0. \]  \hspace{1cm} (22)

The set \( \mathcal{S} \) is nonempty, as it contains the point \( x = 0 \). Suppose now that the point \( x = 0 \) is an isolated point of \( \mathcal{S} \) and remove it from \( \mathcal{S} \), thus considering the set \( \mathcal{S}_v \) defined as

\[ \mathcal{S}_v = \{ x \in \mathbb{R}^n \setminus \{0\} : \dot{v}(x)|_{(20)} = 0 \}. \]

Let

\[ \mathcal{G} = \inf\{v(x) : x \in \mathcal{S}_v\}. \]

Then \( \mathcal{G} > 0 \), since \( v(x) \geq 0 \), where \( v(x) = 0 \) only for \( x = 0 \), and, by the assumption, \( x = 0 \) is the isolated point of the set (22). It should be noted that this assumption always takes place in the case when the coordinate functions of \( A(x) \) and \( B(x) \) are real analytic, e.g. polynomials.

The set

\[ \tilde{\mathcal{S}}_v = \{ x \in \mathbb{R}^n : v(x) < \mathcal{G} \} \]  \hspace{1cm} (23)

is contained in the set of attraction of the state \( x = 0 \) of the closed-loop system (20) and thus can be considered as the guaranteed estimate of the region of attraction.
4. Optimization of estimates and stability region

The set $\tilde{S}_v$ depends on the function $v(x)$ which in turn is uniquely defined by the positive definite matrix $W$. If $W_1$ and $W_2$ are two such matrices that differs only by the constant multiple, $W_1 = kW_2$, $k > 0$, then the sets $\tilde{S}_v[W_1]$ and $\tilde{S}_v[W_2]$ that corresponds to $W_1$ and $W_2$, respectively, will be equal, $\tilde{S}_v[W_1] = \tilde{S}_v[W_2]$. In all other cases, the corresponding sets $\tilde{S}_v$ will be, generally speaking, different. However, these sets can be compared based on their $n$-dimensional volume, choosing then the estimate of stability region with larger volume.

The set $\tilde{S}_v$ is bounded by the ellipsoid $\mathcal{E} = \{ x : v(x) = v(x_0) \}$ which simplifies the computation of the corresponding volume by solving the problem

$$\text{volume} \left( \{ x \in \mathbb{R}^n : v(x) \leq \mathcal{J} \} \right) \rightarrow \sup W \in S,$$

where $S$ denotes the set of symmetric matrices. The restriction $W \in S$ on the matrix $W = [w_{ij}]$ can be represented in the form of the set of conditions: $w_{11} = 1$ and according to the Sylvester’s criterion, $\Delta_i > 0$, $i = 2, 3, \ldots, n$, where $\Delta_i$ are the leading principal minors of the matrix $W$.

The estimate (23) of the set $\tilde{S}_v$ changes under the variation of the matrix $W$, however the closed-loop system (20) and the region of stability of the system (9) under the control $\tilde{u}(x)$, defined in (14), remain unaltered. The feedback coefficients (11) can also be regarded as the variation parameters, though observing the asymptotic stability of linear system (12). In this case, both the closed-loop system (20) and the region of stability of the system (9) will be changed. Nevertheless, this allows one to consider some other objective functions in (24) other than volume and hence reduce the problems of stabilization from the given state or set of states to optimization problems.

5. The role of linear approximation

The solution (14) of the problem of stabilization and the estimate $\tilde{S}_v$ in (23) for the region of stability are obtained on the basis of linear approximation. In fact, the property of the system (9) to be affine does not play any role and the same reasonings works for nonlinear systems of the form $\dot{x} = f(x, u)$, $f(0, 0) = 0$. The transition to linear approximation that underlies the approach presented above is used twice. First, when finding the linear feedback (12); second, when choosing the function $v(x)$. Thus, each time the information related to the nonlinearity of the system (9) is being lost. This essentially limits both the region of stability and its estimate.

However, for a certain class of nonlinear systems, one can avoid linear approximation in both cases: (i) for synthesis of stabilizing control, and (ii) for constructing the estimate of the region of stability, preserving all the other steps when obtaining these two. This is related to a class of nonlinear affine systems that we will consider further in the paper.

Consider the system (9) in a new variable $z$ in the state space $\mathbb{R}^n$ by means of the smooth change of variables $z = Z(x)$ and obtain the affine system

$$\dot{z} = C(z) + D(z)u, \quad z \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad |u| \leq 1$$

with the same control and independent variable $t$, where

$$C(z) = \left. \frac{\partial Z(x)}{\partial x} A(x) \right|_{x = Z^{-1}(z)}, \quad D(z) = \left. \frac{\partial Z(x)}{\partial x} B(x) \right|_{x = Z^{-1}(z)}.$$

Therefore, the class of affine systems is closed under the given change of variables. These transformations and similar ones are used to reduce nonlinear systems with control and, in particular, affine systems, to a simpler form [22]. The transformations of this type for nonlinear systems with control might result in simplified mathematical formulation of problems posed for these systems, which may facilitate finding new problem solving methods.
The above is also related to the problem of stabilization of affine systems. For affine systems, within the elaborated framework, the transformation of affine system to the equivalent affine system of canonical form is efficiently used instead of linear approximation, which is then employed to construct the stabilizing control that turns out to serve as a nonlinear feedback.

The original affine system, which becomes closed-loop system by means of the nonlinear feedback, is turned out to be the system for which it is always possible to construct the Lyapunov function and then use this function for constructing the estimate of the region of stability.

6. The canonical form of affine system

We assume that for affine systems (9) there exists a diffeomorphism

\[ \Phi : D \to Z \subset \mathbb{R}^n, \quad \Phi(x) = z, \quad z = (z_1, z_2, \ldots, z_n) \]  

that transforms the affine system on the set \( D \) of admissible states into an equivalent system of canonical form

\[ \begin{aligned} \dot{z}_1 &= z_2, \\
\dot{z}_2 &= z_3, \\
\vdots \quad & \\
\dot{z}_{n-1} &= z_n, \\
\dot{z}_n &= f(z) + g(z)u \end{aligned} \]  

where \( f(z) \) and \( g(z) \) are smooth, continuous functions.

Hereafter, we consider the affine systems (9) for which the transformation defined by (25) exists and that can be represented in certain variables \( z = (z_1, z_2, \ldots, z_n) \) of the state space \( \mathbb{R}^n \) in the canonical form (26). The affine systems of the form (26) are referred to as canonical form systems, and the variables \( z \in \mathbb{R}^n \) are called canonical variables.

Let \( A \) and \( B \) be vector fields that correspond to the affine system (9), and recursively a vector field \( \mathcal{F}_A^1B = B, \mathcal{F}_A^2B = [A,B], \mathcal{F}_A^kB = [A,\mathcal{F}_A^{k-1}B], \) where the brackets \([\cdot,\cdot]\) denote the Lie bracket of vector fields [32]. The expression \( X^k\varphi(x) = X\left(X^{k-1}\varphi(x)\right) \) for \( k \in \mathbb{N}, \, k > 0, \) denotes the Lie derivative of vector field \( X \) of the function \( X^{k-1}\varphi(x) \), where \( X^0\varphi(x) = \varphi(x) \).

Then, with regard to canonical form systems, we have the following

**Theorem 6.1.** The affine system (9) can be transformed into canonical form system (26) by means of a smooth change of variables \( z = \Phi(x) \) in a region \( \Omega \subset \mathbb{R}^n \) if and only if

(i) there exists a function \( \varphi(x) \in C^\infty(\Omega) \) that satisfies the system of linear homogeneous partial differential equations of first order

\[ \mathcal{F}_A^kB\varphi(x) = 0, \quad k = 0, 1, \ldots, n-2. \]  

(ii) there exists a system of algebraic relations

\[ z_k = A^{k-1}\varphi(x), \quad k = 1, 2, \ldots, n \]  

that determines the change of variables in \( \Omega \).

Moreover, for the canonical form system (26) it holds that

\[ f(z) = A^n\varphi(x) \bigg|_{x=\Phi^{-1}(z)} \quad \text{and} \quad g(z) = (-1)^{n-1}\mathcal{F}_A^{n-1}B\varphi(x) \bigg|_{x=\Phi^{-1}(z)} \]  

where \( x = \Phi^{-1}(z) \) is a solution of the system (28) with respect to \( x \).
If the affine system (9) satisfies the conditions of theorem 6.1 and the equilibrium point \( x = 0 \) belongs to \( \Omega \), then we can always assume \( \varphi(0) = 0 \). Otherwise, we can take the function \( \varphi(x) - \varphi(0) \) instead of \( \varphi(x) \). When one chooses the function \( \varphi(x) \) as described above, the equilibrium \( x = 0 \) will also have zero coordinates \( z = 0 \) in the canonical variables.

The canonical form system (26) is called regular at a point (region) if the coefficient \( g(z) \) at control \( u \) is not equal to zero at that point (at all points in the region).

7. Nonlinear stabilization
Suppose that the affine dynamical system (9) satisfies the conditions of theorem 6.1 in a certain neighborhood \( \Omega \) of the equilibrium point \( x = 0 \). Then the system (9) can be written in the canonical form (26) and, moreover, the system (26) is regular at the equilibrium \( z = 0 \). In this case the linearization of the system (9) is not required for the synthesis of stabilizing control.

Indeed, by applying the feedback control

\[
    u(z) = -f(z) - \sum_{i=1}^{n} \frac{p_{i-1} z_i}{g(z)},
\]

(30)

where \( p_i, i = 0, 1, \ldots, n-1 \), are some constants, the system (26) becomes the closed-loop system. From (30), we get the system

\[
    \dot{z} = Pz, \quad P = \begin{bmatrix}
    0 & 1 & \cdots & 0 \\
    0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1 \\
    -p_0 & -p_1 & \cdots & -p_{n-1}
\end{bmatrix}.
\]

(31)

For the equilibrium \( z = 0 \) of the canonical form system (26) to be stabilized by the control (30) it is sufficient to choose such values of the constants \( p_i, i = 0, 1, \ldots, n-1 \), that make the zero solution of the system (31) asymptotically stable. In this case, as it was already mentioned above, for any positive definite quadratic form

\[
    W(z) = z^T Wz, \quad W \in M_n[\mathbb{R}], \quad W^T = W > 0
\]

one can construct a positive definite quadratic form

\[
    \mathcal{V}(z) = z^T \mathcal{V}z, \quad \mathcal{V} \in M_n[\mathbb{R}], \quad \mathcal{V}^T = \mathcal{V} > 0
\]

such that

\[
    \mathcal{V}(z) \bigg|_{(31)} = -W(z).
\]

Using the relations (28) and (29), we can rewrite the control (30) and the function \( \mathcal{V}(z) \) in the variable \( x \). Thus, we get the following control function

\[
    u_s(x) = (-1)^{n-1} \left( -A^n \varphi(x) - \sum_{i=1}^{n} p_{i-1} A^{i-1} \varphi(x) \right) / \mathcal{F}_A^{n-1} B \varphi(x),
\]

(32)

which is the solution to the stabilization problem for the affine system (9), and find the Lyapunov function \( v_s(x) = \mathcal{V}(\Phi(x)) \) of the following system

\[
    \dot{x} = A(x) + B(x) u_s(x),
\]

(33)
which the control (32) is applied to. Henceforth, the process of finding of stabilizing control (32) is to be referred to as a method of nonlinear stabilization.

As in the case of linear feedback control \( u = c^T x \), in order to take into account the restrictions on the control, we use the bounded control (14), by setting

\[
\tilde{u}_s(x) = \begin{cases} 
\tilde{u}_s(x), & \text{if } |\tilde{u}_s(x)| \leq 1 \\
\text{sign}(\tilde{u}_s(x)), & \text{if } |\tilde{u}_s(x)| > 1.
\end{cases}
\]

(34)

Since the control \( \tilde{u}_s(x) \) is continuous at the point \( x = 0 \) and is equal to zero at this point, it follows from (14) that the controls \( \tilde{u}_s(x) \) and \( \tilde{u}(x) \) coincide in some neighborhood of \( x = 0 \). The set of points

\[ \mathcal{P}_s = \{ x \in \mathbb{R}^n : |u_s(x)| < 1 \} \]

can serve as such a neighborhood.

Hence, the control \( u = \tilde{u}_s(x) \) satisfying the condition \(|u| \leq 1\) is a solution to the stabilization problem for the system (9) in a certain neighborhood of the equilibrium \( x = 0 \).

The affine system (9), which the control (34) is applied to, has the form

\[
\dot{x} = A(x) + B(x)\tilde{u}_s(x).
\]

(35)

We use the Lyapunov function \( v_s(x) \) of the closed-loop system (33) to obtain the guaranteed estimates of the region of stability.

Let

\[ \delta^*_v = \{ x \in \Omega \setminus 0 : \dot{v}_s(x)|_{(35)} = 0 \} \cup \partial\Omega \cup \mathcal{K}, \]

where \( \partial\Omega \) is the boundary of the region \( \Omega \) and

\[ \mathcal{K} = \{ x \in \Omega : F_n^{-1} A B \varphi (x) = 0 \}, \]

and

\[ \mathcal{G}^s = \inf \{ v_s(x) : x \in \delta^*_v \} \]

Then \( \delta^*_v > 0 \), since \( \dot{v}_s(x) \geq 0 \), where \( \dot{v}_s(x) = 0 \) in \( \Omega \) only when \( x = 0 \).

Note that the set

\[ \tilde{\delta}^*_v = \{ x \in \Omega : v_s(x) < \mathcal{G}^s \} \]

is contained in the set of attraction of the state \( x = 0 \) of the closed-loop system (35) and thus can be considered as the guaranteed estimate of the region of attraction.

The optimization of the estimate and the region of stability can be performed with the use of the optimization techniques outlined in section 4.

8. Conclusion

Mathematical models using dynamical systems have proved valuable to understand the interactions and the evolution of different physical phenomena. It is well known that stability analysis is an important performance metric for any dynamical system. In the paper we discussed the optimal control and stability problems of nonlinear control-affine dynamical systems. We proposed the solution to the problem of synthesis of continuous control function that ensures the stability of the system at the origin. The nonlinear stabilization approach for affine dynamical systems has been presented. The presented results contribute to the development of a stability analysis for nonlinear dynamical systems based on Lyapunov function methods. We gave the guaranteed estimates for the region of attraction of a zero equilibrium state in nonlinear feedback control affine systems. This paper has showed how the methods based on Lyapunov functions can be used for smooth stabilizing feedback control design for nonlinear systems. The analysis of stability of nonlinear systems with weaker restrictions on control functions and the extension of the results to prove a larger region of attraction is in the scope of the future research.
References

[1] Betts J T 1998 Math. Comp. 21 193–207.
[2] Polak E 1973 SIAM Rev. 15 553–584.
[3] Mirkin L 2004 Systems Control Lett. 51 331–342.
[4] Bagdasaryan A and Kadry S 2017 In: IEEE Xplore 7th International Conference on Modeling, Simulation, and Applied Optimization (ICMSAO), Sharjah, 2017, pp. 1–3, doi: 10.1109/ICMSAO.2017.7934900.
[5] Rampazzo F and Vinter R 2000 SIAM J. Control Optim. 39 989–1007
[6] Arutyunov A V and Aseev S M 1997 SIAM J. Control Optim. 35 930–952.
[7] Bagdasaryan Armen 2019 J. Phys.: Conf. Ser. 1391 012113.
[8] Bagdasaryan A and Kim T 2011 A model of hierarchically consistent control of nonlinear dynamical systems Communications in Computer and Information Science 256 58–64. ed T Kim, H Adeli, A Stoica and B Kang (Berlin, Heidelberg: Springer)
[9] Isidori A 1995 Nonlinear Control Systems (New York: Springer).
[10] Lyapunov A 1992 The General Problem of the Stability of Motion (Taylor & Francis). transl. by A.T. Fuller.
[11] Krasovskii N N 1959 it Problems of the Theory of Stability of Motion (Stanford University Press).
[12] LaSalle J P 1960 IRE Trans. Circ. Thy. 7 520–527.
[13] Giesl P and Hafstein S 2014 J. Math. Anal. Appl. 410 292–306.
[14] Conley C 1978 Isolated Invariant Sets and the Morse Index, CBMS 38 (American Math. Soc.)
[15] Norton D 1995 Comment. Math. Univ. Carolin. 36 585–597.
[16] Grushkovskaya V, Alexander Zuyev A and Ebenbauer C 2018 Automatica 94 151–160.
[17] Grushkovskaya V and Ebenbauer C 2020 Extremum seeking control of nonlinear dynamic systems using Lie bracket approximations, Internat. J. Adapt. Control Signal Process. to appear
[18] Scheinker A and Krstić M 2017 Model-Free Stabilization by Extremum Seeking (Cham: Springer).
[19] Zhang C and Ordóñez R (2012) Extremum-Seeking Control and Applications (London: Springer).
[20] Dürr H-B, Stankovic M, Ebenbauer C and Johansson K J 2013 Automatica 49 1538–1552.
[21] Grushkovskaya V, Zuyev A and Ebenbauer C 2017 On a class of generating vector fields for the extremum seeking problem: Lie bracket approximation and stability properties arXiv:1703.02348.
[22] Nijmeijer H and van der Schaft A 1990 Nonlinear Dynamical Control Systems (New York: Springer).
[23] Bellman R 1962 SIAM J. Control 1 32–34.
[24] Lakshmikantham V, Matrosov V and Sivasundaram S 1991 Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems (Dordrecht: Kluwer Academic Publishers Group).
[25] Rubagotti M, Zaccarian L and Bemporad A 2016 Internat. J. Control 89 950–959.
[26] Nersesov S and Haddad W 2006 IEEE Trans. Automat. Control 5 203–215.
[27] Nersesov S, Haddad W and Hui Q 2008 J. Franklin Inst. 345 819–837.
[28] Artstein Z 1983 Nonlinear Anal. 7 1163–1173.
[29] Sontag E D 1989 Systems Control Lett. 13 117–123.
[30] Tsianis J 1991 SIAM J. Control Optim. 29 457–473.
[31] Haddad W M and Chellaboina V 2011 Nonlinear dynamical systems and control: a Lyapunov-based approach (Princeton University Press).
[32] Sussmann H J 1982 Lie brackets, real analyticity and geometric control Differential Geometric Control Theory (Houghton, Mich) ed R W Brackett, R S Millman, and H J Sussmann 1983 Progr Math vol. 27 (Boston, MA: Birkhäuser) 1–116.