A $\mathbb{Z}_2$-index of symmetry protected topological phases with reflection symmetry for quantum spin chains

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Abstract

For the classification of SPT phases, defining an index is a central problem. In the famous paper [PTBO1], Pollmann, Tuner, Berg, and Oshikawa introduced $\mathbb{Z}_2$-indices for injective matrix products states (MPS) which have either $\mathbb{Z}_2 \times \mathbb{Z}_2$ dihedral group (of $\pi$-rotations about $x$, $y$, and $z$-axes) symmetry, time-reversal symmetry, or reflection symmetry. The first two are on-site symmetries. In [O1], an index for on-site symmetries, which generalizes the index in [PTBO1], was introduced for general unique gapped ground state phases in quantum spin chains. It was proved that the index is an invariant of the $C^1$-classification of SPT phases. The index for the reflection symmetry, which is not an on-site symmetry, was left as an open question. In this paper, we introduce a $\mathbb{Z}_2$-index for the reflection symmetric unique gapped ground state phases, and complete the generalization problem of index by Pollmann et.al. We also show that the index is an invariant of the $C^1$-classification.

1 Introduction

Classification of unique gapped ground states in quantum many-body systems is an important problem in modern condensed matter physics and quantum information science. In one dimension, it is believed that all unique gapped ground states belong to a single phase, in the sense that any two such ground states can be smoothly connected with each other through a series of models with unique gapped ground states. This conjecture was verified for frustration free models with uniformly bounded degeneracy [O3]. Motivated by the study of the Haldane phenomena in antiferromagnetic quantum spin chains, Gu and Wen [GW] proposed a finer classification based on the notion of symmetry protected topological (SPT) phase. Instead of considering the whole family of Hamiltonians, we fix some symmetry and consider the set of all Hamiltonians with a unique gapped ground state in the bulk, satisfying the symmetry. We then say such two Hamiltonians are equivalent if they can be connected to each other via a continuous path of symmetric Hamiltonians with unique gapped ground state. It can be possible that two symmetric Hamiltonians which can be connected via a path of non-symmetric gapped Hamiltonians fail to be connected via a path of symmetric gapped Hamiltonians. A Hamiltonian which can not be connected to trivial Hamiltonians (i.e, Hamiltonians with on-site interactions) via a symmetry preserving path belongs to the SPT phase. The question is how to show some Hamiltonian is in the SPT phase. One way should be defining some index which is stable along the path of symmetric gapped Hamiltonians. If some Hamiltonian has an index which is different from that of trivial phases, the Hamiltonian should be in a SPT phase. Finding such an index is a non-trivial important question for the classification problem of SPT phases.

In the famous paper [PTBO1], Pollmann, Tuner, Berg, and Oshikawa introduced $\mathbb{Z}_2$-indices for injective matrix products states (MPS) which have either $\mathbb{Z}_2 \times \mathbb{Z}_2$ dihedral group (of $\pi$-rotations

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about \(x, y, \) and \(z\)-axes) symmetry, time-reversal symmetry, or reflection symmetry. The first two are on-site symmetry, and the index is the cohomology class of some projective representation associated to the symmetric injective MPS. It was claimed there, that as the index takes discrete values, it should be stable under the continuous path of gapped Hamiltonians.

The \(\mathbb{Z}_2\)-index beyond the framework of matrix product state was recently introduced by Tasaki for systems satisfying on-site \(U(1)\)-symmetry together with one of \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-onsite symmetry/reflection symmetry/time-reversal symmetry \([\text{Tas1}]\). He showed that these are actually invariant of the classification.

In \([\text{O4}]\), we extended the index of Pollmann et.al. for on-site symmetry with full generality (without asking \(U(1)\)-symmetry). We also proved that our index is an invariant of the \(C^\ast\)-classification of SPT phases. The index for the reflection symmetry, which is not an on-site symmetry, was left as an open question. In this paper, we introduce a \(\mathbb{Z}_2\)-index for the reflection symmetric unique gapped ground state phase, and complete the generalization problem of index by Pollmann et.al.

Now let us state our result more in details. For a Hilbert space \(\mathcal{H}, B(\mathcal{H})\) denotes the set of all bounded operators on \(\mathcal{H}\). If \(V : \mathcal{H}_1 \to \mathcal{H}_2\) is a linear/anti-linear map from a Hilbert space \(\mathcal{H}_1\) to another Hilbert space \(\mathcal{H}_2\), then \(\text{Ad}(V) : B(\mathcal{H}_1) \to B(\mathcal{H}_2)\) denotes the map \(\text{Ad}(V)(x) := VxV^\ast, \ x \in B(\mathcal{H}_1)\).

We start by summarizing standard setup of quantum spin chains on the infinite chain \([\text{BR1}, \text{BR2}]\). Throughout this paper, we fix some \(2 \leq \mu, \nu \) by \(M \) for systems satisfying on-site symmetry, and the index is the cohomology class of some projective representation associated to the symmetric injective MPS. It was claimed there, that as the index takes discrete values, it should be stable under the continuous path of gapped Hamiltonians.

We denote the set of all finite subsets in \(Z\) by \(S\). For each \(n \in \mathbb{N}\), we set \(A_n := [-n, n] \cap \mathbb{Z}\). For each \(z \in \mathbb{Z}\), let \(A_{\{z\}}\) be an isomorphic copy of \(M_d\), and for any finite subset \(\Lambda \subset \mathbb{Z}\), let \(A_\Lambda = \otimes_{z \in \Lambda} A_{\{z\}}\), which is the local algebra of observables in \(\Lambda\). For finite \(\Lambda\), the algebra \(A_\Lambda\) can be regarded as the set of all bounded operators acting on the Hilbert space \(\otimes_{z \in \Lambda} \mathbb{C}^d\). We use this identification freely. If \(\Lambda_1 \subset \Lambda_2\), the algebra \(A_{\Lambda_1}\) is naturally embedded in \(A_{\Lambda_2}\) by tensoring its elements with the identity. The algebra \(A_{\mathbb{Z}}\) (resp. \(A_I\)) representing the half-infinite chain is given as the inductive limit of the algebras \(A_{\Lambda}\) with \(\Lambda \in \mathcal{S}\), \(\Lambda \subset [0, \infty)\) (resp. \(\Lambda \subset (-\infty, 1)\)). The algebra \(A\), representing the two sided infinite chain is given as the inductive limit of the algebras \(A_{\Lambda}\) with \(\Lambda \in \mathcal{S}\). Note that \(A_{\Lambda}\) for \(\Lambda \in \mathcal{S}\), \(A_{\Lambda}\), and \(A_I\) can be regarded naturally as subalgebras of \(\mathcal{A}\). We denote the set of local observables by \(A_{\text{loc}} = \bigcup_{\Lambda \in \mathcal{S}} A_{\Lambda}\). We denote by \(\beta_x\) the automorphisms on \(\mathcal{A}\) representing the space translation by \(x \in \mathbb{Z}\). By \(Q(j), \ j \in \mathbb{Z}\), we denote the element of \(\mathcal{A}\) with \(Q \in M_d\) in the \(j\)-th component of the tensor product of \(\mathcal{A}\) and the unit in any other component. The reflection \(\gamma\) is the unique \(*\)-automorphism on \(\mathcal{A}\) which satisfies

\[
\gamma \left( Q^{(j)} \right) = Q^{(-j-1)}, \quad \text{for all } Q \in M_d\text{ and } j \in \mathbb{Z}.
\]

From \(\gamma\), we define \(*\)-isomorphisms \(\gamma_{R \to L} : \mathcal{A}_R \to \mathcal{A}_L\) and \(\gamma_{L \to R} : \mathcal{A}_L \to \mathcal{A}_R\) by

\[
\gamma (\mathbb{I}_A \otimes A) = \gamma_{R \to L}(A) \otimes \mathbb{I}_{A_R}, \quad A \in \mathcal{A}_R,
\]

and

\[
\gamma (B \otimes \mathbb{I}_{A_R}) = \mathbb{I}_{A_L} \otimes \gamma_{L \to R}(B), \quad B \in \mathcal{A}_L.
\]

We introduce the \(\mathbb{Z}_2\)-index for reflection invariant pure states satisfying the split property (see Definition 2.1 in Section 2 Definition 2.7). Since a unique gapped ground state of a reflection invariant Hamiltonians satisfies these properties, this defines an index for such systems. (See Section 3) The definition of the index is simple. Let \(\omega\) be a reflection invariant pure state which satisfies the split property with respect to \(\mathcal{A}_L\) and \(\mathcal{A}_R\). We then can find its GNS triple of the form \((\mathcal{H}_\omega \otimes \mathcal{H}_\omega, \pi_\omega \otimes \gamma_{L \to R} \otimes \pi_\omega, \Omega_\omega)\) (where \(\pi_\omega\) is an irreducible representation of \(\mathcal{A}_R\) and a unitary operator \(\Gamma_\omega\) on \(\mathcal{H}_\omega \otimes \mathcal{H}_\omega\) implementing \(\gamma\). (Lemma 2.24) From the structure, we either
have \( \Gamma_\omega(\zeta \otimes \eta) = \eta \otimes \zeta \) for all \( \zeta, \eta \in \mathcal{H}_\omega \) or \( \Gamma_\omega(\zeta \otimes \eta) = -\eta \otimes \zeta \) for all \( \zeta, \eta \in \mathcal{H}_\omega \). (Theorem 2.6) This sign \( \sigma_\omega = \pm 1 \) is our \( \mathbb{Z}_2 \)-index. The same index can be obtained from the Tomita-Takesaki modular conjugation.: For the above GNS triple of \( \omega \), let \( \mathbb{I} \otimes s_\omega \) be the support projection of \( \Omega_\omega \) in \( \mathbb{I} \otimes B(\mathcal{H}_\omega) \). Then we can consider modular conjugation \( J_\omega \) associated to \( s_\omega \otimes B(s_\omega \mathcal{H}_\omega) \) and \( \Omega_\omega \). (Lemma 4.2) There exists an anti-unitary \( \theta : s_\omega \mathcal{H}_\omega \rightarrow s_\omega \mathcal{H}_\omega \) such that

\[
J_\omega (s_\omega \otimes x) J_\omega^* = \theta x \theta^* \otimes s_\omega, \quad J_\omega (x \otimes s_\omega) J_\omega^* = s_\omega \otimes \theta x \theta^* ,
\]

for all \( x \in B(s_\omega \mathcal{H}_\omega) \). (Proposition 4.3) This \( \theta \) satisfies \( \theta^2 = \kappa_\omega s_\omega \) with some \( \kappa_\omega \in \{-1, 1\} \), because of \( J_\omega^2 = s_\omega \otimes s_\omega \). It turns out that \( \kappa_\omega \) coincides with our \( \mathbb{Z}_2 \)-index \( \sigma_\omega \). (Theorem 4.3) This \( \theta \) is related to the Schmidt decomposition of \( \Omega_\omega \). (Lemma 4.2) Therefore, considering the Schmidt decomposition can be one way to calculate the index \( \sigma_\omega \). (Remark 4.5)

As stated above, for reflection invariant injective matrix product states, a \( \mathbb{Z}_2 \)-index was introduced in [PTB1]. It turns out that our \( \mathbb{Z}_2 \)-index restricted to such states coincides with that of [PTB1]. This is proven in Section 5 using the relation \( \kappa_\omega = \sigma_\omega \).

The \( \mathbb{Z}_2 \)-index \( \sigma_\omega \) is invariant under automorphic equivalence via an automorphism which allows a reflection invariant decomposition.:  

**Definition.** We say an automorphism \( \alpha \) of \( \mathcal{A} \) allows a reflection invariant decomposition if there is an automorphisms \( \alpha_R \) on \( \mathcal{A}_R \), and a unitary \( W \) in \( \mathcal{A} \) such that

\[
\tilde{\alpha}^{-1} \circ \alpha = \text{Ad}(W), \quad \gamma(W) = W,
\]

where

\[
\tilde{\alpha} := (\gamma_{R \rightarrow L} \circ \alpha_R \circ \gamma_{L \rightarrow R}) \otimes \alpha_R.
\]

From the definition, we can show the following:

**Theorem.** (See Theorem 2.9 for details.) Let \( \omega_0, \omega_1 \) be reflection invariant pure states satisfying the split property. Suppose that \( \omega_0 \) and \( \omega_1 \) are automorphic equivalent via an automorphism which allows a reflection invariant decomposition. Then the \( \mathbb{Z}_2 \)-indices \( \sigma_{\omega_0}, \sigma_{\omega_1} \) associated to \( \omega_0, \omega_1 \) are equal.

Recalling that a unique gapped ground state is pure and satisfies the split property (see Theorem 3.2), our \( \mathbb{Z}_2 \)-index can be understood as an index of of reflection invariant Hamiltonians with unique gapped ground states. It turns out that this \( \mathbb{Z}_2 \)-index is an invariant of the \( C^1 \)-classification.:  

**Corollary.** (See Theorem 3.6 for more precise statement) Let us consider a \( C^1 \)-path of interactions, in the reflection invariant unique gapped ground state phase. Suppose that if we associate some suitable boundary conditions along the path, they give local Hamiltonians which are gapped for an increasing sequence of finite boxes. (See Definition 3.7.) Then the \( \mathbb{Z}_2 \)-index \( \sigma_\omega \) does not change along the path.

This can be shown from the fact that ground states along the \( C^1 \)-path are mutually automorphic equivalent via an automorphism which allows a reflection invariant decomposition. The boundary conditions in the Corollary can be arbitrary, as long as they guarantee the gap. We may take it as periodic boundary condition, for example. Furthermore, the boundary condition itself does not need to be reflection invariant.

Our theorem, along with results in [PTO1,PTO2,CGW,Tas2] about matrix product states, shows that AKLT interaction and trivial interaction belong to different reflection symmetric unique gapped ground state phases. In other words, AKLT interaction and trivial interaction can never be connected by a \( C^1 \)-path of reflection invariant interactions without without closing the gap.
2 The $\mathbb{Z}_2$-index associated to the reflection symmetric split states

We introduce $\mathbb{Z}_2$-index for reflection invariant pure state satisfying the split property. Let us first recall the definition of the split property. Here we give the following definition, which is most suitable for our purpose. It corresponds to the standard definition [DL] in our setting (see [M3]).

**Definition 2.1.** Let $\varphi$ be a pure state on $\mathcal{A}$. Let $\varphi_R$ be the restriction of $\varphi$ to $\mathcal{A}_R$, and $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ be the GNS triple of $\varphi_R$. We say $\varphi$ satisfies the split property with respect to $\mathcal{A}_L$ and $\mathcal{A}_R$, if the von Neumann algebra $\pi_\varphi(\mathcal{A}_R)''$ is a type I factor.

Recall that a type I factor is isomorphic to $B(\mathcal{K})$, the set of all bounded operators on a Hilbert space $\mathcal{K}$. See [T1]. We consider following type of GNS-triple for a reflection invariant pure state which satisfies the split property.

**Definition 2.2.** Let $\omega$ be a reflection invariant pure state on $\mathcal{A}$ which satisfies the split property with respect to $\mathcal{A}_R$ and $\mathcal{A}_L$. We say $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ is a reflection-split representation associated to $\omega$ if setting $\hat{\mathcal{H}}_\omega := \mathcal{H}_\omega \otimes \mathcal{H}_\omega$ and $\hat{\pi}_\omega := (\pi_\omega \circ \gamma_{L \to R}) \otimes \pi_\omega$, following hold:

1. $(\mathcal{H}_\omega, \pi_\omega)$ is an irreducible representation of $\mathcal{A}_R$,
2. $\Omega_\omega$ is a unit vector of $\hat{\mathcal{H}}_\omega$,
3. $(\mathcal{H}_\omega, \hat{\pi}_\omega, \Omega_\omega)$ is a GNS triple of $\omega$,
4. $\Gamma_\omega$ is the unique unitary operator on $H_\omega$ such that

$$\Gamma_\omega \hat{\pi}_\omega(\mathcal{A}) \Omega_\omega = \hat{\pi}_\omega \circ \gamma(A) \Omega_\omega, \quad A \in \mathcal{A}. \quad (7)$$

**Remark 2.3.** Because of $\gamma^2 = \text{id}$, from the definition of $\Gamma_\omega$ (7), we have $\Gamma_\omega^2 = I_{\hat{\mathcal{H}}_\omega}$.

**Remark 2.4.** For the rest of this paper, for any reflection-split representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ we use the notation $\hat{\mathcal{H}}_\omega := \mathcal{H}_\omega \otimes \mathcal{H}_\omega$ and $\hat{\pi}_\omega := (\pi_\omega \circ \gamma_{L \to R}) \otimes \pi_\omega$.

**Lemma 2.5.** For any reflection invariant pure state $\omega$ on $\mathcal{A}$ which satisfies the split property with respect to $\mathcal{A}_R$ and $\mathcal{A}_L$, there exists a reflection-split representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ associated to $\omega$. Furthermore, if $(\mathcal{H}_\omega, \hat{\pi}_\omega, \Omega_\omega, \hat{\Gamma}_\omega)$ is another reflection-split representation of $\omega$, there exists a unitary $V : \mathcal{H}_\omega \to \hat{\mathcal{H}}_\omega$ such that such that

$$\text{Ad}(V) \circ \pi_\omega(A) = \hat{\pi}_\omega(A), \quad A \in \mathcal{A}_R, \quad (8)$$

$$\hat{\Gamma}_\omega = \text{Ad}(V \otimes V)(\Gamma_\omega), \quad (9)$$

$$\big(V \otimes V\big) \Omega_\omega = \hat{\Omega}_\omega. \quad (10)$$

**Proof.** Let $(\mathcal{H}_R, \pi_R, \Omega_R)$ be a GNS triple of $\omega|_{A_R}$. Note that from the reflection invariance of $\omega$, $(\mathcal{H}_R, \pi_R \circ \gamma_{L \to R}, \Omega_R)$ is a GNS triple of $\omega|_{A_L}$. Therefore, $(\mathcal{H}_R \otimes \mathcal{H}_R, \pi_R \circ \gamma_{L \to R} \otimes \pi_R, \Omega_R \otimes \Omega_R)$ is a GNS triple of $\omega|_{A_L \otimes A_R}$.

Since $\omega$ satisfies the split property, there exists a Hilbert space $\mathcal{H}_\omega$ and a $*$-isomorphism $\iota : \pi_R(\mathcal{A}_R)'' \to B(\mathcal{H}_\omega)$. We introduce a representation $\pi_\omega := \iota \circ \pi_R$ of $\mathcal{A}_R$ on $\mathcal{H}_\omega$. Since $\iota$ is a $*$-isomorphism, $(\mathcal{H}_\omega, \pi_\omega)$ is an irreducible representation of $\mathcal{A}_R$.

Set $\hat{\mathcal{H}}_\omega := \mathcal{H}_\omega \otimes \mathcal{H}_\omega$ and let $\hat{\pi}_\omega := (\pi_\omega \circ \gamma_{L \to R}) \otimes \pi_\omega$ be the representation of $\mathcal{A}$ on $\hat{\mathcal{H}}_\omega$. Now we would like to show the existence of a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ such that $(\mathcal{H}_\omega, \hat{\pi}_\omega, \Omega_\omega)$ is a GNS triple of $\omega$. Since there is a $*$-isomorphism $\iota \otimes \iota : (\pi_R \circ \gamma_{L \to R}(\mathcal{A}_L))'' \otimes \pi_R(\mathcal{A}_R)'' \to B(\mathcal{H}_\omega \otimes \mathcal{H}_\omega)$ (Theorem 5.2 IV of [T1]), the representation $\hat{\pi}_\omega = (\iota \otimes \iota) \circ (\pi_R \circ \gamma_{L \to R} \otimes \pi_R)$ is quasi-equivalent to $\pi_R \circ \gamma_{L \to R} \otimes \pi_R$, the GNS representation of $\omega|_{A_L \otimes \omega|_{A_R}}$. (Theorem 2.4.26 [BR1].) On the
other hand, as ω satisfies the split property, by the proof of Proposition 2.2 of [M2], ω|_{A_L} ⊗ ω|_{A_R} is quasi-equivalent to ω. (In Proposition 2.2 of [M2], it is assumed that the state is translationally invariant because of the first equivalent condition (i). However, the proof for the equivalence between (ii) and (iii) does not require translation invariance.) Hence πω is quasi-equivalent to the GNS representation of ω. (See section 2.4 of [BR1].) Therefore, there is a density matrix ρ on \( \hat{\mathcal{H}}_\omega \) such that

\[
\omega(A) = \text{Tr} \rho (\hat{\pi}_\omega(A)), \quad A \in \mathcal{A}.
\]

Since ω is pure and \( \hat{\pi}_\omega(A)' = B(\hat{\mathcal{H}}_\omega) \), this ρ should be a one rank operator onto a one dimensional space \( \mathbb{C} \Omega_\omega \), with some unit vector \( \Omega_\omega \in \hat{\mathcal{H}}_\omega \). Because of \( \hat{\pi}_\omega(A)' = B(\mathcal{H}_\omega) \), \( \Omega_\omega \) is cyclic for \( \hat{\pi}_\omega(A) \), and \( (\hat{\mathcal{H}}_\omega, \hat{\pi}_\omega, \Omega_\omega) \) is a GNS triple of ω. From the γ-invariance of ω, there exists \( \Gamma_\omega \) satisfying (7). (Corollary 2.3.17 of [BR1].)

Now let us show the uniqueness. Let \( (\hat{\mathcal{H}}_\omega, \hat{\pi}_\omega, \Omega_\omega, \Gamma_\omega) \) be another reflection-spilt representation of ω. Since both of \( (\mathcal{H}_\omega \otimes \mathcal{H}_\omega, (\pi_\omega \circ \gamma_{L \rightarrow R}) \otimes \pi_\omega, \Omega_\omega) \) and \( (\hat{\mathcal{H}}_\omega \otimes \hat{\mathcal{H}}_\omega, (\hat{\pi}_\omega \circ \hat{\gamma}_{L \rightarrow R}) \otimes \hat{\pi}_\omega, \hat{\Omega}_\omega) \) are GNS triple of ω, there is a unitary \( U : \mathcal{H}_\omega \rightarrow \hat{\mathcal{H}}_\omega \) such that

\[
U (\pi_\omega \circ \gamma_{L \rightarrow R} \otimes \pi_\omega) (A) U^* = (\hat{\pi}_\omega \circ \hat{\gamma}_{L \rightarrow R} \otimes \hat{\pi}_\omega) (A), \quad A \in \mathcal{A}, \quad \text{and} \quad U \Omega_\omega = \hat{\Omega}_\omega.
\]

(Theorem 2.3.16 of [BR1].) Restricting the first equation to \( \mathcal{A}_R \), we have

\[
U (\mathbb{I}_{\mathcal{H}_\omega} \otimes \pi_\omega) (A) U^* = \mathbb{I}_{\mathcal{H}_\omega} \otimes \hat{\pi}_\omega(A), \quad A \in \mathcal{A}_R.
\]

From this we obtain a *-isomorphism \( \tau \) from \( B(\mathcal{H}_\omega) = \pi_\omega(\mathcal{A}_R)' \) to \( B(\hat{\mathcal{H}}_\omega) = \hat{\pi}_\omega(\mathcal{A}_R)' \) such that

\[
U (\mathbb{I}_{\mathcal{H}_\omega} \otimes x) U^* = \mathbb{I}_{\mathcal{H}_\omega} \otimes \tau(x), \quad x \in \mathcal{B}(\mathcal{H}_\omega),
\]

which satisfies

\[
\tau \circ \pi_\omega (A) = \hat{\pi}_\omega(A), \quad A \in \mathcal{A}_R.
\]

Applying Wigner’s theorem to \( \tau \), there exists a unitary \( \hat{V} : \mathcal{H}_\omega \rightarrow \hat{\mathcal{H}}_\omega \) such that

\[
\tau(x) = \hat{V} x \hat{V}^*, \quad x \in \mathcal{B}(\mathcal{H}_\omega).
\]

In particular, we have

\[
\text{Ad}(\hat{V}) (\pi_\omega(A)) = \hat{\pi}_\omega(A), \quad A \in \mathcal{A}_R.
\]

(17)

From this and (12), we have

\[
\text{Ad}(U) ((\pi_\omega \circ \gamma_{L \rightarrow R} \otimes \pi_\omega) (A)) = (\hat{\pi}_\omega \circ \hat{\gamma}_{L \rightarrow R} \otimes \hat{\pi}_\omega) (A)
\]

\[
= \text{Ad}(\hat{V} \otimes \hat{V}) ((\pi_\omega \circ \gamma_{L \rightarrow R} \otimes \pi_\omega) (A)), \quad A \in \mathcal{A}.
\]

(18)

Since \( ((\pi_\omega \circ \gamma_{L \rightarrow R} \otimes \pi_\omega) (A))' = B(\mathcal{H}_\omega \otimes \mathcal{H}_\omega), (\hat{V} \otimes \hat{V})^* U = c U \mathbb{1}_{\mathcal{H}_\omega} \), for some \( c \in \mathbb{T} \). Choosing one \( c_1 \in \mathbb{T} \) such that \( c_1^2 = c \), we set \( V := c_1 \hat{V} \). Then we have \( U = V \otimes V \), and from (12), (10) holds. The property (9) holds from (12) and (10). By (17), we obtain (8). □

For a reflection invariant pure state which satisfies the split property with respect to \( \mathcal{A}_R \) and \( \mathcal{A}_L \) we can define an index via the reflection-spilt representation of ω.

**Theorem 2.6.** Let ω be a reflection invariant pure state which satisfies the split property with respect to \( \mathcal{A}_R \) and \( \mathcal{A}_L \). Let \( (\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega) \) be a reflection-spilt representation associated to ω. Then we have

\[
\Gamma_\omega (\mathbb{I}_{\mathcal{H}_\omega} \otimes x) \Gamma_\omega^* = x \otimes \mathbb{1}_{\mathcal{H}_\omega}, \quad x \in \mathcal{B}(\mathcal{H}_\omega).
\]

(19)

Furthermore, there exists a constant \( \sigma_\omega = \pm 1 \) such that

\[
\Gamma_\omega (\xi \otimes \eta) = \sigma_\omega \eta \otimes \xi, \quad \text{for all} \quad \xi, \eta \in \mathcal{H}_\omega.
\]

(20)

This \( \sigma_\omega \) is independent of the choice of the reflection-spilt representation \( (\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega) \).
Definition 2.7. From Theorem 2.6 for each reflection invariant pure state \( \omega \) which satisfies the split property with respect to \( A_R \) and \( A_L \), we can define a \( \mathbb{Z}_2 \)-index \( \sigma_\omega \).

Proof. We first prove (19). For any \( A \in A_R \), we have

\[
\Gamma_\omega (\mathbb{I}_{H_\omega} \otimes \pi_\omega (A)) \Gamma^*_\omega = \Gamma_\omega (\hat{\pi}_\omega (\mathbb{I}_{A_L} \otimes A)) \Gamma^*_\omega = \hat{\pi}_\omega \circ \gamma (\mathbb{I}_{A_L} \otimes A) = \hat{\pi}_\omega (\gamma_{R-L} (A) \otimes \mathbb{I}_{H_\omega}) = \pi_\omega (A) \otimes \mathbb{I}_{H_\omega}.
\]

(21)

Since both sides are \( \sigma \)-weak continuous and \( \pi_\omega (A_R)'' = B(H_\omega) \), we obtain (19).

From (19), we derive (20): For any nonzero \( \xi, \eta \in H_\omega \), there exists \( \sigma_{\xi, \eta} \in \mathbb{T} \) such that

\[
\Gamma_\omega (\xi \otimes \eta) = \sigma_{\xi, \eta} (\eta \otimes \xi),
\]

(22)

because

\[
\Gamma_\omega |\xi \otimes \eta\rangle \langle \xi \otimes \eta| \Gamma^*_\omega = |\eta \otimes \xi\rangle \langle \eta \otimes \xi|,
\]

(23)

from (19). Considering the case \( \xi = \eta \neq 0 \) in (22), we have

\[
\xi \otimes \xi = \Gamma^2_\omega (\xi \otimes \xi) = \sigma_{\xi, \xi} \Gamma_\omega (\xi \otimes \xi) = \sigma_{\xi, \xi}^2 \xi \otimes \xi.
\]

(24)

The first equality is from Remark 2.3. From this, we obtain \( \sigma_{\xi, \xi} = \pm 1 \). Again by (19), for nonzero \( \xi, \eta \in H_\omega \), we obtain

\[
\Gamma_\omega |\xi \otimes \eta\rangle \langle \eta \otimes \eta| \Gamma^*_\omega = |\xi \otimes \xi\rangle \langle \eta \otimes \eta|.
\]

(25)

On the other hand, from the above argument, we have

\[
\Gamma_\omega |\xi \otimes \xi\rangle \langle \eta \otimes \eta| \Gamma^*_\omega = \sigma_{\xi, \xi} \sigma_{\eta, \eta} |\xi \otimes \xi\rangle \langle \eta \otimes \eta|.
\]

(26)

Since \( \eta, \xi \) are not zero, we obtain \( \sigma_{\xi, \xi} \sigma_{\eta, \eta} = 1 \). Recalling that \( \sigma_{\xi, \xi}, \sigma_{\eta, \eta} \) take values in \( \pm 1 \), we obtain \( \sigma_{\xi, \xi} = \sigma_{\eta, \eta} \). Therefore, we set \( \sigma_\omega := \sigma_{\xi, \xi} \), which is independent of the choice of nonzero \( \xi \in H_\omega \). To prove (20), we use (19) again and for any nonzero \( \xi, \eta \in H_\omega \), we have

\[
|\eta \otimes \xi\rangle \langle \xi \otimes \xi| = \Gamma_\omega |\xi \otimes \eta\rangle \langle \xi \otimes \eta| \Gamma^*_\omega = \sigma_{\xi, \xi} \sigma_\omega |\eta \otimes \xi\rangle \langle \xi \otimes \xi|.
\]

(27)

The first equality follows from (19) and the second one is the definition of \( \sigma_{\xi, \eta} \) and \( \sigma_\omega \). From this and \( \sigma_\omega = \pm 1 \), we obtain \( \sigma_{\xi, \eta} = \sigma_\omega \), completing the proof of (20).

To show that the sign \( \sigma_\omega \) is independent of the choice of the reflection-split representation, let \( (\hat{\mathcal{H}}_\omega, \hat{\pi}_\omega, \hat{\Omega}_\omega, \hat{\Gamma}_\omega) \) be another reflection-split representations associated to \( \omega \). Let \( V : \mathcal{H}_\omega \rightarrow \hat{\mathcal{H}}_\omega \) be the unitary given in Lemma 2.5. We have

\[
\hat{\Gamma}_\omega (\xi \otimes \eta) = (V \otimes V) \Gamma_\omega (V^* \xi \otimes V^* \eta) = (V \otimes V) \sigma_\omega (V^* \eta \otimes V^* \xi) = \sigma_\omega (\eta \otimes \xi),
\]

(28)

for any \( \xi, \eta \in \mathcal{H}_\omega \), proving the claim. \( \Box \)

Now we prove that the \( \mathbb{Z}_2 \)-index is invariant under automorphic equivalence via an automorphism which allows a reflection invariant decomposition. Let us recall the definition:

Definition 2.8. We say an automorphism \( \alpha \) of \( \mathcal{A} \) allows a reflection invariant decomposition if there is an automorphisms \( \alpha_R \) on \( \mathcal{A}_R \), and a unitary \( W \) in \( \mathcal{A} \) such that

\[
\tilde{\alpha}^{-1} \circ \alpha = \text{Ad}(W), \quad \gamma(W) = W,
\]

(29)

where

\[
\tilde{\alpha} := (\gamma_{R-L} \circ \alpha_R \circ \gamma_{L-R}) \otimes \alpha_R.
\]

(30)

We call \( (\alpha_R, W) \), a reflection invariant decomposition of \( \alpha \).
We prove the following theorem:

**Theorem 2.9.** Let \( \omega_0, \omega_1 \) be reflection invariant pure states satisfying the split property with respect to \( A_R \) and \( A_L \). Suppose that \( \omega_0 \) and \( \omega_1 \) are automorphic equivalent via an automorphism \( \alpha \), i.e., \( \omega_1 = \omega_0 \circ \alpha \), which allows a reflection invariant decomposition \( (\alpha R, W) \). Then the \( \mathbb{Z}_2 \)-indices \( \sigma_{\omega_0}, \sigma_{\omega_1} \) associated to \( \omega_0, \omega_1 \) are equal.

**Proof.** Let \( (\mathcal{H}_{\omega_0}, \pi_{\omega_0}, \Omega_{\omega_0}, \Gamma_{\omega_0}) \) be a reflection-split representation associated to \( \omega_0 \). Set \( \hat{\mathcal{H}}_{\omega_0} := \mathcal{H}_{\omega_0} \otimes \mathcal{H}_{\omega_0}, \hat{\pi}_{\omega_0} := (\pi_{\omega_0} \circ \gamma_{L \rightarrow R}) \otimes \pi_{\omega_0} \). We also set \( \hat{\alpha} := (\gamma_{R \rightarrow L} \circ \alpha R \circ \gamma_{L \rightarrow R}) \otimes \alpha R \) and

\[
\hat{\pi}_{\omega_1} := (\pi_{\omega_0} \circ \alpha R \circ \gamma_{L \rightarrow R}) \otimes (\pi_{\omega_0} \circ \alpha R) = \hat{\pi}_{\omega_0} \circ \hat{\alpha}.
\]

We claim that \( (\mathcal{H}_{\omega_0}, \pi_{\omega_0} \circ \alpha R, \hat{\pi}_{\omega_1}(W^*)\Omega_{\omega_0}, \Gamma_{\omega_0}) \) is a reflection-split representation of \( \omega_1 \). From this, we obtain the statement of the Theorem i.e., \( \sigma_{\omega_0} = \sigma_{\omega_1} \).

The first condition of Definition 2.2 is from \( \pi_{\omega_0} \circ \alpha R (A_R)^{'''} = \pi_{\omega_0}(A_R)^{'''} = B(\mathcal{H}_{\omega_0}) \). The second one is trivial because \( W \) is unitary. To prove the third one, note that \( (\mathcal{H}_{\omega_0}, \hat{\pi}_{\omega_1}, \Omega_{\omega_0}) \) is a GNS triple of \( \omega_0 \circ \hat{\alpha} \). From \( \alpha^{-1} \circ \alpha = \text{Ad}(W) \) and \( \omega_1 = \omega_0 \circ \alpha \), we have

\[
\omega_1 \circ \alpha = \omega_0 \circ \alpha \circ \text{Ad}(W) \tag{31}
\]

Combining these two facts, we have

\[
\omega_1(A) = (\hat{\pi}_{\omega_1}(W^*)\Omega_{\omega_0}, \hat{\pi}_{\omega_1}(A)\hat{\pi}_{\omega_1}(W^*)\Omega_{\omega_0}), \quad A \in \mathcal{A} \tag{32}
\]

Since \( \hat{\pi}_{\omega_1}(W^*)\Omega_{\omega_0} \) is cyclic for \( \hat{\pi}_{\omega_1}(A) \), hence \( (\hat{\mathcal{H}}_{\omega_0}, \hat{\pi}_{\omega_1}(W^*)\Omega_{\omega_0}) \) is a GNS triple of \( \omega_1 \). Finally for the fourth condition of Definition 2.2, note that \( \gamma \) and \( \hat{\alpha} \) commute because

\[
\gamma \circ \hat{\alpha}(A \otimes B) = \gamma \circ ((\gamma_{R \rightarrow L} \circ \alpha R \circ \gamma_{L \rightarrow R}(A)) \otimes \alpha R(B)) = (\gamma_{R \rightarrow L} \circ \alpha R(B)) \otimes (\alpha R \circ \gamma_{L \rightarrow R}(A)) \tag{33}
\]

for any \( A \in \mathcal{A}_L \) and \( B \in \mathcal{A}_R \). From this fact, we obtain

\[
\Gamma_{\omega_0}\hat{\pi}_{\omega_1}(A)\hat{\pi}_{\omega_1}(W^*)\Omega_{\omega_0} = \Gamma_{\omega_0}\pi_{\omega_0} \circ \hat{\alpha}(AW^*)\Omega_{\omega_0} = (\hat{\pi}_{\omega_0} \circ \gamma \circ \hat{\alpha}(AW^*))\Omega_{\omega_0} = (\hat{\pi}_{\omega_0} \circ \hat{\alpha} \circ \gamma(A)) (\hat{\pi}_{\omega_0} \circ \hat{\alpha} (W^*)) \Omega_{\omega_0} = (\hat{\pi}_{\omega_0} \circ \gamma(A)) (\pi_{\omega_0} \circ \hat{\alpha} (W^*)) \Omega_{\omega_0}, \tag{34}
\]

for all \( A \in \mathcal{A} \). For the fourth equality, we used \( \gamma(W) = W \). This completes the proof. \( \square \)

### 3 C\(^1\)-classification of gapped Hamiltonians with the reflection symmetry.

Let us now apply the result in Section 2 to the \( C^1 \)-classification of gapped Hamiltonians preserving the reflection symmetry.

A mathematical model of a quantum spin chain is fully specified by its interaction \( \Phi \). An interaction is a map \( \Phi \) from \( \mathcal{S}_Z \) into \( \mathcal{A}_\text{loc} \) such that \( \Phi(X) \in \mathcal{A}_X \) and \( \Phi(X) = \Phi(X)^* \) for each \( X \in \mathcal{S}_Z \). Let \( R : Z \rightarrow Z \) be the reflection : \( R(i) := -i - 1, i \in Z \). An interaction \( \Phi \) is reflection invariant if \( \gamma(\Phi(X)) = \Phi(R(X)) \) for all \( X \in \mathcal{S}_Z \). An interaction \( \Phi \) is of finite range if there exists an \( m \in \mathbb{N} \) such that \( \Phi(X) = 0 \) for \( X \) with diameter larger than \( m \). We denote by \( B_f \), the set of all finite range interactions \( \Phi \) which satisfy

\[
a_{\phi} := \sup_{X \in \mathcal{S}_Z} \|\Phi(X)\| < \infty. \tag{35}
\]
We may define addition on $B_f$: for $\Phi, \Psi \in B_f$, $\Phi + \Psi$ denotes the interaction given by $(\Phi + \Psi)(X) = \Phi(X) + \Psi(X)$ for each $X \in \mathcal{S}_Z$.

For an interaction $\Phi$ and a finite set $\Lambda \in \mathcal{S}_Z$, we define the local Hamiltonian on $\Lambda$ by

$$
(H_\Phi)_\Lambda := \sum_{X \in \Lambda} \Phi(X).
$$

The dynamics given by this local Hamiltonian is denoted by

$$
\tau_t^{\Phi, \Lambda}(A) := e^{it(H_\Phi)_\Lambda}Ae^{-it(H_\Phi)_\Lambda}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}.
$$

If $\Phi$ belongs to $B_f$, the limit

$$
\tau_t^\Phi(A) = \lim_{\Lambda \to \mathbb{Z}} \tau_t^{\Phi, \Lambda}(A)
$$

exists for each $A \in \mathcal{A}$ and $t \in \mathbb{R}$, and defines a strongly continuous one parameter group of automorphisms $\tau^\Phi$ on $\mathcal{A}$. (See [BR2].) We denote the generator of the $C^*$-dynamics $\tau^\Phi$ by $\delta_\Phi$.

For $\Phi \in B_f$, a state $\varphi$ on $\mathcal{A}$ is called a $\tau^\Phi$-ground state if the inequality $-i\varphi(\Lambda \delta_\Phi(A)) \geq 0$ holds for any element $A$ in the domain $\mathcal{D}(\delta_\Phi)$ of $\delta_\Phi$. Let $\varphi$ be a $\tau^\Phi$-ground state, with the GNS triple $(H_{\varphi}, \pi_{\varphi}, \Omega_{\varphi})$. Then there exists a unique positive operator $H_{\varphi, \Phi}$ on $H_{\varphi}$ such that $e^{itH_{\varphi, \Phi}}\pi_{\varphi}(A)\Omega_{\varphi} = \pi_{\varphi}(\tau_t^\Phi(A))\Omega_{\varphi}$, for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{\varphi, \Phi}$ the bulk Hamiltonian associated with $\varphi$. Note that $\Omega_{\varphi}$ is an eigenvector of $H_{\varphi, \Phi}$ with eigenvalue 0. See [BR2] for the general theory.

The following definition clarifies what we mean by a model with a unique gapped ground state.

**Definition 3.1.** We say that a model with an interaction $\Phi \in B_f$ has a unique gapped ground state if (i) the $\tau^\Phi$-ground state, which we denote as $\varphi$, is unique, and (ii) there exists a $g > 0$ such that $\sigma(H_{\varphi, \Phi}) \setminus \{0\} \subset [g, \infty)$, where $\sigma(H_{\varphi, \Phi})$ is the spectrum of $H_{\varphi, \Phi}$.

Note that the uniqueness of $\varphi$ implies that 0 is a non-degenerate eigenvalue of $H_{\varphi, \Phi}$.

If $\varphi$ is a $\tau^\Phi$-ground state of reflection invariant interaction $\Phi \in B_f$, then its reflection $\varphi \circ \gamma$ is also a $\tau^\Phi$-ground state. In particular, if $\varphi$ is a unique $\tau^\Phi$-ground state, it is pure and reflection invariant.

In [M3], T. Matsui showed that the spectral gap implies the split property.

**Theorem 3.2** (Theorem 1.5, Lemma 4.1, and Proposition 4.2 of [M3]). Let $\varphi$ be a pure $\tau^\Phi$-ground state of $\Phi \in B_f$, and denote by $H_{\varphi, \Phi}$ the corresponding bulk Hamiltonian. Assume that 0 is a non-degenerate eigenvalue of $H_{\varphi, \Phi}$ and there exists $g > 0$ such that $\sigma(H_{\varphi, \Phi}) \setminus \{0\} \subset [g, \infty)$. Then $\varphi$ satisfies the split property with respect to $\mathcal{A}_L$ and $\mathcal{A}_R$.

This theorem, combined with Definition 2.7 allows us to define the $\mathbb{Z}_2$-index for reflection invariant Hamiltonians with unique gapped ground state.

**Definition 3.3.** Let $\Phi \in B_f$ be a reflection invariant interaction which has a unique gapped ground state $\omega$. By Theorem 3.2 $\omega$ satisfies the split property. Hence we obtain the $\mathbb{Z}_2$-index $\sigma_\omega$ in Definition 2.7. In this setting, we denote this $\sigma_\omega$ by $\tilde{\sigma}_\Phi$ and call it the $\mathbb{Z}_2$-index associated to $\Phi$.

Since $\tilde{\sigma}_\Phi$ takes discrete values $\{-1, 1\}$, for a continuous path of interactions $\Phi(s)$, we would expect that $\tilde{\sigma}_{\Phi(s)}$ is constant. We prove this in the setting of $C^1$-classification.

**Definition 3.4.** We say the map $\Phi : [0, 1] \ni s \to \Phi(s) := \{\Phi(X; s)\}_{X \in \mathcal{S}_Z} \in B_f$ is a $C^1$-path of reflection invariant gapped interactions satisfying the Condition $B$, if there exist

(i) numbers $M, R \in \mathbb{N}$, $g > 0$ and an increasing sequence $n_k \in \mathbb{N}$, $k = 1, 2, \ldots$,

(ii) $C^1$-functions $a, b : [0, 1] \to \mathbb{R}$ such that $a(s) < b(s)$,
(iii) a sequence of paths of interactions \( \Psi_k : [0, 1] \ni s \rightarrow \Psi_k(s) := \{\Psi_k(X; s)\}_{X \in \mathcal{E}_z} \in \mathcal{B}_f, \)

\[ k = 1, 2, \ldots, \]

and the following hold:

1. For each \( X \in \mathcal{G}_z \), the map \([0, 1] \ni s \rightarrow \Phi(X; s), \Psi_k(X; s) \in \mathcal{A}_X \) are continuous and piecewise \( C^1 \). We denote by \( \Phi'(X; s), \Psi_k'(X; s) \), the corresponding derivatives.

2. For each \( s \in [0, 1] \), and \( X \in \mathcal{G}_z \) with \( \text{diam}(X) \geq M \), we have \( \Phi(X; s) = 0 \).

3. For each \( s \in [0, 1] \), and \( k \in \mathbb{N} \), we have \( \Psi_k(X; s) = 0 \) unless \( X \subset \Lambda_{n_k} \setminus \Lambda_{n_k} - R \).

4. Interactions are bounded as follows

\[
C_1 := \sup_{s \in [0, 1]} \sup_{k \in \mathbb{N}} \sup_{X \in \mathcal{G}_z} \left( \|\Phi(X; s)\| + |X| \|\Phi'(X; s)\| + \|\Psi_k(X; s)\| + |X| \|\Psi_k'(X; s)\| \right) < \infty. \tag{39}
\]

5. For each \( s \in [0, 1] \), there exists a unique \( \tau^{\Phi(s)} \)-ground state \( \varphi_s \).

6. For each \( s \in [0, 1] \), \( \Phi(s) \) is reflection-invariant.

7. For each \( k \in \mathbb{N} \) and \( s \in [0, 1] \), the spectrum \( \sigma\left((H_{\Phi(s)} + \Psi_k(s))_{\Lambda_{n_k}}\right) \) of \( (H_{\Phi(s)} + \Psi_k(s))_{\Lambda_{n_k}} \) is decomposed into two non-empty disjoint parts \( \sigma\left((H_{\Phi(s)} + \Psi_k(s))_{\Lambda_{n_k}}\right) = \Sigma_1^{(k)}(s) \cup \Sigma_2^{(k)}(s) \) such that \( \Sigma_1^{(k)}(s) \subset \{a(s), b(s)\}, \Sigma_2^{(k)}(s) \subset \{b(s) + g, \infty\} \) and the diameter of \( \Sigma_1^{(k)}(s) \) converges to 0 as \( k \to \infty \).

The interaction \( \Psi_k(s) \) corresponds to a boundary condition. Note that it does not forbid an interaction between intervals \([-n, -n + R] \cap \mathbb{Z} \) and \([n - R, n] \cap \mathbb{Z} \). In particular, the periodic boundary condition is included in this framework. Also, note that we do not require that the boundary term \( \Psi_k(s) \) to be reflection invariant.

By exactly the same way as in Proposition 3.5 of [O4], we can show the following:

**Proposition 3.5.** Let \( \Phi : [0, 1] \ni s \rightarrow \Phi(s) := \{\Phi(X; s)\}_{X \in \mathcal{E}_z} \in \mathcal{B}_f \) be a \( C^1 \)-path of reflection invariant gapped interactions satisfying the Condition B. Let \( \varphi_0 \) be the unique \( \tau^{\Phi(s)} \)-ground state, for each \( s \in [0, 1] \). Then \( \varphi_0 \) and \( \varphi_1 \) are automorphic equivalent via an automorphism, which allows a reflection invariant decomposition.

As a corollary of this proposition and Theorem 3.6, we obtain the following.

**Theorem 3.6.** Let \( \Phi : [0, 1] \ni s \rightarrow \Phi(s) := \{\Phi(X; s)\}_{X \in \mathcal{E}_z} \in \mathcal{B}_f \) be a \( C^1 \)-path of reflection invariant gapped interactions satisfying the Condition B. Then we have \( \hat{\sigma}_{\Phi(0)} = \hat{\sigma}_{\Phi(1)} \).

Namely, the \( \mathbb{Z}_2 \)-index is invariant along the \( C^1 \)-path of reflection invariant gapped interactions, satisfying the Condition B.

## 4 The \( \mathbb{Z}_2 \)-index and the modular conjugation

In this section, we give a characterization of \( \sigma_\omega \) from the point of view of Tomita-Takesaki modular theory. It will be used in Section 5 to prove that our index generalizes the index introduced in [PTB01]. This also allows us to connect \( \sigma_\omega \) with the Schmidt decomposition of \( \Omega_\omega \).

First let us recall Tomita-Takesaki theory. See [BR1] or [T2] for more information. Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \). Let \( \Omega \) be a cyclic (i.e. \( \mathcal{M}\Omega \) is dense in \( \mathcal{H} \)) and
separating (i.e., \(x\Omega = 0, x \in \mathcal{M}\) implies \(x = 0\)) vector for \(\mathcal{M}\). We define an anti-linear operator on \(\mathcal{H}\) with domain \(\mathcal{M}\Omega\) by

\[
Sx\Omega := x^*\Omega, \quad x \in \mathcal{M}.
\]

It turns out that \(S\) is closable. (Proposition 2.5.9 of [BR1].) We denote the closure by the same symbol \(S\). The operator \(S\) has a polar decomposition \(S = J\Delta^{\frac{1}{2}}\) where \(J\) is an anti-unitary \(J\) called the modular conjugation associated to \((\mathcal{M}, \Omega)\) and \(\Delta\) is a nonsingular positive operator called the modular operator associated to \((\mathcal{M}, \Omega)\). For the commutant \(\mathcal{M}'\) of \(\mathcal{M}\), we have

\[
J\Delta^{-\frac{1}{2}}x'\Omega = (x')^*\Omega, \quad x' \in \mathcal{M}'.
\]

We also have

\[
\Delta^{-\frac{1}{2}} = J\Delta^{\frac{1}{2}}J^*, \quad J^2 = 1, \quad \Delta\Omega = J\Omega = \Omega.
\]

(Proposition 2.5.11 of [BR1].) The subspace \(\mathcal{M}'\Omega\) is a core of \(J\Delta^{-\frac{1}{2}}\). The Tomita-Takesaki theory states that

\[
\Delta^it\mathcal{M}\Delta^{-it} = \mathcal{M}, \text{ for all } t \in \mathbb{R} \text{ and } J\mathcal{M}J^* = \mathcal{M}'.
\]

From the first property, we may define a \(W^\ast\)-dynamics (i.e., \(\sigma\)-weak continuous one parameter group of automorphisms) \(\sigma_t(x) := \Delta^itx\Delta^{-it}, t \in \mathbb{R}, x \in \mathcal{M}\) on \(\mathcal{M}\). It is called the modular automorphisms associated to \((\mathcal{M}, \Omega)\).

Set \(\mathcal{D} := \{z \in \mathbb{C} \mid 0 < \Re z < 1\}\) and denote by \(\mathcal{D}'\) its closure. For any \(x, y \in \mathcal{M}\), there exists a bounded and continuous function \(F_{x,y}\) on \(\mathcal{D}\) which is analytic on \(\mathcal{D}\), satisfying

\[
F_{x,y}(t) = \langle \Omega, \sigma_t(x)y\Omega \rangle, \quad F_{x,y}(t+i) = \langle \Omega, y\sigma_t(x)\Omega \rangle,
\]

for all \(t \in \mathbb{R}\). This condition is called the KMS-condition for the positive linear functional \(\mathcal{M} \ni x \mapsto \langle \Omega, x\Omega \rangle\) on \(\mathcal{M}\) and the modular automorphisms are characterized as the unique \(W^\ast\)-dynamics which satisfies the KMS condition for this linear functional. (Theorem 1.2 VIII [12].)

Now let us come back to our problem.

**Lemma 4.1.** Let \(\omega\) be a reflection invariant pure state on \(\mathcal{A}\) which satisfies the split property with respect to \(\mathcal{A}_R\) and \(\mathcal{A}_L\). Let \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)\) be a reflection-split representation associated to \(\omega\). Let \(s_\omega\) be a projection in \(B(\mathcal{H}_\omega)\) such that the support projection of \(\Omega_\omega\) in \(I_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega)\) is \(I_{\mathcal{H}_\omega} \otimes s_\omega\). Set \(\mathcal{M} := s_\omega \otimes B(s_\omega \mathcal{H}_\omega)\), and \(p_\omega = s_\omega \otimes s_\omega\). Then

1. the support projection of \(\Omega_\omega\) in \(B(\mathcal{H}_\omega) \otimes I_{\mathcal{H}_\omega}\) is \(s_\omega \otimes I_{\mathcal{H}_\omega}\),
2. \(p_\omega \Omega_\omega = \Omega_\omega\),
3. the commutant \(\mathcal{M}'\) of \(\mathcal{M}\) in \(p_\omega \hat{\mathcal{H}}_\omega\) is \(\mathcal{M}' = B(s_\omega \mathcal{H}_\omega) \otimes s_\omega\),
4. \(\Omega_\omega\) is cyclic and separating for \(\mathcal{M}\) in \(p_\omega \hat{\mathcal{H}}_\omega\),
5. \(\Gamma_\omega p_\omega = p_\omega \Gamma_\omega\).

**Proof.** To show 1., let \(s'_\omega\) be a projection in \(B(\mathcal{H}_\omega)\) such that the support projection of \(\Omega_\omega\) in \(B(\mathcal{H}_\omega) \otimes I_{\mathcal{H}_\omega}\) is \(s'_\omega \otimes I_{\mathcal{H}_\omega}\). From [19] and [17] with \(A = I_{\mathcal{A}}\), we have

\[
\langle \Omega_\omega, ((1 - s_\omega) \otimes I_{\mathcal{H}_\omega}) \Omega_\omega \rangle = \langle \Omega_\omega, \Gamma_\omega ((1 - s_\omega) \otimes I_{\mathcal{H}_\omega}) \Gamma_\omega^\ast \Omega_\omega \rangle = \langle \Omega_\omega, (I_{\mathcal{H}_\omega} \otimes (1 - s_\omega)) \Omega_\omega \rangle = 0.
\]

(45)

Therefore, we have \(1 - s_\omega \leq 1 - s'_\omega\). Similarly, we obtain \(1 - s'_\omega \leq 1 - s_\omega\). Hence we obtain \(s_\omega = s'_\omega\).

2. is clear from the definition. 3. follows from Tomita’s commutant Theorem (See Theorem 5.9 of [11]). Since \(\Omega_\omega\) is separating for \(\mathcal{M}'\), it is cyclic for \(\mathcal{M}\) in \(p_\omega \hat{\mathcal{H}}_\omega\), 4. (Proposition 2.5.3 [BR1]), 5. is from the definition of \(p_\omega = s_\omega \otimes s_\omega\) and [19].
From 4. of Lemma 4.1, we can define modular conjugation $J_\omega$ and modular operator $\Delta_\omega$ associated to $(M,\Omega_\omega)$. Let us investigate their properties.

**Lemma 4.2.** Let $\omega$ be a reflection invariant pure state on $A$ which satisfies the split property with respect to $A_R$ and $A_L$. Let $(H_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ be a reflection-split representation associated to $\omega$. Let $s_\omega$ be a projection in $B(H_\omega)$ such that the support projection of $\Omega_\omega$ in $I_{H_\omega} \otimes B(H_\omega)$ is $I_{H_\omega} \otimes s_\omega$. Let $\mathcal{M} := s_\omega \otimes B(s_\omega H_\omega)$, and $p_\omega = s_\omega \otimes s_\omega$. Let $J_\omega$, $\Delta_\omega$ be modular conjugation, modular operator on $p_\omega H_\omega$ associated to $(M, \Omega_\omega)$. Let

$$\Omega_\omega = \sum_{k \in \Lambda} \sqrt{\lambda_k} \xi_k \otimes \zeta_k$$

(46)

be a Schmidt decomposition of $\Omega_\omega$. Here $\Lambda$ is a countable set and the sequence $\{\lambda_k\}_{k \in \Lambda} \subset \mathbb{R}_{\geq 0}$ satisfies $\sum_{k \in \Lambda} \lambda_k = 1$. Furthermore, each of $\{\xi_k\}_{k \in \Lambda}$ and $\{\zeta_k\}_{k \in \Lambda}$ are orthonormal sets of $H_\omega$.

We also define a density matrix $\rho_\omega$ by

$$\rho_\omega = \sum_{k \in \Lambda} \lambda_k |\xi_k\rangle \langle \zeta_k|.$$  

(47)

Then we have the following:

1. Both of $\{\xi_k\}_{k \in \Lambda}$ and $\{\zeta_k\}_{k \in \Lambda}$ are orthonormal basis of $s_\omega H_\omega$. There exists a unitary $u$ on $s_\omega H_\omega$ such that $\xi_k = u \xi_k$, for each $k \in \Lambda$.

2. The action of the modular operator is given by

$$\Delta_\omega^{\frac{1}{2}} (s_\omega \otimes x) \Omega_\omega = \sum_{k \in \Lambda} u \xi_k \otimes \rho_\omega^{\frac{1}{2}} x \xi_k, \quad x \in B(s_\omega H_\omega).$$

(48)

In particular, if the rank of $\rho_\omega$ is finite then we have

$$\Delta_\omega^{\frac{1}{2}} (s_\omega \otimes x) \Omega_\omega = \left( s_\omega \otimes \rho_\omega^{\frac{1}{2}} x \rho_\omega^{-\frac{1}{2}} \right) \Omega_\omega, \quad x \in B(s_\omega H_\omega).$$

(49)

3. Let $c$ be the complex conjugation on $s_\omega H_\omega$ given by $c \xi_k = \xi_k$ for $k \in \Lambda$. Then we have

$$(s_\omega \otimes x) \Omega_\omega = \sum_{k \in \Lambda} u \rho_\omega^{\frac{1}{2}} c^* x^* c \xi_k \otimes \zeta_k, \quad x \in B(s_\omega H_\omega).$$

(50)

4. The adjoint of the modular conjugation on $M$ is given by

$$J_\omega (s_\omega \otimes x) J_\omega^* = u c^* x c \otimes s_\omega, \quad x \in B(s_\omega H_\omega).$$

(51)

**Proof.** By the definition of $s_\omega$ and 1. of Lemma 4.1, we see that each of $\{\xi_k\}_{k \in \Lambda}$ and $\{\zeta_k\}_{k \in \Lambda}$ are orthonormal basis of $s_\omega H_\omega$. Therefore, there is a unitary $u$ on $s_\omega H_\omega$ such that $\xi_k = u \xi_k$, for all $k \in \Lambda$. This $u$ is given as $u = \sum_{k \in \Lambda} |\xi_k\rangle \langle \xi_k|$ where the summation converges in the strong topology. This proves 1.

Let $\Lambda_n, n \in \mathbb{N}$ be an increasing sequence of finite subsets of $\Lambda$ such that $\Lambda_n \uparrow \Lambda$. Set

$$Q_n := \sum_{k \in \Lambda_n} |\xi_k\rangle \langle \xi_k|.$$  

(52)

For $x \in B(s_\omega H_\omega)$ and $n \in \mathbb{N}$, set $x_n := Q_n x Q_n$. Note that the sequence $x_n \in B(s_\omega H_\omega)$ approximates $x$ in the $\sigma$-strong topology.
We would like to specify the action of the modular operator \( \Delta \). We claim
\[
\Delta^{\frac{1}{2}} (s_\omega \otimes x) \Omega_\omega = \sum_{k \in \Lambda} u\zeta_k \otimes \rho_k^{\frac{1}{2}} x \zeta_k, \quad x \in B(s_\omega \mathcal{H}_\omega).
\] (53)
Note that the right hand side converges in norm because \( \rho_\omega \) is in the trace class. To prove (53), we first specify the modular automorphism \( \sigma \) with respect to \( (\mathcal{M}, \Omega_\omega) \). We define a \( W^* \)-dynamics \( \alpha \) on \( \mathcal{M} \) by
\[
\alpha_t(s_\omega \otimes x) := s_\omega \otimes \text{Ad}(\rho_\omega^t)(x), \quad t \in \mathbb{R}, \quad x \in B(s_\omega \mathcal{H}_\omega),
\] (54)
and show that \( \alpha = \sigma \). To do that, we recall the uniqueness of the \( W^* \)-dynamics which satisfies the KMS condition. Let \( \phi \) be a state on \( \mathcal{M} \) given by
\[
\phi(s_\omega \otimes x) = \langle \Omega_\omega, (s_\omega \otimes x) \Omega_\omega \rangle, \quad x \in B(s_\omega \mathcal{H}_\omega).
\] (55)
Note that
\[
\text{Tr}_{\mathcal{H}_\omega}(\rho_\omega x) = \langle \Omega_\omega, (s_\omega \otimes x) \Omega_\omega \rangle = \phi(s_\omega \otimes x), \quad x \in B(s_\omega \mathcal{H}_\omega).
\] (56)
From this, we can see that \( \phi \) is \( \alpha \)-invariant. We show that \( \alpha \) satisfies the KMS-condition for \( \phi \). This follows from the standard argument like in Proposition 5.3.7 of [BR2]: For any \( x, y \in B(s_\omega \mathcal{H}_\omega) \), and \( n \in \mathbb{N} \), setting \( x_n := Q_n x Q_n \), we may define an entire analytic function
\[
F_{x_n,y}(z) := \langle \Omega_\omega, \alpha_z(s_\omega \otimes x_n) (s_\omega \otimes y) \Omega_\omega \rangle, \quad z \in \mathbb{C},
\] (57)
because \( \mathbb{R} \ni t \mapsto \alpha_t(s_\omega \otimes x_n) \in \mathcal{M} \) has an analytic continuation \( \alpha_z(s_\omega \otimes x_n) = s_\omega \otimes \text{Ad}(\rho_\omega^z)(x_n) \) \( \in \mathcal{M} \), \( z \in \mathbb{C} \). This \( F_{x_n,y}(z) \) is bounded and continuous on \( \mathcal{D} \) and analytic on \( \overline{\mathcal{D}} \). Furthermore, it satisfies the boundary condition (54) i.e., we have
\[
F_{x_n,y}(t) = \langle \Omega_\omega, \alpha_t(s_\omega \otimes x_n) (s_\omega \otimes y) \Omega_\omega \rangle, \quad F_{x_n,y}(t+i) = \langle \Omega_\omega, (s_\omega \otimes y) \alpha_t(s_\omega \otimes x_n) \Omega_\omega \rangle, \quad t \in \mathbb{R},
\] (58)
for all \( t \in \mathbb{R} \). The second property holds because of (57) and the property of the trace.

By the \( \alpha \)-invariance of \( \phi \) and the \( \alpha \)-strong* convergence of \( x_n \) to \( x \), (using the Cauchy-Schwartz inequality,) one can show from (57) that \( F_{x_n,y}(t) \) and \( F_{x_n,y}(t+i) \), as functions of \( t \in \mathbb{R} \), are Cauchy sequence of continuous bounded functions on \( \mathbb{R} \) with respect to the uniform norm. Therefore, by the Phragmen-Lindelöf theorem, \( \{F_{x_n,y}(z)\}_n \) is a Cauchy sequence of continuous bounded functions on \( \overline{\mathcal{D}} \) with respect to the uniform norm. Therefore, \( F_{x_n,y}(z) \) has a limit \( F_{x,y}(z) \) on \( z \in \mathcal{D} \), which is bounded and continuous on \( \mathcal{D} \) and analytic on \( \overline{\mathcal{D}} \). We also have
\[
F_{x,y}(t) = \lim_{n \to \infty} F_{x_n,y}(t) = \langle \Omega_\omega, \alpha_t(s_\omega \otimes x) (s_\omega \otimes y) \Omega_\omega \rangle, \quad t \in \mathbb{R},
\] (59)
and
\[
F_{x,y}(t+i) = \lim_{n \to \infty} F_{x_n,y}(t+i) = \langle \Omega_\omega, (s_\omega \otimes y) \alpha_t(s_\omega \otimes x) \Omega_\omega \rangle, \quad t \in \mathbb{R},
\] (60)
for any \( t \in \mathbb{R} \). Therefore, \( \alpha \) satisfies the KMS condition for \( \phi \), and from the uniqueness, we get \( \alpha = \sigma \). Now let us prove (53). For each \( x \in B(s_\omega \mathcal{H}_\omega) \), we again consider \( x_n := Q_n x Q_n \), \( n \in \mathbb{N} \). By the entire analyticity of \( s_\omega \otimes x_n \) with respect to \( \alpha = \sigma \), we have
\[
\Delta^{\frac{1}{2}} (s_\omega \otimes x_n) \Omega_\omega = \alpha_{-\frac{1}{2}}(s_\omega \otimes x_n) \Omega_\omega = \left( s_\omega \otimes \text{Ad}(\rho_\omega^{\frac{1}{2}})(x_n) \right) \Omega_\omega = \sum_{k \in \Lambda} u\zeta_k \otimes \rho_k^{\frac{1}{2}} x_n \zeta_k.
\] (61)
(The first equation is a standard argument. See proof of Theorem 5.5 of [DJP] for example.) The left hand side of (61) converges to \( \Delta^{\frac{1}{2}} (s_\omega \otimes x) \Omega_\omega \) as \( n \to \infty \) because of
\[
\| \Delta^{\frac{1}{2}} (s_\omega \otimes x_n) \Omega_\omega - \Delta^{\frac{1}{2}} (s_\omega \otimes x) \Omega_\omega \| = \| J_\omega \Delta^{\frac{1}{2}} (s_\omega \otimes x_n) \Omega_\omega - J_\omega \Delta^{\frac{1}{2}} (s_\omega \otimes x) \Omega_\omega \| = \| (s_\omega \otimes (x_n^* - x^*)) \Omega_\omega \|,
\] (62)
and the $\sigma$-strong*-convergence $x_n \to x$. The right hand of (64) converges to $\sum_{k \in \Lambda} u\zeta_k \otimes \frac{1}{\rho} x\zeta_k$ because of

$$
\left\| \sum_{k \in \Lambda} u\zeta_k \otimes \frac{1}{\rho} (x_n - x)\zeta_k \right\|^2 = \sum_{k \in \Lambda} \left\| \frac{1}{\rho} (x_n - x)\zeta_k \right\|^2,
$$

and the $\sigma$-strong*-convergence $x_n \to x$. Hence we have proven (63).

Next we show 3.

$$(s_\omega \otimes x) \Omega_\omega = \sum_{l \in \Lambda} u \rho \frac{1}{\nu} c^* x^* c\zeta_l \otimes \zeta_l, \quad x \in B(s_\omega \mathcal{H}_\omega).$$

(64)

The right hand side converges in norm. To prove (64), first we consider

$$(s_\omega \otimes Q_n x Q_m) \Omega_\omega = \sum_{k \in \Lambda, l \in \Lambda} \sqrt{\lambda_k} u\zeta_k \otimes | \zeta_l \rangle \langle \zeta_l, x\zeta_k \rangle,$$

(65)

for $n, m \in \mathbb{N}$. Since we have $\langle \zeta_l, x\zeta_k \rangle = (c\zeta_l, xc\zeta_k) = (c^* x c\zeta_k, \zeta_l)$ by $c\zeta_k = \zeta_k$, we have

$$
\sum_{l \in \Lambda, k \in \Lambda} \sum_{l \in \Lambda} \sqrt{\lambda_k} u\zeta_k \otimes | \zeta_l \rangle \langle \zeta_l | c^* x c\zeta_l, \zeta_l \rangle = \sum_{l \in \Lambda} u Q_m \rho \frac{1}{\nu} Q_m c^* x c\zeta_l, \zeta_l.
$$

(66)

Hence we obtain

$$(s_\omega \otimes Q_n x Q_m) \Omega_\omega = \sum_{l \in \Lambda} u Q_m \rho \frac{1}{\nu} Q_m c^* x c\zeta_l, \zeta_l.$$ 

(67)

Taking $m \to \infty$, and then $n \to \infty$, we obtain (64).

Next we consider the action of $J_\omega$. We claim

$$J_\omega (s_\omega \otimes x) J_\omega^* = u c^* x cu^* \otimes s_\omega, \quad x \in B(s_\omega \mathcal{H}_\omega).$$

(68)

To prove this, note that

$$J_\omega (s_\omega \otimes x) \Omega_\omega = J_\omega J_\omega \Delta_\omega^\frac{1}{2} (s_\omega \otimes x^*) \Omega_\omega = \Delta_\omega^\frac{1}{2} (s_\omega \otimes x^*) \Omega_\omega = \sum_{k \in \Lambda} u\zeta_k \otimes \frac{1}{\rho} x^* \zeta_k = \lim_{n \to \infty} \sum_{k \in \Lambda} u\zeta_k \otimes \frac{1}{\rho} x^* \zeta_k,$$

(69)

for any $x \in B(s_\omega \mathcal{H}_\omega)$, by 2. The right hand side converges in norm. Therefore, for any $m \in \mathbb{N}$, we have

$$(s_\omega \otimes Q_m) J_\omega (s_\omega \otimes x) \Omega_\omega = \lim_{n \to \infty} \sum_{k \in \Lambda} u\zeta_k \otimes Q_m \rho \frac{1}{\nu} x^* \zeta_k, \quad x \in B(s_\omega \mathcal{H}_\omega).$$

(70)

Note that as in the proof of (67), we have

$$\sum_{k \in \Lambda} u\zeta_k \otimes Q_m \rho \frac{1}{\nu} x^* \zeta_k = \sum_{k \in \Lambda} u\zeta_k \otimes | \zeta_l \rangle \langle \zeta_l | c^* x^* c\zeta_l, \zeta_l \rangle = \sum_{l \in \Lambda} \sum_{k \in \Lambda} u | \zeta_l \rangle \langle \zeta_l | c^* x^* c\zeta_l, \zeta_l \rangle = \sum_{l \in \Lambda} u Q_m u^* \otimes s_\omega \sum_{l \in \Lambda} u^* c^* x^* c\zeta_l, \zeta_l = (u Q_n u^* \otimes s_\omega) \sum_{l \in \Lambda} \sqrt{\lambda} u c^* x^* c\zeta_l, \zeta_l = (u Q_n u^* \otimes s_\omega) \sum_{l \in \Lambda} \sqrt{\lambda} u \zeta_l \otimes \zeta_l = (u Q_n u^* \otimes s_\omega) (s_\omega \otimes Q_m) \Omega_\omega,$$

(71)
for any $x \in B(s_\omega H_\omega)$. Taking $n \to \infty$, we obtain
\[
(s_\omega \otimes Q_m) J_\omega (s_\omega \otimes x) \Omega_\omega = \lim_{n \to \infty} \sum_{k \in \Lambda_n} u_\zeta_k \otimes Q_m \rho_\omega^k x^* \zeta_k = (uc^* xc^* \otimes Q_m) \Omega_\omega, \tag{72}
\]
for any $x \in B(s_\omega H_\omega)$. Taking $m \to \infty$, we obtain
\[
J_\omega (s_\omega \otimes x) J_\omega^* \Omega_\omega = J_\omega (s_\omega \otimes x) \Omega_\omega = (uc^* xc^* \otimes s_\omega) \Omega_\omega, \quad x \in B(s_\omega H_\omega), \tag{73}
\]
for any $x \in B(s_\omega H_\omega)$. Note that $J_\omega (s_\omega \otimes x) J_\omega^* \in \text{Ad}(J_\omega)(\mathcal{M}) = \mathcal{M}'$ and $uc^* xc^* \otimes s_\omega \in \mathcal{M}'$. Since $\Omega_\omega$ is separating for $\mathcal{M}'$, we have
\[
J_\omega (s_\omega \otimes x) J_\omega^* = uc^* xc^* \otimes s_\omega, \quad x \in B(s_\omega H_\omega), \tag{74}
\]
proving the claim.

The adjoint action of $J_\omega$ on $\mathcal{M}$ introduces a $\mathbb{Z}_2$-index.

**Proposition 4.3.** Let $\omega$ be a reflection invariant pure state on $\mathcal{A}$ which satisfies the split property with respect to $\mathcal{A}_L$ and $\mathcal{A}_R$. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ be a reflection-split representation associated to $\omega$. Let $s_\omega$ be a projection in $B(\mathcal{H}_\omega)$ such that the support projection of $\Omega_\omega$ in $1_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega)$ is $1_{\mathcal{H}_\omega} \otimes s_\omega$. Set $\mathcal{M} := s_\omega \otimes B(s_\omega H_\omega)$, and $p_\omega = s_\omega \otimes s_\omega$. Let $J_\omega$ be the modular conjugation on $p_\omega H_\omega$ associated to $(\mathcal{M}, \Omega_\omega)$. Then we have the following:

1. There exists an anti-unitary $\theta : s_\omega H_\omega \to s_\omega H_\omega$ such that
   \[
   J_\omega (s_\omega \otimes x) J_\omega^* = \theta x^* \otimes s_\omega, \tag{75}
   \]
   for all $x \in B(s_\omega H_\omega)$.

2. For any anti-unitary $\theta : s_\omega H_\omega \to s_\omega H_\omega$ satisfying \eqref{72}, we have
   \[
   J_\omega (x \otimes s_\omega) J_\omega^* = s_\omega \otimes \theta x^*, \tag{76}
   \]
   for all $x \in B(s_\omega H_\omega)$.

3. There exists a $\kappa_\omega \in \{\pm 1\}$ satisfying $\theta^2 = \kappa_\omega s_\omega$ for any anti-unitary $\theta : s_\omega H_\omega \to s_\omega H_\omega$ satisfying \eqref{72}.

**Proof.** Let $\Delta_\omega$ be the modular operator of $(\mathcal{M}, \Omega_\omega)$. From 5. of Lemma 4.1, $\tilde{\Gamma} := \Gamma_\omega p_\omega$ defines a unitary operator on $p_\omega H_\omega$. Note that $\tilde{\Gamma} \Omega_\omega = \Omega_\omega$, because of Lemma 4.1.2, and Definition 2.2.4. We claim that
\[
J_\omega \tilde{\Gamma} = \tilde{\Gamma} J_\omega. \tag{77}
\]
To see this, we recall \eqref{JGJ} and \eqref{JGJtilde}. For any $x \in \mathcal{M}$, we have
\[
\tilde{\Gamma} J_\omega \Delta_\omega^+ x \Omega_\omega = \tilde{\Gamma} x^* \Omega_\omega = \tilde{\Gamma} x^* \tilde{\Gamma}^* \Omega_\omega = \tilde{\Gamma} x^* \tilde{\Gamma}^* \Omega_\omega. \tag{78}
\]
Note that from \eqref{JGJtilde}, the element $\tilde{\Gamma} x^* \tilde{\Gamma}^*$ belongs to $\mathcal{M}'$. Therefore, from \eqref{JGJtilde}, we have
\[
\tilde{\Gamma} J_\omega \Delta_\omega^+ \tilde{\Gamma} x \tilde{\Gamma}^* \Omega_\omega = \tilde{\Gamma} x \tilde{\Gamma}^* \Omega_\omega. \tag{79}
\]
Since $\mathcal{M}\Omega_\omega$ is a core of $J_\omega \Delta_{\omega}^{\frac{1}{2}}$ and $\mathcal{M}'\Omega_\omega$ is a core of $J_\omega \Delta_{\omega}^{\frac{1}{2}}$, this means

$$\hat{\Gamma} J_\omega \Delta_{\omega}^{\frac{1}{2}} = J_\omega \Delta_{\omega}^{\frac{1}{2}} \hat{\Gamma} = J_\omega \tilde{\Gamma}^* \Delta_{\omega}^{\frac{1}{2}} \hat{\Gamma}. \quad (80)$$

By the uniqueness of the polar decomposition, we obtain $\hat{\Gamma} J_\omega = J_\omega \hat{\Gamma}$, proving the claim.

Next we note that there are anti-$*$-automorphisms $\Theta_{L\to R}, \Theta_{R\to L}$ on $B(s_\omega \mathcal{H}_\omega)$ such that

$$J_\omega (s_\omega \otimes x) J_\omega^* = \Theta_{R\to L}(x) \otimes s_\omega, \quad (81)$$

$$J_\omega (x \otimes s_\omega) J_\omega^* = s_\omega \otimes \Theta_{L\to R}(x), \quad (82)$$

for any $x \in B(s_\omega \mathcal{H}_\omega)$. This is because of the Tomita-Takesaki theory, i.e., $J_\omega \mathcal{M} J_\omega^* = \mathcal{M}'$ and $J_\omega \mathcal{M}' J_\omega^* = \mathcal{M}$. Note that $\Theta_{L\to R} \circ \Theta_{R\to L} = \text{id} = \Theta_{R\to L} \circ \Theta_{L\to R}$, because of $J_\omega^2 = s_\omega$. By (19) and (77), this $\Theta_{L\to R}$ and $\Theta_{R\to L}$ coincide. For any $x \in B(s_\omega \mathcal{H}_\omega)$, we have

$$\Theta_{R\to L}(x) \otimes s_\omega = (\text{Ad } J_\omega) (s_\omega \otimes x) = \left( \text{Ad } J_\omega \hat{\Gamma} \right) (x \otimes s_\omega) = \left( \text{Ad } \tilde{\Gamma} J_\omega \right) (x \otimes s_\omega)$$

$$= \text{Ad } \hat{\Gamma} (s_\omega \otimes \Theta_{L\to R}(x)) = \Theta_{L\to R}(x) \otimes s_\omega. \quad (83)$$

Hence we have $\Theta_{R\to L}(x) = \Theta_{L\to R}(x)$ for any $x \in B(s_\omega \mathcal{H}_\omega)$.

Now let us prove 1.-3. of the Lemma. 1. is shown in Lemma 4.2, as $\theta = uc^*$. To prove the second and the third statement, let $\theta : s_\omega \mathcal{H}_\omega \to s_\omega \mathcal{H}_\omega$ be any anti-unitary such that

$$J_\omega (s_\omega \otimes x) J_\omega^* = \theta x \theta^* \otimes s_\omega, \quad x \in B(s_\omega \mathcal{H}_\omega). \quad (84)$$

From this and (81), we obtain

$$\Theta_{R\to L}(x) = \Theta_{L\to R}(x) = \theta x \theta^*, \quad x \in B(s_\omega \mathcal{H}_\omega). \quad (85)$$

Therefore, from (82),

$$J_\omega (x \otimes s_\omega) J_\omega^* = s_\omega \otimes \theta x \theta^*, \quad x \in B(s_\omega \mathcal{H}_\omega). \quad (86)$$

This proves the second statement.

Furthermore, we have

$$s_\omega \otimes x = J_\omega^2 (s_\omega \otimes x) (J_\omega^*)^2 = J_\omega (\theta x \theta^* \otimes s_\omega) J_\omega^* = s_\omega \otimes \theta^2 x (\theta^*)^2, \quad (87)$$

for any $x \in B(s_\omega \mathcal{H}_\omega)$. This means $\theta^2 = \tilde{\kappa}_\theta s_\omega$ with some $\tilde{\kappa}_\theta \in \mathbb{T}$. Then by the anti-linearity of $\theta$, we have

$$\tilde{\kappa}_\theta \theta = \theta \tilde{\kappa}_\theta = \theta^2 \theta^2 = \theta \tilde{\kappa}_\theta = \tilde{\kappa}_\theta \theta. \quad (88)$$

This means $\tilde{\kappa}_\theta$ is real, namely $\tilde{\kappa}_\theta = \pm 1$.

This $\tilde{\kappa}_\theta$ is independent of the choice of $\theta$ satisfying (85) for if $\theta_1$ is another such anti-unitary, we have

$$\theta_1 x \theta_1^* \otimes s_\omega = J_\omega (s_\omega \otimes x) J_\omega^* = \theta x \theta^* \otimes s_\omega, \quad (89)$$

for all $x \in B(s_\omega \mathcal{H}_\omega)$. Hence $\theta^* \theta_1$ is a unitary operator on $s_\omega \mathcal{H}_\omega$ which commutes with any $x \in B(s_\omega \mathcal{H}_\omega)$, i.e., $\theta = c \theta_1$ for some $c \in \mathbb{T}$. We then have

$$\tilde{\kappa}_\theta s_\omega = \theta^2 = c \theta_1 \theta_1 = c \bar{c} \theta_1 \theta_1 = \tilde{\kappa}_\theta s_\omega, \quad (90)$$

and get $\tilde{\kappa}_\theta = \tilde{\kappa}_\theta_1 = : \kappa_\omega$. This proves the third statement.

The sign $\kappa_\omega$ coincides with our $\mathbb{Z}_2$-index $\sigma_\omega$. 

\hspace{1cm} \square
Theorem 4.4. Let $\omega$ be a reflection invariant pure state on $A$ which satisfies the split property with respect to $A_R$ and $A_L$. Let $(H_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ be a reflection-split representation associated to $\omega$. Let $s_\omega$ be a projection in $B(H_\omega)$ such that the support projection of $\Omega_\omega$ in $I_{H_\omega} \otimes B(H_\omega)$ is $I_{H_\omega} \otimes s_\omega$. Set $M := s_\omega \otimes B(s_\omega H_\omega)$, and $p_\omega = s_\omega \otimes s_\omega$. Let $J_\omega$ be the modular conjugation on $p_\omega H_\omega$ associated to $(M, \Omega_\omega)$. Let $\sigma_\omega$ be the $\mathbb{Z}_2$-index associated to $\omega$ in Definition 2.7 and $\kappa_\omega$ the $\mathbb{Z}_2$-index associated to $\omega$ in Proposition 4.3. Then we have $\kappa_\omega = \sigma_\omega$.

Proof. We use the notation used in Lemma 4.2. From $\Gamma_\omega \Omega_\omega = \Omega_\omega$, we obtain $u = \sigma_\omega c^* u^* c$: first we have

$$\sum_{k \in \Lambda} u \rho_\omega^\frac{1}{2} \zeta_k \otimes \zeta_k = \sum_{k \in \Lambda} \sqrt{\lambda_k} u \zeta_k \otimes \zeta_k = \Omega_\omega = \Gamma_\omega \Omega_\omega = \sum_{k \in \Lambda} \sqrt{\lambda_k} \Gamma_\omega (u \zeta_k \otimes \zeta_k) = \sum_{k \in \Lambda} \sqrt{\lambda_k} \sigma_\omega (\zeta_k \otimes u \zeta_k) = \sigma_\omega (u^* \otimes u) \sum_{k \in \Lambda} \sqrt{\lambda_k} (u \zeta_k \otimes \zeta_k) = \sigma_\omega (u^* \otimes s_\omega) (s_\omega \otimes u) \Omega_\omega = \sigma_\omega (u^* \otimes s_\omega) \sum_{l \in \Lambda} u \rho_\omega^\frac{1}{2} c^* u^* c \zeta_l \otimes \zeta_l$$

$$= \sigma_\omega \sum_{l \in \Lambda} \rho_\omega^\frac{1}{2} c^* u^* c \zeta_l \otimes \zeta_l. \quad (91)$$

Here we used Theorem 2.6 and Lemma 4.2 3. From this we obtain

$$u \rho_\omega^\frac{1}{2} \zeta_k = \sigma_\omega \rho_\omega^\frac{1}{2} c^* u^* c \zeta_k,$$

for all $k \in \Lambda$. Hence we have

$$u \rho_\omega^\frac{1}{2} = \sigma_\omega c^* u^* c c^* u \rho_\omega^\frac{1}{2} c^* u^* c. \quad (93)$$

By the uniqueness of the polar decomposition, we obtain the claim

$$u = \sigma_\omega c^* u^* c. \quad (94)$$

Now we are ready to complete the proof of the Theorem. By Proposition 4.3 and Lemma 4.2 4., we have $(u c)^2 = (u c^*)^2 = \kappa_\omega s_\omega$. From this and (94), we have

$$\kappa_\omega = \langle (u c)^2 \zeta_k, \zeta_k \rangle = \langle u c u c^* \zeta_k, \zeta_k \rangle = \langle (u c^*)^2 \zeta_k, u c \zeta_k \rangle = \langle c^* u c \zeta_k, u c \zeta_k \rangle = \langle \sigma_\omega u \zeta_k, u c \zeta_k \rangle = \sigma_\omega,$$

for any $k \in \Lambda$. This completes the proof. \hfill \Box

Remark 4.5. From the proof, we see that one way to derive the index $\sigma_\omega$ for concrete state $\omega$ is to consider the Schmidt decomposition and calculate $u$. Using (94), we can obtain $\sigma_\omega$.

5 $\mathbb{Z}_2$-index in Matrix Product States

In this section, we prove that the $\mathbb{Z}_2$-index $\sigma_\omega$ for a matrix product state $\omega$ is the same as the $\mathbb{Z}_2$-index found in [PTBO1]. First let us recall known facts on matrix product states. Let $k \in \mathbb{N}$ be a number and $v = (v_\mu)_{\mu=1,d} \in M_k^{\times d}$ a $d$-tuple of $k \times k$ matrices. For each $l \in \mathbb{N}$, we set

$$K_l(v) := \text{span} \{ v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{l-1}} | (\mu_0, \mu_1, \cdots, \mu_{l-1}) \subset \{1, \ldots, d\}^{\times l} \}.$$  

We say $v$ is primitive if $K_l(v) = M_k$ for $l$ large enough. We denote by $\text{Prim}_u(d,k)$ the set of all primitive $d$-tuples $v$ of $k \times k$ matrices which are normalized, i.e.,

$$\sum_{\mu=1, \ldots, d} v_{\mu} v_{\mu}^* = 1.$$
For \( \nu \in \text{Prim}_u(d, k) \), there exists a unique \( T_\nu \)-invariant state \( \hat{\rho}_\nu \), and it is faithful. (See [W] for example.) We denote the density matrix corresponding to \( \hat{\rho}_\nu \) by \( \rho_\nu \). Each \( \nu \in \text{Prim}_u(d, k) \) generates a translationally invariant state \( \omega_\nu \) by

\[
\omega_\nu \left( \bigotimes_{i=0}^{l-1} |\psi_{\mu_i} \rangle \langle \psi_{\nu_i}| \right) = \hat{\rho}_\nu \left( v_{\mu_0} \cdots v_{\mu_{l-1}} \nu_{\nu_{l-1}} \cdots \nu_{\nu_0} \right), \quad \mu_i, \nu_i = 1, \ldots, d, \quad i = 0, \ldots, l-1, \quad l \in \mathbb{N}.
\]

(97)

A translationally invariant state which has this representation is called a matrix product state. This representation is unique up to unitary and phase \([\text{FNW2}]\): If both of \( \nu^{(1)} \in \text{Prim}_u(d, k_1) \) and \( \nu^{(2)} \in \text{Prim}_u(d, k_2) \) generate the same matrix product state, then \( k_1 = k_2 \) and there exists a unitary \( U : \mathbb{C}^{k_1} \rightarrow \mathbb{C}^{k_2} \) and \( e^{i\theta} \in \mathbb{T} \) such that

\[
U \nu^{(1)} = e^{i\theta} \nu^{(2)} U, \quad \mu = 1, \ldots, d.
\]

(98)

Let \( \omega \) be a reflection invariant matrix product state generated by \( \nu \in \text{Prim}_u(d, k) \). It is a unique ground state of some translation invariant finite range interaction. i.e., there is an interaction \( \Phi_\nu \) given by some fixed local positive element \( h_\nu \in \mathcal{A}_{[0, m-1]} \) with some \( m \in \mathbb{N} \) as

\[
\Phi_\nu(X) := \begin{cases} 
\beta_x(h_\nu), & \text{if } X = [x, x+m-1] \cap \mathbb{Z} \text{ for some } x \in \mathbb{Z} \\
0, & \text{otherwise}
\end{cases}
\]

(99)

for each \( X \in \mathcal{S}_\mathbb{Z} \) and \( \omega \) is a unique \( \tau^{\Phi_\nu} \)-ground state. (See \([\text{FNW}] \) and \([\text{O3}] \) Corollary 5.6 \([\text{O1}] \) Theorem 1.18, Lemma 3.25.) For this interaction \( h_\nu \), \( 1 - h_\nu \) is equal to the support projection of \( \omega|_{\mathcal{A}_{[0, m-1]}} \). (Note that \( \omega|_{\mathcal{A}_L} = \Xi_L(\hat{\rho}_\nu) \) with the \( \Xi_L \) in Lemma 3.14 of \([\text{O1}] \) and the \( T_\nu \)-invariant state \( \hat{\rho}_\nu \).) From the proof of Lemma 3.19 of \([\text{O1}] \) equation (48), we see that \( 1 - h_\nu \) is equal to the support projection of \( \omega|_{\mathcal{A}_{[0, m-1]}} \). Note that primitive \( \nu \) belongs to ClassA, Remark 1.16 of \([\text{O1}] \). Therefore, from the reflection invariance of \( \omega \), \( \Phi_\nu \) is reflection invariant. The Hamiltonian given by this interaction is frustration-free, i.e., for each finite interval \( I \) with \( |I| \geq m \), the local Hamiltonian \( (H_{\Phi_\nu})_I \) has a nontrivial kernel, which is the ground state space of \( (H_{\Phi_\nu})_I \). We denote by \( G_{I, \nu} \), the orthogonal projection onto this kernel. By Lemma 3.19 of \([\text{O1}] \), and its proof (equation (48)), the support of the restriction \( \omega|_{\mathcal{A}_I} \) is equal to \( G_{I, \nu} \) and there exists some constant \( d_\nu > 0 \) such that

\[
\psi \leq d_\nu \cdot \omega,
\]

(100)

for any frustration free state \( \psi \) on \( \mathcal{A}_R \), i.e., a state \( \psi \) satisfying \( \psi(\beta_x(h_\nu)) = 0 \) for any \( 0 \leq x \in \mathbb{Z} \). (See proof of Lemma 2.3 of \([\text{O2}] \).)

Now let us come back to our problem. With the analogous argument as in \([\text{PTBO1}] \), we obtain the following. See \([\text{Ian2}] \) for a nice description.

**Lemma 5.1.** Let \( \omega \) be a reflection invariant matrix product state generated by \( \nu \in \text{Prim}_u(d, k) \). Let \( \hat{\rho}_\nu \) be the \( T_\nu \)-invariant state given by a density matrix \( \rho_\nu \). Then there exist \( e^{i\theta} \in \mathbb{T} \) and \( \theta : \mathbb{C}^k \rightarrow \mathbb{C}^k \) an anti-unitary such that

\[
v_\mu = e^{i\theta} \rho^{\frac{1}{k}} \psi_{\mu} \cdot \theta \psi_{\mu}^* \rho^{\frac{1}{k}}, \quad \mu = 1, \ldots, d.
\]

(101)

For any \( e^{i\theta} \in \mathbb{T} \) and \( \theta : \mathbb{C}^k \rightarrow \mathbb{C}^k \) an anti-unitary satisfying \([\text{O1}] \), we have

\[
\theta^2 = \zeta_\omega I,
\]

(102)

with some \( \zeta_\omega \in \{\pm 1\} \). The value \( \zeta_\omega \) does not depend on the choice of \( \nu \), \( e^{i\theta} \in \mathbb{T} \) and \( \theta : \mathbb{C}^k \rightarrow \mathbb{C}^k \).

**Definition 5.2.** By this Lemma, we define a \( \mathbb{Z}_2 \)-index \( \zeta_\omega \).
Proof. Let \( \rho_v = \sum_{j=1}^{d} \lambda_j |\xi_j\rangle \langle \xi_j | \) be the spectral decomposition of \( \rho_v \), where \( \lambda_j > 0 \) and \( \{ \xi_j \}_{j=1}^{d} \) is an orthonormal basis of \( \mathbb{C}^k \). Let \( c : \mathbb{C}^k \to \mathbb{C}^k \) be the complex conjugation such that \( c\xi_j = \xi_j \) for all \( j = 1, \ldots, k \). Note that \( c^* = c \).

Set
\[
\tilde{v}_\mu := cv_\mu c, \quad \tilde{\nu}_\mu := \rho_v^{-\frac{1}{2}} (\tilde{v}_\mu)^* \rho_v^{-\frac{1}{2}}, \quad \mu = 1, \ldots, d.
\]

We claim \( \tilde{v} \in \text{Prim}_u(d, k) \) and it generates \( \omega \). Since \( K_l(v) = M_k \) for \( l \) large enough, \( K_l(\tilde{v}) = M_k \) for \( l \) large enough. Hence \( \tilde{v} \) is primitive. Using \( cp_v c = \rho_v \) and \( \sum_\mu \nu_\mu \rho_\nu v_\mu = \rho_v \), we have
\[
\sum_\mu \tilde{v}_\mu \tilde{\nu}_\mu = \sum_\mu \rho_v^{-\frac{1}{2}} cv_\mu c \rho_\nu v_\mu = \sum_\mu \rho_v^{-\frac{1}{2}} cv_\mu c \rho_\nu v_\mu - \frac{1}{2} = \rho_v^{-\frac{1}{2}} cp_v c \rho_v^{-\frac{1}{2}} = \mathbb{I}_k
\]

Hence we have \( \tilde{v} \in \text{Prim}_u(d, k) \). The state \( \rho_v \) is \( T_v \)-invariant because
\[
\sum_\mu \tilde{v}_\mu \tilde{\nu}_\mu = \sum_\mu \rho_v^{-\frac{1}{2}} cv_\mu c \rho_v^{-\frac{1}{2}} \rho_\nu \rho_v^{-\frac{1}{2}} cv_\mu c \rho_v^{-\frac{1}{2}} = \rho_v,
\]
from \( \sum_\mu \nu_\mu v_\mu^* = 1 \). Now we show that \( \tilde{v} \) generates \( \omega \). For any \( l \in \mathbb{N} \) and \( \mu_i, v_i = 1, \ldots , d, i = 0, \ldots , l - 1 \), from the reflection invariance and translation invariance, we have
\[
\omega \left( \bigotimes_{i=0}^{l-1} |\psi_{\mu_i}\rangle \langle \psi_{\nu_i}| \right) = \omega \circ \gamma \left( \bigotimes_{i=0}^{l-1} |\psi_{\mu_i}\rangle \langle \psi_{\nu_i}| \right) = \omega \left( \bigotimes_{i=-l}^{-1} |\psi_{\mu_{-i-1}}\rangle \langle \psi_{\nu_{-i-1}}| \right) = \omega \left( \bigotimes_{i=0}^{l-1} |\psi_{\mu_{-i-1}}\rangle \langle \psi_{\nu_{-i-1}}| \right)
\]

Note that
\[
\langle \xi_j, v_{\mu_{-1}} v_{\mu_{-2}} \cdots v_{\mu_l} v_{\mu_0} v_{\nu_1} v_{\nu_2} \cdots v_{\nu_{l-1}} \rangle = \sum_{j=1}^{k} \lambda_j \langle \xi_j, v_{\mu_{-1}} v_{\mu_{-2}} \cdots v_{\mu_l} v_{\mu_0} v_{\nu_1} v_{\nu_2} \cdots v_{\nu_{l-1}} \rangle
\]

Substituting this to (106), we have
\[
(106) = \tilde{\rho}_v \left( \tilde{v}_{\mu_0} \tilde{v}_{\mu_1} \cdots \tilde{v}_{\mu_{l-1}} \tilde{v}_{\nu_0} \tilde{v}_{\nu_1} \cdots \tilde{v}_{\nu_{l-1}} \tilde{v}_{\mu_{l-1}} \right) = \text{Tr} \left( \tilde{\rho}_v \left( \tilde{v}_{\mu_0} \tilde{v}_{\mu_1} \cdots \tilde{v}_{\mu_{l-1}} \tilde{v}_{\nu_0} \tilde{v}_{\nu_1} \cdots \tilde{v}_{\nu_{l-1}} \right) \right)
\]

Hence \( \tilde{v} \) generates \( \omega \), proving the claim.

Now, as both of \( v \) and \( \tilde{v} \) generates same state \( \omega \), by the uniqueness [98], there exist a unitary \( U \) on \( \mathbb{C}^k \) and \( e^{it} \in \mathbb{T} \) such that
\[
U v_\mu = e^{it} \tilde{v}_\mu U, \quad \mu = 1, \ldots, d.
\]

From the fact that \( \hat{\rho}_v \) is \( T_v \)-invariant and (109), we see that the state \( \hat{\rho}_v \circ \text{Ad} U \) is \( T_v \)-invariant. By the uniqueness of \( T_v \)-invariant state (by \( v \in \text{Prim}_u(d, k) \)), we get
\[
U \rho_v U^* = \rho_v.
\]

Set \( \theta := U^* c : \mathbb{C}^k \to \mathbb{C}^k \) an anti-unitary operator. From (109) and the definition of \( \tilde{v} \), and (110), we obtain (111):
\[
v_\mu = e^{it} U^* \tilde{v}_\mu U = e^{it} U^* \rho_v^{-\frac{1}{2}} (\tilde{v}_\mu)^* \rho_v^{-\frac{1}{2}} U = e^{it} \rho_v^{-\frac{1}{2}} (U^* c) v_\mu (U^* c)^* \rho_v^{-\frac{1}{2}} = e^{it} \rho_v^{-\frac{1}{2}} \theta v_\mu \theta^* \rho_v^{-\frac{1}{2}}.
\]
Now for any \( e^{it_0} \in \mathbb{T} \) and \( \theta_0 : \mathbb{C}^k \to \mathbb{C}^k \) an anti-unitary satisfying
\[
v_\mu = e^{it_0} \rho_v^{-\frac{1}{2}} \theta_0 v_\mu^* \theta_0^* \rho_v^{\frac{1}{2}}, \quad \mu = 1, \ldots, d,
\]
we show \( \theta_0^2 = \mathbb{I} \) or \( \theta_0^2 = -\mathbb{I} \). Taking adjoint of (111), we have
\[
v_\mu^* = e^{-it_0} \rho_v^{\frac{1}{2}} \theta_0 v_\mu^* \theta_0^* \rho_v^{-\frac{1}{2}}.
\]
Substituting this to (101), we obtain
\[
v_\mu = e^{it_0} \rho_v^{-\frac{1}{2}} \theta_0 e^{-it_0} \rho_v^{\frac{1}{2}} \theta_0 v_\mu^* \theta_0^* \rho_v^{-\frac{1}{2}} \rho_v^{\frac{1}{2}} = e^{2it_0} \rho_v^{-\frac{1}{2}} \theta_0 \rho_v^{\frac{1}{2}} \theta_0 v_\mu^* \theta_0^* \rho_v^{-\frac{1}{2}} \rho_v^{\frac{1}{2}} \rho_v^{\frac{1}{2}}
\]
Since \( v \) is primitive, this means \( e^{2it_0} = 1 \) and \( \rho_v^{-\frac{1}{2}} \theta_0 \rho_v^{\frac{1}{2}} \theta_0 = b \mathbb{I} \) for some \( b \in \mathbb{C} \). Decomposing \( b = e^{is}|b| \) with \( e^{is} \in \mathbb{T} \), we have
\[
\theta_0^2 \left( \theta_0^* \rho_v^{\frac{1}{2}} \theta_0 \right) = e^{is}|b| \rho_v^{\frac{1}{2}}.
\]
By the uniqueness of the polar decomposition and the faithfulness of \( \rho_v \), we get \( \theta_0^2 = e^{is} \mathbb{I} \). But then
\[
e^{is} \theta_0 = \theta_0^2 \theta_0 = \theta_0^3 \theta_0 = \theta_0^2 \theta_0 = \theta_0 e^{is} = e^{-is} \theta_0.
\]
Therefore, \( e^{is} \) is real and we get that \( \theta_0^2 = \mathbb{I} \) or \( \theta_0^2 = -\mathbb{I} \).

To prove the independence of this sign of \( \omega, e^{it} \) and \( \theta \), let \( \omega \in \text{Prim}_\mu(d, k') \) be a generator of \( \omega \), with a \( T_\omega \)-invariant state \( \hat{\rho}_\omega \) given by a density matrix \( \rho_\omega \). Let \( e^{iu} \in \mathbb{T} \) and \( \xi : \mathbb{C}^k' \to \mathbb{C}^k' \) an anti-unitary such that
\[
\omega_\mu = e^{iu} \rho_\omega^{-\frac{1}{2}} \xi \omega_\mu^* \xi^* \rho_\omega^{\frac{1}{2}}.
\]
Since both of \( v \) and \( \omega \) generates same \( \omega \), from the uniqueness (113), \( k = k' \) and there exist a unitary \( V \) on \( \mathbb{C}^k \) and \( e^{i\lambda} \in \mathbb{T} \) such that \( v_\mu = e^{i\lambda} V^* \omega_\mu V \). From
\[
\hat{\rho}_v = \hat{\rho}_v \circ T_\omega = \hat{\rho}_v \circ \text{Ad}(V^*) \circ T_\omega \circ \text{Ad}(V),
\]
\( \hat{\rho}_v \circ \text{Ad}(V^*) \) is a \( T_\omega \)-invariant state. By the uniqueness of a \( T_\omega \)-invariant state, we get \( \rho_\omega = V \rho_v V^* \).

Now we have
\[
e^{i\lambda} V^* \omega_\mu V = v_\mu = e^{it} \rho_v^{-\frac{1}{2}} \theta_0 \omega_\mu^* \theta_0^* \rho_v^{\frac{1}{2}} = e^{it} \rho_v^{-\frac{1}{2}} \theta_0 \left( e^{i\lambda} V^* \omega_\mu V \right)^* \theta_0^* \rho_v^{\frac{1}{2}} = e^{it+i\lambda} \rho_v^{-\frac{1}{2}} \theta V^* \omega_\mu^* V \theta^* \rho_v^{\frac{1}{2}}.
\]
From this, (117), and \( \rho_\omega = V \rho_v V^* \), we have
\[
e^{it} \rho_v^{-\frac{1}{2}} \theta V^* \omega_\mu^* V \theta^* \rho_v^{\frac{1}{2}} = e^{it} V \rho_v^{-\frac{1}{2}} \theta V^* \omega_\mu^* V \theta^* \rho_v^{\frac{1}{2}} V^* = \omega_\mu = e^{iu} \rho_\omega^{-\frac{1}{2}} \xi \omega_\mu^* \xi^* \rho_\omega^{\frac{1}{2}}.
\]
Therefore, we get
\[
e^{it} V \theta V^* \omega_\mu^* V \theta^* V^* = e^{iu} \xi \omega_\mu^* \xi^*.
\]
Since \( \omega \) is primitive, this means \( \xi^* V \theta V^* = e^{iu} \mathbb{I} \) for some \( e^{iu} \in \mathbb{T} \). Then we have
\[
V \theta^2 V^* = V \theta V^* V \theta V^* = e^{-iu} \xi V \theta^2 V^* = \xi^2.
\]
This proves the claim. \( \square \)
Since a matrix product state $\omega$ generated by a normalized primitive $d$-tuple is a unique gapped ground state by [FNW, O3], it is pure and satisfies the split property. Therefore, if furthermore $\omega$ is reflection invariant, we can associate $\omega$, our $\mathbb{Z}_2$-index $\sigma_\omega$ in Definition 2.4. We then have the following theorem.

**Theorem 5.3.** For a reflection invariant matrix product state $\omega$ generated by a normalized primitive $d$-tuple of matrices, we have

$$\sigma_\omega = \zeta_\omega.$$  

**Proof.** Let $\omega$ be a reflection invariant matrix product state generated by $v \in \text{Prim}_n(d, k)$. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega, \Gamma_\omega)$ be a reflection-split representation associated to $\omega$. Then

$$\left(\mathcal{H}_\omega := \mathcal{H}_\omega \otimes \mathcal{H}_\omega, \hat{\pi}_\omega := (\pi_\omega \circ \gamma_{L \to R}) \otimes \pi_\omega, \Omega_\omega\right)$$

is a GNS triple of $\omega$. Let $s_\omega$ be a projection in $B(\mathcal{H}_\omega)$ such that the support projection of $\Omega_\omega$ in $\mathbb{1}_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega)$ is $\mathbb{1}_{\mathcal{H}_\omega} \otimes s_\omega$. Set $\mathcal{M} := s_\omega \otimes B(s_\omega \mathcal{H}_\omega)$, and $\rho_\omega = s_\omega \otimes s_\omega$. We define a density matrix $\rho_\omega$ on $\mathcal{H}_\omega$ by

$$\text{Tr}_{\mathcal{H}_\omega}(\rho_\omega x) = \langle \Omega_\omega, (\mathbb{1}_{\mathcal{H}_\omega} \otimes x) \Omega_\omega \rangle , \quad x \in B(\mathcal{H}_\omega).$$

(123)

Since $\omega$ is translation invariant, there is a unitary $V$ on $\hat{\mathcal{H}}_\omega$ such that

$$V \hat{\pi}_\omega(A)V^* = \hat{\pi}_\omega \circ \beta_1(A), \quad A \in \mathcal{A}.$$  

From this, we obtain a homomorphism from $(\hat{\pi}_\omega(A_R))'' = \mathbb{1}_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega)$ onto $(\hat{\pi}_\omega \circ \beta_1(A_R))'' \subset \mathbb{1}_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega)$.

$$\mathbb{1}_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega) \ni \mathbb{1}_{\mathcal{H}_\omega} \otimes x \mapsto V (\mathbb{1}_{\mathcal{H}_\omega} \otimes x) V^* \in \mathbb{1}_{\mathcal{H}_\omega} \otimes B(\mathcal{H}_\omega).$$

(124)

Therefore, there exists an endomorphism $\Theta$ on $B(\mathcal{H}_\omega)$ such that

$$V (\mathbb{1}_{\mathcal{H}_\omega} \otimes x) V^* = \mathbb{1}_{\mathcal{H}_\omega} \otimes \Theta(x), \quad x \in B(\mathcal{H}_\omega),$$

(125)

and $\Theta(B(\mathcal{H}_\omega))' = \pi_\omega(M_d \otimes \mathbb{1}_{A_{1, \infty}})$. (Recall Lemma 2.6.8 of [BK1].) Note that $\Theta \circ \pi_\omega(A) = \pi_\omega \circ \beta_1(A)$ for any $A \in \mathcal{A}_R$. We recall the following fact:

**Lemma 5.4 ([A]).** Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, and $n \in \mathbb{N}$. Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital endomorphism of $B(\mathcal{H})$ such that $(\Phi (B(\mathcal{H})))'$ is isomorphic to $M_n$. Let $\{E_{ij}\}_{i,j=1, \ldots, n}$ be a system of matrix units of $(\Phi (B(\mathcal{H})))'$. Then there exist $S_i \in B(\mathcal{H})$, $i = 1, \ldots, n$ such that

$$S_i^* S_j = \delta_{ij}, \quad E_{ij} = S_i S_j^*, \quad \sum_{j=1}^n S_j x S_j^* = \Phi(x), \quad x \in B(\mathcal{H}).$$

(126)

Applying this to our $\Theta$ in (125), we obtain operators $S_\mu \in B(\mathcal{H}_\omega)$ with $\mu = 1, \ldots d$ satisfying the following:

$$S_\mu^* S_\nu = \delta_{\mu \nu} \mathbb{1},$$

(127)

$$\sum_{\mu=1, \ldots, d} S_\mu \pi_\omega(A) S_\mu^* = \pi_\omega \circ \beta_1(A), \quad A \in \mathcal{A}_R.$$  

(128)

$$\pi_\omega (\epsilon_\mu \otimes \mathbb{1}_{[1, \infty)}) = S_\mu S_\nu^* \quad \text{for all} \quad \mu, \nu = 1, \ldots, d.$$  

(129)
free ground state of the translation invariant finite range interaction $\Phi_H$ dimensional.

We denote $(S_\rho h_{\alpha})_{\alpha}$ a self-adjoint element $\xi$. Hence $\rho \in K$ orthogonal projection onto $\omega$ faithful on $K$. Therefore, for each $x \in \{\text{BR1}\}$ there exists a net $\pi(\cdot)$ of positive elements in the unit ball of $A_R$. By Proposition 4.3 there exists an anti-unitary $\eta$, $\omega$, $\theta$ such that $\omega(\beta_x(h_u)) = 0$ for all $x \in Z$. We consider the following frustration-free subspace of $\mathcal{H}_\omega$:

$$\mathcal{K} := \cap_{x \geq 0} \ker \pi(\beta_x(h_u)).$$

Note that the support of $\rho(\omega)$ defined in (123), is in $K$, because $\omega$ is frustration-free. Let $P_K$ be the orthogonal projection onto $K$. As in [M1] (Lemma 3.2 and the argument in the proof of Lemma 3.6), $K$ is a finite dimensional space, and $S_\rho^*$ preserves $K$:

$$S_\rho^* P_K = P_K S_\rho^* P_K, \quad \mu = 1, \ldots, d.$$  

We denote $(S_\rho^* P_K)^* \in B(K)$ by $B_\mu$, $\mu = 1, \ldots, d$. Note that $\rho_\omega$ is of finite rank because $K$ is finite dimensional.

We claim that $\mathcal{B} = (B_\mu)_{\mu=1,\ldots,d} \in \text{Prim}_\omega(d, \text{dim} K)$. To prove this, it suffices to show that $\rho_\omega$ is faithful on $K$ and for the completely positive unital map $T_\mathcal{B}$ defined by $T_\mathcal{B}(x) = \sum_{\mu=1,\ldots,d} B_\mu x B_\mu^*$, $x \in B(K)$, we have $T_\mathcal{B}(x) \to \text{Tr}_\mathcal{H}_\omega(\rho_\omega x)\mathbb{I}$, as $N \to \infty$, for each $x \in B(K)$. (See Lemma C.5 of [O1].) First we show that $\rho_\omega$ is faithful on $K$. If $\rho_\omega$ is not faithful on $K$, then there exists a unit vector $\xi \in K$ which is orthogonal to the support of $\rho_\omega$. By the definition of $K$, this $\xi$ defines a frustrated free state $\psi = \langle \xi, \pi(\cdot) \xi \rangle$ on $A_R$. Let $p$ be the orthogonal projection onto the one-dimensional space $\mathbb{C}\xi$. As $\pi(AR)^p = B(\mathcal{H}_\omega)$, by Kaplansky’s density theorem, (Theorem 2.4.16 of [BR1]) there exists a net $\{x_\alpha\}_{\alpha}$ of positive elements in the unit ball of $A_R$ such that $\pi(\omega(x_\alpha)) \to p$ in the $\sigma(w)$-topology. For this net, we have $\lim_\alpha \omega(x_\alpha) = 0$ and $\lim_\alpha \psi(x_\alpha) = 1$. This contradicts to (100). Hence $\rho_{\omega_\mu}$ is faithful on $K$. Next we show $T_\mathcal{B}(x) \to \text{Tr}_\mathcal{H}_\omega(\rho_\omega x)\mathbb{I}$, as $N \to \infty$ for all $x \in B(K)$. By $\pi_\omega(AR)^p = B(\mathcal{H}_\omega)$ and the finite dimensionality of $K$, we have $B(K) = P_K \pi_\omega(AR \cap A^\text{loc}) P_K$. Therefore, for each $x \in B(K)$, there is an element $A \in A_R \cap A^\text{loc}$ such that $x = P_K \pi_\omega(A) P_K$. Since $\omega$ is a factor state and translation invariant, we have $\sigma(w) - \lim_{N \to \infty} \pi_\omega \circ \beta_N(A) = \omega(A)\mathbb{I}$. Therefore, for any $\eta \in K$, we have

$$\langle \eta, T_\mathcal{B}(x) \eta \rangle = \langle \eta, T_\mathcal{B}(P_K \pi_\omega(\mathcal{A})(P_K) \eta) \rangle = \langle \eta, \pi_\omega \circ \beta_N(A) \eta \rangle \to \omega(A) \|\eta\|^2 = \text{Tr}_\mathcal{H}_\omega(\rho_\omega x) \|\eta\|^2, \quad N \to \infty.$$  

Hence $\mathcal{B} \in \text{Prim}_\omega(d, \text{dim} K)$.

The above proof for the primitivity also tells us that $\rho_\omega$ is the $T_\mathcal{B}$-invariant state. From (130) and the definition of $\mathcal{B}$ and (131), we see that $\mathcal{B}$ is a $d$-tuple generating $\omega$. Furthermore, as $\rho_\omega$ is faithful on $K$, we have $s_\omega = P_K$.

Let $J_\omega$ (resp. $\Delta_\omega$) be the modular conjugation (resp. modular operator) on $\rho_\omega \mathcal{H}_\omega$ associated to $(\mathcal{M}, \Omega_\omega)$. By Proposition 4.3 there exists an anti-unitary $\theta : s_\omega \mathcal{H}_\omega \to s_\omega \mathcal{H}_\omega$ such that

$$J_\omega (s_\omega \otimes x) J_\omega^* = \theta x \theta^* \otimes s_\omega, \quad J_\omega (x \otimes s_\omega) J_\omega^* = s_\omega \otimes \theta x \theta^*,$$  

for all $x \in B(s_\omega \mathcal{H}_\omega)$. By Proposition 4.3 and Theorem 4.4 we have $\theta^2 = \kappa_\omega s_\omega = \sigma_\omega s_\omega$. Recall also from Lemma 4.2

$$\Delta_\omega (s_\omega \otimes x) \Omega_\omega = \left(s_\omega \otimes \rho_\omega^\frac{1}{2} x \rho_\omega^{-\frac{1}{2}}\right) \Omega_\omega, \quad x \in B(s_\omega \mathcal{H}_\omega).$$
Now we prove that for the $\theta$ in (133), there exist an $a \in \mathbb{T}$ such that
\[ a \rho_{\omega}^{-\frac{1}{2}} \theta B_{\nu}^a \theta^* \rho_{\omega}^\frac{1}{2} = B_{\nu}, \quad \nu = 1, \ldots, d. \] (135)

From this, we obtain the claim of the Theorem:
\[ \sigma_{\omega} s_{\omega} = \theta^2 = \zeta_{\omega} s_{\omega}. \] (136)

To show (135), we first show
\[ ||S_{\mu}^* \otimes \mathbb{I}|| \Omega_{\omega}\rangle \langle (S_{\nu}^* \otimes \mathbb{I}) \Omega_{\omega}|| = ||(\mathbb{I} \otimes S_{\nu}^*) \Omega_{\omega}\rangle \langle (\mathbb{I} \otimes S_{\mu}^*) \Omega_{\omega}||, \quad \mu, \nu = 1, \ldots, d. \] (137)

For any $l \in \mathbb{N}$, $\mu, \nu, \mu_-, \ldots, \mu_{-l}, \nu_-, \ldots, \nu_{-l}, \lambda_0, \ldots, \lambda_{l-1}, \eta_0, \ldots, \eta_{l-1} = 1, \ldots, d$, we have
\[
\langle S_{\mu}^* \otimes \mathbb{I} \rangle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \otimes \pi_{\omega} \left( \bigotimes_{j=0}^{l-1} e_{\lambda_j, \eta_j} \right) \right) \langle S_{\nu}^* \otimes \mathbb{I} \rangle \Omega_{\omega} \rangle = \langle \Omega_{\omega}, S_{\mu} S_{\mu_-} \cdots S_{\mu_{-l}} S_{\nu_-} \cdots S_{\nu_{-l}} S_{\nu}^* \otimes \pi_{\omega} \left( \bigotimes_{j=0}^{l-1} e_{\lambda_j, \eta_j} \right) \rangle \langle \Omega_{\omega} \rangle \rangle
\]
\[
= \langle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_{j+1}, \nu_{j+1}} \otimes e_{\mu \nu} \right) \otimes \pi_{\omega} \left( \bigotimes_{j=0}^{l-1} e_{\lambda_j, \eta_j} \right) \right) \rangle \langle \Omega_{\omega} \rangle \rangle
\]
\[= \omega \left( \bigotimes_{j=-l}^{l-1} e_{\mu_{j+1}, \nu_{j+1}} \otimes e_{\mu \nu} \otimes \bigotimes_{j=0}^{l-1} e_{\lambda_j, \eta_j} \right) \] (138)

In the third and the fourth line $e_{\mu \nu}$ is localized at site $j = -1$. Since $\omega$ is translation invariant, we have
\[ (138) = \omega \left( \bigotimes_{j=-l}^{l-1} e_{\mu_{j+1}, \nu_{j+1}} \otimes e_{\mu \nu} \otimes \bigotimes_{j=1}^{l-1} e_{\lambda_{j-1}, \eta_{j-1}} \right). \] (139)

Here, $e_{\mu \nu}$ is localized at site $j = 0$. Then we have
\[ (139) = \langle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \otimes \pi_{\omega} \left( \bigotimes_{j=1}^{l} e_{\lambda_j, \eta_j} \right) \right) \rangle \langle \Omega_{\omega} \rangle \rangle
\]
\[ = \langle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \otimes S_{\mu} S_{\lambda_0} \cdots S_{\lambda_{l-1}} S_{\eta_{l-1}} \cdots S_{\eta_0} S_{\nu}^* \right) \rangle \langle \Omega_{\omega} \rangle \rangle
\]
\[ = \langle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \otimes S_{\mu} \pi_{\omega} \left( \bigotimes_{j=0}^{l-1} e_{\lambda_j, \eta_j} \right) \right) \rangle \langle \Omega_{\omega} \rangle \rangle
\]
\[ = \langle (\mathbb{I} \otimes S_{\mu}^*) \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \right) \langle \Omega_{\omega} \rangle \rangle
\]
(140)

Hence we obtain
\[
\langle S_{\mu}^* \otimes \mathbb{I} \rangle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \otimes \pi_{\omega} \left( \bigotimes_{j=0}^{l-1} e_{\lambda_j, \eta_j} \right) \right) \rangle \langle S_{\nu}^* \otimes \mathbb{I} \rangle \Omega_{\omega} \rangle = \langle \mathbb{I} \otimes S_{\mu}^* \rangle \Omega_{\omega}, \left( \pi_{\omega} \circ \gamma_{L \rightarrow R} \left( \bigotimes_{j=-l}^{l-1} e_{\mu_j, \nu_j} \right) \right) \langle \Omega_{\omega} \rangle \rangle
\]
(141)
for any \( l \in \mathbb{N}, \mu, \nu, \mu-l, \ldots, \mu-l, \nu-l, \ldots, \nu-l, \lambda_0, \ldots, \lambda_l, \eta_0, \ldots, \eta_{l-1} = 1, \ldots, d \). Since \( \hat{\pi}_\omega(A_{\text{loc}}) \) is dense in \( B(H_\omega) \) with respect to the \( \sigma \)-weak topology, this means

\[
|\langle (S_\nu^* \otimes \mathbb{1}) \Omega_\omega \rangle \langle (1 \otimes S_\mu^*) \Omega_\omega \rangle| = |\langle (1 \otimes S_\mu^*) \Omega_\omega \rangle |, \quad \quad \text{(142)}
\]

proving the claim.

From (137) with \( \mu = \nu = 1, \ldots, d \), we see that there is \( \nu \in \mathbb{T} \) such that

\[
(S_\nu^* \otimes \mathbb{1}) \Omega_\omega = a_\nu (I \otimes S_\nu^*) \Omega_\omega. \quad \text{(143)}
\]

Substituting this to (137), we find \( a_\mu = a_\nu =: a \), if \( (S_\nu^* \otimes \mathbb{1}) \Omega_\omega \) are not zero. Hence we get a constant \( a \in \mathbb{T} \) such that

\[
(S_\nu^* \otimes \mathbb{1}) \Omega_\omega = a (I \otimes S_\nu^*) \Omega_\omega, \quad \nu = 1, \ldots, d. \quad \text{(144)}
\]

By the definition of \( \mathbb{B} \) and recalling \( s_\omega = P_K \), we obtain

\[
(B_\nu^* \otimes \mathbb{1}) \Omega_\omega = a (I \otimes B_\nu^*) \Omega_\omega, \quad \nu = 1, \ldots, d. \quad \text{(145)}
\]

On the other hand, by (111), (112), (134), (133) we have

\[
(B_\nu^* \otimes \mathbb{1}) \Omega_\omega = (B_\nu^* \otimes s_\omega) \Omega_\omega = J_\omega \Delta_\omega^{-\frac{1}{2}} (B_\nu \otimes s_\omega) \Omega_\omega = \Delta_\omega^{-\frac{1}{2}} J_\omega (B_\nu \otimes s_\omega) J_\omega^* \Omega_\omega
\]

\[
= \Delta_\omega^{-\frac{1}{2}} (s_\omega \otimes \theta B_\nu \theta^*) \Omega_\omega = \left(s_\omega \otimes \rho_\omega^{\frac{1}{2}} \theta B_\nu \theta^* \rho_\omega^{-\frac{1}{2}}\right) \Omega_\omega. \quad \text{(146)}
\]

Combining (145) and (146), we obtain

\[
\left(s_\omega \otimes \rho_\omega^{\frac{1}{2}} \theta B_\nu \theta^* \rho_\omega^{-\frac{1}{2}}\right) \Omega_\omega = a (s_\omega \otimes B_\nu^*) \Omega_\omega, \quad \nu = 1, \ldots, d. \quad \text{(147)}
\]

Since \( \Omega_\omega \) is separating for \( \mathcal{M} \), we obtain

\[
\rho_\omega^{\frac{1}{2}} \theta B_\nu \theta^* \rho_\omega^{-\frac{1}{2}} = a B_\nu^*, \quad \nu = 1, \ldots, d. \quad \text{(148)}
\]

Taking adjoint, we obtain (135). This completes the proof of the Theorem. \( \Box \)

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