Anderson localization of few interacting bosons in a continuum with speckle disorder

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The disorder-induced localization of few bosons interacting via a contact potential is investigated through the analysis of the level-spacing statistics familiar from random matrix theory. The model we consider is defined in a continuum and describes one-dimensional bosonic atoms exposed to the spatially correlated disorder due to an optical speckle field. First, we identify the speckle-field intensity required to observe, in the single-particle case, the statistics of Anderson localized systems in a computationally and experimentally feasible system size. Then, we analyze the two-body and the three-body systems, exploring a broad interaction range, from the noninteracting limit up to the vicinity of the strongly-interacting Tonks-Girardeau limit. Our main result is that the contact potential does not induce delocalization, indicating that Anderson localization can occur also in interacting few-body systems in a continuum. We also analyze how the ground-state energy evolves as a function of the interaction strength.

I. INTRODUCTION

Since Anderson’s 1958 seminal article [1], it is known that quenched disorder can induce localization of noninteracting quantum particles, determining the absence of transport of any conserved quantity in macroscopic samples. If and when Anderson localization can be stable against inter-particle interactions has been an outstanding open question ever since [2–4]. In recent years, this question has been addressed in innumerable theoretical articles, putting forward the theory of so-called many-body localization [5, 6]. This phenomenon is expected to occur in isolated one-dimensional systems with disorder. Among other properties, it is characterized by the occurrence of perfect insulating behavior at finite temperature, and by the fact that the many-body localized system is unable to act as its own thermal bath, therefore violating the eigenstate thermalization hypothesis. See Refs. [7, 8] for recent reviews. While some previous theoretical predictions on many-body localization, based mostly on perturbative calculations, considered continuous-space models [9, 10], most numerically-exact simulations considered one-dimensional discrete-lattice models within the tight binding formalism. In fact, whether many-body localization can occur in a continuum is still a controversial issue. In Ref. [11], it is claimed that many-body localization can occur even in continuous-space systems if the (non-deterministic) disorder is in the impurity limit, but that it might be unstable if the correlation length of the disorder is finite. Ref. [12], instead, states that many-body localization cannot occur at all in a continuum. On the other hand, the continuous-space simulations of Ref. [13], which considered fermionic atoms in a quasiperiodic (hence, deterministic) potential and were based on time-dependent density functional theory within the adiabatic approximation, displayed one the experimental hallmarks of many-body localization, namely the long-time persistence of an initially imprinted density pattern. This phenomenon has indeed been observed in the cold-atom experiments on many-body localization [14–16]. Resolving this controversy is essential, given that discrete-lattice models are at most a reasonably good low-energy approximation of experimental systems. In this Article, we shed some light on this issue, considering however a few-body setup.

The model we consider is tailored to describe a setup that can be implemented in cold-atom experiments [17–19]. Specifically, it describes bosonic atoms in a one-dimensional continuum, interacting via a repulsive zero-range interaction. The atoms are exposed to the spatially correlated random potential corresponding to the disorder pattern that is generated when an optical speckle field is shone onto the atomic cloud. Due to the higher computational cost of continuous-space models compared to lattice models – for which one could simulate around 10 itinerant particles or, say, 20 or 30 immobile spins – we focus on one-, two-, and three-boson systems. The main goal of our analysis is to verify whether Anderson localization is stable against the repulsive contact inter-particle interaction, meaning that the few-body analogue of a many-body localized phase may in fact occur. The theoretical tool we employ to identify Anderson localized states is the analysis of the energy-level spacing statistics familiar from quantum chaos and random matrix theories [19, 20]. In this framework, one discerns delocalized ergodic states from Anderson localized states by identifying the Wigner-Dyson statistical distribution and the Poisson distribution of the level spacings, respectively. This approach has been commonly adopted in studies on single-particle Anderson localization in discrete lattice models [21, 22], and more recently also in continuous-space (single-particle) models relevant for cold-atoms experiments [23, 24]. Chiefly, this approach has been established as one of the most sound criteria to identify
many-body localized phases \[27\]. In this context, it has been applied to one-dimensional discrete systems, including spin models \[25-31\], spinless fermion models \[27-32\], and recently also to bosonic models \[33-34\]. Here, we apply it to a continuous-space few-boson model.

The first step we take is to determine, in the single particle case, the disorder intensity required to observe the statistics of Anderson localized systems (i.e. the Poisson level-spacing distribution) for a linear system size that is feasible for our computational approach – and, as a matter of fact, also for cold-atom experiments – in a sufficiently broad low energy region of the spectrum. Then, we analyze if and how inter-particle interactions affect the Anderson localization in the two and in the three-particle cases, again analyzing in an energy resolved manner the level-spacing statistics. This is obtained via large-scale matrix diagonalization calculations of the two-body and three-body Hamiltonians represented in the Fock space corresponding to a suitably chosen basis. A broad range of interaction strengths is considered, ranging from the noninteracting limit, up to strong interaction strengths in the broad vicinity of the Tonks-Girardeau limit where the bosons fermionize, meaning that the system can be mapped to a noninteracting fermion model. Furthermore, we characterize how the ground-state energy evolves in the crossover between the noninteracting and the strongly interacting limits.

The rest of the article is organized as follows: in Section 11 we describe the continuous-space model with speckle disorder and the computational approach we adopt, analyzing in particular the convergence of the energy levels as a function of the basis size. Section 111 focuses on a single particle in the speckle disorder, analyzing the spatial structure of the eigenstates and the level-spacing statistics. The results for two-boson and three-boson systems are presented in Section 1111. Our conclusions and the future perspectives are reported in Section 1V.

II. THE HAMILTONIAN

In the general case, the model we consider consists in \(N\) identical bosons of mass \(m\) in a one-dimensional box of size \(L\), with a random external field \(V(x)\) that describes an optical speckle field \[10\]. The Hamiltonian reads,

\[
\mathcal{H} = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + V(x_i) \right) + \sum_{i<j}^{N} v(|x_i - x_j|). \tag{1}
\]

The variables \(x_i\), with the particle label \(i = 1, \ldots, N\), indicate the particle coordinates. Hard-wall boundary conditions are considered, meaning that the wave functions vanish at the system boundaries. \(v(|x_i - x_j|)\) indicates a zero-range two-body interaction potential between particles \(i\) and \(j\), defined as,

\[
v(|x_i - x_j|) = g \delta(|x_i - x_j|). \tag{2}
\]

The coupling parameter \(g\), which fixes the interaction strength, is related to the one-dimensional scattering length, \(a_0\), as \(g = -\hbar^2/(ma_0)\). In this work, we consider a repulsive interaction, \(g \geq 0\). The one-dimensional Hamiltonian \(\mathcal{H}\) accurately describes ultracold gases in tight cigar-shaped traps, and the interaction parameter \(g\) can be tuned either by varying the radial confining strength and/or tuning the three-dimensional scattering length via Feshbach resonances \[33\].

The external field \(V(x)\) describes the potential experienced by alkali atoms exposed to optical speckle fields. Such fields are generated when coherent light passes through a rough (semitransparent) surface. An efficient numerical algorithm to create speckle fields in computer simulations has been described elsewhere \[33, 35\], and we refer the readers interested in more details about the algorithm to those references.

Fully developed speckle fields in large systems are characterized by an exponential probability distribution of the local intensities \(V\), which reads \(P(V) = \exp(-V/V_0)/V_0\) for \(V \geq 0\), and \(P(V) = 0\) for \(V < 0\) \[40\]. Here, \(V_0\) is the average intensity of the field, and coincides with its standard deviation. \(V_0\) is therefore the unique parameter that characterizes the disorder strength.

The two-point spatial correlation function of local intensities of a speckle field depends on the distance \(d\) between two given points and reads \[57\],

\[
\Gamma(d) = \frac{(V(x + d)V(x))}{V_0^2} - 1 = \langle \sin(d\pi/\ell)/(d\pi/\ell) \rangle^2. \tag{3}
\]

Here, the brackets \(\langle \cdots \rangle\) indicate spatial averages. Notice that in a large enough system, the speckle field is self-averaging, meaning that spatial averages can be replaced by averages of local values over many realizations of the speckle field. The length-scale \(\ell\) is related to the inverse of the aperture width of the optical apparatus employed to create the optical speckle field and to focus it onto the atomic cloud. It characterizes the typical distance over which the local intensities loose statistical correlations, or, in other words, the typical size of the speckle grains. In the following, we will use this spatial correlation length as unit of lengths, setting \(\ell = 1\). This length-scale also allows one to define a characteristic energy scale, often referred to as correlation energy, which reads \(E_c = \hbar^2/(m\ell^2)\). This quantity will be used in the following as the unit for energies, unless explicitly stated.

The interaction parameter \(g\) will be expressed in units of \(\hbar^2/(\ell m)\).

The few-body problem is solved by direct diagonalization of the second-quantized many-body Hamiltonian
in a truncated many-body basis. The Hamiltonian is written in second-quantization as the sum of three terms:

$$\hat{H} = \hat{K} + \hat{V} + \hat{\nu}. \quad (4)$$

Each term can be defined using the standard creation and annihilation operators, $\hat{a}_i^\dagger$ and $\hat{a}_i$, that fulfill the bosonic commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$. These operators create or annihilate bosons in the single-particle states, which are taken as the eigenstates of the free particle moving in a one-dimensional box with hard walls. The box size is chosen large enough compared to the spatial correlation length $\ell$, i.e., $\hbar\omega = E_c$. For the speckle potential $V_0 = 50E_c$.

The interaction term reads

$$\hat{\nu} = \frac{g}{2} \sum_{i,j,k,l} v_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l, \quad (10)$$

with

$$v_{ijkl} = \frac{1}{2L} \left( -\delta_{i,j+k+l} + \delta_{i,j-k+l} + \delta_{i,j+k-l} - \delta_{i,j-k-l} \right). \quad (11)$$

These integrals are determined via numerical quadrature based on the composite five-point Bode’s rule, using a sufficiently fine grid so that the residual numerical error due to the discretization is negligible.

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We follow the criterion to truncate the many-body basis discussed in Ref. 12, where its efficiency in terms of computing resources has been highlighted. The many-body basis is built including all states with a kinetic energy equal or smaller than a given threshold $E_{\text{max}}$. The minimal number of single-particle modes $M$ required to include all such many-body states is retained.

Table I reports the analysis of the convergence with the energy truncation parameter $E_{\text{max}}$ for a few representative setups. $D_{MB}$ in this table indicates the number of states in the many-body basis set. Specifically, we consider the ground-state energy of two bosons in the noninteracting case ($g = 0$) and with a relatively strong interaction ($g = 1$). Here the disorder strength is set to $V_0 = 50E_c$. One notices that with the largest basis set the residual truncation error is much smaller than 0.1%. While the truncation effect becomes somewhat larger at higher energies, we consider in this work an energy range where this effect is negligible. An estimate of the accuracy of our numerical procedure can be obtained by considering the case of two interacting bosons trapped in a harmonic potential, which was exactly solved in Ref. 43.

We choose a harmonic oscillator of length $\ell$, which is the typical size of the minima in the speckle potential, within our finite box of size $L = 100\ell/\sqrt{2}$. For an interaction strength $g = 1$ we reproduce the exact results up to the second decimal. This provides a reasonable estimate of the accuracy of our method. Furthermore, we mention here that the results of the analysis of the (ensemble averaged) level-spacing statistics are less sensitive to the truncation error than the individual energy level of a single realization of the speckle field.

| $E_{\text{max}}$ | $M$ | $D_{MB}$ | $E_{GS}(g = 0)$ | $E_{GS}(g = 1)$ | $E_{GS}^\infty (g = 1)$ |
|------------------|-----|----------|-----------------|----------------|------------------------|
| 20               | 142 | 7941     | 7.0266          | 7.7813         | 1.3249                 |
| 40               | 201 | 15889    | 6.8818          | 7.5954         | 1.3191                 |
| 60               | 246 | 23836    | 6.8328          | 7.5403         | 1.3167                 |
| 100              | 318 | 39747    | 6.8309          | 7.5338         | 1.3143                 |
| 120              | 348 | 47697    | 6.8308          | 7.5319         | 1.3136                 |

These integrals are determined via numerical quadrature based on the composite five-point Bode’s rule, using a sufficiently fine grid so that the residual numerical error due to the discretization is negligible.

The interaction term reads

$$\hat{\nu} = \frac{g}{2} \sum_{i,j,k,l} v_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l, \quad (10)$$

with

$$v_{ijkl} = \frac{1}{2L} \left( -\delta_{i,j+k+l} + \delta_{i,j-k+l} + \delta_{i,j+k-l} - \delta_{i,j-k-l} \right). \quad (11)$$

We follow the criterion to truncate the many-body basis discussed in Ref. 12, where its efficiency in terms of computing resources has been highlighted. The many-body basis is built including all states with a kinetic energy equal or smaller than a given threshold $E_{\text{max}}$. The minimal number of single-particle modes $M$ required to include all such many-body states is retained. The energy threshold $E_{\text{max}}$ represents an algorithmic parameter whose role has to be analyzed. In fact, while the computation is exact in the $E_{\text{max}} \to \infty$ limit, a residual truncation error might occur for finite $E_{\text{max}}$ value. Once the number of bosons and energy threshold are fixed we can build the corresponding many-body Fock basis. Then we construct the Hamiltonian in this Fock basis and diagonalize its low energy part by means of the ARPACK implementation of Lanczos method.

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We choose a harmonic oscillator of length $\ell$, which is the typical size of the minima in the speckle potential, within our finite box of size $L = 100\ell/\sqrt{2}$. For an interaction strength $g = 1$ we reproduce the exact results up to the second decimal. This provides a reasonable estimate of the accuracy of our method. Furthermore, we mention here that the results of the analysis of the (ensemble averaged) level-spacing statistics are less sensitive to the truncation error than the individual energy level of a single realization of the speckle field.
In panel (a), the first eigenfunctions of the speckle potential, \( V(x) \), from bottom to top, in order of increasing energy. A realization of the speckle potential with \( V_0 = 50E_c \) is shown in panel (b). The system size is \( L = 100 \ell/\sqrt{2} \).

### III. ANDERSON LOCALIZATION IN THE SINGLE-PARTICLE CASE

In this Article, the occurrence of the Anderson localization phenomenon is inspected by analyzing the statistics on the energy-level spacings. As a preliminary step, we start by analyzing the general features of the single-particle modes. Fig. 1 displays the low-energy wavefunctions for a given realization of the speckle field of intensity \( V_0 = 50E_c \). One observes that their support is limited to a narrow window, much smaller than the total system size \( L \). This is consistent with the Anderson localization phenomenon. In particular, the lowest-energy states are located at the positions where the speckle field has deeper and broader wells (see the speckle profile in panels (b) of Fig. 1). At higher energies the modes are slightly more extended; furthermore, certain high-energy modes occur in the same well where a lower energy mode is located, but with an additional node that ensures the orthogonality condition.

The analysis of the statistical distribution of the spacings between consecutive energy levels allows one to discern Anderson localized states from delocalized ergodic states. Specifically, localized states are associated to the Poisson distribution of the level spacings, while delocalized states are associated to the Wigner-Dyson distribution typical of random matrices. An efficient procedure to identify these two distributions consists in determining the average over a large ensemble of speckle fields of the following ratio of consecutive level spacings [27]:

\[
    r_i = \min \left\{ \frac{E_{i+1} - E_i}{E_i - E_{i-1}}, \frac{E_{i+1} - E_i}{E_i - E_{i-1}} \right\}.
\]

Notice that the ensemble averaging we perform, indicated in the following with \( \langle r \rangle \), is energy resolved, meaning that only states within a narrow energy window are considered. This allows us to address possible scenarios where both localized states and delocalized states occur, but in different sectors of the energy spectrum. The Poisson distribution translates to the ensemble average \( \langle r \rangle \approx 0.38629 \), while the Wigner-Dyson distribution translates to \( \langle r \rangle \approx 0.53070 \) [44].

Since the seminal work of Ref. [43], it is widely accepted that in infinite one-dimensional disordered systems the Anderson localization occurs for any amount of disorder, even if this amount is vanishingly small. However, in finite-size systems the localization length might be comparable to the system size, hindering the observation of the Anderson localization. This effect is particularly relevant if the disorder is weak or if the energy window under consideration is high. Indeed, in both of these cases the localization length is typically long. As a matter of fact, at energies much larger than the average speckle field intensity the single particle modes are
weakly affected by the disorder, so that they approach the modes corresponding to free particles in a box with hard walls. In fact, while in an infinite system the speckle field has no upper bound, in a finite system the peaks of the speckle field might not be sufficiently high to affect high energy particles. It is, therefore, pivotal for our purposes to identify a disorder strength and an energy range where the Anderson localization can be observed in a system size that is feasible for our computational approach. Fig. 2 displays the energy-resolved analysis of the level-spacings statistics for a few representative setups of the optical speckle field. Specifically, panel (a) shows $\langle r \rangle$ versus $E/E_c$ for a fixed system size and different disorder strengths, while panel (b) shows data corresponding to different system sizes at a fixed disorder strength. One observes that, at low energy, the $\langle r \rangle$ values precisely agree with the prediction for the Poisson distribution, indicating that the low-energy states are Anderson localized. However, significant deviations occur at higher energies. We attribute them to the finite-size effect mentioned above. In fact, one observes that for larger system sizes the Poisson distribution result occurs also at higher energies. The latter is consistent with the expectation that in an infinite system the whole energy spectrum would be localized. In the following, we will consider the system size $L = 100\ell/\sqrt{2}$ and the disorder strength $V_0 = 50E_c$, where the $\langle r \rangle$ values precisely correspond to the statistics of Anderson localized systems in a reasonably broad energy range $0 < E \lesssim 100E_c$. Notice that the upper limit is twice as large as the average speckle-field intensity $V_0$.

It is worth pointing out that the linear system size of typical cold-atom experiments performed with optical speckle field is comparable to the system size considered here; it ranges from a few tens to a few hundred times the speckle correlation length. Therefore, this analysis also serves as a guide for experiments on Anderson localization in atomic gases.

While the next section is devoted to systems with $N = 2$ or $N = 3$ interacting bosons, we address here the special case of $N > 1$ noninteracting particles. Clearly, the system properties in this case can be traced back to the single-particle problem. However, as we discuss here, special care has to be taken in order to correctly extract the correct level spacing statistics.

In fact, in certain circumstances, the $N$-boson energy-level spacings in the noninteracting limit take specific, nonrandom values. For a given realization of the speckle potential, we can distinguish two possible scenarios, depicted in the two panels of Fig. 3 depending on the relative distances of the first and of the second single-particle levels from the single-particle ground-state; they are indicated below as $\Delta_1$ and $\Delta_2$, respectively. For the scenario displayed in panel (a) of Fig. 3 where $2\Delta_1 < \Delta_2$, the three lowest-energy eigenstates of the noninteracting $N$-boson system are

$$|E_0\rangle = |N, 0, ..., 0\rangle,$$
$$|E_1\rangle = |N - 1, 1, 0, ..., 0\rangle,$$
$$|E_2\rangle = |N - 2, 2, 0, ..., 0\rangle,$$

and their associated energies are (see Fig. 3)

$$E_0 = NE_{GS},$$
$$E_1 = NE_{GS} + \Delta_1,$$
$$E_2 = NE_{GS} + 2\Delta_1,$$

where $E_{GS}$ is the single-particle ground state energy. In this situation, the value of $r_1$ associated to the lowest energy of the system is

$$r_1 = \frac{E_1 - E_0}{E_2 - E_1} = \frac{\Delta_1}{\Delta_1} = 1.$$

One notices that this does not randomly fluctuate for different speckle-field realizations.

In the second scenario (see panel (b) of Fig. 3), where $2\Delta_1 > \Delta_2$, the three lowest-energy eigenstates of the system are

$$|E_0\rangle = |N, 0, ..., 0\rangle,$$
$$|E_1\rangle = |N - 1, 1, 0, ..., 0\rangle,$$
$$|E_2\rangle = |N - 1, 0, 1, 0, ..., 0\rangle,$$

and their associated energies are

$$E_0 = NE_{GS},$$
$$E_1 = NE_{GS} + \Delta_1,$$
$$E_2 = NE_{GS} + \Delta_2.$$
FIG. 4: Mean value of $r$ as a function of $E/V_0$ distributed in energy computed with and without a randomness filter for the noninteracting two-boson system. The filter removes the values $r_i \geq 0.999$.

FIG. 5: Energies of the ground state (red solid line) and first excited state (blue short-dashed line) of the system of $N = 2$ bosons in the speckle potential of Fig. 4 panel (b), as a function of the interaction strength $g$. This figure is obtained with $M = 636$, which results in a $D_{MB} = 159069$ for an $E_{\text{max}} = 400$.

Therefore, we have

$$r_1 = \frac{E_2 - E_1}{E_1 - E_0} = \frac{\Delta_2 - \Delta_1}{\Delta_1}.$$ (18)

This is a random variable which depends on the level spacings, and one expects it to follow the Poisson (or eventually the Wigner-Dyson) distribution.

If the data emerging from both scenarios are included in the ensemble average, one obtains, in the low-energy regime, $\langle r \rangle$ values with an upward bias, therefore deviating from the Poisson statistics even in setups where the single-particle modes are Anderson localized. This effect, displayed in Fig. 4, for the representative setup with $N = 2$, $L = 100 \ell/\sqrt{2}$ and $V_0 = 50E_c$, should not be associated to a delocalization phenomenon. For this reason, in our calculations with $N > 1$ noninteracting particles we introduce a filter that removes the $r_i$ values which are numerically indistinguishable from $r_i = 1$, i.e. the data corresponding the the first scenario described above. With this filter, the ensemble-averaged $\langle r \rangle$ values agree with the Poisson distribution result within statistical uncertainties (see Fig. 4). As expected, the filter has no effect at moderate to high energies. It is worth emphasizing that this effect occurs only for noninteracting particles. As soon as $g > 0$, the many-body state is a superposition of many basis states; so, the two scenarios described above do not apply, and the $r_i$ values randomly fluctuate for different speckle field realizations.

IV. TWO AND THREE INTERACTING BOSONS

We start the discussion on the interacting few-boson setup with a qualitative analysis of the interaction effect on the ground-state energy. Specifically, we consider $N = 2$ bosons in a speckle field of intensity $V_0 = 50E_c$, in a $L = 100\ell/\sqrt{2}$ box. As discussed in the previous section, in this setup the single-particle modes are Anderson localized in a broad energy-range $0 < E \lesssim 2V_0$.

In the noninteracting limit, the ground state is the Fock-basis state $|2, 0, \ldots, 0\rangle$, and the corresponding energy equals two times the single-particle ground-state energy.

In the first excited state, one boson is promoted to the ground state and to the first excited state of a representative speckle field instance are displayed in Fig. 4 as a function of the interaction parameter $g$. One notices that, while the ground-state energy increases with $g$, the first excited-state energy is essentially unaltered. This is due to the fact that in the excited state the two bosons are localized in well separated wells; therefore, the zero-range interaction has an almost negligible effect.

In the strongly interacting limit, $g \to \infty$, the lowest-energy state is $|1, 1, \ldots, 0\rangle$. This corresponds to the Tonks-Girardeau scenario, where bosons with infinitely-strong zero-range repulsive interaction can be mapped to a system of noninteracting indistinguishable fermions, which occupy different single-particle modes due to the Pauli exclusion principle. Remarkably, the transition between the noninteracting and the Tonks-Girardeau regimes is extremely sharp. This effect is due to the long separation between the two lowest-energy minima for this realization of the speckle potential. For the speckle field instance analyzed in Fig. 5 this sharp crossover — resembling the two-body analog of a quantum phase transition — occurs at $g \approx 2.8$. Beyond this pseudo-critical point the two-boson system is effectively fermionized. This implies that, in the presence of strong disorder, the Tonks-Girardeau physics occurs at strong but finite values of the interaction parameter $g$, as opposed to homogeneous systems where bosons are fermionized only in the $g \to \infty$ limit.
For different speckle field instances, this fermionization transition occurs at different values of the coupling parameter \( g \). Also the energy levels in the noninteracting limit and in the Tonks-Girardeau limit, as well as in the crossover region, randomly fluctuate. In Fig. 6 the average over many realizations of the speckle field of the two-boson ground-state energy is plotted as a function of the interaction parameter \( g \). Here, we consider interaction strengths ranging from the noninteracting limit to the moderately large interaction parameter \( g = 1 \). One notices that this interaction strength is sufficient to shift the ground-state energy away from the noninteracting-limit result, reaching values in fact closer to the Tonks-Girardeau limit than to the noninteracting limit. The same scenario occurs for the \( N = 3 \) boson system, which is analyzed in Fig. 7. This regime of intermediate interaction strength \( g \approx 1 \) is, in fact, the one where one expects to have more pronounced delocalization effects. Indeed, in the strongly-interacting Tonks-Girardeau limit the system properties are again determined by the single-particle modes. Since the latter are Anderson localized for the disorder strength considered here, one expects the many-body system to be localized, too. This type of re-entrant behavior has been observed in the cold-atom experiments on many-body localization [14]. The experimentalists indeed found that in the Tonks-Girardeau limit the system is many-body localized if the corresponding noninteracting system is localized. In the following, we focus on the interaction regime \( 0 \leq g \leq 1 \), where any interesting interaction effect would take place. Stronger interactions require extremely large basis-set sizes, so that it is not computationally feasible for us to perform averages of many realization of the speckle field.

The analysis of the level-spacings statistics for the interacting two-boson system is displayed in Fig. 8. Specifically, we plot the disorder-averaged \( \langle r \rangle \) values as a function of \( E/V_0 \), for different values of the interaction parameter \( g \). The disorder strength \( V_0 \) and the linear system size \( L \) are the ones discussed above and in the previous section. We focus on the low-energy regime \( E \lesssim V_0 \). Accurately computing more energy levels for many speckle-field instances, in particular, at high energies where larger basis sets are required, exceeds our computational resources.

For the computations of Fig. 8 the basis sets includes 23836 states, namely the ones with a kinetic energy less or equal to \( E_{\text{max}} = 60 E_c \); this corresponds to employing \( M = 246 \) single-particle modes. The disorder ensemble includes 985 realizations of the speckle field. It is clear that the \( \langle r \rangle \) values are always consistent with the prediction corresponding to the Poisson distribution of the level spacings, which is associated to Anderson localized systems. The statistical uncertainty is larger in the \( E \to 0 \) limit due to the low density of states in the low energy regime, which reduces the available statistics. The agreement with the Poisson distribution implies that, for the range of coupling constant considered here, the zero-range interaction does not induce delocalization of the two-boson system. It is possible that a two-body mobility edge, separating low-energy localized states from high energies extended states, would occur at higher energies. However, addressing higher energies requires larger computational resources and is beyond the scope of the present article.

The results for the \( N = 3 \) bosons systems are shown in Fig. 9. Here, the basis-set size is 117977, corresponding to the Fock basis states with a kinetic energy less or equal to \( E_{\text{max}} = 12 E_c \), in turn implying the use of \( M = 110 \) single particle modes. The disorder-ensemble includes 250 realizations of the speckle field. We observe that also in the three-boson system Anderson localization is, in the low energy regime and for the coupling parameters...
has been implemented in early cold-atom experiments correlations of the disorder field. This is the setup that patterns, taking into account the structure of the spatial considered here, stable against the effect of zero-range interactions.

V. SUMMARY AND CONCLUSIONS

We have performed a computational investigation to establish if and how zero-range repulsive interactions affect the Anderson localization phenomenon. While previous computational investigations on this critical issue addressed discrete-lattice models, we focused here on a model defined in continuous space. Specifically, we considered a one-dimensional model which describes ultracold atoms exposed to random optical speckle patterns, taking into account the structure of the spatial correlations of the disorder field. This is the setup that has been implemented in early cold-atom experiments on the Anderson localization.

The computational procedure we employed is based on exact-diagonalization calculations, combined with the statistical analysis of the levels-spacings statistics familiar from random matrix theory. This is, in fact, one of the most sound criteria commonly employed to identify localized phases in noninteracting as well as in interacting disordered systems.

As a preliminary step, we identified the speckle-field intensity required to observe the statistics of Anderson localized systems in a finite linear system size that is feasible for our computational approach and for cold-atom experiments. Our main finding is that, if two or three interacting bosons move in such a speckle field, the Anderson localization is stable against zero-range interparticle interactions in a broad range of interactions strengths, ranging from the noninteracting limit, up to moderately strong interactions half way to the Tonks-Girardeau regime. Addressing even stronger interactions is beyond the scope of this article since, on the one hand, it would require larger computational resources and, on the other hand, delocalization effects due to interactions are not expected in this regime since in

![FIG. 8: Distribution in energy of $\langle \mathbf{r} \rangle$ for $N = 2$ bosons in a 1D box with a speckle potential. The numerical results with different interaction strengths $g$, of a contact potential, are compared with the theoretical value that correspond to a Poisson distribution of the energy gaps.](image1)

![FIG. 9: Distribution in energy of $\langle \mathbf{r} \rangle$ for $N = 3$ bosons in a 1D box with a speckle potential for different interaction strengths. The numerical results are compared with the theoretical predictions corresponding to a Poisson distribution of the energy gaps.](image2)
the Tonks-Girardeau limit the system properties are determined by the single-particle modes. Our results are limited to a low-energy regime, of the order of the speckle-field intensity $E \lesssim V_0$, where the accuracy of the diagonalization results is under control. It is possible that at higher energies two-body or three-body mobility edges would occur. We leave this question to future investigations.

Previous studies on the possible occurrence of many-body localization in continuous-space systems have provided contradictory results. The findings reported here establish that in a few-body system Anderson localization can be stable against zero-range interactions in a continuous-space models relevant for cold-atom experiments.

We conclude by formulating two additional interesting questions, that naturally emerge from the previous discussion: i) would finite-range or long-range interactions induce delocalization in an otherwise Anderson localized system? ii) Would a localization-delocalization transition take place if more particles were included? We leave these two question to future investigations.

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