Relative controllability of impulsive multi-delay differential systems

Zhongli You\textsuperscript{a}, Michal Fečkan\textsuperscript{b,c}, JinRong Wang\textsuperscript{a,d,1}, Donal O’Regan\textsuperscript{e}

\textsuperscript{a}Department of Mathematics, Guizhou University, Guiyang 550025, Guizhou, China
zlyoumath@126.com; jrwang@gzu.edu.cn
\textsuperscript{b}Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia
michal.feckan@fmph.uniba.sk
\textsuperscript{c}Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
\textsuperscript{d}School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China
\textsuperscript{e}School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
donal.oregan@nuigalway.ie

Received: August 30, 2020 / Revised: January 31, 2021 / Published online: January 1, 2022

Abstract. In this paper, relative controllability of impulsive multi-delay differential systems in finite dimensional space are studied. By introducing the impulsive multi-delay Gramian matrix, a necessary and sufficient condition, and the Gramian criteria, for the relative controllability of linear systems is given. Using Krasnoselskii’s fixed point theorem, a sufficient condition for controllability of semilinear systems is obtained. Numerically examples are given to illustrate our theoretically results.

Keywords: impulsive multi-delay differential systems, impulsive multi-delay Gramian matrix, relative controllability.
1 Introduction

In many motion processes of nature, science, and technology, the state of motion may be changed or interfered suddenly in a very short time, and then the system state will be changed. If the state change time of the disturbed system is very short, it can be regarded as instantaneous, and then this kind of instantaneous sudden change phenomenon is called pulse phenomenon. Time-delay systems are systems with aftereffect or dead time, genetic systems, equations with deviating arguments or differential difference equations. They are used to model various phenomena from population systems, viscoelasticity, biological sciences, chemistry, economics, mechanics, physics, physiology, and engineering sciences. In the real world, impulsive phenomena and time-delay effects are intertwined and interact with each other. Impulse technology is widely used in the state control of time-delay systems and has applications in military and civil fields.

The delayed exponential matrix functions approach was presented in [6, 10] for discrete and continuous delay systems with permutable matrices, respectively. This new approach has been used in the stability of solutions and control problems for linear and nonlinear delay systems (see [1–5, 7–9, 11, 13–20]).

Medved’ and Pospíšil extended the idea of deriving the representation of delay differential equations in [6, 10] to multi-delay differential equations with linear parts defined by pairwise permutable matrices in [16] and obtained sufficient conditions for the asymptotic stability of solutions. You and Wang [22, 23] extended the multiple delayed exponential matrix function in [10] to the impulsive case and used it to discuss the representation and stability of solutions in [24]. However, there are still very few results for the relative controllability of impulsive multi-delay differential systems. In this paper, we study the following impulsive multi-delay differential systems:

\[
\nu'(t) = A\nu(t) + \sum_{m=1}^{n} B_m \nu(t - \vartheta_m) + f(t, \nu(t)) + C u(t), \quad t \in J, \ t \notin \mathcal{T},
\]

\[
\Delta \nu(t_i) := \nu(t_i^+) - \nu(t_i^-) = D_i \nu(t_i), \quad t_i \in \mathcal{T},
\]

\[
\nu(t) = \psi(t), \quad -\vartheta \leq t \leq 0, \ \vartheta := \max\{\vartheta_1, \ldots, \vartheta_n\},
\]

where \(\vartheta_m > 0\), \(A, B_m, C, D_i\) are constant \(N \times N\) matrices, \(A B_m = B_m A, B_j B_m = B_m B_j, A D_i = D_i A\), and \(B_m D_i = D_i B_m\) for each \(m, j = 1, 2, \ldots, n\), \(i = 1, 2, \ldots\), \(\psi \in C^{1}_\vartheta := C^{1}([-\vartheta, 0], \mathbb{R}^N)\), and \(\nu(t) \in \mathbb{R}^N\). Now \(f \in C(J \times \mathbb{R}^N, \mathbb{R}^N), \ J := [0, \tau_1]\), \(\tau_1 > 0, 0 < t_1 < t_2 < \cdots < t_h < \tau_1\), and the control function \(u(\cdot)\) takes values from \(L^2(J, \mathbb{R}^N)\). Let \(\nu(t_i^+) = \lim_{\epsilon \to 0^+} \nu(t_i + \epsilon)\) and \(\nu(t_i^-) = \nu(t_i)\) represent respectively the right and left limits of \(\nu(t)\) at \(t = t_i\).

First, we investigate the relative controllability of the linear case of (1), i.e., \(f = 0 \in \mathbb{R}^N\) using the impulsive multi-delayed matrix exponential in (2). Next, we construct a suitable control function for (1), which means that we give a condition (necessary and sufficient) for \(u \in L^2(J, \mathbb{R}^N)\) to lead the solution of (1) with \(f = 0\) to \(\nu_{\tau_1}\) at the time \(\tau_1\). We apply Krasnoselskii’s fixed point theorem to show that (1) is also relatively controllable under suitable conditions.
The rest of this paper is organized as follows. In Section 2, we give some notations, concepts, and important lemmas. In Section 3, we establish relative controllability results for linear and semilinear systems, respectively. Examples are given to illustrate our main results in the final section.

2 Preliminaries

Let \( \mathbb{R}^N \) be the \( N \)-dimensional Euclid space with the vector norm \( \| \cdot \| \), and \( \mathbb{R}^{N \times N} \) be the \( N \times N \) matrix space with real value elements. For \( \nu \in \mathbb{R}^N \) and \( A \in \mathbb{R}^{N \times N} \), we introduce the vector infinite-norm \( \| \nu \| = \max_{1 \leq i \leq N} |\nu_i| \) and the matrix infinite-norm \( \| A \| = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |a_{ij}| \), respectively, where \( \nu_i \) and \( a_{ij} \) are the elements of the vector \( \nu \) and matrix \( A \). Let \( L(\mathbb{R}^N) \) be the space of bounded linear operators in \( \mathbb{R}^N \). Denote by \( C(J, \mathbb{R}^N) \) the Banach space of vector-value bounded continuous functions from \( J \to \mathbb{R}^N \) endowed with the norm \( \| \nu \|_C = \sup_{t \in J} \| \nu(t) \| \). In addition, \( \| \psi \|_C = \sup_{t \in [-\varrho, 0]} \| \psi(t) \| \). We introduce a space \( C^1(J, \mathbb{R}^N) = \{ \nu \in C(\mathbb{R}^+, \mathbb{R}^N): \nu' \in C(\mathbb{R}^+, \mathbb{R}^N) \} \). Denote \( PC(J, \mathbb{R}^N) := \{ \nu: J \to \mathbb{R}^N: \nu \in C((t_i, t_{i+1}], \mathbb{R}^N), \text{there exist } \nu(t_i^-) \text{ and } \nu(t_i^+) \text{ with } \nu(t_i^-) = \nu(t_i) \text{ for any } i = 1, 2, \ldots \} \) and \( PC^1(J, \mathbb{R}^N) := \{ \nu: J \to \mathbb{R}^N: \nu' \in PC(J, \mathbb{R}^N) \} \). Let \( X_1, X_2 \) be two Banach spaces, and \( L_b(X_1, X_2) \) denotes the space of all bounded linear operators from \( X_1 \) to \( X_2 \). Next, \( L^p(J, X_2) \) denotes the Banach space of functions \( y: J \to X_2 \), which are Bochner integrable normed by \( \| y \|_{L^p(J, X_2)} \) for some \( 1 < p < \infty \).

We recall the notation of the multi-delayed matrix exponential given by [16]:

\[
\mathcal{E}^{B_1, \ldots, B_t}_{\vartheta_1, \ldots, \vartheta_j} = \begin{cases} 
\Theta, & t < -\vartheta_j, \\
\mathcal{X}_{j-1}(t + \vartheta_j), & -\vartheta_j \leq t < 0, \\
\mathcal{X}_{j-1}(t + \vartheta_j) + B_j \int_0^t \mathcal{X}_{j-1}(t - s_1) \mathcal{X}_{j-1}(s_1) \, ds_1 + \cdots \\
+ B_j \int_0^t \mathcal{X}_{j-1}(t - s_1) \, ds_1 + \cdots + \int_0^t \mathcal{X}_{j-1}(t - s_1) \, ds_1, \\
\times \prod_{i=1}^{z-1} \mathcal{X}_{j-1}(s_i - s_{i+1}) \mathcal{X}_{j-1}(s_z - (z-1)\vartheta_j) \, ds_z \cdots ds_1, \\
(z-1)\vartheta_j \leq t < z\vartheta_j, & z = 1, 2, \ldots,
\end{cases}
\]

(2)

where \( \mathcal{X}_{j-1}(t) = \mathcal{E}^{B_1, \ldots, B_{j-1}(t-\vartheta_{j-1})}_{\vartheta_1, \ldots, \vartheta_{j-1}}, j = 2, \ldots, n, \) and \( \Theta \) is the zero matrix.

From [24] we know \( \mathcal{Y}(\cdot, \cdot): \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{N \times N} \) and

\[
\mathcal{Y}(t, s) = e^{A(t-s)} \mathcal{X}(t, s + \vartheta), \quad t > s,
\]

(3)

where

\[
\mathcal{X}(t, s) = \mathcal{E}^{B_1, \ldots, B_n}_{\vartheta_1, \ldots, \vartheta_n}(t-s) + \sum_{s-\vartheta < t_j \leq t} D_j \mathcal{E}^{\tilde{B}_1, \ldots, \tilde{B}_n}_{\vartheta_1, \ldots, \vartheta_n}(t-\vartheta-t_j) \mathcal{X}(t_j, s),
\]

\( \tilde{B}_m = e^{-A\vartheta_m} B_m, m = 1, \ldots, n. \)
Next, the solution of (1) has the form
\[
\nu(t) = \mathcal{Y}(t, -\vartheta)\varphi(-\vartheta)
+ \int_{-\vartheta}^{0} \mathcal{Y}(t, s)[\psi'(s) - A\psi(s)] \, ds - \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta} \mathcal{Y}(t, s + \vartheta_m)\psi(s) \, ds
+ \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} \mathcal{Y}(t, s) \left[ f(s, \nu(s)) + Cu(s) \right] \, ds
+ \int_{0}^{t} \mathcal{Y}(t, s) \left[ f(s, \nu(s)) + Cu(s) \right] \, ds.
\]

(4)

Lemma 1. (See [16, Lemma 13].) If \( \|B_i\| \leq b_i e^{b_i \vartheta} \), \( b_i \in \mathbb{R}^+ \), \( i = 1, \ldots, n \), then
\[
\|e^{B_1 \ldots B_n (t-\vartheta)}\| \leq e^{(b_1 + \cdots + b_n) t}, \quad t \in \mathbb{R}.
\]

Lemma 2. Suppose that \( \sum_{j=1}^{\infty} \|D_j\| \) is convergent, \( \|\tilde{B}_m\| \leq \alpha_m e^{\alpha_m \vartheta_m} \), \( \alpha_m \in \mathbb{R}^+ \), \( m = 1, \ldots, n \). For any \( t > s \), we have
\[
\|\mathcal{X}(t, s)\| \leq \left( \prod_{s-\vartheta < t_j \leq t} (\|D_j\| + 1) \right) e^{\alpha(t+\vartheta-s)},
\]
(5)
\[
\|\mathcal{Y}(t, s)\| \leq \left( \prod_{s < t_j \leq t} (\|D_j\| + 1) \right) e^{(\|A\|+\alpha)(t-s)},
\]
(6)
\[\alpha = \alpha_1 + \cdots + \alpha_n.\]

Proof. Without loss of generality, we suppose that \( t_i \leq s - \vartheta < t_{i+1} \) and \( t_{i+l} \leq t < t_{i+l+1} \), \( i, l = 0, 1, 2, \ldots \). We use mathematical induction.

For \( l = 0 \), by Lemma 1,
\[
\|\mathcal{X}(t, s)\| \leq \|\mathcal{E}_{\tilde{B}_1 \ldots \tilde{B}_n (t-s)}\| \leq e^{(\alpha_1 + \cdots + \alpha_n) (t+\vartheta_n-s)} \leq e^{\alpha(t+\vartheta-s)}.
\]

For \( l = 1 \), using Lemma 1, we have
\[
\|\mathcal{X}(t, s)\| \leq \|\mathcal{E}_{\tilde{B}_1 \ldots \tilde{B}_n (t-s)}\| + \|D_{i+1}\| \mathcal{E}_{\tilde{B}_1 \ldots \tilde{B}_n (t-\vartheta_{i+1})} \mathcal{X}(t_{i+1}, s)
\leq e^{\alpha(t+\vartheta-s)} + \|D_{i+1}\| e^{(\alpha_1 + \cdots + \alpha_n) (t+\vartheta_n-t_{i+1})} e^{\alpha(t_{i+1}+\vartheta-s)}
\leq e^{\alpha(t+\vartheta-s)} + \|D_{i+1}\| e^{\alpha(t+\vartheta_n-s)} \leq (\|D_{i+1}\| + 1) e^{\alpha(t+\vartheta-s)}.
\]
For $l = k$, we suppose that
\[
\| \mathcal{X}(t, s) \| \leq \left( \prod_{s - \theta < t_j \leq t} (\| D_j \| + 1) \right) e^{\alpha(t - \theta - s)}.
\]

For $l = k + 1$, using Lemma 1, we have
\[
\| \mathcal{X}(t, s) \| \leq \left( 1 + \sum_{j=1}^{i+k+1} \| D_j \| \prod_{z=i+1}^{j-1} (\| D_z \| + 1) \right) e^{\alpha(t - \theta - s)}
\]
\[
= \left( \prod_{j=i+1}^{i+k+1} (\| D_j \| + 1) \right) e^{\alpha(t - \theta - s)}.
\]

Thus, we obtain (5).

Finally, using (3) and (5) via \( e^{A t} \| \leq e^{\| A \| t} \), one derives (6) immediately. The proof is finished. \qed

Lemma 3 [Krasnoselskii’s fixed point theorem]. (See [12].) Let \( B \) be a bounded closed and convex subset of Banach space \( X \), and let \( F_1, F_2 \) be maps of \( B \) into \( X \) such that \( F_1 x + F_2 y \in B \) for every pair \( x, y \in B \). If \( F_1 \) is a contraction and \( F_2 \) is compact and continuous, then the equation \( F_1 x + F_2 x = x \) has a solution on \( B \).

Theorem 1 [PC-type Ascoli–Arzela theorem]. (See [21, Thm. 2.1].) Let \( \mathcal{D} \subset PC(J, X) \), where \( X \) is a Banach space. Then \( \mathcal{D} \) is a relatively compact subset of \( PC(J, X) \) if:

(i) \( \mathcal{D} \) is a uniformly bounded subset of \( PC(J, X) \);
(ii) \( \mathcal{D} \) is equicontinuous in \((t_i, t_{i+1})\), \( i = 0, 1, 2, \ldots, h \) (here \( t_0 = 0 \) and \( t_{h+1} = \tau_1 \));
(iii) \( \mathcal{D}(t) = \{ \nu(t) : \nu \in \mathcal{D}, t \in J \setminus \mathcal{T} \} \), \( \mathcal{D}(t_i^+) = \{ \nu(t_i^+) : \nu \in \mathcal{D} \} \) and \( \mathcal{D}(t_i^-) = \{ \nu(t_i^-) : \nu \in \mathcal{D} \} \) are relatively compact subsets of \( X \).

3 Relative controllability

Definition 1. (See [11, Def. 4].) System (1) is called relatively controllable if for an arbitrary initial vector function \( \psi \in C^1([-\theta, 0], \mathbb{R}^N) \), the final state of the vector \( \nu_{\tau_1} \in \mathbb{R}^N \) and time \( \tau_1 \), there exists a control \( u \in L^2(J, \mathbb{R}^N) \) such that system (1) has a solution \( \nu \in C^1([-\theta, \tau_1]) \) that satisfies the boundary conditions \( \nu \) and \( \nu(\tau_1) = \nu_{\tau_1} \).

https://www.journals.vu.lt/nonlinear-analysis
3.1 Linear systems

Let \( f(t, \nu(t)) \equiv 0, \ t \in J \). System (1) reduces to the following linear impulsive multi-delay controlled system:

\[
\nu'(t) = A \nu(t) + \sum_{m=1}^{n} B_m \nu(t - \vartheta_m) + C u(t), \quad t \in J, \ t \notin T, \\
\Delta \nu(t_i) = D_i \nu(t_i), \quad t_i \in T, \\
\nu(t) = \psi(t), \quad -\vartheta \leq t \leq 0.
\]

The solution has a form

\[
\nu(t) = \mathcal{Y}(t, -\vartheta) \psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(t, s) \left[ \psi'(s) - A \psi(s) \right] ds \\
- \sum_{m=1}^{n} B_m \int_{-\vartheta}^{0} \mathcal{Y}(t, s + \vartheta_m) \psi(s) ds + \int_{0}^{t} \mathcal{Y}(t, s) C u(s) ds.
\]

Similar to the classical Gramian matrix, we consider the impulsive multi-delay Gramian matrix as follows:

\[
W_{\vartheta_1, \ldots, \vartheta_n}[0, \tau_1] = \tau_1 \int_{0}^{\tau_1} \mathcal{Y}(\tau_1, s) C C^T \mathcal{Y}^T(\tau_1, s) ds.
\]

**Theorem 2.** System (7) is relatively controllable if and only if \( W_{\vartheta_1, \ldots, \vartheta_n}[0, \tau_1] \) is nonsingular.

**Proof.** First, we verify the sufficiency. Since \( W_{\vartheta_1, \ldots, \vartheta_n}[0, \tau_1] \) is nonsingular, its inverse \( W_{\vartheta_1, \ldots, \vartheta_n}^{-1}[0, \tau_1] \) is well defined. For any final state \( \nu_{\tau_1} \in \mathbb{R}^N \), one can select a control function as follows:

\[
u(t) = C^T \mathcal{Y}^T(\tau_1, t) W_{\vartheta_1, \ldots, \vartheta_n}^{-1}[0, \tau_1] \eta,
\]

where

\[
\eta = \nu_{\tau_1} - \mathcal{Y}(\tau_1, -\vartheta) \psi(-\vartheta) - \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s) \left[ \psi'(s) - A \psi(s) \right] ds \\
+ \sum_{m=1}^{n} B_m \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s + \vartheta_m) \psi(s) ds.
\]

Then

\[
\nu(\tau_1) = \mathcal{Y}(\tau_1, -\vartheta) \psi(-\vartheta) + \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s) \left[ \psi'(s) - A \psi(s) \right] ds \\
- \sum_{m=1}^{n} B_m \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s + \vartheta_m) \psi(s) ds + \int_{0}^{\tau_1} \mathcal{Y}(\tau_1, s) C u(s) ds
\]
\[ Y(\tau_1, -\vartheta) \psi(-\vartheta) + \int_{-\vartheta}^{0} Y(\tau_1, s) [\psi'(s) - A\psi(s)] \, ds \]
\[- \sum_{m=1}^{n} B_m \int_{-\vartheta}^{\vartheta_m} Y(\tau_1, s + \vartheta_m) \psi(s) \, ds \]
\[ + \int_{0}^{\tau_1} Y(\tau_1, s) C C^T Y^T(\tau_1, s) W_{\vartheta_1, \ldots, \vartheta_n}^{-1} [0, \tau_1] \eta \, ds \]
\[ = \nu_{\tau_1}. \]

Next, by contradiction we prove the necessity. We assume that \( W_{\vartheta_1, \ldots, \vartheta_n}^{-1} [0, \tau_1] \) is singular matrix, i.e., there exists at least one nonzero state \( \tilde{\nu} \in \mathbb{R}^N \) such that
\[ \tilde{\nu}^T W_{\vartheta_1, \ldots, \vartheta_n}^{-1} [0, \tau_1] \tilde{\nu} = 0. \]

Then one obtains
\[ 0 = \tilde{\nu}^T W_{\vartheta_1, \ldots, \vartheta_n}^{-1} [0, \tau_1] \tilde{\nu} = \int_{0}^{\tau_1} \tilde{\nu}^T Y(\tau_1, s) C C^T Y^T(\tau_1, s) \tilde{\nu} \, ds = \int_{0}^{\tau_1} \| \tilde{\nu}^T Y(\tau_1, s) C \|^2 \, ds, \]
which implies \( \tilde{\nu}^T Y(\tau_1, s) C = 0^T \) for all \( s \in J \).

Since system (7) is relatively controllable, according to Definition 1, there exists a control \( u_1(t) \) that drives the initial state to zero at \( \tau_1 \), i.e.,
\[ \nu(\tau_1) = Y(\tau_1, -\vartheta) \psi(-\vartheta) + \int_{-\vartheta}^{0} Y(\tau_1, s) [\psi'(s) - A\psi(s)] \, ds \]
\[- \sum_{m=1}^{n} B_m \int_{-\vartheta}^{\vartheta_m} Y(\tau_1, s + \vartheta_m) \psi(s) \, ds + \int_{0}^{\tau_1} Y(\tau_1, s) C u_1(s) \, ds \]
\[ = 0. \] (8)

Similarly, there also exists a control \( u_2(t) \) that drives the initial state to \( \tilde{\nu} \) (nonzero) at \( \tau_1 \), i.e.,
\[ \nu(\tau_1) = Y(\tau_1, -\vartheta) \psi(-\vartheta) + \int_{-\vartheta}^{0} Y(\tau_1, s) [\psi'(s) - A\psi(s)] \, ds \]
\[- \sum_{m=1}^{n} B_m \int_{-\vartheta}^{\vartheta_m} Y(\tau_1, s + \vartheta_m) \psi(s) \, ds + \int_{0}^{\tau_1} Y(\tau_1, s) C u_2(s) \, ds \]
\[ = \tilde{\nu}. \] (9)
Theorem 3. Suppose that

\[ a \text{ controllable, provided that} \]

We assume the following:

3.2 Semilinear systems

Relative controllability of impulsive multi-delay differential systems

77

Multiplying both sides of (10) by \( \tilde{v}^T \), we obtain

\[ \tilde{v}^T \tilde{v} = \int_0^{\tau_1} \tilde{v}^T \mathcal{Y}(\tau_1, s) C [u_2(s) - u_1(s)] \, ds = 0. \]

Thus, \( \tilde{v} = 0 \), which conflicts with \( \tilde{v} \neq 0 \). Thus, the impulsive multi-delay Gramian matrix \( W_{\tau_1, \ldots, \tau_n} \) is nonsingular. The proof is complete. \( \square \)

3.2 Semilinear systems

We assume the following:

(H1) The operator \( W : L^2(J, \mathbb{R}^N) \to \mathbb{R}^N \) defined by \( W u = \int_0^{\tau_1} \mathcal{Y}(\tau_1, s) C u(s) \, ds \) has an inverse operator \( W^{-1} \), which takes values in \( L^2(J, \mathbb{R}^N) / \ker W \). Then we set \( M = \| W^{-1} \|_{L^2(J, \mathbb{R}^N) / \ker W} \). From [20, Remark 3.3] we know

\[ M = (\| W_{\tau_1, \ldots, \tau_n}^{-1} [0, \tau_1] \|)_{1/2}. \]

(H2) The function \( f : J \times \mathbb{R}^N \to \mathbb{R}^N \) is continuous, and there exists a constant \( q > 1 \) and \( L_f(\cdot) \in L^q(J, \mathbb{R}^+ \) such that

\[ \| f(\cdot, \nu(\cdot)) - f(\cdot, \nu(\cdot)) \| \leq L_f(\cdot) \| \nu(\cdot) - \nu(\cdot) \|, \quad \nu, \nu \in \mathbb{R}^N. \]

**Theorem 3.** Suppose that (H1) and (H2) are satisfied. Then system (1) is relatively controllable, provided that

\[ M_2 \left[ 1 + a \frac{\| C \| M}{\| A \|} \left( e^{(\| A \| + \alpha) \tau_1} - 1 \right) \right] < 1, \]

where \( a = \prod_{i=1}^N (\| D_j \| + 1) \), and \( M_2 = a \left( (1/(\| A \| + \alpha) p) (e^{(\| A \| + \alpha) \tau_1} - 1) \right)^{1/p} \), \( \| L_f \|_{L^q(J, \mathbb{R}^+)} \), \( 1/p + 1/q = 1, p, q > 1 \).

**Proof.** Using hypothesis (H1), for arbitrary \( \nu(\cdot) \in PC \) and \( t \in J \), we define the control function \( u_\nu(t) \) by

\[
\begin{aligned}
    u_\nu(t) &= W^{-1} \left( \nu_{\tau_1} - \mathcal{Y}(\tau_1, -\vartheta) \psi(-\vartheta) - \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s) \left[ \psi'(s) - A \psi(s) \right] ds 
    
    &+ \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} \mathcal{Y}(\tau_1, s + \vartheta_m) \psi(s) ds 
    
    - \int_{0}^{\tau_1} \mathcal{Y}(\tau_1, s) f(s, \nu(s)) ds \right) (t). 
\end{aligned}
\]
We show that, using this control, the operator \( F : PC \to PC \), defined by
\[
(F \nu)(t) = \mathcal{Y}(t, -\vartheta)\psi(-\vartheta)
+ \int_0^t \mathcal{Y}(t, s)\left[\psi'(s) - A\psi(s)\right] ds - \sum_{m=1}^n B_m \int_{-\vartheta}^{\vartheta_m} \mathcal{Y}(t, s + \vartheta_m)\psi(s) ds
+ \int_0^t \mathcal{Y}(t, s)f(s, \nu(s)) ds + \int_0^t \mathcal{Y}(t, s)Cu_{\nu}(s) ds,
\]
has a fixed point \( \nu \), which is a mild solution of (1).

We check that \( (F \nu)(\tau_1) = \nu \tau_1 \), which means that \( u_\nu \) steers system (1) from \( (F \nu)(0) \) to \( \nu \tau_1 \) in finite time \( \tau_1 \). This implies that system (1) is relatively controllable on \( J \).

For each positive number \( r \), let \( B_r = \{ \nu \in PC : \|\nu\|_{PC} \leq r \} \) (a bounded, closed, and convex set of \( PC \)). Set \( R_f = \sup_{t \in J} \|f(t, 0)\| \).

We divide the proof into three steps.

**Step 1.** We claim that there exists a positive number \( r \) such that \( F(B_r) \subseteq B_r \).

From (H2) and Hölder’s inequality we obtain that
\[
\int_0^t e^{\|A\|+\alpha}(t-s) L_f(s) ds \leq \left( \int_0^t e^{p(\|A\|+\alpha)(t-s)} ds \right)^{1/p} \left( \int_0^t L_f^q(s) ds \right)^{1/q} \leq \left[ \frac{1}{(\|A\|+\alpha)p} (e^{(\|A\|+\alpha)t} - 1) \right]^{1/p} \|L_f\|_{L^q(J, R^+)},
\]
and
\[
\int_0^t e^{\|A\|+\alpha}(t-s) \|f(s, 0)\| ds \leq R_f \int_0^t e^{\|A\|+\alpha}(t-s) ds = \frac{R_f}{\|A\|+\alpha} (e^{(\|A\|+\alpha)t} - 1).
\]

From (12), (H1) and (H2) we have
\[
\|u_\nu(t)\| \leq \|W^{-1}\|_{L_b(R^N, L^2(J, R^N))}^\text{ker} W \left( \|\nu_{\tau_1}\| + \|\mathcal{Y}(\tau_1, -\vartheta)\| \right)
+ \int_0^t \mathcal{Y}(\tau_1, s)\|\psi'(s) - A\psi(s)\| ds
+ \sum_{m=1}^n \|B_m\| \int_{-\vartheta}^{\vartheta_m} \mathcal{Y}(\tau_1, s + \vartheta_m)\|\psi(s)\| ds
+ \int_0^{\tau_1} \mathcal{Y}(\tau_1, s)\|f(s, \nu(s))\| ds.
\]
\[
\begin{align*}
\leq M \left[ \| \nu_{\tau_1} \| + \left( \prod_{j=1}^{h} (\| D_j \| + 1) \right) e^{(\| A \| + \alpha)(\tau_1 + \vartheta)} \| \psi(-\vartheta) \| \right. \\
+ \left( \prod_{j=1}^{h} (\| D_j \| + 1) \right) \int_{-\vartheta}^{0} e^{(\| A \| + \alpha)(\tau_1 - s)} \| \psi'(s) - A\psi(s) \| \, ds \\
+ \left( \prod_{j=1}^{h} (\| D_j \| + 1) \right) \sum_{m=1}^{n} \| B_m \| \int_{-\vartheta}^{\tau_j} e^{(\| A \| + \alpha)(\tau_1 - \vartheta_m - s)} \| \psi(s) \| \, ds \\
\left. + \left( \prod_{j=1}^{h} (\| D_j \| + 1) \right) \int_{0}^{\tau_j} e^{(\| A \| + \alpha)(\tau_1 - s)} (L_f(s) \| \nu(s) \| + \| f(s, 0) \|) \, ds \right]
\leq M \| \nu_{\tau_1} \| + a e^{(\| A \| + \alpha)(\tau_1 + \vartheta)} \| \psi(-\vartheta) \| + \int_{-\vartheta}^{0} a e^{(\| A \| + \alpha)(\tau_1 - s)} \| \psi'(s) - A\psi(s) \| \, ds \\
+ \sum_{m=1}^{n} a \| B_m \| \int_{-\vartheta}^{-\vartheta_m} e^{(\| A \| + \alpha)(\tau_1 - \vartheta_m - s)} \| \psi(s) \| \, ds \\
+ a \left[ \frac{1}{(\| A \| + \alpha)p\tau_1} - 1 \right]^{1/p} \| L_f \| _{L^p(J, \mathbb{R}^+)} \| \nu \| _{PC} \\
+ \frac{a R_f}{\| A \| + \alpha} (e^{(\| A \| + \alpha)\tau_1} - 1) \\
\leq M \| \nu_{\tau_1} \| + M M_1 + M M_2 \| \nu \| _{PC},
\end{align*}
\]

where
\[
M_1 = a e^{(\| A \| + \alpha)(\tau_1 + \vartheta)} \| \psi(-\vartheta) \| + \int_{-\vartheta}^{0} a e^{(\| A \| + \alpha)(\tau_1 - s)} \| \psi'(s) - A\psi(s) \| \, ds \\
+ \sum_{m=1}^{n} a \| B_m \| \int_{-\vartheta}^{-\vartheta_m} e^{(\| A \| + \alpha)(\tau_1 - \vartheta_m - s)} \| \psi(s) \| \, ds \quad + \frac{a R_f}{\| A \| + \alpha} (e^{(\| A \| + \alpha)\tau_1} - 1).
\]

From (H1) and (H2) we have
\[
\| (F \nu)(t) \| \leq \| \mathcal{Y}(t, -\vartheta) \| \| \psi(-\vartheta) \| + \int_{-\vartheta}^{0} \| \mathcal{Y}(t, s) \| \| \psi'(s) - A\psi(s) \| \, ds \\
+ \sum_{m=1}^{n} \| B_m \| \int_{-\vartheta}^{-\vartheta_m} \| \mathcal{Y}(t, s + \vartheta_m) \| \| \psi(s) \| \, ds
\]
\[
+ \int_0^t \left\| \mathcal{Y}(t, s) \right\| f(s, \nu(s)) \, ds 
+ \int_0^t \left\| \mathcal{Y}(t, s) \right\| C \left\| u_\nu(s) \right\| \, ds
\leq \left( \prod_{j=1}^h (\|D_j\| + 1) \right) e^{(\|A\|+\alpha)(t+\vartheta)} \left\| \psi(-\vartheta) \right\| \\
+ \left( \prod_{j=1}^h (\|D_j\| + 1) \right) \int_{-\vartheta}^0 e^{(\|A\|+\alpha)(t-s)} \left\| \psi'(s) - A\psi(s) \right\| \, ds
+ \sum_{m=1}^n \left( \prod_{j=1}^h (\|D_j\| + 1) \right) e^{(\|A\|+\alpha)(t-\vartheta_m-s)} \left\| \psi(s) \right\| \, ds
\leq M_1 + M_2 \|\nu\|_{PC} + \left( \prod_{j=1}^h (\|D_j\| + 1) \right) \|C\| \\
\times \left( M \|\nu_{\tau_1}\| + MM_1 + MM_2 \|\nu\|_{PC} \right) \int_0^t e^{(\|A\|+\alpha)(t-s)} \, ds
\leq M_1 \left[ 1 + \left( \prod_{j=1}^h (\|D_j\| + 1) \right) \frac{\|C\|}{\|A\| + \alpha} \left( e^{(\|A\|+\alpha)t - 1} \right) \right] \\
+ \left( \prod_{j=1}^h (\|D_j\| + 1) \right) \frac{\|C\|}{\|A\| + \alpha} \left( e^{(\|A\|+\alpha)t - 1} \right) \|\nu_{\tau_1}\| \\
+ M_2 \left( \prod_{j=1}^h (\|D_j\| + 1) \right) \frac{\|C\|}{\|A\| + \alpha} \left( e^{(\|A\|+\alpha)t - 1} \right) \|\nu\|_{PC}
\leq M_1 \left[ 1 + \frac{a \|C\|}{\|A\| + \alpha} \left( e^{(\|A\|+\alpha)\tau_1 - 1} \right) \right] \\
+ \frac{a \|C\|}{\|A\| + \alpha} \left( e^{(\|A\|+\alpha)\tau_1 - 1} \right) \|\nu_{\tau_1}\| \\
+ M_2 \left[ 1 + \frac{a \|C\|}{\|A\| + \alpha} \left( e^{(\|A\|+\alpha)\tau_1 - 1} \right) \right] \tau
= r,
\]

https://www.journals.vu.lt/nonlinear-analysis
where
\[ r = \frac{M_1[1 + a\|C\|\|M\|e(\|A\|+\alpha)\tau_1 - 1] + a\|C\|\|M\|e(\|A\|+\alpha)\tau_1 - 1]\|\nu_{\tau_1}\|}{1 - M_2[1 + a\|C\|\|M\|e(\|A\|+\alpha)\tau_1 - 1]} . \]

Hence, we obtain \( \mathcal{F}(B_r) \subseteq B_r \) for such an \( r \).

Now, we define operators \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) on \( B_r \) as
\[
(\mathcal{F}_1 \nu)(t) = Y(t, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} Y(t, s)[\psi'(s) - A\psi(s)] \, ds
- \sum_{m=1}^{n} B_m \int_{-\vartheta}^{-\vartheta_m} Y(t, s + \vartheta_m)\psi(s) \, ds + \int_{0}^{t} Y(t, s)C u_{\nu}(s) \, ds, \quad t \in J,
\]
and
\[
(\mathcal{F}_2 \nu)(t) = \int_{0}^{t} Y(t, s)f(s, \nu(s)) \, ds, \quad t \in J.
\]

Step 2. We claim that \( \mathcal{F}_1 \) is a contraction mapping.
Let \( \nu, \gamma \in B_r \). From (H1) and (H2), for each \( t \in J \), we have
\[
\|u_{\nu}(t) - u_{\gamma}(t)\|
\leq M \int_{0}^{\tau_1} \|Y(t, s)\| \|f(s, \nu(s)) - f(s, \gamma(s))\| \, ds
\leq M \int_{0}^{\tau_1} \left( \prod_{s < t_j \leq \tau_1} (\|D_j\| + 1) \right) e(\|A\|+\alpha)(\tau_1 - s) L_f(s) (\|\nu(s) - \gamma(s)\|) \, ds
\leq M \left( \prod_{j=1}^{n} (\|D_j\| + 1) \right) \int_{0}^{\tau_1} e(\|A\|+\alpha)(\tau_1 - s) L_f(s) \, ds \|\nu - \gamma\|_{PC}
\leq M a \left[ \frac{1}{(\|A\| + \alpha)^p} (e(\|A\|+\alpha)p\tau_1 - 1)^{1/p} \right] \|L_f\|_{L^p(\mathbb{R}^+)} \|\nu - \gamma\|_{PC}
\leq M M_2 \|\nu - \gamma\|_{PC} .
\]
Thus,
\[
\||\mathcal{F}_1 \nu)(t) - (\mathcal{F}_1 \gamma)(t)||
\leq \int_{0}^{t} \|Y(t, s)\| \|C\| \|u_{\nu}(s) - u_{\gamma}(s)\| \, ds
\]

Nonlinear Anal. Model. Control, 27(1):70–90, 2022
\[ \leq \left( \prod_{j=1}^{h} (\|D_j\| + 1) \right) \int_0^{\tau_1} \|D_j\| \|MM_2\| C \|M\| \|\nu - \gamma\|_{PC} \]

\[ \leq \frac{a \|C\| MM_2}{\|A\| + \alpha} \left( e^{\|A\| + \alpha \tau_1} - 1 \right) \|\nu - \gamma\|_{PC} , \]

so we obtain

\[ \|F_1\nu - F_1\gamma\|_{PC} \leq T \|\nu - \gamma\|_{PC} , \quad T = \frac{a \|C\| MM_2}{\|A\| + \alpha} \left( e^{\|A\| + \alpha \tau_1} - 1 \right) . \]

From (11) we have \( T < 1 \), so \( F_1 \) is a contraction.

**Step 3.** We claim that \( F_2 : B_r \rightarrow PC \) is a compact and continuous operator.

Let \( \nu_n \in B_r \) with \( \nu_n \rightarrow \nu \) in \( B_r \). Using (H2), we have \( f(s, \nu_n(s)) \rightarrow f(s, \nu(s)) \) in \( PC \), and thus, using the Lebesgue dominated convergence theorem, we have

\[ \left\| (F_2\nu_n)(t) - (F_2\nu)(t) \right\| \]

\[ \leq \int_0^t \left\| Y(t, s) \right\| \left| f(s, \nu_n(s)) - f(s, \nu(s)) \right| \ ds \]

\[ \leq \left( \prod_{j=1}^{h} (\|D_j\| + 1) \right) \int_0^t e^{\|A\| + \alpha (t-s)} \left| f(s, \nu_n(s)) - f(s, \nu(s)) \right| \ ds \]

\[ \rightarrow 0 \quad \text{as} \quad n \rightarrow 0 , \]

which implies that \( F_2 \) is continuous on \( B_r \).

To check the compactness of \( F_2 : B_r \rightarrow PC \), we prove that \( F_2(B_r) \) is equicontinuous and uniformly bounded. In fact, for any \( \nu \in B_r \), \( t_k < t \leq t + d \leq t_{k+1} \), \( k = 0, 1, \ldots, h \),

\[ (F_2\nu)(t + d) - (F_2\nu)(t) \]

\[ = \int_0^{t+d} Y(t + d, s) f(s, \nu(s)) \ ds - \int_0^t Y(t, s) f(s, \nu(s)) \ ds \]

\[ = \int_0^t Y(t + d, s) f(s, \nu(s)) \ ds \]

\[ + \int_0^t e^{A(t+d-s)} \left( X(t + d, s + \vartheta) - X(t, s + \vartheta) \right) f(s, \nu(s)) \ ds \]

\[ + \int_0^t (e^{A(t+d-s)} - e^{A(t-s)}) X(t, s + \vartheta) f(s, \nu(s)) \ ds . \]

https://www.journals.vu.lt/nonlinear-analysis
Let

\[
S_1(t) = \int_t^{t+d} \mathcal{Y}(t + d, s) f(s, \nu(s)) \, ds,
\]

\[
S_2(t) = \int_0^t e^{A(t+d-s)}(\mathcal{X}(t + d, s + \vartheta) - \mathcal{X}(t, s + \vartheta)) f(s, \nu(s)) \, ds,
\]

\[
S_3(t) = \int_0^t (e^{A(t+d-s)} - e^{A(t-s)}) \mathcal{X}(t, s + \vartheta) f(s, \nu(s)) \, ds
\]

\[
= \int_0^t (e^{Ad} - I) \mathcal{Y}(t, s) f(s, \nu(s)) \, ds,
\]

where \( I \) is the identity matrix.

From above we see that

\[
\left\| (\mathcal{F}_2 \mathcal{V})(t + d) - (\mathcal{F}_2 \mathcal{V})(t) \right\| \leq \left\| S_1(t) \right\| + \left\| S_2(t) \right\| + \left\| S_3(t) \right\|.
\]

Now, we only need to check \( \|S_i(t)\| \to 0 \) as \( d \to 0 \), \( i = 1, 2, 3 \). Clearly,

\[
\left\| S_1(t) \right\| \leq \int_t^{t+d} \left\| \mathcal{Y}(t + d, s) \right\| f(s, \nu(s)) \, ds
\]

\[
\leq \int_t^{t+d} \left( \prod_{s < t_j \leq t+d} (\|D_j\| + 1) \right) e^{(\|A\| + \alpha)(t+d-s)} (L_f(s) \|\nu(s)\| + \|f(s, 0)\|) \, ds
\]

\[
\leq \left[ \frac{1}{(\|A\| + \alpha)p} (e^{(\|A\| + \alpha)pd} - 1) \right]^{1/p} L_f \left\| L^\mathcal{A}(J, \mathbb{R}^+) \right\| F_C
\]

\[
+ \frac{R_f}{\|A\| + \alpha} (e^{(\|A\| + \alpha)d} - 1)
\]

\[
\to 0 \quad \text{as} \quad d \to 0,
\]

\[
\left\| S_2(t) \right\| \leq \int_0^t \left\| e^{A(t+d-s)} \right\| \left\| \mathcal{X}(t + d, s + \vartheta) - \mathcal{X}(t, s + \vartheta) \right\| f(s, \nu(s)) \, ds
\]

\[
\leq e^{\|A\| \tau_1} \int_0^t \left\| \mathcal{E}_{\theta_1, \ldots, \theta_n}(t+d-\vartheta-s) - \mathcal{E}_{\theta_1, \ldots, \theta_n}(t-\vartheta-s)
\]

\[
+ \sum_{s < t_j \leq t} D_j \left( \mathcal{E}_{\theta_1, \ldots, \theta_n}(t+d-\vartheta-t_j) - \mathcal{E}_{\theta_1, \ldots, \theta_n}(t-\vartheta-t_j) \right) \mathcal{X}(t_j, s + \vartheta)
\]

\[
\times (L_f(s) \|\nu(s)\| + \|f(s, 0)\|) \, ds
\]

Nonlinear Anal. Model. Control, 27(1):70–90, 2022
By the continuity of $E_{\varphi_1}, \ldots, \varphi_n$ we have $\|S_2\| \to 0$ as $d \to 0$. Also,

$$
\|S_3(t)\| \leq \int_0^t \left\| e^{Ad} - I \right\| \| \mathcal{Y}(t, s) \| \| f(s, \nu(s)) \| \, ds
$$

\[\leq \left\| e^{Ad} - I \right\| \int_0^t \left( \prod_{s < t_j \leq t} (\|D_j\| + 1) \right) e^{(\|A\| + \alpha)(t-s)}
\times (L_f(s) \| \nu(s) \| + \| f(s, 0) \|) \, ds
\]

\[\leq \left\| e^{Ad} - I \right\| \| a \left( \frac{1}{(\|A\| + \alpha)\rho_1} (e^{(\|A\| + \alpha)\tau_1} - 1) \right)^{1/p} \| L_f \|_{L^q(J, \mathbb{R}^+)}^r
+ \frac{R_f}{\|A\| + \alpha} (e^{(\|A\| + \alpha)\tau_1} - 1) \to 0 \quad \text{as} \quad d \to 0.
\]

As a result, we immediately obtain that

$$
\| (F_2\nu)(t + d) - (F_2\nu)(t) \| \to 0 \quad \text{as} \quad d \to 0
$$

for all $\nu \in \mathcal{B}_r$. Therefore, $F_2(\mathcal{B}_r)$ is equicontinuous in $PC$.

https://www.journals.vu.lt/nonlinear-analysis
Next, repeating the above computations, we have

\[
\| (F_2\nu)(t) \| \\
\leq \int_0^t \| Y(t,s)\| \| f(s,\nu(s))\| \, ds \\
\leq \int_0^t \left( \prod_{s < t_j \leq t} (\|D_j\| + 1) \right) e^{(\|A\|+\alpha)(t-s)} (L_f(s)\|\nu(s)\| + \|f(s,0)\|) \, ds \\
\leq a \left[ \frac{1}{(\|A\|+\alpha)p} \left( e^{(\|A\|+\alpha)p\tau_1} - 1 \right) \right]^{1/p} L_f \|L^r_{(J,\mathbb{R}^+)r} \| \\
+ \frac{aR_f \|A\| + \alpha}{(\|A\|+\alpha)(\|A\|+\alpha)} \left( e^{(\|A\|+\alpha)p\tau_1} - 1 \right) .
\]

Hence, \( F_2(B_r) \) is uniformly bounded. From Theorem 1, \( F_2(B_r) \) is relatively compact in \( PC \). Thus, \( F_2 : B_r \rightarrow PC \) is a compact and continuous operator.

Furthermore, using Theorem 3, \( F \) has a fixed point \( \nu \) on \( B_r \). Obviously, \( \nu \) is a solution of system (1) satisfying \( \nu(\tau_1) = \nu_{\tau_1} \). The boundary condition \( \nu(t) = \psi(t), -\vartheta \leq t \leq 0 \) holds from (4). The proof is complete. \( \square \)

4 Numerical examples

Example 1. Consider the following semilinear impulsive multi-delay differential controlled system:

\[
\nu'(t) = A\nu(t) + \sum_{m=1}^{2} B_m \nu(t - \vartheta_m) \\
+ f(t,\nu(t)) + Cu(t), \quad t \in J, t \notin \mathcal{T}, \\
\Delta \nu(t_i) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} \nu(t_i), \quad t_i \in \mathcal{T}, \\
\nu(t) = (3, 4)^T, \quad -0.3 \leq t \leq 0,
\]

and set \( J = [0, 0.6], \tau_1 = 0.6, \vartheta_1 = 0.3, \vartheta_2 = 0.2 \). Then \( \vartheta = \max\{\vartheta_1, \vartheta_2\} = 0.3 \) and \( \mathcal{T} = \{0.35, 0.7, 1.05, \ldots \} \),

\[
A = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.3 & -0.1 \\ 0 & 0.2 \end{pmatrix}, \\
C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f(t,\nu(t)) = \begin{pmatrix} -0.06t\nu_1(t) \\ 0.04t\nu_2(t) \end{pmatrix}.
\]
Note $A, B_m, D_i$ are mutually permutable for $m = 1, 2, i = 1, 2, 3, \ldots$, and

$$W_{0.3.0.2}[0, 0.6] = \int_0^{0.6} \mathcal{Y}(0.6, s)CC^T \mathcal{Y}^T(0.6, s) \, ds$$

$$= \int_0^{0.6} e^{A(0.6-s)} \chi(0.6, s + 0.3)CC^T \chi^T(0.6, s + 0.3)e^{A^T(0.6-s)} \, ds$$

$$= W_1 + W_2 + W_3 + W_4 + W_5 + W_6,$$

where

$$W_1 = \int_0^{0.05} e^{A(0.6-s)} \left[ I + \tilde{B}_1(0.2 - s) + \tilde{B}_2(0.3 - s) + \frac{\tilde{B}_2^2}{2}(0.1 - s)^2 \\
+ D_1[I + \tilde{B}_2(0.05 - s)] \right] CC^T \left[ I + \tilde{B}_1^T(0.2 - s) + \tilde{B}_2^T(0.3 - s) \\
+ \frac{(\tilde{B}_2^T)^2}{2}(0.1 - s)^2 + [I + \tilde{B}_2^T(0.05 - s)] D_1^T \right] e^{A^T(0.6-s)} \, ds,$$

$$W_2 = \int_0^{0.1} e^{A(0.6-s)} \left[ I + \tilde{B}_1(0.2 - s) + \tilde{B}_2(0.3 - s) + \frac{\tilde{B}_2^2}{2}(0.1 - s)^2 + D_1 \right] \\
\times CC^T \left[ I + \tilde{B}_1^T(0.2 - s) + \tilde{B}_2^T(0.3 - s) + \frac{(\tilde{B}_2^T)^2}{2}(0.1 - s)^2 + D_1^T \right] \\
\times e^{A^T(0.6-s)} \, ds,$$

$$W_3 = \int_0^{0.2} e^{A(0.6-s)} \left[ I + \tilde{B}_1(0.2 - s) + \tilde{B}_2(0.3 - s) + D_1 \right] \\
\times CC^T \left[ I + \tilde{B}_1^T(0.2 - s) + \tilde{B}_2^T(0.3 - s) + D_1^T \right] e^{A^T(0.6-s)} \, ds,$$

$$W_4 = \int_0^{0.25} e^{A(0.6-s)} \left[ I + \tilde{B}_2(0.3 - s) + D_1 \right] \\
\times CC^T \left[ I + \tilde{B}_2^T(0.3 - s) + D_1^T \right] e^{A^T(0.6-s)} \, ds,$$

$$W_5 = \int_0^{0.3} e^{A(0.6-s)} \left[ I + \tilde{B}_2(0.3 - s) \right] CC^T \left[ I + \tilde{B}_2^T(0.3 - s) \right] e^{A^T(0.6-s)} \, ds,$$

$$W_6 = \int_0^{0.3} e^{A(0.6-s)} CC^T e^{A^T(0.6-s)} \, ds.$$
Specifically,

\[
W_1 = \begin{pmatrix} 0.0432 & 0.0427 \\ 0.0427 & 0.1010 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.0449 & 0.0446 \\ 0.0446 & 0.0955 \end{pmatrix}, \\
W_3 = \begin{pmatrix} 0.0955 & 0.0954 \\ 0.0954 & 0.1769 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0.0523 & 0.0525 \\ 0.0525 & 0.0838 \end{pmatrix}, \\
W_5 = \begin{pmatrix} 0.0405 & 0.0405 \\ 0.0405 & 0.0583 \end{pmatrix}, \quad W_6 = \begin{pmatrix} 0.2272 & 0.2272 \\ 0.2272 & 0.2639 \end{pmatrix},
\]

then

\[
W_{0.3,0.2}[0,0.6] = \begin{pmatrix} 0.5036 & 0.5029 \\ 0.5029 & 0.7794 \end{pmatrix}, \quad M = \sqrt{\|W_{0.3,0.2}^{-1}[0,0.6]\|} = 3.0308.
\]

Further, for any \(\nu, \mu \in \mathbb{R}^2\),

\[
\| f(t, \nu) - f(t, \mu) \| = \max\{-0.06t|\nu_1 - \mu_1|, 0.04t|\nu_2 - \mu_2|\} \\
\leq 0.06t \max\{|\nu_1 - \mu_1|, |\nu_2 - \mu_2|\} \\
= 0.06t\|\nu - \mu\|.
\]

Note \(L_f(t) = 0.06t\) and let \(p = q = 2\), \(\|L_f\|_{L^2(J,R^+)} = (\int_0^{0.6} (0.06s)^2 \, ds)^{1/2} = 0.0161\).

Note

\[
\|B_1\| = \|e^{-A\varrho_1}B_1\| = \left\| \begin{pmatrix} 0.3644 & 0.3735 \\ 0 & 0.7379 \end{pmatrix} \right\| \leq \alpha_1 e^{0.3\alpha_1},
\]

choose \(\alpha_1 = 0.61384\),

\[
\|B_2\| = \|e^{-A\varrho_2}B_2\| = \left\| \begin{pmatrix} 0.4475 & -0.0831 \\ 0 & 0.3644 \end{pmatrix} \right\| \leq \alpha_2 e^{0.2\alpha_2},
\]

choose \(\alpha_2 = 0.4820\),

\[
\alpha = \alpha_1 + \alpha_2 = 1.09584, \quad \|A\| + \alpha = 4.09584, \quad a = \prod_{j=1}^h (\|D_j\| + 1) = 1.2.
\]

As a result,

\[
M_2 = a \left[ \frac{1}{(\|A\| + \alpha)p} (e^{(\|A\|+\alpha)\varrho_1} - 1) \right]^{1/p} \|L_f\|_{L^q(J,R^+)} = 0.0785,
\]

\[
M_2 \left[ 1 + \frac{a\|C\|M}{\|A\| + \alpha} (e^{(\|A\|+\alpha)\tau_1} - 1) \right] = 0.0785 \times \left[ 1 + \frac{1.2 \times 3.0308}{4.09584} (e^{4.09584 \times 0.6} - 1) \right] = 0.8226 < 1.
\]

Thus all the conditions of Theorem 3 are satisfied, so (13) is relatively controllable on \([0,0.6]\); see Fig. 1.

Nonlinear Anal. Model. Control, 27(1):70–90, 2022
Example 2. In Example 1, let $f(t, \nu(t)) = 0$, $t \in [0, 0.6]$. Note $W_{0.3,0.2}[0, 0.6]$ is a nonsingular matrix. From Theorem 2 we know that the linear multi-delay system is relatively controllable. Furthermore, one can get

$$
\eta = \nu_{\tau_1} - \mathcal{Y}(\tau_1, -\vartheta) \psi(-\vartheta)
$$

$$
- \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s) \left[ \psi'(s) - A\psi(s) \right] \, ds + \sum_{m=1}^{n} B_m \int_{-\vartheta}^{0} \mathcal{Y}(\tau_1, s + \vartheta_m) \psi(s) \, ds
$$

$$
= \nu_{\tau_1} - \mathcal{Y}(0.6, -0.3) \psi(-0.3)
$$

$$
- \int_{-0.3}^{0} \mathcal{Y}(0.6, s) \left[ \psi'(s) - A\psi(s) \right] \, ds + B_2 \int_{-0.3}^{0} \mathcal{Y}(0.6, s + 0.2) \psi(s) \, ds,
$$

and then the control function is given by

$$
u(t) = C^T \mathcal{Y}^T(\tau_1, t) W_{\vartheta_1, \ldots, \vartheta_n}^{-1}[0, \tau_1] \eta
$$

$$
= C^T \mathcal{X}^T(0.6, t + 0.3) e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}
$$

$$\begin{cases}
C^T[I + \tilde{B}_1^T(0.2 - t) + \tilde{B}_2^T(0.3 - t) + \frac{(\tilde{B}_2)^2}{2}(0.1 - t)^2] + [I + \tilde{B}_2^T(0.05 - t)] D_2^T e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}, & 0 \leq t \leq 0.05, \\
C^T[I + \tilde{B}_1^T(0.2 - t) + \tilde{B}_2^T(0.3 - t) + \frac{(\tilde{B}_2)^2}{2}(0.1 - t)^2 + D_2^T] e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}, & 0.05 < t \leq 0.1, \\
C^T[I + \tilde{B}_1^T(0.2 - t) + \tilde{B}_2^T(0.3 - t) + D_1^T] e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}, & 0.1 < t \leq 0.2, \\
C^T[I + \tilde{B}_2^T(0.3 - t) + D_1^T] e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}, & 0.2 < t \leq 0.25, \\
C^T[I + \tilde{B}_2^T(0.3 - t)] e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}, & 0.25 < t \leq 0.3, \\
C^T e^{A^T(0.6-t)W_{0.3,0.2}^{-1}[0,0.6] \eta}, & 0.3 < t \leq 0.5, \\
\Theta, & 0.5 < t \leq 0.6.
\end{cases}
$$
5 Conclusion

In this paper the relative controllability of impulsive multi-delay differential systems in finite-dimensional space is considered. In [24] the authors construct the index of impulsive multi delay matrix and give the explicit solution of linear impulsive multi delay differential equations. Based on the expression of the solution of linear impulsive multi delay differential equations, necessary and sufficient conditions for the relative controllability of linear systems and the Gramian criteria are given. In Theorem 3, using Krasnoselskii fixed point theorem, we give a sufficient condition for the controllability of semilinear systems.

In Theorem 2 the control function is given, but it is not necessarily optimal, and we hope in the future to study the optimal control problem of impulsive multi-delay differential equations. In Theorem 3, we require the operator $F_2$ to be compact, and we hope to study controllability under noncompact conditions in the future.

Acknowledgment. The authors are grateful to the referees for their careful reading of the manuscript and their valuable comments. We thank the editor also.

References

1. A. Boichuk, J. Diblík, D. Khusainov, M. Růžičková, Fredholm’s boundary-value problems for differential systems with a single delay, *Nonlinear Anal., Theory Methods Appl.*, 72(5):2251–2258, 2010, https://doi.org/10.1016/j.na.2009.10.025.

2. X. Cao, J. Wang, Finite-time stability of a class of oscillating systems with two delays, *Math. Meth. Appl. Sci.*, 41(13):4943–4954, 2018, https://doi.org/10.1002/mma.4943.

3. J. Diblík, M. Fečkan, M. Pospíšil, Representation of a solution of the Cauchy problem for an oscillating system with two delays and permutable matrices, *Ukr. Math. J.*, 65(1):64–76, 2013, https://doi.org/10.1007/s11253-013-0765-y.

4. J. Diblík, M. Fečkan, M. Pospíšil, On the new control functions for linear discrete delay systems, *SIAM J. Control Optim.*, 52(3):1745–1760, 2014, https://doi.org/10.1137/140953654.

5. J. Diblík, D. Ya. Khusainov, M. Růžičková, Controllability of linear discrete systems with constant coefficients and pure delay, *SIAM J. Control Optim.*, 47(3):1140–1149, 2008, https://doi.org/10.1137/070689085.

6. J. Diblík, D.Ya. Khusainov, Representation of solutions of discrete delayed system $x(k + 1) = Ax(k) + Bx(k - m) + f(k)$ with commutative matrices, *J. Math. Anal. Appl.*, 318(1):63–76, 2006, https://doi.org/10.1016/j.jmaa.2005.05.021.

7. J. Diblík, D.Ya. Khusainov, J. Baštinec, A.S. Sirenko, Exponential stability of linear discrete systems with constant coefficients and single delay, *Appl. Math. Lett.*, 51:68–73, 2016, https://doi.org/10.1016/j.aml.2015.07.008.

8. J. Diblík, B. Morávková, Discrete matrix delayed exponential for two delays and its property, *Adv. Difference Equ.*, 2013:139, 2013, https://doi.org/10.1186/1687-1847-2013-139.

Nonlinear Anal. Model. Control, 27(1):70–90, 2022
9. J. Diblík, B. Morávková, Representation of the solutions of linear discrete systems with constant coefficients and two delays, *Abstr. Appl. Anal.*, **2014**:320476, 2014, https://doi.org/10.1155/2014/320476.

10. D. Ya. Khusainov, G. V. Shuklin, Linear autonomous time-delay system with permutation matrices solving, *Stud. Univ. Žilina, Math. Ser.*, **17**(1):101–108, 2003.

11. D. Ya. Khusainov, G. V. Shuklin, Relative controllability in systems with pure delay, *Int. Appl. Mech.*, **41**(2):210–221, 2005, https://doi.org/10.1007/s10778-005-0079-3.

12. M. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, 1964.

13. M. Li, J. Wang, Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations, *Appl. Math. Comput.*, **324**:254–265, 2018, https://doi.org/10.1016/j.amc.2017.11.063.

14. C. Liang, J. Wang, D. O’Regan, Controllability of nonlinear delay oscillating systems, *Electron. J. Qual. Theory Differ. Equ.*, **2017**(47):1–18, 2017, https://doi.org/10.14232/ejqtde.2017.1.47.

15. Z. Luo, W. Wei, J. Wang, Finite time stability of semilinear delay differential equations, *Nonlinear Dyn.*, **89**(1):713–722, 2017, https://doi.org/10.1007/s11071-017-3481-6.

16. M. Medved’, M. Pospíšil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, *Nonlinear Anal., Theory Methods Appl.*, **75**(7):3348–3363, 2012, https://doi.org/10.1016/j.na.2011.12.031.

17. M. Medved’, M. Pospíšil, L. Škripková, Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices, *Nonlinear Anal., Theory Methods Appl.*, **74**(12):3903–3911, 2011, https://doi.org/10.1016/j.na.2011.02.026.

18. M. Pospíšil, Representation and stability of solutions of systems of functional differential equations with multiple delays, *Electron. J. Qual. Theory Differ. Equ.*, **2012**(54):1–30, 2012, https://doi.org/10.14232/ejqtde.2012.1.54.

19. M. Pospíšil, Representation of solutions of delayed difference equations with linear parts given by pairwise permutable matrices via $Z$-transform, *Appl. Math. Comput.*, **294**:180–194, 2017, https://doi.org/10.1016/j.amc.2016.09.019.

20. J. Wang, Z. Luo, M. Fečkan, Relative controllability of semilinear delay differential systems with linear parts defined by permutable matrices, *Eur. J. Control*, **38**:39–46, 2017, https://doi.org/10.1016/j.ejcon.2017.08.002.

21. W. Wei, X. Xiang, Y. Peng, Nonlinear impulsive integro-differential equation of mixed type and optimal controls, *Optimization*, **55**(1–2):141–156, 2006, https://doi.org/10.1080/02331930500530401.

22. Z. You, J. Wang, On the exponential stability of nonlinear delay systems with impulses, *IMA J. Math. Control Inform.*, **35**(3):773–803, 2018, https://doi.org/10.1093/imamci/dnw077.

23. Z. You, J. Wang, Stability of impulsive delay differential equations, *J. Appl. Math. Comput.*, **56**(1–2):253–268, 2018, https://doi.org/10.1007/s12190-016-1072-1.

24. Z. You, J. Wang, D. O’Regan, Asymptotic stability of solutions of impulsive multi-delay differential equations, *Trans. Inst. Meas. Control*, **40**(15):4143–4152, 2018, https://doi.org/10.1177/0142331217742966.

https://www.journals.vu.lt/nonlinear-analysis