A uniqueness theorem for degenerate Kerr-Newman black holes

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February 8, 2010

Abstract
We show that the domains of dependence of stationary, $I^+$-regular, analytic, electrovacuum space-times with a connected, non-empty, rotating, degenerate event horizon arise from Kerr-Newman space-times.

1 Introduction

A classical problem in general relativity is that of classification of domains of outer communication of suitably regular black hole space-times. A complete solution for stationary, $I^+$-regular, analytic, vacuum, connected non-degenerate black holes has been given in [11], building on the fundamental work in [6, 23, 30, 32, 33] and others; see [1, 2] for some progress towards removing the hypothesis of analyticity. The analysis in [11] has been extended to the electrovacuum case in [17, 18] (see [5, 7, 27] for previous results). The aim of this work is to remove the condition of non-degeneracy in the rotating case (here $\langle\mathcal{M}_{\text{ext}}\rangle$ denotes the domain of outer communications; the reader is referred to [11] for terminology and further notation):

**Theorem 1.1** Let $(\mathcal{M}, g)$ be a stationary, $I^+$-regular, analytic, electrovacuum space-time with connected, non-empty, rotating, degenerate future event horizon $I^+(\mathcal{M}_{\text{ext}}) \cap \partial(\langle\mathcal{M}_{\text{ext}}\rangle)$. Then $(\langle\mathcal{M}_{\text{ext}}\rangle, g)$ is isometrically diffeomorphic to the domain of outer communications of a Kerr-Newman space-time.

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Non-rotating, degenerate, vacuum and suitably well-behaved solutions are expected not to exist; here one should keep in mind that while the usual staticity argument for non-rotating configurations applies both for non-degenerate [32] (compare [11, end of Section 7]) and degenerate [13, Section 5] configurations, it requires existence of a maximal surface, which has only been proved in the non-degenerate case so far [16]. Static electrovacuum solutions with degenerate components have been classified in [14], see also [13].

The first element needed to prove Theorem 1.1 and missing in the arguments given in [11,18] under the current assumptions, is the global reduction to a harmonic map problem; equivalently, one needs to prove that the area density of the orbits of the isometry group can be used as one of global coordinates on the domain of outer communications; this is established below in Theorem 3.3. The other missing element is the proof that the harmonic map associated to $(M,g)$ lies a finite distance to a Kerr-Newman one; we do this below in Theorem 3.4 in vacuum and Theorem 3.5 in electrovacuum. The remaining arguments of the proof of Theorem 1.1 are as in [11,18]; for the convenience of the reader we present a few more essential steps in Section 4.

Our analysis below can be used to provide a uniqueness theorem for stationary and axisymmetric space-times with several black hole components, along the lines of Corollary 6.3 of [11]; note that many such vacuum configurations are excluded by the analysis in [29].

2 Adapted coordinates

Assuming $I^\pm$-regularity and analyticity, it follows from the Structure Theorem 4.5 in [11] that Hawking’s rigidity theorem [11, Theorem 4.13] applies, and so for each rotating connected component of the future event horizon $I^+(\mathcal{M}_{\text{ext}})\cap\partial\langle\mathcal{M}_{\text{ext}}\rangle$ there exists on $\mathcal{M}_{\text{ext}}$ a Killing vector field $\xi$ tangent to the generators, without zeros on $I^+(\mathcal{M}_{\text{ext}})\cap\partial\langle\mathcal{M}_{\text{ext}}\rangle$, as well as a second Killing vector field $\eta$, commuting with $\xi$, and generating a $U(1)$ action on $\mathcal{M}$. Introducing null Gaussian coordinates [28] near a connected degenerate component of $I^+(\mathcal{M}_{\text{ext}})\cap\partial\langle\mathcal{M}_{\text{ext}}\rangle$, the metric there takes the form

$$g = -\tilde{r}^2 \tilde{F}(\tilde{r}, \tilde{x}) \ dv^2 - 2 d\tilde{r} d\tilde{v} + 2 \tilde{r} h_a(\tilde{r}, \tilde{x}) dv d\tilde{x}^a + h_{ab}(\tilde{r}, \tilde{x}) d\tilde{x}^a d\tilde{x}^b,$$

(2.1)

where $\xi = \partial_v$, the horizon is at $\tilde{r} = 0$, we write $\tilde{x} = (\tilde{x}^a)$, and the $\tilde{x}^a$’s are coordinates on a two dimensional cross section of the horizon, which is
spherical by the topology theorem [13]. All functions are smooth functions of their arguments near $\tilde{r} = 0$.

It has been shown in [22], and rediscovered in [26] (see also [20]), that, for axisymmetric stationary vacuum metrics, the leading order behaviour of the functions above coincides with that of the extreme Kerr metric. We choose the coordinates $\tilde{x}^a$ at $\tilde{r} = 0$ to coincide with the spherical Boyer-Lindquist coordinates $(\tilde{\theta}, \varphi)$ of the Kerr metric (compare (3.3)-(3.8) below). A similar procedure applies to the electrovacuum situation, using [26]. We will return to the details of those constructions in Sections 3.1 and 3.2. The coordinates on the horizon are then propagated away from the horizon so as to obtain the form (2.1) of the metric. Then $\eta = \partial_{\varphi}$, and since the commutator $[\xi, \eta]$ vanishes, the construction leading to (2.1) can be carried out so that all metric functions are independent of both $v$ and $\varphi$.

It turns out to be convenient to rewrite (2.1) as

$$g = -\tilde{r}^2 F(\tilde{r}, \tilde{\theta}) dv^2 - 2 dv \left( d\tilde{r} + \tilde{r} \lambda(\tilde{r}, \tilde{\theta}) d\tilde{\theta} \right) + h_{\varphi\varphi}(d\varphi + \tilde{r} \alpha(\tilde{r}, \tilde{\theta}) dv) d\tilde{\theta} + h_{\tilde{\theta}\tilde{\theta}} d\tilde{\theta}^2 .$$

To obtain this form of the metric one defines $\alpha$ as

$$\alpha := \frac{g_{\varphi v}}{\tilde{r} g_{\varphi\varphi}} \equiv \frac{g(\partial_{\varphi}, \partial_v)}{\tilde{r} g(\partial_{\varphi}, \partial_{\varphi})} ,$$

and the other functions in (2.2) are then obtained by redefinitions:

$$F = \tilde{F} - g_{\varphi\varphi} \alpha^2 , \quad \lambda = -h_{\tilde{\theta}\tilde{\theta}} + g_{\varphi\varphi} \alpha ,$$

with $h_{\varphi\varphi} = g_{\varphi\varphi}$, etc. Since $g_{\varphi\varphi}$ vanishes at zeros of $\varphi$, smoothness of $\alpha$ at the zero-set of $\partial_{\varphi}$ requires justification; this proceeds as follows:

Since $\partial_{\varphi}$ and $\partial_v$ are Killing vector fields, both $g(\partial_{\varphi}, \partial_v)$ and $g(\partial_{\varphi}, \partial_{\varphi})$, and hence their ratio, are scalar functions on space-time. So smoothness of the ratio is obvious away from zeros of $g(\partial_{\varphi}, \partial_{\varphi})$. To proceed further, we need to understand the nature of the zero-set of $\partial_{\varphi}$.

In the current coordinate system the Killing vector field $\eta$ coincides with $\partial_{\varphi}$. It is well known that a periodic Killing vector field cannot be null on a causal domain of outer communications $\langle \langle M_{\text{ext}} \rangle \rangle$. It further follows from [11, Theorem 4.5] that in $I^+$-regular space-times the Killing vector field $\eta$ cannot be null on $I^+ (\langle \langle M_{\text{ext}} \rangle \rangle \cap \partial (\langle \langle M_{\text{ext}} \rangle \rangle))$. So, under the hypothesis of $I^+$-regularity, on this last region the function $g(\partial_{\varphi}, \partial_{\varphi})$ vanishes only at zeros
of $\eta$. Consider then a point $p$ at which a Killing vector field $\eta$ vanishes. It is also well known (see, e.g., [10, Proposition 7.1]) that, in four dimensional space-times, there exists a normal coordinate system $\{x^\mu\}$ centred at $p$ such that:

1. either there exist constants $\beta_\mu \in \mathbb{R}$, $\mu = 0, 1$, not both zero, such that
   \[ \eta = \beta_0 (x^0 \partial_1 + x^1 \partial_0) + \beta_1 (x^3 \partial_2 - x^2 \partial_3) ; \]  
   (2.5)

2. or there exists a constant $a \in \mathbb{R}^*$ such that
   \[ \eta = a \left( (x^0 - x^2) \partial_1 + x^1 (\partial_0 + \partial_2) \right) . \]  
   (2.6)

Exponentiating, in normal coordinates near $p$ the action of the isometry group $\phi_t$ generated by $\eta$ is linear and takes the form

\[ \begin{pmatrix} \cosh(\beta_0 t) & \sinh(\beta_0 t) & 0 & 0 \\ \sinh(\beta_0 t) & \cosh(\beta_0 t) & 0 & 0 \\ 0 & 0 & \cos(\beta_1 t) & -\sin(\beta_1 t) \\ 0 & 0 & \sin(\beta_1 t) & \cos(\beta_1 t) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \]  
(2.7)

in case 1., while in case 2. the matrix $B^\nu_\mu := \nabla_\mu \eta^\nu$ is nilpotent, with $B^3 = 0$, so that the matrix $\Lambda_t$ associated with the action of $\phi_t$ is

\[ \Lambda_t = \text{Id} + tB + \frac{t^2}{2} B^2, \text{ with } B = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \]

This shows that periodic orbits are not possible in the second case, while in the first they are possible if and only if $\beta_0 = 0$.

Smoothness of $\alpha$ can now be established by adapting the analysis of the proof of [8, Proposition 3.1] to the current setting; we provide the details to exhibit some key factorizations needed in the arguments that follow: Consider a covering of

\[ \mathcal{A} := \{\eta = 0\} \]

by domains of definition $\mathcal{O}$ of smooth coordinate systems $x^A$, $A = 0, 1$, and for $q \in \mathcal{O}$ let $x^a$, $a = 2, 3$, denote normal coordinates on $\exp_q \left( (T_q \mathcal{A})^\perp \right)$. Note that the coordinates $(x^2, x^3)$ here are not identical with the ones in (2.5)-(2.6), but the $x^a |_{\exp_p \left( (T_p \mathcal{A})^\perp \right)}$’s coincide, where $p$ is as in the analysis leading
to (2.5)–(2.6). We have just seen that \( \mathcal{A} \) is a smooth timelike submanifold of \( \mathcal{M} \); and \( \mathcal{A} \) is totally geodesic (in the sense of having vanishing second fundamental form) by standard arguments. Set \((x^h) = (x^A, x^a)\), and
\[
\tilde{\rho} = \sqrt{(x^2)^2 + (x^3)^2}.
\] (2.8)

We have the following local form of the metric
\[
g = \hat{g}_{AB}dx^Adx^B + \sum_{a=2}^3(dx^a)^2 \sum_{A,a} O(\tilde{\rho})dx^Adx^a + \sum_{A,B} O(\tilde{\rho}^2)dx^Adx^B + \sum_{a,b} O(\tilde{\rho}^2)dx^adx^b,
\] (2.9)
with \( \hat{g} \) the (Lorentzian) metric induced by \( g \) on \( \mathcal{A} \). The \( O(\tilde{\rho}^2) \) character of the \( dx^adx^b \) error terms is standard in normal coordinates; the \( O(\tilde{\rho}) \) character of the \( dx^a dx^A \) error terms comes from orthogonality; the \( O(\tilde{\rho}^2) \) character of the \( dx^Adx^B \) error terms follows from the totally geodesic character of \( \mathcal{A} \).

The Killing vector field \( \eta \) takes the form \( \eta = x^3 \partial_2 - x^2 \partial_3 = \partial_\varphi \), where
\[
(x^2, x^3) = (\tilde{\rho} \cos \varphi, \tilde{\rho} \sin \varphi).
\] (2.10)

When expressed in terms of \( \tilde{\rho} \) and \( \varphi \), the functions \( g_{\mu\nu} := g(\partial_{x^\mu}, \partial_{x^\nu}) \) are smooth functions of the \( x^\mu \)'s. Let \( R_\pi \) denote a rotation by \( \pi \) in the \((x^a)\)-planes; \( R_\pi \) is obtained by flowing along \( \eta \) a parameter-time \( \pi \) and is therefore an isometry, leading to
\[
g_{ab}(x^A, -x^2, -x^3) = g_{ab}(x^A, x^2, x^3),
g_{AB}(x^A, -x^2, -x^3) = g_{AB}(x^A, x^2, x^3),
g_{Aa}(x^A, -x^2, -x^3) = -g_{Aa}(x^A, x^2, x^3).
\]

In particular all odd-order derivatives of \( g_{ab} \) with respect to the \( x^a \)'s vanish at \( \{x^a = 0\} \), etc. Those symmetry properties together with Borel’s summation Lemma imply that there exist smooth fields \( b_{AB}(x^C, s), \gamma_A(x^C, s), and \gamma(x^C, s) \) such that
\[
g_{AB}(x^C, x^2, x^3) = b_{AB}(x^C, \tilde{\rho}^2),
(g_{Ab}n^b)(x^B, x^2, x^3) = \tilde{\rho}^2 \gamma_A(x^B, \tilde{\rho}^2),
\]
\[
u(x^A, x^2, x^3) := \sqrt{(g(\eta, \eta))(x^A, x^2, x^3)} = \tilde{\rho} \left(1 + \tilde{\rho}^2 \gamma(x^A, \tilde{\rho}^2)\right).
\] (2.11)

Similarly, let \( n = x^a \partial_a \), then \( g_{ab}n^an^b \) and \( g_{ab}n^an^b \) are smooth functions invariant under the flow of \( \eta \), with \( g_{ab}n^an^b = (g_{ab} - \delta_{ab})n^an^b = O(\tilde{\rho}^3) \),
\( g_{ab}n^a n^b = \bar{\rho}^2 + O(\bar{\rho}^4) \), hence there exist smooth functions \( \zeta(x^A, s) \) and \( \sigma(x^A, s) \) such that

\[
(g_{ab} \eta^a n^b)(x^A, x^2, x^3) = \bar{\rho}^2 \zeta(x^A, \bar{\rho}^2),
\]

\[
(g_{ab} n^a n^b)(x^A, x^2, x^3) = \bar{\rho}^2(1 + \bar{\rho}^2 \sigma(x^A, \bar{\rho}^2)).
\]

We note similar formulae for the Maxwell two-form \( F \) and its Hodge-dual \( \ast F \):

\[
(F_{ab} \eta^a n^b)(x^A, x^2, x^3) = \bar{\rho}^2 \zeta(x^A, \bar{\rho}^2),
\]

\[
(\ast F_{ab} \eta^a n^b)(x^A, x^2, x^3) = \bar{\rho}^2 \bar{\zeta}(x^A, \bar{\rho}^2),
\]

\[
(F_{ab} n^a)(x^A, x^2, x^b) = \bar{\rho}^2 \lambda_A(\bar{\rho}^2, x^b).
\]

In polar coordinates (2.10) one therefore obtains

\[
g(\eta, \cdot) = \bar{\rho}^2 \left( (1 + \bar{\rho}^2 \gamma)^2 d\varphi + \zeta \bar{\rho} d\bar{\rho} + \gamma_A dx^A \right).
\]

Writing \( g \) in the form

\[
g = u^2 (d\varphi + \chi_j dy^j)^2 + \gamma_{jk} dy^j dy^k, \tag{2.15}
\]

with \( y^j = (x^A, \bar{\rho}) \), one has \( g(\eta, \cdot) = u^2 (d\varphi + \chi_j dy^j) \) leading to

\[
\chi = \frac{\bar{\rho} \zeta}{(1 + \bar{\rho}^2 \gamma)^2} d\bar{\rho} + \frac{\gamma_A}{(1 + \bar{\rho}^2 \psi)^2} dx^A,
\]

\[
\gamma_{jk} dy^j dy^k = (1 + \bar{\rho}^2 \sigma) d\bar{\rho}^2 + b_{AB} dx^A dx^B + 2\lambda_A \bar{\rho} d\bar{\rho} dx^A - u^2 \chi_i \chi_j dy^i dy^j,
\]

in particular the functions \( \gamma_{\bar{\rho}\bar{\rho}}, \gamma_{AB} \), and \( \gamma_{A\bar{\rho}}/\bar{\rho} \) are smooth functions of \( \bar{\rho}^2 \) and \( x^A \).

We have proved:

**Proposition 2.1** The one-form \( \chi \) defined in (2.15) extends smoothly to the rotation axis \( \mathscr{A} = \{ \eta = 0 \} \).
In particular
\[
\frac{g_{\varphi\varphi}}{g_{\phi\phi}} = \chi(\partial_v)
\]
is a smooth function on space-time. Since it vanishes at \( \tilde{r} = 0 \), the quotient \( \frac{g_{\varphi\varphi}}{\tilde{r} g_{\phi\phi}} \) is also a smooth function on space-time by Taylor’s theorem. Hence the function \( \alpha \) defined in (2.3) is smooth. Smoothness of \( A \) and \( \lambda \) as in (2.4) follows.

In the coordinate system adapted to the horizon as in (2.2), the intersection of the axis \( \mathcal{A} \) and of the Killing horizon corresponds to \( \sin \tilde{\theta} = 0 \). To see that this remains true in a neighbourhood of the horizon, recall that the construction of the Gauss normal coordinates in (2.1) involves the family of null geodesics normal to the section \( S := \{ \nu = 0 \} \) of the connected component of the future event horizon under consideration: the local coordinates \( (\tilde{\theta}, \varphi) \) on \( S \) are first Lie-propagated to \( \tilde{J}^-(S) \) along the normal null geodesics, and then to a neighbourhood of the Killing horizon along the flow of \( \partial_v \). Since \( S \) is invariant under the action of \( U(1) \), so is its normal bundle. It follows from (2.7) that, at the north and south poles of \( S \), which are fixed points of the rotational Killing vector \( \eta \), those normal geodesics are initially tangent to \( \mathcal{A} \). But \( \mathcal{A} \) is totally geodesic, so in fact those geodesics remain on \( \mathcal{A} \): one of them is the generator of the event horizon, the second one is the one which is used to propagate the coordinates \( (\tilde{\theta}, \varphi) \) away from the horizon. Now, \( \mathcal{A} \) is also invariant under the flow of \( \partial_v \), which is tangent on \( \mathcal{A} \) to that null normal geodesic to \( S \) which coincides with the generator of the horizon. Thus \( \partial_v \) is transversal to the other null geodesic on \( \mathcal{A} \), so flowing this other geodesic along \( \partial_v \) fills out a neighbourhood of this geodesic within \( \mathcal{A} \). Since \( \tilde{\theta} \) is constant along the flow of \( \partial_v \), we conclude that \( \sin \tilde{\theta} = 0 \) on \( \mathcal{A} \). Finally, e.g., by dimension considerations, we obtain that \( \{ \sin \tilde{\theta} = 0 \} \) coincides with \( \mathcal{A} = \{ \tilde{\rho} = 0 \} \) in a collar neighbourhood of \( S \).

So, near \( \tilde{\theta} = 0 \) the function \( \tilde{\rho} \) of (2.8) is equivalent to \( \tilde{\theta} \), which is equivalent to \( \sin \tilde{\theta} \), and by the arguments above for small \( \tilde{\theta} \) we have

\[
\frac{\tilde{\rho}}{\sin \tilde{\theta}} = \tilde{f}(\tilde{r}, \tilde{\theta}) ,
\]
(2.17)

for some function \( \tilde{f} \), smooth in its arguments, bounded away from zero, and which can be smoothly extended to an even function of \( \tilde{\theta} \) across zero. Similarly near \( \tilde{\theta} = \pi \) the function \( \tilde{\rho} \) is equivalent to \( \pi - \tilde{\theta} \), which is again equivalent to \( \sin \tilde{\theta} \) near \( \tilde{\theta} = \pi \), and so the function \( \tilde{f} \) in (2.17) extends
smoothly across $\theta = \pi$ to a function which is bounded away from zero and even in $\pi - \theta$ for $\theta$ close to $\pi$. Since (2.17) is trivial away from the zeros of $\sin \tilde{\theta}$, we conclude that (2.17) holds everywhere.

Functions of $\tilde{\theta} \in [0, \pi]$ with the smooth even extension properties near zero and $\pi$, as just described in the last paragraph, will be called sphere functions: indeed, a function of $\tilde{\theta}$ defines a smooth function on a sphere if and only if it is a sphere-function in the sense just defined.

Equations (2.11) and (2.18) lead us to
\[
\frac{h_{\varphi\varphi}}{\sin^2 \tilde{\theta}} = f(\tilde{r}, \tilde{\theta}) ,
\] (2.18)
for some sphere function $f$, smooth in its arguments, and bounded away from zero.

It also follows from what has been said so far that the functions $\alpha$, $\lambda$, $F$, $h_{\varphi\varphi}/\sin^2 \tilde{\theta}$, $h_{\tilde{\theta}\tilde{\theta}}$, $h_{\varphi\tilde{\theta}}/\sin \tilde{\theta}$, and $\lambda/\sin \tilde{\theta}$ are smooth sphere functions of $\tilde{r}$ and $\tilde{\theta}$.

As the next step, we modify the coordinate $\tilde{r}$ to a new coordinate $\hat{r}$ by setting
\[
d\hat{r} = e^\tilde{\chi}(\tilde{r}, \tilde{\theta})(d\tilde{r} + \tilde{r}\lambda(\tilde{r}, \tilde{\theta})d\tilde{\theta}) ,
\] (2.19)
normalized so that
\[
\hat{r}(\tilde{r} = 0, \tilde{\theta} = 0) = 0 .
\]
Equivalently,
\[
\partial_{\tilde{r}} \hat{r} = e^\tilde{\chi} , \quad \partial_{\tilde{\theta}} \hat{r} = \tilde{r} e^\tilde{\chi} \lambda .
\] (2.20)
The integrability conditions for $\hat{r}$ give
\[
\partial_{\tilde{\theta}} \tilde{\chi} - \tilde{r}\lambda \partial_{\tilde{r}} \tilde{\chi} = \partial_{\tilde{r}}(\tilde{r}\lambda) ,
\] (2.21)
which can be solved by shooting characteristics from the north pole $\tilde{\theta} = 0$, where we impose $\tilde{\chi} = 0$. Again smoothness of $\tilde{\chi}$ and of $\hat{r}$ at the north and south poles requires justification: Since $\lambda/\sin \tilde{\theta}$ is a smooth sphere function of $\tilde{r}$ and $\tilde{\theta}$, by matching powers in a power-series expansion of $\tilde{\chi}$ in (2.21) one finds that $\tilde{\chi}$ is a smooth sphere function of $\tilde{r}$ and $\tilde{\theta}$. In other words, for each $\tilde{r}$, $\tilde{\chi}$ defines naturally a smooth function on $S^2$. A similar argument applies to (2.21).

Since $\partial_{\tilde{\theta}} \hat{r} = 0$ at $\tilde{r} = 0$ from (2.19), we have
\[
\hat{r}(\tilde{r} = 0, \tilde{\theta}) = 0
\]
for all $\tilde{\theta}$. Since $\tilde{\chi}$ is a smooth function on $I_\tilde{r} \times S^2$, where $I_\tilde{r}$ is the interval of definition of $\tilde{r}$, (2.20) implies that both

$$\frac{\tilde{r}}{\tilde{r}} \text{ and } \frac{\tilde{r}}{\tilde{r}}$$

are smooth functions on $I_\tilde{r} \times S^2$ near $\{\tilde{r} = 0\}$.

To summarize, we have shown:

**Proposition 2.2** Near a spherical degenerate Killing horizon in an axially symmetric spacetime the metric can be written in the form

$$g = -\tilde{r}^2 F(\tilde{r}, \tilde{\theta}) \, dv^2 + 2 \psi(\tilde{r}, \tilde{\theta}) \, dv \, d\tilde{r} + h_{\phi\phi}(\tilde{r}, \tilde{\theta}) (d\phi + \tilde{r} \alpha(\tilde{r}, \tilde{\theta}) \, dv)^2$$

$$+ h_{\tilde{\theta} \phi}(\tilde{r}, \tilde{\theta}) (d\phi + \tilde{r} \alpha(\tilde{r}, \tilde{\theta}) \, dv) d\tilde{\theta} + h_{\tilde{\theta} \tilde{\theta}}(\tilde{r}, \tilde{\theta}) \, d\tilde{\theta}^2,$$

(2.23)

where $\partial_\phi$ is the Killing field defining the Killing horizon, $\partial_\phi$ is the axial Killing field, the horizon is at $\tilde{r} = 0$, $(\varphi, \tilde{\theta})$ parameterize a two-dimensional spherical cross section of the horizon, and $F$, $\alpha$, $\psi$, $h_{\phi\phi}/\sin^2 \tilde{\theta}$, $h_{\tilde{\theta}\tilde{\theta}}$, $h_{\phi\tilde{\theta}}/\sin \tilde{\theta}$ (and hence also $\det h_{ab}/\sin^2 \tilde{\theta}$) are smooth sphere functions in a neighbourhood of $\tilde{r} = 0$.

Similarly, for any anti-symmetric tensor $\mathcal{F}$ the functions $\mathcal{F}_{\phi\phi}/\sin^2 \tilde{\theta}$, $\mathcal{F}_{\phi\tilde{\theta}}/\sin^2 \tilde{\theta}$, $\mathcal{F}_{\tilde{\theta}\phi}/\sin \tilde{\theta}$, $\mathcal{F}_{\tilde{\theta}\tilde{\theta}}/\sin \tilde{\theta}$ and $\mathcal{F}_{\phi\tilde{\theta}}$ are smooth sphere functions.

### 3 Geometric analysis near a degenerate horizon

In this section, we would like to extract geometric information near a degenerate horizon $\mathcal{E}_0$ in an axially symmetric and stationary electrovacuum space-time $(\mathcal{M}, g)$ using the metric form (2.23).

More precisely, it has been shown in [22] in vacuum, and in [26] in electrovacuum, that the near-horizon geometry is determined uniquely by the area $A_0$ of a cross section $S_0$ of the horizon $\mathcal{E}_0$, the electric charge $q_e$ and the magnetic charge $q_b$ of the horizon. For convenience of notation we introduce the area radius of the horizon:

$$r_0 = \sqrt{\frac{A_0}{4\pi}}.$$ 

(3.1)

Note that, by the near-horizon analysis in [26],

$$r_0^2 \geq q_e^2 + q_b^2.$$ 

(3.2)
3.1 The near-horizon limit in vacuum

Assume that \((\mathcal{M}, g)\) is a vacuum space-time, so \(q_e = q_b = 0\). The near-horizon geometry of the extreme Kerr solution which has horizon area \(A_0\) is given by (see, e.g., [3]):

\[
g_{\text{NHK}} = \frac{1 + \cos^2 \theta}{2} \left[ -\frac{\dot{r}^2}{r_0^2} dt^2 + \frac{r_0^2}{\dot{r}^2} dr^2 + r_0^2 \, d\theta^2 \right] + \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} \left( d\phi + \frac{\dot{r}}{r_0^2} dt \right)^2 ,
\]

where \((t, \dot{r} + r_0/\sqrt{J}, \theta, \phi)\) is the Boyer-Lindquist coordinate system for the Kerr solution.

By the change of variables

\[
v = t - \frac{r_0^2}{\dot{r}} , \quad \varphi = \phi - \log \left( \frac{\dot{r}}{r_0} \right) ,
\]

the above metric can be rewritten as

\[
g_{\text{NHK}} = \frac{1 + \cos^2 \varphi}{2} \left[ -\frac{\dot{\varphi}^2}{r_0^2} dv^2 + 2 dv \, d\dot{r} + r_0^2 \, d\theta^2 \right] + \frac{2r_0^2 \sin^2 \varphi}{1 + \cos^2 \varphi} \left( d\varphi + \frac{\dot{r}}{r_0^2} dv \right)^2 .
\]

We then use the results in [22] and the analysis in Section 2 to obtain, in a neighbourhood of \(E_0\), a null Gaussian coordinate system \((v, \varphi, \ddot{r}, \tilde{\theta})\) such that the metric \(g\) takes the form \((2.23)\) and the coordinates \((\tilde{\theta}, \varphi)\) agree with the coordinate \((\theta, \varphi)\) of the above Kerr metric at \(\ddot{r} = 0\). Furthermore,

\[
F(0, \tilde{\theta}) = \frac{1}{2r_0^2} (1 + \cos^2 \tilde{\theta}) , \\
\psi(0, \tilde{\theta}) = \frac{1}{2} (1 + \cos^2 \tilde{\theta}) , \\
h_{\varphi \varphi}(0, \tilde{\theta}) = \frac{2r_0^2 \sin^2 \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} , \\
h_{\varphi \tilde{\theta}}(0, \tilde{\theta}) = 0 , \\
h_{\tilde{\theta} \tilde{\theta}}(0, \tilde{\theta}) = \frac{1}{2} r_0^2 (1 + \cos^2 \tilde{\theta}) , \\
\alpha(0, \tilde{\theta}) = \frac{1}{r_0^2} .
\]

Observe that equations \((3.5)-(3.7)\) together with Proposition 2.2 allow us to write

\[
h_{\varphi \varphi} = \frac{2r_0^2 \sin^2 \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} \beta_{\varphi \varphi} , \quad \det h = r_0^4 \sin^2 \tilde{\theta} \beta ,
\]

\[10\]
for some smooth sphere functions \( \beta_{\varphi \varphi} \) and \( \beta \) of \((\hat{r}, \hat{\theta})\), which satisfy \( \beta_{\varphi \varphi}(0, \hat{\theta}) \equiv \beta(0, \hat{\theta}) \equiv 1 \).

### 3.2 The near-horizon limit in electrovacuum

In the general case where \((\mathcal{M}, g)\) is electrovacuum, by [26], the near-horizon geometry is characterized by that of the Kerr-Newman solution which has the same horizon area parameter \( A_0 \) and charge parameters \( q_e \) and \( q_b \). In the Kerr-Newman case, the near-horizon fields can be obtained by first applying a duality rotation to \( F_{\text{KN}} \) (to account for the magnetic charge), and then calculating the near-horizon limit. Using, e.g., [24, pp.79-80] one finds (compare [3]):

\[
g_{\text{NHKN}} = \frac{m_0^2 + a_0^2 \cos^2 \theta}{r_0^2} \left[ -\frac{\hat{r}^2}{r_0^2} dt^2 + \frac{r_0^2}{\hat{r}^2} d\hat{r}^2 + r_0^2 d\hat{\theta}^2 \right] + \frac{r_0^2 \sin^2 \theta}{m_0^2 + a_0^2 \cos^2 \theta} \left( d\phi + \frac{2a_0 m_0 \hat{r}}{r_0^4} dt \right)^2,
\]

\[
F_{\text{NHKN}} = q_e \left\{ -\frac{2a_0 m_0 r_0^2 \sin \theta \cos \theta}{(m_0^2 + a_0^2 \cos^2 \theta)^2} d\phi \wedge d\theta + \frac{(m_0^2 - a_0^2 \cos^2 \theta)}{r_0^2 (m_0^2 + a_0^2 \cos^2 \theta)} d\hat{r} \wedge dt 
+ \frac{4a_0^2 m_0^2 \hat{r} \sin \theta \cos \theta}{r_0^2 (m_0^2 + a_0^2 \cos^2 \theta)^2} d\theta \wedge dt \right\}
+ q_b \left\{ \frac{r_0^2 (m_0^2 - a_0^2 \cos^2 \theta) \sin \theta}{(m_0^2 + a_0^2 \cos^2 \theta)^2} d\phi \wedge d\theta + \frac{2a_0 m_0 \cos \theta}{r_0^2 (m_0^2 + a_0^2 \cos^2 \theta)} d\hat{r} \wedge dt 
- \frac{2a_0 m_0 (m_0^2 - a_0^2 \cos^2 \theta) \hat{r} \sin \theta}{r_0^2 (m_0^2 + a_0^2 \cos^2 \theta)^2} d\theta \wedge dt \right\}.
\]

Here \( r_0 \) is as in (3.1), \( a_0 = \sqrt{(r_0^2 - q_b^2 - q_e^2)/2} \) and \( m_0 = \sqrt{a_0^2 + q_e^2 + q_b^2} \). Note that the sign of \( a_0 \) is not determined in [26], but we can always make it positive using the transformation \( \phi \mapsto -\phi \).

Introducing the change of variables

\[
v = t - \frac{r_0^2}{\hat{r}}, \quad \varphi = \phi - \frac{2a_0 m_0}{r_0^2} \log \hat{r},
\]
we obtain
\[ g_{\text{NHKN}} = \frac{m_0^2 + a_0^2 \cos^2 \theta}{r_0^2} \left[ -\frac{\hat{r}^2}{r_0^2} dv^2 + 2d\hat{r}d\hat{\theta} + \frac{r_0^2}{r_0^2} d\theta^2 \right] \]
\[ + \frac{r_0^4 \sin^2 \theta}{m_0^2 + a_0^2 \cos^2 \theta} \left( d\varphi + \frac{2a_0 m_0 \hat{r}}{r_0^4} dv \right)^2 , \quad (3.10) \]
\[ F_{\text{NHKN}} = q e \left\{ -\frac{2a_0 m_0 r_0^2 \sin \theta \cos \theta}{(m_0^2 + a_0^2 \cos^2 \theta)^2} d\varphi \wedge d\theta + \frac{(m_0^2 - a_0^2 \cos^2 \theta)}{r_0^2(m_0^2 + a_0^2 \cos^2 \theta)} d\hat{r} \wedge dv + \frac{2a_0 m_0 \cos \theta}{r_0^4(m_0^2 + a_0^2 \cos^2 \theta)} d\hat{r} \wedge dv \right\} \]
\[ + \frac{4a_0^2 m_0^2 \hat{r} \sin \theta \cos \theta}{r_0^2(m_0^2 + a_0^2 \cos^2 \theta)^2} d\theta \wedge dv + \frac{2a_0 m_0 (m_0^2 - a_0^2 \cos^2 \theta)}{r_0^2(m_0^2 + a_0^2 \cos^2 \theta)} d\hat{r} \wedge dv \]
\[ + \frac{2a_0 m_0 r_0^2 \cos \theta \sin \theta}{r_0^2(m_0^2 + a_0^2 \cos^2 \theta)^2} \hat{r} \sin \theta \wedge dv \right\} . \quad (3.11) \]

Thus, as already explained, we can select a null Gaussian coordinate system \((v, \hat{r}, \tilde{\theta}, \varphi)\) in a neighbourhood of \(E_0\) in \(\mathcal{M}\) such that \(g\) takes the form (2.23) there, the coordinates \((\tilde{\theta}, \varphi)\) coincide with the coordinates \((\theta, \varphi)\) as in (3.10) on \(E_0\), and

\[ F(0, \tilde{\theta}) = \frac{m_0^2 + a_0^2 \cos^2 \tilde{\theta}}{r_0^2} , \quad (3.12) \]
\[ \psi(0, \tilde{\theta}) = \frac{m_0^2 + a_0^2 \cos^2 \tilde{\theta}}{r_0^2} , \quad (3.13) \]
\[ h_{\varphi\varphi}(0, \tilde{\theta}) = \frac{r_0^4 \sin^2 \tilde{\theta}}{m_0^2 + a_0^2 \cos^2 \tilde{\theta}} , \quad (3.14) \]
\[ h_{\varphi\tilde{\theta}}(0, \tilde{\theta}) = 0 , \quad (3.15) \]
\[ h_{\tilde{\theta}\tilde{\theta}}(0, \tilde{\theta}) = m_0^2 + a_0^2 \cos^2 \tilde{\theta} , \quad (3.16) \]
\[ \alpha(0, \tilde{\theta}) = \frac{2a_0 m_0}{r_0^4} . \quad (3.17) \]

Moreover, by Proposition 2.2 we have
\[ h_{\varphi\varphi} = \frac{r_0^4 \sin^2 \tilde{\theta}}{m_0^2 + a_0^2 \cos^2 \tilde{\theta}} \beta_{\varphi\varphi} , \quad \det h = r_0^4 \sin^2 \tilde{\theta} \beta , \quad (3.18) \]

for some smooth sphere functions \(\beta_{\varphi\varphi}\) and \(\beta\) of \((\hat{r}, \tilde{\theta})\) which satisfy \(\beta_{\varphi\varphi}(0, \tilde{\theta}) \equiv \beta(0, \tilde{\theta}) \equiv 1\).
3.3 The orbit-space metric

In the following, we use $x^A$ as the dummy variable for $r$ and $\tilde{\theta}$ and $x^a$ as the dummy variable for $v$ and $\varphi$. This should not be confused with the coordinates $(x^A, x^a)$ of the proof of Proposition 2.1.

The Killing part of the metric $g$ is defined as

$$g_{\parallel} = -\dot{r}^2 F(\dot{r}, \tilde{\theta}) dv^2 + h_{\varphi\varphi}(\dot{r}, \tilde{\theta})(d\varphi + \dot{r} \alpha(r) dv)^2.$$  \hspace{1cm} (3.19)

Note that

$$\det g_{\parallel} = -\dot{r}^2 F h_{\varphi\varphi}.$$  

In particular, $g_{\parallel}$ is Lorentzian for $\dot{r} \neq 0$ if and only if $Ah_{\varphi\varphi}$ is non-negative. In electrovacuum this follows from (2.18) and from the analysis of the near-horizon geometry in \cite{22, 26} (compare (3.3) and (3.12)). Alternatively, one can simply assume that this is true and carry-on the analysis from there.

The orbit-space metric $q$ is defined as

$$q_{AB} = g_{AB} - g_{\parallel}^{ab} g_{Aa} g_{Bb},$$

where $g_{\parallel}^{ab}$ is the matrix inverse to $g_{\parallel} (\partial_a, \partial_b)$. In matrix notation (3.19) reads

$$g_{\parallel} = \begin{bmatrix} \dot{r}^2(-F + \alpha^2 h_{\varphi\varphi}) & \dot{r} \alpha h_{\varphi\varphi} \\ \dot{r} \alpha h_{\varphi\varphi} & h_{\varphi\varphi} \end{bmatrix},$$

and so its inverse reads

$$g_{\parallel}^{-1} = -\frac{1}{\dot{r}^2 F h_{\varphi\varphi}} \begin{bmatrix} h_{\varphi\varphi} & -\dot{r} \alpha h_{\varphi\varphi} \\ -\dot{r} \alpha h_{\varphi\varphi} & \dot{r}^2(-F + \alpha^2 h_{\varphi\varphi}) \end{bmatrix}.$$  

The orbit-space metric is then

$$q = q_{\dot{r}\dot{r}} d\dot{r}^2 + q_{\tilde{\theta}\tilde{\theta}} d\tilde{\theta}^2 = F^{-1} \psi^2 \frac{d\dot{r}^2}{\dot{r}^2} + \frac{\det h}{h_{\varphi\varphi}} d\tilde{\theta}^2.$$  \hspace{1cm} (3.20)

By (3.12)-(3.17) we have

$$q_{\dot{r}\dot{r}} = \frac{1}{\dot{r}^2} \left(m_0^2 + a_0^2 \cos^2 \tilde{\theta} + O(\dot{r})\right), \quad q_{\tilde{\theta}\tilde{\theta}} = m_0^2 + a_0^2 \cos^2 \tilde{\theta} + O(\dot{r}).$$  \hspace{1cm} (3.21)

with the error terms meant for small $\dot{r}$.  

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Now, define the function $\rho$ by
\[
\rho = \sqrt{-\det g} = \hat{r} \sqrt{F h_{\varphi\varphi}} .
\] (3.22)

By (3.12) and (3.18) we have
\[
\rho = \hat{r} \tilde{\beta}_\rho \sin \tilde{\theta}
\] (3.23)
for some smooth sphere function $\tilde{\beta}_\rho = \tilde{\beta}_\rho(\hat{r}, \tilde{\theta})$ such that
\[
\tilde{\beta}_\rho(\hat{r}, \tilde{\theta}) = 1 + O(\hat{r}) .
\]

By the Einstein-Maxwell electrovacuum equation, $\rho$ is harmonic with respect to $q$ (see, e.g., [35, Section 2]). Let $z$ be minus the harmonic conjugate of $\rho$, i.e. $z$ is defined up to a constant by
\[
z_{,\hat{r}} = q^{\tilde{\theta}\hat{\theta}} \sqrt{\det q} \rho_{,\tilde{\theta}} = \sqrt{\frac{h_{\varphi\varphi}}{F \det h}} \hat{r} \rho_{,\tilde{\theta}} ,
z_{,\tilde{\theta}} = -q^{\hat{r}\tilde{\theta}} \sqrt{\det q} \rho_{,\hat{r}} = -\sqrt{\frac{F \det h \hat{r}}{h_{\varphi\varphi}}} \rho_{,\hat{r}} .
\]

By (3.12)-(3.16) and (3.23), we have
\[
z_{,\hat{r}} = \gamma \tilde{\beta}_\rho \cos \tilde{\theta} , \quad z_{,\tilde{\theta}} = -\frac{1}{\gamma} \hat{r} (\hat{r} \tilde{\beta}_\rho)_{,\hat{r}} \sin \tilde{\theta} ,
\]
where $\gamma$ is some smooth positive function of $(\hat{r}, \tilde{\theta})$ such that $\gamma(0, \tilde{\theta}) \equiv 1$. Thus, up to a shift by a constant,
\[
z = \hat{r} \tilde{\beta}_z \cos \tilde{\theta} , \quad (3.24)
\]
where
\[
\tilde{\beta}_z(\hat{r}, \tilde{\theta}) = \frac{1}{\hat{r}} \int_0^{\hat{r}} \gamma(s, \tilde{\theta}) \tilde{\beta}_\rho(s, \tilde{\theta}) \, ds = 1 + O(\hat{r}) .
\]

Altogether, (3.23) and (3.24) imply that
\[
r^2 := \rho^2 + z^2 = \hat{r}^2 + O(\hat{r}^3) \text{ as } \hat{r} \to 0 .
\]

We have thus proved:
Proposition 3.1 In $I^+$-regular, axisymmetric, stationary and electrovacuum space-times, every degenerate component of the event horizon corresponds to a point lying on the axis $\rho = 0$ in the $(\rho, z)$ plane.

Remark 3.2 For the sake of simplicity we have stated the result under the hypotheses of Theorem 1.1. However, the analysis above only uses the following: the horizon is degenerate, and has a spherical cross-section $S$ on which the Killing vector $\partial_\tau$ has no zeros; the Killing vector field $\partial_\phi$ is spacelike wherever non-zero; the function $\rho$ is harmonic with respect to the orbit-space metric $q$; and finally
\[
\lim_{r \to 0} \frac{F \det h}{\psi^2 h_{\phi\phi}} = 1 = \lim_{r \to 0} \frac{F h_{\phi\phi}}{\sin^2 \tilde{\theta}}.
\]

3.4 Global isothermal coordinates

We wish to show that the functions $\rho$ and $z$ provide global coordinates on the quotient manifold $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle / (\mathbb{R} \times U(1))$, where $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle$ is the domain of outer communications in $(\mathcal{M}, g)$. For this we adopt the strategy in [11], which in turn draws on [9]; the arguments there need to be extended in a non-trivial way to cover the current setting.

Let $\mathcal{B}$ be the manifold obtained from the orbit space $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle / (\mathbb{R} \times U(1))$ by doubling along the axis, as in [9]. The metric $q$ extends smoothly to a smooth metric on $\mathcal{B}$, which we will also denote by $q$. In this section, we show that the functions $\rho$ and $z$, defined in the previous section and appropriately extended to the double, provide global isothermal coordinates for $(\mathcal{B}, q)$ and hence, by restriction, for $\langle\langle \mathcal{M}_{\text{ext}} \rangle\rangle / (\mathbb{R} \times U(1))$.

As shown in Proposition 3.1 in the connected case the horizon corresponds to a point $p$ in a one-point completion $\mathcal{B} := \mathcal{B} \cup \{p\}$ of $\mathcal{B}$. The point $p$ will be denoted by $0$; the reason for this slight abuse of notation will be clear momentarily. For configurations with $N_d$ degenerate components of the horizon and $N_r$ non-degenerate ones, each degenerate horizon will correspond to a point $p_i$ in a completion
\[
\overline{\mathcal{B}} := \mathcal{B} \cup_{i=1}^{N_d} \{p_i\} \cup_{a=1}^{N_r} D^1_a
\]
where the $D^1_a$’s are disks corresponding to smooth boundary components for $\overline{\mathcal{B}}$; see [11] for a detailed description of the non-degenerate components of the event horizon. It should be noted that the point 0 in the former case, and the $p_i$’s in the latter case, are genuinely not points in $\mathcal{B}$. 

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In a \( \mathcal{B} \)-neighbourhood of each \( p_i \), we parameterize \( \mathcal{B} \) by a small punctured disc \( D_{4\epsilon} \setminus \{0\} \subset \mathbb{R}^2 \) via the polar map \((\hat{x}, \hat{y}) \mapsto (\hat{r}, \hat{\theta})\), with \( \hat{r} \in (0, 4\epsilon) \). By (3.21) in this region, \( q \) is conformal to
\[
\hat{q} := d\hat{r}^2 + r^2 f(\hat{r}, \hat{\theta}) \, d\hat{\theta}^2,
\]
where \( f \) is a smooth sphere function such that \( f(0, \hat{\theta}) \equiv 1 \). This can be rewritten as
\[
\hat{q} = d\hat{r}^2 + f(\hat{r}, \hat{\theta}) (d\hat{x}^2 + d\hat{y}^2 - d\hat{r}^2) + \frac{f(\hat{r}, \hat{\theta}) - 1}{\hat{r}^2} (\hat{x} \, d\hat{x} + \hat{y} \, d\hat{y})^2.
\]
So \( \hat{q} \) will extend smoothly across \( \hat{x} = \hat{y} = 0 \) if and only if \( f - 1 \) equals \( \hat{r}^2 \) times a smooth function of \( \hat{x} \) and \( \hat{y} \). If this happens to be the case, we can apply [9, Theorem 2.9] to reach the desired conclusion, Theorem 3.3 below. However, it is not clear that \( f \) will take this form in general, so the above strategy needs to be revised to allow general metrics \( \hat{q} \) as above. For this we need to provide first some preliminary analysis.

Let \( R_{\hat{q}} \) denote the Gaussian curvature of \( \hat{q} \). It is evident that \( R_{\hat{q}} \) is smooth in all sufficiently small punctured discs \( D_{4\epsilon} \setminus \{0\} \), \( 0 < \epsilon < \epsilon_0 \) for some \( \epsilon_0 \), with
\[
R_{\hat{q}} = O(\hat{r}^{-1}) \quad \text{and} \quad |D R_{\hat{q}}| = O(\hat{r}^{-2}) \quad \text{for small} \ \hat{r} > 0. \tag{3.25}
\]
Moreover, the usual formula for the scalar curvature in a frame formalism in dimension two shows that there are functions
\[
\hat{f}_x, \hat{f}_y \in C^\infty(D_{4\epsilon} \setminus \{0\}) \cap L^\infty(D_{4\epsilon}) \quad \text{such that} \quad R_{\hat{q}} = \partial_x \hat{f}_x + \partial_y \hat{f}_y. \tag{3.26}
\]
In particular, \( R_{\hat{q}} \in H^{-1}(D_{4\epsilon}) \). Let \( \hat{u} \in H^1_0(D_{4\epsilon}) \) be the solution to (see e.g. [21, Theorem 8.3])
\[
\begin{cases}
-\Delta \hat{u} = \frac{R_{\hat{q}}}{2} \quad \text{in} \ D_{4\epsilon}, \\
\hat{u} = 0 \quad \text{on} \ \partial D_{4\epsilon},
\end{cases}
\]
so that the metric \( e^{-2\hat{u}} \hat{q} \) is flat in \( D_{2\epsilon} \setminus \{0\} \). By (3.26) and standard elliptic estimates (see e.g. [21, Theorem 8.24]), \( \hat{u} \) is smooth in \( D_{4\epsilon} \setminus \{0\} \), \( \mu \)-Hölder continuous in \( D_{4\epsilon} \) for some \( \mu \in (0, 1) \) with
\[
\|\hat{u}\|_{C^\mu(D_{2\epsilon})} \leq C(\epsilon, \|f\|_{L^\infty(D_{4\epsilon})}, \|\hat{f}_x\|_{L^\infty(D_{4\epsilon})}, \|\hat{f}_y\|_{L^\infty(D_{4\epsilon})}).
\]
Now, pick any \( \hat{u} \in C^\infty(\mathcal{B}) \cap C^\mu(\overline{\mathcal{B}}) \) such that \( \hat{u} \equiv u \) in the region which is parameterized by \( D_\epsilon \). Define \( \tilde{q} = e^{-2\tilde{u}} \hat{q} \). It is readily seen that \( \tilde{q} \) is flat near 0. Since \( \hat{u} \) is continuous, \( \tilde{q} \) has no conical singularity at the origin, and so the metric \( \tilde{q} \) is smooth across 0 in an appropriate differentiable structure (which might, or might not, coincide with the one defined by the coordinates \( \hat{x} \) and \( \hat{y} \), but this turns out to be irrelevant for what follows).

We can now apply [9, Theorem 2.19] to find a function \( \tilde{u} \in C^\infty(\mathcal{B}) \) such that \( q := e^{-2\tilde{u}} \tilde{q} \) is a smooth flat metric on the complete simply connected manifold \( \mathcal{B} \). Since the relevant equations are conformally invariant, one can ignore the possible singularities at the \( p_i \)'s of the conformal factor relating \( q \) and \( \tilde{q} \), and proceed as in [11] (see in particular the argument leading from Equation (6.8) to Equation (6.11) there) to show that \( \rho \) and \( z \) provide a global coordinate system on \( \langle \mathcal{M}_{\text{ext}} \rangle/(\mathbb{R} \times U(1)) \). We conclude that

**Theorem 3.3** Under the hypotheses of Theorem 1.1, the area function \( \rho \) and its harmonic conjugate \( -z \) form a global manifestly asymptotically flat coordinate system on \( \langle \mathcal{M}_{\text{ext}} \rangle/(\mathbb{R} \times U(1)) \).

For the sake of completeness, we note some regularity properties of \( \hat{u} \). Fix some point \( p \in D_\epsilon \setminus \{0\} \). Applying [21, Theorem 8.32] to \( u - u(0) \) in \( D_{|p|/2}(p) \) and recalling \( (3.25) \), we have

\[
|D\hat{u}(p)| \leq C \left[ |p|^{-1} \|u(0)\|_{L^\infty(D_{|p|/2}(p))} + |p| \|R\tilde{q}\|_{L^\infty(D_{|p|/2}(p))} \right] \\
\leq C \left[ |p|^{\mu - 1} + 1 \right].
\]

It follows that

\[
|D\hat{u}(p)| \leq |p|^{\mu - 1} \text{ for any } p \in D_\epsilon \setminus \{0\}. \tag{3.27}
\]

Similarly, applying [21, Theorem 6.2] to \( u - u(0) \) in \( D_{|p|/2}(p) \) and noting that, by \( (3.25) \),

\[
\|R\tilde{q}\|_{C^{\mu'}(D_{|p|/2}(p))} \leq C |p|^{1-\mu'} \text{ for any } \mu' \in (0, 1],
\]

we get

\[
|D^2\hat{u}(p)| \leq C \left[ |p|^{-2} \|u(0)\|_{L^\infty(D_{|p|/2}(p))} + |p|^{\mu'} \|R\tilde{q}\|_{C^{\mu'}(D_{|p|/2}(p))} \right] \\
\leq C \left[ |p|^{\mu - 2} + |p|^{\mu' - 1} \right].
\]

We thus have

\[
|D^2\hat{u}(p)| \leq |p|^{\mu - 2} \text{ for any } p \in D_\epsilon \setminus \{0\}. \tag{3.28}
\]
3.5 Hypersurface-orthogonality

Recall that $\xi = \partial_v$, $\eta = \partial_\varphi$. It is well-known that, in electrovacuum (see, e.g., [25]), the plane distribution $\mathrm{Span}\{\xi, \eta\}$ is integrable; equivalently

$$d\xi \wedge \xi \wedge \eta = d\eta \wedge \xi \wedge \eta = 0 .$$

(3.29)

By direct computations, we find

$$d\xi \wedge \xi \wedge \eta = \left\{ \hat{r}^2 h_{\varphi \varphi} \left[ - \alpha h_{\varphi \varphi} [\alpha \psi] \hat{\theta} + F_{,\hat{\vartheta}} \psi - \psi_{,\hat{\theta}} F \right] \\
+ \hat{r}^3 \alpha F \left[ - h_{\varphi \varphi,\hat{r}} h_{\varphi \hat{\theta}} + h_{\varphi \varphi} h_{\varphi \hat{\theta},\hat{r}} \right] \right\} \, dv \wedge d\hat{r} \wedge d\varphi \wedge d\hat{\theta} ,
$$

$$d\eta \wedge \xi \wedge \eta = \left\{ - \hat{r} \psi h_{\varphi \varphi}^2 \alpha \hat{\theta} \\\n+ \hat{r}^2 F \left[ - h_{\varphi \varphi,\hat{r}} h_{\varphi \hat{\theta}} + h_{\varphi \varphi} h_{\varphi \hat{\theta},\hat{r}} \right] \right\} \, dv \wedge d\hat{r} \wedge d\varphi \wedge d\hat{\theta} .
$$

Thus the hypersurface orthogonality condition (3.29) reads

$$\psi h_{\varphi \varphi}^2 \alpha \hat{\theta} + \hat{r} F \left[ - h_{\varphi \varphi,\hat{r}} h_{\varphi \hat{\theta}} + h_{\varphi \varphi} h_{\varphi \hat{\theta},\hat{r}} \right] = 0 ,$$

(3.30)

$$F_{,\hat{\vartheta}} \psi + (-F + \alpha^2 h_{\varphi \varphi}) \psi_{,\hat{\theta}} = 0 .$$

(3.31)

3.6 The Ernst potential of $\partial_\varphi$ in vacuum

We now turn our attention to the second missing ingredient required for the uniqueness argument. Namely, we will show that, in a neighbourhood of the horizon $E_0$, the harmonic map associated to $(\mathcal{M}, g)$ lies a finite distance from that associated to the Kerr-Newman solution which has the same parameters $A_0$, $q_e$ and $q_b$. We start with the special case where $(\mathcal{M}, g)$ is vacuum. The electrovacuum case will be considered in Section 3.7.

The (complex) Ernst potential associated with the Killing vector $\eta = \partial_\varphi$ is defined as $X + i Y$ where

$$X = g(\eta, \eta) , \quad dY = * (\eta \wedge d\eta) ,$$

where $*$ is the Hodge operator of $g$. Here, by a common abuse of notation, we use the same symbol $\eta$ for the vector $\eta$ and its metric dual $g(\eta, \cdot)$. The existence of the twist potential $Y$ is a consequence of the Einstein vacuum equations; see, e.g., [33] Section 2].
The reference Kerr metric has been chosen to have the same area radius $r_0$ as the metric under consideration, and so from (3.30), we have

$$X = \frac{2r_0^2 \sin^2 \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} (1 + O(\hat{r})). \quad (3.32)$$

To obtain the twist potential $Y$, a computation gives

$$\eta \wedge d\eta = -h_{\varphi\varphi}^2 [\hat{r} \alpha,\hat{r}] dv \wedge d\hat{r} \wedge d\varphi + \hat{r} h_{\varphi\varphi}^2 \alpha,\hat{\vartheta} dv \wedge d\varphi \wedge d\hat{\theta}$$

$$+ \hat{r} \alpha h_{\varphi\varphi} h_{\varphi\varphi,\hat{r}} - h_{\varphi\varphi} [\hat{r} \alpha h_{\varphi\varphi},\hat{r}] dv \wedge d\hat{r} \wedge d\hat{\theta}$$

$$+ \left[ -h_{\varphi\varphi} h_{\varphi\varphi,\hat{r}} + h_{\varphi\varphi} h_{\varphi\varphi,\hat{r}} \right] d\hat{r} \wedge d\varphi \wedge d\hat{\theta}.$$ 

Using

$$\text{det} g_{\mu\nu} = -\psi^2 \text{det} h,$$

we are led to

$$*(\eta \wedge d\eta) = \frac{dv}{\sqrt{\text{det} h}} \left\{ -\hat{r} h_{\varphi\varphi}^2 \alpha,\vartheta + \hat{r}^2 F \psi^{-1} \left[ -h_{\varphi\varphi,\hat{r}} + h_{\varphi\varphi} h_{\varphi\varphi,\hat{r}} \right] \right\}$$

$$- \frac{d\hat{r}}{\sqrt{\text{det} h}} \left[ -h_{\varphi\varphi,\hat{r}} + h_{\varphi\varphi} h_{\varphi\varphi,\hat{r}} \right]$$

$$+ d\hat{\theta} h_{\varphi\varphi} \sqrt{\text{det} h} \psi^{-1} [\hat{r} \alpha,\hat{r}]. \quad (3.33)$$

In the above formula, the $dv$ component must vanish. This is a consequence of one of the hypersurface orthogonality conditions, namely that $d\eta \wedge \xi \wedge \eta = 0$ (see (3.30)). Thus,

$$*(\eta \wedge d\eta) = -\frac{1}{\sqrt{\text{det} h}} \left[ -h_{\varphi\varphi,\hat{r}} + h_{\varphi\varphi} h_{\varphi\varphi,\hat{r}} \right] d\hat{r}$$

$$+ h_{\varphi\varphi} \sqrt{\text{det} h} \psi^{-1}[\hat{r} \alpha,\hat{r}] d\hat{\theta}. \quad (3.33)$$

Now, as $dY = *(\eta \wedge d\eta)$, the relations (3.34), (3.8) and (3.9) imply

$$Y_{,\hat{r}} = \gamma_{\hat{r}} \sin \tilde{\theta}, \quad Y_{,\tilde{\theta}} = 4r_0^2 \gamma_{\tilde{\theta}} \frac{\sin^3 \tilde{\theta}}{(1 + \cos^2 \tilde{\theta})^2}, \quad (3.34)$$

where $\gamma_{\hat{r}}$ and $\gamma_{\tilde{\theta}}$ are smooth sphere function of $(\hat{r},\tilde{\theta})$ with $\gamma_{\tilde{\theta}}(0,\tilde{\theta}) \equiv 1$.

By [11] Section 6], in a sufficiently regular black hole space-time, in a collar neighbourhood of every component of the Killing horizon the axis of
rotation $\mathcal{A}$ has exactly two connected components, each of which meets a cross section of the horizon at exactly one point. Now, by Proposition 2.2, in a neighbourhood of the horizon, $\partial \phi$ vanishes along $\{\tilde{\theta} = 0\}$ and $\{\tilde{\theta} = \pi\}$. Evidently these two sets correspond to different component of $\mathcal{A}$. Denote by $\mathcal{A}_+$ and $\mathcal{A}_ -$ the components of $\mathcal{A}$ that contain $\{\tilde{\theta} = 0\}$ and $\{\tilde{\theta} = \pi\}$, respectively.

It is well-known that $Y$ is constant on each component of $\mathcal{A}$. In a neighbourhood of the horizon, this can be seen readily from the first equation in (3.34). Away from the horizon, see e.g. [12, Eq. (2.6)] or [33]. By (3.34), we have

$$Y|_{\mathcal{A}_-} - Y|_{\mathcal{A}_+} = \int_0^\pi Y_{\tilde{\theta}}(0, \tilde{\theta}) d\tilde{\theta} = 4 r_0^2 \int_0^\pi \frac{\sin^3 \tilde{\theta}}{(1 + \cos^2 \tilde{\theta})^2} d\tilde{\theta} = 4 r_0^2.$$  

Hence, shifting $Y$ by a constant if necessary, we can assume that

$$Y|_{\mathcal{A}_-} = 2 r_0^2, \quad Y|_{\mathcal{A}_+} = -2 r_0^2. \quad (3.35)$$

Then, by integrating (3.34),

$$Y = -\frac{4 r_0^2 \cos \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} + \delta Y(\hat{r}, \tilde{\theta}),$$

where $\delta Y$ is given by

$$\delta Y(\hat{r}, \tilde{\theta}) = 4 r_0^2 \int_0^{\tilde{\theta}} (\gamma_{\tilde{\theta}}(\hat{r}, \tau) - 1) \frac{\sin^3 \tau}{(1 + \cos^2 \tau)^2} d\tau$$

$$= -4 r_0^2 \int_0^\pi (\gamma_{\tilde{\theta}}(\hat{r}, \tau) - 1) \frac{\sin^3 \tau}{(1 + \cos^2 \tau)^2} d\tau.$$  

It thus follows that

$$Y = -\frac{4 r_0^2 \cos \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} + O(\hat{r} \sin^4 \tilde{\theta}). \quad (3.36)$$

To proceed, we recall that the distance $d_b$ between two points $(X_1, Y_1)$ and $(X_2, Y_2)$ in the (real) hyperbolic plane is implicitly given by the formula [4, Theorem 7.2.1]:

$$\cosh d_b - 1 = \frac{1}{2} \left( \left( \sqrt{\frac{X_1}{X_2}} - \sqrt{\frac{X_2}{X_1}} \right)^2 + \frac{(Y_1 - Y_2)^2}{X_1 X_2} \right).$$
Also, recall that we have shown that the functions \( z \) and \( \rho \) defined in Section 3.3 provide global isothermal coordinates on the orbit space. Define \((r, \theta)\) by
\[
(z, \rho) = (r \cos \theta, r \sin \theta).
\]

Now consider a reference Ernst potential \( X_{Kerr} + iY_{Kerr} \) as given in [19]:

\[
X_{Kerr}(r, \theta) = \left( \frac{1}{2} (r \sqrt{2} + r_0)^2 + \frac{r_0^2}{2} + \frac{r_0^3 (r \sqrt{2} + r_0) \sin^2 \theta}{(r \sqrt{2} + r_0)^2 + r_0^2 \cos^2 \theta} \right) \sin^2 \theta ,
\]
\[
Y_{Kerr}(r, \theta) = r_0^2 (\cos^2 \theta - 3 \cos \theta) - \frac{r_0^4 \cos \theta \sin^4 \theta}{(r \sqrt{2} + r_0)^2 + r_0^2 \cos^2 \theta} .
\]

Here \( r \) and \( \theta \) are polar coordinates associated to Kerr’s own \((z, \rho)\) coordinates. It is convenient to rewrite \( X_{Kerr} \) and \( Y_{Kerr} \) as

\[
X_{Kerr}(r, \theta) = \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} + O(r \sin^2 \theta) ,
\]
\[
Y_{Kerr}(r, \theta) = -\frac{4r_0^2 \cos \theta}{1 + \cos^2 \theta} + O(r \sin^4 \theta) .
\]

The leading order term near \( r = 0 \) for \( Y_{Kerr} \) can also be rewritten in the following form

\[
-\frac{4r_0^2 \cos \theta}{1 + \cos^2 \theta} = -2r_0^2 + \frac{r_0^2 \sin^4 \theta}{2(1 + \cos^2 \theta) \cos^4(\theta/2)} ,
\]

useful away from \( \theta = \pi \), or as

\[
-\frac{4r_0^2 \cos \theta}{1 + \cos^2 \theta} = 2r_0^2 - \frac{r_0^2 \sin^4 \theta}{2(1 + \cos^2 \theta) \sin^4(\theta/2)} ,
\]

which is useful away from \( \theta = 0 \). This shows that in either case the deviation from the constant terms \( \pm 2r_0^2 \) in \( Y_{Kerr} \) factors out through \( \sin^4 \theta \).

In the remainder of this section we derive a bound for the hyperbolic distance between \((X, Y)\) and \((X_{Kerr}, Y_{Kerr})\), which are compared after identifying the \((z, \rho)\) coordinates of the solution under consideration with the \((z, \rho)\) coordinates of the reference Kerr solution. This leads to relations between \((r, \theta)\) and \((\hat{r}, \tilde{\theta})\), which we analyze now. By (3.23) and (3.24) we have

\[
r \sin \theta = \rho = \hat{r} \hat{\beta}_\rho \sin \tilde{\theta} , \quad \text{and} \quad r \cos \theta = z = \hat{r} \hat{\beta}_z \cos \tilde{\theta} .
\]
Thus, for small $\hat{r}$,

\begin{align*}
    r^2 &= \rho^2 + z^2 = \hat{r}^2 + O(\hat{r}^3), \\
    \sin \theta &= (1 + O(\hat{r})) \sin \tilde{\theta} . \\
    \cos \theta &= (1 + O(\hat{r})) \cos \tilde{\theta} .
\end{align*}

(3.41)

(3.42)

(3.43)

Substituting (3.41)–(3.43) into (3.39)–(3.40) we get

\begin{align*}
    X_{Kerr}(r, \theta) &= \frac{2r_0^2 \sin^2 \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} (1 + O(\hat{r})) , \\
    Y_{Kerr}(r, \theta) &= -\frac{4r_0^2 \cos \tilde{\theta}}{1 + \cos^2 \tilde{\theta}} + O(\hat{r} \sin^4 \tilde{\theta}) .
\end{align*}

(3.44)

(3.45)

From (3.32), (3.36), (3.44) and (3.45) we arrive at

\[ d_0((X,Y), (X_{Kerr}, Y_{Kerr})) = O(1) \text{ for small } \hat{r}. \]

(3.46)

We have therefore proved (for terminology, see [11]):

**Theorem 3.4**

Let $(\mathcal{M}, g)$ be a vacuum, $I^+$ regular, stationary and axisymmetric asymptotically flat black hole space-time. Let $\mathcal{E}_0$ be a degenerate component of the event horizon $I^+(\mathcal{M}_{ext}) \cap \partial(\mathcal{M}_{ext})$ with cross-section area $A_0$. There exists a neighbourhood of $\mathcal{E}_0$ on which the hyperbolic-plane distance between the complex Ernst potential of $(\mathcal{M}, g)$ and that of the extreme Kerr space-time with the same area of the cross-sections of the horizon is bounded.

### 3.7 The Ernst potential of $\partial \varphi$ in electrovacuum

We continue with the extension of the analysis in Section 3.6 to the electrovacuum case. Let $\mathcal{F}$ be the electro-magnetic two-form in a stationary axisymmetric space-time $(\mathcal{M}, g)$ satisfying the sourceless Einstein-Maxwell equations: thus $\mathcal{F}$ is invariant under both $\xi$ and $\eta$ and satisfies the Maxwell equations:

\[ \mathcal{F} = dA, \quad d* \mathcal{F} = 0. \]

(3.47)

To fix terminology, the vectorial Ernst potential $(U, V, \chi^e, \chi^m)$ of the rotational Killing field $\eta$, is defined as follows. First, $U$ is defined as

\[ U = -\frac{1}{2} \log X = -\frac{1}{2} \log g(\eta, \eta). \]
Next, the electric and magnetic potentials $\chi^e$ and $\chi^m$ of $\eta$ are defined by

\[ d\chi^e = i_\eta \ast F, \quad d\chi^m = i_\eta F. \]

To dispel confusion, we emphasize that these are not the same as the standard electric and magnetic potentials which are defined using the stationary Killing field. The existence of $\chi^e$ and $\chi^m$ is a consequence of (3.47). Note that $\eta$ vanishes on the axis $A$, which implies that $\chi^e$ and $\chi^m$ are constant on each connected component of $A$. It further follows from the Einstein-Maxwell equations that the 1-form $\ast(\eta \wedge d\eta) - 2\chi^m d\chi^e + 2\chi^e d\chi^m$ is closed (see e.g. [35]), and so we can define $V$ by

\[ 2\, dV \equiv dY := \ast(\eta \wedge d\eta) - 2\chi^m d\chi^e + 2\chi^e d\chi^m. \]

Similarly to the vacuum case, $V$ is constant on each connected component of $A$.

In the sequel, we analyze the asymptotic behaviour of $(U, V, \chi^e, \chi^m)$ as $\hat{r} \to 0$. It is desired to relate this potential to that of the reference degenerate Kerr-Newman solution which has the same horizon area $A_0$ and charge parameters $q_e$ and $q_b$. To this end, we introduce the variable $(r, \theta)$ as in Section 3.6 by

\[ (z, \rho) = (r \cos \theta, r \sin \theta), \]

where the functions $z$ and $\rho$ are defined in Section 3.3. The Ernst potential of the reference Kerr-Newman solution then takes the following form in terms of $r$ and $\theta$ (see, e.g., [24]):

\[ \begin{align*}
U_{\text{KN}} &= -\frac{1}{2} \log \frac{\sin^2 \theta ((r + m_0)^2 + a_0^2) - r^2 a_0^2 \sin^2 \theta}{(r + m_0)^2 + a^2 \cos^2 \theta}, \\
V_{\text{KN}} &= -a_0 m_0 (3 \cos^2 \theta - \cos^3 \theta) - \frac{a_0^2 \sin^2 \theta - q_0^2 m_0^{-1} (r + m_0)}{(r + m_0)^2 + a_0^2 \cos^2 \theta}, \\
\chi_{\text{KN}}^e &= -q_e \cos \theta \left( \frac{r + m_0)^2 + a_0^2}{(r + m_0)^2 + a_0^2 \cos^2 \theta} + q_b a_0 \sin^2 \theta \frac{r + m_0}{(r + m_0)^2 + a_0^2 \cos^2 \theta} \right), \\
\chi_{\text{KN}}^m &= -q_b \cos \theta \left( \frac{r + m_0)^2 + a_0^2}{(r + m_0)^2 + a_0^2 \cos^2 \theta} - q_e a_0 \sin^2 \theta \frac{r + m_0}{(r + m_0)^2 + a_0^2 \cos^2 \theta} \right).
\end{align*} \]

Here $q_0$ is the total charge,

\[ q_0 := \sqrt{q_e^2 + q_b^2}. \]
Note that in [24], only the case of vanishing magnetic charge is considered. The general magnetically charged solution can be obtained from this by a duality rotation, $F \mapsto \cos \lambda F + \sin \lambda \ast F$, where $\lambda$ is a real constant. Under this transformation the new potentials $\chi_{\text{KN}}^e$ and $\chi_{\text{KN}}^m$ are obtained from the old ones by a constant rotation in the $(\chi_{\text{KN}}^e, \chi_{\text{KN}}^m)$ plane, while $U_{\text{KN}}$ and $V_{\text{KN}}$ remain unchanged, whence the above formulae.

As shown in Section 3.6, we have the relations

$$\sin \theta = (1 + O(\hat{r})) \sin \tilde{\theta}, \quad \text{and} \quad \cos \theta = (1 + O(\hat{r})) \cos \tilde{\theta} \quad \text{for small } \hat{r}, \quad (3.48)$$

where the error terms are smooth sphere functions. Thus, by a simple calculation, in a neighbourhood of the horizon we have

$$U_{\text{KN}} = - \log \sin \tilde{\theta} + O(1), \quad (3.49)$$

$$V_{\text{KN}} = 2a_0 m_0 s(\tilde{\theta}) + O(\sin^2 \tilde{\theta}), \quad (3.50)$$

$$\chi_{\text{KN}}^e = q_e s(\tilde{\theta}) + O(\sin^2 \tilde{\theta}), \quad (3.51)$$

$$\chi_{\text{KN}}^m = q_b s(\tilde{\theta}) + O(\sin^2 \tilde{\theta}), \quad (3.52)$$

where $s$ is some smooth function of $\tilde{\theta}$ such that $s(\tilde{\theta}) \equiv -1$ in $[0, \pi/6]$ and $s(\tilde{\theta}) \equiv 1$ in $[5\pi/6, \pi]$.

We are ready for the analysis of $(U, V, \chi^e, \chi^m)$ near the horizon. From the flux formulae for the total electric and magnetic charges of the horizon we have

$$q_b = \frac{1}{2} \int_0^\pi \chi_{\tilde{\theta}}^m d\tilde{\theta}, \quad (3.53)$$

$$q_e = \frac{1}{2} \int_0^\pi \chi_{\tilde{\theta}}^e d\tilde{\theta}. \quad (3.54)$$

In view of the above two identities, we can assume without loss of generality that

$$\chi_{\tilde{\theta}}^m|_{\mathcal{A}_\pm} = \mp q_b, \quad \chi_{\tilde{\theta}}^e|_{\mathcal{A}_\pm} = \mp q_e, \quad (3.55)$$

where $\mathcal{A}_\pm$ are the connected components of the axis $\mathcal{A}$ as defined in Section 3.6.

By Proposition 2.2 we have\(^1\)

$$\chi_{\tilde{\theta}}^m(\hat{r}, \tilde{\theta}) = 2F_{\tilde{\phi} \tilde{\theta}}(\hat{r}, \tilde{\theta}) = O(\sin \tilde{\theta}) \quad \text{for small } \hat{r}, \quad (3.56)$$

\(^1\)We use the convention $F = F_{\mu \nu} dx^\mu \wedge dx^\nu$ for the coefficients $F_{\mu \nu}$ of a two-form.
and so, by (3.55), again for small $\hat{r}$,
\[
\chi^m(\hat{r}, \tilde{\theta}) = -q_b + \int_0^\hat{\theta} \chi^m_\hat{\theta}(\hat{r}, \tilde{\theta}) d\tau = q_b - \int_\hat{\theta}^{\pi} \chi^m_\hat{\theta}(\hat{r}, \tilde{\theta}) = q_b s(\tilde{\theta}) + O(\sin^2 \tilde{\theta}) .
\]

Similarly, we have
\[
\chi^e(\hat{r}, \tilde{\theta}) = 2 * F_{\varphi \tilde{\theta}}(\hat{r}, \tilde{\theta}) = O(\sin \tilde{\theta}) \text{ for small } \hat{r} ,
\]
leading to
\[
\chi^e(\hat{r}, \tilde{\theta}) = q_e s(\tilde{\theta}) + O(\sin^2 \tilde{\theta}) \text{ for small } \hat{r} .
\]

Next, applying the result of [28] to $(${\cal M}, g, F$)$ and to $(\mathcal{M}, g, * F)$ we find
\[
F_{\varphi \tilde{\theta}}(0, \tilde{\theta}) = \mathcal{F}_{\text{KN} \varphi \tilde{\theta}}(0, \tilde{\theta}) , \text{ and } * F_{\varphi \tilde{\theta}}(0, \tilde{\theta}) = * \mathcal{F}_{\text{KN} \varphi \tilde{\theta}}(0, \tilde{\theta}) ,
\]
where $\mathcal{F}_{\text{KN}}$ is the electro-magnetic two-form associated to the reference Kerr-Newman solution. Note that here we have used $\theta \equiv \tilde{\theta}$ on the horizon. It thus follows from the definition of $\chi^e$ and $\chi^m$ that
\[
\chi^e(0, \tilde{\theta}) \equiv \chi^e_{\text{KN}}(0, \tilde{\theta}) , \text{ and } \chi^m(0, \tilde{\theta}) \equiv \chi^m_{\text{KN}}(0, \tilde{\theta}) .
\]

By a direct computation, we then get
\[
\int_0^\pi \left[ - \chi^m \chi^e_\hat{\theta} + \chi^e \chi^m_\hat{\theta} \right]_{r=0} d\tilde{\theta} = \frac{q_0^2}{a_0^2 m_0^2} r_0^4 \arctan \frac{a_0}{m_0} - \frac{q_0^4}{a_0^2 m_0} .
\]

To continue, we recall (3.33) which gives
\[
* (\eta \wedge d\eta)_{\tilde{\theta}} = h_{\varphi \varphi} \psi^{-1} \sqrt{\det h} [\hat{r} \alpha] ,
\]
Hence, by (3.13), (3.17) and (3.18),
\[
* (\eta \wedge d\eta)_{\tilde{\theta}} = \frac{2a_0 m_0 r_0^4 \sin^3 \tilde{\theta}}{m_0^2 + a_0^2 \cos^2 \tilde{\theta}} \gamma_{\tilde{\theta}} ,
\]
where $\gamma_{\tilde{\theta}}$ is a smooth sphere function satisfying $\gamma_{\tilde{\theta}}(0, \tilde{\theta}) \equiv 1$. It follows that
\[
\int_0^\pi * (\eta \wedge d\eta)_{\tilde{\theta}} |_{r=0} d\tilde{\theta} = -2 \frac{q_0^2}{a_0^2 m_0^2} r_0^4 \arctan \frac{a_0}{m_0} + 2 \frac{r_0^4}{a_0 m_0} .
\]
Using (3.60), (3.62) and recalling the definition of $V$ we get

$$V|_{\mathcal{A}^-} - V|_{\mathcal{A}^+} = \int_0^\pi V_{\tilde{\sigma}}(0, \tilde{\theta}) \, d\tilde{\theta} = \frac{r_0^4}{a_0 \, m_0} - \frac{q_0^4}{a_0 \, m_0} = 4 \, a_0 \, m_0 .$$

(3.63)

Thus, we can assume without loss of generality that

$$V|_{\mathcal{A}^\pm} = \pm 2 \, a_0 \, m_0 .$$

Now, taking (3.56), (3.57), (3.58), (3.59) and (3.61) into account we get

$$V_{\tilde{\sigma}}(\hat{r}, \tilde{\theta}) = O(\sin \tilde{\theta}) \text{ for small } \hat{r} .$$

(3.64)

Thus

$$V(\hat{r}, \tilde{\theta}) = 2a_0 m_0 \sin(\tilde{\theta}) + O(\sin^2 \tilde{\theta}) \text{ for small } \hat{r} .$$

Finally, by (3.18),

$$U(\hat{r}, \tilde{\theta}) = -\log \sin \tilde{\theta} + O(1) \text{ for small } \hat{r} .$$

(3.65)

We conclude from (3.65), (3.64), (3.59), (3.57) and (3.49)-(3.52) that

$$d_b((U, V, \chi^e, \chi^m), (U_{\text{KN}}, V_{\text{KN}}, \chi_{\text{KN}}^e, \chi_{\text{KN}}^m)) = O(1) \text{ for small } \hat{r} .$$

We have therefore proved (for terminology, see [11]):

**Theorem 3.5** Let $(\mathcal{M}, g, F)$ be an electrovacuum, $I^+$ regular, stationary and axisymmetric asymptotically flat black hole space-time. Let $\mathcal{E}_0$ be a degenerate component of the event horizon $I^+(\mathcal{M}_{\text{ext}}) \cap \partial(\mathcal{M}_{\text{ext}})$ with cross-section area $A_0$, electric charge $q_e$ and magnetic charge $q_b$. There exists a neighbourhood of $\mathcal{E}_0$ on which the complex-hyperbolic-plane distance between the vectorial Ernst potential of the rotational Killing vector field of $(\mathcal{M}, g)$ and that of the extreme Kerr-Newman space-time with the same horizon cross-section area, electric charge and magnetic charge is bounded.
Remark 3.6 A more careful analysis as in Section 3.6 shows the following asymptotic behaviour for small $\hat{r}$:

$$|\chi^e - \chi_{\text{KN}}^e| + |\chi^m - \chi_{\text{KN}}^m| + |V - V_{\text{KN}}| \leq C\hat{r}\sin^2 \tilde{\theta},$$

$$|V - V_{\text{KN}} - \chi_{\text{KN}}^m(\chi^e - \chi_{\text{KN}}^e) + \chi_{\text{KN}}^e(\chi^m - \chi_{\text{KN}}^m)| \leq C\hat{r}\sin^4 \tilde{\theta}.$$ 

4 Proof of Theorem 1.1

Let $(\mathcal{M}, g)$ be a stationary, $I^+$-regular, analytic electrovacuum space-time with connected, non-empty, rotating future event horizon $\mathcal{E}_0$. As justified in detail in [11], we only need to consider the case where the metric is axisymmetric. By Theorem 3.3, the area function $\rho$ and its harmonic conjugate $-z$ form a global manifestly asymptotically flat coordinate system on $(\mathcal{M}_{\text{ext}})/\mathbb{R} \times U(1)$, where $\mathcal{E}_0$ corresponds to the point $\rho = z = 0$. It is well known that the vectorial Ernst potential $(U, V, \chi^e, \chi^m)$ is a harmonic map from $\mathbb{R}^3 \setminus \{\rho = 0\} = \{(\rho, z, \varphi : \rho > 0, z \in \mathbb{R}, \varphi \in [0, 2\pi]\}$ into the complex hyperbolic plane. Define a reference vectorial Ernst potential $(U_{\text{KN}}, V_{\text{KN}}, \chi_{\text{KN}}^e, \chi_{\text{KN}}^m)$ as in Section 3.7. By the asymptotic analysis of [31] (compare [18]) and Theorem 3.5, the hyperbolic distance $d_b$ between the two Ernst potentials is finite and goes to zero as one recedes to infinity. Using the subharmonicity of $d_b$ and [12, Proposition C.4], we conclude that $d_b \equiv 0$ and so $(U, V, \chi^e, \chi^m) \equiv (U_{\text{KN}}, V_{\text{KN}}, \chi_{\text{KN}}^e, \chi_{\text{KN}}^m)$. It is then customary to show that $(\mathcal{M}_{\text{ext}}, g)$ is diffeomorphic to the corresponding domain of outer communications in that Kerr-Newman space-time to which the reference Ernst potential is associated. □

Acknowledgements: Both authors were supported in part by the EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1). PTC was further supported in part by the Polish Ministry of Science and Higher Education grant Nr N N201 372736. LN would like to thank Dr. Willie W.Y. Wong for drawing his attention to the problem studied in the present paper.

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