ON CONWAY’S POTENTIAL FUNCTION
FOR COLORED LINKS

BOJU JIANG

ABSTRACT. The Conway potential function (CPF) for colored links is a convenient version of the multi-variable Alexander-Conway polynomial. We give a skein characterization of CPF which is much simpler than the one by Murakami. This work is based on Hartley’s original model of CPF, without the use of computer algebra tools. An interesting by-product is a characterization of the Alexander-Conway polynomial in the realm of knots, not links.

1. INTRODUCTION

A colored link is an oriented link \( L = L_1 \cup \cdots \cup L_\mu \) in \( S^3 \) together with a map \( \{1, \ldots, \mu\} \to \mathbf{T} \) assigning to each component \( L_i \) a color label \( t_i \in \mathbf{T} \), where \( \mathbf{T} \) is a set of color symbols which will also be regarded as independent variables in polynomials, functions, etc. Different components of a link are allowed to share a color. Two colored links \( L \) and \( L' \) are isotopic if there exists an ambient isotopy from \( L \) to \( L' \) preserving the color and orientation of each component.

The Alexander polynomial \( \Delta_L(t_1, \ldots, t_n) \), introduced by Alexander [A1] in 1928, is an invariant for colored links with \( n \)-colors. It is a Laurent polynomial with integer coefficients, with variables in \( \mathbf{T} \). It is defined only up to sign.

The Conway potential function (CPF) of a colored link \( L \) is a well-defined rational function \( \nabla_L(t_1, \ldots, t_n) \) that is related to the Alexander polynomial in the following way:

\[
\nabla_L(t_1, \ldots, t_n) = \begin{cases} 
\frac{\Delta_L(t^2)}{t - t^{-1}}, & \text{if } n = 1; \\
\Delta_L(t_1^2, \ldots, t_n^2), & \text{if } n > 1.
\end{cases}
\]

This equality is used to remove the ambiguity of sign in the Alexander polynomial, to be called the Alexander-Conway polynomial. CPF was introduced by Conway [C2] in 1970, but without an explicit model until 1983 by Hartley [H1]. When two colors \( t_i \) and \( t_j \) merge, simply identify the variables \( t_i \) and \( t_j \) in \( \nabla_L(t_1, \ldots, t_n) \).

Our main result is

Main Theorem. The Conway potential function \( \nabla_L \) is the invariant of colored links determined uniquely by the following five axioms, where \( t_i, t_j, t_k \) are arbitrary

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color symbols, repetition allowed, $i, j, k$ are just shorthand for them in diagrams.

\[(\text{II}) \quad \nabla \left( \begin{array}{c} i \\ j \\ x \end{array} \right) + \nabla \left( \begin{array}{c} j \\ i \\ x \end{array} \right) = \left( t_i t_j + t_i^{-1} t_j^{-1} \right) \cdot \nabla \left( \begin{array}{c} j \\ y \\ \ell \end{array} \right) ; \]

\[(\text{III}) \quad (t_i^{-1} t_j^{-1} - t_i t_j) \left\{ \nabla \left( \begin{array}{c} i \\ j \\ k \end{array} \right) + \nabla \left( \begin{array}{c} j \\ i \\ k \end{array} \right) \right\} + (t_j t_k - t_j^{-1} t_k^{-1}) \left\{ \nabla \left( \begin{array}{c} j \\ k \\ \ell \end{array} \right) + \nabla \left( \begin{array}{c} k \\ j \end{array} \right) \right\} + (t_i t_k^{-1} - t_i^{-1} t_k) \left\{ \nabla \left( \begin{array}{c} k \\ \ell \\ i \end{array} \right) + \nabla \left( \begin{array}{c} \ell \\ k \\ i \end{array} \right) \right\} = 0 ; \]

\[(\text{IO}) \quad \nabla \left( \begin{array}{c} \ell \\ \ell \end{array} \right) = 0 ; \]

\[(\Phi) \quad \nabla \left( \begin{array}{c} i \\ j \end{array} \right) = (t_i - t_i^{-1}) \cdot \nabla \left( \begin{array}{c} i \\ \ell \end{array} \right) ; \]

\[(\text{H}) \quad \nabla \left( \begin{array}{c} \ell \\ \ell \end{array} \right) = 1. \]

This is a new characterization of the Conway potential function by skein relations, much simpler (cf. Remark 4.5) than the one given by Murakami in [M2]. This also provides a pedestrian’s approach to the main conclusions of the paper [M2] which is hard to read. Murakami’s state model for CPF (Theorem 5.3 of that paper) can be validated by checking our axioms in a straightforward way.

Cimasoni’s model of CPF [C1, Theorem] can also be validated by our axioms instead of Murakami’s. In fact, it was this nice geometric model that first led us to local relations such as (III8.1) and (III6.1).

As a by-product of our approach, we also have

**Corollary for Knots.** The Alexander-Conway polynomial $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$ for knots is the invariant of knots determined uniquely by the following three axioms.

\[((\text{II})_K) \quad \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) + \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) = \left( t + t^{-1} \right) \cdot \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) ; \]

\[((\text{III})_K) \quad \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) + \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) = \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) + \Delta \left( \begin{array}{c} \ell \\ \ell \end{array} \right) ; \]
Note that the relation (III) is different from Conway’s four-term relation (cf. Corollary 2.3).

It is well known that the uncolored Alexander-Conway polynomial is characterized by the classical Conway relation

\[(C) \quad \Delta \left( \begin{array}{c}
\end{array} \right) - \Delta \left( \begin{array}{c}
\end{array} \right) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta \left( \begin{array}{c}
\end{array} \right)\]

and (O). The two sides of (C) have different number of components. The point here is that we now have a characterization within the realm of knots.

The structure of the paper is as follows. New skein relations are gathered in Section 2. The proof in Section 3 are based on the original model of CPF in Hartley [H1]. To prepare for the proof of the Main Theorem in Section 6, we review the language of colored braids in Section 4, and prove a key lemma in Section 5. Some final remarks are given in Section 7.

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### 2. Some ‘triangular’ skein relations for CPF

In the theorem below we consider link projections that are identical except within a disk where they differ in a specific way. The proof is postponed to the next section, after a discussion of corollaries.

**Theorem 2.1.** The Conway potential function satisfies the following skein relations, where \(t_i, t_j, t_k\) are arbitrary color symbols, repetition allowed, \(i, j, k\) are just shorthand for them in diagrams.

\[(\text{III}_{8,1}) \quad t_i t_j \cdot \nabla \left( \begin{array}{c}
\end{array} \right) - t_i^{-1} t_j^{-1} \cdot \nabla \left( \begin{array}{c}
\end{array} \right) + t_j^{-1} t_k^{-1} \cdot \nabla \left( \begin{array}{c}
\end{array} \right) - t_j t_k \cdot \nabla \left( \begin{array}{c}
\end{array} \right) + t_i^{-1} t_k \cdot \nabla \left( \begin{array}{c}
\end{array} \right) - t_i t_k^{-1} \cdot \nabla \left( \begin{array}{c}
\end{array} \right) + \nabla \left( \begin{array}{c}
\end{array} \right) - \nabla \left( \begin{array}{c}
\end{array} \right) = 0.\]
\[(III_{8.2})\]

\[t_i^{-1}t_j^{-1} \cdot \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} - t_it_j \cdot \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + t_jt_k \cdot \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} - t_j^{-1}t_k^{-1} \cdot \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + t_i t_k^{-1} \cdot \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} - t_i^{-1}t_k \cdot \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} - \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} = 0.\]

**Corollary 2.2.** The Conway potential function satisfies the following skein relations:

\[(III_{6.1})\]

\[\begin{align*}
( t_i^{-1}t_j^{-1} - t_it_j ) & \left\{ \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} \right\} \\
+ ( t_jt_k - t_j^{-1}t_k^{-1} ) & \left\{ \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} \right\} \\
+ ( t_it_k^{-1} - t_i^{-1}t_k ) & \left\{ \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} \right\} = 0.
\end{align*}\]

\[(III_{6.2})\]

\[\begin{align*}
( t_it_j^{-1} - t_i^{-1}t_j ) & \left\{ \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} \right\} \\
+ ( t_j^{-1}t_k - t_j^{-1}t_k ) & \left\{ \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} \right\} \\
+ ( t_k t_i^{-1} - t_k^{-1}t_i ) & \left\{ \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} + \nabla \begin{pmatrix} i & j \\ k \end{pmatrix} \right\} = 0.
\end{align*}\]

**Proof.** \((III_{6.1})\) is the difference of \((III_{8.1})\) and \((III_{8.2})\). \((III_{6.2})\) is derived from \((III_{6.1})\), see Example 4.6 as an application of colored braid notations. □

Conway’s four-term relation also comes as a corollary.
Corollary 2.3 (Conway). The Conway potential function satisfies the skein relation

\[(\text{III}_4) \quad \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) = \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right). \]

Proof. Apply a left-right flip to the relation (III\(_{8,2}\)), and interchange the labels \(i\) and \(k\) so that \(i\) appears on the left side again. Comparing the new look of (III\(_{8,2}\)) with (III\(_{8,1}\)), we see many terms are identical by the third Reidemeister move. Their difference gives (III\(_4\)). □

In another direction, what if the \(j\)-arrows in the diagrams go upward?

Corollary 2.4. The Conway potential function satisfies the following skein relations:

\[(\text{III}'\_6,1) \quad (t_i t_j^{-1} - t_j t_i^{-1}) \left\{ \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) \right\} \
+ (t_j t_k^{-1} - t_k t_j^{-1}) \left\{ \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) \right\} \
+ (t_k t_i^{-1} - t_i t_k^{-1}) \left\{ \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) \right\} = 0.\]

\[(\text{III}'\_6,2) \quad (t_i t_j - t_i^{-1} t_j^{-1}) \left\{ \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) \right\} \
+ (t_j^{-1} t_k^{-1} - t_j t_k) \left\{ \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) \right\} \
+ (t_k t_i^{-1} - t_i t_k^{-1}) \left\{ \nabla \left( \begin{array}{c} i \downarrow \; j \downarrow \; k \\ \downarrow \big/ \end{array} \right) + \nabla \left( \begin{array}{c} i \downarrow \; k \downarrow \; j \\ \downarrow \big/ \end{array} \right) \right\} = 0.\]

They are equivalent to the skein relations (III\(_{6,2}\)) and (III\(_{6,1}\)), respectively.

Proof. Observe that when inserted into the frame in Figure 1, the tangles in (III\(_{6,1}'\)) become the tangles in (III\(_{6,2}\)), respectively. For example, Figure 2 shows the case of the last tangle in both relations. Hence (III\(_{6,1}'\)) is equivalent to (III\(_{6,2}\)).

By reversing the \(j\)-arrows in the above argument (for example in Figure 2) we also see that (III\(_{6,1}\)) is equivalent to (III\(_{6,2}'\)). □
The skein relation (III'\textsubscript{6,1}) looks remarkably symmetric.

3. Proof of Theorem 2.1

In this section we assume the reader is familiar with the terminology and notation in Sections 2 and 3 of [H1]. Since we use indices $i, j, k$ for colors, we shall use indices $a, b, c$ for crossing points/generating arcs in a link projection, as well as for rows/columns in a matrix.

Let $K_r, r = 1, \ldots, 8$, denote the eight colored links appearing in the relation (III\textsubscript{8,1}), in that order. Number the crossing points $P_1, \ldots, P_m$, then number the generating arcs so that $u_a$ is the generating arc exiting from $P_a$. Outside of the depicted region, the crossing points and generating arcs should be numbered and colored in the same way for all $K_r$.

Colors are marked by symbols $\{t_1, \ldots, t_n\}$. A coloring of a link is a map $\theta : \{u_1, \ldots, u_m\} \to \{t_1, \ldots, t_n\}$ which takes $u_a$ to $t_i$ if the generating arc $u_a$ has the $i$-th color. We use the same notation for the induced linear map $\theta : \mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}] \to \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

Focus on any one link $K_r$. Around each crossing point $P_a$ consider a small anticlockwise circle, starting at a point point to the right of both over- and under-crossing arcs at $P_a$. This gives a Wirtinger relator $R_a$, a word in the free group with basis $\{u_1, \ldots, u_m\}$. Let $M_r$ be the $m \times m$ Jacobian matrix (in the sense of free differential calculus) $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_m^{\pm 1}] \to \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

For each $i$, let $\kappa_i$ denote the total “curvature” of components of $K_r$ with $i$-th color. Let $\nu'_{i}$ and $\nu''_{i,r}$ be the number of crossing points, outside and inside of the depicted region, respectively, at which the overcrossing arc has $i$-th color in the link $K_r$. Let $\mu_{i,r} := \kappa_i - \nu'_i - \nu''_{i,r}$. Let $\phi : \mathbb{Q}(t_1, \ldots, t_n) \to \mathbb{Q}(t_1, \ldots, t_n)$ be the substitution map sending each $t_i$ to $t_i^2$. Then the Conway potential function of $K_r$
is

\[ \nabla_r(t_1, \ldots, t_n) = D_r(t_1, \ldots, t_n) \phi \cdot t_1^{\mu_{1,r}} \cdots t_n^{\mu_{n,r}}. \]

In Figure 3 we give pictures in the style of [H1], where \( t_i, t_j, t_k \) (repetition allowed) are color labels of the strings \( u_1, u_2, u_3 \), respectively. For \( 1 \leq r \leq 6 \) we add an anticlockwise curl as shown, increasing the number of crossing points to \( m + 1 \). Clearly, adding such a curl does not change \( \mu_{i,r} \) and \( \det(M_r^{(ab)}) \). We can also assume \( a, b > 6 \) (by adding more curls elsewhere if necessary).

In the following computation, the column vectors \( c_1, \ldots, c_6 \) and the \((m - 4) \times (m - 7)\) matrix \( * \) are common to all \( r \). By Laplace expansion along the first four (if \( r \leq 6 \)) or three (if \( r = 7, 8 \)) rows, \( \det(M_r^{(ab)}) \) is a linear combination of \((m - 4) \times (m - 4)\) minors \( S_{\alpha\beta\gamma} := \| c_\alpha c_\beta c_\gamma \| * \) for \( 1 \leq \alpha < \beta < \gamma \leq 6 \). The coefficients are \( 4 \times 4 \) or \( 3 \times 3 \) minors, readily read off from the matrix \( M_r^{(ab)} \). See Table 1. We will use the shorthand \( \delta_r := \det(M_r^{(ab)}) \).

For \( K_1 \), we have Wirtinger relators and Jacobian

\[
R_0 = u_0 u_4 u_5^{-1} u_4^{-1}, \quad R_1 = u_1 u_4^{-1}, \quad R_2 = u_3 u_2 u_3^{-1} u_0^{-1}, \quad R_3 = u_3 u_1 u_6^{-1} u_1^{-1},
\]

\[
\det(M_1^{(ab)}) = \begin{vmatrix}
1 & 0 & 0 & 0 & t_j - 1 & -t_i & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & t_k & 1 - t_j & 0 & 0 & 0 & 0 \\
0 & t_k - 1 & 0 & 1 & 0 & 0 & -t_i & 0 \\
0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & *
\end{vmatrix}.
\]

For \( K_2 \), we have Wirtinger relators

\[
R_0 = u_5 u_0 u_5^{-1} u_4^{-1}, \quad R_1 = u_6 u_4 u_6^{-1} u_0^{-1}, \quad R_2 = u_2 u_5^{-1}, \quad R_3 = u_3 u_2 u_6^{-1} u_2^{-1},
\]
$$\det(M_2^{(ab)}) = \begin{vmatrix}
  t_j & 0 & 0 & 0 & -1 & 1 - t_i & 0 & 0 \\
  -1 & t_k & 0 & 0 & 0 & 0 & 1 - t_i & 0 \\
  0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
  0 & 0 & t_k - 1 & 1 & 0 & 0 & -t_j & 0 \\
  0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & * 
\end{vmatrix}.$$ 

For $K_3$, we have Wirtinger relations 

$$R_0 = u_0 u_1 u_6^{-1} u_1^{-1}, \quad R_1 = u_5 u_1 u_5^{-1} u_4^{-1}, \quad R_2 = u_2 u_5^{-1}, \quad R_3 = u_3 u_2 u_0^{-1} u_2^{-1},$$

$$\det(M_3^{(ab)}) = \begin{vmatrix}
  1 & t_k - 1 & 0 & 0 & 0 & 0 & -t_i & 0 \\
  0 & t_j & 0 & 0 & -1 & 1 - t_i & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
  -t_j & 0 & t_k - 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & * 
\end{vmatrix}.$$ 

For $K_4$, we have Wirtinger relations 

$$R_0 = u_0 u_4 u_5^{-1} u_4^{-1}, \quad R_1 = u_6 u_1 u_6^{-1} u_4^{-1}, \quad R_2 = u_3 u_2 u_3^{-1} u_0^{-1}, \quad R_3 = u_3 u_6^{-1},$$

$$\det(M_4^{(ab)}) = \begin{vmatrix}
  1 & 0 & 0 & 0 & t_j - 1 & -t_i & 0 & 0 \\
  0 & t_k & 0 & 0 & -1 & 0 & 1 - t_i & 0 \\
  -1 & 0 & t_k & 1 - t_j & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
  0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & * 
\end{vmatrix}.$$ 

For $K_5$, we have Wirtinger relations 

$$R_0 = u_5 u_0 u_5^{-1} u_4^{-1}, \quad R_1 = u_6 u_1 u_6^{-1} u_0^{-1}, \quad R_2 = u_3 u_2 u_3^{-1} u_5^{-1}, \quad R_3 = u_3 u_6^{-1},$$

$$\det(M_5^{(ab)}) = \begin{vmatrix}
  t_j & 0 & 0 & 0 & -1 & 1 - t_i & 0 & 0 \\
  -1 & t_k & 0 & 0 & 0 & 0 & 1 - t_i & 0 \\
  0 & 0 & t_k & 1 - t_j & 0 & -1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
  0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & * 
\end{vmatrix}.$$ 

For $K_6$, we have Wirtinger relations and Jacobian 

$$R_0 = u_0 u_1 u_6^{-1} u_1^{-1}, \quad R_1 = u_1 u_4^{-1}, \quad R_2 = u_2 u_4 u_5^{-1} u_4^{-1}, \quad R_3 = u_3 u_2 u_0^{-1} u_2^{-1},$$

$$\det(M_6^{(ab)}) = \begin{vmatrix}
  1 & t_k - 1 & 0 & 0 & 0 & 0 & -t_i & 0 \\
  0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & t_j - 1 & -t_i & 0 & 0 \\
  -t_j & 0 & t_k - 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & * 
\end{vmatrix}.$$ 

For $K_7$, we have Wirtinger relations and Jacobian 

$$R_1 = u_6 u_1 u_6^{-1} u_4^{-1}, \quad R_2 = u_2 u_4 u_5^{-1} u_4^{-1}, \quad R_3 = u_3 u_2 u_6^{-1} u_2^{-1},$$
\[
\det(M_{7}^{(ab)}) = \begin{vmatrix}
  t_k & 0 & 0 & -1 & 0 & 1 - t_i & 0 \\
  0 & 1 & 0 & t_j - 1 & -t_i & 0 & 0 \\
  0 & t_k - 1 & 1 & 0 & 0 & -t_j & 0 \\
  c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & *
\end{vmatrix}.
\]

For \(K_8\), we have Wirtinger relators and Jacobian
\[
R_1 = u_5 u_1 u_5^{-1} u_4^{-1}, \quad R_2 = u_3 u_2 u_3^{-1} u_5^{-1}, \quad R_3 = u_3 u_1 u_6^{-1} u_1^{-1},
\]
\[
\det(M_{8}^{(ab)}) = \begin{vmatrix}
  t_j & 0 & 0 & -1 & 1 - t_i & 0 & 0 \\
  0 & t_k & 1 - t_j & 0 & -1 & 0 & 0 \\
  t_k - 1 & 1 & 0 & 0 & 0 & -t_j & 0 \\
  c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & *
\end{vmatrix}.
\]

Let
\[
\varepsilon := t_1^{\kappa_1} \cdots t_n^{\kappa_n} \cdot ([(-1)^{a+b}/w^\theta_a(u^\theta_b - 1)])^\phi.
\]

Then \(\nabla_r = \varepsilon \cdot t_1^{-\kappa_1} \cdots t_n^{-\kappa_n} \cdot \delta^\phi_r\). Namely,
\[
\nabla_1 = \varepsilon \cdot t_1^{-2} t_k^{-1} \delta^\phi_1, \quad \nabla_2 = \varepsilon \cdot t_j^{-2} t_k^{-1} \delta^\phi_2, \\
\nabla_3 = \varepsilon \cdot t_1^{-1} t_j^{-1} \delta^\phi_3, \quad \nabla_4 = \varepsilon \cdot t_k^{-1} t_j^{-1} \delta^\phi_4, \\
\nabla_5 = \varepsilon \cdot t_1^{-1} t_k^{-1} \delta^\phi_5, \quad \nabla_6 = \varepsilon \cdot t_j^{-1} t_k^{-1} \delta^\phi_6, \\
\nabla_7 = \varepsilon \cdot t_1^{-1} t_j^{-1} \delta^\phi_7, \quad \nabla_8 = \varepsilon \cdot t_k^{-1} t_j^{-1} \delta^\phi_8.
\]

**Proof of Theorem 2.1.** Denote the left hand side of the relation (III_{8,1}) and (III_{8,2}) by \(F_1\) and \(F_2\), respectively. Then
\[
t_{i,j} t_k : F_1 = t_i^2 t_j^2 t_k \nabla_1 - t_k \nabla_2 + t_i \nabla_3 - t_i t_j^2 t_k \nabla_4 \\
+ t_j^2 t_k \nabla_5 - t_i^2 t_j \nabla_6 + t_i t_j t_k \nabla_7 - t_i t_j t_k \nabla_8 \\
= \varepsilon \cdot [t_j \delta_1 - t_j^{-1} \delta_2 + t_j^{-1} \delta_3 - t_j \delta_4 + \delta_5 - \delta_6 + \delta_7 - \delta_8]^\phi,
\]
\[
t_{i,j} t_k : F_2 = t_k \nabla_1 - t_i^2 t_j^2 t_k \nabla_2 + t_i^2 t_j^2 \nabla_3 - t_i \nabla_4 \\
+ t_i^2 t_j \nabla_5 - t_j^2 t_k \nabla_6 + t_i t_j t_k \nabla_7 - t_i t_j t_k \nabla_8 \\
= \varepsilon \cdot [t_i^{-1} \delta_1 - t_i \delta_2 + t_k \delta_3 - t_i^{-1} \delta_4 + t_i t_k^2 \delta_5 - t_i^{-1} t_k \delta_6 + \delta_7 - \delta_8]^\phi.
\]

From Table 1, it is straightforward to check that the two expressions enclosed in brackets equal to 0. Hence \(F_1 = F_2 = 0\).

\[\square\]

### 4. Colored braids

For braids, we use the following conventions: Braids are drawn from top to bottom. The strands of a braid are numbered at the top of the braid, from left to right. Each \(n\)-braid \(\beta\) defines a permutation of \(\{1, \ldots, n\}\), \(i \mapsto i^\beta\) where \(i^\beta\) is the position of the \(i\)-th strand at the bottom of \(\beta\). The product \(\beta_1 \cdot \beta_2\) of two \(n\)-braids is obtained by drawing \(\beta_2\) below \(\beta_1\). The set \(B_n\) of all \(n\)-braids forms a group under this multiplication, with standard generators \(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\).
|   | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $\delta_5$ | $\delta_6$ | $\delta_7$ | $\delta_8$ |
|---|---|---|---|---|---|---|---|---|
| $S_{123}$ | $t^2_i$ | $t_j$ | $t_it_j$ | $t_i$ | 1 | $t^2_it_j$ | $t_it_j$ | $t_i$ |
| $S_{124}$ | 0 | $t_j$ | 0 | $t_i$ | 1 | $0$ | $t_i(1-t_i)$ | $(1-t_i)$ |
| $S_{125}$ | $t_i$ | 0 | 0 | $t_i$ | 1 | 0 | $(t_i-1)$ | $(1-t_j)$ |
| $S_{126}$ | $t_i$ | 1 | 1 | $t_i$ | 1 | $t_i$ | 1 |
| $S_{134}$ | 0 | $t_jt_k$ | 0 | $t_k$ | 0 | $0$ | $t_i(t_k-1)$ | $(t_i-1)$ |
| $S_{135}$ | $-t_it_k$ | $-t_j$ | $-t_it_j$ | $-t_k$ | $-t_it_j$ | 0 | $(t_k-1)$ | $(t_k-1)-t_j$ |
| $S_{136}$ | 0 | $1-t_k$ | 0 | 0 | $t_i$ | 0 |
| $S_{145}$ | 0 | $t_j$ | 0 | $t_jt_k$ | 0 | $t_i-1$ | 0 |
| $S_{146}$ | 0 | $1-t_i$ | 0 | 0 | $t_k$ | 0 |
| $S_{156}$ | $t_k$ | 1 | 1 | $t_k$ | 1 | $t_k$ |
| $S_{234}$ | $t^2_i$ | $t^2_it_k$ | $t_it_j^2$ | $t_it_k$ | $t_jt_k$ | $t^2_it_j$ | $t_it_jt_k$ | $t_it_j$ |
| $S_{235}$ | $t_i$ | 0 | 0 | $t_jt_k$ | 0 | $t_it_j$ | 0 |
| $S_{236}$ | $t_i$ | 0 | $t_j$ | 0 | 0 | $t_it_j$ | 0 |
| $S_{245}$ | $t_i$ | 0 | 0 | $t_jt_k$ | 0 | $t_it_j$ | 0 |
| $S_{246}$ | $-t_i$ | $-t_jt_k$ | $-t_j$ | $-t_it_k$ | $-t_it_k$ | $-t_i$ | 0 |
| $S_{256}$ | $t_k$ | 0 | 0 | $t_k$ | 0 | $1-t_j$ | $(1-t_k)$ | $(1-t_j)$ |
| $S_{345}$ | $t_it_k$ | $t^2_it_k$ | $t^2_it_j$ | $t^2_it_k$ | $t_it_j$ | 0 | 0 |
| $S_{346}$ | 0 | $t_jt_k$ | 0 | $t_jt_k$ | 0 | 0 | 0 |
| $S_{356}$ | $t_k$ | 0 | 0 | $1-t_k$ | 0 | $1-t_k$ | 0 |
| $S_{456}$ | $t_k$ | $t_j$ | $t^2_j$ | $t^2_jt_k$ | 0 | $t_k$ | 0 |

Table 1. Coefficient of $S_{\alpha\beta\gamma}$ in $\delta_r := \det(M^{(ab)}_r)$
It is well known that links can be presented as closed braids. The closure of a braid $\beta \in B_n$ will be denoted $\hat{\beta}$. Two braids (possibly with different numbers of strands) have isotopic closures if and only if they can be related by a finite sequence of two types of moves:

1. Conjugacy $\beta \leftrightarrow \beta'$ in a braid group;
2. Markov move $\beta \in B_n \leftrightarrow \beta \sigma_n^\pm \in B_{n+1}$.

Let $T$ be the set of color symbols. A colored $n$-braid $\beta(t_1, \ldots, t_n)$ is an $n$-braid $\beta \in B_n$ with a sequence $(t_1, \ldots, t_n)$ of color symbols (repetition allowed), where $t_i \in T$ is the color of the $i$-th strand at the top of the braid. For brevity, when we refer to a colored braid $\beta \in B_n$ without specifying its color sequence, it is understood that the color symbol of the $i$-th strand (at the top) is $t_i$.

Every colored link is the closure $\hat{\beta}$ of some colored braid $\beta \in B_n$ whose color sequence at the bottom coincides with that at the top.

**Definition 4.1.** Suppose $\beta_1, \ldots, \beta_k \in B_n$ are colored braids that have the same color sequence $(t_1, \ldots, t_n)$ at the top, and also share a common color sequence at the bottom. (We will say these braids have the same color permutation.) Let $C_1, \ldots, C_k \in \mathbb{C}(t_1, \ldots, t_n)$ be rational functions. An equation

$$C_1 \cdot \nabla L_{\beta_1} + \cdots + C_k \cdot \nabla L_{\beta_k} = 0$$

is called a **skein relation** in colored $B_n$ if it holds true for all links $L_{\beta_1}, \ldots, L_{\beta_k}$ that are identical except in a cylinder where they are represented by the braids $\beta_1, \ldots, \beta_k$, respectively. The formal sum

$$C_1 \cdot \beta_1 + \cdots + C_k \cdot \beta_k$$

is called a **skein relator**.

**Proposition 4.2.** Assume that

$$C_1 \cdot \nabla L_{\beta_1} + \cdots + C_k \cdot \nabla L_{\beta_k} = 0$$

is a skein relation for colored $n$-braids. Then for any given (uncolored) braid $\beta \in B_n$, the following equations are also skein relations:

$$(S) \cdot \beta$$

$$C_1 \cdot \nabla L_{\beta_1} + \cdots + C_k \cdot \nabla L_{\beta_k} = 0;$$

$$(\beta \cdot S)$$

$$\beta C_1 \cdot \nabla L_{\beta \beta_1} + \cdots + \beta C_k \cdot \nabla L_{\beta \beta_k} = 0,$$

where each $\beta C_h(t_1, \ldots, t_n)$ is obtained from $C_h$ by permuting variables, replacing each variable $t_i$ of $C_h$ with the variable $t_j$ such that $j^\beta = i$.

**Proof.** (i) Look at the cylinder where the links $L_{\beta_1, \beta}, \ldots, L_{\beta_k, \beta}$ are represented differently by braids $\beta_1, \ldots, \beta_k, \beta$, respectively. In the upper half cylinder they are represented by braids $\beta_1, \ldots, \beta_k$. So the assumption implies the conclusion.

(ii) Look at the cylinder where the links $L_{\beta \beta_1}, \ldots, L_{\beta \beta_k}$ are represented differently by braids $\beta_1, \ldots, \beta_k, \beta$, respectively. In the lower half cylinder they are represented as braids $\beta_1, \ldots, \beta_k$. So the assumption implies a linear equality $C_1' \cdot \nabla L_{\beta \beta_1} + \cdots + C_k' \cdot \nabla L_{\beta \beta_k} = 0$, in which the coefficients $C_h'$ are essentially the same as $C_h$, but
Definition 4.3. Let $t$ be the variable in $C_h'$, which refers to the color label of the $j$-th strand at the top of the colored braid $(\beta \beta_h)(t_1, \ldots, t_n)$, i.e., the $j$-th strand at the top of the whole cylinder (for $\beta \beta_h$). This strand is exactly the $j$-th strand at the top of the lower half cylinder (for $\beta_h$). So the variable $t_j$ in $C_h'$ is the variable $t_j$ in $C_h$. Hence $C_h' = C_h$. □

Motivated by this Proposition, we define

\[
(C_1 \beta_1) \cdot (C_2 \beta_2) := (C_1 \cdot \beta_1 C_2)(\beta_1 \beta_2).
\]

This is an $F$-algebra. Note that it is not the ordinary group-algebra, but is twisted by the left $B_n$-action on the coefficient field $F$.

Skein relators are regarded as elements of the algebra $F B_n$. It is clear that a scalar (from $F$) multiple of a skein relator is a skein relator. The sum of two skein relators is not a skein relator unless they have the same color permutation. Proposition 4.2 amounts to saying that skein relators can be left- and right-multiplied by braids.

Example 4.4. The skein relations in Section 2 correspond to the following skein relators. (The symbol $e$ stands for the trivial braid.)

\[
(\Pi_B) \quad \sigma_1^2 + \sigma_1^{-2} - (t_1 t_2 + t_1^{-1} t_2^{-1}) \cdot e = (\sigma_1 - t_1^{-1} t_2^{-1} \sigma_1^{-1}) \cdot (\sigma_1 - t_1 t_2 \sigma_1^{-1});
\]

\[
(\Pi_{8,1B}) \quad (\sigma_1 - t_1 t_2 \sigma_1^{-1}) \cdot (t_2^{-1} t_3 \sigma_2 - \sigma_2^{-1}) \cdot (\sigma_1 - t_1^{-1} t_2^{-1} \sigma_1^{-1});
\]

\[
(\Pi_{8,2B}) \quad (\sigma_1 - t_1^{-1} t_2^{-1} \sigma_1^{-1}) \cdot (t_2 t_3 \sigma_2 - \sigma_2^{-1}) \cdot (\sigma_1 - t_1 t_2 \sigma_1^{-1});
\]

\[
(\Pi_{6,1B}) \quad (t_1^{-1} t_2^{-1} - t_1 t_2) \cdot (\sigma_1 \sigma_2 \sigma_1^{-1} + \sigma_1^{-1} \sigma_2^{-1} \sigma_1) + (t_2 t_3 - t_1^{-1} t_3^{-1}) \cdot (\sigma_1 \sigma_2 \sigma_1^{-1} + \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1});
\]

\[
(\Pi_{6,2B}) \quad (t_2 t_3^{-1} + t_2^{-1} t_3) \cdot (\sigma_1 \sigma_2 \sigma_1^{-1} + \sigma_1^{-1} \sigma_2^{-1} \sigma_1) + (t_1 t_2^{-1} - t_1^{-1} t_2) \cdot (\sigma_1^{-1} \sigma_2 \sigma_1^{-1} + \sigma_1 \sigma_2^{-1} \sigma_1^{-1}) + (t_1^{-1} t_3 - t_1 t_3^{-1}) \cdot (\sigma_1^{-1} \sigma_2 \sigma_1^{-1} + \sigma_2 \sigma_1^{-1} \sigma_2).
\]

Remark 4.5. The key axiom in Murakami [22] for CPF says that

\[
(t_1 + t_1^{-1})(t_2 - t_2^{-1}) \cdot \sigma_2 \sigma_1^2 \sigma_2 - (t_2 - t_2^{-1})(t_3 + t_3^{-1}) \cdot \sigma_1 \sigma_2^2 \sigma_1
\]

\[
(\Pi_{7B}) \quad - (t_1^{-1} t_3 - t_1 t_3^{-1}) \cdot (\sigma_2^2 \sigma_2 + \sigma_2^2 \sigma_1^2) + (t_1^{-1} t_2 t_3 - t_1 t_2^{-1} t_3^{-1})(t_3 + t_3^{-1}) \cdot \sigma_1^2
\]

\[
- (t_1 + t_1^{-1})(t_1 t_2 t_3^{-1} - t_1^{-1} t_2^{-1} t_3) \cdot \sigma_2^2 - (t_2 t_3^{-1} - t_2^{-1} t_3) \cdot e
\]

is a skein relator.

Example 4.6. The skein relation $(\Pi_{6,2})$ is proved by observing that

\[
(t_1 t_3 - t_1^{-1} t_3^{-1}) \cdot (\Pi_{6,2B}) = (t_1 t_2 - t_1^{-1} t_2) \cdot \sigma_1^{-1} \cdot (\Pi_{6,1B}) \cdot \sigma_2
\]

\[
- (t_2 t_3^{-1} - t_2^{-1} t_3) \cdot \sigma_2 \cdot (\Pi_{6,1B}) \cdot \sigma_1^{-1}.
\]

Note that the two terms on the right hand side are skein relators and have the same color permutation, so the left hand side is also a skein relator.
5. A KEY LEMMA

Definition 5.1. In the algebra \( \mathbb{F}B_n \), let \( \sim \) be the equivalence relation, compatible with right- and left- multiplications by elements of \( B_n \), and generated by the basic relations:

\[
(A) \quad \sigma_i^2 + \sigma_i^{-2} - (t_1 t_2 + t_1^{-1} t_2^{-1}) \cdot e \sim 0
\]

where \( e \) stands for the trivial braid in \( B_n \), and

\[
(B) \quad (\sigma_i t_{i+1} \sigma_i^{-1} - t_{i+1}^{-1} t_i) \cdot (\sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_k^{i_k} - e) \sim 0
\]

More precisely, two elements of \( \mathbb{F}B_n \) are equivalent if their difference is an \( \mathbb{F} \)-linear combination of elements of the form \( \beta \cdot S \cdot \beta' \) where \( \beta, \beta' \in B_n \) and \( S \) is the left hand side of either (A) or (B).

For example, by conjugation in \( B_n \) we have \( \sigma_i^2 + \sigma_i^{-2} - (t_i t_{i+1} + t_i^{-1} t_{i+1}^{-1}) \cdot e \sim 0 \)
and \( (t_i^{-1} t_{i+1} - t_{i+1} t_i) \cdot (\sigma_i t_{i+1} \sigma_i^{-1} + \sigma_i^{-1} \sigma_i t_{i+1}^{-1}) \sim 0 \).

Clearly, the relations (A) and (B) are meant to mimic the skein relations (II) and (III), respectively.

The statement of the following lemma is taken from Murakami [M1], although the definition of equivalence is quite different.

Lemma 5.2. With respect to \( \sim \), every braid \( b \in B_n \) is equivalent to an \( \mathbb{F} \)-linear combination of braids of the form \( y \sigma_{k_1}^{k_1} \cdots \sigma_{k_r}^{k_r} \).

A word \( b \in B_n \) can be written as

\[
b = b_0 \sigma_{n-1}^{k_1} b_1 \sigma_{n-1}^{k_2} \cdots \sigma_{n-1}^{k_r} b_r
\]

where \( b_j \in B_{n-1} \) and \( k_j \neq 0 \). We allow that \( b_0 \) and \( b_r \) be trivial, but assume other \( b_j \)'s are nontrivial. The number \( r \) will be denoted as \( r(b) \).

The lemma will be proved by an induction on the double index \( (n, r) \). Note that the lemma is trivial when \( n = 2 \), or \( r(b) \leq 1 \).

It is enough to consider the case \( r = 2 \), because induction on \( r \) works beyond 2. Indeed, if \( r(b) > 2 \), let \( b' = b_1 \sigma_{n-1}^{k_2} \cdots \sigma_{n-1}^{k_r} b_r \), then \( r(b') < r(b) \). By inductive hypothesis \( b' \) is equivalent to a linear combination of elements of the form \( y' \sigma_{n-1}^{k'} z' \), hence \( b \) is equivalent to a linear combination of elements of the form \( b_0 \sigma_{n-1}^{k_1} y' \sigma_{n-1}^{k_2} z' \).

This brings the problem back to the \( r = 2 \) case. Henceforth we assume \( r = 2 \).

Since the initial and terminal part of \( b \), namely \( b_0 \) and \( b_r \), do not affect the conclusion of the lemma, we can drop them. So we assume \( b = \sigma_{n-1}^{k_1} b_1 \sigma_{n-1}^{k_2} \), where \( b_1 \in B_{n-1} \).

By the induction hypothesis on \( n \), \( b_1 \in B_{n-1} \) is a linear combination of elements of the form \( y_1 \sigma_{n-2}^{k_1} z_1 \). Note that \( y_1, z_1 \in B_{n-2} \) commute with \( \sigma_{n-1} \). So it suffices to focus on braids of the form \( b = \sigma_{n-1}^{k_1} \sigma_{n-2}^{k_2} \).
To further simplify our notation, we assume \( n = 3 \) below. The proof for a general \( n \) can be obtained by a simple change of subscripts, replacing \( \sigma_1 \) and \( \sigma_2 \) with \( \sigma_{n-2} \) and \( \sigma_n \) respectively.

Thus, Lemma 5.2 has been reduced to the following

**Lemma 5.3.** Every \( \sigma_2^\ell \sigma_1^m \) is equivalent to a linear combination of braids of the form \( \sigma_1^k \sigma_2^\ell \sigma_1^m \).

**Proof.** In view of the relation (A), we may restrict the exponent \( k \) to take values 1, 2 and 3 (we are done if \( k \) is 0). If \( k > 1 \) we can decrease \( k \) by looking at \( \sigma_2^{k-1}(\sigma_2 \sigma_1) \), so it suffices to prove the case \( k = 1 \). Again by (A), we can restrict the exponents \( \ell, m \) to the values ±1 and 2. There are altogether 9 cases to verify.

5 trivial cases (braid identities):

\[
\sigma_2 \sigma_1 = \sigma_1 \sigma_2, \quad \sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1, \quad \sigma_2 \sigma_1^{-1} \sigma_2^{-1} = \sigma_1^{-1} \sigma_2^{-1} \sigma_1, \\
\sigma_2 \sigma_1^2 = \sigma_1^2 \sigma_2, \quad \sigma_2 \sigma_1^2 \sigma_1^{-1} = \sigma_1 \sigma_2 \sigma_1, \\
\sigma_2 \sigma_1^2 \sigma_1^{-1} = \sigma_1 \sigma_2 \sigma_1.
\]

The case \( \sigma_2 \sigma_1^{-1} \sigma_2 \): Multiplying (B) by \( \sigma_2 \) on the right and \( \sigma_1^{-1} \) on the left, and taking braid identities into account, we get the relation

\[
(t_1^{-1} t_2^{-1} - t_1 t_2) \cdot (\sigma_2 \sigma_1^{-1} \sigma_2 + \sigma_1^{-1} \sigma_2 \sigma_1) + (t_1 t_3 - t_1^{-1} t_3^{-1}) \cdot (\sigma_1^{-1} \sigma_2 \sigma_1 + \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_1^{-1}) = 0.
\]

Then \( \sigma_2 \sigma_1^{-1} \sigma_2 \) is verified.

Remark 5.4. Note again that the coefficients are not exactly the same as in the original (B). The variables \( t_1 \) and \( t_2 \) got interchanged. A permutation of variables takes place whenever a relation is multiplied by a braid on the left, as indicated in Definition 4.3.

The case \( \sigma_2 \sigma_1^{-1} \sigma_2^2 \): Multiplying the previous relation by \( \sigma_2 \) on the right, and taking braid identities into account, we see that

\[
(t_1^{-1} t_2^{-1} - t_1 t_2) \cdot (\sigma_2 \sigma_1^{-1} \sigma_2^2 + \sigma_1^{-1} (\sigma_2 \sigma_1^{-1} \sigma_2)) + (t_1 t_3 - t_1^{-1} t_3^{-1}) \cdot (\sigma_2 \sigma_1 + \sigma_1^2 \sigma_2^{-1} \sigma_1^{-1}) = 0.
\]

This reduces \( \sigma_2 \sigma_1^{-1} \sigma_2^2 \) to the verified case \( \sigma_2 \sigma_1^{-1} \sigma_2 \).

The case \( \sigma_2 \sigma_1^{-1} \sigma_2 \): Multiplying (B) on the right by \( \sigma_1 \sigma_2 \sigma_1 \), we get

\[
(B')
(t_1^{-1} t_2^{-1} - t_1 t_2) \cdot (\sigma_1 \sigma_2 \sigma_1 + \sigma_2^2) + (t_2 t_3 - t_2^{-1} t_3^{-1}) \cdot (\sigma_2 \sigma_1^2 \sigma_2 + \sigma_1^2)
+ (t_1 t_3^{-1} - t_1^{-1} t_3) \cdot (\sigma_1^2 (\sigma_2 \sigma_1^2 \sigma_2) + e) \sim 0.
\]

This is a relation among pure braids. We have the following set of linear equations, where the first line is (B'), the second line is (B') multiplied by \( \sigma_1^{-2} \) on the left, the third line is nothing but the relation (A), and \( K_1, K_2 \) are linear combinations of
braids of the form $\sigma^k \sigma'^{\ell} \sigma^m$.

\[
\begin{align*}
(t_1 t_3^{-1} - t_1^{-1} t_3) \cdot \sigma_1^2 (\sigma_2 \sigma_1^2 \sigma_2) + (t_2 t_3 - t_2^{-1} t_3^{-1}) \cdot (\sigma_2 \sigma_1^2 \sigma_2) & \sim K_1, \\
(t_1 t_3^{-1} - t_1^{-1} t_3) \cdot (\sigma_2 \sigma_1^2 \sigma_2) + (t_2 t_3 - t_2^{-1} t_3^{-1}) \cdot \sigma_1^{-2} (\sigma_2 \sigma_1^2 \sigma_2) & \sim K_2, \\
\sigma_1^{-2} (\sigma_2 \sigma_1^2 \sigma_2) - (t_1 t_2 + t_1^{-1} t_2^{-1}) \cdot (\sigma_2 \sigma_1^2 \sigma_2) + \sigma_1^{-2} (\sigma_2 \sigma_1^2 \sigma_2) & \sim 0.
\end{align*}
\]

The determinant
\[
\det \begin{pmatrix}
(t_1 t_3^{-1} - t_1^{-1} t_3) & t_2 t_3 - t_2^{-1} t_3^{-1} & 0 \\
0 & t_1 t_3^{-1} - t_1^{-1} t_3 & t_2 t_3 - t_2^{-1} t_3^{-1} \\
1 & -t_1 t_2 - t_1^{-1} t_2^{-1} & 1
\end{pmatrix} = (t_1 t_2 - t_1^{-1} t_2^{-1})^2 \neq 0.
\]

Solving these equations we see that $\sigma_2 \sigma_1^2 \sigma_2$ is equivalent to a linear combination of braids of the form $\sigma_1^k \sigma'^{\ell} \sigma_1^m$, as desired.

The case $\sigma_2 \sigma_1^2 \sigma_2^2$: Multiplying (B') by $\sigma_2$ on the right, we get
\[
(\sigma_2 \sigma_1^2 \sigma_1^2 + \sigma_1^3) - (t_1 t_2 - t_1^{-1} t_2^{-1}) \cdot (\sigma_2 \sigma_1^2 \sigma_1^2 + \sigma_1^3) + (t_1 t_3^{-1} - t_1^{-1} t_3) \cdot (\sigma_1 \sigma_2 \sigma_1^3 \sigma_2 + \sigma_2) \sim 0.
\]

Since $\sigma_2 \sigma_1^2 \sigma_2$ reduces by (A) to the trivial case $\sigma_2 \sigma_1 \sigma_2$ and the verified case $\sigma_2 \sigma_1^{-1} \sigma_2$, the case $\sigma_2 \sigma_1^2 \sigma_2$ is also verified.

Thus, we have verified all 9 cases. \(\square\)

**Remark 5.5.** The above proof is delicate. Had we defined $\sim$ to mimic the skein relation (III.2) instead of (IV.1), the proof would fail in the case $\sigma_2 \sigma_1^2 \sigma_2$ when $t_1 = t_2 \neq t_3$.

### 6. Proof of the Main Theorem

It is clear that the Conway potential function $\nabla_L$ satisfies all these axioms, axiom (III) being the relation (III.0.1) of Section 2. The focus is the uniqueness.

We will say a colored link $L$ is *computable* if its CPF is uniquely determined by the axioms listed in the Main Theorem. It suffices to prove, by induction on $n$, the following

**Proposition P**. For every colored $n$-braid $\beta \in B_n$, of which the color sequences at the top and bottom coincide, the closure $\hat{\beta}$ is a computable colored link.

When $n = 1$, Proposition P$_1$ is true, because colored $B_1$ has only one element $e$, the trivial 1-braid, with color symbol $t_1$. Its closure $\hat{e}$ is the trivial knot, whose CPF must be $(t_1 - t_1^{-1})^{-1}$ by axioms (Φ) and (H).

Now assume inductively that Proposition P$_{n-1}$ is true, we shall prove that P$_n$ is also true.

Suppose $\beta$ is a colored $n$-braid. By Lemma 5.2, $\beta$ is equivalent to an $\mathbb{F}$-linear combination of braids of the form $y \sigma_{n-1}^k z$ with $y, z \in B_{n-1}$. We can assume that all the latter braids have the same color permutation as $\beta$, because as an $\mathbb{F}$-linear space $\mathbb{F}B_n$ has a direct sum decomposition according to the color permutations of the braids. Then the CPF of $\hat{\beta}$ is an $\mathbb{F}$-linear combination of CPF’s of closures...
of the latter braids. Without loss, we can assume $\beta$ is of the form $\alpha \sigma_{n-1}^k \gamma$ with $\alpha, \gamma \in B_n$. By axiom (II) we can assume the exponent $k$ is 0, $\pm 1$ or 2.

If $k = 0$, the link $\widehat{\beta}$ has a free circle. So its CPF must be 0 by axiom (IO).

If $k = 2$, the link $\widehat{\beta}$ is the link $\widehat{\alpha \gamma}$ (regarded as a closed $(n-1)$-braid) with a ring attached to the last strand. The latter link is computable by inductive hypothesis, so the former is also computable by axiom (Φ).

If $k = \pm 1$, the link $\widehat{\beta}$ is isotopic to the link $\widehat{\alpha \gamma}$ (regarded as a closed $(n-1)$-braid) which is computable by inductive hypothesis. So the former is also computable.

Thus Proposition $P_n$ is proved. The induction on $n$ is now complete. □

7. Final remarks

Remark 7.1. It is interesting to note that we did not use the classical axiom at a same-color crossing

(I) \[ \nabla \frac{\sigma^i j}{\sigma^i j} - \nabla \frac{\sigma^i j}{\sigma^i j} = (t_i - t_i^{-1}) \cdot \nabla \frac{\sigma^i j}{\sigma^i j} \]

used by Conway and Murakami. Note that axiom (I) plus axiom (Φ) implies axiom (IO), because

\[
(t_i - t_i^{-1})(t_j - t_j^{-1}) \cdot \nabla \frac{\sigma^i j}{\sigma^i j} \stackrel{(\Phi)}{=} (t_j - t_j^{-1}) \cdot \nabla \frac{\sigma^i j}{\sigma^i j} \\
= \nabla \frac{\sigma^i j}{\sigma^i j} - \nabla \frac{\sigma^i j}{\sigma^i j} = \nabla \frac{\sigma^i j}{\sigma^i j} - \nabla \frac{\sigma^i j}{\sigma^i j} = 0.
\]

So our axioms in the Main Theorem is weaker than if we use axiom (I) instead of (IO).

Remark 7.2. The inductive argument in Sections 5 and 6 provides a recursive algorithm for computing $\nabla \widehat{\beta}$. This justifies the term ‘computable’ in Section 6. A remarkable feature of this algorithm is that it never increases the number of components of links. In fact, all the reductions in Section 5 are by axioms (II) and (III) which respect the components, while in Section 6 components could get removed, but never added, by axioms (IO) and (Φ). So if we start off with a knot, we shall always get knots along the way, the axioms (IO) and (Φ) becoming irrelevant. Hence the Corollary for Knots stated in the Introduction.

Remark 7.3. We have chosen the relation (III$_{6,1}$) to be axiom (III). Alternatively, we can use two relations, (III$_4$) plus (III$_{8,1}$), because they jointly imply (III$_{8,2}$) (cf. the proof of Corollary 2.3), hence also imply (III$_{6,1}$).

We do not know whether Conway’s four term relation (III$_4$), or the relation (III$_{8,1}$), alone, is powerful enough to play the role of axiom (III) in the Main Theorem.
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DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING 100871, CHINA

E-mail address: bjiang@math.pku.edu.cn