1. Introduction

In this paper we study an order-relation between measures on an $m$-hyperconvex domain $\Omega$ in $\mathbb{C}^n$. Let $\mu$ and $\nu$ be measures on $\Omega$. We say that $\mu$ is $m$-subharmonically greater than $\nu$ if $\int_{\Omega} (-\varphi) d\mu \geq \int_{\Omega} (-\varphi) d\nu$, $\forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ and write $\mu \succcurlyeq \nu$, where $\mathcal{E}_{0,m}(\Omega)$ is the Cegrell class of negative $m$-subharmonic functions defined in Sect. 2. It is easy to see that the condition $\mu \geq \nu$ implies $\mu \succcurlyeq \nu$. But the inverse is not true (see Example 1). We also show that if $u, v$ are functions in the Cegrell class $\mathcal{F}_m(\Omega)$ such that $u \leq v$, then their complex Hessian measures are in the relation $H_m(u) \succcurlyeq H_m(v)$ (see Proposition 2). But the inverse is not true (see Example 2).

In Sect. 4, we study maximality with respect to the $\succcurlyeq$-ordering, and a related notion of minimality for $m$-subharmonic functions in the class $\mathcal{F}_m(\Omega)$. A finite measure $\mu$ on $\Omega$ is said to be maximal if for any measure $\nu$ on $\Omega$ such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succcurlyeq \mu$ implies that $\nu = \mu$. The Dirac measure is a maximal measure. Theorem 9 shows that each finite measure on $\Omega$ with compact support is majorized in the $\succcurlyeq$-ordering by a maximal measure with the same total mass. A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any
function \( v \in \mathcal{F}_m(\Omega) \) with the same total Hessian mass, the relation \( v \leq u \) implies that \( v = u \). We show that if a function \( u \in \mathcal{F}_m(\Omega) \) and \( H_m(u) \) is maximal measure, then \( u \) is minimal function (see Proposition 5). But the converse is still unknown. Theorem 10 shows that if \( u \in \mathcal{F}_m(\Omega) \) is such that \( H_m(u) \) is carried by an \( m \)-polar set, then \( u \) is a minimal function. However, there are functions in \( \mathcal{F}_m(\Omega) \) whose Hessian measure are maximal and are not carried by an \( m \)-polar set. We also prove that each function in \( \mathcal{F}_m(\Omega) \) is minorized by a minimal function with the same total Hessian mass.

In Sect. 5, we apply the \( m \)-subharmonic ordering to the problem of convergence in the weak*-topology. First, we prove that if \( \{\mu_j\} \) is an \( m \)-subharmonically increasing sequence of measures on \( \Omega \) with uniformly bounded total mass then \( \mu_j \) converges to a measure \( \mu \) in the weak*-topology. And finally, we use the notion of maximal measure to prove a sufficient condition of convergence in the weak*-topology for the class \( \mathcal{F}_m(\Omega) \) (see Theorem 14).

2. Preliminaries

Let \( \Omega \) be an open set in \( \mathbb{C}^n \) and let \( m \) be a natural number \( 1 \leq m \leq n \).

As usual let 
\[
d = \partial + \bar{\partial}, \quad d^c = i(\bar{\partial} - \partial),
\]
and let \( \beta = dd^c|z|^2 \) be the canonical Kähler form in \( \mathbb{C}^n \). Denote by \( SH_m(\Omega) \) the set of all \( m \)-subharmonic functions in \( \Omega \), and \( SH_m^-\Omega \) for the set of all nonpositive \( m \)-subharmonic functions in \( \Omega \). For \( u_1, \ldots, u_m \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \), the operator
\[
H_m(u_1, \ldots, u_m) := dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}
\]
\[
= dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m})
\]
is a nonnegative Radon measure. In particular, when \( u = u_1 = \cdots = u_m \), the Hessian measures
\[
H_m(u) := (dd^c u)^m \wedge \beta^{n-m}
\]
are well-defined for \( u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \) (see [4]).

Definition 1. Let \( E \) be a subset of \( \Omega \). The \( m \)-relative extremal function \( h_{m,E,\Omega} \) is defined by
\[
h_{m,E,\Omega}(z) = \sup\{u(z) : u \in SH_m(\Omega), u \leq 0 \text{ and } u \leq -1 \text{ on } E\}.
\]

By [11, Proposition 1.5], we have that \( h_{m,E,\Omega}^* \) is \( m \)-subharmonic on \( \Omega \).

Definition 2. Let \( \Omega \) be an open set. A function \( u \in SH_m(\Omega) \) is called \( m \)-maximal if \( v \in SH_m(\Omega), v \leq u \) outside a compact set subset of \( \Omega \) implies that \( v \leq u \) in \( \Omega \).

Theorem 1 [4]. Assume that \( u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \). Then \( H_m(u) = 0 \) in \( \Omega \) if and only if \( u \) is \( m \)-maximal.

Now let us recall the definition of \( m \)-hyperconvex domain.
Definition 3. A bounded domain $\Omega \subset \mathbb{C}^n$ is called an $m$-hyperconvex if there exists an $m$-subharmonic function $\rho : \Omega \to (-\infty, 0)$ such that the closure of the set \( \{ z \in \Omega : \rho(z) < c \} \) is compact in $\Omega$ for every $c \in (-\infty, 0)$. In other words, the sublevel set \( \{ z \in \Omega : \rho(z) < c \} \) is relatively compact in $\Omega$. Such a function $\rho$ is called the exhaustion function.

Theorem 2 [9, Proposition 1.4.11]. Let $\Omega$ be an $m$-hyperconvex bounded domain and $K \Subset \Omega$ is compact. Then $h^*_{m,K,\Omega}$ is m-maximal in $\Omega \setminus K$.

Let us recall the definition of $m$-polar sets.

Definition 4. A set $E \subset \mathbb{C}^n$ is called $m$-polar if for any $z \in E$ there exists a neighbourhood $V$ of $z$ and $v \in SH_m(V)$ such that $E \cap V \subset \{ v = -\infty \}$.

The following theorem was proved by Lu.

Theorem 3 [9, Theorem 1.6.5]. If $E$ is $m$-polar, then there exists $u \in SH^-_m(\mathbb{C}^n)$ such that $E \subset \{ u = -\infty \}$.

Throughout this paper $\Omega$ will denote a bounded $m$-hyperconvex domain in $\mathbb{C}^n$. Now we recall the definitions of the Cegrell classes.

Definition 5. (1) We let $\mathcal{E}_{0,m}(\Omega)$ denote the class of bounded functions in $SH_m(\Omega)$ such that
\[
\lim_{z \to \partial \Omega} u(z) = 0 \text{ and } \int_{\Omega} H_m(u) < +\infty.
\]

(2) A function $u \in SH_m(\Omega)$ belongs to $\mathcal{E}_m(\Omega)$ if for each $z_0 \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of $z_0$ and a decreasing sequence $\{ u_j \} \subset \mathcal{E}_{0,m}(\Omega)$ such that $u_j \downarrow u$ in $U$ and $\sup_j \int_{\Omega} H_m(u_j) < +\infty$.

(3) Denote $\mathcal{F}_m(\Omega)$ be the class of functions $u \in SH_m(\Omega)$ such that there exists a sequence $\{ u_j \} \subset \mathcal{E}_{0,m}(\Omega)$ decreases to $u$ in $\Omega$ and $\sup_j \int_{\Omega} H_m(u_j) < +\infty$.

We have the following inclusions
\[
\mathcal{E}_{0,m} \subset \mathcal{F}_m \subset \mathcal{E}_m \text{ and } SH^-_m(\Omega) \cap L^\infty_{loc}(\Omega) \subset \mathcal{E}_m.
\]

Below we present some of the basic properties of the Cegrell classes.

Theorem 4 [2,9]. For each $u \in SH^-_m(\Omega)$, there exists a sequence $\{ u_j \} \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ such that $u_j \downarrow u$ in $\Omega$.

Proposition 1. Let $K$ be one of the classes $\mathcal{E}_{0,m}, \mathcal{F}_m, \mathcal{E}_m$. Then $K$ is a convex cone. Moreover, if $u \in K$ and $v \in SH^-_m(\Omega)$ then $\max\{ u, v \} \in K$.

The following lemma explains why the functions in $\mathcal{E}_{0,m}(\Omega)$ are sometimes called test functions.

Theorem 5 [2,9]. For $\varphi \in C^\infty_0(\Omega)$, there exist two functions $u, v$ in $\mathcal{E}_{0,m} \cap C(\overline{\Omega})$ such that $\varphi(z) = u(z) - v(z), \forall z \in \Omega$. 
Following Cegrell’s idea Lu proved that the Hessian operator is well-defined for the functions in the class $\mathcal{E}_m(\Omega)$.

**Theorem 6** [9, Theorem 1.7.14]. Let $u^k \in \mathcal{E}_m(\Omega), k = 1, \ldots, m$ and $\{u_j^k\}_{j}$ be sequences in $\mathcal{E}_{0,m}(\Omega)$ such that $u_j^k \downarrow u^k$, for each $1 \leq k \leq m$. Then the sequence of measures
\[
d^c u_1^1 \wedge \cdots \wedge d^c u_j^m \wedge \beta^{n-m}\]
converge to a Radon measure in weak*-topology independent to the choice of sequences $\{u_j^k\}$. We define $d^c u_1^1 \wedge \cdots \wedge d^c u^m \wedge \beta^{n-m}$ to be this limit.

Integration by parts formula is true for the function from the Cegrell class $F_m(\Omega)$.

**Theorem 7** [9, Theorem 1.7.18]. Assume that $u, v, w, w_1, \ldots, w_{m-1} \in F_m(\Omega)$. Then we have
\[
\int_{\Omega} u d^c v \wedge T = \int_{\Omega} v d^c u \wedge T,
\]
where $T = d^c w_1 \wedge \cdots \wedge d^c w_{m-1} \wedge \beta^{n-m}$ and the equality means that if one of the two terms is finite then they are equal.

The following theorem is sometimes called the Cegrell decomposition theorem.

**Theorem 8.** Let $\mu$ be a finite, positive measure on $\Omega$. Then there exist $\varphi \in \mathcal{E}_{0,m}(\Omega)$ and $0 \leq f \in L^1(H_m(\varphi))$ such that
\[
\mu = f H_m(\varphi) + \nu,
\]
where $\nu$ is carried by a $m$-polar set.

**Proof.** By the proof of [10, Theorem 4.14], we can find a function $u \in \mathcal{E}_{1,m}(\Omega)$ and $0 \leq f \in L^1(H_m(u))$ such that $\mu = f H_m(u) + \nu$, where $\nu$ is charged by an $m$-polar subset of $\Omega$. The rest of the proof goes verbatim as the proof of [10, Theorem 5.3].

\[\square\]

### 3. The $m$-Subharmonic Ordering

Let $\mu_j, \mu$ be measures on $\Omega$. By Theorem 5, we can see that following conditions are equivalent

1. $\lim_{j \to \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \ \forall \varphi \in C_0(\Omega)$;
2. $\lim_{j \to \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \ \forall \varphi \in C_0^\infty(\Omega)$;
3. $\lim_{j \to \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \ \forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C_0(\Omega)$.

If one of above assertion is satisfied, we say that $\mu_j$ tends to $\mu$ on $\Omega$ in the weak*-topology.
Remark 1. (1) If \( \mu_j \to \mu \) in the weak*-topology on \( \Omega \), then
\[
\mu(\Omega) \leq \liminf_{j \to \infty} \mu_j(\Omega).
\]
(2) Assume that \( \{\mu_j\}_j \) is a sequence of measures on \( \Omega \) and \( \sup_j \mu_j(\Omega) < \infty \), then there exists a subsequence \( \{\mu_{j_k}\}_k \subset \{\mu_j\}_j \) such that \( \mu_{j_k} \) converges to a measure \( \mu \) in the weak*-topology as \( k \to \infty \).

Definition 6. Let \( \mu \) and \( \nu \) be measures on \( \Omega \). We write \( \mu \succeq \nu \) if and only if
\[
\int_{\Omega} -\varphi d\mu \geq \int_{\Omega} -\varphi d\nu, \quad \forall \varphi \in E_{0,m}(\Omega) \cap C(\overline{\Omega}).
\]
And we say that \( \mu \) is \( m \)-subharmonically greater than \( \nu \).

Remark 2. (1) If \( \mu \succeq \nu \), then
\[
\int_{\Omega} -\varphi d\mu \geq \int_{\Omega} -\varphi d\nu, \quad \forall \varphi \in \text{SH}^-_{m}(\Omega), \quad \text{by Theorem 4.}
\]
In particular, \( \mu(\Omega) \geq \nu(\Omega) \).
(2) If \( \mu \geq \nu \), then \( \mu \succeq \nu \). But Example 1 shows that the opposite implication is not true.

Example 1. For \( a \in \Omega \), let \( \delta_a \) be the Dirac measure at \( a \). Let \( \sigma_r \) be the normalized measure on the sphere \( \partial B(a, r) \), where \( r \) enough small such that \( B(a, r) \subset \Omega \). Then for each \( \varphi \in \text{SH}^-_{m}(\Omega) \), by the subharmonicity we have
\[
\int_{\Omega} \varphi d\delta_a = \varphi(a) \leq \int_{\partial B(a, r)} \varphi d\sigma_r = \int_{\Omega} \varphi d\sigma_r.
\]
Thus \( \delta_a \succ \sigma_r \), but it is clear that \( \delta_a \) is not greater than \( \sigma_r \) even though \( \delta_a(\Omega) = \sigma_r(\Omega) = 1 \).

Proposition 2. If \( u, v \in \mathcal{F}_{m}(\Omega) \) and \( u \geq v \), then \( H_{m}(v) \succeq H_{m}(u) \).

Proof. For \( \varphi \in E_{0,m}(\Omega) \), by Theorem 7
\[
\int_{\Omega} -\varphi H_{m}(u) = \int_{\Omega} -u d\varphi \wedge (dd^{c}u)^{m-1} \wedge \beta^{n-m}
\leq \int_{\Omega} -v d\varphi \wedge (dd^{c}u)^{m-1} \wedge \beta^{n-m}
= \int_{\Omega} -\varphi d\varphi \wedge (dd^{c}u)^{m-1} \wedge \beta^{n-m}
\leq \cdots \leq \int_{\Omega} -\varphi (dd^{c}u)^{m} \wedge \beta^{n-m} = \int_{\Omega} -\varphi H_{m}(v).
\]
Thus \( H_{m}(v) \succeq H_{m}(u) \).

The following example shows that the converse implication to the statement given in Proposition 2 is not true.

Example 2. Let \( \Omega \) is the unit ball \( \mathbb{B} \) in \( \mathbb{C}^n \), \( n \geq 2 \) and define the functions
\[
v(z) = \frac{2}{3}(t^3 - 1), \quad w(z) = t^2 - 1.
\]
Then \( v, w \in E_{0,2}(\mathbb{B}) \cap C^2(\mathbb{B}) \) and \( w \leq v \) on
so $H_2(w) \succ H_2(v)$ by Proposition 2. For more details, we can compute (see [12])

$$H_2(w)(z) = 4^n n! dV, \quad H_2(v)(z) = 2^{2n-1} (2n+2)(n-1)! |z|^2 dV,$$

where $dV$ is the Lebesgue measure on $\mathbb{C}^n$. By [12] one can compute the solution $u$ to the equation

$$H_2(u) = \frac{H_2(v) + H_2(w)}{2}, \quad u \in \mathcal{E}_{0,2}(\mathbb{B}) \cap C^2(\mathbb{B}).$$

The solution is given by

$$u(z) = \frac{\sqrt{2}}{3} \left( |z|^2 + 1 \right)^{\frac{3}{2}} - \frac{4}{3}.$$  

We have $H_2(u) \succ H_2(v)$ by (1). Otherwise, $u(0) > -1 = v(0)$, so $v \not\prec u$.

**Remark 3.** The relation $\succ$ defines a partial order on the set of positive Borel measures on $\Omega$. But it is not a total order. To see that consider the Dirac measures $\delta_z$ and $\delta_w$, where $z, w \in \Omega$ and $z \neq w$. Choose $\varphi, \psi \in SH_m^{-}(\Omega)$ such that $\varphi(z) < \varphi(w)$ and $\psi(z) > \psi(w)$. Then $\int_{\Omega} -\varphi \delta_z > \int_{\Omega} -\varphi \delta_w$ and $\int_{\Omega} -\psi \delta_z < \int_{\Omega} -\psi \delta_w$, so $\delta_z$ and $\delta_w$ are not comparable with respect to $\succ$.

**Definition 7.** For a set $E \subset \Omega$, we define the convex hull of $E$ in $\Omega$ with respect to the family $SH_m(\Omega) \cap C(\overline{\Omega})$, denoted by $\hat{E}$ as followed

$$\hat{E} = \{ z \in \Omega : \varphi(z) \leq \sup_{E} \varphi, \quad \forall \varphi \in SH_m(\Omega) \cap C(\overline{\Omega}) \}.$$  

**Remark 4.** We have that $\hat{E}$ is closed in $\Omega$. Moreover, if $E$ is relatively compact in $\Omega$, so is $\hat{E}$.

**Proposition 3.** Let $\mu, \nu$ be finite regular measures on $\Omega$ such that $\mu(\Omega) = \nu(\Omega)$. If $\nu \succeq \mu$ then $\text{supp } \nu \subset \overline{\text{supp } \mu}$.

**Proof.** Put $K = \text{supp } \mu$. If $\hat{K} = \overline{\Omega}$ then Proposition 3 is clear. Therefore we assume that $\Omega \setminus \hat{K} \neq \emptyset$. Suppose that $\text{supp } \nu \not\subseteq \hat{K}$. Since $\hat{K}$ is closed in $\Omega$, it follows that $\nu(\Omega \setminus \hat{K}) > 0$. By the regularity of $\nu$, we can find a compact set $L \subset \Omega \setminus \hat{K}$ such that $\nu(L) > 0$. From the definition of $\hat{K}$, for each $z \in L$, there exist a neighborhood $U(z)$ of $z$ and a function $\varphi \in SH_m(\Omega) \cap C(\overline{\Omega})$ such that $\varphi(\xi) > \sup_{\hat{K}} \varphi$, $\forall \xi \in U(z)$. We choose $z_1, \ldots, z_k \in L$ such that $L \subset \bigcup_{i=1}^{k} U(z_i)$. Let $\varphi_1, \ldots, \varphi_k$ be the associated functions and $M_i = \sup_{\hat{K}} \varphi_i$, $M = M_1 + \cdots + M_k$. Define

$$\psi = \max\{ \varphi_1, M_1 \} + \cdots + \max\{ \varphi_k, M_k \}.$$  

Then we have $\psi \in SH_m(\Omega) \cap C(\overline{\Omega})$, $\psi \geq M$ on $\overline{\Omega}$, $\psi = M$ on $K$ and $\psi > M$ on $L$. Define $\psi_0 = \psi - \max_{\Omega} \psi$ and let $M_0 = M - \max_{\overline{\Omega}} \psi$. Then $\psi_0 \in SH_m^{-}(\Omega) \cap C(\overline{\Omega})$, $\psi_0 \geq M_0$ on $\Omega$, $\psi_0 = M_0$ on $K$ and $\psi_0 > M_0$ on $L$. Hence,
\[ \int_{\Omega} -\psi_0 d\nu = -M_0 \nu(\Omega) = -M_0 \mu(\Omega) = \int_{\Omega} -\psi_0 d\mu. \]

Proposition 3 is proved by a contradiction. \qed

4. Maximal Measures and Minimal Functions

We want to study the maximality with respect to the \( m \)-subharmonic ordering by using some kind of normalization.

**Definition 8.** A finite measure \( \mu \) on \( \Omega \) is said to be maximal if for any measure \( \nu \) on \( \Omega \) such that \( \nu(\Omega) = \mu(\Omega) \), the relation \( \nu \gtrsim \mu \) implies that \( \nu = \mu \).

**Example 3.** For \( 1 \leq m < n \), we define
\[ \varphi_j(z) = \max\left\{ -\frac{1}{j}|z|^{2-\frac{2m}{n}}, -1 \right\} \in SH^{-m}(B) \]
and \( \delta_0 \) is the Dirac measure defined on the unit ball \( B \) in \( \mathbb{C}^n \). Then for each measure \( \nu \), \( \nu(\Omega) = 1 \) and \( \nu \gtrsim \delta_0 \) we have
\[ \lim_{j \to \infty} \int_{\mathbb{B}} -\varphi_j d\nu = -\nu(\{0\}) \]
and
\[ -1 \leq \int_{\mathbb{B}} -\varphi_j d\delta_0 \leq \int_{\mathbb{B}} -\varphi_j d\nu \leq 1, \forall j. \]
Thus we get \( \nu(\{0\}) = 1 \), so \( \nu = \delta_0 \) which implies \( \delta_0 \) is maximal.

**Remark 5.**
1. If we can write a maximal measure as the sum \( \mu = \mu_1 + \mu_2 \) of two finite measures, then these are maximal too. To prove this, assume that \( \mu_1 \) is not maximal. Then there is a finite measure \( \nu \neq \mu_1 \) such that \( \nu(\Omega) = \mu_1(\Omega) \) and \( \nu \succ \mu \). We have \( (\nu + \mu_2)(\Omega) = \mu(\Omega) \) and \( \nu + \mu_2 \succ \mu \), but \( \nu + \mu_2 \neq \mu \), which is a contradiction.
2. If \( \mu \) is maximal measure, so is \( c\mu \), for \( c > 0 \).
3. We will show that the condition \( \mu_1, \mu_2 \) are maximal does not imply the maximality of \( \mu_1 + \mu_2 \) (see Example 5). This implies that the set of maximal measures on \( \Omega \) is not a convex cone.

**Definition 9.** We say that a set \( K \subseteq \Omega \) is an interpolation set for \( SH^{-m}(\Omega) \) if for each \( f \in C(K), f < 0 \) there exists a function \( \varphi \in SH^{-m}(\Omega) \) such that \( \varphi = f \) on \( K \).

**Proposition 4.** If \( \mu \) is a finite measure on \( \Omega \) such that \( \text{supp}\mu \) is contained in some interpolation set \( K \) for \( SH^{-m}(\Omega) \), then \( \mu \) is maximal.
Proof. Assume that $\nu$ is a measure on $\Omega$ such that $\nu(\Omega) = \mu(\Omega)$ and $\nu \succcurlyeq \mu$. By Proposition 3, we have $\text{supp}\nu \subset \hat{\text{supp}}\tilde{\mu} \subset K$. For a given $f \in C(K)$, $f \leq 0$, there exists a function $\varphi \in SH_m^{-}(\Omega)$ such that $\varphi = f$ on $K$. We get

$$\int_{\Omega} -fd\nu = \int_{\Omega} -\varphi d\nu \leq \int_{\Omega} -\varphi d\mu = \int_{\Omega} -fd\mu.$$  

This implies that $\int_{\Omega} fd\mu \geq \int_{\Omega} fd\nu$ holds for any $f \in C_0(\Omega), f \leq 0$. Hence $\mu \leq \nu$, so $\mu = \nu$. □

Example 4. Let $a_1, \ldots, a_k \in \Omega$. For $1 \leq j \leq k$, we choose $M_j$ such that

$$\psi_j(z) = \sum_{l \neq j} \ln |z - a_l| + M_j \in SH_m^{-}(\Omega).$$

For each value $c_j < 0$, we take $d_j > 0$ such that $d_j \psi_j(a_j) = c_j$. Define $\varphi = \max(d_1 \psi_1, \ldots, d_k \psi_k)$. Then we have $\varphi \in SH_m^{-}(\Omega)$ and $\varphi(a_j) = c_j$. Thus the finite set $\{a_1, \ldots, a_k\}$ is an interpolation set for $SH_m^{-}(\Omega)$. And Proposition 4 implies that the measure $\sum_{j=1}^{k} b_j \delta_{a_j}$ is maximal, where $\delta_{a_j}$ is the Dirac measure at the point $a_j$ and $b_1, \ldots, b_k$ are given nonnegative numbers.

We will show that each finite measure with compacted support is majorized by a maximal measure with the same total mass.

Lemma 1. Assume that $\mu$ and $\nu$ are measures on $\Omega$ such that $\nu \succcurlyeq \mu$. If $\int_{\Omega} \varphi d\mu = \int_{\Omega} \varphi d\nu > -\infty$ for some negative strictly $m$-subharmonic function $\varphi$. Then $\mu = \nu$.

Proof. For given $f \in C_0^\infty(\Omega)$, choose a constant $c > 0$ so that $(\pm f + c\varphi) \in SH_m^{-}(\Omega)$. Then we have

$$\int_{\Omega} (\pm f + c\varphi) d\mu = \int_{\Omega} \pm f d\mu + c \int_{\Omega} \varphi d\mu \geq \int_{\Omega} (\pm f + c\varphi) d\nu$$

$$= \int_{\Omega} \pm f d\nu + c \int_{\Omega} \varphi d\nu,$$

which implies that $\int_{\Omega} \pm f d\mu \geq \int_{\Omega} \pm f d\nu$. So $\mu = \nu$. □

Theorem 9. Let $\mu$ be a finite measure on $\Omega$ with compact support. Then there is a maximal measure $\mu_0$ such that $\mu_0 \succcurlyeq \mu$ and $\mu_0(\Omega) = \mu(\Omega)$.

Proof. Put $K = \hat{\text{supp}}\tilde{\mu}$ and

$$\mathcal{M}_\mu = \{\nu: \nu \succcurlyeq \mu, \nu(\Omega) = \mu(\Omega)\}.$$  

Because $\mu \in \mathcal{M}_\mu$, so $\mathcal{M}_\mu \neq \emptyset$. By Proposition 3, $\text{supp}\nu \subset K$ for each $\nu \in \mathcal{M}_\mu$. Let $\rho$ be the exhaustion function of $\Omega$ that is negative, continuous strictly $m$-subharmonic. We define

$$A = \sup_{\nu \in \mathcal{M}_\mu} \int_{\Omega} (-\rho) d\nu.$$
Since $\rho$ is bounded on $K$, it follows that $A$ is finite. Let $\{\nu_j\}_j$ be a sequence in $\mathcal{M}_\mu$ such that $\int_\Omega (-\rho) d\nu_j \to A$, as $j \to \infty$. By Remark 1, we may assume that $\nu_j$ tend to some measure $\mu_0$ in the weak*-topology and $\mu_0(\Omega) \leq \mu(\Omega)$. For each $\varphi \in \mathcal{E}_{0,m} \cap C(\Omega)$,
\[
\int_\Omega (-\varphi) d\mu_0 = \lim_{j \to \infty} \int_\Omega (-\varphi) d\nu_j \geq \int_\Omega (-\varphi) d\mu,
\]
which implies that $\mu_0 \succeq \mu$. By Remark 2 and the fact $\mu_0 \leq \mu(\Omega)$, we get $\mu_0(\Omega) = \mu(\Omega)$. Thus $\mu_0 \in \mathcal{M}_\mu$. Take a function $f \in C_0(\Omega)$, $f = 1$ on $K$. We get
\[
\int_\Omega (-\rho) d\mu_0 = \int_\Omega (-\rho) f d\mu_0 = \lim_{j \to \infty} \int_\Omega (-\rho) f d\nu_j = \lim_{j \to \infty} \int_\Omega (-\rho) d\nu_j = A.
\]
Suppose that $\nu$ be any measure on $\Omega$ such that $\nu \geq \mu_0$ and $\nu(\Omega) = \mu(\Omega)$. Then $\nu \in \mathcal{M}_\mu$ and $A \geq \int_\Omega (-\rho) d\nu \geq \int_\Omega (-\rho) d\mu_0 = A$. Hence $\int_\Omega (-\rho) d\nu = \int_\Omega (-\rho) d\mu_0 = A$. Lemma 1 implies that $\nu = \mu_0$, so Theorem 9 is finished. $\square$

**Definition 10.** A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any function $v \in \mathcal{F}_m(\Omega)$, the conditions $H_m(u)(\Omega) = H_m(v)(\Omega)$ and $v \leq u$ imply $v = u$.

**Proposition 5.** Let $u \in \mathcal{F}_m(\Omega)$ be such that $H_m(u)$ is a maximal measure. Then $u$ is minimal.

To prove this proposition we need the following lemma.

**Lemma 2.** If $u, v \in \mathcal{F}_m(\Omega)$, $H_m(u) = H_m(v)$ and $u \leq v$ then $u = v$.

**Proof.** We use a method from [7]. Using integration by parts, we have
\[
\int_\Omega -(u - v)(dd^c \rho)^{m} \land \beta^{n-m} = \int_\Omega d(u - v) \land d^c \rho \land (dd^c \rho)^{m-1} \land \beta^{n-m}
\]
\[
\leq \left[ \int_\Omega d(u - v) \land d^c (u - v) \land (dd^c \rho)^{m-1} \land \beta^{n-m} \right]^{\frac{1}{2}}
\]
\[
\times \left[ \int_\Omega d\rho \land d^c \rho \land (dd^c \rho)^{m-1} \land \beta^{n-m} \right]^{\frac{1}{2}},
\]
where $\rho \in \mathcal{E}_{0,m}(\Omega) \cap C^\infty(\Omega)$ is a strictly $m$-subharmonic exhaustion function of $\Omega$ (see [2]). Hence, to prove $u = v$ it is enough to show that
\[
\int_\Omega d(u - v) \land d^c (u - v) \land (dd^c \rho)^{m-1} \land \beta^{n-m} = 0. \tag{2}
\]
If $m = 1$ then (2) is clear. For $m \geq 2$ and $j + k = m - 1$, we have

$$0 \leq \int_{\Omega} - (u - v)(dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} - \rho dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge \beta^{n-m}$$

$$\leq \int_{\Omega} - (u - v) \sum_{a + b = m - 1} (dd^c u)^a \wedge (dd^c v)^b \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} - \rho dd^c(u - v) \wedge \sum_{a + b = m - 1} (dd^c u)^a \wedge (dd^c v)^b \wedge \beta^{n-m}$$

$$= \int_{\Omega} - \rho (H_m(u) - H_m(v)) = 0.$$ 

Thus, for every couple $j, k, j + k = m - 2$ we have

$$\int_{\Omega} - u dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} - \rho dd^c(u - v) \wedge (dd^c u)^j+1 \wedge (dd^c v)^k \wedge \beta^{n-m} = 0.$$ 

Similarly, $\int_{\Omega} - v dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} = 0$. So

$$\int_{\Omega} -(u - v) dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m}$$

$$= \int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} = 0, \quad (3)$$

for every couple $j, k, j + k = m - 2$. Assume that

$$\int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} = 0 \quad (4)$$

for $j + k = m - l - 1$. By (3), (4) is true for $l = 1$. For $j + k = m - l - 2$ we have

$$\int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^{l+1} \wedge \beta^{n-m}$$

$$= \int_{\Omega} - \rho (dd^c(u - v))^2 \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m}$$

$$= \int_{\Omega} d \rho \wedge d^c(u - v) \wedge dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m}$$

$$\leq \left| \int_{\Omega} d \rho \wedge d^c(u - v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right|$$

$$+ \left| \int_{\Omega} d \rho \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right|$$
\begin{align*}
\leq \left[ \int_{\Omega} d\rho \wedge d^c \rho \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^\frac{1}{2} \\
\times \left[ \int_{\Omega} d(u - v) \wedge d^c (u - v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^\frac{1}{2} \\
+ \left[ \int_{\Omega} d\rho \wedge d^c \rho \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^\frac{1}{2} \\
\times \left[ \int_{\Omega} d(u - v) \wedge d^c (u - v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^\frac{1}{2} \\
= 0,
\end{align*}

by assumption (4). So (2) is true by taking \( l = m - 1 \) in (4).

\textbf{Proof of Proposition 5.} Assume that \( v \in \mathcal{F}_m(\Omega), H_m(v)(\Omega) = H_m(u)(\Omega) \) and \( v \leq u \). Since \( v \leq u \), Proposition 2 implies that \( H_m(v) \geq H_m(u) \). From the assumption \( H_m(u) \) is maximal, we get \( H_m(u) = H_m(v) \). Now Proposition 5 follows from Lemma 1.

\textbf{Lemma 3.} Assume that \( u, v \in \mathcal{E}_m(\Omega) \) and \( u \geq v \). Then \( \chi_{\{u = -\infty\}} H_m(u) \leq \chi_{\{v = -\infty\}} H_m(v) \).

\textbf{Proof.} We use a method from [1]. For \( \epsilon > 0 \) small enough, set \( w_j = \max\{(1 - \epsilon)u - j, v\} \). Then we have \( w_j = (1 - \epsilon)u - j \) on the open set \( \{v < -\frac{j}{\epsilon}\} \). Therefore

\[ H_m(w_j) = (1 - \epsilon)^m H_m(u) \text{ on } \{v < -\frac{j}{\epsilon}\}. \]

Hence \( H_m(w_j) \geq (1 - \epsilon)^m \chi_{\{u = -\infty\}} H_m(u) \). Letting \( j \to \infty \), then we get \( H_m(v) \geq (1 - \epsilon)^m \chi_{\{u = -\infty\}} H_m(u) \). The proof is complete by letting \( \epsilon \to 0^+ \).

\textbf{Lemma 4.} For each \( u \in \mathcal{F}_m(\Omega) \), if \( H_m(u) \) is carried by an \( m \)-polar set, then \( H_m(u) = \chi_{\{u = -\infty\}} H_m(u) \).

\textbf{Proof.} We use the same idea as in [5]. We choose a sequence \( \{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega), u_j \downarrow u \). Then \( \frac{u_j}{1 - u_j} \downarrow \frac{u}{1 - u} \in \mathcal{F}_m(\Omega) \cap L^\infty(\Omega) \). For each \( v \in C^2(\Omega) \),

\[ \frac{\partial}{\partial z_l \partial \bar{z}_k} \left( \frac{v}{1 - v} \right) = \frac{v_l v_k}{(1 - v)^2} + \frac{2v_l v_k}{(1 - v)^3}, \forall \ 1 \leq l, k \leq n. \]

This implies that

\[ \frac{H_m(u_j)}{(1 - u_j)^{2m}} \leq H_m \left( \frac{u_j}{1 - u_j} \right). \]
The function $\frac{1}{(1-u)^{2m}}$ is convex on $[-\infty,0]$, hence by [11, Proposition 2.1], $\frac{1}{(1-u)^{2m}} - 1 \in \overline{SH}_m(\Omega)$. For every fixed $k$,

$$
\left(\frac{1}{(1-u_k)^{2m}} - 1\right) H_m(u) \geq \lim_{j \to \infty} \left(\frac{1}{(1-u_k)^{2m}} - 1\right) H_m(u_j)
$$

$$
\geq \lim_{j \to \infty} \left(\frac{1}{(1-u_j)^{2m}} - 1\right) H_m(u_j) \geq \lim_{j \to \infty} \left(\frac{1}{(1-u)^{2m}} - 1\right) H_m(u_j)
$$

$$
= \left(\frac{1}{(1-u)^{2m}} - 1\right) H_m(u).
$$

Letting $k \to \infty$, we get $\frac{H_m(u_j)}{(1-u_j)^{2m}}$ tends weakly to $\frac{H_m(u)}{(1-u)^{2m}}$. Moreover, $H_m\left(\frac{u_j}{1-u_j}\right)$ tends weakly to $H_m\left(\frac{u}{1-u}\right)$. Hence,

$$
\frac{H_m(u)}{(1-u)^{2m}} \leq H_m\left(\frac{u}{1-u}\right). \tag{5}
$$

Theorem 8 shows that there exist $\varphi \in \mathcal{E}_{0,m}(\Omega)$ and $f \in L^1(H_m(\varphi))$ such that

$$
H_m(u) = fH_m(\varphi) + \nu,
$$

where $\nu$ is carried by an $m$-polar set. Moreover, (5) implies that $\frac{H_m(u)}{(1-u)^{2m}}$ has no mass on $m$-polar sets. Hence, $\frac{\nu}{(1-u)^{2m}} = 0$, so $\nu$ is carried by the set $\{u = -\infty\}$.

**Theorem 10.** Let $u \in \mathcal{F}_m(\Omega)$ be such that $H_m(u)$ is carried by an $m$-polar set. Then $u$ is a minimal function.

**Proof.** Assume that $v \in \mathcal{F}_m(\Omega)$, $v \leq u$ and $H_m(v)(\Omega) = H_m(u)(\Omega)$. By Lemmas 3 and 4,

$$
\int_\Omega H_m(v) \geq \int_\Omega \chi_{\{v = -\infty\}} H_m(v) \geq \int_\Omega \chi_{\{u = -\infty\}} H_m(u) = \int_\Omega H_m(u).
$$

Hence, $H_m(v) = \chi_{\{v = -\infty\}} H_m(v)$. By Lemma 3 again, $H_m(u) \leq H_m(v)$. Combine this with $H_m(u)(\Omega) = H_m(v)(\Omega)$, we get $H_m(u) = H_m(v)$. Lemma 2 implies that $u = v$. \hfill \Box

**Proposition 6.** Assume that $\mu$ is a finite measure on $\Omega$ such that $\overline{\text{supp}\mu}$ is contained in a level set $\{z \in \Omega: \psi(z) = c\}$, where $c > -\infty$ and $\psi < 0$ is a strictly $m$-subharmonic function on $\Omega$. Then $\mu$ is maximal.

**Proof.** Suppose that $\nu \succ \mu$ and $\nu(\Omega) = \mu(\Omega)$. By Proposition 3, $\text{supp}\nu \subset \{z \in \Omega: \psi(z) = c\}$. Thus,

$$
\int_\Omega -\psi d\nu = \int_\Omega -cd\nu = \int_\Omega -cd\mu = \int_\Omega -\psi d\mu < \infty.
$$

Therefore, Lemma 1 implies that $\nu = \mu$, and the proof is complete. \hfill \Box

The following example confirms Remark 5(3).
Example 5 [3, Examples 4.15, 4.16]. We consider the unit disc $\mathbb{D}$ in $\mathbb{C}$. Define the sets $S_1 = \{ z = \frac{1}{2} e^{i\theta} : 0 \leq \theta \leq \pi \}$ and $S_2 = \{ z = \frac{1}{2} e^{i\theta} : \pi < \theta < 2\pi \}$. Let $\sigma$ be the area measure on the circle $\partial \mathbb{D}(0, \frac{1}{2})$ and define $\mu_j = \sigma|_{S_j}$, for $j = 1, 2$. We have $S_j \subset \{ \psi = |z|^2 - 1 = -\frac{3}{4} \}$. Let $h_j = h_{1,s_j,\mathbb{D}}$ be the 1-relative extremal function for $S_j$ over $\mathbb{D}$. Then $h_j \in \mathcal{E}_0,1(\mathbb{D}) \cap C(\mathbb{D})$ and $h_j = -1$ on $S_j$. Moreover, $h_j$ is harmonic on the connected set $\mathbb{D}\setminus S_j$, which implies that $h > -1$ on $\mathbb{D}\setminus S_j$. Hence $\hat{S}_j = S_j$ and Proposition 6 deduces that $\mu_1$ and $\mu_2$ are maximal measures. But $\sigma = \mu_1 + \mu_2$ is not maximal (see Example 1).

We will show that each function in $\mathcal{F}_m(\Omega)$ is minorized by a minimal function with the same total Hessian mass.

**Proposition 7.** Let $\{u_j\}$ be a decreasing sequence in $\mathcal{F}_m(\Omega)$ such that $u_j \downarrow u$ and $H_m(u_j)(\Omega) = H_m(u_{j+1})(\Omega)$ for all $j$. Then $u \in \mathcal{F}_m(\Omega)$ and $H_m(u)(\Omega) = H_m(u_j)(\Omega)$.

**Proof.** We have $u \in \mathcal{S}H_m(\Omega)$, and by Theorem 4, there exists a sequence $\{w_j\} \subset \mathcal{E}_{0,m}(\Omega) \cap C(\mathbb{D})$ such that $w_j \downarrow u$ as $j \to \infty$. Set $v_j = \max(w_j, u_j)$. Then $v_j \geq u_j$, $v_j \in \mathcal{E}_{0,m}(\Omega)$ and $v_j \downarrow u$ as $j \to \infty$. Theorem [10, Theorem 3.22] implies that

$$\sup_j \int \Omega H_m(v_j)(\Omega) \leq \sup_j H_m(u_j) = H_m(u_1) < \infty,$$

Thus, $u \in \mathcal{F}_m(\Omega)$. Since the sequence of measures $H_m(u_j)$ converges to the measure $H_m(u)$ in the weak* topology, we get

$$\lim_{j \to \infty} \inf H_m(v_j)(\Omega) \geq H_m(u)(\Omega).$$

Moreover, by [10, Theorem 3.22] again, we obtain $H_m(u)(\Omega) \geq H_m(u_j)$ since $u_j \in \mathcal{F}_m(\Omega)$, $u \leq u_j$. $\square$

**Theorem 11.** For each $u \in \mathcal{F}_m(\Omega)$, there exists a minimal function $u_0 \in \mathcal{F}_m(\Omega)$ such that $u_0 \leq u$ and $H_m(u_0)(\Omega) = H_m(u)(\Omega)$.

**Proof.** Define $S = \{ v \in \mathcal{F}_m(\Omega) : v \leq u, H_m(v)(\Omega) = H_m(u)(\Omega) \}$. Let $T$ be the totally ordered subset of $S$ and let $t(z) = \inf_{v \in T} v(z)$. We shall prove that $t \in S$. It is obvious that $t \leq u$. Let $\{K_i\}$ be a compact exhaustion sets of $\Omega$ and let $\{t_j\}$ be a sequence of continuous functions such that $t_j \geq t$ and $t_j \downarrow t$ as $j \to \infty$. For each $z \in K_i$, choose $v_z \in T$ such that $v_z(z) < t_j(z)$ and define the open set $U_z = \{ w \in \Omega : v_z(w) < t_j(w) \}$. Take $z_1, \ldots, z_N \in K_i$ such that $\bigcup_{k=1}^N U_{z_k} \supset K_i$. Since $T$ is totally ordered, we may choose $v^1_j$ to be the smallest of the functions $v_{z_1}, \ldots, v_{z_N}$, which implies that $v^1_j < t_j$ on $K_i$. Now let $u_1 = v^1_1$ and $u_j$ be the smallest of the functions $\{u_1, \ldots, u_{j-1}, v^1_j\}$ if $j \geq 2$, since $T$ is totally ordered. Then $\{u_j\}$ is a decreasing sequence of functions in $T$ such that $u_j \leq v^1_j < t_j$ on $K_j$. Therefore $u_j \in \mathcal{F}_m(\Omega)$, $H_m(u_j)(\Omega) = H_m(u)(\Omega)$ and $u_j \downarrow t$, as $j \to \infty$. Proposition 7 implies $t \in \mathcal{F}_m(\Omega)$ and $H_m(t)(\Omega) = H_m(u)(\Omega)$. 

Vol. 73 (2018) On $m$-Subharmonic Ordering of Measures Page 13 of 18
Hence \( t \in S \). Since \( T \) is arbitrary, Zorn’s lemma deduces that there is a minimal element \( u_0 \) of \( S \), so the proof is complete. \( \square \)

5. Convergence in the Weak*-Topology

We will use the \( m \)-subharmonic ordering to obtain some results on weak*-convergence of measures. If \( \Omega \) is a bounded domain in \( \mathbb{C}^n \) and \( \{u_j\} \) is a sequence of locally bounded \( m \)-subharmonic functions on \( \Omega \) which is decreasing to a function \( u \in \text{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \), then \( H_m(u_j) \) converges to \( H_m(u) \) in the weak*-topology (see [4]). The same conclusion holds if \( \text{SH}_m(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) is replaced by the class \( \mathcal{E}_m(\Omega) \), where \( \Omega \) is a bounded \( m \) hyperconvex domain (see [9]).

The following example shows that Hessian operator is discontinuous with respect to the convergence in \( L^1_{\text{loc}} \). This example follows the idea in [8].

**Example 6.** For \( n \geq 2 \), we define

\[
u_j(z_1, \ldots, z_n) = \left| \sum_{k=1}^{n} z_k^{2j} \right|^{\frac{1}{2j}}
\]

We can compute

\[
\frac{\partial^2 u}{\partial z_p \partial \overline{z_q}} = \frac{1}{4} \left| \sum_{k=1}^{n} z_k^{2j} \right|^{\frac{1}{2j} - 2} z_p^{2j-1} z_q^{2j-1}, \forall 1 \leq p, q \leq n.
\]

Thus, \( H_m(u_j) = 0 \), for all \( j \). We have \( 0 \leq u_j \leq n^\frac{1}{2j} u \), where \( u(z_1, \ldots, z_n) = \max\{|z_1|, \ldots, |z_n|\} \). Hence, we get \( u_j \to u \) in \( L^1_{\text{loc}}(\mathbb{C}^n) \) as \( j \to \infty \). We can show that \( H_m(u) \neq 0 \). Assume the contrary. Then \( H_m(u) = 0 \) on the polydisc \( \Delta_n(r) = \mathbb{D}(0,r) \times \cdots \times \mathbb{D}(0,r) \), i.e., \( u \) is \( m \)-maximal function on \( \Delta_n(r) \). Note that \( u \geq r_1 \) outside the compact subset \( \overline{\Delta_n(r_1)} \), where \( r_1 < r \) but we do not have \( u \geq r_1 \) on \( \Delta_n(r) \).

The following theorem give us a sufficient condition for weak*-convergence for the class \( \mathcal{F}_m(\Omega) \).

**Theorem 12.** If \( u_j \to u \) in \( L^1_{\text{loc}}(\Omega) \) and there is a strictly \( m \)-subharmonic function \( v \in \mathcal{E}_{0,m}(\Omega) \) such that

\[
\int_{\Omega} v H_m(u_j) \to \int_{\Omega} v H_m(u) \text{ as } j \to \infty,
\]

then \( H_m(u_j) \) tends to \( H_m(u) \) in the weak*-topology.

**Proof.** We use the idea from [6]. For \( w \in \mathcal{E}_{0,m}(\Omega) \), using integration by parts (Theorem 7) we have
\[ \int_{\Omega} wH_m(u_j) \leq \int_{\Omega} wH_m[(\sup_{s \geq j} u_s)^{\downarrow}] \leq \int_{\Omega} wH_m(u) \quad \text{as } j \to \infty. \]

Hence,

\[ \limsup_{j \to \infty} \int_{\Omega} wH_m(u_j) \leq \int_{\Omega} wH_m(u). \] \tag{6}

Theorem 4 implies that (6) is true for \( w \in SH_m^{-}(\Omega) \). Let \( \varphi \in C_0^{\infty}(\Omega) \) be given. By assumption \( v \) is strictly \( m \)-subharmonic we can choose \( A > 0 \) large enough such that \((\pm \varphi + Av) \in \mathcal{E}_{0,m}(\Omega)\). By (6) we have

\[ \limsup_{j \to \infty} \int_{\Omega} (\pm \varphi + Av)H_m(u_j) \leq \int_{\Omega} (\pm \varphi + Av)H_m(u). \]

Combining this with assumption \( \lim_{j \to \infty} \int_{\Omega} vH_m(u_j) = \int_{\Omega} vH_m(u) \) we obtain

\[ \limsup_{j \to \infty} \int_{\Omega} \pm \varphi H_m(u_j) \leq \int_{\Omega} \pm \varphi H_m(u), \]

which implies the desired result. \( \square \)

**Definition 11.** If \( \{\mu_j\} \) is a sequence of measures such that \( \mu_{j+1} \succcurlyeq \mu_j \) for all \( j \), then we say that \( \{\mu_j\} \) is \( m \)-subharmonically increasing.

**Theorem 13.** Let \( \{\mu_j\} \) be an \( m \)-subharmonically increasing sequence of measures on \( \Omega \) such that \( \sup_j \mu_j(\Omega) < \infty \). Then \( \mu_j \) converges to a measure \( \mu \) in the weak*-topology. Moreover, \( \int_{\Omega} (-\varphi)d\mu_j \uparrow \int_{\Omega} (-\varphi)d\mu \) for each \( \varphi \in SH_m^{-}(\Omega) \).

**Proof.** Let \( \varphi \in SH_m^{-}(\Omega) \cap L^\infty(\Omega) \). Then

\[ 0 \leq \int_{\Omega} (-\varphi)d\mu_1 \leq \int_{\Omega} (-\varphi)d\mu_2 \leq \cdots \leq \sup_{\Omega \setminus \varphi} \sup_j \mu_j(\Omega) < \infty \]

so \( \lim_{j \to \infty} \int_{\Omega} (-\varphi)d\mu_j < \infty \). Thus the limit exists for each \( \varphi \in C_0(\Omega) \). It follows that this defines a measure \( \mu \) on \( \Omega \) that \( \mu_j \) converges to \( \mu \) in the weak*-topology. Moreover, we know that \( \lim_{j \to \infty} \int_{\Omega} (-\varphi)d\mu_j = \int_{\Omega} (-\varphi)d\mu \) for each \( \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega) \). Now, let \( \varphi \in SH_m^{-}(\Omega) \). As above \( \{\int_{\Omega} (-\varphi)d\mu_j\} \) is an increasing sequence. We always have

\[ \lim_{j \to \infty} \int_{\Omega} (-\varphi)d\mu_j \geq \int_{\Omega} (-\varphi)d\mu. \] \tag{7}

To show the equality in (7), we assume the contrary, i.e.,

\[ \lim_{j \to \infty} \int_{\Omega} (-\varphi)d\mu_j > \int_{\Omega} (-\varphi)d\mu. \]

Choose \( j_0 \) enough large such that \( \int_{\Omega} (-\varphi)d\mu_{j_0} > \int_{\Omega} (-\varphi)d\mu \), and a sequence \( \{\varphi_k\} \in \mathcal{E}_{0,m} \cap C(\Omega) \) such that \( \varphi_k \downarrow \varphi \). Then we might choose \( k_0 \) such that \( \int_{\Omega} (-\varphi_{k_0})d\mu_{j_0} > \int_{\Omega} (-\varphi)d\mu \). It follows that
\[
\int_{\Omega} (-\varphi_{k_0})d\mu = \lim_{j \to \infty} \int_{\Omega} (-\varphi_{k_0})d\mu_j \geq \int_{\Omega} (-\varphi_{k_0})d\mu_{j_0} \\
> \int_{\Omega} (-\varphi)d\mu \geq \int_{\Omega} (-\varphi_{k_0})d\mu,
\]
which is a contradiction. \(\square\)

If \(\{u_j\} \subset F_m(\Omega)\) converges to \(u \in F_m(\Omega)\) in \(L^1_{\text{loc}}(\Omega)\), then we can relate the limit measure of sequence \(\{H_m(u_j)\}\) in Theorem 13 to \(H_m(u)\) as follows.

**Corollary 1.** Assume that \(\{u_j\} \subset F_m(\Omega)\) such that
\begin{enumerate}
  
  1. \(u_j\) converges to \(u \in F_m(\Omega)\) in \(L^1_{\text{loc}}(\Omega)\),
  2. \(H_m(u_j)\) is \(m\)-subharmonically increasing,
  3. \(\sup_j H_m(u_j) < \infty\).
\end{enumerate}

Then \(H_m(u_j)\) converges to a measure \(\mu\) in the weak*-topology such that \(\mu \geq H_m(u)\). Moreover, \(\int_{\Omega} (-\varphi)H_m(u_j) \uparrow \int_{\Omega} (-\varphi)d\mu\) for each \(\varphi \in SH_m^{-}(\Omega)\).

**Proof.** By Theorem 13 it remains to show that \(\mu \geq H_m(u)\). By the proof of Theorem 12, assumption (1) implies that \(\lim \inf_{j \to \infty} \int_{\Omega} (-\varphi)H_m(u_j) \geq \int_{\Omega} (-\varphi)H_m(u)\) for each \(\varphi \in SH_m^{-}(\Omega)\). \(\square\)

The following theorem gives us a bridge between convergence in weak*-topology and the concept of maximal measures defined in Sect. 4.

**Theorem 14.** Let \(\{u_j\} \subset F_m(\Omega)\) such that
\begin{enumerate}
  
  1. \(u_j\) converges to \(u \in F_m(\Omega)\) in \(L^1_{\text{loc}}(\Omega)\),
  2. \(H_m(u)\) is a maximal measure,
  3. \(\lim_{j \to \infty} H_m(u_j)(\Omega) = H_m(u)(\Omega)\).
\end{enumerate}

Then \(H_m(u_j)\) converges to \(H_m(u)\) in the weak*-topology.

**Proof.** Assumption (3) implies that there is a subsequence \(\{H_m(u_{j_k})\} \subset \{H_m(u_j)\}\) which converging to a measure \(\mu\) in the weak*-topology. Let \(\varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega)\) be given. As in the proof of Corollary 1, assumption (a) implies that \(\mu \geq H_m(u)\). Moreover, by (3) we have \(\mu(\Omega) \leq \lim \inf_{j \to \infty} H_m(u_{j_k})(\Omega) \leq H_m(u)(\Omega)\). Thus, \(\mu(\Omega) = H_m(u)(\Omega)\). By assumption (2) we can conclude that \(\mu = H_m(u)\). \(\square\)

**Open Question**

One might ask if there is a converse of Proposition 5. The answer is affirmative if \(n = m = 1\) (see [3, Proposition 4.11]). In higher dimension, the answer is unknown.
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