DYNAMICAL ROTATIONAL INSTABILITY AT LOW $T/W$

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ABSTRACT

Dynamical instability is shown to occur in differentially rotating polytropes with $N = 3.33$ and $T/W \gtrsim 0.14$. This instability has a strong $m = 1$ mode, although the $m = 2, 3,$ and 4 modes also appear. Such instability may allow a centrifugally hung core to begin collapsing to neutron star densities on a dynamical timescale. The gravitational radiation emitted by such unstable cores may be detectable with advanced ground-based detectors, such as LIGO.

If the instability occurs in a supermassive star, it may produce gravitational radiation detectable by the space-based detector LISA.

Subject headings: gravitation — hydrodynamics — instabilities — stars: neutron — stars: rotation

On-line material: color figure, mpg animations

1. INTRODUCTION

Rotational instabilities are potentially important in the evolution of massive stellar cores. For example, the core of a massive star that has been prevented from collapsing to neutron star densities by centrifugal forces may rotate rapidly enough for rotational instabilities to develop (Hayashi, Eriguchi, & Hashimoto 1998, 1999). Massive cores spun up by accretion from a binary companion (Wagoner 1984) and the remnants of compact binary mergers (Rasio & Shapiro 1992; Zhuge, Centrella, & McMillan 1994) may also reach fast enough rotation rates to become unstable. Rotational instabilities may produce detectable gravitational radiation (Schutz 1989; Thorne 1996). If enough angular momentum is shed, full collapse to a neutron star or black hole may occur.

We focus here on global rotational instabilities that arise in fluids from growing azimuthal modes $e^{i\omega t}$ (Tassoul 1978). Dynamical instabilities are driven by hydrodynamics and gravity and grow on the order of the dynamical timescales of the system. In objects with Maclaurin-like rotation laws, dynamical instabilities are generally expected to arise at high values of $\beta \equiv T/W \gtrsim 0.27$ in Newtonian gravity (see New, Centrella, & Tohline and references therein), and $\beta \sim 0.25$–0.26 in general relativity (Shibata, Baumgarte, & Shapiro 2000). Here $T$ is the rotational kinetic energy and $W$ is the gravitational potential energy. Dynamical instability may set in at lower values of $\beta$ for thick, self-gravitating disks (Pickett, Durisen, & Davis 1996; Woodward, Tohline, & Hachisu 1994) and in tori (Tohline & Hachisu 1990). Secular instabilities develop on longer timescales and arise from dissipative processes such as viscosity and gravitational radiation reaction. These are expected to set in at lower values of $\beta$, at $\beta \sim 0.14$ for the viscosity-driven instability and at $\beta \sim 0.1$ and $\beta < 0.03$ for gravitational-wave drive $p$-modes and $r$-modes (Lindblom, Owens, & Ushomirsky 2000; Andersson & Kokkotas 2000), although the rotation law, polytropic index, and general relativity can affect the value of the instability limit (Imamura et al. 1995; Stergioulas & Friedman 1998).

Studies of scenarios for the formation of centrifugally hung cores have generally concluded that these objects will have $\beta < 0.27$ (Tohline 1984; Eriguchi & Müller 1985; Müller & Eriguchi 1985). Direct numerical simulations of axisymmetric stellar collapse by Zwerger & Müller (1997) also indicate that, for the cases studied, it is difficult for collapsing cores to reach and exceed $\beta \sim 0.27$. Overall, it is generally assumed that centrifugally hung cores will evolve secularly, as in the “fizzler” scenario (Hayashi et al. 1998, 1999).

In this Letter, we report on new three-dimensional numerical simulations of rotating stellar cores that exhibit dynamical rotational instability at relatively low values of $\beta \sim 0.14$. These models are computed using Newtonian gravity with no back-reaction; this is a reasonable approximation for a stellar core of mass $M \sim 1.4 M_\odot$ hung up at a radius $R \sim 100$ km, so that $(GM/R^2) \lesssim 0.02$. Our results show that compact objects can become dynamically unstable in a region of parameter space previously thought to be solely the province of secular instabilities.

2. NUMERICAL SIMULATIONS

Rotating stellar cores are initially assumed to be axisymmetric equilibrium polytropes with equation of state $P = K \rho^{\Gamma} = K \rho^{1+\Gamma/2}$, with $\Gamma = 1.3 (N = 3.33)$. The angular velocity distribution is given by the so-called $j$-constant rotation law, $\Omega^2 = j_0^2 (d^2 + \varpi^2)^2$, where $\varpi$ is the cylindrical radius and $d$ is an arbitrary constant (Hachisu 1986). As $d \to 0$, the specific angular momentum approaches the constant value $j_0$. Here, we use $d = 0.2$. The models are computed on a uniformly spaced $(\sigma, z)$ grid using the self-consistent field method of Hachisu (1986). Varying the axis ratio yields equilibria with different values of $\beta$; see New, Centrella, & Tohline (2000) for details. Figure 1 shows density contours in the $x$-$z$ plane for the four models that we evolved here, $\beta = 0.090, 0.12, 0.14$, and 0.18. Notice that, with the exception of the $\beta = 0.090$ model, the density maxima are toroidal; similar structures were used as initial data in the collapse models of Rampp, Müller, & Ruffert (1998).

These initial models were introduced into two different three-dimensional hydrocodes. The first of these is the $L$ code discussed in New et al. (2000), with a cylindrical $(\sigma, z, \phi)$ grid of $64 \times 64 \times 128$ zones. The second is the piecewise parabolic method code discussed by Brown (2000), with a Cartesian $(x, y, z)$ grid of $128 \times 128 \times 128$ zones. Both codes have the...
same resolution along the $x$- and $z$-axes, with the equatorial radius $R_e$ of the initial model extending out to zone 48 in the $x$-direction. The cylindrical code imposes equatorial plane symmetry, whereas the Cartesian code imposes no symmetries. Random perturbations of 1% were imposed on the densities, and the models were evolved forward in time.

3. RESULTS

The model with initial $\beta = 0.14$ exhibits a dynamical instability that is very similar in simulations performed with both the cylindrical and Cartesian hydrocodes. These runs were stopped when the loss of mass from the edge of the grid became significant. Figure 2 shows density contours in the equatorial plane from the run with the cylindrical code. The green (lightest gray scale) contour in the first frame delineates the toroidal plane from the run with the cylindrical code. The green (lightest gray scale) contour in the first frame delineates the toroidal plane from the run with the cylindrical code.

For the cylindrical code, the amplitudes were calculated in a circle of radius $0.32$. For the Cartesian code, these amplitudes were calculated in the equatorial plane in a ring of width $\Delta \varpi = 1/48$ at radius $\varpi = 0.32$; see New et al. (2000) for details. For the Cartesian code, the amplitudes were computed on a circle of radius $\varpi = 0.32$ using a nonuniform discretization, which avoids grid boundaries, and a linear interpolation of density.

Similar results were found in rings at other values of $\varpi$. The evolution in both runs is dominated by an exponentially growing $m = 1$ mode (thick solid line). Once this mode is well into its exponential growth phase, an $m = 2$ mode also develops, followed at later times by $m = 3$ and $m = 4$ modes. (The constant amplitude $m = 4$ signal in the Cartesian run is due to the geometry of the numerical grid.) Since these modes grow rapidly on dynamical timescales, they signal dynamical instability. Numerical values for the growth rates, eigenfrequencies, and pattern speeds are shown in Table 1 for $m = 1$ and $m = 2$.

![Fig. 1](image1.png)

![Fig. 2](image2.png)

**Fig. 1**.—Density contours are shown in the $x$-$z$ plane for four models with $d = 0.2$ and $N = 3.33$. The initial maximum density is normalized to unity, and the contours are at levels of 0.9, 0.1, 0.01, and 0.001.

**Fig. 2**.—Two-dimensional density contours in the equatorial plane are shown for the model with $\beta = 0.14$ run on the cylindrical code. The contour levels are 0.01 (purple), 0.1 (blue), 0.9 (green), 2 (yellow), and 4 (off-white) times the maximum density at the initial time, which is normalized to unity. In the gray scale version of this figure, the density decreases as the darkness of the shading increases. [See the electronic edition of the Journal for a color version of this figure.]

\[
\tan^{-1}\left[\frac{\ln(A_m)}{\ln(A_m')}\right].
\]

We write the phase angle $\phi_m = \sigma_m t$, where $\sigma_m = \frac{d\phi_m}{dt}$ is the eigenfrequency and $W_m = \sigma_m / \Omega_m$ is the pattern speed of the $m$th mode.

| Code       | $\beta$ | $s_1$ | $s_2$ | $W_1$ | $W_2$ |
|------------|---------|-------|-------|-------|-------|
| Cylindrical| 0.14    | 0.40  | 0.92  | 3.6   | 3.6   |
| Cartesian  | 0.14    | 0.35  | 0.90  | 3.5   | 3.5   |
| Cylindrical| 0.18    | 0.99  | 1.8   | 3.3   | 3.3   |
| Cartesian  | 0.18    | 0.98  | 1.1   | 3.2   | 3.2   |

*Given in a ring at radius $\varpi = 0.32$ in the equatorial plane.*
was run for $p_m$ see Table 1. In the run on the cylindrical code, the modes at very close to the same rate, starting at about the same time; however, code, the and modes both grew at about the same rate, starting at about the same time. For the evolutions run with both codes, the mode grew further.

reliable values for the and modes. Longer runs with larger grids and higher resolution are needed to obtain a spurious signal (New et al. 2000). For the simulations system center of mass could develop at late times, resulting in significant motion of the cylindrical hydrocode showed that significant motion of the center of mass (less than one zone) had developed in the model. This suggests that this instability may be related to the ones found by Woodward et al. (1994) and Tohline & Hachisu (1990) in their studies of polytropic tori with $N = 3/2$. We note that Pickett et al. (1996) also found a dominant $m = 1$ instability that set in at a relatively low $\beta \approx 0.20$, in a centrally condensed model (without an off-center density maximum). Their model was an $N = 3/2$ polytrope with an $n' = 2$ rotation law. The $n' = 2$ rotation law produces configurations with a strong concentration of angular momentum in the outer regions of the model. We plan to carry out more detailed studies to investigate the character of the unstable modes seen in our simulations and their properties for various values of $N$ and the $j$–constant rotation law parameter $d$.

Previous studies of the $m = 2$ bar-mode instability using the cylindrical hydrocode showed that significant motion of the system center of mass could develop at late times, resulting in a spurious $m = 1$ signal (New et al. 2000). For the simulations reported here, we monitored the position of the overall center of mass and verified that it underwent no systematic motion during the development of the $m = 1$ mode, in both the cylindrical and Cartesian codes. When the runs with $T/|W| = 0.14$ and $T/|W| = 0.18$ were stopped, some small, spurious center of mass motion (less than one zone) had developed in both codes. This effect is possibly related to the loss of mass from the grid as the outer regions expand; runs with larger grids are needed to determine the late-time behavior of these models. Previous studies of the bar instability using the Cartesian code (Brown 2000) showed that angular momentum losses due to numerical inaccuracies can be significant. For the $\beta = 0.14$ and $\beta = 0.18$ simulations with the Cartesian code, the artificial angular momentum losses amounted to less than 0.4% and 1.5%, respectively.

4. DISCUSSION

Our results demonstrate that dynamical instability can occur in differentially rotating polytropes with low values of $\beta \approx 0.14$ and that this instability has a strong, $m = 1$ character. Figure 1 shows that the unstable axisymmetric equilibria each have a torus of dense material centered on the rotation axis within the model. This suggests that this instability may be related to the ones found by Woodward et al. (1994) and Tohline & Hachisu (1990) in their studies of polytropic tori with $N = 3/2$. We note that Pickett et al. (1996) also found a dominant $m = 1$ instability that set in at a relatively low $\beta \approx 0.20$, in a centrally condensed model (without an off-center density maximum). Their model was an $N = 3/2$ polytrope with an $n' = 2$ rotation law. The $n' = 2$ rotation law produces configurations with a strong concentration of angular momentum in the outer regions of the model. We plan to carry out more detailed studies to investigate the character of the unstable modes seen in our simulations and their properties for various values of $N$ and the $j$–constant rotation law parameter $d$.

If such instability occurs in a centrifugally hung core, collapse to neutron star densities may result. Our simulations do show that the density is increasing at the end of the unstable runs. Further studies are needed to see how dense the remnant actually becomes and whether the $m = 1$ mode will result in the dense region moving at a velocity comparable to those of actual neutron stars (Popov et al. 2000).

Dynamical instability will also produce gravitational radiation. Although longer runs are needed to obtain the full gravitational waveforms, we can estimate the properties of the grav-
DYNAMICAL ROTATIONAL INSTABILITY AT LOW $T/W$

Vol. 550

for a massive stellar core with $M \sim 1.4 M_\odot$ and $R_\star \sim 200$ km, the peak amplitude will occur at a frequency of roughly $f \sim 200$ Hz. This peak amplitude will be $h \sim 10^{-24} r_{20}^{-1}$ for $\beta = 0.14$ and $h \sim 10^{-23} r_{20}^{-1}$ for $\beta = 0.18$. Here $r_{20}$ is the distance to the source in units of 20 Mpc. Emission from such unstable cores may be detectable with advanced ground-based interferometers, such as LIGO-II. An even more optimistic scenario for detectable gravitational radiation occurs if these instabilities arise in supermassive stars (New & Shapiro 2001). For example, if the instability occurs when a supermassive star contracts to the point that $(GM/Rc^2) \sim 1/15$, an approximate value for uniformly rotating stars (Baumgarte & Shapiro 1999), we estimate the frequency of the gravitational radiation to be $f \sim 3.5 \times 10^{-3}$ Hz, with amplitude $h \sim 10^{-18} r_{20}^{-1}$ for $\beta = 0.14$ and $h \sim 10^{-17} r_{20}^{-1}$ for $\beta = 0.18$. Such signals would be easily detectable by the space-based LISA detector.

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REFERENCES

Andersson, N., & Kokkotas, K. D. 2000, J. Mod. Phys. D, in press (gr-qc/0010102)
Baumgarte, T., & Shapiro, S. 1999, ApJ, 526, 941
Brown, J. D. 2000, Phys. Rev. D, 62, 084024
Eriguchi, Y., & Müller, E. 1985, A&A, 147, 161
Hachisu, I. 1986, ApJS, 61, 479
Hayashi, A., Eriguchi, Y., & Hashimoto, M. 1998, ApJ, 492, 286
Imamura, J., Toman, J., Durisen, R., Pickett, B., & Yang, S. 1995, ApJ, 444, 363
Lindblom, L., Owens, B. J., & Ushomirsky, G. 2000, Phys. Rev. D, 62, 084030
Müller, E., & Eriguchi, Y. 1985, A&A, 152, 325
New, K., Centrella, J., & Tohline, J. 2000, Phys. Rev. D, 62, 064019
Pickett, B., Durisen, R., & Davis, G. 1996, ApJ, 458, 714
Popov, S., Colpi, M., Treves, A., Turolla, R., Lipunov, V., & Prokhorov, M. 2000, ApJ, 530, 896
Rampp, M., Müller, E., & Ruffert, M. 1998, A&A, 332, 969
Rasio, F., & Shapiro, S. 1992, ApJ, 401, 226
Schutz, B. 1989, Classical Quantum Gravity, 6, 1761
Shibata, M., Baumgarte, T., & Shapiro, S. 2000, ApJ, 542, 453
Stergioulas, N., & Friedman, J. 1998, ApJ, 492, 301
Tassoul, J. 1978, Theory of Rotating Stars (Princeton: Princeton Univ. Press)
Thorne, K. 1996, in IAU Symp. 165, Compact Stars in Binaries, ed. J. van Paradijs, E. P. J. van den Heuvel, & E. Kuulkers (Dordrecht: Kluwer), 153
Tohline, J. 1984, ApJ, 285, 721
Tohline, J., Durisen, R., & McCollough, M. 1985, ApJ, 298, 220
Tohline, J., & Hachisu, I. 1990, ApJ, 361, 394
Wagoner, R. 1984, ApJ, 278, 345
Woodward, J., Tohline, J., & Hachisu, I. 1994, ApJ, 420, 247
Zhuge, X., Centrella, J., & McMillan, S. 1994, Phys. Rev. D, 50, 6247
Zwerger, T., & Müller, E. 1997, A&A, 320, 209