Lower bound on the number of the maximum genus embedding of $K_{n,n}$

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Abstract

In this paper, we provide an method to obtain the lower bound on the number of the distinct maximum genus embedding of the complete bipartite graph $K_{n,n}$ ($n$ be an odd number), which, in some sense, improves the results of S. Stahl and H. Ren.

Key Words: graph embedding; maximum genus; v-type-edge

MSC(2000): 05C10

1. Introduction

Graphs considered here are all connected and finite. A surface $S$ means a compact and connected two-manifold without boundaries. A cellular embedding of a graph $G$ into a surface $S$ is a one-to-one mapping $\psi: G \to S$ such that each component of $S - \psi(G)$ is homomorphic to an open disc. The maximum genus $\gamma_M(G)$ of a connected graph $G$ is the maximum integer $k$ such that there exists an embedding of $G$ into the orientable surface of genus $k$. By Euler’s polyhedron formula, if a cellular embedding of a graph $G$ with $n$ vertices, $m$ edges and $r$ faces is on an orientable surface of genus $\gamma$, the $n - m + r = 2 - 2\gamma$. Since $\gamma \geq 1$, we have $\gamma(G) \leq \frac{1}{2} \lfloor \beta(G) \rfloor$, where $\beta(G) = m - n + 1$ is called the Betti number (or cycle rank) of the graph $G$. It follows that $\gamma_M(G) \leq \frac{1}{2} \lfloor \beta(G) \rfloor$. If $\gamma_M(G) = \frac{1}{2} \lfloor \beta(G) \rfloor$,

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$^2$This work was partially Supported by the China Postdoctoral Science Foundation funded project (Grant No: 20110491248), the National Natural Science Foundation of China (Grant No: 11171114), and the New Century Excellent Talents in University (Grant No: NCET-07-0276).

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then the graph is called *upper embeddable*. It is not difficult to deduced that a graph is upper embeddable if and only if its face number is not greater than two. Since the introductory investigations on the maximum genus of graphs by Nordhaus, Stewart, and White\cite{1}, this parameter has attracted considerable attention from mathematicians and computer scientists. Up to now, the research about the maximum genus of graphs mainly focus on the aspects as characterizations and complexity, the upper embeddability, the lower bound, the enumeration of the distinct maximum genus embedding, etc.. For more detailed information, the reader can be found in a survey in \cite{2}.

It is well known that the enumeration of the distinct maximum genus embedding plays an important role in the study of the genus distribution problem, which may be used to decide whether two given graphs are isomorphic. It was S. Stahl\cite{3} who provides the first result about the lower bound on the number of the distinct maximum genus embedding, which is states as the following:

**Lemma 1**\cite{3} A connected graph (loops and multi-edges are allowed) of order $n$ with degree sequence $d_1, d_2, \ldots, d_n$ has at least

$$(d_1 - 5)!(d_2 - 5)!(d_3 - 5)!(d_4 - 5)! \prod_{i=5}^{n} (d_i - 2)!$$

distinct orientable embeddings with at most two facial walks, where $m! = 1$ whenever $m \leq 0$.

But up to now, except \cite{3} and \cite{4}, there is little result concerning the number of the maximum genus embedding of graphs. In this paper, we will provide a method to enumerate the number of the distinct maximum genus embedding of the complete bipartite graph $K_{n,n}$ ($n$ be an odd number), and offer a lower bound which is better than that of S. Stahl\cite{3} and H. Ren\cite{4} in some sense. Furthermore, the enumerative method below can be used to any maximum genus embedding, other than the method in \cite{3} which is restricted to upper embeddable graphs. Terminologies and notations not explained here can be seen in \cite{5} for general graph theory, and in \cite{6} and \cite{7} for topological graph theory.

### 2. Main results

A simple graph $G$ is called a *complete bipartite graph* if its vertex set can be partitioned into two subsets $X$ and $Y$ so that every edge has one end in $X$ and one end in $Y$, and every vertex in $X$ is joined to every vertex in $Y$. We denote a complete bipartite graph $G$ with bipartition $X$ and $Y$ by $G_{[X][Y]}$. A 2-path is called a *$v$-type-edge*, and is denoted by $V$. Let $\psi(G)$ be an embedding of a graph $G$. We say that a *$v$-type-edge* are inserted into $\psi(G)$ if the three endpoints of the *$v$-type-edge* are inserted into the corners of the faces in $\psi(G)$, yielding an embedding of $G + V$. The embedding $\psi(G)$ of $G$ is called a *one-face-embedding* (or *two-face-embedding*) if the total face number of $\psi(G)$ is one.
(or two). The following observation can be easily obtained and is essential in the proof of the Theorem A.

**Observation** Let $\psi(G)$ be an embedding of a graph $G$. We can insert a $v$-type-edge $V$ to $\psi(G)$ to get an embedding $\rho(G + V)$ of $G + V$ so that the face number of $\rho(G + V)$ is not more than that of $\psi(G)$.

**Theorem A** For $n \equiv 1 \pmod{2}$, the number of the distinct maximum genus embedding of the complete bipartite graph $K_{n,n}$ is at least

$$2^{\frac{n-1}{2}} \times ((n-2)!)^n \times ((n-1)!)^n.$$

**Proof** Let $n = 2s + 1$ and $V(K_{n,n}) = \{x_1, x_2, \ldots , x_n\} \cup \{y_1, y_2, \ldots , y_n\}$, where $X = \{x_1, x_2, \ldots , x_n\}$ and $Y = \{y_1, y_2, \ldots , y_n\}$ are the two independent set of $K_{n,n}$. We denote the $v$-type-edge $y_i x_j y_{n+1}$ by $V_{ji}$, where $i \in \{1, 2, \ldots , s\}$ and $j \in \{1, 2, \ldots , n\}$.

There are 2 different ways to embed $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$ on an orientable surface so that the embedding is a *one-face-embedding*. Select any one of them and denote its face boundary by $W_0$. In $W_0$, there are three *face-corner* containing $x_1$ and $x_2$ respectively. So, there are 3 different ways to put $V_{1,2}$ in $W_0$, and 3 different ways to put $V_{2,2}$ in $W_0$. Therefore, the total number of ways to put $V_{1,2} \cup V_{2,2}$ in $W_0$ is $3 \times 3 = 9$. For each of the above 9 ways, there are 2 different ways to make the embedding of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$ being a *one-face-embedding*. So, for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$, there are $3 \times 3 \times 2$ different ways to add $V_{1,2} \cup V_{2,2}$ to $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$.

Similarly, we can get that for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$, there are $5 \times 5 \times 2$ different ways to add $V_{1,3} \cup V_{2,3}$ to $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_n]}$.

In general, we have that for each of the *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_{2k-1}]}$, there are $(2k-1) \times (2k-1) \times 2$ different ways to add $V_{1,k} \cup V_{2,k}$ to $G_{[x_1,x_2][y_1,y_2,\ldots,y_{2k-1}]}$ to get a *one-face-embedding* of $G_{[x_1,x_2][y_1,y_2,\ldots,y_{2k+1}]}$. 

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From the above we can get that the number of the distinct one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \) is at least
\[
2 \times (3 \times 3 \times 2) \times (5 \times 5 \times 2) \times (7 \times 7 \times 2) \times \ldots \times ((2s - 1) \times (2s - 1) \times 2)
\]
\[
= 2^s \times ((2s - 1)!!)^2.
\]

**Claim 2:** For each of the one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \), there are at least
\[
2 \times (2s - 1)!! \times 2^s
\]
different ways to make \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \) being a one-face-embedding.

Let \( \mathcal{E}_1 \) be an arbitrary one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \). In \( \mathcal{E}_1 \), there are two different face-corner containing \( y_i \) (\( i = 1, 2, 3 \)). So, there are \( 2 \times 2 \times 2 = 8 \) different ways to add \( y_1x_3 \cup V_{3,1} \) to \( \mathcal{E}_1 \) to make \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \) being a one-face-embedding.

For each of the above 8 one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \), there are 3 different face-corner containing \( x_3 \) and 2 different face-corner containing \( y_i \) (\( i = 4, 5 \)). So, for each of the above 27 one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \), there are \( 3 \times 2 \times 2 \) different ways to add \( V_{3,2} \) to \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \) to make \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \cup V_{3,2} \) being a one-face-embedding.

In general, we have that for each of the one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \cup V_{3,2} \cup \ldots \cup V_{3,k-1} \) to get a one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{3,1} \cup V_{3,2} \cup \ldots \cup V_{3,k-1} \cup V_{3,k} \).

From the above we can get that for each of the one-face-embedding of \( G_{[x_1,x_2][y_1,y_2,...,y_n]} \), there are at least
\[
(2 \times 2 \times 2) \times (3 \times 2 \times 2) \times (5 \times 2 \times 2) \times \ldots \times ((2s - 1) \times 2 \times 2)
\]
\[
= 2 \times (2s - 1)!! \times 2^s
\]
different ways to make \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \) being a one-face-embedding.

**Claim 3:** For each of the one-face-embedding of \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \), there are at least
\[
3 \times (2s - 1)!! \times 3^s
\]
different ways to make \( G_{[x_1,x_2,x_3,x_4][y_1,y_2,...,y_n]} \) being a one-face-embedding.

Let \( \mathcal{E}_2 \) be an arbitrary one-face-embedding of \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \). In \( \mathcal{E}_2 \), there are three different face-corner containing \( y_i \) (\( i = 1, 2, 3 \)). So, there are \( 3 \times 3 \times 3 = 27 \) different ways to add \( y_1x_3 \cup V_{4,1} \) to \( \mathcal{E}_2 \) to make \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{4,1} \) being a one-face-embedding.

For each of the above 27 one-face-embedding of \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{4,1} \), there are 3 different face-corner containing \( x_3 \) and 3 different face-corner containing \( y_i \) (\( i = 4, 5 \)). So, for each of the above 27 one-face-embedding of \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \cup y_1x_3 \cup V_{4,1} \), there are \( 3 \times 3 \times 3 \) different ways to add \( V_{4,2} \) to \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \cup y_1x_4 \cup V_{4,1} \) to make \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \cup y_1x_4 \cup V_{4,1} \cup V_{4,2} \) being a one-face-embedding.

In general, we have that for each of the one-face-embedding of \( G_{[x_1,x_2,x_3][y_1,y_2,...,y_n]} \cup y_1x_4 \cup V_{4,1} \cup V_{4,2} \cup \ldots \cup V_{4,k-1} \), there are \( (2k - 1) \times 3 \times 3 \) different ways to add \( V_{4,k} \).
to $G[x_1,x_2,x_3] [y_1,y_2,\ldots,y_n] \cup y_1 x_4 \cup V_{4,1} \cup V_{4,2} \cup \ldots \cup V_{4,k-1}$ to get a one-face-embedding of $G[x_1,x_2,x_3] [y_1,y_2,\ldots,y_n] \cup y_1 x_4 \cup V_{4,1} \cup V_{4,2} \cup \ldots \cup V_{4,k-1} \cup V_{4,k}$.

From the above we can get that for each of the one-face-embedding of $G[x_1,x_2,x_3] [y_1,y_2,\ldots,y_n]$, there are at least
\[
(3 \times 3 \times 3) \times (3 \times 3 \times 3) \times (5 \times 3 \times 3) \times \ldots \times ((2s - 1) \times 3 \times 3) \\
= 3 \times (2s - 1)!! \times 3^{2s}
\]
different ways to make $G[x_1,x_2,x_3] [y_1,y_2,\ldots,y_n]$ being a one-face-embedding.

Similarly, we can get the following general result.

**Claim 4:** For each of the one-face-embedding of $G[x_1,x_2,\ldots,x_{k-1}] [y_1,y_2,\ldots,y_n]$, there are at least $(k - 1) \times (2s - 1)!! \times (k - 1)^{2s}$ different ways to make $G[x_1,x_2,\ldots,x_{k-1},x_k] [y_1,y_2,\ldots,y_n]$ being a one-face-embedding.

Noticing that a one-face-embedding of a graph must be its maximum genus embedding, we can get, from Claim 1 - Claim 4, that the number of the distinct maximum genus embedding of $K_{n,n}$ is at least
\[
\{2^s \times ((2s - 1)!!)^2 \} \times \{2 \times (2s - 1)!! \times 2^{2s}\} \times \{3 \times (2s - 1)!! \} \\
\times 3^{2s} \times \ldots \times \{2s \times (2s - 1)!! \times (2s)^{2s}\} \\
= 2^s \times ((2s - 1)!!)^{2s+1} \times ((2s)!!)^{2s+1} \\
= 2^{n-1} \times ((n - 2)!!)^n \times ((n - 1)!)^n. 
\]

**Remark** Through a comparison we can get that the result in Theorem A is much better than that of Lemma 1 when $n \leq 9$.

In [1], the second author of the present paper obtained that a connected loopless graph of order $n$ has at least $\frac{1}{4M(G)} \prod_{v \in V(G)} (d(v) - 1)!$ distinct maximum genus embedding. Let $f_1(n) = \frac{1}{4M(G)} \prod_{v \in V(G)} (d(v) - 1)! = \frac{1}{4M(G)} \prod_{v \in V(G)} (d(v) - 1)! \times ((n - 1)!)^n$. Through a computation we can get $f_1(3) - f_2(3) = 16$, $f_1(5) - f_2(5) = 6772211712$. So, when $n \leq 5$ the result obtained in Theorem A is much better than that of [1].

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