Cameron-Liebler line classes in AG(3, q)

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Abstract

The study of Cameron-Liebler line classes in PG(3, q) arose from classifying specific collineation subgroups of PG(3, q). Recently, these line classes were considered in new settings. In this point of view, we will generalize the concept of Cameron-Liebler line classes to AG(3, q). In this article we define Cameron-Liebler line classes using the constant intersection property towards line spreads. The interesting fact about this generalization is the link these line classes have with Cameron-Liebler line classes in PG(3, q). Next to giving this link, we will also give some equivalent ways to consider Cameron-Liebler line classes in AG(3, q), some classification results and an example based on the example found in [3] and [6].

1 Introduction

Let $p$ be a prime and $q = p^h$, with $h \geq 1$. Then we can consider PG(3, q) and the corresponding affine space AG(3, q), with $\pi_{\infty}$ the hyperplane at infinity, as the 3-dimensional projective and affine space over $\mathbb{F}_q$. The fact that these spaces are linked, will lead to interesting connections in the study of Cameron-Liebler line classes. We start with the definition of some line sets in both AG(3, q) and PG(3, q).

Definition 1.1. Consider PG(3, q) or AG(3, q).

1. A set of pairwise disjoint lines is called a partial line spread.

2. A set of conjugated switching sets consists out of two disjoint partial line spreads that cover the same set of points.

3. A line spread is a partial line spread that partitions the points of the corresponding space. The size of such line spreads is fixed. In PG(3, q) a line spread has size $q^2 + 1$ and in AG(3, q) it has size $q^2$.

In the following lemma we describe some examples of line spreads in AG(3, q).

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Lemma 1.2. Consider the affine space $AG(3,q)$ and the corresponding projective space $PG(3,q)$. Then the following sets $S$ are line spreads in $AG(3,q)$.

1. (Type I) Every line spread $PG(3,q)$ restricted to the affine space.

2. (Type II) Consider a point $p$ in $\pi_\infty$ and define the set $S$ as the set of all affine lines through $p$.

3. (Type III) Consider a line $\ell$ in $\pi_\infty$. Then there are exactly $q$ planes through $\ell$ not equal to $\pi_\infty$. Call these planes $\pi_i$, for $i \in \{1, \ldots, q\}$. If we choose for every plane $\pi_i$ a point $p_i$ on $\ell$ (not all equal), then we can define the spread

$$S := \{ \ell \mid \ell \text{ a line, such that } p_i \in \ell \subseteq \pi_i \text{ for some } i \in \{1, \ldots, q\} \}.$$ 

Proof. 1. Suppose that $S$ is a line spread in $PG(3,q)$. Then it is immediately clear that its restriction to $AG(3,q)$ defines a partial line spread, since every two distinct lines are still disjoint. Now if we consider an affine point $p$, then we know this point is covered by a line $l \in S$, which will be an affine line, thus lies in the restriction.

2. Trivial.

3. It is clear that all these lines of $S$ are pairwise disjoint. So if we can prove that for every affine point $p$ there exists an element of $S$ that contains it, we are done. Consider for this point $p$ the plane $\langle p, \ell \rangle$. Then this is a plane through $\ell$. Without loss of generality we may assume that it is $\pi_i$. Then the line $\langle p, p_i \rangle$ is a line in $S$ which contains $p$. This proves that $S$ is indeed a line spread.

Remark 1.3. We are well aware that a line spread of type II is in fact a line spread of type III, where we choose all the points $p_i$ equal. But since we will need to make a difference later, we distinguish these cases.

Figure 1: Spread of type III
2 Some useful results

In this section, we will give some useful results that will be used later on. We will give these results as general as possible so that they can be used in other contexts. First of all we choose \( n > 1 \) and \( 1 \leq k \leq n - 1 \). Here we consider the affine space \( \text{AG}(n, q) \) and the corresponding projective space \( \text{PG}(n, q) \). In these spaces, we define \( \Pi_k \) and \( \Phi_k \), as the set of \( k \)-spaces in \( \text{PG}(n, q) \), and \( \text{AG}(n, q) \), respectively.

**Construction 2.1 (Incidence matrix).** Consider the incidence matrix \( P_n \) of \( \text{PG}(n, q) \), where the rows correspond to the points and the columns correspond to the elements of \( \Pi_k \). We order the rows and columns in such a way that the first rows and columns correspond to the affine points and \( k \)-spaces respectively. Then \( P_n \) is of the following form:

\[
P_n = \begin{bmatrix} A & 0 \\ B_2 & P_{n-1} \end{bmatrix}.
\]

(1)

Here \( A \) is the incidence matrix of \( \text{AG}(n, q) \), where again the rows correspond to the points and the columns correspond to the elements of \( \Phi_k \). The matrix \( \bar{0} \) is the zero-matrix and the part that remains unnamed, we call \( B_2 \).

Before we state an important result, we will give a lemma that will be useful.

**Lemma 2.2.** ([4, Theorem 9.5]) The point-(\( k \)-space) incidence matrix of \( \text{PG}(n, q) \) or \( \text{AG}(n, q) \) has full rank.

Using this lemma, we give the following important result.

**Theorem 2.3.** Consider a set \( \mathcal{L} \) of \( k \)-spaces in \( \text{PG}(n, q) \) and let \( P_n \) and \( A \) be as in Construction 2.1. If the characteristic vector \( \chi_{\mathcal{L}} \in (\ker(P_n))^\perp \) and \( \mathcal{L} \) contains no \( k \)-spaces at infinity, then \( \chi_{\mathcal{L}} \) restricted to \( \text{AG}(n, q) \) belongs to \( (\ker(A))^\perp \).

**Proof.** We consider the characteristic vector \( \chi_{\mathcal{L}} \) of the set \( \mathcal{L} \). Since this set misses the hyperplane at infinity, it follows from our chosen ordering in Construction 2.1 that \( \chi_{\mathcal{L}} \) is of the form:

\[
\chi_{\mathcal{L}} = \begin{pmatrix} \chi_{\mathcal{L}} \\ 0 \end{pmatrix} \in (\ker(P_n))^\perp.
\]

Consider now a vector \( v_1 \in \ker(A) \), then we need to prove that \( \chi_{\mathcal{L}} \cdot v_1 = 0 \). We claim that we are able to find a vector \( v_2 \), such that \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \ker(P_n) \). For this vector \( v \) we find that

\[
0 = \chi_{\mathcal{L}} \cdot v = \chi_{\mathcal{L}} \cdot v_1,
\]

and since \( v_1 \) was arbitrarily chosen, the proof is done. But we still need to prove our claim. So for \( v_1 \) as above, we need to find a vector \( v_2 \) such that

\[
\begin{cases}
Av_1 + 0v_2 = 0 \\
B_2v_1 + P_{n-1}v_2 = 0
\end{cases}
\]

holds. Notice that line 1 in Equation (2) is always zero since \( v_1 \in \ker(A) \). Due to Lemma 2.2, the matrix \( P_{n-1} \) has full rank. Hereby, we can always find a vector \( v_2 \) that makes line 2 in Equation (2) zero. \(\square\)
A last result that we want to state is the following. For this result we need to define $k$-spreads in $\text{AG}(n,q)$. A $k$-spread in $\text{AG}(n,q)$ is a set of skew $k$-spaces that partitions the point set of $\text{AG}(n,q)$.

**Lemma 2.4.** Consider a set $\mathcal{L}$ of $k$-spaces in $\text{AG}(n,q)$, such that $\chi_{\mathcal{L}} \in (\ker(A))^\perp$ with $A$ equal to the point-(k-space) incidence matrix of $\text{AG}(n,q)$. Then it follows that for every affine $k$-spread $\mathcal{S}$, it holds that

$$|\mathcal{L} \cap \mathcal{S}| = x,$$

for a certain fixed integer $x$.

**Proof.** Let $\chi_{\mathcal{S}}$ be the characteristic vector of the $k$-spread $\mathcal{S}$, then

$$\chi_{\mathcal{S}} - \begin{bmatrix} n \\ k \end{bmatrix}_q^{-1} \mathbf{j} \in \ker(A).$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ equals the number of $k$-spaces through a point in $\text{AG}(n,q)$ and $\mathbf{j}$ is the vector of the correct dimension that contains 1 on every position. Equation (3) is valid since every point is contained in exactly one element of $\mathcal{S}$, so every row of $A$ and $\chi_{\mathcal{S}}$ have exactly one 1 in common. The proof follows since from $\chi_{\mathcal{L}} \in (\ker(A))^\perp$, we know that

$$\chi_{\mathcal{L}} \cdot \left( \chi_{\mathcal{S}} - \begin{bmatrix} n \\ k \end{bmatrix}_q^{-1} \mathbf{j} \right) = 0,$$

or simplified we get

$$|\mathcal{L} \cap \mathcal{S}| = \begin{bmatrix} n \\ k \end{bmatrix}_q^{-1} |\mathcal{L}|.$$

Here we denote $\begin{bmatrix} n \\ k \end{bmatrix}_q^{-1} |\mathcal{L}|$ by $x$. \hfill \Box

### 3 Cameron-Liebler line classes

Cameron-Liebler line classes were first observed in [2]. In this article, Cameron and Liebler tried to classify the collineation subgroups of $\text{PG}(3,q)$ that have the same number of orbits on the lines as on the points. They noticed that these orbits on the lines have special properties. Line classes that satisfy these properties were later called Cameron-Liebler line classes. We will give the definition of a Cameron-Liebler line class for $\text{PG}(3,q)$ as well as for $\text{AG}(3,q)$.

**Definition 3.1.** A *Cameron-Liebler line class* of parameter $x$ in both $\text{PG}(3,q)$ or $\text{AG}(3,q)$ is a class of lines, such that for every line spread $\mathcal{S}$ it holds that $|\mathcal{L} \cap \mathcal{S}| = x$.

Yet there are many other equivalent definitions known for Cameron-Liebler line classes in $\text{PG}(3,q)$. The following theorem will list some of them.
**Theorem 3.2.** ([10, Theorem 1] and [8, Definition 1.1]) Let \( \mathcal{L} \) be a set of lines of size \( |\mathcal{L}| = x(q^2 + q + 1) \) in \( \text{PG}(3, q) \) with characteristic vector \( \chi_{\mathcal{L}} \). Then the following properties are equivalent:

1. Let \( P_3 \) be the point-line incidence matrix of \( \text{PG}(3, q) \), then \( \chi_{\mathcal{L}} \in (\ker(P_3))^\bot = \text{Im}(P_3^T) \).
2. For every line spread \( \mathcal{S} \), \( |\mathcal{L} \cap \mathcal{S}| = x \).
3. For every pair of conjugated switching sets \( \mathcal{R} \) and \( \mathcal{R}' \), \( |\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'| \).
4. For every line \( \ell \), there are exactly \( (x - \chi_{\mathcal{L}}(\ell))q^2 \) elements of \( \mathcal{L} \) disjoint to \( \ell \). Here \( \chi_{\mathcal{L}}(\ell) = 1 \) if \( \ell \in \mathcal{L} \) and zero otherwise.
5. For every point \( p \) and plane \( \pi \), with \( p \in \pi \),

\[
|\text{star}(p) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q + 1)|\text{pencil}(p, \pi) \cap \mathcal{L}|.
\]

Here \( \text{star}(p) \) is the set of lines through the point \( p \), \( \text{line}(\pi) \) is the set of lines in the plane \( \pi \) and \( \text{pencil}(p, \pi) = \text{star}(p) \cap \text{line}(\pi) \).

Some trivial examples of Cameron-Liebler line classes in \( \text{PG}(3, q) \) are the following:

**Example 3.3.** [2, Proposition 3.4] The following sets are Cameron-Liebler line classes in \( \text{PG}(3, q) \).

1. The empty set, which has parameter \( x = 0 \).
2. All the lines through a fixed point \( p \). This example has parameter \( x = 1 \).
3. All the lines in a fixed plane. Also this example has parameter \( x = 1 \) and is in fact the dual of the previous example.
4. Take a plane \( \pi \) and a point \( p \notin \pi \), then the set of all the lines through \( p \) together with all the lines in \( \pi \) gives a Cameron-Liebler line class of parameter \( x = 2 \).

**Theorem 3.4.** [2, Proposition 3.4] If \( \mathcal{L} \) is a Cameron-Liebler line class of parameter \( x \in \{0, 1, 2\} \) in \( \text{PG}(3, q) \), then \( \mathcal{L} \) is listed in Example 3.3.

It was long thought that these examples together with their complements were the only possible Cameron-Liebler line classes. This was disproven by Drudge in [5] who found an example of parameter \( x = 5 \) in \( \text{PG}(3, 3) \). This example was later generalized in [11] to an infinite family of parameter \( x = \frac{q^2 - 1}{2} \) in \( \text{PG}(3, q) \), for \( q \) odd. More recently another infinite family was found simultaneously in [3] and [6] and has parameter \( x = \frac{q^2 - 1}{2} \) in \( \text{PG}(3, q) \), with \( q \equiv 5 \) or 9 (mod 12).

Our goal is to consider Cameron-Liebler line classes in \( \text{AG}(3, q) \), in a similar way as in \( \text{PG}(3, q) \) and try to obtain similar results.

**Remark 3.5.** We should also remark that due to Lemma 1.2 we know that \( \text{AG}(3, q) \) has significantly more line spreads than \( \text{PG}(3, q) \). This together with Definition 3.1 yields the intuition that Cameron-Liebler line classes in \( \text{AG}(3, q) \) satisfy more conditions than those in \( \text{PG}(3, q) \) and thus are more rare.

A second observation is the size of a Cameron-Liebler line class in \( \text{AG}(3, q) \).
Lemma 3.6. Consider a Cameron-Liebler line class $\mathcal{L}$ of parameter $x$ in $AG(3, q)$. Then $|\mathcal{L}| = x(q^2 + q + 1)$.

Proof. We shall count the pairs $(\ell, S)$, where $S$ is a line spread of type II and $\ell \in S \cap \mathcal{L}$. If we denote the number of line spreads of type II through $i$ fixed lines as $n_i$, for $i \in \{0, 1\}$, then

$$n_0 x = |\mathcal{L}| n_1.$$ 

But we should notice that $n_0$ is the number of line spreads of type II. This is equal to the number of points in the plane $\pi_\infty$, so equal to $q^2 + q + 1$. In fact we also know $n_1$. If we pick an arbitrary line $\ell$, then there is only one line spread of type II through it, namely the spread of type II defined by $\ell \cap \pi_\infty$. So in total we have that

$$(q^2 + q + 1) x = |\mathcal{L}|,$$

which proves the lemma. \hfill $\blacksquare$

Interesting here is that a Cameron-Liebler line class in $AG(3, q)$ has the same size as a Cameron-Liebler line class in $PG(3, q)$ with the same parameter $x$. The analog goes for the following equivalences, which hold for as well $PG(3, q)$ as $AG(3, q)$.

Lemma 3.7. Consider Cameron-Liebler line classes $\mathcal{L}$ and $\mathcal{L}'$ with parameter $x$ and $x'$ (in $PG(3, q)$ or $AG(3, q)$), then the following statements are true.

1. For the parameter $x$, $0 \leq x \leq q^2 + 1$ in $PG(3, q)$ or $0 \leq x \leq q^2$ in $AG(3, q)$.

2. If $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{L}' \setminus \mathcal{L}$ is a Cameron-Liebler line class of parameter $x - x'$. Consequently the complement of $\mathcal{L}$ in $AG(3, q)$ and $PG(3, q)$ is also a Cameron-Liebler line class of parameter $q^2 - x$ and $q^2 + 1 - x$, respectively.

3. If $\mathcal{L} \cap \mathcal{L}' = \emptyset$, then $\mathcal{L} \cup \mathcal{L}'$ is a Cameron-Liebler line class of parameter $x + x'$.

Proof. Property (1) follows from the size of a line spread and Definition [3.1]. The other properties also follow from the same definition. \hfill $\blacksquare$

Inspired by this, we try to make a link between Cameron-Liebler line classes in $AG(3, q)$ and $PG(3, q)$. A first step to do this is by the following theorem.

Theorem 3.8. If $\mathcal{L}$ is a Cameron-Liebler line class of $AG(3, q)$, then $\mathcal{L}$ is a Cameron-Liebler line class in the corresponding projective space $PG(3, q)$ with the same parameter $x$.

Proof. Consider a Cameron-Liebler line class $\mathcal{L}$ in $AG(3, q)$ and consider the corresponding projective space $PG(3, q)$. We know that for every line spread $S$ in $PG(3, q)$ the restriction to the affine space is also a line spread in $AG(3, q)$. Thus this restriction has exactly $x$ elements in common with $\mathcal{L}$. But since $\mathcal{L}$ is a line set in the affine space, it contains no lines in $\pi_\infty$, such that $|\mathcal{L} \cap S| = x$. This proves the theorem. \hfill $\blacksquare$

A second step is to find a result that is sort of a converse theorem for the one we found above.

Theorem 3.9. Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $PG(3, q)$ of parameter $x$. Then $\mathcal{L}$ defines a Cameron-Liebler line class in $AG(3, q)$ with the same parameter $x$ if and only if $\mathcal{L}$ is skew to all the lines in the plane at infinity of $AG(3, q)$. 

6
Proof. Consider Construction 2.1 for \( k = 1 \), then the columns correspond to lines. So we have the point-line incidence matrix of the affine and the projective space denoted by \( A \) and \( P_3 \) respectively. Consider a Cameron-Liebler line class \( \mathcal{L} \) with parameter \( x \) in the projective space that is skew to the set of lines in the plane at infinity. Then we write the characteristic vector corresponding to the projective space as \( \bar{\chi}_L \) and the characteristic vector corresponding to the affine space as \( \chi_L \). Since we know that \( \mathcal{L} \) is disjoint to the plane at infinity and \( \mathcal{L} \) is a projective Cameron-Liebler line class we have that,

\[
\bar{\chi}_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \in (\ker(P_3))^\perp.
\]

If we now use Theorem 2.3 we get that \( \chi_L \in (\ker(A))^\perp \). So due to Lemma 2.4 we get that \( \mathcal{L} \) is also a Cameron-Liebler line class in the \( AG(3, q) \). Note that the parameter stays the same, since \( |\mathcal{L}| = (q^2 + q + 1)x \) as a projective Cameron-Liebler line class. Thus, by Lemma 3.6 \( \mathcal{L} \) also has parameter \( x \) in \( AG(3, q) \).

We now prove the converse. Let \( \mathcal{L} \) be a projective Cameron-Liebler line class of parameter \( x \), such that its restriction to the affine space also defines a Cameron-Liebler line class of parameter \( x \). It follows, by Theorem 3.8, that we can consider this restriction in \( PG(3, q) \) as a projective Cameron-Liebler line class \( \mathcal{L}' \) of parameter \( x \) that misses \( \pi_\infty \). Now we know that \( \mathcal{L}' \subseteq \mathcal{L} \), so due to Lemma 3.7 it follows that \( \mathcal{L} \setminus \mathcal{L}' \) defines a Cameron-Liebler line class of parameter \( x - x = 0 \) in \( PG(3, q) \). In particular, we know that this line class lies inside \( \pi_\infty \). It is clear that this can only be the empty set, such that \( \mathcal{L} = \mathcal{L}' \) and does not contain lines in \( \pi_\infty \).

Remark 3.10. Suppose that \( \mathcal{L} \) is a Cameron-Liebler line class with parameter \( x + 1 \) in \( PG(3, q) \), that contains all the lines in the plane at infinity. Then we can use Example 3.3 and Lemma 3.7 to obtain a Cameron-Liebler line class with parameter \( x \) in \( PG(3, q) \) skew to the set of lines at infinity. Here we can use the previous theorem and we see a 1–1 connection between these projective Cameron-Liebler line classes and Cameron-Liebler line classes in \( AG(3, q) \) of parameter \( x \).

4 Equivalent definitions and non-existence conditions

There are a lot of equivalent definitions for Cameron-Liebler line classes in \( PG(3, q) \). We can ask, keeping Theorem 3.8 in mind, if they correspond to equivalent definitions in the affine case.

Theorem 4.1. Consider in the affine space \( AG(3, q) \) a set of lines \( \mathcal{L} \) such that \( |\mathcal{L}| = x(q^2 + q + 1) \), with \( x \) a positive integer. Let \( A \) be the point-line incidence matrix of \( AG(3, q) \), then the following properties are equivalent.

1. For every line spread \( \mathcal{S} \), \( |\mathcal{L} \cap \mathcal{S}| = x \).
2. The characteristic vector \( \chi_L \in (\ker(A))^\perp \).
3. The characteristic vector \( \chi_L \in \text{Im}(A^T) \).
4. For every pair of conjugated switching sets \( \mathcal{R} \) and \( \mathcal{R}' \), \( |\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'| \).
5. For every line \( \ell \), the number of elements of \( L \) disjoint to \( \ell \) is equal to 

\[(q^2 + 1)(x - \chi_L(\ell)),\]

and through every point at infinity there are exactly \( x \) lines of \( L \). Here \( \chi_L(\ell) = 1 \) if \( \ell \in L \) and zero otherwise.

**Proof.** First of all it is clear that (2) is equivalent with (3), since this is the case for every matrix.

1. From (1) to (2): Note that \( L \) is by definition a Cameron-Liebler line class in \( \text{AG}(3, q) \). So from Theorem 3.3, we know that \( L \) defines a Cameron-Liebler line class in the corresponding projective space. Here we can use Theorem 3.2 to obtain that the characteristic vector corresponding to \( L \) in PG(3, q) lies in \((\ker(P_3))^\perp\), with \( P_3 \) the point-line incidence matrix of PG(3, q) (see Construction 2.1). Since \( L \) does not contain lines in \( \pi_\infty \), we may use Theorem 2.3. Thus we obtain that the characteristic vector restricted to the affine space lies in \((\ker(A))^\perp\). But this restriction is exactly \( \chi_L \).

2. From (2) to (4): Since \( R \) and \( R' \) are conjugated switching sets, they cover the same set of points, so it necessarily holds for the characteristic vectors that 

\[\chi_R - \chi_{R'} \in \ker(A)\].

This implies that 

\[\chi_L \cdot (\chi_R - \chi_{R'}) = 0.\]

From here we find that 

\[|L \cap R| - |L \cap R'| = 0,\]

which proves the statement.

3. From (4) to (1): Consider two spreads \( S_1, S_2 \). Then we know that \( S_1 \setminus S_2 \) and \( S_2 \setminus S_1 \) are conjugated switching sets, since they cover the same set of points and have no elements in common. So, by (4), we know that 

\[|L \cap (S_1 \setminus S_2)| = |L \cap (S_2 \setminus S_1)|\]

which implies that 

\[|L \cap S_1| = |L \cap S_2| = c.\]

But we still have to prove that \( c = x \). This we do by double counting the pairs \((\ell, S)\), with \( S \) a line spread of type II containing the line \( \ell \) and \( \ell \in L \). We get

\[|L| \cdot n_1 = n_0 \cdot c,\]

where \( n_i \) is the number of line spreads of type II through \( i \) fixed lines, with \( i \in \{0, 1\} \).

So we know that \( n_0 = q^2 + q + 1 \) and \( n_1 = 1 \). This gives that 

\[|L| = (q^2 + q + 1) \cdot c.\]

Since we also know that \( |L| = (q^2 + q + 1)x \), we get that \( c = x \).
Till here we already have proven that the first four points are equivalent, so we only need to prove the equivalence with property (5).

First if Property (1) holds, then \( \mathcal{L} \) is a Cameron-Liebler line class in \( \text{AG}(3, q) \). So by Theorem 3.8 this line class is also a Cameron-Liebler line class in the corresponding projective space with the same parameter \( x \). Due to Theorem 3.2 we know that there are exactly \( q^2(x - \chi_\mathcal{L}(\ell)) \) lines of \( \mathcal{L} \) disjoint to \( \ell \) in \( \text{PG}(3, q) \). Since \( \mathcal{L} \) is an affine line set, these \( q^2(x - \chi_\mathcal{L}(\ell)) \) lines are all lines of \( \mathcal{L} \) that are disjoint to \( \ell \) in \( \text{AG}(3, q) \). So the only elements we still need to count are those lines of \( \mathcal{L} \) that are disjoint to \( \ell \) in \( \text{AG}(3, q) \) and intersect in \( \text{PG}(3, q) \). Hence these lines will intersect \( \ell \) in the point \( p = \ell \cap \pi_\infty \) at infinity. If we consider the line spread \( \mathcal{S} \) of type II containing all affine lines through \( p = \ell \cap \pi_\infty \), then \( |\mathcal{L} \cap \mathcal{S}| = x \).

From this we obtain the second part and that there are in total exactly

\[
(q^2 + 1)(x - \chi_\mathcal{L}(\ell))
\]
elements of \( \mathcal{L} \) disjoint to \( \ell \) in \( \text{AG}(3, q) \).

We now prove the converse direction. Consider a set of lines \( \mathcal{L} \) such that Property (5) holds. Then we know that for every affine line \( \ell \) in \( \text{PG}(3, q) \) the number of lines of \( \mathcal{L} \) that are disjoint to \( \ell \) in \( \text{PG}(3, q) \) is equal to \( q^2(x - \chi_\mathcal{L}(\ell)) \). This follows from the fact that we subtracted those \( x - \chi_\mathcal{L}(\ell) \) lines that intersect \( \ell \) at infinity. But if we now look at a line \( \ell' \subseteq \pi_\infty \), then we know that there are \( q^2 \) points in \( \pi_\infty \setminus \ell' \). Through each of those points we have \( x \) affine lines of \( \mathcal{L} \), which are all disjoint to \( \ell' \) in \( \text{PG}(3, q) \). So these are also all the disjoint lines of \( \mathcal{L} \), since if we had another line of \( \mathcal{L} \) that is disjoint to \( \ell' \), it should first be an affine line that then intersects \( \pi_\infty \) in a point. This implies that we in fact already counted it. We get that there are \( q^2 x \) elements of \( \mathcal{L} \) disjoint to \( \ell' \not\in \mathcal{L} \). So we conclude that for any arbitrary projective line \( \ell \) in \( \text{PG}(3, q) \) there are exactly

\[
q^2(x - \chi_\mathcal{L}(\ell))
\]
lines of \( \mathcal{L} \) disjoint to \( \ell \). This is equivalent with definition (5) in Theorem 3.2 of a Cameron-Liebler line class in the projective space. Thus \( \mathcal{L} \) defines a Cameron-Liebler line class in \( \text{AG}(3, q) \), where the parameter \( x \) follows from its size.

For the last part of this paper we want to give some non-existence results and some examples. Let us first state a more recent result of Gavrilyuk and Metsch in [7] about Cameron-Liebler line classes in \( \text{AG}(3, q) \).

**Theorem 4.2.** ([7, Theorem 1.1]) Suppose that \( \mathcal{L} \) is a Cameron-Liebler line class with parameter \( x \) of \( \text{PG}(3, q) \). Then for every plane and every point of \( \text{PG}(3, q) \),

\[
\left(\frac{x}{2}\right) + n(n - x) \equiv 0 \mod (q + 1),
\]

where \( n \) is the number of lines of \( \mathcal{L} \) in the plane, respectively through the point.

If we translate this result with Theorem 3.8 to \( \text{AG}(3, q) \), it will look like this.

**Corollary 4.3.** If \( \mathcal{L} \) defines a Cameron-Liebler line class in \( \text{AG}(3, q) \) with parameter \( x \), then the equation

\[
\frac{x(x - 1)}{2} \equiv 0 \mod (q + 1)
\]

holds.
Proof. Due to Theorem 3.8 every Cameron-Liebler line class $\mathcal{L}$ in $\text{AG}(3, q)$ is a Cameron-Liebler line class in $\text{PG}(3, q)$ of the same parameter $x$. So we may use Theorem 4.2. Here we notice that $\mathcal{L}$ is skew to the set of all lines in a plane, namely the plane at infinity. Thus we may fill in $n = 0$.

One could ask how good this non-existence condition is. What if we for example choose another plane or point in $\text{PG}(3, q)$, could we find more conditions by using Theorem 4.2. Notice first that choosing a point at infinity would not help, since this point defines a line spread of type II and thus would always contain $n = x$ lines of $\mathcal{L}$. This leads to the same result. But what if we choose another plane? It can be proven that this is not helpful either. This is done in the following lemma.

**Lemma 4.4.** Let $\mathcal{L}$ be a Cameron-Liebler line class of parameter $x$ in $\text{AG}(3, q)$, then for every (affine) plane $\pi$ it holds that

$$|\text{line}(\pi) \cap \mathcal{L}| \equiv 0 \mod (q + 1).$$

**Proof.** We consider the affine space $\text{AG}(3, q)$, together with the corresponding projective space $\text{PG}(3, q)$. We also define $\pi_{\infty}$ as the plane at infinity. Consider a Cameron-Liebler line class $\mathcal{L}$ of parameter $x$ in $\text{AG}(3, q)$. Then, by Theorem 3.8 we know that $\mathcal{L}$ defines a Cameron-Liebler line class in $\text{PG}(3, q)$ with the same parameter $x$. Here we can use Theorem 4.2 to obtain that for every point $p$ and plane $\pi$, with $p \in \pi$, we get that

$$|\text{star}(p) \cap \mathcal{L}| + |\text{line}(\pi) \cap \mathcal{L}| = x + (q + 1)|\text{pencil}(p, \pi) \cap \mathcal{L}|.$$

If we now choose $\pi$ as an arbitrary affine plane and $p$ as a point at infinity. Then we get that $|\text{star}(p) \cap \mathcal{L}| = x$ and so the requested result.

This lemma proves that choosing another plane for Theorem 4.2 will not improve Corollary 4.3, since this leads to the same result. So the only improvements that can be made is by looking at different points not at infinity.

**Corollary 4.5.** The only Cameron-Liebler line class of parameter $x \in \{1, 2\}$ in $\text{AG}(3, q)$ consists of all the lines through an affine point. Thus this Cameron-Liebler line class has parameter $x = 1$.

**Proof.** Suppose that $\mathcal{L}$ defines a Cameron-Liebler line class of parameter $x$ in $\text{AG}(3, q)$. Then we know due to Theorem 3.8 that $\mathcal{L}$ also defines a Cameron-Liebler line class in $\text{PG}(3, q)$ of the same parameter $x$. So for $x = 1$ we know that, due to Theorem 3.4, $\mathcal{L}$ consists of all lines through a point or the set of all the lines in a plane. Since it is immediately clear that all the lines in a plane does not satisfy the constant intersection axiom with line spreads of type II, we may conclude that the only example for $x = 1$ left is the set of all lines through a point.

The case $x = 2$ follows trivially from Corollary 4.3 since it is clear that $x = 2$ does never satisfy $\frac{x(x-1)}{2} \equiv 0 \mod (q + 1)$.

**Remark 4.6.** As a consequence of this Corollary and Lemma 3.7 we know that in $\text{AG}(3, q)$ there do not exist Cameron-Liebler line classes of parameter $x = q^2 - 2$. Another consequence is that the only possibility for $x = q^2 - 1$ consists of the complement of all the lines through an affine point.
5 Number of possible parameters $x$ for Cameron-Liebler line classes in $AG(3, q)$

Consider again Equation (5) which is equivalent to

$$x(x - 1) \equiv 0 \mod 2(q + 1).$$

(6)

So we conclude, by Corollary 4.3, that if a Cameron-Liebler line class $L$ with parameter $x \in \{0, 1, \ldots, q^2\}$ exists in $AG(3, q)$, then Equation (6) holds for $x$. Now if we consider the prime factorization $2(q+1) = p_1^{h_1} \cdots p_s^{h_s}$, then we can take a look at the system of equations

$$x(x - 1) \equiv 0 \mod p_i^{h_i}, \text{ for } i \in \{1, \ldots, s\}.$$

(7)

We know that if we find a value $x$ that satisfies this system of equations, we get, due to the fact that all the prime powers are coprime, a solution for (6). In fact the Chinese Remainder Theorem states that this solution will be unique modulo $2(q + 1)$.

Lemma 5.1. Let $L$ be a Cameron-Liebler line class with parameter $x$ in $AG(3, q)$, where the prime factorization of $2(q+1)$ is equal to $p_1^{h_1} \cdots p_s^{h_s}$. Then

$$x \equiv 0 \pmod{p_i^{h_i}} \text{ or } x \equiv 1 \pmod{p_i^{h_i}}, \text{ for } i \in \{1, \ldots, s\}.$$

(8)

Proof. This follows from Equations (6) and (7). □

This lemma will enable us to find an upper bound for the number of parameters $x$ of possible Cameron-Liebler line classes in $AG(3, q)$.

Theorem 5.2. Let $x$ be the parameter of a Cameron-Liebler line class $L$ in $AG(3, q)$, so $x \leq q^2$. Consider now the prime factorization $2(q+1) = p_1^{h_1} \cdots p_s^{h_s}$. Then there are at most

$$\begin{cases} 2^{s-1}q, & \text{if } q \text{ is even} \\ 2^{s-1}q - 2^{s-1} + 2, & \text{if } q \text{ is odd} \end{cases}$$

(9)

possibilities for $x$.

Proof. If $L$ is a Cameron-Liebler line class with parameter $x$ in $AG(3, q)$, then by Corollary 4.3 it follows that Equation (6) holds. So to count the maximal number of possible parameters, we need to count the maximal number of solutions for (6).

Due to our previous observations about the Chinese Remainder Theorem and Lemma 5.1 we only need to count the number of possible solutions for the equations

$$x \equiv 0, 1 \mod p_i^{h_i},$$

for every $i \in \{1, \ldots, s\}$. If we now pick in every equation a 1 or 0, then we know that there is a unique solution for

$$x(x - 1) \equiv 0 \mod 2(q + 1).$$

Note that there are $2^s$ possibilities to pick such a solution. But remark that these solutions are considered in the interval $I = [0, 2(q+1) - 1]$ and adding $2(q+1)$ to a solution gives a new solution. So one can ask how many times the interval $I$ fits inside the interval $[0, q^2]$. This is equal to the number $\left\lfloor \frac{q^2}{2(q+1)} \right\rfloor$.  

11
1. For $q \equiv 1 \pmod{2}$:

$$\frac{q^2}{2(q+1)} = \frac{q-1}{2} + \frac{1}{2(q+1)},$$

where $\frac{q-1}{2} \in \mathbb{N}$, since $q$ is odd. This gives that $I$ fits $\frac{q-1}{2}$ times inside $[0, q^2]$. So in each of the following intervals, there is precisely one solution for every choice we made before

$$[0, 2(q+1)-1], [2(q+1), 4(q+1)-1], \ldots, \left[\left(\frac{q-1}{2} - 1\right)2(q+1), \left(\frac{q-1}{2}\right)2(q+1)-1\right],$$

where the last interval in this row can be simplified as follows

$$\left[\left(\frac{q-1}{2} - 1\right)2(q+1), \left(\frac{q-1}{2}\right)2(q+1)-1\right] = [q^2 - 2q - 3, q^2 - 2].$$

This all gives at most $2^s\left(\frac{q-1}{2}\right)$ solutions in the first $\frac{q-1}{2}$ intervals. So now we only need to add the solutions for $q^2 - 2 < x \leq q^2$. For $x = q^2 - 1$ there is, by Remark 4.6, only one Cameron-Liebler line class: the complement of all lines through a point. If $x = q^2$, we see that the only possibility is to consider every line in $AG(3, q)$. So we get in total

$$2^s\left(\frac{q-1}{2}\right) + 2 = 2^s q - 2^{s-1} + 2$$

solutions for $x \in [0, q^2]$.

2. For $q \equiv 0 \pmod{2}$, we get:

$$\frac{q^2}{2(q+1)} = \frac{q-2}{2} + \frac{q+2}{2(q+1)}.$$

Now $\frac{q+2}{2} \in \mathbb{N}$, since $q$ is even. So we get for all $x$ in one of these intervals at most $2^s\left(\frac{q-2}{2}\right)$ solutions. One can calculate that the last interval that fits inside $[0, q^2]$ is of the form $[q^2 - 3q - 4, q^2 - q - 3]$. So for $q^2 - q - 3 < x \leq q^2$, we can only estimate that there are at most $2^s$ solutions. So now there are at most

$$2^s\left(\frac{q-2}{2}\right) + 2^s = 2^{s-1} q$$

solutions for $x \in [0, q^2]$.

\[\Box\]

6 A non-trivial example and a consequence

Here we give a non-trivial example of a Cameron-Liebler line class in $AG(3, q)$. We will use the example of De Beule, Demeyer, Metsch and Rodgers stated in [3] and the example of Feng, Momihara and Xiang stated in [6]. Both articles simultaneously found a Cameron-Liebler line class with parameter $x = \frac{q^2 - 1}{2}$ in $PG(3, q)$ that is skew to the set of all lines in a plane $\pi$ (for $q \equiv 5, 9 \pmod{12}$).
**Theorem 6.1.** \[3, \text{Theorem 5.1 and Lemma 6.1}\] There exists a Cameron-Liebler line class of parameter \( x = \frac{q^2 - 1}{2} \) in \( \text{PG}(3, q) \), for \( q \equiv 5 \text{ or } 9 \pmod{12} \), which is skew to the set of all lines in a plane.

This example restricted to \( \text{AG}(3, q) \), where we choose the plane at infinity as \( \pi \), gives us by Theorem 3.9 a Cameron-Liebler line class in \( \text{AG}(3, q) \) with the same parameter. So we can conclude the following corollary.

**Corollary 6.2.** Consider the affine space \( \text{AG}(3, q) \). If \( q \equiv 5 \text{ or } 9 \pmod{12} \), then there exists a Cameron-Liebler line class with parameter \( x = \frac{q^2 - 1}{2} \).

**Remark 6.3.** This corollary, together with Remark 4.6, also gives that for \( q \equiv 5 \text{ or } 9 \pmod{12} \), there exists a Cameron-Liebler line class with parameter \( x = \frac{q^2 + 1}{2} \) in \( \text{AG}(3, q) \).

Now we want to give a complete characterization of the parameters for Cameron-Liebler line classes in \( \text{AG}(3, 5) \). This result was based on a similar result for \( \text{PG}(3, 5) \) in [7], where they used Theorem 4.2. In a similar way we will use Corollary 4.3 to achieve this for \( \text{AG}(3, 5) \). Let us first state the result found in [7].

**Theorem 6.4.** \[7, \text{Theorem 1.3}\] A Cameron-Liebler line class with parameter \( x \) exists in \( \text{PG}(3, 5) \) if and only if \( x \in \{0, 1, 2, 10, 12, 13, 14, 16, 24, 25, 26\} \).

**Corollary 6.5.** There exists a Cameron-Liebler line class \( \mathcal{L} \) of parameter \( x \) in \( \text{AG}(3, 5) \) if and only if \( x \in \{0, 1, 12, 13, 24, 25\} \).

**Proof.** Due to Theorem 3.8, every Cameron-Liebler line class in \( \text{AG}(3, 5) \) defines a Cameron-Liebler line class in \( \text{PG}(3, 5) \) with the same parameter. Hence if there exists no Cameron-Liebler line class of parameter \( x \) in \( \text{PG}(3, 5) \), then there exists no Cameron-Liebler line class of parameter \( x \) in \( \text{AG}(3, 5) \).

From Theorem 6.4 it follows that \( x \in \{0, 1, 2, 10, 12, 13, 14, 16, 24, 25, 26\} \) are the only possible parameters for Cameron-Liebler line classes in \( \text{AG}(3, 5) \). But if we consider Corollary 4.3 with \( 2(q + 1) = 4 \cdot 3 \), this then reduces to \( x \in \{0, 1, 12, 13, 16, 24, 25\} \). Notice that if there would exist a Cameron-Liebler line class of parameter \( x = 16 \), then it follows from Lemma 3.7 that the complement of this line class is a Cameron-Liebler line class of parameter \( x = q^2 - 16 = 9 \). This parameter does not occur in the list of remaining possible parameters, so we find a contradiction.

We only need to show that all these cases occur. We list the following examples:

- \( x = 0 \): Put \( \mathcal{L} = \emptyset \).
- \( x = 1 \): Let \( \mathcal{L} \) be all the lines through a fixed affine point.
- \( x = 12 \): See Corollary 6.2
- \( x \in \{13, 24, 25\} \): Use Lemma 3.7 which states that the complement of a Cameron-Liebler line class is also a Cameron-Liebler line class.

\( \Box \)
7 Final remarks

In this last section, we want to compare our results to the result obtained by Penttila in [9]. In his PhD thesis, Penttila not only considered Cameron-Liebler line classes in PG(3, q), but also observed symmetrical tactical decompositions in PG(3, q). Such a symmetrical tactical decomposition is a pair (P, L), with P and L a specific partition of the points and lines of PG(3, q) respectively. For more information, we refer to [2]. Here it was proven that every line class of the partition L defines a Cameron-Liebler line class in PG(3, q). A similar case follows for AG(3, q). Besides this fact, Penttila also observed that if a symmetrical tactical decomposition of PG(3, q) contains the hyperplane at infinity as a line and point class, then this symmetrical decomposition is also a symmetrical tactical decomposition in AG(3, q). This result is comparable with Theorem 3.9. Conversely, every symmetrical tactical decomposition in AG(3, q) can be extended to a symmetrical tactical decomposition in PG(3, q). This fact is comparable with Theorem 3.8. Hence, these results are not exactly the same as Theorems 3.9 and 3.8, since not every Cameron-Liebler line class in PG(3, q) (and possibly in AG(3, q)) is a line class in a symmetrical tactical decomposition. Hence it remains interesting to find such analogous results.

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References

[1] A. A. Bruen and Keldon Drudge. The construction of Cameron-Liebler line classes in PG(3, q). Finite Fields Appl., 5(1):35–45, 1999.
[2] P.J. Cameron and R.A. Liebler. Tactical decompositions and orbits of projective groups. Linear Algebra Appl., 46:91–102, 1982.
[3] Jan De Beule, Jeroen Demeyer, Klaus Metsch, and Morgan Rodgers. A new family of tight sets in $Q^+(5, q)$. Des. Codes Cryptogr., 78(3):655–678, 2016.
[4] Bart De Bruyn. An introduction to incidence geometry. Frontiers in Mathematics. Birkhäuser/Springer, Cham, 2016.
[5] Keldon Drudge. On a conjecture of Cameron and Liebler. European J. Combin., 20(4):263–269, 1999.
[6] Tao Feng, Koji Momihara, and Qing Xiang. Cameron-Liebler line classes with parameter $x = \frac{q^2 - 1}{2}$. J. Combin. Theory Ser. A, 133:307–338, 2015.
[7] Alexander L. Gavrilyuk and Klaus Metsch. A modular equality for Cameron-Liebler line classes. J. Combin. Theory Ser. A, 127:224–242, 2014.
[8] Patrick Govaerts and Leo Storme. On Cameron-Liebler line classes. Adv. Geom., 4(3):279–286, 2004.
[9] Tim Penttila. Collineations and configurations in projective spaces. PhD thesis, Wolfson College, Oxford, 1985.
[10] Tim Penttila. Cameron-Liebler line classes in \( \text{PG}(3,q) \). \textit{Geom. Dedicata}, 37(3):245–252, 1991.