ON THE DIVISOR FUNCTION AND THE Riemann Zeta-Function in Short Intervals

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Abstract. We obtain, for $T^\varepsilon \leq U = U(T) \leq T^{1/2-\varepsilon}$, asymptotic formulas for

\[ \int_T^{2T} (E(t+U) - E(t))^2 \, dt, \quad \int_T^{2T} (\Delta(t+U) - \Delta(t))^2 \, dt, \]

where $\Delta(x)$ is the error term in the classical divisor problem, and $E(T)$ is the error term in the mean square formula for $|\zeta(\frac{1}{2} + it)|$. Upper bounds of the form $O_{\varepsilon} (T^{1+\varepsilon} U^2)$ for the above integrals with biquadrates instead of square are shown to hold for $T^{3/8} \leq U = U(T) \ll T^{1/2}$. The connection between the moments of $E(t+U) - E(t)$ and $|\zeta(\frac{1}{2} + it)|$ is also given. Generalizations to some other number-theoretic error terms are discussed.

1. Introduction

Power moments represent one of the most important parts of the theory of the Riemann zeta-function $\zeta(s)$, defined as

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\sigma = \Re s > 1), \]

and otherwise by analytic continuation. Of particular significance are the moments on the “critical line” $\sigma = \frac{1}{2}$, and a vast literature exists on this subject (see e.g., the monographs [5], [6], and [23]). In this paper we shall be concerned with moments of the error function

\[ E(T) := \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt - T \left( \log \frac{T}{2\pi} + 2\gamma - 1 \right), \]

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where \( \gamma = -\Gamma'(1) \) is Euler’s constant. More specifically, we shall consider the moments

\[
(1.2) \quad \int_T^{2T} (E(t + G) - E(t - G))^k \, dt \quad (k \in \mathbb{N} \text{ fixed}),
\]

where \( G = G(T) \) is “short” in the sense that \( G = O(T) \) as \( T \to \infty \) and \( G \gg 1 \).

To deal with bounds for the expressions like the one in (1.2), it seems convenient to use also results on the moments of the function

\[
E^*(t) := E(t) - 2\pi \Delta^*(\frac{t}{2\pi}),
\]

where

\[
\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2} \Delta(4x) = \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1).
\]

Here as usual \( d(n) = \sum_{\delta | n} 1 \) is the number of positive divisors of \( n \), and

\[
(1.3) \quad \Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1)
\]

is the error term in the classical Dirichlet divisor problem. The function \( E^*(t) \) gives an insight into the analogy between the Dirichlet divisor problem and the mean square of \( |\zeta(\frac{1}{2} + it)| \). It was investigated by several authors, including M. Jutila [15], who introduced the function \( E^*(t) \), and the author [6]–[8]. Among other things, the author (op. cit.) proved that

\[
\int_0^T (E^*(t))^2 \, dt = T^{4/3} P_3(\log T) + O_\varepsilon(T^{5/4 + \varepsilon}),
\]

where \( P_3 \) is a polynomial of degree three in \( \log T \) with positive leading coefficient,

\[
(1.4) \quad \int_0^T |E^*(t)|^5 \, dt \ll_\varepsilon T^{2+\varepsilon}, \quad \int_0^T |E^*(t)|^3 \, dt \ll_\varepsilon T^{3/2 + \varepsilon},
\]

and none of these three results implies any one of the other two. From the bounds in (1.4) and the Cauchy-Schwarz inequality for integrals it follows that

\[
(1.5) \quad \int_0^T |E^*(t)|^4 \, dt \ll_\varepsilon T^{7/4 + \varepsilon}.
\]

Here and later \( \varepsilon (> 0) \) denotes arbitrarily small constants, not necessarily the same ones at each occurrence, and \( a = O_\varepsilon(b) \) (same as \( a \ll_\varepsilon b \)) means that the implied
constant depends only on \( \varepsilon \). In addition to (1.2) it makes sense to investigate the moments

\[
(1.6) \quad \int_T^{2T} \left( \Delta(t + G) - \Delta(t - G) \right)^k dt \quad (k \in \mathbb{N} \text{ fixed}),
\]

as well. The interest in this topic comes from the work of M. Jutila [12], who investigated the case \( k = 2 \) in (1.2) and (1.6). He proved that

\[
(1.7) \quad \int_T^{T+H} \left( \Delta(x + U) - \Delta(x) \right)^2 dx
\]

\[
= \frac{1}{4\pi^2} \sum_{n \leq \frac{T}{e}} \frac{d^2(n)}{n^{3/2}} \int_T^{T+H} x^{1/2} \left| \exp \left( 2\pi iU \sqrt{\frac{n}{x}} \right) - 1 \right|^2 dx + O_\varepsilon(T^{1+\varepsilon} + HU^{1/2}T^{\varepsilon}),
\]

for \( 1 \leq U \ll T^{1/2} \ll H \leq T \), and an analogous result holds also for the integral of \( E(x + U) - E(x) \) (the constants in front of the sum and in the exponential will be \( 1/\sqrt{2\pi} \) and \( \sqrt{2\pi} \), respectively). From (1.7) one deduces (\( a \asymp b \) means \( a \ll b \ll a \))

\[
(1.8) \quad \int_T^{T+H} \left( \Delta(x + U) - \Delta(x) \right)^2 dx \asymp HU \log^3 \left( \frac{\sqrt{T}}{U} \right)
\]

for \( HU \gg T^{1+\varepsilon} \) and \( T^{\varepsilon} \ll U \leq \frac{1}{2} \sqrt{T} \). In [14] Jutila proved that the integral in (1.8) is

\[
\ll_\varepsilon T^{\varepsilon}(HU + T^{2/3}U^{4/3}) \quad (1 \ll H, U \ll X).
\]

This bound and (1.8) hold also for the integral of \( E(x + U) - E(x) \). Furthermore Jutila conjectured that

\[
(1.9) \quad \int_T^{2T} \left( E(t + U) - E(t - U) \right)^4 dt \ll_\varepsilon T^{1+\varepsilon}U^2
\]

holds for \( 1 \ll U \ll T^{1/2} \), and the analogous formula should hold for \( \Delta(t) \) as well. In fact, using the ideas of K.-M. Tsang [24] who investigated the fourth moment of \( \Delta(x) \), it can be shown that one expects the integral in (1.9) to be of order \( TU^2 \log^6(\sqrt{T}/U) \). Jutila also indicated that the truth of his conjecture (1.9) implies

\[
(1.10) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^6 dt \leq_\varepsilon T^{1+\varepsilon}.
\]
This is (a weakened form of) the sixth moment for $|\zeta(\frac{1}{2} + it)|$, and the best known exponent at present on the right-hand side of (1.10) is $5/4$ (see [5], [6]). In view of the bound (op. cit.)

\begin{align}
|\zeta(\frac{1}{2} + it)|^k \ll \log t \int_{t-1}^{t+1} |\zeta(\frac{1}{2} + ix)|^k \, dx + 1, \quad (k \in \mathbb{N} \text{ fixed})
\end{align}

we actually have, using (1.9) with $U = T^\varepsilon$ and (1.11) with $k = 2$,

\begin{align}
\int_T^{2T} |\zeta(\frac{1}{2} + it)|^8 \, dt \ll \varepsilon \int_T^{2T} \{\log T(E(t + T^\varepsilon) - E(t - T^\varepsilon))^4 + T^\varepsilon\} \, dt \ll \varepsilon T^{1+\varepsilon},
\end{align}

and the eighth moment bound (1.12) is notably stronger than (1.10). It may be remarked that the fourth moments of $\Delta(x)$ and $E(T)$ have been investigated by several authors, including Ivić–Sargos [11], K.-M. Tsang [24], and W. Zhai [25], [26].

**2. Statement of results**

Our first aim is to derive from (1.7) (when $H = T$) a true asymptotic formula. The result is

**THEOREM 1.** For $1 \ll U = U(T) \leq \frac{1}{2}\sqrt{T}$ we have $(c_3 = 8\pi^{-2})$

\begin{align}
\int_T^{2T} (\Delta(x + U) - \Delta(x))^2 \, dx = TU \sum_{j=0}^{3} c_j \log^j \left(\frac{\sqrt{T}}{U}\right) \\
+ O_{\varepsilon}(T^{1/2+\varepsilon}U^2) + O_{\varepsilon}(T^{1+\varepsilon}U^{1/2}),
\end{align}

a similar result being true if $\Delta(x + U) - \Delta(x)$ is replaced by $E(x + U) - E(x)$, with different constants $c_j$.

**Remark 1.** For $T^\varepsilon \leq U = U(T) \leq T^{1/2-\varepsilon}$ (2.1) is a true asymptotic formula.

**Corollary 1.** For $1 \ll U \leq \frac{1}{2}\sqrt{T}$ we have $(c_3 = 8\pi^{-2})$

\begin{align}
\sum_{T \leq n \leq 2T} (\Delta(n + U) - \Delta(n))^2 = TU \sum_{j=0}^{3} c_j \log^j \left(\frac{\sqrt{T}}{U}\right) \\
+ O_{\varepsilon}(T^{1/2+\varepsilon}U^2) + O_{\varepsilon}(T^{1+\varepsilon}U^{1/2}),
\end{align}

The formula (2.2) is a considerable improvement over a result of Coppola–Salerno [3], who had $(T^\varepsilon \leq U \leq \frac{1}{2}\sqrt{T}, \, L = \log T)$

\begin{align}
\sum_{T \leq n \leq 2T} (\Delta(n + U) - \Delta(n))^2 = \frac{8}{\pi^2} TU \log^3 \left(\frac{\sqrt{T}}{U}\right) + O(TUL^{5/2}\sqrt{L}).
\end{align}
Corollary 2. For \( T \leq x \leq 2T \) and \( T^{\epsilon} \leq U = U(T) \leq T^{1/2-\epsilon} \) we have

\[
\Delta(x+U) - \Delta(x) = \Omega \left\{ \sqrt{U} \log^{3/2} \left( \frac{\sqrt{x}}{U} \right) \right\}, \quad E(x+h) - E(x) = \Omega \left\{ \sqrt{U} \log^{3/2} \left( \frac{\sqrt{x}}{U} \right) \right\}.
\]

These omega results \( (f(x) = \Omega(g(x)) \) means that \( \lim_{x \to \infty} f(x)/g(x) \neq 0 \) show that Jutila’s conjectures made in [12], namely that

\[
\Delta(x+U) - \Delta(x) \ll_{\epsilon} x^{\epsilon} \sqrt{U}, \quad E(x+U) - E(x) \ll_{\epsilon} x^{\epsilon} \sqrt{U}
\]

for \( x^{\epsilon} \leq U \leq x^{1/2-\epsilon} \) are (if true), close to being best possible. The difficulty of these conjectures may be seen if one notes that from the definition of \( \Delta(x) \) (the analogue of this for \( E(T) \) is not known to hold, in fact it is equivalent to the Lindelöf hypothesis (see [6])) one easily obtains

\[
\Delta(x+U) - \Delta(x) \ll_{\epsilon} x^{\epsilon} \ (1 \ll U \leq x),
\]

which is much weaker than (2.5). However, a proof of (2.6) has not been obtained yet by the classical Voronoï formula. This formula will be needed later for the proof of Theorem 3, and in a truncated form it reads (see e.g., Chapter 3 of [5])

\[
\Delta(x) = \frac{1}{\pi \sqrt{2}} x^{\frac{1}{4}} \sum_{n \leq N} d(n)n^{-\frac{3}{4}} \cos(4\pi \sqrt{nx} - \frac{1}{4} \pi) + O_{\epsilon}(x^{\frac{1}{2}+\epsilon} N^{-\frac{1}{4}}) \quad (2 \leq N \ll x).
\]

One also has (see [5, eq. (15.68)]), for \( 2 \leq N \ll x, \)

\[
\Delta^*(x) = \frac{1}{\pi \sqrt{2}} x^{\frac{1}{4}} \sum_{n \leq N} (-1)^n d(n)n^{-\frac{3}{4}} \cos(4\pi \sqrt{nx} - \frac{1}{4} \pi) + O_{\epsilon}(x^{\frac{1}{2}+\epsilon} N^{-\frac{1}{4}}),
\]

which is completely analogous to (2.7), the only difference is that in (2.8) there appears a factor \((-1)^n\) in the sum.

Remark 2. The analogue of (2.3) for the sum

\[
\sum_{T \leq n \leq 2T} (E(n+U) - E(n))^2
\]

does not carry over, because \( E(T) \) (see (1.1)) is a continuous function, while \( \Delta(x) \) is not, having jumps at natural numbers of order at most \( O_{\epsilon}(x^{\epsilon}) \). The true order of magnitude of the sum in (2.9) seems elusive. Sums of \( E(n) \) were investigated by Y. Bugeaud and the author [2]. By using the irrationality measure of \( e^{2\pi m} \) and for the partial quotients in its continued fraction expansion, a non-trivial bound for \( \sum_{n \leq x} E(n) \) is obtained.
There are several ways in which the asymptotic formula (2.1) of Theorem 1 may be generalized. This concerns primarily number-theoretic terms related to arithmetic functions $f(n)$ whose generating series $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ ($\Re s > 1$) belongs to the so-called Selberg class of degree two (see e.g., the survey work of Kaczorowski–Perelli [16]). Instead of trying to formulate a general result which contains (2.1) as a special case, we shall state the corresponding results for two well-known number-theoretic quantities. Let, as usual, $r(n) = \sum_{n=a^2+b^2} 1$ denote the number of ways $n$ may be represented as a sum of two integer squares, and let $\varphi(z)$ be a holomorphic cusp form of weight $\kappa$ with respect to the full modular group $SL(2, \mathbb{Z})$, and denote by $a(n)$ the $n$-th Fourier coefficient of $\varphi(z)$. We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n) \varphi = a(n) \varphi$ for every $n \in \mathbb{N}$ (see e.g., R.A. Rankin [20] for the definition and properties of the Hecke operators). The classical example is $a(n) = \tau(n)$ ($\kappa = 12$), the Ramanujan function defined by

$$
\sum_{n=1}^{\infty} \tau(n) x^n = x \{(1 - x)(1 - x^2)(1 - x^3) \cdots \}^{24} \quad (|x| < 1).
$$

If $P(x) := \sum_{n \leq x} r(n) - \pi x$ denotes then the error term in the classical circle problem and $A(x) := \sum_{n \leq x} a(n)$, then we have

**THEOREM 2.** For $T^\varepsilon \leq U = U(T) \leq \frac{1}{2} \sqrt{T}$ we have

$$
\int_T^{2T} (P(t + U) - P(t))^2 \, dt = TU \left( A_1 \log \left( \frac{\sqrt{T}}{U} \right) + A_2 \right) + O_\varepsilon(T^{1/2 + \varepsilon U^2}) + O_\varepsilon(T^{1+\varepsilon \sqrt{U}}),
$$

and

$$
\int_T^{2T} (A(t + U) - A(t))^2 \, dt = C T^\kappa U + O_\varepsilon(T^{\kappa-2/5+\varepsilon U^{9/5}}) + O_\varepsilon(T^{\kappa+\varepsilon \sqrt{U}})
$$

with some explicitly computable constants $A_1, C > 0$ and $A_2$.

**Corollary 3.** For $T \leq x \leq 2T$ and $T^\varepsilon \leq U = U(T) \leq T^{1/2-\varepsilon}$ we have

$$
P(x + U) - P(x) = \Omega \left( \sqrt{U \log \left( \frac{\sqrt{x}}{U} \right)} \right), \quad A(x + U) - A(x) = \Omega(x^{\kappa-1/2} \sqrt{U}).
$$

Our next result relates bounds for moments of $|\zeta(\frac{1}{2} + it)|$ to bounds of moments of $E(t+G) - E(t-G)$. This is usually done (see e.g., Chapter 8 of [5]) by counting “large values” of $|\zeta(\frac{1}{2} + it)|$ which occur in $[T, 2T]$. Our result is the following
THEOREM 3. Let \( t_1, \ldots, t_R \) be points in \([T, 2T]\) which satisfy \( T^\varepsilon \leq V \leq |\zeta(1/2 + it_r)| \) and \( |t_r - t_s| \geq 1 \) for \( r, s \leq R \) and \( r \neq s \). Then we have, for \( L = \log T, G = A(V/L)^2 \) with a suitable constant \( A > 0 \), and \( k \in \mathbb{N} \) fixed,

\[
R \ll V^{-2-2k} L^{2+2k} \int_{T/3}^{3T} \left\{ |E(t + 2G) - E(t - 2G)|^k + |E(t + 1/2 G) - E(t - 1/2 G)|^k \right\} \, dt.
\]

Corollary 4. Suppose that the integral on the right-hand side of (2.12) is bounded by \( O_\varepsilon(T^{\alpha + \varepsilon} G^\beta) \) for some real constants \( \alpha = \alpha(k) (> 0) \) and \( \beta = \beta(k) \leq k - 1 \), and \( T^\varepsilon \leq G = G(T) \ll T^{1/3} \). Then we have

\[
\int_0^T |\zeta(1/2 + it)|^{2+2k-2\beta} \, dt \ll_\varepsilon T^{1+\alpha+\varepsilon}.
\]

One obtains Corollary 3 from Theorem 3 in a standard way (see e.g., Chapter 8 of [5]). The condition \( T^\varepsilon \leq G \ll T^{1/3} \) comes from the definition of \( G \) and the classical bound \( \zeta(1/2 + it) \ll t^{1/6} \). The condition \( \beta \leq k - 1 \) is necessary, because we know that \( \int_0^T |\zeta(1/2 + it)|^4 \, dt \ll T \log^4 T \), and the condition in question implies that the exponent of the integral in (2.13) is at least 4.

In connection with Jutila’s conjecture (1.9) one may, in general, consider constants \( 0 \leq \rho(k) \leq 1 \) for fixed \( k > 2 \) which one has

\[
\int_T^{2T} |E(t + G) - E(t - G)|^k \, dt \ll_\varepsilon T^{1+\varepsilon} G^{k/2} \quad (T^{\rho(k)+\varepsilon} \ll G = G(T) \ll T),
\]

and similarly for the moments of \( |\Delta(t + G) - \Delta(t - G)| \). A general, sharp version of Jutila’s conjecture would be that \( \rho(k) = 0 \) for any fixed \( k > 2 \) and (2.14) holds for \( G \ll \sqrt{T} \). The following theorem gives the unconditional value of \( \rho(4) \), and shows that Jutila’s conjecture holds in a certain range. Any improvements of these ranges would be of interest.

THEOREM 4. We have, for \( T^{3/8} \ll G = G(T) \ll T^{1/2} \),

\[
\int_T^{2T} (E(t + G) - E(t - G))^4 \, dt \ll_\varepsilon T^{1+\varepsilon} G^2,
\]

\[
\int_T^{2T} (\Delta(t + G) - \Delta(t - G))^4 \, dt \ll_\varepsilon T^{1+\varepsilon} G^2.
\]
3. The proof of Theorem 1 and Theorem 2

We shall deduce Theorem 1 from Jutila’s formula (1.7) with \( H = T \). First note that the integral on the right-hand side equals

\[
\int_T^{2T} x^{1/2} \left| \exp \left( 2\pi iU \sqrt{\frac{n}{x}} \right) - 1 \right|^2 \, dx
\]

\[
= \int_T^{2T} x^{1/2} \left( 2 - e^{-2\pi iU \sqrt{n/x}} - e^{2\pi iU \sqrt{n/x}} \right) \, dx
\]

\[
= 2 \int_T^{2T} x^{1/2} \left( 1 - \cos \left( 2\pi U \sqrt{\frac{n}{x}} \right) \right) \, dx
\]

\[
= 4 \int_T^{2T} x^{1/2} \sin^2 \left( \pi U \sqrt{\frac{n}{x}} \right) \, dx.
\]

In the last integral we make the change of variable

\[
\pi U \sqrt{\frac{n}{x}} = y, \quad \sqrt{x} = \frac{\pi U \sqrt{n}}{y}, \quad x = \pi^2 U^2 ny^{-2}, \quad dx = -2\pi^2 U^2 ny^{-3}.
\]

Therefore the main term on the right-hand side of (1.7) becomes

\[
(3.1) \quad 2\pi U^3 \sum_{n \leq T/(2U)} d^2(n) \int_{\pi U \sqrt{n/T}}^{\pi U \sqrt{2n/T}} \frac{\sin^2 y}{y^4} \, dy.
\]

Now we change the order of summation and integration: from

\[
1 \leq n \leq \frac{T}{2U}, \quad \pi U \sqrt{n/2T} \leq y \leq \pi U \sqrt{n/T}
\]

we infer that

\[
\frac{\pi U}{\sqrt{2T}} \leq y \leq \pi \sqrt{\frac{U}{2}}, \quad \frac{T y^2}{\pi^2 U^2} \leq n \leq \frac{2T y^2}{\pi^2 U^2}.
\]

Thus (3.1) becomes

\[
(3.2) \quad 2\pi U^3 \int_{\frac{\pi \sqrt{T}}{\pi U^2}}^{\frac{\pi \sqrt{2T}}{\pi U^2}} \sum_{\max(1, \frac{T y^2}{\pi U^2}) \leq n \leq \min(\frac{T}{\pi U^2}, \frac{2T y^2}{\pi U^2})} d^2(n) \cdot \frac{\sin^2 y}{y^4} \, dy.
\]

The range of summation in (3.2) will be

\[
I := \left[ \frac{T y^2}{\pi U^2}, \frac{2T y^2}{\pi U^2} \right], \quad \text{if } y \in J := \left[ \frac{\pi U}{\sqrt{T}}, \frac{1}{2\pi \sqrt{U}} \right].
\]
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By using the elementary bound \(|\sin x| \leq \text{min}(1, |x|)\), it is easily seen that the error made by replacing the interval of integration in (3.2) by \(J\) will be

\[
\ll (TU + T^{1/2}U^2) \log^3 T,
\]

which is absorbed by the error term in (2.1). When \(y \in J\), the sum over \(n \in I\) can be evaluated by the use of the asymptotic formula (see [5] and [10, Lemma 3])

\[
(3.3) \quad \sum_{n \leq x} d^2(n) = x \left( \sum_{j=0}^{3} a_j \log^j x \right) + O_\varepsilon(x^{1/2+\varepsilon}) \quad (a_3 = 1/(\pi^2)).
\]

We note that the value \(a_3 = 1/(\pi^2)\) is easily computed from the residue of \(x^s\zeta^4(s)/s\zeta(2s)\) at \(s = 1\), and the remaining \(a_j's\) in (3.3) can be also explicitly computed. The error term in (3.3) can be improved to \(O(x^{1/2} \log^5 x \log \log x)\) (see Ramachandra–Sankaranarayanan [18]), but the exponent \(1/2\) of \(x\) cannot be improved without assumptions on the zero-free region of \(\zeta(s)\) (such as e.g., the Riemann hypothesis that all complex zeros of \(\zeta(s)\) have real parts equal to 1/2). However, this improvement is not needed in view of the error term \(O(HU^{1/2}T^\varepsilon)\) in (1.7).

To continue with the proof, note that if we use (3.3) to evaluate the expression in (3.2) we shall obtain, with effectively computable constants \(b_j (b_3 = 1/(\pi^2))\), that the major contribution equals

\[
(3.4) \quad 2\pi U^3 \int_{\pi U}^{\frac{\pi U}{\sqrt{T}}} \sin^2 \frac{y}{y^3} \left\{ \frac{T y^2}{\pi^2 U^2} \left( \sum_{j=0}^{3} b_j \log^j \left( \frac{T y^2}{U^2} \right) \right) + O_\varepsilon \left( \frac{T^{1/2+\varepsilon} y}{U} \right) \right\} dy
\]

\[
= \frac{2}{\pi} TU \int_{\pi U}^{\frac{\pi U}{\sqrt{T}}} \sin^2 \frac{y}{y^3} \left( \sum_{j=0}^{3} b_j \log^j \left( \frac{T y^2}{U^2} \right) \right) dy + O_\varepsilon(T^{1/2+\varepsilon}U^2).
\]

The last error term above comes from the fact that

\[
\int_{\pi U}^{\frac{\pi U}{\sqrt{T}}} \sin^2 \frac{y}{y^3} dy = \int_{\pi U}^{1} \sin^2 \frac{y}{y^3} dy + O(1)
\]

\[
\ll \int_{\pi U}^{1} \frac{dy}{y} + 1 \ll \log \frac{\sqrt{T}}{U},
\]

where \(|\sin x| \leq \text{min}(1, |x|)\) was used again. Likewise we deduce that, for \(0 < \alpha \leq 1, \beta \gg 1,\)

\[
(3.5) \quad \int_{\alpha}^{\beta} \frac{\sin^2 y}{y^2} dy = \int_{0}^{\infty} \frac{\sin^2 y}{y^2} dy + O(\alpha) + O(\beta^{-1})
\]

\[
= \frac{\pi}{2} + O(\alpha) + O(\beta^{-1}).
\]
We expand as a binomial
\[ \log^j \left( \frac{T y^2}{U^2} \right) = \left( \log \frac{T}{U^2} + 2 \log y \right)^j \quad (j = 2, 3), \]
and use a relation similar to (3.5) for an integral containing an additional power of \( \log y \). Hence from (3.4) it transpires that the main term on the right-hand side of (1.7) is equal to

\[
\frac{2}{\pi} T U \left\{ \frac{\pi}{2} b_3 \log^3 \left( \frac{T}{U^2} \right) + c_2 \log^2 \left( \frac{T}{U^2} \right) + c'_1 \log \left( \frac{T}{U^2} \right) + c_0' \\
+ O_\varepsilon \left( T^{-1/2} U + T^{-1/2} U \right) \right\} \\
= T U \left\{ \frac{8}{\pi^2} \log^3 \left( \sqrt{T} \right) + c_2 \log^2 \left( \sqrt{T} U \right) + c_1 \log \left( \sqrt{T} U \right) + c_0 \right\} \\
+ O_\varepsilon \left( T^{1/2+\varepsilon U^2} + T^{1+\varepsilon} U^{1/2} \right).
\]

From (3.6) and (1.7) we easily obtain (2.1). The proof of (2.1) with \( E(x+U) - E(x) \) in place of \( \Delta(x+U) - \Delta(x) \) follows verbatim the above argument.

The formula (2.1) of Corollary 1 follows from (2.1) and

\[
\int_T^{2T} (\Delta(x+U) - \Delta(x))^2 \, dx = \sum_{T < n \leq 2T} (\Delta(n+U) - \Delta(n))^2 + O(U^{5/2} \log^{5/2} T),
\]

for \( 1 \ll U \ll \sqrt{T} \). Namely we can assume \( U, T \) are integers (otherwise making an admissible error). Using (1.3) and the mean value theorem, it follows that the left-hand side of (3.7) equals (0 \leq \theta \leq 1)

\[
\sum_{T \leq m \leq 2T-1} \int_{m}^{m+1-0} \left( \sum_{x < n \leq x+U} d(n) - U \left( \log (x + U) + 2\gamma \right) \right)^2 \, dx
\]

\[
= \sum_{T \leq m \leq 2T-1} \int_{m}^{m+1-0} \left( \sum_{m < n \leq m+U} d(n) - U \left( \log (x + U) + 2\gamma \right) \right)^2 \, dx
\]

\[
= \sum_{T \leq m \leq 2T-1} \left( \Delta (m+U) - \Delta (m) + O(U^2T^{-1} \log T) \right)^2.
\]

Now we expand the square, use the Cauchy-Schwarz inequality and (2.3) for the cross terms, replace the range of summation by \([T, 2T] \), and (3.7) follows.
To prove Theorem 2, note first that \( r(n) = 4 \sum_{d|n} \chi(d) \), where \( \chi \) is the non-principal character modulo four. Thus \( 0 \leq r(n) \leq 4d(n) \), and \( \frac{1}{4}r(n) \) is multiplicative. From the functional equation
\[
L(s) = \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} L(1-s),
\]
where \( L(s) \) is the generating Dirichlet series of \( r(n) \) one obtains the explicit formula
(3.8) \[
P(x) = -\frac{1}{\pi} x^{1/4} \sum_{n \leq N} r(n) n^{-3/4} \cos(2\pi \sqrt{nx} + \frac{\pi}{4}) + O(x^{1/2+\varepsilon} N^{-1/2})
\]
for \( 1 \ll N \ll x \), much in the same way as one obtains (2.7) (see e.g., Chapter 13 of [5]). This formula is completely analogous to (2.7), and consequently Jutila’s proof gives the analogue of (1.7), with a different constant in front of the sum, \( d(n) \) replaced by \( r(n) \), and \( \pi i U \sqrt{n/x} \) in the exponential. The proof of Theorem 1 goes through up to (3.3), where instead of the asymptotic formula for sums of \( d^2(n) \) we shall use
(3.9) \[
\sum_{n \leq x} r^2(n) = 4x \log x + Cx + O(x^{1/2} \log^3 x \log \log x),
\]
where \( C = 8.0665 \ldots \) is an explicitly given constant. The asymptotic formula (3.9) is due to Kühllein–Nowak [17]. Note that the main term in (3.9) is somewhat different than the main term in (3.3), which is reflected in different main terms in (2.1) and (2.10). By using (3.9) the proof of (2.10) is essentially the same as the proof of (2.1), so there is no need for the details. For our purposes (3.9) with the error term \( O(x^{12+\varepsilon}) \) suffices, in view of the term \( O(x^{1+\varepsilon} U^{1/2}) \) in the analogue of (2.1).

As to the proof of (2.11), note that by P. Deligne’s bound one has \( |a(n)| \leq n^{(\kappa-1)/2}d(n) \), and the analogue of (2.7) reads (see e.g., M. Jutila [13] for a proof, who has a more general result with exponential factors)
(3.10) \[
A(x) = \sum_{n \leq x} a(n) = \frac{1}{\pi \sqrt{2}} x^{\frac{\kappa}{2} - \frac{1}{4}} \sum_{n \leq N} a(n) n^{-\frac{\kappa}{2} - \frac{1}{4}} \cos \left( 4\pi \sqrt{nx} - \frac{\pi}{4} \right)
+ O(x^{\kappa/2+\varepsilon} n^{-1/2}) \quad (1 \ll N \ll x).
\]
One has the asymptotic formula \((A > 0 \text{ can be explicitly evaluated})\)
(3.11) \[
\sum_{n \leq x} a^2(n) = Ax^\kappa + O(x^{\kappa-2/5}).
\]
The above formulas show that \( a(n) \) behaves similarly to \( n^{(\kappa-1)/2}d(n) \). The bound for the error term in (3.11), one of the longest standing records in analytic number theory is due to R.A. Rankin [19] and A. Selberg [22]. Following Jutila’s proof of (2.1), the proof of (2.10) and using (3.10) instead of (3.3) at the appropriate place, we arrive at (2.11).
4. The proof of Theorem 3

In this section we shall present the proof of Theorem 3. From the definition (1.1) of $E(T)$ we have, for $T \leq u, t \leq 2T$, $1 \ll G \ll T$,

$$E(u + \frac{1}{2}G) - E(u - \frac{1}{2}G) = \int_{u-G/2}^{u+G/2} |\zeta(\frac{1}{2} + ix)|^2 \, dx + O(G \log T).$$

Consequently integration over $u$ gives

$$\int_{t-G/2}^{t+G/2} (E(u + \frac{1}{2}G) - E(u - \frac{1}{2}G)) \, du$$

$$= \int_{t-G/2}^{t+G/2} \int_{u-G/2}^{u+G/2} |\zeta(\frac{1}{2} + ix)|^2 \, dx \, du + O(G^2 \log T)$$

$$\leq \int_{t-G/2}^{t+G/2} \int_{t-G}^{t+G} |\zeta(\frac{1}{2} + ix)|^2 \, dx \, du + O(G^2 \log T)$$

$$= G \int_{t-G}^{t+G} |\zeta(\frac{1}{2} + ix)|^2 \, dx + O(G^2 \log T).$$

Using again (1.1) for the last integral it follows that

$$E(t + G) - E(t - G) \geq \frac{1}{G} \int_{t-G/2}^{t+G/2} (E(u + \frac{1}{2}G) - E(u - \frac{1}{2}G)) \, du - CG \log T$$

for $1 \ll G \ll T$ and a suitable constant $C > 0$. The bound in (4.1) is useful when $E(t + G) - E(t - G)$ is negative. Likewise, from

$$\int_{t-G}^{t+G} (E(u + 2G) - E(u - 2G)) \, du$$

$$= \int_{t-G}^{t+G} \int_{u-2G}^{u+2G} |\zeta(\frac{1}{2} + ix)|^2 \, dx \, du + O(G^2 \log T)$$

$$\geq \int_{t-G}^{t+G} \int_{t-G}^{t+G} |\zeta(\frac{1}{2} + ix)|^2 \, dx \, du + O(G^2 \log T)$$

$$= 2G \int_{t-G}^{t+G} |\zeta(\frac{1}{2} + ix)|^2 \, dx + O(G^2 \log T)$$

$$= 2G(E(t + G) - E(t - G)) + O(G^2 \log T)$$

we obtain a bound which is useful when $E(t + G) - E(t - G)$ is positive. This is

$$E(t + G) - E(t - G) \leq \frac{1}{2G} \int_{t-G}^{t+G} (E(u + 2G) - E(u - 2G)) \, du + CG \log T.$$
Combining (4.1) and (4.2), depending on the sign of \( E(t + G) - E(t - G) \), we obtain, for \( T \leq t \leq 2T, 1 \ll G \ll T, C > 0 \),

\[
|E(t + G) - E(t - G)| \leq CG \log T + \\
\frac{1}{G} \int_{t-G}^{t+G} \left\{ |E(u + 2G) - E(u - 2G)| + |E(u + \frac{1}{2}G) - E(u - \frac{1}{2}G)| \right\} \, du.
\]

(4.3)

Suppose now that the hypotheses of Theorem 3 hold. Then \( (L = \log T) \)

\[
V^2 \leq |\zeta(\frac{1}{2} + it_r)|^2 \ll L \left( \int_{t_r-1/3}^{t_r+1/3} |\zeta(\frac{1}{2} + ix)|^2 \, dx + 1 \right) \quad (r = 1, \ldots, R).
\]

(4.4)

The interval \([T, 2T]\) is covered then with subintervals of length \(2G\), of which the last one may be shorter. In these intervals we group subintegrals over disjoint intervals \([t_r - 1/3, t_r + 1/3]\). Should some intervals fall into two of such intervals of length \(2G\), they are treated then separately in an analogous manner. It follows that

\[
R \ll V^{-2}L^2 \sum_{j=1}^{J} \int_{\tau_j-G}^{\tau_j+G} |\zeta(\frac{1}{2} + ix)|^2 \, dx,
\]

where \( J \leq R, \tau_j \in [T/3, 3T], |\tau_j - \tau_{j+1}| \geq 2G \ (j \neq \ell; j, \ell \leq J) \) by considering separately points with even and odd indices. Now we note that by (1.1)

\[
\int_{\tau_j-G}^{\tau_j+G} |\zeta(\frac{1}{2} + ix)|^2 \, dx = O(GL) + E(\tau_j - G) - E(\tau_j + G).
\]

(4.5)

For \( E(\tau_j - G) - E(\tau_j + G) \) we use (4.3) with \( t = \tau_j \), choosing

\[
G = AV^2L^{-2}
\]

with suitable \( A(>0) \) so that \( O(GL) \leq \frac{1}{2}V^2 \). In this way we obtain, using Hölder’s inequality for integrals, noting that the intervals \([\tau_j - G, \tau_j + G]\) are disjoint (if we consider separately systems of points \( \tau_j \) with even and odd indices \( j \)) and \( J \leq R \),

\[
R \ll V^{-4}L^4 \sum_{j=1}^{J} \int_{\tau_j-G}^{\tau_j+G} \{|E(u + 2G) - \cdots |\} \, du
\]

(4.6)

\[
\ll V^{-4}L^4 \sum_{j=1}^{J} \left( \int_{\tau_j-G}^{\tau_j+G} \{|E(u + 2G) - \cdots |\}^k \, du \right)^{1/k} G^{1-1/k}
\]

\[
\ll V^{-4}L^4(RG)^{1-1/k} \left( \int_{T/3}^{3T} \{|E(u + 2G) - \cdots |\}^k \, du \right)^{1/k} .
\]

If we simplify (4.6), we obtain the assertion (2.12) of Theorem 3.
5. The proof of Theorem 4

For the proof of Theorem 4 we shall need the case $k = 2$ of the following

**Lemma 1.** Let $k \geq 2$ be a fixed integer and $\delta > 0$ be given. Then the number of integers $n_1, n_2, n_3, n_4$ such that $N < n_1, n_2, n_3, n_4 \leq 2N$ and

$$|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll \varepsilon N^{\varepsilon}(N^4 \delta + N^2).$$

Lemma 1 was proved by Robert–Sargos [21]. It represents a powerful arithmetic tool which is essential in the analysis when the biquadrate of sums involving $\sqrt{n}$ appears in exponentials, and was used e.g., in [11].

It is enough to prove (2.14) of Theorem 4 for $\Delta(x)$. Namely because of the analogy between (2.7) and (2.8) (which differs from (2.13) only by the presence of the innocuous factor $(-1)^n$ in the sum), the same bound in the same range for $G$ will hold with the integral of $\Delta^*(x)$ replacing $\Delta(x)$. But then, in view of

$$E(t) = E^*(t) + 2\pi \Delta^*\left(\frac{t}{2\pi}\right)$$

and (1.5), we obtain

$$\int_T^{2T} (E(t + G) - E(t - G))^4 \, dt$$

$$\ll \int_T^{2T} \left(E^*(t + G) - E^*(t - G)\right)^4 \, dt + \int_T^{2T} \left(\Delta^*\left(\frac{t + G}{2\pi}\right) - \Delta^*\left(\frac{t - G}{2\pi}\right)\right)^4 \, dt$$

$$\ll_{\varepsilon} T^{7/4+\varepsilon} + T^{1+\varepsilon} G^2 \ll_{\varepsilon} T^{1+\varepsilon} G^2$$

precisely for $G \geq T^{3/8}$.

For the proof of (2.14) with $\Delta(x)$ we start from (2.7) with $x = t + G$, $x = t - G$, $T \leq t \leq 2T$, $N = T$ in both cases. We split the sum over $n$ into $O(\log T)$ subsums over $M < n \leq M' \leq 2M$, and raise each sum in question to the fourth power and integrate. When $M \geq TG^{-4/3}$ we note that, using twice (2.7), we have

$$S(t, M) := t^{1/4} \sum_{M < n \leq M'} d(n)n^{-3/4} \cos(4\pi \sqrt{n}t - \pi/4) \ll_{\varepsilon} T^{1/2+\varepsilon} M^{-1/2}.$$
Hence from (5.2) and the first derivative test (see e.g., Lemma 2.1 of [5]) we infer that in this range

\[
\int_T^{2T} (S(t + G) - S(t - G))^4 \, dt \\
\ll \varepsilon T^{1+\varepsilon} M^{-1} \int_T^{2T} \left( S^2(t + G) + S^2(t - G) \right) \, dt \\
\ll \varepsilon T^{1+\varepsilon} M^{-1} T^{1/2} \left( \int_T^{2T} \sum_{n > M} d^2(n)n^{-3/2} \\
+ \sum_{M < m \neq n \leq 2M} T^{1/2} d(m)d(n)(mn)^{-3/4} |\sqrt{m} - \sqrt{n}|^{-1} \right) \\
\ll \varepsilon T^{3/2+\varepsilon} M^{-1} (TM^{-1/2} + T^{1/2}) \ll \varepsilon T^{1+\varepsilon} G^2,
\]

as requested, since \( M \geq TG^{-4/3} \).

If

(5.3) \[ M \leq T^{1-\varepsilon} G^{-2}, \]

we proceed as follows. First in \( S(t \pm G) \) we replace \( (t \pm G)^{1/4} \) by \( t^{1/4} \), making a small total error in the process. Then we note that

\[
\cos(4\pi \sqrt{n(t + G)} - \pi/4) - \cos(4\pi \sqrt{n(t - G)} - \pi/4) \\
= -2 \sin \left(2\pi \sqrt{n}(\sqrt{t + G} - \sqrt{t - G})\right) \cos \left(2\pi \sqrt{n}(\sqrt{t + G} + \sqrt{t - G})\right).
\]

Furthermore, since

(5.4) \[ \sqrt{t + G} - \sqrt{t - G} = \sqrt{t} \left( \frac{G}{t} + \sum_{j=2}^{\infty} d_j \left( \frac{G}{t} \right)^j \right) \]

with suitable constants \( d_j \), it follows that in view of (5.3) in the series expansion of

\[
\sin \left(2\pi \sqrt{n}(\sqrt{t + G} - \sqrt{t - G})\right)
\]

the term \( 2\pi G \sqrt{n}/t \) will dominate in size. Hence if we take sufficiently many terms in (5.4) the tail of the series will make a negligible contribution, and we are left with a finite number of integrals, of which the largest contribution will come from (5.5)

\[
T \int_T^{2T} \left| \sum_{M < n \leq M'} d(n)n^{-3/4} Gn^{1/2} t^{-1/2} \exp \left(2\pi i \sqrt{n}(\sqrt{t + G} + \sqrt{t - G})\right) \right|^4 \, dt.
\]
Let now \( \varphi(t) (\geq 0) \) be a smooth function, supported in \([T/2, 5T/2]\) and equal to unity in \([T, 2T]\). Then \( \varphi^{(r)}(t) \ll r^{-r} \) for \( r = 0, 1, 2, \ldots \). We have

\[
\int_T^{2T} |\cdots|^4 \, dt \leq \int_{T/2}^{5T/2} \varphi(t)|\cdots|^4 \, dt
\]

\[
\ll \frac{G^4}{T^2} \int_{T/2}^{5T/2} \varphi(t) \sum_{k,\ell,m,n\geq M} \frac{d(k)d(\ell)d(m)d(n)}{(k\ell mn)^{1/4}} \exp\left(i\Delta(\sqrt{t+G} + \sqrt{t-G})\right) \, dt,
\]

where

\[
\Delta := \Delta(k, \ell, m, n) = 2\pi(\sqrt{k} + \sqrt{\ell} - \sqrt{m} - \sqrt{n}).
\]

In the last integral we perform a large number of integrations by parts. During this process the exponential factor will remain the same, while the integrand will acquire each time an additional factor of order \( \sim 1/(\Delta \sqrt{T}) \). Hence the contribution of integer quadruples \((k, \ell, mn)\) for which \( |\Delta| > T^{\varepsilon - 1/2} \) will be negligible. The contribution of the remaining quadruples is estimated by Lemma 1 (with \( k = 2, \delta = |\Delta|T^{\varepsilon - 1/2} \)) and trivial estimation. In this way it is seen that the expression in (5.5) is

\[
\ll \varepsilon T^\varepsilon G^4 M^{-1}(T^{-1/2} M^{7/2} + M^2)
\]

\[
= T^{\varepsilon - 1/2} G^4 M^{5/2} + T^\varepsilon G^4 M
\]

\[
\ll \varepsilon T^{2+\varepsilon} G^{-1} + T^{1+\varepsilon} G^2 \ll \varepsilon T^{1+\varepsilon} G^2
\]

for \( G \gg T^{1/3} \). It remains to deal with the intermediate range

\[
(5.6)
\]

\[
T^{1-\varepsilon} G^{-2} \ll M \ll T^{1+\varepsilon} G^{4/3}.
\]

This is accomplished similarly as in the previous case, by using the trivial inequality

\[
(S(t+G) - S(t-G))^4 \ll S^4(t+G) + S^4(t-G),
\]

namely by working with two expressions \( S(t \pm G) \), without taking into account the effect of \( t + G \) and \( t - G \) combined. We see that the contribution will be, in view of (5.6),

\[
\ll \varepsilon T^{2+\varepsilon} M^{-3}(T^{-1/2} M^{7/2} + M^2) = T^{3/2+\varepsilon} M^{1/2} + T^{2+\varepsilon} M^{-1}
\]

\[
\ll \varepsilon T^{2+\varepsilon} G^{-2/3} + T^{1+\varepsilon} G^2 \ll \varepsilon T^{1+\varepsilon} G^2
\]

for \( T^{3/8} \leq G \ll T^{1/2} \), as asserted. This proves (2.15) and completes the proof of Theorem 4.
Remark 3. If one had the analogue of (5.1) with $k = 2$, namely the bound
$N^\epsilon(N^6\delta + N^3)$ for six square roots, then the above argument would lead to
\[
\int_T^{2T} (\Delta(t + G) - \Delta(t - G))^6 \, dt \ll \epsilon T^{1+\epsilon} G^3 \quad (T^{6/13} \leq G = G(T) \ll T^{1/2}),
\]
which would still be a non-trivial result.

In concluding, it may be remarked that one can also obtain another proof of the important bound
\[(5.7) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll \epsilon T^{2+\epsilon}.
\]
This bound is due to D.R. Heath-Brown [4], who had $\log^{17} T$ in place of $T^\epsilon$, and still represents the sharpest known bound for high moments of $|\zeta(\frac{1}{2} + it)|$. Namely in (4.5) we immediately choose $G = AV^2L^{-2}$ with $t_j = u$ and then integrate, with an additional smooth weight. Like in the original proof of (5.7) in [4], the sum $\sum_2(T)$ in Atkinson’s formula [1] (or [5, Chapter 15]) for $E(T)$ will make a negligible contribution, while the range of summation in $\sum_1(T)$ will be $1 \leq n \leq T^{1+\epsilon} G^{-2}$. The technical details are as before, while the function $f(t, n)$ in the sum $\sum_1(T)$ is neutralized by using a procedure due to M. Jutila [15, Part II], which was also used in [7]. In this way (5.7) will eventually follow.

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