Graph isomorphism and volumes of convex bodies

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Abstract

We show that a nontrivial graph isomorphism problem of two undirected graphs, and more generally, the permutation similarity of two given $n \times n$ matrices, is equivalent to equalities of volumes of the induced three convex bounded polytopes intersected with a given sequence of balls, centered at the origin with radii $t_i \in (0, \sqrt{n-1})$, where $\{t_i\}$ is an increasing sequence converging to $\sqrt{n-1}$. These polytopes are characterized by $n^2$ inequalities in at most $n^2$ variables. The existence of fpras for computing volumes of convex bodies gives rise to a semi-fpras of order $O^*(n^{14})$ at most to find if given two undirected graphs are isomorphic.

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1 Introduction

Let $G_1 = (V, E_1), G_2 = (V, E_2)$ be two simple undirected graphs, where $V$ is the set of vertices of cardinality $n$ and $E_1, E_2 \subset V \times V$ the set of edges. $G_1$ and $G_2$ are called isomorphic if there exists a bijection $\sigma : V \to V$ which induces the corresponding bijection $\tilde{\sigma} : E_1 \to E_2$. The graph isomorphism problem, abbreviated here as $GIP$, is the computational complexity of determination if $G_1$ and $G_2$ are isomorphic. Clearly the $GIP$ in the class $NP$. It is one of a very small number of problems whose complexity is unknown [8] [10]. For certain graphs it was known that the complexity of $GIP$ is polynomial [1] [2] [5] [9] [17] [18].
The graph isomorphism problem is a special case of permutational similarity of two \( n \times n \) real values matrices \( A, B \in \mathbb{R}^{n \times n} \), or more generally \( n \times n \) matrices with entries in any ring with identity. Namely, let \( \mathcal{P}_n \subset \mathbb{R}^{n \times n} \) be the group of permutation matrices. Does there exists \( P \in \mathcal{P}_n \) such that \( B = P A P^\top \)?

Using the notion of coherent algebras, as a tool to identify nontrivial pairs of matrices \( A, B \in \mathbb{R}^{n \times n} \) which may be permutationally similar [7], we first show that a nontrivial permutational similarity of \( A, B \in \mathbb{R}^{n \times n} \) can be polynomially reduced to an isomorphism problem of two regular undirected connected multi-graphs with the same degree, (self-loops allowed), and the same characteristic polynomial. Assume that these two graphs represented by symmetric \( A, B \in S_n(\mathbb{Z}_+) \) its rows sum equal to \( N \).

Denote by \( 1 = (1, \ldots, 1)^\top \), and let \( M_n \subset \mathbb{R}^{n \times n} \) be the space of matrices with zero row and column sum:

\[
X 1 = X^\top 1 = 0, \quad X = [x_{ij}] \in \mathbb{R}^{n \times n}.
\]  

Note that \( M_n \) is \((n-1)^2\) dimensional subspace of \( \mathbb{R}^{n \times n} \). Denote by \( \Omega_n \subset \mathbb{R}^{n \times n} \) the convex set of doubly stochastic matrices. Note that \( Y \in \Omega_n \) if and only if \( Y = X + J_n, J_n := \frac{1}{n} 11^\top \), where \( X \in M_n \) and each entry of \( X \) satisfies \( t \geq -\frac{1}{n} \).

It is easy to see that \( \|X\|_F := \sqrt{\text{tr}X X^\top} \leq \sqrt{n-1} \) if \( -J_n \leq X \in M_n \), and equality holds if and only if \( X = P - J_n \) for some \( P \in \mathcal{P}_n \).

For any \( S, T \in \mathbb{R}^{n \times n} \) we define the following subspace of matrices and a corresponding bounded polytope:

\[
P_0(S,T) := \{X \in M_n, \ SX - XT = 0\}, \quad P(S,T) := \{X \in P_0(S,T), \ X \geq -J_n\}.
\]  

Denote by \( \|P(S,T)\|_F := \max_{X \in P(S,T)} \|X\|_F \), the radius of \( P(S,T) \). Thus two regular multi undirected graphs \( G, H \), with the the same number of vertices and edges, are isomorphic if and only if \( \|P(A,B)\|_F = \sqrt{n-1} \), where \( A, B \) are the representation matrices of \( G, H \) respectively. However, it is known that finding the radius of a convex set is \( NP \)-hard [15]. (Note that the diameter of a balanced convex set \( K \), i.e. \(-K = K\), is twice its radius.)

The main result of this paper is

**Theorem 1.1** Let \( G, H \) be two regular multi undirected graphs, with the the same number of vertices \( n \) and edges \( e \). Denote by \( A, B \in \mathbb{Z}_+^{n \times n} \) the representation matrices of \( G, H \) respectively. The the following statements are equivalent.

1. \( G \) and \( H \) are isomorphic.

2. The dimension of the convex sets \( P(A,A), P(A,B), P(B,B) \) are equal. Furthermore, for each \( t \in (0, \sqrt{n-1}] \) the volumes of the intersection of the above three polytopes with the ball of radius \( t \) centered at \( 0 \) are the equal.
3. The dimension of the convex sets $P(A, A)$, $P(A, B)$, $P(B, B)$ are equal. Furthermore, for a given sequence of balls $t_i \in (0, \sqrt{n-1})$, where $\{t_i\}$ is an increasing sequence converging to $\sqrt{n-1}$, the volumes of the intersection of the above three polytopes with each ball of radius $t_i$ centered at $0$ are the equal.

The main argument of the proof of this is theorem follows straightforward from the observation that $I_n - J_n \in P(A, A)$, i.e. $\|P(A, A)\|_F = \sqrt{n-1}$.

We show that for the convex sets $P(A, A)$, $P(A, B)$, $P(B, B)$ one can apply the known results, which give fully randomized polynomial approximation scheme for computing the volumes of the intersection of these set with a ball of radius $t \in (\frac{1}{n}, \sqrt{n-1})$, e.g. [4, 15, 11, 16]. Combining these results we obtain some algorithms for testing the volume conditions given by Theorem 1.1. Recall that the problem of finding the exact volume of a polytope in $\mathbb{R}^m$, given by a polynomial number of affine inequalities in $m$, is $\#P$-hard [3]. We hope that this approach will lead to a fpras to determine if given two graphs are isomorphic.

We now summarize briefly the contents of this paper. In §2 we discuss the notion of coherent algebras and their relations to the graph isomorphism problem. In §3 we construct the polytopes $P(A, A)$, $P(A, B)$, $P(B, B)$ which are intimately related to a permutational similarity of $A, B$, assumed to be scaled doubly stochastic with the same row sums. We give an outline of the proof of Theorem 1.1. In §4 we outline a semi-fpras to find if given $A, B$ are permutationally similar, which is based on the fpras for computing volume of convex sets.

## 2 Coherent algebras

A subalgebra $C \subseteq \mathbb{R}^{n \times n}$ is called a coherent algebra, if it is closed under the transposition and entry-wise multiplication of two matrices, and contains $I_n = [\delta_{ij}], J_n = [\frac{1}{n}]$, the identity matrix and the doubly stochastic matrix with equal entries. (Denote by $A \circ B = [a_{ij}] \circ [b_{ij}]$ the entry-wise product $[a_{ij}b_{ij}]$.) We now briefly survey the main properties of coherent algebra used her. Our main source is our paper [7]. Additional references for the properties of coherent algebras cited explicitly, where needed.

A trivial coherent algebra is an algebra of dimension 2 spanned by $I, J$. Coherent algebras of dimension 3 are induced either by strongly regular graphs, or by Hadamard matrices. A coherent algebra $C$ has a canonical basis consisting of $(0, 1)$ matrices $E_1, \ldots, E_d$. Each $E_i$ is either symmetric or asymmetric, i.e. $E_i \circ E_i^\top = 0$, and balanced, i.e. the nonzero rows and columns of $E_i$ are equal to $r_i$ and $c_i$ respectively. Furthermore, $E_i \circ E_j = 0$ for $i \neq j$, and $\sum_{i=1}^d E_i = 11^\top$. $C$ is characterized by the tensor $T(C) = [t_{i,j,k}] \in \mathbb{Z}^{d \times d \times d}$.

$$E_iE_j = \sum_{k=1}^d t_{i,j,k}E_k.$$  \hfill (2.1)
Any \( A \in \mathbb{R}^{n \times n} \) induces the minimal coherent subalgebra \( \mathcal{C}(A) \subset \mathbb{R}^{n \times n} \) containing \( A \). One finds in polynomial time the canonical basis \( E_1, \ldots, E_d \in \{0, 1\}^{n \times n} \) of \( \mathcal{C}(A) \). ([7] Lemma 3.1 yields that one needs at most \( 17n^{10} \) flops.) If \( B \in \mathbb{R}^{n \times n} \) is permutationally similar to \( A \), it follows that \( \mathcal{C}(B) \) is strongly isomorphic to \( \mathcal{C}(A) \). So \( \mathcal{C}(B) \) has the canonical basis \( F_1, \ldots, F_d \), such that \( F_i = PE_iP^\top \), \( i = 1, \ldots, d \) for a corresponding \( P \in \mathcal{P} \). These equalities induces the strong isomorphism \( \iota: \mathcal{C}(A) \to \mathcal{C}(B) \) given by \( \iota(E_i) = F_i, i = 1, \ldots, d \).

Thus for \( A, B \in \mathbb{R}^{n \times n} \) to be permutationally similar we must have an isomorphism \( \iota: \mathcal{C}(A) \to \mathcal{C}(B) \), such that \( \iota(E_i) = F_i, i = 1, \ldots, n \). (\( \iota \) is an isomorphism of two algebras, which preserve the transposition and entry-wise multiplication, i.e. \( \iota(U^\top) = \iota(U)^\top, \iota(U \circ V) = \iota(U) \circ \iota(V) \).) In particular, \( T(\mathcal{C}(A)) = T(\mathcal{C}(B)) \), and this condition is essentially equivalent to isomorphism of \( \mathcal{C}(A) \) and \( \mathcal{C}(B) \).

The existence or nonexistence of such isomorphism is determined in a polynomial time in \( O(n^{10}) \). It is possible that the isomorphism of coherent algebras does not imply the strong isomorphism.

**Theorem 2.1** Let \( A, B \in \mathbb{R}^{n \times n} \). Assume that the coherent algebras \( \mathcal{C}(A) \) and \( \mathcal{C}(B) \) are isomorphic, i.e. \( \iota: \mathcal{C}(A) \to (B) \) is an isomorphism of coherent algebras. Then there exists \( A_1 \in \mathcal{C}(A), B_1 \in \mathcal{C}(B) \) with the following properties: \( \iota(A_1) = \iota(B_1) \); \( A_1 \) and \( B_1 \) are symmetric matrices with positive integer entries whose values are less than \( n^3 \); each row sum of \( A_1 \) and \( B_1 \) is equal to \( n \); \( A_1 \) and \( B_1 \) have the same characteristic polynomial. Furthermore, \( A \) and \( B \) are permutationally similar if and only if \( A_1 \) and \( B_1 \) are permutationally similar.

**Outline of Proof.** Assume \( E_1, \ldots, E_h, h \geq 1 \) are all the diagonal matrices in the canonical basis \( E_1, \ldots, E_d \). Consider \( A_2 := \sum_{i=h+1}^{d} m_i E_i \), where \( m_2, \ldots, m_d \in \mathbb{N} \) satisfy the condition \( m_i \neq m_j \) unless \( E_i^\top = E_j \). (If \( E_i^\top = E_j \) we let \( m_i = m_j \).) The number of distinct integers in \( \{m_{h+1}, \ldots, m_d\} \) is \( p \leq \frac{n(n-1)}{2} \). Hence we can assume that the set of distinct integers in \( \{m_{h+1}, \ldots, m_d\} \) is \( \{1, \ldots, p\} \). Let \( N - 1 \) be the maximal row sum of \( A_2 \). Then \( A_1 = A + D \), where \( D \) is the diagonal matrix such that each row of \( A_2 \) equals to \( N \). The results of [7] imply that \( A_1 \in \mathcal{C}(A) \). Set \( B_1 = \iota(A_1) \). Then all other claims of the theorem follow straightforward from the results in [7]. \( \square \)

Note that \( A_1, B_1 \) are representation matrices of two regular undirected multi-graphs \( G, H \) with self loops, with the same numbers of vertices, edges, and the same characteristic polynomials. Furthermore, \( G \) and \( H \) are connected. In the rest of the paper we assume that \( A = A_1, B = B_1 \).

It is possible to show, that by increasing the entries of \( A_1 \), which are still are of order \( O(n^K) \), that in addition to the above conditions on \( A_1, A_1 \) is generic in \( \mathcal{C}(A) \). That is, the multiplicity of each eigenvalue of \( A_1 \) is the minimal possible for any symmetric matrix \( S \in \mathcal{C}(A) \).

More generally, any set of \( A_1, \ldots, A_k \in \mathbb{R}^{n \times n} \) induces a minimal coherent subalgebra \( \mathcal{C}(A_1, \ldots, A_k) \subset \mathbb{R}^{n \times n} \) which contains these matrices. It is obtained
by the following process. First, express each matrix $A_i$ as a unique linear combination of $(0, 1)$ matrices with pairwise distinct coordinates. This gives rise to $T_1, \ldots, T_k \in \{0, 1\}^{n \times n}$. (At the first step $T_1 = I_n, T_2 = 11^\top$.) By considering the subspace spanned by all nonzero $(0, 1)$ matrices of the form $T_i \circ T_j, T_i^\top \circ T_j, T_i^\top \circ T_j^\top, i, j = 1, \ldots, k$ we obtain a subspace $U_1 \subset \mathbb{R}^{n \times n}$ spanned by $(0, 1)$ matrices $R_1, \ldots, R_l$ with disjoint support such that their sum is equal to $11^\top$. We now consider the set of matrices $R_i R_j, i, j = 1, \ldots, l$, whose span $U_2$ includes $R_1, \ldots, R_l$. Apply the previous algorithm to these $l^2$ matrices to obtain a $(0, 1)$ basis in $U_2$ which is a refinement of the basis $R_1, \ldots, R_l$. After $p \leq n^2$ steps we will have that $U_p = U_{p+1}$. Then $C(A_1, \ldots, A_k) = U_p$.

We say that the set $A_1, \ldots, A_k \in \mathbb{R}^{n \times n}$ is permutationally similar to $B_1, \ldots, B_k \in \mathbb{R}^{n \times n}$ if $B_i = P A_i P^\top$ for $i = 1, \ldots, k$ and some $P \in \mathcal{P}_n$. As in the case $k = 1$ the nontrivial permutational similarity induces an isomorphism of the coherent algebras $\iota : C(A_1, \ldots, A_k) \rightarrow C(B_1, \ldots, B_k)$, such that $\iota(A_i) = B_i, i = 1, \ldots, k$. As in the case $k = 1$, the problem of nontrivial permutational similarity of $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ can be reduced to permutational similarity of two symmetric scaled doubly stochastic matrices $A, B \in \mathbb{N}^{n \times n}$ with the same characteristic polynomial.

3 Convex polytopes associate with GIP

Let $\Omega_n \subset \mathbb{R}^{n \times n}_+$ be the convex set of $n \times n$ doubly stochastic matrices. Recall that $\Omega_n = \{X \in M_n, X \geq -J_n\}$. We say that $A \in \mathbb{R}^{n \times n}_+$ is a scaled doubly stochastic matrix if $A = aC$ for some $a > 0$ and $C$ doubly stochastic.

**Lemma 3.1** Let $A \in \mathbb{R}^{n \times n}_+, n \geq 2$ be an irreducible symmetric scaled doubly stochastic matrix. Assume that $A$ has $\mu + 1 \geq 2$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_\mu$, where $m_i$ is the multiplicity of $\lambda_i$ for $i = 0, \ldots, \mu$. $(m_0 = 1.)$ Then the dimension of the subspace $P_0(A, A)$ and the polytope $P(A, A)$, given by (3.2), is $\Delta := \sum_{i=1}^\mu m_i^2$.

Assume that $B \in \mathbb{R}^{n \times n}_+$ is an irreducible symmetric scaled doubly stochastic matrix having the same row sums as $A$. Then the following are equivalent

1. $A$ and $B$ have the same characteristic polynomial.

2. The three subspaces $P_0(A, A), P_0(A, B), P_0(B, B)$ have the same dimension.

**Proof.** Since $A$ is irreducible, its Perron-Frobenius root, $\lambda_0$ is the largest eigenvalue of multiplicity 1. As $A$ is scaled doubly stochastic, $A = \lambda_0 C$ for some symmetric stochastic matrix $C$. So $A 1 = \lambda_0 1$. Choose an orthonormal basis of $\mathbb{R}^n$ consisting of orthonormal eigenvectors $x_0 = \frac{1}{\sqrt{n}} 1, x_1, \ldots, x_{n-1}$, corresponding to the eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_\mu$. Let $Q \in \mathbb{R}^{n \times n}$ be the orthogonal matrix whose columns are $x_0, \ldots, x_{n-1}$. Then $Q^\top A Q$ is the block diagonal matrix $\oplus_{i=0}^\mu \lambda_i I_{m_i}$. Observe that

$$Q^\top M_n Q = \{0_{1 \times 1} \oplus Z, Z \in \mathbb{R}^{(n-1) \times (n-1)}\}. \quad (3.1)$$
Recall next that any commuting matrix with $Q^T AQ$ has the block diagonal form $\oplus_{i=0}^{\mu} U_i$ where $U_i \in \mathbb{R}^{m_i \times m_i}$ for $i = 0, \ldots, \mu$. Hence

$$Q^T P_0(A,A)Q = \{0_{1 \times 1} \oplus_{i=1}^{\mu} U_i, \quad U_i \in \mathbb{R}^{m_i \times m_i}, \quad i = 1, \ldots, \mu\}. \quad (3.2)$$

Thus $\dim P_0(A,A) = \Delta$. Since $0_{n \times n} \in P_0(A,A)$ is an interior point of $P(A,A)$ it follows that $\dim P(A,A) = \Delta$.

Assume first that $B$ has the same characteristic polynomial as $A$. Since $A, B$ are symmetric, there exists an orthogonal matrix $Q_1$, with he columns $x_0, y_1, \ldots, y_{n-1}$, such that $Q_1^T BQ_1 = \oplus_{i=0}^{\mu} \lambda_i I_{m_i}$ where $t$. Hence

$$P_0(A, B) = P_0(A, A)Q_2^T, \quad Q_2 = Q_1Q^T, \quad B = Q_2AQ_2^T. \quad (3.3)$$

So $\dim P_0(A, B) = \dim P_0(B, B) = \Delta$.

Assume that $\dim P_0(A, A) = \dim P_0(A, B) = \dim P_0(B, B)$. Hence the dimension of the following three subspaces in $\mathbb{R}^{n \times n}$: $\{X, \ AX - XA = 0\}$, $\{X, \ AX - XB = 0\}$, $\{X, \ BX - XB = 0\}$ are equal. Therefore $A$ and $B$ are similar $[6]$. \hfill \Box

**Proof of Theorem** Observe first that if $U, V \in \mathbb{R}^{n \times n}$ are orthogonal matrices, then the transformation $X \mapsto UXV$ is an orthogonal transformation on $\mathbb{R}^{n \times n}$. In particular, the ball $B(0, t) \subset \mathbb{R}^{n \times n}$, centered at 0 of radius $t$ satisfies the equality $UB(0, t)V = B(0, t)$ for any orthogonal $U, V$.

Assume first that $A, B$ are similar, i.e $B = Q_2AQ_2^T$ for some orthogonal $Q_2$. Then for any $t > 0$

$$B(0, t) \cap P_0(A, B) = (B(0, t) \cap P_0(A, A))Q_2^T,$$

$$B(0, t) \cap P_0(B, B) = Q_2(B(0, t) \cap P_0(A, A))Q_2^T.$$

Suppose furthermore that $A$ and $B$ are permutationally permutationally similar, i.e $Q_2 = P \in P_n$. (So [2] holds.) Then (3.3) yields that $P(A, B) = P(A, A)P^T, \quad P(B, B) = PP(A, A)P^T$. In particular

$$B(0, t) \cap P_0(A, B) = (B(0, t) \cap P(A, A))P^T,$$

$$B(0, t) \cap P_0(B, B) = P(B(0, t) \cap P(A, A))P^T.$$

Hence, above intersections have the same volume for any $t > 0$. This proves the conditions [2][3].

Recall that $\|P(A, B)\|_F \leq \sqrt{n - 1}$, and equality holds if and only if $(P_n - J_n) \cap P(A, B) \neq \emptyset$, i.e. $A$ and $B$ are permutationally similar. Observe next that since $I_n - J_n \in P(A, A) \cap P(B, B)$ it follows that $\|P(A, A)\|_F = \|P(B, B)\|_F = \sqrt{n - 1}$. Hence, the volumes of $B(0, t) \cap P(A, A), B(0, t) \cap P(B, B)$ increase in the interval $(0, \sqrt{n - 1})$.

Assume that [3] holds. Since the volumes of $B(0, t_i) \cap P(A, A), i = 1, \ldots$, form an increasing, it follows that the volumes of $B(0, t_i) \cap P(A, B), i = 1, \ldots$, form an increasing sequence. Hence $\|P(A, B)\| = \sqrt{n - 1}$. So $A$ and $B$ are permutationally similar. Clearly, the same arguments imply that the condition [2] implies the permutational similarity of $A$ and $B$. \hfill \Box
4 A semi-fpras for graph isomorphism

We now point out how to apply the existing fully polynomial randomized approximation schemes to compute a volume of a convex sets, e.g. [11]. To do that it would be convenient to map the three convex polytopes $P(A, A)$, $P(A, B)$, $P(B, B)$ of dimension $\Delta$, to one ambient space $\mathbb{R}^\Delta$ by by three different linear transformations:

$$T_1 : P_0(A, A) \rightarrow \mathbb{R}^\Delta, \ T_2 : P_0(A, B) \rightarrow \mathbb{R}^\Delta, \ T_3 : P_0(A, A) \rightarrow \mathbb{R}^\Delta,$$  

such that each $T_i$ preserves the inner product. We demonstrate for $T_2$. Choose an orthonormal basis $W_1, \ldots, W_\Delta$ in $P_0(A, B)$. Then $T(W_i) = (\delta_{1i}, \ldots, \delta_{\Delta i})^T$, $i = 1, \ldots, \Delta$. It is straightforward to show that $T_1(P(A, A)), T_2(P(A, B)), T_3(P(B, B))$ are polytopes $X_1, X_2, X_3 \subset \mathbb{R}^{\Delta \times \Delta}$, which are given as follows. There exists nonzero vectors $u_{(1,1),i}, \ldots, u_{(n,n),l} \in \mathbb{R}^\Delta, l = 1, 2, 3$ such that

$$X_l = \{ x \in \mathbb{R}^\Delta, \ u_{(i,j),l}^T x \geq -\frac{1}{n}, i, j = 1, \ldots, n \}, \ l = 1, 2, 3.$$  

(4.2)

One can compute the vectors $u_{(i,j),l}$ in polynomial time in $n$. Note that the inequality $x_{ij} \geq -\frac{1}{n}$ is equivalent to $u_{(i,j),l}^T x \geq -\frac{1}{n}$ in the orthonormal basis of the corresponding linear space $P_0(A, A), P_0(A, B), P_0(B, B)$.

As we explain in the next section the permutational similarity of $A$ and $B$ is equivalent to the existence of an orthonormal matrix $O \in \mathbb{R}^{\Delta \times \Delta}$ such that $O\{u_{(1,1),1}, \ldots, u_{(n,n),1}\} = \{u_{(1,1),2}, \ldots, u_{(n,n),2}\}$. (Similar statement holds for $\{u_{(1,1),3}, \ldots, u_{(n,n),3}\}, \{u_{(1,1),2}, \ldots, u_{(n,n),2}\}$.)

It is trivial to see that the polytopes $P(A, A), P(A, B), P(B, B)$ contain the ball of radius $\frac{1}{n}$ centered at the origin. Theorem [11] yields that $A$ and $B$ are permutationally similar if

$$\text{vol}(B(0, t)X_1) = \text{vol}(B(0, t)X_2) = \text{vol}(B(0, t)X_3) \text{ for each } t \in (\frac{1}{n}, \sqrt{n} - 1).$$  

(4.3)

We now suggest a probabilistic test of the above equalities, for a finite number of values $t \in (\frac{1}{n}, \sqrt{n} - 1)$ with a relative error $\varepsilon$ and with probability $1 - \eta$. In the random algorithms suggested in [4, 15, 11], adopted to find the volumes of $X_i, i = 1, 2, 3$, one considers the intersection of the sequence of balls of radii:

$$t_j = \frac{2j}{n}, \ j = 0, \ldots, M = \lceil N \log_2 n \sqrt{n} - 1 \rceil.$$  

(4.4)

Here $N$ can be chosen as $\Delta$, as in [4, 15, 11], or if we want more points we can take $N = n^c \Delta$ for some $c > 0$. Let $X_{j,i} := B(t_j) \cap X_i, j = 0, \ldots, M, i = 1, 2, 3$. For each $X_{j,i}$, one generates $p = 400e^{-2}N \log N$ random points from certain distribution, e.g. [11, §6]. Then $\frac{\text{vol}(X_{j,i})}{\text{vol}(X_{j-1,i})}$ is estimated by the fraction of number of the sampled points in $X_{j,i}$ to the number of the sampled in $X_{j,i}$.
which are in $X_{j-1,i}$. By Theorem 1.1 if $A$ and $B$ are permutationally similar, we must have the equalities

$$\frac{\text{vol}(X_{j,1})}{\text{vol}(X_{j-1,1})} = \frac{\text{vol}(X_{j,2})}{\text{vol}(X_{j-1,2})} = \frac{\text{vol}(X_{j,3})}{\text{vol}(X_{j-1,3})}, \quad \text{vol}(X_{j,1}) = \text{vol}(X_{j,2}) = \text{vol}(X_{j,3})$$

(4.5)

for $j = 1, \ldots, M$. Hence, in our process of estimating the volumes of $X_1, X_2, X_3$ we test the above equalities within relative error $\varepsilon$. If all the above equalities hold within the relative error $\varepsilon$, we declare that the matrices $A, B$ are $\varepsilon, \eta$ permutationally similar. If one of the equalities in (4.5) fails with respect to relative error $\varepsilon$, we have two options. Either declare that the matrices $A, B$ are not $\varepsilon, \eta$ permutationally similar, or retest this equality with a smaller $\varepsilon$ and $\eta$. If all the retested equalities hold, then we declare that $A, B$ are $\varepsilon, \eta$ permutationally similar. Otherwise, we declare that the matrices $A, B$ are not $\varepsilon, \eta$ permutationally similar.

Note that each oracle query if a point $x \in B(t) \cap X_i$ needs $\Delta n^2 \leq n^4$ flops, since the dot product in $\mathbb{R}^\Delta$ need $\Delta$ flops. Since the randomized algorithm suggested in [11] is of order $O^*(\Delta^5)$ we see that the our algorithm for checking the graph isomorphism, or permutational similarity of $A, B \in \mathbb{R}^{n \times n}$, is of order $O^*(n^{14})$ at most.

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