Systems of Dyadic Cubes of Complete, Doubling, Uniformly Perfect Metric Spaces without Detours

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Abstract

Systems of dyadic cubes are the basic tools of harmonic analysis and geometry, and this notion had been extended to general metric spaces. In this paper, we construct systems of dyadic cubes of complete, doubling, uniformly perfect metric spaces, such that for any two points in the metric space, there exists a chain of three cubes whose diameters are comparable to the distance of the points. We also give an application of our construction to previous research of potential analysis and geometry of metric spaces.

1 Introduction

The standard system of dyadic cubes of $\mathbb{R}^d$,

$$S_k = \left\{ \prod_{i=1}^{d} \left[ \frac{a_i}{2^k}, \frac{a_i + 1}{2^k} \right] \mid a_i \in \mathbb{Z} \ (1 \leq i \leq d) \right\} \quad (k \in \mathbb{Z}) \quad (1.1)$$

is a basic tool of analysis on the Euclidean spaces. Constructing the counterparts of dyadic cubes of general metric spaces were started by David [7, 8] and Christ [5] for metric measure spaces. Hytönen and Kairema [11] and Kažimäki, Rajala and Suomala [12] extended these results for metric spaces without measures. Systems of generalized dyadic cubes were used for various studies of harmonic analysis (e.g. [1, 2]) and analysis on metric spaces (e.g. [6, 13]).

From the viewpoint of discrete approximation of a metric space, it is important whether the structure of a system of dyadic cubes is comparable to that of the original metric space or not. For example, (1.1) satisfies

for any $x, y \in \mathbb{R}^d$ and $k \in \mathbb{Z}$ with $|x - y|_{\mathbb{R}^d} \leq 2^{-k}$, there exist

$Q_x, Q_y \in S_k$ such that $x \in Q_x, y \in Q_y$ and $Q_x \cap Q_y \neq \emptyset$.

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However, a system of dyadic cubes of a metric space may not satisfy such a condition, for instance, two close points may not have any short chain of dyadic cubes (see Figure 1). Hence it is a natural question when a system of dyadic cubes $S_k$ ($k \in \mathbb{Z}$) of a metric space $(X, d)$, satisfying the following conditions for some $C, C', r > 0$ and $M \in \mathbb{N}$ exist:

- $C^{-1} r^k \leq \text{diam}(Q, d) \leq C r^k$ for any $Q \in S_k$
- for any $x, y \in X$ and $k \in \mathbb{Z}$ with $d(x, y) \leq C' r^k$, there exist $\{Q_i\}_{i=0}^M \subset S_k$ such that $x \in Q_0, y \in Q_M$ and $Q_{i-1} \cap Q_i \neq \emptyset$ ($1 \leq i \leq M$).

The aim of this paper is to give an answer to the question in a constructive way, under certain conditions.

Let $(X, d)$ be a metric space, $B(x, r)$ be an open ball of radius $r$, and $\overline{A}$ (resp. int$A$) be the closure (resp. interior) of a set $A \subset X$.

**Definition 1.1 (Doubling).** $(X, d)$ is called doubling if there exists $N \in \mathbb{N}$ such that for any $x \in X$ and $r > 0$, there exist $\{x_i\}_{i=1}^N \subset X$ with $B(x, 2r) \subset \bigcup_{i=0}^N B(x_i, r)$.

**Definition 1.2 (Uniformly perfect).** $(X, d)$ is called uniformly perfect if there exists $\gamma > 1$ such that $B(x, \gamma r) \setminus B(x, r) \neq \emptyset$ whenever $B(x, r) \neq X$.

Throughout this paper, we assume $(X, d)$ is complete, doubling and uniformly perfect. Our main result is the following.

**Theorem 1.3.** Let $C_*, c_*$ be constants with $0 < c_* < C_* < \infty$. Then there exist $C_1, C_2, C_3, r_0 > 0$ such that if a set of points $\bigcup_{k \in \mathbb{Z}} \{x_\omega\}_{\omega \in \Omega_k}$ with some $\Omega_k$ ($k \in \mathbb{Z}$) satisfies

\begin{align*}
\text{d}(x_\omega, x_\lambda) &\geq c_* r^k \quad \text{for any } \omega, \lambda \text{ with } \omega \neq \lambda \quad (1.2) \\
\min_{\omega \in \Omega_k} \text{d}(y, x_\omega) &< C_* r^k \quad \text{for any } y \in X \quad (1.3)
\end{align*}

for some $r \in (0, r_0]$, then there exist $\{Q_\omega \subset X \mid k \in \mathbb{Z}, \omega \in \Omega_k\}$ such that

![Figure 1: Two close points without short chains](image-url)
\[ \bigcup_{\omega \in \Omega_k} Q_\omega = X \text{ for any } k \in \mathbb{Z}, \]  
(D1)

\[ \text{int} Q_\omega = \text{int}(\overline{Q_\omega}), \quad \text{int} Q_\bar{\omega} = Q_\omega, \]  
(D2)

- if \( l \geq k \), then either \( Q_\omega \subset Q_\lambda \) or \( Q_\omega \cap Q_\lambda = \emptyset \) holds
  
  for any \( \omega \in \Omega_l, \lambda \in \Omega_k \),
(D3)

\[ B(x_\omega, C_1 r^k) \subset Q_\omega \subset B(x_\omega, C_2 r^k), \]  
(D4)

- if \( y, z \in X \) satisfy \( d(y, z) \leq C_3 r^k \), then there exist \( \omega_0, \omega_1, \omega_2 \in \Omega_k \) such that
  
  \( Q_{\omega_i} \cap Q_{\omega_{i+1}} \neq \emptyset \) (\( i = 0, 1 \)) and \( y \in Q_{\omega_0}, z \in Q_{\omega_2} \).
(D5)

This theorem is based on the results and proofs of [5] and [11], and our main contribution is to develop the proof in order to satisfy the additional condition (D5). In [11], Hytönen and Kairema proved that there exist \( C > 0, N \in \mathbb{N}, \{\{\Omega_k^{(i)}\}_{k \in \mathbb{Z}}\}_{1 \leq i \leq N} \) and \( \{Q_\omega \mid \omega \in \bigcup_{k \in \mathbb{Z}} \Omega_k^{(i)}\}_{1 \leq i \leq N} \), which satisfy (D1) to (D4) for each \( i \) and

for any \( x, y \in X \) and \( k \in \mathbb{Z} \) with \( d(x, y) \leq C_1 r^k \),

there exist \( i \leq N \) and \( \omega \in \Omega_k^{(i)} \) such that \( \{x, y\} \subset Q_\omega \) (1.4) only using the doubling condition, but \( i \) in (1.4) may be different for each point.

In order to state an example of previous results which use a system of dyadic cubes with (D5), we prepare some conditions.

**Definition 1.4** (Ahlfors regular). Let \((Y, \rho)\) be a metric space and \(\mu\) be a Borel measure on it. We say \(\mu\) is \(\alpha\)-Ahlfors regular with respect to \((Y, \rho)\) if there exists \(C > 0\) such that

\[ C^{-1} r^\alpha \leq \mu(B_\rho(x, r)) \leq C r^\alpha \quad \text{for any } x \in Y \text{ and } r_x \leq r \leq \text{diam}(Y, \rho), \]

where \( r_x = r_{x, \rho} = \inf_{y \in Y \setminus \{x\}} \rho(x, y) \). \((Y, \rho)\) is called \(\alpha\)-Ahlfors regular if there exists a Borel measure \(\mu\) such that \(\mu\) is \(\alpha\)-Ahlfors regular with respect to \((Y, \rho)\).

Ahlfors regularity is sometimes used in studies of harmonic analysis via dyadic cubes (e.g. [9]). We also note that if \((Y, \rho)\) is an \(\alpha\)-Ahlfors regular space without isolated points, then the Hausdorff dimension of \((Y, \rho)\) coincides to \(\alpha\), and \((Y, \rho)\) is uniformly perfect.

**Definition 1.5** (Quasisymmetry, in the sense of [13]). Let \(Y\) be a set and \(\rho, \delta\) be distances on \(Y\). We say that \(\rho\) is quasisymmetric to \(\delta\) if there exists a homeomorphism \(\theta : [0, \infty) \to [0, \infty)\) such that for any \(x, y, z \in Y\) with \(x \neq z\),

\[ \frac{\delta(x, y)}{\delta(x, z)} \leq \theta(\rho(x, y)/\rho(x, z)). \]
Definition 1.6 (Ahlfors regular conformal dimension). The Ahlfors regular conformal dimension (ARC dimension in short) of \((Y, \rho)\) is defined by

\[
\dim_{\text{AR}}(Y, \rho) = \inf\{\alpha \mid \text{there exists a metric } \delta \text{ on } Y \text{ such that } (Y, \delta) \text{ is } \alpha\text{-Ahlfors regular and } \delta \text{ is quasisymmetric to } \rho\},
\]

where \(\inf \emptyset = \infty\).

Quasisymmetry was introduced by Tukia and Väisälä in [15] as a condition of embedding maps between metric spaces, and it is a generalization of quasiconformal mappings on the complex plane. ARC dimension was implicitly introduced by Bourdon and Pajot [4], and named by Bonk and Kleiner [3]. In [3], the ARC dimension is related to Cannon’s conjecture, which is famous in the study of hyperbolic groups: it claims that for any hyperbolic group \(G\) whose boundary is homeomorphic to \(S^2\), there exists a geometric action of \(G\) on the hyperbolic space \(\mathbb{H}^3\). For compact metric spaces, in [13] Kigami showed that if the counterpart of a system of dyadic cubes satisfying (D1) to (D5), called “partition satisfying basic framework” exists, then \(\dim_{\text{AR}}(Y, \rho)\) is characterized by the \(p\)-energies \((p > 0)\) on the graphs defined by the system of dyadic cubes (see Theorem 3.7). In Section 3, we introduce the detailed conditions and results of [13] and its extension to \(\sigma\)-compact metric spaces and graphs, [14]. We also state the application of our main result to them in that section.

2 Remarks and proof of main theorem

Before the proof of our main theorem, we give some remarks on Theorem 1.3. We first note that the existence of \(\{x_\omega\}_{\omega \in \Omega_k}\) with (1.2) and (1.3) is easily shown as noted in [11, Subsection 2.12].

Lemma 2.1. Let \(A = \{x_\omega\}_{\omega \in \Omega_k}\) be points with (1.2), then there exists \(B = \{x_\omega\}_{\omega \in \Omega_k}\) with (1.2), (1.3) and \(A \subset B\).

Proof. It is shown by the maximal argument with Zorn’s lemma. \(\square\)

Remark. We can avoid Zorn’s lemma if we know \((X, d)\) is separable. A doubling metric space is separable, but this fact is shown by the existence of points satisfying (1.2) and (1.3) in general.

We also note that the results in [3] and [11] were stated for quasimetrics, that is, for \((Y, \rho)\) satisfying the axioms of a metric spaces but replacing the triangle inequality with the following condition:

there exists \(A \geq 1\) such that \(\rho(x, z) \leq A(\rho(x, y) + \rho(y, z))\) for any \(x, y, z \in Y\).

It is known that if \(\rho\) is a quasimetric, then there exists \(c, \beta \in (0, 1)\) and a metric \(\rho'\) such that \(c\rho'(x, y) \leq \rho^3(x, y) \leq c^{-1}\rho'(x, y)\) for any \(x, y \in Y\) (see [10, Proposition 14.5]), so we can apply Theorem 1.3 to quasimetric spaces with this fact.
2.1 Proof of Theorem 1.3

For simplicity of notation, we write \( \max\{a, b\} = a \lor b, \min\{a, b\} = a \land b \) and \( \Omega = \bigcup_{k \in \mathbb{Z}} \Omega_k \). We also write \#A for the number of vertices of a set A. We first prove Theorem 1.3 under strong conditions. Since \((X, d)\) is doubling, we may assume \( \Omega_k \) in the assumption of Theorem 1.3 is countable, so we can write \( \Omega_k = \{(k, n) \mid n \in \mathbb{N}, n < N_k\} \) for some \( N_k \in \mathbb{N} \cup \{\infty\} \).

**Proposition 2.2.** Let \( \alpha_1, \alpha_2, \alpha_3 > 0 \) and \( r_0 \in (0, 1) \) satisfy

\[
\frac{r_0}{1 - r_0} \alpha_1 < \alpha_2 \land (\alpha_3 - C_N).
\]

We assume that for any \( r \in (0, r_0] \) and \( A = \bigcup_{k \in \mathbb{Z}} \{x_\omega \}_{\omega \in \Omega_k} \) with 1.2 and 1.3, there exists a map \( \pi = \pi_A : \Omega \rightarrow \Omega \) such that for any \( k \in \mathbb{Z} \),

- \( \pi(\Omega + 1) = \Omega \) and \( \sup_{\omega \in \Omega} \#\pi^{-1}(\omega) < \infty \),
- \( d(x_{k+1, m}, x_{(k+1, m)}) < \alpha_1 r^k \),
- if \( d(x_{k, n}, x_{k+1, m}) < \alpha_2 r^k \) then \( \pi(k + 1, m) = (k, n) \),
- if \( B(x_{k, n}, \alpha_3 r^k) \cap B(x_{k, n}, \alpha_3 r^k) \neq \emptyset \), then there exist \( m_1, m_2 \in \mathbb{N} \) such that \( B(x_{k+1, m_1}, \alpha_3 r^{k+1}) \cap B(x_{k+1, m_2}, \alpha_3 r^{k+1}) \neq \emptyset \) and \( \pi(k + 1, m_2) = (k, n) \) for any \( k \in \mathbb{Z} \),
- if \( d(x_{k+1, m_1}, x_{(k+1, m_1)}) \geq C_N r^k \) and \( d(x_{k+1, m_1}, x_{k, n}) < \alpha_3 r^k \), then there exist \( m_2, m_3 \) such that \( B(x_{k+1, m_2}, \alpha_3 r^{k+1}) \cap B(x_{k+1, m_3}, \alpha_3 r^{k+1}) \neq \emptyset \) and \( \pi(k + 1, m_3) = \pi(k + 1, m_2) \) and \( (k, n) = \pi(k + 1, m_1) \).

Then, Theorem 1.3 holds.

In order to prove this proposition, we prepare some technical lemmas. We let \( K_\omega = \bigcup_{l \geq 0} \{x_\lambda \mid \lambda \in \pi^{-l}(\omega)\} \) for any \( \omega \in \Omega \).

**Lemma 2.3.** Let \( \alpha_4 = \frac{\alpha_1 - \alpha_2}{1 - r_0} \) and \( \alpha_5 = \alpha_2 - \frac{\alpha_1 r_0}{1 - r_0} \).

1. \( K_{k, n} \subset B(x_{k, n}, \alpha_4 r^k) \) for any \( (k, n) \).
2. \( \bigcup_{\omega \in \Omega_k} K_\omega = X \) for any \( k \).
3. \( B(x_{k, n}, \alpha_5 r^k) \subset K_{k, n} \) for any \( (k, n) \).

**Proof.** (1) If \( \lambda \in \pi^{-l}(\omega) \) for \( l \geq 1 \), then \( d(x_\omega, x_\lambda) \leq \sum_{j=0}^{l-1} d(x_{\pi^j(\lambda)}, x_{\pi^{j+1}(\lambda)}) < \frac{\alpha_1 r^k}{1 - r_0} \leq \alpha_4 r^k \) by 1.11 and 1.12. This suffices to show (1).

(2) Let \( y \in \bigcup_{\omega \in \Omega_k} K_\omega \). Since \((X, d)\) is doubling and \( \{x_\omega \}_{\omega \in \Omega} \) satisfy 1.2 and 1.3, \( \#\{\omega \in \Omega_k \mid d(y, x_\omega) \leq \alpha_4 r^k + 1\} < \infty \). This with (1) shows \( \#\{\omega \in \Omega_k \mid B(y, 1) \cap K_\omega \neq \emptyset\} < \infty \) and so \( y \in \bigcup_{\omega \in \Omega_k} K_\omega \). Therefore \( \bigcup_{\omega \in \Omega_k} K_\omega = \bigcup_{\omega \in \Omega_k} K_\omega \supset \bigcup_{l \geq k} \bigcup_{\omega \in \Omega_k} x_\omega = X \) by 1.3 and 1.1.
(3) Let $y \in B(x_{k,n}, \alpha_3 r^k)$. By (1) and (2), there exists $(l, m) \in \Omega$ such that $l > k$ and $y \in K_{l,m} \subset B(x_{k,n}, \alpha_3 r^k)$. Then
\[
d(x_l^{n-1}(l(m)), x_{n,k}) \leq d(x_{n,k}, x_{l,m}) + d(x_{l,m}, x_{l^{n-1}(l(m))}) < (\alpha_3 + \frac{r\alpha_1}{1-r})^k
\]
similarly to (1). Therefore $\pi^{l-k}(l, m) = (n, k)$ by (1.3) and $y \in K_{k,n}$. □

**Lemma 2.4.** Let $O_{k,n} = X \setminus \bigcup_{l \neq n} K_{k,o}$, then $O_{k,n} = \text{int} K_{k,n}$ and $\overline{O_{k,n}} = K_{k,n}$.

**Proof.** Note that $O_{k,n} = K_{k,n} \setminus \bigcup_{l \neq n} K_{k,o}$ by Lemma 2.3 (3). We first show that
\[
x_{l,m} \in K_{k,n} \text{ for some } m \in \mathbb{N} \text{ and } l \geq k, \text{ then } \pi^{l-k}(l, m) = (k, n). \tag{2.2}
\]
Indeed, we can take $(l', m')$ for sufficiently large $l'$ with $d(x_{l,m}, x_{l',m'}) < \alpha_3 r^l$ and $\pi^{l-k}(l', m') = (k, n)$. Therefore in the similar way to Lemma 2.3 (3), we obtain $\pi^{l-k}(l', m') = (l, m)$ and $\pi^{l-k}(l, m) = (k, n)$.

This shows $\bigcup_{l \geq 1} \{x_\lambda \mid \lambda \in \pi^{-l}(\omega)\} \subset O_\omega$ and so $\overline{O_\omega} = K_\omega$ by definition. Next we show $\text{int} K_{k,n} = O_{k,n}$. By definition of $K_{k,o}$, there exists $(l, m)$ such that $\pi^{l-k}(l, m) = (k, o)$ and $x_{l,m} \in K_{k,n}$. This with (2.2) shows $n = o$ and $\text{int} K_{k,n} \subset O_{k,n}$. On the other hand, by Lemma 2.3 (2) and $\# \{o \mid B(x_{k,n}, \alpha_3 \omega^k + 1) \cap B(x_{k,o}, \alpha_3 r^k)\} < \infty$, it follows that $O_{k,n} = \{y \in X \mid d(y, K_{k,n}) < 1\} \setminus \bigcup_{l \neq n} K_{k,o}$ is open, and so $O_{k,n} \subset \text{int} K_{k,n}$. □

**Lemma 2.5.** If $B(x_{k,n}, \alpha_3 r^k) \cap B(x_{k,o}, \alpha_3 r^k) \neq \emptyset$, then $K_{k,n} \cap K_{k,o} \neq \emptyset$.

**Proof.** By (1.4), there exist $n_l, o_l$ for $l \geq k$ such that $\pi^{l-k}(l, n_l) = (k, n)$,
\[
\pi^{l-k}(l, o_l) = (k, o)
\]
and $d(x_{l,n_l}, x_{l,o_l}) < 2\alpha_3 r^l$. Since both $K_{k,n}$ and $K_{k,o}$ are complete, and both $\{x_{l,n_l}\}_{l \geq k}$ and $\{x_{l,o_l}\}_{l \geq k}$ are Cauchy sequences, these sequences converge to the same limit on $K_{k,o} \cap K_{k,n}$. □

**Proof of Proposition 2.2.** We inductively define $Q_{k,n}$.
\begin{itemize}
  \item For $(0, n) \in \Omega_0$, we define $Q_{0,1} = K_{0,1}$ and $Q_{0,n} = K_{0,n} \setminus \bigcup_{i=1}^{n-1} Q_{0,i}$.
  \item Let $(k, n) \in \bigcup_{l \geq 0} \Omega_l$. If $n = \min \{o \mid \pi(k, n) = \pi(k, o)\}$, then we define $Q_{k,n} = (Q_{\pi(k,n)} \cap K_{k,n})$. Otherwise,
\[
Q_{k,n} := (Q_{\pi(k,n)} \cap K_{k,n}) \setminus \{Q_{k,o} \mid o < n \text{ and } \pi(k, o) = \pi(k, n)\}.
\]
  \item For $(k, n) \in \bigcup_{l < 0} \Omega_l$, let $Q_{k,n} = \bigcup_{\omega \in \pi^{-1}(k,n)} Q_{\omega}$.
\end{itemize}

Since $x_{k,n} \in O_{k,n}$ and $x_{k,n} \in K_{k+1,m}$ for some $m$, $x_{k,n} \in \bigcup_{\omega \in \pi^{-1}(k,n)} K_\omega$. This with $\#(\pi^{-1}(k, w)) < \infty$ shows $K_{k,n} = \bigcup_{\omega \in \pi^{-1}(k,n)} K_\omega$. Therefore (D1) and (D3) follow. Moreover, since $O_\omega \subset O_{\pi(\omega)}$ by $K_\omega = \bigcup_{\lambda \in \pi^{-1}(\omega)} K_{\lambda}$, it inductively follows that for any $k \geq 0$ and $n \in \mathbb{N}$, it holds that $O_{k,n} \subset Q_{k,n} \subset K_{k,n}$.
• Let \( y \in O_{k,n} \) for some \((k,n) \in \bigcup_{1 \leq p} Q_p\). By definition of \( Q_{k,n} \), there exists \( o \in \mathbb{N} \) such that \( y \in Q_{0,o} \subseteq K_{0,o}\). This with \( y \in O_{k,n} \) assures \( \pi^{-k}(0,o) = (k,n) \) and \( O_{k,n} \subseteq Q_{k,n} \subset K_{k,n}\).

Therefore Lemma 2.4 leads to (D2). The condition (D1) immediately follows from Lemma 2.3.

Let \( y, z \in X \) with \( d(y,z) < (\alpha_3 - C_* - \alpha_4 r_0)^k \). By (1.3) and (D1), there exist \( n_0, n_1, n_2 \) such that \( d(y, x_{k,n_1}) < C_* r^k \), \( y \in K_{k,n_0} \), and \( z \in K_{k,n_2} \). Then,

• If \( d(y, x_{k,n_2}) < \alpha_3 r^k \), then \( K_{k,n_1} \cap K_{k,n_2} \neq \emptyset \) by Lemma 2.5.

• Otherwise, for \((k + 1, m_1) \in \pi^{-1}(k, n_2)\) with \( z \in K_{k+1,m_1}\),

\[
d(x_{k,n_2}, x_{k+1,m_1}) \geq d(y, x_{k,n_1}) - d(y, z) - d(z, x_{k+1,m_1}) > (\alpha_3 - (\alpha_3 - C_*) - \alpha_4 r_0) - \alpha_4 r^k \geq C_* r^k
\]

and

\[
d(x_{k,n_1}, x_{k+1,m_1}) \leq d(y, x_{k,n_1}) + d(y, z) + d(z, x_{k+1,m_1}) < (C_* + (\alpha_3 - C_*) - \alpha_4 r_0) + \alpha_4 r^k \leq \alpha_3 r^k.
\]

Therefore by (1.5), there exist \( m_2, m_3 \) such that \( \pi(k + 1, m_2) = (k, n_2) \), \( \pi(k + 1, m_3) = (k, n_1) \) and \( B(x_{k+1,m_2}, \alpha_3 r^k) \cap B(x_{k+1,m_3}, \alpha_3 r^k) \neq \emptyset \). This shows \( K_{k,n_1} \cap K_{k,n_2} \supset K_{k+1,m_2} \cap K_{k+1,m_3} \neq \emptyset \).

Similarly, if \( d(y, x_{k,n_0}) \geq \alpha_3 r^k \), then there exists \( m_4 \) such that \( \pi(k + 1, m_4) = (k, n_0) \), \( y \in K_{k+1,m_4} \), \( d(x_{k,n_0}, x_{k+1,m_4}) > C_* r^k \) and \( d(x_{k,n_1}, x_{k+1,m_4}) < \alpha_3 r^k \). Therefore \( K_{k,n_0} \cap K_{k,n_1} \neq \emptyset \) regardless of whether \( d(y, x_{k,n_0}) < \alpha_3 r^k \) or not, and (D1) follows.

We next prove that the assumptions of Proposition 2.2 actually hold under the assumption of Theorem 1.3. Let \( \gamma > 1 \) be the constant in Definition 1.2. In other words, for any \( y \in X \) and \( r > 0 \) with \( B(y, r) \neq X \) satisfies \( B(x, \gamma r) \setminus B(x, r) \neq \emptyset \).

**Proposition 2.6.** For any \( \alpha_1, \alpha_3 \) with

\[
\alpha_1 > \alpha_3 > (C_* \vee 1) \gamma, \tag{2.3}
\]

let \( \alpha_2 > 0 \) be sufficiently small such that

\[
(1 + \gamma) \alpha_2 < (\alpha_1 - \alpha_3) \wedge c_* \tag{2.4}
\]

and \( r_0 \in (0, 1) \) be also sufficiently small such that (2.1) and the following hold:

\[
r_0 C_* < \alpha_2 \tag{2.5}
\]

\[
(1 + \gamma)(\alpha_2 + \alpha_6 r_0 + C_* r_0) < c_* \tag{2.6}
\]

\[
(1 + \gamma)\alpha_2 + (4 + \gamma)\alpha_6 r_0 + (2 + \gamma)C_* r_0 < \alpha_1 - \alpha_3 \tag{2.7}
\]
where
\[ N = \sup_{y \in X, s > 0} \sup \{ \# A \mid A \subset B(y, \alpha_1 s), d(z_1, z_2) \geq c_s \}, \]
for any \( z_1, z_2 \in A \) with \( z_1 \neq z_2 \).

Remark. (1) \( N < \infty \) because \((X,d)\) is doubling.

(2) If \( r_0 \) satisfies inequalities (2.4), (2.5), (2.6) and (2.7), then any \( r \in (0, r_0) \) satisfies the same inequalities but replacing \( r_0 \) with \( r \).

Proof. We first divide each \( \Omega_{k+1} \) into three parts. We inductively define \( f_{k+1} : N \to N \cup \{ \infty \} \) by
\[
f_{k+1}(1) = \min \{ m \mid x_{k+1,m} \notin \bigcup_{\omega \in \Omega_k} B(x, (\alpha_2 + \alpha_6 r^k)^2) \},
\]
\[
f_{k+1}(i) = \min \{ m \mid x_{k+1,m} \notin \bigcup_{\omega \in \Omega_k} B(x, (\alpha_2 + \alpha_6 r^k)^2) \text{ and } d(x_{k+1,m, f_{k+1}(j)}, j) \geq 2\alpha_6 r^{k+1} \text{ for any } j < i \text{ with } f_{k+1}(j) < \infty \}
\]
and let
\[
\begin{align*}
\Omega_{k+1}^A &= \{ \omega \in \Omega_{k+1} \mid x_\omega \in B(x, \alpha_2 r^k) \text{ for some } \lambda \in \Omega_k \}, \\
\Omega_{k+1}^B &= \{ \omega \in \Omega_{k+1} \mid x_\omega \in B(x_{k+1, f_{k+1}(i)}, \alpha_6 r^{k+1}) \text{ for some } i \text{ with } f_{k+1}(i) < \infty \}, \\
\Omega_{k+1}^C &= \Omega_{k+1} \setminus (\Omega_{k+1}^A \cup \Omega_{k+1}^B).
\end{align*}
\]
Note that by definition of \( f_{k+1} \), \( \Omega_{k+1}^A \cap \Omega_{k+1}^B = \emptyset \). Now we define \( \pi|_{\Omega_{k+1}} \) on each part.

- For any \( \omega \in \Omega_{k+1}^A \), there exists unique \( \lambda \in \Omega_k \) such that \( d(x_\omega, x_\lambda) < \alpha_2 r^k \) due to (1.2) and \( 2\alpha_2 < c_s \) by (2.4). We define \( \pi(\omega) = \lambda \).

- For any \( \omega \in \Omega_{k+1}^C \), let
\[
\begin{align*}
\mathfrak{B}_\omega &= \min \{ n \mid d(x_\omega, x_{k,n}) < C_s r^k \} \quad \text{(2.8)}
\end{align*}
\]
then \( \mathfrak{B}_\omega < \infty \) due to (1.3). We set \( \pi(\omega) = (k, \mathfrak{B}_\omega) \).

- Let \( i \in N \) with \( f_{k+1}(i) < \infty \), then by definition of \( f_{k+1} \),
\[
\Omega_{k+1}^{B,i} := \{ \omega \in \Omega_{k+1} \mid x_\omega \in B(x_{k+1, f_{k+1}(i)}, \alpha_6 r^{k+1}) \}
\]

8
is disjoint. Let
\[
N_k^i = \{ n \in \mathbb{N} \mid d(x_{k+1, f_k+i}, x_{k,n}) < (\alpha_1 - \alpha_0 r)^k \},
\]
\[
P_k^i = \{ (p, q) \mid p, q \in N_k^i \text{ and } p < q \},
\]
then \#P_k^i \leq N(N - 1)/2. If P_k^i \neq \emptyset, we give an arbitrary order and write \( P_k^i = \{ (n_k^{i,2j-1}, n_k^{i,2j}) \mid 1 \leq j \leq \#P_k^i \} \). We also write \( \beta_j \) a sequence defined by the recurrence relation \( \beta_0 = 0 \) and \( \beta_j = \gamma \beta_{j-1} + (1 + 1)(2 + 1)C_* \). Note that \( \beta_j = C_*(1 + \gamma)(2 + 1)(\gamma^j - 1)/(\gamma - 1) \). Since \( \Omega_{k+1}^i \neq \emptyset \) by \([1,3]\) and \([2,7]\), \( B(x_{k+1, f_{k+i}+(i)}, \beta_j r^{j+1}) \neq X \) for any \( j \leq N(N - 1)/2 \).

Let \( 1 \leq j \leq \#P_k^i \), then there exist \( y, z \in X \) and \( a, b \in \mathbb{N} \) such that
\[
(\beta_{j-1} + (2 + 1)C_*) r^{j+1} \leq d((y, x_{k+1, f_{k+i}+(i)}), \gamma (\beta_{j-1} + (2 + 1)C_*) r^{j+1},
\]
\[
d((y, x_{k+1,a}), C_* r^{j+1},
\]
\[
C_* r^{j+1} \leq d((z, x_{k+1,a}), C_* \gamma r^{j+1},
\]
\[
d((z, x_{k+1,b}), C_* \gamma r^{j+1}
\]
because \((X, d)\) is uniformly perfect and \([1,3]\) holds. Note that
\[
z \in B(x_{k+1,a}, C_* \gamma r^{j+1}) \cap B(x_{k+1,b}, C_* \gamma r^{j+1}).
\]
We set \( m_{k+1}^{i,2j-1} = a \) and \( m_{k+1}^{i,2j} = b \).

Now we define \( \pi \) on \( \Omega_{k+1}^i \) (including cases of \( P_k^i = \emptyset \)) by
\[
\pi(k + 1, m) = \begin{cases} (k, n_{k+1}^{i,p}) \text{ if } m = m_{k+1}^{i,p} \text{ for some } p \leq 2\#P_k^i, \\ (k, m_{k+1}^{i,1}) \text{ otherwise} \end{cases}
\]
where \( m_{k+1}^{i,1} \) is as in \([2,8]\). Then \( \pi \) is well-defined because
\[
d(x_{k+1, m_{k+1}^{i,2j-1} + 1}, x_{k+1, m_{k+1}^{i,2j}}) > C_* r^{j+1} - C_* r^{j+1} = 0,
\]
\[
d(x_{k+1, f_{k+i}+(i)}, x_{k+1, m_{k+1}^{i,2j-1} - 1} \sqcup d(x_{k+1, f_{k+i}+(i)}, x_{k+1, m_{k+1}^{i,2j}})
\]
\[
< (\gamma (\beta_{j-1} + (2 + 1)C_*) + (2 + 1)C_*) r^{j+1} = \beta_j r^{j+1},
\]
\[
d(x_{k+1, f_{k+i}+(i)}, x_{k+1, m_{k+1}^{i,2j-1}} \sqcap d(x_{k+1, f_{k+i}+(i)}, x_{k+1, m_{k+1}^{i,2j}})
\]
\[
> (\beta_{j-1} + (2 + 1)C_*) \gamma r^{j+1} = \beta_{j-1} r^{j+1}
\]
hold for any \( 1 \leq j \leq \#P_k^i \), and so \( m_{k+1}^{i,p} \neq m_{k+1}^{i,q} \) for any \( p, q \leq 2\#P_k^i \) with \( p \neq q \). Moreover, \([2,10]\) also shows \( (k + 1, m_{k+1}^{i,p}) \in \Omega_{k+1}^{i,1} \) for any \( p \).

We next show \( \pi : \Omega \to \Omega \) satisfies conditions \([1,1]\) to \([1,5]\).

\([1,2]\) Let \((k + 1, m) \in \Omega_{k+1}^i \). If \( m = m_{k+1}^{i,p} \) for some \( i \) and \( p \), then
\[
d(x_{k+1 + 1, m}, x_{k+1, m}) \leq d(x_{k+1, m}, x_{k+1, f_{k+i}+(i)}) + d(x_{k+1, f_{k+i}+(i)}, x_{k, m_{k+1}^{i,p}})
\]
\[
< (\alpha_1 - \alpha_0 r)^k = \alpha_1 r^k.
\]
Otherwise, \([1,2]\) follows from \( \alpha_1 > C_* \).
It is obvious.

Let \( \omega \in \Omega_k \), then there exists \( \lambda \in \Omega_{k+1} \) with \( d(x_\omega, x_\lambda) < C_r r^{k+1} \) by (1.3). Since (1.3) and (2.5) hold, \( \pi(\lambda) = \omega \) and so \( \pi(\Omega_{k+1}) = \pi(\Omega) \). \( \sup_{\omega \in \Omega} \#\pi^{-1}(\omega) < \infty \) follows from (1.2) and (1.2) because \( (X,d) \) is doubling.

Assume \( y \in B(x_{k,n_1}, \alpha_3 r^k) \cap B(x_{k,n_2}, \alpha_3 r^k) \) for some \( y \in X \) with \( n_1 < n_2 \).

- If \( y \notin \bigcup \omega \in \Omega_k B(x_\omega, (\alpha_2 + \alpha_6 r + C_\ast r)r^k) \), set \( z = y \).
- Otherwise, let \( n_3 = \min\{ n \mid y \in B(x_{k,n}, (\alpha_2 + \alpha_6 r + C_\ast r)r^k) \} \). Since \( \{x_{k,n_1}, x_{k,n_2}\} \notin B(x_{k,n_3}, (\alpha_2 + \alpha_6 r + C_\ast r)r^k) \) by \( 2(\alpha_2 + \alpha_6 r + C_\ast r) < c_\ast \), there exists \( z \in X \) with \( (\alpha_2 + \alpha_6 r + C_\ast r)r^k \leq d(z, x_{k,n_3}) < \gamma(\alpha_2 + \alpha_6 r + C_\ast r)r^k \).

Then for any \( l \neq n_3 \),
\[
d(z, x_{k,l}) \geq d(x_{k,l}, x_{k,n_3}) - d(x_{k,n_3}, z) \\
\geq (c_\ast - \gamma(\alpha_2 + \alpha_6 r + C_\ast r))r^k > (\alpha_2 + \alpha_6 r + C_\ast r)r^k.
\]

In both cases, \( z \) satisfies \( d(x_\omega, z) > (\alpha_2 + \alpha_6 r + C_\ast r)r^k \) for any \( \omega \in \Omega_k \). Let \( m_0 = \min\{ m \mid d(x_{k+1,m}, z) < C_\ast r^{k+1} \} \), then \( (\alpha_2 + \alpha_6 r)r^k < d(x_{k+1,m_0}, x_{k+1,\omega}) \) for any \( \omega \in \Omega_k \), so there exists \( i \in \mathbb{N} \) such that \( 2\alpha_6 r^{k+1} > d(x_{k+1,m_0}, x_{k+1,f_{k+1}(i)}) \). Moreover,
\[
d(x_{k,n_1}, x_{k+1,f_{k+1}(i)}) \\
\leq d(x_{k,n_1}, y) + d(y, z) + d(z, x_{k+1,m_0}) + d(x_{k+1,m_0}, x_{k+1,f_{k+1}(i)}) \\
< (\alpha_3 + (1 + \gamma)(\alpha_2 + \alpha_6 r + C_\ast r) + C_\ast r + 2\alpha_6 r)r^k < (\alpha_1 - \alpha_6 r)r^k
\]
for \( l = 1, 2 \), by (2.7), therefore \( (n_1, n_2) \in P_{k+1}^i \). By definition of \( \pi \) on \( \Omega_{k+1}^B \), there exists \( j \leq \# P_{k+1}^i \) such that \( \pi(k+1, m^{i,2j}_{k+1}) = (k,n_1) \) and \( \pi(k+1, m^{i,2j}_{k+1}) = (k,n_2) \). Then
\[
B(x_{k+1,m^{i,2j-1}_{k+1}}, \alpha_3 r^{k+1}) \cap B(x_{k+1,m^{i,2j}_{k+1}}, \alpha_3 r^{k+1}) \neq \emptyset
\]
because (2.5) and \( C_\ast \gamma < \alpha_3 \) hold.

If \( d(x_{k+1,m_1}, x_{\pi(k+1,m_1)}) \geq C_r r^k \), then there exist \( i, p \) with \( m_1 = m^{i,p}_{k+1} \) by definition of \( \pi \). Additionally, if \( d(x_{k+1,m_1}, x_{k,n}) < \alpha_3 r^k \), then
\[
d(x_{k+1,f_{k+1}(i)}, x_{k,n}) < (\alpha_3 + \alpha_6 r)r^k < (\alpha_1 - \alpha_6 r)r^k.
\]
Therefore \( (n^{i,p}_{k+1}, n) \in P_{k+1}^i \) or \( (n, n^{i,p}_{k+1}) \in P_{k+1}^i \) holds if \( n \neq n^{i,p}_{k+1} \), and so we obtain desired \( m_2 \) and \( m_3 \) by definition of \( \pi \) on \( \Omega_{k+1}^B \), in the same way as the proof of (1.4).

Theorem 1.3 immediately follows from Propositions 2.2 and 2.6.
3 Application for evaluations of the Ahlfors regular conformal dimension

Here we introduce the framework and results of [13, 14] to give an application of our main theorem.

**Definition 3.1.** Let $T$ be a countable set and $\pi : T \to T$ be a map such that the following hold:

- Let $F_\pi = \{w \mid \pi^n(w) = w \text{ for some } n \geq 1\}$, then $\#F_\pi \leq 1$. (H1)
- For any $w, v \in T$, there exist $n, m \geq 0$ such that $\pi^n(w) = \pi^m(v)$. (H2)

Let $\phi \in F_\pi$ if $F_\pi \neq \emptyset$, otherwise we fix any $\phi \in T$. We call the triplet $(T, \pi, \phi)$ a tree with a reference point.

We justify this definition as follows.

**Lemma 3.2.** (1) Let $b(w, v) = \min\{n \geq 0\mid \pi^n(w) = \pi^n(v) \text{ for some } m \geq 0\}$ for $w, v \in T$, then $\pi_{b(w, v)}(w) = \pi_{b(v, w)}(v)$.

(2) Let $\mathcal{A} = \{(w, v) \mid \pi(w) = v \text{ or } \pi(v) = w\} \setminus \{(\phi, \phi)\}$, then $(T, \mathcal{A})$ is a tree.

**Remark.** Since (H2) holds, $b(w, v) < \infty$ for any $w, v \in T$.

**Proof.** (1) Assume $\pi_{b(w, v)}(w) \neq \pi_{b(v, w)}(v)$, then $\pi_{b(w, v)}(w) = \pi_{m_1}(v)$ and $\pi_{m_2}(w) = \pi_{b(v, w)}(v)$ for some $m_1 > b(v, w)$ and $m_2 > b(w, v)$. This shows $\pi_{b(w, v)}(w) = \pi_{m_1}(v) = \pi_{(m_1-b(v, w))+(m_2-b(w, v))}(\pi_{b(v, w)}(v))$ and $\pi_{b(v, w)}(v)$, which contradict (H1).

(2) (H2) assures that $(T, \mathcal{A})$ is connected. Let $(w_i)_{i=0}^n$ be a simple path from $w$ to $v$, that is, $w_0 = w$, $w_n = v$, $(w_{i-1}, w_i) \in \mathcal{A}$ for any $1 \leq i \leq n$ and $w_i \neq w_j$ for any $i \neq j$. Since $(w_i)_{i=0}^n$ is simple, there exist $0 \leq i_0 \leq n$ such that $\pi(w_i) = w_{i+1}$ for any $i < i_0$ and $w_i = \pi(w_{i+1})$ for any $i \geq i_0$. This shows $\pi^{i_0}(w) = \pi^{n-i_0}(v)$, and it also follows that $w_{b(w, v)} = w_{n-b(v, w)}$ because of (1). Therefore $i_0 = b(w, v)$ and $n = b(w, v) + b(v, w)$ by simplicity, so the simple path from $w$ to $v$ is unique.

**Remark.** If $F_\pi \neq \emptyset$, $(T, \mathcal{A}, \phi)$ coincides with “tree with a reference point” in the sense of [13, Definition 2.1.2].

Throughout this section, $T = (T, \pi, \phi)$ is a tree with a reference point. We also let $[w] = b(w, \phi) - b(\phi, w)$ for $w \in T$ and $(T)_k = \{w \in T \mid [w] = k\}$ for $k \in \mathbb{Z}$.

**Definition 3.3 (Partition).** Let $(Y, \rho)$ be a $(\sigma$-compact) metric space without isolated points, and $C(Y, \rho)$ be a set of all nonempty compact subsets of $(Y, \rho)$
except single points. We say \( K : T \to \mathcal{C}(Y, \rho) \) is a partition of \( Y \) (parametrized by \( T \)) if the following conditions hold.

- \( \bigcup_{w \in (T)_0} K(w) = Y \) and for any \( w \in T \), \( \bigcup_{v \in \pi^{-1}(w)} K(v) = K(w) \). (P1)
- For any sequence \( (w_k)_{k \in \mathbb{Z}} \) with \( \pi(w_{k+1}) = w_k \) for any \( k \in \mathbb{Z} \), \( \cap_{k \in \mathbb{Z}} K(w_k) \) is a single point. (P2)

Hereafter, we write \( K_w \) instead of \( K(w) \) for simplicity. In [13, Definition 2.2.1], the following condition is also included in the definition of partition.

- For any \( w \in T \), \( K_w \) has no isolated points. (P*)

**Lemma 3.4.** (P*) follows for any partition \( K \).

**Proof.** Assume \( x \in K_w \) be an isolated point of \( K_w \) for some \( w \in T \). By (P1), we can find \( (w_k)_{k \in \mathbb{Z}} \) with \( \pi(w_{k+1}) = w_k \) and \( x \in K_{w_k} \) for any \( k \in \mathbb{Z} \), and \( w_0 = w \). Since \( x \) is an isolated point of \( K_{w_k} \subset K_{w_0} \) for any \( k \geq 0 \) and \( K_{w_k} \) is not a single point, \( (K_{w_k} \setminus \{x\})_{k \geq 0} \) is a decreasing sequence of nonempty compact sets. Therefore \( \bigcap_{k \in \mathbb{Z}} K_{w_k} \setminus \{x\} = \bigcap_{k \geq 0} (K_{w_k} \setminus \{x\}) \neq \emptyset \), this contradicts (P2). \( \square \)

**Definition 3.5.** For \( w \in T \) and \( s \in (0, \infty) \), \( x, y \in X \) and \( M \geq 1 \), we define \( g_\rho(w) = \text{diam}(K_w, \rho) \),

\[
\Lambda^\rho_s = \begin{cases} 
\emptyset & \text{if } F_\pi \neq \emptyset \text{ and } s > g_\rho(\phi) \\
\{w \in T \mid g_\rho(w) \leq s < g_\rho(\pi(w))\} & \text{otherwise},
\end{cases}
\]

\[E^\rho_s = \{(w, v) \in \Lambda^\rho_s \times \Lambda^\rho_s \mid w \neq v \text{ and } K_w \cap K_v \neq \emptyset\},\]

\( l^\rho_s(\cdot, \cdot) \) is the graph distance of \( (\Lambda^\rho_s, E^\rho_s) \) and

\[
\delta^\rho_M(x, y) = \inf\{s > 0 \mid \text{there exist } w, v \in \Lambda^\rho_s \text{ such that } x \in K_w, y \in K_v \text{ and } l^\rho_s(w, v) \leq M\}.
\]

**Definition 3.6 (Basic framework).** Assume \( \sup_{w \in T} \#\pi^{-1}(w) < \infty \) and let \( K \) be a partition of \( (Y, \rho) \) such that for any \( w \in T \), \( K_w \setminus \bigcup_{v \in (T)_0; v \neq w} K_v \neq \emptyset \), and there exists an open set \( U_w \supset K_w \) with \( \#\{v \in (T)_{[w]} \mid U_w \cap K_v \neq \emptyset\} < \infty \). We say \( \rho \) satisfies basic framework with respect to \( K \) if the following conditions hold.

- (Adapted). There exists \( M \geq 1, \eta_1 > 0 \) such that

\[
\eta_1^{-1} \delta^\rho_M(x, y) \leq \rho(x, y) \leq \eta_1 \delta^\rho_M(x, y) \quad \text{for any } x, y \in Y. \quad \text{(B1)}
\]

- (Thick). There exists \( \eta_2 > 0 \) such that for any \( w \in T \),

\[
\{y \mid \delta^\rho_k(x_w, y) \leq \eta_2 g_\rho(\pi(w))\} \subset K_w \text{ with some } x_w \in K_w. \quad \text{(B2)}
\]

- (Uniformly finite). \( \sup_{w \in (0, \infty), w \in \Lambda^\rho_s} \#\{v \mid v \in \Lambda^\rho_s, l^\rho_s(w, v) \leq 1\} < \infty. \quad \text{(B3)}
\]

- There exist \( \eta_3 > 0 \) and \( r \in (0, 1) \) such that

\[
\eta_3^{-1} r^{[w]} \leq g_\rho(w) \leq \eta_3 r^{[w]} \quad \text{for any } w \in T. \quad \text{(B4)}
\]
The main result of [13] is the following.

**Theorem 3.7** ([13], Theorem 4.6.4, [14], Theorem 3.9). Let $K$ be a partition of $(Y, \rho)$, $E_k = \{(w, v) \in (T)_k \times (T)_k \mid w \neq v \text{ and } K_w \cap K_v \neq \emptyset\}$, $l_k$ be the graph distance of $(T)_k, E_k$ and

$$E_{p, k, w, M} = \inf \left\{ \frac{1}{2} \sum_{(u, v) \in E_{[w] + k}} |f(u) - f(v)|^p \mid f : (T)_{[w] + k} \to \mathbb{R}, f(u) = 1 \right\}$$ \[if \pi^k(w) = w \text{ and } f(u) = 0 \text{ if } l_{[w]}(w, \pi^k(u)) > m\].

If $\rho$ satisfies basic framework with respect to $K$, then

$$\dim_{AR}(Y, \rho) = \inf \{p \mid \limsup_{k \to \infty} \frac{E_{p, k, w, M}}{w \in T} = 0\},$$

where $M$ is the constant appearing in [B1].

We can adapt our main theorem to this framework. Recall that $(X, d)$ is a complete, doubling, uniformly perfect metric space.

**Proposition 3.8.** There exist a tree with a reference point $(T, \pi, \phi)$ and a partition $K$ of $(X, d)$ such that $d$ satisfies the basic framework with respect to $K$.

**Proof.** Fix any $x_* \in X$. Then by Lemma [2.1] we can take $\bigcup_{k \in \mathbb{Z}} \{x_w\}_{\omega \in \Omega_k}$ with [1.2], [1.3] and $x_* \in \{x_w\}_{\omega \in \Omega_k}$ for any $k \in \mathbb{Z}$. Let $Q_{\omega} \subset X (\omega \in \Omega)$ be sets given by Theorem [13]. We also let $k_0 = \max \{k \in \mathbb{Z} \mid \#\Omega_k = 1\}$ with $\max \emptyset = -\infty$ and $T = \bigcup_{k > k_0} \Omega_k$. If $k_0 > -\infty$ and $\omega \in \Omega_{k_0}$, we define $\pi(\omega) = \omega$. Otherwise, let $\pi(\omega)$ be the unique vertex in $\Omega_{k-1}$ such that $Q_{\omega} \subset Q_{\pi(\omega)}$: existence and uniqueness follow from [D1] and [D3]. Then $\pi$ satisfies the hypothesis [H1] and [H2] by [D1] and $x_* \in \bigcap_{k \in \mathbb{Z}} \{x_w \mid w \in \Omega_k\}$.

Let $K_{\omega} = Q_{\omega}$. Since every bounded sets on a doubling metric space is totally bounded, [D1], [D3] and [D4] with the uniformly perfect condition of $(X, d)$ show that $K$ is a partition of $(X, d)$.

[B3] also follows from these conditions. [B3], $\sup_{w \in T} \#\pi^{-1}(w) < \infty$, and $\#\{v \in (T)_{[w]} \mid U_w \cap K_v \neq \emptyset\} < \infty$ for some $U_w \supset K_w$ follow from [D4] and the doubling condition. By [D5], $\delta^2_I(x, y) \leq \eta d(x, y)$ for some $\eta_I > 0$, which imply [B1] because of [B3] Theorem 2.4.5. [B1], [B3] and [D3] also imply [B2]. Finally, $K_w \setminus \bigcup_{v \in (T)_{[w]} \setminus \pi(w)} K_v \neq \emptyset$ follows from [D2] because of [B3] Proposition 2.2.3 and [B4] Lemma 3.10.

**Theorem 3.9.** Let $(Y, \rho)$ be a complete metric space without isolated points. Then the following conditions are equivalent.

1. $(Y, \rho)$ is doubling and uniformly perfect.
2. $\dim_{AR}(Y, \rho) < \infty$.
3. There exist a tree with a reference point and a partition $K$ of $(Y, \rho)$ such that $\rho$ satisfies the basic framework with respect to $K$.  

13
Proof. 

((1) ⇔ (2)) This follows from [10, Theorem 13.3 and Corollary 14.15].

((1) ⇒ (3)) It is shown in Proposition 3.8.

((3) ⇒ (1)) By [13, Proposition 4.3.1], we may assume $r < \eta_3^{-2}$ for constants in the basic framework, and so we may take $\Lambda_{r,\rho, k} = (T)_k$ without loss of generality. If $A \subset B(x, 2r)$ satisfies $d(y, z) \geq r$ for any $y, z \in A$ with $w \neq z$, [11] and [14] imply that there exists $k \in \mathbb{N}$ independent of $x$ and $r$, and there exist $\omega$ and $\lambda_y \in Y(y \in A)$ such that $\lambda_y \in (T)_{[\omega]+k}$, $l_{[\omega]}(\omega, \pi_k(\lambda_y)) \leq M$ and $\lambda_y \neq \lambda_z$ if $y \neq z$. Therefore $\#A \leq (\sup_{\omega \in T} \#\pi^{-1}(\omega))^{k} (\sup_{s,\omega} \#\{v \mid v \in \Lambda_{s, t} \cap \{w, v \leq 1\})^{M}$ and so $(Y, \rho)$ is doubling. On the other hand, [13, Lemma 3.6.4] shows $(Y, \rho)$ is uniformly perfect. \qed

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