Accurate nuclear symmetry energy at finite temperature within a BHF approach

Jia-Jing Lu (陆家靖), Fan Li (李凡), Zeng-Hua Li (李增花),* and Chong-Yang Chen (陈重阳)
Institute of Modern Physics, Key Laboratory of Nuclear Physics and
Ion-beam Application (MOE), Fudan University, Shanghai 200433, P.R. China

G. F. Burgio and H.-J. Schulze
INFN Sezione di Catania, Dipartimento di Fisica, Università di Catania, Via Santa Sofia 64, 95123 Catania, Italy
(Dated: June 1, 2020)

We compute the free energy of asymmetric nuclear matter in a Brueckner-Hartree-Fock approach at finite temperature, paying particular attention to the dependence on isospin asymmetry. The first- and second-order symmetry energies are determined as functions of density and temperature and useful parametrizations are provided. We find small deviations from the quadratic isospin dependence and very small corresponding effects on (proto)neutron star structure.

I. INTRODUCTION

The nuclear symmetry energy, i.e., the energy difference between removing a neutron or a proton from nuclear matter [1], is an important topic of experimental and theoretical nuclear (astro)physics, as it affects a large number of phenomena in nuclear structure physics [2], heavy-ion collisions [3–5], and astrophysics like neutron star (NS) structure [6–8] or recently NS mergers [9–13].

At least under the last two scenarios, the nuclear system is at non-negligible finite temperature of the order of several tens of MeV. This requires to consider the free energy as fundamental thermodynamical quantity. Therefore in recent years some phenomenological methods, such as a momentum-dependent effective interaction [14] and the nuclear energy-density functional theory [15], were applied to the study of the behavior of the free energy of nuclear matter as a function of the baryon density. More recently, microscopic calculations based on the self-consistent Green’s Function method with nuclear forces derived from chiral effective field theory were performed [16]. Moreover, we have computed the free energy up to large nucleon densities $\rho \lesssim 0.8$ fm$^{-3}$ and temperatures $T \lesssim 50$ MeV within the theoretical Brueckner-Hartree-Fock (BHF) method, and provided convenient parametrizations for practical use.

Under these circumstances, the nuclear free (symmetry) energy depends on the partial densities $\rho_n, \rho_p$, and temperature $T$. An important feature is the dependence on isospin asymmetry $\beta \equiv (\rho_n - \rho_p)/(\rho_n + \rho_p)$ for fixed nucleon density $\rho = \rho_n + \rho_p$, and for cold matter it has been demonstrated that a quadratic dependence $\sim \beta^2$ is rather accurate [17, 18]. However, at finite temperature this approximation becomes less reliable [19–22] and one should seek to go beyond this lowest-order parametrization.

This is the focus of the present article, where we study in detail the dependence of the finite-temperature free energy on isospin and provide parametrizations that go beyond the quadratic law. We will also give a simple application to NS structure in order to estimate the magnitude of the effect in practical applications.

We consider in this work two microscopic EOSs that have been derived within the BHF formalism [2, 23–26] based on realistic two-nucleon (NN) and compatible three-nucleon forces (TBF) [27–31], namely those employing the Argonne V18 [32] or the Bonn B [33, 34] NN potentials, respectively. They all feature reasonable properties at (sub)nuclear densities in agreement with nuclear-structure phenomenology [31, 35–37], and are also fully compatible with recent constraints obtained from the analysis of the GW170817 NS merger event [38–40], as well as from NS cooling [41, 42].

Our paper is organized as follows. In Sec. II we briefly review the computation of the free energy in the finite-temperature BHF approach and give some details of the fitting procedure. In Sec. III we present the numerical results for the free energy and some model calculations of hot NS structure. Conclusions are drawn in Sec. IV.

II. FORMALISM

The calculations for hot asymmetric nuclear matter are based on the Brueckner-Bethe-Goldstone (BBG) theory [17, 18, 23–26] and its extension to finite temperature [19, 43–45]. Here we simply give a brief review for completeness. The free energy density in ‘frozen-correlations’ approximation [26, 43, 44, 46–53] is

$$f = \frac{F}{A} = \sum_{i=n,p} \left[ 2 \sum_k n_i(k) \left( \frac{k^2}{2m_i} + \frac{1}{2} U_i(k) \right) - T s_i \right],$$

(1)

where

$$s_i = -2 \sum_k \left( n_i(k) \ln n_i(k) + [1 - n_i(k)] \ln [1 - n_i(k)] \right)$$

(2)

is the entropy density for the component $i$ treated as a free Fermi gas with spectrum $e_i(k)$. At finite temperature,

$$n_i(k) = \left[ \exp \left( \frac{e_i(k) - \mu_i}{T} \right) + 1 \right]^{-1}$$

(3)
TABLE I. Parameters of the fit for the free energy per nucleon $F/A$, Eq. (13), for symmetric nuclear matter (SNM), asymmetric ($\beta = 0.6$) nuclear matter (ANM), and pure neutron matter (PNM) with the V18 and BOB EOSs.

|      | a    | b    | c    | d | $\tilde{a}$ | $\tilde{b}$ | $\tilde{c}$ | $\tilde{d}$ | $\tilde{e}$ |
|------|------|------|------|---|-----------|-----------|-----------|-----------|-----------|
| SNM | -54  | 363  | 2.68 | -8| -149      | 211       | -58       | 81        | 2.40      |
| V18 ANM | -23 | 473  | 2.72 | -3| -140      | 200       | -61       | 82        | 2.36      |
| PNM  | 38  | 668  | 2.78 | 6 | -91       | 153       | -26       | 38        | 2.64      |
| BOB ANM | -60 | 495  | 2.69 | -9| -124      | 203       | -60       | 80        | 2.38      |
| PNM  | 52  | 860  | 2.89 | 4 | -82       | 149       | -25       | 36        | 2.67      |

is a Fermi distribution, where the auxiliary chemical potentials $\mu_{n,p}$ are fixed by the condition $\rho_i = 2 \sum_k n_i(k)$. The single-particle energy

$$e_i = \frac{k_i^2}{2m_i} + U_1,$$

$$U_1(\rho, x_p) = \text{Re} \sum_{n=1}^{\infty} n_\alpha \langle 12 \rangle \langle 12 \rangle \alpha \beta (\rho, x_p; e_1 + e_2) \langle 12 \rangle \alpha \beta .$$

(4)

(5)
is obtained from the interaction matrix $K$, which satisfies the self-consistent equation

$$K(\rho, x_p; E) = V + V \text{Re} \sum_{n=1}^{\infty} n_\alpha \langle 12 \rangle \langle 12 \rangle \alpha \beta (\rho, x_p; e_1 + e_2) \langle 12 \rangle \alpha \beta .$$

(6)

Here $E$ is the starting energy and $x_p = \rho_p/\rho$ is the proton fraction. The multi-indices $1,2$ denote in general momentum, isospin, and spin.

Two choices for the realistic $NN$ interaction $V$ are adopted in the present calculations [31]: the Argonne $V_{18}$ [32] and the Bonn B (BOB) [33, 34] potential. They are supplemented with microscopic TBF employing the same meson-exchange parameters as the two-body potentials. The TBF are reduced to an effective two-body force and added to the bare potential in the BHF calculation, see Refs. [29–31, 54] for details.

The knowledge of the free energy allows to derive all necessary thermodynamical quantities in a consistent way, namely one defines the “true” chemical potentials $\mu$, pressure $p$, and internal energy density $\varepsilon$ as

$$\mu_i = \frac{\partial f}{\partial \rho_i},$$

$$p = \rho s \frac{\partial (f/\rho)}{\partial \rho} = \sum_i \mu_i \rho_i - f,$$

$$\varepsilon = f + Ts, \quad s = -\frac{\partial f}{\partial T}.$$  

(7)

(8)

(9)

For the case of asymmetric nuclear matter, one might expand the free energy for fixed total density and temperature in terms of the asymmetry parameter $\delta = \beta^2 = (1 - 2x_p)^2$,

$$f(\delta) \approx f(0) + \delta f_{\text{sys,2}} + \delta^2 f_{\text{sys,4}}.$$  

(10)

Limiting to the second term, one obtains the symmetry energy as the difference between pure neutron matter (PNM) and symmetric nuclear matter (SNM),

$$f_{\text{sys,2}} = f(1) - f(0),$$

$$f_{\text{sys,4}} = 0,$$  

(11a)

(11b)

which is usually a good approximation at zero temperature [17, 18, 53], and also used at finite temperature [19]. It has, however, been pointed out [20–22, 56–63] that at least the kinetic part of the free energy density [first term in Eq. (11)] violates the parabolic law, in particular at high temperature. We therefore extend the expansion to second order and compute $f_{\text{sys,4}}$ in the following way: Inverting the system of equations for $f(0), f(\alpha), f(1)$, where $\alpha$ is an arbitrarily chosen value (we use $\alpha = 0.6^2$, which corresponds to a typical $x_p = 0.2$ in NS matter), one obtains

$$f_{\text{sys,2}} = \frac{\alpha^2 [f(1) - f(0)] - [f(\alpha) - f(0)]}{\alpha^2 - \alpha},$$

$$f_{\text{sys,4}} = \frac{\alpha [f(1) - f(0)] - [f(\alpha) - f(0)]}{\alpha^2 - \alpha}.$$  

(12a)

(12b)

in which $f(0), f(\alpha), f(1)$ depend on total density and temperature. Following Ref. [64], we provide analytical fits for these dependencies of the numerical results in the required ranges of density $(0.05 \text{ fm}^{-3} \lesssim \rho \lesssim 1 \text{ fm}^{-3})$ and temperature $(5 \text{ MeV} \lesssim T \lesssim 50 \text{ MeV})$ in the following functional form for symmetric nuclear matter (SNM),

$$f_{\text{sys,2}} = f(1) - f(0),$$

$$f_{\text{sys,4}} = 0,$$  

(11a)

(11b)

FIG. 1. Free energy per nucleon as a function of asymmetry for different densities at $T = 0$ (top panels), $50$ MeV (middle panels) for the V18 (left panels) or BOB (right panels) EOS. Dashed lines show the parabolic approximation Eq. (11). The bottom panels show the deviation between numerical results and the linear, [Eq. (11), solid curves], or quadratic, [Eq. (12), dashed curves], $\beta^2$ fits.
the free energy per nucleon

\[ F(A, \rho, T) = a\rho + b\rho^c + d \]
\[ + \alpha t^2 + \beta t^2 \ln(\rho) + (\alpha t^2 + \beta t^2)\rho, \]  

where \( t = T/(100 \text{ MeV}) \) and \( F/A \) and \( \rho \) are given in MeV and \( \text{fm}^{-3} \), respectively. The parameters of the fits are listed in Table I for SNM, asymmetric nuclear matter with \( x_p = 0.2 \) (ANM), and PNM, for the different EOSs we are using. The rms deviations of fits and data are better than \( 0.3 \text{ MeV} \) for all EOSs.

### III. RESULTS

Fig. 1 shows the free energy per nucleon as a function of the asymmetry parameter \( \delta \) for different densities and at temperatures \( T = 0 \) (upper row) and \( T = 50 \text{ MeV} \) (middle row), for both EOSs. The linear approximation Eq. \((10,11)\) is indicated by dashed lines in the figure, and the deviations from the linear [Eq. \((11)\)] or quadratic [Eq. \((12)\)] laws at \( T = 50 \text{ MeV} \) are indicated in the lower row. One observes that in general even the linear law provides a very good fit, even at low density and high temperature, where the deviations might reach a few percent. With the quadratic law, the deviations remain below 2 MeV over the whole parameter space \( [\rho, T, \beta] \). In this case the overall variances are 0.47 and 0.54 MeV for the V18 and BOB EOS, respectively.

In order to compare the magnitude of violation of the linear or quadratic \( \beta^2 \) laws with those of other frequently used

finite-temperature nuclear EOSs, we performed the previous analysis also for the SFHo [65] and the HShen [66, 67] EOS and report the values of the variance \( \langle \Delta F/A \rangle_{\text{rms}} \) for both the linear and quadratic law in Table II. We observe that in all cases the quadratic law is an important improvement by at least a factor three, but also the linear law is a very reasonable approximation.

Fig. 2 shows the derived free symmetry energies per nucleon \( F_{\text{sym},2}/A \), Eqs. \((11a,12a)\), and \( F_{\text{sym},4}/A \), Eq. \((12b)\), as functions of density and temperature. One notes that the dependence on density is more pronounced for \( F_{\text{sym},2}/A \) than for \( F_{\text{sym},4}/A \), while the opposite is the case for the temperature dependence. The \( F_{\text{sym},2}/A \) results in quadratic approximation (solid curves in upper row) are somewhat smaller than in linear approximation (dashed curves) in order to compensate for the finite \( F_{\text{sym},4}/A \), in particular at finite temperature. For comparison, the \( T = 0 \) results for \( F_{\text{sym},4}/A \) obtained by RMF theory with FSU interactions [55] are shown as dotted and dash-dotted curves in the lower row. They are comparable with our BHF results, especially the BOB model.

### TABLE II. Quality \( \langle \Delta F/A \rangle_{\text{rms}} \) (in MeV) of the linear or quadratic \( \beta^2 \) laws for the free energy per nucleon \( F/A \) obtained with different EOSs.

| EOS   | V18 | BOB | SFHo | Shen |
|-------|-----|-----|------|------|
| linear| 1.51| 1.77| 1.12 | 1.53 |
| quadratic| 0.47| 0.54| 0.23 | 0.39 |
expanded around normal density $\rho_0$ in terms of normal values $J_2, J_4$ and slope parameters $L_2, L_4$:

$$F_{\text{sym},2}/A(\rho, T) \approx J_2(T) + L_2(T)x,$$

$$F_{\text{sym},4}/A(\rho, T) \approx J_4(T) + L_4(T)x,$$

with $x = (\rho - \rho_0)/3\rho_0$ and $J_1(T) = J_{\text{sym},1}(\rho_0, T)$, $L_1(T) = 3\partial J_{\text{sym},1}(\rho_0, T)/\partial \rho$. These quantities are shown in Fig. 3. The $T = 0$ values are $J_2(0) = 31.0(32.7)$ MeV and $L_2(0) = 58.5(64.2)$ MeV for V18(BOB), which should be confronted with recent constraints $J_2 = 31.7 \pm 2.7$ MeV and $L_2 = 58.7 \pm 28.1$ MeV [6, 68]. In the same figure we report also the results for the SFHo and Shen EOSs according to our analysis, see also Table II. Reasonable values are obtained in the first case, but too large ones in the latter.

The second-order symmetry energy $J_4(0)$ is theoretically more controversial compared to the first-order one $J_2(0)$. Our results are $J_4(0) = 0.41, 0.93, 1.17, 1.17$ MeV for the V18, BOB, SFHo, Shen EOS, respectively. Within energy density functionals with mean-field approximation, for example Skyrme-Hartree-Fock and Gogny-Hartree-Fock models, the values of $J_4$ reported in the literature are around 1.0 MeV [59], and around 0.66 MeV within RMF models [55], while values extracted from Quantum Molecular Dynamics models could be larger depending on the specific interaction [57]. From the view point of finite nuclei, $J_4$ can be related to the second-order symmetry energy $a_{\text{sym},4}(A)$ in a semi-empirical mass formula, in which the latter can be inferred from the double difference of “experimental” symmetry energies by analyzing the binding energies of a large number of measured nuclei [69, 70]. In this case, the estimates are $J_4 = 20.0 \pm 4.6$ MeV [69] and two possible $J_4 = 8.5 \pm 0.5$ MeV or $J_4 = 3.3 \pm 0.5$ MeV [70], which are significantly different and larger than those deduced from nuclear matter, which points to a great model dependence and to the importance of finite-size effects in nuclei.

Regarding the temperature dependence, from Fig. 3 one can see that $J_2(T)$ and $J_4(T)$ are increasing monotonically with temperature for all models, whereas $L_2(T)$ decreases and $L_4(T)$ exhibits nonmonotonic behavior. It is notable that the $J_4(T)$ results are nearly universal for all EOSs. Note that in our approach the temperature dependence is constrained to be a linear combination of $T^2$ and $T^4$ terms according to Eq. (13). We compare our results with the ones of the chiral effective field theory calculation [22]. Considering also the cutoff dependence of the chiral potentials, we observe that both results are in quantitative agreement in particular in the low temperature region, but the latter predicts a more linear temperature dependence. (At low temperature such behavior is excluded by the condition of vanishing entropy in the $T \to 0$ limit). The temperature dependence of the free symmetry energy is also discussed in Refs. [71, 72], where an isospin- and momentum-dependent interaction constrained by heavy-ion collisions and the Skyrme SLy4 parameters have been employed, respectively. Those investigations show very similar behavior and numerical magnitudes to the present calculations about the free symmetry energy.
In order to assess the relevance of the previous results to practical applications, we perform some model calculations of NS structure employing the different approximations for the symmetry energy. Fig. 4 shows the proton fractions of $\beta$-stable and charge-neutral nuclear matter in the upper panel and the mass-radius relations of NSs in the lower panel at the temperatures $T = 0$ and $T = 50$ MeV. Results using the linear [Eq. (11), thin curves] or the quadratic [Eq. (12), thick curves] $\delta$ laws are compared with both BOB and V18 interactions. One can see that the inclusion of $F_{\text{sym},4}$ in the latter case causes a slight decrease of the proton fraction in particular at high temperature, corresponding to a slight reduction of $F/A$ as seen in Fig. 1. The effect on the mass-radius relations is nearly invisible, even at large finite temperature, which means that the linear law Eq. (11) is a very good approximation for the determination of the stellar structure.

IV. SUMMARY

We have studied the isospin dependence of the free symmetry energy of nuclear matter at zero and finite temperature within the framework of the Brueckner-Hartree-Fock approach at finite temperature with different potentials and compatible nuclear three-body forces. We have compared our results with phenomenological models, i.e., SFHo and Shen EOS, which are widely used in numerical simulations of astrophysical processes.

We have determined the first- and second-order terms in an expansion with respect to isospin asymmetry and provided convenient parametrizations for practical applications. A model study of neutron star structure at finite temperature demonstrated that the often used parabolic law is an excellent approximation and the second-order modifications are very small.

ACKNOWLEDGMENTS

This work is sponsored by the National Natural Science Foundation of China under Grant Nos. 11475045, 11975077 and the China Scholarship Council, No. 201806100066. We further acknowledge partial support from “PHAROS,” COST Action CA16214.

[1] M. Baldo and G. F. Burgio, Progress in Particle and Nuclear Physics 91, 203 (2016), arXiv:1606.08838 [nucl-th].
[2] M. Baldo and G. F. Burgio, Reports on Progress in Physics 75, 026301 (2012), arXiv:1102.1364 [nucl-th].
[3] P. Danielewicz, R. Lacey, and W. G. Lynch, Science 298, 1592 (2002), arXiv:nucl-th/0208016 [nucl-th].
[4] B.-A. Li, L.-W. Chen, and C. M. Ko, Phys. Rep. 464, 113 (2008), arXiv:0804.3580 [nucl-th].
[5] M. B. Tsang, J. R. Stone, F. Camera, P. Danielewicz, S. Gandolfi, K. Hebeler, C. J. Horowitz, J. Lee, W. G. Lynch, Z. Kohley, R. Lemmon, P. Möller, T. Murakami, S. Riordan, X. Roca-Maza, F. Sammarruca, A. W. Steiner, I. Vidaña, and S. J. Yennello, Phys. Rev. C 86, 015803 (2012), arXiv:1204.0466 [nucl-ex].
[6] B.-A. Li, P. G. Krastev, D.-H. Wen, and N.-B. Zhang, European Physical Journal A 55, 117 (2019), arXiv:1905.13175 [nucl-th].
[7] C. Y. Tsang, M. B. Tsang, P. Danielewicz, W. G. Lynch, and F. J. Fattoyev, arXiv e-prints, arXiv:1901.07673 (2019), arXiv:1901.07673 [nucl-ex].
[8] J. M. Latimer and A. W. Steiner, European Physical Journal A 50, 40 (2014), arXiv:1403.1186 [nucl-th].
[9] LIGO Scientific Collaboration and Virgo Collaboration, Phys. Rev. Lett. 119, 161101 (2017), arXiv:1711.05832 [gr-qc].
[10] LIGO Scientific Collaboration and Virgo Collaboration, Phys. Rev. Lett. 121, 161101 (2018), arXiv:1805.11581 [gr-qc].
[11] LIGO Scientific Collaboration and Virgo Collaboration, Physical Review X 9, 011001 (2019), arXiv:1805.11579 [gr-qc].
[12] L. Baiotti and L. Rezzolla, Reports on Progress in Physics 80, 096901 (2017), arXiv:1607.03540 [gr-qc].
[13] L. Baiotti, Progress in Particle and Nuclear Physics 109, 103714 (2019), arXiv:1907.08534 [astro-ph.HE].
[14] C. C. Moustakidis and C. P. Panos, Phys. Rev. C 79, 045806 (2009), arXiv:0805.0353 [nucl-th].
[15] A. F. Fantina, N. Chamel, J. M. Pearson, and S. Goriely, in Journal of Physics Conference Series, 11375045, 11975077 and the China Scholarship Council, No. 201806100066. We further acknowledge partial support from “PHAROS,” COST Action CA16214.
