Competition of superfluidity and density waves in one-dimensional Bose-Fermi mixtures

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We study a mixture of one-dimensional bosons and spinless fermions at incommensurate filling using phenomenological bosonization and Green’s functions techniques. We derive the relation between the parameters of the microscopic Hamiltonian and macroscopic observables. Galilean invariance results in extra constraints for the current current interactions. We obtain the exact exponents for the various response functions, and show that superfluid fluctuations are enhanced by the effective boson-fermion density-density interaction and suppressed by the effective boson-fermion current-current interaction. In the case of a bosonized model with purely density-density interaction, when the effective boson-fermion density-density interaction is weak enough, the superfluid exponent of the bosons has a non-monotonous variation with the ratio of the fermion velocity to the boson velocity. By contrast, density-wave exponent and the exponent for fermionic superfluidity are monotonous functions of the velocity ratio.

I. INTRODUCTION

The recent development of atom trapping technologies [1, 2] has led to the opening of new research fields of many-body physics. In particular, the possibility of controlling to a large degree the shape of the trapping potential has permitted the experimental realization of nearly isolated low dimensional interacting boson systems [3], and the observation of the bosonic Mott insulator [4]. Moreover, the use of Fano-Feshbach resonances [5, 6] allows experimentalist to control interaction strength in the low dimensional atomic gases [7–9]. This has permitted the observation of the Girardeau [10] fermionization of one-dimensional bosons with strongly repulsive interactions [11, 12]. From the theoretical point of view, fermions as well as bosons with repulsive interactions in one dimension are in the Tomonaga-Luttinger liquid state [13–15]. The bosonization technique [13] allows the description of their low-energy physics in terms of collective phonon-like modes as well as the computation of the various correlation functions. It is found that in the Tomonaga-Luttinger liquid state at zero temperature, only quasi-long range order (whether superfluid or density wave) can be obtained, with correlation functions decaying as power laws, the exponent of the power law being a rational function of the Tomonaga-Luttinger parameter $K$. This parameter $K$ depends on interaction. In the case of bosons with contact interaction [17], this parameter can be calculated exactly, and it is found that $K \geq 1$ making superfluidity the dominant correlation. When the interaction becomes extremely repulsive, or the density becomes very low, the Girardeau model is recovered [11] and the exponent $K$ goes to 1. In the case of spinless fermions, perturbative calculations [18, 20] show that the exponent $K$ is smaller than 1 for repulsive interactions (thus favoring density waves) and larger than 1 for attractive interactions (thus favoring a $p$-wave type superfluidity). For finite temperatures, the correlation functions decay exponentially, with a correlation length inversely proportional to temperature. Such behavior gives rise to susceptibilities that diverge as a power law with temperature, with an exponent determined by the zero temperature correlation exponent. Recent experiments [21, 25] with trapped atoms have shown the possibility of realizing many-body systems mixing fermions and bosons. From the theoretical point of view, mixtures of bosons and spinless fermions in one dimension were considered in Refs. [20, 28]. It was found that sufficiently strong repulsion or attraction between fermionic and bosonic atoms could result in respectively a phase separation or a collapse, while interactions of moderate strength would lead (in the case of incommensuration between the atom densities) to the formation of a two-component Tomonaga-Luttinger liquid. In Refs. [27, 28] it was proposed to describe the two-component Tomonaga-Luttinger liquid in terms of polarons, and various exponents of correlation functions were obtained. An integrable model of interacting bosons and fermions was considered in Refs. [29, 33], in which a two-component Tomonaga-Luttinger liquid was found. The mixture has also been investigated in numerical calculations [34–40]. More recent experimental work has considered mixtures of fermionic atoms. Motivated by these experiments, there has been some numerical studies of mixtures of three fermionic species [11].

Although the one-dimensional boson fermion mixture has been studied extensively including analytical expression of the exponents [27, 28] of the equal time correlation function, the previous work has emphasized the polaron correlations over the density wave correlations. It is thus worthwhile to analyze the competition between density-wave and superfluid correlation as interactions in the system are varied. Besides the question of the effect of interaction on the exponents, an important question rele-
vant for experiment is the evolution of the exponents as a function of relative concentration of fermions and bosons in the mixture. Fermions could have two opposite effects. On the one hand, fermionic particles can give rise to retarded attractive interactions between the bosons, which can compensate the direct boson-boson repulsion and favor superfluidity and possibly collapse of the mixture. On the other hand, fermionic particles do not form Bose condensates, and may thus disfavor the formation of a superfluid state. For the simple Bose-Fermi mixture model, it is useful to examine the effect of the doping on the correlated states of respective particles, e.g., how the superfluidity is enhanced or suppressed by the interaction between boson and fermion. A last question, in order to compare numerical and analytical result, is to develop a phenomenological bosonization scheme allowing for the calculation of the parameters of the low energy effective Hamiltonian from ground state energy calculations.

In the present article, we present a study of mixtures of fermionic or bosonic atoms using bosonization and a Green’s function formalism. After discussing briefly in Sec. II the phenomenological bosonized Hamiltonian of the model and its relation with the macroscopic observables, we focus in Sec. III on the calculation by Green’s function techniques of the exponents for superfluidity and density waves. We find a relation between the products of density wave and superfluid exponents for the bosons and the fermions. We also give an expression of the correlation lengths at finite temperature. In Sec. IV, we use the Green’s function technique to obtain the Bragg scattering intensity following Ref. 42. Then, in Sec. V we consider the effect of the variation of the density of one of the two atom species first in the case of the model of Ref. 26. We show that for this model, the variation of the boson superfluid exponent with fermion density is not monotonous in the case of a weak interaction. We also consider the effect of the effective current-current interaction and show that it suppresses superfluidity.

II. HAMILTONIAN

We consider a mixture of spinless fermions and bosons. The Hamiltonian reads:

\[ H = \sum_{a,b} \int \frac{dx}{2\pi} \left[ \pi^2 M_{ab} \Pi_a \Pi_b + N_{ab} \partial_x \phi_a \partial_x \phi_b \right], \]

where \( \phi_a(x), \Pi_b(x') = i \delta_{a,b} \delta(x - x') \) and \( a, b \in \{ B, F \} \).

Indeed, the Hamiltonian is invariant under a parity transformation. Under such a transformation, \( \phi_a(x) \rightarrow -\phi_a(-x) \) and \( \Pi_a(x) \rightarrow -\Pi_a(-x) \), so that quadratic terms \( \Pi_a \partial_x \phi_b \) change sign under parity, and Eq. 2 is the most general Hamiltonian quadratic in \( \Pi_a \) and \( \partial_x \phi_a \) for \( a = B, F \). The matrices \( M_{ab} \) and \( N_{ab} \) in Eq. 2 are real symmetric and are deduced from variations of the ground state energy \( E_{GS} \) of a finite system of size \( L \) with (respectively) changes of boundary conditions.

In the case of a Galilean invariant Hamiltonian such as Eq. 1, it is possible to further constrain the elements of the matrix \( M \) with the relations (see App. A):

\[ \pi \rho_F = M_{FF} \rho_F + M_{BF} \rho_B, \]
\[ \pi \rho_B = M_{BF} \rho_F + M_{BB} \rho_B. \]

So that knowing just one of the parameters \( M_{FF}, M_{BB} \), and \( M_{BF} \) fully determines the matrix \( M \). If one starts from the limit \( V_{BF} = 0 \) and applies bosonization first, and then reintroduces the interaction \( V_{BF}(x - x') = V_{BF} \delta(x - x') \) as a perturbation, to lowest order one obtains the Hamiltonian:

\[ H = \sum_{\nu = F, B} \int \frac{dx}{2\pi} \left[ \frac{u_\nu}{K_\nu} (\pi \Pi_\nu)^2 + \frac{u_\nu}{K_\nu} (\partial_x \phi_\nu)^2 \right] + \frac{V_{BF}^2}{\pi^2} \int dx \partial_x \phi_B \partial_x \phi_F, \]

in which \( u_\nu, K_\nu = \frac{\pi \rho(0)}{M_\nu} \) as a result of Galilean invariance of Eq. 1, \( \rho(0) = \frac{1}{\pi \rho(0)^2}, \) where \( \rho(0) \) is the density of particles, and \( \kappa_\nu \) the compressibility. In the case of non-interacting fermions \( V_{BF} = 0 \), one has \( K_F = 1 \).

For hard core bosons, \( K_B = 1 \), while for the Lieb-Liniger model \( K_B = \frac{1}{2} \), one has \( K_B \geq 1 \), with \( K_B \rightarrow \infty \) when \( V_{BB} \rightarrow 0 \) and \( K_B \rightarrow 1 \) for \( V_{BB} \rightarrow \infty \). Note that at this low order in perturbation theory, no term \( M_{FF} \) is present. This term is expected to appear in second order perturbation theory, along with corrections to the bare terms \( M_{BF} \) and \( M_{BB} \). A Hamiltonian with quadratic interactions similar to Eq. 2 but also comprising interband interactions similar to Eq. 3 reads 26–28:

\[ H = \sum_{a,b} \int \frac{dx}{2\pi} \left[ \pi^2 M_{ab} \Pi_a \Pi_b + N_{ab} \partial_x \phi_a \partial_x \phi_b \right], \]
considered in the context of a two band model of interacting spinless fermions [43]. A Hamiltonian equivalent to Eq. 4 has been studied by path integral methods as a model of one-dimensional electrons interacting with acoustic phonons [44,45]. Due to the quadratic character of the Hamiltonian [26], its spectrum is readily obtained as two branches $\omega_{\pm}(q) = u_{\pm}|q|$ with linear spectrum, showing that its ground state is a two component Tomonaga-Luttinger liquid provided the velocities $u_{\pm}$ of both components are real. The vanishing of the velocity $u_-$ is an indication of instability [20] either towards phase separation (in the case of repulsive boson-fermion interaction) or collapse (in the case of attractive boson-fermion interaction).

III. SUPERFLUID AND DENSITY WAVE CORRELATIONS

From the diagonalization of the Hamiltonian [26], bosonization allows to obtain the exponents of the various correlation functions [25]. In this paper, we will use a different (but equivalent) approach to compute the correlation functions of the system.

By an equation of motion method, we first obtain the Green’s functions:

$$ G_{ab}(x, \tau) = -(T_{\tau}(\phi_a(x, \tau) - \phi_a(0,0))\phi_b(0,0)), $$ (8)

$$ \bar{G}_{ab}(x, \tau) = -(\bar{T}_{\tau}(\theta_a(x, \tau) - \theta_a(0,0))\theta_b(0,0)), $$ (9)

where $\theta_a = \pi \int^x dx' \Pi_a(x')$ and $a,b$ can be $B$ or $F$. From these Green’s functions [8] and [9], we find the correlation functions of exponential fields as:

$$ \langle \tau_{\lambda} e^{i \sum \lambda_a \phi_a(x, \tau)} e^{-i \sum \lambda_a \phi_a(0,0)} \rangle = e^{-\sum_{a,b} \lambda_a \lambda_b G_{ab}((\tau))}, $$ (10)

$$ \langle \tau_{\lambda} e^{i \sum \lambda_a \theta_a(x, \tau)} e^{-i \sum \lambda_a \theta_a(0,0)} \rangle = e^{-\sum_{a,b} \lambda_a \lambda_b \bar{G}_{ab}((\tau))}, $$ (11)

where $\lambda_B$ and $\lambda_F$ are real numbers. Such method is of course applicable to cases with more than two components, as long as the Hamiltonian remains quadratic in the fields $\phi_a$ and $\theta_a$.

To derive the Green’s functions, we start from the equations of motions in Matsubara time of the fields $\phi_a$ and $\Pi_a$ read:

$$ \partial_{\tau} \phi_a(x, \tau) = [H,\phi_a] = -i\pi \sum_b M_{ab} \Pi_b(x, \tau), $$ (12)

$$ \partial_{\tau} \Pi_a(x, \tau) = [H,\Pi_a] = -\frac{i}{\pi} \sum_b N_{ab} \partial^2_{\tau} \phi_b. $$ (13)

The equations of motions for the Green’s functions $G_{ab}(x, \tau)$ thus read:

$$ \partial_{\tau} G_{ab}(x, \tau) = i\pi \sum_c M_{ac} \langle \tau_{\bar{T}} \Pi_c(x, \tau) \phi_b(0,0) \rangle, $$ (14)

$$ \partial^2_{\tau} G_{ab}(x, \tau) = \pi \delta(x) \delta(\tau) M_{ab} - \sum_c (MN)_{ac} \partial^2_{\tau} G_{cb}(x, \tau). $$ (15)

Going to Fourier space, we obtain $(a,b = B,F)$:

$$ G_{ab}(q, \omega_n) = -\pi((\omega^2_n + (MN)q^2)^{-1})M_{ab}, $$ (16)

where $\omega_n = 2n\pi T$. From Eq. 10:

$$ G_{ab}(x, \tau) = \frac{1}{\beta} \sum_{\omega_n} \int \frac{dq}{2\pi} G_{ab}(q, \omega_n)(e^{iqx-i\omega_n \tau} - 1)e^{-|q|^\alpha}, $$ (17)

where we have introduced the cutoff $\alpha$. By using the duality transformation $\partial_{\tau} \phi_a = \pi P_a$, $\partial_{\tau} \theta_a = \pi \Pi_a$, we obtain equations of motion for $G_{ab}$ similar to Eq. 14 with the roles of $M$ and $N$ exchanged. Thus, in Fourier space, we have:

$$ \bar{G}_{ab}(q, \omega_n) = -\pi((\omega^2_n + (MN)q^2)^{-1})N_{ab}. $$ (18)

The expressions 10 and 11 show that the retarded Green’s functions have poles for $i\omega$ equal to $u_{\pm}|q|$ where $u^2_{\pm} \leq u^2_{\pm}$ are the two eigenvalues of $M$. Stability requires that $u^2_+ > 0$ and thus det$(M) > 0$. Since we know from App. A that det$(M) > 0$ in a Galilean invariant model, instabilities occur for det$(N) = 0$. From the definition of $N$, Eq. 4, such instabilities are either collapse or phase separation.

Further, for zero temperature and $\tau = 0$, we can obtain a general form for $G$ using Eq. 17. We find:

$$ G(x, 0) = \frac{1}{2}(MN)^{-1/2} \ln \left( \frac{\sqrt{x^2 + \alpha^2}}{\alpha} \right), $$ (19)

$$ \bar{G}(x, 0) = \frac{1}{2}(NM)^{-1/2} \ln \left( \frac{\sqrt{x^2 + \alpha^2}}{\alpha} \right). $$ (20)

Then equations 10 and 11 lead to:

$$ \langle \tau_{\lambda} e^{i \sum \lambda_a \phi_a(0,0)} e^{-i \sum \lambda_a \phi_a(0,0)} \rangle = \left( \frac{\alpha}{\sqrt{x^2 + \alpha^2}} \right)^{-\frac{1}{2}} \frac{1}{\lambda(MN)^{-1/2} \lambda}, $$ (21)

$$ \langle \tau_{\lambda} e^{i \sum \lambda_a \theta_a(0,0)} e^{-i \sum \lambda_a \theta_a(0,0)} \rangle = \left( \frac{\alpha}{\sqrt{x^2 + \alpha^2}} \right)^{-\frac{1}{2}} \frac{1}{\lambda(NM)^{-1/2} \lambda}, $$ (22)

where $\lambda = (\lambda_B, \lambda_F)$. We can define the matrices $\eta_{\phi} = (MN)^{-1/2} M$ and $\eta_{\theta} = (NM)^{-1/2} N = (NM)^{1/2} M^{-1}$. These matrices yield the exponents for the exponential fields. One can see that the duality relations become $\eta_{\phi}^* \eta_{\phi} = \eta_{\theta}^* \eta_{\theta} = 1$. For the two component system, one has the identity:

$$ (MN)^{-1/2} = \frac{I}{u_+ + u_-} + \frac{u_+u_-}{u_+ + u_-} (MN)^{-1}, $$ (23)

which can be checked by applying the right hand side of the formula to each eigenvector of $MN$. We thus have:

$$ \eta_{\phi} = \frac{1}{u_+ + u_-} (M + u_+ u_- N^{-1}) $$ (24)

$$ \eta_{\theta} = \frac{1}{u_+ + u_-} (N + u_+ u_- M^{-1}) $$ (25)
In the case where \( \det(N) \to 0 \), we see that \( \eta_0 \) will have matrix elements going to infinity as \( u_-^2 \), whereas \( \eta_0 \to N/u_+ \). Therefore, near a collapse or a phase separation the density wave exponents are divergent, while the superfluid exponents remain finite.

For nonzero temperature, the sum Eq. (14) is dominated for long distances by the term with \( n = 0 \). One finds that:

\[
\frac{1}{\beta} \int G(q, \omega_0 = 0)(e^{i\alpha} - 1) \frac{dq}{2\pi} = \pi \frac{2}{\beta} T|x|N^{-1},
\]

(26)

\[
\frac{1}{\beta} \int \tilde{G}(q, \omega_0 = 0)(e^{i\alpha} - 1) \frac{dq}{2\pi} = \pi \frac{2}{\beta} T|x|M^{-1},
\]

(27)

so that the correlation functions decay exponentially for long distances,

\[
\langle T \tilde{e}^{\sum_a \phi_a(x,0)} e^{i \sum_a \phi_a(0,0)} \rangle \sim e^{-\frac{\pi}{\beta} \lambda^N |x|^{-\beta}},
\]

(28)

\[
\langle T \tilde{e}^{\sum_a \phi_a(x,0)} e^{i \sum_a \phi_a(0,0)} \rangle \sim e^{-\frac{\pi}{\beta} \lambda M |x|^{-\beta}}.
\]

(29)

with thermal correlation lengths given respectively by \( \xi_0(\lambda) = 2/\langle T \tilde{e}^{\lambda^N} \rangle \) and \( \xi_0(\lambda) = 2/\langle T \tilde{e}^{\lambda M} \rangle \).

We note that near the instability, the correlation length \( \xi_0(\lambda) \) goes to zero, in accordance with the reduction of density wave ordering found at zero temperature, while the length \( \xi_0 \) remains finite.

**A. Atomic density wave correlations**

To characterize the atomic density-wave (ADW) ordering of the fermions, the field operator of which is given by:

\[
\psi_F(x) = e^{ik_Fx} \psi_F^+ + e^{-ik_Fx} \psi_F^-,
\]

(30)

where \( k_F = \pm \rho_0^{(F)} \alpha \) is the Fermi wavevector, \( \alpha \) is a short-distance cutoff, and \( \pm \) label the two Fermi points, we have to calculate the correlation function of the density operator \( \psi^\dagger(x) \psi(x) \), namely \( \langle \rho_{2k_F}(x) \rho_{2k_F}(0) \rangle \sim e^{-\beta \phi_F(x) + \beta \phi_F(0)} \sim \frac{s(\lambda)}{\alpha} \eta_{ADW}(F) \).

The density wave exponent \( \eta_{ADW}^{(F)} \) is:

\[
\eta_{ADW}^{(F)} = \frac{2}{u_+ + u_-} \left[ M_{FF} + \frac{u_+ u_-}{\det(N)} N_{BB} \right].
\]

(32)

For \( V_{BB} = 0 \), the exponent of Eq. (32) reduces to \( 2k_F \).

Near the collapse (for \( V_{BB} < 0 \)) or the phase separation (for \( V_{BB} > 0 \)) which is obtained at \( u_- \to 0 \), we note that the exponent \( \eta_{ADW}^{(F)} \) is diverging as \( \sim 1/u_- \). The fermionic density-density correlation at small wavevectors can also be obtained from bosonization. We have:

\[
\langle \rho_{F,0}(x) \rho_{F,0}(0) \rangle = \langle \rho_{F}^{(0)} \rangle^2 \eta_{ADW}^{(F)} \frac{4\pi^2 u_-^2}{4\pi^2 x^2},
\]

(33)

where we have defined \( \rho_{F,0}(x) = (\psi^+)^\dagger(x) \psi^+(x) + (\psi^-)^\dagger(x) \psi^-(x) \). For the bosonic density wave fluctuations, using the Haldane expansion of the density,

\[
\rho_B(x) = \rho_B^{(0)} - \frac{1}{\pi} \partial_x \phi_B + \sum_{m \geq 1} A_m \cos(2m\phi_B - 2m\rho_B^{(0)} x),
\]

(34)

we find that the density density correlation function of the bosons reads:

\[
\langle \rho_B(x) \rho_B(0) \rangle = \langle \rho_B^{(0)} \rangle^2 - \frac{\eta_{ADW}^{(B)}}{4\pi^2 x^2} + \sum_{m \geq 1} \frac{A_m^2 \cos(m\rho_B^{(0)} x)}{(x/\alpha)^{m} \eta_{ADW}^{(B)}},
\]

(35)

The dominant correlations are at wavevector \( 2\pi \rho_B^{(0)} \) and are characterized by the exponent:

\[
\eta_{ADW}^{(B)} = \frac{2}{u_+ + u_-} \left[ M_{BB} + \frac{u_+ u_-}{\det(N)} N_{FF} \right].
\]

(36)

This exponent is obtained from the fermionic exponent by the exchange \( (M_{BB}, N_{FF}) \leftrightarrow (M_{FF}, N_{BB}) \). The exponent is also divergent when \( u_- \to 0 \).

It is also interesting to consider the cross correlations between bosonic and fermionic density. One has:

\[
\langle \rho_B(x) \rho_F(0) \rangle = \rho_B^{(0)} \rho_F^{(0)} - \frac{M_{BF} - \frac{u_+ u_- N_{BF}}{\det(N)} \frac{1}{2\pi^2 x^2}}{u_+ + u_-},
\]

(37)

so that the non-uniform components of the densities of bosons and fermions remain uncorrelated. The cross density correlations vanish when \( V_{BF} = 0 \) and are positive when \( V_{BF} < 0 \) as a result of the boson-fermion attraction.

For finite temperature, the correlation functions of Eqs. (31) and (33) decay exponentially, with correlation lengths given by:

\[
\xi_{ADW}^{(F)} = \frac{2\det(N)}{\pi N_{BB} T},
\]

(38)

\[
\xi_{ADW}^{(B)} = \frac{2\det(N)}{\pi N_{FF} T}.
\]

(39)

**B. superfluid correlations**

Fermions, due to their spinless character, can only present p-wave type superfluidity. The order parameter is \(-i\psi(x) \nabla \psi(x)\) and can be expressed using the decomposition of Eq. (30) in the form:

\[
\psi_F^\dagger(x) \psi_F(x) \sim \frac{e^{2\theta_F}}{2\pi^2}.
\]

(40)

The order parameter for p-wave superfluidity exhibits algebraic correlations:

\[
\langle \psi_F^\dagger(x) \psi_F(x) (\psi_F^\dagger(0) \psi_F(0)) \rangle \sim (x/\alpha)^{-\eta_{ADW}^{(F)}},
\]

(41)
with the exponent for fermion superfluidity:

$$\eta^{(F)}_{S} = \frac{2}{u_+ + u_-} \left[ N_{FF} + \frac{u_+ u_-}{\det(M)} M_{BB} \right].$$  \hspace{1cm} (42)

We now turn to the superfluid fluctuations of the bosons. The quasi-long range superfluid order is characterized by the correlation function:

$$\langle \psi_B^+(x) \psi_B(0) \rangle \sim \langle e^{-i \theta_B(x) + i \theta_B(0)} \rangle \sim (x/\alpha)^{-\eta^{(B)}_S}. \hspace{1cm} (43)$$

Our result for the superfluid exponent \( \eta^{(B)}_S \) is:

$$\eta^{(B)}_S = \frac{1}{2(u_+ + u_-)} \left[ N_{BB} + \frac{u_+ u_-}{\det(M)} M_{FF} \right], \hspace{1cm} (44)$$

The superfluid and the density wave exponents are not independent from each other. Indeed, noting that \( u_+ u_- = \det(MN)^{1/2} \), we have that:

$$4\eta^{(B)}_S = \{(\det(N)/\det(M))^{1/2}\eta^{(F)}_{ADW} \text{ and } \eta^{(F)}_{S} = \{(\det(N)/\det(M))^{1/2}\eta^{(B)}_{ADW}. \hspace{1cm} \text{This implies that the exponents satisfy the relation } \eta^{(F)}_{ADW} \eta^{(B)}_{ADW} = 4\eta^{(F)}_{ADW} \eta^{(B)}_{ADW}. \hspace{1cm} \text{Turning to the finite temperature case, the thermal lengths are:}$$

$$\xi^{(F)} = \frac{2\det(M)}{\pi M_{BB} T}, \hspace{1cm} \xi^{(B)} = \frac{2\det(M)}{\pi M_{FF} T}. \hspace{1cm} (45, 46)$$

IV. BRAGG SCATTERING

According to Ref. [42], the Bragg scattering intensity is proportional to the imaginary part of the retarded density-density response functions \( \chi(q, \omega) \). Retarded density response functions \( \chi_{ab}(q, \omega) \) \((a,b = B,F)\) can be obtained from the Fourier transform of Green’s functions \( \Phi \) as:

$$\chi_{ab}(q, \omega) = \frac{q^2}{\pi^2} G_{ab}(q, \omega) |\omega_n + \omega + i\omega_0|. \hspace{1cm} (47)$$

Using the expression of the Fourier transform \( \Phi \) from Sec. [11] we obtain for \( \omega > 0 \):

$$\text{Im} \chi_{ab}(q, \omega) = -\frac{|q|}{2} \left[ \frac{u_-(M_{ab} - u_+^2 (N^{-1})_{ab}) - \delta(\omega - u_- |q|)}{u_+^2 - u_0^2} + \frac{u_(u_+^2 (N^{-1})_{ab} - M_{ab})}{u_+^2 - u_0^2} \delta(\omega - u_+ |q|) \right]. \hspace{1cm} (48)$$

Equation (48) predicts peaks at frequencies \( u_\pm |q| \). The matrices \( M \) and \( N \) can be deduced from the spectral weight of these peaks. One has the following sum rules:

$$\int_0^\infty dq \text{Im} \chi_{ab}(q, \omega) = \frac{|q|}{2\pi} \eta^{(B)}_{ab}, \hspace{1cm} (49)$$

$$\int_0^\infty d\omega \omega \text{Im} \chi_{ab}(q, \omega) = \frac{1}{2} (N^{-1})_{ab}, \hspace{1cm} (50)$$

$$\int_0^\infty d\omega \omega \text{Im} \chi_{ab}(q, \omega) = \frac{q^2}{2} M_{ab}. \hspace{1cm} (51)$$

The first sum rule is simply a restatement of our derivation of the equal time Green’s function in Sec. [11]. The second sum rule is a Kramers-Kronig relation giving the real part of the zero frequency (matrix) density-density response function as an integral of its imaginary part. Since the real part of the density density response function is the compressibility, the second result is not surprising. The last sum rule is a consequence of current conservation. Indeed, using current conservation, one can relate the density-density response function to the current-current response function. Using again a Kramers-Kronig, the last integral is shown to be equal to the static current-current response function.

V. VARIATION OF EXPONENTS

In this section, we first consider the model of Eq. (7) with \( M_{BF} = 0 \). For the two-component case \((a,b \in \{B,F\}\), the velocities are found as [26]:

$$u_{\pm} = \frac{u_F^2 + u_B^2}{2} \pm \sqrt{\left(\frac{u_F^2 - u_B^2}{2}\right)^2 + \left(\frac{V_{BF}}{\pi}\right)^2 u_B K_B u_F K_F}. \hspace{1cm} (52)$$

Using the Green’s function methods of Sec. [11] we obtain the following expressions for the exponents:

$$\eta^{(B)}_{ADW} = \frac{u_B K_B}{u_+ u_-} \left[ u_+ + u_- + \frac{u_F^2 - u_B^2}{u_+ + u_-}\right], \hspace{1cm} (53)$$

$$\eta^{(F)}_{ADW} = \frac{u_F K_F}{u_+ u_-} \left[ u_+ + u_- - \frac{u_F^2 - u_B^2}{u_+ + u_-}\right], \hspace{1cm} (54)$$

$$\eta^{(B)}_{S} = \frac{1}{4u_B K_B} \left[ u_+ + u_- - \frac{u_F^2 - u_B^2}{u_+ + u_-}\right], \hspace{1cm} (55)$$

$$\eta^{(F)}_{S} = \frac{1}{u_F K_F} \left[ u_+ + u_- + \frac{u_F^2 - u_B^2}{u_+ + u_-}\right]. \hspace{1cm} (56)$$

Finally, using that \( u_+^2 + u_-^2 = u_F^2 + u_B^2 \) and \( u_+^2 u_-^2 = u_F^2 u_B^2 - (V_{BF} u)^2/\pi^2 \) where \( u \equiv \sqrt{u_B K_B u_F K_F} \), we can express the exponents entirely as functions of \( u_F, u_B, K_F, K_B \) and \( V_{BF} \). Expanding to second order in \( V_{BF} \) we find:

$$\eta^{(B)}_{S} = \frac{1}{2K_B} - \frac{V_{BF}^2 K_F u_B}{4\pi^2 u_F (u_F + u_B)^2} + O(V_{BF}^4), \hspace{1cm} (57)$$

$$\eta^{(F)}_{S} = \frac{2}{K_F} - \frac{V_{BF}^2 K_B u_B}{\pi^2 u_F (u_F + u_B)^2} + O(V_{BF}^4), \hspace{1cm} (58)$$

$$\eta^{(B)}_{ADW} = 2K_B \left[ 1 + \frac{V_{BF}^2 K_F K_B (2u_B + u_F)}{2\pi^2 (u_F + u_B)^3 u_B} + O(V_{BF}^4) \right], \hspace{1cm} (59)$$

$$\eta^{(F)}_{ADW} = 2K_F \left[ 1 + \frac{V_{BF}^2 K_F K_B (2u_B + u_F)}{2\pi^2 (u_F + u_B)^3 u_F} + O(V_{BF}^4) \right]. \hspace{1cm} (60)$$

This shows that superfluidity is enhanced by the boson-fermion interaction. The enhancement of fermionic superfluidity becomes weaker as \( u_F \) is increased. By contrast, when \( u_F \) increased, the enhancement of bosonic...
Thus the excitation is stable for
\[ u_+^2 u_-^2 = (u_B u_F - M_{BF}^2 / (K_B K_F))(u_B u_F - N_{BF}^2 K_B K_F) > 0 \] (63)

From Eq. (43), the exponent for the superfluidity for boson is calculated as

\[ \eta^{(B)} = \frac{1}{(2 u_B K_B)} \left[ \frac{1}{u_B^2 + u_F^2 + 2 N_{BF} M_{BF} + 2 u_+ u_-} \right] \left[ \frac{u_B^2 u_F (u_B u_F - N_{BF}^2 K_B K_F)}{u_B + u_F} \right] \]

\[ = \frac{1}{2 K_B} \frac{1}{1 + \xi^2 + 2 \xi \tilde{N} \tilde{M} + 2 \xi \sqrt{(1 - \tilde{N}^2)(1 - \tilde{M}^2)}} \left( 1 + \xi \sqrt{\frac{1 - \tilde{N}^2}{1 - \tilde{M}^2}} \right). \] (64)
where $\tilde{N} = N_{BF} \sqrt{K_B K_F}/\sqrt{u_B u_F}$ and $\tilde{M} = M_{BF} (\sqrt{K_B K_F})^{-1}/\sqrt{u_B u_F}$. The second line shows that $2K_B \eta_s^{(B)}$ depends on $\tilde{N}$, $\tilde{M}$, and $\xi = u_F/u_B$. The stable conditions for $u_\perp$ are expressed as $-1 < \tilde{N}, \tilde{M} < 1$. At the boundary of $|\tilde{M}| = 1$, $\eta_s^{(B)}$ becomes infinite while it stays constant at $|\tilde{N}| = 1$ ($|\tilde{M}| \neq 1$). Since $\tilde{N}$ ($\tilde{M}$) has an effect of decreasing (increasing) $\eta_s^{(B)}$, they compete with each other. In Fig. 1, the contour plot of the quantity $2\eta_s^{(B)}$ is shown on the plane of $\tilde{N}$ (horizontal axis) and $\tilde{M}$ (vertical axis) with the fixed $\xi = 2$, where the case of $2K_B \eta_s^{(B)} < 1 (> 1)$ denotes the enhancement (suppression) of the superfluidity. For $\tilde{N} = 0$, the current-current boson-fermion coupling $\tilde{M}$ suppresses the superfluidity. The effect of the suppression reduces and vanishes, i.e., $\eta_s^{(B)} = 1/(2K_B)$ on the solid line in Fig. 1 due to the compensation effect of these two coupling constants. For small $|\tilde{M}|$ and $|\tilde{N}|$, $\eta_s^{(B)} = 1/(2K_B)$ is obtained at $\tilde{M} = \tilde{N}$ for $\tilde{M} \tilde{N} > 0$, and at $\tilde{M} = -\tilde{N} \xi/(1 + \xi)$ for $\tilde{M} \tilde{N} < 0$. In the region of $\tilde{N} < 0$, we find a novel feature for small $|\tilde{M}|$. The superfluidity is enhanced by the coupling $|\tilde{N}|$, and is further enhanced due to the coupling $|\tilde{M}|$. The enhancement becomes optimized, i.e., the minimum value of $2K_B \eta_s^{(B)}$ is obtained at a certain value of $|\tilde{M}|$.

VI. CONCLUSION

We have analyzed the mixture of bosonic and fermionic atoms in one dimension using a Green’s function equation of motion method starting from a phenomenological bosonized Hamiltonian. We have derived expressions of the zero temperature exponents and of the thermal correlation thermal correlation lengths for the density wave and superfluid order parameters. For the exponents, we have recovered the exponents previously obtained for the mixture in Ref. [23] and we have studied their behavior as a function of the fermion density for fixed interaction. We have found that for weak interaction, the behavior of the superfluid exponent of the bosons could be non-monotoneous, although boson superfluid correlations were always enhanced with respect to the system without fermions. Such behavior can lead to a non-monotoneous dependence of the superfluid transition temperature as a function of fermion density in an array of weakly coupled mixtures generalizing the bosonic array considered in Ref. [23]. This behavior is also in contrast with the one observed in experiments with three dimensional interacting systems [24], where fermion doping was seen to reduce superfluidity. In one dimensional systems at incommensurate filling, fermion doping is seen to always enhance superfluidity, the maximal enhancement being obtained near the collapse instability. By contrast, we have seen that density wave exponents are decreasing functions of the fermion density, which recover their value in the absence of fermion-boson interaction only in the limit of large fermion density. We have also been able to predict how the relative weights of the peaks in Bragg scattering intensity depend on the parameters of the phenomenological Hamiltonian.

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Appendix A: Consequences of Galilean invariance

Let us consider a system with $N_p$ different types of particles indexed by $1 \leq a \leq N_p$ with masses $M_a$. Under a Galilean boost, $p_{i,a} \rightarrow p_{i,a} + M_a v$ and $r_{i,a} \rightarrow r_{i,a} + vt$. This transformation is realized by the unitary operator:

$$U = \exp \left[ i \sum_{a=1}^{N_p} \sum_{i=1}^{N_a} M_a vr_{i,a} \right]$$

(A1)

One has:

$$U^\dagger HU = H + vP + \frac{1}{2} Mv^2,$$  

(A2)

where the total mass $M = \sum_a N_a M_a$ and the total momentum:

$$P = \sum_{a=1}^{N_p} \sum_{i=1}^{N_a} p_{i,a},$$

(A3)

is the total momentum of the system. In second quantization, the operator $U$ takes the form:

$$U = \exp \left[ i \int dr \sum_{a=1}^{N_p} M_a vr_\alpha(r) \right] ,$$

(A4)

and one has : $U^\dagger \psi_\alpha(r)U = e^{i M_a vr_\alpha(r)}$. Therefore, in bosonization, a Galilean boost takes the form $\theta_\alpha(r) \rightarrow \theta_\alpha(r) + M_a vr$, i.e., $\pi \Pi_\alpha(r) \rightarrow \pi \Pi_\alpha(r) + M_a v$. From the conservation equation for particles of type $a$, $\partial_x j_a + \partial_t \rho_a = 0$ and the bosonization relations [23], we have that the current density is $j_a = \partial_t \phi_a/\pi$. Using the equation of motion, we can rewrite $j_a = \sum_b M_{ab} \partial_\theta_b/\pi$. Therefore, under a Galilean boost, we will have:

$$\langle j_a \rangle = \frac{1}{\pi} \sum_b M_{ab} M_b v.$$

(A5)

Turning to the particle current, in the rest frame, the particle current will be $\langle \dot{r}_a \rangle = v \rho_a$. This can be seen for
instance by calculating $j_a = \partial_v (\phi_a(x - vt))/\pi$. From this result, it is clear that one must have:

$$\pi \rho_a = \sum_b M_{ab} M_b,$$

leading to the following expressions for $M_{ab}$:

$$M_{BB} = \frac{\pi \rho_B}{M_B} + \frac{2\pi}{L} \frac{\chi}{M_B^2},$$

$$M_{FF} = \frac{\pi \rho_F}{M_F} + \frac{2\pi}{L} \frac{\chi}{M_F^2},$$

$$M_{BF} = \frac{2\pi}{L} \frac{\chi}{M_F M_B},$$

where:

$$\chi = \sum_n \frac{|n| P_B(0)|^2}{E_n - E_0}.$$  

Obviously, $\chi > 0$. Using the expressions (A8), we obtain:

$$\det(M) = \frac{\pi^2 \rho_F \rho_B}{M_F M_B} + \left( \frac{\pi \rho_F}{M_F^2} + \frac{\pi \rho_B}{M_B^2} \right) \chi,$$

so that $\det(M) > 0$.

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