DIFFERENTIAL SUBORDINATIONS FOR FUNCTIONS WITH POSITIVE REAL PART USING ADMISSIBILITY CONDITIONS

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Abstract. Some sufficient conditions on certain constants which are involved in some first, second and third order differential subordinations associated with certain functions with positive real part like modified Sigmoid function, exponential function and Janowski function are obtained so that the analytic function $p$ normalized by the condition $p(0) = 1$, is subordinate to Janowski function. The admissibility conditions for Janowski function are used as a tool in the proof of the results. As application, several sufficient conditions are also computed for Janowski starlikeness.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions $f$ on the open unit disc $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. An analytic function defined on $\mathbb{D}$ is univalent if $f$ is one-to-one in $\mathcal{A}$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of all univalent functions. Denote the class of all analytic functions $f$ having Taylor series expansion $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, for some $a \in \mathbb{C}$ and fixed integer $n$ by $\mathcal{H}[a, n]$. Let $f$ and $g$ be analytic in $\mathbb{D}$. The function $f$ is subordinate to $g$, and write $f \prec g$, if there exists an analytic function $w: \mathbb{D} \to \mathbb{D}$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if $g \in \mathcal{U}$ then, $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Let $\mathcal{P}$ be the class of functions with positive real part of the form $p(z) = 1 + c_1 z + c_2 z + \cdots$ over $\mathbb{D}$. Let $A$ and $B$ be arbitrary fixed numbers which are satisfying the inequality $-1 \leq B < A \leq 1$, then the analytic function $p \in \mathcal{P}$ is known as the Janowski functions associated with right half plane if it satisfies the subordination relation $p(z) \prec (1 + Az)/(1 + Bz)$ for all $z \in \mathbb{D}$. The class of such functions is denoted by $\mathcal{P}[A, B]$. Let $\mathcal{S}^*[A, B]$ be the class of Janowski starlike functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \in \mathcal{P}[A, B]$ for $z \in \mathbb{D}$, introduced by Janowski [17]. Let $\mathcal{S}^*[A, B]$ be the class of the functions $f \in \mathcal{A}$ such that the quantity $zf'(z)/f(z)$ lies in the region $\Delta = \{ w \in \mathbb{C} : |(w - 1)/(A - Bw)| < 1 \}$. As a special case, we note that $\mathcal{S}^*[1 - 2\alpha, -1] = \mathcal{S}^*(\alpha)$ that contains starlike functions of order $\alpha$ [16, 31]. In 2015, authors [24] introduced the class $\mathcal{S}^*_c$ which contains the functions $f \in \mathcal{A}$ satisfying the subordination relation $zf'(z)/f(z) \prec e^z$ for all $z \in \mathbb{D}$. In addition, if $f \in \mathcal{S}^*_c$, then the quantity $zf'(z)/f(z)$ lies in the domain $\{ w \in \mathbb{C} : |\log w| < 1 \}$. Recently, Goel and Kumar [14] introduced and studied the class $\mathcal{S}^*_SG$ which contains starlike functions associated with modified sigmoid function $\phi_{SG}(z) := 2/(1 + e^{-z})$ and satisfy the subordination relation $zf'(z)/f(z) \prec \phi_{SG}$ for all $z \in \mathbb{D}$. In similar way, if the
function \( f \in S_{SG} \), then the quantity \( zf'(z)/f(z) \) lies in the domain \{ \( w \in \mathbb{C} : |\log w/(2-w)| < 1 \} \). For details, see [11, 2, 3].

Goluzin [15] studied initially the first order differential subordination \( zp'(z) < zq'(z) \), whenever \( zq'(z) \) is convex, the subordination \( p < q \) holds and the function \( q \) is the best dominant. After this basic result, many authors established several generalizations of differential subordination implications. In 1981, an article titled ”Differential subordination and univalent functions” by Miller and Mocanu [25] commenced the study of differential subordination as a generalized version of differential inequalities. For more details, see [8, 22, 29, 26]. In 1989, Nunokawa et al. [28] studied the first order differential subordination and proved that \( 1+z p'(z) < 1+z \) implies \( p(z) < 1+z \). They used this result to provide a criterion so that a normalized analytic function is univalent in \( \mathbb{D} \). Then, Ali et al. [4] generalized this result and proved that \( p(z) \) is subordinate to the Janowski function whenever \( 1+\beta z p'(z)/p'(z) < (1+Dz)/(1+Ez) \) for \( j = 0, 1, 2 \). Here, \( A, B, D, E \in [-1, 1] \). Further, Ali et al. [5] determined the estimate on \( \beta \) so that the subordination \( 1+\beta z p'(z)/p'(z) \) is subordinate to the function \( \sqrt{1+z} \), \( (j=0, 1, 2) \) which implies that \( p(z) \) is subordinate to \( \sqrt{1+z} \). Later, Kumar et al. [21] computed a bound on \( \beta \) so that \( p(z) \prec \sqrt{1+z} \), whenever \( 1+\beta z p'(z)/p'(z) \prec (1+Dz)/(1+Ez) \), \( (j=0, 1, 2) \) with \( |D| \leq 1 \) and \( -1 < E < 1 \). Some of these results were not sharp. Also, it was difficult to establish analogous results for certain functions with positive real parts such as \( \phi_0(z) := 1 + (z/k)((k+z)/(k-z)) \), \( (k = 1 + \sqrt{2}) \), \( \phi_{sin}(z) := 1 + \sin z \), \( Q(z) := e^{e^{-z}} \) by the approach used in above discussed research work. Later in 2018, Kumar and Ravichandran [19] used some different approach and were able to established best possible bounds on \( \beta \) so that \( 1+\beta z p'(z)/p'(z) \) is subordinate to \( \sqrt{1+z}, (1+A z)/(1+B z) \) which implies that \( p(z) \prec e^z, (1+A z)/(1+B z) \). In 2018, Ahuja et al. [1] also obtained sharp subordination implications results for the functions associated with lemniscate of Bernoulli. For recent work related to first order differential subordinations, reader may refer [9, 10, 12, 13, 32, 33].

In 2018, Madaan et al. [23] established first and second order differential subordinations associated with the lemniscate of Bernoulli using admissibility technique. Further, Anand et al. [7] also studied the generalized first order differential subordination for the Janowski functions. In 2019, Dorina Raducanu [30] established second order differential subordination implications associated with generalized Mittag-Leffler function. For related work, readers may see [20, 27, 18].

Motivated by the aforesaid work, using admissibility conditions for Janowski functions, we determine certain conditions on \( \beta, \gamma, A \) and \( B \) where \( -1 \leq B < A \leq 1 \) so that \( p \) belongs to the class \( \mathcal{P}[A, B] \) whenever \( 1+\beta z p'(z)/p'(z) < (1+\alpha p(z))p(k(z)), (1-\alpha) p(z)+\alpha p^2(z)+\beta z p'(z)/p'(z) \) \( (\alpha \in [0, 1]) \), \( (1/p(z)) - \beta z p'(z)/p'(z), p(z)+z p'(z)/(\beta p(z)+\gamma)^k \) \( (\gamma > 0) \), \( 1+\gamma z p'(z)+\beta z p''(z) \) and \( p(z)+\gamma z p'(z)+\beta z p''(z) \) are subordinate to some functions with positive real part like \( e^z, 2/(1+e^{-z}) \) and \( (1+A z)/(1+B z) \), where \( k \) is a positive integer. Certain implications of these results are also discussed which gives sufficient conditions for an analytic function \( f \) to be in the class \( S^*[A, B] \).

2. The Admissibility Condition

This section provides some basic facts related to admissibility conditions associated with Janowski function that will be needed in the proving our main results.
Definition 2.1. Let \( \psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{D} \) be analytic and \( h \in \mathbb{S} \). Then a function \( p \in \mathcal{A} \), satisfying following differential subordination relation
\[
\psi(p(z), z p'(z), z^2 p''(z); z) \prec h(z)
\]
is called its solution.

Let \( \Omega \) denote the class of all functions \( q \in \mathbb{S} \) defined on \( \overline{\mathbb{D}} \setminus \mathbb{E}(q) \), where \( \mathbb{E} = \{ \zeta \in \partial \mathbb{D} : \lim_{z \to \zeta} q(z) = \infty \} \) such that \( q'(\zeta) \neq 0 \) for \( \zeta \in \partial \mathbb{D} \setminus \mathbb{E}(q) \).

Definition 2.2. Let \( \Omega \subset \mathbb{C}, q \in \mathcal{Q} \) and \( n \geq 1 \). Consider the class of admissible functions \( \Psi_n[\Omega, q] \), consists of those functions \( \psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C} \) which satisfy the admissibility condition:
\[
\psi(r, s, t; z) \notin \Omega
\]
whenever
\[
r = q(\zeta), s = m \zeta q'(\zeta) \text{ and } \Re\left(\frac{t}{s} + 1\right) \geq m \Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)
\]
for \( z \in \mathbb{D}, \zeta \in \partial \mathbb{D} \setminus \mathbb{E}(q) \) and \( m \geq n \geq 1 \).

We write the class \( \Psi_1[\Omega, q] \) as \( \Psi[\Omega, q] \).

Theorem 2.3. [25] Theorem 2.3b, p.28] Let the function \( \psi \in \Psi_n[\Omega, q] \) and \( q(0) = a \). If \( p \in \mathcal{H}[a, n] \), then
\[
\psi(p(z), z p'(z), z^2 p''(z); z) \in \Omega \implies p(z) \prec q(z).
\] (2.1)

Let \( \Omega \) be a simply connected domain which is not the entire complex plane, then there exists a conformal mapping \( h \) from \( \mathbb{D} \) onto \( \Omega \) with \( h(0) = \psi(a, 0, 0; 0) \). Therefore, if \( p \in \mathcal{H}[a, n] \), equation (2.1) can be written as
\[
\psi(p(z), z p'(z), z^2 p''(z); z) \prec h(z) \implies p(z) \prec q(z).
\] (2.2)

The univalent function \( q \) is called dominant of the solutions of the differential subordination (2.2). The function \( \tilde{q} \) is called the best dominant of (2.2) if \( \tilde{q} \prec q \) for all dominants of (2.2).

Consider the function \( q(z) = (1 + A z) / (1 + B z) \) for \(-1 \leq A < B \leq 1 \). Denote the class \( \Psi_n[\Omega, (1 + A z) / (1 + B z)] \) by \( \Psi_n[\Omega; A, B] \). Therefore, the admissibility conditions for the function \( q \) are given as follows:

Theorem 2.4. [7] Let the function \( p \in \mathcal{H}[1, n] \) such that \( p(z) \neq 1 \) and \( n \geq 1 \) and \( \Omega \) be a subset of \( \mathbb{C} \). The class \( \Psi_n[\Omega; A, B] \) is defined as the class of all those functions \( \psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C} \) such that
\[
\psi(r, s, t; z) \notin \Omega \quad \text{whenever} \quad (r, s, t; z) \in \text{Dom } \psi \quad \text{and}
\]
\[
r = q(\zeta) = \frac{1 + A e^{i \theta}}{1 + B e^{i \theta}}, s = m \zeta q'(\zeta) = \frac{m(A - B)e^{i \theta}}{(1 + B e^{i \theta})^2} \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \geq \frac{m(1 - B^2)}{1 + B^2 + 2B \cos \theta}
\]
for \( z \in \mathbb{D}, \theta \in (0, 2\pi) \) and \( m \geq 1 \).
On taking \( \psi \in \Psi_n[\Omega; A, B] \) in Theorem (2.4), we have

**Corollary 2.5.** If \( (p(z), z p'(z), z^2 p''(z); z) \in \text{Dom}\psi \) and \( \psi(p(z), z p'(z), z^2 p''(z); z) \in \Omega \) for \( z \in \mathbb{D} \), then \( p \) belongs to \( \mathcal{P}[A, B] \).

For \( \zeta = e^{i\theta} \), where \( \theta \in [0, 2\pi) \), let us consider

\[
|q(\zeta)| = \sqrt{\frac{1 + A^2 + 2A \cos \theta}{1 + B^2 + 2B \cos \theta}} := k(\theta) \tag{2.3}
\]

whose minimum value is \( \frac{1 - A}{1 - B} \), attained at \( \theta = \pi \). Also, observe that

\[
|q'(\zeta)| = \frac{A - B}{1 + B^2 + 2B \cos \theta} := d(\theta) \tag{2.4}
\]

and the minimum value of \( d(\theta) \) is \( d(0) = \frac{A - B}{1 + B^2 + 2B} \) for \( B > 0 \) and \( d(\pi) = \frac{A - B}{1 + B^2 - 2B} \) for \( B < 0 \). Note that

\[
\text{Re} \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) = \frac{-2B(B + \cos \theta)}{1 + B^2 + 2B \cos \theta} := g(\theta) \tag{2.5}
\]

and the minimum value of \( g(\theta) \) is \( g(0) = \frac{-2B(B + 1)}{1 + B^2 + 2B} \) for \( B > 0 \) and \( g(\pi) = \frac{-2B(B - 1)}{1 + B^2 - 2B} \) for \( B < 0 \).

Using above values, we get the admissibility condition for third order differential subordination as follows:

In order to prove our main results, we will use the following lemmas extensively.

**Lemma 2.6.** [20] Let \( z \) be a complex number. Then

\[
|\log(1 + z)| \geq 1 \quad \text{if and only if} \quad |z| \geq e - 1.
\]

**Lemma 2.7.** Consider the disc \( \Delta_{\beta} = \{w \in \mathbb{C} : |w| < \beta, 0 < \beta \leq 1\} \). Then the inequality

\[
\left| \log \left( \frac{w}{2 - w} \right) \right| \geq 1
\]

holds if and only if \( |w| \geq \beta_0 \approx 0.473519 \), where \( \beta_0 \) is the positive real root of the equation

\[
(e^2 - 1)\beta^4 - 2(e^2 - 4)\beta^3 + 4(e^2 - 6)\beta^2 + 32\beta - 16 = 0.
\]

**Proof.** For \( \theta \in [0, 2\pi] \), let \( w = \beta e^{i\theta} \) be a boundary point of the disc \( \Delta_{\beta} \). Consider

\[
\left| \log \left( \frac{\beta e^{i\theta}}{2 - \beta e^{i\theta}} \right) \right|^2 = \frac{1}{2} \log \left( \frac{4\beta^2 + 4\beta^3 \cos \theta}{(\beta^2 - 4\beta \cos \theta)^2} \right) + i \arctan \left( \frac{2\beta \sin \theta}{2\beta \cos \theta - \beta^2} \right)^2
\]

\[
= \left( \log \left( \frac{4\beta^2 + 4\beta^3 \cos \theta}{4\beta^2 - 4\beta \cos \theta} \right) \right)^2 + \left( \arctan \left( \frac{2\sin \theta}{2\cos \theta - \beta} \right) \right)^2
\]

\[
:= f(\beta, \theta)
\]

In the interval \([0, 2\pi]\), the function \( f(\beta, \theta) \) attains its absolute minimum at \( \theta = 0 \) and therefore, \( f(\beta, \theta) \geq f(\beta, 0) \) for all \( \theta \in [0, 2\pi] \). Thus, the inequality

\[
\left| \log \left( \frac{\beta e^{i\theta}}{2 - \beta e^{i\theta}} \right) \right| \geq 1
\]
holds if and only if

$$f(\beta, 0) \geq 1$$

or equivalently,

$$(e^2 - 1)\beta^4 - 2(e^2 - 4)\beta^3 + 4(e^2 - 6)\beta^2 + 32\beta - 16 \geq 0. \quad (2.6)$$

Therefore, by Intermediate Value Theorem, the inequality (2.6) holds for \( \beta \geq \beta_0 \approx 0.473519 \) which is the positive real root of the equation in (2.6). \( \square \)

3. First Order Differential Subordination

In order to prove first order differential subordination relations, we need the following result due to Swati et al. [19], which is a specific case of Theorem (2.4).

**Theorem 3.1.** Let \( p \in \mathcal{H}[1, n] \) such that \( p(z) \neq 1 \) and \( n \geq 1 \). Let \( \Omega \) be a set in \( \mathbb{C} \). The class \( \Psi_n[\Omega; A, B] \) is defined as the class of all those functions \( \psi : \mathbb{C}^2 \times \mathbb{D} \rightarrow \mathbb{C} \) such that

$$\psi(r, s; z) \notin \Omega \quad \text{whenever} \quad (r, s; z) \in \text{Dom} \psi \quad \text{and}$$

$$r = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad \text{and} \quad s = \frac{m(A - B)e^{i\theta}}{(1 + Be^{i\theta})^2} \quad (3.1)$$

for \( z \in \mathbb{D}, \theta \in (0, 2\pi) \) and \( m \geq 1 \).

Consequently, when \( \psi \in \Psi_n[\Omega; A, B] \), the above theorem becomes: Let \( z \in \mathbb{D} \). If \( (p(z), zp'(z); z) \in \text{Dom} \psi \) and \( \psi(p(z), zp'(z); z) \in \Omega \), then \( p \in \mathcal{P}[A, B] \).

Using above theorem, we determine the conditions on \( A, B \) and \( \beta \) so that the function \( \psi(p(z), zp'(z); z) \) is subordinate to Modified Sigmoid function \( \phi_{SG} \) and exponential function \( e^z \) implies \( p(z) \) is subordinate to \( (1 + Az)/(1 + Bz) \).

**Theorem 3.2.** Let \(-1 \leq B < A \leq 1 \) and \( k \) be a non-negative integer. Let \( p \) be an analytic function defined on \( \mathbb{D} \), which satisfies \( p(0) = 1 \) and \( \beta_0 \approx 0.475319 \). Then the following are sufficient for \( p \in \mathcal{P}[A, B] \).

(a) \( 1 + \beta \frac{zp'(z)}{p(z)} \prec \phi_{SG} \), where \( |\beta| \geq \begin{cases} \frac{\beta_0(1+|A|)^k(1+|B|)^{2-k}}{(A-B)} & \text{when } 0 \leq k \leq 2 \\ \frac{\beta_0(1+|A|)^k}{(A-B)(1+|B|)^k - 2} & \text{when } k > 2 \end{cases} \)

(b) \( 1 + \beta \frac{zp'(z)}{p(z)} \prec e^z \), where \( |\beta| \geq \begin{cases} \frac{(e-1)(1+|A|)^k(1+|B|)^{2-k}}{(A-B)} & \text{when } 0 \leq k \leq 2 \\ \frac{(e-1)(1+|A|)^k}{(A-B)(1+|B|)^k - 2} & \text{when } k > 2 \end{cases} \)

**Proof.** (a) Let \( \Omega = \phi_{SG}(\mathbb{D}) = \{ w \in \mathbb{C} : |\log \left( \frac{w}{2-w} \right) | < 1 \} \). Consider the analytic function \( \psi : \mathbb{C} \setminus \{0\} \times \mathbb{C} \times \mathbb{D} \rightarrow \mathbb{C} \) defined as

$$\psi(r, s; z) = 1 + \beta \frac{s}{r^k}.$$

In accordance with Theorem (3.1), \( \psi \in \Psi[\Omega; A, B] \), if \( \psi(r, s; z) \notin \Omega \) where \( r \) and \( s \) are given in the equation (3.1). Therefore, it is enough to show that the required subordination holds if

$$\left| \log \left( \frac{\psi(r, s; z)}{2 - \psi(r, s; z)} \right) \right| \geq 1. \quad (3.2)$$
(i) When \( 0 \leq k \leq 2 \), let us consider

\[
|\psi(r, s; z)| = \left| 1 + \beta \frac{m(A - B)e^{i\theta}}{(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{2-k}} \right|
\]

\[
= \frac{\left| (1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{2-k} + \beta m(A - B)e^{i\theta} \right|}{(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{2-k}}
\]

\[
\geq \frac{|\beta|m(A - B) - |(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{2-k}|}{|(1 + Ae^{i\theta})^k|| (1 + Be^{i\theta})^{2-k}|}
\]

\[
\geq \frac{|\beta|m(A - B) - (1 + |A|)^k(1 + |B|)^{2-k}}{(1 + |A|)^k(1 + |B|)^{2-k}}
\]

\[
=: \phi(m).
\]

By First Derivative Test, \( \phi(m) \) is an increasing function for \( m \geq 1 \). This implies \( \phi(m) \geq \phi(1) \) for all \( m \geq 1 \). Hence, the last inequality reduces to

\[
|\psi(r, s; z)| \geq \phi(1)
\]

where

\[
|\phi(1)| = \frac{|\beta|(A - B) - (1 + |A|)^k(1 + |B|)^{2-k}}{(1 + |A|)^k(1 + |B|)^{2-k}}.
\]

Using Lemma (2.7), the inequality (3.2) is true if

\[
|\phi(1)| \geq \beta_0
\]

or

\[
|\beta|(A - B) - (1 + |A|)^k(1 + |B|)^{2-k} \geq \beta_0(1 + |A|)^k(1 + |B|)^{2-k},
\]

which shows that \( \psi(r, s; z) \notin \Omega \) for \( |\beta| \geq \beta_0 \frac{(1 + |A|)^k(1 + |B|)^{2-k}}{(A - B)} \).

(ii) When \( k > 2 \), observe that

\[
|\psi(r, s; z)| = \left| 1 + \beta \frac{m(A - B)(1 + Be^{i\theta})^{k-2}e^{i\theta}}{(1 + Ae^{i\theta})^k} \right|
\]

\[
= \frac{\left| (1 + Ae^{i\theta})^k + \beta m(A - B)(1 + Be^{i\theta})^{k-2}e^{i\theta} \right|}{(1 + Ae^{i\theta})^k}
\]

\[
\geq \frac{|\beta|m(A - B)|(1 + Be^{i\theta})^{k-2} - |(1 + Ae^{i\theta})^k|}{|(1 + Ae^{i\theta})^k|}
\]

\[
\geq \frac{|\beta|m(A - B)(1 - |B|)^{k-2} - (1 + |A|)^k}{(1 + |A|)^k}
\]

\[
=: \phi(m).
\]

As in the earlier case, note that \( \phi(m) \) is an increasing function of \( m \). Hence, \( \phi(m) \geq \phi(1) \) for all \( m \geq 1 \). By Lemma (2.7), (3.2) holds if

\[
|\psi(r, s; z)| \geq |\phi(1)| \geq \beta_0
\]
where
\[ \phi(1) = \frac{|\beta|(A - B)(1 - |B|)k - (1 + |A|)^k}{(1 + |A|)^k}. \]

Therefore, for \( |\beta| \geq \beta_0 \frac{(1 + |A|)^k}{(A - B)(1 - |B|)k - (1 + |A|)^k} \), \( \psi \in \Psi[\Omega; A, B] \) and hence, \( p \in \mathcal{P}[A, B] \).

(b) Consider the domain \( \Omega = \{ w \in \mathbb{C} : |\log w| < 1 \} \). Let \( \psi : \mathbb{C} \setminus \{0\} \times \mathbb{C} \times \mathbb{D} \to \mathbb{C} \) be defined by
\[ \psi(r, s; z) = 1 + \beta \frac{s}{r^k}. \]

By Theorem (3.1), \( \psi \) belongs to \( \Psi[\Omega; A, B] \) if \( \psi(r, s; z) / \in \Omega \) for \( z \in \mathbb{D} \). This implication holds if
\[ |\log(\psi(r, s; z))| \geq 1 \]

Since
\[ |\log(\psi(r, s; z))| = \left| \log \left( 1 + \beta \frac{s}{r^k} \right) \right| = \left| \log \left( 1 + \beta \frac{m(A - B)e^{i\theta}(1 + Be^{i\theta})^k}{(1 + Be^{i\theta})^2(1 + Ae^{i\theta})^k} \right) \right|, \]

by Lemma (2.6), inequality (3.3) holds if and only if
\[ \left| \frac{\beta m(A - B)e^{i\theta}(1 + Be^{i\theta})^k}{(1 + Be^{i\theta})^2(1 + Ae^{i\theta})^k} \right| \geq e - 1. \]

(i) When \( 0 \leq k \leq 2 \), consider
\[ \left| \frac{\beta m(A - B)e^{i\theta}}{(1 + Be^{i\theta})^{2-k}(1 + Ae^{i\theta})^k} \right| \geq \left| \frac{\beta |m(A - B)|}{(1 + Be^{i\theta})^{2-k}(1 + Ae^{i\theta})^k} \right| \geq \left| \beta |(A - B)| \right| / (1 + |B|)^{2-k}(1 + |A|)^k \geq e - 1 \]

if \( |\beta| \geq \frac{(e - 1)(1 + |A|)^k(1 + |B|)^{2-k}}{(A - B)} \),

which shows that \( \psi(r, s; z) \notin \Omega \) and hence the required result holds.

(ii) When \( k > 2 \), observe that
\[ \left| \frac{\beta m(A - B)e^{i\theta}(1 + Be^{i\theta})^{k-2}}{(1 + Ae^{i\theta})^k} \right| \geq \left| \frac{\beta |m(A - B)|(1 + Be^{i\theta})^{k-2}}{(1 + Ae^{i\theta})^k} \right| \geq \left| \beta |(A - B)(1 - |B|)^{k-2} \right| / (1 + |A|)^k \]

Therefore, (3.4) holds if
\[ \frac{|\beta|(A - B)(1 - |B|)^{k-2}}{(1 + |A|)^k} \geq e - 1 \]
or equivalently,
\[ |\beta| \geq \frac{(e-1)(1+|A|)^k}{(A-B)(1-|B|)^{k-2}}. \]
which proves that \( \psi(r, s; z) \notin \Omega \) and hence, \( p \in \mathcal{P}[A, B] \).

**Corollary 3.3.** Let \( f \in \mathcal{A} \) and \( \beta_0 \approx 0.473519 \). Set \( G(z) := \frac{f'(z)}{f(z)} - z \left( \frac{f''(z)}{f(z)} \right)^2 + \frac{zf'(z)}{f(z)}. \) If one of the following subordination holds, then \( f \in S^*[A, B] \).

(a) \( 1 + \beta \bar{z}G(z) \prec \phi_{SC} \) for \( |\beta|(A-B) \geq \beta_0(1+|B|)^2 \),
(b) \( 1 + \beta \left( \frac{f'(z)}{f(z)} \right) G(z) \prec \phi_{SC} \) for \( |\beta|(A-B) \geq \beta_0(1+|A|)(1+|B|) \),
(c) \( 1 + \frac{\beta}{z}G(z) \prec \phi_{SC} \) for \( |\beta|(A-B) \geq \beta_0(1+|A|)^2 \),
(d) \( 1 + \beta zG(z) \prec e^z \) for \( |\beta|(A-B) \geq (e-1)(1+|B|)^2 \),
(e) \( 1 + \beta \left( \frac{f'(z)}{f(z)} \right) G(z) \prec e^z \) for \( |\beta|(A-B) \geq (e-1)(1+|B|)(1+|A|) \),
(f) \( 1 + \frac{\beta}{z}G(z) \prec e^z \) for \( |\beta|(A-B) \geq (e-1)(1+|A|)^2 \).

**Theorem 3.4.** Let \(-1 \leq B < A \leq 1\), \( \beta_0 \approx 0.473519\), \( k \) be a non-negative integer and \( p \in \mathcal{A} \) such that \( p(0) = 1 \). If any of the following subordinations holds true, then \( p(z) \in \mathcal{P}[A, B] \).

(a) \( 1 + \beta \left( \frac{zp'(z)}{p(z)} \right)^2 \prec \phi_{SC} \), where \( |\beta| \geq \begin{cases} \frac{\beta_0(1+|A|)^2(1+|B|)^2}{(A-B)^2}, & \text{when } 0 \leq k \leq 4 \\ \frac{\beta_0(1+|A|)^k}{(A-B)^2(1-|B|)^{k-1}}, & \text{when } k > 4 \end{cases} \)
(b) \( 1 + \beta \left( \frac{zp'(z)}{p(z)} \right)^2 \prec e^z \), where \( |\beta| \geq \begin{cases} \frac{(e-1)(1+|A|)^2(1+|B|)^2}{(A-B)^2}, & \text{when } 0 \leq k \leq 4 \\ \frac{(e-1)(1+|A|)^k}{(A-B)^2(1-|B|)^{k-1}}, & \text{when } k > 4 \end{cases} \)

**Proof.** (a) Consider \( \Omega \) as in Theorem 3.2(a). Define the analytic function \( \psi : \mathbb{C} \setminus \{0\} \times \mathbb{C} \times \mathbb{D} \rightarrow \mathbb{C} \) as
\[ \psi(r, s; z) = 1 + \beta \frac{s^2}{r^k}. \]
Therefore, we have
\[ \psi(r, s; z) = 1 + \beta \frac{m^2(A-B)^2e^{2i\theta}(1+Be^{i\theta})^k}{(1+Be^{i\theta})^k(1+Be^{i\theta})^4}. \]
Proceeding in the similar manner as in Theorem 3.2, we have the following two cases.

(i) When \( 0 \leq k \leq 4 \), let us consider
\[
|\psi(r, s; z)| = \left| 1 + \beta \frac{m^2(A-B)^2e^{2i\theta}(1+Be^{i\theta})^k}{(1+Be^{i\theta})^k(1+Be^{i\theta})^4} \right|
\]
\[ \geq \left| \frac{m^2(A-B)^2}{(1+Be^{i\theta})^k(1+Be^{i\theta})^4} \right| - \beta \frac{|m^2(A-B)^2 - (1+Be^{i\theta})^k||(1+Be^{i\theta})^4-k|}{(1+Be^{i\theta})^k(1+Be^{i\theta})^4-k}\]
\[ \geq \left| \frac{m^2(A-B)^2}{(1+|A|)^k(1+|B|)^4-k} \right| - \beta \frac{|m^2(A-B)^2 - (1+|A|)^k(1+|B|)^4-k|}{(1+|A|)^k(1+|B|)^4-k}\]
\[= \phi(m)\]

Simple observation shows that \(\phi(m)\) in an increasing function for \(m \geq 1\). The required subordination result holds if \(\psi(r, s; z) \notin \Omega\). So, using Lemma \((2.7)\), it is concluded that \(p \in \mathcal{P}[A, B]\) if \(|\beta| \geq \frac{\beta_{0}(1+|A|)^{k}(1+|B|)^{4-k}}{(A-B)^{2}}\).

(ii) When \(k > 4\), observe that
\[
|\psi(r, s; z)| = \left|1 + \frac{\beta m^{2}(A - B)^{2}e^{2i\theta}(1 + Be^{i\theta})^{k}}{(1 + Ae^{i\theta})^{k}(1 + Be^{i\theta})^{4}}\right|
\]
\[
= \frac{|1 + Ae^{i\theta})^{k} + \beta m^{2}(A - B)^{2}e^{2i\theta}(1 + Be^{i\theta})^{k-4}|}{(1 + Ae^{i\theta})^{k}}
\]
\[
\geq \frac{|\beta| m^{2}(A - B)^{2}(1 + Be^{i\theta})^{k-4} - (1 + |A|)^{k}}{(1 + |A|)^{k}}
\]
\[
=: \phi(m)
\]

Noting that \(\phi'(m) > 0\) for \(m \geq 1\) and proceeding as in the part (i), we get the desired subordination result.

(b) Let \(\Omega = \{w \in \mathbb{C} : |\log w| < 1\}\) be the domain. Let \(\psi : \mathbb{C}\setminus\{0\} \times \mathbb{C} \times \mathbb{D} \to \mathbb{C}\) be defined as
\[
\psi(r, s; z) = 1 + \frac{\beta s^{2}}{r^{k}}.
\]

On the similar lines of the proof of Theorem \((3.2)\) and using Lemma \((2.6)\), we get the desired result if
\[
\left|\frac{\beta m^{2}(A - B)^{2}e^{2i\theta}(1 + Be^{i\theta})^{k}}{(1 + Ae^{i\theta})^{k}(1 + Be^{i\theta})^{4}}\right| \geq e - 1. \tag{3.5}
\]

(i) When \(0 \leq k \leq 4\), consider
\[
\frac{|\beta m^{2}(A - B)^{2}e^{2i\theta}}{(1 + Ae^{i\theta})^{k}(1 + Be^{i\theta})^{4-k}} \geq \frac{|\beta| m^{2}(A - B)^{2}}{|(1 + Ae^{i\theta})^{k}||(1 + Be^{i\theta})^{4-k}|}
\]
\[
\geq \frac{|\beta|(A - B)^{2}}{(1 + |A|)^{k}(1 + |B|)^{4-k}} \quad \therefore m \geq 1
\]

Now
\[
\frac{|\beta|(A - B)^{2}}{(1 + |A|)^{k}(1 + |B|)^{4-k}} \geq e - 1
\]

if and only if
\[
|\beta| \geq \frac{(e - 1)(1 + |A|)^{k}(1 + |B|)^{4-k}}{(A - B)^{2}}.
\]

(ii) When \(k > 4\), observe that
\[
\frac{|\beta m^{2}(A - B)^{2}e^{2i\theta}(1 + Be^{i\theta})^{k-4}}{(1 + Ae^{i\theta})^{k}} \geq \frac{|\beta| m^{2}(A - B)^{2}(1 - Be^{i\theta})^{k-4}}{|(1 + Ae^{i\theta})^{k}|}
\]
Let $\Omega$ be same as in Theorem (3.2)(a). Let the function $r, s$ satisfy the following subordination imply $f \in S^*[A, B]$. 

(a) $1 + \beta(z\mathcal{G}(z))^2 \prec \phi_{SG}$ for $|\beta|(A - B)^2 \geq \beta_0(1 + |B|)^4$,

(b) $1 + \beta \left( \frac{f(z)}{f'(z)} \right)^2 (\mathcal{G}(z))^2 \prec \phi_{SG}$ for $|\beta|(A - B)^2 \geq \beta_0(1 + |A|)(1 + |B|)^3$,

(c) $1 + \beta \left( \frac{f(z)}{f'(z)} \right)^2 (\mathcal{G}(z))^2 \prec \phi_{SG}$ for $|\beta|(A - B)^2 \geq \beta_0(1 + |A|)(1 + |B|)^3$,

(d) $1 + \beta(z\mathcal{G}(z))^2 \prec e^z$ for $|\beta|(A - B)^2 \geq (e - 1)(1 + |B|)^4$,

(e) $1 + \beta \left( \frac{f(z)}{f'(z)} \right)^2 (\mathcal{G}(z))^2 \prec e^z$ for $|\beta|(A - B)^2 \geq (e - 1)(1 + |A|)(1 + |B|)^3$,

(f) $1 + \beta \left( \frac{f(z)}{f'(z)} \right)^2 (\mathcal{G}(z))^2 \prec e^z$ for $|\beta|(A - B)^2 \geq (e - 1)(1 + |A|)^2(1 + |B|)^2$.

**Theorem 3.6.** Let $-1 \leq B < A \leq 1$, $\beta_0 \approx 0.475319$ and $\alpha \in [0, 1]$. If $p \in \mathcal{P}$ satisfy the differential subordination

$$(1 - \alpha)p(z) + \alpha p^2(z) + \beta z \frac{p'(z)}{p^k(z)} \prec \phi_{SG}(z),$$

where

$$|\beta| \geq \frac{\beta_0(1 + |A|)^k(1 + |B|)^2}{k} + \frac{(1 - \alpha)(1 + |A|)(1 + |B|) + \alpha(1 + |A|)^{k+2}}{(A - B)(1 - |B|)^k}.$$ 

then $p(z) \prec (1 + Az)/(1 + Bz)$.

**Proof.** Let $\Omega$ be same as in Theorem (3.2)(a). Let the function $\psi$ be defined as

$$\psi(r, s; z) = (1 - \alpha)r + \alpha r^2 + \frac{\beta s}{r^k}.$$ 

Substituting the values of $r$ and $s$ from equation (3.1), we get

$$\psi(r, s; z) = (1 - \alpha) \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} + \alpha \frac{(1 + Ae^{i\theta})^2}{(1 + Be^{i\theta})^2} + \beta m(A - B) e^{i\theta} \frac{(1 + B e^{i\theta})^k}{(1 + A e^{i\theta})^k (1 + B e^{i\theta})^2}.$$ 

Then

$$|\psi(r, s; z)| = \left| (1 - \alpha) \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} + \alpha \frac{(1 + Ae^{i\theta})^2}{(1 + Be^{i\theta})^2} + \beta m(A - B) e^{i\theta} \frac{(1 + Be^{i\theta})^k}{(1 + A e^{i\theta})^k (1 + B e^{i\theta})^2} \right|$$

$$\geq \left| \beta m(A - B)(1 - |B|)^k - (1 - \alpha)(1 + |A|)^{k+1}(1 + |B|) - \alpha(1 + |A|)^{k+2} \right|$$

$$\quad \geq \frac{\beta m(A - B)(1 - |B|)^k - (1 - \alpha)(1 + |A|)^{k+1}(1 + |B|) - \alpha(1 + |A|)^{k+2}}{(1 + |A|)(1 + |B|)^2}$$

$$= : \phi(m)$$
Verify that the function \( \phi(m) \) is increasing \( \forall \ m \geq 1 \) and hence, attains its minimum value at \( m = 1 \). Since

\[
\frac{|\beta|(A - B)(1 - |B|)^k - (1 - \alpha)(1 + |A|)^{k+1}(1 + |B|) - \alpha(1 + |A|)^{k+2}}{(1 + |A|)^k(1 + |B|)^2} \geq \beta_0
\]

by Theorem (3.1) and Lemma (2.7), we get the desired result.

\[\square\]

**Remark 3.7.** For \( \alpha = 0 \), the above theorem reduces to the following result.

**Corollary 3.8.** Let \( p \) be an analytic function satisfying \( p(0) = 1 \) and \( \beta > \beta_0 \approx 0.475319 \). Then each of the following subordinations is sufficient to imply \( p \in \mathcal{P}[A, B] \).

\[
p(z) + \beta z \frac{p'(z)}{p(z)} \prec \phi_{SG}, \text{ where } |\beta| \geq \frac{\beta_0(1 + |A|)^k(1 + |B|)^2 + (1 + |A|)^{k+1}(1 + |B|)}{(A - B)(1 - |B|)^k}.
\]

**Theorem 3.9.** Let \( \beta_0 \approx 0.475319 \) and \( k \) be a non-negative integer. If \( p \in \mathcal{P} \) and satisfies the differential subordination

\[
\left( \frac{1}{p(z)} \right)^{-\beta} z \frac{p'(z)}{p(z)} \prec \phi_{SG}(z), \text{ where } |\beta| \geq \begin{cases} 
\frac{\beta_0(1 + |A|)^k(1 + |B|)^2 + (1 + |A|)^{k+1}(1 + |B|)}{(A - B)(1 - |A|)} & \text{when } 0 \leq k \leq 2, \\
\frac{\beta_0(1 + |A|)^k + (1 + |A|)^{k+1}(1 + |B|)}{(A - B)(1 - |A|)(1 - |B|)^{k-2}} & \text{when } k > 2.
\end{cases}
\]

then \( p(z) \prec (1 + Az)/(1 + Bz) \).

**Proof.** Let \( \Omega \) be same as in Theorem (3.2)(a). Consider the analytic function \( \psi \) defined as

\[
\psi(r, s; z) = \frac{1}{r} - \beta \frac{s}{r^k}.
\]

Substituting the values of \( r \) and \( s \) as given in equation (3.1), we get

\[
\psi(r, s; z) = \frac{1 + Be^{i\theta}}{1 + Ae^{i\theta}} - \beta \frac{m(A - B)e^{i\theta}(1 + Be^{i\theta})^k}{(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^2}
\]

Proceeding as in Theorem (3.2)(a), the following two cases arises.

(i) When \( 0 \leq k \leq 2 \), consider

\[
|\psi(r, s; z)| = \frac{1 + Be^{i\theta}}{1 + Ae^{i\theta}} - \beta \frac{m(A - B)e^{i\theta}(1 + Be^{i\theta})^k}{(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^2}
\]

\[
= \left| (1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{3-k} - \beta m(A - B)e^{i\theta}(1 + Ae^{i\theta}) \right| (1 + Ae^{i\theta})^{k+1}(1 + Be^{i\theta})^{2-k}
\]

\[
= \frac{\beta m(A - B)(1 - |A|) - (1 + |A|)^{k+1}(1 + |B|)^{3-k}}{(1 + |A|)^k(1 + |B|)^2-k}
\]

\[= \phi(m)\]

Observe that \( \phi'(m) > 0 \) for \( m \geq 1 \). In view of above and Lemma (2.7), simple computations gives the desired bound on \( \beta \) in terms of \( A \) and \( B \).
Corollary 3.10. Let \( \beta_0 \approx 0.475319 \), \( f \) be an analytic function and \( G(z) \) be same as in Corollary [3.3]. Then each of the following subordinations imply that \( f \in S^*[A,B] \).

(a) \( \frac{f(z)}{zf'(z)} - \beta zG(z) \prec \phi_{SC} \) for \( |\beta|(A - B)(1 - |A|) \geq \beta_0(1 + |A|)(1 + |B|)^2 + (1 + |B|) \),

(b) \( \frac{f(z)}{zf'(z)} - \beta \frac{f(z)}{f'(z)} G(z) \prec \phi_{SC} \) for \( |\beta|(A - B)(1 - |A|) \geq \beta_0(1 + |A|)^2(1 + |B|) + (1 + |A|)(1 + |B|) \),

(c) \( \frac{f(z)}{zf'(z)} - \beta \left( \frac{f(z)}{f'(z)} \right)^2 G(z) \prec \phi_{SC} \) for \( |\beta|(A - B)(1 - |A|) \geq \beta_0(1 + |A|)^3 + (1 + |A|)^2(1 + |B|) \).

Theorem 3.11. Suppose \(-1 \leq B < A \leq 1\), \( \gamma > 0\), \( \beta_0 \approx 0.475319 \) and \( k \) be a non-negative integer. Let \( p \) be an analytic function satisfying the differential subordination

\[
p(z) + \frac{zp'(z)}{(\beta p(z) + \gamma)^k} \prec \phi_{SC}(z), \quad \text{where}
\]

\[(A - B)(1 - |B|) \geq \beta_0(1 + |B|)^{2-k}(\beta |1 + |A|| + \gamma(1 + |B|))^k(2 + |A| + |B|), \quad \text{when } 0 \leq k \leq 2,
\]

\[(A - B)(1 - |B|)^{k-1} \geq \beta_0(2 + |A| + |B|)((\beta |1 + |A|| + \gamma(1 + |B|))^k, \quad \text{when } k > 2.
\]

Then \( p \in \mathcal{P}[A,B] \).

Proof. Let \( \Omega \) be the domain as defined in Theorem [3.2](a). Define the function \( \psi(r, s; z) : \mathbb{C}\setminus\{0\} \times \mathbb{C} \times \mathbb{D} \to \mathbb{C} \) as

\[
\psi(r, s; z) = r + \frac{s}{(\beta r + \gamma)^k}.
\]

Then using equation [3.1], the function \( \psi \) becomes

\[
\psi(r, s; z) = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} + \frac{m(A - B)(1 + Be^{i\theta})^k e^{i\theta}}{(1 + Be^{i\theta})^2(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k}
\]

In view of Theorem [3.1], the desired subordination \( p \prec (1 + Az)/(1 + Bz) \) will follow if we show that \( \psi \in \Psi[\Omega; A, B] \). For this, it suffices to show that

\[
\left| \log \left( \frac{\psi(r, s; z)}{2 - \psi(r, s; z)} \right) \right| \geq 1.
\]
(i) When $0 \leq k \leq 2$, observe that

$$
|\psi(r, s; z)| = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} + \frac{m(A - B)e^{i\theta}}{(1 + Be^{i\theta})^{2-k}(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k} \\
= \frac{(1 + Ae^{i\theta})(1 + Be^{i\theta})^{2-k}(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k}{(1 + Be^{i\theta})^{3-k}(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k} + m(A - B)e^{i\theta}(1 + Be^{i\theta})^{k-1} \\
m(A - B)|(1 + Be^{i\theta})| - |(1 + Ae^{i\theta})||(1 + Be^{i\theta})^{2-k}| \\
\geq \frac{|(1 + Be^{i\theta})^{3-k}|(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k|}{(1 + |B|)^{3-k}(\beta(1 + |A|) + \gamma(1 + |B|))^k}
$$

Similar analysis as done in Theorem 3.2(a) gives that $\psi(r, s; z) \notin \Omega$ for

$$(A - B)(1 - |B|) \geq \beta_0(1 + |B|)^{2-k}(\beta(1 + |A|) + \gamma(1 + |B|))^k(2 + |A| + |B|).$$

(ii) When $k > 2$, consider

$$
|\psi(r, s; z)| = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} + \frac{m(A - B)e^{i\theta}(1 + Be^{i\theta})^{k-2}}{(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k} \\
= \frac{(1 + Ae^{i\theta})(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k + m(A - B)e^{i\theta}(1 + Be^{i\theta})^{k-1}}{(1 + Be^{i\theta})(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k} \\
m(A - B)|(1 + Be^{i\theta})^{k-1}| - |(1 + Ae^{i\theta})||(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^k| \\
\geq \frac{|(1 + Be^{i\theta})^{k-1}|}{(1 + |B|)^{(k-1)}}(\beta(1 + |A|) + \gamma(1 + |B|))^k
$$

On the similar lines as in proof of part (i), we get the desired result.

\[\square\]

4. Second order differential subordination

In this section, sufficient conditions are obtained so that the subordination implication

$$p(z) < \frac{1 + Az}{1 + Bz}$$

holds whenever $\psi(p(z), zp'(z), z^2p''(z); z)$ is subordinate to Modified Sigmoid function, exponential function and Janowski function.

**Theorem 4.1.** Let $-1 < B < A < 1$, $\gamma > 0$, $\beta > 0$ and $\beta_0 \approx 0.475319$. Let $p$ be an analytic function satisfying $p(0) = 1$. Then, each of the following is sufficient for $p \in P[A, B]$.

(a) $1 + \gamma zp'(z) + \beta z^2p''(z) < \phi_{SG}(z)$, where
Proof. (a) Let $\Omega = \phi_{\text{SG}}(\mathbb{D}) = \{w \in \mathbb{C} : |\log (w/(2-w))| < 1\}$.

Consider the analytic function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined as

$$\psi(r, s, t; z) = 1 + \gamma s + \beta t$$

For $\psi \in \Psi[\Omega; A, B]$, we must have $\psi(r, s, t; z) \notin \Omega$. By Theorem (2.4), this implication is true if

$$\left| \log \left( \frac{\psi(r, s, t; z)}{2 - \psi(r, s, t; z)} \right) \right| \geq 1. \quad (4.1)$$

By Lemma (2.7), the inequality (4.1) holds if and only if $|\psi(r, s, t; z)| \geq \beta_0$. A calculation shows that

$$|\psi(r, s, t; z)| = |1 + \gamma s + \beta t|$$

$$\geq 1 + \gamma |s| \left| 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} \right| - 1$$

$$\geq \gamma |s| \text{Re} \left( 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} \right) - 1$$

$$\geq m \gamma d(\theta) \left( 1 + \left( \frac{\beta}{\gamma} \right) (mg(\theta) + m - 1) \right) - 1$$

$$\geq \begin{cases} 
\frac{m(A-B)}{1+B^2+2B} \left( \gamma + \frac{-2B(B+1)\beta m}{1+B^2+2B} \right) - 1, B > 0 \\
\frac{m(A-B)}{1+B^2-2B} \left( \gamma + \frac{-2B(B-1)\beta m}{1+B^2-2B} \right) - 1, B < 0 
\end{cases}$$

$$:= \phi(m),$$

where $d(\theta)$ and $g(\theta)$ are given by (2.4) and (2.5) respectively. Observe that $\phi(m)$ is increasing function for $m \geq 1$. Therefore, we have $|\psi(r, s, t; z)| \geq \phi(1) \geq \beta_0$ and hence, $\psi \in \Psi[\Omega; A, B]$. By Theorem (2.4) $p(z) < (1 + Az)/(1 + Bz)$. 

(b) $1 + \gamma z p'(z) + \beta z^2 p''(z) \prec e^z$, where

$$\begin{align*}
(A - B)[\gamma(1 + B^2 + 2B - 2B\beta(B + 1)] &\geq (\beta_0 + 1)(1 + B^2 + 2B)^2 \text{ for } B > 0 \text{ and} \\
(A - B)[\gamma(1 + B^2 - 2B - 2B\beta(B - 1)] &\geq (\beta_0 + 1)(1 + B^2 - 2B)^2 \text{ for } B < 0.
\end{align*}$$

(c) $1 + \gamma z p'(z) + \beta z^2 p''(z) \prec (1 + Az)/(1 + Bz)$, where

$$\begin{align*}
(A - B)(1 - B^2)[\gamma(1 + B^2 + 2B - 2B\beta(B + 1)] - |B|(A - B)(1 + B^2 + 2B)^2 &\geq (A - B)(1 + B^2 + 2B)^2, \text{ for } B > 0 \text{ and} \\
(A - B)(1 - B^2)[\gamma(1 + B^2 - 2B - 2B\beta(B - 1)] - |B|(A - B)(1 + B^2 - 2B)^2 &\geq (A - B)(1 + B^2 - 2B)^2, \text{ for } B < 0.
\end{align*}$$
(b) Consider the domain $\Omega = \{ w \in \mathbb{C} : |\log w| < 1 \}$. Let the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ be defined as

$$\psi(r, s, t; z) = 1 + \gamma s + \beta t.$$ 

For $\psi \in \Psi[\Omega; A, B]$, we must have $\psi(r, s, t; z) \notin \Omega$. In order to satisfy this relation, it is sufficient to show that

$$|\log(\psi(r, s, t; z))| \geq 1.$$

Since

$$\left| \gamma s \left( 1 + \frac{\beta t}{\gamma s} \right) \right| \geq \gamma |s| \left| 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} \right| \geq \gamma |s| \Re \left( 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} \right) \geq m \gamma d(\theta) \left( 1 + \left( \frac{\beta}{\gamma} \right) (mg(\theta) + m - 1) \right) \geq \begin{cases} \frac{m(A-B)}{1+B^2+2B} \left( \gamma - \frac{2B(B+1)\beta m}{1+B^2+2B} \right), & B > 0 \\ \frac{m(A-B)}{1+B^2-2B} \left( \gamma - \frac{2B(B-1)\beta m}{1+B^2-2B} \right), & B < 0 \end{cases} := \phi(m)$$

and $\phi(m)$ is increasing function of $\phi$, by Lemma 2.6, $|\gamma s \left( 1 + \frac{\beta t}{\gamma s} \right)| \geq e - 1$. Thus, by Theorem 2.4, $p(z) \prec (1 + Az)/(1 + Bz)$.

(c) Consider the domain

$$\Omega = \left\{ w \in \mathbb{C} : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$ 

Let $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ be defined as $\psi(r, s, t; z) = 1 + \gamma s + \beta t$. Now, $\psi \in \Psi[\Omega; A, B]$, if $\psi(r, s, t; z) \notin \Omega$. On the similar lines on the proof of part(a),

$$\left| \psi(r, s, t; z) - \frac{1 - AB}{1 - B^2} \right| = \left| 1 + \gamma s + \beta t - \frac{1 - AB}{1 - B^2} \right| \geq \gamma |s| \Re \left( 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} \right) - \frac{|B|(A-B)}{1 - B^2} \geq m \gamma d(\theta) \left( 1 + \left( \frac{\beta}{\gamma} \right) (mg(\theta) + m - 1) \right) - \frac{|B|(A-B)}{1 - B^2} := \phi(m).$$

Using the values of $d(\theta)$ and $g(\theta)$ as given in the equations (2.4) and (2.5), and first derivative test for function $\phi$ we have, for $B > 0$,

$$\left| \psi(r, s, t; z) - \frac{1 - AB}{1 - B^2} \right| \geq \frac{(A-B)}{1 + B^2 + 2B} \left( \gamma - \frac{2B(B+1)\beta}{1 + B^2 + 2B} \right) - \frac{|B|(A-B)}{1 - B^2} \geq \frac{A-B}{1 - B^2}.$$
and for $B < 0$,
\[
\left| \psi(r, s, t; z) - \frac{1 - AB}{1 - B^2} \right| \geq \frac{(A - B)(1 + B^2 - 2B)}{1 + B^2 - 2B} \left( \gamma - \frac{2B(B - 1)\beta}{1 + B^2 - 2B} \right) - \frac{|B|(A - B)}{1 - B^2} \geq \frac{A - B}{1 - B^2}.
\]

Therefore, $\psi \in \Psi[\Omega; A, B]$ and hence, by Theorem 2.4, $p(z) \prec (1 + Az)/(1 + Bz)$. □

**Corollary 4.2.** Let $\gamma$ and $\beta$ be positive integers and $f$ be an analytic function. Set
\[
\mathcal{H}(z) = 1 + \gamma \left( \frac{z^2f''(z)}{f(z)} - \left( \frac{zf'(z)}{f(z)} \right)^2 + \frac{zf'(z)}{f(z)} + \beta \left( \frac{z^3f'''(z)}{f(z)} \right) \right) + \frac{2z^2f''(z)}{f(z)} + 2\left( \frac{zf'(z)}{f(z)} \right)^3 - 2\left( \frac{zf'(z)}{f(z)} \right)^2 - \frac{3z^3f''(z)f''(z)}{f(z)^2}.
\]

Then, $f \in S^*[A, B]$ if any one of the following condition hold.

(a) $\mathcal{H}(z) \prec \phi_{SG}(z)$, where
\[
(A - B)[\gamma(1 + B^2 + 2B) - 2B\beta(B + 1)] \geq (\beta_0 + 1)(1 + B^2 + 2B)^2 \text{ for } B > 0 \text{ and } (A - B)[\gamma(1 + B^2 - 2B) - 2B\beta(B - 1)] \geq (\beta_0 + 1)(1 + B^2 - 2B)^2 \text{ for } B < 0.
\]

(b) $\mathcal{H}(z) \prec e^z$, where
\[
(A - B)[\gamma(1 + B^2 + 2B) - 2B\beta(B + 1)] \geq (e - 1)(1 + B^2 + 2B)^2 \text{ for } B > 0 \text{ and } (A - B)[\gamma(1 + B^2 - 2B) - 2B\beta(B - 1)] \geq (e - 1)(1 + B^2 - 2B)^2 \text{ for } B < 0.
\]

(c) $\mathcal{H}(z) \prec (1 + Az)/(1 + Bz)$, where
\[
(A - B)(1 - B^2)[\gamma(1 + B^2 + 2B) - 2B\beta(B + 1)] - |B|(A - B)(1 + B^2 + 2B)^2 \geq (A - B)(1 + B^2 + 2B)^2, \text{ for } B > 0 \text{ and } (A - B)(1 - B^2)[\gamma(1 + B^2 - 2B) - 2B\beta(B - 1)] - |B|(A - B)(1 + B^2 - 2B)^2 \geq (A - B)(1 + B^2 - 2B)^2, \text{ for } B < 0.
\]

**Theorem 4.3.** Suppose $-1 < B < A < 1$, $\beta_0 \approx 0.475319$, $\beta > 0$ and $\gamma > 0$. Let $p$ be an analytic function which satisfies the condition $p(0) = 1$ and the following inequalities holds:
\[
(A - B)(1 + B)[\gamma(1 + B^2 + 2B) - 2B(B + 1)\beta] - (1 + A)(1 + B^2 + 2B)^2 \geq \beta_0(1 + B)(1 + B^2 + 2B)^2 \text{ for } B > 0, \text{ and } (A - B)(1 + B)[\gamma(1 + B^2 - 2B) - 2B(B - 1)\beta] - (1 + A)(1 + B^2 - 2B)^2 \geq \beta_0(1 + B)(1 + B^2 - 2B)^2 \text{ for } B < 0.
\]

Then,
\[
p(z) + \gamma zp'(z) + \beta z^2p''(z) \prec \phi_{SG}(z)
\]
implies
\[
p \prec \frac{1 + Az}{1 + Bz}.
\]
Proof. Let $\Omega$ be the domain defined in Theorem (3.2)(a). Consider the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined as

$$\psi(r, s, t; z) = r + \gamma s + \beta t.$$ 

For $\psi$ to be in $\Psi[\Omega; A, B]$, we must have $\psi(r, s, t; z) \notin \Omega$. By Theorem (2.4), this result is true if

$$\left| \log \left( \frac{\psi(r, s, t; z)}{2 - \psi(r, s, t; z)} \right) \right| \geq 1$$

Using Lemma (2.7), this inequality holds if and only if

$$|\psi(r, s, t; z)| \geq \beta_0 \quad (4.2)$$

where $\beta_0$ is the positive real root of the equation in (2.6). If $k(\theta)$, $g(\theta)$ and $d(\theta)$ are given by the equations (2.3), (2.5) and (2.4) respectively, then

$$|\psi(r, s, t; z)| = |r + \gamma s + \beta t|$$

$$\geq \gamma |s| \left| 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} - |r| \right| $$

$$\geq \gamma |s| \Re \left( 1 + \left( \frac{\beta}{\gamma} \right) \frac{t}{s} - |r| \right) $$

$$\geq m \gamma d(\theta) \left( 1 + \left( \frac{\beta}{\gamma} \right) (mg(\theta) + m - 1) \right) - |k(\theta)|$$

$$\geq m \gamma d(\theta) \left( 1 + \left( \frac{\beta}{\gamma} \right) (mg(\theta) + m - 1) \right) - \frac{1 + A}{1 + B}$$

$$:= \phi(m).$$

Since $\phi(m)$ is increasing function, we have, for $B > 0$,

$$|\psi(r, s, t; z)| \geq \frac{(A - B)}{1 + B^2 + 2B} \left( \gamma - \frac{2B(B + 1)\beta}{1 + B^2 + 2B} \right) - \frac{1 + A}{1 + B}$$

and

for $B < 0$,

$$|\psi(r, s, t; z)| \geq \frac{(A - B)}{1 + B^2 - 2B} \left( \gamma + \frac{2B(B - 1)\beta}{1 + B^2 - 2B} \right) - \frac{1 + A}{1 + B}.$$ 

Therefore, inequality (4.2) is satisfied and hence, we get the required result. \qed

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