LINES OF CURVATURE OF THE DOUBLE TORUS

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ABSTRACT. We describe the $\nu$-lines of curvature of an embedding of the double torus into $\mathbb{R}^4$, defined as the link of the real part of the Milnor fibration of a polynomial, where $\nu$ is its gradient. Through this analysis, we present a complete description of the foliation of lines of curvature of the embedding, defined as the image of the stereographic projection of this link into $\mathbb{R}^3$.

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1. Introduction

The lines of curvature of surfaces in $\mathbb{R}^3$ have been a subject of interest for several centuries, since G. Monge described the case of the triaxial ellipsoid by direct integration of the differential equation of these lines. This is the first example of a foliation with singularities on a compact surface [8]. Despite the fact that many studies have focused on different properties of local and global nature of the lines of curvature of surfaces in $\mathbb{R}^3$, (see for instance [4], [1], [13], [7], [14]), a complete analytic description of them has not been achieved for the case of an oriented compact surface of genus higher than one. A reason for this may be that, contrasting with the ellipsoid case which is a quadric surface, embeddings of compact surfaces of higher genus are quartic or even higher degree algebraic surfaces, [3, 5] and consequently, their lines of curvature are described by rather complicated differential equations. The simplest case in this class of surfaces is the double torus.

In the present article we present a description of the foliation of lines of curvature of an embedding of the double torus into $\mathbb{R}^3$. In order to avoid the problem arising from the degree of the algebraic surface in $\mathbb{R}^3$ we consider an embedding of the double torus into $\mathbb{R}^4$ as the link of the real part of the Milnor fibration of a polynomial with isolated singularity at the origin. Specifically, this surface is embedded into the sphere $S^3 \subset \mathbb{R}^4$ of radius $r$ with center at the origin for which we analyze the lines of curvature with respect to the gradient $\nu$, referred to in our research as $\nu$-lines of curvature, of this polynomial on the link.

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The $\nu$-lines of curvature of surfaces, where $\nu$ is a normal vector field, have been studied in $\mathbb{R}^4$ in [11] and [2], where the expression of the differential equation is given in the Monge chart, in which the surface is defined as the graph of a differentiable function. In this article, we introduce an equation of these lines according to the description of the surface as the intersection of the loci of two polynomials, which simplifies the analysis. This equation, together with the symmetries of the link, allow us to apply elements from both algebraic geometry and dynamical systems to describe the foliation completely. Then, we apply the stereographic projection which transforms the $\nu$-lines of curvature of this spherical embedding into the lines of curvature of the image of the link in $\mathbb{R}^3$.

The article is organized as follows: in section 2 we present some preliminaries and the formulation of the differential equation of $\nu$-lines of curvature that will be used in the study (Proposition 2.3 and Corollary 2.5). In section 3, we analyze the embedding of the double torus into $\mathbb{R}^4$. Namely, we determine a decomposition of the surface parameterized with convenient coordinate charts which allow us to use the symmetries of the surface to simplify the study. In section 4, we prove the main theorem which provides the description of the foliation of the $\nu$-lines of curvature on the double torus embedded into $\mathbb{R}^4$ as a transversal intersection (Theorem 4.2). As a corollary we get the description of the lines of curvature of the image of this embedding under the stereographic projection in $\mathbb{R}^3$ (Corollary 4.19).

2. Preliminaries

2.1. The $\nu$-lines of curvature of a surface in $\mathbb{R}^4$. Let $M$ be a smooth oriented surface immersed in $\mathbb{R}^4$ with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^4$. For each $p \in M$ consider the decomposition $T_p\mathbb{R}^4 = T_pM \oplus N_pM$, where $N_pM$ is the orthogonal complement of $T_pM$ in $\mathbb{R}^4$. Let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^4$. Given local vector fields $X$, $Y$ on $M$, let $\bar{X}$, $\bar{Y}$ be local extensions to $\mathbb{R}^4$. The tangent component of the Riemannian connection in $\mathbb{R}^4$ is the Riemannian connection of $M$:

$$\nabla_X Y = (\bar{\nabla}_X \bar{Y} - \bar{\nabla}_X Y).$$

This map is symmetric and bilinear. So, at each point $p$, $\nabla(X,Y)$ only depends on $X(p)$ and $Y(p)$.

Suppose the $\nu$ is a unitary vector field in $N(M)$. The $\nu$-second fundamental form of $M$ at $p$ is the quadratic form,

$$II : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{N}M, \ II(X,Y) = \bar{\nabla}_X \bar{Y} - \bar{\nabla}_X Y.$$ 

This map is symmetric and bilinear. So, at each point $p$, $II(X,Y)$ only depends on $X(p)$ and $Y(p)$.

Suppose the $\nu$ is a unitary vector field in $N(M)$. The $\nu$-second fundamental form of $M$ at $p$ is the quadratic form,

$$II_{\nu} : T_pM \to \mathbb{R}, \ II_{\nu}(X) = \langle II_{\nu}(X,X), \nu \rangle.$$ 

Recall the $\nu$-shape operator

$$S_{\nu} : T_pM \to T_pM, \ S_{\nu}(X) = -(\bar{\nabla}_X \nu)^	op,$$
where $\tilde{\nu}$ is a local extension to $\mathbb{R}^4$ of the normal vector field $\nu$ at $p$ and $\top$ means the tangent component. This operator is self-adjoint and, for any $X, Y \in T_pM$, satisfies the following equation:

$$< S_\nu(X), Y > = < II(X, Y), \nu > .$$

Thus, for each $p \in M$, there exists an orthonormal basis of eigenvectors of $S_\nu \in T_pM$, for which the restriction of the second fundamental form to the unitary vectors, $II_\nu|_{S^1}$, takes its maximal and minimal values. The corresponding eigenvalues $k_1, k_2$ are the maximal and minimal $\nu$-principal curvatures, respectively. The point $p$ is a $\nu$-umbilic if the $\nu$-principal curvatures coincide. Let $U_\nu$ be the set of $\nu$-umbilics in $M$. For any $p \in M \setminus U_\nu$, there are two $\nu$-principal directions defined by the eigenvectors of $S_\nu$. These fields of directions are smooth and integrable, and therefore they define, in $M \setminus U_\nu$, two families of orthogonal curves, its integrals, which are called the $\nu$-principal lines of curvature, one maximal and the other one minimal. The $\nu$-umbilics are considered as the singularities of these foliations. The differential equation of $\nu$-lines of curvature is

$$S_\nu(X(p)) = \lambda(p)X(p) .$$

2.2. Equation of $\nu$-lines of curvature of surfaces defined implicitly in 4-space. We first determine a characterization of a $\nu$-principal direction at a point of a surface provided with a normal frame $\{\nu, \mu\}$.

Proposition 2.1. Let $M$ be a surface immersed in $\mathbb{R}^4$. Assume that $\{\nu, \mu\}$ is a frame of the normal bundle on $M$. Consider the unitary vector field $\tilde{\nu} = \frac{\nu}{|\nu|}$. If $p \in M$, the vector $X \in T_pM$ is tangent to a $\tilde{\nu}$-principal direction, if and only if the following equation holds at $p$:

$$\langle \tilde{\nabla}_X \nu \wedge X \wedge \mu, \nu \rangle = 0 .$$

(2.1)

Proof. We decompose the covariant derivative as the sum of the following vector fields:

$$\tilde{\nabla}_X \nu = (\tilde{\nabla}_X \nu) \top + p_\mu(X)\mu + p_\nu(X)\nu,$$

where $p_\mu, p_\nu : T_pM \to \mathbb{R}$ are the projections of $\tilde{\nabla}_X \nu$ on the lines determined by the normal frame $\{\nu, \mu\}$ at $p$.

Therefore,

$$\langle \tilde{\nabla}_X \nu \wedge X \wedge \mu, \nu \rangle = \langle (\tilde{\nabla}_X \nu) \top + p_\mu(X)\mu + p_\nu(X)\nu \wedge X \wedge \mu, \nu \rangle = \langle (\tilde{\nabla}_X \nu) \top \wedge X \wedge \mu, \nu \rangle .$$

(2.2)

Observe that the last expression vanishes if and only if $(\tilde{\nabla}_X \nu) \top$ and $X$ are linearly dependent.

If $\nu$ is unitary, $-(\tilde{\nabla}_X \nu) \top = S_\nu(X)$ is the $\nu$-shape operator. Then, equation (2.1) holds if and only if $S_\nu(X)$ and $X$ are linearly dependent; this is equivalent to the fact that $X$ must be an eigenvector of the $\nu$-shape operator.

On the other hand, if $\nu$ is not unitary, then,
\[
\left( \nabla_X \tilde{\nu} \right)^\top = \left( \nabla_X \frac{\nu}{||\nu||} \right)^\top = \left( X \frac{1}{||\nu||} \nu + \frac{1}{||\nu||} (\nabla_X \nu) \right)^\top = \frac{1}{||\nu||} (\nabla_X \nu)^\top.
\]

Therefore, from this equation we have
\[
\langle \nabla_X \nu \wedge X \wedge \mu, \nu \rangle = \langle (\nabla_X \nu)^\top \wedge X \wedge \mu, \nu \rangle = -||\nu|| S_{\tilde{\nu}}(X) \wedge X \wedge \mu, \nu \rangle.
\]

Thus, we conclude that equation (2.1) holds if and only if \( X \) is an eigenvector of the \( \tilde{\nu} \)-shape operator. \(\square\)

**Remark 2.2.** Observe that this proposition allows us to consider any non-unitary normal vector field \( \nu \) on \( M \) in order to define the \( \tilde{\nu} \)-principal directions of curvature. This fact will be used in the setting when we consider non-unitary gradient vector fields to define the lines of curvature.

Consider now a surface \( M \) defined as the transversal intersection of the inverse images of regular values of two differentiable functions, there is a natural frame of the normal bundle of \( M \), given by the gradient vector fields of both functions. Thus, \( M \) is endowed with a natural pair of lines of curvature. A direct computation that uses the multilinearity properties of the left hand side of (2.1) provides the following expression for the equation of these lines of curvature:

**Proposition 2.3.** Let \( F, G : \mathbb{R}^4 \to \mathbb{R} \) be a pair of differentiable functions. Suppose that \( M \) is a surface defined as the transversal intersection of the inverse image of two regular values of these functions. Let \( \nu \) and \( \mu \) be the gradient vector fields of \( F \) and \( G \), respectively. Then, the differential equation of the \( \nu \)-lines of curvature of \( M \) is given by:

\[
(dx_1, dx_2, dx_3, dx_4) \begin{pmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\
\Omega_{12} & \Omega_{22} & \Omega_{23} & \Omega_{24} \\
\Omega_{13} & \Omega_{23} & \Omega_{33} & \Omega_{34} \\
\Omega_{14} & \Omega_{24} & \Omega_{34} & \Omega_{44}
\end{pmatrix} \begin{pmatrix}
dx_1 \\
dx_2 \\
dx_3 \\
dx_4
\end{pmatrix} = 0, \quad (2.3)
\]

where \( \{x_1, x_2, x_3, x_4\} \) is the system of parameters of \( \mathbb{R}^4 \) and

\[
\Omega_{ij} = \frac{1}{2} \left( \langle \nabla \frac{\partial}{\partial x_i} \nu \wedge \frac{\partial}{\partial x_j} \wedge \mu, \nu \rangle + \langle \nabla \frac{\partial}{\partial x_j} \nu \wedge \frac{\partial}{\partial x_i} \wedge \mu, \nu \rangle \right).
\]

**Remark 2.4.** This proposition still holds if \( p, q \) or both are singular values of any of the two functions, but only if \( M = F^{-1}(p) \cap G^{-1}(q) \) lies in the complement of the corresponding critical points. The other hypothesis remain under assumption.
In case we have a frame of the tangent bundle of $M$, equation (2.3) has a simpler expression.

**Corollary 2.5.** Assume that $V_1, V_2$ is a frame of the tangent bundle of $M$. Then, the expression of the differential equation (2.3) is:

$$l_1^2 \Omega_{11} + l_1 l_2 (\Omega_{12} + \Omega_{21}) + l_2^2 \Omega_{22} = 0$$  \hspace{1cm} (2.4)

where $\Omega_{ij} = \langle \nabla_{V_i} \nu \wedge V_j \wedge \mu, \nu \rangle$, $i, j \in \{1, 2\}$ and $l_i$ is the $i$-component of the vectorial solution $X$ of the equation, namely: $X = l_1 V_1 + l_2 V_2$.

3. An embedding of the double torus into $\mathbb{R}^4$

Let us analyze a family of examples where the $\nu$-lines of curvature can be described using the expressions presented above. In [12], J. Seade studied the topology of a class of surfaces obtained as the projections of certain complex hypersurfaces onto a real line through the origin of $\mathbb{C}$. Let us consider an element of that class:

\[\bar{F}: \mathbb{C}^2 \to \mathbb{C}, \quad \bar{F}(z_1, z_2) = z_1^p + z_2^q, \quad 1 < p, q \in \mathbb{N}.\]

The hypersurface defined by the real part of $\bar{F}$: $Re \bar{F} = z_1^p + z_2^q$ is diffeomorphic to the cone over the link $L_r = Re \bar{F}^{-1}(0) \cap S^3_r$, $r > 0$. Moreover, $L_r$ is a closed oriented surface of genus $(p-1)(q-1)$ in $S^3_r \subset \mathbb{R}^4$. For instance, consider the case $p = 2, q = 3$ in which $L_r$ is a double torus in $S^3_r \subset \mathbb{R}^4$. We provide an elementary proof of this fact in the following proposition.

A direct computation with the help of the standard identification between $\mathbb{C}^2$ and $\mathbb{R}^4$ given by

\[(z_1 = x + iy, z_2 = u + iv) \mapsto (x, y, u, v)\]

shows that $Re \bar{F}$ has the following expression as a function defined in $\mathbb{R}^4$:

\[F: \mathbb{R}^4 \to \mathbb{R}, \quad F(x, y, u, v) = x^2 - y^2 + u^3 - 3uv^2. \hspace{1cm} (3.1)\]

Therefore, if we consider the function

\[G_r: \mathbb{R}^4 \to \mathbb{R}, \quad G_r(x, y, u, v) = x^2 + y^2 + u^2 + v^2 - r^2, \hspace{1cm} (3.2)\]

for any positive $r \in \mathbb{R}$, the intersection $F^{-1}(0) \cap G_r^{-1}(0)$ is the link $L_r$. Let us denote it by $T_r(2)$.

The transversality of the intersection is determined in terms of the gradient vector fields. Namely, let $F, G: \mathbb{R}^4 \to \mathbb{R}$ be a pair of differentiable functions. Assume that $p \in F^{-1}(a) \cap G^{-1}(b)$ is a regular point for both functions. The intersection of $F^{-1}(a)$ and $G^{-1}(b)$ is transversal at $p$ if and only if the gradient vectors $\mu$ and $\nu$ of these functions are linearly independent at $p$. Thus, we state the following:

**Proposition 3.1.** The intersection $T_r(2) = F^{-1}(0) \cap G_r^{-1}(0)$, $r > 0$ is a smooth surface.
Proof. We prove that the inverse images of zero \( F^{-1}(0) \) and \( G_r^{-1}(0) \) intersect transversally. A direct computation shows that the origin is the unique singular point of both functions, \( F \) and \( G \). Since it does not lie in \( G_r^{-1}(0), r > 0 \), then it lies off \( T_r(2) \). Now, we show that the gradient vector fields \( \mu(x, y, u, v) = (2x, -2y, 3u^2 - 3v^2, -6uv) \) and \( \nu(x, y, u, v) = 2(x, y, u, v) \) are linearly independent along \( F^{-1}(0) \cap G_r^{-1}(0) \), if \( r > 0 \). To do so, we consider the equation:

\[
(2x, -2y, 3u^2 - 3v^2, -6uv) = \lambda(x, y, u, v), \quad \lambda \in \mathbb{R} \setminus \{0\} \quad (3.3)
\]

Suppose that \( x \neq 0 \). Then, \( \lambda = 2 \) and \( y = 0 \). This implies that \( 2v = -6uv \). Consequently, if \( v \neq 0 \), then \( u = -\frac{1}{3} \) and the relation of the third coordinates above implies that \( 2u = 3u^2 - 3v^2 \), namely, \( v^2 = \frac{1}{9} \frac{3}{2} \). By assuming that \( G_r \) vanishes at \((x, 0, -\frac{1}{3} \frac{3}{2} \pm \sqrt{3})\), we get \( x^2 = r^2 - \frac{9}{4} \). By supposing that \( F \) vanishes at this point, we get \( x^2 < 0 \) which is a contradiction. So, if \( x \neq 0 \) then \( v = 0 \) and \( y = 0 \). This implies that \( u = \frac{2}{3} \). By evaluating \( G_r \) at a point of the form \((x, 0, \frac{2}{3}, 0)\), we get again a contradiction. Therefore, \( x = 0 \). Analogous considerations derived from assuming that points of the form \((0, y, u, v)\) are in the intersection of the loci of \( G_r \) and \( F \) provide straightforward contradictions. Therefore, the gradient vector fields are linearly independent at any point of \( T_r(2) \). □

Now, we consider the symmetries of \( T_r(2) \). It is straightforward to verify that this surface is invariant under the action of the group \( G \) generated by the following applications:

\[
\Gamma_i : C^2 \rightarrow C^2, \quad i = 1, 2, 3,
\]

\[
\Gamma_1(z_1, z_2) = (\bar{z}_1, z_2), \quad \Gamma_2(z_1, z_2) = (-\bar{z}_1, z_2) \quad \text{and} \quad \Gamma_3(z_1, z_2) = (z_1, e^{2\pi i \frac{1}{3}} z_2).
\]

Let us describe this embedding of the double torus.

From the very definition of the functions \( F \) and \( G \) we have four charts of \( T_r(2) \) given by the projection onto the \( uv \)-plane of the following hexagonal regions.

\[
H_{++} = \{(x, y, u, v) \in T_r(2) : x \geq 0, y \geq 0\},
\]

\[
H_{+-} = \{(x, y, u, v) \in T_r(2) : x \geq 0, y \leq 0\}, \quad (3.4)
\]

\[
H_{-+} = \{(x, y, u, v) \in T_r(2) : x \leq 0, y \geq 0\},
\]

\[
H_{--} = \{(x, y, u, v) \in T_r(2) : x \leq 0, y \leq 0\}.
\]

We consider these sets with boundary, since it will be useful to identify corresponding boundaries in the construction below. The projection \( \mathbb{R}^4 \rightarrow \mathbb{R}^2 \)

\[
(x, y, u, v) \mapsto (u, v)
\]

defines the hexagon \( H_{uv} \) on the \( uv \)-plane as follows.
\[ H_{uv} = \{(u,v) \in \mathbb{R}^2 : r^2 - u^3 + 3uv^2 - u^2 - v^2 \geq 0, \quad r^2 + u^3 - 3uv^2 - u^2 - v^2 \geq 0\}. \]

Thus, we consider the coordinate functions
\[
\varphi_{\pm\pm} = \left( \pm \sqrt{\frac{r^2 - u^2 - u^3 - v^2 + 3uv^2}{2}}, \pm \sqrt{\frac{r^2 - u^2 + u^3 - v^2 - 3uv^2}{2}}, u, v \right),
\]
where the domain of these applications is \( H_{uv} \).

\[ \text{Figure 1. Projection of the double torus on the uv-plane.} \]

The boundary of \( H_{uv} \) is constituted by
\[ \partial H_{uv} = X^1 \cup Y^1 \cup X^2 \cup Y^2 \cup X^3 \cup Y^3, \]
where
\[
\begin{align*}
X^1 &= \{(u,v) : r^2 - u^3 + 3uv^2 - u^2 - v^2 = 0, \quad u \geq 0\}, \\
Y^1 &= \{(u,v) : r^2 + u^3 - 3uv^2 - u^2 - v^2 = 0, \quad u \geq 0, \quad v \geq 0\}, \\
X^2 &= \{(u,v) : r^2 - u^3 + 3uv^2 - u^2 - v^2 = 0, \quad u \leq 0, \quad v \geq 0\}, \\
Y^2 &= \{(u,v) : r^2 + u^3 - 3uv^2 - u^2 - v^2 = 0, \quad u \leq 0\}, \\
X^3 &= \{(u,v) : r^2 - u^3 + 3uv^2 - u^2 - v^2 = 0, \quad u \leq 0, \quad v \leq 0\}, \\
Y^3 &= \{(u,v) : r^2 + u^3 - 3uv^2 - u^2 - v^2 = 0, \quad u \geq 0, \quad v \leq 0\}.
\]

Observe that
\[
T(2)_+ = T_r(2) \cap \{x \geq 0\} = H_{++} \cup H_{+-} \\
T(2)_- = T_r(2) \cap \{x \leq 0\} = H_{+-} \cup H_{--}.
\]

Therefore, \( T(2)_+ \) is constructed by gluing the closure of two hexagonal regions through the identification of the curves defined by the equation \( y = \)}
0. That is, $T(2)_+$ is a surface homeomorphic to the sphere minus tree discs (see figure 2).

Since $T(2)_-$ is the image of $T(2)_+$ by the reflection with respect to the hyperplane $x = 0$, we obtain $T(2)$ by gluing two surfaces of this type along the boundary of the missing discs.

**Figure 2.** First identification.

**Figure 3.** A half of a double torus.

**Remark 3.2.** Let us denote by $T_{±±} : \mathbb{C}^2 \to \mathbb{C}^2$, the translation of the origin to any of the points $ϕ_{±±}(0, 0)$, respectively. Then, the conjugation $T_{±±}^{-1} \circ g \circ T_{±±}$, where $g$ belongs to the group generated by $Γ_3$, leaves $T(2)$ invariant.

We describe the action of $G$ on $T_r(2)$. Each point in the interior of $H_{uv}$ represents 4 points identified by the action of the subgroup of reflections of $G$ on the torus. We denote by $I$ the set of the intersections of the curves
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$X_i$ and $Y_j$, $i, j = 1, 2, 3$. Each point in the boundary $H_{uv} \setminus I$ represents 2 points, while, points in $I$ represent only one point of $T_r(2)$.

Observe that it is enough to determine this region only in $H_{++}$, since any of the hexagonal regions described in (3.4) are obtained from this one by one reflection.

We analyze the action of $\Gamma_3$ on the closure of the hexagonal region $H_{++}$. So, consider the following subset of $H_{++}$

$$P_{++} = \{(x, y, u, v) \in H_{++} | v \geq 0, v \geq -\sqrt{3}u\}. \quad (3.5)$$

This subset is the closure of a pentagonal region. Let $P_0$ be the image of the projection of $P_{++}$ in the plane $uv$ (see figure 4). We describe the boundary of $P_0$ as follows: The line segment which goes from the origin to the midpoint of $X^1$ is denoted by $e_1$; the upper half of $X^1$ is denoted by $e_2$; the edge $Y^1$ is denoted by $e_3$; the upper half of $X^2$ is denoted by $e_4$ and finally; $e_5$ denotes the line segment which goes from the midpoint of $X^2$.

![Figure 4. A pentagonal fundamental region](image)

If we rotate this pentagon $P_0$ by $2\pi/3$ we obtain another pentagon $P_1$, and if we rotate $P_1$ we obtain a third pentagon $P_2$. The closure of the projection of $H_{++}$ is the union of closure of these three pentagons. We can define analogously, pentagonal regions in $P_{+-}$, $P_{+-}$ and $P_{-+}$ contained in the other hexagonal regions.

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4.1. The lines of curvature of the double torus in $\mathbb{R}^4$. We consider the unitary vector field

$$\nu = \frac{F}{||F||}, \text{ where } \nabla F = (2x, -2y, 3u^2 - 3v^2, -6uv)$$

is the gradient vector field of $F$ on $T_r(2)$. 

Proposition 2.3 provides the differential equation of \( \nu \)-lines of curvature of the double torus \( T_r(2) \) immersed in \( S^3 \subset \mathbb{R}^4 \).

**Corollary 4.1.** The differential equation of the \( \nu \)-lines of curvature of \( T_r(2) \) has the following expression:

\[
\begin{pmatrix}
0 & 6v(u^2 - 3u^2) & -yu(2 - 18u + 9\rho^2) & y(9(u^2 - v^2) + u(2 + 9\rho^2)) \\
* & 0 & xv(2 + 18u + 9\rho^2) & x(9(u^2 - v^2) - u(2 + 9\rho^2)) \\
* & * & -24xyv & -24xyu \\
* & * & * & 24xyv \\
\end{pmatrix} \ dX^T = 0, \quad (4.1)
\]

where \( dX = (dx, dy, du, dv) \), \( \rho^2 = u^2 + v^2 \) and the values of * are determined by the symmetry of the matrix.

We provide the description of the foliation of \( \nu \)-lines of curvature of the double torus \( T_r(2) \), \( r > 0 \). As is common, we consider the index of an isolated \( \nu \)-umbilic point as the index of an isolated singularity of the field of principal directions on the surface, [7]. According to the classification provided in [4], (see also [1]) we refer to a \( \nu \)-umbilic point of type \( D_3 \) as a generic \( \nu \)-umbilic point with three separatrices and index \( -\frac{1}{2} \). We define a pathwise separatrix of the \( \nu \)-principal foliation as a smooth curve constituted by the union of separatrices, some of them of maximal \( \nu \)-principal curvature and the other of minimal \( \nu \)-principal curvature. In the theorem below, we will omit the prefix \( \nu \) for the elements of the foliation of \( \nu \)-lines of curvature, such as \( \nu \)-umbilic points, \( \nu \)-separatrices, etc.

**Theorem 4.2.** The foliation of \( \nu \)-lines of curvature of the double torus \( T_r(2) \) has the following structure:

i) It only has 4 umbilic points, each one of type \( D_3 \).

ii) It has 3 pathwise separatrices; each one is an immersion of \( S^1 \) containing the whole set of umbilics.

iii) The first homology group of the double torus \( T_r(2) \) is generated by 1-cycles that are pathwise separatrices.

iv) The complement of the closure of the set of separatrices is foliated by cycles, that is, closed curvature lines diffeomorphic to \( S^1 \).

We begin by proving statement i) in the following two propositions.

**Proposition 4.3.** Let \( r \in \mathbb{R} \) such that \( 0 < r < \frac{1}{\sqrt{10}} \). The foliation of \( \nu \)-lines of curvature of \( T_r(2) \) has 4 umbilic points.

**Proof.** Let us consider the open subset \( T_r(2)^1 = \{ (x, y, u, v) \in T_r(2) | xy \neq 0 \} \) of \( T_r(2) \). We define on \( T_r(2)^1 \) the frame:

\[
V_1(x, y, u, v) = -\left( \frac{v - 3uv}{2x} \right) \frac{\partial}{\partial x} - \left( \frac{v + 3uv}{2y} \right) \frac{\partial}{\partial y} + \frac{\partial}{\partial v},
\]

\[
V_2(x, y, u, v) = -\left( \frac{2u + 3u^2 - 3v^2}{4x} \right) \frac{\partial}{\partial x} - \left( \frac{2u - 3u^2 + 3v^2}{4y} \right) \frac{\partial}{\partial y} + \frac{\partial}{\partial u}.
\]
After multiplying the coefficients of the differential equation in Theorem 2.5 of \(\nu\)-lines of curvature on \(T_r(2)^1\) by the monomial \(xy\), they have the expression:

\[
\Omega_{11} = v(-18u^3(x^2 - y^2) + 2u(1 + 18v^2)(x^2 - y^2) + 27u^4(3v^2 + x^2 + y^2) - 3u^2(9v^4 + x^2 + y^2 + v^2(3 - 9x^2 - 9y^2)) + 3(v^4 + 8x^2y^2 + 3v^2(x^2 + y^2)))
\]

\[
\Omega_{12} = \frac{1}{2}(36u^4(x^2 - y^2) - 54u^2v^2(x^2 - y^2) + 2v^2(-2 + 9v^2)(x^2 - y^2) - 27u^5(3v^2 + x^2 + y^2) + 6u^3(-3v^2 + 18v^4 - 2(x^2 + y^2)) - 3u(9v^6 + 16x^2y^2 + 4v^2(x^2 + y^2) - v^4(2 + 9x^2 + 9y^2)))
\]

\[
\Omega_{21} = \frac{1}{2}(-81u^5v^2 + 9u^4(x^2 - y^2) - 4u^2(-1 + 27v^2)(x^2 - y^2) + 9v^4(-x^2 + y^2) + 6u^3(18v^4 - 2(x^2 + y^2) - 3v^2(1 + 3x^2 + 3y^2)) - 3u(9v^6 + 16x^2y^2 + 4v^2(x^2 + y^2) + 2v^4(-1 + 9x^2 + 9y^2)))
\]

\[
\Omega_{22} = \frac{-v}{4}(-81u^6 + 9u^4(4 + 21v^2 - 6x^2 - 6y^2) - 72u^3(x^2 - y^2) + 8u(1 + 18v^2)(x^2 - y^2) - 3u^2(4v^2 + 45v^4 - 20(x^2 + y^2)) + 3(9v^6 + 32x^2y^2 + 4v^2(x^2 + y^2) + 18v^4(x^2 + y^2))).
\]

Taking into account that conditions \(x^2 + y^2 = -(u^2 + v^2) + r^2\) and \(x^2 - y^2 = -(u^3 - 3uv^2)\) hold when restricted to \(T_r(2)\), these coefficients can be expressed only in terms of \(u, v\) as follows:

\[
\Omega_{11} = v(6r^4 - 15u^6 + 3u^2v^2 + u^4(7 - 27v^2) + 3r^2(9u^4 - v^2 + u^2(-5 + 9u^2))),
\]

\[
\Omega_{12} + \Omega_{21} = \frac{u}{2}(-24r^4 - 3r^2(9u^4 + v^2(-8 + 9v^2)) + 2u^2(-4 + 9v^2)(4.2) + 2u^2(3u^4 + v^2(-10 + 9u^2) + u^2(-2 + 36v^2))),
\]

\[
\Omega_{22} = \frac{v}{4}(-24r^4 - 21u^6 + u^4(8 - 27v^2) - 3u^2v^2(-4 + 9v^2) + 3v^4(-4 + 9v^2) + 6r^2(-2u^2 + 9u^4 + 6v^2 - 9v^4)).
\]

Therefore, points in \(T_r(2)\) where \(u = v = 0\) are \(\nu\)-umbilic, namely, points in the intersection of the plane curves \(x^2 - y^2 = 0\) and \(x^2 + y^2 - r^2 = 0\). These points are:

\[
\frac{r}{\sqrt{2}}(1, 1, 0, 0), \frac{r}{\sqrt{2}}(-1, 1, 0, 0), \frac{r}{\sqrt{2}}(1, -1, 0, 0), \frac{r}{\sqrt{2}}(-1, -1, 0, 0).
\]

A simple analysis, which is omitted, shows that there are no umbilic points in the subset \(T_r(2) \setminus T_r^1(2)\).

\[\Box\]

**Proposition 4.4.** The foliation of \(\nu\)-lines of curvature of \(T_r(2)\) only has 4 umbilic points.
Proof. To prove this property we will fix the radius $r = \frac{1}{\sqrt{10}}$ in order to get convenient computations. Once we prove the property for the double torus $T_{\frac{1}{\sqrt{10}}}(2)$, the general case follows the fact that the $\nu$-principal foliations are invariant under homotheties of $\mathbb{R}^4$ centered at the origin.

The $\nu$-umbilic points determined in Proposition 4.3 in this case are:

$$\left\{ \frac{1}{\sqrt{20}} (1, 1, 0, 0), \frac{1}{\sqrt{20}} (-1, 1, 0, 0), \frac{1}{\sqrt{20}} (1, -1, 0, 0), \frac{1}{\sqrt{20}} (-1, -1, 0, 0) \right\}.$$  

Now we show that these points are the unique $\nu$-umbilics on $T_{\frac{1}{\sqrt{10}}}(2)$.

We assume that $uv \neq 0$ and denote the polynomials $\frac{1}{v} \Omega_{11}, \frac{1}{v} (\Omega_{12} + \Omega_{21})$ and $\frac{1}{v} \Omega_{22}$, by $\Omega_1$, $\Omega_2$, $\Omega_3$, respectively. Namely,

$$\Omega_1 = 6r^4 - 15u^6 + 3u^2v^2 + u^4(7 - 27v^2) + 3r^2(9u^4 - v^2 + u^2(-5 + 9v^2)),$$
$$\Omega_2 = \frac{1}{2} (-24r^4 - 3r^2(9u^4 + v^2(-8 + 9v^2) + 2u^2(-4 + 9v^2)) + 2u^2(3u^4 + v^2(-10 + 9v^2)) + u^2(-2 + 36v^2)),$$
$$\Omega_3 = \frac{1}{4} (-24r^4 - 21u^6 + u^4(8 - 27v^2) - 3u^2v^2(-4 + 9v^2) + 3v^4(-4 + 9v^2) + 6r^2(-2u^2 + 9u^4 + 6v^2 - 9v^4)).$$

Straightforward considerations show that $\nu$-umbilic points satisfy the assumption. Thus, $p \in T_{\frac{1}{\sqrt{10}}}(2)$ is umbilic if and only if $p$ lies in the intersection of the zero loci of $\Omega_i$, $i = 1, 2, 3$. Consequently, to determine this intersection we define a reduction of these polynomials regarding the following lemma whose proof is straightforward, and is therefore omitted.

Lemma 4.5. Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_m)$ be a polynomial mapping. Assume that the induced application $\rho^* : \mathbb{R}[y_1, \ldots, y_m] \rightarrow \mathbb{R}[x_1, \ldots, x_n]$, defined by $\rho^*(Q) = P$, where $P = Q \circ \rho$, implies that $\rho(\mathcal{L}(P)) \subset \mathcal{L}(Q)$, where $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are the loci of $P$ and $Q$, respectively.

Let

$$\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (u, v) \mapsto (w = u^2, z = v^2).$$

Thus, the polynomials
\[
\begin{align*}
\Theta_1(w, z) &= \frac{3}{50} \frac{3w}{2} + \frac{97w^2}{10} - \frac{15w^3}{10} - \frac{3z}{10} + \frac{57wz}{10} - 27w^2z, \\
\Theta_2(w, z) &= -\frac{6}{25} \frac{6w}{5} + \frac{67w^2}{5} - 21w^3 + \frac{18z}{5} + 12wz - 27w^2z - \frac{87z^2}{5} - 27wz^2 + 27z^3, \\
\Theta_3(w, z) &= -\frac{6}{25} \frac{12w}{5} - \frac{67w^2}{10} + \frac{6w^3}{5} + \frac{12z}{5} - \frac{127wz}{5} + 72w^2z - \frac{27z^2}{10} + 18wz^2,
\end{align*}
\]

satisfy \( \rho^*(\Theta_i) = \Omega_i, \ i = 1, 2, 3. \)

Since Lemma 4.5 implies that \( \bigcap_{i=1}^3 \rho(L(\Omega_i)) \subseteq \bigcap_{i=1}^3 L(\Theta_i) \), it is enough to show that \( \bigcap_{i=1}^3 L(\Theta_i) = \emptyset \) to complete the proof. We fix the radio of the sphere and compute using Wolfram Mathematica to get a set of generators of the ideal defined by \( \Theta_i, \ i = 1, 2, 3. \) Then, we state

**Lemma 4.6.** Let us assume that \( r = \frac{1}{\sqrt{10}} \). Then, the ideal generated by the polynomials \( \Theta_i, i = 1, 2, 3 \) has the following Groebner basis:

\[
\begin{align*}
G_1(w, z) &= \Gamma_1(z) = 2457 + 355500z - 16427260z^2 + 227180200z^3 - 141847600z^4 + 452401200z^5 - 725112000z^6 + 4665600z^7, \\
G_2(w, z) &= -272440432411875 + 651925265295213w + 388134083787779w^2 - 65583408399640z^2 + 38942278748259700z^3 + 2318821632923662800z^4 - 5434755866556408000z^5 + 4628749229159040000z^6.
\end{align*}
\]

The polynomial \( G_1 \) depends only on \( z \) and its roots are computable. The positive ones are

\[
\begin{align*}
z_1 &= \frac{1}{20}, \ 	ext{and} \\
z_2 &= \frac{1}{240} \left( 67 - \left( \frac{13^5}{\sigma} \right)^{\frac{2}{3}} - (13\sigma)^{\frac{1}{3}} \right),
\end{align*}
\]

where \( \sigma = 1289 - 216\sqrt{35} \). Thus, the intersection of the loci of \( G_1 \) and \( G_2 \) is obtained by substituting these roots in the polynomial \( G_2 \) and solving for \( w \). That is, for \( z_1 \) we get \( w_1 = 3/20 \), while for \( z_2 \) we get

\[
w_2 = \frac{\sigma^{\frac{2}{3}}}{720} (13^{\frac{1}{3}} (556848\sqrt{35} - 3294481) + 13^{\frac{2}{3}} \sigma^{\frac{1}{3}} (2808\sqrt{35} - 16757) - \sigma^{\frac{2}{3}} (14472\sqrt{35} - 86363)).
\]

By a direct substitution we see that none of these points lies on the sphere of radius \( r = \frac{1}{\sqrt{10}} \). This completes the proof under the hypothesis \( xy \neq 0 \).
If $x$ or $y$ vanishes, a similar analysis can be applied. It is simpler because most of the equations above involve easier expressions, so it is omitted. □

Let us analyze the $ν$-lines of curvature near the umbilic points in order to determine the umbilic separatrices and their indexes.

Taking the umbilic point $\frac{r}{\sqrt{2}}(1, 1, 0, 0)$ as the origin, we can introduce, for the local analysis, a system of coordinates $X, Y, Z, W$, in a neighborhood of this point, where the double torus can be parameterized as the graph of a differentiable function $h : U \subset \mathbb{R}^2 \to \mathbb{R}^2$. The parametrization has the form $\mathbf{x}(u, v) = (f(u, v), g(u, v), u, v)$, where:

$$f(u, v) = \frac{r}{\sqrt{2}} - \frac{\sqrt{r^2 - u^2 - v^2 + 3uv^2}}{\sqrt{2}},$$

$$g(u, v) = \frac{r}{\sqrt{2}} - \frac{\sqrt{r^2 - u^2 + u^3 - v^2 - 3uv^2}}{\sqrt{2}}.$$

Lemma 4.7. The separatrices at the umbilic point $\frac{r}{\sqrt{2}}(1, 1, 0, 0)$ have slope $p = 0, \pm \sqrt{3}$.

Proof. In [11] is shown that if we have a local immersion of a surface into $\mathbb{R}^4$ with a parameterization of the form

$$\mathbf{x}(u, v) = \left( \frac{k}{2}(u^2 + v^2) + \frac{a}{6}u^3 + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + R_1, \frac{\alpha}{2}u^2 + \beta uv + \frac{\gamma}{2}v^2 + \frac{\delta}{6}u^3 + \frac{\varepsilon}{2}u^2v + \frac{\zeta}{2}uv^2 + \frac{\eta}{6}v^3 + R_2, u, v \right),$$

then the slopes of the separatrices at the umbilic point are given by the solutions of the equation:

$$(b + \beta n)p^3 + (\beta m - c - (\gamma - \alpha)n)p^2 - (2b - a + (\gamma - \alpha)m + \beta n)p - \beta m = 0,$$

where $p = dv/du$ is the variable to determine, and $k, a, b, c, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ are the local parameters of the immersion.

We use the Taylor expansion of $f(u, v)$ and $g(u, v)$ up to order three, to express the parameterization as follows:

$$\mathbf{x}(u, v) = \left( \frac{1}{2\sqrt{2}r}(u^2 + v^2) + \frac{3}{6\sqrt{2}r}u^3 - \frac{3}{2\sqrt{2}r}uv^2, \frac{1}{2\sqrt{2}r}(u^2 + v^2) - \frac{1}{2\sqrt{2}r}u^3 + \frac{9}{6\sqrt{2}r}uv^2, u, v \right).$$

By solving the previous equation we obtain that $p = 0, \pm \sqrt{3}$. □

Then, by applying the action of the group $\mathcal{G}$ on $T_r(2)$ we conclude the following:

Proposition 4.8. The umbilic points of $T_r(2)$ are of type $D_3$.

Now we prove statement ii).
Proposition 4.9. There exists a pathwise separatrix, homeomorphic to a circle, which goes through all the umbilic points.

Proof. If we consider a curve $S_1(t): I \to \mathbb{R}^4$ given by $S_1(t) = (h_1(t), h_2(t), h_3(t), h_4(t))$, it is easy to show that it satisfies the equation of the curvature lines when the last coordinate vanishes, i.e., for $h_4(t) = 0$. Also, if we consider that the curve is in the double torus, $S_1(t)$ should simultaneously satisfy conditions $F(S_1) = 0$ and $G_r(S_1) = 0$. Consequently, we obtain a curve defined in 4 parts given by:

$$
S_{11}(t) = \left(-\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, t, 0\right),
$$

$$
S_{12}(t) = \left(\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, t, 0\right),
$$

$$
S_{13}(t) = \left(\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, \frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, t, 0\right),
$$

$$
S_{14}(t) = \left(-\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, \frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, t, 0\right).
$$

We have that $U_i \in S_{1i}(t)$, $i = 1, 2, 3, 4$ where $U_1 = (-\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}, 0, 0)$, $U_2 = (\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}, 0, 0)$, $U_3 = (\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0, 0)$, $U_4 = (-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0, 0)$.

We have that $(\Gamma_1 \circ \Gamma_2 \circ \Gamma_1)(S_{11}) = (\Gamma_1 \circ \Gamma_2 \circ \Gamma_1)(S_{12}) = (\Gamma_1 \circ \Gamma_2)(S_{13}) = \Gamma_1(S_{14}) = S_{11}$. In addition, the matrices $\Gamma_2$ and $\Gamma_1$ have as fixed points the points obtained taking $x = 0$ for $\Gamma_2$ and $y = 0$ for $\Gamma_1$. Taking this in the curvature line, we obtain 4 fixed points $p_i$, $i = 1, 2, 3, 4$ where $p_1 = S_{12} \cap S_{13}$, $p_2 = S_{11} \cap S_{14}$, $p_3 = S_{11} \cap S_{12}$ and $p_4 = S_{13} \cap S_{14}$. Thus, since $S_1(t)$ is a differentiable curve and $S_{1i}$, $i = 1, 2, 3, 4$ are connected arcs with just one point in common each two of them, and since we can generate $S_{1i}$, $i = 2, 3, 4$ with the action of $G$ on $S_{11}$, we conclude that $S_1(t)$ is homeomorphic to a circle. $\square$

Proposition 4.10. There exist two pathwise separatrices obtained by rotating $S_1(t)$ $2\pi/3$ and $4\pi/3$ with center at an umbilic point.

Proof. We know that the curvature lines are invariant under rotations. By applying $\Gamma_3$ to the separatrix $S_1$, we obtain a new curvature line $S_2 : I \to \mathbb{R}^4$ associated to $p = \sqrt{3}$ given by:
Observe that since the projection of the umbilic point.

Proof. iii) One 2-cell.

ii) Five 1-cells, all of them are lines of curvature. Among them, two are

i) Five 0-cells, one of them is an umbilic point.

The set $S_{21}(t) = \left( -\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

$S_{22}(t) = \left( \frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, \frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

$S_{23}(t) = \left( -\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, \frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

$S_{24}(t) = \left( \frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

where $U_i \in S_{2i}(t)$, $i = 1, 2, 3, 4$.

In the same way, applying $T_{-U_3} \circ \rho_{\pi/3} \circ T_{U_3}$ we obtain a new curvature line $S_3 : I \to \mathbb{R}^4$ associated to $p = -\sqrt{3}$ and given by:

$S_{31}(t) = \left( -\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

$S_{32}(t) = \left( \frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

$S_{33}(t) = \left( -\frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, \frac{\sqrt{r^2 - t^2 + t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

$S_{34}(t) = \left( \frac{\sqrt{r^2 - t^2 - t^3}}{\sqrt{2}}, -\frac{t}{2}, -\frac{\sqrt{3}}{2}t \right)$,

where $U_i \in S_{3i}(t)$, $i = 1, 2, 3, 4$.

Proof of statement iii). We begin by describing a CW complex structure which includes the $\nu$-lines of curvature and the $\nu$-umbilics as cells of $T_{\nu}(2)$.

Lemma 4.11. The set $P_{++} \subset T_{\nu}(2)$ admits the following CW-complex structure.

i) Five 0-cells, one of them is an umbilic point.

ii) Five 1-cells, all of them are lines of curvature. Among them, two are separatrices containing the umbilic point.

iii) One 2-cell.

Proof. We begin by defining the 2-cell as the interior of $P_{++}$, see (3.5). Then, the 1-cells are described as follows:

$a_1 = \{(x, y, u, v) \in P_{++}; v = 0\}$, $a_2 = \{(x, y, u, v) \in P_{++}; x = 0, u > 0\}$

$a_3 = \{(x, y, u, v) \in P_{++}; y = 0\}$, $a_4 = \{(x, y, u, v) \in P_{++}; x = 0, v > 0\}$

$a_5 = \{(x, y, u, v) \in P_{++}; v = -\sqrt{3}u\}$.

Observe that since the projection of $P_{++}$ onto the $uv$-plane is $P_0$, the projections of these 1-cells are $e_1, ..., e_5$, respectively. We point out that, according to the differential equation of $\nu$-lines of curvature of Corollary 4.1,
a straightforward computation implies that \( a_i, \ i = 1, ..., 5 \) are \( \nu \)-lines of curvature of \( \Gamma_r(2) \). Moreover, the curves \( S_{13} \) defined in Proposition 4.9 and \( S_{23} \) defined in Proposition 4.10 are parameterizations of \( a_1 \) and \( a_5 \), respectively. Thus, these 1-cells are separatrices. Finally, the 0-cells are defined as the preimages of the vertices of the pentagonal region \( P_0 \) under the projection. Among them, the one corresponding to the origin in the \( uv \)-plane is the umbilic \((\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0, 0)\).

The \( CW \)-complex structure of \( P_{++} \) in this lemma allows us to state the following:

**Proposition 4.12.** The double torus \( T_r(2) \) admits a \( CW \)-complex structure such that the 0-skeleton includes the umbilic points and the 1-skeleton the separatrices.

**Proof.** The cells of the \( CW \)-complex structure of \( T_r(2) \) are the images of the cells of the pentagonal region \( P_{++} \) under the action of the group \( G \). We describe them as follows: The 0-skeleton is constituted by sixteen 0-cells. Namely, four of the orbit of \( v_0 \), six for the orbit of \( v_1 \), and three of the orbit \( v_i, \ i = 2, 3 \). Observe that the umbilic points belong to the orbit of \( v_0 \). The 1-skeleton is constituted by thirty 1-cells defined by the orbits of the boundary of the pentagonal region. Namely, the orbit of \( a_1 \) consists of twelve 1-cells including \( a_5 \); all of them are separatrices. The orbit of \( a_i \) consists of six 1-cells for each \( i = 2, 3, 4 \). The 2-skeleton consists of twelve 2-cells defined by the orbit of \( P_{++} \). By a direct computation, it can be verified that this decomposition provides the Euler-characteristic of the double torus:

\[
12 - 30 + 16 = -2 = \chi(T_r(2)).
\]

**Corollary 4.13.** The first homology group of the double torus \( T_r(2) \), \( H_1(T_r(2)) \), with coefficients in \( \mathbb{Z}_2 \) has a basis whose representatives are cycles constituted by pathwise separatrices.

**Proof.** We provide a basis \( \{\alpha_i\}_{i=1}^4 \) of \( H_1(T_r(2)) \) whose representatives are the cycles \( \{c_i\}_{i=1}^4 \), which are defined as follows.

First, we consider

\[
c_1' = b_1 + b_2, \ c_2' = \Gamma_3(c_1'),
\]

where \( b_1 = a_4 + \Gamma_3(a_2) \) and \( b_2 = \Gamma_1(b_1) \). Then, we define

\[
c_1 = d_1 + d_2 + \Gamma_1(d_1 + d_2), \ c_2 = \Gamma_3(c_1),
\]

where, \( d_1 \) is the separatrix whose projection is diagonal of slope \( \sqrt{3} \), \( u, v \geq 0 \) and \( d_2 = \Gamma_3(d_1) \). Thus, \( c_i, \ i = 1, 2 \), is a pathwise separatrix homologous to \( c_i' \), \( i = 1, 2 \). We define \( d_3 = a_3 + \Gamma_2(a_3) \) and \( d_4 = \Gamma_3(d_3) \). Thus, we have defined a basis \( \{c_1', c_2', d_3, d_4\} \) of \( H_1(T_r(2)) \) whose cycles are constituted by
$\nu$-lines of curvature and belong to the 1-skeleton. Finally, by choosing the representatives

$$c_3 = a_1 + a_5 + \Gamma_2(a_1 + a_5)$$

and

$$c_4 = \Gamma_3(c_3),$$

since $c_i$ is homologous to $d_i$, $i = 3, 4$, we get the desired basis of $H_1(T_\tau(2))$ whose cycles $\{c_i\}_{i=1}^4$ are pathwise separatrices.

Proof of statement iv). In the proof of this statement, we will use some argument involving properties of the flow of a planar vector field. The Poincaré Bendixon theorem will play a relevant role. For the basic notions and the proof of this theorem see [6].

**Proposition 4.14.** The lines of curvature on the complement of the separatrices of $T_\tau(2)$ are immersions of $S^1$.

**Proof.** Let $L_1$ and $L_2$ be the foliations of maximal and minimal $\nu$-lines of curvature, respectively, in the closure of the region $P_0$ (see figure 4). We assume that $e_1$ and $e_5$ are separatrices in $L_1$ containing the origin in the $uv$-plane. Thus, $e_2$ and $e_4$ belong to $L_2$ and $e_3$ belongs in turn to $L_1$. In the interior of $P_0$, the equation of maximal (respectively minimal) $\nu$-lines of curvature is defined by a vector field $X_1$ without singularities. Moreover, we can assume that $X_1$ extends to $e_2$ pointing towards the region $P_0$ along $e_2$. Consequently, each solution with initial condition at $e_2$ never returns to this edge of $P_0$.

**Lemma 4.15.** The $\omega$-limit of the solution $\gamma_p$ of $X_1$ with initial condition $\gamma_p(0) = p \in e_2$ does not contain points in $e_1 \cup e_3 \cup e_5$. Moreover, this solution intersects $e_4$.

**Proof of the lemma.** We show that there exists a tubular neighborhood of $e_1 \setminus \{(0,0)\}$ which does not intersect $\gamma_p(t), \ t > 0$. Consider the Poincaré application

$$\pi : \Sigma^\nu_3(0) \to e_2$$

where $\Sigma_3(0)$ is the minimal $\nu$-line of curvature in $P_0$ extending the separatrix in the complement of the closure of $P_0$, and where $\Sigma^\nu_3(0)$ is a neighborhood of the origin in $\Sigma_3(0)$ of ratio $r$. Since this application is continuous, we can choose $\delta > 0$ such that $\pi(\Sigma^\delta_3(0)) \subset e_2^\epsilon = |p|/2(p_0)$, where $p_0 = e_1 \cap e_2$ and $e_2^\epsilon = |p|/2(p_0)$ is the neighborhood of $p_0 \in e_2$ with ratio equal to a half of the distance from $p$ to $p_0$. Since $p \in e_2$ is not in $e_2^\epsilon = |p|/2(p_0)$, the uniqueness of the solutions of $X_1$ implies that $\gamma_p(0)$ never intersects the tubular neighborhood of $e_1$, defined by the solutions of $X_1$ with initial conditions in $e_2^\epsilon = |p|/2(p_0)$. Analogous arguments allow us to extend such a tubular neighborhood to the whole union $e_1 \cup e_3 \cup e_5$.

Now we prove the second statement. Suppose that $\gamma_p$ does not intersect $e_4$. Observe that the $\omega$-limit of the solution $\gamma_p$ is closed and contained in
$P_0$ and thus bounded without singular points. Then the Poincaré-Bendixon theorem implies that $\gamma_p$ is a closed solution. This implies in turn that there is a singular point in the bounded region determined by this solution. So we get a contradiction. \hfill \Box

**Lemma 4.16.** If $p, q \in e_2$ and $r \in e_4$ are such that $\gamma_p(t_0) = r = \gamma_q(t_1)$, where $t_i, i = 0, 1$, is the first value of the parameter where the corresponding solution intersects $e_4$, then $p = q$ and $t_0 = t_1$.

**Proof of the lemma.**

\begin{align*}
\gamma_p : [0, t_0] & \rightarrow \overline{F}_0, \; \gamma_p(0) = p, \; \gamma_p(t_0) = r \\
\gamma_q : [0, t_1] & \rightarrow \overline{F}_0, \; \gamma_q(0) = q, \; \gamma_q(t_1) = r.
\end{align*}

We define $i = \gamma_p[0, t_0] \cap \gamma_q[0, t_1]$. Thus, this set is a non-empty closed subset of each solution. Moreover, the box flow theorem implies that $i$ is an open set. Then, $i = \gamma_p[0, t_0]$ and the proof follows from this equation. \hfill \Box

**Proof of Proposition 4.14.**

When the curve $\gamma_p(t)$ reaches the side $e_4$ at $r$ in the $uv$-plane, the curve $\overline{\gamma}(t)$ reaches the boundary of $H_{++}$ at the same point. Thus, in $T_r(2)$ the curve $\overline{\gamma}(t) = (x(t), y(t), u(t), v(t))$, $x(t) \geq 0$ crosses the boundary of $H_{++}$ and arrives to $H_{-+}$, now with $x(t) < 0$. Under the projection $\pi(x, y, u, v) = (u, v)$ it returns to the interior of the pentagon with negative values in the first entrance. Note that $\pi \circ \overline{\gamma} = \gamma$. That is, the projection of $\pi \circ \gamma_p^-(t)$ is contained in $\gamma([0, t_0])$. Therefore, it returns to the pentagon following the solutions but in the opposite direction. So, Lemma 4.16 implies that this solution reaches $e_2$ at $p$ implying that $\overline{\gamma}_p$ is a closed solution. \hfill \Box

**Remark 4.17.** Observe that each line of curvature of the statement of the proposition is contained in the union $P_{++} \cup P_{-+}$, where these sets are the pentagonal regions defined in (4), or in another pair of these regions.

### 4.2. The lines of curvature of the double torus in $\mathbb{R}^3$.

We consider a surface $M$, defined as the transversal intersection of the sphere of radius $r$ centered at the origin, $S^3_r \subset \mathbb{R}^4$, with the inverse image of a value of a differentiable function $F : \mathbb{R}^4 \rightarrow \mathbb{R}$ with an isolated singularity at the origin. Let $\nu$ be a unit vector field parallel to the gradient of $F$. The lines of curvature transform in a proper way under the stereographic projection. We state the property in a more general setting.

**Proposition 4.18.** Consider a surface $M \subset S^3_r$. The stereographic projection $\sigma : S^3_r \setminus \{p\} \subset \mathbb{R}^4 \rightarrow \mathbb{R}^3$, where $p \in S^3_r \setminus M$, provides an immersion of $M$ into $\mathbb{R}^3$, which transforms the $\nu$-lines of curvature of $M$ into the lines of curvature of $\sigma(M)$, for any field $\nu$ normal to $M$ not parallel to the radial vector field. That is, a curve $l \subset M$ is a $\nu$-line of curvature if and only if $\sigma(l)$ is a line of curvature of $\sigma(M)$.

**Proof.** We begin by recalling that any spherical surface $M \subset \mathbb{R}^4$ is $R$-umbilic, where $R$ is the radial vector field, defined as the unitary vector
field parallel to the gradient of the function \( G \). Moreover, any normal field independent to the radial vector field at each point of \( M \) defines the same principal configuration (see Lemma 2.1 in \([10]\)). Consider this normal field \( \nu \). Its orthogonal projection to \( T_pS^3 \) defines a non-null field \( \hat{\nu} \) normal to \( M \) at each point. The stereographic projection as a conformal map transforms the vector field \( \hat{\nu} \) into one of the normal fields of \( \sigma(M) \subset \mathbb{R}^3 \), and the \( \hat{\nu} \)-lines of curvature on \( M \) into the lines of curvature \( \sigma(M) \). Since the \( \nu \)-lines of curvature of \( M \) coincide with the \( \hat{\nu} \)-lines of curvature, the proof is complete.

We denote the embedding \( \sigma(T_r(2)), r = 1 \) of the double torus into \( \mathbb{R}^3 \) by \( T_2 \).

**Corollary 4.19.** *The principal configuration of the embedded double torus \( T_2 \subset \mathbb{R}^3 \) has the same structure of that of \( T_r(2) \) provided by Theorem 4.2.*

We conclude by observing that the approach applied in this article to study the lines of curvature of the double torus can be extended to describe the lines of curvature of model embeddings of compact oriented surfaces of genus higher than two. Specifically, we have used the differential equation in Corollary 2.5 to describe the \( \nu \)-lines of curvature of the model defined in section 3 as the link \( L_r = \text{Re}\bar{F} \cap S^3, r > 0 \), of genus \((p-1)(q-1), p, q \geq 3\), where the vector field \( \nu \) is the gradient of \( \text{Re}\bar{F} = z_1^p + z_2^q - \bar{z}_1^p + \bar{z}_2^q \). The symmetries of this model allowed us to determine the type of umbilic points. Moreover, the topological decomposition of the link induced a CW-structure which includes umbilic points and separatrices as some of its cells. Furthermore, arguments of dynamical systems, similar to those used in this article, could be applied to prove that the lines of curvature are closed cycles. Finally, as in the case of the double torus, the stereographic projection defines the embedding of \( L_r, r > 0 \) into \( \mathbb{R}^3 \) which transforms the foliations of \( \nu \)-lines of curvature into the foliation of lines of curvature of the image of the link in \( \mathbb{R}^3 \). However, certain problems for the determination of the number of umbilic points appeared in the case of an arbitrary genus, which need to be solved to complete the description.

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