SCHANUEL’S THEOREM FOR HEIGHTS DEFINED VIA EXTENSION FIELDS

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Abstract. Let $k$ be a number field, let $\theta$ be a nonzero algebraic number, and let $H(\cdot)$ be the Weil height on the algebraic numbers. In response to a question by T. Loher and D. W. Masser, we prove an asymptotic formula for the number of $\alpha \in k$ with $H(\alpha \theta) \leq X$.

We also prove an asymptotic counting result for a new class of height functions defined via extension fields of $k$. This provides a conceptual framework for Loher and Masser’s problem and generalizations thereof.

Moreover, we analyze the leading constant in our asymptotic formula for Loher and Masser’s problem. In particular, we prove a sharp upper bound in terms of the classical Schanuel constant.

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1. Introduction

Let $k$ be a number field. A well-known result due to Schanuel [Sch79] shows that the subset of $k^n$ of points with absolute multiplicative Weil height no larger than $X$ has cardinality

$$S_k(n)X^{d(n+1)} + O(X^{d(n+1)-1}\log X),$$

as $X$ tends to infinity. Here $d$ is the degree of $k$ and the positive constant $S_k(n)$ involves all the classical number field invariants; for the definition see (1.2).
In the present article we generalize this result in various respects motivated by a question of Loher and Masser. Let $\theta$ be a nonzero algebraic number, let $H(\cdot)$ denote the absolute multiplicative Weil height on the algebraic numbers $\mathbb{Q}$, and write $N(\theta k, X)$ for the number of $\alpha \in k$ with $H(\theta \alpha) \leq X$.

Evertse was the first one to consider the quantity $N(\theta k, X)$. The proof of his celebrated uniform upper bounds \cite{Eve84} for the number of solutions of $S$-unit equations over $k$ involves the following uniform upper bound

$$N(\theta k, X) \leq 5 \cdot 2^d X^{3d} + 1.$$ 

Later Schmidt \cite{Sch91} Lemma 8B, p.29] refined Evertse’s argument to get the correct exponent on $X$. Schmidt used a different height but elementary inequalities between them imply

$$N(\theta k, X) \leq 36 \cdot 2^{3d} X^{2d}.$$ 

But the constant is fairly large. Indeed, the constant’s exponential dependence on $d$ can be removed, as shown by Loher and Masser. More precisely, they proved

$$N(\theta k, X) \leq 68(d \log d) X^{2d},$$ 

provided $d > 1$, and $N(\theta \mathbb{Q}, X) \leq 17 X^2$. (In the special case $\theta \in k$ a similar result was obtained earlier by Loher in his Ph.D. thesis \cite{Loh01}.) By counting roots of unity they also showed that an upper bound with a constant of the form $o(d \log \log d)$ cannot hold, and hence regarding the degree their result is nearly optimal.

All the proofs of these upper bounds rely in an essential way on the box-principle, which works well for upper bounds but seems inappropriate to produce asymptotic results. This may have motivated Loher and Masser’s following statement \cite{LM04} p.279 regarding their bound on $N(\theta k, X)$: “It would be interesting to know if there are asymptotic formulae like Schanuel’s for the cardinalities here, at least for fixed $\theta$ not in $k$.”

Our Theorem 1 responds to this problem for fixed $\theta$ not in $k$, and our Theorem 3 generalizes Theorem 1 to arbitrary dimensions. (In fact, we prove a more general result, see Theorem 5 in Section 6.) Theorem 2 gives a sharp upper bound for the leading constant in these asymptotics in terms of Schanuel’s constant $S_k(n)$. In Theorem 3 we shall see asymptotic results for varying $\theta$ not in $k$. But first we introduce some notation.

We start with Schanuel’s constant $S_k(n)$ which is defined as follows

$$S_k(n) = \frac{h_k R_k}{w k \zeta_k (n + 1)} \left( \frac{2^n (2\pi)^s}{\sqrt{|\Delta_k|}} \right)^{n+1} (n+1)^{r+s-1}.$$ 

Here $h_k$ is the class number, $R_k$ the regulator, $w_k$ the number of roots of unity in $k$, $\zeta_k$ the Dedekind zeta-function of $k$, $\Delta_k$ the discriminant, $r = r_k$ is the number of real embeddings of $k$ and $s = s_k$ is the number of pairs of complex conjugate embeddings of $k$.

For each place $v$ of $k$ (or $w$ of $K := k(\theta)$ we choose the unique absolute value $| \cdot |_v$ on $k$ (or $| \cdot |_w$ on $K$) that extends either the usual Euclidean absolute value on $\mathbb{Q}$ or a usual $p$-adic absolute value. We also fix a completion $k_v$ of $k$ at $v$ and for each Archimedean place $v$ of $k$ we define a set of points $(z_0, \ldots, z_n) \in k_v^{n+1}$ by

$$\prod_{w|v} \max \{|z_0|_v, |\theta|_w |z_1|_v, \ldots, |\theta|_w |z_n|_v \} \leq \frac{1}{v_{k_v}},$$

where the product runs over all places $w$ of $K = k(\theta)$ extending $v$. As these sets are open, bounded, and not empty, they are measurable and have a finite, positive
This raises the question of the existence of bounds which are uniform in or even uniform in both quantities that is uniform in $\alpha \in O$ and the standard inequalities.

Next we consider the problem of uniformly bounding $g$ to check, the theorem implies even $g$. Let

Theorem 1. Let $\theta$ be a nonzero algebraic number, let $k$ be a number field and denote its degree by $d$. Then, as $X \geq 1$ tends to infinity, we have

$$N(\theta k, X) = g_k(\theta, 1) S_k(1) X^{2d} + O(X^{2d-1} \mathcal{L}),$$

where $\mathcal{L} := \log(X + 1)$ if $d = 1$ and $\mathcal{L} := 1$ otherwise. The implicit constant in the $O$-term depends on $\theta$ and on $k$.

Let us briefly discuss some properties of the constant $g_k(\theta, 1)$ and then illustrate the theorem by some examples.

For any nonzero $\alpha$ in $k$ we have $\theta k = \alpha \theta k$. Also, the height is invariant under multiplication by a root of unity. Therefore $N(\theta k, X) = N(\zeta \alpha \theta k, X)$ for any $\alpha \in k^*$ and any root of unity $\zeta$, in particular we have

$$g_k(\theta, 1) = g_k(\zeta \alpha \theta, 1).$$

This can also be proved directly from the definition as we shall see in Section 2. By Schanuel’s Theorem we conclude that $g_k(\zeta \alpha, 1) = 1$. But, as is straightforward to check, the theorem implies even $g_k(\zeta \alpha, 1) = 1$ for $\zeta$ a root of any unit in $O_k$ and $\alpha \in k^*$.

The fact that $H(\alpha \theta) = H(\alpha^{-1} \theta^{-1})$ implies

$$g_k(\theta, 1) = g_k(\theta^{-1}, 1).$$

This raises the question of the existence of bounds which are uniform in $\theta$ or in $d$, or even uniform in both quantities $\theta$ and $d$. From (1.1) we obtain an upper bound that is uniform in $\theta$, i.e., for $d > 1$

$$g_k(\theta, 1) \leq \frac{68d \log d}{S_k(1)}.$$
Now if we fix \( d > 1 \) and vary the fields \( k \) then by the Siegel-Brauer Theorem the right hand-side tends to infinity, so this bound really depends on \( \Delta_k \) and not only on \( d \). However, intuitively one might guess that for most \( \alpha \in k \) one has \( H(\theta \alpha) \geq H(\alpha) \), so one might even expect that \( g_k(\theta, 1) \leq 1 \) holds true, which, of course, would be best-possible. We shall answer here all of these questions. We start with the upper bound and confirm the intuitive guess.

**Theorem 2.** Let \( \theta \) be a nonzero algebraic number. Then \( g_k(\theta, n) \leq 1 \). Moreover, equality holds if and only if \( \theta \mathcal{O}_K = s^n \theta \mathcal{O}_K \) and for any Archimedean place \( v \in M_k \) and all places \( w \) of \( K \) above \( v \) the \( |\theta|_w \) are all equal.

Let us now illustrate Theorem 1 with an example, and thereby explain also the situation regarding lower bounds for \( g_k(\theta, 1) \). Let us first take \( k = \mathbb{Q} \), and \( \theta = \sqrt[p]{l} \) for a prime number \( p \). Then we get the asymptotics

\[
\frac{2\sqrt{p}}{p+1} S_{\mathbb{Q}(1)} x^2 = \frac{24\sqrt{p}}{\pi^2(p+1)} x^2.
\]

More generally, if \( p \) is inert in \( k \) and \( \theta = \sqrt[p]{l} \) then we get the asymptotics

\[
(1.6) \quad \frac{2p^{d/2}}{p^d+1} S_{h(1)} x^{2d}.
\]

Letting \( p \) tend to infinity shows that there is no lower bound for \( g_k(\sqrt[p]{l}, 1) \) which is uniform in \( \theta \). Likewise, fixing a \( p \) and taking a sequence \( \mathbb{Q}, k_1, k_2, \ldots \) of number fields with \( p \) inert in \( k_i \) and \( |k_i : \mathbb{Q}| \to \infty \) shows that there is no lower bound for \( g_k(\theta, 1) \) which is uniform in \( d \).

The fast decay of \( g_k(\sqrt[p]{l}, 1) \) as \( p \) runs over the set \( \mathbb{P}_k \) (which we define as the set of positive rational primes inert in \( k \)) suggests another problem. Let

\[
\sqrt[p]{\mathbb{P}_k} := \{ \sqrt[p]{\alpha} : p \in \mathbb{P}_k, \alpha \in k \} = \bigcup_{p \in \mathbb{P}_k} \sqrt{pk}.
\]

The above set has uniformly bounded degree, and thus, by Northcott’s Theorem, we may consider its counting function \( N(\sqrt[p]{\mathbb{P}_k}, X) := \{ \beta \in \sqrt[p]{\mathbb{P}_k} : H(\beta) \leq X \} \).

Now if \( d > 2 \) then the sum over the terms in (1.6) converges, so it is natural to ask whether the asymptotics of \( N(\sqrt[p]{\mathbb{P}_k}, X) \) are given simply by summing the asymptotics of \( N(\sqrt[p]{\mathbb{P}_k}, X) \) over \( \mathbb{P}_k \). The following result positively answers this question.

**Theorem 3.** Let \( k \) be a number field of degree \( d \). Then, as \( X \to \infty \) tends to infinity, we have

\[
N(\sqrt[p]{\mathbb{P}_k}, X) = \begin{cases} 
S_h(1) X^{d(\log \log X + O(1))} & \text{if } d = 2, \\
\left( \sum_{p \in \mathbb{P}_k} \frac{2p^{d/2}}{p^d+1} \right) S_h(1) X^{2d} + O(X^{2d-1} \mathcal{L}) & \text{if } d > 2,
\end{cases}
\]

where \( \mathcal{L} = \log X \) if \( d = 3 \) and \( \mathcal{L} = 1 \) if \( d > 3 \). The implicit constant in the \( O \)-term depends on \( k \).

The case \( d = 2 \) is just slightly more difficult than \( d > 2 \) and requires additionally Chebotarev’s density theorem and partial summation. However, it is not clear to us how to handle the case \( d = 1 \).

Finally, let us mention that Theorem 1 can also be used to count the elements in the nonzero, e.g., square classes \( k^*/(k^*)^2 \). Each class has the form \( \gamma \cdot (k^*)^2 \) with some \( \gamma \in k^* \). To count the number \( N(\gamma \cdot (k^*)^2, X) \) of elements in this square class with height no larger than \( X \) we note that \( H(\gamma \alpha^2) = H(\sqrt[p]{\gamma^2}) \), and thus \( N(\gamma \cdot (k^*)^2, X) = (1/2) (N(\sqrt[p]{\gamma}, X) - 1) \). E.g., the square class \((Q^*)^2\) has asymptotically \((6/\pi^2)X\) elements whereas the square class \(11 \cdot (Q^*)^2\) has asymptotically
only $(\sqrt{11}/\pi^2)X$ elements of height bounded by $X$.

Next we generalize Theorem 1 to higher dimensions. Let $N(\theta k^n, X)$ be the number of points $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ with $H((\theta \alpha_1, \ldots, \theta \alpha_n)) \leq X$. Of course, here $H$ is the (affine) absolute multiplicative Weil height on $\mathbb{Q}$ (see, e.g., [BG06], [HS00], and [Lau83]).

**Theorem 4.** Let $\theta$ be a nonzero algebraic number, let $k$ be a number field, denote its degree by $d$, and let $n$ be a positive rational integer. Then, as $X \geq 1$ tends to infinity, we have

$$N(\theta k^n, X) = g_k(\theta, n)S_k(n)X^{d(n+1)} + O(X^{d(n+1)-1} \Sigma),$$

where $\Sigma := \log(X+1)$ if $(n, d) = (1, 1)$, and $\Sigma := 1$ otherwise. The implicit constant in the $O$-term depends on $\theta$, on $k$, and on $n$.

Of course the invariance property holds remains valid for arbitrary $n$ instead of 1.

So far we have counted elements $\theta \alpha$ in $\theta k^n$ of bounded height. What if we replace the set $\theta k$ by $\theta + k$? Or $\theta k^2$ by $\theta_1 k \times \theta_2 k$? More generally, we suppose $L_1, \ldots, L_n$ are linearly independent linear forms in $n$ variables with coefficients in $\mathbb{Q}$ and $\theta_1, \ldots, \theta_n$ are in $\mathbb{Q}$. Suppose we want to count elements of bounded height in the set

$$\{(L_1(\alpha) + \theta_1, \ldots, L_n(\alpha) + \theta_n) : \alpha \in k^n\}.$$

Now let $\alpha := (\omega_1/\omega_0, \ldots, \omega_n/\omega_0) \in k^n$ and define $\omega := (\omega_0, \ldots, \omega_n)$. Then

$$H((L_1(\alpha) + \theta_1, \ldots, L_n(\alpha) + \theta_n)) = \prod_{\omega} \max\{|L_0(\omega)|, \ldots, |L_n(\omega)|\}_{|\omega|}^{\frac{X}{\omega_0}}$$

where $L_0(\omega) = \omega_0$ and $L_i(\omega) = L_i(\omega_1, \ldots, \omega_n) + \theta_i \omega_0$ (for $1 \leq i \leq n$), which give us $n+1$ linearly independent linear forms. Here the right hand-side defines a special case of a so-called adelic Lipschitz height $H_N$ (introduced in [Wid10b]) on $\mathbb{P}^n(K)$, where $K$ is any number field containing $k$, and the coefficients of $L_0, \ldots, L_n$, and the product runs over all places $w$ of $K$. Thus, we need to count the points $P = (\omega_0: \cdots: \omega_n) \in \mathbb{P}^n(k)$ with $\omega_0 \neq 0$ and $H_N(P) \leq X$.

These generalizations of Loher and Masser’s problem naturally motivate our general theorem (Theorem 4) which is as follows. Given two number fields $k \subseteq K$ and an adelic Lipschitz height $H_N$ on $K$, we give an asymptotic formula for the number of points $P \in \mathbb{P}^n(k)$ with $H_N(P) \leq X$, as the parameter $X$ tends to infinity. To be more accurate, we also impose a minor additional assumption on the adelic Lipschitz height $H_N$ which seems fulfilled in all natural applications, in particular, it holds in the aforementioned examples.

The special case $K = k$ of our general theorem follows from a result in [Wid10b]. There, a complementary result was proved, in the sense that points of $\mathbb{P}^n(K)$ defined over a proper subextension of $K/k$ were excluded from the counting (which is insignificant for the main term but was needed to obtain good error terms).

Now already with general linear forms as above it seems unlikely that the main term can be brought into an as civilized form as for Theorem 4 (see also the remark in [Wid10a], p. 1766 third paragraph). Indeed, a considerable part of our work consists of finding the simple representation of the constant in the special case of Theorem 4. However, it turns out that the given representation is not so convenient for theoretical considerations. Indeed, even the most obvious properties, such as the invariance property, are not immediately clear from the present definition. In Section 2 we establish a representation of $g_k(\theta, n)$ as a product of local factors...
(Proposition 2.2), which is a first step in the proof of Theorem 2 and also reveals the invariance property (1.5).

At any rate, a situation involving linear forms similar to the above turns up if we want to count solutions of a system of linear equations with certain restrictions to the coordinates of the solutions. Here is an example. Consider the equation
\[ \sqrt{2}x + \sqrt{3}y + \sqrt{5}z = 0, \]
defined over \( K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) \). Using arguments from [Wid10a] one can easily compute that the number of solutions \((x, y, z) \in K^3 \) with \( H((x, y, z)) \leq X \) is asymptotically given by
\[ \left( \frac{\sqrt{96} - (\sqrt{2} + \sqrt{3} - \sqrt{5})^2}{\sqrt{144}} \right)^8 S_K(2)X^{24} + O(X^{23}). \]

But what about the number of such solutions whose first two coordinates are rational? This question reduces to counting the elements \((\omega_0 : \omega_1 : \omega_2) \in \mathbb{P}^2(\mathbb{Q})\) with bounded adelic Lipschitz height
\[ H_{\mathcal{N}}((\omega_0 : \omega_1 : \omega_2)) = \prod_w \max\{ |\omega_0|_w, |\omega_1|_w, |\omega_2|_w \} \frac{\sqrt{2}\omega_1 + \sqrt{3}\omega_2}{\sqrt{5}} \left( 1 - \frac{|\omega_0|_w}{|\omega_2|_w} \right)^{[K_0:K_2]}. \]
Applying our general theorem gives the following asymptotic formula
\[ N_L(X) = V_{\mathcal{N}} \cdot \frac{1}{624(3)} \cdot (1 + 2 \cdot 5^{1/4} + 4 \cdot 5^{-1/2})X^3 + O(X^2) \]
for the number \( N_L(X) \) of solutions \((x, y, z) \) of (1.7) of height bounded by \( X \) and with \( x, y \in \mathbb{Q} \). Here \( V_{\mathcal{N}} \) denotes the volume of the set of points \((x_0, z_1, z_2) \) in \( \mathbb{R}^3 \) that satisfy the inequality
\[ \max\{|z_0|, |z_1|, |z_2|, |\sqrt{2}z_1 + \sqrt{3}z_2|/\sqrt{5}\} \max\{|z_0|, |z_1|, |z_2|, |\sqrt{2}z_1 - \sqrt{3}z_2|/\sqrt{5}\} < 1. \]

For the computations we refer the reader to the appendix.

Finally, by Northcott’s theorem there is no need to restrict to a fixed number field, and one could also consider all number fields of a given fixed degree simultaneously. Let us define the set
\[ \theta k(n; e) = \{ (\theta \alpha_1, \ldots, \theta \alpha_n) : [k(\alpha_1, \ldots, \alpha_n) : k] = e \}. \]
So Theorem 4 gives the asymptotics for the counting function \( N(\theta k(n; 1), X) = N(\theta k^n, X) \), and more generally, one could ask for the asymptotics of \( N(\theta k(n; e), X) \). The special case \( \theta \in k \) was considered in [Sch93], [Sch95], [Gao95], [MV08], [MV07], and [Wid09]. Indeed, it is likely that the methods from [Wid10a] and [Wid09], combined with those of the present article, are sufficient to solve this problem, provided \( n \) is large enough. On the other hand, it would be interesting to know whether Masser and Vaaler’s approach from [MV07] can be combined with ours to handle the case \( n = 1 \).

The plan of the paper is as follows. In Section 2 we establish a product representation of \( \gamma_k(\theta, n) \) and we use this to deduce some of its properties. This product form is also the starting point in the proof of Theorem 2 which we give in Section 3. Then in Section 4 we state and prove some basic facts about lattice points which are required for the proofs of Theorem 4 and Theorem 6. Section 5 provides the necessary notions such as adelic Lipschitz systems to state our general theorem. Then in Section 6 we state the general theorem (Theorem 5), and in Section 7 we give its proof. From Theorem 5 we deduce Theorem 6 which is done in Section 8. The proof of Theorem 6 is carried out in Section 9. Finally, in the appendix we...
calculate formula \((1.8)\) using Theorem \(5\).

By a prime ideal we always mean a nonzero prime ideal. By \(E \subseteq \mathcal{O}_k\), we mean that \(E\) is a nonzero ideal of \(\mathcal{O}_k\). An empty product is always interpreted as \(1\), and an empty sum is interpreted as \(0\).

2. **Product representation and invariance properties of the constant**

In this section, we use a product representation for the constant \(g_k(\theta, n)\) to derive some of its properties. Let \(\mathfrak{D}, B\) be nonzero ideals of \(\mathcal{O}_K\) or \(\mathcal{O}_k\), respectively. For convenience, we define

\[
q(\mathfrak{D}, B) := q(\mathfrak{D}, B, n) := \frac{\mathfrak{N}_K(\mathfrak{D}, uB)^{(n+1)/[K:k]}}{\mathfrak{N}_k B}.
\]

Clearly, \(q(\mathfrak{D}, B)\) is multiplicative in \(\mathfrak{D}\) and \(B\), and \(q(\mathfrak{D}, B) = q((\mathfrak{D}, uB), B)\). Moreover, if \(B_1 | B_2\) then \(q(B_2, B_1) = \mathfrak{N}_k B_1^n\) and \(q(uB_2, B_1B_2) = \mathfrak{N}_k B_1^n q(\mathfrak{D}, B_1^{-1}B_2)\).

We now define local factors at prime ideals \(P\) of \(\mathcal{O}_k\). If \(P \nmid \mathfrak{D}\) then we define \(g_P(\mathfrak{D}, n) := 1\). If \(P \mid \mathfrak{D}\) then we define

\[
g_P(\mathfrak{D}, n) := \frac{\mathfrak{N}_k p^{n+1} - \mathfrak{N}_k P^n}{\mathfrak{N}_k P^{n+1} - 1} + \frac{(\mathfrak{N}_k P - 1)(\mathfrak{N}_k P^n - 1)}{\mathfrak{N}_k P^{n+1} - 1} \sum_{j=1}^{\nu_P(\mathfrak{D}) - 1} q(\mathfrak{D}, P^j) + \frac{\mathfrak{N}_k P^n - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} q(\mathfrak{D}, P^n).\]

Clearly, \(g_P(\mathfrak{D}, n) = g_P(\mathfrak{D}_P, n)\), where \(\mathfrak{D}_P := \prod_{P|\mathfrak{D}} \mathfrak{P}^{\nu_P(\mathfrak{D})}\) is the part of \(\mathfrak{D}\) lying over \(P\).

**Lemma 2.1.** Let \(\mathfrak{D}\) be a nonzero ideal of \(\mathcal{O}_K\) and \(D := \mathfrak{D}\). Then

\[
\sum_{B|D} q(\mathfrak{D}, B) \sum_{A|B^{-1}D} \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P|AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} = \prod_P g_P(\mathfrak{D}, n).
\]

**Proof.** We start by investigating the expression

\[
S(D, B) := \sum_{A|B^{-1}D} \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P|AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}
\]

for a given ideal \(B\) of \(\mathcal{O}_k\) dividing \(D\). Clearly,

\[
S(D, B) = \prod_{P|B} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} \sum_{A|B^{-1}D} f(A),
\]

where

\[
f(A) := \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P|B} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}.
\]

The function \(f\) is multiplicative and \(f(\mathcal{O}_k) = 1\). For any prime ideal \(P\) dividing \(B^{-1}D\), we have

\[
f(P) = \begin{cases} -\mathfrak{N}_k P^{-1} & \text{if } P \nmid B, \\ -(\mathfrak{N}_k P^n - 1)/(\mathfrak{N}_k P^{n+1} - 1) & \text{if } P \mid B. \end{cases}
\]

Moreover, \(f(P^e) = 0\) if \(e > 1\). We use

\[
\sum_{A|B^{-1}D} f(A) = \prod_{P|B^{-1}D} (1 + f(P))
\]
to obtain
\[ S(D, B) = \prod_{P \mid D} \frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - 1} \prod_{P \mid B, P \mid D} \frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - 1} \prod_{P \mid (B^{-1} D, B)} \frac{\mathfrak{m}_k P - 1}{\mathfrak{m}_k P}. \]

Let \( T(D, B) := S(D, B)/\prod_{P \mid D} \frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - 1} \). Then the expression on the left-hand side of the Lemma is given by
\[
\left( \prod_{P \mid D} \frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - 1} \right) \sum_{B \mid D} q(\mathfrak{D}, B) T(D, B).
\]

Since both \( T(D, B) \) and \( q(\mathfrak{D}, B) \) are multiplicative in \( B \), this is equal to
\[
\prod_{P \mid D} \left( \frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - 1} \sum_{j=0}^{v(P)} q(\mathfrak{D}, P^j) T(D, P^j) \right).
\]

The lemma now follows from the observation that
\[
T(D, P^j) = \begin{cases} 
\frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n} \cdot \frac{\mathfrak{m}_k P - 1}{\mathfrak{m}_k P} & \text{if } 1 \leq j < v_P(D), \\
\frac{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^{n+1} - \mathfrak{m}_k P^n} & \text{if } j = v_P(D).
\end{cases}
\]

Lemma \( \ref{lemma:1} \) with \( \mathfrak{D} := O_K \) yields the following formula for \( g_k(\theta, n) \).

**Proposition 2.2.** If \( \alpha \) is nonzero and in \( \mathcal{O}_k \) with \( \alpha \theta \in \mathcal{O}_K \) then
\[
g_k(\theta, n) = \frac{V}{\mathfrak{m}_k(\mathfrak{a})^n} \prod_{P} g_P(\alpha \mathfrak{O}_K, n).
\]

The next lemma shows that \( g_k(\theta, n) \) does not depend on the choice of \( \alpha \).

**Lemma 2.3.** Let \( A \) be a nonzero ideal of \( \mathcal{O}_k \) and \( \mathfrak{D} \) a nonzero ideal of \( \mathcal{O}_K \). Then
\[
g_P(\mathfrak{a} A \mathfrak{D}, n) = \mathfrak{m}_k P_{n v_P(A)} g_P(\mathfrak{D}, n).
\]

**Proof.** Clearly, \( \mathfrak{a} A \mathfrak{D} = A \mathfrak{D} \mathfrak{D} \). Assume first that \( P \nmid \mathfrak{a} \mathfrak{D} \). The lemma is then obvious if \( P \nmid A \) as well. If \( 1 \leq j \leq v_P(A) \), we have \( q(\mathfrak{a} A \mathfrak{D}, P^j) = \mathfrak{m}_k P^n j \) and a simple computation shows that \( g_P(\mathfrak{a} A \mathfrak{D}, n) = \mathfrak{m}_k P^n j \).

Now assume that \( P \mid \mathfrak{a} \mathfrak{D} \). We have
\[
q(\mathfrak{a} A \mathfrak{D}, P^j) = \begin{cases} 
\mathfrak{m}_k P^{n j} & \text{if } 1 \leq j \leq v_P(A), \\
\mathfrak{m}_k P_{n v_P(A)} q(\mathfrak{D}, P^{j - v_P(A)}) & \text{if } v_P(A) + 1 \leq j \leq v_P(A \mathfrak{D}).
\end{cases}
\]

Thus,
\[
\sum_{j=1}^{v_P(A \mathfrak{D}) - 1} q(\mathfrak{a} A \mathfrak{D}, P^j) = \frac{\mathfrak{m}_k P^{(v_P(A)+1)n} - \mathfrak{m}_k P^n}{\mathfrak{m}_k P^n - 1} + \mathfrak{m}_k P_{n v_P(A)} \sum_{j=1}^{v_P(A \mathfrak{D}) - 1} q(\mathfrak{D}, P^j),
\]

and another simple computation proves the lemma. \( \square \)
Given nonzero $\alpha, \beta \in \mathcal{O}_k$ such that $\alpha \theta, \beta \theta \in \mathcal{O}_K$, we have
\[ \mathfrak{m}_k(\alpha)^n \prod_p g_p(\beta \theta \mathcal{O}_K, n) = \prod_p g_p(\alpha \theta \mathcal{O}_K, n) = \mathfrak{m}_k(\beta)^n, \]
which shows the independence of $g_k(\theta, n)$ from the choice of $\alpha$.

To see invariance property (1.5) directly from (2.1), we need the following lemma.

**Lemma 2.4.** Let $\alpha \in k^*$. Then
\[ V(\alpha \theta, k, n) = \frac{V(\theta, k, n)}{\mathfrak{m}_k(\alpha)^n}. \]

**Proof.** For any Archimedean place $v$ of $k$, the map $\phi_v : k_v^{n+1} \to k_v^{n+1}$ defined by $\phi_v(z_0, z_1, \ldots, z_n) = (z_0, [\alpha]_v z_1, \ldots, [\alpha]_v z_n)$ is a linear automorphism of $k_v^{n+1}$ (considered as $\mathbb{R}^{k_v \times (n+1)}$) of determinant $[\alpha]_v^{[k_v, \mathbb{R}]n}$. Therefore, $[\alpha]_v^{[k_v, \mathbb{R}]n}V_v(\alpha \theta, k, n) = V_v(\theta, k, n). \square$

Lemma 2.3 and Lemma 2.4 imply that
\[ g_k(\alpha \theta, n) = g_k(\theta, n) \]
for every nonzero $\alpha \in \mathcal{O}_k$, and hence for every $\alpha \in k^*$. In particular, it suffices to prove Theorem 2 and Theorem 4 for integral $\theta$.

3. **Proof of Theorem 2**

We start off by estimating the volume $V(\theta, k, n)$.

**Lemma 3.1.** We have
\[ V(\theta, k, n) \leq \mathfrak{m}_K(\theta)^{-n/[K:k]}. \]

Moreover, equality holds if and only if for any Archimedean place $v$ of $k$ the absolute values $|\theta|_v$ are equal for all $w \mid v$.

**Proof.** Let $v$ be an Archimedean place of $K$ and $(z_0, \ldots, z_n) \in k_v^{n+1}$. Now
\[ \prod_{w \mid v} \max\{|z_0|_w, |\theta|_v z_1|_w, \ldots, |\theta|_v z_n|_w\}^{\frac{1}{[w:v]}} \]
\[ \geq \max\{|z_0|_v, p_v z_1|_v, \ldots, p_v z_n|_v\}, \]
where $p_v = p_v(\theta) := \prod_{w \mid v} |\theta|_w^{\frac{1}{[w:v]+1}}$. Hence
\[ V_v \leq p_v^{-n[k_v, \mathbb{R}]} \begin{cases} 2^{n+1} & \text{if } v \text{ is real}, \\ 4^{n+1} & \text{if } v \text{ is complex}. \end{cases} \]

And thus,
\[ V(\theta, k, n) \leq \prod_{w \mid \infty} |\theta|_w^{-\frac{n(k_v, \mathbb{R})}{[k_v, k]}} = \mathfrak{m}_K(\theta)^{-n/[K:k]}. \]

Now if $|\theta|_w$ is constant on $w \mid v$ then we have equality in the statement of the lemma, by a similar argument as in Lemma 2.4. Let us now suppose that there exists a $v$ such that $|\theta|_w$ is not constant on $w \mid v$. Let $w_1$ above $v$ be such that $|\theta|_{w_1}$ is minimal. Then we have
\[ p_v' := \prod_{w \mid v, w \neq w_1} |\theta|_{w_1}^{K_v : k_{w_1}}/A > \prod_{w \mid v} |\theta|_{w_1}^{K_v : k_{w_1} w_1} = p_v, \]
where
\[ A := \sum_{w \mid v, w \neq w_1} [K_w : k_v]. \]
Now $\prod_{w|\nu} \max\{|z_0|_w, |\theta|_w z_1|_w, \ldots, |\theta|_w z_n|_w\}^{(K_{w_k},-1)} < 1$ implies
$$|z_0|^{(K_{w_k},-1)} (p_i'|_{z_i})^{(K_{w_k},-1)} < 1$$
for $1 \leq i \leq n$. As $p_i' > p_i$, there exists $\varepsilon > 0$ such that the open ball of radius $\varepsilon$ centered at $(1, p_i'^{-1}, \ldots, p_i^{-1})$ has empty intersection with the set defined by
$$\prod_{w|\nu} \max\{|z_0|_w, |\theta|_w z_1|_w, \ldots, |\theta|_w z_n|_w\}^{(K_{w_k},-1)} < 1.$$
To prove the lemma, we need to show that the above is bounded by \( \mathfrak{N}_k P^u \) (with strict inequality if \( l = 1 \)). To this end, let \( h \) be the function given by
\[
h(x) := x^{n+1} - x^n + x^{n+1}u - x^{n+1}uL + x^L + x^{1}.
\]
Hence, we need to show that \( h(\mathfrak{N}_k P) \geq 0 \), with a strict inequality if \( l = 1 \). With \( \tilde{u} := u - L \in (0, 1] \) and
\[
h_1(x) := x^{n+1} + x^{n} - x^{n+1}u - x^{n+1}uL + x^L + x^{1}.
\]
we have \( h(x) = x^Lh_1(x) \). If \( \tilde{u} = 1 \) then \( h_1(x) = 0 \). We observe that \( \tilde{u} = 1 \) is impossible if \( l = 1 \), since \( u < L \). Let us assume that \( 0 < \tilde{u} < 1 \) and prove that, in this case, \( h_1(x) > 0 \) for all \( x > 1 \).

The function \( h_1(x) \) is in fact a polynomial in \( x^{1/[K:k]} \). We have
\[
n\tilde{u} + n + 1 > \left( \frac{(n + 1)\tilde{u} + n}{n + 1} \right) > \frac{(n + 1)\tilde{u}}{n} > n\tilde{u}.
\]
By Descartes’ rule of signs, \( h_1(x) \) has at most three positive zeros (with multiplicities). Since \( h_1(1) = h'_1(1) = h''_1(1) = 0 \) and \( \lim_{x \to \infty} h_1(x) = \infty \), we have \( h_1(x) > 0 \) for \( x > 1 \).

We can now easily finish the proof of Theorem 2. After multiplying with a suitable element from \( k^* \) we can assume that \( \theta \) is an algebraic integer and choose \( \alpha := 1 \). From Proposition 2.2, Lemmata 3.1 and 3.2, and the observation that
\[
\mathfrak{N}_K(\theta)^{\text{vol} \mathfrak{O}_K} = \prod_{\rho} \mathfrak{N}_K((\theta \mathfrak{O}_K)_\rho)^{\text{vol} \mathfrak{O}_K},
\]
we immediately get that \( g_k(\theta, n) \leq 1 \). Now if \( g_k(\theta, n) = 1 \) then we must have equality in Lemmata 3.1 and 3.2. So by Lemma 3.1 we get that for each Archimedean place \( v \) of \( k \) the \( |\theta|_w \) for \( w = v \) are all equal. And equality in Lemma 3.2 shows that \( \theta \mathfrak{O}_K = \mathfrak{u} \mathfrak{O}_K \). This concludes the proof of Theorem 2.

4. Preliminaries on lattices

In this section we establish a basic counting result for lattice points, which will be used in the proofs of Theorem 3 and Theorem 4.

For a vector \( \mathbf{x} \) in \( \mathbb{R}^m \) we write \( |\mathbf{x}| \) for the Euclidean length of \( \mathbf{x} \). For a lattice \( \Lambda \) in \( \mathbb{R}^m \) we write \( \lambda_i = \lambda_i(\Lambda) \) \( (1 \leq i \leq m) \) for the successive minima of \( \Lambda \) with respect to the Euclidean distance.

**Definition 4.1.** Let \( M \) and \( m > 1 \) be positive integers and let \( L \) be a non-negative real. We say that a set \( S \) is in \( \text{Lip}(m, M, L) \) if \( S \) is a subset of \( \mathbb{R}^m \), and if there are \( M \) maps \( \phi_1, \ldots, \phi_M : [0, 1]^{m-1} \to \mathbb{R}^m \) satisfying a Lipschitz condition
\[
|\phi_i(x) - \phi_i(y)| \leq L|x - y| \text{ for } x, y \in [0, 1]^{m-1}, i = 1, \ldots, M,
\]
such that \( S \) is covered by the images of the maps \( \phi_i \).

We can now state and prove our counting result.

**Lemma 4.2.** Let \( m > 1 \) be an integer, let \( \Lambda \) be a lattice in \( \mathbb{R}^m \) with successive minima \( \lambda_1, \ldots, \lambda_m \), and let \( a \in \{1, \ldots, m\} \). Let \( S \) be a bounded set in \( \mathbb{R}^m \) such that the boundary \( \partial S \) of \( S \) is in \( \text{Lip}(m, M, L) \), \( S \) is contained in the zero-centered ball of radius \( L \), and 0 \( \notin S \). Then \( S \) is measurable and we have
\[
\left| |S \cap \Lambda| - \frac{\text{Vol} S}{\text{det} \Lambda} \right| \leq c_1(m) M \max \left\{ \frac{L^a}{\lambda_1^{a-1}}, \frac{L^{m-1}}{\lambda_1^{a-1} \lambda_m^{m-a}} \right\}.
\]
The constant \( c_1(m) \) depends only on \( m \).
Proof. Applying [Wid10b, Theorem 5.4] proves measurability and gives
\[
\left|S \cap \Lambda\right| = \frac{\text{Vol} S}{\text{det} \Lambda} \leq c_1(m) M \max_{0 \leq i \leq m-1} \frac{L_i}{\lambda_1 \cdots \lambda_i}.
\]

First we assume \( L/\lambda_1 \geq 1 \).

Then we conclude
\[
\max_{0 \leq i \leq m-1} \frac{L_i}{\lambda_1 \cdots \lambda_i} \leq \max_{0 \leq a \leq m-a-1} \frac{L_{\lambda_1 \cdots \lambda_{a+1}}}{\lambda_1 \cdots \lambda_{a+1}} \leq \frac{L_{\lambda_1} \cdots \lambda_{m-a}}{\lambda_1 \cdots \lambda_{m-a}} \leq \frac{L_{\lambda_1} \cdots \lambda_{m-a}}{\lambda_1 \cdots \lambda_{m-a}} = \max \left\{ \frac{L_{\lambda_1} \cdots \lambda_{m-a}}{\lambda_1 \cdots \lambda_{m-a}} \right\}.
\]

Next we assume \( L/\lambda_1 < 1 \). Then we have \( \left|S \cap \Lambda\right| = 0 \). Moreover, by Minkowski’s second theorem,
\[
\frac{\text{Vol} S}{\text{det} \Lambda} \leq c_1(m) \frac{L_m}{\lambda_1 \cdots \lambda_m}.
\]

Furthermore,
\[
\frac{L_m}{\lambda_1 \cdots \lambda_m} \leq \frac{L_m}{\lambda_1 \cdots \lambda_m} \leq \max \left\{ \frac{L_{\lambda_1} \cdots \lambda_{m-a}}{\lambda_1 \cdots \lambda_{m-a}} \right\}.
\]

The following lemma is an easy consequence of [Wid10b, Lemma 4.1], but we prefer to state it explicitly.

Lemma 4.3. Let \( \Lambda \) be a lattice in \( \mathbb{R}^m \). Then there exist linearly independent vectors \( v_1, \ldots, v_m \) in \( \Lambda \) such that \( |v_i| = \lambda_i \) for \( 1 \leq i \leq m \).

Lemma 4.4. Let \( \Lambda \) be a lattice in \( \mathbb{R}^m \). Then there exists a basis \( u_1, \ldots, u_m \) of \( \Lambda \) such that
\[
|u_i| \leq C_0(m) \lambda_i^{m-1} \lambda_1 \cdots \lambda_m
\]
for \( 1 \leq i \leq m \), where \( C_0(m) \) is an explicit constant depending only on \( m \).

Proof. Let \( v_1, \ldots, v_m \) be linearly independent vectors as in Lemma 4.3. By a lemma of Mahler and Weyl (see [Cas57, Lemma 8, p. 135]) we obtain a basis \( u_1, \ldots, u_m \) of \( \Lambda \) such that \( |u_i| \leq \max\{1, m/2\} \lambda_i \). Observing that \( |u_i| \leq |u_1| \cdots |u_m|/\lambda_1^{m-1} \), the lemma follows from Minkowski’s second theorem.

The following result will be used only for the proof of Theorem 4 in Section 9.

Lemma 4.5. Let \( \Lambda_1 \) and \( \Lambda_2 \) be lattices in \( \mathbb{R}^d \), and consider the lattice \( \Lambda := \Lambda_1 \times \Lambda_2 \) in \( \mathbb{R}^{2d} \). Then we have
\[
\lambda_1(\Lambda) = \min\{\lambda_1(\Lambda_1), \lambda_1(\Lambda_2)\},
\]
\[
\lambda_{d+1}(\Lambda) \geq \max\{\lambda_1(\Lambda_1), \lambda_1(\Lambda_2)\}.
\]

Proof. The first assertion is obvious. For the second assertion we choose, by Lemma 4.3, \( d+1 \) linearly independent elements \( v_j = (w^{(1)}_j, w^{(2)}_j) \in \Lambda \) \((1 \leq j \leq d+1)\) with \( |v_j| = \lambda_j \). Clearly, not all of them can lie in \( \mathbb{R}^d \times \{0\} \), and similarly not all of them can lie in \( \{0\} \times \mathbb{R}^d \). Suppose \( v_1 \notin \mathbb{R}^d \times \{0\} \) and \( v_2 \notin \{0\} \times \mathbb{R}^d \). Hence \( |v_1| \geq |w^{(1)}_2| \geq \lambda_1(\Lambda_2) \) and \( |v_2| \geq |w^{(1)}_1| \geq \lambda_1(\Lambda_1) \). This proves the lemma.
5. Adelic Lipschitz heights

In [MV07] Masser and Vaaler have introduced what one may call Lipschitz heights on $\mathbb{P}^n(K)$. This notion generalizes the absolute Weil height and allows so-called Lipschitz distance functions instead of just the maximum norm at the Archimedean places. Nonetheless, this notion is sometimes too rigid, as one often also needs modification at a finite number of non-Archimedean places. This leads naturally to the concept of adelic Lipschitz heights, introduced in [Wz10b].

5.1. Adelic Lipschitz systems on a number field. Let $K$ be a number field and recall that $M_K$ denotes the set of places of $K$, and that for every place $w$ we have fixed a completion $K_w$ of $K$ at $w$. We write $d_w = [K_w : \mathbb{Q}_w]$ for the local degree, where $\mathbb{Q}_w$ denotes the completion of $\mathbb{Q}$ with respect to the unique place of $\mathbb{Q}$ that extends to $w$. The value set of $w$, $\Gamma_w := \{|\alpha|_w : \alpha \in K_w\}$ is equal to $[0,\infty)$ if $w$ is Archimedean, and to

$$\{0, (\mathfrak{N}_K \mathfrak{P}_w)^0, (\mathfrak{N}_K \mathfrak{P}_w)^{\pm 1/d_w}, (\mathfrak{N}_K \mathfrak{P}_w)^{\pm 2/d_w}, \ldots\}$$

(topologized by the discrete topology) if $w$ is a non-Archimedean place corresponding to the prime ideal $\mathfrak{P}_w$ of $\mathcal{O}_K$. For $w | \infty$ we identify $K_w$ with $\mathbb{R}$ or $\mathbb{C}$, respectively, and we identify $\mathbb{C}$ with $\mathbb{R}^2$.

**Definition 5.1.** An adelic Lipschitz system $\mathcal{N}$ on $K$ (of dimension $n$) is a set of continuous maps

$$N_w : K_w^{n+1} \to \Gamma_w, \quad w \in M_K$$

such that for $w \in M_K$ we have

(i) $N_w(0) = 0$ if and only if $z = 0$,

(ii) $N_w(az) = |a|_w N_w(z)$ for all $a \in K_w$ and all $z \in K_w^{n+1}$,

(iii) if $w | \infty$: $\{z : N_w(z) = 1\}$ is in $\text{Lip}(d_w(n + 1), M_w, L_w)$ for some $M_w, L_w$,

(iv) if $w \not| \infty$: $N_w(z_1 + z_2) \leq \max\{N_w(z_1), N_w(z_2)\}$ for all $z_1, z_2 \in K_w^{n+1}$.

Moreover, we assume that the equality of functions

$$N_w(z) = \max\{|z_0|_w, \ldots, |z_n|_w\}$$

holds for all but a finite number of $w \in M_K$.

If we consider only the functions $N_w$ for $w | \infty$ then we get a Lipschitz system (of dimension $n$) in the sense of Masser and Vaaler [MV07].

For all $w \in M_K$ there are $c_w \leq 1$ in the value group $\Gamma_w^* = \Gamma_w \setminus \{0\}$ with

$$c_w \max(|z_0|_w, \ldots, |z_n|_w) \leq N_w(z) \leq c_w^{-1} \max(|z_0|_w, \ldots, |z_n|_w)$$

for all $z = (z_0, \ldots, z_n)$ in $K_w^{n+1}$. Due to (5.2) we can and will assume that

$$c_w = 1$$

for all but a finite number of places $w$. We define

$$C_N^{\text{fin}} := \prod_{w | \infty} c_w^{-\frac{d_w}{|\mathfrak{P}_w|}} \geq 1,$$

and

$$C_N^{\text{inf}} := \max_{w | \infty} \{c_w^{-1}\} \geq 1.$$
For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, we write $v_\mathfrak{p}$ for the corresponding valuation on $K$, normalized by $v_\mathfrak{p}(K^*) = \mathbb{Z}$. For a nonzero fractional ideal $\mathfrak{A}$ of $K$, a non-Archimedean place $w$ of $K$, associated to the prime $\mathfrak{p}_w$, we define
\[
|\mathfrak{A}|_w := \mathfrak{N}_K(\mathfrak{p}_w)^{-v_\mathfrak{p}(\mathfrak{A})/d_w},
\]
so that $|\alpha|_w = |\alpha \mathcal{O}_K|_w$ for $\alpha \in K^*$. For $w \in M_K$ let $\sigma_w$ be the canonical embedding of $K$ in $K_w$, extended componentwise to $K^{n+1}$. For any nonzero $\omega \in K^{n+1}$, let $i_{\mathcal{X}}(\omega)$ be the unique fractional ideal of $K$ defined by
\[
N_w(\sigma_w \omega) = |i_{\mathcal{X}}(\omega)|_w
\]
for all non-Archimedean $w \in M_K$, and we set by convention $i_{\mathcal{X}}(0) := \{0\}$.
Moreover, set
\[
\mathcal{O}_K(\omega) := \omega_0 \mathcal{O}_K + \cdots + \omega_n \mathcal{O}_K,
\]
so that $\mathcal{O}_K(\omega)$ is simply $i_{\mathcal{X}}(\omega)$ for any adelic Lipschitz system with \((5.2)\) for all finite places. Now by \((5.3)\) we get
\[
(5.7) \quad c_w \max\{|\omega_0|_w, \ldots, |\omega_n|_w\} \leq |i_{\mathcal{X}}(\omega)|_w \leq c_w^{-1} \max\{|\omega_0|_w, \ldots, |\omega_n|_w\}.
\]
Recall that $c_w = 1$ up to finitely many exceptions and let
\[
F_{\mathcal{X}} := \{\mathfrak{A} : \mathfrak{A} \text{ nonzero fractional ideal of } K \text{ and } c_w \leq |\mathfrak{A}|_w \leq c_w^{-1} \text{ for all } w \mid \infty\}.
\]
By unique factorization of fractional ideals, $F_{\mathcal{X}}$ is finite. Moreover, for any $\omega \in K^{n+1}$, we have
\[
(5.8) \quad i_{\mathcal{X}}(\omega) = \mathcal{O}_K(\omega)\mathfrak{H}(\omega)
\]
for some $\mathfrak{H}(\omega) \in F_{\mathcal{X}}$. Taking the product in \((5.7)\) over all finite places with multiplicities $d_w$ shows that
\[
(5.9) \quad C_{\mathcal{X}}^{\inf} [K:Q] \mathfrak{N}_K \mathcal{O}_K(\omega) \leq \mathfrak{N}_K i_{\mathcal{X}}(\omega) \leq C_{\mathcal{X}}^{\inf} [K:Q] \mathfrak{N}_K \mathcal{O}_K(\omega).
\]

5.2. Adelic Lipschitz heights on $\mathbb{P}^n(K)$. Let $\mathcal{N}$ be an adelic Lipschitz system on $K$ of dimension $n$. Then the height $H_{\mathcal{N}}$ on $K^{n+1}$ is defined by
\[
H_{\mathcal{N}}(\omega) := \prod_{w \in M_K} N_w(\sigma_w(\omega))^{\delta_{\mathcal{N}}/d_w}.
\]
Thanks to the product formula and \((ii)\) from Definition \((5.1)\) $H_{\mathcal{N}}(\omega)$ is invariant under scalar multiplication by elements of $K^*$. Therefore $H_{\mathcal{N}}$ is well-defined on $\mathbb{P}^n(K)$ by setting
\[
H_{\mathcal{N}}(P) := H_{\mathcal{N}}(\omega),
\]
where $P = (\omega_0 : \cdots : \omega_n) \in \mathbb{P}^n(K)$ and $\omega = (\omega_0, \ldots, \omega_n) \in K^{n+1}$. We note that by \((5.3)\), \((5.5)\) and \((5.6)\) we have
\[
(5.10) \quad (C_{\mathcal{N}}^{\inf} C_{\mathcal{N}}^{\inf})^{-1} H(P) \leq H_{\mathcal{N}}(P) \leq C_{\mathcal{N}}^{\inf} C_{\mathcal{N}}^{\inf} H(P),
\]
where $H(P)$ denotes the projective absolute multiplicative Weil height of $P$. Hence, by Northcott’s theorem, $\{P \in \mathbb{P}^n(K) : H_{\mathcal{N}}(P) \leq X\}$ is a finite set for each $X$ in $[0, \infty)$. 

6. The general theorem

Let \( k \subseteq K \) be number fields and let \( \mathcal{N} \) be an adelic Lipschitz system of dimension \( n \) on \( K \). Recall that the functions \( N_w, n, \) and \( K \) are all part of the data of \( \mathcal{N} \). From \( \mathcal{N} \) we obtain an adelic Lipschitz height \( H_\mathcal{N} \) on \( \mathbb{P}^n(K) \). Our goal in this section is to derive an asymptotic formula for the counting function

\[
N_\mathcal{N}(\mathbb{P}^n(k), X) := \{ P \in \mathbb{P}^n(k) : H_\mathcal{N}(P) \leq X \}.
\]

Let us set some necessary notation first. For each Archimedean place \( v \) of \( k \) we denote by \( \Lambda(v) \) the set of archimedean places of \( k \) lying above \( v \) and \( \Xi(v) \) the set of non-Archimedean places of \( k \) lying above \( v \).

\[
(6.1) \quad N_v(z) := \prod_{w \mid v} N_w(z) \frac{d_w}{w | \mathcal{H}(K)}.
\]

Let \( \mathcal{N} = \mathcal{N}(N, k) \) be the collection of functions \( N_v \), where \( N_v \) is as in \( (6.1) \) if \( v \) is an Archimedean place of \( k \) and

\[
N_v(z) := \max \{|z_0|_v, \ldots, |z_n|_v\}
\]

if \( v \) is a non-Archimedean place of \( k \). From now on we assume that \( \mathcal{N} \) is an adelic Lipschitz system (of dimension \( n \)) on \( k \) (the conditions (i), (ii) and (iv) are automatically satisfied but (iii) may possibly fail). Hence there exists a positive integer \( M_{\mathcal{N}} \) and a positive real number \( L_{\mathcal{N}} \) such that the sets defined by \( N_v(z) = 1 \) lie in \( \text{Lip}(d_v(n+1), M_{\mathcal{N}}, L_{\mathcal{N}}) \) for all Archimedean places \( v \) of \( k \). The sets defined by \( N_v(z) < 1 \) are measurable and have a finite, positive volume which we denote by \( V_v \), and set

\[
(6.2) \quad V_{\mathcal{N}} := \prod_{v \mid \infty} V_v.
\]

We denote by \( \sigma_1, \ldots, \sigma_d \) the embeddings from \( k \) to \( \mathbb{R} \) or \( \mathbb{C} \) respectively, ordered such that \( \sigma_{r+s+i} = \sigma_{r+i} \) for \( 1 \leq i \leq s \). We define

\[
(6.3) \quad \sigma : k \rightarrow \mathbb{R}^r \times \mathbb{C}^s = \mathbb{R}^d
\]

\[
\sigma(\omega) = (\sigma_1(\omega), \ldots, \sigma_{r+s}(\omega))
\]

and extend \( \sigma \) componentwise to get a map

\[
(6.4) \quad \sigma : k^{n+1} \rightarrow \mathbb{R}^m,
\]

where \( m = d(n+1) \).

For nonzero fractional ideals \( C \) of \( k \), and \( \mathfrak{D} \) of \( K \), we define the following subsets of \( \mathbb{R}^m \):

\[
\begin{align*}
\Lambda_C(\mathfrak{D}) & := \{ \sigma(\omega) : \omega \in k^{n+1}, \mathcal{O}_k(\omega) = C, i_\mathcal{N}(\omega) = \mathfrak{D} \}, \\
\Lambda_C(\mathfrak{D}) & := \{ \sigma(\omega) : \omega \in k^{n+1}, \mathcal{O}_k(\omega) = C, i_\mathcal{N}(\omega) \subseteq \mathfrak{D} \}, \\
\Lambda(\mathfrak{D}) & := \{ \sigma(\omega) : \omega \in k^{n+1}, i_\mathcal{N}(\omega) \subseteq \mathfrak{D} \}.
\end{align*}
\]

Note that by \( (5.8) \) we have

\[
(6.5) \quad \mathfrak{D} \in \mathcal{A}CF_{\mathcal{N}}
\]

whenever \( \Lambda_C(\mathfrak{D}) \) is non-empty, where \( \mathcal{A}CF_{\mathcal{N}} \) denotes the finite set of fractional ideals of the form \( \mathcal{A}\mathfrak{F} \) with \( \mathfrak{F} \in F_{\mathcal{N}} \).

Let \( \mathcal{R} \) be a set of integral representatives for the class group \( C_k \). For any \( C \in \mathcal{R} \), we choose a finite set \( S_C \) of nonzero fractional ideals of \( K \) such that

\[
S_C \text{ contains all } \mathfrak{D} \text{ with } \Lambda_C(\mathfrak{D}) \neq \emptyset.
\]

Moreover, we choose a finite set \( T \) in the following way. For any \( \mathfrak{D} \in S_C \), let \( T_C, \mathfrak{D} \) be the set of all nonzero ideals \( \mathfrak{A} \) of \( \mathcal{O}_K \) such that \( \Lambda_C(\mathfrak{A}\mathfrak{D}) \neq \emptyset \). This set is finite, since, similar as above, we have \( \mathfrak{A}\mathfrak{D}\mathfrak{E} \in \mathcal{A}CF_{\mathcal{N}} \) for some ideal \( \mathfrak{E} \) of \( \mathcal{O}_K \) whenever
Weber. Some of which can be traced back to Schanuel, or even to Dedekind and Weil, norms, as we shall prove in the appendix (see Lemma A1).

The map \( R^L \) of variables \( z_1, \ldots, z_n \) follows that \( g_k^N \) the choices of these sets. From Theorem 5, (5.10), and Schanuel’s theorem it will follow that \( g_k^N > 0 \).

Finally, we define the embedding \( \exp \) the diagonal exponential map from \( \mathbb{R} \) to \( \mathbb{R}^k \). The proof of Theorem 5 makes frequent use of arguments from [MV07] and [Wid10b] (some of which can be traced back to Schanuel, or even to Dedekind and Weber).

Let \( \Lambda_C(\mathfrak{A}) \neq \emptyset \). Then we choose \( T \) to be any finite set of nonzero ideals of \( \mathcal{O}_K \) such that

\[ T \text{ contains all the sets } T_{C,D} \text{ for } C \in \mathcal{R} \text{ and } D \in S_C. \]

We define

\[ g_k^N := \sum_{C \in \mathcal{R}} \sum_{D \in S_C} \sum_{E \in T} \mu_K(\mathfrak{A}) \sum_{E \subseteq \mathfrak{m}_k} \mu_k(E) \frac{\mathfrak{m}_K^{n+1}}{\det \Lambda(\mathfrak{A},(CE))}, \]

where

\[ \Lambda(\mathfrak{A},CE) = \Lambda(\mathfrak{A}) \cap \sigma((CE)^{n+1}). \]

Note that the infinite sum in (6.6) taken over all nonzero ideals \( E \) converges absolutely, as \( \det \Lambda(\mathfrak{A},CE) \geq (2 \cdot q_k^{(n+1)}) \).

Although \( g_k^N \) seems to depend on the choice of \( \mathcal{R}, S_C \) and \( T \), we will see that this is actually not the case. Of course, one could impose a minimality condition to render the choice of the sets \( S_C \) and \( T \) unique, but for the calculation of \( g_k^N \) it is convenient to have more flexibility for the choices of these sets. From Theorem 5, (5.10), and Schanuel’s theorem it will follow that \( g_k^N > 0 \).

Finally, we define

\[ A_N := A_N(k) := |F_N| \mathcal{M}_N^{d+1}((L_N + C_N^{nf})C_N^{fin})^{d(n+1)-1}. \]

We can now state the theorem.

**Theorem 5.** Let \( k \subseteq K \) be number fields, \( d := [k : \mathbb{Q}] \), let \( N \) be an adelic Lipschitz system (of dimension \( n \)) on \( K \), and suppose that \( N' = N'(N,k) \) is an adelic Lipschitz system (of dimension \( n \)) on \( k \). Then, as \( X \to \infty \) tends to infinity, we have

\[ N_N(P^n(k),X) = \omega_K^{-1}(n+1)^{r+s-1} R_k V_N g_k^N X^d(n+1) + O(|T|A_N X^{d(n+1)-1} \mathfrak{L}), \]

where \( \mathfrak{L} = 1 + \log(C_N^{nf}/C_N^{fin} X) \) if \( (n,d) = (1,1) \) and \( \mathfrak{L} = 1 \) otherwise. The implicit constant in the \( O \)-term depends only on \( k \) and on \( n \).

The hypothesis of \( N' \) being an adelic Lipschitz system is a minor one. For instance, this hypothesis is certainly fulfilled when the functions \( N_w \) of \( N \) are norms, as we shall prove in the appendix (see Lemma A1).

**7. Proof of Theorem 5**

The proof of Theorem 5 makes frequent use of arguments from [MV07] and [Wid10b] (some of which can be traced back to Schanuel, or even to Dedekind and Weber).

Let \( q := r + s - 1 \), \( \Sigma \) the hyperplane in \( \mathbb{R}^{q+1} \) defined by \( x_1 + \cdots + x_{q+1} = 0 \) and \( \delta = (d_1, \ldots, d_{q+1}) \) with \( d_i = 1 \) for \( 1 \leq i \leq r \) and \( d_i = 2 \) for \( r + 1 \leq i \leq r + s = q + 1 \). The map \( l(\eta) := (d_1 \log |\sigma_1(\eta)|, \ldots, d_{q+1} \log |\sigma_{q+1}(\eta)|) \) sends \( \omega \) to \( \mathbb{R}^{q+1} \). For \( q > 0 \) the image of the unit group \( \mathcal{O}_K^{n+1} \) under \( l \) is a lattice in \( \Sigma \) with determinant \( \sqrt{q} + 1 R_k \).

We now define a set \( S_F(T) \) using our adelic Lipschitz system \( N' \) on \( k \). Let \( F \) be a bounded set in \( \Sigma \) and for \( T > 0 \) let \( F(T) \) be the vector sum

\[ F(T) := F + \delta(-\infty, \log T). \]

We denote by \( (N_1(z_1))^{d_1}, \ldots, N_{q+1}(z_{q+1})^{d_{q+1}} \) the diagonal exponential map from \( \mathbb{R}^{q+1} \) to \( (0,\infty)^{q+1} \). Any embedding \( \sigma_i \) \( (1 \leq i \leq q + 1) \) corresponds to an Archimedean place \( v \), and thus gives rise to one of our Lipschitz distance functions \( N_i := N_v \) from \( N' \). We use variables \( z_1, \ldots, z_{q+1} \) with \( z_i \) in \( \mathbb{R}^{d(n+1)}. \) Exactly as in [MV07] we define \( S_F(T) \) in \( \mathbb{R}^m \) for \( m = \sum_{i=1}^{q+1} d_i(n+1) = d(n+1) \) as the set of all \( z_1, \ldots, z_{q+1} \) such that

\[ (N_1(z_1)^{d_1}, \ldots, N_{q+1}(z_{q+1})^{d_{q+1}}) \in \exp(F(T)). \]
We note that
\[(7.3) \quad 0 \notin S_F(T) .\]
Using (ii) from Definition 3.1 it is easily seen that \(S_F(T)\) is homogeneously expanding, i.e.,
\[(7.4) \quad S_F(T) = TS_F(1) .\]
Moreover, if \(F\) lies in a zero-centered ball of radius \(r_F\) then
\[S_F(T) \subseteq \{(z_1, \ldots, z_{q+1}) : N_t(z_i) \leq \exp(r_F)T \text{ for } 1 \leq i \leq q + 1}\].
The latter set lies in the the zero-centered ball of radius \(\sqrt{mC_n^{nf}} \exp(r_F)T\), and thus
\[(7.5) \quad S_F(T) \subseteq B_0(\sqrt{mC_n^{nf}} \exp(r_F)T) .\]
Note that for \(q = 0\) we automatically have \(F = \{0\}\), and our set \(S_F(T)\) is precisely
the set defined by \(N_t(\mathbf{z}) \leq T\).

We now specify our set \(F\) when \(q > 0\). We choose a basis \(u_1, \ldots, u_q\) of the lattice
\(l(O_T^q)\) as in Lemma 4.4. Set \(F := [0,1)u_1 + \cdots + [0,1)u_q\). So \(F\) is measurable
of \((q\text{-dimensional})\) volume
\[(7.6) \quad \text{Vol}(F) = \sqrt{q+1}R_k\]
(and this remains true for \(q = 0\)). From the argument in [Wid10b] following (8.2), we see that \(\lambda_i(l(O^q_T)) \geq c_d\) for some positive constant \(c_d\) depending only on \(d\). With the estimate from Lemma 1.3, we get
\[(7.7) \quad |u_i| \leq C_0(q)c_d^{q+1}\text{Vol}(F) \leq C_dR_k, \quad (1 \leq i \leq q)\]
for some positive constant \(C_d\) depending only on \(d\). Note that \(F\) lies in the zero centered ball of radius \(qC_dR_k\), and this remains trivially true for \(q = 0\). Therefore by (7.5)
\[(7.8) \quad S_F(T) \subseteq B_0(\kappa T),\]
where
\[(7.9) \quad \kappa := \sqrt{mC_n^{nf}} \exp(qC_dR_k) .\]

**Lemma 7.1.** There exists a constant \(c_k(n)\) depending only on \(k\) and \(n\), a positive integer \(M\), and a positive real \(\tilde{L}\) with \(M \leq c_k(n)M_n^{q+1}, \tilde{L} \leq c_k(n)(L_{N'} + C_n^{nf})\), such that
\[(7.10) \quad \partial S_F(T) \in \text{Lip}(m, M, \tilde{L}T) \text{ and } S_F(T) \subseteq B_{0}(\tilde{L}T) .\]

**Proof.** The second part follows immediately from (7.8) and (7.9).

Let us now prove the first part. For \(q = 0\) our set \(S_F(T)\) is precisely the set defined by \(N_v(\mathbf{z}) \leq T\), where \(v\) is the single Archimedean place of \(k\). So the boundary of \(S_F(T)\) is the set \(\{z : N_v(\mathbf{z}) = T\} = T\{z : N_v(\mathbf{z}) = 1\}\). By assumption \(N'\) is an adelic Lipschitz system, and thus the latter set lies in \(\text{Lip}(m, M_{N'}, L_{N'}T)\). This proves the lemma for \(q = 0\).

Suppose now that \(q \geq 1\). Then we can find \(2q\) linear maps \(\psi_i : [0,1]^{q-1} \rightarrow \Sigma\) parameterizing \(\partial F\) which, because of (7.7), will satisfy a Lipschitz condition with constant \((q-1)C_2R_k\) (for \(q = 1\) this is simply interpreted as \(|\partial F| \leq 2\)). The claim now follows from [Wid10b, Lemma 7.1] by a simple computation. \(\square\)

We conclude from [MV07, Lemma 4.1, (7.4), and (7.3)] that \(S_F(T)\) is measurable and has volume
\[(7.11) \quad \text{Vol } S_F(T) = (n + 1)^q R_kV_{N'}T^n .\]
Lemma 7.2. We have

\[ N_N(\mathbb{P}^n(k), X) = \omega_k^{-1} \sum_{C \in \mathcal{K}} \sum_{D \in S_C} |\Lambda_{C}(\mathcal{D}) \cap S_F(X \mathfrak{N}_K \mathcal{D}^{1/[K: \mathbb{Q}]})|. \]

Proof. Let \( P \in \mathbb{P}^n(k) \) with homogeneous coordinates \((\omega_0, \ldots, \omega_n) = \omega \in k^{n+1} \setminus \{0\}\). Recall the definition of the adelic Lipschitz system \( N' \). The functions \( N_v \) (or \( N_i \)) will denote those associated with \( N' \), whereas \( N_w \) will denote a function associated with the adelic Lipschitz system \( N \) on \( K \).

Now

\[ (7.12) \]

\[ i_{N'}(\omega) = \mathcal{O}_k(\omega) \]

Suppose \( \varepsilon \in k^* \). Then we have

\[ i_{N'}(\varepsilon \omega) = \varepsilon i_{N'}(\omega). \]

Hence the ideal class of \( i_{N'}(\omega) \) is independent of the coordinates \( \omega \) we have chosen. In particular, we can choose \( \omega \) such that \( i_{N'}(\omega) = C \) for some unique \( C \) in \( \mathcal{R} \). Thus, \( \omega \) is unique up to scalar multiplication by units \( \eta \), and moreover, \( i_{N'}(\omega) := \mathcal{D} \in S_C \).

The set \( F(\infty) = F + \mathbb{R}_d \) is a fundamental set of \( \mathbb{R}^{q+1} \) under the action of the additive subgroup \( l(O'_k) \). Because of Definition 5.2 (ii) we have

\[ \log N_i(\sigma_i(\eta \omega))^{d_i} = \log N_i(\sigma_i(\omega))^{d_i} + d_i \log |\sigma_i| \eta \]

for \( 1 \leq i \leq q + 1 \). Hence, there exist exactly \( \omega_k \) representatives \( \omega \) of \( P \) with

\[ (d_1 \log N_i(\sigma_i \omega), \ldots, d_{q+1} \log N_{q+1}(\sigma_{q+1}(\omega))) \in F(\infty). \]

But the above is equivalent with

\[ (N_1(\sigma_1(\omega))^{d_1}, \ldots, N_{q+1}(\sigma_{q+1}(\omega))^{d_{q+1}}) \in \exp(F(\infty)). \]

Furthermore

\[ \exp(F(T_0)) = \{(X_1, \ldots, X_{q+1}) \in \exp(F(\infty)) : X_1 \cdots X_{q+1} \leq T_0^{d_1}\}. \]

Hence, for all \( \omega_k \) representatives \( \omega \) of \( P \) as above, the inequality

\[ \prod_{v \mid \infty} N_v(\sigma_v(\omega))^{d_v/d} = \prod_{v \mid \infty} \prod_{\omega \mid v} N_w(\sigma_w(\omega))^{d_w/[K: \mathbb{Q}]} \leq T_0 \]

is equivalent to

\[ \sigma \omega \in S_F(T_0). \]

On the other hand,

\[ \prod_{w \mid \infty} N_w(\sigma_w(\omega))^{d_w/[K: \mathbb{Q}]} = \mathfrak{N}_K i_{N'}(\omega)^{-1/[K: \mathbb{Q}]} = \mathfrak{N}_K \mathcal{D}^{-1/[K: \mathbb{Q}]} \]

As

\[ H_N(P) = \prod_{v \mid \infty} \prod_{w \mid v} N_w(\sigma_w(\omega))^{d_w/[K: \mathbb{Q}]} \prod_{w \mid \infty} N_w(\sigma_w(\omega))^{d_w/[K: \mathbb{Q}]}, \]

the claim follows. \( \square \)

Lemma 7.3. We have

\[ N_N(\mathbb{P}^n(k), X) = \omega_k^{-1} \sum_{C \in \mathcal{K}} \sum_{D \in S_C} \mu_K(\mathcal{D}) \sum_{E \subseteq \mathcal{D}_k} \mu_k(E) |\Lambda(\mathfrak{N}_E, CE) \cap S_F(X \mathfrak{N}_K \mathcal{D}^{1/[K: \mathbb{Q}]})|, \]

where \( E \) runs over all nonzero ideals of \( \mathcal{O}_k \).
Proof. We start off from Lemma 7.2 and we apply Möbius inversion twice to get rid of the two coprimality conditions \( C \) and \(*\).

Directly from the definition we get
\[
\Lambda_C(\mathcal{A} \mathcal{D}) = \bigcup_n \Lambda_C(\mathcal{A} \mathcal{B} \mathcal{D}),
\]
where \( \mathcal{B} \) runs over all nonzero ideals of \( \mathcal{O}_K \). This is clearly a disjoint union. Note that \( \Lambda_C(\mathcal{A} \mathcal{B} \mathcal{D}) \neq \emptyset \) only when \( \mathcal{A} \mathcal{B} \mathcal{D} \) lies in the finite set \( S_C \). Möbius inversion leads then to
\[
|\Lambda_C(\mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]})) = \sum_{\mathfrak{a}} \mu_K(\mathfrak{a}) \sum_{\mathfrak{b}} |\Lambda_C(\mathcal{A} \mathcal{B} \mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
\[
= \sum_{\mathfrak{a}} \mu_K(\mathfrak{a}) |\Lambda_C(\mathcal{A} \mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
where the sums run over all nonzero ideals in \( \mathcal{O}_K \). Next note that by definition of \( T_C \mathcal{D} \) we have \( \Lambda_C(\mathcal{A} \mathcal{D}) = \emptyset \) whenever \( \mathfrak{a} \notin T_C \mathcal{D} \). As \( T_C \mathcal{D} \subseteq T \) we can restrict the last sum to \( \mathfrak{a} \in T \) and we get
\[
|\Lambda_C(\mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]})) = \sum_{\mathfrak{a} \in T} \mu_K(\mathfrak{a}) |\Lambda_C(\mathcal{A} \mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
We now deal with the second coprimality condition \( C \). Also directly from the definition we get
\[
\Lambda(\mathcal{A} \mathcal{D}, EC) = \Lambda(\mathcal{A} \mathcal{D}) \cap \sigma((EC)^{n+1}) = \bigcup_{B \subseteq \mathcal{O}_K} \Lambda_{ECB}(\mathcal{A} \mathcal{D}) \cup \{0\}.
\]
Again, \( B \) runs over all nonzero ideals of \( \mathcal{O}_K \) and the union is disjoint. As \( \sigma((EC)^{n+1}) \) is a lattice and \( S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]})) \) is bounded we conclude from the latter equality that \( \Lambda_{ECB}(\mathcal{A} \mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]})) \) is empty for all but finitely many \( B \). Möbius inversion and \( (\mathcal{A} \mathcal{D}) \) lead therefore to
\[
|\Lambda_C(\mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
\[
= \sum_{E \subseteq \mathcal{O}_K} \mu_K(E) \sum_{B \subseteq \mathcal{O}_K} |\Lambda_{ECB}(\mathcal{A} \mathcal{D}) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
\[
= \sum_{E \subseteq \mathcal{O}_K} \mu_K(E) |\Lambda(\mathcal{A} \mathcal{D}, CE) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
In view of Lemma 7.2 this proves the claim. \( \square \)

We choose a positive real \( \Gamma \) such that for any \( C \in \mathcal{R} \) and any \( \mathcal{D} \in S_C \)
\[
(7.13) \quad \Gamma \leq \frac{\eta_K C}{\eta_K(\mathcal{D})^{1/[K:k]}}
\]
Before we proceed note that if \( S_C \) is chosen minimal for all \( C \in \mathcal{R} \) (i.e. \( S_C = \{i_N(\omega) : \omega \in k^{n+1}, \mathcal{O}_k(\omega) = C\} \)) then it follows from \( 5.9 \) that we can choose \( \Gamma = C^{m-d}_K \), and moreover, \( |S_C| \leq |F_N| \).

Lemma 7.4. Let \( \lambda_1 = \lambda_1(\Lambda(\mathcal{A} \mathcal{D}, CE)) \) be the first successive minimum of the lattice \( \Lambda(\mathcal{A} \mathcal{D}, CE) \), and let \( M \) and \( \tilde{F} \) be as in Lemma 7.1. Then we have
\[
|\Lambda(\mathcal{A} \mathcal{D}, CE) \cap S_F(X \mathcal{O}_K \mathcal{D}^{1/[K:Q]}))|
\]
\[
= \frac{Vol_{S_F(1)\mathcal{O}_K \mathcal{D}}^{1/[K:Q]} X^m}{\det \Lambda(\mathcal{A} \mathcal{D}, CE)}
\]
\[
+ O \left( M \frac{\eta_K(\mathcal{D})}{\lambda_1^{m-1}} (\tilde{F}X)^{m-1} \right),
\]
where the constant in the $O$-term depends only on $m$. Moreover, with $\Gamma$ as in (7.13) we have

$$\lambda_1 \geq \mathfrak{N}_k(D)^{1/[K:Q]}(\Gamma \mathfrak{N}_k(E))^{1/d}.$$ 

And finally, with $\kappa$ as in (7.9), if $\mathfrak{N}_k E > (\kappa X)^d / \Gamma$ then

$$\Lambda(\mathfrak{A} \mathfrak{D}, CE) \cap S_F(X \mathfrak{N}_k D^{1/[K:Q]}) = \emptyset.$$ 

Proof. For the first assertion we use (7.3) and apply Lemma 4.2 with $a = D$. Thanks to (7.8) and Lemma 7.1 the required conditions are satisfied, and using (7.4) the first result drops out.

Now for the second statement we first observe that $\lambda_1$ is at least as large as the first successive minimum of the lattice $\sigma(CE)$. But it is well-known that the latter is at least $\mathfrak{N}_k(CE)^{1/d}$, see, e.g., [MV07, Lemma 5]. Now as $\mathfrak{D} \in S_C$ and by the definition of $\Gamma$ we get $\mathfrak{N}_k C \geq \Gamma \mathfrak{N}_k(D)^{1/[K:Q]}$ and this yields the second assertion.

The last claim follows upon combining the above estimate for $\lambda_1$ with (7.3), (7.8). \qed

We can now conclude the proof of Theorem 5. Let us first assume that $(n,d) \neq (1,1)$. Combining Lemma 7.3, Lemma 7.4 and (7.11) gives the main term as in Theorem 5. The error term is bounded by

$$\sum_{C \in \mathfrak{R}} \sum_{D \in S_C} \sum_{A \in T} \sum_{E \leq C_k} O\left(\frac{\mathfrak{N}_k D^{m-1} (\tilde{L}X)^{m-1}}{\lambda_1^{m-1}}\right) \leq \sum_{C \in \mathfrak{R}} \sum_{D \in S_C} \sum_{A \in T} \sum_{E \leq C_k} O\left(\frac{\tilde{M}(\tilde{L}X)^{m-1}}{\Gamma(m-1)/d \mathfrak{N}_k E(n+1)-1/d}\right) \leq \sum_{C \in \mathfrak{R}} \sum_{D \in S_C} \sum_{A \in T} O\left(\frac{\tilde{M}(\tilde{L}X)^{m-1}}{\Gamma(m-1)/d}\right) = O\left(\sum_{C \in \mathfrak{R}} |S_C||T|\frac{\tilde{M}(\tilde{L}X)^{m-1}}{\Gamma(m-1)/d}\right).$$

This proves the Theorem in the case $(n,d) \neq (1,1)$ except that the constant in the error term is different from the one in the statement of the theorem. In particular, it shows that the main term is independent of the particular choice of the sets $S_C$. However, If we choose all the sets $S_C$ to be minimal then, by the remark just after (7.13), we can choose $\Gamma = C_N^{-n/d}$, and $|S_C| \leq |F_N|$. This, and not forgetting the definition of $\tilde{M}$ and $\tilde{L}$ from Lemma 7.4 yields the desired error term.
We now assume \((n,d) = (1,1)\) (which of course means \(k = \mathbb{Q}, E = \{C\}, \omega_k = 2\)). Using also the last part of Lemma 7.4 we conclude

\[
N(\mathbb{P}^1(\mathbb{Q}), X) = \frac{1}{2} \sum_{\mathfrak{A} \in S_C} \sum_{\mathfrak{p} \in T} \mu_K(\mathfrak{A}) \sum_{E \in \mathbb{Z}} \mu_\mathbb{Q}(E) |\Lambda(\mathfrak{A} \mathbb{D}, CE) \cap S_F(X \mathfrak{N}_K \mathbb{D}^{1/[K:\mathbb{Q}]})|
\]

\[
= \frac{1}{2} \sum_{\mathfrak{A} \in S_C} \sum_{\mathfrak{p} \in T} \mu_K(\mathfrak{A}) \sum_{E \in \mathbb{Z}} \mu_\mathbb{Q}(E) \frac{\text{Vol}_F(1) \mathfrak{N}_K \mathbb{D}^{1/[K:\mathbb{Q}]}}{\det \Lambda(\mathfrak{A} \mathbb{D}, CE)} X^2
\]

\[
+ O \left( \sum_{\mathfrak{A} \in S_C} \sum_{\mathfrak{p} \in T} \sum_{E \in \mathbb{Z}} \frac{\text{Vol}_F(1) \mathfrak{N}_K \mathbb{D}^{1/[K:\mathbb{Q}]}}{\det \Lambda(\mathfrak{A} \mathbb{D}, CE)} \right)
\]

\[
+ O \left( \sum_{\mathfrak{A} \in S_C} \sum_{\mathfrak{p} \in T} \sum_{E \in \mathbb{Z}} \frac{\tilde{M} \mathfrak{N}_K \mathbb{D}^{1/[K:\mathbb{Q}]}}{\lambda_1} \frac{\tilde{L}_X}{\lambda_1} \right).
\]

Now the first term gives the main term as before. For the second term we use Minkowski’s second theorem to estimate the determinant in terms of \(\lambda_1\), and then a simple computation using Lemma 7.4 and (7.8) gives the error term \(O(n d)\). For the last error term we use again Lemma 7.4, and again a simple computation yields the error term

\[
O(|S_C| |T| (1 + \log(n d))).
\]

To get the right error term we choose again \(S_C\) to be minimal so that we can take \(\Gamma = C_{N}^{n-1}\), and \(|S_C| \leq |F_N|\). This proves Theorem 4.

8. Proof of Theorem 3

In this section, we deduce Theorem 3 from Theorem 5. Let us recall some simple facts which will be used in the sequel without further notice. Let A, B be ideals of \(\mathcal{O}_k\), and let \(\mathfrak{A}, \mathfrak{B}\) be ideals of \(\mathcal{O}_K\). Moreover, suppose that P is a prime ideal of \(\mathcal{O}_k\) and that P runs over all prime ideals of \(\mathcal{O}_K\) above P. Then

- \(v_P(\mathfrak{A}) = \max_{q|P} \{[v_P(\mathfrak{A})]/e_P]\}\)
- \(\mathfrak{A} \mid u^a \mathfrak{A}\)
- \(u(AB) = uA^a B\)
- \(\mathfrak{A} \mid u^a A\) if and only if \(\mathfrak{A} \mid A\)

As mentioned after Lemma 7.4 we can and will assume that \(\theta\) is an algebraic integer. Let \(K := k(\theta)\), and let \(\mathcal{N}\) be the adelic Lipschitz system on \(\mathcal{K}\) of dimension \(n\) defined by

\[
N_w(\mathfrak{A}) := \max\{|z_0|_w, |\theta|_w z_1|_w, \ldots, |\theta|_w z_n|_w\},
\]

so

\[
i_{\mathcal{N}}(\omega) = \omega_0 \mathcal{O}_K + \theta \omega_1 \mathcal{O}_K + \cdots + \theta \omega_n \mathcal{O}_K.
\]

Lemma 8.1. We have

\[
N(\theta, X) = N_{\mathcal{N}} (\mathbb{P}^n(\mathbb{K}), X) + O(X^{nd}),
\]

where the implicit constant in the error term depends only on \(k, \theta, \) and \(n\).
Proof. The points \( \mathbf{a} = (\omega_1/\omega_0, \ldots, \omega_n/\omega_0) \in k^n \) with \( H(\theta \mathbf{a}) \leq X \) are in one-to-one correspondence with the projective points \( P = (\omega_0 : \cdots : \omega_n) \in \mathbb{P}^n(k) \) with \( \omega_0 \neq 0 \) and \( H_\mathcal{N}(P) \leq X. \)

If \( n > 1 \) then we can apply Theorem 5 with \( n - 1 \) and the adelic Lipschitz system given by the norm functions (see Lemma A1 in the appendix)

\[
N_\mathcal{N}(z_1, \ldots, z_n) := \max (|\theta|_w |z_1|_w, \ldots, |\theta|_w |z_n|_w)
\]

(with \( \mathcal{R}, \mathcal{S}_C \) and \( T \) chosen in such a way that \( |T| \) is minimal) to see that the number of such points \( P \) with \( \omega_0 = 0 \) is \( O(X^{nd}) \). This trivially remains true for \( n = 1 \).

Since the functions \( N_\mathcal{N} \) are norms, the adelic Lipschitz system \( \mathcal{N} \) satisfies the hypothesis of Theorem 5. As our choice of \( \mathcal{R}, \mathcal{S}_C \) and \( T \) in Theorem 5 will depend only on \( k, n \) and \( \theta \), we obtain

\[
N_\mathcal{N}(P^n(k), X) = \omega_k^{-1} (n + 1)^{r+s-1} R_k V_{\mathcal{N}} g^N_k X^{d(n+1)} + O(X^{d(n+1)-1} \varpi),
\]

where \( \varpi := \log(X + 1) \) if \((n, d) = (1, 1) \) and \( \varpi := 1 \) otherwise. The implicit constant in the error term depends only on \( k, \theta, \) and \( n \).

We notice that

\[
V_{\mathcal{N}} = (2^s \pi^s)^{n+1} V(\theta, k, n),
\]

with \( V(\theta, k, n) \) as in (8.3). To prove the theorem, we need to compute \( g^N_k \). First we choose the sets \( \mathcal{R}, \mathcal{S}_C \) and \( T \). Denote

\[
D := \theta(\mathcal{O}_k).
\]

For \( \mathcal{R} \) we choose any system of integral representatives for the class group \( \text{Cl}_k \) with

\[
(C, D) = \mathcal{O}_k \text{ for all } C \in \mathcal{R}.
\]

We will see in Lemma 8.2 (ii), that

\[
S_C := \{^u C(\theta \mathcal{O}_k, ^u B) : B \subseteq \mathcal{O}_k, B \mid D \}
\]

is a valid choice for \( S_C \). For \( T \), we take the finite set

\[
T := \bigcup_{C \in \mathcal{R}} \bigcup_{\mathfrak{A} \in S_C} \{ \mathfrak{A} \subseteq \mathcal{O}_k : \mathfrak{A} \mid \theta \mathcal{O}_k \}.
\]

Lemma 8.2.

(i) Let \( \omega \in k^{n+1} \) with \( \mathcal{O}_k(\omega) = C \). Then \( i_N(\omega) \in S_C \).

(ii) Let \( \mathfrak{A} \) be an ideal of \( \mathcal{O}_k \) and \( B \) an ideal of \( \mathcal{O}_k \). Then \( \theta(\mathfrak{A}, ^u B) = (\theta \mathfrak{A}, B) \).

(iii) Let \( B \) be an ideal of \( \mathcal{O}_k \) with \( B \mid D \). Then \( \theta(\mathcal{O}_k, ^u B) = B \).

Proof. (i): We have \( \omega_0 \mathcal{O}_K + \cdots + \omega_n \mathcal{O}_K = ^u \mathcal{O}_k(\omega) = ^u C \), so

\[
i_N(\omega) = ^u C(\omega_0(\omega)^{-1} + \theta(\omega_1(\omega)^{-1} + \cdots + \omega_n(\omega)^{-1})) = ^u C(\omega_0(\omega)^{-1}, \theta \mathcal{O}_K).
\]

Moreover, since \( \theta \mathcal{O}_K \mid ^u D \), we obtain

\[
(\omega_0(\omega)^{-1}, \theta \mathcal{O}_K) = (\omega_0(\omega)^{-1}, ^u D, \theta \mathcal{O}_K) = (\theta \mathcal{O}_K, ^u B),
\]

for \( B := (\omega_0 C^{-1}, D) \mid D \).

(ii): Let \( P \) be a prime ideal of \( \mathcal{O}_k \) and \( ^u P = \prod \mathfrak{P} \mathfrak{P}^{e} \) its factorization in \( \mathcal{O}_K \). Then

\[
v_P(\theta(\mathfrak{A}, B)) = \max_{\mathfrak{P}} \left\{ \left[ \min_{\mathfrak{P}} \left( v_{\mathfrak{P}}(\mathfrak{A}), v_{\mathfrak{P}}(B) \right) / e_{\mathfrak{P}} \right] \right\}
\]

\[
= \max_{\mathfrak{P}} \{ \min \{ v_{\mathfrak{P}}(\mathfrak{A}) / e_{\mathfrak{P}}, v_{\mathfrak{P}}(B) / e_{\mathfrak{P}} \} \}
\]

\[
= \min \{ \max \{ v_{\mathfrak{P}}(\mathfrak{A}) / e_{\mathfrak{P}}, v_{\mathfrak{P}}(B) \} \} = v_P((\theta \mathfrak{A}, B)).
\]

(iii): By (ii), we have \( \theta(\mathcal{O}_K, ^u B) = (D, B) = B. \)
The first step in our computation of \( q_k^N \) is to evaluate the determinant of the lattice \( \Lambda(\mathfrak{A}, CE) = \Lambda(\mathfrak{A}) \cap \sigma((CE)^{n+1}) \).

Lemma 8.3. Let \( \mathfrak{A}, B \) be nonzero ideals of \( \mathcal{O}_K \) and \( \mathcal{O}_k \), respectively. Then
\[
\det \Lambda(\mathfrak{A}, B) = (2^{-s} \sqrt{|\Delta_k|})^{n+1} \cdot \mathfrak{m}_k (\mathfrak{A} \cap B) \cdot \mathfrak{m}_k \left( (\mathfrak{A}(\mathfrak{O}_K, \mathfrak{A})^{-1}) \cap B \right)^n.
\]

Proof. Let \( \omega = (\omega_0, \ldots, \omega_n) \in k^n \). Clearly, \( \sigma \omega \in \Lambda(\mathfrak{A}, B) \) if and only if \( \omega_i \in \mathfrak{A} \) for all \( 0 \leq i \leq n \), \( \omega_0 \in \mathfrak{A} \), and \( \theta \omega_i \in \mathfrak{A} \) for all \( 1 \leq i \leq n \). For \( \omega_i \in \mathcal{O}_k \), we have
\[
\theta \omega_i \in \mathfrak{A} \quad \text{if and only if} \quad \mathfrak{A}(\mathfrak{O}_K, \mathfrak{A})^{-1} | \omega_i \mathcal{O}_K.
\]
Therefore, we obtain
\[
\Lambda(\mathfrak{A}, B) = \sigma \left( (\mathfrak{A} \cap B) \times \left( (\mathfrak{A}(\mathfrak{O}_K, \mathfrak{A})^{-1}) \cap B \right)^n \right).
\]
\( \Box \)

Let \( \mathfrak{A} \in T \) and let \( B \) be an ideal of \( \mathcal{O}_k \) with \( B \mid D \). To facilitate further notation, we define ideals \( A \) and \( A_1 \) of \( \mathcal{O}_k \) by
\[
A = A(\mathfrak{A}, B) := \mathfrak{A}(\mathfrak{O}_K, uB) \quad \text{and} \quad A_1 = A_1(\mathfrak{A}, B) := \mathfrak{A}(\mathfrak{O}_K, uB)(\mathfrak{O}_K, \mathfrak{A}^uB)^{-1} \mid A.
\]
For any \( \mathfrak{D} = uC(\mathfrak{O}_K, uB) \in \mathcal{S}_C \) and for any nonzero ideal \( E \) of \( \mathcal{O}_k \) we have
\[
\mathfrak{m}_k (\mathfrak{D}) \cap CE = \mathfrak{n}_k C \cdot \mathfrak{n}_k (A \cap E).
\]
Clearly, we have \( (\mathfrak{O}_K, \mathfrak{O}_K, uB) = (\mathfrak{O}_K, \mathfrak{O}_K, uB) \). Furthermore, by our choice of \( \mathcal{R} \) with \( \mathcal{S}_C \), we have \( (\mathfrak{O}_K, \mathfrak{O}_K) = \mathfrak{O}_K \). Therefore, we obtain
\[
\mathfrak{m}_k (\mathfrak{D}) \cap CE = \mathfrak{n}_k C \cdot \mathfrak{n}_k (A_1 \cap E).
\]
Moreover, we have
\[
\mathfrak{m}_K (\mathfrak{D}) \cap CE = \mathfrak{n}_k C \cdot \mathfrak{n}_k (\mathfrak{O}_K, \mathfrak{O}_K, uB)^{(n+1)/[K:k]}.
\]

Lemma 8.4. Let \( B \) be an ideal of \( \mathcal{O}_k \) with \( B \mid D \), let \( \mathfrak{D} = uC(\mathfrak{O}_K, uB) \in \mathcal{S}_C \), let \( \mathfrak{A} \in T \), and let \( E \) be a nonzero ideal of \( \mathcal{O}_k \). Then
\[
\frac{\mathfrak{m}_K (\mathfrak{D}) \mathfrak{n}_k (\mathfrak{D})^{n+1}}{\det \Lambda(\mathfrak{A}, CE)} = (2^{-s} \sqrt{|\Delta_k|})^{-(n+1)} \cdot \frac{\mathfrak{n}_k (\mathfrak{O}_K, uB)^{(n+1)/[K:k]}}{\mathfrak{n}_k (A \cap E) \cdot \mathfrak{n}_k (A_1 \cap E)^n}.
\]

Proof. We apply Lemma 8.3 and use (8.10), (8.11), and (8.12). \( \Box \)

Lemma 8.5. We have
\[
q_k^N = c_0 \sum_{B \mid D} \mathfrak{m}_K (\mathfrak{O}_K, uB)^{n+1} \sum_{\mathfrak{A} \in T} \mu_K (\mathfrak{A}) \sum_{E \mid \mathcal{O}_k} \mathfrak{m}_k (E) \mathfrak{n}_k (A \cap E) \cdot \mathfrak{n}_k (A_1 \cap E)^n,
\]
where \( A = A(\mathfrak{A}, B), A_1 = A_1(\mathfrak{A}, B), \) and \( c_0 := h_k 2^{n+1}(\sqrt{|\Delta_k|})^{-(n+1)} \) and \( E \) runs over all nonzero ideals of \( \mathcal{O}_k \).

Proof. Recall the definition of \( q_k^N \) in (6.4). The expression on the right-hand side in Lemma 8.3 does not depend on \( C \). With (8.3), a simple computation proves the lemma. \( \Box \)

The inner sum over \( E \) in Lemma 8.5 can be handled by the following lemma.

Lemma 8.6. Let \( J_1 \mid J \) be nonzero ideals of \( \mathcal{O}_k \) and let
\[
\xi := \sum_{E \mid \mathcal{O}_k} \mu_k (E) \mathfrak{n}_k (J \cap E) \cdot \mathfrak{n}_k (J_1 \cap E)^n.
\]
If \( J_1 \neq \mathcal{O}_k \) then \( \xi = 0 \). If \( J_1 = \mathcal{O}_k \) then
\[
\xi = \frac{1}{\zeta_k(n+1)\eta_k(n)} \prod_{p|J} \frac{\eta_k P^n + 1 - \eta_k P}{\eta_k P^n + 1 - 1}.
\]

**Proof.** Let \( f(E) := \mu_k(E) \cdot \nu_k(J, E) \cdot \nu_k(J_1, E) \). Then \( f \) is multiplicative and
\[
\xi = \frac{1}{\nu_k(J, J_1)} \sum_{E \subseteq \mathcal{O}_k} \frac{f(E)}{\eta_k E^{n+1}}.
\]

Clearly, this Dirichlet series converges absolutely for all \( n > 0 \). Let us compute its Euler product expansion. For any prime ideal \( P \) of \( \mathcal{O}_k \), we have \( f(P^e) = 0 \) if \( e \geq 2 \). Moreover, \( f(\mathcal{O}_k) = 1 \) and
\[
f(P) = \begin{cases} 
-\eta_k P^{n+1} & \text{if } P \mid J_1, \\
-\eta_k P & \text{if } P \mid J \text{ and } P \nmid J_1, \\
-1 & \text{if } P \nmid J.
\end{cases}
\]

We obtain the formal expansion
\[
\sum_{E \subseteq \mathcal{O}_k} \frac{f(E)}{\eta_k E^{n+1}} = \prod_{P \mid J_1} \left( 1 - \frac{\eta_k P^{n+1}}{\eta_k P} \right) \prod_{P \mid J} \left( 1 - \frac{\eta_k P}{\eta_k P} \right) \prod_{P} \left( 1 - \frac{1}{\eta_k P} \right).
\]

Since the infinite product \( \prod_{p|J} (1 - \eta_k P^{-s}) \) converges absolutely for \( s > 1 \), we obtain \( \xi = 0 \) whenever \( J_1 \neq \mathcal{O}_k \). If \( J_1 = \mathcal{O}_k \) and \( s = n + 1 \), the expression simplifies to
\[
\sum_{E \subseteq \mathcal{O}_k} \frac{f(E)}{\eta_k E^{n+1}} = \frac{1}{\zeta_k(n+1)} \prod_{p|J} \frac{\eta_k P^n + 1 - \eta_k P}{\eta_k P^n + 1 - 1}.
\]

Recall the definition of \( A \) and \( A_1 \) from (8.10) and (8.13). We have \( A_1 = \mathcal{O}_k \) if and only if \( \mathfrak{A}((\theta \mathcal{O}_K, u^B) = (\theta \mathcal{O}_K, \mathfrak{A}^n B) \), which is equivalent to \( \mathfrak{A}((\theta \mathcal{O}_K, u^B) \mid \theta \mathcal{O}_K \), or (8.13) \( \mathfrak{A} \mid \theta \mathcal{O}_K((\theta \mathcal{O}_K, u^B)^{-1}. \)

Recall that, by (8.17), the set \( T \) contains all ideals \( \mathfrak{A} \) of \( \mathcal{O}_K \) with \( \mathfrak{A} \mid \theta \mathcal{O}_K \). Also, for every \( \mathfrak{A} \) with (8.13), we have \( A = \mathfrak{g}(\mathfrak{A}((\theta \mathcal{O}_K, u^B)) \mid D \). We obtain
\[
g_k^N = c_1 \sum_{B \mid D} \mathfrak{g}(\mathfrak{A}((\theta \mathcal{O}_K, u^B)) \frac{n}{\mathfrak{A}} \sum_{A \mid D} \frac{\mathfrak{g}(\mathfrak{A}((\theta \mathcal{O}_K, u^B)) \frac{n}{\mathfrak{A}}}{\mathfrak{A}A} \prod_{P \mid A} \frac{\eta_k P^n + 1 - \eta_k P}{\eta_k P^n + 1 - 1} \mathfrak{s}(A, B),
\]

where \( c_1 := \zeta_k(n+1)^{-1} c_0 = h_k 2^{k(n+1)} \zeta_k(n+1)^{-1}(\sqrt{\Delta_k})^{-1} \) and \( \mathfrak{s}(A, B) := \sum_{\mathfrak{A} \mid A} \mu_k(\mathfrak{A}). \)

If \( \mathfrak{s}(A, B) \) is not zero then there is at least one \( \mathfrak{A} \) with
\[
A = \mathfrak{g}(\mathfrak{A}((\theta \mathcal{O}_K, u^B)) \subseteq \mathfrak{g}(\mathfrak{A}((\theta \mathcal{O}_K, u^B) = B.
\]

For the last equality, we used Lemma 5.2 (iii). We replace \( A \) by \( B^{-1} A \) to obtain
\[
g_k^N = c_1 \sum_{B \mid D} \frac{\mathfrak{g}(\mathfrak{A}((\theta \mathcal{O}_K, u^B)) \frac{n}{\mathfrak{A}}}{\mathfrak{A}B} \sum_{A \mid B^{-1} D} \frac{1}{\mathfrak{A}A} \prod_{P \mid AB} \frac{\eta_k P^n + 1 - \eta_k P}{\eta_k P^n + 1 - 1} \mathfrak{s}(A, B),
\]

where \( \mathfrak{s}(A, B) := \sum_{\mathfrak{A} \mid A} \mu_k(\mathfrak{A}). \)
Clearly, conditions (8.13) and (8.15) imply use that every

\[ (8.13) \]

and the first part of (8.15). For the second part of (8.15), we

In fact, (8.13) and (8.15) are equivalent to (8.16) and (8.17). Inde-

By inclusion-exclusion for (8.17), we obtain

\[ s(A, B) = \sum_{\mathfrak{A} \subseteq \mathcal{O}_K} \mu_K(\mathfrak{A}) = \sum_{\mathfrak{A} \subseteq \mathcal{O}_K} \mu_K(\mathfrak{A}). \]

By inclusion-exclusion for (8.17), we obtain

\[ s(A, B) = \sum_{F \mid A} \mu_k(F) \sum_{\mathfrak{A} \subseteq \mathcal{O}_K} \mu_K(\mathfrak{A}). \]

The last sum is 1 if \( F = A \). Moreover,

\[ F^{-1} A \mid B^{-1} D = \mathfrak{a}(\mathfrak{a}(\theta \mathcal{O}_K, u B))^{-1} \mid \theta \mathcal{O}_K(\theta \mathcal{O}_K, u B)^{-1}, \]

so \( F \not\equiv A \) implies that

\[ \mathfrak{a}(\theta \mathcal{O}_K(\theta \mathcal{O}_K, u B)^{-1}, (F^{-1} A)) \not\equiv \mathcal{O}_K. \]

This shows that the last sum is 0 whenever \( F \not\equiv A \).  

We obtain

\[ g^N_k = c_1 \sum_{B \mid D} \frac{\mathfrak{N}_K(\theta \mathcal{O}_K, u B)^{(n+1)/[K:k]}}{\mathfrak{N}_k} \sum_{A \mid B^{-1} D} \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P \mid AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}, \]

and Theorem 3 follows by substituting this and (8.3) in (8.2).
9. Proof of Theorem \[\text{3}\]

In this section we will use not only Landau's $O$-notation but also Vinogradov's symbol $\ll$. All implied constants depend solely on $k$. As we will encounter expressions like $\log \log X$ we assume throughout the entire section that $X \geq 3$. Our main task will be to prove the following proposition.

**Proposition 9.1.** Suppose $p \in \mathbb{P}_k$. Then, as $X \geq 3$ tends to infinity, we have

$$N(\sqrt{pk^*}, X) = \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} + O\left(\frac{X^{2d-1}}{p(d-1)/2} + X^d \log X + X^d \log p\right).$$

We choose the adelic Lipschitz system $\mathcal{N}$ (of dimension 1) on $K := k(\sqrt{p})$, defined by

$$N_w((z_0, z_1)) := \max\{|z_0|_w, |\sqrt{p}|_w|z_1|_w\}$$

for any place $w$ of $K$. Recall the definition of $C^{\inf}_\mathcal{N}$ and $C^{\inf}_\mathcal{N}$ from (5.3) and (5.6), and note that we can take

$$C^{\inf}_\mathcal{N} = C^{\inf}_\mathcal{N} = \sqrt{p}.$$  

**Lemma 9.2.** We have

$$N(\sqrt{pk^*}, X) = N_\mathcal{N}(\mathbb{P}^1(k); X) - 2.$$

**Proof.** The map $\alpha \mapsto (1 : \alpha)$ is a one-to-one correspondence between $k^*$ and $\mathbb{P}^1(k) \setminus \{(0 : 1), (1 : 0)\}$. Moreover, $H(\sqrt{pk}) = H_\mathcal{N}(1 : \alpha)$. Hence there is a one-to-one correspondence between $\{\alpha \in k^*: H(\sqrt{pk}) \leq X\}$ and $\{P \in \mathbb{P}^1(k) \setminus \{(0 : 1), (1 : 0)\}: H_\mathcal{N}(P) \leq X\}$. As $H_\mathcal{N}((0 : 1)) = H_\mathcal{N}((1 : 0)) = 1$ the claim follows. \hfill \qed

We can now basically follow the proof of Theorem \[\text{3}\] using our specific adelic Lipschitz system. However, to get the good error terms regarding $p$ an additional idea is required. We will use the same notation as in Sections \[\text{6}\] and \[\text{7}\]. In particular, recall the definition of the set $S_F(T)$ introduced in (7.2). As in (8.3), we choose a system $\mathcal{R}$ of integral representatives for $\mathcal{O}_k$ such that $(C, \mathcal{P}\mathcal{O}_k) = \mathcal{O}_k$ for all $C \in \mathcal{R}$.

**Lemma 9.3.** We can choose $S_C := \{aC, \sqrt{p}C\}$.

**Proof.** As in (8.1) we have $i_X(\omega) = \omega_0 \mathcal{O}_K + \sqrt{p}\omega_1 \mathcal{O}_K$. So if $\mathcal{O}_k(\omega) = C$ we get $\sqrt{p}C \subseteq i_X(\omega) \subseteq aC$. As $\sqrt{p}\mathcal{O}_K$ is a prime ideal this proves the lemma. \hfill \qed

With this choice of the sets $S_C$ we directly verify that $T$ from (7.13) can be chosen to be

$$T := p^{-d/2}.$$  

From now on $C$ is always in $\mathcal{R}$, $\mathfrak{D}$ is always in $S_C$, and $\mathfrak{A}$ will always be in $T$.

**Lemma 9.4.** We can choose $T$ such that $|T| \leq 4h_k$.

**Proof.** Recall that we may choose $T = \bigcup_{C \in \mathcal{R}} \bigcup_{\mathfrak{D} \in S_C} T_{C, \mathfrak{D}}$. By definition we have

$$T_{C, \mathfrak{D}} = \{\mathfrak{B} \subseteq \mathcal{O}_K: \lambda_C(\mathfrak{D}\mathfrak{B}) \neq \emptyset\} = \{\mathfrak{B} \subseteq \mathcal{O}_K: \lambda_C^\prime(\mathfrak{D}\mathfrak{B}) \neq \emptyset \text{ for some } \mathfrak{E} \subseteq \mathcal{O}_K\} \subseteq \{\mathfrak{B} \subseteq \mathcal{O}_K: \mathfrak{E}\mathfrak{D}\mathfrak{B} \in S_C \text{ for some } \mathfrak{E} \subseteq \mathcal{O}_K\}.$$

Now using that $S_C = \{aC, \sqrt{p}C\}$ and that $\sqrt{p}\mathcal{O}_K$ is a prime ideal we see that $|T_{C, \mathfrak{D}}| \leq 2$ for any $\mathfrak{D} \in S_C$. Thus $|T| \leq \sum_{C \in \mathcal{R}} \sum_{\mathfrak{D} \in S_C} |T_{C, \mathfrak{D}}| \leq 4h_k$. \hfill \qed
Proof. By Lemma 9.5 we have \( \Lambda \) is as in Lemma 9.6.

Moreover, if \( D = \sqrt[4]{C} \) then we have
\[
\Lambda(\mathfrak{A}, CE) \subseteq \sigma(CE) \times \sigma(CE).
\]

Proof. The first assertion is clear from the definition. For the second assertion we could use the last equality in the proof of Lemma 9.5, but we prefer to give a direct argument here. Note that \( \sigma \omega \in \Lambda(\mathfrak{A}) \) implies \( \mathfrak{D} | i_{\mathfrak{N}}(\omega) = (\omega_0, \sqrt{p} \omega_1) \). As \( D = \sqrt[4]{C} \) we conclude \( \sqrt{p} \mathfrak{O}_K | \omega_0 \mathfrak{O}_K \), and thus \( p \mathfrak{O}_k | \omega_0 \mathfrak{O}_k \). Therefore \( \omega_0 \in C \cap p \mathfrak{O}_k \). This proves the second assertion.

Next we use a trick, simpler but reminiscent of those used in [Wid12a, Section 6]. To this end we introduce a linear automorphism \( \Phi \) of determinant 1 on \((\mathbb{R}^r \times C^s)^2\) by
\[
\Phi(z_0, z_1) := (p^{-1/4}z_0, p^{1/4}z_1).
\]

Lemma 9.6. Write \( \Lambda := \Lambda(\mathfrak{A}, CE) \). If \( D = \sqrt[4]{C} \) then we have
\[
\lambda_1(\Phi \Lambda) \geq p^{-1/4} \mathfrak{N}(CE)^{1/d},
\]
\[
\lambda_{d+1}(\Phi \Lambda) \geq p^{1/4} \mathfrak{N}(CE)^{1/d}.
\]

If \( D = \sqrt[4]{C} \) then we have
\[
\lambda_1(\Phi \Lambda) \geq \begin{cases} 
    p^{-1/4} \mathfrak{N}(CE)^{1/d}, & \text{if } p \mathfrak{O}_k \mid E, \\
    p^{1/4} \mathfrak{N}(CE)^{1/d}, & \text{if } p \mathfrak{O}_k \nmid E.
\end{cases}
\]
\[
\lambda_{d+1}(\Phi \Lambda) \geq \begin{cases} 
    p^{1/4} \mathfrak{N}(CE)^{1/d}, & \text{if } p \mathfrak{O}_k \mid E, \\
    p^{3/4} \mathfrak{N}(CE)^{1/d}, & \text{if } p \mathfrak{O}_k \nmid E.
\end{cases}
\]

Proof. By Lemma 9.5 we have \( \Lambda \subseteq \Lambda_1 \times \Lambda_2 \), where \( \Lambda_2 := p^{1/4} \sigma(CE) \) and \( \Lambda_1 \) is \( p^{-1/4} \sigma(CE) \) if \( D = \sqrt[4]{C} \) and \( p^{-1/4} \sigma(CEp(C, p \mathfrak{O}_k)^{-1}) \) if \( D = \sqrt[4]{C} \). Recall the fact (already used in Lemma 7.4) that \( \lambda_1(\sigma \Lambda) \geq \mathfrak{N}(A)^{1/d} \) for any nonzero ideal \( A \) of \( k \). Using this and applying Lemma 9.5 the result follows from an easy computation.

Lemma 9.7. There exist constants \( c_1 = c_1(k) \) and \( M = M(k) \) depending solely on \( k \) such that, with \( L = c_1 p^{-1/4} T \), we have \( \Psi S_F(T) \subseteq B_0(L) \) and the boundary \( \partial \Phi S_F(T) \in \text{Lip}(2d, M, L) \).

Proof. The adelic Lipschitz system \( \mathcal{N} \) on \( K \) leads to an adelic Lipschitz system \( \mathcal{N}' \) on \( k \) as in Section 6. The latter is used to define \( S_F(T) \).

Now notice that applying \( \Phi \) to \( S_F(T) \) gives the same as defining \( S_F(T) \) using the standard adelic Lipschitz system defined by \( N_v(z_0, z_1) = \max \{ |z_0|^v, |z_1|^v \} \) for all \( v \) and then homogeneously shrinking this set by the factor \( p^{-1/4} \). The claims then follow immediately from Lemma 7.1–7.4, and 7.8 applied to the standard adelic Lipschitz system.

Lemma 9.8. Let \( E_1 := X^{d}/\mathfrak{N}(E) \), and let \( E_2 := X^{2d-1}/(p^{(d-1)/2} \mathfrak{N}(E))^{2-1/d} \).

Then we have
\[
|\Lambda(\mathfrak{A}, CE) \cap S_F(X^{d}/\mathfrak{N}(E))| = \frac{\text{Vol} S_F(1) \mathfrak{N}(D) X^{2d}}{\det \Lambda(\mathfrak{A}, CE)} + O \left( \begin{cases} 
    E_1 + E_2 & \text{if } p \mathfrak{O}_k \mid E \\
    p^{d/2} E_1 + p^{d-1/2} E_2 & \text{if } p \mathfrak{O}_k \nmid E
\end{cases} \right).
\]
Moreover, there is a constant \( \gamma = \gamma(k) \geq 1 \) depending only on \( k \), such that
\[
|\lambda(\mathfrak{A}, CE) \cap S_F(X \mathfrak{N}_K \mathcal{D}^{1/(2d)})| = 0 \text{ whenever } \mathfrak{N}_E > (\gamma p X)^d.
\]

**Proof.** First note that
\[
|\lambda(\mathfrak{A}, CE) \cap S_F(X \mathfrak{N}_K \mathcal{D}^{1/(2d)})| = |\Phi \lambda(\mathfrak{A}, CE) \cap \Phi S_F(X \mathfrak{N}_K \mathcal{D}^{1/(2d)})|.
\]
Now we apply Lemma 4.2 with \( a = d + 1 \) combined with Lemma 9.7 to conclude
\[
\Phi \lambda(\mathfrak{A}, CE) \cap \Phi S_F(X \mathfrak{N}_K \mathcal{D}^{1/(2d)}) = \frac{Vol S_F(1) \mathfrak{N}_K \mathcal{D} X^{2d}}{det \lambda(\mathfrak{A}, CE)} + O \left( \max \left\{ \frac{p^{-d/4} X^d \mathfrak{N}_K \mathcal{D}^{1/2}}{\lambda_1(\Phi \mathfrak{A})^d}, \frac{p^{-(2d-1)/4} X^{2d-1} \mathfrak{N}_K \mathcal{D}^{-1/(2d)}}{\lambda_1(\Phi \mathfrak{A})^d \lambda_{d+1}(\Phi \mathfrak{A})^{d-1}} \right\} \right).
\]
Finally, we use Lemma 9.6 to estimate \( \lambda_1(\Phi \mathfrak{A}) \) and \( \lambda_{d+1}(\Phi \mathfrak{A}) \), and the first claim follows from a simple computation. The second claim follows from Lemma 7.4 combined with 9.2 and 9.3. \( \square \)

We are now in the position to prove Proposition 9.1. In the introduction we already computed the main term, see (1.6). Proceeding exactly as in the proof of Theorem 5 in the case \((n,d) = (1,1)\), we obtain
\[
N_X(p^1(k); X) = 2p^{d/2} \frac{d!}{d^d} S_1(1) X^{2d} + O \left( \sum_{C \in \mathcal{R}} \sum_{D \in S_C} \sum_{A \in T} \sum_{\mathfrak{N}_E \in (\gamma p X)^d} \frac{Vol \Phi S_F(X \mathfrak{N}_K \mathcal{D}^{1/(2d)})}{det \Phi \lambda(\mathfrak{A}, CE)} \right) + O \left( \sum_{C \in \mathcal{R}} \sum_{D \in S_C} \sum_{A \in T} \sum_{\mathfrak{N}_E \in (\gamma p X)^d} \mathcal{E}_1 + \mathcal{E}_2 \right) + O \left( \sum_{C \in \mathcal{R}} \sum_{D \in S_C} \sum_{A \in T} \sum_{\mathfrak{N}_E \in (\gamma p X)^d} \sum_{\mathfrak{N}_E \in (\gamma p X)^d} p^{d/2} \mathcal{E}_1 + p^{d-1/2} \mathcal{E}_2 \right).
\]

For the first error term we apply Minkowski’s second theorem and Lemma 9.7 to get the upper bound
\[
\frac{Vol \Phi S_F(X \mathfrak{N}_K \mathcal{D}^{1/(2d)})}{det \Phi \lambda(\mathfrak{A}, CE)} \ll \frac{L^{2d}}{\lambda_1(\Phi \mathfrak{A})^d \lambda_{d+1}(\Phi \mathfrak{A})^{d-1}};
\]
where \( L \ll p^{-1/4} X \mathfrak{N}_K \mathcal{D}^{1/(2d)} \). Summing the above over the finite sums can be handled by Lemmata 9.3 and 9.4. Now for the infinite sum over the ideals \( E \), we apply Lemma 9.6 and a straightforward computation (using the dichotomy \( P \parallel E \), \( P \nmid E \)) yields the upper bound
\[
\ll \frac{X^d}{p^{d/2}}.
\]
For the second error term we note that
\[
\sum_{\mathfrak{N}_E \in (\gamma p X)^d} \frac{X^d}{\mathfrak{N}_E} \ll X^d \log ((\gamma p X)^d) \ll X^d \log X + X^d \log p.
\]
Then we proceed similarly as for the first error term and we obtain the upper bound
\[ \ll X^d \log X + X^d \log p + \frac{X^{2d-1}}{p^{d-1/2}}. \]

Finally, using precisely the same arguments and the corresponding estimates from Lemma 9.6, we deduce for the last error term the upper bound
\[ \ll X^d \log X + \frac{X^{2d-1}}{p^{d-1/2}}. \]

Combining these estimates and Lemma 9.2 completes the proof of Proposition 9.1.

We can now sum \( N(\sqrt{p^k}, X) \) over all \( p \in P_k \). The next lemma tells us that we can restrict the summation to \( p \leq X^2 \).

**Lemma 9.9.** For any \( \alpha \in k^* \) and any \( p \in P_k \) we have \( H(\sqrt{p^k}) \geq \sqrt{\mathfrak{p}} \).

**Proof.** Let \( x \in K \) and let \( \mathfrak{p} \) be the prime ideal \( \sqrt{\mathfrak{p}O_K} \). Then
\[ H(x) = \max\{1, \mathfrak{p}_k \mathfrak{p}^{-v_p(xO_K)/(2d)} \} = \max\{1, p^{d^2} \}^{-v_p(xO_K)/(2d)}. \]
In particular, if \( v_p(xO_K) < 0 \) we get \( H(x) \geq \sqrt{\mathfrak{p}} \). As \( H(x) = H(1/x) \) for any nonzero \( x \) whatsoever, it suffices to show that the order of \( \sqrt{\mathfrak{p}O_K} \) at \( \mathfrak{p} \) is nonzero. As \( p \) is inert in \( k \) the order of \( \alpha O_K \) at \( \mathfrak{p} \) is even. Hence the order of \( \sqrt{\mathfrak{p}O_K} \) at \( \mathfrak{p} \) is odd.

We can now prove Theorem 3. Clearly, we have
\[ N(\sqrt{P_k}, X) = 1 + \sum_{\substack{p \in P_k \\ p \leq X^2}} N(\sqrt{p^k}, X) \]
\[ = \sum_{\substack{p \in P_k \\ p \leq X^2}} \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} + O\left( \frac{X^{2d-1}}{p^{d-1/2}} + X^d \log X + X^d \log p \right) \]
\[ = \sum_{\substack{p \in P_k \\ p \leq X^2}} \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} + O\left( \sum_{\substack{p \in P_k \\ p \leq X^2}} \frac{X^{2d-1}}{p^{d-1/2}} \right) + O\left( \sum_{\substack{p \in P_k \\ p \leq X^2}} X^d \log X \right). \]

By the prime number theorem we have
\[ \sum_{\substack{p \in P_k \\ p \leq X^2}} X^d \log X \ll X^{d+2}. \]

A very crude estimate gives
\[ \sum_{\substack{p \in P_k \\ p \leq X^2}} \frac{X^{2d-1}}{p^{d-1/2}} \ll \begin{cases} X^{2d-1} & \text{if } d \geq 4, \\ X^5 \log X & \text{if } d = 3, \\ X^4 & \text{if } d = 2. \end{cases} \]
To handle the first term let us start with the simpler case \( d \geq 3 \). Then we have
\[ \sum_{\substack{p \in P_k \\ p \leq X^2}} \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} = \sum_{p \in P_k} \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} + O\left( \sum_{\substack{p \in P_k \\ p \leq X^2}} \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} \right) \]
\[ = \sum_{p \in P_k} \frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d} + O(X^{2d-1}). \]
This finishes the proof of Theorem 3 for \( d \geq 3 \).
Let us now assume $d = 2$. It remains to show that
$$\sum_{p \in \mathcal{P}_X} \frac{2p}{p^2 + 1} = \log \log X + O(1).$$
Clearly, we have
$$\sum_{p \in \mathcal{P}_X} \frac{2p}{p^2 + 1} = \sum_{p \in \mathcal{P}_X} \frac{2}{p} + O(1).$$
By an explicit version of Chebotarev’s density theorem (see, e.g., [LO77]) we know that for $T \geq 3$ (using $\text{Li}(T) = T/\log T + O(T \log \log T/(\log T)^2)$)
$$\sum_{p \in \mathcal{P}_X} 1 = \frac{T}{2 \log T} + O\left(\frac{T \log \log T}{(\log T)^2}\right).$$
Applying partial summation we get
$$\sum_{p \in \mathcal{P}_X} \frac{2}{p} = \sum_{m=2}^{X^2} \frac{1}{(m+1) \log m} + O(1) = \log \log X + O(1).$$
This completes the proof of Theorem 3 for $d = 2$.

Appendix

We will now apply Theorem 5 to deduce the formula (1.8). We start by proving our claim that $\mathcal{N}'$ is an adelic Lipschitz system whenever all the functions $N_w$ of $\mathcal{N}$ are norms.

Lemma A1. Let $\mathcal{N}$ be an adelic Lipschitz system (of dimension $n$) on $K$ and assume that for every Archimedean place $v$ of $k$ there exists a place $w$ of $K$ extending $v$ such that
(a) $K_w = k_v$, or
(b) $N_w$ is a norm.
Then $\mathcal{N}' = \mathcal{N}'(\mathcal{N}, k)$ is an adelic Lipschitz system (of dimension $n$) on $k$.

Proof. The conditions (i), (ii) and (iv) in Definition 5.1 are obviously satisfied. It remains to prove (iii). To this end let $v$ be an Archimedean place of $k$ and let $w_1$ be a place of $K$ extending $v$ with (a) or (b). In a first step, we prove that the set
$$S := \{z \in k_v^{n+1} : N_{w_1}(z) = 1\}$$
is in $\text{Lip}(d_v(n+1), M, L)$, for some values of $M$ and $L$. This is clear if (a) holds, since $\mathcal{N}$ is an adelic Lipschitz system. If (b) holds then the restriction of $N_{w_1}$ to $k_v^{n+1} \subseteq K_w^{n+1}$ is again a norm and our claim follows from [Wid12b, Theorem 2.6].

Let $\psi_j$ be the parameterizing maps for $S$. For positive real $l$, the maps $l\psi_j$ then parameterize the set $N_{w_1}(z) = l$, and this in turn means that the set of all $z \in k_v^{n+1}$ with $N_v(z) = 1$ will be parameterized by the maps
$$\Phi_j : [0, 1)^{d_v(n+1)-1} \to \mathbb{R}^{d_v(n+1)}$$
defined by
$$\Phi_j(t) := \left(\prod_{i=2}^{a} N_{w_1}(\psi_j(t))^{-\frac{d}{d_v(n+1)}}\right) \psi_j(t).$$
As $N_{w_1}(\psi_j(t)) = 1$ we have $N_v(\psi_j(t)) \geq c_\mathcal{N} > 0$. Thus it is easily seen that the maps $\Phi_j$ also satisfy a Lipschitz condition. \qed
Let us now show how the formula (1.8) follows from Theorem 5. We use the adelic Lipschitz system $\mathcal{N}$ (of dimension 2) on $K := \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ defined by

$$N_w(z_0, z_1, z_2) := \max\{|z_0|_w, |z_1|_w, |z_2|_w, \sqrt{2z_1 + \sqrt{3z_2}}|_w\},$$

for any place $w$ of $K$. Hence all the $N_w$ are norms so that, thanks to Lemma A1, we can apply Theorem 5. With the notation from Section 6, we have $N_{K}(X) = N_{\mathcal{N}}(\mathbb{P}^2(\mathbb{Q}), X) + O(X^3)$, as already mentioned in the introduction. Here the error term accounts for the projective points of the form $(0 : \omega_1 : \omega_2)$. With Theorem 5, the only remaining task is to calculate $g_{\omega}^N$.

**Lemma A2.** We have

$$g_{\omega}^N = \frac{1}{31\zeta(3)}(1 + 2 \cdot 5^{1/4} + 4 \cdot 5^{-1/2}).$$

**Proof.** For some tedious computations in $K$, we use the computer algebra system *Sage*. We use the same notation as in Section 6. Clearly, we can choose $R = \{\mathbb{Z}\}$. For any $\omega = (\omega_0, \omega_1, \omega_2) \in \mathbb{Q}^3$, we have

$$i_{\mathcal{N}}(\omega) = \omega_0O_K + \omega_1O_K + \omega_2O_K + \frac{\sqrt{2}\omega_1 + \sqrt{3}\omega_2}{\sqrt{5}}O_K.$$ 

If $O_{\mathbb{Q}}(\omega) = \mathbb{Z}$ then $\omega_0O_K + \omega_1O_K + \omega_2O_K = O_K$, so $i_{\mathcal{N}}(\omega) \supseteq O_K$. On the other hand, we clearly have $i_{\mathcal{N}}(\omega) \subseteq (\sqrt{5})^{-1}O_K$. Thus, we can choose

$$S_2 := \{(\sqrt{5})^{-1}D : D \mid \sqrt{5}O_K\}.$$

Moreover, if $\omega \in \Lambda((\sqrt{5})^{-1}D)$, for some nonzero ideal $\mathfrak{A}$ of $O_K$, then $i_{\mathcal{N}}(\omega) = (\sqrt{5})^{-1}D$, for some nonzero ideal $D_1 \mid \sqrt{5}O_K$. In particular, $D\mathfrak{A} \mid D_1$. This shows that $T_{\mathcal{N},(\sqrt{5})^{-1}D}$ is contained in the finite set

$$T := \{\mathfrak{A} : \mathfrak{A} \mid \sqrt{5}O_K\}.$$

With (6.6), we obtain

$$(9.5) \quad g_{\omega}^N = \sum_{D \mid \sqrt{5}O_K} \mu_K(D) \sum_{\mathfrak{A} \mid \sqrt{5}O_K} \mu_K(\mathfrak{A}) \Sigma(\mathfrak{A}D),$$

where

$$\Sigma(\mathfrak{B}) := \sum_{n \in \mathbb{N}} \frac{\mu(n)}{\det(\Lambda((\sqrt{5})^{-1}\mathfrak{B}, n\mathbb{Z})).}$$

Let us evaluate this sum for any ideal $\mathfrak{B}$ of $O_K$ dividing $5O_K$. Elementary manipulations show that $\Lambda((\sqrt{5})^{-1}\mathfrak{B}, n\mathbb{Z})$ is the sublattice of $\mathbb{Z}^3$ consisting of all

$$\omega = (\omega_0, \omega_1, \omega_2) \in (n\mathbb{Z} \cap (\sqrt{5})^{-1}\mathfrak{B})^3$$

such that $\sqrt{2}\omega_1 + \sqrt{3}\omega_2 \in \mathfrak{B}$.

We have $5O_K = \mathfrak{P}_1^3\mathfrak{P}_2^3$, where $\mathfrak{P}_1, \mathfrak{P}_2$ are distinct prime ideals of $O_K$ with inertia degrees equal to 2.

For $\mathfrak{B} = O_K$, the first condition in (9.6) amounts to $\omega \in (n\mathbb{Z})^3$. Then the second condition is always satisfied, and $\det(\Lambda((\sqrt{5})^{-1}O_K, n\mathbb{Z}) = n^3$. Therefore,

$$\Sigma(O_K) = \sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^3} = \frac{1}{\zeta(3)}.$$ 

If $\mathfrak{B} = \mathfrak{P}_1$, then the first condition in (9.6) is equivalent to $\omega \in (n\mathbb{Z})^3$. For the second condition, we find that $-(\sqrt{3})^{-1}\sqrt{2} \equiv 3 \mod \mathfrak{P}_1$, so this condition

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is equivalent to \( \omega_2 = 3\omega_1 + a \), for an \( a \in \mathfrak{P}_1 \cap n\mathbb{Z} = \text{lcm}(5,n)\mathbb{Z} \). Therefore, \( \Lambda(\sqrt[5]{5}^{-1}\mathfrak{P}_1, n\mathbb{Z}) \) has the basis

\[
\{(n, 0, 0), (0, n, 3n), (0, 0, \text{lcm}(5,n))\}
\]

of determinant \( n^2 \text{lcm}(5,n) \). A similar computation shows that \( -(\sqrt{3})^{-1}\sqrt{2} \equiv 2 \mod \mathfrak{P}_2 \), so

\[
\{(n, 0, 0), (0, n, 2n), (0, 0, \text{lcm}(5,n))\}
\]

is a basis of \( \Lambda(\sqrt[5]{5}^{-1}\mathfrak{P}_2, n\mathbb{Z}) \) of the same determinant. Thus,

\[
\Sigma(\mathfrak{P}_1) = \sum_{n \in \mathbb{N}} \frac{\mu(n)}{n^2 \text{lcm}(5,n)^2} = \frac{1}{\zeta(3)} \frac{5^2 - 1}{5^3 - 1}.
\]

For \( \mathfrak{B} = \mathfrak{P}_1\mathfrak{P}_2 = \sqrt[5]{5}\mathcal{O}_K \), the first condition in (9.6) is again equivalent to \( \omega \in (n\mathbb{Z})^3 \). The second condition is equivalent to \( \omega_2 \equiv -(\sqrt{3})^{-1}\sqrt{2}\omega_1 \mod \mathfrak{P}_1\mathfrak{P}_2 \). By the Chinese remainder theorem and what we have seen before, this is equivalent to

\[
\omega_2 \equiv 2\omega_1 \mod 5 \quad \text{and} \quad \omega_2 \equiv 3\omega_1 \mod 5,
\]

so \( \omega_1 \equiv \omega_2 \equiv 0 \mod 5 \). Thus, \( \Lambda(\sqrt[5]{5}^{-1}\mathfrak{P}_1\mathfrak{P}_2, n\mathbb{Z}) = n\mathbb{Z} \times (\text{lcm}(5,n)\mathbb{Z})^2 \) has determinant \( n \text{lcm}(5,n)^2 \). We obtain

\[
\Sigma(\mathfrak{P}_1\mathfrak{P}_2) = \sum_{n \in \mathbb{N}} \frac{\mu(n)}{n \text{lcm}(5,n)^2} = \frac{1}{\zeta(3)} \frac{5 - 1}{5^3 - 1}.
\]

In the other cases, that is \( \mathfrak{P}_1^2 \mid \mathfrak{B} \) or \( \mathfrak{P}_2^2 \mid \mathfrak{B} \), we have \( \delta((\sqrt[5]{5}^{-1}\mathfrak{B}) = 5\mathbb{Z} \), so the first condition in (9.6) is equivalent to \( \omega \in (\text{lcm}(5,n)\mathbb{Z})^3 \). In this case, the second condition is always satisfied, so we obtain det \( \Lambda((\sqrt[5]{5}^{-1}\mathfrak{B}, n\mathbb{Z}) = \text{lcm}(5,n)^3 \) and

\[
\Sigma(\mathfrak{B}) = \sum_{n \in \mathbb{N}} \frac{\mu(n)}{\text{lcm}(5,n)^3} = 0.
\]

A simple computation shows that

\[
\mathfrak{m}_K((\sqrt[5]{5}^{-1}\mathcal{O}_K)^{3/8} = 5^{3/2}, \quad \mathfrak{m}_K((\sqrt[5]{5}^{-1}\mathfrak{P}_1)^{3/8} = 5^{-3/4}, \quad \mathfrak{m}_K(\mathcal{O}_K)^{3/8} = 1.
\]

To prove the lemma, just substitute this and (9.7) – (9.10) in (9.5). □

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