A HALES–JEWETT TYPE PROPERTY OF FINITE SOLVABLE GROUPS

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ABSTRACT. A conjecture of Leader, Russell and Walters in Euclidean Ramsey theory says that a finite set is Ramsey if and only if it is congruent to a subset of a set whose symmetry group acts transitively. As they have shown the “if” direction of their conjecture follows if all finite groups have a Hales–Jewett type property. In this paper, we show that this property is satisfied in the case of finite solvable groups. Our result can be used to recover the work of Kríž in Euclidean Ramsey theory.

1. Introduction

1.1. Overview. A finite set $X$ in $\mathbb{R}^n$ is called Ramsey if for every $r \in \mathbb{N}$ there exists a positive integer $N = N(X, r)$ such that for every $r$-coloring of $\mathbb{R}^N$ there exists a monochromatic subset of $\mathbb{R}^N$ which is congruent to $X$. The concept of Ramsey set was originally introduced and studied by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus in [1, 2, 3]. Frankl and Rödl in [5] proved that any $n$-dimensional simplex is Ramsey (see also [6] and [14]). Another important result is that of Kríž in [11] saying that a finite subset of $\mathbb{R}^n$ is Ramsey if its symmetry group acts transitively and has a solvable subgroup with at most two orbits. As a consequence, he showed that all regular polygons and all regular polyhedra are Ramsey. For a general survey of Euclidean Ramsey theory see [8].

A central problem in Euclidean Ramsey theory is to determine which sets are Ramsey. In [11] it was shown that every Ramsey set is spherical, that is it lies on the surface of some sphere. A well-known conjecture of Graham [7] says that the Ramsey sets are exactly the spherical sets. In [13], Leader, Russell and Walters proposed an alternative conjecture, stating that a finite set is Ramsey if and only if it is subtransitive in the sense that it is congruent to a subset of a finite set whose symmetry group acts transitively. Their conjecture is a genuine refinement of that of Graham’s, since although every subtransitive set is spherical, almost all four point subsets of the circle are not subtransitive (see [12]). Concerning the one direction of their conjecture, saying that a set is Ramsey if it is subtransitive, they showed that it can be reduced to a list of equivalent conjectures which are free from any geometric notion and hopefully more manageable to prove. One of these conjectures [13] Conjecture C] states that every finite group satisfies a combinatorial statement which resembles the Hales–Jewett theorem [10]. In this paper, we show that in the case of finite solvable groups a stronger statement is satisfied. The motivation to investigate this class of groups comes from the above mentioned work of Kríž [11].

Key words and phrases. Ramsey theory, Euclidean Ramsey theory, Hales–Jewett theorem, finite solvable groups, group actions.

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To state the conjecture and our results we will need an algebraic notion of variable words originally introduced by Graham and Rothschild in [9], which we develop in the next subsection.

1.2. Variable words and groups. First let us state some general notation. In the following by \( \mathbb{N} \) we denote the set of all positive integers and for every \( n \in \mathbb{N} \), let \( [n] = \{1, \ldots, n\} \). Also for a finite set \( X \), \( |X| \) stands for its cardinality. An action of a group \( G \) on a nonempty set \( X \) is a map from \( G \times X \) to \( X \), denoted by \( (g, x) \rightarrow gx \) such that \( ex = x \), for all \( x \in X \) (where \( e \) is the identity element of \( G \)) and \( h(gx) = (hg)x \), for all \( x \in X \) and all \( h, g \in G \). For any \( x \in X \) the set \( Gx = \{gx : g \in G\} \) is an orbit of \( G \) in \( X \). We say that \( G \) acts transitively (or it is transitive) if \( G \) has only one orbit.

Let \( G \) be a finite group acting on a finite set \( X \). We view \( X \) as an alphabet and its elements as letters and we call the finite sequences with values in \( X \), constant words. We also fix a set \( \{v_g : g \in G\} \) of distinct variables indexed by the set \( G \), such that \( v_g \notin X \) for every \( g \in G \). For every nonempty subset \( H \) of \( G \), by the term \( H \)-variable word over \( X \) (of length \( N \)), we mean a finite sequence \( W = (w_i)_{i=1}^{N} \) with \( w_i \in X \cup \{v_h : h \in H\} \) for all \( i \in [N] \), such that the set \( F_h = \{i \in [N] : w_i = v_h\} \) is non empty for all \( h \in H \). The positive integer \( d = \sum_{h \in H} |F_h| \) will be called the degree of \( W \). Given an \( H \)-variable word \( W \) over \( X \) and \( x \in X \), by \( W(x) \) we denote the constant word of the same length obtained by leaving the letters of \( W \) unchanged and replacing each variable \( v_h \) by \( hx \), where \( hx \) is the result of the action of \( h \) on \( x \), i.e. if \( W = (w_i)_{i=1}^{N} \) then \( W(x) = (w_i(x))_{i=1}^{N} \in X^N \), where

\[
(1.1) \quad w_i(x) = \begin{cases} w_i & \text{if } w_i \in X, \\ hx & \text{if } w_i = v_h \text{ for some } h \in H. \end{cases}
\]

**Example 1.1.** Let \( X = [3] \), \( G = S_3 \) be the symmetric group of \( [3] \) and \( H = \{e, \tau, \tau^2\} \) be the subgroup of \( S_3 \) generated by the cycle \( \tau = (1\ 2\ 3) \). The sequence \( W = (v_{e}, v_{1}, v_{2}, v_{\tau}, v_{\tau^2}) \) is a \( H \)-variable word over \( X \) of length \( N = 5 \) and degree \( d = 3 \). Moreover, under the natural action of \( S_3 \) on \([3]\) we have \( W(1) = (1, 1, 2, 3, 2) \), \( W(2) = (2, 1, 2, 1, 3) \) and \( W(3) = (3, 1, 2, 2, 1) \).

**Remark 1.2.** In connection with Euclidean Ramsey theory, it is worth mentioning the following geometric interpretation of the above notion of variable words. Suppose that \( X \) is a finite subset of \( \mathbb{R}^n \) and \( G \) is the symmetry group of \( X \), i.e. the set of all distance preserving maps from \( X \) to \( X \), with group operation the composition of functions. Let \( \emptyset \neq H \subseteq G \) and \( W \) be a \( H \)-variable word over \( X \) of length \( N \) and degree \( d \). Then by (1.1), it is easy to see that the map \( x \rightarrow W(x) \) is a \( d^{1/2} \)-dilation of \( X \) into \( X^N \), that is for every \( x, x' \in X \), \( ||W(x) - W(x')||^2 = d||x - x'||^2 \), where the norm in the left (resp. right) hand-side is the usual Euclidean norm in \( \mathbb{R}^{nN} \) (resp. \( \mathbb{R}^n \)).

In the following, if \( G = X \), whenever we omit to mention any particular action of \( G \) on itself, we will always mean that the action is the natural operation of \( G \). So in this case, if \( W \) is a \( H \)-variable word over \( G \), then for every \( g \in G \), \( W(g) \) is the constant word obtained by substituting in \( W \) each variable \( v_h \) by the ordinary product \( hg \) of \( h \) and \( g \) in \( G \).

For every \( r \in \mathbb{N} \) and for a nonempty set \( Y \), an \( r \)-coloring of \( Y \) is a map from \( Y \) to \( [r] \). A subset \( Z \) of \( Y \) is called monochromatic if the coloring is constant on \( Z \).
Under the above notation, the conjecture of Leader, Russell and Walters [13] Conjecture C] is restated as follows.

**Conjecture 1.** [13] Let \( G \) be a finite group and \( r \in \mathbb{N} \). Then there exist positive integers \( d \) and \( N \) such that for every \( r \)-coloring of \( G^N \) there exist a nonempty \( H \subseteq G \) and a \( H \)-variable word \( W \) over \( G \) of length \( N \) and degree \( d \), such that the set \( \{W(g) : g \in G\} \) is monochromatic.

Assuming that Conjecture [1] holds and taking into account Remark [1.2] it is not difficult to show that any finite subset \( X \) of \( \mathbb{R}^n \) with a transitive symmetry group is Ramsey (for details see [13] Proposition 2.1) or Corollary [1.6] below).

In this paper we will exclusively deal with \( H \)-variable words over \( X \) in which for all \( h \in H \) the variable \( v_h \) appears the same number of times. These variable words will be called uniform. This notion also appears in [13] in a similar setting. Actually, following the circle of the equivalent conjectures in [13], it can be deduced that Conjecture [1] can be equivalently restated for \( H = G \) and \( W \) a uniform \( G \)-variable word over \( G \).

### 1.3. The main results of the paper.

Recall that a group \( G \) is called solvable if it has a subnormal series \( \{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G \) such that \( G_i/G_{i-1} \) is abelian for all \( i \in [n] \). If \( G \) is finite then \( G \) is solvable if and only if it has a subnormal series such that all factors are cyclic. Every abelian group is solvable and by the famous theorem of Feit and Thomson [4], every finite group of odd order is solvable. On the other hand, the symmetric group \( S_n \) is solvable only for \( n \leq 4 \).

**Definition 1.3.** Let \( \{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G \) be a subnormal series with cyclic factors of a finite solvable group \( G \) and let \( p_i \) be the order of the factor group \( G_i/G_{i-1} \), for every \( i \in [n] \). The number

\[
(1.2) \quad \prod_{i=1}^{n} (p_i - 1) \prod_{j>i}^{n} p_j = p_1^{(p_1 - 1)} \prod_{j=2}^{n} p_j^{(p_j - 1)} \prod_{j=3}^{n} p_j^{(p_j - 1)} \cdots p_{n-1}^{(p_{n-1} - 1)}
\]

will be called a HJ-degree of \( G \).

For example, the series \( \{e\} \triangleleft A_3 \triangleleft S_3 \) gives that the number \( 3^4 \cdot 2 \) is a HJ-degree of \( S_3 \). Also, by the series \( \{e\} \triangleleft \{e, (1 2)(3 4)\} \triangleleft V_4 \triangleleft A_4 \triangleleft S_4 \), where \( V_4 \) is the Klein 4-group, we get that the number \( 2^{19} \cdot 3^4 \) is a HJ-degree of \( S_4 \). The above HJ-degrees of \( S_3 \) and \( S_4 \) are unique since the preceding series are the only subnormal series with cyclic factors that these groups have. In general, a solvable group may have more than one HJ-degrees. Indeed, denoting by \( C_n \) a cyclic group of order \( n \), the series \( \{e\} \triangleleft C_6 \triangleleft C_2 \triangleleft C_6 \triangleleft C_6 \triangleleft C_6 \triangleleft C_6 \triangleleft C_6 \) give that the numbers \( 6^5 \), \( 2^3 \cdot 3^2 \) and \( 3^4 \cdot 2 \) are HJ-degrees of \( C_n \). The number \( n^{n-1} \) is the largest HJ-degree of \( C_n \). In general, using (1.2), it can be easily shown that refinements of a subnormal series lead to smaller HJ-degrees.

Our first main result says that finite solvable groups satisfy a stronger form of Conjecture [1] in the sense that the degree \( d \) of the resulting variable word may be assumed to be a HJ-degree of \( G \) and hence it is independent of the number of colors.

**Theorem 1.4.** Let \( G \) be a finite solvable group, \( d \) be a HJ-degree of \( G \) and \( r \in \mathbb{N} \). Then there exists a positive integer \( N \) such that for every \( r \)-coloring of \( G^N \) there exists a uniform \( G \)-variable word \( W \) over \( G \) of length \( N \) and degree \( d \) such that the set \( \{W(g) : g \in G\} \) is monochromatic.
Our second main result concerns actions of finite solvable groups.

**Theorem 1.5.** Let \( G \) be a finite solvable group acting on a finite set \( X \), \( d \) be a HJ-degree of \( G \) and \( r \in \mathbb{N} \). Then there exists a positive integer \( N \) such that for every \( r \)-coloring of \( X^N \) there exists a uniform \( G \)-variable word \( W \) over \( X \) of length \( N \) and degree \( d \), where \( p \) is the number of the orbits of \( G \) in \( X \), such that for every \( x \in X \), the set \( \{W(gx) : g \in G\} \) is monochromatic.

It is clear that Theorem 1.5 includes Theorem 1.4 as a special case. However, as we will see the two theorems are equivalent. To illustrate the connection with Euclidean Ramsey theory, let us present the following consequence which is a refined form of [11, Theorem 4.3].

**Corollary 1.6.** Let \( X \) be a finite non-empty subset of \( \mathbb{R}^n \) and \( G \) be a solvable group of isometries of \( X \). Also let \( d \) be a HJ-degree of \( G \) and \( \lambda = d^{-r/2} \), where \( p \) is the number of the orbits of \( G \) in \( X \). Then for every \( r \in \mathbb{N} \) there exists a positive integer \( N \) such that for every \( r \)-coloring of \( \lambda X^N \) there is an isometric embedding \( f : X \rightarrow \lambda X^N \) such that for every \( x \in X \), the set \( \{f(gx) : g \in G\} \) is monochromatic.

**Proof.** Fix \( r \in \mathbb{N} \). By Theorem 1.5 and Remark 1.2 (for \( G \) and \( d \) in place of \( H \) and \( d \)), there exists a positive integer \( N \) such that for any \( r \)-coloring of \( X^N \) there exists a \( d^{r/2} \)-dilation \( \phi : X \rightarrow X^N \) with the property that for every \( x \in X \) the set \( \{\phi(gx) : g \in G\} \) is monochromatic. Now let \( \epsilon \) be an \( r \)-coloring of \( \lambda X^N \) and let \( c_\lambda : X^N \rightarrow [r] \) defined by \( c_\lambda(x) = c(\lambda x) \), for all \( x \in X^N \). If \( \phi \) is the \( d^{r/2} \)-dilation of \( X \) into \( X^N \) corresponding to \( c_\lambda \), then it is easy to check that the map \( f : X \rightarrow \lambda X^N \) defined by \( f(x) = \lambda \phi(x) \) is as desired. \( \square \)

1.4. Organization of the paper. In Section 2 we define a Hales–Jewett type property for finite groups as well as a generalization of this property for actions on finite sets. Under these definitions our main results are reduced to three propositions. A basic tool that we will use in their proofs is a variant of a well-known lemma due to Shelah [15] which is presented in Section 3. In Section 4 we introduce some extra notation and the proofs are completed in Sections 5-7. We close the paper in Section 8 by stating some notes and remarks.

2. The basic steps of the proof of the main results.

In the following, given a finite group \( G \) acting on a finite set \( X \) and a nonempty subset \( H \) of \( G \), by \( \mathcal{V}^d_{\text{un}}(H; X) \) we will denote the set of all uniform \( H \)-variable words over \( X \) of degree \( d \). In particular, if \( G = H = X \), by \( \mathcal{V}^d_{\text{un}}(G; G) \) we denote the set of all uniform \( G \)-variable words over \( G \) of degree \( d \).

By isolating the property of solvable groups arising from Theorem 1.4 we formulate the following definition.

**Definition 2.1.** Let \( G \) be a finite group and \( d \in \mathbb{N} \). We will say that \( G \) has the \( d \)-uniform Hales–Jewett property (in short, \( d \)-UHJP), if for every \( r \in \mathbb{N} \) there exists a positive integer \( N = N(G, d, r) \) such that for every \( r \)-coloring of \( G^N \) there exists a variable word \( W \in \mathcal{V}^d_{\text{un}}(G; G) \) of length \( N \) such that the set \( \{W(g) : g \in G\} \) is monochromatic.

In view of Definition 2.1 Theorem 1.4 states that if \( G \) is a finite solvable group and \( d \) is a HJ-degree of \( G \) then \( G \) has the \( d \)-UHJP. The trivial group \( \{e\} \) has the 1-UHJP (simply consider the variable word \( W = v_e \)). Moreover, it is easy to see,
using for instance the correspondence \( g \rightarrow (g, \ldots, g) \in G^k \), that if \( G \) has the \( d \)-UHJP then it also has the \( kd \)-UHJP, for all \( k \in \mathbb{N} \). The first step towards the proof of Theorem 1.4 is the following.

**Proposition 2.2.** If \( p \) is any positive integer and \( G \) is a cyclic group of order \( p \) then \( G \) has the \( p^{p-1} \)-UHJP.

For the proof of the above statement we will use a combinatorial argument due to Kríž [11]. We introduce now a generalization of Definition 2.1 which is motivated by our second main result.

**Definition 2.3.** Let \( G \) be a finite group acting on a finite set \( X \). Also let \( H \) be a subgroup of \( G \), \( E \) be an equivalence relation on \( X \) and \( d \in \mathbb{N} \). We will say that \((H, X)\) has the \((E, d)\)-UHJP, if for every \( r \in \mathbb{N} \) there exists a positive integer \( N = N(H, X, E, d, r) \) such that for every \( r \)-coloring of \( X^N \) there exists a variable word \( W \in V_{\text{var}}(H; X) \) of length \( N \) such that for every \( x \in X \) the set \( \{W(x') : x'E \} \) is monochromatic.

If \( H = X = G \) then for simplicity we will say that \( G \) has the \((E, d)\)-UHJP. If a group \( G \) acts on a set \( X \) then the restriction of the action to a subgroup \( H \) of \( G \) yields an equivalence relation on \( X \) with equivalence classes the orbits of \( H \) in \( X \). We will denote this equivalence relation by \( E_{X|H} \), that is

\[
E_{X|H} : x'E_{X|H} x \iff x' \in Hx.
\]

Notice that under Definition 2.3 and (2.1), Theorem 1.5 states that if \((G, X)\) has a \( p \)-HJ-degree of \( G \) in \( X \) then the \( (E_{X|H}, d^p) \)-UHJP, where \( p \) is the number of the orbits of \( G \) in \( X \). One of the basic ingredients of the proof of theorems 1.4 and 1.5 is the next proposition.

**Proposition 2.4.** Let \( G \) be a finite group acting on a finite set \( X \) and let \( H \) be a subgroup of \( G \). If \( H \) has the \( d \)-UHJP then \((H, X)\) has the \((E_{X|H}, d^p)\)-UHJP, where \( p \) is the number of the orbits of \( H \) in \( X \).

If \( X = G \) (and the action is the ordinary multiplication of \( G \)) then \( E_{G|H} \) has as equivalence classes the right cosets of \( H \) in \( G \). In this case Proposition 2.4 takes the following form.

**Corollary 2.5.** Let \( G \) be a finite group and \( H \) be a subgroup of \( G \). If \( H \) has the \( d \)-UHJP then \((H, G)\) has the \((E_{G|H}, d^p)\)-UHJP, where \( p \) is the index of \( H \) in \( G \).

One of the basic properties of the class of solvable groups is that it is closed under group extensions. Our next proposition says that the same holds for the class of finite groups having the \( d \)-UHJP for some \( d \in \mathbb{N} \). Recall that if \( H \) and \( K \) are groups then an extension of \( K \) by \( H \) is a group \( G \) along with a surjective homomorphism \( \pi : G \rightarrow K \) and an injective homomorphism \( \iota : H \rightarrow G \) such that the image of \( \iota \) equals the kernel of \( \pi \). In particular, if \( H \) is a normal subgroup of \( G \) then \( G \) is an extension of \( G/H \) by \( H \) (with \( \iota \) the identity map on \( H \) and \( \pi \) the natural surjective homomorphism \( g \rightarrow gH \) from \( G \) to \( G/H \)).

**Proposition 2.6.** Let \( H \) and \( K \) be finite groups and let \( G \) be an extension of \( K \) by \( H \). If \( H \) has the \( d_H \)-UHJP and \( K \) has the \( d_K \)-UHJP then \( G \) has the \( d \)-UHJP, where \( d = d_H^{\lfloor d_K \rfloor} \cdot d_K \).

Assuming the above propositions we can now give the proofs of our main results.
Proof of Theorem 1.4 (assuming propositions 2.3 and 2.4). Let $G$ be a finite solvable group and $d$ be a HJ-degree of $G$. Let $\{e\} = G_0 < G_1 < \cdots < G_n = G$ be a subnormal series with cyclic factors such that $d = \prod_{i=1}^{n} p_i^{(p_i-1)\Pi \pi_i}$, where $p_i = |G_i/G_{i-1}|$ for all $i \in [n]$. We have to show that $G$ has the $d$-UHJP. Let $d_0 = 1$ and inductively define $d_i = d_{i-1}^{-1} p_i^{p_i-1}$, for every $i \in [n]$. As we have already mentioned, $G_0$ has the $d_0$-UHJP. Let $i \geq 1$ and assume that $G_{i-1}$ has the $d_{i-1}$-UHJP. Since $G_i/G_{i-1}$ is a cyclic group of order $p_i$, by Proposition 2.2, $G_i/G_{i-1}$ has the $p_i^{p_i-1}$-UHJP. Moreover, it is clear that $G_i$ is an extension of $G_i/G_{i-1}$ by $G_{i-1}$ and hence, by Proposition 2.3, and the inductive definition of $d_i$, $G_i$ has the $d_i$-UHJP. By induction $G$ has the $d_n$-UHJP. It is now a matter of a simple calculation to check that $d_n = d$.

Proof of Theorem 1.5 (assuming Theorem 1.4 and Proposition 2.4). Let $G$ be a finite solvable group acting on a finite set $X$ and let $d$ be a HJ-degree of $G$. By Theorem 1.4, $G$ has the $d$-UHJP. Hence, by Proposition 2.4, for $H = G$, $(G, X)$ has the $(E_X G, d^p)$-UHJP, where $p$ is the number of the orbits of $G$ in $X$. As we have already noticed this is the content of Theorem 1.5.

3. A variant of Shelah’s lemma

For the proof of propositions 2.2, 2.4, and 2.6 we will use a variant of a well-known lemma due to Shelah used in his proof of the Hales–Jewett theorem [15]. We fix for the following a finite group $G$ acting on a finite set $X$, a subgroup $H$ of $G$, an equivalence relation $E$ on $X$ and $d \in \mathbb{N}$. For every variable word $W \in V_{un}^d(H; X)$, by $|W|$ we will denote the length of $W$. If $(W_i)_{i=1}^n$ is a sequence in $V_{un}^d(H; X)$ and $(x_i)_{i=1}^n \in X^n$ then for every $j \in [n]$, by $\prod_{i=1}^n W_i(x_i)$, we denote the concatenation $W_j(x_j) \cdots W_n(x_n)$.

Lemma 3.1. If $(H, X)$ has the $(E, d)$-UHJP then for every $n, r \in \mathbb{N}$ there exists a positive integer $N = N(n, r)$ satisfying the following property. For any $r$-coloring of $X^N$ there exists a sequence $(W_i)_{i=1}^n$ in $V_{un}^d(H; X)$ with $\sum_{i=1}^n |W_i| = N$ such that for every $(x_i)_{i=1}^n \in X^n$, the set $\{\prod_{i=1}^n W_i(x_i) : \forall i \in [n] x_i \not\in E x_i\}$ is monochromatic.

Proof. For every $r \in \mathbb{N}$, let $f(r) = N(H, X, E, d, r)$, where $N(H, X, E, d, r)$ is as in Definition 2.3. It is clear that for $n = 1$ we may set $N(1, r) = f(r)$ for all $r \in \mathbb{N}$. Assume that for some $n \in \mathbb{N}$ and all $r \in \mathbb{N}$, the numbers $N(n, r)$ have been defined. Let $r \in \mathbb{N}$ be arbitrary and set

$$N(n + 1, r) = N\left(n, r^{[X]}\right) + f \left(r^{[X]} N(n, r^{[X]})\right).$$

Now let $N = N(n + 1, 1)$ and let $c : X^X \to [r]$ be an $r$-coloring of $X^X$. By (3.1), we have $N = N_1 + N_2$, where $N_1 = N\left(n, r^{[X]}\right)$ and $N_2 = f(r^{[X]} N_1)$. Let $c_2 : X^{N_2} \to [r^{[X]} N_1]$ defined by $c_2(y) = (c(x \bowtie y))_{x \in X^{N_2}}$, for every $y \in X^{N_2}$. By the choice of $N_2$ there exists a variable word $W \in V_{un}^d(H; X)$ with $|W| = N_2$ such that for every $x \in X^{N_1}$,

$$c(x \bowtie W(x')) = c(x \bowtie W(x)) \text{ if } x' \not\in E x.$$  

Now define $c_1 : X^{N_1} \to r^{[X]} |X|$ by $c_1(x) = (c(x \bowtie W(x)))_{x \in X^{N_1}}$, for every $x \in X^{N_1}$. By the choice of $N_1$ there exists a sequence $(W_i)_{i=1}^n$ in $V_{un}^d(H; X)$ with $\sum_{i=1}^n |W_i| = r^{[X]} N_1.$

\[\text{□}\]
$N_1$ such that for every $x \in X$,
\[(3.3) \quad c \left( \prod_{i=1}^{n} W_i(x_i') \right) = c \left( \prod_{i=1}^{n} W_i(x_i) \right) \text{ if } x_i' Ex_i \text{ for all } i \in [n].\]

We set $W_{n+1} = W$. Then $(W_i)_{i=1}^{n+1}$ is a sequence in $V_{un}^d(H; X)$ with $\sum_{i=1}^{n+1} |W_i| = N_1 + N_2 = N$. Also, let $(x_i)_{i=1}^{n+1}, (x_i')_{i=1}^{n+1} \in X^{n+1}$ such that $x_i' Ex_i$ for all $i \in [n+1]$. Then, $c \left( \prod_{i=1}^{n+1} W_i(x_i') \right)$ and the proof is completed.

4. SOME USEFUL NOTATION

We introduce here some notation which will facilitate our proofs in the next sections. Fix for the following a finite group $G$ acting on a finite set $X$ and a nonempty subset $H$ of $G$.

4.0.1. Let $(W_i)_{i=1}^{n}$ be a sequence where each term $W_i$ is either a constant or a uniform $H$-variable word over $X$. For a partition of $[n] = \bigcup_{j=1}^{m} F_j$, by the notation $\prod_{j=1}^{m} \prod_{i \in F_j} W_i$ we simply mean the concatenation $W_1 \ldots \ W_n$.

4.0.2. Let $W = (w_i)_{i=1}^{n} \in V_{un}^d(H; X)$. The product representation
\[(4.1) \quad W = \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} v_h,\]

means that $F = \{ i \in [n] : w_i = x_i \in X \}$ and $F_h = \{ i \in [n] : w_i = v_h \}$, for all $h \in H$. By (1.1), for any $x \in X$, $W(x)$ will be represented as
\[(4.2) \quad W(x) = \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} hx.\]

Also, for every $\tau \in G$, by $W^\tau$ we will denote the $H\tau$-variable word over $G$ resulting from $W$ by leaving its letters unchanged and substituting each variable $v_h$ by $v_{h\tau}$. Notice that if $W$ is represented as in (4.1) then $W^\tau$ is represented as
\[(4.3) \quad W^\tau = \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} v_{h\tau}.\]

Using (4.2) and (4.3), for every $\tau \in G$ and every $x \in X$, we have $W^\tau(x) = \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} (h\tau)x = \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} h(\tau x)$, and hence,
\[(4.4) \quad W^\tau(x) = W(\tau x).\]

Moreover notice that if $H$ is a subgroup of $G$ and $\tau \in H$, then $W^\tau \in V_{un}^d(H; X)$.

5. PROOF OF PROPOSITION 2.2

As we have already mentioned in Section 2 for the proof of Proposition 2.2 we will use a combinatorial argument due to Kříž [11]. This argument requires a Ramsey-type result which, although is a direct consequence of Ramsey’s theorem [16], we state it explicitly below and we give a self-contained proof.

Let $T : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ inductively defined by $T(1, r) = 1$, $T(2, r) = r + 1$ and $T(n + 1, r) = T(n, 2^r)$ for every $n \geq 2$ and every $r \in \mathbb{N}$. Notice that $T(n, r)$ is a tower-type function, for example $T(3, r) = 2^r + 1$, $T(4, r) = 2^{2^r} + 1$, $T(5, r) = 2^{2^{2^r}} + 1$. 
Lemma 5.1. Let \( p, r \in \mathbb{N} \) with \( p \geq 2 \) and \( n = T(p, r) \). Then for any \( r \)-coloring of \( \binom{n}{p-1} \) there exists \( P \subseteq [n] \) with \( |P| = p \) such that \( P_\ast \) and \( P^\ast \) have the same color.

**Proof.** If \( p = 2 \) then \( T(2, r) = r + 1 \) and the lemma follows easily by the pigeonhole principle. We proceed by induction on \( p \). Assume that the lemma is true for some \( p \geq 2 \). Let \( q = p + 1, n = T(q, r) \) and \( c : \binom{n}{q-2} \to [r] \). We look for a subset \( Q \subseteq [n] \) with \( |Q| = q \) and such that \( c(Q_\ast) = c(Q^\ast) \). Let \( \mathcal{P}([r]) \) be the powerset of \([r] \) and let \( c' : \binom{n}{q-2} \to \mathcal{P}([r]) \), defined by

\[
(5.1) \quad c'(B) = \{c(B \cup \{x\}) : x \in [n] \text{ and } max B < x\},
\]

for every \( B \in \binom{n}{q-2} \). We may view \( c' \) as a \( 2^r \)-coloring of \( \binom{n}{q-2} = \binom{n}{p-1} \) and hence, since \( n = T(q, r) = T(p, r + 1) = T(p, 2^r) \), by our inductive assumption, we can find \( P \in \binom{n}{p} \) such that \( c'(P_\ast) = c'(P^\ast) \). Since \( P = P^\ast \cup \text{max } P \), we have \( c(P) \in c'(P^\ast) \) and thus, \( c(P) \in c'(P_\ast) \). Hence, by (5.1) for \( B = P_\ast \), there exists \( x \in [n] \) such that

\[
(5.2) \quad \text{max } P_\ast < x \quad \text{and} \quad c(P_\ast \cup \{x\}) = c(P).
\]

We set \( Q = P \cup \{x\} \). Since \( \text{max } P = \text{max } P_\ast < x \) we have that \( |Q| = p + 1 = q \). Moreover, notice that \( c(Q_\ast) = c(P_\ast \cup \{x\}) = c(P) = c(Q^\ast) \). \(\square\)

Let \( p \in \mathbb{N} \) with \( p \geq 2 \) and let \( G = \{\tau^j : 0 \leq j \leq p - 1\} \) be a cyclic group of order \( p \). Proposition 2.2 is a consequence of the following lemma.

Lemma 5.2. Let \( k \in \{1, \ldots, p-1\} \). Then for every \( r \in \mathbb{N} \) there exists a positive integer \( N \) such that for any \( r \)-coloring of \( G^N \) there exists a variable word \( W \in V_{\text{in}}^p(G; G) \) of length \( N \) such that the set \( \{W(\tau^j) : 0 \leq j \leq k\} \) is monochromatic.

**Proof.** First let \( k = 1 \). Let \( r \in \mathbb{N}, N = T(p, r) \) and \( c : G^N \to [r] \). Then \( c \) induces an \( r \)-coloring on \( \binom{[N]}{p-1} \) as follows. For any \( B = \{n_1 < \cdots < n_{p-1}\} \in \binom{[N]}{p-1} \), we set \( \tau^B = (\tau^B)^N_{i=1} \in G^N \), where

\[
(5.3) \quad \tau^B_i = \begin{cases} 
1 & \text{if } i \not\in B \\
\tau^q & \text{if } i = n_q \text{ for some } q \in \{1, \ldots, p-1\}.
\end{cases}
\]

Now let \( \tilde{c} : \binom{[N]}{p-1} \to [r] \) defined by \( \tilde{c}(B) = c(\tau^B) \). Since \( N = T(p, r) \), by Lemma 5.1, there exists \( P \in \binom{[N]}{p} \) such that

\[
(5.4) \quad \tilde{c}(P_\ast) = \tilde{c}(P^\ast).
\]

Without loss of generality, let \( P = [p] \) and let \( W = \prod_{q=1}^p v_{r_{q-1}} \times \prod_{i=p+1}^N e \). Clearly, \( W \in V_{\text{in}}^p(G; G) \). Also, since \( c(W(e)) = c\left(\prod_{q=1}^p \tau^q \times \prod_{i=p+1}^N e\right) = c(\tau^{P_\ast}) = \tilde{c}(P_\ast) \), and \( c(W(\tau)) = c\left(\prod_{q=1}^p \tau^q \times \prod_{i=p+1}^N e\right) = c(\tau^{P^\ast}) = \tilde{c}(P^\ast) \), by (5.4), we get that \( c(W(e)) = c(W(\tau)) \) and the proof for \( k = 1 \) is completed.
Assume now that the lemma is true for some $k \leq p - 2$. By Definition \ref{2.3} our inductive assumption means that $G$ has the $(E_k, d_k)$-UHJP, where $d_k = p^k$ and $E_k$ is the equivalence relation on $G$ defined by

$$gE_kg' \iff g, g' \in \{ \tau^j : 0 \leq j \leq k \} \text{ or } g = g'.$$

Let $r \in \mathbb{N}$ and let $N = N(n, r)$ be as in Lemma \ref{5.1} for $H = X = G$, $E = E_k$, $d = d_k$ and $n = T(p, p^{k+1})$. Let $c : G^N \rightarrow [r]$ be an $r$-coloring of $G^N$. By the choice of $N$, we can find a sequence $(W_i)_{i=1}^n$ of variable words in $V^r_{\text{un}}(G; G)$ with $\sum_{i=1}^n |W_i| = N$ such that for any $(g_i)_{i=1}^n \in G^n$ the set $\{ \prod_{i=1}^n W_i(g_i) : \forall i \in [n] \ g_i E_k g_i \}$ is monochromatic. For every $j \in \{0, \ldots, k\}$ and every $B = \{n_1 < \cdots < n_{p-1}\} \in \binom{n}{p-1}$, let $\tau^{B,j} = \left( \tau^{B,j}_i \right)_{i=1}^n \in G^n$, where

$$\tau^{B,j}_i = \begin{cases} 
\tau^j & \text{if } i \notin B \\
\tau^{q+j} & \text{if } i = n_q \text{ for some } q \in \{1, \ldots, p-1\}.
\end{cases}$$

Now let $\tilde{c} : \binom{n}{p-1} \rightarrow [p^{k+1}]$ defined by $\tilde{c}(B) = (\tilde{c}_j(B))_{j=0}^{k}$, where

$$\tilde{c}_j(B) = c \left( \prod_{i=1}^n W_i \left( \tau^{B,j}_i \right) \right).$$

Since $n = T(p, p^{k+1})$, by Lemma \ref{5.1} there exists $P \in \binom{n}{p}$ such that $\tilde{c}(P) = \tilde{c}(P^*)$, or equivalently,

$$\tilde{c}_j(P) = \tilde{c}_j(P^*) \text{, for all } j = 0, \ldots, k.$$ (5.7)

As in the case $k = 1$, let us assume that $P = [p]$ and let

$$W = \prod_{q=1}^p W_q^{\tau^{q-1}} \times \prod_{i=p+1}^n W_i(e)$$

where $W_q^{\tau^{q-1}}$ is as in (4.3) for every $q \in [p]$. Since $W_q \in V^r_{\text{un}}(G; G)$, we have that $W_q^{\tau^{q-1}} \in V^r_{\text{un}}(G; G)$ for every $q \in [p]$ and consequently, $W \in V^r_{\text{un}}(G; G)$. It remains to show that the set $\{ W(\tau^j) : 0 \leq j \leq k + 1 \}$ is monochromatic. Indeed, let $j \in \{0, \ldots, k\}$. Then,

$$c \left( W(\tau^{j+1}) \right) \leq c \left( \prod_{q=1}^{p-1} W_q(\tau^{q+j}) \times W_p(\tau^j) \times \prod_{i=p+1}^n W_i(e) \right).$$

$$\tau^{j+E_k} c \left( \prod_{q=1}^{p-1} W_q(\tau^{q+j}) \times W_p(e) \times \prod_{i=p+1}^n W_i(e) \right) \leq c \left( \prod_{i=1}^n W_i \left( \tau^{p,j}_i \right) \right) \leq \tilde{c}_j(P^*).$$

(5.8)
Similarly,
\[
c(W(\tau^j)) = c \left( W_1(\tau^j) \times \prod_{q=2}^{p} W_q(\tau^{q-1}+j) \times \prod_{i=p+1}^{n} W_i(e) \right)
\]
\[
\tau^1E_k c \left( W_1(e) \times \prod_{q=2}^{p} W_q(\tau^{q-1}+j) \times \prod_{i=p+1}^{n} W_i(e) \right)
\]
\[
\equiv c \left( \prod_{i=1}^{n} W_i(\tau_i^{j}+j) \right) = c_{j}(P_k).
\]

By the above and (5.7), we get that \(c(W(\tau^{j+1})) = c(W(\tau^j))\), for every \(j \in \{0, \ldots, k\}\), i.e. the set \(\{W(\tau^j) : 0 \leq j \leq k+1\}\) is monochromatic. \(\square\)

6. PROOF OF PROPOSITION 2.4

We fix for the following a finite group \(G\) acting on a finite set \(X\) and a subgroup \(H\) of \(G\) having the \(d\)-UHJP for some \(d \in \mathbb{N}\).

Lemma 6.1. Let \(y \in X\) and let \(E_y\) be the equivalence relation on \(X\) defined by

\[
x'E_yx \iff x', x \in Hy \text{ or } x' = x.
\]

Then \((H, X)\) has the \((E_y, d)\)-UHJP.

Proof. Let \(r \in \mathbb{N}\) and since \(H\) has the \(d\)-UHJP, let \(N = N(H, d, r)\) be as in Definition 2.1 (for \(H\) in place of \(G\)). Let \(c : X^N \to [r]\) and \(c_H : H^N \to [r]\) defined by

\[
c_H(h_1, \ldots, h_N) = c(h_1y, \ldots, h_Ny).
\]

By the choice of \(N\) there exists \(W_H \in V^d_{\text{un}}(H; H)\) of length \(N\) such that

\[
c_H(W_H(h)) = c_H(W_H(h')) \forall h, h' \in H.
\]

Writing \(W_H\) as \(W_H = \prod_{i \in F} h_i \times \prod_{h \in H} \prod_{i \in F_h} v_h\), we define

\[
W = \prod_{i \in F} h_i y \times \prod_{h \in H} \prod_{i \in F_h} v_h.
\]

It is clear that \(W \in V^d_{\text{un}}(H; X)\) and \(|W| = N\). Hence, to complete the proof it remains to verify that the set \(\{W(x') : x'E_yx\}\) is monochromatic. Since \(E_y\) is the identity relation on \(X \setminus Hy\), it suffices to show that \(c(W(x)) = c(W(x'))\), for every \(x, x' \in Hy\). Indeed, let \(x \in Hy\) and choose \(h_x \in H\) such that \(x = h_x y\). Then,

\[
c(W(x)) = c(W(h_x y)) = c \left( \prod_{i \in F} h_i y \times \prod_{h \in H} \prod_{i \in F_h} h(h_x y) \right)
\]
\[
= c \left( \prod_{i \in F} h_i y \times \prod_{h \in H} \prod_{i \in F_h} (hh_x)y \right)
\]
\[
= c_H \left( \prod_{i \in F} h_i \times \prod_{h \in H} \prod_{i \in F_h} hh_x \right) = c_H(W_H(h_x)),
\]

and thus, by (6.3), the set \(\{W(x) : x \in Hy\}\) is monochromatic. \(\square\)
Lemma 6.2. For every induction on $k$

Proof. Notice that $E_p = E_{X \mid H}$ and hence, Proposition 2.4 follows by the next lemma.

Lemma 6.2. For every $k \in [p]$, $(H, X)$ has the $(E_k, d^k)$-UHJP.

Proof. The case $k = 1$ has already been done in Lemma 6.1. We proceed by induction on $k \in [p]$. Let $k \in [p - 1]$ and assume that $(H, X)$ has the $(E_k, d^k)$-UHJP. Fix $r \in \mathbb{N}$ and set $n = N(H, X, E_k, d^k, r)$. By Lemma 6.1 $(H, X)$ has the $(E_{y_k+1}, d)$-UHJP. Let $N = N(n, r)$ be as in Lemma 3.1 (for $E = E_{y_k+1}$).

Let $c : X^N \to [r]$. By the choice of $N$, there exists a sequence $(W_i)_i^n$ in $V^d_{\text{un}}(H; X)$ with $\sum_{i=1}^n |W_i| = N$ and such that

$$c \left( \prod_{i=1}^n W_i(x_i) \right) = c \left( \prod_{i=1}^n W_i(x_i) \right) \quad \text{whenever } x'_i E_{y_k+1} x_i, \forall i \in [n].$$

Let $c' : X^n \to [r]$ defined by

$$c'(x_1, \ldots, x_n) = c \left( \prod_{i=1}^n W_i(x_i) \right).$$

By the choice of $n$ and (6.5), there exists a variable word $W' \in V^d_{\text{un}}(H; X)$ with $|W'| = n$ such that

$$c'(W'(x)) = c'(W'(y_j)), \quad \text{for every } j \in [k] \text{ and every } x \in H y_j.$$

Let $W' = \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} v_h$, and let

$$W = \prod_{i \in F} W_i(x_i) \times \prod_{h \in H} \prod_{i \in F_h} W^h_i.$$
Also, if \( j \in [k] \) and \( x = h_xy_j \in H_yj \), then,
\[
c(W(x)) = c(W(h_xy_j)) = c \left( \prod_{i \in F} W_i(x_i) \times \prod_{h \in H} \prod_{i \in F_h} W_i(h(h_xy_j)) \right)
\]
\[
= \left( \prod_{i \in F} x_i \times \prod_{h \in H} \prod_{i \in F_h} h(h_xy_j) \right)
\]
\[
= c'(W'(h_xy_j)) \quad \text{(6.8)}
\]

By the above the proof is completed. \( \square \)

### 7. Proof of Proposition 2.6

Let \( H, G, K \) be finite groups such that \( G \) is an extension of \( K \) by \( H \). Let \( \pi : G \to K \) be the surjective homomorphism and \( \iota : H \to G \) be the injective homomorphism such that the image of \( \iota \) equals the kernel of \( \pi \). Assume that \( H \) has the \( d_H \)-UHJP. Identifying \( H \) with \( \iota(H) \), by Corollary 3.1 we have that \( (H, G) \) has the \( (E_{G[H]}, d_{H}^p) \)-UHJP, where \( p = |G/H| \). Hence, Proposition 2.6 is a direct consequence of the above and the following lemma.

**Lemma 7.1.** Let \( H, G, K \) be finite groups such that there exists a surjective homomorphism \( \pi : G \to K \) with kernel \( H \). If \( (H, G) \) has the \( (E_{G[H]}, d_{H}^p) \)-UHJP for some \( d_1 \in \mathbb{N} \) and \( K \) has the \( d_K \)-UHJP then \( G \) has the \( d \)-UHJP, where \( d = d_1d_K \).

**Proof.** Notice that \( H \) is a normal subgroup of \( G \) and

\[
gE_{G[H]}g' \iff \pi(g) = \pi(g').
\]

Let \( r \in \mathbb{N} \) and let \( N = N(n, r) \) be as in Lemma 3.1 for \( X = G, E = E_{G[H]}, d = d_1 \) and \( n = N(K, d_K, r) \). Let \( c : G^N \to [r] \). By the choice of \( N \) and (6.2), there exists a sequence \( (W_i)_{i=1}^n \) of variable words in \( V_{\mathbb{N}}^d(H, G) \), with \( \sum_{i=1}^n |W_i| = N \), and such that

\[
c \left( \prod_{i=1}^n W_i(g_i) \right) = c \left( \prod_{i=1}^n W_i(g'_i) \right) \quad \text{whenever } \pi(g_i) = \pi(g'_i) \forall i \in [n].
\]

Let \( c_K : K^n \to [r] \) defined by

\[
c_K(\kappa_1, \ldots, \kappa_n) = c \left( \prod_{i=1}^n W_i(g_i) \right) \quad \text{if } \kappa_i = \pi(g_i) \forall i \in [n].
\]

By (7.2) and since \( \pi : G \to K \) is onto, the coloring \( c_K \) is well-defined. By the choice of \( n \), there exists a variable word \( W_K \in V_{\mathbb{N}}^{d_K}(K; K) \) of length \( n \) and such that

\[
c_K(W_K(\kappa)) = c_K(W_K(\kappa')) \quad \forall \kappa, \kappa' \in K.
\]

Let \( W_K = \prod_{i \in F} \kappa_i \times \prod_{\kappa \in K} \prod_{i \in F_\kappa} v_\kappa \) and set

\[
W = \prod_{i \in F} W_i(g_{\kappa_i}) \times \prod_{\kappa \in K} \prod_{i \in F_\kappa} W_i^{g_\kappa},
\]

where for every \( \kappa \in K, g_\kappa \in G \) is such that \( \pi(g_\kappa) = \kappa \). It is easy to see that \( |W| = \sum_{i=1}^n |W_i| = N \). Moreover, \( \bigcup_{\kappa \in K} Hg_\kappa = G \) and since for every \( \kappa \in K \),

\[
\begin{align*}
\end{align*}
\]
\(|K| \cdot |F_n| = d_K\) and \(W_{g_k} \in V_{\text{un}}^d(H g_k; G)\), we conclude that \(W \in V_{\text{un}}^d(G; G)\), for \(d = d_1 d_K\). Finally, let \(g \in G\) and let \(\kappa_g = \pi(g)\). Then,

\[
c(W(g)) = c \left( \prod_{i \in F} W_i(g_\kappa_i) \times \prod_{\kappa \in K} \prod_{i \in F_\kappa} W_i(g_{\kappa_i}) \right)
\]

\[
= c_K \left( \prod_{i \in F} \kappa_i \times \prod_{\kappa \in K} \prod_{i \in F_\kappa} \pi(g_{\kappa_i}) \right)
\]

\[
= c_K \left( \prod_{i \in F} \kappa_i \times \prod_{\kappa \in K} \prod_{i \in F_\kappa} \pi(\kappa_g) \right)
\]

\[
= c_K \left( \prod_{i \in F} \kappa_i \times \prod_{\kappa \in K} \prod_{i \in F_\kappa} \kappa_g \right) = c_K(W_K(\kappa_g)),
\]

and hence, by (7.3), the set \(\{W(g) : g \in G\}\) is monochromatic. \(\Box\)

8. Notes and remarks

The class of finite groups having the \(d\)-UHJP for some \(d \in \mathbb{N}\) shares similar properties with the class of solvable groups, namely, by propositions 2.2 and 2.6 it contains all finite cyclic groups and it is closed under extensions. It is well known that the class of solvable groups is also closed under subgroups and quotients. The next proposition says that the same holds for the finite groups with the \(d\)-UHJP.

**Proposition 8.1.** Let \(G\) be a finite group having the \(d\)-UHJP for some \(d \in \mathbb{N}\). Then the following are satisfied. (a) If \(H\) is a subgroup of \(G\) then \(H\) has the \(d\)-UHJP. (b) If \(H\) is a normal subgroup of \(G\) then \(G/H\) has the \(d\)-UHJP.

**Proof.** (a) Let \(H\) be a subgroup of \(G\). Let \(p\) be the index of \(H\) in \(G\) and choose \(\tau_1, \ldots, \tau_{p-1} \in G\) such that \(G/H = \{H, \tau_1 H, \ldots, \tau_{p-1} H\}\). Setting \(\tau_0 = e\), notice that for every \(g \in G\) there exists a unique pair \((i_g, h_g) \in \{0, \ldots, p-1\} \times H\) such that \(g = \tau_{i_g} h_g\). Let \(\varphi : G \rightarrow H\) defined by \(\varphi(g) = h_g\) for all \(g \in G\). It is easy to see that \(\varphi\) satisfies the following properties. (i) It is surjective and \(\varphi(h) = h\) for all \(h \in H\), (ii) for every \(g \in G\) and every \(h \in H\), \(\varphi(gh) = \varphi(g) h\), and (iii) for every \(h \in H\), \(|\{g \in G : \varphi(g) = h\}\| = p\).

Now fix \(r \in \mathbb{N}\) and let \(c : H^N \rightarrow [r]\), where \(N = N(G,d,r)\) is as in Definition 2.1. Let \(\tilde{c} : G^N \rightarrow [r]\) defined by \(\tilde{c}(g_1, \ldots, g_N) = c(\varphi(g_1), \ldots, \varphi(g_N))\). By the choice of \(N\), we can find \(\tilde{W} \in V_{\text{un}}^d(G; G)\) of length \(N\) and \(k_0 \in [r]\) such that \(\tilde{c}(\tilde{W}(g)) = k_0\) for all \(g \in G\). Let \(\tilde{W} = \prod_{i \in F} g_i \times \prod_{g \in G} \prod_{i \in F_g} v_g\) and set \(W = \prod_{i \in F} h_i \times \prod_{h \in H} \prod_{i \in F_h} v_h\), where \(h_i = \varphi(g_i)\), for all \(i \in F\) and \(F_h^g = \cup \{F_g : \varphi(g) = h\}\) for all \(h \in H\). Since \(\tilde{W} \in V_{\text{un}}^d(G; G)\), by (iii) we conclude that \(W \in V_{\text{un}}^d(H; H)\). It remains to show that \(\{W(h) : h \in H\}\) is \(c\)-monochromatic. Indeed, let \(h \in H\). Then, \(k_0 = \tilde{c}(\tilde{W}(h)) = c \left( \prod_{i \in F} \varphi(g_i) \times \prod_{g \in G} \prod_{i \in F_g} \varphi(g) h \right) = c (W(h))\), and the proof is completed. (b) The proof is similar to the above by using the surjective homomorphism \(g \rightarrow gH\) from \(G\) to \(G/H\) in place of \(\varphi\). \(\Box\)
A natural question is whether there exists a non-solvable group having the $d$-UHJP for some $d \in \mathbb{N}$. The first candidate groups here are the alternating group $A_5$ or the symmetric group $S_5$. A more general question is the following.

**Question.** Let $G$ be a finite group. Suppose that $G$ contains two subgroups $H$ and $K$ such that $H \cap K = \{e\}$ and $G = HK = \{hk : h \in H$ and $k \in K\}$. If $H$ has the $d_H$-UHJP and $K$ has the $d_K$-UHJP, is it true that $G$ must have the $d$-UHJP for some $d \in \mathbb{N}$?

Notice that, if we additionally assume that $H$ is normal in $G$, then $G$ is an extension of $K$ by $H$ and the conclusion holds by Proposition 2.6. However, if we drop the assumption of normality, then the proof of Proposition 2.6 (see Lemma 7.1) cannot be carried out.

An affirmative answer to the above question would have as a consequence that every finite group has the $d$-UHJP for some $d \in \mathbb{N}$. To see this, notice that by Proposition 5.1 and Cayley’s theorem, it is enough to show that every $S_n$ has the $d_n$-UHJP for some $d_n \in \mathbb{N}$. Indeed, $S_n = H_nC_n$, where $H_n$ is the set of all permutations on $[n]$ which stabilize a fixed $i \in [n]$ and $C_n$ is the cyclic group generated by the cycle $(1 2 \ldots n)$. By Proposition 2.2, $C_n$ has the $n^{n-1}$-UHJP and clearly $H_n$ is isomorphic to $S_{n-1}$. By induction the conclusion follows.

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