Gauge equivalence among quantum nonlinear many body systems

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Transformations performing on the dependent and/or the independent variables are an useful method used to classify PDE in class of equivalence. In this paper we consider a large class of U(1)-invariant nonlinear Schrödinger equations containing complex nonlinearities. The U(1) symmetry implies the existence of a continuity equation for the particle density $\rho \equiv |\psi|^2$ where the current $\mathbf{j}_\psi$ has, in general, a nonlinear structure. We introduce a nonlinear gauge transformation on the dependent variables $\rho$ and $\mathbf{j}_\psi$ which changes the evolution equation in another one containing only a real nonlinearity and transforms the particle current $\mathbf{j}_\psi$ in the standard bilinear form.

We extend the method to U(1)-invariant coupled nonlinear Schrödinger equations where the most general nonlinearity is taken into account through the sum of an Hermitian matrix and an anti-Hermitian matrix. By means of the nonlinear gauge transformation we change the nonlinear system in another one containing only a purely Hermitian nonlinearity. Finally, we consider nonlinear Schrödinger equations minimally coupled with an Abelian gauge field whose dynamics is governed, in the most general fashion, through the Maxwell-Chern-Simons equation. It is shown that the nonlinear transformation we are introducing can be applied, in this case, separately to the gauge field or to the matter field with the same final result. In conclusion, some relevant examples are presented to show the applicability of the method.

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I. INTRODUCTION

The Schrödinger equation is one of the most studied topics both from a mathematical and physical point of view. A particular interest is related to the possible nonlinear extensions of this equation. Just one year after the discovery of the Schrödinger equation, Fermi proposed its first nonlinear generalization [21].

In the following years, many nonlinear extensions of this equation have been proposed in literature in order either to explore fundamental arguments of the quantum mechanics, with the usual linear theory representing only an approximation, or to describe particular phenomenological physical effects. Among the many attempts made to generalize in a nonlinear manner the Schrödinger equation we recall the Bialynicki-Birula and Mycielski equation [6], with the nonlinear term $-b \ln(|\psi|^2) \psi$; the Guerra and Pusterla model [34], that with the purpose of preserving the superposition principle of the quantum mechanics introduced the nonlinear term $(\Delta |\psi|/|\psi|) \psi$; more recently, the Weinberg model [93, 96], with the introduction of homogeneous nonlinear terms in order to save partially the same fundamental principle.

On a phenomenological basis we recall the well known cubic Schrödinger equation [32, 33, 78], used in the study of the dynamical evolution of a Boson gas with a $\delta$-function pair-wise repulsion or attraction [4] and in the description of the Bose-Einstein condensation of alcali atoms like $^7$Li, $^{23}$Na and $^{87}$Rb [89, 91]: the model introduced by Kostin [57, 58, 86, 87], with the nonlinear term $i \ln(\psi/\psi^*) \psi + i (\ln(\psi/\psi^*)) \psi$ used to describe dissipative systems, and others [15, 28, 31, 64, 69].

Many nonlinear Schrödinger equations (NLSEs) contain complex nonlinearities. For instance, a nonlinearity of the type $a_1 |\psi|^2 \psi + a_2 |\psi|^4 \psi + i a_3 \partial_x (|\psi|^2 \psi) + (a_4 + i a_5) \partial_x |\psi|^2 \psi$ was introduced to describe a single mode wave propagating in a Kerr dielectric guide [23, 24]: the nonlinearity $a_1 |\psi|^2 \psi + i a_2 \psi + i a_3 \partial_x \psi + i a_4 |\psi|^2 \psi$; proposed in [66, 67] to take into account of pumping and dumping effects of the nonlinear media, is used to describe dynamical modes in plasma physics, hydrodynamics, and also solitons in optical fibers (63 and references therein); the nonlinearity $a_1 |\psi|^2 \psi + i a_2 \partial_{xxx} \psi + i a_3 \partial_x (|\psi|^2 \psi) + i a_4 \partial_x |\psi|^2 \psi$ introduced to describe the propagation of high power optical pulses in ultrashort soliton communication systems [29, 14, 47, 61, 88]. In [53, 54] a NLSE with the complex nonlinearity $\kappa (\psi^* \nabla \psi - \psi \nabla \psi^*) \nabla \psi + (\kappa/2) \nabla (\psi^* \nabla \psi - \psi \nabla \psi^*) \psi$ has been introduced to take into account a generalized Pauli exclusion-inclusion principle between the quantum particles constituting the system. In [48], in the stochastic quantization framework, starting from the most general kinetic containing a nonlinear drift term and compatible with a linear diffusion term, a class of NLSEs with a complex nonlinearity was derived whilst, recently [82, 84], a wide class of NLSEs has been obtained starting from the quantization of a classical many body system whose underlying kinetic is
described by a nonlinear Fokker-Planck equation associated to a generalized trace-like entropy. Finally, the Doebner-Goldin equation \[17, 19, 20, 29\] was introduced from topological considerations as the most general class of NLSEs compatible with the linear Fokker-Planck equation for the probability density \( \rho \equiv |\psi|^2 \), where the nonlinear term was derived from the unitary group representation of the infinite-dimensional diffeomorphism group proposed as a universal quantum kinematical group \[30\].

In the recent years, an increasing interest has been also addressed to systems of coupled nonlinear Schrödinger equations (CNLSEs), particularly after the invention of high-intensity lasers which have allowed \[71\] the experimental test of the pioneering theoretical works on the optical fibers propagation in long-distance communications \[96, 97\]. In fact, single-mode optical fibers are not really single-mode type since two possible polarizations exist. A rigorous study of their propagation requires the use of CNLSEs in order to take into account the evolution of the different polarized waves. In 1974 Manakov \[68\] introduced a CNLSE starting from the cubic NLSE by considering that the total field is a superposition of two, left and right polarized fields. When ultrashort pulses are transmitted through fibers, CNLSEs with complex high derivative nonlinearities arise \[63, 72–74, 79, 94\]. CNLSEs are also employed in the study of light propagation through a nonlinear birefringent medium, in systems with nonrelativistic interactions among the different kind of particles, in spinor Bose-Einstein condensation, in the description of micro-polar elastic solids, among the many \[13, 18, 25, 38, 39, 41, 59, 60, 92, 93\].

Finally, when coupled with gauge fields, NLSEs are useful in the study of some interesting phenomenologies in condensed matter physics. For instance, in the Ginzburg-Landau theory of the superconductivity the cubic NLSE is coupled with an Abelian gauge field whose interaction is described by means of the Maxwell equations \[55, 90\]. Some time, gauge field dynamics can be described by the further Chern-Simons term which confers mass to the field without destroy the gauge invariance of the theory. These models have particle-like solutions obeying to a non-conventional statistics, named anyons \[42, 43, 98\], which can find an application in the study of high-

After the brief introductory next section about the notations used in this paper we begin, in section 3, by considering a wide class of canonical NLSEs with complex nonlinearity in a \( n+1 \)-dimensional space-time (throughout this paper we use units \( \hbar = c = e = 1 \) and we set \( m = 1/2 \))

\[
i \frac{\partial \psi}{\partial t} + \Delta \psi + \left( W[\psi^*, \psi] + i W[\psi^*, \psi] \right) \psi = 0 ,
\]  

(1.1)

describing, in the mean field approximation, the dynamics of a nonrelativistic scalar field \( \psi \) conserving the particle number \( N = \int \rho \, d^nx \). The real \( W[\psi^*, \psi] \) and the imaginary \( W[\psi^*, \psi] \) nonlinearities appearing in equation (1.1) are smooth functionals of the fields \( \psi, \psi^* \) and of their spatial derivatives of any order. Since the nonlinearity in the evolution equation is complex the particle current \( j_\psi \) has, in general, a nonlinear structure which differs from the bilinear form of the standard linear quantum mechanics. We introduce the Lagrangian formulation both in the wave-function representation and in the hydrodynamic representation and we study the U(1) symmetry, which plays a relevant role for the purpose of the present work. Then, we introduce a nonlinear unitary transformation \( \psi \rightarrow \phi \) that changes the complex nonlinearity \( W[\psi, \psi^*] + i W[\psi, \psi^*] \) in another one \( \tilde{W}[\phi, \phi^*] \) which turns out to be purely real. As a consequence the new current \( j_\phi \) assumes the bilinear form of the linear Schrödinger theory. In \[8, 13, 18, 25, 38, 39, 41, 59, 60, 92, 93\] we can find some examples of nonlinear gauge transformation of the third
In the following we consider a nonrelativistic canonical system described by the action of the integrals, by requiring that all fields and their spatial derivatives vanish quickly on the boundary of

Hereinafter we use the notation between square brackets \( G \). We introduce the variation of a functional \( \delta G \)

spatial derivative of any order. Since the theory is nonrelativistic the Lagrangian contains only time derivatives of

We assume that the system \( (1.2) \) has a purely Hermitian nonlinearity \( W \), with \( W = \hat{W} \). The functional derivative can be defined by means of the Euler operator \[77\]

where \( \hat{A}\Psi \equiv \hat{W} \Psi \), \( \hat{A} \) is composed by an Hermitian matrix \( \hat{W} \). As a consequence, the transformed currents assume the standard bilinear form.

Finally, in section 5, we generalize the method to a class of NLSEs minimally coupled with an Abelian gauge field \( A_{\mu} \), where the matter field is described by the following equation

with \( \psi \) the scalar charged field and \( D_{\mu} \equiv (D_0, D) \) denotes the standard covariant derivative. The dynamics of the gauge field is provided by the Maxwell-Chern-Simons equation

where \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the electromagnetic field and \( j^\rho_{A_{\alpha}} \equiv (\rho, j^\rho_{A_{\alpha}}) \) is the covariant current. We stress one more that the charged current \( j^\rho_{A_{\alpha}} \) has, in general, a nonlinear structure due to the presence, in the evolution equation \( (1.3) \), of the complex nonlinearity \( W[\psi^*, \psi, A] + i W[\psi^*, \psi, A] \). The gauge transformation can be applied equivalently to the matter field or to the gauge field obtaining the same final result: the nonlinearity in the Schrödinger equation \( (1.3) \) turns out to be purely real and the expression of the charged current reduces to the standard bilinear form.

In section 6, we collect some explicit examples to illustrate the applicability of the method whilst, in the conclusive section 7, we discuss the possible further development about the nonlinear gauge transformations presented in this paper.

II. PRELIMINARY MATHEMATICAL BACKGROUND

Let \( M \) be a complex \( n \)-dimensional smooth manifold labeled by the vector \( x \equiv (x_1, \ldots, x_n) \). Let \( \mathcal{F} : M \rightarrow \mathbb{R} \) be the algebra of the functions on \( M \) and \( F : \mathcal{F} \rightarrow \mathbb{R} \) the algebra of the functionals on \( \mathcal{F} \) of the type \( G = \int G(x, t) \, dt \). Let \( \psi_i(x, t) \in \mathcal{F} \) with \( j = 1, \ldots, p \) a set of \( p \) fields on \( M \), with \( t \) a real parameter and we denote by \( \Omega \equiv (\psi_1, \ldots, \psi_p) \) a \( p \)-dimensional vector on \( M = M \times \ldots \times M \). We assume uniform boundary conditions, to guarantee the convergence of the integrals, by requiring that all fields and their spatial derivatives vanish quickly on the boundary of \( M \).

In the following we consider a nonrelativistic canonical system described by the action \( A = \int \mathcal{L}[\Omega] \, dt \), where \( \mathcal{L}[\Omega] \in F \) is the Lagrangian density which depends on the scalar fields \( \psi_j \in M \) and their space and time derivatives. Hereinafter we use the notation between square brackets \( G[\psi] \) to indicate the dependence of \( G \) on the field \( \psi \) and its spatial derivative of any order. Since the theory is nonrelativistic the Lagrangian contains only time derivatives of the first order.

We introduce the variation of a functional \( G \) with respect to \( \Omega \) as

where the functional derivative can be defined by means of the Euler operator \[77\]

given by

\[\mathcal{E}_{\psi_j} (G) = - \frac{\partial}{\partial t} \left( \frac{\partial G[\Omega]}{\partial (\partial_t \psi_j)} \right) + \sum_{|k|=0} (-1)^k D_k \left( \frac{\partial G[\Omega]}{\partial (D_k \psi_j)} \right),\]
with $D_k = \partial^k / (\partial x_1^{i_1} \ldots \partial x_n^{i_n})$ and $\sum_{k=0}^\infty \equiv \sum_{k=0}^\infty \sum_{l_k}$. The sum $\sum_{l_k}$ is over the multi-index $I_k \equiv (i_1, \ldots, i_n)$ with $0 \leq i_q \leq k$, $\sum_q i_q = k$ and $1 \leq q \leq n$.

It is easy to show, by using equation (2.7), that the Euler operator satisfies the following property

$$\mathcal{E} \left( \frac{\partial B}{\partial t} + \nabla \cdot \mathbf{C} \right) = 0,$$

where $B$ and the components of $\mathbf{C}$ belong to $F$ whilst $\nabla \equiv (\partial_1, \ldots, \partial_n)$ is the $n$-dimensional gradient operator.

### III. NONLINEAR SCHÖDINGER EQUATION FOR SCALAR PARTICLES

We begin by consider nonrelativistic many body systems of scalar interacting particles described, in the mean field approximation, through a very gener general family of NLSEs. It is useful to recall the main aspects of the canonical theory both in the wave-function formulation and in the hydrodynamic formulation.

#### a) Wave-function formulation.

Let us consider the class of canonical NLSEs described by the Lagrangian density

$$\mathcal{L}[\psi^*, \psi] = -\frac{i}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - |\nabla \psi|^2 - U[\psi^*, \psi],$$

(3.1)

which is a functional of the 2-dimensional vector $\Omega \equiv (\psi^*, \psi)$. The first two terms are the same encountered in the standard linear quantum mechanics whilst the last term is the nonlinear potential describing the interaction among the particles of the system. We assume that $U[\psi^*, \psi]$ be a real smooth functional of the fields $\psi$ and $\psi^*$ and their spatial derivatives which leave the Lagrangian density (3.1) invariant under transformations belonging to the U(1) group that assures the conservation of the total number of particles. As we will show, this condition imposes a constraint on the form of the nonlinear potential $U[\psi, \psi^*]$.

By introducing the action of the system

$$\mathcal{A} = \int_{\mathcal{R}} \mathcal{L}[\psi^*, \psi] d^n x dt,$$

(3.2)

where the domain of integration is the whole real region $\mathcal{R} = M \times \mathbb{R}$, the evolution equation for the vector field $\psi$, corresponding to the stationary trajectories of the action, can be obtained from the extremal problem

$$\delta \mathcal{A} = 0,$$

(3.3)

where the variation is performed with respect to the vector $\Omega$. According to equation (2.5), the Euler-Lagrange equations for the fields $\psi$ and $\psi^*$ are given by

$$\frac{\delta}{\delta \psi^*} \int_{\mathcal{R}} \frac{i}{2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) d^n x dt - \frac{\delta}{\delta \psi^*} \int_{\mathcal{R}} |\nabla \psi|^2 d^n x dt$$

$$- \frac{\delta}{\delta \psi^*} \int_{\mathcal{R}} U[\psi^*, \psi] d^n x dt = 0,$$

(3.4)

and its conjugate. We recall that the Lagrangian density (3.1) is defined modulo a total derivative (null Lagrangian) which does not give contribute to the evolution equations because, as stated in equation (2.8), the variation of a total derivative vanishes.

After performing the functional derivatives we obtain the following NLSE

$$i \frac{\partial \psi}{\partial t} + \Delta \psi + \Lambda[\psi^*, \psi] = 0,$$

(3.5)

where $\Delta \equiv \partial_1^2 + \ldots + \partial_n^2$ is the $n$ dimensional Laplacian operator and the complex nonlinear term $\Lambda[\psi^*, \psi]$ is given by

$$\Lambda[\psi^*, \psi] = -\frac{\delta}{\delta \psi^*} \int_{\mathcal{R}} U[\psi^*, \psi] d^n x dt.$$

(3.6)
b) Hydrodynamic formulation.

In the hydrodynamic formulation we introduce two real fields $\rho$ and $S$ related to the wave-function through the polar decomposition

$$\psi(x, t) = \rho^{1/2}(x, t) \exp \left( i S(x, t) \right),$$

or equivalently

$$\rho(x, t) = |\psi(x, t)|^2,$$

$$S(x, t) = \frac{i}{2} \log \left( \frac{\psi^*(x, t)}{\psi(x, t)} \right).$$

By defining the action $A = \int_{\mathcal{R}} L[\rho, S] d^n x d t$ through the Lagrangian density

$$L[\rho, S] = -\frac{\partial S}{\partial t} \rho - (\nabla S)^2 \rho - \frac{(\nabla \rho)^2}{4 \rho} - U[\rho, S],$$

where $U[\rho, S]$ is the nonlinear potential in the hydrodynamic representation, from the variational problem $\delta A = 0$, where now $\Omega \equiv (\rho, S)$, we obtain two real equations

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + U_q[\rho] - W[\rho, S] = 0,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (2 \rho \nabla S) + 2 \rho W[\rho, S] = 0.$$

In equations 3.11 and 3.12 $U_q[\rho] = -\Delta \rho^{1/2}/\rho^{1/2}$ denotes the quantum potential [7, 62] whilst the two real functionals $W[\rho, S]$ and $W[\rho, S]$ are given by

$$W[\rho, S] = -\frac{\delta}{\delta \rho} \int_{\mathcal{R}} U[\rho, S] d^n x d t,$$

$$W[\rho, S] = -\frac{1}{2} \frac{\delta}{\delta S} \int_{\mathcal{R}} U[\rho, S] d^n x d t.$$

According to the relation

$$\frac{\delta}{\delta \psi} = \psi \left( \frac{\delta}{\delta \rho} + \frac{i}{2 \rho} \frac{\delta}{\delta S} \right),$$

the quantities $W[\rho, S]$ and $W[\rho, S]$ are related to the nonlinearity $\Lambda[\psi, \psi^*]$ in

$$\Lambda[\psi, \psi^*] = \left( W[\rho, S] + i W[\rho, S] \right) \psi.$$

Equation 3.11 is an Hamilton-Jacobi-like equation for the field $S$ whilst equation 3.12 describes the evolution equation for the field $\rho$. The last term in equation 3.12 originates from the nonlinear potential $U[\rho, S]$ and is a source of particle which, in the general case, destroys the conservation of the number of particles of the system. Accounting for equation 3.16 we can rewrite the NLSE (3.5) in

$$i \frac{\partial \psi}{\partial t} + \Delta \psi + \left( W[\rho, S] + i W[\rho, S] \right) \psi = 0,$$

where the complex nonlinearity $\Lambda[\rho, S]$, expressed in the hydrodynamic fields $\rho$ and $S$, is separated in the real $W[\rho, S]$ and imaginary $W[\rho, S]$ part.
A. U(1) Symmetry

Differently from the linear Schrödinger equation which is U(1)-invariant, the presence of the nonlinearity $U[\rho, S]$ generally breaks this symmetry. In fact, equation (3.12), in general, is not a continuity equation for the field $\rho$. In the following we study the relevant restrictions to the nonlinear potential $U[\rho, S]$ so that the Lagrangian (3.1) turns out to be U(1)-invariant. Such invariance implies, according to the Noether theorem [76], the conservation of the total number of particles by restoring a continuity equation for the field $\rho$.

To begin with, we rewrite equation (3.12) in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j_{\psi}^{(0)} + 2 \rho W[\rho, S] = 0,$$

(3.18)

where

$$j_{\psi}^{(0)} = -i \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right) \equiv 2 \rho \nabla S,$$

(3.19)

is the bilinear particles current of the standard quantum mechanics. Making use of equation (3.14), equation (3.18) become

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j_{\psi}^{(0)} - \frac{\delta}{\delta S} \int_{\mathcal{R}} U[\rho, S] \, d^nx \, dt = 0,$$

(3.20)

and taking into account the definition of functional derivative (2.6) we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j_{\psi}^{(0)} - \frac{\partial}{\partial S} \left( \int_{\mathcal{R}} U[\rho, S] \, d^nx \, dt \right) = 0.$$

(3.21)

This last equation can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j_{\psi} = \frac{\partial U}{\partial S},$$

(3.22)

where the nonlinear current $j_{\psi}$ has components

$$(j_{\psi})_i = 2 \rho \partial_i S + \frac{\delta}{\delta (\partial_i S)} \int_{\mathcal{R}} U[\rho, S] \, d^nx \, dt,$$

(3.23)

because, according to the definition (2.7), the following relation holds

$$\sum_{[k=1]} (-1)^k D_i \left( \frac{\partial}{\partial (D_i S)} \int_{\mathcal{R}} U[\rho, S] \, d^nx \, dt \right) = 0.$$

(3.24)

Remark that the expression of current (3.23) is always defined modulo the curl of an arbitrary functional $G[\rho, S]$ which does not give contribute to equation (3.22).

As stated before, the Lagrangian (3.1), for a general nonlinear potential $U[\rho, S]$, is not U(1)-invariant. In fact, equation (3.22) describe a very general kinetics process in which the right hand side plays the role of a source of particles. Trivially, if the nonlinear potential $U[\rho, S]$ depends on the field $S$ only through its spatial derivatives the right hand side of equation (3.22) vanishes and it becomes a continuity equation for the field $\rho$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j_{\psi} = 0.$$

(3.25)
The conserved quantity associated to equation (3.25) is the total number of particles

\[ N = \int_M \rho \, d^n x , \]  

(3.26)

where the integral is evaluated on the full real region \( M \) (uniform boundary conditions guarantee the convergence of the integral).

Under the assumption that \( U[\rho, S] \) depends on \( S \) only through its derivatives, according to the relation

\[ \frac{\delta}{\delta S} \int_R U[\rho, S] \, d^n x \, dt = -\nabla \cdot \left( \frac{\delta}{\delta (\nabla S)} \int_R U[\rho, S] \, d^n x \, dt \right) , \]  

(3.27)

we can rewrite the expression of \( W[\rho, S] \) in

\[ W[\rho, S] = \frac{1}{2} \rho \nabla \cdot \mathcal{J}[\rho, S] , \]  

(3.28)

where \( \mathcal{J}[\rho, S] \) is given by

\[ (\mathcal{J})_i [\rho, S] = \frac{\delta}{\delta (\partial_i S)} \int_R U[\rho, S] \, d^n x \, dt , \]  

(3.29)

and the expression of the current (3.23) becomes

\[ j_\psi = j_\psi^{(0)} + \mathcal{J}[\rho, S] . \]  

(3.30)

Summing up, the nonlinear potential \( U[\rho, S] \) generally breaks the U(1) symmetry of the system (3.1). On the other hand, if it depends on the field \( S \) only through its spatial derivative, the U(1) symmetry is restored. This can be clarified if we taking into account that, under a global U(1) transformation (gauge transformation of first kind)

\[ \psi \rightarrow \phi = e^{i \epsilon} \psi , \]  

(3.31)

the phase \( S \) transforms in

\[ S \rightarrow \mathcal{S} = S + \epsilon , \]  

(3.32)

where \( \epsilon \) is the constant parameter of the transformation. As a consequence, if \( S \) appears in the Lagrangian only through its derivatives, the transformation (3.31) does not change the Lagrangian of the system.

**B. Gauge transformation**

We introduce a unitary and nonlinear transformation on the field \( \psi \)

\[ \psi(x, t) \rightarrow \phi(x, t) = U[\rho, S] \psi(x, t) , \]  

(3.33)

whose purpose is to change the NLSE (3.17), which contains a complex nonlinearity \( W[\rho, S] + i \tilde{W}[\rho, S] \), in another one containing only a purely real nonlinearity \( \tilde{W}[\rho, S] \). As a consequence the current \( j_\psi \), given in equation (3.30), is transformed in another one \( j_\psi \rightarrow j_\psi^{(0)} \) having the merely bilinear form of the ordinary quantum mechanics.

Since the transformation is unitary: \( U^* = U^{-1} \), equation (3.33) does not change the quantity

\[ \rho(x, t) = |\psi(x, t)|^2 = |\phi(x, t)|^2 , \]  

(3.34)

representing the density of probability of position of the system.

The functional \( U[\rho, S] \) is defined in

\[ U[\rho, S] = \exp \left( i \sigma[\rho, S] \right) , \]  

(3.35)
where the generator $\sigma[\rho, S]$ is a real functional which relates the phase $S$ of the field $\phi$ with the phase $S$ of the field $\psi$

$$S = S + \sigma[\rho, S], \quad (3.36)$$

since we define

$$\phi(x, t) = \rho^{1/2}(x, t) \exp \left( iS(x, t) \right). \quad (3.37)$$

When equation (3.36) is invertible, we can express the phase $S$ as a functional of the fields $S$ and $\rho$.

The expression of the generator $\sigma[\rho, S]$ is related to the imaginary part $W[\rho, S]$ of the NLSE through

$$\nabla \sigma[\rho, S] = \frac{1}{2\rho} \mathcal{J}[\rho, S], \quad (3.38)$$

which defines $\sigma[\rho, S]$ modulo an arbitrary integration constant. The same equation (3.38) imposes a condition on the form of the nonlinear potential as it follows from the relation $\nabla \times \nabla \sigma = 0$ (where $\nabla \times f$ means $\partial_i f_j - \partial_j f_i$ with $i, j = 1, \ldots, n$)

$$\nabla \times \left( \frac{\mathcal{J}[\rho, S]}{\rho} \right) = 0. \quad (3.39)$$

Equation (3.39) selects the potentials $U[\rho, S]$ and in this way the nonlinear systems in which we can perform the transformation (3.33). For one-dimensional systems this transformation can be always accomplished.

By plunging the expression of $\psi(x, t) = U^{-1}[\rho, S] \phi(x, t)$ in the NLSE (3.17) it is easy to verify that it reduces in the following evolution equation

$$i \frac{\partial \phi}{\partial t} + \Delta \phi + \tilde{W}[\rho, S] \phi = 0, \quad (3.40)$$

which contains only a real nonlinearity $\tilde{W}[\rho, S]$ given by

$$\tilde{W}[\rho, S] = W - (\nabla \sigma)^2 + 2 \nabla S \cdot \nabla \sigma + \frac{\partial \sigma}{\partial t}, \quad (3.41)$$

where $W \equiv W[\rho, S[\rho, S]]$ and $\sigma \equiv \sigma[\rho, S[\rho, S]]$. Because the phase $S$ appears in equation (3.40) only through its spatial derivatives, as required by the U(1)-invariance, the arbitrary integration constant arising from the definition of $\sigma[\rho, S]$, does not produce any effect and can be posed equal to zero. Finally, the last term in equation (3.41) can be solved using the Hamilton-Jacobi equation (3.11) and the continuity equation (3.12) in order to reduce the nonlinearity $\tilde{W}[\rho, S]$ in a quantity containing only space derivatives.

We can easily verify that the continuity equation for the field $\rho$, obtained from equation (3.40), is now given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j^{(0)} = 0, \quad (3.42)$$

where, due to the reality of the nonlinearity $\tilde{W}[\rho, S]$, the current

$$j^{(0)} = 2 \rho \nabla S, \quad (3.43)$$

assumes the standard bilinear form.

Let us briefly discuss the generalization of this transformation to non canonical systems. Firstly, we observe that for a non canonical system the two quantities $W[\rho, S]$ and $W[\rho, S]$ are not derivable from a potential $U[\rho, S]$. In particular, the real nonlinearity $W[\rho, S]$ can assume any arbitrary expression whereas the form of the imaginary nonlinearity $W[\rho, S]$, constrained by the continuity equation for the field $\rho$, is given by

$$W[\rho, S] = \frac{1}{2\rho} \nabla \cdot \mathcal{J}[\rho, S], \quad (3.44)$$

for an arbitrary functional $\mathcal{J}[\rho, S]$. The particle current $j_\psi$ is still given through the relation

$$j_\psi = 2 \rho \nabla S + \mathcal{J}[\rho, S], \quad (3.45)$$
but now the functional \( J[\rho, S] \) is related to \( W[\rho, S] \) only through equation (3.44). Following the same steps described for the canonical case, it is easy to verify that the transformation (3.33) with generator defined in the same way

\[
\nabla \sigma[\rho, S] = \frac{1}{2\rho} J[\rho, S],
\]

(3.46)

eliminates the imaginary part of the nonlinearity in the motion equation (3.17) which assumes the expression (3.40)-(3.41). We observe that, differently from the canonical case, equation (3.39) now constraints only the form of the imaginary part \( W[\rho, S] \) whilst the real part \( W[\rho, S] \) is completely arbitrary.

Finally, let us remark that when the transformation (3.33) is applied to a canonical equation, generally it breaks the canonical structure of the theory. Consequently, the new NLSE is no more expressible in the Lagrangian formalism. Differently, when the transformation is applied to a non canonical system, the new NLSE can acquire a canonical structure. It is not hard to show that this is possible if the transformed nonlinearity \( \tilde{W}[\rho] \) is a functional depending only on the field \( \rho \). In fact, the real quantity \( \tilde{W} \) is related to a nonlinear potential \( \tilde{U} \) through equation (3.13) and because the new nonlinearity is purely real, from equation (3.14) it follows that the potential \( \tilde{U} \) and consequently \( \tilde{W} \) cannot depend on the field \( S \).

IV. COUPLED NONLINEAR SCHRO"DINGER EQUATIONS

Physical systems, whose dynamics is described by means of CNLSEs are ubiquitous in nature. They occur, for instance, in presence of many interacting particle of different spices or in presence of multi polarized laser beams propagating in optical fibers. In this last case, any polarized component of the electric or magnetic field, can be considered like a “particle state” since its evolution is describable through a NLS-like equation (36).

We observe that for many particle systems of different kind, by denoting with \( N_k \) the number of the \( k \)th species, many possible combinations of conserved multiplets can be realized. In particular, two relevant physical situations are given when:

a) All the quantities \( N_k \) are separately conserved, which is relevant, for instance, in nonrelativistic systems of multispecies where process of transmutation from a species to another one is forbidden.

b) Only the quantity \( N_{tot} = \sum_k N_k \) is conserved. Relevant examples are given in the study of light propagation in optical fibers where each species describes a polarization mode and only the total intensity of the beam is conserved.

In the following we introduce a wide class of CNLSEs in the form

\[
i \frac{\partial \Psi}{\partial t} + \hat{A} \Delta \Psi + \tilde{A}[\bar{\rho}, \bar{S}] \Psi = 0,
\]

(4.1)

where \( \Psi = (\psi_1, \ldots, \psi_p) \), \( \bar{\rho} \equiv (\rho_1, \ldots, \rho_p) \) and \( \bar{S} \equiv (S_1, \ldots, S_p) \) are \( p \) dimensional vectors. We denote the operator valued matrix \( \hat{M}[\bar{v}] \) by an hat (the lower case letter \( m[\bar{v}] \) denotes its entries) and use the notation between square brackets to indicate the functional dependence on the components of the vector \( \bar{v} = (v_1, \ldots, v_p) \) and on its spatial derivatives of any order. Without loss of generality we assume the \( p \times p \) matrix \( \hat{A} \) in a diagonal form.

We observe that any system of CNLSEs can be always accommodated in the form given in equation (4.1) with a diagonal nonlinearity \( \tilde{A}[\bar{\rho}, \bar{S}] \). Such nonlinearity can be separated in an Hermitian matrix \( \tilde{W} = (\hat{A} + \hat{A}^{\dagger})/2 \) and an anti-Hermitian matrix \( i \tilde{W} = (\hat{A} - \hat{A}^{\dagger})/2 \). Thus, we can pose \( \tilde{A}[\bar{\rho}, \bar{S}] = \tilde{W}[\bar{\rho}, \bar{S}] + i \tilde{W}[\bar{\rho}, \bar{S}] \), where the diagonal matrices \( \tilde{W}[\bar{\rho}, \bar{S}] \) and \( \tilde{W}[\bar{\rho}, \bar{S}] \) have purely real entries. Such assumption is only for sake of convenience and does not imply any restriction on the form of the nonlinearity.

We will consider a general situation in which the system (4.1) has \( q \) conserved multiplets of order \( p_k \), with \( k = 1, \ldots, q \) and \( \sum_k p_k = p \), where \( 1 \leq q \leq p \). In this way, the two particular cases a) and b) quoted previously are recognized for \( q = p \) and \( q = 1 \), respectively.

Let us organize the fields \( \psi_i \), belonging to the vector \( \Psi \), in

\[
\Psi \equiv (\psi_{11}, \ldots, \psi_{1p_1}; \psi_{21}, \ldots, \psi_{2p_2}; \ldots; \psi_{q1}, \ldots, \psi_{qp_q}),
\]

(4.2)

and, from now on, we relabel the fields \( \psi_i \) in \( \psi_{ki} \) where, the first index \( k \) refers to the \( k \)th multiplet of order \( p_k \), whereas the second index \( l \), with \( 1 \leq l \leq p_k \), refers to the \( l \)th field inside to the multiplet \( k \).

Canonical system (4.1) can be obtain from the Lagrangian density

\[
\mathcal{L}[\Psi^\dagger, \Psi] = \frac{i}{2} \left( \Psi^\dagger \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^\dagger}{\partial t} \Psi \right) - \nabla \Psi^\dagger \cdot \hat{A} \nabla \Psi - U[\Psi^\dagger, \Psi],
\]

(4.3)
where the scalar product in the second term is applied among the gradient operators. The nonlinear potential \( U[\Psi, \Psi^\dagger] \) is a real functional which depends on the vector fields \( \Psi, \Psi^\dagger \) and their spatial derivatives. Accounting for the uniform boundary conditions, the potential \( U[\Psi^\dagger, \Psi] \) vanishes together with all its derivatives, at the spatial infinity.

By introducing the action of the system

\[
\mathcal{A} = \int_\mathbb{R} \mathcal{L}[\Psi^\dagger, \Psi] \, dx \, dt ,
\]

(4.4)

the evolution equation for the vector field \( \Psi \) is given by the stationary trajectories of the action as it follows from the variational problem \( \delta \mathcal{A} = 0 \), where the variation is performed with respect to the 2p-dimensional vector \( \Omega \equiv (\Psi^\dagger, \Psi) \).

In this way we obtain the equation

\[
i \frac{\partial \Psi}{\partial t} + \hat{A} \Delta \Psi - \delta \frac{\delta U}{\delta \Psi^\dagger} \int_\mathbb{R} U[\Psi^\dagger, \Psi] \, dx \, dt = 0 ,
\]

(4.5)

and its Hermitian conjugate, which form a system of 2p-nonlinear coupled Schrödinger equations. Taking in account of the polar decomposition of the fields \( \psi_{kl} \) in the real fields \( \rho_{kl} \) and \( S_{kl} \)

\[
\psi_{kl}(x, t) = \rho_{kl}^{1/2}(x, t) \exp \left( i S_{kl}(x, t) \right) ,
\]

(4.6)

we can express the variation \( \delta/\delta \psi_{kl}^* \) as

\[
\frac{\delta}{\delta \psi_{kl}^*} = \psi_{kl} \left( \frac{\delta}{\delta \rho_{kl}} + \frac{i}{2} \frac{\delta}{\delta S_{kl}} \right) .
\]

(4.7)

In this way, each component of equation (4.5) can be written in

\[
i \frac{\partial \psi_{kl}}{\partial t} + a_{kl} \Delta \psi_{kl} - \left( \frac{\delta}{\delta \rho_{kl}} + \frac{i}{2} \frac{\delta}{\delta S_{kl}} \right) \int_\mathbb{R} U[\rho, S] \, dx \, dt \psi_{kl} = 0 ,
\]

(4.8)

where \( U[\rho, S] \) is the nonlinear potential in the hydrodynamic representation. Equation (4.8) can be posed in the following matrix form

\[
i \frac{\partial \Psi}{\partial t} + \hat{A} \Delta \Psi + \left( \hat{W}[\rho, S] + i \hat{W}[\rho, S] \right) \Psi = 0 ,
\]

(4.9)

where the Hermitian and anti-Hermitian nonlinearities are given by

\[
\hat{W}[\rho, S] = -\text{diag} \left( \frac{\delta}{\delta \rho_{kl}} \int_\mathbb{R} U[\rho, S] \, dx \, dt \right) ,
\]

(4.10)

\[
\hat{W}[\rho, S] = -\text{diag} \left( \frac{1}{2} \frac{\delta}{\delta S_{kl}} \int_\mathbb{R} U[\rho, S] \, dx \, dt \right) .
\]

(4.11)

Finally, by using the polar decomposition (4.6), equation (4.9) can be separate in a system of 2p nonlinear real coupled equations

\[
\frac{\partial S_{kl}}{\partial t} + a_{kl} (\nabla S_{kl})^2 - a_{kl} \frac{\Delta \rho_{kl}^{1/2}}{\rho_{kl}^{1/2}} - w_{kl}[\rho, S] = 0 ,
\]

(4.12)

\[
\frac{\partial \rho_{kl}}{\partial t} + 2 a_{kl} \nabla \cdot (\rho_{kl} \nabla S_{kl}) + 2 \rho_{kl} w_{kl}[\rho, S] = 0 .
\]

(4.13)

The first set of equations (4.12) is a system of \( p \)-coupled Hamilton-Jacobi-like equations for the fields \( S_{kl} \), whilst the second set of equations (4.13) describes the time evolution of the fields \( \rho_{kl} \).
A. $\text{U}(1)$ symmetry

In the following we consider those systems written in the form (4.9) which admit a set of $q$ continuity equations

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot j_{\Psi, k} = 0 ,$$

which assure the conservation of the quantities

$$N_k = \int_M \rho_k \, d^n x .$$

This imposes some restrictions on the functional dependence of the potential $U[\vec{\rho}, \vec{S}]$ with respect to the fields $\vec{\rho}$ and $\vec{S}$. To obtain such restrictions we recall the following relation

$$\frac{\delta}{\delta S_{ki}} = \frac{\partial}{\partial S_{ki}} - \nabla \cdot \frac{\delta}{\delta (\nabla S_{ki})} ,$$

so that, by taking in account the expression of the matrix $\hat{W}[\vec{\rho}, \vec{S}]$, we can rewrite equation (4.13) in

$$\frac{\partial \rho_{ki}}{\partial t} + \nabla \cdot \left( 2 a_{kl} \rho_{kl} \nabla S_{ki} + \frac{\delta}{\delta (\nabla S_{ki})} \int_R U[\vec{\rho}, \vec{S}] \, d^n x \, dt \right) - \frac{\partial}{\partial S_{ki}} \int_R U[\vec{\rho}, \vec{S}] \, d^n x \, dt = 0 .$$

(4.17)

By summing this equation on the index $l$, with $1 \leq l \leq p_k$, we obtain

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot \left( j^{(0)}_{\Psi, k} + J_k[\vec{\rho}, \vec{S}] \right) + I_k[\vec{\rho}, \vec{S}] = 0 ,$$

(4.18)

where

$$\rho_k = \sum_{l=1}^{p_k} \rho_{kl} ,$$

and

$$j^{(0)}_{\Psi, k} = \sum_{l=1}^{p_k} j^{(0)}_{\Psi, kl} ,$$

(4.19)

(4.20)

with

$$j^{(0)}_{\Psi, kl} = 2 a_{kl} \rho_{kl} \nabla S_{kl} .$$

(4.21)

Moreover, we posed

$$J_k[\vec{\rho}, \vec{S}] = \sum_{l=1}^{p_k} J_{kl}[\vec{\rho}, \vec{S}] ,$$

(4.22)

with

$$(J_{kl})_i[\vec{\rho}, \vec{S}] = \frac{\delta}{\delta (\partial_i S_{ki})} \int_R U[\vec{\rho}, \vec{S}] \, d^n x \, dt ,$$

(4.23)

whilst

$$I_k[\vec{\rho}, \vec{S}] = \sum_{l=1}^{p_k} I_{kl}[\vec{\rho}, \vec{S}] ,$$

(4.24)
with
\[ I_{kl} = -\frac{\partial}{\partial S_{kl}} \int_R U[\vec{\rho}, \vec{S}] d^n x dt . \] (4.25)

By comparing equation (4.14) with equation (4.18) we obtain, as a condition, that the functionals \( I_k[\vec{\rho}, \vec{S}] \) must be expressed as the gradient of a set of functionals \( G_k[\vec{\rho}, \vec{S}] \)
\[ I_k[\vec{\rho}, \vec{S}] = \nabla \cdot G_k[\vec{\rho}, \vec{S}] . \] (4.26)

We remark that the expression of the functionals \( G_k[\vec{\rho}, \vec{S}] \) is determined univocally from the nonlinear potential \( U[\vec{\rho}, \vec{S}] \) through equations (4.24), (4.25) and (4.26) which select, in this way, the class of the Lagrangians (4.3) of the family of CNLSEs compatible with the set of continuity equations (4.14). If the conditions (4.26) are accomplished, equations (4.18) become a system of \( q \) continuity equations, where the total currents of the \( k \)th multiplet \( j_{\psi,k} \) is given by
\[ j_{\psi,k} = j^{(0)}_{\psi,k} + J_k[\vec{\rho}, \vec{S}] + G_k[\vec{\rho}, \vec{S}] . \] (4.27)

We recall that, as it follows from the Noether theorem, equations (4.14) are consequence of the invariance of the Lagrangian with respect to a global unitary transformation
\[ \Psi \rightarrow \Phi = \hat{U} \Psi , \] (4.28)
where
\[ \hat{U} = \text{diag} \left( \exp(i \vec{\epsilon}) \right) , \] (4.29)
and
\[ \vec{\epsilon} = \left( \epsilon_1, \ldots, \epsilon_1; \epsilon_2, \ldots, \epsilon_2; \ldots; \epsilon_q, \ldots, \epsilon_q \right) , \] (4.30)
are the constant parameters of the transformation.

In fact, the Lagrangian (4.3) is invariant under the transformation (4.28) if the nonlinear potential \( U[\vec{\rho}, \vec{S}] \) changes according to
\[ \delta U[\vec{\rho}, \vec{S}] = -\sum_{k=1}^q \epsilon_k \nabla \cdot G_k[\vec{\rho}, \vec{S}] , \] (4.31)
where \( G_k[\vec{\rho}, \vec{S}] \) are arbitrary functionals. We recall that in this way the motion equation (4.5) does not change because the Lagrangian density is always defined modulo a total derivative of an arbitrary functional. Taking in account for the independence of the parameters \( \epsilon_k \), from equation (4.31) we obtain
\[ \sum_{l=1}^{p_k} \frac{\partial}{\partial S_{kl}} \int U[\vec{\rho}, \vec{S}] d^n x dt = -\nabla \cdot G_k[\vec{\rho}, \vec{S}] , \] (4.32)
which, according to the definitions (4.24) and (4.26) coincides with the condition (4.26). In addition, because the parameters \( \epsilon_k \) are constants, the potential \( U[\vec{\rho}, \vec{S}] \) can depend on the phases \( S_{kl} \) also through their spatial derivatives of any order.

### B. Gauge transformation

We are ready to generalize the nonlinear gauge transformation described in the section 3.2 to the family of CNLSEs under inspection. Let us introduce the following transformation
\[ \Psi(x, t) \rightarrow \Phi(x, t) = \hat{U}[\vec{\rho}, \vec{S}] \Psi(x, t) , \] (4.33)
where \( \hat{U}[\vec{\rho}, \vec{S}] \) is a diagonal and unitary matrix: \( \hat{U}^\dagger = \hat{U}^{-1} \). This implies
\[
\rho_{kl}(x, t) = |\psi_{kl}(x, t)|^2 = |\phi_{kl}(x, t)|^2 ,
\]
whilst the phases \( S_{kl} \) are related to fields \( \phi_{kl} \) through the relation
\[
S_{kl} = \frac{i}{2} \ln \left( \frac{\phi_{kl}^*(x, t)}{\phi_{kl}(x, t)} \right),
\]
(4.35)
since we define
\[
\phi_{kl}(x, t) = \rho_{kl}^{1/2}(x, t) \exp \left( i S(x, t) \right).
\]
(4.36)
Without lost of generality, the matrix \( \hat{U}[\vec{\rho}, \vec{S}] \) can be written in
\[
\hat{U}[\vec{\rho}, \vec{S}] = \text{diag} \left( \exp \left( i \vec{\sigma}[\vec{\rho}, \vec{S}] \right) \right),
\]
(4.37)
where \( \vec{\sigma} \equiv (\ldots, \sigma_{kl}, \ldots) \) is a \( p \)-dimensional vector with real components. The generators of the transformation \( \sigma_{kl}[\vec{\rho}, \vec{S}] \) relate the phase \( \vec{S} \) of the new field \( \Phi \) with the phase \( \vec{S} \) of the old field \( \Psi \) according to
\[
\vec{S} = \vec{S} + \vec{\sigma}[\vec{\rho}, \vec{S}].
\]
(4.38)
We introduce the generators \( \sigma_{kl}[\vec{\rho}, \vec{S}] \) through the relations
\[
\nabla \sigma_{kl}[\vec{\rho}, \vec{S}] = \frac{1}{2 \rho_{kl}} \left( \mathcal{J}_{kl}[\vec{\rho}, \vec{S}] + \mathcal{R}_{kl}[\vec{\rho}, \vec{S}] \right),
\]
(4.39)
where \( \mathcal{R}_{kl}[\vec{\rho}, \vec{S}] \) are arbitrary real functionals related to \( G_{kl}[\vec{\rho}, \vec{S}] \), introduced in equation (4.26), in
\[
\sum_{l=1}^{p_k} \mathcal{R}_{kl}[\vec{\rho}, \vec{S}] = G_{kl}[\vec{\rho}, \vec{S}].
\]
(4.40)
Consistence of equations (4.39) implies the following constraints
\[
\nabla \times \left[ \frac{1}{\rho_{kl}} \left( \mathcal{J}_{kl}[\vec{\rho}, \vec{S}] + \mathcal{R}_{kl}[\vec{\rho}, \vec{S}] \right) \right] = 0 .
\]
(4.41)
These equations select the potential \( U[\vec{\rho}, \vec{S}] \) and, through equations \( 4.10 \)-\( 4.11 \), the nonlinear system where the transformation can be performed.
We remark that, according to equations (4.39), equation (4.33) defines a wide class of transformations, one for every choice of the set of functionals \( \mathcal{R}_{kl}[\vec{\rho}, \vec{S}] \). Each transformation changes the initial system (4.9), with the nonlinearity \( \hat{W}[\vec{\rho}, \vec{S}] + i \hat{W}'[\vec{\rho}, \vec{S}] \), in another one with a purely Hermitian nonlinearity \( \hat{W}'[\vec{\rho}, \vec{S}] \).

Preliminarily, we observe that, within the notation (4.23) and (4.25), the matrix \( \hat{W}[\vec{\rho}, \vec{S}] \) assumes the expression
\[
\hat{W}[\vec{\rho}, \vec{S}] = -\text{diag} \left( \frac{1}{2 \rho_{kl}} \left( I_{kl}[\vec{\rho}, \vec{S}] + \nabla \cdot \mathcal{J}_{kl}[\vec{\rho}, \vec{S}] \right) \right).
\]
(4.42)
In this way, by performing the gauge transformation, equation (4.39) becomes
\[
i \frac{\partial \Phi}{\partial t} + \bar{A} \Delta \Phi + \left( \hat{W}[\vec{\rho}, \vec{S}] + i \hat{W}'[\vec{\rho}, \vec{S}] \right) \Phi = 0 ,
\]
(4.43)
where
\[
\hat{W}_{kl}[\vec{\rho}, \vec{S}] = \text{diag} \left( w_{kl} - a_{kl} (\nabla \sigma_{kl})^2 + 2 a_{kl} \nabla S_{kl} \cdot \nabla \sigma_{kl} + \partial S_{kl} / \partial t \right),
\]
(4.44)
and
\[
\hat{W}'_{kl}[\vec{\rho}, \vec{S}] = \text{diag} \left( \frac{1}{2 \rho_{kl}} \mathcal{R}_{kl}[\vec{\rho}, \vec{S}] \right).
\]
(4.45)
The functionals $\mathcal{F}_i[\vec{\rho}, \vec{S}]$ are given by

$$\mathcal{F}_i[\vec{\rho}, \vec{S}] = I_i[\vec{\rho}, \vec{S}] - \nabla \cdot R_i[\vec{\rho}, \vec{S}]\ ,$$

and fulfill the relations

$$\sum_{l=1}^{p_k} \mathcal{F}_i[\vec{\rho}, \vec{S}] = 0\ ,$$

as can be verify by using equations (4.27), (4.26) and (4.40).

We remark that, as a consequence of this last relation, equation (4.43) admits the following set of continuity equations

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot j_{\psi,k} = 0\ ,$$

where

$$j_{\psi,k} = \sum_{l=1}^{p_k} j_{\psi,kl}\ ,$$

i.e., the nonlinear currents $j_{\psi,k}$ are transformed in $j_{\psi,k} \rightarrow j_{\psi,k}^{(o)}$ which have the standard bilinear form. On the other hand, the system (4.43) with the nonlinearity $\tilde{W}_i[\vec{\rho}, \vec{S}] + i \tilde{W}_i'[\vec{\rho}, \vec{S}]$, can be rewritten in

$$i \frac{\partial \Phi}{\partial t} + \tilde{A} \Delta \Phi + \tilde{W}'[\vec{\rho}, \vec{S}] \Phi\ ,$$

with a purely Hermitian nonlinearity $\tilde{W}' = (\tilde{W}')^\dagger$ given in the following block-form

$$\tilde{W}'[\vec{\rho}, \vec{S}] = \text{diag} \left( \tilde{W}'_{ik}[\vec{\rho}, \vec{S}] \right)\ .$$

The $p_k \times p_k$ matrices $\tilde{W}'_{ik}[\vec{\rho}, \vec{S}] = \tilde{D}_k[\vec{\rho}, \vec{S}] + \tilde{C}_k[\vec{\rho}, \vec{S}]$ have a diagonal part

$$\tilde{D}_k[\vec{\rho}, \vec{S}] = \text{diag} \left( w_{ki} - a_{ki} (\nabla \sigma_{ki})^2 + 2 a_{ki} \nabla S_{kl} \cdot \nabla \sigma_{kl} + \frac{\partial \sigma_{ki}}{\partial t} \right)\ ,$$

with purely real entries, and an off-diagonal part

$$\left( \tilde{C}_k \right)_{lm} = i \frac{\mathcal{F}_k - \mathcal{F}_m}{2 p_k \sqrt{\rho_{kl}^\dagger \rho_{lm}}} e^{i (S_{kl} - S_{km})}\ ,$$

which result to be Hermitian matrices $\tilde{C}_k = \tilde{C}_k^\dagger$.

We observe that because the Lagrangian (4.3) is U(1)-invariant, the arbitrary integration constants, deriving from the definition (4.39), do not produce any effect and can be posed equal to zero. Moreover, the last term in equation (4.53) can be solved using equations (4.12)-(4.13) reducing the nonlinearity $\tilde{W}'[\vec{\rho}, \vec{S}]$ in a quantity containing only space derivatives.

The extension of the method to the case of non canonical coupled systems is almost immediate and can be performed following the same steps described at the end of section 3.2.

For a non canonical system the matrix $\tilde{W}[\vec{\rho}, \vec{S}]$ can assume any arbitrary expression whereas the form of the matrix $\tilde{W}'[\vec{\rho}, \vec{S}]$ is constrained by the existence of the set of the continuity equations for the fields $\rho_k$. Without loss of generality we can pose

$$w_{ki}[\vec{\rho}, \vec{S}] = -\frac{1}{2 \rho_{ki}} \left( I_{ki}[\vec{\rho}, \vec{S}] + \nabla \cdot \mathcal{F}_{ki}[\vec{\rho}, \vec{S}] \right)\ ,$$

(4.55)
where now the functionals $\mathcal{F}_k[\vec{\rho}, \vec{S}]$ and $I_k[\vec{\rho}, \vec{S}]$ are no more related to the nonlinear potential $U[\vec{\rho}, \vec{S}]$ through equations (4.23) and (4.25). The continuity equations (4.14) require that the functionals $I_k[\vec{\rho}, \vec{S}]$ still fulfill the constraints (4.26) for an arbitrary set of functionals $G_k[\vec{\rho}, \vec{S}]$. The total currents $j_{\psi,k}$ are given in equation (4.27) but now the functionals $I_k[\vec{\rho}, \vec{S}]$ and $J_k[\vec{\rho}, \vec{S}]$ are related to the matrix $\hat{W}[\vec{\rho}, \vec{S}]$ only through equation (4.55). Finally, we introduce the transformation (4.33) with generators (4.39) which eliminates the anti-Hermitian matrix $\hat{W}[\vec{\rho}, \vec{S}]$ of the nonlinearity and transforms the system of CNLSEs in the form given in equation (4.51) with only an Hermitian matrix $\hat{W}'[\vec{\rho}, \vec{S}]$ given still through equations (4.52)-(4.54).

Let us now briefly study separately two particular relevant cases:

a) CNLSEs conserving the number of each species of particles

We assume $p = q$ with $p_k = 1$ and replace the double index $kl \rightarrow k$. From equation (4.5) we obtain the following evolution equation for the quantities $\rho_k$

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot j_{\psi,k} + I_k = 0 \), (4.56)$$

where the currents $j_{\psi,k}$ are given by

$$\left( j_{\psi,k} \right)_i = 2 a_k \rho_k \partial_i S_k + \frac{\delta}{\delta(\partial_i S_k)} \int_R U[\vec{\rho}, \vec{S}] d^nx dt \), (4.57)$$

while the quantities $I_k[\vec{\rho}, \vec{S}]$ assume the expression

$$I_k = -\frac{\partial}{\partial S_k} \int_R U[\vec{\rho}, \vec{S}] d^nx dt \). (4.58)$$

Conservation of the single densities $\rho_k$ implies that all the quantities $I_k[\vec{\rho}, \vec{S}]$ must vanish. This requires that the potential $U[\vec{\rho}, \vec{S}]$ depends on the phases $S_k$ only through their spatial derivatives as it follows also from arguments based on the invariance of Lagrangian under a global unitary transformation. Remark that in this case the matrix $\hat{W}[\vec{\rho}, \vec{S}]$ assumes the simple expression

$$\hat{W}[\vec{\rho}, \vec{S}] = \text{diag} \left( \frac{-1}{2 \rho_k} \nabla \cdot \mathcal{F}_k[\vec{\rho}, \vec{S}] \right) \), (4.59)$$

where the functionals $\mathcal{F}_k[\vec{\rho}, \vec{S}]$ are defined in equations (4.22) and (4.23), after posing $p_k = 1$. All the quantities $G_k[\vec{\rho}, \vec{S}]$ introduced in equation (4.26) reduce to constant vectors which, without lost of generality, can be posed equal to zero. This implies that all the functionals $\mathcal{F}_k[\vec{\rho}, \vec{S}]$ vanish and after the transformation, the matrix $\hat{W}'[\vec{\rho}, \vec{S}]$ is reduced in a diagonal form given by

$$\hat{W}'[\vec{\rho}, \vec{S}] = \text{diag} \left[ w_k - a_k (\nabla \sigma_k)^2 + 2 a_k \nabla \sigma_k \cdot \nabla \sigma_k + \frac{\partial \sigma_k}{\partial t} \right] \), (4.60)$$

which contains now only a purely real nonlinearity since the off-diagonal part $\hat{C}[\vec{\rho}, \vec{S}]$ vanishes. Remark that, in this case, the gauge transformation is univocally defined because the generators are given by $\nabla \sigma_k[\vec{\rho}, \vec{S}] = \mathcal{F}_k[\vec{\rho}, \vec{S}] / 2 a_k \rho_k$.

b) CNLSEs conserving the total number of particles

We pose $q = 1$ and replace the double index $kl \rightarrow l$. From equation (4.5) we obtain the following evolution equation for the density $\rho_{\text{tot}}$

$$\frac{\partial \rho_{\text{tot}}}{\partial t} + \nabla \cdot j + I_{\text{tot}}[\vec{\rho}, \vec{S}] = 0 \), (4.61)$$

where

$$\rho_{\text{tot}} = \sum_{l=1}^p \rho_l \), (4.62)$$
is the total density of particles and the current $j$ is given by

$$
(j)_i = \sum_{l=1}^{p} \left( 2 a_i \rho_i \partial_i \Sigma_l + \frac{\delta}{\delta(\partial_i \Sigma_l)} \int_{\mathcal{R}} U[\tilde{\rho}, \tilde{S}] d^n x \, dt \right) .
$$

(4.63)

Conservation of $\rho_{\text{tot}}$ require that $I_{\text{tot}}[\tilde{\rho}, \tilde{S}]$, defined by

$$
I_{\text{tot}}[\tilde{\rho}, \tilde{S}] = - \sum_{l=1}^{p} \frac{\partial}{\partial S_l} \int_{\mathcal{R}} U[\tilde{\rho}, \tilde{S}] d^n x \, dt ,
$$

(4.64)

and can be expressed in

$$
I_{\text{tot}}[\tilde{\rho}, \tilde{S}] = \nabla \cdot G[\tilde{\rho}, \tilde{S}] ,
$$

(4.65)

so that equation (4.61) becomes a continuity equation

$$
\frac{\partial \rho_{\text{tot}}}{\partial t} + \nabla \cdot j_{\text{tot}} = 0 ,
$$

(4.66)

where the total current $j_{\text{tot}}$ is given in

$$
j_{\text{tot}} = j + G[\tilde{\rho}, \tilde{S}] .
$$

(4.67)

By performing the transformation (4.53) with generator (4.39), where

$$
\sum_{l=1}^{p} R_l[\tilde{\rho}, \tilde{S}] = G[\tilde{\rho}, \tilde{S}] ,
$$

(4.68)

we obtain the new system of CNLSEs (4.51) with an Hermitian nonlinearity $\hat{W}[\tilde{\rho}, \tilde{S}] = \hat{D}[\tilde{\rho}, \tilde{S}] + \hat{C}[\tilde{\rho}, \tilde{S}]$. The diagonal part

$$
\hat{D}[\tilde{\rho}, \tilde{S}] = \text{diag} \left[ w_i - a_i (\nabla \sigma_i)^2 + 2 a_i \nabla S_i \cdot \nabla \sigma_i + \frac{\partial \sigma_i}{\partial t} \right] ,
$$

(4.69)

contains purely real entries whilst the off-diagonal part

$$
\hat{C}_{lm} = i \frac{F_l - F_m}{2 \sqrt{\rho_l \rho_m}} e^{i (s_l - s_m)} ,
$$

(4.70)

results to be Hermitian.

V. NONLINEAR SCHRÖDINGER EQUATION COUPLED WITH GAUGE FIELDS

In this section we generalize the nonlinear transformation to NLSEs coupled with Abelian gauge fields whose dynamic is described by means of the standard Maxwell term with the inclusion of the additional Chern-Simons term.

A. The canonical model

We consider a class of NLSEs describing, in the mean field approximation, a system of interacting charged particles. The model is furnished by the following Lagrangian density

$$
L[\psi^*, \psi, A_\mu] = L_m[\psi^*, \psi, A_\mu] + L_\phi[A_\mu] ,
$$

(5.1)

where the Lagrangian of the matter field $L_m$ is given by

$$
L_m[\psi^*, \psi, A_\mu] = \frac{i}{2} \left[ \psi^* D_\mu \psi - \psi (D_\mu \psi)^* \right] - |D_\mu \psi|^2 - U[\rho, S, A] ,
$$

(5.2)
with \( D_\mu = (\partial_\mu + i A_\mu) \) the covariant derivative and \( U[\rho, S, A] \) is the nonlinear potential in the hydrodynamic representation depending on the abelian gauge field \( A_\mu \equiv (A_0, -\mathbf{A}) \) only through its spatial components. The Lagrangian of the gauge field \( \mathcal{L}_\mathbf{A} \) assumes the expression

\[
\mathcal{L}_\mathbf{A}[A_\mu] = -\gamma \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g}{2} \varepsilon^{\tau\mu\nu} A_\tau F_{\mu\nu} ,
\]

(5.3)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor, with \( \partial_\mu \equiv (\partial/\partial t, \nabla) \).

In the following, Greek indices take the value \( 0, \ldots, n \), the Latin indices assume the value \( 1, \ldots, n \) and denote the spatial coordinates. Indices are lowered and uppered depending on the metric tensor \( \eta_{\mu\nu} \equiv \text{diag}(1, -1, \ldots, -1) \). The Levi-Civita tensor \( \varepsilon^{\tau\mu\nu} \), fully antisymmetric, is defined as \( \varepsilon^{012} = 1 \). The parameters \( \gamma \) and \( g \) weight the contribution of the Maxwell interaction and the Chern-Simons interaction. We recall that the Chern-Simons term gives contribution only when the dynamic of the system is constrained in a manifold with an even number of space dimensions (like in the plane) whilst in an odd number of space dimensions it reduces to a total derivative which does not give contribution to the motion equation.

Starting from the action of the system

\[
\mathcal{A} = \int_{\mathcal{R}} \mathcal{L}[\psi^*, \psi, A_\mu] d^n x d t ,
\]

(5.4)

the evolution equations for the fields \( \psi, \psi^* \) and \( A_\mu \) are obtained by posing \( \delta \mathcal{A} = 0 \) where the variation is performed with respect to the 3-vector \( \Omega \equiv (\psi, \psi^*, A_\mu) \).

The motion equation for the gauge field assumes the expression

\[
\gamma \partial_\mu F^{\mu\nu} + g \varepsilon^{\nu\tau\mu} F_{\tau\mu} = j_\nu^\nu ,
\]

(5.5)

where the covariant current \( j_\nu^\nu \equiv (\rho, j_\nu^\nu) \) has spatial components

\[
\left(j_\nu^\nu\right)_i = 2 \rho \left(\partial_i S + A_i\right) + \frac{\delta}{\delta A_i} \int_{\mathcal{R}} U[\rho, S, A] d^n x d t.
\]

(5.6)

By observing that \( F^{\mu\nu} = -F^{\nu\mu} \), from equation (5.5) we immediately obtain the continuity equation for the field \( \rho \)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j_\rho^\nu = 0 ,
\]

(5.7)

which assures the conservation of the total charge of the system.

On the other hand, the evolution equation for the matter field, as it follows from the Lagrangian density (5.1), is given by

\[
i D_4 \psi + D^2 \psi + \left(W[\rho, S, A] + i \mathcal{W}[\rho, S, A]\right) \psi = 0 ,
\]

(5.8)

where the real and imaginary parts of the nonlinearity are given, respectively, by

\[
W[\rho, S, A] = -\frac{\delta}{\delta \rho} \int_{\mathcal{R}} U[\rho, S, A] d^n x d t ,
\]

(5.9)

and

\[
\mathcal{W}[\rho, S, A] = -\frac{1}{2 \rho} \frac{\delta}{\delta S} \int_{\mathcal{R}} U[\rho, S, A] d^n x d t .
\]

(5.10)

For consistence, equation (5.8) must admit the same continuity equation (5.7).

Following standard arguments, by multiply equation (5.8) by \( \psi^* \) and taking its imaginary part, we obtain

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \left[2 \rho \left(\nabla S - \mathbf{A}\right)\right] - \frac{\delta}{\delta S} \int_{\mathcal{R}} U[\rho, S, A] d^n x d t = 0 ,
\]

(5.11)
which can be written in

$$\frac{\partial \rho}{\partial t} + \mathbf{\nabla} \cdot \mathbf{j}_A = \frac{\partial}{\partial S} \int \mathcal{U}[\rho, S, A] d^nx dt ,$$

(5.12)

where the charged current $j_A$ now becomes

$$\left( j_A \right)_i = 2 \rho (\partial_i S + A_i) + \frac{\delta}{\delta (\partial_i S)} \int \mathcal{U}[\rho, S, A] d^nx dt .$$

(5.13)

By comparing this expression with equation (5.6) it follows that:

1) the nonlinear potential $\mathcal{U}[\rho, S, A]$ must depend on the field $S$ only through its spatial derivatives so that the right hand side of equation (5.12) vanishes becoming, in this way, a continuity equation for the field $\rho$.

2) the fields $\mathbf{\nabla} S$ and $A$ must be present in the nonlinear potential through the combination $\mathbf{\nabla} S - A$. In other words the Lagrangian of the matter field can be obtained consistently from the Lagrangian of the scalar field (3.1) by replacing in it the standard derivatives with the covariant ones $\partial \mu \rightarrow D \mu = \partial \mu + i A_\mu$ (minimal coupling prescription).

Since, as a required, $\mathcal{U}[\rho, S, A]$ depends only through the quantity $\mathbf{\nabla} S - A$ and its highest spatial derivatives, equation (5.10) can be written in

$$W[\rho, S, A] = \frac{1}{2} \rho \mathbf{\nabla} \cdot \mathbf{J}_A[\rho, S, A] ,$$

(5.14)

where the vector $\mathbf{J}_A[\rho, S, A]$ is defined in

$$\left( \mathbf{J}_A \right)_i[\rho, S, A] = \frac{\delta}{\delta (\partial_i S)} \int \mathcal{U}[\rho, S, A] d^nx dt ,$$

(5.15)

and the charged current (5.13) assumes the expression

$$j_A = 2 \rho (\mathbf{\nabla} S - A) + \mathbf{J}_A[\rho, S, A] .$$

(5.16)

B. Gauge transformation

Firstly, we recall that the system described by the Lagrangian (5.1) is invariant over a local U(1) transformation (gauge transformation of second kind), accomplished both on the fields $\psi$ and $A_\mu$, by means of

$$A_\mu(x, t) \rightarrow A_\mu(x, t) - \partial_\mu \omega(x, t) ,$$

$$\psi(x, t) \rightarrow \exp \left( i \omega(x, t) \right) \psi(x, t) ,$$

(5.17)

where $\omega(x, t)$ is a well-behaved function in the sense of $\epsilon^{\mu\nu} \partial_\mu \partial_\nu \omega = 0$, with $\epsilon^{\mu\nu}$ the anti-symmetric tensor $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Remark that, under the transformation (5.17), Lagrangian (5.3) changes according to

$$\mathcal{L}_g \rightarrow \mathcal{L}_g + \frac{g^2}{2} \epsilon^{\mu\nu\tau} \partial_\mu \left( \omega F_{\nu\tau} \right) ,$$

(5.18)

with an extra surface term which does not change the motion of equations for the fields $\psi$ and $A_\mu$.

Let us now introduce the gauge transformation of third kind as a unitary nonlinear transformation performed only on the field $\psi$

$$\psi(x, t) \rightarrow \phi(x, t) = \mathcal{U}[\rho, S, A] \psi(x, t) ,$$

(5.19)

which allows to eliminate the imaginary part $W[\rho, S, A]$ of the nonlinearity in the evolution equation (5.8) and reduces the charged current to the standard bilinear form

$$j_A[\rho, S, A] \rightarrow j^{(0)}_A[\rho, S, A] = 2 \rho (\mathbf{\nabla} S - A) .$$

(5.20)
The unitary functional $\mathcal{U}[\rho, S, A]$ is given by
\[ \mathcal{U}[\rho, S, A] = \exp \left( i \sigma[\rho, S, A] \right), \tag{5.21} \]
where the real generator of the transformation $\sigma[\rho, S, A]$ defined according to
\[ \nabla \sigma[\rho, S, A] = \frac{1}{2 \rho} \mathcal{J}_A[\rho, S, A], \tag{5.22} \]
are constrained by
\[ \nabla \times \left( \frac{\mathcal{J}_A[\rho, S, A]}{\rho} \right) = 0. \tag{5.23} \]

By performing the transformation (5.19), from equation (5.8) we obtain the following NLSE for the charged field $\phi$
\[ i D_t \phi + D^2 \phi + \tilde{W}[\rho, S, A] \phi = 0, \tag{5.24} \]
where the real nonlinearity $\tilde{W}[\rho, S, A]$ assumes the expression
\[ \tilde{W}[\rho, S, A] = W - (\nabla \sigma)^2 + 2(\nabla S - A) \cdot \nabla \sigma + \frac{\partial \sigma}{\partial t}, \tag{5.25} \]
with $W \equiv W[\rho, S[\rho, S, A], A]$ and $\sigma \equiv \sigma[\rho, S[\rho, S, A], A]$. The new phase $S$ of the field $\phi$ is related to the old phase $S$ of the field $\psi$ through the relation
\[ S = S + \sigma[\rho, S, A], \tag{5.26} \]
and because the nonlinearity in equation (5.24) is a purely real quantity the continuity equation for the field $\rho$ becomes
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot j^{(0)}_{A \phi} = 0, \tag{5.27} \]
with the transformed charged current $j^{(0)}_{A \phi}$ given in equation (5.20).

Since the nonlinear transformation has been accomplished only on the matter field, the evolution equation for the gauge field retains formally the same expression given in equation (5.5)
\[ \gamma \partial_\mu F^{\mu \nu} + g \varepsilon^{\nu \tau \mu} F_{\tau \mu} = j^{\nu}_{A \phi}, \tag{5.28} \]
but with the transformed charged source $j^{\nu}_{A \phi} \equiv (\rho, j^{(0)}_{A \phi})$.

On the other hand, the presence of the gauge field enable us to introduce a transformation on it, leaving the matter field unchanged.

In fact, let us introduce the following transformation
\[ A(x, t) \rightarrow \chi(x, t) = A(x, t) - \nabla \sigma[\rho, S, A], \tag{5.29} \]
where $\sigma[\rho, S, A]$ is still defined through equation (5.22).

Accounting for $F_{\mu \nu} = -F_{\nu \mu}$, it follows that
\[ F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \chi_\nu - \partial_\nu \chi_\mu, \tag{5.30} \]
whenever $\sigma[\rho, S, A]$ is a well behaved function fulfilling the relation
\[ \varepsilon^{\mu \nu} \partial_\mu \partial_\nu \sigma[\rho, S, A] = 0. \tag{5.31} \]

This implies that, for $\mu$ and $\nu$ spatial indices, equation (5.30) is trivially satisfied as consequence of condition (5.23), differently, for $\mu$ or $\nu$ equal to zero, equation (5.30) implies the following transformation for the component $A_0(x, t)$ of the gauge field
\[ A_0(x, t) \rightarrow \chi_0(x, t) = A_0(x, t) + \frac{\partial}{\partial t} \sigma[\rho, S, A]. \tag{5.32} \]
By performing the transformation (5.29) and (5.32) in equation (5.5) we obtain

\[ \gamma \partial_\mu F^{\mu \nu} + g \varepsilon^{\nu \tau \rho} F_{\tau \rho} = \tilde{j}_\nu \]

where the new covariant current \( \tilde{j}_\nu \equiv (\rho, \tilde{j}_\nu^{(0)}) \) with

\[ \tilde{j}_\nu^{(0)} = 2 \rho (\nabla S - \chi) \]

fulfills the continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \tilde{j}_\nu = 0 \]  

(5.35)

Differently, from equation (5.8) it follows

\[ i \overline{D}_\mu \psi + \overline{D}^2 \psi + W[\rho, \psi, \chi] \psi = 0 \]

(5.36)

which has the same form of equation (5.24) but now the covariant derivative is defined in \( \overline{D}_\mu = \partial_\mu + i \chi_\mu \), while the real nonlinearity becomes

\[ \overline{W}[\rho, \psi, \chi] = W - (\nabla \sigma)^2 + 2 (\nabla S - \chi) \cdot \nabla \sigma + \frac{\partial \sigma}{\partial t} \]

(5.37)

with \( W \equiv W[\rho, S, A[\rho, S, \chi]] \) and \( \sigma \equiv \sigma[\rho, S, A[\rho, S, \chi]] \).

In conclusion, it is worthy to observe that if we introduce the nonlinear transformation both on the matter field and the gauge filed, by following the prescription given in equation (5.17), the evolution equations (5.5) and (5.8) are not changed in form because the variations due to the matter field are balanced by the variations due to the gauge field. Thus, in this case transformation (5.19) behaves exactly like a gauge transformation of second kind.

VI. APPLICATIONS

To show the applicability of the nonlinear transformation introduced in this paper, we consider some examples for the three cases: scalar NLSEs, coupled NLSEs and gauged NLSEs. Some of the examples here discussed are already known in literature. We show that the nonlinear transformations introduced by different Authors can be obtained, in a unified way, with the method presented in this work.

A. Scalar NLSEs

Let us consider, as a first example, the following 1-dimensional NLSE

\[ i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + a_1 |\psi|^2 \psi + a_2 |\psi|^4 \psi + i a_3 |\psi|^2 \frac{\partial \psi}{\partial x} + i a_4 |\psi|^2 \frac{\partial \psi^*}{\partial x} \psi^2 = 0 \]

(6.1)

where \( a_1, a_2, a_3 \) and \( a_4 \) are real constants. After introducing the hydrodynamic fields \( \rho \) and \( S \) we can write the real and imaginary part of the nonlinearity in

\[ W[\rho, S] = b_1 \rho + b_2 \rho^2 + b_3 \rho \frac{\partial S}{\partial x} \]

(6.2)

and

\[ W[\rho] = b_4 \frac{\partial \rho}{\partial x} \]

(6.3)

where \( b_1 = a_1, b_2 = a_2, b_3 = a_4 - a_3 \) and \( b_4 = (a_3 + a_4) / 2 \).

The canonical subclass of equation (6.1) is given by posing \( b_3 = -2 b_4 \) and admits the following potential

\[ U[\rho, S] = - \left( \frac{b_2}{2} \rho^2 + \frac{b_3}{3} \rho^3 + \frac{b_4}{2} \rho^2 \frac{\partial S}{\partial x} \right) \]

(6.4)
Equation (6.1) conserves the density $\rho$ and the corresponding particles current is given by

$$ j_\psi = 2 \rho \frac{\partial S}{\partial x} + b_4 \rho^2. $$

(6.5)

After performing the transformation (3.33) with

$$ \sigma[\rho] = \frac{b_4}{2} \int \rho \, dx', $$

(6.6)

equation (6.1) is changed in

$$ i \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + \left( b_1 \rho + \tilde{b}_2 \rho^2 + b_3 \rho \frac{\partial S}{\partial x} \right) \phi = 0, $$

(6.7)

where $\tilde{b}_2 = b_2 - b_4 / 2 - b_4^2 / 4$.

Equation (6.1) contains, as particular cases, some known NLSEs. Among them we recall:

1) The Chen-Lee-Liu equation [14] ($b_1 = b_2 = 0, b_3 = -2b_4$) which is transformed in the NLSE with real nonlinearity

$$ \tilde{W}[\rho, S] = \tilde{b}_2 \rho^2 + b_4 \rho \frac{\partial S}{\partial x}, $$

(6.8)

where $\tilde{b}_2 = 3b_2^2 / 4$.

2) The Jackiw-Aglietti equation [2, 41] ($b_1 = 0, b_2 = -3b_4 / 4$ and $b_3 = -b_4$) which is transformed in the NLSE with real nonlinearity

$$ \tilde{W}[\rho, S] = b_3 \rho \frac{\partial S}{\partial x}. $$

(6.9)

3) The Eckaus equation [9, 13] ($b_1 = b_3 = 0$) which is transformed in the NLSE with real nonlinearity

$$ \tilde{W}[\rho] = \tilde{b}_2 \rho^2, $$

(6.10)

with $\tilde{b}_2 = b_2 - b_4^2 / 4$. Remark that, when $b_1 \neq 0$ we obtain, after transformation, the cubic-quintic NLSE with real nonlinearity

$$ \tilde{W}[\rho] = b_1 \rho + \tilde{b}_2 \rho^2, $$

(6.11)

studied in [27].

4) The Kaup-Newell equation [56] ($b_1 = b_2 = 0$ and $b_4 = -3b_3 / 2$) which is transformed in the NLSE with real nonlinearity

$$ \tilde{W}[\rho, S] = \tilde{b}_2 \rho^2 + b_4 \rho \frac{\partial S}{\partial x}, $$

(6.12)

with $\tilde{b}_2 = 3b_2^2 / 16$.

As a second example we consider the canonical NLSE introduced in [53, 54]

$$ i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \kappa \left( \psi \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial \psi^*}{\partial x} \right) \frac{\partial \psi}{\partial x} + \kappa \frac{\partial}{\partial x} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) \psi = 0, $$

(6.13)

where $\kappa$ is a real parameter. The real and imaginary nonlinearities in the hydrodynamic representation are given by

$$ W[\rho, S] = -2\kappa \rho \left( \frac{\partial S}{\partial x} \right)^2, $$

(6.14)

and

$$ \tilde{W}[\rho] = \kappa \frac{\partial}{\partial x} \left( \rho^2 \frac{\partial S}{\partial x} \right). $$

(6.15)
They are obtained from the potential
\[ U[\rho, S] = \kappa \left( \rho \frac{\partial S}{\partial x} \right)^2, \tag{6.16} \]
whereas the particles current assumes the expression
\[ j_\psi = 2\rho (1 + \kappa \rho) \frac{\partial S}{\partial x}. \tag{6.17} \]

By performing the transformation (3.33) with generator
\[ \sigma[\rho, S] = \kappa \int \rho \frac{\partial S}{\partial x'} dx', \tag{6.18} \]
equation (6.13) changes in
\[ i \frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} - \left[ \frac{2 \kappa \rho}{1 + \kappa \rho} \left( \frac{\partial S}{\partial x} \right)^2 - \frac{\kappa}{2} \rho \frac{\partial^2 \log \rho}{\partial x^2} \right] \phi = 0. \tag{6.19} \]
Remark that although equation (6.13) can be generalized in any spatial dimensions [53, 54] condition (3.39) is not satisfied in general and the transformation (3.33) can be applied consistently only in 1-dimensional case.

Another example is given by the class of the Doebner-Goldin equations [18]
\[ i \frac{\partial \psi}{\partial t} + \Delta \psi + \sum_{i=1}^{5} c_i R_i[\rho, S] + i \frac{D}{2} R_5[\rho] \psi = 0, \tag{6.20} \]
where the nonlinear functionals \( R_i \) are given by \( R_1 = \nabla \cdot (\rho \nabla S)/\rho, R_2 = \Delta \rho/\rho, R_3 = (\nabla S)^2, R_4 = \nabla S \cdot \nabla \rho/\rho \) and \( R_5 = (\nabla \rho/\rho)^2 \). The canonical subclass of equation (6.20) is obtained for \( c_1 = -c_3 = D, c_3 = 0 \) and \( c_5 = -2c_5 \) and it follows from the potential
\[ U[\rho, S] = D \nabla \rho \cdot \nabla S + c_5 \frac{(\nabla \rho)^2}{\rho}. \tag{6.21} \]
The particles current is given by
\[ j_\psi = 2\rho \nabla S + D \nabla \rho, \tag{6.22} \]
and the corresponding continuity equation is the well-known Fokker-Planck equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}(0) + D \Delta \rho = 0, \tag{6.23} \]
where \( D \) is the diffusion coefficient.
By performing the transformation (3.33) with generator
\[ \sigma[\rho] = D \frac{\partial \log \rho}{\partial x}, \tag{6.24} \]
equation (6.20) transforms in
\[ i \frac{\partial \phi}{\partial t} + \Delta \phi + \sum_{i=1}^{5} \tilde{c}_i \sigma[\rho] \phi = 0, \tag{6.25} \]
with coefficients \( \tilde{c}_1 = c_1 - D, \tilde{c}_2 = c_2 - c_1 D/2, \tilde{c}_3 = c_3, \tilde{c}_4 = c_4 + (c_1 - 1) D \) and \( \tilde{c}_5 = c_5 - c_3 D - (c_3 - 1) D^2/4 \).
It is easy to verify that the generator (6.24) satisfies condition (3.39) and the nonlinear transformation can be performed in any \( n \geq 1 \) spatial dimensions.
As a final example we consider the following family of NLSE
\[ i \frac{\partial \psi}{\partial t} + \Delta \psi + \left( W(\rho, S) + i W(\rho, S) \right) \psi = 0, \tag{6.26} \]
with nonlinearities

\[ W(\rho, S) = -\frac{D}{2} f(\rho) \nabla \cdot \left( \frac{j^{(0)}}{\rho} \right) + G[\rho], \tag{6.27} \]

and

\[ \mathcal{W}(\rho, S) = -\frac{D}{2 \rho} \nabla \cdot \left( f(\rho) \nabla \rho \right), \tag{6.28} \]

where

\[ f(\rho) = \rho \frac{\partial \ln \kappa(\rho)}{\partial \rho}, \tag{6.29} \]

and \( G[\rho] \) is an arbitrary functional of \( \rho \). Equation (6.26) can be obtained from the potential

\[ U[\rho, S] = -D f(\rho) \nabla \cdot \nabla S + \int_{\rho} G[\rho'] d\rho', \tag{6.30} \]

and was recently derived in the canonical quantization framework from a classical many body systems described by generalized entropies \[83\].

The particles current is given by

\[ j = 2\rho \nabla S = D f(\rho) \nabla \rho, \tag{6.31} \]

which is the sum of a linear drift current \( j_{\text{drift}} = 2\rho \nabla S \) and a nonlinear diffusion current \( j_{\text{diff}} = -D f(\rho) \nabla \rho \) different from Fick’s current \( j_{\text{Fick}} = -D \nabla \rho \) which is recovered by posing \( \kappa(\rho) = \alpha \rho \), with \( \alpha \) a constant. The diffusive term is related to the entropy of the classical system through the relation (with the Boltzmann constant \( k_B = 1 \))

\[ S(\rho) = -\int_{\mathcal{M}} d^n x \int_{\rho} \ln \kappa(\rho') d\rho'. \tag{6.32} \]

By performing the transformation (3.33) with generator

\[ \sigma[\rho] = \frac{D}{2} \ln \kappa(\rho), \tag{6.33} \]

equation (6.26) changes in

\[ i \frac{\partial \phi}{\partial t} + \Delta \phi - \frac{D^2}{2} \left[ f_1(\rho) \Delta \rho + f_2(\rho) (\nabla \rho)^2 \right] \phi + G[\rho] \phi = 0, \tag{6.34} \]

with

\[ f_1(\rho) = \rho \left( \frac{\partial}{\partial \rho} \ln \kappa(\rho) \right)^2, \tag{6.35} \]

\[ f_2(\rho) = \frac{1}{2} \frac{\partial f_1(\rho)}{\partial \rho}, \tag{6.36} \]

which contains a purely real nonlinearity depending only on \( \rho \).

In particular, starting from the entropy \( S = -\int_{\mathcal{M}} \rho \log \rho d^n x \), with \( \kappa(\rho) = e\rho \), equation (6.26) becomes

\[ i \frac{\partial \psi}{\partial t} + \Delta \psi - \frac{D}{2} \nabla \cdot \left( \frac{j^{(0)}}{\rho} \right) \psi - i \frac{D^2}{2} \frac{\Delta \rho}{\rho} \psi = 0, \tag{6.37} \]

which coincides with the canonical sub-family of the Doebner-Goldin equations described in the previous example. After transformation it becomes

\[ i \frac{\partial \phi}{\partial t} + \Delta \phi - \frac{D^2}{2} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \phi = 0, \tag{6.38} \]
which was studied previously in [34]. Remarkably, this equation is equivalent to the following linear Schrödinger equation

$$i \kappa \frac{\partial \chi}{\partial t} + \kappa^2 \Delta \chi = 0 \,,$$

(6.39)

with \( \kappa = \sqrt{1 - D^2} \) where the field \( \chi \) is related to hydrodynamic fields \( \rho \) and \( S \) through the relation \( \chi = \sqrt{\rho} \exp(iS/\kappa) \).

### B. Coupled NLSEs

Let us now consider the following 1-dimensional system of CNLSEs

$$i \frac{\partial \psi_j}{\partial t} + a_j \frac{\partial^2 \psi_j}{\partial x^2} + \Lambda(\psi_j, \psi_j^*) \psi_j = 0 \,,$$

(6.40)

with nonlinearity

$$\Lambda(\psi_j, \psi_j^*) = -i \sum_{i=1}^{p} \left( \alpha_{ij} \frac{\rho_j}{\rho_i} \frac{\partial \psi_i^*}{\partial x} + \beta_{ij} \frac{\partial \rho_i}{\partial x} \right) + \gamma_{ij} \psi_j \frac{\partial \psi_i^*}{\partial x} + \epsilon_{ij} \psi_i \frac{\partial \psi_i^*}{\partial x} + f_j(\bar{\rho}) \,, $$

(6.41)

where \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) and \( \epsilon_{ij} \) are real constants and \( f_j(\bar{\rho}) \) are arbitrary real functionals depending only on the vector field \( \bar{\rho} \).

In the hydrodynamic representation the nonlinearity \( 6.41 \) has Hermitian and anti-Hermitian part given, respectively, by

$$\hat{W}[\bar{\rho}, \bar{S}] = \text{diag} \left[ \sum_{i=1}^{p} \rho_i \left( b_{ij} \frac{\partial S_i}{\partial x} + c_{ij} \frac{\partial S_i}{\partial x} \right) + f_j(\bar{\rho}) \right] \,,$$

(6.42)

$$\hat{\bar{W}}[\bar{\rho}] = \text{diag} \left[ \sum_{i=1}^{p} \left( d_{ij} \frac{\rho_i}{\rho_j} \frac{\partial \rho_j}{\partial x} + e_{ij} \frac{\partial \rho_j}{\partial x} \right) \right] \,,$$

(6.43)

where \( b_{ij} = \alpha_{ij} - \beta_{ij}, c_{ij} = \gamma_{ij} - \epsilon_{ij}, d_{ij} = (\alpha_{ij} + \beta_{ij})/2 \) and \( e_{ij} = (\gamma_{ij} + \epsilon_{ij})/2 \).

Equation \( 6.40 \) includes some cases already known in literature. For instance: the vector generalization of the Kaup-Newell equation [25] \( (a_j = 1, c_{ij} = 0, -b_{ij} = 2 d_{ij} = e_{ij} = \beta \) and \( f_j(\bar{\rho}) = 0) \); the coupled Chen-Lie-Liu equation (Type I) [92] \( (a_j = 1, c_{ij} = e_{ij} = 0, -b_{ij} = 2 d_{ij} = \beta, f_j(\bar{\rho}) = 0) \); the coupled Chen-Lie-Liu equation (Type II) [92] \( (a_j = 1, b_{ij} = d_{ij} = 0, c_{ij} = -2 e_{ij} = \beta, f_j(\bar{\rho}) = 0) \); the hybrid CNLSE [38, 39] \( (a_j = 1, c_{ij} = 0, -b_{ij} = 2 d_{ij} = e_{ij} = \beta \) and \( f_j(\bar{\rho}) = \beta \sum_{k} \rho_k \) ); the vectorial Eckhaus equation [3] \( (\alpha_{ij} = 0, f_j(\bar{\rho}) = \sum_{ik} \lambda_{ij} \rho_i \rho_k) \). Moreover, for \( q = p = 2 \), with \( b_{ij} + 2 d_{ij} = 0 \) and \( f_j(\bar{\rho}) = f \rho_i + g \rho_j \), \( f_j(\bar{\rho}) = g \rho_i + f \rho_j \), equation \( 6.40 \) has been studied in [81].

The canonical sub-family of equation \( 6.40 \) is given by \( b_{ij} = c_{ij} = -2 d_{ij} = -2 e_{ij} \) and can be obtained through the following nonlinear potential

$$U[\bar{\rho}, \bar{S}] = -\sum_{i,j=1}^{p} b_{ij} \rho_i \rho_j \frac{\partial S_i}{\partial x} + F(\bar{\rho}) \,,$$

(6.44)

where the conditions \( \delta F(\bar{\rho})/\delta \rho_j = f_j(\bar{\rho}) \) are assumed.

We observe that:

a) when \( d_{ij} = e_{ij} \), for \( i \neq j \), equation \( 6.40 \) conserves the densities \( \rho_j \) and the currents take the form

$$j_{\psi,j} = 2 a_j \rho_j \frac{\partial S_j}{\partial x} - (d_{jj} + e_{jj}) \rho_j^2 - 2 \sum_{i=1,i \neq j}^{p} d_{ij} \rho_i \rho_j \,,$$

(6.45)

with \( J_\psi(\bar{\rho}) = -(d_{jj} + e_{jj}) \rho_j^2 - 2 \sum_{i \neq j} d_{ij} \rho_i \rho_j \) and \( I_\psi(\bar{\rho}) = 0 \).

b) when \( d_{ij} + e_{ij} = d_{ji} + e_{ji} \), equation \( 6.40 \) conserves the total density \( \rho_{\text{tot}} = \sum_j \rho_j \), and the total current is given
by
\[ j_{\text{tot}} = \sum_{j=1}^{p} \left[ 2a_j \rho_j \frac{\partial S_j}{\partial x} - \sum_{i=1}^{p} (d_{ij} + e_{ij}) \rho_i \rho_j \right], \quad (6.46) \]

with \( J_{\text{tot}} = -(d_{ij} + e_{ij}) \rho_j^2 \) and
\[
I_{\text{tot}}[\hat{\rho}] = -2 \sum_{i \neq j} \left( d_{ij} \rho_i \frac{\partial \rho_j}{\partial x} + e_{ij} \rho_j \frac{\partial \rho_i}{\partial x} \right). \quad (6.47)
\]

If we choose the functionals \( R_{\text{tot}}[\hat{\rho}] = 0 \) in the case a) and
\[
R_{\text{tot}}[\hat{\rho}] = -\sum_{i=1, i \neq j}^{p} \lambda_{ij} \rho_i \rho_j, \quad (6.48)
\]
in the case b), where \( \lambda_{ij} = d_{ij} + e_{ij} \), the generators (4.39) can be written in the unified form
\[
\sigma_{\text{tot}}[\hat{\rho}] = -\frac{1}{2a_j} \sum_{i=1}^{p} \lambda_{ij} \int_{x}^{x^{'}} \rho_i \, dx'. \quad (6.49)
\]

By performing the gauge transformation, from equation (6.40) we obtain a new system of CNLSEs for the field \( \Phi \) with nonlinearity
\[
\hat{W}'[\hat{\rho}, \hat{S}] = \hat{D}[\hat{\rho}, \hat{S}] + \hat{C}[\hat{\rho}, \hat{S}], \quad (6.50)
\]
where the diagonal matrix \( \hat{D}[\hat{\rho}, \hat{S}] \) has entries
\[
\hat{D}[\hat{\rho}, \hat{S}] = \text{diag} \left[ \sum_{i=1}^{p} \rho_i \left( \mu_{ij} \frac{\partial S_j}{\partial x} + \nu_{ij} \frac{\partial S_i}{\partial x} \right) \right.
+ \sum_{i,k=1}^{p} \omega_{ijk} \rho_i \rho_k + f_i(\hat{\rho}) \left. \right], \quad (6.51)
\]
with
\[
\mu_{ij} = b_{ij} + \lambda_{ij}, \quad (6.52)
\]
\[
\nu_{ij} = c_{ij} - \frac{a_j}{a_i} \lambda_{ij},
\]
\[
\omega_{ijk} = \frac{1}{4a_j} \left( \lambda_{ij} \lambda_{jk} + 2b_{ij} \lambda_{kj} + 2 \frac{a_j}{a_i} c_{ij} \lambda_{ki} \right), \quad (6.53)
\]
whereas the off-diagonal matrix \( \hat{C} \) has entries
\[
\left( \hat{C} \right)_{ij} = i \frac{\mathcal{F}_i(\hat{\rho}) - \mathcal{F}_j(\hat{\rho})}{2 \sqrt{\rho_i \rho_j}} e^{i \left( S_i - S_j \right)} , \quad (6.54)
\]
where
\[
\mathcal{F}_i(\hat{\rho}) = \sum_{i=1}^{p} (d_{ij} - e_{ij}) \left( \rho_i \frac{\partial \rho_j}{\partial x} - \frac{\partial \rho_i}{\partial x} \rho_j \right) . \quad (6.55)
\]

We observe that the functionals (6.54) vanish in the case a) and the nonlinearity \( \hat{W}'[\hat{\rho}, \hat{S}] \) reduces to a purely real quantity.
Let us now collect some particular cases belonging to equation (6.40).
1) By choosing \( b_{ij} = -\lambda_{ij} \) and \( a_{j} c_{ij} = 2a, \lambda_{ij} \) we obtain a system of CNLSEs with a purely real nonlinearity which depends only on the fields \( \rho \).

\[
i \frac{\partial \phi_j}{\partial t} + a_j \Delta \phi_j - \left( \sum_{i,k=1}^{p} \omega_{ijk} \rho_i \rho_k + f_j(\bar{\rho}) \right) \psi_j = 0. \tag{6.55}
\]

When \( f_j(\bar{\rho}) = \sum_{i,k} \lambda_{ijk} \rho_i \rho_k \) with \( \lambda_{ijk} = \sum_{i,k} b_{ij} (b_{k} - 2 b_{k}) / 4 a_j \), it reduces to a system of decoupled linear Schrödinger equations

\[
i \frac{\partial \phi_j}{\partial t} + a_j \Delta \phi_j = 0. \tag{6.56}
\]

2) By choosing \( b_{ij} = -\lambda_{ij} \) for \( i \neq j \), \( a_{j} c_{ij} = 2a, \lambda_{ij} \) and \( f_j(\bar{\rho}) = \sum_{i,k} \lambda_{ijk} \rho_i \rho_k \) with

\[
\begin{aligned}
\lambda_{kk} &= \lambda_{kk} \left( b_{kk} + 3 \lambda_{kk}/2 \right) / 2 a_k , \\
\lambda_{kjk} &= \lambda_{kjk} \left( b_{kk} + \lambda_{kk}/2 + \lambda_{jk} \right) / 2 a_k , \\
\lambda_{kkj} &= \lambda_{kkj} \lambda_{kk}/4 a_k , \\
\lambda_{kkj} &= \lambda_{kkj} \left( \lambda_{kk} - \lambda_{kk}/2 \right) / 2 a_k ,
\end{aligned}
\tag{6.57}
\]

we obtain the following system of decoupled Jackiw-like NLSEs

\[
i \frac{\partial \phi_j}{\partial t} + a_j \frac{\partial^2 \phi_j}{\partial x^2} + \eta_j \frac{\partial \phi_j}{\partial x} = 0 , \tag{6.58}
\]

with \( \eta_j = (b_{jj} + \lambda_{jj}) / 2 a_j \).

3) By choosing \( b_{ij} = -\lambda_{ij} \), \( \lambda_{kji} = c_{kji} \lambda_{ij}/2 a_j - \lambda_{kji} \lambda_{ij}/4 a_k \) we obtain the CNLSEs

\[
i \frac{\partial \phi_j}{\partial t} + a_j \frac{\partial^2 \phi_j}{\partial x^2} + \sum_k \eta_{jk} \frac{\partial \phi_j}{\partial x} = 0 , \tag{6.59}
\]

being \( \eta_{jk} = (c_{jk} - a_k \lambda_{jk}/a_j) / 2 a_k \). The nonlinear term in equation (6.59) has been considered in \([10]\).

C. Gauged NLSEs

Let us consider a system of charged particles undergoing to anomalous diffusion and described by the following NLSE

\[
i D_4 \psi + D^2 \psi + \Lambda[\rho, S, A] \psi = 0 , \tag{6.60}
\]

with nonlinearity

\[
\Lambda[\rho, S, A] = \left[ a_1 \nabla \cdot (\nabla S - A) + a_2 \frac{\Delta \rho}{\rho^{1-q}} + a_3 \left( \frac{\nabla \rho}{\rho^{q-3}} \right)^2 \right] + i \frac{D}{2} \frac{\Delta \rho^2}{\rho} , \tag{6.61}
\]

where \( a_1 = q D, a_2 = 2 \alpha \) and \( a_3 = \alpha (2q - 3) \) with \( \alpha, q \) and \( D \) constant parameters. Equation (6.60) must be considered jointly with

\[
\gamma \partial_{\mu} F^{\mu \nu} + g \varepsilon^{\nu \tau \mu} F_{\tau \mu} = j^\nu_{A\nu} , \tag{6.62}
\]

describing the dynamics of the gauge field.

The nonlinearity (6.61) can be obtained from the potential

\[
U[\rho, S, A] = D q \rho^{q-1} \nabla \rho \cdot (\nabla S - A) + \alpha \rho^{2q-3} \frac{\nabla \rho^2}{\rho} , \tag{6.63}
\]
and the charged current \( j_{A\psi} \) is given by

\[
j_{A\psi} = j_{A\psi}^{(0)} + D q \rho^{q-1} \nabla \rho ,
\]

(6.64)

with \( j_{A\psi}^{(0)} = 2 \rho (\nabla S - A) \).

As a consequence the system fulfills the following continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j_{A\psi}^{(0)} + D \Delta \rho = 0 ,
\]

(6.65)

which is a nonlinear Fokker-Planck equation for charged particles.

By performing the transformation (5.19) with

\[
\sigma[\rho] = \frac{D q \rho^{q-1} - 1}{q - 1} ,
\]

(6.66)

where the integration constant has been chosen to avoid the singularity for \( q \to 1 \), equations (6.60) and (6.62) are transformed in

\[
i D_t \phi + D^2 \phi + \beta \rho^{2q-2} \left[ \frac{\Delta \rho}{\rho} + \left( q - \frac{3}{2} \right) \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \phi = 0 ,
\]

(6.67)

with \( \beta = 2 \alpha - q^2 D^2/2 \) and

\[
\gamma \partial_\mu F^{\mu\nu} + g \varepsilon^{\nu\tau\mu} F_{\tau\mu} = j_{A\nu}^{\nu} ,
\]

(6.68)

where \( j_{A\nu}^{\nu} = (\rho, j_{A\psi}^{(0)}) \) with \( j_{A\psi}^{(0)} = 2 \rho (\nabla S - A) \).

Similar equations can be obtained equivalently by means of the transformation

\[
\chi = A - \frac{D q}{2} \rho^{q-2} \nabla \rho ,
\]

(6.69)

\[
\chi_0 = A_0 - \frac{D q}{2} \rho^{q-2} \nabla \cdot j_{A\psi}^{(0)} .
\]

(6.70)

It is worthy to observe that equation (6.60), in the \( q \to 1 \) limit, reduces to the gauged canonical subclass of the Doebner-Goldin family discussed in section 6.1

\[
i D_t \psi + D^2 \psi + \left[ D \nabla \cdot (\nabla S - A) + 2 \alpha \frac{\Delta \rho}{\rho} - \alpha \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \psi
\]

\[
+ \ i \frac{D}{2} \frac{\Delta \rho}{\rho} \psi = 0 ,
\]

(6.71)

which is obtainable from the potential

\[
U[\rho, S, A] = D \nabla \rho \cdot (\nabla S - A) + \alpha \left( \frac{\nabla \rho}{\rho} \right)^2 ,
\]

(6.72)

and the continuity equation (6.65) reduces to the linear Fokker-Planck equation for charged particles

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j_{A\psi}^{(0)} + D \Delta \rho = 0 .
\]

(6.73)

In the same limit the gauge transformation has generator

\[
\sigma[\rho] = \frac{D}{2} \log \rho ,
\]

(6.74)

and reduces equation (6.71) to

\[
i D_t \phi + D^2 \phi + \beta \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \phi = 0 ,
\]

(6.75)

with \( \beta = 2 \alpha - D^2/2 \).
VII. CONCLUSIONS AND COMMENTS

In this paper we have considered a class of canonical NLSEs containing complex nonlinearities and describing U(1)-invariant systems. We have introduced a unitary and nonlinear transformation $\psi \rightarrow \phi$ which reduces the complex nonlinearity in a real one and at the same time transforms the quantum particles current in the standard bilinear form. We have extended the method to U(1)-invariant CNLSEs. For these systems we have generalized the gauge transformation with the purpose to change the initial nonlinearity in another one purely Hermitian. It has been shown that there are many different possibilities to define the generator of the transformation. For any choice we obtain a new CNLSE with a different, but Hermitian, nonlinearity. Finally, we have specialized the method for NLSEs minimally coupled with an Abelian gauged field. We have shown that there are two different ways to reduce the complex nonlinearity in a purely real one: or by a nonlinear unitary transformation on the matter field or, alternatively, by a nonlinear transformation on the gauge field.

In the following let us make some considerations about the transformation studied in the present work. Firstly, the problem of the integrability of a nonlinear evolution equation is one of the most studied topics in mathematical physics. Let us consider the most general U(1)-invariant scalar NLSE in the hydrodynamic representation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (2\rho \nabla S + \mathcal{J}) = 0 \ , \quad (7.1)$$

$$\frac{\partial S}{\partial t} + (\nabla S)^2 + U_q - W = 0 \ . \quad (7.2)$$

In the Calogero picture [8, 11–13], the system of equations (7.1) and (7.2) is $C$-integrable if there exists a transformation of the dependent and/or independent variables: $t \rightarrow T, \ x \rightarrow X, \ \rho \rightarrow R, \ S \rightarrow S$ which changes equations (7.1), (7.2) in

$$\frac{\partial R}{\partial T} + \nabla \cdot (2R \nabla S) = 0 \ , \quad (7.3)$$

$$\frac{\partial S}{\partial T} + (\nabla S)^2 + U_q = 0 \ , \quad (7.4)$$

where $\nabla$ and $U_q$ are the gradient and the quantum potential in the new variables. Equations (7.3) and (7.4) constitute the well known hydrodynamic representation of the standard linear Schrödinger equation.

On the other hand, the transformation on the field $S \rightarrow S$ introduced in this paper, reduces the continuity equation (7.1) in the standard form given by equation (7.3) and can be seen as a first step in the Calogero program.

Secondly, the most general gauge transformation of the kind discussed in the present work can be stated as

$$\psi(t, \ x) \rightarrow \phi(t, \ x) = U[\rho, \ S] \psi(t, \ x) \ , \quad (7.5)$$

which is an infinite dimensional unitary representation of the diffeomorphism group with

$$U[\rho, \ S] = \exp \left( i \omega[\rho, \ S] \right) \ . \quad (7.6)$$

As matter of fact the real generator $\omega[\rho, \ S]$ could be any arbitrary functional depending on the fields $\rho$ and $S$. For instance, in [18] the generator of the transformation has been assumed in

$$\omega(\rho, \ S) = \frac{\gamma(t)}{2} \log \rho + (\lambda(t) - 1)) S + \theta(t, \ x) \ , \quad (7.7)$$

which produces a group of transformations mapping the Doebner-Goldin equation in itself. We observe that the one parameter subclass of this transformation with $\theta(t, \ x) = 0, \ \lambda(t) = 1$ and $\gamma(t) = constant$, coincides with those studied in this work.

Throughout this paper, the generator of the gauge transformation has been chose with the purpose to make real the complex nonlinearity of the NLSE under inspection. Alternatively, nonlinear gauge transformations can be useful generalized with the purpose to classify NLSEs in equivalence classes. Any equation belonging to the same class, in spite of its nonlinearity, is gauge equivalent, by means of equation (7.6), to the others equations belonging to the same class.

For instance, let us consider the following family of NLSEs

$$i \frac{\partial \psi}{\partial t} + \Delta \psi + \Lambda[\rho, \ S] \psi = 0 \ , \quad (7.8)$$
with complex nonlinearity

\[ A[\rho, S] = f_1(\rho) \Delta S + f_2(\rho) \nabla \cdot \nabla S + f_3(\rho) (\nabla \rho)^2 + f_4(\rho) \Delta \rho + \frac{i}{\rho} \nabla \left( f_5(\rho) \nabla \rho \right), \]  

(7.9)

where the expression of the imaginary part guarantees the existence of a continuity equation for \( \rho \).

The quantities \( f_i(\rho) \) are functional parameters fixing the NLSE. Any NLSE belonging to the family of equations \( (7.8) \) can be determined univocally through the vector \( \vec{f} \equiv \{ f_1(\rho), \ldots, f_5(\rho) \} \). By performing a gauge transformation with generator \( \omega(\rho) \) depending only on the field \( \rho \), equation \( (7.8) \) changes in

\[ i \frac{\partial \phi}{\partial t} + \Delta \phi + \tilde{A}[\rho, S] \phi = 0, \]  

(7.10)

with

\[ \tilde{A}[\rho, S] = \tilde{f}_1(\rho) \Delta S + \tilde{f}_2(\rho) \nabla \rho \cdot \nabla S + \tilde{f}_3(\rho) (\nabla \rho)^2 + \tilde{f}_4(\rho) \Delta \rho + \frac{i}{\rho} \nabla \left( \tilde{f}_5(\rho) \nabla \rho \right). \]  

(7.11)

\[ (7.12) \]

What is important is to note that the transformation maintains the same structure in the nonlinearity through the presence of the functional groups \( \Delta S, \nabla S \cdot \nabla \rho, (\nabla \rho)^2 \) and \( \Delta \rho \), whilst the expressions of the new parameters \( \tilde{f}_i(\rho) \) are given by

\[ \tilde{f}_1 = f_1 - 2 \rho \frac{\partial \omega}{\partial \rho}, \]

\[ \tilde{f}_2 = f_2, \]

\[ \tilde{f}_3 = f_3 - \left[ f_2 - \frac{\partial \omega}{\partial \rho} + 2 \frac{\partial f_5}{\partial \rho} + \left( f_1 - 2 \rho \frac{\partial \omega}{\partial \rho} \right) \frac{\partial \omega}{\partial \rho} \right] \frac{\partial \omega}{\partial \rho}, \]

\[ \tilde{f}_4 = f_4 - \left( f_1 + 2 f_5 - 2 \rho \frac{\partial \omega}{\partial \rho} \right) \frac{\partial \omega}{\partial \rho}, \]

\[ \tilde{f}_5 = f_5 - \rho \frac{\partial \omega}{\partial \rho}. \]  

(7.13)

By eliminating \( \omega(\rho) \) among these equations we obtain a set of gauge invariants relations

\[ \tilde{f}_1 - f_1 = 2 \left( \tilde{f}_5 - f_5 \right), \]

\[ \tilde{f}_2 - f_2 = \frac{1}{2} \left[ \left( \frac{\partial f_5}{\partial \rho} + f_1 \frac{\partial \omega}{\partial \rho} \right) \left( \tilde{f}_5 - f_5 \right) \right], \]

\[ \tilde{f}_4 - f_4 = \frac{1}{2} \left( f_1 + 2 \tilde{f}_5 \right) \left( \tilde{f}_5 - f_5 \right). \]  

(7.14)

Given two NLSEs belonging to the family \( \{7.8\} \), labeled by the respective vectors \( \vec{f} \equiv \{ f_1(\rho), \ldots, f_5(\rho) \} \) and \( \vec{f}' \equiv \{ \tilde{f}_1(\rho), \ldots, \tilde{f}_5(\rho) \} \), if the functionals \( f_i(\rho) \) and \( \tilde{f}_i(\rho) \) fulfil the relations \( \{7.14\} \), the two NLSEs are gauge equivalents since there exist a generator \( \omega(\rho) \) such that, by means of equation \( \{7.5\} \), transforms the first NLSE, labeled by the vector \( \vec{f} \), in the second NLSE labeled by the vector \( \vec{f}' \).

In particular, observing that the linear Schrödinger equation is represented by the vector \( \vec{f} \equiv \{ 0, 0, 0, 0, 0 \} \), it follows that any NLSE fulfilling the relations

\[ f_1 = 2 f_3, \]

\[ f_2 = 0, \]

\[ f_3 = \frac{f_1}{\rho} \left( 2 \frac{\partial f_5}{\partial \rho} + \frac{f_1}{\rho} \right), \]

\[ f_4 = 2 \frac{f_1^2}{\rho}, \]  

(7.15)
is gauge equivalent to the linear Schrödinger equation. This sub-family can be linearizable by means of the transformation (7.5) with generator \( \omega(\rho) = \int \psi^*(\rho') \psi(\rho') d\rho' \). In this sense, equations (7.15) define the subclass of the family of equations (7.8) which are \( C \)-integrable.

In conclusion, we have shown that the transformation introduced in the present work allows us to deal, in a unifying scheme, different NLSEs already known in literature, obtaining in a systematic way the transformations introduced by various authors.

A natural continuation of this work could be performed in several ways:

1) Extending the method to the case of NLSEs coupled with non Abelian gauge fields, which are relevant, for instance, in the study of heavy-quark particle systems.

2) Extending the method to relativistic nonlinear equations. In this context, in [16], a relativistic generalization of the transformation introduced in [18] has been proposed to generate nonlinear extensions of the Dirac equation.

3) Extending the method to discrete NLSEs, which are particularly relevant in the study of lattice models in condensed matter.

[1] Ablowitz M.J., Benney D.J.: Evolution of multi-phase modes for nonlinear dispersive waves, Stud. Appl. Math. 49, 225–238 (1979).
[2] Aglietti U., Griguolo L., Jackiw R., Pi S.-Y., Seminara D. Anyons and chiral solitons on a line; Phys. Rev. Lett. 77, 4406–4409 (1996).
[3] Agrawal G.P.: Modulation instability induced by cross-phase modulation, Phys. Rev. Lett. 59, 880–883 (1987).
[4] Barashenkov I., Harin A.: Nonrelativistic Chern-Simons theory for the repulsive Bose gas, Phys. Rev. Lett. 72, 1575–1579 (1994).
[5] Berkhoer A.L., Zakharov V.E.: Self excitation of waves with different polarizations in nonlinear media, Zh. Eksp. Teor. Fiz. 58, 903–911 (1970); [Sov. Phys. JETP 31, 486–490 (1970)].
[6] Bialynicki-Birula I., Mycielski J.: Nonlinear wave mechanics, Ann. Phys. (N.Y.) 100, 62–93 (1976).
[7] Bohm D.: A suggested interpretation of the quantum theory in terms of “hidden” variables, Phys. Rev. 85, 166–193 (1951).
[8] Calogero F., Degasperis A., De Lillo S.: The multicomponent Eckhaus equation, J. Phys. A: Math. Gen. 30, 5805–5814 (1997).
[9] Calogero F.: Universal C-integrable nonlinear partial-differential equation in \( n + 1 \) dimensions, J. Math. Phys. 34, 3197–3209 (1993).
[10] Calogero F.: C-integrable nonlinear partial-differential equations in \( n + 1 \) dimensions, J. Math. Phys. 33, 1257–1271 (1992).
[11] Calogero F., Xiaoda J.: C-integrable nonlinear PDES .2, J. Math. Phys. 32, 875–887 (1991).
[12] Calogero F., Xiaoda J.: C-integrable nonlinear PDES .2, J. Math. Phys. 32, 2703–2717 (1991).
[13] Calogero F., De Lillo S.: The Eckhaus Pde: \( i \psi_t + \psi_{xx} + 2 (|\psi|^2) \psi + |\psi|^4 = 0 \), Inv. Problems 3, 633-681 (1987) (Corrigendum), Inv. Problems 4, 571 (1988).
[14] Chen H.H., Lee Y.C., Liu C.S: Integrability of non-linear Hamiltonian-systems by inverse scattering method, Phys. Scr. 20, 490–492 (1979).
[15] Dodonov V.V., Mizrahi S.S.: Generalized nonlinear Doebner-Goldin Schrödinger equation and the relaxation of quantum systems, Physica A 214, 619–628 (1995).
[16] Doebner H.-D., Zhakanov R.: Nonlinear Dirac equations and nonlinear gauge transformations, (2003); arXiv:quant-ph/0304167.
[17] Doebner H.-D., Goldin G.A., Nettermann P.: Properties of nonlinear Schrödinger equations associated with diffeomorphism group-representations, J. Math. Phys. 40, 49 (1999).
[18] Doebner H.-D., Goldin G.A.: On a general nonlinear Schrödinger equation admitting diffusion currents, Phys. Lett. A 162, 397–401 (1992).
[19] Feynmann R.P., Hibbs A.R.: Quantum Mechanics and Path Integrals, McGraw-Hill, New-York, (1965).
[20] Florja´nczyk M., Gagnon L.: Exact-solutions for a higher-order nonlinear Schrödinger equation, Phys. Rev. A 41, 4478–4485 (1990).
[21] Fordy A.P.: Derivative nonlinear Schrödinger equations and hermitian symmetric-spaces, J. Phys. A, Math. Gen. 17, 1235–1245 (1984).
[22] Gedalin M., Scott T.C.: Optical solitary waves in the higher order nonlinear Schrödinger equation , Band Y.B., Phys. Rev. Lett. 78, 448–451 (1997).
[100] Yip S.-K.: Internal vortex structure of a trapped spinor Bose-Einstein condensate, Phys. Rev. Lett. 83, 4677–4681 (1999).