Boundary dynamics and the statistical mechanics of
the 2+1 dimensional black hole

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Abstract
We calculate the density of states of the 2+1 dimensional BTZ black hole in the
micro- and grand-canonical ensembles. Our starting point is the relation between
2+1 dimensional quantum gravity and quantised Chern-Simons theory. In the micro-
canonical ensemble, we find the Bekenstein–Hawking entropy by relating a Kac-Moody
algebra of global gauge charges to a Virasoro algebra with a classical central charge
via a twisted Sugawara construction. This construction is valid at all values of the
black hole radius. At infinity it gives the asymptotic isometries of the black hole, and
at the horizon it gives an explicit form for a set of deformations of the horizon whose
algebra is the same Virasoro algebra. In the grand-canonical ensemble we define the
partition function by using a surface term at infinity that is compatible with fixing the
temperature and angular velocity of the black hole. We then compute the partition
function directly in a boundary Wess-Zumino-Witten theory, and find that we obtain
the correct result only after we include a source term at the horizon that induces a
non-trivial spin-structure on the WZW partition function.

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1 Introduction

The Chern-Simons formulation of 2+1 dimensional gravity [1] has provided many interesting new insights into the problem of quantum gravity. (For a review see [2].) Most notably, making use of the relationship between 2+1 dimensional Chern-Simons theory and the 1+1 dimensional WZW model [3, 4], Carlip has argued for a statistical mechanical interpretation of the entropy of a 2+1 dimensional black hole, backed up by a pair of tantalising calculations which yielded the correct value for the black hole entropy, both in Lorentzian [5], and Euclidean [6] formalisms. Furthermore, the derivation given in [5] has been recently applied with success to de Sitter space [7], and may provide a tool for understanding black hole entropy in string theory far from extremality [8].

The most important assumption in Carlip’s analysis is that those degrees of freedom responsible for the black hole entropy are located at the horizon. This idea is certainly appealing and has been advocated by many authors. However, at a technical and conceptual level, it is difficult to see what states are counted in Carlip’s calculation, and what is being held fixed. In principle, it should be possible to count states in a micro-canonical ensemble, holding the mass and spin of the black hole fixed, or to infer the number of states in a grand-canonical ensemble, holding fixed the black hole temperature and angular velocity.

A different approach to understanding the statistical mechanical origin of the 2+1 dimensional black hole entropy was recently proposed by Strominger [9]. In this approach, the basic ingredient is the discovery by Brown and Henneaux [10] that the asymptotic symmetry group of 2+1 dimensional gravity with a negative cosmological constant [11] is the conformal group with a (classical) central charge

\[ c = \frac{3l^2}{2G}, \]  

(1.1)

where \(-1/l^2\) is the cosmological constant. Note that in the weak coupling limit \(G \to 0\), \(c\) becomes very large. Strominger has pointed out that if one counts states by regarding the theory as equivalent to \(c\) free bosons, then at a fixed value of \(L_0\) and \(\bar{L}_0\), the degeneracy of states gives rise to exactly the Bekenstein–Hawking entropy. Since in 2+1 dimensions \(lM = L_0 + \bar{L}_0\) and \(J = L_0 - \bar{L}_0\) where \(M\) and \(J\) are, respectively, the black hole mass and angular momentum [12], Strominger’s computation is clearly a micro-canonical calculation. In this approach one would like to find the underlying conformal field theory with a central charge equal to (1.1), and its connection is to a counting of states at the black hole horizon. Since it is clear that one cannot obtain the correct black hole entropy in general by looking only at asymptotic isometries, it seems that Strominger’s calculation succeeds because the trivial nature of gravity in 2+1 dimensions results in an isomorphism between boundary theories at infinity and at the horizon.

This paper has two goals. On the one hand, in Sec. 2 we present a micro-canonical calculation of the black hole entropy. Starting with a Chern-Simons theory, we find the algebra of global charges present at any constant radius boundary surface in the black hole spacetime. We prove that a subset of this infinite set of conserved charges satisfies the
Virasoro algebra with a central charge equal to \((1.1)\). This central charge, as in \((10)\), arises classically \((13)\). The advantage of considering the Chern-Simons formulation is that the underlying conformal field theory is, at least at the classical level, an \(SL(2, R) \times SL(2, R)\) WZW model whose relation to the Virasoro generators is via a twisted Sugawara construction \((13)\). Further, it is possible, to show that this Virasoro algebra arises from a reduction of the WZW theory to a Liouville theory \((13)\). The counting of black hole microstates then follows just as in \((9)\). As stressed before, since this counting needs \(M\) and \(J\) fixed, the relevant states live in a micro-canonical ensemble.

The subset of global charges satisfying the Virasoro algebra are shown to be precisely those charges that, at infinity, leave the leading order form of the metric invariant, agreeing with the asymptotic isometries found in Ref. \((10)\). However, the construction leading to a Virasoro algebra of surface deformations is equally valid at any radius, and in particular at the horizon. Thus, we are able to derive the Bekenstein–Hawking entropy from the algebra of horizon deformations. We give the explicit form of the generators of the Virasoro algebra and of the diffeomorphisms that they generate, valid at a boundary surface located at any radius.

Our second goal is to find the grand-canonical partition function for the 2+1 dimensional black hole. In Sec. 3, we compute the partition function \(Z(\tau)\) for three dimensional Euclidean gravity on a solid torus with a fixed value of the modular parameter \(\tau\) of the torus. We show that this parameter is related to the black hole inverse temperature \(\beta\) and angular velocity \(\Omega\) by

\[
\tau = \hbar \beta \left( \Omega + \frac{i}{\Omega} \right),
\]

and therefore \(Z(\tau)\) is clearly the grand-canonical partition function. The computation of \(Z(\tau)\) is not straightforward because the relevant group is \(SL(2, C)\) which is not compact. We use here the trick of replacing \(SL(2, C) \rightarrow SU(2) \times SU(2)\) \((10, 16)\) and show that the partition function correctly accounts for the 2+1 dimensional black hole entropy, after continuing the spin of the \(SU(2)\) representations back to complex values. In this case, we find the correct answer only if we include a source term at the black hole horizon which has a particularly interesting interpretation in terms of the WZW theory. It tracks the spin of each representation and is the analogue of a \((-1)^F\) operator in fermionic theories, twisting the WZW theory in the time direction.

## 2 The micro-canonical ensemble

In this section we canonically quantise the degrees of freedom associated with the gravitational field, by using the relation between 2+1 dimensional gravity and Chern-Simons theory. We then relate the Kac-Moody algebra of the boundary WZW theory that emerges from the Chern-Simons theory to a Virasoro algebra that describes asymptotic isometries of the metric. This relation, achieved by a twisted Sugawara construction, results in a theory whose degeneracy at fixed mass and spin of the black hole leads to the Bekenstein–Hawking entropy \((9)\).
2.1 Global charges in Chern-Simons theory

Let us begin with some general remarks on Chern-Simons theory, always motivated by an application to the 2+1 dimensional black hole. Consider the Chern-Simons action

\[ I_{CS}[A] = \frac{k}{4\pi} \int \varepsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) d^3x, \]  

(2.1)

where as we shall see below, we shall be interested in \( k < 0 \). Up to a boundary term, (2.1) can be put into the canonical form

\[ I[A_i, A_0] = \frac{k}{4\pi} \int_{\Sigma \times R} \varepsilon^{ij} g_{ab} \left( A_a^i \dot{A}_b^j - A_a^b \dot{F}_{ij}^a \right) dt d^2x, \]  

(2.2)

(here \( \varepsilon_{t\rho\phi} = 1 \) and \( \text{Tr}(J_a J_b) = g_{ab} \)). We take (2.2), without additional boundary terms, as our starting point. The variation of this action leads to

\[ \delta I[A] = \frac{k}{4\pi} \int \varepsilon^{\mu\nu\rho} \text{Tr} (\delta A_\mu F_{\nu\rho}) d^3x - \frac{k}{2\pi} \int_{\partial\Sigma} \text{Tr} A_t \delta A_\phi , \]  

(2.3)

where we have assumed, in \( t, \rho, \phi \) coordinates, that there is a single outer boundary at fixed \( \rho \). In order to make the variation of this action well defined, we choose to fix one or other of the conditions

\[ A_t = \pm A_\phi \]  

(2.4)

at the boundary (in the \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) Chern-Simons theory we will choose \( \pm A_t = \pm \pm A_\pm \)). However, this is not enough to ensure the differentiability of (2.3) since under (2.4) the boundary term reduces to \( \pm \delta \int_{\partial\Sigma} \text{Tr} A_\phi^2 \). We thus need to impose the equation

\[ \pm \delta \int_{\partial\Sigma} \text{Tr} A_\phi^2 = 0 \]  

(2.5)

or, in other words, we need to fix the value of \( \int \text{Tr} A_\phi^2 \) at the boundary. Under (2.4) and (2.5) the action has well defined variations and the variational problem is then well posed. We now analyse the meaning of these boundary conditions.

The chirality conditions (2.4) should be regarded as specifying the class of spacetimes or field configurations that will be considered. In terms of lightlike coordinates \( x^\pm = \phi \pm t \) these conditions read \( A_\pm = 0 \) and it is well known that they are satisfied by the BTZ black hole solutions (see, for example, [15]). These conditions leave a large residual symmetry group, namely chiral (anti chiral) gauge transformations which are generated by Kac-Moody fields, and the corresponding boundary degrees of freedom are described by a chiral \( WZW \) action. Since the black hole is asymptotically anti-de Sitter, it is natural to further reduce the boundary theory by imposing Polyakov’s reduction conditions on the \( SL(2, \mathbb{R}) \) currents leading to an effective Liouville theory, as done in [13]. We shall mention this possibility in Sec. 2.8, but in this paper we shall mainly consider the full Kac-Moody theory. Let us finally point out that the chirality boundary conditions (2.4) are not known to be in one to
one correspondence with the existence of a black hole, or an event horizon. The definition of “black holes states” is a delicate problem which needs some information about the topology as well as local fields. Our strategy here is to consider the chirality conditions, which include the black hole solutions, as a starting point and quantise the theory on that Hilbert space.

Condition (2.3) fixes the ensemble to be micro-canonical. Indeed, in the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ theory, the values of $\int_{\partial \Sigma} Tr^A_\phi^2$ are proportional to linear combinations of the black hole mass and angular momenta. An alternative procedure would be to add the term $\int_{\partial \Sigma} Tr A^2_\phi$ to the action (2.3) and leave its value undetermined. This corresponds to a canonical ensemble and will be studied in detail in Sec. 3.

There will in general also be a boundary variation at any inner boundary, but in the following discussion we shall only consider charges on an outer boundary.

In [17, 13] it was shown that the algebra of gauge constraints leads to a set of global charges at the boundary whose Poisson bracket algebra is a classical Kac-Moody algebra. These global charges are equivalent to the global charges obtained by a reduction of the Chern-Simons theory to a boundary WZW theory.

From the point of view of the three dimensional theory in Hamiltonian form, the global charges arise from considering the generators of gauge transformations

$$G(\eta^a) = -\frac{k}{4\pi} \int_{\Sigma} g_{ab} \eta^a \varepsilon^{ij} F_{ij}^b d^2x + Q(\eta^a), \tag{2.6}$$

where $Q$ is a boundary term. Because of the presence of the boundary, the functional derivative of $G(\eta)$ is well-defined only if the boundary term $Q(\eta)$ has the variation

$$\delta Q(\eta^a) = \frac{k}{2\pi} \int_{\partial \Sigma} g_{ab} \eta^a \delta A^b_k dx^k. \tag{2.7}$$

The Poisson bracket of two generators of the form (2.6) then becomes,

$$\{G(\eta), G(\lambda)\} = \frac{k}{4\pi} \int_{\Sigma} d^2x f^a_{bc} \eta^b \lambda^c g_{ab} \varepsilon^{ij} F_{ij}^c + \frac{k}{2\pi} \int_{\partial \Sigma} g_{ab} \eta^a D_k \lambda^b dx^k \tag{2.8}$$

where $D_k \lambda^a = \partial_k \lambda^a + f^a_{bc} A_k^b \lambda^c$. One expects the boundary term in the right hand side of (2.8) to be equal to the charge $Q(f^a_{bc} \eta^b \lambda^c)$, plus a possible central term [18]. But to check this we first need to give boundary conditions in order to integrate (2.7) and extract the value of $Q$. We shall consider two different classes of boundary conditions.

### 2.1.1 Gauge charges

Assuming that $\eta^a$ is fixed at the boundary the charge is

$$Q(\eta) = \frac{k}{2\pi} \int_{\partial \Sigma} g_{ab} \eta^a A^b_k dx^k. \tag{2.9}$$

One can then check that, indeed, the boundary term in (2.8) contains the charge associated to the commutator $[\eta, \lambda]$ plus a central term. Imposing the constraints, the algebra of global
charges becomes
\[
\{Q(\eta), Q(\lambda)\}_{DB} = -Q([\lambda, \eta]) + \frac{k}{2\pi} \int g_{ab} \eta^a \partial_b \lambda^b dx^k.
\] (2.10)

Since we are interested in the Dirac bracket algebra, we should not only solve the constraint on the bulk but also fix a gauge. It is convenient to use the gauge chosen in [13], and in [15], in a parallel treatment using the WZW formulation,
\[
A_{\rho} = b(\rho)^{-1} \partial_{\rho} b(\rho),
\] (2.11)

where the boundary is taken to be at fixed \(\rho\). This gauge choice together with the constraint \(F_{\nu\phi} = 0\) imply
\[
A_{\phi} = b(\rho)^{-1} A(\phi, t) b(\rho).
\] (2.12)

The gauge choice \(2.11\) is preserved only by gauge transformations whose parameters are of the form
\[
\eta(\rho, \phi, t) = b^{-1}(\rho) \lambda(\phi, t) b(\rho).
\] (2.13)

Since \(\eta\) still contains an arbitrary function of time \(\lambda(\phi, t)\), it seems that the gauge freedom has not been fixed completely yet. The extra requirement that fixes the time dependence of the gauge parameters comes from the boundary condition on \(A_{\mu}\). We have chosen our action in order to impose one or other of the conditions
\[
A_t = \pm A_{\phi}.
\] (2.14)

These conditions remove the gauge invariance since the group of transformations leaving \(2.14\) invariant does not contain any arbitrary function of time. \(\lambda\) is constrained to depend only on \(t + \phi\) or \(t - \phi\). Setting,
\[
A^a(\phi, t) = -\frac{1}{k} \sum_n g^{ab} T_{b n}(t) e^{-in\phi},
\] (2.15)

equation \(2.10\) leads to the classical Kac-Moody algebra
\[
\{T_{a m}, T_{b n}\} = f^c_{ab} T_{c m+n} - i k m g_{ab} \delta_{m+n}
\] (2.16)

Note that the central term has the usual sign for \(k < 0\).

We could have obtained the same algebra by inserting \(2.11\) into the action \(2.2\) and computing the resulting Poisson brackets (see the next section for a detailed discussion of this reduction).

### 2.1.2 Diffeomorphisms

We have seen above that global charges associated to gauge transformations that do not vanish at the boundary give rise to an infinite number of conserved charges satisfying the Kac-Moody algebra. We shall now investigate those charges associated to the group of
diffeomorphisms at the boundary. Since the boundary is a circle, one expects to find the Virasoro algebra. Furthermore, since in Chern-Simons theory diffeomorphisms and gauge transformations are related, one expects the Virasoro and Kac-Moody generators to be related by the Sugawara construction. Actually, for those diffeomorphisms with a non-zero component normal to the boundary, one finds a twisted Sugawara construction which induces a classical central charge in the Virasoro algebra [13]. This central charge was first found in [10] in the ADM formulation of 2+1 dimensional gravity and has recently been shown to play an important role in understanding the statistical mechanical origin of the 2+1 dimensional black hole entropy [9].

Recall that in Chern-Simons theories, diffeomorphisms with parameter $\xi^\mu$ are related to gauge transformations with parameter $\eta^a = A^a_\mu \xi^\mu$ by the equations of motion. In a canonical realisation of gauge symmetries, $A^a_\rho$ is a Lagrange multiplier, and so we must instead consider gauge transformations

$$\eta^a = \xi^i A^a_i.$$  (2.17)

As we did in the last section, we fix the gauge by fixing $A_\rho = b^{-1} \partial_\rho b$. We also choose coordinates for the on-shell solution for which $b = e^{\rho a}$ which implies

$$A^a_\rho = \alpha^a.$$  (2.18)

This will be a good choice of the radial coordinate at infinity for the black hole. As a consequence of this choice, diffeomorphisms $\xi^i$ that preserve the gauge choice (2.11) and (2.13) must be independent of $\rho$. Since the gauge choice only fixes the gauge freedom in the interior of the manifold, we are able to derive the algebra of global boundary diffeomorphisms in complete generality by looking only at $\xi^i(\phi, t)$, subject to the constraints imposed by the boundary condition on the gauge field.

Since from (2.17) the gauge parameter is field-dependent, we replace (2.4) by

$$Q(\xi) = -\frac{k}{4\pi} \int g_{ab} \left( 2\xi^r \alpha^a A^b + \xi^a A^a A^b \right) d\phi.$$  (2.19)

This is a good choice of $Q(\xi)$ since $A_\rho$ is left unchanged by the action of the global charges. In other words $A_\rho = \alpha$ is fixed at the boundary. The algebra of charges then becomes [13]

$$\{Q(\xi), Q(\zeta)\}_{DB} = -Q([\xi, \zeta]) + \frac{k}{2\pi} g_{ab} \alpha^a \alpha^b \int \xi^r \partial_\phi \zeta^r d\phi.$$  (2.20)

If we restrict the diffeomorphisms to be of the specific form [13, 14] (see below for a geometrical justification for this restriction),

$$\xi^i = (-\beta \partial_\phi \zeta^i, \zeta),$$  (2.21)

then the algebra of these restricted diffeomorphisms is the continuous form of the Virasoro algebra with central charge

$$\{Q(\xi), Q(\zeta)\}_{DB} = -Q([\xi, \zeta]) - \frac{k}{2\pi} \alpha^a \beta^2 \int \xi^r \partial_\phi \zeta^r d\phi.$$  (2.22)
where $\alpha^2$ denotes $\alpha^a\alpha^b g_{ab}$.

Defining
\[
\sum_n L_n e^{-i n \phi} = -\frac{k}{2} g_{ab} \left( \alpha^a \alpha^b \beta^2 + 2 \alpha^a \partial_\phi A^b + A^a A^b \right),
\]
(2.23)
or equivalently
\[
L_n = -\frac{1}{2k} \sum_m g^{ab} T_a m T_b n - in \alpha^a T_a n - \frac{k}{2} \alpha^2 \beta^2 \delta_n,
\]
(2.24)
gives the usual Poisson bracket version of the Virasoro algebra
\[
\{L_m, L_n\} = i (m - n) L_{m+n} - i k \alpha^2 \beta^2 m(m^2 - 1) \delta_{m+n},
\]
(2.25)
so that the central charge is
\[
c = -\frac{12 k \alpha^2 \beta^2}{l},
\]
(2.26)
which is positive for $k < 0$. Hence, as expected, those diffeomorphisms that lead to global charges, after the restriction (2.21), induce an infinite number of conserved charges satisfying the Virasoro algebra with a classical central charge. Eq. (2.24) is an example of a twisted Sugawara construction [19]. Note that from (2.23) we see that the boundary condition (2.5) fixes the value of $L_0$ at the boundary.

### 2.2 2+1 dimensional Chern-Simons gravity

In 2+1 dimensional gravity with a negative cosmological constant, the Einstein–Hilbert action is represented by the difference of two Chern-Simons actions
\[
I_{CS} = I \left[ (+) A \right] - I \left[ (-) A \right],
\]
(2.27)
for a pair of $SL(2, R)$ gauge fields $(+) A$ and $(-) A$, where
\[
I \left[ (+) A \right] = \frac{k}{4\pi} \int \varepsilon^{ij} \text{Tr} \left( (+) A_i (+) \dot{A}_j - (-) A_i (-) A_j F_{ij} \right) d^3 x.
\]
(2.28)
The Einstein–Hilbert action is recovered by defining
\[
(+)^a A^a_{\mu} = \omega^a_{\mu} \pm \frac{e^a_{\mu}}{l},
\]
(2.29)
from which it follows that
\[
I_{CS} = \frac{k}{4\pi l} \int \sqrt{-g} \left( R + \frac{2}{l^2} \right) d^3 x,
\]
(2.30)

\footnote{Note that the appearance of the term proportional to $m$ in the Virasoro algebra is due to the shift of the $L_0$ operator by $-\frac{k}{2} g_{ab} \alpha^a \alpha^b \beta^2$ in (2.24).}

\footnote{We take $J_0 = \frac{1}{2} \left( \begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right)$, $J_1 = \frac{1}{2} \left( \begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right)$, $J_2 = \frac{1}{2} \left( \begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right)$ so that $[J_a, J_b] = \varepsilon_{ab} J_c$, and $\text{Tr} (J_a J_b) = \frac{k}{2} \eta_{ab}$ where $\varepsilon_{012} = 1$ and $\eta_{ab} = \text{diag}(-1, 1, 1)$.}
ignoring boundary terms. This relates the level $k$ of the Chern-Simons theories to Newton’s constant,

$$k = -\frac{l}{4G}. \quad (2.31)$$

We see that for the black hole, $k < 0$, explaining why we have developed our arguments for negative $k$.

### 2.3 Diffeomorphisms and gauge transformations in 2+1 dimensional Chern-Simons gravity

In the gauge theory representation of 2+1 dimensional gravity with a negative cosmological constant, it is possible to reproduce the full diffeomorphism transformation properties of $e_i^a$ and $\omega_i^a$ by a gauge transformation in both the covariant and canonical formalisms. In the covariant formalism, this gauge transformation must be chosen so that the gauge parameters for the connections $(\pm)A_i^a$ are equal to

$$(\pm)\chi^a = \xi^\mu (\pm)A_\mu^a, \quad (2.32)$$

where $\xi^\mu$ is the same in both cases.

In the canonical formalism, the situation is a little more complicated, and perhaps not well-known, so we devote some space to it. If in this case we set $(+)^i = (-)^i$, then there are only two arbitrary functions that parametrise diffeomorphisms, and it is easy to see that these two parameters only generate spatial diffeomorphisms. We are thus led to consider the case $(+)^i \neq (-)^i$.

In a canonical theory, diffeomorphisms and Lorentz transformations are realised by the action of the constraints. In this case, the Hamiltonian is equal to

$$H = \frac{1}{16\pi G} \int d^2 x \left[ e_i^a \varepsilon^{ij} \left( R_{aij} + \frac{1}{l^2} \varepsilon_{abc} e_i^b e_j^c \right) + \omega_i^a (D_i e_{aj} - D_j e_{ai}) \right], \quad (2.33)$$

Via the two Lagrange multipliers $e_i^a$ and $\omega_i^a$, the Hamiltonian induces a diffeomorphism defined by $e_i^a = \chi^i n^a + \chi^k e_i^k$ and a Lorentz transformation defined by $j^a = \omega_i^a$. Explicitly,

$$\delta_{(\chi^i,j^a)} e_i^a = \partial_i \left( \chi^i n^a + \chi^k e_i^k \right) + \varepsilon_{bc}^a \omega_i^b \left( \chi^c n^b + \chi^k e_i^k \right) - \varepsilon_{bc}^a \omega_i^b \chi^c, \quad (2.34)$$

$$\delta_{(\chi^i,j^a)} \omega_i^a = \frac{1}{l^2} \varepsilon_{bc}^a \varepsilon_i^b \left( \chi^i n^c + \chi^k e_i^k \right) + \partial_i j^a + \varepsilon_{bc}^a \omega_i^b j^c. \quad (2.35)$$

Let us now compare these transformations laws with those obtained by a gauge transformations parametrised by $(\pm)\eta^a = (\pm)\xi^i (\pm)A_i^a$. It is convenient to define

$$V^i = \frac{1}{2} \left( (+)^i + (-)^i \right), \quad W^i = \frac{l}{2} \left( (+)^i - (-)^i \right). \quad (2.36)$$
The transformation equations for $e^a_i$ and $\omega^a_i$ read

\[ \delta_{(V^i, W^i)} e^a_i = \frac{\partial_i}{l^2} (e^a_j V^j + \omega^a_j W^j) + \varepsilon^a_{bc} \omega^b_i \left( e^c_j V^j + \omega^c_j W^j \right) - \frac{1}{l^2} \varepsilon^a_{bc} \left( e^b_j W^j \right) e^c_i, \]  \hspace{1cm} (2.37)

\[ \delta_{(V^i, W^i)} \omega^a_i = \frac{1}{l^2} \left[ \partial_i \left( e^a_j W^j \right) + \varepsilon^a_{bc} \omega^b_i \left( e^c_j V^j + \omega^c_j W^j \right) \right]. \]  \hspace{1cm} (2.38)

Comparing this with (2.34) and (2.35) we recognise these expressions as the canonical formulae for diffeomorphisms parametrised by

\[ \chi^a \equiv \frac{1}{2} \sqrt{h} \varepsilon^a_{abc} \omega^b_i \left( e^c_j V^j + \omega^c_j W^j \right), \]  \hspace{1cm} (2.39)

along with a Lorentz transformation with parameter $j^a = e^a_j W^j / l^2$. Using

\[ n^a = -\frac{1}{2\sqrt{h}} \varepsilon^a_{bc} e^b_i e^c_j \varepsilon^{ij}, \]  \hspace{1cm} (2.40)

($h_{ij} = e^a_i e_{aj}$ and is used to raise and lower spatial indices), we can pick out

\[ \chi^\perp = \frac{1}{2\sqrt{h}} \varepsilon_{abc} \omega^a_i e^b_j e^c_k W^i \varepsilon^{jk}, \]  \hspace{1cm} (2.41)

and

\[ \chi^i = V^i + e^a_i \omega^a_j W^j. \]  \hspace{1cm} (2.42)

We can now see explicitly that if we had set $W^i = 0$, then the gauge transformations (2.38) would not generate timelike diffeomorphisms.

### 2.4 The 2+1 dimensional black hole

The classical black hole solution \cite{20} in Lorentzian signature can be conveniently written in proper radial coordinates as

\[ ds^2 = -\sinh^2 \rho \left( \frac{r^+ dt}{l} + r^- d\phi \right)^2 + l^2 d\rho^2 + \cosh^2 \rho \left( \frac{r^- dt}{l} + r^+ d\phi \right)^2. \]  \hspace{1cm} (2.43)

In these coordinates, the horizon is at $\rho = 0$. $\phi$ is an angular coordinate with period $2\pi$. Note that the above metric represents only the exterior of the black hole. The inner regions can be obtained by replacing some hyperbolic functions by their trigonometric partners. The mass $M$ and angular momentum $J$ of the black hole are given in terms of $r_\pm$ as

\[ M = \frac{r^2_+ + r^2_-}{8Gl^2}, \hspace{1cm} J = \frac{2r_+ r_-}{8Gl}, \]  \hspace{1cm} (2.44)

and the relation between the Schwarzschild radial coordinate $r$ and the proper radial coordinate $\rho$ is

\[ r^2 = r^2_+ \cosh^2 \rho - r^2_- \sinh^2 \rho. \]  \hspace{1cm} (2.45)
By going to its Euclidean section (see Eq. (3.10) below), the black hole (2.43) can be seen to have a temperature
\[ T = \frac{\hbar (r_+ - r_-)}{2\pi l^2 r_+}. \]  
(2.46)
and, using the first law of black hole mechanics, \(\delta M = T\delta S + \Omega\delta J\), we find an entropy
\[ S = \frac{2\pi r_+}{4\hbar G}. \]  
(2.47)

The metric can be written in first order form as
\[
\begin{align*}
e^0 &= \left(\frac{r_+ dt}{l} + r_- d\phi\right) \sinh \rho, \\
e^1 &= l d\rho, \\
e^2 &= \left(\frac{r_- dt}{l} + r_+ d\phi\right) \cosh \rho,
\end{align*}
\]  
(2.48)
so that after computing the spin connection, the gauge connection representing the black hole is given by
\[
\begin{align*}
^{(\pm)} A^0 &= \pm \frac{r_+ \pm r_-}{l} \left(\frac{dt}{l} \pm d\phi\right) \sinh \rho, \\
^{(\pm)} A^1 &= \pm d\rho, \\
^{(\pm)} A^2 &= \frac{r_+ \pm r_-}{l} \left(\frac{dt}{l} \pm d\phi\right) \cosh \rho,
\end{align*}
\]  
(2.49)
or in matrix form,
\[ (-)^{\pm} A = \frac{1}{2} \begin{pmatrix}
\pm d\rho \\
z_{\pm} e^{\pm \rho} dx_{\pm} \\
\mp d\rho
\end{pmatrix}, \]  
(2.50)
where \(x_{\pm} = t/l \pm \phi\) and \(z_{\pm} = (r_+ \pm r_-)/l\).

We can put this solution into the gauge (2.11),
\[
\begin{align*}
^{(\pm)} \alpha &= \pm J_1, \\
^{(\pm)} b &= \exp \left(^{(\pm)} \alpha \rho \right) = \begin{pmatrix}
e^{\pm \rho/2} & 0 \\
0 & e^{\mp \rho/2}
\end{pmatrix},
\end{align*}
\]  
(2.51)
so that
\[ (-)^{\pm} A = \pm z_{\pm} J_2. \]  
(2.52)
We then see that for the black hole
\[ \alpha^2 = 1/2. \]  
(2.53)
(Note that this value of \(\alpha^2\) can be changed by a rescaling of \(\rho\).)

We can see from (2.52) that the gauge connection \(A_\phi\) leads to a non-trivial holonomy around the closed loops of constant \(\rho\) and \(t\),
\[ \text{Tr} P \exp \int^{(\pm)} A_\phi d\phi = 2 \cosh (\pi z_{\pm}) = 2 \cosh \left(\pi \sqrt{8G(M \pm J/l)}\right). \]  
(2.54)
2.5 Global charges and the 2+1 dimensional black hole

The first step in discussing the algebra of global charges for the 2+1 dimensional black hole is to choose appropriate boundary conditions for the gauge fields \((\pm) A^a_{\mu}\). From the on-shell gauge fields (2.49), it is natural to impose the conditions

\[
(+)^a_i = (+)^a_\phi, \quad (-)^a_i = (-)^a_\phi, \quad (2.55)
\]

which lead to the conditions

\[
\partial_\tau^{(\pm)} \xi^i = 0 \quad (2.56)
\]

on the diffeomorphism parameters. This, along with the constraint

\[
\partial_\rho^{(\pm)} \xi^\phi = 0, \quad (2.57)
\]

then defines the complete set of diffeomorphisms that leave the boundary conditions invariant and preserve the gauge (2.11) and (2.13).

We have seen above that if we impose on these diffeomorphisms the supplementary condition

\[
(\pm) \xi^\rho = -\beta \partial^\phi (\pm) \xi^\phi, \quad (2.58)
\]

then the algebra of global charges associated with the remaining diffeomorphisms,

\[
(+)^i = \left( -\beta \partial^\phi (+)^\phi (x^+), (+)^\phi (x^+) \right), \quad (-)^i = \left( -\beta \partial^\phi (-)^\phi (x^-), (-)^\phi (x^-) \right) \quad (2.59)
\]

leads to a pair of Virasoro algebras, one sector coming from each of the gauge fields, with central charge \(c = -12k\alpha^2\beta^2\).

In terms of the gauge field (2.49) representing the black hole, the global charges (without condition (2.58)) generate the transformations

\[
\delta^{(\pm)} A^a = \frac{1}{2} \left[ \mp \partial^\phi (\pm) \xi^\rho \right] z_\pm e^{\mp \rho} \left( \mp \partial^\phi (\pm) \xi^\phi \right), \quad \delta^{(\pm)} A^\rho = 0. \quad (2.60)
\]

Let us now look at the form of these transformations at infinity (corresponding to placing the boundary at infinity). Then, focusing on the leading order terms of order \(e^\rho\), we find that conditions (2.58) with \(\beta = 1\) are precisely what is required for them to vanish. It is easy to check that rescaling the coordinate \(g\) (this means changing the value \(\alpha^2\)) introduces the (more general) condition

\[
\alpha^2 \beta^2 = \frac{1}{2}, \quad (2.61)
\]

which of course includes the above discussion of the black hole solution in the gauge (2.49). Thus the sub-algebra of global charges defined by (2.59) and the condition (2.61) generates
asymptotic isometries of the black hole metric. Note that these automatically include the anti-de Sitter group $SO(2, 2)$.

Let us make a direct comparison with the asymptotic isometries found in [10]. Using the results of the last subsection, we can translate the action of the transformations (2.59) into the action of a temporal and spatial diffeomorphism. Using the on-shell values of $\omega_i^a$ and $N$, we see from (2.41) and (2.42), and from the appropriate coordinate relations that

$$
\chi^+ = N (3) \chi^t = NW^{\phi}, \\
\chi^i = (3) \chi^i + N^i (3) \chi^t = V^i + N^i W^{\phi},
$$

so that we get the diffeomorphism

$$
(3) \chi^t = l \left( \begin{pmatrix} + \xi^{\phi} (x^+) - (-) \xi^{\phi} (x^-) \\ \partial_x (\phi^+) \xi^{\phi} (x^+) + \partial_x (\phi^-) \xi^{\phi} (x^-) \\ + \xi^{\phi} (x^+) + (-) \xi^{\phi} (x^-) \end{pmatrix} \right)
$$

accompanied by a Lorentz transformation with parameter $e_k^a W^i / l$. Comparing (2.63) with the asymptotic isometries found in Ref. [10],

$$
(3) \chi^t = l \left( T^+ (x^+) + T^- (x^-) \right) + \frac{l^3 e^{-2\rho}}{2} \left( \partial_+ T^+ - \partial_- T^- \right) + O(e^{-4\rho}), \\
(3) \chi^\rho = - \left( \partial_+ T^+ + \partial_- T^- \right) + O(e^{-\rho}), \\
(3) \chi^\phi = T^+ - T^- - \frac{l^2 e^{-2\rho}}{2} \left( \partial_+ T^+ - \partial_- T^- \right) + O(e^{-4\rho}),
$$

we see that there is exact agreement to leading order. (Here $\partial_\pm = \{ l \partial / \partial t, \pm \partial / \partial \phi \} / 2$, and note that $\partial_\pm T^\pm = \pm \partial_\phi T^\pm$. ) The disagreement to sub-leading order is because the diffeomorphisms (2.64) do not preserve our gauge choice in the interior. It seems that up to a choice of gauge in the interior (which is irrelevant since we are trying to isolate the boundary dynamics), (2.64) and (2.59) are equivalent.

We know from the analysis of global charges that they lead to a Virasoro algebra with central charge $c = -12k \alpha^2 \beta^2$. Inserting (2.61) and the value of $k$ given by (2.31), we find that

$$
c = \frac{3l}{2G},
$$

which agrees with the result obtained in [10] for the algebra of asymptotic isometries. As a result, we see that the algebra of diffeomorphisms obtained by Brown and Henneaux is related to the Kac-Moody algebra of the boundary WZW theory at infinity by the twisted Sugawara construction (2.24).
2.6 A Virasoro algebra at all $\rho$

Perhaps the most important point about the analysis of global charges is that it goes through for a boundary located on any surface of constant $\rho$. Thus the set of diffeomorphisms defined by (2.59) and (2.61) leads to a Virasoro algebra of global charges with $c = -6k$ on any such boundary. However, the connection between the Virasoro algebra and isometries of the three metric appears to be valid only at infinity. At finite $\rho$, the Virasoro algebra is a subset of all deformations of that surface, but without any obvious property to distinguish it. In particular, if we take the boundary to be at the horizon, we find that the global charges that generate the Virasoro algebra generate a particular subset of deformations of the horizon with components both tangential and normal to the horizon, described by (2.62) and (2.63). We cannot, of course, rule out the possibility that these deformations may have some distinguishing properties that remain undiscovered.

If at finite $\rho$, the subset of global charges which give rise to the Virasoro algebra are not special in any way, perhaps one should consider all generators

$$(\pm)\xi^i(x^\pm)$$

(2.66)
on an equal footing, and regard the condition (2.58) as a technical step that leads to the Virasoro algebra. The fact that the Virasoro algebra is a subalgebra of the algebra of deformations then suggests that the number of states generated by the larger algebra should be greater than or equal to the Bekenstein value. We shall discuss this issue briefly in the conclusions.

2.7 Density of states

We have so far in this section derived the algebra of global charges on any boundary of fixed $\rho$, and shown that a subset of them leads to the Virasoro algebra with a classical central term. We have also made explicit the relation between the asymptotic isometries of Ref. [10] and this subset of global charges when defined at infinity. We may now count states in the conformal field theory, by looking at representations of the Virasoro algebra, as done in [9]. We must look for representations with a specific value of $L_0$ and $\bar{L}_0$, since according to (2.24), these two quantities are related to the mass and spin of the black hole as

$$L_0 = -\frac{k}{4\pi} \int \left( g_{ab} A^a A^b + \frac{1}{2} \right) d\phi = \frac{1}{2} (lM + J) + \frac{l}{16G},$$

$$\bar{L}_0 = -\frac{k}{4\pi} \int \left( g_{ab} A^a A^b - \frac{1}{2} \right) d\phi = \frac{1}{2} (lM - J) + \frac{l}{16G},$$

(2.67)
or, neglecting the $l/16G$ terms,

$$M = \frac{L_0 + \bar{L}_0}{l}, \quad J = L_0 - \bar{L}_0.$$  

(2.68)

Note that fixing (2.5) is equivalent to fixing $M$ and $J$, justifying our choice of action (2.2) for the micro-canonical ensemble.
As pointed out in [9], since $c$ is large, one can use the degeneracy formula for $c$ free bosons to deduce a density of states equal to

$$\rho(M, J) = \exp \left( \frac{2\pi r_+}{4\hbar G} \right),$$  

which agrees with the Bekenstein–Hawking entropy of the 2+1 dimensional black hole.

It is interesting that now this analysis is not necessarily related to diffeomorphisms at infinity. We can think of these states as living on any surface of constant $\rho$, and in particular they could be defined at the horizon. As far as we are aware, this system then provides the first explicit realisation of a set of deformations of a black hole horizon that can be quantised to yield the correct Bekenstein–Hawking entropy.

2.8 A Liouville action for the Virasoro algebra

We end this section with a brief remark about the relation between the Virasoro algebra (2.25) and the reduction of WZW theory to Liouville theory as discussed in [13] in the context of 2+1 dimensional gravity, and by a number of other authors in a more general context (see [21] for an extensive list of references). As explained in [13], the first step is to join the two chiral WZW theories into a single non-chiral WZW theory. Then the reduction takes place by imposing certain constraints on the currents of the WZW theory and interpreting a second set of constraints as gauge fixing conditions.

Referring back to the conditions (2.58) and (2.56), we see that (2.56) are just the chirality conditions for each $SL(2, R)$ sector. As we shall see below, in the WZW theories, these conditions arise from the dynamics of the WZW currents. Eqs. (2.58) are equivalent to holding fixed

$$\frac{1}{2} \left( (\pm) A_0^0 \pm (\pm) A_0^2 \right) = z_\pm,$$

and are equivalent to the constraints usually imposed in the reduction from WZW to Liouville theory [13, 21]. The reduction is completed by a set of gauge fixing conditions on the currents. A direct application of the results of [13] uses the simplest gauge fixing condition, $(\pm) A_0^3 = 0$. Looking at the set of diffeomorphisms (2.21) that lead to (2.25), one can see from (2.60) that although $(\pm) A_0^3 = 0$ on-shell,

$$\delta (\pm) A_0^3 = \pm \partial_0^{\pm(\pm)} \xi^0.$$

Thus, Dirac brackets will be required to compute the operator algebra for the Liouville theory.

We can invoke Ref. [21] and see that for any gauge fixing condition, the constraints (2.58) lead to a Liouville theory with a central charge that for large $k$ is equal to [22]

$$c = -6k,$$

(since we must use the non-standard convention that has $k < 0$). This agrees with the result we have obtained for the central charge from (2.26).

We conclude that the Virasoro algebra (2.23) can be interpreted as coming from an underlying Liouville theory, as predicted in Refs. [15] and [9].
3 The grand-canonical partition function

The goal of this section is to compute the grand-canonical partition function for three-dimensional gravity

\[ Z(\beta, \Omega) = \int D[e] D[\omega] \exp \left( -\frac{1}{\hbar} I_{EH}[e, \omega; \beta, \Omega] \right) \]

where \( \beta \) and \( \Omega \) are, respectively, the inverse temperature and angular velocity of the black hole. These two (intensive) parameters define the grand-canonical ensemble. The thermodynamic quantities such as average energy and entropy can then be obtained from \( Z \) through the thermodynamic formulae

\[
\langle E \rangle = \frac{\Omega}{\beta} \frac{\partial \log Z}{\partial \Omega} - \frac{\partial \log Z}{\partial \beta}, \tag{3.2}
\]

\[
\langle J \rangle = \frac{1}{\beta} \frac{\partial \log Z}{\partial \Omega}, \tag{3.3}
\]

\[
S = \log Z - \beta \frac{\partial \log Z}{\partial \beta}, \tag{3.4}
\]

since

\[
Z(\beta, \Omega) = \int dE dJ \rho(E, J) e^{-\beta E - \beta \Omega J} = e^{-\beta \langle E \rangle - \beta \Omega \langle J \rangle + S}. \tag{3.5}
\]

The problem now requires two steps: First, we need to impose boundary conditions in the action principle such that the action has well defined variations for \( \beta \) and \( \Omega \) fixed, and second, we need to actually compute \( Z(\beta, \Omega) \).

3.1 Euclidean three dimensional gravity

The grand-canonical partition function will be defined as a sum over Euclidean metrics. The Einstein–Hilbert action for Euclidean gravity with a negative cosmological constant may again be represented by the difference of two Chern–Simons actions, but now for the group \( SL(2, C) \) \footnote{Our conventions are \([J_a, J_b] = \epsilon_{abc} J^c\) and \( \text{Tr}(J_a J_b) = -(1/2)\delta_{ab} \).}. We shall use this property to compute the partition function.

Defining,

\[
A^a = w^a + \frac{i}{\ell} e^a, \quad \bar{A}^a = w^a - \frac{i}{\ell} e^a, \tag{3.6}
\]

and \( A = A^a J_a, \bar{A} = \bar{A}^a J_a \), then up to boundary terms,

\[
I_{EH} = \frac{1}{16\pi G} \int_M \sqrt{g} \left( R + \frac{2}{l^2} \right) = i \left( I[A] - I[\bar{A}] \right), \tag{3.7}
\]

where \( I[A] \) is the Chern-Simons action written in a 2+1 decomposition,

\[
I[A] = \frac{k}{4\pi} \int_M \varepsilon^{ij} \text{Tr} \left( -A_i \dot{A}_j + A_0 F_{ij} \right) d^3 x. \tag{3.8}
\]
The coupling constant $k$ is given by

$$k = -\frac{l}{4G}, \quad (3.9)$$

just as in the Lorentzian case.

### 3.2 The Euclidean 3d black hole and its complex structure

The Euclidean black hole solution is obtained by defining $t = -it_E$ and $r_\pm = i\alpha$ in (2.43), giving

$$ds^2 = \sinh^2 \rho \left( \frac{r_+ dt_E}{l} - \alpha d\phi \right)^2 + l^2 d\rho^2 + \cosh^2 \rho \left( \frac{\alpha dt_E}{l} + r_+ d\phi \right)^2. \quad (3.10)$$

For the Euclidean calculation it is helpful to change coordinates to

$$\varphi = \phi + \Omega t_E, \quad x^0 = \frac{t}{\hbar \beta}, \quad (3.11)$$

where

$$\beta = \frac{2\pi l^2 r_+}{\hbar (r_+^2 - r_-^2)}, \quad \Omega = i\Omega_M = -\frac{\alpha}{lr_+}. \quad (3.12)$$

Here, $\Omega_M = r_-/lr_+$ is the Minkowskian angular velocity.

The angular coordinate $\varphi$ has the standard period $0 \leq \varphi < 2\pi$, while the time coordinate $x^0$, which is also periodic, has the range $0 \leq x^0 < 1$. The $\rho = \text{const.}$ surfaces in the black hole manifold have thus the topology of a torus with the identifications

$$\varphi \sim \varphi + 2\pi n, \quad x^0 \sim x^0 + m, \quad (3.13)$$

with $n, m$ integers. The radial coordinate $\rho$ has the range $0 < \rho < \infty$, with $\rho = 0$ representing the black hole horizon. Thus the Euclidean black hole manifold is represented by a solid torus. The line $\rho = 0$ represents the horizon, and is a circle at the centre of the solid torus. We shall discuss below whether in the sum over metrics in the partition function, this line should be regarded as an inner boundary of the solid torus (see Fig. 1).
The gauge field representing the Euclidean black hole is given by

\begin{align}
A^1 &= \frac{(r_+ - i\alpha)}{l} \sinh \rho \left( d\varphi + \tau dx^0 \right), \\
A^2 &= id\rho, \\
A^3 &= \frac{i(r_+ - i\alpha)}{l} \cosh \rho \left( d\varphi + \tau dx^0 \right),
\end{align}

(3.14)

where \( \tau \) is a complex dimensionless number given by,

\[ \tau = \hbar \beta \left( \Omega + \frac{i}{l} \right). \]

(3.15)

The corresponding formulae for \( \bar{A} \) are obtained simply by complex conjugation.

It is now natural to define a complex coordinate \( z \) by

\[ z = \varphi + \tau x^0. \]

(3.16)

The identifications (3.13) induce in the complex plane the identifications

\[ z \sim z + 2\pi n + \tau m, \quad n, m \text{ integers}. \]

(3.17)

We thus find that the black hole has a natural complex structure with a modular parameter \( \tau \) given by (3.15).

The important point is that by the coordinate transformation (3.11), we have introduced a second pair of parameters into the metric, which by virtue of the periodicity relations (3.13), should be thought of as the intensive parameters \( \beta \) and \( \Omega \). On-shell, they are related to \( r_\pm \) by (3.12), conditions which emerge by either imposing the absence of conical singularities in the Euclidean manifold or using the first law \( \delta M = T \delta S + \Omega \delta J \). Off-shell, \( \tau \) and \( r_\pm \) can be taken to be independent.

### 3.3 Boundary conditions and boundary terms

#### 3.3.1 Spatial infinity

Let us first consider the outer boundary of the solid torus which represents spatial infinity. The correct boundary term at infinity should be consistent with boundary conditions that fix \( \beta \) and \( \Omega \) at infinity. In the complex coordinates \((z, \bar{z})\), the on-shell gauge field (3.14) and its complex conjugate have the property that

\[ A_{\bar{z}} = 0, \quad \bar{A}_z = 0. \]

(3.18)

In terms of the spacetime coordinates \( x^0, \varphi \), these conditions read,

\[ A_0 = \tau A_\varphi, \quad \bar{A}_0 = \bar{\tau} A_\varphi, \]

(3.19)
as can be verified using $A_{\varphi}d\varphi + A_0dx^0 = A_{\bar{z}}dz + A_{\bar{\bar{z}}}d\bar{z}$ and that $\text{Im}(\tau) = \hbar \beta / l \neq 0$. Eqs. (3.18), or (3.19), are just the Euclidean version of the chirality conditions (2.4) used in Sec. 2, and we shall also use them here as part of our boundary conditions. The residual gauge group is generated by the Kac-Moody currents $(A_{\bar{z}})$ $A_{z}$ which are (anti-) holomorphic functions of $(\bar{z})$ $z$.

The main difference with the analysis of Sec. 2 is that we shall not fix the value of $\int \text{Tr} A^2$ because we now work in a grand-canonical ensemble. We thus need to add to the action (3.8) a boundary term at infinity in order to make it well defined. The appropriate action for the grand-canonical ensemble defined by the boundary conditions (3.19) and a fixed $\tau$ (note that this fixes $\beta$ and $\Omega$ at infinity) is

$$I[A_i, A_0, \tau] = \frac{k}{4\pi} \int_M \epsilon^{ij} \text{Tr} (-A_i A_j + A_0 F_{ij}) - \frac{k\tau}{4\pi} \int_{T^\infty} \text{Tr} A^2_\varphi,$$

(3.20)

plus a boundary term at the horizon that we discuss in the next paragraph. [Note that conditions (3.19) were also used in [24], with a slight modification that makes the modular parameter time dependent, in an attempt to obtain a better understanding of Carlip’s original paper [5].]

Using the expressions (2.68) for the mass and spin, it is easy to see that the boundary term (the only term that survives in the semiclassical limit) gives the correct weight factor $-i\hbar \beta (E - \Omega J)$ for the grand-canonical ensemble.

### 3.3.2 The horizon

The boundary conditions at the horizon are more subtle. In the Lorentzian theory it is quite natural to introduce a boundary term at the horizon, since only the ‘outer’ part of the black hole may be viewed as physical. In the Hamiltonian formulation this means that one has to fix the hypersurfaces at the bifurcation point which leads to an isolated, non-smooth boundary, often referred to as joint or edge. In [25] it was shown that such a non-smooth boundary gives rise to additional boundary terms. In the case of black hole spacetimes this joint contribution at the bifurcation point is responsible for the appearance of a non-zero charge located at the horizon, equal to one quarter the black hole area, which then can be interpreted as the entropy of the black hole [26, 27].

In the Euclidean spacetime the situation is different since the ‘inner’ part of the black hole is already cut off from the manifold. Nevertheless, it has been argued (see, for example, [27]) that one also has to introduce a boundary at the origin of the Euclidean spacetime by removing a point from the Euclidean $(r, t)$-plane because the foliation using the vector field $\partial_t$ is not well defined at $r = r_+$. Since we are using a Hamiltonian action, the $t = \text{const}$ surfaces are annuli with two boundaries and it is necessary to give some boundary conditions at the horizon in order to ensure that the Hamiltonian is a well defined functional and its derivatives exist.

We must now decide which boundary conditions to impose at the horizon. Note firstly that the on-shell black hole field (3.14) has a time component at $\varrho = 0$ that does not vanish,

$$A^3|_{\varrho=0} = -2\pi dx^0.$$

(3.21)
Since $x^0$ is an angular coordinate, one may suspect that this indicates the presence of a non-trivial holonomy in the temporal direction. However, an explicit calculation using (3.21) reveals that this is not the case and $A_0^3$ can be set equal to zero by a globally well-defined gauge transformation. The situation changes if one allows a conical singularity at the horizon as advocated in [23]. In this case, the non-vanishing of $A_0$ at the horizon does imply the existence of a holonomy. To handle this situation classically, one can either remove the horizon from the manifold and hence change the topology or, alternatively, one can introduce a source term or Wilson line along the horizon and work with a solid torus topology. The introduction of a Wilson line at the core of the solid torus was also suggested in [23], but as far as we know its consequences were never explored.

The above discussion suggests that we should look to fix $A_0$ at the horizon. This can be achieved using the canonical action with no boundary term. However, this does not then fix the condition (3.21) (as opposed to allowing conical singularities). This can be ensured only by introducing a Wilson line term whose variation, when coupled to the bulk action, fixes (3.21). The Wilson line term is

$$k/4\pi \int_{S^1} \text{Tr} \left( K X^\mu A_\mu - \bar{K} \bar{X}^\mu \bar{A}_\mu \right) \delta^2 (X^\mu (\tau) - x^\mu) d\tau dx^0 d^2x,$$  \hspace{1cm} (3.22)

which is localised along a non-contractible loop in the solid torus defined by $X^\mu (\tau)$. Here $\tau$ is a parameter along the Wilson line. $K$ is an element in the Lie algebra of the group that specifies the vector charge of the source. We could also have included a dynamical source with its own kinetic term [28, 29, 3, 30], but it is not clear that this is necessary.

To conform with the usual choice of coordinates, we shall take the Wilson line to be located along the curve $\rho = 0$, and to be parametrised by the variable $\varphi$. Since we want the action principle to contain the black hole in its space of solutions, the strength of the source is fixed by looking at the classical field (3.14) at the point $\rho = 0$. We take

$$K^a = (0, 0, -4\pi),$$  \hspace{1cm} (3.23)

and the source term becomes

$$k/2 \int_M \left( A^3_\varphi - \bar{A}^3_\varphi \right) \delta^2 (x) d\varphi d^2x,$$  \hspace{1cm} (3.24)

where now $d^2x$ and $\delta^2 (x)$ refer to the $\rho, t$ plane (not $\rho, \varphi$). Note that the choice of orientation of $K$ in the Lie algebra is unimportant, since it only fixes the orientation of the solution in internal space. The source term (3.24) is the analogue of the geometrical horizon term proportional to the area that was mentioned above. Indeed, on-shell, the value of this term (plus the other copy) is equal to one fourth of the area. In the off-shell language of Carlip and Teitelboim [23], the field equations will lead to $\Theta = 2\pi$ and $\Sigma = 0$ as expected, since no conical singularities are allowed.
3.3.3 The action

The total Euclidean action for the black hole is therefore the sum of (3.20) and (3.24) giving,

\[ I[A, \tau; \bar{A}, \bar{\tau}] = I[A, \tau] - I[\bar{A}, \bar{\tau}], \quad (3.25) \]

with

\[ I[A, \tau] = \frac{k}{4\pi} \int_M \epsilon^{ij} \text{Tr} \left( -A_i \dot{A}_j + A_0 F_{ij} \right) \]

\[ - \frac{k\tau}{4\pi} \int_{T_{\rho=\infty}}^\infty \text{Tr} A_\phi^2 + \frac{k}{2} \int_{S_{\rho=0}} A_\phi^3, \]

(3.26)

(where since \( \epsilon^{ij} = \epsilon_{ij} \), in Euclidean space, the sign of the bulk term is different to that in (2.2)). It is worth stressing here, once again, that this action has well defined variations provide \( \tau \) is fixed.

Note, finally, that the micro-canonical action (2.2) and the grand-canonical action (3.26) differ by exactly the boundary term at infinity equal to \( \beta(E + \Omega J) \) that one would expect on general grounds, and that has been advocated by Brown and York [31].

We shall now explore the semi-classical and quantum mechanical consequences of this action.

3.4 Semiclassical partition function

The action (3.20) with the source term (3.24) has the right semiclassical value. Since the canonical action is zero on the classical black hole background (3.14) one obtains,

\[ Z_{\text{semiclassical}} = e^{-\beta(M+\Omega J)+S}, \]

(3.27)

where \( S \) is given by

\[ S = \frac{2\pi r_+}{4\hbar G}, \]

(3.28)

as expected, and comes entirely from the source term at the horizon.

This partition function is grand-canonical because in the action principle only \( \beta \) and \( \Omega \) (or \( \tau \)) were fixed. This means that \( Z \) is a function of \( \beta \) and \( \Omega \). Using (2.44) and (3.12) one can write \( Z \) as a function of \( \beta \) and \( \Omega \),

\[ Z(\beta, \Omega) = \exp \left[ \frac{\pi^2 l^2}{2\hbar^2 G\beta(1 + l^2\Omega^2)} \right]. \]

(3.29)

This is the semiclassical value of the grand-canonical partition function.
3.5 The partition function and the chiral WZW model

In a Chern-Simons formulation of three dimensional Euclidean quantum gravity, the partition function involves a sum over an $SL(2, C)$ gauge field. We must therefore deal with the fact that the group $SL(2, C)$ is not compact and the black hole manifold has a boundary. (For manifolds without boundaries it has been proved in [16] that $Z$ can be understood as a complexified $SU(2)$ problem.) As has been stressed in [6], one can hope to make progress by treating each of the complex connections $A$ and $\bar{A}$ as real, so that the partition function becomes just the product of two, complex conjugate, $SU(2)$ partition functions. Following [6] we shall write

$$Z = |Z_{SU(2)}|^2,$$

(3.30)

and compute $Z_{SU(2)}$, hoping to make sense of its relation to the trace over states in $SL(2, C)$ by some form of analytic continuation.

We thus consider the functional integral,

$$Z_{SU(2)}(\tau) = \int D[A_i]D[A_0] \exp \left( \frac{i}{\hbar} I[A_i, A_0; \tau] \right),$$

(3.31)

where $I$ is given in (3.26), and we integrate over all gauge fields satisfying the boundary conditions (3.19).

Integrating over $A_0$ gives the constraint that $F_{ij} = 0$, which implies that $A_i = \hbar^{-1} \partial_i h$ where $h$ is a map from $M$ to the group. We also have to fix a gauge and we can do this using the gauge fixing choice (2.11) which fixes

$$h(\rho, \varphi, t) = b(\rho)g(\varphi, t), \quad A(\varphi, t) = \hbar^{-1} \partial_t g.$$

(3.32)

Since the first homotopy group of the solid torus is non-trivial, $g$ could be multi-valued. After inserting the gauge fixed and flat $A_i$ into the functional integral one obtains an expression that only depends on the boundary values of the map $g$ [4],

$$Z_{SU(2)}(\tau) = \int Dg \exp \left( \frac{i}{\hbar} I_{CWZW}[g, \tau] \right),$$

(3.33)

where the chiral WZW action is given by,

$$I_{CWZW}[g, \tau] = -\frac{k}{4\pi} \int_{T^2} \text{Tr}(\partial_\varphi g^{-1} \dot{g}) - \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg)^3 - \frac{\tau k}{4\pi} \int \text{Tr}(g^{-1} \partial_t g)^2 + \frac{k}{4\pi} \int \text{Tr}(Kg^{-1} \partial_\varphi g).$$

(3.34)

Here $K$ is related to the original $K$ of (3.23) by conjugation by $b(\rho)$ and so may be taken equal to $K$ without loss of generality.

The reduction of the three dimensional problem to a two dimensional conformal theory is a consequence of the absence of propagating degrees of freedom in the three dimensional field theory. This allowed us to solve the constraint. A second consequence of the absence of
degrees of freedom is that the boundary term at the horizon is now linked to the boundary term at infinity. The conformal field theory lives on a torus with no reference at all to the radial coordinate.

The chiral WZW action (3.34) has two pieces. The kinetic term (first line) defines the commutation relations of the theory. As is well known [32], these commutation relations are given by the SU(2) Kac-Moody algebra,

\[
[T^a_n, T^b_m] = i\hbar \epsilon^{ab}_{\ c} T^c_{n+m} + n\hbar \frac{k}{2} \delta^{ab} \delta_{n+m,0},
\]

where the \(T^a_n\) are the Fourier components of the gauge field,

\[
A_\varphi = g^{-1} \partial_\varphi g = \frac{2}{k} \sum_{n=-\infty}^{\infty} T^a_n e^{i n \varphi}.
\]

Note that it is more convenient to define the \(T^a_n\) in this way rather than as in (2.15) in Euclidean space. The second piece (second line in (3.34)) is the Hamiltonian. Since the Hamiltonian involves \(A^2\) it has to be regularised by choosing a normal ordering. Moreover, it is well known that the coefficient of \(L_0\) in the non-Abelian theory is not \(k^{-1}\) but rather \((k + \hbar)^{-1}\). In the following we shall be interested in the limit of large \(k\) and therefore this shift can be neglected.

The partition function can then be calculated as

\[
Z(\tau) = \sum_{2s=0}^{k/\hbar} \text{Tr}_s \left( q^{L_0/\hbar} e^{i\theta \hat{T}^3_0/\hbar} \right),
\]

where for large \(k\), \(L_0\) is given by

\[
L_0 = \frac{1}{k} \sum_{n=-\infty}^{\infty} :T^a_n :T^b_n : \delta_{ab},
\]

and \(s\) labels the spin of the different SU(2) representations (it can be integer or half-integer). The symbol Tr\(_s\) represents a trace over states belonging to the representation with spin \(s\). The spin structure term can be thought of as being equivalent to having a non-zero flux through the hole created by closing the time direction.

The problem of computing \(\text{Tr}_s \left( q^{L_0/\hbar} e^{i\theta \hat{T}^3_0/\hbar} \right)\) for a given value of \(s, q\) and \(\theta\) is well known and explicit formulae are available. For SU(2), writing \(q = e^{i\tau}\) and taking \(\theta = 2\pi\), one has [33],

\[
\text{Tr}_s \left( q^{L_0/\hbar} e^{i2\pi \hat{T}^3_0/\hbar} \right) = \frac{q^{hs(s+1)/k}}{1-q^m} \prod_{m=1}^{\infty} \left( 1 - q^m \right)^3. \quad (3.39)
\]

The denominator in (3.39) does not depend on \(s\) and it is therefore a global factor in the partition function. This factor provides a quantum correction to the entropy that does not
depend on Newton’s constant. The value of this contribution can be calculated in the limit of small \( \tau \) (large black holes) as
\[
\Pi_{m=1}^{\infty} (1 - q^m) \approx e^{\pi^2/6(\rho - 1)}.
\] (3.40)
Inserting \( q = e^{ir} \) and using (3.4) and (3.30), the contribution to the entropy from this term is equal to
\[
S_0 = \frac{\pi r_+}{l}.
\] (3.41)
This correction, which does not depend on \( G \), has already appeared in the literature [23, 6, 34].

Let us now consider the numerator in (3.39). In the limit of large \( k \), the sum over \( n \) is suppressed because \( e^{-kn^2/\beta l} \to 0 \) exponentially for \( n \neq 0 \). We thus keep only the term \( n = 0 \). This means that the Bekenstein–Hawking entropy does not come from the higher Kac-Moody modes but from the sum over representations. This is quite different from the analysis in [6] in which the entropy comes from the term with zero spin. Setting \( n = 0 \) and defining \( j = 2s \), we arrive at
\[
Z_{SU(2)} = Z_0 Z_{1/k} \sum_{j=1}^{k/\hbar} j (-1)^j e^{i\hbar r j^2/4k},
\] (3.42)
where \( Z_0 \) represents the correction that does not depend on \( G \), while \( Z_{1/k} \) the contribution from the sum over \( n \) whose logarithm vanishes at least as \( 1/k \).

Note that in the quantum calculation the term \( \exp (2\pi i T_3 / \hbar) \), arising from the boundary term at the horizon, produces the factor \( (-1)^{2s} \) which may be interpreted as a \( (-1)^F \) operator that alternates bosonic and fermionic representations. We shall see that this operator has an important role in producing the right contribution to the entropy.

The sum (3.42) is not what we want because the black hole does not belong to the set of states that we are considering when we calculate the \( SU(2) \) partition function. Indeed, \( j \) labels unitary \( SU(2) \) representations for which
\[
L_0 \left| j \right\rangle = \frac{1}{k} T_0 T_0^a T_0^b \delta_{ab} \left| j \right\rangle = \hbar^2 j (j + 2) \frac{e^{i\hbar r j^2/4k}}{4k} \left| j \right\rangle.
\] (3.43)
However, the value of \( L_0 \) on the black hole background is \( L_0 = -k(r_+ + i\alpha)^2/4l^2 \), from which it follows that
\[
j^2 = -\frac{k^2 (r_+ + i\alpha)^2}{\hbar^2 l^2},
\] (3.44)
for large \( j \). Thus, the states we are interested in for the black hole belong to an \( SU(2) \) representation with complex spin. Even in the non-rotating case, \( \alpha = 0 \), \( j^2 \) is negative and thus cannot be real. We shall not attempt to give any interpretation to such a representation.

\[5\text{Note, however, that it is possible that the source term at the horizon could be regarded as an effective action term that arises from integrating out modes that are not seen in this } SU(2) \times SU(2) \text{ calculation.} \]
here, but the reason behind it lies in the fact that Euclidean gravity is a Chern-Simons theory for the group $SL(2, C)$ rather than two copies of $SU(2)$.

Let us analytically continue $j$ to the complex plane and set $j \to i\eta$. The sum reduces to

$$Z_{SU(2)} = Z_0 Z_{1/k} \sum_j j \exp \left( -\frac{i\eta \tau j^2}{4k} + \pi j \right).$$

(3.45)

Note that the $(-1)^F$ operator now has eigenvalues $e^{\pi j}$. Remarkably, this term which produced the right entropy in the semiclassical calculation, also provides the right degeneracy in this quantum mechanical calculation.

Consider the total partition function $Z = |Z_{SU(2)}|^2$. Since $j$ is complex we define $j = j_1 + i j_2$. Using $\tau = \hbar \beta (\Omega + i/l)$ and $k = -l/4G$ we find

$$Z = |Z_0 Z_{1/k}|^2 \sum_{j_1,j_2} \left( j_1^2 + j_2^2 \right) \exp(-\beta(M_{j_1,j_2} + \Omega J_{j_1,j_2}) + 2\pi j_1).$$

(3.46)

with

$$M_{j_1,j_2} = \frac{2G\hbar^2 (j_1^2 - j_2^2)}{l^2}, \quad J_{j_1,j_2} = \frac{4G\hbar^2 j_1 j_2}{l}$$

(3.47)

Since, for a black hole, $M$ and $J$ are related to the inner and outer horizons by (2.44) (or in the Euclidean version by replacing $r - = i\alpha$), we obtain

$$r_+ = 4G\hbar j_1, \quad \alpha = 4G\hbar j_2.$$  

(3.48)

The term $(j_1^2 + j_2^2)$ outside the exponential in (3.46) combines with the sum to give the required measure over $M$ and $J$. The entire partition function becomes

$$Z(\beta, \Omega) \sim \frac{l^3}{16\hbar^3 G^2} \int dM dJ \rho(M, J) \exp(-\beta(M + \Omega J))$$

(3.49)

and implies that the density of states is $\rho(M, J) = \exp(2\pi j_1)$. Using (3.48), we obtain

$$\rho(M, J) = \exp \left( \frac{2\pi r_+(M, J)}{4\hbar G} \right),$$

(3.50)

in complete agreement with the Bekenstein–Hawking value.

Finally, the semiclassical grand-canonical partition function (3.29) and thus the entropy $S$ can be obtained by a simple saddle point approximation (3.45). Noticing that the sum (3.43) has a saddle point at $j = 2\pi ik/\hbar \tau$ we find that

$$Z_{SU(2)}(\tau) = Z_0 Z_{1/k} \exp \left( \frac{i\pi^2 l}{4\hbar G \tau} \right),$$

(3.51)

where we have inserted $k = -l/4G$. Computing the complex modulus of $Z$, and taking into account the value of $\tau$ given in (3.15), we find that

$$Z(\beta, \Omega) = |Z_0 Z_{1/k}|^2 \exp \left( \frac{\pi^2 l^2}{2\hbar^2 G^2 \beta (1 + l^2 \Omega^2)} \right),$$

(3.52)
in complete agreement with (3.29).

It is interesting to note that this calculation can be repeated in the case where \( k \to \infty \), the semiclassical limit, in a purely abelian theory. Details of this calculation are given in the appendix.

4 Conclusions

We have performed two separate calculations of the entropy of the 2+1 dimensional black hole using the relation between 2+1 dimensional gravity and Chern-Simons theory. In Sec. 2, we have worked in the micro-canonical ensemble, and have calculated the density of states starting from the Kac-Moody algebra of global charges (WZW theory). We have computed the correct density of states by relating the global charges to a particular Virasoro algebra via a twisted Sugawara construction, in a way first considered in [13]. This Virasoro algebra turns out to generate the same asymptotic isometries considered in Refs. [4], if the analysis of global charges is performed at infinity. We have shown that it is also present on any other boundary surface at constant radius, including the black hole horizon. In Sec. 3, we have worked in the grand-canonical ensemble, which we have defined by adding an appropriate boundary term at infinity. We have found that in order to obtain the correct partition function we must also add a source term at the horizon. This source term gives the correct value of the partition function both semi-classically and in an exact quantum mechanical calculation. It is the analogue of the term equal to \( \frac{A}{4G} \) that is sometimes added to the canonical Einstein–Hilbert action to yield the correct semi-classical partition function for black holes in arbitrary dimensions.

In the micro-canonical calculation we saw that we obtain the correct density of states at a given value of mass and spin after we make a reduction from the WZW theory to a theory of boundary deformations that satisfies the Virasoro algebra, or equivalently to a Liouville theory. Since this reduction involves additional constraints on the allowed global charges, it one expects that the density of states should be greater in the WZW theory (although possibly equal to leading order). Why, then, can one not calculate the density of states directly in terms of representations of the Kac-Moody algebra? In the WZW theory, it seems clear that there are an insufficient number of states (and for this reason, in the grand-canonical ensemble the correct partition function required the addition of a source term to give a larger apparent degeneracy). The answer presumably lies in the use of a twisted Sugawara construction to connect the Kac-Moody and Virasoro algebras. Although the states we eventually count are unitary states with respect to the standard quantisation of the Virasoro algebra, they most probably correspond to a non-unitary, twisted representation of the Kac-Moody algebra\(^6\). Thus in order to find the correct density of states, we should look

\(^6\)Carlip [4] has pointed out that the correct number of states is obtained through a representation of the Virasoro algebra with \( c = 3l/2G \) only if the vacuum has \( L_0 = 0 \), which translates to a negative eigenvalue for \( L_0 \) defined in an untwisted Sugawara construction from the underlying WZW theory. However, it is interesting to note from (2.67) that the condition \( L_0 = 0 \) corresponds to \( M = -1/8G, J = 0 \) (Anti-de Sitter space) and \( \tilde{L}_0 = -l/16G \), while the black hole vacuum \( M = J = 0 \) has \( \tilde{L}_0 = 0 \) but \( L_0 = l/16G \). This
at a different set of states to those usually constructed in representations of the Kac-Moody algebra. Of course, this calculation is further complicated by the fact that representations of $SL(2, R)$ WZW theory are poorly understood.

In contrast to the micro-canonical case, in the grand-canonical calculation we have managed to obtain the correct density of states directly in a standard (not twisted) WZW theory, using standard expressions for the partition function (and an analytic continuation). However, this result came from a partition function with a “spin-structure” term, twisting the WZW theory in the time direction. It seems likely that the twists in the space and time directions that we have discussed are related by a modular transformation$^7$.

In this context, it is also interesting to speculate on the correct interpretation of the source or Wilson line term that gives rise to the non-trivial “spin-structure”. We saw in our semiclassical and quantum mechanical calculations that this term produces the black hole entropy not as a density of states, but rather as an operator eigenvalue. We conjecture that this source term can be understood in a different context as an effective action term. If we begin with the micro-canonical picture of a twisted WZW model with trivial spin-structure, then it should be possible to get back to an untwisted WZW theory by integrating out the additional degrees of freedom arising from the spatial twisting of the WZW theory. The effect of integrating out these states would be to introduce the spin-structure term. It would be extremely nice to see this connection explicitly.

Finally, we comment on what these various calculations tell us about the location of the degrees of freedom giving rise to the black hole entropy. While Carlip [5, 6] has advocated that these degrees of freedom should be located on the black hole horizon, Strominger [9] has shown that the algebra of asymptotic isometries of the metric leads to the correct density of states. Our discussion of the global charges and of the reduction to Liouville theory in Sec. 2 showed explicitly that the Virasoro algebra responsible for the density of states can live at any value of the radius $\rho$. A similar conclusion is suggested by the $\rho$ independence of the grand-canonical calculation. We were able to obtain an explicit form for a set of deformations of the horizon whose classical algebra is the Virasoro algebra, with the same classical central charge as the set of asymptotic isometries, that gives the correct Bekenstein–Hawking entropy.

The discovery of an algebra of operators at the horizon, whose representations yield the correct density of states, makes an extension of these ideas to higher dimensions look more plausible. Whereas it is unlikely that the algebra of asymptotic isometries of black holes in higher dimensions could lead to the correct density of states, it seems more likely that an algebra of deformations of the horizon could have the required properties. Although these two algebras are identical in our case, this is because of the trivial dynamics of 2+1 dimensional gravity and would certainly not be true in general. In higher dimensional applications, one would then have to relate these charges at the horizon to the mass and spin of the black hole. This problem is solved in our case by the same trivial dynamics of the theory.

$^7$It would be interesting to compute the Euclidean partition function in a canonical framework where $\varphi$ is the time coordinate and to verify that this can yield the same partition function.
What is not so clear from our analysis is the physical interpretation of the subset of deformations that lead to the Virasoro algebra at any finite radius. However, as mentioned above in the context of WZW theory, the restriction to this subset of deformations will reduce the density of states, but it may well not change it to the leading semi-classical order. In that case, the complete algebra of deformations of the horizon would lead to the correct density of states. However, checking this probably requires getting a handle on the problem of state counting in the WZW theory.

We are hopeful that a generalisation to arbitrary dimensions of the calculations that have been developed for the 2+1 dimensional black hole may come about through the algebra of deformations of the horizon. Indeed, this algebra may have a very direct application for black holes in higher dimensions whose near-horizon behaviour is similar to the 2+1 dimensional black hole. In this case we would be able to talk of states localised at the horizon.

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A The Abelian WZW Theory

In the weak coupling limit the non Abelian nature of the theory can be neglected. Therefore it should also be possible to derive the semiclassical value of the partition function from the Abelian theory, and we shall now show that this is indeed the case. If we define

\[ g(t, \varphi) = e^{X^a(t, \varphi)J_a}, \]  

where \( J_a \) are now the generators of the Abelian Lie algebra, the chiral WZW action (3.34) reduces to

\[ I_{CWZW}[X, \tau] = \frac{k}{8\pi} \int \left(-\partial_0 X^a \partial_\varphi X_a + \tau (\partial_\varphi X)^2 - 4\pi \partial_\varphi X^3 \right). \]  

From this action we find the field equations

\[ (\partial_0 - \tau \partial_\varphi)\partial_\varphi X^a = 0. \]  

Thus the general solution of these field equations contain only right moving modes which expresses the chirality of our theory. Motivated by the field equation we may expand \( X^a \) in
normal modes as,
\[
X^a (x^0, \varphi) = 2 \varphi \alpha_0^a (x^0) + 2 \sum_{n \neq 0} \frac{\alpha_n^a (x^0)}{in} e^{in\varphi} + 2 \alpha (x^0) .
\]  
(A.4)

Note that the action (A.2) has a gauge symmetry, \( \delta X^a = \epsilon (x^0) \), which we use to set the function \( \alpha (x^0) \) to zero. Replacing the mode expansion into the chiral WZW action, one obtains
\[
I_{CWZW}[\alpha] = \int dx^0 \left( -\sum_{n \neq 0} \frac{\dot{\alpha}_0^a \alpha_{n}^a}{ikn} + \sum_{n \geq 1} \frac{2 \dot{\alpha}_n^a \alpha_{n}^a}{ikn} + \tau L_0 - 2\pi \alpha_0^3 \right),
\]  
(A.5)

where \( L_0 \) is a Virasoro generator,
\[
L_0 = \frac{1}{k} (\alpha_0^2 + 2 \sum_{n \geq 1} \alpha_n^a \alpha_{n}^a),
\]  
(A.6)

and where we have eliminated total derivative terms. The action (A.5) gives rise to the Abelian Kac-Moody algebra
\[
[\alpha_n^a, \alpha_m^b] = n \hbar k \frac{1}{2} \delta_{ab} \delta_{n+m,0},
\]  
(A.7)

from which we can define a Fock space in the standard manner.

The partition function can be calculated as,
\[
Z(\tau) = \int D[\alpha] \exp \left( \frac{i}{\hbar} I_{CWZW}[\alpha, \tau] \right) = \int d\alpha_0 \text{Tr} \exp \left( \frac{i}{\hbar} L_0 - \frac{i}{\hbar} 2\pi \alpha_0^3 \right). 
\]  
(A.8)

From the field equation (A.3) we deduce again that semi-classically only the zero modes contribute to the partition function. Hence we may split the partition function (A.8) into a leading and a sub-leading part
\[
Z(\tau) = \text{Tr} q^N \int d\alpha_0 \exp \left\{ \frac{i}{\hbar} \left( \frac{\alpha_0^a \alpha_{0}^a}{k} - 2\pi \alpha_0^3 \right) \right\},
\]  
(A.9)

where we have introduced the number operator
\[
N = \frac{1}{\hbar k} \sum_{n=1}^{\infty} \delta_{ab} \alpha_{-n}^a \alpha_{n}^b.
\]  
(A.10)

In the saddle point approximation the integral over \( \alpha_0 \) (\( \alpha_0^1 = \alpha_0^2 = 0 \) and \( \alpha_0^3 = k\pi/\tau \)) is easily evaluated
\[
Z(\tau) = \text{Tr} q^N \exp \left\{ \frac{\pi^2 l^2}{4\hbar^2 G\beta (1 - i\Omega)} \right\}. 
\]  
(A.11)

Note that the integral over \( \alpha_0 \) is only well defined, if we assume \( \alpha_0^3 < 0 \). For the calculation of the prefactor we need to know the number of states at each level \( N = n \). This calculation is well known and can be found in [33]. In the limit \( \tau \to 0 \) (large black holes) one finds
\[
\text{Tr} q^N = \exp \left( \frac{i\pi^2}{2\tau} \right). 
\]  
(A.12)
The total partition function is thus given by

\[
Z(\beta, \Omega) = |Z(\tau)|^2 = \exp \left( \frac{\pi^2 l^2}{2\hbar^2 G\beta (1 + l^2 \Omega^2)} + \frac{\pi^2 l}{\hbar\beta (1 + l^2 \Omega^2)} \right).
\]  

(A.13)

Thus the Abelian calculation not only produces the Bekenstein-Hawking part of the partition function but also leads to the \(Z_0\) correction discussed above.

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