On the numerical radii of $2 \times 2$ complex matrices

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Abstract
The numerical radius of the general $2 \times 2$ complex matrix is calculated.

1 Introduction and preliminaries

Let $A$ be a linear bounded operator, acting in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Denote by $W(A) = \{ \langle Ax, x \rangle : \|x\| = 1 \}$ the numerical range and by $w(A) = \sup_{\lambda \in \mathbb{C}} |\lambda|$ the numerical radius of the operator $A$. As it is well known, the numerical radius defines a norm on the space of all bounded operators, equivalent to the usual one, i.e.

$$w(A) \leq \|A\| \leq 2w(A).$$

The importance of the numerical radius is partially motivated by its role in the investigation of many iterative schemes of solution of operator equations Eiermann [1993]. Despite the great abundance of research papers on the numerical range and the numerical radius, the exact calculations of the numerical radius of general non-normal operator are very scarce. A few known
results concern the $n-$ dimensional Jordan block

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

\((w(J_n) = \cos \frac{\pi}{n+1})\) and the Volterra integration operator

\((V f)(x) = \int_0^x f(t) dt, f \in L^2(0; 1)\)

\((w(V) = 1/2)\).

In the sequel we consider the simplest case of two-dimensional space.

2 Main result

As any operator in a finite-dimensional space may be reduced to the Schur’s upper triangular form, we get the matrix

$$A = \begin{pmatrix} \lambda_1 & \lambda_3 \\
0 & \lambda_2 \end{pmatrix}.$$ 

Recall that the numerical range of an operator, acting in a two-dimensional space is an elliptical disk (Halmos [1982], ch. 22) having as foci two eigenvalues \(\{\lambda_1, \lambda_2\}\) and

\(\{tr(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\}^{1/2}\)

as minor axis \(2b\).

We have \(b = |\lambda_3|/2\), the distance between foci is \(2c = |\lambda_2 - \lambda_1|\), hence major axis is \(2a = \sqrt{|\lambda_2 - \lambda_1|^2 + |\lambda_3|^2}\). The center of symmetry of the ellipse has the affix \(m = \frac{\lambda_2 + \lambda_3}{2}\) and whole ellipse is rotated by the angle \(\varphi = \arg(\lambda_2 - \lambda_1)\). The equation of the ellipse is

\[
\frac{((x - \text{Re} \, m) \cos \phi + (y - \text{Im} \, m) \sin \phi)^2}{a^2} + \frac{((x - \text{Re} \, m) \sin \phi - (y - \text{Im} \, m) \cos \phi)^2}{b^2} - 1 = 0.
\]
After elementary transformations it may be reduced to the form
\[ G(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \]
where
\[
A = \frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2},\]
\[
B = \sin \phi \cos \phi \left( \frac{1}{a^2} - \frac{1}{b^2} \right),\]
\[
C = \frac{\sin^2 \phi}{a^2} + \frac{\cos^2 \phi}{b^2},\]
\[
D = -A \text{Re} m - B \text{Im} m,\]
\[
E = -B \text{Re} m - C \text{Im} m,\]
\[
F = \left( \frac{\cos \phi \text{Re} m + \sin \phi \text{Im} m}{a} \right)^2 + \left( \frac{\cos \phi \text{Im} m - \sin \phi \text{Re} m}{b} \right)^2 - 1.
\]

The square of numerical radius \( w(A) \) may be found as the greatest value of the sum \( x^2 + y^2 \), where \( x \) and \( y \) are constrained by the equation \( G(x, y) = 0 \). According to the Lagrange multiplier method, the auxiliary function
\[
L(x, y, \lambda) = x^2 + y^2 + \lambda G(x, y)
\]
should be introduced and the system of equations
\[
\begin{align*}
\frac{\partial L}{\partial x} &= 0, \\
\frac{\partial L}{\partial y} &= 0, \\
\frac{\partial L}{\partial \lambda} &= 0,
\end{align*}
\]
will be solved.

We prefer another approach. The equation of the circumference is
\[
\Gamma(x, y) = x^2 + y^2 - R^2 = 0.
\]
If the ellipse lies inside the circle and touches the circumference, then they have a common tangent, therefore
\[
\frac{Ax + By + D}{Bx + Cy + E} = \frac{x}{y}.
\]
Finally, we get the set of simultaneous equations
\[
\begin{align*}
Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F &= 0, \\
Bx^2 + (C - A)xy - By^2 + Ex - Dy &= 0,
\end{align*}
\] (1)
Note that the second curve is a hyperbola. The solution of this set may be reduced to a quartic polynomial equation. Using some notions of projective geometry (see Richter-Gebert [2011], ch. 11), it is possible to avoid cumbersome calculations. Introducing the homogeneous coordinates \( (x \ y \ 1)^t \) of the point \((x; y)\), two matrices

\[
G = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} \quad \text{and} \quad H = \frac{1}{2} \begin{pmatrix} 2B & C - A & E \\ C - A & -2B & -D \\ E & -D & 0 \end{pmatrix}
\]

may be associated with (1). Two conics define a pencil of conics \( \alpha H + \beta G; \alpha, \beta \in \mathbb{R} \), passing through the intersection points of \( G \) and \( H \). Solving the generalized eigenvalue problem \( \det (H - \lambda G) = 0 \) we get the degenerated conics \( C = H - \lambda_0 G, \det (C) = 0 \). Now we calculate the adjugate matrix \( J = Ad(C) \). Maybe the simplest way is the use of the formula \( J = -(a_1 I + a_2 C + C^2) \), where \( a_1 = \frac{1}{2} (\text{tr}^2 (C) - \text{tr} (C^2)) \), \( a_2 = -\text{tr} (C) \). Denote \( p_1 = \sqrt{J_{11}}, p_2 = \sqrt{J_{22}}, p_3 = \sqrt{J_{33}} \), where the signs before the radicals are chosen in a such way that \( \{p_1, p_2, p_3\}^T \cdot \{p_1, p_2, p_3\} = J \) and construct the antisymmetric matrix

\[
P = \begin{pmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{pmatrix}.
\]

Now we choose the parameter \( \alpha \) to satisfy the condition \( K = C - \alpha P, \text{rank} (K) = 1 \). Any row (or column) of \( K \) defines a degenerate conics (a straight line) \( l \). Replacing the first equation in (1) by \( l = 0 \) we may found easily (solving a quadratic equation) the coordinates of the intersection points. One of them will supply the minimum of \( \sqrt{x^2 + y^2} \) and the second point is at the distance \( w (A) \) from the origin of the coordinate system.

We append below a MatLab program, which admits as input a \( 2 \times 2 \) upper triangular matrix and calculates its numerical radius.

### Appendix

```matlab
function numradius(A)
    nor=norm(A);
    b=.5*abs(A(1,2));
```

3
\[ c = 0.5 \cdot \text{abs}(A(1,1)-A(2,2)) \]
\[ a = \sqrt{b^2 + c^2} \]
\[ m = 0.5 \cdot (A(1,1) + A(2,2)) \]
\[ k = \text{real}(m); l = \text{imag}(m); \]
\[ \text{phi} = \text{angle}(A(2,2)-A(1,1)); \]
\[ s = \text{linspace}(0,2\pi,2000); \]
\[ u = a \cdot \cos(s); v = b \cdot \sin(s); \]
\[ R = \begin{bmatrix} \cos(\text{phi}) & -\sin(\text{phi}) \\ \sin(\text{phi}) & \cos(\text{phi}) \end{bmatrix}; \]
\[ K = R^* [u; v]; \]
\[ z = k + i* (K(1,.) + l); \]
\[ \text{plot}(\text{real}(z), \text{imag}(z), 'r'); \]
\[ \text{axis 'equal'} \]

\[ A = \cos(\text{phi})^2/a^2 + \sin(\text{phi})^2/b^2; \]
\[ B = \sin(\text{phi}) \cdot \cos(\text{phi}) \cdot (1/a^2 - 1/b^2); \]
\[ C = \cos(\text{phi})^2/b^2 + \sin(\text{phi})^2/a^2; \]
\[ D = -k \cdot A - l \cdot B; \]
\[ E = -k \cdot B - l \cdot C; \]
\[ F = (k \cdot \cos(\text{phi}) + l \cdot \sin(\text{phi}))^2/a^2 + (l \cdot \cos(\text{phi}) - k \cdot \sin(\text{phi}))^2/b^2 - 1; \]
\[ G = \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix}; \]
\[ H = 0.5 \begin{bmatrix} 2 & B & C - A & E; C & B & A - 2 \cdot B & -D & E & D & -D & 0 \end{bmatrix}; \]
\[ e = \text{eig}(H,G); \]
\[ \text{for } j = 1:3 \]
\[ \text{if isreal(e(j))} \]
\[ c = e(1); \]
\[ \text{end} \]
\[ \text{end} \]
\[ C = H - c \cdot G; \]
\[ s1 = \text{trace}(C); s2 = \text{trace}(C^2); a2 = -s1; a1 = -0.5 \cdot s1 \cdot a2 + s2; \]
\[ A_d = -(a1 \cdot \text{eye}(3) + a2 \cdot C + C^2); \]
\[ p(1) = \sqrt{A_d(1,1)}; \]
\[ p(2) = \text{sign}(A_d(1,2)) \cdot \sqrt{A_d(2,2)}; \]
\[ p(3) = \text{sign}(A_d(1,3)) \cdot \sqrt{A_d(3,3)}; \]
\[ P = \begin{bmatrix} 0 & p(3) & -p(2); & p(3) & 0 & p(1); & p(2) & -p(1) & 0 \end{bmatrix}; \]
\[ e = \text{eig}(C, P); \]
\[ e(4) = \sqrt{((C(1,2)^2 - C(1,1) \cdot C(2,2))/p(3)}; \]
\[ e(5) = \sqrt{((C(3,2)^2 - C(3,3) \cdot C(2,2))/p(1)}; \]
\[ e(6) = \sqrt{((C(1,3)^2 - C(1,1) \cdot C(3,3))/p(2)}; \]
\[ \text{for } j = 1:6 \]
K=C_e(j)*P;
r(j)=rank(K);
l=mean(K(:,1));
n=mean(K(:,2));
q=mean(K(:,3));
[x,y]=solve('H(1,1)*x^2+H(2,2)*y^2+2*H(1,2)*x*y+... 2*H(1,3)*x+2*H(2,3)*y=0','l*x+n*y+q=0');
m1=eval(sqrt(x(1)^2+y(1)^2));
m2=eval(sqrt(x(2)^2+y(2)^2));
w(j)=max(m1,m2);
end
for j=1:6
  if isreal(w(j))&& w(j)≤ nor&& (r(j)==1)
    numr=w(j);
  end
end

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