Energy-momentum tensors in classical field theories - a modern perspective

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Abstract

The paper presents a general geometric approach to energy-momentum tensors in Lagrangian field theories, based on a Hilbert-type definition. The approach is consistent with the ones defining energy-momentum tensors in terms of hypermomentum maps given by the diffeomorphism invariance of the Lagrangian - and, in a sense, complementary to these, with the advantage of an increased simplicity of proofs and also, opening up new insights into the topic. A special attention is paid to the particular cases of metric and metric-affine theories.

Keywords: stress-energy-momentum tensor, energy-momentum balance law, Lepage equivalent of a Lagrangian

1 Introduction

The paper presents a geometric approach to energy-momentum tensors in general Lagrangian field theories. Rather than trying to offer a detailed review of the long and intricate history of the topic, it aims to bring more simplicity and clarity - by looking at the problem from a somewhat different perspective.

This perspective is nothing but an extension to arbitrary Lagrangian field theories of the general relativistic one, based on a Hilbert-type definition of the energy-momentum tensor.

The energy-momentum tensor introduced this way is, of course, not new - and neither are its usual properties (generalized covariant conservation law, gauge invariance etc.). When evaluated on critical sections of the matter Lagrangian, the Hilbert-type energy-momentum tensor agrees (up to a sign) with the "improved Noether current", as introduced by Gotay and Marsden, [6]. Still, there are some advantages of using a Hilbert-type approach over a Noether-type one. The first one is simplicity, both in the calculation of the energy-momentum tensor and in the proofs of its properties.

The Hilbert-type formula also opens up the possibility of using results of the inverse problem of variational calculus, such as: the classification of first-order energy-momentum source forms, [9], or the notion of variational completion, [15]. In the situation when we know, e.g., by some empirical method, a term of
an energy-momentum tensor, the latter allows one to recover its full expression, together with the corresponding Lagrangian.

Before passing to the technical details of our story, let us present in brief the main problems surrounding energy-momentum tensors.

- The canonical, or Noether energy-momentum tensor is defined, in special relativity, by means of the Noether current due to the invariance of the Lagrangian to the group of spacetime translations. It is conserved on-shell and its time-time and time-space components give the correct energy and momentum densities of the system. Still, as it is generally neither symmetric, nor gauge-invariant, it requires an "improvement" procedure; the classical special-relativistic recipe (Belinfante&-Rosenfeld, 1940) is based on enlarging the considered symmetry group to the whole Poincaré group.

- General relativity came with a completely different toolkit. The Hilbert, or metric energy-momentum tensor, given by the variational derivative of the matter Lagrangian with respect to the metric, is symmetric, gauge-invariant and, as a result of the diffeomorphism invariance of the Lagrangian, its covariant divergence vanishes on-shell. Hence, it does not require any improvement procedure. But, on the other hand, it is not obvious at all that it gives the correct energy and momentum densities of the system under discussion.

It thus appeared the idea of obtaining the energy-momentum tensor of basically any classical field theory as a kind of improved Noether current, which should coincide, in the case of general relativity, with the Hilbert one. But this proved to be a highly non-trivial task - and gave rise to long-standing debates (a good account of which is given, e.g., in [6]).

The first problem which appears is the one of the symmetry group to be considered. On a general manifold $X$, translations - let alone the Poincaré group - make no sense; the natural choice in this case seems to be the group $Diff(X)$ of diffeomorphisms of $X$. But, as $Diff(X)$ is infinite-dimensional, no non-trivial Noether current can be obtained from it; the corresponding "Noether current" (called a hypermomentum map) is always zero when all the variables are subject to Euler-Lagrange equations, [3], [6]. The way out of this impasse consists, [3], [4], in dividing the variables of the theory into background ones (e.g., a metric and/or a connection, a tetrad etc.) and dynamical (or matter) ones, and, accordingly, in splitting the total Lagrangian $\lambda$ into a sum

$$\lambda = \lambda_b + \lambda_m,$$

where the background Lagrangian $\lambda_b$ only depends on the background variables and their derivatives up to some order, while the matter Lagrangian $\lambda_m$ may depend on all the variables of the theory. For the matter Lagrangian $\lambda_m$, the background variables will no longer be subject to any Euler-Lagrange equations (these are only supposed to obey Euler-Lagrange equations for the total Lagrangian $\lambda$). This way, one can obtain a nonzero hypermomentum map, which solves the problem.
Improving Noether currents is the path taken by the majority of the authors. A general, geometric and systematic approach in this respect is the one by Gotay and Marsden ([6], 2001) - with some refinements brought by Forger and Römer [3]. The approach is extended to higher order variational problems of differential index 1 by Fernández, García and Rodrigo, [2].

Roughly speaking, the main result in [6] states that, if a first-order Lagrangian on a fibered manifold $\pi: Y \to X$ is invariant to the flow of an arbitrary vector field $\xi$ on $X$ and $J_\xi$ is the corresponding Noether current, then, for any solution $\gamma: X \to Y$ of the Euler-Lagrange equations, there uniquely exists a $(1,1)$-tensor density $T(\gamma) = T^i_j(\gamma) \frac{\partial}{\partial x^i} \otimes dx^j$ on $X$ such that, for all compact hypersurfaces $\Sigma \subset X$,

$$\int_{\Sigma} J^i \gamma^* J_\xi = \int_{\Sigma} T^i_j(\gamma) \xi^j \omega_i$$

(here, $\omega_i = dx^0 \wedge ... \wedge dx^{i-1} \wedge dx^{i+1} \wedge ... \wedge dx^n$).

The improved canonical energy-momentum tensor density (1) is gauge invariant and given (up to a sign) by a Hilbert-type formula. Moreover, relation (1) ensures that $T(\gamma)$ is a "physically correct" energy-momentum tensor, in the following sense, [6]. If $\Sigma$ is a Cauchy hypersurface and the vector field $\xi$ is transversal to $\Sigma$, then the Hamiltonian (energy) corresponding to the "direction of evolution" $\xi$, is $H_\xi = -\int_{\Sigma} J^i \gamma^* J_\xi$, i.e., it is given, again up to a sign, by (1).

The latter remark points out that one could hardly overestimate the importance of having the energy-momentum tensor related to Noether currents as in (1). But there is another way of looking at the same relation. Instead of using it as a definition, we will prefer to obtain it as a consequence of a Hilbert-type definition.

We will assume, as in [6], [2], that the differential index of the theory in the background variables is 1. Under this assumption, in the Euler-Lagrange expression of the matter Lagrangian $\lambda_m$ with respect to the background variables, we can isolate a total divergence term - which leads to the correct energy-momentum tensor. The remaining terms give a generalized covariant conservation law (an energy-momentum balance law) - which is obtained as an immediate consequence of the first variation formula.

These results hold true regardless of the order of the Lagrangian or of the nature of the coupling (minimal or non-minimal) between the background and the matter variables.

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1The main advance brought in [3] resides in the fact that the energy-momentum tensor is built from the matter Lagrangian $\lambda_m$ only (while in [6], it is built from the total Lagrangian $\lambda$ - and thus vanishes on-shell). Also, the energy-momentum tensor is regarded as a geometric object on a jet bundle of the total configuration manifold, rather than on the spacetime manifold $X$ - a standpoint which we will also adopt in the following.

2As diffeomorphisms of $X$ act on $X$, not on $Y$, an embedding $Dif f(X) \to Aut(Y)$ is needed in order to correctly define this invariance. A rigorous definition will be given below.
The paper is structured as follows. In Section 2, we present a quite detailed overview of the necessary notions and results. Section 3 is devoted to the definition of the energy-momentum tensor and Section 4, to its main properties, namely, energy-momentum balance law and gauge invariance. Section 5 discusses two major particular cases: metric and metric-affine backgrounds. In the metric-affine case, the obtained energy-momentum balance law has a simpler (and explicitly covariant) expression compared to the known ones, [13], [12]. The last section presents an application to energy-momentum tensors of the notion of variational completion.

2 Variational calculus - the modern framework

The language of differential forms, in which a Lagrangian is regarded as a differential form on a certain jet bundle, allows a concise, coordinate-free formulation of variational calculus on arbitrary manifolds. In this approach, the variation of a Lagrangian \( \lambda \) is understood as its Lie derivative with respect to a vector field on the respective jet bundle. The notions and results presented in this section can be found in more detail, e.g., in the monograph [8].

2.1 Differential forms on jet prolongations of a fibered manifold

Consider a fibered manifold \( Y \) of dimension \( m + n \), with connected, orientable \( n \)-dimensional base \( X \) and projection \( \pi : Y \rightarrow X \). The manifold \( Y \) will be called the configuration manifold and \( X \), the spacetime manifold.

Fibered charts \( (V, \psi) \), \( \psi = (x^i, y^\sigma) \) on \( Y \) induce the fibered charts \( (V^r, \psi^r) \), \( \psi^r = (x^i, y^\sigma, y^\sigma_1, \ldots, y^\sigma_{j_1j_2\ldots j_r}) \) on the \( r \)-jet prolongation \( J^rY \) of \( Y \) and \((U, \phi)\), \( \phi = (x^i) \) on \( X \). We denote by \( \pi^{r,s} \) the canonical projections \( \pi^{r,s} : J^rY \rightarrow J^sY \), \( (x^i, y^\sigma, y^\sigma_1, \ldots, y^\sigma_{j_1j_2\ldots j_r}) \mapsto (x^i, y^\sigma, y^\sigma_1, \ldots, y^\sigma_{j_1j_2\ldots j_s}) \) \((r > s, J^0Y := Y)\) and by \( \pi^r \), the projection \( \pi^r : J^rY \rightarrow X \), \( (x^i, y^\sigma, y^\sigma_1, \ldots, y^\sigma_{j_1j_2\ldots j_r}) \mapsto (x^i) \).

By \( \Omega^r_kY \), we will mean the set of \( k \)-forms of order \( r \) over \( Y \). In particular, \( \mathcal{F}(Y) := \Omega^0_0Y \) is the set of real-valued smooth functions over \( J^0Y \). The set of \( C^\infty \)-smooth sections of \( Y \) will be denoted by \( \Gamma(Y) \); its elements \( \gamma : X \rightarrow Y \), \( (x^i) \mapsto (y^\sigma(x^i)) \) will be called fields. The notation \( \mathcal{X}(Y) \) will mean the module of vector fields on \( Y \).

**Horizontal forms. Lagrangians.** A differential form \( \rho \in \Omega^r_kY \) is called \( \pi^r \)-horizontal, or, simply, horizontal, if \( \rho(\Xi_1, \ldots, \Xi_k) = 0 \) whenever one of the vector fields \( \Xi_i \), \( i = 1, \ldots, k \), is \( \pi^r \)-vertical (i.e., \( T\pi^r(\Xi_i) = 0 \)). Any horizontal form on \( \Omega^r_kY \) is locally expressed as:

\[
\rho = \frac{1}{k!} A_{i_1i_2\ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k},
\]

(2)

where the coefficients \( A_{i_1i_2\ldots i_k} \), \( k \leq n \), are smooth functions of the coordinates \( x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1j_2\ldots j_r} \). Similarly, one can introduce \( \pi^{r,s} \)-horizontal forms, \( 0 \leq r < s \leq n \).
s ≤ r; locally, these are generated by exterior products of the differentials $dx^1, dy^α, ..., dy^α_{j_1...j_s}$.

In particular, a horizontal $n$-form $λ ∈ Ω^n(Y)$, where $n = \dim X$, is called a Lagrangian. Locally, a Lagrangian is expressed as:

$$λ = Lω_0, \quad L = L(x^i, y^α, ..., y^α_{i_1...i_r}),$$

where $ω_0 := dx^1 \wedge ... \wedge dx^n$.

Any differential $k$-form $ρ ∈ Ω^k(Y)$ can be transformed into a horizontal $k$-form of order $r + 1$. This is achieved by means of the horizontalization operator, which is the (unique) morphism of exterior algebras $h : Ω^r(Y) → Ω^{r+1}(Y)$ (where $Ω^rY$ means the set of all differential forms of order $s$ over $Y$) such that, for any $f ∈ F(Y)$ and any fibered chart: $hf = f \circ π^{r+1, r}$ and $hdf = df dx^i$. Here,

$$df := \partial_i f + \frac{∂f}{∂y^α} y^α_{j_1...j_r}.$$ 

On the natural basis 1-forms, it acts as:

$$hdx^i := dx^i, \quad hdy^α = y^α_{i_1 dx^i}, ..., hdy^α_{j_1...j_k} = y^α_{j_1...j_k} dx^i, \quad k = 1, r.$$ (4)

Moreover, for any $ρ ∈ Ω^r_k(Y)$, there holds the equality:

$$J^r γ^* ρ = J^{r+1} γ^*(hρ), \quad ∀ γ ∈ Γ(Y).$$ (5)

If two $π^r$-horizontal forms $ρ, θ ∈ Ω^r_k(Y)$ satisfy $J^r γ^* ρ = J^r γ^* θ$ for any section $γ ∈ Γ(Y)$, then $ρ = θ$.

**Contact forms and first canonical decomposition of a differential form on $J^r Y$.** A form $θ ∈ Ω^r_k Y$ is a contact form of order $r$ on $Y$ if it is annihilated by all jets $J^r γ$ of sections $γ ∈ Γ(Y)$; equivalently, $θ$ is a contact form if and only if it belongs to the kernel of $h$.

For instance,

$$ω^σ = dy^α - y^α_{i_1} dx^i, \quad ω^σ_{i_1} = dy^α_{i_1} - y^α_{i_1, i_j} dx^i, ..., \quad ω^σ_{i_1i_2...i_{r−1}} = dy^α_{i_1i_2...i_{r−1}} - y^α_{i_1i_2...i_{r−1}, i_j} dx^i,$$ (6)

represent contact forms on $J^r Y$. These contact forms can be used in order to construct a local basis of the module of 1-forms over $J^r Y$, called the contact basis: $\{dx^1, ω^σ, ..., ω^σ_{i_1i_2...i_{r−1}}, dy^α_{i_1...i_r}\}$.

A $k$-form $θ ∈ Ω^r_k Y$ is $l$-contact ($l ≤ k$) if, corresponding to any fibered chart on $Y$, in the decomposition of the pulled back form $(π^{r+1, r})^* θ ∈ Ω^{r+1}_k Y$ in the contact basis over $J^{r+1} Y$, each term contains exactly $l$ contact 1-forms (6), i.e.,

$$(π^{r+1, r})^* θ = \frac{1}{l!(k−l)!} θ^{A_1...A_l, i_{i_1+1}...i_{i_l}} ω^{A_1} \wedge ... \wedge ω^{A_l} \wedge dx^{i_1+1} \wedge ... \wedge dx^{i_l},$$

where $A ∈ \{σ, (σ^j), ..., (σ_{j_1...j_s})\}$. 

5
An arbitrary $k$-form $\rho \in \Omega^k_Y$ admits a unique decomposition (called the first canonical decomposition):

$$(\pi^{r+1,r})^* \rho = h\rho + p_1\rho + \ldots + p_k\rho,$$  \hspace{1cm} (7)

where the form $p_l\rho$ is $l$-contact, $l = 1, \ldots, k$. The sum $p\rho = p_1\rho + \ldots + p_k\rho$ is the contact component of $\rho$.

A $\pi^{r,0}$-horizontal, 1-contact $(n + 1)$-form $\eta \in \Omega^{r}_{n+1}Y$ is called a source form. Locally, a source form is expressed as:

$$\eta = \eta_\sigma \omega^\sigma \wedge \omega_0.$$  \hspace{1cm} (8)

With respect to coordinate changes on $J^rY$ induced by fibered coordinate changes $x^i = x^i(x')$, $y^\sigma = y^\sigma(x', y^\sigma')$, we have:

$$dx^i = \frac{\partial x^i}{\partial x'^i} dx'^i, \hspace{0.5cm} \omega^\sigma = \frac{\partial y^\sigma}{\partial y'^\sigma} \omega'^\sigma, \hspace{1cm} (9)$$

$$\omega_0 = \text{det}(\frac{\partial x^i}{\partial x'^i}) \omega_0', \hspace{0.5cm} \omega_i = \frac{\partial x^i}{\partial x'^i} \text{det}(\frac{\partial x^j}{\partial x'^j}) \omega'_j, \hspace{1cm} (10)$$

where $\omega_i = 1_{\partial/\partial x^i} \omega_0 = (-1)^{i-1} dx_1 \wedge \ldots dx_{i-1} \wedge dx_i \wedge \ldots \wedge dx_n$ and

$$\eta_{\sigma'} = \frac{\partial y^\sigma}{\partial y'^\sigma} \text{det}(\frac{\partial x^i}{\partial x'^i}) \eta_{\sigma'} \hspace{1cm} (11)$$

for the components of a (globally defined) source form [8] on $Y$.

### 2.2 Lepage equivalents of a Lagrangian and first variation formula

**Lepage equivalents.** Consider a Lagrangian

$$\lambda = \mathcal{L} \omega_0 \in \Omega^r_Y, \hspace{0.5cm} \mathcal{L} = \mathcal{L}(x^i, y^\sigma, \ldots, y^\sigma_{i_1 \ldots i_r}).$$  \hspace{1cm} (12)

The action attached to the Lagrangian (12) and to a compact domain $D \subset X$ is the function $S : \Gamma(Y) \to \mathbb{R}$, given by:

$$S(\gamma) = \int_D J^r \gamma^* \lambda.$$  

The variation $\delta \Xi : \Gamma(Y) \to \mathbb{R}$ of $S$ under the flow of a vector field $\Xi = \xi^i \partial_i + \xi^\sigma \partial_\sigma$ on $Y$ is given, [8], by the Lie derivative $\partial_{J^r \Xi} \lambda$ of $\lambda$ with respect to the prolongation $J^r \Xi$:

$$\delta \Xi S(\gamma) = \int_D J^r \gamma^* \partial_{J^r \Xi} \lambda. \hspace{1cm} (13)$$
A section $\gamma : X \to Y, (x^i) \mapsto y^\sigma(x^i)$ is called a critical section for $S$, if the variation $\delta_S(\gamma)$ vanishes, i.e., if:

$$\int_D J^r \gamma^* (\partial_J r \Xi \lambda) = 0.$$  \hspace{1cm} (14)

A Lepage equivalent of a Lagrangian $\lambda \in \Omega^r_n Y$ is an $n$-form $\theta_\lambda$ on some jet prolongation $J^s Y$, with the following properties:

1. $\theta_\lambda$ defines the same variational problem as $\lambda$, i.e.:
   $$\pi_q \# (\pi_{q,s+1} \# h \theta_\lambda) = \pi_{q,r} \# \lambda,$$  \hspace{1cm} (15)

   where $q := \max(s + 1, r)$;

2. the first contact component $p_1 d\theta_\lambda$ is a source form (i.e., it is generated by $\omega^\sigma$ alone).

Every Lagrangian $\lambda \in \Omega^r_n Y$ admits Lepage equivalents $\theta_\lambda$. Corresponding to any local chart, these Lepage equivalents are expressed, [8], as:

$$\theta_\lambda = \Theta_\lambda + d\eta + \mu,$$  \hspace{1cm} (16)

where $\eta$ is a contact form, $\mu$ is at least 2-contact and $\Theta_\lambda$ is of order $s \leq 2r - 1$.

The notion of Lepage equivalent generalizes the idea of Poincaré-Cartan form from mechanics and allows one to obtain a coordinate-free description of both Euler-Lagrange equations and Noether theorem, as follows.

**First variation formula.** Let $\lambda \in \Omega^r_n Y$ be a Lagrangian, $\theta_\lambda \in \Omega^s_n Y$, an arbitrary Lepage equivalent of $\lambda$, of some order $s$ and $\Xi \in \mathcal{X}(Y)$, a projectable vector field. Then, there holds the first variation formula, [8],

$$J^r \gamma^* (\partial_J r \Xi \lambda) = J^{s+1} \gamma^* i_{J^{s+1} \Xi} (p_1 d\theta_\lambda) + d(J^{s+1} \gamma^* i_{J^s \Xi} \theta_\lambda).$$  \hspace{1cm} (17)

The terms in the right hand side of (17) have the following meaning:

i) the source form $p_1 d\theta_\lambda$ is the Euler-Lagrange form attached to $\lambda$:

$$p_1 d\theta_\lambda = E(\lambda) \in \Omega^{s+1+1}_{n+1}(Y),$$  \hspace{1cm} (18)

locally given by

$$E(\lambda) = E_\sigma(\lambda) \omega^\sigma \wedge \omega_0,$$

$$E_\sigma(\lambda) = \frac{\delta L}{\delta y^\sigma} = \frac{\partial L}{\partial y^\sigma} - d_i \frac{\partial L}{\partial y_i^\sigma} + \cdots + (-1)^r d_{i_1} \cdots d_{i_r} \frac{\partial L}{\partial y_{i_1 \cdots i_r}}.$$

On-shell, i.e., along critical sections $\gamma$, the functions $E_\sigma(\lambda) \circ J^{s+1} \gamma$ vanish. Denoting equality on-shell by $\approx$, this is written as:

$$E_\sigma(\lambda) \circ J^{s+1} \gamma \approx 0.$$
The Euler-Lagrange form $E(\lambda)$ - though expressed in terms of a Lepage equivalent $\theta^\lambda$ - does not depend on the choice of $\theta^\lambda$.

ii) Let us denote:

$$J^\Xi := -h J^r \gamma^* (\partial J^r \Xi) \in \Omega^{s+1}_n Y$$

(in local writing, this is: $J^\Xi = J^i \omega_i$, $J^i = J^i (x^k, y^\sigma, ..., y^\sigma_{i_1 ... i_r})$).

Integrating on a compact domain $D \subset X$, the first variation formula now reads:

$$\int_D J^{s+1} \gamma^* (\partial J^{s+1} \Xi) = - \int_D J^{s+1} \gamma^* J^\Xi. \quad (20)$$

The vector field $\Xi \in \mathcal{X}(Y)$ is said to be a symmetry generator for $\lambda$ if $\lambda$ is invariant under the 1-parameter group of $J^r \Xi$; this is equivalent to:

$$\partial J^r \Xi = 0. \quad (21)$$

If $\Xi$ is a symmetry generator for $\lambda$, then $J^\Xi$ acquires the meaning of Noether current.

Noether’s first theorem states that, if $\Xi \in \mathcal{X}(Y)$ is a symmetry generator for $\lambda$, then, for any Lepage equivalent $\theta^\lambda$, there holds:

$$J^{s+1} \gamma^* dJ^\Xi \approx 0. \quad (22)$$

2.3 Diffeomorphisms and diffeomorphism invariance

The notion of energy-momentum tensor is tightly related to the invariance of the matter Lagrangian to the group of spacetime diffeomorphisms $Diff(X)$. But, in order to rigorously define this invariance, some preliminary discussions are needed.

A diffeomorphism $\Phi : Y \to Y$ is called an automorphism of $Y$ if there exists a mapping $\varphi \in Diff(X)$ such that

$$\pi \circ \Phi = \varphi \circ \pi; \quad (23)$$

if relation (23) holds, then the automorphism $\Phi$ is said to cover $\varphi$. An automorphism of $Y$ is called strict if it covers the identity of $X$. We will denote in the following by $Aut(Y)$ and $Aut_s(Y)$ the sets of automorphisms of $Y$ and, respectively, strict automorphisms of $Y$.

Passing to infinitesimal generators, any generator $\Xi$ of an automorphism of $Y$ is a $\pi$-projectable vector field; in any fibered chart, it is expressed as:

$$\Xi = \xi^i (x) \partial_i + \Xi^\sigma (x,y) \partial_\sigma. \quad (24)$$

Strict automorphisms are generated by vertical vector fields $\Xi = \Xi^\sigma (x,y) \partial_\sigma$.

As stated above, we are interested in the way that diffeomorphisms of $X$ affect a Lagrangian $\lambda \in \Omega^*_n(Y)$, $\lambda = \mathcal{L}(x^i, y^\sigma, ..., y^\sigma_{i_1 ... i_r}) \omega_0$. But diffeomorphisms of $X$ do not act on the field variables - at least, not directly; more rigorously stated, $Diff(X)$ is not a subgroup of $Aut(Y)$ (it is rather a quotient group, $Diff(X) \simeq Aut(Y)/Aut_s(Y)$).
Consequently, in order to be able to speak about diffeomorphism invariance of the Lagrangian, we need embeddings or liftings \( \text{Diff}(X) \to \text{Aut}(Y) \).

In the following, we will assume that there exists a canonical lifting, given by a group morphism

\[
\text{Diff}(X) \to \text{Aut}(Y), \quad \varphi \mapsto \Phi,
\]

such that, for any \( \varphi \in \text{Diff}(X) : \pi \circ \Phi = \varphi \).

Passing to the infinitesimal level, the canonical lifting gives rise to an \( \mathbb{R} \)-linear module monomorphism

\[
l : \mathcal{X}(X) \to \mathcal{X}_p(Y), \quad \xi \mapsto \Xi,
\]

such that \( \pi^*|\mathcal{X}_p(Y) \circ l = \text{id}_{\mathcal{X}(X)} \). In local writing, this is given by

\[
\xi = \xi^i(x^j) \frac{\partial}{\partial x^i} \mapsto \Xi = \xi^i(x^j) \frac{\partial}{\partial x^i} + \Xi^\sigma(x^j, y^\mu) \frac{\partial}{\partial y^\sigma} \in \mathcal{X}_p(Y).
\]

The lifting \( l \) is required to have the property that \( \pi(\text{supp}(l(\Xi))) \subset \text{supp}(\xi) \), where \( \text{supp}(f) \) denotes the support of a mapping \( f \) (defined as the closure of the set \( f^{-1}(0) \)). As a consequence, \( l \), \( \xi \) in any fibered chart, the local components \( \Xi^\sigma \) are expressible as linear combinations of a finite number (say, \( k \)) of partial derivatives of the components \( \xi^i \):

\[
\Xi^\sigma = C^\sigma_i \xi^i + C^\sigma_i^j \xi^j + \ldots + C^\sigma_i^{j_1 \ldots j_k} \xi^{j_1 \ldots j_k},
\]

where \( C^\sigma_i, C^\sigma_i^j, \ldots, C^\sigma_i^{j_1 \ldots j_k} \) are functions of \( (x^k, y^\mu) \) only. The number \( k \in \mathbb{N} \) is called the differential index (or, simply, the index) of the lifting.

**Particular case.** Assume that the index of the lifting \( l \) is 1. A direct calculation shows that, with respect to fibered coordinate changes \( x^i = x^i(x^i') \), \( y^\sigma = y^\sigma(x^i', y^\sigma') \), the top degree coefficients \( C^\sigma_i = C^\sigma_i(x^k, y^\mu) \) transform by the rule:

\[
C^\sigma_i' = \frac{\partial y^\sigma'}{\partial y^\sigma} \frac{\partial x^i'}{\partial x^i} \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^i'} \frac{\partial y^\sigma}{\partial y^\sigma'},
\]

i.e., they define a section\(^4\) of the tensor product \( TY \otimes TX \otimes T^*X \).

**Example.** A typical example of a canonical lifting of (pure) index 1 is obtained if \( Y = T^{p,q}(X) \) is the bundle of tensors of type \( (p, q) \) over \( X \). In this case, any diffeomorphism \( \varphi \in \text{Diff}(X) \) is canonically lifted into an automorphism of

\(^3\)Such liftings canonically exist, for instance, if \( Y \to X \) is a natural bundle, i.e., if \( Y \) is obtained from \( X \) by applying a covariant functor to the whole category of differentiable manifolds.

\(^4\)This section represents the so-called principal symbol of the differential operator \( \nu \circ l : \mathcal{X}(X) \to \mathcal{X}(Y) \) providing, for any \( \xi \in \mathcal{X}(X) \), the \( \pi \)-vertical component of the lift \( l(\xi) \).
\( T^{p,q}(X) \) by pushforward/pullback. Denoting the fibered coordinates on \( T^{p,q}(X) \) by \( (x^i, y_{j_1 \ldots j_p}^1) \), the corresponding mapping \( l : \mathcal{X}(X) \to \mathcal{X}(Y) \) is given, [4], by relation (24), with.

\[
\Xi_{j_1 \ldots j_q}^{i_1 \ldots i_p} = \xi_{j_1}^{i_1} y_{j_2 \ldots j_q}^{i_2 \ldots i_p} + \ldots + \xi_{j_q}^{i_q} y_{j_1 \ldots j_{q-1}}^{i_1 \ldots i_{p-1} h} - \xi_{j_1}^{h} y_{j_2 \ldots j_q}^{i_1 \ldots i_p} - \xi_{j_q}^{i_q} y_{j_1 \ldots j_{q-1}}^{i_1 \ldots i_{p-1} h}.
\] (29)

A differential form \( \rho \in \Omega^{r,k} Y \) is said to be \textit{diffeomorphism invariant} (or \textit{generally covariant}) if, for any \( \varphi \in Diff(X) \),

\[
J^r \Phi^* \rho = \rho,
\] (30)

where \( \Phi \) denotes the \( \varphi \) through the canonical lifting \( Diff(X) \to Aut(Y) \).

If a differential form \( \rho \in \Omega^{r,k} Y \) is diffeomorphism invariant, then the lift \( \Xi \) of any vector field \( \xi \in X(X) \) is a symmetry generator for \( \lambda \).

### 3 The energy-momentum tensor

Assume, in the following, as in [4], [5], that the configuration manifold \( Y \) is a fibered product

\[ Y = Y^{(b)} \times_X Y^{(m)}, \]

with local coordinates \((x^i, y^A, y^\sigma)\). The coordinates \( y^A \) will be called \textit{background variables} and \( y^\sigma \), \textit{matter variables}. Consider, on \( Y \) a Lagrangian of order \( r \):

\[
\lambda = \lambda_b + \lambda_m, \tag{31}
\]

where the \textit{background Lagrangian} \( \lambda_b \) depends only on \( y^A, y^A_{i_1 \ldots i_r}, \ldots \) and the \textit{matter Lagrangian}

\[
\lambda_m = \mathcal{L}_m(x^i, y^A, \ldots, y^A_{i_1 \ldots i_r}, y^\sigma, y^\sigma_1, \ldots, y^\sigma_{i_1 \ldots i_r}) \omega_0.
\]

may depend on all the coordinates on \( J^r Y \).

We will make the following assumptions:

**A1.** The lift \( \lambda(X) \to \mathcal{X}(Y) \), \( \xi \mapsto \Xi := \xi^i(x^j) \partial_i + \Xi^A(x^j, y^\mu) \partial_A + \Xi^\sigma(x^j, y^\mu) \partial_\sigma \), is of index at most 1 in the background variables \( y^A \), i.e., corresponding to any fibered chart on \( Y \):

\[
\Xi^A = C^A_{i} \xi^i + C^A_{ij} \xi^j. \tag{32}
\]

**A2.** The matter Lagrangian \( \lambda_m \) is generally covariant.

\textit{Note.} There is no restriction upon the index of the lifting \( l \) in the matter variables \( y^\sigma \).

The hypothesis A2 implies that, for any vector field \( \xi \in \mathcal{X}(X) \), \( \partial_J \Xi = \lambda_m = 0 \). Let \( \theta_{\lambda_m} \) be an arbitrary Lepage equivalent (of order \( s \)) of \( \lambda_m \). Using the notations in [18], [19], the first variation formula reads:
for any section \( \gamma := (\gamma^{(b)}, \gamma^{(m)}) : X \rightarrow Y \), locally represented as \( \gamma(x^i) = (x^i, y^A(x^i), y^\sigma(x^i)) \).

The Euler-Lagrange form \( E(\lambda_m) = p_1 d \theta_{\lambda_m} \) splits into background and matter components as:
\[
E(\lambda_m) = E^{(b)}(\lambda_m) + E^{(m)}(\lambda_m),
\]
where:
\[
E^{(b)}(\lambda_m) = p_1^{(b)} d \theta_{\lambda_m} = \frac{\delta L_m}{\delta y^A} \omega^A \land \omega_0,
E^{(m)}(\lambda_m) = p_1^{(m)} d \theta_{\lambda_m} = \frac{\delta L_m}{\delta y^\sigma} \omega^\sigma \land \omega_0.
\]

The background component \( E^{(b)}(\lambda_m) \) in (34) will be used as a "raw material" for the energy-momentum tensor.

**Definition 1** The energy-momentum source form\(^5\) of \( \lambda_m \) is the Euler-Lagrange form of \( \lambda_m \) with respect to the background variables: \( \tau := E^{(b)}(\lambda_m) \).

In local writing,
\[
\tau = \tau_A \omega^A \land \omega_0 \in \Omega_{n+1}^{s+1}(Y),
\]
where:
\[
\tau_A = \frac{\delta L_m}{\delta y^A}.
\]

**Remark.** In [8], p. 118, it is proven the following result. For any automorphism \( \Phi \in Aut(Y) \) and any Lagrangian \( \lambda \in \Omega^r_n Y \),
\[
J^{s+1} \Phi^* E(\lambda) = E(J^r \Phi^* \lambda);
\]
in particular, if \( \lambda \) is \( \Phi \)-invariant, then so is its Euler-Lagrange form \( E(\lambda) \). Using this result, we find out that, if the Lagrangian \( \lambda_m \in \Omega^r_n Y \) is \( \Phi \)-invariant, then \( \tau \) is also \( \Phi \)-invariant.

Taking into account the splitting (34), the first variation formula (33) becomes:
\[
0 = J^{s+1} \gamma^* [i_{J^{s+1} \Xi} E^{(m)}(\lambda_m) + i_{J^{s+1} \Xi} \tau] - J^{s+1} \gamma^* dJ^{\Xi}.
\]
On-shell for the matter variables \( y^\sigma \), i.e., when the matter component \( \gamma^{(m)} \) of the section \( \gamma = (\gamma^{(b)}, \gamma^{(m)}) : X \rightarrow Y, (x^i) \rightarrow (y^A(x^i), y^\sigma(x^i)) \) is critical for \( \lambda_m \), we obtain:
\[
J^{s+1} \gamma^* (i_{J^{s+1} \Xi} \tau) - J^{s+1} \gamma^* dJ^{\Xi} \approx_{y^\sigma} 0,
\]
or, integrating on a compact domain \( D \subset X \) (and applying the horizontalization operator):
\[
\int_D J^{s+2} \gamma^* (h_{J^{s+1} \Xi} \tau) - \int_{\partial D} J^{s+1} \gamma^* dJ^{\Xi} \approx_{y^\sigma} 0.
\]

Let us investigate more closely the volume term in the above integral.

\(^5\)In [8], the source form \( \tau \) is called the energy-momentum tensor. Still, we will prefer here a slightly different terminology, for reasons which will be clarified below.
Lemma 2 If the embedding $l : \mathcal{X}(X) \to \mathcal{X}(Y)$, $\xi \mapsto \Xi$, is of index at most 1 in the background variables $y^A$, then there uniquely exist the $F(X)$-linear mappings $B : \mathcal{X}(X) \to \Omega^{s+2}_n Y$ and $\mathcal{T} : \mathcal{X}(X) \to \Omega^{s+1}_{n-1} Y$ with $\pi^r$-horizontal values, satisfying:

$$h_{iJ^{s+1}} \Xi = B(\xi) + h d(\mathcal{T}(\xi)), \quad \forall \xi \in \mathcal{X}(X).$$  \hfill (40)

**Proof.** We first define $B$ and $\mathcal{T}$ locally. Take an arbitrary fibered chart $(V^{s+2}, \psi^{s+2})$ on $J^{s+2} Y$, with $U = \pi^{s+2}(V^{s+2})$. In this chart, $h_{iJ^{s+1}} \Xi$ is expressed as:

$$h_{iJ^{s+1}} \Xi = (\tilde{\Xi}^A \tau_A) \omega_0, \quad \tilde{\Xi}^A = \Xi^A - y^A \xi^i.$$

With $\Xi^A$ from \eqref{32}, we find

$$h_{iJ^{s+1}} \Xi = \{(C^A_i - y^A_i) \tau_A \xi^i + C^A_{ij} \tau_A \xi^i \xi^j\} \omega_0$$

$$= \{(C^A_i - y^A_i) \tau_A - d_j (C^A_{ij} \tau_A) \xi^i + d_j (C^A_{ij} \tau_A \xi^i)\} \omega_0.$$

Denoting

$$B(\xi) = (C^A_i - y^A_i) \tau_A \xi^i \omega_0, \quad \mathcal{T}(\xi) = (C^A_{ij} \tau_A \xi^i \xi^j) \omega_0,$$

we obtain two linear mappings $B : \mathcal{X}(U) \to \Omega^{s+2}_n Y$ and $\mathcal{T} : \mathcal{X}(U) \to \Omega^{s+1}_{n-1} (Y)$, having horizontal values and obeying $\eqref{10}$.

The uniqueness of the (momentarily, locally defined) mappings $B$ and $\mathcal{T}$ can be established as follows. Assume that the mappings $\tilde{B} : \mathcal{X}(U) \to \Omega^{s+2}_n Y$, $\tilde{\mathcal{T}} : \mathcal{X}(U) \to \Omega^{s+1}_{n-1} (Y)$ also obey the above properties. Then, for any $\xi \in \mathcal{X}(X)$, we have:

$$0 = (B - \tilde{B})(\xi) + h d(\mathcal{T} - \tilde{\mathcal{T}})(\xi).$$

As all, these mappings have horizontal values, they can be expressed as: $B(\xi) = B_i \xi^i \omega_0$, $\tilde{B}(\xi) = \tilde{B}_i \xi^i \omega_0$, $\mathcal{T}(\xi) = \mathcal{T}_i^j \xi^i \omega_j$, $\tilde{\mathcal{T}}(\xi) = \tilde{\mathcal{T}}_i^j \xi^i \omega_j$. Substituting into the above equality,

$$0 = [(B_i - \tilde{B}_i) + d_j (\mathcal{T}_i^j - \tilde{\mathcal{T}}_i^j)] \xi^i + (\mathcal{T}_i^j - \tilde{\mathcal{T}}_i^j) \xi^i.$$  \hfill (44)

As this relation holds for any $\xi$, we obtain $\mathcal{T}_i^j - \tilde{\mathcal{T}}_i^j = 0$ and $B_i - \tilde{B}_i = 0$. Therefore, $B = \tilde{B}$ and $\mathcal{T} = \tilde{\mathcal{T}}$.

Now, take two fibered chart domains $V^{s+2}, V^{s+2'} \subset J^{s+2} Y$, with $U = \pi^{s+2}(V^{s+2}), U' = \pi^{s+2}(V^{s+2'})$ and an arbitrary vector field $\xi \in \mathcal{X}(U \cap U')$. We denote by $\mathcal{T}$ and $\mathcal{T}'$ the mappings corresponding by $\eqref{44}$ to the two domains. Taking into account the rules of transformation $\eqref{11}$, $\eqref{28}$, $\eqref{10}$ of $\tau_A$, $C^A_{ij}$ and $\omega_i$, a brief calculation shows that:

$$\mathcal{T}(\xi) = (C^A_{ij} \tau_A \xi^i) \omega_j = (C^A_{ij} \tau_A \xi^i) \omega_{j'} = \mathcal{T}'(\xi)$$

i.e., the mapping $\mathcal{T}$ can be defined globally on $\mathcal{X}(X)$. As a consequence of $\eqref{40}$, $B$ is also globally well defined. \qed

Therefore, it makes sense
Definition 3 The energy-momentum tensor of $\lambda_m$ is the mapping:

$$T : \mathcal{X}(X) \to \Omega_{n-1}^{s+1}(Y), \quad \xi \mapsto T(\xi), \quad (45)$$

uniquely defined by the splitting

$$h_{i'j'k'\xi} = \mathcal{B}(\xi) + h(\mathcal{T}(\xi)), \quad (46)$$

where the mappings $T : \mathcal{X}(X) \to \Omega_{n-1}^{s+1}(Y), \xi \mapsto T(\xi)$ and $\mathcal{B} : \mathcal{X}(X) \to \Omega_{n-2}^{s+1}(Y), \xi \mapsto \mathcal{B}(\xi)$ are $\mathcal{F}(X)$-linear and have $\pi^r$-horizontal values.

In the following, we will call the mapping $\mathcal{B}$ defined by (46), the balance function.

Corollary 4 In local writing, the energy-momentum tensor is given by:

$$\xi^i \partial_i \mapsto T(\xi) = T_{ij}^i \xi_j \omega_j, \quad (47)$$

where

$$T_{ij}^i = C_{i}^{A_j} \tau_A = C_{i}^{A_j} \frac{\delta L_m}{\delta y_A}. \quad (48)$$

With respect to fibered coordinate changes on $J^{s+1}Y$, the functions $T_{ij}^i$ obey the rule:

$$T_{ij}^i = \frac{\partial x^j}{\partial x'^j} \frac{\partial x'^i}{\partial x^i} \det \left( \frac{\partial x'^k}{\partial x^k} \right) T'_{ij}^i. \quad (49)$$

Remarks.

1) The local expression (48) of $T$ coincides, up to a minus sign (and a pullback by sections of $J^{s+1}Y$), to the one found by Gotay and Marsden, [6], as a result of a different, Noether-type construction. The reason for our choice of the sign in (40) (i.e., also in (48)) is that it gives, in the case of purely metric backgrounds, the usual Hilbert energy-momentum tensor, with a correct sign.

2) Here, the energy-momentum tensor $T$ is regarded as a geometric object on the jet bundle $J^{s+1}Y$ and not on the base manifold $X$. For each section $\gamma \in \Gamma(Y)$, a corresponding linear mapping $T(\gamma) : \mathcal{X}(X) \to \Omega_{n-1}(X)$ can be obtained from $T$ by pullback: $T(\gamma)(\xi) := J^{s+1}Y \mathcal{T}(\xi), \forall \xi \in \mathcal{X}(X)$. This way, the energy-momentum tensor "on $X" T(\gamma)" is regarded as an element of $\Omega_1(X) \otimes \Omega_{n-1}(X)$.

3) The rule of transformation (49) of its coefficients is due to the fact that $T$ is expressed in the non-invariant local basis $dx^i \otimes \omega_j$. If $\omega_j$ are replaced with an invariant basis for horizontal $n-1$ forms, then the corresponding local components of $T$ will obey a tensor-type transformation rule.

4) Equivalent Lagrangians give rise to the same energy-momentum tensor $T$; this is a consequence of the fact that $T$ is a combination of Euler-Lagrange expressions of $\lambda_m$.

5) For energy-momentum source forms $\tau$ whose order in the background variables $y^A$ does not exceed 1, a complete characterization (as a polynomial in these variables) is available, [9], [15]. This gives us immediately a general characterization of the corresponding energy-momentum tensors $T$. 

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4 Properties of the energy-momentum tensor

Using Definition 3 we will prove:

**Theorem 5**

(i) *(The energy-momentum balance law): In any fibered chart of \( J^{s+2}Y \), there hold the relations:

\[
\left( d_i J^j_i - (C^A_i - y_i^A)\tau_A \right) \circ J^{s+2} \gamma \approx y^s 0, \quad i = 1, ..., n.
\]  

(50)

where \( \approx y^s \) means equality on-shell for the matter component \( \gamma_m : (x^i) \mapsto (y^\sigma(x^i)) \) of the section \( \gamma = (\gamma^{(b)}, \gamma^{(m)}) : X \rightarrow Y \).

(ii) For any vector field \( \xi \in \mathcal{X}(X) \) and any compact domain \( D \subset X \):

\[
\int_{\partial D} J^{s+1} \gamma^* \mathcal{T}(\xi) \approx y^s \int_{\partial D} J^{s+1} \gamma^* \mathcal{J}(l(\xi)),
\]  

(51)

where \( l : \mathcal{X}(X) \rightarrow \mathcal{X}(Y) \) denotes the canonical lift.

**Proof.** Take an arbitrary vector field \( \xi \in \mathcal{X}(X) \) and denote \( \Xi := l(\xi) \). Using the splitting (46) together with the integral first variation formula (39), we get:

\[
0 \approx y^s \int_D J^{s+2} \gamma^* \mathcal{B}(\xi) + \int_{\partial D} J^{s+1} \gamma^*(\mathcal{T}(\xi) - \mathcal{J}(\Xi)).
\]  

(52)

Equation (52) holds, in particular, for any vector field \( \xi \) with support contained in \( D \), i.e., \( \xi|_{\partial D} = 0 \). It follows that

\[
\int_D J^{s+2} \gamma^* \mathcal{B}(\xi) \approx y^s 0;
\]  

(53)

using the local writing (43) of the balance function \( \mathcal{B} \), this gives (i). Then, choosing \( \xi \) with \( \xi|_{\partial D} \neq 0 \), and using (i), together with (52), we find (ii). ■

**Remarks.**

1) Property (ii) tells us that \( \mathcal{T}(\xi) \) coincides on-shell with the "improved Noether current" given by the diffeomorphism invariance of \( \lambda_m \).

2) The order of the Lagrangian \( \lambda_m \) either in the background variables or in the matter ones was completely irrelevant in deducing the above results. That is, they are valid also in the case of non-minimal coupling between the background and the matter variables.

3) Since it is defined as a linear combination of Euler-Lagrange expressions \( \tau_A \), the energy-momentum tensor \( \mathcal{T} \) does not depend on the choice of the Lepage equivalent \( \theta \lambda_m \) (on the contrary, Noether currents do depend on the choice of the Lepage equivalent).

4) If we apply the above construction to the background Lagrangian \( \lambda_b \), which only depends on the background variables, then the corresponding energy-momentum balance relations (50) will be identically satisfied for any section \( \gamma \in \Gamma(Y) \); as we will in the next section, in the particular case of general relativity, these are actually, the contracted Bianchi identities.
Theorem 6 (Gauge invariance of $T$): If a strict automorphism $\Phi \in \text{Aut}_s(Y(b) \times X Y(m))$ is a symmetry of $\lambda$ acting trivially on the background manifold $Y(b)$, then:

$$J^{s+1} \Phi^* T(\xi) = T(\xi), \forall \xi \in \mathcal{X}(X).$$

(54)

Proof. According to the hypothesis, $\Phi = (id_X, id_{Y(b)}, \Phi^m)$; locally, $\Phi : (x^i, y^\sigma, y^A) \mapsto (x^i, f^\sigma(x^i, y^p), y^A)$.

Since $\Phi$ is a symmetry of $\lambda$, we deduce that $J^{s+1} \Phi^* \tau = \tau$. Moreover, a brief calculation using the local expression of $\Phi$ shows that:

$$J^{s+2} \Phi^* (h_i J^{s+1} \Xi \tau) = h_i J^{s+1} \Xi \tau.$$ Further, using the uniqueness of the decomposition (46) of $h_i J^{s+1} \Xi \tau$, we find that

$$J^{s+2} \Phi^* B(\xi) = B(\xi), \quad J^{s+1} \Phi^* T(\xi) = T(\xi);$$

the latter equality proves the statement. ■

5 The case of metric and tensor backgrounds

We will study, in the following, the case when the background variables consist of a metric and (optionally), some other tensor quantity. In this case, it is useful to write the energy-momentum balance law (50) in a manifestly covariant form, using Levi-Civita covariant derivatives.

Denoting by $\text{Met}(X)$ the bundle of metrics, defined as the set of all symmetric nondegenerate tensors of type (2,0) on $X$, the background manifold becomes:

$$Y(b) = \text{Met}(X) \times X T^{p,q}(X)$$

and, accordingly, $Y = \text{Met}(X) \times X T^{p,q}(X) \times X Y(m)$. With the notations in the previous sections, the background variables are $y^A \in \{g^{jk}, y_1^{i_1...i_p}, g^{ijk...i_r}, g_{i_1...i_r}^{j_1...j_p}\}$ (alternatively, one can regard $\text{Met}(X)$ as a set of tensors of type (0,2) and use $g^{jk}$ as coordinates). In the following, $dV_g = \sqrt{|\det g|} \omega_0$ will mean the Riemannian volume form on $X$.

Assume that the Lagrangian $\lambda$ is natural, i.e.,

$$\lambda_m = L_m dV_g,$$

where $L_m = L_m(x^i, y^\sigma, g^{jk}, y_1^{i_1...i_p}, ...y_1^{i_1...i_r}, g^{ijk...i_r})$ is a differential invariant (also commonly called in the literature, a scalar); with the notations in the above sections, we have: $\lambda_m = \mathcal{L}_m \omega_0$, where:

$$\mathcal{L}_m = L_m \sqrt{|\det g|}.$$ (55)

As any such Lagrangian is generally covariant, we can apply the above scheme.

The energy-momentum source form $\tau$ can be expressed as:

$$\tau = \tau_A \omega^A \wedge \omega_0 =: \mathcal{I}_A \omega^A \wedge dV_g,$$

(56)
where
\[
\mathcal{F}_A := \frac{\tau_A}{\sqrt{|\det g|}} = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta y^A};
\]  
(57)

Accordingly, the energy-momentum tensor \( T : \mathcal{X}(X) \rightarrow \Omega^{n+1}_n(Y), \xi \mapsto \mathcal{T}(\xi) = T^j_i \xi^i \omega_j \) can be written, in any fibered chart, as:
\[
\mathcal{T}(\xi) =: T^j_i \xi^i \sqrt{|\det g|} \omega_j,
\]

with:
\[
T^j_i = C^{\mathcal{A}j}_i \mathcal{F}_A = \frac{1}{\sqrt{|\det g|}} T^j_i.
\]  
(58)

Using (49), we can see that, with respect to fibered coordinate changes, \( T^j_i \) obey a tensor-type transformation rule: \( T^j_i' = \frac{\partial x'^j}{\partial x^i} \frac{\partial x^i}{\partial y^A} T^j_i \).

The canonical lifting
\[
l^{(b)} : \mathcal{X}(X) \rightarrow \mathcal{X}(Y^{(b)}), \quad \xi^i \frac{\partial}{\partial x^i} \mapsto \xi^i \frac{\partial}{\partial x^i} + (C^{\mathcal{A}j}_i \xi^i + C^{\mathcal{A}j}_i) \frac{\partial}{\partial y^A}
\]
is given by (20), i.e., \( C^{\mathcal{A}j}_i = 0 \) and \( C^{\mathcal{A}j}_i \in \{ C^{(jk)^h}_i, C^{(k_1...k_q)_i} \} \) are as follows:
\[
C^{(jk)^h}_i = \delta^h_i g^{jk} + \delta^h_i \xi^h, \quad C^{(k_1...k_p)_i}_{(j_1...j_q)i} = \delta^i_{j_1} y^{j_2...j_q}_{k_2...k_p} + ... + \delta^i_{j_1} y^{j_2...j_q}_{k_2...k_p} - \delta^j_{j_1} y^{j_2...j_q}_{k_1...k_p} - \delta^j_{j_1} y^{j_2...j_q}_{k_1...k_p}. \]  
(59)

The energy-momentum tensor components (60) are:
\[
T^j_i = 2g^{jh} \mathcal{F}_{hi} + (y^{j_1...j_p}_{k_1...k_p} \mathcal{F}^{j_1...j_p}_{k_1...k_p} + ... - y^{j_2...j_q}_{k_1...k_p} \mathcal{F}^{j_2...j_q}_{k_1...k_p}). \]  
(61)

A direct calculation using the relation \( d_j \sqrt{|\det g|} = \Gamma^i_{ji} \sqrt{|\det g|} \) (where \( \Gamma^i_{jk} \) denote the formal Christoffel symbols of \( g \)) shows that the energy-momentum balance law (57) can be written in terms of formal Levi-Civita covariant derivatives :i, as:
\[
\xi^i [y^{A}_{;i} \mathcal{F}_A + T^j_i] \approx 0.
\]

Taking into account that \( g^{jk}_i \) is 0, in the above relation, the Met(X) part of the expression \( y^{A}_{;i} \mathcal{F}_A \) vanishes, i.e., \( y^{A}_{;i} \mathcal{F}_A = y^{i_1...i_p}_{j_1...j_q} \mathcal{F}^{i_1...i_p}_{j_1...j_q} \). In other words:

**Theorem 7** (manifestly covariant version of energy-momentum balance law):
If the background manifold is \( Y^{(b)} = Met(X) \times X^p T^p q(X) \), then, for any natural

\[\text{Here, formal means the fact that the Christoffel symbols } \Gamma^i_{jk} \text{ (defined by the usual formula) are regarded as functions on } J^1 Met(X); \text{ only when evaluated on a section } (x^i) \mapsto (g^{jk}(x^i)), \text{ they become the usual Christoffel symbols on } X.\]
matter Lagrangian $\lambda_m = \mathcal{L}_m \omega_0 \in \Omega^r_n(Y^{(b)} \times_X Y^{(m)})$ and for any section $\gamma : X \rightarrow Y$:

$$(y^A, \xi_A + T^i_{ij}) \circ J^{s+2} \gamma \approx y^r, \quad i = 1, ..., n,$$

(62)

where: $\xi_A = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta y^A}$, $A = \left(\begin{array}{c} i_1, ..., i_p \\ j_1, ..., j_q \end{array}\right)$, semicolons denote Levi-Civita covariant derivatives and $\approx_{y^r}$ means equality on-shell for the matter variables $y^r$.

Let us investigate, in the following, two particular cases.

5.1 Purely metric theories

Assume that the only background variable is a metric, i.e., $Y^{(b)} = \text{Met}(X)$, $y^A = g^{ij}$. In this case, the energy-momentum source form is

$$\tau = \mathcal{T}_{hl} \omega^{hl} \wedge dV_g, \quad \mathcal{T}_{hl} = \frac{1}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta g^{hl}}. $$

The energy-momentum tensor $T : \mathcal{X}(X) \rightarrow \Omega^{s+1}_{n-1}(Y)$ is locally given by:

$$T^j_i = C^{(hl)}_{ij} \xi_{hl} = 2g^{hl} \xi_{hi}. $$

(63)

Lowering indices by $g$, we get:

**Proposition 8** If the only background variable is a metric tensor $g^{ij}$, then the energy-momentum tensor $\mathcal{T}$ is given by

$$T_{ij} = \frac{2}{\sqrt{|\det g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}}. $$

The energy-momentum balance law (62) reads:

$$T^j_i \circ J^{s+2} \gamma \approx y^r, \quad 0.$$ 

(64)

Applying the same algorithm to the Hilbert Lagrangian $\lambda_g = RdV_g$, the corresponding "energy-momentum tensor" is the Einstein tensor $\mathcal{E} = E_{ij} \sqrt{|\det g|} dx^i \otimes \omega_j$; the covariant conservation law (64) gives the contracted Bianchi identities:

$$E^i_{ij} \circ J^{s+2} \gamma = 0.$$ 

5.2 Metric-affine theories

In a metric-affine theory, the background variables are, a priori, a metric and a connection. Hence, at first sight, the background manifold is $Y^{(b)} = \text{Met}(X) \times_X \text{Conn}(X)$ - and the canonical lift $l$ is, in this case, of index 2, i.e., we cannot apply it the above considerations. But this problem can be overcome if we consider, instead of connections, distortion tensors:

$$N^i_{jk} = K^i_{jk} - \Gamma^i_{jk},$$

(65)
giving the difference between the connection (with coefficients $K^i_{jk}$) of the theory and the Levi-Civita connection of the metric. That is, we can consider as our background manifold

$$Y^{(b)} = \text{Met}(X) \times_X T^{1,2}(X),$$

with fibered coordinates $(x^i, g^{ij}, N^i_{jk})$. Since both factors are tensor spaces, we can use the canonical lift \(\lambda\), of index 1 in the background variables. The corresponding coefficients $C^A_j^i \in \{C^{(h)}_i^j, C^{m}_j^i\}$ are:

$$C^{(h)}_i^j = \delta^i_l g^{hj} + \delta^h_i g^{lj}, \quad C^{m}_j^i = \delta^m_i N^j_{hl} - \delta^j_i N^m_{il} - \delta^j_i N^m_{hl}.$$

The energy-momentum tensor components $T^j_i := C^A_j^i \Sigma_A$ are obtained as:

$$T^j_i = 2\Sigma^j_i + (\Sigma^h_i N^j_{hl} - \Sigma^i_m N^m_{il} - \Sigma^h_j N^m_{hl}),$$

where:

$$\Sigma^j_i = \frac{1}{\sqrt{|\det g|}} \frac{\delta L_m}{\delta g^{ij}}, \quad \Sigma^{jk}_i = \frac{1}{\sqrt{|\det g|}} \frac{\delta L_m}{\delta N^i_{jk}} = \frac{\delta L_m}{\delta N^i_{kh}}.$$

**Remark.** The energy-momentum tensor $T$ contains a contribution from the metric (which is the symmetric, Hilbert energy-momentum tensor $\text{Met}_T^i_j = 2\Sigma^j_i$) plus a term given by the distortion tensor $N$. That is, $T_{ij}$ is, generally, non-symmetric.

Applying (62), we find:

**Proposition 9** In a metric-affine theory, the energy-momentum tensor (66) obeys the balance law:

$$(T^j_i;_j + N^j_{kh};_i \frac{\delta L_m}{\delta N^j_{kh}}) \circ J^{s+2}\gamma \approx g^s \text{'} 0,$$  

where semicolons denote Levi-Civita covariant derivatives.

The precise form of the matter Lagrangian function $L_m$, as well as its order in either the background or the dynamical variables is irrelevant.

Using in the above procedure, instead of $\lambda_m$, the background Lagrangian $\lambda_b = L_b \sqrt{|\det g|} \omega_0$, the obtained energy-momentum balance law (67) becomes an identity.

### 6 Energy-momentum tensor and variational completion

Given a source form $\varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ on a fibered manifold $Y$, a (local) variational completion, \(\Sigma\), of $\varepsilon$ is a source form $\kappa$ on an open set $W \subset Y$, with the property that $\varepsilon + \kappa$ is variational. In particular, the canonical variational completion of
$\varepsilon$ is the source form $\kappa(\varepsilon)$ given by the difference between the Euler-Lagrange form of the Vainberg–Tonti Lagrangian

$$\lambda_\varepsilon = L_\varepsilon \omega_0, \quad L_\varepsilon = y^A \int_0^1 \varepsilon_A(x^i, ty^B_i, ..., ty^B_{i_1} ... i_r) dt$$

of $\varepsilon$ and $\varepsilon$ itself:

$$\kappa(\varepsilon) = E(\lambda_\varepsilon) - \varepsilon.$$  \hspace{1cm} (69)

The source form $\kappa(\varepsilon)$ can be expressed, [14], in terms of the coefficients of the Helmholtz form of $\varepsilon$, which give a measure of the non-variationality of $\varepsilon$.

Assume, in the following, that the background manifold is $Y^{(b)} = \text{Met}(X)$, with fibered coordinates $(x^i, g_{ij})$; in this case, the energy-momentum source form $\tau = \tau^{ij} \omega_{ij} \wedge \omega_0$ and the energy-momentum tensor density $T$ are related by:

$$T^{ij} = -2\tau^{ij}$$

(the minus sign appears because, in the relation $T^i_j = C^l_{(hl)i} \tau^{hl}, C^l_{(hl)i} = -\delta^l_h g_{li} - \delta^l_i g_{hi}$).

If we know a term $\varepsilon$ of the energy-momentum source form $\tau$, a Lagrangian and, accordingly, the full expression of $\tau$ can be found using (68), (69).

**Example:** energy-momentum tensor of a perfect fluid in general relativity.

Assume that $\dim X = 4$, the signature of the metric is $(+, -, -, -)$ and physical measurement units are such that $c = 1$. For an ideal fluid, the full expression of the energy-momentum tensor is

$$T^{ij} = (p + \rho) u^i u^j - p g^{ij},$$  \hspace{1cm} (70)

where $\rho$ denotes the density and $p$, the isotropic pressure of the fluid (both defined in a local inertial frame in which the fluid is at rest, [1]), $u^i = \frac{dx^i}{d\tau}$ are the components of the 4-velocity of the fluid and $d\tau = \sqrt{g_{ij} dx^i dx^j}$.

Let us take, for instance, the term $\rho u^i u^j$ and find its canonical Met$(X)$-variational completion. Build the source form $\varepsilon = \varepsilon^{ij} \omega_{ij} \wedge \omega_0$ on Met$(X)$, with coefficients $\varepsilon^{ij} = \varepsilon^{ij}(x^k, g_{kl})$ given by:

$$\varepsilon^{ij} = \alpha \rho u^i u^j \sqrt{\det(g_{ij})}, \quad \alpha \in \mathbb{R}$$

and let us study the behavior of $\varepsilon^{ij}$ with respect to homotheties $\chi_t: g_{ij} \mapsto t g_{ij}$.

The density $\rho$ is inverse proportional to the (3-dimensional) spatial volume, given, in a local inertial coframe $\{e^i\}$, by the volume element $dV_3 = e^1 \wedge e^2 \wedge e^3$; passing to an arbitrary frame, this is, [10], $dV_3 = \frac{\sqrt{\det(g_{kl})}}{\sqrt{g_{00}}} dx^1 \wedge dx^2 \wedge dx^3$.
Using the relations: $\sqrt{|\det(g_{ij})|} \circ \chi_t = t^2 \sqrt{|\det(g_{ij})|}$, $\sqrt{|g_{00}|} \circ \chi_t = t^{1/2} \sqrt{|g_{00}|}$, we find: $\rho \circ \chi_t = t^{-3/2} \rho$. Using the expression $u^i = \frac{dx^i}{dt}$, we have $u^i \circ \chi_t = t^{-1/2} u^i$.

Substituting into (71),

$$
\varepsilon^{ij} \circ \chi_t = t^{-1/2} \alpha \rho u^i u^j \sqrt{|\det(g_{ij})|};
$$

Since $g_{ij} u^i u^j = 1$, we get the Vainberg-Tonti Lagrangian $\lambda_\varepsilon = \mathcal{L}_\varepsilon \omega_0$, with:

$$
\mathcal{L}_\varepsilon = g_{ij} \int_0^1 \varepsilon^{ij} \circ \chi_t \, dt = \alpha \rho \sqrt{|\det(g_{ij})|} \int_0^1 t^{-1/2} \, dt = 2 \alpha \rho \sqrt{|\det(g_{ij})|}.
$$

Variation of $\lambda_\varepsilon$ with respect to the metric (as in [1], p. 197) gives then: $T^{ij} = -2\alpha \{(p + \rho) u^i u^j - pg^{ij}\}$. The term $\rho u^i u^j$ is obtained for $\alpha = -1/2$; for this value, the Lagrangian is nothing but the known Lagrangian:

$$
\lambda_{\text{fluid}} = -\rho \sqrt{|\det(g_{ij})|} \omega_0. \quad (72)
$$

The energy-momentum tensor (70) and the Lagrangian (72) were found in [15] starting from the term $pg^{ij}$.

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