RECIPROCALS OF EXPONENTIAL POLYNOMIALS AND
PERMUTATION ENUMERATION

IRA M. GESSEL

Abstract. We show that the reciprocal of a partial sum of the alternating exponential series
\[
\left( \sum_{n=0}^{2m-1} (-1)^n \frac{x^n}{n!} \right)^{-1}
\]
is the exponential generating function for permutations in which every increasing run has length congruent to 0 or 1 modulo 2m. More generally we study polynomials whose reciprocals are exponential generating functions for permutations whose run lengths are restricted to certain congruence classes, and extend these results to noncommutative symmetric functions that count words with the same restrictions on run lengths.

1. Introduction

Since
\[
\left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]
we might ask if anything interesting can be said about the coefficients of reciprocals of the partial sums of \( \sum_n (-1)^n x^n/n! \). For partial sums with an odd number of terms there are negative coefficients, but for partial sums with an even number of terms the reciprocals have positive coefficients. We will give here a general result on permutation enumeration that as a special case gives a combinatorial interpretation to the numbers \( u_n \) defined by
\[
\left( \sum_{n=0}^{2m-1} (-1)^n \frac{x^n}{n!} \right)^{-1} = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}.
\]
We will see that \( u_n \) is the number of permutations of \( [n] = \{1, 2, \ldots, n\} \) in which every increasing run has length congruent to 0 or 1 modulo 2m. (The increasing runs of a permutation are its maximal increasing consecutive subsequences; thus,
the increasing runs of the permutation 132576 are 13, 257, and 6, with lengths 2, 3 and 1.)

For example, when $m = 2$, we have
\[
\sum_{n=0}^{\infty} u_n \frac{x^n}{n!} = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)^{-1},
\]
and the first few values of $u_n$ are as follows:

| n | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $u_n$ | 1  | 1  | 1  | 1  | 2  | 10 | 50 | 210 | 840 | 4200 | 29400 | 231000 | 1755600 | 13213200 |

More generally, we consider a class of polynomials whose reciprocals, as exponential generating functions, count permutations whose run lengths lie in certain congruence classes.

2. Noncommutative symmetric functions

A simple class of polynomials with this property is given by the following result, which we will prove later.

**Proposition 2.1.** Let $r$ be a positive integer. Then the coefficient of $x^n/n!$ in
\[
\sum_{k=0}^{r-1} \left( \frac{x^{kr}}{(kr)!} - \frac{x^{kr+1}}{(kr+1)!} \right)^{-1}
\]
is the number of permutations of $\{n\}$ in which every increasing run has length congruent to $0, 1, \ldots, r - 1$ modulo $mr$.

Note that if we take the limit as $m \to \infty$ we recover the known result [1, pp. 156–157] that the coefficient of $x^n/n!$ in
\[
\left(1 - x + \frac{x^r}{r!} - \frac{x^{r+1}}{(r+1)!} + \frac{x^{2r}}{(2r)!} - \frac{x^{2r+1}}{(2r+1)!} + \cdots\right)^{-1}
\]
is the number of permutations of $\{n\}$ in which every increasing run has length at most $r - 1$.

Proposition 2.1 is a consequence of a more refined result on noncommutative symmetric functions. We define the complete noncommutative symmetric function $h_n$ by
\[
h_n = \sum_{i_1 \leq \cdots \leq i_n} X_{i_1}X_{i_2} \cdots X_{i_n},
\]
where the $X_i$ are noncommuting variables. (The algebra of noncommutative symmetric functions is generated by the $h_n$. Noncommutative symmetric functions are
studied extensively in [2], though their definitions and notation are different from ours.)

For a composition \( L = (L_1, L_2, \ldots, L_k) \), we define the *ribbon noncommutative symmetric function* \( r_L \), by

\[
r_L = \sum_{L} X_{i_1} X_{i_2} \cdots X_{i_n}
\]

where the sum is over all \((i_1, \ldots, i_n)\) satisfying

\[
i_1 \leq \cdots \leq i_{L_1} > i_{L_1+1} \leq \cdots \leq i_{L_1+L_2} > \cdots > i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n.
\]

The connection between noncommutative symmetric functions and exponential generating functions is given by a well-known homomorphism \( \Psi \) from noncommutative symmetric functions to power series in \( x \): The homomorphism is determined by \( \Psi(h_n) = x^n/n! \), but a more enlightening description of \( \Psi \) is that if \( f \) is a noncommutative symmetric function then the coefficient of \( x^n/n! \) in \( \Psi(f) \) is the coefficient of \( x_1 x_2 \cdots x_n \) in the result of replacing \( X_1, X_2, \ldots \) in \( f \) with commuting variables \( x_1, x_2, \ldots \). It follows that for a composition \( L = (L_1, \ldots, L_k) \) of \( n \), \( \Psi(r_L) = \beta(L)x^n/n! \), where \( \beta(L) \) is the number of permutations of \([n]\) whose sequence of increasing run lengths is \( L \). (See, e.g., [4, Section 4.2].)

We will state our main results for sums of ribbon noncommutative symmetric functions and the corresponding exponential generating functions.

Our main tool is the “run theorem” as stated by Zhuang [7, Theorem 1], which allows us to count words with restrictions on the lengths of their increasing runs. The result was first proved in an equivalent form by Gessel [3, Theorem 5.2]; a closely related result was proved by Jackson and Aleliunas [5, Theorem 4.1].

**Lemma (The run theorem).** Suppose that sequences \( a_n \) and \( w_n \) are related by

\[
\left( \sum_{n=0}^{\infty} a_n x^n \right)^{-1} = \sum_{n=0}^{\infty} w_n x^n,
\]

where \( a_0 = w_0 = 1 \). Then

\[
\left( \sum_{n=0}^{\infty} a_n h_n \right)^{-1} = \sum L w_L r_L
\]

where the sum on the right is over all compositions \( L \), and \( w_L = w_{L_1} \cdots w_{L_k} \) where \( L = (L_1, \ldots, L_k) \).

We are interested in the case in which only finite many of the \( a_n \) are nonzero and each \( w_n \) is either 0 or 1. The next proposition results from applying the run theorem to this case, and then applying the homomorphism \( \Psi \).
Proposition 2.2. Let \( a(x) = a_0 + a_1 x + \cdots + a_d x^d \), where \( a_0 = 1 \), and suppose that every power series coefficient of \( 1/a(x) \) is 0 or 1. Let \( S \) be the set of positive integers \( s \) such that the coefficient of \( x^s \) in \( 1/a(x) \) is 1. Then

\[
\left( \sum_{n=0}^{d} a_n h_n \right)^{-1} = \sum_{L} r_L,
\]

where the sum on the right is over all compositions \( L \) with every part in \( S \), and the coefficient of \( x^n/n! \) in

\[
\left( \sum_{n=0}^{d} a_n \frac{x^n}{n!} \right)^{-1}
\]

is the number of permutations of \([n]\) in which every increasing run length is in \( S \).

A simple application of Proposition 2.2 is the following.

Proposition 2.3. Let \( m \) and \( r \) be positive integers. Then

\[
\left( \sum_{k=0}^{m-1} (h_{kr} - h_{kr+1}) \right)^{-1} = \sum_{L} r_L,
\]

where the sum on the right is over compositions \( L \) in which every part is congruent to 0, 1, \ldots, or \( r-1 \) modulo \( mr \).

Proof. In Proposition 2.2 we take

\[
a(x)^{-1} = \frac{1 + x + x^2 + \cdots + x^{r-1}}{1 - x^r} = \frac{1 - x^r}{(1-x)(1-x^{mr})}\]

Then

\[
a(x) = (1-x) \frac{1-x^{mr}}{1-x^r} = (1-x)(1 + x^r + x^{2r} + \cdots + x^{(m-1)r}) \]

\[
= \sum_{k=0}^{m-1} (x^{kr} - x^{kr+1}).\]

The limiting case \( m \to \infty \) of Proposition 2.3 was given by Gessel [3, Example 3, pp. 51–52]. Results closely related to this limiting case were proved by Jackson and Aleliunas [5, Corollary 7.2] and Yang and Zeilberger [6].

Proposition 2.1 follows directly from Proposition 2.3 by applying the homomorphism \( \Psi \). By the same reasoning, we have a slight generalization.

Proposition 2.4. Let \( b \) and \( r \) be positive integers. Then

\[
\left( \sum_{k=0}^{m-1} (h_{kr} - h_{kr+1}) \right)^{-1} = \sum_{L} r_L,
\]
where the sum is over compositions \( L \) in which every part is congruent to 0, \( b \), . . . , or \((r - 1)b\) modulo \( mrb\).

These results suggest the following problem: Find all finite sequences \( a_1, a_2, \ldots, a_n \) such that in the ribbon expansion
\[
(1 + a_1 h_1 + \cdots + a_n h_n)^{-1} = \sum_L w_L r_L, \tag{3}
\]
each coefficient \( w_L \) is 0 or 1. By the run theorem, this is equivalent to finding all polynomials \( a(x) \) (with constant term 1) such that every coefficient of \( a(x)^{-1} \) is 0 or 1.

First we prove a lemma that gives a necessary condition.

**Lemma 2.5.** Let \( R(x) \) be a rational function with bounded integer power series coefficients. Then \( R(x) = N(x)/(1 - x^m) \) for some polynomial \( N(x) \) and some positive integer \( m \).

**Proof.** Let \( R(x) = \sum_{n=0}^{\infty} r_n x^n \). Then the numbers \( r_n \) satisfy a linear homogeneous recurrence with constant coefficients, so by the pigeonhole principle and the fact that the set \( \{r_0, r_1, r_2, \ldots \} \) is finite, the sequence \( r_0, r_1, \ldots \) must be eventually periodic, and the result follows. \( \square \)

We can now give a necessary and sufficient condition. Let \( \Phi_d(x) \) denote the \( d \)th cyclotomic polynomial. We will need four well-known properties of cyclotomic polynomials:

(i) The polynomials \( \Phi_d(x) \) are irreducible.

(ii) We have the factorization \( x^m - 1 = \prod_{d|m} \Phi_d(x) \).

(iii) Every cyclotomic polynomial has leading coefficient 1.

**Proposition 2.6.** Let \( a(x) \) be a nonconstant polynomial. Then every power series coefficient of \( a(x)^{-1} \) is 0 or 1 if and only if for some integer \( m \)
\[
a(x) = \frac{1 - x^m}{N(x)}
\]
where \( N(x) \) is a product of distinct cyclotomic polynomials, \( N(x) = \Phi_{d_1}(x) \cdots \Phi_{d_s}(x) \), each \( d_i \) divides \( m \), all coefficients of \( N(x) \) are 0 or 1, and the degree of \( N(x) \) is less than \( m \).

**Proof.** Suppose that \( a(x) \) is a nonconstant polynomial such that every power series coefficient of \( a(x)^{-1} \) is 0 or 1. Then by Lemma 2.5, \( a(x) = (1 - x^m)/N(x) \) for some polynomial \( N(x) \).

Since \( a(x) \) is a polynomial, by (i) and (ii), \( N(x) \) must be a product \( \pm \Phi_{d_1}(x) \cdots \Phi_{d_s}(x) \) for distinct divisors \( d_1, \ldots, d_s \) of \( m \).
Since \( a(x) \) is not a constant, \( N(x) \) must have degree less than \( m \). Then
\[
a(x)^{-1} = \frac{N(x)}{1 - x^m} = N(x) + x^m N(x) + x^{2m} N(x) + \cdots,
\]
so since the degree of \( N(x) \) is less than \( m \), the coefficients of \( N(x) \) are all 0 or 1.
Finally, since \( N(x) \) has all nonnegative coefficients, by (iii) we must have the plus sign in \( N(x) = \pm \Phi_{d_1}(x) \cdots \Phi_{d_s}(x) \).

The proof of the converse is easy, and is omitted. \( \square \)

Proposition 2.6 is not completely satisfactory, since it does not give a simple criterion for determining when a product of cyclotomic polynomials has all coefficients 0 and 1, and there is probably no simple criterion.
The simplest examples not included in Proposition 2.4 are
\[
a(x) = 1 - x^2 - x^3 + x^4 + x^5 - x^7 = (1 - x)(1 + x + x^2)(1 - x^2 + x^4),
\]
for which
\[
a(x)^{-1} = \frac{(1 + x^2)(1 + x^3)}{1 - x^{12}} = \frac{1 + x^2 + x^3 + x^5}{1 - x^{12}}
\]
and
\[
a(x) = 1 - x^2 - x^3 + x^5 + x^6 - x^8 = (1 - x)(1 + x)(1 - x^3 + x^6)
\]
for which
\[
a(x)^{-1} = \frac{(1 + x^2 + x^4)(1 + x^3 + x^6)}{1 - x^{18}} = \frac{1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^{10}}{1 - x^{18}}.
\]

Many examples of allowable \( N(x) \), including all of our examples so far, can be obtained from products of the polynomials
\[
1 + x^k + x^{2k} + \cdots + x^{(r-1)k} = \frac{1 - x^{rk}}{1 - x^k} = \prod_{d|k; d\nmid r} \Phi_d(x).
\]

Any such product will have positive coefficients, but there does not seem to be a simple criterion for determining when they are all 0 or 1.

Not all products of cyclotomic polynomials with coefficients 0 or 1 are of this form. For example, we have
\[
\Phi_5(x) \Phi_6(x) = (1 + x + x^2 + x^3 + x^4)(1 - x + x^2) = 1 + x^2 + x^3 + x^4 + x^6,
\]
which gives the polynomial
\[
a(x) = \frac{1 - x^{30}}{1 + x^2 + x^3 + x^4 + x^6}
\]
\[
= 1 - x^2 - x^3 + 2x^5 + x^6 - x^7 - 2x^8 - x^9 + 2x^{10} + 2x^{11} - 2x^{13} - 2x^{14} + x^{15} + 2x^{16} + x^{17} - x^{18} - 2x^{19} + x^{21} + x^{22} - x^{24}
\]
and
\[ a(x)^{-1} = \frac{1 + x^2 + x^3 + x^4 + x^6}{1 - x^{30}}. \]

Combining Proposition 2.2 with Proposition 2.6 gives a more explicit version of Proposition 2.2.

**Theorem 2.7.** Let \( a(x) = a_0 + a_1x + \cdots + a_dx^d \), where \( a_0 = 1 \), and suppose that every power series coefficient of \( 1/a(x) \) is 0 or 1. Then there exists a positive integer \( m \) and a finite set \( T \subseteq \{0, 1, \ldots, m - 1\} \), with \( 0 \in T \), such that
\[ a(x)^{-1} = \sum_{t \in T^*} x^t, \]
where \( T^* \) is the set of nonnegative integers congruent modulo \( m \) to an element of \( T \).

Moreover
\[ \left( \sum_{n=0}^{d} a_n h_n \right)^{-1} = \sum_{L} r_L, \]
where the sum on the right is over all compositions \( L \) with every part in \( T^* \), and the coefficient of \( x^n / n! \) in
\[ \left( \sum_{n=0}^{d} a_n \frac{x^n}{n!} \right)^{-1} \]
is the number of permutations of \( [n] \) in which every increasing run length is in \( T^* \).

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Department of Mathematics, Brandeis University, Waltham, MA 02453-2700

E-mail address: gessel@brandeis.edu