Coherent and semiclassical states of a charged particle in electromagnetic fields

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Abstract

In the present article we extend our study (BJP 45 (2015) 369) of generalized coherent states (GCS) of a one-dimensional particle considering such important physical system as a 3-dimensional charged particle in electric and magnetic fields. Constructing GCS in many-dimensional case, we meet nontrivial technical complications that make the consideration nontrivial and instructive. The GCS of a system under consideration are constructed. We study properties of these GCS such as completeness relations, minimization of uncertainty relations and so on. We point out which family of the obtained GCS of a charged particle in magnetic field is related with the CS constructed first by Malkin and Man’ko. We obtain conditions under which some of the GCS can be considered as semiclassical states.

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I. INTRODUCTION

Coherent states (CS) play an important role in modern quantum theory as states that provide a natural relation between quantum mechanical and classical descriptions. They have a number of useful properties and as a consequence a wide range of applications, e.g., in semiclassical description of quantum systems, in quantization theory, in condensed matter physics, in quantum computations, and so on, see e.g., [1–7]. Starting by the works [8, 9] CS are defined as eigenvectors of some annihilation operators that are at the same time integrals of motion. Of course such defined CS have to satisfy the corresponding Schrödinger equation. In the frame of such a definition one can in principle construct CS for a general quadratic system. In the article [10] we, following these scheme, constructed different families of CS (GCS) for one-dimensional systems with general time-dependent quadratic Hamiltonian, see too [11, 12]. In the article [12], we have demonstrated that the GCS of a free particle can be treated as quantum states that describe a semiclassical motion, whereas there exist GCS that describe pure quantum motion.

In the present article we extend our study beyond one-dimensional systems, considering such important physical system as a 3-dimensional charged particle in electric and magnetic fields. Considering many-dimensional systems we meet nontrivial technical complications that makes the consideration nontrivial and instructive. We discuss properties of the constructed GCS such as completeness relations, minimization of uncertainty relations, and so on. As in the one-dimensional case, we succeeded to find conditions that allow one to attribute the GCS either to the class of semiclassical quantum states or to purely quantum states.

This article is organized as follows. In Section 2, we outlook classical and quantum descriptions of a charged particle in parallel electric and magnetic fields and obtain for such a system integrals of motion which are creation and annihilation operators. In section 3, we construct GCS that satisfies the Schrödinger equation and calculate the mean values, standard deviations and uncertainty relations, as well as we discuss properties of the constructed GCS such as completeness relations, minimization of uncertainty relations, and so on. We also demonstrated that the GCS of a charged particle in magnetic field is related with the CS constructed first by Malkin and Man’ko. In section 4, we obtain conditions under the electric and magnetic field such that the GCS can be considered as semiclassical states or
purely quantum states.

II. CHARGED PARTICLE IN PARALLEL ELECTRIC AND MAGNETIC FIELDS

A. Classical and quantum equations of motion

Consider a charged particle\textsuperscript{1}, with total charge $e$, moving in a 3-dimensional Euclidean space,

$$ H = \frac{P^2}{2m} + eA_0, \quad P = p - \frac{e}{c}A, \quad (1) $$

interacting with constant, uniform and parallel electric $E$ and magnetic $B$ fields

$$ E = (0, 0, E), \quad B = (0, 0, B), \quad A_0 = -zE \sin^2 \alpha, \quad A = \frac{1}{2} (-By, Bx, -2ctE \cos^2 \alpha), \quad \alpha \in [0, \pi/2]. \quad (2) $$

The corresponding Hamiltonian can be written in the following form

$$ H = H_{xy} + H_z, \quad H_{xy} = \frac{p_{\perp}^2}{2m} + \frac{m\omega_r^2 r_{\perp}^2}{8} - \frac{\omega}{2} L, $$$$ $H_z = \frac{p_z^2}{2m} - m\xi z \sin^2 \alpha + \xi p_z t \cos^2 \alpha + \frac{mc^2}{2} t^2 \cos^4 \alpha, $$

$$ L = xp_y - yp_x, \quad \omega = \frac{eB}{mc}, \quad \xi = \frac{eE}{m}. \quad (3) $$

Here $p$ is the canonical momenta conjugated to the coordinates $r$, $\omega$ is the cyclotron frequency, and $\xi$ denotes the acceleration along the $z$-axis. The velocity $v$ is related to the canonical variables as $mv = P$.

The division of $H$ into two separate parts, $H_{xy}$ and $H_z$, indicates implicitly the independence between particle’s motion on the $xy$-plane (hereafter referred as $xy$-motion) from the motion along the $z$-direction (hereafter referred as $z$-motion). This fact is explicitly

\[1\] Throughout the text, $e$ denotes the algebraic electric charge ($e = -|e|$ for an electron), bold letters represent vectors, e. g., $r = (x, y, z)$, latin indices are $j, k, \ldots = 1, 2, 3$ and greek indices are $\beta, \eta, \ldots = 1, 2$. 
confirmed by the structure of canonical equations of motion\textsuperscript{2}

\[
\begin{align*}
v_x(t) &= \dot{x}(t) = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} + \frac{\omega}{2}y, \quad \dot{p}_x(t) = -\frac{\partial H}{\partial x} = \frac{\omega}{2}p_y - \frac{m\omega^2}{4}x, \\
v_y(t) &= \dot{y}(t) = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} - \frac{\omega}{2}x, \quad \dot{p}_y(t) = -\frac{\partial H}{\partial y} = -\frac{\omega}{2}p_x - \frac{m\omega^2}{4}y, \\
v_z(t) &= \dot{z}(t) = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} + \xi t \cos^2 \alpha, \quad \dot{p}_z(t) = -\frac{\partial H}{\partial z} = m\xi \sin^2 \alpha. \numberthis (4)
\end{align*}
\]

The general solution for the \(z\)-motion follows from the third line of Eq. (4),

\[
\begin{align*}
z(t) &= z_0 + v^0_z t + \frac{1}{2} \xi t^2, \quad p_z(t) = m \left( v^0_z + \xi t \sin^2 \alpha \right), \\
z_0 &= z(0), \quad v^0_z = v_z(0) = \frac{p^0_z}{m}, \quad p_z = p_z(0), \numberthis (5)
\end{align*}
\]

corresponds to an accelerated motion along the \(z\)-direction. As for the \(xy\)-motion, it can be presented in two equivalent ways. The first one is given in terms of the initial Cauchy data \((r_\perp(0) = r_{0\perp}, v_\perp(0) = v_{0\perp})\),

\[
\begin{align*}
x(t) &= x_0 + v^0_y \frac{1 - \cos(\omega t)}{\omega} + v^0_x \frac{\sin(\omega t)}{\omega}, \quad p_x(t) = \frac{v^0_x \cos(\omega t) + v^0_y \sin(\omega t) + v^0_x - \omega y_0}{2}, \\
y(t) &= y_0 - v^0_x \frac{1 - \cos(\omega t)}{\omega} + v^0_y \frac{\sin(\omega t)}{\omega}, \quad p_y(t) = \frac{v^0_y \cos(\omega t) - v^0_x \sin(\omega t) + v^0_y + \omega x_0}{2}, \numberthis (6)
\end{align*}
\]

which can be equivalently given in terms of the perpendicular velocity components,

\[
\begin{align*}
x(t) &= x_0 - \frac{v_y(t) - v^0_y}{\omega}, \quad y(t) = y_0 + \frac{v_x(t) - v^0_x}{\omega}, \\
p_x(t) &= m\frac{v_x(t) + v^0_x - \omega y_0}{2}, \quad p_y(t) = m\frac{v_y(t) + v^0_y + \omega x_0}{2}, \\
v_x(t) &= v^0_y \sin(\omega t) + v^0_x \cos(\omega t), \quad v_y(t) = v^0_y \cos(\omega t) - v^0_x \sin(\omega t). \numberthis (7)
\end{align*}
\]

In a second representation, the \(xy\)-motion is parametrized in terms of the coordinates of the center of the orbit \((x_c, y_c)\), its radius \(R\) and the initial phase \(\theta_0\) instead the initial Cauchy data,

\[
\begin{align*}
x(t) &= x_c + R \cos \theta, \quad y(t) = y_c + R \sin \theta, \quad \theta = \omega t + \theta_0, \\
x_c &= R_c \cos \theta_c, \quad y_c = R_c \sin \theta_c, \quad R^2 = x_c^2 + y_c^2, \quad R^2 = (x - x_c)^2 + (y - y_c)^2, \numberthis (8)
\end{align*}
\]

\textsuperscript{2} Quantities associated with the \(xy\)-motion are labelled with the symbol \(\perp\) (e. g., \(r_\perp = (x, y)\)) while quantities associated with the \(z\)-motion are labelled with the symbol \(\parallel\) (e. g., \(r_\parallel = z\)).
FIG. 1: Particle’s classical circular motion along the $xy$-plane. The $3D$ trajectory correspond to a helix, with a circular $xy$-motion and accelerated $z$-motion, whose direction point out of the picture, aligned with the external electromagnetic fields.

as illustrated in Fig. 1

In this representation, the distance from the origin $r (t)$ can be expressed in terms of $R$, $R_c$ and the relative angle $\theta - \theta_c$ as

$$r^2 = x^2 + y^2 = R^2 + R_c^2 + 2RR_c \cos (\theta - \theta_c),$$

whose maximum $r_{\text{max}} = R + R_c$ and minimum $r_{\text{min}} = |R - R_c|$ correspond to configurations in which $\theta = \theta_c$ and $\theta = \theta_c + \pi$, respectively. From the perpendicular velocities $\mathbf{v}_\perp (t) = (\dot{x} (t), \dot{y} (t))$,

$$\dot{x} (t) = v_x (t) = -R \omega \sin \theta = -\omega (y - y_c),$$

$$\dot{y} (t) = v_y (t) = R \omega \cos \theta = \omega (x - x_c),$$

(10)
one can readily see that both sets of solutions can be mapped to each other through the following relations between the initial conditions \((x_0, y_0, v^0_x, v^0_y)\) and the integration constants \((x_c, y_c, \theta_0, R)\) as

\[
x_0 = x_c + R \cos \theta_0, \quad y_0 = y_c + R \sin \theta_0, \\
v^0_x = -R \omega \sin \theta_0, \quad v^0_y = R \omega \cos \theta_0.
\] (11)

Eqs. (8) and (10) may be written as follows

\[
x(t) = x_c + \frac{1}{\omega} \sqrt{\frac{2E_\perp}{m}} \cos \theta, \quad y(t) = y_c + \frac{1}{\omega} \sqrt{\frac{2E_\perp}{m}} \sin \theta, \\
v_x(t) = -\sqrt{\frac{2E_\perp}{m}} \sin \theta, \quad v_y(t) = \sqrt{\frac{2E_\perp}{m}} \cos \theta,
\] (12)

where the perpendicular energy \(E_\perp\) is a conserved quantity,

\[
E_\perp = \frac{m}{2} \left( v^2_x + v^2_y \right) = \frac{m}{2} \left[ (v^0_x)^2 + (v^0_y)^2 \right] = \frac{mR^2 \omega^2}{2}.
\] (13)

It is worth noting that in the limit of zero magnetic field, the \(xy\)-motion tends to a uniform motion according to Eq. (6), while diverges according to Eqs. (12), due to the \(\omega^{-1}\) singularity. At a first sight, this might be a source of discrepancy but, to solve the Cauchy problem, one has to specify the initial coordinates \(x_0, y_0\) and velocities \(v^0_x, v^0_y\). This means that the integration constants \(x_c, y_c\) and \(\theta_0\) must be expressed in terms of the latter constants. In fact such, solving Eqs. (11) for \(x_c, y_c,\) and \(\theta_0\) and substituting into Eqs. (12), the divergence is eliminated.

The quantum nonrelativistic motion of the system under consideration is described by the corresponding Schrödinger equation

\[
\hat{H} \Psi(t) = \hat{H} \Psi(t), \quad \hat{H} = \hat{H}_{xy} + \hat{H}_z, \quad \partial_t = \frac{\partial}{\partial t}, \\
\hat{H}_{xy} = \frac{\hat{p}^2_x}{2m} + \frac{m\omega^2}{8} r^2_x - \frac{\omega}{2} \hat{L}, \quad \hat{L} = x \hat{p}_y - y \hat{p}_x, \quad \hat{p} = -i\hbar \nabla, \\
\hat{H}_z = \frac{\hat{p}^2_z}{2m} - m \xi z \sin^2 \alpha + \xi \hat{p}_z t \cos^2 \alpha + \frac{m\xi^2 t^2}{2} \cos^4 \alpha.
\] (14)

Introducing dimensionless variables \(q, \hat{\pi},\) and \(\tau\) as

\[
q = t^{-1} \mathbf{r}, \quad \tau = \frac{\hbar}{m l^2 t}, \quad \hat{\pi} = \frac{l}{\hbar} \hat{\mathbf{p}}, \quad \hat{\Pi} = \frac{l}{\hbar} \hat{\mathbf{P}} \to \mathbf{\hat{v}} = \frac{\hbar}{l m} \hat{\Pi}, \\
\Omega = \frac{m l^2}{\hbar} \omega, \quad \Xi = \frac{m^2 l^3}{\hbar^2} \xi, \quad [q_k, \hat{\Pi}_j] = [q_k, \hat{\pi}_j] = i \delta_{kj},
\] (15)
we pass to an equivalent Schrödinger equation

\[ \hat{S} \Phi (q, \tau) = 0, \quad \hat{S} = \partial_\tau + i \left( \hat{H}_\perp + \hat{H}_\parallel \right), \quad \Phi (q, \tau) = \sqrt{\hbar} \Psi \left( q, \frac{ml^2}{\hbar} \tau \right), \]

\[ \hat{H}_\perp = \frac{\hat{\pi}_\perp^2}{2} + \frac{\Omega^2 q_1^2}{8} - \frac{\Omega}{2} \hat{L}, \quad \hat{L} = q_1 \hat{\pi}_2 - q_2 \hat{\pi}_1, \quad \hat{H}_{xy} = \frac{\hbar^2}{ml^2} \hat{H}_\perp, \]

\[ \hat{H}_\parallel = \frac{\hat{\pi}_\parallel^2}{2} - \Xi q_3 \sin^2 \alpha + \Xi \tau \hat{\pi}_3 \cos^2 \alpha + \frac{\Xi^2 \tau^2}{2} \cos^4 \alpha, \quad \hat{H}_z = \frac{\hbar^2}{ml^2} \hat{H}_\parallel. \] (16)

### B. Special sets of integrals of motion

Following well-known method [8,10], we are going to find special sets \( \hat{A}_j (\tau) \) of quantum integrals of motion in the problem under consideration. They have to be linear combinations of basic operators \( q, \hat{\pi} \) and \( \hat{A} \), at the same time, annihilation and creation operators,

\[ \hat{A}_j (\tau) = f_{jk} (\tau) q_k + i g_{jk} (\tau) \hat{\pi}_k + \varphi_j (\tau), \] (17)

\[ \left[ \hat{A}_j (\tau), \hat{A}_k^\dagger (\tau) \right] = \delta_{jk}, \quad \left[ \hat{A}_j (\tau), \hat{A}_k (\tau) \right] = \left[ \hat{A}_j^\dagger (\tau), \hat{A}_k^\dagger (\tau) \right] = 0, \quad \forall \tau. \] (18)

Here, the coefficients \( f_{jk} (\tau), g_{jk} (\tau) \) and \( \varphi_j (\tau) \) are some time-dependent functions.

One can easily see that both \( \hat{A}_j (\tau) \) and \( \hat{A}_j^\dagger (\tau) \) are integrals of motion if

\[ \partial_\tau \hat{A}_j (\tau) = \left[ \hat{S}, \hat{A}_j (\tau) \right] = 0. \] (19)

Since the \( xy \)-motion is independent from the \( z \)-motion, the functions \( f_{3k} (\tau) \) and \( g_{3k} (\tau) \) can be chosen as

\[ f_{3k} (\tau) = f_3 (\tau) \delta_{3k}, \quad g_{3k} (\tau) = g_3 (\tau) \delta_{3k}, \] (20)

which means that the operator \( \hat{A}_3 (\tau) \) depends on the operators \( q_3 \) and \( \hat{\pi}_3 \) only. Under such a condition the unknown functions \( f_{jk} (\tau), g_{jk} (\tau) \) and \( \varphi_j (\tau) \) must obey the following equations

\[ \dot{f}_{31} (\tau) = \frac{\Omega}{2} f_{32} (\tau) + \frac{i \Omega^2}{4} g_{31} (\tau), \quad \dot{g}_{31} (\tau) = i f_{31} (\tau) + \frac{\Omega}{2} g_{32} (\tau), \quad \varphi_3 (\tau) = 0, \]

\[ \dot{f}_{32} (\tau) = -\frac{\Omega}{2} f_{31} (\tau) + \frac{i \Omega^2}{4} g_{32} (\tau), \quad \dot{g}_{32} (\tau) = i f_{32} (\tau) - \frac{\Omega}{2} g_{31} (\tau), \]

\[ \dot{f}_3 (\tau) = 0, \quad \dot{g} (\tau) = i f_3 (\tau), \quad \varphi_3 (\tau) = \frac{\Xi \left[ g_3 (\tau) \sin^2 \alpha - i \tau f_3 (\tau) \cos^2 \alpha \right]}{i \sqrt{2}}. \] (21)

---

[3] The summation convention over repeated scripts is assumed throughout, unless otherwise explicitly stated.
In addition, it follows from Eqs. (18) and (19) that functions $f_{jk}(\tau)$ and $g_{jk}(\tau)$ are subjected to the following conditions

\[
\begin{align*}
  f_{jk}(\tau)g_{k'k}(\tau) + f^*_{k'k}(\tau)g_{jk}(\tau) &= f_{jk}(0)g^*_{k'k}(0) + f^*_{k'k}(0)g_{jk}(0) = 2\delta_{jk'}, \\
  f_{jk}(\tau)g_{k'k}(\tau) - f^*_{k'k}(\tau)g_{jk}(\tau) &= f_{jk}(0)g_{k'k}(0) - f^*_{k'k}(0)g_{jk}(0) = 0.
\end{align*}
\]

(22)

The general solution of (21) has the form

\[
\begin{align*}
  f_{\beta 1}(\tau) &= \frac{i\Omega}{2}c_{\beta 1} - ib_{\beta 2} \frac{1 + \cos (\Omega \tau)}{2} = \frac{ib_{\beta 1}}{2} \sin (\Omega \tau), \\
  f_{\beta 2}(\tau) &= \frac{-i\Omega}{2}c_{\beta 2} - ib_{\beta 1} \frac{1 + \cos (\Omega \tau)}{2} = \frac{ib_{\beta 2}}{2} \sin (\Omega \tau), \\
  g_{\beta 1}(\tau) &= c_{\beta 2} + b_{\beta 1} \frac{1 - \cos (\Omega \tau)}{\Omega} + b_{\beta 2} \sin (\Omega \tau), \\
  g_{\beta 2}(\tau) &= c_{\beta 1} - b_{\beta 2} \frac{1 - \cos (\Omega \tau)}{\Omega} + b_{\beta 1} \sin (\Omega \tau), \\
  f_{3}(\tau) &= f_{0}, \\
  g_{3}(\tau) &= g_{0} + if_{0}\tau, \\
  \varphi_{3}(\tau) &= -\left[ \frac{f_{0}\tau}{2} + ig_{3}(\tau)\sin^{2}\alpha \right] \Xi_{\tau} \sqrt{2},
\end{align*}
\]

where $c_{\beta q}, \ b_{\beta q}, \ f_{0}$ and $g_{0}$ are some constants. Without any loss of generality, we can set $\varphi_{j}(0) = 0$.

It follows from (19) that a common basis for $\hat{S}$ and $\hat{A}_{j}(\tau)$ can be found in the form

\[
\begin{align*}
  \hat{S}|\zeta, \tau\rangle &= \lambda_{\zeta}(\tau)|\zeta, \tau\rangle, \\
  \hat{A}_{j}(\tau)|\zeta, \tau\rangle &= \zeta_{j}|\zeta, \tau\rangle,
\end{align*}
\]

wherein $\zeta = (\zeta_{1}, \zeta_{2}, \zeta_{3})$ are complex numbers and $\lambda_{\zeta}(\tau)$ is an arbitrary time-dependent function.

### III. GENERALIZED COHERENT STATES

We define the GCS as solutions of the Schrödinger equation that are eigenstates of the operators $\hat{A}_{j}(\tau)$ with eigenvalues $\zeta_{j} = \zeta_{\perp}, \zeta_{3}, \ z_{\perp} = \zeta_{1}, \zeta_{2}$. One can represent the GCS as

\[
\Phi_{\zeta}(\mathbf{q}, \tau) = \langle \mathbf{q}|\zeta, \tau\rangle = \Phi_{\zeta_{\perp}}(\mathbf{q}_{\perp}, \tau) \Phi_{\zeta_{3}}(q_{3}, \tau).
\]

(26)

Then, to obey three conditions (25), the functions $\Phi_{\zeta_{\perp}}(\mathbf{q}_{\perp}, \tau)$ and $\Phi_{\zeta_{3}}(q_{3}, \tau)$ have to satisfy the following equations

\[
\begin{align*}
  \hat{A}_{\beta}\Phi_{\zeta_{\perp}}(\mathbf{q}_{\perp}, \tau) &= \frac{f_{\beta 1}q_{1} + f_{\beta 2}q_{2} + g_{\beta 1}\partial_{q_{1}} + g_{\beta 2}\partial_{q_{2}}}{\sqrt{2}} \Phi_{\zeta_{\perp}}(\mathbf{q}_{\perp}, \tau) = \zeta_{\beta}\Phi_{\zeta_{\perp}}(\mathbf{q}_{\perp}, \tau), \\
  \hat{A}_{3}\Phi_{\zeta_{3}}(q_{3}, \tau) &= \left( \frac{f_{0}q_{3} + g_{3}\partial_{q_{1}}}{\sqrt{2}} + \varphi_{3} \right) \Phi_{\zeta_{3}}(q_{3}, \tau) = \zeta_{3}\Phi_{\zeta_{3}}(q_{3}, \tau).
\end{align*}
\]

(27)

(28)
The general solution of Eq. (28) reads
\[ \Phi_{\xi_3}(q_3, \tau) = \exp \left[ -\frac{f_0 q_3^2}{g_3^2} + \sqrt{2} \frac{\zeta_3 - \varphi_3}{g_3} q_3 + i \phi_3(\tau) \right], \] (29)
where \( \phi_3(\tau) \) is an arbitrary function of \( \tau \).

The general solution of the set (27), see Appendix A, has the form
\[ \Phi_{\xi_\perp}(q_\perp, \tau) = \exp \left[ -\frac{G_1 q_1^2 + G_2 q_2^2}{2} + F q_1 q_2 + \sqrt{2} (Q_1 q_1 - Q_2 q_2) + i \phi_\perp(\tau) \right], \] (30)
where
\[ G_1 = \frac{f_{11} g_{22} - f_{21} g_{12}}{g_{11} g_{22} - g_{12} g_{21}}, \quad G_2 = \frac{f_{22} g_{11} - f_{12} g_{21}}{g_{11} g_{22} - g_{12} g_{21}}, \]
\[ Q_1 = \frac{g_{22} \zeta_1 - g_{12} \zeta_2}{g_{11} g_{22} - g_{12} g_{21}}, \quad Q_2 = \frac{g_{21} \zeta_1 - g_{11} \zeta_2}{g_{11} g_{22} - g_{12} g_{21}}, \quad F = \frac{f_{11} g_{21} - f_{21} g_{11}}{g_{11} g_{22} - g_{12} g_{21}}, \]
and \( \phi_\perp(\tau) \) is an arbitrary time-dependent function. Thus,
\[ \Phi_{\zeta}(q, \tau) = \Phi_{\xi_\perp}(q_\perp, \tau) \Phi_{\xi_3}(q_3, \tau) = \exp \left[ -\frac{G_1 q_1^2 + G_2 q_2^2}{2} + F q_1 q_2 + \sqrt{2} (Q_1 q_1 - Q_2 q_2) + \frac{f_0 q_3^2}{g_3^2} + \sqrt{2} \frac{\zeta_3 - \varphi_3}{g_3} q_3 + i \phi(\tau) \right], \] (32)
being \( \phi(\tau) \equiv \phi_3(\tau) + \phi_\perp(\tau) \) some function of \( \tau \), such that \( \Phi_{\zeta}(q, \tau) \) satisfies the Schrödinger equation (16). This means that the function \( \phi(\tau) \) must be given by
\[ \phi(\tau) = \tilde{\phi}_1(\tau) + \frac{i}{2} \ln g_3 - i \ln C, \]
\[ \tilde{\phi}_1(\tau) = \int \left[ Q_1^2 + Q_2^2 - \frac{G_1 + G_2}{2} + \left( \frac{\zeta_3 - \varphi_3}{g_3} \right)^2 + \frac{i \xi_3}{\sqrt{2}} \cos^2 \alpha \right]^2 d\tau, \] (33)
where \( C \) is a normalization constant, which can be found from the normalization condition. The normalized function \( \Phi_{\zeta}(q, \tau) \) reads:
\[ \Phi_{\zeta}(q, \tau) = \left( \frac{\text{Re} G_1 \text{Re} G_2 - \text{Re}^2 F}{\pi^3 g_3^3} \right)^{1/4} \exp \left[ i \text{Re} \tilde{\phi}_1(\tau) - |g_3|^2 \text{Re}^2 \left( \frac{\zeta_3 - \varphi_3}{g_3} \right) \right] \]
\[ - \frac{G_1 q_1^2 + G_2 q_2^2}{2} + \frac{f_0 q_3^2}{g_3^2} + F q_1 q_2 + \sqrt{2} (Q_1 q_1 - Q_2 q_2) + \sqrt{2} \frac{\zeta_3 - \varphi_3}{g_3} q_3 \]. (34)

The family of states \( \Phi_{\zeta}(q, \tau) \) is parametrized by constants \( c_{3\eta}, b_{3\eta}, f_0 \) and \( g_0 \).
They imply, in turn, relations

\[ g_{12}(\tau) = ig_{11}(\tau) \iff f_{12}(\tau) = if_{11}(\tau) \quad \& \quad g_{21}(\tau) = ig_{22}(\tau) \iff f_{21}(\tau) = if_{22}(\tau). \quad (35) \]

They follow from Eqs. (22) that

\[ c_{12} = -ic_{11}, \quad c_{21} = -ic_{22}, \quad b_{12} = -ib_{11}, \quad b_{21} = -ib_{22}, \quad (36) \]

and

\[
\begin{align*}
 f_{11}(\tau) &= \frac{i\Omega}{2} c_{11} - b_{11} \frac{1 + e^{i\Omega\tau}}{2}, \\
 f_{22}(\tau) &= -\frac{i\Omega}{2} c_{22} - b_{22} \frac{1 + e^{-i\Omega\tau}}{2}, \\
 g_{11}(\tau) &= -ic_{11} + b_{11} \frac{1 - e^{i\Omega\tau}}{\Omega}, \\
 g_{22}(\tau) &= -ic_{22} - b_{22} \frac{1 - e^{-i\Omega\tau}}{\Omega}. \quad (37)
\end{align*}
\]

In addition, it follows from Eqs. (22) that

\[
\begin{align*}
 2 \text{Re}(f_{11}g_{11}^*) &= 2 \text{Im}(b_{11}c_{11}^*) - \Omega |c_{11}|^2 = 1, \\
 2 \text{Re}(f_{22}g_{22}^*) &= 2 \text{Im}(b_{22}c_{22}^*) + \Omega |c_{22}|^2 = 1, \\
 f_{11}g_{22} - f_{22}g_{11} &= ib_{22}c_{11} - (ib_{11} + \Omega c_{11}) c_{22} = 0. \quad (38)
\end{align*}
\]

On this stage, the GCS \( \Phi_\zeta(q, \tau) \) can be rewritten as follows

\[
\Phi_\zeta(q, \tau) = \frac{1}{\pi^{3/4} \sqrt{2g_{11}g_3}} \exp \left[ i \text{Re} \tilde{\phi}_2(\tau) - 2 |g_{11}|^2 \left( \text{Re}^2 Q_1 + \text{Re}^2 Q_2 \right) - |g_3|^2 \text{Re}^2 \left( \frac{\zeta_3 - \varphi_3}{g_3} \right) \right] \\
+ \frac{i\Omega\tau}{2} - \frac{f_{11}q_1^2 + q_2^2}{g_{11}} - \frac{f_0 q_3^2}{g_3} + \sqrt{2} \tilde{\zeta}_3 - \varphi_3 \frac{q_3}{g_3} + \sqrt{2} (Q_1 q_1 - Q_2 q_2), \quad (39)
\]

where

\[
\tilde{\phi}_2(\tau) = \int \left[ Q_1^2 + Q_2^2 + \left( \frac{\zeta_3 - \varphi_3}{g_3} + \frac{i\xi_3}{\sqrt{2}} \cos^2 \alpha \right)^2 \right] d\tau,
\]

\[
Q_1 = \frac{g_{22}\zeta_1 - ig_{11}\zeta_2}{2g_{11}g_{22}}, \quad Q_2 = \frac{ig_{22}\zeta_1 - g_{11}\zeta_2}{2g_{11}g_{22}}. \quad (40)
\]

It follows from Eqs. (17), (25) and (35) that

\[
\begin{align*}
\zeta_1 &= \frac{f_{11}(\tau) [q_1(\tau) + iq_2(\tau)] + ig_{11}(\tau) [\pi_1(\tau) + i\pi_2(\tau)]}{\sqrt{2}}, \\
\zeta_2 &= \frac{if_{22}(\tau) [q_1(\tau) - iq_2(\tau)] - g_{22}(\tau) [\pi_1(\tau) - i\pi_2(\tau)]}{\sqrt{2}}, \\
\zeta_3 &= \frac{f_0 q_3(\tau) + ig_3(\tau) \pi_3(\tau)}{\sqrt{2}} + \varphi_3(\tau), \quad (41)
\end{align*}
\]
wherein the mean values \( q_j (\tau) = \langle \zeta, \tau | q_j | \zeta, \tau \rangle \) and \( \pi_j (\tau) = \langle \zeta, \tau | \pi_j | \zeta, \tau \rangle \) obey the classical equations of motion \[4\] written in dimensionless variables,

\[
\begin{align*}
\dot{\pi}_1 (\tau) &= \frac{\Omega}{2} \pi_2 (\tau) - \frac{\Omega^2 q_1 (\tau)}{4}, \\
\dot{\pi}_2 (\tau) &= -\frac{\Omega}{2} \pi_1 (\tau) - \frac{\Omega^2 q_2 (\tau)}{4}, \\
\dot{q}_1 (\tau) &= \frac{\Omega}{2} q_2 (\tau) + \pi_1 (\tau), \\
\dot{q}_2 (\tau) &= -\frac{\Omega}{2} q_1 (\tau) + \pi_2 (\tau), \\
\dot{\pi}_3 (\tau) &= \Xi \sin^2 \alpha, \\
\dot{q}_3 (\tau) &= \pi_3 (\tau) + \Xi \tau \cos^2 \alpha.
\end{align*}
\]

In terms of classical trajectories \( q_j (\tau) \) and \( \pi_j (\tau) \), the functions \( \Phi (\mathbf{q}, \tau) \) take the form

\[
\Phi (\mathbf{q}, \tau) = \frac{1}{\pi^{3/4} g_{11} \sqrt{2} g_3} \exp \left[ -\frac{3}{g_{11}} \frac{[q_1 - q_1 (\tau)]^2 + [q_2 - q_2 (\tau)]^2}{2} - \frac{f_0 [q_3 - q_3 (\tau)]^2}{g_3} \right]
\]

\[
- \frac{\pi_1 (\tau) [2q_1 - q_1 (\tau)] + \pi_2 (\tau) [2q_2 - q_2 (\tau)] + \pi_3 (\tau) [2q_3 - q_3 (\tau) + i \Omega \tau + i j_3]}{2i}
\]

where

\[
\tilde{A}_3 (\tau) = \tilde{\phi}_3 (\tau) = \Xi \int \frac{q_3 (\tau) \sin^2 \alpha - \tau (\pi_3 (\tau) + \Xi \tau \cos^2 \alpha) \cos^2 \alpha \, d\tau}{2}.
\]

Let us consider a different representation for the GCS. Using the operators \( \hat{A}_j (\tau) \) and \( \hat{A}^\dagger_j (\tau) \), we can construct a Fock space \( | \mathbf{n}, \tau \rangle \), \( n = (n_1, n_2, n_3) \), \( n_j = 0, 1, 2, \ldots \), at any time instant \( \tau \),

\[
| \mathbf{n}, \tau \rangle = \frac{[\hat{A}_1^\dagger (\tau)]^{n_1} [\hat{A}_2^\dagger (\tau)]^{n_2} [\hat{A}_3^\dagger (\tau)]^{n_3}}{\sqrt{n_1! n_2! n_3!}} | \mathbf{0}, \tau \rangle, \quad \hat{A}_j (\tau) | \mathbf{0}, \tau \rangle = 0,
\]

\[
\langle \mathbf{m}, \tau | \mathbf{n}, \tau \rangle = \delta_{m_1, n_1} \delta_{m_2, n_2} \delta_{m_3, n_3}, \quad \sum_{n_1, n_2, n_3 = 0}^\infty | \mathbf{n}, \tau \rangle \langle \mathbf{n}, \tau | = 1.
\]

Then,

\[
| \zeta, \tau \rangle = \exp \left( -\frac{| \zeta |^2}{2} \right) \sum_{n_1, n_2, n_3 = 0}^\infty \frac{\zeta_1^{n_1} \zeta_2^{n_2} \zeta_3^{n_3}}{\sqrt{n_1! n_2! n_3!}} | \mathbf{n}, \tau \rangle = \hat{D} (\zeta, \tau) | \mathbf{0}, \tau \rangle,
\]

where the displacement operator \( \hat{D} (\zeta, \tau) \) reads

\[
\hat{D} (\zeta, \tau) \equiv e^{-\frac{| \zeta |^2}{2}} e^{ \zeta_j \hat{A}_j (\tau)} e^{-\zeta_j \hat{A}^\dagger_j (\tau)} = e^{\zeta_j \hat{A}_j (\tau) - \zeta_j \hat{A}^\dagger_j (\tau)}.
\]

Using the completeness property of the states \( | \mathbf{n}, \tau \rangle \), we can find their overlapping and prove a completeness relation for the GCS,

\[
\langle \zeta', \tau | \zeta, \tau \rangle = \exp \left( \langle \zeta' | \hat{D} (\zeta, \tau) | \zeta \rangle \right) = \frac{1}{\pi^3} \int \int | \zeta, \tau \rangle \langle \zeta, \tau | d^2 \zeta = 1,
\]

\[
d^2 \zeta = d^2 \zeta_1 d^2 \zeta_2 d^2 \zeta_3, \quad d^2 \zeta_j = d \text{Re} \zeta_j d \text{Im} \zeta_j, \quad \forall \tau.
\]
The states $|n, \tau\rangle$ (43) in $q$-representation $\Phi_n(q, \tau) = \langle q|n, \tau\rangle$ have the form (see appendix B)

$$
\Phi_n(q, \tau) = \frac{1}{\sqrt{n_1!n_2!n_3!}} \left(\frac{g_{11}^*}{g_{11}}\right)^{n_1} \left(\frac{g_{22}^*}{i g_{11}}\right)^{n_2} H_{n_1,n_2} \left(\frac{q_1 - iq_2 \sqrt{2} g_{11}^*}{|g_{11}|\sqrt{2}}, \frac{q_1 + iq_2 \sqrt{2} g_{11}^*}{|g_{11}|\sqrt{2}}\right) \Phi_0(q, \tau).
$$

Using the relations

$$
\exp(\mu Z + \nu Z^* - \mu \nu) = \sum_{m_1, m_2=0}^{\infty} H_{m_1,m_2}(Z, Z^*) \frac{\mu^{m_1} \nu^{m_2}}{m_1! m_2!},
$$

$$
\exp(2\chi h - h^2) = \sum_{m=0}^{\infty} \frac{H_m(\chi)}{m!} h^m,
$$

we can calculate the sum in Eq. (46) and rewrite the CS as follows

$$
\Phi_\chi(q, \tau) = \Phi_0(q, \tau) \exp \left[\sqrt{2} \frac{q_3 + \sqrt{2} \text{Re}(\varphi_3 g_3^*)}{g_3} \zeta - \frac{\zeta^2 g_3^*}{2g_3}\right] + \zeta_1 \frac{q_1 - iq_2 \sqrt{2} g_{11}^*}{|g_{11}|\sqrt{2}} \zeta_2 - \frac{|\zeta|^2}{2},
$$

where the vacuum state $\Phi_0(q, \tau)$ reads

$$
\Phi_0(q, \tau) = \frac{1}{\pi^{3/4} g_{11} \sqrt{2} g_3} \exp \left[\frac{i \text{Re} \tilde{\varphi}_4(\tau)}{\sqrt{2} g_3} - |g_3|^2 \frac{\text{Re}^2(\varphi_3)}{g_3} + \frac{i \Omega \tau}{2}\right] \frac{f_{11} q_1^2 + q_2^2}{2} - \frac{f_0 q_3^2}{2} - \sqrt{2} \frac{\varphi_3}{g_3} q_3 \right],
$$

$$
\tilde{\varphi}_4(\tau) = \int \left(\frac{i \Xi \tau}{\sqrt{2} \cos^2 \alpha - \varphi_3^2} g_3\right) d\tau.
$$

Using the relations

$$
|f_{11}(\tau)| = |f_{22}(\tau)|, \quad |g_{11}(\tau)| = |g_{22}(\tau)|,
$$

one can easily verify that the states (51) coincide with ones (39).

A. Standard deviations and uncertainty relations

We recall that the standard deviation $\sigma_\chi(\tau)$ of a some physical quantity $\chi$ in the states $|\zeta, \tau\rangle$ is calculated via the corresponding operator $\hat{\chi}$ as follows

$$
\sigma_\chi(\tau) \equiv \sqrt{\langle \zeta, \tau | (\hat{\chi} - \langle \zeta, \tau | \hat{\chi} | \zeta, \tau \rangle)^2 | \zeta, \tau \rangle} = \sqrt{\chi^2(\tau) - (\chi(\tau))^2};
$$

$$
(\chi(\tau))^2 \equiv \langle \zeta, \tau | \hat{\chi}^2 | \zeta, \tau \rangle, \quad \chi^2(\tau) \equiv \langle \zeta, \tau | \hat{\chi} | \zeta, \tau \rangle.
$$

(54)
Below, we calculate standard deviations of some physical quantities of the problem under discussion. To this end, it is convenient to express the operators $q_j$ and $\hat{q}_j$ in terms of the annihilation and creation operators $\hat{A}_j(\tau)$ and $\hat{A}_j^\dagger(\tau)$. It follows from (17), (18) and (35) that

$$q_1 = \frac{g_{11}^* \hat{A}_1 + g_{11} \hat{A}_1^\dagger - i \left( g_{22}^* \hat{A}_2 - g_{22} \hat{A}_2^\dagger \right)}{\sqrt{2}}, \quad \pi_1 = \frac{f_{11}^* \hat{A}_1 - f_{11} \hat{A}_1^\dagger - i \left( f_{22}^* \hat{A}_2 + f_{22} \hat{A}_2^\dagger \right)}{i\sqrt{2}},$$

$$q_2 = \frac{g_{22}^* \hat{A}_2 + g_{22} \hat{A}_2^\dagger - i \left( g_{11}^* \hat{A}_1 - g_{11} \hat{A}_1^\dagger \right)}{\sqrt{2}}, \quad \pi_2 = \frac{f_{22}^* \hat{A}_2 - f_{22} \hat{A}_2^\dagger - i \left( f_{11}^* \hat{A}_1 + f_{11} \hat{A}_1^\dagger \right)}{i\sqrt{2}},$$

$$q_3 = \frac{g_3^* \hat{A}_3 + g_3 \hat{A}_3^\dagger - 2 \Re (g_3^* \varphi_3)}{\sqrt{2}}, \quad \pi_3 = \frac{f_0^* \hat{A}_3 - f_0 \hat{A}_3^\dagger - 2 \Im (f_0^* \varphi_3)}{i\sqrt{2}}. \quad (55)$$

Then, using Eq. (25) we can easily to find

$$q_1(\tau) = \sqrt{2} \left[ \Im (g_{22}^* \zeta_2) + \Re (g_{11}^* \zeta_1) \right] = q_1^0 - \frac{\Pi_2(\tau) - \Pi_2^0}{\Omega},$$

$$q_2(\tau) = \sqrt{2} \left[ \Im (g_{11}^* \zeta_1) + \Re (g_{22}^* \zeta_2) \right] = q_2^0 + \frac{\Pi_1(\tau) - \Pi_1^0}{\Omega},$$

$$\pi_1(\tau) = \sqrt{2} \left[ \Im (f_{11}^* \zeta_1) - \Re (f_{22}^* \zeta_2) \right] = \frac{\Pi_1(\tau) + \Pi_1^0 - \Omega q_2^0}{2},$$

$$\pi_2(\tau) = \sqrt{2} \left[ \Im (f_{22}^* \zeta_2) - \Re (f_{11}^* \zeta_1) \right] = \frac{\Pi_2(\tau) + \Pi_2^0 + \Omega q_1^0}{2},$$

$$q_3(\tau) = \sqrt{2} \Re [g_3^* (\zeta_3 - \varphi_3)] = q_0 + \Pi_3^0 \tau + \frac{\Xi}{2} \tau^2,$$

$$\pi_3(\tau) = \sqrt{2} \Im [f_0^* (\zeta_3 - \varphi_3)] = \Pi_3^0 + \Xi \tau \sin^2 \alpha, \quad (56)$$

where

$$\Pi_1(\tau) = \Pi_1^0 \cos (\Omega \tau) + \Pi_2^0 \sin (\Omega \tau), \quad \Pi_2(\tau) = \Pi_2^0 \cos (\Omega \tau) - \Pi_1^0 \sin (\Omega \tau),$$

$$q_1^0 = \sqrt{2} \left[ \Re (c_{22}^* \zeta_2) - \Im (c_{11}^* \zeta_1) \right], \quad q_2^0 = \sqrt{2} \left[ \Re (b_{11}^* \zeta_1) - \Im (b_{22}^* \zeta_2) \right],$$

$$q_0 = \sqrt{2} \Re (g_3^* \zeta_3), \quad \Pi_3^0 = \sqrt{2} \Im (f_0^* \zeta_3). \quad (57)$$

Mean values of the operators $q_j^2$ and $\pi_j^2$ in the states $|\zeta, \tau\rangle$ are given by

$$q_\beta^2(\tau) = (q_\beta(\tau))^2 + \frac{|g_{11}|^2 + |g_{22}|^2}{2}, \quad \pi_\beta^2(\tau) = (\pi_\beta(\tau))^2 + \frac{|f_{11}|^2 + |f_{22}|^2}{2},$$

$$q_3^2(\tau) = (q_3(\tau))^2 + \frac{|g_3|^2}{2}, \quad \pi_3^2(\tau) = (\pi_3(\tau))^2 + \frac{|f_0|^2}{2}. \quad (58)$$
Thus, we can find standard deviations of the position $\sigma_q(\tau)$ and momentum $\sigma_\pi(\tau)$,

$$\sigma_q^3(\tau) = \sigma_q^3(\tau) = \sqrt{|g_{11}|^2 + |g_{22}|^2}, \quad \sigma_\pi(\tau) = \sigma_\pi(\tau) = \sqrt{|f_{11}|^2 + |f_{22}|^2},$$

$$\sigma_{q_3}(\tau) = \frac{|g_3|}{\sqrt{2}}, \quad \sigma_{\pi_3}(\tau) = \sigma_{\pi_3}(0) = \sigma_{\pi_3} = \frac{|f_0|}{\sqrt{2}}. \quad (59)$$

By mean from (38) and (59), the constants $c_{\beta\beta}, b_{\beta\beta}, f_0$ and $g_0$ can be related to the initial standard deviations $\sigma_{q_j}(0) \equiv \sigma_{q_j}$ and $\sigma_{\pi_j}(0) \equiv \sigma_{\pi_j}$ in the form

$$|c_{11}| = |c_{22}| = \sigma_q, \quad |b_{11}| = \sigma_\pi \sqrt{1 + \frac{2\Omega + \Omega^2 \sigma_q^2}{4\sigma_\pi^2}} - \frac{2\Omega - \Omega^2 \sigma_q^2}{4\sigma_\pi^2},$$

$$\text{Im}(c_{11}b_{11} + c_{22}b_{22}) = -\Omega \sigma_q^2, \quad \text{Re}(c_{11}b_{11} + c_{22}b_{22}) = \sqrt{4\sigma_q^2 \sigma_\pi^2 - 1},$$

$$\cos(\arg f_0 - \arg f_0) = \frac{1}{2\sigma_{\pi_3} \sigma_{q_3}}, \quad \text{Im}(f_0^* g_0) = \sqrt{4\sigma_{q_3}^2 \sigma_{\pi_3}^2 - 1}. \quad (60)$$

Taking the relations (37) and (60) into account, we can write (59) as follows

$$\sigma_q(\tau) = \sigma_q \sqrt{\cos(\Omega \tau) + \frac{4\sigma_q^2 + \Omega^2 \sigma_q^2}{2\sigma_q^2} \left(1 - \cos(\Omega \tau)\right) + \frac{4\sigma_q^2 \sigma_\pi^2}{2\sigma_q^2} - \frac{\Omega}{2\sigma_q^2} \sqrt{4\sigma_q^2 \sigma_\pi^2 - 1} \sin(\Omega \tau)},$$

$$\sigma_\pi(\tau) = \sigma_\pi \sqrt{1 + \left(\frac{\Omega^2 \sigma_\pi^2}{4\sigma_q^2} - 1\right) \frac{1 - \cos(\Omega \tau)}{2} - \frac{\Omega}{4\sigma_q^2} \sqrt{4\sigma_q^2 \sigma_\pi^2 - 1} \sin(\Omega \tau)},$$

$$\sigma_{q_3}(\tau) = \sigma_{q_3} \sqrt{1 + \frac{4\sigma_{q_3}^2}{2\sigma_{q_3}^2} \left(\frac{1}{\sigma_{q_3}^2} + \frac{\sigma_{\pi_3}^2}{\sigma_{q_3}^2} \tau^2\right)}. \quad (61)$$

One can see that $\sigma_q(\tau), \sigma_\pi(\tau)$ and $\sigma_{q_3}(\tau)$ are real functions if

$$\sigma_q \sigma_\pi \geq \frac{1}{2}, \quad \sigma_{q_3} \sigma_{\pi_3} \geq \frac{1}{2}, \quad (62)$$

which correspond the Heisenberg uncertainty relation in $\tau = 0$.

In what follows, we consider the conditions

$$\sigma_\pi = \frac{1}{2\sigma_q}, \quad \sigma_{\pi_3} = \frac{1}{2\sigma_{q_3}}. \quad (63)$$

Thus, the Heisenberg uncertainty relation is minimal at $\tau = 0$. For any $\tau$, we have that

$$\sigma_q(\tau) \sigma_\pi(\tau) = \frac{1}{2} \sqrt{1 + 1 - \frac{(2 - \Omega^2 \sigma_q^4) \Omega^2 \sigma_q^4}{4\Omega^2 \sigma_q^4} \sin^2(\Omega \tau)} \geq \frac{1}{2},$$

$$\sigma_{q_3}(\tau) \sigma_{\pi_3} = \frac{1}{2} \sqrt{1 + \frac{\tau^2}{4\sigma_{q_3}^4}} \geq \frac{1}{2}. \quad (64)$$
The product $\sigma_q(\tau)\sigma_\pi(\tau)$ is minimal for any $\tau$ if
\[
\sigma_q = \frac{1}{\sqrt{\Omega}}.
\] (65)

One can see that the product $\sigma_q(\tau)\sigma_\pi(\tau)$ is limited from above for any $\tau$,
\[
\frac{1}{2} \leq \sigma_q(\tau)\sigma_\pi(\tau) \leq \frac{1}{4} \sqrt{(2 + \Omega^2\sigma_q^4)\Omega^2\sigma_\pi^4 + 1}. \tag{66}
\]

To obtain the Robertson-Schrödinger relation [13, 14] we need calculate the covariance
\[
\sigma_\chi\kappa(\tau) = \langle (\hat{\chi} - \langle \hat{\chi} \rangle)(\hat{\kappa} - \langle \hat{\kappa} \rangle) + (\hat{\kappa} - \langle \hat{\kappa} \rangle)(\hat{\chi} - \langle \hat{\chi} \rangle) \rangle. \tag{67}
\]

Then,
\[
\sigma_{q_3\pi_3}(\tau) = \langle \hat{\pi}_3q_3 \rangle - \langle \hat{\pi}_3 \rangle \langle q_3 \rangle + \frac{i}{2} = \frac{f_0^*g_3 - 1}{2i}, \tag{68}
\]

Thus, we have that
\[
\sigma_{q_j}^2(\tau)\sigma_\pi^2_j(\tau) - \sigma_{q_j\pi_j}(\tau) = \frac{1}{4}. \tag{69}
\]

The Robertson-Schrödinger uncertainty relations are minimized for any $\tau$, this means that
the GCS are squeezed states, see e.g., [15].

Now, let us study mean values and standard deviations of the velocities,
\[
\hat{\Pi}_1 = \hat{\pi}_1 + \frac{\Omega}{2}q_2, \quad \hat{\Pi}_2 = \hat{\pi}_2 - \frac{\Omega}{2}q_1, \quad \hat{\Pi}_3 = \hat{\pi}_3 + \Xi\tau\cos^2\alpha. \tag{70}
\]

It follows from (56) that
\[
\Pi_1(\tau) = \pi_1(\tau) + \frac{\Omega}{2}q_2(\tau), \quad \Pi_2(\tau) = \pi_2(\tau) - \frac{\Omega}{2}q_1(\tau), \quad \Pi_3(\tau) = \pi_3(\tau) + \Xi\tau\cos^2\alpha. \tag{71}
\]

Using relations (56), (58) and (59), we can calculate mean values of the operators $\hat{\Pi}_1^2$, $\hat{\Pi}_2^2$ and $\hat{\Pi}_3^2$,
\[
\Pi_\beta^2(\tau) = (\Pi_\beta(\tau))^2 + \sigma_\pi^2(\tau) + \frac{\Omega^2\sigma_q^4(\tau)}{4} = (\Pi_\beta(\tau))^2 + \frac{1 + \Omega^2\sigma_q^4}{4\sigma_q^4},
\]
\[
\Pi_3^2(\tau) = (\Pi_3(\tau))^2 + \frac{|f_0|^2}{2}. \tag{72}
\]
Thus, we find the Standard deviations of the velocities in the form

\[ \sigma_{\Pi_j(\tau)} = \sqrt{\frac{1 + \Omega^2 \sigma_q^2}{2 \sigma_q}} \equiv \sigma_{\Pi_j}, \quad \sigma_{\Pi_3(\tau)} = \frac{|f_0|}{\sqrt{2}} \equiv \sigma_{\pi_3}. \quad (73) \]

Taking into account relation (63), we see that in (43) exist families of GCS which differ one from another by values of the parameters \( \sigma = \sigma_q, \sigma_{q_3}, \)

\[
\Phi_{\sigma}(q, \tau) = \exp \left[ -\frac{f_{11}^{\sigma}(\tau)|q_1 - q_1(\tau)|^2 + |q_2 - q_2(\tau)|^2}{2} - \frac{|q_3 - q_3(\tau)|^2}{2(2\sigma_{q_3}^2 + i\tau)} + i\tilde{\phi}_3 + i\Omega \tau}{2} \right. 
- \frac{g_{11}^{\sigma_q}(\tau)\sqrt{2^3\pi^3(\sigma_{q_3}^2 + i\tau)}}{2i} 
\left. \pi_1(\tau)[2q_1 - q_1(\tau)] + \pi_2(\tau)[2q_2 - q_2(\tau)] + \pi_3(\tau)[2q_3 - q_3(\tau)] \right], \quad (74)
\]

where

\[
f_{11}^{\sigma_q}(\tau) = \frac{i \Omega \sigma_q}{2} + \frac{(1 + \Omega \sigma_q^2)(1 + e^{i\Omega \tau})}{4i \sigma_q}, \quad g_{11}^{\sigma_q}(\tau) = -i \sigma_q - \frac{(1 + \Omega \sigma_q^2)(1 - e^{i\Omega \tau})}{2i \Omega \sigma_q}. \quad (75)
\]

Setting electric \( E \) and magnetic \( B \) fields to zero in Eq. (74), we obtain the GCS for the free 3-dimensional particle,

\[
\Phi_{\zeta}(q, \tau) = \exp \left[ -\frac{|q_1 - q_1(\tau)|^2}{2(2\sigma_{q_1}^2 + i\tau)} - \frac{|q_2 - q_2(\tau)|^2}{2(2\sigma_{q_2}^2 + i\tau)} - \frac{|q_3 - q_3(\tau)|^2}{2(2\sigma_{q_3}^2 + i\tau)} \right. 
\left. \pi_1(\tau)[2q_1 - q_1(\tau)] + \pi_2(\tau)[2q_2 - q_2(\tau)] + \pi_3(\tau)[2q_3 - q_3(\tau)] \right], \quad (76)
\]

where \( q_j(\tau) = q_j^0 + \Pi_0^0 \tau \) and \( \pi_j = \Pi_0^0 \).

The function (76) is a product of three one-dimensional GCS that depend on \( q_1, q_2 \) and \( q_3 \) respectively. These one-dimensional GCS coincide with ones obtained and studied in our previous publications [12].

B. CS of a charged particle in constant magnetic field

Quantum nonrelativistic motion of a charged particle in a constant magnetic field was studied in a number of articles, see e.g., [16-20]. CS of such a system were obtained first by Malkin and Man’ko [21], see too [22-26].

The states (51) for zero electric field are GCS for such a system. To compare the latter states with ones by Malkin and Man’ko one has to projet them on the \( xy \)-plane and to use
the relation (65). In Ref. [21], the CS of $\hat{H}_{xy}$ (14) are given by ($\hbar = 1$)

$$\Phi_{\alpha\beta}(\varepsilon) = \sqrt{m\omega/2\pi} \exp \left( -|\varepsilon|^2 + \sqrt{2}\beta\varepsilon + i\sqrt{2}\alpha\varepsilon^* - i\alpha\beta - \frac{|\alpha|^2 + |\beta|^2}{2} \right), \quad (77)$$

where

$$\varepsilon = \sqrt{m\omega x + iy}, \quad \hat{a} = -\frac{i}{\sqrt{2}} \left( \varepsilon + \frac{\partial}{\partial \varepsilon^*} \right), \quad \hat{b} = \frac{1}{\sqrt{2}} \left( \varepsilon^* + \frac{\partial}{\partial \varepsilon} \right)$$

$$\hat{a}\Phi_{\alpha\beta} = \alpha\Phi_{\alpha\beta}, \quad \hat{b}\Phi_{\alpha\beta} = \beta\Phi_{\alpha\beta}. \quad (78)$$

Being rewritten in term of the operators $\hat{a}$ and $\hat{b}$ the Hamiltonian $\hat{H}_{xy}$ takes the form

$$\hat{H}_{xy} = \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (79)$$

Then, the time evolution of $\Phi_{\alpha\beta}(\varepsilon)$ is given by means of $\alpha(t) = \alpha e^{-i\omega t}$ as follows

$$\Phi_{\alpha\beta}(\varepsilon, t) = \exp \left( -i\hat{H}_{xy}t \right) \Phi_{\alpha\beta}(\varepsilon)$$

$$= \sqrt{m\omega/2\pi} \exp \left[ -|\varepsilon|^2 + \sqrt{2} \left( i\alpha e^{-i\omega t} \varepsilon^* + \beta \varepsilon \right) - i\alpha e^{-i\omega t} \beta - \frac{|\alpha|^2 + |\beta|^2}{2} - \frac{i\omega t}{2} \right]. \quad (80)$$

Now, let us consider the states (51) in dimensional variables (15), with $\hbar = 1$, on the plane $xy$ as follows

$$\Psi_{\sigma_\perp}(\mu, t) = \frac{1}{\sqrt{\pi g_{11}}^\sigma} \exp \left[ - \frac{f_{11}^\sigma}{g_{11}^\sigma} |\mu|^2 + \frac{1}{\sqrt{2}} \left( \frac{\zeta_1^* \mu}{g_{11}^\sigma} + \frac{\zeta_2^* \mu}{ig_{22}^\sigma} \right) \right.$$\n$$\left. - \frac{g_{22}^\sigma}{ig_{11}^\sigma} \zeta_1 \zeta_2 - \frac{|\zeta_1|^2 + |\zeta_2|^2}{2} + \frac{i\omega t}{2} \right], \quad (81)$$

where

$$f_{11}^\sigma(t) = \frac{(1 + m\omega \sigma_{1 \perp}^2)(1 + e^{i\omega t}) - 2m\omega \sigma_{1 \perp}^2}{4i\sigma_{1 \perp}}, \quad f_{22}^\sigma(t) = \frac{2m\omega \sigma_{1 \perp}^2 + (1 - m\omega \sigma_{2 \perp}^2)(1 + e^{-i\omega t})}{4i\sigma_{1 \perp}},$$

$$g_{11}^\sigma(t) = \frac{2m\omega \sigma_{1 \perp}^2 - (1 + m\omega \sigma_{2 \perp}^2)(1 - e^{i\omega t})}{2im\omega \sigma_{1 \perp}}, \quad g_{22}^\sigma(t) = \frac{2m\omega \sigma_{1 \perp}^2 + (1 - m\omega \sigma_{2 \perp}^2)(1 - e^{-i\omega t})}{2im\omega \sigma_{1 \perp}},$$

$$\mu = x + iy, \quad \sigma_q = l^{-1}\sigma_{\perp}, \quad \Phi_{\sigma_\perp}(q_{\perp}, \tau) = l\Psi_{\sigma_\perp}(\mu, t). \quad (82)$$

Using the condition (65) as follows

$$\sigma_{\perp} = \frac{1}{\sqrt{m\omega}}. \quad (83)$$
We find that
\begin{align}
    f_{\sigma \perp 11}^\sigma (t) &= \frac{l e^{i \omega t}}{2i} \sqrt{m \omega}, \quad g_{\sigma \perp 11}^\sigma (t) = \frac{e^{i \omega t}}{il \sqrt{m \omega}}, \\
    f_{\sigma \perp 22}^\sigma &= \frac{l \sqrt{m \omega}}{2i}, \quad g_{\sigma \perp 22}^\sigma = \frac{1}{il \sqrt{m \omega}}. 
\end{align}

Thus, the states \( \Psi_{\sigma \perp} (\mu, t) \rightarrow \Psi (\mu, t) \), take the form
\begin{align}
    \Psi (\mu, t) &= \sqrt{\frac{m \omega}{\pi}} \exp \left[ -\frac{m \omega |\mu|^2}{4} + \sqrt{\frac{m \omega}{2}} (i \zeta_1 \mu^* e^{i \omega t} + \zeta_2 \mu) \\
    &- ie^{-i \omega t} \zeta_1 \zeta_2 - \frac{|\zeta_1|^2 + |\zeta_2|^2}{2} - \frac{i \omega t}{2} \right]. 
\end{align}

Using the relations
\begin{align}
    \mu &= \frac{2}{\sqrt{m \omega}} \xi, \quad \zeta_1 = \alpha, \quad \zeta_2 = \beta,
\end{align}
we can show that
\begin{align}
    \Psi (\mu, t) &= \Phi_{\alpha \beta} (\xi, t). 
\end{align}

Thus, we find that the GCS \((51)\) coincides with the CS of Malkin and Man’ko given by the condition \((83)\).

IV. GCS AS SEMICLASSICAL STATES

In general case, the GCS cannot be considered as semiclassical states (SS) because the standard deviations can grow over the time or by means of any other parameter of the system. The study of the SS have attracted attention of theoretical physicists and chemical, see e.g., \([27–36]\). We have that the GCS can considered as SS if the conditions below are satisfied:

I - Position and velocity mean value must propagate along the classical trajectories.

II - Position and velocity standard deviation must have a short interval of variation.

In our previous publication \([12]\), we find that SS of a free particle is given by condition
\begin{align}
    v \gg \frac{h}{2m \sigma_x},
\end{align}
wherein \( v \) is the velocity and \( \sigma_x \) is the position standard deviation. This means that the velocity mean value must be much larger than its corresponding standard deviation. Hence, the variation of the position mean value is much larger than the variation of its corresponding standard deviation. In a certain sense, this implies that the mean values propagate with a
constant standard deviation, which is consistent with the conditions I and II listed above. Of course that in a long instant time we will not have SS anymore.

First, let us study the conditions of SS on the $z$-motion. To this, we return to the initial dimensional variables and rewrite the Eqs. (56) (61), (71) and (73) in these variables, as follows

$$z(t) = z_0 + v_0^z t + \frac{\xi}{2} t^2, \quad v_z(\tau) = v_0^z + \xi t,$$

$$\sigma_z(t) = \sigma_z \sqrt{1 + \frac{\hbar^2}{4m^2\sigma_z^2} t^2}, \quad \sigma_{v_z} = \frac{\hbar}{2m\sigma_z}.$$  

Consider the inequality between the velocity $v_z(\tau)$ and its corresponding standard deviation $\sigma_{v_z}$ in the form

$$v_0^z + \xi t \gg \frac{\hbar}{2m\sigma_z}. \quad (89)$$

One can see that at any time instant $t$ the inequality holds true provided that the condition below is satisfied

$$v_0^z \gg \frac{\hbar}{2m\sigma_z}. \quad (90)$$

The condition (90) implies that the variation of $z(t)$ is much larger than the variation of $\sigma_z(t)$. Thus, we can write

$$\frac{\Delta z(t)}{\Delta \sigma_z(t)} = \frac{z(t) - z(0)}{\sigma_z(t) - \sigma_z(0)} \gg 1 \Rightarrow \frac{v_0^z t + \frac{\xi}{2\sigma_z} t^2}{\sqrt{1 + \frac{\hbar^2}{4m^2\sigma_z^2} t^2}} \gg 1. \quad (91)$$

The electric field allows to have SS when the condition (90) is no longer true. However, this is possible after an instant of time, given by

$$t \gg \left( \frac{\hbar}{2m\sigma_z} - \frac{v_0^z}{\xi} \right) \frac{1}{\xi}. \quad (92)$$

We can see that much larger are the parameters $\xi, v_0^z$ is smaller the time required for the conditions of SS to be satisfied. Then, it follows from (91) that

$$\frac{\xi}{2\sigma_z} \gg \frac{\hbar^2}{4m^2\sigma_z^4} \Rightarrow E \gg \frac{\hbar^2}{2m\sigma_z^3}. \quad (93)$$

On the $xy$-plane, the standard deviation of the position $\sigma_\perp(t)$ and velocity $\sigma_{v_\perp}$ are given by

$$\sigma_x(t) = \sigma_y(t) \equiv \sigma_\perp(t) = \sigma_\perp \sqrt{\cos(\omega t) + \left(1 + \frac{\hbar^2}{m^2\omega^2\sigma_\perp^4}\right) \frac{1 - \cos(\omega t)}{2}},$$

$$\sigma_{v_x} = \sigma_{v_y} \equiv \sigma_{v_\perp} = \frac{\hbar}{2m\sigma_\perp} \sqrt{1 + \frac{m^2\sigma_\perp^4\omega^2}{\hbar^2}}. \quad (94)$$
In this case, the motion is limited and therefore a separate analysis must be carried out beyond those discussed for $z$-motion. One can see that the standard deviations depends on the magnetic field $B$. Thus, we must analysis the following cases:

**Case I**

$$\frac{m^2\sigma^2_\perp \omega^2}{\hbar^2} \gg 1 \Rightarrow B \gg \frac{hc}{e\sigma^2_\perp},$$

(95)

the velocity standard deviation becomes much large and the position standard deviation becomes the smallest possible.

**Case II**

$$\frac{m^2\sigma^2_\perp \omega^2}{\hbar^2} = 1 \Rightarrow B = \frac{hc}{e\sigma^2_\perp},$$

(96)

the position standard deviation becomes a constant

$$\sigma_\perp (t) = \sigma_\perp (0) = \sigma_\perp,$$

(97)

and the velocity standard deviation is given by

$$\sigma_{v\perp} = \frac{\hbar}{\sqrt{2m}\sigma_\perp}.$$    

(98)

**Case III**

$$\frac{m^2\sigma^2_\perp \omega^2}{\hbar^2} = 0 \Rightarrow B = 0,$$

(99)

we have that the velocity standard deviation assumes its lowest value

$$\sigma_{v\perp} = \frac{\hbar}{2m\sigma_\perp},$$

(100)

But, the position standard deviation begins to grow over time

$$\sigma_\perp (t) = \sigma_\perp \sqrt{1 + \frac{\hbar^2}{4m^2\sigma^4_\perp}t^2}.$$    

(101)

**Case IV**

$$0 < \frac{m^2\sigma^4_\perp \omega^2}{\hbar^2} < 1 \Rightarrow 0 < B < \frac{hc}{e\sigma^2_\perp},$$

(102)

The simultaneous measurements in the position and velocity have the standard deviation with the lowest value possible. In addition, we must impose the condition below, such that in the limit of null field the conditions of SS holds true

$$v^0_x \gg \frac{\hbar}{2m\sigma_\perp} \sqrt{1 + \frac{m^2\sigma^4_\perp \omega^2}{\hbar^2}}, \quad v^0_y \gg \frac{\hbar}{2m\sigma_\perp} \sqrt{1 + \frac{m^2\sigma^4_\perp \omega^2}{\hbar^2}}.$$    

(103)
We can see that the condition of SS is not satisfied only in Case I because the standard deviation of the velocity is a function of the magnetic field. The Case IV presents the best description of the SS because the position and velocity mean values presented simultaneously the lowest value in their measurements. In addition, it admit the free particle limit.

Taking into account the limit \( E = B = 0 \), we find the conditions of SS of a free particle in 3-dimension from (89) and (103) as follows

\[
\begin{align*}
v_0^x & \gg \frac{\hbar}{2m\sigma_x}, \\
v_0^y & \gg \frac{\hbar}{2m\sigma_y}, \\
v_0^z & \gg \frac{\hbar}{2m\sigma_z}.
\end{align*}
\]

\[\text{(104)}\]

V. CONCLUDING REMARKS

In this work, we obtain the classical equations of motion which are reduced to the free particle case in the limit of null field by mean of the initial Cauchy data. We find the corresponding GCS that are solutions of the Schrödinger equation, parametrized by the standard deviations \( \sigma_x, \sigma_y \) and \( \sigma_z \) at the initial time instant. These states are squeezed states which present potential application in optical. In addition, we can obtain the GCS of a free particle taking into account the limit of null field, such result it is important to the scattering process. We obtain conditions under which the GCS can be considered as SS and these conditions hold on the limit of null field.

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Appendix A: Solution from the set \((27)\)

To solve equations from the set \((27)\), we are going to use the fact (see Ref. [37]) that

\[
w (x, y) = \Upsilon (z) \exp \left[ \frac{1}{a} \int f (x) \, dx + \frac{1}{b} \int g (y) \, dy \right], \quad z = bx - ay, \tag{A1}
\]

with an arbitrary function \( \Upsilon (z) \) is the general solution of the equation

\[
(a \partial_x + b \partial_y) w (x, y) = \left[ f (x) + g (y) \right] w (x, y). \tag{A2}
\]
Consider the Eq. (28) with $\beta = 2$,

$$
(g_{21} \partial q_1 + g_{22} \partial q_2) \Phi_{\zeta} (q_\perp, \tau) = \left( \sqrt{2} \zeta_2 - f_{21} q_1 - f_{22} q_2 \right) \Phi_{\zeta} (q_\perp, \tau).
$$

(A3)

Comparing the Eqs. (A3) and (A2) we have only one possibility to make the following identifications

$$
w (x, y) = \Phi_{\zeta} (q_\perp, \tau), \quad a = g_{21}, \quad b = g_{22}, \quad q_1 = x, \quad q_2 = y.
$$

(A4)

But, for the functions $f (x) \rightarrow f (q_1)$ and $g (y) \rightarrow g (q_2)$ there is more than one possibility, for example,

I. $f (q_1) = -f_{21} q_1, \quad g (q_2) = \sqrt{2} \zeta_2 - f_{22} q_2$,

II. $f (q_1) = \sqrt{2} \zeta_2 - f_{21} q_1, \quad g (q_2) = -f_{22} q_2$,

III. $f (q_1) = \frac{\zeta_2}{\sqrt{2}} - f_{21} q_1, \quad g (q_2) = \frac{\zeta_2}{\sqrt{2}} - f_{22} q_2$.

(A5)

Any of the possibilities listed above leads to the same solution of the Eqs. (27). Let us consider the situation III. Thus, the general solution of the Eq. (A3) takes the form

$$
\Phi_{\zeta} (q_\perp, \tau) = \Upsilon (u) \exp \left( \frac{\zeta_2 q_1}{\sqrt{2} g_{21}} + \frac{\zeta_2 q_2}{\sqrt{2} g_{22}} - f_{21} \frac{q_1^2}{g_{21}^2} - f_{22} \frac{q_2^2}{g_{22}^2} \right), \quad u = g_{22} q_1 - g_{21} q_2.
$$

(A6)

At the same time, the solution (A6) must be satisfy the Eq. (27) with $\beta = 1$,

$$
(f_{11} q_1 + f_{12} q_2 + g_{11} \partial q_1 + g_{12} \partial q_2) \Phi_{\zeta} (q_\perp, \tau) = \sqrt{2} \zeta_1 \Phi_{\zeta} (q_\perp, \tau).
$$

(A7)

Using the relation $f_{11} g_{21} - f_{21} g_{11} = f_{22} g_{12} - f_{12} g_{22}$ and the transformations

$$
\partial q_1 = g_{22} \partial u + g_{22}^* \partial u^*, \quad \partial q_2 = -g_{21} \partial u - g_{21}^* \partial u^*.
$$

(A8)

We find that (A6) is a solution of the Eq. (A7) if the function $\Upsilon (u)$ satisfies the following equation

$$
\left( \partial_u + \frac{F}{g_{21} g_{22}} - \frac{F_0}{\sqrt{2} g_{21} g_{22}} \right) \Upsilon (u) = 0,
$$

$$
F = \frac{f_{11} g_{21} - f_{21} g_{11}}{g_{11} g_{22} - g_{12} g_{21}}, \quad F_0 = \frac{(g_{11} g_{22} + g_{12} g_{21}) \zeta_2 - 2 g_{21} g_{22} \zeta_1}{g_{11} g_{22} - g_{12} g_{21}},
$$

(A9)

whose general solution reads

$$
\Upsilon (u) = \exp \left( -\frac{F}{g_{21} g_{22}} \frac{u^2}{2} - \frac{F_0}{\sqrt{2} g_{21} g_{22}} u + i \phi_\perp \right),
$$

where $\phi_\perp (\tau)$ is an arbitrary time-dependent function.
Appendix B: Calculating the states $\Phi_n(q, \tau)$

The number states $\Phi_n(q, \tau) = \langle q | n, \tau \rangle$ we find from (45) as follows

$$
\Phi_n(q, \tau) = \frac{\left( \hat{A}_1^n \right)_{n_1} \left( \hat{A}_2^n \right)_{n_2} \left( \hat{A}_3^n \right)_{n_3}}{\sqrt{n_1! n_2! n_3!}} \Phi_0(q, \tau),
$$

(B1)

where the vacuum state $\Phi_0(q, \tau)$ is given in (52).

The creation operators $\hat{A}^\dagger_j(\tau)$ we can write as follows

$$
\hat{A}_1^\dagger = \frac{f^*_j (q_1 - i q_2) - g^*_j (\partial_{q_1} - i \partial_{q_2})}{\sqrt{2}} = -\frac{g^*_j \Phi_\perp^{-1} (\partial_{q_1} - i \partial_{q_2}) \Phi_\perp}{\sqrt{2}},
$$

$$
\hat{A}_2^\dagger = \frac{f^*_j (q_1 + i q_2) - g^*_j (\partial_{q_1} + i \partial_{q_2})}{i \sqrt{2}} = -\frac{g^*_j \Phi_\perp^{-1} (\partial_{q_1} + i \partial_{q_2}) \Phi_\perp}{i \sqrt{2}},
$$

$$
\hat{A}_3^\dagger = \frac{f^*_j q_3 - g^*_j \partial_{q_3}}{\sqrt{2}} + \phi^*_3(\tau) = -\frac{g^*_3 \Phi_3^{-1} \partial_{q_3} \Phi_3}{\sqrt{2}},
$$

(B2)

wherein the functions $\Phi_\perp$ and $\Phi_3$ are given by

$$
\Phi_\perp = \exp \left( -\frac{f^*_j q_1^2 + q_2^2}{2g^*_j} \right), \quad \Phi_3 = \exp \left( -\frac{f^*_j q_3^2}{2g^*_j} - \frac{\sqrt{2} \phi^*_3}{g^*_3} \right).
$$

(B3)

Using the relations, see [38–41], below

$$
\begin{align*}
H_{n,m}(Z, Z^*) &= (-1)^{n+m} e^{Z Z^* \partial_{Z^*} \partial_{Z} - Z Z^*}, \quad Z \in \mathbb{C}, \\
H_n(x) &= (-1)^n e^{x^2} d_x^n e^{-x^2},
\end{align*}
$$

(B4)

where $H_n(x)$ are the Hermite polynomials and $H_{n,m}(Z, Z^*)$ are the Hermite polynomials of two variables, we find that

$$
\begin{align*}
\left( \hat{A}_1^\dagger \right)_{n_1} \left( \hat{A}_2^\dagger \right)_{n_2} \Phi_\perp &= \left( \frac{g^*_j}{g^*_j} \right)_{n_1} \left( \frac{g^*_j}{ig^*_j} \right)_{n_2} H_{n_1,n_2}(\frac{q_1 - i q_2}{\sqrt{2g^*_j}}, \frac{q_1 + i q_2}{\sqrt{2g^*_j}}) \Phi_\perp, \\
\left( \hat{A}_3^\dagger \right)_{n_3} \Phi_3 &= \left( \frac{g^*_j}{2g^*_j} \right)_{n_3} H_{n_3} \left( \frac{q_3 + \sqrt{2} \text{Re}(\phi^*_3)}{|g^*_3|} \right) \Phi_3.
\end{align*}
$$

(B5)

Then, the sum (B1) takes the form

$$
\begin{align*}
\Phi_n(q, \tau) &= \frac{1}{\sqrt{n_1! n_2! n_3!}} \left( \frac{g^*_j}{2g^*_j} \right)_{n_3} H_{n_1,n_2}(\frac{q_1 - i q_2}{\sqrt{2g^*_j}}, \frac{q_1 + i q_2}{\sqrt{2g^*_j}}) \Phi_0(q, \tau) \\
&\times \left( \frac{g^*_j}{g^*_j} \right)_{n_1} \left( \frac{g^*_j}{ig^*_j} \right)_{n_2} H_{n_1,n_2}(\frac{q_1 - i q_2}{\sqrt{2g^*_j}}, \frac{q_1 + i q_2}{\sqrt{2g^*_j}}) \Phi_0(q, \tau).
\end{align*}
$$

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