EXTENSIONS OF COPSON’S INEQUALITIES

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Abstract. We extend the classical Copson’s inequalities so that the values of parameters involved go beyond what is currently known.

1. Introduction

Let \( p > 0 \) and \( \mathbf{x} = (x_n)_{n \geq 1} \) be a non-negative sequence. Let \( (\lambda_n)_{n \geq 1} \) be a non-negative sequence with \( \lambda_1 > 0 \) and let \( \Lambda_n = \sum_{i=1}^{n} \lambda_i \). The classical Copson’s inequalities are referred as the following ones [3, Theorem 1.1, 2.1]:

\[
\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left( \sum_{k=1}^{n} \lambda_k x_k \right)^p \leq \left( \frac{p}{c - 1} \right)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{p-c} x_n^p, \quad 1 < c \leq p;
\]
\[
\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left( \sum_{k=n}^{\infty} \lambda_k x_k \right)^p \leq \left( \frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{p-c} x_n^p, \quad 0 \leq c < 1.
\]

When \( \lambda_k = 1 \) for all \( k \) and \( c = p \), inequality (1.1) becomes the celebrated Hardy’s inequality ([8, Theorem 326]). We note that the reversed inequality of (1.2) holds when \( c \leq 0 < p \) and the constants are best possible in all these cases.

It is easy to show that inequalities (1.1) and (1.2) are equivalent to each other by the duality principle [10, Lemma 2] for the norms of linear operators. It’s observed by Bennett [1, p. 411] that inequality (1.1) continues to hold for \( c > p \) with constant \( \left( \frac{p}{c - 1} \right)^p \). A natural question to ask now is whether inequality (1.1) itself continues to hold for \( c > p \). Note that in this case the constant \( \left( \frac{p}{c - 1} \right)^p \) is best possible by considering the case \( \lambda_n = 1, x_n = n^{(c-p-1-\epsilon)/p} \) with \( \epsilon \to 0^+ \).

As analogues to Copson’s inequalities, the following inequalities are due to Leindler [9, (1)]:

\[
\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left( \sum_{k=1}^{n} \lambda_k x_k \right)^p \leq \left( \frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{p-c} x_n^p, \quad 0 \leq c < 1;
\]
\[
\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left( \sum_{k=n}^{\infty} \lambda_k x_k \right)^p \leq \left( \frac{p}{c - 1} \right)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{p-c} x_n^p, \quad 1 < c \leq p,
\]

where we assume \( \sum_{n=1}^{\infty} \lambda_n < \infty \) and we set \( \Lambda_n^* = \sum_{k=n}^{\infty} \lambda_k \). We point out here that Leindler’s result corresponds to case \( c = 0 \) of inequality (1.3), after a change of variables. Inequalities (1.3) and (1.4) are given in [11 Corollary 5, 6, p. 412]. Again it is easy to see that inequalities (1.3) and (1.4) are equivalent to each other by the duality principle. Moreover, the constants are best possible.

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As an application of Copson’s inequalities, we note the following result of Bennett and Grosse-Erdmann [2, Theorem 8] that asserts for \( p \geq 1, \alpha \geq 1, \)

\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=n}^{\infty} \Lambda_k^p \right)^n \leq (\alpha p + 1)^p \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{\alpha p} \left( \sum_{k=n}^{\infty} \xi_k \right)^p.
\]

(1.5)

Here the constant is best possible. They also conjectured [2, p. 579] that inequality (1.5) (resp. its reverse) remains valid with the same best possible constant when \( p \geq 1, 0 < a < 1 \) (resp. \( -1/p < a < 0 \)). Weaker constants are given for these cases in [2, Theorem 9, 10].

It is our goal in this paper to show in the next section that the method developed in [4–7] can be applied to extend Copson’s inequality (1.2) to some \( c > p \) (or equivalently, by the duality principle, to extend Copson’s inequality (1.2) to some \( c < 0 \)). In Section 3 we extend inequality (1.5) to some \( 0 < \alpha < 1 \).

2. Main Result

Before we prove our main result, we need a lemma first.

**Lemma 2.1.** Let \( p > 0 \) be fixed. In order for the following inequality (resp. its reverse)

\[
\frac{1-c}{p} x \leq \left( 1 + \frac{1-c}{p} \right)^{1-p} - (1-x)^{1-c}
\]

(2.1)

to be valid when \( c < 0, p > 1 \) (resp. when \( 0 < c < 1, 0 < p < 1 \)) for all \( 0 \leq x \leq 1 \), it suffices that it is valid when \( x=1 \).

**Proof.** As the proofs for both cases are similar, we only consider the case \( c < 0, p > 1 \) here. Let

\[
f_{p,c}(x) = \left( 1 + \frac{1-c}{p} \right)^{1-p} - (1-x)^{1-c} - \frac{1-c}{p} x.
\]

Note that \( f_{p,c}(0) = 0 \) and we have

\[
f'_{p,c}(x) = \frac{(1-c)(1-p)}{p} \left( 1 + \frac{1-c}{p} \right)^{-p} + (1-c)(1-x)^{-c} - \frac{1-c}{p},
\]

\[
f''_{p,c}(x) = \frac{(1-c)^2(p-1)}{p} \left( 1 + \frac{1-c}{p} \right)^{-p} + (1-c)c(1-x)^{-c-1}.
\]

It is easy to see that \( f''_{p,c}(x) = 0 \) is equivalent to the equation \( g_{p,c}(x) = 0 \) where

\[
g_{p,c}(x) = \left( \frac{-pc}{(p-1)(1-c)} \right)^{-1/(p+1)} (1-x)^{(1+c)/(p+1)} - 1 - \frac{1-c}{p} x.
\]

If \( 0 > c > -1 \), then it is easy to see that \( f''_{p,c}(x) < 0 \) so that \( f''_{p,c}(x) = 0 \) has at most one root in \((0,1)\). As \( \lim_{x \to 1-} f''_{p,c}(x) = -\infty \), it follows that if \( f''_{p,c}(x) \leq 0 \) for all \( 0 \leq x < 1 \), then \( f_{p,c}(x) \) is concave down and the assertion of the lemma follows. Otherwise we have \( f''_{p,c}(0) > 0 \) and this combined with the observation that \( f'_{p,c}(0) = 0, f'_{p,c}(1) < 0 \) implies that there exists an \( x_0 \in [0,1] \) such that \( f'_{p,c}(x) \geq 0 \) for \( 0 \leq x \leq x_0 \) and \( f'_{p,c}(x) \leq 0 \) for \( x_0 \leq x \leq 1 \) and the assertion of the lemma follows. The case \( c = -1 \) can be similarly discussed.

If \( c < -1 \), then \( g'_{p,c}(x) = 0 \) has at most one root in \((0,1)\) so that \( f''_{p,c}(x) = 0 \) has at most two roots in \((0,1)\). If \( f''_{p,c}(0) < 0 \), then as \( f'_{p,c}(1) > 0 \), it follows that \( f''_{p,c}(x) = 0 \) has exactly one root in \((0,1)\), and as \( f'_{p,c}(0) = 0, f'_{p,c}(1) < 0 \), it follows that \( f'_{p,c}(x) < 0 \) for all \( x \in [0,1] \) and the assertion of the lemma follows. If \( f''_{p,c}(0) > 0 \), then \( f''_{p,c}(x) = 0 \) has either no root or two roots in \((0,1)\). If \( f''_{p,c}(x) = 0 \) has no root in \((0,1)\), then \( f''_{p,c}(x) \geq 0 \) for \( x \in [0,1] \). As \( f''_{p,c}(0) = 0, f''_{p,c}(1) < 0 \), we see that this is not possible. If \( f''_{p,c}(x) = 0 \) has two roots in \((0,1)\), it follows that \( f_{p,c}(x) \) is first increasing, then decreasing and then increasing again for \( x \in [0,1] \) and it follows from \( f'_{p,c}(1) < 0 \)
that there exists an \( x'_0 \in [0, 1] \) such that \( f'_{p,c}(x) \geq 0 \) for \( 0 \leq x \leq x'_0 \) and \( f'_{p,c}(x) \leq 0 \) for \( x'_0 \leq x \leq 1 \) and the assertion of the lemma again follows. The case \( f''_{p,c}(0) = 0 \) can be discussed similarly as above and this completes the proof.

We now consider extending inequality (1.2) to \( c < 0 \). For two fixed two positive sequences \( \{a_n\}, \{b_n\} \), we recall that it is shown in \([5, \text{Section 6}]\) that we have the following inequality:

\[
\sum_{k=1}^{\infty} \left( \sum_{n=k}^{\infty} \lambda_{n}^{1/p} \Lambda_{n/p} \sum_{k=n}^{\infty} \lambda_{k}^{1-1/p} \Lambda_{k}^{-1-c/p} x_{k} \right)^{p} \leq \left( \frac{p}{1-c} \right)^{p} \sum_{n=1}^{\infty} x_{n}^{p},
\]

where \( \{w_{n}\} \) is a positive sequence, \( N \) is a large integer and for \( 1 \leq n \leq N \), we set \( S_{n} = \sum_{k=1}^{n} b_{k} x_{k} \) and \( A_{n} = S_{n}/a_{n} \).

We now recast inequality (1.2) as

\[
\sum_{n=1}^{\infty} \left( \sum_{k=n}^{\infty} \lambda_{n}^{1/p} \Lambda_{n/p} \sum_{k=n}^{\infty} \lambda_{k}^{1-1/p} \Lambda_{k}^{-1-c/p} x_{k} \right)^{p} \leq \left( \frac{p}{1-c} \right)^{p} \sum_{n=1}^{\infty} x_{n}^{p}.
\]

It remains to establish inequality (2.3). For this, it suffices to establish inequality (2.3) with the infinite sums replaced by finite sums from 1 to \( N \). We may also assume \( \lambda_{n} > 0 \) for all \( n \). We then set

\[
a_{n} = \lambda_{n}^{-1/p} \Lambda_{n}^{-c/p}, \quad b_{n} = \lambda_{n}^{-1/p} \Lambda_{n}^{1-c/p},
\]

in inequality (2.2) to see that in order to establish inequality (2.3), it suffices to find a positive sequence \( \{w_{n}\} \) such that

\[
\left( \sum_{k=n}^{\infty} w_{k} \right)^{p-1} \leq \left( \frac{p}{1-c} \right)^{p} \lambda_{n}^{1-1/p} \Lambda_{n}^{c/p} \left( \frac{w_{n}^{p-1} \Lambda_{n}^{p-c} - w_{n-1}^{p-1} \Lambda_{n-1}^{p-c}}{\lambda_{n-1}^{p-1}} \right), \quad n \geq 2;
\]

\[
\left( \sum_{k=1}^{\infty} w_{k} \right)^{p-1} \leq \left( \frac{p}{1-c} \right)^{p} \lambda_{1}^{1-1/p} \Lambda_{1}^{c/p} \left( \frac{w_{1}^{p-1} \Lambda_{1}^{p-c}}{\lambda_{1}^{p-1}} \right).
\]

Upon a change of variables: \( w_{n} \to \lambda_{n} w_{n} \), we can recast the above inequalities as

\[
\left( 1 \right) \sum_{k=n}^{\infty} \lambda_{k} w_{k} \right)^{p-1} \leq \left( \frac{p}{1-c} \right)^{p} \lambda_{n} \Lambda_{n}^{-1} \left( w_{n}^{p-1} - w_{n-1}^{p-1} \right) \Lambda_{n}^{p-c}, \quad n \geq 2;
\]

\[
\left( 1 \right) \sum_{k=1}^{\infty} \lambda_{k} w_{k} \right)^{p-1} \leq \left( \frac{p}{1-c} \right)^{p} w_{1}^{p-1}.
\]

We now define the sequence \( \{w_{n}\} \) inductively by setting \( w_{1} = 1 \) and for \( n \geq 2 \),

\[
\sum_{k=n}^{\infty} \lambda_{k} w_{k} = \frac{p}{1-c} \Lambda_{n-1} w_{n-1}.
\]

This implies that for \( n \geq 2 \),

\[
w_{n} = \frac{\Lambda_{n-1} \Lambda_{n}}{p} \left( 1 + \frac{1-c}{p} \lambda_{n} \right)^{-1} w_{n-1}.
\]

Using the above relations, we can simplify inequalities (2.4), (2.5) to see that inequality (2.4) is equivalent to inequality (2.1) with \( x = \lambda_{n}/\Lambda_{n} \) while inequality (2.5) is equivalent to

\[
\left( 1 + \frac{1-c}{p} \right)^{1-p} - \frac{1-c}{p} \geq 0.
\]
It is easy to see that the above inequality is just the case $x = 1$ of inequality (2.1), we then conclude from Lemma 2.1 that inequality (1.2) is valid for $c < 0$ as long as the above inequality holds.

Next, we consider extending inequality (1.3) to $c < 0$. For two fixed two positive sequences $\{a_n\}, \{b_n\}$, we recall that it is shown in [6, (3.6)] (see also the discussion in Section 5 of [7]) that in order for the following inequality

$$\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} a_n b_k x_k \right)^p \leq U_p \sum_{n=1}^{\infty} x_n^p,$$

to be valid for a given constant $U_p, p > 1$, it suffices to find a positive sequence $\{w_n\}$ such that

$$\left( \sum_{k=1}^{n} w_k \right)^{p-1} \leq U_p a_n^p \left( \frac{w_n^{p-1}}{b_n^p} - \frac{w_{n+1}^{p-1}}{b_{n+1}^p} \right).$$

(2.7)

Without loss of generality, we may assume $\lambda_n > 0$ for all $n$. By a change of variables, we recast inequality (1.3) as

$$\sum_{n=1}^{\infty} \left( \Lambda_n^{1/p} x_n^{c/p} \sum_{k=1}^{n} \lambda_k^{1-1/p} x_k \right)^p \leq \left( \frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} x_n^p.$$  

It follows from (2.7) that in order to establish the above inequality, it suffices to find a positive sequence $\{w_n\}$ such that

$$\left( \sum_{k=1}^{n} w_k \right)^{p-1} \leq \left( \frac{p}{1-c} \right)^p \Lambda_n \left( w_n^{p-1} - w_{n+1}^{p-1} \right).$$

(2.8)

By a change of variables: $w_n \mapsto \lambda_n w_n$, we can recast the above inequality as

$$\left( \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k w_k \right)^{p-1} \leq \left( \frac{p}{1-c} \right)^p \Lambda_n \left( w_n^{p-1} - w_{n+1}^{p-1} \right).$$

We now define the sequence $\{w_n\}$ inductively by setting $w_1 = 1$ and for $n \geq 1$,

$$\sum_{k=1}^{n} \lambda_k w_k = \frac{p}{1-c} \Lambda_{n+1} w_{n+1}.$$  

This implies that for $n \geq 2$,

$$w_n = \frac{\Lambda_{n+1}}{\Lambda_n} \left( 1 + \frac{1-c}{p} \frac{\lambda_n}{\Lambda_n} \right)^{-1} w_{n+1}.$$  

Using the above relations, we can simplify inequality (2.8) to see that the $n \geq 2$ cases are equivalent to inequality (2.1) with $x = \lambda_n/\Lambda_n^*$. It is also easy to see that the $n = 1$ case of (2.8) corresponds to the following inequality:

$$\frac{1-c}{p} x \leq \left( \frac{1-c}{p} x \right)^{1-p} - (1-x)^{1-c}.$$  

It is easy to see that the above inequality is implied by inequality (2.1), we then conclude from Lemma 2.1 that inequality (1.3) holds for $c < 0$ as long as inequality (2.6) holds.

Note that for fixed $p > 0$, the function $(1 + x)^{1-p} - x$ is a decreasing function of $x$. Moreover, it is easy to see that inequality (2.6) (resp. its reverse) always holds with $c = 0$ when $p > 1$ (resp. when $0 < p < 1$). We note that our discussions above for inequality (1.2) can be carried out for the case $0 < p < 1, 0 < c < 1$ with the related inequalities reversed. We therefore obtain the following
Theorem 2.1. Let $p > 0$ be fixed. Let $c_0$ denote the unique number satisfying
\[
\left(1 + \frac{1 - c_0}{p}\right)^{1-p} - \frac{1 - c_0}{p} = 0.
\]
Then inequalities (1.2) and (1.3) hold for all $c_0 \leq c < 1$ when $p > 1$ and the reversed inequality (1.2) holds for all $c < c_0$ when $0 < p < 1$.

We leave it to the reader for the corresponding extensions to $c > p$ of inequalities (1.1) and (1.4) by the duality principle.

3. Some related results

In this section we first consider the conjecture of Bennett and Grosse-Erdmann on inequality (1.5) for the case $0 < \alpha < 1$. We may assume $\lambda_n > 0$ for all $n$. We note here that it is shown in [2, (153), (156)] that it suffices to show that
\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=n}^{\infty} \left( \Lambda_0^\alpha - \Lambda_{k-1}^\alpha \right) x_k \right)^{p} \leq (\alpha p)^p \sum_{n=1}^{\infty} \lambda_n (\Lambda_n^\alpha x_n)^p,
\]
where we set $\Lambda_0 = 0$. By the duality principle, it is easy to see that the above inequality is equivalent to
\[
\sum_{n=1}^{\infty} \left( \frac{\Lambda_n^\alpha - \Lambda_{n-1}^\alpha}{\lambda_n^{1-1/p} \Lambda_n^\alpha} \sum_{k=1}^{n} \lambda_k^{1-1/p} x_k \right)^{p} \leq \left( \frac{\alpha p}{p-1} \right)^p \sum_{n=1}^{\infty} x_n^p.
\]

It follows from (2.7) that in order to establish the above inequality, it suffices to find a positive sequence $\{w_n\}$ such that
\[
\left( \sum_{k=1}^{n} w_k \right)^{p-1} \leq \left( \frac{\alpha p}{p-1} \right)^p \left( \frac{\Lambda_n^\alpha - \Lambda_{n-1}^\alpha}{\lambda_n^{1-1/p} \Lambda_n^\alpha} \right)^{-p} \left( \frac{w_n^{p-1}}{\lambda_n^{p-1} - \lambda_{n+1}^{p-1}} \right),
\]
By a change of variables: $w_n \mapsto \lambda_n w_n$, we can recast the above inequality as
\[
\left( \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k^{1/p} \right)^{p-1} \leq \left( \frac{p}{p-1} \right)^p \left( \frac{\alpha_\lambda \Lambda_n^{\alpha-1}}{\alpha_\lambda \Lambda_n^{\alpha-1}} \right)^p \frac{\Lambda_n}{\alpha_\lambda} \left( \frac{w_n^{p-1}}{\lambda_n^{p-1} - \lambda_{n+1}^{p-1}} \right).
\]

We now define the sequence $\{w_n\}$ inductively by setting $w_1 = 1$ and for $n \geq 1,$
\[
\sum_{k=1}^{n} \lambda_k w_k = \frac{p}{p-1} \Lambda_n w_{n+1}.
\]
This implies that
\[
w_{n+1} = \left( 1 - \frac{1}{p} \frac{\lambda_n}{\Lambda_n} \right) w_n.
\]
Using the above relations, we can simplify inequality (3.1) to see that it is equivalent to the following:
\[
\left( \frac{p}{p-1} \right)^p \left( \left( 1 - \frac{x}{p} \right)^{1-p} - 1 \right) \geq x \left( \frac{1 - (1-x)^\alpha}{\alpha x} \right)^p,
\]
where we set $x = \lambda_n / \Lambda_n$ so that $0 \leq x \leq 1$.

By Hadamard’s inequality, which asserts for a continuous convex function $h(u)$ on $[a,b],$
\[
\frac{1}{b-a} \int_a^b h(u) \, du \geq h\left( \frac{a+b}{2} \right),
\]
we see that
\[
\left( \frac{p/x}{p-1} \right) \left( \left( \frac{1-x}{p} \right)^{1-p} - 1 \right) = \frac{1}{1-(1-x/p)} \int_{1-x/p}^{1} u^{-p} du \geq \left( \frac{1-x}{2p} \right)^{-p}.
\]

Thus, it remains to show that
\[
\left( \frac{1-x}{2p} \right)^{-1} \geq \frac{1-(1-x)\alpha}{\alpha x},
\]
Equivalently, we need to show \( f_{\alpha,p}(x) \geq 0 \) where
\[
f_{\alpha,p}(x) = \alpha x - \left( \frac{1-x}{2p} \right) (1-(1-x)^{\alpha}).
\]

It’s easy to see that \( f_{\alpha,p}(0) = f'_{\alpha,p}(0) = 0 \) and \( f''_{\alpha,p}(x) \) has a most one root in \((0,1)\). It follows that \( f'_{\alpha,p}(x) \) has a most one root in \((0,1)\). Suppose \( \alpha > 1 - 1/p \) so that \( f''_{\alpha,p}(0) > 0 \). This together with the observation that \( \lim_{x \to 1^{-}} f'_{\alpha,p}(x) = -\infty \) implies that in order for \( f_{\alpha,p}(x) \geq 0 \) for all \( x \in [0,1] \), it suffices to have \( f_{\alpha,p}(1) \geq 0 \). We then deduce that we need to have
\[
\alpha \geq 1 - \frac{1}{2p}.
\]

We then obtain the following

**Theorem 3.1.** Inequality (1.5) is valid for \( p > 1, \alpha \geq 1 - \frac{1}{2p} \).

We now consider the following analogue to inequality (1.5):
\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \Lambda_{n}^{*\alpha} y_k \right)^{p} \leq (\alpha p + 1)^{p} \sum_{n=1}^{\infty} \lambda_n \Lambda_{n}^{*\alpha} \left( \sum_{k=1}^{n} x_n \right)^{p}.
\]
Again we may assume \( \lambda_n > 0 \) for all \( n \). We set
\[
y_n = \sum_{k=1}^{n} x_k
\]
to recast inequality (3.3) as
\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \left( \Lambda_{k}^{*\alpha} - \Lambda_{k+1}^{*\alpha} \right) y_k + \Lambda_{n}^{*\alpha} y_n \right) \leq (\alpha p + 1)^{p} \sum_{n=1}^{\infty} \lambda_n \Lambda_{n}^{*\alpha} y_n^{p}.
\]
By Minkowski’s inequality, we have
\[
\left( \sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n-1} \left( \Lambda_{k}^{*\alpha} - \Lambda_{k+1}^{*\alpha} \right) y_k + \Lambda_{n}^{*\alpha} y_n \right) \right)^{\frac{1}{p}} \leq \left( \sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n-1} \left( \Lambda_{k}^{*\alpha} - \Lambda_{k+1}^{*\alpha} \right) y_k \right) \right)^{\frac{1}{p}} + \left( \sum_{n=1}^{\infty} \lambda_n \left( \Lambda_{n}^{*\alpha} y_n \right) \right)^{\frac{1}{p}}.
\]
Thus, it suffices to show that
\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \left( \Lambda_{k}^{*\alpha} - \Lambda_{k+1}^{*\alpha} \right) y_k \right) \leq (\alpha p)^{p} \sum_{n=1}^{\infty} \lambda_n \Lambda_{n}^{*\alpha} y_n^{p}.
\]
When \( 0 < \alpha \leq 1 \), we note that we have
\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \left( \Lambda_{k}^{*\alpha} - \Lambda_{k+1}^{*\alpha} \right) y_k \right) \leq \sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \left( \alpha \lambda_k \Lambda_{k}^{*\alpha-1} \right) y_k \right).
\]
It then follows from inequality (1.3) with \( c = 0, x_k = \Lambda_k^{\alpha} - y_k \) that
\[
\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \lambda_k \Lambda_k^{\alpha - 1} y_k \right) \leq p^p \sum_{n=1}^{\infty} \Lambda_n \alpha y_n^p.
\]
Thus, inequality (3.4) is valid when \( 0 < \alpha \leq 1 \).

We now consider the case \( \alpha \geq 1 \). By the duality principle, it is easy to see that inequality (3.5) is equivalent to
\[
\sum_{n=1}^{\infty} \left( \frac{\Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha}}{\Lambda_n^{\alpha - 1} / \Lambda_n^{\alpha}} \sum_{k=1}^{n} \lambda_k^{1/p} y_k \right) \leq \left( \frac{\alpha p}{p - 1} \right)^p \sum_{n=1}^{\infty} y_n^p.
\]

We then see that upon setting
\[
a_n = \frac{\lambda_1^{1/p} \Lambda_n^{\alpha}}{\Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha}}, \quad b_n = \lambda_1^{1/p},
\]
in inequality (2.2) that one can establish inequality (3.5) as long as one can find a positive sequence \( \{w_n\} \) such that
\[
\left( \sum_{k=n}^{\infty} w_k \right)^{p - 1} \leq \left( \frac{\alpha p}{p - 1} \right)^p \left( \frac{\Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha}}{\lambda_1^{p - 1} / \lambda_n^{p - 1}} \right) \left( \Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha} \right)^{p - 1} \Lambda_n^{\alpha} w_n^{p - 1}, \quad n \geq 2;
\]
\[
\left( \sum_{k=1}^{n} w_k \right)^{p - 1} \leq \left( \frac{\alpha p}{p - 1} \right)^p \left( \frac{\Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha}}{\lambda_1^{p - 1} / \lambda_n^{p - 1}} \right) \left( \Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha} \right)^{p - 1} \Lambda_n^{\alpha} w_n^{p - 1}.
\]

Upon a change of variables: \( w_n \rightarrow \lambda_n w_n \), we can recast the above inequalities as
\[
\left( \frac{1}{\Lambda_n^{\alpha}} \sum_{k=n}^{\infty} \lambda_k w_k \right)^{p - 1} \leq \left( \frac{p}{p - 1} \right)^p \left( \frac{\alpha \Lambda_n^{\alpha} \lambda_1^{p - 1}}{\Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha}} \right) \left( \Lambda_n^{\alpha} - \Lambda_{n+1}^{\alpha} \right)^{p - 1} \Lambda_n^{\alpha} w_n^{p - 1}, \quad n \geq 2;
\]
\[
\left( \frac{1}{\Lambda_1^{\alpha}} \sum_{k=1}^{\infty} \lambda_k w_k \right)^{p - 1} \leq \left( \frac{p}{p - 1} \right)^p \left( \frac{\alpha \Lambda_1^{\alpha} \lambda_1^{p - 1}}{\Lambda_1^{\alpha} - \Lambda_2^{\alpha}} \right) \left( \Lambda_1^{\alpha} - \Lambda_2^{\alpha} \right)^{p - 1} \Lambda_1^{\alpha} w_1^{p - 1}.
\]

We now define the sequence \( \{w_n\} \) inductively by setting \( w_1 = 1 \) and for \( n \geq 2 \),
\[
\sum_{k=n}^{\infty} \lambda_k w_k = \frac{p}{p - 1} \Lambda_n^{\alpha} w_{n-1}.
\]
This implies that for \( n \geq 2 \),
\[
w_n = \left( 1 - \frac{\lambda_n}{p \Lambda_n^{\alpha}} \right)^{-1} w_{n-1}.
\]
Using the above relations, we can simplify inequality (3.6) to see that it is equivalent to inequality (3.2) with \( x = \lambda_n / \Lambda_n^{\alpha} \) while inequality (3.7) is equivalent to the following inequality
\[
\left( \frac{p}{p - 1} \right) \left( 1 - \frac{x}{\alpha x} \right)^{1-p} \geq x \left( \frac{1 - (1 - x)^\alpha}{\alpha x} \right)^p.
\]
As the above inequality is implied by inequality (3.2), it suffices to establish inequality (3.2) for all \( \alpha \geq 1 \). For this, we note that it is easy to show that the right-hand side expression of (3.2) is a decreasing function of \( \alpha \) and inequality (3.2) is valid when \( \alpha = 1 \). It therefore follows that inequality (3.2) is valid for all \( \alpha \geq 1 \). As it is easy to check that the constant in (3.3) is best possible by considering \( \lambda_n = n^{-\alpha}, a > 1, x_k = n^{\alpha}, b = ((a - 1)(a\alpha + 1) - \epsilon) / p - 1 \) with \( \epsilon \rightarrow 0^+ \), we conclude the paper with the following

**Theorem 3.2.** Inequality (3.3) is valid for \( p > 1, \alpha > 0 \). The constant is best possible.
References

[1] G. Bennett, Some elementary inequalities, *Quart. J. Math. Oxford Ser. (2)* 38 (1987), 401–425.
[2] G. Bennett and K.-G. Grosse-Erdmann, On series of positive terms, *Houston J. Math.*, 31 (2005), 541-586.
[3] E. T. Copson, Note on series of positive terms, *J. London Math. Soc.*, 3 (1928), 49-51.
[4] P. Gao, Hardy-type inequalities via auxiliary sequences, *J. Math. Anal. Appl.*, 343 (2008), 48-57.
[5] P. Gao, On weighted remainder form of Hardy-type inequalities, arXiv:0907.5285
[6] P. Gao, On $l^p$ norms of weighted mean matrices, *Math. Z.*, 264 (2010), 829-848.
[7] P. Gao, On a result of Levin and Stečkin, *Int. J. Math. Math. Sci.*, 2011 (2011), Art. ID 534391, 15pp.
[8] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1952.
[9] L. Leindler, Generalization of inequalities of Hardy and Littlewood, *Acta Sci. Math. (Szeged)*, 31 (1970), 279–285.
[10] H. L. Montgomery, The analytic principle of the large sieve, *Bull. Amer. Math. Soc.* 84 (1978), 547–567.

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