Big Bang Nucleosynthesis constraints on higher-order modified gravities

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We use Big Bang Nucleosynthesis (BBN) data in order to impose constraints on higher-order modified gravity, and in particular on: (i) $f(G)$ Gauss-Bonnet gravity, and $f(P)$ cubic gravities, arising respectively through the use of the quadratic-curvature Gauss-Bonnet $G$ term, and the cubic-curvature combination, (ii) string-inspired quadratic Gauss-Bonnet gravity coupled to the dilaton field, (iii) models with string-inspired quartic curvature corrections, and (iv) running vacuum models. We perform a detailed investigation of the BBN epoch and we calculate the deviations of the freeze-out temperature $T_f$ in comparison to $\Lambda$CDM paradigm. We then use the observational bound on $\left| \frac{\delta T_f}{T_f} \right|$ in order to extract constraints on the involved parameters of various models. We find that all models can satisfy the BBN constraints and thus they constitute viable cosmological scenarios, since they can additionally account for the dark energy sector and the late-time acceleration, in a quantitative manner, without spoiling the formation of light elements during the BBN epoch. Nevertheless, the obtained constraints on the relevant model parameters are quite strong.

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I. INTRODUCTION

Modified gravity is one of the two main ways that are being followed in order to explain the early and late accelerated phases of universe expansion [1–4], with the other one being the introduction of inflaton or dark energy sectors [5, 6]. Amongst the various classes of gravitational modifications that can fulfill the above cosmological motivation, theories that incorporate higher-order corrections to the Einstein-Hilbert Lagrangian have an additional motivation, namely the potential for improving the renormalizability of General Relativity [7, 8]. Such theories may naturally arise as (ghost-free) low-energy effective field-theory limits of String Theory [9] and include Einstein gravity in the lowest-order in a derivative expansion. A particularly interesting subclass of such ghost-free higher-derivative theories that are equivalent to General Relativity at the linearized level in the vacuum, with only a transverse and massless propagating graviton, are the (Lovelock) theories [10]. The most general, ghost-free covariant gravitational action in a Minkowski vacuum (that is, up to and including quadratic-order terms in fluctuations $h_{\mu\nu}$ of the graviton field $g_{\mu\nu}$ in the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\eta_{\mu\nu}$ the Minkowski metric), which also involves higher-curvature as well as non-local terms with improved Ultraviolet behaviour (UV), but recovers Einstein’s General Relativity in the Infrared (IR), has been given in [11].

The construction of higher-order gravities is based on the addition of extra terms in the Einstein-Hilbert Lagrangian, such as in $f(G)$ gravity [12–14], in Lovelock gravity [10, 15], in Weyl gravity [16, 17], in Galileon theory [18, 19], etc. Restricting to quadratic-in-curvature corrections a well-studied class is obtained by using functions of the Gauss-Bonnet combination, resulting to the $f(G)$ gravity [20], which proves to have interesting cosmological phenomenology [21–35]. Similarly, using cubic terms one may construct the particular cubic curvature invariance $P$, which is a combination that is neither topological nor trivial in four dimensions and when used as a Lagrangian leads to a spectrum identical to that of General Relativity [36]. Cubic and $f(P)$ gravity have been also showed to lead to interesting cosmological [37–43] and black-hole applications [44–51].

Additionally, in the context of microscopic string theory models [52, 53], one faces situations where Gauss-Bonnet higher-curvature combinations couple to non-trivial dilaton fields, which also lead to interesting black hole solutions with secondary scalar hair [54] or modified cosmologies with dilatons [55–63]. Finally, another class of such modification is the “running vacuum models” [64–68], according to which the vacuum energy density is a
function of even powers of the Hubble parameter and its derivative. Such scenarios are shown to have interesting cosmology and phenomenology [69–77] and can additionally accept a microscopic string-inspired origin [78–80].

An important and necessary test of any modified gravity is the confrontation with cosmological observations, since such a confrontation provides information on the involved unknown functions, as well as the allowed regions of the model parameters. Although, investigations related to late-time cosmological data [81] have been performed in some detail in the case of higher-order gravities [21, 24, 33], the use of early-time, and in particular, of Big Bang Nucleosynthesis (BBN) considerations has not been done as yet. Hence, in the present work we address this crucial issue, namely we impose constraints on \( f(G) \), Gauss-Bonnet-Dilaton and \( f(P) \) gravities, and combinations thereof, as well as on running vacuum scenarios, through BBN analysis.

The plan of the article is the following: In Section II we briefly present higher-order gravity, and in particular \( f(G) \), Gauss-Bonnet-Dilaton \( f(P) \) gravity, as well as string-inspired models with quartic curvature corrections, and running vacuum models, and we apply them in a cosmological framework. In Section III, after a brief introduction to the basics of BBN, we examine in detail the BBN constraints on various specific models, extracting the bounds on the involved model parameters. Finally, Section IV is devoted to the Conclusions.

II. HIGHER-ORDER GRAVITY AND COSMOLOGY

In this section we present higher-order gravity and we apply it in a cosmological framework. As we mentioned in the Introduction, such theories are obtained through the addition of higher-order terms, that are constructed by contractions of Riemann tensors, in the Einstein-Hilbert Lagrangian [10]. Throughout the work we consider the flat homogeneous and isotropic Friedmann-Robertson-Walker (FRW) geometry with metric

\[
    ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j ,
\]

where \( a(t) \) is the scale factor. In the following subsections we examine the quadratic and cubic cases separately.

A. \( f(G) \) gravity and cosmology

Let us first consider quadratic terms in the Riemann tensor. The corresponding combination is the Gauss-Bonnet one, given as

\[
    G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{2}
\]

Although this term is topological in four dimensions and thus it cannot lead to any corrections in the field equations, the extended action

\[
    S = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R + f(G) \right], \tag{3}
\]

with \( M_P \equiv 1/\sqrt{8\pi G_N} = 2.4 \times 10^{18} \text{ GeV} \) the (reduced) Planck mass, and \( G_N \) the gravitational constant, corresponds to a new gravitational modification, namely \( f(G) \) gravity. Variation of the action with respect to the metric leads to

\[
    M_P^2 G^{\mu\nu} = \frac{1}{2} g^{\mu\nu} f(G) - 2f'(G) R R^{\mu\nu} + 4f'(G) R_{\mu}^{\rho} R_{\rho}^{\nu} \\
    - 2f'(G) R^{\mu\rho\sigma\tau} R_{\nu}^{\rho\sigma\tau} - 4f'(G) R_{\mu}^{\rho\sigma\nu} R_{\nu}^{\rho\sigma} + 2\n\]

\[
    \nabla_{\mu} \nabla_{\nu} f'(G) R - 2g^{\mu\nu} \nabla^2 f'(G) R + 4 \nabla^2 f'(G) R^{\mu\nu} \\
    - 4 \nabla_{\mu} \nabla_{\nu} f'(G) R_{\rho}^{\rho\nu} - 4 \nabla_{\rho} \nabla_{\nu} f'(G) R_{\mu}^{\rho\nu} + 4g^{\mu\nu} \nabla_{\rho} \nabla_{\sigma} f'(G) R^{\rho\sigma}, \tag{4}
\]

with \( f_G \equiv \partial f(G)/\partial G \). Applying it to a cosmological framework, namely to the metric (1), and considering additionally the matter and radiation perfect fluids, we find the Friedmann equations

\[
    3M_P^2 H^2 = \rho_m + \rho_r + \rho_{DE} \tag{5}
\]

\[
    -2M_P^2 \dot{H} = \rho_m + \rho_m + \rho_r + \rho_r + \rho_{DE} + \rho_{DE}, \tag{6}
\]

with \( \rho_m \) and \( \rho_m \) respectively the energy density and pressure of the matter fluid, \( \rho_r \) and \( \rho_r \) the corresponding quantities for radiation sector, and where we have introduced the corresponding quantities of the effective dark energy sector as

\[
    \rho_{DE} \equiv \frac{1}{2} \left[ -f'(G) + 2AH^2 \left( H^2 + \dot{H} \right) f'(G) \\
    - 2A^2 H^4 \left( 2H^2 + H \dot{H} + 4H^2 \dot{H} \right) f''(G) \right], \tag{7}
\]

\[
    \rho_{DE} \equiv f(G) - 2AH^2 \left( H^2 + \dot{H} \right) f'(G) \\
    + 8(24)^2 \left( 2H^2 + H \dot{H} + 4H^2 \dot{H} \right)^2 f''(G) \\
    + 192H^2 \left( 6H^3 + 8H \dot{H} \dot{H} + 24H^2 \dot{H} \\
    + 6H^3 \ddot{H} + 8H^3 \ddot{H} + H^2 \dot{H} \right) f''(G), \tag{8}
\]

where primes denote differentiation with respect to the argument. Note that in FRW metric the Gauss-Bonnet combination becomes

\[
    G = 2AH^2 \left( H^2 + \dot{H} \right), \tag{9}
\]

\[
    \partial_{\nu} \Gamma^\lambda_{\mu\sigma} + \Gamma^\gamma_{\mu\sigma} \Gamma^\lambda_{\gamma\nu} - \left( \nu \leftrightarrow \sigma \right), \text{ Ricci tensor } R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \text{ and } \text{Ricci scalar } R = R_{\mu\nu}g^{\mu\nu}. \text{ We also work in units } h = c = 1.
\]

\[
    \Delta
\]

1 Our notation and conventions throughout this work are: signature of metric \((- + + +)\), Riemann Curvature tensor \( R^\lambda_{\mu\nu\sigma} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\sigma} - \frac{\partial \Gamma^\lambda_{\mu\sigma}}{\partial x^\nu} + \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\mu\rho} \Gamma^\lambda_{\nu\sigma} \).
which has indeed squared powers comparing to the Ricci scalar \( R = 6(2H^2 + H) \).

**B. Gauss-Bonnet-Dilaton Gravity**

An interesting situation, which stems directly from microscopic string theory models [52, 53], is the modified gravity with quadratic curvature terms, which are coupled to non-trivial (dimensionless) dilatons \( \Phi \), with potential \( V(\Phi) \) [54]. The effective action in this case reads

\[
\int d^4x \sqrt{-g} M_p^2 \left[ \frac{R}{2} - \frac{1}{4} \partial_\mu \Phi \partial^\mu \Phi + c_1 e^\Phi (G - V(\Phi)) \right], \tag{10}
\]

where the four-dimensional Gauss-Bonnet invariant \( G \) is given in (2). In string theory models the coefficient \( c_1 \) is given by

\[
c_1 = \frac{\alpha'}{8 g_s^{(0)}}, \tag{11}
\]

where \( g_s^{(0)} \) is the string coupling and \( \alpha' = 1/M_s^2 \) is the Regge slope of the string, with \( M_s \) the string scale, which is in general different from the (reduced) Planck mass \( M_P \), entering the (3+1)-dimensional Einstein-Hilbert part of the low-energy string effective action (10).

Einstein’s and dilaton equations of motion in the presence of dilaton-Gauss-Bonnet coupling, stemming from (10), read [54]:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{4} g_{\mu\nu} \left( \partial_\alpha \Phi \right)^2 + 4 V(\Phi) - \alpha' K_{\mu\nu},
\]

\[
K_{\mu\nu} = 2 \left( g_{\mu\rho} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\rho} \right) \varepsilon^{\lambda\sigma\alpha\beta} D_\gamma \left( \tilde{R}^\rho_{\gamma\sigma\alpha\beta} \partial_\sigma f \right),
\]

\[
\partial_\mu \left( \sqrt{-g} \partial^\mu \Phi \right) = -2 \sqrt{-g} c_1 e^\Phi G - 2 \frac{\delta V(\Phi)}{\delta \Phi}, \tag{12}
\]

with \( f \equiv \frac{\alpha'}{\alpha' g_s^{(0)}} \). In the above equations \( D_\mu \) denotes a gravitational covariant derivative, \( \tilde{R}^\nu_{\alpha\beta} = \frac{1}{2} \varepsilon^{\mu\rho\sigma\tau} R_{\mu\rho\sigma\beta} \) is the dual Riemann tensor. The gravitationally covariant Levi-Civita tensor density \( \varepsilon_{\mu\rho\sigma\tau} \) is defined as \( \varepsilon_{\mu\rho\sigma\tau} = \sqrt{-g} \varepsilon_{\mu\rho\sigma\tau} \), with \( \varepsilon_{\mu\rho\sigma\tau} \) the flat-Minkowski-spacetime totally antisymmetric Levi-Civita symbol. The contravariant tensor density \( \varepsilon^{\mu\rho\sigma\tau} \) is defined accordingly by raising the indices with the appropriate curved metric tensor. The conserved stress tensor of the theory reads

\[
T_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{4} g_{\mu\nu} \left( \partial_\alpha \Phi \right)^2 + 4 V(\Phi) - \alpha' K_{\mu\nu}, \tag{13}
\]

with

\[
D_\mu T^{\mu\nu} = 0. \tag{14}
\]

Applying the above general field equations in the case of the flat FRW metric (1) we obtain the Friedmann equations

\[
3H^2 = \frac{1}{4} \dot{\Phi}^2 + V(\Phi) - 24c_1 e^\Phi \dot{\Phi} H^3 + M_p^{-2} (\rho_m + \rho_r), \tag{15}
\]

\[
2\dot{H} + 3H^2 = - \left[ \frac{1}{4} \dot{\Phi}^2 - V(\Phi) + 16c_1 e^\Phi \dot{\Phi} (\dot{H} + H^2) + 8c_1 e^\Phi (\dot{\Phi}^2 H^2 + M_p^{-2} (\rho_m + \rho_r)) \right], \tag{16}
\]

as well as the dilaton equation

\[
\ddot{\Phi} + 3H \dot{\Phi} + 2V'(\Phi) - 48c_1 e^\Phi H^2 (\dot{H} + H^2) = 0. \tag{17}
\]

Hence, we can identify the effective dark energy density from (15) as

\[
\rho_{DE} \equiv M_P^2 \left[ \frac{1}{4} \dot{\Phi}^2 + V(\Phi) - 24c_1 e^\Phi \dot{\Phi} H^3 \right], \tag{18}
\]

and the dark energy pressure as

\[
p_{DE} \equiv M_P^2 \left[ \frac{1}{4} \dot{\Phi}^2 - V(\Phi) + 16c_1 e^\Phi \dot{\Phi} (\dot{H} + H^2) + 8c_1 e^\Phi (\dot{\Phi}^2 H^2 + M_p^{-2} (\rho_m + \rho_r)) \right]. \tag{19}
\]

**C. \( f(P) \) gravity and cosmology**

We now proceed to the investigation of cubic terms. A general such combination is written as [10]

\[
P = \beta_1 R^\rho_{\mu\nu} R^\mu_{\rho\sigma} R^\gamma_{\delta\sigma} - \beta_2 R^\rho_{\mu\nu} R^\mu_{\rho\sigma} R^\gamma_{\delta\sigma} R^\delta_{\gamma\sigma} + \beta_3 R^\rho_{\mu\nu} R^\mu_{\rho\sigma} + \beta_4 R R^\rho_{\mu\nu} R^\mu_{\rho\sigma} + \beta_5 R R^\rho_{\mu\nu} R^\mu_{\rho\sigma}, \tag{20}
\]

Hence, using it as an argument of an arbitrary function we can construct the action of \( f(P) \) gravity as [37]

\[
\int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R + f(P) \right]. \tag{21}
\]

The above cubic combination possesses many coupling parameters. Nevertheless, we can significantly reduce their number by requiring that in the case of simple cubic theory (i.e. with \( f(P) = P \)) the resulting theory possesses a spectrum identical to that of general relativity, that this combination is neither topological nor trivial in four dimensions, and that its definition is independent of the dimensions [36]. Focusing additionally on FRW geometry we finally find that [37]

\[
P = 6\beta_2 H^4 \left( 2H^2 + 3H \right), \tag{22}
\]

which has only one free parameter. As expected is cubic in terms comparing to the Ricci scalar.
The two Friedmann equations of $f(P)$ gravity in the case of FRW geometry take the standard form (5), (6), however now the energy density and pressure of the effective dark energy fluid are written as

$$
\rho_{DE} = -f(P) - 18\dot{\Omega}H^4(H\dot{\theta} - H^2 - \dot{\theta}) f'(P),
$$
(23)

$$
\rho_{DE} = f(P) + 6\dot{\Omega}^3 \left[ H\dot{\theta}^2 + 2(H^2 + 2H\dot{\theta})\dot{\theta} - 3H^3 - 5HH\rho \right] f'(P).
$$
(24)

### D. String-inspired quartic curvature corrections

We next proceed to discuss models inspired from string theory, which involve quartic curvature corrections [57]. In the notation of that work the effective low-energy string action is given in the form:

$$
S = \int d^Dx \sqrt{-g} \left[ R + \mathcal{L}_c + \ldots \right],
$$
(25)

with $R$ the scalar curvature, and where for simplicity we use units where $M_P^2 = 1$. In the above expression $\mathcal{L}_c$ denote the the string-inspired correction terms given by

$$
\mathcal{L}_c = c_1 \alpha' e^{-2\phi} \mathcal{L}_2 + c_2 \alpha'^2 e^{-4\phi} \mathcal{L}_3 + c_3 \alpha'^3 e^{-6\phi} \mathcal{L}_4,
$$
(26)

where $\alpha'$ is the string Regge slope, serving as an expansion parameter, $\Phi$ is the dilaton field, and

$$
\mathcal{L}_2 = \Omega_2,
$$
(27)

$$
\mathcal{L}_3 = 2\Omega_3 + R^{\mu\nu} R_{\alpha\beta} R^\alpha_{\mu\beta} R^\mu_{\alpha\beta},
$$
(28)

$$
\mathcal{L}_4 = \mathcal{L}_{41} - \delta H \mathcal{L}_{42} - \frac{\delta B}{2} \mathcal{L}_{43},
$$
(29)

with

$$
\Omega_2 = R^2 - 4R_{\mu\nu} R^{\mu\nu} + 4R_{\mu\alpha\beta} R^{\mu\alpha\beta} + 2R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} + 3R_{\mu\nu} R_{\alpha\beta} R^{\mu\nu} R_{\alpha\beta},
$$
(30)

$$
\Omega_3 = R_{\mu\nu} R_{\alpha\beta} R^\mu_{\gamma\delta} R^\nu_{\alpha\beta} R^\beta_{\mu\gamma\delta} - 2R_{\mu\nu} R_{\alpha\beta} R^\mu_{\gamma\delta} R^\nu_{\alpha\beta} R^\beta_{\mu\beta\gamma\delta} + \frac{3}{4} R^2 R_{\mu\alpha\beta} R_{\alpha\beta} + R_{\mu\alpha\beta} R_{\alpha\beta} R^\mu_{\alpha\beta} + 4R_{\mu\nu} R_{\alpha\beta} R^\mu_{\alpha\beta} R^\nu_{\alpha\beta} - 6R R_{\alpha\beta}^2 + \frac{R^2}{4},
$$
(31)

$$
\mathcal{L}_{41} = \zeta(3) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \left( R_{\beta\gamma\delta} R^\beta_{\alpha\gamma\delta} - 2 R_{\beta\gamma\delta} R^\beta_{\gamma\delta} \right),
$$
(32)

$$
\mathcal{L}_{42} = \left( R_{\mu\alpha\beta} R_{\mu\alpha\beta} \right) - 1 + \frac{1}{4} R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R_{\mu\nu},
$$
(33)

$$
\mathcal{L}_{43} = \left( R_{\mu\alpha\beta} R_{\mu\alpha\beta} \right)^2 - 10 R_{\mu\alpha\beta} R_{\mu\alpha\beta} R_{\sigma\delta\rho} R_{\sigma\delta\rho} + R_{\mu\rho\sigma} R_{\mu\rho\sigma} R_{\sigma\delta\rho} - R_{\mu\alpha\beta} R_{\mu\alpha\beta} R_{\sigma\delta\rho} R_{\sigma\delta\rho} + R_{\mu\alpha\beta} R_{\mu\alpha\beta} R_{\sigma\delta\rho} R_{\sigma\delta\rho}.
$$
(34)

The coefficient $\delta H(B)$ is equal to 1 for the case of the heterotic (bosonic) string theory and zero otherwise. The Gauss-bonnet term, $\Omega_2$, as well as the Euler density, $\Omega_3$, do not contribute to the background equation of motion for $D = 4$, unless the dilaton is dynamically evolving, which we shall not consider here. The values of the coefficients ($c_1, c_2, c_3$) depend on the kind of the underlying string theories [57]. In particular, we have $(c_1, c_2, c_3) = (0, 0, 1/8), (1/8, 0, 1/8), (1/4, 1/48, 1/8)$ for type II, heterotic, and bosonic strings, respectively.

Following [57], we consider a spatially flat $(3+1)$-dimensional Friedmann-Robertson-Walker metric with a lapse function $N(t)$, namely $\alpha^2 = -N(t)^2 dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2$, and varying the effective action (25) with respect to $N$ yields

$$
6\dot{H}^2 = \rho_c + \rho_m,
$$
(35)

where

$$
\rho_c = \frac{d}{dt} \left( \frac{\partial \mathcal{L}_c}{\partial \dot{N}} \right) + 3H \frac{\partial \mathcal{L}_c}{\partial N} - \frac{\partial \mathcal{L}_c}{\partial N} - \mathcal{L}_c \bigg|_{N=1}.
$$
(36)

The quantity $\rho_m$ denotes the energy density of the cosmic fluid, which in [57] was assumed to be of barotropic type, with an equation of state $w_m = p_m/\rho_m$ satisfying the conservation equation

$$
\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0.
$$
(37)

From Eq. (36) we find that for bosonic strings, the energy density $\rho_c$ is given by [57]

$$
\rho_c = A(5H^6 + 2I^3 - 6HJ) + B\left\{ -21\zeta(3) + 210H^8 \right\} - \left\{ -3\zeta(3) - 90I^4 - 12\zeta(3) + 48H^4I^2 \right\} + A\left\{ -4(3) + 120H^2I^3 - 24(3) - 96H^6I \right\} + J\left\{ 8(3) - 32H^5 + 12(3) - 360H^2 \right\}
$$
(38)

$$
+ 24(3)H^3I \right\},
$$
(39)

where $I \equiv H^2 + \dot{H}$, $J \equiv \dot{H} + 3HH^3 + H^3$, $A = 24c_2c^2e^{-4\phi}$ and $B = 6c_3c^3e^{-6\phi}$.

For type II & heterotic strings we have [57]

$$
\rho_c = B\left\{ a_8H^8 + a_4I^4 + a_2H^2I^2 + a_0H^6I \right\} - J\left\{ a_5H^5 + a_1H^2I^2 + a_3H^3I \right\},
$$
(39)

with $B = 6c_3c^3e^{-6\phi}(3)$, $a_8 = -21, a_c = -3, a_4 = -12, a_2 = 4, a_6 = -24, a_5 = -8, a_1 = -12, a_3 = -24$ for type II string, and $B = 6c_3c^3e^{-6\phi}$, $a_8 = -21\zeta(3) + 35, a_c = -3\zeta(3) + 15, a_4 = -12\zeta(3) - 6, a_2 = 4\zeta(3) + 20, a_6 = -24\zeta(3) + 12, a_5 = -8\zeta(3) + 4, a_1 = -12\zeta(3) + 66, a_3 = -24\zeta(3)$ for the heterotic string.

Let us make a comment here on the relation of this model with the case of cubic corrections of the previous sections. In the case of cubic and $f(P)$ gravity the coefficient parameters in (20) have been chosen in order for $P$ not to have higher than second-order derivatives, while in the present model this condition is not required (one can see that (38) and (39) contain higher derivatives). Hence, even in the limit $c_3 = 0$ the scenario with string-inspired quartic curvature corrections does not recover the cubic scenario.

Finally, in [57] the authors considered the case of phantom energy, with $w_m < -1$. The analysis showed, for
instance, that for the case of type II string theories, universes with \( w_m \lesssim -1.2 \) approach the Big Crunch singularity, while for \(-1.2 \lesssim w_m < -1\), they tend to approach the Big Rip singularity.

In this work we are interested in examining these cases from the point of view of BBN constraints, as we will see in subsection III E below.

E. Running vacuum cosmology

In this subsection we briefly review cosmology with running vacuum [64–77]. In such a scenario the vacuum energy density is expressed, as a result of general covariance, in terms of even integer powers of the Hubble parameter. We mention that possible dependence in terms of \( H^4 \) during the brief epoch of BBN in which we are interested in, can be expressed in terms of \( H^2 \) using the approximately constant value of the deceleration parameter for that period, and hence we shall not consider it explicitly here (nonetheless, the current-era phenomenology of such \( H \) terms has been examined with precision (for latest work see, e.g. [75]).

The corresponding running vacuum model (RVM) energy density, which plays the role of the dark energy, reads as

\[
\rho_{\text{RVM}}^{\text{vac}} = 3M_P^2 \left( c_0 + \nu H^2 + \frac{\alpha}{H^2} H^4 + \ldots \right),
\]

while the RVM equation of state is [64–67]

\[
P_{\text{RVM}}^{\text{vac}} = -\rho_{\text{RVM}}^{\text{vac}},
\]

in the standard parametrisation where \( H_I \) is the inflationary scale, which according to Planck data [81] is of the order of \( H_I \sim 10^{-5} M_P \). Furthermore, the coefficients \( \nu, \alpha > 0 \) are dimensionless, and the neglected terms in (40) denote terms of order \( H^6 \) and higher since they do not play a significant role [72, 73]. The constant \( c_0 \) in (40) cannot be determined from first principles, given that the model is based on an expansion in even powers of \( H \) of the “renormalisation-group-like scaling” \( \frac{d \ln \rho_{\text{RVM}}^{\text{vac}}}{d \ln H} = \sum_{n=1}^{\infty} c_n H^{2n} \). In cosmological terms \( c_0 \) plays the role of the cosmological constant. In such a case, the term \( \nu H^2 \) yields observable deviations from \( \Lambda \)CDM scenario, and the current phenomenology requires \( \nu = \mathcal{O}(10^{-3}) > 0 \) [69, 71, 76]. Moreover, RVM-like models can be used to alleviate the tensions [82] that characterise the current-era cosmological data [69, 70, 75, 77] (for the phenomenology of more general models of \( \Lambda(t) \)-varying cosmologies, see [83, 84]).

Microscopic models for RVM are constructed within the context of string-inspired cosmological scenarios [78–80]. In such models, which involve gravitational anomalies (such as Chern-Simons terms) coupled to axion fields, the term \( H^4 \) term, which may drive dynamical inflation, is induced by a CP-violating primordial gravitational-wave condensate of the Chern-Simons terms. In the post-inflationary epoch gravitational anomalies are assumed to be cancelled, and therefore the term \( H^4 \) will be absent during BBN. Nonetheless, in general, there may be other, non-geometric, reasons for having such \( H^4 \) terms, for instance as a result of integrating out interacting quantum fields of some mass scale \( m \).

In general, as discussed in [78, 85, 86] (see also [87, 88] in the case of (dynamically broken) supergravity, which can be linked to early-universe running vacuum [78, 85]), quantum graviton corrections around a background cosmological space-time might also lead to a renormalisation of the Newton constant, which becomes “running” too, exhibiting a mild logarithmic dependence on the Hubble parameter. The precise form of such a running will depend on the underlying theory of quantum gravity.\(^2\) In this respect, for our phenomenological purposes here, we may parametrise the effective RVM vacuum energy density during the post-inflationary eras, including the BBN epoch, as [78, 85, 86]:

\[
\rho_{\text{RVM}}^{\text{vac}} \equiv \rho_{\text{DE}} \simeq 3M_P^2 \left[ c_0 + \nu + d_1 \ln(M_P^{-2} H^2) \right] H^2

+ \frac{\alpha}{H^2} \left[ 1 + d_2 \ln(M_P^{-2} H^2) \right] H^4 + \ldots \right),
\]

where \( c_0 > 0, \nu > 0, \alpha > 0 \) and \( d_i, i = 1, 2 \) are phenomenological parameters.

III. BIG BANG NUCLEOSYNTHESIS CONSTRAINTS

In this section we will investigate the Big Bang Nucleosynthesis (BBN) constraints on scenarios that are governed by higher-order modified gravity. BBN is realised during the radiation epoch [92–95]. In the case of standard cosmology, i.e. in the case of Standard Model radiation in the framework of general relativity, during the BBN the first Friedmann equation is approximated as

\[
H^2 \simeq \frac{M_P^{-2}}{3} \rho_c \equiv H_{\text{cr}}^2.
\]

\(^2\) The supergravity models of [87] involve logarithmic corrections of the one-loop induced de-Sitter cosmological constant \( \Lambda > 0 \), which in a general relativity setting is proportional to the curvature scalar \( R \), and thus proportional to \( H^2 \) in cosmological models. For our purposes we assume a mild cosmic time dependence of \( H(t) \). In this sense, the quantum graviton integration in such models leads to \( \ln(R) R^n, n \in \mathbb{Z}^+ \) terms in an effective action, which in a cosmological setting leads to energy densities of the form [78, 85, 86]. It should be stressed that the above computation in the model of [87] refers to weakly-coupled graviton fluctuations about a (de Sitter) background in a fixed gauge. The issue of gauge invariance in a full quantum gravity theory is a complicated issue, and will not be discussed here. We refer the reader to [89, 90] for more details on this important issue. We also notice that modified gravity models involving \( \ln(R) \) terms have been considered in [91], but from a different perspective than ours. Moreover, in our case the corrections have a smooth flat limit \( R \rightarrow 0 \).
Additionally, we know that the energy density of relativistic particles is

$$\rho_r = \frac{\pi^2}{30} g_* T^4,$$  (44)

with $g_* \sim 10$ the effective number of degrees of freedom and $T$ the temperature. Hence, we obtain

$$H(T) \approx \left( \frac{4\pi^3 g_*}{45} \right)^{1/2} \frac{T^2}{M_{Pl}},$$  (45)

where $M_{Pl} = (8\pi)^{1/2} M_P = 1.22 \times 10^{19}$ GeV is the Planck mass.

Since the radiation conservation equation finally leads to a scale factor evolution of the form $a \sim t^{1/2}$, we can finally extract the expression between temperature and time, namely

$$\frac{1}{t} \approx \left( \frac{32\pi^3 g_*}{90} \right)^{1/2} \frac{T^2}{M_{Pl}} \text{ (or } T(t) \approx (t/\text{sec})^{-1/2} \text{ MeV}).$$

During the BBN, the calculation of the neutron abundance arises from the protons-neutron conversion rate

$$\lambda_{pn}(T) = \lambda_{(n+\nu_e-\nu_e-p+e^-)} + \lambda_{(n+e^+\rightarrow p+\nu_e)} + \lambda_{(n\rightarrow p+e^-+\bar{\nu}_e)},$$

and its inverse $\lambda_{np}(T)$, and therefore for the total rate we have $\lambda_{\text{tot}}(T) = \lambda_{np}(T) + \lambda_{pn}(T)$. Assuming that the various particles (neutrinos, electrons, photons) temperatures are the same, and low enough in order to use the Boltzmann distribution instead of the Fermi-Dirac one), and neglecting the electron mass compared to the electron and neutrino energies, straightforward calculations lead to the expression [94, 95]

$$\lambda_{\text{tot}}(T) = 4AT^3(4T^2 + 2 \times 3!QT + 2!Q^2),$$  (47)

where $Q = m_n - m_p = 1.29 \times 10^{-3}$ GeV is the mass difference between neutron and proton and $A = 1.02 \times 10^{-11}$ GeV$^{-4}$.

Let us now calculate the corresponding freeze-out temperature. This will arise from the comparison of the universe expansion rate $\frac{1}{t}$ with $\lambda_{\text{tot}}(T)$. In particular, if $\frac{1}{t} \ll \lambda_{\text{tot}}(T)$, namely if the expansion time is much smaller than the interaction time we can consider thermal equilibrium [92, 93]. On the contrary, if $\frac{1}{t} \gg \lambda_{\text{tot}}(T)$ then particles do not have enough time to interact and therefore they decouple. Thus, the freeze-out temperature $T_f$, in which the decoupling takes place corresponds to $H(T_f) = \lambda_{\text{tot}}(T_f) \approx c_q T_f^4$, with $c_q \equiv 4A4! \approx 9.8 \times 10^{-10}$ GeV$^{-4}$ [96–100]. Using (45) and (47), the above requirement gives

$$T_f = \left( \frac{4\pi^3 g_*}{45M_{Pl}c_q^{29/6}} \right)^{1/6} \sim 0.0006 \text{ GeV}.$$  (48)

Now, in any modified cosmological scenario one obtains extra terms in the Friedmann equations. During the BBN era these extra contributions need to be small, compared to the radiation sector of standard cosmology, in order not to spoil the observational facts. In particular, from a general modified Friedmann equation of the form (5) we obtain

$$H = H_{GR} \sqrt{1 + \frac{\rho_{DE}}{\rho_r}} = H_{GR} + \delta H,$$  (49)

where $H_{GR}$ is the Hubble parameter of standard cosmology. Thus, we have

$$\delta H = \left( \sqrt{1 + \frac{\rho_{DE}}{\rho_r}} - 1 \right) H_{GR}.$$  (50)

This deviation from standard cosmology, i.e form $H_{GR}$, will lead to a deviation in the freeze-out temperature $\delta T_f$. Since $H_{GR} = \lambda_{\text{tot}} \approx c_q T_f^4$, we easily find

$$\frac{\delta T_f}{T_f} \sim \frac{\rho_{DE}}{\rho_r} \frac{H_{GR}}{10c_q T_f^4},$$  (52)

where we used that $\rho_{DE} \ll \rho_r$ during BBN. This theoretically calculated $\frac{\delta T_f}{T_f}$ should be compared with the observational bound

$$\left| \frac{\delta T_f}{T_f} \right| < 4.7 \times 10^{-4},$$  (53)

which is obtained from the observational estimates of the baryon mass fraction converted to $^4\text{He}$ [101–107]. In the following subsections we use the above formalism, and in particular expression (52), in order to impose constraints on $\rho_{DE}$ and thus on the underlying modified gravity, in specific models.

### A. $f(G)$ Gravity

In this section we apply the above formalism in order to extract the BBN constraints on $f(G)$ gravity. In particular, we will use the dark energy density expression (7) that holds in the case of $f(G)$ cosmology. In order to proceed we need to examine specific $f(G)$ forms. We will focus on three well-studied models that are known to lead to interesting late-time cosmology.

We stress at this point that, since we are interested in scenarios where the $f(G)$ term gives rise to the effective dark energy sector, we do not consider the case where $f(G)$ is linear in $G$, since in this case there is no contribution from $G$ to the Friedmann equations, given its topological (total-derivative) nature in $(3+1)$-dimensional space times.
1. \( f(G) \) Model I

As a first example, namely \( f(G) \) Model I, we consider the power-law model \( [20] \) where

\[
f(G) = \alpha G^n,
\]
with \( n \neq 1 \). In this expression \( n \) is the only free model parameter, since as long as \( n \neq 1 \) then \( \alpha \) can be expressed in terms of the present value of the Hubble parameter \( H_0 \) and the present value of the dark energy density parameter \( \Omega_{DE0} \equiv \rho_{DE0}/(3M_{Pl}^2H_0^2) \) (in the case \( n = 1 \) the above model cannot account for dark energy, in view of the aforementioned total derivative nature of the four-dimensional Gauss-Bonnet invariant). In particular, by applying (7) at present we find

\[
\alpha = \frac{3H_0^2\Omega_{DE0}}{M_{Pl}^{-2}[(n-1)G_0^n - n(n-1)\gamma_0G_0^{n-2}]},
\]
where

\[
G_0 = 24H_0^2\left(H_0^2 + \dot{H}_0\right),
\]
and

\[
\gamma_0 = 24^2H_0^6\left(2H_0^2 + H_0\dot{H}_0 + 4H_0^2\ddot{H}_0\right).
\]

Inserting (54) into (7) and then into (52) we finally obtain

\[
\frac{\delta T_f}{T_f} = -\Omega_{DE0}(\zeta)^{4n-1}(T_f)^{8n-7}
\]

\[
\cdot \left[(-1)^n + n(1)\cdot n^{-1} + 8n(n-1)(\gamma_0 n-2)\right]
\]

\[
\cdot (H_0)^{2-2n}\left(H_0^2 + \dot{H}_0\right)^{-n}[10c_q(n-1)]^{-1}
\]

\[
\cdot \left[(1-2n)\left(H_0^2 + 2H_0\dot{H}_0\right) - n\dot{H}_0H_0 + H_0\right]^{-1},
\]

(58)

with

\[
\zeta = \left(\frac{4\pi^3g_*}{45}\right)^\frac{1}{2} M_{Pl}^{-1}.
\]

In this expression we set \( [81] \)

\[
\Omega_{DE0} \approx 0.7, \quad H_0 \approx 1.4 \times 10^{-42} \text{ GeV},
\]
and the derivatives of the Hubble function at present are calculated through \( H_0 = -H_0^2(1 + q_0) \) and \( \dot{H}_0 = H_0^2(j_0 + 3q_0 + 2) \) with \( q_0 = -0.503 \) the current deceleration parameter of the Universe \([81]\], and \( j_0 = 1.011 \) the current jerk parameter \([108, 109]\). In Fig. 1 we plot \( \delta T_f/T_f \) appearing in (58) vs the model parameter \( n \), as well as the upper bound inferred from (53). As becomes evident from the figure, the expression (58) satisfies the bound (53) for \( n \lesssim 0.45 \).

We stress here that in this work we desire to impose BBN constraints on higher-order modified gravities models that can describe dark energy. Hence, concerning the present \( f(G) \) model we require the fulfillment of condition (55), which imposes a dependence of the model parameters \( \alpha \) and \( n \). That is why BBN analysis leads to a strong constraint on \( n \). If we relax condition (55) the BBN constraints can always be fulfilled for every \( n \) by suitably constraining \( \alpha \), and equivalently the BBN constraints can always be fulfilled for every \( \alpha \) by suitably constraining \( n \). However, under the condition (55), namely under the requirement that the Gauss-Bonnet terms describe dark energy at the late Universe, then \( n \) is constrained close to zero, in which case \( f(G) \) correction becomes a constant and the scenario becomes \( \Lambda \text{CDM} \).

2. \( f(G) \) Model II

As a second concrete example we consider the \( f(G) \) Model II, namely \([21]\)

\[
f(G) = \lambda \frac{G}{\sqrt{G_*}} \arctan\left(\frac{G}{G_*}\right) - \lambda \frac{G}{2\sqrt{G_*}} \ln\left(1 + \frac{G^2}{G_*^2}\right) - \alpha \lambda \sqrt{G_*},
\]

(61)

where \( \alpha, \lambda > 0 \) and \( G_* > 0 \) are the model parameters. Applying (7) at present time we can express \( \lambda \) as

\[
\lambda = 3M_{Pl}^2H_0^2\Omega_{DE0}\left\{\arctan\left(\frac{G_0}{G_*}\right)\sqrt{G_0} \left(1 - \sqrt{\frac{G_0}{G_*}}\right) + \sqrt{G_*}\left[\alpha + \frac{1}{2} \ln\left(1 + \frac{G_*^2}{G_*^2}\right) - \frac{\gamma_0}{G_*^2 + G_*^2}\right]\right\}^{-1},
\]

(62)
and thus we remain with only two free parameters $\alpha$ and $G_\ast$. Inserting (61) into (7) and then into (52) we obtain

$$\frac{\delta T_f}{T_f} = \frac{1}{20} \Omega_{DE 0} H_0^2 \sqrt{G_\ast} \left[ c_q \zeta T_f^7 \right]^{-1} \cdot \left\{ 2\alpha + \ln \left[ 1 + \frac{c_q T_f^16}{G_\ast^2} \right] - \frac{9216 c_q T_f^16}{576 \zeta c_q T_f^16 + G_\ast^2} \right\}^{\frac{1}{2}} \left( 1 - \sqrt{G_\ast} \right)^{-1} \left( 1 + \frac{G_0 \zeta T_f^7}{G_\ast} \right) \left( 1 + \frac{G_0^2}{G_\ast^2} \right),$$

(63)

with $\zeta$ given in (59).

Interestingly enough, expression (64) always satisfies the bound (53) for the $\alpha$ regions that are needed in order to have a stable de Sitter point [21] (that is, $G_\ast^2 \sim 10^{-42}$ GeV and $\alpha \approx 100$ and $\lambda \ll 1$). Hence, BBN cannot impose any constraints on this model beyond those already obtained from theoretical consistency.

3. $f(G)$ Model III

We proceed to the investigation of the $f(G)$ Model III, for which [21]

$$f_2(G) = \frac{\lambda}{\sqrt{G_\ast}} \arctan \left( \frac{G}{G_\ast} \right) - \alpha \lambda \sqrt{G_\ast}. \quad (65)$$

Applying (7) at present time we eliminate $\lambda$, and then, on inserting into (52) we find

$$\frac{\delta T_f}{T_f} = \frac{1}{10} \Omega_{DE 0} H_0^2 \left[ c_q \zeta T_f^7 \right]^{-1} \cdot \left\{ \left( \frac{G_0 + G_\ast}{G_0^2 + G_\ast^2} \right) \left( 1 + \alpha \right) \left( \frac{G_0}{G_\ast} + \alpha \frac{G_\ast}{G_0} \right) \cdot 2 \gamma_0 G_\ast^2 \right\}^{-1} \cdot \left\{ \left( \frac{G_0^2 + G_\ast^2}{G_0^2 + G_\ast^2} \right) \left( 1 + \alpha \right) \right\}^{-1} \left( 1 + \frac{G_0^2}{G_\ast^2} \right)$$

(66)

with $G_0$ given in (56) and $\zeta$ in (59). Similarly to the previous model, we find that expression (66) always satisfies the bound (53) for the parameter regions that are needed in order to have a stable de Sitter point [21]. Thus, BBN cannot impose stinger constraints on this model, i.e. it always satisfies BBN requirements.

4. $f(G)$ Model IV

As a last model we consider [21]

$$f(G) = \lambda \sqrt{G_\ast} \ln \left[ \cosh \left( \frac{G}{G_\ast} \right) \right] - \alpha \lambda \sqrt{G_\ast}. \quad (67)$$

Applying (7) at present time we eliminate $\lambda$ and then inserting into (52) we find

$$\frac{\delta T_f}{T_f} = \frac{1}{10} \Omega_{DE 0} H_0^2 \left\{ 4608 \zeta \cosh(\xi^2) T_f^{16} \right\} \cdot \ln \left( \cosh(\xi^2) \right) - \alpha \mid G_\ast^2 - 24 \zeta^4 T_f^8 G_\ast \tanh(\xi^2) \left\{ \left( \frac{G_0}{G_\ast} \right)^2 \right\}^{-1} \left( 1 + \frac{G_\ast}{G_\ast} \right)^{-1} \ln \left( \cosh \left( \frac{G_0}{G_\ast} \right) \right) - \alpha \left( \frac{G_\ast}{G_\ast} \right)^2$$

(68)

with $\zeta$ defined in (59) and $\xi = \frac{24 \zeta^4 T_f^8 G_\ast}{G_\ast}$. Similarly to the previous models, expression (66) always satisfies the bound (53) for the parameter regions required in order to have a stable de Sitter point [21], and therefore BBN requirements are always fulfilled.

B. Gauss-Bonnet-Dilaton Gravity

The case of Gauss-Bonnet-Dilaton gravity is more complicated than $f(G)$ and $f(P)$ gravity, since one has the additional dilaton field $\Phi$, which follows its own evolution (equation (17)). Thus, in general one cannot extract any analytical expressions and one needs to solve numerically the whole system of Friedmann and dilaton equations. Nevertheless, in the well-studied case of an exponential potential of the form [58–63]:

$$V(\Phi) = V_0 e^{\lambda \Phi}, \quad (69)$$

with $\lambda < 0$ a constant, equation (17) leads to the lowest-order solution during the radiation epoch $\Phi \approx c_3 \ln a$, implying $\Phi \approx c_3 H$, where $a(t) = a(t)^{1/2}$, $c_3 = -\frac{4}{\lambda}$ and $a_3 = (2 \lambda^2 V_0)^{1/4}$. Inserting these into (18) we find that during the radiation era we have

$$\rho_{DE} = M_p^2 \left[ \frac{H^2}{\lambda^2} + \frac{96 \lambda}{2} c_1 (\lambda^2 V_0)^{-\frac{1}{2}} H^4 + \frac{1}{2} \right]. \quad (70)$$

Applying it at present time, we can express the parameter $c_1$ in terms of $H_0$ and $\Omega_{DE 0}$ as

$$c_1 = \frac{\left( \frac{2}{\lambda^2} - \Omega_{DE 0} \right) \sqrt{\lambda} (\lambda^2)^{\frac{1}{2}} H_0^{2 - I - \frac{1}{2}}}{32} \left( \frac{V_0}{H_0^2} \right)^{\frac{1}{2}}. \quad (71)$$
Hence, inserting these into (52) we acquire
\[
\frac{\Delta T_f}{T_f} = \frac{\zeta \left[ \frac{2}{\lambda^2} - \left( \frac{\zeta T_f}{T_0} \right)^2 (\lambda+1) \right]}{10 c_q T_f^3},
\]
(72)
with \(\zeta\) given in (59). Interestingly enough we find that the constraint \(\left| \frac{\Delta T_f}{T_f} \right| < 4.7 \times 10^{-4}\) is satisfied in the narrow window
\[
\lambda \in (-1.00561, -1.00558),
\]
(73)
where we used (60). This result was expected, since in any model that can describe the dark energy sector at present times the BBN requirements restrict its parameters to suitably narrow ranges, since general parameter values would lead to unacceptable large early dark energy during the BBN.

Finally, using (71) we can find the range of the Gauss-Bonnet coupling parameter \(c_1\), for given values of the \(V_0/H_0^2\), viewing the model as a phenomenological modified-gravity model, independent of string theory. For instance, for \(V_0/H_0^2 \sim 10^{94}\) (since \(H_0 \approx 10^{-42}\) GeV this implies that \(\sqrt{V_0} \approx 10^5\) GeV) we find \(c_1 H_0^2 \in (2.629 \times 10^{-95}, 2.646 \times 10^{-95})\), i.e. \(c_1 \in (1.341 \times 10^{-11}, 1.350 \times 10^{-11})\) GeV\(^{-2}\).

We now remark that, when we concentrate on string theory models [52–54], we need to use the expression (11) for the Gauss-Bonnet coefficient \(c_1\). Taking into account that in standard string phenomenology \(g_s (02)/4\pi \approx 1/20\), and that the current collider searches imply \(M_s > O(10)\) TeV, we find \(c_1 \lesssim 10^{-9}\) GeV\(^{-2}\). Hence, applying the above procedure the other way around, we find from (71) and (73) that the BBN constraints are satisfied for \(V_0/H_0^2 \gtrsim 10^{83}\), which implies in order of magnitude the condition \(\sqrt{V_0} \gtrsim 10^8\) GeV.

In the above results should be interpreted as implying that such dilaton dominance in string-inspired cosmologies should end long before the BBN era [110], unless the relevant parameter \(\lambda\) in the dilaton quintessence-like potential lies in the aforementioned narrow window. In that case, however, the dilaton presence during BBN might be in conflict with other phenomenological consequences of the dilaton cosmology, e.g. supersymmetry searches for dark matter at colliders [110–112], provided one attributes the dominant dark matter species to supersymmetry. Such issues fall beyond our purposes in the current work.

C. \(f(P)\) Gravity

As a first example, namely \(f(P)\) Model I, we consider the power-law model
\[
f(P) = \alpha P^n,
\]
(74)
where \(n\) is the only free model parameter, since \(\alpha\) can be expressed in terms of \(H_0\) and \(\Omega_{DE0}\) given in (60) by applying (23) at the present epoch. Inserting (74) into (23) and then into (52) we acquire
\[
\frac{\delta T_f}{T_f} = 2.1 (\zeta)^{6n-1} (T_f)^{12n-7}
\]
\[
\cdot \left[ (-24)^n - 216 n (n-1) (-24)^{n-1} + 18 n (-24)^{n-1} \right]
\]
\[
\cdot (30c_q)^{-1} (6)^{-1-n} (H_0)^{-4n} \left( 2H_0^2 + 3H_0 \right)^{2-n}
\]
\[
\cdot \left[ 216 n (n-1) - 18 \right] \hat{H}_0 H_0^2
\]
\[
+ 54 n (n-1) \left( 4H_0^2 + \hat{H}_0 H_0 \right) - 12 H_0^2 \right]^{-1}.
\]
(75)

In Fig. 2 we draw \(\delta T_f/T_f\) from (58) vs the model parameter \(n\) (blue solid curve) in the case of \(f(P)\) Model of (74), and the upper bound for \(\delta T_f/T_f\) from (53) (red dashed line). As we observe, constraints from BBN require \(n \lesssim 0.31\).

D. \(f(G) + f(P)\) gravity and cosmology

For completeness, let us examine a more realistic case, namely the combination of \(f(G)\) and \(f(P)\) gravity. The action is
\[
S = \int d^4x \left[ \frac{M_0^2}{\alpha} R + f(G) + f(P) \right],
\]
(76)
where
\[ f(G) + f(P) = \alpha G^n + b P^n. \]  
(77)

The dark energy component is
\[ \rho_{DE} = \frac{1}{2} \left[ -f(G) + 2a H^2 \left( H^2 + \dot{H} \right) + f'(G) \right. \]
\[ \left. - 24^2 H^4 \left( 2\dot{H}^2 + 2\ddot{H} + 4H^2 \dot{H} \right) f''(G) \right] \]
\[ - \left[ f(P) + 18\beta H^4 (H\partial_h - H^2 - \dot{H}) f'(P) \right]. \]  
(78)

Using the above analysis for this model we get
\[ \frac{\Delta T_f}{T_f} = -\Omega_{DE} \alpha \zeta^6 n - T_f^{8n-7} \left( 2 - 24 \right)^n \]
\[ \times \left[ 10 c_q (a_0 - b_0) \right]^{-1} \left\{ 8n(n - 1) - n \right\} \]
\[ + \frac{3b}{2} \alpha \zeta^6 n T_f^{4n} \left[ 12(n - 1) - n \right], \]  
(79)

where
\[ a_0 = (24)^n (n - 1) \left( H_0^2 + \dot{H}_0 \right)^{n-2} \]
\[ \times \left[ (1 - 2n) \left( \dot{H}_0^2 + 2H_0H_0^{\prime} \right) - n \ddot{H}_0 H_0 + H_0^2 \right], \]  
(80)

and
\[ b_0 = 108 \beta \zeta^6 (n - 1) H_0^{2n} \left( 2H_0^2 + 3\dot{H}_0 \right)^{n-2} \]
\[ \times \left[ 4\dot{H}_0 H_0^2 + 4\dot{H}_0^2 + \dot{H}_0 H_0 - 12 H_0^2 \right]. \]  
(81)

E. String-inspired quartic curvature correction models

In this subsection we discuss BBN constraints on models containing quartic curvature corrections, obtained in the low-energy limit of string theory, (25) [57]. The first Friedmann equation (35) can be re-written as
\[ 6H^2 = \rho_m + \rho_r + \rho_{DE}, \]  
(82)

in units where \( M_P/2 = 1 \), where we have absorbed all the extra terms in an effective dark energy sector, and we have added the radiation sector for completeness. Hence, according to (39), in the case of bosonic strings we have
\[ \rho_{DE} = B [a_8 H^8 + a_4 H^4 I^2 + a_2 H^2 I^2 + a_6 H^6 I^2 \]
\[ - J(a_5 H^5 + a_1 H I^2 + a_3 H^3 I)] \].  
(83)

Since in action (25) the dilaton field does not evolve dynamically, we can choose the same fixed value in the past and present. Performing the analysis described above we find
\[ \frac{\Delta T_f}{T_f} = (60 c_q)^{-1} \zeta^3 T_f^5 B \left[ a_8 + a_4 - a_2 - a_6 \right] \]
\[ - 3(a_5 + a_1 - a_3), \]  
(84)

where \( B \) is the free parameter. For type II strings this satisfies the bound (53) for \(-2.7 \times 10^{-109} < B < 2.7 \times 10^{-109} \) or \(-3.8 \times 10^{-110} < c_3 \alpha'^3 e^{-6\phi} < 3.8 \times 10^{-110} \) in \( M_P/2 = 1 \) units. For heterotic type we have
\[ -3.6 \times 10^{-110} < B < 3.6 \times 10^{-110} \) or \(-6 \times 10^{-111} < c_3 \alpha'^3 e^{-6\phi} < 6 \times 10^{-111} \) in \( M_P/2 = 1 \) units. Hence, transforming to standard units we have
\[ -5.4 \times 10^{-37} \text{GeV}^{-4} < B < 5.4 \times 10^{-37} \text{GeV}^{-4}. \]

Similarly, for the case of bosonic strings, using (88) we have
\[ \rho_{DE} = A (5H^6 + 2I^3 - 6HIJ) + B \left[ [21\zeta(3) + 210] H^8 \right. \]
\[ \left. - [3\zeta(3) - 90] I^4 - [12\zeta(3) + 48] H^4 I^2 \right] \]
\[ + [4\zeta(3) + 120] H^2 I^3 - [24\zeta(3) - 96] H^6 I \]
\[ + J \left[ 8\zeta(3) - 32] H^8 + [12\zeta(3) - 360] HI^2 \right. \]
\[ \left. + 24\zeta(3) H^3 I \right]. \]  
(85)

Repeating the above analysis we find
\[ \frac{\Delta T_f}{T_f} = (60 c_q)^{-1} \zeta^5 T_f^5 \]
\[ \times \left[ 21 A - 28 B \zeta^2 T_f \left( \zeta(3) + \frac{285}{7} \right) \right]. \]  
(86)
Now, imposing an indicative value for the parameter $B$ from the above range, i.e. $B = 10^{-109}$ in $M_p/2 = 1$ units, we find that the range of $A$ satisfies the bound (53) for $2 \times 10^{-74} < A < 3.1 \times 10^{-74}$ or $8.4 \times 10^{-76} < c_2 \alpha^2 e^{-46} < 1.3 \times 10^{-75}$. Hence, transforming to standard units we have $4.1 \times 10^{-38} \text{ GeV}^{-2} < A < 4.5 \times 10^{-36} \text{ GeV}^{-2}$. Finally, imposing the indicative value $B = -10^{-109}$ we find $-3.1 \times 10^{-73} < A < -2.0 \times 10^{-74}$ or $-1.3 \times 10^{-74} < c_2 \alpha^2 e^{-46} < -8.4 \times 10^{-75}$, i.e. $-4.5 \times 10^{-36} \text{ GeV}^{-2} < A < -2.9 \times 10^{-36} \text{ GeV}^{-2}$.

F. Running vacuum cosmology

We use the effective dark energy density (42) in the case of (an extended in general) running-vacuum type cosmology in order to evaluate the BBN freeze-out point. We start by considering the simplest (and more standard) cosmology in order to evaluate the BBN freeze-out point.

We use the effective dark energy density (42) in the current era, yields

$$0 < \nu = \mathcal{O}(10^{-3}).$$

We stress that the presence of the non-zero parameter $\nu$ affects the running of the matter and energy densities in the current era, leading to observable (in principle) deviations from the $\Lambda$CDM paradigm [69–71, 83, 84]. The cosmological constant term $c_0$ in (42) is then fitted using the relation

$$c_0 = H_0^2 (\Omega_{DE0} - \nu),$$

where $H_0$ is the current value of the Hubble parameter, and

$$\Omega_{DE0} = \rho_{DE0} / (3M_p^2 H_0^2),$$

with $\rho_{DE0}$ the dark-energy contribution to the vacuum energy budget today. This is constrained by the available current-era data [81] to the value given in (60). Note that $\alpha$ does not appear in (98), since it multiplies $H^4$ which as we mentioned is negligible at late times.

In what follows, we shall concentrate on constraining $\nu$ solely from BBN constraints, without further attempts to fit it using the plethora of the available data, simply by requiring that the running vacuum energy density is associated with the dark energy contribution, which are the basic assumption of our analysis. This will provide additional constraints in the range of $\nu$, which can then be compared with the value obtained from the analysis of [69–71, 83, 84].

- First, we examine models with $\frac{\alpha}{M_p} \equiv c_2 = d_1 = d_2 = 0$, $c_0 \neq 0$, $\nu \neq 0$ which are also relevant for the current-era phenomenology, since, as already mentioned, the $H^4$ terms in the vacuum energy density (42) are not dominant at late epochs [69–73]. Using the constraint (98) to eliminate $c_0$, we find

$$\frac{\Delta T_f}{T_f} \equiv (10 \zeta c_q T_f^2)^{-1} [H_0^2 (\Omega_{DE0} - \nu) + \nu \zeta^2 T_f^4].$$

Hence, on account of (60), the BBN bound (53) implies that

$$-0.0023 \lesssim \nu \lesssim 0.0023.$$  

Notice that the satisfaction of the BBN constraints is possible even with negative or zero values of $\nu$. On the other hand, as mentioned above, fitting the plethora of the other modern-era cosmological data requires [69] $\nu > 0$. Nonetheless, the order of $10^{-3}$ for the parameter $\nu$ found here (cf. (91)) is in agreement with those fits. Hence, the conventional running vacuum model with a vacuum energy density that fits the current-era constraints [69, 71], with $\nu = \mathcal{O}(10^{-3}) > 0$, also satisfies the BBN constraints.

- We continue with the case $d_1 = d_2 = c_0 = 0$, $\nu \neq 0$ and $\frac{\alpha}{M_p} \equiv c_2 \neq 0$. Since, as we have mentioned above, we do not expect the $H^4$ terms to play any role in the late eras of the Universe evolution, we do not expect any strong constraints on the parameter $\alpha$ in this case. Indeed, we have

$$\rho_{DE} = 3M_p^2 H^2 (\nu + c_2 H^2).$$

Inserting it in (89) we extract the condition

$$\nu = \Omega_{DE0} - c_2 H_0^2.$$  

Using (93) to eliminate $\nu$ in terms of $c_2$ we finally result to

$$\frac{\Delta T_f}{T_f} = \frac{1}{10} \zeta c_q T_f^{-3} e^{-1} \left( [\Omega_{DE0} - c_2 \left( H_0^2 - \zeta^2 T_f^4 \right)] \right).$$

Note that this is a linear expression in terms of $c_2$, and thus we easily deduce, using (60), that for $c_2 \geq 0$ (required from the running vacuum model) it does not satisfy the BBN bound (53).

- In the case $d_1 = d_2 = \nu = 0$, $c_0 \neq 0$ and $\frac{\alpha}{M_p} \equiv c_2 \neq 0$, we have the constraint

$$c_0 = H_0^2 (\Omega_{DE0} - c_2 H_0^2),$$

which can be used to eliminate $c_0$. Hence, concerning $\frac{\Delta T_f}{T_f}$ we finally find

$$\frac{\Delta T_f}{T_f} = (10 \zeta c_q T_f^2)^{-1} \left( [H_0^2 (\Omega_{DE0} - c_2 H_0^2) + c_2 \zeta^4 T_f^4] \right).$$

Thus, in this case the bound (53) leads to

$$0 \lesssim c_2 \lesssim 9.7 \times 10^{46} \text{ GeV}^{-2}. $$
Now inserting the above range of $c_2$ into (95), we find that
\[
\frac{c_0}{H_0^2} = \Omega_{DE0} - c_2H_0^2 \approx 0.7, \quad (98)
\]
in agreement with the fit of the RVM model to the other cosmological data, performed in [69, 70].

- In the more general case $d_1 = d_2 = 0$ with $c_0 \neq 0$, $\nu \neq 0$, $c_2 \neq 0$, we have the constraint
\[
c_0 = H_0^2\Omega_{DE0} - \nu H_0^2 - c_2H_0^4, \quad (99)
\]
which allows us to eliminate $c_0$ in terms of $\nu$ and $c_2$. We extract
\[
\frac{\Delta T_f}{T_f} = \left(10c_4T_f^7\right)^{-1} \left[\Omega_{DE0}H_0^2 + \nu \left(\zeta^2T_f^4 - H_0^2\right) + c_2 \left(\zeta^2T_f^4 - H_0^2\right)\right]. \quad (100)
\]
Thus, imposing $-0.0023 < \nu < 0.0023$ (which was found in a previous case, (91)) we conclude that the expression (100) satisfies the bound (53) for $0 < c_2 < 9.7 \times 10^{46}$ GeV$^{-2}$ (cf. (97)). Using the constraint (99) we thus find $\frac{\Delta T_f}{T_f} = \Omega_{DE0} - \nu - c_2H_0^2 \approx \Omega_{DE0} - \nu$ which then gives
\[
0.6977 < \frac{c_0}{H_0^2} < 0.7023, \quad (101)
\]
again in the ball park of the RVM fit to the other cosmological data [69, 70].

We proceed with examining the case where the $\ln(H)H^{2n}$ terms in (42) are present, namely the cases with $d_1 \neq 0, d_2 \neq 0$.

- We focus first on $d_1$, i.e. we consider $d_2 = \nu = c_2 = 0$, $d_1 \neq 0$, $c_0 \neq 0$. In this case we extract the constraint
\[
c_0 = H_0^2\left[\Omega_{DE0} - d_1 \ln \left(M_P^{-2}H_0^2\right)\right], \quad (102)
\]
which allows us to eliminate $c_0$ in terms of $d_1$. Hence, we find
\[
\frac{\Delta T_f}{T_f} = \left(10c_4\zeta T_f^7\right)^{-1} \left[H_0^2\left[\Omega_{DE0} - d_1 \ln \left(M_P^{-2}H_0^2\right)\right] + d_1 \ln \left(M_P^{-2}\zeta^2T_f^4\right)\zeta^2T_f^4\right], \quad (103)
\]
and therefore deduce that
\[
d_1 \in (-1.2 \times 10^{-5}, 1.2 \times 10^{-5}). \quad (104)
\]
Inserting the range (104) of $d_1$ into (102) we find
\[
\frac{c_0}{H_0^2} \in (0.697, 0.703), \quad (105)
\]
that is, the presence of non-polynomial $H^2\ln(H^2M_P^{-2})$ terms in the RVM energy density is consistent with BBN for a range of the relevant parameter (104), which affects only marginally the standard RVM parameters. This is consistent with the fact that such corrections have been argued above to arise from quantum-graviton fluctuations [78, 85, 86].

- Focusing on $d_2$, i.e. considering $\nu = d_1 = 0$, and $c_0 \neq 0, c_2 \neq 0, d_2 \neq 0$, we first find the constraint
\[
c_0 = H_0^2\left[\Omega_{DE0} - c_2H_0^2\left[1 + d_2 \ln \left(M_P^{-2}H_0^2\right)\right]\right], \quad (106)
\]
which allows us to eliminate $c_0$ in terms of $c_2$ and $d_2 \neq 0$. In this case we find
\[
\frac{\Delta T_f}{T_f} = \left(10c_4\zeta T_f^7\right)^{-1} \left[c_2\zeta^4T_f^4 \left[1 + d_2 \ln \left(M_P^{-2}\zeta^2T_f^4\right)\right] + H_0^2\left[\Omega_{DE0} - c_2H_0^2\left[1 + d_2 \ln \left(M_P^{-2}H_0^2\right)\right]\right]\right]. \quad (107)
\]
Hence, imposing a typical value $c_2 = 10^{46}$ GeV$^{-2}$, that was found above (cf. (97)), we deduce that
\[
d_2 \in (-4.4 \times 10^{-2}, 5.4 \times 10^{-2}). \quad (108)
\]
Inserting this range of $d_2$ into (106) we find again $\frac{\Delta T_f}{T_f} \approx 0.7$. The reader should notice that, although $H^4$ terms in the vacuum energy density are not affecting the current-era phenomenology, nonetheless the terms $d_2H^4\ln(M_P^{-2}H^2)$ do affect BBN in general, and thus only a narrow window (108) (but considerably wider than the corresponding allowed range of $d_1$, (104)) is consistent with standard BBN, under the assumption that the RVM provides an alternative to dark energy.

- Finally, let us examine the cases where both $d_1$ and $d_2$ are not zero and $\nu = 0$. Thus, we consider $d_1 \neq 0, d_2 \neq 0, c_0 \neq 0, c_2 \neq 0, \nu = 0$. We first find the constraint
\[
c_0 = H_0^2\left[\Omega_{DE0} - d_1 \ln \left(M_P^{-2}H_0^2\right) - c_2H_0^2\left[1 + d_2 \ln \left(M_P^{-2}H_0^2\right)\right]\right], \quad (109)
\]
Then we find
\[
\frac{\Delta T_f}{T_f} = \left(10c_4\zeta T_f^7\right)^{-1} \left[H_0^2\left[\Omega_{DE0} - d_1 \ln \left(M_P^{-2}H_0^2\right) - c_2H_0^2\left[1 + d_2 \ln \left(M_P^{-2}H_0^2\right)\right]\right] + \zeta^2T_f^4 \left[d_1 \ln \left(M_P^{-2}\zeta^2T_f^4\right) + c_2\zeta^4T_f^4 \left[1 + d_2 \ln \left(M_P^{-2}\zeta^2T_f^4\right)\right]\right]\right]. \quad (110)
\]
Hence, taking the typical values $c_2 = 10^{46}$ GeV$^{-2}$ and $d_2 = 10^{-5}$ we find $d_1 \in (-1.0 \times 10^{-5}, 1.3 \times 10^{-5})$, and then according to (109) we acquire $\frac{c_0}{H_0^2} \in (0.697, 0.704)$. On the other hand, imposing $c_2 = 10^{46}$ GeV$^{-2}$ and $d_1 \sim 10^{-5}$ we acquire $d_2 \in (-8.5 \times 10^{-2}, 1.2 \times 10^{-2})$ and
then $\frac{d_2}{d_1} \approx 0.7$. Hence, $d_1$ and $d_2$ are restricted in narrow windows in order for BBN constraints to be satisfied. Nonetheless, such windows do not affect significantly the rest of the RVM parameters, which are in the ballpark of the data fits of the standard RVM [69, 70].

In summary, (extended) running vacuum scenarios can satisfy the BBN constraints, however the corresponding model parameters are constrained around their standard, $\Lambda$CDM values. This was expected since such models are natural extensions of $\Lambda$CDM paradigm, and hence one can always find a parameter region for which the early time behavior is close to $\Lambda$CDM evolution.

IV. CONCLUSIONS

In this work we investigated the implications of higher-order modified gravity to the formation of light elements in the early Universe, namely on the Big Bang Nucleosynthesis (BBN). Such gravitational modifications are proved to be both theoretically motivated as well as phenomenologically very efficient in describing the later times evolution of the universe. Nevertheless, in order for such scenarios to be able to be considered as viable, one should examine that they do not spoil the early universe behaviour, and in particular the BBN epoch.

We investigated various classes of higher-order modified gravity and in particular models of $f(G)$ gravity, Gauss-Bonnet-Dilaton gravity, $f(P)$ cubic gravity, and running vacuum cosmology, under the assumption that these models can quantitatively describe the current dark energy sector. In the case of the well studied power-law $f(G)$ model, excluding the case where the exponent is $n = 1$ since it cannot describe dark energy, we found the constraint: $n \leq 0.45$. On the other hand, for other $f(G)$ models in the literature that include trigonometric functions and logarithms, we found that, as long as one imposes the conditions necessary for their theoretical consistency, BBN bounds are always fulfilled.

In the case of Gauss-Bonnet-Dilaton gravity with an exponential potential we found that the BBN constraints are satisfied in a narrow window for the potential exponent, which subsequently leads to narrow constraints on the Gauss-Bonnet - dilaton coupling parameter. For the case of $f(P)$ gravity with a power-law form, we found that the exponent is bounded by $n \leq 0.31$. For the more realistic case of $f(G) + f(P)$ gravity, we found that the exponent is bounded in the more narrow window $n \leq 0.22$. Moreover, for completeness we extracted the constraints on the string-inspired quartic curvature corrections models, too. Finally, we examined many sub-cases of running vacuum scenarios and we found that they can satisfy the BBN constraints. However, the corresponding model parameters are constrained around their standard $\Lambda$CDM values, which was to be expected, given that these models are extensions of $\Lambda$CDM cosmology.

The present analysis shows that models of higher-order modified gravity, apart from being closer to a renormalizable gravitational theory, they can be viable candidates of the description of Nature too, since they can quantitatively account for the dark energy sector and the late-time acceleration of the Universe, without altering the successes of the BBN epoch and the formation of light elements. Nonetheless, we mention that in most of the cases the corresponding model parameters are constrained in narrow windows, which is expected since it is well known that BBN analysis imposes strong constraints on possible deviations from standard cosmology. However, even in this case the results of the present work reveal the capabilities of such constructions and offers a motivation for further investigation, at a more detailed level, of the evolution of cosmic perturbations and their role in the large-scale structure of the Universe.

Finally, we mention that in this work we did not consider Chern-Simons modifications to gravity, involving coupling of axion fields to gravitational anomaly terms [80, 113, 114]. Such terms arise naturally in some string theory models, and they can become non trivial in the presence of CP violating perturbations around cosmological backgrounds, such as those due to gravitational waves [80, 115] (although it must be noted that, in the string-inspired model of [80], such gravitational anomalies are supposed to be absent in the post inflationary era, and hence in the epoch of BBN we are interested in here). We hope to study such more general cases in future works.

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