SCAFFOLDS AND INTEGRAL HOPF GALOIS MODULE STRUCTURE ON PURELY INSEPARABLE EXTENSIONS

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Abstract. Let $p$ be prime. Let $L/K$ be a finite, totally ramified, purely inseparable extension of local fields, $[L : K] = p^n$, $n \geq 2$. It is known that $L/K$ is Hopf Galois for numerous Hopf algebras $H$, each of which can act on the extension in numerous ways. For a certain collection of such $H$ we construct “Hopf Galois scaffolds” which allow us to obtain a Hopf analogue to the Normal Basis Theorem for $L/K$. The existence of a scaffold structure depends on the chosen action of $H$ on $L$. We apply the theory of scaffolds to describe when the fractional ideals of $L$ are free over their associated orders in $H$.

1. Introduction

Let $L/K$ be a totally ramified extension of local fields of degree $p^n$, where the residue field of $K$ has characteristic $p$. Suppose further that $L/K$ is Galois with $G = Gal(L/K)$. Let $\mathcal{O}_K$ and $\mathcal{O}_L$ denote the valuation rings of $K$ and $L$ respectively. There are two natural ways to describe the elements of $L$, namely by using its valuation $v_L$ or by using its Galois action. If $\pi \in L$ is a uniformizing parameter, then every element of $L$ is a $K$-linear combination of powers of $\pi$; computing its valuation is a simple process. A drawback of the valuation representation of $L$ is that the Galois action is not necessarily transparent.

Alternatively, we have the Normal Basis Theorem, which asserts that there exists a $\rho \in L$ whose Galois conjugates form a $K$-basis for $L/K$; equivalently, $L$ is a free rank one module over the group algebra $KG$. Here, every element of $L$ is a $K$-linear combination of $\{\sigma(\rho) : \sigma \in G\}$, which allows for a simple description of the Galois action; however, the valuation representation is not transparent, making certain Galois module theory questions difficult to answer. For example, $\mathcal{O}_L$ is an $\mathcal{O}_K G$-module, however by Noether’s Theorem $\mathcal{O}_L$ is not free of rank one if $L/K$ is wildly ramified. The $\mathcal{O}_K G$-module structure of $\mathcal{O}_L$ when $\mathcal{O}_L$ does not possess a normal integral basis can be more difficult. A typical strategy, thanks to Leopoldt [Leo59] is to replace $\mathcal{O}_K G$ with a larger $\mathcal{O}_K$-subalgebra of $KG$, namely $\mathfrak{A} = \{\alpha \in KG : \alpha(\mathcal{O}_L) \subset \mathcal{O}_L\}$, which also acts on $\mathcal{O}_L$; the structure of $\mathcal{O}_L$ as an $\mathfrak{A}$-module can be simpler to describe.

In an attempt to unite these representations, G. Griffith Elder [Eld09] first developed a theory of “Galois scaffolds”. In that work a Galois scaffold consists of a subset $\{\theta_1, \theta_2, \ldots, \theta_n\}$ of $KG$, together with a positive integer $v$, called an integer

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certificate, such that \( \{ v_L(\theta_i^j(\rho)) : 1 \leq i \leq n, \ 0 \leq j \leq p-1 \} \) is a complete set of residues mod \( p^n \) where \( \rho \in L \) is any element of valuation \( v \). Certainly, \( \{ \theta_i^j(\rho) \} \) forms a \( K \)-basis for \( L \), and this basis facilitates the study of both valuation and Galois action, particularly if \( \theta_i^j = 0 \) for all \( i \). A simple example of a Galois scaffold arises when \( n = 1 \) and the break number \( b \) is relatively prime to \( p \); in this case, if \( G = \langle \sigma \rangle \) then \( \theta_i^1 = \sigma - 1, \ v = b \) is an example of a Galois scaffold. Such scaffolds do not always exist – in fact, integer certificates may not exist, for example if \( L/K \) is unramified and \( p^b = 1 \) [Bry71]. This notion of scaffold was refined in [BE13], and then again in [BCE14], the latter version being the most useful for describing the integral Galois module structure.

The version in [BCE14] is also the most general as it does not insist that \( L/K \) be Galois, merely that there is a \( K \)-algebra \( A \) which acts on \( L \) in a very reasonable way. A classic example of such an algebra is a \( K \)-Hopf algebra. There are many more Hopf Galois extensions than Galois extensions. For example, any Galois extension is Hopf Galois for at least one Hopf algebra (namely, \( H = KG \)) and, if \( n \geq 2 \), many more: the exact determination of the number of such \( H \) is a group theory problem thanks to [GP87], which covers all separable extensions. At the other extreme, if the extension \( L/K \) is purely inseparable, then it is also Hopf Galois [Char70]; if \( [L : K] \geq p^2 \), then there are numerous Hopf algebras which make \( L/K \) Hopf Galois [Koc13].

In the setting where \( L/K \) is Hopf Galois with Hopf algebra \( H \), one can study the structure of \( \mathcal{O}_L \) as an \( H \)-module. Given [BCE14], a natural approach would be an attempt to construct an \( H \)-scaffold which, loosely, consists of \( \{ \lambda_t : t \in \mathbb{Z} \} \subset L \) with \( v_L(\lambda_t) = t \), along with \( \{ \Psi_i : 0 \leq i \leq n-1 \} \subset H \) such that \( \Psi_i \) acts on \( \lambda_t \) in a manner which makes \( v_L(\Psi_i(\lambda_t)) \) easy to compute.

Here, we focus on the case where \( L/K \) is a totally ramified, purely inseparable extension of local fields, \( [L : K] = p^n, \ n \geq 2 \). We take a collection of Hopf algebras \( H \) which make \( L/K \) Hopf Galois and describe the generalized integral Hopf Galois module structure of \( \mathcal{O}_L \). The integral Hopf Galois module structure we seek is a description of all of the fractional ideals of \( L \) as \( H \)-modules. In detail, each fractional ideal of \( L \) is of the form \( \mathfrak{P}_L^h \) for \( h \in \mathbb{Z} \), where \( \mathfrak{P}_L \) is the maximal ideal of \( \mathcal{O}_L \). In other words, \( \mathfrak{P}_L^h = \{ x \in L : v_L(x) \geq h \} \). For each \( h \) we let \( \mathfrak{A}_h \) be the largest subset of \( H \) which acts on \( \mathfrak{P}_L^h \), i.e.,
\[
\mathfrak{A}_h = \{ \alpha \in H : \alpha \mathfrak{P}_L^h \subset \mathfrak{P}_L^h \}.
\]
We call \( \mathfrak{A}_h \) the associated order of \( \mathfrak{P}_L^h \) in \( H \): it is clearly an \( \mathcal{O}_K \)-subalgebra of \( H \) and \( \mathfrak{A}_h \otimes_{\mathcal{O}_K} K \cong H \). By construction, \( \mathfrak{A}_h \) acts on \( \mathfrak{P}_L^h \); the existence of the scaffold allows for a numerical criterion for determining whether \( \mathfrak{P}_L^h \) is a free \( \mathfrak{A}_h \)-module. The criterion itself is independent of the scaffold, provided the scaffold exists.

The paper is organized as follows. After giving a definition of an \( H \)-scaffold, a simpler version than the one in [BCE14], we consider the family of monogenic \( K \)-Hopf algebras \( H_{n,r,f} \), \( 1 \leq r \leq n-1, \ f \in K^\times \) introduced in [Koc14], which make \( L \) an \( H_{n,r,f} \)-Galois object. We examine the case where \( 2r \geq n \) and consider actions of the linear dual \( H := H_{n,r,f}^* \) which give \( L/K \) the structure of a Hopf Galois extension. A subtlety that arises is that \( H \) possesses an infinite number of actions on \( L \); in each case, \( L/K \) is \( H \)-Galois. The different actions correspond with different choices for \( K \)-algebra generator \( x \in L \); and for each choice of \( x \) we
Definition 2.1. Let $\mathbb{K}$ be an integer. An $H$-scaffold will allow us to consider the effect of the action on the valuation of some specially chosen elements, and using [BCE14 Th 3.1, 3.7] we will use it to describe the integral Hopf Galois module structure. We will then focus on a specific action for which an $H$-scaffold exists, and explicitly describe which fractional ideals $\mathfrak{P}_L^b$ are free over their associated orders. We conclude with some remarks concerning selecting the “best” choices of $r$ and $f$, and the action on $L$, for answering integral Hopf Galois module theory questions.

The evident purpose of this work is to construct $H$-scaffolds. However, our results contribute to the bigger picture of scaffolds. The definition of a scaffold has evolved significantly since Elder’s 2009 paper, which required many scaffolds exist. But we will see that in the finite purely inseparable case, $\mathfrak{P}_L^b$-scaffold will allow us to consider the effect of the action on the valuation of some specially chosen elements, and using [BCE14 Th 3.1, 3.7] we will use it to describe the integral Hopf Galois module structure. We will then focus on a specific action for which an $H$-scaffold exists, and explicitly describe which fractional ideals $\mathfrak{P}_L^b$ are free over their associated orders. We conclude with some remarks concerning selecting the “best” choices of $r$ and $f$, and the action on $L$, for answering integral Hopf Galois module theory questions.

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Throughout, we fix an integer $n \geq 2$ and $L$ a totally ramified purely inseparable extension of $K = \mathbb{F}_{q^n}((T))$ of degree $p^n$. Let $v_K$ be the $T$-adic valuation, $v_L$ the extension of $v_K$ to $L$. Write $L = K(x)$, $x^p^n = \beta \in K$, $v_K(\beta) = -b < 0$, $p \mid b$. We let $H$ and $H_{n,r,f}$ be as above, and we assume $2r \geq n$.

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2. Scaffolds

The definition of an $A$-scaffold in [BCE14] is very general – more so than we need here. We will simplify this definition as much as possible, and since our acting $K$-algebra is a Hopf algebra we will refer to it as an $H$-scaffold.

Definition 2.1. Let $a$ be an integer such that $ab \equiv -1 \mod p^n$. Let $\Sigma > 1$ be an integer. An $H$-scaffold on $L$ of tolerance $\Sigma$ consists of:

1. A set \{\lambda_j : j \in \mathbb{Z}, v_L(\lambda_j) = j\} of elements of $L$ such that $\lambda_{j_1}^{-1}\lambda_{j_2} \in K$ when $j_1 \equiv j_2 \mod p^n$.
2. A collection \{\Psi_s : 0 \leq s \leq n - 1\} of elements in $H$ such that $\Psi_s(1_K) = 0$ for all $s$ and, mod $\lambda_{j+p^n}\mathfrak{P}_L^k$,

$$\Psi_s(\lambda_j) = \begin{cases} u_{s,j}^{-1}\lambda_{j+p^n} & \text{res}(\lambda_j)_s > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $u_{s,j} \in \mathfrak{O}_K$, $\text{res}(\lambda_j)$ is the least nonnegative residue of $\lambda_j \mod p^n$, and

$$\text{res}(\lambda_j) = \sum_{s=0}^{n-1} \text{res}(\lambda_j)_s p^s, \ 0 \leq \text{res}(\lambda_j)_s \leq p - 1$$

is the $p$-adic expansion of $\text{res}(\lambda_j)$.

Given an $H$-scaffold, we know the effect of applying $\Psi_s$ to $\lambda_j$, provided $\text{res}(\lambda_j)_s > 0$. For $0 < i < p - 1$ it can be readily seen that $\text{res}(\lambda_{j+p^n})_s = p - i > 0$, hence $\Psi_s^i(\lambda_j) = u\lambda_{j+p^n+s} \mod \lambda_{j+p^n}^i\mathfrak{P}_L^k$ for some $u \in \mathfrak{O}_K^\times$. More generally,

$$v_L(\Psi_s^i_0^2\Psi_s^1^1\cdots\Psi_s^{n-1}) = b + b \sum_{s=0}^{n-1} s p^s, \ 0 \leq s \leq p - 1.$$
By allowing the \( \{i_s\} \) to vary, we obtain \( p^n \) elements of \( L \), pairwise incongruent modulo \( p^n \), hence \( \{ \Psi_0^i \Psi_1^j \cdots \Psi_{n-1}^{i_{n-1}} (\lambda_s) : 0 \leq i_s \leq p - 1 \} \) is a \( K \)-basis for \( L \).

We will use the result below to construct our \( H \)-scaffolds.

**Lemma 2.2.** Suppose we have \( \{ \Psi_s : 0 \leq s \leq n - 1 \} \subset H \) such that, for \( i \leq p^n - 1 \),
\[
i = \sum_{s=0}^{n-1} i_s p^n,
\]
for some \( \xi > 1 \). Let
\[
\lambda_j = T^{(j + b \text{res}(a_j))/p^n} x^{\text{res}(a_j)}.
\]
Then \( \{ \lambda_j \}, \{ \Psi_s \} \) form a scaffold of tolerance \( \xi \).

**Proof.** First, since \( v_L (x) = -b \),
\[
v_L (\lambda_j) = j + b \text{res}(a) - b \text{res}(a) = j,
\]
and clearly \( v_L (\lambda_j, \lambda_j^{-1}) = j_1 - j_2 \), so condition (1) of the definition above is satisfied. Next, we have
\[
\Psi_s (\lambda_j) = \Psi_s \left( T^{(j + b \text{res}(a_j))/p^n} x^{\text{res}(a_j)} \right)
= T^{(j + b \text{res}(a_j))/p^n} \Psi_s \left( x^{\text{res}(a_j)} \right)
= T^{(j + b \text{res}(a_j))/p^n} \text{res}(a_j) x^{\text{res}(a_j) - p^s} \mod x^{\text{res}(a_j) - p^s} \Psi_L^\xi
\]
If \( \text{res}(a_j) = 0 \) then \( \Psi_s (\lambda_j) = 0 \). Otherwise, \( a (j + bp^s) \equiv a - p^s \mod p^n \) and
\[
\text{res} (a (j + bp^s)) = \text{res} (a - p^s)
= \text{res} (a) - p^s,
\]
the latter equality since \( \text{res}(a_j) \geq p^s \). Thus \( \text{res}(a_j) = p^s + \text{res} (a (j + bp^s)) \) and so
\[
 j + b \text{res}(a) = j + b (p^s + \text{res}(a (j + bp^s)))
= j + b \text{res}(a (j + bp^s)) + bp^s,
\]
giving
\[
\text{res}(a_j) T^{(j + b \text{res}(a_j))/p^n} x^{\text{res}(a_j) - p^s} = \text{res}(a_j) T^{(j + b \text{res}(a (j + bp^s)) + bp^s)/p^n} x^{\text{res}(a (j + bp^s))}
= \text{res}(a_j) \lambda_j + bp^s.
\]
Setting \( u_{s,j} = \text{res}(a_j) \) shows that (2) is also satisfied. \( \square \)

**Remark 2.3.** By adjusting each \( \lambda_j \) by a scalar it is possible to have \( u_{s,j} = 1 \). This is the primary difference between the construction above and the one found in [BCE14, Sec. 5.3].

In the work to follow, we will use the definition of \( H \)-scaffold given by the description in Lemma 2.2. As the choice of \( \{ \lambda_j \} \) will remain fixed (assuming a constant \( b \)), we will refer to the scaffold as \( \{ \Psi_s \} \).
3. The Hopf Algebra Structure

In this section, we introduce the class of Hopf algebras we will use to construct our \( H \)-scaffolds. To do so, we first recall a family of Hopf algebras introduced in \( [Koc14] \). For \( 0 < r < n \leq 2r \) and \( f \in K^\times \), let \( H_{n,r,f} \) be the \( K \)-Hopf algebra whose \( K \)-algebra structure is \( H_{n,r,f} = K[t]/(t^r) \); whose counit and antipodal map are \( \varepsilon(t) = 0 \) and \( \lambda(t) = -t \) respectively; and whose comultiplication is

\[
\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} t^{p'\ell} \otimes t^{p'(p-\ell)}.
\]

Let us fix values for \( r, n, \) and \( f \) as above, and let \( H = H^*_{n,r,f} \). Certainly, \( H \) has a \( K \)-basis \( \{z_0 = 1, z_1, \ldots, z_{p^n-1}\} \) with \( z_i : H \to K \) given by

\[
z_j(t^i) = \delta_{i,j},
\]

where \( \delta_{i,j} \) is the Kronecker delta. The algebra structure on \( H \) is induced from the coalgebra structure on \( H_{n,r,f} \); explicitly,

\[
z_{j_1} z_{j_2}(h) = \text{mult}(z_{j_1} \otimes z_{j_2}) \Delta(h).
\]

In this section we will show that \( \{z_p^s : 0 \leq s \leq n - 1\} \) generate \( H \) as a \( K \)-algebra. This set will be (part of) the scaffolds we develop.

We start by recalling a result which will facilitate the study of the algebra structure of \( H \) as well as the action of \( H \) on \( L \).

Lemma 3.1. Let

\[
S_f(u,v) = u + v + \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} u^{p'\ell} v^{p'(p-\ell)}.
\]

Then, for every positive integer \( i \), \( S_f(u,v)^i \) is an \( K^\times \)-linear combination of elements of the form

\[
f^{i_3} u^{i_1} v^{i_2} + p' \ell' v^{i_2} + p'' \ell'',
\]

where

\[
i = i_1 + i_2 + i_3,
\]

\[
\ell' = i_{3,1} + 2i_{3,2} + \cdots + (p-1)i_{3,p-1}
\]

\[
\ell'' = (p-1)i_{3,1} + (p-2)i_{3,2} + \cdots + i_{3,p-1},
\]

and \( i_{3,1} + i_{3,2} + \cdots + i_{3,p-1} = i_3 \).

Proof. This is a straightforward calculation from \( [Koc14] \) Lemma 5.1 – we recall it here for the reader’s convenience.

We have

\[
S_f(u,v)^i = \left(u + v + \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} u^{p'\ell} v^{p'(p-\ell)} \right)^i
\]

\[
= \sum_{i_1+i_2+i_3=i} \binom{i}{i_1,i_2,i_3} (u^{i_1} v^{i_2}) \left( f \sum_{\ell=1}^{p-1} \frac{1}{\ell!(p-\ell)!} u^{p'\ell} v^{p'(p-\ell)} \right)^{i_3}.
\]
The last factor in each summand can be expanded as

\[ f_{i_3} \sum_{i_3, i_3 + \cdots + i_{3,p} - 1 = i_3} \left( \left( \begin{array}{c} i_3 \\ i_{3,1}, \ldots, i_{3,p} - 1 \end{array} \right) \prod_{j=1}^{p-1} \frac{1}{i_{3,j}! (p - i_{3,j})!} \right) \left( t_{i_1 + p^r _2 \ell + i_2 + p^r _2 \ell''} \right). \]

The result follows. \[ \square \]

Next, we consider powers of the \( z_{p^r} \)'s.

**Lemma 3.2.** For \( 0 \leq s \leq r \), \( 1 \leq m \leq p - 1 \); or \( 0 \leq s \leq r - 1 \), \( m = p \) we have \( z_{p^r}^m = m! z_{mp^r} \). In particular, \( z_{p^r}^p = 0 \).

**Proof.** See \[\text{Koc14}\] Lemmas 5.2, 5.3. While the result there was for \( n = r + 1 \), its validity depended on the form of the comultiplication; the more general \( 2r \geq n \) case the comultiplication has the same form, and hence a nearly identical proof. \[ \square \]

The result above does not hold for \( s > r \). However, we do have

**Lemma 3.3.** For \( 0 \leq s \leq n - 1 \), \( 1 \leq j, m \leq p - 1 \), we have \( z_{p^r}^j \left( t_{mp^r} \right) = m! \delta_{j,m} \).

Furthermore, if \( s \geq r \) then \( z_{p^r}^j \left( t_{mp^r} \right) = f^{p^{r-1}} \delta_{i,s-r} \). In particular, \( z_{p^r}^p \neq 0 \).

**Proof.** Certainly, if \( s < r \) then the result follows from the previous lemma. Thus, we will assume that \( s \geq r \). The statement \( z_{p^r}^j \left( t_{mp^r} \right) = m! \delta_{j,m} \) is clearly true for \( j = 1 \). Suppose \( z_{p^r}^{j-1} \left( t_{(m-1)p^r} \right) = \delta_{j,m-1} \). Since \( s + r \geq n \) we have that \( t_{mp^r} \) is a primitive element, hence

\[ z_{p^r}^j \left( t_{mp^r} \right) = \text{mult} \left( z_{p^r}^{j-1} \otimes z_{p^r} \right) \left( t_{mp^r} \otimes 1 + 1 \otimes t_{p^r} \right)^m \]

\[ = \sum_{i=0}^{m} \binom{m}{i} z_{p^r}^{j-1} \left( t_{ip^r} \right) z_{p^r} \left( t_{p^r} \right). \]

Recalling that \( z_i \left( t^j \right) = 0 \) for \( i \neq j \), for this to be nonzero, we require \( i = j - 1 \) and \( m - i = 1 \). Thus, \( m = j \) and

\[ z_{p^r}^m \left( t_{mp^r} \right) = \binom{m}{m-1} z_{p^r}^{m-1} \left( t_{p^r}^{(m-1)} \right) z_{p^r} \left( t_{p^r} \right) \]

\[ = m! \]

proving the first statement of the lemma.

For the second, we have

\[ z_{p^r} \left( t_{p^r} \right) = \text{mult} \left( z_{p^r}^{p-1} \otimes z_{p^r} \right) \left( t \otimes 1 + 1 \otimes t + f^{p-1} \sum_{j=1}^{p-1} \frac{1}{j! (p-j)!} t_{p^r}^j \otimes t_{p^r}^{(p-j)} \right)^p \]

\[ = z_{p^r}^{p-1} \left( t_{p^r} \right) z_{p^r} \left( 1 \right) + z_{p^r}^{p-1} \left( 1 \right) z_{p^r} \left( t_{p^r} \right) + f^{p} \sum_{j=1}^{p-1} \frac{1}{j! (p-j)!} z_{p^r}^{p-1} \left( t_{p^r}^{j+1} \right) z_{p^r} \left( t_{p^r}^{j+1} \right). \]

Since \( z_{p^r} \left( 1 \right) = 0 \) we may ignore the first two terms, and so

\[ z_{p^r} \left( t_{p^r} \right) = f^{p} \sum_{j=1}^{p-1} \frac{1}{j! (p-j)!} z_{p^r}^{p-1} \left( t_{p^r}^{j+1} \right) z_{p^r} \left( t_{p^r}^{j+1} \right). \]
In order that a summand be nonzero we require \( p^{r+i} (p-\ell) = p^s \), i.e. \( \ell = p-1 \), and hence \( i = s - r \). We have, since \( \frac{z_{p^r}^{p-1}(\ell p^r)}{(p-1)!} = (p-1)! \delta_{p-1,m} \),

\[
\begin{align*}
z_{p^r}^{p^s} (p^r) &= \frac{1}{(p-1)! \cdot p^{s-1}(p-1)!} \\
&= \frac{1}{(p-1)!} \\
&= z_{p^r}^{p^{s+r}}.
\end{align*}
\]

For \( i \neq s - r \) we have \( z_{p^r}^{p^s} (t^p) = 0 \).

It can be shown that the set

\[
\{ \prod_{s=0}^{n-1} z_{p^r}^{j_s} : 0 \leq j_s \leq p-1 \}
\]

is a \( K \)-basis for \( H \). A formal proof will be given in section 5. By counting dimensions, it is clear that \( z_{p^r}^{p^s} = 0 \) for \( r \leq s \leq n-1 \).

The coalgebra structure on \( H \) is induced from the multiplication on \( H_{n,r,f} \) and is simply

\[
\Delta(z_j) = \sum_{i=0}^{j} z_{j-i} \otimes z_i.
\]

4. The Hopf Galois Action

In [Koc14] we describe how \( L \) can be viewed as an \( H_{n,r,f} \)-Galois object. Since \( 2r \geq n \) the \( K \)-algebra map \( \alpha : L \to L \otimes H_{n,r,f} \) given by

\[
(2) \quad \alpha(x) = x \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} x^{p^\ell} \otimes t^{p^r (p-\ell)}
\]

provides an \( H_{n,r,f} \)-comodule structure on \( L \); furthermore, the map \( \gamma : L \otimes L \to L \otimes H_{n,r,f} \) given by \( \gamma(x^i \otimes x^j) = x^i \alpha(x^j) \) is an isomorphism, hence \( L \) is an \( H_{n,r,f} \)-Galois object. In this section, we describe the induced action of \( H = H_{n,r,f}^* \) on \( L \) which makes \( L/K \) an \( H \)-Galois extension.

Before proceeding, notice that this action depends on two choices: the choice of \( x \), the \( K \)-algebra generator for \( L \), and the choice of \( t \), the \( K \)-algebra generator for \( H_{n,r,f} \).

By replacing \( x \) with \( x' \), \( p \uparrow v_L (x') \) we may define

\[
\alpha^{x'}(x') = x' \otimes 1 + 1 \otimes t + \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} (x')^{p^\ell} \otimes t^{p^r (p-\ell)}
\]

and obtain a different coalgebra structure. Alternatively, if we replace \( t \) with, say, \( t_g := gt \), \( g \in K^* \) we may define

\[
\alpha_g(x) = x \otimes 1 + 1 \otimes t_g + \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} x^{p^\ell} \otimes t_g^{p^r (p-\ell)}
\]

which also results in a different coalgebra structure. Furthermore, each of the coalgebra structures here give \( L \) the structure of an \( H_{n,r,f} \)-Galois object. Combined,
we have coactions given by

\[ \alpha^y_h (y) = y \otimes 1 + 1 \otimes t_h + f h^{1-p^{r+1}} \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} r^{p^\ell \otimes t^p (p-\ell)}, \quad y \in L^r, \quad p \nmid v_L (y), \quad h \in K^x \]

although some choices of \( h, y \) produce the same actions, e.g. \( \alpha^x_h = \alpha^x_j \). By fixing \( x \in L \) we eliminate some of the ambiguity as to which coaction is being used. For the rest, notice that \( K \{ t_g \}/(t^p_{\infty}) = H_{n,r,f} \), and so \( H_{n,r,f} = H_{n,r,fg}^{1-p^{r+1}} \) for any choice of \( g \in K^x \), hence choosing the \( K \)-algebra generator for the Hopf algebra is equivalent to choosing a representative of a coset in \( K^x/(K^x)^{p^{r+1}-1} \). Once such a choice \( f \) is made, it is assumed that the coaction of \( H_{n,r,f} \) follows the coaction given in eq. (2). In other words, we will always use the action \( \alpha^x_h \).

Generally, if \( A \) is a \( K \)-Hopf algebra such that \( L \) is an \( A \)-Galois object, then \( A^* \) acts on \( L \) by

\[ h(y) = \text{mult} (1 \otimes h) \alpha (y), \quad h \in A^*, \quad y \in L. \]

As \( H \) is generated by \( \{ z_{p^r} : 0 \leq s \leq n-1 \} \), it suffices to compute \( z_{p^r} \left( x^t \right) \) for \( 0 \leq s \leq n-1, \quad 1 \leq i \leq p^n - 1 \).

\[ \textbf{Proposition 4.1.} \] For \( 0 \leq i \leq p^n - 1 \), write

\[ i = \sum_{s=0}^{n-1} is^{p^s}. \]

Then, for \( 0 \leq s \leq r - 1 \) we have

\[ z_{p^r} \left( x^i \right) = i_s x^{i-p^s}. \]

Additionally,

\[ z_{p^r} \left( x^i \right) = i_r x^{i-p^r} - i f x^{p^r (p-1)+i-1}. \]

\[ \textbf{Remark 4.2.} \] Note that if \( i < p^s \) then \( z_{p^r} \left( x^i \right) = 0 \), and if \( i < p^r \) then \( z_{p^r} \left( x^i \right) = -i f x^{p^r (p-1)+i-1}. \)

\[ \textbf{Proof.} \] We have

\[ z_{p^r} \left( x^i \right) = \text{mult} (1 \otimes z_{p^r}) \alpha (x^i) \]

\[ = \text{mult} (1 \otimes z_{p^r}) S_f \left( x \otimes 1, 1 \otimes t \right)^i \]

\[ = \text{mult} (1 \otimes z_{p^r}) \left( x \otimes 1 + 1 \otimes t + f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} r^{p^\ell \otimes t^p (p-\ell)} \right)^i \]

\[ = \text{mult} (1 \otimes z_{p^r}) \sum_{i_3} \left( x^{i_3} \otimes t^{i_3} \right) \left( f \sum_{\ell=1}^{p-1} \frac{1}{\ell! (p-\ell)!} r^{p^\ell \otimes t^p (p-\ell)} \right)^i. \]

When simplified, the tensors are of the form \( x^{i_1+i_3} p^\ell \otimes t^{i_2+i_3} p^\ell \), \( \ell, \ell' \) as before. Applying \( 1 \otimes z_{p^r} \) to each tensor will give 0 unless

\[ p^s = i_2 + i_3 p^\ell \ell'. \]
Assume first that \( s < r \). Since \( p^r > p^s \) we see that \( \ell'' = 0 \). This can only occur if \( i_3 = 0 \). Thus \( i_2 = p^s \) and \( i_1 = i - p^r \), giving
\[
z_{p^r}(x^i) = \left( \frac{i}{i - p^r, p^s, 0} \right) x^{i - p^r} z_{p^r}(t^{p^r}) = \left( \frac{i}{p^s} \right) x^{i - p^s} = i_s x^{i - p^s},
\]
the last equality following from Lucas’ Theorem (see [Fin47]). Thus \( z_{p^r}(x^i) = i_s x^{i - p^s} \), as desired.

Now we consider the case \( s = r \). Then \( i_3 = 0 \), \( i_2 = p^r \), \( i_1 = i - p^r \) certainly satisfies eq. (4). However, we get an additional solution to this equation, namely \( i_3 = 1 \), \( \ell' = p - 1 \), \( \ell'' = 1 \), \( i_2 = 0 \), \( i_1 = i - 1 \) as
\[
i_2 + i_3 p^r \ell'' = p^r (p - \ell),
\]
with this solution we have the left-hand side equal \( 0 + p^r (1) = p^r \), hence \( \ell = p - 1 \).

Thus
\[
z_{p^r}(x^i) = \left( \frac{i}{i - p^r, p^s, 0} \right) x^{i - p^r} z_{p^r}(t^{p^r}) = \left( \frac{i}{i - p^r, p^s, 0, 1} \right) x^{i - 1} f \frac{1}{(p - 1)! (p - (p - 1))!} x^{p^r (p - 1)} z_{p^r}(t^{p^r}) = i_s x^{i - p^r} - i f x^{p^r (p - 1) + i - 1}.
\]

\[\square\]

Much like it was for the algebra structure, describing the action for \( s > r \) is more complicated as eq. (4) can have numerous solutions. However, in the sequel we will be able to effectively study how the valuation of an element of \( L \) changes when \( z_{p^r} \) is applied.

5. A Scaffold on \( H \)

Recall that \( L = K(x), x^p = \beta, v_L(x) = v_K(\beta) = -b, p \nmid b \). In this section we build an \( H \)-scaffold for \( L \) using the action above. Initially, we will insist on a restriction on \( f \), however this restriction will ultimately not be necessary.

We start by determining the effect of applying \( z_{p^r} \) to powers of \( x \). The first result is fundamental.

**Proposition 5.1.** Let \( 0 \leq s \leq n - 1 \), \( 1 \leq i \leq p^n - 1 \). Write \( i = \sum_{s=0}^{n-1} i_s p^s \). If \( v_K(f) \geq bp^{r+1-n} \) then
\[
z_{p^r}(x^i) \equiv i_s x^{i - p^r} \mod x^{i - p^s} \Psi_L^S
\]
where \( \Psi = p^n v_K(f) - b (p^{r+1} - 1) \).

**Proof.** Since \( z_{p^r}(x^i) = (1 \otimes z_{p^r})(\alpha(x)^i) \) we have
\[
z_{p^r}(x^i) = \sum_{i_1 + i_2 + i_3 = i} f^{i_3} \sum_{i_1, \ldots, i_3, p - 1 = i_3} c_{i_1, i_2, i_3} x^{i_1 + p^r \ell'} z_{p^r}(t^{i_2 + p^r \ell''})
\]
where \( c_{i_1, i_2, i_3} \in K^\times \) and, as before,
\[
\ell' = i_{3,1} + 2i_{3,2} + \cdots + (p - 1) i_{3,p-1}
\]
\[
\ell'' = (p - 1) i_{3,1} + (p - 2) i_{3,2} \cdots + i_{3,p-1}.
\]
For a summand to be nontrivial we require $i_2 + p'r'' = p^s$, in which case the summand is a $K^\times$-multiple of $f^i x^i + p'p''$.

If $s < r$ then Lemma 1.1 gives

$$z_{p^r} (x^i) = i_s x^{i-p^s},$$

and clearly the desired congruence holds.

Now suppose $s \geq r$. Then $z_{p^r} (x^i)$ will again contain the summand $i_s x^{i-p^s}$ arising from $i_3 = 0$, however there may be positive choices of $i_3$ which make $i_2 + p'r'' = p^s$. Since $i_3 \leq r'' \leq (p-1)i_3$ it follows that $i_3 \leq p^{s+r}$. For an $p''$ in this interval we have $i_2 = p^s - p'r''$ and $i_1 = i - (p^s - p'r'') - i_3$. Since $l' + l'' = p'i_3$, the $i_3 > 0$ terms in the summand are all of the form

$$c_{i_1,i_2,i_3} f^{i_1} x^{i_1 - (p^{s-r} - p'r'' - i_3 + p'(p^s - i_1))} = c_{i_1,i_2,i_3} f^{i_3} x^{i_3 (p^{s-r} + i_3 - p(r'' + i_1 - 1))} = c_{i_1,i_2,i_3} f^{i_3} x^{i_3 (p^{s-r} + i_3)} x^{i-p^s},$$

Thus

$$z_{p^r} (x^i) = i_s x^{i-p^s} + \sum (c_{i_1,i_2,i_3} f^{i_3} x^{i_3 (p^{s-r} + i_3 - p(r'' + i_1 - 1))} x^{i-p^s},$$

where the sum is taken over all $i_1,i_2,i_3$ with $i_3 > 0$. Now for $i_3 \geq 1$, $i_3 \geq 1$.

$$v_L \left( f^{i_3} x^{i_3 (p^{s-r} + i_3 - p(r'' + i_1 - 1))} \right) = p^n v_K (f) - b \left( i_3 (p^s - i_1) \right) = i_3 (p^n v_K (f) - b p^{s-r} + b),$$

and since $v_K (f) \geq b p^{s-r} + n$ this expression is minimized when $i_3$ is minimized, i.e., $i_3 = 1$. Thus

$$v_L \left( f^{i_3} x^{i_3 (p^{s-r} + i_3 - p(r'' + i_1 - 1))} \right) \geq p^n v_K (f) - b \left( p^{s-r} + 1 \right),$$

so $f^{i_3} x^{i_3 (p^{s-r} + i_3 - p(r'' + i_1 - 1))} \in \mathcal{P}_L, \exists = p^n v_K (f) - b \left( p^{s-r} + 1 \right)$. Hence,

$$z_{p^r} (x^i) = i_s x^{i-p^s} \left( 1 + \sum c_{i_1,i_2,i_3} f^{i_3} x^{i_3 (p^{s-r} + i_3 - p(r'' + i_1 - 1))} \right)$$

and so

$$z_{p^r} (x^i) \equiv i_s x^{i-p^s} \mod x^{i-p^s} \mathcal{P}_L^\exists.$$

As we have seen, the restriction on $v_K (f)$ is not a restriction on the Hopf algebra, merely on the ways in which this Hopf algebra can act on $L$. We must write $H = H^s_{n,r,f}$, $v_K (f) \geq b p^{s-r} + n$ for the action (induced from the coaction in eq. (2)) for this choice of $f$ to provide an $H$-scaffold. As $H_{n,r,f} = H_{n,r,T_{p^n}^{p+s-r} - f}$, it is clear that there will be an infinite number of actions of $H$ on $L$ which produce scaffolds. To ensure a scaffold of tolerance $\exists > 1$ we require a slight increase in the lower bound for $v_K (f)$. For the rest of the section, we shall assume $v_K (f) > b p^{s-r} + n$.

**Theorem 5.2.** For $v_K (f) > b p^{s-r} + n$, the set $\left\{ z_{i_1}^{j_1} \cdots z_{i_n}^{j_n} : 0 \leq j_s \leq p-1 \right\}$ constructed above is an $H$-scaffold on $L$ with tolerance $\exists = p^n v_K (f) - b \left( p^{s-r} + 1 \right) > 1$.

The presentation of the scaffold above follows the form given in Lemma 2.2. To obtain a scaffold which follows Definition 2.1, we pick an integer $a$ such that $ab \equiv -1 \pmod{p^n}$ and set

$$\lambda_j = T^{(j+a) \bmod{\text{res}(a)}} / p^{n \cdot \text{res}(a)}, j \in \mathbb{Z}.$$
This set, together, with \( \{ \Psi_s = z_{p^s} : 0 \leq s \leq n - 1 \} \), forms the scaffold on \( L \) of tolerance \( \mathcal{T} \) as in the sense of Definition 2.1. In particular,
\[
\lambda_b = T^{(b + b \text{res}(ab))}/p^n \gamma_{x \text{res}(ab)} = T^{(b + b(p^n - 1))}/p^n z_{p^{n-1}} = T^{b_p z_{p^{n-1}}}. 
\]
(5)

As an immediate consequence, we get:

**Corollary 5.3.** The set
\[
\left\{ \prod_{s=0}^{n-1} z_{p^s}^{j_s} (\lambda_b) : 0 \leq j_s \leq p - 1 \right\}
\]
is a \( K \)-basis for \( L \).

**Proof.** This follows from the discussion between Definition 2.1 and Lemma 2.2. In particular, note that
\[
\left\{ v_L \left( \prod_{s=0}^{n-1} z_{p^s}^{j_s} (\lambda_b) \right) : 0 \leq s \leq n - 1, 0 \leq j_s \leq p - 1 \right\}
\]
forms a complete set of residues mod \( p^n \). □

We devote the remainder of this section to showing that the action of \( H \) on \( L \) has an “integer certificate”. In classical Galois module theory, a number \( c \in \mathbb{Z} \) is called an integer certificate if, for all \( \rho \in L \) with \( v_L (\rho) = c \), the set \( \{ \sigma(\rho) : \sigma \in \text{Gal}(L/K) \} \) is a \( K \)-basis for \( L \). We modify that here: a number \( c \in \mathbb{Z} \) is an integer certificate if whenever \( v_L (\rho) = c \) the set
\[
\left\{ z_{p^0}^{j_0} \cdot \cdots \cdot z_{p^{n-1}}^{j_{n-1}} (\rho) : 0 \leq j_s \leq p - 1 \right\}
\]
is a \( K \)-basis for \( L \).

As an immediate consequence to Proposition 5.1 we get

**Corollary 5.4.** Let \( 0 \leq s \leq n - 1 \), \( 1 \leq i \leq p^n - 1 \). Suppose \( z_{p^r} (x^i) \neq 0 \). Then \( v_L (z_{p^r} (x^i)) = b(p^s - i) = v_L (x^i) + bp^s \).

As each application of \( z_{p^r} \) increases valuation by \( bp^s \), the above result allows us to determine the effect, on valuation, of applying our basis elements of \( H \) to the standard \( K \)-basis of \( L \).

**Corollary 5.5.** Let \( 1 \leq i \leq p^n - 1 \), and let \( 0 \leq j_s \leq p - 1 \) for all \( 0 \leq s \leq n - 1 \). If \( z_{p^0}^{j_0} \cdot \cdots \cdot z_{p^{n-1}}^{j_{n-1}} (x^i) \neq 0 \) then
\[
v_L \left( z_{p^0}^{j_0} \cdot \cdots \cdot z_{p^{n-1}}^{j_{n-1}} (x^i) \right) = v_L (x^i) + b \sum_{s=0}^{n-1} j_s p^s.
\]

To set some notation, given \( 0 \leq j \leq p^n - 1 \), we define \( 0 \leq j_0, \ldots, j_{n-1} \leq p - 1 \) to be the unique integers such that
\[
j = \sum_{s=0}^{n-1} j_s p^s.
\]
Conversely, given a collection \( \{ j_0, \ldots, j_{n-1} \} \) with \( 0 \leq j_s \leq p - 1 \) for all \( 0 \leq s \leq n - 1 \) we define \( j \) using the summation above.
We claim that if \( v_L (\rho) = b \) then
\[
\left\{ z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (\rho) : 0 \leq j_\ell \leq p - 1 \right\}
\]
forms a basis for \( L/K \). The crucial step to establishing this is the following.

**Proposition 5.6.** Pick \( \rho \in L \) with \( v_L (\rho) = b \). Then
\[
v_L \left( z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (\rho) \right) = b (1 + j).
\]

**Proof.** Any \( \rho \in L \) with \( v_L (\rho) = b \) has the form
\[
\rho = g \left( x^{-1} + \sum_{\ell=1}^{p^n} a_\ell x^{-1-\ell} \right)
\]
with \( g \in K \), \( a_\ell \in K \), \( v_K (g) = 0 \), and \( v_L (a_\ell) > -b \ell \) for all \( 1 \leq \ell \leq p^n \). Let us write \( g = g_0 T^b x^{p^n} \), and for simplicity we assume \( g_0 = 1 \). Then
\[
\rho = T^b x^{p^n-1} + T^b \sum_{\ell=1}^{p^n} a_\ell x^{p^n-1-\ell}
\]
(note that \( T^b x^{p^n-1} \) is the element \( \lambda_b \) from eq. 5 and thus is part of the scaffold in the Definition 2.1 sense) and
\[
z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (\rho) = T^b z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1}) + T^b \sum_{\ell=1}^{p^n} a_\ell z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1-\ell}).
\]

Applying Corollary 5.5 to the case where \( i = p^n-1-\ell \), either \( z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1-\ell}) = 0 \) or
\[
v_L \left( z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1-\ell}) \right) = -b (p^n - 1 - \ell) + bj.
\]
Furthermore, observe that
\[
z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1}) \neq 0, \quad 0 \leq j_\ell \leq p - 1
\]
since \( p^n - 1 = (p - 1) + (p - 1) p + \cdots + (p - 1) p^{n-1} \). Thus,
\[
v_L \left( T^b z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1}) \right) = bp^n + b (p^n - 1) + bj
\]
since
\[
v_L \left( T^b a_\ell z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1-\ell}) \right) \geq p^n b + v_L (a_\ell) - b (p^n - 1 - \ell) + bj
\]
and, since \( v_L (a_\ell) > -b \ell \),
\[
p^n b + v_L (a_\ell) - b (p^n - 1 - \ell) + bj = p^n b + v_L (a_\ell) - bp^n + b + b \ell + bj
\]
\[
= v_L (a_\ell) + b \ell + b (1 + j)
\]
\[
> b (1 + j) = v_L \left( T^b z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (x^{p^n-1}) \right),
\]
hence
\[
v_L \left( z_1^{j_0} z_2^{j_1} \ldots z_p^{j_{p-1}} (\rho) \right) = \min \left\{ b (1 + j) \right\}
\]
since the minimum is uniquely achieved. \( \Box \)
Lemma 6.1. Proof. As we can write $L/K$.

Remark 5.7. Generally, it is not the case that if $z_{p^r}(y) \neq 0$ then $v_L(z_{p^r}(y)) = v_L(y) + bp^s$, i.e., that an application of $z_{p^r}$ universally increases valuation by $bp^s$. For example, $v_L(x^{p-1} + T x^p) = -(p-1)b$ but $v_L(z_p(x^{p-1} + T x^p)) = v_L(T) = p^n$. However, it is always true that $v_L(z_p(y)) \geq v_L(y) + bp^s$.

Corollary 5.8. The set $\left\{ z_1^{j_1} \cdots z_n^{j_n} : 0 \leq j_s \leq p - 1 \right\}$ forms a $K$-basis for $L$, i.e., $b$ is an integer certificate.

Proof. Observe that $\left\{ v_L\left( z_1^{j_1} \cdots z_n^{j_n} \right) : 0 \leq j_s \leq p - 1 \right\} = \left\{ b(1 + j) : 0 \leq j \leq p^n - 1 \right\}$.

Now $\left\{ b(1 + j) : 0 \leq j \leq p^n - 1 \right\}$ is a complete set of residues mod $p^n$ since $p \nmid b$.

Thus, $\left\{ z_1^{j_1} \cdots z_n^{j_n} : 0 \leq j_s \leq p - 1 \right\}$ is $K$-linearly independent, and hence a basis for $L$.

6. Integral Hopf Galois Module Structure

In this section we describe the Hopf Galois module structure of $\Omega_L$ and of all of the fractional ideals $\mathfrak{P}_L^b$ of $L$. Given a high enough tolerance level, the results of [BCE14] enable us to describe the $H$-module structure of $\mathfrak{P}_L^b$. We apply their work below, and then we will take a look at a specific action of $H$ on $L$.

Let $h \in \mathbb{Z}$. Since $\mathfrak{P}_L^{h+p^n} = T \mathfrak{P}_L^h$ and $\mathfrak{A}_{h+p^n} = \mathfrak{A}_h$ it suffices to consider the Hopf Galois module structure on a complete set of residues mod $p^n$. We will pick the set of residues $h$ such that $0 \leq b - h \leq p^n - 1$.

We start with:

Lemma 6.1. There exists actions of $H$ on $L$ which produce $H$-scaffold structures on $L/K$ with arbitrarily high tolerance.

Proof. As we can write $H = H_{n.r.f}^*$ with $v_K(f)$ of arbitrarily high valuation, this is clear since $\mathfrak{T} = p^n v_K(f) - b \left( p^{r+1} - 1 \right)$ for $v_K(f) \geq b p^{r+1} - n$.

For the remainder of this section, pick $f$ such that

$$ v_K(f) \geq \frac{2p^n - 1 + b \left( p^{r+1} - 1 \right)}{p^n}, $$

so $\mathfrak{T} \geq 2p^n - 1$. This level of tolerance allows us to determine integral Hopf Galois module structure.

Remark 6.2. This new bound on $v_K(f)$ is larger than the one we imposed in section 5. While we could have simply assumed $v_K(f) \geq \left( 2p^n - 1 + b \left( p^{r+1} - 1 \right) \right) p^{-n}$ throughout, we wanted to also provide examples of $H$-scaffolds for which Hopf Galois module structure could not be completely determined.

We will now introduce numerical data from [BCE14]. For each $0 \leq j \leq p^n - 1$, let

$$ d_h(j) = \left\lfloor \frac{bj + b - h}{p^n} \right\rfloor $$

$$ w_h(i) = \min \left\{ d_h(i + j) - d_h(i) : 0 \leq i \leq p^n - 1, \ i_s + j_s \leq p - 1 \text{ for all } s \right\}, $$

using our convention that $j = \sum j_s p^s$, $i = \sum i_s p^s$ as before. Then, using Theorem 3.1, Theorem 3.7, and Corollary 3.2 of [BCE14] we get all of the following.
Remark 6.4. It is important to note that the determination as to whether $\mathfrak{P}_L^h$ is free over $\mathfrak{A}_h$ does not depend on the $H$-scaffold itself, merely on the behavior of $d_h$ and $w_h$.

Remark 6.5. Note that if $\text{res}(b) \mid (p^n - 1)$ then $\mathfrak{O}_K$ is free over $\mathfrak{A}$, but in general the converse does not hold. But since 2 is a special case of 3 where $h = 0$ we do have necessary and sufficient conditions for when $\mathfrak{O}_K$ is free over $\mathfrak{A}$.

Let us interpret these results in the case where $b = 1$, which requires that $v_K(f) \geq 3$. (Note that scaffolds exist for $v_K(f) = 2$, as well as for $v_K(f) = 1$ unless $n = r + 1$.) Then $2 - p^n \leq h \leq 1$ and

$$d_h(j) = \left\lfloor \frac{j + 1 - h}{p^n} \right\rfloor = \begin{cases} 1 & j \geq p^n - 1 + h \\ 0 & j < p^n - 1 + h \end{cases}.$$ 

Since $w_h(j) \leq d_h(j)$, which is readily seen by setting $i = 0$ in the definition of $w_h(j)$, the statement $w_h(j) = d_h(j)$ for all $0 \leq j \leq p^n - 1$ is true if and only if $w_h(j) = 1$ whenever $j \geq p^n - 1 + h$. Suppose $h > (1 - p^n)/2$ and $d_h(j) = 1$. Then $j > p^n - 1 + (1 - p^n)/2 = (p^n - 1)/2$. Now assume there exists an $i$ such that $d_h(i + j) - d_h(i) = 0$ and $i + j \leq p - 1$ for all $s$. Then $d_h(i + j) = d_h(j) = 1$ so $d_h(i) = 1$ as well. Thus $i > (p^n - 1)/2$. But then $i + j \geq p^n$, contradicting the fact that $i + j \leq p - 1$ for all $s$. Therefore, no such $i$ can occur, hence $w_h(j) = d_h(j)$ for all $j$ and $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot \rho$.

Now suppose that $h \leq (1 - p^n)/2$ and let $j = p^n + h - 1$. Then $d_h(j) = 1$. Let

$$i = p^n - 1 - j = p^n - 1 - (p^n + h - 1) = -h.$$

Then $i + j = p - 1$ for all $s$. As above, $d_h(i + j) = 1$. But $i = -h < p^n - 1 - h$ so $d_h(i) = 0$. Thus $w_h(j) = w_h(p^n + h - 1) = 0$ and $\mathfrak{P}_L^h$ is not free over $\mathfrak{A}_h$.

We summarize, generalizing to all $h \in \mathbb{Z}$.

Theorem 6.6. Let $H = H^*_{n,r,f}$, $0 < r < n \leq 2r$, $f \in K^\times$. Suppose $v_L(x) = -1$ and $v_L(\rho) = 1$. Let $h \in \mathbb{Z}$, and let $m = [h/p^n]$. Then $\mathfrak{P}_L^h$ is free over $\mathfrak{A}_h$ if and only if $\text{res}(h - 2) > (p^n - 3)/2$; under this restriction, $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot (T^{m\rho})$.

Remark 6.7. Notice that we do not need $v_K(f) \geq 2(1 - p^{-n}) + p^{r+1-n}$ in the statement above since, for any $f \in K^\times$, an $H^*_{n,r,f}$ of suitably high tolerance exists.

Proof. Consider first the case $2 - p^n \leq h \leq 1$. Then, $0 \leq h - 2 + p^n \leq p^n - 1$. We have seen that $\mathfrak{P}_L^h$ is $\mathfrak{A}_h$-free if and only if $h > (1 - p^n)/2$, and since $h \leq 1$ this inequality holds if and only if

$$\frac{p^n - 3}{2} < h - 2 + p^n \leq p^n - 1.$$ 

Thus, $\mathfrak{P}_L^h$ is $\mathfrak{A}_h$-free if and only if $\text{res}(h - 2) > (p^n - 3)/2$. 

Proposition 6.3. With the notation as above:

1. $\mathfrak{A}_h$ has $\mathfrak{O}_K$-basis $\left\{ T^{-w_h(j)} z^j_1 z^{j_1}_2 \cdots z^{j_{n-1}}_p : 0 \leq j \leq p^n - 1 \right\}$.

2. $\mathfrak{O}_K$ is a free $\mathfrak{A}$-module of rank one—explicitly, $\mathfrak{O}_L = \mathfrak{A} \cdot \rho$, $v_L(\rho) = b$ if $\text{res}(b) \mid (p^n - 1)$ for some $1 \leq m \leq n$.

3. $\mathfrak{P}_L^h$ is a free $\mathfrak{A}_h$-module if and only if $w_h(j) = d_h(j)$ for all $0 \leq j \leq p^n - 1$; furthermore if this equality holds then $\mathfrak{P}_L^h = \mathfrak{A}_h \cdot \rho$, $v_L(\rho) = h$.

4. If $w_h(j) \neq d_h(j)$, then $\mathfrak{P}_L^h$ can be generated over $\mathfrak{A}_h$ using $\ell$ generators, where $\ell = \# \left\{ i : d_h(i) > d_h(i - j) + w_h(j) \text{ for all } 0 \leq j \leq p^n - 1 \text{ with } j_s \leq i_s \right\}$.
Now for more general $h$, $\mathcal{P}_L^h$ is free over $\mathfrak{A}_h$ if and only if $\mathcal{P}_L^{\text{res}(h)}$ is free over $\mathfrak{A}_{\text{res}(h)} = \mathfrak{A}_h$, so we have freeness if and only if
\[
\text{res}(\text{res}(h) - 2) > (p^n - 3)/2,
\]
and since the left-hand side reduces to $\text{res}(h - 2)$ we get the inequality desired. That $\mathcal{P}_L^h = \mathfrak{A}_h \cdot (T^n p)$ is immediate since $\mathcal{P}_L^h = T^n \mathcal{P}_L^{\text{res}(h)}$.

In particular, notice that $\mathcal{O}_L$ is free over $\mathfrak{A}$ when $b = 1$.

7. Picking the Best Hopf Algebra and Action

In the examples provided here – with $L = K(x)$, $v_L(x) = b$, $p \nmid b$ – questions concerning the Hopf Galois module structure of $\mathcal{O}_L$ have little to do with the exact Hopf algebra chosen. For any choice of $0 < r < n \leq 2r$ and $v_K(f) \geq 2 - p^n(1 - b(p^{r+1} - 1))$ we have scaffolds of sufficiently high tolerance, and their existence allows us to apply the numerical data of Proposition 6.3. So, if $\mathcal{P}_L^h$ is free over $\mathfrak{A}_h$ for $H = H^*_{r,f}$, then $\mathcal{P}_L^h$ is free over $\mathfrak{A}_h$ for any $H = H^*_{r,f}$, $0 < r' < n \leq 2r$ and $v_K(f') \geq 2 - p^n(1 - b(p^{r+1} - 1))$. Additionally, the description of $\mathfrak{A}_h$ given in Proposition 6.3 is independent of which Hopf algebra $H$ is chosen since the value of $T^{-\nu_h(j)}$ is independent of $H$; of course, the actual elements $z_{pr^e}$ depend on the chosen $H$.

In addition to the family constructed here, the divided power $K$-Hopf algebra $A$ of rank $p^n$ found in ([Mon93] Ex. 5.6.8), where it is denoted $H$) acts on $L$: in terms of its dual, $A^*$ represents the $n$th Frobenius kernel of the additive group scheme, and its simple coaction is given by Chase in [Cha70]. In [BCE14] Sec. 5.2 a scaffold of infinite tolerance (so the congruences are replaced by equalities) is constructed for $A$. Their scaffold is similar to our constructions – indeed, for large values of $v_K(f)$, $A^*$ and $H_{r,f}$ act very similarly on $L$, and we can view $H_{r,f}$ as a deformation of $A^*$.

Thus, it is natural to ask: which Hopf algebra is “best”? As the determination of integral Hopf Galois module structure does not depend on the choice of $H$, there would need to be further properties of interest to make a distinction.

For a single choice of $H_{r,f}$, different actions lead to scaffolds of different tolerances, though we can always make $\mathfrak{A}$ arbitrarily large. So here, we may ask: which action is the “best”? If one is primarily interested in describing $\mathcal{O}_L$ as an $\mathfrak{A}$-module then the action where $L = K(x)$, $v_L(x) = -1$ appears to be a good choice since $\mathcal{O}_L$ is free over $\mathfrak{A}$ whenever $v_K(f) \geq 3$. If, on the other hand, one is primarily interested in describing $\mathcal{P}_L^h$ for a specific value of $h$ there may be better choices. For example, in an unpublished work by Jelena Sundukova, she states that $\mathcal{P}_L^h$ is a free $\mathfrak{A}_h$-module if $v_L(x) = -h$. Her work also describes choices of $v_L(x)$ which make $\mathcal{P}_L^h$ free over $\mathfrak{A}_h$ reasonably rare, for example $v_L(x) = p^n - 2$. As with choosing the Hopf algebra, we would need to have more properties of this action which we deem “desirable” in order to pick one action over another.

References

[BCE14] Nigel P. Byott, Lindsay N. Childs, and G. Griffith Elder, Scaffolds and generalized integral Galois module structure, preprint (2014).

[BE13] Nigel P. Byott and G. Griffith Elder, Galois scaffolds and Galois module structure in extensions of characteristic $p$ local fields of degree $p^e$, J. Number Theory 133 (2013), no. 11, 3598–3610.
[Byo11] Nigel P. Byott, *A valuation criterion for normal basis generators of Hopf-Galois extensions in characteristic $p$*, J. Théor. Nombres Bordeaux **23** (2011), no. 1, 59–70.

[Cha76] Stephen U. Chase, *Infinitesimal group scheme actions on finite field extensions*, Amer. J. Math. **98** (1976), no. 2, 441–480.

[Eld09] G. Griffith Elder, *Galois scaffolding in one-dimensional elementary abelian extensions*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1193–1203.

[Fin47] N. J. Fine, *Binomial coefficients modulo a prime*, Amer. Math. Monthly **54** (1947), 589–592.

[GP87] Cornelius Greither and Bodo Pareigis, *Hopf Galois theory for separable field extensions*, J. Algebra **106** (1987), no. 1, 238–258.

[Koc14] Alan Koch, *Hopf Galois structures on purely inseparable extensions*, New York J. Math. **20** (2014), 779–797.

[Leo59] Heinrich-Wolfgang Leopoldt, *Über die Hauptordnung der ganzen Elemente eines abelschen Zahlkörpers*, J. Reine Angew. Math. **201** (1959), 119–149.

[Mon93] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[Noe31] Emmy Noether, *Normalbasis bei körpern ohne höhere Verzweigung*, J. Reine Angew. Math. **167** (1931), 147–152.