LIMITING ABSORPTION PRINCIPLE FOR DISCRETE SCHRÖDINGER OPERATORS WITH A WIGNER-VON NEUMANN POTENTIAL AND A SLOWLY DECAYING POTENTIAL

SYLVAIN GOLÈNIA AND MARC-ADRIEN MANDICH

ABSTRACT. We consider discrete Schrödinger operators on \( \mathbb{Z}^d \) for which the perturbation consists of the sum of a long range type potential and a Wigner-von Neumann type potential. Still working in a framework of weighted Mourre theory, we improve the limiting absorption principle (LAP) that was obtained in [Ma1]. To our knowledge, this is a new result even in the one-dimensional case. The improvement consists in a weakening of the assumptions on the long range potential and better LAP weights. The improvement relies only on the fact that the generator of dilations (which serves as conjugate operator) is bounded from above by the position operator. To exploit this, Loewner’s theorem on operator monotone functions is invoked.

1. Introduction

The limiting absorption principle (LAP) is an important resolvent estimate in the spectral and scattering theory of quantum mechanical Hamiltonians, in particular it implies the existence of purely absolutely continuous spectrum. For Schrödinger operators on \( \mathbb{R}^d \), the LAP was derived for a large class of short and long range potentials, see e.g. [A], [PSS], [ABG], as well as for decaying oscillatory potentials such as the Wigner-von Neumann potential, see e.g. [DMR], [RT1], [RT2]. Schrödinger operators with Wigner-von Neumann potentials [NW] are of interest because when appropriately calibrated they may produce eigenvalues embedded in the absolutely continuous spectrum of the Hamiltonian, and are linked with the phenomenon of resonance. The Wigner-von Neumann type potentials are still very much an active area of research. For the multi-dimensional case, see e.g. [GJ2], [J], [JM], [Mar1], [Mar2], [Mar3] and [Ma1]; for the one-dimensional case, see e.g. [Li2], [L1], [L2], [L3], [Sim], [JS], [KN], [NS] and [KS].

Regarding the LAP, historically a lot of effort was put into lessening the decay assumptions on the short/long range perturbations. In this regard a breakthrough was made in 1981 by E. Mourre, see [Mo1] and [Mo2]. Very roughly speaking, assuming the short range perturbation satisfies \( V_{sr}(x) = O(|x|^{-2}) \) and the long range perturbation satisfies \( x \cdot \nabla V_{lr}(x) = O(|x|^{-1}) \), as well as several other technicalities, Mourre’s theory implied the following LAP for the Schrödinger operator \( H = -\Delta + V_{sr} + V_{lr} \) on \( \mathbb{R}^d \) over some open real interval \( I \):

\[
\sup_{\text{Re}(z) \in I, \text{Im}(z) > 0} \| \langle A \rangle^{-s}(H - z)^{-1}\langle A \rangle^{-s} \| < \infty, \quad s > 1/2.
\]

Here \( \langle x \rangle := \sqrt{1 + |x|^2} \) and \( A \) is some self-adjoint operator. The theory was improved and by the end of the 20th century the book [ABG] presented a more sophisticated but optimal abstract framework to treat the short/long range potentials. This framework now goes by the name of classical Mourre theory. The Wigner-von Neumann potential is not in the scope of this framework because its decay is at best \( O(|x|^{-1}) \), see e.g. [GJ2, Proposition 5.4] for the continuous case and [Ma1, Proposition 4.2] for the discrete case.

2010 Mathematics Subject Classification. 39A70, 81Q10, 47B25, 47A10.

Key words and phrases. limiting absorption principle, discrete Schrödinger operator, Wigner-von Neumann potential, Mourre theory, weighted Mourre theory, Loewner’s theorem, polylogarithms.
The beginning of the 21st century sees the emergence of the new approaches to Mourre’s commutator method with [G] and [GJ1], and the so-called weighted Mourre theory, see [GJ2]. While the classical theory revolves around the Mourre estimate

\[ E_I(H)[H, iA], E_I(H) \gtrsim \gamma E_I(H) + K, \]

where \( E_I(H) \) is the spectral projection of \( H \) onto the interval \( I \), \( K \) is a compact operator, and \( \gamma > 0 \), the weighted theory is centered around an estimate of the type

\[ E_I(H)[H, i\varphi(A)], E_I(H) \gtrsim \gamma E_I(H)\langle A \rangle^{-2s} E_I(H) + K, \quad s > 1/2, \]

where \( \varphi \) is the real-valued function

\[ \varphi(t) := \int_{-\infty}^{t} \frac{dx}{\langle x \rangle^{2s}}, \quad t \in \mathbb{R}. \]

Note that \( \varphi \) is bounded for \( s > 1/2 \) and \( \varphi'(A) = \langle A \rangle^{-2s}, \) i.e. the derivative of \( \varphi \) yields the appropriate weights for the LAP as in (1.1).

In [GJ2] ideas of weighted Mourre theory are used to derive the LAP for the continuous Schrödinger operator \( H = -\Delta + V_{\text{sr}} + V_{\text{tl}} + W \) on \( \mathbb{R}^d \), where \( W(x) = w \sin(k|x|)/|x|, \) \( w, k \in \mathbb{R} \), is the Wigner-von Neumann potential, and the short/long range potentials respectively satisfy \( V_{\text{sr}}(x) = O(|x|^{-1-\varepsilon}) \) and \( x \cdot \nabla V_{\text{tl}}(x) = O(|x|^{-\varepsilon}) \) for some \( \varepsilon > 0 \). The cost of including the \( W \) was that the decay assumptions for \( V_{\text{sr}} \) and \( V_{\text{tl}} \) are suboptimal as per [ABG, Chapter 7]. The article [Ma1] was an application of the ideas of [GJ2] to corresponding (albeit non-radial) discrete Schrödinger operators on \( \mathbb{Z}^d \), see Theorem 1.4 below for a statement.

Let \( \log_q(x) := 1, \log_1(x) := \log(1 + x), \) and for integer \( k \geq 2 \), \( \log_k(x) := \log \left( 1 + \log_{k-1}(x) \right) \). Thus \( k \) is the number of times the function \( \log(1 + x) \) is composed with itself. Also denote \( \log_p^k(x) := (\log_k(x))^p, \) \( p \in \mathbb{R} \). This article is a sequel to [Ma1]. The aim is to explore the case where the function \( \varphi \) in (1.3) is not given by (1.4) but rather by

\[ \varphi(t) := \int_{-\infty}^{t} \frac{dx}{\langle x \rangle^{2m+1} \log_k^2(\langle x \rangle) \prod_{k=0}^{m} \log_k(\langle x \rangle)}, \quad t \in \mathbb{R}, \]

for some \( m \in \mathbb{N} \) and \( p > 1/2 \). Note that \( \varphi \) as in (1.5) is an increasing function that is asymptotically equal to a constant minus a \( O(\log_{m+1}^1(t)) \) term for \( t \to +\infty \), hence a bounded function.

More notation is needed to present our results. The position space is the lattice \( \mathbb{Z}^d \) for integer \( d \geq 1 \). For \( n = (n_1, ..., n_d) \in \mathbb{Z}^d \), set \( |n|^2 := n_1^2 + ... + n_d^2 \). Consider the Hilbert space \( \mathcal{H} := l^2(\mathbb{Z}^d) \).

Let \( S_i \) be the shift operator to the right on the \( i \)th coordinate:

\[ (S_i \psi)(n) := \psi(n_1, ..., n_i - 1, ..., n_d), \quad \forall n \in \mathbb{Z}^d, \quad \psi = (\psi(n))_{n \in \mathbb{Z}^d} \in \mathcal{H}, \quad 1 \leq i \leq d. \]

The shifts to the left on the \( i \)th coordinate are \( S_i^* = S_i^{-1} \). The discrete Schrödinger operator

\[ H := \Delta + V + W \]

acts on \( \mathcal{H} \), where \( \Delta \) is the discrete Laplacian operator defined by

\[ \Delta := \sum_{i=1}^{d} 2 - S_i - S_i^* = \sum_{i=1}^{d} \Delta_i, \]

\( W \) is a Wigner-von Neumann potential, parametrized by \( w \in \mathbb{R}, \) \( k \in (0, 2\pi) \setminus \{\pi\} \) and defined by

\[ (W \psi)(n) := \frac{w \sin(k(n_1 + ... + n_d))}{|n|} \psi(n), \quad \text{for all } n \in \mathbb{Z}^d \text{ and } \psi \in \mathcal{H}, \]

and \( V \) is a multiplication operator by a bounded real-valued sequence \( (V(n))_{n \in \mathbb{Z}^d} \) such that \( (V \psi)(n) := V(n)\psi(n) \). Denote \( \tau_i V, \tau_i^* V \) the operators of multiplication acting by

\[ (\tau_i V) \psi(n) := V(n_1, ..., n_i - 1, ..., n_d) \psi(n), \quad (\tau_i^* V) \psi(n) := V(n_1, ..., n_i + 1, ..., n_d) \psi(n). \]
The conjugate operator to $\sigma$ is purely absolutely continuous and $\sigma(\Delta) = [0, 4d]$. The formulation of the LAP requires a conjugate self-adjoint operator. Let $N_i$ be the position operator on the $i$th coordinate defined by

$$ (N_i \psi)(n) := n_i \psi(n), \quad \text{Dom}[N_i] := \{ \psi \in L^2(\mathbb{Z}^d) : \sum_{n \in \mathbb{Z}^d} |n_i \psi(n)|^2 < \infty \}. $$

The conjugate operator to $H$ is the generator of dilations denoted $A$ and it is the closure of

$$ A_0 := i \sum_{i=1}^d 2^{-1} (S_i^* + S_i) - (S_i^* - S_i) N_i = \frac{i}{2} \sum_{i=1}^d (S_i - S_i^*) N_i + N_i (S_i - S_i^*) $$

with domain the compactly supported sequences. $A$ is self-adjoint, see e.g. [GGo]. Let

$$ \mu(H) := (0, 4) \setminus \{ E_{\pm}(k) \} \quad \text{for} \quad d = 1, $$

$$ \mu(H) := (0, E(k)) \cup (4d - E(k), 4d) \quad \text{for} \quad d \geq 2. $$

Here $E_{\pm}(k) := 2 \pm 2\cos(k/2)$ and $E(k)$ is defined in Proposition 3.2. The sets $\mu(H)$ are real numbers where the classical Mourre estimate (1.2) holds for $H$ — assuming the presence of the oscillatory perturbation, i.e. $W \neq 0$.

**Proposition 1.1.** Let $d \geq 1$. If $V$ satisfies hypothesis (H) for some $m \in \mathbb{N}$ and $0 \leq r < q$, and $E \in \mu(H)$, then there is an open interval $I$ of $E$ that contains finitely many eigenvalues of $H$ and these are of finite multiplicity. Also more precisely, if $d = 1$ and $E \in (E_{-}(k), E_{+}(k))$, then the point spectrum of $H$ is empty in $I$ provided $I \subset (E_{-}(k), E_{+}(k))$.

Denote $P_{\perp} := 1 - P$, where $P$ is the projection onto the pure point spectral subspace of $H$. The main result of this article is:

**Theorem 1.2.** Let $d \geq 1$. Let $V$ satisfy hypothesis (H) for some $m \in \mathbb{N}$ and $2 \geq r < q$. Let $E \in \mu(H)$. Then there is an open interval $I$ of $E$ such that for any compact interval $J \subset I$, any integer $M \geq m$, and any $p > 1/2$,

$$ \sup_{\text{Re}(z) \in J} \mathbb{I}_{\text{Im}(z) > 0} \left\| W_{M+1}^{-p} (A) (H - z)^{-1} P_{\perp} W_{M+1}^{-p} (A) \right\| < \infty, \quad \text{where} $$

$$ W_{M+1}^{-p}(x) := \langle x \rangle^{-\frac{p-1}{2}} w_{M+1}^{-p}(x), \quad w_{M+1}^{\alpha, \beta}(x) := \log_{M+1}(\langle x \rangle) \prod_{k=0}^M \log_k^\beta(\langle x \rangle), \quad \alpha, \beta \in \mathbb{R}. $$

The following local decay estimate also holds for all $\psi \in \mathcal{H}$, $M \geq m$ and $p > 1/2$:

$$ \int_{\mathbb{R}} \left\| W_{M+1}^{-p} (A) e^{-itH} P_{\perp} E_J(H) \psi \right\|^2 dt < \infty. $$

By Lemma 4.9 the above estimates also hold for $\langle N \rangle = (1 + |N_1|^2 + ... + |N_d|^2)^{1/2}$ instead of $\langle A \rangle$. Finally, the spectrum of $H$ is purely absolutely continuous on $J$ whenever $P = 0$ on $J$.
Remark 1.2. We can take \( q = r \) by applying Theorem 1.2 with \( m + 1 \).

Remark 1.3. If \( W = 0 \), then Theorem 1.2 holds for \( E \in [0, 4d] \setminus \{0, 4, \ldots, 4d - 4, 4d\} \).

Example 1.3. Suppose \( V(n) = O \left( |n|^{-1} \log^{-q} |n| \right) \) for some \( q > 2 \). Then hypothesis \( (H) \) holds with \( m = 0 \) and \( r = 2 \). The LAP in Theorem 1.2 holds with the weights \( W_{M+1}^p(A) \) for any \( M \geq 0 \) and \( p > 1/2 \), so in particular for \( W_1^p(A) = \langle A \rangle^{-\frac{1}{2}} \log^{-p}(1 + \langle A \rangle) \), \( p > 1/2 \).

For comparison, the statement of the principal result of [Ma1] is added below, taking into account the technical improvement [Ma2, Theorem 1.8] :

**Theorem 1.4.** [Ma1] Let \( d \geq 1 \). Let \( V \) satisfy \( V(n) = O(|n|^{-\varepsilon}) \) and \( n_i(V - \tau V)(n) = O(|n|^{-\varepsilon}) \) for some \( \varepsilon > 0 \) and \( \forall 1 \leq i \leq d \). Let \( E \in \mu(H) \). Then there exists an open interval \( I \) containing \( E \) such that for any compact interval \( J \subset I \) and any \( s > 1/2 \),

\[
\sup_{\text{Re}(z) \in J, \text{Im}(z) > 0} \| \langle A \rangle^{-s} (H - z)^{-1} P^\perp \langle A \rangle^{-s} \| < \infty.
\]

Thus, Theorem 1.2 improves Theorem 1.4 by weakening the decay assumptions on \( V \), as well as improving the LAP weights. The article [Ma1] considers a second Wigner-von Neumann potential, namely \( W'(n) = \prod_{i=1}^p \sin(k_i n_i)/n_i \). Although \( W' \) could have been included in this article and similar results would have been derived, we have not done so to keep the size of the article reasonable. Another obvious comment is that we do not know how to effectively use commutator methods to study spectral properties of the discrete Hamiltonian \( H \) (with \( W \neq 0 \)) on \( [0, 4d] \setminus \mu(H) \), see the comments that follow [Ma1, Proposition 4.5]. For the continuous Schrödinger operators on \( \mathbb{R}^d \), with \( W \neq 0 \), a similar limitation existed, see [GJ2] and [JM], in that the LAP was established only on \( (0, k^2/4) \) for \( d \geq 2 \). But interestingly, an argument was recently found to justify a LAP on \( (0, +\infty) \setminus \{k^2/4\} \) when the Wigner-von Neumann potential is radial, i.e. \( W(x) = q \sin(k|x|)/|x| \), see [J] and [Mb, Section 3.5].

Fix \( d = 1 \) to discuss Theorem 1.2 in relation to other one-dimensional results in the literature. In [Sim], the perturbation consists of a Wigner-von Neumann potential plus a \( V \in \ell^1(\mathbb{Z}) \), and it is proved that the spectrum of \( H \) is purely absolutely continuous on \( (0, 4) \setminus \{E_\pm(k)\} \). Note that our assumption (1.10), with \( 2 = r < q \), implies \( (V - \tau V) \in \ell^1(\mathbb{Z}) \) but not necessarily \( V \in \ell^1(\mathbb{Z}) \), (although the weaker assumption \( 1 = r < q \) is sufficient to have \( (V - \tau V) \in \ell^1(\mathbb{Z}) \)). Another recent result is [Lil], where it is shown that if the perturbation is \( O(|n|^{-1}) \), then \( H \) does not have singular continuous spectrum. Although this criteria applies to the Wigner-von Neumann potential, it does not apply to all potentials \( V \) in the scope of this article, since for example hypothesis \( (H) \) allows for \( V(n) = \log^{-q}(1 + \langle n \rangle) \), \( q > 2 \). Now suppose the absence of oscillations, i.e. \( W = 0 \). Because \( (V - \tau V) \in \ell^1(\mathbb{Z}) \), \( V \) is of bounded variation, and so the spectrum of \( H \) is purely absolutely continuous on \( (0, 4) \), by a result due to B. Simon in [Si].

Consider now \( d \geq 1 \) to discuss Theorem 1.2 in relation to the framework of Mourre theory as exposed in [ABG]. First, it is important to cite [BSa], where the details of this framework are worked out for the multi-dimensional discrete Schrödinger operators on \( \mathbb{Z}^d \). From this perspective, this article is an attempt to bridge the gap between [Ma1] and [BSa].

Recall the regularity classes \( C^{1,1}_1(A) \subset C^{1,1}_k(A) \subset C^1(A) \) that structure the classical Mourre theory, see Section 2.1. In [ABG, Chapter 7], it is shown that \( C^{1,1}(A) \) is an optimal class in that if the Hamiltonian \( H \in C^{1,1}(A) \) then a LAP holds for \( H \), whereas there is an example where \( H \in C^{1,1}_1(A) \) but \( H \notin C^{1,1}(A) \) and no LAP holds. In [Ma1, Proposition 4.2] it is proved that the Wigner von-Neumann potential \( W \), and thereby \( H \), are not even of class \( C^1(A) \). Thus our Hamiltonian does not fall under the framework of classical Mourre theory.
Let us discuss the assumptions on the short/long range perturbations $V_{sr}, V_{lr}$. The criteria in the classical theory (see [BSa, Theorem 2.1], also [ABG, Theorem 7.6.8]) are:

(1.18) \[ \int_1^\infty \sup_{|\kappa| < 2\kappa} |V_{sr}(n)| d\kappa < \infty, \quad \forall \ 1 \leq i \leq d, \]

and

(1.19) \[ V_{lr}(n) = o(1) \quad \text{and} \quad \int_1^\infty \sup_{|\kappa| < 2\kappa} |(V_{lr} - \tau_{\kappa} V_{lr})(n)| d\kappa < \infty, \quad \forall \ 1 \leq i \leq d. \]

Table 1 contrasts these criteria with our hypothesis in Theorem 1.2.

| If $V$ satsifies ... then | crit. (1.18) holds | crit. (1.19) holds | hyp. of Thm 1.2 holds |
|--------------------------|-----------------|-----------------|-------------------|
| $V(n) = O(g(n))$         | $\forall m \in \mathbb{N}, 1 = r < q$ | $\forall m \in \mathbb{N}, 1 = r < q$ | $\forall m \in \mathbb{N}, 2 = r < q$ |
| $(V - \tau_{\kappa} V)(n) = O(g(n)), \forall i$ | no               | $\forall m \in \mathbb{N}, 1 = r < q$ | $\forall m \in \mathbb{N}, 2 = r < q$ |
| $V(n) = \sum_{i=1}^d \log_2^2(2^{j(n+1)})$ | no               | no              | $\forall q > 2$ |

Table 1. Examples. $g(n) := O\left(|n|^{-1} \log_m^{-q} |n| \prod_{k=0}^n \log_k^{-r} |n| \right)$

The last line in Table 1 shows that it is possible to concoct a non-radial potential $V$ that verifies the hypothesis of Theorem 1.2 but neither (1.18) nor (1.19). It is an open problem for us to verify if this potential is of class $C^{1,1}(A)$. If the potential $V$ is radial the first two lines of Table 1 suggest that our assumptions on $V$ may be slightly suboptimal, at least in the absence of oscillations, i.e. $W = 0$, because Theorem 1.2 requires $r = 2$, rather than $r = 1$.

To the best of our understanding, our requirement of "an entire extra logarithm" ($r = 1$ vs. $r = 2$) is due to a technical limitation in the construction of the almost analytical extension of the function $\varphi$, see (1.5), which is employed to apply the Helffer-Sjöstrand formula and associated functional calculus. Indeed, although $\varphi'(t) = \langle t \rangle^{-1} \log_2^{-q} \langle t \rangle \prod_{k=0}^n \log_k^{-1} \langle t \rangle$, the extension only verifies $|\hat{\varphi}| \leq c (\text{Re}(z))^{-1-\ell} |\text{Im}(z)|^\ell$, $\ell \in \mathbb{N}$. Clearly all logarithmic decay is lost in the process of extending to the complex plane.

Let us briefly discuss the LAP weights. Let $T$ be a self-adjoint operator, $E_{2j}(T)$ its spectral projection onto a set $\Sigma$. Let $s, p, p' \in \mathbb{R}, M \in \mathbb{N}$. Let $\Sigma_j = \{x \in \mathbb{R} : 2^j - 1 \leq |x| \leq 2^j\}, j \geq 1$ and $\Sigma_0 := \{x \in \mathbb{R} : |x| \leq 1\}$. Define the following Banach spaces with the obvious norms:

$$L^2_{s,p,p',M}(T) := \left\{ \psi \in \mathcal{H} : \langle \psi, \mathcal{H} \rangle^* \log_{M+1}^p \langle \psi, \mathcal{H} \rangle \prod_{k=0}^M \log_k^{p'} \langle \psi, \mathcal{H} \rangle \psi \rangle \langle \psi, \mathcal{H} \rangle < \infty \right\},$$

and

$$B(T) := \left\{ \psi \in \mathcal{H} : \sum_{j=0}^\infty \sqrt{2^j} \|E_{\Sigma_j}(T)\psi\| < \infty \right\}.$$

The dual $(L^2_{s,p,p',M}(T))^*$ of $L^2_{s,p,p',M}(T)$ with respect to the inner product on $\mathcal{H}$ is $(L^2_{s,p,p',M}(T))^* = L^2_{-s,-p,-p',M}(T)$ and the dual $B^*(T)$ of $B(T)$ is the Banach space obtained by completing $\mathcal{H}$ in the norm

$$\|\psi\|_{B^*(T)} = \sup_{j \in \mathbb{N}} \sqrt{2^{-j}} \|E_{\Sigma_j}(T)\psi\|.$$

We refer to [JP] and the references therein for these definitions. Write $L^2_s(T) := L^2_{s,0,0,0}(T)$. For any $s, p > 1/2$ and $M \in \mathbb{N}$, the following inclusions hold:

$$L^2_s(T) \subseteq L^2_{1/2,p,1/2,M}(T) \subseteq B(T) \subseteq L^2_{1/2}(T),$$
and
\[ L^2_{-1/2}(T) \subseteq B^s(T) \subseteq L^2_{-1/2,-p,-1/2,M}(T) \subseteq L^2_{-s}(T). \]
Also clear is that \( L^2_{1/2,p,1/2,M}(T) \subset L^2_{1/2,p',1/2,M'}(T) \) for \( p' \leq p \) and \( M' \geq M \).

Let \( \mathcal{E}(H) \) be the set of eigenvalues and thresholds of \( H \). Classical Mourre theory says that under appropriate conditions for \( V \), and \( W = 0 \), then the LAP
\[
\sup_{\eta > 0} \|(H - \lambda - i\eta)^{-1}\psi\|_{\mathcal{K}^\ast} \leq c(\lambda)\|\psi\|_{\mathcal{K}}
\]
holds for some appropriate pair of Banach spaces \( (\mathcal{K}, \mathcal{K}^\ast) \), \( c(\lambda) > 0 \) and \( \lambda \in \mathbb{R}\setminus\mathcal{E}(H) \). Also, \( c(\lambda) \) can be chosen uniform in \( \lambda \) over fixed compact subsets of \( \mathbb{R}\setminus\mathcal{E}(H) \). On the one hand, the spaces \( (\mathcal{K}, \mathcal{K}^\ast) = (B(A), B^*(A)) \) are optimal in a certain sense, see [JP], [AH] and the discussion at the beginning of [ABG, Chapter 7], but these are not Besov spaces. On the other hand, the Besov spaces \( (\mathcal{K}, \mathcal{K}^\ast) = (\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty}) \) which appear in [ABG] and [BSa] are not as optimal as \( (B(A), B^*(A)) \), but are optimal in the scale of Besov spaces and allow for a larger class of potentials \( V \). Note that our new result Theorem 1.2 implies (1.20) for \( \mathcal{K} = L^2_{1/2,p,1/2,M}(A) \), any \( p > 1/2, M \geq m \), while Theorem 1.4 implies (1.20) for \( \mathcal{K} = L^2_{2}(A) \), any \( s > 1/2 \).

Finally a few comments about the proof of Theorem 1.2. The skeleton and heart remain the same as in [GJ2], or [Ma1]. The difference comes from using the function \( \varphi \) in (1.5) rather than (1.4). Although this appears like a minor difference it creates the non-trivial task of bounding functions of the generator of dilations \( A \) by functions of the operator of position \( |N| \). More specifically, while the proof of Theorem 1.4 required only \( \langle A \rangle \langle N \rangle^{-\varepsilon} \in \mathcal{B}(\mathcal{H}) \), the bounded operators on \( \mathcal{H} \), the proof of Theorem 1.2 requires \( w^{\alpha,\beta}_M(A) \cdot w^{-\alpha,-\beta}_M(N) \in \mathcal{B}(\mathcal{H}) \) for some appropriate \( \alpha, \beta \geq 0 \), which is equivalent to \( w^{2\alpha,2\beta}_M(N) \leq w^{-2\alpha,-2\beta}_M(N) \) in the sense of forms. To achieve this we make a detour via the polylogarithm functions \( \text{Li}_\sigma(z) \) (specifically the ones of order \( 2 < \sigma \leq 3 \) are used). The reason for doing this is that higher powers of the logarithm are not Nevanlinna functions, but the functions \( \text{Li}_\sigma(z) \), \( \text{Re}(\sigma) > 0 \), are Nevanlinna functions which continue analytically across \((-1, +\infty)\) from \( \mathbb{C}_+ \) to \( \mathbb{C}_- \), and are thus operator monotone functions, see Section 8.

The plan of the article is as follows. In Section 2, we recall the relevant definitions and results of classical and weighted Mourre theory. In Section 3 we derive the classical Mourre estimate for our Hamiltonian \( H \) and prove Proposition 1.1. In Section 4 we derive operator norm bounds involving logarithms of the operator \( A \). In Section 5 we prove several key Lemmas that are needed for the proof of Theorem 1.2. In Section 6 we prove Theorem 1.2. The last 3 Sections are appendices. In appendix A (Section 7) we review Loewner’s theorem which makes the connection between Nevanlinna functions and operator monotone functions, and prove an extension of Loewner’s theorem that is suitable for unbounded self-adjoint operators. In appendix B (Section 8) we briefly review polylogarithms and explain that the ones of positive order are Nevanlinna functions. In appendix C (Section 9) we recall a few basics about almost analytic extensions and the Helffer-Sjöstrand formula.

**Acknowledgements**: We thank Thierry Jecko for fruitful conversations on the topic.

2. Basics of the abstract classical and weighted Mourre theories

2.1. **Operator Regularity**. We consider two self-adjoint operators \( T \) and \( A \) acting in some complex Hilbert space \( \mathcal{H} \), and for the purpose of this brief overview \( T \) will be bounded. Given \( k \in \mathbb{N} \), we say that \( T \) is of class \( C^k(A) \), and write \( T \in C^k(A) \), if the map
\[
\mathbb{R} \ni t \mapsto e^{itA}T_e^{-itA} \in \mathcal{B}(\mathcal{H})
\]
has the usual $C^k(\mathbb{R})$ regularity with $\mathcal{B}(\mathcal{H})$ endowed with the strong operator topology. The form $[T, A]$ is defined on $\text{Dom}[A] \times \text{Dom}[A]$ by $\langle \psi, [T, A] \phi \rangle := \langle T\psi, A\phi \rangle - \langle A\psi, T\phi \rangle$. Recall the following convenient result:

**Proposition 2.1.** [ABG, Lemma 6.2.9] Let $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent:

1. $T \in C^1(A)$.
2. The form $[T, A]$ extends to a bounded form on $\mathcal{H} \times \mathcal{H}$ defining a bounded operator denoted by $\text{ad}^1(T) := [T, A]_\circ$.
3. $T$ preserves $\text{Dom}[A]$ and the operator $TA - AT$, defined on $\text{Dom}[A]$, extends to a bounded operator.

Consequently, $T \in C^1(A)$ if and only if the iterated commutators $\text{ad}^p_A(T) := [\text{ad}^{p-1}_A(T), A]_\circ$ are bounded for $1 \leq p \leq k$. We say that $T \in C^k(A)$ if the map (2.1) has the $C^k(\mathbb{R})$ regularity with $\mathcal{B}(\mathcal{H})$ endowed with the norm operator topology. We say that $T \in C^{1,1}(A)$ if

$$
\int_0^1 \|[T, e^{itA}]_\circ, e^{itA}]_\circ\|t^{-2}dt < \infty.
$$

It turns out that $C^2(A) \subset C^{1,1}(A) \subset C^1(A) \subset C^1(A)$.

2.2. The Mourre estimate. Let $I$ be an open interval and assume $T \in C^1(A)$. We say that the **Mourre estimate** holds for $T$ on $I$ if there is $\gamma > 0$ and a compact operator $K$ such that

$$
E_I(T)[T, iA]_\circ E_I(T) \geq \gamma E_I(T) + K
$$

in the form sense on $\text{Dom}[A] \times \text{Dom}[A]$. We say that the **strict Mourre estimate** holds for $T$ on $I$ if (2.2) holds with $K = 0$. Assuming the estimate holds over $I$, $T$ has at most finitely many eigenvalues in $I$, and they are of finite multiplicity, while if the strict estimate holds $T$ has no eigenvalues in $I$. This is a direct consequence of the Virial Theorem, see [ABG, Proposition 7.2.10]. Let $I(E; \varepsilon)$ be the open interval of radius $\varepsilon > 0$ centered at $E \in \mathbb{R}$. If the strict Mourre estimate holds for $T$ on some interval containing $E$, one considers the function $\varphi^A_T : \mathbb{R} \to \mathbb{R}$:

$$
\varphi^A_T : E \to \sup \{ \gamma \in \mathbb{R} : \exists \varepsilon > 0 \text{ such that } E_{I(E;\varepsilon)}(T)[T, iA]_\circ E_{I(E;\varepsilon)}(T) \geq \gamma \cdot E_{I(E;\varepsilon)}(T) \}.
$$

It is known for example that $\varphi^A_T$ is lower semicontinuous and $\varphi^A_T(E) < \infty$ if and only if $E \in \sigma(T)$. For more properties of this function, see [ABG, chapter 7].

2.3. The weighted Mourre estimate & LAP. In [GJ2] it is explained in a more general setting that a **projected weighted estimate** of the form

$$
P^\perp E_I(T)[T, iB]_\circ E_I(T)P^\perp \geq cP^\perp E_I(T)C^2E_I(T)P^\perp,
$$

where $c > 0$, $B, C$ are any linear operators satisfying $CBC^{-1} \in \mathcal{B}(\mathcal{H})$, and $C$ self-adjoint and injective, will imply a LAP of the form

$$
\sup_{\Re(z) \in I, \ \Im(z) > 0} \|C(T - z)^{-1}P^\perp C\| < \infty.
$$

The proof is an argument by contradiction and therefore does not provide a description of the continuity properties of the LAP resolvent. Nonetheless, the resolvent estimate (2.4) implies the absence of singular continuous spectrum for $H$ in $I$, and if $B$ is $T$-bounded then (2.3) also gives the time decay estimate

$$
\int_\mathbb{R} \|Ce^{itT}E_I(T)\psi\|^2 < \infty, \ \psi \in \mathcal{H}.
$$
3. Verifying the classical Mourre estimate

In this Section we derive the Mourre estimate described in the previous Section for $T = H$ given by (1.6). We use freely Proposition 2.1. Let

\[
\Theta(\Delta) := \{t_0, t_1, ..., t_d\}, \quad \text{where} \quad t_k = 4k, k = 0, 1, ..., d.
\]

These are the thresholds of $\Delta$. Recall that our choice of conjugate operator $A$ is the closure of (1.11). This choice is justified by the fact that the commutator between $\Delta$ and $A$ is:

\[
[\Delta, iA] = \sum_{k=1}^{d} \Delta_k(4 - \Delta_k).
\]

In particular $\Delta \in \mathcal{C}^1(A)$. Note that, since $\sigma(\Delta_k) = [0, 4], \forall k = 1, ..., d$, the strict Mourre estimate holds for $\Delta$ and $A$ over any interval $I \subseteq \sigma(\Delta) \setminus \Theta(\Delta) = [0, 4d] \setminus \Theta(\Delta)$. In fact, one can be more specific and prove that if $E \in \sigma_k(\Delta) := [t_{k-1}, t_k]$, for some $k = 1, ..., d$, then

\[
g_\Delta^k(E) = -(E - t_{k-1})(E - t_k).
\]

This can be proved by induction on $d$ and using the nifty result [ABG, Theorem 8.3.6]. One can further show that for all $k = 1, ..., d$, $[\Delta, iA]$ decomposes into the sum of a positive operator and a remainder $b_k(\Delta)$, namely

\[
[\Delta, iA] = -(\Delta - t_{k-1})(\Delta - t_k) + b_k(\Delta),
\]

where

\[
b_k(\Delta) := -8(k - 1)\Delta + 16k(k - 1) + \sum_{1 \leq i, j \leq d} \Delta_i \Delta_j.
\]

The remainder $b_k(\Delta)$ is a non-negative operator from which no strict positivity can be extracted by localizing in energy. Finally, it is not hard to prove that $\Delta \in \mathcal{C}^2(A)$, or $\mathcal{C}^\infty(A)$ for that matter. Moving on to the commutator between the potential $V$ and $A$, we have:

\[
[V, iA] = \sum_{i=1}^{d} (2^{-1} - N_i)(V - \tau_i V)S_i + (2^{-1} + N_i)(V - \tau^*_i V)S^*_i.
\]

It is readily seen that the assumption (1.10), for some $m \in \mathbb{N}, 0 \leq r < q$, implies that $[V, iA]$ is a compact operator. In particular $V \in \mathcal{C}^1(A)$. Finally, regarding the commutator between the Wigner-von Neumann potential $W$ and $A$, we have $[W, iA] = K_W + B_W$ where

\[
K_W := 2^{-1}W \sum_{i=1}^{d} (S^*_i + S_i) + 2^{-1} \sum_{i=1}^{d} (S^*_i + S_i)W,
\]

\[
B_W := \sum_{i=1}^{d} U_i \tilde{W}(S^*_i - S_i) - \sum_{i=1}^{d} (S^*_i - S_i)\tilde{W}U_i,
\]

$\tilde{W}$ is the operator $(\tilde{W}\psi)(n) := w \sin(k(n_1 + ... + n_d))\psi(n)$ and $U_i$ is the operator $(U_i\psi)(n) := n_i |n|^{-1}\psi(n)$. Note that $K_W$ is compact and $B_W$ is bounded. Thus $W \in \mathcal{C}^1(A)$.

The following Lemma applies to the one-dimensional Laplacian ($d = 1$).

**Lemma 3.1.** [Ma1, Lemma 3.4] Recall that $E_\pm(k) := 2 \pm 2 \cos(k/2)$. Let $E \in [0, 4] \setminus \{E_\pm(k)\}$. Then there exists $\varepsilon = \varepsilon(E) > 0$ such that for all $\theta \in \mathcal{C}^\infty(\mathbb{R})$ supported on $I = (E - \varepsilon, E + \varepsilon)$, $\theta(\Delta)\tilde{W}\theta(\Delta) = 0$. Thus $\theta(\Delta)B_W\theta(\Delta)$ is a compact operator.

The following Lemma applies to the multi-dimensional Laplacian ($d \geq 2$).
Proposition 3.2. [Ma1, Proposition 4.5] Let

\[
E(k) := \begin{cases} 
4 - 4 \cos(k/2) & \text{for } k \in (0, \pi) \\
4 + 4 \cos(k/2) & \text{for } k \in (\pi, 2\pi)
\end{cases}
\quad \text{and} \quad \mu(H) := (0, E(k)) \cup (4d - E(k), 4d).
\]

For each \( E \in \mu(H) \) there exists \( \varepsilon = \varepsilon(E) > 0 \) such that for all \( \theta \in C_c^\infty(\mathbb{R}) \) supported on \( I := (E - \varepsilon, E + \varepsilon) \), \( \theta(\Delta)\bar{W}\theta(\Delta) = 0 \). In particular, \( \theta(\Delta)\bar{W}\theta(\Delta) \) is a compact operator.

Putting together everything discussed in this Section we get the Mourre estimate for \( H \):

Proposition 3.3. Let \( d \geq 1 \), \( H = \Delta + V + W \) and \( \mu(H) \) be as defined in the Introduction. Suppose that \( V \) satisfies (1.10) for some \( m \in \mathbb{N} \) and \( 0 \leq r < q \). Then \( H \in C^1(A) \), and for any \( I \in \mu(H) \), there are \( \gamma > 0 \) and compact \( K \) such that the Mourre estimate \( E_I(H) [H, iA], E_I(H) \geq \gamma E_I(H) + K \) holds. \( I \) contains at most finitely many eigenvalues and these are of finite multiplicity. The corresponding eigenfunctions, if any, decay sub-exponentially. If \( d = 1 \) and \( E \in (E_-(k), E_+(k)) \), then the point spectrum of \( H \) is empty in \( I \) provided \( I \subset (E_-(k), E_+(k)) \).

Proposition 3.3 is basically a reformulation of Proposition 1.1. The proof is easy and goes along the lines of [Ma1, Proposition 3.5]. The decay of the eigenfunctions is a consequence of [Ma2, Theorem 1.5], while the absence of eigenvalues is a consequence of [Ma2, Theorem 1.2]. Note that if \( H = \Delta + V \), with \( V \) as in Proposition 3.3 and \( W = 0 \), then the above shows that we have a Mourre estimate for \( H \) on any \( I \in [0, 4d] \setminus \Theta(\Delta) \).

4. Operator norm bounds for functions of \( A \)

For self-adjoint operators \( T, S, T \leq S \) means \( \langle \psi, T\psi \rangle \leq \langle \psi, S\psi \rangle \) for all \( \psi \in \text{Dom}[T] \cap \text{Dom}[S] \).

For this Section, it is useful to know that \( \langle A \rangle \) and \( \langle N \rangle \) are unbounded operators. This is seen directly from the graph norm and using an appropriate sequence of unit vectors. The following result appears in [Ma1, Lemma 5.1] but we repeat the proof for convenience.

Lemma 4.1. Let \( c_4 := \max(d^3 + d^2 + 1, 4(d + 1)) \). \( \forall 0 \leq \alpha \leq 1 \), \( (A^2 + 1)^\alpha \leq (c_4)^\alpha (N^2 + 1)^\alpha \).

Proof. Let \( \psi \in \text{Dom}[N^2] \subset \text{Dom}[A^2] \). First we calculate for \( \alpha = 1 \):

\[
\langle \psi, (A^2 + 1)\psi \rangle = \|\psi\|^2 + \|A\psi\|^2 \leq \|\psi\|^2 + \left( \sum_{j=1}^d \|\psi\|^2 + 2\|N_j\psi\| \right)^2
\]

\[
\leq \|\psi\|^2 + \left( d^2 \|\psi\|^2 + \sum_{j=1}^d 4\|N_j\psi\|^2 \right)(d + 1) \leq c_4 \left( \|\psi\|^2 + \sum_{j=1}^d \|N_j\psi\|^2 \right)
\]

\[
= c_4 \langle \psi, (N^2 + 1)\psi \rangle.
\]

Pass to exponent \( \alpha \) by invoking the Heinz inequality, see [RS, Ex. 51 of Chapter VIII] or [H].

For convenience we will also be writing (in this Section only)

\[
A := \langle A \rangle = \sqrt{A^2 + 1}, \quad \text{and} \quad N := \sqrt{c_4} \langle N \rangle = \sqrt{c_4} \sqrt{N^2 + 1}.
\]

Note that \( 1 \leq A \) and \( \sqrt{S} = \sqrt{c_1} \leq N \). By Lemma 4.1, \( A^{2\alpha} \leq N^{2\alpha} \), \( 0 \leq \alpha \leq 1 \). But this is equivalent to saying that \( \|A^\alpha\psi\| \leq \|N^\alpha\psi\| \) for all \( \psi \in \text{Dom}[A^\alpha] \cap \text{Dom}[N^\alpha] \). Let \( \ell_0(\mathbb{Z}^d) \) be the sequences in \( \mathbb{Z}^d \) having compact support. Note that \( \ell_0(\mathbb{Z}^d) \subset \text{Dom}[N^{\alpha}] \subset \text{Dom}[A^\alpha] \), and \( \ell_0(\mathbb{Z}^d) \) is dense in \( \ell^2(\mathbb{Z}^d) \). Since \( N^\alpha : \ell_0(\mathbb{Z}^d) \mapsto \ell_0(\mathbb{Z}^d) \) is surjective, it follows that \( \|A^\alpha N^{-\alpha}\psi\| \leq \|\psi\| \), at least for all \( \psi \in \ell_0(\mathbb{Z}^d) \). Thus \( A^\alpha N^{-\alpha} \in \mathcal{B}(\mathcal{H}) \). In what follows we shall be using this small argument repeatedly. We will also use the Helffer-Sjöstrand formula extensively, see Appendix 9. Let \( \Phi_n, n \in \mathbb{N}^*, \) be the polylogarithmic functions from (8.3). Denote \( \Phi_n(x) := (\Phi_n(x))^\alpha \).
Lemma 4.2. For all $0 \leq \alpha \leq 1/2$, $\Phi_n^\alpha(A) \cdot \Phi_n^{-\alpha}(N) \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}^*$. 

Proof. As discussed in Section 8 the $\Phi_n$, $n \in \mathbb{N}^*$, are Nevanlinna functions that continue analytically across $(-1, +\infty)$ from $\mathcal{C}_+$ to $\mathcal{C}_-$. Since $1 \leq A \leq N$, Theorem 7.3 implies $\Phi_n(A) \leq \Phi_n(N)$. By the Heinz inequality, $\Phi_n^{2\alpha}(A) \leq \Phi_n^{2\alpha}(N)$, $0 \leq \alpha \leq 1/2$. The result follows. □

Lemma 4.3. For all $k \in \mathbb{N}$ and $0 \leq \alpha \leq 1/2$,

$$\Phi_n^\alpha(\log_k(A)) \cdot \Phi_n^{-\alpha}(\log_k(N)) \in \mathcal{B}(\mathcal{H}), \quad n \in \mathbb{N}^*.$$

Proof. $\Phi_n(\log_k(x))$ belongs to $P(-1, +\infty)$ because it is the composition of functions all belonging to $P(-1, +\infty)$. Also, $1 \leq A \leq N$. By Loewner’s Theorem, $\Phi_n(\log_k(A)) \leq \Phi_n(\log_k(N))$. The result follows from the Heinz inequality. □

Lemma 4.4. For all $0 \leq p \leq 3/2$, $\log^p(1 + \langle A \rangle) \cdot \log^{-p}(1 + \langle N \rangle) \in \mathcal{B}(\mathcal{H})$.

Remark 4.1. For $0 \leq p \leq 1/2$, the result follows from well-known results on the logarithm, see e.g. [RS1, Exercise 51 of Chapter VIII]. For $p > 1/2$ we don’t know how prove the result without the use of the polylogarithmic functions.

Proof. For $0 \leq \alpha \leq 1/2$, $\log^3(1 + \langle A \rangle) \cdot \log^{-3}(1 + \langle N \rangle)$ equals

$$\log^3(1 + \langle A \rangle) \Phi_3^\alpha(A) \cdot \Phi_3^{-\alpha}(N) \leq \Phi_3^\alpha(N) \log^{-3}(1 + \langle N \rangle),$$

bounded by (8.4) bounded by Lemma 4.2 bounded by definition of $N$. □

Lemma 4.5. For all $k \in \mathbb{N}$ and $0 \leq p \leq 3/2$, $\log^p_k(\langle A \rangle) \cdot \log^{-p}_k(\langle N \rangle) \in \mathcal{B}(\mathcal{H})$.

Proof. The statement is trivial for $k = 0$ and $k = 1$ is Lemma 4.4. Now let $k \geq 1$. For $0 \leq \alpha \leq 1/2$, $\log^3_k(\langle A \rangle) \cdot \log^{-3}_k(\langle N \rangle)$ equals

$$\log^3_k(\langle A \rangle) \Phi_3^\alpha(\log_k(A)) \Phi_3^{-\alpha}(\log_k(N)) \leq \Phi_3^\alpha(\log_k(A)) \Phi_3^{-\alpha}(\log_k(N)).$$

bounded by (8.4) bounded by Lemma 4.3 bounded by (8.4). □

The following Lemma is used explicitly in the proof of Theorem 1.2.

Lemma 4.6. For all $m \in \mathbb{N}$ and $0 \leq p \leq 3/2$,

$$\log^p_{m+1}(\langle A \rangle) \prod_{k=0}^m \log_k(\langle A \rangle) \cdot \log^{-p}_{m+1}(\langle N \rangle) \prod_{k=0}^m \log^{-1}_k(\langle N \rangle) \in \mathcal{B}(\mathcal{H}).$$

Proof. By induction on $m$. The base case $m = 0$ is Lemma 4.4 (recall that $\log_1(x) := \log(1 + x)$). For the inductive step, $m \rightarrow m + 1$, we have

$$\log^p_{m+2}(\langle A \rangle) \prod_{k=0}^{m+1} \log_k(\langle A \rangle) \cdot \log^{-p}_{m+2}(\langle N \rangle) \prod_{k=0}^{m+1} \log^{-1}_k(\langle N \rangle).$$

After commuting $\log^p_{m+2}(\langle A \rangle)$ with $\prod_{k=0}^{m+1} \log^{-1}_k(\langle N \rangle)$ this operator is equal to:

$$\prod_{k=0}^{m+1} \log_k(\langle A \rangle) \prod_{k=0}^{m+1} \log^{-1}_k(\langle N \rangle) \cdot \log^p_{m+2}(\langle A \rangle) \log^{-p}_{m+2}(\langle N \rangle)$$

bounded by the Induction hypothesis

$$+ \prod_{k=0}^{m+1} \log_k(\langle A \rangle) \left[ \log^p_{m+2}(\langle A \rangle), \prod_{k=0}^{m+1} \log^{-1}_k(\langle N \rangle) \right] \log^{-p}_{m+2}(\langle N \rangle).$$

to be developed.
It is enough to show that the latter term with the under bracket is a bounded operator. It is equal to
\[
\frac{i}{2\pi} \int_C \frac{\partial}{\partial z} \prod_{k=0}^{m+1} \log_k (\langle A \rangle) (z - A)^{-1} \left[ A, \prod_{k=0}^{m+1} \log_k (\langle N \rangle) \right] (z - A)^{-1} dz \wedge d\bar{z},
\]
where \( f(x) = \log^{p+2}_m (\langle x \rangle) \in S^2(\mathbb{R}), \forall \varepsilon > 0 \). By Lemma 4.7, \( [A, \prod_{k=0}^{m+1} \log_k (\langle N \rangle)] \in \mathcal{B}(\mathcal{H}) \). To keep things simple, one can use Lemma 4.10 to get
\[
\left\| \prod_{k=0}^{m+1} \log_k (\langle A \rangle) (z - A)^{-1} \right\| \leq c \| A \| \delta (z - A)^{-1} \leq c \langle x \rangle^{\delta} |y|^{-1}
\]
for some \( \delta > 0 \) and \( c > 0 \). Thus the above integral converges in norm to a bounded operator. □

**Lemma 4.7.** \( \langle N \rangle^{-\sigma} \) and \( \prod_{k=0}^{m} \log_k (\langle N \rangle) \) are of class \( \in \mathcal{C}^1 (A) \), \( \forall \sigma > 0 \) and \( m \in \mathbb{N} \).

The proof of Lemma 4.7 is a direct application of Proposition 2.1 and (3.3). The following two Lemmas allow to pass from \( \langle A \rangle \) to \( \langle N \rangle \) in the LAP (1.14) and the local decay estimate (1.16).

**Lemma 4.8.** For all \( 0 \leq p \leq 3/2, \langle A \rangle^{\frac{1}{2}} \log^p (1 + \langle A \rangle) \cdot \langle N \rangle^{-\frac{1}{2}} \log^{-p} (1 + \langle N \rangle) \in \mathcal{B}(\mathcal{H}) \).

**Proof.** Since \( \langle A \rangle^{1/2} \langle N \rangle^{-1/2} \in \mathcal{B}(\mathcal{H}) \) and \( \log^p (1 + \langle A \rangle) \log^{-p} (1 + \langle N \rangle) \in \mathcal{B}(\mathcal{H}) \), it is enough to prove that \( \langle A \rangle^{1/2} \langle N \rangle^{-1/2} \in \mathcal{B}(\mathcal{H}) \). But the latter operator is equal to
\[
\frac{i}{2\pi} \int_C \frac{\partial}{\partial z} \langle A \rangle^{1/2} (z - A)^{-1} [A, \langle N \rangle^{-1/2}] (z - A)^{-1} dz \wedge d\bar{z},
\]
where \( f(x) = \log^p (1 + \langle x \rangle) \in S^2(\mathbb{R}), \forall \varepsilon > 0 \). By Lemma 4.7 \( [A, \langle N \rangle^{-1/2}] \in \mathcal{B}(\mathcal{H}) \). Applying Lemma 4.10 shows that the above integral converges in norm to a bounded operator. □

**Lemma 4.9.** For all \( m \in \mathbb{N}, 0 \leq p, q \leq 3/2, \)
\[
\langle A \rangle^{\frac{1}{2}} \log_{m+1}^p (\langle A \rangle) \prod_{k=0}^{m} \log_k^q (\langle A \rangle) \cdot \langle N \rangle^{-\frac{1}{2}} \log_{m+1}^{-p} (\langle N \rangle) \prod_{k=0}^{m} \log_k^{-q} (\langle N \rangle) \in \mathcal{B}(\mathcal{H}).
\]

**Proof.** Again by induction on \( m \). The base case \( m = 0 \) is Lemma 4.8. The inductive step is handled in the same way as in Lemma 4.6. One commutes \( \log_{m+2}^p (\langle A \rangle) \) with \( \langle N \rangle^{-\frac{1}{2}} \prod_{k=0}^{m+1} \log_k^{-q} (\langle N \rangle) \). One applies the inductive hypothesis to first term, and for the second term (the one with the commutator) it is enough to show that
\[
\langle A \rangle^{\frac{1}{2}} \prod_{k=0}^{m+1} \log_k^q (\langle A \rangle) \left[ \log_{m+2}^p (\langle A \rangle), \langle N \rangle^{-\frac{1}{2}} \prod_{k=0}^{m+1} \log_k^{-q} (\langle N \rangle) \right] \in \mathcal{B}(\mathcal{H}).
\]
To this end one performs a similar integral expansion as in the proof of Lemma 4.6, with the same function \( f(x) = \log_{m+2}^p (\langle x \rangle) \in S^2(\mathbb{R}), \forall \varepsilon > 0 \). Then, to keep things simple, one can use the fact that there are \( \delta > 0 \) and \( c > 0 \) (by Lemma 4.10) such that
\[
\left\| \langle A \rangle^{\frac{1}{2}} \prod_{k=0}^{m+1} \log_k^q (\langle A \rangle) (z - A)^{-1} \right\| \leq c \| A \|^{1/2+\delta} (z - A)^{-1} \leq c \langle x \rangle^{1/2+\delta} |y|^{-1}
\]
One sees that the integral converges in norm to a bounded operator. □

The next lemma has been proved in many places, e.g. [DG, G, GJ2]. We propose a proof by contradiction that we will generalise in the next lemma.

**Lemma 4.10.** Let \( A \) be a self-adjoint operator. Let \( \Omega := \{(x, y) \in \mathbb{R}^2 : 0 < |y| < c \langle x \rangle \}, \) for some \( c > 0 \). Then for every \( 0 \leq s \leq 1 \) there exists \( C > 0 \) such that for all \( z = x + iy \in \Omega \):
\[
\| A \|^s (A - z)^{-1} \| \leq C \langle x \rangle^s |y|^{-1}.
\]
Proof. If $s = 0$ one can take $C = 1$. Suppose that $s > 0$. By the spectral theorem,

$$
\| \langle A \rangle ^s (A - z)^{-1} \| ^2 \leq \sup _{t \in \mathbb{R}} \langle t \rangle ^{2s} \left( (t - x)^2 + y^2 \right)^{-1}.
$$

Let

$$
f_s(x, y, t) := \frac{\langle t \rangle ^{2s} y^2}{\langle x \rangle ^{2s} (t - x)^2 + y^2}, \text{ defined on } (x, y, t) \in \overline{\Omega} \times \mathbb{R} \setminus \Lambda,
$$

where $\overline{\Omega} = \{ (x, y) \in \mathbb{R}^2 : |y| \leq c \langle x \rangle \}$ and $\Lambda := \{ (x, y, t) \in \overline{\Omega} \times \mathbb{R} : y = 0, x = t \}$. To prove the Lemma, it is enough to show that $f_s$ is uniformly bounded on the entire region $\overline{\Omega} \times \mathbb{R}$. Suppose that there is a sequence $(x_n, y_n, t_n) \in \overline{\Omega} \times \mathbb{R}$ such that $f_s(x_n, y_n, t_n) \to +\infty$. Since we have

$$
0 \leq \frac{y^2}{(t - x)^2 + y^2} \leq 1, \quad \forall (x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},
$$

and since $s > 0$, we infer that $\langle t_n \rangle / \langle x_n \rangle \to \infty$, as $n \to \infty$. In particular, since $s \leq 1$, we have

$$
\langle t_n \rangle ^s / \langle x_n \rangle ^s \leq \langle t_n \rangle / \langle x_n \rangle \text{ for } n \text{ large enough and } |t_n| \to \infty, \text{ as } n \to \infty.
$$

Next, we have:

$$
0 \leq f_s(x_n, y_n, t_n) \leq \frac{\langle t_n \rangle ^2}{\langle x_n \rangle ^2} \left( \frac{\langle t_n \rangle}{\langle t_n \rangle - \langle x_n \rangle} \right)^2 \frac{y_n^2}{\langle x_n \rangle ^2} \leq \frac{c}{\left( \frac{\langle t_n \rangle}{\langle t_n \rangle - \langle x_n \rangle} \right)^2}.
$$

We infer:

$$
\limsup _{n \to \infty} f_s(x_n, y_n, t_n) \leq c,
$$

which is a contradiction. \qed

The next lemma is new and proved by contradiction. It seems complicated to give a direct proof.

**Lemma 4.11.** Let $A$ and $\Omega$ be as in Lemma 4.10. For every $0 < s < 1$ and $p \in \mathbb{R}$ there exists $C > 0$ such that for all $z = x + iy \in \Omega$:

$$
\| \log ^p (1 + \langle A \rangle) \langle A \rangle ^s (A - z)^{-1} \| \leq C \log ^p (1 + \langle x \rangle) \langle x \rangle ^s |y| ^{-1}.
$$

**Proof.** The case $p = 0$ is covered by the previous lemma. By the spectral theorem,

$$
\| \log ^p (1 + \langle A \rangle) \langle A \rangle ^s (A - z)^{-1} \|^2 \leq \sup _{t \in \mathbb{R}} \log ^{2p} (1 + \langle t \rangle) \langle t \rangle ^{2s} \left( (t - x)^2 + y^2 \right)^{-1}.
$$

Therefore, we consider the function

$$
g_s(x, y, t) := \frac{\log ^{2p} (1 + \langle t \rangle) \langle t \rangle ^{2s} y^2}{\log ^{2p} (1 + \langle x \rangle) \langle x \rangle ^{2s} (t - x)^2 + y^2}, \text{ defined on } (x, y, t) \in \overline{\Omega} \times \mathbb{R} \setminus \Lambda,
$$

where $\overline{\Omega} = \{ (x, y) \in \mathbb{R}^2 : |y| \leq c \langle x \rangle \}$ and $\Lambda := \{ (x, y, t) \in \overline{\Omega} \times \mathbb{R} : y = 0, x = t \}$. We aim at proving that $g_s$ is uniformly bounded on the entire region $\overline{\Omega} \times \mathbb{R}$. Suppose that there is a sequence $(x_n, y_n, t_n) \in \overline{\Omega} \times \mathbb{R}$ such that $g_s(x_n, y_n, t_n) \to +\infty$. Recalling (4.3), since $s > 0$, up to a subsequence, we infer that

- either $1) \frac{\log ^{2p} (1 + \langle t_n \rangle)}{\log ^{2p} (1 + \langle x_n \rangle)} \to \infty$,
- or $2) \frac{\langle t_n \rangle}{\langle x_n \rangle} \to \infty$,

as $n \to \infty$. 

We start with the case 2). Given $\alpha > 0$, there are $C_\alpha, C'_\alpha > 0$ such that $\forall a, b \in \mathbb{R}$ we have

\[(4.6) \quad 0 \leq \frac{\log(1 + \langle a \rangle)}{\log(1 + \langle b \rangle)} = \frac{\log \left(\frac{1 + \langle a \rangle}{1 + \langle b \rangle}\right)}{\log(2)} + 1 \leq \frac{1}{\log(2)} \log \left(\frac{1 + \langle a \rangle}{1 + \langle b \rangle}\right) + 1 \leq C_\alpha \left(\frac{1 + \langle a \rangle}{1 + \langle b \rangle}\right) \leq C'_\alpha \left(\frac{\langle a \rangle}{\langle b \rangle}\right)^{\alpha}.
\]

Recalling $s \in (0, 1)$ and by considering the case $p > 0$ and $p < 0$, by choosing $\alpha$ small enough, there is a finite constant $C$ such that

\[0 \leq g_s(x_n, y_n, t_n) \leq C f_1(x_n, y_n, t_n)\]

Repeating (4.4), we obtain a contradiction.

We turn to the case 1). If $p > 0$, (4.6) ensures that $\frac{1 + \langle t_n \rangle}{1 + x_n} \to \infty$, as $n \to \infty$. Hence the case 2) holds which in turn gives a contradiction.

Suppose now that 1) holds and $p < 0$. Unlike before, we obtain this time that $\frac{\langle x_n \rangle}{\langle t_n \rangle} \to \infty$, as $n \to \infty$. Using (4.3) and (4.6), there is $C, s' > 0$ such that

\[0 \leq g_s(x_n, y_n, t_n) \leq C \left(\frac{\langle t_n \rangle}{\langle x_n \rangle}\right)^{s'} \to 0,
\]
as $n \to \infty$. This is a contradiction. \qed

**Lemma 4.12.** Let $A$ and $\Omega$ be as in Lemma 4.10. For every $0 < s < 1$ and $p, q \in \mathbb{R}$ there exists $C > 0$ such that for all $z = x + iy \in \Omega$

\[(4.7) \quad \left\|\langle A \rangle^s (A - z)^{-1} \log_{m+1}^p (\langle A \rangle) \prod_{k=0}^m \log_k^q (\langle A \rangle)\right\| \leq C \langle x \rangle^s |y|^{-1} \log_{m+1}^p (\langle x \rangle) \prod_{k=0}^m \log_k^q (\langle x \rangle).
\]

**Proof.** Thanks to (4.6), the proof is analogous to that of Lemma 4.11. \qed

5. Preliminary lemmas for the Proof of Theorem 1.2

Let $P$ denote the orthogonal projection onto the pure point spectral subspace of $H$.

**Proposition 5.1.** [GJ1] $\forall u, v \in \text{Dom}[A]$, the rank one operator $|u\rangle\langle v| : \psi \mapsto \langle v, \psi| u \in \mathcal{C}^1(A)$.

**Lemma 5.2.** Assume $V$ satisfies (1.9) and (1.10) for some $q > r = 2$, $m \in \mathbb{N}$. Then for any open interval $I \subset \mu(H)$ and any $\eta \in C^\infty_c(\mathbb{R})$ with supp$\eta \subset I$, $PE_I(H)$ and $P^\perp \eta(H) \in \mathcal{C}^1(A)$.

**Proof.** $I \subset \mu(H)$ so the Mourre estimate holds for $H$ and $A$ on $I$. In particular the point spectrum of $H$ is finite on $I$, see [ABG, Corollary 7.2.11]. Therefore $PE_I(H)$ is a finite rank operator. By [Ma2, Theorem 1.5], the eigenfunctions of $H$, if any, belong to the domain of $A$. We may therefore apply Lemma 5.1 to get $PE_I(H) \in \mathcal{C}^1(A)$. As for $P^\perp \eta(H)$ it is equal to $\eta(H) - PE_I(H)\eta(H)$, and so belongs to $\mathcal{C}^1(A)$. \qed

**Lemma 5.3.** Let $R > 1$ and recall that $w^{a,b}_M(A/R)$ is given by (1.15). Consider

\[\tilde{K}_0 = \eta(\Delta)[V, iA]\eta(\Delta) + (\eta(H) - \eta(\Delta))[H, iA]\eta(\Delta) + \eta(H)[H, iA]\eta(\Delta) - \eta(\Delta),\]

and

\[K_0 := \tilde{K}_0 + \eta(\Delta)[W, iA]\eta(\Delta) = \tilde{K}_0 + \eta(\Delta)K_W\eta(\Delta) + \eta(\Delta)B_W\eta(\Delta),\]

where $\eta \in C^\infty_c(\mathbb{R})$ is supported on an open connected interval $I \subset [0, 4d]\Theta(\Delta)$. Assume $V$ satisfies assumptions (1.9) and (1.10) for some $q > r = 2$, $m \in \mathbb{N}$. Then for all $1/2 < p < q/4$, $M \geq m$, $w^{p,1}_M(A/R)\tilde{K}_0w^{p,1}_M(A/R)$ is a compact operator whose norm is uniformly bounded w.r.t $R$.

Furthermore, if $I \subset \mu(H)$, then further shrinking the size of the interval $I$ also allows $w^{p,1}_M(A/R)\tilde{K}_0w^{p,1}_M(A/R)$ to be a compact operator whose norm is uniformly bounded w.r.t $R$. 
Assume Lemma 5.4. 

Proof. Write \( w_{M}^{2p,1}(A/R) \tilde{K}_0 w_{M}^{2p,1}(A/R) \) as 

\[
\begin{align*}
  w_{M}^{2p,1}(A) & = w_{M}^{2p-1}(A) \tilde{K} w_{M}^{2p-1}(A) w_{M}^{2p,1}(A), \\
  \tilde{K} & := w_{M}^{2p,1}(A) \tilde{K}_0 w_{M}^{2p,1}(A).
\end{align*}
\]

Thus we want to show that \( \tilde{K} \) is a compact operator. \( \tilde{K} \) is the sum of the following 3 terms:

\[
\begin{align*}
  \tilde{K}_1 & := w_{M}^{2p,1}(A) \eta(\Delta)[V, iA]_o \eta(\Delta) w_{M}^{2p,1}(A), \\
  \tilde{K}_2 & := w_{M}^{2p,1}(A) (\eta(H) - \eta(\Delta))[H, iA]_o \eta(\Delta) w_{M}^{2p,1}(A), \\
  \tilde{K}_3 & := w_{M}^{2p,1}(A) \eta(H)[H, iA]_o (\eta(H) - \eta(\Delta)) w_{M}^{2p,1}(A).
\end{align*}
\]

Each of these terms are compact, as explained below.

- For \( \tilde{K}_1 \) : we want to commute \( w_{M}^{2p,1}(A) \) with \( \eta(\Delta) \). Since \( \eta(\Delta) \in C^1(A) \) and \( w_{M}^{2p,1}(x) \in \mathcal{S}^e(\mathbb{R}) \), \( \forall \varepsilon > 0 \), \( [w_{M}^{2p,1}(A), \eta(\Delta)]_o \in \mathcal{B}(H) \) by Proposition 9.3. Also, \( w_{M}^{2p,1}(A) w_{M}^{2p-1}(N) \in \mathcal{B}(H) \) by Lemma 4.6, and \( w_{M}^{2p,1}(N)[V, iA]_o w_{M}^{2p,1}(N) \) is a compact operator, by the assumption (1.10) \((1/2 < p < q/4)\). Thus \( \tilde{K}_1 \) is a compact operator.

- For \( \tilde{K}_2 \) : write it as

\[
\begin{align*}
  w_{M}^{2p,1}(A)(\eta(H) - \eta(\Delta))[\Delta, iA]_o \eta(\Delta) w_{M}^{2p,1}(A) + w_{M}^{2p,1}(A)(\eta(H) - \eta(\Delta))[V, iA]_o \eta(\Delta) w_{M}^{2p,1}(A).
\end{align*}
\]

For the first term commute \( [\Delta, iA]_o \eta(\Delta) \) with \( w_{M}^{2p,1}(A) \) and then apply (5.2). For the second term commute \( \eta(\Delta) \) with \( w_{M}^{2p,1}(A) \) and then apply (5.1) and assumption (1.9). We leave the details to the reader.

- For \( \tilde{K}_3 \) : same idea as for \( \tilde{K}_2 \).

Now the second part of the statement where we assume \( I \subset \mu(H) \). Recall \( K_W \) and \( B_W \) are given by (3.4) and (3.5). It is enough to show that

\[
\tilde{K}_4 := w_{M}^{2p,1}(A) \eta(\Delta) K_W \eta(\Delta) w_{M}^{2p,1}(A)
\]

and

\[
\tilde{K}_5 := w_{M}^{2p,1}(A) \eta(\Delta) B_W \eta(\Delta) w_{M}^{2p,1}(A)
\]

are compact. For \( \tilde{K}_4 \), commute \( w_{M}^{2p,1}(A) \) with \( \eta(\Delta) \) and use the fact that \( w_{M}^{2p,1}(A) K_W w_{M}^{2p,1}(A) \) is compact. For \( \tilde{K}_5 \), use the fact that \( \eta(\Delta) \tilde{W} \eta(\Delta) = 0 \) if the the support of \( \eta \) is sufficiently small, by Lemma 3.1 and Proposition 3.2, and also that \( w_{M}^{2p,1}(A)[\eta(\Delta), U_I] \) is compact. \( \square \)

Lemma 5.4. Assume \( V \) satisfies (1.9) for some \( q > r = 2 \), \( m \in \mathbb{N} \). Let \( \eta \in C_c^{\infty}(\mathbb{R}) \) be supported on an open interval \( I \). Then \( \forall \ 0 < p < q/4, M \geq m \),

\[
(5.1) \quad (\eta(H) - \eta(\Delta)) w_{M}^{2p,1}(A),
\]

\[
(5.2) \quad w_{M}^{2p,1}(A)(\eta(H) - \eta(\Delta)) w_{M}^{2p,1}(A)
\]

are compact operators.

Proof. We prove the first one and leave the second one for the reader. By Proposition 9.3, \( \Delta \in C^1(w_{M}^{2p,1}(A)) \), since \( w_{M}^{2p,1}(x) \in \mathcal{S}^e(\mathbb{R}) \), \( \forall \varepsilon > 0 \), and \( \Delta \in C^1(A) \). Thus \( [\Delta, w_{M}^{2p,1}(A)]_o \in \mathcal{B}(H) \).
By the Helffer-Sjöstrand formula and the resolvent identity, \((\eta(H) - \eta(\Delta))w_{M}^{2p,1}(A)\) equals

\[
\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \hat{\eta}}{\partial \bar{z}}(z - H)^{-1}(V + W)(z - \Delta)^{-1}w_{M}^{2p,1}(A)dz \wedge d\bar{z}
\]

\[
= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \hat{\eta}}{\partial \bar{z}}(z - H)^{-1}(V + W)w_{M}^{2p,1}(A)(z - \Delta)^{-1}dz \wedge d\bar{z}
\]

\[
+ \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \hat{\eta}}{\partial \bar{z}}(z - H)^{-1}(V + W)(z - \Delta)^{-1}[\Delta, w_{M}^{2p,1}(A)]_{\circ}dz \wedge d\bar{z}
\]

The integrands of the last two integrals are compact operators. With the support of \(\eta\) bounded, the integrals converge in norm, and so the compactness is preserved in the limit. \(\square\)

6. Proof of Theorem 1.2

We are now ready to prove the projected weighted Mourre estimate (2.3), which in turn will imply the LAP (2.4). The proof makes use of almost analytic extensions of \(C^{\infty}(\mathbb{R})\) bounded functions and the class of functions \(\mathcal{S}^{0}(\mathbb{R})\) with \(p = 0\), see Appendix 9. We also mention that the proof is essentially the same as that of [GJ2, Theorem 4.15] (see also [Ma1, Theorem 5.4]), but we display it in detail for the reader’s convenience.

![Figure 1. From left to right : \(E_{f}(x)\), \(\theta(x)\), \(\eta(x)\), \(\chi(x)\), \(E_{f}(x)\).](image)

Proof of Theorem 1.2. Let \(E \in \mu(H)\) so let \(I \subset \mu(H)\) be an open interval containing \(E\) such that the Mourre estimate holds over \(I\). By Lemma 5.2, \(PE_{f}(H)\) and \(P^{\perp}\eta(H)\) are of class \(C^{1}(A)\), for any \(\eta \in C^{\infty}_{c}(\mathbb{R})\) with \(\text{supp}(\eta) \subset I\). Let \(\theta, \eta, \chi \in C^{\infty}_{c}(\mathbb{R})\) be bump functions such that \(\eta\theta = \theta\), \(\chi\eta = \eta\) and \(\text{supp}(\chi) \subset I\), see Figure 1. Later in the proof we will shrink the interval \(I\) around \(E\).

We aim to derive (2.3) on \(I\) for \(B = \varphi(A/R)\), \(C = \sqrt{\varphi'(A/R)}\), some \(R > 1\) and \(\varphi\) given by

\[
\varphi : \mathbb{R} \mapsto \mathbb{R}, \quad \varphi : t \mapsto \int_{-\infty}^{t} \frac{dx}{\langle x \rangle w_{M}^{2p,1}(x)}, \quad t \in \mathbb{R}, \quad p > 1/2.
\]

Note that \(\varphi \in \mathcal{S}^{0}(\mathbb{R})\), so that \(\varphi(A/R) \in \mathcal{B}(\mathfrak{H})\) for all \(R > 1\). Consider the bounded operator

\[
F := P^{\perp} \theta(H)[H, i\varphi(A/R)]_{\circ} \theta(H)P^{\perp}
\]

\[
= \frac{i}{2\pi} \int_{\mathbb{R}} \frac{\partial \hat{\varphi}}{\partial z}(z)P^{\perp} \theta(H)(z - A/R)^{-1}[H, iA]_{\circ}(z - A/R)^{-1} \theta(H)P^{\perp}dz \wedge d\bar{z}.
\]

To save room, henceforth we drop the element \(dz \wedge d\bar{z}\). By Lemma 5.2 \(P^{\perp}\eta(H) \in C^{1}(A)\), so

\[
[P^{\perp}\eta(H), (z - A/R)^{-1}]_{\circ} = (z - A/R)^{-1}[P^{\perp}\eta(H), A/R]_{\circ}(z - A/R)^{-1}
\]
Next to \( P^\perp \theta(H) \) we introduce \( P^\perp \eta(H) \) and commute it with \((z - A/R)^{-1}\). We get:

\[
F = \frac{i}{2\pi} \frac{1}{R} \int_C \frac{\partial \hat{\psi}}{\partial z}(z) P^\perp \theta(H)(z - A/R)^{-1} P^\perp \eta(H)[H, i A]_\circ \eta(H) \frac{1}{R} P^\perp (z - A/R)^{-1} \theta(H) P^\perp
\]

\[
+ P^\perp \theta(H) \frac{A}{R} \left( \frac{B_1 + B_2 + B_3}{R^2} \right) \frac{1}{R} P^\perp (z - A/R)^{-1} \theta(H) P^\perp,
\]

where \( B_1, B_2, B_3 \in \mathcal{B}(\mathcal{H}) \) are uniformly bounded in \( R \). We refer to [Ma1, Proof of Theorem 5.4] for extra details. Next to each \( \eta(H) \) we insert \( \chi(H) \), and then decompose \( \eta(H)[H, i A]_\circ \eta(H) \) as follows:

\[
\eta(H)[H, i A]_\circ \eta(H) = \eta(\Delta)[\Delta, i A]_\circ \eta(\Delta) + K_0,
\]

where

\[
K_0 = \eta(\Delta)[V + W, i A]_\circ \eta(\Delta) + (\eta(H) - \eta(\Delta)) \eta(\Delta) + \eta(H)[H, i A]_\circ (\eta(H) - \eta(\Delta)).
\]

Note that \( K_0 \) is the compact operator that appears in Lemma 5.3. We also let

\[
M_0 := P^\perp \chi(H) \eta(\Delta)[\Delta, i A]_\circ \eta(\Delta) \chi(H) P^\perp.
\]

Thus:

\[
F = \frac{i}{2\pi} \frac{1}{R} \int_C \frac{\partial \hat{\psi}}{\partial z}(z) P^\perp \theta(H)(z - A/R)^{-1} M_0 (z - A/R)^{-1} \theta(H) P^\perp
\]

\[
+ \frac{i}{2\pi} \frac{1}{R} \int_C \frac{\partial \hat{\psi}}{\partial z}(z) \frac{1}{R} P^\perp \chi(H) K_0 \chi(H) P^\perp (z - A/R)^{-1} \theta(H) P^\perp
\]

\[
+ P^\perp \theta(H) \frac{A}{R} \left( \frac{B_1 + B_2 + B_3}{R^2} \right) \frac{1}{R} P^\perp (z - A/R)^{-1} \theta(H) P^\perp.
\]

We work on the 2nd integral term given by (6.2).

**CLAIM.**

\[
\mathcal{K} := \frac{i}{2\pi} \int_C \frac{\partial \hat{\psi}}{\partial z}(z) \frac{A}{R} \left( \frac{B_1 + B_2 + B_3}{R^2} \right) \frac{1}{R} P^\perp \chi(H) K_0 \chi(H) P^\perp (z - A/R)^{-1} w^p M \left( \frac{A}{R} \right) \frac{1}{R} P^\perp (z - A/R)^{-1} w^p M \left( \frac{A}{R} \right) \frac{1}{R}
\]

is a compact operator for \( 1/2 < p < q/4 \), where \( q \) is the exponent that appears in (1.9) and (1.10) - Hypothesis (H). Also the norm of \( \mathcal{K} \) goes to zero as the support of \( \chi \) gets tighter around \( E \).

**PROOF OF CLAIM.** Write \( \mathcal{K} \) as

\[
\mathcal{K} = \frac{i}{2\pi} \int_C \frac{\partial \hat{\psi}}{\partial z}(z) \left( \frac{A}{R} \right) \left( \frac{B_1 + B_2 + B_3}{R^2} \right) (z - A/R)^{-1} \times
\]

\[
\text{bounded by Lemma 4.12}
\]

\[
w^p M \left( \frac{A}{R} \right) \frac{1}{R} P^\perp \chi(H) K_0 \chi(H) P^\perp (z - A/R)^{-1} w^p M \left( \frac{A}{R} \right) \frac{1}{R} P^\perp (z - A/R)^{-1} w^p M \left( \frac{A}{R} \right) \frac{1}{R} P^\perp (z - A/R)^{-1} w^p M \left( \frac{A}{R} \right) \frac{1}{R} P^\perp (z - A/R)^{-1} w^p M \left( \frac{A}{R} \right) \frac{1}{R}
\]

\[
\text{bounded by Lemma 4.12}
\]

Now we want to commute \( w^p M \left( \frac{A}{R} \right) \) with \( P^\perp \chi(H) \). Since \( P^\perp \chi(H) \in \mathcal{C}^1(A) \) and \( w^{p, 1} M(x) \in \mathcal{S}^\varepsilon(\mathbb{R}) \), \( \forall \varepsilon > 0 \),

\[
\left[ w^{p, 1} M \left( \frac{A}{R} \right), P^\perp \chi(H) \right] \in \mathcal{B}(\mathcal{H})
\]

by Proposition 9.3. So \( \mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 \), where
\[
K_1 = \frac{i}{2\pi} \int_C \frac{\partial \bar{\phi}}{\partial \bar{z}}(z) \left( z - A/R \right) \left( z - A/R \right)^{-1} \times \]
bounded by Lemma 4.12
\[
P_{\perp} \chi(H) w_M^{2p,1} \left( A/R \right) K_0 \chi(H) P_{\perp} \left( z - A/R \right)^{-1} w_M^{p,1} \left( A/R \right) \left( \frac{A}{R} \right) \left( \frac{A}{R} \right)^{1/2}
\]
bounded by Lemma 4.12

and
\[
K_2 = \frac{i}{2\pi} \int_C \frac{\partial \bar{\phi}}{\partial \bar{z}}(z) \left( z - A/R \right) \left( z - A/R \right)^{-1} \times \]
bounded by Lemma 4.12
\[
w_M^{2p,1} \left( A/R \right) \left[ w_M^{2p,1} \left( A/R \right) \cdot P_{\perp} \chi(H) \right] \cdot K_0 \chi(H) P_{\perp} \left( z - A/R \right)^{-1} w_M^{p,1} \left( A/R \right) \left( \frac{A}{R} \right) \left( \frac{A}{R} \right)^{1/2}.
\]
bounded uniformly in \( R \) by Proposition 9.3
bounded by Lemma 4.12

By Lemma 4.12, there is \( C > 0 \) such that
\[
\| A \left( \frac{A}{R} \right)^{1/2} w_M^{-3p, -\frac{3}{2}} \left( \frac{A}{R} \right) \left( z - A/R \right)^{-1} \| \leq C w_M^{-3p, -\frac{3}{2}}(x)\langle x \rangle^\frac{1}{2}|y|^{-1},
\]
and
\[
\| A \left( \frac{A}{R} \right)^{1/2} w_M^{p,1} \left( \frac{A}{R} \right) \left( z - A/R \right)^{-1} \| \leq C w_M^{p,1}(x)\langle x \rangle^\frac{1}{2}|y|^{-1}.
\]

Now apply Lemma 4.12 and (9.2) with \( \rho = 0 \) and \( \ell = 2 \). Note that \( \| K_2 \| \) is bounded above by a multiple of \( \| K_0 \chi(H) P_{\perp} \| \) times
\[
\int_{\Omega} \langle x \rangle^{\rho - 1 - \ell}|y|^\ell \cdot w_M^{-3p, -\frac{3}{2}}(x)\langle x \rangle^\frac{1}{2}|y|^{-1} \cdot w_M^{p,1}(x)\langle x \rangle^\frac{1}{2}|y|^{-1} \ dx \ dy = \int_{\Omega} \frac{dx \ dy}{\langle x \rangle^2 w_M^{2p,1}(x)} < \infty,
\]
thanks to \( p > 1/2 \). Because \( K_0 \) is compact, it follows that \( K_2 \) is a compact operator and \( \| K_2 \| \) goes to zero as the support of \( \chi \) gets tighter around \( E \). We repeat a similar operation for \( K_1 \) : write \( w_M^{p,1} \left( A/R \right) = w_M^{2p,1} \left( A/R \right) w_M^{-3p, -\frac{3}{2}} \left( A/R \right) \) and commute \( w_M^{2p,1} \left( A/R \right) \) with \( \chi(H) P_{\perp} \). Doing this gives
\[
K_1 = \frac{i}{2\pi} \int_C \frac{\partial \bar{\phi}}{\partial \bar{z}}(z) \left( z - A/R \right) \left( z - A/R \right)^{-1} \times \]
bounded by Lemma 4.12
\[
P_{\perp} \chi(H) w_M^{2p,1} \left( A/R \right) K_0 w_M^{2p,1} \left( A/R \right) \chi(H) P_{\perp} \left( z - A/R \right)^{-1} w_M^{-3p, -\frac{3}{2}} \left( A/R \right) \left( \frac{A}{R} \right) \left( \frac{A}{R} \right)^{1/2} + K_3 \]
compact by Lemma 5.3
bounded by Lemma 4.12

One shows that \( \| K_3 \| \leq C \| P_{\perp} \chi(H) w_M^{2p,1} \left( A/R \right) K_0 w_M^{2p,1} \left( A/R \right) \| \) for some constant \( C > 0 \) and goes to zero as the support of \( \chi \) gets tighter around \( E \), notably because \( w_M^{2p,1} \left( A/R \right) K_0 w_M^{2p,1} \left( A/R \right) \) is compact by Lemma 5.3. As for \( K_3 \), it is compact and satisfies \( \| K_3 \| \leq C \| P_{\perp} \chi(H) w_M^{2p,1} \left( A/R \right) K_0 \| \) for some constant \( C > 0 \). All in all, we see that \( K \) is a compact operator such that \( \| K \| \) goes to zero as the support of \( \chi \) gets tighter around \( E \). This proves the claim.
We now proceed with the proof of the Theorem. Thanks to the claim we have:

\[
F = \frac{1}{2\pi R} \int_{C} \frac{\partial \varphi}{\partial z}(z) P P \theta(H)(z - A/R)^{-1} M_0(z - A/R)^{-1} \theta(H) P precedent.
\]

The next thing to do is to commute \((z - A/R)^{-1}\) with \(M_0\):

\[
F = \frac{1}{2\pi R} \int_{C} \frac{\partial \varphi}{\partial z}(z) P P \theta(H)(z - A/R)^{-2} M_0 \theta(H) P precedent.
\]

We apply (9.3) to the first integral (which converges in norm), while for the second integral we use the fact that \(M_0 \in C^1(A)\) to conclude that there exists a \(B_4 \in \mathcal{B}(\mathcal{H})\) whose norm is uniformly bounded in \(R\) such that

\[
F = R^{-1} P P \theta(H) \varphi'(A/R) M_0 \theta(H) P precedent.
\]

Now \(\varphi'(A/R) = (A/R)^{-1} M_0 w^{2p-1}(A/R)\). By Proposition 9.3,

\[
\langle A/R \rangle^{-\frac{1}{2}} w^{p-\frac{1}{2}}(A/R), M_0 \rangle \langle A/R \rangle^{\frac{1}{2}} w^{p-\frac{1}{2}}(A/R) = R^{-1} B_5
\]

for some \(B_5 \in \mathcal{B}(\mathcal{H})\) whose norm is uniformly bounded in \(R\). Thus

\[
F = R^{-1} P P \theta(H) \langle A/R \rangle^{-\frac{1}{2}} w^{p-\frac{1}{2}}(A/R) M_0 w^{p-\frac{1}{2}} \langle A/R \rangle^{-\frac{1}{2}} \theta(H) P precedent.
\]

where \(\gamma > 0\) comes from applying the Mourre estimate for \(\Delta\) and \(A\). Let

\[
\tilde{K} := \gamma P^{\perp} \chi(H)(\eta^2(\Delta) - \eta^2(H)) \chi(H) P precedent.
\]

Note that \(\tilde{K}\) is compact with \(\|\tilde{K}\|\) vanishing as the support of \(\chi\) gets tighter around \(E\). Thus

\[
F \geq \gamma R^{-1} P P \theta(H) \langle A/R \rangle^{-\frac{1}{2}} w^{p-\frac{1}{2}}(A/R) P^{\perp} \chi(H)(\eta^2(\Delta) - \eta^2(H)) P^{\perp} w^{p-\frac{1}{2}} \langle A/R \rangle^{-\frac{1}{2}} \theta(H) P precedent.
\]

Finally, we commute \(P^{\perp} \chi(H)\eta^2(H)\chi(H) P precedent\) with \(w^{p-\frac{1}{2}}(A/R)\langle A/R \rangle^{-\frac{1}{2}}\), and see that

\[
\gamma \langle P^{\perp} \eta^2(H) P^{\perp}, w^{p-\frac{1}{2}}(A/R)\langle A/R \rangle^{-\frac{1}{2}} \rangle \omega^{p-\frac{1}{2}}(A/R) \langle A/R \rangle^{-\frac{1}{2}} = R^{-1} B_6
\]
for some $B_6 \in \mathcal{B}(\mathcal{H})$ whose norm is uniformly bounded in $R$. Thus we have
\[
F \geq \gamma R^{-1} P^\perp \theta(H) \left( \frac{A}{R} \right)^{-1} w_{M}^{-2p-1} \theta(H) P^\perp + P^\perp \theta(H) \left( \frac{A}{R} \right)^{-\frac{1}{2}} w_{M}^{-p} \left( \frac{A}{R} \right) \left( \frac{1}{R^2} \sum_{i=1}^{6} B_i + \frac{K}{R} \right) w_{M}^{-p} \left( \frac{A}{R} \right)^{-\frac{1}{2}} \theta(H) P^\perp.
\]

To conclude, we shrink the support of $\chi$ to ensure that $\|K + \tilde{K}\| < \gamma / 3$ and choose $R > 1$ so that $\|\sum_{i=1}^{6} B_i \| / R < \gamma / 3$. Then $K + \tilde{K} \geq -\gamma / 3$ and $\sum_{i=1}^{6} B_i / R \geq -\gamma / 3$, so
\[
F = P^\perp \theta(H) \left[ H, i\varphi \left( \frac{A}{R} \right) \right] \theta(H) P^\perp \geq \gamma R^{-1} \theta(H) \left( \frac{A}{R} \right)^{-1} w_{M}^{-2p-1} \theta(H) P^\perp.
\]
Let $J$ be any open interval with $J \subset I$. Applying $E_{\theta(H)}$ on both sides of this inequality yields the projected weighted Mourre estimate \((2.3)\), with $c = \gamma / (3R)$ and $C = \langle A / R \rangle^{-\frac{1}{2}} w_{M}^{-p} \left( A / R \right)$, $1/2 < p < q / 4$. The proof of Theorem 1.2 is complete, as explained in Section 2.3. \hfill \Box

7. APPENDIX A. NEVANLINNA FUNCTIONS, OPERATOR MONOTONE FUNCTIONS AND LOEWNER’S THEOREM

We revisit Loewner’s theorem on matrix operator monotone functions, see e.g. [L], [Do], [Si2] and [Ha]. This wonderful theorem makes a striking connection between the operator monotone functions and the Nevanlinna functions.

Let $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) denote the complex numbers with strictly positive (resp. negative) imaginary part. A Nevanlinna function (also known as Herglotz, Pick or R function) is an analytic function that maps $\mathbb{C}_+$ to $\mathbb{C}_+$. A function $f$ is Nevanlinna if and only if it admits a representation
\[
f(z) = \alpha + \beta z + \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda), \quad z \in \mathbb{C}_+
\]
where $\alpha \in \mathbb{R}$, $\beta \geq 0$, and $\mu$ is a positive Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} (\lambda^2 + 1)^{-1} d\mu(\lambda) < \infty$. We refer to [Do, Theorem 1 of Chapter II] for a proof of this wonderful result. The integral representation is unique. The measure $\mu$ is recovered from $f$ by the Stieltjes inversion formula
\[
\mu((\lambda_1, \lambda_2]) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} \operatorname{Im}(f(\lambda + i\varepsilon)) d\lambda.
\]
Standard examples of Nevanlinna functions given in the literature include $z^p$ for $0 \leq p \leq 1$, $-z^p$ for $-1 \leq p \leq 0$, and the logarithm $\log(z)$ with the branch cut $(-\infty, 0]$.

**Notation.** Let $(a, b)$ be an open interval (finite or infinite). Denote $P(a, b)$ the Nevanlinna functions that continue analytically across $(a, b)$ into $\mathbb{C}_-$ and where the continuation is by reflection.

Functions in $P(a, b)$ are real-valued on $(a, b)$ and their measure satisfies $\mu((a, b]) = 0$, see [Do, Lemma 2, Chapter II]. Functions in $P(a, b)$ are strictly increasing on $(a, b)$, unless they are constant. Indeed, if $f$ is not a constant function $\beta$ and $\mu$ cannot be simultaneously zero and so
\[
f'(x) = \beta + \int_{\mathbb{R}} \frac{d\mu(\lambda)}{(\lambda - x)^2} > 0, \quad x \in (a, b).
\]
Let $\mathcal{M}_n(\mathbb{C})$ be the set of $n \times n$ matrices with entries in $\mathbb{C}$, and consider a function $f : (a, b) \mapsto \mathbb{R}$. **Definition.** $f$ is matrix monotone of order $n$ in $(a, b)$ if $f(A) \leq f(B)$ holds whenever $A, B$ in $\mathcal{M}_n(\mathbb{C})$ are hermitian matrices with spectrum in $(a, b)$ and $A \leq B$.

In 1934 Karl Loewner proved the following remarkable theorem that characterizes the matrix monotone functions:
**Theorem 7.1.** [L] Let \( f : (a, b) \to \mathbb{R} \), where \((a, b)\) is a finite or infinite open interval. Then \( f \) is matrix operator monotone of order \( n \) in \((a, b)\) for all \( n \in \mathbb{N} \) if and only if \( f \) admits an analytic continuation that belongs to \( P(a, b) \).

Loewner’s theorem is a truly wonderful result and has been reproved in several different ways, see e.g. [Do, Theorem I of Chapter VII and Chapter IX for the converse], [Si2] or [Ha] and references therein for a concise historical exposition. For the purpose of this article, we need a version of Loewner’s theorem that applies to unbounded self-adjoint operators. In B. Simon’s book [Si2, Chapter 2] it is explicitly discussed how Loewner’s theorem extends to unbounded operators. We propose below yet another proof of the extension to the semi-bounded operators (which is the case for \( \langle A \rangle \) and \( \langle N \rangle \)).

**Definition.** \( f \) is operator monotone in \((a, b)\) if \( f(A) \leq f(B) \) holds whenever \( A, B \) (possibly unbounded) are self-adjoint operators in \( \mathcal{H} \) with spectrum contained in \((a, b)\) and \( A \leq B \).

We start with a Lemma:

**Lemma 7.2.** Let \( A \) be a self-adjoint operator with \( A > a > 0 \). Let \( f \) belong to \( P(a, +\infty) \) be such that \( \lim_{\varepsilon \to 0} f(a + \varepsilon) > 0 \). Then \( \text{Dom}[A] \subseteq \text{Dom}[f(A)] \) with equality if and only if \( \beta > 0 \).

**Proof.** \( f \) belongs to \( P(a, +\infty) \) so \( f \) admits an integral representation as in (7.1). Denote the supremum of the support of \( \mu \) by \( \Sigma_\mu \). Note that \( \Sigma_\mu \leq a \). We start by showing that \( \lim_{x \to +\infty} f(x) = \beta \), or equivalently

\[
\lim_{x \to +\infty} \int_{-\infty}^{\Sigma_\mu} \frac{1 + \lambda x}{x(\lambda - x)(\lambda^2 + 1)} d\mu(\lambda) = 0.
\]

We wish to exchange the order of the limit and integration. We have

\[
(7.2) \quad \left| \frac{1 + \lambda x}{x(\lambda - x)} \right| \leq 1, \quad \forall (\lambda, x) \in (-\infty, 0] \times [1, +\infty) \cup [0, a] \times [a + \sqrt{a^2 + 1}, +\infty).
\]

We may therefore apply the dominated convergence theorem, and the above limit follows. This limit implies that \( f(x) \) is a bounded function on \((a, +\infty)\) and hence \( \text{Dom}[A] \subset \text{Dom}[f(A)] \). For the reverse inclusion, note that \( f(x) \) is a well defined bounded function on \((a, +\infty)\) iff \( \beta > 0 \). \( \square \)

We are now ready to prove the extension of Theorem 7.1 to semi-bounded operators:

**Theorem 7.3.** Let \( f : (a, b) \to \mathbb{R} \), where \( 0 < a < b \leq +\infty \). Then \( f \) is operator monotone in \((a, b)\) if and only if \( f \) admits an analytic continuation that belongs to \( P(a, b) \).

**Proof.** If \( f \) is operator monotone in \((a, b)\), then in particular it is matrix operator monotone of order \( n \) in \((a, b)\) for all \( n \in \mathbb{N} \), and so \( f \) admits an analytic continuation that belongs to \( P(a, b) \) by Loewner’s theorem. This direction is in fact the hard direction in Loewner’s theorem, but the extension is trivial!

For the converse, we suppose that \( f \) admits an analytic continuation that belongs to \( P(a, b) \). Consider the 2 separate cases \( b < +\infty \) and \( b = +\infty \). If \( b < +\infty \), then \( f \) is matrix operator monotone of order \( n \) in \((a, b)\) for all \( n \in \mathbb{N} \) by Loewner’s theorem. But, since the interval \((a, b)\) is finite, this is equivalent to being operator monotone in \((a, b)\), see e.g. [BS, Lemma 2.2] or [Si2, Chapter 2]. Now the case \( b = +\infty \). We have \( \Sigma_\mu \leq a \), and

\[
f(x) = \alpha + \beta x + \int_{-\infty}^{\Sigma_\mu} \left( \frac{1}{\lambda - x} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda), \quad x \in (a, +\infty).
\]

Let

\[
g_r(x) := \alpha + \int_{-r}^{\Sigma_\mu} \left( \frac{1}{\lambda - x} - \frac{\lambda}{\lambda^2 + 1} \right) d\mu(\lambda), \quad g(x) := f(x) - \beta x, \quad x \in (a, +\infty).
\]


Note that \( \{g_r\}_{r \in \mathbb{R}^+} \) is of a sequence of functions of the real variable \( x \) that converges pointwise to \( g(x) \) as \( r \to +\infty \) for all \( x \in (a, +\infty) \). Moreover, since for every fixed \( r \), \( g'_r(x) \geq 0 \), all functions in the sequence are increasing in the variable \( x \). We need a Lemma.

**Lemma 7.4.** For every fixed \( \varepsilon > 0 \), the sub-sequence \( \{g_r(x)\}_{r > 1/\varepsilon} \) has the property that \( g_r(x) \to g(x) \) for every \( x > \max(\Sigma_\mu + \varepsilon, a) \).

**Proof.** Clearly \( g_r(x) \to g(x) \) as \( r \to +\infty \) for all \( x \in (a, +\infty) \). What needs to be shown is that the sequence \( g_r(x) \) is increasing pointwise. Let \( \varepsilon > 0 \) be given and fix \( x > \max(\Sigma_\mu + \varepsilon, a) \). Recall \( a \) is assumed to be strictly positive. The integrand in (7.3) is equal to

\[
\frac{1 + \lambda x}{(\lambda - x)(\lambda^2 + 1)}
\]

On the one hand, \( x > \Sigma_\mu + \varepsilon \) implies that \( (\lambda - x) \) is negative for all \( \lambda \in \text{supp} \mu \). On the other hand, the assumption \( x > a \) implies that \( (1 + \lambda x) \) is negative for all \( \lambda < -1/x \). Thus, for all \( x > \max(\Sigma_\mu + \varepsilon, a) \) and for all \( \lambda < -1/\varepsilon \), the integrand in (7.3) is positive. Thus, as \( r \) increases above \( 1/\varepsilon \), the value of the integral in (7.3) can only increase. This completes the proof of the Lemma. \( \square \)

**Continuation of the proof of Theorem 7.3.** Now fix \( A, B \) self-adjoint operators with spectrum contained in \( (a, +\infty) \) and \( A \leq B \). Because the spectrum is a closed set there exists \( a' \) with \( a < a' < \inf \sigma(A) \leq \inf \sigma(B) \). Note that adding a constant to \( f \) does not alter the assumptions of the Theorem and leads to the same conclusion, so we may assume that \( g(a') > 0 \).

Now let \( \psi \in \mathcal{H} \). \( A \leq B \) implies \( \langle \psi, (\lambda - A)^{-1}\psi \rangle \leq \langle \psi, (\lambda - B)^{-1}\psi \rangle \) for all \( \lambda \leq \Sigma_\mu \), cf. e.g. [RS1, Exercise 51, Chapter VIII] or [Si2, Theorem 2.8]. Integrating we get

\[
\int_{-\infty}^{\Sigma_\mu} \langle \psi, (\lambda - A)^{-1}\psi \rangle \, d\mu(\lambda) \leq \int_{-\infty}^{\Sigma_\mu} \langle \psi, (\lambda - B)^{-1}\psi \rangle \, d\mu(\lambda), \quad \text{for every finite } r > |\Sigma_\mu|.
\]

Since we integrate over the compact \([-r, \Sigma_\mu]\) and the integrand is norm continuous, we infer

(7.4) \[
\langle \psi, g_r(A)\psi \rangle \leq \langle \psi, g_r(B)\psi \rangle, \quad \text{for every finite } r > |\Sigma_\mu|.
\]

Each \( g_r \) is a bounded Borel function. By Lemma 7.4, \( g_r(x) \to g(x) \) as \( r \to +\infty \) for all \( x \in [a', +\infty) \).

The choice earlier that \( g(a') > 0 \), together with the fact that each \( g_r \) is an increasing function of \( x \) on \( (a, +\infty) \), ensures that \( 0 \leq g_r(x) \leq g(x) \) holds for all \( r \) sufficiently large and all \( x \in [a', +\infty) \).

By the spectral theorem [RS1, Theorem VIII.5], it follows that \( g_r(A)\psi \to g(A)\psi \) as \( r \to +\infty \) for all \( \psi \in \text{Dom}[g(A)] \), and similarly for \( B \). Taking limits in (7.4) gives

\[
\langle \psi, g(A)\psi \rangle \leq \langle \psi, g(B)\psi \rangle, \quad \psi \in \text{Dom}[g(A)] \cap \text{Dom}[g(B)].
\]

Thus, invoking Lemma 7.2, if \( \beta = 0 \) we have

\[
\langle \psi, f(A)\psi \rangle \leq \langle \psi, f(B)\psi \rangle, \quad \psi \in \text{Dom}[g(A)] \cap \text{Dom}[g(B)],
\]

whereas if \( \beta > 0 \) we have

\[
\langle \psi, f(A)\psi \rangle \leq \langle \psi, f(B)\psi \rangle, \quad \psi \in \text{Dom}[A] \cap \text{Dom}[B].
\]

Finally, note that \( \text{Dom}[f(A)] = \text{Dom}[g(A)] \) if \( \beta = 0 \), while \( \text{Dom}[f(A)] = \text{Dom}[A] \) if \( \beta > 0 \), and similarly for \( B \). \( \square \)

To close this Section, we have a remark about results in the literature on order relations for general self-adjoint operators. In [O], Olson introduced the spectral order for self-adjoint operators \( A \leq B \) if and only if \( E_{(-\infty,t]}(A) \geq E_{(-\infty,t]}(B) \) for all \( t \in \mathbb{R} \). Here \( E_{(-\infty,t]}(A) \) and \( E_{(-\infty,t]}(B) \) are the spectral resolutions of the identity for \( A \) and \( B \). He showed that \( A^n \leq B^n \) for every \( n \in \mathbb{N} \) is equivalent to \( A \leq B \), see also [U, Proposition 5]. Furthermore, it is shown in [FK] that this order relation is equivalent to \( f(A) \leq f(B) \) for any continuous monotone nondecreasing function \( f \) defined on an interval which contains \( \sigma(A) \cup \sigma(B) \). For the purpose of this article,
we do not know if \( A^n \leq N^n \) holds \( \forall n \in \mathbb{N} - A \) and \( N \) as in Section 4 — but if it does it would considerably simplify this article. To check \( A^n \leq N^n \) by brute force is unbearable even for small values of \( n \).

8. Appendix B: Polylogarithms of positive order are Nevanlinna functions

That logarithm is a Nevanlinna function follows from the identity \( \log(z) = \log(r) + i\theta \), where \( z = re^{i\theta}, r > 0, \) and \( \theta \in (-\pi, \pi) \). The integral representation of the logarithm is

\[
\log(z) = \int_{-\infty}^{0} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) d\lambda.
\]

The composition of Nevanlinna functions produces another Nevanlinna function. So for example \( \log^p(z) \) is Nevanlinna for \( 0 \leq p \leq 1 \). What about higher powers of the logarithm? Certainly the square and cube of the logarithm are not Nevanlinna functions. Indeed writing

\[
\log^2(z) = \log^2(r) - \theta^2 + i2\theta \log(r), \quad \text{and} \quad \log^3(z) = \log^3(r) - 3\theta^2 \log(r) + i\theta(3\log^2(r) - \theta^2)
\]

reveals that these functions do not map \( \mathbb{C}_+ \) to \( \mathbb{C}_+ \). In spite of this there are functions that are Nevanlinna and are "almost" equal to the logarithms. To motivate the idea, we note that the Stieltjes inversion formula gives

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda, \delta} \Im \left( \log^2(\lambda + i\epsilon) \right) d\lambda = \begin{cases} 0 & \lambda_1 \geq 0, \\
\lambda_2^2 2\log(|\lambda|) d\lambda & \lambda_2 \leq 0 \end{cases}
\]

when applied to \( \log^2(z) \) and

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\lambda, \delta} \Im \left( \log^3(\lambda + i\epsilon) \right) d\lambda = \begin{cases} 0 & \lambda_1 \geq 0, \\
\lambda_2^2 (3\log^2(|\lambda|) - 2^2) d\lambda & \lambda_2 \leq 0 \end{cases}
\]

when applied to \( \log^3(z) \). This suggests to calculate the Nevanlinna functions corresponding to the measures \( d\mu(\lambda) = 1_{(\lambda < -1)} \log(-\lambda) d\lambda \) and \( d\mu(\lambda) = 1_{(\lambda < -1)} \log^2(-\lambda) d\lambda \).

We introduce polylogarithms. We refer to [Le] and [PBM] for formulas and a detailed exposition. The polylogarithm of order \( \sigma \in \mathbb{C} \) is defined by the power series

\[
\text{Li}_\sigma(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\sigma}.
\]

The definition is valid for complex \( |z| < 1 \) and is extended to the complex plane by analytic continuation. For the purpose of this article, we are interested in the polylogarithms with \( \sigma > 2 \), or \( \sigma = 3 \) if we want to simplify by taking the smallest integer above 2. The standard branch cut is \([1, +\infty)\) for \( \text{Li}_1(z) \) and \((1, +\infty)\) for \( \text{Li}_2(z) \) and \( \text{Li}_3(z) \). The polylogarithm of order 1 can be written in terms of a logarithm as \( \text{Li}_1(z) = -\log(1-z) \). The polylogarithm of order 2 is called the dilogarithm or Spencer function while the polylogarithm of order 3 is called the trilogarithm. On p. 494 of [PBM] the following integral representation is given without proof:

\[
\text{Li}_{\sigma+1}(z) = \frac{z}{\Gamma(\sigma + 1)} \int_{1}^{\infty} \frac{\log^\sigma(\lambda)}{\lambda(\lambda - z)} d\lambda,
\]

for \( |\arg(1-z)| < \pi, \Re(\sigma) > -1 \), or for \( z = 1, \Re(\sigma) > 0 \). Here \( \Gamma \) is the Gamma function. Obviously (8.1) is equivalent to

\[
\text{Li}_{\sigma+1}(z) = -\frac{1}{\Gamma(\sigma + 1)} \int_{1}^{\infty} \frac{\log^\sigma(\lambda)}{\lambda(\lambda^2 + 1)} d\lambda + \frac{1}{\Gamma(\sigma + 1)} \int_{1}^{\infty} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) \log^\sigma(\lambda) d\lambda.
\]

This means that \( \text{Li}_{\sigma+1}(z) \) is a Nevanlinna function for \( \Re(\sigma) > -1 \). Although not difficult to prove, it is not clear where a proof of (8.1) can be found in the literature. Thus we prove it:
Proposition 8.1. (8.1) is true.

Proof. Let \( \lambda \geq 1 \). Writing \( 1/(\lambda(\lambda - z)) \) as a power series in \( z \) we have \( 1/(\lambda(\lambda - z)) = \sum_{k=0}^{\infty} \lambda^{-k-2} z^k \) for \( |z| < 1 \). Then the rhs of (8.1) is equal to

\[
\frac{z}{\Gamma(\sigma + 1)} \int_1^{\infty} \sum_{k=0}^{\infty} \frac{\log^k(\lambda)}{\lambda^{k+2}} z^k d\lambda = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\sigma + 1)} \int_{-\infty}^{\infty} \frac{\log^k(\lambda)}{\lambda^{k+1}} d\lambda = \sum_{k=1}^{\infty} \frac{z^k}{k^{\sigma+1}}, \quad \text{Re}(\sigma) > -1.
\]

To evaluate the last integral the change of variable \( k \log(\lambda) = t \) was performed, followed by the definition of the Gamma function. Thus the rhs of (8.1) is equal to \( \text{Li}_{\sigma+1}(z) \) for \( |z| < 1 \) and \( \text{Re}(\sigma) > -1 \). The result follows by the uniqueness of the analytic continuation. \( \square \)

While we’re at it we note that \( \text{Li}_0(z) = z/(1 - z) \) is also a Nevanlinna function.

Definition. For \( \sigma \in \mathbb{C} \),

\[
\Phi_\sigma(z) := -\text{Li}_\sigma(-z), \quad z \in \mathbb{C}\setminus(-\infty, -1].
\]

Clearly (8.2) implies that the \( \Phi_\sigma \) are Nevanlinna for \( \text{Re}(\sigma) > -1 \) with integral representations given by:

\[
\Phi_{\sigma+1}(z) = \frac{1}{\Gamma(\sigma + 1)} \int_1^{\infty} \frac{\log^k(\lambda)}{\lambda^{k+1}} d\lambda + \frac{1}{\Gamma(\sigma + 1)} \int_{-\infty}^{1} \left( \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right) \log^k(-\lambda) d\lambda,
\]

for \( |\arg(1 + z)| < \pi \). Finally, the other reason we resort to polylogarithms is because they decay at the same rate as the logarithms, at least for positive integer order (this follows directly from the inversion/reflection formula [Le, (6) of Appendix A.2.7] together with \( \text{Li}_n(0) = 0 \), namely:

\[
\lim_{x \to +\infty} \frac{\Phi_n(x)}{\log^n(x)} = \frac{1}{n!}, \quad n \in \mathbb{N}.
\]

9. Appendix C. Almost analytic extensions and Helffer-Sjöstrand calculus

We refer to [D], [DG], [GJ1], [GJ2], [MS] for more details. Let \( \rho \in \mathbb{R} \) and denote by \( S^\rho(\mathbb{R}) \) the class of functions \( \varphi \in C^\infty(\mathbb{R}) \) such that

\[
|\varphi^{(k)}(x)| \leq C_k \langle x \rangle^{\rho-k}, \quad \text{for all } k \in \mathbb{N}.
\]

Lemma 9.1. [D] and [DG] Let \( \varphi \in S^\rho(\mathbb{R}) \), \( \rho \in \mathbb{R} \). Then for every \( N \in \mathbb{Z}^+ \) there exists a smooth function \( \tilde{\varphi}_N : \mathbb{C} \to \mathbb{C} \), called an almost analytic extension of \( \varphi \), satisfying:

\[
\tilde{\varphi}_N(x + i0) = \varphi(x), \quad \forall x \in \mathbb{R};
\]

\[
\text{supp } (\tilde{\varphi}_N) \subset \{x + iy : |y| \leq \langle x \rangle \};
\]

\[
\tilde{\varphi}_N(x + iy) = 0, \quad \forall y \in \mathbb{R} \text{ whenever } \varphi(x) = 0;
\]

\[
\forall \ell \in \mathbb{N} \cap [0, N], \left| \frac{\partial^{\ell} \tilde{\varphi}_N}{\partial z^\ell}(x + iy) \right| \leq c_\ell \langle x \rangle^{\rho-1-\ell} |y|^\ell \text{ for some constants } c_\ell > 0.
\]

Lemma 9.2. Let \( \rho < 0 \) and \( \varphi \in S^\rho(\mathbb{R}) \). Then for all \( k \in \mathbb{N} \) and \( N \in \mathbb{N} \):

\[
\varphi^{(k)}(A) = \frac{i^{(k)!}}{2\pi} \int_{\mathbb{C}} \frac{\partial^k \tilde{\varphi}_N}{\partial z^k}(z)(z - A)^{-1-k} dz \wedge d\overline{z}
\]

where the integral exists in the norm topology. For \( \rho \geq 0 \), the following limit exists:

\[
\varphi^{(k)}(A) f = \lim_{R \to \infty} \frac{i^{(k)!}}{2\pi} \int_{\mathbb{C}} \frac{\partial^k (\varphi \tilde{\theta}_R)_N}{\partial z^k}(z)(z - A)^{-1-k} f dz \wedge d\overline{z}, \quad \text{for all } f \in \text{Dom}[\langle A \rangle^\rho].
\]

In particular, if \( \varphi \in S^\rho(\mathbb{R}) \) with \( 0 \leq \rho < k \) and \( \varphi^{(k)} \) is a bounded function, then \( \varphi^{(k)}(A) \) is a bounded operator and (9.3) holds (with the integral converging in norm).
Proposition 9.3. [GJ1] Let $T$ be a bounded self-adjoint operator satisfying $T \in \mathcal{C}^1(A)$. Then:

$$(9.5) \quad [T, (z - A)^{-1}]_0 = (z - A)^{-1}[T, A]_0 (z - A)^{-1},$$

and for any $\varphi \in \mathcal{S}^p(\mathbb{R})$ with $p < 1$, $T \in \mathcal{C}^1(\varphi(A))$ and

$$(9.6) \quad [T, \varphi(A)]_0 = \frac{i}{2\pi} \int_C \frac{\partial \varphi_N}{\partial z} (z - A)^{-1}[T, A]_0 (z - A)^{-1} dz \wedge d\bar{z}.$$

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