Existence and Multiplicity Results for Nonlocal Boundary Value Problems with Strong Singularity

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Received: 6 April 2020; Accepted: 27 April 2020; Published: 1 May 2020

Abstract: In this paper, we study singular $\varphi$-Laplacian nonlocal boundary value problems with a nonlinearity which does not satisfy the $L^1$-Carathéodory condition. The existence, nonexistence and/or multiplicity results of positive solutions are established under two different asymptotic behaviors of the nonlinearity at $\infty$.

Keywords: multiplicity of positive solutions; sup-multiplicative-like function; singular weight; nonlocal boundary conditions

1. Introduction

Consider the following singular $\varphi$-Laplacian boundary value problem (BVP)

$$
(w(t)\varphi(u'(t)))' + \lambda h(t)f(t,u(t)) = 0, \quad t \in (0,1),
$$

$$
u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r),
$$

where $w \in C([0,1],[0,\infty))$, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $\lambda \in [0,\infty)$ is a parameter, $h \in C([0,1],[0,\infty))$ and $f \in C([0,1] \times [0,\infty), \mathbb{R})$.

Throughout this paper, the following hypotheses are assumed, unless otherwise stated.

(A1) there exist increasing homeomorphisms $\psi_1, \psi_2 : [0,\infty) \to [0,\infty)$ such that

$$
\varphi(x)\psi_1(y) \leq \varphi(xy) \leq \varphi(x)\psi_2(y) \quad \text{for all } x, y \in [0,\infty).
$$

(A2) For $i = 1,2$, $\alpha_i$ is monotone increasing on $[0,1]$ satisfying

$$
\hat{\alpha}_i := \alpha_i(1) - \alpha_i(0) \in [0,1).
$$

All integrals in (2) are meant in the sense of Riemann-Stieljes. By a solution $u$ to BVP (1) and (2), we mean $u \in C^1([0,1]) \cap C([0,1])$ with $w\varphi(u') \in C^1([0,1])$ satisfies the Equation (1) and the boundary conditions (2).

The condition (A1) on the odd increasing homeomorphism $\varphi$ was first introduced by Wang in [1] where the existence, nonexistence and/or multiplicity of positive solutions to quasilinear elliptic equations were studied. Later on, the condition (A1) was weakened by some researchers. For example, Karakostas ([2,3]) introduced a sup-multiplicative-like function as an odd increasing homeomorphism $\varphi$ satisfies the following condition.

(F1) There exists an increasing homeomorphism $\psi_1 : [0,\infty) \to [0,\infty)$ such that

$$
\varphi(x)\psi_1(y) \leq \varphi(xy) \quad \text{for all } x, y \in [0,\infty).
$$
Webb and Infante [16] considered problem (1) with various nonlocal boundary conditions involving
\[ p \] (see, e.g., [2,4]). Lee and Xu ([5,6]) generalized the condition
\[ \text{not requiring that } p_2(0) = 0 \] and studied the existence of positive solutions to singularly weighted nonlinear systems. In [7], it was pointed out that the condition (A1) is equivalent to the one \((F_1)\). Consequently, the condition (A1) is equivalent to those in [2,3,5,6].

Due to a wide range of applications in mathematics and physics (see, e.g., [8–14]), \(p\)-Laplacian or more generalized Laplacian problems have been extensively studied. For example, when \(\varphi(s) = |s|^{p-2}s\) for some \(p \in (1, \infty), w \equiv 1 \) and \(h \in \mathcal{H}_\varphi\), Agarwal, Lü and O’Regan [15] investigated the existence and multiplicity of positive solutions to BVP (1) and (2) with \(\hat{\lambda}_1 = \hat{\lambda}_2 = 0\) under various assumptions on the nonlinearity \(f = f(t,u)\) at \(u = 0\) and \(\infty\). When \(\varphi(s) = s, w \equiv 1 \) and \(\lambda = 1\), Webb and Infante [16] considered problem (1) with various nonlocal boundary conditions involving a Stieltjes integral with a signed measure and gave several sufficient conditions on the nonlinearity \(f = f(t,u)\) for the existence and multiplicity of positive solutions to problem (1) with multi-point boundary conditions.

Xu, Qin and Li [18] studied the following three-point boundary value problem
\[
\begin{align*}
\left\{ \begin{array}{l}
(\varphi_p(u'(t)))' + \lambda g(u(t)) + k(u(t)) = 0, \quad t \in (0,1), \\
u(0) = 0, \quad u(1) = u(\eta),
\end{array} \right.
\end{align*}
\] (5)
where \(p > 1\), \(\varphi_p(s) = |s|^{p-2}s\), \(\eta \in (0,1)\) and \(g, h \in C([0, \infty), [0, \infty])\) are strictly increasing. Under the suitable assumptions on \(g\) and \(k\) such that \(g\) is \(p\)-sublinear at \(0\) and \(k\) is \(p\)-superlinear at \(\infty\), the exact number of pseudo-symmetric positive solutions to problem (5) was studied.

Recently, Son and Wang [19] considered the following \(p\)-Laplacian system with nonlinear boundary conditions
\[
\begin{align*}
\left\{ \begin{array}{l}
(\varphi_p(u_i'))' + \lambda h_i(t)f_i(u_i) = 0, \quad t \in (0,1), \\
u_i(0) = 0 = a_iu_i(1) + c_i(\lambda, u_i(1), u_i(1))u_i(1),
\end{array} \right.
\end{align*}
\] (6)
where \(i, j \in \{1, 2\}, i \neq j\), \(\varphi_p(s) = |s|^{p-2}s\) for some \(p \in (1, \infty), c_i = c_i(\lambda, r, s)s\) is nondecreasing for \(s \in (0, \infty)\) and \(f_i \in C([0, \infty), \mathbb{R})\). Under several assumptions on \(h_i \) and \(f_i\), the existence and multiplicity of positive solutions to problem (6) were shown.

Bachhouche, Djebali and Moussaoui [20] considered the following \(p\)-Laplacian problem with nonlocal boundary conditions involving bounded linear operators \(L_0, L_1\)
\[
\begin{align*}
\left\{ \begin{array}{l}
(\varphi(u'))' + \lambda f(t,u,u') = 0, \quad t \in (0,1), \\
u(0) = L_0(u), \quad u(1) = L_1(u).
\end{array} \right.
\end{align*}
\] (7)
Here \(\varphi\) satisfies the following inequality
\[ \varphi(sx) \leq \varphi(s)\varphi(x) \text{ for all } s, x \in [0, \infty) \]
and the nonlinearity \(f = f(t,u,v)\) satisfies \(L^1\)-Carathéodory condition. The authors showed the existence of a positive solution or a nonnegative solution to problem (7).
For more general \( \varphi \) which does not satisfy \((A1)\), Kaufmann and Milne [21] considered BVP (1) and (2) with \( \hat{\alpha}_1 = \hat{\alpha}_2 = 0 \) and \( h \in L^1(0,1) \) with \( h \neq 0 \) and \( f = f(u) \in C([\mathbb{R}_+, \mathbb{R}_+]) \) and showed the existence of a positive solution for all \( \lambda > 0 \) under the assumptions on the nonlinearity \( f \) which induce the sublinear nonlinearity provided \( \varphi(s) = |s|^{p-1} s \) with \( p > 1 \). Recently, for an odd increasing homeomorphism \( \varphi \) satisfying \((A1)\), Kim and Jeong [4] studied various existence results for positive solutions to BVP (1) and (2) with \( \lambda = 1 \). For other interesting results, we refer the reader to [22–47] and the references therein.

Let \( \xi : [0, \infty) \to [0, \infty) \) be an increasing homeomorphism. Then we denote by \( H_\xi \) the set

\[
\left\{ g \in C((0,1),(0,\infty)) : \int_0^1 \xi^{-1} \left( \int_s^1 g(t)dt \right) ds < \infty \right\}.
\]

It is well known that

\[
\varphi^{-1}(x)\psi_2^{-1}(y) \leq \varphi^{-1}(xy) \leq \varphi^{-1}(x)\psi_1^{-1}(y)
\]

for all \( x, y \in \mathbb{R}_+ \) (8) and

\[
L^1(0,1) \cap C(0,1) \subseteq H_{\psi_1} \subseteq H_\varphi \subseteq H_{\psi_2}
\]

(see, e.g., ([7], Remark 1)).

Recall that we say that \( g : (0,1) \times [0, \infty) \to \mathbb{R} \) satisfies \( L^1 \)-Carathéodory condition if

(i) \( g(\cdot, u) \) is measurable for all \( u \in [0, \infty) \);
(ii) \( g(t, \cdot) \) is continuous for almost all \( u \in [0, \infty) \);
(iii) for every \( r > 0 \), there exists \( h_r \in L^1(0,1) \) such that

\[
|g(t, u)| \leq h_r(t) \text{ for a.e. } t \in (0,1) \text{ and all } u \in [0, r].
\]

Throughout this paper, we assume \( h \in H_\varphi \). Since there may be a function \( h \in H_\varphi \setminus L^1(0,1) \) (see, e.g., Remark 2 below), the nonlinearity \( h(t)f(t, u) \) in the equation (1) may not satisfy the \( L^1 \)-Carathéodory condition. Consequently, the solution space should be taken as \( C[0,1] \), since the solutions to BVP (1) and (2) may not be in \( C^1[0,1] \) unlike References [20–22] where the nonlinearity satisfies the \( L^1 \)-Carathéodory condition. The lack of solution regularity and the boundary conditions (2) make it difficult to get the desired result.

The rest of this article is organized as follows. In Section 2, we give some preliminaries which are crucial for proving the main results in this paper. In Section 3, the main results (Theorems 2–4) are proved and some examples which illustrate the main results are given. Finally, the summary of this paper is given in Section 4.

2. Preliminaries

Throughout this section, we assume that \((A1),(A2)\), \( f \in C([0,1] \times [0, \infty),(0,\infty)) \) and \( h \in H_\varphi \) hold. The usual maximum norm in a Banach space \( C[0,1] \) of continuous functions on \([0,1]\) is denoted by

\[
\|u\|_\infty := \max_{t \in [0,1]} |u(t)| \text{ for } u \in C[0,1],
\]

and let

\[
K := \{ u \in \mathcal{P} : u(t) \geq \rho \|u\|_\infty \text{ for } t \in [\alpha, \beta] \text{ and } u \text{ satisfies (2) } \}
\]

be a cone in \( C[0,1] \). Here, \( \mathcal{P} := C([0,1] \times [0,\infty)) \), \( \alpha \) and \( \beta \) are any fixed constants satisfying \( 0 < \alpha < \beta < 1 \), \( w_0 := \min_{t \in [0,1]} w(t) > 0 \) and
\[ \rho_w := \min \{ \alpha, 1 - \beta \} \psi_2^{-1} \left( \frac{1}{\|w\|_\infty} \right) \left[ \psi_1^{-1} \left( \frac{1}{\|w_0\|} \right) \right]^{-1} \in (0, 1]. \]

For \( r > 0 \), let
\[ K_r := \{ u \in \mathcal{K} : \|u\|_\infty < r \}, \partial K_r := \{ u \in \mathcal{K} : \|u\|_\infty = r \} \]
and
\[ \mathcal{K}_r := K_r \cup \partial K_r. \]

Now, we introduce a solution operator related to BVP (1) and (2). Let \( (\lambda, u) \in (0, \infty) \times \mathcal{K} \) be given. Define functions \( v_{\lambda,u}^1, v_{\lambda,u}^2 : (0, 1) \to (-\infty, \infty) \) by, for \( x \in (0, 1) \),
\[ v_{\lambda,u}^1(x) = A_1 \int_0^1 \int_0^1 I_{\lambda,u}(s,x)dsda_1(r) + \int_0^1 I_{\lambda,u}(s,x)ds \]
and
\[ v_{\lambda,u}^2(x) = -A_2 \int_0^1 \int_r^1 I_{\lambda,u}(s,x)dsda_2(r) - \int_r^1 I_{\lambda,u}(s,x)ds. \]
Here
\[ A_i := (1 - \lambda_i)^{-1} \in [1, \infty) \text{ for } i = 1, 2 \]
and
\[ I_{\lambda,u}(s,x) = \varphi^{-1} \left( \frac{\lambda}{w(s)} \int_s^x h(\tau)f(\tau,u(\tau))d\tau \right). \]

**Remark 1.** We give the properties of \( I_{\lambda,u} \) for any given \( (\lambda, u) \in (0, \infty) \times \mathcal{K} \) as follows.

1. \( I_{\lambda,u}(x,y) > 0 \) and \( I_{\lambda,u}(y,x) < 0 \) for any \( x, y \) satisfying \( 0 < x < y < 1 \).
2. \( I_{\lambda,u}(s,x_1) < I_{\lambda,u}(s,x_2) \) for any \( s \in (0,1) \) and \( 0 < x_1 < x_2 < 1 \).
3. Let \( x \in (0,1) \) be given. Then \( I_{\lambda,u}(.,x) \in L^1(0,1) \). Moreover, for any \( \epsilon \in [0,\min\{x,1-x\}] \), there exists \( C^* = C^*(x,\epsilon,\lambda,u) > 0 \) satisfying
\[ \int_0^1 |I_{\lambda,u}(s,x)|ds \leq C^*. \tag{9} \]

Indeed, by (8),
\[ \int_0^1 |I_{\lambda,u}(s,x)|ds \]
\[ = \int_0^x |I_{\lambda,u}(s,x)|ds + \int_x^1 |I_{\lambda,u}(s,x)|ds \]
\[ = \int_0^x \varphi^{-1} \left( \frac{\lambda}{w(s)} \int_s^x h(\tau)f(\tau,u(\tau))d\tau \right)ds + \int_x^1 \varphi^{-1} \left( \frac{\lambda}{w(s)} \int_x^h h(\tau)f(\tau,u(\tau))d\tau \right)ds \]
\[ \leq \int_0^x \varphi^{-1} \left( \frac{\lambda M_u}{w_0} \int_s^x h(\tau)d\tau \right)ds + \int_x^1 \varphi^{-1} \left( \frac{\lambda M_u}{w_0} \int_x^h h(\tau)d\tau \right)ds \]
\[ \leq \varphi^{-1} \left( \frac{\lambda M_u}{w_0} \right) \left[ \int_0^{x+\epsilon} \varphi^{-1} \left( \int_s^{x+\epsilon} h(\tau)d\tau \right)ds + \int_{x-\epsilon}^1 \varphi^{-1} \left( \int_{x-\epsilon}^{x-\epsilon} h(\tau)d\tau \right)ds \right] =: C^*. \]

Here
\[ M_u := \max\{f(x,u(x)) : x \in [0,1]\} > 0. \]

The following lemmas (Lemmas 1–3) can be proved by the similar arguments in [4] (Section 2) and [39] (Section 2). For the sake of completeness, we give the proofs of them.
Lemma 1. Assume that (A1), (A2), \( f \in C([0, 1] \times [0, \infty), (0, \infty)) \) and \( h \in \mathcal{H}_\varphi \) hold, and let \((\lambda, u) \in (0, \infty) \times \mathcal{K}\) be given. Then there exists a unique point \( \sigma = \sigma(\lambda, u) \in (0, 1) \) satisfying

\[
v^1_{\lambda, u}(\sigma) = v^2_{\lambda, u}(\sigma).
\]

Proof. From Remark 1, it follows that \( v^1_{\lambda, u} \) is a strictly increasing continuous function on \((0, 1)\) and \( v^2_{\lambda, u} \) is a strictly decreasing continuous function on \((0, 1)\).

Next, we prove

\[
\lim_{x \to 0^+} v^1_{\lambda, u}(x) \in [-\infty, 0].
\]

In order to show it, we rewrite \( v^1_{\lambda, u}(x) \) by, for \( x \in (0, 1) \),

\[
v^1_{\lambda, u}(x) = A_1 \left( \int_0^1 \int_0^r I_{\lambda, u}(s, x)d\alpha_1(r) + \left( 1 - \int_0^1 d\alpha_1(r) \right) \int_0^x I_{\lambda, u}(s, x)ds \right)
= A_1 \left( \int_0^1 \int_0^r I_{\lambda, u}(s, x)d\alpha_1(r) + \int_0^x I_{\lambda, u}(s, x)ds \right).
\]

For any \( x \in (0, 1) \), by Remark 1 (1),

\[
\int_0^1 \int_0^r I_{\lambda, u}(s, x)d\alpha_1(r)
= - \int_0^1 \int_0^x I_{\lambda, u}(s, x)d\alpha_1(r) + \int_0^1 \int_0^x I_{\lambda, u}(s, x)ds \leq 0,
\]

which implies

\[
v^1_{\lambda, u}(x) \leq A_1 \int_0^x I_{\lambda, u}(s, x)ds \text{ for any } x \in (0, 1).
\]

By (8), for any \( x \in (0, 1/2) \),

\[
0 \leq \int_0^x I_{\lambda, u}(s, x)ds = \int_0^x \varphi^{-1} \left( \frac{\lambda}{w(s)} \int_s^x h(\tau, u(\tau))d\tau \right)
\leq \varphi^{-1} \left( \frac{\lambda M_u}{w_0} \right) \int_0^x \varphi^{-1} \left( \int_s^x h(\tau, u(\tau))d\tau \right)ds,
\]

where

\[
M_u = \max\{f(x, u(x)) : x \in [0, 1]\} > 0.
\]

From \( h \in \mathcal{H}_\varphi \), it follows that

\[
\lim_{x \to 0^+} \int_0^x I_{\lambda, u}(s, x)ds = 0.
\]

Combining this and (10) yields

\[
\lim_{x \to 0^+} v^1_{\lambda, u}(x) \in [-\infty, 0].
\]

Next we will show

\[
\lim_{x \to 1^-} v^1_{\lambda, u}(x) \in (0, \infty].
\]
For any $x \in (0, 1)$,
\[
v_{\lambda,u}(x) = A_1 \left[ \int_0^x \int_0^r I_{\lambda,u}(s,x)dsda_1(r) \\
+ \int_x^1 \int_0^r I_{\lambda,u}(s,x)dsda_1(r) + \int_1^x \int_0^r I_{\lambda,u}(s,x)dsda_1(r) \\
+ \int_0^x I_{\lambda,u}(s,x)ds. \right]
\]
From
\[
\lambda h(\tau)f(\tau, u(\tau)) > 0 \text{ for any } \tau \in (0, 1),
\]
it follows that
\[
\lim_{x \to 1^-} \int_0^x I_{\lambda,u}(s,x)ds = \lim_{x \to 1^-} \int_0^x \varphi^{-1}\left( \frac{\lambda}{\psi(s)} \int_s^x h(\tau)f(\tau, u(\tau))d\tau \right) \in (0, \infty].
\]
For any $x \in (0, 1)$,
\[
\int_0^x \int_0^s I_{\lambda,u}(s,x)dsda_1(r) + \int_x^1 \int_0^s I_{\lambda,u}(s,x)dsda_1(r) > 0.
\]
For any $x > 1/2$, by (8),
\[
\left| \int_x^1 \int_x^r I_{\lambda,u}(s,x)dsda_1(r) \right| = \int_x^1 \int_x^r \varphi^{-1}\left( \frac{\lambda}{\psi(s)} \int_s^x h(\tau)f(\tau, u(\tau))d\tau \right)dsda_1(r) \\
\leq \varphi^{-1}\left( \frac{\lambda M}{\psi \psi_0} \right) \int_x^1 \int_x^r \varphi^{-1}\left( \int_s^x h(\tau)d\tau \right)dsda_1(r) \\
\leq \varphi^{-1}\left( \frac{\lambda M}{\psi \psi_0} \right) \int_x^1 da_1(r) \int_x^r \varphi^{-1}\left( \int_s^x h(\tau)d\tau \right)dsda_1(r) \\
\leq \varphi^{-1}\left( \frac{\lambda M}{\psi \psi_0} \right) \int_x^1 da_1(r) \int_x^r \varphi^{-1}\left( \int_s^x h(\tau)d\tau \right)ds.
\]
Combining this and the fact $h \in \mathcal{H}_{\psi}$ yields
\[
\lim_{x \to 1^-} \int_x^1 \int_x^r I_{\lambda,u}(s,x)dsda_1(r) = 0.
\]
Consequently
\[
\lim_{x \to 1^-} v_{\lambda,u}^1(x) \in (0, \infty].
\]
Similarly, it can be shown that
\[
\lim_{x \to 0^+} v_{\lambda,u}^1(x) \in (0, \infty] \text{ and } \lim_{x \to 1^-} v_{\lambda,u}^2(x) \in [-\infty, 0].
\]
Thus, by continuity and strict monotonicity of $v_{\lambda,u}^1$ and $v_{\lambda,u}^2$, there exists a unique $\sigma \in (0, 1)$ satisfying
\[
v_{\lambda,u}^1(\sigma) = v_{\lambda,u}^2(\sigma).
\]

\[
\square
\]
Define an operator $T : [0, \infty) \times \mathcal{K} \to C[0, 1]$ by
\[
T(0, u) = 0 \text{ for } u \in \mathcal{K},
\]
and for $(\lambda, u) \in (0, \infty) \times \mathcal{K}$,
\[
T(\lambda, u)(t) = \begin{cases} 
A_1 \int_0^t \int_0^r I_{\lambda,u}(s,\sigma)dsda_1(r) + \int_0^t I_{\lambda,u}(s,\sigma)ds, & \text{if } 0 \leq t \leq \sigma, \\
-2A_2 \int_0^t \int_0^r I_{\lambda,u}(s,\sigma)dsda_2(r) - \int_0^t I_{\lambda,u}(s,\sigma)ds, & \text{if } \sigma \leq t \leq 1,
\end{cases} \quad (11)
\]
where \( \sigma = \sigma(\lambda, u) \) is the unique point satisfying \( v_{\lambda,u}^1(\sigma) = v_{\lambda,u}^2(\sigma) \) in Lemma 1. By the definition of \( \sigma = \sigma(\lambda, u) \), \( T \) is well defined and
\[
T(\lambda, u)(\sigma) = v_{\lambda,u}^1(\sigma).
\]
Moreover, \( T(\lambda, u) \) is strictly increasing on \([0, \sigma)\) and is strictly decreasing on \((\sigma, 1]\).

**Lemma 2.** Assume that \((A1), (A2), f \in C([0, 1] \times [0, \infty), (0, 0))\) and \( h \in H_\phi \) hold. Then
\[
T(\lambda, u) \in \mathcal{K} \text{ for any } (\lambda, u) \in [0, \infty) \times \mathcal{K}
\]
and
\[
T(\lambda, u)(\sigma) = \| T(\lambda, u) \|_\infty > 0 \text{ for any } (\lambda, u) \in (0, \infty) \times \mathcal{K}.
\]
Moreover, \( u \) is a positive solution to BVP \((1)\) and \((2)\) if and only if \( T(\lambda, u) = u \) for some \((\lambda, u) \in (0, \infty) \times \mathcal{K}\).

**Proof.** First, we show that
\[
T(\lambda, u) \in \mathcal{K} \text{ for any } (\lambda, u) \in [0, \infty) \times \mathcal{K}.
\]
Clearly,
\[
T(0, u) = 0 \in \mathcal{K} \text{ for any } u \in \mathcal{K}.
\]
Let \((\lambda, u) \in (0, \infty) \times \mathcal{K}\) be given. Then, by \((11)\),
\[
(T(\lambda, u))'(s) = I_{\lambda,u}(s, \sigma) \text{ for } s \in (0, 1),
\]
which implies, for \( r \in [0, 1]\),
\[
T(\lambda, u)(r) = T(\lambda, u)(0) + \int_0^r I_{\lambda,u}(s, \sigma)ds. \tag{12}
\]
Since
\[
T(\lambda, u)(0) = A_1 \int_0^1 \int_0^r I_{\lambda,u}(s, \sigma)dsd\alpha_1(r)
\]
\[
= \frac{1}{1 - \dot{\alpha}_1} \int_0^1 \int_0^r I_{\lambda,u}(s, \sigma)dsd\alpha_1(r),
\]
integrating \((12)\) from 0 to 1,
\[
\int_0^1 T(\lambda, u)(r)dr = \int_0^1 T(\lambda, u)(0)dr + \int_0^1 \int_0^r I_{\lambda,u}(s, \sigma)dsd\alpha_1(r)
\]
\[
= \dot{\alpha}_1 T(\lambda, u)(0) + (1 - \dot{\alpha}_1) T(\lambda, u)(0)
\]
\[
= T(\lambda, u)(0).
\]
Similarly, it can be shown that
\[
T(\lambda, u)(1) = \int_0^1 T(\lambda, u)(r)dr.
\]
Thus \( T(\lambda, u) \) satisfies the boundary conditions \((2)\). Since \( T(\lambda, u) \) is strictly increasing on \([0, \sigma)\) and is strictly decreasing on \((\sigma, 1]\),
\[
T(\lambda, u)(t) \geq \min\{T(\lambda, u)(0), T(\lambda, u)(1)\} \text{ for } t \in [0, 1].
\]
We only consider the case
\[
\min\{T(\lambda, u)(t) : 0 \leq t \leq 1\} = T(\lambda, u)(0),
\]
since the case
\[
\min \{ T(\lambda, u)(t) : 0 \leq t \leq 1 \} = T(\lambda, u)(1)
\]
is similar. Then
\[
T(\lambda, u)(0) = \int_0^1 T(\lambda, u)(r) \, d\alpha_1(r) \geq \hat{\alpha}_1 T(\lambda, u)(0),
\]
which implies
\[
T(\lambda, u)(0) \geq 0,
\]
since
\[
\hat{\alpha}_1 = \int_0^1 d\alpha_1(r) \in [0, 1).
\]
Consequently,
\[
T(\lambda, u)(t) \geq 0 \text{ for all } t \in [0, 1], \text{i.e., } T(\lambda, u) \in \mathcal{P}.
\]
Clearly
\[
T(\lambda, u)(\sigma) = \| T(\lambda, u) \|_{\infty} > 0,
\]
since \( T(\lambda, u) \) is strictly increasing on \([0, \sigma]\) and is strictly decreasing on \((\sigma, 1] .\)
For \( t \in [0, \sigma] \), by (8),
\[
T(\lambda, u)(t) = T(\lambda, u)(0) + \int_0^t \varphi^{-1} \left( \lambda \int_s^\sigma h(\tau) f(\tau, u(\tau)) \, d\tau \right) \, ds
\]
\[
\geq T(\lambda, u)(0) + \psi_2^{-1} \left( \frac{1}{\| w \|_{\infty}} \right) q_1(t).
\]
(13)
Here
\[
q_1(t) := \int_0^t \varphi^{-1} \left( \lambda \int_s^\sigma h(\tau) f(\tau, u(\tau)) \, d\tau \right) \, ds \text{ for } t \in [0, \sigma].
\]
Similarly,
\[
\| T(\lambda, u) \|_{\infty} = T(\lambda, u)(\sigma) \leq T(\lambda, u)(0) + \psi_1^{-1} \left( \frac{1}{w_0} \right) q_1(\sigma).
\]
(14)
Since
\[
q_1'(t) = \varphi^{-1} \left( \lambda \int_s^\sigma h(\tau) f(\tau, u(\tau)) \, d\tau \right) > 0 \text{ for } t \in (0, \sigma),
\]
\( q_1' \) is a strictly decreasing function on \([0, \sigma]\). Consequently, \( q_1 \) is a strictly increasing concave function
on \([0, \sigma]\) with \( q_1(0) = 0 \), so that
\[
q_1(t) \geq t q_1(\sigma) \text{ for } t \in [0, \sigma].
\]
Consequently, by (13) and (14),
\[
T(\lambda, u)(t) - T(\lambda, u)(0) \geq \psi_2^{-1} \left( \frac{1}{\| w \|_{\infty}} \right) q_1(t)
\]
\[
\geq t \psi_2^{-1} \left( \frac{1}{\| w \|_{\infty}} \right) q_1(\sigma)
\]
\[
\geq t \rho_1(\| T(\lambda, u) \|_{\infty} - T(\lambda, u)(0)),
\]
where
\[
\rho_1 := \psi_2^{-1} \left( \frac{1}{\| w \|_{\infty}} \right) \psi_1^{-1} \left( \frac{1}{w_0} \right)^{-1} \in (0, 1].
\]
Consequently, for $t \in [0, \sigma]$, \[
T(\lambda, u)(t) \geq \rho_1 t \|T(\lambda, u)\|_{\infty} + (1 - \rho_1 t)T(\lambda, u)(0) \\
\geq \rho_1 t \|T(\lambda, u)\|_{\infty}.
\]

Similarly, it can be shown that \[
T(\lambda, u)(t) \geq \rho_1 (1 - t) \|T(\lambda, u)\|_{\infty} \text{ for } t \in [\sigma, 1].
\]

Then \[
T(\lambda, u)(t) \geq \rho_1 \min\{t, 1 - t\} \|T(\lambda, u)\|_{\infty} \text{ for } t \in [0, 1],
\]
and consequently \[
T(\lambda, u)(t) \geq \rho_1 \|T(\lambda, u)\|_{\infty} \text{ for } t \in [\alpha, \beta],
\]
i.e., \[
T(\lambda, u) \in \mathcal{K}.
\]

Assume that \[
T(\lambda, u) = u \text{ for some } (\lambda, u) \in (0, \infty) \times \mathcal{K}.
\]

From direct differentiation and the definition of $\mathcal{K}$, it follows that $u$ is a nonnegative solution to BVP (1) and (2). Since $\lambda > 0$, $T(\lambda, u) \neq 0$, and by (15), \[
u(t) = T(\lambda, u)(t) > 0 \text{ for } t \in (0, 1).
\]

Consequently, $u$ is a positive solution to BVP (1) and (2) with $\lambda > 0$. Let $u_\lambda$ be a positive solution to BVP (1) and (2). Then \[
0 \leq u_\lambda(0) < \|u_\lambda\|_{\infty}.
\]

Indeed, assume on the contrary that $u_\lambda(0) = \|u_\lambda\|_{\infty} > 0$. Since \[
0 \leq u_\lambda(0) = \int_0^1 u_\lambda(r) \alpha_1(r) \, dr \leq \hat{\alpha}_1 \|u_\lambda\|_{\infty} = \hat{\alpha}_1 u_\lambda(0).
\]

Then $\|u_\lambda\|_{\infty} = u_\lambda(0) = 0$, which contradicts the fact that $u_\lambda$ is a positive solution to BVP (1) and (2). Similarly, it can be shown that \[
0 \leq u_\lambda(1) < \|u_\lambda\|_{\infty}.
\]

Consequently, there exists a point $\sigma_\lambda \in (0, 1)$ satisfying \[
\|u_\lambda\|_{\infty} = u_\lambda(\sigma_\lambda).
\]

Integrating the Equation (1) with $u = u_\lambda$ yields \[
u_\lambda(r) = u_\lambda(0) + \int_0^r I_{\lambda, u_\lambda}(s, \sigma_\lambda) \, ds = u_\lambda(1) - \int_r^1 I_{\lambda, u_\lambda}(s, \sigma) \, ds \text{ for } r \in [0, 1].
\]

By boundary conditions (2) with $u = u_\lambda$, \[
u_\lambda(0) = A_1 \int_0^1 \int_0^1 I_{\lambda, u_\lambda}(s, \sigma_\lambda) \, ds \, d\alpha_1(r)
\]
and \[
u_\lambda(1) = -A_2 \int_0^1 \int_r^1 I_{\lambda, u_\lambda}(s, \sigma_\lambda) \, ds \, d\alpha_2(r).
\]
Consequently

\[ u_\lambda \equiv T(\lambda, u_\lambda) \in K. \]

Clearly \( \lambda > 0 \), since

\[ T(0, u) = 0 \] for all \( u \in K. \]

Thus, the proof is complete. \( \square \)

**Lemma 3.** Assume that \((A1), (A2), f \in C([0, 1] \times [0, \infty), (0, \infty))\) and \( h \in H_{\psi} \) hold. Let \( L > 0 \) be given and let \((\lambda_n, u_n)\) be a bounded sequence in \((0, \infty) \times K\) with

\[ |\lambda_n| + \|u_n\|_\infty \leq L. \]

If \( \lim_{n \to \infty} \sigma_n \in \{0, 1\}, \) then

\[ T(\lambda_n, u_n)(\sigma_n) = \|T(\lambda_n, u_n)\|_\infty \to 0 \]

and

\[ \lambda_n \to 0 \] as \( n \to \infty. \)

Here \( \sigma_n = \sigma(\lambda_n, u_n) \) is the unique point satisfying

\[ \nu^1_{\lambda_n, u_n}(\sigma_n) = \nu^2_{\lambda_n, u_n}(\sigma_n) \] for each \( n \in \mathbb{N}. \)

**Proof.** We only prove the case

\[ \lim_{n \to \infty} \sigma_n = 0, \]

since the case \( \lim_{n \to \infty} \sigma_n = 1 \) can be dealt similarly. Since there exist positive constants \( N_1, N_2 \) satisfying

\[ \lambda_n N_1 \leq \lambda_n f(t, u) \leq N_2 \] for all \((t, u) \in [0, 1] \times [0, L] \) and all \( n, \)

by \((8)\) and \((10), \)

\[
\|T(\lambda_n, u_n)\|_\infty = A_1 \int_0^{\sigma_n} \int_0^{r \sigma_n} I_{\lambda_n, u_n}(s, \sigma_n) ds \, d\alpha_1(r) + \int_{\sigma_n}^\infty I_{\lambda_n, u_n}(s, \sigma_n) ds \\
\leq A_1 \int_0^{\sigma_n} I_{\lambda_n, u_n}(s, \sigma_n) ds \\
\leq A_1 \varphi_1^{-1} \left( \frac{N_2}{\omega_0} \right) \int_0^{\sigma_n} \varphi^{-1} \left( \int_s^{\sigma_n} h(\tau) \, d\tau \right) ds.
\]

Then, from \( h \in H_{\psi}, \) it follows that

\[
\|T(\lambda_n, u_n)\|_\infty \to 0 \] as \( n \to \infty. \quad (16)
\]

Since \( T(\lambda_n, u_n)(1) \geq 0 \) for all \( n, \) by \((8)\),

\[
\|T(\lambda_n, u_n)\|_\infty = T(\lambda_n, u_n)(\sigma_N) \\
= T(\lambda_n, u_n)(1) - \int_0^{\sigma_n} I_{\lambda_n, u_n}(s, \sigma_n) ds \\
\geq - \int_0^{\sigma_n} I_{\lambda_n, u_n}(s, \sigma_n) ds \\
= \int_0^{\sigma_n} \varphi^{-1} \left( \frac{\lambda_n}{\omega(s)} \right) \int_s^{\sigma_n} h(\tau) f(\tau, u_n(\tau)) \, d\tau \, ds \\
\geq \varphi_2^{-1} \left( \frac{\lambda_n N_1}{\|\omega\|_\infty} \right) \int_0^{\sigma_n} \varphi^{-1} \left( \int_s^{\sigma_n} h(\tau) \, d\tau \right) ds \geq 0.
\]
Since \( h(t) > 0 \) for all \( t \in (0, 1) \), by (16),
\[
\lambda_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Using Lemma 3 and (8), by the similar arguments in the proof of [17] (Lemma 2.4) and [48] (Lemma 3.3), one can prove the complete continuity of the operator \( T = T(\lambda, u) \). We only state the result as follows.

**Lemma 4.** Assume that \((A1), (A2), f \in C([0, 1] \times [0, \infty), (0, \infty))\) and \( h \in \mathcal{H}_\psi \) hold. Then the operator \( T : [0, \infty) \times K \to K \) is completely continuous, i.e., compact and continuous.

We recall a well-known theorem for the existence of a global continuum of solutions by Leray and Schauder [49]:

**Theorem 1.** (see, e.g., [50] (Corollary 14.12)) Let \( X \) be a Banach space with \( X \neq \{0\} \) and let \( K \) be a cone in \( X \). Consider

\[
x = T(\lambda, x),
\]

where \( \lambda \in [0, \infty) \) and \( u \in K \). If \( T : [0, \infty) \times K \to K \) is completely continuous and \( T(0, u) = 0 \) for all \( u \in K \), there exists an unbounded solution component \( C \) of (17) in \([0, \infty) \times K \) emanating from \((0, 0)\).

Since \( T(0, u) = 0 \) for all \( u \in K \), by Lemmas 2–4 and Theorem 1, one has the following proposition.

**Proposition 1.** Assume that \((A1), (A2), f \in C([0, 1] \times [0, \infty), (0, \infty))\) and \( h \in \mathcal{H}_\psi \) hold. Then there exists an unbounded solution component \( C \) of (17) in \([0, \infty) \times K \) satisfying (i) \( C \cap \{(0 \times K) = \{(0, 0)\}\) and (ii) for any \((\lambda, u) \in C \setminus \{(0, 0)\}\), \( u \) is a positive solution to BVP (1) and (2) with \( \lambda > 0 \).

### 3. Main Results

First, we give a list of hypotheses on \( f = f(t, s) \) which are used in this section:

- \((F_0)\) \( \lim_{s \to \infty} \min_{t \in [0, 1]} \frac{f(t, s)}{\psi_1(s)} = 0 \).
- \((F'_0)\) \( \lim_{s \to \infty} \max_{t \in [0, 1]} \frac{f(t, s)}{\phi(s)} = 0 \).
- \((F_\infty)\) There exists a nondegenerate interval \([\alpha, \beta] \subseteq (0, 1)\) satisfying
  \[
  \lim_{s \to \infty} \min_{t \in [\alpha, \beta]} \frac{f(t, s)}{\phi(s)} = \infty.
  \]

For convenience, let
\[
\gamma := \frac{\alpha + \beta}{2}.
\]

Since \( \alpha \) and \( \beta \) are any fixed constants in the cone \( K \) satisfying \( 0 < \alpha < \beta < 1 \),
\[
0 < \alpha < \gamma < \beta < 1.
\]

When we need the assumption \((F_\infty)\), let \( \alpha \) and \( \beta \) in the cone \( K \) be the same constants in the assumption \((F_\infty)\).
Lemma 5. Assume that (A1), (A2), \( f \in C([0, 1] \times [0, \infty), (0, \infty)) \), \((F_{\infty})\) and \( h \in \mathcal{H}_{\rho} \) hold. Then there exists \( \lambda > 0 \) such that BVP (1) and (2) has no positive solutions for any \( \lambda > \lambda \).

Proof. Let \( u \) be a positive solution to BVP (1) and (2) with \( \lambda > 0 \) and let \( \sigma \in (0, 1) \) be the unique point satisfying \( u(\sigma) = \|u\|_{\infty} \). Since \( f \in C([0, 1] \times [0, \infty), (0, \infty)) \), by \((F_{\infty})\), there exists \( \hat{C} > 0 \) satisfying

\[
f(t, s) > \hat{C} \varphi(s) \text{ for } (t, s) \in [\alpha, \beta] \times [0, \infty).
\]

We only give the proof for the case \( \sigma \geq \gamma \), since the case \( \sigma < \gamma \) can be dealt similarly. Then

\[
u(t) \geq u(\alpha) \text{ for } t \in [\alpha, \gamma],
\]

which implies

\[
f(t, u(t)) > \hat{C} \varphi(u(t)) \geq \hat{C} \varphi(u(\alpha)) \text{ for } t \in [\alpha, \gamma].
\]

By Lemma 2 and (8),

\[
u(\alpha) = u(0) + \int_0^\alpha I_{\lambda, u}(s, \sigma) ds
\]

\[
\geq \int_0^\alpha \varphi^{-1} \left( \frac{\lambda}{\varphi(s)} \int_s^\sigma h(\tau)f(\tau, u(\tau)) d\tau \right) ds
\]

\[
\geq \int_0^\alpha \varphi^{-1} \left( \frac{\lambda}{\varphi(s)} \int_s^\gamma h(\tau)f(\tau, u(\tau)) d\tau \right) ds
\]

\[
\geq \int_0^\gamma \varphi^{-1} \left( \int_s^\gamma h(\tau)d\tau \varphi(u(\alpha)) \right) ds \varphi^{-1} \left( \frac{\lambda \hat{C}}{\|w\|_{\infty}} \right) u(\alpha)
\]

\[
\geq C_h \varphi^{-1} \left( \frac{\lambda \hat{C}}{\|w\|_{\infty}} \right) u(\alpha).
\]

Here

\[
C_h := \min \left\{ \int_\alpha^\gamma \varphi^{-1} \left( \int_s^\gamma h(\tau)d\tau \right) ds, \int_\gamma^\beta \varphi^{-1} \left( \int_s^\beta h(\tau)d\tau \right) ds \right\} > 0.
\]

Thus

\[
\lambda \leq \varphi_2 \left( \frac{1}{C_h} \right) \frac{\|w\|_{\infty}}{\hat{C}} =: \lambda.
\]

\[\square\]

Lemma 6. Assume that (A1), (A2), \( f \in C([0, 1] \times [0, \infty), (0, \infty)) \), \((F_{\infty})\) and \( h \in \mathcal{H}_{\rho} \) hold. Let \( I > 0 \) be given. Then there exists \( M_I > 0 \) such that \( \|u\|_{\infty} \leq M_I \) for any positive solutions \( u \) to BVP (1) and (2) with \( \lambda \in [I, \infty) \).

Proof. Suppose to the contrary that there exists a sequence \( \{(\lambda_n, u_n)\} \) satisfying \( u_n \) is a positive solutions to BVP (1) and (2) with \( \lambda = \lambda_n \in [I, \infty) \) and \( \|u_n\|_{\infty} \to \infty \) as \( n \to \infty \).

Take

\[
C^* = \frac{\|w\|_{\infty} \varphi_2(\alpha^{-1})}{(\gamma - \alpha) h_0} + 1,
\]

where

\[
h_0 := \min\{h(t) : t \in [\alpha, \beta]\} > 0.
\]
By \((F_\infty)\), there exists \(K > 0\) such that
\[ f(t, s) > C^* \varphi(s) \text{ for } (t, s) \in [a, b] \times (K, \infty). \]
By Lemma 2,
\[ u_N(t) \geq \rho_u \|u_N\|_{\infty} \text{ for } t \in [a, b]. \]
Then, for sufficiently large \(N > 0\),
\[ u_N(t) \geq K \text{ for } t \in [a, b], \]
which implies
\[ \lambda_N h(t) f(t, u_N(t)) \geq I_0 C^* \varphi(u_N(t)) \text{ for all } t \in [a, b]. \]
Let \(\sigma_N \in (0, 1)\) be a unique point satisfying
\[ u_N(\sigma_N) = \|u_N\|_{\infty}. \]
We only consider the case \(\sigma_N \geq \gamma\), since the case \(\sigma_N < \gamma\) can be dealt in a similar manner. By (8) and the fact that
\[ u_N(t) \geq u_N(\sigma) \text{ for } t \in [a, \sigma_N], \]
one has
\[
\begin{align*}
u_N(\sigma) &= u_N(0) + \int_0^\sigma \varphi^{-1} \left( \frac{\lambda_N}{w(s)} \int_s^{\sigma_N} h(\tau)f(\tau, u_N(\tau))d\tau \right)ds \\
&\geq \int_0^\sigma \varphi^{-1} \left( \frac{\lambda_N}{w(s)} \int_s^{\gamma} h(\tau)f(\tau, u_N(\tau))d\tau \right)ds \\
&\geq a\varphi^{-1}(\|w\|_{\infty}^{-1}(\gamma - a) I_0 C^* \varphi(u_N(\sigma))) \\
&\geq a\psi_2^{-1}(\|w\|_{\infty}^{-1}(\gamma - a) I_0 C^*) u_N(\sigma),
\end{align*}
\]
which implies
\[ C^* \leq \frac{\|w\|_{\infty} \psi_2(a^{-1})}{(\gamma - a) I_0}. \]
However, this contradicts the choice of \(C^*\). Thus the proof is complete. \(\square\)

**Theorem 2.** Assume that \((A1), (A2), f \in C([0, 1] \times [0, \infty), (0, \infty)), (F_\infty)\) and \(h \in H_\varphi\) hold. Then there exists \(\lambda_* > 0\) such that BVP \((1)\) and \((2)\) has at least two positive solutions \(u^1_\lambda\) and \(u^2_\lambda\) for \(\lambda \in (0, \lambda_*)\), at least one positive solution for \(\lambda = \lambda_*\) and no positive solutions for \(\lambda > \lambda_*\). Moreover, for \(\lambda \in (0, \lambda_*),\) two positive solutions \(u^1_\lambda\) and \(u^2_\lambda\) satisfy
\[ \|u^1_\lambda\|_{\infty} \to 0 \text{ and } \|u^2_\lambda\|_{\infty} \to \infty \text{ as } \lambda \to 0^+. \]

**Proof.** Set
\[ \lambda_* := \sup \{ \lambda > 0 : \text{BVP \((1)\) and \((2)\) has at least two positive solution for all } \lambda \in (0, \lambda) \}. \]
Then, by Proposition 1, Lemmas 5 and 6, \(\lambda_* \in (0, \infty)\) is well-defined. Indeed, let \(\{(\lambda_n, u_n)\}\) be a sequence in the unbounded solution component \(\mathcal{C}\) defined in Proposition 1 satisfying
\[ \lambda_n + \|u_n\|_{\infty} \to \infty \text{ as } n \to \infty. \]
By Lemma 5,
\[ \lambda_n \leq \lambda, \]
which implies
\[\|u_n\|_\infty \to \infty \text{ as } n \to \infty.\]

From Lemma 6, it follows that \(\lambda_n \to 0^+\) as \(n \to \infty\). Consequently, the shape of the continuum of \(C\) is determined, so that BVP (1) and (2) has two positive solutions \(u_1, u_2\) for all \(\lambda > 0\) such that
\[\|u_1\|_\infty \to 0 \text{ and } \|u_2\|_\infty \to \infty \text{ as } \lambda \to 0^+.\]

By Lemma 5, there are no positive solutions to BVP (1) and (2) for all \(\lambda \geq \bar{\lambda}\). Thus, \(\lambda_\ast \in (0, \infty)\) is well-defined.

By the definition of \(\lambda_\ast\), BVP (1) and (2) has at least two positive solutions for \(\lambda \in (0, \lambda_\ast)\). Let \(\{(\lambda_n, u_n)\}\) be a sequence such that
\[u_n = T(\lambda_n, u_n) \text{ for each } n \text{ and } \lambda_n \to \lambda_\ast \text{ as } n \to \infty.\]

By the compactness of \(T\) and Lemma 5, there exists a subsequence, say it again \(\{(\lambda_n, u_n)\}\), satisfying
\[u_n = T(\lambda_n, u_n) \to u_\ast \text{ in } C[0, 1] \text{ as } n \to \infty.\]

Since
\[\lambda_n, u_n \to (\lambda_\ast, u_\ast) \text{ in } [0, \infty) \times K,\]
from the continuity of \(T\), it follows that
\[u_\ast = T(\lambda_\ast, u_\ast).\]

Thus BVP (1) and (2) has at least one positive solution for \(\lambda = \lambda_\ast\).

To complete the proof of Theorem 2, it suffices to show that there are no positive solutions to BVP (1) and (2) for \(\lambda > \lambda_\ast\). Assume on the contrary that there exists \(\lambda_1 \in (\lambda_\ast, \infty)\) such that BVP (1) and (2) has a positive solution \(u_1\) for \(\lambda = \lambda_1\). We will show that there are two positive solutions to BVP (1) and (2) for all \(\lambda \in (0, \lambda_1)\), which contradicts the definition of \(\lambda_\ast\).

Let \(\lambda \in (0, \lambda_1)\) be fixed and set
\[\epsilon = \frac{1}{2} \left( \frac{\lambda_1}{\lambda} - 1 \right) \min_{t \in [0,1]} f(t, u_1(t)) > 0.\]

By the continuity of \(f = f(t, s)\), there exists \(\delta = \delta(\lambda) > 0\) such that if \(x, y \in [0, \|u_1\| + 1]\) and \(|x - y| < 2\delta\), then
\[|f(t, x) - f(t, y)| < \epsilon, \quad t \in [0, 1].\]

We claim that \(\beta(t) = u_1(t) + \delta\) satisfies
\[(w(t)\varphi(\beta'(t)))' + \lambda h(t)f(t, \beta(t)) < 0, \quad t \in (0, 1).\]

Indeed, assume on the contrary that \(\beta\) does not satisfy (18), i.e., there exists \(t_0 \in (0, 1)\) such that
\[(w(t_0)\varphi(\beta'(t_0)))' + \lambda h(t_0)f(t_0, \beta(t_0)) \geq 0.\]

Since \(\beta'(t) = u_1'(t)\) for all \(t \in (0, 1)\),
\[\lambda h(t_0)f(t_0, \beta(t_0)) \geq -(w(t_0)\varphi(\beta'(t_0)))' = -(w(t_0)\varphi(u_1'(t_0)))' = \lambda_1 h(t_0)f(t_0, u_1(t_0)),\]
which implies
\[f(t_0, \beta(t_0)) \geq \frac{\lambda_1}{\lambda} f(t_0, u_1(t_0)).\]
From 
\[ |\beta(t_0) - u_1(t_0)| = \delta < 2\delta, \]
it follows that 
\[ \epsilon + f(t_0, u_1(t_0)) > f(t_0, \beta(t_0)). \]
Consequently, by (19),
\[ \epsilon \geq \left( \frac{\lambda_1}{\lambda} - 1 \right) f(t_0, u_1(t_0)), \]
which contradicts the choice of \( \epsilon \). Thus, \( \beta(t) = u_1(t) + \delta \) satisfies (18).

Consider the following modified problem
\[
\begin{cases}
(w(t)\varphi(u'(t)))' + \lambda h(t)f(t, \gamma(t, u(t))) = 0, & t \in (0, 1), \\
u(0) = \int_0^1 u(r)da_1(r), & u(1) = \int_0^1 u(r)da_2(r),
\end{cases}
\] (20)
where \( \gamma : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is defined by, for \( t \in [0, 1] \),
\[ \gamma(t, s) = \begin{cases} 
\beta(t), & \text{if } s \geq \beta(t), \\
s, & \text{if } 0 < s < \beta(t), \\
0, & \text{if } s \leq 0.
\end{cases} \]

Let \( u \) be a positive solution to problem (20). We show that \( u(t) \leq \beta(t) \) for \( t \in [0, 1] \). If not, there exists \( t_0 \in [0, 1] \) satisfying
\[ x(t_0) = \max\{x(t) : t \in [0, 1]\} > 0, \]
where
\[ x(t) = u(t) - \beta(t) \text{ for } t \in [0, 1]. \]
If \( \hat{\alpha}_1 = 0 \), then \( u(0) = 0 < \delta = \beta(0) \) and \( x(0) < 0 < x(t_0) \). If \( \hat{\alpha}_1 \in (0, 1) \), then
\[
x(0) = u(0) - \beta(0) = u(0) - (u_1(0) + \delta) \\
= \int_0^1 u(r)da_1(r) - \left( \int_0^1 u_1(r)da_1(r) + \delta \right) \\
< \int_0^1 x(r)da_1(r) \leq \hat{\alpha}_1 x(t_0) < x(t_0).
\]
Similarly, \( x(1) < x(t_0) \). Consequently, \( t_0 \in (0, 1) \) and \( x'(t_0) = 0 \), i.e.,
\[ u'(t_0) = \beta'(t_0). \] (21)
For some \( t^* \in (0, t_0) \),
\[ x(t^*) < x(t_0) \] (22)
and \( x(t) > 0 \) for \( t \in [t^*, t_0] \), i.e.,
\[ u(t) > \beta(t), \ t \in [t^*, t_0]. \] (23)

By (18) and (23), for \( t \in [t^*, t_0] \),
\[
-(w(t)\varphi(u'(t)))' = \lambda h(t)f(t, \gamma(t, u(t))) \\
= \lambda h(t)f(t, \beta(t)) \\
< -(w(t)\varphi(\beta'(t)))'.
\]
Integrating this from $t$ to $t_0$, by (21),
\[ u'(t) \leq \beta'(t) \text{ for } t \in [t^*, t_0). \]

Integrating it again from $t^*$ to $t_0$,
\[ u(t_0) - u(t^*) \leq \beta(t_0) - \beta(t^*), \]
which contradicts (22). Thus
\[ u(t) \leq \beta(t) \text{ for } t \in [0, 1], \]
which implies
\[ \gamma(t, u(t)) = u(t) \text{ for all } t \in [0, 1]. \]

Consequently $u$ is a positive solution to BVP (1) and (2).

Since $\hat{\alpha}_1 \in [0, 1]$, it is easy to see that $u(0) < \beta(0)$ and $u(1) < \beta(1)$. Indeed,
\[
\begin{align*}
    u(0) &= \int_0^1 u(r)\,d\alpha_1(r) \\
    &\leq \int_0^1 \beta(r)\,d\alpha_1(r) = \int_0^1 (\beta_1(r) + \delta)\,d\alpha_1(r) = \int_0^1 u_1(r)\,d\alpha_1(r) + \delta \kappa_1 \\
    &< u_1(0) + \delta = \beta(0).
\end{align*}
\]

Similarly, it can be shown that $u(1) < \beta(1)$.

Set
\[ \Omega = \{ u \in C[0,1] : -1 < u(t) < \beta(t), \, t \in [0,1] \}. \]

Then $\Omega$ is a bounded open subset in $C[0,1]$. We claim that $u \in \Omega \cap \mathcal{K}$. Assume on the contrary that there exist $t_1, t_2$ and $\delta_1 > 0$ such that
\[ 0 < t_1 - \delta_1 < t_1 \leq t_2 < t_2 + \delta_1 < 1, \]
\[ u(t) = \beta(t) \text{ for } t \in [t_1, t_2] \]
and
\[ u(t) < \beta(t) \text{ for } t \in [t_1 - \delta_1, t_1] \cup (t_2, t_2 + \delta_1]. \]

Since $\beta$ satisfies (18),
\[
\max\{(w(t)\varphi(\beta'(t)))' + \lambda h(t)f(t, \beta(t)) : t \in [t_1 - \delta_1, t_2 + \delta_1]\} =: -\epsilon_1 < 0. \tag{24}
\]

Set
\[ \epsilon_2 = \frac{\epsilon_1}{\lambda h^*} > 0, \tag{25} \]
where
\[ h^* := \max\{h(t) : t \in [t_1 - \delta_1, t_2 + \delta_1]\}. \]

Then there exists $\delta_2 > 0$ such that if $|x - y| < \delta_2$ and $x, y \in [0, ||\beta||_\infty + 1]$, then
\[ |f(t, x) - f(t, y)| < \epsilon_2, \]
and there exists an interval $[a, b] \subset (t_1 - \delta_1, t_2 + \delta_1)$ such that
\[ (u - \beta)'(a) > 0, \quad (u - \beta)'(b) < 0 \]
and
\[ -\delta_2 < \gamma(t, u(t)) - \beta(t) = u(t) - \beta(t) \leq 0, \, t \in [a, b]. \]
Consequently
\[ w(a)[\phi(u'(a)) - \phi(\beta'(a))] > 0, \quad w(b)[\phi(u'(b)) - \phi(\beta'(b))] < 0 \]
and
\[ f(t, \gamma(t, u(t))) = f(t, u(t)) < f(t, \beta(t)) + \epsilon_2, \quad t \in [a, b]. \]

Then, by (24) and (25),
\[
0 > w(b)[\phi(u'(b)) - \phi(\beta'(b))] - w(a)[\phi(u'(a)) - \phi(\beta'(a))],
\]
\[
= \left[ w(b)\phi(u'(b)) - w(a)\phi(u'(a)) \right] - \left[ w(b)\phi(\beta'(b)) - w(a)\phi(\beta'(a)) \right]
\]
\[
= \int_a^b ((w(t)\phi(u'(t)))' - (w(t)\phi(\beta'(t)))')dt
\]
\[
= \int_a^b (-\lambda h(t)f(t, \gamma(t, u(t))) - (w(t)\phi(\beta'(t)))')dt
\]
\[
> \int_a^b (-\lambda h(t)[f(t, \beta(t)) + \epsilon_2] - (w(t)\phi(\beta'(t)))')dt
\]
\[
= \int_a^b (-\lambda h(t)\epsilon_2 - [(w(t)\phi(\beta'(t)))' + \lambda h(t)f(t, \beta(t))] dt
\]
\[
\geq \int_a^b (-\lambda\epsilon_2 h(t) + \epsilon_1)dt \geq \int_a^b (-\lambda\epsilon_2 h^* + \epsilon_1)dt = 0.
\]

This is a contradiction. Thus \( u \in \Omega \cap \mathcal{K} \).

Since BVP (1) and (2) is equivalent to problem (20) on \( \Omega \cap \mathcal{K} \), by Lemmas 5 and 6 and the same argument in the proof of [51] (Theorem 1.1), one can conclude that BVP (1) and (2) has at least two positive solutions for \( \lambda_n < \lambda < \lambda_1 \). Thus the proof is complete. \( \square \)

**Lemma 7.** Assume that \((A1), (A2), f \in C([0, 1] \times [0, \infty), (0, \infty)), \) either \((F_0)\) and \( h \in \mathcal{H}_\phi \) or \((F'_0)\) and \( h \in \mathcal{H}_{\phi_1} \) hold. Let \( L > 0 \) be given. Then there exists \( M_L > 0 \) such that \( \|u\|_\infty \leq M_L \) for any positive solutions \( u \) to BVP (1) and (2) with \( \lambda \in [0, L] \).

**Proof.** We give the proof for the case that \((F_0)\) and \( h \in \mathcal{H}_\phi \), since the case \((F'_0)\) and \( h \in \mathcal{H}_{\phi_1} \) can be proved in a similar manner.

Set
\[ M := (4L)^{-1}w_0\psi_1(h^{-1}_s) > 0, \]
where
\[ h_s = \max \left\{ A_1 \int_0^\gamma \phi^{-1} \left( \int_s^\gamma h(\tau)d\tau \right) ds, A_2 \int_0^1 \phi^{-1} \left( \int_\gamma^b h(\tau)d\tau \right) ds \right\}. \]

By \((F_0)\), there exists \( s_M > 0 \) such that
\[ f(t, s) \leq M\psi_1(s) \quad \text{for} \quad (t, s) \in [0, 1] \times [s_M, \infty). \] \( (26) \)

Assume to the contrary that there exists a sequence \( \{\lambda_n, u_n\} \) such that \( u_n \) is a positive solution to BVP (1) and (2) with \( \lambda = \lambda_n \in (0, L) \) and \( \|u_n\|_\infty \to \infty \) as \( n \to \infty \). Set
\[ C_M = \max \{f(t, s) : (t, s) \in [0, 1] \times [0, s_M] \} > 0. \]

Then there exists \( N > 0 \) satisfying
\[ \|u_N\|_\infty \geq \psi_1^{-1} \left( \frac{C_M}{M} \right), \]
which implies
\[ C_M \leq M\psi_1(\|u_N\|_\infty). \]
Consequently, by the definition of $C_M$ and (26),

$$f(t, s) \leq C_M + M\psi_1(s) \leq 2M\psi_1(\|u_N\|_{\infty}) \text{ for } (t, s) \in [0,1] \times [0, \|u_N\|_{\infty}].$$

(27)

Let $\sigma_N$ be a unique point satisfying $\|u_N\|_{\infty} = u_N(\sigma_N)$. Assume that $\sigma_N \leq \gamma$, since the case $\sigma_N > \gamma$ can be dealt in a similar manner. Then, by (8), (10) and (27),

$$\|u_N\|_{\infty} = u_N(\sigma_N) = T(\lambda_N, u_N)(\sigma_N) = \nu_1^{\lambda_N, u_N}(\sigma_N)
\leq A_1 \int_0^{\sigma_N} I_{\lambda_N, u_N}(s, \sigma_N) ds
\leq A_1 \int_0^{\sigma_N} \varphi^{-1} \left( \frac{\lambda_N}{w(s)} \int_s^{\sigma_N} h(\tau)f(\tau, u_N(\tau))d\tau \right) ds
\leq A_1 \int_0^{\gamma} \varphi^{-1} \left( \frac{2LM}{\bar{w}_0} \int_s^{\gamma} h(\tau)d\tau \varphi_1(\|u_N\|_{\infty}) \right) ds
\leq A_1 \int_0^{\gamma} \varphi^{-1} \left( \int_s^{\gamma} h(\tau)d\tau \varphi_1^{-1} \left( \frac{2LM}{\bar{w}_0} \right) \right) ds
\leq h_0 \varphi_1^{-1} \left( \frac{2LM}{\bar{w}_0} \right) \|u_N\|_{\infty} < \|u_N\|_{\infty}.
$$

Here the choice of $M$ is used in the last inequality. This contradiction completes the proof. \hfill \Box

**Remark 2.** The assumptions $(F_0)$ and $h \in \mathcal{H}_\phi$ are different from the ones $(F'_0)$ and $h \in \mathcal{H}_{\psi_1}$ in Theorem 3. Indeed, let

$$\varphi(s) = s + s^2 \text{ and } \psi_1(s) = \min\{s, s^2\} \text{ for } s \in [0, \infty).$$

Then the first inequality in (3) is satisfied. Clearly, $(F_0)$ implies $(F'_0)$, since

$$\varphi(1) \psi_1(s) \leq \varphi(s) \text{ for all } s \in [0, \infty).$$

Let $f(t, s) = 1 + s^2$ for $(t, s) \in [0,1] \times [0, \infty)$. Then

$$\lim_{t \to \infty} \frac{1 + s^3}{\varphi(s)} = 0, \text{ but } \lim_{s \to \infty} \frac{1 + s^3}{\psi_1(s)} = \infty.$$

Consequently, $(F'_0)$ does not imply $(F_0)$. Since $\mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\phi$, we give an example of $h \in \mathcal{H}_\phi \setminus \mathcal{H}_{\psi_1}$. Let

$$h(t) = t^{-2} \text{ for } t \in (0, 1].$$

From

$$\varphi^{-1}(s) = \frac{-1 + \sqrt{1 + 4s}}{2} \text{ and } \psi^{-1}_1(s) = \max\{\sqrt{s}, s\} \text{ for } s \in [0, \infty),$$

it follows that

$$\varphi^{-1} \left( \int_s^{\gamma} t^{-2}d\tau \right) = \varphi^{-1} \left( s^{-1} - 2 \right) = \frac{-1 + \sqrt{1 + 4(s^{-1} - 2)}}{2} \in L^1 \left( 0, \frac{1}{2} \right)$$

and

$$\psi^{-1}_1 \left( \int_s^{\gamma} t^{-2}d\tau \right) = \psi^{-1}_1 \left( s^{-1} - 2 \right) = s^{-1} - 2 \notin L^1 \left( 0, \frac{1}{3} \right).$$

Consequently
since \( h \in C(0, 1) \).

**Theorem 3.** Assume that \( (A1), (A2), f \in C([0, 1] \times [0, \infty), (0, \infty)), \) either \( \hat{F}_0 \) and \( h \in H_\varphi \) or \( \hat{F}_0^* \) and \( h \in H_{\varphi_1} \) hold. Then for any \( \lambda \in (0, \infty) \), there exists a positive solution \( u_\lambda \) to BVP (1) and (2) such that
\[
\| u_\lambda \|_\infty \to 0 \text{ as } \lambda \to 0^+ \text{ and } \| u_\lambda \|_\infty \to \infty \text{ as } \lambda \to \infty.
\]

**Proof.** Set
\[
\lambda^* = \inf \{ \lambda \in [0, \infty) : (\lambda, u_\lambda) \in C \}.
\]
Here \( C \) is the unbounded solution component in Proposition 1. Then, by Lemma 7, \( \lambda^* = \infty \). Indeed, assume on the contrary that \( \lambda^* < \infty \). Then, by Lemma 7, all solutions \( u_\lambda \) to problem (1) satisfying \( (\lambda, u_\lambda) \in C \) are bounded in \( C[0, 1] \). This contradicts the fact that the solution component \( C \) is unbounded in \([0, \infty) \times K \). Thus, \( \lambda^* = \infty \), and for any \( \lambda \in (0, \infty) \), there exists a positive solution \( u_\lambda \) to BVP (1) and (2) satisfying
\[
(\lambda, u_\lambda) \in C \text{ and } \| u_\lambda \|_\infty \to 0 \text{ as } \lambda \to 0^+.
\]

Next we show that
\[
\| u_\lambda \|_\infty \to \infty \text{ as } \lambda \to \infty.
\]
Assume to the contrary that there exists a sequence \( \{ (\lambda_n, u_n) \} \) in \( C \) such that
\[
\lambda_n \to \infty \text{ as } n \to \infty,
\]
but there exists \( m > 0 \) satisfying
\[
\| u_n \|_\infty \leq m \text{ for all } n.
\]
Since \( f \in C([0, 1] \times [0, \infty), (0, \infty)) \), there exists \( \delta_m > 0 \) satisfying
\[
f(t, u_n(t)) \geq \delta_m \text{ for all } t \in [0, 1] \text{ and all } n.
\]
For each \( n \), let \( \sigma_n \) be the unique point satisfying \( u_n(\sigma_n) = \| u_n \|_\infty \). Suppose that \( \sigma_n \geq \gamma \) (the case \( \sigma_n < \gamma \) is similar). Then, by (8),
\[
\| u_n \|_\infty \geq u_n(\gamma) = u_n(0) + \int_0^\gamma \varphi^{-1} \left( \frac{1}{w(s)} \int_\delta^{\sigma_n} \lambda_n b(\tau) f(\tau, u_n(\tau)) d\tau \right) d\sigma
\]
\[
\geq \int_0^\gamma \varphi^{-1} \left( \frac{1}{w(s)} \int_0^\gamma h(\tau) d\tau \right) d\sigma \lambda_n \delta_m
\]
\[
\geq \int_0^\gamma \varphi^{-1} \left( \frac{1}{w(s)} \int_0^\gamma h(\tau) d\tau \right) ds \psi_2^{-1}(\lambda_n \delta_m) \to \infty \text{ as } n \to \infty,
\]
which contradicts the fact that \( \| u_n \|_\infty \leq m \) for all \( n \). Thus, the proof is complete. \( \square \)

**Remark 3.** Assume that \( f \in C([0, 1] \times \mathbb{R}_+, (0, \infty)) \) and \( \hat{\alpha}_i \in (0, 1) \) for \( i = 1, 2 \). Then, for any positive solutions \( u \) to BVP (1) and (2),
\[
u(t) \geq \rho \| u \|_\infty \text{ for all } t \in [0, 1]. \quad (28)
\]

Here
\[
\rho := \rho_1 \min \left\{ \int_0^1 \min\{r, 1 - r\} d\alpha_i(r) : i = 1, 2 \right\} \in (0, 1)
\]
and
\[
\rho_1 = \psi_2^{-1} \left( \frac{1}{\| w \|_\infty} \right) \left[ \psi_1^{-1} \left( \frac{1}{w_0} \right) \right]^{-1} \in (0, 1].
\]
In fact, by (2) and (15), for $t \in [0,1]$,
\[
    u(t) \geq \min\{u(0), u(1)\} \\
    = \min\left\{ \int_0^1 u(r)d\alpha_i(r) : i = 1, 2 \right\} \\
    = \min\left\{ \int_0^1 T(\lambda, u)(r)d\alpha_i(r) : i = 1, 2 \right\} \\
    \geq \min\left\{ \int_0^1 \rho_1 \min\{r, 1-r\}\|T(\lambda, u)\|_{\infty}d\alpha_i(r) : i = 1, 2 \right\} \\
    = \rho_1 \min\left\{ \int_0^1 \min\{r, 1-r\}d\alpha_i(r) : i = 1, 2 \right\} \|u\|_{\infty} = \tilde{\rho}\|u\|_{\infty}.
\]

**Theorem 4.** Assume that $(A1), (A2), \tilde{\alpha}_i \in (0,1)$ for $i = 1, 2$ and $f \in C([0,1] \times [0,\infty), \mathbb{R})$ satisfies $f(t,s) > 0$ for all $(t,s) \in [0,1] \times [M, \infty)$ and for some $M > 0$.

1. Assume that $(F_0)$ and $h \in \mathcal{H}_0^\varphi$ hold. Then there exists $\lambda_0 > 0$ such that BVP (1) and (2) has at least one positive solution $u_\lambda$ for any $\lambda \in (0,\lambda_0)$ satisfying
   \[
   \|u_\lambda\|_{\infty} \to \infty \text{ as } \lambda \to 0^+.
   \]

2. Assume that either $(F_0)$ and $h \in \mathcal{H}_0^\varphi$ or $(F_0')$ and $h \in \mathcal{H}_1^\varphi$ hold. Then there exists $\lambda_0 > 0$ such that BVP (1) and (2) has at least one positive solution $u_\lambda$ for any $\lambda \in (\lambda_0, \infty)$ satisfying
   \[
   \|u_\lambda\|_{\infty} \to \infty \text{ as } \lambda \to \infty.
   \]

**Proof.** We only give the proof of (2) with the case $(F_0)$ and $h \in \mathcal{H}_0^\varphi$, since other cases can be proved in a similar manner.

Consider the following modified problem
\[
\begin{align*}
(w(t)\varphi(u'(t)))' + \lambda h(t)f_1(t, u(t)) &= 0, \quad t \in (0,1), \\
\tag{29}
u(0) &= \int_0^1 u(r)d\alpha_1(r), \quad u(1) = \int_0^1 u(r)d\alpha_2(r),
\end{align*}
\]

where
\[
f_1(t,s) = \begin{cases} f(t,M), & \text{ for } (t,s) \in [0,1] \times [0,M), \\
f(t,s), & \text{ for } (t,s) \in [0,1] \times [M,\infty). \end{cases}
\]

Then, by $(F_0)$, $f_1 \in C([0,1] \times [0,\infty), (0,\infty))$ satisfies
\[
\lim_{s \to \infty} \min_{t \in [0,1]} \frac{f_1(t,s)}{\varphi_1(s)} = 0.
\]

By Theorem 3, problem (29) has at least one positive solution $u_\lambda$ for any $\lambda \in (\lambda_0, \infty)$ satisfying
\[
\|u_\lambda\|_{\infty} \to \infty \text{ as } \lambda \to \infty.\]

Since
\[
\|u_\lambda\|_{\infty} \to \infty \text{ as } \lambda \to \infty,
\]

there exists $\lambda_0 > 0$ such that positive solutions $u_\lambda$ satisfy
\[
\|u_\lambda\|_{\infty} \geq \tilde{\rho}^{-1}M \text{ for any } \lambda \in (\lambda_0, \infty).
\]

By Remark 3, for $\lambda \in (\lambda_0, \infty)$,
\[
u_\lambda(t) \geq M \text{ for } t \in [0,1].
\]

Consequently
\[
f_1(t, u_\lambda(t)) = f(t, u_\lambda(t)) \text{ for } t \in [0,1].
\]
and \( u_\lambda \) becomes the positive solution to BVP (1) and (2) for \( \lambda \in (\lambda_0, \infty) \). Thus the proof is complete. \( \square \)

Finally, we give some examples to illustrate the main results (Theorem 2, Theorems 3 and 4) obtained in this section.

**Example 1.** Consider the following problem

\[
\begin{cases}
(t + 1)^{-1} \varphi(u'(t))' + \lambda h(t) f(t, u(t)) = 0, \quad t \in (0, 1), \\
u(0) = \int_0^1 u(r) d\alpha_1(r), \quad u(1) = \int_0^1 u(r) d\alpha_2(r),
\end{cases}
\]  

(30)

where \( \varphi \) is defined by

\[ \varphi(s) = s + |s|s \quad \text{for} \quad s \in (-\infty, \infty) \]

and

\[ \alpha_1(r) = \frac{1}{2} r^2 \quad \text{and} \quad \alpha_2(r) = \frac{1}{3} r^3 \quad \text{for} \quad r \in [0, 1]. \]

Then it is easy to see that (A1) is satisfied with

\[ \psi_1(y) = \min\{y, y^2\} \quad \text{and} \quad \psi_2(y) = \max\{y, y^2\} \quad \text{for} \quad y \in [0, \infty) \]

and (A2) holds with

\[ \hat{\alpha}_1 = \frac{1}{2} \quad \text{and} \quad \hat{\alpha}_2 = \frac{1}{3} \]

Note that

\[ \psi_1^{-1}(s) = \max\{\sqrt{s}, s\} \quad \text{and} \quad \psi_2^{-1}(s) = \min\{\sqrt{s}, s\} \quad \text{for} \quad s \in [0, \infty). \]

From

\[ w_0 = \frac{1}{2} \quad \text{and} \quad \|w\|_{\infty} = 1, \]

it follows that

\[ \psi_2^{-1}(\|w\|_{\infty}) = \psi_2^{-1}(1) = 1 \quad \text{and} \quad \psi_1^{-1}(w_0) = \psi_1^{-1}(2) = 2. \]

Consequently

\[ \rho_w = \frac{1}{2} \min\{\alpha, 1 - \beta\} \in (0, 1) \quad \text{for any} \quad \alpha, \beta \in (0, 1). \]

(1) Let

\[ h(t) = t^{-2} \quad \text{for} \quad t \in (0, 1]. \]

Then \( h \in \mathcal{H}_\varphi \setminus \mathcal{H}_{\psi_1} \) (see Remark 2).

(i) Let \( f \) be any positive continuous function satisfying

\[ f(t, s) = \left( \frac{7}{8} + \sin t + t \right) (1 + s^3) \quad \text{for} \quad (t, s) \in \left[ \frac{1}{4}, \frac{3}{4} \right] \times [0, \infty). \]

Then (F\(_\infty\)) is satisfied with

\[ \alpha = \frac{1}{4} \quad \text{and} \quad \beta = \frac{3}{4}. \]

By Theorem 2, there exists \( \lambda_* > 0 \) such that problem (30) has at least two positive solutions \( u_1^{\lambda_*}, u_2^{\lambda_*} \) for any \( \lambda \in (0, \lambda_*) \), at least one positive solution for \( \lambda = \lambda_* \) and no positive solutions for \( \lambda > \lambda_* \). Moreover, two positive solutions \( u_1^{\lambda} \) and \( u_2^{\lambda} \) for \( \lambda \in (0, \lambda_*) \) satisfy
\[ \|u_1^\lambda\|_\infty \to 0 \text{ and } \|u_2^\lambda\|_\infty \to \infty \text{ as } \lambda \to 0^+. \]

(ii) Let \( f : (0, \infty) \to \mathbb{R} \) be defined by
\[
f(t, s) = \begin{cases} \frac{3s^2 + t}{s^2}, & \text{for } (t, s) \in [0, 1] \times (0, 2], \\ \frac{4t}{t^2} + (s - 2)^3, & \text{for } (t, s) \in [0, 1] \times (2, \infty). \end{cases}
\]

Then \( (F_\infty) \) is satisfied for any \( \alpha, \beta \) satisfying
\[ 0 < \alpha < \beta < 1 \text{ and } f(t, s) > 0 \text{ for all } (t, s) \in [0, 1] \times [2, \infty). \]

By Theorem 4 (1), there exists \( \lambda_\infty > 0 \) such that problem (30) has at least one positive solution \( u_\lambda \) for all \( \lambda \in (\lambda_\infty, \infty) \) satisfying
\[ \|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to 0^+. \]

(2) Let
\[ h(t) = t^{3/2} \text{ for } t \in (0, 1], \]
Then \( h \in \mathcal{H}_{\psi_1} \), since \( \psi_1^{-1}(s) = s \) for \( s \geq 1 \).

(i) Let \( f \) be defined by
\[ f(t, s) = \left(\cos t + s^3\right) \text{ for } (t, s) \in [0, 1] \times (0, \infty). \]

Then \( (F'_0) \) are satisfied. By Theorem 3, problem (30) has at least one positive solution \( u_\lambda \) for any \( \lambda \in (0, \infty) \) satisfying
\[ \|u_\lambda\|_\infty \to 0 \text{ as } \lambda \to 0^+ \text{ and } \|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty. \]

(ii) Let \( f \) be defined by
\[ f(t, s) = \frac{3s^2 - 2 + t}{2s} \text{ for } (t, s) \in [0, 1] \times (0, \infty). \]

Then \( (F'_0) \) is satisfied and \( f(t, s) > 0 \) for all \( (t, s) \in [0, 1] \times [1, \infty) \). By Theorem 4 (2), there exists \( \lambda_0 > 0 \) such that problem (30) has at least one positive solution \( u_\lambda \) for any \( \lambda \in (\lambda_0, \infty) \) satisfying
\[ \|u_\lambda\|_\infty \to \infty \text{ as } \lambda \to \infty. \]

4. Conclusions

In this work, the existence, nonexistence and/or multiplicity of positive solutions to BVP (1) and (2) were studied. If the nonlinearity \( f = f(t, u) \in C([0, 1] \times [0, \infty), (0, \infty)) \) is superlinear at \( u = \infty \), it is not hard to show the result that, for some \( \lambda_1^1, \lambda_2^2 > 0 \), BVP (1) and (2) has at least two positive solutions \( u_1^\lambda \) and \( u_2^\lambda \) for \( \lambda \in (0, \lambda_1^1) \), at least one positive solution for \( \lambda \in [\lambda_1^1, \lambda_2^2] \) and no positive solutions for \( \lambda > \lambda_2^2 \). This result is partial since there is no information on the multiplicity of positive solutions for \( \lambda \in [\lambda_1^1, \lambda_2^2] \). By the lack of solution regularity and the boundary conditions (2), it is not obvious to show \( \lambda_1^1 = \lambda_2^2 \). In Theorem 2, when the nonlinearity \( f = f(t, u) \in C([0, 1] \times [0, \infty), (0, \infty)) \) is superlinear at \( u = \infty \), the global result for positive solutions to BVP (1) and (2) with respect to the parameter \( \lambda \) (i.e., \( \lambda_1^1 = \lambda_2^2 \)) was shown. In Theorem 3, when the nonlinearity \( f = f(t, u) \in C([0, 1] \times [0, \infty), (0, \infty)) \) is sublinear at \( u = \infty \), the existence of one positive solution for all \( \lambda > 0 \) was shown. Theorems 2 and 3 extend the results in [7] for problem (1) with Dirichlet boundary conditions (\( \hat{a}_1 = \hat{a}_2 = 0 \)) to the ones for problem (1) with Riemann-Stieltjes integral boundary conditions in some ways. In Theorem 4, when \( \hat{a}_1 \hat{a}_2 \neq 0 \) and the sign-changing nonlinearity \( f = f(t, u) \in C([0, 1] \times (0, \infty), \mathbb{R}) \) may be singular at \( u = 0 \), the existence of one positive solution was shown for all small \( \lambda > 0 \) when \( f = f(t, u) \) is
superlinear at $u = \infty$, and the existence of one positive solution was shown for all large $\lambda > 0$ when $f = f(t,u)$ is sublinear at $u = \infty$.

Funding: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2017R1D1A1B03035623).

Acknowledgments: The author would like to thank the anonymous reviewers for their useful comments.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Wang, H. On the structure of positive radial solutions for quasilinear equations in annular domains. Adv. Differ. Equ. 2003, 8, 111–128.
2. Karakostas, G.L. Positive solutions for the $\Phi$-Laplacian when $\Phi$ is a sup-multiplicative-like function. Electron. J. Differ. Equ. 2004, 68, 1–12.
3. Karakostas, G.L. Triple positive solutions for the $\Phi$-Laplacian when $\Phi$ is a sup-multiplicative-like function. Electron. J. Differ. Equ. 2004, 69, 1–13.
4. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular $\varphi$-Laplacian Nonlocal Boundary Value Problems when $\varphi$ is a Sup-multiplicative-like Function. Mathematics 2020, 8, 420. [CrossRef]
5. Xu, X.; Lee, Y.H. Some existence results of positive solutions for $\varphi$-Laplacian systems. Abstr. Appl. Anal. 2014, 2014, 814312. [CrossRef]
6. Xu, X.; Lee, Y.H. On singularly weighted generalized Laplacian systems and their applications. Adv. Nonlinear Anal. 2018, 7, 149–165. [CrossRef]
7. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular Boundary Value Problems Involving $\varphi$-Laplacian. Mathematics 2019, 7, 654. [CrossRef]
8. Baxley, J.V. A singular nonlinear boundary value problem: Membrane response of a spherical cap. SIAM J. Appl. Math. 1988, 48, 497–505. [CrossRef]
9. Cannon, J.R. The solution of the heat equation subject to the specification of energy. Quart. Appl. Math. 1963, 21, 155–160. [CrossRef]
10. Chegis, R.Y. Numerical solution of a heat conduction problem with an integral condition. Liet. Mat. Rink. 1984, 24, 209–215.
11. Guidotti, P.; Merino, S. Gradual loss of positivity and hidden invariant cones in a scalar heat equation. Differ. Integral Equ. 2000, 13, 1551–1568.
12. Infante, G.; Pietramala, P.; Tenuta, M. Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory. Commun. Nonlinear Sci. Numer. Simul. 2014, 19, 2245–2251. [CrossRef]
13. Infante, G.; Webb, J.R.L. Nonlinear non-local boundary-value problems and perturbed Hammerstein integral equations. Proc. Edinb. Math. Soc. 2006, 49, 637–656. [CrossRef]
14. O’Regan, D. Upper and lower solutions for singular problems arising in the theory of membrane response of a spherical cap. Nonlinear Anal. 2001, 47, 1163–1174. [CrossRef]
15. Agarwal, R.P.; Lü, H.; O’Regan, D. Eigenvalues and the one-dimensional $p$-Laplacian. J. Math. Anal. Appl. 2002, 266, 383–400. [CrossRef]
16. Webb, J.R.L.; Infante, G. Positive solutions of nonlocal boundary value problems involving integral conditions. NoDEA Nonlinear Differ. Equ. Appl. 2008, 15, 45–67. [CrossRef]
17. Kim, C.G. Existence of positive solutions for multi-point boundary value problem with strong singularity. Acta Appl. Math. 2010, 112, 79–90. [CrossRef]
18. Feng, M.; Zhang, X.; Ge, W. Exact number of pseudo-symmetric positive solutions for a $p$-Laplacian three-point boundary value problems and their applications. J. Appl. Math. Comput. 2010, 33, 437–448. [CrossRef]
19. Son, B.; Wang, P. Analysis of positive radial solutions for singular superlinear $p$-Laplacian systems on the exterior of a ball. Nonlinear Anal. 2020, 192, 111657. [CrossRef]
20. Bachouche, K.; Djebali, S.; Moussaoui, T. $\varphi$-Laplacian BVPS with linear bounded operator conditions. Arch. Math. (Brno) 2012, 48, 121–137. [CrossRef]
21. Kaufmann, U.; Milne, L. Positive solutions for nonlinear problems involving the one-dimensional $\varphi$-Laplacian. J. Math. Anal. Appl. 2018, 461, 24–37. [CrossRef]
22. Ding, Y. Positive solutions for integral boundary value problem with \( \phi \)-Laplacian operator. *Bound. Value Probl.* **2011**, *2011*, 827510. [CrossRef]

23. Ko, E.; Lee, E.K. Existence of multiple positive solutions to integral boundary value systems with boundary multiparameters. *Bound. Value Probl.* **2018**, *2018*, 155, 1–16. [CrossRef]

24. Ma, S.; Zhang, X. Positive solutions to second-order singular nonlocal problems: existence and sharp conditions. *Bound. Value Probl.* **2019**, *2019*, 173, 1–18. [CrossRef]

25. Dogan, A. Positive solutions of nonlinear multi-point boundary value problems. *Positivity* **2018**, *2018*, 22, 1387–1402. [CrossRef]

26. Goodrich, C.S. On nonlinear boundary conditions involving decomposable linear functionals. *Proc. Edinb. Math. Soc.* **2015**, *58*, 421–439. [CrossRef]

27. Goodrich, C.S. On semipositone non-local boundary-value problems with nonlinear or affine boundary conditions. *Proc. Edinb. Math. Soc.* **2017**, *60*, 635–649. [CrossRef]

28. Jeong, J.; Kim, C.G.; Lee, E.K. Multiplicity of positive solutions to a singular \( (p_1, p_2) \)-Laplacian system with coupled integral boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *32*, 1–23. [CrossRef]

29. Jiang, J.; Liu, L.; Wu, Y. Existence of symmetric positive solutions for a singular system with coupled integral boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2018**, *94*, 1–19. [CrossRef]

30. Karakostas, G.L.; Palaska, K.G.; Tsamatos, P.C. Positive solutions for a second-order \( \Phi \)-Laplacian equations with limiting nonlocal boundary conditions. *Electron. J. Differ. Equ.* **2016**, *251*, 1–17.

31. Karakostas, G.L.; Tsamatos, P.C. Existence of multiple positive solutions for a nonlocal boundary value problem. *Topol. Methods Nonlinear Anal.* **2002**, *19*, 109–121. [CrossRef]

32. Shivaji, R.; Sim, I.; Son, B. A uniqueness result for a semipositone \( p \)-Laplacian problem on the exterior of a ball. *J. Math. Anal. Appl.* **2017**, *445*, 459–475. [CrossRef]

33. Kim, C.G.; Lee, E.K. Multiple positive solutions for singular multi-point boundary-value problems with a positive parameter. *Electron. J. Differ. Equ.* **2014**, *38*, 1–13.

34. Webb, J.R.L. Existence of positive solutions for a thermostat model. *Nonlinear Anal. Real World Appl.* **2012**, *13*, 923–938. [CrossRef]

35. Webb, J.R.L.; Infante, G. Positive solutions of nonlocal boundary value problems: A unified approach. *J. Lond. Math. Soc.* **2006**, *74*, 673–693. [CrossRef]

36. Webb, J.R.L. Positive solutions of nonlinear differential equations with Riemann-Stieltjes boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *86*, 1–13. [CrossRef]

37. Sim, I. On the existence of nodal solutions for singular one-dimensional \( \phi \)-Laplacian problem with asymptotic condition. *Commun. Pure Appl. Anal.* **2008**, *7*, 905–923. [CrossRef]

38. García-Huidobro, M.; Manásevich, R.; Ward, J.R. Positive solutions for equations and systems with \( p \)-Laplace-like operators. *Adv. Differ. Equ.* **2009**, *14*, 401–432.

39. Kim, C.G. Existence, nonexistence and multiplicity of positive solutions for singular boundary value problems involving \( \phi \)-Laplacian. *Mathematics* **2019**, *7*, 953. [CrossRef]

40. An, Y.; Kim, C.G.; Shi, J. Exact multiplicity of positive solutions for a \( p \)-Laplacian equation with positive convex nonlinearity. *J. Differ. Equ.* **2016**, *260*, 2091–2118. [CrossRef]

41. Son, B.; Wang, P. Positive radial solutions to classes of nonlinear elliptic systems on the exterior of a ball. *J. Math. Anal. Appl.* **2020**, *488*, 124069. [CrossRef]

42. Jeong, J.; Kim, C.G.; Lee, E.K. Existence and Nonexistence of Solutions to \( p \)-Laplacian Problems on Unbounded Domains. *Mathematics* **2019**, *7*, 438. [CrossRef]

43. Cabana, A.; Wanasssi, O. Existence Results for Nonlinear Fractional Problems with Non-Homogeneous Integral Boundary Conditions. *Mathematics* **2020**, *8*, 255. [CrossRef]

44. Agarwal, R.P.; Luca, R. Positive solutions for a semipositone singular Riemann-Liouville fractional differential problem. *Int. J. Nonlinear Sci. Numer. Simul.* **2019**, *20*, 823–831. [CrossRef]

45. Yan, B.; O’Regan, D.; Agarwal, R.P. Existence of solutions for Kirchhoff-type problems via the method of lower and upper solutions. *Electron. J. Differ. Equ.* **2019**, *54*, 1–19.

46. Benciobra, M.; Rezoug, N.; Samet, B.; Zhou, Y. Second Order Semilinear Volterra-Type Integro-Differential Equations with Non-Instantaneous Impulses. *Mathematics* **2019**, *7*, 1134. [CrossRef]

47. Chu, K.D.; Hai, D.D.; Shivaji, R. Uniqueness of positive radial solutions for infinite semipositone \( p \)-Laplacian problems in exterior domains. *J. Math. Anal. Appl.* **2019**, *472*, 510–525. [CrossRef]
48. Kim, C.G.; Lee, Y.H. Existence of multiple positive solutions for $p$-Laplacian problems with a general indefinite weight. *Commun. Contemp. Math.* 2008, 10, 337–362. [CrossRef]

49. Leray, J.; Schauder, J. Topologie et équations fonctionnelles. *Ann. Sci. École Norm. Supérieure* 1934, 51, 45–78. [CrossRef]

50. Zeidler, E. *Nonlinear Functional Analysis and Its Applications. I;* Springer: New York, NY, USA, 1986. [CrossRef]

51. Kim, C.G. Existence of positive solutions for singular boundary value problems involving the one-dimensional $p$-Laplacian. *Nonlinear Anal.* 2009, 70, 4259–4267. [CrossRef]

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