Global and Local Horizon Quantum Mechanics

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Abstract

Horizons are classical causal structures that arise in systems with sharply defined energy and corresponding gravitational radius. A global gravitational radius operator can be introduced for a static and spherically symmetric quantum mechanical matter state by lifting the classical “Hamiltonian” constraint that relates the gravitational radius to the ADM mass, thus giving rise to a “horizon wave-function”. This minisuperspace-like formalism is shown here to be able to consistently describe also the local gravitational radius related to the Misner-Sharp mass function of the quantum source, provided its energy spectrum is determined by spatially localised modes.

1 Introduction

A black hole can be viewed as a gravitationally bound state confining all possible signals within its horizon. According to General Relativity, such extreme causal structures occur in very compact gravitating systems, and understanding black holes from the perspective of

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particle physics is akin to solving QCD in the strongly coupled regime. Like the strongly coupled regime of QCD is usually investigated by means of effective descriptions, one could analogously conceive effective quantum descriptions for the observables of relevance in the physics of black holes, and processes that lead to their formation [1], rather than insisting in (or in preparation for) the search of the ultimate quantum theory of gravity.

The semiclassical picture, in which matter fields are quantised on classical black hole manifolds, has led to the discovery of the Hawking radiation [2], and paradoxes indicating a possibly fundamental incompatibility between the quantum theory (of fields) and General Relativity. Both of these pillars of physics emerged from the rethinking of the interplay between a physical system and the observer. It might therefore be useful to investigate first and foremost which variables could best capture the crucial (and hopefully observable) physics of black holes, despite of what we have come to regard as otherwise fundamental. Examples of this approach are already found among the attempts at quantising canonically the Einstein-Hilbert action [3, 4, 5, 6]. DeWitt himself already realised the extreme complication of this programme in his seminal 1967 paper [5], and immediately reverted to a simplified formulation based on preserving isotropy and homogeneity of the universe at the quantum level. These symmetries reduce the superspace of all possible metrics to the Friedman-Robertson-Walker minisuperspace for the cosmic scale factor.

Black holes, and the collapse of compact objects leading to their formation [1], cannot be realistically modelled as homogeneous systems, and their quantum description will necessarily be more involved from the onset. However, one can proceed similarly and consider a reduced superspace by imposing isotropy or axial symmetry, or selecting directly a family of metrics, like, for instance, in Refs. [7, 8, 9]. Alternatively, one could study the general properties of causal structures that typically appear in such space-times, and then quantise their intrinsic (gravitational) degrees of freedom [10]. In any case, it is important to remark that these ways of quantisation concern some (reduced) degrees of freedoms, mostly in a manner independent of the (conventional) quantum state of the matter source.

In the following, we shall further develop an effective quantum description of static horizons originally called “horizon wave-function” (HWF) [11, 12] and later on “horizon quantum mechanics” (HQM) in Ref. [13] 1. The peculiarity of this approach is that it aims at describing quantum mechanically the existence of trapping surfaces (which reduce to horizons in the static case) from the quantum state of the source that produces the black hole. The HQM can therefore be viewed as complementary to approaches which consider the horizon and black holes as purely gravitational (or metric) objects, in that the gravitational degrees of freedom are taken “on-shell” with respect to a suitable constraint with the matter state. The HQM can then be naturally applied to systems which do not turn out to be black holes, that is, to matter systems with a negligible probability of being black holes, or to objects “on the verge” of being black holes [15].

In more details, our construction is based on the classical key concept of the gravitational radius of a static and spherically symmetric self-gravitating source, for which this quantity determines the existence of horizons. We recall that we can always write a spherically

\footnote{1For an attempt at including time evolution, see Refs. [13, 14].}
symmetric metric $g_{\mu\nu}$ as
\[
ds^2 = g_{ij} \, dx^i \, dx^j + r^2(x^i) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]
where and $x^i = (x^1, x^2)$ are coordinates on surfaces of constant $\theta$ and $\phi$. A horizon then exists where the expansion of null geodesics vanishes, $g^{ij} \nabla_i r \nabla_j r = 0$, $\nabla_i r$ being perpendicular to surfaces of constant area $A = 4\pi r^2$. If we set $x^1 = t$ and $x^2 = r^2$, and denote the static matter density as $\rho = \rho(r)$, Einstein equations tell us that $g^{rr} = 1 - r_H(r)/r$, where
\[
r_H(r) = 2\ell_p \frac{m(r)}{m_p}.
\]
is the gravitational radius determined by the Misner-Sharp mass function
\[
m(r) = 4\pi \int_0^r \rho(\bar{r}) \, \bar{r}^2 \, d\bar{r}.
\]
A horizon then exists where $g^{rr} = 0$, or where the gravitational radius satisfies
\[
r_H(r) = r,
\]
for $r > 0$. In the vacuum outside the region where the source is located, the Misner-Sharp mass approaches the Arnowitt-Deser-Misner (ADM) mass of the source,
\[
\lim_{r \to \infty} m(r) = M,
\]
and the gravitational radius likewise becomes the Schwarzschild radius
\[
R_H = 2\ell_p \frac{M}{m_p}.
\]

If the source is described by quantum physics, the quantities that define the Misner-Sharp mass $m$ (and ADM mass $M$) should become quantum variables and one expects the gravitational radius will undergo the same fate. The HQM was precisely proposed \cite{11} in order to describe the “fuzzy” Schwarzschild (or gravitational) radius of a localised (but likewise fuzzy) quantum source. It is important to emphasise once more that the HQM differs from most previous attempts in which the gravitational degrees of freedom of the horizon, or of the black hole metric, are quantised independently of the state of the source. In our case, the gravitational radius is instead quantised along with the matter source that produces it, somewhat more in line with the highly non-linear general relativistic description of the gravitational interaction in the strong regime.

Clearly, the HQM becomes particularly relevant for sources of the Planck size \cite{15, 18}, for which quantum effects may not be neglected. The Heisenberg principle of quantum

\footnote{Let us remark this is the frame in which the Tolman-Oppenheimer-Volkoff equation is usually derived \cite{16}.}

\footnote{We shall use units with $c = 1$, and the Newton constant $G = \ell_p/m_p$, where $\ell_p$ and $m_p$ are the Planck length and mass, respectively, and $\hbar = \ell_p m_p$.}
mechanics introduces an uncertainty in the particle’s spatial localisation of the order of the Compton-de Broglie length, $\lambda_M \simeq \ell_p m_p / M$, and $R_H$ only makes sense if

$$R_H \gtrsim \lambda_M \quad \iff \quad M \gtrsim m_p .$$

(1.7)

In fact, this is the argument that grants the Planck mass (and Planck length) a remarkable role in the search for a quantum theory of gravity [19]. It is comforting that the HQM predicts a particle is very likely a black hole only if (1.7) holds [12, 20, 21]. Remarkably, it also predicts that a truly macroscopic black hole cannot be produced by a very localised source [13, 12], but could be associated with an extended source [22, 23, 24].

In the next Section, we shall first review the general foundations of the HQM, and then show that the same physical states that describe the global horizon also describe local horizons, provided the spectral decomposition involves spatially localised energy eigenmodes. The consequences of the discrete spectrum for a GUP and Hawking radiation will also be briefly considered. We shall finally comment on our results in Section 3.

2 Horizon Quantum Mechanics

The HQM emerges from relating the matter source to its gravitational radius at the quantum level [11, 13], and allows us to put on more quantitative grounds the condition (1.7) that should distinguish black holes from regular particles.

Before reviewing and extending this formalism, let us clarify the underlying viewpoint by noting that one could describe formally the state of a quantum system for which there exist two relevant sets of variables, say $X$ and $Y$, as

$$| \Psi \rangle = \sum_{\alpha, \beta} c_{\alpha \beta} | X_\alpha, Y_\beta \rangle .$$

(2.1)

Without loss of generality, we can group terms in the above superposition as

$$| \Psi \rangle = \sum_{\alpha, \beta} (A_{\alpha \beta} | X_\alpha, Y_\beta \rangle + B_{\alpha \beta} | X_\alpha \rangle | Y_\beta \rangle + D_{\alpha \beta} | Y_\beta \rangle | X_\alpha \rangle + C_{\alpha \beta} | X_\alpha \rangle | Y_\beta \rangle )$$

(2.2)

where

$$\hat{X} | X_\alpha \rangle = X_\alpha | X_\alpha \rangle \quad \text{and} \quad \hat{Y} | Y_\beta \rangle = Y_\beta | Y_\beta \rangle .$$

(2.3)

In the sum in Eq. (2.2), the first term has no specific features; the second (third) term is of the kind that admits the Born-Oppenheimer approximation with $X$ (respectively $Y$) representing slow degrees of freedom compared to $Y$ (respectively $X$); finally, the fourth term contains the contribution from the direct product of the two separate Hilbert spaces of the eigenstates (2.3). We can now view $X$ as “matter” degrees of freedom, such as the usual standard model fields, and $Y$ as “gravitational” degrees of freedom. Upon further assuming the state $| \Psi \rangle$ only contains $\sum_{\beta} | Y_\beta \rangle \sim | Y_s \rangle$ which reproduces a (semi-)classical
configuration, the third term in Eq. (2.2) would reduce to the usual approach of QFT on a
given (curved) background [25], \(| \Psi \rangle \sim \sum_\alpha | Y_s \rangle | X_\alpha Y_s \rangle\), in which only the matter fields
retain their full quantum properties 4.

One could also do without the (semi)classical approximation. We shall in fact see below
that the states of relevance for the HQM are of the fourth kind in the sum (2.2), provided we
suitably reduce the matter degrees of freedom to \(X = H\) (the “matter energy”) and \(Y = R_H\)
(the gravitational radius). This is not very different from the usual quantum mechanical
treatment of the hydrogen atom, in which one quantises the (reduced) electron’s position,
whereas terms containing \(| Y_s \rangle\) replaced by an electron’s energy level yield the Lamb shift
due to the QFT description of the Coulomb field.

### 2.1 Global gravitational radius

We only consider spherically symmetric sources which are both localised in space and at
rest in the chosen reference frame. If the specific source is not at rest, one should therefore
change coordinates accordingly before applying the following analysis. We denote with \(\alpha\) the
set of (discrete or continuous) quantum numbers parametrising the spectral decomposition
of the source, so that our matter state can be written as

\[
| \psi_S \rangle = \sum_\alpha C_S(E_\alpha) | E_\alpha \rangle ,
\]

where the sum formally represents the spectral decomposition in Hamiltonian eigenmodes,

\[
\hat{H} = \sum_\alpha E_\alpha | E_\alpha \rangle \langle E_\alpha | ,
\]

and \(H\) should be specified depending on the system we wish to consider. We can then replace
the r.h.s. of Eq. (1.5) defining the ADM mass with the expectation value of the Hamiltonian,

\[
M \rightarrow \langle \psi_S | \hat{H} | \psi_S \rangle = \langle \psi_S | \sum_\alpha E_\alpha | E_\alpha \rangle \langle E_\alpha | \psi_S \rangle
\]

\[
= \sum_\alpha |C_S(E_\alpha)|^2 E_\alpha ,
\]

which follows from the orthonormality of the energy eigenstates,

\[
\langle E_\alpha | E_\beta \rangle = \delta_{\alpha\beta} ,
\]

where \(\delta\) is the Kronecker delta for a discrete energy spectrum and the Dirac delta for a
continuous spectrum 5. Let us then introduce the gravitational radius eigenstates

\[
\hat{R}_H \ | R_{H\beta} \rangle = R_{H\beta} \ | R_{H\beta} \rangle .
\]

4 Conversely, but perhaps of less interest, the second term would be useful in order to describe states in
which matter can be approximated classically but gravity remains fully quantum.

5 This point is purely technical in the global approach, but will become crucial in the local analysis.
We can now show that a physical state for our system can be described by linear combinations like the fourth term in Eq. (2.2),

\[ |\Psi\rangle = \sum_{\alpha,\beta} C(E_\alpha, R_{H\beta}) |E_\alpha\rangle |R_{H\beta}\rangle. \tag{2.9} \]

In fact, the algebraic (Hamiltonian) constraint (1.6) now reads

\[ 0 = \left(\hat{H} - \frac{m_p}{2\ell_p} \hat{R}_H\right) |\Psi\rangle = \sum_{\alpha,\beta} \left(E_\alpha - \frac{m_p}{2\ell_p} R_{H\beta}\right) C(E_\alpha, R_{H\beta}) |E_\alpha\rangle |R_{H\beta}\rangle, \tag{2.10} \]

and is clearly solved by the combination in Eq. (2.9) with

\[ C(E_\alpha, R_{H\beta}) = C(E_\alpha, 2\ell_p E_\alpha/m_p) \delta_{\alpha\beta}. \tag{2.11} \]

By tracing out the gravitational radius part, we should recover the matter state, that is

\[ |\psi_S\rangle = \sum_{\alpha} C(E_\alpha, 2\ell_p E_\alpha/m_p) |E_\alpha\rangle, \tag{2.12} \]

which implies

\[ C(E_\alpha, 2\ell_p E_\alpha/m_p) = C_S(E_\alpha). \tag{2.13} \]

Now, by integrating out the matter states, we will obtain

\[ |\psi_H\rangle = \sum_{\alpha} C_S(m_p R_{H\alpha}/2\ell_p) |R_{H\alpha}\rangle, \tag{2.14} \]

where \( m_p R_{H\alpha}/2\ell_p = E(R_{H\alpha}) \). We have thus recovered the HWF [11]

\[ \psi_H(R_{H\alpha}) = \langle R_{H\alpha} | \psi_H \rangle = C_S(m_p R_{H\alpha}/2\ell_p), \tag{2.15} \]

where the values \( R_{H\alpha} \) form a discrete (continuous) spectrum if \( E_\alpha \) is discrete (continuous) in the quantum number \( \alpha \).

If the spectral decomposition (2.4) is continuous, so will be the HWF, and we can write

\[ \psi_H(R_H) = N_H C_S(m_p R_H/2\ell_p), \tag{2.16} \]

with \( N_H^{-2} = 4\pi \int_0^{\infty} |C_S(m_p R_H/2\ell_p)|^2 R_H^2 dR_H \) determined by the scalar product \(^6\)

\[ \langle \psi_H | \phi_H \rangle = 4\pi \int_0^{\infty} \psi_H^*(R_H) \phi_H(R_H) R_H^2 dR_H. \tag{2.17} \]

In this continuous case, the normalised wave-function (2.16) yields the probability density

\[ P_H(R_H) = 4\pi R_H^2 |\psi_H(R_H)|^2 \tag{2.18} \]

\(^6\)Note the integration is formally extended from zero to infinity, although it will be naturally limited to a smaller range if the spectral decomposition of the source is limited above and/or below.
that we would detect a gravitational radius of size $R_H$ associated with the particle in the quantum state $|\psi_S\rangle$. Moreover, we can define the conditional probability density that the particle lies inside its own gravitational radius $R_H$ as

$$\mathcal{P}_<(r < R_H) = P_S(r < R_H) \mathcal{P}_H(R_H),$$

(2.19)

where $P_S(r < R_H) = 4\pi\int_0^{R_H} |\psi_S(r)|^2 r^2 dr$ is the usual probability that the particle is found inside a sphere of radius $r = R_H$. One can also view $\mathcal{P}_<(r < R_H)$ as the probability density that the sphere $r = R_H$ is a trapping surface. Finally, the probability that the particle described by the state $|\psi_S\rangle$ is a black hole (regardless of the horizon size), will be obtained by integrating (2.19) over all possible values of $R_H$, namely

$$P_{BH} = \int_0^\infty \mathcal{P}_<(r < R_H) dR_H,$$

(2.20)

which will depend on the observables and parameters of the specific matter state.

### 2.2 Local gravitational radius

We have seen that the global gravitational radius can be described irrespectively of whether the spectral decomposition is discrete or continuous. In order to show that the same physical quantum states (2.9) with coefficients given in Eq. (2.11) also allow for a local description of the gravitational radius, we shall instead need localised energy eigenmodes and correspondingly discrete energy quantum numbers.

Instead of the ADM mass (1.5), we now start from Misner-Sharp mass at finite radius, and again assume the system is static, so that $m = m(r)$. We first observe that the total Hamiltonian (2.5) associated with the ADM mass can also be written as

$$\hat{H} = \sum_\alpha E_\alpha |E_\alpha\rangle \langle E_\alpha | E_\alpha\rangle \langle E_\alpha |$$

$$= \sum_\alpha E_\alpha \sum_{r=0}^\infty |E_\alpha\rangle \langle E_\alpha | r\rangle \langle r | E_\alpha\rangle \langle E_\alpha |$$

$$= 4\pi \sum_\alpha E_\alpha \int_0^\infty |\psi_{E_\alpha}(\bar{r})|^2 \bar{r}^2 d\bar{r} |E_\alpha\rangle \langle E_\alpha |. $$

(2.21)

which follows from the discrete orthogonality condition (2.7) and the (continuous) decomposition of the identity $\sum_{r=0}^\infty |r\rangle \langle r | = 4\pi \int_0^\infty \bar{r}^2 d\bar{r} |r\rangle \langle r | = \hat{I}$. We can analogously introduce the radius-dependent Hamiltonian

$$\hat{H}(r) = \sum_\alpha E_\alpha \sum_{r'=0}^r |E_\alpha\rangle \langle E_\alpha | r'\rangle \langle r' | E_\alpha\rangle \langle E_\alpha |$$

$$= \sum_\alpha E_\alpha P_\alpha(r) |E_\alpha\rangle \langle E_\alpha |,$$

(2.22)
where we defined
\[
P_\alpha(r) = 4\pi \int_0^r |\psi_{E_\alpha}(\bar{r})|^2 \bar{r}^2 \, d\bar{r} ,
\] (2.23)
and note that \( \lim_{r \to \infty} P_\alpha(r) = 1 \). This property only holds for square integrable (that is, localised) energy eigenmodes, a restriction which was not necessary in the global case, since the norm of these modes never entered explicitly in that calculation. However, the condition \( \langle E_\alpha \mid E_\alpha \rangle = 1 \) is now necessary in order to obtain (2.21) above, which means the local construction requires the existence of localised (bound) Hamiltonian eigenmodes.

Assuming again the spectral decomposition (2.4), the Misner-Sharp mass function can now be replaced by the radius-dependent quantity
\[
m(r) \to \langle \psi_S \mid \hat{H}(r) \mid \psi_S \rangle = \sum_\alpha |C_S(E_\alpha)|^2 E_\alpha P_\alpha(r) .
\] (2.24)
The gravitational radius (1.2) analogously depends on \( r \), and we introduce the local gravitational radius eigenstates,
\[
\hat{r}_H(r) \mid R_{H\beta} \rangle = r_{H\beta}(r) \mid R_{H\beta} \rangle .
\] (2.25)
where it is again the operator that carries a radial label. If we now recall a physical state that satisfies the global (Hamiltonian) constraint can be written as
\[
\mid \Psi \rangle = \sum_\alpha C_S(E_\alpha) \mid E_\alpha \rangle \mid R_{H\alpha} \rangle ,
\] (2.26)
it immediately follows that it will also satisfy the local (Hamiltonian) constraint (1.2) for all values of \( r \), that is
\[
0 = \left[ \hat{H}(r) - \frac{m_p}{2\ell_p} \hat{R}_H(r) \right] \mid \Psi \rangle = \sum_\alpha \left[ E_\alpha(r) - \frac{m_p}{2\ell_p} r_{H\alpha}(r) \right] C_S(E_\alpha) \mid E_\alpha \rangle \mid R_{H\alpha} \rangle ,
\] (2.27)
provided the local eigenvalues
\[
r_{H\alpha}(r) = P_\alpha(r) R_{H\alpha} .
\] (2.28)
Since the spectral decomposition must now be discrete, so are the above eigenvalues, and one has
\[
\langle \psi_H \mid \hat{r}_H(r) \mid \psi_H \rangle = \sum_\alpha |C_S(E_\alpha)|^2 P_\alpha(r) R_{H\alpha}
\]
\[
= \sum_\alpha |\psi_H(R_{H\alpha})|^2 P_\alpha(r) R_{H\alpha} .
\] (2.29)
where $\psi_H$ is the (discrete) global HWF. Finally, the classical local condition (1.4) for the existence of a (static) trapping surface at the radius $r$ can now be directly replaced by

$$\langle \psi_H | \hat{r}_H(r) | \psi_H \rangle = r,$$  \hspace{1cm} (2.30)

which therefore defines quantum local horizons.

It is worth discussing further what would go wrong if the spectral decomposition (2.4) did not contain only localised energy eigenmodes but also, say spatially homogenous modes, like the plane waves. The function $P_\alpha = P_\alpha(r)$ would increase monotonically with $r$ without bounds, and any attempt at regularising it, for example by enclosing the system in a "box" of size $R$, would lead to expressions like $P_\alpha \sim r/R$ that explicitly depend on the (arbitrary) cut-off $R$, a clear sign of inconsistency. This is of course implicitly seen already from Eq. (2.21) which again only makes sense if $\langle E_\alpha | E_\alpha \rangle = 1$, as we remarked above.

### 2.3 GUP and Hawking radiation

In Ref. [12], a GUP was obtained by combining (linearly) the uncertainty in the source size $\langle \Delta \hat{r}^2 \rangle$ encoded in the matter state (2.12) with the uncertainty in the horizon size $\langle \Delta \hat{R}_H^2 \rangle$ given by a (continuous) global HWF (2.16). In particular, for Gaussian matter states,

$$\psi_S(r) \simeq e^{-r^2/\ell^2},$$ \hspace{1cm} (2.31)

where $\ell \simeq \ell_p m_p/m$ is the Compton width of the source of mass $m$, the GUP was shown to take the form

$$\Delta r \simeq \ell_p \frac{m_p}{\Delta p} + \gamma \ell_p \frac{\Delta p}{m_p},$$ \hspace{1cm} (2.32)

with $\gamma$ an arbitrary coefficient,

$$\langle \Delta \hat{r}^2 \rangle = 4 \pi \int_0^\infty |\psi_S(r)|^2 r^4 \, dr - \left( 4 \pi \int_0^\infty |\psi_S(r)|^2 r^3 \, dr \right)^2 \simeq \ell^2,$$ \hspace{1cm} (2.33)

and

$$\langle \Delta \hat{R}_H^2 \rangle = 4 \pi \int_0^\infty |\psi_H(R_H)|^2 R_H^4 \, dR_H - \left( 4 \pi \int_0^\infty |\psi_H(R_H)|^2 R_H^3 \, dR_H \right)^2 \simeq \frac{\ell^2}{\ell_p^2}.$$ \hspace{1cm} (2.34)

Finally, the global uncertainty in radial momentum is given by

$$\Delta p^2 = 4 \pi \int_0^\infty |\psi_S(p)|^2 p^4 \, dp - \left( 4 \pi \int_0^\infty |\psi_S(p)|^2 p^3 \, dp \right)^2 \simeq m_p^2 \frac{\ell_p^2}{\ell_p^2},$$ \hspace{1cm} (2.35)

where

$$\psi_S(p) \simeq e^{-p^2/m_p^2}.$$ \hspace{1cm} (2.36)
As we have just shown, were one to employ the local description of Section 2.2, the spectrum of the source should be discrete, and consequently so would be the HWF. In particular, we can now define a local uncertainty in the source size

\[ \langle \Delta \hat{\Delta r}^2(r) \rangle = 4\pi \int_0^r |\psi_S(\bar{r})|^2 \bar{r}^4 \, d\bar{r} - \left( 4\pi \int_0^r |\psi_S(\bar{r})|^2 \bar{r}^3 \, d\bar{r} \right)^2. \]  

(2.37)

Likewise, and replacing integrals with sums over the spectral index \( \alpha \), we can also define local uncertainties for the gravitational radius

\[ \langle \Delta \hat{\Delta r_H}^2(r) \rangle = \sum_\alpha |C_S(E_\alpha)|^2 R_{\alpha a}^2 P_\alpha(r) - \left( \sum_\alpha |C_S(E_\alpha)|^2 R_{\alpha a} P_\alpha(r) \right)^2, \]  

(2.38)

and for the radial momentum

\[ \Delta p^2(r) = \sum_\alpha |C_S(E_\alpha)|^2 p_\alpha^2 P_\alpha(r) - \left( \sum_\alpha |C_S(E_\alpha)|^2 p_\alpha P_\alpha(r) \right)^2, \]  

(2.39)

where \( p_\alpha = p(E_\alpha) \).

It is interesting to note that, for spectral eigenmodes, the above sums would reduce to single terms and one finds

\[ \frac{\langle \Delta \hat{\Delta r_H}^2(r) \rangle}{R_{\alpha H}^2} = P_\alpha(r) [1 - P_\alpha(r)] = \frac{\Delta p^2(r)}{p_\alpha^2}. \]  

(2.40)

By again combining linearly the size uncertainty (2.37) with the uncertainty in the gravitational radius (2.38), one obtains a local GUP of the form (2.32) at each (finite) value of \( r \). Since \( P_\alpha(r \to \infty) = 1 \), one also finds \( \langle \Delta \hat{\Delta r_H}^2(r \to \infty) \rangle = \langle \Delta \hat{\Delta r_H}^2 \rangle = 0 = \Delta p^2(r \to \infty) \), in agreement with the fact that the global GUP reduces to the standard Heisenberg uncertainty relation for spectral eigenmodes. Of course, Gaussian states (2.31) may now not be in the discrete Hilbert space \( \mathcal{H} \), or they could instead be energy eigenstates, depending on the details of the system at hand. In any case, it is hard to conceive that macroscopic black holes are simple spectral eigenmodes, and some form of global GUP should therefore apply.

A more drastic consequence follows for the Hawking radiation, as one expects only quanta corresponding to transitions between states of the discrete spectrum be allowed. Moreover, this argument would further support the idea that macroscopic black holes cannot be spectral eigenmodes, as those would not support any Hawking emission. It is in fact tempting to draw a connection with corpuscular models of black holes [22], as they appear to be bound states in the sense shown in Ref. [26], and do not suffer of the paradoxes related to the standard description of the Hawking radiation.

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7 More technically, one can view \( 4\pi \langle \Delta \hat{\Delta r}^2(r) \rangle \) as the uncertainty in the area of a sphere of coordinate radius \( r \).

8 Of course, they might still be obtained as the limit of suitable series.
3 Conclusions

We have analysed the quantum constraint that relates the gravitational radius of a spherically symmetric source to its spectral decomposition, and shown that the same quantum state can be employed in order to describe both the global radius associated with the ADM mass and the local radius associated with the Misner-Sharp mass function.

A crucial difference that emerges between the local and global gravitational radius at the quantum level is that the former requires the spectral decomposition is done in terms of localised energy eigenmodes, whereas the global radius can be defined in any case. From the physical point of view, one can argue the global gravitational radius is an asymptotic property of a self-gravitating system and should therefore be rather insensitive to the details of its internal structure, whereas the local gravitational radius should be determined by the precise internal structure of the source. It therefore appears consistent that the local gravitational radius can be defined only provided the inner structure of the source is properly characterised as well. Finally, the fact the spectral decomposition must be discrete does not constitute a real limitation in most practical situations, since any realistic astrophysical sources, like stars, should have very finely-spaced energy levels.

So far we have only considered the formal extension of the HQM to the local gravitational radius, and some general considerations regarding the GUP and Hawking radiation, but it will next be important to apply this approach to specific models of self-gravitating objects, and also to compare our findings with specific proposals of quantum black holes [9, 15, 22, 24, 27, 28].

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