Soliton, double layer and shock formation in pair-ion plasmas

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Abstract. Nonlinear electrostatic structures are studied in unmagnetized pair-ion plasmas. The low amplitude solitons and double layer structures are obtained using reductive perturbation method in non-dissipative and ideal plasmas. It is found that both electrostatic potential hump (compressive) and dip (rarefactive) solitons and double layers structures are obtained depending on the temperature ratio between and positive and negative ion species. The Kadomtsev-Petviashvili-Burger (KPB) is also derived by taking into account the weak transverse perturbations and kinematic viscosity of both positive and negative ions plasmas. Both two-dimensional rarefactive and compressive shocks and solitons solutions are obtained using Tan hyperbolic method. The structure dependence on temperature ratios between two ion species is also discussed. The present study may have some relevance for understanding of the formation of electrostatic structure in laboratory produced pair-ion plasmas.

1. Introduction
The collective modes in relativistic and nonrelativistic electron-positron (EP) pair plasmas have been studied with its application to astrophysical plasmas [1-5] and references therein. However, it is not an easy task to generate and maintain EP pair plasma in laboratory experiments for longer times due to their short annihilation time in comparison with plasma period to study collective effects. Therefore, in order to study the collective modes for same mass but oppositely charged particle plasma, a pair-ion (PI) fullerene plasma was produced in the laboratory [6-7]. The PI plasma is quite different from the usual electron-ion (EI) plasmas, where difference in masses of ions and electrons induces different time and length scales. The space time symmetry is maintained in pure pair-ion (PI) and electron-positron (EP) plasmas, because mobility of charge particles is the same in electromagnetic fields in this case. Three electrostatic modes have been observed parallel to the magnetic field in pure fullerene plasma experiment. A lot of theoretical research work has already been published on electrostatic waves in PI plasmas under the condition that temperatures of the species are not equal [8-12].

The manuscript has been organized in the following way: The nonlinear dynamic equations for electrostatic waves in PI plasmas are presented in Sec.II. The Kortewge-de Vries (KdV) equation is obtained using reductive perturbation method in Sec. III. The extended KdV (EKdV) equation is also obtained for low amplitude double layer (DL) structures in PI plasmas in Sec. IV. The Kadomtsev-Petviashvili-Burgers (KPB) equation for two dimensional soliton and dissipative shock solutions are derived for PI plasmas in Sec. V. Finally, conclusion on nonlinear electrostatic structures has been presented in Sec.VI.
II. Nonlinear set of equations for pair-ion plasma

We are considering pure pair-ion (PI) unmagnetized plasma. Two fluid model has been used to study the electrostatic waves for isothermally heated positive and negative ions of same mass. The normalized set of continuity and momentum equations for positive and negative ions in one dimension can be written as follows,

\[
\begin{align*}
\frac{\partial n_+}{\partial t} + \frac{\partial}{\partial x}(n_+ v_+) &= 0, \\
\frac{\partial n_-}{\partial t} + \frac{\partial}{\partial x}(n_- v_-) &= 0, \\
\frac{\partial v_+}{\partial t} + (v_+ \frac{\partial}{\partial x})v_+ &= -\frac{\partial \Phi}{\partial x} - \frac{1}{n_+} \frac{\partial n_+}{\partial x}, \\
\frac{\partial v_-}{\partial t} + (v_- \frac{\partial}{\partial x})v_- &= -\frac{\partial \Phi}{\partial x} - \frac{\beta}{n_-} \frac{\partial n_-}{\partial x},
\end{align*}
\]

The Poisson equation is given by

\[
\frac{\partial^2 \Phi}{\partial x^2} = (n_- - n_+),
\]

where the normalized electrostatic potential \( \Phi = e\phi/T_+ \) has been defined. The perturbed velocities for positive and negative ions are normalized by ion thermal velocity due to positive ions \( v_{T_+} = (T_+/m)^{1/2} \) and perturbed number density by unperturbed densities \( n_0 \). The temperature ratio of negative and positive ion species \( \beta = T_-/T_+ \) has been defined. The space and time co-ordinates, \( x \) and \( t \) are normalized by Debye length \( \lambda_{D_+} = (T_+/4\pi n_0 e^2)^{1/2} \) and ion plasma frequency by \( \omega_p = (4\pi n_0 e^2/m)^{1/2} \) respectively. In equilibrium, we have \( n_0 = n_0 \) (say \( n_0 \)). The isothermal ion pressure is defined as \( p_\pm = \gamma n_\pm T_\pm \) and \( \gamma = 1 \).

III. Derivation of Korteweg-de Vries equation in pair-ion plasmas

In this section, we apply the reductive perturbation method to solve nonlinear Eqs.(1-5) and obtain Korteweg-de Vries (KdV) equation for electrostatic soliton in unmagnetized pure PI plasmas. The perturbed quantities can be expanded in the power series of \( \varepsilon \) as follows:

\[
\begin{align*}
n_j &= 1 + \varepsilon n_j^{(1)} + \varepsilon^2 n_j^{(2)} + \ldots \\
v_j &= \varepsilon v_j^{(1)} + \varepsilon^2 v_j^{(2)} + \ldots \\
\Phi &= \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \ldots
\end{align*}
\]

Here all perturbed quantities are functions of \( x \) and \( t \), while \( \varepsilon \) is a small \( 0 < \varepsilon < 1 \) expansion parameter characterizing the strength of the nonlinearity, such that the stretched variables are introduced in the standard fashion as follows:

\[
\xi = \varepsilon^{1/2}(x - \lambda t), \quad \tau = \varepsilon^{3/2} t
\]

Using Eqs. (6-7) and (9) in Eqs (1-5) and collecting the lowest order terms, we obtain

\[
\lambda = \sqrt{\frac{1 + \beta}{2}}.
\]

which is the linear phase velocity of acoustic wave in PI plasmas. Now collecting, the next higher
order and after some simplification we obtain the following Korteweg-de Vries (KdV) equation for the pure PI plasmas as follows [11],

\[ \partial_t \Phi^{(1)} + A \partial_x \Phi^{(1)} + B \partial_x^3 \Phi^{(1)} = 0, \quad (11) \]

where nonlinear and dispersive coefficients are defined by \( A \) and \( B \) respectively,

\[ A = \frac{[(3\lambda^2 - 1)(\lambda^2 - \beta)^3 - (3\lambda^2 - \beta)(\lambda^2 - 1)^3]}{2\lambda(\lambda^2 - 1)(\lambda^2 - \beta)[(\lambda^2 - \beta)^2 + (\lambda^2 - 1)^2]}, \]

\[ B = \frac{(\lambda^2 - 1)^2(\lambda^2 - \beta)^2}{2\lambda[(\lambda^2 - \beta)^2 + (\lambda^2 - 1)^2]}. \]

The soliton solution of Eq. (11) is given by,

\[ \Phi^{(1)} = \phi_m \text{sech}^2 \left( \frac{\eta}{W} \right), \quad (12) \]

Where \( \phi_m = 3U/A \) and \( W = (4B/U)^{1/2} \) are amplitude and the width of electrostatic soliton in pure PI plasmas, and \( \eta \) is the transformed coordinate with respect to a wave frame moving with velocity \( U \) \( \eta = \xi - Ut \). The weakly nonlinear electrostatic excitation in pure PI plasmas parallel to magnetic field is possible only when a slight difference of temperature between the positive and negative ions (i.e., \( T_+ \neq T_- \)) exist. The numerical values of PI plasma density \( 1 - 2 \times 10^8 \text{ cm}^{-3} \) at \( E_e = 100 \text{ eV} \) while temperatures of \( C_{60}^+ \) and \( C_{60}^- \) ions i.e., \( T_+ \) and \( T_- \) respectively lies in the range of \( 0.3 - 0.5 \text{ eV} \) have been reported in the laboratory experiments [7].

**IV. Derivation of extended KdV equation for double layers in pair-ion plasmas**

An electrostatic double layer is a structure in plasma which consists of two parallel layers with opposite electrical charge. These sheets of charge cause a strong electric field and correspondingly a sharp change in voltage (electrical potential) across the double layer. The positive and negative ions, which enter the double layer are accelerated, decelerated or reflected by the electric field. In order to find DL we use the following stretching for independent variables described as follows,

\[ \xi = \varepsilon(x - \lambda t), \quad \tau = \varepsilon^3 t \quad (13) \]

The perturbed quantities are expanded in the powers of \( \varepsilon \) as defined earlier. Now collecting the higher order terms of continuity and momentum equations of ions \( -\varepsilon^3 \) and Poisson equation \( -\varepsilon^4 \) and using relation of parameter given in Eq.(10) after some simplification, we obtain the EKdV equation given by :

\[ A \partial_t \varphi_1 + Q \partial_x \varphi_1^2 + B \partial_x \varphi_1^3 + \partial_x^3 \varphi_1 = 0 \quad (14) \]

where the coefficients are defined as

\[ A = 2\lambda \left[ \frac{1}{(\lambda^2 - \beta)^2} + \frac{1}{(\lambda^2 - 1)^2} \right] \]

\[ B = \left[ \frac{(3\lambda^2 - 1)^2}{2(\lambda^2 - 1)^5} - \frac{2\lambda^2 - \frac{1}{3}}{(\lambda^2 - 1)^4} + \frac{(3\lambda^2 - 1)^2}{2(\lambda^2 - \beta)^5} - \frac{(2\lambda^2 - \beta)^2}{(\lambda^2 - \beta)^4} \right] \]
\[ Q = \frac{(3\lambda^2 - \beta)}{2(\lambda^2 - \beta)^3} - \frac{(3\lambda^2 - 1)}{2(\lambda^2 - 1)^3} \]

Now using the transformation \( \eta = \xi - U\tau \), where \( U \) is the speed of the co-moving frame with the speed of the soliton. The above EKdV equation can be described in the form of energy integral equation as follows,

\[ \frac{1}{2} \left( \frac{\partial \varphi_1}{\partial \eta} \right)^2 + V(\varphi_1) = 0 \]  (15)

where the Sagdeev potential is defined as

\[ V(\varphi_1) = -\frac{UA}{2} \varphi_1^2 + \frac{Q}{3} \varphi_1^3 + \frac{B}{4} \varphi_1^4 \]  (16)

For the formation of DLs, the Sagdeev potential must satisfy the following conditions,

\[ V(\varphi_1) = 0, \quad \frac{dV}{d\varphi_1} = 0 \text{ at } \varphi_1 = 0 \text{ and } \varphi_1 = \varphi_m, \]

\[ \frac{d^2 V}{d\varphi_1^2} < 0 \text{ at } \varphi_1 = 0 \text{ and } \varphi_1 = \varphi_m \]  (17)

The above conditions are satisfied only if, coefficients \( Q \) and \( U \) described as

\[ Q = -\frac{3}{2} B\varphi_m, \quad U = -\frac{1}{2} \frac{B}{A} \varphi_m^2 \]

The Sagdeev potential satisfying the DLs boundary conditions is given as follows,

\[ V(\varphi_1) = \frac{B}{4} \varphi_1^2 (\varphi_m - \varphi_m)^2 \]  (18)

Using Eq.(38), the solution of energy integral equation (35) can be written as,

\[ \varphi_1 = \frac{\varphi_m}{2} \left[ 1 - \tanh \left\{ \frac{-B}{8} \varphi_m (\eta - U\tau) \right\} \right] \]  (19)

where amplitude and width of the DLs are defined as

\[ \varphi_m = \frac{-2}{3} \frac{Q}{B}, \quad \text{width} = \sqrt{\frac{8}{B} \varphi_m} \]

In order to find the real solution of Eq.(19) \( B<0 \) must hold. In case, \( Q<0 \) compressive and for \( Q>0 \) rarefactive DLs are formed, where \( \beta=\frac{T_+}{T_-} \) has been defined. It is clear from Eq.(19) that DL structures will be formed only when there occurs a small difference of thermal energies between pair ion species.

V. Derivation of Kadomtsev-Petviashvili-Burgers (KPB) equation for pair-ion plasmas

In order to derive the KPB equation in PI fullerene plasmas the dissipation in the system is taken into account through kinematic viscosity of the ions in the system. The normalized form of continuity, momentum, and Poisson's equations in two dimensions are [12]:

\[ \frac{\partial n_+}{\partial t} + \frac{\partial}{\partial x}(n_+u_+) + \frac{\partial}{\partial y}(n_+v_+) = 0, \]
\frac{\partial n_-}{\partial t} + \frac{\partial}{\partial x}(n_- u_-) + \frac{\partial}{\partial y}(n_- v_-) = 0,
\frac{\partial u_+}{\partial t} + u_+ \frac{\partial u_+}{\partial x} + v_+ \frac{\partial u_+}{\partial y} = -\frac{1}{n_+} \frac{\partial n_+}{\partial x} + \eta_+ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_+,
\frac{\partial v_+}{\partial t} + u_+ \frac{\partial v_+}{\partial x} + v_+ \frac{\partial v_+}{\partial y} = -\frac{1}{n_+} \frac{\partial n_+}{\partial y} + \eta_+ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v_+,
\frac{\partial u_-}{\partial t} + u_- \frac{\partial u_-}{\partial x} + v_- \frac{\partial u_-}{\partial y} = -\frac{\beta}{n_-} \frac{\partial n_-}{\partial x} + \eta_- \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_-,
\frac{\partial v_-}{\partial t} + u_- \frac{\partial v_-}{\partial x} + v_- \frac{\partial v_-}{\partial y} = -\frac{\beta}{n_-} \frac{\partial n_-}{\partial y} + \eta_- \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v_-,
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = n_- - n_+.

Here, the ion fluid velocities along x-axis and y-axis i.e., \( u_a \) and \( v_a \) respectively, are normalized by the ion-thermal speed \( v_{\text{Th}}=(T_{\text{i}}/m)^{1/2} \), ion density \( n_a \) is normalized by equilibrium density \( n_0 \) and electrostatic wave potential \( \phi \) is normalized by \( T_{\text{i}}/e \). The space and time variables are normalized in the units of positively charged ion Debye length \( \lambda_{\text{D}+}=(T_{\text{i}}/(4\pi n_0 e^2))^{1/2} \) and its plasma frequency \( \omega_{\text{p}+}=(4\pi n_0 e^2/m)^{1/2} \) respectively. The temperature ratios of the ion species are defined as \( \beta=T_{\text{i}}/T_+ \) whereas the normalized viscosity coefficients are \( \eta_+ = \nu_+/\left(\lambda_+ V_{\text{Th}_+}\right) \) and \( \eta_- = \nu_-/\left(\lambda_- V_{\text{Th}_-}\right) \) for positively and negatively charged ion fluid, respectively and \( \nu_{\pm} = \mu_{\pm}/mn_0 \) have been defined. The viscosity coefficients of the same mass fluids are different since their temperatures are assumed to be different.

The kinematic viscosity is defined as \( \nu_a = \mu_a/mn_0 \) (where \( \mu_a \) is defined as the dynamic viscosity of the \( a \) species in PI plasmas). The dynamic viscosity of fully ionized gas has been analyzed by Braginskii [13] and in the absence of magnetic field, the dynamic viscosity of ions is given by
\[
\mu_{\alpha} = 0.406 m_\alpha^2 \left( k_B T_{\alpha} \right)^2 \frac{2.21 \times 10^{-15} \ T_{\alpha}^2 A_\alpha^2 \ gm}{Z^4 \ln \Lambda \ cm \ sec}
\]
where \( A_\alpha \) is the atomic weight of ions, \( Z \) is the charge number and \( \ln \Lambda \) is the Coulomb logarithm.
The above defined dynamic viscosity of ions by Braginskii will be used for numerical studies in our model for PI species.

In order to study the two dimensional electrostatic waves in unmagnetized pair-ion plasmas, we define the stretching of the independent variables as:
\[
\zeta = \epsilon^{1/2} (x - \lambda t), \ \chi = \epsilon y, \ \tau = \epsilon^{3/2} t,
\]
where \( \epsilon \) is a small expansion parameter and \( \lambda \) is wave phase velocity. Using reductive perturbative technique, we expand the perturbed quantities about their equilibrium values in powers of \( \epsilon \) such that
\[ n_a = 1 + \epsilon n_a^{(1)} + \epsilon^2 n_a^{(2)} + \epsilon^3 n_a^{(3)} + \cdots, \]
\[ u_a = \epsilon u_a^{(1)} + \epsilon^2 u_a^{(2)} + \epsilon^3 u_a^{(3)} + \cdots, \]
\[ v_a = \epsilon^2 v_a^{(1)} + \epsilon^3 v_a^{(2)} + \epsilon^4 v_a^{(3)} + \cdots, \]
\[ \phi = \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \epsilon^3 \phi^{(3)} + \cdots. \]

The value of \( \eta \) is assumed to be small, i.e., \( \eta = \epsilon^{1/2} \eta_{0\pm} \) where \( \eta_{0\pm} \) is \( O(1) \). Substituting above relations in the dynamics equations of PI plasmas and collecting different powers of \( \epsilon \), the lowest order equation yields the dispersion relation of acoustic wave in PI plasmas is given in Eq.(10).

From next higher order equations in \( \epsilon \) we obtain the KPB equation for pair-ion plasmas as follows
\[
\frac{\partial}{\partial \zeta} \left( \frac{\partial \Phi}{\partial \tau} + A \Phi \frac{\partial \Phi}{\partial \zeta} + B \frac{\partial^3 \Phi}{\partial \zeta^3} - C \frac{\partial^2 \Phi}{\partial \zeta^2} \right) + D \frac{\partial^2 \Phi}{\partial \chi^2} = 0,
\]
\[
A = \frac{[(3\lambda^2 - 1)(\lambda^2 - \beta)^3 - (3\lambda^2 - \beta)(\lambda^2 - 1)^3]}{2\lambda(\lambda^2 - 1)(\lambda^2 - \beta)[(\lambda^2 - 1)^2 + (\lambda^2 - \beta)^2]},
\]
\[
B = \frac{(\lambda^2 - 1)^2(\lambda^2 - \beta)^2}{2\lambda[(\lambda^2 - 1)^2 + (\lambda^2 - \beta)^2]},
\]
\[
C = \frac{\eta_{0\uparrow}(\lambda^2 - \beta)^2 - \eta_{0\downarrow}(\lambda^2 - 1)^2}{2[(\lambda^2 - 1)^2 + (\lambda^2 - \beta)^2]},
\]
\[
D = \frac{\lambda}{2},
\]

where \( \Phi \equiv \phi^{(1)} \), \( A \) and \( C \) are the coefficients of non-linearity and dissipation whereas \( B \) and \( D \) are the coefficients of predominant and weak dispersion respectively. Using the transformation \( \tilde{\xi} = k(\zeta + \chi - U\tau) \), where \( k \) is the dimensionless nonlinear wave number. There are number of methods to solve the nonlinear partial differential equations (NLPDE’s). However, when the partial differential equation in a system is formed by the combined effect of dispersion and dissipation, the most convenient and efficient method to solve the NLPDE is tanh method [13], which gives the solution of KPB equation as follows,
\[
\Phi(\zeta, \chi, \tau) = \frac{6}{25} \frac{C^2}{AB} \left[1 - \tanh \frac{C}{10B} \left\{ \zeta + \chi - \left( \frac{6C^2}{25B} + D \right) \tau \right\} \right] + \frac{3}{25} \frac{C^2}{AB} \left[ \sec h^2 \frac{C}{10B} \left\{ \zeta + \chi - \left( \frac{6C^2}{25B} + D \right) \tau \right\} \right].
\]

Where \( k = C/10B \). The above solution will exist only if \( T_+ \neq T_- \) holds. The coefficients \( A \), \( B \) and \( C \) become undetermined at \( T_+ = T_- \) and therefore no shock and soliton structures can be obtained. If we assume the dissipative coefficient \( C \) equal to zero, then the KPB equation reduces to KP equation which admits soliton solution given as
\[
\Phi(\zeta, \chi, \tau) = \frac{12B}{A} \sec h^2 [\zeta + \chi - U\tau].
\]

Where \( U = -(4B + D) \). The parametric dependence of the electrostatic shocks and solitons for above
equations has been shown and discussed in detail in [12]. Both compressive and rarefactive two dimensional solitons can be obtained from Eq.(22) depending on the value of $\beta$ i.e., the value of the temperature ratio between between positive and negatively charged ions in PI plasmas (i.e., $T_+ < T_-$ and $T_+ > T_-$ ). Similarly the shock structures from Eq.(21)can be obtained for the cases $T_+ < T_-$ and $T_+ > T_-$ respectively. It can be seen from the figures that for $T_- > T_+$ case, only compressive shocks are obtained while for $T_+ > T_-$ case, rarefactive shocks are formed. This happens due to the fact that for $T_+ > T_-$, the nonlinearity changes sign (it becomes negative in this case contrary to the positive nonlinearity sign in the $T_- > T_+$ case) and hence the formation of the rarefactive shocks. The situation reverses for $T_+ > T_-$ case owing the change of the sign of the nonlinearity.

VI. Conclusion

To conclude, linear and nonlinear electrostatic waves have been studied in unmagnetized PI plasmas. The nonlinear partial differential equations such as KdV, EKdV and KPB equations are obtained using reductive perturbation method. It is found that KdV equation gives both electrostatic hump and dip soliton solutions in unmagnetized pure PI plasmas. The EKdV equation gives both the compressive and rarefactive electrostatic DLs structures in PI plasma depending on the value of temperature ratio of both PI species. The study of DLs in PI plasma is important due to its application in accelerating or decelerating the charged particles in plasmas. The two dimensional soliton and shock structures are also obtained from KPB equation by taking into account the dissipation through kinematic viscosity of the PI species. It is found that electrostatic structures are formed only when there is a slight difference of temperature between positive and negative fullerene ions remains in PI plasma. The results are general and may be applicable to explain some salient features of the formation of nonlinear electrostatic structures in laboratory produced PI plasmas.

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