Abstract

Recovering a signal from its Fourier magnitude is referred to as phase retrieval, which occurs in different fields of engineering and applied physics. This paper gives a new characterization of the phase retrieval problem. Particularly useful is the analysis revealing that the common gradient-based regularization does not contain more information other than the magnitude measurements for phase retrieval. Focusing on binary signals, we show that a box relaxation to the binary constraint is equivalent to the original problem. We further prove that binary signals can be recovered uniquely up to trivial ambiguities under certain conditions. Finally, we use the characterization theorem to develop an efficient denoising algorithm.

1 Introduction

In many fields of physics and engineering, one can only measure the magnitude of the Fourier Transform of a discrete signal $x \in \mathbb{C}^N$. Denote the discrete Fourier Transform by $F$. Recovering $x$ from $|Fx|$ is referred to as phase retrieval (PR), since the phase information is completely lost in measurements. The PR problem occurs in various applications, e.g., crystallography \[1, 2\], astronomy \[3\], and laser optics \[4\]; please refer to \[5\] for a contemporary overview.

Phase retrieval is known as a challenging problem, largely due to its nonconvexity and solutions being non-unique \[6\]. Specifically for the nonuniqueness, a.k.a., ambiguities, there are trivial ambiguities and non-trivial ambiguities.
Trivial ambiguities of $|\mathcal{F} y| = |\mathcal{F} x|$ can be summarized as:

- **Global phase shift:** $y_k = x_k \cdot e^{i\phi_0}$
- **Conjugate inverse:** $y_k = x^*_{-k}$
- **Spatial shift:** $y_k = x_{k+k_0}$

where $\phi_0 \in [0, 2\pi)$, $k_0 \in \mathbb{Z}$ are the phase shift and spatial shift respectively, $*$ denotes the complex conjugate. Note that every combination of (1.1) is also a trivial ambiguity. The non-trivial ambiguities of one-dimensional signals can be classified by the roots of the $\mathbb{Z}$-transform of the autocorrelation of the signal [7], while almost all multi-dimensional signals only have non-trivial ambiguities [8], since the $\mathbb{Z}$-transform of their autocorrelation being reducible is of measure zero in the space of all polynomials [7, 9].

For unique recovery of a real $N$-dimensional signal up to trivial ambiguities, at least $2N - 1$ random measurements are needed, provided the sampling matrix has full spark [10]. This result was later extended to the complex case in [11], requiring at least $4N - 4$ measurements. Other sufficient conditions for unique recovery include minimum phase signals [12], sparse signals with non-periodic support [13], and signals with autocorrelation sequence being collision free and $\|x\|_0 \neq 6$ [14]. For $s$-sparse signals in $\mathbb{R}^N$, $O(s \log(N/s))$ phaseless measurements are sufficient to recover [15, 16].

In addition to taking more measurements than the ambient dimension, one often relies on regularization to refine the solution space with an attempt to reduce ambiguities. Stemmed from image processing, a common choice is a gradient-type formalism. For example, Chang et al. [17] considered the total variation, which is the $\ell_1$ norm of the gradient for phase retrieval.

Despite lack of theoretical guarantees, many algorithms are used to solve the phase retrieval problems, including alternating projections [18], Wirtinger flow [19], alternating direction method of multipliers (ADMM) [17], and a preconditioned proximal algorithm [20].

This paper contributes to a new set of characterization theorems for phase retrieval, indicating that gradient-based regularization is redundant to the magnitude measurements. We also consider to impose additional information on the underlying signal in order to resolve the ambiguities. Specifically, we focus on binary signals due to its simplicity and a wide variety of applications such as bar code [21, 22] and obstacle detection [23]. A major difficulty in the reconstruction of binary signals is that the binary constraint is nonconvex. It was observed empirically in [24] that incorporating a box constraint into the ADMM framework, referred to ADMMB, often gives an exact recovery of binary signal, which motivates us to give a theoretical explanation. In particular, we prove that the phase retrieval problem with binary constraint is equivalent to phase retrieval with box relaxation. Different to the discussion on the uniqueness of binary signals in [25], we describe a new type of trivial ambiguities and show that unique recovery is possible for binary signals under certain conditions. Finally, we take the noise in consideration and develop a denoising algorithm.

Our contributions are three-fold: (1) We give a characterization theorem
(Theorem 3.2) to general phase retrieval problem, leading to the fact that $\|\nabla^n x\|_2$ is completely determined by $|F x|$. (2) We give thorough analysis of phase retrieval problem in a binary setting. We show that the box relaxation to binary constraint is equivalent to the original binary phase retrieval problem (Theorem 4.1). We then describe a new type of ambiguities and guarantee the uniqueness of binary phase under certain conditions. (3) We conduct a series of error analysis (Proposition 5.1 and Corollaries 5.2-5.3) of phase retrieval, which motivates a new denosing scheme.

The rest of the paper is organized as follows. In Section 2 we set up notations and review some ways of taking magnitude measurements. In Section 3 we give a characterization theorem and its consequences. In Section 4 we show that the phase retrieval of binary signals can be relaxed to the box constraint. Specifically in Section 4.1 we describe a new ambiguity for binary signals and show that the unique recovery of binary signals is possible under some special circumstances. We then extend our results to more general two-valued signals and other sampling schemes in Section 4.2 and 4.3 respectively. In Section 5 we estimate recover accuracy with respect to noise and propose a denoising algorithm that empirically yields better performance compared to a naïve approach. Section 6 concludes the paper. All the proofs can be found in Appendix.

2 Preliminaries

2.1 Notations

Let $x, y \in \mathbb{C}^N$ be arbitrary signals, we define some notations that will be used throughout the paper,

- $x_k$ denotes the $k$-th entry of $x$, i.e. $x = (x_0, x_1, x_2, \ldots, x_{N-1})^T$

- $\|x\|_p$ denotes the $\ell_p$-norm of $x$, i.e. $\|x\|_p = \left(\sum_{k=0}^{N-1} |x_k|^p\right)^{\frac{1}{p}}$. Specifically for $p = 0$, $\|x\|_0$ is the $\ell_0$ norm by counting the number of nonzero element of $x$.

- $e_k$’s denotes the standard basis in $\mathbb{C}^N$, i.e. the vector with a 1 in the $k$-th coordinate and 0’s elsewhere, e.g., $e_0 = (1, 0, 0, \ldots, 0)^T$ and $e_1 = (0, 1, 0, \ldots, 0)^T$.

- $F_{N \rightarrow M} : \mathbb{C}^N \rightarrow \mathbb{C}^M$ denotes the matrix representing discrete Fourier transform (DFT), i.e.

\[
F_{N \rightarrow M} = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{M-1} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)(N-1)}
\end{bmatrix}, \quad (2.1)
\]
where $\omega = e^{-\frac{2\pi i}{M}}$. Note that $\frac{1}{\sqrt{N}} \mathcal{F}_{N\to N}$ is unitary. If $M > N$, we refer it as the oversampling Fourier matrix.

- We define
  \[ x \odot y = (x_0y_0, x_1y_1, \ldots, x_{N-1}y_{N-1}), \]
  where $\odot$ denotes the Hadamard product (i.e. entrywise multiplication).

- The discrete (periodic) convolution $x \ast y$ is defined by
  \[ (x \ast y)_j = \sum_{k=0}^{N-1} x_k y_{(j-k) \mod N}, \]  
  for $j = 0, 1, \ldots, N - 1$.

- The (regular) autocorrelation is defined by
  \[ \text{Aut}(x)_j = \sum_{k=0}^{N-1} x_{(k+j) \mod N} x_k, \]  
  where $j = -N+1, -N+2, \ldots, N-1$ and $x_k = 0, \forall k < 0$ and $k > N-1$.

- By replacing the zero boundary condition in the regular autocorrelation with periodic boundary condition, we consider periodic autocorrelation defined as
  \[ \text{Aut}_p(x)_j = \sum_{k=0}^{N-1} x_{(k+j) \mod N} x_k, \]
  for $j = 0, 1, \ldots, N - 1$.

For the rest of the paper, we denote $\mathcal{F}_{N\to N}$ by $\mathcal{F}$, $\mathcal{F}_{N\to M}$ by $\mathcal{F}_M$, and omit $\mod N$ if the context is clear.

### 2.2 Sampling Schemes

In practice, there are numerous ways \cite{26, 27, 28, 29, 30, 31} to measure the magnitude of a signal. This paper develops new theoretical characterizations in PR, specifically focusing on the following sampling schemes.

- **Classic Fourier Transform.** One aims to find an unknown signal $x \in \mathbb{C}^N$ from the magnitude measurements $b := |\mathcal{F}x|$, i.e.
  \[ b_n = \left| \sum_{k=0}^{N-1} x_k e^{-\frac{2\pi kn}{N}} \right|, \quad \forall n = 0, 1, \ldots, N - 1. \]
• **Oversampling Fourier Transform.** An $M$-point ($M > N$) oversampling discrete Fourier Transform (DFT) of a signal $x \in \mathbb{C}^N$ is defined by

$$b_n = \left| \sum_{k=0}^{N-1} x_k e^{-\frac{2\pi k n i}{N}} \right|, \quad \forall n = 0, 1, \ldots, M - 1.$$ 

One wants to recover $N$-dimensional signal $x$ based on $M$-dimensional measurements of $|F_M x|$. A typical choice of $M$ is $M = 2N$ [32], which is experimentally adopted by Miao et al [33]. We will show in Theorem 3.3 that a larger $M$ does not provide more information to guarantee improvement in the PR problem.

• **Short-Time Fourier Transform (STFT)** [27, 28]. Let $x \in \mathbb{C}^N$ be a signal of length $N$ and $w \in \mathbb{C}^W$ be a window of length $W$. The Short-Time Fourier Transform (STFT) of $x$ with respect to $w$, defined as

$$z_{n,m} = \sum_{k=0}^{N-1} x_k w_{mL-k} e^{-\frac{2\pi k n i}{N}},$$

for $n = 0, 1, \ldots, N - 1$ and $m = 0, 1, \ldots, R - 1$, where $L$ denotes the separation in time between adjacent short-times sections, $R = \lceil \frac{N+W-1}{L} \rceil$ denotes the number of short-time sections considered, and $w_k := 0$ for all $k < 0$ and $k > W - 1$.

• **Frequency-resolved optical gating trace (FROG)** [26, 27, 31]. Let $z_{n,m} = x_n x_n + m L$, where $L$ is a fixed integer. The FROG trace is equivalent to the one-dimensional Fourier magnitude of $z_{n,m}$ for each fixed $m$, i.e.,

$$|\hat{z}_{n,m}|^2 = \left| \sum_{k=0}^{N-1} x_k x_{k+mL} e^{-\frac{2\pi k n i}{N}} \right|^2,$$

for $n = 0, \ldots, N - 1, m = 0, \ldots, \lceil \frac{N}{T} \rceil - 1$.

3 **New Characterization on Phase Retrieval**

In the classic setting of PR, i.e., recovering $x$ from $|F_M x|$, it is natural to ask what information is contained in the measurements $|F_M x|$. It was shown in [8, 25] that $\text{Aut}(x)$ is determined by $|F_{2N-1} x|$, which is rephrased in Theorem 3.1.

**Theorem 3.1** ([8] [25]). Given $x, y \in \mathbb{C}^N$, the following statements are equivalent:

1. $|F_{2N-1} x| = |F_{2N-1} y|$;
(2) $\text{Aut}(x) = \text{Aut}(y)$.

We aim to extend this analysis to arbitrary number of measurements (not just $2N - 1$) and period autocorrelation. Specifically in Theorem 3.2, we show that when $M = N$, $\text{Aut}_p(x)$ and $\|v \ast x\|_2$ for $v \in \mathbb{C}^N$ are determined by $|\mathcal{F}x|$ and vice versa. A similar result for $M \geq 2N - 1$ is presented in Theorem 3.3, which suggests that taking measurements more than $2N - 1$ theoretically provides no other information than the autocorrelation to the PR problem. To the best of our knowledge, the equivalence to $\|v \ast x\|_2$ is novel in the literature, which leads to an useful consequence as characterized in Corollaries 3.5-3.6.

**Theorem 3.2.** Given $x, y \in \mathbb{C}^N$, the following statements are equivalent:

1. $|\mathcal{F}x| = |\mathcal{F}y|$;
2. $\text{Aut}_p(x) = \text{Aut}_p(y)$;
3. $\|v \ast x\|_2 = \|v \ast y\|_2 \ \forall v \in \mathbb{C}^N$.

**Theorem 3.3.** Given $x, y \in \mathbb{C}^N$, $M \geq 2N - 1$, the following statements are equivalent:

1. $|\mathcal{F}_M x| = |\mathcal{F}_M y|$;
2. $\text{Aut}(x) = \text{Aut}(y)$

Also, either (1) and (2) implies that $\text{Aut}_p(x) = \text{Aut}_p(y)$ and $\|v \ast x\|_2 = \|v \ast y\|_2 \ \forall v \in \mathbb{C}^N$. The reverse does not necessarily hold.

**Remark 3.4.** For $M < 2N - 1$ and $M \neq N$, we cannot determine the autocorrelation, i.e., $\text{Aut}(x)$, from $M$ magnitude measurements of $|\mathcal{F}_M(x)|$, due to an insufficient number of measurements.

When the measurements are limited, it is desirable and often necessary to impose some regularization term in order to regularize the solution and avoid ambiguities in PR as much as possible. Stemmed from imaging processing, a common choice of such regularization is the use of $\|\nabla^n x\|_2$ for an integer $n$ to enforce $n$-order smoothness of the underlying signal $x$. In other word, a regularized PR problem can be expressed as

$$\minimize_{x} \|\nabla^n x\|_2 \ \text{s.t.} \ |\mathcal{F}x| = b.$$ 

For example, Chang et al. [17] considered the total variation with $n = 1$. Unfortunately, Theorem 3.2 implies that such gradient-based regularization cannot resolve any ambiguities. Using the fact that $\nabla^n x = v_n \ast x$ for some $v_n \in \mathbb{C}^N$, we show in Corollary 3.5 that $\|\nabla^n x\|_2$ can be determined by $|\mathcal{F}x|$ and hence imposing such regularization does not contain additional information. The oversampling case is presented in Corollary 3.6.

**Corollary 3.5.** Given $x, y \in \mathbb{C}^N$, if $|\mathcal{F}x| = |\mathcal{F}y|$, then $\|\nabla^n x\|_2 = \|\nabla^n y\|_2, \forall n \in \mathbb{N}$. 

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**Corollary 3.6.** Let $M \geq 2N - 1$, given $x, y \in \mathbb{C}^N$, if $|\mathcal{F}_M x| = |\mathcal{F}_M y|$, then $\|\nabla^n x\|_2 = \|\nabla^n y\|_2$, $\forall n \in \mathbb{N}$.

**Remark 3.7.** When $N < M < 2N - 1$, gradient-based regularization may be helpful. For example, let

$$
\begin{align*}
x &= (0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1), \\
y &= (0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1).
\end{align*}
$$

Both of them are of 11-dimension and have the same $|\mathcal{F}_M x| = |\mathcal{F}_M y|$ for $M = 12$, but $\|\nabla^3 x\|_2^2 = 7.5 \neq 7 = \|\nabla^3 y\|_2^2$, where the third order finite scheme $\nabla^3 x$ is defined by $(\nabla^3 x)_k := -\frac{1}{2}x_{k-2} + x_{k-1} - x_{k+1} + \frac{1}{2}x_{k+2}$.

## 4 Box Relaxation to Binary Constraint

We now restrict our attention to binary signals $x \in \{0, 1\}^N$, as another way of imposing additional prior knowledge to facilitate phase retrieval. Mathematically, we formulate this binary phase retrieval problem as follows,

$$\text{Find } x \in \{0, 1\}^N, \text{ s.t. } |\mathcal{F}x| = b. \quad (P)$$

Since binary constraint is nonconvex, we relax it by a box constraint in a similar way as a linear Fourier problem:

$$\text{Find } x \in [0, 1]^N, \text{ s.t. } |\mathcal{F}x| = b. \quad (Q)$$

If $(P)$ has a solution, then $(Q)$ also has a solution. The question is whether we can recover $x$ from $b$ through $(Q)$. Unfortunately, there are still numerous (trivial and non-trivial) ambiguities that make this problem impossible. Our contribution here is to prove that all the solutions to $(Q)$ are solutions to $(P)$, i.e. lie in $\{0, 1\}^N$ and have the same number of 1’s as the ground-truth signal; see Theorem 4.1.

**Theorem 4.1.** Given $x \in \{0, 1\}^N$ and $y \in [0, 1]^N$, if $|\mathcal{F}x| = |\mathcal{F}y|$, then $y \in \{0, 1\}^N$ and $\|y\|_0 = \|x\|_0$.

We then characterize trivial ambiguities specifically for binary phase retrieval in Section 4.1, followed by extensions to arbitrarily two-valued signals in Section 4.2 and other sampling schemes in Section 4.3.

### 4.1 Ambiguities and Uniqueness

In addition to trivial ambiguities (1.1) for general PR, there is another type of trivial ambiguity in the binary setting. For example, one has

$$|\mathcal{F}(1, 1, 1, 0, 0, 1, 0, 0, 0, 0)| = |\mathcal{F}(0, 0, 0, 0, 0, 1, 0, 1, 1, 1)|,$$

in which the two signals are not related by (1.1), but rather by switching zeros and ones. We present this trivial ambiguity for binary phase retrieval in Proposition 4.2.
Proposition 4.2. Given \( x \in \{0,1\}^N \) and \( N \) is even, if \( \|x\|_0 = \frac{N}{2} \), then \( |\mathcal{F}x| = |\mathcal{F}(1-x)| \), where \( 1 \) denotes the vector of all one’s, i.e. \( 1 = (1,1,\ldots,1)^T \).

As a by-product from the proof of Proposition 4.2, we reveal an interesting fact, stating that if \( x \) and \( y \) has the same Fourier magnitude, then so do \((1-x)\) and \((1-y)\):

Proposition 4.3. Given \( x, y \in \{0,1\}^N \), \( |\mathcal{F}x| = |\mathcal{F}y| \) if and only if \( |\mathcal{F}(1-x)| = |\mathcal{F}(1-y)| \).

Note that the trivial ambiguity in Proposition 4.2 requires \( \|x\|_0 = \frac{N}{2}. \) We show in Proposition 4.4 that the exact recovery of \( x \) up to trivial ambiguities \((1,1)\) is guaranteed when \( \|x\|_0 \leq 3 \) and \( \|x\|_0 \geq N - 3. \) The idea is to make use of \( \text{Aut}_p \) defined in (2.4). Specifically if we restrict the ground-truth signal \( x \) to lie in \( \{0,1\}^N \), then \( (\text{Aut}_p(x))_k \) is the number of pair of 1’s with distance \( k \). The combinatorial nature of \( \text{Aut}_p(x) \) guarantees the uniqueness of \( x \) up to trivial ambiguities.

Proposition 4.4. Given \( x \in \{0,1\}^N \), if \( \|x\|_0 = 0, 1, 2, 3, N - 3, N - 2, N - 1 \) or \( N \), then we can uniquely recover \( x \) from \( |\mathcal{F}x| \) up to the trivial ambiguities \((1,1)\).

Remark 4.5. The above does not hold for \( 4 \leq \|x\|_0 \leq N - 4 \) in general. For example, \((0,0,0,0,0,1,0,1,0,0,1,1)^T\) and \((0,0,0,0,0,1,0,0,0,1,0,1,1)^T\) have the same magnitude after Fourier Transform, but they are not related to each other by trivial ambiguities. These two vectors also have the same FROG trace, which suggests that FROG with more measurements than in the classic Fourier case can not resolve ambiguities either.

4.2 Extensions to two-valued signals

Theorem 4.1 can be naturally extended to \( \{0,\alpha\}^N \) by scaling. After slightly modifying the proof, we can have the similar statements for \( \{-\alpha,\alpha\}^N (\alpha > 0) \) and \( \{\alpha,\beta\}^N (0 \leq \alpha < \beta) \).

Theorem 4.6. Given \( \alpha > 0 \), \( x \in \{0,\alpha\}^N \) and \( y \in [0,\alpha]^N \), if \( |\mathcal{F}x| = |\mathcal{F}y| \), then \( y \in \{0,\alpha\}^N \) and \( \|y\|_0 = \|x\|_0 \).

Theorem 4.7. Given \( 0 \leq \alpha < \beta \), \( x \in \{\alpha,\beta\}^N \) and \( y \in [\alpha,\beta]^N \), if \( |\mathcal{F}x| = |\mathcal{F}y| \), then \( y \in \{\alpha,\beta\}^N \) and \( y \) has the same number of \( \alpha \)'s and \( \beta \)'s as \( x \).

Theorem 4.8. Given \( x \in \{-1,1\}^N \) and \( y \in [-1,1]^N \), if \( |\mathcal{F}x| = |\mathcal{F}y| \), then \( y \in \{-1,1\}^N \), and the number of 1’s in \( y \) is the same as the number of 1’s in \( x \) or the number of -1 in \( x \).

Theorem 4.9. Given \( \alpha > 0 \), \( x \in \{-\alpha,\alpha\}^N \) and \( y \in [-\alpha,\alpha]^N \), if \( |\mathcal{F}x| = |\mathcal{F}y| \), then \( y \in \{-\alpha,\alpha\}^N \), and the number of \( \alpha \)'s in \( y \) is the same as the number of \( \alpha \) in \( x \) or the number of -\( \alpha \)'s in \( x \).

\(^1\)Note that it is a wrap-around distance. For example, \( x_0 \) and \( x_{N-1} \) are considered of distance 1.
4.3 Extensions to other sampling schemes

We can also extend the analysis to the oversampling case, STFT, and FROG in Theorems 4.10 and 4.12-4.13, respectively. Similar to Theorems 4.6-4.9, we can also extend our results to \{0, \alpha\}^N, \{-\alpha, \alpha\}^N, which are omitted.

**Theorem 4.10.** Let \(M \geq N\), given \(x \in [0, 1]^N, y \in \{0, 1\}^N\), if \(|\mathcal{F}_{N \rightarrow M}x| = |\mathcal{F}_{N \rightarrow M}y|\), then \(y \in \{0, 1\}^N\) and \(\|y\|_0 = \|x\|_0\).

**Theorem 4.11.** Given \(x \in \{0, 1\}^N\) and \(y \in [0, 1]^N\), if \(x\) and \(y\) has the same STFT under non-zero constant window, with \(W \geq L\), as defined in (2.5), then \(y \in \{0, 1\}^N\).

**Theorem 4.12.** Given \(x \in \{0, 1\}^N\) and \(y \in [0, 1]^N\), if \(x\) and \(y\) has the same FROG trace (2.6), then \(y \in \{0, 1\}^N\).

**Theorem 4.13.** Given \(x \in \{-1, 1\}^N\) and \(y \in [-1, 1]^N\), if \(x\) and \(y\) has the same FROG trace, then \(y \in \{-1, 1\}^N\).

**Remark 4.14.** Unlike Theorem 4.8, the number of 1’s in \(x\) is not necessarily the same as the number of 1’s nor -1’s in \(y\). For example, if we take \(x = (1, 1)^T\) and \(y = (1, -1)^T\), then \(x\) and \(y\) have the same FROG trace.

Recall that the Z-transform of a signal \(x \in \mathbb{C}^N\) is defined by

\[
P_x(z) = \sum_{k=0}^{N-1} x_k z^k,
\]

which is a complex polynomial. The proof of [24, Theorem 2.1] shows that if the Z-transform of an unknown binary signal \(x \in \{0, 1\}^N\) is either reciprocal or irreducible, then \(x\) can be recover uniquely up to conjugate inverse. Using this fact, the exact recovery up to trivial ambiguities in the oversampling case is characterized in Propositions 4.15-4.16.

**Proposition 4.15.** Given \(M \geq 2N - 1\) in the setting of the oversampling Fourier PR, \(x \in \{0, 1\}^N\), if \(x_n = x_{N-n}\) for all \(n = 1, 2, \ldots, N - 1\), i.e. if \(x\) is equal to its conjugate inverse, then we can recover \(x\) uniquely.

**Proposition 4.16.** In the setting of the oversampling Fourier PR, given \(M \geq 2N - 1\), we can recover a random unknown binary \(x \in \{0, 1\}^N\) uniquely up to conjugate inverse with probability at least \(\frac{c}{\log N}\) for some fixed constant \(c > 0\).

Note that the factor \(\frac{c}{\log N}\) in Proposition 4.16 is just a lower bound. In fact, there is a conjecture in [35] that most of all polynomial with 0, 1 coefficients are irreducible. This conjecture will provide a much better lower bound.

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\(^2\hat{P}(z)\) is called a reciprocal polynomial of \(P(z)\) if \(\hat{P}_n(z) = z^n P_n(z^{-1})\), where \(n\) is the degree of the polynomial \(P(z)\).
5 Denoising

The preceding sections focus on the noiseless case, where the measured data we obtain is \( \mathbf{b} = |F\mathbf{x}| \). However, noise is inevitable in practice and there is a need to develop denoising techniques. For this purpose, we consider a corrupted measurement \( \tilde{\mathbf{b}} = \mathbf{b} + \eta \) with a noise term \( \eta \). In the proof of Theorem 3.2 (specifically Lemma 7.2), we reveal that \( F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) = \text{Aut}_p(\mathbf{x}) \). If the noise \( \eta \) is small enough, then \( F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) \approx F^{-1}(\mathbf{b} \odot \mathbf{b}) = \text{Aut}_p(\mathbf{x}) \).

Proposition 5.1. Given \( \epsilon > 0 \), \( \mathbf{x} \in \mathbb{C}^N \setminus \{0\} \), \( \mathbf{b} = |F\mathbf{x}| \), \( \tilde{\mathbf{b}} = \mathbf{b} + \eta \) for some noise \( \eta \in \mathbb{C}^N \), if \( \|\eta\|_{\infty} < \min\{\frac{\epsilon}{4\|\mathbf{b}\|_{\infty}}, \frac{\epsilon}{2} \} \), then \( \|F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) - \text{Aut}_p(\mathbf{x})\|_{\infty} < \epsilon \).

Note that Proposition 5.1 gives an upper bound for errors in estimating autocorrelation with and without noise. It would be ideal to quantify the error to the ground-truth signal, which is unfortunately impossible due to trivial and non-trivial ambiguities. In the following, we restrict the ground-truth signal \( \mathbf{x} \in \{0, 1\}^N \) and observe a denoising scheme based on Proposition 5.1 often gives good results.

For binary signal \( \mathbf{x} \), we know \( \text{Aut}_p(\mathbf{x}) \in \mathbb{Z}^N \). If the noise \( \eta \) is small such that \( \|\eta\|_{\infty} < \frac{1}{8N} \), we can round off each entry of \( F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) \) to the nearest integer to perform denoising. Corollary 5.2 gives an upper bound of the noise \( \eta \) for sparse signals, while Corollary 5.3 presents a general case.

Corollary 5.2. Given \( \mathbf{x} \in \{0, 1\}^N \setminus \{0\} \), \( \mathbf{b} = |F\mathbf{x}| \), \( \tilde{\mathbf{b}} = \mathbf{b} + \eta \) for some noise \( \eta \in \mathbb{C}^N \), if \( \|\eta\|_{\infty} < \frac{1}{8\|\mathbf{x}\|_0} \), then \( \|F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) - \text{Aut}_p(\mathbf{x})\|_{\infty} < \frac{1}{2} \).

Corollary 5.3. Given \( \mathbf{x} \in \{0, 1\}^N \), \( \mathbf{b} = |F\mathbf{x}| \), \( \tilde{\mathbf{b}} = \mathbf{b} + \eta \) for some noise \( \eta \in \mathbb{C}^N \), if \( \|\eta\|_{\infty} < \frac{1}{8N} \), then \( \|F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) - \text{Aut}_p(\mathbf{x})\|_{\infty} < \frac{1}{2} \).

Recall the signal-to-noise ratio (SNR) is defined by

\[
\text{SNR}_{\text{DB}} = 10 \log_{10} \frac{\|\mathbf{x}\|_2^2}{\|\eta\|_2^2}.
\]

Corollary 5.2 suggests that if \( \text{SNR}_{\text{DB}} \) is larger than \( 10 \log_{10}(64) + 40 \log_{10} \|\mathbf{x}\|_0 \), then it is safe to round off each entry to the nearest 0 and 1.

Corollary 5.4. Given \( \mathbf{x} \in \{0, 1\}^N \setminus \{0\} \), if

\[
\text{SNR}_{\text{DB}} > 10 \log_{10}(64) + 40 \log_{10} \|\mathbf{x}\|_0,
\]

then \( \|F^{-1}(\tilde{\mathbf{b}} \odot \tilde{\mathbf{b}}) - \text{Aut}_p(\mathbf{x})\|_{\infty} < \frac{1}{2} \).

The proposed denoising scheme, referred to as rounding scheme, is described as follows: given a corrupted measurement \( \tilde{\mathbf{b}} \in \mathbb{C}^N \),
1. Round off each entry $F^{-1}(b \odot \hat{b})$ to nearest integer to get the autocorrelation $\text{Aut}_p(x)$.

2. Calculate $b = \sqrt{F(\text{Aut}_p(x))}$, where square root is taken entrywisely.

3. Solve the minimization problem:
   \[
   x^* = \arg\min_x \| |F|x| - b \|_2^2 \quad \text{s.t.} \quad x \in [0, 1]^N. \quad (5.5)
   \]

4. Round off each entry of $x^*$ to be either 0 or 1

We compare the proposed scheme with a naïve scheme: given a corrupted measurement $\hat{b} \in \mathbb{C}^N$,

1. Solve the minimization problem
   \[
   x^{*} = \arg\min_{x} \| |F|x| - \hat{b} \|_2^2 \quad \text{s.t.} \quad x \in [0, 1]^N. \quad (5.6)
   \]

2. Round off each entry of $x^*$ to be either 0 or 1

Both schemes involve a general phase retrieval problem subject to a box constraint, which can be solved efficiently via alternating direction methods of multiplier with a box constraint (ADMMB); please refer to [24] for more details.

We compare the performance of the naïve scheme and the rounding scheme in terms of success rates. We consider the ground-truth signal $x_{\text{true}}$ is a binary vector of 50-dimensional with different sparsity contaminated by different noise. We consider ten sparsity levels ($1, 2, \ldots, 10$) and generate the noisy measurements $\hat{b}$ by adding Gaussian noise with SNR = (36, 32, \ldots, 0) dB. In Figure 1, we plot the success rates based on 1000 random realizations and we declare a trial is successful if $\| |F|x_{\text{recovered}}| - b \| < 10^{-6}$. Compared to the naïve scheme, rounding scheme works much better when the signal is sparse, which is expected by Corollary 5.2 that sparser signals allow for larger tolerance of the noise. According to Corollary 5.4, the exact recovery bound of SNR is calculated as $18 + 40 \log_{10} \|x\|_0$, which aligns well with Figure 1.
6 Conclusions

In this paper, we improved upon a autocorrelation-based characterization of Fourier phase retrieval theorem. Our analysis suggested that the standard gradient-based regularization, i.e., $\|\nabla^n x\|_2$, is redundant to the magnitude measurements, thus not helpful to phase retrieval. Furthermore, we proved that binary signals can be recovered by imposing a box constraint. We also presented ambiguities and uniqueness for binary phase retrieval. Finally, we proposed a denoising scheme suggested by one of characterization theorems. Since the proposed denoising scheme involves rounding, it is interesting to extend to 2D images, in which the measured data are often integer-valued.

7 Appendix

Proof of Theorem 3.1. Although the basic idea of the proof is in [8], we present it here to make the paper self-contained. Denote $M = 2N - 1$.

(1) $\Rightarrow$ (2) Define $A_x(z) = z^{N-1} \sum_{n=-(N-1)}^{N-1} (\text{Aut}(x))_n z^n$, and similarly for $A_y(z)$. Note that

$$e^{2\pi i M^{-1}} A_x(e^{-2\pi i k M}) = |(\mathcal{F}_M x)_k|^2 = |(\mathcal{F}_M y)_k|^2 = e^{2\pi i M^{-1}} A_y(e^{-2\pi i k M}),$$

for all $k = 0, 1, \ldots, M - 1$. Since $A_x$ and $A_y$ are polynomials of degree at most $2N - 1$, their coefficients are determined by $|\mathcal{F}_M x| = |\mathcal{F}_M y|$, which is a system of $M \geq 2N - 1$ linear equations with $M \geq 2N - 1$. Thus, $\text{Aut}(x) = \text{Aut}(y)$.

(2) $\Rightarrow$ (1). Suppose $\text{Aut}(x) = \text{Aut}(y)$. Then $A_x(z) = A_y(z)$. Since $M \geq 2N - 1$, we have

$$|(\mathcal{F}_M x)_k|^2 = e^{2\pi i M^{-1}} A_x(e^{-2\pi i k M}) = e^{2\pi i M^{-1}} A_y(e^{-2\pi i k M}) = |(\mathcal{F}_M y)_k|^2,$$

for all $k = 0, 1, \ldots, M - 1$. □

To prove Theorem 3.2, we introduce Lemma 7.2 and 7.3. Note that Lemma 7.2 is a periodic version of a similar result in [36, P. 215].

Lemma 7.2. $\mathcal{F}(\text{Aut}_p(x)) = |\mathcal{F}x| \odot |\mathcal{F}x|$, $\forall x \in \mathbb{C}^N$. 

12
Proof. It is straightforward that for all $j = 0, 1, \ldots, N - 1$, we have
\[
(F(\text{Aut}_p(x)))_j = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x_{n+m} x_n \omega^{mj}
= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x_n \omega^{(m-n)j} = \left( \sum_{m=0}^{N-1} x_m \omega^{mj} \right) \left( \sum_{n=0}^{N-1} x_n \omega^{nj} \right) = (Fx)_j (Fx)_j = (|Fx|)^2.
\]

Lemma 7.3 (Sphere constraint). Given $x, y \in \mathbb{C}^N$, if $|Fx| = |Fy|$, then $\|x\|_2 = \|y\|_2$.

Proof. Since $\frac{1}{\sqrt{N}}F$ is unitary, we have
\[
\|x\|_2 = \left\| \frac{1}{\sqrt{N}}Fx \right\|_2 = \frac{1}{\sqrt{N}} \|Fx\|_2.
\]
Similarly, we have
\[
\|y\|_2 = \frac{1}{\sqrt{N}} \|Fy\|_2 = \frac{1}{\sqrt{N}} \|Fx\|_2 = \|x\|_2.
\]

Proof of Theorem 3.2. (1) ⇒ (2). Suppose $|Fx| = |Fy|$, by Lemma 7.2, $F(\text{Aut}_p(x)) = |Fx| \circ |Fx|$. Hence, $\text{Aut}_p(x) = F^{-1}(|Fx| \circ |Fx|) = F^{-1}(\text{Aut}_p(y)) = \text{Aut}_p(y)$.

(2) ⇒ (1). Suppose $\text{Aut}_p(x) = \text{Aut}_p(y)$. By Lemma 7.2 we have $|Fx| = \sqrt{\text{Aut}_p(x)} = \sqrt{\text{Aut}_p(y)} = |Fy|$, where the square root is taken entrywisely.

(1) ⇒ (3). By the Convolution Theorem, we have $\forall v \in \mathbb{C}^N$ and $j = 0, 1, \ldots, N - 1$,
\[
(F(v * x))_j = (Fv)_j \times (Fx)_j,
\]
thus leading to,
\[
|(F(v * x))_j| = |(Fv)_j|(Fx)_j|.
\]
Similar result holds for $F(v * y)$. Since $|Fx| = |Fy|$ (by assumption), we have $|F(v * x)| = |F(v * y)|$, which implies that $\|v * x\|_2 = \|v * y\|_2$ by Lemma 7.3.

(3) ⇒ (1). Suppose $\|v * x\|_2 = \|v * y\|_2$ for all $v \in \mathbb{C}^N$. Since $F$ is invertible,
we can choose \( \mathbf{v}_k = \mathcal{F}^{-1} \mathbf{e}_k \in \mathbb{C}^N \). Then we have
\[
\| \mathbf{v}_k \ast \mathbf{x} \|_2^2 = \left\| \frac{1}{\sqrt{N}} \mathcal{F}(\mathbf{v}_k \ast \mathbf{x}) \right\|_2^2
\]
\[
= \frac{1}{N} \sum_{j=0}^{N-1} |(\mathcal{F}(\mathbf{v}_k \ast \mathbf{x}))_j|^2 = \frac{1}{N} \sum_{j=0}^{N-1} |(\mathcal{F}\mathbf{v}_k)_n|^2 |(\mathcal{F}\mathbf{x})_j|^2
\]
\[
= \frac{1}{N} \sum_{j=0}^{N-1} |(\mathbf{e}_k)_n|^2 |(\mathcal{F}\mathbf{x})_j|^2 = \frac{1}{N} |(\mathcal{F}\mathbf{x})_k|^2.
\]

Similarly, we have \( \| \mathbf{v}_k \ast \mathbf{x} \|_2^2 = \| \mathbf{v}_k \ast \mathbf{y} \|_2^2 = \frac{1}{N} |(\mathcal{F}\mathbf{y})_k|^2 \). Therefore, we have \( |\mathcal{F}\mathbf{x}| = |\mathcal{F}\mathbf{y}| \). \( \square \)

**Proof of Theorem 4.1**. (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (1) are the same as the proof of Theorem 3.1 by considering a suitable \( M \). It remains to prove that (1) implies \( \text{Aut}_p(\mathbf{x}) = \text{Aut}_p(\mathbf{y}) \) and \( \| \mathbf{v} \ast \mathbf{x} \|_2 = \| \mathbf{v} \ast \mathbf{y} \|_2 \) \( \forall \mathbf{v} \in \mathbb{C}^N \). This directly follows from (1) \( \Rightarrow \) (2) and (3) in Theorem 3.2 by considering \( M = N \) in equation (7.1). \( \square \)

To prove Theorem 4.1, we need Lemmas 7.3-7.4.

**Lemma 7.4** (Plane constraint). Given \( \mathbf{x}, \mathbf{y} \in [0, \infty)^N \), if \( |(\mathcal{F}\mathbf{x})_0| = |(\mathcal{F}\mathbf{y})_0| \), then
\[
\sum_{n=0}^{N-1} x_n = \sum_{n=0}^{N-1} y_n.
\]

**Proof.** It follows from the assumption of \( x_i \geq 0 \) \( \forall i \) that
\[
|(\mathcal{F}\mathbf{x})_0| = |(1, 1, \ldots, 1)\mathbf{x}| = x_0 + x_1 + \cdots + x_{N-1}.
\]

Similarly, we have \( |(\mathcal{F}\mathbf{y})_0| = y_0 + y_1 + \cdots + y_{N-1} \), which completes the proof. \( \square \)

For any \( \mathbf{y} \) having the same Fourier magnitude as \( \mathbf{x} \), Lemma 7.3 implies that \( \mathbf{y} \) must lie on a sphere, while Lemma 7.4 implies that \( \mathbf{y} \) must lie on a plane. The binary assumption states that \( \mathbf{y} \) must be lie on \([0, 1]^N\). Therefore, the solution \( \mathbf{y} \) must lie on the intersection of these three sets. Figure 2 illustrates these 3 sets when \( \mathbf{x} = (1, 0) \). Below we present a rigorous proof to Theorem 4.1.

**Proof of Theorem 4.1**. Write \( \mathbf{x} = (x_0, x_1, \ldots, x_{N-1})^T \) and \( \mathbf{y} = (y_0, y_1, \ldots, y_{N-1})^T \). By Lemma 7.3 we have
\[
\sum_{n=0}^{N-1} y_n^2 = \|\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 = \sum_{n=0}^{N-1} x_n^2 = \sum_{n=0}^{N-1} x_n = \|\mathbf{x}\|_0 .
\]
Figure 2: Illustration of Theorem 4.1 when $x = (1, 0)$. The plane constraint is $y_1 + y_2 = 1$, indicated by the red line. The sphere constraint is $y_1^2 + y_2^2 = 1$, indicated by the blue circle. The box constraint is $0 \leq y_1, y_2 \leq 1$, indicated by the green square. The solutions to the problem (Q) lie on the intersection of these 3 sets, indicated by the black points. Both black points are ambiguous due to Proposition 4.2.

where the second last and last equality come from the fact that $x$ is a binary signal, $x_i$ is either 0 or 1. It follows from Lemma 7.4 that

$$
\sum_{n=0}^{N-1} y_n = \sum_{n=0}^{N-1} x_n = \|x\|_0.
$$

By assumption, we have $y_i \in [0,1]$ and hence $y_i^2 \leq y_i$ for all $i = 0, 1, \ldots, N-1$ with the equality holds if and only if $y_i = 0$ or 1. Since

$$
\|x\|_0 = \sum_{n=0}^{N-1} y_n^2 \leq \sum_{n=0}^{N-1} y_n = \|x\|_0,
$$

we have $\sum_{n=0}^{N-1} y_n^2 = \sum_{n=0}^{N-1} y_n$, which forces $y_i^2 = y_i$, i.e. $y_i = 0$ or 1. Therefore, we have $y \in \{0,1\}^N$ and

$$
\|y\|_0 = \sum_{n=0}^{N-1} y_n = \sum_{n=0}^{N-1} x_n = \|x\|_0.
$$

Proof of Theorem 4.6. Theorem 4.6 is a simple consequence of scaling based on Theorem 4.1. Since $\alpha^{-1}x \in \{0,1\}^N$, $\alpha^{-1}y \in \{0,1\}^N$, and $|F(\alpha^{-1}x)| = |F(\alpha^{-1}y)|$, we have $\alpha^{-1}y \in \{0,1\}^N$ and $\|\alpha^{-1}y\|_0 = \|\alpha^{-1}x\|_0$. Therefore, $y \in \{0, \alpha\}^N$ and $\|y\|_0 = \|\alpha^{-1}y\|_0 = \|\alpha^{-1}x\|_0 = \|x\|_0$. □

Proof of Theorem 4.7. Lemma 7.4 states that $y$ lies on the plane $P : \sum x_i = \sum y_i$, which is convex. The box constraint $y \in [\alpha, \beta]^N$ is also convex. Therefore,
Figure 3: Illustration of Theorem 4.6 in 2D. The *double plane* constraint is \( y_1 + y_2 = 2 \) and \( y_1 + y_2 = -2 \), indicated by the red line, the *sphere* constraint is \( y_1^2 + y_2^2 = 1 \), indicated by the blue circle. The *box* constraint is \(-1 \leq y_1, y_2 \leq 1\), indicated by the green square. The solutions to the our problem lies on the intersection of these 3 sets, indicated by the black points.

we denote \( C := P \cap \{\alpha, \beta\}^N \), as a convex compact set. By Krein-Milman Theorem [37, Theorem 3.23], \( C \) is the closure of the convex hull of its extreme points.

We claim that the set of extreme points \( E = \{z_i\}_{i \in I} \) is a subset of points \( \{\alpha, \beta\} \) with the same number of \( \alpha \)'s and \( \beta \)'s as \( x \). Given \( w \) be an extreme point of \( C \), assume that \( w \) does not belong to \( \{\alpha, \beta\}^N \). Since \( \sum w_i = \sum x_i \) and \( x \in \{\alpha, \beta\}^N \), there exists some \( i < j \) such that \( w_i, w_j \neq \alpha \) and \( \beta \) (otherwise, we will have \( w \in \{\alpha, \beta\}^N \)). Choose small \( \epsilon > 0 \) such that \( w_i, w_j > \alpha + \epsilon \) and \( w_i, w_j < \beta - \epsilon \). Let \( w_1 = (w_0, w_1, \ldots, w_i + \epsilon, \ldots, w_j - \epsilon, \ldots, w_{N-1})^T \) and \( w_2 = (w_0, w_1, \ldots, w_i - \epsilon, \ldots, w_j + \epsilon, \ldots, w_{N-1})^T \). Then \( w_1, w_2 \in C \) and \( w = \frac{1}{2}(w_1 + w_2) \), contradicting the fact that \( w \) is an extreme point of \( C \). Hence, we have \( w \in \{\alpha, \beta\}^N \). It follows from \( \sum w_i = \sum x_i \) that \( w \) has the same number of \( \alpha \)'s and \( \beta \)'s as \( x \). Since \( E \) is a finite set, the convex hull of \( E \) is compact and thus equal to \( C \).

Since \( y \in C \), we write \( y = \sum \lambda_i z_i \) for some \( 0 \leq \lambda_i \leq 1 \), \( \sum \lambda_i = 1 \). Since \( z_i \) has the same number of \( \alpha \)'s and \( \beta \)'s as \( x \), then \( f(x) = f(z_i) \) for all \( i \in I \), where \( f(w) := \|w\|_2^2 \), which is a strictly convex function. By Lemma 7.3 we have \( f(y) = f(x) \). If \( y \) does not belong to \( E \), then we have

\[
f(y) < \sum \lambda_i f(z_i) = \sum \lambda_i f(x) = f(x) \sum \lambda_i = f(x),
\]

which is a contradiction. So \( y \in E \), i.e. \( y \in \{\alpha, \beta\}^N \) and has the same number of \( \alpha \)'s and \( \beta \)'s as \( x \). 

To prove Theorem 4.8 we notice that Lemma 7.4 (plane constraint) is no longer applicable. Fortunately, we have a *double plane* constraint, as shown in Figure 3 and Lemma 7.5.

16
Lemma 7.5. Given $x, y \in [-1, 1]^N$, if $|\langle Fx \rangle_0| = |\langle Fy \rangle_0|$, then
\[
\sum_{n=0}^{N-1} x_n = \sum_{n=0}^{N-1} y_n \text{ or } \sum_{n=0}^{N-1} x_n = -\sum_{n=0}^{N-1} y_n.
\] (7.6)

Proof. $|\langle Fx \rangle_0| = |\langle Fy \rangle_0|$ is equivalent to $|\sum_{n=0}^{N-1} x_n| = |\sum_{n=0}^{N-1} y_n|$. Since $x$ and $y$ are real-valued, the relationship in (7.6) holds. □

Proof of Theorem 4.8. Since $y \in \{-1, 1\}^N$, we have $y_i^2 \leq 1$ for all $i$ and $\sum_{n=0}^{N-1} y_n^2 \leq N$ with equality holds if and only if $y_i = \pm 1$. By Lemma 7.3 and the assumption that $x \in \{-1, 1\}^N$, we have
\[
\sum_{n=0}^{N-1} y_n^2 = \|y\|^2 = \|x\|^2 = \sum_{n=0}^{N-1} x_n^2 = N,
\]
which forces $y_i = \pm 1$, i.e. $y \in \{-1, 1\}^N$.

Denote the numbers of 1’s in $x, y$ be $k_x$ and $k_y$, respectively. Then the number of $-1$’s in $x$ and $y$ are $(N - k_x)$ and $(N - k_y)$. It follows from Lemma 7.3 that
\[
k_x + (N - k_x) \times (-1) = |\sum_{n=0}^{N-1} x_n| = |\sum_{n=0}^{N-1} y_n| = \pm (k_y + (N - k_y) \times (-1)),
\]
i.e. $k_x = k_y$ or $k_x = N - k_y$. In other word, the number of 1’s in $x$ is the same of the number of 1’s in $y$ or the number of $-1$’s in $y$. □

We omit the proof of Theorem 4.9 since it is basically the same as Theorem 4.6.

Proof of Theorem 4.10. Write $|\langle F_{N\rightarrow M}x \rangle| = |\langle F_{M\rightarrow M}\hat{x} \rangle|$, where $\hat{x} = (x_0, x_1, \ldots, x_{N-1}, 0, 0, \ldots, 0)^T \in [0, 1]^M$. Similarly, we write $|\langle F_{M\rightarrow M}y \rangle| = |\langle F_{M\rightarrow M}\hat{y} \rangle|$ and define $\hat{y}$. Note that $\hat{x} \in [0, 1]^M$, $\hat{y} \in \{0, 1\}^M$ and $|\langle F_{M\rightarrow M}\hat{x} \rangle| = |\langle F_{M\rightarrow M}\hat{y} \rangle|$. By Theorem 4.1 one has $\hat{y} \in \{0, 1\}^M$ and $||\hat{x}||_0 = ||\hat{y}||_0$. Since $\hat{x}$ and $\hat{y}$ are obtained by adding zeros to $x$ and $y$, we have $y \in \{0, 1\}^N$ and $||x||_0 = ||\hat{x}||_0 = ||\hat{y}||_0 = ||y||_0$. □

Proof of Theorem 4.11. Without loss of generality, we may assume the windows $w$ is an all one vector $\mathbb{1}$ by scaling. Recall the STFT of $x$ is defined by
\[
z_{n,m} = \sum_{k=0}^{N-1} x_k w_{mL-k} e^{-\frac{2\pi kmn}{N}}.
\]

Since $W \geq L$, for each $l = 0, 1, \ldots, N - 1$, there is some $m$ such that $w_{mL-l} = 1$. For such $m$, define $\hat{x}_k = x_k w_{mL-k}$ for all $k = 0, 1, \ldots, N - 1$ and define $\hat{y}$ in a similar way. Then, $\hat{x} \in \{0, 1\}^N$ and $\hat{y} \in \{0, 1\}^N$ by our assumption on $w$.

Now, $|\langle F\hat{x} \rangle| = z_{., m} = |\langle F\hat{y} \rangle|$. Applying Theorem 4.1, we have $\hat{y} \in \{0, 1\}^N$. In particular, $y_i = y_{mL-l} = \hat{y}_l \in \{0, 1\}$. Since $l$ is arbitrary, we have $y \in \{0, 1\}^N$. □
Proof of Theorem 4.12. Denote $|\hat{z}_{k,m}|^2$ and $|\hat{w}_{k,m}|^2$ be the FROG trace (2.6) of $x$ and $y$, respectively. We consider $m = 0$ and define $z_0 = (z_{0,0}, z_{1,0}, \ldots, z_{N-1,0})^T$ and similarly for $w_0$. As $x_n \in \{0, 1\}$, we obtain $z_{n,0} = x_n^2 = x_n$ and $w_{n,0} = y_n^2 = y_n$. Now, our assumption translates to $|\hat{z}_{k,0}| = |\hat{w}_{k,0}|$ for $k = 0, \ldots, N-1$, i.e., $|Fz_0| = |Fw_0|$. Since $z_0 \in \{0, 1\}^N$ and $w_0 \in \{0, 1\}^N$, we have $w_0 \in \{0, 1\}^N$ by Theorem 4.1, i.e., $w_{n,0} = y_n \in \{0, 1\}$ for all $n = 0, \ldots, N-1$. Therefore, we obtain $y \in \{0, 1\}^N$, which implies that $y = w_0$. Since $x = z_0$, we have $|Fx| = |Fz_0| = |Fw_0| = |Fy|$ and $\|x\|_0 = \|y\|_0$ by Theorem 4.1. □

Proof of Theorem 4.13. The proof is similar to the proof of Theorem 4.12 by noting that $z_{n,0} = x_n^2 = 1 \in \{-1, 1\}$ and using Theorem 4.8. □

Proof of Proposition 4.2. It is straightforward that
\[
Fx + F(\mathbb{1} - x) = F\mathbb{1} = Ne_0.
\]
Therefore, we obtain $(F(x)_j + (F(\mathbb{1} - x))_j = 0$ for $j = 1, 2, \ldots, N-1$, particularly that $|(F(x)j| = |(F(\mathbb{1} - x))j|$.

In addition, we have $|(Fx)_0| = \|x\|_0 = N$ and
\[
|(F(\mathbb{1} - x))_0| = \|(\mathbb{1} - x)\|_0 = (N - N) = N = \|Fx\|_0,
\]
which completes the proof. □

Proof of Proposition 4.3. Similar to Proposition 4.2, we have
\[
|(F(\mathbb{1} - x))_j| = |(Fx)_j| = |(Fy)_j| = |(F(\mathbb{1} - y))_j|
\]
for $j = 1, 2, \ldots, N-1$. When $j = 0$, we get
\[
(F(\mathbb{1} - x))_0 = N - (Fx)_0 = N - (Fy)_0 = (F(\mathbb{1} - y))_0.
\]
Therefore, $|F(\mathbb{1} - x)| = |F(\mathbb{1} - y)|$. Similar analysis for the other direction. □

Proof of Proposition 4.4. When $\|x\|_0 = 0$, $x$ is a zero vector. In particular, the recovery is unique.

When $\|x\|_0 = 1$, we get $x = e_k$ for some $k$, which is related by spatial shifts to each other. Hence, the recovery is unique up to trivial ambiguities.

When $\|x\|_0 = 2$, we obtain the $\text{Aut}_p(x)$ from $F_x$ by Theorem 3.2. Without loss of generality, up to spatial shift, we assume $x_0 = 1$. Let $k$ be the smallest positive number such that $(\text{Aut}_p(x))_k$ is nonzero. Since $(\text{Aut}_p(x))_k$ is equal to the number of pair of 1’s with distance $k$ and there is only two 1’s in $x$, i.e. only one pair of 1’s. This pair must contain $x_0$. Say the pair contains $x_0$ and $x_j$. We know that $x_j$ and $x_0$ has distance $k$. Hence, $j = k$ or $N - k$, i.e. we either have $x_0 = x_k = 1$ or $x_0 = x_{N-k} = 1$, which are spatial shifts of each other.

When $\|x\|_0 = 3$, given $|F_x|$, we obtain the $\text{Aut}_p(x)$. Let $k$ be the smallest positive number such that $(\text{Aut}_p(x))_k$ is nonzero. Since there is three 1’s in $x$,
there is $3C_2$, i.e. 3 pairs of 1’s in $x$. Thus, $(\text{Aut}_p(x))_k = 1, 2 \text{ or } 3$. By spatial shift, we may assume one of the pairs contains $x_0$ and $x_k$.

If $(\text{Aut}_p(x))_k = 2 \text{ or } 3$, then there is still at least one pair of 1’s containing $x_0$ or $x_k$ and the remaining 1. If it contains $x_0$, then the 1 should lie in $x_{N-K}$ since $x_k$ is already occupied. If the pair contains $x_k$, by similar reasoning, the 1 should lie in $x_{2k}$. In both cases, all three 1’s are placed and this 2 cases are spatial shift of each other.

If $(\text{Aut}_p(x))_k = 1$, let $l$ be the smallest positive number greater than $k$ such that $(\text{Aut}_p(x))_l$ is nonzero. By considering the position of 1, we have 4 cases:

- $x_{N-l} = 1$, $x_{N-l+k} = 1$, $x_l = 1$ or $x_{l+k} = 1$. The cases that $x_{N-l+k} = 1$ and $x_l = 1$ are impossible, otherwise it will contradicts the minimality of $l$, $k$ and the fact that $(\text{Aut}_p(x))_k = 1$, i.e., there is a pair of 1 with distance $(l - k) < l$ while this pair is not the pair corresponding to the pair of distance $k$. Hence, we either have $x_0 = x_k = x_{l+k} = 1$ or $x_0 = x_k = x_{N-l} = 1$. Note that these two cases are equivalent to each other through conjugate inverse and spatial shift.

The cases when $||x||_0 = N - 3, N - 2, N - 1$ or $N$ now follow from above. If $||x||_0 = N - 3, N - 2, N - 1$ or $N$, then $||1 - x||_0 = 0, 1, 2$ or 3. Hence, we can recover $(1 - x)$ up to trivial ambiguities. Since $x = 1 - (1 - x)$, the recovery of $x$ is unique up to trivial ambiguities.

**Proof of Proposition [4.12]** By Theorem [3.3] $\text{Aut}(x)$ is uniquely determined when $M \geq 2N - 1$. It suffices to note that $P_x(z)$, the Z-transform of $x$, is a reciprocal polynomial since $x$ is equal to its conjugate inverse. According to [25] Theorem 2.1, if $P_x(z)$ is reciprocal, then there does not exist $y \neq x$ such that $\text{Aut}(y) = \text{Aut}(x)$. Therefore, we can uniquely recover $x$ from $\text{Aut}(x)$ up to trivial ambiguities.

**Proof of Proposition [4.16]** [38] Theorem 1] shows that for a random binary $x$ with $x_0 = x_{N-1} = 1$, the $Z$-transform of $x$ ($P_x(z)$) is irreducible with probability at least $\frac{\epsilon'}{\log N}$ for some fixed constant $\epsilon' > 0$. Note that we have $2^{N-2}$ binary signals under the constraint $x_0 = x_{N-1} = 1$ while we have $2^N$ binary signals in total. For a random binary $x$, $P_x(z)$ is irreducible with probability at least $\frac{1}{4 \log N} = \frac{\epsilon'}{2 \log N}$, where $c = \frac{\epsilon'}{4} > 0$ is a fixed constant. It was further proved in [25] Theorem 2.1] that if $P_x(z)$ is irreducible, then we can recover $x$ uniquely up to conjugate inverse, which completes the proof.

**Proof of Proposition [5.1]**

\[
\|\mathcal{F}^{-1}(\tilde{b} \odot \tilde{b}) - \text{Aut}_p(x)\|_\infty = \|\mathcal{F}^{-1}(\tilde{b} \odot \tilde{b}) - \mathcal{F}^{-1}(b \odot b)\|_\infty \\
\leq \|\mathcal{F}^{-1}(\tilde{b} \odot \tilde{b}) - \mathcal{F}^{-1}(b \odot b)\|_2 = \frac{1}{\sqrt{N}} \|\tilde{b} \odot \tilde{b} - b \odot b\|_2 \\
\leq \|\tilde{b} \odot \tilde{b} - b \odot b\|_\infty = \|2b \odot \eta + \eta \odot \eta\|_\infty \\
\leq 2 \|b\|_\infty \|\eta\|_\infty + \|\eta\|_\infty^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon,
\]

19
where the first and second inequalities come from the fact that $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{N} \|x\|_\infty$ for all $x \in \mathbb{C}^N$, the third inequality comes from the fact that $\|x \odot y\|_\infty \leq \|x\|_\infty \|y\|_\infty$ for all $x, y \in \mathbb{C}^N$.

To prove Corollary 5.2, we introduce Lemma 7.7.

**Lemma 7.7.** Let $x \in \{0, 1\}^N$, $b := |F^x|$, we have $\|b\|_\infty = \|x\|_0$.

**Proof.**

\[
\begin{align*}
b_n &= \left| \sum_{k=0}^{N-1} x_k e^{-2\pi ki n/N} \right| \\
&\leq \sum_{k=0}^{N-1} |x_k| e^{-2\pi ki n/N} \\
&= \sum_{k=0}^{N-1} |x_k| = \|x\|_1 = \|x\|_0,
\end{align*}
\]

which implies that $\|b\|_\infty \leq \|x\|_0$. On the other hand, we have

\[
\|b\|_\infty \geq b_0 = \left| \sum_{k=0}^{N-1} x_k \right| = \|x\|_0.
\]

Therefore, $\|b\|_\infty = \|x\|_0$.

**Proof of Corollary 5.2** Let $\epsilon = \frac{1}{2}$. Since $x \neq 0$, we have $\|x\|_0 \geq 1$. By Lemma 7.7, we have $\|\eta\|_\infty < \frac{1}{8} \|x\|_0 = \frac{1}{8} \|b\|_\infty = \frac{1}{4} \|b\|_\infty$. Also, $\|\eta\|_\infty < \frac{1}{8} \|x\|_0 \leq \frac{1}{8} \leq \min\{\frac{\epsilon}{2}, 1\}$. The remaining follows from Proposition 5.1.

**Proof of Corollary 5.3** When $x = 0$, then $b = |F^x| = 0$, $\text{Aut}_p(x) = 0$ and $\tilde{b} = b + \eta = \eta$.

\[
\frac{1}{N} \left| \sum_{k=0}^{N-1} x_k e^{-2\pi ki n/N} \right| = \frac{1}{N} \left| \sum_{k=0}^{N-1} x_k \right| = \|x\|_1 = \|x\|_0.
\]

Thus, we have $\|\eta\|_\infty \leq \frac{1}{16} \|x\|_0$. The rest is straightforward from Corollary 5.2.

**Proof of Corollary 5.4** The inequality

\[
\text{SNR}_{\text{dB}} > 10 \log_{10}(64) + 40 \log_{10}(\|x\|_0),
\]

is equivalent to

\[
\frac{\|x\|_2^2}{\|\eta\|_2^2} > 64 \|x\|_0^4.
\]

Since $x \in \{0, 1\}^N$, $\|x\|_2 = \|x\|_0$. Thus, we have $\|\eta\|_\infty^2 \leq \|\eta\|_2^2 < \frac{1}{64 \|x\|_0^2}$. 

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