Inbound Replenishment and Outbound Dispatch Decisions under Hybrid Shipment Consolidation Policies: An Analytical Model and Comparison

Bo Wei Sıla Çetinkaya Daren B.H. Cline

Abstract

We consider a distribution warehouse where both the inbound inventory replenishment and outbound dispatch decisions are subject to fixed (as well as per-unit) transportation charges and demand is stochastic. In order to realize scale economies associated with transportation operations both on the outbound and inbound sides, dispatch schedules must be synchronized over time with inventory replenishment decisions. An immediate delivery policy on the outbound side does not make economic sense because outbound dispatch operations and costs will benefit from a temporal shipment consolidation policy. Our particular interest in this setting is the exact modeling and analysis of hybrid shipment consolidation policies, in comparison to the time- and quantity-based counterparts. Since shipment consolidation prolongs customer waiting and inventory holding, we investigate average delay penalty per order and average inventory per time unit as critical measures of performance of the distribution operation, along with the annual cost. By fixing the expected inbound replenishment and outbound dispatch frequencies, we compare these measures among the alternative ways of operation under hybrid, time-based, and quantity-based policies. This comparison then lends itself to an explicit analytical comparison of average costs under these three policies, without a need for solving the corresponding optimization problems.

Keywords: Supply chain integration, coordination of inventory and transportation decisions, shipment consolidation, hybrid/time-and-quantity consolidation policy, renewal theory, martingales, truncated random variables, stochastic calculus.

1 Introduction and Motivations

A growing body of academic publications, along with a particular industry focus, have placed an emphasis on the value of coordination of different functional specialties (e.g., finance, marketing, production/inventory, logistics/transportation, etc.) in the context of supply chain management practices (Axsäter 2015, Büyükkaramikli et al. 2014, Çetinkaya and Lee 2000, ...)
Notably, coordination of production/inventory and logistics/transportation operations has received a significant attention over the past two decades as more companies have implemented specific programs enabling such coordination, including but not limited to vendor-managed inventory (VMI), third-party logistics (3PL), and time definite delivery (TDD) practices (Alumur et al. [2012], Çetinkaya [2005], Çetinkaya et al. [2006, 2008], Çetinkaya and Lee [2000], Gürler et al. [2014], Jaruphongsa et al. [2007], Kaya et al. [2013b,a], Lee et al. [2003], Ponce et al. [2020]).

The existing body of work supporting the implementation of VMI, 3PL, TDD and similar programs can be categorized as deterministic (e.g., see Çetinkaya and Lee [2002], Çetinkaya et al. [2009], Hwang [2010], Jaruphongsa et al. [2007], Kaya et al. [2013b], Lee et al. [2003]) and stochastic (e.g., see Axsäter [2001], Çetinkaya et al. [2006, 2008], Çetinkaya and Lee [2000], Katariva et al. [2014], Kaya et al. [2013a], Marklund [2011], Mutlu and Çetinkaya [2010], Stenius et al. [2016]) demand models where several practical generalizations related to the latter category remain as open and challenging research problems. More specifically, despite a notable body of literature on the integration of inventory and transportation decisions, there is still a need for exact analytical decision making models considering stochastic demand. In this paper, we revisit a stylized, yet, fundamental, problem setting with the goal of addressing this need and filling a void in the existing literature.

In particular, we consider a stock-keeping point, e.g., a distribution warehouse, where both the inbound inventory replenishment and outbound dispatch decisions are subject to fixed (as well as per-unit) transportation charges. These charges are associated with lump-sum transportation (e.g., set-up, loading, bundling, equipment, etc.) costs of each inbound and outbound delivery, regardless of quantity delivered to satisfy stochastic demands realized over time. Hence, outbound dispatch decisions must be synchronized with inbound inventory replenishment decisions over time so as to realize scale economies associated with logistical operations both on the inbound and outbound sides.

Unlike the traditional approaches to inventory control, an immediate delivery policy on
the outbound side does not make economic sense in the context of the problem at hand. This is because of the fixed costs of outbound dispatch operations which will benefit from the implementation of a temporal shipment consolidation policy. Such a policy is characterized by active efforts to merge several smaller stochastic demands realized over time into larger dispatch quantities to be shipped as combined loads realizing scale economies (Çetinkaya and Bookbinder [2003], Higginson and Bookbinder [1994, 1995]).

Three kinds of temporal shipment consolidation policies are common in the logistics practice and previous literature (e.g., see Cai et al. [2014], Çetinkaya [2005], Çetinkaya and Bookbinder [2003, 2006, 2014, 2008], Çetinkaya and Lee [2000], He et al. [2020], Higginson and Bookbinder [1994, 1995], Mutlu et al. [2010], Satır et al. [2018], Wei et al. [2020]), and they are revisited in this paper: (i) Quantity-based policy (QP) under which stochastic demands are held until a target dispatch quantity $q$ is accumulated; (ii) Time-based policy (TP) under which a consolidated dispatch quantity is released by a pre-determined time (say, periodically, every $T$ time-units; and (iii) Hybrid policy (HP) or time-and-quantity policy under which the shipper attempts to accumulate a target dispatch quantity $q$ but releases the existing consolidated load by a predetermined time (say, at most after $T$ time-units since the last dispatch) if the size of the consolidated load has not reached $q$ in a timely fashion. Clearly, QPs assure transportation scale economies are fully realized whereas TPs ensure timely delivery. HPs are aimed at balancing the economic benefits versus timely delivery trade-offs associated with QPs and TPs (Çetinkaya [2005], Çetinkaya et al. [2006, 2014], Higginson and Bookbinder [1994], Mutlu et al. [2010], Wei et al. [2020]). For the case where demand is a Poisson process, as in this paper, by assumption, the policy parameters are such that $T > 0$ and $q$ is a positive integer.

As noted in Çetinkaya [2005], temporal shipment consolidation may be implemented on its own without coordination, a.k.a., a pure consolidation practice. However, for all practical purposes, it is imperative to consider the impact of temporal shipment consolidation on inventory and dispatch decisions throughout the supply chain. Thus, a more informed approach targeting supply chain coordination is to integrate/synchronize temporal ship-
ment consolidation with production/inventory and logistics/transportation decisions with a holistic vision. This latter approach is referred to as an integrated inventory/consolidation practice (Çetinkaya [2005]).

Our focus in this paper is the exact modeling of HPs in an integrated framework for making inbound replenishment and outbound dispatch decisions simultaneously. An accompanying goal is to offer an analytical comparison of the resulting model with the previously developed counterpart models that consider TPs (Çetinkaya and Lee [2000]) and QPs (Çetinkaya et al. [2006]). That is, the overall analysis presented here is based on three alternative models of operation to include the integrated inbound replenishment and outbound dispatch models under QPs, TPs, and HPs.

More specifically, we propose exact and approximate analytical methods to compute and then compare the (i) cost performance, (ii) delivery performance, and (iii) inventory stagnancy/flow under HPs relative to its alternatives, i.e., QPs and TPs. Clearly, all three types of shipment consolidation policies are implemented at the expense of prolonged customer waiting and inventory holding before an outbound dispatch is made to satisfy stochastic demands realized over time. Hence, in addition to annual cost, we investigate average delay penalties (including both linear and squared delay) per order and average inventory per time unit as additional critical measures of interest associated with the performance of distribution operation under investigation. By fixing the expected replenishment cycle length (i.e., inbound replenishment frequency) and/or the expected shipment consolidation cycle length (i.e., outbound dispatch frequency), we are able to compare these two measures among the three alternative models of operation.

This comparison then lends itself to an explicit analytical comparison of average cost criteria of the three operational models, without a need for solving the corresponding three different optimization problems. Notably, our comparative results lead to a full analytical characterization of cost performance under the alternative models of operation with shipment consolidation. Thus, we fill a gap in literature via a set of analytically provable results which were only illustrated by numerical tests previously in Çetinkaya et al. [2006]. Our an-
alytically provable results justify the impact and value of alternative shipment consolidation policies regardless of the values of model parameters and are of practical value in the context of design and operation of an integrated framework for inventory-transportation systems.

The remainder of this paper is organized as follows. An overview of previously established results and our specific contributions relative to the existing literature are summarized in Section 2. In Section 3, we introduce the underlying stochastic processes and operational system characteristics of the integrated model under HP, and then we provide an exact formulation and analysis of the resulting model. Multiple critical performance measures applicable in the context of the three alternative models of operation (to include the integrated inventory replenishment and outbound dispatch models under QPs, TPs, and HPs) are proposed and the main comparison results in terms of these measures along with long-run average cost are given in Section 4. Finally, Section 5 concludes the study with a discussion of further generalizations and related directions for future research.

2 Previous Literature and Our Contributions

Research in pure consolidation has been inspirational and instrumental for the development and implementation of integrated practices and models. Hence, we begin with a review of the previous quantitative modeling efforts in this area in Section 2.1 which is followed by a review of the relevant work on integrated consolidation practices in Section 2.2. An overview of our results and the accompanying practical insights are then discussed in Section 2.3.

2.1 Related Literature on Pure Consolidation Practices

A growing body of literature has focused on development of quantitative models supporting pure consolidation practices since mid 90s. Notably, three related streams of research contribute to this literature: (i) numerical and analytical models aimed at computation of operational policy parameters for implementing pure shipment consolidation practices in industrial logistics (Cai et al. [2014], Cetinkaya and Bookbinder [2003], Cetinkaya et al. [2014], Bookbinder and Higginson [2002], Higginson and Bookbinder [1995], He et al. [2020]) and
disaster relief (Cook and Lodree [2017], Zhang et al. [2019]) applications; (ii) admission control, cost allocation, and pricing models for managing pure shipment consolidation practices (Satır et al. [2018], Ülkü and Bookbinder [2012], Wei et al. [2020]); and (iii) performance analysis of pure policies for improving shipment consolidation operations (Higginson and Bookbinder [1994], Mutlu et al. [2010], Wei et al. [2020]). Among these three, the results offered by the last stream of research is the most relevant for our purposes, and, thus, summarized here.

A detailed simulation study in Higginson and Bookbinder [1994] reveals that that QPs achieve the least cost compared with the other policies. However, in terms of average waiting time, HPs outperform both TPs and QPs. Later, the analytical and numerical analyses in Çetinkaya et al. [2014] provide a comparison of the distribution of maximum waiting time (MWT) and the average order delay (AOD) among QPs, TPS, and HPs, where demand is modeled as a Poisson process and

\[
AOD = \frac{\mathbb{E}[\text{Cumulative waiting per consolidation cycle}]}{\mathbb{E}[\text{Number of orders arriving in a consolidation cycle}]}.
\]  

(1)

Here, the term delay refers to the time between arrival and clearing of a realized (i.e., outstanding) demand due to outbound dispatch delay associated with shipment consolidation policy in place. Hence, both MWT and AOD are treated as indicators of timely service (i.e., service performance). Under fixed policy parameters, \( q \) and/or \( T \), the analytical results in Çetinkaya et al. [2014] indicate that HPs outperform QPs and TPs, not only in terms of the \( P(MWT > t) \) for a given \( t \); but, also, in terms of \( AOD \). Also, the numerical results in Çetinkaya et al. [2014] demonstrate that for a fixed expected consolidation cycle length (i.e., dispatch frequency), while QPs perform the best in terms of \( AOD \), the general class of HPs performs better than the general class of counterpart TPs. More recently, the full set of analytical results in Wei et al. [2020] provide a general analytical characterization of \( AOD \), leading to a complete comparative analysis of alternative renewal-type clearing policies including but not limited to QPs, TPs, and HPs. That is, regardless of the parametric setting at hand, for a fixed expected consolidation cycle length, QPs are the best, and HPs outperform counterpart TPs in terms of \( AOD \). Hence, the analysis in Wei et al. [2020] closes the gaps in literature regarding service performance of pure shipment consolidation practices.
via a complete set of analytically provable results regarding \textit{AOD} which were only illustrated through numerical tests in Çetinkaya et al. [2014].

In light of these recent analytical findings illuminating the clear advantages of HPs over TPs, along with the cost versus service trade-offs between QPs and HPs, in the context of pure consolidation practices, there is value in comparing these policies in an integrated framework. Such a comparison happens to be the core objective of the current paper. In order to support this objective, we now discuss additional results from the relevant literature on integrated consolidation practices.

\subsection*{2.2 Related Literature on Integrated Consolidation Practices}

The existing work on integrated consolidation practices was inspired by the problem setting in Çetinkaya and Lee [2000] and focuses on computation of policy parameters for simultaneous optimization of inventory replenishment (Axsäter [2001], Çetinkaya et al. [2006, 2008], Çetinkaya and Lee [2000], Muthu and Çetinkaya [2010]) or production (Kaya et al. [2013b,a]) quantities along with outbound dispatch decisions to minimize the expected long run average cost per time unit. Hence, we also consider the problem setting of Çetinkaya and Lee [2000] where the problem was motivated in the context of a VMI application in which the upper echelon is a vendor that is (i) operating the distribution warehouse; (ii) implementing the TP in order to serve a group of downstream retailers of a market area with Poisson demands; and (iii) aiming to optimally schedule the upstream replenishment quantities and the downstream consolidated shipments.

In traditional inventory control applications, demands are assumed to be satisfied immediately; but, under the stipulations of a VMI program, the vendor has the authority to combine and hold on to the delivery of small orders from the market area for a reasonable amount of time in order to achieve transportation scale economies. This mode of operation leads to a basic push-pull system where the initial stage of the supply chain (vendor) is operated in a push-based manner \textbf{while the remaining stages (retailers in the market area) employ a pull-based strategy (Simchi-Levi et al. [2000])} taking advantage of the potential
benefits of temporal shipment consolidation practices.

The analysis in Çetinkaya and Lee [2000] focuses on the implementation of a TP on the outbound side and relies on an approximate expression of the cost function. The problem setting of Çetinkaya and Lee [2000] have been revisited in Çetinkaya et al. [2006] in order to compare the impact of HPs with QPs and TPs, again under Poisson demand. However, the comparative results presented therein also rely on approximate cost expression of Çetinkaya and Lee [2000] along with a simulation study without any analytical results applicable to HPs. Two significant generalizations of the problem to consider general renewal demand processes and common-carrier costs have been studied in Çetinkaya et al. [2008] and Mutlu and Çetinkaya [2010], respectively; again, developing approximate expressions of cost functions and without any consideration of HPs. Hence, our focus here is the exact modeling and analysis under a HP in an integrated framework for making inbound replenishment and outbound dispatch decisions simultaneously. The resulting model is referred to as the integrated model under HP in the remainder of the paper and has been compared with the integrated models under TP (Çetinkaya and Lee [2000]) and QP (Çetinkaya et al. [2006]). Our comparative results include both cost-based and service-based indicators of performance for all three models of interest as explained in the next section.

According to the detailed numerical study of Çetinkaya et al. [2006] with approximate and simulation results, HPs remain advantageous in the context of integrated consolidation practices, too, because they are cost-wise superior to TPs and service-wise superior to QPs. By providing an exact analytical model and analysis of the integrated consolidation practice under HPs, we aim to validate and offer an analytical justification for these observations revealed only numerically in Çetinkaya et al. [2006]. Also, in Çetinkaya et al. [2006], the service aspect is measured by the long run average cumulative waiting time only whereas we consider more specific performance indicators, as explained in the next section, leading to a more comprehensive set of comparative results.
2.3 Overview of Our Results and Accompanying Practical Insights

As we have noted earlier, shipment consolidation policies are in place at the expense of both prolonged customer waiting and inventory holding. Hence, in addition to studying the exact cost expressions of integrated models under TPs, QPs, and HPs, we also (i) investigate average delay penalties per order (through $AOD$ as well as $AOSD$ as defined below) and (ii) average inventory per time unit (through $AIR$ as defined below), as critical measures of performance of the integrated distribution operations. More specifically, we are interested in a comparative analysis of the following criteria.

- **Cost:** All three integrated models of interest in this paper take into account for four cost components; namely, inventory replenishment, inventory carrying, dispatch, and customer waiting costs. Our comparative results rely on the computation of exact expressions of these cost components for obtaining the exact expression of expected long run average cost per time unit.

- **Order delay:** We consider two measures in order to quantify order delay.

  1. In the spirit of previous work on pure consolidation practices (Çetinkaya et al. [2014], Wei et al. [2020]), we consider $AOD$, defined in (1), as an indicator of service performance of integrated consolidation practices, too.

  2. In $AOD$, given by Eq. (1), the waiting penalty is assumed to be linear to the delivery delay. However, in some situations, due to the impatience of the customer, it also makes sense to consider the case where the waiting penalty is proportional to the square of the delivery delay encountered due to shipment consolidation practice. The corresponding service criterion we propose for this purpose is then given by

$$AOSD = \frac{\mathbb{E}[\text{Cumulative squared waiting per consolidation cycle}]}{\mathbb{E}[\text{Number of orders arriving in a consolidation cycle}]}.$$  

- **Inventory stagnancy/flow:** Last but not least, we investigate average inventory per
time unit, i.e., average inventory rate, denoted by \( AIR \) and given by

\[
AIR = \frac{E[\text{Cumulative inventory carried per replenishment cycle}]}{E[\text{Replenishment cycle length}]},
\]

as an additional measure of performance.

As we have noted earlier, our comparative investigation relies on the computation of exact expressions of the components of expected long run average cost per time unit along with the exact expressions of \( AOD \), \( AOSD \), and \( AIR \). These expressions include terms involving truncated random variables, and their development rely on the use of a combination of renewal theory, martingales, and stochastic calculus. Our analysis of the resulting expressions also leads to the discovery of new characteristics of truncated random variables involved.

Throughout this paper, for a non-negative integer-valued random variable \( X \) and a positive integer \( q \), \( X_q \triangleq \min(X, q) \) denotes the truncated random variable of interest.

The main contributions of our work can be summarized as follows:

1. By applying a renewal theoretical framework and utilizing martingales and stochastic calculus, we offer an exact formulation of the integrated model under HP, which was studied via simulation only in the existing literature.

2. By developing and using new and refined properties of truncated Poisson random variables, we are able to show that for a fixed expected consolidation cycle length under the integrated models of interest,

   - QP performs the best and TP performs the worst in terms of \( AOD \), while HP lies between QP and TP;
   - both QP and HP lead to smaller \( AOSD \) than TP; but,
   - QP does not necessarily achieve the smallest \( AOSD \), which is different from the corresponding result in terms of \( AOD \) (see Theorems [7] and [9]).

3. Moreover, relying on a reasonable approximation technique, we are able to show that for a fixed expected replenishment cycle length and a fixed expected consolidation cycle
length under the integrated practice of interest, QP has smaller AIR than both HP and TP in an approximate fashion, and AIR under HP is approximately the same as that under TP (see Theorem 11).

4. Building on the analytical comparison results in terms of AOD/AOSD and AIR, we are also able to compare the expected long run average cost per time unit for the three integrated models of operation regardless of the parametric setting, without a need to solve the corresponding optimization problems explicitly and, hence, eliminating the burden of a numerical solution approach completely.

5. Considering the critical measures of performance, including cost-based and service-based criteria, we present insightful guidelines informing the design and operation of integrated inventory-transportation systems.

3 Integrated Model under HP

We begin with a more detailed description of the integrated problem setting under HP and introduce the underlying stochastic processes of interest in Section 3.1. Then we proceed with the model formulation in Section 3.2. Throughout the remainder of this paper, for two real numbers \(a\) and \(b\), let \(a \wedge b \triangleq \min(a, b)\).

3.1 Problem Setting and Underlying Stochastic Processes

We consider a distribution warehouse (e.g., vendor) serving a market area, potentially consisting of multiple customers (e.g., retailers) with stochastic demands realized over time. The demand process of each customer is assumed to follow a Poisson process so that the aggregate demand process of the market area, denoted by \(N(t)\), is also Poisson, say with rate \(\lambda\). The inventory at the distribution warehouse is replenished instantaneously by incurring a fixed replenishment cost of \(A_R\) and a per unit replenishment cost of \(c_R\). Any inventory held at the distribution warehouse is subject to an inventory holding cost of \(h\) per unit per time unit. The outbound dispatch decisions for transporting goods to satisfy the stochastic
demand of market area are also subject to a fixed dispatch cost of $A_D$ along with a per unit dispatch cost of $C_D$ at the expense of the distribution warehouse. Hence, outbound dispatch schedules must be synchronized over time with inbound inventory replenishment decisions, and outbound dispatch operations and costs take advantage of an explicit shipment consolidation practice under a HP. Since shipment consolidation is implemented at the expense of prolonged customer waiting, outstanding orders waiting to be delivered to the market are subject to a waiting cost penalty of $\omega$ per unit per time unit.

Let $q_H$ and $T_H$ denote the parameters associated with the HP of interest, under which a dispatch decision is made every $\tau_{q_H} \wedge T_H$ time units, where $\tau_{q_H}$ is first hitting time of $q_H$ with respect to the aggregate Poisson demand process. By assumption, the policy parameters are such that $T_H > 0$ and $q_H$ is a positive integer.

A shipment consolidation cycle is then defined as the time between two successive outbound dispatch decisions. Since the distribution warehouse is able to replenish its inventory in bulk immediately from an external source with plentiful supply, it suffices to review the inventory level at the end of each shipment consolidation cycle. At the review instant, if the on-hand inventory is not sufficient to clear all outstanding orders, the distribution center first replenishes its stock and then releases the entire consolidated load such that the remaining on-hand inventory is a positive integer, say $Q_H$.

An inventory replenishment cycle is then defined as the time between two successive inventory replenishment decisions. Thus, the decision variables are $q_H$ and $T_H$ which trigger a dispatch decision, and the order-up-to level $Q_H$ which indicates the base-stock quantity.

Let $L(t)$ and $I(t)$ denote the size of the consolidated load waiting to be released and the inventory level at the distribution warehouse at time $t$, respectively. Then, at the end of a shipment consolidation cycle, we have one of the two cases:

- If $I(t) < L(t)$ at the end of a consolidation cycle then a replenishment quantity of size $Q_H + L(t) - I(t)$ is ordered and received immediately. Next, the entire consolidated load is dispatched immediately so that a new replenishment cycle (also a new consolidation cycle) starts with $Q_H$ units of on hand inventory.
• If $I(t) \geq L(t)$ at the end of a consolidation cycle, then the consolidated load is dispatched using the on-hand inventory. The replenishment cycle continues but a new shipment consolidation cycle starts with $I(t) - L(t)$ units of on-hand inventory.

That is, it suffices for the distribution warehouse to employ an $(s, S)$ policy with $s = -1$ and $S = Q_H$ for replenishing the inventory. After dispatching the consolidated demand of the previous consolidation cycle, the inventory on hand is always between 0 and $Q_H$. Thus, the inventory process $I(t)$ is a regenerative process. The regeneration points are the epochs at which the vendor’s inventory is replenished (the inventory level is renewed to be $Q_H$) so that $I(t)$ consists of i.i.d. replenishment cycles.

We let $L^R_n$ denote the length of the $n$-th replenishment cycle and $T_{Cost_n}$ denote the total cost incurred during the $n$-th replenishment cycle, where $n = 1, 2, \ldots$. Then, the pairs $(L^R_n, T_{Cost_n})$, $n \geq 1$, are i.i.d.. Since the order inter-arrival times have a finite mean, and the costs are assumed to be finite, $\mathbb{E}[L^R_n] = \mathbb{E}[L^R_{HP}] < \infty$ and $\mathbb{E}[T_{Cost_n}] = \mathbb{E}[T_{Cost_{HP}}] < \infty$, $n \geq 1$. It then follows from the renewal reward theorem (Ross [1996]) that the expected long run average cost per time unit associated with the operation of the distribution warehouse, denoted by $AC_{HP}(q_H, T_H, Q_H)$ can be computed by using

$$AC_{HP}(q_H, T_H, Q_H) = \frac{\mathbb{E}[T_{Cost_{HP}}]}{\mathbb{E}[L^R_{HP}]).}$$

The optimization problem at hand is to minimize this cost function in order to compute the optimal values of $q_H, T_H$ and $Q_H$.

### 3.2 Model Formulation: Fundamental Expressions of Interest

Next, we proceed with deriving an explicit expression of $AC_{HP}(q_H, T_H, Q_H)$ by calculating the expected replenishment cycle length $\mathbb{E}[L^R_{HP}]$ and expected cost of a replenishment cycle $\mathbb{E}[T_{Cost_{HP}}].$

Clearly, the consolidation process $L(t)$ is also a regenerative process where successive outbound dispatches represent regeneration epochs. Let $L^C_i$ denote the length of the $i$-th consolidation cycle (i.e., time between two dispatch decisions), and let $N_i$ denote the size
of the consolidated load accumulated during the $i$-th consolidation cycle. In our setting, \( \{L_i^C\}_{i\geq 1} \) and \( \{N_i\}_{i\geq 1} \) are two sequences of i.i.d. random variables. It follows that for all $i \geq 1$, $L_i^C$ has the same probability distribution as $\tau_{q_H} \wedge T_H$. For all $i \geq 1$, $N_i$ has the same probability distribution as $N(\tau_{q_H} \wedge T_H)$. Note that $N(\tau_{q_H} \wedge T_H)$ can be rewritten as a truncated random variable $Y_{q_H} = \min(Y, q_H)$, where $Y \sim \text{Poisson}(\lambda T_H)$.

Observe that $N(t) - \lambda t$ is a martingale and $\tau_{q_H} \wedge T_H$ is a bounded stopping time. Using the martingale stopping theorem in [Ross, 1996, p.300], we have $E[N(\tau_{q_H} \wedge T_H)] = \lambda E[\tau_{q_H} \wedge T_H]$. Thus, the expected consolidation cycle length is

\[
E[L_i^C] = E[L_{HP}^C] = \frac{1}{\lambda} E[N(\tau_{q_H} \wedge T_H)] = \frac{1}{\lambda} E[Y_{q_H}],
\]

and the expected number of orders arriving in one consolidation cycle is

\[
E[N_i] = E[Y_{q_H}].
\]

### 3.2.1 Expected Replenishment Cycle Length

Each inventory replenishment cycle consists of at least one shipment consolidation cycle. Let $K_H$ denote the number of shipment consolidation cycles within a replenishment cycle, which is defined as

\[
K_H \triangleq \min \left\{ k \text{ is a positive integer : } \sum_{j=1}^{k} N_j > Q_H \right\}.
\]

The length of an inventory replenishment cycle in the integrated model under HP with parameters $q_H$ and $T_H$ is $L_{HP}^R = \sum_{j=1}^{K_H} L_j^C$. We observe that $\sum_{j=1}^{K_H} L_j^C$ is a finite stopping time with respect to $N(t)$, and for any $t > 0$,

\[
\left| N \left( \sum_{j=1}^{K_H} L_j^C \wedge t \right) - \lambda \sum_{j=1}^{K_H} L_j^C \wedge t \right| \leq q_H K_H + \lambda T_H K_H,
\]

where the random variable in the right hand side is integrable. Thus, by Proposition 2.5.7(iii) in [Athreya and Lahiri, 2006, p.65], \( \left\{ N \left( \sum_{j=1}^{K_H} L_j^C \wedge t \right) - \lambda \sum_{j=1}^{K_H} L_j^C \wedge t \right\}_{t \geq 0} \) is a uniformly integrable martingale. Moreover, by the martingale stopping theorem and Vitali convergence theorem (the uniform integrability convergence theorem) in [Rosenthal, 2006, p.105], we have

\[
E \left[ N \left( \sum_{j=1}^{K_H} L_j^C \right) \right] = \lambda E \left[ \sum_{j=1}^{K_H} L_j^C \right].
\]
It then follows that
\[
E \left[ L_{HP}^R \right] = E \left[ \sum_{j=1}^{K_H} L_j^C \right] = \frac{1}{\lambda} E \left[ N \left( \sum_{j=1}^{K_H} L_j^C \right) \right] = \frac{1}{\lambda} E \left[ \sum_{j=1}^{K_H} N_j \right] = \frac{1}{\lambda} E \left[ K_H \right] E \left[ Y_{qh} \right], \tag{8}
\]
where the last equality follows from the Wald’s equation (Ross [1996]) because \( K_H \) is a stopping time with respect to the sequence \( \{ N_j \}_{j \geq 1} \). With this observation in mind, we compute \( E[ K_H ] \) next.

Using the definition of \( K_H \), we then have \( \{ K_H \geq k \} \Leftrightarrow \{ \sum_{j=1}^{k-1} N_j \leq Q_H \} \) for all \( k \in \{1, 2, \ldots \} \). Let \( G_H(\cdot) \) as the distribution function of \( Y_{qh} \) and \( G_H^{(k)}(\cdot) \) as the \( k \)-fold convolution of \( G_H(\cdot) \). Moreover, we define \( M_{G_H}(i) \triangleq \sum_{k=0}^{\infty} G_H^{(k)}(i) \). Then we have \( P( K_H \geq k ) = G_H^{(k-1)}(Q_H) \) for all \( k \in \{1, 2, \ldots \} \) and
\[
E[K_H] = \sum_{k=1}^{\infty} P( K_H \geq k ) = \sum_{k=1}^{\infty} G_H^{(k-1)}(Q_H) = 1 + \sum_{k=1}^{\infty} G_H^{(k)}(Q_H) = M_{G_H}(Q_H). \tag{9}
\]
We substitute Eq. (9) into Eq. (8) and obtain
\[
E \left[ L_{HP}^R \right] = \frac{1}{\lambda} E \left[ Y_{qh} \right] M_{G_H}(Q_H). \tag{10}
\]

### 3.2.2 Expected Inventory Replenishment Cost per Replenishment Cycle

Let \( E[ RCost_{HP} ] \) denote the replenishment cost per replenishment cycle. Since \( K_H \) is a stopping time with respect to the sequence \( \{ N_j \}_{j \geq 1} \), by the Wald’s equation, we have
\[
E[ \text{Replenishment Quantity} ] = E \left[ \sum_{j=1}^{K_H} N_j \right] = E[ K_H ] E \left[ Y_{qh} \right],
\]
so that
\[
E[ RCost_{HP} ] = A_R + c_R E[ \text{Replenishment Quantity} ] = A_R + c_R E[ K_H ] E \left[ Y_{qh} \right]. \tag{11}
\]

### 3.2.3 Expected Inventory Carrying Cost per Replenishment Cycle and AIR

Under the HP, the inventory dynamics within a replenishment cycle is as follows,
\[
I(t) = \begin{cases} 
Q_H, & 0 \leq t \leq L_1^C, \\
Q_H - N_1, & L_1^C < t \leq \sum_{j=1}^{2} L_j^C, \\
\vdots \\
Q_H - \sum_{j=1}^{K_H-1} N_j, & \sum_{j=1}^{K_H-1} L_j^C < t \leq \sum_{j=1}^{K_H} L_j^C.
\end{cases}
\]
Letting $\mathbb{E}[H_{HP}]$ denote the expected cumulative inventory carrier per replenishment cycle, we have

$$
\mathbb{E}[H_{HP}] \triangleq H(Q_H, q_H, T_H) = \mathbb{E} \left[ \int_0^{\sum_{j=1}^{\infty} I_j} I(t)dt \right].
$$

Using the renewal argument,

$$
H(Q_H, q_H, T_H|N_1 = i) = \begin{cases} 
\mathbb{E}[\tau_{q_H} \wedge T_H] Q_H, & \text{if } i \geq Q_H + 1, \\
\mathbb{E}[\tau_{q_H} \wedge T_H] Q_H + H(Q_H - i, q_H, T_H), & \text{if } i \leq Q_H,
\end{cases}
$$

so that

$$
H(Q_H, q_H, T_H) = \mathbb{E}[\tau_{q_H} \wedge T_H] Q_H + \sum_{i=0}^{Q_H} H(Q_H - i, q_H, T_H) g_H(i),
$$

where $g_H(\cdot)$ denotes the probability mass function of $Y_{q_H}$. We further denote $g^{(k)}_H(\cdot)$ as the $k$-fold convolution of $g_H(\cdot)$ and define $m_{g_H}(i) \triangleq \sum_{k=0}^{\infty} g^{(k)}_H(i)$. The expression for $H(Q_H, q_H, T_H)$ is a renewal type equation, its solution is given as

$$
H(Q_H, q_H, T_H) = \mathbb{E}[\tau_{q_H} \wedge T_H] Q_H + \sum_{i=0}^{Q_H} (Q_H - i) m_{g_H}(i).
$$

Remark 1. Note that for $i = 0, 1, \ldots, Q_H$, $m_{g_H}(i)$ is the sum of the probabilities to reach inventory $Q_H - i$ after $k = 0, 1, 2, \ldots$ dispatches. In the integrated model under HP (see Axsäter [2001, 2015], Zheng and Federgruen [1991] for details),

$$
\frac{m_{g_H}(i)}{M_{G_H}(Q_T)} = \frac{m_{g_H}(i)}{\sum_{i=0}^{Q_H} m_{g_H}(i)}
$$

is the steady state probability of inventory level $Q_H - i$, where $i = 0, 1, \ldots, Q_H$.

Now, letting $\mathbb{E}[H{Cost}_{HP}]$ denote the expected inventory holding cost within one replenishment cycle under HP, we have

$$
\mathbb{E}[H{Cost}_{HP}] = hH(Q_H, q_H, T_H) = \frac{h}{\lambda} \mathbb{E}[Y_{q_H}] \sum_{i=0}^{Q_H} (Q_H - i) m_{g_H}(i).
$$

Also, substituting Eqs. (10) and (12) in Eq. (3), it then follows that the AIR associated with the integrated model under HP is given by

$$
AIR_{HP} = \frac{\sum_{i=0}^{Q_H} (Q_H - i) m_{g_H}(i)}{M_{G_H}(Q_H)}.
$$
Based on a visual inspection of the right hand side of Eq. (14), it is easy to conclude that $AIR_{HP}$ is difficult to evaluate both analytically and numerically. This difficulty is rooted at the same issue with the form of Eq. (10) which is used for obtaining Eq. (14). Hence, the approximations presented below are useful proxies when an exact comparative analysis is not possible as in the case of evaluating $AIR$ expressions (e.g., see Section 4.3).

From the definition of $K_H$ in Eq. (7),

$$
\mathbb{E} \left[ \sum_{j=1}^{K_H} N_j \right] = \mathbb{E}[K_H] \mathbb{E}[Y_{qh}] \geq Q_H + 1 \quad \text{and} \quad \mathbb{E} \left[ \sum_{j=1}^{K_H-1} N_j \right] = \mathbb{E}[K_H] \mathbb{E}[Y_{qh}] - \mathbb{E}[Y_{qh}] \leq Q_H.
$$

As result,

$$
\frac{Q_H}{\mathbb{E}[Y_{qh}]} + 1 \geq \mathbb{E}[K_H] \geq \frac{Q_H}{\mathbb{E}[Y_{qh}]} + \frac{1}{\mathbb{E}[Y_{qh}]}.
$$

Then, treating $K_H$ as a continuous random variable, we have

$$
\mathbb{E} \left[ \sum_{j=1}^{K_H} N_j \right] = \mathbb{E}[K_H] \mathbb{E}[Y_{qh}] \approx Q_H + 1,
$$

which, in turn, implies that

$$
\mathbb{E}[K_H] \approx \frac{Q_H + 1}{\mathbb{E}[Y_{qh}]}.
$$

(15)

It then follows from from Eq. (8) that

$$
\mathbb{E} \left[ L_{HP}^R \right] = \frac{1}{\lambda} \mathbb{E}[K_H] \mathbb{E}[Y_{qh}] \approx \frac{Q_H}{\lambda} + 1.
$$

(16)

Moreover, using Eq. (9), we also have

$$
M_{G_H}(Q_H) \approx \frac{Q_H + 1}{\mathbb{E}[Y_{qh}]},
$$

and

$$
m_{qH}(i) = M_{G_H}(i) - M_{G_H}(i-1) \approx \frac{1}{\mathbb{E}[Y_{qh}]},
$$

so that Eq. (12) leads to

$$
\mathbb{E}[H_{HP}] \approx \frac{1}{\lambda} \mathbb{E}[Y_{qh}] Q_H + \frac{1}{2\lambda} (Q_H + 1) Q_H.
$$

(17)

Substituting Eqs. (16) and (17) in Eq. (3), it then follows that

$$
AIR_{HP} \approx \frac{Q_H (2\mathbb{E}[Y_{qh}] + Q_H + 1)}{2(Q_H + 1)}.
$$

(18)
3.2.4 Expected Dispatch Cost per Replenishment Cycle

Let $E[DCost_{HP}]$ denote the dispatch cost per replenishment cycle. Since all outstanding demands are dispatched every $\tau_{qH} \wedge T_H$ units of time and $K_H$ is the number of shipment consolidation cycles within one replenishment cycle, we have

$$E[DCost_{HP}] = A_D E[K_H] + c_D E\left[ \sum_{j=1}^{K_H} N_j \right] = A_D E[K_H] + c_D E[K_H] E[Y_{qH}]. \tag{19}$$

3.2.5 Expected Waiting Cost per Replenishment Cycle, $AOD$, and $AOSD$

In the spirit of previous work in shipment consolidation, we associate a linear, time-dependent penalty, denoted by $\omega$, associated with customer waiting. Since the shipment consolidation cycle length under a HP with parameters $q_H$ and $T_H$ is $\tau_{qH} \wedge T_H$ and the corresponding cumulative linear delay within one shipment consolidation cycle is $W_{HP} = \int_0^{\tau_{qH} \wedge T_H} N(t)dt$, the expected cumulative waiting within one shipment consolidation cycle is

$$E[W_{HP}] = E\left[ \int_0^{\tau_{qH} \wedge T_H} N(t) \right].$$

While the results in Mutlu et al. [2010] already provide a useful, yet, involved method to compute the expected cumulative waiting within one consolidation cycle under HP in the context of pure consolidation practices, here we propose a drastically simplified approach based on a martingale argument associated with the Poisson demand process.

**Lemma 2.** Let $N(t)$ be a Poisson process with rate $\lambda$, and define $W(t) \triangleq \int_0^t N(u)du$. Then

$$W(t) - \frac{1}{2\lambda} N^2(t) + \frac{1}{2\lambda} N(t)$$

is a martingale with respect to the natural filtration $\{\mathcal{G}_t\}$, which is the $\sigma$-field generated by the family of random variables $\{N(s), s \in [0, t]\}$.

**Proof.** See Appendix A.2 for all proofs. \[\square\]

Using Lemma 2 noting $\tau_{qH} \wedge T_H$ is a bounded stopping time with respect to the demand process $N(t)$, and applying the martingale stopping theorem, we obtain

$$E[W_{HP}] = E[W(\tau_{qH} \wedge T_H)] = \frac{1}{2\lambda} E[N^2(\tau_{qH} \wedge T_H) - N(\tau_{qH} \wedge T_H)] = \frac{1}{2\lambda} E[Y_{qH} (Y_{qH} - 1)]. \tag{20}$$
Now, letting $\mathbb{E}[W_{Cost_{HP}}]$ denote the expected waiting cost per replenishment cycle and recalling that $K_H$ is the number of shipment consolidation cycles within one replenishment cycle, we have

$$\mathbb{E}[W_{Cost_{HP}}] = \omega \mathbb{E}[K_H] \mathbb{E}[W_{HP}] = \frac{\omega}{2\lambda} \mathbb{E}[K_H] \mathbb{E}[Y_{qh}(Y_{qh} - 1)].$$

(21)

Also, substituting Eqs. (6) and (20) in Eq. (1), it then follows that the $AOD$ associated with the integrated model under HP is given by

$$AOD_{HP} = \frac{\mathbb{E}[Y_{qh}(Y_{qh} - 1)]/(2\lambda)}{\mathbb{E}[Y_{qh}]}.$$ 

(22)

While the cost penalty considered in $\mathbb{E}[W_{Cost_{HP}}]$ is assumed to be linear to the delivery delay, in practice, due to customer impatience, this assumption may not be realistic. Hence, we also consider the case where this penalty may be proportional to the square of the delivery delay encountered by the customer. The counterpart expected cumulative squared waiting within one shipment consolidation cycle is then

$$\mathbb{E}[W'_{HP}] = \mathbb{E}\left[\int_0^{\tau_{qh} \wedge T_H} (\tau_{qh} \wedge T_H - t)^2 dN(t)\right],$$

and as we show in Appendix A.1

$$\mathbb{E}[W'_{HP}] = \frac{1}{3\lambda^2} \mathbb{E}[Y_{qh+1}(Y_{qh+1} - 1)(Y_{qh+1} - 2)].$$

(23)

Noting that HP with parameters $q$ and $T$ reduces

- to QP with parameter $q$ when $T \rightarrow \infty$, and
- to TP with parameter $T$ when $q \rightarrow \infty$,

and using the index TP or QP to represent the consolidation policy type to generalize the notation we have adopted so far, we also have the expected cumulative squared waiting within one consolidation cycle under QP with parameter $q$ is

$$\mathbb{E}[W'_{QP}] = \lim_{T \rightarrow \infty} \mathbb{E}[W'_{HP}] = \frac{q^3 - q}{3\lambda^2},$$

(24)
and the expected cumulative squared waiting within one consolidation cycle under TP with parameter $T$ is

$$
E[W_{TP}'] = \lim_{q \to \infty} E[W_{HP}'] = \frac{\lambda q^3}{3\lambda^2} - \frac{\lambda T^3}{3}. \tag{25}
$$

Clearly, one can then associate a cost penalty, say $\omega'$, with $E[W']$ leading to alternative expressions of the expected long run average cost functions of the three integrated models. Our primary purpose here in developing the above expressions of $E[W']$ is to utilize them in examining the counterpart $AOSD$ expressions as indicators of service performance with impatient customers which have not been investigated in the previous literature. For this purpose, substituting Eqs. (6) and (23) in Eq. (2), we have the following $AOSD$ expression associated with the integrated model under HP:

$$
AOSD_{HP} = \frac{E[Y_{qH+1}(Y_{qH+1} - 1)(Y_{qH+1} - 2)]/(3\lambda^2)}{E[Y_{qH}]} \tag{26}
$$

Likewise, using Eqs. (24) and (25) and noting that expected number of orders arriving in one consolidation cycle under QP with parameter $q$ and TP with parameter $T$ is $q$ and $\lambda T$, respectively, it can be easily verified that the $AOSD$ expressions associated with the integrated models under QP and TP are:

$$
AOSD_{QP} = \frac{q^2 - 1}{3\lambda^2} \quad \text{and} \quad AOSD_{TP} = \frac{T^2}{3}. \tag{27}
$$

Since $AOSD$ has not been studied in Çetinkaya et al. [2006] and Çetinkaya and Lee [2000], the counterpart expressions are developed here.

### 3.2.6 Exact Expression of the Cost Function under HP

Last but not least, using Eqs. (11), (13), (19), and (21), in $E[TCost_{HP}] = E[RCost_{HP}] + E[HCost_{HP}] + E[DCost_{HP}] + E[WCost_{HP}]$, and recalling Eqs. (11) and (10), we have

$$
AC_{HP}(Q_H, q_H, T_H) = \lambda(c_R + c_D) + \frac{\lambda A_R}{M_G(Q_H)E[Y_{qH}]} + \frac{\lambda A_D}{E[Y_{qH}]} + h \sum_{i=0}^{Q_H} (Q_H - i)m_{qH}(i) \frac{M_G(Q_H)}{M_G(Q_H)} + \frac{\omega E[Y_{qH}(Y_{qH} - 1)]}{2E[Y_{qH}]} \tag{28}
$$
4 Comparative Analysis

We now set the stage for a detailed comparative analysis of the operation of the distribution warehouse considering the integrated models under QP (as presented in Çetinkaya et al. [2006]), TP (as presented in Çetinkaya and Lee [2000]), and HP (as developed in the previous section). Since our comparative analysis draws from Çetinkaya et al. [2006] and Çetinkaya and Lee [2000], first in Section 4.1 we recall the relevant results therein directly without derivations and provide a summary of the key expressions we use for comparisons. Because the subsequent analysis relies on some new and refined properties of truncated Poisson random variables, next in Section 4.2 we discuss such relevant formal results. Finally, in Section 4.3 we compare AOD, AOSD and AIR as critical measures of interest of the three integrated models of investigation, and then we focus on a comparison of these models in terms of the expected long run average cost per time unit. Interestingly, we are able to offer a complete analytical comparison of the cost criteria without a need for explicitly solving the three underlying optimization problems.

4.1 Summary of Expressions Utilized for Comparisons

In this subsection, we directly cite the relevant results from Çetinkaya et al. [2006] and Çetinkaya and Lee [2000] for the integrated models under QP and TP, respectively. We adopt a self-explanatory and mnemonic notation using the policy identifier (QP, TP, HP) as an index.

We let $\tau$ and $Q_T$ denote the consolidation cycle length and the order-up-to level in the integrated model under TP for which $K_T$ is the number of consolidation cycles within one replenishment cycle. From the results in Çetinkaya and Lee [2000], $\mathbb{E}[K_T] = M_{G_T}(Q_T)$, where

$$M_{G_T}(i) \triangleq \sum_{k=0}^{\infty} G^{(k)}(i), \quad g^{(k)}(i) = \frac{(k\lambda T)^i e^{-k\lambda T}}{i!}, \quad G^{(k)}(j) = \sum_{i=0}^{j} g^{(k)}(i).$$

Also, the expected cumulative inventory carrying per replenishment cycle under TP is $\mathbb{E}[H_{TP}] = T \sum_{i=0}^{Q_T} (Q_T - i) m_{g_T}(i)$, where $m_{g_T}(i) \triangleq \sum_{k=0}^{\infty} g^{(k)}(i)$, and the counterpart ex-
expected cumulative waiting time per consolidation cycle is $\mathbb{E}[W_{TP}] = \lambda T^2/2$. Since $\mathbb{E}[K_T]$ cannot be computed in closed form, using a technique similar to the one used to obtain Eq. (18), the results in Çetinkaya and Lee [2000] indicate that

$$\mathbb{E}[K_T] \approx \frac{Q_T + 1}{\lambda T}$$

and

$$\mathbb{E}[H_{TP}] \approx TQ_T + \frac{Q_T(Q_T + 1)}{2\lambda},$$

whereas

$$\mathbb{E}[L^C_{TP}] = T, \quad \mathbb{E}[L^R_{TP}] = M_{G_T}(Q_T)T \approx \frac{Q_T + 1}{\lambda},$$ (29)

$$\mathbb{E}[RCost_{TP}] = A_R + c_R\mathbb{E}[K_T]T, \quad \text{and} \quad \mathbb{E}[DCost_{TP}] = A_D\mathbb{E}[K_T] + c_D\mathbb{E}[K_T]T.$$ (30)

Also, using the definitions of $AOD$ and $AIR$ given by Eqs. (1) and (3), it is easy to show that

$$AOD_{TP} = T/2, \quad \text{and} \quad AIR_{TP} = \sum_{i=0}^{Q_T}(Q_T - i)m_{g_T}(i)M_{G_T}(Q_T) \approx \frac{Q_T(2\lambda T + Q_T + 1)}{2(Q_T + 1)}.$$ (31)

In the integrated model under QP in Çetinkaya et al. [2006], $n$ denotes the number of consolidation cycles within an inventory replenishment cycle and $q$ denotes the targeted consolidation size. The corresponding expected cumulative inventory carrying per replenishment cycle and the expected cumulative waiting per consolidation cycle are given by

$$\mathbb{E}[H_{QP}] = \frac{1}{2\lambda}n(n-1)q^2$$

and

$$\mathbb{E}[W_{QP}] = \frac{1}{2\lambda}(q-1)q,$$

respectively, whereas

$$\mathbb{E}[L^C_{QP}] = \frac{q}{\lambda}, \quad \mathbb{E}[L^R_{QP}] = \frac{q}{\lambda}n, \quad \mathbb{E}[RCost_{QP}] = A_R + c_Rnq, \quad \text{and} \quad \mathbb{E}[DCost_{QP}] = nA_D + c_Dnq.$$ (32)

Also, using the definitions of $AOD$ and $AIR$ given by Eqs. (1) and (3), it is easy to show that

$$AOD_{QP} = \frac{q - 1}{2\lambda}, \quad \text{and} \quad AIR_{QP} = (n-1)q/2.$$ (33)

Recalling Eqs. (5), (11), (16), and (19) developed in Eqs. (29), (31), and (32) that were previously developed in Çetinkaya et al. [2006] and Çetinkaya and Lee [2000], we summarize
\[ \mathbb{E}[L_{QP}^R] = nq/\lambda \]
\[ \mathbb{E}[L_{QP}^C] = q/\lambda \]
\[ \mathbb{E}[L_{TP}^R] = M_G T (Q_T) T \approx (Q_T + 1)/\lambda \]
\[ \mathbb{E}[L_{TP}^C] = T \]
\[ \mathbb{E}[L_{HP}^R] = M_G H (Q_H) \mathbb{E}[Y_{q_H}]/\lambda \approx (Q_H + 1)/\lambda \]
\[ \mathbb{E}[L_{HP}^C] = \mathbb{E}[Y_{q_H}]/\lambda \]
\[ \mathbb{E}[RCost_{QP}] = A_R + c_R nq \]
\[ \mathbb{E}[DCost_{QP}] = n A_D + c_D nq \]
\[ \mathbb{E}[RCost_{TP}] = A_R + c_R \mathbb{E}[K_T] \lambda T \]
\[ \mathbb{E}[DCost_{TP}] = A_D \mathbb{E}[K_T] + c_D \mathbb{E}[K_T] \lambda T \]
\[ \mathbb{E}[RCost_{HP}] = A_R + c_R \mathbb{E}[K_H] \mathbb{E}[Y_{q_H}] \]
\[ \mathbb{E}[DCost_{HP}] = A_D \mathbb{E}[K_H] + c_D \mathbb{E}[K_H] \mathbb{E}[Y_{q_H}] \]

Table 1: Expected cycle lengths and replenishment and dispatch costs per replenishment cycle.

Moreover, recalling Eqs. (14), (18), (22), (26), (27), (31), and (33) developed here, we summarize the exact and approximate expressions of \(AOD\), \(AOSD\), and \(AIR\) in Table 2. We use both Tables 1 and 2 for the purposes of our comparative analysis in the remainder of the paper.

4.2 New and Refined Properties of Truncated Poisson Random Variables

The following lemma offers the key technical result which is useful to prove the new and refined properties of truncated Poisson random variables revealed by Lemmas 4 and 6.

**Lemma 3.** Suppose \(X \sim \text{Poisson}(\mu), V_n \sim \text{gamma}(n, 1)\) for integer \(n \geq 1\), we have

\[ \mathbb{E}[X_q^{(k)}] = \mathbb{E}[X_q(X_q - 1) \cdots (X_q - k + 1)] = \mathbb{E}[V_{q-k+1}^k \wedge \mu^k], \]

\[ \frac{d}{d\mu} \mathbb{E}[X_q^{(k)}] = k \mu^{k-1} \mathbb{P}(X \leq q - k) \]

where \(q\) and \(k\) are two positive integers and \(k \leq q\).

The property of truncated Poisson random variables revealed by the following lemma is fundamental for the purpose of comparing HP and TP in terms of \(AOD\) under a fixed expected consolidation cycle length.
\[
AOD_{QP} = \frac{q-1}{2\lambda}
\]
\[
AOD_{TP} = T/2
\]
\[
AOD_{HP} = \frac{\mathbb{E}[Y_{qH}(Y_{qH}-1)]/(2\lambda)}{\mathbb{E}[Y_{qH}]}
\]
\[
AOSD_{QP} = \frac{q^2-1}{3\lambda^2}
\]
\[
AOSD_{TP} = T^2/3
\]
\[
AOSD_{HP} = \frac{\mathbb{E}[Y_{qH+1}(Y_{qH+1}-1)(Y_{qH+1}-2)]/(3\lambda^2)}{\mathbb{E}[Y_{qH}]}
\]
\[
AIR_{QP} = (n-1)q/2
\]
\[
AIR_{TP} = \sum_{i=0}^{Q_T}(Q_T-i)m_{gT}(i)_{M_{G_T}(Q_T)} \approx \frac{Q_T(2\lambda T+Q_T+1)}{2(Q_T+1)}
\]
\[
AIR_{HP} = \sum_{i=0}^{Q_H}(Q_H-i)m_{gH}(i)_{M_{G_H}(Q_H)} \approx \frac{Q_H(2\mathbb{E}[Y_{qH}]+Q_H+1)}{2(Q_H+1)}
\]

Table 2: Expressions of AOD, AOSD, and AIR.

Lemma 4. Suppose \( X \sim \text{Poisson}(\mu) \), then \( \mathbb{E}^2[X_q]/\mathbb{E}^2[X_q^2] \) is increasing in \( \mu \), and \( \mathbb{E}^2[X_q] > \mathbb{E}[X_q^2] \) for all \( \mu > 0 \), where \( \mathbb{E}[X_q^2] \triangleq \mathbb{E}[X_q(X_q-1)] \).

Before we present another new property of truncated Poisson random variables in Lemma 6, we state an intuitive prerequisite result supporting its proof.

Lemma 5. Suppose \( X \sim \text{Poisson}(\mu) \), and \( A \subset \mathbb{Z}_+ \), then

\[
\frac{d}{d\mu} \mathbb{E}[X|X \in A] = \frac{1}{\mu} \text{VAR}[X|X \in A] \geq 0,
\]

where the equality holds only if \( A \) is a singleton set.

Last but not least, the following additional property of truncated Poisson random variables is useful when we compare HP and TP in terms of AOSD for a fixed expected consolidation cycle length.
Lemma 6. Suppose $X \sim \text{Poisson}(\mu)$, then there exists some $\mu' > 0$, such that $\frac{E[X^3]}{E[X^{(3)}_{q+1}]}$ is increasing on $(0, \mu')$ and decreasing on $(\mu', \infty)$, and $E[X_q] > E[X^{(3)}_{q+1}]$ for all $\mu > 0$, where $E[X^{(3)}_{q+1}] \triangleq E[X_{q+1}(X_{q+1} - 1)(X_{q+1} - 2)]$.

4.3 Comparative Results

We now proceed with a comparison of the three integrated models in terms of $AOD$, $AOSD$ and $AIR$ as well as the cost performance.

4.3.1 Order Delay Penalty Comparison

We note that, for the purpose of a sensible comparison in terms of service frequency, the comparative results for $AOD$ and $AOSD$ consider a fixed expected consolidation cycle length, and, hence a fixed dispatch/service frequency, among the three shipment consolidation policies.

We begin with our first main result which demonstrates that under a fixed expected consolidation cycle length, QP performs the best, TP performs the worst in terms of $AOD$, and HP lies between QP and TP.

Theorem 7. Under a fixed expected consolidation cycle length, $AOD_{QP} < AOD_{HP} < AOD_{TP}$. As a result, under a fixed expected consolidation cycle length and a fixed expected replenishment cycle length, we have $E[WCost_{QP}] < E[WCost_{HP}] < E[WCost_{TP}]$.

Remark 8. The intuition behind Theorem 7 is as follows. Under a QP, each shipment is released immediately after the last order of the cycle arrives. However, under a HP, random some time elapses between the arrival of the last order and the shipment release epoch. That is, the entire consolidated load of a HP cycle has to wait for this extra time unlike the case under a QP. As a result, for a given expected consolidation cycle length, QP dominates the counterpart HP in terms of $AOD$.

We proceed with a formal result demonstrating that TP performs worse than QP and HP in terms of $AOSD$, under a fixed expected consolidation cycle length. However, under
a fixed expected consolidation cycle length, QP does not necessarily achieve smaller $AOSD$ than HP, which is different from the corresponding result for $AOD$.

**Theorem 9.** Under a fixed expected consolidation cycle length, $AOSD_{QP} < AOSD_{TP}$ and $AOSD_{HP} < AOSD_{TP}$; but, neither $AOSD_{QP}$ nor $AOSD_{HP}$ dominates the other one.

**Remark 10.** Under a fixed expected consolidation cycle length, the fact that neither $AOSD_{QP}$ nor $AOSD_{HP}$ dominates the other one is different from the corresponding result for $AOD$. The reason is that $AOSD$ assigns a disproportional weight (via the squared delay) to large waiting times. While HP has a maximum limit on the waiting time, QP does not. Hence, depending on the value of this limit imposed by HP relative to the demand volume, QP or HP may be preferable in terms of $AOSD$.

### 4.3.2 Inventory Stagnancy/Flow Comparison

Next, we proceed with an examination of $AIR$ as a critical measure of interest indicating inventory stagnancy/flow. For the purpose of a sensible comparison, along with the dispatch/service frequency, we also fix the inventory turnover frequency, i.e., we consider fixed values of expected consolidation and replenishment cycle lengths. Since the exact expressions of $AIR$ we have obtained in this paper do not lend themselves for an exact comparison, the comparison results of Theorem 11 hold in an approximate fashion.

**Theorem 11.** Under a fixed expected consolidation cycle length and a fixed expected replenishment cycle length, $AIR_{TP} \approx AIR_{HP} \gtrsim AIR_{QP}$. As a result, under a fixed expected consolidation cycle length and a fixed expected replenishment cycle length, $E[HCost_{TP}] \approx E[HCost_{HP}] \gtrsim E[HCost_{QP}]$.

**Remark 12.** Under a fixed expected consolidation cycle length, the means of the consolidated loads of each outbound dispatch under QP, HP, and TP are the same ($q = E[Y_{qH}] = \lambda T$), and the variances are 0, $VAR[Y_{qH}]$, and $\lambda T$, respectively. From Lemma 4, $VAR[Y_{qH}] < E[Y_{qH}]$, i.e., under a fixed expected consolidation cycle length, the variance of dispatch quantity in each outbound under HP lies between those under QP and TP, and this result offers intuition to explain Theorem 11.
Remark 13. Recalling the exact expressions of expected consolidation and replenishment cycle lengths in Table I, we note that, by assumption, Theorem 11 requires
\[ \lambda T = \mathbb{E}[Y_{q_H}] = q \quad \text{and} \quad M_{G_T}(Q_T) = M_{G_H}(Q_H) = n. \] (34)
Clearly, given values of expected consolidation and replenishment cycle lengths, say \( \mathbb{E}[L^C] \) and \( \mathbb{E}[L^R] \), respectively, there may or may not exist a solution
\[ (n, q, Q_T, T, Q_H, q_H, T_H) \]
that satisfies Eq. (34) such that \( n, q, Q_T, Q_H \) and \( q_H \) are all integers. Our proof of Theorem 11 in Appendix A.2.8, however, sets the approximate expression of the replenishment cycle length in Table I equal to \( \mathbb{E}[L^R] \) so that we work with the solution such that
\[ \lambda T = \mathbb{E}[Y_{q_H}] = q \quad \text{and} \quad Q_T + 1 \approx Q_H + 1 \approx nq. \]
Clearly, the above equations can always be satisfied by some carefully selected values of \( n, q, Q_T, T, Q_H, q_H, \) and \( T_H \) which may or not be implementable (e.g., due to non-integer values) in the context of the three integrated models (e.g., due to non-integer solutions). Hence, we say that the theorem is applicable in practice whenever such an implementable solution exists.

4.3.3 Cost Comparison

We let \( AC_p \) denote the expected long run average cost per time unit for \( p = QP, TP, HP, \) and we again consider fixed dispatch/service and inventory turnover frequencies, i.e., fixed values of expected consolidation and replenishment cycle lengths. Then, the expected number of outbound dispatches within a replenishment cycle is the same under the three policies. As a result, both the resulting inventory replenishment costs per replenishment cycle and the resulting dispatch costs per replenishment cycle under the three policies are the same. This scenario, in turn, allows us to compare the resulting \( AC_p \) values for \( p = QP, TP, HP \) without explicitly solving the three underlying optimization problems.
Theorem 14. In the linear delay penalty case, under a fixed expected consolidation cycle length and a fixed replenishment cycle length,

\[ AC_{QP} \lesssim AC_{HP} \lesssim AC_{TP}. \]

Observe that the theorem relies on the approximation result of Eq. (15) which is obtained by treating \( K_H \) as a continuous random variable, an assumption which has been used in Çetinkaya and Lee [2000], Wald [1944] as well for obtaining similar approximations. As mentioned in Axsäter [2001] and demonstrated in Çetinkaya et al. [2006], the approximation extremely well except in the cases where the distribution warehouse is operated as a transshipment point (i.e., without holding any inventory so that there is only a single consolidation cycle in a replenishment cycle). Hence, the approximation is very effective for all practical purposes in which the warehouse carries inventory. If there is indeed only one consolidation cycle within a replenishment cycle, so that no inventory is held at the vendor’s warehouse, then all of our comparative results are still true. This is because (i) there is no inventory holding cost term to worry about and (ii) the comparison results in terms of the service measures \( AOD \) and \( AOSD \) still hold, so that the comparison results in terms of the average cost criteria are remain true.

Remark 15. In the squared delay penalty case, by a similar argument (using Theorems 9 and 11), we have that under a fixed expected consolidation length and a fixed replenishment cycle length, \( AC_{QP} \lesssim AC_{TP} \) and \( AC_{HP} \lesssim AC_{TP} \) but neither \( AC_{QP} \) nor \( AC_{HP} \) beats the other one.

5 Conclusion

In this paper, we have focused on the exact modeling of HPs in an integrated framework for making inbound replenishment and outbound dispatch decisions simultaneously. The accompanying goal of our modeling effort has been offering an analytical comparison of the resulting model with the previously developed counterpart models that consider TPs and QPs. Considering the \( AOD, AOSD \) and \( AIR \), as well as the cost, as potential evaluation
criteria, we have obtained some insightful comparative results in terms of the relative performance of the models of interest. We have also offered some new and refined properties of truncated Poisson random variables. Our comparative results regarding the cost performance of the models is notable because, unlike the previous work on the topic, it does not require numerical validation or optimization regardless of the input parameters of the underlying models.

Overall, our results demonstrate the relative impact and value of HPs over QPs and TPs, analytically. One fundamental finding in this regards is the analytically provable value of HPs over TPs, in general, as well as its potential value over QPs in terms of AOSD, i.e., when dealing with impatient customers. Collectively, these findings are of practical value in the context of design and operation of an integrated framework for inventory-transportation systems and in selecting an operational policy for shipment consolidation practices in an integrated framework.

Interesting and challenging areas of future research motivated by and extending our results include consideration of more general demand processes (renewal processes or Brownian motion) and investigation of the structure of exact optimal policies relative to QPs, TPs, and HPs.

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A Online Appendix

A.1 Squared Delay Penalty

Let $\tau_n$ be the first hitting time of $n$ with respect to $N(t)$, where $n$ is a positive integer, so that $\tau_n$ is distributed as gamma($n, \lambda$). Let the positive integer $q$ and $T > 0$ denote two parameters of the HP under our consideration. Then, the expected cumulative squared waiting per shipment consolidation cycle is

$$
\mathbb{E}[W_{HP}^2] = \mathbb{E} \left[ \int_0^{\tau_q \wedge T} (\tau_q \wedge T - t)^2 dN(t) \right]
$$

$$
= \mathbb{E} \left[ \int_0^{\tau_q \wedge T} (\tau_q \wedge T)^2 dN(t) \right] - 2 \mathbb{E} \left[ \int_0^{\tau_q \wedge T} t(\tau_q \wedge T)dN(t) \right] + \mathbb{E} \left[ \int_0^{\tau_q \wedge T} t^2 dN(t) \right] .
$$

(35)

In order to compute the three terms on the right hand side of Eq. (35), first, let us use

$$
\mathbb{E}[\tau_q^2 1_{\tau_q \leq T}] = \frac{q(q + 1)}{\lambda^2} \mathbb{P}(N(T) \geq q + 2),
$$

so that

$$
\mathbb{E} \left[ \int_0^{\tau_q \wedge T} (\tau_q \wedge T)^2 dN(t) \right] = \mathbb{E} \left[ (\tau_q \wedge T)^2 N(\tau_q \wedge T) \right]
$$

$$
= q \mathbb{E} \left[ \tau_q^2 1_{\tau_q \leq T} \right] + T^2 \mathbb{E} \left[ N(T) 1_{N(T) \leq q-1} \right]
$$

$$
= \frac{q^2(q + 1)}{\lambda^2} \mathbb{P}(N(T) \geq q + 2) + T^2 \sum_{n=0}^{q-1} n \mathbb{P}(N(T) = n).
$$

(36)

Second, observe that $2 \mathbb{E} \left[ \int_0^{\tau_q \wedge T} t(\tau_q \wedge T)dN(t) \right]$ is given by

$$
2 \int_0^T \mathbb{E} \left[ \int_0^{\tau_q} t \tau_q dN(t) | \tau_q = s \right] f_{\tau_q}(s) ds + 2T \mathbb{E} \left[ \int_0^T t dN(t) 1_{N(T) \leq q-1} \right]
$$

$$
= 2 \int_0^T s \left( \mathbb{E} \left[ \sum_{i=1}^{q-1} \tau_i | \tau_q = s \right] + s \right) f_{\tau_q}(s) ds + 2T \sum_{n=0}^{q-1} \mathbb{E} \left[ \sum_{i=1}^n \tau_i | N(T) = n \right] \mathbb{P}(N(T) = n)
$$

$$
= 2 \int_0^T s((q-1)\frac{s}{2} + s) f_{\tau_q}(s) ds + 2T \sum_{n=0}^{q-1} n \frac{T}{2} \mathbb{P}(N(T) = n)
$$

$$
= (q + 1) \mathbb{E} \left[ \tau_q^2 1_{\tau_q \leq T} \right] + T^2 \sum_{n=0}^{q-1} n \mathbb{P}(N(T) = n)
$$

$$
= \frac{q(q + 1)^2}{\lambda^2} \mathbb{P}(N(T) \geq q + 2) + T^2 \sum_{n=0}^{q-1} n \mathbb{P}(N(T) = n),
$$

(37)

where $f_{\tau_q}(\cdot)$ is the density of $\tau_q \sim \text{gamma}(q, \lambda)$, and the third equality is derived from Lemma 4.5.1 and Theorem 4.5.2 in [Resnick, 2002, p. 322, 325]. Third, noting that $\int_0^t s^2 dN(s) - \frac{1}{3} \lambda t^3$
is a martingale, and \( \tau_q \wedge T \) is a bounded stopping time, we apply the martingale stopping theorem to obtain

\[
\mathbb{E} \left[ \int_0^{\tau_q \wedge T} t^2 dN(t) \right] = \frac{1}{3} \lambda \mathbb{E} \left[ (\tau_q \wedge T)^3 \right] = \frac{1}{3} \lambda \mathbb{E} \left[ \tau_q^3 1_{\tau_q \leq T} \right] + \frac{1}{3} \lambda T^3 \mathbb{P}(N(T) \leq q - 1)
\]

\[
= \frac{(q + 2)(q + 1)q}{3\lambda^2} \mathbb{P}(N(T) \geq q + 3) + \frac{1}{3\lambda^2} \sum_{m=0}^{q+2} m(m - 1)(m - 2) \mathbb{P}(N(T) = m)
\]

\[
= \frac{1}{3\lambda^2} \mathbb{E} \left[ Y_{q+2}(Y_{q+2} - 1)(Y_{q+2} - 2) \right],
\]

where we use

\[
\mathbb{E} \left[ \tau_q^3 1_{\tau_q \leq T} \right] = \frac{(q + 2)(q + 1)q}{\lambda^3} \mathbb{P}(N(T) \geq q + 3),
\]

and

\[
\lambda T^3 \mathbb{P}(N(T) = n) = \frac{(n + 3)(n + 2)(n + 1)}{\lambda^2} \mathbb{P}(N(T) = n + 3).
\]

Substituting Eqs. (36), (37) and (38) in Eq. (35), we obtain

\[
\mathbb{E} [W_{HP}'] = \frac{(q + 1)q(q - 1)}{3\lambda^2} \mathbb{P}(N(T) \geq q + 2) + \frac{1}{3\lambda^2} \sum_{m=0}^{q+1} m(m - 1)(m - 2) \mathbb{P}(N(T) = m)
\]

\[
= \frac{1}{3\lambda^2} \mathbb{E} \left[ Y_{q+1}(Y_{q+1} - 1)(Y_{q+1} - 2) \right].
\]

A.2 Proofs

A.2.1 Proof of Lemma 2

Proof. Since the Poisson process has stationary independent increments, for \( s < t \), we have

\[
\mathbb{E} \left[ \int_0^t N(u) du \mid G_s \right] = \int_0^s N(u) du + \mathbb{E} \left[ \int_s^t N(u) du \mid G_s \right]
\]

\[
= \int_0^s N(u) du + (t - s)N(s) + \mathbb{E} \left[ \int_0^{t-s} N(u) du \right]
\]

\[
= \int_0^s N(u) du + (t - s)N(s) + \frac{1}{2}\lambda(t - s)^2,
\]

\[
\frac{1}{2\lambda} \mathbb{E} [N^2(t) \mid G_s] = \frac{1}{2\lambda} \left( N^2(s) + 2\lambda(t-s)N(s) + \lambda(t-s) + \lambda^2(t-s)^2 \right), \quad \text{and}
\]

\[
\frac{1}{2\lambda} \mathbb{E} [N(t) \mid G_s] = \frac{1}{2\lambda} (N(s) + \lambda(t-s)).
\]

Therefore,

\[
\mathbb{E} \left[ \int_0^t N(u) du - \frac{1}{2\lambda} N^2(t) + \frac{1}{2\lambda} N(t) \mid G_s \right] = \int_0^s N(u) du - \frac{1}{2\lambda} N^2(s) + \frac{1}{2\lambda} N(s),
\]

\[
\int_0^t N(u) du - \frac{1}{2\lambda} N^2(t) + \frac{1}{2\lambda} N(t) \mid G_s = \int_0^s N(u) du - \frac{1}{2\lambda} N^2(s) + \frac{1}{2\lambda} N(s).
\]
which shows that $W(t) - \frac{1}{2}N^2(t) + \frac{1}{2}N(t)$ is a martingale.

\[ \text{A.2.2 Proof of Lemma 3} \]

**Proof.** Using the relationship between Poisson and gamma distribution,

\[
\mathbb{E}[X_q^{(k)}] = \sum_{x=0}^{q} x^{(k)} \frac{e^{-\mu} \mu^{x}}{x!} + \sum_{x=q+1}^{\infty} q^{(k)} \frac{e^{-\mu} \mu^{x}}{x!} = \mu^k \mathbb{P}(X \leq q - k) + q^{(k)} \mathbb{P}(X \geq q + 1)
\]

\[
= \mu^k \mathbb{P}(V_{q-k+1} > \mu) + q^{(k)} \mathbb{P}(V_{q+1} \leq \mu) = \int_{\mu}^{\infty} \frac{\mu^k v^{q-k-1} e^{-v}}{(q-k)!} dv + \int_{0}^{\mu} \frac{\mu^k v^{q-k-1} e^{-v}}{(q-k)!} dv
\]

\[
= \mathbb{E}[V_{q-k+1}^k \wedge \mu^k].
\]

Also, recalling the following two properties of Poisson random variable,

\[
\mu^k \mathbb{P}(X = q - k) = q^{(k)} \mathbb{P}(X = q) \quad \text{and} \quad \frac{d}{d\mu} \mathbb{P}(X \leq q) = -\mathbb{P}(X = q),
\]

it is straightforward to show that

\[
\frac{d}{d\mu} \mathbb{E}[X_q^{(k)}] = k \mu^{k-1} \mathbb{P}(X \leq q - k) - \mu^k \mathbb{P}(X = q - k) + q^{(k)} \mathbb{P}(X = q) = k \mu^{k-1} \mathbb{P}(X \leq q - k).
\]

\[ \text{A.2.3 Proof of Lemma 4} \]

**Proof.** Using Lemma 3, it is straightforward to show that

\[
\frac{d}{d\mu} \mathbb{E}^2[X_q] = \frac{2\mathbb{E}[X_q]}{\mathbb{E}^2[X_q]} \left( \mathbb{P}(X \leq q - 1) \mathbb{E}[X_q^{(2)}] - \mu \mathbb{E}[X_q] \mathbb{P}(X \leq q - 2) \right).
\]

(39)

Noting that $\mu^k \mathbb{P}(X = q - k) = q^{(k)} \mathbb{P}(X = q)$, we have

\[
\mathbb{E}[X_q^{(2)}] = q^{(2)} \mathbb{P}(X \geq q + 1) + \sum_{n=0}^{q} n^{(2)} \mathbb{P}(X = n) = q^{(2)} \mathbb{P}(X \geq q + 1) + \mu^2 \mathbb{P}(X \leq q - 2),
\]

(40)

and

\[
\mathbb{E}[X_q] = q \mathbb{P}(X \geq q + 1) + \sum_{n=0}^{q} n \mathbb{P}(X = n) = q \mathbb{P}(X \geq q + 1) + \mu \mathbb{P}(X \leq q - 1).
\]

(41)
Substituting Eqs. (40) and (41) in Eq. (39), and using $\mu \mathbb{P}(X = n - 1) = n \mathbb{P}(X = n)$, we have

$$
\frac{d}{d\mu} \mathbb{E}^2 \left[ X_q \right] = \frac{2q \mathbb{E} \left[ X_q \right] \mathbb{P}(X \geq q + 1)}{\mathbb{E}^2 \left[ X_q^{(2)} \right]} \frac{((q - 1)\mathbb{P}(X \leq q - 1) - \mu \mathbb{P}(X \leq q - 2))}{\mathbb{P}(X \leq q - 1)}
$$

$$
= \frac{2q \mathbb{E} \left[ X_q \right] \mathbb{P}(X \geq q + 1)}{\mathbb{E}^2 \left[ X_q^{(2)} \right]} \sum_{n=0}^{q-1} (q - 1 - n) \mathbb{P}(X = n) > 0,
$$

which implies that $\frac{\mathbb{E}^2 \left[ X_q \right]}{\mathbb{E} \left[ X_q^{(2)} \right]}$ is increasing in $\mu$. Moreover, using Lemma 3,

$$
\lim_{\mu \downarrow 0} \frac{\mathbb{E}^2 \left[ X_q \right]}{\mathbb{E} \left[ X_q^{(2)} \right]} = \lim_{\mu \downarrow 0} \frac{\mathbb{E}^2 \left[ (V_q/\mu) \wedge 1 \right]}{\mathbb{E} \left[ (V_q/\mu)^2 \wedge 1 \right]} = 1.
$$

Thus, $\mathbb{E}^2 \left[ X_q \right] > \mathbb{E} \left[ X_q^{(2)} \right]$. \hfill \Box

A.2.4 Proof of Lemma 5

Proof. It can be easily verified that

$$
\frac{d}{d\mu} \mathbb{E} \left[ X \mid X \in A \right] = \frac{d}{d\mu} \sum_{x \in A} x \frac{\mu^x}{x!} = \sum_{x \in A} \frac{x^2 \mu^{x-1}}{x!} \sum_{x \in A} \frac{\mu^x}{x!} - \sum_{x \in A} \frac{x \mu^x}{x!} \sum_{x \in A} \frac{\mu^x}{x!} \sum_{x \in A} \frac{\mu^x}{x!} = \frac{1}{\mu} \mathbb{V}_{A|\mathbb{R}} \left[ X \mid X \in A \right] \geq 0.
$$

\hfill \Box

A.2.5 Proof of Lemma 6

Proof. Using Lemma 3, it is straightforward to obtain

$$
\frac{d}{d\mu} \mathbb{E}^3 \left[ X_q \right] = \frac{3 \mathbb{E}^2 \left[ X_q \right]}{\mathbb{E}^2 \left[ X_q^{(2)} \right]} \left( \mathbb{P}(X \leq q - 1) \mathbb{E} \left[ X_q^{(3)} \right] - \mu^2 \mathbb{E} \left[ X_q \right] \mathbb{P}(X \leq q - 2) \right).
$$

Noting that $\mu^k \mathbb{P}(X = q - k) = q^{(k)} \mathbb{P}(X = q)$, we have

$$
\mathbb{E} \left[ X_q^{(3)} \right] = (q^2 - 1) \mathbb{P}(X \geq q + 2) + \sum_{n=0}^{q+1} n^{(3)} \mathbb{P}(X = n) = (q^2 - 1) \mathbb{P}(X \geq q + 2) + \mu^3 \mathbb{P}(X \leq q - 2),
$$

(43)
\[ \mathbb{E} [X_q] = q \mathbb{P}(X \geq q + 1) + \sum_{n=0}^{q} n \mathbb{P}(X = n) = q \mathbb{P}(X \geq q + 1) + \mu \mathbb{P}(X \leq q - 1). \] (44)

Substituting Eqs. (43) and (44) in Eq. (42), we have

\[
\frac{d}{d\mu} \frac{\mathbb{E}^3 [X_q]}{\mathbb{E}^2 \left[ X^{(3)}_{q+1} \right]} = 3\mu q \mathbb{E}^2 [X_q] \mathbb{P}(X \geq q + 2) \mathbb{P}(X \leq q - 2) \left( \frac{(q^2 - 1) \mathbb{P}(X \leq q - 1)}{\mu \mathbb{P}(X \leq q - 2)} - \frac{\mu \mathbb{P}(X \geq q + 1)}{\mathbb{P}(X \geq q + 2)} \right). \] (45)

From Lemma 5, we note that

\[
\frac{\mu \mathbb{P}(X \leq q - 2)}{\mathbb{P}(X \leq q - 1)} = \frac{\mu}{\sum_{x=0}^{q-2} \frac{\mu^x}{x!}} = \frac{\sum_{x=0}^{q-1} \frac{\mu^x}{x!}}{\sum_{x=0}^{q-1} \frac{\mu^x}{x!}} = \mathbb{E} [X | X \leq q - 1]
\]

is increasing in \(\mu\) (from 0 to \(q - 1\)). Likewise,

\[
\frac{\mu \mathbb{P}(X \geq q + 1)}{\mathbb{P}(X \geq q + 2)} = \mathbb{E} [X | X \geq q + 2]
\]

is increasing in \(\mu\) (from \(q + 2\) to \(\infty\)). Therefore, the last expression in parenthesis in Eq. (45) is decreasing in \(\mu\), positive (and unbounded) for small \(\mu\) and negative (and unbounded) for large \(\mu\), which implies that there exists some \(\mu'\), \(\mathbb{E}^3 [X_q] / \mathbb{E} \left[ X^{(3)}_{q+1} \right] \) is increasing on \((0, \mu')\) and decreasing on \((\mu', \infty)\). Moreover, using Lemma 3,

\[
\lim_{\mu \downarrow 0} \frac{\mathbb{E}^3 [X_q]}{\mathbb{E} \left[ X^{(3)}_{q+1} \right]} = \lim_{\mu \downarrow 0} \frac{\mathbb{E}^3 [(V_q / \mu) \wedge 1]}{\mathbb{E} [(V_{q-1} / \mu)^3 \wedge 1]} = 1,
\]

and

\[
\lim_{\mu \to \infty} \frac{\mathbb{E}^3 [X_q]}{\mathbb{E} \left[ X^{(3)}_{q+1} \right]} = \frac{q^3}{(q + 1)^3} = \frac{q^2}{q^2 - 1} > 1.
\]

Thus, \(\mathbb{E}^3 [X_q] > \mathbb{E} \left[ X^{(3)}_{q+1} \right]\) for all \(\mu > 0\). \(\square\)

### A.2.6 Proof of Theorem 7

**Proof.** The proof follows from the proof of the counterpart result in [Wei et al., 2020] which considers pure consolidation policies only; but, it is given here for the sake of completeness. We consider a fixed \(\mathbb{E} [L^C]\) and use the following notations for the corresponding policy
parameters under this $\mathbb{E}[L^C]$ value: QP with parameter $q$, TP with parameter $T$, HP with parameters $q_H$ and $T_H$. Recalling the $\mathbb{E}[L^C]$ expressions in Table 1, we note that, by assumption,

\[ \frac{1}{\lambda} \mathbb{E}[Y_{qh}] = \frac{q}{\lambda}, \tag{46} \]

and

\[ \frac{1}{\lambda} \mathbb{E}[Y_{qh}] = T. \tag{47} \]

Next, recalling the results in Table 2 and reiterating the assumption of fixed $\mathbb{E}[L^C]$ values for all the policies of interest, we proceed with showing that 

\[ (q - 1)q < \mathbb{E}[Y_{qh}(Y_{qh} - 1)], \]

and

\[ \mathbb{E}[Y_{qh}(Y_{qh} - 1)] < \lambda^2 T^2. \]

In fact, recalling the assumption in Eq. (46),

\[ \mathbb{E}[Y_{qh}(Y_{qh} - 1)] = \mathbb{E}[Y_{qh}^2] - q = \text{VAR}[Y_{qh}] + \mathbb{E}^2[Y_{qh}] - q > q^2 - q. \]

From Lemma 4 and recalling the assumption in Eq. (47), we have

\[ \mathbb{E}[Y_{qh}(Y_{qh} - 1)] < \mathbb{E}^2[Y_{qh}] = \lambda^2 T^2. \]

The proof of the second part of the theorem is straightforward, and, hence, it is omitted.

\[ \square \]

### A.2.7 Proof of Theorem 9

**Proof.** We consider a fixed $\mathbb{E}[L^C]$ and use the following notation for the corresponding policy parameters under this $\mathbb{E}[L^C]$ value: QP with parameter $q$, TP with parameter $T$, HP with parameters $q_H$ and $T_H$. Recalling the $\mathbb{E}[L^C]$ expressions in Table 1, we note that, by assumption,

\[ \frac{q}{\lambda} = T, \tag{48} \]

and

\[ \frac{\mathbb{E}[Y_{qh}]}{\lambda} = T. \tag{49} \]

Next, recalling the results in Table 2 and reiterating the assumption of fixed $\mathbb{E}[L^C]$ value for all the policies of interest, we proceed with showing that

\[ \frac{q^2 - q}{3\lambda^2} < \lambda T^3 / 3, \tag{50} \]
and
\[ \frac{E[Y_{q_H+1}^{(3)}]}{3\lambda^2} < \lambda T^3/3. \] (51)

Recalling the assumption in Eq. (48), we can easily see that inequality (50) holds. From Lemma 6 and recalling the assumption in Eq. (49), we have
\[ E[Y_{q_H+1}^{(3)}] < E^3[Y_{q_H}] = (\lambda T)^3, \]
which verifies inequality (51).

We need to demonstrate that under a fixed expected consolidation cycle length \( E[L^C] \), neither \( AOSD_{QP} \) nor \( AOSD_{HP} \) dominates the other one. To see this, letting \( q = E[Y_{q_H}] \), we only need to compare \( q^3 - q \) with \( E[Y_{q_H+1}^{(3)}] \). The numerical tests demonstrate that \( q^3 - q > E[Y_{q_H+1}^{(3)}] \) when \( q_H \) is small, but \( q^3 - q < E[Y_{q_H+1}^{(3)}] \) when \( q_H \) is large. For example, let the arrival process \( N(t) \) be a Poisson process with rate 1. We select \( q = 5 \) as the parameter under QP, \( q_H = 6 \) and \( T_H = 5.9199 \) as the parameters under HP. We can numerically verify that QP and HP have the same expected consolidation cycle length, i.e., \( q = E[Y_{q_H}] \). In this case, \( q^3 - q = 120 \), and \( E[Y_{q_H+1}^{(3)}] = E[Y_{q_H+1}(Y_{q_H+1} - 1)(Y_{q_H+1} - 2)] = 112.8573 \). In this example, we demonstrate that under a fixed expected consolidation cycle length, QP does not necessarily achieve smaller \( AOSD \) than HP, which is different from the \( AOD \) case.

**A.2.8 Proof of Theorem 11**

**Proof.** We consider fixed \( E[L^C] \) and \( E[L^R] \), and all possible policies under the \( E[L^C] \) and \( E[L^R] \) values. Recalling the \( E[L^C] \) and \( E[L^R] \) expressions in Table 1 we note that, by assumption,
\[ \lambda T = E[Y_{q_H}] = q, \quad Q_T + 1 \approx Q_H + 1 \approx nq. \] (52)

Next, recalling the \( AIR \) expressions in Table 2 it’s sufficient to show
\[ \frac{Q_T(2\lambda T + Q_T + 1)}{2(Q_T + 1)} \approx \frac{Q_H(2E[Y_{q_H}] + Q_H + 1)}{2(Q_H + 1)}, \] (53)
and
\[ \frac{Q_H(2E[Y_{q_H}] + Q_H + 1)}{2(Q_H + 1)} \geq \frac{(n - 1)q}{2}. \] (54)
Obviously, Eq. (53) follows from Eq. (52). Moreover, using Eq. (52), we have

\[
\frac{Q_H (2\mathbb{E}[Y_{qH}] + Q_H + 1)}{2(Q_H + 1)} - \frac{(n - 1)q}{2} \approx \frac{(nq - 1)(n + 2)}{2n} - \frac{(n - 1)q}{2} \\
= \frac{2(nq - 1) + n(q - 1)}{2n} > 0,
\]

which implies Eq. (54) holds. In sum, we have \( AIR_{TP} \approx AIR_{HP} \geq AIR_{QP} \).

The proof of the second part of the theorem is straightforward, and, hence, it is omitted.

\[\square\]

A.2.9 Proof of Theorem 14

Proof. Considering a fixed replenishment cycle length \( \mathbb{E}[L^R] \) and a fixed expected consolidation cycle length \( \mathbb{E}[L^C] \), and recalling the results in Table 1, we have

\[\mathbb{E}[K_T] = \mathbb{E}[K_H] = n,\]

\[\mathbb{E}[RCost_{QP}] = \mathbb{E}[RCost_{HP}] = \mathbb{E}[RCost_{TP}], \quad \text{and} \]

\[\mathbb{E}[DCost_{QP}] = \mathbb{E}[DCost_{HP}] = \mathbb{E}[DCost_{TP}].\]

Recalling the average cost rate

\[AC_p = \frac{\mathbb{E}[HCost_p] + \mathbb{E}[WCost_p] + \mathbb{E}[RCost_p] + \mathbb{E}[DCost_p]}{\mathbb{E}[L^R]}, \quad p = QP, TP, HP,\]

along with Theorems 7 and 11 it can be easily shown that \( AC_{QP} \preceq AC_{HP} \preceq AC_{TP} \). \[\square\]