A Riemann-Hilbert approach to asymptotic analysis of Toeplitz+Hankel determinants

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September 26, 2019

Abstract. In this paper we will formulate $4 \times 4$ Riemann-Hilbert problems for Toeplitz+Hankel determinants and the associated system of orthogonal polynomials, when the Hankel symbol is supported on the unit circle and also when it is supported on an interval $[a, b]$, $0 < a < b < 1$. The distinguishing feature of this work is that in the formulation of the Riemann-Hilbert problem no specific relationship is assumed between the Toeplitz and Hankel symbols. We will develop nonlinear steepest descent methods for analysing these problems in the case where the symbols are smooth (i.e. in the absence of Fisher-Hartwig singularities) and admit an analytic continuation in a neighborhood of the unit circle (if the symbol’s support is the unit circle). We will finally introduce a model problem and will present its solution requiring certain conditions on the ratio of Hankel and Toeplitz symbols. This in turn will allow us to find the asymptotics of the norms $h_n$ of the corresponding orthogonal polynomials and, in fact, the large $n$ asymptotics of the polynomials themselves. We will explain how this solvable case is related to the recent operator-theoretic approach in [6] to Toeplitz+Hankel determinants. At the end we will discuss the prospects of future work and outline several technical, as well as conceptual, issues which we are going to address next within the $4 \times 4$ Riemann-Hilbert framework introduced in this paper.

Notation. Throughout the paper we will frequently use the notation $\tilde{f}(z)$, to denote $f(z^{-1})$.

1 Introduction and preliminaries

The $n \times n$ Toeplitz and Hankel matrices associated respectively to the symbols $\phi$ and $w$, supported on the unit circle $\mathbb{T}$ are respectively defined as

$$T_n[\phi; r] := \{ \phi_{j-k+r} \}, \quad j, k = 0, \ldots, n-1, \quad \phi_k = \int_{\mathbb{T}} z^{-k} \phi(z) \frac{dz}{2\pi i z}, \quad (1.1)$$

and

$$H_n[w; s] := \{ w_{j+k+s} \}, \quad j, k = 0, \ldots, n-1, \quad w_k = \int_{\mathbb{T}} z^{-k} w(z) \frac{dz}{2\pi i z}, \quad (1.2)$$

for fixed offset values $r, s \in \mathbb{Z}$. If the Hankel symbol $w$ is supported on a subset $I$ of the real line, then $w_k$ in (1.2) are instead given by
\[ w_k = \int_1 x^k w(x) dx. \quad (1.3) \]

The Toeplitz and Hankel determinants characterize important objects particularly in random matrix theory, statistical mechanics, theory of orthogonal polynomials, theory of Fredholm determinants, etc. For more on the history of the development of the theory of Toeplitz and Hankel determinants and their numerous applications we refer the reader to the review articles [15] and [21]. We also refer to the monographs [8] and [9] as the main sources for general facts concerning the theory of Toeplitz matrices and operators. The asymptotic results concerning the Hankel determinants and their applications - both recent and classical, are featured in the papers [15], [10], [11], [20], and [19] and in the references therein.

There has been a growing interest in the asymptotics of Toeplitz+Hankel determinants in recent years. A Toeplitz+Hankel matrix is naturally the sum \( T_n[\phi; r] + H_n[w; s] \), and thus it has a determinant of the form

\[ D_n(\phi, w; r, s) := \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} \end{pmatrix} \]

where, naturally, \( \phi \) and \( w \) respectively denote the Toeplitz and Hankel symbols. Although there are no results in the literature for the Toeplitz+Hankel determinants where \( w \) is supported on the line, the case where \( w \) is supported on the unit circle has been considered under specific assumptions. E.Basor and T.Ehrhardt have studied different aspects of these determinants in a series of papers [2] [3] [4] [5] [6] via operator-theoretic tools over the last 20 years or so. In [13], the Riemann-Hilbert technique which has already been proven very effective to study the asymptotics of Toeplitz and Hankel determinants was extended for the first time to the determinants of Toeplitz + Hankel matrices generated by the same symbol \( w = \tilde{\phi} \), where the Hankel weight is supported on \( T \). In that work the symbol was assumed to be of Fisher-Hartwig type and it was further required that the symbol be even, i.e. \( w = \tilde{\phi} \). In [9], by employing the relevant results in [13], the authors managed for the first time to find the asymptotics of Toeplitz+Hankel determinants for certain non-coinciding symbols. Indeed, they considered

\[ \phi(z) = c(z)\phi_0(z), \quad \text{and} \quad w(z) = c(z)d(z)\phi_0(z), \quad (1.5) \]

where the functions \( c \) and \( d \) are assumed to be smooth and nonvanishing on the unit circle with zero winding number. Neither \( c \) nor \( d \) are assumed to be even functions but it is further required that \( d \) satisfies the conditions \( d \tilde{d} = 1 \) (on the unit circle) and \( d(\pm 1) = 1 \). Furthermore, \( \phi_0 \) is assumed to be an even function of FH type and \( w_0 \) is related to \( \phi_0 \) in one of the following four ways: a) \( w_0(z) = \pm \phi_0(z) \), b) \( w_0(z) = z\phi_0(z) \) and c) \( w_0(z) = -z^{-1}\phi_0(z) \).

Since the Riemann-Hilbert analysis carried out in [13] does not allow for different symbols, the primary goal of this paper is to develop a Riemann-Hilbert framework for asymptotic analysis of Toeplitz+Hankel determinants \( D_n(\phi, w; r, s) \) where \( \phi \) and \( w \) are not a priori related, at least at the level of formulation of the problem. Indeed, asymptotics of Toeplitz+Hankel determinants with different symbols are interesting for several reasons that prompts this research project. For example, the type (1.5) of Toeplitz+Hankel determinants has appeared in the very recent work [12] in connection with the analysis of Ising model on the 45° rotated half-plane, or the so-called zig-zag half-plane.

Perhaps, our most important motivation behind studying Toeplitz+Hankel determinants is to study the large \( n \) asymptotics of the eigenvalues of the Hankel matrix \( H_n[w] \) associated to the
symbol \( w \). Specifically, we want to extend the recent results \([14]\) concerning the spectral asymptotics of the Toeplitz matrices\(^1\) to the Hankel case. The key feature which allows an effective asymptotic spectral analysis of Toeplitz matrices and, in particular, the use of the Riemann-Hilbert method, is that the characteristic polynomial of a Toeplitz matrix is again a Toeplitz determinant with a symbol of general Fisher-Hartwig type (i.e. no conditions on the \( \beta \)-parameters). The asymptotics of such Toeplitz determinants is given by the Basor-Tracy formula (first conjectured by E. Basor and C. Tracy and then proved in \([13]\)). However, in the case of Hankel matrices, and this is the crux of the matter, their characteristic polynomials are not Hankel determinants. Indeed, the characteristic polynomial of Toeplitz matrices is again a Toeplitz determinant with a symbol of general Fisher-Hartwig type (i.e. no conditions on the \( \beta \)-parameters). The asymptotics of the Toeplitz matrices has to be addressed at a fundamental level by formulation of a suitable Riemann-Hilbert problem.

We also show that for analysis of Toeplitz+Hankel determinants based on a certain 4 × 4 analogue of the Deift-Zhou non-linear steepest descent method and arrive at a 4 × 4 model Riemann-Hilbert problem on the unit circle which does not contain the parameter \( n \). It is significant to note that one arrives at the same model Riemann-Hilbert in the fundamentally different case where \( w \) is supported on the unit circle. The key feature which allows an effective asymptotic analysis of Toeplitz+Hankel determinants is given by the Basor-Tracy formula (first conjectured by E. Basor and C. Tracy and then proved in \([13]\)). However, in the case of Hankel matrices, and this is the crux of the matter, their characteristic polynomials are not Hankel determinants. Indeed, the characteristic polynomial of a Toeplitz matrix is again a Toeplitz determinant with a symbol of general Fisher-Hartwig type (i.e. no conditions on the \( \beta \)-parameters). The asymptotics of the Toeplitz matrices has to be addressed at a fundamental level by formulation of a suitable Riemann-Hilbert problem.

In this paper, we are proposing a version of the Riemann-Hilbert formalism for the asymptotic analysis of Toeplitz+Hankel determinants based on a certain 4 × 4 Riemann-Hilbert problem. When the Hankel symbol is supported on the unit circle, we introduce the following system of monic orthogonal polynomials \( \{P_n(z)\} \), deg \( P_n(z) = n \), associated to \( D_n(\phi, w; r, s) \):

\[
\int \mathcal{P}_n(z)z^{-k-r}\phi(z)\frac{dz}{2\pi i} + \int \mathcal{P}_n(z)z^{k+s}\tilde{w}(z)\frac{dz}{2\pi i} = h_n\delta_{n,k}, \quad k = 0, 1, \ldots, n.
\]

We also show that for \( r = s = 1 \), if the symbols are analytic in a neighborhood of the unit circle, one can proceed with a 4 × 4 analogue of the Deift-Zhou non-linear steepest descent method and arrive at a 4 × 4 model Riemann-Hilbert problem on the unit circle which does not contain the parameter \( n \). It is significant to note that one arrives at the same model Riemann-Hilbert in the fundamentally different case where \( w \) is supported on the interval \([a, b]\), with \( 0 < a < b < 1 \). In this situation we consider the following system of monic orthogonal polynomials \( \{P_n(z)\} \), deg \( P_n(z) = n \), associated to \( D_n(\phi, w; r, s) \):

\[
\int \mathcal{P}_n(z)z^{-k-r}\phi(z)\frac{dz}{2\pi i} + \int_a^b \mathcal{P}_n(x)x^{k+s}w(x)dx = h_n\delta_{n,k}, \quad k = 0, 1, \ldots, n.
\]

In this case, we can proceed with the Riemann-Hilbert analysis with \( r = 1 \) and an arbitrary value for \( s \in \mathbb{Z} \).

We have been able to solve the model problem for the class of symbols \([15]\) considered in \([6]\), in the absence of Fisher-Hartwig singularities. It is important to discuss the relevancy of the two conditions assumed to be satisfied by the function \( d \) in \([6]\), in our Riemann-Hilbert framework (see \([1,5]\) and below). Unlike the condition \( d(e^{i\theta})d(e^{-i\theta}) = 1 \) which is, remarkably, a simplifying condition for the factorization of the model Riemann-Hilbert problem, it should be noticed that the condition \( d(\pm 1) = 1 \) is not required in the entirety of our Riemann-Hilbert approach. Solving the model problem allows us to find the asymptotics for the norm \( h_n \) of the associated orthogonal polynomials.

We provide the details of this calculation for the case where \( w \) is supported on the unit circle. The following theorem is a precise statement of this asymptotic result.

**Theorem 1.1** Suppose that \( \phi(e^{i\theta}) \) is smooth and nonzero on the unit circle with zero winding number, which admits an analytic continuation in a neighborhood of the unit circle. Let \( w = d\phi \), where \( d \) satisfies all the properties of \( \phi \) in addition to \( d(e^{i\theta})d(e^{-i\theta}) = 1 \), for all \( \theta \in [0, 2\pi] \). Then the asymptotics of

\[
h_{n-1} \equiv \frac{D_n(\phi, w; 1, 1)}{D_{n-1}(\phi, w; 1, 1)}.
\]

\(^1\) The large \( n \) behavior of the individual eigenvalues of Toeplitz matrices has been also addressed in a number of works - see \([2]\) and references therein.
\[ h_{n-1} = -\alpha(0) \frac{\mathcal{E}(n)}{\mathcal{E}(n-1)} (1 + \mathcal{O}(e^{-2cn})), \quad n \to \infty, \]  

where \( c > 0 \), and

\[ \mathcal{E}(n) = \frac{2}{\alpha(0)} R_{1,43}(0; n) - C_{\rho}(0) R_{1,23}(0; n), \]  

\[ R_{1,23}(z; n) = \frac{1}{2\pi i} \int_{\Gamma_n^1} \frac{\mu^n g_{23}(\mu)}{\mu - z} d\mu, \quad R_{1,43}(z; n) = \frac{1}{2\pi i} \int_{\Gamma_n^1} \frac{\mu^n g_{43}(\mu)}{\mu - z} d\mu, \]  

\[ g_{23}(z) = -\frac{\alpha(0)\tilde{d}(z)\beta(z)}{\tilde{\alpha}(z)}, \quad g_{43}(z) = -\alpha^2(0)\beta(z) \left( \frac{\alpha(z)}{\phi(z)} + \frac{\tilde{d}(z)C_{\rho}(z)}{\tilde{\alpha}(z)} \right), \]  

\[ C_{\rho}(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\beta^-(\tau)\beta^+(\tau)\tilde{\alpha}^-(\tau)\tilde{\alpha}^+(\tau)(\tau - z)} d\tau, \]  

and finally

\[ \alpha(z) = \exp \left[ \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln(\phi(\tau))}{\tau - z} d\tau \right], \quad \beta(z) = \exp \left[ \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln(d(\tau))}{\tau - z} d\tau \right]. \]  

In (1.13), the contour \( \Gamma_n^1 \) is a circle, oriented counter-clockwise, with radius \( r < 1 \) so that the functions \( \phi \) and \( d \) are analytic in the annulus \( \{ z : r < |z| < 1 \} \).

**Remark 1.2** The analyticity of \( \phi \) in the neighborhood of the unit circle is a technical condition. It can be lifted and replaced by certain smoothness conditions using the approximation type arguments similar to the ones used in [13] in subsection 6.2. We shall address this issue together with several other technical points in the forthcoming publications.

### 1.1 Outline

In section 2 we will analyze the case where Hankel symbol \( w \) is supported on the unit circle. We will propose a \( 2 \times 2 \) Riemann-Hilbert problem with a shift for the associated orthogonal polynomials. For an effective Riemann-Hilbert analysis, we will then propose a \( 2 \times 4 \) Riemann-Hilbert problem whose jump conditions could be written in the usual form of matrix multiplications. We will then formulate a \( 4 \times 4 \) Riemann-Hilbert problem which is the suitable framework for our analysis. In the formulation of the \( 4 \times 4 \) Riemann-Hilbert problem, for technical reasons, we will restrict to the particular offset values \( r = s = 1 \). Following the natural steps of steepest descent analysis we will arrive at a model problem in section 2.6 which we will refer to as the model Riemann-Hilbert problem for the pair \((\phi, w)\).

Section 3 is devoted to analysis of the case where the Hankel symbol is supported on the interval \([a, b]\), with \( 0 < a < b < 1 \). Similar to section 2 we will propose a \( 2 \times 2 \) Riemann-Hilbert problem with a shift for the associated system of orthogonal polynomials and for the same reasons mentioned above we pass through a \( 2 \times 4 \) to arrive at a suitable \( 4 \times 4 \) Riemann-Hilbert problem. In this case, our methods allow for considering an arbitrary offset for the Hankel part, more precisely, we can pursue the steepest descent analysis for \( r = 1 \) and an arbitrary \( s \in \mathbb{Z} \). This steepest descent analysis leads us to a model problem, which is the same as the model problem of section 2 except that it is for the pair \((\phi, -\tilde{u})\), where

\[ u(z) = \int_{a}^{b} \frac{x^s w(x)}{x - z} dx. \]
Although there are similarities between the steepest descent analysis of section 2 and section 3, we feel obliged to lay out a thorough exposition to illustrate the remarkable fact that the same model RH problem emerges in both cases.

In section 4, we present the factorization of the model RH problem for the pair \((\phi, d\phi)\), where the functions \(\phi\) and \(d\) are smooth and nonvanishing on the unit circle with zero winding number. The function \(d\) further satisfies \(d\tilde{d} = 1\) on the unit circle. We will then use this solution to construct the solutions to the global parametrix and the small-norm Riemann-Hilbert problems, which finally enables us to find the asymptotics of the norms \(h_n\) of the associated monic orthogonal polynomials.

Finally, in section 5 we summarize the still open technical and conceptual questions which we are going to address in our future work.

2 Toeplitz + Hankel determinants: Hankel weight supported on \(\mathbb{T}\)

In this section we assume that \(w\) is supported on the unit circle and that both symbols \(\phi\) and \(w\), admit analytic continuations to a neighborhood of the unit circle. A key observation is that the determinant (1.4) is related to the system of monic polynomials, \(\{P_n(z)\}\), determined by the orthogonality relations

\[
\int_{\mathbb{T}} P_n(z)z^{-k-r}\phi(z)\frac{dz}{2\pi i z} + \int_{\mathbb{T}} P_n(z)z^{k+s}\tilde{w}(z)\frac{dz}{2\pi i z} = h_n\delta_{n,k}, \quad k = 0, 1, \ldots, n. \tag{2.1}
\]

These polynomials exist and are unique if the Toeplitz+Hankel determinants (1.4) are non-zero. The uniqueness of the polynomial \(P_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0\) satisfying (2.1), simply follows from the fact that one has the following linear system for the coefficients \(a_j\), \(1 \leq j \leq n-1\):

\[
\begin{pmatrix}
T_{n+1}[\phi; r] + H_{n+1}[w; s] \\
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n-1} \\
1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
h_n
\end{pmatrix}
\tag{2.2}
\]

So if \(D_{n+1} \neq 0\), the coefficients \(a_j\) and hence \(P_n\), can be uniquely determined by inverting the Toeplitz+Hankel matrix in (2.2). Expectedly, the polynomials \(P_n\) can be written as the following determinants

\[
P_n(z) := \frac{1}{D_n} \det \begin{pmatrix}
\phi_r + ws \\
\phi_{r+1} + ws + 1 \\
\vdots \\
\phi_{r+n-1} + ws + n - 1 \\
1
\end{pmatrix}
\begin{pmatrix}
\phi_r + ws + 1 \\
\phi_{r+1} + ws + 2 \\
\vdots \\
\phi_{r+n-1} + ws + n - 1 \\
z
\end{pmatrix}
\begin{pmatrix}
\phi_{r-n} + ws + n \\
\phi_{r-n+1} + ws + n - 1 \\
\vdots \\
\phi_{r-n+n-1} + ws + n - 2 \\
z^{n-1}
\end{pmatrix} \tag{2.3}
\]
where \( D_n \equiv D_n(\phi, w; r, s) \). Indeed, for the polynomials defined by (2.3) we have that

\[
\int_{\mathbb{T}} P_n(z)z^{k+s}w(z) \frac{dz}{2\pi iz} = \frac{1}{D_n} \det \begin{pmatrix}
\phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\
\phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\
w_{k+s} & w_{k+s+1} & \cdots & w_{k+s+n-1} & w_{k+s+n}
\end{pmatrix},
\]

hence

\[
\int_{\mathbb{T}} P_n(z)z^{-k-r}w(z) \frac{dz}{2\pi iz} = \frac{1}{D_n} \det \begin{pmatrix}
\phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\
\phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\
w_{k+r} & w_{k+r+1} & \cdots & w_{k+r+n-1} & w_{k+r+n}
\end{pmatrix},
\]

and

\[
\int_{\mathbb{T}} P_n(z)z^{-k-r}w(z) \frac{dz}{2\pi iz} = \frac{1}{D_n} \det \begin{pmatrix}
\phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\
\phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\
w_{n+r} & w_{n+r+1} & \cdots & w_{n+r+n-1} & w_{n+r+n}
\end{pmatrix} = \frac{D_{n+1}}{D_n} \delta_{n,k},
\]

hence

\[
\int_{\mathbb{T}} P_n(z)z^{-k-r}w(z) \frac{dz}{2\pi iz} + \int_{\mathbb{T}} P_n(z)z^{k+s}w(z) \frac{dz}{2\pi iz} = \frac{1}{D_n} \det \begin{pmatrix}
\phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\
\phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\
\phi_{n+r} + w_{n+s} & \phi_{n+r+1} + w_{n+s+1} & \cdots & \phi_{n+r+n-1} + w_{n+s+n-1} & \phi_{n+r+n}
\end{pmatrix} = \frac{D_{n+1}}{D_n} \delta_{n,k}.
\]

So the polynomials defined by (2.3) are the unique polynomials satisfying (2.1), and

\[
h_n = \frac{D_{n+1}(\phi, w; r, s)}{D_n(\phi, w; r, s)}.
\]

Now, we consider the function \( \mathcal{Y} \) defined as

\[
\mathcal{Y}(z; n) = \frac{P_n(z)}{\frac{1}{h_{n-1}}P_{n-1}(z)} - \frac{1}{\frac{1}{h_{n-1}}P_{n-1}(z)} \int_{\mathbb{T}} \frac{\xi^s \bar{w}(\xi)P_n(\xi) + \xi^r \tilde{\phi}(\xi)\bar{P}_n(\xi)}{\xi - z} \frac{d\xi}{2\pi i\xi},
\]

built from the orthogonal polynomials \( P_n \) satisfying (2.1). Consider the following Riemann-Hilbert problem for finding the \( 2 \times 2 \) matrix \( \mathcal{Y} \) satisfying:

- **RH-\( \mathcal{Y} \)** \( \mathcal{Y} \) is holomorphic in \( \mathbb{C} \setminus \mathbb{T} \).
- **RH-\( \mathcal{Y} \)** For \( z \in \mathbb{T} \) we have

\[
\mathcal{Y}_+^{(1)}(z; n) = \mathcal{Y}_-^{(1)}(z; n), \quad z \in \mathbb{T},
\]

and

\[
\mathcal{Y}_+^{(2)}(z; n) = \mathcal{Y}_-^{(2)}(z; n) + z^{-1+s}w(z)\mathcal{Y}_+^{(1)}(z; n) + z^{-1+r}\tilde{\phi}(z)\mathcal{Y}_-^{(1)}(z^{-1}; n), \quad z \in \mathbb{T},
\]
• RH-$\mathcal{Y}$3  As $z \to \infty$, $\mathcal{Y}$ satisfies

$$\mathcal{Y}(z; n) = (I + O(z^{-1})) z^{|a_3} = \left( z^n + O(z^{-1}) \right) / z^{n-1} + O(z^{-n-1}) \right),$$

where $\mathcal{Y}^{(1)}$ and $\mathcal{Y}^{(2)}$ are the first and second columns of $\mathcal{Y}$ respectively. The next theorem establishes the association of the function (2.7) with the Riemann-Hilbert problem above.

**Theorem 2.1** The function $\mathcal{Y}$ given by (2.7) satisfies RH-$\mathcal{Y}$1 through RH-$\mathcal{Y}$3.

**Proof.** It is clear that RH-$\mathcal{Y}$1 is satisfied due to general properties of Cauchy integrals. From (2.8) we see that $\mathcal{Y}_{11}$ and $\mathcal{Y}_{21}$ are entire functions, and from (2.10) we know that $\mathcal{Y}_{11}$ has to be a monic polynomial of degree $n$ and $\mathcal{Y}_{21}$ has to be a polynomial of degree $n - 1$. From (2.9) and what we just mentioned about $\mathcal{Y}_{11}$ we would have

$$\mathcal{Y}_{12}(z; n) - \mathcal{Y}_{12}(z; n) = z^{-1+s \tilde{w}}(z) \mathcal{Y}_{11}(z; n) + z^{-1+r \tilde{\phi}(z) \mathcal{Y}_{11}(z; n)}. \quad (2.11)$$

So by Plemelj-Sokhotskii formula we have

$$\mathcal{Y}_{12}(z; n) = \frac{1}{2\pi i} \int_T \frac{\xi^{-1+s \tilde{w}}(\xi) \mathcal{Y}_{11}(\xi; n) + \xi^{-1+r \tilde{\phi}(\xi) \mathcal{Y}_{11}(\xi; n)} d\xi. \quad (2.12)$$

Using the identity

$$\frac{1}{\xi - z} = -\sum_{k=0}^{n} \frac{\xi^k}{z^{k+n+1}}, \quad (2.13)$$

we get

$$\mathcal{Y}_{12}(z; n) = -\sum_{k=0}^{n} \frac{1}{z^{k+n+1}} \int_T \left[ \xi^{-1+s \tilde{w}}(\xi) \mathcal{Y}_{11}(\xi; n) \xi^{k} + \xi^{-1+r \tilde{\phi}(\xi) \mathcal{Y}_{11}(\xi; n) \xi^{k}} \right] \frac{d\xi}{2\pi i}$$

$$+ \frac{1}{z^{n+1}} \int_T \left[ \xi^{-1+s \tilde{w}}(\xi) \mathcal{Y}_{11}(\xi; n) + \xi^{-1+r \tilde{\phi}(\xi) \mathcal{Y}_{11}(\xi; n)} \right] \frac{d\xi}{2\pi i}. \quad (2.14)$$

Note that since $\mathcal{Y}_{12}(z; n) = O(z^{-n-1})$, we must have :

$$\int_T \tilde{w}(\xi) \mathcal{Y}_{11}(\xi; n) \xi^{k+s} \frac{d\xi}{2\pi i \xi} + \int_T \tilde{\phi}(\xi) \mathcal{Y}_{11}(\xi; n) \xi^{k+r} \frac{d\xi}{2\pi i \xi} = 0, \quad 0 \leq k \leq n - 1. \quad (2.15)$$

In the second integral we make the change of variable $\xi \mapsto \tau := \xi^{-1}$ and as a result we will arrive at

$$\int_T \mathcal{Y}_{11}(\xi; n) \xi^{k+s} \tilde{w}(\xi) \frac{d\xi}{2\pi i \xi} + \int_T \mathcal{Y}_{11}(\tau; n) \tau^{-k-r} \phi(\tau) \frac{d\tau}{2\pi i \tau} = 0, \quad 0 \leq k \leq n - 1. \quad (2.16)$$

Since $\mathcal{Y}_{11}$ satisfies the orthogonality relations (2.1) we necessarily have

$$\mathcal{Y}_{11}(z; n) = \mathcal{P}_n(z). \quad (2.17)$$

In a similar fashion one can show that

$$\mathcal{Y}_{22}(z; n) = \frac{1}{2\pi i} \int_T \frac{\xi^{-1+s \tilde{w}}(\xi) \mathcal{Y}_{21}(\xi; n) + \xi^{-1+r \tilde{\phi}(\xi) \mathcal{Y}_{21}(\xi; n)}}{\xi - z} \frac{d\xi}{2\pi i \xi} \quad (2.18)$$

and

$$\mathcal{Y}_{21}(z; n) = -\frac{1}{h_{n-1}} \mathcal{P}_{n-1}(z). \quad (2.19)$$
It is important to notice that if the solution to the \( \mathcal{Y}^{-}\text{RHP} \) exists, it is unique, because \( \mathcal{Y}_{ij}, i,j = 1,2 \) are all uniquely identified with the unique orthogonal polynomials satisfying the orthogonality conditions (2.21). Also note that

\[
\lim_{z \to \infty} z^n Y_{21}(z; n+1) = -h_n.
\]

As in the pure Toeplitz or Hankel cases, this formula in conjunction with (2.6) reduces the asymptotic analysis of the Toeplitz+Hankel determinants to the asymptotic analysis of the \( \mathcal{Y}^{-}\text{Riemann-Hilbert} \) problem.

2.1 The associated \( 2 \times 4 \) and \( 4 \times 4 \) Riemann-Hilbert problems

In the rest of this section we will develop a \( 4 \times 4 \) analogue of the Deift/Zhou non-linear steepest descent method for the Toeplitz+Hankel determinants (1.4). For technical reasons that will be elaborated later, we will focus on the case where \( r = s = 1 \). We are positive that our method has the capacity to allow for analyzing general values of \( r \) and \( s \) and we will briefly discuss the prospects of such possible generalizations in section 5.2. We shall only consider the so-called Szegő-type smooth symbols \( \phi \) and \( w \), assuming their analyticity in a neighborhood of the unit circle.

The \( \mathcal{Y}^{-}\text{RHP} \) is a particular case of the matrix Riemann-Hilbert problem with a shift, or the matrix analytical boundary problem of the Carleman type. Indeed, the matrix form of the equations (2.8) - (2.9) reads as follows,

\[
\mathcal{Y}_{\pm}(z; n) = \mathcal{Y}_{\mp}(z; n) G_1(z) + \mathcal{Y}_{\mp}(\kappa(z); n) G_2(z),
\]

where

\[
G_1(z) = \begin{pmatrix} 1 & \tilde{w}(z) \\ 0 & 1 \end{pmatrix}, \quad G_2(z) = \begin{pmatrix} 0 & \tilde{\phi}(z) \\ 0 & 0 \end{pmatrix},
\]

and the “shift” \( \kappa \) is the mapping

\[
\kappa(z) = \frac{1}{z}.
\]

The presence of the shift makes it impossible to directly apply the usual \( 2 \times 2 \) version of the Deift-Zhou nonlinear steepest descent method to the \( \mathcal{Y}^{-}\text{RHP} \). However, the mapping \( \kappa \) satisfies the Carleman condition, \( \kappa(\kappa(z)) = z \), and hence we can translate the \( 2 \times 2 \) \( \mathcal{Y}^{-}\text{RHP} \) to the usual matrix form by doubling the relevant matrix sizes. More precisely, we first propose the associated \( 2 \times 4 \) and then the associated \( 4 \times 4 \) Riemann-Hilbert problems. Although more complicated, the analysis of the proposed \( 4 \times 4 \) Riemann-Hilbert problem follows in the same spirit as the lower dimensional RHPs until we get to the model Riemann-Hilbert problem for Toeplitz+Hankel determinants introduced in section 2.6.

Let us define the \( 2 \times 4 \) matrix \( \tilde{\mathcal{X}} \) out of the columns of \( \mathcal{Y} \) as follows

\[
\tilde{\mathcal{X}}(z; n) := \begin{pmatrix} \mathcal{Y}^{(1)}(z; n), \tilde{\mathcal{Y}}^{(1)}(z; n), \mathcal{Y}^{(2)}(z; n), \tilde{\mathcal{Y}}^{(2)}(z; n) \end{pmatrix},
\]

From (2.8), (2.9) and (2.10) we obtain the following Riemann-Hilbert problem for \( \tilde{\mathcal{X}} \):

- \textbf{RH-}\( \tilde{\mathcal{X}} \text{1} \) \( \tilde{\mathcal{X}} \) is holomorphic in \( \mathbb{C} \setminus (\mathbb{T} \cup \{0\}) \).
- \textbf{RH-}\( \tilde{\mathcal{X}} \text{2} \) For \( z \in \mathbb{T} \), \( \tilde{\mathcal{X}} \) satisfies

\[
\tilde{\mathcal{X}}_{+}(z; n) = \tilde{\mathcal{X}}_{-}(z; n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
\[ \mathcal{K}(z; n) = \begin{pmatrix} 1 + O(z^{-1}) & C_1(n) + O(z^{-1}) & O(z^{-1}) & C_3(n) + O(z^{-1}) \\ C_2(n) + O(z^{-1}) & 1 + O(z^{-1}) & C_4(n) + O(z^{-1}) \end{pmatrix} \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

- \textbf{RH-1'} As \( z \to \infty \) we have

\[ \mathcal{K}(z; n) = \begin{pmatrix} C_1(n) + O(z) & 1 + O(z) & C_3(n) + O(z) & O(z) \\ C_2(n) + O(z) & O(z) & C_4(n) + O(z) & 1 + O(z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \]

- \textbf{RH-1''} As \( z \to 0 \) we have

\[ \mathcal{K}(z; n) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

- \textbf{RH-3'} As \( z \to \infty \) we have

\[ \mathcal{K}(z; n) = P(n)(I + O(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \]

\textbf{Remark 2.2} The uniqueness of the solution of \( \mathcal{K} \)-RHP is established using the standard Liouville theorem-based arguments. We also note that the matrix factor \( P(n) \) in (2.29) is not a priori prescribed.

\textbf{Remark 2.3} It is easy to see that the solution \( \mathcal{K}(z; n) \) of the \( \mathcal{K} \) - RHP satisfies the symmetry relation,

\[ (\sigma_1 \otimes I_2) P^{-1}(n) \mathcal{K}(z^{-1}; n) (\sigma_1 \otimes I_2) = \mathcal{K}(z; n), \]

which in turn yields the following symmetry equation for \( P(n) \),

\[ P(n) = (\sigma_1 \otimes I_2) P^{-1}(n) (\sigma_1 \otimes I_2). \]
2.2 Relation of the $2 \times 4$ and the $4 \times 4$ Riemann-Hilbert problems

Put

$$R(z; n) := X(X^{-1}(z; n)).$$

(2.32)

From (2.23) and (2.27) it is clear that $R$ has no jumps. From (3.12) and (2.29) we can obtain the behavior of $R$ near zero:

$$R(z; n) = (C_1(n) + O(z) \ 0 \ \ 0 \ \ 0) + O(z) = (C_2(n) + O(z) \ 0 \ \ 0 \ \ 0).$$

(2.33)

Therefore $R$ is an entire function. Also note that from (2.24) and (2.28) we have

$$R(z; n) = \begin{pmatrix} 1 \ C_1(n) \ 0 \ 0 \\ 0 \ C_2(n) \ 1 \ C_3(n) \\ 0 \ 0 \ 1 \ C_4(n) \\ 0 \ 0 \ 0 \ 1 \end{pmatrix} P^{-1}(n),$$

(2.34)

Taking into account the symmetry equation (2.31), this is a well-defined linear system on $C_j(n)$ which is generically uniquely solvable.

Once we asymptotically solve the $X$-RHP, the large-$n$ asymptotic expression for $P(n)$ can be found from

$$P(n) = X(X^{-1}(z; n)) \begin{pmatrix} 1 \ 0 \ 0 \ 0 \\ 0 \ z^n \ 0 \ 0 \\ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ z^{-n} \end{pmatrix} \bigg|_{z=0},$$

(2.37)

which enables us to find asymptotic expressions for the constants $C_i$, $1 \leq i \leq 4$ via (2.36). This allows for construction of the asymptotic solution to the $X$-RHP through (2.32).

2.3 The primary opening of the lenses

Let us consider the contour $\Gamma := \Gamma_i \cup \bar{\Delta} \cup \Gamma_o$ shown in Figure 1. Define the function $Z$ as

$$Z(z; n) := X(z; n) \begin{pmatrix} J_{X, i}^{-1}(z), & z \in \Omega_1, \\ J_{X, o}(z), & z \in \Omega_2, \\ I, & z \in \Omega_0 \cup \Omega_\infty, \end{pmatrix}$$

(2.38)

where $J_{X, i}$ and $J_{X, o}$ are defined in the following factorization for the jump matrix of the $X$-RHP, which we denote by $J_X$:
Figure 1: The jump contour $\Gamma$ for the $\mathcal{Z}$, $T$ and the global parametrix Riemann-Hilbert Problems

\[
J_X(z) := \begin{pmatrix}
1 & 0 & \tilde{w}(z) & -\phi(z) \\
0 & 1 & \tilde{\phi}(z) & -w(z) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -w(z) \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & \tilde{w}(z) & 0 \\
1 & \tilde{\phi}(z) & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\equiv J_{X,o}(z)J_{X,T}(z)J_{X,i}(z).
\]

(2.39)

We remind that the symbol $w$, and hence $\tilde{w}$, are analytic in a neighborhood of $T$ which is supposed to include the domains $\Omega_1$ and $\Omega_2$.

The function $\mathcal{Z}$ satisfies the following Riemann-Hilbert problem:

- **RH-Z1** $\mathcal{Z}$ is holomorphic in $\mathbb{C} \setminus (T \cup \{0\})$.

- **RH-Z2** $\mathcal{Z}_+(z;n) = \mathcal{Z}_-(z;n)J_\mathcal{Z}(z)$, where

\[
J_\mathcal{Z}(z) = \begin{cases}
J_{X,T}(z), & z \in T, \\
J_{X,i}(z), & z \in \Gamma_i, \\
J_{X,o}(z), & z \in \Gamma_o.
\end{cases}
\]

(2.40)

- **RH-Z3** As $z \to \infty$ we have

\[
\mathcal{Z}(z;n) = (I + O(z^{-1})) \begin{pmatrix}
z^n & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & z^{-n} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(2.41)

- **RH-Z4** As $z \to 0$ we have

\[
\mathcal{Z}(z;n) = P(n)(I + O(z)) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & z^{-n} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^n
\end{pmatrix}.
\]

(2.42)
Remark 2.4 The term "opening of the lenses" is usually used to describe situations where the jump matrix on the added contours is exponentially close to the identity matrix for large values of the parameter $n$. The passage $T \mapsto \Gamma$, corresponding to the RH transformation $X \mapsto Z$, is clearly not of this type. However, our secondary opening of the lenses (the passage $\Gamma \mapsto \Gamma_S$ which corresponds to the RH transformation $T \mapsto S$) in section 2.4 is an example of a usual opening of the lenses.

Remark 2.5 The primary opening of the lenses is essential for the progression of the RH analysis in the following sections. This is due to a technical reason that will be elaborated at the end of next section. Since the structure of jump matrices is different in section 3, we do not have an analogous step when the Hankel symbol is supported on $[a, b]$, $0 < a < b < 1$.

2.4 Normalization of behaviours at 0 and $\infty$

Following the natural steps of Riemann-Hilbert analysis, we will normalize the behavior of $Z$ at 0 and $\infty$; to this end let us define

\[
T(z; n) := Z(z; n) = \begin{cases} 
\begin{pmatrix} z^{-n} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & z^n & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix}, & |z| > 1, \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & z^n & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^{-n} 
\end{pmatrix}, & |z| < 1. 
\end{cases} \tag{2.43}
\]

It is very important to note that in order to have a suitable Riemann-Hilbert analysis, the normalization of behaviors at 0 and $\infty$ can only be carried out only after the undressing $X \mapsto Z$; this is due to technical reasons that will be further commented about at the end of this section. We have the following RHP for $T$:

- **RH-T1** $T$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \Gamma_i \cup \Gamma_o)$.
- **RH-T2** $T_+(z; n) = T_-(z; n)J_T(z; n)$, where

\[
J_T(z; n) = \begin{cases} 
\hat{J}(z; n), & z \in \mathbb{T}, \\
J_{X,i}(z), & z \in \Gamma_i, \quad \text{where} \quad \hat{J}(z; n) = \begin{pmatrix} z^n & 0 & 0 & -\phi(z) \\
0 & z^n & \phi(z) & 0 \\
0 & 0 & z^{-n} & 0 \\
0 & 0 & 0 & z^{-n} 
\end{pmatrix}, \tag{2.44}
\end{cases}
\]

and the matrices $J_{X,i}$ and $J_{X,o}$ are defined by the

- **RH-T3** As $z \to \infty$, we have $T(z; n) = (I + O(z^{-1}))$.

We observe that for $z \in \mathbb{T}$, $J_T$ can be factorized as follows

\[
\hat{J}(z; n) = \begin{pmatrix} I_2 \\
z^{-n}\Phi^{-1}(z) \\
0_2 \\
-I_2 \end{pmatrix} \begin{pmatrix} 0_2 \\
\Phi(z) \\
-I_2 \\
z^n\Phi^{-1}(z) \end{pmatrix} \equiv J_{T,o}(z; n)J(z)J_{T,i}(z; n), \tag{2.45}
\]

where $0_2$ and $I_2$ are respectively $2 \times 2$ zero and identity matrices and

\[
\Phi(z) = \begin{pmatrix} 0 & -\phi(z) \\
\phi(z) & 0 \end{pmatrix}. \tag{2.46}
\]
Note that $J_{T,i}$ is exponentially close to the identity matrix for $z$ inside of the unit circle and $J_{T,o}$ is exponentially close to the identity matrix for $z$ outside of the unit circle.

Now we are in a position to address remark $2.44$ in the previous section. Indeed, if one normalizes the behaviors at $0$ and $\infty$ without the undressing transformation $X \mapsto Z$; i.e. by directly defining the function $\mathcal{T}$ as

$$
\mathcal{T}(z; n) := \mathcal{X}(z; n) \begin{cases}
(z^n 0 0 0) & , |z| > 1, \\
(0 1 0 0) & , |z| < 1.
\end{cases} \tag{2.47}
$$

then the jump matrix $J_T := T_n^{-1}T_+^-$ on the unit circle would be

$$
J_T(z; n) = \begin{pmatrix}
z^n & 0 & z^n w(z) & -\phi(z) \\
0 & z^n & \phi(z) & -z^{-n} w(z) \\
0 & 0 & z^{-n} & 0 \\
0 & 0 & 0 & z^{-n}
\end{pmatrix}, \tag{2.48}
$$

for which finding a factorization like $2.45$ remains a challenge, mainly due to presence of the large parameter $n$ in the 13 and 24 elements of $J_T$. This fact justifies the necessity of the undressing step $\mathcal{X} \mapsto \mathcal{Z}$. Indeed, due to the specific matrix structure of the jump matrices $J_{X,i}$ and $J_{X,o}$, they do not change under the transformation $2.43$.

### 2.5 The secondary opening of the lenses

The next Riemann-Hilbert transformation $T \mapsto S$, provides us with a problem with jump conditions on five contours where three jump matrices do not depend on $n$ and the other two converge exponentially fast to the identity matrix as $n \to \infty$. Let us define the function $S$, suggested by $2.45$, as

$$
S(z; n) := T(z; n) \times \begin{cases}
J_{T,i}^{-1}(z; n), & z \in \Omega'_1, \\
J_{T,o}(z; n), & z \in \Omega'_2, \\
1, & z \in \Omega''_1 \cup \Omega''_2 \cup \Omega_0 \cup \Omega_\infty,
\end{cases} \tag{2.49}
$$

where the regions $\Omega'_1$, $\Omega'_2$, $\Omega''_1$ and $\Omega''_2$ are shown in Figure $2$. we have the following Riemann-Hilbert problem for $S$

- **RH-S1** $S$ is holomorphic in $\mathbb{C} \setminus (\mathcal{T} \cup \Gamma_i \cup \Gamma_o \cup \Gamma'_i \cup \Gamma'_o)$.
- **RH-S2** $S_+(z; n) = S_-(z; n)JS(z; n)$, where

$$
JS(z; n) = \begin{pmatrix}
J(z), & z \in \mathbb{T}, \\
J_{T,i}(z; n), & z \in \Gamma'_i, \\
J_{T,o}(z; n), & z \in \Gamma'_o, \\
J_{X,i}(z), & z \in \Gamma_i, \\
J_{X,o}(z), & z \in \Gamma_o.
\end{pmatrix} \tag{2.50}
$$
• **RH-S3** As \( z \to \infty \), we have \( S(z; n) = I + \mathcal{O}(z^{-1}) \).

In the usual way, we will first try to solve this Riemann-Hilbert problem by disregarding the jump matrices which depend on \( n \), this solution is denoted by \( \hat{S} \) and will be referred to as the global parametrix. Once we construct the global parametrix, we will consider the small-norm Riemann-Hilbert problem for the ratio \( R := S(\hat{S})^{-1} \) and discuss its solvability in the forthcoming sections.

2.6 The global parametrix and the model Riemann-Hilbert problem for the pair \((\phi, w)\)

The \( S \)-RHP reduces to the following Riemann-Hilbert problem for the global parametrix \( \hat{S} \), when we ignore the jump matrices which are exponentially close to the identity matrix:

- **RH-\( \hat{S} \)1** \( \hat{S} \) is holomorphic in \( \mathbb{C} \setminus (T \cup \Gamma_i \cup \Gamma_o) \).

- **RH-\( \hat{S} \)2** \( \hat{S}_+(z) = \hat{S}_-(z)J_{\hat{S}}(z) \), where

\[
J_{\hat{S}}(z) = \begin{cases}
  J(z), & z \in T, \\
  J_{\chi,i}(z), & z \in \Gamma_i, \\
  J_{\chi,o}(z), & z \in \Gamma_o.
\end{cases}
\]  

(2.51)

- **RH-\( \hat{S} \)3** As \( z \to \infty \), we have \( \hat{S}(z) = I + \mathcal{O}(z^{-1}) \).

And we finally dress the \( \hat{S} \)-RHP to obtain a model problem for the global parametrix having jumps only on the unit circle. We define the function \( \Lambda \) as
Λ(\(z\)) := \begin{cases} J_{\Lambda_1}(z), & z \in \Omega_1, \\ J_{\Lambda_2}(z), & z \in \Omega_2, \\ I, & z \in \Omega_0 \cup \Omega_{\infty}. \end{cases}

(2.52)

Now we arrive at the following Riemann-Hilbert problem for Λ that from now on we will refer to as the model Riemann-Hilbert problem for the pair \((\phi, w)\):

- **RH-Λ1**  \(\Lambda\) is holomorphic in \(\mathbb{C} \setminus \mathbb{T}\).
- **RH-Λ2** \(\Lambda_+ (z) = \Lambda_-(z) J_\Lambda(z)\), for \(z \in \mathbb{T}\), where

\[
J_\Lambda(z) = \begin{pmatrix}
0 & 0 & -\phi(z) & 0 \\
-w(z) & 0 & -\phi(z) & 0 \\
0 & -\frac{1}{\phi(z)} & 0 & 0 \\
\frac{1}{\phi(z)} & 0 & -\bar{w}(z) & 0 \\
\end{pmatrix}.
\]

(2.53)

- **RH-Λ3** As \(z \to \infty\), we have \(\Lambda(z) = I + O(z^{-1})\).

The conditions on \(w\) and \(\phi\) which ensure the solvability of this model problem are not completely known and categorized at this point. We also want to stress that the appearance of the 4 \(\times\) 4 model Λ-problem in the asymptotic analysis of the original \(X\)-RHP is the crucial difference of the Toeplitz+Hankel case we consider in this work comparing to the pure Toeplitz or pure Hankel or Toeplitz +Hankel with the same symbols cases. Indeed, even if the pair \((\phi, w)\) is such that the Λ-RHP is solvable it does not mean that it is explicitly solvable. Hence, one should not expect the closed form of the asymptotic answer in the case of the generic pair \((\phi, w)\). However, in section 4 we will present a detailed analysis of this model problem for a specific family of pairs \((\phi, w)\) within the broader class of Toeplitz and Hankel weights considered by E. Basor and T. Ehrhardt in \([6]\) for which the model Λ-problem is explicitly solvable.

Remarkably, we arrive at the same model Λ-Riemann-Hilbert problem, if we start with a Hankel weight supported on the interval \([a, b]\), \(0 < a < b < 1\). This will be shown in the next section.

### 3  Toeplitz + Hankel determinants: Hankel weight supported on the interval \([a, b]\), \(0 < a < b < 1\).

Let us consider the determinant \((3.1)\) where \(w_k\) are given by \((1.3)\) with \(I = [a, b]\), \(0 < a < b < 1\). The Riemann-Hilbert approach outlined in this section can be naturally extended to the three other cases: i) \(-1 < a < b < 0\), ii) \(-\infty < a < b < -1\), and iii) \(1 < a < b < \infty\). We consider the system of orthogonal polynomials \(\{P_n(z)\}\), \(\deg P_n(z) = n\), satisfying the following orthogonality conditions

\[
\int_a^b P_n(x) x^{k+s} w(x) dx + \int_{\mathbb{T}} P_n(z) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} = \delta_{n,k}, \quad k = 0, 1, \ldots, n.
\]

(3.1) 

In this respect the Toeplitz+Hankel determinants with generic symbols are similar to the block Toeplitz determinants, where the explicit answers can be obtained only in two cases: (a) the Fourier expansion of the corresponding matrix symbol is one side truncated or (b) one can produce an explicit Wiener-Hopf factorization of the symbol.
One can write a determinantal formula for $P_n$, like in (3.3), which yields

$$h_n = \frac{D_{n+1}(\phi, w; r, s)}{D_n(\phi, w; r, s)}.$$  

(3.2)

Now, we consider the function $Y$ defined as

$$Y(z; n) = \begin{pmatrix} P_n(z) \\ -\frac{1}{h_{n-1}}P_{n-1}(z) \end{pmatrix} \begin{pmatrix} \int_a^b \frac{P_n(x) x^s w(x)}{x-z} dx + \int_T \frac{\tilde{\phi}(\xi) \xi^s \tilde{P}_n(\xi)}{\xi-z} \frac{d\xi}{2\pi i\xi} \\ \int_a^b \frac{P_{n-1}(x) x^s w(x)}{x-z} dx + \int_T \frac{\phi(\xi) \xi^s \tilde{P}_{n-1}(\xi)}{\xi-z} \frac{d\xi}{2\pi i\xi} \end{pmatrix},$$  

(3.3)

built from the orthogonal polynomials $P_n$ satisfying (3.1). Consider the following Riemann-Hilbert problem for finding the $2 \times 2$ matrix $Y$ satisfying

- **RH-Y1** $Y$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup [a, b])$.
- **RH-Y2** For $z \in \mathbb{T}$ we have
  $$Y_+^{(1)}(z; n) = Y_-^{(1)}(z; n),$$  

(3.4)

and

$$Y_+^{(2)}(z; n) = Y_-^{(2)}(z; n) + z^{r-1}\phi(z)Y_-^{(1)}(z^{-1}; n).$$  

(3.5)

- **RH-Y3** For $x \in (a, b)$ we have
  $$Y_+^{(1)}(x; n) = Y_-^{(1)}(x; n),$$  

(3.6)

and

$$Y_+^{(2)}(x; n) = Y_-^{(2)}(x; n) + 2\pi i x^s w(x)Y_-^{(1)}(x; n).$$  

(3.7)

- **RH-Y4** As $z \to \infty$
  $$Y(z) = \left( I + O\left(\frac{1}{z}\right) \right) z^{n\sigma_3} = \begin{pmatrix} z^n + O(z^{n-1}) & O(z^{n-1}) \\ O(z^{-n}) & z^{-n} + O(z^{-n-1}) \end{pmatrix}.$$  

(3.8)

where $Y^{(1)}$ and $Y^{(2)}$ are the first and second columns of $Y$, respectively. We have the analogue of Theorem 2.1 here as well.

**Theorem 3.1** The function $Y$ given by (3.3) satisfies RH-Y1 through RH-Y4.

We omit the proof here as it is similar to the proof of Theorem 2.1

### 3.1 The associated $2 \times 4$ and $4 \times 4$ Riemann-Hilbert problems

In this section we will consider $D_n(\phi, w; 1, s)$ for technical reasons that will be elaborated later and again assume that the symbol $\phi$ is analytic in a neighborhood of $\mathbb{T}$. The formulation of the $2 \times 4$ and $4 \times 4$ Riemann-Hilbert problems are very similar to those of section 2.1, however there are minor differences that convinces us to practice clarity in our exposition. Let us consider the following $2 \times 4$ matrix function, constructed from the columns of $Y$ given by (3.3):

$$\tilde{\mathcal{X}}(z; n) := \begin{pmatrix} Y^{(1)}(z; n), \tilde{Y}^{(1)}(z; n), Y^{(2)}(z; n), \tilde{Y}^{(2)}(z; n) \end{pmatrix}.$$  

(3.9)

Let us define $\Sigma := \mathbb{T} \cup [a, b] \cup [b^{-1}, a^{-1}]$, and $\Sigma' := \Sigma \setminus \{a, b, b^{-1}, a^{-1}\}$. $\tilde{\mathcal{X}}(z; n)$ satisfies the following Riemann-Hilbert problem
Figure 3: The jump contour $\Sigma$. 

- **RH-$\hat{X}$1**  
  $\hat{X}$ is analytic in $\mathbb{C} \setminus (\Sigma \cup \{0\})$, 

- **RH-$\check{X}$2**  
  For $z \in \Sigma'$, we have $\check{X}(z; n) = \check{X}(z; n)J_X(z)$, where 
  $$ 
  J_X(z) = \begin{cases} 
  \begin{pmatrix} 
  1 & 0 & 0 & -\phi(z) \\
  0 & 1 & \phi(z) & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 
  \end{pmatrix}, & z \in T, \\
  \begin{pmatrix} 
  1 & 0 & 2\pi i x^s w(x) & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 
  \end{pmatrix}, & z \equiv x \in (a, b), \\
  \begin{pmatrix} 
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 \ -2\pi i x^{-s} \check{w}(x) \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 
  \end{pmatrix}, & z \equiv x \in (b^{-1}, a^{-1}). 
  \end{cases} 
  $$ (3.10)

- **RH-$\hat{X}$3**  
  As $z \to \infty$ 
  $$ 
  \hat{X}(z; n) = \begin{pmatrix} 
  1 + O(z^{-1}) & E_1(n) + O(z^{-1}) & O(z^{-1}) & E_3(n) + O(z^{-1}) \\
  O(z^{-1}) & E_2(n) + O(z^{-1}) & 1 + O(z^{-1}) & E_4(n) + O(z^{-1}) 
  \end{pmatrix} \begin{pmatrix} 
  z^n & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & z^{-n} & 0 \\
  0 & 0 & 0 & 1 
  \end{pmatrix}. 
  $$ (3.11)

- **RH-$\check{X}$4**  
  As $z \to 0$ 
  $$ 
  \check{X}(z; n) = \begin{pmatrix} 
  E_1(n) + O(z) & 1 + O(z) & E_3(n) + O(z) & O(z) \\
  E_2(n) + O(z) & O(z) & E_4(n) + O(z) & 1 + O(z) 
  \end{pmatrix} \begin{pmatrix} 
  1 & 0 & 0 & 0 \\
  0 & z^{-n} & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & z^n 
  \end{pmatrix}, 
  $$ (3.12) 

where

$$
E_1(n) = Y_{11}(0; n), \quad E_3(n) = Y_{12}(0; n), \quad E_2(n) = Y_{21}(0; n), \quad E_4(n) = Y_{22}(0; n). 
$$ (3.13)
It is straightforward to check that \( \tilde{X} \) given by (3.9) and (3.3) satisfies the Riemann-Hilbert problem \( RH-\tilde{X}_1 \) through \( RH-\tilde{X}_4 \).

Similar to our approach in section 2.1, we introduce the following Riemann-Hilbert problem of finding the \( 4 \times 4 \) matrix function \( X \) satisfying:

- **RH-X1** \( X \) is analytic in \( \mathbb{C} \setminus (\Sigma \cup \{0\}) \).
- **RH-X2** For \( z \in \Sigma' \), we have \( X_4(z;n) = X_-(z;n)J_X(z) \), where \( J_X \) is given by (3.10).
- **RH-X3** As \( z \to \infty \)
  \[
  X(z;n) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.14}
  \]
- **RH-X4** As \( z \to 0 \)
  \[
  X(z;n) = Q(n)(I + \mathcal{O}(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \tag{3.15}
  \]

Exact similar arguments as in section 2.2 can be used to show that the solutions to the \( X \)-RHP and the \( \tilde{X} \)-RHP are related by

\[
\tilde{X}(z;n) = \begin{pmatrix} 1 & E_1(n) & 0 & E_3(n) \\ 0 & E_2(n) & 1 & E_4(n) \end{pmatrix} X(z;n), \tag{3.16}
\]

and moreover,

\[
\begin{pmatrix} 1 & E_1(n) & 0 & E_3(n) \\ 0 & E_2(n) & 1 & E_4(n) \end{pmatrix} = \begin{pmatrix} E_1(n) & 1 & E_3(n) & 0 \\ E_2(n) & 0 & E_4(n) & 1 \end{pmatrix} Q^{-1}(n). \tag{3.17}
\]

### 3.2 Normalization of behaviors at 0 and \( \infty \)

Unlike the situation in section 2 where we had to make the transformation \( \mathcal{X} \to \mathcal{Z} \) before normalization of behaviors at zero and infinity, when the Hankel symbol is supported on the interval \( [a, b] \) we can immediately normalize the asymptotic behaviors at 0 and infinity, due to the desired structure of jump matrices. Indeed, it is natural to define

\[
T(z;n) := X(z;n)
\]

for

\[
\begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad |z| > 1,
\]

and

\[
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, \quad |z| < 1.
\]

The function \( T \) satisfies the following Riemann-Hilbert problem:
Figure 4: The jump contour $\Sigma_S \equiv \Sigma \cup \Sigma_o \cup \Sigma_i$ of the S-RHP.

- **RH-T1** $T$ is holomorphic in $\mathbb{C} \setminus \Sigma$.

- **RH-T2** For $z \in \Sigma'$, we have $T_+(z;n) = T_-(z;n)J_T(z;n)$, where

$$J_T(z;n) = \begin{cases} \hat{J}(z;n), & z \in \mathbb{T}, \\ J_X(z), & z \in (a,b) \cup (b^{-1},a^{-1}). \end{cases}$$  \hfill (3.19)

We recall that $\hat{J}$ is given by (2.44) and the matrices $J_X$ for $z \in (a,b)$ and $z \in (b^{-1},a^{-1})$ are given by (3.10).

- **RH-T3** As $z \to \infty$, we have $T(z;n) = (I + O(z^{-1}))$.

We bring the reader’s attention to the fact that the transformation (3.18) does not change the jump matrices $J_X$.

### 3.3 Opening of the lenses

Using (2.45), we open the lenses off the unit circle as shown in the Figure 4 and we define

$$S(z;n) := T(z;n) \times \begin{cases} J_{T,i}^{-1}(z;n), & z \in \Omega_1, \\ J_{T,o}(z;n), & z \in \Omega_2, \\ I, & z \in \mathbb{C} \setminus (\Omega_1 \cup \Omega_2 \cup [a,b] \cup [b^{-1},a^{-1}]). \end{cases}$$  \hfill (3.20)

where $J_{T,i}$ and $J_{T,o}$ are defined in (2.45). Let $\Sigma_S \equiv \Sigma \cup \Sigma_o \cup \Sigma_i$ and $\Sigma'_S \equiv \Sigma' \cup \Sigma_o \cup \Sigma_i$ (see Figure 4). It is straightforward to check that $S$ satisfies the following Riemann-Hilbert problem

- **RH-S1** $S$ is holomorphic in $\mathbb{C} \setminus \Sigma_S$.

- **RH-S2** For $z \in \Sigma'_S$ we have $S_+(z;n) = S_-(z;n)J_S(z;n)$, where

$$J_S(z;n) = \begin{cases} J(z), & z \in \mathbb{T}, \\ J_{T,i}(z;n), & z \in \Sigma_i, \\ J_{T,o}(z;n), & z \in \Sigma_o, \\ J_X(z), & z \in (a,b) \cup (b^{-1},a^{-1}), \end{cases}$$  \hfill (3.21)

where these matrices are defined in (2.45) and (3.10).
3.4 The global parametrix and a model Riemann-Hilbert problem

Let us consider the following Riemann-Hilbert for \( \hat{S} \), or the global parametrix, which is expected to be a good approximation to \( S \) for large parameter \( n \). This RHP is simply obtained from the \( S \)-RHP by ignoring the jumps on \( \Sigma_i \) and \( \Sigma_o \):

- **RH-\( \hat{S} \)\(_1\)** \( \hat{S} \) is holomorphic in \( \mathbb{C} \setminus \Sigma \), (see Figure 3).
- **RH-\( \hat{S} \)\(_2\)** For \( z \in \Sigma' \), we have \( \hat{S}_+(z) = \hat{S}_-(z) J_{\hat{S}}(z) \), where
  \[
  J_{\hat{S}}(z) = \begin{cases} 
  \hat{J}(z), & z \in T, \\
  \hat{J}_X(z), & z \in (a, b) \cup (b^{-1}, a^{-1}).
  \end{cases}
  \] (3.22)
- **RH-\( \hat{S} \)\(_3\)** As \( z \to \infty \), we have \( \hat{S}(z) = I + O(z^{-1}) \).

Let \( u \) be defined by

\[
\begin{align*}
  u(z) := & \int_a^b \frac{t^s w(t)}{t - z} \, dt.
\end{align*}
\] (3.23)

The Plemelj-Sokhotskii formula implies that

\[
\begin{align*}
  u_+(x) - u_-(x) &= 2\pi i x^s w(x), & x \in (a, b), \\
  \tilde{u}_+(x) - \tilde{u}_-(x) &= -2\pi i x^{-s} \tilde{w}(x), & x \in (b^{-1}, a^{-1}).
\end{align*}
\] (3.24)

Put

\[
\begin{align*}
  \Theta(z) := & \hat{S}(z) \begin{cases} 
  1 & -u(z) & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  1 & 0 & 0 \\
  0 & 1 & -\tilde{u}(z) \\
  0 & 0 & 1 \\
  0 & 0 & 0
  \end{cases}
  , & \quad |z| < 1, \\
  \begin{cases} 
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & \tilde{u}(z) \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0
  \end{cases}
  , & \quad |z| > 1.
\end{align*}
\] (3.25)

It can be checked that \( \Theta \) does not have jumps on the intervals \( (a, b) \) and \( (b^{-1}, a^{-1}) \). We have arrived at the following *model Riemann-Hilbert problem* on the unit circle:

- **RH-\( \Theta \)\(_1\)** \( \Theta \) is holomorphic in \( \mathbb{C} \setminus T \).
- **RH-\( \Theta \)\(_2\)** \( \Theta_+(z) = \Theta_-(z) J_{\Theta}(z) \), for \( z \in T \), where

\[
J_{\Theta}(z) = \begin{pmatrix} 
  0 & 0 & -\phi(z) \\
  \frac{\tilde{u}(z)}{\phi(z)} & 0 & 0 \\
  0 & -\frac{1}{\phi(z)} & 0 \\
  0 & \frac{\tilde{u}(z)}{\phi(z)} & 0 \\
  0 & 0 & \frac{u(z)}{\phi(z)}
\end{pmatrix}.
\] (3.26)
RH-Θ3  As $z \to \infty$, we have $\Theta(z) = I + O(z^{-1})$.

Recalling section 2.6, we note that this is exactly the model Riemann-Hilbert problem for the pair $(\phi, \tilde{\tilde{u}})$. Hence, it can be concluded that the study of the Toeplitz+Hankel determinants both when the Hankel symbol is supported on the unit circle and also when it is supported on the interval $[a, b]$, reduces to the study of the model Riemann-Hilbert problem RH-A1 through RH-A3. For a specific choice of the symbols $\phi$ and $w$, we will present the solution to the model Riemann-Hilbert problem for the pair $(\phi, w)$ in the next section.

## 4 Analysis of the model problem and a solvable pair

As mentioned before, it is an ambitious task to classify all the pairs $(\phi, w)$ for which the $\Lambda$-model Riemann-Hilbert problem is solvable. However, it is reasonable to start our analysis with the class of symbols (1.5) considered in [6]. Since in our work the symbols are not assumed to be of the Fisher-Hartwig type (which needs a more delicate treatment, see section 5.4), we should still expect that the model Riemann-Hilbert problem be solvable for the class of symbols (1.5) when there is no Fisher-Hartwig singularity $a_0(z) = b_0(z) \equiv 1$. Indeed this is the case as will be elaborated in this section.

As commented in the beginning of section 2.1, asymptotics of $D_n(\phi, d\phi; r, s)$, for general $r$ and $s$ requires a more delicate approach (see section 5.2) and we do not discuss the details here. So let us consider $D_n(\phi, d\phi; 1, 1)$, where $d$

- a) is analytic in a neighborhood of the unit circle,
- b) has a zero winding number,
- c) does not vanish on the unit circle, and
- d) satisfies the condition $d(z)\tilde{d}(z) = 1$ on the unit circle.

For instance, a class of functions satisfying these conditions is given by

$$d(z) = \prod_{j=1}^{m} d_j(z), \quad d_j(z) = \pm \left(\frac{z - b_j}{z - a_j}\right)^{\alpha_j} \left(\frac{a_j z - 1}{b_j z - 1}\right)^{\alpha_j}, \quad (4.1)$$

where $\alpha_j \in \mathbb{C}$, all factors are defined by their principal branch, and

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < 1.$$  

Note that a similar construction can be found for $-1 < b_m < a_m < \cdots < b_1 < a_1 < 0$, and thus a larger class of functions can be found from multiplying functions of the first class with those of the second class. Although we have a class of functions satisfying the required properties, a complete categorization of functions satisfying the four required properties for $d$ is yet to be found. We emphasize that the conditions $d(\pm 1) = 1$ required in [6] do not play a role in the Riemann-Hilbert analysis. Indeed, for $d$ as defined in (4.1) one can check that $d(\pm 1) = (-1)^{\epsilon_d}$, where $\epsilon_d$ is the number of the $d_j$-factors in whose definition the sign"$-$" is taken. So in this sense we are considering functions $d$ which are slightly more general than those considered in [6]. At the same time, we have our technical assumption of analyticity of the symbols in a neighborhood of the unit circle which is not needed in the analysis of [6].

Note that the condition $d\tilde{d} = 1$ on the unit circle renders the 23-element of the jump matrix $J_\Lambda$ zero; indeed

$$J_{\Lambda,23}(z) = \tilde{\phi}(z) - \frac{w(z)\tilde{w}(z)}{\phi(z)} = \tilde{\phi}(z)(1 - d(z)\tilde{d}(z)) = 0.$$
Hence, for the particular choices made above, the jump matrix $G_{\Lambda}$ reduces to

$$J_{\Lambda}(z) = \begin{pmatrix}
0 & 0 & 0 & -\phi(z) \\
-d(z) & 0 & 0 & 0 \\
0 & -\frac{1}{\phi(z)} & 0 & 0 \\
\frac{1}{\phi(z)} & 0 & \tilde{w}(z) & 0
\end{pmatrix}. \quad (4.2)$$

In order to factorize $J_{\Lambda}$, let us first consider the following Szegő functions

$$\alpha(z) = \exp \left[ \frac{1}{2\pi i} \int_{T} \frac{\ln(\phi(\tau))}{\tau - z} d\tau \right], \quad \beta(z) = \exp \left[ \frac{1}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau - z} d\tau \right]. \quad (4.3)$$

By Plemelj-Sokhotskii formula, $\alpha$, $\beta$, $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy the following jump conditions on the unit circle:

$$\alpha_+(z) = \alpha_-(z)\phi(z), \quad \beta_+(z) = \beta_-(z)d(z),$$

$$\tilde{\alpha}_-(z) = \tilde{\alpha}_+(z)\tilde{\phi}(z), \quad \tilde{\beta}_-(z) = \tilde{\beta}_+(z)\tilde{d}(z). \quad (4.4)$$

It turns out that knowing the value of $\beta(0)$ is crucial for finding an asymptotic expression for $h_n$ (see section 4.2) and the condition $d\tilde{d} \equiv 1$ on the unit circle allows us to evaluate $\beta(0)$ easily. Indeed

$$\int_{T} \ln(d(\tau)) \frac{d\tau}{\tau} = \int_{T} \ln(d^{-1}(\tau)) \frac{d\tau}{\tau} = -\int_{T} \ln(d(\tau)) \frac{d\tau}{\tau}. \quad (4.5)$$

Thus

$$\int_{T} \ln(d(\tau)) \frac{d\tau}{\tau} = 0, \quad \text{and therefore,} \quad \beta(0) = 1.$$

Next, we show that $\beta = \tilde{\beta}$. Note that

$$\tilde{\beta}(z) = \exp \left[ \frac{1}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau - z} d\tau \right] = \exp \left[ \frac{-z}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau - z} \frac{d\tau}{\tau} \right]$$

$$= \exp \left[ \frac{-z}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau - z} \frac{d\tau}{\tau} \right] = \exp \left[ \frac{z}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau - z} \frac{d\tau}{\tau} \right],$$

Where we have again used the fact that $d\tilde{d} \equiv 1$ on the unit circle. Using

$$\frac{1}{(\tau - z)\tau} = \frac{z^{-1}}{\tau - z} - \frac{z^{-1}}{\tau},$$

we can write the last expression for $\tilde{\beta}$ as

$$\tilde{\beta}(z) = \exp \left[ \frac{1}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau - z} d\tau - \frac{1}{2\pi i} \int_{T} \frac{\ln(d(\tau))}{\tau} d\tau \right] = \frac{\beta(z)}{\beta(0)} = \beta(z), \quad \text{by (1.3)}.$$

To show that $\beta = \tilde{\beta}$ one could also argue that they both solve the same scalar RHP which has a unique solution. We also note that $\alpha(z), \beta(z) = 1 + O(z^{-1})$, and $\tilde{\alpha}(z) = \alpha(0)(1 + O(z^{-1}))$ as $z \to \infty$. Now we can write the solution of the $\Lambda$-RHP (in the case $d\tilde{d} \equiv 1$ on $\mathbb{T}$) as

22
\[ \Lambda(z) = \Lambda^{-1}_\infty \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ C_\rho(z) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \times \begin{cases} \left( \begin{array}{cccc} -\beta(z) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\tilde{\alpha}(z) & 0 & 0 \\ 0 & 0 & 0 & -\alpha(z) \end{array} \right), & |z| < 1, \\
\left( \begin{array}{cccc} 0 & \beta(z) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta(z)\tilde{\alpha}(z)\alpha(z)} \\ 0 & \tilde{\alpha}(z) & 0 & 0 \\ \alpha(z) & 0 & 0 & 0 \end{array} \right), & |z| > 1. \end{cases} \] (4.6)

where \( C_f(z) \) is the Cauchy-transform of \( f(z) \):

\[ C_f(z) = \frac{1}{2\pi i} \int_T \frac{f(\tau)}{\tau - z} d\tau, \]

and

\[ \Lambda^{-1}_\infty = \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha(0) & 0 & 0 \end{array} \right), \quad \rho(z) = -\frac{1}{\beta_-(z)\beta_+(z)\tilde{\alpha}_-(z)\alpha_+(z)}. \] (4.7)

Using (4.4), the Plemelj-Sokhotskii formula and general properties of the Cauchy integral, it can be checked that \( \Lambda \) given by (4.6) satisfies the \( \Lambda \)-RHP.

### 4.1 The small-norm Riemann-Hilbert problem associated to \( D_n(\phi, d\phi, 1, 1) \)

Let us consider

\[ R(z; n) := S(z; n)\overset{\circ}{S}(z)^{-1}. \] (4.8)

This function clearly has no jumps on \( \Gamma_i, \Gamma_o \) and \( T \), since \( S \) and \( \overset{\circ}{S} \) have the same jumps on these contours. Thus, \( R \) satisfies the following small-norm Riemann-Hilbert problem

- **RH-R1** \( R \) is holomorphic in \( \mathbb{C} \setminus \Gamma_R \).
- **RH-R2** \( R_+(z; n) = R_-(z; n)J_R(z; n) \), for \( z \in \Gamma_R \).
- **RH-R3** As \( z \to \infty \), \( R(z; n) = I + \mathcal{O}(z^{-1}) \),

where \( \Gamma_R := \Gamma'_i \cup \Gamma'_o \), and \( J_R \) is given by

\[ J_R(z; n) = \overset{\circ}{S}(z)G_S(z; n)\overset{\circ}{S}(z)^{-1} = \begin{cases} \overset{\circ}{S}(z)G_{T,i}(z; n)\overset{\circ}{S}(z)^{-1}, & z \in \Gamma'_i, \\
\overset{\circ}{S}(z)G_{T,o}(z; n)\overset{\circ}{S}(z)^{-1}, & z \in \Gamma'_o. \end{cases} \] (4.9)

Using (4.6), (4.7) and the definitions of \( G_{T,i}, G_{T,o}, G_{X,i} \) and \( G_{X,i} \) given by (2.39) and (2.45) we find
By standard theory of small-norm Riemann-Hilbert problems [16, 17], there exists $n_*$ so that

\[
R(z) = I + R_1(z) + R_2(z) + R_3(z) + \cdots, \quad z \in \mathbb{C} \setminus \Gamma_R, \quad n \geq n_*,
\]

where $R_k$ can be found recursively from

\[
R_k(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{[R_{k-1}(\mu)]_+ (J_R(\mu) - I)}{\mu - z} d\mu, \quad z \in \mathbb{C} \setminus \Gamma_R, \quad k \geq 1.
\]

Therefore we have

\[
R_1(z; n) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{J_R(\mu; n) - I}{\mu - z} d\mu = 
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & R_{1,12}(z; n) & 0 & 0 \\
0 & 0 & R_{1,14}(z; n) & 0 \\
0 & 0 & 0 & R_{1,14}(z; n)
\end{pmatrix}, \quad z \in \mathbb{C} \setminus \Gamma_R
\]

where

\[
R_{1,j}(z; n) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{\mu^n g_{j,k}(\mu)}{\mu - z} d\mu, \quad jk = 12, 14, 23, 43,
\]

\[
R_{1,j}(z; n) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{\mu^n g_{j,k}(\mu)}{\mu - z} d\mu, \quad jk = 21, 32, 34, 41.
\]

Also from [16, 17] we can write an expression for $R_2$:

\[
R_2(z; n) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{[R_1(\mu; n)]_+ (J_R(\mu; n) - I)}{\mu - z} d\mu = 
\begin{pmatrix}
R_{2,11}(z; n) & 0 & R_{2,13}(z; n) & 0 \\
0 & R_{2,22}(z; n) & 0 & R_{2,24}(z; n) \\
0 & 0 & R_{2,33}(z; n) & 0 \\
0 & 0 & 0 & R_{2,44}(z; n)
\end{pmatrix},
\]

\[\text{where}
\]

\[
g_{12}(z) = -\frac{\alpha(z)}{\phi(z)\beta(z)} - \frac{\tilde{w}(z)C_\rho(z)}{\phi(z)\beta(z)\tilde{\alpha}(z)}, \quad g_{14}(z) = \frac{\tilde{w}(z)}{\phi(z)\beta(z)\tilde{\alpha}(z)\alpha(0)},
\]

\[
g_{23}(z) = -\frac{\alpha(0)\tilde{w}(z)\beta(z)}{\phi(z)\tilde{\alpha}(z)}, \quad g_{34}(z) = -\frac{1}{\alpha(0)\phi(z)} \left( \frac{\alpha(0)\beta(z)}{\phi(z)} + \frac{\beta(z)\tilde{w}(z)C_\rho(z)}{\tilde{\alpha}(z)\phi(z)} \right),
\]

\[
g_{21}(z) = \frac{w(z)\beta(z)}{\phi(z)\alpha(z)}, \quad g_{32}(z) = -\frac{1}{\alpha(0)\phi(z)} \left( \frac{\beta(z)}{\phi(z)} - \frac{w(z)\tilde{\alpha}(z)\beta(z)\alpha(z)C_\rho(z)}{\tilde{\alpha}(z)\alpha(z)} \right),
\]

\[
g_{41}(z) = -\frac{1}{\alpha(0)\phi(z)} \left( \frac{1}{\tilde{\alpha}(z)\beta(z)\alpha(z)} - \frac{w(z)\beta(z)C_\rho(z)}{\alpha(z)} \right).
\]
Therefore by (2.32), (2.35), (2.36) and (4.20) we can write
\[
R_{2,k,j}(z; n) = \left\{ \begin{array}{ll}
\sum_{\ell \in \{2,4\}} \frac{1}{2\pi i} \int_{\Gamma_{\ell}} \frac{\mu^{-n} \cdot [R_{1,k,\ell}(\mu; n)]_j g_{\ell j}(\mu)}{\mu - z} d\mu, & j = 1, k = 1, 3, \\
\sum_{\ell \in \{2,4\}} \frac{1}{2\pi i} \int_{\Gamma_{\ell}} \frac{\mu^n \cdot [R_{1,k,\ell}(\mu; n)]_j g_{\ell j}(\mu)}{\mu - z} d\mu, & j = 3, k = 1, 3, \\
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{\mu^{-n} \cdot [R_{1,k,1}(\mu; n)]_j g_{1 j}(\mu)}{\mu - z} d\mu + \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\mu^{-n} \cdot [R_{1,k,2}(\mu; n)]_j g_{2 j}(\mu)}{\mu - z} d\mu, & k = 2, 4.
\end{array} \right. 
\]

Moreover, using (4.13) and a straightforward calculation one can justify that the matrix structure (i.e. the location of zero and nonzero elements) of \(R_{2k+1}\) and \(R_{2k}, k \geq 1\), are similar to that of \(R_1\) and \(R_2\), respectively. It is also straightforward to show that
\[
R_{k,i,j}(z; n) = O(e^{-cn}), \quad n \to \infty, \quad k \geq 1,
\]
for some positive constant \(c\).

### 4.2 Asymptotics of \(h_n\)

From (2.20) we have
\[
-\frac{1}{h_{n-1}} = \lim_{z \to 0} z^{-1/2} Y_{21}(z^{-1}; n).
\]

Let us denote
\[
\mathcal{A}(z; n) := P^{-1}(n) \mathcal{X}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & z^n & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^{-n} \end{pmatrix},
\]
and also let us define the matrix \(\mathcal{B}(n)\) in the following expansion for \(\mathcal{A}(z; n)\), which is equivalent to RH-\(\mathcal{A}4\):
\[
\mathcal{A}(z; n) = I + \mathcal{B}(n) z + O(z^2), \quad z \to 0.
\]

Therefore by (2.32), (2.35), (2.36) and (4.20) we can write
\[
\mathcal{X}(z, n) = \begin{pmatrix} C_1(n) & 1 & C_3(n) & 0 \\
C_2(n) & 0 & C_4(n) & 1 \end{pmatrix} \mathcal{A}(z; n) \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & z^{-n} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^n \end{pmatrix}.
\]

Using (2.22) and (4.22) we can write
\[
Y_{21}(z^{-1}; n) = \mathcal{X}_{22}(z; n) = C_2(n) A_{12}(z; n) z^{-n} + C_4(n) A_{32}(z; n) z^{-n} + A_{42}(z; n) z^{-n}.
\]

From (4.21) we have
\[
z^{-n} A(z; n) = z^{-n} \cdot I + z^{-n+1} \mathcal{B}(n) + O(z^{-n+2}), \quad z \to 0.
\]

Therefore, as \(z \to 0\)
\[
z^{-n}A_{ij}(z; n) = \begin{cases} 
z^{-n+1}A_{ij}(n) + O(z^{-n+2}), & i \neq j, \\
z^{-n} + z^{-n+1}B_{ii}(n) + O(z^{-n+2}), & i = j.
\end{cases}
\] (4.25)

Therefore by (4.19), (4.23) and (4.25) we have
\[
-\frac{1}{h_{n-1}} = C_2(n)B_{12}(n) + C_4(n)B_{32}(n) + B_{42}(n).
\] (4.26)

Tracing back the Riemann-Hilbert transformations, we find that for \(z \in \Omega_0\) we have
\[
\mathcal{X}(z; n) = R(z; n)\Lambda(z)
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & z^{-n} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^n
\end{pmatrix},
\]
hence,
\[
A(z; n) = P^{-1}(n)R(z; n)\Lambda(z),
\] (4.27)

by (4.20). Also from (4.27) and (4.27) we conclude that
\[
P(n) = R(0; n)\Lambda(0).
\] (4.28)

Let us denote the coefficients in the expansions of \(R(z; n)\) and \(\Lambda(z)\), as \(z \to 0\), by
\[
R(z; n) = R(0; n) + R^{(1)}(n) \cdot z + R^{(2)}(n) \cdot z^2 + O(z^3), \quad \Lambda(z) = \Lambda(0) + \Lambda^{(1)} z + \Lambda^{(2)} \cdot z^2 + O(z^3).
\] (4.29)

Therefore from (4.21), (4.27), and (4.28) we have
\[
B(n) = \Lambda^{-1}(0)R^{-1}(0; n)R^{(1)}(n)\Lambda(0) + \Lambda^{-1}(0)\Lambda(0).
\] (4.30)

Note that
\[
R^{(1)}(n) = \frac{1}{2\pi i} \int_{\Gamma_n} (J_R(\mu; n) - I) \frac{d\mu}{\mu^2} + O(e^{-2cn}), \quad R^{-1}(0; n) = I - R_{14}(0; n) + O(e^{-2cn}),
\] (4.31)
as \(n \to \infty\). More explicitly we have
\[
R^{(1)}(n) = \begin{pmatrix}
0 & R_{12}^{(1)}(n) & 0 & R_{14}^{(1)}(n) \\
R_{21}^{(1)}(n) & 0 & R_{23}^{(1)}(n) & 0 \\
0 & R_{32}^{(1)}(n) & 0 & R_{34}^{(1)}(n) \\
R_{41}^{(1)}(n) & 0 & R_{43}^{(1)}(n) & 0
\end{pmatrix}, \quad n \to \infty,
\] (4.32)

where
\[
R_{jk}^{(1)}(n) = \frac{1}{2\pi i} \int_{\Gamma_n} \mu^{-n-2} g_{jk}(\mu) d\mu, \quad jk = 12, 14, 23, 43,
\] (4.33)
\[
R_{jk}^{(1)}(n) = \frac{1}{2\pi i} \int_{\Gamma_n} \mu^{-n-2} g_{jk}(\mu) d\mu, \quad jk = 21, 32, 34, 41,
\] (4.33)

and
\[
R^{-1}(0; n) = \begin{pmatrix}
1 & -R_{12}(0; n) & 0 & -R_{14}(0; n) \\
-R_{12}(0; n) & 1 & -R_{13}(0; n) & 0 \\
-R_{13}(0; n) & 0 & 1 & -R_{14}(0; n) \\
-R_{14}(0; n) & 0 & -R_{13}(0; n) & 1
\end{pmatrix} + O(e^{-2cn}), \quad n \to \infty.
\] (4.34)
From (4.6) and (4.7) we have

\[
\Lambda(0) = \begin{pmatrix}
0 & 0 & 0 & -\alpha(0) \\
-1 & 0 & 0 & 0 \\
0 & -\frac{1}{\alpha(0)} & 0 & 0 \\
-C_\rho(0)\alpha(0) & 0 & 1 & 0
\end{pmatrix},
\Lambda^{(1)} = \begin{pmatrix}
0 & 0 & 0 & \Lambda^{(1)}_{14} \\
\Lambda^{(1)}_{21} & 0 & 0 & 0 \\
0 & \Lambda^{(1)}_{32} & 0 & 0 \\
\Lambda^{(1)}_{41} & 0 & \Lambda^{(1)}_{43} & 0
\end{pmatrix},
\]

(4.35)

where

\[
\Lambda^{(1)}_{14} = -\frac{\alpha(0)}{2\pi i} \int_T \log \phi(\mu) \frac{d\mu}{\mu^2}, \quad \Lambda^{(1)}_{21} = -\frac{1}{2\pi i} \int_T \log d(\mu) \frac{d\mu}{\mu^2}, \quad \Lambda^{(1)}_{32} = \frac{1}{2\pi i \alpha(0)} \int_T \log \phi(\mu) d\mu,
\]

\[
\Lambda^{(1)}_{41} = -\alpha(0) \left\{ \frac{1}{2\pi i} \int_T \rho(\mu) \frac{d\mu}{\mu^2} - \frac{1}{4\pi^2} \left( \int_T \rho(\mu) \frac{d\mu}{\mu^2} \right) \left( \int_T \log d(\mu) \frac{d\mu}{\mu^2} \right) \right\},
\]

\[
\Lambda^{(1)}_{43} = \frac{1}{2\pi i} \left\{ \int_T \log w(\mu) d\mu - \int_T \log \phi(\mu) \frac{d\mu}{\mu^2} \right\}.
\]

(4.36)

From (4.30), (4.32), (4.34) and (4.35) we find that

\[
B_{12}(n) = \frac{R^{(1)}_{23}(n)}{\alpha(0)}, \quad B_{32}(n) = C_\rho(0)R^{(1)}_{23}(n) - \frac{R^{(1)}_{23}(n)}{\alpha(0)},
\]

\[
B_{42}(n) = -\frac{1}{\alpha(0)} \left( R_{1,12}(0;n)R^{(1)}_{23}(n) + R_{1,14}(0;n)R^{(1)}_{43}(n) \right).
\]

(4.37)

Note that \(B_{12}(n), B_{32}(n)\) are of order \(O(e^{-cn})\), while \(B_{42}(n)\) is of order \(O(e^{-2cn})\). From (4.28) we can write the asymptotic expansion for \(P(n)\)

\[
P(n) = \begin{pmatrix}
-C_\rho(0)\alpha(0)R_{1,14}(0;n) - R_{1,12}(0;n) & 0 & R_{1,14}(0;n) & -\alpha(0) \\
-1 & -R_{1,24}(0;n) & 0 & -\alpha(0)R_{1,21}(0;n) \\
-C_\rho(0)\alpha(0)R_{1,34}(0;n) - R_{1,32}(0;n) & 0 & 0 & 0 \\
-C_\rho(0)\alpha(0) & -R_{1,43}(0;n) & 1 & -\alpha(0)R_{1,41}(0;n)
\end{pmatrix} + O(e^{-2cn}),
\]

(4.38)

as \(n \to \infty\). Revisiting (2.36) we have

\[
\begin{pmatrix}
1 & C_1(n) & 0 & C_3(n) \\
0 & C_2(n) & 0 & C_2(n)
\end{pmatrix} P(n) = \begin{pmatrix}
C_1(n) & 1 & C_3(n) & 0 \\
C_2(n) & 0 & C_4(n) & 1
\end{pmatrix},
\]

(4.39)

From this equation, in particular, we find the following two equations for the constants \(C_2\) and \(C_4\)

\[
C_2(n)P_{21}(n) + P_{31}(n) + C_4(n)P_{41}(n) = C_2(n), \quad C_2(n)P_{24}(n) + P_{32}(n) + C_4(n)P_{42}(n) = 0.
\]

(4.40)

Solving for \(C_2\) and \(C_4\) we find

\[
C_2(n) = \frac{P_{42}(n)P_{31}(n) - P_{31}(n)P_{32}(n)}{(1 - P_{21}(n))P_{42}(n) + P_{41}(n)P_{22}(n)}, \quad C_4(n) = \frac{P_{22}(n)P_{31}(n) + |1 - P_{21}(n)|P_{32}(n)}{(1 - P_{21}(n))P_{42}(n) + P_{41}(n)P_{22}(n)}.
\]

(4.41)
From (4.38) we have
\[ C_2(n) = \frac{C_0(0)}{\left(\frac{2}{\alpha(0)}\right)R_{1,43}(0;n) - C_0(0)R_{1,23}(0;n)(1 + \mathcal{O}(e^{-2cn}))}, \] (4.42)
and
\[ C_4(n) = \frac{-\frac{2}{\alpha(0)}R_{1,43}(0;n) - C_0(0)R_{1,23}(0;n)}{\left(\frac{2}{\alpha(0)}\right)R_{1,43}(0;n) - C_0(0)R_{1,23}(0;n)(1 + \mathcal{O}(e^{-2cn}))}. \] (4.43)
Combining (4.26), (4.37), (4.42) and (4.43) we obtain
\[ h_{n-1} = -\alpha(0) \cdot \frac{2}{\alpha(0)}R_{1,43}(0;n) - C_0(0)R_{1,23}(0;n) \frac{1}{\left(\frac{2}{\alpha(0)}\right)R_{1,43}(0;n) - C_0(0)R_{1,23}(0;n)(1 + \mathcal{O}(e^{-2cn}))}, \quad n \to \infty. \] (4.44)
Note that from (4.15) and (4.33) we have
\[ R_{1,jk}(0;n) = R_{jk}(n + 1), \quad \text{for} \quad jk = 12, 14, 23, 43, \]
\[ R_{1,jk}(0;n) = R_{jk}(n - 1), \quad \text{for} \quad jk = 21, 32, 34, 41. \] (4.45)
Denoting
\[ \mathcal{E}(n) := \frac{2}{\alpha(0)}R_{1,43}(0;n) - C_0(0)R_{1,23}(0;n), \] (4.46)
and using (4.43) we can write (4.44) as
\[ h_{n-1} = -\alpha(0) \cdot \frac{\mathcal{E}(n)}{\mathcal{E}(n-1)}(1 + \mathcal{O}(e^{-2cn})), \quad n \to \infty. \] (4.47)
This concludes the proof of theorem 1.1.

Remark 4.1 From (2.7), (2.22), (2.32), and (2.35) we have the following representation for the orthogonal polynomials \( P_n(z) \) in terms of the solution \( X(z; n) \) of the X - RHP:
\[ P_n(z) = X_1(z; n) + C_1(z)X_2(z; n) + C_4(z)X_4(z; n). \] (4.48)
The asymptotic results concerning the function \( X(z; n) \) obtained in this section can be translated to the large \( n \) asymptotic formulae for the polynomials \( P_n(z) \). Indeed, skipping the rather tedious though straightforward calculations, we arrive at the following asymptotics for \( P_n(z) \) on the unit circle.
\[ P_n(z) = \frac{\beta(z)(2\alpha(0)C_0(z) - C_0(0)\alpha(0))}{\mathcal{E}(n)} + \alpha(z)z^n \left(1 + \frac{C_0(0)\alpha(0)R_{1,21}(z; n) - 2R_{1,41}(z; n)}{\mathcal{E}(n)}\right) + \mathcal{O}(e^{-cn}), \] (4.49)
as \( n \to \infty \). While in the interior and exterior of the unit circle we have the following asymptotic formulae for \( P_n(z) \):
\[ P_n(z) = \frac{\beta(z)(2\alpha(0)C_0(z) - C_0(0)\alpha(0))}{\mathcal{E}(n)} + \mathcal{O}(e^{-cn}), \quad |z| < 1, \] (4.50)
\[ P_n(z) = \alpha(z)z^n \left(1 + \frac{C_0(0)\alpha(0)R_{1,21}(z; n) - 2R_{1,41}(z; n)}{\mathcal{E}(n)}\right) + \mathcal{O}(e^{-2cn}), \quad |z| > 1, \] (4.51)
as \( n \to \infty \) (compare with (4.49)).
5 Remaining open questions

We consider this work as a starting point of a long term research project. There are many challenging technical as well as conceptual open questions related to the Riemann-Hilbert formalism we are suggesting in this paper. Here we highlight some of them that we consider the most pressing.

5.1 Derivation of the relevant Christoffel-Darboux formulae and differential identities

Our main objective in this paper has been to develop a $4 \times 4$ steepest descent analysis for certain Toeplitz+Hankel determinants and we have achieved that. However, to obtain the asymptotics of $D_n(\phi, d\phi, 1, 1)$ one has to derive suitable differential identities. We propose that the differential identity has to be with respect to the parameters $\alpha_i$ in the function $d$ given by (4.1). Thus, one has to perform $m$ integrations in the parameters $\alpha_i$, $1 \leq i \leq m$. Note that for $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, we have $d \equiv 1$ and hence $\phi = w$. Hence the starting point of the integration in $\alpha_1$ could be taken from the results of [6]. Integration of the differential identity in $\alpha_1$ will provide us with an asymptotic expression for $D_n(\phi, d\phi, 1, 1)$, which also serves as the starting point of integration in $\alpha_2$. Thus we can find asymptotics of $D_n(\phi, d_1d_2\phi, 1, 1)$ which also serves as the starting point of integration in $\alpha_3$, and so on. Repeating this procedure will finally lead us to the asymptotics of $D_n(\phi, d\phi, 1, 1)$.

In order to derive the differential identities mentioned above, one has to find recurrence relations and prove a Christoffel-Darboux formula for the polynomials (2.1) and follow a path similar to that introduced by I.Krasovsky in [20]. Note that the recurrence relations can be found by analyzing the function $M(z; n) := \mathcal{X}(z; n + 1)\mathcal{X}^{-1}(z; n)$, which is holomorphic in $\mathbb{C} \setminus \{0\}$ and can be globally determined by its singular parts at zero and infinity.

5.2 Extension of the analysis to general offset values $r, s \neq 1$.

If we lift the restriction $r = s = 1$ then in the jump matrix of the $\mathcal{X}$-RHP the functions $\phi(z)$ and $w(z)$ should be replaced by $z^{1-r}\phi(z)$ and $z^{1-s}w(z)$, respectively. This, in turn would raise a serious question about solvability of the $\Lambda$-RHP. Indeed, for instance, we would not be able to define the function $\alpha$ in (4.3) and hence to factorize the jump matrix (4.2). The way out of this difficulty could be the use of the relation between the determinants $D_n(\phi, w; r, s)$ with different values of $r, s$ or both. Such relations are well known in the pure Toeplitz case (for example see Lemma 2.4 in [13]). However, for general Toeplitz+Hankel determinants they are yet to be found. Another way to approach the problem could be to develop the so-called Bäcklund-Schlesinger transformations of the $\mathcal{X}$-RHP itself. That is, the transformations of the form,

$$\mathcal{X}(z) \mapsto R(z)\mathcal{X}(z),$$

where $R(z)$ is a properly chosen rational function for which the above transformation results in the desired shifting of the parameters $r$ and $s$. Also, one can try to allow the matrix $P(n)$ in the setting of the $\mathcal{X}$-RHP to depend on $z$.

It is worth to point out that the problem with the extension of our scheme to the general values of $r$ and $s$ is not actually a problem of the setting of the relevant Riemann-Hilbert problem. Indeed, it is rather the question of the correct way to approach to its asymptotic solution. Let us demonstrate this point by considering the pure Toeplitz case.

\footnote{Although the authors in [6] do not particularly study the asymptotics of $D_n(\phi, \phi, 1, 1)$, this asymptotics is achievable by their methods.}
Assume that \( D_n(\phi, 0; r, 0) \neq 0 \) for all \( n \), so that \( \mathcal{P}_n(z) \) and, correspondingly, the solution \( \mathcal{Y}(z; n) \) of the \( \mathcal{Y} \)-RHP exist for all \( n \). Put,

\[
\mathcal{X}(z; n) := P_{\infty}^{-1}(n) \begin{pmatrix}
\mathcal{Y}_{11}(z; n) & 0 & 0 & \mathcal{Y}_{12}(z; n) \\
0 & \mathcal{Y}_{11}(z; n) & \mathcal{Y}_{12}(z; n) & 0 \\
0 & \mathcal{Y}_{21}(z; n) & \mathcal{Y}_{22}(z; n) & 0 \\
\mathcal{Y}_{21}(z; n) & 0 & 0 & \mathcal{Y}_{22}(z; n)
\end{pmatrix},
\]

where the normalizing matrix \( P_{\infty}(n) \) is given by

\[
P_{\infty}(n) = \begin{pmatrix}
1 & 0 & 0 & \mathcal{Y}_{12}(0; n) \\
0 & \mathcal{Y}_{11}(0; n) & 0 & 0 \\
0 & \mathcal{Y}_{21}(0; n) & 1 & 0 \\
0 & 0 & 0 & \mathcal{Y}_{22}(0; n)
\end{pmatrix},
\]

which is invertible for generic \( \phi \). It is straightforward to check that the so defined \( 4 \times 4 \) matrix-valued function \( \mathcal{X} \) solves the \( \mathcal{X} \)-Riemann-Hilbert problem, \( \text{RH-\mathcal{X}1 - RH-\mathcal{X}4} \) with \( w \equiv 0 \), and \( \phi(z) \) replaced by \( z^{1-r} \phi(z) \). Take now \( r = 0 \). That is, let us consider the standard orthogonal polynomials on the circle with the weight \( \phi(z) \) having zero winding number. Then, from the standard \( 2 \times 2 \) Riemann-Hilbert formalism \( \mathcal{I} \), we know everything about the asymptotic behavior of the corresponding orthogonal polynomials \( \mathcal{P}_n(z) \) and hence we know asymptotic solution of the \( \mathcal{X} \)-Riemann-Hilbert problem corresponding to \( D_n(\phi, 0; 0, 0) \). And, we know this in spite of the fact that the approach developed in the body of this work can not be satisfactorily used for the case \( r = s = 0 \). We believe that this observation might entail a useful hint on how to modify our Riemann-Hilbert approach to deal with general values of \( r \) and \( s \).

### 5.3 Extension of the Riemann-Hilbert analysis of section 3 for more general choices of \( I \)

We recall that our Riemann-Hilbert analysis of section 3 with minimal modifications, naturally extends to the following three cases as well: i) \( -1 < a < b < 0 \), ii) \( -\infty < a < b < -1 \), and iii) \( 1 < a < b < \infty \). A natural first step in generalizing the results of section 3 beyond the above cases is considering the case where \( I \) has 0 as an end point. This would slightly affect the analysis as one has to take into account the behavior of \( w \) at 0 in the set up of the \( 2 \times 4 \) and the subsequent Riemann-Hilbert problems.

The other interesting case to be studied is when \( I \) intersects the unit circle. Clearly, in this case one has to perform local analysis in a neighborhood of the intersection point(s) of \( I \) and the unit circle. Although these local constructions are reminiscent of what one does near the Fisher-Hartwig singularities or the endpoints of the support of the symbol, here even if the possible intersection points \( \pm 1 \), are regular points for non-FH symbols \( \phi \) and \( w \), one has to still perform local analysis due to collision of the supports of \( \phi \) and \( w \).

Another possible generalization would be to consider \( I \) to be the union of two symmetric intervals with respect to the unit circle, i.e. \( I = [a, b] \cup [b^{-1}, a^{-1}] \). This generalization should be accessible by slight modification of our approach explained in section 3 However, generalization to the case where \( I \) is a union of two non-symmetrical intervals with respect to the unit circle needs a more special treatment.

### 5.4 Extension to Fisher-Hartwig symbols

One can study the large-\( n \) asymptotics of determinant \( D_n(\phi, d\phi, 1, 1) \) (and with increasing effort \( D_n(\phi, d\phi, r, s) \) for fixed \( r, s \in \mathbb{Z} \)) assuming that \( \phi \) possesses Fisher-Hartwig singularities \( \{z_i\}_{i=1}^{m} \) on
the unit circle. It is in fact in this level of generality that E. Basor and T. Ehrhardt have been able to compute the asymptotics of $D_n(\phi, d\phi, 0, 1)$, $D_n(-\phi, d\phi, 0, 1)$, $D_n(\phi, d\phi, 0, 1)$, and $D_n(-z\phi, d\phi, 0, 1)$ via the operator-theoretic methods in [6]. However, the authors in [6] further require that the Fisher-Hartwig part of $\phi$ be even. In fact they used some results of the work [13] to prove their asymptotic formulas for Toeplitz+Hankel determinants, and for this reason they inherited the evenness assumption from the work [13] where the authors needed evenness of $\phi$ in their $2 \times 2$ setting to relate Hankel and Toeplitz+Hankel determinants to a Toeplitz determinant with symbol $\phi$.

From a Riemann-Hilbert perspective, in the presence of Fisher-Hartwig singularities, one has to construct the $4 \times 4$ local parametrices near the points $z_i$. Expectedly, these local parametrices must be expressed in terms of confluent hypergeometric functions as suggested by [13]. We have not yet worked out the details of this construction but we believe that it should be well within reach. It would be methodologically important to achieve the results obtained from operator-theoretic tools via the Riemann-Hilbert approach as well. Moreover, we expect that the evenness of the Fisher-Hartwig part of $\phi$ would not play a role in our $4 \times 4$ setting, and in that sense there are reasonable prospects of generalizing the results of [6] to symbols $\phi$ with non-even Fisher-Hartwig part.

5.5 Characteristic polynomial of a Hankel matrix

As mentioned in the Introduction, arguably the most important motivation behind studying the asymptotics of Toeplitz+Hankel determinants is to study the large $n$ asymptotics of the eigenvalues of the matrix $H_n[w]$. We recall that the characteristic polynomial $\det(H_n[w] - \lambda I)$ is indeed the Toeplitz+Hankel determinant $D_n(-\lambda, w, 0, 0)$. In this case the associated $\Lambda$-model Riemann-Hilbert problem needs a special treatment. In a sense it is a simpler problem as the symbol $\phi$ is identically equal to a constant, but more complicated - compared to the situation in section 4 - as it does not enjoy $J_{\Lambda,23}(z) = 0$. In any case, the solution to this model problem provides us with the constant term in the asymptotics of $D_n(-\lambda, w, 0, 0)$, and in the case of Fisher-Hartwig weight $w$, one can hope to obtain the leading terms of this asymptotic expansion (up to the constant term, viz. the solution of the $\Lambda$-model problem) from the local analysis near the Fisher-Hartwig singularities. This last point is yet another motivation to pursue the goals of section 5.4.

Acknowledgements

We are very grateful to Estelle Basor, Thomas Bothner, Percy Deift and Igor Krasovsky for their interest in this work and for many very useful comments and suggestions. R. Gharakhloo acknowledges support by NSF-grant DMS-1700261. A. Its acknowledges support by NSF-grant DMS-1700261 and by Russian Science Foundation grant No.17-11-01126.

\footnote{The very recent work [13] shows that indeed one can explicitly describe the leading terms of the asymptotics of $D_n(-\lambda, w, 0, 0)$, in the case of $w$ having jump discontinuities and $\lambda$ sufficiently large.}

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