On One Problem of Optimization of Approximate Integration *

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Abstract

It is proved that interval quadrature formula of the form

\[ q(f) = \frac{1}{2h} \sum_{k=1}^{n} c_k \int_{x_k-h}^{x_k+h} f(t) dt \]

\((c_k \in \mathbb{R}, x_1 + h < x_2 - h < x_2 + h < \ldots < x_n - h < x_n + h < x_1 + 2\pi - h)\) with equal \(c_k\) and equidistant \(x_k\) is optimal among all such formulas for the class \(K * F_1\) of convolutions of a \(CVD\)-kernel \(K\) with functions from the unite ball of the space \(L_1\) of \(2\pi\)-periodic integrable functions.

Key words: interval quadrature formula, \(CVD\)-kernel, classes of convolutions.

1. Let \(C\) and \(L_p\) \((1 \leq p \leq \infty)\) be the spaces of \(2\pi\)-periodic functions endowed with corresponding norms; \(\|\cdot\|_p\) – norm in \(L_p\). The convolution of functions \(K \in L_1\) (kernel of convolution) and \(\phi \in L_1\) is defined by equality

\[ K * \phi(x) = \int_0^{2\pi} K(x - t) \phi(t) dt. \]

Given a kernel \(K\) set \(\mu = \mu(K) = 1\), if \(\int_0^{2\pi} K(t) dt = 0\), and \(\mu = \mu(K) = 0\), if \(\int_0^{2\pi} K(t) dt \neq 0\). Denote by \(\nu(f)\) the number of sign changes over a period of a \(2\pi\)-periodic function \(f\).

A kernel \(K\) is called a \(CVD\)-kernel (denoted by \(K \in CVD\)), if for any function of the form

\[ f(x) = a\mu + K * \phi(x) \]  

\((a \in \mathbb{R}, \phi \in C, \phi \perp \mu)\) the inequality \(\nu(f) \leq \nu(\phi)\) holds.

Let \(\psi(x)\) be an entire function of the form

\[ \psi(x) = x^l e^{-\gamma x^2} \prod_{k=1}^{\infty} (1 + \delta_k x) e^{\delta_k x} \]

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where $l \in \mathbb{Z}_+, \gamma, \delta, \delta_k \in \mathbb{R}, 0 < \gamma^2 + \sum_{k=1}^{\infty} |\delta_k| < \infty$. Then (see [1], [2]) the kernel

$$K(x) = \sum_{k=-\infty}^{\infty} e^{ikx} \psi(ik)$$

($\sum_{k=-\infty}^{\infty}$ denotes that the summation is carried out over all $k$ such that $\psi(ik) \neq 0$) is a CVD-kernel. In particular, Bernoulli's kernels

$$B_r(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} (ik)^r$$

and, more generally, kernels of the form

$$B_p(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} \frac{(ik)^r}{\mathcal{P}(ik)},$$

where $\mathcal{P}(x)$ is an algebraic polynomial having real zeros (integral operator of convolution with such a kernel inverses differential operators of the form $\mathcal{P} \left( \frac{d}{dx} \right)$) are CVD-kernels.

Denote by $F_p$ ($1 \leq p \leq \infty$) the unite ball in the space $L_p$. Given a kernel $K$ denote by $K \ast F_p$ the class of functions $f$ of the form (1) where $a \in \mathbb{R}, \phi \in F_p, \phi \perp \mu$. Note that $B_r \ast F_p$ is standard for the theory of quadrature formulas class $W^{(r)}_p$ of real-valued, $2\pi$-periodic functions $f$ having locally absolutely continuous derivative $f^{(r-1)} (f^{(0)}) = f$ and such that $\|f^{(r)}\|_p \leq 1$.

Consider the set $Q_n (n = 1, 2, \ldots)$ of all possible quadrature formulas of the form

$$q(f) = \sum_{k=1}^{n} c_k f(x_k),$$

where $c_k \in \mathbb{R}, x_1 < x_2 < \ldots < x_n < x_1 + 2\pi$. The problem about optimal for the class $K \ast F_p$ quadrature formula from $Q_n$ is formulated in the following way. Find the value

$$R_n(K \ast F_p) = \inf_{q \in Q_n} \sup_{f \in K \ast F_p} \left\| \int_0^{2\pi} f(x) dx - q(f) \right\|,$$  \hspace{1cm} (3)

and parameters (knots $x_k$ and coefficients $c_k$) of a quadrature formula $q$ that realizes $\inf$ in the right hand part of (3).

This problem was completely solved for the classes $W^{(r)}_p$ ($r = 1, 2, \ldots; 1 \leq p \leq \infty$) in the papers of V. P. Motornyi [3], A. A. Ligun [4], A. A. Zhensykybaev [5], [6]. It was proved that for any $n$ the optimal formula has $n$ equidistant knots and equal coefficients. In the papers [7], [8], [9], [10] these results were generalized to the cases of more general function classes $K \ast F_p$.

We will consider the following problem. Let CVD-kernel $K; p \in [1, \infty]; n = 1, 2, \ldots; h \in (0, \pi/n)$ be given. Denote by $Q_{n,h}$ the set of all functionals of the form

$$q(f) = \sum_{k=1}^{n} c_k \frac{1}{2h} \int_{x_k-h}^{x_k+h} f(t) dt$$  \hspace{1cm} (4)
where \( c_k \in \mathbb{R},\ x_1 + h < x_2 - h < x_2 + h < ... < x_n - h < x_n + h < x_1 + 2\pi - h. \) Set
\[
R(f, q) = \int_0^{2\pi} f(t) dt - q(f),
\]
\[
R(K \ast F_p, q) = \sup_{f \in K \ast F_p} |R(f, q)|,
\]
\[
R_{n,h}(K \ast F_p) = \inf_{q \in Q_{n,h}} R(K \ast F_p, q).
\]
(5)

The problem is formulated as follows. Find the value (5) and parameters \( x_k \) and \( c_k \) \((k = 1, ..., n)\) of a functional \( q \) that realizes inf in the right hand part of (5).

From the applications point of view, interval quadrature formulae are more natural than the usual quadrature formulae based on values at points, since quite often the result of measuring physical quantities, due to the structure of the measurement devices, is an average values of the function, describing the studied quantities, over some interval. Note that one can obtain the usual quadrature formula from the corresponding interval quadrature formula as a limit case, setting \( h \to 0 \).

In this paper we solve the problem on optimal interval quadrature formula for the class \( K \ast F_1 \) with \( p = 1 \). Thus we will essentially use Ligun's idea from [4].

2. Suppose that \( f \in K \ast F_1, q \in Q_{n,h}, \) and obtain for \( R(f, q) \) an integral representation. Given \( q \in Q_{n,h} \) let
\[
H(q; t) = \frac{1}{2h} \sum_{k=1}^{n} c_k \chi_{(x_k-h,x_k+h)}(t)
\]
where \( \chi_A \) is the indicator of a set \( A \subset [0, 2\pi] \) continued with the period \( 2\pi \) on the entire real axis. We will have
\[
R(f, q) = a \cdot \mu \left( 2\pi - \sum_{k=1}^{n} c_k \right) + \int_0^{2\pi} \phi(u) \left[ \int_0^{2\pi} K(t-u) dt - \int_0^{2\pi} K(t-u) H(q; t) dt \right] du
\]
\[
= a \cdot \mu \left( 2\pi - \sum_{k=1}^{n} c_k \right) + \int_0^{2\pi} \phi(u) \int_0^{2\pi} K(t-u) [1 - H(q; t)] dt du.
\]
Let
\[
M(q, u) = \int_0^{2\pi} K(t-u) [1 - H(q; t)] dt.
\]
We obtain
\[
R(f, q) = a \cdot \mu \left( 2\pi - \sum_{k=1}^{n} c_k \right) + \int_0^{2\pi} \phi(u) M(q, u) du.
\]
(6)

In the case \( \mu(K) = 0 \) the first term in the right-hand part of (6) is equal to zero. Solving the problem about optimal interval quadrature formula for the class \( K \ast F_1 \) in the case \( \mu(K) = 1 \) we can consider functionals \( q \) such that \( \sum_{k=1}^{n} c_k = 2\pi \) only (otherwise...
\( R(K \ast F_1, q) = +\infty \). Thus in the case \( \mu(K) = 1 \) we can assume that the first term in the right-hand part of (6) is equal to zero also.

Taking into account the relations (6), above presented facts, and S. M. Nikol’skii’s duality theorem (see [11], [12, Chapt. 2, Theorem 2.2.1]) we obtain

\[
R(K \ast F_1; q) = \begin{cases} 
\inf_{\lambda \in \mathbb{R}} \| M(q; \cdot) - \lambda \|_\infty, & \text{if } \mu(K) = 1, \\
\| M(q; \cdot) \|_\infty, & \text{if } \mu(K) = 0. 
\end{cases}
\]

For \( n = 1, 2, \ldots, \lambda \in \mathbb{R}, \) and \( 0 < h < \pi/n \) set

\[
q_{n,h,\lambda}(f) = \lambda \sum_{k=1}^{n} \frac{1}{2h} \int_{2k\pi n^{-1} + h}^{2k\pi n^{-1} - h} f(t) dt.
\]

**Theorem 1.** Let \( K \) be a CVK-d kernel, \( n = 1, 2, \ldots, \) and \( 0 < h < \pi/n. \) Then

\[
R_{n,h}(K \ast F_1) = \begin{cases} 
R(K \ast F_1; q_{n,h,2\pi/n}) = \inf_{\lambda \in \mathbb{R}} \| M(q_{n,h,2\pi/n}; \cdot) - \lambda \|_\infty, & \text{if } \mu(K) = 1, \\
R(K \ast F_1; q_{n,h,\lambda}) = \inf_{\lambda \in \mathbb{R}} \| M(q_{n,h,\lambda}; \cdot) \|_\infty, & \text{if } \mu(K) = 0. 
\end{cases}
\]

**Proof.** Case \( \mu(K) = 1. \) Denote by \( \lambda_q (q \in Q_{n,h}) \) the constant of the best \( L_\infty \)-approximation of the function \( M(q; \cdot) \). Suppose that for some \( q \in Q_{n,h} \) the inequality

\[
\| M(q; \cdot) - \lambda_q \|_\infty < \| M(q_{n,h,2\pi/n}; \cdot) - \lambda_{q_{n,h,2\pi/n}} \|_\infty
\]

holds true. Let

\[
\Delta_\tau(t) = M(q; t - \tau) - M(q_{n,h,2\pi/n}; t) - \lambda_q + \lambda_{q_{n,h,2\pi/n}}, \tau \in \mathbb{R}.
\]

Since the function \( M(q_{n,h,2\pi/n}; \cdot) \) is \( 2\pi/n \)-periodic, we will have that for any \( \tau \)

\[
\nu(\Delta_\tau(\cdot)) \geq 2n.
\]

It is clear that \( K(\cdot) \in CVK \) with \( \mu = 1. \) Taking into account this fact and representation (6) it is easy to verify that for any \( \tau \) we will have

\[
\nu(\Delta_\tau'(\cdot)) \geq 2n
\]

where

\[
\Delta_\tau'(t) = -H(q_{n,h,2\pi/n}; t) - H(q; t - \tau).
\]

Among the coefficients \( c_k \) of the functional \( q \) choose such that \( |c_k| \leq 2\pi/n \) and denote by \( k_0 \) its number. Set \( \tau_0 = 2\pi/n - x_{k_0} \).

**Lemma 1.**

\[
\nu(\Delta_{\tau_0}(\cdot)) \leq 2n - 2.
\]
If Lemma 1 will be proved, we obtain the contradiction with the above presented statement about the number of the sign changes of $\Delta'_\tau$. After that the part of Theorem 1 related to the case $\mu(K) = 1$ will be proved.

**Proof of the Lemma 1.** Suppose that $\nu(\Delta'_\tau_0) \geq 2n$. It means that there exist points $y_1 < y_2 < \ldots < y_{2n} < y_1 + 2\pi$ such that difference $\Delta'_\tau_0$ has at these points nonzero values of alternating sign. Let (for definiteness) $\Delta'_\tau_0$ has negative values at the points $y_2, y_4, \ldots, y_{2n}$. It is easily seen that such points must belong to the different intervals of positivity of the function $H(q; t - \tau_0)$. But the number of such intervals inside an interval of the length $2\pi$ is less than or equal to $n$. Really, the difference $\Delta'_\tau_0$ can not change sign inside the common part of an interval of positivity of the function $H(q_n,h,2\pi/n;t)$ and an interval of positivity of the function $H(q; t - \tau_0)$ (this difference is constant inside such an part). Thus at least one of the point of negativity (say $y_2^l$) must belong to the interval $(2\pi/n - h, 2\pi/n + h)$. But in view of the choice of $\tau_0$, the difference $\Delta'_\tau_0$ is nonnegative on this interval. This is a contradiction.

Lemma is proved.

The case $\mu(K) = 0$. We need the following analog of the Lemma 1 from [8].

**Lemma 2.** Let $\lambda^*$ be such that

$$\inf_{\lambda \in \mathbb{R}} \| M(q_n,h,\lambda; \cdot) \|_\infty = \| M(q_n,h,\lambda^*; \cdot) \|_\infty.$$ 

Then

$$\max_t M(q_n,h,\lambda^*; t) = -\min_t M(q_n,h,\lambda^*; t).$$

**Proof of Lemma 2.** We have

$$M(q_n,h,\lambda^*; t) = \int_0^{2\pi} K(u)du - \lambda \int_0^{2\pi} K(u - t)H(q_n,h,1; u)du.$$ 

It is well known that $CVD$-kernel $K$ with $\mu(K) = 0$ does not change sign (let it is nonnegative for definiteness). Then it is easily seen that function

$$\psi(t) = \int_0^{2\pi} K(u - t)H(q_n,h,1; u)du$$

is strictly positive. If for a given $\lambda_0 > 0$

$$\max_t \left[ \int_0^{2\pi} K(u)du - \lambda_0\psi(t) \right] > -\min_t \left[ \int_0^{2\pi} K(u)du - \lambda_0\psi(t) \right], \quad (8)$$

then for $\lambda < \lambda_0$ and close enough to $\lambda_0$ we will have

$$\left\| \int_0^{2\pi} K(u)du - \lambda\psi(\cdot) \right\|_\infty < \left\| \int_0^{2\pi} K(u)du - \lambda_0\psi(\cdot) \right\|_\infty. \quad (9)$$
If instead of (8) we have
\[
\max_t \left[ \int_0^{2\pi} K(u)du - \lambda_0 \psi(t) \right] < - \min_t \left[ \int_0^{2\pi} K(u)du - \lambda_0 \psi(t) \right],
\]
then the inequality (9) will hold true for all \( \lambda > \lambda_0 \) and close enough to \( \lambda_0 \). Since the existence of \( \lambda^* \) and its positiveness are obvious, the Lemma is proved.

Suppose that for some \( q \in Q_{n,h} \)
\[
R(K \ast F_1; q) < \inf_{\lambda \in \mathbb{R}} R(K \ast F_1; q_{n,h,\lambda})
\]  \hspace{1cm} (10)
or that is equivalent (in view of (7))
\[
\|M(q; \cdot)\|_{\infty} < \|M(q_{n,h,\lambda^*}; \cdot)\|_{\infty}.
\]  \hspace{1cm} (11)
Taking into account the Lemma 2 and the fact that \( M(q_{n,h,\lambda^*}; \cdot) \) is \( 2\pi/n \)-periodic, we conclude that
\[
\nu(\Delta_{\tau}(\cdot)) \geq 2n.
\]
where
\[
\Delta_{\tau}(t) = M(q; t - \tau) - M(q_{n,h,\lambda^*}; t)
\]
But
\[
\Delta_{\tau}(t) = \int_0^{2\pi} K(u - t)[H(q_{n,h,\lambda^*}; u) - H(q; u - \tau)]du.
\]
Since \( K \in CVD \) for any \( \tau \) then
\[
\nu(\Delta'_{\tau}) := \nu(H(q_{n,h,\lambda^*}; u) - H(q; u - \tau)) \geq 2n.
\]
However analogously to the case \( \mu(K) = 1 \) it is possible to choose \( \tau_0 \) such that \( \nu(\Delta'_{\tau_0}) \leq 2n - 2 \). We omit the details.

Therefore the relation (11), and consequently the relation (10), is impossible. Theorem 1 is proved.

Let us show that Theorem 1 implies the optimality of the quadrature formula with equidistant knots and equal coefficients on the class \( K \ast F_1 \) among the all quadrature formulas from \( Q_n \). Let a CVD-kernel \( K \) be such that the class \( K \ast F_1 \) is a relatively compact subset of the space \( C \) (in the case \( \mu(K) = 0 \) or can be obtained by shifts to the constants from relatively compact subset of \( C \) (in the case \( \mu(K) = 1 \)). For any quadrature formula \( q \) of the form (2) let \( \{q_h, h > 0\} \) be the family of functionals of the form (4) having the same \( c_k \) and \( x_k \). It is easily seen that \( q_h(f) \rightarrow q(f) \) uniformly on the set \( f \in K \ast F_1 \) as \( h \rightarrow 0 \) (remind that in the case \( \mu(K) = 1 \) we consider formulas \( q \) and functionals \( q_h \) such that \( \sum_{k=1}^{n} c_k = 2\pi \) only). Set
\[
q_{n,0,\lambda}(f) = \lambda \sum_{k=1}^{n} f \left( \frac{2k\pi}{n} \right).
\]
Then (we restrict ourself by the case $\mu(K) = 1$) for any $q \in Q_n$ we will have

$$R(K \ast F_1; q) = \lim_{h \to 0} R(K \ast F_1; q_h) \geq \lim_{h \to 0} R(K \ast F_1; q_{n,h,2\pi/n}) = R(K \ast F_1; q_{n,0,2\pi/n})$$

that is the optimality of the rectangle formula on the class $K \ast F_1$.

Finally we note that the statement of the Theorem 1 can be easily generalized to the classes of convolutions of the functions from $F_1$ with $O(M, \Delta)$-kernels (see [8]).

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