Policy Iteration Method for Time-Dependent Mean Field Games Systems with Non-separable Hamiltonians

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Abstract

We introduce two algorithms based on a policy iteration method to numerically solve time-dependent Mean Field Game systems of partial differential equations with non-separable Hamiltonians. We prove the convergence of such algorithms in sufficiently small time intervals with Banach fixed point method. Moreover, we prove that the convergence rates are linear. We illustrate our theoretical results by numerical examples, and we discuss the performance of the proposed algorithms.

Keywords Mean Field Games · Numerical methods · Policy iteration · Convergence

Mathematics Subject Classification 49N70 · 91A13 · 35Q80 · 65M06

1 Introduction

Mean Field Games (MFG for short) theory has been introduced in [42, 46] to characterize Nash equilibria for differential games involving a large (infinite) number of agents. The corresponding mathematical formulation leads to the study of a system of partial differential equations (PDEs), composed by a Hamilton–Jacobi–Bellman (HJB for short) equation, characterizing the value function and the optimal control for the agents; and a Fokker–Planck (FP for short) equation, governing the distribution of the

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population when the agents behave in an optimal way. For a comprehensive introduction to the applications of MFG theory we refer to the monographs by Carmona and Delarue [30], Bensoussan et al. [19] and the lecture notes [3]. In the case of a problem with finite horizon $T > 0$ and periodic boundary conditions, the MFG system reads as

\begin{align}
\begin{cases}
-\partial_t u - \epsilon \Delta u + H(m, Du) = 0 & \text{in } Q \\
\partial_t m - \epsilon \Delta m - \text{div}(mH_p(m, Du)) = 0 & \text{in } Q \\
m(x, 0) = m_0(x), u(x, T) = u_T(x) & \text{in } \mathbb{T}^d,
\end{cases}
\end{align}

where $Q := \mathbb{T}^d \times [0, T]$ and $\mathbb{T}^d$ stands for the flat torus $\mathbb{R}^d / \mathbb{Z}^d$, and $\epsilon > 0$.

The numerical solution to the system (1) is of paramount importance for applications of the MFG theory. Since the two equations in (1) are strongly coupled in a forward-backward structure they can be solved neither independently of each other nor jointly with a simple time-marching method. This is an intensive area of research and many methods have been proposed with successful applications, see e.g., [6, 45] and the references therein. Convergence of finite difference schemes or semi-Lagrangian schemes has been studied in [1, 2] and [28, 29] respectively. But to the best of our knowledge, convergence of algorithms to solve the discrete problems has been proved only for a few methods. Convergence of fictitious play [26, 49] and online mirror descent [40, 48] has been proved for monotone MFG with separable Hamiltonian. An augmented Lagrangian method and primal-dual methods have been studied respectively in [16, 18] and in [22, 23, 47], and convergence holds when the MFG has a variational structure. The convergence of a monotone-flow methods for MFG satisfying a monotonicity condition has been considered in [12, 36] using a contraction argument. However, none of these methods cover the case of non-separable Hamiltonian with a generic structure. To solve MFG with such Hamiltonians, in the absence of a more sophisticated method, a possible natural approach is to use fixed point iterations, i.e., to alternatively update the population distribution and the individual player’s (optimal) value function. However, the computation of the value function boils down to the resolution of a HJB equation, which is in general costly. Policy iteration (in the context of MFG) can be viewed as a modification the fixed point procedure in which, at each iteration the HJB is solved for a fixed control, which is updated separately after the update of the value function. The policy iteration method for MFG can also be viewed as an extension of the usual policy iteration method for HJB equations: here, the update of the population is intertwined with the updates of the value function and the control.

In [24], Cacace, Camilli and Goffi introduced the policy iteration method to study the numerical approximation of the solution to the mean field games system (1), when the Hamiltonians have a separable structure, i.e.,

$$H(x, m, Du) = \mathcal{H}(x, Du) + F(x, m(x, t)).$$

They introduced suitable discretizations to numerically solve both stationary and time-dependent MFGs and they proved the convergence of the policy iteration method for
the continuous and the discrete problems. Moreover, the performance of the algorithm on examples in dimension one and two has been discussed. The rate of convergence of this method has been considered in [25].

The policy iteration method, introduced by Bellman [17] and Howard [41], is a method to solve nonlinear HJB equations arising in discrete and continuous optimal control problems. The general principle is to replace the HJB equation by a sequence of linearized equations each with a fixed policy. The policy is then updated at each step by solving an optimization problem given the current value function. Recently, a policy iteration algorithm has been used in [32] for a mean field games model in mathematical finance.

The convergence of policy iteration algorithm for HJB equations has been widely studied (see [11, 34, 50, 51, 53]). However, the convergence analysis of policy iteration for MFGs has distinct features from that of HJB equations. The policy iteration scheme for HJBs is known to be improving the solution monotonically [20, 43]. In general, this monotonicity property is lost in the policy iteration for MFGs, as already observed in [24]. Heuristically, this is because the Nash equilibrium problem is a fixed point problem, even though each single agent solves an optimization problem.

In many applications, the separable Hamiltonian assumption (2) is considered to be too restrictive. Typical examples without separable Hamiltonians are MFGs with congestion effects (see e.g., [7, 8]) or MFGs arising in macroeconomics (see e.g., [4]). From the PDE point of view, the short time existence of solutions to general MFGs with non-separable Hamiltonians has been studied in [27, 31], a series of papers by Ambrose et al. [13–15] and Gangbo et al. in [35]. The probabilistic approach to this problem has been considered in [30]. The existence and uniqueness of solution the congestion type MFGs has been studied by Achdou and Porretta in [10], Gomes et al. [33, 37] and Graber [38]. For the numerical solution of congestion type MFGs we refer to [5, 6, 45].

Based on the ideas from [24], we propose the following two policy iteration algorithms for the MFG system (1).

We first define the Lagrangian as the Legendre transform of $H$:

$$L(m, q) = \sup_{p \in \mathbb{R}^d} \{ p \cdot q - H(m, p) \}. \quad (3)$$

The first policy iteration algorithm consists in iteratively updating the population distribution, the value function and the control. We introduce a bound $R$ on the control. In applications this may be understood as, for example, some financial constraints [32]. All of our convergence results hold when $R$ is large enough and the control is unconstrained.

**Policy Iteration Algorithm 1 (PI1)** Given $R > 0$ and given a bounded, measurable vector field $q^{(0)} : \mathbb{T}^d \times [0, T] \rightarrow \mathbb{R}^d$ with $|q^{(0)}| \leq R$ and $\|\text{div}q^{(0)}\|_{L^1(Q)} \leq R$, iterate:

(i) Solve

\[
\begin{aligned}
\partial_t m^{(n)} - \epsilon \Delta m^{(n)} - \text{div}(m^{(n)}q^{(n)}) &= 0, \quad \text{in } Q \\
m^{(n)}(x, 0) &= m_0(x), \quad \text{in } \mathbb{T}^d.
\end{aligned}
\]

(4)
(ii) Solve
\begin{equation}
\begin{aligned}
-\partial_t u^{(n)} - \epsilon \Delta u^{(n)} + q^{(n)} Du^{(n)} - L(m^{(n)}, Du^{(n-1)}, q^{(n)}) = 0 & \quad \text{in } Q \\
\left[u^{(n)}(x, T) = u_T(x) \right] & \quad \text{in } \mathbb{T}^d,
\end{aligned}
\end{equation}

where for \((m, p, q) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d\),
\begin{equation}
L(m, p, q) = p \cdot q - H(m, p).
\end{equation}

(iii) Update the policy
\begin{equation}
q^{(n+1)}(x, t) = \arg \max_{|q| \leq R} \left\{ q \cdot Du^{(n)}(x, t) - L(m^{(n)}, q) \right\} \quad \text{in } Q.
\end{equation}

Since a change of the population distribution might induce a change in the control, a variant of the above method consists in updating the control between each update of the population distribution and the value function. This leads to a second version of the policy iteration algorithm.

Policy iteration algorithm 2 (PI2) Given \(R > 0\) and given a bounded, measurable vector field \(q^{(0)} : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d\) with \(|q^{(0)}| \leq R\) and \(\|\text{div}q^{(0)}\|_{L^1(Q)} \leq R\), iterate:

(i) Solve
\begin{equation}
\begin{aligned}
\partial_t m^{(n)} - \epsilon \Delta m^{(n)} - \text{div}(m^{(n)}q^{(n)}) = 0 & \quad \text{in } Q \\
m^{(n)}(x, 0) = m_0(x) & \quad \text{in } \mathbb{T}^d.
\end{aligned}
\end{equation}

(ii) Update the policy
\begin{equation}
\tilde{q}^{(n)}(x, t) = \arg \max_{|q| \leq R} \left\{ \tilde{q} \cdot D\tilde{u}^{(n-1)}(x, t) - L(m^{(n)}, \tilde{q}) \right\} \quad \text{in } Q.
\end{equation}

(iii) Solve
\begin{equation}
\begin{aligned}
-\partial_t \tilde{u}^{(n)} - \epsilon \Delta \tilde{u}^{(n)} + \tilde{q}^{(n)} D\tilde{u}^{(n)} - L(m^{(n)}, \tilde{q}^{(n)}) = 0 & \quad \text{in } Q \\
\left[\tilde{u}^{(n)}(x, T) = u_T(x) \right] & \quad \text{in } \mathbb{T}^d.
\end{aligned}
\end{equation}

(iv) Update the policy
\begin{equation}
q^{(n+1)}(x, t) = \arg \max_{|q| \leq R} \left\{ q \cdot D\tilde{u}^{(n)}(x, t) - L(m^{(n)}, q) \right\} \quad \text{in } Q.
\end{equation}

It is important to notice that while \(L\) appearing in (1), (8), (9), and (10) is the Lagrangian, the term \(\mathcal{L}\) defined in (6) and appearing in (5) can only be regarded as a “perturbed Lagrangian” since it can not be obtained by the Legendre transform (3). The difference between the algorithms (PI1) and (PI2) is specific to the non-separable Hamiltonian structure. For a MFG system with (2) the control depends on \(m\) only implicitly via \(u\) and hence \(\tilde{q}^{(n)} = q^{(n)}\). However, in the non-separable case the dependence is explicit. Therefore it can be helpful to update the control again after each update of \(m\).
If we use a separable Hamiltonian, then both algorithms (PI1) and (PI2) will be the same as the one proposed in [24], where the authors proved the convergence using a compactness argument under the Lasry Lions monotonicity condition. In this paper, we concentrate on the case of MFGs with non-separable Hamiltonians. The existence and uniqueness of solutions to such systems, in many cases, can only be obtained by restricting to a short time interval, or assuming smallness of data (e.g., [13–15]). This is particularly true when we consider Hamiltonians which can be degenerate at \( m = 0 \). Assuming the time horizon \( T \) is sufficiently small, we prove via the contraction fixed point method the convergence of both algorithms (PI1) and (PI2) to the solution of the MFG system (1). Furthermore, we prove that the convergence takes place at a linear rate for both schemes without the additional assumptions on the Hamiltonians as in [25]. In our paper, the notion of solution for the Fokker-Planck equation is more regular than the one used for [24].

As in [24] for the separable Hamiltonian case, an important advantage of our method is that at each iteration we only need to solve two PDEs that are linear and decoupled. The advantage in terms of computational time is illustrated in the numerical examples by comparing with a fixed point algorithm combined with Newton-type method to solve the non-linear HJB equation.

The paper is organized as follows. In Sect. 2, we introduce some notations, assumptions and preliminary results. In Sect. 3 we prove the convergence of the policy iteration algorithms in a sufficiently small time interval (see Theorems 3.3 and 3.6 for each policy iteration method). Restricting our attention to a small time horizon is justified by the fact that we want to avoid making restrictive assumptions on the structure of the MFG (see also Remark 2.5). In Sect. 4, we prove the linear rates of convergence for the two MFG policy iteration schemes under additional assumptions on the time interval (see Theorems 4.1 and 4.3). In Sect. 5 we provide numerical examples to illustrate our results.

2 Preliminaries

We recall some basic facts on Legendre transform that are repeatedly used throughout the paper. We denote

\[
f^*(p) = \sup_{|q| \leq R} \{ p \cdot q - f(q) \}.
\]

For a strictly convex \( f(q) \) with suitable regularity assumptions, the supremum for \( f^*(p) \) is attained at \( q^* \), where

\[
q_i^*(x, t) = \partial_{p_i} f^*(p)(x, t) = \begin{cases} (\partial_q f(\cdot))^{-1} p(x, t) & \text{if } |(\partial_q f(\cdot))^{-1} p_i(x, t)| \leq R, \\ R \text{sign}(p_i(x, t)) & \text{if } |(\partial_q f(\cdot))^{-1} p_i(x, t)| > R. \end{cases}
\]

(11)

For more details about the quadratic Hamiltonian with control constraints and its numerical approximation, we refer to Sects. 5 and 6 in [9].
We now introduce some useful anisotropic Sobolev spaces to handle time-dependent problems. First, given a Banach space \( X \), \( L^p(0, T; X) \) denotes the usual vector-valued Lebesgue space. For any \( r \geq 1 \), we denote by \( W_r^{2, 1}(Q) \) the space of functions \( f \) such that \( \partial_t^\delta D_x^\sigma f \in L^r(Q) \) for all multi-indices \( \sigma \) and \( \delta \) such that \( |\sigma| + 2\delta \leq 2 \), endowed with the norm
\[
\| u \|_{W_r^{2, 1}(Q)} = \left( \int_Q \sum_{|\sigma| + 2\delta \leq 2} |\partial_t^\delta D_x^\sigma u|^r dx dt \right)^{\frac{1}{r}}.
\]
We recall that, by classical results in interpolation theory, the sharp space of initial (or terminal) trace of \( W_r^{2, 1}(Q) \) is given by the fractional Sobolev class \( W_r^{2-\frac{2}{r}}(\mathbb{T}^d) \).

\( |u| \) denotes the usual \( L^\infty(Q) \) norm for \( u(x, t) \) with \((x, t) \in Q\). \( C^{1,0}(Q) \) with the norm \( |u|_{C^{1,0}(Q)} \) will be the space of continuous functions on \( Q \) with continuous derivatives in the \( x \)-variable, up to the parabolic boundary, since the spatial variable varies in the torus, up to \( t = 0 \).

We recall the definition of parabolic Hölder spaces on the torus (we refer to [44] for a more comprehensive discussion). For \( 0 < \alpha < 1 \), we denote
\[
[u]_{C^{\alpha, \frac{q}{2}}(Q)} := \sup_{(x_1, t_1), (x_2, t_2) \in Q} \frac{|u(x_1, t_1) - u(x_2, t_2)|}{(d(x_1, x_2)^2 + |t_1 - t_2|)^{\frac{\alpha}{2}}},
\]
where \( d(x, y) \) stands for the geodesic distance from \( x \) to \( y \) in \( \mathbb{T}^d \). The parabolic Hölder space \( C^{\alpha, \frac{q}{2}}(Q) \) is the space of functions \( u \in L^\infty(Q) \) for which \( [u]_{C^{\alpha, \frac{q}{2}}(Q)} < \infty \). It is endowed with the norm:
\[
|u|_{C^{\alpha, \frac{q}{2}}(Q)} := |u| + [u]_{C^{\alpha, \frac{q}{2}}(Q)}.
\]
The space \( C^{1+\alpha, \frac{1+\alpha}{2}}(Q) \) is endowed with the semi-norm
\[
[u]_{C^{1+\alpha, \frac{1+\alpha}{2}}(Q)} := \sum_{i=1}^d |\partial_{x_i} u|_{C^{\alpha}}(Q) + \sup_{(x_1, t_1), (x_2, t_2) \in Q} \frac{|u(x_1, t_1) - u(x_2, t_2)|}{|t_1 - t_2|^{\frac{1+\alpha}{2}}},
\]
and the norm
\[
|u|_{C^{1+\alpha, \frac{1+\alpha}{2}}(Q)} := |u| + [u]_{C^{1+\alpha, \frac{1+\alpha}{2}}(Q)}.
\]
Likewise, \( |.|_{C^{\alpha, \frac{q}{2}}(Q)} \) and \( |.|_{C^{1,0}(Q)} \) are used to define the analogous norms on spaces of functions defined on \( \mathbb{T}^d \).

For a vector field \( b(x, t) \) in \( \mathbb{R}^d \) with \( 1 \leq i \leq d \) we denote
\[
\|b\|_{L^r(Q)} = \sup_i \|b_i\|_{L^r(Q)}, \quad |b| = \sup_i |b_i|.
\]
Lemma 2.1 (Lemma 2.3 of [31]) Let \( \alpha \in (0, 1) \). For any \( f \in C^{1+\alpha,(1+\alpha)/2}(Q) \),

\[
|f|_{Q}^{(1)} \leq |f(\cdot, T)|_{T^d}^{(1)} + T^{\alpha/2} |f|_{Q}^{(1+\alpha)}, \tag{15}
\]

\[
|f|_{Q}^{(1)} \leq |f(\cdot, 0)|_{T^d}^{(1)} + T^{\alpha/2} |f|_{Q}^{(1+\alpha)}. \tag{16}
\]

Lemma 2.2 (Lemma 2.4 of [31]) Let \( r > 1 \), \( f \in W^{2,1}_{2r}(Q) \). Then

\[
\| f \|_{W^{2,1}_{2r}(Q)} \leq T^{\frac{1}{2r}} \| f \|_{W^{2,1}_{2r}(Q)}. \tag{17}
\]

Proposition 2.3 (Inequality (2.21) of [39], or Proposition 2.5 of [31]) Let \( f \in W^{2,1}_{r}(Q) \). Then,

\[
|f|_{Q}^{(2-\frac{d+2}{2r})} \leq C(\| f \|_{W^{2,1}_{r}(Q)} + \| f(\cdot, 0) \|_{W^{2-\frac{2}{r}}(T^d)}) , \quad r > \frac{d + 2}{2}, \quad r \neq d + 2, \tag{18}
\]

where \( C \) remains bounded for bounded values of \( T \).

This embedding result is distinct from the classic result (Corollary p.342 of [44]) in that the constant \( C \) remains bounded when \( T \) tends to 0.

We can then easily obtain the following from Lemma 2.1 and Proposition 2.3.

Lemma 2.4 Let \( r > d + 2 \), \( \bar{f} \in W^{2,1}_{r}(Q) \). We assume either \( \bar{f}(x, 0) = 0 \) or \( \bar{f}(x, T) = 0 \). Then

\[
|\bar{f}|_{Q}^{(1)} \leq C T^{\frac{1}{2} - \frac{d+2}{2r}} \| \bar{f} \|_{W^{2,1}_{r}(Q)}, \tag{19}
\]

where \( C \) remains bounded for bounded values of \( T \).

We describe the assumptions on the data of the problem.

(H1) \( H \) is continuous with respect to \( p, m \). \( H, H_{p_1}, H_{p_1 p_j}, H_{m p_i} \) are locally Lipschitz continuous functions with respect to \( (p, m) \in \mathbb{R}^d \times \mathbb{R}^+ \).

(H2) \( H \) is strictly convex in the \( p \)-entry.

Remark 2.5 Typical examples we are going to consider are MFGs with congestion effects. For instance, with \( \gamma > 1, \beta \) and \( c \) non-negative constants and \( f(m) \) a locally Lipschitz function of \( m \in \mathbb{R}^+ \), we consider:

\[
H(m, p) = \frac{|p|^\gamma}{(c + m)^\beta} + f(m).
\]

Note that when \( \beta > 2 \), the Hamiltonian is super-quadratic and when \( c = 0 \) the Hamiltonian is degenerate at \( m = 0 \). Covering such situations is one of the reasons why we restrict our attention to short time horizons for the convergence results we prove in the sequel.
Remark 2.6 In this paper we only consider the case \( H(m, p) \). The results can be naturally extended to include Hamiltonians of the form \( H(x, m, p) \) with suitable additional assumptions.

In the following we recall a classical result of the linear parabolic equation.

\[
\begin{aligned}
-\partial_t u - \epsilon \Delta u + b(x, t) \cdot Du + c(x, t)u &= f(x, t) \quad \text{in } Q, \\
u(x, T) &= u_T(x) \quad \text{in } T^d.
\end{aligned}
\]  

(20)

Proposition 2.7 Let \( r > d + 2 \) and suppose that \( b \in L^\infty(Q; \mathbb{R}^d) \), \( c \in L^\infty(Q) \), \( f \in L^r(Q) \), and \( u_T \in W^{2-\frac{2}{r}}_r(T^d) \). Then the problem (20) admits a unique solution \( u \in W^{2,1}_r(Q) \) and it holds

\[
\|u\|_{W^{2,1}_r(Q)} \leq C \left( \|f\|_{L^r(Q)} + \|u_T\|_{W^{2-\frac{2}{r}}_r(T^d)} \right),
\]  

(21)

where \( C \) depends on the \( L^\infty(Q) \) norms of the coefficients \( b \) and \( c \) as well as on \( r, d \), \( T \) and remains bounded for bounded values of \( T \).

This result has been proved with other boundary conditions ( [44], Chapter 9, Theorem 9.1) and proved in the periodic setting in Appendix A of [31], where the proof is based on the local estimate of [44], eq. (10.12), p.355. The case when \( c \in L^r(Q) \) has been considered in the Appendix of [21].

Proposition 2.8 (Theorem 4, Appendix of [21]) Let \( r > d + 2 \). Suppose that \( b \in L^r(Q; \mathbb{R}^d) \), \( c \in L^r(Q) \), \( f \in L^r(Q) \), and \( u_T \in W^{2-\frac{2}{r}}_r(T^d) \). The problem (20) admits a unique solution \( u \in W^{2,1}_r(Q) \) such that

\[
\|u\|_{W^{2,1}_r(Q)} \leq C,
\]

with \( C \) depending only on the \( L^r(Q) \) norms of \( b, c, f \) and \( \|u_T\|_{W^{2-\frac{2}{r}}_r(T^d)} \).

Remark 2.9 Proposition 2.8 only established the existence and uniqueness of solutions \( u \in W^{2,1}_r(Q) \) but not the estimate (21). With the technique developed in [31] one can show that (21) holds under the assumptions of Proposition 2.8 and \( T \) sufficiently small.

3 Policy Iterations Methods for the MFG System

We will use the following assumption.

\begin{enumerate}
\item[(I1)] \( u_T \in W^2_\infty(T^d) \); for some \( r > d + 2 \), \( m_0 \in W^2_r(T^d) \), \( m_0 \geq \underline{m} > 0 \) and \( \int_{T^d} m_0(x)dx = 1 \).
\end{enumerate}

We define the space

\[
X^T_M = \{(u, m) : u \in C^{1,0}(Q) \cap W^{2,1}_r(Q), m \in C^{1,0}(Q),
\|u\|_{W^{2,1}_r(Q)} + |u|^{(1)}_Q + |m|^{(1)}_Q \leq M\}.
\]  

(22)
3.1 Policy Iteration (PI1)

Let us start with the analysis of the policy iteration method (PI1). Define the operator $T$ on $X_M^T$ by: $T(u, m) = (\hat{u}, \hat{m})$ such that

$$
\begin{align*}
\partial_t \hat{m} - \epsilon \Delta \hat{m} - H_p(m, Du)D\hat{m} - H_{pm}(m, Du)(Dm)\hat{m} \\
- \sum_{i,j} H_{p_j p_i}(m, Du)(\partial^2_{x_i x_j} u)\hat{m} = 0 \\
- \partial_t \hat{u} - \epsilon \Delta \hat{u} + H_p(m, Du)D\hat{u} - \mathcal{L}(\hat{m}, Du, H_p(m, Du)) = 0,
\end{align*}
$$

where we recall that the perturbed Lagrangian $\mathcal{L}$ is defined in (6). The following theorem and its proof are highly based on Theorem 1.1 of [31]. However, since it is central to the theoretical study of the policy iteration algorithms, we provide the full details of the proof.

**Theorem 3.1** Suppose (H1), (H2) and (H1) hold. Let $K$ be such that

$$
K \geq \max \left\{ \frac{2}{m}, 2|m_0|^{(1)}_{\text{mp}}, 2|u(\cdot, T)|^{(1)}_{\text{mp}} \right\}.
$$

Let

$$
M_1 = 2\left( |m_0|^{(1)}_{\text{mp}} + |u_T|^{(1)}_{\text{mp}} \right).
$$

Then there exists $\bar{T}$ sufficiently small such that for all $T \in (0, \bar{T}]$, $T$ is a contraction on the space

$$
X_M^T \cap \{(u, m) : |u|^{(1)}_Q \leq K, 1/K \leq m \leq K\}.
$$

**Proof** Step 1: Lipschitz regularization. Let $\varphi$ be a global Lipschitz function such that $\varphi(z) = z$ for all $z \in [1/K, K]$, $\varphi(z) \in [1/(2K), 2K]$ for all $z \in \mathbb{R}$. Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a globally Lipschitz function such that $\psi(z) = z$ for all $|z| \leq K$ and $|\psi(z)| \leq 2K$ for all $z \in \mathbb{R}^d$.

We then consider the regularized fixed point operator $T_K$ defined on $X_M^T$ by:

$$
T_K(u, m) = (\hat{u}, \hat{m})
$$

such that

$$
\begin{align*}
\partial_t \hat{m} - \epsilon \Delta \hat{m} - H_p(\varphi(m), \psi(Du))D\hat{m} - (H_{pm}(\varphi(m), \psi(Du))(Dm)\hat{m} \\
- \sum_{i,j} H_{p_j p_i}(\varphi(m), \psi(Du))(\partial^2_{x_i x_j} u)\hat{m} = 0 \\
- \partial_t \hat{u} - \epsilon \Delta \hat{u} + H_p(\varphi(m), \psi(Du))D\hat{u} - \mathcal{L}(\hat{m}, \psi(Du), H_p(\varphi(m), \psi(Du))) = 0,
\end{align*}
$$

where $\mathcal{L}$ is defined in (6).

From (H1) and (24) we have $H_p(\varphi(m), \psi(Du))$, $H_{pp}(\varphi(m), \psi(Du))$ and $H_{pm}(\varphi(m), \psi(Du))$ are globally Lipschitz with respect to the pair $(m, Du)$. In fact,
there exist two positive constants $C_H$ and $C'_H$ depending only on the data of the problem and $K$, such that for all $(u_1, m_1), (u_2, m_2) \in X^T_M$,

\[
\sup_{(x,t) \in \mathcal{Q}} \left\{ |H(\varphi(m_1), \psi(Du_1)) - H(\varphi(m_2), \psi(Du_2))| + |H_p(\varphi(m_1), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2))| + |H_{pp}(\varphi(m_1), \psi(Du_1)) - H_{pp}(\varphi(m_2), \psi(Du_2))| \right\} \\
\leq C_H(m_1 - m_2)_{\mathcal{Q}}^{(1)} + |u_1 - u_2|_{\mathcal{Q}}^{(1)},
\]

and for all $(u, m) \in X^T_M$,

\[
\sup_{(x,t) \in \mathcal{Q}} \left\{ |H(\varphi(m), \psi(Du))| + |H_p(\varphi(m), \psi(Du))| + |H_{pp}(\varphi(m), \psi(Du))| \right\} \leq C'_H.
\]

From (27) we obtain that there exists $C_L$ depending only on $K$ such that

\[
\sup_{(x,t) \in \mathcal{Q}} \left\{ |\mathcal{L}(\varphi(\hat{m}_1), \psi(Du_1), H_p(\varphi(m_1), \psi(Du_1))) - \mathcal{L}(\varphi(\hat{m}_2), \psi(Du_2), H_p(\varphi(m_2), \psi(Du_2)))| \right\} \\
\leq \sup_{(x,t) \in \mathcal{Q}} \left\{ |H_p(\varphi(m_1), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2))| \cdot |\psi(Du_1)| + |\psi(Du_1) - \psi(Du_2)| \cdot |H_p(\varphi(m_2), \psi(Du_2))| \\
+ |H(\varphi(\hat{m}_1), \psi(Du_1)) - H(\varphi(\hat{m}_2), \psi(Du_2))| \right\} \\
\leq C_L((m_1 - m_2)_{\mathcal{Q}}^{(1)} + |\hat{m}_1 - \hat{m}_2|_{\mathcal{Q}}^{(1)} + |u_1 - u_2|_{\mathcal{Q}}^{(1)}).
\]

We denote by $C'_L$ a constant such that:

\[
\sup_{(x,t) \in \mathcal{Q}} |\mathcal{L}(\varphi(\hat{m}), \psi(Du), H_p(\varphi(m), \psi(Du)))| \leq C'_L. \tag{30}
\]

**Step 2:** $\mathcal{T}_K$ maps $X^T_{M_1}$ into itself.

Suppose that $(u, m) \in X^T_{M_1}$, then from (28) and $\|u\|_{W^{2,1}_r(\mathcal{Q})} \leq M_1$, we have

\[
\|H_{p, p_1}(\varphi(m), \psi(Du))\partial_{x,x_i} u\|_{L^r(\mathcal{Q})} \leq \|H_{p, p_1}(\varphi(m), \psi(Du))\|_{L^\infty(\mathcal{Q})} \cdot \|\partial_{x,x_i} u\|_{L^r(\mathcal{Q})} \leq C'_H M_1. \tag{31}
\]

Likewise we have

\[
\|H_p(\varphi(m), \psi(Du))\|_{L^r(\mathcal{Q})} \leq T^{1/r} C'_H, \tag{32}
\]
\[ \| H_{pm}(\varphi(m), \psi(Du)) Dm \|_{L^r(Q)} \leq T^{1/r} C'_H M_1, \] (33)

where \( C'_H \) and \( M_1 \) are defined in (28) and (25).

From (31), (32) and (33), we can obtain using Proposition 2.8 that there exists a unique solution \( \hat{m} \) to the first equation in (26) such that

\[ \| \hat{m} \|_{W^{2,1}(Q)} \leq C_1, \]

where \( C_1 \) depends only on \( K, M_1, C'_H \) and \( \| m_0 \|_{W^{2,2}_{r,T^d}} \).

From Proposition 2.3 we have

\[ |\hat{m}|_Q^{(1)} \leq |m_0|_{T^d}^{(1)} + T^{\frac{1}{r} - \frac{d+2}{2r}} C'_2. \] (34)

Together with Lemma 2.1 this yields

\[ |\hat{m}|_Q^{(1)} \leq T^{\frac{1}{r} - \frac{d+2}{2r}} C'_2. \] (35)

Here both \( C_2 \) and \( C'_2 \) remain bounded for bounded \( T \).

Next, we consider the linearized HJB equation in the system (26). Again from Proposition 2.7 we have

\[ \| \hat{u} \|_{W^{2,1}_{r,T^d}(Q)} \leq C_3 \left( \| L(\varphi(\hat{m}), \psi(Du), H_p(\varphi(m), \psi(Du))) \|_{L^r(Q)} + \| u_T \|_{W^{2,2}_{r,T^d}(T^d)} \right) \]

\[ \leq T^{\frac{1}{r}} C_3 C_L + C_3 \| u_T \|_{W^{2,2}_{r,T^d}(T^d)} \]

\[ \leq C'_3. \]

Using Lemmas 2.1 and 2.2 we obtain

\[ \| \hat{u} \|_{W^{2,1}_{r,T^d}(Q)} \leq T^{\frac{1}{r}} C'_3, \quad |\hat{u}|_Q^{(1)} \leq |u_T|_T^{(1)} + T^{\frac{1}{r} - \frac{d+2}{2r}} C'_3. \] (36)

Here again, both \( C_3 \) and \( C'_3 \) remain bounded for bounded \( T \). Recall (25) and \( r > d + 2 \).

Then there exists \( T \) so small that

\[ |\hat{m}|_Q^{(1)} + |\hat{u}|_Q^{(1)} + \| \hat{u} \|_{W^{2,1}_{r,T^d}(Q)} \]

\[ \leq |m_0|_{T^d}^{(1)} + T^{\frac{1}{r} - \frac{d+2}{2r}} C'_2 + |u_T|_T^{(1)} + T^{\frac{1}{r} - \frac{d+2}{2r}} C'_3 + T^{\frac{1}{r}} C'_3 \]

\[ < M_1. \]

Therefore, we have \( T : X^{T}_{M_1} \rightarrow X^{T}_{M_1} \).
\textbf{Step 3:} $T_K : X_{M_1}^T \to X_{M_1}^T$ is a contraction operator. 
Let $(\hat{u}_1, \hat{m}_1) := \hat{T}(u_1, m_1)$ and $(\hat{u}_2, \hat{m}_2) := \hat{T}(u_2, m_2)$. We aim at showing

$$\|\hat{u}_1 - \hat{u}_2\|_{W^{2,1}_r(Q)} + |\hat{u}_1 - \hat{u}_2|_Q + |\hat{m}_1 - \hat{m}_2|_Q \leq \Gamma \left( \|u_1 - u_2\|_{W^{2,1}_r(Q)} + |u_1 - u_2|_Q + |m_1 - m_2|_Q \right),$$

for some $0 < \Gamma < 1$.

Denote $\hat{U} = \hat{u}_1 - \hat{u}_2$ and $\hat{M} = \hat{m}_1 - \hat{m}_2$. From system (26) we have

$$\partial_t \hat{M} - \varepsilon \Delta \hat{M} - H_p(\varphi(m_1), \psi(Du_1))D \hat{M}
- \left( H_{pm}(\varphi(m_1), \psi(Du_1)) Dm_1 + \sum_{i, j} H_{pj pi}(\varphi(m_1), \psi(Du_1)) \partial^2_{x_j x_i} u_1 \right) \hat{M}
- \left( H_p(\varphi(m_1), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2)) \right) \hat{D} \hat{m}_2
- \left( H_{pm}(\varphi(m_1), \psi(Du_1)) Dm_1 + \sum_{i, j} H_{pj pi}(\varphi(m_1), \psi(Du_1)) \partial^2_{x_j x_i} u_1 \right) \hat{m}_2
+ \left( H_{pm}(\varphi(m_2), \psi(Du_2)) Dm_2 + \sum_{i, j} H_{pj pi}(\varphi(m_2), \psi(Du_2)) \partial^2_{x_j x_i} u_2 \right) \hat{m}_2$$

$$= 0.$$

From (27) and the fact that $\hat{m}_2$ remains in $X_{M_1}^T$ we have

$$\sup_{(x, t) \in Q} \left| \left( H_p(\varphi(m_1), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2)) \right) \hat{D} \hat{m}_2 \right| \leq C_H M_1 |u_1 - u_2|^{(1)}_Q + |m_1 - m_2|^{(1)}_Q.$$ (39)

Moreover,

$$\sup_{(x, t) \in Q} \left\{ |H_{pm}(\varphi(m_1), \psi(Du_1)) Dm_1 - H_{pm}(\varphi(m_2), \psi(Du_2)) Dm_2| \right\}
\leq \sup_{(x, t) \in Q} \left\{ |H_{pm}(\varphi(m_1), \psi(Du_1)) - H_{pm}(\varphi(m_2), \psi(Du_2))| \cdot |Dm_1| \right.
+ |H_{pm}(\varphi(m_2), \psi(Du_2))| \cdot |Dm_1 - Dm_2| \right\}
\leq (M_1 C_H + C'_H) (|u_1 - u_2|^{(1)}_Q + |m_1|^{(1)} - |m_2|^{(1)}).$$ (40)

Since $|\hat{m}_2| \leq M_1$, $\|u_1\|_{W^{2,1}_r(Q)} \leq M_1$ and $\|u_2\|_{W^{2,1}_r(Q)} \leq M_1$, we have

$$\|H_{pj pi}(\varphi(m_1), \psi(Du_1)) \partial^2_{x_j x_i} u_1 - H_{pj pi}(\varphi(m_2), \psi(Du_2)) \partial^2_{x_j x_i} u_2\|_{L^r(Q)}
\leq M_1 \|H_{pj pi}(\varphi(m_1), \psi(Du_1)) - H_{pj pi}(\varphi(m_2), \psi(Du_2))\|_{L^\infty(Q)} \cdot \|\partial^2_{x_j x_i} u_1\|_{L^r(Q)}
+ M_1 \|H_{pj pi}(\varphi(m_1), \psi(Du_1))\|_{L^\infty(Q)} \cdot \|\partial^2_{x_j x_i} u_1 - \partial^2_{x_j x_i} u_2\|_{L^r(Q)}$$
\[ \leq C_H M_1^2 (|m_1 - m_2|^1_Q + |u_1 - u_2|^1_Q) + M_1 C'_H \|u_1 - u_2\|_{W_r^{2,1}(Q)}. \]

Therefore from Proposition 2.8 we obtain that there exists a unique solution \( \tilde{M} \) to (38) such that

\[ \|\tilde{M}\|_{W_r^{2,1}(Q)} \leq C_4, \]

where \( C_4 \) depends only on \( K, M_1, C'_H, C_H \) and \( \|m_0\|_{W_r^{2,2}(T')}. \)

Moreover, from Proposition 2.7 one can obtain that

\[ \|\tilde{M}\|_{W_r^{2,1}(Q)} \leq C'_4 (|m_1 - m_2|^1_Q + |u_1 - u_2|^1_Q + |u_1 - u_2|^1_{W_r^{2,1}(Q)} + |\tilde{M}|^1_Q), \]

where \( C'_4 \) remains bounded for bounded \( T \). From Lemma 2.4 we have

\[ |\tilde{M}|^1_Q \leq T^{\frac{1}{2} - \frac{d+2}{2r}} C'_4 (\|u_1 - u_2\|_{W_r^{2,1}(Q)} + |m_1 - m_2|^1_Q + |u_1 - u_2|^1_Q + |\tilde{M}|^1_Q). \]

Since \( C'_4 \) remains bounded for bounded \( T \), it is then clear that for sufficiently small \( T \), such that \( T^{\frac{1}{2} - \frac{d+2}{2r}} C'_4 < 1 \), we can obtain

\[ |\tilde{M}|^1_Q \leq T^{\frac{1}{2} - \frac{d+2}{2r}} C''_4 (\|u_1 - u_2\|_{W_r^{2,1}(Q)} + |m_1 - m_2|^1_Q + |u_1 - u_2|^1_Q), \]

where \( C''_4 \) remains bounded for bounded \( T \).

We next turn to the linearized HJB equation. From (26) we have

\[
0 = -\partial_t \hat{U} - \varepsilon \Delta \hat{U} + H_p(\varphi(m_1), \psi(Du_1)) Du_1 \\
+ (H_p(\varphi(m_2), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2))) Du_2 \\
- \mathcal{L}(\varphi(\hat{m}_1), \psi(Du_1), H_p(\varphi(m_1), \psi(Du_1))) \\
+ \mathcal{L}(\varphi(\hat{m}_2), \psi(Du_2), H_p(\varphi(m_2), \psi(Du_2))).
\]

From (29) and (42) we have

\[
\|\mathcal{L}(\varphi(\hat{m}_1), \psi(Du_1), H_p(\varphi(m_1), \psi(Du_1))) \\
- \mathcal{L}(\varphi(\hat{m}_2), \psi(Du_2), H_p(\varphi(m_2), \psi(Du_2)))\|_{L^r(Q)} \\
\leq T^{1/r} C_{\mathcal{L}}(|\hat{m}_1 - \hat{m}_2|^1_Q + |m_1 - m_2|^1_Q + |u_1 - u_2|^1_Q) \\
\leq C_{\mathcal{L}}(T^{1/r} + T^{\frac{1}{2} - \frac{d+2}{2r}} C''_4 (\|u_1 - u_2\|_{W_r^{2,1}(Q)} + |m_1 - m_2|^1_Q + |u_1 - u_2|^1_Q)).
\]

From (27),
\[ \| (H_p(\varphi(m_1), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2)) Du_2 \|_{L^r(Q)} \leq T^{1/r} M_1 C_H (|m_1 - m_2|_Q^{(1)} + |u_1 - u_2|_Q^{(1)}). \]

Again, from Proposition 2.7 we have
\[
\| \tilde{U} \|_{W^{2,1}_r(Q)} \leq C_5 \| (H_p(\varphi(m_1), \psi(Du_1)) - H_p(\varphi(m_2), \psi(Du_2)) Du_2 \|_{L^r(Q)} \\
+ \| L(\varphi(\hat{m}_1), \psi(Du_1), H_p(\varphi(m_1), \psi(Du_1)) \\
- L(\varphi(\hat{m}_2), \psi(Du_2), H_p(\varphi(m_2), \psi(Du_2))) \|_{L^r(Q)} \\
\leq C_5 (C_L(T^{1/r} + T^{1/2} - \frac{d}{2} C_4) + T^{1/r} M_1 C_H) (\| u_1 - u_2 \|_{W^{2,1}_r(Q)} \\
+ |m_1 - m_2|_Q^{(1)} + |u_1 - u_2|_Q^{(1)}).
\]

By using Lemma 2.1 and Proposition 2.3 we have
\[
|\tilde{U}_1|_Q^{(1)} \leq T^{1/2} - \frac{d + 2}{2r} C_6 (C_L(T^{1/r} + T^{1/2} - \frac{d}{2} C_4) + T^{1/r} M_1 C_H) (\| u_1 - u_2 \|_{W^{2,1}_r(Q)} \\
+ |m_1 - m_2|_Q^{(1)} + |u_1 - u_2|_Q^{(1)}).
\]

Therefore we obtain
\[
\| \hat{u} - \hat{u}_2 \|_{W^{2,1}_r(Q)} + |\hat{u}_1 - \hat{u}_2|_Q^{(1)} + |\hat{m}_1 - \hat{m}_2|_Q^{(1)} \\
\leq (T^{1/r} + T^{1/2} - \frac{d}{2} + T^{1/2} - \frac{d + 2}{2r}) C_7 (\| u_1 - u_2 \|_{W^{2,1}_r(Q)} \\
+ |u_1 - u_2|_Q^{(1)} + |m_1 - m_2|_Q^{(1)}) \tag{44}
\]

where \( C_7 \) here remains bounded for bounded \( T \). By making \( T \) small enough we can obtain that \( T_K \) is a strict contraction in \( X_{M_1}^T \).

Hence, from Banach fixed point theorem, if \( T \) is small enough, \( T_K : X_{M_1}^T \to X_{M_1}^T \)

admits a unique fixed point. We denote it by \((u^*, m^*)\).

**Step 4: Sobolev Regularity.** Note that \((u^*, m^*)\) satisfies
\[
\partial_t m^* - \epsilon \Delta m^* - \text{div}(H_p(\varphi(m^*), \psi(Du^*)))m^* - H_p(\varphi(m^*), \psi(Du^*)) Dm^* = 0,
\]

and \( \partial_x (H_p(\varphi(m^*), \psi(Du^*))) \in L^r(Q) \) and \( H_p(\varphi(m^*), \psi(Du^*)) \in L^r(Q) \). Hence we have \( m^* \in W^{2,1}_r(Q) \) from Proposition 2.8.

Moreover, from system (26) we have \((\hat{u}, \hat{m})\) bounded in \( W^{2,1}_r(Q) \) with the bound independent of \((u, m)\), \( T_K(u, m) = (\hat{u}, \hat{m}) \).

**Step 5: Back to the Initial Problem.** We conclude by showing that \( T \) maps the space
\[
X_{M_1}^T \cap \{|u|_Q^{(1)} \leq K, 1/K \leq m \leq K\},
\]

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into itself, if $T$ is sufficiently small. It is then a contraction in this space.

From (13) and (14) we have, taking $t_1, t_2 \in [0, T]$,

$$
\sup_{(x, t_1) \neq (x, t_2) \in Q} \left| \hat{m}(x, t_1) - \hat{m}(x, t_2) \right| \leq \sup_{(x, t_1) \neq (x, t_2) \in Q} \left| \hat{m}(x, t_1) - \hat{m}(x, t_2) \right| T^{1 - \frac{d + 2}{2r}}
$$

we have

$$
\hat{m}(\hat{x}, 0) - \hat{m}(\hat{x}, \hat{t}) \leq |\hat{m}(\hat{x}, \hat{t}) - \hat{m}(\hat{x}, 0)| \leq T^{1 - \frac{d + 2}{2r}} |\hat{m}|_{Q}^{(2 - \frac{d + 2}{r})}.
$$

From (34) we have then

$$
\min_{(x, t) \in Q} \hat{m}(x, t) \geq m(\hat{x}, 0) - |\hat{m}|_{Q}^{(2 - \frac{d + 2}{r})} T^{1 - \frac{d + 2}{2r}}
$$

Likewise, we can get

$$
\max_{(x, t) \in Q} \hat{m} \leq |\hat{m}|_{Q}^{(1)} + C_{2}T^{1 - \frac{d + 2}{2r}}.
$$

Moreover, as in (36),

$$
|\hat{u}|_{Q}^{(1)} \leq |u_{T}|_{T_d}^{(1)} + T^{\frac{1}{2} - \frac{d + 2}{2r}} C_{3}.
$$

We recall that both $C_{2}$ and $C_{3}$ remain bounded for bounded $T$, therefore we can choose $T$ so small that

$$
C_{2}T^{2 - \frac{d + 2}{r}} < \min\{1/K, K/2\}, \quad T^{\frac{1}{2} - \frac{d + 2}{2r}} C_{3} < K/2.
$$

This and (24) yield $|\hat{u}|_{Q}^{(1)} \leq K$ and $1/K \leq \hat{m} \leq K$ for all $(x, t) \in Q$. □

From Theorem 3.1 we obtain the following short time existence and uniqueness result for the MFG system (1) with non-separable Hamiltonian.
Theorem 3.2 Let \((H1), (H2) and (II)\) be in force. There exists a sufficiently small \(\tilde{T}\) such that for all \(T \in (0, \tilde{T}]\) the system (I) admits a unique solution \((u^*, m^*) \in W_{r}^{2,1}(Q) \times W_{r}^{2,1}(Q)\).

This result is a special case of Theorem 4.1 from [31]. In our case, the terminal cost \(u_T\) does not depend on \(m(\cdot, T)\). In Theorem 4.1 of [31] it may be a regularizing function of \(m(\cdot, T)\), but excludes the case where \(u_T\) depends locally on \(m(\cdot, T)\) (see Section 3.1 of [31]). Here we only consider terminal cost \(u_T\) which does not depend on \(m(\cdot, T)\), as the numerical approximation for nonlocal coupling with rigorous convergence analysis is beyond the scope of this paper.

The next result extends to the case with non-separable Hamiltonians the results [24, Theorems 2.3 and 2.5].

Theorem 3.3 Let \((H1), (H2) and (II)\) be in force. Let \(\tilde{T}\) be as in Theorem 3.1. Then, there exists a \(\tilde{T}_1 \leq \tilde{T}\) such that for all \(T \in (0, \tilde{T}_1]\) and \(R\) sufficiently large, the sequence \((u^{(n)}, m^{(n)})\), generated by the policy iteration algorithm (PII), converges to the solution \((u^*, m^*) \in W_{r}^{2,1}(Q) \times W_{r}^{2,1}(Q)\) of (I).

**Proof** We start with an initial guess \(q^{(0)} : \mathbb{T}^d \times [0, T] \to \mathbb{R}^d\) with \(|q^{(0)}| < R\) and \(\|\text{div} q^{(0)}\|_{L^r(Q)} < R\). We perform the same regularization as in Step 1 of Theorem 3.1 with \(\phi\) and \(\psi\), \(K\) is defined by (24). From (4), (5), Proposition 2.7 and Proposition 2.8, using similar arguments as in Step 5 of Theorem 3.1, there exists a sufficiently small \(T_1\) such that, for \(T \in (0, T_1]\) we have \(m^{(n)} \in [1/K, K]\) and \(|Du^{(n)}| \leq K\). Then we start with the regularized iteration system, for \(n \geq 0\):

\[
\begin{aligned}
\partial_t m^{(n+1)} - \epsilon \Delta m^{(n+1)} &= H_p(\varphi(m^{(n)}), \psi(Du^{(n)})) m^{(n+1)} - (H_{pm}(\varphi(m^{(n)}), \psi(Du^{(n)})) \partial_{x} m^{(n+1)} = 0 \\
- \sum_{i,j} H_{pj} p_i(\varphi(m^{(n)}), \psi(Du^{(n)})) \partial_{x,i} u^{(n+1)} &= 0 \\
-L(\varphi(m^{(n+1)}), \psi(Du^{(n)}), H_p(\varphi(m^{(n)}), \psi(Du^{(n)}))) &= 0, \\
u^{(n+1)}(x, T) &= u_T(x), \quad m^{(n+1)}(x, 0) = m_0(x).
\end{aligned}
\]

(48)

From the proof of Theorem 3.1, for \(T \in (0, \tilde{T}]\), we have that \((u^{(n)}, m^{(n)})\) converges to the solution \((u^*, m^*) \in W_{r}^{2,1}(Q) \times W_{r}^{2,1}(Q)\) of (I). We aim to show the iteration system (48) is the same as (PII).

We can argue inductively. For each \(n\), assuming \(m^{(n)} \in [1/K, K]\) and \(|Du^{(n)}| \leq K\), we can follow the argument in Step 5 of Theorem 3.1 to obtain: for all \(T \in (0, T]\) we have \(m^{(n+1)} \in [1/K, K]\) and \(|Du^{(n+1)}| \leq K\).

By (H1) and the above remark, there exists a bound on

\[|H_p(m^{(n)}, Du^{(n)})| + |H_{pm}(m^{(n)}, Du^{(n)})| + |H_{pp}(m^{(n)}, Du^{(n)})|\]

which depends only on \(K\). From Propositions 2.7 and 2.8, we can also obtain a bound on \(|m^{(n)}(1)_{Q} + u^{(n)}_{Q}\|_{W_{r}^{2,1}(Q)}\), which depends only on data of the problem, \(K\) and \(T\), remains bounded for bounded \(T\). Therefore we can obtain a bound for \(|q^{(n)}| + \|\text{div} q^{(n)}\|_{L^r(Q)}\) independent of \(n\). Then the system (48) is exactly the algorithm (PII).

Finally we can conclude by choosing \(\tilde{T}_1 = \min\{T_1, \tilde{T}\}\). \(\square\)
Remark 3.4 In Theorem 3.3, we needed to introduce $\hat{T}_1 \leq \hat{T}$ for considerations related to the initial guess. If the initial guess is sufficiently well chosen, then we may have $\hat{T}_1 = \hat{T}$. In practice, we found that $q^{(0)} = 0$ is usually a good initial guess. This can be partially explained as follows. In this case, (4) becomes a simple heat equation with initial condition $m(x, 0) \in [2/K, K/2]$. By the maximum principle of heat equation we have directly that $m^{(0)}(x, t) \in [1/K, K]$, for all $(x, t) \in Q$ and $T \in (0, \hat{T})$.

3.2 Policy Iteration (PI2)

We now turn our attention to the algorithm (PI2). Recall that $X^T_M$ is defined in (22). We define the operator $T_2$ on this set by: $T_2(u, m) = (\hat{u}, \hat{m})$ such that

\[
\begin{aligned}
\begin{cases}
\partial_t \hat{m} - \epsilon \Delta \hat{m} - H_p(m, Du) D\hat{m} - H_{pm}(m, Du)(Dm)\hat{m} \\
- \sum_{i,j} H_{p_{ij}}(m, Du) (\partial_{x_i} x_j u) \hat{m} = 0 \\
- \partial_t \hat{u} - \epsilon \Delta \hat{u} + H_p(\hat{m}, Du) D\hat{u} - L(\hat{m}, H_p(\hat{m}, Du)) = 0,
\end{cases}
\end{aligned}
\]

(49)

where $L(\hat{m}, H_p(\hat{m}, Du)) = H_p(\hat{m}, Du) Du - H(\hat{m}, Du)$.

Theorem 3.5 Let (H1), (H2) and (II) be in force. Then there exists $M_2$ sufficiently large and $\tilde{T}_2$ sufficiently small such that for all $T \in (0, \tilde{T}_2)$, $T_2$ is a contractive operator in the space $X^T_M$.

We omit the proof since it is quite similar to the proof of Theorem 3.1.

Theorem 3.6 Let (H1), (H2) and (II) be in force and $\tilde{T}_2$ be defined as in Theorem 3.5. Then, there exists a $\tilde{T}_2 \leq \tilde{T}_2$, such that for all $T \in (0, \tilde{T}_2]$ and $R$ sufficiently large, the sequence $(u^{(n)}, m^{(n)})$, generated by the policy iteration algorithm (PI2), converges to the solution $(u^*, m^*) \in W^{2,1}_r(Q) \times W^{2,1}_r(Q)$ of (1).

We sketch the proof of Theorem 3.6 to stress the main differences with the proof of Theorem 3.3.

Proof We start with an initial guess $q^{(0)} : \mathbb{T}^d \times [0, T] \rightarrow \mathbb{R}^d$ with $|q^{(0)}| \leq R$ and $\|\text{div}q^{(0)}\|_{L^\infty(Q)} \leq R$. We perform the same regularization as in Step 1 of Theorem 3.1 with $\varphi$ and $\psi$, $K$ is defined by (24). Using the same argument as in Theorem 3.3 there exists a sufficiently small $T_2$ such that, for $T \in (0, T_2]$ we have $m^{(0)} \in [1/K, K]$ and $|Du^{(0)}| \leq K$. Then we start with the regularized iteration system, for $n \geq 0$:

\[
\begin{aligned}
\begin{cases}
\partial_t m^{(n+1)} - \epsilon \Delta m^{(n+1)} - H_p(m^{(n+1)}, \psi(Du^{(n)})) Du^{(n+1)} \\
- (H_{pm}(m^{(n)}), \psi(Du^{(n)}))(Dm^{(n+1)} m^{(n+1)}) \\
- \sum_{i,j} H_{p_{ij}}(m^{(n)})(\psi(Du^{(n)})) \partial_{x_i} x_j u^{(n)} m^{(n+1)} = 0, \\
- \partial_t u^{(n+1)} - \epsilon \Delta u^{(n+1)} + H_p(m^{(n+1)}, \psi(Du^{(n)})) Du^{(n+1)} \\
- L(\varphi(m^{(n+1)}), H_p(m^{(n+1)}, \psi(Du^{(n)}))) = 0, \\
u^{(n+1)}(x, T) = u_T(x), \\
m^{(n+1)}(x, 0) = m_0(x).
\end{cases}
\end{aligned}
\]

(50)
We can argue by induction: if \( m^{(n)} \in [1/K, K], |Du^{(n)}| \leq K \) and \( T \in (0, \hat{T}_2) \), then we have \( m^{(n+1)} \in [1/K, K] \) and \( |Du^{(n+1)}| \leq K \).

There exist bounds depending only on \( K \) for

\[
|H_p(m^{(n)}, Du^{(n)})| + |H_{pm}(m^{(n)}, Du^{(n)})| + |H_{pp}(m^{(n)}, Du^{(n)})|
\]

and

\[
|H_p(m^{(n+1)}, Du^{(n)})| + |H_{pm}(m^{(n+1)}, Du^{(n)})| + |H_{pp}(m^{(n+1)}, Du^{(n)})|.
\]

We can then follow the same arguments as Theorem 3.3 and obtain bounds on

\[
|q^{(n)}| + \|\text{div}q^{(n)}\|_{L^r(Q)} \quad \text{and} \quad |\tilde{q}^{(n)}| + \|\text{div}\tilde{q}^{(n)}\|_{L^r(Q)}.
\]

independent of \( n \). Then the system (50) is exactly the algorithm (PI2). We can conclude by choosing \( \hat{T}_2 = \min\{T_2, \hat{T}_2\}. \)

\( \Box \)

### 4 A Rate of Convergence for the Policy Iteration Method

**Theorem 4.1** Let (H1), (H2) and (I1) be in force. Let \( \hat{T}_1 \) and \( R \) be as in Theorem 3.3. Then, there exists a constant \( C \), which depends only on the data of problem and remains bounded for all \( T \in (0, \hat{T}_1] \), such that, if \((u^{(n)}, m^{(n)})\) is the sequence generated by the policy iteration method (PI1), we have

\[
\|m^{(n+1)} - m^*\|_{W^2_r(Q)} \leq C \left( \|q^{(n+1)} - q^*\|_{L^r(Q)} + \|\text{div}(q^{(n+1)} - q^*)\|_{L^r(Q)} \right),
\]

(51)

and

\[
\|u^{(n+1)} - u^*\|_{W^2_r(Q)} \leq CT^{1/2 - \frac{d}{2r}} \left( \|u^{(n)} - u^*\|_{W^2_r(Q)} + \|m^{(n+1)} - m^*\|_{W^2_r(Q)} \right) - \frac{d}{2r}.
\]

(52)

**Proof** Along the proof, the constant \( C \) can change from line to line, but it is always independent of \( n \) and remains bounded for all \( T \in (0, \hat{T}_1] \). We start with the proof of (51) for the FP equation. For all \( n \) we have \( u^{(n)}, m^{(n)} \in W^2_r(Q) \). As in Theorem 3.1 we take \((u, m) \in X^T_{M_1} \), so that

\[
\|u\|_{W^2_r(Q)} + |u^{(1)}|_Q + |m|^{(1)}_Q \leq M_1.
\]

From

\[
q^{(n)} = H_p(m^{(n-1)}, Du^{(n-1)}),
\]

we have

\[
\partial_{x_i} q^{(n)}_j = H_{mpj}(m^{(n-1)}, Du^{(n-1)})\partial_{x_i} m^{(n-1)} + H_{pj}(m^{(n-1)}, Du^{(n-1)})\partial_{x_i x_j}^2 u^{(n-1)}.
\]

(53)
Hence for all $n$, $\text{div} q^{(n)} \in L^r(Q)$.

Set $M^{(n+1)} = m^{(n+1)} - m^*$. Then $M^{(n+1)}$ satisfies the equation

$$
\partial_t M^{(n+1)} - \epsilon \Delta M^{(n+1)} - \text{div}(q^{(n+1)} M^{(n+1)}) = \text{div}((q^{(n+1)} - q^*)m^*),
$$
with $M^{(n+1)}(\cdot, 0) = 0$. This can be reformulated as

$$
\partial_t M^{(n+1)} - \epsilon \Delta M^{(n+1)} - q^{(n+1)} D M^{(n+1)} - \text{div}(q^{(n+1)} M^{(n+1)})
= \text{div}(q^{(n+1)} - q^*)m^* + (q^{(n+1)} - q^*)Dm^*.
$$

Since the $L^r(Q)$ norms of both $q^{(n+1)}$ and $\text{div}q^{(n+1)}$ are bounded independently of $n$, from Proposition 2.8, assuming $T$ small, we have

$$
\|M^{(n+1)}\|_{W^{2,1}_r(Q)} \leq C\|\text{div}(q^{(n+1)} - q^*)m^* + (q^{(n+1)} - q^*)Dm^*\|_{L^r(Q)}
\leq C|m^*_Q|_Q \|q^{(n+1)} - q^*\|_{L^r(Q)} + \|\text{div}(q^{(n+1)} - q^*)\|_{L^r(Q)}
\leq C(\|q^{(n+1)} - q^*\|_{L^r(Q)} + \|\text{div}(q^{(n+1)} - q^*)\|_{L^r(Q)}).
$$

We now prove the estimate (52) for the HJB equation. The function $U^{(n+1)} = u^{(n+1)} - u^*$ satisfies the equation

$$
-\partial_t U^{(n+1)} - \epsilon \Delta U^{(n+1)} + q^{(n+1)} D U^{(n+1)} = \mathcal{F}(x, t)
$$
with $U^{(n+1)}(\cdot, T) = 0$, where

$$
\mathcal{F}(x, t) = H(m^*, Du^*) - (q^{(n+1)} Du^* - L(m^{(n+1)}, Du^{(n)}, q^{(n+1)}))
= H(m^*, Du^*) - H(m^{(n+1)}, Du^{(n)}) + q^{(n+1)}(Du^{(n)} - Du^*).
$$

Hence, recalling that $q^{(n+1)} = H_\rho(m^{(n)}, Du^{(n)})$ is bounded, again from Proposition 2.7 we have

$$
\|U^{(n+1)}\|_{W^{2,1}_r(Q)} \leq C\|\mathcal{F}\|_{L^r(Q)}.
$$

As we have assumed that conditions for Theorem 3.1 are satisfied, then we have $m^*, m^{(n+1)} \in [\frac{1}{K}, K], |Du^*| \leq K, |Du^{(n)}| \leq K$ and $|q^{(n+1)}| \leq R$. Therefore from (27) we have

$$
H(m^*, Du^*) - H(m^{(n+1)}, Du^{(n)}) \leq C_H\|Du^{(n)} - Du^*\|_{L^\infty(Q)}
+\|m^{(n+1)} - m^*\|_{L^\infty(Q)},
$$
and then
\[ \| \mathcal{F} \|_{L^\prime(Q)} \leq T^\frac{1}{\gamma} \| \mathcal{F} \|_{L^\infty(Q)} \]
\[ \leq T^\frac{1}{\gamma} (\| Du^{(n)} - Du^* \|_{L^\infty(Q)} + m^{(n+1)} - m^* \|_{L^\infty(Q)}) \]
\[ \leq T^\frac{1}{\gamma} (\| u^{(n)} - u^{*1(1)} \|_{Q} + m^{(n+1)} - m^* \|_{Q}) \]
\[ \leq T^\frac{1}{\gamma} \cdot CT^\frac{1}{2} \| u^{(n)} - u^* \|_{W^{2,1}_r(Q)} + m^{(n+1)} - m^* \|_{W^{2,1}_r(Q)} \]

(From Lemma 2.4)
\[ \leq CT^\frac{1}{2} - \frac{d}{\gamma} (\| u^{(n)} - u^* \|_{W^{2,1}_r(Q)} + m^{(n+1)} - m^* \|_{W^{2,1}_r(Q)}) . \]

Then we can get (52) from (55).

\section*{Corollary 4.2}

Denote \( M^{(n)} = m^{(n)} - m^* \), \( U^{(n)} = u^{(n)} - u^* \) for all \( n \geq 1 \). Under the same assumptions as in Theorem 4.1, the following estimate holds for \( n \neq 1 \):

\[ \| U^{(n+1)} \|_{W^{2,1}_r(Q)} + \| M^{(n+1)} \|_{W^{2,1}_r(Q)} \leq CT^\frac{1}{2} - \frac{d}{\gamma} (\| U^{(n)} \|_{W^{2,1}_r(Q)} + \| M^{(n)} \|_{W^{2,1}_r(Q)} + \| U^{(n-1)} \|_{W^{2,1}_r(Q)}) . \]

Moreover, there exist \( \ell > 1 \), \( n_0 \) sufficiently large and \( \hat{T} \in (0, \hat{T}_1) \) sufficiently small such that, for all \( T \in (0, \hat{T}) \), we have a linear rate of convergence, i.e.,

\[ \| U^{(n)} \|_{W^{2,1}_r(Q)} + \| M^{(n)} \|_{W^{2,1}_r(Q)} \leq \left( \frac{1}{\ell} \right) ^{n-n_0} (\| U^{(n_0)} \|_{W^{2,1}_r(Q)} + \| M^{(n_0)} \|_{W^{2,1}_r(Q)}) . \]

\section*{Proof}

Along the proof, the constant \( C \) can change from line to line, but it is always independent of \( n \) and remains bounded for all \( T \in (0, \hat{T}_1) \).

First note that, from

\[ q^{(n+1)} - q^* = H_p(m^{(n)}, Du^{(n)}) - H_p(m^*, Du^*), \]

and (H1) we have

\[ \| q^{(n+1)} - q^* \|_{L^\prime(Q)} \leq CT^\frac{1}{2} (\| u^{(n)} - u^{*1(1)} \|_{Q} + m^{(n)} - m^* \|_{Q}) \]
\[ \leq CT^\frac{1}{2} - \frac{d}{\gamma} (\| U^{(n)} \|_{W^{2,1}_r(Q)} + \| M^{(n)} \|_{W^{2,1}_r(Q)}) . \]

From (53) and (H1) we have

\[ \| \partial_{x_i} q_j^{(n+1)} - \partial_{x_i} q_j^* \|_{L^\prime(Q)} \]
\[ = \| H_{mp_j}(m^{(n)}, Du^{(n)}) \partial_{x_i} m^{(n)} - H_{mp_j}(m^*, Du^*) \partial_{x_i} m^* \|_{L^\prime(Q)} \]
\[ + \| H_{p_j}(m^{(n)}, Du^{(n)}) \partial^2_{x_i x_j} u^{(n)} - H_{p_j}(m^*, Du^*) \partial^2_{x_i x_j} u^* \|_{L^\prime(Q)} \]
\[ \leq CT^{\frac{1}{2}} |m^{(n)} - m^{**}|_{Q} + M_1 \|H_{m_{p_j}}(m^{(n)}, Du^{(n)}) - H_{m_{p_j}}(m^*, Du^*)\|_{L^r(Q)} \\
+ M_1 \|H_{p_{i_j}}(m^{(n)}, Du^{(n)}) - H_{p_{i_j}}(m^*, Du^*)\|_{L^r(Q)} \\
+ C\|\nabla^2_{x_i x_j} u^{(n)} - \nabla^2_{x_i x_j} u^*\|_{L^r(Q)}, \]

and

\[ \|\text{div} q^{(n+1)} - \text{div} q^*\|_{L^r(Q)} \]
\[ \leq C(\|u^{(n)} - u^*\|_{W^{2,1}_{r}} + T^{\frac{1}{2}} |m^{(n)} - m^{**}|_{Q}) \]
\[ \leq C(\|u^{(n)} - u^*\|_{W^{2,1}_{r}} + T^{\frac{1}{2}} - \frac{d}{\ell} \|m^{(n)} - m^*\|_{W^{2,1}_{r}}). \]

By (52) and the fact that \( r > d + 2 \) we have

\[ \|m^{(n+1)} - m^*\|_{W^{2,1}_{r}} \]
\[ \leq C(\|q^{(n+1)} - q^*\|_{L^r(Q)} + \|\text{div}(q^{(n+1)} - q^*)\|_{L^r(Q)}) \]
\[ \leq C(\|u^{(n)} - u^*\|_{W^{2,1}_{r}} + T^{\frac{1}{2}} - \frac{d}{\ell} \|m^{(n)} - m^*\|_{W^{2,1}_{r}}) \]
\[ \leq CT^{\frac{1}{2}} - \frac{d}{\ell} \left( \|u^{(n-1)} - u^*\|_{W^{2,1}_{r}} + \|m^{(n)} - m^*\|_{W^{2,1}_{r}} \right). \]

From the previous estimates, (56) follows.

Since \((U^{(n)}, M^{(n)})\) converges to \((0, 0)\), there exist \( \ell > 1 \) and \( n_0 \) sufficiently large such that

\[ \|U^{(n_0+1)}\|_{W^{2,1}_{r}} + \|M^{(n_0+1)}\|_{W^{2,1}_{r}} \leq \frac{1}{\ell} \left( \|U^{(n_0)}\|_{W^{2,1}_{r}} + \|M^{(n_0)}\|_{W^{2,1}_{r}} \right), \]

By choosing \( T \) so small that \( CT^{\frac{1}{2}} - \frac{d}{\ell} \leq \frac{1}{\ell + \ell^2} \) and \( T \leq \tilde{T}_1 \), by induction, we have for all \( n \geq n_0 \)

\[ \|U^{(n+2)}\|_{W^{2,1}_{r}} + \|M^{(n+2)}\|_{W^{2,1}_{r}} \leq \frac{1}{\ell} \left( \|U^{(n)}\|_{W^{2,1}_{r}} + \|M^{(n)}\|_{W^{2,1}_{r}} \right). \]

**Theorem 4.3** Let \((H1), (H2)\) and \((II)\) be in force and \( \tilde{T}_2 \) and \( R \) be defined as in Theorem 3.6. Then, there exists a constant \( C \), which depends only on the data of problem and remains bounded for all \( T \in (0, \tilde{T}_2] \), such that, if \((u^{(n)}, m^{(n)})\) is the sequence generated by the policy iteration method \((PI2)\), we have

\[ \|m^{(n+1)} - m^*\|_{W^{2,1}_{r}} \leq C(\|q^{(n+1)} - q^*\|_{L^r(Q)} + \|\text{div}(q^{(n+1)} - q^*)\|_{L^r(Q)}), \]

(57)

and
\[ \| \tilde{u}^{(n+1)} - u^* \|_{W^2_1(Q)} \leq C T^{\frac{1}{2} - \frac{d}{d'}} \left( \| \tilde{u}^{(n)} - u^* \|_{W^2_1(Q)} + \| m^{(n+1)} - m^* \|_{W^2_1(Q)} \right). \]

(58)

**Proof** Along the proof, the constant \( C \) can change from line to line, but it is always independent of \( n \) and remains bounded for all \( T \in (0, \hat{T}_2) \).

Again, set \( M^{(n+1)} = m^{(n+1)} - m^* \). We repeat the exact same reasoning as Theorem 4.1 to obtain

\[ \| M^{(n+1)} \|_{W^2_1(Q)} \leq C (\| q^{(n+1)} - q^* \|_{L^r(Q)} + \| \text{div}(q^{(n+1)} - q^*) \|_{L^r(Q)}). \]

The function \( \tilde{u}^{(n+1)} = \tilde{u}^{(n+1)} - u^* \) satisfies the equation

\[-\partial_t \tilde{u}^{(n+1)} - \epsilon \Delta \tilde{u}^{(n+1)} + \tilde{q}^{(n+1)} D\tilde{u}^{n+1} = \tilde{F}(x, t)\]

with \( \tilde{u}^{(n+1)}(x, T) = 0 \), where

\[ \tilde{F}(x, t) = H(m^*, Du^*) - (\tilde{q}^{(n+1)} Du^* - L(m^{(n+1)}), \tilde{q}^{(n+1)}) \]

\[ = H(m^*, Du^*) - H(m^{(n+1)}, D\tilde{u}^{(n)}) + \tilde{q}^{(n+1)} (D\tilde{u}^{(n)} - Du^*). \]

Hence, recalling that \( \tilde{q}^{(n+1)} = H_p(m^{(n+1)}, D\tilde{u}^{(n)}) \) is bounded, again from Proposition 2.7 we have

\[ \| \tilde{u}^{(n+1)} \|_{W^2_1(Q)} \leq C \| \tilde{F} \|_{L^r(Q)}, \]

(59)

and (following the same arguments as in Theorem 4.1)

\[ \| \tilde{F}(x, t) \|_{L^r(Q)} \leq C T^{\frac{1}{2} - \frac{d}{d'}} \left( \| \tilde{u}^{(n)} - u^* \|_{W^2_1(Q)} + \| m^{(n+1)} - m^* \|_{W^2_1(Q)} \right). \]

Then we can get (58) from (59).

\( \square \)

**Corollary 4.4** Denote \( M^{(n)} = m^{(n)} - m^* \), \( \tilde{u}^{(n)} = \tilde{u}^{(n)} - u^* \) for all \( n \geq 1 \). Under the same assumptions of Theorem 4.3, the following estimate holds

\[ \| \tilde{u}^{(n+1)} \|_{W^2_1(Q)} + \| M^{(n+1)} \|_{W^2_1(Q)} \]

\[ \leq C T^{\frac{1}{2} - \frac{d}{d'}} \left( \| \tilde{u}^{(n)} \|_{W^2_1(Q)} + \| M^{(n)} \|_{W^2_1(Q)} + \| \tilde{u}^{(n-1)} \|_{W^2_1(Q)} \right). \]

(60)

Moreover, there exists \( \ell > 1, n_0 \) sufficiently large and \( \hat{T} \in (0, \hat{T}_2) \) sufficiently small such that, for all \( T \in (0, \hat{T}) \), we have a linear rate of convergence

\[ \| \tilde{u}^{(n)} \|_{W^2_1(Q)} + \| M^{(n)} \|_{W^2_1(Q)} \leq \left( \frac{1}{\ell} \right)^{n-n_0} \left( \| \tilde{u}^{(n_0)} \|_{W^2_1(Q)} + \| M^{(n_0)} \|_{W^2_1(Q)} \right). \]

(61)
Proof Along the proof, the constant $C$ can change from line to line, but it is always independent of $n$ and remains bounded for all $T \in (0, \hat{T}_2]$. From the results of Theorem 4.3 we have

$$
\|\text{div} q^{(n+1)} - \text{div} q^*\|_{L^r(Q)} \\
\leq C(\|\tilde{u}^{(n)} - u^*\|_{W^{2,1}_r(Q)} + T^{\frac{1}{2} - \frac{d}{2r}} \|m^{(n)} - m^*\|_{W^{2,1}_r(Q)}),
$$

and

$$
\|m^{(n+1)} - m^*\|_{W^{2,1}_r(Q)} \\
\leq (\|q^{(n+1)} - q^*\|_{L^r(Q)} + \|\text{div}(q^{(n+1)} - q^*)\|_{L^r(Q)}) \\
\leq C(\|\tilde{u}^{(n)} - u^*\|_{W^{2,1}_r(Q)} + T^{\frac{1}{2} - \frac{d}{2r}} \|m^{(n)} - m^*\|_{W^{2,1}_r(Q)}) \\
\leq CT^{\frac{1}{2} - \frac{d}{2r}}(\|\tilde{u}^{(n-1)} - u^*\|_{W^{2,1}_r(Q)} + \|m^{(n)} - m^*\|_{W^{2,1}_r(Q)}).
$$

From the previous estimates, it follows (61).

Since $(\tilde{u}^{(n)}, M^{(n)})$ converges to $(0, 0)$, there exists a $\ell > 1, n_0$ sufficiently large such that

$$
\|\tilde{u}^{(n_0+1)}\|_{W^{2,1}_r(Q)} + \|M^{(n_0+1)}\|_{W^{2,1}_r(Q)} \leq \frac{1}{\ell} \left( \|\tilde{u}^{(n_0)}\|_{W^{2,1}_r(Q)} + \|M^{(n_0)}\|_{W^{2,1}_r(Q)} \right),
$$

by choosing $T$ so small that $CT^{\frac{1}{2} - \frac{d}{2r}} \leq \frac{1}{\ell + \ell^2}$ and $T \leq \hat{T}_2$, by induction, we have for all $n \geq n_0$

$$
\|\tilde{u}^{(n+2)}\|_{W^{2,1}_r(Q)} + \|M^{(n+2)}\|_{W^{2,1}_r(Q)} \leq \left( \frac{1}{\ell} \right)^2 \left( \|\tilde{u}^{(n)}\|_{W^{2,1}_r(Q)} + \|M^{(n)}\|_{W^{2,1}_r(Q)} \right).
$$

\[\square\]

5 Numerical Simulations

In this section, we illustrate with numerical examples the two policy iteration methods analyzed in the previous sections. To this end we rely on a finite difference scheme introduced and analyzed in [1, 2] for MFG PDE systems. We consider the following examples.

Example 1 We first consider the following one dimensional example in which the agents are encouraged to move towards one of two possible targets. They are penalized at the terminal time based on the distance to the nearest target. This is reminiscent of the min-LQG MFG of [52], except that we do not consider mean field interactions that encourage the agents to follow the mean position of the population. Instead, the dynamics of a typical agent are subject to congestion effects in the spirit of [10]: it requires more effort to move in a crowded region than in a non-crowded region. To be consistent with the above theoretical analysis, we consider that the domain is the
one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The two targets are located at 0.3 and 0.7. The Hamiltonian is:

$$H(m, Du) = \sup_{q \in \mathbb{R}^d} \left\{ q \cdot Du - \frac{1}{2} (1 + 4m) \beta |q|^2 - \zeta m \right\}$$

$$= \frac{1}{2(1 + 4m) \beta} |Du|^2 - \zeta m .$$

Here, the argmax is given by $q^*(x, t) = \frac{Du(x, t)}{(1 + 4m(x, t)) \beta}$ in $Q$, and $\beta, \zeta$ are positive constants. The last term corresponds to a crowd aversion cost which discourages the agents from being in a very crowded region (independently of whether they move or not). We take a uniform distribution over $[0.375, 0.625]$ for $m_0$.

The corresponding MFG PDE system is:

$$
\begin{cases}
- \partial_t u - 0.05 \Delta u + \frac{1}{2(1+4m)^\beta} |Du|^2 - \zeta m = 0 & \text{in } Q \\
\partial_t m - 0.05 \Delta m - \text{div} \left( \frac{m Du}{(1 + 4m)^\beta} \right) = 0 & \text{in } Q \\
u_T(x) = 10 \min\{(x - 0.3)^2, (x - 0.7)^2\} & \text{in } \mathbb{T} \\
m_0(x) = 4 \text{ for } x \in [0.375, 0.625], \, m_0(x) = 0 \text{ otherwise} .
\end{cases}
$$

We implement the two policy iteration methods on a finite-difference approximation of the above PDE system. We fix a grid $G$ on $\mathbb{T}^d$. Then, we denote by $U, M$ and $Q$ the vectors on $G$ approximating respectively the solution and the policy. We will use the symbol $\#\neq$ to denote suitable discretizations of the linear differential operators at the grid nodes. Here we use uniform grids and the centered second order finite differences for the discrete Laplacian, whereas the Hamiltonian and the divergence term in the FP equation are both computed via the Engquist-Osher numerical flux for conservation laws as in [24]. To be more precise, in the present example which is in dimension $d = 1$, we consider a uniform discretization of $\mathbb{T}^d$ with $I$ nodes $x_i = i \, h$, for $i = 0, \ldots, I - 1$, where $h = 1/I$ is the space step. We then introduce the discrete operators:

$$
(\Delta^\# u)_i = \frac{1}{h^2} \left( U_{[i-1]} - 2U_i + U_{[i+1]} \right) ,
$$

$$
(D^\# u)_i = (D_L U_i, \, D_R U_i) = \frac{1}{h} \left( U_i - U_{[i-1]}, \, U_{[i+1]} - U_i \right) ,
$$

where the index operator $[-] = \{(\cdot + I) \mod I\}$ accounts for the periodic boundary conditions. When updating the policy, we have

$$Q_t = (Q_{t,L}, \, Q_{t,R}) = \frac{1}{(1 + 4M_t)^\beta} (D_L U_t, \, D_R U_t) .$$
Using the notation $(\cdot)^+ = \max \{\cdot, 0\}$ and $(\cdot)^- = \min \{\cdot, 0\}$ for the positive and negative part respectively, we denote $Q_{\pm} = (Q^{+}_{L}, Q^{-}_{R})$, and we have

$$
(|Q_{\pm}|^2)_i = \left(Q^{+}_{i,L}\right)^2 + \left(Q^{-}_{i,R}\right)^2.
$$

The discrete divergence operator is such that:

$$
\left(\text{div}_x(M \cdot Q)\right)_i = \frac{1}{h} \left(M_{i+1}^n Q_{i+1,n,L}^+ - M_i^n Q_{i,L}^+\right) + \frac{1}{h} \left(M_i^n Q_{i,R}^- - M_{i-1}^n Q_{i-1,n,R}^-\right).
$$

For the time discretization, we employ an implicit Euler method for both the time-forward FP equation and the time-backward HJB equation. To this end, we introduce a uniform grid on the interval $[0, T]$ with $N + 1$ nodes $t_n = n \Delta t$, for $n = 0, \ldots, N$, and time step $\Delta t = T/N$. Then, we denote by $U_n, M_n$ and $Q_n$ the vectors on $G$ approximating respectively the solution and the policy at time $t_n$. In particular, we set on $G$ the initial condition $M_{0,i} = m_0(x_i)/\sum_j h m_0(x_j)$ and the final condition $U_N = u_T(\cdot)$.

The policy iteration algorithm (PI1) for the fully discretized system is the following: Given an initial guess $Q_n^{(0)} : G \rightarrow \mathbb{R}^{2d}$ for $n = 0, \ldots, N - 1$, initial and final data $M_0, U_N : G \rightarrow \mathbb{R}$, iterate on $k \geq 0$:

(i) Solve on $G$

$$
\begin{cases}
M_{n+1}^{(k)} - \Delta t \left(0.05 \Delta_x M_{n+1}^{(k)} + \text{div}_x(M_{n+1}^{(k)} Q_n^{(k)})\right) = M_n^{(k)}, & n = 0, \ldots, N - 1 \\
M_0^{(k)} = M_0
\end{cases}
$$

(ii) Solve on $G$

$$
\begin{cases}
U_n^{(k)} - \Delta t \left(0.05 \Delta_x U_n^{(k)} - Q_{n,\pm}^{(k)} \cdot D_x U_n^{(k)}\right) \\
\quad = U_{n+1}^{(k)} + \Delta t \left(\frac{1}{2}(1 + 4M_{n+1}^{(k)})^\beta |Q_{n,\pm}^{(k)}|^2 + \xi M_n^{(k)}\right), & n = 0, \ldots, N - 1 \\
U_N^{(k)} = U_N
\end{cases}
$$

(iii) Update the policy $Q_n^{(k+1)} = \frac{D_x U_n^{(k)}}{(1+4M_n^{(k)})^\beta}$ on $G$ for $n = 0, \ldots, N - 1$.

Consistently with our theoretical convergence result, in the implementation we do not put any bound on the control. For the following results, we take $\beta = 1.5, \xi = 1$ and $T = 1$ for the final horizon. We used a number of nodes $I = 200$ in space and $N = 200$ in time. The initial policy was set to $Q_n^{(0)} = (0, 0)$ on $G$ for all $n$. Here we present results for (PI1).

In Fig. 1, we report the time evolution of the density, by plotting, for several fixed $n$, the solution density $M_n$ and the policy $Q_n = Q^{+}_{n,L} + Q^{-}_{n,R}$. We can see that the distribution splits into two parts, one moving towards the left target and one moving...
towards the right target. However, due to the congestion cost as well as the crowd aversion cost, each part can not concentrate exactly on the target.

In Fig. 2, we report results on the convergence with respect to the number of iterations: the residuals of the discrete MFG system, as well as the discrete $L^\infty$ distance between $Q^{(k)}$, $M^{(k)}$ and $U^{(k)}$ computed by the policy iteration and the final solution $Q^*$, $M^*$ and $U^*$ from the fixed point iteration algorithm. Here we use the fixed point iteration method to obtain a benchmark solution. This algorithm has been previously used for solving mean field games with non-separable Hamiltonians or mean field type control problems e.g. in [6]. The details are provided below. Here we observe that the solution via our policy iteration method is consistent with the benchmark solution. Moreover, consistently with our theoretical findings (see Sect. 4), the convergence rate is linear, except for the first few iterations and after a lower bound is reached due to the limitations on the approximation accuracy of the discrete system.

We now give the details of the fixed point iteration algorithm for solving the discrete MFG system with a non-separable Hamiltonian. It iterates over the distribution and the value function. The main difference with the policy iteration method (PI1) is that at

---

**Fig. 1** Example 1. Solution for the MFG system (63) obtained with policy iteration (PI1). a The density $M$ and b the policy $Q = Q_L^+ + Q_R^-$ at several time steps.

**Fig. 2** Example 1. Convergence of policy iteration (PI1) for the MFG system (63). a Residuals of MFG system, b $L^\infty$ distance between $Q^{(k)}$, $M^{(k)}$ and $U^{(k)}$ from policy iteration and the final solution $Q^*$, $M^*$ and $U^*$ from fixed point algorithm.
each iteration, we solve an HJB equation instead of solving a linear equation with a
given control. As explained below, for this step we rely on Newton method.

The main iteration of the fixed point method is the following outer loop: Given
an initial guess \( M_n^{(0)} : \mathcal{G} \to \mathbb{R}^{2d} \) for \( n = 1, \cdots, N \), and \( U_n^{(0)} : \mathcal{G} \to \mathbb{R}^{2d} \) for
\( n = 0, \ldots, N - 1 \), iterate on \( k \geq 1 \) up to convergence,

(i) Solve on \( \mathcal{G} \):

\[
\begin{align*}
M_n^{(k)} &= M_0 \\
\frac{M_{n+1}^{(k)} - M_n^{(k)}}{\Delta t} - 0.05 \Delta_x M_n^{(k)} - \text{div}_x(M_n^{(k)} DU_{n}^{(k-1)} \frac{1}{(1 + 4M_n^{(k-1)})^\beta}) &= 0,
\end{align*}
\]

\( n = 0, \ldots, N - 1 \)

(ii) Solve on \( \mathcal{G} \):

\[
\begin{align*}
U_n^{(k)} &= U_N \\
\frac{U_{n+1}^{(k)} - U_n^{(k)}}{\Delta t} - 0.05 \Delta_x U_n^{(k)} + \frac{|D_x U_n^{(k)}|^2}{2(1 + 4M_n^{(k)})^\beta} - \zeta M_n^{(k)} &= 0,
\end{align*}
\]

for \( n = N - 1, \ldots, 0 \).

We use a forward time marching method and backward time marching method
respectively for step (i) and (ii). In step (ii) we need to solve a nonlinear system for
every time step \( n \). We do this by Newton method, which consists in the following inner
loop. For given \( n \) and \( k \), set initial guess \( \tilde{U}_n^{(\tilde{k} = 0)} = U_n^{(k - 1)} \), and then iterate on \( \tilde{k} \geq 0 \):

(iii) Compute the residual of HJB system:

\[
\mathcal{F}_n^{(k)}(\tilde{U}_n^{(k)}) = \frac{U_{n+1}^{(k)} - \tilde{U}_n^{(k)}}{\Delta t} - 0.05 \Delta_x \tilde{U}_n^{(k)} + \frac{|D_x \tilde{U}_n^{(k)}|^2}{2(1 + 4M_n^{(k)})^\beta} - \zeta M_n^{(k)},
\]

(ii) Compute the Jacobian matrix:

\[
\mathcal{J}_n^{(k)}(\tilde{U}_n^{(k)}) = \frac{1}{\Delta t} I - 0.05 \Delta_x + \frac{D_x \tilde{U}_n^{(k)}}{(1 + 4M_n^{(k)})^\beta} \cdot D_x,
\]

(iii) Update: \( \tilde{U}_n^{(\tilde{k} + 1)} = \tilde{U}_n^{(\tilde{k})} + (\mathcal{J}_n^{(\tilde{k})})^{-1}(-\mathcal{F}_n^{(\tilde{k})}) \).

For step (iii), instead of computing the inverse of the Jacobian matrix \( \mathcal{J}_n^{(\tilde{k})} \), an
alternative method is to first solve a linear system to find \( (\tilde{U}_n^{(\tilde{k} + 1)} - \tilde{U}_n^{(\tilde{k})}) \) and then
deduce \( \tilde{U}_n^{(\tilde{k})} \) from here.

The aforementioned benchmark solution \( Q^*, M^* \) and \( U^* \) is obtained by running
the fixed point method until convergence (up to numerical approximations).

We perform some numerical experiments on the maximal time horizon \( T \) with
which the algorithm converges. The results are listed in Table 1 with different values
of \( \beta \) and \( \zeta \), without changing the step size \( \Delta t \), \( \Delta x \) or any other parameters. In some
cases the algorithm converges even when $T = 50$. The fact that convergence depends heavily on the value of the constant $\zeta$ is consistent with the theoretical findings of [25].

**Example 2** We now give an example in dimension $d = 2$ in which the domain is $\mathbb{T}^2$. The running cost represents congestion effects, but in this example the Hamiltonian $H(m, Du)$ is singular at $m = 0$. There is a terminal cost that encourages the agents to move towards some sub-regions of the domain. The initial distribution is a truncated Gaussian distribution centered around $(0.25, 0.25)$. The MFG PDE system is:

\[
\begin{cases}
-\partial_t u - 0.3 \Delta u + \frac{1}{2m_n \pi^2} |Du|^2 = 0 & \text{in } Q \\
\partial_t m - 0.3 \Delta m - \text{div}(\frac{m_D}{m_n \pi^2}) = 0 & \text{in } Q \\
u_T(x_1, x_2) = 1.2 \cos(2\pi x_1) + \cos(2\pi x_2) & \text{in } \mathbb{T}^2 \\
m_0(x_1, x_2) = C \exp[-10((x_1 - 0.25)^2 + (x_2 - 0.25)^2)] & \text{in } \mathbb{T}^2,
\end{cases}
\]

(64)

where $C$ is a constant such that $\int_{\mathbb{T}^2} m_0(x) dx = 1$. We set the terminal time $T = 0.5$. The finite-difference scheme described above can be adapted to this two-dimensional example in a straightforward way. See e.g., [1, 2] for more details. For the numerical results provide below, we used $I = 50$ nodes in each space dimension and $N = 50$ nodes in time.

We compare the two policy iteration methods that we proposed, namely (PI1) and (PI2). For both methods, we used the initial policy $Q_n^{(0)} \equiv (0, 0, 0, 0)$ on $G$ for $n = 0, \ldots, N - 1$. In Fig. 3, we give residuals of MFG system and the discrete $L^\infty$ distance between $Q^{(k)}$, $M^{(k)}$ and $U^{(k)}$ at each iteration and the final solution $Q^*$, $M^*$ and $U^*$ from the fixed point iteration algorithm. We see that it takes about 37 iterations with (PI1) to decrease $\max_{n,i,j} |M_{n,i,j}^{(k+1)} - M_{n,i,j}^{(k)}|$ to $10^{-8}$ whereas it takes only 29 iterations with algorithm (PI2). In Fig. 4, we report the contours of density $M_n$ at different time $t$ with the algorithm (PI2). Since both methods yield similar results, we omit the contours of $M_n$ obtained with the algorithm (PI2).

Using the same setting as for the policy iteration algorithm, and the initial guess $U_n^{(0)} \equiv 0$, $M_n^{(0)} \equiv 1$, the fixed point algorithm converges with 27 (outer) iterations. In Fig. 5, we report the residual of discrete MFG system with the fixed point iteration and (PI2). For the latter, we count the number of outer iterations. In the fixed point iteration algorithm, at iteration $k$ we have a fixed $M^{(k)}$ and solve the HJB equation using Newton method, hence the HJB residual is very small. We see that the residual for the FP equation is slightly smaller than the one obtained with (PI2), but it roughly decays.
at the same rate. However, remember that here we are comparing iterations of (P12) with outer iterations of the fixed point method, but each iteration of the latter involves an inner loop for the Newton method. Furthermore there is no clear way to parallelize this inner loop. As a consequence, the fixed point method is overall more expensive from a computational viewpoint. For the sake of illustration, we provide in Table 2 computational times obtained with each method on a computer with Intel(R) Xeon(R) processor running at 2.20 GHz. Note that after 60 iterations, the fixed point method has basically converged and hence the inner loop with Newton method converges much faster than during the first iterations because the initial guess for the non-linear HJB equation is already quite correct.

Example 3 We conclude with a variant of Example 2, where we take a super-quadratic nonlinearity for the gradient term in the Hamiltonian. Note that our theoretical results also apply to this setting. We take the following Hamiltonian:

\[
H(m, Du) = \max_q \left\{ q Du - \frac{2}{3} m^{1/4} |q|^3 \right\} = \frac{|Du|^3}{3m^{1/2}}.
\]
Fig. 4 Example 2. Solution obtained by (PI2) for the 2d MFG system (64) : contours of density at several time steps.

Table 2 Policy iteration (PI2) vs Fixed Point iteration total CPU times with different number of iterations

| Iterations | (PI2) Total CPU (s) | Fixed Point Total CPU (s) |
|------------|---------------------|--------------------------|
| 10         | 18.82               | 33.51                    |
| 20         | 37.23               | 55.42                    |
| 30         | 56.59               | 77.30                    |
| 60         | 114.43              | 132.41                   |

where the maximizer is: $q(x, t) = \frac{|Du|}{m^{1/2}} Du$ in $Q$. The corresponding PDE system is:

$$
\begin{align*}
-\partial_t u - 0.3\Delta u + \frac{1}{3m^{1/2}} |Du|^3 &= 0 & \text{in } Q \\
\partial_t m - 0.3\Delta m - \text{div}(\frac{mDu|Du|}{m^{1/2}}) &= 0 & \text{in } Q \\
u_T(x_1, x_2) &= 1.2 \cos(2\pi x_1) + \cos(2\pi x_2) & \text{in } \mathbb{T}^2 \\
m_0(x_1, x_2) &= C \exp\{-10[(x_1 - 0.25)^2 + (x_2 - 0.25)^2]\} & \text{in } \mathbb{T}^2
\end{align*}
$$

(65)
Fig. 5  Example 2.  
(a) the residual of HJB equation,  
(b) the residual of FP equation

Fig. 6  Example 3. Solution obtained by \((\text{PI1})\) for the 2d MFG system (65): contours of density at several time steps

\begin{align*}
t & = 0 \\
t & = 0.16 \\
t & = 0.33 \\
t & = 0.5
\end{align*}
Using the same setting as in Example 2, the policy iteration algorithm (PI1) with 46 iterations leads to \( \max_{n,i,j} |M_{n,i,j}^{(k+1)} - M_{n,i,j}^{(k)}| \) smaller than \( 10^{-8} \). The contours of density \( M_{n} \) at different time \( t_{n} \) are displayed in Fig. 6, which is to be compared with Fig. 4. We see that in the present example, the mass is much more concentrated at the terminal time. This can be explained by the fact that a super-quadratic Hamiltonian corresponds to a lower congestion cost. Hence the agents can move faster and get closer to a desired position.

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