Proximal algorithms with Bregman distances for bilevel equilibrium problems with application to the problem of “how routines form and change” in Economics and Management Sciences

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Abstract

In this paper we present the bilevel equilibrium problem under conditions of pseudomonotonicity. Using Bregman distances on Hadamard manifolds we propose a framework for to analyse the convergence of a proximal point algorithm to solve this bilevel equilibrium problem. As an application, we consider the problem of “how routines form and change” which is crucial for the dynamics of organizations in Economics and Management Sciences.

Keywords: Proximal algorithms; equilibrium problem; Hadamard manifold; variational rationality.

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1 Introduction

The bilevel equilibrium problem (BEP) has been widely studied and is a very active field of research. One of the motivations is that it covers optimization problems and mathematical programs with equilibrium constraints. These problems were addressed by Luo et al. in [16] and Migdalas et al. in [18]. Bilevel problems have first been formalized as optimization problems in the early 1970s by Bracken and McGill in [3].

In the linear setting, several authors have presented iterative processes to approximate a solution of bilevel problems. Cabot in [4] built an algorithm which is able to minimize hierarchically several functions over their successive argmin sets. Dinh and Muu in [10] presented an algorithm for solving bilevel variational inequality problem when the lower problem is pseudomonotone with respect to its solution set. Moudafi in [19] presented proximal methods for a class of bilevel monotone equilibrium problems. More recently, Ding in [9] used the auxiliary problem principle to BEPs. In both papers, the bifunctions are required to be monotone. In this paper, under the hypothesis of pseudomonotonicity, we present a proximal algorithm for BEP on Hadamard manifolds with Bregman distances. We point out that our algorithm retrieves and generalizes the proximal point method for bilevel equilibrium problems presented in [19], to the case where the bifunctions of BEP are not necessarily monotone, as well as for equilibrium problems on Hadamard manifolds presented by Bento et al. in [2].

Motivated by game theory, the BEP was studied by various authors; see Fudenberg and Tirole [13]. As an application, we consider the problem of “how routines form and change” which is crucial for the dynamics of organizations in Economics and Management Sciences. There is a large litterature about it, starting with the work of Schumpeter [26] [27] in Economics, and Nelson and Winter [20] in Management Sciences, within an evolutionary perspective inspired by the theory of evolution in biology. At the organizational level, they said “…organizations and individual employees increasingly are pursuing change in how work is
organized, how it is managed and in who is carrying it out. At the same time, there are numerous individual, organizational, and societal forces promoting stability in work and employment relations”. In this article, the authors examine “change and stability and the forces pushing individuals and organizations to pursue both”. . . and “some level of tension between stability and change is an inevitable part of organizational life . . . ”. To hope to solve this very important problem for the survival and dynamic efficiency of organizations, the most important step is to embedd this problem in a larger one. That is, routine formation and break is an example, and a leading one, of a lot of stay/stability and change dynamics where some things change and other things stay, in parallel or in sequence, at the individual, organizational or interacting agents levels. We use the recent “Variational rationality” approach presented by Soubeyran in [28, 29] as a required enlarged framework which modelizes and unifies a lot of stability and change dynamics. Then, we show how habits and routines can be seen as the ends of worthwhile successions of variational traps.

The organization of our paper is as follows. In Section 2, we give some elementary facts on Riemannian manifolds, convexity and Bregman distances and functions needed for reading this paper. In Section 3, we present a sufficient condition for existence of a solution for the bilevel equilibrium problem on Hadamard manifolds under similar conditions required in the linear case, and using the proximal point algorithm to solve bilevel equilibrium problems on Hadamard manifold, we derived a convergence analysis. In Section 4, we consider the problem of “how routines form and change” which is crucial for the dynamics of organizations in Economics and Management Sciences. In Section 5, we present a conclusion.
2 Preliminary

2.1 Riemannian Geometry

In this section, we recall some fundamental and basic concepts needed for reading this paper. These results and concepts can be found in the books on Riemannian geometry; see do Carmo [11] and Sakay [25].

Let $M$ be a $n$-dimensional connected manifold. We denote by $T_x M$ the $n$-dimensional tangent space of $M$ at $x$, by $T M = \cup_{x \in M} T_x M$ the tangent bundle of $M$ and by $\mathcal{X}(M)$ the space of smooth vector fields over $M$. When $M$ is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$, with the corresponding norm denoted by $\| \cdot \|$, then $M$ is a Riemannian manifold. Recall that the metric can be used to define the length of piecewise smooth curves $\gamma : [a, b] \to M$ joining $x$ to $y$, i.e., $\gamma(a) = x$ and $\gamma(b) = y$, by $l(\gamma) := \int_a^b \| \gamma'(t) \| \, dt$, and moreover, by minimizing this length functional over the set of all such curves, we obtain a Riemannian distance $d(x, y)$ inducing the original topology on $M$. We denote by $B(x, \epsilon)$ the Riemannian ball on $M$ with center $x$ and radius $\epsilon > 0$. The metric induces a map $f \mapsto \text{grad} f \in \mathcal{X}(M)$ which, for each smooth function over $M$, associates its gradient via the rule $\langle \text{grad} f, X \rangle = df(X)$, $X \in \mathcal{X}(M)$. Let $\nabla$ be the Levi-Civita connection associated with $(M, \langle \cdot, \cdot \rangle)$. The parallel transport along $\gamma$ from $\gamma(0) = y$ to $\gamma(1) = x$, denoted by $P_{\gamma_y,x}$, is an application $P_{\gamma_y,x} : T_y M \to T_x M$ defined by $P_{\gamma_y,x}(v) = V(1)$, where $V$ is the unique vector field along $\gamma$ such that $DV/dt = 0$ and $V(0) = v$. Since that $\nabla$ is a Riemannian connection, $P_{\gamma_y,x}$ is a linear isometry, furthermore $P_{\gamma_y,x}^{-1} = P_{\gamma_x,y}$ and $P_{\gamma_t,x} = P_{\gamma_y,x} \circ P_{\gamma_t,y}$, for all $t \in [0,1]$. A vector field $V$ along $\gamma$ is said to be parallel iff $\nabla_{\gamma'} V = 0$. If $\gamma'$ itself is parallel we say that $\gamma$ is a geodesic. Given that the geodesic equation $\nabla_{\gamma'} \gamma' = 0$ is a second-order nonlinear ordinary differential equation, we conclude that the geodesic $\gamma = \gamma_v(., x)$ is determined by its position $x$ and velocity $v$ at $x$. It is easy to verify that $\| \gamma' \|$ is constant. We say that $\gamma$ is normalized iff $\| \gamma' \| = 1$. The restriction of a geodesic to a closed bounded interval is called a geodesic segment. Given points $x, y \in M$, we denote the geodesic segment from $x$ to $y$ by $[x, y]$. 

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We usually do not distinguish between a geodesic and its geodesic segment, as no confusion can arise. A geodesic segment joining \( x \) to \( y \) in \( M \) is said to be \textit{minimal} iff its length equals \( d(x, y) \) and the geodesic in question is said to be a \textit{minimizing geodesic}.

A Riemannian manifold is \textit{complete} iff the geodesics are defined for any values of \( t \). The Hopf-Rinow’s Theorem ([11, Theorem 2.8, page 146] or [25, Theorem 1.1, page 84]) asserts that, if this is the case, then any pair of points in \( M \) can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, \((M, d)\) is a complete metric space and, bounded and closed subsets are compact. From the completeness of the Riemannian manifold \( M \), the \textit{exponential map} \( \exp_x : T_x M \to M \) is defined by \( \exp_x v = \gamma_v(1, x) \), for each \( x \in M \). A complete simply-connected Riemannian manifold of nonpositive sectional curvature is called an Hadamard manifold. It is known that if \( M \) is a Hadamard manifold, then \( M \) has the same topology and differential structure as the Euclidean space \( \mathbb{R}^n \); see, for instance, [11, Lemma 3.2, page 149] or [25, Theorem 4.1, page 221]. Furthermore, are known some similar geometrical properties to the existing in Euclidean space \( \mathbb{R}^n \), such as, given two points there exists an unique geodesic segment that joins them. Now, we present a geometric property which will be very useful in the convergence analysis.

Let us recall that a \textit{geodesic triangle} \( \Delta(x_1x_2x_3) \) of a Riemannian manifold is the set consisting of three distinct points \( x_1, x_2, x_3 \) called the \textit{vertices} and three minimizing geodesic segments \( \gamma_{i+1} \) joining \( x_{i+1} \) to \( x_{i+2} \) called the \textit{sides}, where \( i = 1, 2, 3 \pmod{3} \).

**Theorem 2.1.** Let \( M \) be a Hadamard manifold, \( \Delta(x_1x_2x_3) \) a geodesic triangle and \( \gamma_{i+1} : [0, l_{i+1}] \to M \) geodesic segments joining \( x_{i+1} \) to \( x_{i+2} \) and consider \( l_{i+1} := l(\gamma_{i+1}), \theta_{i+1} = \frac{1}{2} (\gamma'_{i+1}(0), -\gamma'_{i}(l_{i})) \), for \( i = 1, 2, 3 \pmod{3} \). Then,

\[
\theta_1 + \theta_2 + \theta_3 \leq \pi,
\]
\[ l_{i+1}^2 + l_{i+2}^2 - 2l_{i+1}l_{i+2}\cos\theta_{i+2} \leq l_i^2, \]

\[ d^2(x_{i+1}, x_{i+2}) + d^2(x_{i+2}, x_i) - 2\langle \exp_{x_{i+2}}^{-1} x_{i+1}, \exp_{x_{i+2}}^{-1} x_i \rangle \leq d^2(x_i, x_{i+1}). \]  

(2.1)

**Proof.** See, for example, [25, Theorem 4.2, page 161].

As a consequence of (2.1), we have

\[ \frac{1}{2} d^2(x, z) - \frac{1}{2} d^2(y, z) - \langle \exp_{y}^{-1} z, \exp_{y}^{-1} x \rangle \geq \frac{1}{2} d^2(x, y), \]  

where we take \( x = x_1, z = x_2, y = x_3 \).

**In this paper, all manifolds** \( M \) **are assumed to be Hadamard and finite dimensional.**

### 2.2 Convexity

A set \( \Omega \subset M \) is said to be convex iff any geodesic segment with end points in \( \Omega \) is contained in \( \Omega \), that is, iff \( \gamma : [a, b] \rightarrow M \) is a geodesic such that \( x = \gamma(a) \in \Omega \) and \( y = \gamma(b) \in \Omega \), then \( \gamma((1-t)a + tb) \in \Omega \) for all \( t \in [0, 1] \). Given \( B \subset M \), we denote by \( \text{conv}(B) \) the convex hull of \( B \), that is, the smallest convex subset of \( M \) containing \( B \). Let \( \Omega \subset M \) be a convex set. A function \( f : \Omega \rightarrow \mathbb{R} \) is said to be convex iff for any geodesic segment \( \gamma : [a, b] \rightarrow \Omega \) the composition \( f \circ \gamma : [a, b] \rightarrow \mathbb{R} \) is convex. Take \( p \in \Omega \). A vector \( s \in T_pM \) is said to be a subgradient of \( f \) at \( p \) iff

\[ f(q) \geq f(p) + \langle s, \exp_p^{-1} q \rangle, \quad q \in \Omega. \]  

(2.3)

The set of all subgradients of \( f \) at \( p \), denoted by \( \partial f(p) \), is called the subdifferential of \( f \) at \( p \). It is known that if \( f \) is convex and \( M \) is an Hadamard manifold, then \( \partial f(p) \) is a nonempty set, for each \( p \in \Omega \); see Udriste [30, Theorem 4.5, page 74].
2.3 Bregman Distances and Functions

Let $M$ be a Hadamard manifold and $S$ a nonempty open convex set of $M$ with a topological closure $\overline{S}$. Let $h : M \to \mathbb{R}$ be a strictly convex function on $\overline{S}$ and differentiable in $S$. Then, the Bregman distance associated to $h$, denoted by $D_h$, is defined as a function $D_h(\cdot, \cdot) : \overline{S} \times S \to \mathbb{R}$ such that

$$D_h(x, y) := h(x) - h(y) - \langle \text{grad} h(y), \exp_y^{-1} x \rangle. \quad (2.4)$$

Notice that the expression of the Bregman distance depends on the definition of the metric. For some examples from different manifolds; see [22].

Let us adopt the following notation for the partial level sets of $D_h$. For $\alpha \in \mathbb{R}$, take:

$$\Gamma_1(\alpha, y) := \{ x \in \overline{S} : D_h(x, y) \leq \alpha \} \quad \text{and} \quad \Gamma_2(x, \alpha) := \{ y \in S : D_h(x, y) \leq \alpha \}.$$

**Definition 2.1.** A real function $h : M \to \mathbb{R}$ is called a Bregman function if there exists a nonempty open convex set $S$ such that

(a) $h$ is continuous on $\overline{S}$;
(b) $h$ is strictly convex on $\overline{S}$;
(c) $h$ is continuously differentiable on $S$;
(d) For all $\alpha \in \mathbb{R}$ the partial level sets $\Gamma_1(\alpha, y)$ and $\Gamma_2(x, \alpha)$ are bounded for every $y \in S$ and $x \in \overline{S}$, respectively;
(e) If $\lim y^k = y^* \in \overline{S}$, then $\lim D_h(y^*, y^k) = 0$;
(f) If $\lim D_h(z^k, y^k) = 0$, $\lim y_k = y^* \in \overline{S}$ and $\{z^k\}$ is bounded, then

$$\lim z^k = y^*.$$
In the remainder of this paper, \( h \) denotes a Bregman function with zone \( S \), where \( S \) is a nonempty, open, convex set.

**Proposition 2.1.** For all \( y, z \in S \) and \( x \in \bar{S} \), holds

\[
\langle \text{grad} D_h(z, y), \exp_{z^{-1}}^{-1} x \rangle = D_h(x, y) - D_h(x, z) - D_h(z, y).
\]

*Proof.* See \[23, Proposition 3.1\].

**Lemma 2.1.** The following statements hold:

(i) \( \text{grad} D_h(\cdot, y)(x) = \text{grad} h(x) - P_{\gamma_{y,x}} \text{grad} h(y) \), for all \( x, y \in S \), where the geodesic curve \( \gamma : [0, 1] \to M \) is such that \( \gamma(0) = y \) and \( \gamma(1) = x \);

(ii) \( D_h(\cdot, y) \) is strictly convex on \( \bar{S} \) for all \( y \in S \);

(iii) For all \( x \in \bar{S} \) and \( y \in S \), \( D_h(x, y) \geq 0 \) and \( D_h(x, y) = 0 \) if and only if \( x = y \).

*Proof.* See \[22, Lemma 4.1\].

3 Proximal Point for Bilevel Equilibrium Problem

3.1 Bilevel Equilibrium Problem

From now on \( \Omega \subset S \) will denote a nonempty closed convex set, unless explicitly stated otherwise. Given a bifunction \( F : \Omega \times \Omega \to \mathbb{R} \), the *equilibrium problem* in the Riemannian context (denoted by EP) is:

\[
\text{Find } x^* \in \Omega : \quad F(x^*, y) \geq 0, \quad y \in \Omega.
\]  

(3.1)

As far as we know, this problem was considered firstly, in this context by Colao et al. in \[6\], where the authors pointed out important problems, which are retrieved from \((3.1)\). For example, if \( V \in \mathcal{X}(M) \), in the
particular case where

\[ F(x,y) = \langle V(x), \exp_x^{-1} y \rangle, \quad x, y \in \Omega, \]

(3.1) reduces to the variational inequality problem; see, for instance, Németh [21].

Our interest is in finding a solution to the following problem (denoted by BEP)

\[
\text{Find } x^* \in EP(F, \Omega) : \quad Q(x^*, y) \geq 0, \quad y \in EP(F, \Omega),
\]

(3.2)

where \( Q : \Omega \times \Omega \to \mathbb{R} \) is a bifunction and \( EP(F, \Omega) \) denotes the set of solutions to (3.1).

**Definition 3.1.** Let \( K : \Omega \times \Omega \to \mathbb{R} \) be a bifunction. \( K \) is said to be

1. monotone iff \( K(x, y) + K(y, x) \leq 0 \), for all \( (x, y) \in \Omega \times \Omega \);

2. pseudomonotone iff, for each \( (x, y) \in \Omega \times \Omega \),

\[
K(x, y) \geq 0 \quad \text{implies} \quad K(y, x) \leq 0;
\]

3. \( \theta \)-undermonotone iff, there exists \( \theta \geq 0 \) such that,

\[
K(x, y) + K(y, x) \leq \theta (-\nabla h(x) + P_{y,x} \nabla h(y), \exp_x^{-1} y),
\]

for all \( (x, y) \in \Omega \times \Omega \).

**Remark 3.1.**

(i) Clearly, monotonicity implies pseudomonotonicity, but the converse does not hold even in a linear context; see, for instance, Iusem and Sosa [15];

(ii) We emphasize that the Definition 3.1 (3) was presented in the linear context by Iusem and Nasri in [14].

Note that Lemma 2.1 implies

\[
\nabla D_h(x, z) = \nabla h(x) - P_{y,z} \nabla h(z), \quad x, z \in S,
\]
and

\[ P_{γ_{y,x}} \text{grad}D_h(y, z) = P_{γ_{y,x}} \text{grad}h(y) - P_{γ_{z,y}} \circ P_{γ_{y,z}} \text{grad}h(z), \quad x, y, z \in S. \]

Hence, for \( x, y, z \in S \),

\[ (-\text{grad}D_h(x, z) + P_{γ_{y,x}} \text{grad}D_h(y, z), \exp_x^{-1} y) = (-\text{grad}h(x) + P_{γ_{y,x}} \text{grad}h(y), \exp_x^{-1} y). \]

Unless stated to the contrary, in the remainder of this paper we assume that \( F, Q : \Omega \times \Omega \to \mathbb{R} \) are bifunctions satisfying the following assumptions:

**H1)** \( K(x, x) = 0 \) for each \( x \in \Omega \);

**H2)** For every \( x \in \Omega \), \( y \mapsto K(x, y) \) is convex and lower semicontinuous;

**H3)** For every \( y \in \Omega \), \( x \mapsto K(x, y) \) is upper semicontinuous.

For each fixed \( k \in \mathbb{N} \) and \( z_0 \in M \), let us define:

\[ \Omega_k := \{ x \in \Omega : d(x, z_0) \leq k \} \]

and for each \( y \in \Omega \)

\[ L_K(y) := \{ x \in \Omega : K(y, x) \leq 0 \}. \]

**Assumption 3.1.** Given \( k \in \mathbb{N} \), for all finite set \( \{y_1, \ldots, y_m\} \subset \Omega_k \), one has

\[ \text{conv}(\{y_1, \ldots, y_m\}) \subset \bigcup_{i=1}^m L_K(k, y_i). \]

**Remark 3.2.** Note that, in the particular case where \( K \) is pseudomonotone, the property described by the previous assumption is naturally verified; see proof in [2].

**Assumption 3.2.** Given \( z_0 \in M \) fixed, consider a sequence \( \{z^k\} \subset \Omega \) such that \( \{d(z^k, z_0)\} \) converges to infinity as \( k \) goes to infinity. Then, there exists \( x^* \in \Omega \) and \( k_0 \in \mathbb{N} \) such that

\[ K(z^k, x^*) \leq 0, \quad k \geq k_0. \]
The following is the main result of this section.

**Theorem 3.1.** Under Assumptions 3.1 and 3.2, EP admits a solution.

**Proof.** See [2].

3.2 Proximal Point Algorithm

In this section, following some ideas presented by Moudafi in [19], we present an approach of the proximal point algorithm for bilevel equilibrium problems on Hadamard manifolds, where the convergence result is obtained for bifunctions not necessarily monotone.

For \( \lambda > 0, \mu > 0, z \in \Omega \) fixed and \( \Omega \subset S \), consider the bifunction

\[
L_{\mu,\lambda,z}(x, y) := F(x, y) + \mu Q(x, y) + \lambda (\nabla D_h(x, z), \exp_x^{-1} y), \ x, y \in \Omega.
\]  (3.3)

Next we describe a proximal point algorithm to solve the equilibrium problem (3.2).

**Algorithm 1.** Take \( \{\mu_k\} \) and \( \{\lambda_k\} \) two bounded sequences of positive real numbers.

**Initialization.** Choose an initial point \( x^0 \in \Omega \);

**Stopping criterion.** Given \( x^k \), if \( x^{k+1} = x^k \) and \( x^k \in EP(F, \Omega) \), STOP. Otherwise;

**Iterative Step.** Given \( x^k \), take as the next iterate any \( x^{k+1} \in \Omega \) such that:

\[
x^{k+1} \in EP(L_k, \Omega), \quad L_k := L_{\mu_k,\lambda_k,x^k}.
\]  (3.4)

**Remark 3.3.**

(a) Notice that if \( Q \equiv 0 \) in (3.3) it is sufficient to require, as a stopping criterion for the Algorithm 1, that

\[ x^{k+1} = x^k; \]

(b) If \( \{x^k\} \) terminates after a finite number of iterations, then it terminates at a solution of (3.2). Indeed, take \( k \) such that \( x^{k+1} = x^k \) and \( x^k \in EP(F, \Omega) \). From definition \( x^{k+1} \) and \( L_k \), and since that
\[ \text{grad}D_h(x^{k+1}, x^k) = 0, \text{ we obtain:} \]
\[ F(x^{k+1}, y) + \mu_k Q(x^{k+1}, y) \geq 0, \quad y \in \Omega. \tag{3.5} \]

Now, using that \( x^{k+1} = x^k \in EP(F, \Omega) \) and \( F \) is pseudomonotone, it follows that \( F(y, x^{k+1}) \leq 0, \) for all \( y \in \Omega. \) Hence, \( F(x^{k+1}, y) = 0, \) for all \( y \in EP(F, \Omega) \) and the desired result follows from \( (3.5) \) for considering that \( \mu_k > 0. \)

Next results are useful to ensure the well-definition of Algorithm 1. In the remainder of this section, we assume that \( \lambda, \mu \) are positive real numbers, \( z \in \Omega, \) both fixed, and that \( \Omega \subset S. \)

**Lemma 3.1.** Let \( F, Q \) be \( \theta \)-undermonotone bifunctions with \( \theta \leq \lambda. \) Then, \( L_{\mu, \lambda, z} \) is monotone.

**Proof.** From \( (3.3) \) and taking \( \bar{Q} = F + \mu Q, \) it is easy to see that
\[
L_{\lambda, \mu, z}(x, y) + L_{\lambda, \mu, z}(y, x) = \bar{Q}(x, y) + \bar{Q}(y, x) + \lambda(\text{grad}D_h(x, z), \exp_x^{-1} y) + \lambda(\text{grad}D_h(y, z), \exp_y^{-1} x). \tag{3.6}
\]

Since \( \lambda(\text{grad}D_h(y, z), \exp_y^{-1} x) = \lambda(-\gamma_{y, x} \text{grad}D_h(y, z), \exp_y^{-1} y), \) from the equality \( (3.6), \) we obtain
\[
L_{\lambda, \mu, z}(x, y) + L_{\lambda, \mu, z}(y, x) = \bar{Q}(x, y) + \bar{Q}(y, x) + \lambda(\text{grad}D_h(x, z) - \gamma_{y, x} \text{grad}D_h(y, z), \exp_x^{-1} y). \]

Now, taking into account that \( (x, y) \mapsto \bar{Q}(x, y) \) is a \( \theta \)-undermonotone and \( \Omega \subset S, \) from item (ii) of Remark 3.1 we get
\[
L_{\lambda, \mu, z}(x, y) + L_{\lambda, \mu, z}(y, x) \leq (\theta - \lambda)(\text{grad}D_h(x, z) - \gamma_{y, x} \text{grad}D_h(y, z), \exp_x^{-1} y),
\]
and the desired result follows by combining last inequality with Lemma 2.1 (ii), monotonicity of \( \text{grad}D_h(., z) \) and assumption \( \theta \leq \lambda. \)

\[ \square \]
Lemma 3.2. Let $F, Q$ be $\theta$-undermonotone bifunctions with $\theta < \lambda$ and assume that $\lim D_h(z^k, z) - \rho d(z^k, z) = +\infty$, for some $\rho > 0$. If $F, Q$ satisfies Assumption 3.2, then $L_{\mu, \lambda, z}$ also satisfies.

Proof. First of all, given $z_0 \in M$, consider a sequence $\{z^k\} \subset \Omega$ such that $\{d(z^k, z_0)\}$ converges to infinity as $k$ goes to infinity. Using (3.3) and $\bar{Q} = F + \mu Q$, we get

$$L_{\mu, \lambda, z}(z^k, z) = \bar{Q}(z^k, z) + \lambda(\text{grad}D_h(z^k, z), \exp^{-1} z),$$

$$= \bar{Q}(z^k, z) + \lambda(\text{grad}h(z^k) - P_{\gamma_{z, z^k}} \text{grad}h(z), \exp^{-1} z),$$

$$\leq -\bar{Q}(z, z^k) + (\theta - \lambda)(\text{grad}h(z^k) + P_{\gamma_{z, z^k}} \text{grad}h(z), \exp^{-1} z),$$

where the last inequality follows from $\theta$-undermonotonicity of $\bar{Q}$. Let us show that $L_{\mu, \lambda, z}(z^k, z) \leq 0$, for all $k \geq k_0$. Define the function $f_z : \Omega \to \mathbb{R}$ by $f_z(y) = \bar{Q}(z, y)$. Since $M$ is an Hadamard manifold and $y \mapsto \bar{Q}(z, y)$ is a convex function, there exists $v' \in \partial f_z(x')$. So, applying inequality (2.3) with $f = f_z$, $s = v'$, $p = x'$ and $q = z^k$, we have

$$\langle v', \exp^{-1} z^k \rangle \leq f_z(z^k) - f_z(x') = \bar{Q}(z, z^k) - \bar{Q}(z, x'), \quad k = 0, 1, \ldots$$

(3.7)

From (3.8) and Cauchy-Schwarz inequality,

$$-\bar{Q}(z, z^k) \leq ||v'|| d(z^k, x') - \bar{Q}(z, x').$$

From (2.4), we get

$$\langle -\text{grad}h(z^k) + P_{\gamma_{z, z^k}} \text{grad}h(z), \exp^{-1} z \rangle = D_h(z^k, z) + D_h(z, z^k) \geq D_h(z^k, z).$$

Using (3.8) and the last two inequalities, we have

$$L_{\mu, \lambda, z}(z^k, z) \leq ||v'|| d(z^k, x') - \bar{Q}(z, x') + (\theta - \lambda)D_h(z^k, z), \quad k = 0, 1, \ldots$$

$$= \frac{||v'||}{\rho} [\rho d(z^k, z) - \rho \bar{Q}(z, x') - D_h(z^k, z)], \quad k = 0, 1, \ldots,$$

(3.9)
where $\rho = (\lambda - \theta)$. Now, taking into account that $\theta < \lambda$, \( \{d(z^k, z_0)\} \) converges to infinity and \([\rho d(z^k, z) - D_h(z^k, z)] \to -\infty \) as $k$ goes to infinity, the desired results follows from inequality (3.9) which concludes the proof.

**Remark 3.4.** The assumption: \( \lim D_h(z^k, z) - \rho d(z^k, z) = +\infty \) is a form of coercivity which is similar to what was introduced in linear setting in [14]. The sake of illustration, we present to follow a class of Bregman functions in which this assumption is verified. Before, consider the following proposition treated in the linear setting by Chuong et al. in [5].

**Proposition 3.1.** Let $h_0 : M \to \mathbb{R}$ be a Bregman function. Fixed $z \in M$, define

\[
h(x) = h_0(x) + \frac{1}{2}d^2(x, z).
\]

Then,

\[
D_h(x, y) \geq D_{h_0}(x, y) + \frac{1}{2}d^2(x, y).
\]

**Proof.** From Definition 2.4 and relation (2.2), we get

\[
D_h(x, y) = D_{h_0}(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{2}d^2(y, z) - \langle -\exp^{-1}_y z, \exp^{-1}_y x \rangle
\geq D_{h_0}(x, y) + \frac{1}{2}d^2(x, y).
\]

(3.10)

**Example 3.1.** Given a Riemannian distance $d$ in $M$. Let $h_0 : M \to \mathbb{R}$ be a Bregman function. Take $h(x) = h_0(x) + \frac{1}{2}d^2(x, z)$, with $z \in M$ fixed. Then Proposition 3.1 implies

\[
D_h(x, y) \geq D_{h_0}(x, y) + \frac{1}{2}d^2(x, y).
\]

Since $d(z^k, z) \to +\infty$ as $k \to +\infty$, we obtain

\[
\lim_{k \to \infty} [D_h(z^k, z) - \rho d(z^k, z)] = +\infty.
\]
Theorem 3.2. Assume that Assumption \textit{3.2} holds and \(F, Q\) are \(\theta\)-undermonotone bifunctions with \(\theta < \lambda\) and \(\lim D_h(z^k, z) - \rho d(z^k, z) = +\infty\). Then there exists an unique \(\bar{x}^* \in \Omega\) such that

\[ L_{\mu,\lambda,z}(\bar{x}^*, y) \geq 0, \quad y \in \Omega. \]

Proof. From Lemma \textit{3.1} it follows that \(L_{\mu,\lambda,z}\) is monotone and, in particular, pseudomotone (this follows from Remark \textit{3.1}). Moreover, Remark \textit{3.2} implies that \(L_{\mu,\lambda,z}\) satisfies Assumption \textit{3.1} and Lemma \textit{3.2} implies that \(L_{\mu,\lambda,z}\) satisfies Assumption \textit{3.2}. Hence, from Theorem \textit{3.1} there exists a point \(x^*_1 \in \Omega\) such that

\[ L_{\mu,\lambda,z}(x^*_1, y) \geq 0, \quad y \in \Omega. \]

Let us suppose, by contradiction, that there exists \(\bar{x}^*_2\) satisfying the last inequality. Then,

\[ 0 \leq L_{\mu,\lambda,z}(x^*_1, x^*_2) = \bar{Q}(x^*_1, x^*_2) + \lambda (\text{grad} D_h(x^*_1, z), \exp_{x^*_1}^{-1} x^*_2) \]

and

\[ 0 \leq L_{\mu,\lambda,z}(x^*_2, x^*_1) = \bar{Q}(x^*_2, x^*_1) + \lambda (\text{grad} D_h(x^*_2, z), \exp_{x^*_2}^{-1} x^*_1) \]

By summing the last two inequalities, we get

\[ 0 \leq L_{\mu,\lambda,z}(x^*_1, x^*_2) + L_{\mu,\lambda,z}(x^*_2, x^*_1) \leq (\theta - \lambda) (-\text{grad} D_h(x^*_1, z) - P_{x^*_2, x^*_1} \text{grad} D_h(x^*_2, z), \exp_{x^*_1}^{-1} x^*_2). \]

From monotonicity of \(\text{grad} D_h(., z)\) and assumption \(\theta < \lambda\) we get the desired result.

Corollary 3.1. Assume that Assumption \textit{3.2} holds and \(F, Q\) are \(\theta\)-undermonotone bifunction. If \(\{\lambda_k\}\) is a bounded sequence of positive real numbers such that \(\theta < \lambda_k, k \in \mathbb{N}\), then Algorithm \textit{1} is well-defined.

Proof. This follows immediately from Theorem \textit{3.2}.

In the remainder of this paper we assume that the assumptions of the previous corollary hold and, in view of item (b) of Remark \textit{3.3} that \(\{\mu_k\}, \{\lambda_k\}\) and \(\{x^k\}\) are infinite sequences generated from Algorithm \textit{1}.
3.3 Convergence Analysis

In this subsection we present the convergence of the sequence \( \{x^k\} \).

**Lemma 3.3.** Let \((\xi_k)\) and \((\rho_k)\) be nonnegative sequences of real numbers satisfying:

(a) \( \xi_{k+1} \leq \xi_k + \rho_k \)

(b) \( \sum_{k=0}^{+\infty} \rho_k < \infty \).

Then the sequence \((\xi_k)\) converges.

**Proof.** See [24, Lemma 9, page 49].

**Definition 3.2.** A sequence \( \{z^k\} \subset M \) is said to be \( D_h \)-quasi-Fejér convergent to a nonempty set \( U \subset M \) iff for every \( z \in U \), there exists \( \{\rho_k\} \subset \mathbb{R}_+ \) such that

\[
\sum_{k=0}^{+\infty} \rho_k < \infty, \quad \text{and} \quad D_h(z^{k+1}, z) \leq D_h(z^k, z) + \rho_k \quad k = 0, 1, \ldots.
\]

**Proposition 3.2.** Let \( U \subset M \) be a nonempty set and \( \{z^k\} \) be a sequence \( D_h \)-quasi-Fejér convergent to \( U \). Then \( \{z^k\} \) is bounded. If, furthermore, an accumulation point \( \bar{z} \) of \( \{z^k\} \) belongs to \( U \), then the whole sequence \( \{z^k\} \) converges to \( \bar{z} \) as \( k \) goes to \(+\infty\).

**Proof.** Note that

\[
D_h(z^{k+1}, z) \leq D_h(z^k, z) + \rho_k \leq D_h(z^{k-1}, z) + \rho_{k-1} + \rho_k \leq \cdots \leq D_h(z^0, z) + \sum_{j=0}^{+\infty} \rho_j.
\]

Thus \( z^{k+1} \in \Gamma_1(\alpha, z) \) with \( \alpha = D_h(z^0, z) + \sum_{j=0}^{+\infty} \rho_j \). From Definition 2.1 (d), we obtain that \( \{z^k\} \) is bounded. Let \( \bar{z} \in U \) be an accumulation point of \( \{z^k\} \), then there exists a subsequence \( \{z^{k_j}\} \) such that \( z^{k_j} \to \bar{z} \). Definition 2.1 (e) implies \( D_h(z^{k_j}, \bar{z}) \to 0 \). Given an real number \( \epsilon > 0 \), there exists \( k_{j_0} \in \mathbb{N} \) such that \( D_h(z^{k_{j_0}}, \bar{z}) < \epsilon/2 \) and \( \sum_{j=k_{j_0}}^{+\infty} \rho_j < \epsilon/2 \). Hence, for \( k \geq k_{j_0} \), we get
\[ D_h(z^k, \bar{z}) \leq D_h(z^{k-1}, \bar{z}) + \rho_{k-1} + \rho_k \leq \cdots \leq D_h(z^k_{j_0}, \bar{z}) + \sum_{j = k_{j_0}}^{\infty} \rho_j < \epsilon. \]

\[ \square \]

**Theorem 3.3.** Assume that \( F \) is a pseudomonotone function and satisfies (H1)-(H3) and \( EP(F, \Omega) \neq \emptyset \). If the function \( x \mapsto Q(x, y) \) is bounded for each \( y \in EP(F, \Omega) \) and \( \sum_{k=0}^{+\infty} (\mu_k / \lambda_k) < \infty \), then, for all \( x^* \in EP(F, \Omega) \):

(i) The sequence \( \{D_h(x^*, x^k)\} \) is convergent;

(ii) The sequence \( \{x^k\} \) is bounded;

(iii) \( \lim_{k \to \infty} D(x^{k+1}, x^k) = 0 \) and \( \lim_{k \to \infty} d(x^{k+1}, x^k) = 0 \).

**Proof.** From (3.4),

\[ F(x^{k+1}, y) + \mu_k Q(x^{k+1}, y) + \lambda_k \langle \text{grad} D_h(x^{k+1}, x^k), \exp_{x^{k+1}}^{-1} y \rangle \geq 0, \quad \forall y \in \Omega. \]

Taking arbitrary \( x^* \in EP(F, \Omega) \) and doing \( y = x^* \) in the last inequality, we get

\[ F(x^{k+1}, x^*) + \mu_k Q(x^{k+1}, x^*) + \lambda_k \langle \text{grad} D_h(x^{k+1}, x^k), \exp_{x^{k+1}}^{-1} x^* \rangle \geq 0. \] (3.11)

The pseudomonotonicity of \( F \) implies \( F(x^{k+1}, x^*) \leq 0 \). Thus,

\[ \frac{\mu_k}{\lambda_k} Q(x^{k+1}, x^*) + \langle \text{grad} D_h(x^{k+1}, x^k), \exp_{x^{k+1}}^{-1} x^* \rangle \geq 0. \]

Using Proposition 2.1 and last inequality,

\[ \frac{\mu_k}{\lambda_k} Q(x^{k+1}, x^*) + D_h(x^*, x^k) - D_h(x^*, x^{k+1}) - D_h(x^{k+1}, x^k) \geq 0. \] (3.12)

From the last inequality, we obtain

\[ \frac{\mu_k}{\lambda_k} Q(x^{k+1}, x^*) + D_h(x^*, x^k) \geq D_h(x^*, x^{k+1}). \]
Taking \( \xi_k = D_h(x^*, x^k) \) and \( \rho_k = \frac{\mu_k}{\lambda_k} Q(x^{k+1}, x^*) \) in Lemma 3.3 and using that \( \sum_{k=0}^{+\infty} \frac{\mu_k}{\lambda_k} Q(x^{k+1}, x^*) < \infty \), it follows that \( \{D_h(x^*, x^k)\} \) converges. Hence \( \{x^k\} \subseteq \Gamma_2(x^*, \alpha) \), in view Definition 2.1 (d), \( \{x^k\} \) is bounded.

From (3.12), we get
\[
\frac{\mu_k}{\lambda_k} Q(x^{k+1}, x^*) + D_h(x^*, x^k) - D_h(x^{k+1}, x^k) \geq 0
\]
and since the leftmost expression in (3.13) converges to 0, we obtain
\[
\lim_{k \to \infty} D_h(x^{k+1}, x^k) = 0.
\]
Thus, by Definition 2.1 (f), \( \lim_{k \to \infty} d(x^{k+1}, x^k) = 0 \).

Now, we present our main convergence result.

**Theorem 3.4.** Under assumptions of Theorem 3.3 the whole sequence \( \{x^k\} \), generated by Algorithm (1), converges to a solution of (3.1). If in addition,
\[
\lim \sup_{k \to \infty} \left[ \frac{\lambda_k}{\mu_k} (\langle \text{grad} D_h(x^{k+1}, x^k), \exp^{-1}_{x^{k+1}} y \rangle \right] \leq 0, \quad \forall y \in EP(F, \Omega),
\]
then \( \{x^k\} \) converges to a solution of (3.2).

**Proof.** From Theorem 3.3 we have that \( \{x^k\} \) is bounded. Let \( \bar{x} \) be a cluster point of \( \{x^k\} \) and \( \{x^{kj}\} \subseteq \{x^k\} \) such that \( \{x^{kj}\} \) converges to \( \bar{x} \). Now fix any \( y \in \Omega \). Since \( x^{kj+1} \) solves \( EP(L_{kj}, \Omega) \), we have
\[
F(x^{kj+1}, y) + \mu_{kj} Q(x^{kj+1}, y) + \lambda_{kj} \langle \text{grad} D_h(x^{kj+1}, x^{kj}), \exp^{-1}_{x^{kj+1}} y \rangle \geq 0,
\]
for all \( y \in \Omega \). Using Proposition 2.1 we get
\[
F(x^{kj+1}, y) \geq -\mu_{kj} Q(x^{kj+1}, y) + \lambda_{kj} [D_h(y, x^{kj+1}) - D_h(y, x^{kj})] \\
+ \lambda_{kj} D_h(x^{kj+1}, x^{kj}), \quad \forall y \in \Omega.
\]

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From Definition 2.1 upper continuity of $F$, limitation of $\{\lambda_k\}$ and (3.14), it follows

$$F(\bar{x}, y) \geq \limsup_{j \to \infty} F(x^k_j, y) \geq 0, \ \forall \ y \in \Omega.$$ 

Therefore, $\bar{x} \in EP(F, \Omega)$. By Proposition 3.2 we conclude that the sequence $\{x^k\}$ converges to $\bar{x}$. It remains to show that $Q(\bar{x}, y) \geq 0, \ \forall \ y \in EP(F, \Omega)$. Taking (3.11) again with $y \in EP(F, \Omega)$ and using pseudomonotonicity of $F$, we obtain

$$Q(x^{k+1}, y) + \frac{\lambda_k}{\mu_k} \langle \text{grad} D_h(x^{k+1}, x^k), \exp^{-1}_{x^{k+1}} y \rangle \geq -F(x^{k+1}, y) \geq 0, \ \forall \ y \in EP(F, \Omega).$$

for all $y \in EP(F, \Omega)$. Since

$$\limsup_{k \to \infty} \left[ \frac{\lambda_k}{\mu_k} \langle \text{grad} D_h(x^{k+1}, x^k), \exp^{-1}_{x^{k+1}} y \rangle \right] \leq 0, \ \forall y \in EP(F, \Omega)$$

and $x \mapsto Q(x, y)$ is upper semicontinuous, we obtain $Q(x^*, y) \geq 0$, for all $y \in EP(F, \Omega).$ \hfill \Box \hfill \Box

Remark 3.5. Let us emphasize that the assumption

$$\limsup_{k \to \infty} \left[ \frac{\lambda_k}{\mu_k} \langle \text{grad} D_h(x^{k+1}, x^k), \exp^{-1}_{x^{k+1}} y \rangle \right] \leq 0, \ \forall y \in EP(F, \Omega).$$

is similar to one shown in linear setting by Moudafi in [19, Theorem 2.3].

Remark 3.6.

(i) Note that when $Q \equiv 0$ and $M$ is an $n$-dimensional space, we recover the main result presented by Mastroeni in [17];

(ii) In case that $F(x, y) = \phi(x) - \phi(y), Q(x, y) = \Phi(x) - \Phi(y)$ and $M$ is an $n$-dimensional space, we recover the main result presented by Cabot in [4];

(iii) When $M$ is an $n$-dimensional space, $D_h$ is the euclidian distance, we recover the main result presented by Moudafi in [19];
(iv) When $M$ is an $n$-dimensional Hadamard manifold, $Q \equiv 0$ and, fixed $z \in M$, $h(x) = (1/2)d^2(x,z)$, where $d$ is the distance in $M$, we recover the main convergence results presented by Bento et al. in [2].

4 Application to stability and change dynamics

4.1 How Routines Form and Break

As an application, we consider the problem of “how routines form and change” which is crucial for the dynamics of organizations in Economics and Management Sciences. There is a large litterature about it, starting with the work in Economics [26, 27], and in Management Sciences [20], within an evolutionary perspective inspired by the theory of evolution in biology. To hope to solve this very important problem for the survival and dynamic efficiency of organizations, the most important step is to embedd this problem in a larger one. That is, routine formation and break is an example, and a leading one, of a lot of stay/stability and change dynamics where some things change and other things stay, in parallel or in sequence, at the individual, organizational or interacting agents levels. The litterature on stability and change dynamics is very large and very dispersed, in different disciplines in Behavioral Sciences, including Psychology, Economics, Game theory, Decision theory, Management Sciences, Sociology, Philosophy, Artificial Intelligence, Political Sciences . . . which makes them very difficult to handle. The problem is to know how to mix, in the right way, static and dynamic aspects of human behaviors. However, a recent variational rationality approach of human behavior presented in [28, 29], the author modelized and unified in a unique and simple model which admits a lot of variants, a lot of such stay/stability and change dynamics. In this last section on applications, we will use, as a tool, this variational rationality approach to make the link between two points: i) the mathematical part of the paper which uses a proximal point algorithm to show convergence to a bilevel equilibrium and, ii) the theories on routine formation and break.
To save space let us focus only on three main references (among a myriad!) which consider the problem of “how routines form and break”.

i) First the nice survey of Becker [1] summarizes the main properties of routines. It clarifies a large and passionate debate on, both, the definition of routines and their importance. Routines are patterns (including actions, activities, behaviors and interactions), recurrent, collective, mindlessness or effortful accomplishments, procedural, context dependent, embedded, specific, path dependent and triggered by internal goals, feedbacks and external cues.

ii) Second a very recent special issue of a leading journal in Management Sciences, D’Adderio et al. in [8] focuses the attention on this stability and change routine dynamic, exploring sources of stability and change in organizations. Let us list the following aspects: coordination, interdependence, multiplicity of routines, actants and artifacts, routines and institutions, mechanisms for feedback and change, recombinations and mashups, granularity and levels of analysis, time scales, performance, generativity and novelty.

iii) If we compare habits (at the individual level) to routines (at the collective level), then, we can know how routines form and break because it turns out that every habit starts with a psychological pattern called a “habit loop”, which is a three-part process (Duhigg [12]). “First, there’s a cue, or trigger, that tells your brain to go into automatic mode and let a behavior unfold . . . . Then there’s the routine, which is the behavior itself . . . . The third step, he says, is the reward: something that your brain likes that helps it remember the habit loop in the future”.

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4.2 The Variational Rationality (VR) Model Of An Agent/Organization

The variational rationality approach of human stability and change dynamics (see, [28][29]) rests on three main concepts: worthwhile changes, worthwhile transitions and stationary or variational traps. The definition and modelization of these leading concepts requires to define a list of intermediary concepts which allow a lot of variants and extensions. Each of them needs lengthy discussions for suitable applications in different disciplines. Let us consider the simple prototype given in [28][29] as a reduced form of the general variational rationality approach.

4.2.1 The motivation-resistance to change structure

Consider an entity, say an agent or an organization, who performs some individual or collective action and gets a related numerical payoff. The following list helps to define what is a worthwhile change.

1) $X$ is the space of actions of the entity, $x, y \in X$ are two actions, $x \sim y \neq x$ is a change and $x \sim y = x$ is a stay where $x$ refers to the statu quo and $y$ is an action/goal.

2) $e \in E$ refers to the perception of the past and current experiences/environments of the entity, as well as beliefs about her futur experiences/environments. The term experience/environment means an experience in a related environment. Experience depends, among other things, of the succession of past actions and past environments. Environments include internal or external things, as parameters and actions done by other agents. Past, current and futur experiences/environments can be given or chosen, depending on how agents choose what to retrieve from their memory, how they form beliefs, attitudes and expectations about their futur experiences/environments, and how they set futur goals.

3) $A_e = A_e(x, y) \in \mathbb{R}$ is her advantage to change from $x$ to $y$. If $A_e(x, y) \geq 0$, there is an advantage to change from $x$ to $y$. If $A_e(x, y) \leq 0$ there is a disadvantage to change from $x$ to $y$, i.e a loss.
\( F_e(x, y) = -A_e(x, y) \geq 0 \). Then, advantages to change and loss functions are opposite. For example if actions \( x \) and \( y \) generate the payoffs (profits) \( g(x) \in \mathbb{R} \) and \( g(y) \in \mathbb{R} \), there is an advantage to change from doing \( x \) to do \( y \) if \( g(y) \geq g(x) \), i.e., \( A_e(x, y) = g(y) - g(x) \geq 0 \). There is a loss to change if \( g(y) \leq g(x) \), i.e., \( F_e(x, y) = -A_e(x, y) = g(x) - g(y) \geq 0 \). Let \( f(x) \) be the unsatisfied need of the entity. Then, after having done action \( x \), this entity will have an advantage to change from \( x \) to \( y \) if \( A_e(x, y) = f(x) - f(y) \geq 0 \). This means that her unsatisfied need will decrease, \( f(y) \leq f(x) \). If the entity have a disadvantage \( A_e(x, y) = f(x) - f(y) \leq 0 \) to move from \( x \) to \( y \), her loss is \( F_e(x, y) = -A_e(x, y) = f(y) - f(x) \geq 0 \) because her unsatisfied need has increased when moving from \( x \) to \( y \).

Consider the simplest vertical organization, with a leader \( l \) and a follower \( f \). Let \( x = (x^l, x^f) \in X = X^l \times X^f \) be the collective action carried out by the leader and the follower. The leader performs action \( x^l \in X^l \) and the follower carries out action \( x^f \in X^f \). Let \( x \in X \) \( \cong y \in X \) be a planned change from the current collective action \( x \) to the future collective action \( y \). Let \( A^l(x, y) \geq 0 \) be the leader numerical advantage to change and let \( A^f(x, y) \geq 0 \) be the follower advantage to change when they are positive or zero, and their losses to changes \( A^l(x, y) \leq 0 \) and \( A^f(x, y) \leq 0 \) when they are negative or zero. Then, the joint advantage to change of the organization is \( A_e(x, y) = A^l(x, y) + \mu A^f(x, y) \) where \( \mu > 0 \) is the weight allowed to the follower payoff.

4) \( C_e = C_e(x, y) \in \mathbb{R}_+ \) represents the cost to be able to change of the entity, from having the capability to do the (individual or collective) action \( x \) to the capability to do action \( y \). Then, \( C_e(x, x) = 0 \) is the capability to do action \( x \), having initially the capability to do it. We have \( C_e(x, y) = 0 \) iff \( y = x \).

5) \( I_e(x, y) = C_e(x, y) - C_e(x, x) = C_e(x, y) \geq 0 \) defines her inconvenients to change.

6) \( M_e(x, y) = U_e[A_e(x, y)] \) is the motivation to change of the entity, which is the utility \( M_e = U_e[A_e] \in \mathbb{R}_+ \)
of her advantages to change $A_e$, where $U_e : A_e \in \mathbb{R} \rightarrow U_e[A_e] \in \mathbb{R}_+$ is strictly increasing and zero at zero.

7) $R_e(x, y) = D_e[I_e(x, y)]$ is the resistance to change of the entity, which refers to the desutility $D_e[I_e] \in \mathbb{R}_+$ of her inconvenients to change $I_e$, strictly increasing and zero at zero.

8) $\Delta_{e,\lambda}(x, y) = M_e(x, y) - \lambda R_e(x, y) = U_e[A_e(x, y)] - \lambda D_e[I_e(x, y)]$ is the worthwhile to change payoff of the entity, where $\lambda > 0$ balances her motivation and resistance to change.

### 4.3 Worthwhile single stays and changes

Given the current environment $e \in E$, a change $x \curvearrowright y \neq x$ is worthwhile iff $\Delta_{e,\lambda}(x, y) \geq 0$. In this case motivation to change is higher than $\lambda > 0$ time resistance to change. Then, $\lambda > 0$ represents a satisficing worthwhile to change ratio: $\Delta_{e,\lambda}(x, y) \geq 0 \iff M_e(x, y)/R_e(x, y) \geq \lambda$ for $y \neq x$. A change is not worthwhile when $\Delta_{e,\lambda}(x, y) < 0$. The subset of worthwhile changes rather than to stay which starts from $x$ is $W_{e,\lambda}(x) = \{y \in X, \Delta_{e,\lambda}(x, y) \geq 0\}$.

#### 4.3.1 Stationary traps

Given $e_* \in E$ and $\lambda_* > 0$, $x^* \in X$ is a stationary trap if $\Delta_{e,\lambda_*}(x^*, y) < 0$ for all $y \neq x^*$.

#### 4.3.2 Worthwhile transitions

Given the succession of given experiences/environments $\{e_k : e_k \in E\}$ and a succession of chosen satisficing worthwhile to change ratio $\{\lambda_{k+1} : \lambda_{k+1} > 0\}$, a succession $\{x^k\}$ of worthwhile changes defines a worthwhile transition

$$x^{k+1} \in W_{e_k,\lambda_{k+1}}(x^k), k \in \mathbb{N}.$$
4.3.3 Aspiration points

Given a worthwhile transition $x^{k+1} \in W_{e_k,\lambda_{k+1}}(x^k), k \in \mathbb{N}$, $x^* \in X$ is a strong aspiration point if $x^* \in W_{e_k,\lambda_{k+1}}(x^k), k \in \mathbb{N}$. This means that, starting from any position of the transition, it is worthwhile to directly reach this aspiration point. Aspiration points are weak if it exists $k_0 \in \mathbb{N}$ such that $x^* \in W_{e_k,\lambda_{k+1}}(x^k), k \geq k_0$.

4.3.4 Variational traps

Given $e_\ast \in E$ and $\lambda_\ast > 0$, $x^* \in X$ is a variational trap of the worthwhile transition $x^{k+1} \in W_{e_k,\lambda_{k+1}}(x^k), k \in \mathbb{N}$ if $x^* \in X$ is both a stationary trap and an aspiration point.

Then, a variational trap refers, both, to a stability issue (as a stationary trap) and to a reachability issue, the existence of a feasible and worthwhile transition from the initial position to the end (an aspiration issue).

4.3.5 Variational rationality problem

Starting from $x^0 \in X$, find when a given worthwhile transition

$$x^{k+1} \in W_{e_k,\lambda_{k+1}}(x^k), k \in \mathbb{N}$$

converges to a variational trap $x^* \in X$. The sequence of satisficing worthwhile to change ratio $\{\lambda_k > 0\}$ can be given ex ante, or chosen, in an adaptive way, each step.

4.3.6 Authority, delegation and decentralization

For a hierarchical organization with a leader $l$ and a follower $f$, there are two polar cases:

i) authority: the leader $l$ chooses the joint action $x = (x^l, x^f) \in X$ where the leader performs action $x^l$ and the follower accepts to carry out action $x^f$. 

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ii) delegation: the leader $l$ chooses first action $x^l$ and the follower carries out action $x^f$ in his interest, or best interest.

For a non hierarchical organization, both agents $l$ and $f$ choose their action in a Nash (decentralized) way.

4.4 The Specific Case of This Paper

The mathematical part of our paper considers a specific but very important version of the previous variational rationality prototype. It makes the following list of specific behavioral hypothesis.

4.4.1 From mathematics to behavioral sciences notations

$A^l(x, y) = -Q(x, y) \geq 0$ and $A^f(x, y) = -F(x, y) \geq 0$ are the leader and follower numerical advantages to change from $x$ to $y$ when they are positive or zero. Then, the mathematical payoffs $Q(x, y) = -A^l(x, y)$ and $F(x, y) = -A^f(x, y)$ represent leader and follower numerical losses to change when they are positive or zero. The organizational losses to change $F_e(x, y) = [F(x, y) + \mu Q(x, y)] = -A_e(x, y)$ is the opposite of the organizational advantage to change $A_e(x, y) = A^f(x, y) + \mu A^l(x, y)$. In the present mathematical part of the paper, the past (have been done), current (to be done) and next future actions are $z = x^k$, $x = x^{k+1}$ and $y$. Then, we have $e = (\mu, z, x) = e_k = (\mu_k, x^k, x^{k+1}) \in \mathbb{R}_+ \times X \times X$ is the past and current experience/environment of the organization at period $k$. The future environment is $\lambda = \lambda_{k+1} > 0$. In the mathematical part appears $\lambda_k$ instead of $\lambda_{k+1}$. This means that, each period $k$, the satisficing worthwhile to change ratio $\lambda_k$ is set by an external institution or an external agent. It is chosen by the organization when we consider $\lambda_{k+1}$.
4.4.2 A linear motivation-resistance to change structure

The present mathematical part of the paper considers a simple and linear motivation and resistance to change structure. In this case the utility of advantages to change and the desutility of inconvenients to change are identical to advantages to change and inconvenients to change, $U_e[A_e(x, y)] = A_e(x, y)$ and $D_e[I_e(x, y)] = I_e(x, y)$. Then, the worthwhile to change payoff of the organization is $\Delta_{e,\lambda}(x, y) = \lambda I_e(x, y)$, in this case $L_{e,\lambda}(x, y) = F_e(x, y) + I_e(x, y)$.

4.4.3 Costs to be able to change as Bregman distances

Let $d(x, x) \geq 0$ the distance between actions $x$ and $y$, which defines a dissimilarity index between them. Costs to be able to change from the capability to do an action $x$ to the capability to do an action $y$ are given by $C_h(x, y) = c_e[\rho_h(x, y)]d(x, y) = C_h(x, y)$ for all $e \in E$, where,

i) $h : x \in X \mapsto h(x) \in \mathbb{R}$ is a strictly convex and differentiable function, with mean variation given by $[h(y) - h(x)]/d(x, y)$ and marginal variation $(\nabla h(x), (y - x)/d(x, y))$.

ii) $\rho_h(x, y) = [h(y) - h(x)]/d(x, y) - (\nabla h(x), (y - x)/d(x, y))$ is the difference between the mean and marginal variations. This represents an index of curvature-convexity of the function $h$.

iii) $c_e[\rho_h(x, y)] \geq 0$ defines the per unit of dissimilarity cost to be able to change, $c_e[\rho_h(x, y)] = C_e(x, y)/d(x, y)$ if $y \neq x$. Then, will take this cost as $c_e[\rho_h(x, y)] = \rho_h(x, y)$ for all $x, y \in X$ and $e \in E$. Hence,

$$C_e(x, y) = C_h(x, y) = [h(y) - h(x)] - (\nabla h(x), (y - x))$$

is a Bregman distance $\Lambda_h(y, x) = C_e(x, y)$ when $M = X = \mathbb{R}^n$, and $C_e(x, y) = C_h(x, y) = [h(y) - h(x)] - (\nabla h(x), \exp^{-1}_y x)_y$ on a Riemannian manifold.
4.4.4 Inconvenients to change

They are defined as the marginal costs to move from the past action $x$ to the current action $z$ (to be done right now), $\text{grad} \, C_h(z, x)$, time (as a scalar product) the difference $y - x$ between the current action $x$ and the future action $y$, i.e., $I_e(x, y) = \langle \text{grad} C_h(z, x), y - x \rangle$. This represents the linear approximation of the future costs to be able to change from $x$ to $y$.

4.4.5 A bilevel equilibrium concept with authority

It refers to the mathematical problem: Find $x^* \in EP(F, \Omega)$ (i.e., such that $F(x^*, y)$ for all $y \in \Omega$) such that $Q(x^*, y) \geq 0$ for all $y \in EP(F, \Omega)$. In behavioral terms of a hierarchical organization, where a leader $l$ and a follower $f$ perform the collective action $x = (x^l, x^f) \in X = X^l \times X^f$, the problem is: how the leader, whose advantages to change from $x$ to $y$ are $A^l(x, y) = -Q(x, y)$, can choose a collective action $x^* = (x^{*l}, x^{*f}) \in \Omega$, which is an equilibrium for the follower and is also an equilibrium for him, within the follower subset of equilibria $EP(F, \Omega)$. This formulation of a bilevel equilibrium problem implies authority: the leader can choose both his action $x^{*l}$ and the action $x^{*f}$ of the follower. This is a specific instance of a hierarchical and decentralized bilevel problem. In the general case the follower chooses, in an optimal way, in a second stage, his own action $x^f$, given the observable action $x^l$ chosen in an optimal way by the leader in the first stage. This traditional formulation does not requires authority.

4.4.6 Convergence of a worthwhile stays and changes transition

The main result of this paper shows that proximal point algorithm 1, as defined in this paper, $x^{k+1} \in EP(L_k, \Omega), k \in \mathbb{N}$, modelizes a succession of worthwhile changes from a current stationary trap to the next one which converges to a bilevel equilibrium on a Riemannian manifold. Each period, $e_k = (\mu_k, x^k, x^{k+1})$ and

$$L_{e_k, \lambda_{k+1}}(x^{k+1}, y) = F(x^{k+1}, y) + \mu_k Q(x^{k+1}, y) + \lambda_{k+1} \langle \text{grad} C_h(x^k, x^{k+1}), y - x^{k+1} \rangle \geq 0$$
for all \( y \in \Omega \). Hence \( x^{k+1} \) is a variational trap. Then \( L_{e_k}(x^{k+1}, y) \geq 0 \) for all \( y \in \Omega \). implies \( L_{e_k}(x^{k+1}, y = x^k) \geq 0 \). Lemma 3.1 implies \( L_{e_k}(x^k, x^{k+1}) \leq 0 \), i.e., it is worthwhile to change from the variational trap \( x^k \) to the next \( x^{k+1} \). Theorem 3.4 shows that it converges to a bilevel equilibrium.

Then, along this stays and changes dynamic, stays are difficult to break. They require a change in the satisficing worthwhile to change level. This dynamic is a specific instance of a worthwhile stays and changes transition which does not suppose that each action \( x^k \) is a stationary trap where the agent finds it worthwhile to stay rather than to move; see [28, 29]. In the present paper, what pushes the agent to move again is that the satisficing worthwhile to change ratio \( \lambda_k \) change from \( \lambda_k \) to \( \lambda_{k+1} \neq \lambda_k \). Then the old action \( x^k \) is no longuer a stationary trap.

4.4.7 Action spaces as Riemannian manifolds

Riemannian manifolds are very useful to examine stability and change dynamics because they allow to introduce dynamical resource constraints, like the necessary regeneration of energy, power to do, . . . to escape to ego-depletion (fatigue, stress, . . .). To save space; see Cruz Neto et al. [7].

4.4.8 The variational rationality flavor of the hypothesis

The behavioral content of Definition 3.1, assumptions \( \mathcal{H}(1), \mathcal{H}(2), \) and \( \mathcal{H}(3), \) Assumptions 3.1 and 3.2 have been given by Bento et al. in [2]. To save space we will not repeat these comments.

4.4.9 The Dynamics of Routines Formation And Change

The proximal point algorithm given in this paper helps to understand and modelize how routines form and change as follows. Using the variational rationality model in [28, 29], the author defines routines as stationary traps. Then, routines change when i) the context change, in the present paper, both the experience/environment \( e_k \in E \) and the satisficing worthwhile to change ratio \( \lambda_{k+1} \), and ii) the organization
can find, in this new context, a worthwhile change which leads to a new stationary trap, the new routine. The context becomes more and more similar as this stay and change dynamic converges to an end which is a bilevel equilibrium where motivation to change disappears. Then, at the very end, resistance to change does not matter.

5 Conclusions

In this paper, we presented a proximal algorithm with Bregman distances for BEP on an Hadamard manifold whose iterative process generalizes one proposed in [19]. As an application we show that our proximal convergence result helps to modelize the very important problem of “how routines form and break”. This opens the door to inexact proximal algorithm formulations where, as an other important application, agents, being bounded rational, satisfice, each step, instead to optimize in a changing context.

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