ON THE LINEARITY OF THE HOLOMORPH GROUP OF A FREE GROUP ON TWO GENERATORS

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ABSTRACT. Let \( F_n \) denote the free group generated by \( n \) letters. The purpose of this article is to show that \( \text{Hol}(F_2) \), the holomorph of the free group on two generators, is linear. Consequently, any split group extension \( G = F_2 \rtimes H \) for which \( H \) is linear has the property that \( G \) is linear. This result gives a large linear subgroup of \( \text{Aut}(F_3) \). A second application is that the mapping class group for genus one surfaces with two punctures is linear.

1. Introduction and Preliminaries

The purpose of this paper is to consider whether certain families of discrete groups given by natural semi-direct products are linear. The holomorph of a group \( G \), \( \text{Hol}(G) \), is the universal split extension of \( G \):

\[
1 \to G \to \text{Hol}(G) \xrightarrow{p} \text{Aut}(G) \to 1,
\]

where \( \text{Aut}(G) \) acts on \( G \) in the obvious way. Furthermore, the symbol \( H \rtimes G \) denotes the semi-direct product given by the split extension

\[
1 \to G \to G \rtimes H \xrightarrow{p} H \to 1
\]

with the precise action of \( H \) on \( G \) suppressed. The group \( \text{Hol}(G) \) is universal in the sense that any semi-direct product \( G \rtimes H \) is given by the pullback obtained from a homomorphism \( H \to \text{Aut}(G) \).

Recall that a group \( G \) is called linear if it admits a faithful, finite dimensional representation in \( \text{Gl}(m, k) \) for a field \( k \) of characteristic zero.

The main results here addresses a special case of the following general question stated in [4]:

**Question 1.** Let \( \Gamma \) and \( \pi \) be linear groups. Let

\[
1 \to \Gamma \to G \to \pi \to 1
\]

be a split extension. Give conditions which imply that \( G \) is linear.

It should be noted that the answer is not always positive with a basic example given by Formanek and Procesi’s “poison group” ([8]). Many geometrically interesting groups fit into the scheme given above. More specifically, examples are given by pure braid groups (and thus braid groups), McCool subgroups of \( \text{Aut}(F_n) \), certain fundamental groups of complements of hyperplane arrangements, certain mapping class groups, just to mention a few.

Our main interest in the above problem is when the normal subgroup \( \Gamma \) is a linear group and \( \pi \) is \( \text{Aut}(\Gamma) \). When \( G = F_n \), \( n > 2 \), the free group on \( n \) generators, then

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Hol($F_n$) is not linear because it contains Aut($F_n$) which is not linear ([8]). The main result in this paper is that Hol($F_2$) is linear.

**Theorem (Main Theorem).** The group Hol($F_2$) is linear.

The method of proof is to show that Hol($F_2$) contains a finite index subgroup $\pi$ which is linear. That is done by exhibiting explicit maps from $\pi$ to a product of two groups such that (i) each of the two groups is linear as subgroups of Aut($F_2$) and (ii) the product map is an embedding, thus showing that $\pi$ is linear. Since linearity is preserved under finite extensions, Hol($F_2$) is linear. In addition, the main theorem has the following consequence.

**Corollary 1.1.** Let $\pi$ be a linear group. Then the semidirect product $F_2 \rtimes \pi$ is linear. In particular, any group extension $G$ given by

$$1 \to F_2 \to G \to \mathbb{Z} \to 1$$

is linear.

A second corollary implies that certain mapping class groups are linear. Let $T$ denote the torus $S^1 \times S^1$. Let $\Gamma_1^k$ denote that mapping class group for genus one surfaces with $k$ punctured points. The linearity of the case of $\Gamma_1^1$ is given by the fact that this group is $SL(2,\mathbb{Z})$.

**Corollary 1.2.** The group $\Gamma_1^2$ is linear.

Moreover, in [6], it was shown that Hol($F_2$) is a subgroup of Aut($F_3$). Thus the main theorem implies that Hol($F_2$) is a large, natural, linear subgroup of Aut($F_3$). That is a partial answer to a more general question.

**Question 2.** Find large linear subgroups of Aut($F_n$), $n \geq 3$.

The methods here do not generalize for $n > 3$ as Aut($F_3$), and thus Aut($F_n$), $n > 3$, are not linear by Formanek and Procesi [8]. Nonetheless, they suggest a possible extension for $n = 4$.

**Conjecture 1.** The group $G$ defined by the natural extension

$$1 \to F_3 \to G \to \text{Hol}(F_2) \to 1$$

with the natural action of Hol($F_2$) on $F_3$, is linear.

A positive answer to Conjecture 1 has an interesting consequence. Let $M_n$ be the McCool subgroup of Aut($F_n$) i.e. the subgroup generated by basis-conjugation automorphisms [13]. Let $M_n^+$ be the upper-triangular McCool subgroup as defined in section 2 below or in [5]. An easy calculation shows that

$$M_3^+ \cong P_3 \cong F_2 \rtimes \mathbb{Z} < \text{Hol}(F_2).$$

Also, in [5], it was shown that there is a split exact sequence for all $n$:

$$1 \to F_{n-1} \to M_n^+ \to M_{n-1}^+ \to 1.$$ 

Combining all the above, we see that a positive answer to Conjecture 1 implies the linearity of $M_4^+$.

Notice that Hol($F_2$) fits into a split exact sequence:

$$1 \to F_2 \to \text{Hol}(F_2) \to \text{Aut}(F_2) \to 1.$$
The linearity question of $\text{Aut}(F_2)$ was reduced to the linearity of the braid group on four strands. In [7], it was shown that $\text{Aut}(F_2) \times \mathbb{Z}$ is commensurable with $B_4$. The linearity of $B_4$ and the other braid groups was settled ([1], [10], [11]), proving the linearity of $\text{Aut}(F_2)$.

The following conjecture was formulated in [4] which addresses the linearity question of split extensions with kernel a free group.

**Conjecture 2.** Let $G$ be a linear group and $1 \to F_n \to \Gamma \to G \to 1$ a split exact sequence with $F_n$ a finitely generated free group. If $G$ acts trivially on the homology of $F$, then $\Gamma$ is linear.

The homological condition is needed in that generality because of the counterexample in [8]. It should be noted that the situation in the Main Theorem is different because the action of $\text{Aut}(F_2)$ on $F_2$ is not trivial on the homology. One consequence of this conjecture is that $M_n$ is linear.

**Corollary 1.1** implies that $F_2 \rtimes \mathbb{Z}$ is linear. The above remarks suggest the following conjecture.

**Conjecture 3.** Let $F$ be a free group and $\Gamma = F \rtimes \mathbb{Z}$. Then $\Gamma$ is linear. That is if $1 \to F \to \Gamma \to \mathbb{Z} \to 1$ is exact, then $\Gamma$ is linear.

2. **Proof of the main theorem**

Let $F_n$ be the free group with basis $\{x_1, x_2, \ldots, x_n\}$. Let $\chi_{k,i}$ ($1 \leq i, k \leq n$) denote the elements of $\text{Aut}(F_n)$ defined by:

$$\chi_{k,i}(x_j) = \begin{cases} x_j, & \text{if } j \neq k \\ x_i^{-1}x_kx_i, & \text{if } j = k. \end{cases}$$

The McCool subgroup $M_n$ of $\text{Aut}(F_n)$ is the subgroup generated by $\chi_{k,i}$:

$$M_n = \langle \chi_{k,i} : k, i = 1, 2, \ldots, n, k \neq i \rangle.$$  

The upper triangular McCool subgroup is the subgroup generated by:

$$M_n^+ = \langle \chi_{k,i} : k, i = 1, 2, \ldots, n, k < i \rangle.$$  

For the main properties of $M_n$ and $M_n^+$ see [5]. There is a natural map $\text{Aut}(F_n) \to GL(n, \mathbb{Z})$ which is an epimorphism with kernel denoted $IA_n$. It is known that $IA_2$ is a free group with two generators given by $\chi_{2,1}$ and $\chi_{1,2}$. Thus there is a (non-split) group extension

$$1 \to F_2 \to \text{Aut}(F_2) \to GL(2, \mathbb{Z}) \to 1.$$  

The fact that the kernel is $F_2$ was shown in [2] and [14]. Here $F_2$ is identified with the subgroup of inner automorphisms of $\text{Aut}(F_2)$. We write $F_2 = \langle \tau_a, \tau_b \rangle$, the inner automorphisms of $F_2$.

A basis for $F_2$ will occur in three distinct ways below. Thus a free group with basis $\{\alpha, \beta\}$ will be named $F_2 = \langle \alpha, \beta \rangle$. (Thus to alert the reader, there are 3 distinct choices of bases for $F_2$ given below by $\{x_1, x_2\}$, $\{a, b\}$, and $\{\tau_a, \tau_b\}$.)

The natural map induced by the mod-2 reduction is an epimorphism that induces an exact sequence:

$$1 \to \Gamma(2, 2) \to SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/2\mathbb{Z}) \to 1.$$
The group $\Gamma(2, 2)$ is a free group of rank 2. Also, we consider the extension

$$1 \to \Gamma(2, 2) \to GL(2, \mathbb{Z}) \xrightarrow{(r, \text{det})} GL(2, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} \to 1.$$ 

Form the pull-back diagram:

$$
\begin{array}{ccc}
1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \{1\} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \Gamma(2, 2) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
$$

to obtain a morphism of extensions

$$
\begin{array}{ccc}
1 & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \{1\} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & GL(2, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & GL(2, \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & 1
\end{array}
$$

This middle exact sequence is the extension

$$1 \to IA_2 \to \text{Aut}(F_2) \to GL(2, \mathbb{Z}) \to 1.$$ 

In this case, $IA_2$ is isomorphic to the inner automorphism group of $F_2$ generated by two elements $\chi_{i,j}$ with $i \neq j$ and $1 \leq i, j \leq 2$.

**Lemma 2.1.** The group $\mathcal{F}$ is a subgroup of $\text{Aut}(F_2)$ of index 12. Furthermore, $\mathcal{F}$ is generated by the inner automorphisms of $F_2$ and the automorphisms $x_i$, $i = 1, 2$,

$$
\begin{align*}
&x_1(a) = ab^2, & x_1(b) = b \\
&x_2(a) = a, & x_2(b) = ba^2.
\end{align*}
$$

**Proof.** The result follows because the group $\Gamma(2, 2)$ is a free group on two generators $(9, 12)$

$$
A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\
A_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.
$$

Notice that the image of $x_i$ is $A_i$, for $i = 1, 2$. Also, $\mathcal{F}$ has index 12 in $\text{Aut}(F_2)$ because $GL(2, \mathbb{Z}/2\mathbb{Z})$ has order 6. 

The following describes the structure of $\mathcal{F}$.

**Lemma 2.2.** The group $\mathcal{F}$ can be written as a semi-direct product $\mathcal{F} = \langle \tau_a, \tau_b \rangle \rtimes \langle x_1, x_2 \rangle$ with the action of $x_i$ being exactly as the action of $\langle x_1, x_2 \rangle$ on $\langle a, b \rangle$.

**Proof.** Since the group $\Gamma(2, 2)$ is a free group, the extension is split and thus a semi-direct product. Furthermore, the extension is classified by the map $P\Gamma(2, 2) \to \text{Aut}(F_2)$ which sends $A_1$ to $x_1$ and $A_2$ to $x_2$. The proof of the Lemma follows by inspection. 

Remember that there is a split exact sequence

$$1 \to F_2 \to \text{Hol}(F_2) \xrightarrow{\pi} \text{Aut}(F_2) \to 1.$$ 

Also, $\mathcal{F} < \text{Aut}(F_2)$ and thus it is linear. Lemma 2.1 implies that the group $\pi = p^{-1}(\mathcal{F})$ is of index 6 in $\text{Hol}(F_2)$ and it fits into an exact sequence

$$1 \to F_2 = \langle a, b \rangle \to \pi \to \mathcal{F} \to 1.$$
The next Lemma is the main tool used in the proof of the Main Theorem.

**Lemma 2.3.** There are two maps

\[ f_1, f_2 : \pi \rightarrow \mathcal{F} \]

such that the product

\[ f_1 \times f_2 : \pi \rightarrow \mathcal{F} \times \mathcal{F} \]

is a monomorphism. Thus \( \pi \) and \( \text{Hol}(F_2) \) are linear.

Notice that Lemma 2.3 implies that \( \pi \) and thus \( \text{Hol}(F_2) \) are linear. Thus the Main Theorem follows and it suffices to prove Lemma 2.3, the subject of the next section.

3. **Proof of Lemma 2.3**

Recall from Lemma 2.1 that the group \( \mathcal{F} \) is a subgroup of \( \text{Aut}(F_2) \) of index 6 generated by the inner automorphisms of \( F_2 \) and the automorphisms \( x_i, i = 1, 2 \),

\[
\begin{align*}
x_1(a) &= ab^2, & x_1(b) &= b \\
x_2(a) &= a, & x_2(b) &= ba^2.
\end{align*}
\]

This action means that the elements \( x_i \) are acting by conjugation. So a restatement of this action is given by

\[
\begin{align*}
x_1 \cdot a \cdot x_1^{-1} &= x_1(a) = ab^2, & x_1 \cdot b \cdot x_1^{-1} &= x_1(b) = b \\
x_2 \cdot a \cdot x_2^{-1} &= x_2(a) = a, & x_2 \cdot b \cdot x_2^{-1} &= x_2(b) = ba^2.
\end{align*}
\]

Furthermore, by Lemma 2.2 there is an extension

\[ 1 \rightarrow \langle \tau_a, \tau_b \rangle \rightarrow \mathcal{F} \rightarrow \langle x_1, x_2 \rangle \rightarrow 1 \]

where the action of is specified by regarding \( a = \tau_a \) and \( b = \tau_b \):

\[
\begin{align*}
x_1 \tau_a x_1^{-1} &= \tau_a \tau_b^2, \\
x_1 \tau_b x_1^{-1} &= \tau_b, \\
x_2 \tau_a x_2^{-1} &= \tau_a, \\
x_2 \tau_b x_2^{-1} &= \tau_b \tau_a^2.
\end{align*}
\]

Furthermore, the group \( \pi \) is a split extension

\[ 1 \rightarrow F_2 = \langle a, b \rangle \rightarrow \pi \rightarrow \mathcal{F} \rightarrow 1 \]

with generators for \( \mathcal{F} \) specified above.

The additional data specifying the action of \( \mathcal{F} \) on \( F_2 = \langle a, b \rangle \) is given next.

\[
\begin{align*}
x_1 ax_1^{-1} &= ab^2 \\
x_2 ax_2^{-1} &= a \\
x_1 bx_1^{-1} &= b, \\
x_2 bx_2^{-1} &= ba^2 \\
\tau_a \tau_a^{-1} &= a, \\
\tau_b \tau_a^{-1} &= bab^{-1}, \\
\tau_a \tau_b^{-1} &= aba^{-1}, \text{ and} \\
\tau_b \tau_b^{-1} &= b
\end{align*}
\]

By a direct comparison, the above gives two distinct isomorphic copies of \( \mathcal{F} \) in \( \pi \).
These relations are summarized as follows:

\[
\begin{align*}
x_1 a x_1^{-1} &= ab^2, & x_2 a x_2^{-1} &= a \\
x_1 b x_1^{-1} &= b, & x_2 b x_2^{-1} &= ba^2 \\
x_1 \tau_a x_1^{-1} &= \tau_a \tau_b^2, & x_2 \tau_a x_2^{-1} &= \tau_a \\
x_1 \tau_b x_1^{-1} &= \tau_b, & x_2 \tau_b x_2^{-1} &= \tau_b \tau_a^2 \\
\tau_a a \tau_a^{-1} &= a, & \tau_b \tau_a^a &= bab^{-1} \\
\tau_a b \tau_a^{-1} &= aba^{-1}, & \tau_b \tau_a &= b
\end{align*}
\]

Rewrite the last two pairs of relations as follows:

\[
\begin{align*}
a^{-1} \tau_a a^{-1} a &= a, & b^{-1} \tau_a a^{-1} b &= a \\
a^{-1} \tau_a b \tau_a^{-1} a &= b, & b^{-1} \tau_b \tau_a^{-1} b &= b
\end{align*}
\]

The following hold:

\[
\begin{align*}
\tau_a a &= a \tau_a, & b^{-1} \tau_a a^{-1} b &= a \\
a^{-1} \tau_a b \tau_a^{-1} a &= b, & \tau_b b &= b \tau_b
\end{align*}
\]

Change of generators by setting \( t_a = a^{-1} \tau_a \) and \( t_b = b^{-1} \tau_b \). Notice that the previous relations are equivalent to

\[
\begin{align*}
[t_a, a] &= 1, & [t_b, a] &= 1 \\
[t_a, b] &= 1, & [t_b, b] &= 1
\end{align*}
\]

Thus the group \( \pi \) is generated by the set \( \{a, b, t_a, t_b, x_1, x_2\} \) with relations above equivalent to the following:

\[
\begin{align*}
x_1 a x_1^{-1} &= ab^2, & x_2 a x_2^{-1} &= a \\
x_1 b x_1^{-1} &= b, & x_2 b x_2^{-1} &= ba^2 \\
x_1 t_a x_1^{-1} &= t_a t_b, & x_2 t_a x_2^{-1} &= t_a \\
x_1 t_b x_1^{-1} &= t_b, & x_2 t_b x_2^{-1} &= t_b t_a \\
[t_a, a] &= 1, & [t_b, a] &= 1 \\
[t_a, b] &= 1, & [t_b, b] &= 1
\end{align*}
\]

Thus the group \( \pi \) has a normal subgroup \( N(\pi) \) generated by the set \( \{a, b, t_a, t_b\} \) with the following properties.

1. The subgroup \( N(\pi) \) is isomorphic to a direct product of two free groups \( \langle a, b \rangle \times \langle t_a, t_b \rangle \).
2. The cokernel \( \pi/ N(\pi) \) is isomorphic to a free group \( \langle x_1, x_2 \rangle \).
3. There is a homomorphisms \( h : \pi \rightarrow F \) specified by sending
   (a) \( t_a \) and \( t_b \) to \( 1 \)
   (b) \( x_i \) to \( x_i \)
   (c) \( a \) to \( a \) and \( b \) to \( b \).
4. The kernel of \( h \) is the free group \( \langle t_a, t_b \rangle \).

Notice that the intersection of kernels of \( \ker(h) \cap \ker(p) \) is the intersection of

\[
\langle t_a, t_b \rangle \cap \langle a, b \rangle = \{1\}.
\]

Furthermore, the maps \( f_1, f_2 \) of Lemma 2.3 are given by \( f_1 = h \) and \( f_2 = p \).

Therefore \( \pi = (\langle a, b \rangle \times \langle t_a, t_b \rangle) \cdot \langle x_1, x_2 \rangle \). Then \( \langle a, b \rangle \) and \( \langle t_a, t_b \rangle \) are normal subgroups of \( \pi \) and thus \( \pi \) admits two epimorphisms:

\[
\begin{align*}
f_1 : \pi &\rightarrow (\langle a, b \rangle \times \langle x_1, x_2 \rangle) \cong F \\
f_2 : \pi &\rightarrow (\langle t_a, t_b \rangle \times \langle x_1, x_2 \rangle) \cong F.
\end{align*}
\]
But $\mathcal{F}$ is linear, as a subgroup of $\text{Aut}(F_2)$ and $\ker(f_1) = \langle t_a, t_b \rangle$ and $\ker(f_2) = \langle a, b \rangle$. Since $\ker(f_1) \cap \ker(f_2) = \{1\}$, the composition

$$\pi \overset{\Delta}{\rightarrow} \pi \times \pi \overset{f_1 \times f_2}{\rightarrow} \mathcal{F} \times \mathcal{F}$$

is a monomorphism, where $\Delta$ is the diagonal map. Since $\mathcal{F} \times \mathcal{F}$ is linear, $G$ is linear. But $\pi$ has index 6 in $\text{Hol}(F_2)$. Thus $\text{Hol}(F_2)$ is linear, completing the proof of Lemma 2.3 and the Main Theorem.

4. Proof of Corollary 1.1

Let $\pi$ be a linear group. Let

$$1 \rightarrow F_2 \rightarrow G \overset{p}{\rightarrow} \pi \rightarrow 1$$

be a split extension. The result to be proven is that $G$ is linear. The split extension induces a commutative diagram of exact sequences:

$$\begin{array}{cccccc}
1 & \rightarrow & F_2 & \rightarrow & G & \overset{p}{\rightarrow} & \pi & \rightarrow & 1 \\
\| & & \| & & \downarrow i & & \downarrow j & & \\
1 & \rightarrow & F_2 & \rightarrow & \text{Hol}(F_2) & \rightarrow & \text{Aut}(F_2) & \rightarrow & 1
\end{array}$$

where $j$ is the map induced by the action of $\pi$ on $F_2$. Notice that the right-hand diagram is a pull-back diagram. Thus the map

$$i \times p : G \rightarrow \text{Hol}(F_2) \times \pi$$

is an injection. Since $\pi$ and $\text{Hol}(F_2)$ are linear, $G$ is linear.

5. Proof of Corollary 1.2

Let $\Pi$ denote the group of orientation preserving homeomorphisms $\text{Top}^+(T)$, and

$$\text{Conf}(T, k) = \{(z_1, \ldots, z_k) \in T^k | z_i \neq z_j \text{ if } i \neq j\}$$

the configuration space of $k$ points in $T$. Write $\text{Top}^+(T, Q_k)$ for the topological group of the orientation preserving self-homeomorphisms of $T$ that leave $Q_k$, a set of $k$ distinct points in $T$, invariant. Similarly, we write $\text{PTop}^+(T, Q_k)$ for the orientation-preserving homeomorphisms of $T$ that fix $Q_k$ pointwise. Denote

$$\Gamma^k_1 = \pi_0(\text{Top}^+(T, Q_k)) \text{ and } \text{PT}^k_1 = \pi_0(\text{PTop}^+(T, Q_k)),$$

for the corresponding mapping class groups.

Recall the following facts [3].

(1) If $k \geq 2$, then the spaces

$$\mathbb{E}\Pi \times_\Pi \text{Conf}(T, k),$$

and

$$\mathbb{E}\Pi \times_\Pi \text{Conf}(T, k)/\Sigma_k$$

are respectively $K(\text{PT}^k_1, 1)$, and $K(\Gamma^k_1, 1)$. 

(2) Furthermore, $E\Pi \times \Pi \operatorname{Conf}(T, k)$ is homotopy equivalent to
\[ ESL(2, \mathbb{Z}) \times_{SL(2, \mathbb{Z})} \operatorname{Conf}(T - Q_1, k - 1) \]
where $Q_1 = \{(1, 1)\} \subset T$. Thus there is a fibration
\[ ESL(2, \mathbb{Z}) \times_{SL(2, \mathbb{Z})} \operatorname{Conf}(T, 2) \to BSL(2, \mathbb{Z}) \]
with fibre $T - Q_1$.

Using the above one can easily see that the group $P\Gamma_1^2$ is isomorphic to $SL(2, \mathbb{Z})$.

Also, the kernel of the natural mod-2 reduction map
\[ SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/2\mathbb{Z}) \]
denoted $S\Gamma(2, 2)$ here is a free group on two letters. So, since the fundamental group of $T^2 - Q_1$ is free on two letters, the fundamental group of
\[ \pi_1(ESL(2, \mathbb{Z}) \times_{SL(2, \mathbb{Z})} \operatorname{Conf}(T, 2)) = P\Gamma_1^2 \]
has an index six subgroup $K$ which admits an extension
\[ 1 \to F_2 \to K \to \Gamma(2, 2) \to 1 \]
and is split. Therefore, by the main Theorem 1 the group $K$ is linear and thus $P\Gamma_1^2$ is linear. Notice that $P\Gamma_1^2$ has index two in $\Gamma_1^2$ and therefore $\Gamma_1^2$ is linear.

6. ON LARGE LINEAR SUBGROUPS OF $\operatorname{Aut}(F_n)$

In this final section we present a small step towards understanding Question 2. For any group $G$, there is a group homomorphism defined
\[ E : \operatorname{Hol}(G) \to \operatorname{Aut}(G \ast F) \]
and shown to be a monomorphism where $F$ is a free group [6]. Explicitly, the homomorphism is defined as follows:

- For $f \in \operatorname{Aut}(G)$,
  \[ E(f)(z) = \begin{cases} 
  f(z), & \text{if } z \in G \\
  z, & \text{if } z \in F.
  \end{cases} \]

- For $h \in G$,
  \[ E(h)(z) = \begin{cases} 
  z, & \text{if } z \in G \\
  hzh^{-1}, & \text{if } z \in F.
  \end{cases} \]

It is known that $\operatorname{Aut}(F_3)$ is not linear [8]. But by the above, it obvious that $\operatorname{Hol}(F_2)$ is a subgroup of $\operatorname{Aut}(F_3)$. Thus $\operatorname{Aut}(F_3)$ is not linear but contains a large, ‘natural’, linear subgroup.

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