Quantum error correction and fault-tolerance have provided the possibility for large scale quantum computations without a detrimental loss of quantum information. A very natural class of gates for fault-tolerant quantum computation is the Clifford gate set and as such their usefulness for universal quantum computation is of great interest. Clifford group gates augmented by magic state preparation give the possibility of simulating universal quantum computation. However, experimentally one cannot expect to perfectly prepare magic states. Nonetheless, it has been shown that by repeatedly applying operations from the Clifford group and measurements in the Pauli basis, the fidelity of noisy prepared magic states can be increased arbitrarily close to a pure magic state [1].

We investigate the robustness of magic state distillation to perturbations of the initial states to arbitrary locations in the Bloch sphere due to noise. Additionally, we consider a depolarizing noise model on the quantum gates in the decoding section of the distillation protocol and demonstrate its effect on the convergence rate and threshold value. Finally, we establish that faulty magic state distillation is more efficient than fault-tolerance-assisted magic state distillation at low error rates due to the large overhead in the number of quantum gates and qubits required in a fault-tolerance architecture. The ability to perform magic state distillation with noisy gates leads us to conclude that this could be a realistic scheme for future small-scale quantum computing devices as fault-tolerance need only be used in the final steps of the protocol.

**Keywords:** Universal quantum computation, magic state distillation, quantum fault-tolerance

Communicated by: R Jozsa & B Terhal
1 Introduction

Processes such as imperfect control of quantum operations or unintended coupling between qubit systems and their environment lead to errors in any realistic implementation of a quantum computing device. As such, quantum error correction has been developed in order to recover the quantum information that would otherwise be lost due to these faults [2, 3, 4]. However, quantum error correction itself is not enough for the implementation of a robust quantum computing device as errors can propagate badly between the qubits. Propagating errors could prove to be detrimental to the recovery of quantum information and need to be avoided in order to implement any realistic error correction scheme. Fault-tolerant quantum computation aims to address this concern by encoding the information of each qubit into a larger Hilbert space of many qubits and performing encoded quantum gates in such a way that errors do not propagate through multiple qubit blocks [5]. Fault-tolerance allows the faults to remain tractable at the cost of needing a polylogarithmic increase in the number of qubits and quantum gates to perform encoded operations when the error rate of the quantum gates is below a certain target threshold [6, 7, 8, 9, 10, 11].

A desired property of encoded gate operations is transversality, which limit the propagation of errors in the encoded states of the fault-tolerant architecture. The Clifford gate set, the group of gates generated by the Hadamard gate, the phase gate, and CNOT gate, has been shown to be transversal for many quantum codes [12]. The Clifford gates, along with measurement and state preparation in the $Z$-eigenbasis form the class of stabilizer operations, which have been shown to be efficiently classically simulatable [13]. In order to use Clifford gates for universal quantum computation, one requires the ability to produce a pure, single-qubit, non-stabilizer state [14], known as a magic state. Perfect magic state preparation is difficult due to experimental errors, however magic state distillation allows for the creation of an arbitrarily high fidelity magic state from noisy ancillas by repeatedly applying stabilizer operations [1, 15, 16]. In this work we investigate the effect of noise, in the state preparation and quantum gate application, on the convergence of the magic state distillation protocol.

Bravyi and Kitaev developed the theory of magic state distillation and presented two distillation protocols of five and fifteen qubits [1]. In this work we focus on the five qubit distillation routine, where five copies of a noisy magic state are used to extract a single state of higher fidelity with respect to the magic state. The process is then iterated to increase the fidelity to be arbitrarily high. The input states to the distillation scheme are assumed to be along the magic state axis in the Bloch sphere, the axis connecting the magic state and its orthogonal complement. This assumption is based upon the ability to perform a dephasing channel that collapses all points of the Bloch sphere to their projection along this axis. In this work, we extend the analysis of the distillation scheme to one where the location of the input state is an arbitrary state in the Bloch sphere. The motivation for this is twofold, performing a dephasing operation in an experimental setup would introduce errors that would manifest themselves as perturbations off the magic state axis; moreover, there may be states off the magic state axis that converge faster to the magic state after multiple iterations than the states with the same fidelity on the magic state axis. Performing such an analysis will enable us to conclude that the magic state distillation protocol is robust to slight perturbations about the magic state axis in the Bloch sphere due to noise.

In Sec. 3 we turn our attention to the effect of noise present in the quantum gates of the
We investigate the consequences of depolarizing noise, a one-parameter noise model, on the one and two-qubit Clifford gates in the five-qubit decoder of the protocol. Such noise will affect the rate of convergence, increase the fidelity threshold for the input states of the protocol (the quality of the state preparation will have to be increased), and pose a restriction on the ability to prepare a magic state with arbitrarily high fidelity. As such, with the development of fault-tolerant schemes for the reduction of noise, one can ask if it is more efficient in the total number of quantum gates to use faulty gates or fault-tolerant encoded gates to perform the distillation protocol. In Sec. 3.2 we show that using faulty Clifford gates is the more efficient scheme when the desired error rate of the universal gate requiring magic state distillation is slightly below that of the faulty Clifford gates. As such, faulty magic state distillation would be very useful at lower levels of encodings where the Clifford gates may have higher fidelities due to transversality properties that the universal $\pi/8$ or $\pi/12$-phase gate would not have for certain quantum error correcting codes. The ability to perform magic state distillation without the use of a fault-tolerant encoding is promising for future experimental realizations of multiple rounds of state distillation on small-scale quantum computers.

2 The evolution of quantum states under magic state distillation with perfect Clifford gates

The input states to Bravyi and Kitaev’s magic state distillation protocol are assumed to be along the magic state axis [1]. However, preparing such states may prove to be difficult experimentally. In this section we study the convergence of the distillation scheme under perturbations about the magic state axis. Moreover, we show that for low fidelity input states, the convergence to the magic state may be improved for states away from the magic state axis. This suggests that while performing a dephasing operation to initialize the input states to be along the axis may be useful, it is not absolutely necessary in certain fidelity regimes.

The Clifford gate set is generated by the following gates,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad CNOT = \begin{pmatrix} I & 0 \\ 0 & \sigma^x \end{pmatrix},$$

where each individual gate can be performed on any qubit, and CNOT can be performed on any pair of qubits where $I$ and $\sigma^x$ are 2-by-2 Pauli matrices. The power of magic state distillation is that it requires only the use of Clifford gate operations, along with $|0\rangle$ state ancilla preparation and measurement in the $Z$-basis, to distill multiple copies of noisy magic states to one of higher fidelity, provided the initial state is above a given threshold. This is appealing as Clifford gates have been shown to have transversality in many quantum error correcting codes [12], and as such can be implemented fault-tolerantly in order to reduce their error rate.

Let $\sigma_i$ denote the $i^{th}$ Pauli matrix in the computational basis. There are two types of magic states, up to one-qubit Clifford operators, the H-type magic state,

$$|H\rangle\langle H| = \frac{1}{2} \left[ I + \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z) \right],$$
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which can be used to implement the $\pi/8$-phase gate, $P_{\pi/8} = e^{i\pi/8}|0\rangle\langle 0| + e^{-i\pi/8}|1\rangle\langle 1|$, and the T-type magic state,

$$|T\rangle\langle T| = \frac{1}{2} \left[ I + \frac{1}{\sqrt{3}}(\sigma^x + \sigma^y + \sigma^z) \right],$$

which can be used to implement the $\pi/12$-phase gate $P_{\pi/12}$ [1], both of which, along with the Clifford gates, provide universal quantum computation. Many efforts have contributed to building protocols for magic state distillation and achieving tight noise thresholds for the noisy input ancillas to the distillation protocol in order to understand the transition from classically simulatable quantum computation to genuine quantum computation [1, 15, 16, 17, 18, 19, 20, 21, 22]. Additionally, an experimental demonstration of a single round of Bravyi and Kitaev’s distillation protocol of T-type magic states has been performed in Nuclear Magnetic Resonance (NMR) [23].

The first step of Bravyi and Kitaev’s distillation protocol [1] is to perform a dephasing operation $D$ on five copies of the initial state of the quantum system,

$$D(\rho) = \frac{1}{3} \left( \rho + T\rho T^{\dagger} + T^{\dagger}\rho T \right),$$

where $T = KH$ is a Clifford group gate. If the initial state of the system is expressed according to its Bloch sphere coordinates $(x, y, z)$,

$$\rho = \frac{1}{2} \left[ I + x\sigma^x + y\sigma^y + z\sigma^z \right],$$

the transformation $D$ is equivalent to projecting the state $(x, y, z)$ onto the magic state axis connecting the states $|T_0\rangle$ and $|T_1\rangle$ in the Bloch sphere, where $|T_1\rangle$ is the state orthogonal to $|T_0\rangle$,

$$D(\rho) = \frac{1}{2} \left[ I + \frac{x + y + z}{3} (\sigma^x + \sigma^y + \sigma^z) \right].$$

The dephasing operation leads to the ability to derive a clean threshold for the input fidelity of the initial states in order for the magic state distillation protocol to be beneficial. However, errors in the implementation of the quantum information processor could lead to a preparation of states away from the magic state axis. In this section, we provide an analysis of the effectiveness of the magic state distillation protocol for states prepared at an arbitrary location in the Bloch sphere and give modified target fidelities for state distillation under such noisy state preparation.

The distillation protocol consists of the above dephasing transformation on five prepared initial states, followed by a measurement of the stabilizers of the five-qubit code [23, 24], $XZZXI$, $IXZZX$, $XIXZZ$, $ZXIXZ$, and decoding upon obtaining the “$+1$” eigenstate of all the stabilizers [1], where $X = \sigma^x$ and $Z = \sigma^z$. The roles of the measurement and decoding can be reversed, and the overall circuit can be described by the diagram in Fig. 1. Various encoding/decoding circuits can be developed to encode the five qubit code, we chose to analyze the circuit presented in Fig. 1 as once a qubit is used as a control qubit in a two-qubit gate, it is no longer used in any further two-qubit gates, thus minimizing the propagation of errors through the circuit [26]. Upon following the steps outlined in the above procedure, the initial
noisy states must have a fidelity greater than $F_T = \frac{1}{2} (1 + \sqrt{3}/7) \approx 0.8273$ with respect to the magic state in order for the output state to be of higher fidelity \[1\], where the fidelity with respect to the magic state is defined as $F(\rho) = \langle T_0 | \rho | T_0 \rangle$. Repeating the protocol to obtain multiple copies of the state of increased fidelity $\rho_m$, the process can then be iterated to obtain magic states with arbitrarily high fidelity.

We shall consider the scenario where the dephasing gates are omitted from the protocol, and only the gates in the dashed box in Fig. 1 are implemented, followed by post-selection upon obtaining outcomes of “+1” for $Z$-basis measurements on the top four qubits. If the input state is now located at an arbitrary position in the Bloch sphere given by coordinates $(x, y, z)$,
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then the output state coordinates are as follows:

\[
\begin{align*}
    x_{\text{out}} &= -z(z^4 - 5x^2 + 5y^2(x^2 - 1)) \\
    y_{\text{out}} &= -y(y^4 - 5z^2 + 5x^2(z^2 - 1)) \\
    z_{\text{out}} &= -x(x^4 - 5z^2 + 5y^2(z^2 - 1))
\end{align*}
\]

(5)

The plane of states that have fidelity \( F \) are the states in the Bloch sphere that satisfy,

\[
x + y + z = \sqrt{3}(2F - 1).
\]

(6)

For a given plane of constant input fidelity \( F_{\text{in}} \), define a new coordinate system for that plane where \( r \) is the radial distance of the input state from the magic state axis, and \( \theta \) as the angle between the distance vector and the modified \( x \) axis, as shown in Fig. 2. The difference between the input fidelity and output fidelity can then be expressed according to these coordinates as

\[
d = F_{\text{out}} - F_{\text{in}} = -\frac{2\left(a(54 - 60a^2 + 14a^4 + 135r^4) + 15\sqrt{6}(-3 + a^2)r^3 \cos 3\theta\right)}{2\sqrt{3}\left(108 + 20a^4 + 135r^4 + 60\sqrt{6}ar^3 \cos 3\theta\right)},
\]

(7)

where \( a \) is related to the input fidelity, \( a = \sqrt{3}(2F_{\text{in}} - 1) \). As such, the ability of the distillation protocol to increase the fidelity of the input states depends on the distance of the initial states from the magic state axis and on the spatial angle with respect to the modified \( x \) axis as well as the input fidelity.

Fig. 3. Final states of convergence after multiple rounds of distillation for states on the fidelity plane \( F = 0.886 \), that is a cut of the Bloch sphere with states of a fixed fidelity, as given by Eq. (6). The states in red represent the states that converge to the magic state after multiple rounds of distillation. The states in light and dark blue converge to states with the coordinates \((\pm 1, \pm 1, \pm 1)/\sqrt{3}\) in the Bloch sphere, where the number of sign changes from the coordinates of the \(|T\rangle\) magic state \((1, 1, 1)/\sqrt{3}\) is given by the shading of blue. The states in pink converge to the state orthogonal to \(|T\rangle\), and black to the maximally mixed state.
Fig. 4. Convergence of the states around the perfect Clifford gate threshold $F = 0.8273$ for states along the magic state axis. All states much below the theoretical threshold ($F = 0.823$) converge away from the magic state axis, yet some states slightly below the threshold for on-axis convergence converge to the magic state ($F = 0.8270$). This convergence only appears for states close to the three angles of maximum output fidelity on the fidelity plane. All states close to the magic state axis converge to the magic state for fidelities above the threshold ($F = 0.830$).

Fig. 3 demonstrates the dependence on the distance from the magic state axis and angle for convergence to the magic state after many iterations for the set of states on the plane with fidelity $F = 0.886$ in the positive octant ($x, y, z > 0$). States close to the magic state axis converge to the magic state, while states far away from the axis converge to other undesirable states in the Bloch sphere. Perhaps more surprisingly, there are states below the distillation threshold set by Bravyi and Kitaev [1] that can converge to the magic state. As Fig. 4 shows, states away from the magic state axis on the fidelity plane of $F = 0.8270$ converge to the magic state, while those on the axis converge to the maximally mixed state, as expected as $F$ is below the fidelity threshold for states on the axis. Notice in Eq. (7) that the maximal increase in the fidelity of the state after the distillation procedure occurs for angles $\theta = 0, 2\pi/3, 4\pi/3$. Fixing $\theta = 0$, the difference in fidelity can show an increase as a function of the distance from the magic state axis $r$ for small distances before dropping off rapidly as the distance from the axis grows. For an input fidelity just below the fidelity threshold set in Ref. [1], the fidelity difference can be negative for states close to the axis and increase to be positive for certain distances away from the axis, as Fig. 5(a) shows, but only for states whose angle is close to the angles of maximal increase, as plotted in Fig. 5(b). Therefore at angles close to the angles of maximal increase, one can obtain states whose fidelity threshold can be below the on-axis threshold of $\frac{7}{3}(1 + \sqrt{\frac{3}{7}}) \approx 0.8273$. As such, by performing the magic state distillation routine without the dephasing transformation, one can slightly lower the threshold for the fidelity of input states to be $0.825$ for states at the angle of maximal increase.

We may thus conclude that the dephasing operation that projects the noisy magic states onto the magic state axis is not necessarily needed as there are regions of convergence around this axis. As such, noisy dephasing processes that would have slight perturbations off the magic state axis would not be detrimental to the convergence to the magic state. Furthermore,
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for certain states off the magic state axis and of fidelity just below that derived by Bravyi and Kitaev, omitting the dephasing operation would be beneficial as it would allow such states to remain useful for magic state distillation.

3 Magic state distillation with noisy Clifford gates

3.1 Error rate of faulty magic state distillation

In an experimental realization of the implementation of magic state distillation, any quantum gate will introduce noise. As such, an interesting experimental question would be: to what level of noise is the distillation still beneficial? In order to address such a question, we consider the case where all the gates in the decoding circuit, shown by the boxed region in Fig. 1 are subjected to depolarizing noise. Depolarizing noise is a common noise model for physical implementations of quantum information processing. A noisy one-qubit Clifford gate $G$ will be modelled as follows:

$$G \rightarrow G \Lambda_1,$$

where the gate $\Lambda_1$ performs the depolarizing transformation with a noise parameter $p_1$,

$$\Lambda_1(\rho, p_1) \rightarrow (1 - p_1)\rho + p_1 \frac{I}{2} = (1 - \frac{3p_1}{4})\rho + \frac{p_1}{4}(\sigma^x\rho\sigma^x + \sigma^y\rho\sigma^y + \sigma^z\rho\sigma^z).$$ (8)

Similarly, we introduce noise to a two-qubit controlled Clifford gate by adding a two-qubit depolarizing gate,
where the two-qubit depolarizing gate $\Lambda_2$ is defined by the transformation with a two-qubit noise parameter $p_2$,

$$
\Lambda_2(\rho, p_2) \rightarrow (1 - p_2)\rho + p_2 \frac{I_2}{4}
$$

$$
= \left( 1 - \frac{15p_2}{16} \right) \rho + \frac{p_2}{16} \left( (I \otimes \sigma_x)\rho(I \otimes \sigma_x) + (I \otimes \sigma_y)\rho(I \otimes \sigma_y) + \ldots \right) 
+ (\sigma_z \otimes \sigma_y)\rho(\sigma_z \otimes \sigma_y) + (\sigma_z \otimes \sigma_z)\rho(\sigma_z \otimes \sigma_z). \quad (9)
$$

We shall only consider errors affecting the gates in the decoding procedure. In order to compare the fidelity threshold for the noisy decoding procedure with the ideal magic state distillation protocol proposed by Bravyi and Kitaev \[1\], we assume the input states to the decoding procedure will introduce off diagonal terms in the output density matrix. Additionally, a depolarizing fault on the one-qubit error that would change a single measurement outcome does not contribute off-diagonal errors to the final output density matrix. However, such matrix elements will be on the order of $p_2$, which for low levels of noise will be negligible compared to the strength of the coefficients along the diagonal terms. The reason there are no first-order terms proportional to $p_1$ is that all the one-qubit gates in the protocol are at the end of the circuit, preceding the syndrome measurement, as shown in Fig. \[1\]. It turns out that any one-qubit error that would change a single measurement outcome does not contribute off-diagonal errors to the final output density matrix. Additionally, a depolarizing fault on the output density matrix will only contribute errors to the diagonal coefficients in the output

$$
|T_0\rangle\langle T_0|:
\begin{align*}
&= \frac{1 - 5p_1 - 8p_2}{6} \left( (1 - \epsilon)^5 + 5\epsilon^5(1 - \epsilon)^2 \right) + \frac{p_1}{36} \left[ 19 - 87\epsilon + 197\epsilon^2 - 164\epsilon^3 + 6\epsilon^4 + 32\epsilon^5 \right] \\
&\quad + \frac{p_2}{54} \left[ 20 - 44\epsilon + 107\epsilon^2 - 106\epsilon^3 + 28\epsilon^4 + 8\epsilon^5 \right].
\end{align*}

(10)
$$

$$
|T_1\rangle\langle T_1|:
\begin{align*}
&= \frac{1 - 5p_1 - 8p_2}{6} \left( \epsilon^5 + 5\epsilon^2(1 - \epsilon)^3 \right) + \frac{p_1}{36} \left[ 3 + \epsilon + 61\epsilon^2 - 180\epsilon^3 + 166\epsilon^4 - 32\epsilon^5 \right] \\
&\quad + \frac{p_2}{54} \left[ 13 - 4\epsilon + 37\epsilon^2 - 86\epsilon^3 + 68\epsilon^4 - 8\epsilon^5 \right].
\end{align*}

(11)
$$

$$
|T_0\rangle\langle T_1|:
\begin{align*}
&= \frac{p_2(1 + i)(-1 + 2i)}{432} \left[ (-2 + 3i) - (1 + 5i)\sqrt{3} + ((-2 - 9i) + (3 + 10i)\sqrt{3})\epsilon \\
&\quad + ((6 + 9i) - (3 + 6i)\sqrt{3})\epsilon^2 + ((-8 - 6i) + (2 - 8i)\sqrt{3})\epsilon^3 + (4 + 4i\sqrt{3})\epsilon^4 \right],
\end{align*}

(12)
$$

where $\epsilon = 1 - \langle T_0|\rho_{in}|T_0\rangle$ is the error of the initial state. Note that depolarizing noise in the decoding procedure will introduce off diagonal terms in the $|T_0\rangle\langle T_1|$ basis, that is, will produce an output state that deviates away from the magic state axis. However, such matrix elements will be on the order of $p_2$, which for low levels of noise will be negligible compared to the strength of the coefficients along the diagonal terms. The reason there are no first-order terms proportional to $p_1$ is that all the one-qubit gates in the protocol are at the end of the circuit, preceding the syndrome measurement, as shown in Fig. \[1\]. It turns out that any one-qubit error that would change a single measurement outcome does not contribute off-diagonal errors to the final output density matrix. Additionally, a depolarizing fault on the output density matrix will only contribute errors to the diagonal coefficients in the output.
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density matrix when written in the $|T_0⟩-|T_1⟩$ basis. Therefore, we may assume that upon iterating the distillation protocol, the input states will always remain along the magic state axis. This assumption is also well motivated from the results of Sec. 2 that show that slight deviations off the magic state axis will not affect the convergence of the distillation scheme. For small values of $\epsilon$, the output error $\epsilon_{out}$, the normalized coefficient of the $|T_1⟩⟨T_1|$ term, can be approximated to be $5\epsilon^2 + p_1/2 + 13p_2/9$. Thus in the limit of infinite iterations of the distillation protocol the error rate will be linear in the terms $p_1$ and $p_2$.

Motivated by the average one and two-qubit gate errors in a recent benchmarking experiment in nuclear NMR [27], we have chosen different values for the noise parameters $p_1$ and $p_2$. The graph in Fig. 6 shows the output fidelity of the magic state distillation protocol for different error strengths of the one and two-qubit gates. The results for error strengths given by the benchmarking NMR results, $E_1 = 1.3 \times 10^{-4}$ and $E_2 = 4.7 \times 10^{-3}$, are given by the black curve. In order to obtain an appreciation of the decrease in output fidelity cause by the strength of both of these errors, we have also plotted the output fidelity corresponding to the case when the one and two-qubit errors have the same strength, namely $1.3 \times 10^{-4}$ and $4.7 \times 10^{-3}$. Note that when both errors are on the order of $10^{-4}$ the decrease in fidelity is minimal, this leads one to conclude that the decrease in fidelity in the black curve is caused mostly by the larger error on the two-qubit gates. Finally, note that for the case of choosing an error model based on the results from Ref. [27], the new threshold for the minimal input fidelity is 0.842, which is larger than the theoretical noiseless threshold of 0.8273, and the maximum output fidelity that can be reached through repeated applications of the distillation protocol is 0.9895 with respect to the magic state.

Due to the errors on the applied Clifford gates in the decoding procedure, distilling states
\[ T. Jochym-O'Connor, Y. Yu, B. Helou, and R. Laflamme \]

Fig. 7. Gate sequence in order to apply the $P_{\pi/12}$ gate to an arbitrary input state $|\psi\rangle$. The states $\rho_m$ are the magic states obtained from the magic state distillation protocol. $M_1$ and $M_2$ are measurements of the observable $Z \otimes Z$. In the case of $M^+_1$, one needs to post-select upon obtaining the “+1” outcome. Depending of the result of the measurement $M_2$, either a $P_{\pi/12}$ or $P_{-\pi/12}$ gate is applied and the output state is $\phi$. The remaining two qubits are discarded.

arbitrarily close to the magic state is no longer possible, thus the distilled output state will have the form,

\[ \rho'_m = (1 - \epsilon') |T_0\rangle\langle T_0| + \epsilon' |T_1\rangle\langle T_1|, \tag{13} \]

and the value of the error $\epsilon'$ will be fixed away from 0. However, the application of a universal non-Clifford gate is still possible with such a state, albeit with a certain level of noise. Following the procedure laid out in Ref. [1], shown in Fig. 7, one can achieve the application of the gate $P_{\pi/12}$ to an arbitrary state $|\psi\rangle = a |0\rangle + b |1\rangle$ by using multiple copies of the noisy magic state $\rho'_m$. The fidelity of the applied gate is

\[ F = 1 - \frac{12\epsilon' |a|^2(1 - |a|^2)^2}{3 + (1 - 2\epsilon')^2} \geq 1 - \epsilon', \]

where $\epsilon'$ is the error rate of the distilled magic state given in Eq. (13). After multiple iterations of the distillation protocol, if the error of initial state was below the threshold for state distillation, the error rate $\epsilon'$ will be approximately $p_1/2 + 13p_2/9$. Thus, in order to apply the universal gate with fidelity above $1 - \epsilon'$, one would typically have to apply a final fault-tolerant iteration in order to reduce the error rates of the original Clifford gates to a smaller logical error, as will be discussed in Sec. 3.2.

3.2 Comparing faulty magic state distillation and fault-tolerant magic state distillation

Fault-tolerant quantum computation is a method to reduce the error rate of a given quantum operation using states and gates encoded into a higher-dimensional Hilbert space. Such encodings, combined with the ability of projective measurement and post-selection, provide a means to increase the fidelity of our quantum gates at the expense of using additional qubits and quantum gates to perform the desired encodings of states and encoded operations [5, 6, 7, 8, 9, 10].

Consider a scenario where we are presented with a physical device that is not restricted in the number of qubits at its disposal but limited in the number of gates that can be applied coherently. We could then envision two different methods for applying the magic state distillation procedure, either by applying the protocol with the faulty gates at our disposal or first performing a fault-tolerant encoding of the qubits and then performing the distillation protocol with less noisy encoded logical quantum gates. The faulty distillation protocol, as
described and analyzed in Sec. 3.1 suffers from a reduced convergence rate and a fundamental limit in the ability to distill a magic state with high fidelity. Thus using a fault-tolerance encoding of the Clifford gates appears to be advantageous. However the cost associated with encoding each of the Clifford gates in a fault-tolerant encoding outweighs the savings one would obtain in the number of Clifford gates one would need to apply at lower noise rates in the faulty magic state distillation scheme unless the desired target fidelity is very close to the limit set by the error in the Clifford gates, as discussed in Sec. 3.1.

Fig. 8. Total number of two-qubit gates as a function of input fidelity for the state distillation scheme to achieve a target output fidelity of (a) 0.997 and (b) 0.998. The red dots denote the number of gates required for perfect gates and can be thought of as a lower bound in the number of fault-tolerant encoded logical gates that would be required. The blue dots represent the total number of gates for the faulty distillation protocol, where the gate error probability \( p_1, p_2 \) is 0.001. Note that the y-axis in this figure is plotted logarithmically in order to capture that the increase in the number of gates for each iteration of the distillation scheme scales approximately by a factor of 30.

Fig. 8 plots the total number of two-qubit gates that would be required for either scheme to achieve a desired final fidelity for the magic state. The red dots plotted in Fig. 8 are the number of gates in the ideal setting with no noise, this can be thought of as a lower bound of the number of logical fault-tolerant gates that would need to be applied for any
fault-tolerant encoding. As expected, the number of gates exhibit a step function behaviour, characteristic of the number of iterations of the distillation subroutine that would be required to achieve the final fidelity target. The skewing of the steps is due to the probability of measuring the trivial syndrome, which is reduced for initial states with lower fidelity. One notices that the number of iterations for the fault-tolerant magic state distillation is lower than that of the faulty procedure, this is due to the fact that noise in faulty Clifford gates can decrease the convergent rate for states along the magic state axis and will lower the probability of measuring the trivial syndrome. Thus if one were to be presented with faulty Clifford gates and were to encode the states such that the error rate of the encoded operations were negligible, then the total number of encoded \textit{logical} gates required would be lower than the number of faulty Clifford gates to increase the fidelity of the magic state to a desired level. However, in the regime of typical one and two-qubit gate errors, $10^{-3}$–$10^{-2}$, the fault-tolerant encodings whose threshold rates are above these levels typically use on the order of $100$–$1000$ two-qubit physical gates per encoded logical gate \cite{28}. As such, in order for a fault-tolerant encoded scheme to show an improvement in the total number of gates required for the full distillation protocol, the encoded scheme would have to use on the order of $100$–$1000$ times less logical gates than that of the faulty distillation scheme. Consider the plot given in Fig. 8(a) which compares the number of faulty gates, with error probability $10^{-3}$, with the ideal case of encoded gates, with no error, required for the distillation of a state with fidelity $0.997$ with respect to the magic state. Such a scenario would be of great interest when the error rate of the Clifford gates has been reduced using a fault-tolerant encoding, while the universal $\pi/12$-phase gate has a higher error rate as it is not encoded with the same error correction properties as the Clifford gates. Performing magic state distillation to this target fidelity reduces the error rate of the $\pi/12$-phase gate to $0.003$ due to the distilled magic state, which is below the fault-tolerance threshold for many codes. As the figure shows, there are certain fidelity regimes where the number of rounds of perfect distillation (denoted by the red dots) is one less than the number of rounds of distillation for the faulty distillation (denoted by the blue dots). In such cases, the overhead of the extra round of distillation is approximately 30, as one needs 30 states to perform a round of distillation, since each individual distillation scheme required 5 qubits and the probability of success is $1/6$, thus on average needing 6 different sequences of distillation. Consider now Fig. 8(b) which performs the same analysis with a target fidelity of $0.998$, which corresponds to an error rate that is just above the limit set by the one- and two-qubit Clifford gate errors, as discussed in Sec. 3.1. In this case, the noisy magic state distillation scheme requires, for most input states, two extra rounds of distillation, which corresponds to an overhead of approximately 900 gates. A reduced convergence rate is due to the fact that preparing a magic state of such a fidelity is at the limit of the capabilities of distillation with Clifford gates with an error probability of $10^{-3}$. As mentioned above, encoding into a higher level of fault-tolerant encoding at the given Clifford gate error rate would use $100$–$1000$ gates, as such performing faulty magic state distillation to an error rate of $0.003$, which costs at most an overhead of 30 times more gates, would be more advantageous than fault-tolerantly performing magic state distillation. The case of distilling to an error rate of $0.002$ does not provide as clear an advantage, as it would cost on the order of 900 more gates for certain cases to perform faulty magic state distillation as opposed to perfect magic state distillation. As such, if the fault-tolerant encoding reduces the Clifford
The robustness of magic state distillation against errors in Clifford gates

gate error rate to be sufficiently small without using too expensive an overhead of gates, the
fault-tolerant encoding might prove to be more advantageous. However, reducing the error
rate of the universal gates to be 0.003 would still prove to be very useful as it would be well
below the fault-tolerance threshold for many quantum computation architectures. We thus
conclude that in general using faulty Clifford gates is more efficient for magic state distillation
at noise levels comparable to those in current quantum information implementations, paving
the way for near-future implementations of the distillation protocol. Additionally, this work
concludes that access to the set of Clifford gates below the target fidelity threshold for fault-
tolerant quantum computation could be used in a faulty distillation scheme to reduce the error
rate of an additional gate, providing universality below the target fidelity for fault-tolerant
computation, without having to encode the faulty Clifford gates to a further level in the
fault-tolerant hierarchy. This analysis considers the number of two-qubit gates that would be
required in the distillation protocol for either scheme as typical numbers of two-qubit gates
have been studied extensively in past works [28], however such an analysis could be extended
to one-qubit gates and the authors believe that the behaviour will be equivalent.

Finally, one should note that fault-tolerance would be required if the errors in the one
and two-qubit gates were too high, preventing the distilled state to achieve the desired target
fidelity for a prepared magic state after multiple rounds of the distillation. This is due to the
error of the distilled state always being approximately bounded from below by \( p_1/2 + 13p_2/9 \),
where \( p_1 \) and \( p_2 \) are the error probabilities of the one and two-qubit gates, respectively.
Therefore, if one requires to reduce the error of the distilled magic state, and subsequently
the applied universal gate using the magic states, to be below some small target threshold, one
would need to reduce the error rates of the one and two-qubit gates through a method such
as fault-tolerance. However, this step would only have to be applied in the final iterations of
the protocol in order to get the distilled state over the fidelity hump posed by the errors in
the Clifford gates at one’s disposal.

4 Conclusion

We have both analytically and numerically analyzed the evolution of quantum states under
the five-qubit magic state distillation protocol without the presence of the dephasing trans-
formation for perfect Clifford gates. We have characterized which input states converge to the
magic state under repeated application of the distillation subroutine. Sec. 2 provides results
showing that not all states on a plane with fidelity over the threshold converge to magic state
after multiple iterations of subroutine without dephasing. However, some quantum states
which are undistillable become distillable with the addition of the dephasing transformation
for high enough fidelity, as was shown by Bravyi and Kitaev [1]. Therefore, the fact that some
quantum states contribute to universal quantum computation is dependent on the ability to
access the dephasing transformation. Thus, on a more fundamental level, one may ask what
is the role of the dephasing transformation in universal quantum computation? Also noted
in Sec. 2 there are states below the theoretical threshold derived for states along the magic
state axis, which converge to the magic state after many repetitions of the distillation rou-
tine, and as such if one had access to such states off the magic state axis, one could lower
the threshold for distillation to 0.825. An interesting direction for future research would be
to investigate whether other distillation schemes [15, 16, 17, 18, 20] are also robust to such
noise perturbations.

Additionally, we studied the effect of noisy Clifford gates on the output fidelity, and the error rate in the universal gate induced by noisy distillation of magic states. Fortunately, based on the average error strength given in a recent benchmarking experiment [27], the state after absorbing all the noise in the Clifford gates is not far away from the pure magic state. This means that magic state distillation is robust against the typical noise levels in current experimental implementations. Nevertheless, due to the errors on the applied Clifford gates in the distillation procedure, one cannot produce an output state arbitrarily close to the magic state. In order to further reduce the error on the magic state, and the subsequent universal gate application, one would need to introduce a fault-tolerant encoding of the Clifford gates to reduce their noise. However, it is only at the point when one would want to reduce the error rate of the universal gate to be below the Clifford gate error rate that a fault-tolerant encoding would be beneficial. We showed in Sec. 3.2 that although introducing a fault-tolerant encoding at the beginning of the distillation protocol may seem appealing due to the increased convergence rate of the protocol, it is highly inefficient due to the number of gates that are typically required in any fault-tolerant scheme to reduce the error rate of current implementations of quantum information processors [28]. Since fault-tolerance would not be required to perform magic state distillation, other than in the final iterations of the protocol in order to apply a universal gate with very high fidelity, this work confirms that multiple rounds of faulty magic state distillation is a realistic scheme for future small-scale quantum information processors.

Acknowledgements

Author Contributions – All authors contributed to the development and discussion of the results in this work. T.J. and Y.Y. co-wrote the manuscript.

T.J. acknowledges the support of NSERC through the Alexander Graham Bell CGS–M scholarship. Y.Y acknowledges the support of the China Scholarship Council. B.H. acknowledges the support of NSERC through the NSERC Undergraduate Student Research Award (USRA). The authors would like to thank Joseph Emerson and Ben Reichardt for stimulating insightful discussion throughout this project. This work was additionally funded through QuantumWorks, CIFAR, and Industry Canada.

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Appendix A

In this Appendix, we give a more in depth analysis of the calculations of Eqs. (10)–(12). This calculation provides the output density matrix components for the noisy magic state distillation procedure, where the one- and two-qubit gates are subject to depolarizing noise with parameters $p_1$ and $p_2$, respectively. Let the input state density matrix be written in the
The matrix elements are written in the \(|T_0\rangle\langle T_1|\) basis. For completeness, we have included off-diagonal terms in the input state matrix, as opposed to the assumed input state in Eqs. (10)–(12), signifying a deviation from the magic state axis. Errorless \(T\)-type magic state distillation requires high fidelity of the input states with the magic state \(|T_0\rangle\), as such throughout this calculation we assume \(\epsilon, \alpha, \) and \(\beta\) small. Similarly, since we are analyzing the convergence of magic states under small error rates, \(p_1\) and \(p_2\) are assumed to be small. As such, we shall provide the output density matrix up to second order in any of these terms (see as it is important to keep the leading order in \(\epsilon\) of the error term \(|T_1\rangle\langle T_1|\)). The output of the distillation protocol has the following matrix coefficients:

\[
|T_0\rangle\langle T_0| : \frac{1}{6} + \frac{p_1}{36} \left[ -\frac{11 + 63\epsilon - 8\alpha - 8\beta}{3} \right] + \frac{p_2}{432} \left[ -416 + 2528\epsilon - (79 + 9\sqrt{3})\alpha - (79 - 9\sqrt{3})\beta \right],
\]

\[
|T_1\rangle\langle T_1| : \frac{5\epsilon^2}{6} + \frac{p_1}{36} \left[ 3 + \epsilon \right] + \frac{p_2}{432} \left[ 104 - 32\epsilon + (1 + 3\sqrt{3})\alpha + (1 - 3\sqrt{3})\beta \right],
\]

\[
|T_0\rangle\langle T_1| : \frac{1}{12} (-1 + i)(i + \sqrt{3})\alpha\beta + \frac{p_1}{9} \left[ -i\alpha + (9 + 8i)\beta \right] + \frac{p_2(1 + i)}{864} \left[ (1 + 5\sqrt{3}) - 2(-1 + 13\sqrt{3})\epsilon + ((14 - 24\sqrt{3}) + (-18 - 20\sqrt{3})i)\alpha + (14 + 24\sqrt{3}) + (18 - 20\sqrt{3})i)\beta \right].
\]

The important conclusions that can be drawn from these equations is that if one considers the error term \(|T_1\rangle\langle T_1|\), recall that the fidelity with respect to that magic state is \(F(\rho) = 1 - \langle T_1 | \rho | T_1 \rangle\), it has first order terms that are a function of the one- and two-qubit error rates \(p_1\) and \(p_2\), and all remaining terms are of second order. Moreover, the off-diagonal terms, that is deviations away from the magic state axis, also only contribute terms that are a product of the off-diagonal coefficient along with one- and two-qubit error rates. Such terms could be problematic if the off-diagonal terms were large and growing, however, as motivated in the following paragraph, the off-diagonal terms shall remain small after subsequent iterations of the protocol. Thus, the leading contributions to the error terms in the limit of a large number of iterations are \(p_1/12 + 13p_2/54\), which upon normalization of approximately 1/6 in the limit of small error rate yields \(p_1/2 + 13p_2/9\).

Consider now the off-diagonal terms in the output density matrix. The lowest order term in the output state’s diagonal matrix elements is a first order term of the two-qubit error rate \(p_2\). All other terms are of second order with respect to two of the variables which are assumed to be small. Thus, after multiple iterations of the protocol the contribution to the diagonal term of the output density matrix will be dominated by the two-qubit error rate \(p_2\) and will have little contribution from the one-qubit error rate or the deviations \(\epsilon, \alpha,\) and \(\beta\) from the input state. Such a behaviour should be unsurprising as states that are close to the magic state axis do converge to the magic state after multiple iterations of the protocol with
no error, as shown in Sec. 2, thus it should be of no surprise that similar properties exist for low error rate Clifford gates. Therefore, assuming that the off-diagonal contributions to the error rate of the output density matrix, the $|T_1\rangle\langle T_1|$ contribution, is minimal is a good assumption in the limit where the parameters $p_1, p_2, \epsilon, \alpha,$ and $\beta$ are small.