A Riemann-Hilbert correspondence for Cartier crystals

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Abstract

For a variety $X$ separated over a perfect field of characteristic $p > 0$ which admits an embedding into a smooth variety, we establish an anti-equivalence between the bounded derived categories of Cartier crystals on $X$ and constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves on the étale site $X_{\text{ét}}$. The key intermediate step is to extend the category of locally finitely generated unit $\mathcal{O}_{F,X}$-modules for smooth schemes introduced by Emerton and Kisin to embeddable schemes. On the one hand, this category is equivalent to Cartier crystals. On the other hand, by using Emerton-Kisin’s Riemann-Hilbert correspondence, we show that it is equivalent to Gabber’s category of perverse sheaves in $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$.

Introduction

The Riemann-Hilbert correspondence gave a general answer to Hilbert’s 21st problem from a modern point of view by connecting $D$-modules to constructible $\mathbb{C}_X$-sheaves on a smooth complex variety $X$. For a variety $X$ over a field of positive characteristic, Emerton and Kisin established an analogue to this correspondence in [EK04]. However, the smoothness of $X$ is essential for this approach because it ensures that the objects Emerton and Kisin consider instead of $D$-modules behave nicely.

In this paper we relax the smoothness assumption. We establish a Riemann-Hilbert type correspondence for a variety $X$ over a perfect field of positive characteristic which is possibly singular but we require that $X$ admits an embedding into a smooth variety $Y$. Thereby we suggest the category of so-called Cartier crystals as a replacement for $D$-modules because Cartier crystals seem to be more suitable for generalizations of the correspondence to singular varieties or even more general schemes. In particular, Cartier crystals are indeed closely related to perverse constructible étale sheaves on $X$, as conjectured by Blickle and Böckle in [BB11].

Let us take a brief look at the development of the Riemann-Hilbert correspondence throughout history. A fundamental step to the modern version of the Riemann-Hilbert correspondence was the result by Deligne in 1970 ([Del70]), which states that for a smooth variety $X$ over the complex numbers, there is an equivalence

$$\text{Conn}^{\text{reg}} \rightarrow \text{Loc}(X^{\text{an}})$$
between the categories of regular integrable connections on $X$ and local systems on $X^{\text{an}}$, i.e. $\mathbb{C}_X$-modules which are locally free of finite rank for the analytic topology. Here an integrable connection is a $D_X$-module $M$ which is locally free of finite rank as an $\mathcal{O}_X$-module. This is nothing but a locally free $\mathcal{O}_X$-module of finite rank together with a $\mathbb{C}$-linear map $\nabla: M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$. For the above equivalence, we have to pass to a certain subcategory, namely the regular integrable connections, and the underlying functor of the equivalence is given by taking the kernel of $\nabla$. Note that if $d_X$ denotes the dimension of $X$, this is the cohomology in degree $-d_X$ of the de Rham complex located between the degrees $-d_X$ and 0 and whose differentials are induced by $\nabla$. Let $DR_X(M)$ denote this complex.

Both categories $\text{Conn}^\text{reg}$ and $\text{Loc}(X^{\text{an}})$ are not closed under push-forwards. For instance, the push-forward of a local system on the origin to the affine line is obviously not a local system. The correct extensions are (regular holonomic) $D_X$-modules on the left and constructible $\mathbb{C}_X$-sheaves on the right. However, the functor $H^{-d_X}(DR_X(\_))$ does not yield an equivalence between these larger categories. Again considering the example of the inclusion $i: \{0\} \to \mathbb{A}^1_\mathbb{C}$, we see that $H^{-1}(DR_{\mathbb{A}^1}(i_!\mathbb{C})) = 0$. This is due to the fact that we lose too much information by only taking into account the $-d_X$-th cohomology of $DR_X(\_)$.

To avoid this problem, one considers the derived functor $DR_X(\_) = \Omega^\text{L}_X \otimes_{\mathcal{O}_X} \_\text{L}$ between the derived categories of $D_X$-modules and constructible $\mathbb{C}_X$-sheaves. In the context of complex manifolds, Kashiwara ([Kas80] and [Kas84]) passed to a suitable subcategory called regular holonomic $D_X$-modules – more precisely the full subcategory of the bounded derived category $D^b(D_X)$ consisting of complexes whose cohomology sheaves are regular holonomic – and proved that the de Rham functor is an equivalence $D^b_{\text{rh}}(D_X) \to D^\text{L}_c(\mathbb{C}_X)$, which is compatible with the six operations $f_*, f^!, f^*, f^!, \text{RHom}^\text{!}$ and $\otimes$. Here $D^\text{L}_c(\mathbb{C}_X)$ denotes the full subcategory of $D(\mathbb{C}_X)$ of bounded complexes with constructible cohomology sheaves. This result from 1980 and 1984 is known as the Riemann-Hilbert correspondence. Around the same time, Mebkhout ([Meb84b] and [Meb84a]) gave a proof, which is independent of Kashiwara’s work. Later on, Beilinson and Bernstein developed the Riemann-Hilbert correspondence for algebraic $D$-modules on complex algebraic varieties. Their work is explained in the unpublished notes ([Ber]).

Deligne’s result, which is a special case of the Riemann-Hilbert correspondence, applied to $X = \mathbb{P}^1(\mathbb{C})\setminus S$, the Riemann sphere without a finite set $S$ of points, gives an answer to Hilbert’s 21st problem. For this recall that the sheaf of solutions of a system of linear differential equations is a local system. Via analytically continuing of local solutions along closed paths in $X$, we obtain a transition matrix and therefore a representation of the fundamental group of the Riemann sphere without $S$. The group of such matrices is called the monodromy group of the system of differential equations. Conversely, Hilbert’s 21st problem asks for the existence of a system of linear differen-
tial equations on the Riemann sphere with Fuchsian singularities in $S$ and with a given monodromy.

The functor $DR_X(\_)$ is closely related to the so-called solution functor $Sol_X = R\text{Hom}_{D_X}(\_ , \mathcal{O}_X)$: for every bounded complex $M^\bullet$ of $D_X$-modules, we have

$$DR_X(M^\bullet) \cong Sol_X(\mathcal{D}_X M^\bullet)[d_X],$$

where $\mathcal{D}_X$ is a certain duality. For a coherent $D_X$-module $M$, the sheaf $\text{Hom}_{D_X}(M, \mathcal{O}_X)$ can be identified with the solutions of the system of differential equations corresponding to $M$. Furthermore, there is an equivalence between representations of the fundamental group of $X$ and locally constant $\mathbb{C}_X$-sheaves. The Riemann-Hilbert correspondence in turn is a far reaching generalization of Deligne’s result.

Of course the essential image of the abelian category of regular holonomic $D_X$-modules under the equivalence $DR_X$ is an abelian category inside $D^b_c(\mathbb{C}_X)$, but it turns out that this category differs from the category of constructible $\mathbb{C}_X$-sheaves. The example of the immersion of the origin into the affine line from above already is a first sign of this phenomenon. The abelian subcategory of $D^b_c(\mathbb{C}_X)$ given by the essential image of regular holonomic $D_X$-modules under the de Rham functor is called perverse sheaves.

There is a general tool for describing abelian subcategories of triangulated categories: the theory of $t$-structures. A $t$-structure on a triangulated category $D$ consists of two subcategories $D^{\leq 0}$ and $D^{\geq 0}$ with certain properties. The intersection $D^{\leq 0} \cap D^{\geq 0}$ is called the heart of the $t$-structure. It is an abelian category. For example, the so-called canonical $t$-structure of $D^b_rh(D_X)$ is given by the two subcategories $D^{\leq 0}_{rh}(D_X)$ and $D^{\geq 0}_{rh}(D_X)$ of complexes whose cohomology is zero in positive or negative degrees. In the same way, the category of perverse sheaves on $X$ is obtained as the heart of a $t$-structure on $D^b_c(\mathbb{C}_X)$ which is called the perverse $t$-structure. Indeed, the development of the theory of perverse sheaves by Beilinson, Bernstein, Deligne and Gabber was motivated by the Riemann-Hilbert correspondence. A standard reference for this is [BBD82].

At the beginning of the 21st century, the time was right for a positive characteristic version of the Riemann-Hilbert correspondence. The de Rham theory for varieties over a field of positive characteristic $p$ differs strongly from the one on complex varieties. Instead of the Poincaré lemma, we have the Cartier isomorphism and as a consequence, for a smooth variety $X$, the kernel of the map $\mathcal{O}_X \to \Omega^1_X$ is not a locally constant $\mathbb{Z}/p\mathbb{Z}$-sheaf but given by the $p$-th powers $(\mathcal{O}_X)^p$. Therefore, one has to find a different approach. The Frobenius endomorphism $F$ is a major tool in characteristic $p$. Especially sheaves with an action of the Frobenius turned out to be very useful. The starting point of these objects is the sheaf $\mathcal{O}_{F,X} = \mathcal{O}_X[F]$ of non-commutative rings given on an affine open subset $U \subseteq X$ by the polynomial ring $\mathcal{O}_X(U)[F]$ with the relation $Fr = r^pF$ for local sections $r \in \mathcal{O}_X(U)$. A simple calculation shows that left $\mathcal{O}_X[F]$-modules are identified with $\mathcal{O}_X$-modules $\mathcal{F}$ together with a morphism $F^*\mathcal{F} \to \mathcal{F}$. In [Kat73, Proposition 4.1.1], Katz proved that there is an equivalence between the category of locally free étale $F$-sheaves and the category of coherent, locally free $\mathcal{O}_X$-modules $\mathcal{E}$ together with an isomorphism $F^*_\mathcal{E} \to \mathcal{E}$ of $\mathcal{O}_X$-modules. This may be considered as an analogue of Deligne’s result that there is a natural equivalence $\text{Conn}_{\text{reg}} \to \text{Loc}(X^\text{an})$. 

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It is this result of Katz that motivated Emerton and Kisin to consider left $\mathcal{O}_{F,X}$-modules for establishing an analogue of the Riemann-Hilbert correspondence for smooth varieties over a field $k$ of positive characteristic $p$. As Katz’ work already suggested, certain unit left $\mathcal{O}_{F,X}$-modules, i.e. $\mathcal{O}_{F,X}$-modules $\mathcal{F}$ whose structural morphism $F^*\mathcal{F} \to \mathcal{F}$ is an isomorphism together with some finiteness condition, is the subcategory to look at. In 2004, Emerton and Kisin published [EK04], where they proved that the functor $\text{Sol} = \text{RHom}_{\mathcal{O}_{F,X,\text{ét}}} (\underline{\cdot}_{\text{ét}}, \mathcal{O}_{X,\text{ét}})[d_X]$ yields an anti-equivalence

$$D_{\text{lfgu}}^b (\mathcal{O}_{F,X}) \to D^b_c (X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

between the bounded derived categories of locally finitely generated unit (lfgu for short) left $\mathcal{O}_{F,X}$-modules on the one hand, and the bounded derived category of constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves on the étale site $X_{\text{ét}}$ of $X$ on the other hand. Their correspondence is shown to be compatible with half of the six cohomological operations, namely $f^!$, $f_+$ and $\otimes$. They also prove that under the correspondence the abelian category $\mu_{\text{lfgu}}(X)$ of locally finitely generated unit modules corresponds to the category of perverse sheaves $\text{Perv}(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ defined by Gabber in [Gab04] on $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$. In this Riemann-Hilbert type correspondence, the sheaf of partial differential operators is substituted by the sheaf $\mathcal{O}_{F,X}$. Every $\mathcal{O}_{F,X}$-module naturally has the structure of a $\mathcal{D}_X$-module. The crucial point is that the ring $\mathcal{D}_X$ of arithmetic differential operators introduced by Berthelot equals the union $\bigcup \text{End}_{\mathcal{O}_{X,\text{ét}}}(\mathcal{O}_{X,\text{ét}})$ ([Ber96], [Ber00]). The details of the $\mathcal{D}_X$-module structure of an $\mathcal{O}_{F,X}$-module are explained in [Bli03]. It follows that the category considered by Emerton and Kisin is a subcategory of the category of left modules over the sheaf of rings of differential operators.

The sequence

$$0 \to \mathcal{O}_{X,\text{ét}} \xrightarrow{1-F} \mathcal{O}_{X,\text{ét}} \to 0$$

in some sense plays the role of the de Rham complex for varieties over $\mathbb{C}$. For instance, we can compute $\text{Sol}(\mathcal{O}_X) = \text{RHom}_{\mathcal{O}_{F,X,\text{ét}}} (\mathcal{O}_{X,\text{ét}}, \mathcal{O}_{X,\text{ét}})[d_X]$ using the resolution

$$0 \to \mathcal{O}_{F,X,\text{ét}} \xrightarrow{1-F} \mathcal{O}_{F,\text{ét}} \xrightarrow{\otimes} \mathcal{O}_{X,\text{ét}}$$

of $\mathcal{O}_{X,\text{ét}}$ by free left $\mathcal{O}_{F,X,\text{ét}}$-modules. As a consequence of Artin-Schreier theory, the sequence

$$0 \to (\mathbb{Z}/p\mathbb{Z})_X \to \mathcal{O}_{X,\text{ét}} \xrightarrow{1-F} \mathcal{O}_{X,\text{ét}} \to 0$$

is exact and therefore $\text{Sol}_k(\mathcal{O}_X) \cong (\mathbb{Z}/p\mathbb{Z})_X[d_X]$. This observation is fundamental in the proof of Emerton and Kisin’s Riemann-Hilbert correspondence.

In [BB11], Blickle and Böckle show that if $X$ is smooth and $F$-finite (i.e. the Frobenius morphism is a finite map), then Emerton-Kisin’s category $\mu_{\text{lfgu}}(X)$ is equivalent to their category $\text{Crys}_k(X)$ of Cartier crystals on $X$. This category is obtained by localizing the category of coherent sheaves $M$ on $X$ equipped with a right action by Frobenius, i.e. a map $F_*M \to M$, at the Serre subcategory consisting of those $M$ where the structural map is nilpotent.
The category of Cartier crystals is also defined on singular schemes, and a Kashiwara type equivalence holds in this context [BB13, Theorem 4.1.2], showing that Cartier crystals on a closed subscheme $Z \subseteq X$ are “the same” as Cartier crystals on $X$ supported in $Z$. This suggests that for singular schemes, the category of Cartier crystals should be a reasonable replacement for Emerton-Kisin’s theory, which was only developed for $X$ smooth. Hence one expects a natural equivalence of categories \[
\text{Crys}_\kappa(X) \to \text{Perv}(X_{\text{et}}, \mathbb{Z}/p\mathbb{Z})
\] for any $F$-finite scheme $X$. In this paper we show this result under the assumption that $X$ is a variety over a perfect field $k$, embeddable into a smooth variety. Note that a variety over a perfect field is $F$-finite. The closed immersion of $X$ into a smooth variety $Y$ enables us to employ the Kashiwara equivalence to show that the category of Cartier crystals on $X$ is equivalent to the category of $\text{lfgu}$ modules on $Y$ supported in $X$. This equivalence on the level of abelian categories then extends to a derived equivalence \[
D_b^{\text{crys}}(\text{QCrys}_\kappa(X)) \sim D_b^{\text{lfgu}}(\mathcal{O}_{F,Y})_X,
\] where $D_b^{\text{lfgu}}(\mathcal{O}_{F,Y})_X$ denotes the full subcategory of $D_b^{\text{lfgu}}(\mathcal{O}_{F,Y})$ consisting of complexes whose cohomology sheaves are supported in $X$. The details of this equivalence are worked out in Section 2 and involve showing that the equivalence sketched by Blickle and Böckle between Cartier crystals and $\mu_{\text{lfgu}}(X)$ alluded to above is compatible with pull-back functors for immersions of smooth, $F$-finite schemes and push-forward functors for arbitrary morphisms between smooth, $F$-finite schemes.

In Section 3 we give an intrinsic proof of the fact that for a variety $X$ over a perfect field $k$ the category $D_b^{\text{lfgu}}(\mathcal{O}_{F,X}) := D_b^{\text{lfgu}}(\mathcal{O}_{F,Y})_X$ is well-defined, i.e. independent of the embedding of $X$ into a smooth scheme $Y$. If one had resolution of singularities in characteristic $p$, one would have natural isomorphisms of functors $\text{Sol} f_* \cong f_! \text{Sol}$ for every morphism $f$ between smooth $k$-schemes [EK04, Theorem 9.7.1]. This would enable us to work with derived categories of constructible étale sheaves, which are defined on singular schemes as well, turning the independence of a chosen embedding into an easy exercise. As resolution of singularities is an open problem in higher dimensions, we are required to extend the adjunction between the functors $f^!$ and $f_*$ for proper $f$ from Emerton-Kisin to the case that $f$ is proper over some closed subset, which is somewhat technical. The source of this is a general adjunction statement for quasi-coherent sheaves provided in [Sch18]. It says that for a separated morphism of finite type $f: X \to Y$ of Noetherian schemes, closed immersions $i: Z \to Y$ and $i': Z' \to X$ and a proper morphism $f': Z' \to Z$ making the diagram

$$
\begin{array}{ccc}
Z' & \xrightarrow{i'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Z & \xrightarrow{i} & Y
\end{array}
$$

commutative, there exists a morphism $\text{tr}_f: Rf_*R\Gamma_Z f^! \to \text{id}$ which acts as the counit of an adjunction between $Rf_*$ and $R\Gamma_Z f^!$ regarded as functors between certain derived
categories with cohomology sheaves supported on $Z$ and $Z'$. Here $R\Gamma_{Z'}$ is the local cohomology functor.

Combining these steps, the following theorem summarizes the main results in this paper:

**Theorem.** Let $X$ be a variety over a perfect field and assume that $X$ is embeddable into a smooth variety $Y$. Then there are natural equivalences of categories

$$D^b_{\text{crys}}(\text{QCrys}_\kappa(X)) \sim \rightarrow D^b_{\text{lfgu}}(O_{F,Y})_X \sim \rightarrow D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}).$$

Here the middle category is independent of the embedding. These equivalences are compatible with the respectively defined push-forward and pull-back functors for immersions. Furthermore, the standard $t$-structure on the left corresponds to Gabber’s perverse $t$-structure on the right.

**Corollary.** The abelian category $\text{Crys}_\kappa(X)$ of Cartier crystals on a variety $X$ embeddable into a smooth variety is naturally equivalent to the category $\text{Perv}(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ of perverse constructible étale $p$-torsion sheaves.

While in the final stages of writing up these results, the preprint [Ohk16] appeared. Therein the author shows that Emerton-Kisin’s Riemann-Hilbert correspondence can be extended to the case that $X$ is embeddable into a proper smooth $W_n$-scheme. The case $n = 1$ hence also implies the right half of the just stated theorem in the case that $X$ is embeddable into a proper smooth scheme.

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**Notation and conventions**

Unless otherwise stated, all schemes are locally Noetherian and separated over the field $\mathbb{F}_p$ for some fixed prime number $p > 0$. For such a scheme $X$, we let $F_X$ or $F$, if no ambiguity is possible, denote the Frobenius endomorphism $X \rightarrow X$ which is the identity on the underlying topological space and which is given by $r \mapsto r^p$ on local sections. Often we will deal with $F$-finite schemes, i.e. $F$ is a finite morphism. For instance, a variety over a perfect field is $F$-finite.

Working with Emerton and Kisin’s category of locally finitely generated unit modules forces us at some points to restrict to varieties, i.e. to schemes which are of finite type over a field $k$ containing $\mathbb{F}_p$. With “schemes over $k$” or “$k$-scheme” we always mean
schemes which are separated and of finite type over $k$. For a smooth scheme $X$ over a perfect field $k$, the sheaf of top differential forms $\omega_X$ is an invertible sheaf with a canonical morphism $\omega_X \to F^!\omega_X$ of $\mathcal{O}_X$-modules given by the Cartier operator, see Example 1.2 for the affine space. One can check that it is an isomorphism. In general, if $X$ is regular and $F$-finite, we will assume that there is a dualizing sheaf $\omega_X$ with an isomorphism $\kappa_X : \omega_X \to F^!\omega_X$. For example, this assumption holds if $X$ is a scheme over a local Gorenstein scheme $S = \text{Spec} R$ ([BB11, Proposition 2.20]). Moreover, we assume that $\omega_X$ is invertible.

As in [EK04], for a smooth $k$-scheme $X$, we let $d_X$ denote the function

$$x \mapsto \text{dimension of the component of } X \text{ containing } x.$$ 

If $f : X \to Y$ is a morphism of smooth $k$-schemes, the relative dimension $d_{X/Y}$ is given by $d_{X/Y} = d_X - d_Y \circ f$.

1 Review of Cartier crystals and locally finitely generated unit modules

We begin by reviewing the definitions and results from the theory of Cartier crystals as developed by Blickle and Böckle in [BB11] and [BB13]. In short, a coherent Cartier module $M$ on $X$ is a coherent $\mathcal{O}_X$-module together with a right action of the Frobenius $F$. These form an abelian category and the category of Cartier crystals is obtained by localizing at the full Serre subcategory of those $M$ on which $F$ acts nilpotently. The resulting localized category is an abelian category, which has been shown in [BB11] to enjoy strong finiteness properties: All objects have finite length and all endomorphism sets are finite dimensional $\mathbb{F}_p$-vector spaces.

1.1 Cartier modules and Cartier crystals

**Definition 1.1.** A Cartier module on $X$ is a quasi-coherent $\mathcal{O}_X$-module $M$ together with a morphism of $\mathcal{O}_X$-modules

$$\kappa : F_* M \to M.$$ 

Equivalently, a Cartier module $M$ is a sheaf of right $\mathcal{O}_{F,X}$-modules whose underlying sheaf of $\mathcal{O}_X$-modules is quasi-coherent. Here $\mathcal{O}_{F,X}$ is the sheaf of (non-commutative) rings $\mathcal{O}_X[F]$, defined affine locally on $\text{Spec} R$ as the ring

$$R[F] := R\{F\} / \langle Fr - r^p F \mid r \in R \rangle.$$ 

On the level of abelian sheaves, $M$ and $F_* M$ are equal, hence we may view the structural map $\kappa$ of a Cartier module $M$ as an additive map $\kappa : M \to M$ which satisfies $\kappa(r^p \cdot m) = r\kappa(m)$ for all local sections $r \in \mathcal{O}_X$ and $m \in M$. In this way it is clear that defining the right action of $F$ on $M$ via $\kappa$ defines a right action of $\mathcal{O}_X[F]$ on $M$, and vice versa.
Iterations of $\kappa$ are defined inductively: $\kappa^n := \kappa \circ F_* \kappa^{n-1}$. Considering $\kappa$ as an additive map of abelian sheaves, $\kappa^n$ is the usual $n$-th iteration.

For a finite morphism $f: X \to Y$ of schemes, the functor $f_*$ is left adjoint to the functor $f^* := \mathcal{F} \text{Hom}_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \_)$, where $\mathcal{F}$ is the flat morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, f_* \mathcal{O}_X)$, see [Har66, III. 6]. Hence the structural morphism of a Cartier module $M$ on an $F$-finite scheme may also be given in the form $\tilde{\kappa}: M \to F^* M$.

**Example 1.2.** The prototypical example of a Cartier module is the sheaf $\omega_X$ of top differential forms on a smooth variety over a perfect field $k$. If $X = \text{Spec} k[x_1, \ldots, x_n]$, then $\omega_X$ is the free $k[x_1, \ldots, x_n]$-module of rank 1 generated by $dx_1 \wedge \cdots \wedge dx_n$. This module has a natural homomorphism $\kappa: F^* \omega_X \to \omega_X$ called the *Cartier operator* given by the formula

$$x_1^{(i_1+1)} \cdots x_n^{(i_n+1)} dx_1 \wedge \cdots \wedge dx_n \mapsto \left(\frac{i_1+1}{p}\right) \cdots \left(\frac{i_n+1}{p}\right) x_1 \wedge \cdots \wedge dx_n$$

where a non-integral exponent anywhere renders the whole expression zero.

A morphism $\varphi: M \to N$ of Cartier modules is a morphism of the underlying quasi-coherent sheaves making the following diagram commutative:

$$
\begin{array}{ccc}
F_* M & \xrightarrow{F_* \varphi} & F_* N \\
\kappa_M \downarrow & & \kappa_N \\
M & \xrightarrow{\varphi} & N.
\end{array}
$$

As $F_*$ is exact, one immediately verifies that Cartier modules form an abelian category, the kernels and cokernels being just the underlying kernels and cokernels in $\mathcal{O}_X$-modules with the induced structural morphism. We denote the category of Cartier modules on $X$ by $\text{QCoh}_\kappa(X)$. The full subcategory of coherent Cartier modules $\text{Coh}_\kappa(X)$ consists of those Cartier modules whose underlying $\mathcal{O}_X$-module is coherent. A Cartier module $(M, \kappa)$ is called nilpotent if some power of $\kappa$ is zero; $(M, \kappa)$ is called locally nilpotent if it is the union of its nilpotent Cartier submodules. By $\text{LNil}_\kappa(X)$ we denote the full subcategory of $\text{QCoh}_\kappa(X)$ consisting of locally nilpotent Cartier modules, and $\text{Nil}_\kappa(X)$ denotes the intersection $\text{Coh}_\kappa(X) \cap \text{LNil}_\kappa(X)$. The full subcategory of $\text{QCoh}_\kappa(X)$ consisting of extensions of coherent and locally nilpotent Cartier modules (in either order) we denote by $\text{LNilCoh}_\kappa(X)$. One has the following inclusions

$$
\text{Nil}_\kappa(X) \hookrightarrow \text{LNil}_\kappa(X) \hookrightarrow \text{LNilCoh}_\kappa(X) \hookrightarrow \text{Coh}_\kappa(X) \hookrightarrow \text{QCoh}_\kappa(X),
$$

and each of the full subcategories are Serre subcategories in their ambient category\(^1\). This leads us to our key construction.

\(^1\) A Serre subcategory is a full abelian subcategory which is closed under extensions.
**Definition 1.3.** The category of Cartier quasi-crystals is the localization of the category of quasi-coherent Cartier modules $\text{QCoh}_κ(X)$ at its Serre subcategory $\text{LNil}_κ(X)$. It is an abelian category, which we denote by $\text{QCrys}_κ(X)$.

Similarly, the category of Cartier crystals on $X$ is the localization of the category $\text{Coh}_κ(X)$ of coherent Cartier modules at its Serre subcategory $\text{Nil}_κ(X)$. It is an abelian category, which we denote by $\text{Crys}_κ(X)$. Cartier crystals also can be obtained by localizing $\text{LNilCoh}_κ(X)$ at the subcategory $\text{LNil}_κ(X)$.

In order to define derived functors we have to make sure that the considered categories have enough injectives.

**Proposition 1.4.** The category $\text{QCoh}_κ(X)$ is a Grothendieck category with enough injectives whose underlying $\mathcal{O}_X$-module is injective. Its Serre subcategory $\text{LNil}_κ(X)$ is localizing and hence $\text{QCrys}_κ(X)$ has enough injectives.

**Proof.** The first statements were shown in [BB13, Theorem 2.0.9 and Proposition 3.3.17]. That $\text{LNil}_κ(X)$ is localizing now follows from Corollaire 1 on p. 375 of [Gab62] and from the fact that each $M ∈ \text{QCoh}_κ(X)$ has a maximal locally nilpotent $κ$-subsheaf $M_{nil}$, see [BB13, Lemma 2.1.3]. Then Corollaire 2 of [Gab62] shows that the associated quotient category $\text{QCrys}_κ(X)$ has enough injectives.

Concretely, if $T: \text{QCoh}_κ(X) → \text{QCrys}_κ(X)$ denotes the exact localization functor, then the fact that $\text{LNil}_κ(X)$ is localizing asserts the existence of a right adjoint $V: \text{QCrys}_κ(X) → \text{QCoh}_κ(X)$. If $M/M_{nil} ↪ I$ is an injective hull in $\text{QCoh}_κ(X)$, then it is shown in op. cit. that $TI$ is an injective hull of $T(M/M_{nil})$.

The following finiteness statements are the main results of [BB11]:

**Theorem 1.5** ([BB11, Corollary 4.7, Theorem 4.17]). Let $X$ be a locally Noetherian, $F$-finite scheme of positive characteristic $p$.

(a) Every object in the category of Cartier crystals $\text{Crys}_κ(X)$ satisfies the ascending and descending chain condition on its subobjects.

(b) The Hom-sets in $\text{Crys}_κ(X)$ are finite dimensional $\mathbb{F}_p$-vector spaces.

These finiteness properties are precisely the ones one expects from a category of perverse constructible sheaves in the topological context. It is this result (and the related statement [EK04, Theorem 11.5.4] in the smooth case) that prompted our investigation of a connection between Cartier crystals and Gabber’s category of perverse constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves on $X_{ét}$, which is the content of this article.

### 1.2 Cartier crystals and morphisms of schemes

Up to now, we studied different categories stemming from quasi-coherent sheaves on a single scheme $X$. In this subsection, we consider morphisms $f: X → Y$ of schemes and construct functors between the categories of Cartier (quasi-)crystals on $X$ and on $Y$. With the notation $D^*(\mathcal{A})$ for an abelian category $\mathcal{A}$ and $* ∈ \{+, -, b\}$, we mean the
subcategory of the derived category $D(A)$ of bounded below, bounded above or bounded complexes.

The first result is concerned with the derived functor $Rf_*$ for quasi-coherent $\mathcal{O}_X$-modules. For a large class of morphisms it behaves well with the additional structure of Cartier modules and with localization at nilpotent objects. In principle, to any quasi-coherent Cartier module $M$ with structural map $\kappa_M$, we assign the quasi-coherent $\mathcal{O}_Y$-module $f^*M$ together with the composition

$$F_Y f_* M \xrightarrow{\sim} f_* F_X M \xrightarrow{f^* \kappa_M} f_* M.$$  

This is the underived functor $f_* : \text{QCoh}_\kappa(X) \rightarrow \text{QCoh}_\kappa(Y)$.

**Theorem 1.6 ([BB13, Corollary 3.2.12]).** Let $f : X \rightarrow Y$ be a morphism of $F$-finite schemes. Suppose $* \in \{+,-,b\}$. The functor $Rf_*$ on quasi-coherent sheaves induces a functor

$$Rf_* : D^*(\text{QCoh}_\kappa(X)) \rightarrow D^*(\text{QCoh}_\kappa(Y)).$$

It preserves local nilpotence and hence induces a functor

$$Rf_* : D^*_\text{crys}(\text{QCrysh}_\kappa(X)) \rightarrow D^*_\text{crys}(\text{QCrysh}_\kappa(Y)).$$

If $f$ is of finite type (but not necessarily proper!) then it restricts to a functor

$$Rf_* : D^*_\text{cris}(\text{QCrysh}_\kappa(X)) \rightarrow D^*_\text{cris}(\text{QCrysh}_\kappa(Y))$$

where the subscript crys indicates that the cohomology lies in $\text{LNilCrysh}_\kappa$.

For essentially étale morphisms and for closed immersions there are pull-back functors.

**Theorem 1.7.** Let $f : X \rightarrow Y$ be a morphism of schemes.

(a) Suppose $* \in \{+,-,b\}$. If $f$ is essentially étale, the exact functor $f^*$ induces a functor

$$f^! : D^*_\text{crys}(\text{QCrysh}_\kappa(Y)) \rightarrow D^*_\text{crys}(\text{QCrysh}_\kappa(X)),$$

which is left adjoint to $Rf_*$.  

(b) Suppose $* \in \{+,-\}$. If $f$ is a closed immersion of $F$-finite schemes, the functor $f^! = f^\flat = \mathcal{F} \text{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_X, \_)$, where $\mathcal{F}$ denotes the flat morphism $(X, \mathcal{O}_X) \rightarrow (Y, f_* \mathcal{O}_X)$ of ringed spaces, induces a functor

$$f^! : D^*_\text{crys}(\text{QCrysh}_\kappa(Y)) \rightarrow D^*_\text{crys}(\text{QCrysh}_\kappa(X)),$$

which is right adjoint to $Rf_*$.  

**Proof.** Let $M$ be a quasi-coherent Cartier module on $Y$. For essentially étale $f$, there is a canonical isomorphism $bc : F_X f^* \xrightarrow{\sim} f^* F_Y$. Hence we may equip $f^* M$ with the structural morphism given by the composition

$$F_X f^* M \xrightarrow{bc} f^* F_Y M \xrightarrow{f_\kappa M} f^* M.$$
As $f^*$ preserves coherence, we obtain a functor $\text{Coh}_\kappa(Y) \to \text{Coh}_\kappa(X)$. It is easy to see that $f^*$ preserves nilpotency. Therefore, and by exactness of $f^*$, we obtain the desired functor $D^b_{\text{crys}}(\text{QCrys}_\kappa(Y)) \to D^b_{\text{crys}}(\text{QCrys}_\kappa(Y))$.

In the case of a closed immersion $f$, the composition

$$f^! M \xrightarrow{f^! \tilde{\kappa}_*} f^! F_Y^M \xrightarrow{\tilde{\kappa}^*} F_X^M,$$

where $\tilde{\kappa}$ is the adjoint of $\kappa$, is a natural Cartier structure for the $\mathcal{O}_X$-module $f^! M$. Once again it remains to check that it gives rise to a functor $f^! D^b_{\text{crys}}(\text{QCrys}_\kappa(Y)) \to D^b_{\text{crys}}(\text{QCrys}_\kappa(X))$. The adjunctions of $Rf_*$ and $f^*$ or $f^!$ follow from the corresponding adjunctions for quasi-coherent sheaves. For more details see [BB13, Proposition 3.3.19] and [BB13, Corollary 3.3.24].

Now let $i: Z \to X$ be a closed immersion and $j: U \to X$ the open immersion of the complement $X \setminus Z$. Note that for a closed immersion $i$, the functor $i_*$ is exact and therefore we drop the $R$ indicating derived functors. The units and counits of the adjunctions between $i^*$ and $i^!$ and between $Rj_*$ and $j^!$ lead to a familiar distinguished triangle.

**Theorem 1.8** ([BB13, Theorem 4.1.1]). In $D^+_{\text{crys}}(\text{QCrys}_\kappa(X))$ there is a distinguished triangle

$$i_* i^! \to \text{id} \to Rj_* j^! \to i_* i^![1].$$

This theorem shows the equivalence mentioned in the following definition.

**Definition 1.9.** A complex $\mathcal{M}^\bullet$ of $D^b_{\text{crys}}(\text{QCrys}_\kappa(X))$ is supported on $Z$ if $j^! j^\bullet \mathcal{M} \to \mathcal{M}$ is an isomorphism. We let $D^b_{\text{crys}}(\text{QCrys}_\kappa(X))_Z$ denote the full triangulated subcategory consisting of complexes supported in $Z$.

For Cartier crystals, there is a natural isomorphism of functors $i_* i^! \cong R\Gamma_Z$ where $R\Gamma_Z$ is the local cohomology functor, see [BB11, Proposition 2.5] for the basic result concerning the abelian categories of Cartier modules and the proof of [BB13, Theorem 4.1.1]. This isomorphism identifies the distinguished triangle of Theorem 1.8 with the fundamental triangle of local cohomology. The following theorem is a formal consequence of Theorem 1.8:

**Theorem 1.10** ([BB13, Theorem 4.1.2]). Let $i: Z \to X$ be a closed immersion. The functors $i_*$ and $i^!$ are a pair of inverse equivalences

$$D^b_{\text{crys}}(\text{QCrys}_\kappa(Z)) \xrightarrow{i_*} D^b_{\text{crys}}(\text{QCrys}_\kappa(X))_Z.$$

We call this equivalence the Kashiwara equivalence. If $Z$ is a singular scheme which is embeddable into a smooth scheme $X$, the Kashiwara equivalence enables us to work with objects in $D^b_{\text{crys}}(\text{QCrys}(X))$ instead of $D^b_{\text{crys}}(\text{QCrys}(Z))$. 11
1.3 Review of locally finitely generated unit modules

In [EK04], Emerton and Kisin consider left $\mathcal{O}_{F,X}$-modules, i.e. $\mathcal{O}_X$-modules $\mathcal{M}$ with a structural morphism $F^*\mathcal{M} \to \mathcal{M}$. Instead of localizing, they pass to a certain subcategory. If we speak of $\mathcal{O}_{F,X}$-modules we mean left $\mathcal{O}_{F,X}$-modules. In this subsection, all schemes are separated and of finite type over a field $k$ containing $\mathbb{F}_p$.

**Definition 1.11.** Let $X$ be a variety over $k$. A quasi-coherent $\mathcal{O}_{F,X}$-module is an $\mathcal{O}_{F,X}$-module whose underlying $\mathcal{O}_X$-module is quasi-coherent. If the structural morphism $F^*\mathcal{M} \to \mathcal{M}$ of a quasi-coherent $\mathcal{O}_{F,X}$-module $\mathcal{M}$ is an isomorphism, then $\mathcal{M}$ is called unit. We let $\mu(X)$ and $\mu_u(X)$ denote the abelian categories of quasi-coherent and quasi-coherent unit $\mathcal{O}_{F,X}$-modules.

The term “locally finitely generated” for an $\mathcal{O}_{F,X}$-module $\mathcal{M}$ means that $\mathcal{M}$ is locally finitely generated as a left $\mathcal{O}_{F,X}$-module. Emerton and Kisin’s focus is on locally finitely generated unit modules, lfgu for short, on smooth schemes, where they form an abelian category.

**Definition 1.12.** We let $\mathcal{M}_{\text{lfgu}}(X)$ denote the abelian category of locally finitely generated unit $\mathcal{O}_{F,X}$-modules. We let $D_{\text{lfgu}}(\mathcal{O}_{F,X})$ denote the derived category of complexes of $\mathcal{O}_{F,X}$-modules whose cohomology sheaves are lfgu.

**Proposition 1.13.** Let $f: X \to Y$ be a morphism of smooth $k$-schemes. The functor $f^!: D(\mathcal{O}_{F,Y}) \to D(\mathcal{O}_{F,X})$ defined by

$$f^!\mathcal{M}^* = \mathcal{O}_{F,X} \otimes^{L}_{f^{-1}\mathcal{O}_{F,Y}} f^{-1}\mathcal{M}^*[d_X/Y]$$

restricts to a functor

$$f^!: D_{\text{lfgu}}(\mathcal{O}_{F,Y}) \to D_{\text{lfgu}}(\mathcal{O}_{F,X}).$$

Here $\mathcal{O}_{F,X} \to Y$ denotes $\mathcal{O}_{F,X}$ with the natural $(\mathcal{O}_{F,X}, f^{-1}\mathcal{O}_{F,Y})$-bimodule structure.

**Proof.** This is Lemma 2.3.2 and Proposition 6.7 of [EK04].

**Example 1.14.** Let $f: U \to X$ be an open immersion of smooth $k$-schemes. Then we have $d_{U/X} = 0$ and the inverse image of $\mathcal{O}_{F,X}$ is the restriction to $U$:

$$f^{-1}\mathcal{O}_{F,X} = \mathcal{O}_{F,X}|_U = \mathcal{O}_{F,U}.$$ 

Hence we regard $\mathcal{O}_{F,U \to X}$ as $\mathcal{O}_{F,U}$ with the usual $(\mathcal{O}_{F,U}, \mathcal{O}_{F,U})$-bimodule structure. It follows that $f^!\mathcal{M} = f^*\mathcal{M}$ with the natural structure as a left $\mathcal{O}_{F,U}$-module for every left $\mathcal{O}_{F,X}$-module $\mathcal{M}$.

The construction of the push-forward is more involved. Emerton and Kisin first show that $\mathcal{O}_{F,Y \to X} = f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_Y} \omega_{X/Y}$ is naturally an $(f^{-1}\mathcal{O}_{F,Y}, \mathcal{O}_{F,X})$-bimodule.
We summarize the construction of the right $\mathcal{O}_{F,X}$-module structure from Proposition-Definition 1.10.1, Proposition-Definition 3.3.1 and Appendix A.2 of [EK04]: The relative Frobenius diagram is the diagram

$$
\begin{array}{c}
X \\ F_{X/Y}
\end{array}
\xymatrix{ & X' \ar[l]_{f} & X \\
& Y \ar[u]^{f'} \ar[l]_{F_{Y}} & Y .}
$$

(1)

Here $X'$ is the fiber product of $X$ and $Y$ considered as a $Y$-scheme via the Frobenius and $F_{X/Y}$ is the map obtained from the Frobenius $F_{X}: X \to X$ and the morphism $f$. We call $F_{X/Y}$ the relative Frobenius. Unlike $F_{X}$, it is a morphism of $Y$-schemes. Locally, for $X = \text{Spec} \, S$ and $Y = \text{Spec} \, R$, the structure sheaf of $X'$ is given by the tensor product $R \otimes_{R} S$, where $R$ is viewed as an $R$-module via the Frobenius $F_{R}$. Globally we have an isomorphism $\mathcal{O}_{X'} \cong f^{-1}\mathcal{O}_{X}F \otimes_{f^{-1}\mathcal{O}_{Y}} \mathcal{O}_{Y}$ where $\mathcal{O}_{X}F$ denotes the submodule of $\mathcal{O}_{X}[F] = \mathcal{O}_{F,X}$ generated as a left $\mathcal{O}_{X}$-module by $F$. Consequently, for any $\mathcal{O}_{X}$-module $M$, $F_{Y}^{\bullet}M$ may be viewed as $f^{-1}\mathcal{O}_{X}F \otimes_{f^{-1}\mathcal{O}_{Y}} M$.

Let $\gamma: \mathcal{O}_{Y} \to F_{Y}^{\bullet}\mathcal{O}_{Y}$ be the canonical isomorphism. The adjoint of the composition

$$f^{\bullet}\mathcal{O}_{Y} \xrightarrow{\sim} F_{X/Y}^{\bullet}f^{\bullet}\mathcal{O}_{Y} \xrightarrow{F_{X/Y}^{\bullet}f^{\bullet}\gamma} F_{X/Y}^{\bullet}f^{\bullet}F_{Y}^{\bullet}\mathcal{O}_{Y} \xrightarrow{\sim} F_{X/Y}^{\bullet}F_{Y}^{\bullet}f^{\bullet}\mathcal{O}_{Y}$$

yields a morphism $C_{X/Y}: F_{X/Y}^{\bullet}\omega_{X/Y} \to F_{Y}^{\bullet}\omega_{X/Y}$ called the relative Cartier operator. Note that $F_{X/Y}$ is the identity on the underlying topological spaces of $X$ and $X'$. Therefore $C_{X/Y}$ defines a map of abelian sheaves $\omega_{X/Y} \to F_{Y}^{\bullet}\omega_{X/Y}$. Together with the identification $F_{Y}^{\bullet}\omega_{X/Y} \cong f^{-1}\mathcal{O}_{X}F \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y}$ and the inclusion $\mathcal{O}_{X}F \subset \mathcal{O}_{F,X}$ the relative Cartier defines a map $\omega_{X/Y} \to f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y}$. Now we can state the structure of $f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y}$ as a right $\mathcal{O}_{F,X}$-module. The endomorphism on $f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y}$ induced by multiplication with $F \in \mathcal{O}_{F,X}$ on the right is given by the composition

$$f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y} \xrightarrow{C_{X/Y}} f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y} \xrightarrow{m} f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} \omega_{X/Y},$$

where $m$ is the the multiplication $a \otimes b \mapsto ab$ in the sheaf of rings $f^{-1}\mathcal{O}_{F,Y}$. The functor $f_{+}: D(\mathcal{O}_{F,X}) \to D(\mathcal{O}_{F,Y})$ is then defined by

$$f_{+}\mathcal{M}^{\bullet} = Rf_{+}\mathcal{O}_{F,Y} \otimes_{\mathcal{O}_{F,X}} \mathcal{M}^{\bullet}.$$

**Proposition 1.15.** The functor $f_{+}: D(\mathcal{O}_{F,X}) \to D(\mathcal{O}_{F,Y})$ restricts to a functor

$$f_{+}: D_{\text{fgu}}^{b}(\mathcal{O}_{F,X}) \to D_{\text{fgu}}^{b}(\mathcal{O}_{F,Y}).$$

**Proof.** This is Theorem 3.5.3 and Proposition 6.8.2 of [EK04]. □
Example 1.16. Once again, let \( f : U \rightarrow X \) be an open immersion. Then \( F_{U/X} \) is an isomorphism, identifying \( X' \) with the open subset \( U \) of \( X \), and \( \omega_{U/X} = f^! \mathcal{O}_X = f^* \mathcal{O}_X = \mathcal{O}_U \). Therefore we have
\[
\mathcal{O}_{F,X \leftarrow U} = f^{-1} \mathcal{O}_{F,X} \otimes_{f^{-1} \mathcal{O}_X} \mathcal{O}_U = \mathcal{O}_{F,U} \otimes_{\mathcal{O}_U} \mathcal{O}_U.
\]
The left \( \mathcal{O}_{F,U} \)-module structures of \( \mathcal{O}_{F,U} \otimes_{\mathcal{O}_U} \mathcal{O}_U \) and \( \mathcal{O}_{F,U} \) are obviously compatible with the natural isomorphism \( \mathcal{O}_{F,U} \otimes_{\mathcal{O}_U} \mathcal{O}_U \cong \mathcal{O}_{F,U} \). One verifies that this isomorphism identifies the right \( \mathcal{O}_{F,U} \)-module structure on \( \mathcal{O}_{F,U} \otimes_{\mathcal{O}_U} \mathcal{O}_U \) with the natural one on \( \mathcal{O}_{F,U} \).

Depending on \( f \), there are adjunction relations between \( f^! \) and \( f_+ \). If \( f \) is a closed immersion, a Kashiwara-type equivalence for unit modules holds.

Lemma 1.17 ([EK04, Lemma 4.3.1]). If \( f : X \rightarrow Y \) is an open immersion of smooth \( k \)-schemes, then, for any \( \mathcal{M}^* \in D^-(\mathcal{O}_{F,Y}) \) and any \( \mathcal{N}^* \in D^+(\mathcal{O}_{F,X}) \), there is a natural isomorphism
\[
\mathcal{R} \text{Hom}_{\mathcal{O}_{F,Y}}^\bullet (\mathcal{M}^*, f_+ \mathcal{N}^*) \sim \mathcal{R} f_* \mathcal{R} \text{Hom}_{\mathcal{O}_{F,X}}^\bullet (f^! \mathcal{M}^*, \mathcal{N}^*)
\]
in \( D^+(X, \mathbb{Z}/p\mathbb{Z}) \).

Theorem 1.18 ([KEK04, Theorem 4.4.1]). Let \( f : X \rightarrow Y \) be a proper morphism of smooth \( k \)-schemes. For every \( \mathcal{M}^* \in D^b_{qc}(\mathcal{O}_{F,X}) \) and every \( \mathcal{N}^* \in D^b_{qc}(\mathcal{O}_{F,Y}) \), there is a natural isomorphism in \( D^+(X, \mathbb{Z}/p\mathbb{Z}) \):
\[
\mathcal{R} \text{Hom}_{\mathcal{O}_{F,Y}}^\bullet (f_+ \mathcal{M}^*, \mathcal{N}^*) \sim \mathcal{R} f_* \mathcal{R} \text{Hom}_{\mathcal{O}_{F,X}}^\bullet (\mathcal{M}^*, f^! \mathcal{N}^*)
\]
Here \( D^b_{qc}(\mathcal{O}_{F,X}) \) denotes the subcategory of \( D^b(\mathcal{O}_{F,X}) \) of complexes whose cohomology sheaves are quasi-coherent and analogously for \( D^b_{qc}(\mathcal{O}_{F,Y}) \).

For the proof, Emerton and Kisin show that the trace map \( f_* f^\Delta E^* \rightarrow E^* \) for the residual complex \( E^* \) of \( \mathcal{O}_X \) is compatible with the natural map \( E^* \rightarrow F_X^* E^* \). Here \( f^\Delta \) denotes the functor \( f^! \) for residual complexes, see [Har66, VI.3]. Thus it induces a morphism \( f_+ \mathcal{O}_{F,X}[d_{X/Y}] \rightarrow \mathcal{O}_{F,Y} \), and with the isomorphisms
\[
f_+ f^! \mathcal{F}^* \rightarrow f_+ (\mathcal{O}_{F,X} \otimes_{\mathcal{O}_{F,X}} f^! \mathcal{F}^*) \rightarrow f_+ \mathcal{O}_{F,X}[d_{X/Y}] \otimes_{\mathcal{O}_{F,Y}} \mathcal{F}^*;
\]
the second one being a projection formula ([EK04, Lemma 4.4.7]), we obtain a trace map \( \text{tr} : f_+ f^! \mathcal{F}^* \rightarrow \mathcal{F}^* \) for every \( \mathcal{F}^* \in D^b_{lfgu}(\mathcal{O}_{F,Y}) \). Similarly, as in the case of the adjunction between \( Rf_* \) and \( f^! \) in Grothendieck-Serre duality, the natural transformation of the theorem is obtained by the composition
\[
\mathcal{R} f_* \mathcal{R} \text{Hom}_{\mathcal{O}_{F,X}}^\bullet (\mathcal{M}^*, f^! \mathcal{N}^*) \xrightarrow{\text{tr}} \mathcal{R} \text{Hom}_{\mathcal{O}_{F,Y}}^\bullet (f_+ \mathcal{M}^*, f_+ f^! \mathcal{N}^*)
\]
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where the horizontal arrow is a natural transformation constructed in [EK04, Proposition 4.4.2]. It is this adjunction between $f_+$ and $f^!$ that we want to extend to morphisms which are only proper over the support of the considered complexes. This will be done in section 4.

Finally, for a closed immersion of smooth varieties, we have a Kashiwara type equivalence.

**Theorem 1.19** ([EK04, Theorem 5.10.1]). If $f : X \to Y$ is a closed immersion of smooth $k$-schemes, then the adjunction of Theorem 1.18 provides an equivalence between the category of unit $\mathcal{O}_{F,X}$-modules and the category of unit $\mathcal{O}_{F,Y}$-modules supported on $X$. The fact that the natural map $f_+f^!M \to M$ is an isomorphism implies that $H^0(f^!M) \cong f^!M$.

2 From Cartier crystals to locally finitely generated unit modules

In order to construct an equivalence between Cartier crystals and locally finitely generated unit modules, one uses an equivalence between Cartier modules and so-called $\gamma$-sheaves. It will induce an equivalence between Cartier crystals and $\gamma$-crystals. The latter in turn are known to be equivalent to lfgu modules.

2.1 Cartier modules and $\gamma$-sheaves

We note that for a regular scheme $X$, the Frobenius $F_X : X \to X$ is a flat morphism and hence $F_X^*$ is exact ([Kun69, Theorem 2.1]).

**Definition 2.1.** A $\gamma$-sheaf on a regular, $F$-finite scheme $X$ is a quasi-coherent $\mathcal{O}_X$-module $N$ together with a morphism $\gamma_N : N \to F^*N$.

The theory of $\gamma$-sheaves is very similar to that of Cartier modules. With the obvious morphisms, $\gamma$-sheaves form an abelian category with the nilpotent $\gamma$-sheaves being a Serre subcategory. We obtain $\gamma$-crystals in the same way as we obtained Cartier crystals and so forth. In this section we revisit the connection between Cartier modules and $\gamma$-sheaves as explained in section 5.2.1 of [BB11] and give some details of the proof.

**Definition 2.2.** For any isomorphism $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ of invertible $\mathcal{O}_X$-modules, while $\varphi^{-1} : \mathcal{E}_2 \to \mathcal{E}_1$ denotes the inverse, let $\varphi^!$ denote the induced isomorphism $\mathcal{E}_2^{-1} \to \mathcal{E}_1^{-1}$ between the duals.

If we speak of the $\gamma$-sheaf $\mathcal{O}_X$ we mean the structure sheaf of $X$ together with the natural isomorphism $\gamma_X : \mathcal{O}_X \to F^*\mathcal{O}_X$. By abuse of notation we call this isomorphism the Frobenius.

For a regular, $F$-finite scheme $X$, let $\kappa_X$, or $\kappa_R$ if $X = \text{Spec} R$ is affine, denote the natural isomorphism $\omega_X \cong F^0\omega_X$, which is the adjoint of the Cartier operator if $X$ is a smooth variety.
The next lemma makes explicit a fundamental isomorphism, which will be used repeatedly.

**Lemma 2.3 ([BB11, Lemma 5.7]).** Let \( f : X \rightarrow Y \) be a finite and flat morphism of schemes. For every quasi-coherent \( \mathcal{O}_Y \)-module \( \mathcal{F} \), there is a natural isomorphism

\[
\text{can} : f^! \mathcal{O}_X \otimes_{\mathcal{O}_X} f^* \mathcal{G} \sim f^! \mathcal{G}.
\]

**Proof.** It suffices to construct a natural isomorphism locally and therefore we can identify \( f \) with a ring homomorphism \( R \rightarrow S \) and \( \mathcal{F} \) with an \( R \)-module \( M \). Define the homomorphism

\[
\text{can} : \text{Hom}_R(S, R) \otimes_S (M \otimes_R S) \rightarrow \text{Hom}_R(S, M)
\]

of \( S \)-modules by mapping \( \alpha \otimes (m \otimes t) \) to the homomorphism \( s \mapsto \alpha(st)m \). Since \( f \) is finite flat, we can assume that \( S \) is a free \( R \)-module and choose a basis \( s_1, s_2, \ldots, s_n \). Let \( \varphi_1, \ldots, \varphi_n \) be the dual basis, i.e. \( \varphi_i \in \text{Hom}_R(S, R) \) and \( \varphi_i(j) = \delta_{ij} \). One easily checks that the map

\[
\varphi \mapsto \sum_{i=1}^n \varphi_i \otimes \varphi(s_i).
\]

is inverse to \( \text{can} \).

The following definition is extracted from [BB11, Theorem 5.9].

**Definition 2.4.** Let \( X \) be a regular, \( F \)-finite scheme.

(a) For every Cartier module \( M \) with structural morphism \( \kappa \), the sheaf \( M \otimes \omega_X^{-1} \) has a natural \( \gamma \)-structure given by the composition

\[
\begin{array}{ccc}
M \otimes \omega_X^{-1} & \xrightarrow{\kappa_M \otimes (\kappa_X^{-1})^{-1}} & F^\phi M \otimes (F^\phi \omega_X)^{-1} \\
& \downarrow \sim & \downarrow \sim \\
& \xrightarrow{\text{can}^{-1} \otimes \text{can}^\vee} & F^\phi \mathcal{O}_X \otimes F^* M \otimes (F^\phi \mathcal{O}_X)^{-1} \otimes F^* \omega_X^{-1} \\
& \xrightarrow{\text{ev}} & F^* (M \otimes \omega_X^{-1}) \\
\end{array}
\]

where the vertical arrow on the right is the permutation and \( \text{ev}_L : \mathcal{L} \otimes \mathcal{O}_X \mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{O}_X \) is the evaluation map \( l \otimes \varphi \mapsto \varphi(l) \). This morphism is called the \( \gamma \)-structure of \( M \otimes \omega_X^{-1} \) induced by \( \kappa_M \).
Lemma 2.3. The contains explicit formulas

Definition 2.4. Lemma 2.17

Proof. By construction, the structural morphism of Lemma 2.6.

Let upper horizontal and the rightmost vertical arrow of the following diagram:

\[ N \otimes \omega_X \xrightarrow{\gamma_N \otimes \kappa_X} F^s N \otimes F^s \omega_X \xrightarrow{\text{id} \otimes \text{can}^{-1}} F^s N \otimes F^s \mathcal{O}_X \otimes F^s \omega_X \xrightarrow{\sim} F^b(\gamma N \otimes \omega_X) \xrightarrow{\text{can}} F^b \mathcal{O}_X \otimes F^s N \otimes F^s \omega_X, \]

where the vertical arrow on the right is the permutation. This morphism is called the Cartier structure of \( N \otimes \omega_X \) induced by \( \gamma_N \).

Remark 2.5. Thanks to the fact that the proof of Lemma 2.3 contains explicit formulas for the isomorphism can and its inverse, we can concretely describe the induced Cartier structure of \( N \otimes \omega_R \) for a \( \gamma \)-module \( N \) over a regular ring \( R \) such that \( F_* R \) is free with basis \( s_1, \ldots, s_r \). For \( m \in \omega_R \) set \( \varphi_m := \kappa_R(m) \) and let \( \varphi_1, \ldots, \varphi_r \in \text{Hom}_R(F_* R, R) \) be the dual basis of \( s_1, \ldots, s_r \), this means \( \varphi_i(s_j) = \delta_{ij} \). Following the arrows of Definition 2.4, we see that the Cartier structure \( N \otimes \omega_X \rightarrow F^b(N \otimes \omega_X) \) is given by

\[
\begin{align*}
    n \otimes m &\mapsto \gamma(n) \otimes \varphi_m \\
    &\mapsto \sum_i \gamma(n) \otimes \varphi_i \otimes \varphi_m(s_i) \\
    &\mapsto (s \mapsto \sum_i \gamma(n) \otimes \varphi_i(s) \otimes \varphi_m(s_i)).
\end{align*}
\]

We will need this concrete version later on to prove that, for affine schemes, assigning a Cartier module to a \( \gamma \)-sheaf commutes with certain pullbacks, see Lemma 2.17. The use of the isomorphism \( F^b \mathcal{O}_X \otimes (F^b \mathcal{O}_X)^{-1} \cong \mathcal{O}_X \) involves the concrete formula for the structural morphism of the \( \gamma \)-sheaf associated to a Cartier module.

Lemma 2.6. Let \((N, \gamma_N)\) be a \( \gamma \)-sheaf on a regular, \( F \)-finite scheme \( X \). The adjoint \( F_* (N \otimes \omega_X) \rightarrow N \otimes \omega_X \) of the structural morphism of the Cartier module \( N \otimes \omega_X \) is given by the composition

\[ F_* (N \otimes \omega_X) \xrightarrow{\gamma_N} F_* (F^s N \otimes \omega_X) \xrightarrow{\sim} N \otimes F_* \omega_X \xrightarrow{\tilde{\kappa}_X} N \otimes \omega_X, \]

where the isomorphism in the middle is given by the projection formula.

Proof. By construction, the structural morphism of \( N \otimes \omega_X \) is the composition of the upper horizontal and the rightmost vertical arrow of the following diagram:

\[
\begin{array}{ccc}
N \otimes \omega_X & \xrightarrow{\gamma_N} & F^s N \otimes \omega_X \\
\text{adj} & & \text{adj} \\
F^b F_* (N \otimes \omega_X) & \xrightarrow{\gamma_N} & F^b F_* (F^s N \otimes \omega_X) \xrightarrow{\text{proj}^{-1}} F^b (N \otimes F_* \omega_X) \xrightarrow{\tilde{\kappa}_X} F^b (N \otimes \omega_X). \\
\end{array}
\]
Here \( \text{adj} \) denotes the respective adjunction morphism and \( \text{proj} \) is the isomorphism from the projection formula. The third and the fourth vertical morphism are isomorphisms stemming from \( \text{can} \). For example, the morphism \( F^* N \otimes F^\partial \omega_X \to F^\partial (N \otimes F_* \omega_X) \) is the composition
\[
F^* N \otimes F^\partial \omega_X \xrightarrow{\text{id} \otimes \text{can}^{-1}} F^* N \otimes F^* \omega_X \otimes F^\partial \mathcal{O}_X \xrightarrow{\text{can}} F^\partial (N \otimes \omega_X).
\]

Following the leftmost vertical and the lower horizontal arrows we obtain the adjoint of the morphism which is claimed to be the adjoint of the Cartier structure of \( N \otimes \omega_X \). Hence it suffices to show that the diagram above is commutative.

The first and the last square commute by functoriality. The commutativity of the square in the middle can be checked locally on affine open subsets of \( X \) because \( F \) is an affine morphism. \( \square \)

For the proof of Proposition 2.10, we need the isomorphism \( \omega_X \otimes \omega_X^{-1} \cong \mathcal{O}_X \) of \( \gamma \)-sheaves, which is a consequence of the following general lemma.

**Lemma 2.7.** Let \( f: X \to Y \) be a morphism of schemes and \( \mathcal{L} \) an invertible \( \mathcal{O}_Y \)-module.

(a) If \( \rho: \mathcal{L} \sim \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2 \) is an isomorphism with invertible \( \mathcal{O}_X \)-modules \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), then the diagram
\[
\begin{array}{ccc}
\mathcal{L} \otimes \mathcal{L}^{-1} & \xrightarrow{\rho \otimes (\rho^{-1})} & \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} \\
\text{ev}_\mathcal{L} & & \text{ev}_{\mathcal{L}_1} \times \text{ev}_{\mathcal{L}_2}
\end{array}
\]
commutes.

(b) The diagram of canonical isomorphisms
\[
f^*(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1}) \xrightarrow{\text{id} \otimes \text{ev}_{\mathcal{L}}} f^* \mathcal{L} \otimes_{\mathcal{O}_X} (f^* \mathcal{L})^{-1} \]
\[
f^* \mathcal{O}_X \xrightarrow{\text{id}} \mathcal{O}_Y
\]
commutes.

**Proof.** It suffices to verify the claims for an affine scheme \( X = \text{Spec} \, R \) and, for (b), for a morphism of affine schemes \( \text{Spec} \, S \to \text{Spec} \, R \) and an \( R \)-module \( M \). In this affine situation the claims follow from straightforward calculations. \( \square \)

**Example 2.8.** The \( \gamma \)-sheaf \( \omega := \omega_X \otimes \omega_X^{-1} \) on a regular, \( F \)-finite scheme \( X \) is canonically isomorphic to the structure sheaf \( \mathcal{O}_X \) equipped with the natural morphism \( \mathcal{O}_X \to \mathcal{O}_X \).
We have to show that the diagram

\[
\begin{array}{ccc}
\omega_X \otimes \omega_X^{-1} & \xrightarrow{ev_{\omega_X}} & O_X \\
F^\flat O_X \otimes F^\ast \omega_X \otimes (F^\flat O_X)^{-1} \otimes F^\ast \omega_X^{-1} & \xrightarrow{id} & F^\ast O_X \\
& \sim & \sim \\
F^\ast \omega_X \otimes F^\ast \omega_X^{-1} & \xrightarrow{ev_{F^\ast \omega_X}} & O_X \\
& \sim & \sim \\
F^\ast (\omega_X \otimes \omega_X^{-1}) & \xrightarrow{F^\ast ev_{\omega_X}} & F^\ast O_X
\end{array}
\]

commutes. The commutativity of both the top and the bottom rectangle follows from Lemma 2.7. Note that the horizontal isomorphisms of the lower square are the inverses of the natural isomorphisms of part (b) of Lemma 2.7. By definition, the \(\gamma\)-structure of \(\omega_X \otimes \omega_X^{-1}\) is given by the composition of the vertical arrows on the right. Hence the \(\gamma\)-structure of \(O_X\) with respect to the natural isomorphism \(O_X \xrightarrow{ev_{\omega_X}} \omega_X \otimes \omega_X^{-1}\) is the Frobenius.

Similarly, starting with the \(\gamma\)-module \(O_X\), the induced Cartier structure of \(O_X \otimes \omega_X\) is compatible with \(\kappa_X\) with respect to the isomorphism \(O_X \otimes \omega_X \cong \omega_X\).

**Definition 2.9.** Let \(\text{QCoh}_\gamma(X)\) denote the category of \(\gamma\)-sheaves and let \(\text{Coh}_\gamma(X)\) denote the category of \(\gamma\)-sheaves whose underlying \(O_X\)-module is coherent. We let \(\text{QCrys}_\gamma(X)\) and \(\text{Crys}_\gamma(X)\) denote the corresponding categories of crystals, see Definition 1.3.

**Proposition 2.10.** If \(X\) is a regular, \(F\)-finite scheme, then tensoring with \(\omega_X\) and its inverse induces inverse equivalences of categories between Cartier modules and \(\gamma\)-sheaves on \(X\):

\[
\begin{array}{ccc}
\text{QCoh}_\kappa(X) & \xrightarrow{\otimes_{O_X} \omega_X^{-1}} & \text{QCoh}_\gamma(X) \\
\text{Coh}_\kappa(X) & \xrightarrow{\otimes_{O_X} \omega_X^{-1}} & \text{Coh}_\gamma(X)
\end{array}
\]

In terms of this equivalence, the Cartier module \((\omega_X, \kappa_X)\) corresponds to the \(\gamma\)-sheaf \((O_X, \gamma_X)\).

**Proof.** Let \(\omega\) denote the \(\gamma\)-sheaf \(\omega_X^{-1} \otimes \omega_X\) and \(\gamma_\omega\) its structural morphism. (Note that there is no considerable difference between \(\omega_X \otimes \omega_X^{-1}\) and \(\omega_X^{-1} \otimes \omega_X\).) We start with a Cartier module \((M, \kappa_M)\). Consider the diagram (3) on page 23. Passing through the top arrow we follow the construction of the structural morphism of \(M \otimes \omega_X^{-1} \otimes \omega_X\) while the structural morphism of \(\kappa_M\) is given by the composition of the horizontal arrows on the
\[ \kappa_M = \text{can} \circ (\text{can}^{-1} \circ \kappa_M). \] Hence we have to show that (3) is commutative. Here \( \mu \) denotes a permutation of the tensor product followed by the evaluation map, similar to the top most horizontal arrow. More precisely, it is the composition

\[
\begin{array}{c}
\mathcal{O}_X \otimes F^* \mathcal{M} \otimes (F^\flat \mathcal{O}_X)^{-1} \otimes F^* \mathcal{M} \otimes F^* \mathcal{O}_X \\
\sim \\
\mathcal{O}_X \otimes F^* \mathcal{M} \otimes F^* \mathcal{M}^{-1} \otimes ((F^\flat \mathcal{O}_X)^{-1} \otimes F^\flat \mathcal{O}_X) \otimes F^* \mathcal{O}_X \\
\sim \\
\mathcal{O}_X \otimes F^* (\mathcal{M} \otimes \mathcal{M}^{-1} \otimes \mathcal{O}_X)
\end{array}
\]

of natural isomorphisms and \( \text{ev} \). For simplicity, we will not distinguish between \( F^*(\mathcal{M} \otimes \mathcal{O}_X) \) and \( F^* \mathcal{M} \otimes \mathcal{O}_X \). The commutativity of the upper square is an easy computation.

In the lower left square the map from \( \mathcal{M} \otimes \mathcal{O}_X \otimes \mathcal{O}_X^{-1} \) to \( \mathcal{O}_X \otimes F^*(\mathcal{M} \otimes \mathcal{O}_X) \) is the composition

\[
\begin{array}{c}
\mathcal{M} \otimes \mathcal{O}_X \otimes \mathcal{O}_X^{-1} \\
\sim \\
\mathcal{O}_X \otimes F^*(\mathcal{M} \otimes \mathcal{O}_X) \\
\sim
\end{array}
\]

Hence it suffices to show that (2) is commutative. The commutativity of the left square is Example 2.8 tensored with \( F^* \mathcal{M} \). That the right square commutes can easily be checked by hand: For an arbitrary commutative ring \( R \), an \( R \)-algebra \( S \) and an \( R \)-module \( \mathcal{M} \), the diagram

\[
\begin{array}{ccc}
F^* \mathcal{M} \otimes \mathcal{O}_X & \sim & F^* \mathcal{M} \\
\sim & & \sim \\
F^* \mathcal{M} \otimes F^* (\mathcal{O}_X^{-1} \otimes \mathcal{O}_X) & \sim & F^* (\mathcal{M} \otimes \mathcal{O}_X)
\end{array}
\]

is commutative. The commutativity of the left square is Example 2.8 tensored with \( F^* \mathcal{M} \). That the right square commutes can easily be checked by hand: For an arbitrary commutative ring \( R \), an \( R \)-algebra \( S \) and an \( R \)-module \( \mathcal{M} \), the diagram

\[
\begin{array}{ccc}
(M \otimes_R S) \otimes_S S & \rightarrow & M \otimes_R S \\
\downarrow & & \downarrow \\
(M \otimes_R S) \otimes_S (R \otimes_R S) & \rightarrow & (M \otimes_R R) \otimes_R S
\end{array}
\]

of natural homomorphisms is commutative. Along both ways an element \( (m \otimes s_1) \otimes s_2 \) is mapped to \( (m \otimes 1) \otimes s_1s_2 \). Locally the right square of (2) is just a special case of this diagram.

Now let \( (N, \gamma_N) \) be a \( \gamma \)-sheaf. By definition, the structural morphism \( \gamma'_N \) of \( N \otimes \mathcal{O}_X \otimes \mathcal{O}_X^{-1} \) is the line in the middle of the diagram (4). The upper squares of this diagram
commute by construction. Here the horizontal morphism to the top right corner is given by \( \text{id} \otimes (\text{can}^{-1} \circ \overline{\kappa}_X) \) and the horizontal morphism below is the unique morphism making the upper right square commute. Therefore we see that \( \gamma'_X \) is the tensor product of \( \gamma_N \) and \( \gamma_\omega \), i.e. the bottom rectangle of (4) is commutative.

Now consider the diagram

\[
\begin{array}{cccccc}
N \otimes \omega & \xrightarrow{\text{id} \otimes \gamma_\omega} & N \otimes F^*\omega & \xrightarrow{\gamma_N \otimes \text{id}} & F^*N \otimes F^*\omega & \sim & F^*(N \otimes \omega) \\
\downarrow{\text{id} \otimes \delta} & & \downarrow{\text{id} \otimes F^*\delta} & & \downarrow{\text{id} \otimes F^*\delta} & & \downarrow{F^*(\text{id} \otimes \delta)} \\
N \otimes O_X & \xrightarrow{\text{id} \otimes \gamma_X} & N \otimes F^*O_X & \xrightarrow{\gamma_N \otimes \text{id}} & F^*N \otimes F^*O_X & \sim & F^*(N \otimes O_X) \\
\downarrow{\sim} & & \downarrow{\gamma_N \otimes \text{id}} & & \downarrow{\gamma_N \otimes \text{id}} & & \downarrow{\sim} \\
N & \xrightarrow{\text{id} \otimes \gamma_X} & F^*N \otimes O_X & \sim & F^*N. \\
\end{array}
\]

The upper left square is the commutative diagram of Example 2.8 tensored with \( N \). The upper square in the middle and the bottom left rectangle are clearly commutative. The square to the left of it commutes because of the naturality of the isomorphism \( F^*N \otimes F^*(\_ \_\_) \sim \rightarrow F^*(N \otimes \_\_) \). We already have seen that the bottom right square commutes: It is the same square as the left one of diagram (2) with \( M \) replaced by \( N \). Moreover, the composition of the leftmost arrow is \( N \otimes \delta \) and the composition of the rightmost arrow is \( F^*(N \otimes \delta) \).

Hence the structural morphism of \( N \) is compatible with \( \gamma_N \otimes \gamma_\omega \), which turned out to be compatible with the structural morphism induced from the Cartier module \( N \otimes \omega_X \). Thus we can extract the commutative diagram

\[
\begin{array}{cccccc}
N \otimes \omega_X \otimes \omega_X^{-1} & \xrightarrow{\gamma'_X} & F^*(N \otimes \omega_X \otimes \omega_X^{-1}) \\
\downarrow{\text{id} \otimes \delta} & & \downarrow{\gamma_N} & & \downarrow{F^*(\text{id} \otimes \delta)} \\
N & \xrightarrow{\gamma_N} & F^*N. \\
\end{array}
\]

It follows that the functors \( \_ \_ \otimes \omega_X^{-1} \) and \( \_ \_ \otimes \omega_X \) are inverse equivalences. From Example 2.8 we know that \( \_ \_ \otimes \omega_X^{-1} \) maps \( \omega_X \) with the structural morphism \( \kappa_X \) to \( O_X \) with the structural morphism \( \gamma_X \). Consequently, \( \_ \_ \otimes \omega_X \) maps the \( \gamma \)-sheaf \((O_X, \gamma_X)\) to the Cartier module \((\omega_X, \kappa_X)\).

**Corollary 2.11.** Tensoring with \( \omega_X \) and with \( \omega_X^{-1} \) induces equivalences of categories

\[
\begin{array}{ccc}
\text{QCrys}_\kappa(X) & \xrightarrow{\otimes_{O_X} \omega_X^{-1}} & \text{QCrys}_\gamma(X) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Crys}_\kappa(X) & \xrightarrow{\otimes_{O_X} \omega_X^{-1}} & \text{Crys}_\gamma(X) \\
\end{array}
\]

for every regular, \( F \)-finite scheme \( X \).
**Corollary 2.12.** If $X$ is regular and $F$-finite, the categories $\text{QCoh}_\gamma(X)$ and $\text{QCrys}_\gamma(X)$ have enough injectives.

**Proof.** This follows from Proposition 1.4 and Corollary 2.11. □
2.2 Compatibility with pull-back

The pull-back of quasi-coherent sheaves defines a pull-back functor on $\gamma$-sheaves:

**Definition 2.13.** Let $f: X \to Y$ be a morphism of regular schemes and $N$ a $\gamma$-sheaf on $Y$ with structural morphism $\gamma_N$. The $\gamma$-structure for $f^*N$ is defined as the composition

$$f^*N \xrightarrow{\gamma_N} f^*F_Y^*N \xrightarrow{\sim} F_Y^*f^*N.$$ 

First we consider a closed immersion $i: X \to Y$ of regular, $F$-finite schemes. The aim is to prove the following theorem:

**Theorem 2.14.** Let $i: X \to Y$ be a closed immersion of regular, $F$-finite schemes with codimension $n$. Then there is a canonical isomorphism of functors

$$\otimes \omega_Y^{-1} \circ R^n i^\flat \cong i^* \circ \otimes \omega_Y^{-1}$$

inducing a corresponding isomorphism of functors of crystals, i.e. the diagram

$$\begin{array}{ccc}
\text{Crys}_{\kappa}(Y) & \xrightarrow{\otimes \omega_Y^{-1}} & \text{Crys}_{\gamma}(Y) \\
| & | & | \\
\text{Crys}_{\kappa}(X) & \xleftarrow{\otimes \omega_X^{-1}} & \text{Crys}_{\gamma}(X)
\end{array}$$

is commutative.

We begin with the affine case. Noting that any closed immersion of regular schemes is a local complete intersection morphism, it suffices to consider the case of a complete intersection, where the pull-back of a Cartier module can be computed by using the Koszul complex.

**Lemma 2.15.** Let $f = f_1, f_2, \ldots, f_n$ be a regular sequence of elements of a commutative ring $R$. Let $I$ be the ideal generated by the $f_i$. Then for every Cartier module $M$ with structural map $\kappa$, there is an isomorphism

$$\varphi_f: \text{Ext}_{R/I}^n(M, M) \xrightarrow{\sim} M/IM$$

where $M/IM$ is viewed as an $R/I$-module with the Cartier structure

$$\kappa_{M/IM}: F_\ast M/IM \to M/IM \quad m + IM \mapsto \kappa_M((f_1 \cdot f_2 \cdots f_n)^{p-1}m) + IM.$$ 

**Proof.** By definition we have to compute $\text{RHom}_R(R/I, M)$. The structural morphism $\kappa_M^{\flat}: F_\ast \delta^\flat M \to \delta^\flat M$ equals the composition

$$F_\ast \text{RHom}_R(R/I, M) \to \text{RHom}_R(F_\ast R/I, F_\ast M) \to \text{RHom}_R(R/I, M),$$

24
where the first morphism is the canonical one and the second is induced by $F: R/I \rightarrow F_* R/I$ in the first and $\kappa: F_* M \rightarrow M$ in the second argument. A free resolution of the $R$-module $R/I$ is given by the Koszul chain complex $K(f)$. It is the total tensor product complex in the sense of [Wei94, 2.7.1] of the following complexes $K(f_i)$

$$0 \rightarrow R \overset{f_i}{\rightarrow} R \rightarrow 0$$

concentrated in degrees $-1$ and $0$. Each complex $K(f_i)$ admits a lift of the Frobenius $F: R \rightarrow F_* R$ in degree $0$ by mapping $r$ to $r^{p^i} f_i - 1_i$. This means the diagrams

$$
\begin{array}{ccc}
0 & \rightarrow & R \\
\downarrow & & \downarrow F \\
0 & \rightarrow & F_* R
\end{array}
\quad
\begin{array}{ccc}
0 & \rightarrow & R \\
\downarrow f_i^{p-1} & & \downarrow F \\
0 & \rightarrow & F_* R
\end{array}
$$

commute. The maps $R \overset{f_i^{p-1}}{\rightarrow} F_* R$ give rise to a map of complexes $F: K(f) \rightarrow F_* K(f)$ lifting the Frobenius in degree zero: In general, if $\varphi: M \rightarrow F_* M$ and $\psi: N \rightarrow F_* N$ are $R$-linear maps, it is easy to check that the map $M \otimes_R N \rightarrow M \otimes_R N$ of abelian groups is $p$-linear, i.e. it is an $R$-linear map $M \otimes N \rightarrow F_* (M \otimes N)$. Thus we inductively obtain an $R$-linear morphism of complexes $F: K(f) \rightarrow F_* K(f)$. Unwinding the definition of the tensor product of complexes, we see that the left end of this map is the square

$$
\begin{array}{ccc}
0 & \rightarrow & R \\
\downarrow \prod_{i=1}^n f_i^{p-1} & & \downarrow \prod_{j \neq i} f_j^{p-1} \\
0 & \rightarrow & F_* R
\end{array}
\quad
\begin{array}{ccc}
((-1)^{i+1} f_i)_i & \rightarrow & R^n \\
\downarrow & & \downarrow \\
((-1)^{i+1} f_i)_i & \rightarrow & F_* R^n
\end{array}
$$

Consequently, the $n$-th degree of the composition

$$F_* \operatorname{Hom}_R(K(f), M) \xrightarrow{\text{canonical}} \operatorname{Hom}_R(F_* K(f), F_* M) \xrightarrow{F \circ \kappa} \operatorname{Hom}_R(K(f), M)$$

maps $m$ to $\kappa((\prod f_i^{p-1}) m)$ by the identification $\operatorname{Hom}_R(R, M) \cong M$ via $\varphi \mapsto \varphi(1)$. The differential $\operatorname{Hom}_R(K_{n-1}(f), M) \rightarrow \operatorname{Hom}_R(K_n(f), M)$ corresponds to the map

$$M^n \rightarrow M$$

$$(m_i)_i \mapsto \sum f_i m_i.$$ 

The image is the submodule $IM$ and thus the $n$-th cohomology of $\operatorname{Hom}_R(K(f), M)$ is isomorphic to $M/IM$, equipped with the claimed Cartier structure. \qed
Remark 2.16. The isomorphism $\text{Ext}_R^n(R/I, M) \rightarrow M/IM$ of the underlying sheaves is not canonical. It depends on the choice of the regular sequence $f$. From the construction of this isomorphism we see that if $g = g_1, \ldots, g_n$ is another regular sequence of $R$ generating $I$ and $g_i = \sum c_{ij} f_j$, the automorphism $\tau$ on $M/IM$ making the diagram

$$
\begin{array}{ccc}
\varphi_f & & M/IM \\
\downarrow & & \downarrow \tau \\
\varphi_g & \rightarrow & M/IM
\end{array}
$$

commutative is given by multiplication with $\det(c_{ij})$.

Nevertheless, interpreting the top-Ext-groups as quotient modules in the case we are interested in, namely $\text{Ext}_R^n(R/I, M) \otimes \text{Ext}_R^n(R/I, \omega_R) \cong N/IN \otimes \omega_{R/I}$, leads to isomorphisms, which are independent of the regular sequence generating $I$, since the correcting factors from both terms cancel.

Lemma 2.17. Let $R$ be a commutative ring such that $F_* R$ is finite free, $N$ a $\gamma$-module over $R$ and $I \subseteq R$ an ideal which is generated by a regular sequence of length $n$. There is a canonical isomorphism between Cartier modules

$$\text{Ext}_R^n(R/I, N \otimes \omega_R) \cong N/IN \otimes \omega_{R/I}.$$ 

Proof. Choose a regular sequence $f = f_1, \ldots, f_n$ such that $I$ is generated by the $f_i$. Also choose a basis $r_1, \ldots, r_t$ of $R$ viewed as a free $R$-module via the Frobenius. The dual basis $\varphi_1, \ldots, \varphi_t$ is given by $\varphi_i(r_j) = \delta_{ij}$. Let $\kappa$ denote the intrinsic Cartier structure of $\text{Ext}_R^n(R/I, N \otimes \omega_R)$ and $\tilde{\kappa}$ denote the Cartier structure of $N/IN \otimes \omega_{R/I}$ as explained in Proposition 2.10. Identifying $\text{Ext}_R^n(R/I, N \otimes \omega_R)$ with $(N \otimes \omega_R)/I(N \otimes \omega_R)$ via the isomorphism $\varphi_f$ from Lemma 2.15, we obtain the map

$$\kappa': F_*((N \otimes \omega_R)/I(N \otimes \omega_R)) \rightarrow (N \otimes \omega_R)/I(N \otimes \omega_R)$$

$$n \otimes m + I(N \otimes \omega_R) \mapsto \sum_{i=1}^t \varphi_i(1)\gamma_N(n) \otimes \kappa_R(r_i f^{p-1} m) + I(N \otimes \omega_R)$$

as the induced Cartier structure on $(N \otimes \omega_R)/I(N \otimes \omega_R)$. Here $\gamma_N$ is the $\gamma$-structure of $N$.

Also identifying $\omega_{R/I} \cong \text{Ext}_R^n(R/I, \omega_R)$ with $\omega_R/I\omega_R$ via $f$, its Cartier structure $\kappa_{R/I}$ is given by $m + I\omega_Y \mapsto \kappa_R(f^{p-1} m) + I(\omega_Y)$. From this perspective, the Cartier structure
of \( \tilde{\kappa} \) of \( N/IN \otimes \omega_{R/I} \) induces the structural morphism

\[
\tilde{\kappa}': F_*(N/IN \otimes \omega_{R/I}) \to (N/IN \otimes \omega_{R/I})
\]

\[
\tilde{\kappa}': (\sum_{i=1}^n \phi_i(1) \cdot \gamma_N(n) \otimes \kappa_R(r_i f^{p-1}m))
\]

\[
= \sum_{i=1}^t \phi_i(1) \gamma_N(n) \otimes \kappa_R(r_i f^{p-1}m) + I(N \otimes \omega_R)
\]

on \( N/IN \otimes \omega_{R/I} \). Finally, there is a natural isomorphism

\[
\tau: (N \otimes \omega_R)/I(N \otimes \omega_R) \cong N/IN \otimes \omega_{R/I} \omega_R
\]

of \( R \)-modules, mapping \( n \otimes m \) to \( n \otimes m \). The explicit formulas for \( \kappa' \) and \( \tilde{\kappa}' \) show that \( \tau \) makes the square in the middle of the diagram

\[
\begin{array}{ccc}
\Ext^n_R(R/I,N \otimes \omega_R) & \xrightarrow{\kappa} & F^b \Ext^n_R(R/I,N \otimes \omega_R) \\
\downarrow \phi_L & & \downarrow F^b \phi_L \\
(N \otimes \omega_R)/I(N \otimes \omega_R) & \xrightarrow{n'} & F^b((N \otimes \omega_R)/I(N \otimes \omega_R)) \\
\downarrow \gamma & & \downarrow F^b \tau \\
N/IN \otimes \omega_{R/I} \omega_R & \xrightarrow{\tilde{\kappa}'} & F^b(N/IN \otimes \omega_{R/I} \omega_R) \\
\downarrow \id \otimes \phi_L & & \downarrow F^b(\id \otimes \phi_L) \\
i^* N \otimes \omega_{R/I} & \xrightarrow{\kappa} & F^b(i^* N \otimes \omega_{R/I})
\end{array}
\]

commutative. The squares above and below commute by construction. Let \( \Phi \) be the composition \( (\id \otimes \phi_L)^{-1} \circ \tau \circ \phi_L \). We have just seen that the diagram

\[
\begin{array}{ccc}
\Ext^n_R(R/I,N \otimes \omega_R) & \xrightarrow{\kappa} & F^b \Ext^n_R(R/I,N \otimes \omega_R) \\
\downarrow \Phi & & \downarrow F^b \Phi \\
i^* N \otimes \omega_{R/I} & \xrightarrow{\kappa} & F^b(i^* N \otimes \omega_{R/I})
\end{array}
\]

commutes. Furthermore, \( \Phi \) is natural: Let \( g = g_1, \ldots, g_n \) be another regular sequence generating \( I \) with \( g_i = \sum c_{ij} f_j \). Then, by Remark 2.16,

\[
(\id \otimes \phi_L)^{-1} \circ \tau \circ \phi_L^{N \otimes \omega_R} = \det(c_{ij})^{-1} (\id \otimes \phi_L^{N \otimes \omega_R})^{-1} \circ \tau \circ \det(c_{ij}) \phi_L^{N \otimes \omega_R}
\]

\[
= (\id \otimes \phi_L^{N \otimes \omega_R})^{-1} \circ \tau \circ \phi_L^{N \otimes \omega_R}.
\]

\( \Box \)
Proposition 2.18. Let \( i: X \hookrightarrow Y \) be a closed immersion of regular, \( F \)-finite schemes and let \( N \) be a \( \gamma \)-sheaf on \( Y \). There is a canonical isomorphism
\[
\Phi: i^* \text{Ext}^n_{O_Y}(i_* O_X, N \otimes_{O_Y} \omega_Y) \sim i^* N \otimes_{O_X} \omega_X
\]
of Cartier modules, which is functorial in \( N \). Here \( \Phi \) denotes the flat morphism of ringed spaces \((X, O_X) \to (Y, i_* O_X)\).

Proof. Choose an affine open covering \( \{ U_k \}_k = \text{Spec } R_k \) of \( Y \) such that \( i|_{U_k}: i^{-1}(U_k) \hookrightarrow U_k \) is a complete intersection. By refining the covering we can assume that \( F_* R_k \) is free. Let \( I_k \subseteq R_k \) be the ideal such that \( i|_{U_k} \) corresponds to the ring homomorphism \( R_k \to R_k/I_k \). By Lemma 2.17, we have an isomorphism \( \text{Ext}^n_{R_k}(R_k/I_k, N|_{U_k} \otimes \omega_{R_k}) \cong i^* N|_{U_k} \otimes \omega_{R_k/I_k} \), which is natural and therefore, we can glue the local isomorphisms to the desired global map \( \Phi \).

Proposition 2.18 shows that there is a natural isomorphism of functors \( R^n i^\flat \circ \_ \otimes_{O_Y} \omega_Y \cong \_ \otimes_{O_X} \omega_X \circ i^* \). This enables us to prove Theorem 2.14 because \( \_ \otimes_{O_Y} \omega_Y \) and \( \_ \otimes_{O_X} \omega_X \) are equivalences of categories.

Proof of Theorem 2.14. For every \( \gamma \)-sheaf \( N \) on \( Y \), there is an isomorphism
\[
R^n i^\flat(N \otimes_{O_Y} \omega_Y) \cong i^{-1} \text{Ext}^n_{O_Y}(i_* O_X, N \otimes_{O_Y} \omega_Y) \\
\cong i^* N \otimes_{O_X} \omega_X
\]
by Proposition 2.18, which is functorial in \( N \). As \( \_ \otimes_{O_Y} \omega_Y \) and \( \_ \otimes_{O_X} \omega_X \) are equivalences of categories, even the diagram
\[
\begin{array}{ccc}
\text{Coh}_Y(Y) & \xrightarrow{\_ \otimes_{O_Y} \omega_Y^{-1}} & \text{Coh}_Y(Y) \\
\downarrow R^n i^\flat & & \downarrow i^* \\
\text{Coh}_X(X) & \xrightarrow{\_ \otimes_{O_X} \omega_X^{-1}} & \text{Coh}_Y(X)
\end{array}
\]
commutes. Passing to open crystals finishes the proof.

Now we turn to open immersions.

Proposition 2.19. Let \( j: U \to X \) be an open immersion of regular, \( F \)-finite schemes and \( M \) a Cartier module on \( X \). Then there is a natural isomorphism of \( \gamma \)-sheaves
\[
j^! M \otimes_{O_U} \omega_U^{-1} \cong j^*(M \otimes \omega_X^{-1}).
\]

Proof. One easily checks that, for a Cartier module \( M \) on \( X \) with structural morphism \( \tilde{k}: M \to F_X^! M \), the Cartier structure on \( j^* M \) is the composition
\[
j^* M \xrightarrow{j^* \tilde{k}} j^* F^! M \xrightarrow{F^! U} j^* M.
\]
The dualizing sheaf $\omega_U$ of $U$ is given by $j^*\omega_X$. Therefore we have $j^*M \otimes \omega_U^{-1} \cong j^*(M \otimes \omega_X^{-1})$ and the diagram

\[
\begin{array}{ccc}
  j^*M \otimes j^*\omega_X^{-1} & \longrightarrow & F^b j^*M \otimes F^b j^*\omega_X^{-1} \\
  \downarrow \sim & & \downarrow \sim \\
  j^*(M \otimes \omega_X^{-1}) & \longrightarrow & j^*(F^b M \otimes F^b \omega_X^{-1}) \cong j^*F^b (M \otimes \omega_X^{-1})
\end{array}
\]

commutes. Here the horizontal arrows are the $\gamma$-structures of $j^*M \otimes \omega_U^{-1}$ and $j^*(M \otimes \omega_X^{-1})$.

2.3 Compatibility with push-forward

For the construction of a push-forward for $\gamma$-sheaves, we follow the construction given in subsection 6.3 of [BB]. Then we show that the equivalence between Cartier modules and $\gamma$-sheaves given by tensoring with the dualizing sheaf is compatible with push-forward for morphisms of regular schemes. This proof is also mainly the one given in ibid. By abuse of notation, let $\kappa_X : F_{\chi} \omega_X \rightarrow \omega_X$ be the adjoint of the Cartier structure of $\omega_X$.

**Definition 2.20** ([BB, Definition 6.3.1]). Let $f : X \rightarrow Y$ be a morphism of smooth, $F$-finite $k$-schemes. Let $N$ be a $\gamma$-sheaf on $X$. Then we define the push-forward $f_+ N$ as the twist of the push-forward of Cartier modules, i.e.

$$f_+ N = f_*(N \otimes \omega_X) \otimes \omega_Y^{-1}.$$ 

The push-forward for $\gamma$-crystals is the one induced by the just given push-forward of $\gamma$-sheaves.

By construction, the push-forward for $\gamma$-sheaves is compatible with the push-forward for Cartier modules. In order to show that $f_+$ is compatible with the equivalence between $\gamma$-crystals and lfgu modules, we need a different description of $f_+$ for $\gamma$-sheaves based on the relative Cartier operator.

We recall two general constructions, which are repeatedly used in this subsection. For this we consider a morphism $f : X \rightarrow Y$ of arbitrary schemes over $\text{Spec} \, \mathbb{Z}$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_Y$-module. The adjoint of the composition

$$f^*(f_* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}) \cong f^* f_* \mathcal{F} \otimes f^* \mathcal{E} \xrightarrow{\text{adj}_f \otimes \text{id}} \mathcal{F} \otimes f^* \mathcal{E},$$

where $\text{adj}_f : f^* f_* \rightarrow \text{id}$ is the counit of the adjunction, yields a natural morphism

$$\text{proj} : f_* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E} \rightarrow f_*(\mathcal{F} \otimes f^* \mathcal{E}).$$

As a consequence of the projection formula ([Har77, Exercise III.8.3]), it is an isomorphism if $f$ is quasi-compact and separated and if $\mathcal{E}$ is locally free.
For two morphisms $f : X \to S$ and $g : Y \to S$, let $f' : X \times_S Y \to Y$ and $g' : X \times_S Y \to X$ be the projections such that the square

$$
\begin{array}{c}
\begin{array}{c}
X 
\times_S Y \\
\downarrow^{g'} \\
X 
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
Y \\
\downarrow^g \\
S 
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

is cartesian. There is a canonical morphism of functors

$$
bc : f^*g_* \to g'_*f'^*
$$

of quasi-coherent sheaves given by the adjoint of the composition

$$
g_* \xrightarrow{g_* \text{ad}_{f'}} g_*f'_*f'^* \cong f_*g'_*f'^*,
$$

where $\text{ad}_{f'} : \text{id} \to f'_*f'^*$ is the unit of the adjunction. If $g$ is affine, $bc$ is an isomorphism.

To see this, we can assume that $S$, $X$ and $Y$ are affine, because $g$ and therefore $g'$ is an affine morphism. Then the claim is a well known property of the tensor product.

The morphism $bc$ is also an isomorphism if $X$ and $Y$ are Noetherian, $f$ is flat and $g$ is separated of finite type ([Har77, Proposition III.9.3]). The next lemma relates these two isomorphisms.

**Lemma 2.21.** Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes and let $f' : X \times_S X \to Y$ and $g' : X \times_S Y \to X$ be the projections. Then, for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$, the diagram

$$
\begin{array}{c}
\begin{array}{c}
f_*\mathcal{F} \otimes g_*\mathcal{E} \\
\downarrow^{\text{proj}} \\
f_*(\mathcal{F} \otimes f^*g_*\mathcal{E})
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f_*\mathcal{F} \otimes g_*\mathcal{E} \\
\downarrow^{\text{proj}} \\
g_*(g^*f_*\mathcal{F} \otimes \mathcal{E})
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f_*(\mathcal{F} \otimes g'_*f'^*\mathcal{E}) \\
\downarrow^{\text{bc}} \\
g_*f'_*(g'^*\mathcal{F} \otimes f'^*\mathcal{E})
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f_*(\mathcal{F} \otimes g'_*f'^*\mathcal{E}) \\
\downarrow^{\text{proj}} \\
g_*(f'_*g'^*\mathcal{F} \otimes \mathcal{E})
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
g_*(g'^*f'^*\mathcal{F} \otimes \mathcal{E}) \\
\downarrow^{\text{bc}} \\
g_*f'_*(g'^*\mathcal{F} \otimes f'^*\mathcal{E})
\end{array}
\end{array}
\end{array}
$$

commutes.

**Proof.** For a morphism $h$ of schemes, let $\tilde{\text{ad}}_h$ denote the counit of adjunction $h^*h_* \to \text{id}$. The diagram of which we want to prove the commutativity is obtained by adjunction
from the diagram

\[
g'^* f^* F \otimes g'^* g^* E \xrightarrow{bc \otimes id} f'^* g^* F \otimes f'^* g^* E
\]

Both parts of the diagram are commutative by construction of the morphism bc. □

We turn back to the situation of a morphism \( f : X \to Y \) of smooth schemes over a field \( k \) containing \( \mathbb{F}_p \). For simplicity, let \( \omega_f \) denote the relative dualizing sheaf \( \omega_{X/Y} = f^! O_Y \).

**Lemma 2.22.** Let \( f : X \to Y \) be a morphism of smooth, \( F \)-finite schemes over \( k \). For every \( \gamma \)-sheaf \( N \) on \( X \), there is a natural isomorphism

\[
f_*(N \otimes \omega_X) \otimes \omega_Y^{-1} \to f_*(N \otimes \omega_X \omega_f)
\]

of quasi-coherent sheaves.

**Proof.** Since \( X \) and \( Y \) are smooth, any morphism \( X \to Y \) is regular, i.e. it is a composition of a closed immersion \( X \to W \) such that \( X \) is a local complete intersection in \( W \), followed by a smooth morphism \( W \to Y \). (For a smooth morphism \( f \), the graph factorization

\[
X \xrightarrow{(\text{id}, f)} X \times_k Y \xrightarrow{\text{pr}_Y} Y,
\]

where \( \text{pr}_Y \) denotes the projection, satisfies this requirement.) For a closed immersion \( i \) we have the isomorphism

\[
i^!(\omega_Y) \cong L^i \omega_Y \otimes \omega_f[-n]
\]

of [Har66, Corollary III.7.3] and a smooth morphism is quasi-perfect, see [Sch18, Definition 3.3]. Overall, we see that there are natural isomorphisms

\[
\omega_X \cong f^! \omega_Y \xrightarrow{\sim} f^! O_Y \otimes Lf^* \omega_Y \xrightarrow{\sim} \omega_f \otimes f^* \omega_Y.
\]

Now we obtain the desired isomorphism as the composition

\[
f_*(N \otimes \omega_X) \otimes \omega_Y^{-1} \xrightarrow{\text{proj}} f_*(N \otimes \omega_X \otimes f^* \omega_Y^{-1}) \cong f_*(N \otimes \omega_f)
\]

□
With the relative Frobenius diagram

see diagram (1) and below for the notation, we can define a \( \gamma \)-structure \( \gamma_{N,f} \) for \( f_*(N \otimes \omega_f) \) by the composition

\[
\begin{align*}
  f_*(N \otimes \omega_f) & \sim f'_*(F_{X/Y} N \otimes \omega_f) \\
  & \xrightarrow{\gamma_N} f'_*(F_{X/Y}^r N \otimes \omega_f) \\
  & \xrightarrow{\text{proj}^{-1}} f'_*(F_Y^r N \otimes F_{X/Y} \omega_f) \\
  & \xrightarrow{C_{X/Y}} f'_*(F_Y^r N \otimes F_Y^r \omega_f) \\
  & \xrightarrow{\text{bc}} F_Y^r f_*(N \otimes \omega_f).
\end{align*}
\]

We will show that \( \gamma_{N,f} \) is the structural morphism of \( f_+ N \) via the isomorphism of Lemma 2.22. But first, we clarify how the relative Cartier operator is related to \( \kappa_X \) and \( \kappa_Y \).

**Lemma 2.23.** With the notation of the preceding lemma, the composition

\[
\begin{align*}
  F_Y^r(\omega_f \otimes f^* \omega_Y) & \sim F_Y^r F_{X/Y}^r(\omega_f \otimes F_{X/Y}^r f^* \omega_Y) \\
  & \xrightarrow{\text{proj}^{-1}} F_Y^r(\omega_f \otimes f^* \omega_Y) \\
  & \xrightarrow{C_{X/Y}} F_Y^r(\omega_f \otimes f^* \omega_Y) \\
  & \xrightarrow{\text{proj}} \omega_f \otimes f_Y^r \omega_Y \\
  & \xrightarrow{\text{bc}} \omega_f \otimes f_Y^r \omega_Y \\
  & \xrightarrow{\kappa_Y} \omega_f \otimes f_Y^r \omega_Y
\end{align*}
\]

is compatible with the Cartier structure of \( \omega_X \) under the canonical isomorphism \( \omega_X \cong \omega_f \otimes f^* \omega_Y \).

**Proof.** In the appendix A.2.3. (iii) of [EK04], Emerton and Kisin explain how the relative Cartier operators \( C_{X/Y} \), \( C_{Y/Z} \) and \( C_{X/Z} \) are related for a composition \( X \xrightarrow{f} Y \xrightarrow{g} Z \) of morphisms. Our lemma is the special case where \( Z = \text{Spec} \, k \) and \( g \) is the structural morphism of the \( k \)-scheme \( Y \).

**Proposition 2.24.** Let \( f : X \rightarrow Y \) be a morphism of smooth, \( F \)-finite schemes over \( k \). Let \( N \) be a \( \gamma \)-sheaf on \( X \). The canonical isomorphism

\[
\begin{align*}
  f_*(N \otimes \omega_f) & \sim f_*(N \otimes \omega_X) \otimes \omega_Y^{-1}
\end{align*}
\]
of quasi-coherent $\mathcal{O}_Y$-modules from Lemma 2.22 is an isomorphism of $\gamma$-sheaves.

Proof. As $\otimes \omega_X$ and $\otimes \omega_Y^{-1}$ are equivalences between the categories of $\gamma$-sheaves and Cartier modules on $X$ and on $Y$, it suffices to show that the canonical isomorphism $f^*(N \otimes \omega_f) \otimes \omega_Y \sim f^*(N \otimes \omega_X)$ is an isomorphism of Cartier modules on $Y$. The left hand side of the diagram

$$
\begin{align*}
& F_Y(f_*(N \otimes \omega_f) \otimes \omega_Y) \xrightarrow{\text{proj}} F_Y f_*(N \otimes \omega_f \otimes f^*\omega_Y) \\
& F_Y(f_*(F_Y^* N \otimes \omega_Y) \otimes \omega_Y) \xrightarrow{\text{proj}} F_Y f_*(F_Y^* N \otimes \omega_f \otimes f^*\omega_Y) \\
& F_Y(f_*(\mathcal{F}_{X/Y} \otimes \omega_Y) \otimes \omega_Y) \xrightarrow{\sim} F_Y f_*(\mathcal{F}_{X/Y} \otimes \omega_f \otimes f^*\omega_Y) \\
& F_Y(f_*(\mathcal{F}_Y \otimes \omega_f) \otimes \omega_Y) \xrightarrow{\text{proj}^{-1}} F_Y f_*(\mathcal{F}_Y \otimes f^*\omega_Y \otimes F_{X/Y} \omega_f) \\
& F_Y(f_*(\mathcal{F}_Y \otimes \omega_f) \otimes \omega_Y) \xrightarrow{\text{proj}^{-1}} F_Y f_*(\mathcal{F}_Y \otimes f^*\omega_Y \otimes F_{X/Y} \omega_f) \\
& F_Y(f_*(\mathcal{F}_Y \otimes \omega_f) \otimes \omega_Y) \xrightarrow{\sim} F_Y f_*(\mathcal{F}_Y \otimes f^*\omega_Y) \\
& f_*(\mathcal{F}_Y \otimes \omega_f) \otimes f^*\omega_Y \xrightarrow{\text{proj}^{-1}} f_*(\mathcal{F}_Y \otimes f^*\omega_Y) \\
& f_*(\mathcal{F}_Y \otimes \omega_f) \otimes f^*\omega_Y \xrightarrow{\text{proj}^{-1}} f_*(\mathcal{F}_Y \otimes f^*\omega_Y)
\end{align*}
$$

is the structural morphism of the Cartier module $f_*(N \otimes \omega_f) \otimes \omega_Y$. It is easy to see that the right hand side is the structural morphism of the Cartier module $f_*(N \otimes \omega_X) \cong f_*(N \otimes \omega_f \otimes f^*\omega_Y)$. Hence we have to show that the diagram above commutes. The three upper squares and the bottom square commute by the functoriality and the compatibility of the projection formula with compositions of morphisms. The commutativity of the fourth square from above follows from Lemma 2.21. \qed
2.4 Cartier crystals and locally finitely generated unit modules

The category of $\gamma$-sheaves was just an intermediate step on the way to locally finitely generated unit modules. Recall that there is a functorial way of associating a unit $O_X[F]$-module to a $\gamma$-sheaf $N$ on $X$.

**Definition 2.25.** Let $\mu_u(X)$ denote the category of unit left $O_{F,X}$-modules whose underlying $O_X$-module is quasi-coherent. For a smooth $k$-scheme $X$, let $\text{Gen}$ be the functor

$$\text{Qcoh}_\gamma(X) \to \mu_u(X)$$

which assigns to any quasi-coherent $\gamma$-sheaf $N$ with structural morphism $\gamma: N \to F^*N$ the direct limit $N$ of

$$N \xrightarrow{\sim} F^*N \xrightarrow{F^*\gamma} F^{2*}N \xrightarrow{F^{2*}\gamma} \ldots$$

together with the inverse of the induced isomorphism $N \xrightarrow{\sim} F^*N$.

**Lemma 2.26.** Let $X$ be a smooth, $F$-finite $k$-scheme. The functor $\text{Gen}$ is essentially surjective and induces an equivalence of categories

$$\text{QCrys}_\gamma(X) \xrightarrow{\sim} \mu_{\text{lfgu}}(X).$$

**Proof.** Let $\text{Neg}: \mu_u(X) \to \text{Qcoh}_\gamma(X)$ be the functor which assigns to a quasi-coherent unit $O_{F,X}$-module $M$ with structural morphism $u: F^*M \to M$ the quasi-coherent $\gamma$-sheaf $\mathcal{M}$ whose structural morphism is given by the inverse of $u$. Obviously there is a natural isomorphism

$$\text{Gen} \circ \text{Neg} \xrightarrow{\sim} \text{id},$$

whence the surjectivity of $\text{Gen}$.

For a quasi-coherent $\gamma$-sheaf $N$, the corresponding crystals $\mathcal{N}$ is nil-isomorphic to the corresponding crystal of $\text{Neg} \circ \text{Gen}(N)$. The reason for this is the fact that the structural morphism $\gamma: N \to F^*N$ of a quasi-coherent $\gamma$-sheaf $N$ is a nil-isomorphism: It is immediate that the structural map of the kernel and the cokernel of $\gamma$, interpreted as a morphism of $\gamma$-sheaves, is the zero map.

The image of $\text{Gen}$ of the subcategory $\text{Coh}_\gamma(X)$ of coherent $\gamma$-sheaves on $X$ is the category $\mu_{\text{lfgu}}(X)$. Indeed, after localizing at nilpotent $\gamma$-sheaves and considering $\gamma$-crystals, $\text{Gen}$ induces an equivalence of categories.

**Proposition 2.27** ([BB11, Proposition 5.12]). For a smooth, $F$-finite $k$-scheme $X$, the functor

$$\text{Gen}_X: \text{Coh}_\gamma(X) \to \mu_{\text{lfgu}}(X)$$

factors through $\text{Crys}_\gamma(X)$, inducing an equivalence of categories:

$$\xymatrix{ \text{Coh}_\gamma(X) \ar[dd]_{\text{Gen}} & \text{Crys}_\gamma(X) \ar[r]_{\sim} & \mu_{\text{lfgu}}(X).}$$
Theorem 2.28. Let \( X \) be an \( F \)-finite, smooth \( k \)-scheme. Let \( G \) denote the composition of the exact functors \( \otimes \omega^{-1}_Y \) and \( \text{Gen} \). It induces an equivalence of derived categories

\[
G: D^b_{\text{cris}}(\text{QCrys}_k(X)) \rightarrow D^b_{\text{flugu}}(\mathcal{O}_{F,X}).
\]

Proof. Combining Corollary 2.11 and Lemma 2.26, we see that \( G \) induces an equivalence of abelian categories \( \text{QCrys}_k(X) \rightarrow \mu(X) \) and therefore an equivalence of derived categories \( D^b(\text{QCrys}(X)) \rightarrow D^b(\mu(X)) \). Since \( G \) is exact and restricts to an equivalence \( \text{Crys}_k(X) \rightarrow \mu_{\text{flugu}}(X) \), we obtain an equivalence \( D^b_{\text{cris}}(\text{QCrys}_k(X)) \rightarrow D^b_{\text{flugu}}(\mu(X)) \).

It remains to show that \( D^b_{\text{flugu}}(\mu(X)) \) is naturally equivalent to \( D^b_{\text{flugu}}(\mathcal{O}_{F,X}) \). The inclusion \( \mu_{\text{flugu}}(X) \rightarrow \mu(X) \) induces an equivalence \( D^b(\mu_{\text{flugu}}) \rightarrow D^b(\mu_{\text{flugu}}(\mathcal{O}_{F,X})) \). As the inclusion \( \mu_{\text{flugu}}(X) \rightarrow \mu(X) \) factors through \( \mu_u(X) \), this implies an equivalence \( D^b_{\text{flugu}}(\mu_u(X)) \rightarrow D^b_{\text{flugu}}(\mathcal{O}_{F,X}) \).

Finally, we prove that the equivalence \( G \) of derived functors is compatible with pullbacks. Note that for a morphism \( f: X \rightarrow Y \) of smooth schemes, the functor

\[
f^1: D^b_{\text{flugu}}(\mathcal{O}_{F,Y}) \rightarrow D^b_{\text{flugu}}(\mathcal{O}_{F,X})
\]

is obtained from a right-exact functor of abelian categories.

Definition 2.29. Let \( f: X \rightarrow Y \) be a morphism of smooth \( k \)-schemes. The \((\text{underived})\) pull-back \( f^* \mathcal{M} \) of an \( \mathcal{O}_{F,Y} \)-module \( \mathcal{M} \) is given by

\[
f^* \mathcal{M} = \mathcal{O}_{F,X \rightarrow Y} \otimes_{f^{-1} \mathcal{O}_{F,Y}} f^{-1} \mathcal{M}
\]

\[
= \mathcal{O}_{F,X} \otimes_{f^{-1} \mathcal{O}_{F,Y}} f^{-1} \mathcal{M},
\]

cf. Proposition 1.13. The pull-back \( f^! \) for complexes \( \mathcal{M}^* \) of \( \mathcal{O}_{F,Y} \)-modules from Definition 2.3.1 of [EK04] is the left derived functor of \( f^* \), shifted by \( d_{X/Y} \):

\[
f^! \mathcal{M}^* = \mathcal{O}_{F,X \rightarrow Y} \otimes_{f^{-1} \mathcal{O}_{F,Y}} f^{-1} \mathcal{M}^*[d_{X/Y}].
\]

Corollary 2.30. Let \( f: X \rightarrow Y \) be a closed immersion of smooth, \( F \)-finite \( k \)-schemes of relative dimension \( d_{X/Y} = n \). There is a natural equivalence of functors \( \text{Crys}_k(Y) \rightarrow \mu_{\text{flugu}}(X) \):

\[
f^* \circ G_Y \cong G_X \circ R^n f^!.
\]

Proof. Consider the following diagram of functors:

\[
\begin{array}{ccc}
\text{Crys}_k(Y) & \xrightarrow{\otimes \omega^{-1}_Y} & \text{Crys}_k(Y) & \xrightarrow{\text{Gen}_Y} & \mu_{\text{flugu}}(Y) \\
\downarrow R^n f^! & & \downarrow f^* & & \downarrow f^* \\
\text{Crys}_k(X) & \xrightarrow{\otimes \omega^{-1}_X} & \text{Crys}_k(X) & \xrightarrow{\text{Gen}_X} & \mu_{\text{flugu}}(X).
\end{array}
\]

The left square commutes by Theorem 2.14. The right square also commutes because there is a natural isomorphism \( \text{Gen}_X \circ f^* \cong f^* \circ \text{Gen}_Y \). For a \( \gamma \)-sheaf \( N \) on \( Y \), let \( \mathcal{N} \)
denote $\text{Gen}_Y(N)$, which is the direct limit $\lim F_{i*}^Y N$. As direct limits commute with pull-back of quasi-coherent sheaves, we have a natural isomorphism

$$\text{Gen}_X f^*(N) = \lim (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} F_{i*}^Y N)$$

$$\xrightarrow{\sim} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} (\lim F_{i*}^Y N)$$

$$= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(N).$$

One checks that for a left $\mathcal{O}_{F,Y}$-module $\mathcal{M}$, the underived pullback $F^* \mathcal{M} = \mathcal{O}_{F,X} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} \mathcal{M}$ is the quasi-coherent sheaf $f^* \mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1} \mathcal{M}$ with the natural morphism $F_X^* f^* \mathcal{M} \to f^* \mathcal{M}$ induced by the structural morphism $F_Y^* \mathcal{M} \to \mathcal{M}$. Hence $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(N)$ is isomorphic to $f^* \text{Gen}_Y(N)$. □

**Lemma 2.31.** Let $i: X \to Y$ be a closed immersion of smooth, $F$-finite schemes over $k$. Let $P$ be a locally free left $\mathcal{O}_{F,Y}$-module. Then

$$R^n((\_ \otimes \omega_X^{-1}) \circ i^! \circ (\_ \otimes \omega_Y))P = 0$$

where $(\_ \otimes \omega_X^{-1}) \circ i^! \circ (\_ \otimes \omega_Y)$ is understood as the composition of functors

$$D^b_lfgu(\mathcal{O}_{F,Y}) \xrightarrow{=\omega_Y} D^b_{crys}(\text{QCrys}_k(Y)) \xrightarrow{i^!} D^b_{crys}(\text{QCrys}_k(X)) \xrightarrow{=\omega_X^{-1}} D^b_lfgu(\mathcal{O}_{F,X}).$$

**Proof.** Locally free left $\mathcal{O}_{F,Y}$-modules are in particular locally free as quasi-coherent $\mathcal{O}_Y$-modules. Thus we have

$$R^n((\_ \otimes \omega_X^{-1}) \circ i^! \circ (\_ \otimes \omega_Y))P \cong (\_ \otimes \omega_X^{-1}) \circ R^n i^! \circ (\_ \otimes \omega_Y) \oplus_{j \in J} \mathcal{O}_Y$$

$$\cong (\_ \otimes \omega_X^{-1}) \circ R^n i^! \oplus_{j \in J} \\omega_Y$$

$$\cong 0$$

locally for all $n \neq -d_{X/Y}$ on the underlying quasi-coherent sheaves. □

**Theorem 2.32.** For closed immersions $i: X \to Y$ of smooth, $F$-finite $k$-schemes, the equivalences $D^b_{crys}(\text{QCrys}_k(X)) \to D^b_lfgu(\mathcal{O}_{F,X})$ and $D^b_{crys}(\text{QCrys}_k(Y)) \to D^b_lfgu(\mathcal{O}_{F,Y})$ of derived categories induced by $G_X$ and $G_Y$ are compatible with the pull-backs $i^!$, i.e. we have a canonical isomorphism

$$G_X \circ i^! \cong i^! \circ G_Y$$

of functors from $D^b_{crys}(\text{QCrys}_k(Y))$ to $D^b_lfgu(\mathcal{O}_{F,X})$.

**Proof.** This is an application of the following general result concerning derived functors:

**Proposition 2.33 ([Har66, Proposition I.7.4]).** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, where $\mathcal{A}$ has enough injectives, and let $F_i: \mathcal{A} \to \mathcal{B}$ be an additive functor which has cohomological dimension $\leq n$ on $\mathcal{A}$. Let $P$ be the set of objects $X$ of $\mathcal{A}$ such that $R^i F_1(X) = 0$ for all $i \neq n$, and assume that every object of $\mathcal{A}$ is a quotient of an element of $P$. Let $F_2 = R^n F_1$. Then $RF_1$ and $LF_2$ exist, and there is a functorial isomorphism

$$RF_1 \xrightarrow{\sim} LF_2[-n].$$
First we have to check the requirements. Let $F$ denote the functor $(\_ \otimes \omega_X^{-1}) \circ R^b_{j_+} \circ (\_ \otimes \omega_Y)$. As done in the proof of Lemma 2.15, the derived functors of $R^b_{j_+} = \Gamma \text{Hom}_{\mathcal{O}_Y}(i_*\mathcal{O}_X, \_)$ may be computed locally by resolving $i_*\mathcal{O}_X$ by the Koszul complex. Since this complex has length $-d_{X/Y}$, the cohomological dimension of $F$ is smaller or equal $-d_{X/Y}$. As $Y$ is smooth, every left $\mathcal{O}_{F,Y}$-module is the quotient of a locally free left $\mathcal{O}_{F,Y}$-module ([EK04, Lemma 1.6.2]). Finally, for every locally free left $\mathcal{O}_{F,Y}$-module $P$, Lemma 2.31 states that $R^n F(P) = 0$ for all $n \neq -d_{X/Y}$.

It follows from [Har66, Proposition I.7.4] that $RF \cong Li^*[d_{X/Y}] = i^!$ because $i^! \cong R^{-d_{X/Y}} F$ (Theorem 2.14). Thus

$$(\_ \otimes \omega_X^{-1}) \circ i^! \circ (\_ \otimes \omega_Y) \cong i^!,$$

i.e. the diagram

$$
\begin{array}{ccc}
D^b_c(Q\text{Crys}_k(Y)) & \xrightarrow{\otimes\omega_Y^{-1}} & D^b_{\text{lfgu}}(\mathcal{O}_{F,Y}) \\
\downarrow i^! & & \downarrow i^! \\
D^b_c(Q\text{Crys}_k(X)) & \xrightarrow{\otimes\omega_X^{-1}} & D^b_{\text{lfgu}}(\mathcal{O}_{F,X})
\end{array}
$$

is commutative.

**Corollary 2.34.** For every closed immersion of smooth, $F$-finite $k$-schemes $i: X \to Y$, there is a canonical isomorphism

$$G_Y \circ i_* \cong i_* \circ G_X.$$

**Proof.** This follows formally as $G_X$ and $G_Y$ are equivalences of categories and since $i_*$ is uniquely determined as a left adjoint functor of $i^!$. 

**Proposition 2.35.** Let $j: X \to Y$ be an open immersion of smooth, $F$-finite schemes. Then there are natural isomorphisms

$$G_X \circ j^* \cong j^! \circ G_Y \quad \text{and} \quad G_Y \circ Rj_* \cong j_+ \circ G_X.$$

**Proof.** We already have seen that $\_ \otimes \omega_X^{-1} \circ j^* \cong j^* \circ \_ \otimes \omega_Y^{-1}$ (Proposition 2.19) and that Gen$_X \circ j^* \cong j^! \circ \text{Gen}_Y$ (see the proof of Corollary 2.30, this part holds for an arbitrary flat morphism of smooth $k$-schemes). Therefore $G_X \circ j^* \cong j^! \circ G_Y$. The rest follows from the adjunction of $Rj_*$ or $j_+$ and $j^*$ or $j^!$. 

Up to now, we have seen that the equivalence $G$ between Cartier crystals and lfgu modules is compatible with the (derived) push-forward for open and closed immersions by showing the compatibility for the adjoint pull-back functors. In fact, $G$ is compatible.
with push-forward for arbitrary morphisms of smooth schemes, but we can give a proof only up to the following theorem\(^2\), which is a result of Lurie, see [Lur16, Theorem 1.3.3.2].

**Theorem 2.36.** Let \( F : D(\mathcal{A}) \to D(\mathcal{B}) \) be a functor between derived categories of abelian categories \( \mathcal{A} \) and \( \mathcal{B} \), which is a morphism of triangulated categories. If \( F \) lifts to an exact functor of the stable \( \infty \)-categories whose homotopy categories are the cohomologically bounded below derived categories \( D^+(\mathcal{A}) \) and \( D^+(\mathcal{B}) \), if \( F \) is \( t \)-left exact for the canonical \( t \)-structure, i.e. \( F \) maps \( D^{\geq 0}(\mathcal{A}) \) to \( D^{\geq 0}(\mathcal{B}) \), and if the cohomology of \( F(I) \) is concentrated in degree 0 for every injective object \( I \) of \( \mathcal{A} \), then \( F \) arises as a right derived functor between the abelian categories \( \mathcal{A} \) and \( \mathcal{B} \).

**Proposition 2.37.** Let \( f : X \to Y \) be a morphism of smooth, \( F \)-finite \( k \)-schemes. There is a natural isomorphism

\[
G_Y \circ Rf_* \to f_+ \circ G_X
\]

from \( D^b_{\text{crys}}(\text{QCrys}_s(X)) \) to \( D^b_{\text{lg}}(\mathcal{O}_{F,Y}) \).

**Proof.** As \( \otimes \omega_X^{-1} \circ Rf_* \cong Rf_* \otimes \omega_Y^{-1} \) by construction, it suffices to show that there is a natural isomorphism of functors \( \text{Gen} Rf_* \to f_+ \text{Gen} \) from \( D^b_{\text{crys}}(\text{QCrys}_s(X)) \) to \( D^b_{\text{lg}}(\mathcal{O}_{F,Y}) \). For every complex \( N^\bullet \) of \( \gamma \)-sheaves, the complex \( \text{Gen} N^\bullet \) of quasi-coherent \( \mathcal{O}_{F,X} \)-modules has a two-term resolution by induced modules, namely the short exact sequence

\[
0 \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} N^\bullet \xrightarrow{1-\beta'} \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} N^\bullet \to \text{Gen} N^\bullet \to 0
\]

of [EK04, Proposition 5.3.3]. Here \( \beta' : \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} N^\bullet \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} N^\bullet \) denotes the morphism corresponding to \( \beta \) via the identification

\[
\text{Hom}_{\mathcal{O}_{F,X}}(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} A, \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} B) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_X}(A, \oplus_{n=0}^{\infty} (F^X)^n B)
\]

for \( \mathcal{O}_X \)-modules \( A \) and \( B \) described in 1.7.3 of ibid.

First we verify that the requirements of Theorem 2.36 are satisfied. Let \( I^\bullet \) be a bounded below complex of injective \( \gamma \)-sheaves with \( H^i(I^\bullet) = 0 \) for \( i < 0 \). Let \( \beta : I^\bullet \to F^X I^\bullet \) be the morphism of complexes induced by the structural morphisms of the \( I^i \).

The complex \( f_+ I^\bullet \) represents \( Rf_* I^\bullet \) and, as explained above, we have a short exact sequence

\[
0 \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} f_+ I^\bullet \xrightarrow{1-f_+ \beta'} \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} f_+ I^\bullet \to \text{Gen} f_+ I^\bullet \to 0.
\]

Applying \( f_+ \) to the two-term resolution of \( \text{Gen} I^\bullet \) yields a distinguished triangle

\[
f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet) \xrightarrow{f_+(1-\beta')} f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet) \to f_+(\text{Gen} I^\bullet) \to f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet)[1].
\]

\(^2\)We will not discuss this theorem here as its theoretical background, for example \( \infty \)-categories, goes beyond the scope of this work. We just note that the requirement that \( f_+ \) \( \text{Gen} \) lifts to a functor of the corresponding stable \( \infty \)-categories is satisfied, because \( f_+ \) is a composition of left and right derived functors, which have this property ([Lur16, Example 1.3.3.4]).
The sheaf $\mathcal{O}_{F,Y\leftarrow X}$ is locally free as an $\mathcal{O}_X$-module. It follows that locally $\mathcal{O}_{F,Y\leftarrow X}\otimes_{\mathcal{O}_X} I^\bullet$ is a direct sum of flasque sheaves and hence flasque. We have

$$f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet) = Rf_+((\mathcal{O}_{F,Y\leftarrow X} \otimes_{\mathcal{O}_X} (\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet)))$$

$$\xrightarrow{\sim} Rf_+((\mathcal{O}_{F,Y\leftarrow X} \otimes_{\mathcal{O}_X} I^\bullet))$$

$$\xrightarrow{\sim} f_+(\mathcal{O}_{F,Y\leftarrow X} \otimes_{\mathcal{O}_X} I^\bullet),$$

see also [EK04, Lemma 3.5.1] and its proof. In particular, the complex $f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet)$ is represented by the complex whose $i$-th degree equals the sheaf $f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^i)$.

The canonical isomorphism $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} f_+ I^\bullet \xrightarrow{\sim} f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet)$ of the proof of [EK04, Theorem 3.5.3] makes the left hand square of the diagram commute ([EK04, Proposition 3.6.1]). This shows that the cohomology sheaves of $f_+(\text{Gen} I^\bullet)$ vanish in negative degrees, i.e. $f_+ \text{Gen}$ is left $t$-exact for the canonical $t$-structure of the bounded derived category of $\gamma$-sheaves on $X$. Furthermore, for a single injective $\gamma$-sheaf $I$ on $X$, the upper row of the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} f_+ I^\bullet & \xrightarrow{1-f_+ \beta'} & \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} f_+ I^\bullet \\
\sim & \sim & \sim \\
f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet) & \xrightarrow{f_+(1-\beta')} & f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} I^\bullet) \\
\end{array}
\quad (5)$$

is a short exact sequence when adding 0 at the ends. Consequently, the cohomology of $f_+ \text{Gen} I$ is concentrated in degree 0.

To see that there is an isomorphism of functors $\text{Gen} f_+ \cong H^0(f_+ \text{Gen})$, let $M$ be a $\gamma$-sheaf on $X$. Choose a resolution $I^\bullet$ of $M$ by injective $\gamma$-sheaves. The long exact cohomology sequences for the triangles of the diagram Equation 5 yield a unique isomorphism

$$\text{Gen} f_+ M \cong H^0(\text{Gen} Rf_+ M) = H^0(\text{Gen} f_+ I^\bullet) \xrightarrow{\sim} H^0(f_+ \text{Gen} I^\bullet) \cong H^0(f_+ \text{Gen} M).$$

By Theorem 2.36, the functor $f_+ \text{Gen}$ is the right derived functor of $H^0(f_+ \text{Gen})$. Furthermore, as $\text{Gen}$ is exact, $\text{Gen} Rf_+$ is the right derived functor of $f_+$. Thus, there is a natural equivalence $\text{Gen} Rf_+ \cong f_+ \text{Gen}$ of functors from the bounded derived category of $\gamma$-sheaves on $X$ to the bounded derived category of quasi-coherent unit left $\mathcal{O}_{F,Y}$-modules. It induces an isomorphism of functors between $D^b_{\text{crys}}(\text{QCrys}(X))$ and $D^b_{\text{flgu}}(\mathcal{O}_{F,Y})$ because $D^b(\mu_u(Y)) \xrightarrow{\sim} D^b_{\text{flgu}}(\mathcal{O}_{F,Y})$ ([EK04, Corollary 17.2.5]).

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3 Locally finitely generated unit modules on singular schemes

For a proper map $f: X \to Y$ of smooth $k$-schemes, Emerton and Kisin proved that there is a natural isomorphism

$$R\text{Hom}^\bullet_{O,Y}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simto Rf_*R\text{Hom}^\bullet_{O,X}(\mathcal{M}^\bullet, f'_!\mathcal{N}^\bullet)$$

for $\mathcal{M}^\bullet \in D^b_{\text{qc}}(O_{X})$ and $\mathcal{N}^\bullet \in D^b_{\text{qc}}(O_{F,Y})$ ([EK04, Theorem 4.4.1]) by constructing a trace map acting as the counit of adjunction. We generalize this trace map to separated and finite type morphisms $f: X \to Y$ between smooth $k$-schemes sitting in a commutative diagram

$$\begin{array}{ccc}
Z' & \xrightarrow{i'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Z & \xrightarrow{i} & Y
\end{array}$$

where $i$ and $i'$ are closed immersions and $f'$ is proper. This generalized trace map induces an adjunction between $f_+$ and $R\Gamma_{Z'}f'_!$ considered as functors between the derived categories $D_{\text{lfgu}}^b(O_{X})$ and $D^b_{\text{lfgu}}(O_{F,Y})_Z$ of complexes whose cohomology sheaves are supported in $Z'$ or $Z$.

The base for this more general trace for lfgu modules is a corresponding generalized trace map $\text{tr}_{Z,f} = \text{tr}_f: Rf_*R\Gamma_{Z'}f'_! \to \text{id}$ for quasi-coherent sheaves established in [Sch18] in the situation of the diagram above. First, let us fix some notation: Let $D_{\text{qc}}^-(O_X)_Z$ denote the subcategory of the derived category $D_{\text{qc}}(O_X)$ of quasi-coherent sheaves on $X$ whose objects have bounded above cohomology supported on $Z$ and similar for $D^+_{\text{qc}}(O_Y)_Z$. The generalized trace has many compatibilities of the classical one, for example it behaves well with residually stable base change\(^3\). But most important, it gives rise to the following adjunction:

**Theorem 3.1** ([Sch18, Theorem 3.2]). Let $f: X \to Y$ be a separated and finite type morphism of Noetherian schemes and let $i: Z \to Y$ and $i': Z' \to X$ be closed immersions with a proper morphism $f': Z' \to Z$ such that the diagram

$$\begin{array}{ccc}
Z' & \xrightarrow{i'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Z & \xrightarrow{i} & Y
\end{array}$$

commutes. Then there is a natural transformation $\text{tr}_f: Rf_*R\Gamma_{Z'}f'_! \to \text{id}$ such that, for all $\mathcal{F}^\bullet \in D_{\text{qc}}^-(O_X)_Z$ and $\mathcal{G}^\bullet \in D^+_{\text{qc}}(O_Y)_Z$, the composition

$$Rf_*R\text{Hom}^\bullet_{O_X}(\mathcal{F}^\bullet, R\Gamma_{Z'}f'_!\mathcal{G}^\bullet) \xrightarrow{\text{tr}_f} R\text{Hom}^\bullet_{O_Y}(Rf_*\mathcal{F}^\bullet, Rf_*R\Gamma_{Z'}f'_!\mathcal{G}^\bullet)$$

\(^3\)Here a morphism $f$ is called residually stable if it is flat, integral and the fibers of $f$ are Gorenstein.
is an isomorphism. In particular, taking global sections, the functor $Rf_*$ is left adjoint to the functor $R\Gamma_{Z'}f$.

### 3.1 Generalization of Emerton-Kisin’s adjunction

**Proposition 3.2.** Let $f : X \to Y$ be a separated and finite type morphism of smooth $k$-schemes and let $i : Z \to Y$ and $i' : Z' \to X$ be closed immersions with a proper morphism $f' : Z' \to Z$ such that $f \circ i' = i \circ f'$.

(a) There is a natural morphism

$$\text{tr}_{F,f} : f_+R\Gamma_{Z'}\mathcal{O}_{F,X}[d_{X/Y}] \to \mathcal{O}_{F,Y}$$

of $(\mathcal{O}_{F,Y}, \mathcal{O}_{F,Y})$-bimodules which, as a morphism of left $\mathcal{O}_{F,Y}$-modules, is the trace $\mathcal{O}_{F,Y} \otimes_{\mathcal{O}_Y} Rf_*R\Gamma_{Z'}\omega_{X/Y}[d_{X/Y}] \to \mathcal{O}_{F,Y}$ of Theorem 3.1.

(b) For every $\mathcal{M}^\bullet \in D^b_{qc}(\mathcal{O}_{F,Y})$, the trace map $\text{tr}_{F,f}$ induces a morphism

$$\text{tr}_{F,f}(\mathcal{M}^\bullet) : f_+R\Gamma_{Z'}f'^!\mathcal{M}^\bullet \to \mathcal{M}^\bullet$$

in $D^b_{qc}(\mathcal{O}_{F,Y})$.

**Proof.** (a) This is an analogue of [EK04, Proposition 4.4.9 (i)]. A careful reading of the proof shows that we can adopt it. Consider the relative Frobenius diagram (diagram 1 on page 13):

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/Y}} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{F_Y} & Y.
\end{array}
\]

Since $X$ and $Y$ are assumed to be smooth $k$-schemes, we still have flatness of the Frobenius $F_Y$ and therefore of $F_Y'$ because flatness is stable under base change. Note that $F_{X/Y}$ is finite ([EK04, A.2]). First Emerton and Kisin explain how the relative Cartier operator

$$C_{X/Y} : F_{X/Y}\omega_{X/Y} \to F_Y^\omega\omega_{X/Y}$$

is realized for the residual complex $f^\Delta E^\bullet$. Here $E^\bullet$ denotes the Cousin complex $E^\bullet(\mathcal{O}_X)$. For our result we replace $f^\Delta E^\bullet$ by the subcomplex $\Gamma_Zf^\Delta E^\bullet$ of flasque sheaves which computes $R\Gamma_{Z'}\omega_{X/Y}$. We obtain the relative Cartier operator with support on $Z'$:

$$C_{X/Y}^Z : F_{X/Y}\Gamma_Zf^\Delta E^\bullet \to F_Y^\omega\Gamma_Zf^\Delta E^\bullet.$$
\(f^{-1}\mathcal{O}_Y[F] \rightarrow \mathcal{O}_X[F],\) with a \((f^{-1}\mathcal{O}_F,Y, f^{-1}\mathcal{O}_F,Y)\)-bimodule structure. Finally this endows \(f_*(f^{-1}\mathcal{O}_F \otimes f^{-1}\mathcal{O}_Y \Gamma_Z' f^\Delta E^*)\) with the structure of a \((\mathcal{O}_F,Y, \mathcal{O}_F,Y)\)-bimodule, the one from the definition of \(f_+ R\Gamma Z'/\mathcal{O}_F,X.\)

But there is another way to look at this bimodule: The map \(\mathcal{O}^Z_{X/Y}\) gives rise to a morphism

\[\sigma: f_* \Gamma_Z' f^\Delta E^* \rightarrow F_Y^* f_* \Gamma_Z' f^\Delta E^*\]

by the composition

\[f_* \Gamma_Z' f^\Delta E^* \xrightarrow{\sim} f_* F_{X/Y}^* \Gamma_Z' f^\Delta E^* \xrightarrow{C_{X/Y}^Z} f_* F_Y^* F_{X/Y}^* \Gamma_Z' f^\Delta E^* \xrightarrow{bc^{-1}} F_Y^* f_* \Gamma_Z' f^\Delta E^*,\]

where the first isomorphism is deduced from \(f = f' \circ F_{X/Y}\) and the last isomorphism is flat base change. Now Proposition-Definition 1.10.1 of ibid. in the special case of the morphism \(\text{id}_Y\) yields a \((\mathcal{O}_F,Y, \mathcal{O}_F,Y)\)-bimodule structure on \(\mathcal{O}_F,Y \otimes \mathcal{O}_Y f_* \Gamma_Z' f^\Delta E^*.\) The isomorphism

\[f_*(f^{-1}\mathcal{O}_F \otimes f^{-1}\mathcal{O}_Y \Gamma_Z' f^\Delta E^*) \cong \mathcal{O}_F,Y \otimes \mathcal{O}_Y f_* \Gamma_Z' f^\Delta E^*\]

stemming from the projection formula is compatible with the constructed bimodule structure for both complexes by Lemma 1.10.6 of ibid. Hence it suffices to show that \(\text{tr}_{F,f}\) induces a morphism between the \((\mathcal{O}_F,Y, \mathcal{O}_F,Y)\)-bimodule \(\mathcal{O}_F,Y \otimes \mathcal{O}_Y f_* \Gamma_Z' f^\Delta E^*\) and \(E^*\) equipped with the structure of a \((\mathcal{O}_F,Y, \mathcal{O}_F,Y)\)-bimodule via the canonical isomorphism \(E^* \rightarrow F_Y^* E^*\) induced from the Frobenius \(\mathcal{O}_Y \rightarrow F_Y^* \mathcal{O}_Y.\) Lemma 1.10.2 of ibid. applied to the identity morphism on \(Y\) reduces to the commutativity of the diagram

\[
\begin{array}{ccc}
  f_* \Gamma_Z' f^\Delta E^* & \xrightarrow{\text{tr}_{F,f}} & E^* \\
  | & | & | \\
  F_Y^* f_* \Gamma_Z' f^\Delta E^* & \xrightarrow{F_Y^* \text{tr}_{F,f}} & F_Y^* E^* \\
\end{array}
\]
of complexes. For this we have to see that the following bigger diagram commutes:

\[
\begin{array}{cccccc}
  f_* \Gamma_Z f^! E^* & \xrightarrow{\text{tr}_f} & E^* \\
  \sim & & \sim \\
  f'_* F_{X/Y} \Gamma_Z F_{X/Y}^! f'^! E^* & \xrightarrow{\text{tr}_{F_{X/Y}}} & f'_* \Gamma_Z f'^! E^* & \xrightarrow{\text{tr}_{f'}} & E^* \\
  \sim & & \sim \\
  f'_* F_{X/Y} \Gamma_Z F_{X/Y}^! f'^! E^* & \xrightarrow{\beta} & f'_* \Gamma_Z f'^! E^* & \xrightarrow{\text{tr}_{f'}} & F_Y^! E^* \\
  \sim & & \sim \\
  f'_* F_{X/Y} \Gamma_Z F_{X/Y}^! f'^! E^* & \xrightarrow{\beta} & f'_* \Gamma_Z F_Y^* f^\Delta E^* \\
  \xrightarrow{\text{bc}^{-1}} & & \sim \\
  F_Y^* f_* \Gamma_Z f^\Delta E^* & \xrightarrow{\text{tr}_f} & F_Y^! E^*. \\
\end{array}
\]

The two squares in the middle and the lower left square commute by functoriality of the trace maps \(\text{tr}_{F_{X/Y}}\) and \(\text{tr}_{f'}\). The other squares are commutative because the generalized trace is compatible with compositions of morphisms and with base change by the residually stable map \(F_Y\) ([Sch18, Propositions 2.10 and 2.11]).

(b) Once we know that \(\text{tr}_f\) is a morphism in \(D^{b}_{qc}(\mathcal{O}_{F,Y})\), we can define \(\text{tr}_f(\mathcal{M}^*)\) as the following composition:

\[
f_+ R\Gamma_Z f^! \mathcal{M}^* \xrightarrow{\sim} f_+(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_{F,X}} R\Gamma_Z f^! \mathcal{M}^*) \\
\xrightarrow{\sim} f_+(R\Gamma_Z \mathcal{O}_{F,X} \otimes_{\mathcal{O}_{F,X}} f^! \mathcal{M}^*) \\
\xrightarrow{\sim} f_+ R\Gamma_Z \mathcal{O}_{F,X}[d_{X/Y}] \otimes_{\mathcal{O}_{F,Y}} \mathcal{M}^* \\
\xrightarrow{\text{tr}_f \otimes \text{id}} \mathcal{O}_{F,Y} \otimes_{\mathcal{O}_{F,Y}} \mathcal{M}^* \\
\xrightarrow{\sim} \mathcal{M}^*.
\]

Here the second morphism is the one of [Sch18, Lemma 1.11] and the third morphism is the one of [EK04, Lemma 4.4.7].

**Lemma 3.3.** We keep the notation of the preceding proposition. For an open immersion \(j: U \to Y\), let \(f'\) and \(j'\) denote the projections of \(U' = U \times_Y X\). Assume that \(Z\) and \(Z'\) are the closures of the locally closed subsets \(Z_U = Z \cap U\) and \(Z'_U = Z' \cap U'\) in \(Y\) and in \(X\).
(a) There is a functorial isomorphism $e_{j,f} : f_+ R\Gamma_Z f^j j_+ \xrightarrow{\sim} j_+ f'_+ R\Gamma_Z f^d$ such that the diagram

$$
\begin{array}{ccc}
f_+ R\Gamma_Z f^j j_+ & \xrightarrow{e_{j,f}} & j_+ f'_+ R\Gamma_Z f^d \\
\downarrow \tr{f} & & \downarrow \tr{f'} \\
j_+ & \xrightarrow{\sim} & j_+
\end{array}
$$

commutes.

(b) Let $\tr{f}$ denote the unit id $\rightarrow R\Gamma_Z f^j f_+$ of the adjunction. Then there is a functorial isomorphism $e_{j,f}' : R\Gamma_Z f^j f_+ j'_+ \xrightarrow{\sim} j_+ f'_+ R\Gamma_Z f^d f'_+$ such that the diagram

$$
\begin{array}{ccc}
R\Gamma_Z f^j f_+ j'_+ & \xrightarrow{e_{j,f}'} & j_+ f'_+ R\Gamma_Z f^d f'_+ \\
\downarrow \tr{f} & & \downarrow \tr{f'} \\
j'_+ & \xrightarrow{\sim} & j'_+
\end{array}
$$

commutes.

Proof. Let $Z_U$ and $Z'_u$ denote the closed subsets $U \cap Z$ and $U' \cap Z'$ of $U$ and $U'$. From [Sch18, Proposition 1.13] we know that the functors $j_+$ and $j'_+$ are equivalences

$$
D^b_{\text{log}}(O_{F,U}) Z_U \xrightarrow{\sim} D^b_{\text{log}}(O_{F,Y}) Z_U \quad \text{and} \quad D^b_{\text{log}}(O_{F,U'}) Z'_U \xrightarrow{\sim} D^b_{\text{log}}(O_{F,Y'}) Z'_U.
$$

Furthermore, there are natural isomorphisms

$$
f_+ j'_+ \xrightarrow{\sim} j_+ f'_+ \quad \text{and} \quad R\Gamma_Z f^j j_+ \xrightarrow{\sim} j'_+ R\Gamma_Z f^d f'_+,
$$

where the second one is obtained from the composition

$$
j^d R\Gamma_Z f^j \xrightarrow{\sim} R\Gamma_Z j'_+ f^d f'_+ \xrightarrow{\sim} R\Gamma_Z f^d j^d
$$

of natural isomorphisms. Moreover, together with the canonical isomorphism $f'_+ j^d \cong j^d f_+$ of [EK04, Proposition 3.8], this composition yields a canonical isomorphism

$$
\bar{e}_{j,f} : f'_+ R\Gamma_Z f^d j^d \xrightarrow{\sim} j'_+ f_+ R\Gamma_Z j^d.
$$

For $\mathcal{M} \in D^b_{\text{qc}}(O_{F,Y})$, the diagram

$$
\begin{array}{ccc}
f'_+ R\Gamma_Z f^d j^d \mathcal{M} & \xrightarrow{\bar{e}_{j,f}} & j'_+ f_+ R\Gamma_Z j^d \mathcal{M} \\
\downarrow \sim & & \downarrow \sim \\
\sim f'_+ (O_{F,U'} \otimes O_{F,U'} R\Gamma_Z f^d j^d \mathcal{M}) & \xrightarrow{\sim} & j'_+ f_+ (O_{F,X} \otimes O_{F,X} R\Gamma_Z j^d \mathcal{M}) \\
\downarrow \sim & & \downarrow \sim \\
\sim f'_+ R\Gamma_Z f^d O_{F,U'} \otimes O_{F,U'} j^d \mathcal{M} & \xrightarrow{\bar{e}_{j,f}} & j'_+ f_+ R\Gamma_Z f^d O_{F,X} \otimes O_{F,X} j^d \mathcal{M} \\
\tr{f'} & & \tr{f'} \\
\sim j^d \mathcal{M} & \xrightarrow{\sim} & j^d \mathcal{M}
\end{array}
$$
of natural isomorphisms and the trace commutes: While the first square commutes simply by functoriality, the commutativity of the second square follows from [EK04, Lemma 4.4.7 (ii)]. The commutativity of the lower triangle follows from the compatibility of the trace with residually stable base change ([Sch18, Proposition 2.10]). In summary the diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{f^+} & \mathcal{M} \\
\xrightarrow{j} & & \xrightarrow{j} \\
\mathcal{N} & \xrightarrow{Rf} & \mathcal{N} \\
\end{array}
\]

is commutative. Since \(j^!\) and \(j'^!\) are quasi-inverses of \(j^\_\) and \(j'^\_\) and \(f^+\) and \(Rf\) restrict to the functors \(f'^+\) and \(Rf'\) between \(D(bqc)(\mathcal{O}_{F,X})\) and \(D(bqc)(\mathcal{O}_{F,Y})\) with respect to the equivalences \(j^!\) and \(j'^!\), the claims of the lemma are formal consequences.

**Theorem 3.4.** Let \(f: X \rightarrow Y\) be a separated and finite type morphism of smooth schemes and let \(i: Z \rightarrow Y\) and \(i': Z' \rightarrow X\) be closed immersions with a morphism \(f': Z' \rightarrow Z\) such that the diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & X \\
\xrightarrow{f'} & & \xrightarrow{f} \\
Z & \xrightarrow{i} & Y \\
\end{array}
\]

commutes. Then, for any \(\mathcal{M}^\bullet \in D^b(bqc)(\mathcal{O}_{F,X})\) and any \(\mathcal{N}^\bullet \in D^b(bqc)(\mathcal{O}_{F,Y})\), there is a natural isomorphism

\[
\mathcal{RHom}^{\bullet}_{F,Y}(f^+_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} Rf_+\mathcal{RHom}^{\bullet}_{F,X}(\mathcal{M}^\bullet, Rf'f^!_+\mathcal{N}^\bullet).
\]

In particular, \(f^+_+: D^b(bqc)(\mathcal{O}_{F,X}) \rightarrow D^b(bqc)(\mathcal{O}_{F,Y})\) is left adjoint to \(Rf\).

**Proof.** The morphism \(f\) factors through the graph morphism \(X \times_k Y\), which is a closed immersion, followed by the projection \(X \times_k Y \rightarrow Y\), which is smooth. Therefore, we may assume that \(f\) is an essentially perfect morphism. We show that the natural transformation \(\tau\) given by the composition

\[
\begin{array}{ccc}
Rf_+\mathcal{RHom}^{\bullet}_{F,X}(\mathcal{M}^\bullet, Rf'f^!_+\mathcal{N}^\bullet) & \xrightarrow{\sim} & \mathcal{RHom}^{\bullet}_{F,Y}(f^+_+\mathcal{M}^\bullet, f^+_+Rf'f^!_+\mathcal{N}^\bullet) \\
\xrightarrow{\tau_{f,f}} & & \xrightarrow{\tau_{f,f}} \\
\mathcal{RHom}^{\bullet}_{F,Y}(f^+_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) & & \mathcal{RHom}^{\bullet}_{F,Y}(f^+_+\mathcal{M}^\bullet, \mathcal{N}^\bullet)
\end{array}
\]

is an isomorphism in \(D^+(X, \mathbb{Z}/p\mathbb{Z})\). Here the horizontal arrow is the natural morphism of [EK04, Proposition 4.4.2]. Let \(\mathcal{O}_{F,f}\) denote the \((f^{-1}\mathcal{O}_{F,Y}, \mathcal{O}_{F,X})\)-bimodule \(\mathcal{O}_{F,Y^{-X}}\) and let \(\omega_f\) denote the \(\mathcal{O}_X\)-module \(\omega_{X/Y}\). We set \(d = d_{X/Y}\). First we replace \(\mathcal{M}^\bullet\) by...
a bounded above complex of quasi-coherent induced left $\mathcal{O}_{F,X}$-modules, i.e. left $\mathcal{O}_{F,X}$-
modules of the form $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M$ with quasi-coherent $\mathcal{O}_X$-
modules $M$, see Definition 1.7 and Lemma 1.7.1 of [EK04]. Now by the Lemma on Way-out Functors ([Har66, Proposition 1.7.1]), we reduce to the case of a single sheaf $\mathcal{M}^\bullet = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M$. For such an induced module we have an isomorphism

$$f_+ \mathcal{M} \xrightarrow{\sim} \mathcal{O}_{F,Y} \otimes_{\mathcal{O}_Y} Rf_*(\omega_{X/Y} \otimes_{\mathcal{O}_X} M),$$

which is based on the projection formula, see the proof of [EK04, Theorem 3.5.3]. Note that in this proof $f^!$ always denotes Emerton-Kisin’s pull-back of left $\mathcal{O}_{F,Y}$-modules, sometimes considered as an $\mathcal{O}_Y$-module. It is connected to the functor $f^!$ for quasi-coherent sheaves by the canonical isomorphisms

$$f^! N^\bullet \xrightarrow{\sim} Lf^* N^\bullet [d] \xrightarrow{\sim} \omega_{X/Y}^{-1} \otimes_{\mathcal{O}_X} f^! N^\bullet$$

in $D_{qC}(X)$, where ‘$f^!$’ denotes the classical $f^!$. One can show that there is a commutative diagram

$$\begin{array}{ccc}
Rf_* \mathcal{RHom}_{\mathcal{O}_X}(M, R\Gamma_Z f^! N^\bullet) & \xrightarrow{\sim} & Rf_* \mathcal{RHom}_{\mathcal{O}_{F,X}}(\mathcal{M}, R\Gamma_Z f^! N^\bullet) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{RHom}_{\mathcal{O}_Y}(Rf_*(\omega_{X/Y} \otimes_{\mathcal{O}_X} M), N^\bullet) & \xrightarrow{\sim} & \mathcal{RHom}_{\mathcal{O}_{F,Y}}(f_+ \mathcal{M}, N^\bullet)
\end{array}$$

with an isomorphism $t$ and where the horizontal arrows are the natural isomorphisms induced by the isomorphism

$$\text{Hom}_{\mathcal{O}_X}(M, ___) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_{F,X}}(\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M, ___)$$

of [EK04, 1.7.2] and (6). For this we consider the bigger diagram of natural maps on page 49. Let $t$ be the composition of the left vertical arrows. It is an isomorphism by [Sch18, Proposition 3.4] and Theorem 3.1. Recall that $\mathcal{O}_{F,Y^{-}X}$ is locally free as a right $\mathcal{O}_X$-module and that $\mathcal{O}_{F,Y^{-}X} \otimes_{\mathcal{O}_F,X} M \cong \mathcal{O}_{F,Y^{-}X} \otimes_{\mathcal{O}_Y} M$, which is computed in the proof of Lemma 3.5.1 of [EK04]. In particular, induced modules are acyclic for the functor $\mathcal{O}_{F,Y^{-}X} \otimes_{\mathcal{O}_F,X} __$. For the first square, we consider the diagram without the outer $Rf_*$, resolve $M$ by a complex $\mathcal{P}^\bullet$ of locally free $\mathcal{O}_X$-modules and $R\Gamma_Z f^! N^\bullet$ by a complex $\mathcal{J}^\bullet$ of left $\mathcal{O}_{X,F}$-modules which are acyclic for the functor $\mathcal{O}_{F,Y^{-}X} \otimes_{\mathcal{O}_F,X} __$. For the second square, we consider the diagram without the outer $Rf_*$, resolve $M$ by a complex $\mathcal{P}^\bullet$ of locally free $\mathcal{O}_X$-modules and $R\Gamma_Z f^! N^\bullet$ by a complex $\mathcal{J}^\bullet$ of left $\mathcal{O}_{X,F}$-modules which are acyclic for the functor $\mathcal{O}_{F,Y^{-}X} \otimes_{\mathcal{O}_F,X} __$ as in the proof of Proposition 4.4.2 of ibid. Now $\mathcal{P}^\bullet = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet$ is a complex of locally
free $\mathcal{O}_{F,X}$-modules. We obtain a commutative diagram

\[
\begin{array}{ccc}
\text{RHom}^{*}_{\mathcal{O}_{X}}(P^*, J^*) & \overset{\sim}{\longrightarrow} & \text{RHom}^{*}_{\mathcal{O}_{F,X}}(P^*, J^*) \\
\text{Hom}^{*}_{\mathcal{O}_{X}}(P^*, J^*) & \overset{\sim}{\longrightarrow} & \text{Hom}^{*}_{\mathcal{O}_{F,X}}(P^*, J^*) \\
\text{Hom}^{*}_{\mathcal{O}_{X}}(\omega_f \otimes \mathcal{O}_{X} P^*, \omega_f \otimes \mathcal{O}_{X} J^*) & \longrightarrow & \text{Hom}^{*}_{\mathcal{O}_{F,Y}}(\mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} P^*, \mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} J^*) \\
\text{RHom}^{*}_{\mathcal{O}_{X}}(\omega_f \otimes \mathcal{O}_{X} P^*, \omega_f \otimes \mathcal{O}_{X} J^*) & \longrightarrow & \text{RHom}^{*}_{\mathcal{O}_{F,Y}}(\mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} P^*, \mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} J^*)
\end{array}
\]

of canonical maps. The last two vertical arrows are the canonical morphisms from a functor to its right derived functor. Here the left one is an isomorphism because $\omega_{X/Y} \otimes P^*$ is a locally free $\mathcal{O}_X$-module.

For the second square, we check that the natural map

\[
\text{RHom}^{*}_{\mathcal{O}_{X}}(\omega_f \otimes \mathcal{O}_{X} M, \omega_f \otimes \mathcal{O}_{X} R\Gamma_{Z'}N^*) \\
\text{RHom}^{*}_{\mathcal{O}_{F,Y}}(\mathcal{O}_{F,f} \otimes \mathcal{O}_{X} M, \mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} R\Gamma_{Z'}N^*)
\]

factors through $\text{RHom}^{*}_{\mathcal{O}_{F,Y}}(\omega_f \otimes \mathcal{O}_{X} M, \omega_f \otimes \mathcal{O}_{X} R\Gamma_{Z'}N^*)$. For this we replace $\omega_f \otimes \mathcal{O}_{X} R\Gamma_{Z'}f^!N^*$ by a complex $I^*$ of injective $f^{-1}\mathcal{O}_{Y}$-modules and $\mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} R\Gamma_{Z'}f^!N^*$ by a complex $I^*$ of injective $f^{-1}\mathcal{O}_{F,Y}$-modules. The functor $f^{-1}\mathcal{O}_{F,Y} \otimes_{f^{-1}\mathcal{O}_{Y}} -$ is exact because the right $\mathcal{O}_{Y}$-module $\mathcal{O}_{F,Y}$ is free ([EK04, Lemma 1.3.1]). Furthermore, it is left adjoint to the forgetful functor from $f^{-1}\mathcal{O}_{F,Y}$-modules to $f^{-1}\mathcal{O}_{Y}$-modules. Hence the latter functor preserves injectives. This implies that $I^*$ is a complex of injective $f^{-1}\mathcal{O}_{Y}$-modules and the canonical morphism $\omega_f \otimes \mathcal{O}_{X} R\Gamma_{Z'}f^!N^* \to \mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} R\Gamma_{Z'}f^!N^*$ yields a map $I^* \to I^*$. After replacing $M$ by a complex $P^*$ of locally free $\mathcal{O}_X$-modules as above we have reduced the three $\text{RHom}$ to $\text{Hom}$ and the claimed factorization is trivial.

We return to the second square of the diagram on page 49, where we replace $M$ by a complex $F^*$ of flasque $\mathcal{O}_X$-sheaves. The complexes $\omega_f \otimes \mathcal{O}_{X} F^*$ and $\mathcal{O}_{F,f} \otimes \mathcal{O}_{X} F^*$ are also flasque because they locally are direct sums of flasque sheaves. Hence $f_*(\omega_f \otimes F^*)$ and $f_*(\mathcal{O}_{F,f} \otimes F^*)$ represent $Rf_*(\omega_f \otimes F^*)$ and $Rf_*(\mathcal{O}_{F,f} \otimes F^*)$. As above, we resolve $\omega_f \otimes \mathcal{O}_{X} R\Gamma_{Z'}f^!N^*$ by $I^*$ and $\mathcal{O}_{F,f} \otimes \mathcal{O}_{F,X} R\Gamma_{Z'}f^!N^*$ by $\hat{I}^*$. The injectivity of $I^*$ and $\hat{I}^*$ implies that $\text{Hom}^{*}_{\mathcal{O}_{Y}}(\omega_f \otimes \mathcal{O}_{X} F^*, I^*)$ and $\text{Hom}^{*}_{\mathcal{O}_{F,Y}}(\mathcal{O}_{F,f} \otimes \mathcal{O}_{X} F^*, \hat{I}^*)$ are flasque ([God58, Lemme II.7.3.2]) and hence may be used to compute $Rf_*$. As $f_*$ is right adjoint to the exact functor $f^{-1}$, the complex $f_*\mathcal{I}^*$ is a complex of injective $\mathcal{O}_Y$-modules and
$f_*\tilde{I}^\bullet$ is a complex of injective $\mathcal{O}_{F,Y}$-modules. Therefore

$$\text{RHom}^\bullet_{\mathcal{O}_X}(\_ , f_*\mathcal{I}^\bullet) \cong \text{Hom}^\bullet_{\mathcal{O}_X}(\_ , f_*\mathcal{I}^\bullet)$$

and

$$\text{RHom}^\bullet_{\mathcal{O}_{F,X}}(\_ , f_*\tilde{I}^\bullet) \cong \text{Hom}^\bullet_{\mathcal{O}_{F,X}}(\_ , f_*\tilde{I}^\bullet).$$

This finishes the proof of the commutativity of the second square because the diagram

\[
\begin{array}{ccc}
\text{f_*Hom}^\bullet_{\mathcal{O}_Y}(\omega_f \otimes_{\mathcal{O}_X} F^*, \mathcal{I}^\bullet) & \longrightarrow & \text{f_*Hom}^\bullet_{\mathcal{O}_{F,Y}}(\mathcal{O}_{F,f} \otimes_{\mathcal{O}_X} F^*, \tilde{I}^\bullet) \\
\downarrow & & \downarrow \\
\text{Hom}^\bullet_{\mathcal{O}_{F,Y}}(f_* (\mathcal{O}_{F,f} \otimes_{\mathcal{O}_X} F^*), f_*\tilde{I}^\bullet) & & \text{Hom}^\bullet_{\mathcal{O}_Y}(f_* (\omega_f \otimes_{\mathcal{O}_X} F^*), f_*\mathcal{I}^\bullet) \\
\downarrow & & \downarrow \\
\text{Hom}^\bullet_{\mathcal{O}_{F,Y}}(f_* (\omega_f \otimes_{\mathcal{O}_X} F^*), f_*\tilde{I}^\bullet) & \longrightarrow & \text{Hom}^\bullet_{\mathcal{O}_{F,Y}}(\mathcal{O}_{F,Y} \otimes_{\mathcal{O}_Y} f_* (\omega_f \otimes_{\mathcal{O}_X} F^*), f_*\tilde{I}^\bullet)
\end{array}
\]

of natural morphisms commutes.

The commutativity of the third and the fifth square can be shown similarly. The fourth square commutes by the functoriality of the corresponding horizontal isomorphisms.

For the adjunction of $f_+$ and $R\Gamma_Z f^!$ we proceed as in the proof of [Sch18, Theorem 3.2].
3.2 Definition of $lfgu$ modules on singular schemes

As mentioned earlier, for a regular scheme $X$, the Frobenius $F_X: X \to X$ is a flat morphism and hence $F^*_X$ is exact ([Kun69, Theorem 2.1]). For varieties, the exactness of $F^*_X$ plays an important role in the definition of (locally finitely generated) unit $\mathcal{O}_{F,X}$-modules. For example, it implies that the category of unit $\mathcal{O}_{F,X}$-modules is abelian.

In this section we define the abelian category $\mu_{lfgu}$-scheme as a certain subcategory of $\mathcal{O}_{F,X}$-modules for schemes $X$ which admit a closed immersion $i: X \to Y$ into a smooth $k$-scheme as a certain subcategory of $\mu_{lfgu}(Y)$. Note that this definition generally works for unit $\mathcal{O}_{F,X}$-modules. We restrict to locally finitely generated modules due to our application to Cartier crystals and perverse constructible étale $p$-torsion sheaves.

For the motivation of our approach to $\mu_{lfgu}(X)$ for embeddable $X$, recall the Kashiwara equivalence:

**Theorem 3.5.** Let $i: Z \to X$ be a closed immersion of smooth $k$-schemes. If $\mathcal{M}$ is a unit $\mathcal{O}_{F,Z}$-module supported on $Z$, the adjunction $i_+i^!\mathcal{M} \to \mathcal{M}$ is an isomorphism. Consequently, $H^0(i^!\mathcal{M}) \cong i^!\mathcal{M}$ and the functors $i_+$ and $i^!$ are equivalences between the categories of unit $\mathcal{O}_{F,Z}$-modules and unit $\mathcal{O}_{F,X}$-modules supported on $Z$.

**Proof.** This is Theorem 5.10.1 of [EK04].

Hence, keeping the notation of the preceding theorem, we can canonically interpret unit $\mathcal{O}_{F,Z}$-modules as a certain subcategory of unit $\mathcal{O}_{F,X}$-modules, namely the subcategory of unit $\mathcal{O}_{F,X}$-modules with support on (the image of) $Z$. If $Z$ is not smooth this subcategory still exists because it may be characterized as the subcategory of unit $\mathcal{O}_{F,X}$-modules $\mathcal{M}$ with $j^!\mathcal{M} \cong 0$, where $j$ is the immersion of the open complement of $Z$ in $X$. This motivates the definition of unit $\mathcal{O}_{F,Z}$-modules for $Z$ possibly not smooth but embeddable into a smooth scheme. But first we introduce some notation.

**Definition 3.6.** We call a $k$-scheme $X$ **embeddable** if there is a closed immersion $i: X \to Y$ of $k$-schemes where $Y$ is smooth.

**Example 3.7.** Let $X = \text{Spec} k[x_1, \ldots, x_n]/I$ be an affine variety. Then $X$ is embeddable into the affine space $\mathbb{A}^n_k$ by the closed immersion corresponding to the canonical projection

$$k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I.$$

**Example 3.8.** Let $X$ be a quasi-projective $k$-scheme. By definition, there exists an open immersion $j: X \to Z$ and a projective morphism $p: Z \to \text{Spec} k$ such that $f = p \circ j$. In turn, the morphism $p$ factors into a closed immersion $i: Z \to \mathbb{P}^n_k$ followed by the natural morphism $\mathbb{P}^n_k \to \text{Spec} k$. Let $U$ be an open subset of $\mathbb{P}^n_k$ such that $U \cap i(Z) = i(j(X))$. Then $X \cong U \times_{\mathbb{P}^n_k} Z$ and the projection $X \to U$ is a closed immersion of $X$ into an open subset of the projective space. Thus $X$ is embeddable.

**Definition 3.9.** Assume that $k$ is perfect. Let $X$ be an embeddable $k$-scheme. Let $i: X \to Y$ be a closed immersion into a smooth $k$-scheme $Y$. The category of $lfgu$ $\mathcal{O}_{F,X}$-modules is defined as the full subcategory of $lfgu \mathcal{O}_{F,Y}$-modules $\mathcal{M}$ supported
on the image of \(X\), i.e. \(j^!\mathcal{M} \cong 0\), where \(j : Y \setminus X \to X\) is the open immersion of the complement of \(X\).

The category \(\mathcal{D}_{\text{lg}}^b(O_{F,X})\) is the full subcategory \(\mathcal{D}_{\text{lg}}^b(O_{F,Y})_X\) of those objects in \(\mathcal{D}_{\text{lg}}^b(O_{F,Y})\) whose cohomology sheaves are supported on \(X\).

**Remark 3.10.** With the notation of the preceding definition, let \(\mathcal{M}\) be an \(\text{lg}\) module on \(Y\). Whether \(\mathcal{M}\) is supported in \(X\) only depends on the closed subset \(i(X)\) in \(Y\). For example, the preceding definition does not distinguish between the categories \(\mathcal{D}_{\text{lg}}^b(O_{F,X})\) and \(\mathcal{D}_{\text{lg}}^b(O_{F,X_{\text{red}}})\), where \(X_{\text{red}}\) is the unique closed subscheme of \(X\) whose underlying topological space equals the one of \(X\) and which is reduced.

By Theorem 3.5, it is clear that this definition generalizes the already existing notion of \(\text{lg}\) \(O_{F,X}\)-modules for smooth \(X\). Of course the crucial point is to see that the definition for not-necessarily smooth \(X\) is – up to natural equivalence – independent of a chosen embedding into a smooth scheme.

**Theorem 3.11.** Assume that \(k\) is a perfect field. Let \(f : X \to Y\) be a flat morphism between smooth \(k\)-schemes and let \(i_X : Z \to X\) and \(i_Y : Z \to Y\) be closed immersions of \(k\)-schemes such that the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i_X} & X \\
\downarrow{i_Y} & & \downarrow{f} \\
Y & & 
\end{array}
\]

commutes. Then there are natural isomorphisms of functors

\[
(i) \quad f_+ \circ R\Gamma_Z f^! \cong \text{id}_{\mathcal{D}_{\text{lg}}^b(O_{F,Y})_Z},
\]

\[
(ii) \quad R\Gamma_Z f^! \circ f_+ \cong \text{id}_{\mathcal{D}_{\text{lg}}^b(O_{F,X})_Z}.
\]

**Proof.** The proof proceeds by an excision argument, in a similar way as the proof of [Ohk16, Theorem 4.5]. In the case of a smooth scheme \(Z\) we can use the isomorphism of functors \(R\Gamma_Z \cong i_X^! + i_X^!\) from \(\mathcal{D}_{\text{lg}}^b(O_{F,X})\) to \(\mathcal{D}_{\text{lg}}^b(O_{F,X})_Z\) and \(R\Gamma_Z \cong i_Y^! + i_Y^!\) from \(\mathcal{D}_{\text{lg}}^b(O_{F,Y})\) to \(\mathcal{D}_{\text{lg}}^b(O_{F,Y})_Z\) ([EK04, Proposition 5.11.5]):

\[
f_+ R\Gamma_Z f^! \cong f_+ i_X^! + i_X^! f^! \cong i_Y^! + i_Y^! \cong R\Gamma_Z \cong \text{id}.
\]

We may assume that \(Z\) is reduced, see Remark 3.10. Since a finite set of closed points with the reduced scheme structure is always smooth, this verifies the claim if \(Z\) is 0-dimensional. For the general case, i.e. \(Z\) is not necessarily smooth, let \(V\) be a smooth and dense\(^4\) open subscheme of \(Z\) and assume that the claim holds for all closed subschemes \(Z'\) with \(\dim Z' < \dim Z\). Let \(g\) denote the immersion \(V \hookrightarrow Y\). After choosing an open subset \(U \subseteq Y\) with \(U \cap Z = V\), we can factor \(g\) as \(g = u \circ i'\) where \(u\) is the open

\(^4\)In order to guarantee the existence of a smooth, dense subset, we assumed that \(k\) is perfect.
immersion of $U$ into $Y$ and $i'$ is the closed immersion of $V$ into $U$, i.e. the base change of $i_Y$.

For an object $\mathcal{M}^\bullet$ of $D^b_{\text{lgv}}(O_{F,Y})$, there is a natural morphism $\varphi: \mathcal{M}^\bullet \to g_+g_!\mathcal{M}^\bullet$ whose cone $\mathcal{N}^\bullet$ is supported on $Z \setminus U$ ([EK04, Proposition 5.12.1]). This means that there is a distinguished triangle

$$\mathcal{N}^\bullet \to \mathcal{M}^\bullet \xrightarrow{\varphi} g_+g_!\mathcal{M}^\bullet \to \mathcal{N}^\bullet[1]$$

in $D^b_{\text{lgv}}(O_{F,Y})$. Applying $f_+R\Gamma_Zf_!^1$, the trace yields a morphism of triangles

$$f_+R\Gamma_Zf_!^1\mathcal{N}^\bullet \to f_+R\Gamma_Zf_!^1\mathcal{M}^\bullet \xrightarrow{\varphi} f_+R\Gamma_Zf_!^1g_+g_!\mathcal{M}^\bullet$$

$$\downarrow \quad \uparrow \quad \downarrow$$

$$\text{tr}_f(\mathcal{N}^\bullet) \quad \text{tr}_f(\mathcal{M}^\bullet) \quad \text{tr}_f(g_+g_!\mathcal{M}^\bullet)$$

$$\mathcal{N}^\bullet \quad \mathcal{M}^\bullet \quad g_+g_!\mathcal{M}^\bullet.$$ 

Since $f$ is the identity on $Z$, i.e. $i_Y = f \circ i_X$, we have $Z \cap f^{-1}(Z \setminus U) = Z \setminus U$. Therefore, $R\Gamma_Zf_!^1\mathcal{N}^\bullet \cong R\Gamma_Zf_!^1\mathcal{M}^\bullet$ and $\text{tr}_f(\mathcal{N}^\bullet)$ factors through $f_+R\Gamma_Z(U)f_!^1\mathcal{N}^\bullet$. This means that the diagram

$$f_+R\Gamma_Zf_!^1\mathcal{N}^\bullet \to \mathcal{N}^\bullet \quad \sim \quad \sim$$

$$\text{tr}_{f,Z}(\mathcal{N}^\bullet) \quad \text{tr}_{f,Z}(\mathcal{M}^\bullet)$$

$$f_+R\Gamma_Zf_!^1\mathcal{N}^\bullet$$

is commutative. The dimension of the support of $Z \setminus U$ of $\mathcal{N}^\bullet$ is less than that of $Z$ as $V$ is dense in $Z$. By induction hypothesis, $\text{tr}_{f,Z}(\mathcal{N}^\bullet)$ is an isomorphism and hence $\text{tr}_{f,Z}(\mathcal{M}^\bullet)$ is an isomorphism.

It remains to show that $\text{tr}_f(g_+g_!\mathcal{M}^\bullet) \cong \text{tr}_f(u_+i'_!u'^!\mathcal{M}^\bullet)$ is an isomorphism. By the Kashiwara equivalence, the object $\mathcal{M}^\bullet_U := i'_!i'^!u'^!\mathcal{M}^\bullet$ of $D^b_{\text{lgv}}(O_{F,U})$ is supported on $V$. Let $f'$ denote the projection $U \times_Y X \to U$. The map $\text{tr}_f(u_+\mathcal{M}^\bullet_U)$ equals the composition

$$f_+R\Gamma_Zf_!^1u_+\mathcal{M}^\bullet_U \to u_+f'_+R\Gamma_Vf_!^1\mathcal{M}^\bullet_U \xrightarrow{u_+\text{tr}_f'(\mathcal{M}^\bullet_U)} u_+\mathcal{M}^\bullet_U$$

(Lemma 3.3 (a)). Here the second map is an isomorphism because $V$ is smooth. Consequently, the map $\text{tr}_f(\mathcal{M}^\bullet)$ is an isomorphism. This proves (i). The isomorphism of (ii) can be constructed similarly, using the unit of the adjunction between $f_+$ and $R\Gamma_Zf_!^1$ (i.e. the cotrace) instead of the trace map, and applying Lemma 3.3 (b).

The next corollary shows that the definition of $D^b_{\text{lgv}}(O_{F,X})$ for embeddable varieties $X$ is independent of the chosen embedding.

**Corollary 3.12.** If $i_1: X \to Y_1$ and $i_2: X \to Y_2$ are two embeddings of a $k$-scheme $X$ into smooth $k$-schemes $Y_1$ and $Y_2$, where $k$ is perfect, then there exists a natural equivalence

$$D^b_{\text{lgv}}(O_{F,Y_1})_X \sim D^b_{\text{lgv}}(O_{F,Y_2})_X.$$
Proof. The universal property of $Y_1 \times_k Y_2$ yields a morphism $(i_1, i_2) : X \to Y_1 \times_k Y_2$. It equals the composition

$$X \xrightarrow{(\text{id}, \text{id})} X \times_k X \xrightarrow{(i_1, \text{id})} Y_1 \times_k X \xrightarrow{(\text{id}, i_2)} Y_1 \times_k Y_2,$$

where all maps are closed immersions, the first one because $X$ is assumed to be separated over $k$. Hence $(i_1, i_2)$ is a closed immersion. We obtain a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i_1} & Y_1 \\
\downarrow{(i_1, i_2)} & & \downarrow{p_1} \\
Y_1 \times_k Y_2 & \xrightarrow{i_2} & Y_2,
\end{array}
$$

where $p_1$ and $p_2$ are the projections. By Theorem 3.11, the compositions $p_2 + R\Gamma_{Z/p} \Gamma_1$ and $p_1 + R\Gamma_{Z/p} \Gamma_2$ are inverse equivalences between $D^b_{\text{lgf}}(\mathcal{O}_F, Y_1)_X$ and $D^b_{\text{lgf}}(\mathcal{O}_F, Y_2)_X$. □

4 The Riemann-Hilbert correspondence for Cartier crystals

As its title suggests, one of the main results of Emerton and Kisin’s “The Riemann-Hilbert correspondence for unit $F$-crystals” ([EK04]) is a characteristic $p$-analogue of the Riemann-Hilbert correspondence for $D$-modules. More precisely, for a smooth $k$-scheme $X$, the authors construct inverse equivalences of categories

$$D^b_{\text{lgf}}(\mathcal{O}_F, X) \xrightarrow{\text{Sol}} D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}).$$

Furthermore, $\text{Sol}(D^i_{\text{lgf}}(\mathcal{O}_F, X)) \subseteq p^i D^\geq$ and $\text{Sol}(D^i_{\text{lgf}}(\mathcal{O}_F, X)) \subseteq p^i D^\leq$ where $p^i D^\geq$ and $p^i D^\leq$ are two subcategories of $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ defining the perverse $t$-structure of [Gab04]. Hence Sol establishes an equivalence between the hearts of the corresponding $t$-structures, namely the locally finitely generated unit $\mathcal{O}_F, X$-modules and the so-called perverse constructible $p$-torsion sheaves.

Using this correspondence of Emerton and Kisin, we will establish a Riemann-Hilbert correspondence between Cartier crystals and perverse constructible étale $\mathbb{Z}/p\mathbb{Z}$-sheaves on a scheme which admits an embedding into a smooth scheme. For this we extend the equivalences $G : D^b_{\text{crys}}(\mathbb{Q}\text{Crys}_X(X)) \to D^b_{\text{lgf}}(\mathcal{O}_F, X)$ and $\text{Sol} : D^b_{\text{lgf}}(\mathcal{O}_F, X) \to D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ to singular varieties embeddable into a smooth variety.

4.1 Review of Emerton and Kisin’s Riemann-Hilbert correspondence

Let $X_{\text{ét}}$ denote the small étale site of a scheme $X$. A reference for the étale topology is, for example, [Mil80], Chapter II. A $\mathbb{Z}/p\mathbb{Z}$-sheaf on $X_{\text{ét}}$ is an étale sheaf of modules over the constant sheaf $\mathbb{Z}/p\mathbb{Z}$. Let $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ denote the derived category of complexes of $\mathbb{Z}/p\mathbb{Z}$-sheaves on $X_{\text{ét}}$ whose cohomology sheaves are constructible.
**Definition 4.1.** A sheaf $\mathcal{L}$ of $\mathbb{Z}/p\mathbb{Z}$-modules on $X_{\text{ét}}$ is called **constructible** if there is a stratification $X = \bigsqcup_{i \in I} S_i$ such that the restrictions of $\mathcal{L}$ to the $S_i$ are locally constant sheaves of $\mathbb{Z}/p\mathbb{Z}$-modules for the étale topology with finite stalks.

For $x \in X$, let $i_x : x \to X$ be the inclusion, which is the composition of the inclusion of the closed point of $\text{Spec} \ O_{X_{\text{ét}},x}$ followed by the canonical morphism $\text{Spec} \ O_{X_{\text{ét}},x} \to X$. In [Gab04], Gabber showed that the two subcategories

$$pD^\leq 0 = \{ \mathcal{L}^* \in D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) | H^i(i_x^* \mathcal{L}^*) = 0 \text{ for } i > -\dim \overline{x} \},$$

$$pD^{\geq 0} = \{ \mathcal{L}^* \in D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) | H^i(i_x^! \mathcal{L}^*) = 0 \text{ for } i < -\dim \overline{x} \}$$

define a $t$-structure on $D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$.

**Remark 4.2.** Indeed, Gabber shows that these subcategories define a $t$-structure on the ambient category $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$. For a closed immersion $i : Z \to X$ and the open immersion $j : U \to X$ of the complement $U$ of $Z$, it is obtained from the perverse $t$-structures on $D^b(U_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ and $D^b(Z_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ by recollement:

$$pD^\leq 0 = \{ \mathcal{L}^* \in D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) | i^* \mathcal{L}^* \in pD^\leq 0(Z_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \text{ and } j^* \mathcal{L}^* \in pD^\leq 0(U_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \},$$

$$pD^{\geq 0} = \{ \mathcal{L}^* \in D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) | i^! \mathcal{L}^* \in pD^{\geq 0}(Z_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \text{ and } j^* \mathcal{L}^* \in pD^{\geq 0}(U_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \}.$$ 

This follows directly from the construction of the perverse $t$-structure on $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$.

In this subsection let $X$ be a smooth $k$-scheme. The Riemann-Hilbert correspondence between $D^b_{\text{rig}}(O_{F,X})$ and $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ is realized in two steps: first passing to the étale site and then applying a certain duality functor.

**Theorem 4.3.** (a) For every smooth $k$-scheme $X$, the functor

$$\text{Sol} = \text{RHom}_{O_{F,X_{\text{ét}}}}^\bullet (\underline{\quad}_{\text{ét}}, O_{X_{\text{ét}}})[dX] : D^b_{\text{rig}}(O_{F,X}) \to D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

is an equivalence of categories. A quasi-inverse is given by

$$M = \text{RHom}_{\mathbb{Z}/p\mathbb{Z}}^\bullet (\underline{\quad}, O_{X_{\text{ét}}})[dX].$$

(b) For a morphism $f : X \to Y$ of smooth $k$-schemes, there is a natural isomorphism of functors

$$\text{Sol} \circ f^! \cong f^* \circ \text{Sol}. $$

For an allowable morphism $f : X \to Y$, i.e. a morphism $f$ which factors as $g \circ h$, where $h$ is an immersion and $g$ is a proper smooth morphism, there is also a natural isomorphism of functors

$$\text{Sol} \circ f_+ \cong f_! \circ \text{Sol}. $$

(c) The essential image of the full subcategory $D^b_{\text{rig}}(O_{F,X})$ is equal to the full subcategory $pD^{\geq 0}$ of $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ while the essential image of $D^b_{\text{rig}}(O_{F,X})$ is equal to the full subcategory $pD^{\geq 0}$ of $D^b_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$.

**Proof.** This is [EK04, Theorem 11.4.2 and Theorem 11.5.4].
4.2 Cartier crystals and Ifgu modules on singular schemes

We show that the equivalence

\[ G: D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(X)) \sim \rightarrow D^{b}_{\text{lfgu}}(\mathcal{O}_{F,X}) \]

for smooth \( X \) extends to an equivalence for embeddable \( X \). As a consequence, for a morphism \( f \) between smooth schemes, the inverse equivalences \( f^{+} \) and \( R\Gamma_{Z}f^{!} \) between the subcategories of complexes supported on a closed subscheme are \( t \)-exact.

**Proposition 4.4.** Let \( k \) be a perfect field and let \( X \) be an embeddable \( k \)-scheme. The functor \( G \) induces an equivalence of categories

\[ G: D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(X)) \rightarrow D^{b}_{\text{lfgu}}(\mathcal{O}_{F,X}). \]

**Proof.** Choose a closed immersion \( i: X \rightarrow Y \) into a smooth \( k \)-scheme \( Y \) and let \( j \) denote the open immersion of the complement of \( X \) in \( Y \). The Kashiwara equivalence (Theorem 1.10) identifies \( D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(X)) \) with the subcategory \( D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y))_{X} \) of \( D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y)) \). For \( M^{\bullet} \in D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y))_{X} \) we have

\[ (j^{!} \circ G_{Y})M^{\bullet} \cong (G_{U} \circ j^{*})M^{\bullet} \cong 0 \]

by Proposition 2.35. As \( G \) is an equivalence of categories, there is also a natural isomorphism of functors \( j^{*} \circ G_{Y}^{\prime} \cong G_{U}^{-1} \circ j^{*} \) for the inverse \( G^{-1} \) of \( G \). It follows that \( G \) induces an equivalence of subcategories

\[ G: D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y))_{X} \rightarrow D^{b}_{\text{lfgu}}(\mathcal{O}_{F,Y})_{X}. \]

It remains to show that this equivalence is independent of the choice of the embedding. In the same way as in the proof of Corollary 3.12 we can reduce to the case of two closed immersions \( i_{1}: X \rightarrow Y_{1} \) and \( i_{2}: X \rightarrow Y_{2} \) into smooth \( k \)-schemes \( Y_{1} \) and \( Y_{2} \) together with a morphism \( f: Y_{1} \rightarrow Y_{2} \) such that \( i_{2} = f \circ i_{1} \). The composition \( i_{2*} \circ i_{1}^{!} \) is a natural equivalence between \( D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y_{1}))_{X} \) and \( D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y_{2}))_{X} \). Note that

\[ i_{2*}i_{1}^{!}M^{\bullet} \cong Rf_{*}i_{1*}i_{1}^{!}M^{\bullet} \cong Rf_{*}M^{\bullet} \]

for \( M^{\bullet} \in D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y_{1}))_{X} \). Hence \( Rf_{*} \) is a natural equivalence of categories

\[ Rf_{*}: D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y_{1}))_{X} \rightarrow D^{b}_{\text{crys}}(\text{QCrys}_{\kappa}(Y_{2}))_{X} \]

which is compatible with \( G \), i.e.

\[ f^{+} \circ G_{Y_{1}} \cong G_{Y_{2}} \circ Rf_{*} \]

by Proposition 2.37.

**Remark 4.5.** This also implies that \( f^{+} \) provides a natural equivalence

\[ f^{+}: D^{b}_{\text{lfgu}}(\mathcal{O}_{F,Y_{1}})_{X} \rightarrow D^{b}_{\text{lfgu}}(\mathcal{O}_{F,Y_{2}})_{X} \]

because \( f^{+} \cong G_{Y_{2}} \circ Rf_{*} \circ G_{Y_{1}}^{-1} \).
Keeping the notation of the proof of Proposition 4.4, the canonical $t$-structure of $D^b_{\text{fgu}}(O_{F,Y})$ obviously induces a $t$-structure on the subcategory $D^b_{\text{fgu}}(O_{F,Y})_X$ defined by the two subcategories

$$D^b_{\text{fgu}}(O_{F,Y})_X \cap D^0_{\text{fgu}}(O_{F,Y}) \text{ and } D^b_{\text{fgu}}(O_{F,Y})_X \cap D_0(\text{O}_{F,Y}).$$

**Corollary 4.6.** Let $f: Y_1 \to Y_2$ be a morphism between smooth schemes over a perfect field $k$. Let $i_1: X \to Y_1$ and $i_2: X \to Y_2$ be closed immersions such that $i_2 = f \circ i_1$. The equivalence $f_+$ of Corollary 3.12 between $D^b_{\text{fgu}}(O_{F,Y})_X$ and $D^b_{\text{fgu}}(O_{F,Y})_X$ is $t$-exact for the canonical $t$-structures of both derived categories. In particular, by taking 0-th cohomology, it gives rise to an equivalence of abelian categories

$$\{\mu_{\text{fgu}}(Y_1)_X\} \overset{\sim}{\to} \{\mu_{\text{fgu}}(Y_2)_X\}.$$

**Proof.** The functor $f_+$ is a composition of $t$-exact functors:

$$f_+ \cong G_{Y_2} \circ Rf_* \circ G_{Y_1}^{-1} \cong G_{Y_2} \circ i_{2*} \circ i_1^! \circ G_{Y_1}^{-1},$$

where $Rf_*$ denotes the restricted functor from $D^b_{\text{crys}}(\text{QCrys}_{\kappa}(Y_1))_X$ to $D^b_{\text{crys}}(\text{QCrys}_{\kappa}(Y_2))_X$. It is exact because $Rf_* \cong Rf_* i_1^! i_1^! \cong i_2^* i_1^!$.

### 4.3 A Riemann-Hilbert correspondence on singular schemes

Now we extend the Riemann-Hilbert correspondence between lfgu modules and constructible étale $\mathbb{Z}/p\mathbb{Z}$-sheaves to embeddable schemes. The corresponding equivalence of categories

$$D^b_{\text{fgu}}(O_{F,X}) \overset{\sim}{\to} D^b_{\text{et}}(X_{\text{et}}, \mathbb{Z}/p\mathbb{Z})$$

for embeddable $X$ will be $t$-exact for the canonical $t$-structure on $D^b_{\text{fgu}}(O_{F,X})$ and Gabber’s perverse $t$-structure on $D^b(X_{\text{et}}, \mathbb{Z}/p\mathbb{Z})$. Again, for a closed subscheme $Z$ of $X$, let $j: U \to X$ denote the open immersion of the complement of $Z$ into $X$.

Recall that there are distinguished triangles

$$jh^* \to \text{id} \to i_* i^* \to jh^* [1]$$

and

$$i_* i^! \to \text{id} \to j_* j^* \to i_* i^! [1]$$

in $D^+(X_{\text{et}}, \mathbb{Z}/p\mathbb{Z})$ ([BBD82, 1.4.1.1]). Defining $\Gamma_Z: D(X_{\text{et}}, \mathbb{Z}/p\mathbb{Z}) \to D(X_{\text{et}}, \mathbb{Z}/p\mathbb{Z})$ as the composition $i_* i^*$ of exact functors we obtain a fundamental triangle of local cohomology

$$j_! i_* \to \text{id} \to \Gamma_Z \to j_\ast j^* [1].$$

Note that $i_! = i_*$ because $i$ is a closed immersion.

**Lemma 4.7.** Let $Z$ be a closed subscheme of a smooth $k$-scheme $X$. Then there is a natural isomorphism of functors

$$\text{Sol} \circ R\Gamma_Z \overset{\sim}{\to} \Gamma_Z \circ \text{Sol}.$$
Proof. We show that there is a natural isomorphism

\[ M \circ \Gamma_Z \sim \sim \sim \sim \sim R \Gamma_Z \circ M, \]

where \( M \) is the quasi-inverse of Sol, see Theorem 4.3. The natural isomorphism \( j^! \circ M_X \cong M \circ j^* \) implies that \( M(\Gamma_Z \mathcal{L}^\bullet) \) is supported on \( Z \) for every complex \( \mathcal{L}^\bullet \). Consequently, the morphism \( M(\Gamma_Z \mathcal{L}^\bullet) \to M(\mathcal{L}^\bullet) \) induced by the natural map \( \mathcal{L}^\bullet \to \Gamma_Z \mathcal{L}^\bullet \), which is defined by the fundamental triangle of local cohomology above, factors through \( R \Gamma_Z M(\mathcal{L}^\bullet) \). This gives rise to a morphism of distinguished triangles

\[
\begin{align*}
\xymatrix{ M(\Gamma_Z \mathcal{L}^\bullet) \ar[r] & M(\mathcal{L}^\bullet) \ar[r] & M(j_! j^* \mathcal{L}^\bullet) \\
R \Gamma_Z M(\mathcal{L}^\bullet) \ar[r] & M(\mathcal{L}^\bullet) \ar[r] & j_! j^! M(\mathcal{L}^\bullet), }
\end{align*}
\]

where the horizontal arrows are the natural morphisms and the second and third vertical arrow is an isomorphism. Hence the vertical arrow on the left is an isomorphism. \( \Box \)

Lemma 4.8. For a closed subscheme \( Z \) of a scheme \( X \), Gabber’s perverse t-structure on \( D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \) induces a t-structure on \( D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z \) given by

\[
\begin{align*}
p D^{\geq 0}(X_{\text{ét}})_Z &= D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z \cap p D^{\geq 0}(X_{\text{ét}}), \\
p D^{\leq 0}(X_{\text{ét}})_Z &= D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z \cap p D^{\leq 0}(X_{\text{ét}}).
\end{align*}
\]

Proof. We consider the construction of the perverse truncation functor \( p^! \mathcal{L}^\bullet \leq 0 \) in [Gab04] in more detail. It will turn out that \( p^! \mathcal{L}^\bullet \leq 0 \) is supported on \( Z \) for all \( \mathcal{L}^\bullet \in D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z \).

For simplicity we write \( \tau_{\leq p} \) for \( p^! \mathcal{L}^\bullet \leq 0 \) where \( p \) is a perversity function, see the first section of [Gab04]. For a complex \( F^\bullet \), \( C(F^\bullet) \) denotes the total complex of the double complex \( C^\bullet(F^\bullet) \), where \( C^\bullet(F^\bullet) \) is the Godement resolution of \( F^\bullet \).

Let \( c = - \dim X \). It is a lower bound for the perversity function \( p(x) = - \dim \{ x \} \).

For a complex \( \mathcal{L}^\bullet \), \( d \geq c \) and \( p_d(x) = \min(d, p(x)) \), Gabber iteratively constructs a direct system \( \tau_{\leq p_d} \mathcal{L}^\bullet \) and defines \( \tau_{\leq p} \mathcal{L}^\bullet \) as the direct limit. We start with \( p_c = c \) and the usual truncation \( \tau_{p_c} \mathcal{L}^\bullet = \tau_{\leq c} \mathcal{L}^\bullet \). Clearly, if \( \mathcal{L}^\bullet \) is supported on \( Z \), then so is \( \tau_{p_c} \mathcal{L}^\bullet \). Now for \( \tau_{\leq p_d} F^\bullet \) of some complex \( F^\bullet \), we construct \( \tau_{\leq p_d+1} F^\bullet \) as a subcomplex of \( C(F^\bullet) \). By the construction of the Godement resolution, \( C(F^\bullet) \) is supported on \( Z \) if \( F^\bullet \) is supported on \( Z \). It follows that for every \( d \geq c \), the complex \( \tau_{\leq p_d} \mathcal{L}^\bullet \) is supported on \( Z \) and therefore the direct limit \( \tau_{\leq p} \mathcal{L}^\bullet \) is supported on \( Z \). \( \Box \)

Proposition 4.9. Let \( i : Z \to X \) be a closed immersion of schemes.

(a) The exact functors \( i_* \) and \( i^! \) are inverse equivalences of categories

\[
D_c^b(Z_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{i_*} D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z.
\]

(b) These functors \( i_* : D_c^b(Z_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \to D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z \) and \( i^* : D_c^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})_Z \to D_c^b(Z_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) \) are also t-exact with respect to the perverse t-structures of both categories.
Proof. (a) This is a formal consequence of the distinguished triangle

$$j_! j^* \longrightarrow \text{id} \longrightarrow i_! i^* \longrightarrow j_! j^*[1]$$

in $D^b(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ and the fact that the natural map $\text{id} \rightarrow i^* i_*$ is always an isomorphism.

(b) It suffices to show that $i^*$ is $t$-exact with respect to the perverse $t$-structures, i.e. the essential image of $pD^{\leq 0}(Z_{\text{ét}})$ under $i_*$ is contained in $pD^{\leq 0}$ and the essential image of $pD^{\geq 0}(Z_{\text{ét}})$ under $i_*$ is contained in $pD^{\geq 0}$. For $x \in Z$ let $\tilde{i}_x$ denote the composition

$$\{x\} \rightarrow \text{Spec} \mathcal{O}_{Z_{\text{ét}}, x} \rightarrow Z$$

of canonical morphisms. For $\mathcal{L}^\bullet$ in $pD^{\leq 0}(Z_{\text{ét}})$, i.e. $H^n(\tilde{i}_x^* \mathcal{L}^\bullet) = 0$ for every $x \in Z$ and every $n > -\dim \{x\}$, we have

$$H^n(i_!^* i_* \mathcal{L}^\bullet) \cong H^n(\tilde{i}_x^* i^* \mathcal{L}^\bullet)$$
$$\cong H^n(\tilde{i}_x^* \mathcal{L}^\bullet)$$
$$\cong 0$$

for every $x \in Z$ and every $n > -\dim \{x\}$. For $x \in U = X \setminus Z$, we even have $i_!^* i_* \mathcal{L}^\bullet \cong 0$ because $j^* i_* \mathcal{L}^\bullet \cong 0$ and hence $\tau^* i_* \mathcal{L}^\bullet \cong 0$, where $\tau : \text{Spec} \mathcal{O}_{X_{\text{ét}}, x} \rightarrow X$ is the natural morphism.

Now let $\mathcal{L}^\bullet$ be in $pD^{\geq 0}(Z_{\text{ét}})$, i.e. $H^n(\tilde{i}_x^* \mathcal{L}^\bullet) = 0$ for every $x \in Z$ and every $n < -\dim \{x\}$. There is a natural isomorphism of functors

$$i_!^* \cong i^* i_*$$

given by the composition of the natural isomorphisms $i_!^* \rightarrow \text{id}$ and $\text{id} \rightarrow i^* i_*$ of [BBD82, 1.4.1.2]. Whence

$$H^n(i_!^* i_* \mathcal{L}^\bullet) \cong H^n(\tilde{i}_x^* i_!^* i_* \mathcal{L}^\bullet)$$
$$\cong H^n(\tilde{i}_x^* i^* i_* \mathcal{L}^\bullet)$$
$$\cong H^n(\tilde{i}_x^* \mathcal{L}^\bullet)$$
$$\cong 0$$

for every $x \in Z$ and every $n < -\dim \{x\}$. We have already seen that there is nothing to show for $x \in U$. \hfill \Box

Definition 4.10. For a perfect field $k$ and a $k$-scheme $X$ which admits an embedding into a smooth $k$-scheme $Y$, we define

$$D^b_{\text{fgu}}(\mathcal{O}_{F,Y}) = D^b_{\text{fgu}}(\mathcal{O}_{F,Y})_X \cap D^b_{\text{fgu}}(\mathcal{O}_{F,Y}),$$
$$D^b_{\text{fgu}}(\mathcal{O}_{F,X}) = D^b_{\text{fgu}}(\mathcal{O}_{F,Y})_X \cap D^b_{\text{fgu}}(\mathcal{O}_{F,Y}).$$

These subcategories of $D^b_{\text{fgu}}(\mathcal{O}_{F,X})$ form a natural $t$-structure.
The independence of these subcategories of the embedding into a smooth scheme follows from the fact that for a morphism \( f : Y_1 \to Y_2 \) between two smooth schemes over \( k \), together with closed immersions \( i_1 : X \to Y_1 \) and \( i_2 : X \to Y_2 \) with \( i_2 = f \circ i_1 \), the equivalence
\[
R\Gamma_X f^! \cong M \circ \text{Sol}_\kappa \circ R\Gamma_X f^!
\]
\[
\cong M \circ \Gamma_X f^* \circ \text{Sol}_\kappa
\]
\[
\cong M \circ i_{1*} i_1^* f_2^* i_2^* \circ \text{Sol}_\kappa
\]
\[
\cong M \circ i_{1*} i_1^* \circ \text{Sol}_\kappa
\]
is a composition of \( t \)-exact functors, where \( D^b_c(Y_1,\text{ét},\mathbb{Z}/p\mathbb{Z}) \) and \( D^b_c(Y_2,\text{ét},\mathbb{Z}/p\mathbb{Z}) \) are equipped with the perverse \( t \)-structures (Theorem 4.3 and Proposition 4.9). Therefore, \( R\Gamma_X f^! \) is \( t \)-exact.

**Theorem 4.11.** Let \( k \) be a perfect field and \( X \) an embeddable \( k \)-scheme.

(a) The equivalence \( \text{Sol} \) for smooth schemes induces an anti-equivalence of categories
\[
\text{Sol} : D^b_{\text{lfgu}}(\mathcal{O}_{F,X}) \to D^b_c(X_\text{ét},\mathbb{Z}/p\mathbb{Z})
\]
(b) The essential image of \( D^b_{\text{lfgu}}(\mathcal{O}_{F,X}) \) under this equivalence equals \( pD^{\leq 0} \) and the essential image of \( D^b_{\text{lfgu}}(\mathcal{O}_{F,X}) \) equals \( pD^{\geq 0} \).

**Proof.** After choosing a closed immersion \( i : X \to Y \) into a smooth \( k \)-scheme \( Y \), we see that Sol restricts to an anti-equivalence
\[
\text{Sol} : D^b_{\text{lfgu}}(\mathcal{O}_{F,Y})_X \to D^b_c(Y_\text{ét},\mathbb{Z}/p\mathbb{Z})_X
\]
in the same way as in the proof of Proposition 4.4. For the proof of the independence of the choice of an embedding we again reduce to the situation of two closed immersions \( i_1 : X \to Y_1 \) and \( i_2 : X \to Y_2 \) together with a morphism \( f : Y_1 \to Y_2 \) such that \( i_2 = f \circ i_1 \).

We obtain natural equivalences of categories
\[
R\Gamma_X f^! : D^b_{\text{lfgu}}(\mathcal{O}_{F,Y_2})_X \to D^b_{\text{lfgu}}(\mathcal{O}_{F,Y_1})_X
\]
and
\[
f^{-1}\Gamma_X : D^b_c(Y_2,\text{ét},\mathbb{Z}/p\mathbb{Z})_X \to D^b_c(Y_1,\text{ét},\mathbb{Z}/p\mathbb{Z})_X,
\]
which are compatible with Sol because
\[
\text{Sol} \circ R\Gamma_X f^! \overset{\sim}{\to} \Gamma_X \circ \text{Sol} \circ f^! \overset{\sim}{\to} \Gamma_X f^* \circ \text{Sol}
\]
by Theorem 4.3 and Lemma 4.7. This proves (a).

The equivalence Sol not only restricts to an equivalence between \( D^b_{\text{lfgu}}(\mathcal{O}_{F,Y})_X \) and \( D^b_c(Y_\text{ét},\mathbb{Z}/p\mathbb{Z})_X \) but also between \( D^b_{\text{lfgu}}(\mathcal{O}_{F,Y})_X \) and \( pD^{\leq 0}(Y_\text{ét}) \) (Theorem 4.3). Therefore, Sol induces an equivalence
\[
D^b_{\text{lfgu}}(\mathcal{O}_{F,Y})_X \cap D^b_{\text{lfgu}}(\mathcal{O}_{F,Y})_X \overset{\sim}{\to} D^b_c(Y_\text{ét},\mathbb{Z}/p\mathbb{Z})_X \cap pD^{\leq 0}(Y_\text{ét}).
\]
By Proposition 4.9, \( D^b(Y_\text{ét},\mathbb{Z}/p\mathbb{Z})_X \cap pD^{\leq 0}(Y_\text{ét}) \) is canonically equivalent to \( pD^{\leq 0}(X_\text{ét}) \).

Analogously, one can show that the essential image of \( D^b_{\text{lfgu}}(\mathcal{O}_{F,X})_X \) equals \( pD^{\geq 0} \). \( \square \)
Theorem 4.12. Let $X$ be an embeddable $k$-scheme and assume that $k$ is perfect.

(a) The equivalences $G$ and $Sol$ for smooth schemes induce equivalences

$$G : D^b_{\text{crys}}(\text{QCrys}_\kappa(X)) \to D^b_{\text{Igu}}(\mathcal{O}_{F,X}) \text{ and } Sol : D^b_{\text{Igu}}(\mathcal{O}_{F,X}) \to D^b_\ell(X_{\et}, \mathbb{Z}/p\mathbb{Z}).$$

This means that for a closed immersion $i : X \to Y$ with $Y$ smooth, there is a commutative diagram

$$\begin{array}{ccc}
D^b_\ell(Y_{\et}, \mathbb{Z}/p\mathbb{Z}) & \xleftarrow{h_Y} & D^b_\ell(X_{\et}, \mathbb{Z}/p\mathbb{Z}) \\
\downarrow{\text{Sol Y}} & & \downarrow{\text{Sol X}} \\
D^b_{\text{crys}}(\text{QCrys}_\kappa(Y)) & \xrightarrow{G_Y} & D^b_{\text{Igu}}(\mathcal{O}_{F,Y}) \\
\downarrow{h_X} & & \downarrow{\text{Sol}_X} \\
D^b_{\text{crys}}(\text{QCrys}_\kappa(X)) & \xrightarrow{G_X} & D^b_{\text{Igu}}(\mathcal{O}_{F,X}).
\end{array}$$

(b) Let $Sol_\kappa$ be the composition $Sol \circ G$. The essential image of $D^b_{\text{crys}}^\geq 0(\text{QCrys}_\kappa(X))$ under $Sol_\kappa$ equals the subcategory $^pD^\leq 0$, while the essential image of $D^b_{\text{crys}}^\geq 0(\text{QCrys}_\kappa(X))$ equals the subcategory $^pD^\geq 0$.

(c) If $h : W \to X$ is an open or a closed immersion, then there are natural isomorphisms of functors

$$Sol_\kappa \circ Rh_* \sim h_! \circ Sol_\kappa \text{ and } Sol_\kappa \circ h^! \sim h^* \circ Sol_\kappa$$

where $h^!$ denotes the functor $h^*$ if $h$ is an open immersion and $Rh_* = h_*$ for a closed immersion.

Proof. (a) and (b) follow from Proposition 4.4 and Theorem 4.11.

It remains to prove (c). Let $h : W \to X$ be an open immersion. We will construct the natural transformations by choosing embeddings of $X$. Since we have to make sure that this construction is independent of the embedding, we will consider two closed immersions $i_1 : X \to Y_1$ and $i_2 : X \to Y_2$ of $X$ into smooth $k$-schemes ab initio. We may assume that there is a morphism $f : Y_1 \to Y_2$ with $i_2 = f \circ i_1$, see the proof of Corollary 3.12. Let $h'_2 : V_2 \to Y_2$ be an open immersion such that $(i_2 \circ h)(W) = h'_2(V_2) \cap i_2(X)$ and hence $W \cong X \times_{Y_2} V_2$. Let $i'_2 : W \to V_2$ be the closed immersion induced by $i_2$, i.e. the projection $X \times_{Y_2} V_2 \to V_2$. We have a natural equivalence $Rh_{i'_2} : D^b_{\text{crys}}(\text{QCrys}_\kappa(W)) \to D^b_{\text{crys}}(\text{QCrys}_\kappa(Y_2))_X$. Moreover, we have a natural equivalence $h_{2+} : G_{V_2} \cong G_{Y_2} Rh_{2+}$ of functors from $D^b_{\text{crys}}(\text{QCrys}_\kappa(W))_W$ to $D^b_{\text{Igu}}(\mathcal{O}_{F,Y_2})_X$. Therefore, $Rh_* : D^b_{\text{crys}}(\text{QCrys}_\kappa(W)) \to D^b_{\text{crys}}(\text{QCrys}_\kappa(X))$ induces a functor $h_+ : D^b_{\text{crys}}(\mathcal{O}_{F,W}) \to D^b_{\text{Igu}}(\mathcal{O}_{F,X})$ such that $G_X h_* \cong h_+ G_W$. Note that this functor $h_+$ does not depend on the choice of $V_2$. In particular, to show the independence of the embedding of $X$, we may choose an open immersion $h'_1 : V_1 \to Y_1$ with $(i_1 \circ h)(W) = h'_1(V_1) \cap i_1(X)$ and such that $(f \circ h'_1)(V_1) \subseteq h'_2(V_2)$. Here we set
$V_1 = f^{-1}(V_2)$, which means $V_1 \cong Y_1 \times_{Y_2} V_2$. Let $f': V_1 \to V_2$ be the projection and let $i'_1$ be the projection of $W \cong X \times_{Y_2} V_2 \cong X \times_{Y_1} (Y_1 \times_{Y_2} V_2)$ to $V_1 \cong Y_1 \times_{Y_2} V_2$. It is a closed immersion because it is the base change of the morphism $i_1$. We obtain the following commutative diagram:

$$
\begin{array}{ccc}
W & \xrightarrow{i'_1} & V_1 \\
\downarrow & & \downarrow f' \\
X & \xrightarrow{i_1} & V_2 \\
\end{array}
$$

The following cube demonstrates the natural equivalences, which we have by Proposition 2.37:

$$
\begin{array}{ccc}
D^b_{\text{cris}}(\mathcal{O}_{F,Y_1}) & \xrightarrow{\text{Sol}} & D^b_{\text{lg}}(\mathcal{O}_{F,V_2}) \\
\downarrow \text{R}f'^* & & \downarrow \text{R}f_+ \\
D^b_{\text{cris}}(\mathcal{O}_{F,Y_2}) & \xrightarrow{\text{Sol}} & D^b_{\text{lg}}(\mathcal{O}_{F,V_2}) \\
\downarrow \text{R}f^*_\text{crys} & & \downarrow \text{R}f_\text{crys} \\
D^b_{\text{cris}}(\mathcal{O}_{F,Y_1}) & \xrightarrow{\text{Sol}} & D^b_{\text{lg}}(\mathcal{O}_{F,Y_2}) \\
\end{array}
$$

Here every cube face indicates a natural equivalence, for example, the front refers to the isomorphism of functors $G_{V_2} \circ \text{R}f'_+ \cong h'^+_1 \circ G_{V_1}$. This shows the independence of the isomorphism of functors $G_X h^*_+ \cong h^*_+ \circ G_W$ from the chosen embedding of $X$. By adjunction, we obtain a canonical isomorphism $G_W h^* \cong h^! G_X$ as well.

Similarly, one shows that we have a natural isomorphism $\text{Sol} h^! \cong h^* \text{Sol}$, using the fact that $\text{Sol}$ commutes with the local cohomology functors (Lemma 4.7) and with pullbacks for morphisms between smooth schemes (Theorem 4.3). Again, by adjunction, we obtain a natural isomorphism $\text{Sol} h^*_+ \cong h^! \text{Sol}$. Composing these isomorphisms of functors yields the desired one:

$$\text{Sol}_\kappa \circ \text{R}h^*_+ \cong \text{Sol} \circ G_X \circ \text{R}h^*_+ \cong \text{Sol} \circ h^*_+ \circ G_W \cong h^! \circ \text{Sol} \circ G_W \cong h^! \circ \text{Sol} \circ \text{Sol}_\kappa$$

and analogously for $\text{Sol}_\kappa \circ h^* \cong h^* \circ \text{Sol}_\kappa$.

If $h: W \to X$ is a closed immersion, we proceed similarly, but the proof is simpler because a closed immersion $i: X \to Y$ of $X$ into a smooth $k$-scheme $Y$ yields a closed immersion of $W$ into $Y$ by composing $h$ and $i$.

\[\square\]

**Definition 4.13.** The abelian category $\text{Perv}_{\text{c}}(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$ of perverse constructible étale $p$-torsion sheaves is the heart $pD^{\leq 0} \cap pD^{\geq 0}$ of the perverse $t$-structure on $D^b_{\text{c}}(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$.  

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**Corollary 4.14.** For a perfect field $k$ and an embeddable $k$-scheme $X$, the functor $\text{Sol}_k$ induces an anti-equivalence

$$\text{Crys}_k(X) \rightarrow \text{Perv}_c(X_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$$

between the abelian categories of Cartier crystals on $X$ and perverse constructible $\mathbb{Z}/p\mathbb{Z}$-sheaves on $X_{\text{ét}}$.

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