GETTING INTO THE VORTEX:
ON THE CONTRIBUTIONS OF JAMES MONTALDI

JAIR KOILLER

Departamento de Matemática da UFJF
Cidade Universitária, Juiz de Fora, 36036, Brazil

Abstract. James Montaldi’s expertises span many areas on pure and applied mathematics. I will discuss here just one, his contributions to the motion of point vortices, specially the role of symmetries in the bifurcations and stability of equilibrium configurations in surfaces of constant curvature. This approach leads, for instance, to a very elegant proof of a classical result, the nonlinear stability of Thompson’s regular heptagon in the plane. Here the plane appears “in passing”, just as the transition between positive and negative curvatures.

James Montaldi’s expertises span many areas on pure and applied mathematics but modestly, he mentions only five of them in his web page: (i) bifurcations and symmetry breaking, (ii) Lie groups, (iii) Hamiltonian systems with group symmetry, (iv) momentum maps and (v) conservation laws (Fig.1).

However there are many more areas of his interest: (vi) fractals; (vii) singularity theory and caustics, (viii) Hodge theory, (ix) Equivariant cohomology, (x) flag manifolds, (xi) measure theory in infinite dimensions, (xii) Lyusternik-Schnirelmann category and, last but not least, several themes related to direct applications. Among them, (xiii) nonlinear normal modes, (xiv) nonholonomic mechanical systems, (xv) point vortices, and (xvi) celestial mechanics.

I will discuss here just one, motion of point vortices on surfaces. James’ papers have focused on the role of symmetries for the sphere and for the hyperboloid. The plane appears “in passing”, just a transition between positive and negative curvatures. This approach leads, for instance, to a very elegant proof of the nonlinear stability of Thompson’s heptagon. The cylinder with the flat metric was also studied by James, with new results on vortex streets.

For those non familiar with the area, I suggest starting with the survey [3] Aref (2007), frequently cited in James’ papers as basic reference, a commented bibliography [35] Meleshko and Aref (2007a), and with focus on foundations, the survey [33] Llewellyn Smith (2011). See also the short article [46] by Paul Newton (2014) with some of the current open problems, and his book [45]. For vortices on surfaces, see [7] Boatto and Koiller (2015). Geometric function theory aspects are discussed in [18] Gustafsson (2018), namely the role of the chosen metric on the conformal class of the complex structure of the Riemann surface. Let’s now get into the Vortex!

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Figure 1. James’ interests and eight Erdős paths

Figure 2. An ninth Erdős path
Relative equilibria of point vortices on the sphere [MR1811389
[Chjan Lim, James Montaldi and Mark Roberts (2001)]

Point vortices on the sphere: stability of symmetric relative
equilibria [MR2888015]
[Frédéric Laurent-Polz, Mark Roberts, and James Montaldi (2011)]

These two papers above stand out in a (by now) vast bibliography, and keep
producing many descendants, see e.g., [15] García-Azpeitia (2019). For obvious
reasons, studying vortices on the sphere is very important.¹

Classifying in detail what are the possible patterns in the presence of symmetries,
their stability and bifurcations, is like the work of a botanist. A botanist is a highly
educated and skilled scientist. To be able to do it requires an appropriate education,
persistence, attention to detail and evaluating numerous aspects of life.

The first paper, marking James Montaldi’s première as vorticitist, is recognized
(see e.g., [1] Aref et al. (2003)) as a fundamental work. The latter, a much awaited
sequel, appeared ten years later in the special volume of JGM in honor of the 60th
birthday of Tudor Ratiu.

This second paper develops stability results from Laurent-Polz’s dissertation
under James’s direction [28] Laurent-Poltz (2002). A number of other results from
his thesis were published years before, as a sole author [30] Laurent-Polz (2001),
[31] Laurent-Polz (2004), [29] Laurent-Polz (2005).

The novel techniques for stability analysis introduced in the thesis were further
refined in [44] Montaldi and Tokieda (2013), and about this I will comment below.

I will just briefly comment on some aspects of the first paper. Firstly, the time-
reversing reflectional symmetry allows, with due care, extension of the continuous
symmetries from SO(3) to O(3), acting diagonally.

Indeed, accounting for the finite symmetry group of permutations S of the identi-
tical vortices present in the system, one can extend the group of symmetries to
\( \hat{G} = O(3) \times S \).

An important ingredient for the analysis are the invariant submanifolds: the
fixed-point sets of subgroups \( \Sigma \subset O(3) \times S \). Orbit type lattices also play another
important role for the bifurcation diagrams.

Tables 1–5 are impressive. The examples for \( N \leq 6 \) equal vortices in section 5
speak for themselves. The figures in the next page are meant just to illustrate the
delicate analysis, that applies also to other problems having \( \hat{G} \) as symmetry group
[39] Montaldi (1997), [38] Montaldi and Roberts (1999).

I urge the reader to look also at the supplementary information in his web site

https://www.maths.manchester.ac.uk/~jm/wiki/Vortices/Stability

that I advertise in the format of a haiku:

One ring of \( n \) point vortices
One ring and a pole, or two poles
Two rings, aligned or staggered.

¹But it would be an exaggeration to say that a Plano-vorticitist is also a Plano-terrestrialist.
| Equilibrium | Bifurcating relative equilibria | No. | Orbit relations |
|------------|--------------------------------|-----|----------------|
| $C_{nr}$   | $C_{nr}(k_a, k_b, k_c, k'_f)$ | 1, 1| $k'_f = \frac{1}{2}(n-1)(k_a + k_b) + n k_c$ |
|            | $C_{a}(k_a', k'_b)$ $n$ odd   |     | $k'_b = k'_f + k'_a$ |
|            | $C_{b}(k'_a', k'_b)$ $n$ even  |     | $k'_a = \frac{1}{2}n k_a - \frac{1}{2}k_c + k_c$ |
|            | $C_{h}(k'_a', k'_b, k'_c)$ $n$ even |     | $k'_b = 2k_a + k_c$ |
| $D_{4h}$   | $C_{nr}(k_a, k_b, k_c, k'_f)$ | 2   | $k'_f = -2k_a + k_c$, $k'_b = 2k_a$ |
|            | $C_{a}(k_a', k'_b)$ $n$ odd   |     | $k'_b = n k_a + \frac{1}{2}(n-1)(k_a + k'_b)$ |
|            | $C_{b}(k'_a', k'_b)$ $n$ even  |     | $k'_a = k_a + k_b + k_c + k_f - k_f - k_c$ |
|            | $C_{h}(k'_a', k'_b, k'_c)$ $n$ even |     | $k'_b = n k_a + \frac{1}{2}(n-2)k_c + \frac{1}{2}n k'_c$ |
| $T_d$      | $C_{nr}(k_a, k_b, k_c, k'_f)$ | 4, 4| $k'_f = 4k_a + k_c$, $k'_b = k_c$ |
|            | $C_{a}(k_a', k'_b)$ $n$ odd   |     | $k'_b = 2k_a + k_c + k_c$, $k'_b = k_c$ |
|            | $C_{b}(k'_a', k'_b)$ $n$ even  |     | $k'_a = k_a + 2k_c + 4k_f + k_f - 2k_a$ |
|            | $C_{h}(k'_a', k'_b, k'_c)$ $n$ even |     | $k'_b = k_a' + k_c$ |
| $G_2$      | $C_{nr}(k_a, k_b, k_c, k'_f)$ | 6   | $k'_f = 6k_a + 4k_c + 2k_f$, $k'_b = 2k_a$ |
|            | $C_{a}(k_a', k'_b)$ $n$ odd   |     | $k'_b = 2k_a + 2k_c + 4k_f$ |
|            | $C_{b}(k'_a', k'_b)$ $n$ even  |     | $k'_a = 8k_a + 4k_c + 2k_f + k_c$, $k'_f = 2k_f$ |
|            | $C_{h}(k'_a', k'_b, k'_c)$ $n$ even |     | $k'_b = 2k_a + 2k_c + k_f + k_c$ |
| $I_4$      | $C_{nr}(k_a, k_b, k_c, k'_f)$ | 12  | $k'_f = 12k_a + 4k_c + 4k_f + k_c$, $k'_f = 4k_f$ |
|            | $C_{a}(k_a', k'_b)$ $n$ odd   |     | $k'_b = 4k_a + 2k_c + 2k_f$ |
|            | $C_{b}(k'_a', k'_b)$ $n$ even  |     | $k'_a = 12k_a + 4k_c + 4k_f + k_c$, $k'_f = 4k_f$ |
|            | $C_{h}(k'_a', k'_b, k'_c)$ $n$ even |     | $k'_b = 2k_a + 2k_c + 2k_f$ |
| $I_8$      | $C_{nr}(k_a, k_b, k_c, k'_f)$ | 12  | $k'_f = 2k_a + 2k_c + 2k_f$ |
|            | $C_{a}(k_a', k'_b)$ $n$ odd   |     | $k'_b = 2k_a + 2k_c + 2k_f$ |
|            | $C_{b}(k'_a', k'_b)$ $n$ even  |     | $k'_a = 2k_a + 2k_c + 2k_f$ |
|            | $C_{h}(k'_a', k'_b, k'_c)$ $n$ even |     | $k'_b = 2k_a + 2k_c + 2k_f$ |

**Figure 3.** Types of relative equilibria bifurcating from equilibria. Table 5 from [Lim, Montaldi and Roberts (2001)].
Figure 4. Also Fig. 4 in the paper. The caption says: "Orbit type lattice for the action of $O(3) \times S_N$ for $N = 3 - 5$ identical vortices. The underlined strata are those which we prove contain relative equilibria for any $G$-invariant Hamiltonian. The strata marked with a dagger ($\dagger$) do not contain any relative equilibria for the point-vortex Hamiltonian. See Section 5 [of the paper] for summaries of these existence and non-existence results."
Vortex dynamics on a cylinder
[Montaldi, Soulière, and Tokieda (2003)] MR2031280

This paper, published in SIAM J. Applied Dynamical Systems, was part of Anik Soulière’s thesis ([50], 2007) under Tadashi Tokieda.

The word “cylinder” is used in a metaphorical sense. One thinks of a finite number of point vortices inside a strip in the $z = x + iy$ plane, say, $0 \leq x < L$, and then one repeats the strip by translations $x \rightarrow x + nL$, $n \in \mathbb{Z}$.

So each vortex on the strip becomes a street.

First of all, a bit of motivation. The history and importance of Kármán’s vortex street may not be well known in our community. Kármán streets are ubiquitous, not only in Nature (see e.g., how trouts swim [32] Liao et al. (2003)) — but also in technology [17] Govardhan and Ramesh (2005) — there is an enormous amount of experimental work about them.

![Figure 5. Image from NASA](https://earthobservatory.nasa.gov/images/90734/two-views-of-von-karman-vortices)

Streets formed by two opposite vortices were first studied by Theodore von Kármán in 1911 and 1912 [53] motivated by a troubled experiment about the wake on flows past a (here a real) cylinder that was being done in L. Prandtl’s lab.

Kirchhoff and Rayleigh conceived the wake behind the cylinder as two vortex sheets bounding an infinite mass of “dead fluid” following the body. In hindsight this situation is clearly not physically realizable.

“What I really contributed to the aerodynamic knowledge of the observed phenomenon is twofold: I think I was the first to show that the symmetric arrangement of vortices (upper), which would be an obvious possibility to replace the vortex sheet is unstable. I found that only the asymmetric arrangement (lower) could be stable, and only for a certain ratio of the distance between the rows and the distance between two consecutive vortices of each row. Also, I connected the momentum carried by the vortex system with the drag and showed how the creation of such a vortex system can represent the mechanism of the wake drag” [54] von Kármán (1963).
In his 1911–1912 papers von Kármán computed the linearization of an infinite discrete system in a simplified but very clever way.\textsuperscript{3}

In the section \textit{Wake drag} of von Kármán’s delicious book (pages 67-73) he tells how he got interested in the problem — this is mandatory reading!

Von Kármán, working as a graduate assistant with Prandtl in Göttingen, was thirty years old. Hiemenz, a graduate student was having trouble in producing a steady flow behind the cylinder. It was necessary to know the pressure distribution around the cylinder in a steady flow in order to find the separation point in the boundary layer.

Sadly, Hiemenz found instead that the flow oscillated violently.

Well then, if the flow always oscillates, von Kármán said to himself, then “this phenomenon must have a natural and intrinsic reason.”

Von Kármán also mentions in his book of having seen, many years later, the vortex street phenomenon depicted in a painting of a church in Bologna.

“St. Christopher is shown carrying the child Jesus across a flowing stream. Behind the saint’s naked foot the painter indicated alternating vortices.”

Which church and when was this? There is a controversy concerning this \textsuperscript{[37]} Mizota et al. (2000), \textsuperscript{[16]} Gibbs (2000).

Going now to mathematics. One finds in Lamb’s \textit{Hydrodynamics} the flow due to an infinite number of equally separated vortices on a line with the same strength. The method goes back to Maxwell’s \textit{Electricity and Magnetism}, and is equivalent to defining the Green function for the cylinder. Using this approach, two important papers from the 1990’s are \textsuperscript{[2]} Aref (1995), \textsuperscript{[4]} Aref and Stremler (1996). In the first, a \textit{single} vortex street is considered, a discrete version of a vortex sheet.

In the second paper, \textit{three} streets are considered, under the assumption that the sum of the three vorticities vanish, a phenomenon also seen in the far-wake of an uniform flow past an \textit{oscillating} cylinder. The zero total vorticity constraint allows one to reduce the problem to 2-degrees of freedom, and the translational symmetry guarantees Arnold–Liouville integrability. Aref and Stremler showed that there is a great abundance of motion regimes.

\textsuperscript{3}It is discussed in some little more detail in \S156 of \textsuperscript{[27]} Lamb (1993 (1932). Von Karman first showed that a \textit{single} row is always unstable. This later became another a classic problem in fluid mechanics, the \textit{rolling} of a vortex sheet, going back to \textsuperscript{[47]} Rosenhead (1931); a recent reference is \textsuperscript{[13]} DeVoria and Mohseni (2018).
The article by Montaldi, Soulière and Tokieda takes a new, quite different direction. They apply general principles and use insightful geometric ideas to infer a number of qualitative results for an arbitrary number of vortex streets.

In fact, this article discusses more generally the possible types of relative equilibria and periodic solutions of $N$ vortices on the cylinder.

First, they recall in section 2 how one gets the Hamiltonian for the cylinder. In section 3 they show that, if the sum of the vorticities of the base vortices is nonzero, then the relative equilibria are in fact static equilibria. They give a classification of static equilibria when all vorticities have the same sign. Section 4 deals with relative equilibria when the sum of the vorticities is zero. For $N = 3$ they give conditions for relative equilibria.

In the case of four vortices, several results are established for “leapfrogging” periodic solutions. The numerical simulations are quite interesting.

The proofs combine a clever mix of topological and analytical arguments: the interplay of local behavior (inside a strip) and global, more intricate behavior, for instance, jumping between strips.

Stroboscopically, one does not know (or care) what is really happening. In other words, “que hacen las gentes cuando las luces se apagan?”

One final comment. Forcing the vortex streets to keep the periodic pattern, i.e., to have the whole vortex system always covering the cylinder, the ultimate symmetry breaking bifurcation is precluded, by default. This is actually implied in von Kármán’s work as well.

Therefore, there is a natural question more or less indicated in the paper. Consider a parameter dependent system with (continuous or discrete) symmetry that is imbedded in a more general physically motivated system.

As the parameter varies, a series of bifurcation instabilities may take place. Is symmetry breaking always the last bifurcation to occur?

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**Dynamics of poles with position-dependent strengths and its optical analogues** [43] Montaldi and Tokieda (2011)

This is a both conceptual and somewhat heuristic paper, which in a way reflects the authors’ personalities. Some of the results appeared in chapter 3 of Anik Soulière’s Ph.D. dissertation under Tokieda [50].

Optical models are given in correspondence to planar flow motions with pole singularities. Besides point vortices, the analogy also works in the presence of sinks or sources and higher order poles. The strengths of those singularities may vary along the flow. A fluid flow would be a superposition of vectorfields of the form

$$\mathbf{z} = \mu(z)(z - z_0)^n, \quad n \in \mathbb{Z}.$$  

Except at the singularities, such flows are irrotational and incompressible. The dynamics of a system of poles can be set up in the usual way, each influencing all the others except itself. Allowing the strengths to vary extends previous work by [8] Borisov and Mamaev (2006, see also [5], [6]), and by [26] Lacomba (2009)\(^4\).

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\(^4\)Historically, the first paper in which this problem is addressed seems to be [14] Fridman and Polubarinova 1928. One of the authors is Alexander Friedmann, who obtained the well-known solution to the Einstein equations for the universe. I thank I. Mamaev for this information.
After some general facts and special solutions in sections 2 and 3, the optical analogies are presented in section 4. Montaldi and Tokieda consider the case of “usual” vortices \( n = -1 \) with a common “seabed,” so that

\[
\mu_i(z) = S(z) \mu_i, \quad \mu_i \in \mathbb{C}.
\]

The word “seabed” suggests a variable topography, and in fact the basic motivation is from [19] Hinds et al. (2007). They briefly discuss the difficulties to make a Hamiltonian description, even when the \( \mu_i \) are purely imaginary.

Consider a pair of opposite poles \( \pm \mu S \) moving on a constant seabed \( S(z) = \text{const} \). Their motion will be on straight lines. But there is a subtlety. If \( \mu \) is purely imaginary, they will move in the direction perpendicular to the line segment connecting them. If \( \mu \) is real, they will move in the same direction of the line segment. In hindsight this makes sense. All intermediate lines are possible by choosing an appropriate complex \( \mu \).

In a first analogy (section 4.1) they derive a peculiar version of Snell’s law, when there is an interface between two “media” \( S = s_1 \) and \( S = s_2 \) with the same sign. But I will not be a spoiler... Look at it! Try to work this out and then compare with eqs. (10–12). I got a kick out of this one, and also from the analogue for the law of reflection (section 4.2; here \( s_2 \) and \( s_2 \) have opposite signs). Again, try to do it for yourself. The result is on (17–18). A trapping phenomenon occurs when the pair hits the interface head on: something analogous occurs with waves in the ocean.

Montaldi and Tokieda argue that these two analogies are the only ingredients they would need to get almost all of the dynamics of small pole pairs on an arbitrary seabed \( S \). The idea actually goes back to Bernoulli’s solution of the brachistochrone (see [9] Broer (2014) for a review).

There is a curious phenomenon, of leapfrogging of two pairs of opposite vortices [52] Tophøj and Aref(2013). It can be emulated with just one pair of equal vortices that hit alternately the interface. They advance by waltzing in piecewise circular paths. On the other hand, when a pair of opposite vortices encounters circular craters (or bumps), motions reminiscent of rainbows and caustics are produced.

There is an opportunity to revisit the foundations and look for applications. Varying vortex strengths is common in geophysical fluid dynamics, see e.g., [34] McDonald (1999), specially with regards to shallow water models with varying bottom and in atmospheric models. Recently it has been also proposed by physicists working in Bose–Einstein condensates, [48], [49] Ruban (2017ab). One can also merge with multipoles and blobs see e.g. [12], [20] (D. Holm and associates, 2016).
Deformation of geometry and bifurcations of vortex rings
[44] Montaldi and Tokieda (2013) MR3110136

This is an important work (a 36 pages paper). In my opinion, the entry in
MathReviews does not give justice to the depth of the ideas they introduce, since
it just copies the somewhat modest abstract:

“We construct a smooth family of Hamiltonian systems, together with a family of
group symmetries and momentum maps, for the dynamics of point vortices on sur-
faces parametrized by the curvature of the surface. Equivariant bifurcations in this
family are characterized, whence the stability of the Thomson heptagon is deduced
without recourse to the Birkhoff normal form, which has hitherto been a necessary
tool.”

We begin with some history of the problem. In 1867 Lord Kelvin was advocating
vortex models to explain the properties of matter [24] Kragh (2002). Experiments by
A. M. Meyer in the late 1870’s, with floating magnets subject to a strong magnetic
field, indicated that up to \(n = 5\) they would form a regular polygon. In 1882
the Cambridge University Adams prize theme was about vortex motion. It was
awarded to Joseph John Thomson, then 25 years old, for his Treatise on the motion
of vortex rings.

The relevant sections in the treatise are \(\S\) 48 to \(\S\) 53 (pages 94 to 107). Thomson
showed stability up to \(n = 6\) vortices, and for \(n = 7\) he got a small positive value for
the square of two eigenvalues (p. 106). However Havelock in 1931 found a flaw in
that calculation (see Fig. 10): this pair was actually zero! There is always another
zero pair which does not hurt stability, reflecting the linear change in the frequency
of rotation relative to the size of the system.

In 1977 G.J. Mertz claimed a proof that the heptagonal configuration is nonlin-
eary stable. But it seems that some controversy still remained [25] Kurakin and
Yudovich (2002).

Finally, Cabral and Schmidt [11] settled the problem in 2000, using computer
algebra to do Birkhoff normal form up to fourth order. They showed that it is
positive definite near the origin so the seven vortex problem is indeed orbitally
Lyapunov stable.

Figure 9. Thomson’s problem

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5 A chapter from the very nice book Recent Trends in Dynamical Systems [22], in which the
authors in particular revisit the classical Thomson heptagon problem, proving its stability by a
new method.

6 It is one of the University’s oldest and most prestigious prizes. Named after the mathematician
John Couch Adams, it commemorates his role in the discovery of Neptune.

7 https://archive.org/details/treatiseonmotion00thomuoft/page/n6.

8 A M.Sc. dissertation from the University of Toronto Physics Department
https://open.library.ubc.ca/cIRcle/collections/ubctheses/831/items/1.0085746
If we substitute the values of $e, f, g$ we shall find that this
\[ = 0.02 \frac{m^2}{(2\pi r)^2}. \]
Thus one root of the equation in $\lambda^2$ is positive, therefore the steady motion is unstable. We therefore conclude that six is the greatest number of vortices which can be arranged at equal intervals round the circumference of the circle.

**Figure 10.** Thomson’s mistake

In their paper Montaldi and Tokieda vindicate Mertz’s approach. Through the eyes of Geometric Mechanics, it is basically the energy-momentum method (see eq. (33) of [36] Mertz (1978)).

As we now describe, they go much further. James and Tadashi start by imbedding Thomson’s seven vortex problem $P_0$ in a parametric family $P_\lambda$.

“The spirit of the approach goes back to Poincaré. In the theory of dynamical systems, the simplest solution methods to problems require some nondegeneracy condition (nonzero determinant, nonresonant frequencies, \textellipsis). When the problem, call it $P$, is degenerate, we have to mobilize heavy machinery.

\textellipsis

But there is an alternative approach. Embed $P_0$ in a parametric family $P_\lambda$ of problems and deform it away from degeneracy. The problem can become tractable when regarded as $\lim_{\lambda \to 0} P_\lambda$ if we happen to understand well enough the bifurcations that occur in such a deformation. Thus, this approach trades one machinery for another, of bifurcation theory. The point is that the latter sometimes sheds an unusual light on the problem compared to the former.

\textellipsis

Explicitly, we take $\lambda$ to be the Gaussian curvature and deform the original plane to $\lambda > 0$ (family of spheres) and to $\lambda < 0$ (family of hyperbolic planes). Corresponding to this family of surfaces parametrized by $\lambda$, we must write a whole parametric family of Hamiltonian systems for point vortices: a family of symplectic (Kähler) forms depending on $\lambda$, a family of symmetry groups and momentum maps depending on $\lambda$, a family of invariant Hamiltonians depending on $\lambda$ - all dependences arranged to be smooth. \textellipsis We carry out the stability analysis for the parametric family.”

Montaldi and Tokieda classified the bifurcations that arise in the process, and as a side bonus, proved the nonlinear stability of the heptagon.

The development of the paper is masterful. First they construct the family of surfaces using parametric Lie algebras, getting simultaneously the metrics and consequently the complex and the symplectic structures.

The parameter $\lambda$ can be taken as the curvature of the surface. The phase space $\mathcal{M}_\lambda$ is the product of $n$ copies of the surface minus the diagonals. As the vortices are assumed to have equal strengths, the symplectic form takes equal weights.

The vortex Hamiltonians are defined via Green functions of the corresponding Laplacians (they make a digression about possible choices, opting for the one that makes a smooth parametric family.) They found it convenient to use stereographic projections as in Fig. 11.

The computations are done in the frame rotating with angular velocity $\omega$ of the $n$-polygon. As it is well known, this entails subtracting $\omega J$ from the Hamiltonian.
The energy momentum method, where $J = J_\lambda$ is the momentum map of the $S^1$ action.

The Hessian on the symplectic slice determines the stability. The slice is the complement to the group orbit in $\text{Ker} \, dJ_\lambda$.

An important technical point is the need of a refined criterion for stability: if this Hessian is merely non-negative, but the augmented Hamiltonian admits a local extremum at the relative equilibrium, then the relative equilibrium is stable. This follows from previous work on Hamiltonian systems with symmetry [39] Montaldi (1997).

The final conclusions of the analysis on the interplay of the number $n$ of vortices with the parameter $\lambda$ are given on pages 349-350 (specifically Fig. 14.2). An important quantity is

$$\frac{1 + \lambda^2 r_o^4}{1 + \lambda r_o^2} - \frac{1}{2(n-1)} \left[ \frac{n^2}{4} \right],$$

where $r_o$ is the radius of the ring seen in the stereographic representation over the complex plane. They show that the relevant eigenvalues are all strictly positive if and only this quantity is positive. The conclusion is that negative curvature is stabilizing, and in the planar geometry up to the heptagon (inclusive) the rings are stable.

Well done!! But they wrote, quite non-challantly, in the beginning of the paper:

“The idea of deforming the geometry underlying the dynamics of point vortices, in particular as a route to a better understanding of the Thomson heptagon problem, arose during an evening conversation between the two authors in Peyresq, in the summer of 2003. We have since discussed it in seminars and conferences, and part of it has leaked into the literature.”

The italics are mine. Other researchers have pursued the ideas they suggested in their talks and have published in the meantime. Montaldi and Tokieda seemed not to care about priority. This shows the genuine and generous attitude in Mathematics that I mentioned before. But this is not the whole story!

There is a second part of the paper. In my view it is even more impressive. James and Tadashi scrutinize the bifurcations that occur across the degeneracy.
Their work is indeed a tour-de-force, involving the discrete symmetries of the system. For a regular $n$-polygon ring vortices this discrete symmetry group is given by the dihedral group $D_n$. The parity of $n$ plays a distinct role in the possible bifurcations.

As it has been studied since the work of Golubitsky and Stewart work from the late 1980’s, discrete symmetry breaking bifurcations happen in any type of ODEs. For Hamiltonian equations, there are special features [10] Buono et al. (2005).

Calculations with Maple lead to the main result (Theorem 14.6), that characterizes the first bifurcation, involving a $[n/2]$ symmetry breaking. The ring undergoes subsequent symmetry breaking bifurcations in each of the modes $[n/2] > \ell$, yielding solutions with $D_{(n, \ell)}$ symmetry.

There are avenues for young researchers to pursue. James and Tadashi pointed out several loose ends. One may attempt to carry over the same program for non-constant curvature families of surfaces of revolution. Nowadays one can comfortably write the Hamiltonian for vortex motion on any surface of revolution. A more ambitious project would be to study bifurcations that break the revolution symmetry.

### Point vortices on the hyperbolic plane
Montaldi and Nava-Gaxiola (2014) [40] MR3390464

This work is part of the Ph.D. thesis of Citlalitl Nava-Gaxiola in 2013. The aim of this paper is to investigate the stability of relative equilibria of a system of vortices in the universal cover for the negative constant curvature metric, building specially on [21] Hwang and Kim (2013), Other authors had only touched briefly this question: [23] Kimura (1999), [1] Aref et al. (2003), [44] Montaldi and Tokieda (2013).

Montaldi and Nava-Gaxiola prefer to use the “physical” hyperboloid instead of its abstract representation as the Poincaré disk. The symmetry group is $SL(2, \mathbb{R})$ giving rise to non-compact momentum isotropy subgroups.

The three conserved quantities form the components of the momentum map, and the symmetry properties of this map allow one to divide relative equilibria with non-zero momentum into three principal classes: elliptic, parabolic, and hyperbolic, according to their momentum value, and this plays an important role in questions of stability. The relative equilibrium conditions were found and the trajectories of relative equilibria with non-zero momentum value were described. Stability criteria were given for a number of cases, focusing on two and three vortices. They show that every two point vortex configuration is stable relative to $SL(2, \mathbb{R})$.

However, there is a finer notion of stability, namely stability relative to the subgroup $SL(2, \mathbb{R})_\mu$, the isotropy subgroup for the momentum value $\mu$.

Stability only holds when the momentum value is elliptic, which in turn is true if the vortex strengths are of the same sign or, if they are of opposite signs, the vortices are not too far apart.

They show that every relative equilibrium of three point vortices in the hyperbolic plane is either an equilateral triangle or a geodesic configuration.

Actually, this is entirely analogous to the situation on the plane or the sphere. They also proved several nice results for the relative equilibria whose reconstruction moves on geodesics.

The zero momentum case and the geodesic case with three different lengths are treated briefly in the end of the paper.
For the stability of relative equilibria of three point vortices, remarkably, they found that an equilateral three vortex configuration has the exact same stability conditions of those for systems on the plane and on the sphere, namely, that they are stable relative to the isotropy subgroup whenever $\sum_{i<j} \Gamma_i \Gamma_j > 0$. There is here an opportunity to pursue this investigation further, which already required intensive computer algebra.

**Generalized point vortex dynamics on $\mathbb{CP}^2$**

**Non-Abelian momentum polytopes for products of $\mathbb{CP}^2$**

Montaldi and Shaddad (2019a) Montaldi and Shaddad (2019b) [41] [42]

These two papers contain results from Amna Shaddad’s 2018 thesis, and appeared in a J. Geometric Mechanics special volume in honor of Darryl Holm. The ambient, $\mathbb{CP}^2 = S^5/U(1)$, has dimension 4. It is a very nice symplectic manifold, having a $SU(3)$-invariant symplectic form, the Fubini–Study 2-form.

They consider the Hamiltonian action of $SU(3)$ on $\mathbb{CP}^2$, given by $A[v] = [Av]$, extended diagonally to $N$ copies $\mathbb{CP}^2$, with a weighted sum of the invariant symplectic form. This is a beautiful toy to play with!

Montaldi and Shaddad apply Atiyah, Guillemin/Sternberg and Kirwan’s momentum polytope (convexity) theorem. They start the first paper with an apt discussion of the convexity of momentum maps, that generalizes observations of numerical analysts since Schur in the 1920’s. Recall that this is concerned with inequalities relating the eigenvalues of $n \times n$ hermitian matrices with the diagonal elements.

Then they proceed to the classification of the polytopes associated to $N = 2, 3$, using the weights as parameters. This is a tour-de-force, requiring a mastery of Lie group and symplectic techniques (Weyl chambers, slicing, KKS, etc.)

In turn, this study allows beautiful applications to numerical analysis. For instance, they consider sums of three Hermitian matrices each with a double eigenvalue, and produce Schur type inequalities. Other situations clearly can be addressed as well.

In the second paper they classify the reduced spaces for $N = 2, 3$ for a general $SU(3)$ invariant Hamiltonian. For $N = 3$, the possibilities are a sphere or a sphere with singularities. For $N = 2$ the reduced spaces are just points, and the motion is governed by a collective Hamiltonian. In both cases the system will be completely integrable in the non-Abelian sense.

*But do we have vortices here? The dimension of $\mathbb{CP}^2$ is four, not two!*

In two dimensions, if one has Riemannian metric, one also gets for free a complex structure together with an area form. The Hamiltonian for the vortex interaction, extending the theory the planar domains done by C. C. Lin in the 1940’s can be constructed with the Green function for the Laplace–Beltrami operator, plus its desingularization to account for the self interactions. How about higher dimensions?

In fact, the same construction can be emulated on any Kähler manifold [7] Boatto and Koiller (2015). Calabi–Yau manifolds are important objects in mathematical physics. Is this generalization of vortices in line with singular solutions of known field theories?

In the case of two opposite generalized vortices, infinitesimally closed together, would they move along geodesics of the Kähler metric in the same fashion as Kimura has conjectured to happen for $n = 1$ [23] Kimura (1999)?
Tribute.

As we saw, James Montaldi introduced into the vortex playground the techniques of bifurcation theory, momentum maps and Lie groups, in a way that both experts and non-experts can not only appreciate, but in fact also pursue the many avenues he opened. James found eight paths to his Erdős number, to which we found one more (Fig. 2). No wonder Google Scholar has accounted for 1300+ citations, with around one-forth of them in the last 6 years. One is impressed by the rich details and skilled craftsmanship of James’ papers. They reflect his gregarious attitude towards math, both as a colleague and as a mentor for his students – James has encouraged several women students to become engaged in mathematical research.

Keep it up, King James. Many more years of great work on vortices!

Midori uzumaki nagareru
naka wo kakeru (Ogiwara Seisensui).

Geen runs in the swirl of a vortex
(Il grande libro degli haiku, p. 687 [51] Starace (2018))

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E-mail address: jairkoiller@gmail.com