Obstructions to weak approximation for reductive groups over $p$-adic function fields

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Abstract

We establish arithmetic duality theorems for short complexes associated to reductive groups over $p$-adic function fields. Using dualities, we deduce obstructions to weak approximation for quasi-split reductive groups. Finally, we give an application to homogeneous spaces with connected stabilizers under such groups.

Introduction

This article is a subsequent work of [HSS15] on the investigation of weak approximation for a connected reductive group when its derived subgroup has a quasi-split simply connected covering satisfying weak approximation. Let $X$ be a smooth projective and geometrically integral curve defined over some $p$-adic field $k$ and let $K$ be the function field of $X$. Since each closed point $v \in X^{(1)}$ induces a discrete valuation of $K$ (recall that the local ring $\mathcal{O}_{X,v}$ is a 1-dimensional regular local ring, hence a discrete valuation ring), the completion $K_v$ of $K$ with respect to $v$ makes sense and hence we can ask typical questions concerning the arithmetic of algebraic $K$-groups. For example, Harari and Szamuely studied the cohomological obstruction to the kernel of $H^1(K,G) \to \prod_{v \in X^{(1)}} H^1(K_v,G)$ being trivial in [HS16, Section 6] for connected linear reductive groups. Also, Harari, Scheiderer and Szamuely described the closure of $T(K)$ in $\prod_{v \in X^{(1)}} T(K_v)$ with respect to the product of $v$-adic topologies for a $K$-torus $T$ in [HSS15]. We want to extend the results of [HSS15] to a certain class of connected linear reductive groups and to homogeneous spaces over $K$ under such groups.

The first main step is to establish global duality for the Tate–Shafarevich group of short complexes of $K$-tori.

**Theorem.** Let $C = [T_1 \to T_2]$ be an arbitrary complex of $K$-tori concentrated in degree $-1$ and $0$. Let $T_1'$ and $T_2'$ be the respective dual torus of $T_1$ and $T_2$, and let $C' = [T_2' \to T_1']$. Let $\III(C) := \ker \left( \prod_{v \in X^{(1)}} H^1(K_v,C) \right)$ be the Tate–Shafarevich group of the complex $C$. There is a perfect, functorial in $C$, pairing of finite groups:

$$\III(C) \times \III(C') \to \mathbb{Q}/\mathbb{Z}.$$

The development of the global duality theorem is parallel to Izquierdo’s work [Izq16] where he considered duality theorems for groups of multiplicative type over higher local fields. Recall that a group $M$ of multiplicative type may be identified with the kernel of an epimorphism $T_1 \to T_2$ of tori over the base field. As a consequence, one may use the complex $[T_1 \to T_2]$ to describe the arithmetic of $M$ and part of the dualities established by Izquierdo [Izq16] is based on the surjectivity of $T_1 \to T_2$. In our context, we get rid of the surjectivity assumption on $T_1 \to T_2$ but the price is to restrict ourselves to $p$-adic function fields (which are of cohomological dimension $3$). This refinement is important in our situation since the short complex of tori associated with a connected reductive group (see the paragraph below) is not an epimorphism in general. Finally, we recall that just analogous to global duality results between $\III(C)$ and $\III(C)$ (where $C = [\widetilde{T}_2 \to \widetilde{T}_1]$) over number fields [Dem11, Théorème 5.7], we do not need the finiteness of $\ker(T_1 \to T_2)$.

Now we consider a connected reductive linear group $G$ over the function field $K$. Following Deligne [Del79, 2.4.7] and Borovoi [Bor98], we consider the composite $\rho : G^{sc} \to G^{ss} \to G$, where $G^{ss} = \mathcal{P}G$ is the derived subgroup of $G$ (it is semi-simple) and $G^{sc} \to G^{ss}$ is the simply connected covering of $G^{ss}$ (it is simply connected). For a maximal torus $T$ of $G$, its inverse image $T^{sc} := \rho^{-1}(T)$ is a maximal torus of $G^{sc}$. To each reductive group $G$, we associate a short complex $C = [T^{sc} \to T]$ of $K$-tori concentrated in degree $-1$ and $0$. The next theorem says that in general there is an obstruction to weak approximation for $G$ which is controlled by some sort of Tate–Shafarevich group of $C'$.
Theorem. Let $\Xi_1^1(C')$ be the group of elements in $\Xi_1^1(K,C')$ being trivial in $\Xi_1^1(K_v,C')$ for all but finitely many $v \in X(1)$ and let $\Xi_1^1(C')^D$ be the group of homomorphisms $\Xi_1^1(C') \rightarrow \mathbb{Q}/\mathbb{Z}$. Suppose $G$ is a connected reductive group such that $G^{sc}$ is quasi-split. There is an exact sequence of groups

$$1 \rightarrow \overline{G(K)} \rightarrow \prod_{v \in X(1)} G(K_v) \rightarrow \Xi_1^1(C')^D \rightarrow \Xi_1^1(C) \rightarrow 1.$$  

Here $G(K)$ denotes the closure of the diagonal image of $G(K)$ in $\prod_{v \in X(1)} G(K_v)$ for the product of $v$-adic topologies.

Note that unlike the case of number field (see Kneser [Kne62,Kne65], Harder [Har68] and [PR94, Proposition 7.9]), currently it is not known that semi-simple simply connected groups verify weak approximation over $p$-adic function fields. However, thanks to a theorem of Thăng [Thă96, Theorem 1.4], quasi-split semi-simple and simply connected groups have weak approximation.

Actually the group $G^{sc}$ being quasi-split is necessary because the proof of the theorem relies on the existence of a quasi-trivial maximal torus in $G^{sc}$, which enables us to conclude certain maps are surjective (see Lemma 2.4 and Lemma 2.6, diagram (9) as well). Let us briefly indicate other reasons why we cannot drop the assumption that $G^{sc}$ is quasi-split.

- Consider a quasi-trivial connected reductive group $H$ (its derived subgroup $H^{sc} := \mathcal{D}H$ is simply connected and the maximal toric quotient $H^{tor}$ is quasi-trivial) over a field $L$ of arithmetic type such that $H^{sc}$ satisfies weak approximation. If $L$ is a number field, then we know $H^1(L^v,H^{sc}) = 1$ for non-Archimedean places $v$ and $H^1(L,H^{sc}) \approx \prod_v H^1(L_v,H^{sc})$ by [PR94, Theorem 6.4 and 6.6]. Subsequently the method of Sansuc [San81] (see also the proof of Lemma 2.6) implies that $H$ satisfies weak approximation as well. But if $L$ is a $p$-adic function field, we do not have the vanishing of $H^1(L_v,H^{sc})$ for all but finitely many places and so it is not clear that $H$ satisfies weak approximation.

- If our reductive $K$-group $G$ is semi-simple, then there is an exact sequence $1 \rightarrow F \rightarrow G^{sc} \rightarrow G \rightarrow 1$ with $F$ being a commutative finite étale group scheme. In this case, Sansuc’s method does not give a defect to weak approximation for $G$ for the same reason.

The exact sequence (1) tells us that the group $\Xi_1^1(C')^D$ can be viewed as a defect of weak approximation to the group $G$. Actually we may rephrase the exact sequence (1) in terms of the reciprocity obstruction to weak approximation. More precisely, there is a pairing which annihilates the closure of the diagonal image of $G(K)$ on the left:

$$(-,-) : \prod_{v \in X(1)} G(K_v) \times H^3_{nr}(K(G),\mathbb{Q}/\mathbb{Z}(2)) \rightarrow \mathbb{Q}/\mathbb{Z}.$$  

See [CT95, §4.1] for general definitions and properties of $H^3_{nr}(K(G),\mathbb{Q}/\mathbb{Z}(2))$ and see [HSS15, pp. 18, pairing (17)] for the construction of the pairing (2) above.

Theorem. Let $G$ be a connected reductive group over $K$ such that $G^{sc}$ is quasi-split. There exists a homomorphism $u : \Xi_1^1(C') \rightarrow H^3_{nr}(K(G),\mathbb{Q}/\mathbb{Z}(2))$ such that each family $(g_v) \in \prod_{v \in X(1)} G(K_v)$ satisfying $(g_v,\text{Im } u) = 0$ under the pairing (2) lies in the closure $\overline{G(K)}$ with respect to the product topology.

Finally, we apply the above results to analyse obstructions to weak approximation for homogeneous spaces under $G$ with connected stabilizers. We will construct, for a separated $K$-scheme $Y$ of finite type and a finite set $S \subset X(1)$, a continuous map

$$\text{OBS}_{Y,S} : \prod_{v \in S} Y(K_v) \rightarrow \left(\mathcal{B}^3_S(Y,\mathbb{Q}/\mathbb{Z}(2))/\mathcal{B}^3(Y,\mathbb{Q}/\mathbb{Z}(2))\right)^D$$  

which plays the role of Brauer–Manin obstruction to weak approximation in the context of number fields. Put

$$\mathcal{B}^3_S(Y,\mathbb{Q}/\mathbb{Z}(2)) := \text{Ker} \left( H^3(Y,\mathbb{Q}/\mathbb{Z}(2)) \rightarrow \prod_{v \in S} H^3(Y_v,\mathbb{Q}/\mathbb{Z}(2)) / H^3(K_v,\mathbb{Q}/\mathbb{Z}(2)) \right)$$

where $Y_v := Y \times_K K_v$. The group $\mathcal{B}^3_S(Y,\mathbb{Q}/\mathbb{Z}(2))$ is the analogue in the $p$-adic function field case of the group $\mathcal{B}_S(Y) := \text{Ker} \left( \mathcal{B}_S(Y) \rightarrow \prod_{v \in S} \mathcal{B}_v(Y_v) \right)$ in the number field context (for example, see [Bor96, Section 1 and Theorem 2.2, 2.3, 2.4]). Recall that $\mathcal{B}_0(Y) = \mathcal{B}_1(Y)/\mathcal{B}_0(Y)$ where $\mathcal{B}_1(Y) := \text{Ker} \left( \mathcal{B}_Y \rightarrow (\mathcal{B}_Y) \right)$ and $\mathcal{B}_0(Y) := \text{Im}(\mathcal{B}_K \rightarrow \mathcal{B}_Y)$. 

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Theorem. Let $G$ be a connected reductive group over $K$ such that $G^\circ$ is quasi-split. Let $Y$ be a homogeneous space over $K$ under $G$. Suppose that $Y$ contains a $K$-point $y_0$ and that the stabilizer $\text{Stab}_G(y_0)$ is connected. Let $S \subset X^{(1)}$ be a finite set of places and let $y_S := (y_v) \in \prod_{v \in S} Y(K_v)$ be a family of local points. If $\text{OBS}_{Y,S}(y_S) = 0$, then $y_S \in \overline{Y(K)}_S$. Here $\overline{Y(K)}_S$ denotes the closure of $Y(K)$ in $\prod_{v \in S} Y(K_v)$ with respect to the product topology.

On the other hand, it is not in general true that a homogeneous space (or a torsor) under $G$ has a $K$-point. So it is interesting to know obstructions to the Hasse principle if there is no $K$-points on the homogeneous space $Y$ under $G$. Under good reduction hypothesis on the $p$-adic curve $X$, it is set forth in [HS16, Section 6] that there are cohomological obstructions to the Hasse principle for a reductive linear algebraic group $G$.

As some sort of complement to the present paper, we will establish global duality between $\text{III}^0(C)$ and $\text{III}^2(C')$, which enables one to construct a $12$-term (resp. $15$-term) Poitou–Tate exact sequence of topological abelian groups associated to the complex $C = [T_1 \to T_2]$ (resp. $C \otimes \mathbb{Z}/n$) in an upcoming paper.

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Notation and conventions

Unless otherwise stated, all (hyper)cohomology groups will be taken with respect to the étale topology. In particular, (hyper)cohomology groups over fields are identified with Galois (hyper)cohomology groups. In the sequel, varieties over a field $L$ will always mean separated schemes of finite type over $L$.

Function fields. Throughout this article, $K$ will be the function field of a smooth proper and geometrically integral curve $X$ over a $p$-adic field. For $v \in X$, we write $\mathcal{O}_{X,v}$ for the local ring at $v$ and $\kappa(v)$ for its residue field. Since $X,v$ is a discrete valuation ring for each $v \in X^{(1)}$, closed points will also be referred to places in the sequel. Moreover, $K_v$ (resp. $K_v^h$) will be the completion (resp. Henselization) of $K$ with respect to $v$ and $\mathcal{O}_v$ (resp. $\mathcal{O}_{X,v}$) will be the ring of integers in $K_v$ (resp. $K_v^h$). Note that the fields $K$ and $K_v$ have cohomological dimension 3.

Abelian groups. Let $A$ be an abelian group. We shall denote by $nA$ (resp. $A\{\ell\}$) for the $n$-torsion subgroup (resp. $\ell$-primary subgroup with $\ell$ prime) of $A$. Moreover, let $A_{\text{tors}}$ be the torsion subgroup of $A$, so $A_{\text{tors}} = \varprojlim nA$ is the direct limit of $n$-torsion groups of $A$. We write $A^\wedge$ for the profinite completion of $A$ (that is, the inverse limit of its finite quotients), $A_{\lambda} := \varprojlim A/nA$ and $A^{(\ell)} := \varprojlim A/\ell^n$ for the $\ell$-adic completion with $\ell$ a prime number. A torsion abelian group $A$ is of cofinite type if $nA$ is finite for each $n \geq 1$. If $A$ is $\ell$-primary torsion of cofinite type, then $A/\text{Div}A \simeq A^{(\ell)}$ where the former group is the quotient of $A$ by its maximal divisible subgroup. For a topological abelian group $A$, we write $A^D := \text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ for the group of continuous homomorphisms.

Tori. Let $L$ be a field of characteristic zero and let $\overline{L}$ be a fixed algebraic closure of $L$. We write $\hat{T}$ (resp. $\overline{T}$) for the character module (resp. cocharacter module) of a $L$-torus $T$. These are finitely generated free abelian groups endowed with a $\text{Gal}(\overline{L}/L)$-action, and moreover $\overline{T}$ is the $L$-linear dual of $\hat{T}$. The dual torus $T'$ of $T$ is the torus with character group $\hat{T}$, that is, $\hat{T'} = \hat{T}$. We say a torus $T$ is flasque if $H^1(L', \hat{T}) = 0$ for each finite extension $L'/L$ contained in $L$. A torus $T$ is quasi-trivial if it admits a $\text{Gal}(\overline{L}/L)$-invariant $L$-basis. Equivalently, a torus $T$ is quasi-trivial if it is of the form $\prod R_{L_i/L}G_m$ for some finite extensions $L_i/L$ contained in $L$, where the $R_{L_i/L}$ are various Weil restrictions.

Linear algebraic groups and homogeneous spaces. Let $L$ be a field of characteristic zero and let $\overline{L}$ be a fixed algebraic closure of $L$. Let $H$ be an algebraic group defined over $L$ and let $Y$ be an $L$-scheme endowed with an $H$-action. Then $Y$ is called a homogeneous space under $H$ if the $H(\overline{L})$-action on $Y(\overline{L})$ is transitive. A homogeneous space under $H$ is a $K$-torsor if the action is simply transitive. By definition, reductive algebraic groups will mean connected reductive groups. If $H$ is reductive, then we denote $H^{ss} := \mathcal{O}H$ for its derived subgroup (it is semi-simple). Let $H^\circ \to H^{ss}$ be the universal covering of $H^{ss}$ (it is a finite covering) with $H^\circ$ being simply connected. Finally, we say $H$ is quasi-split if it contains a Borel subgroup defined over the base field $L$. Equivalently, $H$ is quasi-split if and only if some parabolic subgroup is solvable (see [SGA3III, Section 3.9], or [Mil17, Chapter 17, Section I]).

Fundamental diagram associated to a flasque resolution. Let $G$ be a connected reductive linear group over a field $L$ of characteristic zero. By [CT08], there is an exact sequence of connected reductive groups: $1 \to R \to H \to G \to 1$
called flasque resolution of $G$, where $H$ is an extension of a quasi-trivial torus by the semi-simple simply connected $L$-group $G^{sc}$, and $R$ is a flasque $L$-torus central in $H$. Thus there is an exact sequence $1 \rightarrow G^{sc} \rightarrow H \rightarrow Q \rightarrow 1$ where $Q$ is a quasi-trivial torus. Note that we have identifications $G^{sc} \simeq H^{ss}$ and $Q \simeq H^{tor}$ by [CT08, 0.3].

Let $1 \rightarrow R \rightarrow H \rightarrow G$ be a flasque resolution of a connected reductive group $G$. Recall [CT08, pp. 15] that there is a commutative diagram associated with such a flasque resolution

\[
\begin{array}{c c c c}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & F & G^{sc} & G^{ss} & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & R & H & G & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & M & H^{tor} & G^{tor} & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 \\
\end{array}
\]

(3)

with exact rows and columns. In the bottom row of diagram (3), $G^{tor}$ and $H^{tor}$ are respective maximal toric quotient of $G$ and $H$, and kernel $M$ of $H^{tor} \rightarrow G^{tor}$ is a group of multiplicative type. Moreover, the torus $H^{tor}$ is quasi-trivial by construction. Finally, $F$ is the kernel of $R \rightarrow H^{tor}$. Note that being the kernel of $G^{sc} \rightarrow G^{ss}$, $F$ is finite and central in $G^{sc}$. It follows that $M$ is a torus, being the quotient of $R$ by the finite group $F$.

Let $\rho : G^{sc} \rightarrow G^{ss} \rightarrow G$ be the composition and let $T \subset G$ be a maximal $L$-torus. Then $T^{sc} := \rho^{-1}(T)$ is a maximal torus of $G^{sc}$. Applying [CT08, Appendice A] to the maximal torus $T$, we obtain a commutative diagram

\[
\begin{array}{c c c c}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & F & T^{sc} & T \cap G^{ss} & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & R & T^{tor} & T & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & M & H^{tor} & G^{tor} & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 \\
\end{array}
\]

(4)

with exact rows and columns, where $T^{tor} \subset H$ is a maximal torus of $H$. Recall that $H^{tor}$ is a quasi-trivial torus.

**Special coverings.** Let $L$ be a field of characteristic zero. Recall [San81] that an isogeny $G_0 \rightarrow G$ of connected reductive $L$-groups is called a special covering if $G_0$ is the product of a semi-simple simply connected group and a quasi-trivial torus. A special covering $G_0 \rightarrow G$ may be visualized by the following short exact sequence over $L$

\[
1 \rightarrow F \rightarrow H \times Q \rightarrow G \rightarrow 1,
\]

where $G_0 = H \times Q$ is the product of $H$ and $Q$ over $L$, $F$ is a finite $L$-group (thus central in $G_0$), $H$ is a semi-simple simply connected group, and $Q$ is a quasi-trivial torus. For each reductive $L$-group $G$, there exist an integer $m \geq 1$ and a quasi-trivial $L$-torus $Q$ such that $G^m \times Q$ admits a special covering by [San81, Lemme 1.10].

**Motivic complexes.** Let $Y$ be a variety over a field $L$. Bloch introduced a so-called cycle complex $z^i(Y, \bullet)$ in [Blo86]. When $Y$ is smooth, we denote the étale motivic complex over $Y$ by the complex of sheaves $\mathbb{Z}(i) := z^i(-, \bullet)[-2i]$ on the small étale site of $Y$. For example, we have quasi-isomorphisms $\mathbb{Z}(0) \simeq \mathbb{Z}$ and $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$ by [Blo86, Corollary 6.4]. We write $A(i) := A \otimes \mathbb{Z}(i)$ for any abelian group $A$. Finally, if $n$ is an integer invertible in $L$, then [GL01, Theorem 1.5] gives a quasi-isomorphism $\mathbb{Z}/n\mathbb{Z}(i) \simeq \mu_n^{\otimes i}$ where $\mu_n$ is concentrated in degree $0$. We shall write $\mathbb{Q}/\mathbb{Z}(i) := \lim_{\rightarrow} \mu_n^{\otimes i}$ for the direct limit of the sheaves $\mu_n^{\otimes i}$ for all $n \geq 1$.

\section{Dualities for short complex of tori}

Let $C = [T_1 \rightarrow T_2]$ be a short complex of $K$-tori concentrated in degree $-1$ and $0$ and let $C' = [T_2' \rightarrow T_1']$ be its dual. We fix some sufficiently small non-empty open subset $X_0$ of $X$ such that $T_1$ and $T_2$ extends to $X_0$-tori $T_i$.
and $T_2$ in the sense of [SGA3II], respectively. The complexes $C = [T_i \to T_2]$ and $C' = [T_2' \to T_1']$ over $X_0$ are defined analogously (these short complexes are concentrated in degree $-1$ and $0$). We put

$$\Pi^1_i(C) := \text{Ker} \left( \mathbb{H}^1(K, C) \to \prod_{v \in X^{(i)}} \mathbb{H}^1(K_v, C) \right).$$

For a finite set $S$ of places, we put

$$\Pi^1_S(C) := \text{Ker} \left( \mathbb{H}^1(K, C) \to \prod_{v \in S} \mathbb{H}^1(K_v, C) \right).$$

We write $\Pi^1_i(C)$ for elements in $\mathbb{H}^1(K, C)$ being trivial in all but finitely many $\mathbb{H}^1(K_v, C)$, that is, $\Pi^1_i(C) := \bigcup_S \Pi^1_S(C)$ with $S$ running through all finite set of places. For example, if $C = [0 \to T]$, then $C$ is quasi-isomorphic to $T$ and we have $\Pi^1(T) \simeq \Pi^1(C)$. If $C = [T \to 0]$, then $C \simeq T[1]$ and hence $\Pi^1(T) \simeq \Pi^1(C)$. Finally, by [Izq16, Lemme 1.4.3] there are respective natural pairings of complexes over $K$ and $X_0$:

$$C \otimes^L C' \to \mathbb{Z}(2)[3] \quad \text{and} \quad C \otimes^L C' \to \mathbb{Z}(2)[3].$$

We begin with a list of groups under consideration in the sequel.

**Lemma 1.1.** Let $T$ be a $K$-torus. Let $C$ and $C'$ as above. Let $U \subset X_0$ be a non-empty open subset.

1. The groups $\mathbb{H}^i(U, C \otimes^L \mathbb{Z}/n)$ and $\mathbb{H}^i_c(U, C \otimes^L \mathbb{Z}/n)$ are finite for $i \in \mathbb{Z}$.
2. The torsion groups $\mathbb{H}^i(U, C)_{\text{tors}}$ and $\mathbb{H}^i_c(U, C)_{\text{tors}}$ are of cofinite type.
3. The groups $\mathbb{H}^1(U, C)$ and $\mathbb{H}^1_c(U, C)$ are torsion of cofinite type for $i \geq 2$.
4. The group $H^1(K_v, T)$ is finite and the groups $\Pi^1(T)$ are finite for $i = 1, 2$.

**Proof.**

1. This is proved in [Izq16, Proposition 1.4.4].
2. The first statement is a consequence of the sequence

$$0 \to \mathbb{H}^i(U, C)/n \to \mathbb{H}^i(U, C \otimes^L \mathbb{Z}/n) \to n\mathbb{H}^i(U, C) \to 0$$

induced by the distinguished triangle $C \to C' \to C \otimes^L \mathbb{Z}/n \to C[1]$. The same argument works for $\mathbb{H}^i_c(U, C)_{\text{tors}}$.

3. By [HS16, Corollary 3.3 and Proposition 3.4(1)], the groups $H^i(U, T_2)$ and $H^{i+1}(U, T_1)$ are torsion of cofinite type for $i \geq 2$. Now we deduce that $\mathbb{H}^i(U, C)$ is torsion by the exactness of $H^i(U, T_2) \to \mathbb{H}^i(U, C) \to H^{i+1}(U, T_1)$. The group $\mathbb{H}^i(U, C)$ is of cofinite type thanks to the short exact sequence $0 \to \mathbb{H}^{i-1}(U, C)/n \to \mathbb{H}^{i-1}(U, C \otimes^L \mathbb{Z}/n) \to n\mathbb{H}^i(U, C) \to 0$. The same argument works for $\mathbb{H}^i_c(U, C)$.

4. The group $H^1(K_v, T)$ is finite because it has finite exponent and is of cofinite type (see [HS16, Proposition 2.2] for more details). The second statement is part of [HS16, Proposition 3.4(2)].

**1.1 An Artin–Verdier style duality**

Let $j_0 : X_0 \to X$ be the open immersion. We denote $\mathbb{H}^i_c(X_0, C) := \mathbb{H}^i(X_0, j_0^* C)$ for the compact support cohomology. The following result is some sort of variation of the classical Artin–Verdier duality theorem, which provides a more precise statement concerning the $\ell$-primary part.

**Proposition 1.2.** Let $U \subset X_0$ be any non-empty open subset. There is a pairing with divisible left kernel for each prime number $\ell$,

$$\mathbb{H}^1(U, C)(\ell) \times \mathbb{H}^1_c(U, C')(\ell) \to \mathbb{Q}/\mathbb{Z}.$$  

**Proof.** First, recall [Izq16, Proposition 1.4.4] that there is a perfect pairing of finite groups

$$\mathbb{H}^i(U, C \otimes^L \mathbb{Z}/n) \times \mathbb{H}^{i-1}_c(U, C' \otimes^L \mathbb{Z}/n) \to \mathbb{Q}/\mathbb{Z}$$

for $i \in \mathbb{Z}$. The pairing $C \otimes^L C' \to \mathbb{Z}(2)[3]$ induces a pairing $\mathbb{H}^i(U, C) \times \mathbb{H}^{i-1}_c(U, C') \to \mathbb{Q}/\mathbb{Z}$ by [HS16, Lemma 1.1]. In particular, we obtain pairings

$$\epsilon^n \mathbb{H}^1(U, C) \times \mathbb{H}^1_c(U, C')/\ell^n \to \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad \mathbb{H}^0(U, C)/\ell^n \times \epsilon^n \mathbb{H}^2(U, C') \to \mathbb{Q}/\mathbb{Z}$$
which fit into the following commutative diagram with rows being Kummer exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{H}^0(U, C)/\ell^n & \longrightarrow & \mathbb{H}^0(U, C \otimes \mathbb{L}/\ell^n) & \longrightarrow & \ell^n \mathbb{H}^1(U, C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & (\ell^n \mathbb{H}^2(U, C'))^D & \longrightarrow & \mathbb{H}^1_c(U, C' \otimes \mathbb{L}/\ell^n) & \longrightarrow & (\mathbb{H}^1_c(U, C')/\ell^n)^D & \longrightarrow & 0.
\end{array}
\]

Since the middle vertical arrow is an isomorphism by the pairing (5), we obtain an isomorphism by snake lemma

\[\mathbb{K}_n(U) \simeq \text{Coker} \left( \mathbb{H}^0(U, C)/\ell^n \to (\ell^n \mathbb{H}^2(U, C'))^D \right),\]

where \(\mathbb{K}_n(U) = \text{Ker} \left( \ell^n \mathbb{H}^1(U, C) \to (\mathbb{H}^1_c(U, C')/\ell^n)^D \right).\) Taking direct limit over all \(n\) yields an isomorphism

\[\lim_{\longrightarrow n} \mathbb{K}_n(U) \simeq \lim_{\longrightarrow n} \text{Coker} \left( \mathbb{H}^0(U, C)/\ell^n \to (\ell^n \mathbb{H}^2(U, C'))^D \right).\]

The latter limit is a quotient of the divisible group \(\lim_{\longrightarrow n} (\ell^n \mathbb{H}^2(U, C'))^D \simeq (\lim_{\longrightarrow n} \mathbb{H}^2(U, C'))^D\), so it is also divisible. Indeed, since \(\mathbb{H}^2(U, C')\) is a torsion group of cofinite type, the dual of its Tate module \(\lim_{\longrightarrow n} \mathbb{H}^2(U, C')\) is a direct sum of copies of \(\mathbb{Q}_\ell/\mathbb{Z}_\ell\), that is, \((\lim_{\longrightarrow n} \mathbb{H}^2(U, C'))^D\) is divisible. Being isomorphic to a quotient of the divisible group \((\lim_{\longrightarrow n} \mathbb{H}^2(U, C'))^D\), we see that \(\lim_{\longrightarrow n} \mathbb{K}_n(U)\) is divisible as well. Passing to the direct limit over all \(n\) yield an exact sequence (by definition of \(\mathbb{K}_n(U)\) and exactness of direct limit) of abelian groups

\[0 \to \lim_{\longrightarrow n} \mathbb{K}_n(U) \to \mathbb{H}^1(U, C)\{\ell\} \to (\mathbb{H}^1_c(U, C')^{(\ell)})^D\]

which guarantees the required pairing having divisible left kernel. \(\square\)

**Remark 1.3.** We shall see later in Theorem 1.17 that the direct limit \(\lim_{\longrightarrow n} \mathbb{K}_n(U)\) is contained in a finite group. So it vanishes (being finite and divisible), and thus there exists a non-empty open subset \(U_0\) of \(X_0\) such that the induced map \(\mathbb{H}^1(U, C)\{\ell\} \to (\mathbb{H}^1_c(U, C')^{(\ell)})^D\) is an isomorphism for each \(U \subset U_0\).

### 1.2 Local dualities for short complex of tori

In this subsection, we prove local dualities for the completion \(K_v\) and the Henselization \(K_v^h\) with respect to \(v\).

**Proposition 1.4** (Local dualities). Let \(\ell\) be a prime number.

1. There is a perfect pairing functorial in \(C\) between discrete and profinite groups:

   \[\mathbb{H}^1(K_v, C) \times \mathbb{H}^0(K_v, C')^\wedge \to \mathbb{Q}/\mathbb{Z}.\]

2. There is a perfect pairing functorial in \(C\) between finite groups:

   \[\ell^n \mathbb{H}^1(K_v, C) \times \mathbb{H}^0(K_v, C')/\ell^n \to \mathbb{Q}/\mathbb{Z}.\]

**Proof.**

1. The distinguished triangle \(T'_2 \to T'_1 \to C' \to T_2[1]\) induces an exact sequence

   \[H^0(K_v, T_2') \to H^0(K_v, T_1') \to \mathbb{H}^0(K_v, C') \to H^1(K_v, T_2') \to H^1(K_v, T_1').\]

   Since \(H^1(K_v, T_2')\) is finite by Lemma 1.1(4), there is an exact sequence

   \[H^0(K_v, T_2')^\wedge \to \mathbb{H}^0(K_v, C')^\wedge \to H^1(K_v, T_2') \to H^1(K_v, T_1').\]

   and a complex

   \[H^0(K_v, T_2')^\wedge \to H^0(K_v, T_1')^\wedge \to \mathbb{H}^0(K_v, C')^\wedge.\]

   Now the statement follows by exactly the same argument as [Izq16, Proposition 1.4.9(ii)].
(2) Consider the following exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{H}^0(K_v, C)/\ell^n & \longrightarrow & \mathbb{H}^0(K_v, C \otimes \mathbb{Z}/\ell^n) & \longrightarrow & \ell^n \mathbb{H}^1(K_v, C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\ell^n \mathbb{H}^1(K_v, C'))^D & \longrightarrow & \mathbb{H}^0(K_v, C' \otimes \mathbb{Z}/\ell^n)^D & \longrightarrow & (\mathbb{H}^0(K_v, C')/\ell^n)^D & \longrightarrow & 0
\end{array}
\]

with the middle vertical arrow being an isomorphism of finite groups (see the proof of [Izq16, Proposition 1.4.9(i)]). It follows that the right vertical arrow is surjective. Moreover, we have a commutative diagram

\[
\begin{array}{ccc}
\ell^n \mathbb{H}^1(K_v, C) & \longrightarrow & (\mathbb{H}^0(K_v, C')/\ell^n)^D \\
\downarrow & & \downarrow \\
\mathbb{H}^1(K_v, C) & \longrightarrow & (\mathbb{H}^0(K_v, C')\wedge)^D
\end{array}
\]

where the lower horizontal arrow is injective by (1). Therefore the upper horizontal arrow is also injective and hence it is an isomorphism.

\[\square\]

**Remark 1.5.** Taking direct limit over all \( n \) in Proposition 1.4(2) yields isomorphisms

\[\lim_{\longleftarrow} \ell^n \mathbb{H}^1(K_v, C) \simeq \left( \lim_{\longleftarrow} \mathbb{H}^0(K_v, C')/\ell^n \right)^D = (\mathbb{H}^0(K_v, C')\wedge)^D,\]

i.e. we can identify \( \mathbb{H}^0(K_v, C')\wedge \) with the profinite completion \( \mathbb{H}^0(K_v, C')\wedge \) by Proposition 1.4(1).

**Corollary 1.6.** Let \( \ell \) be a prime number.

1. There is a perfect pairing between discrete and profinite groups:

\[\mathbb{H}^1(K_v, C)\{\ell\} \times \mathbb{H}^0(K_v, C')^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.
\]

2. There is a perfect pairing between locally compact groups:

\[\left( \prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C) \right)\{\ell\} \times \left( \bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v, C') \right)^{(\ell)} \rightarrow \mathbb{Q}/\mathbb{Z}.
\]

More precisely, the former group is a direct limit of profinite groups and the latter is a projective limit of discrete torsion groups.

**Proof.** We apply the local duality Proposition 1.4(2), i.e. the isomorphism \( \ell^n \mathbb{H}^1(K_v, C) \simeq (\mathbb{H}^0(K_v, C')/\ell^n)^D \).

1. Passing to the direct limit over all \( n \) yields the isomorphism \( \mathbb{H}^1(K_v, C)\{\ell\} \simeq (\mathbb{H}^0(K_v, C')^{(\ell)})^D \).

2. Taking product over all places gives isomorphisms

\[\ell^n \left( \prod_v \mathbb{H}^1(K_v, C) \right) \simeq \prod_v (\mathbb{H}^0(K_v, C')/\ell^n)^D \simeq \left( \bigoplus_v (\mathbb{H}^0(K_v, C')/\ell^n)^D \right).
\]

Thus the desired perfect pairing follows by passing to the direct limit over all \( n \geq 1 \).

\[\square\]

**Lemma 1.7.** Let \( \ell \) be a prime number. Let \( T \) be a \( K \)-torus and let \( C = [T_1 \rightarrow T_2] \) be as above.

1. The natural map \( H^0(K_v^h, T) \rightarrow H^0(K_v, T) \) induces an isomorphism \( H^0(K_v^h, T)^{(\ell)} \simeq H^0(K_v, T)^{(\ell)}. \) Moreover, there is an isomorphism \( H^0(K_v^h, C)^{(\ell)} \simeq H^0(K_v, C)^{(\ell)}. \)

2. For \( i \geq 1 \), there is an isomorphism \( \mathbb{H}^i(K_v^h, C) \rightarrow \mathbb{H}^i(K_v, C). \)

**Proof.**
(1) The same argument as [Dem11, Lemme 3.7] yields an isomorphism $H^0(K_v h, T)/\ell^n \simeq H^0(K_v, T)/\ell^n$. Therefore the first assertion follows by passing to the inverse limit over all $n$. For the second statement, since $H^1(K_v h, T_i) \simeq H^1(K_v, T_i)$ is finite for $i = 1, 2$, there is a commutative diagram of complexes with rows exact at the last four terms

$$
\begin{array}{cccccc}
H^0(K_v h, T_1) & \to & H^0(K_v h, T_2) & \to & H^0(K_v h, C) & \to & H^1(K_v h, T_1) & \to & H^1(K_v h, T_2) \\
\downarrow & & & & & & & & \\
H^0(K_v, T_1) & \to & H^0(K_v, T_2) & \to & H^0(K_v, C) & \to & H^1(K_v, T_1) & \to & H^1(K_v, T_2).
\end{array}
$$

Now all the vertical arrows except the middle one are isomorphisms, and hence the middle one is also an isomorphism by the 5-lemma.

(2) By [HS16, Corollary 3.2], we know that $H^i(K_v h, T) \simeq H^i(K_v, T)$ for each $i \geq 1$ and for each $K$-torus $T$. Thus the isomorphism $\mathbb{H}^i(K_v h, C) \simeq \mathbb{H}^i(K_v, C)$ for each $i \geq 1$ follows after applying dévissage to the distinguished triangle $T_1 \to T_2 \to C \to T_1[1]$. □

**Corollary 1.8.** There is a perfect pairing between direct limit of profinite groups and projective limit of discrete torsion groups:

$$
\left( \prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C) \right) \ell \times \left( \bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v h, C') \right)^{\ell} \to \mathbb{Q}/\mathbb{Z}.
$$

**Proof.** The same argument as Proposition 1.4 yields a perfect pairing $\ell^n \mathbb{H}^1(K_v h, C) \times \mathbb{H}^0(K_v h, C')/\ell^n \to \mathbb{Q}/\mathbb{Z}$ of finite groups. Therefore Lemma 1.7(2) implies that $\mathbb{H}^0(K_v h, C')/\ell^n \simeq \mathbb{H}^0(K_v, C')/\ell^n$. The desired perfect pairing is an immediate consequence by the same argument as Corollary 1.6(2). □

### 1.3 Global dualities for short complex of tori

The goal of this subsection is to establish a perfect pairing $\text{III}^1(C) \times \text{III}^1(C') \to \mathbb{Q}/\mathbb{Z}$ between finite groups. We first prove the finiteness of $\text{III}^1(C)$ and $\text{III}^1(C')$.

**Lemma 1.9.** Let $C = [T_1 \to T_2]$ be a short complex of tori. The group $\text{III}^1(C)$ is of finite exponent.

**Proof.** Let $L/K$ be a finite Galois extension that splits both $T_1$ and $T_2$. Then for $i = 1, 2$, the $L$-tori $T_i, L = T_i \times_K L$ are products of $\mathbb{G}_m$, and $H^1(L, T_{2, L}) = 0$ by Hilbert’s theorem 90. The distinguished triangle $T_1 \to T_2 \to C \to T_1[1]$ induces a commutative diagram

$$
\begin{array}{ccc}
\mathbb{H}^1(K, C) & \to & H^2(K, T_1) \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
0 & \to & \mathbb{H}^1(L, C_L) & \to & H^2(L, T_{1, L})
\end{array}
$$

where $C_L = [T_{1, L} \to T_{2, L}]$. Recall [HSS15, Lemma 3.2(2)] that $\text{III}^2(T_1)$ is of finite exponent, and hence a restriction-corestriction argument shows that $\text{III}^1(C)$ is of finite exponent. □

The following statement on compact support hypercohomology is the analogue of [HS16, Proposition 3.1].

**Proposition 1.10.** Let $U \subset X_0$ be a non-empty open subset.

1. Let $V \subset U$ be a further non-empty open subset. There is an exact sequence

$$
\cdots \to \mathbb{H}^i_c(V, \mathcal{C}) \to \mathbb{H}^i_c(U, \mathcal{C}) \to \bigoplus_{v \in U \setminus V} \mathbb{H}^i(K(v), i_v^* \mathcal{C}) \to \mathbb{H}^{i+1}_c(V, \mathcal{C}) \to \cdots
$$

where $i_v : \text{Spec} \, K(v) \to U$ is the closed immersion.

2. There is an exact sequence of hypercohomology groups

$$
\cdots \to \mathbb{H}^i_c(U, \mathcal{C}) \to \mathbb{H}^i(U, \mathcal{C}) \to \bigoplus_{v \notin U} \mathbb{H}^i(K_v^h, \mathcal{C}) \to \mathbb{H}_c^{i+1}(U, \mathcal{C}) \to \cdots
$$

where $K_v^h$ is the Henselization of $K$ with respect to the place $v$ and by abuse of notation we write $C$ for the pull-back of $\mathcal{C}$ by the natural morphism $\text{Spec} \, K_v^h \to U$. 

Lemma 1.11. Let $U \subset X_0$ be a non-empty open subset, and put $D_K^2(U, T) = \operatorname{Im}(H^2(U, T) \to H^2(K, T))$ for any $U$-torus $T$. Then there exists a non-empty open subset $U_2 \subset X_0$ such that $D_K^2(U, T)$ is of finite exponent for any non-empty open subset $U \subset U_2$.

Proof. By a restriction-corestriction argument, it will be sufficient to show that $D_K^2(U, \mathbb{G}_m)$ is of finite exponent. By [Gro68, pp. 96, (2.9)], there is an exact sequence

$$H^0(k, \operatorname{Pic}_{X/k}) \to \text{Br } k \to \text{Br } X \to H^1(k, \operatorname{Pic}_{X/k}) \to H^1(k, \mathbb{G}_m).$$

Since $k$ is a $p$-adic local field, we conclude $\text{Br } X / \text{Br } k \simeq H^1(k, \operatorname{Pic}_{X/k})$ for cohomological dimension reasons. Recall that there is a canonical short exact sequence $0 \to \operatorname{Pic}_{X/k}^0 \to \operatorname{Pic}_{X/k} \to \mathbb{Z} \to 0$, thus $H^1(k, \operatorname{Pic}_{X/k})$ is a quotient of $H^1(k, \operatorname{Pic}_{X/k}^0)$ where $\operatorname{Pic}_{X/k}^0$ is an abelian variety. But $H^1(k, \operatorname{Pic}_{X/k}^0)$ is dual to $\operatorname{Pic}_{X/k}^0(k)$ by Tate duality over local fields [Mil06, Chapter I, Corollary 3.4], we deduce that $H^1(k, \operatorname{Pic}_{X/k}^0) \simeq F_0 \bigoplus (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus r}$ with $F_0$ a finite abelian group by Mattuck’s theorem (see [Mat55] and [Mil06, pp. 41]).

Suppose first there is a rational point $e \in X(k)$ on $X$. Let $e^* : \text{Br } X \to \text{Br } k \simeq \mathbb{Q} / \mathbb{Z}$ be the induced map and put $\text{Br}_e X = \{ \alpha \in \text{Br } X \mid e^*(\alpha) = 0 \}$. Note that in this case the map $\text{Br } k \to \text{Br } X$ induced by the structural morphism $X \to \text{Spec } k$ is injective and there is an isomorphism $H^1(k, \operatorname{Pic}_{X/k}) \simeq H^1(k, \operatorname{Pic}_{X/k})$. Thus there is a split short exact sequence $0 \to \operatorname{Pic}_{X/k}^0 \to \operatorname{Pic}_{X/k} \to \mathbb{Z} \to 0$, and consequently $\text{Br}_e X \simeq \text{Br } X / \text{Br } k$. Moreover, if $e \notin U$, then $D_K^2(U, \mathbb{G}_m) \subset \ker(Br U \to \bigoplus_{\ell \notin U} \text{Br } k_e) \subset \text{Br } X$. It follows that there is an injective map $D_K^2(U, \mathbb{G}_m) \to F_0 \bigoplus (\mathbb{Q}_p / \mathbb{Z}_p)^{\oplus r}$. Next, we show that there is a non-empty open subset $U_2 \subset X_0$ such that the decreasing sequence $\{D_K^2(U, \mathbb{G}_m)\{\ell\}\}$ is stable for $U \subset U_2$. By [HS16, Proposition 3.4], the group $H^2(U, \mathbb{G}_m)$ is of cofinite type and hence so is $D_K^2(U, \mathbb{G}_m)$. Since $F_0$ is finite, there exists only finitely many $\ell \neq p$ such that $\ell$ divides the order of $F_0$. As a consequence, there exists a non-empty open subset $U_2 \subset X_0$ (which is independent of $\ell$) such that $D_K^2(U, \mathbb{G}_m)\{\ell\} = D_K^2(U_2, \mathbb{G}_m)\{\ell\}$ holds for any non-empty open subset $U \subset U_2$ and for each $\ell \neq p$ by [HS16, Lemma 3.7]. Again the decreasing sequence $\{D_K^2(U, \mathbb{G}_m)\{p\}\}_{U \subset U_2}$ stabilizes, so there exists some $U_2 \subset U_1$ such that $D_K^2(U_2, \mathbb{G}_m) = D_K^2(U_2, \mathbb{G}_m)$ for all $U \subset U_2$. Letting $U$ run through all non-empty open subsets of $U_2$ yields $D_K^2(U_2, \mathbb{G}_m) = \operatorname{III}^2(\mathbb{G}_m) = 0$, where the vanishing of $\operatorname{III}^2(\mathbb{G}_m)$ is a consequence of [HSS15, Lemma 3.2].

Therefore, there exists a finite Galois extension $k'/k$ such that $X(k') \neq \emptyset$. Put $X' := X \times_k k'$ and $U' := U \times_k k'$. We know that $D_K^2(U', \mathbb{G}_m) \subset \operatorname{Br } K'$ is zero by the previous paragraph (where $K'$ is the function field of $X'$). Therefore a restriction-corestriction argument implies that $D_K^2(U, \mathbb{G}_m) \subset \text{Br } K$ is of finite exponent.
Proposition 1.12. We put $\mathbb{D}_K(U,C) := \text{Im} \left( \mathbb{H}^1_c(U,C) \to \mathbb{H}^1(K,C) \right)$. Then there exists a non-empty open subset $U_0$ of $X_0$ such that

$$\mathbb{D}^1_K(U_0,C) = \mathbb{D}^1_K(U_0,C) = \mathbb{III}^1(C).$$

for each non-empty open subset $U \subset U_0$. Moreover, the group $\mathbb{III}^1(C)$ is finite.

Proof. By Lemma 1.11, the group $\mathbb{D}_K^1(U,T_i)$ is of finite exponent for $U$ sufficiently small. Since $H^1(K,T_2)$ is of finite exponent, it follows that $\mathbb{D}^1_K(U,T_i)$ is of finite exponent (say $N$) by d\'evissage. In particular, the epimorphism $\mathbb{H}^1_c(U,C) \to \mathbb{H}^1_c(U,C)$ factors through $\mathbb{H}^1_c(U,C) \to \mathbb{H}^1_c(U,C)/N$. Recall that $\mathbb{H}^1_c(U,C)/N$ is a subgroup of the finite group $\mathbb{H}^1_c(U,C) \otimes_{\mathbb{Z}/N}$, hence its quotient $\mathbb{D}_K^1(U,C)$ is finite.

For non-empty open subsets $V \subset U \subset X_0$ of $X_0$, we have $\mathbb{D}^1_K(V,C) \subset \mathbb{D}^1_K(U,C)$ by covariant functoriality of $\mathbb{H}^1_c(-,C)$. The decreasing sequence $\{\mathbb{D}^1_K(U,C)\}_{U \subset X_0}$ of finite abelian groups must be stable, hence there exists a non-empty open subset $U_0$ of $X_0$ such that $\mathbb{D}^1_K(U_0,C) = \mathbb{D}^1_K(U_0,C)$ for each non-empty open subset $U \subset U_0$. Note that $\mathbb{D}^1_K(U_0,C) \subset \text{Ker} \left( \mathbb{H}^1(K,C) \to \prod_{U \notin U_0} \mathbb{H}^1(K_v,C) \right)$ by Proposition 1.10(3). Letting $U$ run through all non-empty open subset $U_0$ implies that $\mathbb{D}^1_K(U_0,C) = \mathbb{D}^1_K(U_0,C) = \mathbb{III}^1(C)$. Since the former two groups are finite, so is $\mathbb{III}^1(C)$.

Applying Proposition 1.12 to both $C$ and $C'$, we obtain a non-empty open subset $U_0$ of $X$ such that (6) holds for both $C$ and $C'$. In the sequel, we fix such a non-empty open subset $U_0$ of $X$.

We will need an auxiliary Grothendieck–Serre conjecture style result (for example, see [CTS87, Theorem 4.1]). Let $\mathcal{O}$ be a regular local integral domain with fraction field $\Omega$. Let $\mathcal{J} = [\mathcal{T}_1 \to \mathcal{T}_2]$ be a complex of $\mathcal{O}$-tori concentrated in degree $-1$ and $0$. Let $\mathcal{T}_i = \mathcal{T} \times_{\mathcal{O}} \Omega$ be the generic fibre of $\mathcal{T}_i$ for $i = 1, 2$ and let $\mathcal{J} = [\mathcal{T}_1 \to \mathcal{T}_2]$ be the associated complex in degree $-1$ and $0$.

Proposition 1.13. The natural homomorphism $\mathbb{H}^1(\mathcal{O}, \mathcal{J}) \to \mathbb{H}^1(\Omega, \mathcal{J})$ induced by $\mathcal{O} \subset \Omega$ is injective.

Proof. Let $q : Q \to \mathcal{T}_2$ be an epimorphism of $\mathcal{O}$-tori with $Q$ being quasi-trivial (for example, we may take a flasque resolution of $\mathcal{T}_2$, see [CTS87, (1.3.3)]). Let $Q \times_{\mathcal{T}_2} \mathcal{T}_1 \to \mathcal{T}_2$ be the map $(r, t_1) \mapsto q(r)\rho(t_1)^{-1}$ and let $M$ be its kernel. Note that $M$ is a group of multiplicative type. Let $p_{r_1} : M \to Q \times_{\mathcal{T}_2} \mathcal{T}_1 \to Q$ and $p_{r_2} : M \to Q \times_{\mathcal{T}_2} \mathcal{T}_1 \to \mathcal{T}_1$ be the respective canonical projections. By construction of $M$, we have $q \circ p_{r_1} \simeq \rho \circ p_{r_2}$. A direct verification yields isomorphisms $\text{Ker}p_{r_1} \simeq \text{Ker}\rho$ and $\text{Coker}p_{r_1} \simeq \text{Coker}\rho$, that is, $\mathcal{J}_0 = [M \to \mathcal{Q}]$ is quasi-isomorphic to $\mathcal{J} = [\mathcal{T}_1 \to \mathcal{T}_2]$. Note that $Q = Q \times_{\mathcal{O}} \Omega$ is a quasi-trivial $\Omega$-torus and being faithfully flat is stable under base change, the same argument as above yields that $\mathcal{J}_0 = [M \to \mathcal{T}]$ with $M = M \otimes_{\mathcal{O}} \Omega$ is quasi-isomorphic to the complex $J = [\mathcal{T}_1 \to \mathcal{T}_2]$. Thus it suffices to show $\mathbb{H}^1(\mathcal{O}, \mathcal{J}_0) \to \mathbb{H}^1(\Omega, \mathcal{J}_0)$ is injective.

By construction $Q$ is a quasi-trivial $\mathcal{O}$-torus, thus $H^1(Q, Q) = H^1(\Omega, Q) = 0$ by Shapiro’s lemma and Hilbert’s theorem 90. Consequently it will be sufficient to show $H^2(\mathcal{O}, M) \to H^2(\Omega, M)$ is injective by d\'evissage to the distinguished triangle $M \to Q \to \mathcal{J}_0 \to M[1]$. Take an exact sequence $1 \to M \to Q \to \mathcal{J}_0 \to M[1]$ over $\mathcal{O}$ with $Q_1$ a quasi-trivial $\mathcal{O}$-torus and $Q_2$ an $\mathcal{O}$-torus (for example, see [CTS87, pp. 158, (1.3.1)]). It induces the commutative diagram below with exact rows

$$
\begin{align*}
0 & \to H^1(\mathcal{O}, Q_2) \to H^2(\mathcal{O}, M) \to H^2(\Omega, M) \\
0 & \to H^1(\Omega, Q_2) \to H^2(\Omega, M) \to H^2(\Omega, Q_1)
\end{align*}
$$

where $Q_i = Q_i \times_{\mathcal{O}} \Omega$ for $i = 1, 2$. The left vertical arrow is injective by [CTS87, Theorem 4.1] and the right one is injective by Shapiro’s lemma and the injectivity for Brauer groups, therefore the middle one is also injective.

Corollary 1.14. The homomorphism $\Phi_v : \mathbb{H}^1(\mathcal{O}^h_{U,v}, C) \to \mathbb{H}^1(K^h_v, C)$ induced by the inclusion $\mathcal{O}^h_{U,v} \to K^h_v$ is injective for $v \in U^{(1)}$.

Proof. Taking $\mathcal{O} = \mathcal{O}_v^h$, $\Omega = K_v^h$ and $\mathcal{J} = \mathcal{C}$ in Proposition 1.13 yields the desired injectivity.

To state a key step, we first construct a map $\mathbb{D}^1_{\mathcal{O}_v^h}(K^h_v, C) \to \mathbb{H}^1(\mathcal{O}_v^h, C)$ for some non-empty open subset $U$ of $X_0$. Take $a \in \mathbb{D}^1_{\mathcal{O}_v^h}(K^h_v, C)$ supported outside some non-empty open subset $V$ of $U$, i.e., $a_v = 0$ for $v \in V$. Applying Proposition 1.10(2) to $V$ sends $a$ to $\mathbb{H}^1(\mathcal{O}_v^h, C)$, and so $a$ is sent to $\mathbb{H}^1(\mathcal{O}_v^h, C)$ by the covariant functoriality of $\mathbb{H}^1_c(-, C)$. The construction is independent of the choice of $V$ by the same argument as [BS16, pp. 11, (12)].
Proposition 1.15. There is an exact sequence

$$\bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v^h, C) \rightarrow \mathbb{H}^1_c(U, C) \rightarrow \mathbb{D}_K^1(U, C) \rightarrow 0.$$  

Proof. The sequence is a complex by exactly the same argument of [HS16, Proposition 4.2]. The surjectivity of the last arrow is just the definition of $\mathbb{D}_K^1(U, C)$. Take $\alpha \in \text{Ker} (\mathbb{H}^1_c(U, C) \rightarrow \mathbb{D}_K^1(U, C))$ and a non-empty open subset $V \subset U$. Consider the diagram

$$\begin{array}{ccc}
\mathbb{H}^1_c(V, C) & \rightarrow & \mathbb{H}^1_c(U, C) \\
\downarrow & & \downarrow \\
\mathbb{H}^1(K, C) & \rightarrow & \bigoplus_{v \in U \setminus V} \mathbb{H}^1(K_v, C)
\end{array}$$

where the right vertical arrow is constructed as the composite

$$\mathbb{H}^1(K, C) \rightarrow \mathbb{H}^1(K, C) \rightarrow \bigoplus_{v \in U \setminus V} \mathbb{H}^1(K_v, C).$$

The diagram commutes for the same reason as in the proof of [HS16, Proposition 4.2]. Since the right vertical arrow is injective, a diagram chasing shows that $\alpha$ comes from $\mathbb{H}^1_c(V, C)$. But $\alpha$ goes to zero in $\mathbb{H}^1(K, C)$, we may take $V$ sufficiently small such that $\alpha$ already goes to zero in $\mathbb{H}^1(V, C)$, i.e. $\alpha$ comes from $\bigoplus_{v \in V} \mathbb{H}^0(K_v^h, C)$ by Proposition 1.10(2) and hence the desired sequence is indeed exact. \qed

The next lemma is probably well-known:

Lemma 1.16. Let $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be an exact sequence of abelian groups. If $A_3$ is finite, then $A_1^{(\ell)} \rightarrow A_2^{(\ell)} \rightarrow A_3^{(\ell)} \rightarrow 0$ is exact for each prime number $\ell$. In particular, there is an exact sequence

$$\left( \bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v^h, C) \right)^{\ell} \rightarrow \mathbb{H}^1_c(U, C)^{\ell} \rightarrow \mathbb{D}_K^1(U, C)^{\ell} \rightarrow 0.$$  

Proof. Let’s say $f : A_1 \rightarrow A_2$, $g : A_2 \rightarrow A_3$ and $g_n : A_2/\ell^n \rightarrow A_3/\ell^n$. Thus there is a short exact sequence $0 \rightarrow \text{Ker} g_n \rightarrow A_2/\ell^n \rightarrow A_3/\ell^n \rightarrow 0$. Since $\text{Ker} g_n$ is a quotient of $A_1/\ell^n$, $\{\text{Ker} g_n\}$ forms a surjective system in the sense of [AM69, Proposition 10.2] and it follows that $0 \rightarrow \varinjlim \text{Ker} g_n \rightarrow A_2^{(\ell)} \rightarrow A_3^{(\ell)} \rightarrow 0$ is exact. By the snake lemma, there is an exact sequence $0 \rightarrow \ell \cdot \text{Ker} g \rightarrow \ell \cdot A_2 \rightarrow \ell \cdot A_3 \rightarrow (\text{Ker} g)/\ell^n \rightarrow \text{Ker} g_n \rightarrow 0$. But $\ell \cdot A_3$ is finite by assumption, we conclude that $(\text{Ker} g)^{(\ell)} \rightarrow \varinjlim \text{Ker} g_n$ is surjective by Mittag-Leffler condition. Finally, let $\text{Ker} f_n := \text{Ker} (A_1/\ell^n \rightarrow (\text{Ker} g)/\ell^n)$. Then $\{\text{Ker} f_n\}$ is a surjective system (because $\text{Ker} f/\ell^n \rightarrow \text{Ker} f_n$ is surjective), and hence $A_1^{(\ell)} \rightarrow (\text{Ker} g)^{(\ell)}$ is surjective. Summing up, the sequence $A_1^{(\ell)} \rightarrow A_2^{(\ell)} \rightarrow A_3^{(\ell)} \rightarrow 0$ is exact. \qed

Now we arrive at the global duality of the short complex $C$.

Theorem 1.17. There is a perfect, functorial in $C$, pairing of finite groups:

\[ \mathbb{H}^1(C) \times \mathbb{H}^1(C') \rightarrow \mathbb{Q}/\mathbb{Z}. \]

Proof. We proceed by constructing a perfect pairing of finite groups $\mathbb{H}^1(C) \times \mathbb{H}^1(C') \rightarrow \mathbb{Q}/\mathbb{Z}$ for a fixed prime $\ell$. Define $\mathbb{D}_K^1(U, C)$ by the exact sequence $0 \rightarrow \mathbb{D}_K^1(U, C) \rightarrow \mathbb{H}^1(U, C) \rightarrow \prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C)$ for each $U \subset U_0$. Dualizing the exact sequence in Proposition 1.15 yields the following commutative diagram with exact rows

$$\begin{array}{cccc}
0 & \rightarrow & \mathbb{D}_K^1(U, C)^{\ell} & \rightarrow & \mathbb{H}^1(U, C)^{\ell} & \rightarrow & \left( \prod_{v \in X^{(1)}} \mathbb{H}^1(K_v, C)^{\ell} \right) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (\mathbb{D}_K^1(U, C')^{\ell})^D & \rightarrow & (\mathbb{H}^1_c(U, C')^{\ell})^D & \rightarrow & (\left( \bigoplus_{v \in X^{(1)}} \mathbb{H}^0(K_v^h, C') \right)^{\ell})^D.
\end{array}$$

The first vertical arrow is induced by $\mathbb{H}^1(U, C)^{\ell} \rightarrow (\mathbb{H}^1_c(U, C')^{\ell})^D$ in view of the commutativity of the right square. By local duality Corollary 1.8, the right vertical arrow is an isomorphism, and it follows that the kernels of
the first two vertical arrows are identified. Passing to the direct limit of the dashed arrow induces an exact sequence of abelian groups

\[ 0 \to \lim_{U} \mathbb{K}_{n}(U) \to \lim_{U} \mathbb{D}^{1}_{\text{sh}}(U, \mathcal{C})(\ell) \to \lim_{U} \left( \mathbb{D}^{1}_{K}(U, \mathcal{C}')(\ell) \right)^{D}. \]

Recall that \( \lim_{U} \mathbb{K}_{n}(U) \) is the divisible kernel of the middle vertical arrow introduced in Proposition 1.2. Note that the second limit is just \( \mathbb{III}^{1}(C)(\ell) \) by definition of \( \mathbb{D}^{1}_{\text{sh}}(U, \mathcal{C}) \). In particular, the first limit is trivial being a divisible subgroup of a finite abelian group. Now we can conclude the following isomorphisms of finite abelian groups

\[ \mathbb{D}^{1}_{K}(U, \mathcal{C}')(\ell) = \mathbb{D}^{1}_{K}(U, \mathcal{C}')(\ell) \cong \mathbb{III}^{1}(C')(\ell) \cong \mathbb{III}^{1}(C'(\ell)^{D}) \text{ follows by exchanging the role of } C \text{ and } C'. \]

2 Obstruction to weak approximation via special covering

Let \( G \) be a connected reductive group. Let \( G^{ss} = \mathcal{G} \) be the derived subgroup of \( G \) and let \( q : G^{sc} \to G^{ss} \) be the universal cover of \( G^{ss} \). We consider the composition \( \rho : G^{sc} \to G^{ss} \to G \). Let \( T \subset G \) be a maximal torus over \( K \) and let \( T^{sc} = \rho^{-1}(T) \). Recall that \( T^{sc} \) is a maximal torus of \( G^{sc} \). We apply the above dualities to the morphism \( \rho : T^{sc} \to T \), i.e. to the complexes \( C = [T^{sc} \to T] \) and \( C' = [T' \to (T^{sc})'] \) concentrated in degree \(-1\) and \( 0 \). We first observe that the group \( G^{sc} \) satisfies weak approximation with respect to any finite set \( S \subset X^{(1)} \) of places.

Proposition 2.1. Let \( H \) be a quasi-split semi-simple and simply connected group over \( K \). Then \( H \) satisfies weak approximation with respect to any finite set \( S \subset X^{(1)} \) of places.

Proof. Let \( B \) be a Borel subgroup of \( H \) defined over \( K \) and let \( P \) be a maximal \( K \)-torus contained in \( B \). Applying [HS16, Lemma 6.7 and its proof] implies that \( \tilde{P} \simeq \tilde{B} \simeq \text{Pic}(H/B) \) is a permutation module, i.e. \( P \) is a quasi-trivial torus. Moreover, \( P \) is a Levi subgroup of \( B \) by [BT65, Corollaire 3.14]. Now [Tha96, Corollary 1.5] yields a bijection from the defect of weak approximation \( \prod_{v \in S} H(K_{v})/H(K)_{S} \) for \( H \) to that \( \prod_{v \in S} P(K_{v})/P(K)_{S} \) for \( P \) with respect to any finite set \( S \) of places. Here \( H(K)_{S} \) (resp. \( P(K)_{S} \)) denotes the closure of \( H(K) \) (resp. \( P(K) \)) in \( H(K) \) (resp. \( P(K) \)) with respect to the product of \( v \)-adic topologies.

If \( G^{sc} \) is a quasi-split simply connected reductive group, then we may choose a quasi-trivial maximal torus \( T^{sc} \subset G^{sc} \) by [HS16, Lemma 6.7]. Because \( q : G^{sc} \to G^{ss} \) is an epimorphism, we conclude that \( q(T^{sc}) \) is a maximal torus of \( G^{ss} \) by [Hum75, §21.3, Corollary C]. Therefore, we may choose a maximal torus \( T \subset G \) of \( G \) such that \( T \cap G^{ss} = q(T^{sc}) \), i.e. \( T^{sc} = \rho^{-1}(T) \) for \( \rho : G^{sc} \to G^{ss} \to G \). By [Bor98, Section 2.4], different choice of \( [T^{sc} \to T] \) gives rise to the same hypercohomology group and thus we are allowed to chose a quasi-trivial maximal torus \( T^{sc} \).

Theorem 2.2. Let \( G \) be a connected reductive group such that \( G^{sc} \) is quasi-split.

(1) Let \( S \subset X^{(1)} \) be a finite set of places. There is an exact sequence

\[ 1 \to \overline{G(K)}_{S} \to \prod_{v \in S} G(K_{v}) \to \mathbb{III}^{1}_{\omega}(C')^{D} \to \mathbb{III}^{1}(C) \to 1. \]  

Here \( \overline{G(K)}_{S} \) denotes the closure of the diagonal image of \( G(K) \) in \( \prod_{v \in S} G(K_{v}) \) for the product topology.

(2) There is an exact sequence

\[ 1 \to \overline{G(K)} \to \prod_{v \in X^{(1)}} G(K_{v}) \to \mathbb{III}^{1}_{\omega}(C')^{D} \to \mathbb{III}^{1}(C) \to 1. \]  

Here \( \overline{G(K)} \) denotes the closure of the diagonal image of \( G(K) \) in \( \prod_{v \in X^{(1)}} G(K_{v}) \) for the product topology.

Example 2.3. Let us first look at two special cases of the sequence (7).

(1) If \( G \) is semi-simple, then there is an exact sequence \( 1 \to F \to G^{sc} \to G \to 1 \) with \( F \) finite and central in \( G^{sc} \). In particular, there are exact sequences

\[ 1 \to F \to T^{sc} \to T \to 1 \text{ and } 1 \to F' \to T' \to (T^{sc})' \to 1. \]
Here $F' = \text{Hom}(F, \mathbb{Q}/\mathbb{Z}(2))$ and the latter sequence is obtained from the dual isogeny of $T^\text{sc} \to T$. Consequently there are quasi-isomorphisms $C \simeq F[1]$ and $C' \simeq F'[1]$, and hence $\text{III}_{S}(C) \simeq \text{III}_{S}^{2}(F)$ and $\text{III}_{S}(C') \simeq \text{III}_{S}^{2}(F')$. Therefore the exact sequence (7) reads as

$$1 \to G(K)_{S} \to \prod_{v \in S} G(K_{v}) \to \text{III}_{S}^{2}(F') \to \text{III}_{S}^{2}(F) \to 1.$$ 

Here the second arrow is given by the composite of the coboundary map $G(K_{v}) \to H^{1}(K_{v}, F)$ and the local duality $H^{1}(K_{v}, F) \times H^{2}(K_{v}, F') \to \mathbb{Q}/\mathbb{Z}$, and the last one is given by the global duality $\text{III}_{S}^{2}(F) \times \text{III}_{S}^{2}(F') \to \mathbb{Q}/\mathbb{Z}$ for finite Galois modules (see [HS16, (10) and Theorem 4.4] for details).

(2) If $G = T$ is a torus, then $C = [T^\text{sc} \to T]$ is quasi-isomorphic to the complex $[0 \to T] \simeq T$ and its dual $C' = [T' \to (T'^{\text{sc}})]$ is quasi-isomorphic to the complex $[T' \to 0] \simeq T'[1]$. So we have $\text{III}_{S}(C) \simeq \text{III}_{S}(T)$ and $\text{III}_{S}(C') \simeq \text{III}_{S}^{2}(T')$. Now the exact sequence (7) is of the following form

$$1 \to G(K)_{S} \to \prod_{v \in S} G(K_{v}) \to \text{III}_{S}^{2}(T') \to \text{III}_{S}^{2}(T) \to 1.$$ 

This is the obstruction to weak approximation for tori given by Harari and Szamuely in [HSS15].

The rest of this section is devoted to the proof of Theorem 2.2. For a field $L$ and $i = 0, 1$, we shall denote by $\text{ab}^{i} : H^{i}(L, G) \to \mathbb{H}^{i}(L, C)$ the abelianization map in the sequel. The construction of the abelianization map is set forth in [Bor98, Section 3].

**Lemma 2.4.** Let $L$ be a field of characteristic zero and let $G$ be a connected reductive group over $L$ such that $G^{\text{sc}}$ is quasi-split. Then the canonical map $\text{ab}^{0} : H^{0}(L, G) \to \mathbb{H}^{0}(L, C)$ is surjective.

**Proof.** Let $A \to B$ be a complex of (not necessarily abelian) $\text{Gal}(\overline{L}/L)$-groups concentrated in degree $-1$ and $0$. We write $\mathbb{H}_{\text{rel}}^{0}(L, [A \to B])$ for the non-abelian hypercohomology in the sense of [Bor98, section 3.7]. Now we view $\rho : T^{\text{sc}} \to T$ and $\rho : G^{\text{sc}} \to G$ as complexes of $\text{Gal}(\overline{L}/L)$-groups concentrated in degree $-1$ and $0$. Recall that we may choose $T^{\text{sc}}$ to be quasi-trivial. In particular, we have a commutative diagram

$$
\begin{array}{ccc}
[T^{\text{sc}} \to T] & \longrightarrow & [T^{\text{sc}} \to 1] \\
\downarrow & & \downarrow \\
[G^{\text{sc}} \to G] & \longrightarrow & [G^{\text{sc}} \to 1]
\end{array}
$$

of complexes of $\text{Gal}(\overline{L}/L)$-groups. Applying the functor $\mathbb{H}_{\text{rel}}^{0}(\text{Gal}(\overline{L}/L), -)$ with values in the category of pointed sets and taking into account the identification $\mathbb{H}_{\text{rel}}^{0}(\text{Gal}(\overline{L}/L), [A \to 1]) \simeq H^{1}(\text{Gal}(\overline{L}/L), A)$ (see [Bor98, Example 3.1.2(2)]), we obtain the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{H}_{\text{rel}}^{0}(L, [T^{\text{sc}} \to T]) & \longrightarrow & H^{1}(L, T^{\text{sc}}) = 0 \\
\downarrow & & \downarrow \\
\mathbb{H}_{\text{rel}}^{0}(L, [G^{\text{sc}} \to G]) & \longrightarrow & H^{1}(L, G^{\text{sc}}).
\end{array}
$$

By [Bor98, Lemma 3.8.1], there are isomorphisms

$$\mathbb{H}^{0}(L, C) \simeq \mathbb{H}_{\text{rel}}^{0}(L, [T^{\text{sc}} \to T]) \simeq \mathbb{H}_{\text{rel}}^{0}(L, [G^{\text{sc}} \to G]).$$

Since $T^{\text{sc}}$ is quasi-trivial, we conclude that $\mathbb{H}^{0}(L, C) \to H^{1}(L, G^{\text{sc}})$ is the trivial map, that is, the canonical map $\text{ab}^{0} : G(L) \to \mathbb{H}^{0}(L, C)$ is surjective (see [Bor98, section 3.10]).

We now proceed as in [San81]. The first step is to show the following:

**Lemma 2.5.** Let $m \geq 1$ be an integer and let $Q$ be a quasi-trivial $K$-torus. If the sequence (7) is exact for $G^{m} \times_{K} Q$, then it is also exact for $G$.
Proof. If the sequence (7) is exact for some finite direct product $G^m$, then (7) is exact for $G$ as well. We claim if (7) is exact for the product $G \times_K Q$ of $G$ by some quasi-trivial $K$-torus $Q$, then (7) is also exact for $G$. Since $T \subset G$ is a maximal torus of $G$, $T \times_K Q$ is a maximal torus of $G \times_K Q$. Moreover, $\mathcal{D}(G \times_K Q) = G^m$ holds, so we have a composite $\rho : G^\text{sc} \to G^m \to G \times_K Q$. We introduce the complex $C_Q = [T^\text{sc} \to T \times_K Q]$ which is concentrated in degree $-1$ and $0$. Consider the following commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H^1(K, T^\text{sc}) \\
H^1(K, T^\text{sc})
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\approx}
H^1(K, T)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H^1(K, C)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{v}
H^2(K, T^\text{sc})
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\approx}
H^2(K, T)
\end{array}
\end{array}
\end{array}
\end{array}
$$

where the second vertical map is an isomorphism since $Q$ is a quasi-trivial $K$-torus. Applying [HSS15, Lemma 3.2(a)] yields isomorphisms $\text{III}^1(C_Q) \simeq \text{III}^1(T) \simeq \text{III}^1(C)$. Similarly, $\text{III}^1(C_{Q'})^D \simeq \text{III}^1(C')^D$ holds for any finite subset $S$ of places. Finally, recall that quasi-trivial $K$-tori are $K$-rational, hence in particular $Q$ satisfies weak approximation. It follows that the cokernel of the first map in (7) is stable under multiplying $G$ by a quasi-trivial torus and hence the exactness of (7) for $G \times_K Q$ yields the exactness of (7) for $G$. Summing up, to prove the exactness of (7), we are free to replace $G$ by $G^m \times_K Q$ for some integer $m$ and some quasi-trivial $K$-torus $Q$. $\square$

By [BT65, Proposition 2.2] and Ono’s lemma [San81, Lemma 1.7], there exist an integer $m \geq 1$, quasi-trivial $K$-tori $Q$ and $Q_0$ such that $G^m \times_K Q \to G^m \times_K Q_0$ is a central $K$-isogeny. Recall that $G^m$ satisfies weak approximation by Proposition 2.1. We may therefore assume that $G$ has a special covering $1 \to F_0 \to G_0 \to G \to 1$ where $G_0$ satisfies weak approximation and has derived subgroup $\mathcal{D}G_0 = G^m$. Moreover, by construction $G_0$ contains a quasi-trivial maximal torus $T_0$ over $K$ (again by [HS16, Lemma 6.7]) such that $T_0 \cap G^\text{sc} = T^\text{sc}$ and that the sequence $1 \to F_0 \to T_0 \to T \to 1$ is exact. Therefore we may assume $G$ admits a special covering in the sequel.

The second step is to show the exactness at the first three terms:

Lemma 2.6. There is an exact sequence $1 \to \overline{G(K)}_S \to \prod_{v \in S} G(K_v) \to \text{III}^1_{S}(C')^D$.

Proof. After passing to the dual isogeny of the exact sequence $1 \to F_0 \to T_0 \to T \to 1$, we obtain $\text{III}^2_{S}(F'_0) \simeq \text{III}^2(T')$ since $T'_0$ is quasi-trivial and $\text{III}^2_{S}(T'_0) = 0$ by [HSS15, Lemma 3.2(a)]. Moreover, the distinguished triangle $T' \to (T^\text{sc}')' \to C' \to T'[1]$ induces an isomorphism $\text{III}^1_{S}(C') \simeq \text{III}^2_{S}(T')$ for the same reason. In particular, we obtain an isomorphism $\text{III}^2_{S}(F'_0) \simeq \text{III}^1_{S}(C')$ which fits into the commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
G_0(K)
\end{array}
\begin{array}{c}
\xrightarrow{\partial}
H^1(K, F_0)
\end{array}
\begin{array}{c}
\xrightarrow{\partial_v}
H^1(K_v, F_0)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\prod_{v \in S} G(K_v)
\end{array}
\begin{array}{c}
\prod_{v \in S} H^1(K_v, F_0)
\end{array}
\begin{array}{c}
\prod_{v \in S} H^1(K_v, G_0)
\end{array}
\end{array}
\end{array}
$$

where $F'_0 = \text{Hom}(F_0, \mathbb{Q}/\mathbb{Z}(2))$. Recall that the third column is exact by [HSS15, Lemma 3.1]. We claim the coboundary map $\partial$ is surjective. Since $F_0$ is finite and central in $G_0$, $F_0$ is contained in one, hence every, maximal torus of $G_0$. In particular, $F_0$ is contained in the quasi-trivial maximal torus $T_0$ of $G_0$ and consequently the map $H^1(K, F_0) \to H^1(K, G_0)$ factors through $H^1(K, F_0) \to H^1(K, T_0)$. By construction $T_0$ is quasi-trivial, so the vanishing $H^1(K, T_0) = 0$ implies that $H^1(K, F_0) \to H^1(K, G_0)$ is the trivial map, i.e. $\partial$ is surjective. Similarly, the coboundary map $\partial_v : G(K_v) \to H^1(K_v, F_0)$ is surjective for each place $v \in X^{(1)}$. Since $G_0$ satisfies weak approximation by assumption, $G_0(K)$ has dense image in $\prod_{v \in S} G_0(K_v)$. A diagram chasing now yields the desired exact sequence. $\square$

In order to prove Theorem 2.2(1), the last step is to show the exactness of the last three terms. By definition there is an exact sequence

$$1 \to \text{III}^1(C') \to \text{III}^1_{S}(C') \to \bigoplus_{v \in S} H^1(K_v, C')$$
of discrete abelian groups. Dualizing the sequence, we obtain an exact sequence of profinite groups:

$$\prod_{v \in S} \hat{H}^0(K_v, C)^{\wedge} \to \Pi^1_\omega(C')^D \to \Pi^1_\omega(C) \to 1.$$ 

Since \(\Pi^1_\omega(C') \simeq \Pi^1_\omega(T')\) is a finite group, the groups \(\prod_{v \in S} \hat{H}^0(K_v, C)\) and \(\prod_{v \in S} \hat{H}^0(K_v, C)^{\wedge}\) have the same image in \(\Pi^1_\omega(C')^D\). By Lemma 2.4, the canonical abelianized map \(\prod_{v \in S} G(K_v) \to \Pi_{v \in S} \hat{H}^0(K_v, C)\) is surjective which guarantees the desired exactness.

**Proof of Theorem 2.2(2).** Passing to the projective limit of (7) over all finite subset \(S \subset X^{(1)}\) yields an exact sequence of groups

$$1 \to \overline{G(K)} \to \prod_{v \in X^{(1)}} G(K_v) \to \Pi^1_\omega(C')^D.$$ 

Dualizing the exact sequence of discrete groups

$$1 \to \Pi^1_\omega(C') \to \Pi^1_\omega(C') \to \bigoplus_{v \in X^{(1)}} H^1(K_v, C')$$

yields an exact sequence of profinite groups

$$\prod_{v \in X^{(1)}} \hat{H}^0(K_v, C)^{\wedge} \to \Pi^1_\omega(C')^D \to \Pi^1_\omega(C) \to 0.$$ 

Thus it will be sufficient to show the image of \(\prod_{v \in X^{(1)}} G(K_v)\) is closed in \(\Pi^1_\omega(C')^D\). In view of diagram (9), the quotient of \(\prod_{v \in X^{(1)}} G(K_v) / \overline{G(K)}\) by \(G(K)\) is isomorphic to the quotient of the profinite group \(\prod_{v \in X^{(1)}} H^1(K_v, F_0)\) by the closure of the image of \(H^1(K, F_0)\). Consequently, the quotient of \(\prod_{v \in X^{(1)}} G(K_v) / \overline{G(K)}\) by \(G(K)\) is compact hence the image of \(\prod_{v \in X^{(1)}} G(K_v)\) in \(\Pi^1_\omega(C')^D\) is closed, as required.

Actually the defect of weak approximation for \(G\) can also be given by a simpler group \(\Pi^1_\omega(G^*)\) where \(G^*\) is the group of multiplicative type whose character module is \(\hat{G}^* = \pi^\alg_1(G)\). Here \(\pi^\alg_1\) (see [Bor98, §1] or [CT08, §6] for more details) denotes the algebraic fundamental group of a connected linear algebraic group.

**Proposition 2.7.** Let \(G^*\) be the group of multiplicative type whose character module is \(\pi^\alg_1(G)\). There is an isomorphism of groups \(\Pi^1_\omega(C') \simeq \Pi^1_\omega(G^*)\).

**Proof.** Thanks to the distinguished triangle \(T' \to (T^{\text{sc}})' \to C' \to T'[1]\), there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \hat{H}^1(K, C') & \to & H^2(K, T') & \to & H^2(K, (T^{\text{sc}})') & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \Pi^1_\omega(C') & \to & \Pi^1_\omega(T') & \to & \Pi^1_\omega((T^{\text{sc}})'). & & \\
\end{array}$$

with exact upper row. Since \((T^{\text{sc}})')\) is quasi-trivial, \(\Pi^1_\omega((T^{\text{sc}})') = 0\) by [HSS15, Lemma 3.2(a)] and it follows that \(\Pi^1_\omega(C') \simeq \Pi^1_\omega(T')\) after a diagram chasing. Since \(T^{\text{sc}}\) and \(T^{\text{tor}}\) are quasi-trivial torus, we obtain \(H^1(K, (T^{\text{tor}})') = 0\) and \(\Pi^1_\omega((T^{\text{tor}})') = 0\) by the associated long exact sequence of \(1 \to (H^{\text{tor}})' \to (T^{\text{tor}})' \to (T^{\text{tor}})' \to 1\). Let \(1 \to R \to H \to G \to 1\) be a flasque resolution of \(G\) and consider the associated fundamental diagrams (3) and (4). Dualizing the middle row of the diagram (4) and considering the associated long exact sequence, we obtain

$$\Pi^1_\omega(R') \simeq \Pi^1_\omega(T') \simeq \Pi^1_\omega(C').\ (10)$$

Next, we identify \(\Pi^1_\omega(G^*)\) with \(\Pi^1_\omega(R')\). Recall [CT08, Proposition 6.8] that \(\pi^\alg_1\) is an exact functor from the category of connected linear \(K\)-groups to that of \(\text{Gal}(\overline{K}/K)\)-modules of finite type. Recall also that \(\pi^\alg_1(\overline{R})\) of a \(K\)-torus \(R\) is its module of cocharacters \(\overline{R}\). Thus there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & \overline{R} & \to & \overline{T}^{\text{H}} & \to & \overline{T} & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \overline{R} & \to & \pi^\alg_1(\overline{H}) & \to & \pi^\alg_1(\overline{G}) & \to & 1 \end{array}$$

(11)
of Galois modules of finite type with exact lower row by the exactness of $\pi_1^{alg}$. Recall there is an anti-equivalence from the category of groups of multiplicative type to that of Galois modules of finite type which respects exact sequences, thus diagram (11) corresponds to a commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & T' & \longrightarrow & T'_H & \longrightarrow & R' & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & G^* & \longrightarrow & H^* & \longrightarrow & R' & \longrightarrow & 1
\end{array}
$$

of groups of multiplicative type over $K$. Taking Galois cohomology gives rise to commutative diagrams

$$
\begin{array}{ccc}
H^1(K, R') & \longrightarrow & H^2(K, T') \\
\downarrow & & \downarrow \\
H^1(K, R') & \longrightarrow & H^2(K, G^*)
\end{array}
\quad
\begin{array}{ccc}
\Pi_1^1(R') & \longrightarrow & \Pi_2^2(T') \\
\downarrow & & \downarrow \\
\Pi_1^1(R') & \longrightarrow & \Pi_2^2(G^*)
\end{array}
$$

with $\Pi_1^1(R') \simeq \Pi_2^2(T')$ by (10). In particular, we obtain $\Pi_2^2(G^*) \simeq \Pi_2^2(T') \simeq \Pi_1^1(C')$.

Thus there is an exact sequence of groups by Theorem 2.2(2)

$$1 \to G(K) \to \prod_{v \in X^{(1)}} G(K_v) \to (\Pi_2^2(G^*))^D.$$ Consequently, the defect of weak approximation may also be given by the group $\Pi_2^2(G^*)$.

### 3 Reciprocity obstruction to weak approximation

The next theorem is the promised generalization of [HSS15, Theorem 4.2] to the non-commutative case. Let us briefly recall the construction of the pairing [HSS15, (17)] concerning unramified cohomology groups. Let $Y$ be a smooth integral variety over $K$ with function field $K(Y)$. Thus $Y$ admits a smooth compactification over $K$. Let $y_v : \text{Spec} \ K_v \to Y$ be a $K_v$-point on $Y$. Take $\alpha \in H^3_{nr}(K(Y), \mu_n^{\otimes 2})$ and lift it uniquely to $\alpha_v \in H^3(Y, y_v, \mu_n^{\otimes 2})$. Now $\alpha_v$ goes to $H^3(K_v, \mu_n^{\otimes 2})$ via $H^3(Y, y_v, \mu_n^{\otimes 2}) \to H^3(K_v, \mu_n^{\otimes 2})$. Summing up, we obtain an evaluation pairing

$$Y(K_v) \times H^3_{nr}(K(Y), \mu_n^{\otimes 2}) \to H^3(K_v, \mu_n^{\otimes 2}).$$

Taking the isomorphism $H^3(K_v, Q/Z(2)) \simeq Q/Z$ for each $v \in X^{(1)}$ into account, we can construct a pairing

$$\prod_{v \in X^{(1)}} Y(K_v) \times H^3_{nr}(K(Y), Q/Z(2)) \to Q/Z$$

(12)

**Theorem 3.1.** Let $G$ be a connected reductive group such that $G^{sc}$ is quasi-split. There exists a homomorphism $u : \Pi_1^1(C') \to H^3_{nr}(K(G), Q/Z(2))$ such that each family $(g_v) \in \prod_{v \in X^{(1)}} G(K_v)$ satisfying $(g_v), \text{Im } u = 0$ under the pairing (12) lies in the closure $G(K)$ with respect to the product topology.

**Proof.** We first construct the homomorphism $u : \Pi_1^1(C') \to H^3_{nr}(K(G), Q/Z(2))$. Let $\rho : G^{sc} \to G$ be as before. Let $T^{sc} \subset G^{sc}$ be a quasi-trivial maximal torus and let $T \subset G$ be a maximal torus such that $T^{sc} = \rho^{-1}(T)$. Recall that there is a fundamental diagram (4) associated with the flasque resolution $1 \to R \to H \to G \to 1$, and recall also that $H^{tor}$ is a quasi-trivial torus. Moreover, there is a homomorphism $\Pi_1^1(C') \to H^1(K, R')$ via the inclusion $\Pi_1^1(R') \to H^1(K, R')$ in view of (10).

Because $R$ is a flasque $K$-torus, applying [CT87, Theorem 2.2(i)] implies that the class $[H] \in H^1(G, R)$ comes from a class $[Y] \in H^1(G^{c}, R)$, where $G^{c}$ is a smooth compactification of $G$. The pairing $R \otimes L R' \to Z(2)[2]$ now induces a homomorphism

$$H^1(K, R') \to H^1(G^c, Z(2)), \ a \mapsto a_{G^c} \cup [Y]$$

with $a_{G^c}$ denoting the image of $a$ under $H^1(K, R') \to H^1(G^c, R')$. The same argument as in [HSS15, Theorem 4.2] shows that there is an isomorphism

$$H^2(G^c, Z(2)) \simeq H^3_{nr}(K(G), Q/Z(2))$$

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fitting into a commutative diagram

\[
\begin{array}{c}
H^4(G^c, \mathbb{Z}(2)) \longrightarrow H^3_{tr}(K(G), \mathbb{Q}/\mathbb{Z}(2)) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H^4(K(G), \mathbb{Z}(2)) \longrightarrow H^3(K(G), \mathbb{Q}/\mathbb{Z}(2)).
\end{array}
\]

Now take an element \((g_v) \in \prod_{v \in X^{(1)}} G(K_v)\). By Theorem 2.2(2), \((g_v)\) lies in the closure \(\overline{G(K)}\) if and only if \((g_v)\) is orthogonal to \(\Pi^3_1(C')\). We consider the commutative diagram of various cup-products

\[
\begin{array}{c}
\mathbb{H}^0(K_v, C) \times \mathbb{H}^1(K_v, C') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H^0(K_v, T) \times H^2(K_v, T') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
\delta_v \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H^1(K_v, R) \times H^1(K_v, R') \longrightarrow \mathbb{Q}/\mathbb{Z}.
\end{array}
\]

Since \(H^1(K_v, T^s) = 0\) by the quasi-trivialness of \(T^s\), \(H^0(K_v, T) \rightarrow \mathbb{H}^0(K_v, C)\) is surjective. In particular, there exists \(t_v \in H^0(K_v, T)\) such that its image in \(\mathbb{H}^0(K_v, C)\) equals \(ab_0^0(g_v)\). The diagram together with Theorem 2.2 imply that \((g_v) \in \overline{G(K)}\) if and only if \((t_v)\) is orthogonal to \(\Pi^3_1(R')\). Recall we have isomorphisms (10). More explicitly, it means that

\[
0 = \sum_{v \in X^{(1)}} \langle a_v, ab_0^0(g_v) \rangle_v = \sum_{v \in X^{(1)}} ab_0^0(g_v) \cup a_v = \sum_{v \in X^{(1)}} \delta_v t_v \cup a_v
\]

for each \(a \in \Pi^3_1(C') \cong \Pi^3_1(R')\). Note that \(\delta_v t_v\) is given by \(t_v^*: H^1(T, R) \rightarrow H^1(K_v, R), [T_H] \mapsto [T_H](t_v) = [Y](g_v)\).

It follows that

\[
\sum_{v \in X^{(1)}} \delta_v t_v \cup a_v = \sum_{v \in X^{(1)}} (\{T_H\} \cup a_T)(t_v) = \sum_{v \in X^{(1)}} ([Y] \cup a_{C^s})(g_v)
\]

(13) holds thanks to the commutative diagram

\[
\begin{array}{c}
H^1(G^c, R) \times H^1(G^c, R') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H^1(G, R) \times H^1(G, R') \longrightarrow \mathbb{Q}/\mathbb{Z} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H^1(T, R) \times H^1(T, R') \longrightarrow \mathbb{Q}/\mathbb{Z}.
\end{array}
\]

Note that the vanishing of the last term in (13) means that \((g_v)\) is orthogonal to the image of \(u\) under the pairing \((-,-)\) which completes the proof. \(\square\)

4 Weak approximation for homogeneous spaces

We begin with a cohomological obstruction to weak approximation analogous to the Brauer–Manin obstruction in the context of number fields (for example, see [Bor96]). Let \(Y\) be a variety over \(K\) and let \(Y_v := Y \times_K K_v\) for each place \(v\). For any finite set \(S \subset X^{(1)}\) of places, we define

\[
\mathcal{B}_S^3(Y, \mathbb{Q}/\mathbb{Z}(2)) := \text{Ker} \left( \frac{H^3(Y, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im} H^3(K, \mathbb{Q}/\mathbb{Z}(2))} \longrightarrow \prod_{v \notin S} \frac{H^3(Y_v, \mathbb{Q}/\mathbb{Z}(2))}{\text{Im} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))} \right).
\]

Let \(\mathcal{B}(Y, \mathbb{Q}/\mathbb{Z}(2)) := \mathcal{B}_\emptyset(Y, \mathbb{Q}/\mathbb{Z}(2))\) and let \(\mathcal{B}_S^3(Y, \mathbb{Q}/\mathbb{Z}(2)) := \bigcup_S \mathcal{B}_S^3(Y, \mathbb{Q}/\mathbb{Z}(2))\) where \(S\) runs through all finite set of places. We endow these groups with discrete topologies. Now we suppose there is a \(K\)-point \(y_0 : \text{Spec} K \rightarrow Y\) on \(Y\). Since \(y_0^*: H^3(Y, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2))\) admits a retraction, there is a (non-canonical) isomorphism
Ker $y_0^* \simeq H^3(Y, Q/Z(2))/H^3(K, Q/Z(2))$. In particular, $y_0^*(\tilde{b}) = 0$ for any $b \in B^3_S(Y, Q/Z(2))$ and any lift $\tilde{b}$ of $b$ in $H^3(Y, Q/Z(2))$. We consider the following pairing

$$\langle -, - \rangle : \prod_{v \in S} Y(K_v) \times B^3_S(Y, Q/Z(2)) \to Q/Z, \quad \langle (y_v), b \rangle_S = \sum_{v \in S} y_v^*(\tilde{b})$$

where $y_v^* : H^3(Y, Q/Z(2)) \to H^3(K_v, Q/Z(2)) \simeq Q/Z$ are the maps induced by $y_v$ for $v \in S$ and $\tilde{b} \in H^3(Y, Q/Z(2))$ is a representative of $b$. Moreover, for $b \in B^3(Y, Q/Z(2))$ we have $\langle (y_v), b \rangle_S = 0$ for any $(y_v)$. Thus we obtain a map

$$\text{OBS}_{Y,S} : \prod_{v \in S} Y(K_v) \to (B^3_S(Y, Q/Z(2))/B^3(Y, Q/Z(2)))^D.$$

The map $\text{OBS}_{Y,S}$ is continuous because the pairing $\langle -, - \rangle_S$ is continuous on the left. Moreover, if $y \in Y(K)$, then $\text{OBS}_{Y,S}(y) = 0$ since $y^* : H^3(Y, Q/Z(2))/H^3(K, Q/Z(2)) \to H^3(Y, Q/Z(2))/H^3(K, Q/Z(2))$ coincide for any place $v$. By continuity of $\text{OBS}_{Y,S}$, we deduce that $\text{OBS}_{Y,S}(y_S) = 0$ for $y_S \in \overline{Y}(K)$. In particular, if $Y(K)$ is dense in $Y(K_S)$, then $\text{OBS}_{Y,S}$ is identically zero.

Let $f : Y \to Z$ be a morphism to another $K$-schemes $Z$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\prod_{v \in S} Y(K_v) & \xrightarrow{\text{OBS}_{Y,S}} & (B^3_S(Y, Q/Z(2))/B^3(Y, Q/Z(2)))^D \\
\downarrow f & & \downarrow f \\
\prod_{v \in S} Z(K_v) & \xrightarrow{\text{OBS}_{Z,S}} & (B^3_S(Z, Q/Z(2))/B^3(Z, Q/Z(2)))^D.
\end{array}
$$

(14)

Remark 4.1. Let $T$ be a $K$-torus and let $Y$ be a $K$-torsor under $T$. Then the group $B^3(Y, Q/Z(2))$ is exactly the group $H^3_T(Y, Q/Z(2))$ used in [HS16] for describing cohomological obstruction to the Hasse principle for $Y$. Moreover, there is a canonical homomorphism $H^3(K, T') \to H^3(T, Q/Z(2))/H^3(K, Q/Z(2))$ obtained from the Hochschild–Serre spectral sequence (see [HS16, pp. 15-16]). In particular, for a finite set $S \subset X^{(1)}$ of places, there is an induced homomorphism $\prod_{v \in S} H^3(T') \to B^3_S(T, Q/Z(2))$. That is, the cohomological obstruction $\text{OBS}_{T,S}$ to weak approximation with respect to $S$ can be described by $\prod_{v \in S} H^3(T')^D$. If $\text{OBS}_{T,S} \equiv 0$, then $T(K)$ is dense in $\prod_{v \in S} T(K_v)$ by [HSS15, Theorem 3.3(a)].

Lemma 4.2. Let $G$ be a connected reductive group over $K$ such that $G^{sc}$ is quasi-split. Let $1 \to R \to H \to G \to 1$ be a flasque resolution of $G$ and let $Y$ be a homogeneous space over $K$ under $H$. Then the quotient $Z := Y/H^{ss}$ exists and the map $Y(K) \to Z(K)$ induced by the quotient $Y \to Z$ is surjective.

Proof. Since $H^{ss} \simeq G^{sc}$ is simply connected and quasi-split, it contains a quasi-trivial maximal torus $T^{sc}$. Let $T_H \subset H$ be a maximal torus containing $T^{sc}$. Recall that there is an isomorphism $T_H/T^{sc} \simeq H^{tor} \simeq H/H^{ss}$ by the third column in the diagrams (3) and (4). Moreover, the quotient $Z := Y/H^{ss}$ exists (in the sense of [Bor91, II.6.3]) by [Bor96, Lemma 3.1].

Let $H_0 := \text{Stab}_H(y_0)$ and let $T_0 := T_H \times_K H_0 \simeq \text{Stab}_{T_H}(y_0)$ be the respective stabilizers. Then $Y' := T_H/T_0$ is a homogeneous space under the torus $T_H$ with stabilizer $T_0$. Moreover, the inclusion $T_H \subset H$ induces a morphism $T_H/T_0 \to H/H_0$. Again by [Bor96, Lemma 3.1], the quotients $Y/H^{ss}$ and $Y'/T^{ss}$ exist. We claim the morphism $Y'/T^{ss} \to Y/H^{ss}$ induced by $T_H/T_0 \to H/H_0$ is an isomorphism of schemes. Indeed, $H^{ss}$ is a normal subgroup of $H$ contained in the group $H_0 \rtimes H^{ss}$, hence there is an isomorphism

$$Y/H^{ss} \simeq H/(H_0 \rtimes H^{ss}) \simeq (H/H^{ss})/(H_0 \rtimes H^{ss}/H^{ss}).$$

Similarly, we obtain $Y'/T^{ss} \simeq (T_H/T^{ss})/(T_0 \rtimes T^{ss}/T^{ss})$. Clearly $T_0 \times T^{ss}/T^{ss}$ can be identified with a closed subscheme of $H_0 \times H^{ss}/H^{ss}$. Conversely, each fibre of $H_0 \times H^{ss} \to (H_0 \times H^{ss})/H^{ss}$ has non-empty intersection with $T_H$ inside $H$ because $T_H/T^{ss} \simeq H/H^{ss}$. In particular, each such fibre has non-empty intersection with $T_0 \rtimes T^{ss}$, i.e. $T_0 \times T^{ss}/T^{ss} \to H_0 \times H^{ss}/H^{ss}$ is an isomorphism. Consequently, there is an isomorphism $Y'/T^{ss} \simeq Y/H^{ss}$.

By construction of $Z := Y/H^{ss}$, $Y$ is a $Z$-torsor under $H^{ss}$. Thus for any $z \in Z(K)$, the fibre $Y_z$ is a $K$-torsor under $H^{ss}$ and so there is an exact sequence $Y(K) \to Z(K) \to H^1(K, H^{ss})$ of pointed sets. Here the exactness means that if the fibre $Y_z$ is a trivial $K$-torsor, then $z$ comes from a $K$-point on $Y$. Applying the same argument for the $T^{ss}$-torsor $Y' \to Z$, we obtain a commutative diagram of exact sequences

$$
\begin{array}{ccc}
Y'(K) & \to & Z(K) \to & H^1(K, T^{ss}) \\
\downarrow & & \downarrow & \\
Y(K) & \to & Z(K) \to & H^1(K, H^{ss}).
\end{array}
$$
Again $H^1(K, T^{sc}) = 0$ because we have chosen $T^{sc}$ to be quasi-trivial, so $Y(K) \to Z(K)$ is surjective. □

**Theorem 4.3.** Let $G$ be a connected reductive group over $K$ such that $G^{sc}$ is quasi-split. Let $Y$ be a homogeneous space over $K$ under $G$. Suppose that $Y$ contains a $K$-point $y_0 : \text{Spec} K \to Y$ and that the stabilizer $\text{Stab}_G(y_0)$ is connected. Let $S$ be a finite set of places and let $y_S := (y_v) \in \prod_{v \in S} Y(K_v)$ be a family of local points. If $\text{OBS}_S(Y_S(y_S)) = 0$, then $y_S \in Y(K)_S$. Here $Y(K)_S$ denotes the closure of $Y(K)$ in $\prod_{v \in S} Y(K_v)$ with respect to the product topology.

**Proof.** Let $1 \to R \to H \to G \to 1$ be a flasque resolution of $G$. Recall [CT08, 0.3] that $H^{ss} := \mathcal{D}H$ is identified with the simply connected group $G^{sc}$. Since for $y_0 \in Y(K)$, there is a short exact sequence

$$1 \to R \to \text{Stab}_H(y_0) \to \text{Stab}_G(y_0) \to 1,$$

we conclude that $\text{Stab}_H(y_0)$ is also connected. We view $Y$ as a homogeneous space under $H$ via the isomorphisms $H/\text{Stab}_H(y_0) \simeq G/\text{Stab}_G(y_0) \simeq Y$. Recall that $H^{ss} \simeq G^{sc}$ is simply connected and satisfies weak approximation.

Let $U_Y \subseteq \prod_{v \in S} Y(K_v)$ be any non-empty open neighbourhood of $y_S$. We claim $U_Y(K) \neq \emptyset$. Let $q : Y \to Z$ be the quotient map with fibres being $H^{ss}$-orbits. Then $Z$ turns into a homogeneous space over $K$ under the torus $H^{tor} := H/H^{ss}$. Moreover, diagram (14) yields the vanishing $\text{OBS}_{Z, S}(q(y_S)) = 0$. Note that $q(y_0)$ is a $K$-point on $Z$, so $Z(K) \neq \emptyset$. Consequently $Z$ is a quotient of $H^{ss}$ and hence $Z$ is a torus. Since $\text{OBS}_{Z, S}(q(y_S)) = 0$, we conclude $q(y_S) \in Z(K)$. By [HSS15, Theorem 3.3] and the Remark 4.1. By the smoothness of $q : Y \to Z$, $U_Z := q(U_Y)$ is an open neighbourhood of $(y_S)$ and hence it contains a $K$-point $z$. Let $Z_z := q^{-1}(z)$ be the fibre over $z$. It is a $K$-torsor under $H^{ss}$. By Lemma 4.2 $Y(K) \to Z(K)$ is surjective, so $Y_z(K) \neq \emptyset$ and it follows that $Y_z$ is a trivial $H^{ss}$-torsor. But $H^{ss}$ satisfies weak approximation by assumption, $Y_z$ satisfies it as well. Thus $U_Y \cap \prod_{v \in S} Y_z(K_v)$ contains a $K$-point being a non-empty open subset of $\prod_{v \in S} Y_z(K_v)$. □

**Remark 4.4.** As we have seen in the proof of Theorem 4.3, it will be sufficient to assume there is no obstruction to weak approximation for the quotient $Z := Y/H^{ss}$ which is a torus over $K$. In particular, the obstruction described by $\mathcal{B}_S^3(Z, \mathbb{Q}/\mathbb{Z}(2))$ may be replaced by a finer group $\mathbb{H}_S^3(Z')$ which is unramified by Corollary 5.2 (see below). From this point of view, Theorem 4.3 is a generalization of Theorem 3.1.

5 Appendix

Let $T$ be a $K$-torus and let $T'$ be its dual torus. There is a canonical map constructed in [HS16, pp. 15, (20)]

$$H^2(K, T') \to H^3(T, \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

via the Hochschild–Serre spectral sequence $H^p(K, H^q(T, \mathbb{Q}/\mathbb{Z}(2))) \Rightarrow H^{p+q}(T, \mathbb{Q}/\mathbb{Z}(2))$. Then we send $\mathbb{H}_S^3(T')$ to $H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. Our goal is to show the image of $\mathbb{H}_S^3(T')$ lies in $H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$, the unramified part of $H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$.

Let $1 \to R \to Q \to T \to 1$ be a flasque resolution with $R$ a flasque $K$-torus and $Q$ a quasi-trivial $K$-torus.

**Proposition 5.1.** There is a commutative diagram

$$\begin{array}{ccc}
H^1(K, R') & \longrightarrow & H^4(T, \mathbb{Z}(2))/H^4(K, \mathbb{Z}(2)) \\
\downarrow & & \downarrow \\
H^2(K, T') & \longrightarrow & H^3(T, \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))
\end{array}$$

(15)

where the upper horizontal map is defined by the cup-product $H^1(K, R') \times H^1(T^c, R) \to H^1(T^c, Z(2))$ (see [HSS15, pp. 19] for details), and the right vertical map is induced by the exact sequence $0 \to Z(2) \to Q(2) \to Q(2)/Z(2) \to 0$.

**Corollary 5.2.** The image of $\mathbb{H}_S^3(T')$ in $H^3(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ lies in the unramified part.

**Proof.** The map $H^1(K, R') \to H^4(T, \mathbb{Z}(2))$ factors through $H^4(T^c, \mathbb{Z}(2)) \to H^4(T, \mathbb{Z}(2))$ by construction [HSS15, pp. 19-20], so the image of $\mathbb{H}_S^3(R') \simeq \mathbb{H}_S^3(T')$ lies in the unramified part $H^3_{\text{ur}}(K(T), \mathbb{Q}/\mathbb{Z}(2))/H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. □
The rest of the appendix is devoted to the proof of Proposition 5.1. We begin with some observations on torsion groups under consideration. Suppose $R$ splits over some finite Galois extension $L|K$. The vanishing of $H^1(T_L, R_L) = 0$ implies that the class $[Q] \in H^1(T, R)$ is torsion by a restriction-corestriction argument. The spectral sequence $H^p(K, \text{Ext}_R^q(R, T)) \Rightarrow H^{p+q}(R', T')$ together with the vanishing $\text{Ext}_R^p(R, T) = 0$ implies that $H^1(K, \text{Hom}_R(R, T)) \to H^1(K, R', T')$ is an isomorphism. Thus the group $\text{Ext}_K^1(R', T')$ is both $n$-torsion. We choose a suitable integer $n$ such that the classes $[Q] \in H^1(T, R)$ and $[Q'] \in H^1(T, R')$ are both $n$-torsion.

**Step 1:** We verify the commutativity of diagram (15) with a different construction of the left vertical arrow as in diagram (18).

The Kummer sequence $1 \to nR \to R \to R \to 1$ yields a surjection $H^1(T, nR) \to nH^1(T, R)$, i.e. $[Q] = \iota_n([Q_n])$ for some class $[Q_n] \in H^1(T, nR)$ with $\iota_n : H^1(T, nR) \to H^1(T, R)$ induced by the Kummer sequence.

Let $p : T \to \text{Spec} K$ be the structural morphism. Let $D(K)$ be the derived category of bounded complexes of Galois modules. We consider the object $ND(T) = (\tau_{\leq 1} \text{R}_p, \mu_n^\otimes)[1]$ in $D(K)$ which fits into a distinguished triangle

$$
\mu_n^\otimes [1] \to ND(T) \to H^1(T, R) \to \mu_n^\otimes [2].
$$

(16)

We will follow [HS13, Proposition 1.1] to construct a map

$$
\chi : H^1(T, nR) \to H^1(K, nR', ND(T)).
$$

The pairing $nR \otimes L_n R' \to \mu_n^\otimes$ yields a map $H^1(T, nR) \to H^1(T, \text{Hom}(nR', \mu_n^\otimes))$. Moreover, we obtain a map $H^1(T, \text{Hom}(nR', \mu_n^\otimes)) \to \text{Ext}_1^n(nR', \mu_n^\otimes)$ from the exact sequence in low degrees associated to the local-to-global spectral sequence

$$
H^p(T, \text{Ext}_T^n(nR', \mu_n^\otimes)) \Rightarrow \text{Ext}_T^{p+q}(nR', \mu_n^\otimes).
$$

Since $\text{R}\text{Hom}_T(nS', -) = \text{R}\text{Hom}_K(nS', -) \circ \text{R}_p(-)$ is a composition, formally there is a canonical isomorphism

$$
\text{Ext}_1^n(nR', \mu_n^\otimes) \simeq \text{R}^1\text{Hom}_K(nR', \text{R}_p, \mu_n^\otimes).
$$

Because $\tau_{\geq 2} \text{R}_p, \mu_n^\otimes$ is acyclic in degrees 0 and 1, we obtain an isomorphism

$$
\text{R}^1\text{Hom}_K(nR', \tau_{\leq 1} \text{R}_p, \mu_n^\otimes) \simeq \text{R}^1\text{Hom}_K(nR', \text{R}_p, \mu_n^\otimes)
$$

from the distinguished triangle $\tau_{\leq 1} \text{R}_p, \mu_n^\otimes \to \text{R}_p, \mu_n^\otimes \to \tau_{\geq 2} \text{R}_p, \mu_n^\otimes \to ND(T)$. Now $\chi$ is just the composition

$$
H^1(T, nR) \to \text{Ext}_1^n(nR', \mu_n^\otimes) \simeq \text{R}^1\text{Hom}_K(nR', \tau_{\leq 1} \text{R}_p, \mu_n^\otimes) = \text{Hom}_K(nR', ND(T)).
$$

All the above constructions yield a diagram of cup-products

$$
\begin{array}{ccc}
H^1(K, R') & \times & H^1(T, R) \\
\partial & \downarrow & \partial \\
H^2(K, nR') & \times & H^1(T, nR) \\
\downarrow & \downarrow & \downarrow \\
H^2(K, nR') & \times & \text{Hom}_K(nR', \text{R}_p, \mu_n^\otimes[1]) \\
\downarrow & & \downarrow \\
H^2(K, nR') & \times & \text{Hom}_K(nR', ND(T)) \\
\downarrow & & \downarrow \\
H^2(K, nR') & \times & H^2(K, ND(T))
\end{array}
$$

(17)

where the upper diagram commutes by [HS16, diagram (26)], and the lower two diagrams commute by [Mil80, Proposition V.1.20]. Diagram (17) gives the commutativity of the left two squares of the following diagram:

$$
\begin{array}{ccc}
H^1(K, R') & \rightarrow & H^2(K, nR') \\
& & \rightarrow \\
H^4(T, \mathbb{Z}(2)) & \leftarrow & H^3(T, \mu_n^\otimes) \\
& & \rightarrow \\
H^3(T, \mu_n^\otimes) & \rightarrow & H^3(T, \mu_n^\otimes) \\
& & \rightarrow \\
H^4(T, \mathbb{Z}(2)) & \triangleleft & H^3(T, \mu_n^\otimes) \\
& & \rightarrow \\
H^3(T, \mu_n^\otimes) & \rightarrow & H^3(T, \mu_n^\otimes) \\
& & \rightarrow \\
H^4(T, \mathbb{Z}(2)) & \rightarrow & H^4(T, \mathbb{Z}(2)).
\end{array}
$$

(18)
The right two squares in diagram (18) commute by construction of the Hochschild–Serre spectral sequences. Passing to the quotient by respective subgroup of constants and taking limits over all $n$ imply the commutativity of diagram (15). Consequently, we are done if the upper row of diagram (18) gives the coboundary map $H^1(K, R') \to H^2(K, T')$ induced by the short exact sequence $1 \to T' \to Q' \to R' \to 1$ of tori.

**Step 2:** We check that composite of arrows in the upper row of diagram (18) is just the desired coboundary map in diagram (15).

The Kummer sequence $1 \to \pi_1 \to \pi_2 \to \pi_3 \to 1$ induces a surjection $\text{Ext}_K^1(R', n, R') \to \pi_2 \text{Ext}_K^1(R', T')$, so the class $[Q']$ lifts to a class $[\pi_n] \in \pi_1 \text{Ext}_K^1(R', n, T')$. Similarly, the Kummer sequence $1 \to \pi_2 \to \pi_3 \to \pi_4 \to 1$ induces a surjection $\text{Hom}_K(n, R', T') \to \pi_3 \text{Ext}_K^1(R', n, T')$ by the vanishing of $\text{Hom}_K(R', n, T') = 0$. Hence there is a commutative diagram

$$
\begin{array}{c}
0 \longrightarrow nT' \longrightarrow T' \longrightarrow T' \longrightarrow 1
\end{array}
\begin{array}{c}
0 \longrightarrow nT' \longrightarrow M_n \longrightarrow R' \longrightarrow 0
\end{array}
\begin{array}{c}
0 \longrightarrow T' \longrightarrow Q' \longrightarrow R' \longrightarrow 0.
\end{array}
$$

(19)

Applying the functor $H^n(K, -)$ to diagram (19) yields a commutative diagram

$$
\begin{array}{c}
H^1(K, R') \longtwoheadrightarrow H^2(K, n, R')
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
H^2(K, T') \longtwoheadrightarrow H^2(K, n, T')
\end{array}
$$

which tells us the composite $H^1(K, R') \to H^2(K, n, R') \to H^2(K, n, T') \to H^2(K, T')$ is exact the coboundary map $H^1(K, R') \to H^2(K, T')$ induced by the bottom row.

It remains to show the map $H^2(K, n, R') \to H^2(K, n, T')$ obtained from $\text{Ext}_K^1(R', n, T')$ coincides with the composition $H^2(K, n, R') \to H^2(K, n, T')$ obtained from the identification $\text{Ext}_K^1(T, n, R) \simeq \text{Hom}_K(T, n, \mu^{\otimes 2})$. Again there is a commutative diagram by [Mil80, Proposition V.1.20]:

$$
\begin{array}{c}
\pi_1 \text{Hom}_K(n, R'(K), \Omega^1_n[1]) \times nR'(K) \longrightarrow \pi_1 \text{Hom}_K(n, R'(K), \Omega^1_n[1])
\end{array}
\begin{array}{c}
\text{Hom}_K(n, R'(K), \Omega^1_n[1]) \times nR'(K) \longrightarrow \text{Hom}_K(n, R'(K), \Omega^1_n[1])
\end{array}
$$

which may be rewritten into the following commutative diagram

$$
\begin{array}{c}
\text{Hom}_K(n, R'(K), \Omega^1_n[1]) \longrightarrow \pi_1 \text{Hom}_K(n, R'(K), \Omega^1_n[1])
\end{array}
\begin{array}{c}
\Phi(-, -) \longrightarrow \text{Hom}_K(n, R'(K), \Omega^1_n[1])
\end{array}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
$$

The arrow $\alpha$ is induced by the distinguished triangle (16). Now $\alpha \circ \chi = \Phi(-, -)$ says that $\pi_1 \text{Hom}_K(n, R'(K), \Omega^1_n[1])$ is the same as $\Phi(\pi_2, -)$. In particular, $H^2(K, n, R') \to H^2(K, n, T')$ obtained from the identification $\text{Ext}_K^1(T, n, R) \simeq \text{Hom}_K(n, R', T')$. 

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References

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. MR0242802

[Blo86] Spencer Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. 61 (1986), no. 3, 267–304, DOI 10.1016/0001-8708(86)90081-2. MR8552815

[BT65] Armand Borel and Jacques Tits, *Groupes réguliers*, Inst. Hautes Études Sci. Publ. Math. 27 (1965), 55–150 (French). MR0207712

[Bor91] Armand Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR1102012

[Bor96] Mikhail Borovoi, *The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer*, J. Reine Angew. Math. 473 (1996), 181–194, DOI 10.1515/crll.1995.473.181. MR1390687

[Bor98] , *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. 132 (1998), no. 626, viii+50, DOI 10.1090/memo/0626. MR1401491

[CT95] Jean-Louis Colliot-Thélène, Birational invariants, purity and the Gersten conjecture, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), Proc. Sympos. Pure Math., vol. 58, Amer. Math. Soc., Providence, RI, 1995, pp. 1–64. MR1327280

[CT08] , Résolutions flasques des groupes linéaires connexes, J. Reine Angew. Math. 618 (2008), 77–133, DOI 10.1515/CRELLE.2008.034 (French). MR2404747

[CTS87] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra 106 (1987), no. 1, 148–205, DOI 10.1016/0021-8693(87)90026-3. MR878473

[Del79] Pierre Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289 (French). MR546620

[Dem11] Cyril Demarche, *Suites de Poitou-Tate pour les complexes de tores à deux termes*, Int. Math. Res. Not. IMRN 1 (2011), 135–174, DOI 10.1093/imrn/rnq060 (French, with English and French summaries). MR2755486

[GL01] Thomas Geisser and Marc Levine, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Algebra 250 (2001), 55–103, DOI 10.1016/S0021-8693(01)00141-7. MR1807268

[Har68] Alexander Grothendieck, *Le groupe de Brauer. III. Exemples et compléments*, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188 (French). MR244271

[H13] David Harari and Alexei N. Skorobogatov, *Descent theory for open varieties*, J. Algebraic Geom. 25 (2016), no. 3, 571–605, DOI 10.1090/jag/661. MR3493592

[Har68] Günter Harder, Eine Bemerkung zum schwachen Approximationssatz, Arch. Math. (Basel) 19 (1968), 465–471, DOI 10.1007/BF01898766 (German). MR0241427

[Hum57] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21. MR0396773

[Ig16] Diego Izquierdo, *Dualité et principe local-global sur les corps de fonctions*, Thèse de doctorat de l’Université Paris-Sud, 2016 (French, with English and French summaries).

[Kne02] Martin Kneser, *Approximationssätze für algebraische Gruppen*, J. Reine Angew. Math. 209 (1962), 96–97, DOI 10.1515/crll.1962.209.96 (German). MR0139663

[Kne05] , Starke Approximation in algebraischen Gruppen. I, J. Reine Angew. Math. 218 (1965), 190–203, DOI 10.1515/crll.1965.218.190 (German). MR0184945

[Matt55] Arthur Mattuck, *Abelian varieties over p-adic ground fields*, Ann. of Math. (2) 62 (1955), 92–119, DOI 10.2307/2007101. MR0071116

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR559531

[Mil97] , *Arithmetic duality theorems*, 2nd ed., BookSurge, LLC, Charleston, SC, 2006. MR2261462

[Mil17] , *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field. MR3729270

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. MR1278263

[San81] Jean-Jacques Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. 327 (1981), 12–80, DOI 10.1515/crll.1981.327.12 (French). MR631309

[Tha96] Nguyệt Quốc Thắng, *On weak approximation in algebraic groups and related varieties defined by systems of forms*, J. Pure Appl. Algebra 113 (1996), no. 1, 67–90, DOI 10.1016/0022-4049(95)00141-7. MR1411647
[SGA3II] Michael Artin, Alexander Grothendieck, and Michel Raynaud, *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970 (French). MR0274459 (43 #223b)

[SGA3III] Philippe Gille and Patrick Polo (eds.), *Schémas en groupes (SGA 3). Tome III. Structure des schémas en groupes réductifs*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 8, Société Mathématique de France, Paris, 2011 (French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J.-P. Serre; Revised and annotated edition of the 1970 French original. MR2867622