LIL type behavior of multivariate Lévy processes at zero

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Abstract

We study the almost sure behavior of suitably normalized multivariate Lévy processes as \( t \downarrow 0 \). Among other results we find necessary and sufficient conditions for a law of a very slowly varying function which includes a general law of the iterated logarithm in this setting. We also look at the corresponding cluster set problem.

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1 Introduction

Let \( \{X_t : t \geq 0\} \) be a \( d \)-dimensional Lévy process with \( X_0 = 0 \) and characteristic triplet \((\gamma, \Sigma, \Pi)\), where \( \gamma \in \mathbb{R}^d \) and \( \Sigma \) is a symmetric, non-negative definite \( d \times d \) matrix. \( \Pi \) is the Lévy measure which is a measure defined on the \( \sigma \)-algebra of all \( d \)-dimensional Borel subsets of \( \mathbb{R}^d \) satisfying \( \Pi(\{0\}) = 0 \) and

\[
\int (1 \wedge |y|^2) \Pi(dy) < \infty,
\]

where \( |\cdot| \) will always denote the Euclidean norm on \( \mathbb{R}^d \).

Moreover, if we set \( \Pi(x) = \Pi(\{y : |y| > x\}, x > 0 \), condition (1.1) can be also written as

\[
\int_0^1 \Pi(\sqrt{t}) dt < \infty.
\]

In this paper we are interested in the almost sure behavior of suitably normalized Lévy processes as \( t \downarrow 0 \). Our starting point is the following \( d \)-dimensional version of the law of the iterated logarithm.

We have with probability one,

\[
\limsup_{t \downarrow 0} \frac{|X_t|}{\sqrt{2t \log \log 1/t}} = \sigma,
\]

where \( \sigma^2 \) is the largest eigenvalue of the matrix \( \Sigma \).
This follows easily from the 1-dimensional case (see Proposition 47.11 in [16]). To see that just write \((X_t)_{t \geq 0}\) as a sum of a Gaussian process and a jump process which is possible by the Lévy-Itô decomposition (see, for instance, Theorem 1 on p. 13 in [1]). Then applying the 1-dimensional result for the \(d\) components of the jump process we see that this process is of almost sure order \(o(\sqrt{t \log \log 1/t})\) as \(t \downarrow 0\) and the above result follows from the LIL for \(d\)-dimensional Brownian motion. So if \(\Sigma\) is non-trivial the almost sure behavior of the Lévy process is completely determined by its Gaussian part.

If we have a purely non-Gaussian Lévy process, that is, if \(\Sigma\) is the zero-matrix, the above \(\lim\sup\) is equal to 0 and it is natural to ask whether one can find a different (and necessarily smaller) function \(b(t), 0 \leq t \leq 1\) in this case such that with probability one,

\[
0 < \limsup_{t \downarrow 0} \frac{|X_t|}{b(t)} < \infty.
\]  

(1.3)

We speak in this case of LIL behavior. This problem has been studied in dimension 1 and let us give a short summary of what is already known in this case: The first result is classical and it is due to Khintchine (see Proposition 47.13 in [16]). It states that for any positive, continuous and increasing function \(g\) satisfying \(g(t)/\sqrt{t \log \log 1/t} \to 0\) as \(t \to 0\) there exists a 1-dimensional Lévy process such that with probability one,

\[
\limsup_{t \downarrow 0} \frac{|X(t)|}{g(t)} = \infty.
\]

This shows that the above function \(b(t)\) provided that it exists can be arbitrarily close to \(\sqrt{t \log \log 1/t}\). Fristedt [11] found an example where one has for \(\beta > 0\) with probability one,

\[
0 < \limsup_{t \downarrow 0} \frac{|X_t|}{\sqrt{t(\log \log 1/t)^{(1-\beta)/2}}} = \sqrt{2}.
\]

Note that if we choose \(\beta = 1\) we get the normalizer \(\sqrt{t}\) for which Bertoin, Doney and Maller [2] provide a more complete result by showing that one has with probability,

\[
\limsup_{t \downarrow 0} \frac{|X_t|}{\sqrt{t}} = \lambda,
\]

where the constant \(\lambda\) can be explicitly determined by a certain integral condition. The authors give examples where \(\lambda\) is finite and positive, but it is also possible that it is zero or infinity. Later this result was extended to a functional limit theorem (see [4]).

The next step was done in Savov [17] where the author found an extension of the integral condition in [2] to more general functions \(b(t), 0 \leq t \leq 1\). He also provided a method for calculating a possible normalizing function \(b\) in terms of the Lévy measure \(\Pi\) which is related to the well known LIL of Klass [13] in the random walk case. As in this classical case, the \(\lim\sup\) results for this general normalizing function require extra integrability conditions and consequently there are certain cases where the general normalizing sequence cannot be used and one has to rely on other methods for finding a suitable normalizing function. (See Proposition 3.1 in [17] for an interesting example.) Finally, Savov [17] also indicates a possible link with the paper [9] where LIL type results for the random walk in the infinite variance case are considered.
Given the work in \[9, 10\] it appears now very natural to ask whether and when one can find "nice" functions $b$ such that \((1.3)\) holds. In view of the results in \[9\] where among other things a "law of a very slowly varying function" has been proven and the afore-mentioned results of \[11\] and \[2\] one could simply ask when one has with probability one,

$$0 < \limsup_{t \downarrow 0} h(1/t) \frac{|X_t|}{\sqrt{t \log \log 1/t}} < \infty,$$

where $h : [0, \infty] \rightarrow [0, \infty]$ is non-decreasing and slowly varying at infinity.

Another interesting question is finally whether one can establish analogous results in the $d$-dimensional case. Both questions will be addressed in the present paper.

2 Statement of Main Results

Unless otherwise indicated we assume from now on that $\{X_t : t \geq 0\}$ is a purely non-Gaussian $d$-dimensional Lévy process with $X_0 = 0$. Thus, the the matrix $\Sigma$ in the characteristic triplet $(\gamma, \Sigma, \Pi)$ is equal to the zero-matrix. Furthermore by a standard argument we can ignore the “big jumps” (see, for instance, \[2, 17\]) so that we can assume that the Lévy measure $\Pi$ is supported by the unit ball $D$ in $\mathbb{R}^d$.

Consequently, $X_t$ has characteristic function $\theta \mapsto \mathbb{E} \exp(i\langle \theta, X_t \rangle) = \exp(t\Psi(\theta))$, where

$$\Psi(\theta) = i\langle \gamma, \theta \rangle + \int_D \left( e^{i\langle \theta, y \rangle} - 1 - i\langle \theta, y \rangle \right) \Pi(dy), \theta \in \mathbb{R}^d.$$

Next we define a function $V(t), t \geq 0$ via the Lévy measure $\Pi$ as follows,

$$V(t) = \sup_{|z| \leq 1} \int_{|y| \leq t} \langle y, z \rangle^2 \Pi(dy), t \geq 0.$$

Note that

$$V(t) \leq \int_{|y| \leq t} |y|^2 \Pi(dy), t > 0$$

Recalling \((1.1)\) we see via the dominated convergence theorem that

$$V(t) \searrow 0 \text{ as } t \downarrow 0.$$

To formulate our first results we still have to introduce some function classes. As in \[9\] we denote the class of the continuous and non-decreasing slowly varying functions $h : [0, \infty] \rightarrow [0, \infty]$ by $\mathcal{H}_1$ and we further look at subclasses $\mathcal{H}_q, 0 \leq q < 1$, consisting of functions $h$ satisfying the condition

$$h(xf_{\tau}(x))/h(x) \rightarrow 1 \text{ as } x \rightarrow \infty, 0 \leq \tau < 1 - q,$$

where $f_{\tau}(x) := \exp((\log x)^\tau), x \geq 1$.

We call the functions in $\mathcal{H}_0$ also “very slowly varying”. Examples for such functions are the functions $x \mapsto (\log \log x)^\alpha$ and $t \mapsto (\log x)^\alpha, x \geq e^e$, where $\alpha > 0$.

Our first result gives an upper bound for $\limsup_{t \downarrow 0} h(1/t)|X_t|/\sqrt{2t \log \log 1/t}$ if $h$ is slowly varying at infinity.
**Theorem 2.1** Let \( \{X_t : t \geq 0\} \) be a purely non-Gaussian \( d \)-dimensional Lévy process. Let \( \lambda \geq 0 \) and let \( h \in \mathcal{H}_1 \). Suppose that \( b(t) = \sqrt{t \log \log 1/t} h(1/t) \) for small \( t \). Assume that

\[
\int_0^1 \prod_{i=1}^d (b(t)) dt < \infty \tag{2.1}
\]

and

\[
\limsup_{t \downarrow 0} V(b(t)) h^2(1/t) \leq \lambda^2/2. \tag{2.2}
\]

Then we have with probability one:

\[
\limsup_{t \downarrow 0} h(1/t) \frac{|X_t|}{\sqrt{t \log \log 1/t}} \leq \lambda
\]

The corresponding lower bound result is as follows,

**Theorem 2.2** Let \( \{X_t : t \geq 0\} \) be a purely non-Gaussian \( d \)-dimensional Lévy process. Let \( \lambda \geq 0 \), \( h \in \mathcal{H}_q \) and let \( b \) be as in Theorem 2.1. Assume that

\[
\limsup_{t \downarrow 0} V(b(t)) h^2(1/t) \geq \lambda^2/2. \tag{2.3}
\]

Then we have with probability one,

\[
\limsup_{t \downarrow 0} h(1/t) \frac{|X_t|}{\sqrt{t \log \log 1/t}} \geq (1 - q)^{1/2} \lambda
\]

Combining the two above results we get the following result which we could call the law of a very slowly varying function for Lévy processes.

**Corollary 2.1** Let \( \{X_t : t \geq 0\} \) be a purely non-Gaussian \( d \)-dimensional Lévy process. Suppose that \( b(t) = \sqrt{t \log \log 1/t} h(1/t) \) for small \( t \), where \( h \in \mathcal{H}_0 \). Assume that condition (2.1) is satisfied. If \( \lambda \geq 0 \), the following are equivalent,

(a)

\[
\limsup_{t \downarrow 0} h(1/t) \frac{|X(t)|}{\sqrt{t \log \log 1/t}} = \lambda \quad \text{with prob. 1}
\]

(b)

\[
\limsup_{t \downarrow 0} V(b(t)) h^2(1/t) = \lambda^2/2.
\]

Condition (2.1) is not required if \( h(x) = O(\sqrt{\log \log x}) \). In this case it already follows from (1.2). So if we choose \( h(x) = \sqrt{\log \log x}, x \geq e \), we get:

\[
\limsup_{t \downarrow 0} \frac{|X_t|}{\sqrt{t}} = \lambda \quad \text{with prob. 1} \iff \limsup_{t \downarrow 0} V(t) \log \log 1/t = \lambda^2/2.
\]
Also note that we can choose functions $h(x)$ which converge extremely slowly to infinity as $x \to \infty$. This gives us, as in the classical result of Khintchine, normalizers $b(t)$ which are very close to $\sqrt{t \log \log 1/t}$ and this will happen if $V(t)$ converges extremely slowly to 0 as $t \to 0$.

The next lemma shows that condition (2.1) is actually redundant for any function $h$ satisfying condition (2.4) below (and not only for functions of order $O(\sqrt{\log \log x})$).

**Lemma 2.1** Let $h \in H_0$ be a function such that we have for some $x_0 \geq e^e$ and $x \geq x_0$,

$$\exists \vartheta \in ]0, 1[: h(x) \leq \exp((\log \log x)\vartheta)$$

Then $V(b(t)) = O(h^{-2}(1/t))$ as $t \to 0$ implies: $\int_0^{e^{-2} \Pi(\sqrt{t \log \log 1/t} / h(1/t))} dt < \infty$.

Condition (2.4) is sharp. The assertion of Lemma 2.1 is no longer true if $\vartheta = 1$. There are examples where $V(t) \sim (\log 1/t)^{-2}$ as $t \to 0$ and we still have $\int_0^{e^{-2} \Pi(\sqrt{t \log \log 1/t} / \log 1/t)} dt = \infty$. (See, for instance, Example 2 in [11].) So we cannot apply Lemma 2.1 if $h(x) = \log x, x \geq 1$.

Condition (2.4) is satisfied for all functions $h(x) = (\log \log x)^q, x > e$, where $q > 0$.

We arrive at the following general LIL for Lévy processes from which one can easily re-obtain the afore-mentioned result of [11] and find many other examples.

**Corollary 2.2** Let $\{X_t : t \geq 0\}$ be a purely non-Gaussian $d$-dimensional Lévy process. Given any $-\infty < p < 1/2$ and any $\lambda \geq 0$, the following are equivalent:

(a) $$\limsup_{t \downarrow 0} \frac{|X(t)|}{\sqrt{t (\log \log 1/t)^p}} = \lambda \text{ a.s.}$$

(b) $$\limsup_{t \downarrow 0} V(t)(\log \log 1/t)^{1-2p} = \lambda^2/2.$$ 

An important tool for proving these results will be a general result on the almost sure behavior of normalized $d$-dimensional Lévy processes which extends Theorem 2.1 in [17] to this more general setting. We weaken assumption (2.2) in this paper slightly by still assuming that $b(t)/t$ converges to infinity as $t \to 0$, but we do not require monotonicity. Our condition (2.6) holds in this case as well, but also if $b(t) = \sqrt{t \log \log 1/t} / h(1/t), t > 0$, where $h : [0, \infty[ \to ]0, \infty[$ can be any function which is slowly varying at infinity. This easily follows from the Karamata representation for slowly varying functions (see, for instance, [11], page 9).

**Theorem 2.3** Let $\{X_t : t \geq 0\}$ be a purely non-Gaussian $d$-dimensional Lévy process and let $b(t), 0 \leq t \leq 1$ be a continuous and increasing real-valued function such that $b(t)/t \to \infty$ as $t \to 0$. Assume also that the following two conditions hold on a suitable interval $[0, t_0]$:

$$b(t)/t^\rho \text{ is non-decreasing for some } \rho > 1/3$$

$$\forall \epsilon > 0 \exists 0 < \delta_\epsilon < t_0: b(s)/b(t) \geq (1 - \epsilon)s/t, 0 \leq s \leq t \leq \delta_\epsilon$$
Under condition (2.1) we have with probability one,
\[ \limsup_{t \downarrow 0} \frac{|X_t|}{b(t)} = \alpha_0, \]
where
\[ \alpha_0 := \sup \left\{ \alpha \geq 0 : \int_0^1 \frac{1}{t} \exp \left( -\frac{\alpha^2 b^2(t)}{2tV(b(t))} \right) dt = \infty \right\}. \]

We mention that Theorem 2.3 and all the previous results remain true if we replace the function \( V(t) \) by the larger function
\[ V_1(t) := \sup_{|z|=1} \int_{|\langle y, z \rangle| \leq t} \langle y, z \rangle^2 \Pi(dy). \]

We will prove this at the end of Section 3. This function plays an important role in the weak convergence theory for matrix normalized Lévy processes (see [15]).

We now turn to the cluster set in Theorem 2.3, that is, the set of all limit points of \( \frac{X_t}{b(t)} \) for sequences \( t_n \downarrow 0 \). We denote this set by \( \mathcal{C}(\{\frac{X_t}{b(t)} : t \downarrow 0\}) \). It is well known that this set is equal to a deterministic set \( A \subset \mathbb{R}^d \) with probability one. (For a proof of a much more general version of this fact the reader is referred to Sect. 2 in [4].)

**Theorem 2.4**

(a) Assume that the conditions of Theorem 2.3 are satisfied and that \( \alpha_0 < \infty \). Then the deterministic cluster set \( A = \mathcal{C}(\{\frac{X_t}{b(t)} : t \downarrow 0\}) \) is compact, symmetric about zero and star-like with respect to zero. Moreover, we have \( \alpha_0 = \sup_{x \in A} |x| \) and there exists a unit vector \( z \in \mathbb{R}^d \) such that \( \alpha_0 z \in A \).

(b) Suppose that \( 0 < \alpha_0 < \infty \) and that \( A \) is a subset of \( \mathbb{R}^d \) as in (a). Let \( h \in \mathcal{H}_0 \) be a function such that \( \lim_{x \to \infty} h(x) = \infty \). Set
\[ b(t) = t \log \log \frac{1}{t/h(1/t)}, t \leq e^{-e}. \]

There exists a \( d \)-dimensional Lévy process \( \{X_t : t \geq 0\} \) such that we have with probability one,
\[ \limsup_{t \downarrow 0} \frac{|X_t|}{b(t)} = \alpha_0 \text{ and } \mathcal{C}(\{\frac{X_t}{b(t)} : t \downarrow 0\}) = A. \]

All the results in this section have counterparts for the \( d \)-dimensional random walk in the infinite second moment case (see [10] for Theorems 2.1-2.3). Cluster sets in the random walk case have been studied in [7] and [8] where it also has been shown that any bounded and closed set which is symmetric and star-like w.r.t. zero can occur as cluster set, but this is done in these two papers only for a very specific normalizing sequence. A new feature is here that Theorem 2.4(b) holds for a large class of normalizing sequences and we give a somewhat easier proof since we can define a suitable Lévy measure directly via a certain representation of the set \( A \).

Theorem 2.3 will be proven in Section 3. Our proof follows essentially the method developed in [2].
and which was further refined in [17]. A new element is that we use exponential inequalities for sums of independent random vectors instead of Berry-Esseen type results. This should make it possible to extend our method to the infinite-dimensional case should this be needed. In Section 4 we then show how Theorems 2.1 and 2.2 follow from Theorem 2.3 and we prove Lemma 2.1. In Section 5 we first provide a general result on clustering (see Theorem 5.1) which is valid for any Lévy process and we then derive a criterion for purely non-Gaussian Lévy processes from it (see Lemma 5.2). We finally prove Theorem 2.4 in Subsection 5.3.

3 Proof of Theorem 2.3

Recall that we assume that the Lévy measure Π is supported by the unit ball D in $\mathbb{R}^d$. Since any function $b$ we consider is continuous and increasing, its inverse function $(t) \mapsto b^{-1}(t)$ is well defined for $0 \leq t \leq b(1)$. As we are only interested in the local behavior at zero, we can assume w.l.o.g. that $b(1) = 1$ by redefining the function $b$ on a suitable interval $[s_0, 1]$, where $0 < s_0 < 1$.

By a standard argument from measure theory, we also have that condition (2.1) holds if and only if

$$\int_{0 < |y| \leq 1} b^{-1}(|y|) \Pi(dy) < \infty. \quad (3.1)$$

We need the following lemma.

**Lemma 3.1** Assume that $b : [0, 1] \to [0, 1]$ satisfies the above assumptions and that condition (2.1) holds. Then we have:

(a) $$\int_0^1 \frac{1}{b(t)^3} \int_{0 < |y| \leq b(t)} |y|^3 \Pi(dy) dt < \infty.$$  

(b) $$\int_0^1 \Pi(\epsilon b(t)) dt < \infty, \quad 0 < \epsilon < 1.$$  

(c) $$\frac{t}{b(t)} \int_{b(t) < |y| \leq 1} |y| \Pi(dy) \to 0 \quad \text{as} \quad t \to 0.$$  

**Proof** (a) Using the fact that

$$\frac{|y|}{b(t)} \leq \frac{b^{-1}(|y|)^\rho}{t^\rho}, \quad b^{-1}(|y|) \leq t \leq 1,$$

which follows from (2.5) we find that

$$\int_0^1 \frac{1}{b(t)^3} \int_{0 < |y| \leq b(t)} |y|^3 \Pi(dy) dt = \int_{0 < |y| \leq 1} \int_{b^{-1}(|y|)}^1 \frac{1}{b(t)^3} dt |y|^3 \Pi(dy) \leq \int_{0 < |y| \leq 1} t^{-3\rho} dt b^--(|y|)^{3\rho} \Pi(dy) \leq \frac{1}{1 - 3\rho} \int_{0 < |y| \leq 1} b^--(|y|) \Pi(dy),$$
where the last integral is finite by condition \((3.1)\). This clearly shows that (a) holds.

(b) Note that
\[
\Pi(\epsilon b(t)) - \Pi(b(t)) = \int_{\epsilon b(t) \leq |y| < b(t)} \Pi(dy) \leq \epsilon^{-3} \frac{1}{b(t)^3} \int_{0 < |y| \leq b(t)} |y|^3 \Pi(dy).
\]
Combining this inequality with part (a), we see that (b) holds as well.

(c) Observe that
\[
t \left| \int_{b(t) < |y| \leq 1} y \Pi(dy) \right| / b(t) \leq t \int_{b(t) < |y| \leq 1} |y| \Pi(dy) / b(t).
\]
On account of condition \((2.6)\) we can find \(0 < \bar{t} \leq 1\) such that
\[
b(t_1)/b(t_2) \geq t_1/(2t_2), 0 \leq t_1 \leq t_2 \leq \bar{t}.
\]
It follows that
\[
\frac{b(t)}{|y|} = \frac{b(t)}{b(b^{-1}(|y|))} \geq \frac{t}{2b^{-1}(|y|)}, b(t) \leq |y| \leq b(\bar{t}) =: C'.
\]
We can conclude that for any \(0 < C < C'\),
\[
\frac{t}{b(t)} \int_{b(t) < |y| \leq 1} |y| \Pi(dy) \leq 2 \int_{0 < |y| \leq C} b^{-1}(|y|) \Pi(dy) + \frac{t}{b(t)} \int_{|y| > C} |y| \Pi(dy).
\]
As \((3.1)\) holds, we can choose for a given \(\epsilon > 0\) a positive constant \(C = C_\epsilon < C'\) so that the first integral will become less than \(\epsilon\).

On the other hand, the second integral is finite for any fixed \(C > 0\) (as we have \(\int_{0 < |y| \leq 1} |y|^2 \Pi(dy) < \infty\)).

Recalling that \(t/b(t) \to 0\) as \(t \to 0\), we see that
\[
\limsup_{t \to 0} \frac{t}{b(t)} \int_{b(t) < |y| \leq 1} |y| \Pi(dy) \leq 2\epsilon,
\]
and part (c) of the lemma has been proven. ■

We next need a \(d\)-dimensional version of Lemma 4.3.(i) in [2] which we prove for general Lévy processes.

**Lemma 3.2** Let \(Y_t = (Y_{t,1}, \ldots, Y_{t,d}), t \geq 0\) be a \(d\)-dimensional Lévy process with characteristic triplet \((\gamma, \Sigma, \Pi)\) and suppose that its Lévy measure \(\Pi\) has support \(D\). Then all moments of \(Y_t\) exist and we have
\[
\mathbb{E}|Y_t|^3 / t \to \int |y|^3 \Pi(dy) \text{ as } t \to 0.
\]

**Proof** The existence of the moments follows, for instance, from Theorem 25.3 in [16].

To prove the other assertion of the lemma we first show for any \(x > 0\) satisfying \(\Pi\{y : |y| = x\} = 0\),
\[
\frac{1}{t} \mathbb{P}\{|Y_t| > x\} \to \Pi(x) \text{ as } t \to 0. \tag{3.2}
\]
We apply Corollary 8.9. in [16]. Set for \(x > 0\) and \(0 < \epsilon < x\),
\[
f_{x,\epsilon} = 1 \wedge \frac{\text{dist}(\cdot, (x - \epsilon)D)}{\epsilon},
\]

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where, as usual, \( \text{dist}(y, A) = \inf \{ |y - z| : z \in A \} \) is the distance of \( y \) to the set \( A \subset \mathbb{R}^d \).

It is easy to see that \( f_{x, \epsilon} \) is continuous on \( \mathbb{R}^d \) and we have,

\[
I(x-\epsilon)_{D} \leq 1 - f_{x, \epsilon} \leq I_{xD}.
\]

The conclusion is that

\[
\limsup_{t \to 0} \frac{1}{t} \mathbb{P} \{|Y_t| > x\} \leq \limsup_{t \to 0} \frac{1}{t} \mathbb{E} f_{x, \epsilon}(Y_t) = \int f_{x, \epsilon} d\Pi \leq \Pi(x - \epsilon).
\]

Letting \( \epsilon \) converge to zero, it follows that

\[
\limsup_{t \to 0} \frac{1}{t} \mathbb{P} \{|Y_t| > x\} \leq \Pi \{ y : |y| \geq x \} = \Pi(x).
\]

A similar argument gives that

\[
\liminf_{t \to 0} \frac{1}{t} \mathbb{P} \{|Y_t| > x\} \geq \Pi \{ y : |y| \geq x \} = \Pi(x)
\]

and relation (3.2) has been proven.

After some calculation one obtains from Theorem 25.17 in [16] for \( 1 \leq i \leq d \),

\[
\mathbb{E} Y_{t,i}^2 = t(m_{2,i} + \Sigma_{i,i}) + t^2 \gamma_i^2
\]

and

\[
\mathbb{E} Y_{t,i}^4 = m_{4,i} t + (3(m_{2,i} + \Sigma_{i,i})^2 + 4m_{3,i} \gamma_i)t^2 + 6(m_{2,i} + \Sigma_{i,i}) \gamma_i^2 t^3 + \gamma_i^4 t^4 =: \sum_{j=1}^{4} c_{j,i} t^j,
\]

where \( m_{k,i} = \int y^k \Pi(dy), k \geq 2 \).

Arguing as on page 175 of [2] we conclude that

\[
\frac{1}{t} \mathbb{P} \{|Y_t| > x\} \leq \frac{1}{t} \sum_{i=1}^{d} \mathbb{P} \{|Y_{t,i}| \geq x/\sqrt{d}\} \leq dx^{-2} \sum_{i=1}^{d} (m_{2,i} + \Sigma_{i,i} + t \gamma_i^2) I_{[0,1]}(x) + d^2 x^{-4} \sum_{i=1}^{d} (m_{4,i} + \sum_{j=2}^{4} c_{j,i} t^{j-1}) I_{[1,\infty]}(x),
\]

As there are at most countably many \( x > 0 \) for which \( \Pi \{ y : |y| = x \} > 0 \), we have convergence almost everywhere in (3.2) so that we can apply the dominated convergence theorem. We see that as \( t \to 0 \),

\[
\frac{1}{t} \mathbb{E} |Y_t|^3 = \int_{0}^{\infty} 3x^2 \mathbb{P} \{|Y_t| > x\} dx \to \int |y|^3 \Pi(dy).
\]

The lemma has been proven. \( \square \)

We return to the special case where the Lévy process \( \{ X_t : t \geq 0 \} \) is purely non-Gaussian.

As in [2] [17] we then can write the stochastic process \( X_t \) as a sum of two (independent) stochastic processes \( Y_t(b) \) and \( Z_t(b) \) plus a deterministic term \( \nu(b) \).

Letting \( \Delta X_t = X_t - X_{t-}, t > 0 \) be the jumps of \( X_t \), it follows from the Lévy-Itô decomposition that for any \( 0 < b \leq 1 \),

\[
X_t = t \nu(b) + Y_t(b) + Z_t(b), t > 0,
\]

(3.3)
where
\[
\nu(b) = \gamma - \int_{b < |y| \leq 1} y\Pi(dy),
\]
\[
Y_t^{(b)} = \text{a.s. lim}_{\epsilon \downarrow 0} \left\{ \sum_{0 < s \leq t} \Delta X_s I_{\epsilon < |\Delta X_s| \leq b} - t \int_{\epsilon < |y| \leq b} y\Pi(dy) \right\}
\]
and
\[
Z_t^{(b)} = \sum_{0 < s \leq t} \Delta X_s I_{|\Delta X_s| > b}.
\]
As in Lemma 4.1 of [17] it follows that under condition (2.1) one has for any \(0 < r < 1\),
\[
P\left\{ \sup_{0 \leq s \leq r^n} |Z_{s}^{(b(r^n))}| > 0 \text{ i.o.} \right\} = 0. \tag{3.4}
\]
This will enable us to reduce the proof of Theorem 2.3 to studying the processes
\[
Y_t^{(b(r^n))}, r^n + 1 < t \leq r^n, n \geq 0.
\]
We first look at the upper bound in Theorem 2.3, that is, we show that the \(\lim\sup\) is \(\leq \alpha_0\).

### 3.1 The upper bound part

Using that for any sequence \(\{\xi_n : n \geq 1\}\) of i.i.d. mean zero random vectors with \(\mathbb{E}|\xi_1|^2 < \infty\),
\[
\mathbb{E}\left| \sum_{i=1}^{n} \xi_i \right|^2 \leq (\mathbb{E}\left| \sum_{i=1}^{n} \xi_i \right|^2)^{1/2} = \sqrt{n}\mathbb{E}|\xi_1|^2)^{1/2}
\]
we can infer from Theorem 3.1 in [10],

**Theorem 3.1** Let \(\xi_1, \ldots, \xi_n\) be i.i.d. mean zero random vectors with finite third absolute moments. Then we have for any fixed \(\delta > 0\) and all \(x > 0\),
\[
P\left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{n} \xi_i \right| \geq 2\sqrt{n}(\mathbb{E}|\xi_1|^2)^{1/2} + x \right\} \leq \exp\left( - \frac{x^2}{(2 + \delta)n\Lambda} \right) + Cn\mathbb{E}|\xi_1|^3/x^3,
\]
where \(\Lambda = \sup \{\mathbb{E}(z, \xi_1)^2 : |z| \leq 1\}\) and \(C\) is a positive constant depending on \(\delta\).

This implies the following inequality for the Lévy process \(\{Y_s^{(b)}, s \geq 0\}\),

**Theorem 3.2** Given \(\delta > 0\) we have for all \(t, x > 0\),
\[
P\left\{ \sup_{0 \leq s \leq t} |Y_s^{(b)}| \geq 2(\mathbb{E}|Y_t^{(b)}|^2)^{1/2} + x \right\} \leq \exp\left( - \frac{x^2}{(2 + \delta)tV(b)} \right) + Ct \int_{0 < |y| \leq b} |y|^3\Pi(dy)/x^3,
\]
where \(C > 0\) is a constant depending on \(\delta\).
Proof. By the right continuity of the sample paths \( s \mapsto Y_s^{(b)} \) we have,
\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |Y_s^{(b)}| \geq 2(\mathbb{E}|Y_t^{(b)}|^2)^{1/2} + x \right\} = \lim_{n \to \infty} \mathbb{P} \left\{ \max_{1 \leq k \leq n} |Y_{tk/n}^{(b)}| \geq 2(\mathbb{E}|Y_{t/n}^{(b)}|^2)^{1/2} + x \right\}.
\]

Note that
\[
Y_{tk/n}^{(b)} = \sum_{j=1}^{k} (Y_{t_j/n}^{(b)} - Y_{t(j-1)/n}^{(b)}), 1 \leq k \leq n,
\]
where the random vectors \( \xi_{j,n,b} := Y_{t_j/n}^{(b)} - Y_{t(j-1)/n}^{(b)} \), \( 1 \leq j \leq n \) are i.i.d.
Moreover we have
\[
\mathbb{E}\xi_{1,n,b} = \mathbb{E}Y_{t/n}^{(b)} = 0 \quad \text{and} \quad n\mathbb{E}|\xi_{1,n,b}|^2 = \mathbb{E}|Y_{t/n}^{(b)}|^2,
\]
By Example 25.12 in [16] we also know that
\[
\mathbb{E}\langle z, \xi_{1,n,b} \rangle^2 = \left( \frac{t}{n} \right)^3 \int_{|y| \leq b} \langle z, y \rangle^2 \Pi(dy), \quad z \in \mathbb{R}^d.
\]
We can infer that \( \Lambda \) in the above inequality is equal to \( V(b)t/n \).
It follows that
\[
\mathbb{P} \left\{ \max_{1 \leq k \leq n} |Y_{tk/n}^{(b)}| \geq 2(\mathbb{E}|Y_{t/n}^{(b)}|^2)^{1/2} + x \right\} \leq \exp \left( -\frac{x^2}{(2 + \delta)tV(b)} \right) + Ct\frac{n}{t}\mathbb{E}|Y_{t/n}^{(b)}|^3/x^3.
\]
Letting \( n \) go to infinity and recalling Lemma 3.2, the inequality follows. □

Note also that
\[
\mathbb{E}|Y_{t/n}^{(b)}|^2 \leq dtV(b), t \geq 0, 0 < b \leq 1.
\] (3.5)
We are ready to establish the upper bound in Theorem 2.3, that is we now can show, that with probability one,
\[
\limsup_{t \downarrow 0} \frac{|X_t|}{b(t)} \leq \alpha_0,
\] (3.6)
where w.l.o.g we can and do assume that \( \alpha_0 < \infty \).
By definition of \( \alpha_0 \) we have for any \( \alpha > \alpha_0 \),
\[
\infty > \sum_{n=0}^{\infty} \int_{r^{n+1}}^{r^n} t^{-1} \exp \left( -\frac{\alpha^2b^2(t)}{2tV(b(t))} \right) dt \geq \log(r^{-1}) \sum_{n=0}^{\infty} \exp \left( -\frac{\alpha^2b^2(r^n)}{2r^{n+1}V(b(r^{n+1}))} \right).
\] (3.7)
In particular, we have \( r^nV(b(r^n))/b^2(r^{n-1}) \to 0 \) as \( n \to \infty \).
From this observation it further follows (see 3.5) that
\[
\mathbb{E}\left| Y_{r^n(r^n)}^{(b(r^n))} \right|^2 = o(b^2(r^{n-1})) \quad \text{as} \quad n \to \infty.
\] (3.8)
Let $\delta > 0$. Then it follows from Theorem 3.2 that for large $n$

$$P \left\{ \sup_{0 \leq s \leq r^n} \left| Y_s(b(r^n)) \right| \geq (1 + \delta)(\alpha_0 + \delta)b(r^{n-1}) \right\}$$

$$\leq P \left\{ \sup_{0 \leq s \leq r^n} \left| Y_s(b(r^n)) \right| \geq (\alpha_0 + \delta)(1 + \delta/2)b(r^{n-1}) + 2 \left( \mathbb{E} \left| Y_{r^n}(b(r^n)) \right|^2 \right)^{1/2} \right\}$$

$$\leq \exp \left( -\frac{(\alpha_0 + \delta)^2 b^2(r^{n-1})}{2r^n V(b(r^n))} \right) + C' r^n \int_{0 < |y| \leq b(r^n)} |y|^3 \Pi(dy)/b^3(r^{n-1}),$$

where $C' > 0$ is a constant depending on $\delta$.

From relation (3.7) it follows that

$$\sum_{n=1}^{\infty} \exp \left( -\frac{(\alpha_0 + \delta)^2 b^2(r^{n-1})}{2r^n V(b(r^n))} \right) < \infty.$$ 

Moreover employing the same argument already used for proving (3.7), we can infer from Lemma 3.1(a) that

$$\sum_{n=1}^{\infty} r^n \int_{0 < |y| \leq b(r^n)} |y|^3 \Pi(dy)/b^3(r^{n-1}) < \infty.$$ 

By the Borel-Cantelli lemma we then have for any $\delta > 0$ with probability one,

$$\sup_{r^n+1 < s \leq r^{n+1}} \left| Y_s(b(r^n)) \right| \leq (1 + \delta)(\alpha_0 + \delta)b(r^{n-1}),$$

eventually

Combining this with Lemma 3.1(c) and relation (3.4), we see that with probability one,

$$\limsup_{t \to 0} \frac{|X_t|}{b(t/r^2)} \leq \alpha_0.$$ 

From condition (2.6) in Theorem 2.3 it finally follows that $\limsup_{t \to 0} b(t/r^2)/b(t) \leq r^{-2}$ for any fixed $0 < r < 1$. We see that with probability one,

$$\limsup_{t \to 0} \frac{|X_t|}{b(t)} \leq \alpha_0 r^{-2}.$$ 

Since this holds for any $r < 1$, relation (3.6) follows.

### 3.2 The lower bound part

W.l.o.g. we assume $\alpha_0 > 0$. We show that we have for any $0 < \alpha < \alpha_0$ with probability 1,

$$\limsup_{t \to 0} \frac{|X_t|}{b(t)} \geq \alpha.$$ (3.9)

If $\int_0^1 \Pi(b(t))dt = \infty$, it follows from Lemma 3.1(b) that we have

$$\int_0^1 \Pi(Cb(t))dt = \infty, \forall C > 0$$
and we can infer from Proposition 4.2 in [2] that with probability one \( \limsup_{t \to 0} |X_t|/b(t) = \infty \) and (3.9) is trivially true. This is also the case if \( \limsup_{t \to 0} \mathbb{P}\{|X_t| \geq \alpha b(t)\} > 0 \). For this implies that

\[
\mathbb{P}\{\limsup_{t \to 0} |X_t|/b(t) \geq \alpha\} \geq \limsup_{t \to 0} \mathbb{P}\{|X_t| \geq \alpha b(t)\} > 0.
\]

Then the first probability has to be equal to 1 by Blumenthal’s 0-1 law (see, [16], Proposition 40.4) and (3.9) holds.

To that end we first show for \( \alpha < \alpha_0 \),

\[
\sum_{n=0}^{\infty} \mathbb{P}\{|X_{r^n}| \geq \alpha b(r^n)\} = \infty,
\]

where \( 0 < r < 1 \) has to be chosen so that \( \alpha/r < \alpha_0 \).

Using the same argument as in the proof of relation (3.7), we find that for any \( 0 < \tilde{\alpha} < \alpha_0 \),

\[
\sum_{n=0}^{\infty} \exp\left(-\frac{\tilde{\alpha}^2 b^2(r^n+1)}{2r^n V(b(r^n))}\right) = \infty.
\]

As we have by condition (2.6) for \( 0 < \epsilon < 1 \), \( b(r^{n+1}) \geq rb(r^n)(1-\epsilon) \) if \( n \) is large, we can conclude that for \( 0 < r_1 < r \),

\[
\sum_{n=0}^{\infty} \exp\left(-\frac{\tilde{\alpha}^2 r_1^2 b^2(r^n)}{2r^n V(b(r^n))}\right) = \infty.
\]

Let \( \alpha < \alpha_1 < \alpha_0 \) be chosen so that we still have \( \alpha_1/r < \alpha_0 \). Then there exist an \( r_1 \) satisfying \( 0 < r_1 < r \) and \( \delta > 0 \) small so that \( \tilde{\alpha} := \alpha_1(1+\delta)/r_1 < \alpha_0 \). It follows from (3.12) that

\[
\sum_{n=0}^{\infty} \exp\left(-\frac{\alpha_1^2(1+\delta)^2 b^2(r^n)}{2r^n V(b(r^n))}\right) = \infty.
\]

Next observe that

\[
\mathbb{P}\{|X_{r^n}| \geq \alpha b(r^n)\} \geq \mathbb{P}\{|Y_{r^n}^{(b(r^n))}| \geq \alpha_1 b(r^n)\} - \mathbb{P}\{|Z_{r^n}^{(b(r^n))} + r^n \nu(b(r^n))| \geq (\alpha_1 - \alpha) b(r^n)\}.
\]

From Lemma 3.1(c) in combination with relation (3.4), we readily obtain that

\[
\sum_{n=0}^{\infty} \mathbb{P}\{|Z_{r^n}^{(b(r^n))} + r^n \nu(b(r^n))| \geq (\alpha_1 - \alpha) b(r^n)\} < \infty.
\]

Therefore, (3.11) is proven once we have shown that

\[
\sum_{n=0}^{\infty} \mathbb{P}\{|Y_{r^n}^{(b(r^n))}| \geq \alpha_1 b(r^n)\} = \infty.
\]

To establish this relation, we first derive an inequality which gives lower bounds for the probabilities \( \mathbb{P}\{|Y_t^{(b)}| \geq x\}, x > 0 \).
Lemma 3.3 Let $\xi_1, \ldots, \xi_n$ be i.i.d. mean zero random vectors with finite third absolute moments. Then we have for any $0 < \delta < 1$ and all $x > 0$,
\[
\mathbb{P}\{|\sum_{j=1}^{n} \xi_j| \geq x\} \geq C_1 \exp\left(-\frac{x^2(1 + \delta)^2}{2n\Lambda}\right) - C_2n\mathbb{E}|\xi_1|^3/x^3,
\]
where $\Lambda = \sup\{\mathbb{E}(\xi_1, z)^2 : |z| \leq 1\}$ and $C_i, i = 1, 2$ are positive constants depending on $\delta$ only.

Proof Applying Lemma 5 in [8] with $s = x$ and $t = x\delta/2$, we find that
\[
\mathbb{P}\{|\sum_{j=1}^{n} \xi_j| \geq x\} \geq \mathbb{P}\{|\sum_{j=1}^{n} \eta_j| \geq x(1 + \delta/2)\} \geq 8A\delta^{-3}n\mathbb{E}|X_1|^3x^{-3},
\]
where the random vectors $\eta_1, \ldots, \eta_n$ are i.i.d. with $\mathcal{N}(0, \text{cov}(\xi_1))$-distribution, $\text{cov}(\xi_1)$ is the covariance matrix of $\xi_1$ and $A$ is an absolute constant.

Next choose a unit vector $z \in \mathbb{R}^d$ so that $\Lambda = \mathbb{E}(\xi_1, z)^2 = \mathbb{E}(\eta_1, z)^2$. Then we have of course
\[
\mathbb{P}\{|\sum_{j=1}^{n} \eta_j| \geq x(1 + \delta/2)\} \geq \mathbb{P}\{|\sum_{j=1}^{n} \eta_j, z| \geq x(1 + \delta/2)\} \geq \mathbb{P}\{|\sqrt{n}\eta'| \geq x(1 + \delta/2)/\Lambda^{1/2}\},
\]
where $\eta'$ is a 1-dimensional standard normal random variable. Using the inequality
\[
\mathbb{P}\{\eta' \geq x\} \geq x(x^2 + 1)^{-1} \exp(-x^2/2)/\sqrt{2\pi}, x > 0,
\]
we easily obtain the above lower bound. $\square$

By the same reasoning as in the proof of Theorem 3.2 we obtain the following result for the Lévy processes $\{Y_t^{(b)} : t \geq 0\}$ from Lemma 3.3.

Theorem 3.3 Given $\delta > 0$ we have for all $t, x > 0$,
\[
\mathbb{P}\{|Y_t^{(b)}| \geq x\} \geq C_1 \exp\left(-\frac{x^2(1 + \delta)^2}{2tV(b)}\right) - C_2t \int_{0<|y|\leq b} |y|^3\Pi(dy)/x^3,
\]
where $C_1, C_2 > 0$ are constant depending on $\delta$ only.

Recalling Lemma 3.1(a) and (3.13), we now see that
\[
\sum_{n=0}^{\infty} \mathbb{P}\{|Y_t^{(b)(r^n)}| \geq \alpha_1b(r^n)\} \geq C_1 \sum_{n=0}^{\infty} \exp\left(-\frac{\alpha_1^2(1 + \delta)^2b^2(r^n)}{2tV(b(r^n))}\right) - C_2\alpha_1^{-3} \sum_{n=0}^{\infty} r^n \int_{0<|y|\leq b(r^n)} |y|^3\Pi(dy)/b(r^n)^3 = \infty.
\]
This shows that (3.14) holds and we thus have proven (3.11).

As in [17] (see formula (4.7)) we can infer from (3.11) that for any fixed natural number $m \geq 1$, there exists a $k \in \{0, \ldots, m - 1\}$ such that
\[
\sum_{n=0}^{\infty} \mathbb{P}\{|X_{r^{nm+k}}| \geq \alpha b(r^{nm+k})\} = \infty.
\]
Recall that $0 < r < 1$ had to be chosen so that $\alpha/r < \alpha_0$. We will assume that $r^m < 1/2$ which holds if $m$ is bigger than some finite positive number $m_0 = m_0(r)$. Further set $t_n = r^{m+k}/(1-r^m)$, $n \geq 0$ which implies that $t_{n+1} - t_{n+1} = r^{m+k}$ and also $t_{n+1} \leq (1-r^m)t_n$. Let $0 < \delta < 1$ be fixed. Consider the events
\[ A_n := \{|X_{t_n} - X_{t_{n+1}}| \geq \alpha b(t_n (1-r^m))\}, \quad B_n := \{|X_{t_{n+1}}| \leq \delta b(t_n (1-r^m))\}, \quad n \geq 1. \]
Then we have by condition (2.5),
\[ b(t_n (1-r^m))/b(t_{n+1}) \geq (r-\frac{m}{2}) - \rho \text{ for large } n. \]
Choosing $m \geq m_1$ for a suitable number $m_1 = m_1(r, \alpha, \delta, \rho)$, it follows that for large $n$,
\[ \mathbb{P}(B_n^c) \leq \mathbb{P}(\{|X_{t_{n+1}}| \geq \alpha b(t_{n+1})\}), \]
which converges to zero by (3.10).

The events $A_n$ are independent and we have by (3.16)
\[ \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty. \]
Moreover, $A_1, \ldots, A_n, B_n$ are independent for any $n \geq 1$. Thus, we can infer from the Feller-Chung lemma (see Lemma 3(i) on page 70 in [5]) that
\[ \mathbb{P}(A_n \cap B_n \text{ infinitely often}) = \mathbb{P}(A_n \text{ infinitely often}) = 1. \]
Finally, note that
\[ \{A_n \cap B_n \text{ infinitely often}\} \supset \{\limsup_{t \to 0} |X_t|/b(t(1-r^m)) \geq \alpha - \delta\}. \]
In view of condition (2.6), we have $\liminf_{t \to 0} b(t)/b(t(1-r^m)) \geq 1 - r^m$ which is $\geq (1-\delta)$ if $m$ is big enough. We now can conclude that with probability one,
\[ \limsup_{t \to 0} |X_t|/b(t) \geq (1-\delta)(\alpha - \delta). \]
As $\delta$ can be chosen arbitrarily small, we see that (3.9) holds. Theorem 2.3 has been proven. \(\square\)

To conclude this section we show that Theorem 2.3 remains true if we replace the function $V$ by $V_1$. To prove this it is enough to show that $\alpha_1 = \alpha_0$, where
\[ \alpha_1 := \sup \left\{ \alpha \geq 0 : \int_0^1 \frac{1}{t} \exp \left( -\frac{\alpha^2b^2(t)}{2tV_1(b(t))} \right) dt = \infty \right\}. \] (3.17)
It is obvious that $\alpha_0 \leq \alpha_1$ (since $V(b(t)) \leq V_1(b(t))$).

For the reverse inequality we can assume w.l.o.g. that $\alpha_0 < \infty$.

Observe that we have for any $t > 0$,
\[ \sup_{|z| \leq 1} \int_{\{|y,z| \leq t\}} \langle y, z \rangle^2 \Pi(dy) \leq V(t) + \sup_{|z| \leq 1} \int_{\{|y| > t, \langle y, z \rangle | \leq t\}} \langle y, z \rangle^2 \Pi(dy) \leq V(t) + t^2 \Pi(t). \]
Given $0 < \delta < 1$, set $\alpha = (1 + \delta)(\alpha_0 + \delta)$. Then we have

$$
\int_0^1 \frac{1}{t} \exp \left( - \frac{\alpha^2 b^2(t)}{2tV_1(b(t))} \right) dt \\
\leq \int_0^1 \frac{1}{t} \exp \left( - \frac{\alpha^2 b^2(t)}{2tV(b(t)) + b^2(t)\Pi(b(t))} \right) dt \\
\leq \int_0^1 \frac{1}{t} \exp \left( - \frac{\alpha^2 b^2(t)}{2(1 + \delta) tV(b(t))} \right) dt + \int_0^1 \frac{1}{t} \exp \left( - \frac{\alpha^2 b^2(t)}{2t(1 + \delta^{-1}) b^2(t)\Pi(b(t))} \right) dt \\
\leq \int_0^1 \frac{1}{t} \exp \left( - \frac{(\alpha_0 + \delta)^2 b^2(t)}{2tV(b(t))} \right) dt + \frac{2(1 + \delta^{-1})}{\alpha^2} \int_0^1 \Pi(b(t)) dt < \infty,
$$

where we have used in the last step the trivial inequality $\exp(-x) \leq x^{-1}, x > 0$.

We see that $\alpha_1 \leq (\alpha_0 + \delta)(1 + \delta), \delta > 0$ and consequently we also have $\alpha_0 \geq \alpha_1$.

## 4 Proofs of Theorems 2.1, 2.2 and Lemma 2.1

To simplify our notation, we set for any $\alpha \geq 0$,

$$J(\alpha) := \int_0^1 \frac{1}{t} \exp \left( - \frac{\alpha^2 b^2(t)}{2tV(b(t))} \right) dt.$$

Note that $J(\alpha)$ is finite if there exists a $0 < u_0 < 1$ such that the integral over $[0, u_0]$ is finite.

### Proof of Theorem 2.1

W.l.o.g. we assume that $\lambda < \infty$. In view of Theorem 2.3 it is enough to show that $J(\lambda + \delta) < \infty, \delta > 0$.

Choosing $0 < t_\delta < e^{-1}$ small enough so that

$$b(t) = \sqrt{t \log \log 1/t/h(1/t)}, 0 < t \leq t_\delta$$

and

$$V(b(t)) \leq \frac{(\lambda + \delta/2)^2}{2} h^{-2}(1/t), 0 < t < t_\delta,$$

we readily obtain that

$$\int_0^{t_\delta} \exp(-(\lambda + \delta)^2 b^2(t)/(2tV(b(t)))t^{-1}dt \leq \int_0^{t_\delta} t^{-1}(\log 1/t)^{-(1+\epsilon)^2} dt$$

where $\epsilon = (\lambda + \delta)/(\lambda + \delta/2) - 1 > 0$. It follows that $J(\lambda + \delta) < \infty$. □

### Proof of Theorem 2.2

It is obviously enough to prove this result if $\lambda > 0$. Moreover, as in the lower bound part proof of Theorem 2.3 we can assume w.l.o.g. that

$$\int_0^1 \Pi(b(t)) dt < \infty$$

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since otherwise the lim sup is infinite and Theorem 2.2 is certainly true. Then Theorem 2.3 applies and it is enough to show that

\[ J(\alpha) = \infty, \quad 0 < \alpha < (1 - q)^{1/2} \lambda. \]

Given \( 0 < \alpha < (1 - q)^{1/2} \lambda \), let \( \tau' \) be defined by \( \alpha^2 = \tau' \lambda^2 \) and take a \( \tau \) satisfying

\[ 0 < \tau' < \tau < (1 - q). \]

Next choose a sequence \( e^{-1} > t_k \downarrow 0 \) such that

\[ V(b(t_k)) \geq \frac{\lambda^2}{2} h^{-2}(1/t_k)(1 - 1/k), \quad k \geq 1 \]

and set

\[ \tilde{t}_k = t_k \exp((\log 1/t_k)^{\tau'/2}), \quad k \geq 1. \]

A small calculation gives that

\[ \tilde{t}_k^{-1} f_{\tau}(\tilde{t}_k^{-1}) \geq t_k^{-1}, \quad k \geq 1. \]

Using the fact that

\[ h(\tilde{t}_k^{-1} f_{\tau}(\tilde{t}_k^{-1}))/h(\tilde{t}_k^{-1}) \to 1 \text{ as } k \to \infty, \]

we can conclude that

\[ h(1/t_k)/h(1/\tilde{t}_k) \to 1 \text{ as } k \to \infty. \]

By monotonicity of the functions \( V \) and \( b \) we obtain for some \( \delta_k \to 0 \),

\[ V(b(t)) \geq \frac{\lambda^2}{2} h^{-2}(1/t)(1 - \delta_k), \quad t_k \leq x \leq \tilde{t}_k, \]

Consequently, we have

\[ J(\alpha) \geq \int_{t_k}^{\tilde{t}_k} t^{-1} \log(1/t)^{-\tau'(1-\delta_k)} \, dt \geq \log(1/t_k)(\log 1/t_k)^{-\tau'(1-\delta_k)} \]

which is by definition of \( \tilde{t}_k \),

\[ \geq \log(1/t_k)^{\tau'/2}(1-\delta_k)/2. \]

This sequence converges to infinity and thus \( J(\alpha) = \infty \), whenever \( 0 < \alpha < (1 - q)^{1/2} \). Theorem 2.2 has been proven. \( \square \)

To prove Lemma 2.1 we need a further lemma.

**Lemma 4.1** Assume that \( V(t) := \int_{|y| \leq t} |y|^2 \Pi(dy) > 0, \ t > 0 \). Let \( g : [0, \infty[ \to [0, \infty[ \) be a non-decreasing function such that \( \int_0^c g(t)^{-1} \, dt < \infty \) for all \( c > 0 \). Then we have,

\[ \int_{|y| \leq 1} g(V(|y|)) \Pi(dy) < \infty. \]

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**Proof** We first note that

\[
\int_{|y| \leq 1} \frac{|y|^2}{g(V(|y|))} \Pi(dy) = \int_0^1 \frac{t^2}{g(V(t))} Q(dt),
\]

where \( Q \) is the image measure \( \Pi_f \) with \( f : \mathbb{R}^d \to \mathbb{R} \) being the Euclidean norm.

Set \( V(1) := K \). Let further \( R \) be the p-measure on the Borel subsets of \( \mathbb{R} \) with \( Q \)-density \( K^{-1}t^2I_{[0,1]}(t) \) and note that \( F(t) := V(t)/K \) is the distribution function of \( R \). We then have

\[
\int_0^1 \frac{t^2}{g(V(t))} Q(dt) = K \int_0^1 \frac{1}{g(KF(t))} R(dt).
\]

Consider the generalized inverse function of \( F \), that is

\[
\phi(u) = \inf\{t \geq 0 : F(t) \geq u\}, 0 < u < 1.
\]

Then it is easy to see that \( F(\phi(u)) \geq u, 0 < u < 1 \). Moreover, if \( U : \Omega \to [0,1[ \) is a uniform \((0,1)\)-distributed random variable on a p-space \((\Omega, \mathcal{F}, \mathbb{P})\), the random variable \( \phi \circ U \) has distribution function \( F \). It follows that

\[
\int_0^1 g(KF(t))^{-1} R(dt) = \int g(KF(\phi(U)))^{-1} d\mathbb{P} = \int_0^1 g(KF(\phi(u)))^{-1} du \leq \int_0^1 g(Ku)^{-1} du < \infty
\]

and the lemma has been proven. \( \square \)

Recalling that \( V(t) \leq \bar{V}(t) \leq dV(t) \), we can infer that if \( V(t) > 0, t > 0 \), we have for any \( \delta > 0 \),

\[
\int_{|y| \leq 1} \frac{|y|^2}{\bar{V}(|y|)(\log_+(1/\bar{V}(|y|)))^{1+\delta}} \Pi(dy) < \infty. \tag{4.1}
\]

Here we set \( \log_+(x) = \max\{1, \log x, x > 0\} \).

**Proof of Lemma 2.1**

W. l. o. g. we assume that \( V(t) > 0, t > 0 \) and that \( h(x) \to \infty \) as \( x \to \infty \). Otherwise the lemma is trivial.

The function \( t \mapsto \sqrt{t \log \log 1/t} / h(1/t) = b(t) \) is increasing and invertible on a subinterval \([0, t_0]\). We then clearly have for \( 0 \leq x \leq b(t_0) \),

\[
b^{-}(x) = x^2 h^2(1/b^{-}(x))/\log \log(1/b^{-}(x)).
\]

Since \( V(b(t)) = O(h^{-2}(1/t)) \) as \( t \to 0 \) it follows that \( V(x) \leq C_1 h^{-2}(1/b^{-}(x)), 0 < x \leq b(t_0) \), where \( C_1 \geq 1 \) is a positive constant. Combining this inequality with \( \text{(2.4)} \), we find that for small enough \( x \),

\[
V(x) (\log_+ 1/V(x))^{1/\vartheta} \leq C_2 h^{-2}(1/b^{-}(x))(\log(h(1/b^{-}(x))))^{1/\vartheta} \leq C_2 h^{-2}(1/b^{-}(x))(2 \log \log 1/b^{-}(x))^{\vartheta}^{1/\vartheta} \leq C_2 h^{-2}(1/b^{-}(x)) \log \log 1/b^{-}(x),
\]

where \( C_2 \) is a positive constant. Thus we have for some \( a_0 \leq e^{-2} \),

\[
\int_{0 <|y| \leq a_0} b^{-}(|y|) \Pi(dy) \leq C_2 \int_{0 <|y| \leq a_0} \frac{|y|^2}{\bar{V}(|y|)(\log_+(1/\bar{V}(|y|)))^{1/\vartheta}} \Pi(dy),
\]

where the second integral is finite by \( \text{(4.1)} \). (Recall that \( \vartheta < 1 \).

This implies that \( \int_{0 <|y| \leq 1} b^{-}(|y|) \Pi(dy) < \infty \) and our proof of Lemma 2.1 is complete. \( \square \)
5 Cluster sets

5.1 A general result

Throughout this subsection $X_t, t \geq 0$ will be a (general) $d$-dimensional Lévy process such that $X_t/b(t)$ is stochastically bounded as $t \to 0$, that is,

$$\forall \delta > 0 \exists K_\delta > 0, 0 < t_\delta < 1 : \mathbb{P}\{|X_t| > K_\delta b(t)\} < \delta, 0 < t < t_\delta.$$  \hspace{1cm} (5.1)

As in the previous sections $b : [0, 1] \to [0, 1]$ is a continuous and increasing function such that $b(1) = 1$, $b(t)/t \to \infty$ as $t \to 0$ and conditions (2.5), (2.6) are satisfied.

**Lemma 5.1** Let $0 < r < 1$ be a fixed number and let $x \in \mathbb{R}^d$. The following are equivalent:

(a) $x \in C(\{X_t/b(t) : t \downarrow 0\})$ with probability one

(b) $\sum_{n=1}^{\infty} \mathbb{P}\{|X_t/b(t) - x| < \epsilon \text{ for some } r^{n+1} \leq t < r^n\} = \infty$ for all $\epsilon > 0$.

**Proof** (a) $\Rightarrow$ (b) This follows directly from the Borel-Cantelli lemma.

(b) $\Rightarrow$ (a) It is obviously enough to show that we have for any $\epsilon > 0$,

$$\mathbb{P}(\limsup_{n \geq 1} \{|X_t/b(t) - x| < \epsilon \text{ for some } t_{n+1} \leq t < t_n\}) = 1,$$

where we set $t_n = ar^n$ for some constant $a > 0$ which will be specified later on. This is equivalent to proving

$$\mathbb{P}(\bigcup_{n=N}^{\infty} \{|X_t/b(t) - x| < \epsilon \text{ for some } t_{n+1} \leq t < t_n\}) = 1, \forall N \geq 1.$$

Take an $m \geq 2$. Then the probability of this union is

$$\geq \mathbb{P}\left(\bigcup_{n=N}^{\infty} \left\{|(X_t - X_{t_{n+1}})/b(t) - x| < \epsilon/2 \text{ for some } t_{n+1} \leq t < t_n\} \cap \{|X_{t_{n+1}}/b(t_{n+1})| < \epsilon b(t_{n+1})/2\}\right)$$

which is by the Feller-Chung lemma (see Lemma 3(i) on page 70 in [5])

$$\geq \mathbb{P}\left(\bigcup_{n=N}^{\infty} \{|(X_t - X_{t_{n+1}})/b(t) - x| < \epsilon/2 \text{ for some } t_{n+1} \leq t < t_n\}\right) \inf_{n \geq N} \mathbb{P}\{|X_{t_{n+1}}/b(t_{n+1})| < \epsilon b(t_{n+1})/2\}.$$.

To simplify notation, we set

$$A_n := \{|(X_t - X_{t_{n+1}})/b(t) - x| < \epsilon/2 \text{ for some } t_{n+1} \leq t < t_n\}, n \geq 1$$

and we observe that

$$\mathbb{P}(A_n) = \mathbb{P}\{|X_{t-t_{n+1}}/b(t) - x| < \epsilon/2 \text{ for some } t_{n+1} \leq t < t_n\} \geq \mathbb{P}\{|X_{t-t_{n+1}}/b(t) - x| < \epsilon/2 \text{ for some } t_{n+1} \leq t < t_n\} \geq \mathbb{P}\left\{\frac{X_s}{b(s)} - x < \frac{\epsilon}{2} - |x|/(c_n - 1) \text{ for some } s \in [t_{n+1} - t_{n+m}, t_n - t_{n+m}]\right\},$$
where \( c_{n,m} := \sup_{t_n+1 \leq t \leq t_n} b(t)/b(t - t_n + m) \).

If \( x \neq 0 \) and \( t_{n+1} \leq t \leq t_n \) we have for \( \delta = \epsilon|x|^{-1}/8 \) and large enough \( n \),

\[
b(t)/b(t - t_{n+m}) \leq (1 + \delta)t/(t - t_{n+m}) \leq (1 + \delta)(1 - r^{-m-1})^{-1} \leq 1 + \epsilon|x|^{-1}/4,
\]

provided that \( m \) is bigger than some \( m_2 \) (which depends on \( x \) and \( \epsilon \)).

We can conclude that

\[
\mathbb{P}(A_n) \geq \mathbb{P}\{|X_s/b(s) - x| < \epsilon/4 \text{ for some } t_{n+1} - t_{n+m} \leq s < t_n - t_{n+m}\}.
\]

This also holds if \( x = 0 \) (in which case the last argument is unnecessary).

Setting \( a = (1 - r^{-m-1})^{-1} \), we have \( t_{n+1} - t_{n+m} = r^{n+1} \) and \( t_n - t_{n+m} \geq r^n \) and we find that

\[
\mathbb{P}(A_n) \geq \mathbb{P}\{|X(s)/b(s) - x| < \epsilon/4 \text{ for some } r^{n+1} \leq s < r^n\}
\]

and consequently we have

\[
\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty.
\]

The events \( A_n \) are defined so that \( A_i \) and \( A_j \) are independent whenever \(|i - j| \geq m\). Choosing a \( k \in \{1, \ldots, m - 1\} \) such that \( \sum_{j=1}^{\infty} \mathbb{P}(A_{jm+k}) = \infty \) it easily follows from the Borel-Cantelli lemma for pairwise independent events that \( \mathbb{P}(\lim \sup_{j \geq 1} A_{jm+k}) = 1 \), which of course implies that

\[
\mathbb{P}(\bigcup_{n=N}^{\infty} A_n) = 1, \forall N \geq 1.
\]

Finally, given \( 0 < \delta < 1 \), we can find for \( K_\delta > 0 \) as defined in (5.1) a natural number \( N_\delta \) such that

\[
\mathbb{P}\{|X_{t_n}| \leq K_\delta b(t_n)\} \geq (1 - \delta), \forall n \geq N_\delta.
\]

As we have \( b(t_{n+1})/b(t_{n+m}) \geq r^{-\rho(m-1)} \geq 2K_\delta/\epsilon \) if \( m \) is large enough, we can conclude that

\[
\mathbb{P}\{|X_{t_{n+m}}| \leq \epsilon b(t_{n+1})/2\} \geq 1 - \delta, n \geq N_\delta.
\]

It follows that for \( N \geq N_\delta \) and then trivially for all \( N \geq 1 \),

\[
\mathbb{P}(\bigcup_{n=N}^{\infty} \{|X_t/b(t) - x| < \epsilon \text{ for some } t_{n+1} \leq t < t_n\}) \geq 1 - \delta
\]

and the lemma has been proven. \( \Box \)

We are now able to prove the following criterion for clustering of normalized Lévy processes at zero which is analogous to a well known result of Kesten [12] for random walks.

**Theorem 5.1** Let \( \{X_t : t \geq 0\} \) be a \( d \)-dimensional Lévy process satisfying condition (5.1) and let the function \( b(t), 0 \leq t \leq 1 \) be as in Theorem 2.3. Given \( x \in \mathbb{R}^d \), the following are equivalent:

(a) \( x \in C(\{X_t/b(t) : t \downarrow 0\}) \) with probability 1.
(b) \( \int_0^1 t^{-1} \mathbb{P}\{|X_t/b(t) - x| < \epsilon\} dt = \infty, \forall \epsilon > 0. \)

\textbf{Proof} \((a) \implies (b)\) From Lemma 5.1 we know that (a) implies for any \( \epsilon > 0 \) and any \( 0 < r < 1 \),

\[ \sum_{k=1}^{\infty} p(\epsilon, r, k) = \infty, \]

where \( p(\epsilon, r, k) := \mathbb{P}\{|X_t/b(t) - x| < \epsilon \text{ for some } t \in I_k\} \) and \( I_k = [r^{k+1}, r^k[, k \geq 0. \)

Writing the integral in (b) as

\[ \sum_{k=0}^{\infty} \int_{I_k} t^{-1} \mathbb{P}\{|X_t/b(t) - x| < \epsilon\} dt \]

which is

\[ \geq \log(1/r) \sum_{k=0}^{\infty} \inf_{t \in I_k} \mathbb{P}\{|X_t/b(t) - x| < \epsilon\}, \]

we see that it enough to show that given \( 0 < \epsilon < 1 \), we can find an \( r_{\epsilon} \in [0, 1[ \) such that for large \( k \),

\[ \mathbb{P}\{|X_t/b(t) - x| < \epsilon\} \geq p(\epsilon/4, r_{\epsilon}, k + 1)/2, \forall t \in I_k. \] \hspace{1cm} (5.2)

Consider the following stopping time

\[ \tau_k := \inf\{s \geq r^{k+2} : |X_s/b(s) - x| < \epsilon/4\}. \]

Then clearly, \( p(\epsilon/4, r, k + 1) = \mathbb{P}(\tau_k < r^{k+1}). \) Moreover, we have on the event \( \{\tau_k < r^{k+1}\} : \)

\[ |X_{\tau_k}/b(\tau_k)| \leq |x| + \epsilon/4. \]

We thus can conclude that we have on this event for any \( t \in I_k, \)

\[ |X_t/b(t) - X_{\tau_k}/b(\tau_k)| \leq |X_t - X_{\tau_k}|/b(t) + (|x| + \epsilon/4)(1 - b(\tau_k)/b(t)) \leq |X_t - X_{\tau_k}|/b(r^{k+1}) + \epsilon/4 + |x|(1 - b(r^{k+2})/b(r^k)). \]

If \( r = r_{\epsilon} = (1 - \epsilon/(4|x| + 1))^{1/2}/(1 - \epsilon/(8|x| + 1))^{1/2} \) and we can infer from condition (2.6) that there exists a natural number \( k_{\epsilon,x} \) such that we have for \( k \geq k_{\epsilon,x}, \)

\[ b(r^{k+2})/b(r^k) \geq (1 - \epsilon/(8|x| + 1))r^2 \geq (1 - \epsilon/(4|x| + 1)) \]

and it follows that

\[ |X_t/b(t) - X_{\tau_k}/b(\tau_k)| \leq |X_t - X_{\tau_k}|/b(r^{k+1}) + \epsilon/2, t \in I_k. \]

Next consider the stochastic process

\[ X^*_s := X_{\tau_k+s} - X_{\tau_k}, s \geq 0 \]

which is defined on the event \( \{\tau_k < r^{k+1}\}. \) We then have since \( t - \tau_k \leq r^k - r^{k+2} =: s_k, \)

\[ \{\tau_k < r^{k+1}\} \cap \{\sup_{s \leq s_k} |X_s^*| \leq \epsilon b(r^{k+1})/4\} \subset \bigcap_{t \in I_k} \{\{X_t/b(t) - x| < \epsilon\} \]

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By the strong Markov property of the Lévy process $X_t, t \geq 0$ (see, for instance, Prop. 6 on p. 20 in [1]) the two left-hand events are conditionally independent on $\{\tau_k < \infty\}$ and the conditional distribution of $\{X^*_t : t \geq 0\}$ is equal to the $\mathbb{P}$-distribution of $\{X_t : t \geq 0\}$. We thus have,

$$
\inf_{t \in I_k} \mathbb{P}\{|X_t/b(t) - x| < \epsilon\} \geq \mathbb{P}\{\tau_k < r^{k+1}\}\mathbb{P}\{\sup_{s \leq s_k} |X_s| \leq c b(r^{k+1})/4\}
$$

$$
= p(\epsilon/4, r, k + 1)\mathbb{P}\{\sup_{s \leq s_k} |X_s| \leq c b(r^{k+1})/4\}
$$

We now need an upper bound for $p_k := \mathbb{P}\{\sup_{s \leq s_k} |X_s| > c b(r^{k+1})/4\}$. By the Etemadi inequality (see, for instance, Theorem 22.5 in [3]) which also holds for Lévy processes we have that

$$
p_k \leq 3\mathbb{P}\{|X_{s_k}| > c b(r^{k+1})/12\}
$$

Note that $r^{k+1}/s_k = r/(1 - r^2) \to \infty$ if $r \not\to 1$. So if $r$ is sufficiently large we have

$$
b(r^{k+1})/b(s_k) \geq r^\rho/(1 - r^2)^\rho \geq 12K/\epsilon,
$$

where we choose $K > 0$ so that for small enough $t$,

$$
\mathbb{P}\{|X_t| \geq Kb(t)\} \leq 1/6.
$$

We can conclude that $p_k \leq 1/2$ for large $k$ and we see that (5.2) holds.

(b) $\Rightarrow$ (a) This follows since we have for any $0 < r < 1$,

$$
\int_0^1 t^{-1}\mathbb{P}\{|X_t/b(t) - x| < \epsilon\}dt \leq \sum_{k=0}^{\infty} \int_{r^{k+1}}^{r^k} t^{-1}\mathbb{P}\{|X_t/b(t) - x| < \epsilon\}dt
$$

$$
\leq \sum_{k=0}^{\infty} \log(1/r)\mathbb{P}\{|X_t/b(t) - x| < \epsilon\text{ for some } r^{k+1} \leq t < r^k\}
$$

whence condition (b) of Lemma 5.1 is satisfied which implies that $x$ is in the cluster set. □

### 5.2 Lévy processes without Gaussian part

Under extra assumptions the last criterion for clustering can be further simplified as follows. Let $A(b)$ be the symmetric non-negative definite matrix satisfying

$$
A^2(b) = \left(\int_{|y| \leq b} y_i y_j \Pi(dy)\right)_{1 \leq i, j \leq d}, 0 < b < 1.
$$

**Lemma 5.2** Let $X_t, t \geq 0$ be a purely non-Gaussian Lévy process and let $b(t), 0 \leq t \leq 1$ be a function as in Theorem 2.3. Assume that condition (2.1) is satisfied and that $\alpha_0 < \infty$.

Then the following are equivalent:
(a) $x \in C(\{X_t/b(t) : t \downarrow 0\})$ with probability 1

(b) $\int_0^1 t^{-1} \mathbb{P}\{|Y^{(b(t))}_t/b(t) - x| < \varepsilon\} dt = \infty, \forall \varepsilon > 0$.

(c) $\int_0^1 t^{-1} \mathbb{P}\{|\sqrt{t}A(b(t))\eta/b(t) - x| < \varepsilon\} dt = \infty, \forall \varepsilon > 0$,

where $\eta$ is $N(0, I)$-distributed and $A(b)$ is defined as above.

**Proof** We first prove that (a) and (b) are equivalent. Since we are assuming that $\alpha_0 < \infty$ it is clear that $X_t/b(t)$ is stochastically bounded as $t \downarrow 0$. Consequently, Theorem 5.1 applies so that (a) holds if and only if

$$
\int_0^1 t^{-1} \mathbb{P}\{|X_t/b(t) - x| < \varepsilon\} dt = \infty
$$

for all $\varepsilon > 0$. Recall (see (3.3)) that

$$X_t = tv(b(t)) + Y^{(b(t))}_t + Z^{(b(t))}_t, 0 < t < 1,$$

where we have $tv(b(t))/b(t) \to 0$ as $t \to 0$ (by Lemma 3.1 (c) and since $t/b(t) \to 0$ as $t \to 0$.) Moreover, we have,

$$\mathbb{P}\{Z^{(b(t))}_t \neq 0\} = 1 - \exp(-t\Pi(b(t)) \leq t\Pi(b(t)).$$

It follows that given $\varepsilon > 0$ there exist $0 < t_\varepsilon < 1$ such that

$$\mathbb{P}\{|X_t - Y^{(b(t))}_t| \geq \varepsilon b(t)/2\} \leq t\Pi(b(t)), 0 < t < t_\varepsilon$$

which in turn implies by (2.1) that

$$\int_0^1 t^{-1} \mathbb{P}\{|X_t - Y^{(b(t))}_t| \geq \varepsilon b(t)/2\} dt < \infty, \varepsilon > 0.$$

It is easy now to see that

$$\int_0^1 t^{-1} \mathbb{P}\{|X_t/b(t) - x| < \varepsilon\} dt = \infty, \forall \varepsilon > 0 \iff \int_0^1 t^{-1} \mathbb{P}\{|Y^{(b(t))}_t/b(t) - x| < \varepsilon\} dt = \infty, \forall \varepsilon > 0.$$

Thus (a) and (b) are equivalent.

To see that (b) and (c) are equivalent it is enough to show that we have for $0 < t < 1$ and $\varepsilon > 0$:

$$\mathbb{P}\{|Y^{(b(t))}_t/b(t) - x| < \varepsilon\} \leq \mathbb{P}\{|\sqrt{t}A(b(t))\eta/b(t) - x| < 2\varepsilon\} + \Delta_1(t, \varepsilon) \quad (5.3)$$

and

$$\mathbb{P}\{|\sqrt{t}A(b(t))\eta/b(t) - x| < \varepsilon\} \leq \mathbb{P}\{|Y^{(b(t))}_t/b(t) - x| < 2\varepsilon\} + \Delta_2(t, \varepsilon) \quad (5.4)$$

where $\int_0^1 t^{-1} \Delta_i(t, \varepsilon) dt < \infty, i = 1, 2$.

We only prove the first inequality. The proof of the second one is then an obvious modification of the first proof.

Note that for any $n \geq 1$,

$$Y^{(b(t))}_t = \sum_{j=1}^n Y^{(b(t))}_{t_{j/n}} - Y^{(b(t))}_{t_{(j-1)/n}} =: \sum_{j=1}^n \xi_{j,t,n},$$

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where the random vectors \( \xi_{j,t,n} \) are i.i.d. with mean zero. Moreover we have, \( \text{Cov}(\xi_{j,t,n}) = \frac{1}{n} A^2(b(t)) \). Since we also have \( \mathbb{E}|\xi_{1,t,n}|^3 < \infty \) we can apply Lemma 13 in [7] which actually is a corollary of Theorem 2 in [14].

Setting \( \eta_{j,t,n} := \sqrt{\frac{t}{n}} A(b(t)) \eta_j, 1 \leq j \leq n \), where \( \eta_1, \ldots, \eta_n \) are independent \( \mathcal{N}(0, I) \)-distributed random vectors, we obtain for a suitable constant \( C_t > 0 \),

\[
\mathbb{P}\{|Y_t^{(b(t))}/b(t) - x| < \epsilon \} \leq \mathbb{P}\{|\sqrt{\frac{t}{n}} A(b(t)) \sum_{j=1}^n \eta_j/b(t) - x| < 2\epsilon \} + C_t n \mathbb{E}|\xi_{1,t,n}|^3/b(t)^3
\]

\[
= \mathbb{P}\{|\sqrt{\frac{t}{n}} A(b(t)) \eta_1/b(t) - x| < 2\epsilon \} + C_t t/n \mathbb{E}|Y_t^{(b(t))}/b(t)|^3/b(t)^3.
\]

Recalling Lemma 3.2 and letting \( n \) go to infinity, we finally find that

\[
\mathbb{P}\{|Y_t^{(b(t))}/b(t) - x| < \epsilon \} \leq \mathbb{P}\{|\sqrt{t} A(b(t)) \eta_1/b(t) - x| < 2\epsilon \} + C_t t \int_0^{b(t)} \|y\|^3 \Pi(dy)/b(t)^3
\]

which implies (5.3) via Lemma 3.1(c). \( \square \)

### 5.3 Proof of Theorem 2.4

In the sequel the topological closure of a subset \( B \subset \mathbb{R}^d \) will always be denoted by \( \text{cl}(B) \).

(i) We first prove part (a). It is trivial that \( C(\{X_t/b(t) : t \downarrow 0\}) \) is closed since such cluster sets are always closed. Note that the cluster set \( C(\{g(t) : t \downarrow 0\}) \) for a mapping \( g :[0,1] \rightarrow \mathbb{R}^d \) can be written as \( \cap_{n \geq 1} \text{cl}(\{g(s) : 0 < s < 1/n\}) \), which is closed as an intersection of closed sets in \( \mathbb{R}^d \). Next since \( \limsup_{t \downarrow 0} X_t/b(t) = \alpha_0 < \infty \) with probability one, we can conclude that the cluster set \( A \) has to be bounded and consequently it is compact.

Moreover, there exists an \( \omega \) such that \( \limsup_{t \downarrow 0} X_t(\omega)/b(t) = \alpha_0 \) and \( C(\{X_t(\omega)/b(t) : t \downarrow 0\}) = A \) as both properties hold with probability one.

Take a sequence \( t_n = t_n(\omega) \downarrow 0 \) such that \( \lim_{n \to \infty} |X_{t_n}(\omega)/b(t_n)| = \alpha_0 \). Then the sequence \( X_{t_n}(\omega)/b(t_n) \) is bounded and consequently there exists a subsequence \( n_k = n_k(\omega) \to \infty \) such that \( X_{t_{n_k}}(\omega)/b(t_{n_k}) \) converges to a vector in \( \mathbb{R}^d \) which has norm \( \alpha_0 \). This vector is of the form \( \alpha_0 z \) with \( |z| = 1 \) and it is in the cluster set \( A \).

Finally, it follows directly from Lemma 5.2(c) that the set \( A \) is symmetric about zero and also star-like at zero. Concerning the latter property we recall the following well known corollary of the classical T.W. Anderson inequality:

If \( \eta : \Omega \rightarrow \mathbb{R}^d \) is a \( d \)-dimensional normal random vector we have for \( x \in \mathbb{R}^d \) and \( \delta > 0 \),

\[
\mathbb{P}\{|\eta - x| < \delta \} \leq \mathbb{P}\{|\eta - sx| < \delta \}, 0 \leq s \leq 1.
\]

(ii) We turn to the proof of (b) where we assume w.l.o.g. that \( \alpha_0 = 1 \). It is divided into three steps.

In the first step we define a suitable discrete Lévy measure \( \Pi_0 \) and we show that we have for any Lévy process with characteristic triplet \((\gamma, 0, \Pi_0)\) \( \limsup_{t \downarrow 0} |X_t/b(t)| \leq 1 \) with probability 1.

In the second step we prove that the cluster set of \( X_t/b(t) \) at \( t \downarrow 0 \) contains the set \( A \).

In the third step we finally show that this cluster set is also a subset of \( A \).

**Step 1** \( A \) is symmetric and star-like w.r.t. zero and there is a unit vector \( z_1 \) in \( A \). (We are assuming \( \alpha_0 = 1 \).) Then the whole line segment \( \mathcal{L}_1 := \{t z : |t| \leq 1\} \) has to be in \( A \). Moreover, in view of the
separability of \( \mathbb{R}^d \), we can write \( A \) as the closure of \( \mathcal{L}_1 \) and at most countably many additional line segments, that is,

\[
A = \text{cl}(\bigcup_{j=1}^{\infty} \mathcal{L}_j),
\]

where \( \mathcal{L}_j = \{ t z_j : |t| \leq \sigma_j \} \) with \( z_j \) being a unit vector in \( \mathbb{R}^d \) and \( \sigma_j \in [0, 1], j \geq 2 \). (If \( A \) consists only of finitely many line segments we set \( \sigma_j = 0 \) for large \( j \)).

We set \( L_k := \{ \ell \in \{1, \ldots, k\} : \sigma_k^2 \geq 1/k \}, k \geq 1 \) and we denote the elements in \( L_k \) by \( 1 = j_1(k) < \ldots < j_{\ell_k}(k) \), where \( \ell_k = \#L_k \geq 1 \) for all \( k \geq 1 \). Next we set

\[
\sigma_{k, \ell} := \sigma_{j_\ell(k)} \text{ and } z_{k, \ell} := z_{j_\ell(k)}, 1 \leq \ell \leq \ell_k, k \geq 1.
\]

Our measure \( \Pi_0 \) will have discrete support \( \{ y_{k, \ell}, 1 \leq \ell \leq \ell_k, k \geq k_0 \} \subset D \), where again \( D \) is the Euclidean unit ball and \( k_0 \) will be specified below. The support points are defined as follows,

\[
y_{k, \ell} = b(1/a_{k, \ell}) z_{k, \ell}, 1 \leq \ell \leq \ell_k, k \geq k_0,
\]

where \( a_{k, \ell} := h(\exp(\exp(k^3) + \ell k)), 1 \leq \ell \leq \ell_k, a_{k, \ell+1} := a_{k+1, 1}, k \geq k_0 \) and \( k_0 \geq 2 \) is chosen so that \( a_{k_0, 1} \geq e^2 \).

Since \( a_{k, \ell}, k \geq k_0, 1 \leq \ell \leq \ell_k \) is obviously increasing with respect to the lexicographical order on \( \mathbb{N}^2 \), it follows from the monotonicity of \( b \) that \( (k, \ell) \mapsto |y_{k, \ell}| = b(1/a_{k, \ell}) \) is decreasing w.r.t to this order.

We define the discrete measure \( \Pi_0 \) on \( D \) with support as indicated above by setting

\[
\Pi_0 \{ y_{k, \ell} \} = \frac{1}{b^2(1/a_{k, \ell})} \left[ \frac{\sigma_{k, \ell}^2}{2h^2(a_{k, \ell})} - \frac{\sigma_{k, \ell+1}^2}{2h^2(a_{k, \ell+1})} \right], 1 \leq \ell \leq \ell_k, k \geq k_0,
\]

where \( \sigma_{k, \ell+1} := \sigma_{k+1, 1} \).

It then follows for any given pair \( (k, \ell) \) with \( k \geq k_0 \) and \( \ell \leq \ell_k \) that

\[
\int_{|y| \leq |y_{k, \ell}|} |y|^2 \Pi_0(dy) = \sum_{(k', \ell') \geq (k, \ell)} |y_{k', \ell'}|^2 \Pi_0 \{ y_{k', \ell'} \}
\]

\[
= \sum_{(k', \ell') \geq (k, \ell)} \left[ \frac{\sigma_{k', \ell'}^2}{2h^2(a_{k', \ell'})} - \frac{\sigma_{k', \ell'+1}^2}{2h^2(a_{k', \ell'+1})} \right] = \frac{\sigma_{k, \ell}^2}{2h^2(a_{k, \ell})} \quad \text{ (5.5)}
\]

Applying this inequality with \( |y_{k_0, 1}| = b(1/a_{k_0, 1}) \), we see that

\[
\int_D |y|^2 \Pi_0(dy) < \infty.
\]

Furthermore, we readily obtain from (5.5) that

\[
V(b(t)) \leq \int_{|y| \leq b(t)} |y|^2 \Pi_0(dy) \leq (2h^2(1/t))^{-1}, 0 < t < e^{-2}.
\]

(5.6)

Under the extra assumption (2.4) this already implies via Theorem 2.2 and Lemma 2.4 that with probability one,

\[
\limsup_{t \downarrow 0} |X_t|/b(t) \leq 1.
\]

(5.7)
For the general case we have to show directly that
\[ \int_D b^-(|y|)\Pi_0(dy) < \infty. \]

It is easy to see from the definition of \( \Pi_0 \) and the function \( b \) that this integral is
\[ \leq \sum_{k=k_0}^{\infty} \sum_{\ell=1}^{\ell_k} \frac{1/a_{k,\ell}}{2b^2(1/a_{k,\ell})h^2(a_{k,\ell})} = \sum_{k=k_0}^{\infty} \sum_{\ell=1}^{\ell_k} (2\log \log a_{k,\ell})^{-1}. \]

By the Karamata representation of the function \( h \) (see, for instance, [1], p.9), we have for some \( c_0 > 1 \) and \( a_2 \geq a_1 \geq h(c_0) \),
\[ \frac{a_2}{a_1} = \frac{h(h^-(a_2))}{h(h^-(a_1))} \leq 2h^-(a_2)/h^-(a_1) \]
which implies that for large \( k \), \( a_{k,\ell} \geq \exp(\exp(k^3)) \), \( 1 \leq \ell \leq \ell_k \). Recalling that \( \ell_k \leq k \), we find that for some \( k_1 \geq k_0 \),
\[ \sum_{k=k_1}^{\infty} \sum_{\ell=1}^{\ell_k} (2\log \log a_{k,\ell})^{-1} \leq \sum_{k=k_1}^{\infty} k^{-2} < \infty. \]

Thus (5.7) holds in general.

**Step 2** We show that \( C(\{X_t/b(t): t \downarrow 0\}) \supset A \).

Since we already know that the cluster set is closed, symmetric about zero and star-like at zero, it is enough to show that we have for any fixed \( j \geq 1 \),
\[ \sigma_j z_j \in C(\{X_t/b(t): t \downarrow 0\}) \] with probability one.

It is obviously sufficient to prove this for the \( j \)'s for which \( \sigma_j > 0 \).

In view of Lemma 5.2 (which we can apply on account of (5.7)) this is equivalent to showing
\[ \int_0^1 t^{-1}P\{|\sqrt{\eta_t}/b(t) - \sigma_j z_j| < \epsilon\}dt = \infty, 0 < \epsilon < \sigma_j, \quad (5.8) \]

where \( \eta_t \sim N(0, A^2(b(t))) \), \( 0 \leq t \leq 1 \) and \( A^2(b) \) is the \((d,d)\)-matrix such that
\[ A^2(b)_{i,j} = \int_{|y| \leq b} y_i y_j \Pi_0(dy), 1 \leq i,j \leq d, 0 < b < 1. \]

Given any vector \( v \in \mathbb{R}^d \), \( \langle \eta_t, v \rangle \) is a (1-dimensional) normal random variable with mean zero and
\[ \text{Var}(\langle \eta_t, v \rangle) = \langle v, A^2(b(t))v \rangle = \int_{|y| \leq b} \langle y, v \rangle^2 \Pi_0(dy), 0 < t < 1. \quad (5.9) \]

If \( k \) is large enough so that \( \sigma_j^2 \geq 1/k \), we can find an index \( r_k(j) \in \{1, \ldots, \ell_k\} \) for which \( \sigma_{k,r_k(j)} = \sigma_j \) and \( z_{k,r_k(j)} = z_j \). From the definition of \( \Pi_0 \) and (5.9) combined with the fact that \( \sigma_{j'}^2 \leq 1 \) for all \( j' \), we get for \( t \geq 1/a_{k,r_k(j)} \)
\[ \langle z_j, A^2(b(t))z_j \rangle \geq \sigma_j^2 [h^{-2}(a_{k,r_k(j)}) - kh^{-2}(a_{k,r_k(j)+1})]/2 \geq \sigma_j^2 (1 - ke^{-2k})/(2h^2(a_{k,r_k(j)})) \quad (5.10) \]
Next we define for a \( \tau \in [0,1] \) which be specified later on
\[
\tilde{a}_{k,r_k(j)} = a_{k,r_k(j)} \exp(-\log a_{k,r_k(j)}) \tau / 2
\]
and we set \( I_k(j) = [1/a_{k,r_k(j)}, 1/\tilde{a}_{k,r_k(j)}] \).
Letting \( t_k = 1/a_{k,r_k(j)} \) and \( \tilde{t}_k = 1/\tilde{a}_{k,r_k(j)} \) in the proof of Theorem 2.2, we can conclude that
\[
h(a_{k,r_k(j)})/h(\tilde{a}_{k,r_k(j)}) \to 1 \quad \text{as} \quad k \to \infty \tag{5.11}
\]
As we also have \( h(a_{k,r_k(j)})/h(a_{k,r_k(j)-1}) \to \infty \) as \( k \to \infty \), where we set \( a_{k,0} = a_{k-1,\tilde{t}_{k-1}}, k \geq 2 \), we see that for sufficiently large \( k \), \( \tilde{a}_{k,r_k(j)} > a_{k,r_k(j)-1} \) so that
\[
I_k(j) \subset [1/a_{k,r_k(j)}, 1/a_{k,r_k(j)-1}].
\]

If \( w \in \mathbb{R}^d \) is a vector such that \( \langle w, z_j \rangle = 0 \) and \( t \in I_k(j) \), it follows then that
\[
\langle w, A^2(b(t))w \rangle \leq e^{-2k}(2h^2(a_{k,r_k(j)}))^{-1} \tag{5.12}
\]
Choosing an orthonormal basis \{\( w_{j,i} : 1 \leq i \leq d \)\} of \( \mathbb{R}^d \) with \( w_{j,1} = z_j \) and setting \( \epsilon' = \epsilon / \sqrt{d} \), we can conclude that for any \( t \in I_k(j) \)
\[
\mathbb{P}\{ |\sqrt{t}\eta_{t}/b(t) - \sigma_j z_j | < \epsilon \} \geq \mathbb{P}\left\{ \bigcap_{i=2}^d \{ |\sqrt{t}\langle \eta_{t}, z_j \rangle / b(t) - \sigma_j | < \epsilon' \} \right\} 
\]
\[
\geq \mathbb{P}\{ |\sqrt{t}\langle \eta_{t}, z_j \rangle / b(t) - \sigma_j | < \epsilon' \} - \sum_{i=2}^d \mathbb{P}\{ |\langle \eta_{t}, w_{j,i} \rangle | \geq \epsilon' b(t)/\sqrt{t} \}.
\]
As we have \( \text{Var}(\sqrt{t}\langle \eta_{t}, z_j \rangle / b(t)) = t \langle z_j, A^2(b(t)) z_j \rangle / b^2(t) \leq \int_{|y| \leq b(t)} |y|^2 \Pi_0(dy) h^2(1/t) / \log \log 1/t \to 0 \) as \( t \to 0 \) (recall \( 5.6 \)), one easily sees that for large \( k \),
\[
\mathbb{P}\{ |\sqrt{t}\langle \eta_{t}, z_j \rangle / b(t) - \sigma_j | < \epsilon' \} = \mathbb{P}\{ \langle \eta_{t}, z_j \rangle \leq (\sigma_j - \epsilon')b(t)/\sqrt{t} \} - \mathbb{P}\{ \langle \eta_{t}, z_j \rangle \geq (\sigma_j + \epsilon')b(t)/\sqrt{t} \} 
\]
\[
\geq \mathbb{P}\{ \langle \eta_{t}, z_j \rangle \geq (\sigma_j - \epsilon')b(t)/\sqrt{t} \}/2.
\]
Let \( \eta' \) be a standard normal random variable. Applying inequality \( 3.15 \), we obtain from \( 5.10 \) and \( 5.11 \) that for \( t \in I_k(j) \) and large \( k \),
\[
\mathbb{P}\{ \langle \eta_{t}, z_j \rangle \geq (\sigma_j - \epsilon')b(t)/\sqrt{t} \} \geq \mathbb{P}\{ \eta' \geq (1 - \epsilon')^{1/2}(2 \log \log 1/t)^{1/2} \}
\]
\[
\geq C(\log 1/t)^{-1/2}(\log \log 1/t)^{-1/2} \tag{5.13}
\]
where \( C > 0 \) is a constant.
A similar argument using \( 5.12 \) along with the bound \( \mathbb{P}\{ |\eta' | > t \} \leq 2 \exp(-t^2/2), t > 0 \) shows that for \( t \in I_k(j) \) and large enough \( k \),
\[
\mathbb{P}\{ |\langle \eta_{t}, w_{j,i} \rangle | \geq \epsilon' b(t)/\sqrt{t} \} \leq (\log 1/t)^{-2}, 2 \leq i \leq d. \tag{5.14}
\]
Combining relations \( 5.13 \) and \( 5.14 \), we finally find that for \( t \in I_k(j) \) and large \( k \)
\[
\mathbb{P}\{ |\sqrt{t}\eta_{t}/b(t) - \sigma_j z_j | < \epsilon \} \geq C(\log 1/t)^{-(1-\epsilon')}(\log \log 1/t)^{-1/2} / 4 \tag{5.15}
\]
which in turn implies that
\[
\int_{I_k(j)} t^{-1} \mathbb{P}\{|\sqrt{t} \eta(t)/b(t) - \sigma_{jz}| < \epsilon\} dt \\
\geq C \log(a_k,r_k(j)/\tilde{a}_k,r_k(j))(\log a_k,r_k(j))^{-1}(1-\epsilon')(\log \log a_k,r_k(j))^{-1/2}/4 \\
\geq C(\log a_k,r_k(j))^\tau (\log a_k,r_k(j))^{-1}(1-\epsilon')(\log \log a_k,r_k(j))^{-1/2}/4.
\]
Choosing \( \tau > 1 - \epsilon' \), the last term converges to infinity as \( k \) goes to infinity. Thus (5.8) holds which means that \( \sigma_{jz} \in C(\{X_t/b(t) : t \downarrow 0\}) \).

**Step 3** We show that \( x \not\in A \) implies that \( x \not\in C(\{X_t/b(t) : t \downarrow 0\}) \).

Set \( \epsilon := \text{dist}(x, A)/2 \). This is a positive number since \( A \) is closed. In view of Lemma [5.2] it is sufficient to prove for this choice of \( \epsilon \),
\[
\int_{0}^{1} t^{-1} \mathbb{P}\{|\sqrt{t} \eta(t)/b(t) - x| < \epsilon\} dt < \infty,
\] (5.16)

where, as in Step 2, \( \eta \sim \mathcal{N}(0, A^2(b(t))) \), \( 0 < t < 1 \).

Consider the intervals
\[
J_{k,\ell} := [1/a_{k,\ell}, 1/a_{k,\ell-1}], 1 \leq \ell \leq \ell_k, k \geq 2.
\]
Recall that \( a_{k,0} = a_{k-1,\ell}, k \geq 2 \). Similarly as in (5.10) and (5.12), we then can conclude that for \( t \in J_{k,\ell} \),
\[
\langle z_{k,\ell}, A^2(b(t))z_{k,\ell} \rangle \leq (\sigma_{k,\ell}^2 + e^{-2k})(2h^2(1/t))^{-1} \tag{5.17}
\]
\[
\langle w, A^2(b(t))w \rangle \leq e^{-2k}(2h^2(1/t))^{-1} \text{ if } \langle w, z_{k,\ell} \rangle = 0. \tag{5.18}
\]

Furthermore, we have by definition of \( \epsilon \) that
\[
\mathbb{P}\{|\sqrt{t} \eta(t)/b(t) - x| < \epsilon\} \leq \mathbb{P}\{\text{dist}(\sqrt{t} \eta(t)/b(t), A) > \epsilon\} \leq \mathbb{P}\{\text{dist}(\sqrt{t} \eta(t)/b(t), L_{k,\ell}) > \epsilon\}, \tag{5.19}
\]

where \( L_{k,\ell} = \{tz_{k,\ell} : |t| \leq \sigma_{k,\ell}, 1 \leq \ell \leq \ell_k, k \geq 1\} \). Let \( \{w_{k,\ell,i} : 1 \leq i \leq d\} \) be an orthonormal basis of \( \mathbb{R}^d \) with \( w_{k,\ell,1} = z_{k,\ell} \). Writing
\[
\eta = \langle \eta, z_{k,\ell} \rangle z_{k,\ell} + \sum_{i=2}^{d} \langle \eta, w_{k,\ell,i} \rangle w_{k,\ell,i},
\]
it is easy to see that
\[
\mathbb{P}\{\text{dist}(\sqrt{t} \eta(t)/b(t), L_{k,\ell}) > \epsilon\} \leq \mathbb{P}\{|\langle \eta, z_{k,\ell} \rangle| > (\sigma_{k,\ell} + \epsilon')b(t)/\sqrt{t}\} + \sum_{i=2}^{d} \mathbb{P}\{|\langle \eta, w_{k,\ell,i} \rangle| \geq \epsilon'b(t)/\sqrt{t}\},
\]

where, as in Step 2, \( \epsilon' = \epsilon/\sqrt{d} \).

Using the same exponential inequality for the normal distribution as in (5.14) along with relations (5.17) and (5.18), we can conclude that for large enough \( k \) and \( t \in J_{k,\ell} \),
\[
\mathbb{P}\{\text{dist}(\eta(t), L_{k,\ell}) > \epsilon\} \leq 2d(\log 1/t)^{-1-\epsilon'} \tag{5.20}
\]

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This implies via (5.19) that for some \( k_0 \geq 2 \),

\[
\int_1^{a_{k_0}} t^{-1} \mathbb{P}\{|\sqrt{tr\eta /b(t)} - x| < \epsilon\} dt = \sum_{k=k_0}^{\infty} \sum_{\ell=1}^{\ell_k} \int_{J_{k,\ell}} t^{-1} \mathbb{P}\{|\sqrt{tr\eta /b(t)} - x| < \epsilon\} dt
\]

\[
\leq 2d \int_0^e t^{-1} \log(1/t)^{-1-\epsilon'} dt < \infty.
\]

Thus (5.16) holds and Theorem 2.4 has been proven. □

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