AN EXACT GEOMETRIC MASS FORMULA

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Abstract. We show an exact geometric mass formula for superspecial points in the reduction of any quaternionic Shimura variety modulo at a good prime $p$.

1. Introduction

Let $p$ be a rational prime number. Let $B$ be a totally indefinite quaternion algebra over a totally real field $F$ of degree $d$, together with a positive involution $\ast$. Assume that $p$ is unramified in $B$. Let $O_B$ be a maximal order stable under the involution $\ast$. Let $(V,\psi)$ be a non-degenerate $\mathbb{Q}$-valued skew-Hermitian (left) $B$-module with dimension $2g$ over $\mathbb{Q}$. Put $m := \frac{g}{d}$, a positive integer. A polarized abelian $O_B$-variety $A = (A,\lambda,\iota)$ is a polarized abelian variety $(A,\lambda)$ together with a ring monomorphism $\iota : O_B \to \text{End}(A)$ such that $\lambda \circ \iota(b^\ast) = \iota(b)^t \circ \lambda$ for all $b \in O_B$. Let $k$ be an algebraically closed field of characteristic $p$. An abelian variety over $k$ is said to be superspecial if it is isomorphic to a product of supersingular elliptic curves. Denote by $\Lambda^B_g$ the set of isomorphism classes of $g$-dimensional superspecial principally polarized abelian $O_B$-varieties over $k$. Define the mass of $\Lambda^B_g$ to be

$$\text{Mass}(\Lambda^B_g) := \sum_{\Delta \in \Lambda^B_g} \frac{1}{|\text{Aut}(A,\lambda,\iota)|}.$$  

The mass $\text{Mass}(\Lambda^B_g)$ is studied in Ekedahl [1] (Ekedahl’s result relies on an explicit volume computation in Hashimoto-Ibukiyama [4, Proposition 9, p. 568]) in the special case $B = M_2(\mathbb{Q})$. He proved

**Theorem 1.1** (Ekedahl, Hashimoto-Ibukiyama). One has

$$\text{Mass}(\Lambda_g) = \frac{(-1)^{g(g+1)/2}}{2g} \prod_{i=1}^{g} \zeta(1-2i) \cdot \prod_{i=1}^{g} p^i + (-1)^i,$$

where $\Lambda_g$ is the set of isomorphism classes of $g$-dimensional superspecial principally polarized abelian varieties over $k$ and $\zeta(s)$ is the Riemann zeta function.

Let $B_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ ramified exactly at $\{p, \infty\}$. Let $B'$ be the quaternion algebra over $F$ such that $\text{inv}_v(B') = \text{inv}_v(B_{p,\infty} \otimes \mathbb{Q} B)$ for all $v$. Let $\Delta'$ be the discriminant of $B'$ over $F$.

In this paper we prove

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\textbf{Theorem 1.2.} One has

\begin{equation}
\text{Mass}(\Lambda_g^B) = \frac{(-1)^{dm(m+1)/2}}{2^{md}} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta'} N(v)^i + (-1)^i \prod_{v|p,\nu|\Delta'} N(v)^i + 1 \right\},
\end{equation}

where \( \zeta_F(s) \) is the Dedekind zeta function.

Let \( N \geq 3 \) be a prime-to-\( p \) positive integer. Choose a primitive \( n \)-th root of unity \( \zeta_N \in \overline{\mathbb{Q}} \subset \mathbb{C} \) and fix an embedding \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p \). Let \( \mathcal{M} \) be the moduli space over \( \mathbb{F}_p \) of \( g \)-dimensional principally polarized abelian \( O_B \)-varieties with a symplectic \( O_B \)-linear level-\( N \) structure w.r.t. \( \zeta_N \). Let \( L_0 \) be a self-dual \( O_B \)-lattice of \( V \) with respect to \( \psi \). Let \( G_1 \) be the automorphism group scheme over \( \mathbb{Z} \) associated to the pair \((L_0, \psi)\). As an immediate consequence of Theorem 1.2 we get

\textbf{Theorem 1.3.} The moduli space \( \mathcal{M} \) has

\begin{equation}
|G_1(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)^{dm(m+1)/2}}{2^{md}} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta'} N(v)^i + (-1)^i \prod_{v|p,\nu|\Delta'} N(v)^i + 1 \right\}
\end{equation}
superspecial points.

We divide the proof of Theorem 1.2 into 4 parts; each part is treated in one section. The first part is to express the weighted sum in terms of an arithmetic mass; this is done by Shimura [17] (re-obtained by Gan and J.-K. Yu [3, 11.2, p. 522]) using the theory of Bruhat-Tits Buildings). The third part is to compare the derived arithmetic mass in Section 1 with “the” standard mass in Section 2. This reduces the problem to computing a local index at \( p \). The last part uses Dieudonné theory to compute this local index. A crucial step is choosing a good basis for the superspecial Dieudonné module concerned; this makes the computation easier.

\textbf{Notation.} \( \mathbb{H} \) denotes the Hamilton quaternion algebra over \( \mathbb{R} \). \( \mathbb{A}_f \) denotes the finite adele ring of \( \mathbb{Q} \) and \( \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \). For a number field \( F \) and a finite place \( v \), denote by \( O_F \) the ring of integers, \( F_v \) the completion of \( F \) at \( v \), \( \kappa_v \) the residue field, \( f_v := [\kappa_v : \mathbb{F}_p] \) and \( q_v := N(v) = |\kappa_v| \). For an \( O_F \)-module \( A \), write \( A_v \) for \( A \otimes_{O_F} O_{F_v} \). For a scheme \( X \) over \( \text{Spec} \, A \) and an \( A \)-algebra \( B \), write \( X_B \) for \( X \times_{\text{Spec} \, A} \text{Spec} \, B \). For a linear algebraic group \( G \) over \( \mathbb{Q} \) and an open compact subgroup \( U \) of \( G(\mathbb{A}_f) \), denote by \( \text{DS}(G,U) \) the double coset space \( G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/U \), and write \( \text{Mass}(G,U) := \sum_{i=1}^{h} |\Gamma_i|^{-1} \) if \( G \) is \( \mathbb{R} \)-anisotropic, where \( \Gamma_i := G(\mathbb{Q}) \cap c_i U c_i^{-1} \) and \( c_1,\ldots,c_h \) are complete representatives for \( \text{DS}(G,U) \). For a central simple algebra \( B \) over \( F \), write \( \Delta(B/F) \) for the discriminant of \( B \) over \( F \). If \( B \) a central division algebra over a non-archimedean local field \( F_v \), denote by \( O_B \) the maximal order of \( B \), \( m(B) \) the maximal ideal and \( \kappa(B) \) the residue field. \( \mathbb{Q}_p^n \) denotes the unramified extension of \( \mathbb{Q}_p \) of degree \( n \) and write \( \mathbb{Z}_p^n := O_{\mathbb{Q}_p^n} \).

\section{Simple mass formulas}

Let \( B \) be a finite-dimensional semi-simple algebra over \( \mathbb{Q} \) with a positive involution *, and \( O_B \) be an order of \( B \) stable under *. Let \( k \) be any field.
To any polarized abelian $O_B$-varieties $\mathcal{A} = (A, \lambda, \iota)$ over $k$, we associate a pair $(G_x, U_x)$, where $G_x$ is the group scheme over $\mathbb{Z}$ representing the functor

$$R \mapsto \{ h \in (\text{End}_{O_B}(A_k) \otimes R)^\times \mid h'h = 1 \},$$

where $h \mapsto h'$ is the Rost involution, and $U_x$ is the open compact subgroup $G_x(\hat{\mathbb{Z}})$. For any prime $\ell$, we write $\mathcal{A}(\ell)$ for the associated $\ell$-divisible group with additional structures $\big(A[\ell^\infty], \lambda_\ell, \iota_\ell\big)$, where $\lambda_\ell$ is the induced quasi-polarization of $A[\ell^\infty]$. We also write $\lambda_\ell^* \lambda_\ell = \lambda_1$, and $\text{Isom}_k(\mathcal{A}_1(\ell), \mathcal{A}_2(\ell))$ the set of quasi-isomorphisms $\varphi: A_1 \to A_2$ such that $\varphi^* \lambda_2 = \lambda_1$. Let $\Lambda_x$ be a fixed polarized abelian $O_B$-variety over $k$. Denote by $\Lambda_x(k)$ the set of isomorphism classes of polarized abelian $O_B$-varieties $\mathcal{A}$ over $k$ such that

$$(I_1): \text{Isom}_k(\mathcal{A}_0(\ell), \mathcal{A}(\ell)) \neq \emptyset \text{ for all primes } \ell.$$

Let $\Lambda_x^*(k) \subset \Lambda_x(k)$ be the subset consisting of objects such that

$$(Q): \text{Q-isom}_k(\mathcal{A}_0, \mathcal{A}) \neq \emptyset.$$

Let $\text{ker}^1(Q, G_x)$ denote the kernel of the local-global map $H^1(Q, G_x) \to \prod_v H^1(Q_v, G_x)$.

**Theorem 2.1.** (S Theorem 2.3) Suppose that $k$ is a field of finite type over its prime field.

1. There is a natural bijection $\Lambda_x^*(k) \simeq DS(G_x, U_x)$. Consequently, $\Lambda_x^*(k)$ is finite.
2. One has $\text{Mass}(\Lambda_x^*(k)) = \text{Mass}(G_x, U_x)$.

**Theorem 2.2.** (S Theorem 4.6 and Remark 4.7) Notation as above. If $k \supset \mathbb{F}_p$ is algebraically closed and $A_0$ is supersingular, then $\text{Mass}(\Lambda_x^*(k)) = \text{Mass}(G_x, U_x)$ and $\text{Mass}(\Lambda_x(k)) = |\text{ker}^1(Q, G_x)| \cdot \text{Mass}(G_x, U_x)$.

**Remark 2.3.** The statement of Theorem 2.2 is valid for basic abelian $O_B$-varieties in the sense of Kottwitz (see [5] for the definition). The present form is enough for our purpose.

3. An exact mass formula of Shimura

Let $D$ be a totally definite quaternion division algebra over a totally real field $F$ of degree $d$. Let $(\text{bar}) d \mapsto \hat{d}$ denote the canonical involution. Let $(V', \varphi)$ be a $D$-valued totally definite quaternion Hermitian $D$-module of rank $m$. Let $G^\varphi$ denote the unitary group attached to $\varphi$. This is a reductive group over $F$ and is regarded as a group over $\mathbb{Q}$ via the Weil restriction of scalars from $F$ to $\mathbb{Q}$. Choose a maximal order $O_D$ of $D$ stable under the canonical involution $\cdot$. Let $L$ be an $O_D$-lattice in $V'$ which is maximal among the lattices on which $\varphi$ takes its values in $O_B$. Let $U_0$ be the open compact subgroup of $G^\varphi(\mathbb{A}_f)$ which stabilizes the adelic lattice $L \otimes \mathbb{Z}$.

The following is deduced from a mass formula of Shimura [7] (also see Gan - J.-K. Yu [3] 11.2, p. 522). This form is more applicable to prove Theorem 1.2.
Theorem 3.1 (Shimura). One has

\[(3.1) \quad \text{Mass}(G^\varphi, U_0) = \frac{(-1)^{dm}(m+1)/2}{2md} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta(D/F)} N(v)^i + (-1)^i \right\}.\]

Deduction. In [7, Introduction, p. 68] Shimura gives the explicit formula

\[(3.2) \quad \text{Mass}(G^\varphi, U_0) = |D_F|^{m^2} \prod_{i=1}^{m} D_F^{1/2} \left\{ (2i-1)!/(2\pi)^{2i} \right\} \zeta_F(2i) \prod_{v|\Delta(D/F)} N(v)^i + (-1)^i,\]

where $D_F$ is the discriminant of $F$ over $\mathbb{Q}$. Using the functional equation for $\zeta_F(s)$, we deduce (3.1) from (3.2).

4. Global comparison

Keep the notation as in Section 1. Fix a $g$-dimensional superspecial principally polarized abelian $O_B \otimes \mathbb{Z}_p$-lattices of $(V_{\mathbb{Q}_p}, \psi)$ are isomorphic.

Proof. The proof is elementary and omitted.

Lemma 4.1. Any two self-dual $O_B \otimes \mathbb{Z}_p$-lattices of $(V_{\mathbb{Q}_p}, \psi)$ are isomorphic.

Proof. (1) The inclusion $\Lambda_x \subset \Lambda^B_x$ is clear. We show the other direction. Let $\Lambda_x := \Lambda_x(k)$ as in Section 2. Let $(G_x, U_x)$ be the pair associated to $x$.

Lemma 4.2. One has (1) $\Lambda_x = \Lambda^B_x \cap \ker^1(\mathbb{Q}, G_x) = \{1\}$.

Proof. (1) The inclusion $\Lambda_x \subset \Lambda^B_x$ is clear. We show the other direction. Let $\Delta \in \Lambda^B_g$. It follows from Lemma 4.1 that the condition (Ip) is satisfied for primes $\ell \neq p$. Let $M$ be the covariant Dieudonné module of $A$. One chooses an isomorphism $O_{B,p} \simeq M_2(\mathbb{O}_{F,p})$ so that $\ast : (a_{ij}) \mapsto (a_{ij})^t$. Using the Morita equivalence, it suffices to show that any two superspecial principally quasi-polarized Dieudonné modules with compatible $O_{F,p}$-action are isomorphic. This follows from Theorem 5.1.

4.1. We compute that

- (i) $G_x(\mathbb{R}) = \{ h \in M_m(\mathbb{H})^d \mid \mathcal{H} h = 1 \}$,
- (ii) for $\ell \neq p$, we have $G_x(\mathbb{Q}_\ell) = \prod_{v|\ell} G_{x,v}$ and $U_{x,\ell} = \prod_{v|\ell} U_{x,v}$, where

\[
G_{x,v} = \begin{cases} 
\text{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta(B/F), \\
\{ h \in M_m(B_v) \mid \mathcal{H} h = 1 \}, & \text{otherwise,} 
\end{cases}
\]

\[(4.1) U_{x,v} = \begin{cases} 
\text{Sp}_{2m}(O_{F_v}), & \text{if } v \nmid \Delta(B/F), \\
\{ h \in M_m(O_{B_v}) \mid \mathcal{H} h = 1 \}, & \text{otherwise,} 
\end{cases}
\]

- (iii) $G_x(\mathbb{Q}_p) = \prod_{v|p} G_{x,v}$, where

\[
G_{x,v} = \begin{cases} 
\text{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta^t, \\
\{ h \in M_m(B_v^t) \mid \mathcal{H} h = 1 \}, & \text{otherwise.} 
\end{cases}
\]

Take $D = B^t$ and $V' = D^{\oplus m}$ with $\varphi(x, y) = \sum x_i y_i$, and take $L = O_D^{\oplus m}$. We compute that

- (i) $G^\varphi(\mathbb{R}) = \{ h \in M_m(\mathbb{H})^d \mid \mathcal{H} h = 1 \}$,
(ii)' for any \( \ell \), we have \( G_x(\mathbb{Q}_\ell) = \prod_{v|\ell} G^\varphi_v \) and \( U_{0,\ell} = \prod_{v|\ell} U_{0,v} \), where
\[
G^\varphi_v = \begin{cases} 
\text{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta', \\
\{ h \in M_m(B'_v) | \bar{h} h = 1 \}, & \text{otherwise}, 
\end{cases}
\]
(4.3)
\[
U_{0,v} = \begin{cases} 
\text{Sp}_{2m}(O_{F_v}), & \text{if } v \nmid \Delta', \\
\{ h \in M_m(O_{B'_v}) | \bar{h} h = 1 \}, & \text{otherwise}. 
\end{cases}
\]

For \( \ell \neq p \) and \( v|\ell \), one has \( B_v = B'_v \) and that \( v \nmid \Delta(B/F) \) if and only if \( v \nmid \Delta' \). It follows from the computation above that \( G_{x,\mathbb{R}} \simeq G^\varphi_{\mathbb{R}} \) and \( G_{x,\mathbb{Q}_p} \simeq G^\varphi_{\mathbb{Q}_p} \) for all \( \ell \). Since the Hasse principle holds for the adjoint group \( G_{x}^{\text{ad}} \), we get \( G_x \simeq G^\varphi \) over \( \mathbb{Q} \). We fix an isomorphism and write \( G_x = G^\varphi \). For \( \ell \neq p \) and \( v|\ell \), the subgroups \( U_{0,v} \) and \( U_{x,v} \) are conjugate, and hence they have the same local volume.

4.2. Applying Theorem 2.2 in our setting (Section 1) and using Lemma 4.2, we get \( \text{Mass}(\Lambda^B_g) = \text{Mass}(G_x, U_x) \). Using the result in Subsection 4.1, we get
\[
\text{Mass}(\Lambda^B_g) = \text{Mass}(G^\varphi, U_0) \cdot \mu(U_{0,p}/U_{x,p}),
\]
(4.4)
where \( \mu(U_{0,p}/U_{x,p}) = [U_{x,p} : U_{0,p} \cap U_{x,p}]^{-1}[U_{0,p} : U_{0,p} \cap U_{x,p}] \).

5. Local index \( \mu(U_{0,p}/U_{x,p}) \)

Let \( (M', \langle \cdot, \cdot \rangle', \iota') \) be the covariant Dieudonné module associated to the point \( x = (A_0, \lambda_0, \eta_0) \) in the previous section. Choose an isomorphism \( O_B \otimes \mathbb{Z}_p \simeq M_2(O_F \otimes \mathbb{Z}_p) \) so that \( * \) becomes the transpose. Let \( M := eM', \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle|M \) and \( \iota := \iota'|O_F \), where \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) in \( M_2(O_F \otimes \mathbb{Z}_p) \). The triple \( (M, \langle \cdot, \cdot \rangle, \iota) \) is a superspecial principally quasi-polarized Dieudonné module with compatible \( O_F \otimes \mathbb{Z}_p \)-action of rank \( g = 2dm \). Let \( M = \oplus_{v|p} M_v \) be the decomposition with respect to the decomposition \( O_F \otimes \mathbb{Z}_p = \oplus_{v|p} O_v \); here we write \( O_v \) for \( O_{F_v} \). By the Morita equivalence, we have
\[
U_{x,p} = \text{Aut}_{DM, O_B}(M', \langle \cdot, \cdot \rangle') = \text{Aut}_{DM, O_F}(M, \langle \cdot, \cdot \rangle) = \prod_{v|p} U_{x,v},
\]
(5.1)
where \( U_{x,v} := \text{Aut}_{DM, O_v}(M_v, \langle \cdot, \cdot \rangle) \).

Let \( W := W(k) \) be ring of Witt vectors over \( k \) and \( \sigma \) the absolute Frobenius map on \( W \). Let \( J := \text{Hom}(O_v, W) \) be the set of embeddings; write \( I := \{ \sigma_i \}_{i \in \mathbb{Z}/f v Z} \) so that \( \sigma \sigma_i = \sigma_{i+1} \) for all \( i \). We identify \( \mathbb{Z}/f v Z \) with \( J \) through \( i \mapsto \sigma_i \). Decompose \( M_v = \oplus_{i \in \mathbb{Z}/f v Z} M^i_v \) into \( \sigma_i \)-isotypic components \( M^i_v \). One has (1) each component \( M^i_v \) is a free \( W \)-module of rank \( 2m \), which is self-dual with respect to the pairing \( \langle \cdot, \cdot \rangle \), (2) \( \langle M^i_v, M^j_v \rangle = 0 \) if \( i \neq j \), and (3) the operations \( F \) and \( V \) shift by degree 1 and degree -1, respectively.

**Theorem 5.1.** Let \( (M_v, \langle \cdot, \cdot \rangle, \iota) \) be as above. There is a symplectic basis \( \{ X_j^i, Y_j^i \}_{j=1, \ldots, m} \) for \( M^i_v \) such that
\begin{enumerate}
  \item \( Y_j^i \in VM^{i+1}_v \),
  \item \( FX_j^i = -Y_j^{i+1} \) and \( FY_j^i = pX_j^{i+1} \),
\end{enumerate}
for all \( i \in \mathbb{Z}/f v Z \) and all \( j \).
We construct a symplectic basis \( \{ X_j, Y_j \}_{j=1, \ldots, m} \) for \( N^0 \) such that \( Y_j \in VN^1 \) for all \( j \). Define \( X_j \) and \( Y_j \) recursively for \( j = 1, \ldots, j \):

\[
X_{j+1} = p^{-1} FY_j, \quad Y_{j+1} = -FX_j.
\]

One has \( X_{j+2} = \frac{1}{p} F^2 X_j \) and \( Y_{j+2} = \frac{1}{p} F^2 Y_j \), hence

\[
X_j = (-1)^e p^{-c} F^2 X_0, \quad Y_j = (-1)^e p^{-c} F^2 Y_0 = Y_j,
\]

for all \( j \). It is easy to see that \( \{ X_j, Y_j \}_{j=1, \ldots, m} \) forms a symplectic basis for \( N^i \).

Suppose that \( f = 2e + 1 \) is odd. Let \( N := \{ x \in M \mid F^2 x = (1)^c V^c x \} \). We construct a symplectic basis \( \{ X_0^j, Y_0^j \}_{j=1, \ldots, m} \) for \( N^0 \) with the properties:

\[
X_j \not\in VN^1, \quad Y_j \in VN^1 \quad \text{and} \quad Y_j = (-1)^{c+1} p^{-c} F^j X_1 \quad \text{for all} \quad j.
\]

We can choose \( X_0^j \in N^0 \setminus VN^1 \) so that \( \langle X_1, (-1)^{c+1} p^{-c} F^j X_1 \rangle \in \mathbb{Z}_q^2 \). This follows from the fact that the form \( \langle x, y \rangle := \langle x, p^{-c} F^j y \rangle \) mod \( p \) is a non-degenerate Hermitian form on \( N^0 / VN^1 \). Set \( Y_0^j = (-1)^{c+1} p^{-c} F^j X_1^j \) and let \( \mu := \langle X_1^j, Y_0^j \rangle \). From \( \langle F^j X_1^j, F^j Y_0^j \rangle = (-1)^{c+1} p^{-c} F^j Y_0^j, \langle (-1)^{c+1} p^{-c} F^j Y_0^j, X_0^j \rangle \), we get \( \mu = \mathbb{Z}_q^2 \). Since \( \mathbb{Q}_p^1 / \mathbb{Q}_q \) is unramified, replacing \( X_1^j \) by a suitable \( \lambda X_0^j \), we get \( \langle X_1^j, Y_0^j \rangle = 1 \). Do the same construction for the complement of the submodule \( \langle X_0^j, Y_0^j \rangle \) and use induction; we exhibit such a basis for \( N^0 \).

Define \( X_j^f \) and \( Y_j^f \) recursively for \( i = 1, \ldots, f \) as [5.2]. We verify again that \( X_j^f = X_0^j \) and \( Y_j^f = Y_0^j \). It follows from the relation [5.2] that \( \{ X_j^f, Y_j^f \}_{j=1, \ldots, m} \) forms a symplectic basis for \( N^i \) for all \( i \). This completes the proof.

\begin{proposition}
Notation as above.

(1) If \( f_v \) is even, then

\[
U_{x, v} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2m}(\mathbb{Z}_{q_v}) \mid B \equiv 0 \mod p \right\}.
\]

(2) If \( f_v \) is odd, then

\[
U_{x, v} \simeq \{ h \in M_{m}(O_{B_v}) \mid \tilde{h} h = 1 \}.
\]

\end{proposition}

\begin{proof}
Let \( \phi \in U_{x, v} \). Choose a symplectic basis \( B \) for \( M_v \) as in Theorem 5.1. Since \( \phi \) commutes with the \( O_F \)-action, we have \( \phi = (\phi_i) \), where \( \phi_i \in \text{Aut}(M_v, \langle \cdot, \cdot \rangle) \).

Write \( \phi_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \text{Sp}_{2m}(W) \) using the basis \( B \). Since the map \( F \) is injective, \( \phi_0 \) determines the remaining \( \phi_i \). From \( \phi F^2 = F^2 \phi \), we have \( \phi_i^{+2} = \phi_i^{(2)} \) (as matrices). Here we write \( \phi_i^{(n)} \) for \( \phi_0^{n} \). From \( \phi F = F \phi \) we get \( A_i^{(1)} = D_i^{+1} \), \( B_i^{(1)} = -pC_i^{+1}, C_i^{(1)} = -B_i^{+1} \) and \( D_i^{(1)} = A_i^{+1} \).

(1) If \( f_v \) is even, then \( A_0, B_0, C_0, D_0 \in \mathbb{Z}_{q_v} \) and \( B_0 \equiv 0 \mod p \). This shows [5.3].

(2) Suppose \( f_v \) is odd. From \( \phi_0^{(f_v+1)} = \phi_1 \) we get \( A_0^{(f_v)} = D_0, B_0^{(f_v)} = -pC_0, pC_0^{(f_v)} = -B_0, D_0^{(f_v)} = A_0 \). Hence

\[
U_{x, v} = \left\{ \begin{pmatrix} A & -pC^T \\ C & A \end{pmatrix} \right\} \in \text{Sp}_{2m}(\mathbb{Z}_{q_v^2}).
\]

\end{proof}
where \( \tau \) is the involution of \( \mathbb{Q}_{q_2} \) over \( \mathbb{Q}_{q_v} \). Note that \( O_{B'} = \mathbb{Z}_{q_2} \Pi \) with \( \Pi^2 = -p \) and \( \Pi a = a^\tau \Pi \) for all \( a \in \mathbb{Z}_{q_2} \). The map \( A + C \Pi \mapsto \begin{pmatrix} A & -pC^\tau \\ C & A^\tau \end{pmatrix} \) gives rise to an isomorphism \( \text{[5.4]} \). This proves the proposition.

Let \((V_0 = \mathbb{F}_q^{2m}, \psi_0)\) be a standard symplectic space. Let \( P \) be the stabilizer of the standard maximal isotropic subspace \( \mathbb{F}_q < e_1, \ldots, e_m > \).

**Lemma 5.3.** \[ |\text{Sp}_{2m}(\mathbb{F}_q)/P| = \prod_{i=1}^{m} (q^{2i} + 1). \]

**Proof.** We have a natural bijection between the group \( \text{Sp}_{2m}(\mathbb{F}_q) \) and the set \( B(m) \) of ordered symplectic bases \( \{v_1, \ldots, v_{2m}\} \) for \( V_0 \). The first vector \( v_1 \) has \( q^{2m} - 1 \) choices. The first companion vector \( v_{m+1} \) has \( q^{2m-1} \) choices as it does not lie in the hyperplane \( v_1^\perp \) and we require \( \psi_0(v_1, v_{m+1}) = 1 \). The remaining ordered symplectic basis can be chosen from the complement \( \mathbb{F}_q < v_1, v_{m+1} >^\perp \). Therefore, we have proved the recursive formula \[ |\text{Sp}_{2m}(\mathbb{F}_q)| = (q^{2m} - 1)q^{2m-1}|\text{Sp}_{2m-2}(\mathbb{F}_q)|. \]

From this, we get

\[ (5.5) \quad |\text{Sp}_{2m}(\mathbb{F}_q)| = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1). \]

We have

\[ P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}; AD^t = I_m, \ BA^t = AB^t \right\}. \]

This yields

\[ (5.6) \quad |P| = q^{m^2 + m^2} |\text{GL}_m(\mathbb{F}_q)| = q^{m^2} \prod_{i=1}^{m} (q^i - 1). \]

From \( (5.5) \) and \( (5.6) \), we prove the lemma.

By Proposition 5.2 and Lemma 5.3, we get

**Theorem 5.4.** One has

\[ (5.7) \quad \mu(U_0,p/U_{x,v}) = \prod_{v|p} \mu(U_0,v/U_{x,v}) = \prod_{v|p, v \nmid \Delta} m \prod_{i=1}^{m} (q_i^2 + 1). \]

Plugging the formula \( (5.7) \) in the formula \( (4.4) \), we get the formula \( (1.3) \). The proof of Theorem 1.2 is complete.

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