UNIMODULAR ROWS OVER MONOID EXTENSIONS
OF OVERRINGS OF POLYNOMIAL RINGS

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ABSTRACT. Let \( R \) be a commutative Noetherian ring of dimension \( d \) and \( M \) a commutative cancellative torsion-free seminormal monoid. Then (1) Let \( A \) be a ring of type \( R[d, m, n] \) and \( P \) be a projective \( A[M] \)-module of rank \( r \geq \max\{2, d+1\} \). Then the action of \( E(A[M] \oplus P) \) on \( Um(A[M] \oplus P) \) is transitive and (2) Assume \((R, m, K)\) is a regular local ring containing a field \( k \) such that either \( \text{char } k = 0 \) or \( \text{char } k = p \) and \( \text{tr-deg } K/\mathbb{F}_p \geq 1 \). Let \( A \) be a ring of type \( R[d, m, n]^\ast \) and \( f \in R \) be a regular parameter. Then all finitely generated projective modules over \( A[M] \), \( A[M]f \) and \( A[M] \otimes_R R(T) \) are free. When \( M \) is free both results are due to Keshari and Lokhande [10].

1. Introduction. In this paper, rings are commutative Noetherian with unity and modules are finitely generated.

Let \( R \) be a ring of dimension \( d \). A ring \( A \) is called of type \( R[d, m, n] \) if it is an overring of a polynomial ring \( R[X_1, \ldots, X_m] \) given by \( A = R[X_1, \ldots, X_m, f_1(l_1)^{-1}, \ldots, f_n(l_n)^{-1}] \), where \( f_i \in R[T] \) and either \( l_i \) is some indeterminate \( X_{ij} \) for all \( i \) or \( R \) contains a field \( k \) and \( l_i = \sum_{j=1}^m \alpha_{ij} X_j - r_i \), where \( (\alpha_{i1}, \ldots, \alpha_{im}) \in k^m - \{0\} \) and \( r_i \in R \) for all \( i, j \). We say that \( A \) is a ring of type \( R[d, m, n]^\ast \) if we further assume that \( f_i(T) \in k[T] \) and \( r_i \in k \) for all \( i \).

A monoid \( M \) is cancellative if \( ax = ay \) implies \( x = y \) for \( a, x, y \in M \). A cancellative monoid is torsion-free if for \( x, y \in M \) and \( n > 0 \), \( x^n = y^n \) implies \( x = y \).

A projective \( R \)-module \( P \) is cancellative if for any projective \( R \)-module \( Q \), \( P \oplus R^n \simeq Q \oplus R^n \) implies \( P \simeq Q \). Equivalently, by Bhatwadekar ([1], Proposition 2.17), \( \text{Aut}(P \oplus R^n) \) acts transitively on the set \( Um(P \oplus R^n) \) of unimodular elements for all \( n \geq 1 \).

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In [10], Keshari and Lokhande proved that if $A$ is a ring of type $R[d, m, n]$ and $P$ is a projective $A$-module of rank $\geq \max \{2, \dim R + 1\}$, then $E(A \oplus P)$ acts transitively on $Um(A \oplus P)$. This result was proved by Dhorajia and Keshari [4], when $l_i$ is some indeterminate $X_{i_j}$ for all $i$. We extend above result to monoid algebra $A[M]$ as follows.

**Theorem 1.1.** Let $A$ be a ring of type $R[d, m, n]$ and $M$ be a commutative cancellative torsion-free seminormal monoid. Let $P$ be a projective $A[M]$-module of rank $r \geq \max \{2, d + 1\}$. Then $E(A[M] \oplus P)$ acts transitively on $Um(A[M] \oplus P)$. In particular, $P$ is cancellative.

Gabber [6] proved that for a field $k$, all finitely generated projective $k[0, m, n]^*$-modules are free. It was generalized by Keshari and Lokhande [10] for projective $R[d, m, n]^*$-modules, where $(R, m, K)$ is a regular local ring containing a field $k$ such that either $\text{char } k = 0$ or $\text{char } k = p$ and $\text{tr-deg } K/F \geq 1$.

Gubeladze [7] proved that for a principal ideal domain $R$ and a commutative cancellative torsion-free seminormal monoid $M$, all finitely generated projective $R[M]$-modules are free. In developing an algebraic approach to Gubeladze’s proof, Swan [15] extended Gubeladze result to rings of higher dimension via introduction of a class of domains $\mathcal{R}_n$ with $n > 0$ such that if $R \in \mathcal{R}_n$, then

1. the localization of $R$ with respect to any maximal ideal of $R$ belongs to $\mathcal{R}_n$,
2. $R(X) \in \mathcal{R}_n$, where $R(X)$ is the localization of $R[X]$ with respect to the multiplicative set consisting of all monic polynomials in $X$,
3. if $R$ is local, then all projective modules of rank $n$ over $R[X, X^{-1}]$ are free.

Let $\mathcal{P}_n(R)$ denote the isomorphism class of finitely generated projective $R$-modules of constant rank $n$. Swan [15] proved the result: Let $R \in \mathcal{R}_n$ and $M$ be a commutative cancellative torsion-free seminormal monoid with $U$ as the group of units of $M$. If $\mathcal{P}_n(R) \rightarrow \mathcal{P}_n(R[U])$ is onto, then $\mathcal{P}_n(R) \rightarrow \mathcal{P}_n(R[M])$ is onto, i.e., all projective $R[M]$-modules of rank $n$ are extended from $R$.

Let us recall Quillen’s conjecture [13], $Q_n$: If $(A, m)$ is a regular local ring of dimension $n$ and $u \in m \setminus m^2$, then all projective $A_u$-modules are
free. Bhatwadekar and Rao [2] proved that $Q_n$ is true when $R$ is a regular $k$-spot, i.e., when $R$ is the localization of some affine $k$-algebra at a regular prime ideal. More generally, they proved the result: Let $(R, m)$ be a regular $k$-spot with infinite residue field and $f \in m \setminus m^2$ be a regular parameter. If $B$ is one of $R$, $R(T)$ or $R_f$, then projective modules over $B[X_1, \ldots, X_n, Y_1^{\pm1}, \ldots, Y_m^{\pm1}]$ are free.

Swan [15] used the above result and proved that all localizations of regular affine algebras over fields belong to $R_n$ for all $n > 0$. This in conjugation with Popescu’s result ([12], Theorem 3.1) gives an even bigger class of rings $R[d, m, n]^*$ in $R_n$. Our fourth section deals with such class of domains and yields the following

**Theorem 1.2.** Let $A$ be a ring of type $R[d, m, n]^*$, where $(R, m, K)$ is a regular local ring containing a field $k$ such that either $\text{char } k = 0$ or $\text{char } k = p$ and $\text{tr-deg } K/F_p \geq 1$. Let $M$ be a commutative cancellative torsion-free seminormal monoid. Then

1. all projective $A[M]$ and $A[M] \otimes_R R(T)$-modules are free;
2. if $f, g \in R$ form a part of a regular system of parameters, then all projective $A[M]_f$ and $A[M]_{fg}$-modules are free;
3. if $g_1, \ldots, g_t \in R$ form a part of a regular system of parameters, then projective $A[M]_{g_1 \cdots g_t}$-modules of rank $\geq t$ are free.

Though one should note that the above result doesn’t hold for rings of the type $R[d, m, n]$, counterexamples have been provied in [10].

2. Preliminaries. Given a projective $R$-module $P$, $Um(P)$ denotes the set of unimodular elements of $P$ and $E(P \oplus R)$ denotes the subgroup of $\text{Aut}(P \oplus R)$ generated by transvections. See [10] for basic definitions.

Transvections define an action on the set of unimodular elements $Um(P)$. For $p, q \in Um(P)$, the notation $p \sim_R q$ means that they are in the same orbit via the $E(P)$ action. When $P$ is free, the action translates to that of multiplication on the right. The equivalence of $p$ and $q$ is loosely written as $p \sim_R q$, where $P$ is understood.

This paper generalizes the following theorem by Gubeladze ([9], Theorem 1.1)
Theorem 2.1. Let $R$ be a ring of dimension $d$ and $M$ be a commutative cancellative monoid. Then $E_r(R[M])$ acts transitively on $Um_r(R[M])$, if $r \geq \max\{3, d + 2\}$.

The following is a slightly modified version of a lemma due to Lindel ([11], Lemma 1.1) and will follow from ([4], Lemma 3.9).

Lemma 2.2. Let $P$ be a projective $R[d,m,n]$-module of rank $r$. Then there exists an $s \in R$, which satisfies the following

1. $P_s$ is free,
2. there exists $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in P^*$ such that $(\phi_i(p_j)) = \text{diagonal}(s, \ldots, s)$,
3. $sP \subset p_1R + \ldots + p_rR$,
4. the image of $s$ in $R_{\text{red}}$ is a non-zerodivisor,
5. $(0 : sR) = (0 : s^2R)$.

Following is due to Dhorajia and Keshari ([5], Lemma 3.3)

Lemma 2.3. Let $B$ be a ring and $P$ be a projective $B$-module of rank $r$. Let $s \in B$ be as in the above lemma and $B' = \frac{B[x]}{(x^2 - s^2x)}$. Assume that $E_{r+1}(B')$ acts transitively on $Um_{r+1}(B')$. Then given $(b, p) \in Um(B \oplus P, s^2B)$, there exists $\varepsilon \in E(B \oplus P)$ such that $\varepsilon(b, p) = (1, 0)$.

A commutative diagram of rings

\[
\begin{array}{ccc}
R & \xrightarrow{g_1} & R_1 \\
\downarrow{g_2} & & \downarrow{f_1} \\
R_2 & \xrightarrow{f_2} & R'
\end{array}
\]

has Milnor patching property for unimodular rows if for every $r \geq 3$ and $v_1 \in Um_r(R_1)$ such that $f_1(v_1) \sim e_1$, there exists a pullback $v \in R$ such that $g_1(v) \sim R_1 v_1$. 
A Generalized Karoubi square is a commutative square where $R_2 = S^{-1}R$, $R' = g_1(S)^{-1}R_1$ and the map $g_1$ is an analytic epimorphism along $S \subset R$.

Gubeladze ([8], Proposition 9.1) proved that a generalized Karoubi square has Milnor patching property for unimodular rows. This plays a vital role in patching of unimodular elements.

A monoid $M$ is a set with an associative operation $M \times M \to M$ with unity. We will use multiplicative notation for the operations in $M$. Monoids isomorphic to $\mathbb{Z}_r^+$ are called free monoids of rank $r$. One can refer to ([3], Chapter 2) for a detailed read. Given a ring $R$, similar to that of a polynomial algebra, we talk about the monoid algebra $R[M]$, generated as a free $R$-module with basis as elements of $M$ and coefficients in $R$. By $R[d, m, n][M]$, we mean a monoid algebra with coefficients in a ring $A$ of type $R[d, m, n]$, i.e. $A[M]$.

3. Cancellation of Projective Modules.

Proposition 3.1. Let $A$ be a ring of type $R[d, m, n]$ and $M$ be a commutative cancellative monoid. Then $E_r(A[M])$ acts transitively on $Um_r(A[M])$, if $r \geq \max\{3, d+2\}$.

Proof. We use induction on $n$. For $n = 0$, the result follows from Theorem 2.1 by choosing the monoid as $M[X_1, \ldots, X_m] \simeq M \oplus \mathbb{Z}_m^n$.

Assume $n > 0$. For the ring $A$ of the first type where each $l_i$ is a variable $X_i$, we may assume that $l_n = X_m$. Consider the multiplicative subset $S = 1 + \sum_{i=1}^{m-1} f_i R[X_m]$ and write $A_S = R'[d, m - 1, n - 1] = R'[X_1, \ldots, X_{m-1}, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$, where $R' = R[X_m]_{f, S}$. Let $a = (a_1, \ldots, a_r) \in Um_r(A[M])$. As $\dim R' = d$, by induction on $n$, $a \sim_{A_S[M]} e_1 = (1, 0, \ldots, 0)$. Choose an $s \in S$ such that there exists $\sigma' \in E_r(A_s[M])$ with $\sigma'(a) = e_1$.

Consider the following fibre product diagram

$$
\begin{array}{ccc}
C[M] & \longrightarrow & A[M] \\
\downarrow & & \downarrow \\
C_s[M] & \longrightarrow & A_s[M]
\end{array}
$$
where \( C = R[X_1, \ldots, X_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}] \) is of type \( R[d, m, n-1] \) and \( A = C_{f_n} \). As the diagram above has Milnor patching property for unimodular rows, there exists a \( c \in Um_r(C[M]) \) such that \( p(c) \sim a \). By induction on \( n \), \( c \sim e_1 \) and hence their respective images \( a \sim e_1 \).

The proof for the ring of the second type follows through if the following reduction is considered. Let \( l_n = \sum k_i X_i - r \), where \( k_i \in k \) are not all zero and \( r \in R \). Choose a \( \phi \in E_m(k) \), such that \( (k_1, \ldots, k_m) = (0, \ldots, 0, 1) \phi \). By changing the variables \( (X_1, \ldots, X_m) \) to \( \phi(X_1, \ldots, X_m) \), we can assume \( l_n = X_m + r \). Again transforming the variable \( X_m \) using the translation \( X_m \mapsto X_m - r \), we may assume that \( l_n = X_m \). Use the above arguments to complete the proof. \( \square \)

Following is a direct consequence of the above result, proof of which follows in spirit to that of (4, Theorem 3.8)

**Corollary 3.2.** Let \( A \) be a ring of type \( R[d, m, n] \) and \( M \) be a commutative cancellative monoid. Then the canonical map \( \phi_r : GL_r(A[M])/E_r(A[M]) \to K_1(A[M]) \) is surjective for \( r \geq \{2, \dim R + 1\} \).

The following corollary follows directly from Lemma 2.2 by utilizing Proposition 3.1.

**Corollary 3.3.** Let \( A \) be a ring of type \( R[d, m, n] \) and \( M \) be a commutative cancellative torsion-free seminormal monoid. Let \( P \) be a projective \( A[M] \)-module of rank \( r \). Then there exists \( s \in R \) satisfying the properties of Lemma 2.2.

**Proof.** It is enough to show that there exists an \( s \in R \) such that \( P_s \) is free. We can assume that \( R \) is reduced. Let \( S \) be the set of non-zerodivisors of \( R \). Then \( S^{-1}R \) is a direct product of fields. Hence without loss of generality, we may assume that \( S^{-1}R \) is a field. By Gabber ([6], Theorem 2.1), all projective \( S^{-1}A \) modules are free. By Swan ([14], Corollary 1.3), projective modules over \( S^{-1}A[M] \) are extended from \( S^{-1}A \), hence are free. Thus we can choose an \( s \in R \) such that \( P_s \) is free. \( \square \)
Theorem 3.4. Let $A$ be a ring of type $R[d, m, n]$ and $M$ be a commutative cancellative torsion-free seminormal monoid. Let $P$ be a projective $A[M]$-module of rank $r \geq \max\{2, d + 1\}$. Then $E(A[M] \oplus P)$ acts transitively on $Um(A[M] \oplus P)$. In particular, $P$ is cancellative.

Proof. Without loss of generality, we can assume $A$ is reduced with connected spectrum. If $d = 0$, then $P$ is free by Corollary 3.3 and the result follows from Proposition 3.1.

Assume $d > 0$. Let $(a, p) \in Um(A[M] \oplus P)$. By Corollary 3.3 choose a non-zerodivisor $s \in R$ satisfying the hypothesis of the Lemma 2.2 and let “−” denote reduction modulo $s^2A[M]$. Then by induction on $d$, $(\bar{a}, \bar{p}) \sim (1, 0)$. Since an element of $E(A[M] \oplus P)$ can be lifted to an element of $E(A[M] \oplus P)$, we may assume that $(a, p) \in Um(A[M] \oplus P, s^2A[M])$. By Lemma 2.3 $(a, p) \sim (1, 0)$ and hence $E(A[M] \oplus P)$ acts transitively on $Um(A[M] \oplus P)$. □

4. Generalization of Swan’s result. Using techniques similar to that of the previous section, the following can be derived.

Proposition 4.1. Let $R$ be a UFD of dimension 1, $M$ be a commutative cancellative torsion-free seminormal monoid and $A = R[1, m, n]$. Then all projective $A[M]$-modules are free.

Proof. Let $P$ be a projective $A[M]$-module. We will induct on $n$. If $n = 0$, then $A[M] = A[M \oplus \mathbb{Z}_m^+].$ By [7], $P$ is free.

Assume $n > 0$. First assume that $A$ is of the type where all $l_i$ are variables. We can assume $l_n = X_m$. Consider the multiplicative subset $S = 1 + f_nR[X_m]$ and rewrite $A_S = R'[1, m - 1, n - 1] = R'[X_1, \ldots, X_{m-1}, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}, f_n]$, where $R' = R[X_m]_{f_nS}$ is a 1-dimensional UFD. By induction on $n$, $P_S$ is free, choose a $g \in R[X_m]$ such that $P_{1+f_ng}$ is free.

Let $C = R[X_1, \ldots, X_m, f_1(l_1)^{-1}, \ldots, f_{n-1}(l_{n-1})^{-1}]$ be the subring of $A$ of type $R[1, m, n-1]$. Then $A = C_{f_n}$. By Milnor patching, we get $P$ is extended from $C[M]$. By induction on $n$, projective $C[M]$-modules are free. Therefore, $P$ is free.
The proof when \( l_i \) are of the second type follows in a similar fashion to that of Proposition 3.1.

Swan's criterion for a non local ring \( R \) can be condensed and simply put as: A commutative domain \( R \) is an element of the collection \( \mathcal{R}_n \) if all projective modules of rank \( n \) over \( R_m[x, x^{-1}] \) and \( R(t)_n[x, x^{-1}] \) are free, where \( m \in \text{max}(R) \) and \( n \in \text{max}(R(t)) \). The following theorem due to Popescu ([12], Theorem 3.1) helps us visualize this collection better using Theorem 4.5. A ring \( R \) is essentially of finite type over a ring \( S \), if \( R \) is the localization of an affine \( S \)-algebra \( T \) at a multiplicatively closed subset of \( T \).

**Theorem 4.2.** Let \( R \) be a regular local ring containing a field \( k \). Then \( R \) is a filtered inductive limit of regular local rings essentially of finite type over \( \mathbb{Z} \).

**Theorem 4.3.** The class of regular domains containing a field belongs to \( \mathcal{R}_n \) for all \( n > 0 \).

**Proof.** It is enough to show that if \( R \) is a regular local ring containing a field \( k \), then projective \( R[X, X^{-1}] \)-modules are free. If \( P \) is a projective \( R[X, X^{-1}] \)-module, then by Theorem 4.2, we may assume that \( R \) is a regular \( \mathbb{Z} \)-spot and in particular a regular spot over the prime subfield of \( k \). By Swan's result [15], \( P \) is free.

The following theorem can be found in ([15], Theorem 1.2)

**Theorem 4.4.** Let \( R \in \mathcal{R}_n \) and \( M \) be a commutative cancellative torsion-free seminormal monoid with \( U \) as its group of units. Then the following is a patching diagram

\[
\begin{array}{ccc}
\mathcal{P}_n(R) & \longrightarrow & \mathcal{P}_n(R[M]) \\
\downarrow & & \downarrow \\
\mathcal{P}_n(R) & \longrightarrow & \mathcal{P}_n(RU)
\end{array}
\]
This theorem gives us a way to see when projective modules over $R[M]$ are extended from $R$. If $U$ is trivial, then all projective $R[M]$-modules of rank $n$ are extended from $R$. Also, $U$ being torsion free, is a filtered limit of finite rank free abelian groups. This leads to applications, when sufficient information regarding $P_n(R[Z^r])$ is provided.

Most results of [10] proved for the regular ring $B$ can be generalized to $B[M]$, where $M$ is a commutative cancellative torsion-free seminormal monoid. We state some results generalizing the results in Section 4 and 5 of [10].

**Theorem 4.5.** Let $A$ be a ring of type $R[d, m, n]^*$, where $(R, m, K)$ is a regular local ring containing a field $k$ such that either char $k = 0$ or char $k = p$ and tr-deg $K/F_p \geq 1$. Let $M$ be a commutative cancellative torsion-free seminormal monoid. Then

1. all projective $A[M]$ and $A[M] \otimes_R R(T)$-modules are free;
2. if $f, g \in R$ form a part of a regular system of parameters, then all projective $A[M]_f$ and $A[M]_{fg}$-modules are free;
3. if $g_1, \ldots, g_t \in R$ form a part of a regular system of parameters, then projective $A[M]_{g_1,\ldots,g_t}$-modules of rank $\geq t$ are free.

**Proof.** Let $B$ be any one of the rings $A, A \otimes R(T), A_f, A_{fg}, A_{g_1,\ldots,g_t}$. Then $B$ is a regular ring containing a field, hence belongs to $R_n$ for all $n > 0$ by Theorem 4.3. When $M = 0$, we are done by [10]. Let $U$ be the group of units of $M$. If we show that projective $B[U]$-modules are extended from $R$, then by Theorem 4.3 projective $B[M]$-modules are extended from $B$. Hence the conclusion will again follow from [10].

Since $M$ and hence $U$ is torsion-free, $U$ is direct limit of finite rank free abelian groups. Then $B[Z^r]$ is of the type $R[d, m + r, n + r]^*$, the result follows from [10]. □

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