Decomposing Björner’s Matrix
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Abstract: We give an alternative proof of a (former) conjecture of Björner stating that the matrix expressing face numbers in terms of g numbers is totally non-negative. We briefly discuss the case of simple flag polytopes.

Let $f(t) = \sum f_i t^i$ denote the f-polynomial of a $d$-dimensional simple polytope. The coefficients ($f_i$ denotes the number of codimension $i$ faces of the polytope) are called face numbers of the polytope. It is a consequence of Dehn-Somerville relations that $f(t)$ can be written as a linear combination of polynomials $u_i(t) = \sum_{q=i}^{d-i} (1 + t)^q$. The coefficients of the corresponding expansions are denoted $g_i$, i.e.

$$f(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i u_i(t).$$

In particular there exists a matrix $M_{d}^{(g)}$ such that

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_d \end{pmatrix} = M_{d}^{(g)} \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{\lfloor d/2 \rfloor} \end{pmatrix}.$$  

It was conjectured by Björner that $M_{d}^{(g)}$ is totally non-negative, i.e. all its minors are non-negative. This was proven by Björklund and Engström [BE].

Björner already proved [Bj] that two-by-two minors of $M_{d}^{(g)}$ are non-negative and used it to refine lower and upper bounds for simple polytopes.

In this note we present another proof which shows that $M_{d}^{(g)}$ is totally non-negative.

Since face and g-numbers are related as follows

$$\sum f_i t^{d-i} = \sum_{k=0}^{\lfloor d/2 \rfloor} g_k \sum_{j=k}^{d-k} (1 + t)^j$$

$$= \sum_{k=0}^{\lfloor d/2 \rfloor} g_k \sum_{i} \sum_{j=k}^{d-k} \binom{j}{d-i} t^{d-i}$$

$$= \sum_{i} t^{d-i} \sum_{k=0}^{\lfloor d/2 \rfloor} g_k \left( \binom{d-k+1}{d-i+1} - \binom{k}{d-i+1} \right).$$

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We have

\[ f_i = \sum_{k=0}^{\lfloor d/2 \rfloor} M_d^{(g)}(i, k)g_k \]

with

\[ M_d^{(g)}(i, k) = \binom{d-k+1}{d-i+1} - \binom{k}{d-i+1}. \]

1. Four matrices.

Let us define four infinite matrices:

\[ A\pm(i, j) = \binom{j+1}{i-j} \mp \binom{j}{i-j-1}, \]
\[ G_+(j, k) = \frac{2k+1}{2j+1} \binom{k+j}{2j}, \]
\[ G_-(j, k) = \binom{k+j+1}{2j+1}. \]

In all the cases we assume that \( i, j, \) and \( k \) run through natural numbers including zero.

**Proposition.** The matrices \( A_+, A_-, G_+, \) and \( G_- \) are totally non-negative.

**Proof:** First notice that

\[ A_+(i, j) = \binom{j}{i-j}, \quad A_-(i, j) = \frac{i+1}{j+1} \binom{j+1}{i-j}. \]

It is quite standard that

\[ \binom{j}{i-j}, \quad \binom{j+1}{i-j}, \quad \binom{k+j}{2j}, \quad \binom{k+j+1}{2j+1} \]

are totally non-negative. For example \( \binom{k+j}{2j} \) counts the number of paths from \((-2j, j)\) to \((0, k)\) where only steps \((1, 0)\) or \((0, 1)\) are allowed. (Cf. [FZ, Lemma 1].)

When \( X(i, j) \) is totally non-negative, then \( a(i)X(i, j)b(j) \) is totally non-negative provided \( a \) and \( b \) are non-negative sequences. Thus the proof. □
2. Decomposition of the Matrix $M^{(g)}$

**Theorem.** Let $\epsilon = (-)^d$ and $n = [d/2]$. One has

$$M^{(g)}_d(i, k) = \sum_{j=0}^{n} A_\epsilon(i, j) G_\epsilon(n - j, n - k).$$

**Remark.** Since total non-negativeness is preserved under reversing the order of both indices, and multiplication of such matrices total non-negativity of $M^{(g)}_d$ follows immediately. We believe that the above formula is of independent interest.

**Proof of the Theorem:** It reduces to:

$$\left(\frac{2n - k + 1}{2n - i + 1}\right) - \left(\frac{k}{2n - i + 1}\right) = \sum_{j=k}^{n} \left(\frac{j}{i - j}\right) \frac{2n - 2k + 1}{2n - 2j + 1} \left(\frac{2n - (k + j)}{2n - 2j}\right)$$

(1)

$$\left(\frac{2n - k + 2}{2n - i + 2}\right) - \left(\frac{k}{2n - i + 2}\right) = \sum_{j=k}^{n} \left(\frac{j + 1}{i - j}\right) \frac{2n - (k + j) + 1}{2n - 2j + 1}$$

(2)

Let us provide a proof of the first identity. The second can be obtained in a very similar way.

Multiply both sides by $s^i$ and sum over $0 \leq i \leq 2n + 1$. Left-hand side of (1) becomes

$$s^{2n+1}(1 + 1/s)^{2n-k+1} - s^{2n+1}(1 + 1/s)^k = (1 + s)^{2n+1} \left(\frac{s}{1 + s}\right)^k - s^{2n+1} \left(\frac{1 + s}{s}\right)^k.$$

The right-hand side of (1) becomes

$$\sum_{j=k}^{n} (s(s + 1))^j \frac{2n - 2k + 1}{2n - 2j + 1} \left(\frac{2n - (k + j)}{2n - 2j}\right).$$

Let

$$F_a(z) = \sum_{j=0}^{a} z^j \frac{2a + 1}{2a - 2j + 1} \left(\frac{2a - j}{2a - 2j}\right).$$

Then left-hand side of (1) equals to $(s(s + 1))^k F_{n-k}(s(s + 1))$. It is straightforward to check that

$$z(4z + 1)F_a''(z) + a(z - a(4z + 1))F_a'(z) + 2a(2a + 1)F_a(z) = 0.$$

If $G_a(s) = F_a(s(s + 1))$ then

$$s(s + 1)G_a''(s) - 2a(1 + 2s)G_a'(s) + 2a(2a + 1)G(s) = 0.$$
The above equation is solved also by $(s + 1)^{2a+1}$ and $s^{2a+1}$. Since $G_a(s)$ and $(s + 1)^{2a+1} - s^{2a+1}$ have the same (non-zero) coefficients at $s$ and $s^{2a}$ we conclude that

$$G_a(s) = (s + 1)^{2a+1} - s^{2a+1},$$

and, what follows the left-hand side of (1) is equal to

$$(s(s + 1))^k G_{n-k}(s) = \left(\frac{s}{1 + s}\right)^k (s + 1)^{2n+1} - \left(\frac{1 + s}{s}\right)^k s^{2n+1}.$$

Which completes the proof. □

**Question.** The matrices $G_{e(n-j,n-k)}$ depend on dimension mildly. And they provide, applied to g-numbers another (natural) expansion of of f-polynomial. Does this expansion is combinatorially/geometrically useful?

3. **Flag polytopes and $\gamma$-matrix.**

If the polytope in question is flag (ie. all pairwise intersecting families of faces are centered) an interesting expansion of the f-vector is given in terms of polynomials $v_i(t) = (t + 1)^i(t + 2)^{d-2i}$, ie.

$$f(t) = \sum \gamma_i v_i(t)$$

(cf [Br,G].)

Unsolved question is whether the coefficients $\gamma_i$ are non-negative in this case ([G, Conjecture 2.1.7]). As before we can write down the matrix which computes the coefficients of f-polynomial in terms of $\gamma_i$:

$$\sum f_i t^{d-i} = \sum_{k=0}^{[d/2]} \gamma_k (1 + t)^k (2 + t)^{2k}$$

$$= \sum_{k=0}^{[d/2]} \gamma_k \sum_{p} \sum_{q} \binom{k}{p} \binom{d - 2k}{q} 2^q t^{k-p+d-2k-q}$$

$$= \sum_i t^{d-i} \sum_{k=0}^{[d/2]} \gamma_k \sum_{q} \binom{k}{i-k-q} \binom{d - 2k}{q} 2^q.$$

Thus we have

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_d \end{pmatrix} = M_d^{(\gamma)} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{[d/2]} \end{pmatrix},$$

with

$$M_d^{(\gamma)}(i,k) = \sum_j \binom{k}{i-k-j} \binom{d - 2k}{j} 2^j.$$
One can also express \( g_i \) by \( \gamma_j \):

\[
g_i = \sum_{0 \leq j \leq i} \left( \binom{n - 2j}{i - j} - \binom{n - 2j}{i - 1 - j} \right) \gamma_i = \sum_{0 \leq j \leq i} \binom{n - 2j + 1}{i - j} \frac{n - 2i + 1}{n - 2j + 1} \gamma_i.
\]

Since the matrix with entries \( \binom{n-2j+1}{i-j} \) is easily seen to be totally non-negative we see that \( M_d^{(\gamma)} \) is totally non-negative as \( M_d^{(g)} \) is such. This could be also proven directly by checking that

\[
M_d^{(\gamma)}(i, k) = \sum_{j=0}^{n} A_\varepsilon(i, j) \Gamma(n - j, n - k),
\]

where \( n = \lfloor d/2 \rfloor \), \( \varepsilon = (-)^d \), and

\[
\Gamma(j, k) = 4^{k-j} \binom{k}{j}.
\]

**Question.** Can one use non-negativity of (say, two-by-two) minors of \( M_d^{(\gamma)} \) to get a (depending on conjectural non-negativity of \( \gamma \) numbers) upper/lower bound for face numbers of flag simple polytopes?

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