Partial Supersymmetric ABJM Theory with Flux

Yoonbai Kim, O-Kab Kwon, D. D. Tolla

1Department of Physics, BK21 Physics Research Division, Institute of Basic Science,
2University College,
Sungkyunkwan University, Suwon 440-746, Korea
yoonbai@skku.edu, okab@skku.edu, ddtolla@skku.edu

Abstract

Starting with generic Wess-Zumino type coupling to constant four-form and the dual seven-form field strengths in the ABJM theory, we obtain mass-deformed theories with $\mathcal{N} = 2, 4$ supersymmetries. These theories contain massless scalar fields and allow the implementation of the Mukhi-Papageorgakis Higgsing procedure. Using this procedure, we connect the Higgsed theories to three-dimensional mass-deformed SYM theories. These are also connected by the four-dimensional $\mathcal{N} = 1^*, 2^*$ mass-deformed SYM theories through dimensional reduction. We classify the three-dimensional mass-deformed SYM theories of $\mathcal{N} = 1, 2, 4$ supersymmetry, of which a few cases of $\mathcal{N} = 1, 2$ are connected neither by MP Higgsing procedure nor dimensional reduction.
1 Introduction

Various three-dimensional supersymmetric gauge theories have attracted much interest as the theories describing the low energy dynamics of multiple M2/D2-branes with and without background fluxes. Much of recent interests are focused on the superconformal Chern-Simons matter theory of Aharony-Bergman-Jafferis-Maldacena (ABJM) \[1\], which describes the dynamics of M2-branes on \( \mathbb{C}^4/\mathbb{Z}_k \) orbifold singularity. It was known that the circle compactification of this theory via the Mukhi-Papageorgakis (MP) Higgsing procedure \[2\] leads to the three-dimensional \( \mathcal{N} = 8 \) super Yang-Mills (SYM) theory \[3,4\], which is the low energy effective theory of multiple D2-branes. (See also Refs. \[5,6\].) Though the circle compactification of the \( \mathcal{N} = 6 \) supersymmetry-preserving mass-deformed ABJM (mABJM) theory \[7,8\] can also be taken into account, the MP Higgsing procedure cannot be implemented as a method of the circle compactification. This is because in the \( \mathcal{N} = 6 \) mABJM theory all the scalar fields are massive and the bosonic potential does not involve any flat direction allowing an infinitely large vacuum expectation value of the scalar fields. The latter is a crucial requirement for the application of MP Higgsing procedure.

The origin of the mass-deformation in the \( \mathcal{N} = 6 \) mABJM theory is identified by the presence of Wess-Zumino (WZ) type coupling to special type of constant four-form and the dual seven-
form field strengths \([9,10]\) in the infinite M2-brane tension limit. As we discussed in the previous paragraph, one cannot apply the MP Higgsing procedure to the \(\mathcal{N} = 6\) mABJM theory. In this regard, we construct some supersymmetric mABJM theories with flat directions, which let the MP Higgsing procedure possible. Subsequently, we relate the resulting theories after the MP Higgsing to three-dimensional mass-deformed super Yang-Mills (mSYM) theories. To be specific, we start from the gauge-invariant WZ-type coupling \([10,11]\) in the ABJM theory and then apply the formalism to a generic constant field strength in the infinite M2-brane tension limit.

By appropriate choices of the fluxes we construct mABJM theories preserving \(\mathcal{N} = 2, 4\) supersymmetries. An intriguing aspect of the partially supersymmetric mABJM theories is the fact that they always contain certain number of massless scalar fields which result in some flat directions of the bosonic potential. We show that the MP Higgsing of the \(\mathcal{N} = 2\) mABJM theory leads to a mSYM theory, with the same number of supersymmetry. This mass-deformed theory is equivalent to one of the three distinct three-dimensional mSYM theories, which contain one massless vector multiplet and three massive matter multiplets \([4]\). The three distinct theories are obtained by making different choices of the mass parameters of the six massive fermionic fields of the matter multiplets. Similarly, we show that the \(\mathcal{N} = 4\) mABJM theory is equivalent to a unique \(\mathcal{N} = 4\) mSYM in three dimensions. We also notice that one of the three distinct \(\mathcal{N} = 2\) mSYM theories, but not the one obtained from the \(\mathcal{N} = 2\) mABJM theory, is equivalent to the one from the dimensional reduction of the four-dimensional \(\mathcal{N} = 1^*\) mSYM theory studied by Polchinski-Strassler \([12]\). The \(\mathcal{N} = 4\) mSYM is also equivalent to the one from the dimensional reduction of the \(\mathcal{N} = 2^*\) mSYM theory in four-dimensions. In the framework of gauge/gravity correspondence, three-dimensional mSYM theories have been studied in Refs. \([13–17]\).

The remaining part of the paper is organized as follows. In section 2 we study the deformation of the ABJM theory with generic WZ-type couplings to constant background fluxes. For later convenience we single out only the WZ-type coupling which survives in the limit of infinite tension of M2-brane. In section 3 we appropriately choose the fluxes in order to preserve certain amount of supersymmetry. In section 4 we apply the MP Higgsing procedure to the partially supersymmetric mABJM theories and obtain the corresponding mSYM theories. We then study the classification of these theories in relation with the dimensional reductions of four-dimensional mSYM theories. The detailed procedure of the dimensional reduction of the \(\mathcal{N} = 1^*, 2^*\) theories is included in appendix A. Section 5 is devoted to discussions and future research directions.
2 ABJM Theory with Constant Flux

The ABJM action \([1] \) is given by a Chern-Simons matter theory with \( N = 6 \) supersymmetry and \( U(N) \times U(N) \) gauge symmetry,

\[
S = \int d^3x \mathcal{L}_{\text{ABJM}} = \int d^3x \left( \mathcal{L}_0 + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{ferm}} + \mathcal{L}_{\text{bos}} \right),
\]

where

\[
\mathcal{L}_0 = \text{tr} \left( -D_\mu Y_A^\dagger D^\mu Y^A + i\Psi^\dagger \gamma^\mu D_\mu \Psi_A \right),
\]

\[
\mathcal{L}_{\text{CS}} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} \text{tr} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho - \dot{A}_\mu \partial_\nu \dot{A}_\rho - \frac{2i}{3} \dot{A}_\mu \dot{A}_\nu \dot{A}_\rho \right),
\]

\[
\mathcal{L}_{\text{ferm}} = -\frac{2\pi i}{k} \text{tr} \left( Y_A^\dagger Y^A \Psi^\dagger B \Psi_B - Y^A Y_A^\dagger \Psi^\dagger B \Psi_B + 2Y^A Y_A^\dagger \Psi^\dagger B \Psi_B - 2Y^A Y_B^\dagger \Psi^\dagger A \Psi_B \right.
\]

\[
+ \epsilon^{ABCD} Y_A^\dagger \Psi_B^\dagger Y_C^\dagger \Psi_D - \epsilon^{ABCD} Y^A \Psi^B Y^C \Psi_D \right),
\]

\[
\mathcal{L}_{\text{bos}} = \frac{4\pi^2}{3k^2} \text{tr} \left( Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C + Y^A Y_A^\dagger Y_B^\dagger Y^B Y_C^\dagger Y^C + 4Y^A Y_B^\dagger Y_A^\dagger Y^C Y_C^\dagger \right.
\]

\[
- 6Y^A Y_B^\dagger Y_A^\dagger Y^C Y_C^\dagger \right). \tag{2.4}
\]

The four complex scalar fields \( Y^A \) \((A = 1, 2, 3, 4)\) represent the eight directions \( X^I \) \((I = 1, \cdots, 8)\) transverse to the M2-branes with

\[
Y^A = X^A + iX^{A+4}. \tag{2.6}
\]

This action has \( N = 6 \) supersymmetry with the following transformation rules

\[
\delta Y^A = i\omega^{AB} \Psi_B, \quad \delta Y_A^\dagger = i\Psi^\dagger B \omega_{AB},
\]

\[
\delta \Psi_B = -\gamma^\mu \omega_{AB} D_\mu Y^A + \frac{2\pi}{k} \omega_{BC} \left( Y^C Y_A^\dagger Y^A - Y^A Y_A^\dagger Y^C \right) + \frac{4\pi}{k} \omega_{AC} Y_A^\dagger Y_B^\dagger Y^C,
\]

\[
\delta \Psi^\dagger B = D_\mu Y_A^\dagger \omega^{AB} \gamma^\mu + \frac{2\pi}{k} \omega_{BC} \left( Y_A^\dagger Y^A Y_B^\dagger Y^C - Y_C^\dagger Y^A Y_A^\dagger \right) - \frac{4\pi}{k} \omega^{AC} Y_A^\dagger Y_B^\dagger Y^C,
\]

\[
\delta A_\mu = -\frac{2\pi}{k} \left( Y^A Y_A^\dagger \psi^\dagger \gamma_\mu \omega_{AB} + \omega^{AB} \gamma^\mu \Psi_B \right) Y_A^\dagger,
\]

\[
\delta \dot{A}_\mu = -\frac{2\pi}{k} \left( \Psi^\dagger B \gamma^\mu \omega_{AB} Y^A + Y_A^\dagger \omega^{AB} \gamma^\mu \Psi_B \right), \tag{2.7}
\]

where \( \omega^{AB} = -\omega^{BA} = (\omega_{AB})^* = \frac{1}{2} \epsilon^{ABCD} \omega_{CD} \).

Since the ABJM theory describes low energy dynamics of \( N \) stacked M2-branes, it is intriguing to consider this theory in the background of a constant transverse four-form and the dual seven-form field strengths. Interaction between the M2-branes and the background three-form gauge
fields is depicted by the WZ-type coupling. In the presence of a constant transverse four-form field strength $F_4$, the components of the corresponding three-form gauge field $C_3$ have the following transverse scalar dependence:

$$C_{\mu\nu\rho}, \ C_{\mu A}, \ C_{\mu AB}, \ C_{\mu A\bar{B}} \quad \text{and their complex conjugate (c.c.) are constants},$$

$$C_{ABC}, \ C_{A\bar{B}C} \quad \text{and their c.c. are linear in transverse scalars.} \quad (2.8)$$

Here we employed the index notations of [4], where the unbarred indices are contracted with bifundamental fields while the barred ones are contracted with anti-bifundamental fields. We can set the constant components of $C_3$ in (2.8) to zero by using the gauge transformation of the three-form gauge field, $\delta C_3 = d\Lambda_2$. In addition, one cannot construct $U(N) \times U(N)$ gauge-invariant WZ-type coupling with linear $C_{ABC}$ [10]. Therefore, the only gauge-invariant WZ-type coupling for this particular choice of the three-form gauge field is read from the equation (2.3) of Ref. [10],

$$S^{(3)}_C = \lambda \int d^3x \frac{1}{3!} e^{\mu\nu\rho} \{ \text{tr} \left[ C_{ABC} D_\mu Y^A D_\rho Y^B D_\nu Y^C + \text{(c.c.)} \right] \}, \quad (2.9)$$

where $\lambda = \frac{2\pi l_P^{3/2}}{k \lambda}$ and $l_P$ is the Planck length.

The dual seven-form field strength $F_7$ is expressed in terms of $F_4$ as

$$F_7 = *F_4 + \frac{1}{2} C_3 \wedge F_4. \quad (2.10)$$

According to the argument of the previous paragraph, in the presence of the constant transverse $F_4$, the $C_3 \wedge F_4$ term in (2.10) is linear in the transverse scalar, while the $*F_4$ term is constant. Keeping this in mind, we notice the following transverse scalar dependence for the six-form gauge field $C_6$:

$$C_{\mu\rho ABDE}, \ C_{\mu\rho ABDE}, \cdots, C_{\mu\rho ABDE}, \ C_{\mu\rho ABDE}, \cdots \quad \text{are constants},$$

$$C_{\mu\rho\mu\rho ABC}, C_{\mu\rho\mu\rho ABC}, \cdots \quad \text{are linear in transverse scalars},$$

$$C_{\mu\rho\mu\rho ABC}, C_{\mu\rho\mu\rho ABC}, \cdots \quad \text{are quadratic in transverse scalars.} \quad (2.11)$$

Setting the constant components of $C_6$ in (2.11) to zero using gauge degrees of freedom, we read the gauge-invariant WZ-type coupling from the equation (2.8) of Ref. [10],

$$S^{(6)}_C = -\frac{\pi}{k \lambda} \int d^3x \frac{1}{3!} e^{\mu\nu\rho} \{ \text{tr} \left[ C_{\mu\rho\mu\rho ABC} \beta^{AB}_C + \lambda^3 \left( C_{\mu\rho\mu\rho ABC} D_\mu Y^A D_\rho Y^B D_\nu Y^C \beta^{AB}_F + C_{\mu\rho\mu\rho ABC} D_\mu Y^A D_\rho Y^B D_\nu Y^C \beta^{AB}_F \right) + \text{(c.c.)} \right] \}, \quad (2.12)$$

where $\beta^{AB}_C \equiv \frac{1}{2} (Y^A Y^B - Y^B Y^A)$. 

---

5
In this paper, we consider the infinite tension limit of the M2-brane \((\lambda \to 0)\), which was also considered in Ref. [9], in order to turn off the coupling to gravity modes. In this limit, the three-form coupling in (2.9) and all the six-form couplings in (2.12) except the first term can be neglected. Then it is enough to take into account the following WZ-type coupling,

\[
S_{\text{WZ}} = -\frac{\pi}{\lambda k} \int d^3 x \frac{1}{3!} \epsilon^{\mu\nu\rho} \text{tr} \left[ C_{\mu\nu\rho}^{ABC} \beta_C^{AB} + C_{\mu\nu\rho}^{ABC} (\beta_C^{AB})^\dagger \right].
\]  

(2.13)

The six-form gauge fields which are linear in the transverse scalars are given by

\[
C_{\mu\nu\rho}^{ABC} = -2\lambda \epsilon_{\mu\nu\rho} T_{ABCD} Y^D, \quad C_{\mu\nu\rho}^{ABC} = -2\lambda \epsilon_{\mu\nu\rho} T_{CDAB} Y^D,
\]  

(2.14)

where the complex-valued constant parameters \(T_{ABCD} = (T_{CDAB})^*\) are antisymmetric in the last two barred indices as well as the first two unbarred indices. Therefore, the action in (2.13) is simplified as

\[
S_{\text{WZ}} = \frac{4\pi}{k} \int d^3 x \text{tr} (T_{ABCD} Y^D Y^A Y^B Y^C Y^C Y^B).
\]  

(2.15)

As we will see later, the quartic flux term of the \(N = 6\) mABJM theory can be expressed by the WZ-type coupling (2.15). In addition, different choice of constant flux can be taken into account in M-theory. If the masses of the fermionic and bosonic fields are appropriately chosen, the supersymmetry is partially preserved.

### 3 Supersymmetry-preserving Mass-deformations

In this section we discuss possible mass deformations of the ABJM theory in the presence of the constant flux term (2.15), which preserve some amount of supersymmetry. We start by introducing general gauge-invariant mass terms for scalar and fermion fields in addition to the quartic WZ-type coupling (2.15),

\[
L_{\text{bos}}^m = -\text{tr} (M_A B Y^A Y^B) \quad \text{with} \quad M_A B = (M_B A)^*,
\]

\[
L_{\text{ferm}}^m = -i \text{tr} (\mu_B A \Psi^A \Psi_B) \quad \text{with} \quad \mu_B A = (\mu_A B)^*.
\]  

(3.16)

where \(M_A B\) and \(\mu_A B\) are constant mass matrices. Then the total Lagrangian is written as

\[
L_{\text{tot}} = L_{\text{ABJM}} + L_{\text{WZ}} + L_{\text{bos}}^m + L_{\text{ferm}}^m.
\]  

(3.17)

The corresponding supersymmetry transformation rules in (2.7) for the scalar and gauge fields are unaffected by the mass-deformation while those for the fermionic fields are modified by

\[
\delta' \Psi_A = \mu_A B \omega_{BC} Y^C, \quad \delta' \Psi^A = \mu_B A \omega^{BC} Y^C.
\]  

(3.18)
From the invariance of the total Lagrangian \((3.17)\) under the total supersymmetry transformation \(\delta + \delta'\), we fix the values of \(T_{ABCD}, M^B_A\), and \(\mu^B_A\) according to the number of supersymmetry. Since \(\delta L_{ABJM} = \delta' L_{WZ} = \delta' L^m_{\text{bos}} = 0\), we need to verify only the following invariance,

\[
\delta L_{ABJM} + \delta (L_{WZ} + L^m_{\text{bos}} + L^m_{\text{term}}) + \delta' L^m_{\text{term}} = 0
\]  

(3.19)

up to total derivative. Using the supersymmetry transformation rules in (2.7) and (3.18), one can verify (3.19) under the conditions,

\[
\mu^A_A = 0,
\]

(3.20)

\[
\mu^B_A \mu^C_B \omega_{CD} - M^D_B \omega_{AB} = 0,
\]

(3.21)

\[
\mu^B_A \omega_{CD} - \mu^C_B \omega_{AD} - \mu^E_C \delta_B^A \omega_{ED} + \mu^E_C \delta_A^B \omega_{ED} - 2T_{ACBE} \omega_{ED} = 0.
\]

(3.22)

In order to check the validity of this general setup, we apply it to the well-known maximal supersymmetry preserving case \([7, 8]\). In this case \(SU(4)\) R-symmetry of the ABJM theory is broken to \(SU(2) \times SU(2) \times U(1)\) due to the mass matrix \(\mu^B_A = \text{diag}(m, m, -m, -m)\) with a mass parameter \(m\). Then we determine the bosonic mass matrix and the nonvanishing components of the constant four-form tensor from the conditions (3.21) – (3.22) as

\[
M^B_A = m^2 \delta^B_A, \quad T_{1212} = -m, \quad T_{3434} = m.
\]

(3.23)

This result exactly matches the known result for the case of maximally supersymmetric mABJM theory \([7, 8]\) and the choice (3.23) is unique up to field redefinitions \([18]\).

In the subsequent two subsections, we consider the cases with flat directions in bosonic potentials, where some of scalar fields and corresponding superpartners are massless. In those cases some of the supersymmetries are necessarily broken. The models with \(\mathcal{N} = 2\) and \(\mathcal{N} = 4\) supersymmetries are constructed.

### 3.1 \(\mathcal{N} = 2\)

Let us consider a bosonic potential which is flat along only one complex scalar field. By supersymmetry, the corresponding single complex fermion field should be massless while the other three fermion fields remains to be massive. Without loss of generality, we choose the fermionic mass matrix of the three massive fermionic fields as

\[
\mu^B_A = \text{diag}(0, m_2, m_3, m_4),
\]

(3.24)

where \(m_A\)'s \((A = 2, 3, 4)\) are real mass parameters. Then we notice that \(m_2 + m_3 + m_4 = 0\) due to the condition (3.20). In order to satisfy the conditions in (3.21) and (3.22), we should keep only
one complex component of $\omega_{AB}$ and its complex conjugate nonvanishing. To be specific, we choose nonvanishing $\omega_{14}$ and then $\omega_{23}$ is also nonvanishing by the reality condition of $\omega_{AB}$. Substitution of these into $^{(3.21)}$ determine $M_A^B$ as

$$M_A^B = \text{diag}(m_4^2, m_3^2, m_2^2, 0),$$  

and then nonvanishing components of $T_{ABC\bar{D}}$ are determined by $^{(3.22)}$ as

$$T_{1212} = -T_{3434} = \frac{m_2}{2}, \quad T_{1313} = -T_{2424} = \frac{m_3}{2}, \quad T_{1414} = -T_{2323} = \frac{m_4}{2}. \quad (3.26)$$

One may also choose different nonvanishing components, for instance, $\omega_{12}$ or $\omega_{13}$, however, the results are equivalent to the aforementioned case of the nonvanishing $\omega_{14}$, up to field redefinition.

### 3.2 $\mathcal{N} = 4$

In order to obtain the mass-deformed theory with $\mathcal{N} = 4$ supersymmetry, we have to turn on the mass term for two complex scalar fields. Then two complex fermionic fields become massive while the other two are massless. This implements an appropriate choice for the corresponding fermionic mass matrix

$$\mu_A^B = \text{diag}(0, 0, m, -m).$$  

The conditions in $^{(3.21)} - ^{(3.22)}$ are satisfied only when we keep two nonvanishing complex supersymmetric parameters and their complex conjugates. One possible choice is nonvanishing $\omega_{13}$ and $\omega_{14}$ and then $\omega_{24}$ and $\omega_{23}$ are also nonvanishing. With this choice we read the bosonic mass matrix from $^{(3.21)}$,

$$M_A^B = \text{diag}(m_2^2, m_2^2, 0, 0), \quad (3.28)$$

and the following nonvanishing components of $T_{ABCD}$ from $^{(3.22)}$

$$T_{1313} = -T_{1414} = T_{2323} = -T_{2424} = \frac{m_2}{2}. \quad (3.29)$$

Like the $\mathcal{N} = 2$ case in subsection $^{3.1}$ this choice is unique up to field redefinition.

In the original ABJM theory it was conjectured that the $\mathcal{N} = 6$ supersymmetry is enhanced to $\mathcal{N} = 8$ at Chern-Simons levels $k = 1, 2$ $^[1]$. The existence of such additional $\mathcal{N} = 2$ supersymmetries was verified in terms of the monopole operators $^[19-22]$. For $k > 2$, the supersymmetry enhancement is not possible due to orbifolding. On the other hand, in order to implement the MP Higgsing procedure one has to move the M2-branes away from the orbifold singularity and this
leads to an enhancement of the supersymmetry \[4\]. For instance, after the MP Higgsing procedure, the \( \mathcal{N} = 6 \) supersymmetry of the ABJM theory is enhanced to the \( \mathcal{N} = 8 \) supersymmetry of the three-dimensional SYM theory. The latter theory flows to the supersymmetry enhanced ABJM theory on flat transverse space \((k = 1)\) at the IR fixed point \[23\]. However, as we shall show in the next section, there is no supersymmetry enhancement after the MP Higgsing procedure in the \( \mathcal{N} = 2, 4 \) mABJM theories. This implies the absence of the supersymmetry enhancement in the \( \mathcal{N} = 2, 4 \) mABJM theories, unlike the \( \mathcal{N} = 6 \) mABJM theory.

4 Classification

The dimensional reduction of the ABJM theory with \( U(N) \times U(N) \) gauge symmetry \[1\] via the MP Higgsing procedure \[2\] leads to the three-dimensional \( \mathcal{N} = 8 \) SYM theory with \( U(N) \) gauge symmetry \[3, 4\]. In Ref. \[4\] we have shown that the Higgsing of the ABJM theory deformed by WZ-type couplings of constant fluxes results in effective theories of D2-branes in the background of constant RR fluxes. By supersymmetric completion, for few choices of constant fluxes, we obtained \( \mathcal{N} = 2, 4 \) mSYM theories. In the pervious section we have found the \( \mathcal{N} = 2, 4 \) mABJM theories. Since these theories possess bosonic potentials with flat direction, the MP Higgsing procedure can be carried out for these cases. The resultant theories are compared with the aforementioned \( \mathcal{N} = 2, 4 \) mSYM theories as discussed in Ref. \[4\]. The \( \mathcal{N} = 1 \) mSYM theory in Ref. \[4\] cannot be obtained from the MP Higgsing procedure of mABJM theory due to the following reason. In order to apply this procedure the bosonic potential is required to involve at least one massless complex scalar field. After the Higgsing, this complex field turns to one dynamical real massless scalar field and one would-be Goldstone boson. In fact the \( \mathcal{N} = 1 \) mSYM theory of Ref. \[4\] does not possess any massless scalar field.

4.1 MP Higgsing of the \( \mathcal{N} = 2, 4 \) mABJM Theories

To pursue the MP Higgsing procedure \[2\] we proceed by introducing vacuum expectation value \( v \) for the massless scalar \( Y^A \) along a transverse direction

\[
Y^A = \frac{v}{2} T^0 \delta^{A4} + \tilde{X}^A + i \tilde{X}^{A+4},
\]

(4.30)

where \( \tilde{X}^I \)'s \((I = 1, 2, \cdots, 8)\) are Hermitian scalar fields. Correspondingly we introduce Hermitian fermionic fields \( \tilde{\psi}_r \) \((r = 1, 2, \cdots, 8)\) as

\[
\Psi_A = \tilde{\psi}_A + i \tilde{\psi}_{A+4}.
\]

(4.31)
When the vacuum expectation value $v$ is turned on, in the MP Higgsing procedure, the $U(N) \times U(N)$ gauge symmetry is broken to $U(N)$ and the Hermitian scalar and fermionic fields transform in adjoint representation of the unbroken $U(N)$. Then taking double scaling limit of the large $v$ and large Chern-Simons level $k$ with finite $v/k$, the Yang-Mills coupling $g$ is identified as $g = 2\pi v/k$ and the matter fields are rescaled as $\phi \rightarrow \phi/g$ for dimensional reason. The detailed procedures are explained in Ref. [4].

Application of the Higgsing procedure to the total Lagrangian (3.17) results in

$$\mathcal{L}_{YM}^{=2,4} = \mathcal{L}_{YM}^{=8} + \frac{1}{g^2} \text{tr} \left( i\bar{T}_{ijk} \bar{X}^i [\bar{X}^j, \bar{X}^k] - \bar{M}_{ij} \bar{X}^i \bar{X}^j - i\bar{\mu}_{rs} \bar{\psi}_r \bar{\psi}_s \right), \quad (4.32)$$

where $i, j, k = 1, 2, \cdots, 7$, and

$$\mathcal{L}_{YM}^{=8} = \frac{1}{g^2} \text{tr} \left( -\frac{1}{2} \bar{F}_{\mu\nu} \bar{F}^{\mu\nu} - \bar{D}_\mu \bar{X}^i \bar{D}^{\mu} \bar{X}^i + i\bar{\psi}_r \gamma^\mu \bar{D}_\mu \bar{\psi}_r + \frac{1}{2} [\bar{X}^i, \bar{X}^j]^2 - \Gamma^r_{rs} \bar{\psi}_r [\bar{X}^i, \bar{\psi}_s] \right). \quad (4.33)$$

The cubic interaction term in (4.32) is the result of the MP Higgsing of the WZ-type coupling (2.15) and the antisymmetric tensors $\bar{T}_{ijk}$ are related to the constant four-form tensor $T_{ABCD}$ in (2.14) as follows:

$$\begin{align*}
\bar{T}_{ab4} &= \bar{T}_{a+4b+4} = \frac{2i}{3} (T_{ad\bar{a}4} - T_{btd\bar{a}4}), \quad \bar{T}_{a+4b+4} = -\frac{2}{3} (T_{ad\bar{a}4} + T_{bd\bar{a}4}), \\
\bar{T}_{abc} &= -i (T_{ab\bar{c}4} - T_{c\bar{a}d\bar{b}4}), \quad \bar{T}_{a+4b+4+4} = T_{ab\bar{c}4} + T_{c\bar{a}d\bar{b}4}, \\
\bar{T}_{abc+4} &= \frac{1}{3} (2T_{a+4b} - T_{c\bar{a}d\bar{b}} - 2T_{a\bar{c}b} + T_{b\bar{c}d\bar{a}}), \\
\bar{T}_{ab+4c+4} &= \frac{i}{3} (2T_{c\bar{a}d\bar{b}} - T_{a\bar{d}b} - 2T_{a\bar{b}c} + T_{b\bar{c}d\bar{a}}), \quad (a, b, c = 1, 2, 3). \quad (4.34)
\end{align*}$$

The bosonic and fermionic mass terms in (4.32) are obtained from the MP Higgsing of the mass terms (3.16).

For the $N = 2$ theory of subsection 3.11 we read the nonvanishing components of $\bar{T}_{ijk}$ as well as the fermionic and bosonic mass matrices from (3.24)–(3.26),

$$\begin{align*}
\bar{\mu}_{rs} &= \text{diag}(0, m_2, m_3, m_4, 0, m_2, m_3, m_4), \quad \bar{M}_{ij} = \text{diag}(m_2^2, m_3^2, m_4^2, 0, m_2^2, m_3^2, m_4^2), \\
\bar{T}_{145} &= \frac{2}{3} m_4, \quad \bar{T}_{246} = -\frac{2}{3} m_3, \quad \bar{T}_{347} = -\frac{2}{3} m_2 \quad \text{with} \ m_2 + m_3 + m_4 = 0. \quad (4.35)
\end{align*}$$

Similarly for the $N = 4$ theory of subsection 3.2 we have those quantities from (3.27)–(3.29),

$$\begin{align*}
\bar{\mu}_{rs} &= \text{diag}(0, 0, m, -m, 0, 0, m, -m), \quad \bar{M}_{ij} = \text{diag}(m^2, m^2, 0, 0, m^2, m^2, 0), \\
\bar{T}_{145} &= -\frac{2}{3} m, \quad \bar{T}_{246} = -\frac{2}{3} m. \quad (4.36)
\end{align*}$$
These results can be compared with the corresponding mSYM theories in Ref. [1] with suitable field redefinitions and parameter choices. More precisely, the $\mathcal{N} = 2$ mSYM theory we obtained in this paper is equivalent to that of Ref. [1] if we make the following field redefinitions and identifications of the mass parameters of the two theories:

$$\tilde{X}_1 \rightarrow \tilde{X}_1, \quad \tilde{X}_2 \rightarrow \tilde{X}_3, \quad \tilde{X}_3 \rightarrow \tilde{X}_6, \quad \tilde{X}_4 \rightarrow \tilde{X}_7, \quad \tilde{X}_5 \rightarrow \tilde{X}_2, \quad \tilde{X}_6 \rightarrow \tilde{X}_4, \quad \tilde{X}_7 \rightarrow \tilde{X}_5,$$

$$m_2 \rightarrow \mu_3 = \mu_4, \quad m_3 \rightarrow \mu_5 = \mu_6, \quad m_4 \rightarrow \mu_7 = \mu_8, \quad \text{with } \mu_3 + \mu_5 + \mu_7 = 0,$$

where $\mu_i$'s are the mass parameters used in Ref. [1]. We call this theory ‘$D = 3 \mathcal{N} = 2$ mSYM I’. Actually, in the case of Ref. [1] one can make other two more choices of mass parameters satisfying all the constraints imposed by supersymmetry. We call these theories ‘$D = 3 \mathcal{N} = 2$ mSYM II & III’. However, these choices cannot be related with the Higgsing of the $\mathcal{N} = 2$ mABJM theory by field redefinition. The reason is the fact that in the $\mathcal{N} = 2$ mABJM theory we have only three mass parameters of the three massive complex fields while in the case of Ref. [1] we have six mass parameters of the six massive real fields. Therefore, in the latter case there are more freedoms in choosing the mass parameters. See the next subsection for the details.

The comparison for the $\mathcal{N} = 4$ theories obtained here and Ref. [1] can be made by setting $m_2 = 0 \rightarrow \mu_3 = \mu_4 = 0$ and using the same field redefinitions and parameter choices as in (4.37). In this case other possible choices of mass parameters in Ref. [1] are also identical to the choice in (4.36) up to field redefinitions. The dimensional reduction of the four dimensional $\mathcal{N} = 2^*\text{ mSYM}$ theory [12] also gives this $\mathcal{N} = 4$ mSYM theory. (For the details see Appendix A) It is also important to recall that we started with a general setting of mass deformation in ABJM theory to obtain the $\mathcal{N} = 4$ mABJM theory. In these regards, the $\mathcal{N} = 4$ mABJM and mSYM theories discussed in this paper seem unique.

### 4.2 Classification of the $\mathcal{N} = 2$ mSYM theories

In Ref. [1] we have shown that introduction of the mass deformation to the $\mathcal{N} = 8$ SYM theory preserves $\mathcal{N} = 1, 2, 4$ supersymmetries depending on the choices of the fermionic and bosonic mass parameters and components of the antisymmetric tensor $\tilde{T}_{ijk}$. In this subsection we fully classify $\mathcal{N} = 2$ mSYM theories according to mass parameter choices.

The $\mathcal{N} = 1$ mSYM theory contains one massless gauge boson and seven massive scalar fields. Together with their superpartners which are one massless and seven massive fermionic fields, these sets of fields form one $\mathcal{N} = 1$ vector multiplet and seven massive matter multiplets. Since all the massive scalar fields belong to different multiplets, they are allowed to have different masses of which the parameters are unrestricted unlike the higher supersymmetry cases. In this reason, there is no candidate in the mABJM theory to be linked with this $\mathcal{N} = 1$ mSYM theory.
In the case of the $\mathcal{N} = 2$ mSYM theory, the supersymmetry invariance of the action requires

$$\tilde{\mu}_{rs} = \text{diag}(0, 0, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8), \quad \tilde{M}_{ij} = \text{diag}(\mu_8^2, \mu_7^2, \mu_6^2, \mu_5^2, \mu_4^2, \mu_3^2, 0),$$

$$\tilde{T}_{145} = \frac{1}{3}(\mu_3 + \mu_6 + \mu_7), \quad \tilde{T}_{246} = \frac{1}{3}(\mu_3 + \mu_5 + \mu_7), \quad \tilde{T}_{347} = \frac{1}{3}(\mu_5 + \mu_6),$$

$$\tilde{T}_{127} = -\frac{1}{3}(\mu_7 + \mu_8), \quad \tilde{T}_{136} = -\frac{1}{3}(\mu_4 + \mu_5 + \mu_7), \quad \tilde{T}_{235} = -\frac{1}{3}(\mu_3 + \mu_5 + \mu_8),$$

$$\tilde{T}_{567} = -\frac{1}{3}(\mu_3 + \mu_4),$$

(4.38)

where the mass parameters are constrained as,

$$\mu_4^2 = \mu_3^2, \quad \mu_6^2 = \mu_5^2, \quad \mu_8^2 = \mu_7^2, \quad \mu_3 + \mu_4 + \mu_5 + \mu_6 + \mu_7 + \mu_8 = 0.$$  

(4.39)

There are three independent mass parameter choices satisfying the constraints in (4.39),
\textbf{case I}:
\[ \mu_3 = \mu_4, \quad \mu_5 = \mu_6, \quad \mu_7 = \mu_8 \quad \text{with} \quad \mu_3 + \mu_5 + \mu_7 = 0, \]
\[ \tilde{T}_{145} = \tilde{T}_{246} = \tilde{T}_{136} = \tilde{T}_{235} = 0, \quad \tilde{T}_{347} = \frac{2}{3}\mu_5, \quad \tilde{T}_{127} = -\frac{2}{3}\mu_7, \quad \tilde{T}_{567} = -\frac{2}{3}\mu_3, \]

\textbf{case II}:
\[ \mu_4 = -\mu_3, \quad \mu_6 = -\mu_5, \quad \mu_8 = -\mu_7, \]
\[ \tilde{T}_{145} = \frac{1}{3}(\mu_3 - \mu_5 + \mu_7), \quad \tilde{T}_{246} = \frac{1}{3}(\mu_3 + \mu_5 + \mu_7), \quad \tilde{T}_{136} = \frac{1}{3}(\mu_3 - \mu_5 - \mu_7), \]
\[ \tilde{T}_{235} = \frac{1}{3}(-\mu_3 - \mu_5 + \mu_7), \quad \tilde{T}_{347} = \tilde{T}_{127} = \tilde{T}_{567} = 0, \]

\textbf{case III}:
\[ \mu_4 = -\mu_3, \quad \mu_6 = -\mu_8 = -\mu_7 = \mu_5, \]
\[ \tilde{T}_{145} = \tilde{T}_{246} = \tilde{T}_{136} = -\tilde{T}_{235} = \frac{1}{3}\mu_3, \quad \tilde{T}_{347} = \tilde{T}_{127} = \frac{2}{3}\mu_5, \quad \tilde{T}_{567} = 0. \quad (4.40) \]

As discussed previously the case I is identical to the Higgsed $\mathcal{N} = 2$ mABJM theory of the previous subsection through the field redefinitions in (4.37). As shown in appendix A, the case II is obtained as a result of the dimensional reduction of the four-dimensional $\mathcal{N} = 1^*$ mSYM theory [12]. The case III is a $\mathcal{N} = 2$ mSYM theory which can be connected to neither $\mathcal{N} = 2$ mABJM theory through the MP Higgsing nor the $\mathcal{N} = 1^*$ mSYM theory through dimensional reduction. The reason why the cases II and III are not related with the Higgsed $\mathcal{N} = 2$ mABJM theory of subsection 3.1 is the following. For the latter case, any set of two fermionic fields belonging to the same supermultiplet is inherited from the real and imaginary components of a complex fermionic field in the original mABJM theory. Therefore, they have the same masses. For the former cases, the fermionic fields in the same multiplet have either the same or opposite signs for their mass parameters as indicated in (4.40). For convenience we summarize classification of the mSYM theories in the diagram of Fig. 1.

\section{Conclusion}

In this paper we classified some parity-preserving three-dimensional supersymmetric mass-deformed gauge theories. In the ABJM theory, we introduced a generic WZ-type coupling to constant four-form and dual seven-form field strengths in the limit of infinite M2-brane tension. We showed, with appropriate choice of the fermionic and bosonic mass terms, such deformed ABJM theory possesses $\mathcal{N} = 2, 4, 6$ supersymmetries. In Ref. [4] we already constructed three distinct mSYM theories in three dimensions, which are one $\mathcal{N} = 1$, three $\mathcal{N} = 2$, and one $\mathcal{N} = 4$ mSYM theories. Here we verified that one of the three $\mathcal{N} = 2$ theories and the $\mathcal{N} = 4$ theory are obtained through the MP Higgsing of the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ mABJM theories, respectively. One of the remaining
\( \mathcal{N} = 2 \) theories is obtained by dimensional reduction of the four-dimensional \( \mathcal{N} = 1^* \) theory, while the \( \mathcal{N} = 4 \) mSYM theory is also obtained by the dimensional reduction to the \( \mathcal{N} = 2^* \) theory. The third \( \mathcal{N} = 2 \) and the \( \mathcal{N} = 1 \) mSYM theories are not connected by the MP Higgsing of the mABJM theory or the dimensional reduction of the four-dimensional mSYM theory.

We may extend our analysis in this paper to the cases of the parity-violating three-dimensional gauge theories, such as the \( \mathcal{N} = 3 \) level-deformed ABJM theory developed by Gaiotto and Tomasiello (GT) \[24\] (see also Ref. \[25\]). As the ABJM theory does, the GT theory allows the supersymmetry-preserving mass-deformation \[26\] and the circle compactification via the MP Higgsing procedure \[27\]. Utilizing these properties, one can construct the less supersymmetric mass-deformed GT theories with flat directions which implement the MP Higgsing procedure. This analysis may shed some light on M-theory brane configuration of the GT theory.

Holographic dual of the \( \mathcal{N} = 6 \) mABJM theory is proposed in Ref. \[28\], which is the \( \mathbb{Z}_k \)-quotient of the Lin-Lunin-Maldacena (LLM) geometry \[29\] (see also Ref. \[30\]). The proposal of the dual gravity gets much insights from the structure of the vacuum space of the gauge theory. The dual gravity theories are not yet understood for the partially supersymmetric mABJM theories. The \( \mathcal{N} = 4 \) mABJM theory does not contain any Higgs vacuum solution and does not seem to have a dual gravity theory related to the LLM geometry. On the other hand, after the MP Higgsing and the dimensional uplift, the resulting \( \mathcal{N} = 2^* \) mSYM theory turns out to be dual to the Pilch-Warner geometry in type IIB supergravity \[31\]. It is interesting to figure out the M-theory uplifting of this geometry and to identify the dual geometry of the \( \mathcal{N} = 4 \) mABJM theory. The \( \mathcal{N} = 2 \) mABJM theory has Higgs vacuum solutions but it is still unclear how to modify the LLM geometry to obtain the corresponding dual gravity.

**Acknowledgements**

This work was supported by the Korea Research Foundation Grant funded by the Korean Government with grant numbers 2011-0011660 (Y.K.), 2011-0009972 (O.K.), and 2009-0077423 (D.D.T.).
\section*{A Four-dimensional Mass-deformed SYM Theories}

The four-dimensional $\mathcal{N} = 1^*$ theory by Polchinski and Strassler \cite{12} is constructed by introducing a mass-deformation to the $\mathcal{N} = 4$ SYM theory. The action for the latter is given by

\[
\mathcal{L} = \text{tr} \left[ -\frac{1}{2} F_{\alpha \beta} F^{\alpha \beta} - \bar{D}^a \Phi_a D_a \Phi_a + \frac{\bar{g}^2}{2} [\Phi_a, \Phi_b]^2 + (i \bar{\psi}_p \gamma^a \bar{D}_a \psi_p) - \bar{g} \left( \bar{\psi}_p \left( \Delta^{pq} \frac{1 + \gamma_5}{2} + \bar{\Delta}^{pq} \frac{1 - \gamma_5}{2} \right) \Phi_a, \psi_q \right) \right], \tag{A.41}
\]

where $\alpha, \beta = 0, \ldots, 3$, $a, b = 1, \ldots, 6$, $p, q = 1, \ldots, 4$, $\psi_p$'s are Majorana fermions and $\Phi_a$'s are Hermitian scalar fields. $\Delta^{pq} = g_p \Delta_a g_q$ and $\bar{\Delta}^{pq} = g^*_p \Delta_a g^*_q$ are constants. $\Delta_a$ are the gamma matrices of the six-dimensional Euclidean space, and $g_p, g^*_p$ are the eigenvectors of $\Gamma_* = -i \Delta_1 \ldots \Delta_6$ with eigenvalues +1, -1, respectively. The covariant derivative is given by $\bar{D}_a = \partial_a + i \bar{g}[A_a, \cdot]$. The Clifford algebra for the gamma matrices is given by: $\{\tilde{\gamma}_\alpha, \tilde{\gamma}_\beta\} = -2\eta_{\alpha \beta}$ with the signature $\eta_{\alpha \beta} = \text{diag}(-1, 1, 1, \ldots)$.

The $\mathcal{N} = 4$ supersymmetry transformation rules are

\[
\delta_\epsilon A_a = i \bar{\epsilon}_p \tilde{\gamma}_\alpha \psi_p,
\delta_\epsilon \Phi_a = i \bar{\epsilon}_p (\Delta^{pq} \frac{1 + \gamma_5}{2} + \bar{\Delta}^{pq} \frac{1 - \gamma_5}{2}) \psi_q,
\delta_\epsilon \psi_p = i F_{\alpha \beta} \Sigma^{\alpha \beta} \epsilon_p + \bar{\gamma}^a \bar{D}_a \Phi_a (\Delta^{pq} \frac{1 + \gamma_5}{2} + \bar{\Delta}^{pq} \frac{1 - \gamma_5}{2}) \epsilon_q - \bar{g}[\Phi_a, \Phi_b] (\Delta^{pq} \frac{1 + \gamma_5}{2} + \bar{\Delta}^{pq} \frac{1 - \gamma_5}{2}) \epsilon_q, \tag{A.42}
\]

where the supersymmetry parameters $\epsilon_p$'s are Majorana fermions, and $\Delta^{pq} = g_p \Sigma^{ab} g_q$, $\bar{\Delta}^{pq} = g^*_p \Sigma^{ab} g^*_q$ with $\Sigma^{ab} = -\frac{i}{4}[\Delta^a, \Delta^b]$. After some algebra we obtain

\[
\begin{align*}
\Delta^{pq} &= \frac{i}{4} (\Delta^{pq} \Delta^{pq} - \bar{\Delta}^{pq} \bar{\Delta}^{pq}), \\
\bar{\Delta}^{pq} &= \frac{i}{4} (\Delta^{pq} \bar{\Delta}^{pq} - \bar{\Delta}^{pq} \Delta^{pq}), \tag{A.43}
\end{align*}
\]

where the components of $\Delta_a$'s are given by

\[
\begin{align*}
\Delta^{1q} - \bar{\Delta}^{1q} &= 2i(\delta_{p1} \delta_{q4} - \delta_{p4} \delta_{q1} + \delta_{p2} \delta_{q3} - \delta_{p3} \delta_{q2}), \quad \Delta^{pq} + \bar{\Delta}^{pq} = 0, \\
\Delta^{2q} - \bar{\Delta}^{2q} &= 2i(\delta_{p1} \delta_{q2} - \delta_{p2} \delta_{q1} + \delta_{p3} \delta_{q4} - \delta_{p4} \delta_{q3}), \quad \Delta^{pq} + \bar{\Delta}^{pq} = 0, \\
\Delta^{3q} - \bar{\Delta}^{3q} &= 2i(\delta_{p1} \delta_{q3} - \delta_{p3} \delta_{q1} - \delta_{p2} \delta_{q4} + \delta_{p4} \delta_{q2}), \quad \Delta^{pq} + \bar{\Delta}^{pq} = 0, \\
\Delta^{4q} + \bar{\Delta}^{4q} &= -2(\delta_{p1} \delta_{q4} - \delta_{p4} \delta_{q1} - \delta_{p2} \delta_{q3} + \delta_{p3} \delta_{q2}), \quad \Delta^{pq} + \bar{\Delta}^{pq} = 0, \\
\Delta^{5q} + \bar{\Delta}^{5q} &= 2(\delta_{p1} \delta_{q2} - \delta_{p2} \delta_{q1} - \delta_{p3} \delta_{q4} + \delta_{p4} \delta_{q3}), \quad \Delta^{pq} + \bar{\Delta}^{pq} = 0, \\
\Delta^{6q} + \bar{\Delta}^{6q} &= -2(\delta_{p1} \delta_{q3} - \delta_{p3} \delta_{q1} + \delta_{p2} \delta_{q4} - \delta_{p4} \delta_{q2}), \quad \Delta^{pq} + \bar{\Delta}^{pq} = 0. \tag{A.44}
\end{align*}
\]
In the $\mathcal{N} = 1^*$ theory, without loss of generality we can choose $\epsilon = \epsilon_4$ as the unbroken supersymmetry parameter with the other supersymmetry parameters set to zero. Then the supersymmetry transformation rules in (A.42) are reduced to

$$
\delta_\epsilon A_\alpha = i\bar{\epsilon}\gamma_\alpha\lambda,
$$

$$
\delta_\epsilon \tilde{\Phi}_a = i\bar{\epsilon}(\Delta_a^{t4}\frac{1+\gamma_5}{2} + \tilde{\Delta}_a^{t4}\frac{1-\gamma_5}{2})\psi_t,
$$

$$
\delta_\epsilon \psi_t = \bar{\gamma}_a^\alpha\tilde{D}_a\tilde{\Phi}_a(\Delta_a^{t4}\frac{1+\gamma_5}{2} + \tilde{\Delta}_a^{t4}\frac{1-\gamma_5}{2})\epsilon - \tilde{g}[\tilde{\Phi}_a, \tilde{\Phi}_b](\Delta_a^{t4}\frac{1+\gamma_5}{2} + \tilde{\Delta}_a^{t4}\frac{1-\gamma_5}{2})\epsilon,
$$

$$
\delta_\epsilon \lambda = iF_{\alpha\beta}\Sigma^{\alpha\beta}\epsilon - \tilde{g}[\tilde{\Phi}_a, \tilde{\Phi}_b](\Delta_a^{t4}\frac{1+\gamma_5}{2} + \tilde{\Delta}_a^{t4}\frac{1-\gamma_5}{2})\epsilon,
$$

where $\lambda = \psi_4$ and $t = 1, 2, 3$.

The mass-deformation preserving the $\mathcal{N} = 1$ supersymmetry is

$$
\mathcal{L}_\mu = \text{tr}(-i\mu_{pq}\bar{\psi}_p\psi_q - M_{ab}\tilde{\Phi}_a\tilde{\Phi}_b + i\tilde{g}T_{abc}\tilde{\Phi}_a[\tilde{\Phi}_b, \tilde{\Phi}_c]),
$$

where

$$
\mu_{pq} = \text{diag}(\mu_1, \mu_2, \mu_3, 0), \quad M_{ab} = \text{diag}(\mu_1^2, \mu_2^2, \mu_1^2, \mu_3^2, \mu_2^2)
$$

and the nonvanishing components of $T_{abc}$ are

$$
T_{234} = \frac{1}{3}(\mu_1 - \mu_2 - \mu_3), \quad T_{126} = \frac{1}{3}(\mu_1 - \mu_2 + \mu_3),
$$

$$
T_{135} = \frac{1}{3}(\mu_1 + \mu_2 - \mu_3), \quad T_{456} = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3).
$$

The modification to the fermionic variation is

$$
\delta'_\epsilon \psi_p = \mu_{pq}(\Delta_a^{q4}\frac{1+\gamma_5}{2} + \tilde{\Delta}_a^{q4}\frac{1-\gamma_5}{2})\epsilon\tilde{\Phi}_a.
$$

When $\mu_1 = \mu_2$ and $\mu_3 = 0$, we easily notice that the supersymmetry is enhanced to $\mathcal{N} = 2$. This gives the $\mathcal{N} = 2^*$ theory discussed in Ref [12].

### A.1 Reduction to three dimensions

In order to reduce the $\mathcal{N} = 1^*$ theory to three dimensions we assume that the fields do not depend on the compactified direction. For the bosonic part, by introducing $V^{\frac{1}{2}}A_\alpha = (A_\mu, \phi)$ with $V$ the volume of the compactified direction and $\mu, \nu = 0, 1, 2$, we obtain

$$
VF_{\alpha\beta}F^{\alpha\beta} = F_{\mu\nu}F^{\mu\nu} + 2F_{3\mu}F^{3\mu} = F_{\mu\nu}F^{\mu\nu} + 2D^\mu\phi D_\mu\phi,
$$

where $\Delta_a^{t4}\frac{1+\gamma_5}{2} + \tilde{\Delta}_a^{t4}\frac{1-\gamma_5}{2}$ are

$$
\begin{align*}
\Delta_a^{t4}\frac{1+\gamma_5}{2} &= \frac{1}{3}(\mu_1 - \mu_2 - \mu_3), \\
\Delta_a^{t4}\frac{1-\gamma_5}{2} &= \frac{1}{3}(\mu_1 - \mu_2 + \mu_3), \\
\tilde{\Delta}_a^{t4}\frac{1+\gamma_5}{2} &= \frac{1}{3}(\mu_1 + \mu_2 - \mu_3), \\
\tilde{\Delta}_a^{t4}\frac{1-\gamma_5}{2} &= \frac{1}{3}(\mu_1 + \mu_2 + \mu_3).
\end{align*}
$$

These transformations are used to reduce the $\mathcal{N} = 1^*$ theory to three dimensions.
and, by setting \( V \frac{1}{2} \Phi_a = \Phi_a \) and \( V \frac{1}{2} \tilde{g} = g \), we have
\[
V \tilde{D}^a \tilde{\Phi}_a \tilde{D}_a \tilde{\Phi}_a = D^a \Phi_a D_\mu \Phi_a - g^2 [\phi, \Phi_a]^2, \quad V M_{ab} \tilde{\Phi}_a \tilde{\Phi}_b = M_{ab} \Phi_a \Phi_b,
\]
\[
i V \tilde{g} T_{abc} \tilde{\Phi}_a [\tilde{\Phi}_b, \tilde{\Phi}_c] = ig T_{abc} [\Phi_b, \Phi_c], \quad V \tilde{g}^2 [\tilde{\Phi}_a, \tilde{\Phi}_b]^2 = g^2 [\Phi_a, \Phi_b]^2,
\]
where the covariant derivative is given by \( D_\mu = \partial_\mu - ig [A_\mu, \cdot] \). Using the relation \( \int d^4 x \mathcal{L}_{\text{bos}} = \int d^3 x \mathcal{L}_{\text{bos}} \) and substituting the obtained results into the bosonic part of the four-dimensional action in \( (A.41) \) and \( (A.46) \), we write the bosonic part of the Lagrangian density in three dimension as
\[
\mathcal{L}_{\text{bos}} = \text{tr} \left[ - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D^\mu \Phi D_\mu \phi - D^\mu \Phi_a D_\mu \Phi_a + \frac{1}{2} g^2 ( [\phi, \Phi_a]^2 + [\Phi_a, \phi]^2 ) \right.
\]
\[
\left. + \frac{1}{2} g^2 \Phi_a \Phi_b] - M_{ab} \Phi_a \Phi_b + ig T_{abc} [\Phi_b, \Phi_c] \right]
\]
\[
= - \text{tr} \left[ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D^\mu \tilde{X}_i D_\mu \tilde{X}_i - \frac{1}{2} g^2 [\tilde{X}_i, \tilde{X}_j]^2 + M_{ij} \tilde{X}_i \tilde{X}_j - ig T_{ijkl} \tilde{X}_i [\tilde{X}_j, \tilde{X}_k] \right],
\]
where \( \tilde{X}_i = (\Phi_a, \phi) \) for \( i = 1, \ldots, 7 \) are the seven transverse scalar fields and \( M_{ij} = T_{ijkl} = 0 \).

For the fermionic part, we split the four-dimensional gamma matrices as follows
\[
\gamma^0 = \sigma_3 \otimes i \sigma_2, \quad \gamma^1 = \sigma_3 \otimes \sigma_1, \quad \gamma^2 = \sigma_3 \otimes \sigma_3, \quad \gamma^3 = \sigma_1 \otimes \mathbb{1},
\]
whereas the three-dimensional gamma matrices are given by
\[
\gamma^0 = i \sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_3.
\]
The four-dimensional Majorana spinor has the expansion
\[
V \frac{1}{2} \psi_p = \sum_{r=1}^{2} e^r \otimes \psi^r_p,
\]
where \( e^r \)'s form the basis of \( \mathbb{R}^2 \) and \( \psi^r_p \)'s are Majorana spinors in three dimensions. With \( \gamma_5 = -i \tilde{\gamma}_0 \tilde{\gamma}_1 \tilde{\gamma}_2 \tilde{\gamma}_3 \), their chiral components are written in terms of the three-dimensional Majorana spinors as
\[
V \frac{1}{2} \psi_p^+ = V \frac{1}{2} \frac{1 + \gamma_5}{2} \psi_p = \frac{1 - \sigma_2 \otimes \mathbb{1}}{2} e^r \otimes \psi^r_p,
\]
\[
V \frac{1}{2} \psi_p^- = V \frac{1}{2} \frac{1 - \gamma_5}{2} \psi_p = \frac{1 + \sigma_2 \otimes \mathbb{1}}{2} e^r \otimes \psi^r_p.
\]
Finally, the covariant derivatives are given by
\[
V \frac{1}{2} \tilde{D}_\mu \psi_p = e^r \otimes D_\mu \psi^r_p, \quad V \frac{1}{2} \tilde{D}_3 \psi_p = ig e^r \otimes [\phi, \psi^r_p],
\]
and the Dirac conjugation becomes

\[
V \bar{\psi}_p^\gamma \gamma^\alpha \bar{D}_\alpha \psi_p = V \bar{\psi}_p^\gamma \gamma^\alpha \bar{D}_\alpha \psi_p = (e^r \otimes \psi^r_p)^\dagger \sigma_3 \otimes i \sigma_2.
\]

(A.58)

Similar to the bosonic part, we substitute the results \( (A.53)-(A.58) \) into every term of the fermionic part in \( (A.41) \) and \( (A.46) \) and use \( \int d^4 x \tilde{\mathcal{L}}_{\text{term}} = \int d^3 x \mathcal{L}_{\text{term}} \). Computation of the kinetic term leads to

\[
V \bar{\psi}_p^\gamma \gamma^\alpha \bar{D}_\alpha \psi_p = V \bar{\psi}_p^\gamma \gamma^\alpha \bar{D}_\alpha \psi_p + V \bar{\psi}_p^\gamma \gamma^3 \bar{D}_3 \psi_p
\]

\[
= (e^r \otimes \psi^r_p) \left( (\sigma_3 \otimes \gamma^0) (\sigma_3 \otimes \gamma^\mu) e^s \otimes D_\mu \psi^s_p + ig (\sigma_3 \otimes \gamma^0) (\sigma_1 \otimes \mathbb{I}) e^s \otimes [\phi, \psi^s_p] \right)
\]

\[
= \bar{\psi}_p^{r^\gamma} \gamma^\mu D_\mu \psi^r_p + ig \gamma^r_p \psi^r_p [\phi, \psi^r_p],
\]

(A.59)

where \( \bar{\psi}_p^r = \psi^{r^\dagger} \gamma^0 \) is the Dirac conjugation in three dimensions, \( \gamma^0_{rs} = e^{r^\dagger} \gamma^0 e^s = i e^{r^\dagger} \sigma_2 e^s \), and we have used \( e^{r^\dagger} e^s = \delta^{rs} \). The fermionic mass term and the Yukawa-type interaction terms are

\[
i V \mu_p q \bar{\psi}_p \psi_q = i \mu_p q \bar{\psi}_p^1 \gamma^0 \psi_q^1 - i \mu_p q \bar{\psi}_p^2 \gamma^0 \psi_q^2,
\]

\[
V \bar{g} \Delta^p_a \bar{\psi}_p [\Phi_a, \psi^r_p] = \frac{g}{2} \Delta^p_a \left( i \gamma^1_{rs} + \gamma^2_{rs} \right) \bar{\psi}_p [\Phi_a, \psi^r_q],
\]

\[
V \bar{g} \Delta^p_a \bar{\psi}_p [\Phi_a, \psi^r_q] = \frac{g}{2} \Delta^p_a \left( - i \gamma^1_{rs} + \gamma^2_{rs} \right) \bar{\psi}_p [\Phi_a, \psi^r_q],
\]

(A.60)

where \( \gamma^1_{rs} = e^{r^\dagger} \gamma^1 e^s = e^{r^\dagger} \sigma_1 e^s \) and \( \gamma^2_{rs} = e^{r^\dagger} \gamma^2 e^s = e^{r^\dagger} \sigma_3 e^s \).

Collecting all the terms in \( (A.59)-(A.60) \) we have

\[
\mathcal{L}_{\text{term}} = \text{tr} \left[ i \bar{\psi}_p^{r^\gamma} \gamma^\mu D_\mu \psi^r_p - g \gamma^r_p \bar{\psi}_p [\phi, \psi^r_p] - i \mu_p q \bar{\psi}_p^1 \gamma^0 \psi_q^1 + i \mu_p q \bar{\psi}_p^2 \gamma^0 \psi_q^2 \right.
\]

\[
- \frac{g}{2} \left[ \gamma^1_{rs} (\Delta^p_a - \bar{\Delta}^p_a) + \gamma^2_{rs} (\Delta^p_a + \bar{\Delta}^p_a) \right] \bar{\psi}_p [\Phi_a, \psi^r_q].
\]

(A.61)

This is rewritten as

\[
\mathcal{L}_{\text{term}} = \text{tr} \left[ i \bar{\psi}_p^{r^\gamma} \gamma^\mu D_\mu \psi^r_p - i \mu_p q \bar{\psi}_p^1 \gamma^0 \psi_q^1 + i \mu_p q \bar{\psi}_p^2 \gamma^0 \psi_q^2 - g (M_i)^p q r s \bar{\psi}_p [\tilde{X}_i, \psi^r_q],
\]

(A.62)

by using seven-dimensional index \( i = 1, ..., 7 \) and the corresponding Clifford algebra of the Euclidean Gamma matrices

\[
\{ M_i, M_j \}_{rs}^{pq} = -2 \delta_{ij} \delta_{rs} \delta^{pq},
\]

(A.63)

defined by

\[
(M_i)^{pq}_{rs} = \gamma^{0}_{rs} \delta^{pq},
\]

\[
(M_i)^{pq}_{rs} = i \gamma^1_{rs} \Delta^p_i = -i \gamma^1_{rs} \bar{\Delta}^p_i, \quad \text{for } i = 1, 2, 3,
\]

\[
(M_i)^{pq}_{rs} = \gamma^{2}_{rs} \Delta^p_i = \gamma^{2}_{rs} \bar{\Delta}^p_i, \quad \text{for } i = 4, 5, 6.
\]

(A.64)

The Lagrangians in \( (A.52) \) and \( (A.62) \) are identical to the three-dimensional \( N = 2 \) mSYM theory with the mass parameters adjusted to the ‘case II’ of \( (4.40) \). Similarly, the dimensional reduction of the \( N = 2^* \) theory gives the \( N = 4 \) mSYM theory discussed in subsection 4.1.
A.2 Reduction of the supersymmetry variations

Let us subsequently discuss the supersymmetry variation after dimensional reduction in this subsection. The supersymmetry parameters in (A.42) and their chiral components have an expansion in three-dimensions,

$$\epsilon_p = e^r \otimes e^r_p, \quad \epsilon^+_p = \frac{1 - \sigma_2 \otimes \mathbb{I}}{2} e^r \otimes e^r_p, \quad \epsilon^-_p = \frac{1 + \sigma_2 \otimes \mathbb{I}}{2} e^r \otimes e^r_p,$$

(A.65)

where $e^r_p$'s are three-dimensional Majorana spinors. Since the case of our consideration sets only $\epsilon_4$ nonvanishing, the supersymmetry variation of the bosonic and fermionic fields in (A.45) becomes

$$\delta_t \phi = i \gamma^0_{rs} e^r \lambda^s,$$

$$\delta_t A_\mu = i e^r \gamma_\mu \lambda^r,$$

$$\delta_t \Phi_a = \frac{i}{2} \left( i \gamma^1_{rs} (\Delta^a - \bar{\Delta}^a) + \gamma^2_{rs} (\Delta^a + \bar{\Delta}^a) \right) e^r \psi^a_t,$$

$$\delta_t \psi^a_t = \frac{1}{2} \left( i \gamma^1_{rs} (\Delta^a - \bar{\Delta}^a) + \gamma^2_{rs} (\Delta^a + \bar{\Delta}^a) \right) D_\mu \Phi_a \gamma^\mu \epsilon^s$$

$$- \frac{g}{4} \left( i \gamma^1_{rs} (\Delta^a + \bar{\Delta}^a) + \gamma^2_{rs} (\Delta^a - \bar{\Delta}^a) \right) \left( [\Phi_a, \phi] - [\phi, \Phi_a] \right) e^s$$

$$- \frac{g}{2} \left( [\phi, \Phi_a], [\phi, \Phi_b] \right) \left( (\Delta^a + \bar{\Delta}^a) e^r + i \gamma^0_{rs} (\Delta^a - \bar{\Delta}^a) e^s \right),$$

$$\delta_t \lambda^r = i F_{\mu \nu} \sigma^{\mu \nu} e^r + \gamma^0_{rs} D_\mu \phi \gamma^\mu e^s - \frac{g}{2} (\Phi_a, \Phi_b) \left( (\Delta^a + \bar{\Delta}^a) e^r + i \gamma^0_{rs} (\Delta^a - \bar{\Delta}^a) e^s \right),$$

(A.66)

where $\sigma^{\mu \nu} = \frac{1}{4} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$. By using the compact notation introduced in the previous subsection (A.66) is simplified as

$$\delta_t A_\mu = i e^r \gamma_\mu \lambda^r,$$

$$\delta_t X^i = i (M_{ij})^{rs}_{pq} e^r \psi^s,$$

$$\delta_t \psi^a_t = (M_{ij})^{rs}_{pq} D_\mu \bar{X}^i \gamma_\mu \epsilon^s - g (M_{ij})^{rs}_{pq} [\bar{X}^i, \bar{X}^j] e^s,$$

$$\delta_t \lambda^r = i F_{\mu \nu} \sigma^{\mu \nu} e^r - g (M_{ij})^{rs}_{pq} [\bar{X}^i, \bar{X}^j] e^s,$$

(A.67)

where $(M_{ij})^{pq}_{rs} = \frac{i}{4} (M_i M_j - M_j M_i)_{pq}^{rs}$. Similar computation to the fermionic mass part (A.49) also provides simpler expression,

$$\delta_t^\prime \psi^a_t = \mu_{pq} (\Delta^a - \bar{\Delta}^a) \frac{1 + \gamma_5}{2} \epsilon \bar{\Phi}_a = \mu_{pq} (M_{ij})^{rs}_{pq} \bar{X}^i \epsilon^s.$$

(A.68)

References

[1] O. Aharony, O. Bergman, D. L. Jafferis, J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810, 091 (2008) arXiv:0806.1218 [hep-th].
[2] S. Mukhi, C. Papageorgakis, “M2 to D2,” JHEP 0805, 085 (2008) [arXiv:0803.3218 [hep-th]].

[3] Y. Pang, T. Wang, “From N M2’s to N D2’s,” Phys. Rev. D78, 125007 (2008) [arXiv:0807.1444 [hep-th]].

[4] Y. Kim, O-K. Kwon, D. D. Tolla, “Mass-Deformed Super Yang-Mills Theories from M2-Branes with Flux,” JHEP 1109, 077 (2011). [arXiv:1106.3866 [hep-th]].

[5] S. Mukhi, “Unravelling the novel Higgs mechanism in (2+1)d Chern-Simons theories,” JHEP 1112, 083 (2011) [arXiv:1110.3048 [hep-th]].

[6] I. Jeon, N. Lambert and P. Richmond, “Periodic Arrays of M2-Branes,” [arXiv:1206.6699 [hep-th]].

[7] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, J. Park, “N=5,6 Superconformal Chern-Simons Theories and M2-branes on Orbifolds,” JHEP 0809, 002 (2008) [arXiv:0806.4977 [hep-th]].

[8] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, H. Verlinde, “A Massive Study of M2-brane Proposals,” JHEP 0809, 113 (2008) [arXiv:0807.1074 [hep-th]].

[9] N. Lambert, P. Richmond, “M2-Branes and Background Fields,” JHEP 0910, 084 (2009) [arXiv:0908.2896 [hep-th]].

[10] Y. Kim, O. K. Kwon, H. Nakajima and D. D. Tolla, “Interaction between M2-branes and Bulk Form Fields,” JHEP 1011, 069 (2010) [arXiv:1009.5209 [hep-th]].

[11] J. P. Allen and D. J. Smith, “Coupling M2-branes to background fields,” JHEP 1101, 069 (2011) [arXiv:1104.5397 [hep-th]].

[12] J. Polchinski, M. J. Strassler, “The String dual of a confining four-dimensional gauge theory,” [hep-th/0003136].

[13] I. Bena, “The M theory dual of a three-dimensional theory with reduced supersymmetry,” Phys. Rev. D62, 126006 (2000) [hep-th/0004142].

[14] I. Bena, A. Nudelman, “Warping and vacua of (S)YM(2+1),” Phys. Rev. D62, 086008 (2000) [hep-th/0005163].

[15] I. Bena, A. Nudelman, “Exotic polarizations of D2-branes and oblique vacua of (S)YM(2 +1),” Phys. Rev. D62, 126007 (2000). [hep-th/0006102].
[16] C. h. Ahn and T. Itoh, “Dielectric branes in nonsupersymmetric SO(3) invariant perturbation of three-dimensional N=8 Yang-Mills theory,” Phys. Rev. D 64, 086006 (2001) [arXiv:hep-th/0105044].

[17] S. Hyun, J.-H. Park and S.-H. Yi, “3-D N=2 massive superYang-Mills and membranes / D2-branes in a curved background,” JHEP 0303, 004 (2003) [hep-th/0301090].

[18] C. Kim, Y. Kim, O-K. Kwon and H. Nakajima, “Vortex-type Half-BPS Solitons in ABJM Theory,” Phys. Rev. D 80, 045013 (2009) [arXiv:0905.1759 [hep-th]].

[19] A. Gustavsson and S. J. Rey, “Enhanced N=8 Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2),” arXiv:0906.3568 [hep-th].

[20] O. K. Kwon, P. Oh and J. Sohn, “Notes on Supersymmetry Enhancement of ABJM Theory,” JHEP 0908, 093 (2009) [arXiv:0906.4333 [hep-th]].

[21] D. Bashkirov and A. Kapustin, “Supersymmetry enhancement by monopole operators,” JHEP 1105, 015 (2011) [arXiv:1007.4861 [hep-th]].

[22] H. Samtleben and R. Wimmer, “N=6 Superspace Constraints, SUSY Enhancement and Monopole Operators,” JHEP 1010, 080 (2010) [arXiv:1008.2739 [hep-th]].

[23] A. Kapustin, B. Willett and I. Yaakov, “Nonperturbative Tests of Three-Dimensional Dualities,” JHEP 1010, 013 (2010) [arXiv:1003.5694 [hep-th]]; D. Gang, E. Koh, K. Lee and J. Park, “ABCD of 3d $\mathcal{N} = 8$ and 4 Superconformal Field Theories,” arXiv:1108.3647 [hep-th].

[24] D. Gaiotto, A. Tomasiello, “The gauge dual of Romans mass,” JHEP 1001, 015 (2010) [arXiv:0901.0969 [hep-th]].

[25] M. Fujita, W. Li, S. Ryu and T. Takayanagi, “Fractional Quantum Hall Effect via Holography: Chern-Simons, Edge States, and Hierarchy,” JHEP 0906, 066 (2009) [arXiv:0901.0924 [hep-th]].

[26] O-K. Kwon and D. D. Tolla, “On the Vacua of Mass-deformed Gaiotto-Tomasiello Theories,” JHEP 1108, 043 (2011) [arXiv:1106.3700 [hep-th]].

[27] G. Go, O-K. Kwon and D. D. Tolla, “$\mathcal{N} = 3$ Supersymmetric Effective Action of D2-branes in Massive IIA String Theory,” Phys. Rev. D 85, 026006 (2012) [arXiv:1110.3902 [hep-th]].
[28] S. Cheon, H. C. Kim and S. Kim, “Holography of mass-deformed M2-branes,” arXiv:1101.1101 [hep-th].

[29] H. Lin, O. Lunin and J. M. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP 0410, 025 (2004) [hep-th/0409174].

[30] I. Bena and N. P. Warner, “A Harmonic family of dielectric flow solutions with maximal supersymmetry,” JHEP 0412, 021 (2004) [hep-th/0406145].

[31] K. Pilch and N. P. Warner, “N=2 supersymmetric RG flows and the IIB dilaton,” Nucl. Phys. B 594, 209 (2001) [hep-th/0004063].