Minimal even sets of nodes
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Abstract

We extend some results on even sets of nodes which have been proved for surfaces up to degree 6 to surfaces up to degree 10. In particular, we give a formula for the minimal cardinality of a nonempty even set of nodes.

1 Setup

Let \( S \subset \mathbb{P}^3(C) \) be a hypersurface of degree \( s \) with \( \mu \) ordinary double points (nodes) as its only singularities. Such a surface will be called a nodal surface in the sequel. Denote by \( N = \{P_1, \ldots, P_\mu\} \subset S \) the set of nodes of \( S \). The maximum number of nodes of a nodal surface of degree \( d \) is denoted classically by \( \mu(d) \). There is a lot of (old) literature on nodal surfaces and estimates for \( \mu(d) \) (see [E]). For \( d = 1, 2, \ldots, 6 \) the numbers \( \mu(d) \) are 0, 1, 4, 16, 31, 65 and for every \( k \in \{0, 1, \ldots, \mu(d)\} \) there exists at least one nodal surface of degree \( d \) with exactly \( k \) nodes. In the case of cubic nodal surfaces (\( d = 3 \)), this follows from Cayley’s and Schl"afli’s classification of singular cubic surfaces [Cay], [S]. For quartic nodal surfaces (\( d = 4 \)) the fact that \( \mu(4) = 16 \) is due to Kummer [K], whereas the construction of arbitrary nodal quartics goes back to Rohn [R]. The first quintic nodal surface (\( d = 5 \)) with 31 nodes has been constructed by Togliatti in 1940 [T]. In 1971, Beauville [Be] showed that this is in fact the maximal number. The construction of sextic nodal surfaces (\( d = 6 \)) with 1, \ldots, 64 nodes has been given by Catanese and Ceresa [CC]. In 1994, Barth [Bar1] constructed a sextic nodal surface with 65 nodes. Shortly afterwards, Jaffe and Ruberman [JR] proved that 65 is the maximal number. Both Beauville and Jaffe/Ruberman use the code of a nodal surface in their proofs. This code is a \( \mathbb{F}_2 \) vector space which carries the information of the low degree contact surfaces of the nodal surface. If a nodal surface has “nearly” \( \mu(d) \) nodes, its code often becomes accessible.

Let \( v \in \mathbb{N} \) and denote \( \delta(v) = 2(v/2 - \lfloor v/2 \rfloor) \). This number is 0 if \( v \) is even and 1 if \( v \) is odd. We want to study surfaces \( V \subset \mathbb{P}^3 \) of degree \( v \) with \( S.V = 2D \) for a (not necessarily smooth or reduced) curve \( D \). In other words, surfaces \( V \) which have contact to \( S \) along a curve. Let \( \pi: \tilde{\mathbb{P}}_3 \to \mathbb{P}_3 \) be the embedded resolution of all nodes of \( S \). Given such a surface \( V \), the proper transforms of \( S \) and \( V \) are calculated as

\[
\tilde{S} = \pi^*S - 2 \sum_{i=1}^{\mu} E_i \quad \text{and} \quad \tilde{V} = \pi^*V - \sum_{i=1}^{\mu} \nu_i E_i,
\]

where \( E_i = \pi^{-1}(P_i) \) is the exceptional divisor corresponding to \( P_i \) and \( \nu_i = \text{mult}(V, P_i) \) for every node \( P_i \in N \). On the smooth surface \( \tilde{S} \) we have \( \tilde{V} \sim_{\text{lin}} 2\tilde{D} + \sum_{i=1}^{\mu} \theta_i E_i \), where \( \tilde{D} \) is the proper transform of \( D \) and the \( \theta_i \)'s are nonnegative integers. Let \( H \in \text{Div}(\mathbb{P}_3) \) be a hyperplane section, then

\[
2\tilde{D} \sim_{\text{lin}} v\pi^*H - \sum_{i=1}^{\mu} (\nu_i + \theta_i) E_i,
\]

where \( \tilde{D}.E_i = \nu_i + \theta_i = \text{mult}(D, P_i) = \eta_i \). This shows that in \( \text{Pic}(\tilde{S}) \) the divisor class \( [\delta(v) \pi^*H + \sum_{i=1}^{n} \eta_i \text{ odd} E_i] \) is divisible by 2. This is a remarkable fact, since
every \( E_i \) is on \( S \) a smooth, rational curve with self intersection \(-2\). In particular \( E_i \neq E_j \) for \( i \neq j \).

For any set of nodes \( M \subseteq N \), let \( E_M = \sum_{P_i \in M} E_i \) be the sum of exceptional curves corresponding to the nodes in \( M \).

**Definition 1.1** A set \( M \subseteq N \) of nodes of \( M \) is called strictly even, if the cocycle class \( [E_M] \in H^2(S, \mathbb{Z}) \) is divisible by 2. \( M \) is called weakly even, if the cocycle class \( [\pi^*H + E_M] \in H^2(S, \mathbb{Z}) \) is divisible by 2. \( M \) is called even if \( M \) is strictly or weakly even.

So the set of nodes \( M = \{P_i \in N \mid \text{mult} (D, P_i) \text{ is odd}\} \) through which \( D \) passes with odd multiplicity is strictly even if \( v \) is even and weakly even if \( v \) is odd.

**Definition 1.2** Let \( M \subseteq N \) be an even set of nodes of \( S \). If \( V \subseteq \mathbb{P}_3 \) is a surface with \( S.V = 2D \) and \( M \) is the set of nodes of \( S \) through which \( D \) passes with odd multiplicity, we say that \( M \) is cut out by \( V \) via \( D \).

Conversely, if \( M \subseteq N \) is even, consider the linear system \( |(v \pi^*H - E_M)/2| \) for \( v \in \mathbb{N} \) even if \( M \) is strictly even and odd if \( M \) is weakly even. For \( v \gg 0 \) this linear system is nonempty by R.R. and Serre duality. Then for every \( v \) such that \(|(v \pi^*H - E_M)/2| \neq 0 \) and for every divisor \( D \in |(v \pi^*H - E_M)/2| \) we can find a surface \( V \subseteq \mathbb{P}_3 \) of degree \( v \) which cuts out \( M \). The construction is as follows: \( D \) is effective, so it admits a decomposition \( D = \overline{D} + \sum_{i=1}^{\mu} \tau_i E_i \) such that \( \overline{D} \) is effective and contains no exceptional component and all the numbers \( \tau_i \) are nonnegative. In particular we have for all \( j \in \{1, \ldots, \mu\} \) that

\[
\overline{D}.E_j = \left( \frac{1}{2} (v \pi^*H - E_M) - \sum_{i=1}^{\mu} \tau_i E_i \right).E_j = \begin{cases} 
2\tau_j & \text{is even if } P_j \notin M, \\
2\tau_j + 1 & \text{is odd if } P_j \in M.
\end{cases}
\]

But \( 2\overline{D} \in |v \pi^*H - E_M| \) on the surface \( S \), so \( 2\overline{D} \) is cut out by a surface \( \overline{V} \in |v \pi^*H - E_M| \) in \( \mathbb{P}_3 \). Let \( V = \pi_* (\overline{V}) \) and \( D = \pi_* (\overline{D}) = \pi_* (\overline{D}) \), then by construction \( S.V = 2D \) and \( \text{mult} (D, P_i) = \overline{D}.E_i \) for all \( i \). So \( M \) is exactly the set of nodes of \( S \) through which \( D \) passes with odd multiplicity. This shows that \( M \) is cut out by \( V \) via \( D \). Furthermore we see that only nodal surfaces of even degree do admit weakly even sets of nodes.

If the surface \( V \) cuts out an even set of nodes \( M \) on \( S \) via \( D \), then in general \( D \) is not unique with respect to \( M \). The set of these contact curves is parameterized by the linear system \( L_M = |(v \pi^*H - E_M)/2| \) which is a projective space of dimension \( h^0(O_S((v \pi^*H - E_M)/2)) - 1 \). In particular, if \( h^0(O_S((v \pi^*H - E_M)/2)) \geq 1 \) then there exists a surface of degree \( v \) which cuts out \( M \). It is funny to compute these dimensions, though often not possible.

The canonical divisor of \( S \) is \( K_S \sim_{lin} (s-4) \pi^*H \). Define \( (\frac{n}{k}) = 0 \) for \( n < k \), then Riemann Roch for the bundle \( O_S((v \pi^*H - E_w)/2) \) reads as

\[
\chi(O_S((v \pi^*H - E_w)/2)) = \frac{sv}{8} (v - 2s + 8) + \binom{s-1}{3} + 1 - \frac{|w|}{4}.
\]

The symmetric difference of two strictly even sets of nodes is strictly even again, so the set \( C_S = \{M \subseteq N \mid M \text{ is strictly even}\} \) carries the natural structure of a \( \mathbb{F}_2 \) vector space sitting inside \( \mathbb{F}_2^N \). Hence \( C_S \) is a binary linear code, which is called the code of \( S \). The symmetric difference of two weakly even sets of nodes is strictly even and the symmetric difference of a strictly even set and a weakly even set is weakly even. Thus the set \( \overline{C} = \{M \subseteq N \mid M \text{ is even}\} \) is a binary code of dimension \( \dim_{\mathbb{F}_2} (C_S) \leq \dim_{\mathbb{F}_2} (\overline{C}) \leq \dim_{\mathbb{F}_2} (C_S) + 1 \) sitting also inside \( \mathbb{F}_2^N \).

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The elements of $\mathcal{C}_S$ are called words, and for every word $w \in \mathcal{C}_S$ its weight $|w|$ is its number of nodes. Let $e_1, \ldots, e_n, h$ be the canonical basis of $\mathbb{F}_2^n \oplus \mathbb{F}_2$ and consider

$$
\mathbb{F}_2^{\mu} \xrightarrow{j} \mathbb{F}_2^{\mu} \oplus \mathbb{F}_2 \xrightarrow{\lambda} H^2(\tilde{S}, \mathbb{F}_2) \xrightarrow{\pi} \text{cl}[E_i] \mod 2
$$

$$
e_i \rightarrow \pi_s \text{dim}_2 (C_S) \geq \mu - \frac{1}{2}b_2(\tilde{S}),$$

$$h \rightarrow \pi_s \text{dim}_2 (\mathcal{C}_S) \geq \mu + 1 - \frac{1}{2}b_2(\tilde{S}) \quad (s \text{ even}).$$

The weight of every word $w \in C_S$ is divisible by 4. If $s = \deg S$ is even, then the weight of every word is divisible by 8 [Cat].

### 1.1 Coding theory

We recall some definitions and facts from coding theory [L], [W]. Let $C \subseteq \mathbb{F}_2^n$ be a linear code and let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{F}_2^n$. $C$ is called even if $2 \mid |w|$ for every $w \in C$ and doubly even if $4 \mid |w|$ for every $w \in C$. The dual code of $C$ is defined as

$$C^\perp = \{ v \in \mathbb{F}_2^n \mid (v, w)_{\mathbb{F}_2} = 0 \ \forall w \in C \}.$$ 

If $C$ is doubly even, then $C \subseteq C^\perp$. Since $n = \text{dim}_2 (C) + \text{dim}_2 (C^\perp)$ we also get $2 \dim (C) \leq n$ with equality iff $C$ is self dual. For $w \in C$ the support of $w$ is the linear subspace of $\mathbb{F}_2^n$ which is spanned by the ones of $w$, i.e.

$$\text{supp} (w) = \text{span}_{\mathbb{F}_2} \{ e_i \mid (e_i, w) = 1 \}.$$ 

The image of the projection $p_w : C \rightarrow \text{supp} (w)$ is called projection of $C$ onto the support of $w$ and denoted by $C_w$. Assume that $2d \mid \dim (C)$ for every $w \in C$ for some $d \in \mathbb{N}$. Since $|v + w| + 2|v \cap w| = |v| + |w|$ and $p_w(v) = v \cap w$ we see that $d \mid v'$ for all $v' \in C_w$. Now the code $C_S$ of the nodal surface $S$ is always doubly even. If $s = \deg (S)$ is even, then $(C_S)_w$ is doubly even for all $w \in C_S$.

A $[n, k, d]$-code is a $k$-dimensional linear code $C \subseteq \mathbb{F}_2^n$ with $|w| \geq d$ for all $w \in C \setminus \{0\}$. Many methods have been found to give bounds on $k$ for fixed $n$ and $d$. One of the simplest to apply is the

**Theorem 1.3** (Griesmer bound) For a $[n, k, d]$ code always $n \geq \sum_{i=0}^{k-1} \lceil d/2^i \rceil$.

### 1.2 Examples

The following examples exhibit the trivial and some of the the well known cases of even sets of nodes [L]

**Example 1.4** Let $S$ be a quadratic cone and let $P_1$ be its node. Every line $L \subset S$ runs through $P_1$ and there exists exactly one plane $H$ with $S.H = 2L$. So $\mathcal{C}_S$ is spanned by $w = \{ P_1 \}$ and $h^0(O_S((\pi^*H-E_w)/2)) = 2$. 

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Example 1.5 Let $S$ be a cubic nodal surface, then $C_S$ can only be non trivial if $S$ has exactly $\mu(3) = 4$ nodes $P_1, \ldots, P_4$. But $b_2(\tilde{S}) = 7$, so $\dim_{\mathbb{F}}(C_S) \geq 1$. It follows that $\dim_{\mathbb{F}}(C_S) = 1$ and $C_S$ is spanned by $w = \{P_1, \ldots, P_4\}$. But $w$ is cut out by a quadric: Riemann-Roch on $\tilde{S}$ gives $h^2(\mathcal{O}_\tilde{S}((2\pi^*H-E_w)/2)) = 3$. From Serre duality we get $h^2(\mathcal{O}_\tilde{S}((2\pi^*H-E_w)/2)) = h^0(\mathcal{O}_\tilde{S}((-4\pi^*H+E_w)/2)) = 0$. One easily checks that $\mathcal{O}_\tilde{S}((4\pi^*H-E_w)/2)$ is ample, so by Kodaira vanishing also $h^1(\mathcal{O}_\tilde{S}((2\pi^*H-E_w)/2)) = h^1(\mathcal{O}_\tilde{S}((-4\pi^*H+E_w)/2)) = 0$. This implies that $h^0(\mathcal{O}_\tilde{S}((2\pi^*H-E_w)/2)) = 3$, so there exists a two parameter family of quadric surfaces which cut out $w$.

Example 1.6 A quartic nodal surface $S$ with $\mu(4) = 16$ nodes is a Kummer surface. Since $b_2(\tilde{S}) = 22$, we have $\dim_{\mathbb{F}}(C_S) \geq 5$. On the other hand all nonzero words of $C_S$ must have weight 8 or 16. So $C_S$ is a $[16, k, 8]$ code for some $k \geq 5$. The Griesmer bound implies $k \leq 5$, so $C_S$ is a $[16, 5, 8]$ code. Every such code has exactly one word of weight 16 and 30 words of weight 8. Moreover $C_S$ is (up to permutation of columns) spanned by the rows of the following table.

Example 1.7 A quintic nodal surface $S$ with $\mu(5) = 31$ nodes is called Togliatti surface. One computes $b_2(\tilde{S}) = 53$, so again $\dim_{\mathbb{F}}(C_S) \geq 5$. By [12], all even sets of nodes on $S$ have weight 16 or 20. So $C_S$ is a $[31, k, 16]$ code for some $k \geq 5$. The Griesmer bound gives $31 \geq 16 + 8 + 4 + 2 + 1 + (k - 5)$, so $k \leq 5$. This shows that $C_S$ is a $[31, 5, 16]$ code. Every such code has exactly 31 words of weight 16 and no word of weight 20. Moreover, $C_S$ is (up to a permutation of columns) spanned by the rows of the following table.

Example 1.8 Let $S$ be a nodal sextic surface with $\mu(6) = 65$ nodes. Every nonzero word $w \in C_S$ must have weight 24, 32, 40 or 56 [JR]. We have $b_2(\tilde{S}) = 106$, so $\dim_{\mathbb{F}}(C_S) \geq 12$.

If $C_S$ contains no word of weight 56, then $\dim_{\mathbb{F}}(C_S) = 12$ [JR], [W]. A short argument runs as follows: By the Griesmer bound $C_S$ contains a word $w$ of weight 24. Clearly $p_w: C_S \to (C_S)_w$ has trivial kernel, so $(C_S)_w$ is a doubly even $[24, \dim_{\mathbb{F}}(C_S), 4]$ code. Hence $\dim_{\mathbb{F}}(C_S) \leq 12$.

It is not clear if $C_S$ is unique up to permutation. It is also not known if any nodal sextic surface can have even sets of 56 or 64 nodes.

1.3 The theorem
For nodal surfaces of degree 6, Jaffe and Ruberman proved that the smallest possible nonzero strictly even sets of nodes are the ones cut out by quadrics. This seems to be true for nodal surfaces of arbitrary degree, though we only can prove a few cases. For weakly even sets of nodes, the corresponding statement is proved.
Definition 1.9 For $s \in \mathbb{N}$ the minimal cardinality of an even set of nodes on a nodal surface of degree $s$ is defined as

$$e_{\min}(s) = \min \{|w| \mid w \in C_S, \ S \text{ nodal of degree } s\},$$

$$\tau_{\min}(s) = \min \{|w| \mid w \in \overline{C}_S, \ S \text{ nodal of degree } s\}.$$ 

Our main result is the following

Theorem 1.10 i) (Strictly even sets of nodes) Let $s \in \{3, 4, 5, 6, 7, 8, 10\}$. Then

$$e_{\min}(s) = \begin{cases} s(s-2) & \text{if } s \text{ is even}, \\ (s-1)^2 & \text{if } s \text{ is odd}. \end{cases}$$

Moreover $|w| = e_{\min}(s)$ if and only if $w$ is cut out by a quadric surface.

ii) (Weakly even sets of nodes) Let $s \in \{2, 4, 6, 8\}$. Then

$$\tau_{\min}(s) = \frac{s(s-1)}{2}.$$ 

Moreover $|w| = \tau_{\min}(s)$ if and only if $w$ is cut out by a plane.

A close examination of the proof of theorem 1.10 exhibits that certain weights strictly greater than $e_{\min}(s)$ and $\tau_{\min}(s)$ cannot appear.

Corollary 1.11 For any nodal surface $S$ of degree $s$, there exist no even sets of nodes with the following weights.

| $s$ | 6 | 7 | 8 | 10 |
|-----|---|---|---|----|
| weakly even | 19, 23 | 32, 36, ..., 56 |
| strictly even | 40 | 56 | 88, 96, 104, 112 |

If $w \in \overline{C}_S$ is cut out by a smooth cubic surface, then $|w| = 3s(s-3)/2$ [Cat]. The corollary states that all weights in the open interval $[\tau_{\min}(s), 3s(s-3)/2]$ do not appear for weakly even set of nodes. In the case of strictly even sets of nodes, the gap is the interval $[e_{\min}(s), 2s(s-4)]$. Note that if $w \in C_S$ is cut out by a smooth quartic surface, then $|w| = 2s(s-4)$.

Remark 1.12 It follows from example 1.4 and example 1.5 that the theorem is true for $s = 2, 3$.

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2 The formula of Gallarati

The contact of hypersurfaces in $\mathbb{P}_r$ along a $r-2$ dimensional variety has been (to our knowledge) studied first by D. Gallarati [Ga]. He stated the following

Theorem 2.1 Let $F_m, G_n \subset \mathbb{P}_r$ be hypersurfaces of degree $m$ and $n$ with $F_m \cdot G_n = qC$ for some $r-2$ dimensional variety $C$. Assume that $F_m$ and $G_n$ have at most double points on $C$. If the singular locus of $F_m$ on $C$ (resp. $G_n$ on $C$) is a $r-3$ dimensional variety of degree $t$ (resp. $s$), then

$$q(t-s) = mn(m-n).$$
Proof: There exists a neighborhood $S$ of the origin $0$ such that $S$ is a nodal surface recall that for every even set of nodes $w$ on $S$ there exists a surface $V$ such that $S.V = 2D$ and $w$ is just the set of nodes of $S$ through which $D$ passes with odd multiplicity. We estimate the number of nodes through which $D$ passes with multiplicity one.

For a slightly more general setup, let $M$ be a smooth projective threefold and let $S \subset M$ be a nodal surface. Assume that a surface $V \subset M$ intersects $S$ as $S.V = rD + D'$, $r \geq 2$, for an irreducible curve $D$ which is not contained in the support of $D'$.

**Definition 2.2** A node $P$ of $S$ is called $D$-smooth if $P \in D$ and $P$ is a smooth point of $V$.

This definition is justified by the following

**Lemma 2.3** Let $P$ be a node of $S$. If $P$ is $D$-smooth, then $P$ is a smooth point of $D$. Moreover $r = 2$ and $P \notin \text{supp}(D')$.

**Proof:** There exists a neighborhood $U$ of $P$ in $M$ which is biholomorphic to some open neighborhood of the origin $0 \in \mathbb{C}^3$, so it suffices to prove the lemma for two affine hypersurfaces $S, V \subset \mathbb{C}^3$. We study the intersection with a general plane through $0$.

Let $L \cong \mathbb{P}^2$ be the set of all planes $H \subset \mathbb{C}^3$ through $0$ and let $T = T_0V \in L$ be the tangent plane to $V$ in $0$. Then for all $H \in L \setminus \{T\}$, the curve $C_H = V.H$ is smooth in $0$. The set of all planes $H \in L$ which have contact to the tangent cone $C_0S$ of $S$ in $0$ is parameterized by a smooth conic $Q \subset L$. For all $H \in L \setminus Q$, the curve $F_H = S.H$ has an ordinary double point in $0$. While varying $H$ in $L \setminus (Q \cup \{T\})$, the tangent lines $T_0C_H$ sweep out $T_0V$, while the tangent lines to both branches of $F_H$ in $0$ sweep out $C_0S$. So there exists a plane $\tilde{H} \in L \setminus (Q \cup \{T\})$ such that $T_0C_{\tilde{H}}$ is not contained in $C_0S$. Therefore $C_{\tilde{H}}$ and $F_{\tilde{H}}$ meet transversal in $0$, hence on $\tilde{H}$ we have local intersection multiplicity $(F_{\tilde{H}}, C_{\tilde{H}})_0 = 2$. Then of course

$$2 = (F_{\tilde{H}}, C_{\tilde{H}})_0 = (S|_{\tilde{H}}, V|_{\tilde{H}})_0 = (S.V, \tilde{H})_0 = ((rD + D').H)_0 = r(D.\tilde{H})_0 + (D'.\tilde{H})_0.$$  

Now $0 \in D$ implies $(D.\tilde{H})_0 \geq 1$. Since $r \geq 2$ we get $(D.\tilde{H})_0 = 1$, $r = 2$ and $(D'.\tilde{H})_0 = 0$. This proves the lemma.$\square$

Now we give the lower bound for the number of $D$-smooth nodes of $S$.

**Proposition 2.4** Assume that $D \notin \text{sing}(V)$ and let $\beta$ be the number of singular points of $V$ on $D$ which are smooth points of $S$. Then $S$ has at least $D.(S - V) + \beta$ nodes which are $D$-smooth.

**Proof:** To prove the theorem we would like to have everything smooth. There exists a sequence of blowups (embedded resolution of the singular locus of $S, V$ and $D$)

$$\tilde{M} = M_n \to \cdots \tilde{M}_1 \to M_0 = M.$$
Let $S_i, V_i$ and $D_i$ denote the proper transforms of $S, V$ and $D$ with respect to $\pi_1 \circ \pi_1^{-1} \circ \ldots \circ \pi_1$. We can define divisors $D'_i$ by $S_i, V_i = rD_i + D'_i$ with $D_i \not\subset \text{supp}(D'_i)$, $1 \leq i \leq n$. Moreover we can arrange the maps $\pi_i$ in such a way that the following conditions hold.

i) $\pi_1: M_1 \to M$ is the blowup of $M$ in all points which are singular for both $S$ and $V$.

ii) $\pi = \pi_{n-1} \circ \ldots \circ \pi_2: M_{n-1} \to M_1$ is the embedded resolution of the singular locus of $V_1$, i.e., every map $\pi_{i+1}$ is a blowup of $M_i$ centered in a smooth variety $Z_i \subset M_i$ such that $Z_i$ is either a point or a smooth curve, $1 \leq i \leq n-2$.

iii) $\pi_n: \tilde{M} \to M_{n-1}$ is the embedded resolution of the singularities of $S_{n-1}$ and $D_{n-1}$.

Now one has to keep track of the intersection numbers $D_i, (S_i - V_i)$ as $i$ increases. We study each of the three maps separately.

i) Let $Z = \{P_1, \ldots, P_\alpha\} = \text{sing}(S) \cap \text{sing}(V)$, then $M_1 = \text{Bl}_Z M$. Denote by $E_j = \pi_0^{-1}(P_j)$ the exceptional divisor corresponding to $F_j$. The proper transforms are calculated as $S_1 = \pi_1^* S - 2\sum_{j=1}^{\alpha} E_j$ and $V_1 = \pi_1^* V - \sum_{j=1}^\alpha m_j E_j$, where $m_j = \text{mult}(V, P_j) \geq 2$, $1 \leq j \leq \alpha$. Then the intersection number can be estimated as

$$D_1, (S_1 - V_1) = D_1, \pi_1^* (S - V) + \sum_{j=1}^{\alpha} (m_j - 2) D_1, E_j \geq D_1, (S - V).$$

This is just the information we need, so let us consider the second case.

ii) Every blowup $\pi_{i+1}$ gives rise to an exceptional divisor $E_{i+1} = \pi_{i+1}^{-1}(Z_i)$. In the $(i+1)$-st step always $S_i$ is smooth in all points of $S_i \cap Z_i$, whereas $V_i$ is singular in all points of $Z_i$. So the proper transforms are

$$S_{i+1} = \pi_{i+1}^* S_i - n_i E_{i+1} \quad \text{and} \quad V_{i+1} = \pi_{i+1}^* V_i - p_i E_{i+1}$$

where $n_i = \text{mult}(S_i, Z_i) \in \{0, 1\}$ and $p_i = \text{mult}(S_i, Z_i) \geq 2$. So this time the intersection number in question is just

$$D_{i+1}, (S_{i+1} - V_{i+1}) = D_{i+1}, \pi_{i+1}^* (S_i - V_i) + (p_i - n_i) D_{i+1}, E_{i+1}$$

$$= D_i, (S_i - V_i) + \begin{cases} (p_i - 1) \text{mult}(D_i, Z_i) & \text{if } Z_i \text{ is a point}, \\ p_i \sum_{P \in Z_i \cap D_i} \text{mult}(D_i, P) & \text{if } Z_i \text{ is a curve}, \end{cases}$$

$$\geq D_i, (S_i - V_i) + \#(Z_i \cap D_i).$$

But every singularity of $V$ on $D$ outside the singular locus of $S$ counts at least once. So by induction

$$D_{n-1}, (S_{n-1} - V_{n-1}) \geq D_1, (S_1 - V_1) + \beta \geq D_1, (S - V) + \beta.$$ 

where $\beta = \#(\text{sing}(V) \cap D) \setminus \text{sing}(S)$.

iii) As for the third case we note that $V_{n-1}$ is smooth and $S_{n-1}$ is nodal with $S_{n-1} V_{n-1} = rD_{n-1} + D'_{n-1}$. Either $D_{n-1} \cap \text{sing}(S_{n-1}) = \emptyset$ or $D_{n-1}$ contains at least one node of $S$. But then $r = 2$ by lemma 3.3 and $D_{n-1}$ is smooth in all nodes of $S_{n-1}$. In both cases, $S_{n-1}$ and $D_{n-1}$ do not have common singularities.

Let $P_{\alpha+1}, \ldots, P_{\alpha+\eta}$ be the nodes of $S_{n-1}$ on $D_{n-1}$ and let $P_{\alpha+\eta+1}, \ldots, P_{\alpha+\eta+\tau}$ be the remaining nodes of $S_{n-1}$. Moreover let $E_j = \pi_n^{-1}(P_j)$, $\alpha + 1 \leq j \leq \alpha + \eta + \tau$. But the embedded resolution of the singularities of $D_{n-1}$ on $V_{n-1}$ is the same as
on $S_{n-1}$, so the proper transforms are

$$\tilde{S} = S_n = \pi_n^* S_{n-1} - E_D - 2 \sum_{j=\alpha+1}^{\alpha+\eta+\tau} E_j,$$

$$\tilde{V} = V_n = \pi_n^* V_{n-1} - E_D - \sum_{j=\alpha+1}^{\alpha+\eta} E_j - \sum_{k=\alpha+\eta+1}^{\alpha+\eta+\tau} q_k E_k,$$

where $q_k = \text{mult} (V_k, P_k) \in \{0, 1\}$ and $E_D$ is a sum of exceptional divisors corresponding to the singularities of $D_{n-1}$. Set $\tilde{D} = D_n$ and calculate

$$\tilde{D}. (\tilde{S} - \tilde{V}) = \tilde{D}. \pi_n^* (S_{n-1} - V_{n-1}) - \sum_{j=\alpha+1}^{\alpha+\eta} \tilde{D}. E_j$$

$$= D_{n-1}. (S_{n-1} - V_{n-1}) - \eta$$

$$\geq D. (S - V) - \eta + \beta.$$ 

On the other hand the smooth surfaces $\tilde{S}$ and $\tilde{V}$ have contact of order $r - 1 \geq 1$ along the smooth curve $\tilde{D}$. So the tangent bundles $T_{\tilde{S}}$ and $T_{\tilde{V}}$ agree along $\tilde{D}$. This implies that the normal bundles $N_{\tilde{D}|\tilde{S}}$ and $N_{\tilde{D}|\tilde{V}}$ coincide, thus

$$(\tilde{D}^2)_{\tilde{S}} = \deg (N_{\tilde{D}|\tilde{S}}) = \deg (N_{\tilde{D}|\tilde{V}}) = (\tilde{D}^2)_{\tilde{V}}.$$ 

Now by adjunction formula $\tilde{D}. K_{\tilde{V}} = \tilde{D}. K_{\tilde{S}}$. Using the adjunction formula again we see that

$$0 = \tilde{D}. K_{\tilde{S}} - \tilde{D}. K_{\tilde{V}} = \tilde{D}. (K_{\tilde{M}} + \tilde{S})|_{\tilde{S}} - \tilde{D}. (K_{\tilde{M}} + \tilde{V})|_{\tilde{V}} = \tilde{D}. (\tilde{S} - \tilde{V}).$$

This gives the desired formula $\eta \geq D. (S - V) + \beta$. If $V$ is also a nodal surface one can see easily that we have equality. \(\square\)

The application to surfaces in $M = \mathbb{P}_3$ gives the following

**Corollary 2.5** Let a nodal surface $S \subset \mathbb{P}_3$ of degree $s$ and an irreducible surface $V \subset \mathbb{P}_3$ of degree $v$ intersect as $S.V = 2D$ for curve $D$ on $S$. Assume that $V$ is not singular along a curve contained in $S$ and let $\beta$ be the number of singular points of $V$ which are smooth for $S$. If $s > v$, then $D$ is reduced. Moreover $V$ cuts out an even set of at least $sv (s - v)/2 + \beta$ nodes on $S$ with equality if $V$ is also nodal.

**Proof:** Just run proposition 2.4 on every irreducible component of $D$. \(\square\)

It is possible to extend proposition 2.4 to the case when the surface $V$ is not irreducible, but reduced. The proof however works different.

**Proposition 2.6** Let $S \subset \mathbb{P}_3$ be a nodal surface, $n \in \mathbb{N}$ and let $V_1, \ldots, V_n \subset \mathbb{P}_3$ be different irreducible surfaces of degrees $v_1, \ldots, v_n$ satisfying the following conditions:

i) $V_i$ is not singular along a curve contained in $S$, $1 \leq i \leq n$,

ii) $v_i = \text{deg} (V_i) < s$, $1 \leq i \leq n$ and

iii) $S.(V_1 + \ldots + V_n) = 2D$ for a (not necessarily reduced) divisor $D$ on $S$.

Then the reduced surface $V = V_1 + \ldots + V_n$ of degree $v = v_1 + \ldots + v_n$ cuts out an even set of nodes $w \in C_S$ of weight $|w| \geq sv (s - v)/2$. 

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Proof: Since \( v_i < s \) and \( V_i \) is not singular along a curve contained in \( S \) there exist reduced divisors \( D_i \) and \( R_i \) on \( S \) which do not have a common component such that \( S.V_i = 2D_i + R_i, 1 \leq i \leq n \). But

\[
S.V = S.(V_1 + \ldots + V_n) = 2(D_1 + \ldots + D_n) + R_1 + \ldots + R_n.
\]

This implies that \( R_i \subset \bigcup_{j \neq i} V_j \) and thus \( R_i \) has a decomposition \( R_i = \sum_{j \neq i} R_{i,j} \) such that \( R_{i,j} \subset V_i \cap V_j \). Now we count the nodes of \( S \) through which \( D \) passes with multiplicity 1. Denote \( d_i = \deg(D_i), r_i = \deg(R_i) \) and \( r_{i,j} = \deg(R_{i,j}) \). By corollary \( 2.3 \), \( V_i \) contains at least \( d_i (s - v_i) \) nodes of \( S \) through which \( D_1 \) passes with multiplicity 1. All these nodes lie outside \( R_i \). We cannot simply add these numbers: some nodes might be counted more than once. But every node \( P \in w \) on \( V_i \) which is counted more than once is contained also in some \( V_j \) for a \( j \neq i \), hence in \( F_{i,j} = V_i \cap V_j \). Let \( f_{i,j} = \deg(F_{i,j}) \). Let \( C \) be an irreducible component of \( F_{i,j} \) and let \( c = \deg(C) \). We have the following possibilities:

- \( C \not\subset S \). In this case \( C \) contains at most \( cs/2 \) nodes of \( S \).
- \( C \subset S \) is a component of \( R_i \). Here \( C \) does not contain any node that we counted.
- \( C \subset S \) is a component of \( D_i \) and \( D_j \). Here \( V_i \) and \( V_j \) have contact to \( S \) along \( C \) and thus \( C \) appears in \( F_{i,j} \) with multiplicity \( \geq 2 \). Clearly \( C \) contains at most \( c(s - 1) \) nodes of \( S \).
- \( C \subset S \) is a component of \( D_i \), but not of \( D_j \). Then \( V_i \) and \( V_j \) meet transversal along \( C \), so \( S.V_i + V_j = 3C + \) other curves. So there exist a \( k \not\in \{i, j\} \) such that \( C \in F_{i,k} \). So \( C \) appears with multiplicity \( \geq 2 \) in \( \sum_{j \neq i} F_{i,j} \). Again \( C \) contains at most \( c(s - 1) \) nodes of \( S \).

Since every component of \( R_{i,j} \) is contained in \( F_{i,j} \), this shows that \( F_{i,j} \) contains at most \( (f_{i,j} - r_{i,j})s/2 \) nodes that we counted. So \( V_i \) contains at least

\[
d_i (s - v_i) - \sum_{j \neq i} (f_{i,j} - r_{i,j})
\]
	nodes through which \( D \) passes with multiplicity 1. This implies

\[
|w| \geq \sum_{i=1}^{n} d_i (s - v_i) - \frac{s}{2} \sum_{j \neq i} (f_{i,j} - r_{i,j})
\]

\[
= \frac{1}{2} \left( \sum_{i=1}^{n} (s v_i - r_i) (s - v_i) + s r_i - s \sum_{j \neq i} v_i v_j \right)
\]

\[
= \frac{1}{2} \left( s^2 (v_1 + \ldots + v_n) - s \left( v_1^2 + \ldots + v_n^2 + 2 \sum_{i<j} v_i v_j \right) + r_1 v_1 + \ldots + r_n v_n \right)
\]

\[
\geq \frac{su}{2} (s - v)
\]

This completes the proof. \( \Box \)

3 Contact surfaces and quadratic systems

In this section we apply the previous results to our initial situation. So let again \( S \subset \mathbb{P}_3 \) be a nodal surface of degree \( s \) and \( V \subset \mathbb{P}_3 \) a reduced surface of degree \( v \) such
that $S.V = 2D$ for some curve $D$. We give a complete analysis of the situation when $V$ is a plane or a quadric. Using the notation of the first paragraph, $V$ cuts out an even set of nodes $w \in \mathcal{C}_S$. Recall that the linear system $L_w = [(v\pi^*H-E_w)/2]$ parametrizes all contact curves of the form $D' = (1/2)S.V'$ where $V'$ is a surface of degree $v$ which cuts out $w$. In some cases, $V$ will be the unique surface of degree $v$ which cuts out $w$.

Now $2D \in \mathbb{P}(H^0(\mathcal{O}_S(vH)))$ is the restriction of $V \in \mathbb{P}(H^0(\mathcal{O}_{P_3}(vH)))$ to $S$. Consider the exact sequence

$$0 \to \mathcal{O}_{P_3}((v-s)H) \to \mathcal{O}_{P_3}(vH) \to \mathcal{O}_S(vH) \to 0.$$ 

Since $H^i(\mathcal{O}_{P_3}((v-s)H)) = 0$ for $s > v$, $i = 0, 1$, the induced map $H^0(\mathcal{O}_{P_3}(vH)) \to H^0(\mathcal{O}_S(vH))$ is an isomorphism. So if $v < s$, then $V$ is the unique surface of degree $v$ cutting out $w$ via $D$ and $L_w = [(v\pi^*H-E_w)/2]$ in fact parametrizes the space of all surfaces of degree $v$ which cut out $w$. This space is not a linear system, but the quadratic system

$$Q_w = \{V' \mid S.V' = 2D' \text{ with } D' \in L_w\}.$$ 

It is constructed as follows: If $h^0(\mathcal{O}_S((v\pi^*H-E_w)/2)) = n + 1 \geq 2$ we can find $n + 1$ linearly independent sections $s_0, \ldots, s_n \in H^0(\mathcal{O}_S((v\pi^*H-E_w)/2))$. Clearly all products $s_is_j \in H^0(\mathcal{O}_S((v\pi^*H-E_w)))$ for $0 \leq i \leq j \leq n$. So there exist sections $g_{i,j} \in H^0(\mathcal{O}_{P_3}(v\pi^*H-E_w))$ over $\mathbb{P}_3$ which restrict to $s_is_j$ under the identification $\mathcal{O}_{P_3}(v\pi^*H-E_w) \otimes \mathcal{O}_S \cong \mathcal{O}_S(v\pi^*H-E_w)$. Outside the exceptional locus we can view the $g_{i,j}$ as sections of $\mathcal{O}_{P_3}(vH)$. Since $w$ has codimension $\geq 2$ in $\mathbb{P}_3$, these sections extend also to $w$. This implies that

$$Q_w = Q(g_{i,j} \mid 0 \leq i \leq j \leq n)$$

$$= \left\{ \sum_{i=0}^n \lambda_i^2 g_{i,i} + 2 \sum_{0 \leq i < j \leq n} \lambda_i \lambda_j g_{i,j} = 0 \mid (\lambda_0 : \ldots : \lambda_n) \in \mathbb{P}_n \right\}.$$ 

Therefore the quadratic system $Q_w$ is the image of an embedding of Veronese type of $\mathbb{P}_n$ into the space $\mathbb{P}^{\binom{n+3}{2}}$ parameterizing all surfaces of degree $v$. In general, $Q_w$ will not contain any linear subspace.

The quadratic system $Q_w$ admits a decomposition $Q_w = B_w + F_w$ where $B_w$ is a reduced surface of degree $b \leq v$ and the base locus of $F_w$ (if any) consists only of curves and points. If $F_w$ has no basepoints then $B_w$ cuts out $w$ and so $h^0(\mathcal{O}_S((b\pi^*H-E_w)/2)) = 1$.

**Definition 3.1** An even set of nodes $w \in \mathcal{C}_S$ is called

| semi-stable | stable | unstable |
|-------------|--------|----------|

in degree $v$ if

- $F_w$ is basepointfree,
- $F_w = \emptyset$,
- $F_w$ has basepoints.

The base locus of $Q_w$ is $B(Q_w) = \{g_{i,j} = 0 \mid 0 \leq i \leq j \leq n\}$. It is contained in the discriminant locus $Z(Q_w) = \{g_{i,i}g_{j,j} = g_{i,j}^2 \mid 0 \leq i < j \leq n\}$. There is also a Bertini type theorem for quadratic systems.

**Lemma 3.2** (Bertini for quadratic systems) The general element of $Q_w$ is smooth outside $Z(Q_w)$.

**Proof:** The proof runs like the proof of the Bertini theorem in [GH]. □

Next we give a different characterization of stability.
Proposition 3.3 Let \( w \in \mathcal{C}_g \). The surface \( B_w \) is always reduced and

i) \( w \) is stable in degree \( v \) if and only if \( h^0(\mathcal{O}_g((v\pi^*H-E_w))/2)) = 1 \).

ii) \( w \) is semi stable in degree \( v \) if and only if \( F_w \) contains a square. Then \( B_w \) cuts out \( w \) and either every surface in \( F_w \) is a square or the general surface in \( F_w \) is reduced.

iii) \( w \) is unstable in degree \( v \) if and only if \( F_w \) contains no square. Then \( B_w \) does not cut out \( w \) and the general surface in \( Q_w \) is reduced.

Proof: i) follows from the definition. So let \( w \) be not stable in degree \( v \). We use induction on \( n \).

\( n = 1 \): By construction \( \gcd(g_{0,0}, g_{0,1}, g_{1,1}) = g \) is reduced. Let \( \overline{g}_{i,j} = g_{i,j}/g \) and let \( \overline{Q}_w = Q(\overline{g}_{0,0}, \overline{g}_{0,1}, \overline{g}_{1,1}) \). Now we have two cases.

a) If \( \overline{g}_{0,0} \overline{g}_{1,1} = \overline{g}_{0,1}^2 \), then \( \overline{g}_{0,0} \) and \( \overline{g}_{1,1} \) must be squares. So \( \overline{g}_{0,0} = a_0^2 \), \( \overline{g}_{1,1} = a_1^2 \) and thus \( \overline{g}_{0,1} = a_0 a_1 \). Hence \( B_w = \{g = 0\} \) cuts out \( w \). So by construction the quadratic system \( F_w = \{((\lambda_0 a_0 + \lambda_1 a_1)^2 | (\lambda_0 : \lambda_1) \in \mathbb{P}_1\} \) contains only squares. Then \( F_w \) is free and \( w \) is semi stable in degree \( v \).

b) \( Z(\overline{Q}_w) = \{g_{0,0} \overline{g}_{1,1} = \overline{g}_{0,1}^2\} \) is a surface. If all surfaces of \( Q_w \) are not reduced, then by lemma 3.2 all surfaces of \( Q_w \) contain a component of \( Z(Q_w) \). So this component is constant for all surfaces in \( Q_w \), which contradicts \( \gcd(\overline{g}_{0,0}, \overline{g}_{0,1}, \overline{g}_{1,1}) = 1 \). Therefore the general surface in \( Q_w \) is reduced. Now assume \( B_w \) cuts out \( w \). Then by construction \( F_w \) contains squares, so \( F_w \) is free and \( w \) is semi stable in degree \( v \). Otherwise \( B_w \) does not cut out \( w \), so \( F_w \) must have basepoints in \( w \). Then \( w \) is unstable in degree \( v \).

\( n - 1 \Rightarrow n \): Again \( \gcd(g_{i,j} | 0 \leq i \leq j \leq n) = g \) is reduced. Consider the quadratic system \( Q = Q(g_{i,j} | 0 \leq i \leq j \leq n - 1) \). Either \( \gcd(g_{i,j} | 0 \leq i \leq j \leq n - 1) \) is reduced and we’re done or it’s not reduced. For \( \lambda = (\lambda_0 : \ldots : \lambda_{n-1}) \in \mathbb{P}_{n-1} \) let

\[
g_\lambda = \sum_{i=0}^{n-1} \lambda_i^2 g_{i,i} + 2 \sum_{0 \leq i < j \leq n-1} \lambda_i \lambda_j g_{i,j} \quad \text{and} \quad h_\lambda = \sum_{i=0}^{n-1} \lambda_i g_{i,n,i}.
\]

Now consider the quadratic system

\[
R_\lambda = \{t^2 g_\lambda + 2t \lambda_n h_\lambda + \lambda_n^2 g_{n,n} = 0 | (t : \lambda_n) \in \mathbb{P}_1\}.
\]

While varying \( \lambda \in \mathbb{P}_{n-1} \), \( \gcd(g_\lambda, h_\lambda, g_{n,n}) \) is constant on an open dense subset, since it contains only factors of \( g_{n,n} \). So for general \( \lambda \), \( \gcd(g_\lambda, h_\lambda, g_{n,n}) = \gcd(g_{i,j} | 0 \leq i \leq j \leq n) = g \) is reduced. By the first part either \( g_\lambda g_{n,n} = h_\lambda^2 \) for all \( \lambda \), so \( g_\lambda \) and \( g_{n,n} \) are always squares modulo \( g \). Then \( w \) is semi stable in degree \( v \). Or the general surface in \( R_\lambda \) and hence in \( Q_w \) is reduced. Again either \( B_w \) cuts out \( w \) and we’re in the semi stable case or \( B_w \) does not cut out \( w \). Then \( w \) is unstable in degree \( v \). □

Corollary 3.4 If \( 2v < s \) then \( w \) is semi stable in degree \( v \).

Proof: On \( S \) we have \( g_{0,0} g_{1,1} - g_{0,1}^2 = s_0^2 s_1^2 - (s_0 s_1)^2 = 0 \). Let \( S = \{f = 0\} \), then either \( g_{0,0} g_{1,1} = g_{0,1}^2 \) or \( f | g_{0,0} g_{1,1} - g_{1,1}^2 \). But the second case implies \( 2v \geq s \), so we are in the first case. Then we always run into case a) in the proof of proposition 3.3. □

Corollary 3.5 If \( w \) is semi stable in degree \( v \) and \( 2 \deg(F_w) < s \), then \( F_w \) contains only squares.
Proof: $F_w$ contains a square $W_0 = \{g_{0,0} = g_{0,1}^2 = 0\}$. Now take any other $W_1 = \{g_{1,1} = 0\} \in F_w$ and consider the quadratic system generated by $g_{0,0}$ and $g_{1,1}$: $g_{0,0}$, $g_{1,1}$ give rise to sections $s_0^2$, $s_1^2$ over $S$. Then $s_0s_1$ is the restriction of a section $g_{0,1}$ to $\tilde{S}$. The quadratic system in question is just $Q = Q(g_{0,0}, g_{0,1}, g_{1,1})$. But $g_{0,0}g_{1,1} - g_{0,1}^2$ vanishes on $S$. Since $\deg (g_{0,0}g_{1,1} - g_{0,1}^2) = 2 \deg (F_w) < s$, we have $g_{0,0}g_{1,1} = g_{0,1}^2 = g_{0,1}^2$. This implies that also $g_{1,1}$ is a square. $\square$

**Proposition 3.6** If $w$ is unstable in degree $v$, then there exists a surface $W$ of degree $2v - s$ such that $w$ is cut out by a reduced surface $V$ of degree $v$ satisfying:

i) $V$ is not singular on $S$ outside $W$.

ii) If $V$ is singular along a curve $C \subset S$, then $C$ is a curve of triple points of $W$.

Proof: $w$ is cut out by a reduced surface, so we can assume that $g_{0,0}$ is square free. Again $g_{0,0}g_{1,1} - g_{0,1}^2$ vanishes on $S$ and $g_{0,0}$, $g_{1,1}$ are linearly independent. So there exists a polynomial $\alpha$ of degree $2v - s$ such that $\alpha f = g_{0,0}g_{1,1} - g_{0,1}^2$. Let $W = \{\alpha = 0\}$ and let $V_0 = \{\lambda_0^2g_{0,0} + 2\lambda_0\lambda_1g_{0,1} + \lambda_1^2g_{1,1} = 0 \mid \lambda = (\lambda_0 : \lambda_1) \in \mathbb{P}_1\}$. For every point $P \in \mathbb{P}_0$ we can choose affine coordinates $(z_1, z_2, z_3)$ on an affine neighborhood $U$ of $P$. For any function $h$ on $U$, we identify the total derivative $Dh$ with the gradient $\nabla h$ and $D^2h$ with the Hesse matrix $H(h)$. We find that

$$D(\alpha f) = \alpha \nabla f + f \nabla \alpha = g_{0,0} \nabla g_{1,1} + g_{1,1} \nabla g_{0,0} - 2g_{0,1} \nabla g_{0,1},$$

$$D^2(\alpha f) = \alpha H(f) + f H(\alpha) + \alpha \nabla f \alpha + \nabla f \alpha^t = g_{0,0} H(g_{1,1}) + g_{1,1} H(g_{0,0}) - 2g_{0,1} H(g_{0,1}) + \nabla g_{0,0} \nabla g_{1,1} + \nabla g_{1,1} \nabla g_{0,0} - 2 \nabla g_{0,1} \nabla g_{0,1}.$$

Now let $P \in S$. We have to consider two different cases:

a) $P \in \text{sing}(S) \setminus W$, so $f(P) = 0$, $\nabla f(P) = 0$ and $\text{rk}(H(f)(P)) = 3$. If $P$ is a basepoint of $Q$ then

$$H(\alpha f)(P) = \alpha(P) H(f)(P) = (\nabla g_{0,0} \nabla g_{1,1} + \nabla g_{1,1} \nabla g_{0,0} - 2 \nabla g_{0,1} \nabla g_{0,1})(P).$$

But $P \notin W$, so $\alpha(P) \not= 0$ and $\text{rk}(H(\alpha f)(P)) = 3$. This is only possible if $\nabla g_{0,0}$, $\nabla g_{1,1}$ and $\nabla g_{0,1}$ are linearly independent in $P$. So every surface $V_\lambda$ is smooth in $P$.

b) Let $P \in \text{smooth}(S) \setminus W$. Here $f(P) = 0$, $\nabla f(P) \not= 0$ and $\alpha(P) \not= 0$. Then $\nabla (\alpha f)(P) = \alpha(P) \nabla f(P) \not= 0$, so $P$ is not a basepoint of $Q$. Assume now we have chosen $\lambda$ such that $P \in V_\lambda$. After a permutation of indices we can assume $\lambda_0 = 1$, so $V_\lambda = \{g_{0,0} + 2\lambda_1 g_{0,1} + \lambda_1^2 g_{1,1} = 0\}$. Since $P$ is not a basepoint we have $g_{1,1}(P) \not= 0$. Together with $(g_{0,0}g_{1,1} - g_{0,1}^2)(P) = 0$ we get $\lambda_1 = -(g_{0,1}/g_{1,1})(P)$. Then

$$\nabla (g_{0,0} + 2\lambda_1 g_{0,1} + \lambda_1^2 g_{1,1})(P) = \frac{1}{g_{1,1}(P)} (g_{1,1} \nabla g_{0,0} - 2g_{0,1} \nabla g_{0,1} + g_{0,0} \nabla g_{1,1})(P)$$

$$= \frac{1}{g_{1,1}(P)} \alpha(P) \nabla f(P).$$

We see that $P$ is a smooth point of $V_\lambda$, so together with a) we have proved i).

c) Assume that $V_\lambda$ is singular along a curve $C_\lambda \subset S$ and let $m_\lambda = \text{mult}(V_\lambda, C_\lambda)$. Then $C_\lambda$ is a continuous family of curves and $m = \min \{m_\lambda \mid \lambda \in \mathbb{P}_1\}$ is equal to
This section is devoted entirely to the proof of theorem 1.10. Let $w \in \overline{C}_{S}$. 

i) If $w$ is unstable in degree $s/2$, then $|w| = s^{3}/8$. 

ii) If $w$ is unstable in degree $(s + 1)/2$ (resp. $(s + 2)/2$), then $|w| \geq s(s - 1)^{2}/8$ (resp. $s(s - 2)^{2}/8$).

**Proof:** In the first case $W = \emptyset$. So the general surface in $Q_{w}$ is not singular on $S$, hence irreducible. Now apply corollary 2.3. In the second case deg ($W$) $\leq 2$, so $W$ has no triple curve. Now apply proposition 2.4. 

Now here comes our analysis what happens if $V$ is a plane or a quadric.

**Proposition 3.8** Let $w \in \overline{C}_{S}$. 

i) If $w$ is cut out by a plane $H$, then $|w| = s(s - 1)/2$. Moreover $w$ is stable in degree 1 if $s > 2$ and unstable in degree 1 otherwise.

ii) If $w$ is cut out by a reduced quadric $Q$, then 

$$|w| = \begin{cases} s(s - 2) & \text{if } s \text{ is even,} \\ (s - 1)^{2} & \text{if } s \text{ is odd.} \end{cases}$$

Moreover $w$ is stable in degree 2 if $s > 4$ and unstable in degree 2 otherwise.

**Proof:** i) $H$ is smooth, so $|w| = s(s - 1)/2$ by corollary 2.3. If $s > 2$ then 2deg ($H$) $= 2 < s$, so $w$ is semi stable in degree 1 by lemma 3.4. But then $w$ is stable in degree 1. In the case $s = 2$ example 1.4 shows that $w$ is unstable in degree 1.

ii) Assume first $Q$ is nodal. Then $|w| \in \{s(s - 2), s(s - 2) + 1\}$ by corollary 2.4. But $s(s - 2) + 1 = (s - 1)^{2}$ and 4 $|w|$ imply the above formula for $|w|$.

Now let $Q = H_{1} + H_{2}$ where $H_{1} \neq H_{2}$ are planes and set $L = H_{1} \cap H_{2}$. If $L \subset S$, then $S.H_{i} = 2D_{i} + L$ and each $D_{i}$ contains exactly $(s - 1)^{2}/2$ nodes which are $D_{i}$-smooth by proposition 2.4. Clearly $L$ cannot contain any node of $w$. So $|w| = (s - 1)^{2}$. If $L \not\subset S$, then $S.H_{i} = 2D_{i}$. Every $D_{i}$ is reduced and contains exactly $s(s - 1)/2 D_{i}$-smooth nodes. In every point $P \in L \cap S$ both $H_{1}$ and $H_{2}$ are tangent to $S$, so $P$ is a node of $S$. Both $H_{1}$ and $H_{2}$ have contact to the tangent cone $C_{P}S$ of $S$ at $P$. This implies $L \not\subset C_{P}S$, hence mult ($S, L; P$) $= 2$. Therefore $L$ contains exactly $s/2$ such nodes and $|w| = s(s - 2)$.

If $s > 4$, then $w$ is stable in degree 2. Now let $s \leq 4$. In any case $F_{w}$ cannot contain a square. So $w$ is unstable if $h^{0}(O_{S}((2\pi^{*}H-E_{w})/2)) > 1$. For $s = 3$ this follows from example 1.4. If $s = 4$ then we find using Serre duality that $h^{2}(O_{S}((2\pi^{*}H-E_{w})/2)) = h^{0}(O_{S}((-2\pi^{*}H+E_{w})/2)) = 0$. Therefore it follows that $h^{0}(O_{S}((2\pi^{*}H-E_{w})/2)) \geq \chi(O_{S}((2\pi^{*}H-E_{w})/2)) = 2$. ☐

4 The proof of theorem 1.10

This section is devoted entirely to the proof of theorem 1.10. Let $S$ and $V$ with $S.V = 2D$ as in the first section.
Proof of theorem 1.10: i) By proposition 3.3 the even sets \( w \in \mathcal{T}_S \) cut out by planes satisfy \( |w| = s(s-1)/2 \). We show that no smaller even sets can occur. The proof also explains the “gaps” of corollary 1.11. So let \( w = (s-1)/2 \) be weakly even.

\[ h^i(\mathcal{O}_S((n^*H+E_w)/2)) = h^i(\mathcal{O}_S((n^*H-E_w)/2)) \]

Proof of theorem 1.10: ii) This is essentially a copy of the methods of i). Let \( w \in \mathcal{T}_S \setminus \{0\} \) be strictly even.

\[ s = 4: \text{ We have } |w| \in \{8, 16\}. \text{ If } |w| = 9 \text{ then } \chi(\mathcal{O}_S((2^*H-E_w)/2)) = 2. \text{ But } h^2(\mathcal{O}_S((2^*H-E_w)/2)) = h^0(\mathcal{O}_S((2^*H-E_w)/2)) = 0, \text{ so by Serre duality and lemma } 4.1, h^0(\mathcal{O}_S((2^*H-E_w)/2)) \geq 2 \text{ and } w \text{ is cut out by a quadric.} \]

\[ s = 5: \text{ Now } |w| \in \{8, 16, 24, \ldots \} \text{ and } \chi(\mathcal{O}_S((2^*H-E_w)/2)) = 5 - |w|/4. \text{ As usual we find that } h^2(\mathcal{O}_S((2^*H-E_w)/2)) = h^0(\mathcal{O}_S((2^*H-E_w)/2)) = 0. \text{ If } |w| \leq 16 \text{ then } h^0(\mathcal{O}_S((2^*H-E_w)/2)) \geq 1 \text{ and } w \text{ is cut out by a quadric.} \text{ Then } |w| = 16 \text{ by corollary 3.8.} \]

\[ s = 6: \text{ Here } |w| \in \{8, 16, 24, \ldots \} \text{ and } \chi(\mathcal{O}_S((2^*H-E_w)/2)) = 2 - |w|/4. \text{ This time we find that } h^2(\mathcal{O}_S((2^*H-E_w)/2)) = h^0(\mathcal{O}_S((2^*H-E_w)/2)) = 0. \text{ If } |w| \leq 24 \text{ then } h^0(\mathcal{O}_S((2^*H-E_w)/2)) \geq 1 \text{ and } w \text{ is cut out by a quadric.} \text{ But then } |w| = 24. \]

\[ s = 7: \text{ We modify the proof as follows. One calculates } \chi(\mathcal{O}_S((2^*H-E_w)/2)) = \chi(\mathcal{O}_S((4^*H-E_w)/2)) = 14 - |w|/4. \text{ Let } |w| < 44, \text{ so } |w| \leq 40. \text{ By proposition 3.8, } h^0(\mathcal{O}_S((2^*H-E_w)/2)) \in \{0, 1\}, \text{ so } h^0(\mathcal{O}_S((4^*H-E_w)/2)) \geq 3. \text{ If } w \text{ is unstable in degree 4 then } |w| \geq 42, \text{ contradiction. So } w \text{ is semi stable in degree 4 and stable in degree 2. Now } w \text{ is cut out by a quadric and thus } |w| = 36. \text{ In particular, there is no even set of 40 nodes on a nodal sextic surface.} \]

\[ s = 8: \text{ Using the same argument as for } s = 7, \text{ we get } \chi(\mathcal{O}_S((4^*H-E_w)/2)) = 20 - |w|/4 \text{ and } h^2(\mathcal{O}_S((4^*H-E_w)/2)) = h^0(\mathcal{O}_S((4^*H-E_w)/2)) = 0. \text{ Let } |w| < 64, \text{ then } |w| \leq 56 \text{ and } h^0(\mathcal{O}_S((4^*H-E_w)/2)) \geq 3. \text{ Again by corollary 3.7 } w \text{ cannot be unstable in degree 4. Hence } w \text{ is semi stable in degree 4 and stable in degree 2. In particular } |w| = 48 \text{ and there is no even set of 56 nodes on a nodal octic surface.} \]

\[ s = 10: \text{ Finally we calculate } \chi(\mathcal{O}_S((6^*H-E_w)/2)) = 40 - |w|/4 \text{ and as before } h^2(\mathcal{O}_S((6^*H-E_w)/2)) = h^0(\mathcal{O}_S((6^*H-E_w)/2)) = 0. \text{ Let } |w| < 120, \text{ so } |w| \leq 112 \text{ and} \]


$h^0(\mathcal{O}_{\tilde{S}}((6\pi^*H-E_w)/2)) \geq 6$. As before $w$ is semi stable in degree 6. If $w$ was stable in degree 4 then $h^0(\mathcal{O}_{\tilde{S}}((6\pi^*H-E_w)/2)) = 4$, contradiction. So $w$ is semi stable in degree $w$ and stable in degree 2. Again $|w| = 80$, hence there are no strictly even sets of 88, 96, 104 and 112 nodes on a nodal surface of degree 10.\qed

5 Examples revisited

We want to go a little more into the example of quartics. Many of the facts stated in this example can be found in [Ga].

Example 5.1 Let $S$ be a nodal quartic surface and let $w \in C_S$ with $|w| = 8$. We have seen that $h^0(\mathcal{O}_{\tilde{S}}((2\pi^*H-E_w)/2)) \geq 2$ and that $w$ is unstable in degree 2. Let $Q_w$ be the quadratic system of quadrics which cut out $w$. The base locus of $Q_w$ is contained in the surface $W$ of proposition 3.6. But here $W = \emptyset$, so the only basepoints of $Q_w$ are the nodes of $S$. It follows from lemma 2.3 and Bertini that the general element in $L_w = |(2\pi^*H-E_w)/2|$ is a smooth elliptic curve. In fact $Q_w$ defines an elliptic fibration of $\tilde{S}$. If $h^0(\mathcal{O}_{\tilde{S}}((2\pi^*H-E_w)/2)) > 2$, then we will find two smooth elliptic curves on $\tilde{S}$ intersecting in at least one point, contradiction. This shows that $h^0(\mathcal{O}_{\tilde{S}}((2\pi^*H-E_w)/2)) = 2$. Taking into account that $(2\pi^*H-E_w)/2$ is nef and big on $\tilde{S}$, we can calculate the numbers of the next table. If $|w| = 16$ one computes $h^i(\mathcal{O}_{\tilde{S}}((4\pi^*H-E_w)/2))$ in the same fashion.

| $|w| = 8$ | $h^0$ | $h^1$ | $h^2$ | $|w| = 16$ | $h^0$ | $h^1$ | $h^2$ |
|----------------|-----|-----|-----|----------------|-----|-----|-----|
| $(2\pi^*H-E_w)/2$ | 2   | 0   | 0   | $(2\pi^*H-E_w)/2$ | 0   | 0   | 0   |
| $(4\pi^*H-E_w)/2$ | 8   | 0   | 0   | $(4\pi^*H-E_w)/2$ | 6   | 0   | 0   |

Now let $w \in C_S$ be weakly even. Here $(3\pi^*H-E_w)/2$ is big and nef, so we find the following table.

| $|w| = 6$ | $h^0$ | $h^1$ | $h^2$ | $|w| = 10$ | $h^0$ | $h^1$ | $h^2$ |
|----------------|-----|-----|-----|----------------|-----|-----|-----|
| $(\pi^*H-E_w)/2$ | 1   | 0   | 0   | $(\pi^*H-E_w)/2$ | 0   | 0   | 0   |
| $(3\pi^*H-E_w)/2$ | 5   | 0   | 0   | $(3\pi^*H-E_w)/2$ | 4   | 0   | 0   |

6 Concluding remarks

It is very likely that theorem 1.10 is true for surfaces of arbitrary degree. Unfortunately I cannot prove this. The main obstruction is to exclude the possibility that an irreducible contact surface is singular along a curve which is contained in the nodal surface.

In [Ba2] Barth gave a construction of nodal surfaces admitting even sets of nodes. The surfaces are constructed as degeneracy locus of a generic quadratic form on a globally generated vector bundle on $\mathbb{P}_3$. For convenience we give a list of the strictly even sets of nodes which have been obtained so far. Note that Barth’s construction gives exactly the gap of corollary 1.11.

| degree | even sets |
|-------|-----------|
| 3     | 4         |
| 4     | 8,16      |
| 5     | 16,20     |
| 6     | 24,32,40  |
| 8     | 48,64,72,80, \ldots, 128 |
| 10    | 80,120,128,136, \ldots, 208 |
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