SAFFRON: an adaptive algorithm for online control of the false discovery rate

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Abstract

In the online false discovery rate (FDR) problem, one observes a possibly infinite sequence of \( P_1, P_2, \ldots \), each testing a different null hypothesis, and an algorithm must pick a sequence of rejection thresholds \( \alpha_1, \alpha_2, \ldots \) in an online fashion, effectively rejecting the \( k \)-th null hypothesis whenever \( P_k \leq \alpha_k \). Importantly, \( \alpha_k \) must be a function of the past, and cannot depend on \( P_k \) or any of the later unseen \( p \)-values, and must be chosen to guarantee that for any time \( t \), the FDR up to time \( t \) is less than some pre-determined quantity \( \alpha \in (0, 1) \).

In this work, we present a powerful new framework for online FDR control that we refer to as “SAFFRON”. Like older alpha-investing (AI) algorithms, SAFFRON starts off with an error budget, called alpha-wealth, that it intelligently allocates to different tests over time, earning back some wealth on making a new discovery. However, unlike older methods, SAFFRON’s threshold sequence is based on a novel estimate of the alpha fraction that it allocates to true null hypotheses. In the offline setting, algorithms that employ an estimate of the proportion of true nulls are called adaptive methods, and SAFFRON can be seen as an online analogue of the famous offline Storey-BH adaptive procedure. Just as Storey-BH is typically more powerful than the Benjamini-Hochberg (BH) procedure under independence, we demonstrate that SAFFRON is also more powerful than its non-adaptive counterparts, such as LORD and other generalized alpha-investing algorithms. Further, a monotone version of the original AI algorithm is recovered as a special case of SAFFRON, that is often more stable and powerful than the original. Lastly, the derivation of SAFFRON provides a novel template for deriving new online FDR rules.

1 Introduction

It is now commonplace in science and technology to make thousands or even millions of related decisions based on data analysis. As a simplified example, to discover which genes may be related to diabetes, we can formulate the decision-making problem in terms of hypotheses that take the form “gene X is not associated with diabetes,” for many different genes X, and test for which of these null hypotheses can be confidently rejected by the data. As first identified by Tukey in a seminal 1953 manuscript [13], the central difficulty when testing a large number of null hypotheses is that several of them may appear to be false, purely by chance. Arguably, we would like the set of rejected null hypotheses \( \mathcal{R} \) to have high precision, so that most discovered genes are indeed truly correlated with diabetes and further investigations are not fruitless. Unfortunately, separately controlling the false positive rate for each individual test actually does not provide any guarantee on the precision. This motivated the development of procedures that can provide guarantees on an error metric called the false discovery rate (FDR) [3], defined as:

\[
\text{FDR} = \mathbb{E} [\text{FDP}(\mathcal{R})] = \mathbb{E} \left[ \frac{\mathcal{H}_0 \cap \mathcal{R}}{\mathcal{R}} \right],
\]

where \( \mathcal{H}_0 \) is the unknown set of truly null hypotheses, and \( 0/0 \equiv 0 \). Here the FDP represents the ratio of falsely rejected nulls to the total number of rejected nulls, and since the set of discoveries \( \mathcal{R} \) is data-dependent, the FDR takes an expectation over the underlying randomness. The evidence from a hypothesis test can typically be summarized in terms of a \( p \)-value, and so offline multiple testing algorithms take a set of \( p \)-values \( \{P_i\} \) as their
input, and a target FDR level $\alpha \in (0, 1)$, and produce a rejected set $R$ that is guaranteed to have $\text{FDR} \leq \alpha$. Of course, one also desires a high recall, or equivalently a low false negative rate, but without assumptions on many uncontrollable factors like the frequency and strength of signals, additional guarantees on the recall are impossible.

While the offline paradigm previously described is the classical setting for multiple decision-making, the corresponding online problem is emerging as a major area of its own. For example, large information technology companies run thousands of A/B tests every week of the year, and decisions about whether or not to reject the corresponding null hypothesis must be made without knowing the outcomes of future tests; indeed, future null hypotheses may depend on the outcome of the current test. The current standard of setting all thresholds $\alpha_k$ to a fixed quantity such as 0.05 does not provide any control of the FDR. Hence, the following hypothetical scenario is entirely plausible: a company conducts 1000 tests in one week, each with a target false positive rate of 0.05; it happens to make 80 discoveries in total of which 50 are accidental false discoveries, ending up with an FDP of $5/8$. Such uncontrolled error rates can have severe financial and social consequences.

The first method for online control of the FDR was the alpha-investing algorithm of Foster and Stine [5], later extended to generalized alpha-investing (GAI) algorithms by Aharoni and Rosset [1]. Recently, Javanmard and Montanari [6] proposed variants of GAI algorithms that control the FDR (as opposed to the modified FDR controlled in the original paper [5]), including a new algorithm called LORD. The GAI++ algorithms by Ramdas et al. [10] improved the earlier GAI algorithms (uniformly), and the improved LORD++ (henceforth LORD) method arguably represents the current state-of-the-art in online multiple hypothesis testing.

The current paper’s central contribution is the derivation and analysis of a powerful new class of online FDR algorithms called “SAFFRON” (Serial estimate of the Alpha Fraction that is Futilely Rationed On true Null hypotheses). As an instance of the GAI framework, the SAFFRON method starts off with an error budget, referred to as $\text{alpha-wealth}$, that it allocates to different tests over time, earning back some alpha-wealth whenever it makes a new discovery. However, unlike earlier work in the online setting, SAFFRON is an adaptive method, meaning that it is based on an estimate of the proportion of true nulls. In the offline setting, adaptive methods were proposed by Storey [11, 12], who showed that they are more powerful than the Benjamini-Hochberg (BH) procedure [3] under independence assumptions; this advantage usually increases with the proportion of non-nulls and the signal strength. Thus, the SAFFRON method can be viewed as an online analogue of Storey’s adaptive version of the BH procedure. As shown in Figure 1, our simulations show that SAFFRON demonstrates the same types of advantages over its non-adaptive counterparts, such as LORD and alpha-investing. Furthermore, the ideas behind SAFFRON’s derivation can provide a natural template for the design and analysis of a suite of other adaptive online methods.

The rest of this paper is organized as follows. In Section 2 we derive the SAFFRON algorithm from first principles, leaving the proof of a central technical lemma for Section 4. In Section 5 we investigate the practical choice of tuning parameters, and demonstrate the effectiveness of our recommended choice using simulations. We provide proofs of the results of this paper in Section 6 and at the end present a short summary in Section 7.
2 Deriving the SAFFRON algorithm

Before deriving the SAFFRON algorithm, it is useful to recap a few concepts. By definition of a p-value, if the hypothesis $H_i$ is truly null, then the corresponding p-value is stochastically larger than the uniform distribution ("super-uniformly distributed," for short), meaning that:

$$\text{If the null hypothesis } H_i \text{ is true, then } \Pr\{P_i \leq u\} \leq u \text{ for all } u \in [0, 1].$$  \hspace{1cm} (1)

For any online FDR procedure, let the rejected set after $t$ steps be denoted by $\mathcal{R}(t)$. More precisely, this set consists of all $p$-values among the first $t$ ones for which the indicator for rejection is equal to 1; i.e., $R_j := \mathbb{1}\{P_j \leq \alpha_j\} = 1$, for all $j \leq t$. While we have already defined the classical FDP and FDR in the introduction, several authors, including Foster and Stine [5], have considered a modified FDR, defined as:

$$\text{mFDR}(t) := \frac{E[|\mathcal{H}^0 \cap \mathcal{R}(t)|]}{E[|\mathcal{R}(t)|]}.$$

(2)

In the sequel, we provide guarantees for both mFDR and FDR. Our guarantees on mFDR hold under the following weakening of (1). Define the filtration formed by the sequence of sigma-fields $\mathcal{F}^t := \sigma(R_1, \ldots, R_t)$, and let $\alpha_t := f_t(R_1, \ldots, R_{t-1})$, where $f_t$ is an arbitrary function of the first $t-1$ indicators for rejection. Then, we say that the null $p$-values are conditionally super-uniformly distributed if the following holds:

$$\text{If the null hypothesis } H_i \text{ is true, then } \Pr\{P_t \leq \alpha_t \mid \mathcal{F}^{t-1}\} \leq \alpha_t.$$

(3)

2.1 An oracle estimate of the FDP and a naive overestimate

To understand the motivation behind the new procedure, it is necessary to expand on an perspective on existing online FDR procedures, recently suggested by Ramdas et al. [10]. We begin by defining an oracle estimate of the FDP as:

$$\text{FDP}^*(t) := \sum_{j \leq t, j \in \mathcal{H}^0} \alpha_j.$$

The word oracle indicates that FDP$^*$ cannot be calculated by the scientist, since $\mathcal{H}^0$ is unknown. Intuitively, the numerator $\sum_{j \leq t, j \in \mathcal{H}^0} \alpha_j$ overestimates the number of false discoveries, and FDP$^*(t)$ overestimates the FDP, as formalized in the claim below:

**Proposition 1.** If the null $p$-values are conditionally super-uniformly distributed [5], then we have

(a) $E\left[\sum_{j \leq t, j \in \mathcal{H}^0} \alpha_j\right] \geq E[|\mathcal{H}^0 \cap \mathcal{R}(t)|]$;

(b) If $\text{FDP}^*(t) \leq \alpha$ for all $t \in \mathbb{N}$, then $\text{mFDR}(t) \leq \alpha$ for all $t \in \mathbb{N}$.

Further, if the null $p$-values are independent of each other and of the non-nulls, and $\{\alpha_t\}$ is a monotone function of past rejections, then:

(c) $E[\text{FDP}^*(t)] \geq E[\text{FDP}(t)] \equiv \text{FDR}(t)$ for all $t \in \mathbb{N}$;

(d) The condition $\text{FDP}^*(t) \leq \alpha$ for all $t \in \mathbb{N}$ implies that $\text{FDR}(t) \leq \alpha$ for all $t \in \mathbb{N}$.

To clarify, the word monotone means that $\alpha_t$ is a coordinatewise non-decreasing function of the vector $R_1, \ldots, R_{t-1}$. Proposition [10] follows from the results of Ramdas et al. [10], and we prove it in Section 6 for completeness. Even though FDP$^*(t)$ cannot be directly calculated and used, Proposition [10] is a useful way to identify what would be ideally possible. One natural way to convert FDP$^*(t)$ to a truly empirical overestimate of FDP(t) is to define:

$$\text{FDP}_{\text{LORD}}(t) := \frac{\sum_{j \leq t} \alpha_j}{|\mathcal{R}(t)|}.$$
Since it is trivially true that $\tilde{\text{FDP}}_{\text{LORD}}(t) \geq \text{FDP}^*(t)$, we immediately obtain that Proposition 1 also holds with $\text{FDP}^*(t)$ replaced by $\tilde{\text{FDP}}_{\text{LORD}}(t)$. The subscript LORD is used because Ramdas et al. [10] point out that their variant of the LORD algorithm of Javanmard and Montanari [6] can be derived by simply assigning $\alpha_j$ in an online fashion to ensure that the condition $\tilde{\text{FDP}}_{\text{LORD}}(t) \leq \alpha$ is met for all times $t$.

2.2 A better estimate of the alpha-wealth spent on testing nulls

The main drawback of $\tilde{\text{FDP}}_{\text{LORD}}$ is that if the underlying (unknown) truth is such that the proportion of non-nulls (true signals) is non-negligible, then $\tilde{\text{FDP}}_{\text{LORD}}(t)$ is a very crude and overly conservative overestimate of $\text{FDP}^*(t)$, and hence also of the true unknown FDP. With this drawback in mind, and knowing that we would expect non-nulls to typically have smaller $p$-values, we propose the following novel estimator:

$$\text{FDP}_{\text{SAFFRON}}(\lambda)(t) \equiv \text{FDP}_\lambda(t) : = \frac{\sum_{j \leq t} \alpha_j \mathbb{1}\{P_j > \lambda_j\}}{|\mathcal{R}(t)|},$$

where $\{\lambda_j\}_{j=1}^\infty$ is a predictable sequence of user-chosen parameters in the interval $(0, 1)$. Here the term predictable means that $\lambda_j$ is a deterministic function of the information available from time 1 to $j-1$, which will be formalized later. For simplicity, when $\lambda_j$ is chosen to be a constant for all $j$, we will drop the subscript and just write $\lambda$, and we will consider $\lambda = 1/2$ as our default choice. SAFFRON is based on the idea that the numerator of $\tilde{\text{FDP}}_\lambda$ is a much better estimator of the quantity $\sum_{j \leq t, j \in \mathcal{H}^0} \alpha_j$ than LORD’s naive estimate $\sum_{j \leq t} \alpha_j$.

So as to provide some intuition for why we expect $\text{FDP}_\lambda$ to be a fairly tight estimate of $\text{FDP}^*$, note that $\frac{1\{P_j > \lambda_j\}}{1-\lambda_j}$ has a unit expectation whenever $P_j$ is uniformly distributed (null), but would typically have a much smaller expectation whenever $P_j$ is stochastically much smaller than uniform (non-null). The following theorem shows that, even though $\text{FDP}_\lambda(t)$ is not necessarily always larger than $\text{FDP}^*(t)$, a direct analog of Proposition 1 is nonetheless valid. In order to state this claim formally, we need to slightly modify the assumption [1]. As before, denote by $R_j := 1 \{P_j \leq \alpha_j\}$ the indicator for rejection, and let $C_j := 1 \{P_j \leq \lambda_j\}$ be the indicator for candidacy. Accordingly, we refer to the $p$-values for which $C_j = 1$ as candidates. Moreover, we let $\alpha_j := f_t(R_1, \ldots, R_{t-1}, C_1, \ldots, C_{t-1})$, where $f_t$ denotes an arbitrary function of the first $t-1$ indicators for rejection and candidacy, and define the filtration generated from sigma-fields $\mathcal{F}^t := \sigma(R_1, \ldots, R_t, C_1, \ldots, C_t)$. With respect to this filtration, we introduce a conditional super-uniformity condition on the null $p$-values similar to (3):

If the null hypothesis $H_i$ is true, then $\Pr\{P_t \leq \alpha_t \mid \mathcal{F}^{t-1}\} \leq \alpha_t,$

which can be rephrased as:

$$\mathbb{E}\left[ \frac{1\{P_t \geq \alpha_t\}}{1-\alpha_t} \mid \mathcal{F}^{t-1} \right] \geq 1 \geq \mathbb{E}\left[ \frac{1\{P_t \leq \alpha_t\}}{\alpha_t} \mid \mathcal{F}^{t-1} \right].$$

Note that again marginal super-uniformity [4] implies this condition, provided that the $p$-values are independent.

**Theorem 1.** If the null $p$-values are conditionally super-uniformly distributed [4], then we have:

(a) $\mathbb{E}\left[ \sum_{j \leq t, j \in \mathcal{H}^0} \alpha_j \frac{1\{P_j > \lambda_j\}}{1-\lambda_j} \right] \geq \mathbb{E}\left[ |\mathcal{H}^0 \cap \mathcal{R}(t)| \right]$;

(b) The condition $\tilde{\text{FDP}}_\lambda(t) \leq \alpha$ for all $t \in \mathbb{N}$ implies that $\text{mFDR}(t) \leq \alpha$ for all $t \in \mathbb{N}$.

Further, if the null $p$-values are independent of each other and of the non-nulls, and $\{\alpha_t\}$ is a monotone function of the vector $R_1, \ldots, R_{t-1}, C_1, \ldots, C_{t-1}$, then we additionally have:

(c) $\mathbb{E}\left[ \tilde{\text{FDP}}_\lambda(t) \right] \geq \mathbb{E}[\text{FDP}(t)] \equiv \text{FDR}(t)$ for all $t \in \mathbb{N}$;

(d) The condition $\tilde{\text{FDP}}_\lambda(t) \leq \alpha$ for all $t \in \mathbb{N}$ implies that $\text{FDR}(t) \leq \alpha$ for all $t \in \mathbb{N}$.
In words, SAFFRON starts off with an alpha-wealth attempting to ensure that interesting that the original AI algorithm of Foster and Stine \([5]\) is recovered by choosing \(\lambda\). Even though the motivation that we have presented for SAFFRON relates it to the LORD algorithm, we find it

3.1 Alpha-investing (AI)

Here, we compare SAFFRON to existing procedures in the literature, emphasizing commonalities that allow us to

3.2 Relationship to other procedures

We discuss reasonable default choices in the experimental section.

2.3 The SAFFRON algorithm for constant \(\lambda\)

We now present the SAFFRON algorithm at a high level. For simplicity, we consider the constant \(\lambda\) setting, which performs well in experiments, though it may be a useful direction for future work to construct good heuristics for time-varying sequences \(\{\lambda_j\}_{j=1}^\infty\).

1. Given a target FDR level \(\alpha\), the user first picks a constant \(\lambda \in (0, 1)\), an initial wealth \(W_0 < (1 - \lambda)\alpha\), and a positive non-increasing sequence \(\{\gamma_j\}_{j=1}^\infty\) of summing to one. For example, given a parameter \(s > 1\), we might pick \(\gamma_j \propto j^{-s}\) for some \(s > 1\).

2. We use the term “candidates” to refer to \(p\)-values smaller than \(\lambda\), since SAFFRON will never reject a \(p\)-value larger than \(\lambda\). Recalling the indicator for candidacy \(C_t := \mathbb{1}\{P_t \leq \lambda\}\), and denoting by \(\tau_j\) be the time of the \(j\)-th rejection (and setting \(\tau_0 = 0\)), define the candidates after the \(j\)-th rejection as \(C_{j+} = C_{j+}(t) = \sum_{i=\tau_j+1}^t C_i\).

3. SAFFRON begins by allocation \(\alpha_1 = \min\{\gamma_1 W_0, \lambda\}\), and then at time \(t = 2, 3, \ldots\), it allocates:

\[
\alpha_t := \min\{\lambda, \tilde{\alpha}_t\}, \quad \text{where} \quad \tilde{\alpha}_t := W_0\gamma_t - C_{0+} + ((1 - \lambda)\alpha - W_0)\gamma_{t-\tau_1} - C_{1+} + \sum_{j \geq 2} (1 - \lambda)\alpha\gamma_{t-\tau_j} - C_{j+}.
\]

In words, SAFFRON starts off with an alpha-wealth \(W_0 < (1 - \lambda)\alpha\), never loses wealth when testing candidate \(p\)-values, gains wealth of \((1 - \lambda)\alpha\) on every rejection except the first. If there is a significant fraction of non-nulls, and the signals are fairly strong, then SAFFRON may make more rejections than LORD.

To clarify, SAFFRON guarantees FDR control for any \(\lambda \in (0, 1)\) and any chosen sequence \(\{\gamma_j\}_{j=1}^\infty\), but the algorithm’s power, or ability to detect signals, varies as a function of these parameters. Given the minimal nature of our assumptions, there is no universally optimal constant or sequence: specifically, we do not make assumptions on the frequency of true signals, or on how strong they are, or on their order, all of which are factors that affect the power. We discuss reasonable default choices in the experimental section.

3 Relationship to other procedures

Here, we compare SAFFRON to existing procedures in the literature, emphasizing commonalities that allow us to give a unified view of seemingly disparate algorithms.

3.1 Alpha-investing (AI)

Even though the motivation that we have presented for SAFFRON relates it to the LORD algorithm, we find it interesting that the original AI algorithm of Foster and Stine \([5]\) is recovered by choosing \(\lambda_j = \alpha_j\) in \(\overline{\mathcal{FDP}}\), and attempting to ensure that \(\overline{\mathcal{FDP}}_\lambda(t) \leq \alpha\) for all times \(t \in \mathbb{N}\). In order to see this fact, first note that with this choice of \(\lambda_j\), the indicator \(\mathbb{1}\{P_t > \lambda_j\}\) simply indicates when the \(j\)-th hypothesis is not rejected. Consequently, the numerator of \(\overline{\mathcal{FDP}}\) reads as \(\sum_{j \leq t} \frac{\alpha_j}{1 - \alpha_j} \mathbb{1}\{j \notin \mathcal{R}(t)\}\). Hence, ensuring that \(\overline{\mathcal{FDP}}_\lambda(t) \leq \alpha\) at all times \(t \in \mathbb{N}\), is equivalent to ensuring that \(\sum_{j \leq t} \frac{\alpha_j}{1 - \alpha_j} \mathbb{1}\{j \notin \mathcal{R}(t)\}\) never exceeds \(\alpha(\mathcal{R}(t) \lor 1)\), which, in the language of alpha-investing, is equivalent to ensuring that the algorithm’s wealth never becomes negative Just as Ramdas

\[\text{Recall that the AI algorithm starts off with an alpha-wealth of } \alpha, \text{ reduces its alpha-wealth by } \frac{\alpha_j}{1 - \alpha_j} \text{ after tests that fail to reject, and increase the wealth by } \alpha \text{ on rejections.} \]
et al. [10] were able to reinterpret and rederive LORD in terms of a particular estimate of the FDP, the current work allows us to reinterpret and rederive AI in terms of SAFFRON’s FDP. Nevertheless, despite these similarities, SAFFRON’s update rule for \( \alpha_j \) as stated in Section 2.3 is different from the update used in AI. Originally [5], \( \alpha_j \) was set to a fraction of the available wealth \( W_j \); however, this simple update prevents alpha-investing from being a monotone procedure, meaning that there is no guarantee that \( f_j : (R_1, ..., R_{j-1}) \mapsto \alpha_j \) is a coordinatewise nondecreasing function. For this reason, the original alpha-investing provably controls only mFDR, and not the FDR. However, we may derive a novel monotone version of AI by using \( \lambda_j = \alpha_j \) in SAFFRON’s update rule from Section 2.3, immediately yielding FDR control under independence. Simulations indicate that this new monotone SAFFRON-AI algorithm performs comparably to the original non-monotone AI, or sometimes even outperforms it, as demonstrated in the first subplot of Figure 2. Further, as a consequence of monotonicity, SAFFRON-AI allocates \( \alpha_j \) in a more stable manner compared to the non-monotone AI, as shown in the third subplot of Figure 2.

![Figure 2](image-url)

**Figure 2.** Statistical power and FDR versus fraction of non-null hypotheses \( \pi_1 \) (left), and allocated \( \alpha_j \) versus hypothesis index (right), for SAFFRON with \( \lambda_j = \alpha_j \) and the original alpha-investing (at target level \( \alpha = 0.05 \)). The observations under the alternative are Gaussian with \( \mu_i \sim N(3, 1) \) and standard deviation 1, and are converted into one-sided \( p \)-values as \( P_i = \Phi(-Z_i) \). SAFFRON-AI is sometimes more powerful than AI (first subplot), and also more stable (third subplot), across a variety of choices of tuning parameters for both algorithms.

### 3.2 Storey-BH

In offline multiple testing, where all \( p \)-values are immediately available to the scientist, the Benjamini-Hochberg (BH) procedure [3] is a classical method for guaranteeing FDR control. Although the initial motivation for the BH method was different, it was reinterpreted by Storey et al. [11, 12] in the following manner. Since the small \( p \)-values are more likely to be non-null, suppose that one rejects all \( p \)-values below some fixed threshold \( s \in (0, 1) \), meaning that \( \mathcal{R}(s) = \{i : P_i \leq s\} \). Then, an oracle estimate for the FDP is given by

\[
\text{FDP}^*_\text{BH}(s) := \frac{|\mathcal{H}^0| \cdot s}{|\mathcal{R}(s)|}.
\]

The numerator is a sensible estimate because the nulls are uniformly distributed, and hence we would expect about \( |\mathcal{H}^0| \cdot s \) many nulls to be below \( s \). This is an “oracle” estimate because the scientist does not know \( |\mathcal{H}^0| \). Ideally, one would like to choose a data-dependent \( s \) using the rule

\[
s^* := \max\{s : \text{FDP}^*_\text{BH}(s) \leq \alpha\},
\]

and then reject the set \( \mathcal{R}(s^*) \). Given \( n \) \( p \)-values, the BH procedure overestimates the oracle FDP by the empirically computable quantity

\[
\text{FDP}_{\text{BH}}(s) := \frac{n \cdot s}{|\mathcal{R}(s)|},
\]

and then rejecting the set \( \mathcal{R}(\tilde{s}_{\text{BH}}) \), where \( \tilde{s}_{\text{BH}} := \max\{s : \text{FDP}_{\text{BH}}(s) \leq \alpha\} \). On interpreting the BH procedure in terms of an estimated FDP, Storey et al. [11, 12] noted that when the \( p \)-values are independent, the estimate \( \text{FDP}_{\text{BH}} \) is unnecessarily conservative. Indeed, when the \( p \)-values are exactly uniform, it is known [4, 9] to satisfy
the stronger bound $\text{FDR} = \alpha |\mathcal{H}_0|/n$, which demonstrates that BH underutilizes the FDR budget of $\alpha$ provided to it. Instead, Storey et al. pick a constant $\lambda \in (0, 1)$, and calculate

$$\hat{\text{FDP}}_{\text{StBH}}(s) := \frac{n \cdot s \cdot \hat{\pi}_0}{|\mathcal{R}(s)|},$$

where the unknown proportion of nulls $\pi_0 = |\mathcal{H}_0|/n$ is estimated as

$$\hat{\pi}_0 := \frac{1}{n(1 - \lambda)} \sum_{i=1}^{n} \mathbb{1}\{P_i > \lambda\}.$$

Then, this procedure, which we refer to as “Storey-BH,” calculates $\hat{\pi}_0$ it. Instead, Storey et al. pick a constant $\lambda$. Clear that we had earlier defined the indicator for candidacy as $C_t$. The follows, prove Proposition 1 and Theorem 1 in Section 6. Let us first recall and set up some preliminary notation. In what

Here, we present a lemma that is central to the proof of FDR control for SAFFRON. We later use this lemma to

3.3 Accumulation tests, like SeqStep

Note that $E[2I(P > 1/2)] \geq 1$ for null $p$-values (with equality when they are exactly uniformly distributed, simply because $\int_0^1 2I(p > 1/2)dp = 1$). One may actually use any non-decreasing function $h$ such that $\int_0^1 h(p)dp$ in the formula for $\text{FDP}_\lambda$. Such accumulation functions were studied in the (offline) context of ordered testing [8], and may seamlessly be transferred to the online setting considered here, yielding mFDR control using the same proof. In initial experiments, the use of other functions is not advantageous, and under some additional assumptions in the offline ordered testing setting, the aforementioned authors argued that the step function $(1 - \lambda)^{-1}I(I > \lambda)$ is asymptotically optimal for power. In this light, SAFFRON can also be seen as an online analog of adaptive SeqStep [7], which is a variant of Selective SeqStep [2] and SeqStep [8].

4 A reverse super-uniformity lemma

Here, we present a lemma that is central to the proof of FDR control for SAFFRON. We later use this lemma to prove Proposition 1 and Theorem 1 in Section 6. Let us first recall and set up some preliminary notation. In what follows, $\alpha_t$, $\lambda_t$ are random variables in $(0, 1)$ that always satisfy $\alpha_t \leq \lambda_t$. We denote the indicator for rejection at the $t$-th step by $R_t := \mathbb{1}\{P_t \leq \alpha_t\}$, and recall that since only $p$-values smaller than $\lambda_t$ are candidates for rejection, we had earlier defined the indicator for candidacy as $C_t := \mathbb{1}\{P_t \leq \lambda_t\}$. If we denote $C_t = 1 - C_t$, then it is clear that $R_t C_t = 0$, since $R_t$ and $C_t$ cannot both equal one simultaneously. Also let $R_{1:t} := \{R_1, \ldots, R_t\}$
and $C_{1:t} := \{C_1, \ldots, C_t\}$. As before, we consider the filtration $\mathcal{F}^t := \sigma(R_{1:t}, C_{1:t})$. In what follows, we insist that the sequences $\{\alpha_i\}_{i=1}^\infty$ and $\{\lambda_i\}_{i=1}^\infty$ are predictable, meaning that they are functions of the information available from time 1 to $t-1$ only; specifically, we insist that $\alpha_i, \lambda_i$ are measurable with respect to the sigma-field $\mathcal{F}^{t-1}$. We will also require that the $\{\alpha_i\}$ sequence is monotone, meaning that $\alpha_i = f_i(R_{1:i-1}, C_{1:i-1})$ for some coordinatewise non-decreasing function $f_i : \{0,1\}^{t-1} \to [0, \lambda_i]$. Section 2.3 contains a proof that SAFRON, as described in Section 2.3 satisfies this requirement.

To provide a more interpretable context for the following technical lemma, the reader is encouraged to recall the definition of conditional super-uniformity, as well as its equivalent rephrased form. Lemma 1 guarantees that for independent $p$-values, statement (5) holds true more generally.

**Lemma 1.** Assume that the $p$-values $P_1, P_2, \ldots$ are independent and let $g : \{0,1\}^T \to \mathbb{R}$ be any coordinatewise non-decreasing function. Then, for any index $t \leq T$ such that $H_t \in \mathcal{H}^t$, we have:

$$\mathbb{E} \left[ \frac{f_t(R_{1:t-1}, C_{1:t-1}) \mathbf{1}\{P_t > \lambda_t\}}{(1-\lambda_t)g(R_{1:T})} \left| \mathcal{F}^{t-1} \right. \right] \geq \mathbb{E} \left[ \frac{f_t(R_{1:t-1}, C_{1:t-1})}{g(R_{1:T})} \left| \mathcal{F}^{t-1} \right. \right] \geq \mathbb{E} \left[ \frac{1\{P_t \leq f_t(R_{1:t-1}, C_{1:t-1})\}}{g(R_{1:T})} \left| \mathcal{F}^{t-1} \right. \right].$$

**Proof.** The second inequality is a consequence of super-uniformity lemmas from past work [10, 6], so we only prove the first inequality. At a high level, the proof strategy is inverted, and we will hallucinate a vector with one element being set to 1, instead of being set to 0 in the aforementioned works.

Letting $P_{1:T} = (P_1, \ldots, P_T)$ be the original vector of $p$-values, we define a “hallucinated” vector of $p$-values $\tilde{P}_{1:T} := (P_1, \ldots, P_T)$ that equals $P_{1:T}$, except that the $t$-th component is set to one:

$$\tilde{P}_i = \begin{cases} 1 & \text{if } i = t \\ P_i & \text{if } i \neq t. \end{cases}$$

Define hallucinated candidate and rejection indicators as $\tilde{C}_i = 1 \left\{ \tilde{P}_i \leq \lambda_i \right\}$ and $\tilde{R}_i = 1 \left\{ \tilde{P}_i \leq f_i(\tilde{R}_{1:i-1}, \tilde{C}_{1:i-1}) \right\}$ respectively. Let $R_{1:T} = (R_1, \ldots, R_T)$ and $\tilde{R}_{1:T} := \{\tilde{R}_1, \ldots, \tilde{R}_T\}$ denote the vector of rejections using $P_{1:T}$ and $\tilde{P}_{1:T}$, respectively. Similarly, let $C_{1:T} = (C_1, \ldots, C_T)$ and $\tilde{C}_{1:T} := \{\tilde{C}_1, \ldots, \tilde{C}_T\}$ denote the vector of candidates using $P_{1:T}$ and $\tilde{P}_{1:T}$, respectively.

By construction, we have the following properties:

1. $\tilde{R}_i = R_i$ and $\tilde{C}_i = C_i$ for all $i < t$, hence $f_i(R_{1:i-1}, C_{1:i-1}) = f_i(\tilde{R}_{1:i-1}, \tilde{C}_{1:i-1})$ for all $i \leq t$.

2. $\tilde{R}_i = \tilde{C}_i = 0$, and hence $\tilde{R}_i \leq R_i$ for all $i \geq t$, due to monotonicity of the functions $f_i$.

Hence, on the event $\{P_t > \lambda_t\}$, we have $R_t = \tilde{R}_t = 0$ and $C_t = \tilde{C}_t = 0$, and hence also $R_{1:T} = \tilde{R}_{1:T}$. This allows us to conclude that:

$$\frac{f_t(R_{1:t-1}, C_{1:t-1}) \mathbf{1}\{P_t > \lambda_t\}}{(1-\lambda_t)g(R_{1:T})} = \frac{f_t(R_{1:t-1}, C_{1:t-1}) \mathbf{1}\{P_t > \lambda_t\}}{(1-\lambda_t)g(R_{1:T}^{t-1})}.$$ 

Since $\tilde{R}_{1:T}^{t-1}$ is independent of $P_t$, we may take conditional expectations to obtain:

$$\mathbb{E} \left[ \frac{f_t(R_{1:t-1}, C_{1:t-1}) \mathbf{1}\{P_t > \lambda_t\}}{(1-\lambda_t)g(R_{1:T})} \left| \mathcal{F}^{t-1} \right. \right] = \mathbb{E} \left[ \frac{f_t(R_{1:t-1}, C_{1:t-1}) \mathbf{1}\{P_t > \lambda_t\}}{(1-\lambda_t)g(R_{1:T}^{t-1})} \left| \mathcal{F}^{t-1} \right. \right] \geq \mathbb{E} \left[ \frac{f_t(R_{1:t-1}, C_{1:t-1})}{g(R_{1:T}^{t-1})} \left| \mathcal{F}^{t-1} \right. \right] \geq \mathbb{E} \left[ \frac{1\{P_t \leq f_t(R_{1:t-1}, C_{1:t-1})\}}{g(R_{1:T})} \left| \mathcal{F}^{t-1} \right. \right],$$

where inequality (i) follows by taking an expectation only with respect to $P_t$ by invoking the conditional super-uniformity property; and inequality (ii) follows because $g(R_{1:T}) \geq g(R_{1:T}^{t-1})$ since $R_i \geq \tilde{R}_i$ for all $i$ by monotonicity of the online FDR rule. This concludes the proof of the lemma. □
5 Numerical simulations

In this section, we provide the results of some numerical experiments that compare the performance of SAFFRON with current state-of-the-art algorithms for online FDR control, namely the aforementioned LORD and alpha-investing procedures. In particular, for each method, we provide empirical evaluations of its power while ensuring that the FDR remains below a chosen value. We consider two settings, one in which the p-values are computed from Gaussian observations, and another in which the p-values under the alternative are drawn from a beta distribution [6]. The following two subsections separately analyze these experimental settings. In both cases, SAFFRON outperforms the competing algorithms, with mild dependence on the exact choice of sequence \( \{ \gamma_j \} \).

In all our experiments we control the FDR under \( \alpha = 0.05 \) and estimate the FDR and power by averaging over 200 independent trials. As was previously mentioned, the constant sequence \( \lambda_j = 1/2 \) for all \( j \) was found to be particularly successful, so this is our default choice in this section and we drop the index for simplicity.

5.1 Testing with Gaussian observations

We use the simple experimental setup of testing the mean of a Gaussian distribution with \( T = 1000 \) components. More precisely, for each index \( i \in \{ 1, \ldots, T \} \), the null hypothesis takes the form \( H_i : \mu_i = 0 \). The observations consist of independent Gaussian variates \( Z_i \sim N(\mu_i, 1) \), which are converted into one-sided p-values using the transform \( P_i = \Phi(-Z_i) \), where \( \Phi \) is the standard Gaussian CDF. The motivation for one-sided conversion lies in A/B testing, where one wishes to detect larger effects, not smaller. The parameter \( \mu_i \) is chosen according to the following mixture model:

\[
\mu_i = \begin{cases} 
0 & \text{with probability } 1 - \pi_1 \\
F_1 & \text{with probability } \pi_1,
\end{cases}
\]

where the random variable \( F_1 \) is of the form \( N(\mu_c, 1) \) for some constant \( \mu_c \). We ran simulations for \( \mu_c \in \{ 2, 3 \} \), thus seeing how changing signal strength affects the performance of SAFFRON.

In what follows, we compare SAFFRON’s achieved power and FDR to those of LORD and alpha-investing. The constant infinite sequence \( \gamma_j \propto \frac{\log(j/2)}{je^{\sqrt{\log j}}} \), where the proportionality constant is determined so that the sequence sums to one, was shown to be asymptotically optimal for testing Gaussian means via the LORD method in the paper [6]. Since SAFFRON loses wealth only when testing non-candidates whereas LORD loses wealth at every step, it is expected to behave more conservatively and not use up its wealth at the same rate, conditioned on both using the same sequence \( \{ \gamma_j \} \). For this reason, informally speaking, it can reuse this leftover wealth, hence the sequence \( \{ \gamma_j \} \) chosen for SAFFRON is more aggressive, in the sense that more wealth is concentrated around the beginning of the sequence. In particular, we choose sequences of the form \( \gamma_j \propto j^{-s} \), where the parameter \( s > 1 \) controls the aggressiveness of the procedure; the greater the constant \( s \), the more wealth is concentrated around small values of \( j \). We also consider these sequences for LORD, thus observing the difference in performance resulting from using a more aggressive sequence in the regime of a finite sequence of p-values.

In Figure 3 and Figure 4, we consider \( F_1 = N(2, 1) \), and show how the level of aggressiveness of the sequence \( \{ \gamma_j \} \) affects the power and FDR of SAFFRON and LORD respectively. Figure 5 compares alpha-investing, SAFFRON and LORD, the latter two using the highest performing sequence chosen among six possible sequences, in the same testing scenario. Figure 6, Figure 7 and Figure 8 demonstrate these results in the same order for a similar but somewhat easier testing problem, with \( F_1 = N(3, 1) \). Experiments indicate that increasing the fraction of non-null hypotheses allows SAFFRON to achieve a faster increase of power than LORD, thus performing considerably better than both LORD and the alpha-investing procedure in settings with a great number of non-null observations.
Figure 3. Statistical power and FDR versus fraction of non-null hypotheses $\pi_1$ for SAFFRON (at target level $\alpha = 0.05$) using four different sequences $\{\gamma_j\}$ of increasing aggressiveness. The observations under the alternative are $N(\mu_i, 1)$ with $\mu_i \sim N(2, 1)$, and are converted into one-sided $p$-values as $P_i = \Phi(-Z_i)$. (See also Figure 5.)

Figure 4. Statistical power and FDR versus fraction of non-null hypotheses $\pi_1$ for LORD (at target level $\alpha = 0.05$) using four different sequences $\{\gamma_j\}$ of increasing aggressiveness. The LORD1 method uses the sequence proposed in the paper [6]. The observations under the alternative are $N(\mu_i, 1)$ with $\mu_i \sim N(2, 1)$, and are converted into one-sided $p$-values as $P_i = \Phi(-Z_i)$. (See also Figure 5.)

Figure 5. Statistical power and FDR versus fraction of non-null hypotheses $\pi_1$ for SAFFRON, LORD and alpha-investing (at target level $\alpha = 0.05$), the first two using the sequence $\{\gamma_j\}$ which achieves the highest power for each of them (chosen over six sequences of varying aggressiveness). The observations under the alternative are $N(\mu_i, 1)$ with $\mu_i \sim N(2, 1)$, and are converted into one-sided $p$-values as $P_i = \Phi(-Z_i)$.
Figure 6. Statistical power and FDR versus fraction of non-null hypotheses $\pi_1$ for SAFFRON (at target level $\alpha = 0.05$) using four different sequences $\{\gamma_j\}$ of increasing aggressiveness. The observations under the alternative are $N(\mu_i, 1)$ with $\mu_i \sim N(3, 1)$, and are converted into one-sided $p$-values as $P_i = \Phi(-Z_i)$. (See also Figure 1.)

Figure 7. Statistical power and FDR versus fraction of non-null hypotheses $\pi_1$ for LORD (at target level $\alpha = 0.05$) using four different sequences $\{\gamma_j\}$ of increasing aggressiveness. The LORD1 method uses the sequence proposed in the paper [6]. The observations under the alternative are $N(\mu_i, 1)$ with $\mu_i \sim N(3, 1)$, and are converted into one-sided $p$-values as $P_i = \Phi(-Z_i)$. (See also Figure 1.)

5.2 Testing with beta alternatives

In this setting we generate the $p$-value sequence according to the following model:

$$P_i \sim \begin{cases} \text{Unif}[0, 1], & \text{with probability } 1 - \pi_1 \\ \text{Beta}(m, n), & \text{with probability } \pi_1, \end{cases}$$

where $i \in [T]$ and $T = 1000$, as before. Again we compare the performance of SAFFRON, alpha-investing and LORD in terms of the achieved power with the FDR controlled under a chosen level. For LORD, the asymptotically optimal sequence $\{\gamma_j\}$ was derived in the paper [6] and is of the form $\gamma_j \propto (\frac{1}{j} \log j)^{1/m}$ for $m < 1$ and $n \geq 1$. As in the Gaussian case, for SAFFRON and additionally for LORD we consider the sequence $\gamma_j \propto j^{-s}$ with varying $s$, which, unlike the previously mentioned sequence, does not depend on the parameters of the distribution. For the particular distribution of the observed $p$-values we choose $m = 0.5$ and $n = 5$. The following plots compare the achieved power and FDR of SAFFRON, LORD and alpha-investing, the first two with several different sequences $\{\gamma_j\}$ obtained by varying the parameter $s$. In particular, Figure 8 and Figure 9 show the changes in performance of SAFFRON and LORD respectively with increasing $s$, i.e., increasing aggressiveness of the sequence $\{\gamma_j\}$. Figure 10 compares the performance of SAFFRON, LORD and alpha-investing, where the first two use the highest performing sequence chosen among six considered sequences, as in the setting with Gaussian tests. Although
the simulation results show SAFFRON performing similarly to LORD and alpha-investing for small fractions of non-null hypotheses, it significantly outperforms its competitors in terms of power and using up available wealth with a higher number of \( p \)-values coming from the alternative.

**Figure 8.** Statistical power and FDR versus fraction of non-nulls \( \pi_1 \) for SAFFRON (at target level \( \alpha = 0.05 \)) using four different sequences \( \{\gamma_j\} \) of increasing aggressiveness. Non-null \( p \)-values are distributed as Beta(0.5, 5). (See also Figure 10.)

**Figure 9.** Statistical power and FDR versus fraction of non-null hypotheses \( \pi_1 \) for LORD (at target level \( \alpha = 0.05 \)) using four different sequences \( \{\gamma_j\} \) of increasing aggressiveness. The LORD1 method uses the sequence proposed in the paper [6]. Under the alternative the \( p \)-values are distributed as Beta(0.5, 5). (See also Figure 10.)

**Figure 10.** Statistical power and FDR versus fraction of non-null hypotheses \( \pi_1 \) for SAFFRON, LORD and alpha-investing (at target level \( \alpha = 0.05 \)), using the sequence \( \{\gamma_j\} \) which achieves the highest power for each of them (chosen over six sequences of varying aggressiveness). Under the alternative the \( p \)-values are distributed as Beta(0.5, 5).
6 Proofs

Here we provide the proofs of Proposition 1 and Theorem 1 using the reverse super-uniformity lemma proved in Section 4, as well as the proof of SAFFRON’s monotonicity.

6.1 Proof of Proposition 1

Statement (a) is proved by noting that for any time \( t \in \mathbb{N} \), we have:

\[
\mathbb{E} \left[ |H_0^0 \cap R(t)| \right] = \sum_{j \leq t, j \in H_0^0} \mathbb{E} \left[ 1 \{ P_j \leq \alpha_j \} \right] \leq \sum_{j \leq t, j \in H_0^0} \mathbb{E} [\alpha_j],
\]

where the inequality follows after taking iterated expectations by conditioning on \( F_{j-1} \), and then applying the conditional super-uniformity property (3). If we have \( \text{FDP}^\ast (t) := \frac{1}{|R(t)|} \sum_{j \leq t, j \in H_0^0} \alpha_j \leq \alpha \), as assumed in statement (b), then it follows that:

\[
\sum_{j \leq t, j \in H_0^0} \mathbb{E} [\alpha_j] = \mathbb{E} \left[ \sum_{j \leq t, j \in H_0^0} \alpha_j \right] \leq \alpha \mathbb{E} [|R(t)|],
\]

using linearity of expectation and the assumption on \( \text{FDP}^\ast (t) \). Using part (a) and rearranging yields the inequality \( m\text{FDR}(t) := \frac{\mathbb{E} [|H_0^0 \cap R(t)|]}{\mathbb{E} [|R(t)|]} \leq \alpha \), which concludes the proof of part (b).

If, in addition, the null \( p \)-values are independent of each other and of the non-nulls and the sequence \( \{ \alpha_t \} \) is monotone, we can use the following argument to prove claims (c) and (d). These claims establish that the procedure controls the FDR at any time \( t \in \mathbb{N} \). Still assuming the inequality \( \text{FDP}^\ast (t) \leq \alpha \), we have:

\[
\text{FDR}(t) = \mathbb{E} \left[ \frac{|H_0^0 \cap R(t)|}{|R(t)|} \right] = \sum_{j \leq t, j \in H_0^0} \mathbb{E} \left[ \frac{1 \{ P_j \leq \alpha_j \}}{|R(t)|} \right] \leq \sum_{j \leq t, j \in H_0^0} \mathbb{E} \left[ \frac{1}{|R(t)|} \right] = \mathbb{E} [\text{FDP}^\ast (t)] \leq \alpha,
\]

where the first inequality follows after taking iterated expectations by conditioning on \( F_{j-1} \), and then applying the super-uniformity lemma (10), the following equality uses linearity of expectation, and the final inequality follows by the assumption on \( \text{FDP}^\ast (t) \). This concludes the proof of both statements (c) and (d).

6.2 Proof of Theorem 1

First note that, for any time \( t \in \mathbb{N} \), we have:

\[
\mathbb{E} \left[ |H_0^0 \cap R(t)| \right] = \sum_{j \leq t, j \in H_0^0} \mathbb{E} [1 \{ P_j \leq \alpha_j \}] \leq \sum_{j \leq t, j \in H_0^0} \mathbb{E} [\alpha_j] \leq \mathbb{E} \left[ \sum_{j \leq t, j \in H_0^0} \alpha_j \frac{1 \{ P_j > \lambda_j \}}{1 - \lambda_j} \right],
\]

where the first inequality follows after taking iterated expectations by conditioning on \( F_{j-1} \), and then applying the super-uniformity lemma (10), the following equality uses linearity of expectation, and the final inequality follows by the assumption on \( \text{FDP}^\ast (t) \). This concludes the proof of both statements (c) and (d).
where inequality (i) first uses the law of iterated expectations by conditioning on \( F^{j-1} \), and then both (i) and (ii) apply the conditional super-uniformity property (4), which concludes the proof of part (a). To prove part (b), we drop the condition \( j \in \mathcal{H}^0 \) from the last expectation, and use the assumption that \( \text{FDP}_{\lambda}(t) := \frac{\sum_{j \leq t} \alpha_j 1 \{ P_j > \lambda_j \}}{|\mathcal{R}(t)|} \leq \alpha \) to obtain:

\[
\mathbb{E} \left[ \sum_{j \leq t} \alpha_j \frac{1 \{ P_j > \lambda_j \}}{1 - \lambda_j} \right] \leq \alpha \mathbb{E}[|\mathcal{R}(t)|].
\]

Combining this inequality with the result of part (a), and rearranging the terms, we reach the conclusion that \( \text{mFDR}(t) \leq \alpha \), as desired.

Under the independence and monotonicity assumptions of parts (c, d), we have:

\[
\text{FDR}(t) = \mathbb{E} \left[ \frac{|\mathcal{H}^0 \cap \mathcal{R}(t)|}{|\mathcal{R}(t)|} \right]
= \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \frac{1 \{ P_j \leq \alpha_t \}}{|\mathcal{R}(t)|} \right]
\leq \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \frac{\alpha_j}{|\mathcal{R}(t)|} \right]
\leq \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \frac{\alpha_j 1 \{ P_j > \lambda_j \}}{(1 - \lambda_j)|\mathcal{R}(t)|} \right],
\]

where inequality (iii) first uses iterated expectations by conditioning on \( F^{j-1} \), and then both (iii) and (iv) apply Lemma 1. Assuming that the inequality \( \text{FDP}_{\lambda}(t) \leq \alpha \) holds, it follows that:

\[
\sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \frac{\alpha_j 1 \{ P_j > \lambda_j \}}{(1 - \lambda_j)|\mathcal{R}(t)|} \right] \leq \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \frac{\alpha_j 1 \{ P_j > \lambda_j \}}{(1 - \lambda_j)|\mathcal{R}(t)|} \right]
= \mathbb{E} \left[ \frac{\text{FDP}_{\lambda}(t)}{|\mathcal{R}(t)|} \right]
\leq \alpha,
\]

where inequality (v) follows by linearity of expectation and summing over a larger set of indices; equality (vi) simply uses the definition of \( \text{FDP}_{\lambda}(t) \), and inequality (vii) follows by the assumption, hence proving parts (c,d).

### 6.3 Proof of monotonicity of SAFFRON

In applying the reverse super-uniformity lemma in Section 3 to prove that SAFFRON controls the FDR, it is assumed that SAFFRON is a monotone rule, meaning that \( f_t : (R_{1:T}, C_{1:T}) \rightarrow \alpha_t \) is a coordinatewise non-decreasing function. Here we provide a proof of this claim. We prove it assuming \( \lambda \) is constant, however the same arguments can be applied if it changes at every step, i.e. if it is predictable as stated in Section 3.

Consider some \((R_{1:T}, C_{1:T})\) and \((\hat{R}_{1:T}, \hat{C}_{1:T})\) for a fixed \( T \). We will accordingly denote all relevant variables in the SAFFRON procedures which result in \((R_{1:T}, C_{1:T})\) and \((\hat{R}_{1:T}, \hat{C}_{1:T})\), e.g. \( \alpha_t \) and \( \hat{\alpha}_t \), respectively. Taking into account the possible relations between indicators for rejection and candidacy, \((\hat{R}_{1:T}, \hat{C}_{1:T}) \succeq (R_{1:T}, C_{1:T})\) if and only if, for every \( t \leq T \), one of the following holds:

(i) \( R_t = \hat{R}_t \) and \( C_t = \hat{C}_t \),
(ii) \( R_t = 0, C_t = 1 \) and \( \hat{R}_t = 1, \hat{C}_t = 1 \),
(iii) \( R_t = 0, C_t = 0 \) and \( \hat{R}_t = 0, \hat{C}_t = 1 \),
(iv) \( R_t = 0, C_t = 0 \) and \( \hat{R}_t = 1, \hat{C}_t = 1 \).
From this it is clear that the procedure which generated \((R_1:T, C_1:T)\) up to time \(T\) could not have made more rejections or encountered more candidate \(p\)-values. Further, at each time that it made a rejection, the procedure that generated \((\tilde{R}_1:T, \tilde{C}_1:T)\) also made a rejection. Looking into the SAFFRON update rule for the rejection thresholds, recall that \(\alpha_t\) is computed as:

\[
\alpha_t := \min\{\lambda, \sigma_t\}, \quad \text{where} \quad \sigma_t := W_0\gamma_t-C_0+ \nonumber \\
((1-\lambda)\alpha - W_0)\gamma_t-C_1+ \sum_{j \geq 2}(1-\lambda)\alpha\gamma_{t-\tau_j}-C_j+. \nonumber
\]

Note that, by construction, the terms \(((1-\lambda)\alpha - W_0)\) and \((1-\lambda)\alpha\) are strictly positive. Therefore, since the sequence \(\{\gamma_j\}\) is non-increasing, the sum of the terms \((1-\lambda)\alpha\gamma_{t-\tau_j}-C_j+\) contributing to \(\alpha_t\) is at most as great as the the sum of the terms \((1-\lambda)\alpha\gamma_{t-\tau_j}-\tilde{C}_j+\), because \(\tilde{\alpha}_t\) considers at least all the rejection times in \(\alpha_t\), and has \(\tilde{C}_j+ \geq C_j+\) for all \(j\) (the same holds for the term \(((1-\lambda)\alpha - W_0)\)).

7 Conclusion

This paper introduces SAFFRON, a new algorithmic framework for online mFDR and FDR control. We show empirically that SAFFRON is more powerful than existing algorithms. The derivation and proof of SAFFRON is based on a novel reverse super-uniformity lemma that allows us to estimate the fraction of alpha-wealth that an algorithm spends on testing null hypotheses. One may interpret SAFFRON as an adaptive version of LORD, just as Storeyy-BH is an adaptive version of the Benjamini-Hochberg algorithm. Also, a monotone version of the alpha-investing algorithm, that is often more stable and powerful than the original, is recovered as a special case of the SAFFRON framework. Lastly, the derivation of SAFFRON is rather different from that of earlier generalized alpha-investing (GAI) algorithms, and as such provides a template for the derivation of new algorithms.

References

[1] Ehud Aharoni and Saharon Rosset. Generalized \(\alpha\)-investing: definitions, optimality results and application to public databases. Journal of the Royal Statistical Society, Series B (Statistical Methodology), 76(4):771–794, 2014.

[2] Rina Foygel Barber and Emmanuel J. Candès. Controlling the false discovery rate via knockoffs. The Annals of Statistics, 43(5):2055–2085, 2015.

[3] Yoav Benjamini and Yosef Hochberg. Controlling the false discovery rate: a practical and powerful approach to multiple testing. Journal of the Royal Statistical Society, Series B (Methodological), 57(1):289–300, 1995.

[4] Yoav Benjamini and Daniel Yekutieli. The control of the false discovery rate in multiple testing under dependency. The Annals of Statistics, 29(4):1165–1188, 2001.

[5] Dean Foster and Robert Stine. \(\alpha\)-investing: a procedure for sequential control of expected false discoveries. Journal of the Royal Statistical Society, Series B (Statistical Methodology), 70(2):429–444, 2008.

[6] Adel Javanmard and Andrea Montanari. Online rules for control of false discovery rate and false discovery exceedance. The Annals of Statistics, to appear, 2017.

[7] Lihua Lei and William Fithian. Power of ordered hypothesis testing. In International Conference on Machine Learning, pages 2924–2932, 2016.

[8] Ang Li and Rina Foygel Barber. Accumulation tests for fdr control in ordered hypothesis testing. Journal of the American Statistical Association, 112(518):837–849, 2017.

[9] Aaditya Ramdas, Rina Foygel Barber, Martin Wainwright, and Michael Jordan. A unified treatment of multiple testing with prior knowledge. arXiv preprint arXiv:1703.06222, 2017.
[10] Aaditya Ramdas, Fanny Yang, Martin Wainwright, and Michael Jordan. Online control of the false discovery rate with decaying memory. In Advances In Neural Information Processing Systems, pages 5655–5664, 2017.

[11] John Storey. A direct approach to false discovery rates. Journal of the Royal Statistical Society, Series B (Statistical Methodology), 64(3):479–498, 2002.

[12] John Storey, Jonathan Taylor, and David Siegmund. Strong control, conservative point estimation and simultaneous conservative consistency of false discovery rates: a unified approach. Journal of the Royal Statistical Society, Series B (Statistical Methodology), 66(1):187–205, 2004.

[13] John Tukey. The Problem of Multiple Comparisons: Introduction and Parts A, B, and C. Princeton University, 1953.

[14] Fanny Yang, Aaditya Ramdas, Kevin Jamieson, and Martin J. Wainwright. Multi-A(rmed)/B(andit) testing with online FDR control. Advances in Neural Information Processing Systems, pages 5959–5968, 2017.