MAGNITUDES, SCALABLE MONOIDS AND QUANTITY SPACES

DAN JONSSON

Abstract. In ancient Greek mathematics, magnitudes such as lengths were strictly distinguished from numbers. In modern quantity calculus, a distinction is made between quantities and scalars that serve as measures of quantities. It can be argued that quantities should play a more prominent, independent role in modern mathematics, as magnitudes earlier.

The introduction includes a sketch of the development and structure of the pre-modern theory of magnitudes and numbers. Then, a scalable monoid over a ring is defined and its basic properties are described. Congruence relations on scalable monoids, direct and tensor products of scalable monoids, subalgebras and homomorphic images of scalable monoids, and unit elements of scalable monoids are also defined and analyzed.

A quantity space is defined as a commutative scalable monoid over a field, admitting a finite basis similar to a basis for a free abelian group. The mathematical theory of quantity spaces forms the basis of a rigorous quantity calculus and is developed with a view to applications in metrology and foundations of physics.

1. Introduction and historical background

Equations such as \( E = \frac{mv^2}{2} \) or \( \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \), used to express physical laws, describe relationships between scalars, commonly real numbers. An alternative interpretation is possible, however. Since the scalars assigned to the variables in these equations are numerical measures of certain quantities, the equations express relationships between these quantities as well. For example, \( E = \frac{mv^2}{2} \) can also be interpreted as describing a relation between an energy \( E \), a mass \( m \) and a velocity \( v \) – three underlying physical quantities, whose existence and properties do not depend on scalars used to represent them. With this interpretation, though, \( \frac{mv^2}{2} \) and similar expressions will have meaning only if operations on quantities, corresponding to operations on numbers, are defined. In other words, an appropriate way of calculating with quantities, a quantity calculus, needs to be available.

In a useful survey [2], de Boer described the development of quantity calculus until the late 20th century, starting with Maxwell’s [22] concept of a physical quantity \( q \) comprised of a unit quantity \( [q] \) of the same kind as \( q \) and a scalar \( \{q\} \) which is the measure of \( q \) relative to \([q]\). So that we can write \( q = \{q\}[q] \). Like Lodge [20], in a seminal article in Nature 1888, and Wallot [29], in an influential article in Handbuch der Physik 1926, de Boer argued, however, that the physical quantity should be seen as a primitive notion: ontologically, the quantity precedes the measure used to describe it, and the assignment \( q = \{q\}[q] \) can be used to specify a particular quantity but not to define the notion of a quantity [14 pp. 1–2].
Actually, the roots of quantity calculus go far deeper in the history of mathematics than to W allot, Lodge, Maxwell or even other scientists of the modern era, such as Fourier [9]; its origins can be traced back to ancient Greek geometry and arithmetic, as codified in Euclid’s *Elements* [6].

Of fundamental importance in the *Elements* is the distinction between *numbers* (multitudes) and *magnitudes*. The notion of a number (*arithmos*) is based on that of a “unit” or “monad” (*monas*); a number is “a multitude composed of units”. Thus, a number is essentially a positive integer. (A collection of units containing just one unit was not, in principle, considered to be a multitude of units in Greek arithmetic, so strictly speaking 1 was not a number.) Numbers can be compared, added and multiplied, and a smaller number can be subtracted from a larger one, but the ratio of two numbers $m,n$ is not itself a number but just a pair $m:n$ expressing relative size. (The ratio of integers is not necessarily an integer.) Ratios can, however, be compared; $m : n = m' : n'$ means that $mn' = nm'$. A bigger number $m$ is said to be "measured" by a smaller number $k$ if $m = rk$ for some number $r$; a prime number is a number that is not measured by any other number (or measured only by 1), and $m,n$ are relatively prime when there is no number (except 1) measuring both.

Magnitudes (*megethos*), on the other hand, are phenomena such as lengths, areas, volumes or times. Unlike numbers, magnitudes are of different kinds, and while the magnitudes of a particular kind correspond loosely to numbers, making measurement of magnitudes possible, the magnitudes form a continuum, and there are no distinguished "unit magnitudes". In Greek mathematics, magnitudes of the same kind can be compared and added, and a smaller magnitude can be subtracted from a larger one of the same kind, but magnitudes cannot, in general, be multiplied or divided. One can form the ratio of two magnitudes of the same kind, $p$ and $q$, but this is not a magnitude but just a pair $p:q$ expressing relative size. A greater magnitude $q$ is said to be measured by a smaller magnitude $u$ if there is a number $n$ such that $q = n × u$ taken $n$ times; we may write this as $q = n × u$ here.

Remarkably, the first three propositions about magnitudes proved by Euclid in the *Elements* are, in the notation used here,

$$n × (u_1 + ⋯ + u_k) = n × u_1 + ⋯ + n × u_k,$$

$$n_1 + ⋯ + n_k × u = n_1 × u + ⋯ + n_k × u,$$

$$m × (n × u) = (mn) × u,$$

where $m,n,n_1,⋯,n_k$ are numbers (*arithmoi*), $u$ is a magnitude, $u_1,⋯,u_k$ are magnitudes of the same kind, $+$ denotes the sum of magnitudes of the same kind, and $×$ denotes the product of a number and a magnitude. As shown in Section 2.6 these identities are fundamental in modern quantity calculus as well.

If $p$ and $q$ are magnitudes of the same kind, and there is some magnitude $u$ of this kind and some numbers $m,n$ such that $p = m × u$ and $q = n × u$, then $p$ and $q$ are said to be "commensurable". The ratio of magnitudes $p:q$ can then be represented by the ratio of numbers $m:n$, assumed to be unique (unlike the two numbers specifying the ratio). However, magnitudes may also be "relatively prime"; it may happen that $p:q$ cannot be expressed as $m:n$ for any numbers $m,n$ because there are no $m,n,u$ such that $p = m × u$ and $q = n × u$. In view of the Pythagorean philosophical conviction of the primacy of numbers, the discovery of examples of such "incommensurable" magnitudes created a deep crisis in early Greek mathematics [13], a crisis that also affected the foundations of geometry.
If ratios of *arithmoi* do not always suffice to represent ratios of magnitudes, it seems that it would not always be possible to express in terms of *arithmoi* the fact that two ratios of magnitudes are equal, as are the ratios of the lengths of corresponding sides of similar triangles. This difficulty was resolved by Eudoxos, who realized that a "proportion", that is, a relation among magnitudes of the form "$p$ is to $q$ as $p'$ is to $q'$", conveniently denoted $p : q :: p' : q'$, can be defined numerically even if there is no pair of ratios of *arithmoi* $m : n$ and $m' : n'$ corresponding to $p : q$ and $p' : q'$, respectively, so that $p : q :: p' : q'$ cannot be inferred from $m : n = m' : n'$. Specifically, as described in Book V of the *Elements*, Eudoxos invented an ingenious indirect way of determining if $p : q :: p' : q'$ in terms of nothing but *arithmoi* by means of a construction similar to the Dedekind cut [13]. Using modern terminology, one can say that Eudoxos defined an equivalence relation :: between pairs of magnitudes of the same kind in terms of positive integers, and as a consequence it became possible to conceptualize in terms of *arithmoi* not only ratios of magnitudes corresponding to rational numbers but also ratios of magnitudes corresponding to irrational numbers. Eudoxos thus reconciled the continuum of magnitudes with the discrete *arithmoi*, but in retrospect this feat reduced the incentive to rethink the Greek notion of number, to generalize the *arithmoi*.

To summarize, Greek mathematicians used two notions of muchness, and built a theoretical system around each notion. These systems were connected by relationships of the form $q = n \times u$, where $q$ is a magnitude, $n$ a number and $u$ a magnitude of the same kind as $q$, foreboding from the distant past Maxwell’s quantity formula $q = \{q\}|q|$, although Euclid wisely did not propose to define magnitudes in terms of units and numbers.

The modern theory of numbers dramatically extends the theory of numbers in the *Elements*. Many types of numbers other than positive integers have been added, and the notion of a number as an element of an algebraic system has come to the forefront. The modern notion of number was not developed by a straight-forward extension of the concept of *arithmos*, however; the initial development of the new notion of number during the Renaissance was strongly inspired by the ancient theory of magnitudes.

The beginning of the Renaissance saw renewed interest in the classical Greek theories of magnitudes and numbers as known from Euclid’s *Elements*, but later these two notions gradually fused into that of a real number. Malet [21] remarks:

> As far as we know, not only was the neat and consistent separation between the Euclidean notions of numbers and magnitudes preserved in Latin medieval translations [...], but these notions were still regularly taught in the major schools of Western Europe in the second half of the 15th century. By the second half of the 17th century, however, the distinction between the classical notions of (natural) numbers and continuous geometrical magnitudes was largely gone, as were the notions themselves. [pp. 64–65]

The force driving this transformation was the need for a continuum of numbers as a basis for computation; the discrete *arithmoi* were not sufficient. As magnitudes of the same kind form a continuum, the idea emerged that numbers should be regarded as an aspect of magnitudes. "Number is to magnitude as wetness is to water" said Stevin in *L’Arithmétique* [28], published 1585, and defined a number as "that by which one can tell the quantity of anything" (cela, par lequel s’explique la quantité de chascune chose) [Definition II]. Thus, numbers were seen to form a continuum by virtue of their intimate association with magnitudes.
Stevin’s definition of a number is rather vague, and it is difficult to see how a magnitude can be associated with a definite number, considering that the numerical measure of a magnitude depends on a choice of a unit magnitude. The notion of number was, however, refined during the 17th century. In *La Geometrie* [4], where Descartes laid the groundwork for analytic geometry, he implicitly identified numbers with *ratios* of two magnitudes, namely lengths of line segments, one of which was considered to have unit length, and in *Universal Arithmetick* [23] Newton, who had studied both Euclid and Descartes, defined a number as follows:

By *Number* we mean, not so much a Multitude of Unities, as the abstracted *Ratio* of any Quantity, to another Quantity of the same Kind, which we take for Unity. [p. 2]

By assigning the number 1 to a unit quantity, the representation of quantities by numbers is normalized, addressing a problem with Stevin’s definition. Also, a ratio of quantities of the same kind is a ”dimensionless” quantity. Systems of such quantities contain a canonical unit quantity $1$, and addition, subtraction, multiplication and division of dimensionless quantities yield dimensionless quantities. Hence, a number and the corresponding dimensionless quantity are quite similar, though Newton hints at a difference by calling numbers ”abstracted” ratios of quantities.

Magnitudes, or ”dimensionful” quantities, were thus needed only as a scaffolding for the new notion of numbers, and when this notion had been established its origins fell into oblivion and magnitudes fell out of fashion. The tradition from Euclid paled away, but the idea that numbers specify quantities relative to other quantities remained, as in [7]. A new theory of quantities originated from this idea.

While the Greek theory of magnitudes derived from geometry, the new theory of quantities found applications in mathematical physics, a branch of science that emerged in the 18th century. In *The Analytic Theory of Heat* [10], published in 1822 as *Théorie analytique de la Chaleur*, Fourier explains how physical quantities relate to the numbers in his equations:

In order to measure these quantities and express them numerically, they must be compared with different kinds of units, five in number, namely, the unit of length, the unit of time, that of temperature, that of weight, and finally the unit which serves to measure quantities of heat. [pp. 126–127]

We recognize here the ideas that there are quantities of different kinds and that the number associated with a quantity depends on the choice of a unit quantity of the same kind.

Using the modern notion of, for example, a real number, we can generalize relationships of the form $q = n \times u$, where $n$ is an *arithmos* and $u$ is a magnitude that measures (divides) $q$, to relationships of the form $q = \mu \cdot u$, where $u$ is a freely chosen unit quantity of the same kind as $q$, and $\mu$ is the measure of $q$ relative to $u$, a number specifying the size of $q$ compared to $u$. If $q = \mu \cdot u$ then $\mu$ is determined by $q$ and $u$, and we may write $\mu = f(q, u)$ as $\mu = q/u$.

Fourier realized that the measure of a quantity may be defined in terms of measures of other quantities, in turn dependent on the units for these quantities. For example, the measure of a velocity depends on a unit of length $u_l$ and a unit of time $u_t$ since a velocity is defined in terms of a length and a time, and the measure of an area indirectly depends on a unit of length $u_l$. 
Formally, let the measure $\mu_v$ of a velocity $v$ relative to $u_v$ be given by $\mu_v = F(\mu_\ell, \mu_t) = F(\ell/u_\ell, t/u_t)$, where $F(x, y) = xy^{-1}$, and let the measure $\mu_a$ of the area $a$ of a rectangle relative to $u_a$ be given by $\mu_a = G(\mu_\ell, \mu_w) = G(\ell/u_\ell, w/u_t)$, where $G(x, y) = xy$. Generalizing the magnitude identity $m \times (n \times u) = mn \times u$, we have $M \cdot (N \cdot u) = MN \cdot u$ for any real numbers $M, N$. Thus, if $q = \mu \cdot u$ and $M > 0$ then $q = (M \mu M^{-1}) \cdot u = M \mu \cdot (M^{-1} \cdot u)$, so $M \mu = q/(M^{-1} \cdot u)$, so it follows from the definitions of $F$ and $G$ that, for any non-zero numbers $L, T$,

$\mu'_v = F(\ell/(L^{-1} \cdot u_\ell), t/(T^{-1} \cdot u_t)) = F(L\mu_\ell, T\mu_t) = LT^{-1}F(\mu_\ell, \mu_t) = LT^{-1}\mu_v$,

$\mu'_a = G(\ell/(L^{-1} \cdot u_\ell), w/(L^{-1} \cdot u_t)) = G(L\mu_\ell, L\mu_w) = L^2G(\mu_\ell, \mu_w) = L^2\mu_a$.

The two equations show how the measures $\mu_v$ and $\mu_a$ are affected by a change of units $u_v \mapsto L \cdot u_\ell$ and $u_t \mapsto T \cdot u_t$. Reasoning similarly \cite{10} pp. 128–130, Fourier pointed out that quantity terms can be equal or combined by addition or subtraction only if they agree with respect to each *exposant de dimension*, having identical patterns of exponents in expressions such as $LT^{-1}$, $LT^{-2}$ or $L^2$, since otherwise the validity of numerical equations corresponding to quantity equations would depend on an arbitrary choice of units. He thus introduced the principle of dimensional homogeneity for equations that relate quantities.

Note that if $q = \mu \cdot u$ then $M \cdot q = M \cdot (\mu \cdot u) = M \mu \cdot u$, so $(M \cdot q)/u = M \mu = M(q/u)$. Thus, in a sense turning Fourier’s argument around, we also have

$\mu'_v = F((L \cdot \ell)/u_\ell, (T \cdot t)/u_t) = F(L(\ell/u_\ell), T(t/u_t)) = LT^{-1}F(\ell/u_\ell, t/u_t) = LT^{-1}\mu_v$,

$\mu'_a = G((L \cdot \ell)/u_\ell, (L \cdot w)/u_t) = G(L(\ell/u_\ell), L(w/u_t)) = L^2G(\ell/u_\ell, w/u_t) = L^2\mu_a$.

These equations show how $\mu_v$ and $\mu_a$ are affected when quantities change according to $\ell \mapsto L \cdot \ell$, $t \mapsto T \cdot t$. For any fixed units $u_\ell$ and $u_t$, we can express this as

$\Phi(L \cdot \ell, T \cdot t) = LT^{-1}\Phi(\ell, t)$,

$\Gamma(L \cdot \ell, L \cdot w) = L^2\Gamma(\ell, w)$,

where $\Phi$ and $\Gamma$ are the quantity-valued functions given by $\Phi(\ell, t) = F(\ell/u_\ell, t/u_t) \cdot u_v$ and $\Gamma(\ell, w) = G(\ell/u_\ell, w/u_t) \cdot u_a$, respectively; note that $u_v$ and $u_a$ are also fixed since they depend on $u_\ell$ and $u_t$.

The bilinearity properties of $\Phi$ and $\Gamma$ suggest that we write $\Phi(\ell, t)$ as $\alpha \ell t^{-1}$ and $\Gamma(\ell, w)$ as $\beta \ell w$, where $\alpha$ and $\beta$ are numerical constants. Generalizing this heuristic argument, we may introduce the idea that quantities of the same or different kinds can be multiplied and divided, suggesting that we can form arbitrary expressions of the form $\mu \prod_{i=1}^{n} q_i k_i$, where $\mu$ is any number, $q_i$ are quantities and $k_i$ are integers, thus coming close to the quantity calculus set out below. Note, however, that Fourier did not actually define multiplication or division of quantities as such. This came later, with Lodge \cite{20}, Wallot \cite{29} and others.

In retrospect, one may say that Fourier reinvented magnitudes as protoquantities and extended the range of applications. While Fourier reasoned in terms of multiplication and division of *measures* of quantities, he made a clear distinction between a quantity and its measure relative to a unit, this measure being a real number rather than an arithmos, he distinguished different kinds of quantities, and he considered new kinds of quantities such as temperatures and amounts of heat. Essential elements of a modern quantity calculus treating general quantities as mathematical objects (almost) as real as numbers were thus recognized early in the 19th century.
Subsequent progress in this area of mathematics has not been fast and straightforward, however. A Euclidean synthesis did not emerge; in his survey from 1994 de Boer concluded that "a satisfactory axiomatic foundation for the quantity calculus" had not yet been formulated [2].

Gowers [11] points out that many mathematical objects are not defined directly by describing their essential properties, but indirectly by construction-definitions, specifying constructions that can be shown to have these properties. For example, an ordered pair \((x, y)\) may be defined by a construction-definition as a set \(\{x, \{y\}\}\); it can be shown that this construction has the required properties, namely that \((x, y) = (x', y')\) if and only if \(x = x'\) and \(y = y'\). Many contemporary formalizations of the notion of a quantity (e.g., [3, 19]) use definitions relying on constructions, often defining quantities in terms of scalar-unit pairs in the tradition from Maxwell. (See also the survey in Appendix B.) However, this is rather like defining a vector as a coordinates-basis pair rather than as an element of a vector space, the modern definition.

Although magnitudes are illustrated by line segments in the Elements, the notion of a magnitude is abstract and general. Remarkably, Euclid, following Eudoxos, dealt with this notion in a very modern way. While Euclid carefully defined other important objects such as points, lines and numbers in terms of inherent properties, there is no statement about what a magnitude "is". Instead, magnitudes are characterized by how they relate to other magnitudes through their roles in a system of magnitudes, to paraphrase Gowers [12].

In the same spirit, that of modern algebra, quantities are defined in this article simply as elements of a "quantity space". Thus, the focus is moved from individual quantities and operations on them to the systems to which the quantities belong, meaning that the notion of quantity calculus will give way to that of a quantity space. This article considers the notion of a quantity space introduced in [14] and developed further in [15].

In the conceptual framework of universal algebra, a quantity space is just a special scalable monoid \((X, *, (\omega_\lambda)_{\lambda \in R}, 1_X)\), where \(X\) is the underlying set of the algebra, \((X, *, 1_X)\) is a monoid, \(R\) is a fixed ring and every \(\omega_\lambda\) is unary operation on \(X\). Writing \(* (x, y)\) as \(xy\) and denoting \(\omega_\lambda(x)\) by \(\lambda \cdot x\), we have \(1 \cdot x = x\), \(\lambda \cdot (\kappa \cdot x) = \lambda \kappa \cdot x\) and \(\lambda \cdot xy = (\lambda \cdot x) y = x (\lambda \cdot y)\) for all \(\lambda \in R, x, y \in X\).

The relation \(\sim\) on a scalable monoid \(X\) defined by \(x \sim y\) if and only if \(\alpha \cdot x = \beta \cdot y\) for some \(\alpha, \beta \in R\) is a congruence on \(X\), so \(X\) is partitioned into corresponding equivalence classes. There is no global operation \((x, y) \mapsto x + y\) defined on \(X\), but within each equivalence class that contains a "unit element" addition of its elements is induced by the addition in \(R\) (see Section 2.6), and multiplication of equivalence classes is induced by the multiplication of elements of \(X\) (see Section 2.3).

Quantity spaces are to scalable monoids as vector spaces are to modules. Specifically, a quantity space \(Q\) is a commutative scalable monoid over a field, such that there exists a finite basis for \(Q\), similar to a basis for a free abelian group. As noted, quantities are just elements of quantity spaces, and dimensions are equivalence classes in quantity spaces.

The remainder of this article is divided into two main sections, namely Section 2 which deals with scalable monoids and Section 3 where scalable monoids are specialized to quantity spaces. There are also two Appendices, one of which relates the theory presented here to contemporary research on quantity calculus.
2. Scalable monoids

A scalable monoid is a monoid whose elements can be multiplied by elements of a ring, and where multiplication in the monoid, multiplication in the ring, and multiplication of monoid elements by ring elements are compatible operations.

Scalable monoids are formally defined and compared to rings and modules in Section 2.1, and some basic facts about them are presented in Section 2.2. Sections 2.3 and 2.5 are concerned with congruences on scalable monoids and related notions such as commensurability, orbitoids, homomorphisms and quotient algebras, while direct and tensor products of scalable monoids are defined in Section 2.4. Scalable monoids with unit elements are investigated in Sections 2.6 and 2.7. In particular, addition of elements in the same equivalence class is defined, and coherent systems of unit elements are discussed.

2.1. Mathematical background, main definition and simple examples. A unital associative algebra $X$ over a (unital, associative but not necessarily commutative) ring $R$ can be defined as a set, also denoted $X$, with three operations:

1. **Addition** of elements of $X$, a binary operation $+ : (x, y) \mapsto x + y$ on $X$ such that $X$ equipped with $+$ is an abelian group;
2. **Multiplication** of elements of $X$, a binary operation $\cdot : (x, y) \mapsto xy$ on $X$ such that $X$ equipped with $\cdot$ is a monoid;
3. **Scalar multiplication** of elements of $X$ by elements of $R$, a monoid action $(\alpha, x) \mapsto \alpha \cdot x$ where the multiplicative monoid of $R$ acts on $X$ so that $1 \cdot x = x$ and $\alpha \cdot (\beta \cdot x) = \alpha \beta \cdot x$ for all $\alpha, \beta \in R$ and $x \in X$.

There are identities specifying a link between each pair of operations:

(a) addition and multiplication of elements of $X$ are linked by the distributive laws $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(b) addition of elements of $X$ or $R$ and scalar multiplication of elements of $X$ by elements of $R$ are linked by the distributive laws $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ and $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$;
(c) multiplication of elements of $X$ and scalar multiplication of elements of $X$ by elements of $R$ are linked by the bilinearity laws $\alpha \cdot xy = (\alpha \cdot x)y$ and $\alpha \cdot xy = x(\alpha \cdot y)$.

Related algebraic structures can be obtained from unital associative algebras by removing one of the operations (1) – (3) and hence the links between the removed operation and the two others. Two cases are very familiar: a ring has only addition and multiplication of elements of $X$, linked as described in (a), and a (left) module has only addition of elements of $X$ and scalar multiplication of elements of $X$ by elements of $R$, linked as described in (b). The question arises whether it would be meaningful and useful to define an “algebra without an additive group”, with only multiplication of elements of $X$ and scalar multiplication of elements of $X$ by elements of $R$, linked as described in (c).

The answer is affirmative. It turns out that this notion, a "scalable monoid", formally related to rings and in particular modules, makes sense mathematically and is remarkably well suited for modeling systems of quantities. The ancient arithmos-megethos pair of notions receives a modern interpretation: while numbers can be formalized as elements of rings, typically fields, quantities can be formalized as elements of scalable monoids, specifically quantity spaces.
Definition 2.1. Let $R$ be a (unital, associative) ring. A scalable monoid over $R$ is a monoid $X$ equipped with a scaling action

$$\omega : R \times X \to X, \quad (\alpha, x) \mapsto \alpha \cdot x,$$

such that $1 \cdot x = x$, $\alpha \cdot (\beta \cdot x) = \alpha \beta \cdot x$ and $\alpha \cdot xy = (\alpha \cdot x)y = x(\alpha \cdot y)$.

We denote the identity element of $X$ by $1_X$, and set $x^0 = 1_X$ for any $x \in X$. An invertible element of a scalable monoid $X$ is an element $x \in X$ that has a (necessarily unique) inverse $x^{-1} \in X$ such that $xx^{-1} = x^{-1}x = 1_X$.

It is easy to verify that the trivial scaling action of a ring $R$ on a monoid $X$ defined by $\lambda \cdot x = x$ for all $\lambda \in R$ and $x \in X$ is indeed a scaling action according to Definition 2.1. We call a monoid equipped with a trivial scaling action a trivially scalable monoid. A scalable monoid of this kind is essentially just a monoid, since the operation $(\lambda, x) \mapsto \lambda \cdot x$ can be disregarded in this case.

Example 2.2. A trivial scalable monoid is a trivial monoid $\{1_X\}$ with a trivial scaling action.

Example 2.3. Let $M(n)$ be the multiplicative monoid of all $n \times n$ matrices with entries in $\mathbb{R}$. Then $M(n)$ is a scalable monoid over the corresponding matrix ring $R(n)$, with the scaling action defined by $A \cdot X = (\det A)X$.

Example 2.4. Let $R[\{x_1; \ldots; x_n\}]$ denote the set of all monomials of the form

$$\lambda x_1^{k_1} \ldots x_n^{k_n},$$

where $R$ is a commutative ring, $\lambda \in R$, $x_1, \ldots, x_n$ are uninterpreted symbols and $k_1, \ldots, k_n$ are non-negative integers. We can define the operations $(s, t) \mapsto st$, $(\alpha, t) \mapsto \alpha \cdot t$ and $(\alpha) \mapsto 1_{R[\{x_1; \ldots; x_n\}]}$ on $R[\{x_1; \ldots; x_n\}]$ by setting

$$(\alpha \cdot \lambda x_1^{k_1} \ldots x_n^{k_n})(\beta x_1^{j_1} \ldots x_n^{j_n}) = (\lambda \alpha x_1^{(j_1+k_1)} \ldots x_n^{(j_n+k_n)},$$

$$\alpha \cdot (\lambda x_1^{k_1} \ldots x_n^{k_n}) = (\alpha \lambda) x_1^{k_1} \ldots x_n^{k_n},$$

$$1_{R[\{x_1; \ldots; x_n\}]} = x_1^0 \ldots x_n^0.$$ 

$R[\{x_1; \ldots; x_n\}]$ equipped with these operations is a commutative scalable monoid over $R$.

2.2. Some basic facts about scalable monoids. A not necessarily commutative scalable monoid over a not necessarily commutative ring nevertheless exhibits certain commutativity properties as described in the following useful lemma:

Lemma 2.5. Let $X$ be a scalable monoid over $R$. For any $x, y \in X$ and $\alpha, \beta \in R$ we have

$$(\alpha \cdot x)(\beta \cdot y) = \alpha \beta \cdot xy, \quad \alpha \cdot (\beta \cdot x) = \beta \cdot (\alpha \cdot x) = \beta \alpha \cdot x.$$ 

Proof. By Definition 2.1

$$(\alpha \cdot x)(\beta \cdot y) = \alpha \cdot x(\beta \cdot y) = \alpha \cdot (\beta \cdot xy) = \alpha \beta \cdot xy,$$

$$\alpha \beta \cdot x = \alpha \cdot (\beta \cdot x) = \alpha \cdot (\beta \cdot 1_X x) = \alpha \cdot (\beta \cdot 1_X) x = (\beta \cdot 1_X)(\alpha \cdot x)$$

$$= \beta \alpha \cdot 1_X x = \beta \alpha \cdot x = \beta \cdot (\alpha \cdot x),$$

where the first identity is used in the proof of the second. \hfill \Box
For example, \((\det \mathbf{A}) \mathbf{X} (\det \mathbf{B}) \mathbf{Y} = (\det \mathbf{A} \mathbf{B}) \mathbf{X} \mathbf{Y}\) is obviously the identity \((\mathbf{A} \cdot \mathbf{X})(\mathbf{B} \cdot \mathbf{Y}) = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{X} \cdot \mathbf{Y}\) for the scalable monoid in Example 2.3.

Since every monoid \(\mathfrak{M}\) has a unique identity element \(1_\mathfrak{M}\), the class of all monoids forms a variety of algebras with a binary operation \(* : (x, y) \mapsto xy\), a nullary operation \(1_\mathfrak{M} : () \mapsto 1_\mathfrak{M}\) and identities

\[
x(yz) = (xy)z, \quad 1_\mathfrak{M}x = x = x1_\mathfrak{M}.
\]

The class of all scalable monoids over a fixed ring \(R\) is a variety in addition equipped with a set of unary operations \(\{\omega_\lambda \mid \lambda \in R\}\), derived from the scaling action \(\omega\) in Definition 2.1 by setting \(\omega_\lambda(x) = \lambda \cdot x\) for all \(\lambda \in R\) and \(x \in X\), and with the additional identities

\[
\omega_1(x) = x, \quad \omega_\lambda(\omega_\kappa(x)) = \omega_{\lambda\kappa}(x), \quad \omega_\lambda(xy) = \omega_\lambda(x) \cdot \omega_\lambda(y) = x \cdot \omega_\lambda(y), \quad (\lambda, \kappa \in R),
\]

corresponding to \(1 \cdot x = x\), \(\alpha \cdot (\beta \cdot x) = \alpha \beta \cdot x\), and \(\alpha \cdot xy = (\alpha \cdot x)y = x(\alpha \cdot y)\).

The scalable monoids is thus a variety of algebras belonging to the class of all \((R, 1)\)-magmas

\[
(X, *, (\omega_\lambda)_{\lambda \in R}, 1_X).
\]

where \(X\) is a carrier set, \(*\) a binary operation, \(\omega_\lambda\) a unary operation and \(1_X\) a nullary operation. The general definitions of subalgebras, homomorphisms and products of algebras in the theory of universal algebras apply to \((R, 1)\)-magmas. In particular, a subalgebra of an \((R, 1)\)-magma \(X\) is a subset \(Y\) of \(X\) such that \(1_X \in Y\) and \(xy, \lambda \cdot x \in Y\) for any \(x, y \in Y\) and \(\lambda \in R\). Also, a homomorphism \(\phi : X \to Y\) of \((R, 1)\)-magmas \(X\) and \(Y\) is a function such that \(\phi(1_X) = 1_Y\), and we have \(\phi(xy) = \phi(x)\phi(y)\) and \(\phi(\lambda \cdot x) = \lambda \cdot \phi(x)\) for any \(x, y \in X\) and \(\lambda \in R\).

Recall, furthermore, that varieties are closed under the operations of forming subalgebras, homomorphic images and products since the defining identities are replicated by these operations [1]. Thus, a subalgebra of a scalable monoid over \(R\), a homomorphic image of a scalable monoid over \(R\), and a direct product of scalable monoids over \(R\) are all scalable monoids over \(R\). Results related to these and other constructions will be considered in the remainder of Section 2.

2.3. **Commensurability classes and canonical quotients.** In ancient Greek mathematics, the notions of a sum, difference or ratio of magnitudes did not apply to magnitudes of different kinds, so in particular these could not be commensurable in the Greek (Pythagorean) sense. Moreover, magnitudes of the same kind, for example, two lengths, could nevertheless be incommensurable. In this section, we introduce a seemingly more radical idea: quantities are of the same kind if and only if they are commensurable.

**Definition 2.6.** Given a scalable monoid \(X\) over \(R\), let \(\sim\) be the relation on \(X\) such that \(x \sim y\) if and only if \(\alpha \cdot x = \beta \cdot y\) for some \(\alpha, \beta \in R\). We say that \(x\) and \(y\) are **commensurable** if and only if \(x \sim y\); otherwise \(x\) and \(y\) are **incommensurable**.

Let \(R \cdot x\) denote the set \(\{\lambda \cdot x \mid \lambda \in R\}\), that is, the orbit of \(x \in X\) for the scaling action \(\omega : R \times X \to X\), and let \(\approx\) denote the relation on \(X\) such that \(x \approx y\) if and only if there is some \(t \in X\) such that \(x, y \in R \cdot t\). Note that \(\approx\) is a not an equivalence relation; it is reflexive since \(x \in 1 \cdot x\) for all \(x \in X\) and symmetric by construction but not transitive, meaning that the orbits for \(\omega\) may overlap. On the other hand, \(x \sim y\) if and only if \((R \cdot x) \cap (R \cdot y) \neq \emptyset\), and this relation is indeed transitive.
Proposition 2.7. The relation ~ on a scalable monoid $X$ over $R$ is an equivalence relation.

Proof. The relation ~ is reflexive since $1 \cdot x = 1 \cdot x$ for all $x \in X$, symmetric by construction, and transitive because if $\alpha \cdot x = \beta \cdot y$ and $\gamma \cdot y = \delta \cdot z$ for some $x,y,z \in X$ and $\alpha, \beta, \gamma, \delta \in R$ then it follows from Lemma 2.5 that
\[\gamma \alpha \cdot x = \gamma \cdot (\alpha \cdot x) = \gamma \cdot (\beta \cdot y) = \beta \cdot (\gamma \cdot y) = \beta \cdot (\delta \cdot z) = \beta \delta \cdot z,\]
where $\gamma \alpha, \beta \delta \in R$. $\Box$

Definition 2.8. A commensurability class or orbitoid $C$ is an equivalence class for ~. The orbitoid that contains $x$ is denoted by $[x]$, and $X/\sim$ denotes the set $\{[x] \mid x \in X\}$.

For example, the commensurability classes of $R[x_1; \ldots; x_n]$ in Example 2.4 are the sets $\{\lambda x_1^{k_1} \ldots x_n^{k_n} \mid \lambda \in R\}$ for fixed non-negative integers $k_1, \ldots, k_n$.

Remark 2.9. The orbits corresponding to an action of a group $G$ on a set $X$ are precisely the equivalence classes given by the equivalence relation $\sim_G$ defined by $x \sim_G y$ if and only if $\alpha \cdot x = y$ for some $\alpha \in G$; we clearly have $G \cdot x = G \cdot y$ if and only if $x \sim_G y$. Similarly, orbitoids – generalized orbits in $X$ under a monoid action satisfying $\alpha \cdot (\beta \cdot x) = \beta \cdot (\alpha \cdot x)$ – are given by the equivalence relation $\sim$ defined by $x \sim y$ if and only if $\alpha \cdot x = \beta \cdot y$ for some $\alpha, \beta \in R$. One may say that orbitoids generalize orbits as $\sim$ generalizes $\sim_G$.

Proposition 2.10. If $x \sim y$ then $\lambda \cdot x \sim y$, $x \sim \lambda \cdot y$ and $\lambda \cdot x \sim \lambda \cdot y$ for all $\lambda \in R$.

Proof. If $x \sim y$ then $\alpha \cdot x = \beta \cdot y$ for some $\alpha, \beta \in R$, so by Lemma 2.5
\[\alpha \lambda \cdot x = \alpha \cdot (\lambda \cdot x) = \lambda \cdot (\alpha \cdot x) = \lambda \cdot (\beta \cdot y) = \beta \cdot (\lambda \cdot y) = \beta \lambda \cdot y,\]
where $\alpha \lambda, \beta \lambda \in R$. $\Box$

Corollary 2.11. $\lambda \cdot x \sim x$ for all $x \in X$ and $\lambda \in R$.

It is instructive to compare the present notion of commensurability with the classical one. If $x = \alpha \cdot t$ and $y = \beta \cdot t$ then $\beta \cdot x = \beta \cdot (\alpha \cdot t) = \alpha \cdot (\beta \cdot t) = \alpha \cdot y$, so if $x \approx y$ then $x \sim y$. We say that $x$ and $y$ are strongly commensurable if and only if $x \approx y$; otherwise, $x$ and $y$ are said to be weakly incommensurable.

Incommensurability of magnitudes in the Pythagorean sense obviously corresponds to weak incommensurability, so it is implied by, but does not imply, incommensurability in the present sense. Conversely, we have weakened the classical notion of commensurability here, at the same time making commensurability into an equivalence relation. The present concept of commensurability corresponds to the intuitive notion of magnitudes of the same kind, or the somewhat fuzzy notion of quantities of the same kind in modern theoretical metrology [32].

Remark 2.12. The mathematical quantities defined here are size-properties of certain objects or phenomena. Through one or more abstraction steps, concrete properties can be reduced to more abstract properties. The level of abstraction chosen affects the categorization of quantities into kinds of quantities. For example, it would seem that there are no scalars $\alpha, \beta$ such that $\alpha \cdot x = \beta \cdot y$, where $x$ is a planar angle and $y$ a solid angle. A plane angle cannot be resized to a solid angle, or vice versa, so plane and solid angles would appear to be quantities of different kinds.
Proposition 2.13. Let $X$ be a scalable monoid over $R$. The relation $\sim$ is a congruence on $X$ with regard to the operations $(x, y) \mapsto xy$ and $(\lambda, x) \mapsto \lambda \cdot x$.

Proof. If $\alpha \cdot x = \alpha' \cdot x'$ and $\beta \cdot y = \beta' \cdot y'$ for some $x, x', y, y' \in X$ and $\alpha, \alpha', \beta, \beta' \in R$ then $(\alpha \cdot x)(\beta \cdot y) = (\alpha' \cdot x')(\beta' \cdot y')$, so $\alpha \beta \cdot xy = \alpha' \beta' \cdot x'y'$ by Lemma 2.5. As $\alpha \beta, \alpha' \beta' \in R$, this means that if $x \sim x'$ and $y \sim y'$ then $xy \sim x'y'$. Also, if $x \sim x'$ then $\lambda \cdot x \sim \lambda \cdot x'$ for any $\lambda \in R$ by Proposition 2.10. □

In view of Proposition 2.13 we can define operations on $X/\sim$ as follows:

Definition 2.14. Set $[x][y] = [xy]$, $\lambda \cdot [x] = [\lambda \cdot x]$ and $1_{X/\sim} = [1_X]$ for any $[x], [y] \in X/\sim$ and $\lambda \in R$.

Given these definitions, $X/\sim$ is an $(R, 1)$-magma and the surjective function $\phi : X \rightarrow X/\sim$ given by $\phi(x) = [x]$ satisfies the conditions

$$\phi(xy) = \phi(x)\phi(y), \quad \phi(\lambda \cdot x) = \lambda \cdot \phi(x), \quad \phi(1_X) = 1_{X/\sim},$$

so $\phi$ is a homomorphism of $(R, 1)$-magmas and thus of scalable monoids.

Proposition 2.15. If $X$ is a scalable monoid over $R$ then $X/\sim$ is a scalable monoid over $R$, and the function

$$\phi : X \rightarrow X/\sim, \quad x \mapsto [x],$$

is a surjective homomorphism of scalable monoids.

We call $X/\sim$ the canonical quotient of $X$.

Proposition 2.16. If $X$ is a scalable monoid then $X/\sim$ is a trivially scalable monoid.

Proof. By Corollary 2.11, $\lambda \cdot [x] = [\lambda \cdot x] = [x]$ for all $\lambda \in R, x \in X$. □

In many situations, it is natural to regard $X/\sim$ as a monoid with operations inherited from $X$ by setting $[x][y] = [xy]$ and $1_{X/\sim} = [1_X]$.

2.4. Direct and tensor products of scalable monoids. Consider an $(R, 1)$-magma

$$(X \times Y, \ast, (\omega_\lambda)_{\lambda \in R}, 1_{X \times Y})$$

where $X$ and $Y$ denote the underlying sets of two scalable monoids $X$ and $Y$ over $R$, $\ast$ is a binary operation given by $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, each $\omega_\lambda$ is a unary operation given by $\omega_\lambda(x, y) = (\lambda \cdot x, \lambda \cdot y)$, where $x \in X$ and $y \in Y$, and $1_{X \times Y}$ is a nullary operation given by $1_{X \times Y} = (1_X, 1_Y)$. Straight-forward calculations (or the HSP theorem [1]) show that this $(R, 1)$-magma, likewise denoted $X \times Y$, is a scalable monoid over $R$. We call $X \times Y$ the direct product of $X$ and $Y$.

The direct product of scalable monoids is a generic product, applicable to any universal algebra. Another kind of product, which exploits the fact that $(\lambda \cdot x)y =
\(\lambda \cdot xy = x(\lambda \cdot y)\) in scalable monoids, namely the tensor product, turns out to be more useful in many cases.

**Definition 2.17.** Given scalable monoids \(X\) and \(Y\) over \(R\), let \(\sim_\otimes\) be the binary relation on \(X \times Y\) such that \((x_1, y_1) \sim_\otimes (x_2, y_2)\) if and only if \((\alpha \cdot x_1, \beta \cdot y_1) = (\beta \cdot x_2, \alpha \cdot y_2)\) for some \(\alpha, \beta \in R\).

**Proposition 2.18.** Let \(X\) and \(Y\) be scalable monoids over \(R\). Then \(\sim_\otimes\) is an equivalence relation on \(X \times Y\).

**Proof.** \(\sim_\otimes\) is reflexive since \((1 \cdot x, 1 \cdot y) = (1 \cdot x, 1 \cdot y)\), and symmetric by construction. If \((\alpha \cdot x_1, \beta \cdot y_1) = (\beta \cdot x_2, \alpha \cdot y_2)\) and \((\gamma \cdot x_2, \delta \cdot y_2) = (\delta \cdot x_3, \gamma \cdot y_3)\) then
\[
(\gamma \cdot (\alpha \cdot x_1), \delta \cdot (\beta \cdot y_1)) = (\gamma \cdot (\beta \cdot x_2), \delta \cdot (\alpha \cdot y_2)),
\]
and thus
\[
(\gamma \alpha \cdot x_1, \delta \beta \cdot y_1) = (\beta \delta \cdot x_3, \alpha \gamma \cdot y_3) = (\delta \beta \cdot x_3, \gamma \alpha \cdot y_3),
\]
where \(\gamma \alpha, \delta \beta \in R\), so \(\sim_\otimes\) is transitive as well. \(\square\)

**Definition 2.19.** Let \(X\) and \(Y\) be scalable monoids, let \(x \otimes y\) denote the equivalence class
\[
\{(s, t) \mid (s, t) \sim_\otimes (x, y)\},
\]
where \(x \in X, y \in Y\), and let \(X \otimes Y\) denote the set
\[
\{x \otimes y \mid x \in X, y \in Y\},
\]
equipped with the operations given by
\[
(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2, \quad \lambda \cdot x \otimes y = (\lambda \cdot x) \otimes y, \quad 1_{X \otimes Y} = 1_X \otimes 1_Y.
\]
We call \(X \otimes Y\) the tensor product of \(X\) and \(Y\).

**Proposition 2.20.** Let \(X\) and \(Y\) be scalable monoids over \(R\), \(x \in X\) and \(y \in Y\). Then \((\lambda \cdot x) \otimes y = x \otimes (\lambda \cdot y)\) for every \(\lambda \in R\).

**Proof.** We have \((1 \cdot (\lambda \cdot x), \lambda \cdot y) = (\lambda \cdot x, 1 \cdot (\lambda \cdot y))\), so \((\lambda \cdot x, y) \sim_\otimes (x, \lambda \cdot y)\). \(\square\)

**Proposition 2.21.** Let \(X\) and \(Y\) be scalable monoids over \(R\). Then \(X \otimes Y\) is a scalable monoid over \(R\).

**Proof.** \(X \otimes Y\) is a monoid since
\[
(1_X \otimes 1_Y)(x \otimes y) = 1_X x \otimes 1_Y y = x \otimes y = x 1_X \otimes y 1_Y = (x \otimes y)(1_X \otimes 1_Y),
\]
\[
((x_1 \otimes y_1)(x_2 \otimes y_2))(x_3 \otimes y_3)
= (x_1 x_2 \otimes y_1 y_2)(x_3 \otimes y_3) = (x_1 x_2 x_3) \otimes (y_1 y_2 y_3)
= x_1(x_2 x_3) \otimes y_1(y_2 y_3) = (x_1 \otimes y_1)(x_2 x_3 \otimes y_2 y_3)
= (x_1 \otimes y_1)((x_2 \otimes y_2)(x_3 \otimes y_3)).
\]
Furthermore,
\[
1 \cdot x \otimes y = (1 \cdot x) \otimes y = x \otimes y,
\]
\[
\alpha \cdot (\beta \cdot x \otimes y) = \alpha \cdot ((\beta \cdot x) \otimes y) = (\alpha \cdot (\beta \cdot x)) \otimes y = (\alpha \beta \cdot x) \otimes y = \alpha \beta \cdot x \otimes y,
\]
λ \cdot (x_1 \otimes y_1)(x_2 \otimes y_2)
= \lambda \cdot x_1x_2 \otimes y_1y_2 = (\lambda \cdot x_1x_2) \otimes y_1y_2
= ((\lambda \cdot x_1)x_2) \otimes y_1y_2
= (\lambda \cdot (x_1 \otimes y_1))x_2 \otimes y_2 = (\lambda \cdot x_2 \otimes y_1)(x_2 \otimes y_2),

\lambda \cdot (x_1 \otimes y_1)(x_2 \otimes y_2)
= \lambda \cdot x_1x_2 \otimes y_1y_2 = (\lambda \cdot x_1x_2) \otimes y_1y_2
= x_1x_2 \otimes (\lambda \cdot y_1y_2) = (x_1 \otimes (\lambda \cdot y_2))(x_2 \otimes y_2)
= (x_1 \otimes y_1)(\lambda \cdot x_2 \otimes y_2),

so \(X \otimes Y\) is a scalable monoid.

It follows that if \(X, Y, Z\) are scalable monoids over \(R\) then \((X \otimes Y) \otimes Z\) and \(X \otimes (Y \otimes Z)\) are scalable monoids over \(R\): the tensor product of \(X \otimes Y\) and \(Z\) in the first case and of \(X\) and \(Y \otimes Z\) in the second case. It can also be shown that \((X \otimes Y) \otimes Z\) and \(X \otimes (Y \otimes Z)\) are isomorphic scalable monoids.

The tensor product can be used to "glue" scalable monoids together in a natural way so as to combine them into more inclusive scalable monoids. For example, \(R[[x; y]]\) is isomorphic to the tensor product \(R[[x]] \otimes R[[y]]\) but not to the direct product \(R[[x]] \times R[[y]]\).

### 2.5. Quotients of scalable monoids by normal submonoids

In a monoid we have \(x(yz) = (xy)z\) and \(1_X x = x 1_X\), so a submonoid \(\mathcal{M}\) of a scalable monoid \(X\) can act as a monoid on \(X\) by left or right multiplication. In particular, we can define an action \(\pi : \mathcal{M} \times X \to X\) by setting \(\pi(m, x) = mx\) for any \(m \in \mathcal{M}\) and \(x \in X\). This action can be used to define further notions in the same way that \(\sim\), \([x]\) and \(X/\sim\) were defined in terms of the scaling action \(\omega : R \times X \to X\).

**Definition 2.22.** Let \(X\) be a scalable monoid and \(\mathcal{M}\) a submonoid of \(X\). Then \(\sim_{\mathcal{M}}\) is the relation on \(X\) such that \(x \sim_{\mathcal{M}} y\) if and only if \(mx = ny\) for some \(m, n\) in \(\mathcal{M}\).

A normal submonoid of a scalable monoid \(X\) is a submonoid \(\mathcal{M}\) of \(X\) such that \(x\mathcal{M} = \mathcal{M}x\) for every \(x \in X\). It is clear that if \(\mathcal{M}\) is a central submonoid of \(X\), that is, if every element of \(\mathcal{M}\) commutes with every element of \(X\), then \(\mathcal{M}\) is normal, and every submonoid of a commutative scalable monoid is normal.

**Proposition 2.23.** If \(X\) is a scalable monoid and \(\mathcal{M}\) a normal submonoid of \(X\) then \(\sim_{\mathcal{M}}\) is an equivalence relation on \(X\).

**Proof.** The relation \(\sim_{\mathcal{M}}\) is reflexive since \(1_X x \sim_{\mathcal{M}} 1_X x\) for all \(x \in X\), symmetric by construction, and transitive because if \(m x = n y \equiv m'y = n'z\) for \(x, y, z \in X\) and \(m, n, m', n' \in \mathcal{M}\) then there is some \(n_0 \in \mathcal{M}\) such that \(m'mx = m'ny = n_0 m'y = n_0 n'z\), where \(m'm, n_0 n' \in \mathcal{M}\).

**Definition 2.24.** We denote the equivalence class of \(x\) for \(\sim_{\mathcal{M}}\) by \([x]_{\mathcal{M}}\), and the set of equivalence classes \([x]_{\mathcal{M}} \mid x \in X\) by \(X/\mathcal{M}\).

Results analogous to Propositions 2.13 and 2.15 hold for scalable monoids with normal submonoids.

**Proposition 2.25.** If \(X\) is a scalable monoid over \(R\) and \(\mathcal{M}\) a normal submonoid of \(X\) then the relation \(\sim_{\mathcal{M}}\) is a congruence on \(X\) with regard to the operations \((x, y) \mapsto xy\) and \((\lambda, x) \mapsto \lambda \cdot x\).
Proof. If \( mx = nx' \) and \( m'y = n'y' \) for some \( x, x', y, y' \in X \) and \( m, n, m', n' \in \mathfrak{M} \) then \( (mx)(m'y) = (nx')(n'y') \), so \( (mn_0')(xy) = (n'n_0')(x'y') \). Hence, if \( x \sim_{\mathfrak{M}} x' \) and \( y \sim_{\mathfrak{M}} y' \) then \( xy \sim_{\mathfrak{M}} x'y' \) since \( m'n_0', mn_0', mn_0 \in \mathfrak{M} \).

Also, if \( mx = nx' \) for some \( m, n \in M \) then \( \lambda \cdot mx = \lambda \cdot nx' \) for all \( \lambda \in R \), so \( m(\lambda \cdot x) = n(\lambda \cdot x') \). Hence, if \( x \sim_{\mathfrak{M}} x' \) then \( \lambda \cdot x \sim_{\mathfrak{M}} \lambda \cdot x' \).

In view of Proposition 2.27, we can define operations on \( X/\mathfrak{M} \) as follows.

Definition 2.26. Set \( [x]_{\mathfrak{M}}[y]_{\mathfrak{M}} = [xy]_{\mathfrak{M}}, \lambda \cdot [x]_{\mathfrak{M}} = [\lambda \cdot x]_{\mathfrak{M}} \) and \( 1_{X/\mathfrak{M}} = [1_X]_{\mathfrak{M}} \) for any \( [x], [y] \in X/\sim \) and \( \lambda \in R \).

With these definitions, \( X/\mathfrak{M} \) is an \((R, 1)\)-magma and the surjective function \( \phi_{\mathfrak{M}} : X \to X/\mathfrak{M} \) defined by \( \phi_{\mathfrak{M}}(x) = [x]_{\mathfrak{M}} \) satisfies the conditions
\[
\phi_{\mathfrak{M}}(xy) = \phi_{\mathfrak{M}}(x)\phi_{\mathfrak{M}}(y), \quad \phi_{\mathfrak{M}}(\lambda \cdot x) = \lambda \cdot \phi_{\mathfrak{M}}(x), \quad \phi_{\mathfrak{M}}(1_X) = 1_{X/\mathfrak{M}},
\]
so \( \phi_{\mathfrak{M}} \) is a homomorphism of \((R, 1)\)-magmas and hence of scalable monoids.

Proposition 2.27. If \( X \) is a scalable monoid over \( R \) and \( \mathfrak{M} \) a normal submonoid of \( X \) then \( X/\mathfrak{M} \) is a scalable monoid over \( R \) and the function
\[
\phi_{\mathfrak{M}} : X \to X/\mathfrak{M}, \quad x \mapsto [x]_{\mathfrak{M}}
\]
is a surjective homomorphism of scalable monoids.

Let us now consider the special case where the submonoid \( \mathfrak{M} \) of \( X \) considered above is a scalable submonoid \( M \) so that \( x \in M \) implies \( \lambda \cdot x \in M \) for every \( \lambda \in R \).

Proposition 2.28. If \( M \) is a normal scalable submonoid of a scalable monoid \( X \) over \( R \) then \( X/M \) is a trivally scalable monoid over \( R \).

Proof. If \( M \) is a scalable submonoid of \( X \) then \( (\lambda \cdot x) \sim_M x \) for any \( \lambda \in R \) and \( x \in X \) since \( 1_X(\lambda \cdot x) = (\lambda \cdot 1_X)x \), where \( 1_X, \lambda \cdot 1_X \in M \). Hence, \( \lambda \cdot [x]_M = [\lambda \cdot x]_M = [x]_M \) for any \( \lambda \in R \) and \( [x]_M \in X/M \).

Proposition 2.29. If \( M \) is a scalable submonoid of a scalable monoid \( X \) over \( R \) and \( x, y \in X \) then \( x \sim y \) implies \( x \sim_M y \).

Proof. If if \( \alpha \cdot x = \beta \cdot y \) for some \( \alpha, \beta \in R \) then
\[
(\alpha \cdot 1_X)x = \alpha \cdot 1_X x = \beta \cdot 1_X y = (\beta \cdot 1_X)y.
\]
This implies the assertion since \( \alpha \cdot 1_X, \beta \cdot 1_X \in M \).

Proposition 2.30. Let \( X \) be a scalable monoid over \( R \) and \( x, y \in X \). Then \( R \cdot 1_X \) is a normal scalable submonoid of \( X \), and \( x \sim_{R \cdot 1_X} y \) implies \( x \sim y \).

Proof. Consider any \( \alpha, \beta, \lambda \in R, x \in X \). \( R \cdot 1_X \) is (1) a submonoid of \( X \) since \( 1_X = 1 \cdot 1_X \) and \( (\alpha \cdot 1_X)(\beta \cdot 1_X) = \alpha \beta \cdot 1_X \) where \( 1, \alpha \beta \in R \), (2) a scalable submonoid of \( X \) since \( \lambda \cdot (\alpha \cdot 1_X) = \lambda \alpha \cdot 1_X \) where \( \lambda \alpha \in R \), and (3) a normal submonoid of \( X \) since \( (\alpha \cdot 1_X)x = \alpha \cdot 1_X x = \alpha \cdot x1_X = x(\alpha \cdot 1_X) \). Also, if \( x \sim_{R \cdot 1_X} y \) then
\[
\alpha \cdot 1_X x = (\alpha \cdot 1_X) x = (\beta \cdot 1_X)y = \beta \cdot 1_X y
\]
for some \( \alpha, \beta \in R \), so \( x \sim_{R \cdot 1_X} y \) implies \( x \sim y \).

It follows from Propositions 2.29 and 2.30 that \( x \sim_M y \) generalizes \( x \sim y \).

Corollary 2.31. Let \( X \) be a scalable monoid over \( R \) and \( x, y \in X \). Then \( R \cdot 1_X \) is a normal scalable submonoid of \( X \), and \( x \sim_{R \cdot 1_X} y \) if and only if \( x \sim y \).
For example, \( \kappa \cdot \lambda x_1^{k_1} \ldots x_n^{k_n} = (\kappa \cdot 1) x_1^{0} \ldots x_n^{0} ) (\lambda x_1^{k_1} \ldots x_n^{k_n}) \) for any \( \kappa \in R \), so \( R\langle x_1; \ldots ; x_n \rangle \) and \( R\langle x_1; \ldots ; x_n \rangle / (R \cdot 1_x) \) are isomorphic monoids.

2.6. A non-trivial orbitoid with a unit element is a free module of rank 1.

Recall the principle that magnitudes of the same kind can be added and subtracted, whereas magnitudes of different kinds cannot be combined by these operations. Also recall the idea that a quantity can be represented by a "unit" and a number (measure) specifying "[how many] times the [unit] is to be taken in order to make up" that quantity \([22, p. 41]\). As shown below, there is a connection between these two notions.

Specifically, it may happen that \( R \cdot u \supseteq [u] \) for some \( u \in [u] \), and if in addition a natural uniqueness condition is satisfied we may regard \( u \) as a unit of measurement for \([u]\). If such a unit exists then a sum of elements of \([u]\) can be defined by the construction described in Definition 2.35 below.

Definition 2.32. Let \( C \) be an orbitoid in a scalable monoid over \( R \). A generating element for \( C \) is some \( u \in C \) such that for every \( x \in C \) there is some \( \lambda \in R \) such that \( x = \lambda \cdot u \). A unit element for \( C \) is a generating element \( u \) for \( C \) such that if \( \lambda \cdot u = \lambda' \cdot u \) then \( \lambda = \lambda' \).

By this definition, if \( u \) is a generating element for \( C = [u] \) then \( R \cdot u \supseteq C \). On the other hand, \( \lambda \cdot u \sim u \) for any \( \lambda \in R \), so \( \lambda \cdot u \in [u] \) for any \( \lambda \in R \), so \( R \cdot u \subseteq [u] \). Thus, actually \( R \cdot u = C \).

We now need to consider zero elements in scalable monoids.

Proposition 2.33. Let \( X \) be a scalable monoid over \( R \). For every \( C \in X/\sim \) there is a unique \( 0_C \in C \) such that \( 0_C = 0 \cdot x \) for all \( x \in C \).

Proof. Only uniqueness needs to be proved. If \( x, y \in C \) then \( \alpha \cdot x = \beta \cdot y \) for some \( \alpha, \beta \in R \), so \( 0 \cdot x = 0 \cdot (\alpha \cdot x) = (0 \cdot \alpha) \cdot x = 0 \cdot (\beta \cdot y) = 0 \beta \cdot y = 0 \cdot y \). \( \square \)

We call \( 0_C \) the zero element of \( C \); note that distinct orbitoids have distinct zero elements since \( 0_A = 0_B \) implies \( A = 0_A = 0_B = B \). It is clear that \( \lambda \cdot 0_C = 0_C \) for all \( \lambda \in R \), and that \( 0_{[x]} y = 0_{[y]} x \) and \( 0_{[x]} 0_{[y]} = 0_{[y]} 0_{[x]} \) for all \( x, y \in X \).

A trivial orbitoid is an orbitoid \( C = \{0_C\} \).

We now turn to a lemma and a definition leading to Proposition 2.36.

Lemma 2.34. Let \( X \) be a scalable monoid over \( R \). If \( u \) and \( u' \) are unit elements for \( C \in X/\sim, \rho, \sigma, \rho', \sigma' \in R, \rho \cdot u = \rho' \cdot u' \) and \( \sigma \cdot u = \sigma' \cdot u' \) then \( (\rho + \sigma) \cdot u = (\rho' + \sigma') \cdot u' \).

Proof. As \( u' \in C \), there is a unique \( \tau \in R \) such that \( u' = \tau \cdot u \). Thus,
\[
\begin{align*}
\rho \cdot u &= \rho' \cdot u' = \rho' \cdot (\tau \cdot u) = \rho' \tau \cdot u, \\
\sigma \cdot u &= \sigma' \cdot u' = \sigma' \cdot (\tau \cdot u) = \sigma' \tau \cdot u,
\end{align*}
\]

so \( (\rho' + \sigma') \cdot u' = (\rho' + \sigma') \cdot (\tau \cdot u) = (\rho' + \sigma') \cdot u = (\rho' \tau + \sigma' \tau) \cdot u, \) since \( \rho = \rho' \tau \) and \( \sigma = \sigma' \tau. \) \( \square \)

Hence, the sum of two elements of a scalable monoid can be defined as follows.

Definition 2.35. Let \( X \) be a scalable monoid over \( R \), and let \( u \) be a unit element for \( C \in X/\sim \). If \( x = \rho \cdot u \) and \( y = \sigma \cdot u \), where \( \rho, \sigma \in R \), we set
\[ x + y = (\rho + \sigma) \cdot u. \]
The sum $x + y$ is given by Definition 2.35 if and only if $x$ and $y$ are commensurable and their orbitoid has a unit element. This suggests again that the concept of commensurability introduced in Definition 2.32 can be used to define the ancient Greek notion of magnitudes of the same kind, and to clarify the modern notion of quantities of the same kind.

It follows immediately from Definition 2.35 that

$$(x + y) + z = x + (y + z), \quad x + y = y + x$$

for all $x, y, z \in C$, and that

$$x + 0_C = 0_C + x = x$$

for any $x \in C$ since $0_C = 0 \cdot u$.

If $x = \rho \cdot u$ so that $\lambda \cdot x = \lambda \rho \cdot u$ and $\kappa \cdot x = \kappa \rho \cdot u$ then

$$(\lambda + \kappa) \cdot x = (\lambda + \kappa) \cdot (\rho \cdot u) = (\lambda + \kappa) \rho \cdot u = (\lambda \rho + \kappa \rho) \cdot u = \lambda \cdot x + \kappa \cdot x,$$

and if $x = \rho \cdot u$ and $y = \sigma \cdot u$ so that $\lambda \cdot x = \lambda \rho \cdot u$ and $\lambda \cdot y = \lambda \sigma \cdot u$ then

$$\lambda \cdot (x + y) = \lambda \cdot ((\rho + \sigma) \cdot u) = \lambda (\rho + \sigma) \cdot u = (\lambda \rho + \lambda \sigma) \cdot u = \lambda \cdot x + \lambda \cdot y.$$

A unital ring $R$ has a unique additive inverse $-1$ of $1 \in R$, and we set

$$-x = (-1) \cdot x$$

for all $x \in X$. If $C$ has a unit element $u$ and $x = \rho \cdot u$ for some $\rho \in R$ then $-x = (-1) \cdot (\rho \cdot u) = (-\rho) \cdot u$, and using this fact it is easy to verify that

$$x + (-x) = -x + x = 0_C.$$

As usual, we may write $x + (-y)$ as $x - y$, and thus $x + (-x)$ as $x - x$.

While a trivial orbitoid is a zero module $\{0_C\}$ with $0_C + 0_C = 0_C$ and $\lambda \cdot 0_C = 0_C$ for all $\lambda \in R$, a non-trivial orbitoid with a unit element is a well-behaved module.

**Proposition 2.36.** Let $X$ be a scalable monoid over $R$. If $C \in X/\sim$ is a non-trivial orbitoid with a unit element then $R$ is a non-trivial commutative ring, and $C$, with appropriate definitions of $x + y$ and $\lambda \cdot x$, is a free module of rank 1 over $R$.

**Proof.** Let $u$ be a unit element for $C$. If $0_C \neq x \in C$, $0_C = \lambda \cdot u$ and $x = \kappa \cdot u$ for some $\lambda, \kappa \in R$ then $\lambda \neq \kappa$, so $R$ is non-trivial. We also have $\alpha \beta \cdot u = \beta \alpha \cdot u$ for any $\alpha, \beta \in R$ by Lemma 2.5 so $\alpha \beta = \beta \alpha$ since $u$ is a unit element.

We have seen that $C$ is a module with addition given by Definition 2.35 and scalar multiplication inherited from the scalar multiplication in $X$. Also, if $u$ is a unit element for $C$ then $\{u\}$ is a basis for $C$, and $R$ has the invariant basis number property since it is non-trivial and commutative [27].

Thus, if every orbitoid $C \in X/\sim$ contains a non-zero unit element for $C$ then $X$ is the union of disjoint isomorphic free modules of rank 1 over a non-trivial commutative ring, a result that may be compared to definitions of systems of quantities in terms of unions of one-dimensional vector spaces by Quade [24] and Raposo [25].

Recall that identities corresponding to $(\lambda + \kappa) \cdot x = \lambda \cdot x + \kappa \cdot x$, $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$ and $\lambda \cdot (\kappa \cdot x) = \lambda \kappa \cdot x$ were proved in Propositions 1–3 in Book V of the *Elements*, so rudiments of Proposition 2.36 were present already in the Greek theory of magnitudes.
2.7. Scalable monoids with sets of unit elements. In this section, we build on the discussion in the two previous sections about unit elements and quotients of scalable monoids by normal monoids.

Definition 2.37. A dense set of elements of a scalable monoid $X$ is a set $U$ of elements of $X$ such that for every $x \in X$ there is some $u \in U$ such that $u \sim x$. A sparse set of elements of $X$ is a set $U$ of elements of $X$ such that $u \sim v$ implies $u = v$ for any $u, v \in U$. A closed set of elements of $X$ is a set $U$ of elements of $X$ such that if $u, v \in U$ then $uv \in U$.

We call a (dense) sparse set of unit elements of $X$ a (complete) system of unit elements for $X$.

Definition 2.38. A distributive scalable monoid $X$ is a scalable monoid such that for all $A, B \in X/\sim$ we have

\[(x + y)z = xz + yz, \quad z(x + y) = zx + zy,\]

for all $x, y \in A$ and all $z \in B$.

Proposition 2.39. Let $X$ be a scalable monoid. If $X$ is equipped with a dense closed set of unit elements $U$ then $X$ is a distributive scalable monoid.

Proof. For all $x, y \in A$ and $z \in B$ there are $u, v \in U$ such that $[x] = [y] = [u]$ and $[z] = [v]$ since $U$ is dense in $X$, so $x = \rho \cdot u$, $y = \sigma \cdot u$ and $z = \tau \cdot v$ for some $\rho, \sigma, \tau \in \mathbb{R}$, so $xz = \rho \tau \cdot uv$, $yz = \sigma \tau \cdot uv$, $zx = \tau \rho \cdot vu$ and $zy = \tau \sigma \cdot vu$, so

\[
(x + y)z = ((\rho + \sigma) \cdot u)(\tau \cdot v) = (\rho + \sigma)\tau \cdot uv = (\rho \tau + \sigma \tau) \cdot uv = xz + yz, \\
z(x + y) = (\tau \cdot v)((\rho + \sigma) \cdot u) = \tau(\rho + \sigma) \cdot vu = (\tau \rho + \tau \sigma) \cdot vu = zz + zy,
\]

using the fact that $uv$ and $vu$ are unit elements since $U$ is closed. \qed

We also define a natural notion which is fundamental in metrology.

Definition 2.40. A coherent system of unit elements for $X$ is a submonoid of $X$ which is a complete system of unit elements for $X$.

Recall that if $T$ is a normal submonoid of a scalable monoid $X$ then $X/T$ is a scalable monoid by Proposition 2.27. It is proved in Appendix A that if $S \supseteq T$ is a coherent system of unit elements for $X$ then $S/T$ is a coherent system of unit elements for $X/T$.

3. Quantity spaces

In this section, we specialize scalable monoids in order to obtain a mathematical model suitable for calculation with quantities, a quantity space. The formal definition of a quantity space is given in Section 3.1, and some basic facts about quantity spaces are presented in Section 3.2. Coherent systems of unit quantities for quantity spaces are discussed in Section 3.3. The notion of a measure of a quantity is formally defined in Section 3.4, and ways in which measures serve as proxies for quantities are described. In Section 3.5, we show that the monoid of dimensions $Q/\sim$ corresponding to a quantity space $Q$ is a free abelian group and derive some related results.
3.1. Canonical construction and main definition. It is possible to give an abstract definition of scalable monoids of the form $R[[x_1; \ldots; x_n]]$ (Example 2.4). Let $X$ be a commutative scalable monoid over a commutative ring $R$. A finite scalable-monoid basis for $X$ is a finite set $\{e_1, \ldots, e_n\}$ of elements of $X$ such that every $x \in X$ has a unique expansion

$$x = \mu \prod_{i=1}^{n} e_i^{k_i},$$

where $\mu \in R$ and $k_i$ are non-negative integers. In abstract terms, $R[[x_1; \ldots; x_n]]$ is a commutative scalable monoid $X$ over a commutative ring, such that there exists a finite scalable-monoid basis for $X$.

Now, consider instead the set $K[[x_1, x_1^{-1}; \ldots; x_n, x_n^{-1}]]$ of all Laurent monomials of the form $\lambda x_1^{k_1} \ldots x_n^{k_n}$, where $\lambda \in K$, $K$ is a field, $x_1, \ldots, x_n$ are uninterpreted symbols and $k_1, \ldots, k_n$ are integers, together with essentially the same operations as in Example 2.4, namely

$$\begin{align*}
(\lambda x_1^{j_1} \ldots x_n^{j_n})(\kappa x_1^{k_1} \ldots x_n^{k_n}) &= (\lambda\kappa) x_1^{(j_1+k_1)} \ldots x_n^{(j_n+k_n)}, \\
\alpha \cdot \lambda x_1^{k_1} \ldots x_n^{k_n} &= (\alpha\lambda) x_1^{k_1} \ldots x_n^{k_n}.
\end{align*}$$

Any such $K[[x_1, x_1^{-1}; \ldots; x_n, x_n^{-1}]]$ is likewise a scalable monoid, which can again be characterized abstractly.

**Definition 3.1.** Let $Q$ be a commutative scalable monoid over a field $K$. A finite quantity-space basis for $Q$ is a finite set $\{e_1, \ldots, e_n\}$ of invertible elements of $Q$ such that every $x \in Q$ has a unique expansion

$$x = \mu \cdot \prod_{i=1}^{n} e_i^{k_i},$$

where $\mu \in K$ and $k_i$ are integers.

It is easy to show that any commutative scalable monoid $Q$ over a field, such that there exists a finite quantity-space basis for $Q$, can be represented by some $K[[x_1, x_1^{-1}; \ldots; x_n, x_n^{-1}]]$ [15]. On the other hand, we have the following abstract characterization of this kind of scalable monoid, corresponding to a finitely generated free abelian group and well suited for applications in theoretical metrology, dimensional analysis etc.

**Definition 3.2.** A finitely generated quantity space is a commutative scalable monoid $Q$ over a field, such that there exists a finite quantity-space basis for $Q$.

Although finitely generated quantity spaces can be readily generalized to quantity spaces with infinite bases, only the finite case will be considered here. Below, "basis" and "quantity space" will be understood to mean "finite quantity-space basis" and "finitely generated quantity space", respectively.

Elements of a quantity space are called quantities, unit elements are called unit quantities, and orbitoids in a quantity space are called dimensions.

Note that $K[x_1; \ldots; x_n]$, where $K$ is a field, is not a quantity space, so a commutative scalable monoid over a field is not necessarily a quantity space: the relationship between a scalable monoid and a quantity space is not as close as that between a module and a vector space.
3.2. Some basic properties of quantity spaces.

**Proposition 3.3.** Let $Q$ be a quantity space over $K$ with a basis $\{e_1, \ldots, e_n\}$ and $x, y \in Q$. Then

1. $1_Q = 1 \cdot \prod_{i=1}^n e_i^0$;
2. if $x = \mu \cdot \prod_{i=1}^n e_i^{k_i}$ and $y = \nu \cdot \prod_{i=1}^n e_i^{\ell_i}$ then $xy = \mu \nu \cdot \prod_{i=1}^n e_i^{(k_i + \ell_i)}$;
3. if $x = \mu \cdot \prod_{i=1}^n e_i^{k_i}$ and $\mu \neq 0$ then $x^{-1}$ exists and $x^{-1} = \mu^{-1} \cdot \prod_{i=1}^n e_i^{-k_i}$.

**Proof.** (1) Note that $e_i^0 = 1_Q$ for all $e_i$. (2) This follows from Lemma 2.5 and the fact that $Q$ is commutative. (3) Note that $\mu^{-1} \cdot \prod_{i=1}^n e_i^{-k_i} \in Q$ since $\mu \in K$ and $e_1, \ldots, e_n \in Q$, so this follows from (1) and (2).

**Proposition 3.4.** Let $Q$ be a quantity space over $K$ with a basis $\{e_1, \ldots, e_n\}$ and $x = \mu \cdot \prod_{i=1}^n e_i^{k_i}$. Then the following conditions are equivalent:

1. $x$ is a non-zero quantity;
2. $\mu \neq 0$;
3. $x$ is invertible.

**Proof.** (1) $\iff$ (2). Note that $0 \cdot x = 0 \cdot (\mu \cdot \prod_{i=1}^n e_i^{k_i}) = 0 \cdot \prod_{i=1}^n e_i^{k_i}$. Thus, if $\mu = 0$ then $0 \cdot x = x$, so $x$ is a zero quantity. Conversely, if $0 \cdot x = x$ then $0 \cdot \prod_{i=1}^n e_i^{k_i} = 0 \cdot (\mu \cdot \prod_{i=1}^n e_i^{k_i}) = \mu \cdot \prod_{i=1}^n e_i^{k_i}$, so $\mu = 0$ since the expansion of $x$ is unique.

(2) $\iff$ (3). If $\mu \neq 0$ then $x$ has an inverse by Proposition 3.3. Conversely, if $\mu = 0$ then $\mu \nu = 0 \neq 1$ for all $\nu \in K$, so $x$ does not have an inverse $\nu \cdot \prod_{i=1}^n e_i^{\ell_i}$.

Thus, $1_Q$ is a non-zero quantity, and all elements of a basis are non-zero quantities. Also, it follows from Proposition 3.4 that $Q$ has no zero divisors.

**Corollary 3.5.** The product of non-zero quantities is a non-zero-quantity, and the non-zero quantities in a dimension $C$ form an abelian group.

**Lemma 3.6.** Let $Q$ be a quantity space over $K$ with a basis $\{e_1, \ldots, e_n\}$, and consider $x = \mu \cdot \prod_{i=1}^n e_i^{k_i}$ and $y = \nu \cdot \prod_{i=1}^n e_i^{\ell_i}$. The following conditions are equivalent:

1. $x \sim y$, or equivalently $\mu \cdot \prod_{i=1}^n e_i^{k_i} \sim \nu \cdot \prod_{i=1}^n e_i^{\ell_i}$;
2. $k_i = \ell_i$ for $i = 1, \ldots, n$;
3. $\prod_{i=1}^n e_i^{k_i} = \prod_{i=1}^n e_i^{\ell_i}$;
4. $\nu \cdot x = \mu \cdot y$, or equivalently $\nu \cdot (\mu \cdot \prod_{i=1}^n e_i^{k_i}) = \mu \cdot (\nu \cdot \prod_{i=1}^n e_i^{\ell_i})$.

**Proof.** Implications (2) $\implies$ (3) and (4) $\implies$ (1) are trivial, while (3) $\implies$ (4) follows from Lemma 2.5. To prove (1) $\implies$ (2), note that if $x \sim y$ then

$$\alpha \mu \cdot \prod_{i=1}^n e_i^{k_i} = \alpha \cdot (\mu \cdot \prod_{i=1}^n e_i^{k_i}) = \beta \cdot (\nu \cdot \prod_{i=1}^n e_i^{\ell_i}) = \beta \nu \cdot \prod_{i=1}^n e_i^{\ell_i}$$

for some $\alpha, \beta \in K$, so $k_i = \ell_i$ for $i = 1, \ldots, n$ because of the uniqueness of the expansion of $\alpha \cdot x$.

It follows immediately from Lemma 3.6 that if not $k_i = \ell_i$ for $i = 1, \ldots, n$ then $x \not\sim y$ since $x \sim y$; this is the essence of the principle of dimensional homogeneity formulated by Fourier [9].
3.3. Quantity spaces and unit quantities.

**Proposition 3.7.** If $Q$ is a quantity space then every non-zero $u \in Q$ is a unit quantity for $[u]$.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a basis for $Q$ and set $u = \mu \cdot \prod_{i=1}^{n} e_i^{k_i}$, $x = \nu \cdot \prod_{i=1}^{n} e_i^{l_i}$. Then $\mu \neq 0$ by Proposition 3.3 and if $x \sim u$ then $\mu \cdot x = \nu \cdot u$ by Lemma 3.6 so $x = \mu^{-1} \cdot (\mu \cdot x) = \mu^{-1} \cdot (\nu \cdot u) = \mu^{-1} \nu \cdot u$. Also, if $\lambda \cdot u = \lambda' \cdot u$ then $\lambda \mu \cdot \prod_{i=1}^{n} e_i^{k_i} = \lambda \cdot u = \lambda' \cdot u = \lambda' \mu \cdot \prod_{i=1}^{n} e_i^{k_i}$, so $\lambda \mu = \lambda' \mu$ since the expansion of $\lambda \cdot u$ is unique, so $\lambda = \lambda'$ since $\mu \neq 0$. \qed

**Proposition 3.8.** If $Q$ is a quantity space then every $C \in Q/\sim$ contains a non-zero unit quantity.

**Proof.** If $x = \mu \cdot \prod_{i=1}^{n} e_i^{k_i} \in C$ then $u = 1 \cdot \sum_{i=1}^{n} e_i^{k_i}$ is non-zero by Proposition 3.3 and $u \in C$ by Lemma 3.6 so $u$ is a unit quantity for $C$ by Proposition 3.7. \qed

**Proposition 3.9.** Let $Q$ be a quantity space over $K$. Then $C \in Q/\sim$, with $x + y$ and $\lambda \cdot x$ appropriately defined, is a one-dimensional vector space over $K$.

**Proof.** $C$ is a free module of rank 1 over the field $K$ by Propositions 2.39 and 3.8. \qed

**Proposition 3.10.** Let $Q$ be a quantity space over $K$ with a basis $E = \{e_1, \ldots, e_n\}$. The subset

$$U = \left\{ 1 \cdot \prod_{i=1}^{n} e_i^{k_i} \mid k_i \in \mathbb{Z} \right\}$$

of $Q$ is a coherent system of unit quantities for $Q$.

**Proof.** By Proposition 3.3 all elements of $U$ are non-zero and hence unit quantities by Proposition 3.7. Also, $U$ is dense in $Q$ since it follows from $x = \mu \cdot \prod_{i=1}^{n} e_i^{k_i}$ that $1 \cdot x = \mu \cdot \left( 1 \cdot \prod_{i=1}^{n} e_i^{k_i} \right)$. Finally, if $u = 1 \cdot \prod_{i=1}^{n} e_i^{k_i} \sim 1 \cdot \prod_{i=1}^{n} e_i^{l_i} = v$ then $\prod_{i=1}^{n} e_i^{k_i} = \prod_{i=1}^{n} e_i^{l_i}$ by Lemma 3.6 so $u = v$, meaning that $U$ is sparse in $Q$.

It remains to prove that $U$ is a monoid. Clearly, $1_Q \in U$ since $1_Q = 1 \cdot \prod_{i=1}^{n} e_i^{0_i}$, and we have

$$(1 \cdot \prod_{i=1}^{n} e_i^{k_i}) \left( 1 \cdot \prod_{i=1}^{n} e_i^{l_i} \right) = 1 \cdot \prod_{i=1}^{n} e_i^{(k_i + l_i)} ,$$

so if $u, v \in U$ then $uv \in U$. Thus, $U$ is a submonoid of $Q$. \qed

In other words, every basis can be extended to a coherent system of unit quantities, consisting of quantities that are expressed as products of basis quantities and their inverses. As a direct consequence, we have the following result.

**Proposition 3.11.** If $Q$ is a quantity space over $K$ then $Q$ is distributive.

**Proof.** The assertion follows from Propositions 3.10 and 2.39. \qed

3.4. Measures of quantities.

**Definition 3.12.** Let $Q$ be a quantity space over $K$ with a basis $E = \{e_1, \ldots, e_n\}$. The uniquely determined scalar $\mu \in K$ in the expansion

$$x = \mu \cdot \prod_{i=1}^{n} e_i^{k_i}$$

is called the *measure* of $x$ relative to $E$ and will be denoted by $\mu_E(x)$.

For example, $1_Q = 1 \cdot \prod_{i=1}^{n} e_i^{0_i}$ for any $E$, so we have the following simple but useful fact.
Proposition 3.13. If \( Q \) is a quantity space over \( K \) then \( \mu_E(1_Q) = 1 \) for any basis \( E \) for \( Q \).

Relative to a fixed basis, measures of quantities can be used as proxies for the quantities themselves.

Proposition 3.14. Let \( Q \) be a quantity space over \( K \) with a basis \( E = \{e_1, \ldots, e_n\} \). Then

1. \( \mu_E(xy) = \mu_E(x)\mu_E(y) \) for all \( x, y \in Q \);
2. \( x^{-1} \) exists and \( \mu_E(x^{-1}) = \mu_E(x)^{-1} \) for all \( x \in Q \) such that \( \mu_E(x) \neq 0 \);
3. \( \mu_E(\lambda \cdot x) = \lambda \mu_E(x) \) for all \( \lambda \in K \) and \( x \in Q \);
4. \( \mu_E(x + y) = \mu_E(x) + \mu_E(y) \) for all \( x, y \in X \) such that \( x \sim y \).

Proof. (1) This follows immediately from Proposition 3.3(2). (2) This follows similarly from Proposition 3.3(3).
(3) If \( x = \mu_E(x) \cdot \prod_{i=1}^n e_i^{k_i} \) then \( \lambda \cdot x = \lambda \cdot (\mu_E(x) \cdot \prod_{i=1}^n e_i^{k_i}) = \lambda \mu_E(x) \cdot \prod_{i=1}^n e_i^{k_i} \).
(4) If \( x = \mu_E(x) \cdot \prod_{i=1}^n e_i^{k_i} \) and \( y = \mu_E(y) \cdot \prod_{i=1}^n e_i^{k_i} \) are the expansions of \( x \) and \( y \) then \( \prod_{i=1}^n e_i^{k_i} = \prod_{i=1}^n e_i^{l_i} \) by Lemma 3.6.

As \( \prod_{i=1}^n e_i^{k_i} \) is non-zero, and thus a unit quantity for \( \prod_{i=1}^n e_i^{k_i} \) by Proposition 3.15, we have \( x + y = (\mu_E(x) + \mu_E(y)) \cdot \prod_{i=1}^n e_i^{k_i} \) by Definition 2.35.

□

Proposition 3.15. Let \( Q \) be a quantity space over \( K \). If \( E = \{e_1, \ldots, e_n\} \) is a basis for \( Q \) and \( x = \mu_E(x) \cdot \prod_{i=1}^n e_i^{k_i} \) then \( E' = \{\lambda_1 \cdot e_1, \ldots, \lambda_n \cdot e_n\} \), where \( \lambda_i \neq 0 \), is a basis for \( Q \) and \( x = \mu_{E'}(x) \cdot \prod_{i=1}^n (\lambda_i \cdot e_i)^{k_i} \), where \( \mu_{E'}(x) = \prod_{i=1}^n \lambda_i^{-k_i} \mu_E(x) \).

Proof. We have
\[
\begin{align*}
x &= \mu_E(x) \cdot \prod_{i=1}^n e_i^{k_i} = \mu_E(x) \cdot \prod_{i=1}^n (\lambda_i^{-1} \cdot (\lambda_i \cdot e_i))^{k_i} \\
&= \mu_E(x) \cdot \left( \prod_{i=1}^n \lambda_i^{-k_i} \cdot \prod_{i=1}^n (\lambda_i \cdot e_i)^{k_i} \right) = \mu_E(x) \prod_{i=1}^n \lambda_i^{-k_i} \cdot \prod_{i=1}^n (\lambda_i \cdot e_i)^{k_i}.
\end{align*}
\]
Hence, \( x \) has an expansion in terms of \( E' \). To prove uniqueness, assume that \( x = \mu \prod_{i=1}^n \lambda_i^{-\ell_i} \cdot \prod_{i=1}^n (\lambda_i \cdot e_i)^{\ell_i} \). Changing this expansion in terms of \( E' \) to an expansion in terms of \( E'' = \{\lambda_1^{-1} \cdot (\lambda_1 \cdot e_1), \ldots, \lambda_n^{-1} \cdot (\lambda_n \cdot e_n)\} \) gives
\[
\begin{align*}
x &= \mu \prod_{i=1}^n \lambda_i^{-k_i} \prod_{i=1}^n \lambda_i^{k_i} \cdot \prod_{i=1}^n (\lambda_i^{-1} \cdot (\lambda_i \cdot e_i))^{\ell_i} = \mu \cdot \prod_{i=1}^n e_i^{\ell_i},
\end{align*}
\]
so \( \mu = \mu_E(x) \) and \( \ell_i = k_i \) for \( i = 1, \ldots, n \) by the uniqueness of the expansion of \( x \) in terms of \( E = E'' \).

□

In general, the measure of a quantity thus depends on a choice of basis, but there is an important exception to this rule.

Proposition 3.16. Let \( Q \) be a quantity space over \( K \). For every \( x \in [1_Q] \), \( \mu_E(x) \) does not depend on \( E \).

Proof. \( 1_Q \) is a unit quantity for \([1_Q] \) by Proposition 3.7, so there is a unique \( \lambda \in K \) such that \( x = \lambda \cdot 1_Q \), so \( \mu_E(x) = \lambda \mu_E(1_Q) = \lambda \) for any basis \( E \) for \( Q \) by Propositions 3.13 and 3.14(3).

□

Remark 3.17. The \( \pi \) theorem in dimensional analysis depends on this result [16]. It is common to refer to any \( x \in [1_Q] \) as a "dimensionless quantity", although \( x \) is not really dimensionless – it belongs to, or "has", the dimension \([1_Q] \). Also, many authors (e.g., [29, 5, 31]) identify "dimensionless quantities" with numbers, but
Proposition 3.10 does not justify this identification. A "dimensionless quantity" does not correspond to a unique number, but to a number that depends on the choice of a quantity unit for $[1_Q]$. For example, plane angles can be measured in both radians and degrees. However, if we have a coherent system of units then $[1_Q]$ contains exactly one unit $1_Q$ since $U$ is a submonoid of $Q$ and $u \sim 1_Q$ implies $u = 1_Q$. Also, by Proposition 3.10 each choice of basis for $Q$ that is, each choice of so-called base units $[32]$ — gives rise to a coherent system of units. Note that for a plane angle $1_Q$ corresponds to the radian.

3.5. $Q/\sim$ is a free abelian group. In this section, we show that $Q/\sim$ regarded as a monoid has additional properties derived from the quantity space $Q$. Below, let $\bar{x}$ be given by $\bar{x} = 1 \cdot \prod_{i=1}^n e_i^{k_i}$, where $x = \mu \cdot \prod_{i=1}^n e_i^{k_i}$ is the expansion of $x \in Q$ relative to a basis for $Q$. Note that, irrespective of the choice of basis, $\bar{x}$ is a non-zero quantity by Proposition 3.4 and such that $\bar{x} \sim x$ by Lemma 3.6.

Proposition 3.18. If $Q$ is a quantity space then $Q/\sim$ is an abelian group.

Proof. $Q/\sim$ is a commutative monoid since $[x][y] = [xy] = [yx] = [y][x]$ for all $[x], [y] \in Q/\sim$. Also, $[x][\bar{x}^{-1}] = [\bar{x}^{-1}] = [\mu \cdot 1_Q] = [\bar{x}^{-1}]$ since $[\mu \cdot 1_Q] = [1_Q] = 1_{Q/\sim}$. Thus, $Q/\sim$ is an abelian group. □

Recall that a basis for a finitely generated abelian group $G$ is a set $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of elements of $G$ such that every $x \in G$ has a unique expansion $x = \prod_{i=1}^n e_i^{k_i}$, where $k_i$ are integers.

Proposition 3.19. Let $Q$ be a quantity space with a basis $E = \{e_1, \ldots, e_n\}$. Then $E = \{e_1, \ldots, [e_n]\}$ is a basis for $Q/\sim$ with the same cardinality as $E$.

Proof. The unique expansions of $e_i, e_j \in E$ relative to $E$ are $e_i = 1 \cdot (\cdots e_{i-1}^0 e_i^1 e_{i+1}^0 \cdots), e_j = 1 \cdot (\cdots e_{j-1}^0 e_j^1 e_{j+1}^0 \cdots)$. Hence, if $e_i \neq e_j$ then $[e_i] \neq [e_j]$ by Lemma 3.6. This means that the surjective mapping $\phi : E \to \{[e_1], \ldots, [e_n]\}$ given by $\phi(e_i) = [e_i]$ is injective as well and thus a bijection. It remains to show that $E$ is a basis for $Q/\sim$.

First, let $[x]$ be an arbitrary dimension in $Q/\sim$. As $E$ is a basis for $Q$, we have $x = \mu \cdot \prod_{i=1}^n e_i^{k_i}$ for some $\mu \in K$ and some integers $k_1, \ldots, k_n$, so $[x] = \left[\mu \cdot \prod_{i=1}^n e_i^{k_i}\right] = \prod_{i=1}^n [e_i]^{k_i}$. Also, if $[x] = \prod_{i=1}^n [e_i]^{k_i} = \prod_{i=1}^n [e_i]^{k_i}$, then $\prod_{i=1}^n e_i^{k_i} \sim \prod_{i=1}^n e_i^{k_i}$, so $\prod_{i=1}^n e_i^{k_i} \sim \prod_{i=1}^n e_i^{k_i}$. □

A (finitely generated) abelian group for which there exists a basis is said to be free abelian (of finite rank). Hence, corresponding to the fact that if $X$ is a scalable monoid then $X/\sim$ is a monoid, we have the following much stronger result.

Proposition 3.20. If $Q$ is a quantity space then $Q/\sim$ is a free abelian group of finite rank.

Recall that any two bases for a free abelian group $G$ have the same cardinality, the rank of $G$. Proposition 3.19 thus leads to an analogue of the dimension theorem for finite-dimensional vector spaces.

Proposition 3.21. If $Q$ is a quantity space then any two bases for $Q$ have the same cardinality.
Proof: If $E = \{e_1, \ldots, e_n\}$ and $E' = \{e_1', \ldots, e_m'\}$ are bases for $Q$, so that $E = \{[e_1], \ldots, [e_n]\}$ and $E' = \{[e_1'], \ldots, [e_m']\}$ are bases for $Q/\sim$, then $|E| = |E'| = |E'|$ by Proposition 3.19 and the equicardinality of bases for a free abelian group. □

A quantity space with bases of cardinality $n$ is said to be of rank $n$.

Example 3.22. The dimensions corresponding to base quantities in the International System of Units (SI) $^{[33]}$ such as the dimensions of length, time and mass, denoted $L$, $T$ and $M$, respectively, are elements of a basis for some free abelian group $Q/\sim$. For example, $\{L, T, M\}$ is a basis for $Q/\sim$, where $Q$ is a quantity space for classical mechanics. This is not the only possible basis, however. For example, $\{L, T, F\}$, where $F = MLT^{-2}$, is another three-element basis for $Q/\sim$, and another possible set of base dimensions for classical mechanics.

Let us consider quantity spaces $Q$ and $Q'$ over $K$ with bases $E = \{e_1, \ldots, e_n\}$ and $E' = \{e_1', \ldots, e_m'\}$. It is easy to verify that a bijection $\phi : E \to E'$ can be extended to an isomorphism $\phi^* : Q \to Q'$ by setting $\phi^* \left( \mu \cdot \prod_{i=1}^{n} e_i^{k_i} \right) = \mu \cdot \prod_{i=1}^{n} \phi(e_i)^{k_i}$. Conversely, if $\phi^* : Q \to Q'$ is an isomorphism then $\{\phi^*(e_1), \ldots, \phi^*(e_n)\}$ is clearly a basis for $Q'$ of the same cardinality as $E$. These observations lead to the following classification theorem, similar to a theorem in linear algebra:

**Proposition 3.23.** Quantity spaces over the same field are isomorphic if and only if they are of the same rank (cf. [24]).

There is a reciprocal connection between bases for $Q$ and bases for $Q/\sim$.

**Proposition 3.24.** Let $Q$ be a quantity space, and let $E = \{e_1, \ldots, e_n\}$ be a basis for $Q/\sim$. Then there is a subset $E = \{e_1, \ldots, e_n\}$ of $Q$ such that $e_i \in e_i$, and $E$ is a basis for $Q$.

Proof. We can choose a function $\psi : E \to \{\psi(e_1), \ldots, \psi(e_n)\}$ such that we have $0_{e_i} \neq \psi(e_i) \in e_i$ for all $e_i$. This is a surjective function, and $e_i \neq e_j$ implies $e_i \cap e_j = \emptyset$, so $\psi$ is injective as well and hence a bijection. For convenience, we write $\psi(e_i)$ as $e_i$. Each $e_i$ is invertible by Proposition 3.24.

Let $x$ be an arbitrary quantity in $Q$. As $E$ is a basis for $Q/\sim$, we have $[x] = \prod_{i=1}^{n} e_i^{k_i} = \prod_{i=1}^{n} [e_i]^{k_i} = \prod_{i=1}^{n} e_i^{k_i}$ for some integers $k_1, \ldots, k_n$, and as $e_i \neq 0_{e_i}$ for each $e_i$, $\prod_{i=1}^{n} e_i^{k_i}$ is non-zero and thus a unit quantity for $[x]$ by Proposition 3.7. Hence, there exists a unique $\mu \in K$ for $\prod_{i=1}^{n} e_i^{k_i}$ such that $x = \mu \cdot \prod_{i=1}^{n} e_i^{k_i}$. Also, if $x = \mu \cdot \prod_{i=1}^{n} e_i^{k_i} = \nu \cdot \prod_{i=1}^{n} e_i^{k_i}$, then $\prod_{i=1}^{n} e_i^{k_i} = \prod_{i=1}^{n} e_i^{k_i}$, so $\prod_{i=1}^{n} [e_i]^{k_i} = \prod_{i=1}^{n} [e_i]^{k_i}$, for $\ell_i = k_i$ for $i = 1, \ldots, n$, since $E$ is a basis for $Q/\sim$, so $\nu = \mu$. □

We can now extend to quantity spaces the theorem that a subgroup of a free abelian group is free abelian, using this fact.

**Proposition 3.25.** If a subalgebra $Q'$ of a quantity space $Q$ regarded as a scalable monoid contains all inverses of elements of $Q'$ then $Q'$ is a quantity space.

Proof. First note that $Q'$ is a scalable monoid, so $Q'/\sim$ is a monoid. Also, recall from the proof of Proposition 3.18 that $[x^{-1}] = [x]^{-1}$ so if $\bar{x} \in Q'$ implies $\bar{x}^{-1} \in Q'$ then $[x] \in Q'/\sim$ implies $[x]^{-1} \in Q'/\sim$ since $x \in Q'$ implies $\bar{x} \in Q'$. Hence, $Q'/\sim$ is a subgroup of $Q/\sim$, so $Q'/\sim$ is a free abelian group with a basis $E$ corresponding to a basis $E$ for $Q'$ by Proposition 3.24. □
This result is analogous also to the simple fact that a submodule of a vector space is a vector space, so we have found yet another similarity between free abelian groups, quantity spaces and vector spaces.

References

[1] Birkhoff, G. (1935). On the structure of abstract algebras. Proc. Cambridge Philos. Soc. 31, 433–454.
[2] de Boer, J. (1994). On the history of quantity calculus and the international system, Metrologia 31, 405–429.
[3] Carlson, D.E. (1979). A mathematical theory of physical units, dimensions and measures. Archive for Rational Mechanics and Analysis, 70, 289–304.
[4] Descartes, R. (1637). La Geometrie, Discours de la Méthode. Leiden.
[5] Drobot, S. (1953). On the foundations of dimensional analysis. Stud. Math. 14, 84–99.
[6] Euclid (of Alexandria) (c 300 BC). Stoicheia (The Elements).
[7] Euler, L. (1740). Einleitung zur Rechen-Kunst. Bd. 2. St Petersburg.
[8] Fleischmann, R. (1951). Die Struktur des physikalischen Begriffssystemes, Z. Phys. 129, 377–400.
[9] Fourier, J.B.J. (1822). Théorie analytique de la Chaleur. Paris.
[10] Fourier, J.B.J (2009). The Analytical Theory of Heat (translation of [7]). Cambridge University Press.
[11] Gowers, W.T. Two definitions of ’definition’. https://www.dpmms.cam.ac.uk/~wtg10/definition.html, retrieved 25-09-2019.
[12] Gowers W.T. (2002). Mathematics. A Very Short Introduction. Oxford University Press.
[13] Hasse, H. & Schols, H. (1928). Die Grundlagenkrise der griechischen Mathematik, Kants Studien 33, 4–34.
[14] Jonsson, D. (2014). Quantities, Dimensions and Dimensional Analysis. arXiv:1408.5024 [math.HO].
[15] Jonsson, D. (2019). Magnitudes Reborn: Quantity Spaces as Scalable Monoids. arXiv:1911.07236 [math.RA].
[16] Jonsson, D. (2022). Theory and Application of Augmented Dimensional Analysis. arXiv:2211.04267 [math-ph].
[17] Kitano, M. (2013). Mathematical structure of unit systems. J. Math. Phys. 54, 052901.
[18] Kock, A. (1989). Mathematical structure of physical quantities. Archive for Rational Mechanics and Analysis, 107, 99–104.
[19] Landolt, M. (1943). Grösse, Masszahl und Einheit. Zürich.
[20] Lodge, A. (1888). The multiplication and division of concrete quantities. Nature 38 281–283.
[21] Malet, A. (2006). Renaissance notions of number and magnitude, Hist. Math. 33, 63–81.
[22] Maxwell, J. (1873). Treatise on Electricity and Magnetism, Oxford University Press.
[23] Newton, I. (1720). Universal Arithmetick. London.
[24] Quade, W. (1961). Über die algebraische Struktur des Größenkalküls der Physik, Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft 13, 24–65.
[25] Raposo, A.R. (2018). The algebraic structure of quantity calculus, Measurement Science Review 18, 147–157.
[26] Raposo, A.R. (2021), Equivalence of two algebraic structures underlying quantity calculus. Manuscript.
[27] Richman, F. (1988). Nontrivial uses of trivial rings. Proc. Am. Math. Soc. 103, 1012–1014.
[28] Stevin (de Bruges), S. (1585). L’Arithmétique. Leiden.
[29] Wallot, J. (1926). Dimensionen, Einheiten, Masseysteme, Handbuch der Physik II. Springer.
[30] van der Waerden, B. L. (1930). Moderne Algebra. Teil I. Springer.
[31] Whitney, H. (1968). The mathematics of physical quantities: Part II: Quantity structures and dimensional analysis. Amer. Math. Monthly 75, 227–256.
[32] (2012). International Vocabulary of Metrology – Basic and General Concepts and Associated Terms (VIM). 3rd edition. Bureau International des Poids et Mesures.
[33] (2019) International System of Units (SI). 9th edition. Bureau International des Poids et Mesures.
Appendix A. Coherent systems of units in quotient spaces

Proposition A.1. Let \( X \) be a scalable monoid over \( R \), a coherent system of unit elements for \( X \), and \( T \subseteq S \) a normal submonoid of \( X \). Then \( X/T \) is a scalable monoid, \( [t]_T = [1x]_T \) for any \( t \in T \), and \( S/T = \{ [s]_T \mid s \in S \} \) is a coherent system of unit elements for \( X/T \).

Proof. By Proposition 2.27, \( X/T \) is a scalable monoid since \( T \) is a normal submonoid of \( X \), and if \( t \in T \) then \( t \sim_X 1_X \) since \( 1XT = t1X \) and \( 1x \in T \).

Thus, \( [t]_T = [1x]_T = 1x/T \) since

\[
[1x]_T[x]_T = [1x1x]_T = [x]_T = [x1x]_T = [x]_T[1x]_T
\]

for any \( x \in X \). Also, if \( [s]_T, [s']_T \in S/T \), meaning that \( s, s' \in S \), then \( [s]_T[s']_T = [ss']_T \in S/T \) since \( ss' \in S \). Hence, \( S/T \) is a submonoid of \( X/T \).

Assume that \( [s]_T \sim [s']_T \), where \( s, s' \in S \). Then \( \rho \cdot [s]_T = \sigma \cdot [s']_T \) for some \( \rho, \sigma \in R \), so \( [\rho \cdot s]_T = [\sigma \cdot s']_T \), so \( \rho \cdot s \sim_T \sigma \cdot s' \), so \( t(\rho \cdot s) = t'(\sigma \cdot s') \) for some \( t, t' \in T \), so \( \rho \cdot ts = \sigma \cdot t's \), so \( ts \sim t's \) where \( ts, t's \in S \). Hence, \( ts = t's \) since \( S \) is sparse in \( X \), so \( s \sim_T s' \), meaning that \( [s]_T = [s']_T \). Thus, \( S/T \) is a sparse set of elements of \( X/T \).

Consider any \( C \in X/T \) and let \( x \in X \) be such that \( [x]_T \in C \). Then there is some \( s \in S \) and some \( \rho \in R \) such that \( x = \rho \cdot s \) since \( S \) is a dense set of unit elements in \( X \), and hence \( [x]_T = [\rho \cdot s]_T = \rho \cdot [s]_T \). If \( \rho \cdot [s]_T = \sigma \cdot [s]_T \) then \( [\rho \cdot s]_T = [\sigma \cdot s]_T \), so \( \rho \cdot s \sim_T \sigma \cdot s \), and as in the preceding paragraph this implies that \( \rho \cdot ts = \sigma \cdot t's \) and \( ts = t's \) for some \( t, t' \in T \). Hence, \( \rho = \sigma \) since \( ts \in S \) is a unit element for \([ts] \). This means that \( [s]_T \) is a unit element for \( C \), so \( S/T \) is dense in \( X/T \) as well as sparse.

As a simple example, \( S = \{ 1x^k_1 y^k_2 z^k_3 \mid k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0} \} \) is a coherent system of unit elements for \( \mathbb{R}[x; y; z] \), \( T = \{ 1x^{k_1} y^{k_2} z^{k_3} \mid k_1, k_2 \in \mathbb{Z}_{\geq 0} \} \) is a normal submonoid of \( \mathbb{R}[x; y; z] \), \( T \subseteq S \), and \( \mathbb{R}[x; y; z]/T \) is isomorphic to \( \mathbb{R}[z] \).

A typical application of Proposition A.1 in physics is described by Raposo [25]:

The mechanism of taking quotients is the algebraic tool underlying what is common practice in physics of choosing “systems of units” such that some specified universal constants become dimensionless and take on the numerical value 1. [...] But it has to be remarked that the mechanism goes beyond a change of system of units; it is indeed a change of space of quantities. [p. 153]

For example, one may decide to measure both time and length by means of a measure for length, using the universal constant \( c \), thus introducing a new system of units such that times and lengths are not distinguishable (see further [14]). Such an operation amounts to a projection \( x \mapsto [x]_T \) of the original space of quantities \( X \) onto a quotient space \( X/T \). In terms of \( S \) and \( T \), if \( S = \{ tk_1 t k_2 \mid k_1, k_2 \in \mathbb{Z} \} \), where \( t \) is a unit for time and \( \ell \) a unit for length, is a coherent system of unit elements for \( X \) and we set \( T = \{ t k \mid k \in \mathbb{Z} \} \) then \( S/T \) corresponds to \( \{ \ell k \mid k \in \mathbb{Z} \} \), meaning that both time and length are measured by reference to a unit in \( X/T \) corresponding to a unit for length in \( X \).
Appendix B. Notes on related work and fields of application

B.1. Theoretical approaches. In 1945, Landolt [19], apparently inspired by the development of abstract algebra during the interwar period in Europe [30], called attention to group operations on systems of quantities. Specifically, he pointed out that the invertible quantities form a group under “qualitative Verknüpfung”, that is, multiplication of quantities, and that quantities of the same kind form a group under “intensive Verknüpfung”, that is, addition of quantities.

In a seminal article, Fleischmann [8] shifted the focus from quantities (Grössen) to kinds of quantities (Grössenarten), and suggested that kinds of quantities can themselves be multiplied, requiring that if \( q \) is of kind \( K \) and \( q' \) is of kind \( K' \) then \( qq' \) is of kind \( KK' \). He also proposed that a set of kinds of quantities with a product defined in this way would be a finitely generated free abelian group.

Quade [24] defines systems of quantities by means of a rather complicated construction with one-dimensional vector spaces as building blocks. As a first step, he defines a quantity system as the union \( U \) of all vector spaces \( V_i \) in a countably infinite set \( V \) of pairwise disjoint one-dimensional vector spaces over the real or complex numbers. He then assumes that for any \( x, x' \in U \) there exists an associative, commutative product \( xx' \) satisfying \( \lambda(xx') = (\lambda x)x' \) and such that \( VV = \{xx' \mid x \in V, x' \in V'\} \) is a one-dimensional vector space. Next, he selects a finite number of vector spaces \( V_1, \ldots, V_n \in V \) and considers the set \( \mathcal{G} \) of all products of vector spaces of the form \( \prod_{j=1}^n V_j \), where \( k_j \) are integers. \( \mathcal{G} \) is a finitely generated free abelian group of rank \( n \). To ensure the supply of inverses of quantities, he embeds the set of non-zero elements of the selected vector spaces in a group of fractions \( \mathcal{G} \), using the fact that \( U \) is a commutative semigroup.

Quade’s construction is actually even more complicated than sketched here. The point is that \( \mathcal{G} \) corresponds to a set of quantities, while \( \mathcal{G} \) corresponds to a set of kinds of quantities (or dimensions). Landolts and Fleischmann’s ideas are elaborated formally, but unfortunately not clarified.

Carlson’s [3] definition of quantities is based on a set of “pre-units” which is in effect a predefined basis \( \Xi \) for a finite-dimensional vector space \( V_\Xi \) over \( \mathbb{Q} \) with multiplication as binary operation, so that \( \xi + \eta \) is written as \( \xi\eta \) is \( \xi^\lambda \). Carlson then defines a quantity as a pair \((r, \xi)\), where \( r \) is a real number and \( \xi \) a pre-unit. Multiplication of quantities by real numbers is defined by setting \( a(r; \xi) = (ar, \xi) \), multiplication of quantities is defined by setting \( (r, \xi)(s, \eta) = (rs, \xi\eta) \) and fractional powers of quantities are defined by setting \((r, \xi)^{m/n} = (\sqrt[n]{r^m}, \xi^{m/n})\) if a (unique) positive real nth root \( \sqrt[n]{r^m} \) of \( r^m \) exists; we obtain a quantity structure \( \mathbb{R} \times V_\Xi \). A “fundamental system of units” for \( \mathbb{R} \times V_\Xi \) is a set \( \{(u_1, \xi_1), \ldots, (u_m, \xi_m)\} \) such that \( u_i \in \mathbb{R}_{>0} \) and \( \xi_i \in \Xi \), where every \( q \in \mathbb{R} \times V_\Xi \) has a unique expansion \( q = ru_1\xi_1^{\epsilon_1} \cdots u_m\xi_m^{\epsilon_m} \), where \( r \in \mathbb{R} \). The sequence of exponents \( \epsilon_1, \ldots, \epsilon_m \) associated with \( q \), which is clearly the same for all fundamental systems of units, Carlson calls the “dimensions” of \( q \), and he says that quantities are of the same kind if they have the same dimensions.

Carlson’s construction is incomplete in the sense that vector space operations for quantities of the same kind are not considered, and multiplication of kinds of quantities is not defined. Instead, we have multiplication of pre-units, which can be seen as units in a fixed, coherent system of units.
The approach introduced by Drobot [3] and developed by Whitney [31] is based on the idea that the set of quantities itself — rather than a set of pre-units — is just a vector space $V_Q$ under multiplication of quantities and over a field $R$, so there is a scalar product $q^\lambda$, where $\lambda \in R, q \in V_Q$. $V_Q$ is also assumed to contain a set $R$ of scalars, so another scalar product can be defined as a usual product $rq$, where $r \in R, q \in V_Q$. (Thus, the authors identify dimensionless quantities with scalars.)

Both Drobot and Whitney define, in slightly different ways, sets of quantities of the same kind, called “dimensions” by Drobot and “birays” by Whitney, and both define addition of quantities of the same kind, called “dimensions” by Drobot and “birays” by Whitney, and both define addition of quantities of the same kind, called “dimensions” by Drobot and “birays” by Whitney, and both define addition of quantities of the same kind, and that $\lambda q$ implies $q, r, s$ for any quantities $q, r, s$.

Unfortunately, the assumptions that $V_Q$ is a vector space over $R$ with scalar multiplication $(\lambda, q) \mapsto q^\lambda$ and that $V_Q$ contains a set of scalars $R$ are not fully compatible. In particular, $q^\lambda$ is not a real number for all $q, \lambda \in R$, and if we require that $v \subseteq R_{>0}$ to avoid this problem then all quantities must be positive. Anyway, while integral powers of quantities make sense in physics, it is not clear how to interpret $q^{0.2}$ or $q^i$, where $q$ is a “dimensionful” quantity rather than a number.

Kock [18] proposed a limited but elegant construction also based on the ‘vector-space-with-embedded-scalars’ idea. Accepting the restriction to positive quantities, he pointed out that in the short exact sequence of vector spaces over $Q$,

$$Q_{>0} \xrightarrow{\iota} P \xrightarrow{d} D,$$

where $\iota$ is an inclusion map, $d$ a surjective $Q$-linear map, and the kernel of $d$ is the image of $\iota$, $P$ can be interpreted as a set of quantities, $D$ as a set of dimensions, $Q_{>0}$ as the “dimensionless” quantities in $P$ and, for every $M \in D$, $d^{-1}(M)$ as a set of quantities of the same kind. The operations on $Q_{>0}$, $P$ and $D$ are the usual operations in (multiplicatively written) vector spaces over $Q$, and the identities in the algebraic structure containing $Q_{>0}$, $P$ and $D$ are those that follow from the vector space axioms and the $Q$-linearity of $\iota$ and $d$.

More recently, Raposo [29] has proposed a definition of a system or “space” of quantities somewhat similar to Quade’s but more concise and elegant. By this definition, a space of quantities $Q$ is an algebraic fiber bundle, with fibers of quantities attached to dimensions (kinds of quantities) in a base space assumed to be a finitely generated free abelian group. Each fiber is again a one-dimensional vector space, with scalar product $\lambda q$. Multiplication of quantities and multiplication of dimensions are defined independently, but are assumed to be compatible in the same sense as for Quade. The quantities constitute a commutative monoid, and it is assumed that $q(r + s) = qr + qs$ for any quantities $q, r, s$, where $r, s$ are of the same kind, and that $\lambda(qr) = (\lambda q)r$ for any scalar $\lambda$ and quantities $q, r$. Although this is technically not part of the definition of a space of quantities, Raposo also assumes that if $q$ and $r$ are non-zero quantities then $qr$ is a non-zero quantity.

Raposo’s theory of quantity spaces (supplemented with a ‘no zero divisors’ condition) is complete and free from anomalies. Compared to the theory presented in this article, it contains some redundant elements, since there are more primitive
notions. However, the two theories have been shown to be completely equivalent [26]. This lends credence to both theories, since they were developed independently.

The table below contains a simplified comparison of some aspects of six of the approaches to quantity calculus reviewed above.

| Aspects                      | Authors          |
|------------------------------|------------------|
|                              | Drobot (1953)    |
|                              | Quade (1961)     |
|                              | Whitney (1968)   |
|                              | Carlson (1979)   |
|                              | Jonsson (2014)   |
|                              | Raposo (2016)    |
| Addition of quantities       | Derived          |
| Scalar product of quantities | Primitive        |
| Product of quantities        | Derived          |
| Product of dimensions        | Derived          |
| Exponent of quantities       | Derived          |
| Exponent of dimensions       | Derived          |

Note: For Carlson, we consider products and exponents of pre-units instead of dimensions.

B.2. Rules of quantity calculus. In [2], de Boer lists some fundamental rules for calculation with quantities, drawn from the literature on quantity calculus. All these rules can be derived from the theory of quantity spaces. The list below includes references to de Boer’s rules and relevant definitions or results from this article.

1. (A2.1; Definition 2.1, Corollary 3.5). A quantity can be multiplied by a quantity as in a monoid, and the non-zero quantities form an abelian group under multiplication.
2. (A2.2; Definition 2.1). A quantity $q$ can be multiplied by a number $\lambda$, and we have the identities $1 \cdot q = q$ and $\alpha \cdot (\beta \cdot q) = (\alpha \beta) \cdot q$.
3. (A2.2; Definition 2.1). Multiplication of quantities by numbers and by quantities are related by $\lambda \cdot (pq) = (\lambda \cdot p)q = p(\lambda \cdot q)$.
4. (A3.1, A6.1; Definition 2.1). A system of quantities can be partitioned into equivalence classes of quantities of the same kind. (de Boer makes a distinction between such equivalence classes and dimensions, but in the present theory these two notions coincide.)
5. (A3.2; Proposition 2.20). Kinds of quantities can be multiplied, forming a finitely generated free abelian group under multiplication.
6. (A3.3; Proposition 2.9). If $p$ and $q$ are of the same kind then $\lambda \cdot (p + q) = \lambda \cdot p + \lambda \cdot q$ and $(\alpha + \beta) \cdot q = \alpha \cdot q + \beta \cdot q$.
7. (A3.4; Remark 3.17) All “dimensionless” quantities are of the same kind.
8. (A4.1, A6.2; Proposition 3.20). Kinds of quantities can be multiplied, forming a finitely generated free abelian group under multiplication.
9. (A4.2, A6.2; Proposition 2.10, Definition 2.14). If $q$ is a quantity of kind $K$ and $q'$ a quantity of kind $K'$ then $qq'$ is a quantity of kind $KK'$.
(10) (A5.1; Proposition 3.8). For every kind of quantities and every quantity \( q \) of this kind there is a quantity \( u \) of the same kind such that \( q = \mu \cdot u \), where \( \mu \) is a uniquely determined number. Such a quantity \( u \) is called a unit.

(11) (A5.1, A5.2; Proposition 3.10). It is possible to select exactly one unit from each kind of quantities in such a way that if \( u \) is the unit of kind \( K \) and \( u' \) is the unit of kind \( K' \) then \( uu' \) is the unit of kind \( KK' \). A set of units satisfying this condition is said to be coherent.

We can derive some more fundamental rules not considered by de Boer in [2].

(i) (Corollary 2.11). \( \lambda \cdot q \) is a quantity of the same kind as \( q \).

(ii) (Proposition 3.11). If \( q \) is a quantity and \( r, s \) are quantities of the same kind then \( q(r + s) = qr + qs \).

(iii) (Corollary 3.5). If \( q \) and \( r \) are non-zero quantities then \( qr \) is non-zero.

B.3. On applications of the theory of quantity spaces. At the heart of theoretical metrology is the relationship between measures and what is measured—quantities. Quantity space theory can clarify this relationship by elucidating the nature of quantities and systems of quantities.

In the International Vocabulary of Metrology from 2012 (VIM3) [32], one sense of "quantity" (1.1) is a generic one, corresponding to \( \text{Grössenart} \) or kind of quantity, while a "quantity value" (1.19) represents a particular \( \text{Grösse} \) or a quantity as defined here. Unit quantities are called "(measurement) units" (1.9) in VIM3. A set of "base quantities" (1.4) is in effect a set \( E \) of selected kinds of quantities, or equivalently dimensions, which is a basis for some \( Q/\sim \); the elements of a corresponding basis \( E = \{e_1, \ldots, e_n\} \) for \( Q \) are "base units" (1.10). Further, in VIM3 a "derived unit" (1.11) is some \( u \in Q \), other than a base unit, with an expansion \( u = \mu \cdot \prod_{i=1}^{n} e_i^{k_i} \), where \( \mu \neq 0 \), while a "coherent derived unit" (1.12) is a derived unit \( v \) with an expansion \( v = 1 \cdot \prod_{i=1}^{n} e_i^{k_i} \). Following Fourier [9], VIM3 defines a "quantity dimension" (1.7) of a quantity \( q \in Q \) in terms of an integer tuple \( (k_1, \ldots, k_n) \) describing how \( q \) is expressed as \( \mu_E(q) \cdot \prod_{i=1}^{n} e_i^{k_i} \), where \( \{e_1, \ldots, e_n\} \) is a quantity-space basis for \( Q \).

It would be of interest to fully analyze VIM3 (or the upcoming VIM4) in the light of the theory of quantity spaces. The distinction between concrete and abstract quantities [15, p. 8–12] should be taken into account in this connection.

The relationship between measures and the quantities that they represent is fundamental also in dimensional analysis, which is based on a principle of covariance: a relation between scalars representing a relation between quantities relative to a system of unit quantities must continue to hold when that system is changed in a legitimate way, although individual scalars may change as unit quantities change.

Reference [16] presents an approach to dimensional analysis explicitly based on this principle and expressed in terms of quantities, dimensions and quantity functions rather than scalars, units and scalar functions.

A measure of a quantity is usually assumed to be a real number, but in a quantity space over \( K \) a measure is an element of \( K \), where \( K \) is any field. For example, a measure can be a complex number. This makes the present theory of quantity spaces well suited for applications to problems of quantum physics.

Dan Jonsson, University of Gothenburg, Gothenburg, Sweden.

Email address: dan.jonsson@gu.se