Theory of the Room-Temperature QHE in Graphene

Shigeji Fujita\textsuperscript{a}, Akira Suzuki\textsuperscript{b}

\textsuperscript{a}Department of Physics, University at Buffalo, State University of New York, Buffalo, New York 14260-1500, USA
\textsuperscript{b}Department of Physics, Faculty of Science, Tokyo University of Science, Shinjyuku-ku, Tokyo 162-8601, Japan

Abstract

The unusual quantum Hall effect (QHE) in graphene is often discussed in terms of Dirac fermions moving with a linear dispersion relation. The same phenomenon will be explained in terms of the more traditional composite bosons, which move with a linear dispersion relation. The “electron” (wave packet) moves easier in the direction \([110\text{ c-axis}] \equiv [110]\) of the honeycomb lattice than perpendicular to it, while the “hole” moves easier in \([001]\). Since “electrons” and “holes” move in different channels, the number densities can be high especially when the Fermi surface has “necks”. The strong QHE arises from the phonon exchange attraction in the neighborhood of the “neck” Fermi surfaces. The plateau observed for the Hall conductivity and the accompanied resistivity drop is due to the Bose-Einstein condensation of the c-bosons, each forming from a pair of one-electron–two-fluxons c-fermions by phonon-exchange attraction.

Keywords: quantum Hall effect, linear dispersion relation, c-boson, c-fermion, fluxon, Hall conductivity, magnetoresistivity, supercurrent, Bose-Einstein condensation.

1. Introduction

In 2005 Novoselov et al. \cite{Novoselov2005} discovered a quantum Hall effect (QHE) in graphene. Figure 1 is reproduced after Ref. \cite{Novoselov2005}, Figure 4. The longitudinal magnetoresistivity \(\rho_{xx}\) and the Hall conductivity \(\sigma_{xy}\) in graphene at \(B = 14\) T and \(T = 4\) K are plotted as a function of the conduction electron density \(n\) \((\sim 10^{12}\text{ cm}^{-2})\). The plateau values of the Hall conductivity \(\sigma_{xy}\) are quantized.
Figure 1: QHE in graphene. Hall conductivity \( \sigma_{xy} \) and longitudinal resistivity \( \rho_{xx} \) are indicated by red and green lines, respectively. After Novoselov et al. [1].

in the units of

\[
\frac{4e^2}{h}
\]

within experimental errors, where \( h \) is the Planck constant, \( e \) the electronic charge (magnitude). The longitudinal resistivity \( \rho_{xx} \) reaches zero at the middle of the plateaus. These two are the major signatures of the QHE in graphene.

In 2007 Novoselov et al. [2] reported a discovery of a room temperature QHE in graphene. We reproduced their data in Fig. 2 after Ref. [2], Figure 1. The Hall resistivity \( \rho_{xy} \) for “electrons” and “holes” indicate precise quantization within experimental errors in units of \( h/e^2 \) at magnetic field 29 T and temperature 300 K. This is an extraordinary jump in the observation temperatures since the QHE in heterojunction GaAs/AlGaAs was reported below 1 K. Figure 2 is similar to those in Fig. 1 although the abscissas are different, one in gate voltage and the other in carrier density, and hence the physical conditions are different. We give an explanation later. Notice that the quantization in \( \rho_{xy} \) appears at \( h/4e^2 \), which is a little strange since the most visible quantization for GaAs/AlGaAs appears at \( h/e^2 \).
Figure 2: Room-temperature QHE in graphene after Novoselov et al. [2]. Hall conductivity $\sigma_{xy}(e^2/h)$ (red) and resistance $\rho_{xx}$ (blue) as a function of gate voltage ($V_g$) at temperature 300 K and magnetic field 29 T. Positive values of $V_g$ induce “electrons”, and negative values of $V_g$ induce “holes”, in concentrations $n = (7.2 \times 10^{10} \text{ cm}^{-2} \text{V}^{-1}) V_g$.

From the QHE behaviors in Fig. 1 and Fig. 2 we observe that the quantization occurs at a set of points:

$$\frac{h}{e^2} \frac{(2P + 1)}{4} = P = 0, 1, 2, \cdots .$$

We shall show the quantization rule (2) based on the c-particles (fermions, bosons) model in the present work.

2. Electron Dynamics in Graphene

The normal carriers in solids are “electrons” (“holes”), which spiral around the applied magnetic field $B$ counterclockwise (clockwise) viewed from the tip of the field vector $B$. The “electrons” (“holes”) are excited above (below) the metal’s Fermi energy. These quasi-particles are quotation marked throughout the text. Following Ashcroft and Mermin [3] we regard the conduction electrons are wave packets extending over the unit cells.

We consider a graphene, which forms a 2D honeycomb lattice. The Wigner-Seitz (WS) unit cell, rhombus (shaded) shown in Fig. 3(a), con-
tains two C’s. We showed in our earlier work[5–7] that graphene has “electrons” and “holes” based on the rectangular unit cell (dotted lines) shown in Fig.3(b). We briefly review our calculations. We must choose the rectangular unit cell to establish the Bloch plane waves[7,8] in 2D.

We assume that the “electron” (“hole”) wave packet has the charge $-e$ ($+e$) and a size of the rectangular unit cell, generated above (below) the Fermi energy $\varepsilon_F$. We could show[3,6] that (a) the “electron” and the “hole” have different charge distribution and different effective masses, (b) that the “electron” and “hole” move in different easy channels, (c) that the “electrons” and “holes” are thermally excited with different activation energies, and (d) that the “electron” activation energy $\varepsilon_1$ is smaller than the “hole” activation energy $\varepsilon_2$:

$$\varepsilon_1 < \varepsilon_2$$

Thus, “electrons” are the majority carriers in graphene. The thermally activated electron densities are then given by

$$n_j = n_j e^{-\varepsilon_j/k_BT}, \quad n_j = \text{constant},$$

where $j = 1$ and 2 represent the “electron” and “hole”, respectively. Magnetotransport experiments by Zhang et al.[9] indicate that the “electrons”
are the majority carriers in graphene. Thus, our theory is agreement with experiments.

3. Theory of Quantum Hall Effect

The prevalent theories[10–15] based on the Laughlin wavefunction deal with the QHE at 0 K and immediately above. To describe a room temperature QHE we need a finite temperature theory. Fujita and Okamura developed a quantum statistical theory of the QHE[16]. We follow this theory. See this reference for more details. It is convenient to introduce composite (c-) particles. The c-boson (fermion), each containing an electron and an odd (even) number of flux quanta (fluxons), were introduced by Zhang et al.[17] (Jain[11]) for the description of the fractional QHE (Fermi liquid).

There is a remarkable similarity between the QHE and the High-Temperature Superconductivity (HTSC), both occurring in 2D systems. We regard the phonon exchange attraction as the causes of both QHE and superconductivity. Starting with a reasonable Hamiltonian, we calculate everything using the standard statistical mechanics.

The countability concept of the fluxons, known as the flux quantization:

\[ B = \frac{N_\phi}{A} \frac{\hbar}{e} = \frac{N_\phi}{A} \Phi_0 = n_\phi \Phi_0, \]

where \( A \) = sample area, \( N_\phi = \) fluxon number (integer), \( \Phi_0 = \hbar/e = \) flux quantum, is originally due to Onsager[18]. The magnetic (electric) field is an axial (polar) vector and the associated fluxon (photon) is a half-spin fermion (full-spin boson). The magnetic (electric) flux line cannot (can) terminate at a sink, which supports the fermionic (bosonic) nature of the associated fluxon (photon). No half-spin fermion can annihilate by itself because of angular momentum conservation. The electron spin originates in the relativistic quantum equation (Dirac’s theory of electron)[19]. The discrete (two) quantum numbers (\( \sigma_z = \pm 1 \)) cannot change in the continuous non-relativistic limit, and hence the spin must be conserved. The countability and statistics of the fluxon is the fundamental particle properties. We postulate that the fluxon is a half-spin fermion with zero mass and zero charge.

We assume that the magnetic field \( B \) is applied perpendicular to the interface. The 2D Landau level energy,

\[ \epsilon = \hbar \omega_c \left( N_L + \frac{1}{2} \right), \quad \omega_c \equiv eB/m^*, \]
with the states \((N_L, k_y)\), \(N_L = 0, 1, 2, \ldots\), have a great degeneracy. \(m^*\) is the effective mass of an “electron”. The Center-of-Mass (CM) of any \(c\)-particle moves as a fermion (boson). The eigenvalues of the CM momentum are limited to 0 or 1 (unlimited) if it contains an odd (even) number of elementary fermions. This rule is known as the Ehrenfest-Oppenheimer-Bethe’s (EOB’s) rule [20]. Hence the CM motion of the composite containing an electron and \(Q\) fluxons is bosonic (fermionic) if \(Q\) is odd (even). The system of the \(c\)-bosons condenses below some critical temperature \(T_c\) and exhibits a superconducting state while the system of \(c\)-fermions shows a Fermi liquid behavior.

A longitudinal phonon, acoustic or optical, generates a density wave, which affects the electron (fluxon) motion through the charge displacement (current). The exchange of a phonon between electron and fluxon generates an attractive transition.

BCS [21] assumed the existence of Cooper pairs [22] in a superconductor, and wrote down a Hamiltonian containing the “electron” and “hole” kinetic energies and the pairing interaction Hamiltonian with the phonon variables eliminated. We start with a BCS-like Hamiltonian \(\mathcal{H}\) for the QHE [16]:

\[
\mathcal{H} = \sum_k \varepsilon_k n_k + \sum_k \varepsilon_{k'} n_{k'} - \sum_{q} v_0 \left[ B_{k'q}' B_{kq} + B_{kq}' B_{k'q} + B_{kq} B_{k'q}' \right],
\]

(7)

where \(n_k = c_{k}^\dagger c_{k}\) is the number operator for the “electron” (1) ['hole' (2), fluxon (3) at momentum \(k\) and spin \(s\) with the energy \(\varepsilon_{k,s}\), with annihilation (creation) operators \(c (c^\dagger)\) satisfying the Fermi anti-commutation rules:

\[
\{c_{k,s}, c_{k'}^{\dagger}_{s'}\} = \delta_{k,k'} \delta_{s,s'} \delta_{i,j}, \quad \{c_{k,s}, c_{k'}^{\dagger}_{s'}\} = 0 .
\]

(8)

The fluxon number operator \(n_{k_s}^{(3)}\) is represented by \(a_{k_s}^\dagger a_{k_s}\) with \(a (a^\dagger)\) satisfying the anti-commutation rules:

\[
\{a_{k,s}, a_{k'}^{\dagger}_{s'}\} = \delta_{k,k'} \delta_{s,s'} , \quad \{a_{k,s}, a_{k'}^{\dagger}_{s'}\} = 0 .
\]

(9)

The phonon exchange can create electron-fluxon composites, bosonic or fermionic, depending on the number of fluxons. The center-of-mass of any composite moves as a fermion (boson) if it contains an odd (even) numbers of elementary fermions. We call the conduction-electron composite with an
odd (even) number of fluxons \textit{c-boson} (\textit{c-fermion}). The electron (hole)-type \textit{c}-particles carry negative (positive) charge. We expect that electron (hole)-type Cooper-pair-like \textit{c}-bosons are generated by the phonon-exchange attraction from a pair of electron (hole)-type \textit{c}-fermions. The pair operators \(B\) are defined by

\[
B^{(1)}_{kq,s} \equiv c^{(1)\dagger}_{k+q/2,s} c^{(1)}_{-k+q/2,-s}, \quad B^{(2)}_{kq,s} \equiv c^{(2)\dagger}_{-k+q/2,-s} c^{(2)}_{k+q/2,s}.
\] (10)

The prime on the summation in Eq. (7) means the restriction: \(0 < \varepsilon^{(j)}_{ks} < \hbar \omega_D\), \(\omega_D =\) Debye frequency. The pairing interaction terms in Eq. (7) conserve the charge. The term \(-v_0 B^{(1)\dagger}_{k's} B^{(1)}_{ks}\), where \(v_0 \equiv |V_q V'_{q'}| (\hbar \omega_0 A)^{-1}\), \(A =\) sample area, is the pairing strength, generates a transition in the electron-type \textit{c}-fermion states. Similarly, the exchange of a phonon generates a transition between the hole-type \textit{c}-fermion states, represented by \(-v_0 B^{(2)\dagger}_{k's} B^{(2)}_{ks}\).

The phonon exchange can also pair-create (pair-annihilate) electron (hole)-type \textit{c}-boson pairs, and the effects of these processes are represented by \(-v_0 B^{(1)\dagger}_{k's} B^{(2)}_{ks} \left(-v_0 B^{(1)}_{k's} B^{(2)}_{ks}\right)\).

The Cooper pair is formed from two “electrons” (or “holes”). Likewise the \textit{c}-bosons may be formed by the phonon-exchange attraction from two like-charge \textit{c}-fermions. If the density of the \textit{c}-bosons is high enough, then the \textit{c}-bosons will be condensed and exhibit a superconductivity.

The pairing interaction terms in Eq. (7) are formally identical with those in the generalized BCS Hamiltonian \[23\]. Only we deal here with \textit{c}-fermions instead of conduction electrons.

The \textit{c}-bosons, having the linear dispersion relation, can move in all directions in the plane with the constant speed \((2/\pi)v^{(j)}_F\). The supercurrent is generated by \(\mp \textit{c}-\text{bosons} \) monochromatically condensed, running along the sample length. The supercurrent density (magnitude) \(j\), calculated by the rule: \(j = (\text{carrier charge: } e^*) \times (\text{carrier density: } n_0) \times (\text{carrier drift velocity: } v_d)\), is given by

\[
j \equiv e^* n_0 v_d = e^* n_0 \frac{2}{\pi} \left| v^{(1)}_F - v^{(2)}_F \right|,
\] (11)

where \(e^*\) is the \textit{effective} charge of carriers. The induced Hall field (magnitude) \(E_H\) equals \(v_d B\). The magnetic flux is quantized as in Eq. (5). Hence we obtain

\[
\rho_H \equiv \frac{E_H}{j} = \frac{v_d B}{e^* n_0 v_d} = \frac{1}{e^* n_0} n_0 \Phi_0 = \frac{n_0}{e^* n_0} \left( \frac{h}{e} \right).
\] (12)
We assume that the c-fermion containing an electron and an even number of fluxons has a charge magnitude $e$. For the integer QHE, $e^* = e$, $n_\phi = n_0$ for the carriers, thus we obtain $\rho_H = h/e^2$, explaining the plateau value observed for the integer QHE.

The supercurrent generated by equal numbers of $\pm$ c-bosons condensed monochromatically is neutral. This is reflected in the calculations in Eq. (11). The supercondensate whose motion generates the supercurrent must be neutral. If it has a charge, it would be accelerated indefinitely by the external field because the impurities and phonons cannot stop the supercurrent to grow. That is, the circuit containing a superconducting sample and a battery must be burnt out if the supercondensate is not neutral. In the calculation of $\rho_H$ in Eq. (12), we used the unaveraged drift velocity difference $(2/\pi)|v_F^{(1)} - v_F^{(2)}|$, which is significant. Only the unaveraged drift velocity cancels out exactly from numerator/denominator, leading to an exceedingly accurate plateau value.

We now extend our theory to include elementary fermions (electron, fluxon) as members of the c-fermion set. We can then treat the superconductivity and the QHE in a unified manner. The c-boson containing one electron and one fluxon can be used to describe the integer QHE.

Important pairings and the effects are listed below.
(a) a pair of conduction electrons, superconductivity
(b) fluxon and c-fermions, QHE
(c) a pair of like-charge conduction electrons with two fluxons, QHE in graphene.

4. The Room Temperature QHE

The QHE behavior observed for graphene is remarkably similar to that for GaAs/AlGaAs. The physical conditions are different however since the gate voltage and the applied magnetic field are varied in the experiments. The present authors regard the QHE in GaAs/AlGaAs as the superconductivity induced by the magnetic field. Briefly, the magnetoresistivity for a QHE system reaches zero (superconducting) and the accompanied Hall resistivity reveals a plateau by kind of the Meissner effect. The QHE state is not easy to destroy because of the superconducting energy gap in the c-boson excitation spectrum. If an extra magnetic field is applied to the system at optimum QHE state (the center of the plateau), then the system remains in the same superconducting state by expelling the extra field. If the field is reduced,
then the system stays in the same state by sucking in extra field fluxes, thus generating a Hall resistivity plateau. In the graphene experiments, the gate voltage applied perpendicular to the plane is varied. A little extra voltage polarizes the system without changing the superconducting state. This state has an extra electric field energy:

$$\frac{A}{2} \varepsilon_0 (\Delta E)^2,$$  \hspace{1cm} (13)

where $A$ is the sample area, $\varepsilon_0$ the dielectric constant, and $\Delta E$ is the extra electric field, positive or negative, depending on the field direction. If the gate voltage is further increased (or decreased), then it will eventually destroy the superconducting state, and the resistivity will rise from zero. This explains the flat $\rho_{xy}$ plateau and the rise in resistivity from zero.

We now examine the data shown in Fig. 2. We first observe that the right-left symmetry is broken. In Section 2 we saw that “electrons” and “holes” move in different channels with different masses, which means a broken symmetry. The applied gate voltage induce the conduction electrons and hence changes the Fermi surface. A relatively high voltage 20 V may bring the system to the van Hove singularity points in the neighborhood of which the conductn electron densities are high. We believe that this is where the prominent QHE is observed. We note that such discussions are possible only with the rectangular unit cell model, and not with the WS unit cell model.

We wish to derive the quantization rule (2). Let us first consider the case: $P = 0$. The QHE requires a BEC of $c$-bosons. Its favorable environment is near the van Hove singularities, where the Fermi surface changes its curvature sign. For graphene, this happens when the 2D Fermi surface just touches the Brillouin zone boundary and “electrons” or “holes” are abundantly generated. It is difficult to predict what set of points, where the QHE occur. However, the quantization rule given by Eq. (2) is realized if the $c$-bosons are formed by the phonon exchange attraction from a pair of like-charge $c$-fermions, each containing a conduction electron and two (2) fluxons. We postulate that each $c$-fermion has the effective charge $e$:

$$e^* = e \quad \text{for any } c\text{-fermion.}$$  \hspace{1cm} (14)

In the case of the $c$-fermion composed of an electron and two fluxons, we then must have

$$n_\phi = n_e / 2,$$  \hspace{1cm} (15)
where \( n_e \) is the electron density. The c-boson contains two c-fermions. Thus the fluxon density \( n_\phi \) in Eq. (15) can be expressed in terms of the density of c-bosons \( n_0 \): \( n_\phi = n_0/4 \). Using Eq. (12), we obtain

\[
\sigma_H \equiv \rho_H^{-1} = \frac{j}{E_H} = \frac{e^* n_0 v_d}{v_d B} = \frac{en_0}{n_\phi \Phi_0} = \frac{4e^2}{\hbar}.
\]  

(16)

The QHE states with integers \( P = 1, 2, \cdots \) are generated on the weaker field side. Their strengths decreases with increasing \( P \). Thus, we have obtained the rule (2) within the framework of the traditional fractional QHE theory.

In summary, we have successfully described the QHE in graphene without introducing Dirac fermions. In solid state physics we deal with “electrons” and “holes”, which move in crystals and respond to the Lorentz force. These charged particles are considered as the Bloch wave packets having sizes and charges (not a point particle). The bare point-like Dirac particle, if exists, would be dressed with charge clouds, and it would then acquire a mass. It is very difficult to theoretically argue that Dirac fermions appear only in graphene but in no others. The relativistic Dirac electron moves with the speed of light, \( c \). The observed particle in graphene moves with the speed of the order \( 10^8 \) m/s much lower than the light speed \( c = 3 \times 10^{10} \) m/s. It is difficult to explain this difference of two orders of magnitude from the first principles. Dirac fermion model is inherently connected with the WS cell model, which is rejected in favor of the rectangular unit cell model in our theory. The plateau observed for the Hall conductivity \( \sigma_{xy} \) is caused by the c-bosons condensed. This plateau behavior arises from the superconducting state and hence it is unlikely to be explained based on the Dirac Fermion model.

References

[1] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos and A.A. Firsov, Nature 438, 197 (2005).

[2] K.S. Novoselov, Z. Jiang, Y. Zhang, S.V. Morozov, H.I. Stormer, U. Zeitler, J.C. Maan, G.S. Boebinger, P. Kim, and A.K. Gaim, Science 315, 1379 (2007).
[3] N.W. Ashcroft and N.D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976). pp. 2–7, pp. 114–119, pp. 133–135, p. 137, pp. 214–218, pp. 256–258, pp. 290–293, p. 300.

[4] E. Wigner and F. Seitz, Phys. Rev. 43, 804 (1933).

[5] S. Fujita and A. Suzuki, J. Appl. Phys., 107, 013711 (2010).

[6] S. Fujita, Y. Takato and A. Suzuki, Mod. Phys. Lett. B, 25, 223 (2011).

[7] S. Fujita, A. Jovaini, S. Godoy, and A. Suzuki, Phys. Lett., A376, 2808 (2012).

[8] F. Bloch, Zeits. Phys. 52, 555 (1928).

[9] Y. Zhang, Y.-W. Tan, H.L. Stormer, and P. Kim, Nature 438, 201 (2005).

[10] R.E. Prange and S.M. Girvin, eds., *Quantum Hall Effect* (Springer Verlag, New York, 1990); M. Stone, eds., *Quantum Hall Effect* (World Scientific, Singapore, 1992); S. Das Sarma and A. Pinczuk, *Perspectives in Quantum Hall Effects* (John Wiley, New York, 1997); T. Chakraborty and P. Pietiläinen, *Quantum Hall Effect* (Springer-Verlag, Berlin, 1998); Z.F. Ezawa, *Quantum Hall Effects* (World Scientific, Singapore, 2000); B.I. Halperin, P.A. Lee and H. Read, Phys. Rev. B 47, 7312 (1993).

[11] J.K. Jain, Phys. Rev. Lett. 63, 199 (1989); Phys. Rev. B 40, 8079 (1989); *ibid.* B 41, 7653 (1990); Surf. Sci. 263, 65 (1992).

[12] R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983); Science 242, 525 (1988).

[13] S.M. Girvin and A.H. MacDonald, Phys. Rev. Lett. 58, 1252 (1987).

[14] N. Read, Phys. Rev. Lett. 62, 86 (1989).

[15] R. Shankar and G. Murthy, Phys. Rev. Lett. 79, 4437 (1997).

[16] S. Fujita and Y. Okamura, Phys. Rev. 369, 155313 (2004).

[17] S.C. Zhang, T.H. Hansson, and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989).

[18] L. Onsager, Phil. Mag. 43, 1006 (1952).
[19] P.A.M. Dirac, *Principles of Quantum Mechanics*, 4th ed. (Oxford Univ. Press, Oxford, 1958), pp. 248 - 252, pp. 253 - 263, p. 267.

[20] P. Ehrenfest and J. R. Oppenheimer, Phys. Rev. **37**, 333 (1931), H. A. Bethe and R. Jackiw, *Intermediate Quantum Mechanics*, 2nd ed. (Benjamin, New York, 1968), p. 23, S. Fujita, S-P Gau and A. Suzuki, J. Korean Phys. Soc. **38**, 456 (2001).

[21] J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[22] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).

[23] S. Fujita, K. Ito and S. Godoy, *Quantum Theory of Conducting Matter, Superconductivity* (Springer, New York, 2009), pp. 73 - 75.