BIFURCATIONS, ULTIMATE BOUNDEDNESS AND SINGULAR ORBITS IN A UNIFIED HYPERCHAOTIC LORENZ-TYPE SYSTEM

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Abstract. In this note, by using the theory of bifurcation and Lyapunov function, one performs a qualitative analysis on a novel four-dimensional unified hyperchaotic Lorenz-type system (UHLTS), including stability, pitchfork bifurcation, Hopf bifurcation, singularly degenerate heteroclinic cycle, ultimate bound estimation, global exponential attractive set, heteroclinic orbit and so on. Numerical simulations not only are consistent with the results of theoretical analysis, but also illustrate singularly degenerate heteroclinic cycles with distinct geometrical structures and nearby hyperchaotic attractors in the case of small $b > 0$, i.e. conjugate hyperchaotic Lorenz-type attractors (CHCLTA) and nearby a short-duration transient of singularly degenerate heteroclinic cycles consisting of normally hyperbolic saddle-foci and stable node-foci, etc. In particular, by a linear scaling, a possibly new forming mechanism behind the creation of well-known hyperchaotic attractor with $(a_1, a_2, c, d, e, f, b, p, q) = (-12, 12, 23, -1, -1, 1, 2, -6, -0.2)$, consisting of occurrence of degenerate pitchfork bifurcation at $S_z$, the change in the stability index of the saddle at the origin as $b$ crosses the null value, explosion of normally hyperbolic stable node-foci, collapse of singularly degenerate heteroclinic cycles consisting of normally hyperbolic saddle-foci or saddle-nodes, and stable node-foci, is revealed. The findings and results of this paper may provide theoretical support in some future applications, since they improve and complement the known ones.

1. Introduction. Landmark for the study of hyperchaotic behavior is the work of Rössler [29]. Dating to 1979, it can be taken as the starting point for the theory of hyperchaos with the introduction of the so called hyperchaotic Rössler system. Due to its great potential in some nontraditional engineering and technological applications

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in the fields of biological networks [26, 31], coupled map lattices [38], encryption [8], secure communications [36], liquid mixing [50], nonlinear circuits [2], electronic oscillators [3], lasers [37] and so on, hyperchaos has been considerable researched with increasing interest in the last three decades. Among the research with applications to the control and synchronization of various hyperchaotic systems, how to generate hyperchaos has been considered as a hot topic.

Refer to [29], a chaotic system with more than one positive Lyapunov exponent is defined to be a hyperchaotic system. Therefore, dynamics of a hyperchaotic attractor are expending in more than one direction, giving rise to “messier” and “more complicate” chaotic dynamics. To the best of our knowledge, the lowest possible dimension for the onset of hyperchaos in an autonomous system is four. Representative examples of four-dimensional hyperchaotic systems are hyperchaotic Rössler system [29], hyperchaotic Lorenz-type system [15], hyperchaotic Chua’s circuit [11], hyperchaotic modified Chua’s circuit [33], hyperchaotic Lorenz-Haken system [27], etc. Therefore, four dimensional autonomous systems play an important role in researches and applications related to hyperchaos.

Despite the fact that a huge amount of hyperchaotic systems with complex dynamics have been proposed, there exists an open question that has not been completely answered so far: is there a four dimensional hyperchaotic system which retains most of main dynamics of the classic unified Lorenz-type system [39, 49], such as pitchfork bifurcation, Hopf bifurcation, infinitely many singularly degenerate heteroclinic cycles with classic and conjugate hyperchaotic Lorenz-type attractors near them, ultimate bound estimation, global exponential attractive set, a pair of symmetrical heteroclinic orbits, etc?

The research of hyperchaos is at the preliminary stage at present. The main obstacles are as follows: First, the unified theory to guide an engineering design of a new hyperchaotic system is not available in the current literature. Second, generation and analysis of hyperchaos heavily rely on a combination of basic bifurcation analysis and computer simulations. Third, the rigorous proof of singular orbits and bifurcations, such as singularly degenerate heteroclinic cycles, homoclinic and heteroclinic orbits, etc, assuming their existence, remains a very difficult task today. Last, how to rigorously prove the existence and global boundedness of a truly hyperchaotic attractor from such a hyperchaotic system is extremely challenging.

However, some noticeable progresses have been achieved. Chen and Yang [5] rigorously proved that a 4D hyperchaotic Lorenz-type system has two symmetrical heteroclinic orbits. The novel 5D hyperchaotic system [41] has been rigorously proved to have infinitely many heteroclinic orbits and singular attractors which are bifurcated from the corresponding singularly degenerate heteroclinic cycles. In the 4D Lorenz-type hyperchaotic system [4], Chen coined a pair of homoclinic orbits to the origin utilizing the Fishing Principle [14]. Based on Shilnikov’s boundary-value problem [32], Shilnikov et al studied the existence of homoclinic orbits for high dimensional systems. Nowadays, Tlelo-Cuautle et al [6] made many contributions on chaotic/hyperchaotic systems, which include implementation by using digital hardware. Pathak et al [28] demonstrated the effectiveness of using machine learning for model-free prediction of spatiotemporally chaotic systems of arbitrarily large spatial extent and attractor dimension purely from observations of the system’s past evolution. Recently, theoretical efforts [18, 22, 42, 43, 46, 51, 52] have been performed to find localization domains containing all compact invariant sets of the hyperchaotic systems possessing complex behavior.
Therefore, the preliminary remarkable achievements provide insight into subsequent studies. In addition, most of hyperchaotic systems are derived from chaotic systems and some dynamics may depend on the original models, especially the forming mechanism and relationships between hyperchaotic attractors and algebraic structures. More importantly, how to further extend these studies to hyperchaotic systems is therefore not only theoretically significant but also practically important, motivating the work to be presented in this paper.

More precisely, in this paper, a unified hyperchaotic Lorenz-type system is introduced, which includes the unified Lorenz-type system [49] and hyperchaotic Lorenz-type system [15] as special cases. Combining theoretical analysis and numerical techniques we fix the following issues.

**Issue 1.1:** Numerical simulations illustrated that not only classic and conjugate chaotic Lorenz-type attractors, but also the hyperchaotic ones were generated from the collapse of the corresponding singularly degenerate heteroclinic cycles, see [23, 25, 44, 45, 48, 49]. But it is currently unclear whether the conjugate hyperchaotic Lorenz-type attractors (CHCLTA) exist.

**Issue 1.2:** Referring to [5, Theorem 1, p. 573; Theorem 2, p. 576], the authors only considered the pitchfork and Hopf bifurcation at the origin of the hyperchaotic Lorenz-type system in the case of the bifurcation parameter $b$. How about the other cases?

**Issue 1.3:** Particularly, the authors in [5, 39] rigorously proved that there exists a pair of symmetrical heteroclinic orbits in the unified Lorenz-type system and hyperchaotic Lorenz-type system by aid of Lyapunov function, concepts of both $\alpha$-limit set and $\omega$-limit set. Whether does the system (1) have this kind of dynamics in the region of parameters other than the one given in [5, Theorem 5, p. 578]?

**Issue 1.4:** In addition, the authors in [15, 22, 46] only investigated the ultimate bound estimation and global exponential attractive set of that hyperchaotic Lorenz-type system with all positive parameters. How about the dynamics in the other range of parameters?

To the best of our knowledge, Issue 1.1-1.4 are not discussed in any public literatures. Therefore, it is worthwhile to propose and study the unified hyperchaotic Lorenz-type system.

The main innovations are as follows: (1) Finding classic and conjugate hyperchaotic Lorenz-type attractors through the collapse of the corresponding singularly degenerate heteroclinic cycles near them. (2) Extending the recent results reported in [5, 15, 22, 46] in a wider range of parameters, i.e., pitchfork bifurcation, Hopf bifurcation, ultimate bound estimation, global exponential attractive sets, heteroclinic orbits, etc.

The remainder of this paper is organized as follows. Section 2 introduces the unified hyperchaotic Lorenz-type system. Section 3 discusses dynamics of equilibria of the proposed system. Section 4 numerically illustrates the existence of different shapes of infinitely many singularly degenerate heteroclinic cycles and together with hyperchaotic attractors near them. Section 5 formulates ultimate bound estimation and global exponential attractive sets. In Sect. 6, one rigorously proves that the system has a pair of symmetric heteroclinic orbits under the case $a_2 \neq 0$, $b > 0$, $2a_1 + b \geq 0$, $e < 0$, $a_1 + d < 0$, $q < 0$, $fp(-a_1 + q) > 0$ and $a_2cq - a_1(dq - fp) < 0$. Finally, some conclusions are drawn in Sect. 7.
2. The unified hyperchaotic Lorenz-type system. In this section, one introduces a unified hyperchaotic Lorenz-type system characterized as

\[
\begin{align*}
\dot{x} &= a_1 x + a_2 y, \\
\dot{y} &= cx + dy + exz + fw, \quad a_2 p \neq 0, \\
\dot{z} &= -bz + xy, \quad a_1, b, c, d, e, f, p \in \mathbb{R}, \\
\dot{w} &= py + qw,
\end{align*}
\]

(1)

by generalizing a hyperchaotic Lorenz-type system [15],

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= bx - cy - xz + w, \quad (a, b, c, d, k, r) \in (\mathbb{R}^+)^6, \\
\dot{z} &= -dz + xy, \\
\dot{w} &= -ky - rw,
\end{align*}
\]

(2)

or adding a simple state-feedback dynamical controller to the classic unified Lorenz-type system [49]

\[
\begin{align*}
\dot{x} &= a_1 x + a_2 y, \\
\dot{y} &= cx + dy + exz, \quad (a_1, a_2, b, c, d, e) \in \mathbb{R}^6, \\
\dot{z} &= -bz + xy.
\end{align*}
\]

(3)

Remark 1. It follows from the algebraic structure of the system (1) that it contains the system (2)-(3) as special cases. In particular, the system (1) is reduced to the original Lorenz system when \(-a_1 = a_2 > 0, c > 0, e = -1, f = 0\) and \(p = q = 0\). Therefore, the research being carried out on the system (1) may be helpful for a deep understanding upon the nature of the original Lorenz system.

Fix \((x_0, y_0, z_0, w_0) = (\pm 1.618, \pm 3.14, \pm 2.718, \pm 0.618) \times 1 e - 4\) and

\((a_1, a_2, c, d, e, f, p, q) = (1, -0.3, 0.8, -1.25, 1, 0.5, 0.02, 0.02)\).

For \(b = 0\), Figure 1(a) displays that solutions of the system (1) ultimately approach the infinity after a short-duration transient of singularly degenerate heteroclinic cycles, which converge to conjugate hyperchaotic Lorenz-type attractors (CHCLTA) while \(b = 0.07\), as depicted in Figure 1(b).

In this case, the system (1) has two positive Lyapunov exponents, \(L_1 = 0.07190, L_2 = 0.01254\), and the other two are \(L_3 = 0.000002\) and \(L_4 = -0.384442\), respectively. Figure 2 clearly shows the distinct geometrical structures of the conjugate hyperchaotic Lorenz-type attractors (CHCLTA), similar to the conjugate chaotic Lorenz/Chen/Lü attractor [48].

Remark 2. Referring to [15, 22, 46], the authors mainly revealed the important dynamics of the system (2) in the case of all positive parameters as follows: pitchfork bifurcation, Hopf bifurcation, ultimate bound estimation, global exponential attractive sets and heteroclinic orbits, etc. However, the results obtained in this paper include the ones given in [15, 22, 46] as particular cases. Therefore, the investigation for the system (1) is at least theoretically meaningful on its own right.

3. Dynamical behavior of \(S_0, S_{\pm}\) and \(S_z\). From the algebraic structure of the system (1), one has the statement on the main equilibria.

Proposition 1. (1) When \(b = 0\), the system (1) has the non-isolated equilibria \(S_z = (0, 0, z, 0) (z \in \mathbb{R})\).
A UNIFIED HYPERCHAOTIC LORENZ-TYPE SYSTEM

5
-50
0
50
40
100
150 z
200
250
20
300
x
0
y
0
0 -20 -5-40

(a) $b = 0$

(b) $b = 0.07$

Figure 1. Phase portrait of the system (1) in the projection space $x$-$y$-$z$ with $(a_1, a_2, c, d, e, f, p, q) = (1, -0.3, 0.8, -1.25, 1, 0.5, 0.02, 0.02)$.

(a) $x$-$y$-$z$

(b) $x$-$w$-$z$

Figure 2. Phase portrait of the system (1) in the projection spaces (a) $x$-$y$-$z$, (b) $y$-$w$-$z$ with $(a_1, a_2, b, c, d, e, f, p, q) = (1, -0.3, 0.07, 0.8, -1.25, 1, 0.5, 0.02, 0.02)$.

(2) When $b \neq 0$, $S_0 = (0, 0, 0, 0)$ is the single equilibrium point of the system (1) for $\Delta = a_1 b e q [a_1 (f p - d q) + a_2 c q] \leq 0$; while $\Delta > 0$, the system (1) still has a pair of symmetric equilibria

$$S_\pm = (\pm \Upsilon, \mp \frac{a_1}{a_2} \Upsilon, \frac{a_1 (d q - f p) - a_2 c q}{a_2 c q} \mp \frac{a_1 p}{a_2 q} \Upsilon),$$

where $\Upsilon = \sqrt{\frac{b [a_1 (f p - d q) + a_2 c q]}{a_1 c q}}$.

In the following, one performs the analysis of qualitative behavior of $S_0$, $S_{\pm}$ and $S_z$, respectively. First of all, the characteristic equations of the matrix associated with linearized vector field about $S_0$, $S_{\pm}$ and $S_z$ are

$$(\lambda + b)(\lambda^3 - \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3) = 0$$  \hspace{1cm} (4)
with
\[ \rho_1 = a_1 + d + q, \]
\[ \rho_2 = a_1d + dq + a_1q - fp - a_2c, \]
\[ \rho_3 = -a_1dq + a_1fp + a_2cq, \]
\[ \lambda^2 + (b - \alpha)\lambda^3 + (-b\beta + \varphi)\lambda^2 + b \varepsilon \lambda - b \psi = 0, \]  \eqno (5)\]

where
\[ \alpha = a_1 + d + q, \]
\[ \beta = a_1 + q + \frac{a_2cq + a_1fp}{a_1q}, \]
\[ \varphi = q(a_1 + d) - fp(1 - \frac{a_1}{q}), \]
\[ \varepsilon = a_1(q - 2d + \frac{3fp}{q}) + a_2(2c + \frac{3c}{a_1}), \]
\[ \psi = 2(a_2qc - a_1(dq - fp)), \]

and
\[ \lambda(\lambda^3 + \tau_1\lambda^2 + \tau_2\lambda + \tau_3) = 0, \]  \eqno (6)\]

with
\[ \tau_1 = -(a_1 + d + q), \]
\[ \tau_2 = a_1d + dq + a_1q - pf - a_2(c + \varepsilon), \]
\[ \tau_3 = -a_1dq + a_1pf + a_2qd(c + \varepsilon), \]

respectively.

3.1. Behavior of \( S_0 \). The main dynamics of \( S_0 \) is concluded in the following propositions.

**Proposition 2.** (a) \( S_0 \) is locally asymptotically stable if and only if \( b > 0, \rho_1 < 0, \rho_2 > 0, \rho_3 > 0 \) and \( \rho_1\rho_2 + \rho_3 < 0 \).

(b) \( S_0 \) is globally asymptotically stable either (b1) \( a_1 < 0, e < 0, d < 0, q < 0, b > 0, a_2c < 0 \) and \( pf < 0 \); or (b2) \( a_2 \neq 0, a_1 < 0, 2a_1 + b > 0, e < 0, a_1 + d < 0, q < 0, fp(a_1 + q) > 0 \) and \( a_2c - a_1(dq - fp) \geq 0 \).

**Proposition 3.** The generic pitchfork bifurcation occurs at \( S_0 \) for the critical value
\[ f_0 = \frac{q(a_1d - a_2c)}{qa_1f}, \]
\[ p_0 = \frac{q(a_1d - a_2c)}{qa_1f}, \]
\[ q_0 = \frac{a_1fp}{a_1d - a_2c}, \]
\[ d_0 = \frac{a_1fp + a_2cq}{a_1 + q}, \]
\[ c_0 = \frac{a_1(dq - fp)}{aq}. \]

Take the case \( f_0 = \frac{q(a_1d - a_2c)}{qa_1f} \) for example. The generic pitchfork bifurcation happens at \( S_0 \) when \( p_1^2 - 4(a_1d + dq + a_1q - fp - a_2c) \geq 0 \) and \( a_2c^2b = a_1^2 \). Moreover, the discussions for the other cases are similar and omitted here.

**Proposition 4.** The Hopf bifurcation occurs at \( S_0 \) for the critical value
\[ f_* = \frac{(a_1 + d)(d + q)(a_1 + q) - a_2c}{p(d + q)}, \]
\[ p_* = \frac{(a_1 + d)(d + q)(a_1 + q) - a_2c}{f(d + q)}, \]
\[ c_* = \frac{(a_1 + d)(d + q)(a_1 + q) - a_2c}{a_2(a_1 + d)}. \]

Take the case \( f_* = \frac{(a_1 + d)(d + q)(a_1 + q) - a_2c}{p(d + q)} \) for example. When \( (d + q)[a_1^2(d + q) - a_2c(a_1 - q)] < 0, a_1 + d + q < 0 \) and \( b > 0 \), the system (1) undergoes a Hopf bifurcation at \( S_0 \) if \( f \) passes through the critical value \( f_* \). Furthermore, the corresponding first Lyapunov coefficient is given by
\[ l_1(a_1, a_2, b, d, e, c, p, q) = \frac{-a_2c(a_1 - q)[(a_1 + q)(d + q) - a_2c]}{2bd_1d_2}, \]  \eqno (7)\]
where
\[ N = 8a_1^4d + 8a_1^4q + 8a_1^3d^2 + 8a_1^3dq - 8a_1^3a_2c - 2a_1^2b^2d - 2a_1^2b^2q - 2a_1^2bdq - 2a_1^2bd - 2a_1^2bq - a_1b^2d - 2a_1b^2q - a_1b^2dq - 2a_1b^2cd - 2a_1bq^2 + 8a_1a_2cdq - b^2d^2q - b^2q^2 + a_2b^2cq - 2a_2bcdq, \]
\[ D_1 = (d + q)(4a_1^2 - b^2) - 4a_1a_2c + 4a_2cq \]
and
\[ D_2 = d^3 + 3d^2q + 2a_1d^2 + 3dq^2 + 4a_1dq + q^3 + 2a_1q^2 - a_2cq + a_1a_2c. \]

Moreover,
(i) If \( p(d + q) < 0 \) and \( l_1 > 0 \) (or \( p(d + q) > 0 \) and \( l_1 < 0 \)), then the Hopf bifurcation at \( S_0 \) is subcritical, and the unstable periodic orbit exists for \( f < f^* \).
(ii) If \( p(d + q) < 0 \) and \( l_1 < 0 \) (or \( p(d + q) > 0 \) and \( l_1 > 0 \)), then the Hopf bifurcation at \( S_0 \) is supercritical, and the stable periodic orbit exists for \( f > f^* \).

For the other cases, the discussions are similar and omitted here.

The sketches of proofs of Proposition 2-4 are presented as follows.

**Proof of Proposition 2.**

a) The statement directly follows from the Routh-Hurwitz criterion.

b) Define \( (x, y, z, w) \) to be any one solution of the system (1).

**Case (b1).** \( a_1 < 0, e < 0, d < 0, q < 0, b > 0, a_2c < 0 \) and \( pf < 0 \).

Set the Lyapunov-like function
\[ U_1 = \frac{1}{2} \left( \frac{c}{a_2e} x^2 - \frac{1}{e} y^2 + z^2 + \frac{f}{pe} w^2 \right). \]
Calculating the derivative of \( V \) w.r.t. time \( t \) along the solution of the system (1) leads to
\[ \frac{dU_1}{dt} = \frac{a_1c}{a_2e} x^2 - \frac{d}{e} y^2 - b z^2 + \frac{f q}{pe} w^2 \leq 0. \]
\[ U_1 = 0 \iff x = y = z = w = 0 \iff (x, y, z, w) = S_0. \]

By LaSalle theorem [10], \( S_0 \) is globally asymptotically stable.

**Case (b2).** \( a_2 \neq 0, a_1 < 0, 2a_1 + b \geq 0, e < 0, a_1 + d < 0, q < 0, pf(-a_1 + q) > 0 \) and \( a_2cq - a_1(dq - fp) \geq 0 \).

Define \( Q = z - \frac{1}{2a_2} x^2 \). Then \( \dot{Q} = -bz - \frac{a_1}{a_2} x^2 = bQ - \frac{b^2 + 2a_1}{2a_2} x^2 \). Take the Lyapunov-like function
\[ U_2 = \dot{x}^2 - \frac{a_2f}{pq(-a_1 + q)} (-p \frac{2a_1}{a_2} x + qw)^2 + \frac{a_2^2 e}{2a_1 b} (b(a_2cq - a_1 dq + a_1 fp) - a_2 e x^2)^2 \]
with
\[ \frac{dU_2}{dt} \bigg|_{(1)} = 2(a_1 + d) \dot{x}^2 - \frac{2a_2 f}{p(-a_1 + q)} (-p \frac{a_1}{a_2} x + qw)^2 + \frac{2a_2^2 e}{(b + 2a_1)} Q^2 \leq 0 \]
for the subcase \( 2a_1 + b > 0 \), and another one
\[ U_3 = \dot{x}^2 - \frac{a_2 f}{pq(-a_1 + q)} (-p \frac{a_1}{a_2} x + qw)^2 - \frac{e}{4} \left[ 2(a_2 cq - a_1 dq - fp) \right] \frac{1}{qe} + x^2 |^2 \]
Proof of Proposition 3. Set \( \bar{S} \) with \( S \) for the subcase 2. \( \bar{U}_{2,3} = 0 \Leftrightarrow \dot{x} = \dot{y} = \dot{z} = \dot{w} = 0 \Leftrightarrow (x, y, z, w) = S_0. \)

According to LaSalle theorem [10], \( S_0 \) is also globally asymptotically stable. The proof is completed.

**Proof of Proposition 3.** Set \( \bar{f} = f - \frac{\alpha_1d - a_2c}{a_2p} \). The system (1) has the following equivalent form

\[
\begin{align*}
\dot{x} &= a_1x + a_2y, \\
\dot{y} &= cx + dy + exz + (\bar{f} + \frac{\alpha_1d - a_2c}{a_2p})w, \\
\dot{z} &= -bz + xy, \\
\dot{w} &= py + qw.
\end{align*}
\]

If \( \rho^2 - 4(a_1d + dq + a_1q - f_0p - a_2c) \geq 0 \) and \( a_2^2a_1^2b^2c^2(a_3^2(q + d) - a_2c(a_2 - q)) \neq 0 \), the eigenvalues and corresponding eigenvectors of the associated linear vector field are \( \lambda_1 = 0, \lambda_2 = -b, \lambda_{3,4} = \frac{a_1 \pm \sqrt{\rho^2 - 4(a_1d + dq + a_1q - f_0p - a_2c)}}{2} \), and

\[
\begin{align*}
\xi_1 &= (-a_2, a_1, 0, -\frac{a_1p}{q})^T, \\
\xi_2 &= (0, 0, 1, 0)^T, \\
\xi_3 &= (a_2(\lambda_3 - q), (\lambda_3 - a_1)(\lambda_3 - q) - 0, p(\lambda_3 - a_1))^T, \\
\xi_4 &= (a_2(\lambda_4 - q), (\lambda_4 - a_1)(\lambda_4 - q) - 0, p(\lambda_4 - a_1))^T.
\end{align*}
\]

Taking the following homothetic transformation

\[
(x, y, z, w)^T = (\xi_1, \xi_2, \xi_3, \xi_4)(u, v, r, s)^T
\]

converts the system (1) into

\[
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{r} \\
\dot{s}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -b & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
r \\
s
\end{pmatrix} +
\begin{pmatrix}
-\frac{\rho(\xi_1)}{\lambda_3^2} \\
\lambda_{3,4} \Omega_2 \\
\lambda_{3,4} \Omega_2 \\
\lambda_{3,4} \Omega_2
\end{pmatrix},
\]

where

\[
\Omega_1 = -\bar{f}[-\frac{a_1p}{q}u + pr(\lambda_3 - a_1) + ps(\lambda_4 - a_1)] \\
- cev[-a_2u + a_2r(\lambda_3 - q) + a_2s(\lambda_4 - q)],
\]

\[
\Omega_2 = [-a_2u + a_2r(\lambda_3 - q) + a_2s(\lambda_4 - q)] \\
\times [a_1u + r(\lambda_3 - q)(\lambda_3 - a_1) + r(s(\lambda_4 - q)(\lambda_4 - a_1)).
\]

In order to determine the stability of \( S_0 \) near \( \bar{f} = 0 \) by the center manifold theory, one needs to study the two-parameter family of first-order ordinary differential equations on its center manifold that represents as a graph over \( u \) and \( \bar{f} = 0 \), i.e.

\[
W_{loc}(S_0) = \{(u, \bar{f}, v, r, s) \in \mathbb{R}^5 | v = V(u, \bar{f}), r = R(u, \bar{f}), s = S(u, \bar{f}), \}
\]

\[
V(0, 0) = R(0, 0) = S(0, 0) = 0,
\]

\[
DV(0, 0) = DR(0, 0) = DS(0, 0) = 0\}.
\]
Substituting the following expanded expressions of \(V(u, \tilde{f})\), \(S(u, \tilde{f})\) and \(R(u, \tilde{f})\) into the system (11),

\[
\begin{align*}
V(u, \tilde{f}) &= \sum_{i+j=2}^{\infty} v_{ij} u^i \tilde{f}^j, \\
R(u, \tilde{f}) &= \sum_{i+j=2}^{\infty} r_{ij} u^i \tilde{f}^j, \\
S(u, \tilde{f}) &= \sum_{i+j=2}^{\infty} s_{ij} u^i \tilde{f}^j,
\end{align*}
\]

obtains

\[
\begin{align*}
V(u, \tilde{f}) &= -\frac{a_1 a_2}{b} u^2 + \frac{a_1 a_2 p^2 [2q_1 a_1 (\lambda_3 + \lambda_4) - (2a_1 + p) \lambda_3 \lambda_4]}{b q \lambda_3^2 \lambda_4^2} \tilde{f} \\
&\quad + O(\| (u, \tilde{f}) \|^4), \\
R(u, \tilde{f}) &= \frac{a_1 p}{q \lambda_3^2 (\lambda_4 - \lambda_3)} \tilde{f}, \\
S(u, \tilde{f}) &= \frac{a_1 p}{q \lambda_3^2 (\lambda_4 - \lambda_3)} \tilde{f} + O(\| (u, \tilde{f}) \|^4).
\end{align*}
\]

(12)

Consequently, the restricted vector field of system (11) on its center manifold, i.e.

\[
\begin{align*}
\dot{u} &= \frac{q a_1 a_2}{b \lambda_4} u^3 - \frac{a_1 p}{q} \tilde{f} + \frac{a_2 p [2a_1 (\lambda_3 + \lambda_4) - \lambda_3 \lambda_4]}{q (\lambda_3 \lambda_4)^2} \tilde{f}^2 + O(\| (u, \tilde{f}) \|^4), \\
\dot{\tilde{f}} &= 0,
\end{align*}
\]

(13)

is obtained by substituting expressions in (12) into the system (11).

Since

\[
\begin{align*}
&\frac{\partial U}{\partial u} \bigg|_{u=0, \tilde{f}=0} = \frac{\partial \tilde{f}}{\partial \tilde{u}} \bigg|_{u=0, \tilde{f}=0} = \frac{\partial^2 U}{\partial u^2} \bigg|_{u=0, \tilde{f}=0} = 0, \\
&\frac{\partial^2 U}{\partial u \partial \tilde{f}} \bigg|_{u=0, \tilde{f}=0} = -\frac{a_1 p}{\lambda_3}, \\
&\frac{\partial^2 U}{\partial \tilde{f}^2} \bigg|_{u=0, \tilde{f}=0} = \frac{a_2 p q}{b \lambda_3 \lambda_4},
\end{align*}
\]

the system (1) undergoes a pitchfork bifurcation at \(S_0\) if \(f_0 = \frac{a_1 a_2 p^2}{a_1 p} (d - a_2 c)\) according to the pitchfork bifurcation theory [9, 47].

Furthermore, \(S_0\) is unstable (resp. stable) on its 1D center manifold for \(a_1 a_2 b c q [a_1 (d + q) - a_2 c (a_1 - q)] > 0\) (resp. < 0). The proof is over. \(\square\)

Proof of Proposition 4. 1) Transversality

Assume \(\lambda_1 = -b < 0\), \(\lambda_2 < 0\) and \(\lambda_{3,4} = \pm \omega i\) (\(\omega > 0\)) are the roots of Eq. (4).

From the relation between the roots and coefficients of characteristic equation (4), one arrives at

\[
\begin{align*}
\lambda_2 &= a_1 + d + q, \\
\omega^2 &= a_1 d + dq + a_1 q - fp - a_2 c, \\
\lambda_2 \omega^2 &= -a_1 (fp - dq) - a_2 cq,
\end{align*}
\]

which leads to

\[
f = \frac{(a_1 + d) [(d + q)(a_1 + q) - a_2 c]}{p (d + q)} \Delta f_* \quad \text{and} \quad \omega = \sqrt{-\frac{a_1^2 (d + q)(a_1^2 - q)}{d + q}}.
\]
Calculating the derivatives on both sides of Eq. (4) with respect to \( f \) and substituting \( \lambda \) with \( \omega_i \) yield
\[
\frac{d \text{Re}(\lambda_3)}{df} \bigg|_{f = f_*} = -\frac{p(d + q)}{2[\omega^2 + (a_1 + d + q)^2]} \neq 0,
\]
which verifies the condition of the transversality.

2) Nondegeneracy

In order to validate the nondegeneracy condition, one has to compute the Lyapunov coefficients by using the project method [13].

When \( f = f_* \), the matrix associated with linearized vector field about \( S_0 \)
\[
A = \begin{pmatrix}
    a_1 & a_2 & 0 & 0 \\
    c & d & 0 & f_* \\
    0 & 0 & -b & 0 \\
    0 & p & 0 & q
\end{pmatrix}
\]
satisfies \( Aq = i\omega q \), \( A^T p = -i\omega p \), \( \langle p, q \rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1 \), where
\[
p = \frac{1}{L} \begin{pmatrix}
    -c(q + \omega i) \\
    (a_1 + \omega i)(q + \omega i) \\
    0 \\
    -f_* (a_1 + \omega i)
\end{pmatrix}
\]
and
\[
q = \begin{pmatrix}
    -a_2(q - \omega i) \\
    (a_1 - \omega i)(q - \omega i) \\
    0 \\
    -p(a_1 - \omega i)
\end{pmatrix}
\]
with
\[
L = (q + \omega i)^2[a_2 f_* + (a_1 + \omega i)^2] + f_* p(a_1 + \omega i)^2.
\]

Performing further computations, one obtains
\[
h_{11} = \begin{pmatrix}
    0 \\
    0 \\
    0 \\
    \frac{-2a_2(\omega^2 + q^2)}{b}
\end{pmatrix}
\]
and
\[
h_{20} = \begin{pmatrix}
    0 \\
    0 \\
    \frac{2a_2(a_1 - \omega i)(\omega + q)^2}{b + 2\omega i} \\
    0
\end{pmatrix}.
\]

Finally, one gets the first Lyapunov coefficient as given by (7) by substituting the above calculated results into the expression in [13, Definition, Eq. (3.20), p. 99].

According to [13, Theorem 3.3, (B.1), p. 100], the inequality \( l_1 \neq 0 \) just verifies the nondegeneracy of Hopf bifurcation at \( S_0 \).

Therefore, if \( p(d + q) < 0 \) and \( l_1 > 0 \) (or \( p(d + q) > 0 \) and \( l_1 < 0 \)), then the Hopf bifurcation at \( S_0 \) is subcritical. Otherwise, if \( p(d + q) < 0 \) and \( l_1 < 0 \) (or \( p(d + q) > 0 \) and \( l_1 > 0 \)), then the Hopf bifurcation at \( S_0 \) is supercritical.

This completes the proof. \( \square \)

For convenience of discussion in the next subsection, one introduces the sets
\[
W = \{(a_1, a_2, b, c, d, e, f, p, q) \in \mathbb{R}^9 | a_1 b c [a_2 g c - a_1 (d q - f p)] > 0 \},
\]
\[
W_1 = \{(a_1, a_2, b, c, d, e, f, p, q) \in W : b > \alpha, -b \beta + \varphi > 0, b \epsilon > 0, b \psi < 0, (b - \alpha)(-b \beta + \varphi) > b \epsilon \},
\]
W_2 = W \setminus W_1 and
\[ W^1 = \{(a_1, a_2, b, c, d, e, f, p, q) \in W_1 : \Sigma < 0\}, \]
\[ W^2 = \{(a_1, a_2, b, c, d, e, f, p, q) \in W_1 : \Sigma = 0\}, \]
\[ W^3 = \{(a_1, a_2, b, c, d, e, f, p, q) \in W_1 : \Sigma > 0\}, \]
where \( \Sigma = b\varepsilon[(b - \alpha)(-b\beta + \varphi) - b\varepsilon] + (b - \alpha)^2b\psi. \)

3.2. Behavior of \( S_\pm \). The dynamics of \( S_\pm \) on stability and Hopf bifurcation is summarised as follows.

**Proposition 5.** (i) \( S_\pm \) are unstable when \((a_1, a_2, b, c, d, e, f, p, q) \in W^1_\cup W^2_\) whereas \( S_\pm \) are asymptotically stable when \((a_1, a_2, b, c, d, e, f, p, q) \in W^3_1. \)

(ii) While \((a_1, a_2, b, c, d, e, f, p, q) \in W^2_1\), the system (1) simultaneously undergoes Hopf bifurcation at \( S_\pm \).

**Proof.** i) The stability of \( S_\pm \) easily follows from the Routh-Hurwitz criterion.

ii) If \((a_1, a_2, b, c, d, e, f, p, q) \in W^2_1\), then Eq. (5) has a pair of conjugate purely imaginary roots \( \lambda_{3,4} = \pm \omega i \) with \( \omega = \sqrt{-\frac{b\varepsilon}{a\beta}} \) and two negative real roots \( \lambda_{1,2} = \frac{a - b_0 \pm \sqrt{(a-b_0)(a-b_0 - 4\varepsilon)}}{2} \) < 0, where \( b_0 = b(a_1, a_2, c, d, e, f, p, q) \) is the bifurcation parameter satisfying \( \Sigma = 0. \)

Calculating the derivatives on Eq. (5) with respect to \( b \) and substituting the \( \lambda \) with \( \omega i \) result in
\[
\left. \frac{d\text{Re}(\lambda_3)}{db} \right|_{b=b_0} = -\frac{b_0\varepsilon(\psi - \beta \omega^2) + \omega^2(\omega^2 - \varepsilon)[-b_0\beta + \varphi - 2\omega^2]}{2[(b_0\varepsilon)^2 + \omega^2(-b_0\beta + \varphi - 2\omega^2)^2]} \neq 0,
\]
which validates the transversal condition.

Consequently, Hopf bifurcation simultaneously happens at \( S_\pm \). The proof is over.

3.3. Behavior of \( S_z \). The dynamics of \( S_z \) associated with stability and degenerate pitchfork bifurcation [19, 24, 30] is concluded as follows.

**Proposition 6.** (a) Every one of \( S_z \) is a stable normally hyperbolic node or node-focus if \( \tau_1 > 0, \tau_2 > 0, \tau_3 > 0, \tau_1\tau_2 - \tau_3 > 0. \)

(b) The system (1) undergoes a degenerate pitchfork bifurcation at \( S_z \) when \( b \) crosses the null value and \( a_1beq[a_2qc - a_1(dq - fp)] > 0. \)

The proof of Proposition 6 follows from the Routh-Hurwitz criterion and is similar to the one [19, Lemma 2.2.2.1., p. 2703] and omitted here.

4. Singularly degenerate heteroclinic cycle. Coining singularly degenerate heteroclinic cycles is an inevitable issue when analyzing a chaotic/hyperchaotic system, since the collapse of them is one route to chaos/hyperchaos [7, 12, 20, 21, 23, 25, 40, 41, 44, 45, 49]. In this effort, one in this section tries to give some kind of the forming mechanism of the hyperchaotic attractor [15, p. 867] of the system (1) with \((a_1, a_2, c, d, e, f, b, p, q) = (-12, 12, 23, -1, -1, 1, 2.1, -6, -0.2)\) combining numerical techniques and the dynamics of \( S_z \).

First of all, the following linear scaling with \( k > 0 \)
\[
T_1 : (x, y, z, w, t) \rightarrow (kx, ky, kz, kw, \frac{1}{k}t)
\]
converts the system (1) into the resulting equivalent system
\[
\begin{align*}
\dot{x} &= \frac{a}{k} x + \frac{c}{k} y, \\
\dot{y} &= \frac{c}{k} x + \frac{d}{k} y + e x z + \frac{f}{k} w, \\
\dot{z} &= -\frac{b}{k} z + x y, \\
\dot{w} &= \frac{e}{k} y + \frac{f}{k} w.
\end{align*}
\] (14)

Therefore, the hyperchaotic attractor with
\[(a_1, a_2, c, d, e, f, b, p, q) = (-12, 12, 23, -1, -1, 1, 2.1, -6, -0.2)\]
for system (1) corresponds to the solution of system (14) with
\[(a_1, a_2, c, d, e, f, b, p, q) = (-12, 12, 23, -1, -1, 1, 2.1, -6, -0.2).\]

Here, set \(k = 40\). Thus the aforementioned hyperchaotic attractor corresponds to the solution of system (14) with
\[(a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, 0.0525, -0.15, -0.005).\]

Set \(b = 0\) and fix
\[(a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, -0.15, -0.005).\]

Choosing two groups of initial conditions \((x_0^{1.2}, y_0^{1.2}, w_0^{1.2}) = (\pm 1.618 \times 1e-4, \pm 3.14 \times 1e-4, \pm 0.618 \times 1e-4), (i) z_0^{1.2.3} = 0.3, 0.35, 0.4, (ii) z_0^{1.2.3} = 0.425, 0.465, 0.525,\) one has the following numerical result.

**Numerical Result 4.1** The 2D \(W^u(S_z^1)\) (resp. 2D \(W^u(S_z^2)\)) of each normally hyperbolic saddle-node \(S_z^1\) (resp. saddle-focus \(S_z^2\)) tend toward one of the normally hyperbolic stable node-foci \(S_z^3\) as \(t \to \infty\), forming singularly degenerate heteroclinic cycles with different geometrical structures, which further also collapse into the hyperchaotic attractors when \(b = 0.0525\). See Figure 3-8.

**Remark 3.** The eigenvalues of \((0, 0, 0.3, 0), (0, 0, 0.35, 0)\) and \((0, 0, 0.4, 0)\) are \(0, -0.4787, 0.1373, 0.0114; 0, -0.4543, 0.1073, 0.0169\) and \(0, -0.4276, 0.0654, 0.0322\). And the ones of \((0, 0, 0.425, 0), (0, 0, 0.465, 0)\) and \((0, 0, 0.525, 0)\) are \(0, -0.4123, 0.0416 \pm 0.0232i; 0, -0.3884, 0.0292 \pm 0.0414i\) and \(0, -0.3453, 0.0076 \pm 0.0556i\).

Set \(b = 0.07, (a_1, a_2, c, d, e, f, p, q) = (1, -0.3, 0.8, -1.25, 1, 0.5, 0.02, 0.02)\) and \((x_0^{1.2}, y_0^{1.2}, w_0^{1.2}) = (\pm 1.618 \times 1e-4, \pm 3.14 \times 1e-4, \pm 0.618 \times 1e-4), z_0^{1.2.3} = 3.3, 3.4, 3.5\). The following numerical result is presented.

**Numerical Result 4.2** The 2D \(W^u(S_z^2)\) of each normally hyperbolic saddle-focus \(S_z^2\) tend toward one of the normally hyperbolic stable node-foci \(S_z^3\) as \(t \to \infty\), forming singularly degenerate heteroclinic cycles, which further also collapse into the conjugate hyperchaotic Lorenz-type attractors when \(b = 0.07\). See Figure 9-12.

**Remark 4.** The eigenvalues of \((0, 0, 3.3, 0), (0, 0, 3.4, 0)\) and \((0, 0, 3.5, 0)\) are \(0, -0.3888, 0.0794 \pm 0.143i; 0, -0.3332, 0.0516 \pm 0.1635i\) and \(0, -0.266, 0.018 \pm 0.1851i\).

A detailed numerical study of the solutions of the system (1) with \(b = 0\) has been made, which clearly indicates that the system presents an infinite set of singularly degenerate heteroclinic cycles. Each one of these cycles is formed by one of the two-dimensional unstable manifolds of saddle-node \(S_z^1\) (resp. saddle-focus \(S_z^2\)), which connects \(S_z^1\) (resp. \(S_z^2\)) with the normally hyperbolic stable node-focus \(S_z^3\), as \(t \to \infty\). As the system presents an infinite number of normally hyperbolic saddle-nodes \(S_z^1\) (resp. saddle-focus \(S_z^2\)) and stable node-foci \(S_z^3\), there exists an infinite set of singularly degenerate heteroclinic cycles. In Figure 3(a), Figure 6(a) and Figure 9,
A UNIFIED HYPERCHAOTIC LORENZ-TYPE SYSTEM

0.3
0.4
0.5
0.6
0.7
0.8
y
0 0.2
0.1
x
0 -0.1
-0.5
-0.2

A
1
A
2
A
3

(a) \( b = 0 \)

\( z^0_1 = 0.3 \), (A₂) \( z^0_2 = 0.35 \), (A₃) \( z^0_3 = 0.4 \).

\( (x_0, y_0, w_0) = (\pm 1.618 \times 1e-4, \pm 3.14 \times 1e-4, \pm 0.618 \times 1e-4) \), showing the sensitive dependence on initial conditions.

\( (1.618 \times 1e-4, 3.14 \times 1e-4, 0.3 \times 0.618 \times 1e-4 - 4, 0.31, 0.618 \times 1e-4) \), showing the sensitive dependence on initial conditions.

some of them are shown: for each initial condition considered sufficiently close to
Lyapunov exponents

-0.46
-0.44
-0.38
-0.36
-0.34
-0.32

Lyapunov exponents of the system (1) with $(a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, 0.0525, -0.15, -0.005)$ and $(x_0, y_0, z_0, w_0) = (1.618 \times 1e-4, 3.14 \times 1e-4, 0.3, 0.618 \times 1e-4)$.

The saddle-node $S^1_2$ (resp. saddle-focus $S^2_2$) at the z-axis, a singularly degenerate heteroclinic cycle is created.

Figure 5. Lyapunov exponents of the system (1) with $(a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, 0.0525, -0.15, -0.005)$ and $(x_0, y_0, z_0, w_0) = (1.618 \times 1e-4, 3.14 \times 1e-4, 0.3, 0.618 \times 1e-4)$.

Figure 6. Phase portrait of the system (1) in the projection space $x$-$y$-$z$ with $(a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, 0.0525, -0.15, -0.005)$ and $(x_0, y_0, z_0) = (\pm 1.618 \times 1e-4, \pm 3.14 \times 1e-4, \pm 0.618 \times 1e-4)$, $(B_1) z_0^1 = 0.425$, $(B_2)$ $z_0^2 = 0.465$.

The solutions differ in their colors (or gray scale) and actually show this sensitive dependence. The $y$, $z$, and $w$-coordinates have the same behaviors. Then, in order to ensure that the system (1) presents hyperchaotic behavior for the same parameter value,
A UNIFIED HYPERCHAOTIC LORENZ-TYPE SYSTEM

Fig. 7. The x-coordinate \((t, x(t))\) with the parameter \((a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, 0.0525, -0.15, -0.005)\), initial conditions \((1.618 \times 1e^{-4}, 3.14 \times 1e^{-4}, 0.515, 0.618 \times 1e^{-4}, 4)\) and \((1.618 \times 1e^{-4}, 3.14 \times 1e^{-4}, 0.515, 0.618 \times 1e^{-4}, 4)\), showing the sensitive dependence on initial conditions.

Fig. 8. Lyapunov exponents of the system \((1)\) with \((a_1, a_2, c, d, e, f, b, p, q) = (-0.3, 0.3, 0.575, -0.025, -1, 0.025, 0.0525, -0.15, -0.005)\) and \((x_0, y_0, z_0, w_0) = (1.618 \times 1e^{-4}, 3.14 \times 1e^{-4}, 0.515, 0.618 \times 1e^{-4}, 4)\).

we have calculated its four Lyapunov exponents as follows:

\[ L_1 = 0.004366, L_2 = 0.003157, L_3 = -0.000003, L_4 = -0.390020. \]
This displays that two positive Lyapunov exponents exist, indicating the hyperchaotic behavior of the system for the parameter value. The result is also illustrated in Figure 5 (resp. Figure 8).

Also, Figure 11-12 verify the dependence on initial values and ensure the existence of hyperchaotic behavior for the system (1) when \((a_1, a_2, c, d, e, f, b, p, q) = (1, -0.3, 0.8, -1.25, 1, 0.5, 0.07, 0.02, 0.02)\), initial conditions \((1.618 \times 1e - 4, 3.14 \times 1e - 4, 3.5, 0.618 \times 1e - 4)\) and \((1.618 \times 1e - 4, 3.14 \times 1e - 4, 3.51, 0.618 \times 1e - 4)\).

\[
L_1 = 0.004568, L_2 = 0.003247, L_3 = -0.000001, L_4 = -0.390314.
\]
Figure 11. The $x$-coordinate $(t, x(t))$ with the parameter $(a_1, a_2, c, d, e, f, b, p, q) = (1, -0.3, 0.8, -1.25, 1, 0.5, 0.07, 0.02, 0.02)$, initial conditions $(1.618 \times 10^{-4}, 3.14 \times 10^{-4} - 4, 3.51, 0.618 \times 10^{-4})$, showing the sensitive dependence on initial conditions.

Figure 12. Lyapunov exponents of the system (1) with $(a_1, a_2, c, d, e, f, b, p, q) = (1, -0.3, 0.8, -1.25, 1, 0.5, 0.07, 0.02, 0.02)$ and $(x_0, y_0, z_0, w_0) = (1.618 \times 10^{-4}, 3.14 \times 10^{-4} - 4, 3.51, 0.618 \times 10^{-4})$.

Remark 5. Numerical Result. 4.1-2 only imply that the system (1) has an infinite set of singularly degenerate heteroclinic cycles and the hyperchaotic attractors near
them. The rigorous theoretical proof for the existence of singularly degenerate heteroclinic cycles will be done in the future work.

5. Ultimate bound estimation and global attractive set. By aid of the unified approach for the ultimate bound estimation of a class of HDQADS [43, Theorem 1, p. 2681] and Lyapunov functions, one in this section considers the global boundedness of the system (1). The main results are summarized as follows.

Proposition 7. Suppose $a_1, d, e \in \mathbb{R}^-$, $b, p_{11}, p_{22}, p_{44} \in \mathbb{R}^+$, $p_{14}^2 < p_{11}p_{44}$, $\det (Q) > 0$. Denote

$$\Omega = \left\{ X \in \mathbb{R}^4 \mid p_{11}(x + \frac{p_{14}}{p_{11}}w)^2 + p_{22}y^2 - ep_{22}(z - \frac{a_2p_{11} + cp_{22} + pp_{44}}{ep_{22}})^2 + p_{11}p_{44} - p_{14}^2 w^2 \leq R_{\max} \right\},$$

where $X = (x, y, z, w)^T$ and $R_{\max}$ can be found by calculating the maximum optimization question:

$$\max \quad p_{11}(x + \frac{p_{14}}{p_{11}}w)^2 + p_{22}y^2 - ep_{22}(z - \frac{a_2p_{11} + cp_{22} + pp_{44}}{ep_{22}})^2 + p_{11}p_{44} - p_{14}^2 w^2,$$

$$\text{s.t.} \quad p_{11}(x + \frac{p_{14}}{p_{11}}w)^2 + p_{22}y^2 - ep_{22}(z - \frac{a_2p_{11} + cp_{22} + pp_{44}}{ep_{22}})^2 + p_{11}p_{44} - p_{14}^2 w^2 = 0.$$ (16)

Then, $\Omega$ is the ultimate bound set of the system (1).

Proposition 8. For any $\lambda > 0$, $m > 0$, $a_1 < 0$, $d < 0$, $e < 0$, $q < 0$, $f_p < 0$, $b > 0$, with $V_{\lambda,m}(x, y, z, w) = \lambda x^2 + my^2 - me(z + \frac{me + a_2\lambda}{me})^2 - \frac{mf}{p}w^2$, $L_{\lambda,m} = \frac{b(m + a_2\lambda)^2}{me^2}$, $0 > \theta = \max\{a_1, d, q, -b\}$, $X(t) = (x(t), y(t), z(t), w(t))$, $X(t_0) = (x(t_0), y(t_0), z(t_0), w(t_0))$. When $V_{\lambda,m}(X(t)) > L_{\lambda,m}$, $V_{\lambda,m}(X(t_0)) > L_{\lambda,m}$, there exists an exponential estimate in the system (1), given by $V_{\lambda,m}(X(t)) - L_{\lambda,m} \leq [V_{\lambda,m}(X(t_0)) - L_{\lambda,m}]e^{\theta(t-t_0)}$. Namely, the set

$$\Psi_{\lambda,m} = \{X \mid V_{\lambda,m}(X) \leq L_{\lambda,m}\}$$

$$= \left\{ \lambda x^2 + my^2 - me(z + \frac{me + a_2\lambda}{me})^2 - \frac{mf}{p}w^2 \leq L_{\lambda,m}, \forall \lambda > 0, \forall m > 0 \right\}$$

is the global exponential attractive set of the system (1).

Proposition 9. When $a_2 > 0$ and $b + 2a_1 < 0$, the system (1) has the following inequality

$$\lim_{t \to \infty} [2a_2z - x^2] \geq 0.$$ (17)

Proposition 10. For $a_2 > 0$, $b + 2a_1 < 0$, $\lambda > 0$, $q < 0$, $d > 0$,

(i) $h = \frac{3a_2}{2a_2} \in \mathbb{R}^-$, $e = -q$, $b + 3q > 0$, $d < \frac{h + 3q}{2}$, or

(ii) $h = \frac{2d - q}{2a_2} \in \mathbb{R}^-$, $e = h + a_2 - d$, $b + 3q > 0$, $d < \frac{h + 3q}{2}$, or

(iii) $h = \frac{b + 2d}{2a_2} \in \mathbb{R}^-$, $e = h + a_2 - d$, $b > 2d, d > \frac{b + 3q}{2}$, $H = c + hd - a_2h^2 - ha_1$, $F = q - a_1 - ha_2$, $N = -p_{44}(q - a_1)(q - a_1 - ha_2) > 0$, $G = -\frac{hpf + (H + \lambda a_2)(-q + a_1 + ha_2)}{e(-q + a_1 + ha_2)}$, $\lambda(e + a_1) - hH + Nh^2p^3(a_1 - q) < 0$, with
\[ V_\lambda(X) = \frac{1}{2}[\lambda x^2 + (y - h)x^2 - e(z - G)^2 + N(hpx + Fw)^2], \quad L_\lambda = -\frac{eG^2(8a_2hx - b^2)}{8e^2(z + 2a_1h - b)}, \]

\[ X = (x, y, z, w). \]

When \( V_\lambda(X) > L_\lambda, \ V_\lambda(X_0) > L_\lambda, \) we can get the exponential inequality for the system (1), given by

\[ V_\lambda(X) - L_\lambda \leq [V_\lambda(X_0) - L_\lambda]e^{-2\epsilon(t-t_0)}. \]

By the definition, the set \[ \Psi_\lambda = \{ X | V_\lambda(X) \leq L_\lambda \} \]
\[ = \{ \lambda x^2 + (y - h)x^2 - e(z - G)^2 + N(hpx + Fw)^2 \leq 2L_\lambda \} \]

is the global exponential attractive set of the system (1).

The key issue of proofs of Proposition 7-10 is to construct appropriate Lyapunov-like functions. The sketches of them are given as follows.

1) Proof of Proposition 7. For convenience of employing the method [43, Theorem 1, p. 2681], one has to do some preparation work ahead.

First of all, rewrite the system (1) into the form as follows

\[ \dot{X} = AX + \sum_{i=1}^{4} x_i B_i X + C, \]

where

\[ X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ c & d & 0 & k \\ 0 & 0 & -b & 0 \\ 0 & p & 0 & q \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\epsilon}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_4 = 0_{4 \times 4}, \quad C = 0_{4 \times 1}. \]

Let \( P = (p_{ij})_{4 \times 4} \) with \( p_{ij} = p_{ji} \) (i, j = 1, 2, 3, 4) and u = (u_1, u_2, u_3, u_4) = 2\mu^T P, \)
where \( P, \mu \) satisfy Eqs. (6) and (7) in [43, Theorem 1, p. 2681]. One can calculate

\[ \sum_{i=1}^{3} x_i X^T(B_i^T P + PB_i^T)X = 2x[p_{23}(y^2 + z^2) + p_{13}xy + (p_{33} + e p_{22})yz + p_{44}yw + p_{12}xz + ep_{42}zw]. \]

Since

\[ \sum_{i=1}^{3} x_i X^T(B_i^T P + PB_i^T)X = 0 \quad (19) \]

holds for any \( x_i \in \mathbb{R}(i = 1, 2, 3, 4), \) let

\[ p_{12} = p_{13} = p_{23} = p_{24} = p_{34} = 0, \quad p_{33} = -ep_{22}. \quad (20) \]

Thereout, one gets

\[ P = \begin{bmatrix} p_{11} & 0 & 0 & p_{14} \\ 0 & p_{22} & 0 & 0 \\ 0 & 0 & -ep_{22} & 0 \\ p_{14} & 0 & 0 & p_{14} \end{bmatrix}. \]
From [43, Theorem 1, p. 2681], the proof of Proposition 7 is finished.

2) Proof of Proposition 8.

(1).

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and its derivative along the solution of the system (1):

\[ V = \frac{1}{2} u P - u^T \]

where \( q_{12} = a_{2} p_{11} + c p_{22} + p p_{14} + \frac{u_3}{2} \) and \( q_{24} = a_{2} p_{14} + k p_{22} + p p_{14} \).

For simplifying \( Q \), let

\[ q_{12} = a_{2} p_{11} + c p_{22} + p p_{14} + \frac{u_3}{2} = 0, \quad \frac{e u_2}{2} = 0, \quad u_1 = u_4 = 0; \]

that is

\[ u_3 = -2(a_{2} p_{11} + c p_{22} + p p_{14}), \quad u_1 = u_2 = u_4 = 0, \]

then

\[ u = (0, 0, -2(a_{2} p_{11} + c p_{22} + p p_{14}), 0). \]

So one has

\[
Q = \begin{bmatrix}
2 a_{1} p_{11} & 0 & 0 & (a_{1} + q) p_{14} \\
0 & 2 d p_{22} & 0 & q_{24} \\
0 & 0 & 2 b e p_{22} & 0 \\
(a_{1} + q) p_{14} & q_{24} & 0 & 2 q p_{44}
\end{bmatrix},
\]

\[ M = u A + 2 C^T P = [0, 0, 2 b (a_{2} p_{11} + c p_{22} + p p_{14}), 0]. \]

Since \( a_1, d, e \in \mathbb{R}^-, b, p_{11}, p_{22}, p_{44} \in \mathbb{R}^+, p_{14}^2 < p_{11} p_{44} \), \( \det(Q) > 0 \), one gets \( P > 0 \) and \( Q < 0 \). According to [43, Remark 1, p. 2682], one obtains the Lyapunov-like quadratic function

\[
V(X) = X^T P X + u X + \frac{1}{2} u P - u^T
\]

\[
= p_{11} (x + \frac{p_{11} q}{p_{11}})^2 + p_{22} y^2 + c p_{22} (z - \frac{a_{2} p_{11} + c p_{22} + p p_{14}}{c p_{22}})^2
\]

and its derivative along the solution of the system (1):

\[
\dot{V}(X) = 2 a_{1} p_{11} x^2 + 2 d p_{22} y^2 + 2 b e p_{22} z^2 + 2 p_{44} w^2 + 2 p_{14} (a_{1} + q) x w
\]

\[
+ 2 (a_{2} p_{14} + k p_{22} + p p_{14}) y w + 2 b (a_{2} p_{11} + c p_{22} + p p_{14}) z
\]

\[
= 2 a_{1} p_{11} [x + \frac{p_{11} (a_{1} + q)}{2 a_{1} p_{11}} w]^2 + 2 d p_{22} [y + \frac{a_{2} p_{14} + k p_{22} + p p_{14}}{2 d p_{22}} w]^2
\]

\[
+ 2 b e p_{22} [z + \frac{a_{2} p_{11} + c p_{22} + p p_{14}}{2 e p_{22}}]^2 -\frac{2 b (a_{2} p_{14} + k p_{22} + p p_{14})}{4 e p_{22}} w^2
\]

\[
+ 2 [p_{44} q - \frac{(a_{2} p_{11} + c p_{22} + p p_{14})^2}{4 a_{1} p_{11}}] w^2.
\]

From [43, Theorem 1, p. 2681], the proof of Proposition 7 is finished. \( \square \)

In the following, one reveals the global exponential attractive set of the system (1).

2) Proof of Proposition 8. Set the Lyapunov-like function

\[
V_{\lambda, m}(X) = \lambda x^2 + m y^2 - m c (z + \frac{m c + a_{2} \lambda}{m})^2 - \frac{m f}{p} w^2.
\]
Computing the derivative of $V_{\lambda,m}(X(t))$ along the trajectory of the system (1) leads to
\[
\frac{dV_{\lambda,m}}{dt}(t)_{(1)} = 2a_1\lambda x^2 + 2dmy^2 + 2bmcz^2 - \frac{2mfaz}{p}u^2 + 2b(mc + a_2\lambda)z
\]
\[
\leq a_1\lambda x^2 + dmy^2 - (b)me(z + \frac{mc + a_2\lambda}{me})^2 - \frac{mfaz}{p}u^2 - \frac{b(mc + a_2\lambda)^2}{me}
\]
\[
\leq \theta V_{\lambda,m} - \frac{b(mc + a_2\lambda)^2}{me}
\]
\[
= \theta(V_{\lambda,m} - L_{\lambda,m}) < 0
\]
when $V_{\lambda,m}(X(t)) > L_{\lambda,m}$, $V_{\lambda,m}(X(t_0)) > L_{\lambda,m}$.
In other words,
\[
\frac{dV_{\lambda,m}}{dt}(t)|_{(1)} \leq \theta(V_{\lambda,m} - L_{\lambda,m}). \tag{26}
\]
Integrating both sides of formula (26) and applying comparison principle, one arrives at
\[
V_{\lambda,m}(X(t)) \leq V_{\lambda,m}(X(t_0))e^{\theta(t-t_0)} + \int_{t_0}^t (-\theta)L_{\lambda,m}e^{\theta(t-t_0)}d\tau
\]
\[
= V_{\lambda,m}(X(t_0))e^{\theta(t-t_0)} + L_{\lambda,m}(1 - e^{\theta(t-t_0)}).
\]
If $V_{\lambda,m}(X(t)) > L_{\lambda,m}$ and $V_{\lambda,m}(X(t_0)) > L_{\lambda,m}$, the exponential estimation for the system (1) $V_{\lambda,m}(X(t)) - L_{\lambda,m} \leq [V_{\lambda,m}(X(t_0)) - L_{\lambda,m}]e^{\theta(t-t_0)}$ is derived.
According to the definition and taking limit on both sides of the above inequality as $t \to \infty$ results in $\lim_{t \to \infty} V_{\lambda,m}(X(t)) \leq L_{\lambda,m}$.
Equivalently, the set
\[
\Psi_{\lambda,m} = \{ X(t) | \lim_{t \to \infty} V_{\lambda,m}(X) \leq L_{\lambda,m}, \forall \lambda > 0, \forall m > 0 \}
\]
\[
= \left\{ (x, y, z, w) | \lambda x^2 + my^2 - me(z + \frac{mc + a_2\lambda}{me})^2 - \frac{mfaz}{p}u^2 \leq L_{\lambda,m}, \forall \lambda > 0, \forall m > 0 \right\}
\]
is the global exponential attractive set of the system (1). The proof is over. \(\square\)

3) Proof of Proposition 9. Define
\[
V(x, z) = 2a_2z - x^2.
\]
Then, its derivative along trajectories of the system (1) is $\dot{V} = 2a_2\dot{z} - 2x\dot{x} = -2a_2bz - 2a_1x^2$, i.e., $\dot{V} + bV = -(b + 2a_1)x^2$.
As $b + 2a_1 < 0$, one has $\dot{V} + bV \geq 0$.
For any initial value $V(t_0) = V_0$, according to the comparison theorem, one has $V(t) \geq V_0e^{-b(t-t_0)} \to 0(t \to \infty)$.
Thus
\[
\lim_{t \to \infty} V(t) = \lim_{t \to \infty} [2a_2z - x^2] \geq 0
\]
when $a_2 > 0$ and $b + 2a_1 < 0$.
This completes the proof. \(\square\)

4) Proof of Proposition 10. Similar to the proof of Proposition 9, one defines another different Lyapunov-like function
\[
V_\lambda(X) = \frac{1}{2} [\lambda x^2 + (y - hx)^2 - e(z - G)^2 + N(hpx + Fw)^2]
\]
and computes its derivative along the trajectory of the system (1)

\[
\frac{dV_\lambda}{dt} \bigg|_{(1)} = \lambda x \frac{dx}{dt} + (y - hx)(\frac{dy}{dt} - h \frac{dz}{dt}) - \varepsilon(z - G) \frac{dz}{dt} + N(hpx + Fw) \\
\times (hp \frac{dx}{dt} + F \frac{dz}{dt})
\]

\[
= \lambda x (a_1 x + a_2 y) + (y - hx)(cx + dy + exz + f w - h(a_1 x + a_2 y)) \\
- \varepsilon(z - G)(-bz + xy) \\
+ N(hpx + Fw)(hp(a_1 x + a_2 y) + F(py + qw))
\]

\[
= (\lambda a_1 - hH + Nh^2p^2(a_1 - q))x^2 + (d - ha_2)(y - hx)^2 \\
+ Nq(hpx + Fw)^2 - hex^2 z + bez^2 - Gbez.
\]

According to Proposition 9, one has

\[
\lim_{t \to \infty} [2a_2 z - x^2] > 0.
\]

Since \(\lim_{t \to \infty} [2a_2 z - x^2] > 0\), so there exists a positive constant \(T_0 > 0\), when \(t > T_0\), one arrives at

\[
\frac{dV_\lambda}{dt} \bigg|_{(1)} \leq (\lambda a_1 - hH + Nh^2p^2(a_1 - q))x^2 + (d - ha_2)(y - hx)^2 \\
+ Nq(hpx + Fw)^2 - 2a_2 e h z^2 + bez^2 - Gbez
\]

\[
= -\varepsilon[\lambda x^2 + (y - hx)^2 - \varepsilon(z - G)^2 + N(hpx + Fw)^2] \\
+ \lambda(\varepsilon + a_1) - hH + Nh^2p^2(a_1 - q))x^2 + (\varepsilon + d - ha_2)(y - hx)^2 \\
+ N(\varepsilon + q)(hpx + Fw)^2 - e \varepsilon(z - G)^2 - 2a_2 e h z^2 + bez^2 - Gbez
\]

\[
= -2\varepsilon V_\lambda(X) + [\lambda(\varepsilon + a_1) - hH + Nh^2p^2(a_1 - q))x^2 \\
+ (\varepsilon + d - ha_2)(y - hx)^2 + N(\varepsilon + q)(hpx + Fw)^2 - e f(z).
\]

If \(a_2 > 0\), \(b + 2a_1 < 0\), \(\lambda > 0\), \(q < 0\), \(d > 0\),

(i) \(h = \frac{b + d}{2a_2}, \varepsilon = -q, b + 3q > 0, d < \frac{b + 3q}{2}\), or

(ii) \(h = \frac{2d - q}{2a_2}, \varepsilon = ha_2 - d, b + 3q > 0, d < \frac{b + 3q}{2}\), or

(iii) \(h = \frac{b + 2d}{2a_2}, \varepsilon = ha_2 - d, b > 2d, d > \frac{b + 3q}{2}\),

then \(\varepsilon + d - ha_2 < 0, \varepsilon + q < 0\) and \(\varepsilon + 2a_2 h - b < 0\) hold.

Since \(\varepsilon < 0\) and \(\lambda(\varepsilon + a_1) - hH + Nh^2p^2(a_1 - q) < 0\), one has

\[
\frac{dV_\lambda}{dt} \bigg|_{(1)} \leq -2\varepsilon V_\lambda(X) - e f(z)
\]

\[
= -2\varepsilon V_\lambda(X) - e \max_{z \in \mathbb{R}} f(z)
\]

\[
= -2\varepsilon V_\lambda(X) - \frac{e G^2(8a_2 h \varepsilon - b^2)}{4(\varepsilon + 2a_2 h - b)}
\]

\[
= -2\varepsilon V_\lambda(X) - \frac{e G^2(8a_2 h \varepsilon - b^2)}{8\varepsilon(\varepsilon + 2a_2 h - b)}
\]

\[
= -2\varepsilon [V_\lambda(X) - L_\lambda],
\]

which yields

\[
V_\lambda(X) - L_\lambda \leq [V_\lambda(X_0) - L_\lambda] e^{-2\varepsilon(t-t_0)}.
\]
When \( V_\lambda(X) > L_\lambda \) and \( V_\lambda(X_0) > L_\lambda \). By the definition, taking upper limit on both sides of the above inequality (27) as \( t \to +\infty \) results in
\[
\lim_{t \to +\infty} V_\lambda(X) \leq L_\lambda.
\]
Namely, the set
\[
\Omega_\lambda = \left\{ \left. X(t) \right| \lim_{t \to \infty} V_\lambda(X) \leq L_\lambda \right\}
\]
\[
= \{(x, y, z, w) | ax^2 + (y - hx)^2 - e(z - G) + N(hpx + Fw)^2 \leq 2L_\lambda \}
\]
is the global exponential attractive set of the system (1). The proof is finished. □

**Remark 6.** Using the method [1], one may handle the boundedness of the Chen-type or Lü-type subsystem of the system (1), which are not contained in Proposition 7-10.

In the following Section 6, one studies the global bifurcation on the existence of homoclinic and heteroclinic orbits for the system (1). First of all, one introduces some notations for the convenience of statement in the discussion in the sequel. Denote in the following by \( \phi_i(q_0) = (x(t; q_0), y(t; q_0), z(t; q_0), w(t; q_0)) \) a solution of the system (1) through the initial point \( q_0 = (x_0, y_0, z_0, w_0) \) and by \( W^u_+ \) (resp. \( W^s_+ \)) the positive (resp. negative) branch of the unstable manifold \( W^u(S_0) \) corresponding to \( x > 0 \) (resp. \( x < 0 \)) for large negative \( t \). In addition, let \( \gamma^\pm = \{ \phi^+_i(t) | \phi^+_i(t) = (\pm x_+, t), \pm y_+, \pm z_+, \pm w_+)(t) \in W^u_+, t \in \mathbb{R} \} \).

**6. Existence of heteroclinic orbit.** Utilizing two different Lyapunov functions, concepts of both \( \alpha \)-limit set and \( \omega \)-limit set [5, 16, 17, 19, 20, 21, 24, 34, 35, 39, 40, 41, 44, 45], this section devotes to investigating the existence of heteroclinic orbits of the system (1), aiming at complementing and extending the obtained results in [5, Theorem 5, p.579]. The fundamental work of the proof is how to construct suitable Lyapunov functions for the system (1).

Set the first Lyapunov function
\[
V_1(\phi_i(q_0)) = V_1(x, y, z, w) = -\frac{h(2a_1+b)}{a_2^2}(a_1x + a_2y)^2 + \frac{bf(2a_1+b)}{pq(-a_1+q)}(-p\frac{a_1}{a_2}x + qw)^2
- \frac{h}{2a_1} \left[ \frac{b(2a_2q-a_1(dq-fp))}{a_2qr} - \frac{a_2}{a_2^2}x^2 \right]^2 + (b_z + \frac{a_1}{a_2^2}x^2).
\]
for \( 2a_1 + b > 0 \), and the second one
\[
V_2(\phi_i(q_0)) = V_2(x, y, z, w) = -\frac{pq(-a_1+q)}{a_2^2}((a_1x + a_2y)^2 + (-p\frac{a_1}{a_2}x + qw)^2)
+ \frac{pq(-a_1+q)}{4a_1^2f}(a_1^2x + a_2^2y)^2 - \frac{a_1^2}{a_2^2}y^2.
\]
for \( 2a_1 + b = 0 \).

It follows that
\[
\frac{dV_1(\phi_i(q_0))}{dt} \bigg|_{(1)} = -\frac{2b(2a_1+b)(a_1+a_2)}{a_2^2}(a_1x + a_2y)^2 - 2b(bz + \frac{a_1}{a_2^2}x^2)^2
+ 2bf(2a_1+b) \left[ -p\frac{a_1}{a_2}x + qw \right)^2 \leq 0
\]
and
\[
\frac{dV_2(\phi_i(q_0))}{dt} \bigg|_{(1)} = -\frac{2pq(-a_1+q)(a_1+a_2)}{a_2^2}(a_1x + a_2y)^2
+ 2q(-p\frac{a_1}{a_2}x + qw)^2 \leq 0,
\]
respectively.

Proceeding as in [5, 16, 17, 19, 20, 21, 24, 34, 35, 39, 40, 41, 44, 45], the following results are derived for the both cases \( 2a_1 + b > 0 \) and \( 2a_1 + b = 0 \).
Proposition 11. For $a_2 \neq 0$, $a_1 < 0$, $2a_1 + b \geq 0$, $e < 0$, $a_1 + d < 0$, $q < 0$, $fp(-a_1 + q) > 0$ and $a_2cq - a_1(dq - fp) < 0$, the following assertions are derived.

(i) If there exist $t_1$ and $t_2$ such that $t_1 < t_2$ and $V_{1,2}(\phi_1(q_0)) = V_{1,2}(\phi_1(q_0))$, then $q_0$ is one of the equilibria of the system (1).

(ii) If $\phi_1(q_0) \to S_0$ as $t \to -\infty$, and $x(t; q_0) < 0$ for some $t \in \mathbb{R}$, then $V_{1,2}(S_0) > V_{1,2}(\phi_1(q_0))$ and $x(t; q_0) < 0$ for all $t \in \mathbb{R}$. Consequently, $q_0 \in W^u$.

Proposition 12. Consider $a_2 \neq 0$, $a_1 < 0$, $2a_1 + b \geq 0$, $e < 0$, $a_1 + d < 0$, $q < 0$, $fp(-a_1 + q) > 0$ and $a_2cq - a_1(dq - fp) < 0$ and the Lyapunov functions $V_{1,2}$. Then the following statements hold.

(a) The system (1) has neither homoclinic orbits nor heteroclinic orbits to $S_+$ and $S_-$. 

(b) The system (1) has exactly two heteroclinic orbits to $S_0$ and $S_\pm$.

Proofs of Proposition 11-12 are similar as in [5, 16, 17, 19, 20, 21, 24, 34, 35, 39, 40, 41, 44, 45]. One only sketches them here.

Proof of Proposition 11. i) Since $\frac{dV_{1,2}(\phi_1(q_0))}{dt} \leq 0$, it follows that $\frac{dV_{1,2}(\phi_1(q_0))}{dt} = 0$ for all $t \in (t_1, t_2)$, which yields that $q_0$ is one of the equilibria, i.e.

$$x'(\phi_1(q_0)) \equiv y'(\phi_1(q_0)) \equiv z'(\phi_1(q_0)) \equiv u'(\phi_1(q_0)) \equiv 0. \quad (30)$$

In fact, $x'(\phi_1(q_0)) = a_1x + a_2y = 0$ implies $x(t) \equiv x_0$ and $y'(t, q_0) = 0$, $\forall t \in \mathbb{R}$.

The condition $2a_1 + b \geq 0$ leads to $Q(\phi_1(q_0)) = x^2 - 2a_2z = 0$. Namely, $\frac{d}{dt}Q(\phi_1(q_0)) = 2a_1Q(\phi_1(q_0))$ suggests

$$Q(\phi_1(q_0)) = Q(\phi_1(0))e^{2a_1(t-0)} \quad \text{for all} \quad t \in \mathbb{R} \quad (31)$$

which further yields $Q(\phi_1(q_0)) \equiv 0$, as $\phi_1(q_0)$ is bounded as $t \to -\infty$.

In either case, $\phi_1(q_0) \in \{a_1x + a_2y = 0\} \cap \left\{bz + \frac{a_1}{a_2}x^2 = 0\right\} \cap \left\{-p\frac{a_2}{a_2}x + qw = 0\right\}$

leads to (30).

ii) The hypothesis of (ii) implies that $q_0$ is not an equilibrium point since $q_0 = S_0$ satisfies the first condition $\phi_1(q_0) \to S_0$ as $t \to -\infty$ but does not satisfies the second one $x(t; q_0) < 0$ for some $t \in \mathbb{R}$. Assume by contrary that there exists a $t_0 \in \mathbb{R}$ such that $0 < V_{1,2}(S_0) \leq V_{1,2}(\phi_1(q_0))$. But there exists a $t_1$ such that $V_{1,2}(\phi_1(q_0)) \leq V_{1,2}(\phi_1(q_0))$.

Since $V_{1,2}(t)$ are decreasing with respect to $t$, it follows that $V_{1,2}(\phi_1(q_0)) = V_{1,2}(\phi_1(0))$, using i), it follows that $q_0$ is an equilibrium point of the system. The hypothesis $\lim_{t \to -\infty} \phi_1(q_0) = S_0$ results that $q_0 = S_0$ and $x(t; q_0) = 0$, $\forall t \in \mathbb{R}$, which contradicts the hypothesis. Hence, $V_{1,2}(S_0) > V_{1,2}(\phi_1(q_0)), \forall t \in \mathbb{R}$.

Next, let us prove now that $x(t, q_0) < 0$, $\forall t \in \mathbb{R}$. Otherwise, there exists at least a $t' \in \mathbb{R}$ such that $x(t', q_0) \geq 0$ and using $x(t', q_0) < 0$ for some $t' \in \mathbb{R}$ from the hypothesis of (ii), one gets that there exists a $t \in \mathbb{R}$ such that $x(t, q_0) = 0$. As $V_{1,2}(S_0) > V_{1,2}(\phi_1(q_0)), \forall t \in \mathbb{R}$, it follows that $\phi_1(q_0) \in \Omega \cap P$, where $\Omega = \{(x,y,z,w) : V_{1,2}(S_0) > V_{1,2}(x,y,z,w)\}$ and $P$ is the plane $\{x = 0\}$. However, $\Omega \cap P$ is given by

$$-\frac{b(2a_1 + b)}{2} y^2 - \frac{(2a_1 + b)^2[a_2c - a_1(dq - fp)]^2}{2a_1(a_2q^2)} + b^2z^2 + \frac{bfq(2a_1 + b)}{2p(-a_1 + q)}w^2$$

$$< -\frac{(2a_1 + b)^2[a_2c - a_1(dq - fp)]^2}{2a_1(a_2q^2)}.$$
for \( V_1 \) and
\[
-pq(-a_1+q)y^2 + q^2w^2 + p(-a_1+q)(2cq-a_1(dq-fp))^2 < p(-a_1+q)(2cq-a_1(dq-fp))^2
\]
for \( V_2 \). Any one of the above two cases leads to \( \Omega \cap P = \emptyset \), which is a contradiction. Therefore \( x(t,q_0) < 0, \forall t \in \mathbb{R}. \) This completes the proof of the proposition.

**Proof of Proposition 12.** a) Firstly, let us show that there is neither homoclinic orbits nor heteroclinic orbits to \( S_+ \) and \( S_- \) for \( a_2 \neq 0, a_1 < 0, 2a_1 + b \geq 0, e < 0, a_1 + d < 0, q < 0, fp(-a_1 + q) > 0 \) and \( a_2cq - a_1(dq - fp) < 0 \). Assume \( \gamma(t,q_0) \) is a homoclinic orbit or a heteroclinic orbit to \( S_+ \) and \( S_- \) of the system (1) through an initial point \( q_0 \notin \{S_0, S_-, S_+\} \), i.e.
\[
\lim_{t \to -\infty} \phi_t(q_0) = s_-, \lim_{t \to \infty} \phi_t(q_0) = s_+,
\]
where \( s_- \) and \( s_+ \) satisfy either \( s_- = s_+ \in \{S_0, S_-, S_+\} \) or \( s_- = s_+ = \{S_-, S_+\} \). It follows from (28)-(29) that
\[
V_{1,2}(s_-) \geq V_{1,2}(\gamma(t,q_0)) \geq V_{1,2}(s_+).
\]
In either case, we have the relation \( V_{1,2}(s_-) = V_{1,2}(s_+), \) which suggests \( V_{1,2}(\gamma(t,q_0)) = V_{1,2}(s_+) \). The assertion (i) of Proposition 11 results that \( q_0 \) is one of the equilibria of the system (1). Hence, neither homoclinic orbits nor heteroclinic orbits to \( S_+ \) and \( S_- \) exist in the system (1).

b) Next, we prove that the system (1) has a heteroclinic orbit to \( S_0 \) and \( S_- \). Since \( W^u \) is the negative branch with respect to \( z \) of the unstable manifold at \( S_0 \), there exists a \( t_1 \in \mathbb{R} \) such that \( x(t_1,q_0) < 0 \) for \( q_0 \in W^u \), which by Proposition 11(ii), gives \( x(t,q_0) < 0 \) for all \( t \in \mathbb{R} \) and \( q_0 \in W^u \), i.e. any orbit on \( W^u \) never approach \( S_+ \), which lies in \( x > 0 \). From Proposition 12(a), it also does not tend to \( S_0 \) but one of the equilibria. Therefore, denoting by \( \gamma^- (t) \) an orbit on \( W^u \), it follows that \( \lim_{t \to -\infty} \gamma^- (t) = S_- \), i.e. \( \gamma^- (t) \) is a heteroclinic orbit to \( S_0 \) and \( S_- \) lying in \( x < 0 \). Let us show now that this heteroclinic orbit is unique in \( x < 0 \). Assume that \( \phi_t(q_0) \) is a solution of the system with \( q_0 \) arbitrary, not necessarily on \( W^u \), with \( s_- = s_+ \) as above.
\[
\lim_{t \to -\infty} \phi_t(q_0) = s-, \lim_{t \to \infty} \phi_t(q_0) = s_+,
\]
where \( s_- = s_+ \in \{S_0, S_-\} \), i.e. \( \phi_t(q_0) \) is another heteroclinic orbit to \( S_0 \) and \( S_- \).

As \( V_{1,2} \) are decreasing on the orbits, it follows that
\[
V_{1,2}(s_-) \geq V_{1,2}(\phi_t(q_0)) \geq V_{1,2}(s_+)
\]
for all \( t \in \mathbb{R} \). As \( V_{1,2}(S_0) > V_{1,2}(S_-) \), one gets that \( s_- = S_0 \) and \( s_+ = S_- \), which gives further that \( q_0 \in W^u \), by Proposition 11 (ii), i.e. the orbit \( \phi_t(q_0) \) coincides with \( \gamma^- (t) \). As the orbits are symmetrical with respect to the \( z \)-axis, there exits an unique heteroclinic orbit \( \gamma^+ (t) \) symmetrical to \( \gamma^- (t) \) with respect to the \( z \)-axis.

Moreover, numerical simulations are presented to show the correctness of the theoretical result, as depicted in Figure 13-14. This completes the proof.

**Remark 7.** Except for the results formulated in Proposition 12, numerical simulations also demonstrate that there are some other new heteroclinic orbits when \( (a_1, a_2, b, c, d, e, f, p, q) \in W_1^3 \), see Figure 15-16.
Remark 8. When $a_1b[a_2qc - a_1(dq - fp)] < 0$, it follows from Proposition 1 that $S_0$ is a single equilibrium point of the system (1), in which case there is no heteroclinic orbit.

7. Conclusion. Based on a 3D unified Lorenz-type system and a hyperchaotic Lorenz-type system, a novel 4D unified hyperchaotic Lorenz-type system (UHLTS) is proposed and analyzed. The complex dynamics of it, such as stability and bifurcation of equilibria, the existence of singularly degenerate heteroclinic cycles,
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The forming mechanism of conjugate hyperchaotic Lorenz-type attractors, ultimate bound estimation, global exponential attractive sets, homoclinic and heteroclinic orbits, and so on, are investigated by the linear analysis, center manifold theorem, Routh-Hurwitz criterion, bifurcation theory, Lyapunov function, numerical simulation, etc. In particular, the following important but distinct properties are coined:

(i) conjugate hyperchaotic Lorenz-type attractors (CHCLTA) which are bifurcated from singularly degenerate heteroclinic cycles or the solutions approaching infinity after a short-duration transient of singularly degenerate heteroclinic cycles;
(ii) some kind of forming mechanism of the well-known hyperchaotic attractor that is collapse of singularly degenerate heteroclinic cycles with different geometrical structures;
(iii) ultimate bound estimation and a family of mathematical expressions of global exponential attractive sets;
(iv) a pair of heteroclinic orbits.

Since the findings and results obtained in this paper extend some published results in some known literatures, we hope them will have a good potential in control and synchronization of hyperchaos and their engineering applications. It is expected that the basic ideas and the self-contained approach presented in this paper can be applied to explore similar hyperchaotic systems.

Owing to the great potential of hyperchaos in such nontraditional engineering and technological applications as lasers and electronics, encryption and secure communications, and biological networks, among others, it is insightful and important to develop the future work that circles around the further inquiry into the subjects of generating, controlling, synchronizing and applying hyperchaos.

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