THE IMAGE MILNOR NUMBER AND EXCELLENT UNFOLDINGS

R. GIMÉNEZ CONEJERO AND J.J. NUÑO-BALLESTEROS

Abstract. We show three basic properties on the image Milnor number $\mu_I(f)$ of a germ $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ with isolated instability. First, we show the conservation of the image Milnor number, from which one can deduce the upper semi-continuity and the topological invariance for families. Second, we prove the weak Mond’s conjecture, which says that $\mu_I(f) = 0$ if and only if $f$ is stable. Finally, we show a conjecture by Houston that any family $f_t: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ with $\mu_I(f_t)$ constant is excellent in Gaffney’s sense. By technical reasons, in the two last properties we consider only the corank 1 case.

1. Introduction

The image Milnor number is an invariant introduced by D. Mond in [12] for map-germs $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ with isolated instability. Based on results of Lê and Siersma (cf. [17] and [16]), he showed that the image of a stable perturbation of $f$ has the homotopy type of a wedge of $n$-spheres and that the number of such spheres is independent of the stabilisation. He called this number, denoted by $\mu_I(f)$, the image Milnor number by its analogy with the classical Milnor number $\mu(X, 0)$ of a hypersurface $(X, 0)$ with isolated singularity. In order to ensure the existence of a stabilisation of $f$, it was considered in [12] only the case where $(n, n+1)$ are nice dimensions in the sense of Mather (cf. [9]). But when $f$ has corank 1, it always admits a stabilisation, so there is no reason to put any restriction on the dimensions in such case.

In this paper we will show three basic results on the image Milnor number. The first one is in Section 2 and is about the conservation of the image Milnor number. If $F(x, u) = (f_u(x), u)$ is any $r$-parameter unfolding of $f$, then for all $u$ in $\mathbb{C}^r$ close to 0,

$$\mu_I(f) = \beta_n(X_u) + \sum_{y \in X_u} \mu_I(f_u; y),$$

where $\beta_n(X_u)$ is the number of spheres (i.e., the $n$-th Betti number) of the image $X_u$ of $f_u$ and $\mu_I(f_u; y)$ is the image Milnor number of $f_u$ at $y \in X_u$ (see Theorem 2.6). Two immediate consequences of this conservation are that $\mu_I(f)$ is upper semi-continuous (Corollary 2.7) and also that $\mu_I(f)$ is a topological invariant for families of germs (Corollary 2.10).

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The second result is what we call the weak Mond’s conjecture in Section 3. The original Mond’s Conjecture, cf. [12], says that

\[ \mathcal{A}_e - \text{codim}(f) \leq \mu_I(f), \]

with equality if \( f \) is weighted homogeneous. Here \( \mathcal{A}_e - \text{codim}(f) \) is the \( \mathcal{A}_e - \)codimension of \( f \), that is, the minimal number of parameters in a versal unfolding of \( f \). This conjecture is known to be true for \( n = 1, 2 \) (see [2, 12] for \( n = 2 \) and [13] for \( n = 1 \)) but it remains open for \( n \geq 3 \). In our weak version of the conjecture (Theorem 3.9) we prove that \( \mu_I(f) = 0 \) if and only if \( \mathcal{A}_e - \text{codim}(f) = 0 \) or, equivalently, \( f \) is stable (by Mather’s criterion of infinitesimal stability). Since we use here the results of Houston on the image computing spectral sequence (cf. [7]), we have to restrict ourselves to the corank 1 case.

Finally, in Section 4 we prove a conjecture by Houston in [7] relative to excellent unfoldings. Following Gaffney in [5], an origin-preserving one-parameter unfolding \( F(x, t) = (f_t(x), t) \) is excellent if it admits a stratification by stable types such that the parameter axes are the only 1-dimensional strata (see 4.2 for a more precise definition). Excellent unfoldings play an important role in the theory of equisingularity of mappings. In fact, if the unfolding is excellent, then the polar multiplicity theorem of Gaffney states that the Whitney equisingularity of family is equivalent to the constancy of the polar multiplicities associated to all the strata in the source and target (see [5]). The conjecture of Houston is that the constancy of the image Milnor number \( \mu_I(f_t) \) is also a sufficient condition for an unfolding to be excellent. We prove this result in Theorem 4.3 (also provided that \( f \) has corank 1).

We refer to the book [14] for basic definitions and properties about singularities of mappings, such as stability, finite determinacy, versal unfoldings, etc.

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2. Conservation of the Image Milnor number

We recall the definition of the Milnor fibration (see [10, Theorem 4.8 and Theorem 5.8]). Let \( g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) be a holomorphic non-zero function which defines a hypersurface \( X = g^{-1}(0) \) in \( (\mathbb{C}^{n+1}, 0) \) with arbitrary singularities (either isolated or non-isolated). We fix a Whitney stratification on \( X \). We denote by \( B_\epsilon \) the closed ball of radius \( \epsilon \) centred at 0 in \( \mathbb{C}^{n+1} \), with boundary \( S_\epsilon = \partial B_\epsilon \) and interior \( \bar{B}_\epsilon = B_\epsilon \setminus S_\epsilon \).

A Milnor radius is a number \( \epsilon > 0 \) such that \( S_\epsilon \) is transverse to \( X \), for all \( \epsilon' \) such that \( 0 < \epsilon' \leq \epsilon \). This implies that \( X \cap B_\epsilon \) is homeomorphic to the cone on \( X \cap S_\epsilon \).

Once we have fixed \( \epsilon > 0 \), there exists \( \eta > 0 \) such that

\[ g : g^{-1}(\hat{D}_\eta) \cap B_\epsilon \to \hat{D}_\eta \]

is a locally trivial fiber bundle over \( \hat{D}_\eta \setminus \{0\} \). Here, \( \hat{D}_\eta \) is the open disk of radius \( \eta \) centred at 0 in \( \mathbb{C} \). The choice of \( \eta \) has to be made in such a way
that for all \( t \) such that \( 0 < |t| < \eta \), then \( t \) is a regular value of \( g \) and also \( S_t \) is transverse to \( g^{-1}(t) \). This is called the Milnor fibration and the fibres are called Milnor fibres.

Now we consider an \( r \)-parameter deformation of \( g \), that is, a holomorphic germ \( G : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}, 0) \) written as \( G(y, u) = g_u(y) \) and such that \( g_0 = g \). Then \( G \) defines a hypersurface \( X = G^{-1}(0) \) in \( (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \) which is a deformation of \( X \). We assume that \( X \) also has a Whitney stratification whose restriction to \( \{u = 0\} \) coincides with that of \( X \).

**Definition 2.1** (cf. [16], page 2). We say that the deformation \( G \) is topologically trivial over the Milnor sphere \( S_\epsilon \) if, for \( \eta \) and \( \rho \) small enough,

\[
(1) \quad (S_\epsilon \times \hat{B}_\rho) \cap G^{-1}(\hat{D}_\eta) \xrightarrow{(G, \text{id})} \hat{D}_\eta \times \hat{B}_\rho
\]

is a stratified submersion with strata \( \{0\} \times \hat{B}_\rho \) and \( (\hat{D}_\eta \setminus \{0\}) \times \hat{B}_\rho \) on \( \hat{D}_\eta \times \hat{B}_\rho \) and the induced stratification on \( (S_\epsilon \times \hat{B}_\rho) \cap G^{-1}(\hat{D}_\eta) \).

Since we have a Whitney stratification on \( X \), the restriction of (1) to each stratum in the target is a locally trivial \( \mathcal{C}^0 \)-fibration, by the Thom-Mather first isotopy lemma (cf. [9], Theorem 5.2).

**Theorem 2.2** (cf. [16], Theorem 2.3). With the notation above, let \( G \) be a deformation of \( g \) which is topologically trivial over a Milnor sphere. Let \( u \in \hat{B}_\rho \) and suppose that all the fibres of \( g_u \) are smooth or have isolated singularities except for one special fibre \( X_u := g_u^{-1}(0) \cap B_\epsilon \). Then \( X_u \) is homotopy equivalent to a wedge of spheres of dimension \( n \) and its number is the sum of the Milnor numbers over all the fibres different from \( X_u \).

**Example 2.3.** The condition that \( G \) is topologically trivial over a Milnor sphere is necessary in Theorem 2.2. For instance, consider \( G : (\mathbb{C}^3 \times \mathbb{C}, 0) \to (\mathbb{C}, 0) \) given by \( G(x, y, z, u) = xy - u \). For \( u \neq 0 \), \( X_u = g_u^{-1}(0) \cap B_\epsilon \) has not the homotopy type of a wedge of 2-spheres (in fact, it has the homotopy type of \( S^1 \)).

Let \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) be an \( \mathcal{A} \)-finite germ, that is, with finite \( \mathcal{A} \)-codimension. By the Mather-Gaffney criterion (see e.g. [14], Theorem 4.5), this is equivalent to that \( f \) has isolated instability. In particular, \( f \) is finite and hence, its image is an analytic hypersurface \( (X, 0) \) in \( (\mathbb{C}^{n+1}, 0) \). We take a holomorphic function \( g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) such that \( g = 0 \) is a reduced equation for \( X \). We will assume that either \( (n, n + 1) \) are nice dimensions in Mather’s sense (cf. [9], page 208) or \( f \) has corank 1. In both cases, \( X \) has a natural stratification given by the stable types. This stratification is analytically trivial and hence a Whitney stratification (see [14], Corollary 7.5]).

Consider now an unfolding \( F : (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) \to (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \) of \( f \). Write \( F(x, u) = (f_u(x), u) \), as usual, with \( f_0 = f \). We denote by \( (X, 0) \) the image of \( F \) in \( (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \) and choose a holomorphic function \( G : (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \to (\mathbb{C}, 0) \) such that \( G = 0 \) is a reduced equation of \( X \) and \( g_0 = g \), where \( g_u(y) = G(y, u) \). We also consider in \( X \) the natural Whitney stratification by stable types, which has the property that its restriction to \( u = 0 \) coincides with the stratification of \( X \). We say that \( G \) is a deformation of \( g \) induced by the unfolding \( F \).
Lemma 2.4. Let $f$ be $\mathcal{C}$-finite such that either $(n,n+1)$ are nice dimensions or $f$ has corank 1. Any deformation $G$ induced by an unfolding $F$ is topologically trivial over a Milnor sphere.

Proof. The proof of this lemma is basically the same that appears in [12 proof of Theorem 1.4] in the particular case that $F$ is a stabilisation of $f$. On one hand, $f$ is $\mathcal{C}$-finite, hence it has isolated instability, so $f$ is locally stable on $S_e$. On the other hand, $g$ is regular on $S_e$ by definition of Milnor radius. Since $S_e$ is compact, we can assume, after shrinking $\rho$ if necessary, that $f_u$ is locally stable on $S_e$ and $g_u$ has no critical points on $S_e$, for all $u \in B_\rho$. Now we prove that

$$(S_e \times \hat{B}_\rho) \cap G^{-1}(\hat{D}_\eta) \xrightarrow{(G,\text{id})} \hat{D}_\eta \times \hat{B}_\rho$$

is a stratified submersion.

In fact, let $(y,u) \in (S_e \times \hat{B}_\rho) \cap G^{-1}(\hat{D}_\eta)$. If $y \in X_u$ then $f_u$ is stable at $y$ and hence, $F$ is (analytically) trivial in a neighbourhood of $(y,u)$. This implies that the induced stratification in $(S_e \times \hat{B}_\rho) \cap X$ is also (analytically) trivial in a neighbourhood of $(y,u)$. In particular, the map

$$(S_e \times \hat{B}_\rho) \cap X \xrightarrow{0 \times \text{id}} \{0\} \times \hat{B}_\rho$$

is a stratified submersion at $(y,u)$. Otherwise, if $y \notin X_u$, then $y$ is a regular point of $g_u$ and hence, $(y,u)$ is a regular point of $(G,\text{id})$. It follows that

$$(S_e \times \hat{B}_\rho) \cap G^{-1}(\hat{D}_\eta \setminus \{0\}) \xrightarrow{(G,\text{id})} (\hat{D}_\eta \setminus \{0\}) \times \hat{B}_\rho$$

is a submersion at $(y,u)$. □

It follows from Theorem 2.2 that for all $u$ small enough, $X_u$ is homotopy equivalent to a wedge of spheres of dimension $n$ and its number is the Betti number

$$\beta_n(X_u) = \sum_{y \in B_\rho \setminus X_u} \mu(g_u;y).$$

Since $f$ is finite, it has finite singularity type and hence, there exists a stable unfolding $F$ of $f$. The bifurcation set $B(F)$ is the set germ of parameters $u \in B_\rho$ for $\rho > 0$ small enough, such that $f_u$ is not a locally stable mapping. When $(n,n+1)$ are nice dimensions or $f$ has corank 1 we know that $B(F)$ is a proper analytic subset of $(\mathbb{C}^r,0)$ (see [14 Propositions 5.5 and 5.6]).

Definition 2.5. Let $f$ be $\mathcal{C}$-finite such that either $(n,n+1)$ are nice dimensions or $f$ has corank 1. Take $F$ a stable unfolding of $f$ and $u \in B_\rho \setminus B(F)$, for $\rho$ small enough. The mapping $f_u$ is called a stable perturbation of $f$, its image $X_u$ is called the disentanglement of $f$ and the number of spheres $\beta_n(X_u)$ is called the image Milnor number and is denoted by $\mu_I(f)$.

The image Milnor number $\mu_I(f)$ is well defined, that is, it is independent of the choice of the parameter $u$, of the representatives and of the stable unfolding $F$ (see [14] Section 8 for details).

Remark 2.1. When $(n,n+1)$ are not nice dimensions and $f$ has corank $> 1$, the definition of $\mu_I(f)$ can be done analogously by taking Mather’s canonical stratification of the image instead of the stratification by stable
types and taking a parameter \( u \) such that \( f_u \) is topologically stable instead of stable. However, we will not consider these cases in this paper.

**Remark 2.2.** A stabilisation of \( f \) is a 1-parameter unfolding \( F(x,t) = (f_t(x), t) \) of \( f \) with the property that \( f_t \) is a locally stable mapping for all \( t \neq 0 \) close to the origin. A stabilisation of \( f \) always exists when \( (n, n + 1) \) are nice dimensions or \( f \) has corank 1 (cf. [14, Corollary 5.4]). Moreover, given a second stable unfolding \( F''(x,u) = (f''(x), u) \) of \( f \) we can take the sum of unfoldings

\[
F''(x,u,t) = (f''(x) + f_t(x) - f(x), u, t),
\]

which is also stable. For all \( t \neq 0 \), \( f_t \) is stable, so \( (0, t) \notin B(F'') \) and hence \( \mu(f) = \beta_n(X_t) \), where \( X_t \) is the image of \( f_t \). This is in fact the definition of \( \mu(f) \) given originally by D. Mond in [12, Theorem 1.4] in terms of a stabilisation instead of a stable unfolding.

The following property can be seen as the conservation of the image Milnor number.

**Theorem 2.6.** Let \( f \) be \( \mathcal{A} \)-finite such that either \( (n, n + 1) \) are nice dimensions or \( f \) has corank 1. Let \( F \) be any unfolding of \( f \). Take \( u \in B_\rho \), with \( \rho > 0 \) small enough. Then,

\[
\mu(f) = \beta_n(X_u) + \sum_{y \in X_u} \mu(f_u; y),
\]

where \( \mu(f_u; y) \) is the image Milnor number of \( f_u \) at \( y \in X_u \).

**Proof.** By taking the sum of \( F \) with a stable unfolding, we can assume that \( F \) is itself stable. Since \( f \) is \( \mathcal{A} \)-finite, \( f \) has isolated instability at the origin by the Mather-Gaffney criterion. This implies that \( f_u \) has only finitely many unstable singularities which we denote by \( y_1, \ldots, y_k \in X_u \) and hence,

\[
\sum_{y \in X_u} \mu(f_u; y) = \sum_{i=1}^{k} \mu(f_u; y_i).
\]

Also by Theorem 2.2, \( g_u \) has only finitely many (isolated) critical points on \( B_\varepsilon \setminus X_u \), which we denote by \( z_1, \ldots, z_m \), so that

\[
\beta_n(X_u) = \sum_{j=1}^{m} \mu(g_u; z_j).
\]

For each \( i = 1, \ldots, k \) we choose a Milnor ball \( B_{\epsilon_i} \) for \( g_u \) at \( y_i \) contained in \( B_\varepsilon \). Analogously, for each \( j = 1, \ldots, m \) we choose also a Milnor ball \( B_{\delta_j} \) for \( g_u \) at \( z_j \) contained in \( B_\varepsilon \setminus X_u \). We will assume that the balls \( B_{\epsilon_1}, \ldots, B_{\epsilon_k}, B_{\delta_1}, \ldots, B_{\delta_m} \) are pairwise disjoint (see fig. 1, right).

Again by Theorem 2.2, for each \( i = 1, \ldots, k \), there exists an open ball \( \tilde{B}_{\rho_i} \) centered at \( u \) and contained in \( B_\rho \) such that

\[
\mu_i(f_u; y_i) = \beta_n(X_u' \cap B_{\varepsilon}) = \sum_{w \in B_{\varepsilon} \setminus X_u'} \mu(g_u; w),
\]

for all \( u' \in \tilde{B}_{\rho_i} \setminus B(F) \). We set \( U_1 = \tilde{B}_{\rho_1} \cap \cdots \cap \tilde{B}_{\rho_k} \) (see fig. 1, left).
For each $j = 1, \ldots, m$, $z_j$ is an isolated critical point of $g_u$ and $X_u \cap B_{\delta_j} = \emptyset$. By the conservation of the Milnor number of a function, there exists another open ball $\hat{B}_{\tau_j}$ centered at $u$ and contained in $\hat{B}_\rho$ such that
\[
\mu(g_u; z_j) = \sum_{w \in B_{\delta_j}} \mu(g_u'; w),
\]
and also $X_{u'} \cap B_{\delta_j} = \emptyset$, for all $u' \in \hat{B}_{\tau_j}$. As above, we set $U_2 = \hat{B}_{\tau_1} \cap \cdots \cap \hat{B}_{\tau_m}$.

Consider the compact set
\[
K = B_\epsilon \setminus \left( \bigcup_{i=1}^k \hat{B}_{\epsilon_i} \cup \bigcup_{j=1}^m \hat{B}_{\delta_j} \right) .
\]
Since $g_u$ has no critical points on $K \setminus X_u$, there exists another open neighbourhood $U_3$ of $u$ in $\hat{B}_\rho$ such that $g_{u'}$ has no critical points on $K \setminus X_{u'}$, for all $u' \in U_3 \setminus B(F)$.

Finally, again by Theorem 2.2
\[
\mu_I(f) = \beta_n(X_{u'}) = \sum_{w \in B_\epsilon \setminus X_{u'}} \mu(g_{u'}; w)
\]
\[
= \sum_{i=1}^k \sum_{w \in B_{\epsilon_i} \setminus X_{u'}} \mu(g_{u'}; w) + \sum_{j=1}^m \sum_{w \in B_{\delta_j}} \mu(g_{u'}; w)
\]
\[
= \sum_{i=1}^k \mu_I(f_u; y_i) + \sum_{j=1}^m \mu(g_u; z_j),
\]
for all $u' \in U_1 \cap U_2 \cap U_3 \setminus B(F)$.

A straightforward consequence of Theorem 2.6 is that the image Milnor number is upper semi-continuous.

**Corollary 2.7.** With the conditions and notation of Theorem 2.6,

\[
\mu_I(f) \geq \mu_I(f_u; y),
\]

for all \( y \in X_u \).

The upper semi-continuity of \( \mu_I(f) \) has been also obtained by Houston in [7, Theorem 4.3] but in the particular case that \( f \) has corank 1 and either \( s(f_u) \leq d(f_u) \) or \( s(f_u) \) and \( d(f_u) \) are constant (see Section 3 for the definitions of \( s(f_u) \) and \( d(f_u) \)).

Another consequence of the conservation is the topological invariance of the image Milnor number for unfoldings. We say that an unfolding \( F \) is **topologically trivial** if it is \( \mathcal{A} \)-equivalent as an unfolding to the constant unfolding. That is, if there exist homeomorphisms \( \Phi \) and \( \Psi \) which are unfoldings of the identity in \((\mathbb{C}^n, S)\) and \((\mathbb{C}^{n+1}, 0)\), respectively, such that

\[
\Psi \circ F \circ \Phi^{-1} = f \times \text{id}.
\]

**Corollary 2.8.** With the conditions and notation of Theorem 2.6, if \( F \) is topologically trivial

\[
\mu_I(f) = \sum_{y \in X_u} \mu_I(f_u; y).
\]

**Proof.** Write \( F(x, u) = (f_u(x), u), \Phi(x, u) = (\phi_u(x), u) \) and \( \Psi(y, u) = (\phi_u(y), u). \)

Then \( \psi_u \circ f_u \circ \phi_u^{-1} \), for all \( u \). Hence, \( X_u \) is homeomorphic to \( X \) which is contractible. \( \square \)

A particular case is when \( F \) is good in the sense of Gaffney [5, Definition 2.1]. Roughly speaking it means that \( F \) has isolated instability uniformly. We will assume that \( F \) is a one-parameter unfolding which is origin-preserving, that is, \( f_t(S) = \{0\} \), for all \( t \).

**Definition 2.9.** We say that an origin-preserving one-parameter unfolding \( F(x, t) = (f_t(x), t) \) is good if there exists a representative \( F : U \to W \times T \), where \( U \) is an open neighbourhood of \( S \times \{0\} \) in \( \mathbb{C}^n \times \mathbb{C} \) and \( W, T \) are open neighbourhoods of the origin in \( \mathbb{C}^{n+1}, \mathbb{C} \) respectively, such that

(i) \( F \) is finite,
(ii) \( f_t^{-1}(0) = S \), for all \( t \in T \),
(iii) \( f_t \) is locally stable on \( W \setminus \{0\} \), for all \( t \in T \).

**Corollary 2.10.** If \( F \) is a topologically trivial and good unfolding of an \( \mathcal{A} \)-finite germ \( f \), then \( \mu_I(f_t) \) is constant for the family of germs \( f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \).

### 3. Weak Mond’s conjecture

In this section we prove the weak version of Mond’s conjecture. We first recall the definition of the multiple point spaces \( D^k(f) \) following [15, Proposition-definition 2.5].

**Definition 3.1.** The \( k \)-th multiple point space \( D^k(f) \) of a mapping or a map germ \( f \) is defined as follows:
Let \( f: X \to Y \) be a locally stable mapping between complex manifolds. Then \( D^k(f) \) is equal to the Zariski closure of the set of points \((x^{(i)}), x^{(j)}\) in \( X^k \) such that \( f(x^{(i)}) = f(x^{(j)}) \), but \( x^{(i)} \neq x^{(j)} \) for all \( i \neq j \).

When \( f: (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0) \) is a stable germ, then \( D^k(f) \) is defined analogously but in this case it is a set germ in \(((\mathbb{C}^n)^k, S^k)\).

Let \( f: (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0) \) be finite and let \( F(x, u) = (f_u(x), u) \) be a stable unfolding of \( f \). Then \( D^k(f) \) is the complex space germ in \(((\mathbb{C}^n)^k, S^k)\) given by

\[
D^k(f) = D^k(F) \cap \{u = 0\}.
\]

The fact that \( D^k(f) \) is independent of the choice of the stable unfolding \( F \) can be found in [15, Lemma 2.3 and Proposition-definition 2.5]. In the particular case of a corank 1 mono-germ \( f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \), we have explicit equations for the multiple point spaces \( D^k(f) \). These are given by the so called divided differences of \( f \), which were introduced by Mond in [11, Section 3]. The multi-germ version (also in corank 1) can be found in [8, page 555].

Suppose \( F: X \times U \to Y \times U \) is a mapping of the form \( F(x, u) = (f_u(x), u) \). Then \( D^k(F) \) contains only \( k \)-tuples \((x^{(1)}, u, \ldots, x^{(k)}, u)\) with the same parameter \( u \). So, it is more convenient to embed \( D^k(F) \) in \( X^k \times U \) by identifying such a \( k \)-tuple with the point \((x^{(1)}, \ldots, x^{(k)}, u)\).

Given a mapping \( f: X \to Y \), the symmetric group \( \Sigma_k \) acts on \( D^k(f) \) by permuting the points \( x^{(i)} \). This induces also an action of \( \Sigma_k \) on the homology and the cohomology of \( D^k(f) \). In general, if \( G \) is a subgroup of \( \Sigma_k \) acting linearly on a vector space \( V \), then the \( G \)-alternating part of \( V \) is the subspace

\[
\{v \in V: \sigma v = \text{sign}(\sigma)v, \text{ for all } \sigma \in G\},
\]

and if the group \( G \) is \( \Sigma_k \) we omit the group and denote this by \( V^{\text{Alt}} \).

The following definition is due to Houston [7, Definition 3.9]:

**Definition 3.2.** Let \( f: (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0) \), \( n < p \), be \( \mathcal{A} \)-finite of corank 1 and let \( F(x, t) = (f_t(x), t) \) be a stabilisation of \( f \). We set the following notation:

- \( s(f) = |S| \), the number of branches of the multi-germ;
- \( d(f) = \sup \{k: D^k(f_t) \neq \emptyset\} \), where \( f_t \) is a stable perturbation of \( f \).

The \( k \)-th alternating Milnor number of \( f \), denoted by \( \mu_k^{\text{Alt}}(f) \), is defined as

\[
\mu_k^{\text{Alt}}(f) = \begin{cases} 
\dim_Q H_{n+1-k+1}^\text{Alt}(D^k(F), D^k(f_t); \mathbb{Q}), & \text{if } k \leq d(f), \\
\sum_{l=d(f)+1}^{s(f)} (-1)^l \binom{s(f)}{l}, & \text{if } k = d(f) + 1 \text{ and } s(f) > d(f), \\
0, & \text{otherwise}.
\end{cases}
\]
The value of \( \mu_k^{\text{Alt}}(f) \) when \( k = d(f) + 1 \) and \( s(f) > d(f) \) can be simplified in the following way:

\[
\left| \sum_{l=d+1}^s (-1)^l \binom{s}{l} \right| = \binom{s-1}{d}.
\]

This equality can be proven easily by using elementary properties of binomial numbers. Another useful property is the following lemma, which gives a relation between \( s(f) \) and \( d(f) \).

**Lemma 3.3.** In terms of the Definition 3.2, the inequality \( s(f) > d(f) \) can only happen when \( d(f) \) has the maximal possible value.

**Proof.** Suppose that the maximal possible value for \( d(f) \), for map-germs of the type \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0) \), is \( m \). We will assume, for the sake of the contradiction, that for a map-germ \( f \) in this pair of dimensions \( d(f) < m \) but \( d(f) < s(f) \).

Let \( k = \min \{s(f), m\} \), hence \( d(f) < k \). If we prove that \( D^k(f_1) \) is not empty, for \( f_1 \) a stable perturbation of \( f \), we arrive to a contradiction. To simplify the argument, assume that we have a stabilization \( F(x, t) = (f_t(x), t) \) of \( f \), so that \( f_1 \) is a stable perturbation for every \( t \neq 0 \), and this unfolding is inside another unfolding \( F \) of \( f \) that is stable as a map-germ, i.e., \( F(x, t, u) = (f_{t,u}(x), t, u) \) such that \( f_{t,0} = f_t \).

Given that \( k \leq s(f) \), necessarily \( D^k(f) \) has at least a point. In fact, since \( f \) has \( s(f) \) branches passing through the origin of \( \mathbb{C}^p \), any subset of \( S \) with \( k \) distinct points determines a point in \( D^k(f) \). However, notice that

\[
D^k(f) = D^k(F) \cap \{t = 0, u = 0\} \quad \text{and} \quad D^k(f_{t_0}) = D^k(F) \cap \{t = t_0, u = 0\} = D^k(F) \cap \{t = t_0\}
\]

as well, by definition. Hence, if \( D^k(F) \) has bigger dimension than \( D^k(f) \) we finish because then the intersection with \( \{t = t_0\} \) will contain at least a point, otherwise the dimensions would be equal.

In fact, since \( k \leq m \) and \( f \) is \( \mathcal{A} \)-finite, by [8 Theorem 2.14] it follows that \( \dim D^k(f) = nk - p(k-1) \). The stabilisation \( F \) of \( f \) is also \( \mathcal{A} \)-finite (see [14 Exercise 5.4.2]), so that \( \dim D^k(F) = (n+1)k - (p+1)(k-1) \). Since both sets are non-empty, \( \dim D^k(F) > \dim D^k(f) \), and this finishes the proof.

For instance, for a germ \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \), we have \( s(f) > d(f) \) only when \( d(f) = n + 1 \).

The motivation for the definition of \( \mu_k^{\text{Alt}}(f) \) is the following result by Houston which shows that, for a corank 1 germ \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \), the image Milnor number \( \mu_1(f) \) is equal to the sum of all the alternating Milnor numbers.

**Proposition 3.4** (cf. [7 Definition 3.11]). Let \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) be \( \mathcal{A} \)-finite of corank 1. Then,

\[
\mu_1(f) = \sum_k \mu_k^{\text{Alt}}(f).
\]
The proof is based on the analysis of the image computing spectral sequence associated to the multiple point spaces. Moreover, the above equality can be taken as a definition when we consider the more general situation of a germ \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^0, 0) \), with \( p \geq n + 1 \). In that case, \( \mu_i(f) \) can be also interpreted in terms of the homology of the distengagement of \( f \) (see \cite{7}, Remark 3.12 for details).

On the other hand, the following result, due to Wall, will be crucial. Suppose \( g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0) \) has isolated singularity at 0. Let \( U = O_{n+1}/J_g \) be the Milnor algebra, where \( J_g \) is the Jacobian ideal, generated by the partial derivatives \( \partial g/\partial y_i, 1 \leq i \leq n + 1 \). Denote by \( X_t = g^{-1}(t) \cap B_\epsilon \) the Milnor fiber, where \( 0 < \delta \ll \epsilon \ll 1 \) and \( 0 < |t| < \delta \). We assume \( G \) is a finite group of automorphisms of \( (\mathbb{C}^{n+1}, 0) \) that leaves \( g \) invariant. This implies that we have induced actions of \( G \) on \( X_t \) and on \( U \).

**Theorem 3.5** (see \cite{18} Theorem of page 170). With the above notation, we have an isomorphism of \( \mathbb{C}G \)-modules

\[
H^n(X_t; \mathbb{C}) \cong U \otimes_\mathbb{C} \Lambda^{n+1}(\mathbb{C}^{n+1})^*,
\]

where \( \Lambda^{n+1}(\mathbb{C}^{n+1})^* \) is the \((n + 1)\)th exterior power of the dual \( (\mathbb{C}^{n+1})^* \).

Obviously, the same is true if we replace \( \mathbb{C} \) by \( \mathbb{Q} \) and consider homology instead of cohomology.

We are now able to state and prove the following essential lemma about the structure of the alternating homology of the multiple point spaces:

**Lemma 3.6.** Let \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) be unstable of corank 1, \( \mathcal{A} \)-finite and which admits a 1-parameter stable unfolding \( F(x, t) = (f_t(x), t) \). Take \( f_t \), a stable perturbation of \( f \) and \( k = 2, \ldots, d(f) \). Then, \( H_{n-k+1}^{\text{Alt}}(D^k(f_t); \mathbb{Q}) \neq 0 \) if and only if \( D^k(f) \) is singular. Furthermore, if \( H_{n-k+1}^{\text{Alt}}(D^k(f_t); \mathbb{Q}) \neq 0 \), then \( H_{n-k'+1}^{\text{Alt}}(D^{k'}(f_t); \mathbb{Q}) \neq 0 \) for all \( k' = k, \ldots, d(f) \).

**Proof.** To prove the first part we begin by studying the case \( S = \{0\} \). We use the Marar-Mond criterion, \cite{8} Theorem 2.14. Since \( F \) is stable, \( D^k(F) \) is smooth and \( D^k(f) \) is a hypersurface in \( D^k(F) \) with isolated singularity and with Milnor fibre \( D^k(f) \). Moreover, the symmetric group \( \Sigma_k \) leaves invariant the defining equation of \( D^k(f) \) in \( D^k(F) \). By Theorem 3.5, we have an isomorphism of \( \mathbb{C} \Sigma_k \)-modules

\[
H^{n-k+1}(D^k(f_t); \mathbb{C}) \cong U \otimes_\mathbb{C} \Lambda^{n-k+2}V^*,
\]

where \( U \) is the Milnor algebra of \( D^k(f) \) in \( D^k(F) \) and \( V = T_0 D^k(F) \) is the tangent space of \( D^k(F) \) at the origin. If \( D^k(f) \) is singular, then \( U \neq 0 \) and contains the constants. Now we will see that these constants, after tensoring with \( \Lambda^{n-k+2}V^* \), are contained in the alternating part.

From \cite{8} Proposition 2.3 we can take \( \Sigma_k \)-invariant equations for \( D^k(F) \) in \( \mathbb{C}^n \times \mathbb{C}^k \). Since \( F \) has corank 1, we assume that \( D^k(F) \) is embedded in \( \mathbb{C}^n \times \mathbb{C}^k \) with coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_k \) and that \( \Sigma_k \) acts by permuting \( y_1, \ldots, y_k \). It follows that the tangent space \( V \) has \( \Sigma_k \)-invariant linear equations of the form

\[
a_i(y_1 + \cdots + y_k) + \sum_{j=1}^n b_{ij} x_j = 0, \text{ for } i = 1, \ldots, n.
\]
Hence, we can split $V$ as $V = V_1 \oplus V_2$, where

$V_1 = \{x_1 = 0, \ldots, x_n = 0, y_1 + \cdots + y_k = 0\}$,

$V_2 = V \cap \{y_i = y_j, 1 \leq i < j \leq k\}$.

If $\omega_1, \ldots, \omega_\ell$ is any basis of $V_2^*$, then

$$\lambda = (dy_1 - dy_2) \wedge \cdots \wedge (dy_{k-1} - dy_k) \wedge \omega_1 \wedge \cdots \wedge \omega_\ell$$

generates $\Lambda^{n-k+2}V^*$ and is $\Sigma_k$-alternating. This shows $H^{n-k+1}(D^k(f_i); \mathbb{C})$ has non-zero alternating part in the mono-germ case.

Suppose now that $S$ is any finite set. Let $D^k_1(F), \ldots, D^k_m(F)$ be the connected components of $D^k(F)$. Each $D^k_i(F)$ is a mono-germ at a point $(z^{(1)}, \ldots, z^{(k)}, 0) \in S^k \times \{0\}$. We also denote by $D^k_i(f), \ldots, D^k_m(f)$ the connected components of $D^k(f)$ such that $D^k_i(f) \subset D^k(F)$ for any $i$.

As $D^k(f)$ is singular, without loss of generality we can suppose that $D^k_1(f)$ is singular. Assume that $D^k_1(f)$ is a mono-germ at $(z^{(1)}, \ldots, z^{(k)}) \in S^k$ and let $G \leq \Sigma_k$ be the stabilizer of this point. By following the same argument as in the mono-germ case, but with $D^k_1(F), D^k_1(f)$ and $G$ instead of $D^k(F), D^k(f)$ and $\Sigma_k$, respectively, we find a non-zero element $v$ in the homology of $D^k_1(f)$ which is $G$-alternating.

Now for each $i = 1, \ldots, m$ we choose a permutation $\sigma_i \in \Sigma_k$ that takes $D^k_i(f)$ into $D^k_1(f)$. We claim that $\omega = \sum_i \text{sign}(\sigma_i)\sigma_i v$ is a non-zero element in the homology of $D_k(f_1)$ which is alternating.

Let $\tau \in \Sigma_k$. For each $i = 1, \ldots, m$, $\tau$ takes $\sigma_i(z^{(1)}, \ldots, z^{(k)})$ into some other $\sigma_j(i)(z^{(1)}, \ldots, z^{(k)})$, where $j(i) = 1, \ldots, m$. We can write $\tau \sigma_i$ as $\tau \sigma_i = \sigma_{j(i)} \left(\sigma_{i(j)}^{-1} \tau \sigma_i\right)$, and $\left(\sigma_{i(j)}^{-1} \tau \sigma_i\right) \in G$. Hence,

$$\tau \omega = \tau \sum_i \text{sign}(\sigma_i)\sigma_i v = \sum_i \text{sign}(\sigma_i)^2 \text{sign}(\tau) \text{sign}(\sigma_{j(i)})\sigma_{j(i)} v = \text{sign}(\tau) \sum_i \text{sign}(\sigma_{j(i)})\sigma_{j(i)} v.$$ But if $j(i_1) = j(i_2)$, for some $i_1 \neq i_2$, then

$$g = (\tau \sigma_{i_1})^{-1} (\tau \sigma_{i_2}) = \sigma_{i_1}^{-1} \sigma_{i_2}$$

is in $G$ as it fixes $(z^{(1)}, \ldots, z^{(k)})$. We have $\sigma_{i_2} = \sigma_{i_1} g$ and both $\sigma_{i_1}$ and $\sigma_{i_2}$ take $D^k_i(f)$ to the same component, which is absurd. Hence, $\tau \omega = \text{sign}(\tau)\omega$.

This concludes the proof that if $D^k(f)$ is singular, then $H^{n-k+1}(D^k(f_1); \mathbb{C})$ has non-zero alternating part. The converse is obvious: if $D^k(f)$ is smooth then $H^{n-k+1}(D^k(f_1); \mathbb{C}) = 0$, which has no alternating part.

For the second part, take $k$ such that $D^k(f)$ is singular. Then $D^k(f)$ is a subspace of $(\mathbb{C}^n)^k \setminus S^k$, with coordinates $x_{ij}^{(k)}$, with $i = 1, \ldots, n$ and $j = 1, \ldots, k$, and whose equations are the divided differences, which we represent by $\phi_1, \ldots, \phi_r$ with $r = (n + 1)(k - 1)$. Moreover, $D^k(f)$ has codimension $r$ and, by the Jacobian criterion, the Jacobian matrix $A$ of the functions $\phi_1, \ldots, \phi_r$ has rank less than $r$ at some point in $S^k$. 

Now $D^{k+1}(f)$ is defined in $((\mathbb{C}^n)^{k+1}, S^{k+1})$, by adding $n$ new coordinates $x_1^{(k+1)}, \ldots, x_n^{(k+1)}$ and $n + 1$ new equations $\phi_{r+1}, \ldots, \phi_{r+n+1}$. Since the old equations do not depend on the new variables, the Jacobian matrix of $\phi_1, \ldots, \phi_{r+n+1}$ is

$$
\begin{pmatrix}
A & 0 \\
B & 
\end{pmatrix},
$$

where $B$ is the Jacobian matrix of the new equations with respect to the new variables. Obviously, this matrix has rank $< r + n + 1$ at some point in $S^{k+1}$ and thus, $D^{k+1}(f)$ is also singular, since it has codimension $r + n + 1$. We can proceed recursively for $D^{k'}(f)$, with $k \leq k' \leq d(f)$. \hfill \square

Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite. Take $F$ be a stable unfolding and choose $G: (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \rightarrow (\mathbb{C}, 0)$ such that $G(y, u) = 0$ is a reduced equation of the image of $F$. The relative Jacobian ideal is the ideal $J_y(G)$ generated by the partial derivatives of $G$ with respect to the variables $y_1, \ldots, y_{n+1}$.

**Lemma 3.7.** We have:

$$
\mu_l(f) = 0 \iff G \in \sqrt{J_y(G)}.
$$

**Proof.** We follow the notation of Section 2. If $G \in \sqrt{J_y(G)}$ then $V(J_y(G)) \subseteq V(G)$. Hence, for any $(y, u)$ such that $y$ is a singular point of $g_u$, we have $g_u(y) = 0$. In particular, for $u \notin B(F)$,

$$
\mu_l(f) = \beta_u(X_u) = \sum_{y \in B_y \setminus X_u} \mu(g_u; y) = 0.
$$

Conversely, if $G \notin \sqrt{J_y(G)}$, then $V(J_y(G)) \not\subseteq V(G)$. Hence, there exists $(y, u)$ such that $y$ is a singular point of $g_u$ and $g_u(y) \neq 0$. This gives

$$
\mu_l(f) \geq \beta_u(X_u) = \sum_{y \in B_y \setminus X_u} \mu(g_u; y) \geq 1.
$$

\hfill \square

**Lemma 3.8.** Let $h: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite and let $f$ be any unfolding of $h$ which is also $\mathcal{A}$-finite. If $\mu_l(f) > 0$ then $\mu_l(h) > 0$.

**Proof.** Assume that $f(x, v) = (h_v(x), v)$ and denote by $(y, v)$ the coordinates of $f$ in the target. Let $F$ be a stable unfolding of $f$. If $\mu_l(h) = 0$, then $G \in \sqrt{J_y(G)} \subseteq \sqrt{J_{y,v}(G)}$, so $\mu_l(f) = 0$. \hfill \square

Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite and assume that either $(n, n + 1)$ are nice dimensions or $f$ has corank 1. Here we prove the following weak version of the Mond’s Conjecture in the corank 1 case (see the introduction for the original version of the conjecture).

**Theorem 3.9** (Weak Mond’s Conjecture). Let $f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite of corank 1. Then $\mu_l(f) = 0$ if and only if $f$ is stable.

**Proof.** Obviously, $\mu_l(f) = 0$ when $f$ is stable. Assume that $f$ is not stable. If $s(f) > d(f)$ we know that $d(f) = n + 1$, and also that $\mu_{n+2}^*(f) > 0$. Hence, we can suppose that $s(f) \leq d(f)$. 


By the Marar-Mond criterion either $D^k(f)$ is singular for some $k = 2, \ldots, d(f)$ or $D^k(f)$ is a $k$-tuple of points of $S$ for some $k \geq n + 2$. We suppose first that $D^k(f)$ is singular, for some $k < n + 1$.

If $f$ admits a 1-parameter stable unfolding $F(x,t) = (f_t(x),t)$, then $H_{n-k+1}(D^k(f_t))$ has non-zero alternating part for $t \neq 0$, by Lemma 3.6. Since $D^k(F)$ is contractible and $k < n + 1$, it follows from the exact sequence of the pair $(D^k(F), D^k(f_t))$ that

\[ H_{n-k+2}^{\text{Alt}}(D^k(F), D^k(f_t); \mathbb{Q}) \cong H_{n-k+1}^{\text{Alt}}(D^k(f_t); \mathbb{Q}), \]

so $\mu_k^{\text{Alt}}(f) > 0$.

If $f$ does not admit a 1-parameter stable unfolding, then we consider a minimal stable unfolding $F$. By taking a generic section on the parameter space, we get a finitely determined germ $F_0$ which is an unfolding of $f$ and which admits the 1-parameter stable unfolding $F$. Now $\mu_1(F_0) > 0$ by the above argument and hence also $\mu_1(f) > 0$ by Lemma 3.8.

The next case to consider is when $D^{n+1}(f)$ is singular. Again, we use the exact sequence of the pair $(D^k(F), D^k(f_t))$, but in this case

\[ H_0^{\text{Alt}}(D^{n+1}(F), D^{n+1}(f_t); \mathbb{Q}) \]

is isomorphic to the kernel of the mapping

\[ H_0^{\text{Alt}}(D^{n+1}(f_t); \mathbb{Q}) \to H_0^{\text{Alt}}(D^{n+1}(F); \mathbb{Q}) \]

induced by the inclusion. Take a singular 0-dimensional component of $D^{n+1}(f)$, with multiplicity $m > 1$. Such component will split into $m$ distinct points in $D^{n+1}(f_t)$, which correspond to $m$ distinct generators of $H_0^{\text{Alt}}(D^{n+1}(f_t); \mathbb{Q})$. But these $m$ points are in the same connected component of $D^{k+1}(F)$, for $F(x,t) = (f_t(x),t)$. Hence, we get a non-trivial element of the kernel of (2) and thus $\mu_{n+1}(f) > 0$.

Finally, it only remains to consider the case where $D^{n+1}(f)$ is smooth but $D^k(f)$ is a $k$-tuple of points of $S$ for some $k \geq n + 2$. Since $s(f) \leq d(f)$, $D^{n+1}(f)$ necessarily must contain a point $(x^{(1)}, \ldots, x^{(n+1)})$ such that $x^{(i)} = x^{(j)}$ for some $i \neq j$, as the projections from the previous $D^k(f)$ to this $D^{n+1}(f)$ cover all the possible points in the last space and we have less than $n + 2$ points in $S$. This point will also split into several distinct points in $D^{n+1}(f_t)$, which is not possible if $D^{k+1}(f)$ is smooth. We deduce that this case cannot occur when $s(f) \leq d(f)$.

Note. The proof of Theorem 3.9 is inspired in the proof of [11 Proposition 4.4]. Here it is proved that a corank 1 mono-germ of $\mathcal{A}^\ell$-codimension 1 has image Milnor number equal to 1, based on the same result of Wall (Theorem 3.5).

The following corollary can be deduced easily from Lemma 3.6, Theorem 3.9 and their proofs and it gives a sharper estimate of $\mu_1(f)$ when $f$ is unstable.

**Corollary 3.10.** Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathcal{A}$-finite of corank 1 and unstable. Assume $H_{n-k+1}(D^k(f_t); \mathbb{Q})$ has non-zero alternating part for some $k$:
(i) If \( s(f) \leq d(f) \) then \( \mu_I(f) \geq d(f) - k + 1 \).
(ii) If \( s(f) > d(f) \) then \( \mu_I(f) \geq d(f) - k + 1 + \left( \frac{s(f)}{d(f)} \right) \).

In case (i) there always exists such a \( k \) and in case (ii) \( d(f) \) has to be equal to \( n + 1 \) and such a \( k \) could not exist.

A straightforward consequence of the weak Mond’s conjecture is about the dimension of the relative Jacobian module of \( f \) considered in [3]. It is defined as

\[
M_y(G) = \frac{J(G) + (G)}{J_y(G)}
\]

where \( G: (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \rightarrow (\mathbb{C}, 0) \) is a function such that \( G(y, u) = 0 \) is a reduced equation of the image of a stable unfolding of \( f \). It is not difficult to see that the dimension of \( M_y(G) \) is always \( \leq r \) when \( f \) is \( \mathcal{A} \)-finite. Moreover, it is shown in [3, Theorem 6.1] that the Mond’s conjecture holds for \( f \) when \( M_y(G) \) is Cohen-Macaulay of dimension \( r \).

**Corollary 3.11.** Let \( f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0) \) be \( \mathcal{A} \)-finite of corank 1 and unstable. Then \( M_y(G) \) has dimension \( r \).

**Proof.** It follows from [3, Theorem 6.1] that

\[
\mu_I(f) = e_{O_r} ((u_1, \ldots, u_r); M_y(G)),
\]

the Samuel multiplicity of the \( O_r \)-module \( M_y(G) \) with respect to the parameter ideal \( (u_1, \ldots, u_r) \). But it is well known that an \( R \)-module has multiplicity \( > 0 \) if and only if it has dimension equal to \( \dim R \). \( \square \)

4. Houston’s Conjecture on Excellent unfoldings

It is not difficult to see that if we add a new branch to an unstable multi-germ \( f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0) \) then its \( \mathcal{A}_{e} \)-codimension increases strictly (see for instance [14, Exercise 3.4.1]). We show the same property for the image Milnor number, instead of the \( \mathcal{A}_{e} \)-codimension. The idea of the proof is easy to visualize, as we can see in fig. 2.

![Figure 2. Real representation of the creation of more homology via the addition of more branches. Note that in the complex case this happens in middle dimension.](image)

Given two germs \( f: (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^{n+1}, 0) \) and \( g: (\mathbb{C}^n, z) \rightarrow (\mathbb{C}^{n+1}, 0) \), we denote by \( \{f,g\}: (\mathbb{C}^n, S \sqcup \{z\}) \rightarrow (\mathbb{C}^{n+1}, 0) \) the new multi-germ obtained
as the disjoint union of \( f \) and \( g \). If \( f \) and \( g \) are both of corank 1 and \( \mathscr{A} \)-finite, then
\[
\mu_k(f) \leq \mu_k(\{f, g\}),
\]
for all \( k \), since adding a new branch does not kill the corresponding alternating homology of the \( k \)-multiple point space because the new branch just adds more connected components disjoint from the ones we had before. By Proposition 3.4, this implies that
\[
\mu(f) \leq \mu(\{f, g\}).
\]
We may have \( \mu(f) = \mu(\{f, g\}) \) when \( f \) is stable and \( g \) is transverse to \( f \), so that \( \{f, g\} \) is also stable. In the next lemma, we show that if \( f \) is unstable, then the inequality is strict.

**Lemma 4.1.** Let \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) and \( g : (\mathbb{C}^n, z) \to (\mathbb{C}^{n+1}, 0) \) be \( \mathscr{A} \)-finite. If \( f \) has corank 1 and \( \mu_I(f) > 0 \) then
\[
\mu_I(f) < \mu_I(\{f, g\}).
\]

**Proof.** By the upper semi-continuity of the image Milnor number (see Corollary 2.7), we can assume that the image of \( g \) is a generic hyperplane \( H \) in \( \mathbb{C}^{n+1} \) through the origin. Let \( f_t \) be a stable perturbation of \( f \) with image \( X_t \). Since \( H \) is a generic hyperplane, the disjoint union \( \{f_t, g\} \) gives a stable perturbation of \( \{f, g\} \), with image \( X_t \cup H \).

Furthermore, \( X_t \cap H \) is also the image of a stable perturbation of the restriction \( \tilde{f} : (f^{-1}(H), S) \to (H, 0) \). Since \( H \) is generic and \( f \) is \( \mathscr{A} \)-finite of corank 1, \( (f^{-1}(H), S) \) is smooth and \( \tilde{f} \) is also \( \mathscr{A} \)-finite of corank 1. Moreover, \( \tilde{f} \) cannot be stable because \( f \) is a 1-parameter unfolding of \( \tilde{f} \). Hence \( \mu_I(f) > 0 \), by the weak Mond’s conjecture (Theorem 3.9).

Now, just apply the Mayer-Vietoris sequence:
\[
0 \to H_n(X_t) \to H_n(X_t \cup H) \to H_{n-1}(X_t \cap H) \to 0,
\]
so
\[
\mu_I(f, g) = \mu_I(f) + \mu_I(\tilde{f}) > \mu_I(f).
\]

We recall now the notion of excellent unfolding following Gaffney (cf. [5, Definition 6.2]). Excellent unfoldings play an important role in the theory of equisingularity of families of germs. In fact, when \( F \) is excellent then we can stratify \( F \) in such a way that the parameter axes in the source and target are the only 1-dimensional strata (see fig. 5).

**Definition 4.2.** A one-parameter origin-preserving unfolding \( F \) is called excellent if it is good and it has a representative as in Definition 2.9 such that, in addition, \( f_t \) has no 0-stable singularities on \( W \setminus \{0\} \) (i.e., stable singularities whose isosingular locus is 0-dimensional).

The above lemma together with the conservation of the image Milnor number and the weak Mond’s conjecture allow us to prove Houston’s Conjecture on excellent unfoldings (cf. [7, Conjecture 6.2]) which we state now.
**Figure 3.** The pictures show the stratifications of the image of a non-excellent unfolding (left), due to the presence of a 1-dimensional stratum distinct from the parameter axis (bold line), and an excellent unfolding (right).

**Theorem 4.3.** Let \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) be \( \mathcal{A} \)-finite of corank 1 and let \( F(x, t) = (f_t(x), t) \) be an origin-preserving one-parameter unfolding. Consider the family of germs \( f_t : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \). Then \( \mu_I(f_t) \) constant implies \( F \) excellent.

**Proof.** We will use [7, Corollary 5.9], so we only need to show that \( F \) is good and that either \( s(\tilde{f}_t) \leq d(\tilde{f}_t) \) for all \( t \) or \( s(\tilde{f}_t) \) and \( d(\tilde{f}_t) \) are both constant, where \( \tilde{f}_t \) is the germ at \( f_t^{-1}(0) \) (we keep the notation \( f_t \) for the germ at \( S \)).

We can suppose that \( f \) is not stable, otherwise the result is trivial. We first prove that \( s(\tilde{f}_t) \) is constant, that is, \( f_t^{-1}(0) = S \) and hence, \( \tilde{f}_t = f_t \). We have \( S \subseteq f_t^{-1}(0) \) and if the inclusion was strict, then \( \mu_I(f_t) < \mu_I(\tilde{f}_t) \) by Lemma 4.1. But the upper semi-continuity of Corollary 2.7 implies that \( \mu(\tilde{f}_t) \leq \mu_I(f_t) \), in contradiction with the constancy of \( \mu_I(f_t) \).

If \( s(f_{t_0}) > d(f_{t_0}) \) for some \( t_0 \) then this can only happen when \( d(f_{t_0}) = n + 1 \). But \( s(f_t) \) is constant so \( s(f_t) > n + 1 \geq d(f_t) \), and again we have \( d(f_t) = n + 1 \). This shows that either \( s(f_t) \leq d(f_t) \) for all \( t \) or \( s(f_t) \) and \( d(f_t) \) are both constant.

Finally, we use the conservation of the image Milnor number, Theorem 2.6 to show that \( F \) is good. In fact, we get

\[
\mu_I(f_t; 0) = \mu_I(f) \geq \sum_{y \in X_t} \mu_I(f_t; y),
\]

so \( \mu_I(f_t; y) = 0 \) for all \( y \in X_t \setminus \{0\} \). By the weak Mond’s conjecture Theorem 3.9, \( f_t \) is locally stable on \( X_t \setminus \{0\} \). \( \square \)

One can ask if the converse is true, that is, if an excellent unfolding implies constant image Milnor number. We have the following partial result:

**Proposition 4.4.** Let \( f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) be \( \mathcal{A} \)-finite with \( n = 1, 2 \) and let \( F(x, t) = (f_t(x), t) \) be an origin-preserving one-parameter unfolding. Then \( F \) excellent implies \( \mu_I(f_t) \) constant.
Proof. Let \( n = 1 \). We have \( \mu_I(f_t) = \delta(f_t) - s(f_t) + 1 \), where \( \delta(f_t) \) is the delta invariant (see, for example, [13, Lemma 2.2]). Obviously \( s(f_t) = |S| \) is constant and we also have conservation of the delta invariant, which means that

\[
\delta(f) = \sum_{y \in \Sigma(X_t)} \delta(f_t; y),
\]

where \( \Sigma(X_t) \) is the singular locus of the image of \( f_t \) and \( \delta(f_t; y) \) is the delta invariant of the germ of \( f_t \) at \( f_t^{-1}(y) \). Since \( F \) is excellent, we have \( \Sigma(X_t) = \{0\} \) and \( f_t^{-1}(0) = S \), so \( \delta(f_t) = \delta(f_t; 0) \) is also constant.

Let \( n = 2 \). We consider the double point curve in the source \( D(f_t) \), defined as \( p_1(D^2(f_t)) \), where \( p_1 : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \) is the projection onto the first component. Then \( D(f_t) \) is a family of germs of plane curves in \((\mathbb{C}^2, S)\). Since \( F \) is excellent, we can choose representatives of \( D(f_t) \) on some open neighbourhood \( U \) of \( S \) in \( \mathbb{C}^2 \) such that \( \Sigma(D(f_t)) = S \) for all \( t \). This implies that the (usual) Milnor number \( \mu(D(f_t); x) \) at each point \( x \in S \) must be constant. By a theorem of Fernández de Bobadilla and Pe-Pereira, cf. [4, Theorem C], the unfolding \( F \) is topologically trivial. So, \( \mu_I(f_t) \) is constant by Corollary 2.10.

Example 4.5. The family \( f_t(x, y) = (x, y^2, yp_t(x, y)) \) with

\[
p_t(x, y) = \left(x - \frac{t}{2}\right)^2 + \left(y^2 - \frac{t}{2}\right)^2 - \frac{t^2}{8}
\]
yields an excellent unfolding over \( \mathbb{R} \), but not over \( \mathbb{C} \) because \( y = 0 \) and \( x = \frac{1}{2} \left(t \pm \frac{1}{2} \sqrt{t^2}\right) \) are curves of non-immersive points of \( f_t \). Furthermore its image Milnor number is not constant, \( \mu_I(f_0) > \mu_I(f_t) \) for \( t \neq 0 \) (cf. fig. 4).

![Figure 4](image.png)

Figure 4. From left to right, \( f_0 \) and \( f_t \) with \( t \neq 0 \) as real maps.

Theorem 4.3 and Proposition 4.4 motivate the following more general conjecture, where we consider not only the converse of 4.3 in higher dimensions, but also drop the corank 1 condition.

Conjecture 4.6. For every \( f: (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0) \) \( \mathcal{A} \)-finite germ, and every \( F(x, t) = (f_t(x), t) \) origin-preserving one-parameter unfolding, \( F \) is excellent if and only if \( \mu_I(f_t) \) is constant.

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Departament de Matemàtiques, Universitat de València, Campus de Burjassot, 46100 Burjassot SPAIN

Email address: Roberto.Gimenez@uv.es

Email address: Juan.Nuno@uv.es