Exact solutions of time-dependent three-generator systems

Jian-Qi Shen, Hong-Yi Zhu, and Pan Chen

State Key Laboratory of Modern Optical Instrumentation, Center for Optical and Electromagnetic Research, College of Information Science and Engineering, Zhejiang Institute of Modern Physics and Department of Physics, Zhejiang University, Hangzhou 310027, People’s Republic of China

(November 5, 2018)

Abstract

There exist a number of typical and interesting systems or models which possess three-generator Lie-algebraic structure in atomic physics, quantum optics, nuclear physics and laser physics. The well-known fact that all simple 3-generator algebras are either isomorphic to the algebra $sl(2, C)$ or to one of its real forms enables us to treat these time-dependent quantum systems in a unified way. By making use of the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation, the present paper obtains exact solutions of the time-dependent Schrödinger equations governing various three-generator quantum systems. For some quantum systems whose time-dependent Hamiltonians have no quasialgebraic structures, we show that the exact solutions can also be obtained by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the conserved generator (i.e., the time-independent invariant that commutes with the time-dependent Hamiltonian). The topological property of geometric phase factors in time-dependent systems is briefly discussed.
I. INTRODUCTION

Exact solutions and geometric phase factor [1,2] of time-dependent spin model have been extensively investigated by many authors. Bouchiat and Gibbons discussed the geometric phase for the spin-1 system [3]. Datta et al found the exact solution for the spin-1/2 system [4] by means of the classical Lewis-Riesenfeld theory, and Mizrahi calculated A-A phase for the spin-1/2 system [5] in a time-dependent magnetic field. The more systematic approach to obtaining the formally exact solutions for the spin- j system was proposed by Gao et al [6] who made use of the Lewis-Riesenfeld quantum theory [7]. In this spin- j system, the three generators of the Hamiltonian satisfy the commutation relations of SU(2) Lie algebra. In addition to the spin model, there exist many quantum systems whose Hamiltonian is also constructed in terms of three generators of various Lie algebras which we will illustrate in the following.

The invariant theory which can be better applied to solutions of the time-dependent Schrödinger equation was first proposed by Lewis and Riesenfeld in 1969 [7]. This theory is very appropriate for treating the geometric phase factor. In 1991, Gao et al generalized this theory and put forward the invariant-related unitary transformation method [8]. Exact solutions for time-dependent systems obtained by using the generalized invariant theory contain the geometric and dynamical phase [9–11]. This formulation was developed from the Lewis-Riesenfeld’s formal theory into a powerful tool for treating exact solutions of the time-dependent Schrödinger equation and geometric phase factor. In the present paper, we obtain exact solutions of various time-dependent three-generator systems based on these invariant theories.

This paper is organized as follows: In section 2, we set out several quantum systems or models which have three-generator algebraic structure. In section 3, use is made of the invariant theories and exact solutions of various time-dependent three-generator systems are obtained. In section 4, there are some discussions concerning the closure property of the Lie algebraic generators in the sub-Hilbert-space. In section 5, we conclude with some remarks.
II. THE ALGEBRAIC STRUCTURES OF VARIOUS THREE-GENERATOR SYSTEMS

In our previous work [12] we showed that the time-dependent Schrödinger equation is solvable if its Hamiltonian is constructed in terms of the generators of Lie algebra. Then analyzing the algebraic structures of Hamiltonians plays significant role in obtaining exact solutions of the time-dependent systems. To our knowledge, a large number of quantum systems which have three-generator Hamiltonians have been investigated in the literature.

In the following we set out these systems and discuss the algebraic structures of their Hamiltonians.

(1) Spin model. The time evolution of the wavefunction of a spinning particle in a magnetic field was studied by regarding it as a spin model [5] whose Hamiltonian can be written

\[
H(t) = c_0 \left\{ \frac{1}{2} \sin \theta \exp[-i\varphi]J_+ + \frac{1}{2} \sin \theta \exp[i\varphi]J_- + \cos \theta J_3 \right\}
\] (2.1)

with \(J_\pm = J_1 \pm iJ_2\) satisfying the commutation relations \([J_3, J_\pm] = \pm J_\pm, [J_+, J_-] = 2J_3\). Analogous to this case, in the gravitational theory of general relativity both spin-gravitomagnetic interaction [13] and spin-rotation coupling [14,15,12] are proved spin models. It can be verified that the investigation of the propagation of a photon inside the noncoplanarly curved optical fiber [16–18] is also equivalent to that of a spin model. The Hamiltonian of spin model is composed of three generators which constitute \(SU(2)\) algebra.

(2) Two-coupled harmonic oscillator. The Hamiltonian of the two-coupled harmonic oscillator is of the form (in the units \(\hbar = 1\))

\[
H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + ga_1^\dagger a_2 + g^* a_2^\dagger a_1,
\] (2.2)

where \(a_1^\dagger, a_2^\dagger, a_1, a_2\) are the creation and annihilation operators for these two harmonic oscillators, respectively; \(g\) and \(g^*\) are the coupling coefficients. Set \(J_+ = a_1^\dagger a_2, J_- = a_2^\dagger a_1, J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2), N = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2)\), then we can show that the generators of this Hamiltonian satisfy the commutation relations of \(SU(2)\) algebra. Since \(N\) commutes with \(H\),
i.e., $[N,H] = 0$, we consequently say $N$ is an invariant (namely, it is a conserved generator whose eigenvalue is time-independent). In terms of $J_\pm, J_3$ and $N$, the Hamiltonian in the expression \((2.2)\) can be rewritten as follows

$$H = \omega_1(N + J_3) + \omega_2(N - J_3) + gJ_+ + g^*J_-.$$  \hfill (2.3)

Another interesting Hamiltonian of the two-coupled harmonic oscillator is written in the form

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + ga_1 a_2 + g^*a_1^\dagger a_2^\dagger.$$  \hfill (2.4)

If we take $K_+ = a_1^\dagger a_2^\dagger, K_- = a_1 a_2, K_3 = \frac{1}{2}(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger), N = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$, the generators of the $SU(1,1)$ group are thus realized. The commutation relations are immediately inferred as

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_3.$$  \hfill (2.5)

(3) $SU(1,1) \oplus h(4)$ Lie-algebra system. A good number of quantum systems whose Hamiltonian is some combinations of the generators of a Lie algebra, e.g., $SU(1,1) \oplus h(4)$ ($\oplus$ denotes a semidirect sum) \([19, 20]\) which is used to discuss both the non-Poissonian effects in a laser-plasma scattering and the pulse propagation in a free-electron laser \([20]\). The $SU(1,1) \oplus h(4)$ Hamiltonian is

$$H = AK_3 + FK_+ + F^*K_- + Ba^\dagger + B^*a + G,$$  \hfill (2.6)

where $a$ and $a^\dagger$ are harmonic-oscillator annihilation and creation operators, respectively.

(4) General harmonic oscillator. The Hamiltonian of the general harmonic oscillator is given by \([8]\)

$$H = \frac{1}{2}[Xq^2 + Y(qp + pq) + Zp^2] + Fq,$$  \hfill (2.7)

where the canonical coordinate $q$ and the canonical momentum $p$ satisfy the commutation relation $[q, p] = i$. The following three-generator Lie algebra is easily derived
\[ [q^2, p^2] = 2\{i(qp + pq)\}, \quad [i(qp + pq), q^2] = 4q^2, \quad [i(qp + pq), p^2] = -4p^2. \quad (2.8) \]

(5) Charged particle moving in a magnetic field. The motion of a particle with mass \( \mu \) and charge \( e \) in a homogeneous magnetic field \( \vec{B} = (0, 0, B) \) is described by the following Hamiltonian in the spherical coordinates

\[
H = -\frac{1}{2\mu} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} L^2 \right) + \frac{1}{8} \mu \omega^2 r^2 - \frac{\omega}{2} L_z \quad (2.9)
\]

with \( \omega = \frac{B}{\mu} \). Since both \( L_z \) and \( L^2 \) commute with \( H \) and thus they are called invariants, only the operators associated with \( r \) should be taken into consideration. We can show that if the following operators are defined

\[
K_1 = \mu r^2, \quad K_2 = -\frac{1}{\mu} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} L^2 \right), \quad K_3 = -2i(\frac{3}{2} + r \frac{\partial}{\partial r}), \quad (2.10)
\]

\( K_1, K_2 \) and \( K_3 \) form an algebra

\[
[K_1, K_2] = 2iK_3, \quad [K_3, K_2] = 4iK_2, \quad [K_3, K_1] = -4iK_1. \quad (2.11)
\]

Apparently, \( H \) can be rewritten in terms of the generators of this Lie algebra.

(6) Two-level atomic coupling. The model under consideration is consisted of two-level atom driven by the photons field \(^{21}\). The interaction part of the Hamiltonian contains the transition operator \( |1 \rangle \langle 2 | \) and \( |2 \rangle \langle 1 | \), where \( |1 \rangle \) and \( |2 \rangle \) are the atomic operators of the two-level atom. Simple calculation yields

\[
[[|1 \rangle \langle 1 | - |2 \rangle \langle 2 |, |1 \rangle \langle 2 |] = 2 |1 \rangle \langle 2 |, \quad [[|1 \rangle \langle 1 | - |2 \rangle \langle 2 |, |2 \rangle \langle 1 |] = -2 |2 \rangle \langle 1 |,
\]

\[
[[|1 \rangle \langle 2 |, |2 \rangle \langle 1 |] = |1 \rangle \langle 1 | - |2 \rangle \langle 2 |, \quad (2.12)
\]

which unfolds that the Hamiltonian contains a \( SU(2) \) algebraic structure.

(7) Supersymmetric Jaynes-Cummings model. In addition to the ordinary Jaynes-Cummings models \(^{22}\), there exists a two-level multiphoton Jaynes-Cummings model which possesses supersymmetric structure. In this generalization of the Jaynes-Cummings model, the atomic transitions are mediated by \( k \) photons \(^{23\ 24}\). Singh has shown that this model
can be used to study multiple atom scattering of radiation and multiphoton emission, absorption, and laser processes \[26\]. The Hamiltonian of this model under the rotating wave approximation is given by

\[
H(t) = \omega(t)a^\dagger a + \frac{\omega_0(t)}{2}\sigma_z + g(t)(a^\dagger)^k\sigma_- + g^*(t)a^k\sigma_+, \tag{2.13}
\]

where \(a^\dagger\) and \(a\) are the creation and annihilation operators for the electromagnetic field, and obey the commutation relation \([a, a^\dagger] = 1\); \(\sigma_{\pm}\) and \(\sigma_z\) denote the two-level atom operators which satisfy the commutation relation \([\sigma_z, \sigma_{\pm}] = \pm 2\sigma_{\pm}\). We can verify that this model is solvable and the complete set of exact solutions can be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue of the supersymmetric generator \(N'\)

\[
N' = \begin{pmatrix}
 a^k(a^\dagger)^k & 0 \\
 0 & (a^\dagger)^k a^k
\end{pmatrix}. \tag{2.14}
\]

It can be verified that \(N'\) commutes with the Hamiltonian in (2.13), \(N'\) is therefore called the time-independent invariant. Vogel and Welsch have studied the \(k\)-photon Jaynes-Cummings model with coherent atomic preparation which is time-independent \[27\]. In the framework of the formulation presented in this paper, we can study the totally time-dependent cases of work done by Vogel and Welsch.

(8) Two-level atom interacting with a generalized cavity. Consider the following Hamiltonian \[28\]

\[
H = r(A_0) + s(A_0)\sigma_z + gA_-\sigma_+ + g^*A_+\sigma_-
\]

where \(r(A_0)\) and \(s(A_0)\) are well-defined real functions of \(A_0\), and \(A_0, A_{\pm}\) satisfy the commutation relations \([A_0, A_{\pm}] = \pm mA_{\pm}\) \[29\]. One can show that this Hamiltonian possesses a three-generator algebraic structure.

(9) The interaction between a hydrogenlike atom and an external magnetic field. This model is described by

\[
\begin{align*}
H &= \alpha \vec{L} \cdot \vec{S} + \beta(L_z + 2S_z) \\
&= \beta L_z + \left(\frac{1}{2}\alpha L_z + \beta\right)\sigma_z + \frac{1}{2}\alpha(L_-\sigma_+ + L_+\sigma_-)
\end{align*}
\]

\(2.16\)
with \( L_\pm = L_x \pm iL_y \). It is evidently seen that this form of Hamiltonian is analogous to that in (2.15).

(10) Coupled two-photon lasers. The Hamiltonian of this model is in fact the combination of that of the two-coupled harmonic oscillator and general harmonic oscillator. One can show that there exists a \( SU(2) \) algebraic structure in this model [30].

From what has been discussed above we can draw a conclusion that a number of typical and useful systems or models in laser physics, atomic physics and quantum optics can be attributed to various three-generator type. The analysis of these algebraic structures shows the solvability of these systems. It should be noted that, in the literature, most of above systems or models are only investigated in the stationary cases where the coefficients of the Hamiltonians are totally time-independent. Some systems are investigated for the Hamiltonian with partly time-dependent coefficients. In the present paper, we will give exact solutions of the time-dependent Schrödinger equation of all these systems or models where the coefficients of the Hamiltonians are totally time-dependent.

III. EXACT SOLUTIONS OF TIME-DEPENDENT SCHRÖDINGER EQUATION

Time evolution of most above systems or models is governed by the Schrödinger equation

\[
i \frac{\partial \Psi(t)}{\partial t} = H(t) \Psi(t),
\]

where the Hamiltonian is constructed by three generators \( A, B \) and \( C \) and is often given as follows

\[
H(t) = \omega(t) \left\{ \frac{1}{2} \sin \theta(t) \exp[-i\phi(t)]A + \frac{1}{2} \sin \theta(t) \exp[i\phi(t)]B + \cos \theta(t)C \right\}
\]

(3.2)

with \( A, B \) and \( C \) satisfying the general commutation relations of a Lie algebra

\[
[A, B] = nC, \quad [C, A] = mA, \quad [C, B] = -mB,
\]

(3.3)

where \( m \) and \( n \) are the structure constants of this Lie algebra. Since all simple 3-generator algebras are either isomorphic to the algebra \( sl(2, C) \) or to one of its real forms, we treat
these time-dependent quantum systems in a unified way. According to the Lewis-Riesenfeld invariant theory, an operator $I(t)$ that agrees with the following invariant equation [7]

$$\frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), H(t)] = 0 \quad (3.4)$$

is called an invariant whose eigenvalue is time-independent, i.e.,

$$I(t)|\lambda, t\rangle_I = \lambda|\lambda, t\rangle_I, \quad \frac{\partial \lambda}{\partial t} = 0. \quad (3.5)$$

It is seen from Eq. (3.4) that $I(t)$ is the linear combination of $A, B$ and $C$ and may be generally written

$$I(t) = y\left\{\frac{1}{2}\sin a(t) \exp[-ib(t)]A + \frac{1}{2}\sin a(t) \exp[ib(t)]B\right\} + \cos a(t)C, \quad (3.6)$$

where the constant $y$ will be determined below. It should be pointed out that it is not the only way to construct the invariants. Since the product of two invariants also satisfies Eq. (3.4) [8], there are infinite invariants of a time-dependent quantum system. But the form in Eq. (3.6) is the most convenient and useful one. Substitution of (3.6) into Eq.(3.4) yields

$$y \exp(-ib)(\dot{a}\cos a - \dot{b}\sin a) - im\omega[\exp(-i\phi) \cos a \sin \theta - y \exp(-ib) \sin a \cos \theta] = 0,$$

$$\dot{a} + ny \omega \sin \theta \sin(b - \phi) = 0. \quad (3.7)$$

where dot denotes the time derivative. The time-dependent parameters $a$ and $b$ are determined by these two auxiliary equations.

It is easy to verify that the particular solution $|\Psi(t)\rangle_s$ of the Schrödinger equation can be expressed in terms of the eigenstate $|\lambda, t\rangle_I$ of the invariant $I(t)$, namely,

$$|\Psi(t)\rangle_s = \exp\left[\frac{1}{i}\varphi(t)\right]|\lambda, t\rangle_I \quad (3.8)$$

with

$$\varphi(t) = \int_0^t \langle \lambda, t' | \left[ H(t') - i\frac{\partial}{\partial t'} \right] | \lambda, t' \rangle_I dt'. \quad (3.9)$$
The physical meanings of \( \int_{0}^{t} \langle \lambda, t' \mid H(t') \mid \lambda, t' \rangle \, dt' \) and \( \int_{0}^{t} \langle \lambda, t' \mid -i \frac{\partial}{\partial t'} \mid \lambda, t' \rangle \, dt' \) are dynamical and geometric phase, respectively.

Since the expression (3.8) is merely a formal solution of the Schrödinger equation, in order to get the explicit solutions we make use of the invariant-related unitary transformation formulation \[8\] which enables one to obtain the complete set of exact solutions of the time-dependent Schrödinger equation (3.1). In accordance with the invariant-related unitary transformation method, the time-dependent unitary transformation operator is often of the form

\[
V(t) = \exp[\beta(t)A - \beta^{*}(t)B] \tag{3.10}
\]

with \( \beta(t) = -\frac{a(t)}{2} x \exp[-ib(t)], \ \beta^{*}(t) = -\frac{a(t)}{2} x \exp[ib(t)] \). By making use of the Glauber formulae, lengthy calculation yields

\[
I_{V} = V^\dagger(t)I(t)V(t) = \left\{ \frac{y}{2} \exp(-ib) \sin a \cos\left[\left(\frac{mn}{2}\right)^{\frac{1}{2}}ax\right] - \left(\frac{mn}{2}\right)^{\frac{1}{2}} \exp(-ib) \cos a \sin\left[\left(\frac{mn}{2}\right)^{\frac{1}{2}}ax\right]\right\} A + \left\{ \frac{y}{2} \exp(ib) \sin a \cos\left[\left(\frac{mn}{2}\right)^{\frac{1}{2}}ax\right] - \left(\frac{mn}{2}\right)^{\frac{1}{2}} \exp(ib) \cos a \sin\left[\left(\frac{mn}{2}\right)^{\frac{1}{2}}ax\right]\right\} B + \{ \cos a \cos\left[\left(\frac{mn}{2}\right)^{\frac{1}{2}}ax\right] + \left(\frac{mn}{2}\right)^{\frac{1}{2}} m y \sin a \sin\left[\left(\frac{mn}{2}\right)^{\frac{1}{2}}ax\right]\} C. \tag{3.11}
\]

It can be easily seen that when

\[
y = \frac{m}{\left(\frac{mn}{2}\right)^{\frac{1}{2}}}, \ \ x = \frac{1}{\left(\frac{mn}{2}\right)^{\frac{1}{2}}}, \tag{3.12}
\]

one may derive that \( I_{V} = C \) which is time-independent. Thus the eigen-equation of the time-independent invariant \( I_{V} \) may be written in the form

\[
I_{V} |\lambda\rangle = \lambda |\lambda\rangle, \ \ |\lambda\rangle = V^\dagger(t) |\lambda, t\rangle. \tag{3.13}
\]

Under the transformation \( V(t) \), the Hamiltonian \( H(t) \) can be changed into

\[
H_{V}(t) = V^\dagger(t)H(t)V(t) - V^\dagger(t)i \frac{\partial V(t)}{\partial t} = \{ \omega \cos a \cos \theta + \left(\frac{mn}{2}\right)^{\frac{1}{2}} m \sin a \sin \theta \cos(b - \phi) \} + \left(\frac{b}{m}(1 - \cos a)\right) C \tag{3.14}
\]
by the aid of Baker-Campbell-Hausdorff formula [31]
\[
V(t) \frac{\partial}{\partial t} V(t) = \frac{\partial}{\partial t} L + \frac{1}{2!} [\frac{\partial}{\partial t} L, L] + \frac{1}{3!} [[\frac{\partial}{\partial t} L, L], L] + \frac{1}{4!} [[[\frac{\partial}{\partial t} L, L], L], L] + \cdots \quad (3.15)
\]
with \( V(t) = \exp[L(t)] \). Hence, with the help of Eq. (3.8) and Eq. (3.13), the particular solution of the Schrödinger equation is obtained
\[
|\Psi(t)\rangle_s = \exp\left[\frac{1}{i} \varphi(t)\right] V(t) |\lambda\rangle \quad (3.16)
\]
with the phase
\[
\varphi(t) = \int_0^t \langle \lambda | [V(t') H(t') V(t') - V(t') i \frac{\partial}{\partial t'} V(t')] |\lambda\rangle \, dt' = \varphi_d(t) + \varphi_g(t)
\]
where the dynamical phase is \( \varphi_d(t) = \lambda \int_0^t \omega \left[ \cos a \cos \theta + \left( \frac{mn}{2} \right)^\frac{1}{2} \sin a \sin \theta \cos(b - \phi) \right] + \frac{b}{m} (1 - \cos a) \right] \, dt' \) and the geometric phase is \( \varphi_g(t) = \lambda \int_0^t b/m (1 - \cos a) \, dt' \). It is seen that the former phase is in connection with the coefficients of the Hamiltonian such as \( \omega, \cos \theta, \sin \theta, \) etc., whereas the latter is not immediately related to these coefficients. If the parameter \( a \) is taken to be time-independent, \( \varphi_g(T) = \lambda \int_0^T b/m (1 - \cos a) \, dt' = \frac{\lambda}{m} \left[ 2\pi (1 - \cos a) \right] \) where \( 2\pi (1 - \cos a) \) is an expression for the solid angle over the parameter space of the invariant. It is of interest that \( \frac{\lambda}{m} \left[ 2\pi (1 - \cos a) \right] \) is equal to the magnetic flux produced by a monopole of strength \( \frac{\lambda}{4\pi m} \) existing at the origin of the parameter space. This, therefore, implies that geometric phase differs from dynamical phase and it involves the global and topological properties of the time evolution of a quantum system. This fact indicates the geometric or topological meaning of \( \varphi_g(t) \).

The expression (3.16) is a particular exact solution corresponding to \( \lambda \) and the general solutions of the time-dependent Schrödinger equation are easily obtained by using the linear combinations of all these particular solutions. Generally speaking, in Quantum Mechanics, solution with chronological-product operator (time-order operator) \( P \) is often called the formal solution. In the present paper, however, the solution of the Schrödinger equation governing a time-dependent system is sometimes termed the explicit solution, for reasons of
the fact that it does not involve time-order operator. But, on the other hand, by using Lewis-Riesenfeld invariant theory, there always exist time-dependent parameters, for instance, \( a(t) \) and \( b(t) \) in this paper which are determined by the auxiliary equations (3.7). In the traditional practice, when employed in experimental analysis and compared with experimental results, these nonlinear auxiliary equations should be solved often by means of numerical calculation. From above viewpoints, the concept of explicit solution is understood in a relative sense, namely, it can be considered explicit solution when compared with the time-evolution operator \( U(t) = P \exp\left[ \frac{1}{i} \int_0^t H(t') dt' \right] \) involving time-order operator, \( P \); whereas, it cannot be considered completely explicit solution for it is expressed in terms of some time-dependent parameters which should be obtained via the auxiliary equations. Hence, conservatively speaking, we regard the solution of the time-dependent system presented in the paper as exact solution rather than explicit solution.

IV. DISCUSSIONS

In the previous section, we obtain exact solutions of some time-dependent three-generator systems or models by using these invariant theories. In what follows there is the closure property of the Lie algebraic generators in the sub-Hilbert-space that should be discussed.

The generalized invariant theory can only be applied to the study of the system for which there exists the quasialgebra defined in Ref. [3]. It is easily seen from (2.15) that there is no such quasialgebra for the Hamiltonian (2.15) of example (7)(supersymmetric Jaynes-Cummings model) and example (8)(two-level atom interacting with a generalized cavity). We generalize the method which has been used for finding the dynamical algebra \( O(4) \) of the hydrogen atom to treat this type of time-dependent models. In the case of hydrogen, the dynamical algebra \( O(4) \) was found by working in the sub-Hilbert-space corresponding to a particular eigenvalue of the Hamiltonian [32]. In this paper, we will show that a generalized quasialgebra can also be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue of \( \Delta = A_0 + m \frac{1+\sigma_z}{2} \) in the time-dependent model of two-level atom.
interacting with a generalized cavity. This generalized quasialgebra enables one to obtain the complete set of exact solutions for the Schrödinger equation. It is easily verified that ∆ commutes with $H(t)$ and is a time-independent invariant according to Eq. (3.4). Then in order to unfold the algebraic structure of the Hamiltonian (2.13), the following three operators are defined \[28\]

$$
\begin{align*}
\Sigma_1 &= \frac{1}{2|\chi(\Delta)|^\frac{1}{2}}(A_- \sigma_+ + A_+ \sigma_-), \\
\Sigma_2 &= \frac{i}{2|\chi(\Delta)|^\frac{1}{2}}(A_+ \sigma_- - A_- \sigma_+), \\
\Sigma_3 &= \frac{1}{2} \sigma_z,
\end{align*}
$$

(4.1)

where $\chi = \langle n| A_+ A_- |n\rangle, |n\rangle$ denotes the eigenstates of $A_0$. It is easy to see all these operators commute with ∆ and the quasialgebra \{H, $\Sigma_1, \Sigma_2, \Sigma_3$\} is thus found. Therefore, this type of time-dependent models is proved solvable by working in a sub-Hilbert-space corresponding to the eigenstates of the time-independent invariant.

For the supersymmetric Jaynes-Cummings model, the commutation relations of its supersymmetric Lie-algebraic structure are

$$
\begin{align*}
[Q^\dagger, Q] &= \lambda \sigma_z, \\
[N, Q] &= Q, \\
[N, Q^\dagger] &= -Q^\dagger, \\
[Q, \sigma_z] &= 2Q, \\
[Q^\dagger, \sigma_z] &= -2Q^\dagger,
\end{align*}
$$

(4.2)

where

$$
N = a^\dagger a + \frac{k - 1}{2} \sigma_z + \frac{1}{2}, \quad Q = (a^\dagger)^k \sigma_-,
$$

(4.3)

and $\lambda$ denotes the eigenvalue of the time-independent invariant $N'$. By the aid of (4.2) and (4.3), the Hamiltonian (2.13) of this supersymmetric Jaynes-Cummings model can be rewritten as

$$
H(t) = \omega(t) N + \frac{\omega(t) - \delta(t)}{2} \sigma_z + g(t) Q + g^*(t) Q^\dagger - \frac{\omega(t)}{2}.
$$

(4.4)

Then this time-dependent model can be exactly solved by using the invariant-related unitary transformation formulation where the unitary transformation operator is of the form

$$
V(t) = \exp[\beta(t)Q - \beta^*(t)Q^\dagger].
$$

(4.5)
It should be noted that the above approach to the time-dependent Jaynes-Cummings model is also appropriate for treating the periodic decay and revival of some multiphoton-transitions models which has been investigated by Sukumar and Buck [24].

It is readily verified that in the sub-Hilbert-space corresponding to a particular eigenvalue of the conserved generator (the time-independent invariant), the Hamiltonian of original two-level Jaynes-Cummings model (mono-photon case) \[ SU(2) \] possesses the Lie-algebraic structure, and three-level two-mode mono-photon model possesses the \( SU(3) \) structure. The solution of the time-dependent case of \( SU(2) \) Jaynes-Cummings model is easily obtained by taking the number of photons mediating in the process of atomic transitions \( k = 1 \). Since Shumovsky et al have considered the three-level two-mode multiphoton Jaynes-Cummings model [33] whose Hamiltonian is time-independent, it is also of interest to exactly solve the time-dependent supersymmetric three-level two-mode multiphoton Jaynes-Cummings model by means of invariant theories.

V. SUMMARY

On the basis of the fact that all simple 3-generator algebras are either isomorphic to the algebra \( sl(2, C) \) or to one of its real forms, exact solutions of the time-dependent Schrödinger equation of all three-generator systems or models in quantum optics, nuclear physics, solid state physics, molecular and atomic physics and laser physics are offered by making use of the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation in the present paper. Since it appears only in systems with time-dependent Hamiltonian, the geometric phase factor would be easily investigated if the exact solutions of time-dependent systems had been obtained. In view of above discussions, the invariant-related unitary transformation formulation is a useful tool for treating the geometric phase factor and the time-dependent Schrödinger equation. This formulation replaces the eigenstates of the time-dependent invariants with those of the time-independent invariants through the unitary transformation. Apparently, it is also applicable to the algebraic structure whose
number of generators is more than three. Additionally, it should be pointed out that the
time-dependent Schrödinger equation is often investigated in the literature, whereas less
attention is paid to the time-dependent Klein-Gordon equation. Work in this direction is
under consideration and will be published elsewhere.

Acknowledgments This project is supported by the National Natural Science Foundation
of China under the project No.19775040 and 30000034. The authors thank Prof. Gao
Xiao-Chun for offering the knowledge concerning the closure property of the Lie algebraic
generators in sub-Hilbert-space.
REFERENCES

[1] Berry M V 1984 Proc.R.Soc.London, Ser. A 392 45

[2] Vinet L 1988 Phys.Rev.D 37 2369

[3] Bouchiat C and Gibbons G W 1988 J.Phys. (Paris) 49 187

[4] Datta N et al. 1989 Phys.Rev. A 40 526

[5] Mizrahi S S 1989 Phys.Lett. A 138 465

[6] Gao X C, Xu J B and Qian T Z 1991 Phys.Lett. A 152 449

[7] Lewis H R and Riesenfeld W B 1969 J.Math.Phys. 10 1458

[8] Gao X C, Xu J B and Qian T Z 1991 Phys.Rev. A 44 7016

[9] Gao X C, Gao J and Qian T Z and Xu J B 1996 Phys.Rev. D 53 4374

[10] Gao X C, Fu J, Li X H and Gao J 1998 Phys.Rev. A 57 753

[11] Gao X C, Fu J and Shen J Q 2000 Eur.Phys.J. C 13 527

[12] Shen J Q, Zhu H Y and Li J 2001 Acta Phys.Sin. 50 1884

[13] Kleinert H 2000 Gen.Rel.Gra. 32(7) 1271

[14] Mashhoon B 1995 Phys.Lett. 198 9

[15] Mashhoon B 1999 Gen.Rel.Gra. 31(5) 681

[16] Chiao R Y and Wu Y S 1986 Phys.Rev.Lett. 57 933

[17] Tomita A and Chiao R Y 1986 Phys.Rev.Lett. 57 937

[18] Kwiat P G, Chiao R Y 1991 Phys.Rev.Lett. 66 588

[19] Dattoli G, Lazzaro P D and Torre A 1987 Phys.Rev. A 35 1582

[20] Dattoli G, Richetta M and Torre A 1988 Phys.Rev. A37 2007
[21] Zhou P and Swain S 1996 Phys.Rev.Lett. **77** 3995

[22] Jaynes E T and Cummings F W 1963 Proc.IEEE. **51** 89

[23] Sukumar C V and Buck B 1981 Phys.Lett. A **83** 211

[24] Sukumar C V and Buck B 1984 J.Phys. A **17** 885

[25] Kien F L, Kozierowki M and Quany T 1988 Phys.Rev. A **38** 263

[26] Singh S 1982 Phys.Rev. A **25** 3206

[27] Vogel W and Welsch D -G 1989 Phys.Rev. A **40** 7113

[28] Yu S X and Rauch H 1995 Phys.Rev. A **52** 2585

[29] Bonatsos D, Daskaloyannis C and Lalazissis G A 1993 Phys.Rev. A **47** 3448

[30] Gomes A R 1995 Mod.Phys.Lett. B **9** 999

[31] Wei J and Norman E 1963 J.Math.Phys.(N.Y) **4575**

[32] Schiff L I 1968 Quantum Mechanics, 3rd ed.(New York: McGraw-Hill Book Company) 234

[33] Shumovsky A S, Aliskenderov E I, Kien F L and Vinh N D 1986 J.Phys. A **19** 3607