Piecewise linear unimodal maps with non-trivial continuous piecewise linear commutator

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Abstract

Let \( g : [0, 1] \to [0, 1] \) be piecewise linear unimodal map. We say that \( g \) has non-trivial piecewise linear commutator, if there is a continuous piecewise linear \( \psi : [0, 1] \to [0, 1] \) such that \( g \circ \psi = \psi \circ g \), and, moreover, \( \psi \) is neither an iteration of \( g \), not a constant map.

We prove that if \( g \) has a non-trivial piecewise linear commutator, then \( g \) is topologically conjugated with the tent map by a piecewise linear conjugacy.

1 Introduction

We will call a continuous map \( g : [0, 1] \to [0, 1] \) unimodal, if it can be written in the form

\[
  g(x) = \begin{cases} 
  g_l(x), & 0 \leq x \leq v, \\
  g_r(x), & v \leq x \leq 1,
  \end{cases}
\]

where \( v \in (0, 1) \) is a parameter, the function \( g_l \) is increasing, the function \( g_r \) is decreasing, and

\[
  g(0) = g(1) = 1 - g(v) = 0.
\]

The fundamental example of unimodal map is the tent map

\[
  f : x \mapsto 1 - |1 - 2x|.
\]

Definition 1.1. Continuous surjective solution \( \eta \) of the function equation

\[
  \eta \circ f = g \circ \eta,
\]

where \( f \) is the tent map and \( g \) is unimodal map, is called semi conjugation from \( f \) to \( g \).

Definition 1.2. Let \( g \) be unimodal map. We will call a continuous surjective solution \( \psi \) of the functional equation

\[
  \psi \circ g = g \circ \psi,
\]

a self semi conjugation of \( g \).

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We will prove the next Theorem, which permits to reduce the number of maximal parts of monotonicity of the map, which commutes with the Tent map.

**Proposition 1.** Let \( \psi : [0, 1] \to [0, 1] \) be a piecewise linear non-constant map such that (1.2) holds. If \( \psi \) has \( 2t \) maximal intervals of monotonicity, then there exists non-constant piecewise linear \( \tilde{\psi} : [0, 1] \to [0, 1] \), which has \( t \) maximal intervals of monotonicity, and

\[
\tilde{\psi} \circ g = g \circ \tilde{\psi}.
\] (1.3)

The main result of this work is the next

**Theorem 1.** Let \( g \) be a unimodal map. If there exists a non-trivial piecewise linear commutator of \( g \), then \( g \) is topologically conjugated with the Tent map and, moreover, the conjugacy is piecewise linear.

We will first prove Theorem 1 for the case when \( g \) is topologically conjugated with the tent map.

**Proposition 2.** Let \( g \) be a unimodal map, which is topologically conjugated with the tent map. If there exists a non-trivial piecewise linear commutator of \( g \), then the conjugacy of \( g \) and the tent map is piecewise linear.

The we will use Proposition 2 to prove Theorem 1.

2 Preliminaries and notations

We will use the following facts in the proof of the main result of this work.

**Theorem 2.** [1, p. 53] A unimodal map \( g \) is topologically conjugated to the tent map if and only if the complete pre-image of 0 under the action of \( g \) is dense in \([0, 1]\).

Remind that the set \( g^{-\infty}(a) = \bigcup_{n \geq 1} g^{-n}(a) \), where

\[
g^{-n}(a) = \{x \in [0, 1] : g^n(x) = a\},
\]

is called the complete pre-image of \( a \) (with respect to the action of \( g \)).

The next function will play a crucial role in our reasonings. For every \( t \in \mathbb{N} \) denote

\[
\xi = \xi_t : x \mapsto \frac{1 - (-1)^{[tx]}}{2} + (-1)^{[tx]}\{tx\},
\] (2.1)

where \( \{\cdot\} \) denotes the function of the fractional part of a number and \( [\cdot] \) is the integer part.

We are now ready to formulate the result, which will be used in our calculations.
Theorem 3. [2, Theorem 1], [3] 1. Any self semi conjugacy $\xi$ of the tent map is $\xi_t$ of the form (2.1) for some $t \geq 1$.

2. For every $t \in \mathbb{N}$ the $\xi_t$ of the form (2.1) is self semi conjugacy of the tent map.

Notice that (2.1) describes a piecewise linear function $\xi_t : [0, 1] \to [0, 1]$, whose tangents are $\pm t$, which passes through origin, and all whose kinks belong to lines $y = 0$ and $y = 1$. The graphs of $\xi_5$ and $\xi_6$ are given at Fig. 1.

![Figure 1:](image)

Theorem 4. [4, Theorem 1] Assume that piecewise linear map $g$ is topologically conjugate to the tent map via the homeomorphism $h$. If $h$ is continuously differentiable in a subinterval of $[0, 1]$, then it is piecewise linear.

Remark 2.1. [5, Remark 2.1] Recall that the graph of the $n$th iteration $g^n$ of an arbitrary continuous function $g$ of the form (1.1) has the following properties:

1. The graph consists of $2^n$ monotone curves.

2. Each maximal part of monotonicity of $g^n$ connects the line $y = 0$ and $y = 1$.

3. If $x_1, x_2$ are such that \{ $g^n(x_1), g^n(x_2)$ \} = \{0, 1\} and $g^n$ is monotone on $[x_1, x_2]$, then $g^{n+1}(x_1) = g^{n+1}(x_2) = 0$ and there is $x_3 \in (x_1, x_2)$ such that $g^{n+1}(x_3) = 1$. Moreover in this case $g^{n+1}$ is increasing on $[x_1, x_3]$ and is decreasing on $[x_3, x_2]$.

Since, by Remark 2.1, the set $g^{-n}(0)$ consists of $2^{n-1}$ + 1 points, the notation follows.

Notation 2.2. [5] Notation 2.1] For every map $g : [0, 1] \to [0, 1]$ of the form (1.1) and for every $n \geq 1$ denote \{ $\mu_{n,k}(g)$, $0 \leq k \leq 2^{n-1}$ \} such that $g^n(\mu_{n,k}(g)) = 0$ and $\mu_{n,k}(g) < \mu_{n,k+1}(g)$ for all $k$.

Remark 2.3. [5, Remark 2.2] Notice that $\mu_{n,k}(g) = \mu_{n+1,2k}(g)$ for all $k$, $0 \leq k \leq 2^{n-1}$. 

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Lemma 2.4. [5, Lemma 1] For each map $g$ of the form (1.1), every $n \geq 2$ and $k, 0 \leq k \leq 2^{n-2}$, the equalities

$$g(\mu_{n,k}(g)) = \mu_{n-1,k}(g)$$

and

$$g(\mu_{n,k}(g)) = g(\mu_{n,2^{n-1}-k}(g))$$

hold.

Lemma 2.5. [5, Lemma 3] Let $g_1$ and $g_2$ be unimodal maps, and $h : [0, 1] \to [0, 1]$ be the conjugacy from $g_1$ to $g_2$. Then

$$h(\mu_{n,k}(g_1)) = \mu_{n,k}(g_2)$$

for all $n \geq 1$ and $k, 0 \leq k \leq 2^{n-1}$.

3 Results

3.1 Basic properties of commutative maps

During this section suppose that $g$ is a unimodal map of the form (1.1) and $\psi : [0, 1] \to [0, 1]$ is a continuous piecewise linear unimodal map such that (1.2) holds.

Lemma 3.1. $\psi(0) \in \{0; x_0\}$, where $x_0$ is the unique positive fixed point of $g$.

Proof. Plug $x_0$ into (1.2) and the necessary fact follows. □

Lemma 3.2. If $\psi(0) = x_0$, where $x_0$ is a positive fixed point of $g$, then $\psi(1) \leq x_0$.

Proof. Plug 1 into (1.2) and the necessary fact follows. □

Lemma 3.3. $\psi(0) = 0$.

Proof. By Lemma 3.1 if $\psi(0) \neq 0$, then $\psi(0) = x_0$, where $x_0$ is a fixed point of $g$.

Suppose that $\psi$ increase at 0, i.e. there is $\varepsilon > 0$ such that $\psi$ increase at $(0, \varepsilon)$. If $1 \notin \psi \circ (0, \varepsilon)$, then it follows from (1.2) that $\psi$ increase at $g \circ (0, \varepsilon)$. Plug 1 into (1.2) and obtain that $g \circ \psi(1) = x_0$. Thus, by Lemma 3.2 there exists $x^* \in (0, 1)$ such that $\psi(x^*) = 1$ and $\psi$ increase on $(0, x^*)$. Without loss of generality suppose that $x^*$ is the first zero of $\psi$. Now, plug $x^*$ into (1.2), whence $\psi(x^{**}) = 0$, where $x^{**} = g(x^*)$. Thus, there is $\tilde{x} \in (0, x^*)$ such that $g_1(\tilde{x}) = x^*$. This $\tilde{x}$ transforms (1.2) to a wrong equality, because $\psi(g(\tilde{x})) = 1$, but $g(\psi(x)) < 1$ for all $x \in (0, x^*)$ (see Figure 2a.).

We need only to consider the case when $\psi$ decrease at 0. This case is analogical (see Figure 2b.). This proves the lemma. □
Lemma 3.4. \( \psi(1) \in \{0; 1\} \).

**Proof.** Using Lemma 3.3 plug 1 into (1.2), and the fact follows. \( \square \)

Denote \( \{\mu_{2,k}; k \geq 0\} \) the complete set of elements of \( \psi^{-2}(0) \), such that \( \mu_{2,k} < \mu_{2,k+1} \) for all admissible \( k \). By (i) of Lemma 3.5 the map \( \psi \) is monotone on \( (\mu_{2,k}, \mu_{2,k+1}) \) for each \( k \). Denote \( n \) such that \( \psi_{2,n} = 1 \). The next fact follows from Lemmas 3.3 and 3.4.

**Lemma 3.5.** (i) For every maximal interval \( I \) of monotonicity of \( \psi \) we have that \( \psi(I) = [0, 1] \).

(ii) For every \( k \in \{0, \ldots, n\} \) we have that \( g(\mu_{2,k}) = g(\mu_{2,n-k}) \) and, moreover,

\[
\max\{1_{[0,v]}(\mu_{2,k}) + 1_{[v,1]}(\mu_{2,n-k}); 1_{[0,v]}(\mu_{2,n-k}) + 1_{[v,1]}(\mu_{2,k})\} = 2.
\]

**Lemma 3.6.** If \( n = 4t \) for some \( t \in \mathbb{N} \), then \( \psi(v) = 0 \).

If \( n = 4t + 2 \) for some \( t \in \mathbb{N} \), then \( \psi(v) = 0 \).

If \( n \) is odd, then \( \psi(v) = v \).

**Proof.** If \( n = 2t \) is even, then, by (ii) of Lemma 3.5 \( \mu_{2,t} = v \). Moreover, if \( t = 2s \) is even, than \( \psi(\mu_{2,t}) = 0 \) and, otherwise \( \psi(\mu_{2,t}) = 1 \).

If \( n = 2t + 1 \), then, by (ii) of Lemma 3.5 \( g(\mu_{2,k}) = g(\mu_{2,n-k+1}) \) for \( k = \frac{n+2}{2} = t + 1 \). Thus, \( v \in (\mu_{2,t}, \mu_{2,t+1}) \). Now plus \( v \) into (1.2) and obtain \( \psi(g(v)) = g(\psi(v)) \). Since \( \psi(1) = 1 \) for odd \( n \), then \( 1 = g(\psi(v)) \), whence \( \psi(v) = v \). \( \square \)

Denote \( \psi_k \) the restriction of \( \psi \) on \( [\mu_{2,k}, \mu_{2,k+1}] \). Denote \( \psi_{k,0} \) the restriction of \( \psi_k \) on \( (\mu_{2,k}, \psi_k^{-1}(v)) \) and \( \psi_{k,1} \) the restriction of \( \psi_k \) on \( (\psi_k^{-1}(v), \mu_{2,k+1}) \).
Lemma 3.7. (i) for every \( k \leq t - 1 \) we have that

\[
g(I_{k,s}) = I_{2k+s}, \quad s \in \{0;1\}.
\]

(ii) for every \( k, t \leq k \leq 2t - 1 \) we have

\[
g(I_{k,s}) = I_{2t-1-2(k-t)-s} = I_{4t-1-2k-s}, \quad s \in \{0;1\}.
\]

Lemma 3.8. For any \( k, 0 \leq k \leq 2t - 1 \) the map \( \psi_{2k,s} \) increase, and the map \( \psi_{2k+1,s} \) decrease for \( s \in \{0;1\} \). Moreover,

\[
\psi(I_{2k,0}) = \psi(I_{2k+1,1}) = (0,v), \quad (3.1)
\]

and

\[
\psi(I_{2k,1}) = \psi(I_{2k+1,0}) = (v,1). \quad (3.2)
\]

Proof. Lemma is obvious. \( \square \)

Lemma 3.9. (i) Suppose that \( k \) is such that \( I_{2k} \subset (0,v) \). Then the equality (1.2) for \( x \in I_{2k+p,s} \) holds if and only if

\[
g_{s+p+1-2ps} \circ \psi_{2k+p,s} = \psi_{4k+2p+s} \circ g_1, \quad p, s \in \{0;1\}. \quad (3.3)
\]

(ii) Suppose that \( k \) is such that \( I_{2k} \subset (v,1) \). Then the equality (1.2) for \( x \in I_{2k+p,s} \) holds if and only if

\[
g_{s+p+1-2ps} \circ \psi_{2k+p,s} = \psi_{4t-1-4k-2p-s} \circ g_2, \quad p, s \in \{0;1\}. \quad (3.4)
\]

Proof. By (3.1), write

\[
g \circ \psi_{2k,0} = g_1 \circ \psi_{2k,0}, \quad (3.5)
\]

\[
g \circ \psi_{2k+1,1} = g_1 \circ \psi_{2k+1,1}. \quad (3.6)
\]

By (3.2) write

\[
g \circ \psi_{2k,1} = g_2 \circ \psi_{2k,0}, \quad (3.7)
\]

\[
g \circ \psi_{2k+1,0} = g_2 \circ \psi_{2k+1,1}. \quad (3.8)
\]

Generalize (3.5) and (3.7),

\[
g \circ \psi_{2k,s} = g_{s+1} \circ \psi_{2k,s}, \quad s \in \{0;1\}. \quad (3.9)
\]

Generalize (3.6) and (3.8),

\[
g \circ \psi_{2k+1,s} = g_{2-s} \circ \psi_{2k+1,s}, \quad s \in \{0;1\}. \quad (3.10)
\]
At last, generalize (3.9) and (3.10) as

\[ g \circ \psi_{2k+p,s} = g_{p(2-s)+(1-p)(1+s)} \circ \psi_{2k+1,s}, \; p,s \in \{0; 1\}, \]

or

\[ g \circ \psi_{2k+p,s} = g_{1+p+s-2ps} \circ \psi_{2k+1,s}, \; p,s \in \{0; 1\}, \quad (3.11) \]

Now part (i) of Lemma follows from (i) of Lemma 3.7 and (3.11).

Part (ii) of Lemma follows from (i) and from part (ii) of Lemma 3.7.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** For any \( k, \; 0 \leq k < t \) let \( \tilde{\psi}_{4k+p} \), for \( p \in \{0; 1; 2; 3\} \), be defined on \( I_{4k+p} \) as

\[
\begin{align*}
\tilde{\psi}_{4k} &= g_{1}^{-1} \circ \psi_{4k}, \\
\tilde{\psi}_{4k+1} &= g_{2}^{-1} \circ \psi_{4k+1}, \\
\tilde{\psi}_{4k+2} &= g_{2}^{-1} \circ \psi_{4k+2}, \\
\tilde{\psi}_{4k+3} &= g_{1}^{-1} \circ \psi_{4k+3}.
\end{align*}
\]

(3.12)

It follows from Lemma 3.8 that \( \tilde{\psi}_{4k} \) increase \( I_{4k} \to (0, v) \), the map \( \tilde{\psi}_{4k+1} \) increase \( I_{4k+1} \to (1, v) \), the map \( \tilde{\psi}_{4k+2} \) decrease \( I_{4k+2} \to (1, v) \) and, finally, the map \( \tilde{\psi}_{4k+3} \) decrease \( I_{4k+3} \to (0, v) \).

Define \( \tilde{\psi} : \bigcup_{k} I_{k} \to [0, 1] \) as \( \tilde{\psi} = \tilde{\psi}_{k} \) on \( I_{k} \) for each \( k, \; 0 \leq k \leq 2t - 1 \). By Lemma 3.8 the map \( \tilde{\psi} \) can be continuously extended to the entire \([0, 1]\), whence denote by the same letter its continuation.

It follows from the construction that

\[ g \circ \tilde{\psi} = \psi. \]

(3.13)

Notice, that equations (3.12) can be written as

\[ \tilde{\psi}_{4k+2p+s} = g_{s+p+1-2ps}^{-1} \circ \psi_{4k+2p+s}, \; s,p \in \{0; 1\}. \]

(3.14)

Thus, for any \( k, \; 0 \leq k \leq 2t - 1 \) and \( s \in \{0; 1\} \) write

\[
\begin{align*}
\tilde{\psi} \circ g \bigg|_{I_{2k+p,s}} \quad &\overset{\text{by (i) of Lema 3.7}}{=} \tilde{\psi}_{4k+2p+s} \circ g_{1} \bigg|_{I_{2k+p,s}} &\overset{\text{by (3.14)}}{=} g_{s+p+1-2ps}^{-1} \circ \psi_{4k+2p+s} \circ g_{1}.
\end{align*}
\]

(3.15)

By (3.13) and (3.15), the equality (1.3) is equivalent to

\[ \psi = g_{s+p+1-2ps}^{-1} \circ \psi_{4k+2p+s} \circ g_{1} \]

for all \( k, \; 0 \leq k \leq t - 1 \) and \( s \in \{0; 1\} \). The last follows from (i) of Lemma 3.9.\]
For any $k$, $t \leq k \leq 2t - 1$ and $s \in \{0; 1\}$ write
\[ \tilde{\psi} \circ g \bigg|_{I_{2k+p,s}} = \tilde{\psi}_{4t-1-4k-2p-s} \circ g_2 \bigg|_{I_{2k+p,s}}. \] (3.16)

If $s = 0$, then write
\[ \tilde{\psi}_{4t-1-4k-2p-s} = \tilde{\psi}_{4t-4k+2(1-p)-4+1} \]
\[ = g_{1+(1-p)+1-2(1-p)}^{-1} \circ \psi_{4t-1-4k-2p-s} = g_{p+1}^{-1} \circ \psi_{4t-1-4k-2p-s} = \]
\[ = g_{1+s+p-2p-s}^{-1} \circ \psi_{4t-1-4k-2p-s}. \] (3.17)

If $s = 1$, then write
\[ \tilde{\psi}_{4t-1-4k-2p-s} = \tilde{\psi}_{4t-4k+2(1-p)-4} \]
\[ = g_{1+(1-p)}^{-1} \circ \psi_{4t-4k-2p-s} = g_{2-p}^{-1} \circ \psi_{4t-4k-2p-s} = \]
\[ = g_{1+s+p-2p-s}^{-1} \circ \psi_{4t-1-4k-2p-s}. \] (3.18)

Thus, by (3.17) and (3.18),
\[ \tilde{\psi}_{4t-1-4k-2p-s} = g_{1+s+p-2p-s}^{-1} \circ \psi_{4t-1-4k-2p-s} \circ g_{1+s+p-2p-s} \circ \psi_{4t-1-4k-2p-s} \] (3.19)

Now, by (3.16) and (3.19),
\[ \tilde{\psi} \circ g \bigg|_{I_{2k+p,s}} = g_{1+s+p-2p-s}^{-1} \circ g_2 \bigg|_{I_{2k+p,s}}. \] (3.20)

By (3.13) and (3.20), the equality (1.3) is equivalent to

\[ \psi = g_{1+s+p-2p-s}^{-1} \circ \psi_{4t-1-4k-2p-s} \circ g_2 \]

for all $k$, $t \leq k \leq 2t - 1$ and $s \in \{0; 1\}$. The last follows from (ii) of Lemma 3.9.

\[ \square \]

### 3.2 The case when unimodal map is conjugated with the tent map

Let piecewise linear unimodal map $g$, such that $g^{-\infty}(0) = [0, 1]$, will be fixed till the end of this section. By Theorem 2 let $h : [0, 1] \rightarrow [0, 1]$ be the conjugacy from $f$ to $g$.

**Lemma 3.10.** Let $a \in (0, 1)$ be the first positive kink of $g$. If $\psi'(0) > g'(0)$ for a piecewise linear self-semiconjugation $\psi$ of $g$, then $\frac{a \cdot g'(0)}{\psi'(0)}$ is the first positive kink of $\psi$.

**Proof.** It follows from Lemma 3.15 that $\psi'(0) > 1$.

Since $\psi$ is piecewise linear, then there is $\varepsilon > 0$ such that $\psi(x) = \psi'(0) \cdot x$ for all $x \in (0, \varepsilon)$.
Suppose that \( \varepsilon > 0 \) is such that \( \psi'(0) \cdot \varepsilon < a \). Notice that in this case \( x < a \) for all \( x \in (0, \varepsilon) \), because \( \psi'(0) > 1 \). Thus, for all \( x \in (0, \varepsilon) \) we have that \( \psi'(x) = \psi'(0) \cdot x \), \((g \circ \psi)(x) = g'(0) \cdot \psi'(0) \cdot x\) and \( g(x) = g'(0) \cdot x \). Now, by (1.2),
\[
\psi(x) = \psi'(0) \cdot x
\]
for all \( x \in g \circ (0, \varepsilon) \).

Thus, for every \( x \in g \circ \left(0, \frac{a}{\psi'(0)}\right)\) we have that \( \psi(x) = \psi'(0) \cdot x \). Notice that \( g \circ \left(0, \frac{a}{\psi'(0)}\right) = \left(0, \frac{a}{\psi'(0)}\right) \). Remark that \( \frac{a}{\psi'(0)} < a \).

Take an arbitrary \( \delta, 0 < \delta < a \cdot (g'(0) - 1) \) is such that \( g \) is linear on \((a, a + \delta)\). Then for every \( x \in \left(\frac{a + \delta}{\psi'(0)}, \frac{a \cdot g'(0)}{\psi'(0)}\right) \) we have that \( \psi(x) = \psi'(0) \cdot x \), because \( x < \frac{a \cdot g'(0)}{\psi'(0)} \); also \( g(x) = g'(0) \cdot x \), because \( x < a \), and \((g \circ \psi)(x) = g'(a+) \cdot (\psi'(0) \cdot x - a) + g(a)\).

Now, it follows from (1.2) that
\[
\psi(g'(0) \cdot x) = g'(a+) \cdot (\psi'(0) \cdot x - a) + g(a).
\]
Denote \( u = g'(0) \cdot x \) and remark that \( u \in \left(\frac{(a + \delta) \cdot g'(0)}{\psi'(0)}, \frac{a \cdot (g'(0))^2}{\psi'(0)}\right) \), whenever \( x \in \left(\frac{a + \delta}{\psi'(0)}, \frac{a \cdot g'(0)}{\psi'(0)}\right) \).
Rewrite (3.21) as
\[
\psi(u) = g'(a+) \cdot \left(\psi'(0) \cdot u \frac{g'(0)}{g'(0)} - a\right) + g(a).
\]
Clearly, if \( g'(0) \neq g'(a+) \), then there is \( u \in \left(\frac{(a + \delta) \cdot g'(0)}{\psi'(0)}, \frac{a \cdot (g'(0))^2}{\psi'(0)}\right) \) such that \( \psi(u) \neq \psi'(0) \cdot u \).
Remark that if \( \delta \approx 0 \), then \( \frac{(a + \delta) \cdot g'(0)}{\psi'(0)} \approx \frac{a \cdot g'(0)}{\psi'(0)} \).
\[

\text{Remark 3.11.} \quad \text{For any } n \geq 1 \text{ and } k, 0 \leq k \leq 2^{n-1} \text{ we have that}
\]
\[
\mu_{n,k}(f) = \frac{k}{2^{n-1}}.
\]
The next is the direct corollary from Remark 3.11.

\[
\text{Remark 3.12.} \quad \text{Let } t \geq 1, \text{ and } \xi_t \text{ be defined by (2.1). For any } n \geq 1 \text{ and } k, \frac{k}{2^{n-1}} \leq t \text{ we have that}
\]
\[
\xi_t(\mu_{n,k}(f)) = \mu_{n,k}(f).
\]
\[
\text{Lemma 3.13.} \quad \text{For any self semi conjugacy } \psi \text{ of } g \text{ there exists a self semi conjugacy } \xi \text{ of } f \text{ such that}
\]
\[
\psi = h \circ \xi \circ h^{-1}.
\]
\[
\text{Proof.} \quad \text{Notice that}
\]
\[
\begin{cases}
\psi = \eta \circ h^{-1}, \\
\eta = \psi \circ h.
\end{cases}
\]
Figure 3: Illustrations

provides the correspondence between self semi conjugations $\psi$ of $g$, and the semi conjugations $\eta$ from $f$ to $g$ (see Fig. 3a).

Now the correspondence

$$
\begin{align*}
\xi &= h^{-1} \circ \eta, \\
\eta &= h \circ \xi
\end{align*}
$$

relates conjugacies $\eta$ from $f$ to $g$, and the self semi conjugacies $\xi$ of the tent map (see Fig. 3b). These relations gives the necessary equality

$$
\psi = \eta \circ h^{-1} = (h \circ \xi) \circ h^{-1}.
$$

The next corollary follows from Theorem 3 and Lemma 3.13.

**Corollary 3.14.** For any self semi conjugacy $\psi$ of $g$ there exists $t \in \mathbb{N}$ such that

$$
\psi = h \circ \xi_t \circ h^{-1},
$$

where $\xi_t$ is determined by (2.1).

Due to Corollary 3.14, for every $t \geq 1$ denote

$$
\psi_t = h \circ \xi_t \circ h^{-1}. \tag{3.22}
$$

**Lemma 3.15.** For any $t \in \mathbb{N}$ the equality

$$
\psi_t(\mu_{n,k}(g)) = \mu_{n,kt}(g)
$$

holds for all $n \geq 1$ and $k, k \leq \left\lfloor \frac{n-1}{t} \right\rfloor$, where $\psi_t$ is determined by (3.22).

**Proof.** For every $n \in \mathbb{N}$ and $k, k \leq \left\lfloor \frac{n-1}{t} \right\rfloor$ we have that

$$
\psi_t(\mu_{n,k}(g)) \overset{\text{Lemma 3.15}}{=} (h \circ \xi_t \circ h^{-1})(\mu_{n,k}(g)) = (h \circ \xi_t)(\mu_{n,k}(f)) \overset{\text{Rem. 3.12}}{\text{by Rem. 3.12}}
$$
which is necessary.

\textbf{Lemma 3.16.} Let \( a \) be the first kink of \( g \). For every \( n, k \) such that \( \mu_{n,k}(g) < a \) we have that

\[
\mu_{n,2k}(g) = g'(0) \cdot \mu_{n,k}(g).
\]

\textit{Proof.} Since \( g \) is linear on \((0, a)\) and \( g(0) = 0 \), then for every \( n, k \) such that \( \mu_{n,k}(g) < a \) we have that \( g(\mu_{n,k}(g)) = g'(0) \cdot \mu_{n,k}(g) \). From another hand,

\[
g(\mu_{n,k}(g)) \text{ by Lemma 2.4} = \mu_{n-1,k}(g) \text{ by Rem. 2.3} = \mu_{n,2k}(g),
\]

and the lemma follows.

The following lemma is similar to Lemma 3.16.

\textbf{Lemma 3.17.} Suppose that \( \psi_t \) is piecewise linear for some \( t \geq 1 \). Let \( p \) be the first kink of \( \psi_t \). For every \( n, k \) such that \( \mu_{n,k}(g) < \psi(p) \) we have that

\[
\mu_{n,tk}(g) = \psi_t'(0) \cdot \mu_{n,k}(g).
\]

\textit{Proof.} It follows from the linearity of \( \psi_t \) at 0, and from Lemma 3.15.

We will need the next technical fact.

\textbf{Lemma 3.18.} For every \( n, k \) the set

\[
\left\{ \frac{k \cdot t^p}{2^n + m}, p \in \mathbb{Z}_+, m \in \mathbb{Z} \right\} \cap \left[ \frac{1}{2^n}, \frac{1}{2^n - 1} \right]
\]

is dense in \( \left[ \frac{1}{2^n}, \frac{1}{2^n - 1} \right] \).

\textit{Proof.} For every \( p \geq 1 \) there is the unique \( m \geq 1 \) such that

\[
\frac{k \cdot t^p}{2^n + m} \in \left[ \frac{1}{2^n}, \frac{1}{2^n - 1} \right).
\]

Thus, denote \( m_p, t \geq 1 \) be such that

\[
\frac{k \cdot t^p}{2^n + m_p} \in \left[ \frac{1}{2^n}, \frac{1}{2^n - 1} \right).
\]

The latter means that \( m_p, t \geq 1 \) be such that

\[
k \cdot t^p \in \left[ 2^{m_p}, 2 \cdot 2^{m_p} \right].
\]

Apply \( \log_2 \) and obtain

\[
p \cdot \log_2 t + \log_2 k \in \left[ m_p, m_p + 1 \right],
\]
whence
\[ p \mapsto p \cdot \log_2 t + \log_2 k - m_p \]
will define the \( p \)-th iteration of the rotation by \( \log_2 t \) of the point \( \log_2 k \) on the unit circle. Since \( t \) is not a power of 2, then \( \log_2 t \) is irrational, and lemma follows from the density of any trajectory of the irrational rotation of the unit circle.

\[ \text{Lemma 3.19.} \] If \( \psi_t \) is piecewise linear, then
\[ \psi_t'(0) = (g'(0))^{\log_2 t}. \]

Proof. For every \( n \geq 1 \) and \( k \in \{0, \ldots, 2^n - 1\} \) it follows from Lemma 3.18 that there exist sequences \((s_i)\), \( i \geq 1 \) and \((p_i)\), \( i \geq 1 \) such that
\[ \frac{k \cdot t^{p_i}}{2^{n+s_i}} \to \frac{k}{2^{n-1}} \quad (3.23) \]
for \( i \to \infty \). By Remark 3.11 it means that
\[ \mu_{n,k,2^{-s_i}}(f) \to \mu_{n,k}(f). \]

By continuity of \( h \) obtain
\[ h(\mu_{n,k,2^{-s_i}}(f)) \to h(\mu_{n,k}(f)). \]
and Lemma 2.5
\[ \mu_{n,k,2^{-s_i}}(g) \to \mu_{n,k}(g). \]

Now simplify
\[ \mu_{n,k,2^{-s_i}}(g) \quad \text{by Lemma 3.16} = (g'(0))^{-s_i} \cdot \mu_{n,k}(g) \quad \text{by Lemma 3.17} = (g'(0))^{-s_i} \cdot (\psi_t'(0))^{p_i} \mu_{n,k}(g), \]
whence
\[ (g'(0))^{-s_i} \cdot (\psi_t'(0))^{p_i} \to 1. \quad (3.24) \]

It follows from (3.23) that
\[ t^{p_i} \cdot 2^{-s_i} \to 1. \]

Take log\(_2\) of both sides of the last limit and obtain that
\[ p_i \log_2 t - s_i \to 0, \]
whence the sequence
\[ \varepsilon_i = p_i \log_2 t - s_i \quad (3.25) \]
is such that
\[ \varepsilon_i \to 0, \text{ when } i \to \infty. \quad (3.26) \]
Plug (3.25) into (3.24), and obtain

\[(g'(0))^{\xi_i - p_i \log_2 t} \cdot (\psi'_t(0))^{p_i} \to 1,\]

whence

\[(g'(0))^{\xi_i} \cdot \left(\frac{\psi'(0)}{(g'(0))^{\log_2 t}}\right)^{p_i} \to 1.\]

By (3.26), the sequence \((g'(0))^{\xi_i}\) is bounded by positive numbers, whence

\[\frac{\psi'(0)}{(g'(0))^{\log_2 t}} = 1,\]

and the lemma follows.

We are ready now to prove Proposition 2.

**Proof of Proposition 2.** It follows from Lemmas 2.5 and 3.16 that

\[h(\mu_{n,2k}(f)) = g'(0) \cdot h(\mu_{n,k}(f))\]

for all \(n \geq 1\) and all \(k\), such that \(\mu_{n,k}(g) < a\), where \(a\) is the first kink of \(g\). Now, since, by Remark 3.11, the set \(\{\mu_{n,k}(f), n \geq 1, 0 \leq k \leq 2^{n-1}\} \cap [0,a]\) is dense in \([0,a]\), and \(\mu_{n,2k}(f) = 2 \cdot \mu_{n,k}(f)\) for all \(n, k\), then

\[h(2x) = g'(0) \cdot h(x)\]  

(3.27)

for all \(x < h^{-1}(a)\). We can analogously use Lemma 3.17 to conclude that

\[h(tx) = \psi'_t(0) \cdot h(x)\]  

(3.28)

for all \(x \in [0,h^{-1}(p)]\), where \(p\) is the first kink of \(\psi_t\). There there exists \(x^*\) such that (3.27) and (3.28) holds for all \(x \in [0,x^*]\).

We can find the solution of (3.27) as

\[h(x) = x^{\log_2 g'(0)} \cdot \omega(\log_2 x),\]  

(3.29)

where \(\omega : \mathbb{R} \to \mathbb{R}\) is an unknown continuous function that that

\[\omega(x + 1) = \omega(x)\]  

(3.30)

for all \(x \in \mathbb{R}\).

By Lemma 3.19 rewrite the functional equation (3.28) as

\[h(tx) = (g'(0))^{\log_2 t} \cdot h(x).\]
The latter equality and (3.29) give

\[(tx)^{\log_2 g'(0)} \cdot \omega(\log_2 tx) = (g'(0))^{\log_2 t} \cdot x^{\log_2 g'(0)} \cdot \omega(\log_2 x).\]

After the cancellation by \(x^{\log_2 g'(0)}\), we obtain

\[t^{\log_2 g'(0)} \cdot \omega(\log_2 tx) = (g'(0))^{\log_2 t} \cdot \omega(\log_2 x).\]

Notice, that \((g'(0))^{\log_2 t} = t^{\log_2 g'(0)}\), whence, reminding (3.30), get

\[
\begin{align*}
\omega(x + 1) &= \omega(1), \\
\omega(x + \log_2 t) &= \omega(x).
\end{align*}
\]

(3.31)

Notice that (3.31) means \(x \in [0, 1]\) the values of \(\omega\) are the same on the entire trajectory of \(x\) under irrational rotation of the unit circle \([0, 1]\) by the angle \(\log_2 t\). Thus, \(\omega = \text{const}\) and, finally,

\[h(x) = \omega \cdot x^{\log_2 g'(0)}.\]

(3.32)

Now we are done by Theorem 4, because (3.32) defines a continuously differentiable function.

\[\square\]

### 3.3 The case when the pre-image of 0 under the action of the tent map is dot dense in \([0, 1]\)

Remind that, by Theorem 2, a unimodal surjective map is topologically conjugated with the Tent map if and only if the complete pre-image of 0 under its action is dense in \([0, 1]\).

Thus, let \(Z \subset [0, 1]\) be an open interval such that \(g^n(x) \neq 0\) for all \(x \in Z\) and \(n \geq 0\).

**Lemma 3.20.** Suppose that \(g^{-\infty}(0) \cap Z = \emptyset\) and \(\psi : [0, 1] \to [0, 1]\) be continuous surjective map, which commutes with \(g\). Then for any connected open interval \(J \subset [0, 1]\) such that \(\psi(J) = Z\) we have that \(g^{-\infty}(0) \cap J = \emptyset\).

**Proof.** It follows from (12) that

\[\psi \circ g^n = g^n \circ \psi\]

for all \(n \geq 0\). Suppose that \(x \in J\) is such that \(g^n(x) = 0\). That \((\psi \circ g^n)(x) = \psi(0) = 0\), but

\[(g^n \circ \psi)(x) \neq 0, \text{ because } \psi(x) \in Z.\]

\[\square\]

**Remark 3.21.** It follows from definitions that if \(g^{-\infty}(0) \cap Z = \emptyset\) for some open interval \(Z \subset [0, 1]\), then \(g^{-\infty}(0) \cap g^{-1}(Z) = \emptyset\).
Proof of Theorem 1. As above, let $Z \subset [0, 1]$ be an open interval such that $g^n(x) \neq 0$ for all $x \in Z$ and $n \geq 0$.

Denote $\psi_l$ the maximal monotone part of $\psi$, whose domain contains 0. Then, by Lemma 3.20 and Remark 3.21 we have that $\psi_l^{-n}(Z) \cap g^{-\infty}(0) = \emptyset$ and $g_l^{-n}(Z) \cap g^{-\infty}(0) = \emptyset$.

Denote $x_0 \in (0, 1)$ such that $g(x_0) = 1$, let $x_i = g_l(x_{i-1})$ for $i \geq 1$ and $I_i = (x_{i+1}, x_i)$ for all $i \geq 1$.

Let $k$ be such that both $g$ and $\psi$ are linear on $(0, x_k)$. Then

$$I_{k+t} = \left( \frac{x_k}{(g'(0))^{t+1}}, \frac{x_k}{(g'(0))^t} \right), \ t \geq 0.$$

Define $\nu : (0, x_k) \to [0, 1]$ as

$$\nu(x) = \frac{1}{x_k} \cdot \{\log g'(0) x\},$$

where $\{\cdot\}$ denotes a fractional part of a number. Clearly,

$$\nu \circ g_l^{-1} = \nu,$$

and

$$\nu \circ \psi_l^{-1} = \frac{1}{x_k} \cdot \{\log g'(0) x - \log g'(0) \psi'(0)\},$$

which is irrational rotation of a unit circle, whenever $\psi$ is not an iteration of $g$.

Since any trajectory is dense in $[0, 1]$ under an irrational, then rotation, then $[0, x_k) \cap g^{-\infty}(0) = g_l^{-\infty}(0)$, which is a contradiction, because the bigger pre-image of the maximum point of $g$ has also have pre-images in $(0, x_k)$. \qed

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