EXPOSNENTIAL STABILIZATION OF CASCADE ODE-LINEARIZED KDV SYSTEM BY BOUNDARY DIRICHLET ACTUATION

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ABSTRACT. In this paper, we solve the problem of exponential stabilization for a class of cascade ODE-PDE system governed by a linear ordinary differential equation and a linearized Korteweg-de Vries equation (KdV) posed on a bounded interval. The control for the entire system acts on the left boundary with Dirichlet condition of the KdV equation whereas the KdV acts in the linear ODE by a Dirichlet connection. We use the so-called backstepping design in infinite dimension to convert the system under consideration into an exponentially stable cascade ODE-PDE system. Then, we use the invertibility of such design to achieve the exponential stability for the ODE-PDE cascade system under consideration by using Lyapunov analysis.

CONTENTS

1. Introduction
2. Problem Formulation and Main Result
3. Control Design
4. Proof of Theorem 2.1
   4.1. First step: Invertibility of \( \Omega \)
   4.2. Second step: Well Posedness
   4.3. Third step: Exponential Stability
References

1. INTRODUCTION

It is well known that the Korteweg-de Vries (KdV) equation in bounded domain models the dynamics of various types of extreme waves in shallow water, more particularly tsunami waves and freak waves. From theoretical point of view, the KdV controlled equation has

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some interesting control properties depending on where the controls are located \[8\], \[20\].

In the past decades, stabilization of coupled ODE-PDE systems is widely studied in the literature. Such systems can be used to model various processes such as road traffic \[11\], gas flow pipeline \[10\], power converters connected to transmission lines \[9\], oil drilling \[4\] and many other systems. Many problems of state and output feedback stabilization for coupled ODE-Heat has been solved \[21\], \[22\], \[14\] and ODE- Wave \[13\], \[23\], to cite few.

The problem of controllability of coupled ODE-PDE systems has been discussed in \[25\] and \[26\]. Some nonlinear extensions are studied in \[24\], \[2\], \[6\] where the non linearity is assumed to be global Lipshitz, and in \[5\], \[1\], \[7\] for more general nonlinear ODE. In this paper, we deal with the state stabilization problem for a cascade ODE-KdV system. We applied the infinite dimensional backstepping method to build a stabilizing feedback control for system (1)-(6). The backstepping method was introduced firstly for finite dimensional control systems governed by ODE \[12\]. The first extensions to PDE have appeared in \[3\] and \[17\]. Later, in \[19\] and \[15\], the authors have introduced an invertible integral transformation that transforms the original parabolic PDE into an asymptotically stable one. Recently, the backstepping method is used to design a feedback control law for coupled PDE-ODE (see the textbook \[16\] and references therein). As far as we know, problem of stabilization by backstepping design for such system when the PDE subsystem is governed by the linearized KdV equation has not yet been tackled in the literature. This paper is organized as follows. In Section 2, we present the main result of this paper which is summarized in Theorem 2.1. In section 3, we formulate the backstepping design of the feedback control law. Section 4 is devoted to prove Theorem 2.1 through three steps as follows. Firstly, we prove the invertibility of the transformation given in previous section. Secondly, we establish the well posedness of system (1)-(6). Finally, we prove the exponential stability in the sens of the $H$-norm of solutions of (1)-(6) around the origin.
2. Problem Formulation and Main Result

Let \( l > 0 \), we consider the following cascade ODE-PDE system

\[
\begin{align*}
\dot{X}(t) &= AX(t) + Bu(l,t), \quad t > 0, \\
u_t(x,t) &= -u_x(x,t) - u_{xxx}(x,t), \quad t > 0, \quad x \in (0,l), \\
&\quad u(0,t) = U(t), \quad t > 0, \\
u_x(l,t) &= 0, \quad t > 0, \\
u_{xx}(l,t) &= 0, \quad t > 0. \\
X(0) &= X_0, u(x,0) = u_0(x), \quad x \in (0,l).
\end{align*}
\]

where \( X(t) \in \mathbb{R}^n \) is the state of the ODE subsystem, \( u(x,t) \in \mathbb{R} \) is the state of the linear KdV subsystem, \( U(t) \in \mathbb{R} \) is the control input to the entire system acting in the left boundary \( x = 0 \) of the PDE domain \((0,l)\), and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1} \) such that the pair \((A, B)\) is controllable. The control objective is to exponentially stabilize the system (1)-(6) around its zero equilibrium. Along this paper, the Euclidean norm of a vector \( X \) in \( \mathbb{R}^n \) and the \( L^2 \)-norm of a function \( u \) in \( L^2(0,l) \) are denoted by \( |X| \) and

\[
\|u\| = \left( \int_0^l u^2(x) \, dx \right)^{\frac{1}{2}},
\]

respectively. Let \( H = \mathbb{R}^n \times L^2(0,l) \) the state space of the system (1)-(6). It is obvious that the vector space \( H \) equipped with its norm

\[
\|(X, u)\|_H = \left( |X|^2 + \|u\|^2 \right)^{\frac{1}{2}},
\]

is a Hilbert space. The infinite dimensional backstepping design for coupled ODE-PDE system is to seek a continuous and invertible integral transformation

\[
\Omega : H \to H
\]

\[
(X, u) \mapsto (X, w)
\]
to convert system (1)-(6) into the exponentially stable target system

\[ \dot{X}(t) = (A + BK)X(t) + Bw(l, t), \quad t > 0, \]
\[ w_t(x, t) = -w_x(x, t) - w_{xxx}(x, t) - \lambda w(x, t), \quad t > 0, \quad x \in (0, l), \]
\[ w(0, t) = 0, \quad t > 0, \]
\[ w_x(l, t) = 0, \quad t > 0, \]
\[ w_{xx}(l, t) = 0, \quad t > 0. \]
\[ X(0) = X_0, w(x, 0) = w_0, \quad x \in (0, l), \]

where \( K \in \mathbb{R}^{1 \times n} \) such that \( A + BK \) is Hurwitz, and \( \lambda \) is an arbitrary positive number. Define the dense subspace \( \Lambda \) of \( H \) by

\[ \Lambda = \mathbb{R}^n \times \{ u \in H^3(0, l) \mid u(0) = u'(l) = u''(l) = 0 \}. \]

Our main result is stated in the following theorem which asserts that the cascade ODE-PDE system (1)-(6) is exponentially stable in the sense of the norm \( || \cdot ||_H \).

**Theorem 2.1.** For any initial condition \((X_0, u_0) \in \Lambda\), the closed loop system (1)-(6) with the feedback law (31) admits a unique classical solution

\[ (X, u) \in C([0, +\infty), \Lambda) \cap C^1([0, +\infty); H) \]
and if \((X_0, u_0) \in H\) then the system admits a unique mild solution

\[ (X, u) \in C([0, +\infty), H). \]

Moreover, there exists two constants \( c_1 > 0 \) and \( c_2 > 0 \) such that for all \((X_0, u_0) \in H\) then

\[ ||(X(t), u(., t))||_H \leq c_1 ||(X_0, u_0)||_H e^{-c_2 t}, \quad \forall t \geq 0. \]

3. **CONTROL DESIGN**

The transformation \( \Omega \) is postulated in the following form:

\[ X(t) = X(t), \]
\[ w(x, t) = u(x, t) - \int_x^l u(y, t)q(x,y)dy - \varphi(x)X(t), \]
where the kernel $q(x, y) \in \mathbb{R}$ and the gain function $\varphi(x)^T \in \mathbb{R}^n$ are to be determined. From (17), the first three derivatives of $w(x, t)$ with respect to $x$ are given by

$$w_x(x, t) = u_x(x, t) + q(x, x)u(x, t) - \int_x^l u(y, t)q_x(x, y)dy - \varphi'(x)X(t),$$

(18)

$$w_{xx}(x, t) = u_{xx}(x, t) + u(x, t)\frac{d}{dx}q(x, x) + q(x, x)u_x(x, t) + q_x(x, x)u(x, t) - \int_x^l u(y, t)q_{xx}(x, y)dy - \varphi''(x)X(t),$$

(19)

and

$$w_{xxx}(x, t) = u_{xxx}(x, t) + u(x, t)\frac{d^2}{dx^2}q(x, x) + 2u_x(x, t)\frac{d}{dx}q(x, x) + q(x, x)u_{xx}(x, t) + u(x, t)\frac{d}{dx}q_x(x, x)$$

$$+ q_x(x, x)u_x(x, t) + q_{xx}(x, x)u(x, t) - \int_x^l u(y, t)q_{xxx}(x, y)dy - \varphi'''(x)X(t),$$

(20)

respectively. Furthermore, the derivative of (17) with respect to $t$ is given by

$$w_t(x, t) = u_t(x, t) - \int_x^l u_y(y, t)q(x, y)dy - \varphi(x)X(t),$$

(21)

$$= u_t(x, t) + \int_x^l (u_y(y, t) + u_{yyy}(y, t))q(x, y)dy - \varphi(x)(AX(t) + Bu(l, t)).$$

Integrating by parts the above identity, we get

$$w_t(x, t) = u_t(x, t) - \int_x^l u(y, t)(q_y(x, y) + q_{yyy}(x, y))dy - q(x, x)(u(x, t) + u_x(x, t))$$

$$+ q_y(x, x)u_t(x, t) - q_{yy}(x, x)u(x, t) + (q(x, t) + q_{yy}(x, l))u(l, t)$$

$$- \varphi(x)(AX(t) + Bu(l, t)),$$

(22)

where the boundary conditions (4) and (6) has been used. Thus, from (17), (18), (20) and (22), for all $\lambda > 0$, the identity

$$w_t(x, t) + w_x(x, t) + w_{xxx} + \lambda w(x, t) = - \int_x^l u(y, t)(q_{xxx}(x, y) + q_{yyy}(x, y) + q_x(x, y) + q_y(x, y) + \lambda q(x, y))dy$$

$$+ (q(x, l) + q_{yy}(x, l) - \varphi(x)B)u(l, t)$$

$$+ u_x(x, t)(q_x(x, x) + q_x(x, x) + 2\frac{d}{dx}q(x, x))$$

$$+ u(x, t)(\lambda + q_{xx}(x, y) + \frac{d}{dx}q_x(x, x) + \frac{d^2}{dx^2}q(x, x))$$

$$- (\varphi(x)(A + \lambda I_n) + \varphi''(x) + \varphi'''(x))X(t),$$

(23)
holds. Moreover, setting \( x = 0 \) in (17) and \( x = l \) in (17), (18) and (19), it follows

\[
(24) \quad w(0,t) = U(t) - \int_0^t q(0,y)u(y,t)dy - \varphi(0)X(t),
\]

\[
(25) \quad w(l,t) = u(l,t) - \varphi(l)X(t),
\]

\[
(26) \quad w_x(l,t) = q(l,l)u(l,t) - \varphi'(l)X(t),
\]

\[
(27) \quad w_{xx}(l,t) = \frac{dq}{dx}(l,l)u(l,t) + q_x(l,l)u(l,t) - \varphi''(l)X(t).
\]

Assume that the gain function \( \varphi(x)^T \) defined in \([0,l]\) and the kernel \( q(x,y) \) defined in the triangle

\[
T = \{(x,y) \mid x \in [0,l], y \in [x,l]\}
\]

satisfy

\[
\varphi'''(x) + \varphi'(x) + \varphi(x)(A + \lambda I_n) = 0, \quad x \in [0,l],
\]

\[
\varphi(l) = K,
\]

\[
\varphi'(l) = 0,
\]

\[
(29)
\]

and

\[
q_{xxx}(x,y) + q_{yyy}(x,y) + q_x(x,y) + q_y(x,y) = -\lambda q(x,y), \quad (x,y) \in T,
\]

\[
q(x,l) + q_{yy}(x,l) = \varphi(x)B, \quad x \in [0,l],
\]

\[
q(x,x) = 0, \quad x \in [0,l],
\]

\[
q_x(x,x) = \frac{\lambda}{3}(l-x), \quad x \in [0,l],
\]

(30)

respectively. Then, from the identities (23)-(27), we obtain the target system (9)-(14) for every solution of the closed loop (1)-(6) with the feedback law

\[
(31) \quad U(t) = \int_0^t q(0,y)u(y,t)dy + \varphi(0)X(t).
\]

Notice that the existence of the control law (31) is a direct consequence of the existence of kernels \( q(x,y) \) and \( \varphi(x)^T \) that satisfy (30) and (29), respectively. In the following, we compute explicitly the kernel \( \varphi(x)^T \) and we prove the existence of the kernel \( q(x,y) \). To begin with, the solution of the ODE (29) is

\[
(32) \quad \varphi(x) = (K,0,0)e^{(x-l)M}E,
\]
where \( M \) and \( E \) are the constant matrices

\[
M = \begin{pmatrix}
0 & 0 & -\left( A + \lambda I_n \right)
\end{pmatrix}, \quad E = \begin{pmatrix}
I_n
\end{pmatrix}.
\]

On the other hand, to prove the existence of the kernel \( q(x,y) \), let us make the change of variables

\[
s = x + y, \quad t = y - x,
\]

and define

\[
G(s,t) := q(x,y) = q\left( \frac{s-t}{2}, \frac{s+t}{2} \right).
\]

Then, the function \( G \) defined in the triangle \( T_0 = \{(s,t) \in [0, l], s \in [t, 2l-t]\} \), satisfies

\[
(6G_{tt} + 2G_{ss} + 2G_s)(s,t) = -\lambda G(s,t), \quad (s,t) \in T_0,
\]

\[
(G + G_{st} + 2G_{s2l-s}) = \phi(s-l)B, \quad s \in [l, 2l],
\]

\[
G(s,0) = 0, \quad s \in [0, 2l],
\]

\[
G_t(s,0) = \frac{\lambda}{6} (s-2l), \quad s \in [0, 2l].
\]

To prove the existence of the kernel \( q(x,y) \), we adopt the proof done by \[8, \text{Cerpa-Coron}\] with slight modification. At this stage, it is conventional to put the previous system in an integral form (see for example \[8\]). Using (34), we rewrite (33) in variable \((\eta, \xi)\) as follows

\[
(6G_{tt} + 6G_{ss} + 12G_{st} + 6G_s)(\eta, \xi) = (-\lambda G + 4G_{ss} + 12G_{st} + 4G_s)(\eta, \xi).
\]

Next, we integrate (37) with respect to the variable \( \eta \) over the interval \((s, 2l - \xi)\) and use the boundary condition (34). Then, we integrate the obtained identity with respect to \( \xi \) over the interval \((0, \tau)\) and use (35) and (36). Finally, by integrating the acquired identity with respect to \( \tau \) over the interval \((0, t)\) and use again the boundary condition (35), it follows

\[
G(s,t) = \int_0^t \int_0^{s} \phi(l - \xi)Bd\xi d\tau - 2 \int_0^t G_s(s,\tau)d\tau - \int_0^t \int_0^{s} (G_{ss} + G)(s,\xi)d\xi d\tau \]

\[
- \frac{1}{6} \int_0^t \int_0^{2l-\xi} (4G_{ss} + 4G_s - \lambda G + 12G_{st})(\eta, \xi)d\eta d\xi d\tau.
\]
In order to achieve the existence of the function $G$, we use the classical method of successive approximations \cite{18}. To this end, set

$$
G^1(s, t) = -\frac{\lambda t}{6} (2l - s),
$$

$$
G^{n+1}(s, t) = \theta(t) - 2 \int_0^t G^n(s, \tau) d\tau - \int_0^t \int_0^\tau (G^n(s + G^n(s, \xi)) d\xi d\tau
$$

(39)

$$
\frac{1}{6} \int_0^t \int_0^\tau \int_s^\tau (4G^n_{ss} + 4G^n_s - \lambda G^n + 12G^n_{ss}) \phi(l, \xi) d\eta d\xi d\tau,
$$

where

(40) \quad \theta(t) = \int_0^t \int_0^\tau \phi(l, \xi) B d\xi d\tau.

Using an appropriate calculation, we get

$$
G^2(s, t) = \theta(t) + \frac{1}{6} \left( -\lambda t^2 + \left( \frac{\lambda l}{9} - \frac{\lambda^2 t^2}{18} \right) t^4 + \frac{\lambda^2}{18} t^4 + \frac{\lambda^2}{240} t^5 + \frac{186 - \lambda}{18} \right) s^3 - \frac{\lambda^2}{72} s^2 t^3.
$$

Since $\phi$ is a continuous function on $[0, l]$, from (40), there exists a constant $\rho > 0$ such that $|\theta(t)| \leq \rho t^2$. Keeping this in mind, and using the classical procedure as in \cite{3}, we get the following inequality

$$
|G^k(s, t)| \leq \sum_{i=0}^{k} \left( \sum_{j=0}^{3k-i} a_{ij} t^j \right) s^i,
$$

for all $k \in \mathbb{N}^*$, where the positive coefficients $a_{ij}$ have appropriate decay properties so that the series $\sum_{n=1}^{\infty} G^n(s, t)$ is uniformly convergent in $T_0$. We underline that the uniform convergence of the previous series on the domain $T_0$ is due to the fact that the recursive identity is a function of some integral operators. Therefore, the series defines a continuous function

$$
G(s, t) = \sum_{n=1}^{\infty} G^n(s, t), \forall (s, t) \in T_0.
$$

Thus, the proof of the existence of $G(s, t)$ solution of the integral equation (38) is achieved. Having proved the existence of $G(s, t)$ in $T_0$, that of $q(x, y)$ in $T$ follows immediately. In consequence, the state feedback controller (31) is well defined.

Let’s move to the proof of the main Theorem 2.1.

4. PROOF OF THEOREM 2.1

The proof is divided in three steps. In the first step, we prove the invertibility of the backstepping transformation $\Omega$ defined by \cite{8}. The existence and the global exponential...
stabilization of the solution to the closed loop system with the state feedback are established in the second and the third step, respectively.

4.1. First step: Invertibility of $\Omega$. It is straightforward that the transformation $\Omega$ is invertible, and

\[ \Omega^{-1} : H \to H \]
\[ (X, w) \mapsto (X, u) \]

has the following form

\[ X(t) = X(t) \]
\[ u(x, t) = w(x, t) + \int_x^l w(y, t) h(x, y) dy + \psi(x) X(t), \]

where the kernel function $h(x, y) \in \mathbb{R}$ and the gain function $\psi(x)^T \in \mathbb{R}^n$ are to be determined. As is done in the study of the direct transformation $\Omega$, the derivatives $u_t, u_x, u_{xxx}$ are computed and system holds if $h(x, y)$ satisfies

\[ h_{xxx}(x, y) + h_{yyy}(x, y) + h_x(x, y) + h_y(x, y) = \lambda h(x, y), \]
\[ h(x, l) + h_{xy}(x, l) = -\psi(x) B, \]
\[ h(x, x) = 0, \]
\[ h_x(x, x) = \frac{\lambda}{3} (l - x), \]

in the triangle $T$ defined by $|x| \leq |y| \leq |x| + |y|$ and $\psi(x)$ satisfies in $[0, l]$

\[ \psi'''(x) + \psi''(x) + \varphi(x)(A + BK) = 0, \]
\[ \psi(l) = K, \]
\[ \psi'(l) = 0, \]
\[ \varphi''(l) = 0. \]

First of all, the solution of the ODE is

\[ \psi(x) = (K, 0, 0) e^{(x-l)N} E, \]
where $N$ and $E$ are two constant matrices given by

$$
N = \begin{pmatrix}
0 & 0 & -(A + BK) \\
I_n & 0 & -I_n \\
0 & I_n & 0
\end{pmatrix},
E = \begin{pmatrix}
I_n \\
0 \\
0
\end{pmatrix}.
$$

Then, in the same way as for the kernel $q(x, y)$, one can easily prove the existence and the continuity of the kernel $h(x, y)$ in $T$. Moreover, $\Omega^{-1}$ is continuous operator on the Hilbert space $H$. This achieves the first step of the proof of Theorem 2.1.

### 4.2. Second step: Well Posedness.

Since $\Omega : H \rightarrow H$ is a continuous invertible transformation, $\Omega^{-1}$ maps a trajectory of (9)-(13) into a trajectory of (1)-(6) with the state feedback controller (31). Hence, to prove that (1)-(6) is well posed, it suffices to establish the well posedness of the target system (9)-(14). To this reason, we consider the target KdV subsystem (10)-(14) in the Hilbert state space $L^2(0, l)$ and, we define the unbounded linear operator $\Gamma : \mathcal{D}(\Gamma) \rightarrow L^2(0, l)$ by

$$
\Gamma(w) = -\lambda w - w' - w''
$$

with domain $\mathcal{D}(\Gamma) = \{w \in H^3(0, l) \mid w(0) = w'(l) = w''(l) = 0\}$. A simple computation shows that the adjoint operator $\Gamma^*$ of $\Gamma$ is

$$
\Gamma^*(w) = -\lambda w + w' + w'',$n

with domain $\mathcal{D}(\Gamma^*) = \{w \in H^3(0, l) \mid w(0) = w'(0) = 0, w(l) + w''(l) = 0\}$. It is obvious that the domains $\mathcal{D}(\Gamma)$ and $\mathcal{D}(\Gamma^*)$ are dense subspaces in $L^2(0, l)$. Moreover, by integrations by parts, we get

$$
< \Gamma(w), w > = -\frac{1}{2} (w(l)^2 + w'(0)^2) - \lambda \|w\|^2 \leq 0, \forall w \in \mathcal{D}(\Gamma),
$$

$$
< \Gamma^*(v), v > = -\frac{1}{2} (v(l)^2 + v'(l)^2) - \lambda \|v\|^2 \leq 0, \forall v \in \mathcal{D}(\Gamma^*),
$$

where $<,>$ stands for the $L^2(0, l)$ standard inner product. Thus, both operators $\Gamma$ and $\Gamma^*$ are dissipative. Therefore, according to the Lumer-Phillips Theorem’s, the operator $\Gamma$ generates a $C^0$ semigroup of contraction in $L^2(0, l)$. Consequently, for all initial condition $w_0 \in \mathcal{D}(\Gamma)$, the system (10)-(14) has a unique classical solution

$$
w \in C\left([0, +\infty); \mathcal{D}(\Gamma)\right) \cap C^4\left([0, +\infty); L^2(0, l)\right).
$$
Furthermore, by Duhamel formula’s, for all initial condition \( X_0 \in \mathbb{R}^n \), the ODE (9) has a unique global solution
\[
X(t) = e^{(A+BK)}X_0 + \int_0^t e^{(t-\tau)(A+BK)}Bw(l,\tau)d\tau.
\]
We conclude that the target system (9)-(14) is well posed in \( H \), and for all initial condition \( (X_0,w_0) \in \Lambda \), system (9)-(14) has a unique classical solution
\[
(X,w) \in C([0, +\infty); \Lambda) \cap C^1([0, +\infty); H),
\]
and for \( (X_0,w_0) \in H \), the system (9)-(14) has a unique mild solution
\[
(X,w) \in C([0, +\infty); H).
\]

4.3. Third step: Exponential Stability. Consider the Lyapunov function candidate
\[
V(t) = X(t)^T PX(t) + \frac{\mu}{2}\|w(.,t)\|_{L^2(0,l)}^2,
\]
where \( \mu > 0 \) is a constant to be designed later and the positive definite matrix \( P = P^T > 0 \) is the solution of the Lyapunov equation
\[
P(A + BK) + (A + BK)^TP = -Q,
\]
for some positive definite matrix \( Q = Q^T > 0 \). From (49), it can be obtained that for all \( t \geq 0 \),
\[
\alpha_1\|X(t),w(.,t)\|_{H}^2 \leq V(t) \leq \alpha_2\|X(t),w(.,t)\|_{H}^2,
\]
where
\[
\alpha_1 = \min \left( \lambda_{\text{min}}(P), \frac{\mu}{2} \right) \text{ and } \alpha_2 = \max \left( \lambda_{\text{max}}(P), \frac{\mu}{2} \right).
\]
The derivative of \( V \) along the solutions of (9)-(14) is given by
\[
\dot{V}(t) = -X^T(t)QX(t) + 2X(t)^TPBw(l,t) - \mu w(l,t)^2 - \mu \lambda \|w(.,t)\|^2.
\]
By Young’s inequality, we get
\[
2X(t)^TPBw(l,t) \leq \frac{\lambda_{\text{min}}(Q)}{2}\|X(t)\|^2 + \frac{2}{\lambda_{\text{min}}(Q)}|PB|^2 w(l,t)^2.
\]
Thus,
\[
\dot{V}(t) \leq -\frac{\lambda_{\text{min}}(Q)}{2}\|X(t)\|^2 - \left( \mu - \frac{2}{\lambda_{\text{min}}(Q)}|PB|^2 \right)w(l,t)^2 - \mu \lambda \|w(.,t)\|^2.
\]
Now, using (50) and choose
\[ \mu > \text{Max} \left( \frac{2}{\lambda_{\text{min}}(Q)} |PB|^2, 2\lambda_{\text{max}}(P) \right). \]

Then, the inequality
\[ \dot{V}(t) \leq -\delta V(t), \]
holds for all \( t \geq 0 \), where \( \delta = \min(\lambda_{\text{min}}(Q), 4\lambda) \). Therefore,
\[ V(t) \leq V(0)e^{-\delta t}, \forall t \geq 0. \]

Using (50) and (52), by tacking \( \alpha = \frac{\alpha}{\alpha_1} \), it follows that
\[ ||(X(t), w(\cdot, t))||_H^2 \leq \alpha ||(X_0, w_0)||_H^2 e^{-\delta t}, \forall t \geq 0. \]

Recall that the transformations \( \Omega \) and \( \Omega^{-1} \) are linear and continuous, then there exist two positive constants \( d_1 \) and \( d_2 \) such that
\[ \|(X, w)||_H = \|\Omega(X, u)||_H \leq d_1 ||(X, u)||_H, \]
\[ \|(X, u)||_H = \|\Omega^{-1}(X, w)||_H \leq d_2 ||(X, w)||_H. \]

Hence, for all initial condition \( (X_0, u_0) \in H \), for all \( t \geq 0 \), we obtain from (53), (54) and (55)
\[ ||(X(t), u(\cdot, t))||_H \leq c_1 ||(X_0, u_0)||_H e^{-c_2 t}, \]
where \( c_1 = d_1 d_2 \sqrt{\alpha} \) and \( c_2 = \delta \). Thus, the proof of Theorem 2.1 is complete.

**Remark 4.1.** Since the positive definite matrix \( Q \) and the positive parameter \( \lambda \) are arbitrary, the decay rate \( c_2 \) in (15) is arbitrary. Thus, the system (1)-(6) is rapidly exponentially stable.

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