Klein-Gordon equations with homogeneous time-dependent electric fields

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Abstract

We consider a system associated to Klein-Gordon equations with homogeneous time-dependent electric fields. The upper and lower boundaries of a time-evolution propagator for this system were proven by Veselić in 1991 for electric fields that are independent of time. We extend this result to time-dependent electric fields.

Keywords: Klein-Gordon Equation, Time-Dependent Electric Fields, Non-Selfadjoint Operators.

1 Introduction

We investigate the dynamics of a relativistic charged particle with charge $q \neq 0$ that moves on $\mathbb{R}^n$, $n \in \mathbb{N}$, and is influenced by homogeneous time-dependent electric fields $E(t) = (E_1(t), \ldots, E_n(t))$, which satisfy $E_j(t) \in C^1(\mathbb{R})$ for all $j \in \{1, \ldots, n\}$ and

$$\sup_{t \in \mathbb{R}} \sum_{j=0}^{n} |E_j^{(k)}(t)| < E_{0,k}, \quad E_j^{(k)}(t) = \frac{d^k E_j(t)}{dt^k}, \quad k \in \{0, 1\}, \quad j \in \{1, \ldots, n\};$$

where $0 < E_{0,k} < \infty$ is a constant. The wave functions under consideration satisfy the following Klein-Gordon equations:

$$\begin{cases}
(i\partial_t + qE)^2 \psi_0(t, x) = L(0, p)\psi_0(t, x), \\
\psi_0(0, x) = \psi_{0,0}, \quad \{(i\partial_t + qE)(\psi_0(t, x))\} |_{t=0} = \psi_{0,1}, \\
L(0, p) = c^2 p^2 + (mc^2)^2, \quad qE = qE(t, x) = qE(t) \cdot x,
\end{cases}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $p = (p_1, \ldots, p_n) = -i(\partial x_1, \ldots, \partial x_n) = -i \nabla$, $m > 0$, and $q \in \mathbb{R}\setminus\{0\}$ are the position, momentum, mass, and charge of the charged particle, respectively. We let $c > 0$ denote the speed of light; the inner product of $a, b \in \mathbb{R}^n$ is denoted by $a \cdot b$. To introduce the main theorem, we consider the system of Veselić [15].

Let $\psi_0(t, x), \psi_{0,0},$ and $\psi_{0,1}$ be equivalent to those in [2]. The substitutions $\psi_{0,1}(t, x) = (i\partial_t + qE)\psi_0(t, x)$ and

$$\Psi_0(t, x) := \begin{pmatrix} \psi_0(t, x) \\ \psi_{0,1}(t, x) \end{pmatrix}, \quad \Psi_0 = \Psi_0(0, x) = \begin{pmatrix} \psi_{0,0} \\ \psi_{0,1} \end{pmatrix}$$

yield the following (Hamilton) system:

$$i\frac{\partial}{\partial t} \Psi_0(t, x) = A_0(t)\Psi_0(t, x), \quad A_0(t) := \begin{pmatrix} -qE & 1 \\ L(0, p) & -qE \end{pmatrix}, \quad \Psi_0(0, x) = \Psi_0.$$
By substituting $\mathcal{K}_\alpha$, defined in (12) (see also [15], (1.3)), and by using the same scheme as that found in [15], we arrive at the following system on $\mathcal{H} = L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$:

$$i\frac{\partial}{\partial t}\Phi_{0,\alpha}(t, x) = H_{0,\alpha}(t)\Phi_{0,\alpha}(t, x), \quad \Phi_{0,\alpha}(0, x) = \Phi_{0,\alpha} \in \mathcal{H},$$

where $H_{0,\alpha}(t) = \hat{H}_0(t) + P_{0,\alpha}(t)$ with

$$\hat{H}_0(t) = \begin{pmatrix} -qE & (L(0, p))^{1/2} \\ (L(0, p))^{1/2} & -qE \end{pmatrix},$$

$$P_{0,\alpha}(t) = \frac{i\hbar^2}{2} \begin{pmatrix} (1 - 2\alpha)qE(t) \cdot p(L(0, p))^{-1} & 0 \\ 0 & -(1 + 2\alpha)qE(t) \cdot p(L(0, p))^{-1} \end{pmatrix},$$

where $i = \sqrt{-1}$ and

$$\Phi_{0,\alpha}(t, x) = \begin{pmatrix} (L(0, p))^{1/4 - \alpha/2} \psi_0(t, x) \\ (L(0, p))^{-1/4 - \alpha/2} \psi_{0,1}(t, x) \end{pmatrix}.$$

The construction scheme of $H_{0,\alpha}(t)$ can be found in Appendix A or [15]. Here, we call $U_{0,\alpha}(t)$ the propagator for $H_{0,\alpha}(t)$ if $U_{0,\alpha}(t)$ satisfies the following equations:

$$i\partial_t U_{0,\alpha}(t) = H_{0,\alpha}(t)U_{0,\alpha}(t), \quad i\partial_t (U_{0,\alpha}(t))^{-1} = -(U_{0,\alpha}(t))^{-1}H_{0,\alpha}(t),$$

$$U_{0,\alpha}(t)U_{0,\alpha}(t)^{-1} = U_{0,\alpha}(0) = \text{Id}_{\mathcal{H}}.$$  

The solution of (4) is denoted by $\Phi_{0,\alpha}(t, x) = U_{0,\alpha}(t)\Phi_{0,\alpha}$. The main theorem of this paper proves that $0 < \|U_{0,\alpha}(t)\|_{\mathcal{B}(\mathcal{H})} < \infty$ as $t \to \infty$ and that for any $\alpha \neq 0$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$, $\|U_{0,\alpha}(t)\varphi(p)\|_{\mathcal{B}(\mathcal{H})} \to 0$ or $\infty$ as $t \to \infty$, where $\mathcal{B}(\mathcal{H})$ is the operator norm on $\mathcal{H}$. First, we analyze the asymptotic behavior of $U_{0,\alpha}(t)$ in $t$. Unfortunately, $U_{0,\alpha}(t)$ is difficult to control for general electric fields satisfying only (4). Hence, we impose the following additional condition (E1) on electric fields:

(E1): Let $E(t)$ satisfy (4), and define $b(t) = \int_0^t qE(s) ds$. Then $b(t)$ satisfies $\lim_{t \to \infty} |b(t)| \to \infty$. Moreover, for any vector $a \in \mathbb{R}^n$, there exist constants $e_0$ and $e_1$, independent of $t$ and $a$, such that

$$\int_{|a + b(s)| \leq 2E_{0,\alpha}/(mc^2)} |b'(s)| ds \leq e_0, \quad \int_0^t \frac{|b'(s)|^2 + |b''(s)|}{c^2(a + b(s))^2 + (mc^2)^2} ds \leq e_1$$

holds.

Models of electric fields satisfying Assumption (E1) and remarks regarding this assumption can be found in Appendix B.

We define the Fourier transform $\hat{\mathcal{F}}_1^{-1}$ and inverse Fourier transform $\mathcal{F}_1$ on $L^2(\mathbb{R}^n)$ as follows:

$$\hat{\mathcal{F}}_1^{-1}[\phi](q) := \hat{\mathcal{F}}_1^{-1}[\phi](q), \quad \mathcal{F}_1[\phi](q) := (2\pi i)^{-n/2} \int_{\mathbb{R}^n} e^{\pm i q \cdot \eta} \phi(\eta) d\eta.$$  

We now state the main theorem in this paper.
Theorem 1.1. Set $\alpha = 0$ in \([1]\) and \([6]\), and suppose Assumption (E1) holds. Then for all $t \in \mathbb{R}$, there exist $0 < \Gamma_1 < \Gamma_2$, independent of $t$, such that

$$
\Gamma_1 \leq \|U_{0,0}(t)\|_{\mathcal{B}(\mathcal{H})} \leq \Gamma_2
$$

holds, where $\mathcal{B}(\mathcal{H})$ is the operator norm on $\mathcal{H}$. Conversely, for any $\alpha \neq 0$ and $\Phi_{0,\alpha} \in \mathcal{F}_1^{-1}C_0(\mathbb{R}^n) \times \mathcal{F}_1^{-1}C_0(\mathbb{R}^n)$,

$$
\lim_{t \to \infty} \|U_{0,\alpha}(t)\Phi_{0,\alpha}\|_{\mathcal{H}} = \begin{cases} 
0, & \alpha > 0, \\
\infty, & \alpha < 0,
\end{cases}
$$

holds.

Herein, we say $U_{0,0}(t)$ is stable on $\mathcal{H}$ and $U_{0,\alpha}(t)$, $\alpha \neq 0$ is unstable on $\mathcal{H}$. As a corollary to Theorem 1.1, we obtain the following inequality.

Corollary 1.2. Suppose Assumption (E1) holds. Then for all $t \in \mathbb{R}$, $\psi_{0,0} \in H^{1/2}(\mathbb{R}^n)$, and $\psi_{0,1} \in H^{-1/2}(\mathbb{R}^n)$, there exist $0 < \gamma_1 < \gamma_2$ independent of $t$ such that

$$
\gamma_1 \left( \left\| (L(0,p))^{1/4}\psi_{0,0} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| (L(0,p))^{-1/4}\psi_{0,1} \right\|_{L^2(\mathbb{R}^n)}^2 \right)
$$

$$
< \left\| (L(0,p))^{1/4}\psi_0(t,x) \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| (L(0,p))^{-1/4}(i\partial_t + q_E)\psi_0(t,x) \right\|_{L^2(\mathbb{R}^n)}^2
$$

$$
< \gamma_2 \left( \left\| (L(0,p))^{1/4}\psi_{0,0} \right\|_{L^2(\mathbb{R}^n)}^2 + \left\| (L(0,p))^{-1/4}\psi_{0,1} \right\|_{L^2(\mathbb{R}^n)}^2 \right)
$$

holds, where $\psi_0(t,x)$, $\psi_{0,0}$, and $\psi_{0,1}$ are the same as those in \([2]\).

If $q_E$ is independent of time and satisfies

$$
\|q_Eu\|_{L^2(\mathbb{R}^n)} \leq \delta \left\| (L(0,p))^{1/2}u \right\|_{L^2(\mathbb{R}^n)}, \quad 0 < \delta < 1,
$$

then Najman \([11]\) showed that $A_0(t)$ generates a uniformly bounded propagator on $\mathcal{H}_\alpha$. Veselić subsequently applied Najman’s scheme to non-decreasing constant electric fields $qE \cdot x$ and obtained stability in the time-evolution operator on $\mathcal{H}_\alpha$ and instability on $\mathcal{H}_\alpha$, where $\alpha \neq 0$. Essential to the proof is the factorization of propagator $U_{0,\alpha}(t) = V_\alpha e^{-itqE \cdot x}V_\alpha^{-1}$, where $V_\alpha$ is a time-independent linear operator satisfying differential equations (see \([15]\)). By virtue of this factorization, $\|U_{0,\alpha}(t)\Phi\|$ can be estimated by analyzing $V_\alpha$ instead of $e^{-itH_\alpha}$. We try to extend this approach to time-dependent electric fields. First, we form another factorization of $U_{0,\alpha}(t)$ since the aforementioned $V_\alpha$ depends on time if the electric fields depend on time (i.e., $V_\alpha(t)e^{bt(t) \cdot x}V_\alpha(t)^{-1}$ is different from $U_{0,\alpha}(t)$). In order to form a new factorization, we focus on the so-called Avron-Herbst formula. We refer to Avron-Herbst \([2]\) and Cycon-Froese-Kirsch-Simon \([5]\), Theorem 7.1., which consider the study of the Schrödinger equations with time-dependent (and constant) electric fields:

$$
\begin{cases} 
\partial_t u(t,x) = \left(\frac{p^2}{2m} - qE(t) \cdot x\right)u(t,x) =: H^S_0(t)u(t,x), \\
u(0,x) = u_0,
\end{cases}
$$

(10)
where $H_0^S(t)$ is the Stark Hamiltonian. For a solution $u(t, x)$ to (10), substituting $u(t, x) = e^{ib(t)x}v(t, x)$ yields $i\partial v(t, x) = (p + b(t))^2v(t, x)/(2m)$. Thus, by letting

$$v(t, x) = e^{-ia(t)}e^{-ic(t)p}e^{-ipt^2/(2m)}u_0,$$

one obtains $u(t, x) = e^{-ia(t)}e^{ib(t)x}e^{-ic(t)p}e^{-ipt^2/(2m)}u_0$, i.e., a propagator for $H_0^S(t)$ can be described by $e^{-ia(t)}e^{ib(t)x}e^{-ic(t)p}e^{-ipt^2/(2m)}$. This factorization of the propagator is called the Avron-Herbst formula. This factorization has been applied to many research areas such as quantum scattering theory and non-linear analysis (see Adachi-Ishida [1], Avron-Herbst [2], Möller [10], and Carles-Nakamura [6]). We attempt to apply this scheme to (2): in this process, we analyze the differential equation and non-linear analysis (see Adachi-Ishida [1], Avr on-Herbst [2], Möller [10], and Carles-Nakamura [6]). We attempt to apply this scheme to (2); in this process, we analyze the differential equation $-\partial_t^2 u_0(t, x) = (c^2(p + b(t))^2 + (mc^2)^2)u_0(t, x)$. To consider the asymptotic behavior of solutions to this equation, we use the approach of Hochstadt [8]. At the conclusion of this paper (§4.1 (34)), we obtain a new factorization of the propagator $U_{0, \alpha}(t)$.

Our first approach to prove Theorem 1.1 is to reduce (2) to the ordinary differential equation in (14) through the Fourier transform. A similar approach to the case where the potential $q_E$ is dependent on time but independent of $x$, was studied by Böhme-Ressig [3], [4]. Time-decaying dissipative wave equations were studied by Wirth [16], [17]. Our approach may be applicable to such equations and other open problems such as those discussed by Todorova-Yordanov [14].

2 Definitions and notation

In this section, we introduce definitions and notation. Let $C$ be a constant where $C > 0$. For $h(\tau, \eta), \tau \in \mathbb{R}$, and $\eta \in \mathbb{R}^n$, let $h'(\tau, \eta)$ and $h''(\tau, \eta)$ be defined by

$$h'(\tau, \eta) := h^{(1)}(\tau, \eta), \quad h''(\tau, \eta) := h^{(2)}(\tau, \eta), \quad h^{(l)}(\tau, \eta) := \frac{\partial^l h}{\partial \tau^l}(\tau, \eta),$$

for $l \in \{0, 1, 2\}$. Moreover, let $\mathcal{H} = L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. For $\Phi = (\phi_1, \phi_2)^T$ and $\Psi = (\psi_1, \psi_2)^T$, the norm of the Hilbert space $\mathcal{H}$ is defined by $\|\Phi\|^2_{\mathcal{H}} = \|\phi_1\|^2_{L^2(\mathbb{R}^n)} + \|\phi_2\|^2_{L^2(\mathbb{R}^n)}$ and inner product of $\mathcal{H}$ is defined by

$$(\Phi, \Psi)_{\mathcal{H}} := (\phi_1, \psi_1)_{L^2(\mathbb{R}^n)} + (\phi_2, \psi_2)_{L^2(\mathbb{R}^n)}.$$ 

Let $A, B, C$, and $D$ be linear operators on $L^2(\mathbb{R}^n)$, and let

$$A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

Then for $\Phi = (\phi_1, \phi_2)^T$, we define

$$A\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} := \begin{pmatrix} A\phi_1 + B\phi_2 \\ C\phi_1 + D\phi_2 \end{pmatrix},$$

so that $A$ is a linear operator on $\mathcal{H}$. Furthermore, for $\Phi_1 \in \mathcal{H}$ and $\Phi_2 \in \mathcal{H}$, if there exists $\Phi_3 \in \mathcal{H}$ such that

$$(A\Phi_1, \Phi_2)_{\mathcal{H}} = (\Phi_1, \Phi_3)_{\mathcal{H}}$$
holds, then we define $\Phi_3 = \mathcal{A}^*\Phi_2$; it can be easily obtained by

$$\mathcal{A}^* = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^* = \left( \begin{array}{cc} A^* & C^* \\ B^* & D^* \end{array} \right),$$

where $A^*, B^*, C^*$, and $D^*$ are the adjoint operators of $A$, $B$, $C$, and $D$, respectively, on $L^2(\mathbb{R}^n)$. Finally, $(A)_M\Phi$ for $\Phi \in \mathcal{H}$ means

$$\begin{bmatrix} A \\ 0 \\ A \end{bmatrix} \Phi \quad (11)$$

for some linear operator $A$ on $L^2(\mathbb{R}^n)$.

**3 Estimates of solutions to (2)**

First, we define $b(t) = (b_1(t), b_2(t), \ldots, b_n(t))$ as $\int_0^t qE(s)ds$ and take

$$u_0(t, x) = e^{-i \int_0^t qE(s) \cdot ds} \psi_0(t, x) = e^{-ib(t) \cdot x} \psi_0(t, x).$$

Then $u_0(t, x)$ satisfies equations

$$\begin{cases} -\partial_t^2 u_0(t, x) = L(t, p)u_0(t, x), \\ u_0(0, x) = \psi_{0,0}, \quad (i\partial_t u_0)(0, x) = \psi_{0,1}, \end{cases} \quad (12)$$

$$L(t, p) := e^2(p + b(t))^2 + (mc^2)^2,$$

where $e^{-ib(t) \cdot x}i\partial_t e^{ib(t) \cdot x} = i\partial_t - qE(t) \cdot x$ and $e^{-ib(t) \cdot x}pe^{ib(t) \cdot x} = p + b(t)$ hold on the test function. By the Fourier transform in [8], (12) is transformed into

$$(\partial_t^2 \hat{u}_0)(t, \xi) + L(t, \xi)\hat{u}_0(t, \xi) = 0, \quad \hat{u}_0(0, \xi) = \hat{\psi}_{0,0}, \quad (i\partial_t \hat{u}_0)(0, \xi) = \hat{\psi}_{0,1}. \quad (13)$$

Hence, for any fixed $\xi \in \mathbb{R}^n$, define $\zeta_j(t, \xi), \ j \in \{0, 1\}$, as the solution to

$$\zeta_j''(t, \xi) + L(t, \xi)\zeta_j(t, \xi) = 0, \quad \begin{cases} \zeta_0(0, \xi) = 1, \\ \zeta_0'(0, \xi) = 0, \\ \zeta_1(0, \xi) = 0, \\ \zeta_1'(0, \xi) = 1. \end{cases} \quad (14)$$

Note that the solutions of (2) can be written as

$$\psi_0(t, x) = e^{ib(t) \cdot x} \mathcal{F}^{-1}_1\zeta_0(t, \xi)\hat{\psi}_{0,0} + e^{ib(t) \cdot x}\mathcal{F}^{-1}_1\zeta_1(t, \xi)\hat{\psi}_{0,1}. \quad (15)$$

**3.1 Hochstadt type solutions**

Let $\psi_0(t, x)$ be a solution to the Klein-Gordon equations in (2). Noting (15), it is equivalent to analyze the asymptotic behavior of the solution to (14) and analyze the asymptotic behavior of the solution to (2). To analyze (14), we consider the approach of Hochstadt [8] (also, see Hochstadt [9]). For simplicity, we denote

$$L(t, \xi)^{1/2} = Q(t, \xi), \quad L(t, p)^{1/2} = Q(t, p) \quad (16)$$
in the following. Suppose that \( \zeta_j(t, \xi) \) and \( \zeta_j'(t, \xi) \) are represented by

\[
\begin{align*}
\zeta_0(t, \xi) &= A(t, \xi) \cos (B(t, \xi)), \quad \zeta_0'(t, \xi) = -A(t, \xi)Q(t, \xi) \sin (B(t, \xi)), \\
A(0, \xi) &= 1, \quad B(0, \xi) = 0, \\
\zeta_1(t, \xi) &= C(t, \xi) \sin (D(t, \xi)), \quad \zeta_1'(t, \xi) = C(t, \xi)Q(t, \xi) \cos (D(t, \xi)), \\
C(0, \xi) &= Q(0, \xi)^{-1}, \quad D(0, \xi) = 0,
\end{align*}
\tag{17}
\]

and

\[
\begin{align*}
\zeta_1(t, \xi) &= C(t, \xi) \sin (D(t, \xi)), \quad \zeta_1'(t, \xi) = C(t, \xi)Q(t, \xi) \cos (D(t, \xi)), \\
C(0, \xi) &= Q(0, \xi)^{-1}, \quad D(0, \xi) = 0,
\end{align*}
\tag{18}
\]

respectively, for functions \( A(t, \xi), B(t, \xi), C(t, \xi), \) and \( D(t, \xi) \). Considering (14), (17), and (18), we obtain differential equations

\[
\begin{align*}
A'(t, \xi) &= -Q(t, \xi)^{-1}Q'(t, \xi) \sin^2(B(t, \xi))A(t, \xi), \\
B'(t, \xi) &= Q(t, \xi) - Q(t, \xi)^{-1}Q'(t, \xi) \sin (B(t, \xi)) \cos (B(t, \xi)), \\
C'(t, \xi) &= -Q(t, \xi)^{-1}Q'(t, \xi) \cos^2(D(t, \xi))C(t, \xi), \\
D'(t, \xi) &= Q(t, \xi) + Q(t, \xi)^{-1}Q'(t, \xi) \sin (D(t, \xi)) \cos (D(t, \xi)).
\end{align*}
\tag{19}
\]

Equations (19) and (20) yield

\[
\begin{align*}
A(t, \xi) &= e^{-\int_0^t Q(s, \xi)^{-1}Q'(s, \xi) \sin(2B(s, \xi))ds}, \\
C(t, \xi) &= Q(0, \xi)^{-1} e^{-\int_0^t Q(s, \xi)^{-1}Q'(s, \xi) \cos^2(D(s, \xi))ds},
\end{align*}
\tag{21}
\]

and

\[
\begin{align*}
B(t, \xi) &= \int_0^t Q(s, \xi) - Q(s, \xi)^{-1}Q'(s, \xi) \sin (B(s, \xi)) \cos (B(s, \xi))ds, \\
D(t, \xi) &= \int_0^t Q(s, \xi) + Q(s, \xi)^{-1}Q'(s, \xi) \sin (D(s, \xi)) \cos (D(s, \xi))ds.
\end{align*}
\tag{22}
\]

**Lemma 3.1.** Functions \( B(t) = B(t, \cdot) \) and \( D(t) = D(t, \cdot) \) \((A(t, \cdot) \) and \( C(t, \cdot) \)) are in \( C^2(\mathbb{R}) \). Moreover, \( B(t) \) and \( D(t) \) satisfying the integral equation (22) are unique.

**Proof.** It is obvious that \( B(t) \) and \( D(t) \) are included in \( C^2(\mathbb{R}) \) since \( b(t) \in C^2(\mathbb{R}) \) (i.e., \( Q(t, \cdot) \) and \( Q'(t, \cdot)/Q(t, \cdot) \) are in \( C^1(\mathbb{R}) \)). Hence, we only prove the uniqueness of \( B(t) \) and \( D(t) \). Further, we only prove the uniqueness of \( D(t) \) since the uniqueness of \( B(t) \) can be proven in the same way.

First, we prove that for all \( t \) and \( \xi \), if \( B_1(t, \xi) \) and \( B_2(t, \xi) \) satisfy (22), then \( B_1(t, \xi) = B_2(t, \xi) \). Let \( \varepsilon_E < mc/(E_{0, \xi}) \) and \( 0 \leq t \leq \varepsilon_E \). Then by (22) and

\[
\left| \frac{Q'(s, \xi)}{Q(s, \xi)} \right| = \left| \frac{c^2(\xi + b(s)) \cdot b'(s)}{c^2(\xi + b(s))^2 + (mc^2)^2} \right| \leq \frac{c|b'(s)|}{Q(s, \xi)} \leq \frac{c|b'(s)|}{mc^2},
\tag{23}
\]

we have

\[
\begin{align*}
|B_1(t, \xi) - B_2(t, \xi)| &= \left| \int_0^t \frac{Q'(s, \xi)}{2Q(s, \xi)} \left( \int_{B_1(s, \xi)}^{B_2(s, \xi)} \frac{d}{d\tau} \sin(2\tau) d\tau \right) ds \right| \\
&\leq \frac{E_{0, \xi}}{mc} \int_0^t \left| \int_{B_1(s, \xi)}^{B_2(s, \xi)} \cos(2\tau) d\tau \right| ds.
\end{align*}
\tag{24}
\]
With \((24)\) and since \(t \leq \epsilon E < mc/E_{0,0}\), it follows that \(\sup_{\epsilon E \leq t \leq 2\epsilon E, \xi \in \mathbb{R}} |B_1(t, \xi) - B_2(t, \xi)| = 0.\) For \(t \leq 2\epsilon E\), we have

\[
\sup_{\epsilon E \leq t \leq 2\epsilon E, \xi \in \mathbb{R}} |B_1(t, \xi) - B_2(t, \xi)| 
\leq \frac{E_{0,0}}{mc} \left(0 + \sup_{\epsilon E \leq t \leq 2\epsilon E, \xi \in \mathbb{R}} \int_{\epsilon E}^{t} |B_1(s, \xi) - B_2(s, \xi)| ds \right).
\]

This also implies \(\sup_{\epsilon E \leq t \leq 2\epsilon E, \xi \in \mathbb{R}} |B_1(t, \xi) - B_2(t, \xi)| = 0.\) By repeating the same calculation for \(t \in [n\epsilon E, (n+1)\epsilon E]\) with \(n \in \mathbb{N}\), the lemma holds. □

First, we impose \(\hat{\psi}_{0,j} \in C_0(\mathbb{R}^n), \ j \in \{0, 1\}, \) in \((2)\) and define \(\varphi_j \in C_0^\infty(\mathbb{R}^n)\) such that

\[
\varphi_0(\xi) = 1, \text{ on the support of } \hat{\psi}_{0,0}(\xi), \quad \varphi_1(\xi) = 1, \text{ on the support of } \hat{\psi}_{0,1}(\xi).
\]

(25)

Noting \((14), (17), (18), (21),\) \(|Q(t_0, \xi)|^{-1} \leq C, \ |Q(0, \xi)|^{-1} \leq C,\) and the fact that \(|Q(t_0, \xi)| \leq C\) holds on the support of \(\varphi_j(\xi),\) the following proposition immediately holds.

**Proposition 3.2.** Let \(\zeta_0(t, \xi)\) and \(\zeta_1(t, \xi)\) be equal to those defined in \((17)\) and \((18)\), respectively, and let \(\varphi_0(\xi)\) and \(\varphi_1(\xi)\) be equal to those defined in \((25).\) Then for every fixed \(t_0 \in \mathbb{R},\)

\[
\sup_{\xi \in \mathbb{R}^n} |\zeta_j^{(N)}(t_0, \xi)\varphi_j(\xi)| \leq C_{j,N}
\]

holds, where \(j \in \{0, 1\}\) and \(N \in \{0, 1, 2\}.\)

By this proposition, \(\zeta_j^{(N)}(t_0, p)\varphi_j(p)\) can be defined as a bounded operator on \(L^2(\mathbb{R}^n)\) through the Fourier transform since

\[
\left\| \hat{\mathcal{F}}^{-1}_{t_0} \zeta_j^{(N)}(t_0, \xi) \hat{\mathcal{F}}_{1}^{+1} \varphi_j(\xi) \psi_{0,j} \right\|_{L^2(\mathbb{R}^n)} = \left\| \zeta_j^{(N)}(t_0, \xi)\varphi_j(\xi) \hat{\mathcal{F}}_{1}^{+1} \psi_{0,j} \right\|_{L^2(\mathbb{R}^n)} 
\leq C_{j,N} \left\| \psi_{0,j} \right\|_{L^2(\mathbb{R}^n)}
\]

holds. It also follows that for any fixed \(t,\) \(\zeta_j(t, p)\) can be defined on \(\hat{\mathcal{F}}_{1}^{-1}C_0(\mathbb{R}^n)\) since \(\varphi_j(\xi)\) is independent of \(t\) and satisfies \(\zeta_j^{(N)}(t, \xi)\varphi_j(p)\psi_{0,j} = \zeta_j^{(N)}(t, \xi)\psi_{0,j}.\) The following proposition extends the domains of \(\zeta_1(t, p)\) and \(\zeta_2(t, p)\) from \(\hat{\mathcal{F}}_{1}^{-1}C_0(\mathbb{R}^n)\) to \(H^{1/2}(\mathbb{R}^n)\) and \(H^{-1/2}(\mathbb{R}^n),\) respectively.

**Proposition 3.3.** Suppose Assumption \((E1)\) holds. Let \(A(t, \xi)\) and \(C(t, \xi)\) be equal to those defined in \((21).\) Then there exist \(0 < C_0 < \infty\) and \(0 < C_1 < \infty,\) independent of \(t\) and \(\xi,\) such that

\[
\frac{Q(0, \xi)^{1/2}}{Q(t, \xi)^{1/2}} e^{-c_0} \leq |A(t, \xi)| \leq \frac{Q(0, \xi)^{1/2}}{Q(t, \xi)^{1/2}} e^{c_0}, \quad \frac{1}{Q(t, \xi)^{1/2}Q(0, \xi)^{1/2}} e^{-c_1} \leq |C(t, \xi)| \leq \frac{1}{Q(t, \xi)^{1/2}Q(0, \xi)^{1/2}} e^{c_1}
\]

hold.
Proof. For simplicity, we denote $Q(t, \xi) = Q(t)$ and $B(t, \xi) = B(t)$. We only calculate the term $A(t, \xi)\varphi(\xi)$; the term $C(t, \xi)\varphi(\xi)$ can be calculated in a similar manner.

By simple calculations, it follows that

$$
\int_0^t \frac{Q'(s)}{Q(s)} \sin^2(B(s))ds \leq \frac{1}{2} \left( \log (Q(t)) - \log (Q(0)) - \int_0^t \frac{Q'(s)}{Q(s)} \cos(2B(s))ds \right).
$$

(29)

Hence, to prove Proposition 3.3, it suffices to show that the last term of the right-hand side of the above equation is uniformly bounded in $t$ and $\xi$. Noting (23), we have

$$
B'(s) = Q(s) - Q'(s)Q(s)^{-1}\sin(2B(s))/2 \geq Q(s)/2 + (Q(s)/2 - E_{0,0}/(2mc))
$$

$$
\geq Q(s)/2 + (c|\xi + b(s)| - E_{0,0}/(mc)) / 2.
$$

(30)

Next, we define

$$
\Omega := \{ s \in [0, t] : |\xi + b(s)| \leq 2E_{0,0}/(mc^2) \}.
$$

Then by Assumption (E1) and (23), we obtain that

$$
\left| \int_{\Omega} \frac{Q'(s)}{2Q(s)} \cos(B(s))ds \right| \leq C \int_{\Omega} |b'(s)|ds \leq Ce_0
$$

is bounded and independent of $t$ and $\xi$. Conversely, on the region $[0, t]\setminus\Omega$, by (30), it always follows that

$$
B'(s) \geq Q(s)/2;
$$

(31)

hence, it also follows that

$$
\int_{[0, t]\setminus\Omega} \frac{Q'(s)}{Q(s)} \cos(2B(s))ds
$$

$$
= \left[ \frac{Q'(s)}{2Q(s)B'(s)} \sin(2B(s)) \right]_{\partial([0, t]\setminus\Omega)} - \frac{1}{2} \int_{[0, t]\setminus\Omega} z_1(s) \sin(2B(s))ds,
$$

where

$$
z_1(s) = \frac{1}{(B'(s))^2Q(s)^2} \left\{ Q''(s)Q(s)B'(s) - (Q'(s))^2B'(s) - Q(s)Q'(s)B''(s) \right\}
$$

$$
= \frac{Q''(s)}{(B'(s))^2} - \frac{2(Q'(s))^2}{Q(s)(B'(s))^2} + \frac{(Q'(s))^2}{2Q(s)^2B'(s)} \cos(2B(s)).
$$

Since (31) holds and

$$
|Q'(s)| \leq C|b'(s)|, \quad |Q''(s)| \leq C \left( |b''(s)| + |b'(s)|^2 \right),
$$

it follows that on $[0, t]\setminus\Omega$,

$$
|z_1(s)| \leq C \left( |b''(s)|Q(s)^{-2} + |b'(s)|^2Q(s)^{-2} \right),
$$

8
where $|Q(s)|^{-1} \leq C$ and $|b'(s)| = |E(s)| \leq E_{0.0}$. Hence, by Assumption (E1),
\[
\left| \int_0^t \frac{Q'(s) \cos(2B(s))}{Q(s)} \, ds \right| \leq C + Ce_0 + C \int_0^t \frac{|b'(s)|^2 + |b''(s)|}{c^2(\xi + b(s))^2 + (mc^2)^2} \, ds \\
\leq C(1 + e_0 + e_1).
\]
Therefore, the proposition holds.

By analyzing $\zeta_1$ and $\zeta_2$, we arrive at the following theorem.

**Theorem 3.4.** Let $\psi_0(t, x)$, $\psi_{0,0}$, and $\psi_{0,1}$ be equal to those defined in (2). Suppose Assumption (E1) holds and that $\hat{\psi}_{0,0} \in C_0(\mathbb{R}^n)$ and $\hat{\psi}_{0,1} \in C_0(\mathbb{R}^n)$. Then for all $\theta \in \mathbb{R}$, there exists $0 < C_{0,\theta} < \infty$ such that
\[
\left\| (L(0, p))^{\theta} \psi_0(t, x) \right\|_{L^2(\mathbb{R}^n)} \leq C_{0,\theta}|b(t)|^{(2\theta - 1/2)}, \quad |t| \to \infty
\]
holds in particular,
\[
\left\| \psi_0(t, x) \right\|_{L^2(\mathbb{R}^n)} \leq C_{0,0}|b(t)|^{-1/2}, \quad |t| \to \infty
\]
holds, where $C_{0,\theta}$ is a constant depending only on the volume of the support of $\hat{\psi}_{0,0}$ and $\hat{\psi}_{0,1}$.

Solutions to (2) when the electric fields are independent of time have been investigated (see Narozhnyi and Nikishov [12], Tanji [13], and [15]); rotating electric fields were investigated by Eliezer, Raicher, and Zigler [7]. However, time-decay estimates (32) and (33) have not been considered.

**Proof.** On the support of $\hat{\psi}_{0,0}$ and $\hat{\psi}_{0,1}$, $Q(0, \xi)^{1/2}$ is bounded and
\[
C|b(t)|^2 \leq L(t, \xi)^{1/2} \leq C|b(t)|^2
\]
holds for $t \gg 1$. Thus, the inequality
\[
\left\| L(t, \xi)^{\theta} \zeta_j(t, \xi) \varphi_j(\xi) \hat{\psi}_{0,1}(t, \xi) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| L(t, \xi)^{\theta - 1/4} L(0, \xi)^{1/4} \varphi_j(\xi) \hat{\psi}_j \right\|_{L^2(\mathbb{R}^n)} \\
\leq C|b(t)|^{(2\theta - 1/2)} \left\| \hat{\psi}_{0,1}(t, \xi) \right\|_{L^2(\mathbb{R}^n)}
\]
holds from (27) and (28), where $j \in \{0, 1\}$. Therefore, Theorem 3.4 holds.

## 4  Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First, we decompose $U_{0, \alpha}(t)$ by using Hochstadt type representations (17), (18), and (22). Then, by using this factorization of $U_{0, \alpha}(t)$, we prove the stability and instability properties.
4.1 Factorization of $U_{0,\alpha}(t)$

Noting the definition of $U_{0,\alpha}(t)$ (see (25)), $U_{0,\alpha}(t)$ can be factorized by

$$U_{0,\alpha}(t) := K_{\alpha}^{1/2}U_{A_0}(t)(K_{\alpha}^{1/2})^{-1}$$

$$= (e^{i(t-x)}M \left( \begin{array}{cc} L(t,p)^{1/4-\alpha/2} & 0 \\ 0 & L(t,p)^{-1/4-\alpha/2} \end{array} \right)$$

$\times \left( \begin{array}{cc} \xi_0(t,p) & \xi_1(t,p) \\ i\xi_0'(t,p) & i\xi_1'(t,p) \end{array} \right) \left( \begin{array}{cc} L(0,p)^{-1/4+\alpha/2} & 0 \\ 0 & L(0,p)^{1/4+\alpha/2} \end{array} \right)$$

$$= (e^{i(t-x)}F_1^{-1}L_\alpha(t,\xi) 0 e^{i(t-x)}F_1^{-1}L_\alpha(t,\xi))$$

$$\times \left( \begin{array}{cc} \mathcal{G}_0(t,\xi) \cos(B(t,\xi)) & \mathcal{G}_1(t,\xi) \sin(D(t,\xi)) \\ -i\mathcal{G}_0(t,\xi) \sin(B(t,\xi)) & i\mathcal{G}_1(t,\xi) \cos(D(t,\xi)) \end{array} \right) \left( \begin{array}{cc} F_1^+ & 0 \\ 0 & F_1^+ \end{array} \right), \quad (34)$$

where $L_\alpha(t,\xi) = L(t,\xi)^{-\alpha/2}L(0,\xi)^{\alpha/2}$,

$$\mathcal{G}_0(t,\xi) = e^{\int_0^t(Q^*(s,\xi)\cos(2B(s,\xi))/(2Q(s,\xi)))ds}$$

and

$$\mathcal{G}_1(t,\xi) = e^{-\int_0^t(Q^*(s,\xi)\sin(2D(s,\xi))/(2Q(s,\xi)))ds}.$$

This formula is a natural extension of the Avron-Herbst formula.

4.2 Stability of $U_{0,0}(t)$ on $H$

Here, we prove the first statement of Theorem [11]. Noting that $F_1^{-1}C_0(\mathbb{R}^n) \times F_1^{-1}C_0(\mathbb{R}^n)$ is dense on $H$, every calculation is done on $\Phi_{0,0} \in F_1^{-1}C_0(\mathbb{R}^n) \times F_1^{-1}C_0(\mathbb{R}^n)$. By (34) with $\alpha = 0$, together with the fact that

$$e^{-C_0} \leq |\mathcal{G}_0(t,\xi)| \leq e^{C_0}, \quad e^{-C_1} \leq |\mathcal{G}_1(t,\xi)| \leq e^{C_1}, \quad (35)$$

holds by (27) and (28), we have that there exists $C > 0$ independent of $t$ and the support of $\Phi_{0,0}$ such that $\|U_{0,0}(t)\Phi_{0,0}\|_H \leq C \|\Phi_{0,0}\|_H$ holds. By the density argument, we also have $\|U_{0,0}(t)\|_{B(H)} \leq C$.

Next, we prove $\|U_{0,0}(t)\Phi_{0,0}\|_H \geq C > 0$. Letting $\Phi_{0,0} = (\phi_0, \phi_1)^T$, we have

$$\|U_{0,0}(t)\Phi_{0,0}\|_H^2 = \|\mathcal{G}_0(t,\xi)\phi_0\|_{L^2(\mathbb{R}^n)}^2 + \|\mathcal{G}_1(t,\xi)\phi_1\|_{L^2(\mathbb{R}^n)}^2$$

$$\quad - 2\text{Re} \left( \mathcal{G}_0(t,\xi)\mathcal{G}_1(t,\xi) \sin(B(t,\xi) - D(t,\xi))\hat{\phi}_0, \hat{\phi}_1 \right)_{L^2(\mathbb{R}^n)}.$$

Using the fact that

$$\zeta_0(t,\xi)\zeta_1'(t,\xi) - \zeta_0'(t,\xi)\zeta_1(t,\xi) = 1,$$

we obtain

$$\mathcal{G}_0(t,\xi)\mathcal{G}_1(t,\xi) \cos(B(t,\xi) - D(t,\xi)) = 1. \quad (37)$$
Inequalities (35) and (37) imply that for all \( t \in \mathbb{R} \) and \( \xi \in \mathbb{R}^n \), there exists \( 0 < \delta \leq 1 \) such that

\[
|\cos(B(t, \xi) - D(t, \xi))| > \delta
\]

holds, i.e.,

\[
|\sin (B(t, \xi) - D(t, \xi))| < \sqrt{1 - \delta^2}
\]

holds. Using this inequality, (35), (36), and

\[
\|U_{0,0}(t)\Phi_{0,0}\|^2_{\mathcal{F}} \geq (1 - \sqrt{1 - \delta^2}) \left( \|\mathcal{G}_0(t, \xi)\hat{\psi}_0\|^2_{L^2(\mathbb{R}^n)} + \|\mathcal{F}_1(t, \xi)\hat{\phi}_1\|^2_{L^2(\mathbb{R}^n)} \right)
\]

\[
\geq (1 - \sqrt{1 - \delta^2}) \left( \min\{e^{-2c_0}, e^{-2c_1}\} \right) \|\Phi_{0,0}\|^2_{\mathcal{F}}
\]

we obtain Theorem 1.1.

### 4.3 Instability of \( U_{0,\alpha}(t) \), \( \alpha \neq 0 \), on \( \mathcal{H} \)

We now complete the proof of Theorem 1.1. By (34), for \( \Psi_0 = (\psi_0, \psi_1)^T \in \mathcal{F}^{-1}C_0(\mathbb{R}^n) \times \mathcal{F}^{-1}C_0(\mathbb{R}^n) \), simple calculations show that

\[
\|U_{0,\alpha}(t)\Psi_0\|^2_{\mathcal{H}} = \left\|\sigma_{0,\alpha}(t, \xi)\hat{\psi}_0\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\sigma_{1,\alpha}(t, \xi)\hat{\psi}_1\right\|^2_{L^2(\mathbb{R}^n)} - 2\text{Re}(\sigma_{0,\alpha}(t, \xi)\sigma_{1,\alpha}(t, \xi)\sin(B(t, \xi) - D(t, \xi))\hat{\psi}_0, \hat{\psi}_1)_{L^2(\mathbb{R}^n)}
\]

holds, where \( \sigma_{0,\alpha} \) and \( \sigma_{1,\alpha} \) are defined by

\[
\sigma_{0,\alpha}(t, \xi) = \mathcal{G}_0(t, \xi)\mathcal{L}_\alpha(t, \xi), \quad \sigma_{1,\alpha}(t, \xi) = \mathcal{F}_1(t, \xi)\mathcal{L}_\alpha(t, \xi).
\]

In the same way as the proof of the stability of \( U_{0,0}(t, 0) \), we have that there exist \( c_{00} > 0 \) and \( \delta_{00} > 0 \) such that

\[
\|U_{0,\alpha}(t)\Psi_0\|^2_{\mathcal{H}} \begin{cases} \leq c_{00} \left( \left\|\sigma_{0,\alpha}(t, \xi)\hat{\psi}_0\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\sigma_{1,\alpha}(t, \xi)\hat{\psi}_1\right\|^2_{L^2(\mathbb{R}^n)} \right), \\ \geq \delta_{00} \left( \left\|\sigma_{0,\alpha}(t, \xi)\hat{\psi}_0\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\sigma_{1,\alpha}(t, \xi)\hat{\psi}_1\right\|^2_{L^2(\mathbb{R}^n)} \right), 
\end{cases}
\]

holds. On the other hand, by (27) and (28), note that for \( j \in \{0, 1\} \), there exist \( 0 < \tilde{c}_j < \tilde{C}_j \) such that

\[
\tilde{c}_j \left\|\mathcal{L}_\alpha(t, \xi)\hat{\psi}_j\right\|^2_{L^2(\mathbb{R}^n)} \leq \left\|\sigma_{j,\alpha}(t, \xi)\hat{\psi}_j\right\|^2_{L^2(\mathbb{R}^n)} \leq \tilde{C}_j \left\|\mathcal{L}_\alpha(t, \xi)\hat{\psi}_j\right\|^2_{L^2(\mathbb{R}^n)}
\]

holds, where \( \mathcal{L}_\alpha(t, \xi) := (L(t, \xi))^{-\alpha/2}(L(0, \xi))^{\alpha/2} \). Clearly, \( L(t, \xi) = c^2(\xi + b(t))^2 + (mc^2)^2 \to \infty \) as \( t \to \infty \) holds on \( C_0(\mathbb{R}^n) \times C_0(\mathbb{R}^n) \); hence, it follows that for \( t \to \infty \),

\[
\left\|\sigma_{j,\alpha}(t, \xi)\hat{\psi}_j\right\|^2_{L^2(\mathbb{R}^n)} \to \begin{cases} 0, & \text{if } \alpha > 0, \\ \infty, & \text{if } \alpha < 0,
\end{cases}
\]

holds.
APPENDIX A  Klein-Gordon systems with electric fields

In this section, we construct the (Hamilton) system equation in (2). This construction is the same one in [15]. Denote

\[ \Psi_0(t, x) = \begin{pmatrix} \psi_0(t, x) \\ \psi_{0,1}(t, x) \end{pmatrix}, \quad \psi_{0,1}(t, x) := (i\partial_t + qE)\psi_0(t, x), \quad \Psi_0 = \begin{pmatrix} \psi_{0,0} \\ \psi_{0,1} \end{pmatrix}, \]

where \( \psi_0(t, x) \), \( \psi_{0,0} \), and \( \psi_{0,1} \) are the same as those defined in (2). Then \( \Psi_0(t, x) \) satisfies the following equations:

\[ i\frac{\partial}{\partial t} \Psi_0(t, x) = A_0(t)\Psi_0(t, x), \quad A_0(t) = \begin{pmatrix} -qE & 1 \\ L(0, p) & -qE \end{pmatrix}, \quad \Psi_0(0, x) = \Psi_0. \quad (38) \]

Here, we set \( \zeta_j(t, \xi) \) to be that defined in (14) (or (17) and (18)). Focusing on \( \zeta_j'(t, \xi) = -L(t, \xi)\zeta_j(t, \xi) \), \( j \in \{0, 1\} \), a propagator for \( A_0(t) \), \( U_{A_0}(t) \) can be described by

\[ U_{A_0}(t) = (e^{ib(t)x})_M \begin{pmatrix} \zeta_0(t, p) \\ i\zeta_0'(t, p) \end{pmatrix} = (e^{ib(t)x})_M \begin{pmatrix} \zeta_1(t, p) \\ i\zeta_1'(t, p) \end{pmatrix}. \quad (39) \]

Indeed,

\[
\begin{aligned}
&i\frac{\partial}{\partial t} U_{A_0}(t) = (e^{ib(t)x})_M (i\partial_t - qE(t) \cdot x)_M \mathcal{F}^{-1}_1 \begin{pmatrix} \zeta_0(t, \xi) \\ i\zeta_0'(t, \xi) \end{pmatrix} (e^{ib(t)x})_M \\
&= (e^{ib(t)x})_M \mathcal{F}^{-1}_1 \begin{pmatrix} -qE(t) \cdot x & 0 \\ 0 & -qE(t) \cdot x \end{pmatrix} \begin{pmatrix} \zeta_0(t, \xi) \\ i\zeta_0'(t, \xi) \end{pmatrix} \\
&= (e^{ib(t)x})_M \begin{pmatrix} -qE(L(t, \xi)\zeta_0(t, \xi) \\ L(t, \xi)i\zeta_0'(t, \xi) \end{pmatrix} \begin{pmatrix} \zeta_0(t, \xi) \\ i\zeta_0'(t, \xi) \end{pmatrix} = A_0(t)U_{A_0}(t),
\end{aligned}
\]

where \( e^{ib(t)x}L(t, \xi)e^{-ib(t)x} = L(0, p) \) and \( (i\partial_t)e^{ib(t)x} = e^{ib(t)x}(i\partial_t - b'(t) \cdot x) \).

Next, we define

\[ \mathcal{F} = (\mathcal{F}^{-1}_1)_M, \quad \mathcal{F} = (\mathcal{F}^{-1}_1)_M, \quad (40) \]

and set

\[ K_\alpha(0, p) = \begin{pmatrix} (L(0, p))^{1/2-\alpha} & 0 \\ 0 & (L(0, p))^{-1/2-\alpha} \end{pmatrix}, \quad K_\alpha(0, p)\Phi = \mathcal{F}^{-1}K_\alpha(0, \xi)\mathcal{F}\Phi, \quad (41) \]

\[ \mathcal{K}_\alpha = L^{1/4-\alpha/2}L^2(\mathbb{R}^n) \times L^{-1/4-\alpha/2}L^2(\mathbb{R}^n), \quad (42) \]

for \( \alpha \in \mathbb{R} \) and \( \Phi \in \mathcal{D}(K_\alpha) \), where \( L^jL^2(\mathbb{R}^n) \), \( j \in \mathbb{R} \) is defined as the norm space with respect to the norm

\[ \|u\|_{L^jL^2(\mathbb{R}^n)} := \|(L(0, \xi))^j\hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}, \quad u \in \mathcal{F}^{-1} \mathcal{D}((L(0, \xi))^j). \]

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Furthermore, we define

\begin{equation}
(u,v)_{\mathcal{K}_\alpha} := (K_\alpha(0,\xi)\mathcal{F}u,\mathcal{F}v)_{\mathcal{H}}, \quad \mathcal{F}u, \mathcal{F}v \in \mathcal{D}(K_\alpha(0,\xi)),
\end{equation}

\begin{equation}
K^{1/2}_\alpha := (K_\alpha(0,p))^{1/2} = 
\begin{pmatrix}
(L(0,p))^{1/4-\alpha/2} & 0 \\
0 & (L(0,p))^{-1/4+\alpha/2}
\end{pmatrix},
\end{equation}

\Phi_{0,\alpha}(t,x) = K^{1/2}_\alpha\Psi_{0,\alpha}(t,x), \quad \Phi_{0,\alpha}(0,x) = \Phi_{0,\alpha} = K^{1/2}_\alpha\Psi_0.

It can be shown that for \( u = (u_1, u_2)^T \),

\begin{equation}
(u,u)_{\mathcal{K}_\alpha} = 
\left\| (L(0,\xi))^{1/4-\alpha/2}\hat{u}_1,\hat{u}_1 \right\|_{L^2(\mathbb{R}^n)}^2 + 
\left\| (L(0,\xi))^{-1/4+\alpha/2}\hat{u}_2,\hat{u}_2 \right\|_{L^2(\mathbb{R}^n)}^2 = \|u\|^2_{\mathcal{K}_\alpha}.
\end{equation}

Thus, \((\cdot,\cdot)_{\mathcal{K}_\alpha}\) is the inner product of \(\mathcal{K}_\alpha\). Moreover, notice that for \(\Psi_0 \in \mathcal{K}_\alpha\), \(\|\Phi_{0,\alpha}\|_{\mathcal{K}_\alpha}^2 = (K_\alpha\Psi_0,\Psi_0)_{\mathcal{H}} = \|\Psi_0\|_{\mathcal{K}_\alpha}\), i.e., \(\Phi_{0,\alpha} \in \mathcal{H}\). We then define the system

\begin{equation}
i\frac{\partial}{\partial t}\Phi_{0,\alpha}(t,x) = H_{0,\alpha}(t)\Phi_{0,\alpha}(t,x), \quad \Phi_{0,\alpha}(0,x) = \Phi_{0,\alpha}, \quad H_{0,\alpha}(t) = K^{1/2}_\alpha A_0(t)(K^{1/2}_\alpha)^{-1}.
\end{equation}

on the Hilbert space \(\mathcal{H}\). In the same way, \(U_{0,\alpha}(t)\), the propagator for \(H_{0,\alpha}(t)\), can be written as

\begin{equation}
U_{0,\alpha}(t) = K^{1/2}_\alpha U_{A_0}(t)(K^{1/2}_\alpha)^{-1}, \quad U_{0,\alpha}(t)^{-1} = K^{1/2}_\alpha U_{A_0}(t)^{-1}(K^{1/2}_\alpha)^{-1},
\end{equation}

and we obtain the system

\begin{equation}
i\frac{d}{dt}U_{0,\alpha}(t)\Phi_{0,\alpha} = H_{0,\alpha}(t)U_{0,\alpha}(t)\Phi_{0,\alpha}, \quad \Phi_{0,\alpha} \in \mathcal{H}
\end{equation}

with Hilbert space \(\mathcal{H}\) and complex valued energy \(H_{0,\alpha}(t)\). Straightforward calculations show that \(H_{0,\alpha}(t)\) can be written as

\begin{equation}
\begin{pmatrix}
(L(0,p))^{1/4-\alpha/2}(-q_E)(L(0,p))^{-1/4+\alpha/2} & (L(0,p))^{1/2} \\
(L(0,p))^{1/2} & (L(0,p))^{-1/4+\alpha/2}(-q_E)(L(0,p))^{1/4+\alpha/2}
\end{pmatrix}.
\end{equation}

Noting that for an invertible smooth function \(F\) and its inverse \(F^{-1}\),

\begin{equation}
F(p)^{-1}xF(p) = \mathcal{F}^{-1}F(\xi)^{-1}\mathcal{F}^{-1}_1xF(\xi)\mathcal{F}^{-1}_1 = (\mathcal{F}^{-1}_1x\mathcal{F}^{-1}_1 = i\nabla \xi),
\end{equation}

\begin{equation}
F(p)^{-1}F(\xi)^{-1}(i\nabla F(\xi)\mathcal{F}^{-1}_1 + \mathcal{F}^{-1}F(\xi)^{-1}F(\xi)\mathcal{F}^{-1}_1x\mathcal{F}^{-1}_1 = iF(p)^{-1}(\nabla F(p)) + x
\end{equation}

holds. Hence, \((L(0,p))^{-\theta}q_E \cdot x(L(0,p))^{\theta} = q_E \cdot x + 2ic^2\theta q_E \cdot p(L(0,p))^{-1}\), and \(H_{0,\alpha}(t)\) can be decomposed into \(\hat{H}_{0,\alpha}(t) = H_0(t) + P_{0,\alpha}(t)\); \(H_0(t)\) and \(P_{0,\alpha}(t)\) are the same as those defined in (5) and (6), respectively. Here, \(H_0(t)\) is a symmetric operator (self-adjoint operator for every fixed \(t\), see Lemma 2.1. of [15]), but \(P_{0,\alpha}(t)\) is a non-symmetric operator (clearly, it is a complex valued operator).
APPENDIX B  Models of time-dependent electric fields

Here, we give examples of electric fields satisfying Assumption (E1). First, we assume that \( b(t) \) satisfies \( b(t) = (0, 0, \ldots, 0, b_j(t), 0, \ldots, 0) \), \( j \in \{1, 2, \ldots, n\} \), and \( b_j(t) \) can be written as

\[
    b_j(t) = \begin{cases} 
        C_1 t^\gamma + \rho_\gamma(t) & 0 < \gamma < 1, \\
        C_1 t + \rho_1(t) + \theta_1(t) & \gamma = 1, 
    \end{cases} 
\]

(47)

where \( C_\gamma \neq 0 \) is a constant, \( \rho_\gamma \in C^2(\mathbb{R}^n) \) satisfies \( |\rho_\gamma(t)| = o(t^{\gamma-1}) \) for \( l \in \{0, 1, 2\} \), and \( |\theta_1(t)| \leq C \)

for \( l \in \{0, 1, 2\} \). It can easily be shown that

\[
    \int_{|a+b(s)| \leq 2E_{0,0}/(mc^2)} |b'(s)| ds \leq \int_{|a+b_j(s)| \leq 2E_{0,0}/(mc^2)} |b'_j(s)| ds 
\]

\[
\leq \left| \int_{|\tau| \leq 2E_{0,0}/(mc^2)} \frac{|b'_j(s)|}{|b'_j(s)|} d\tau \right| \leq C 
\]

and

\[
\int_0^t \frac{|b'(s)|^2 + |b''(s)|}{Q(s,a)^2} ds 
\leq C_R + \int_R^t \frac{|b'_j(s)|^2 + |b''_j(s)|}{c^2(a_j + C_\gamma s^\gamma + \rho_\gamma(s) + \theta_\gamma(s))^2 + (mc^2)^2} ds 
\leq C_R + C \sup_{s > R} |s^{1-\gamma}(|b'_j(s)|^2 + |b''_j(s)|)| \int_{-\infty}^\infty \frac{d\tau}{c^2(\tau + \theta_\gamma(s))^2 + (mc^2)^2} 
\]

(48)

hold, where \( \theta_\gamma(s) \equiv 0 \) for \( \gamma < 1 \). By dividing the limits of integration into two regions, \( |\tau| \leq 2|\theta_\gamma(s)| \leq C \) and \( |\tau| \geq 2|\theta_\gamma(s)| \), notice that the last term of the above inequality is smaller than

\[
C \sup_{s > R} |s^{1-\gamma}(|b'_j(s)|^2 + |b''_j(s)|)| \left( \int_{|\tau| \leq C} d\tau + \int_{|\tau| \geq 2|\theta_\gamma(s)|} \frac{d\tau}{c^2\tau^2/4 + (mc^2)^2} \right) \leq C, 
\]

where (47) is utilized.

Next, assume \( b(t) = (0, \ldots, 0, b_j(t), 0, \ldots, 0) \) and \( b_j(t) \) can be written as

\[
    b_j(t) = e_3(\log(1 + e_4|\tau|)), 
\]

where \( e_3 \neq 0 \) and \( e_4 > 0 \) are constants. By the same approach as (48), we obtain the left-hand side of (47) for this particular \( b(t) \). Moreover, by using the fact that \( (b'_j(s))^2 \) and \( b''_j(s) \) are integrable on \([R, \infty)\), the right-hand side of (47) can also be obtained for this \( b(t) \).

Remark APPENDIX B .1. Suppose \( b(t) \) satisfies \( b(t) = (0, \ldots, 0, b_{j1}(t), 0, \ldots, 0, b_{j2}(t), 0, \ldots, 0) \) and \( b_{j1}(t) \) and \( b_{j2}(t) \) are written in the same form as (47) by replacing \( \gamma \to \gamma_1 \) and \( \gamma \to \gamma_2 \), respectively. Then it is sufficient to consider the same approach as above for the maximum of \( \{\gamma_1, \gamma_2\} \); indeed, suppose \( \gamma_1 \geq \gamma_2 \). Noting that

\[
\int_{|a+b(s)| \leq 2E_{0,0}/(mc^2)} |b'(s)| ds \leq C_R + C \int_{|a+b_{j1}(s)| \leq 2E_{0,0}/(mc^2)} |b'_{j1}(s)| ds 
\]

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and
\[
\int_{t}^{R} \frac{|b'(s)|^2 + |b''(s)|}{Q(s,a)^2} ds \leq C \int_{t}^{R} \frac{|b_{j1}'(s)|^2 + |b_{j1}''(s)|}{c^2(a_{j1} + b_{j1})(s)^2 + (mc)^2} ds,
\]
it is straightforward to prove that (7) mimics the above approach. Similarly, we consider the case when \(b(t) = (b_1(t), \ldots, b_n(t))\). However, if AC electric fields are included in \(E(t)\), (7) is difficult to prove. For example, consider the case when \(b_{j1}(t) = t\gamma\) and \(b_{j2}(t) = t\gamma/2 + \cos t\) with \(0 < \gamma < 1\), i.e., \(|b_{j1}(t)| \geq |b_{j2}(t)|\) holds for \(t \gg 1\), but \(|b_{j1}^{(l)}(t)| \geq |b_{j2}^{(l)}(t)|, l \in \{1, 2\}\), is not always true. Clearly, \(s^{1-\gamma}(|b''(s)| + |b'(s)|)\) is not bounded; hence, our proof fails. Other approaches must be established to consider more general electric fields including AC electric fields.

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