Normal Forms for Flat Two-input Control Systems Linearizable via a Two-fold Prolongation

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Abstract: We present normal forms for nonlinear two-input control systems that become static feedback linearizable after a two-fold prolongation of a suitably chosen control, which is one of the simplest dynamic feedback. They form a particular class of flat systems, namely those of differential weight $n + 4$, where $n$ is the number of states. We also show that the dynamic feedback creates singularities in the control space depending on the state and we discuss them.

Keywords: Normal forms, nonlinear control systems, flatness, dynamic linearization.

1. INTRODUCTION

In this paper, we give normal forms for flat control-affine systems of the form

$$
\Sigma : \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2,
$$

(1)

where $x$ is the state defined on an open subset $X$ of $\mathbb{R}^n$ and $u = (u_1, u_2)$ is the control taking values in an open subset $U$ of $\mathbb{R}^2$ (more generally, an $n$-dimensional manifold $X$ and a two-dimensional manifold $U$, resp.), and where $g_1$ and $g_2$ are smooth. The word smooth will always mean $C^\infty$-smooth. The notion of flatness was introduced in control theory in the 1990s, by Fliess, Lévine, Martin and Rouchon (Fliess et al. [1995], see also Isidori et al. [1986], Jakubczyk [1993], Aranda-Bricaire et al. [1995], Pomet [1995]) and has attracted a considerable interest Fliess et al. [1999], Pomet [1997], Van Nieuwstadt et al. [1998], Pereira da Silva and Corrêa Filho [2001] because of its important applications in the problem of motion planning and constructive controllability (see, e.g., Martin et al. [2003], Lévine [2009], Tang et al. [2011], Kolar et al. [2017]). The system $\Sigma : \dot{x} = F(x,u)$, where $x \in X \subset \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$, is flat if we can find locally $m$ functions $\varphi_1(x,u,\ldots,u^{(r)})$, for some $r \geq 0$, such that

$$
x = \gamma(\varphi,\ldots,\varphi^{(s-1)}) \quad \text{and} \quad u = \delta(\varphi,\ldots,\varphi^{(s)}),
$$

for a certain integer $s$ and suitable smooth maps $\gamma$ and $\delta$, where $\varphi = (\varphi_1,\ldots,\varphi_m)$ is called a flat output. Therefore, the evolution in time of all state and control variables can be recovered from that of flat outputs without integration and all trajectories of the system can be completely parameterized.

Systems linearizable via invertible static feedback are flat and their normal forms are well known: they are static feedback equivalent to the Brunovsky canonical form. Flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see Fliess et al. [1995], Pomet [1995, 1997]. In Nicolau and Respondek [2019], the authors presented normal forms for the class of flat systems that are the closest to static feedback linearizable ones, namely those that are feedback linearizable via the simplest dynamic feedback, which is a one-fold prolongation of a suitably chosen control. The goal of this paper is to generalize those results to the case of a two-fold prolongation. We will consider the case of two-input control systems only. Solving that problem in the simplest case of two controls is interesting for few reasons; first, it yields a complete analysis for a well defined class of flat systems, and second, it shows what kind of difficulties one may face when trying to give normal forms or to characterize flatness in the general case. Our aim is to give normal forms for nonlinear flat control systems of differential weight $n + m + 2 = n + 4$ (see Respondek [2003], and Section 2 for the notion of differential weight) and to discuss how the geometry of that class of systems is reflected by the normal forms (necessary and sufficient geometric conditions describing flatness of control-affine differential weight $n + m + 2 = n + 4$ were presented in Nicolau and Respondek [2016a]).

It is well known (see, e.g., Jakubczyk and Respondek [1980], Hunt and Su [1981]) that any static feedback linearizable and controllable system is feedback equivalent to the Brunovsky canonical form that consists of $m$ independent chains of integrators. In Nicolau and Respondek [2019], we proposed for multi-input systems dynamically linearizable via a one-fold prolongation (or, equivalently, flat systems of differential weight $n + m + 1$) a modification of the Brunovsky canonical form that contains at most $m - 1$ nonlinearities (at most only one nonlinearity per each chain). For the particular case of two-input control systems, one (and only one) nonlinearity is present. In this paper, we show that two-input systems dynamically linearizable via a two-fold prolongation can be brought into a normal form generalizing that of Brunovsky as well as that characterizing flatness of differential weight $n + 3$. Namely, at most two nonlinearities (at most one more than for flatness of differential weight $n + 3$) are present. Interest in those normal forms is three-fold. First, to understand that

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systems linearizable dynamically via a two-fold prolongation differ from static feedback linearizable ones (resp., from dynamically linearizable via a one-fold prolongation) by at most two (resp., one) non-removable nonlinearities and to identify where those nonlinearities may appear and on which variables they may depend. Second, for a flat system we can express all state and control variables with the help of flat outputs and their derivatives and the proposed normal forms allow to express all but at most two special variables of the transformed system by pure derivations as well as to identify those special variables and to compute them via the implicit function theorem. Third, like for flatness of differential weight \( n + 3 \), the proposed normal forms allow to describe the singularities that the two-fold dynamic prolongation may create in the input space and thus to identify the control values at which the system ceases to be flat. Fourth, like the Brunovsky canonical form, the presented normal forms are compatible with flat outputs: if \((\varphi_1, \varphi_2)\) is a flat output, then there exists an invertible static feedback transformation bringing the system into that normal form with \((\varphi_1, \varphi_2)\) playing the role of the top variables.

In Nicolau and Respondek [2016a], we gave a geometric characterization of control-affine systems that become static feedback linearizable after a two-fold prolongation. The proposed normal forms apply to all systems described there, but we do not use those results to construct our normal forms. The paper is self-contained and all presented results can be proved independently of those of Nicolau and Respondek [2016a] although can be seen as their illustration and their continuation. The paper is organized as follows. In Section 2, we recall the definitions of flatness and of differential weight. In Section 3, we give our main results and illustrate them by two examples in Section 4.

2. FLATNESS

For \( l \geq -1 \), denote \( u^l = (u, u, \ldots, u^{(l)}) \), with \( u^{-1} \) empty. 

**Definition 1.** The system \( \Xi : \dot{x} = F(x, u), \ x \in X \subset \mathbb{R}^n \), \( u \in U \subset \mathbb{R}^m \), is flat at \((x_0, u_0) \in X \times U \times \mathbb{R}^m \), for \( l \geq -1 \), if there exist a neighborhood \( O^l \) of \((x_0, u_0)\) and \( m \) smooth functions \( \varphi_i = \varphi_i(x, u, \dot{x}, \ldots, u^{(l)}) \), \( 1 \leq i \leq m \), defined in \( O^l \), having the following property: there exist an integer \( s \) and smooth functions \( \gamma_i \), \( 1 \leq i \leq n \), and \( \delta_j \), \( 1 \leq j \leq m \), such that 

\[
\begin{align*}
\dot{x}_i &= \gamma_i(\varphi_1, \varphi_2, \ldots, \varphi^{(s-1)}), & \dot{u}_j &= \delta_j(\varphi_1, \varphi_2, \ldots, \varphi^{(s)})
\end{align*}
\]

for any \( C^{l+s} \)-control \( t \) and corresponding trajectory \( x(t) \) that satisfy \((x(t), u(t), \ldots, u^{(l)}(t)) \in O^l \), where \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi^{(s)}) \) and is called a flat output.

If \( \varphi_i = \varphi_i(x) \), for all \( 1 \leq i \leq m \), we say that the system is \( x \)-flat. The minimal number of derivatives of components of a flat output, needed to express \( x \) and \( u \), is called differential weight [Respondek 2003] of that flat output and is formalized as follows. By definition, for any flat output \( \varphi \) of \( \Xi \) there exist integers \( s_1, \ldots, s_m \) such that 

\[
\begin{align*}
x &= \gamma(\varphi_1, \varphi_2, \ldots, \varphi^{(s_1)}, \ldots, \varphi_m, \varphi_m, \ldots, \varphi^{(s_m)}) \\
u &= \delta(\varphi_1, \varphi_1, \varphi_1, \ldots, \varphi_m, \varphi_m, \ldots, \varphi_m).
\end{align*}
\]

Moreover, we can choose \((s_1, \ldots, s_m)\), \( \gamma \) and \( \delta \) such that (see Respondek [2003]) if for any other \( m \)-tuple \((s_1, \ldots, s_m)\) and functions \( \bar{\gamma} \) and \( \bar{\delta} \) we have 

\[
\begin{align*}
x &= \bar{\gamma}(\varphi_1, \varphi_1, \varphi_1, \ldots, \varphi_m, \varphi_m, \ldots, \varphi_m) \\
u &= \bar{\delta}(\varphi_1, \varphi_1, \varphi_1, \ldots, \varphi_m, \varphi_m, \ldots, \varphi_m),
\end{align*}
\]

then \( s_i \leq \bar{s}_i \), for \( 1 \leq i \leq m \). We will call \( \sum_{i=1}^{m}(s_i + 1) = m + \sum_{i=1}^{m} s_i \) the differential weight of \( \varphi \). A flat output of \( \Xi \) is called minimal if its differential weight is the lowest among all flat outputs of \( \Xi \). The differential weight of a flat system equals the differential weight of a minimal flat output. The differential weight is \( n + m + p \), where \( p \geq 0 \) can be interpreted as the minimal dimension of a precompensator that dynamically linearizes the system. Indeed, \( p = 0 \) corresponds to static feedback linearizable systems, while the case \( p = 1 \) corresponds to systems linearizable via a one-fold prolongation of a suitably chosen control Nicolau and Respondek [2016b, 2017]. Presenting normal forms for the case \( p = 1 \) was the subject of Nicolau and Respondek [2019]. The goal of this paper is to give normal forms for two-input flat control systems of differential weight \( n + m + 2 = n + 4 \).

We say that control-affine systems \( \Sigma \) and \( \bar{\Sigma} \) given, resp., by 

\[
\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in X, \ u \in \mathbb{R}^m
\]

and 

\[
\dot{x} = \bar{f}(x) + \sum_{i=1}^{m} \bar{u}_i \bar{g}_i(x), \quad \bar{x} \in \bar{X}, \ \bar{u} \in \mathbb{R}^m
\]

are of \( \varphi \)-flat output, needed to express \( s \varphi_1 \) for any \( \varphi \) differential weight. In Section 3, we give our main results and illustrate them by two examples in Section 4.

Assumption (A1). We will work under constant ranks assumption implying that all results are valid on an open and dense subset of \( X \) and hold locally, around any given point \( x_0 \) of that set, where all involved ranks are constant.
We study flat systems of differential weight \( n + m + 2 \), thus \( \Sigma \) is not static feedback linearizable (not flat of differential weight \( n + m \)). It follows that there exists the smallest integer \( 0 \leq k \leq n - 1 \) such that the linearizability conditions (either involutivity or constant rank) are not satisfied for \( D^k \) (flat systems are always accessible so \( D^{n-1} = TX \) holds). Under Assumption (A1), only the case \( D^k \) noninvolutive can occur and we denote by \( \overline{D}^k \) its involutive closure. Any two-input flat system of differential weight \( n + m + 2 = n + 4 \) becomes static feedback linearizable after a two-fold prolongation of a suitably chosen control, as asserted by the following result.

**Proposition 1.** The following are equivalent:

(i) \( \Sigma \) is flat at \((x_0, u_0, \bar{u}_0, \ldots, u_0^{(1)})\), of differential weight \( n + 4 \), for a certain \( l \geq -1 \);

(ii) \( \Sigma \) is \( x \)-flat at either \( x_0 \) or \((x_0, u_0)\), of differential weight \( n + 4 \);

(iii) There exists, around \( x_0 \), an invertible static feedback transformation \( u = \alpha(x) + \beta(x)\bar{u} \), bringing \( \Sigma \) into the form \( \tilde{\Sigma} : \dot{x} = f(x) + \bar{u}_1 \xi_1(x) + \bar{u}_2 \xi_2(x) \) such that the prolongation

\[
\Sigma^{(2,0)} : \begin{cases}
\dot{x} = f(x) + y_1 \xi_1(x) + y_2 \xi_2(x) \\
y_1 = u_2 \\
y_2 = y_1 \\
\end{cases}
\]

is locally static feedback linearizable around \((x_0, y_0)\), with \( y_1 = \bar{u}_1, y_2 = \bar{u}_2, f = f + \alpha g \) and \( \bar{g} = \beta \), where \( g = (g_1, g_2) \) and \( \bar{g} = (g_1, g_2) \).

A system \( \Sigma \) satisfying (iii) will be called dynamically linearizable via an invertible two-fold prolongation. \( \Sigma^{(2,0)} \) is, indeed, obtained by applying an invertible static feedback \( u = \alpha + \beta \bar{u} \) and then prolonging the first control \( u_1 \) twice as \( v_1 = \bar{u}_1 \) and not prolonging \( u_2 \) (which explains the notation \( \Sigma^{(2,0)} \)). Before giving our main results, we introduce the notion of corank, and state Proposition 2 needed in the proofs, but also having an independent interest.

**Notation 1.** (Corank). Let \( \mathcal{A} \) and \( \mathcal{B} \) be two distributions of constant rank. Denote \( [\mathcal{A}, \mathcal{B}] = \{a \in \mathcal{A}, b \in \mathcal{B}\} \). If \( \mathcal{A} \subseteq \mathcal{B} \), the corank of the inclusion \( \mathcal{A} \subseteq \mathcal{B} \) equals the rank of the quotient \( \mathcal{B}/\mathcal{A} \), i.e., cork \( (\mathcal{A} \subseteq \mathcal{B}) = \text{rk} (\mathcal{B}/\mathcal{A}) \).

**Proposition 2.** Suppose that \( \Sigma \) is dynamically linearizable via invertible two-fold prolongation and let \( D^k \) be its first noninvolutive distribution. Then the distribution \( D^k \) is feedback invariant and satisfies \( \text{cork}(D^k \subset \overline{D}^k) \leq 2 \). Moreover, if \( \text{cork}(D^k \subset \overline{D}^k) = 2 \), then \( \text{rk} D^k = 2k + 2 \).

According to Proposition 2, at most two independent directions of \( \overline{D}^k \) stick out of \( D^k \). In this paper, we study only the case when the noninvolutivity of \( D^k \) is maximal, i.e., \( \text{cork}(D^k \subset \overline{D}^k) = 2 \). The normal forms for the particular case \( \text{cork}(D^k \subset \overline{D}^k) = 1 \) remind those for flatness of differential weight \( n + 3 \) (for which the first noninvolutive distribution necessarily satisfies \( \text{cork}(D^k \subset \overline{D}^k) = 1 \)), but are slightly different and will be treated elsewhere. To sum up, we make the following assumption:

**Assumption (A2).** The integer \( k \) is the smallest such that \( D^k \) is not involutive and, moreover, we suppose \( \text{cork}(D^k \subset \overline{D}^k) = 2 \).

### 3. MAIN RESULTS: NORMAL FORMS

The main results are given by Theorems 1 and 2 that present four normal forms for the class of flat two-input control-affine systems of differential weight \( n + m + 2 = n + 4 \).

#### 3.1 Normal forms

Given a system \( \Sigma : \dot{x} = f(x) + u_1 \xi_1(x) + u_2 \xi_2(x) \) that is flat at \( x_0 \) (at \((x_0, u_0)\), if \( k = 0 \) or \( k = 1 \)), the normal forms are obtained under local static feedback transformations

\[
z = \phi(x), \quad u = \alpha(x) + \beta(x)\bar{u}
\]

and are flat at \( z_0 \) (at \((z_0, \bar{u}_0)\), if \( k = 0 \) or \( k = 1 \)), where

\[
z = \phi(x_0), \quad u_0 = \alpha(x_0) + \beta(x_0)\bar{u}_0.
\]

For \( k \geq 2 \), we will give two normal forms NF1 and NF2, that are static feedback equivalent, each of them having its advantage: for NF1 we see immediately the control to be prolonged, whereas for NF2 the role of \( k \) is explicit.

The integers \( \rho_i \) and \( \mu_i \), that show up in the normal forms are such that \( \rho_1 + \rho_2 = 2 + n \) and \( \mu_1 + \mu_2 + 2k = n \).

For \( i = 1, 2 \), denote \( z_i^j = (z_i^1, \ldots, z_i^j) \) and \( \bar{w}_i^j = (w_i^1, \ldots, w_i^j) \).

**Theorem 1.** Suppose \( k \geq 2 \). The following are equivalent:

(i) \( \Sigma \) is flat at \( x_0 \) of differential weight \( n + 4 \); (ii) \( \Sigma \) is locally around \( x_0 \), static feedback equivalent in a neighborhood of \( \bar{w}_0 \in \mathbb{R}^n \) to:

**NF1:**

\[
\begin{align*}
\bar{z}_1^{j+1} &= \bar{z}_1^{j+1} \\
\bar{z}_2^{j} &= \bar{z}_2^{j+1} + b(z)\bar{u}_1 \\
\bar{z}_2^{j+2} &= \bar{z}_2^{j+2} + d(z)\bar{u}_1 \\
\end{align*}
\]

where \( 1 \leq j \leq \rho_i - 1, \rho_i \geq k + 1 \), and the functions \( b = b(z^{p_i-2k+2}, z^{p_i-k+2}) \) and \( d = d(z^{p_i-k+3}, z^{p_i-k+3}) \) and are such that \( k \) is as in Assumption (A2);

(iii) \( \Sigma \) is locally around \( x_0 \), static feedback equivalent in a neighborhood of \( \bar{w}_0 \in \mathbb{R}^n \) to:

**NF2:**

\[
\begin{align*}
\bar{w}_1^{j+1} &= \bar{w}_1^{j+1} \\
\bar{w}_2^{j+1} &= \bar{w}_2^{j+1} + p(w) + q(w)\bar{w}_2^{j+2} \\
\bar{w}_1^{j+2} &= \bar{w}_1^{j+2} + \bar{w}_2^{j+2} \\
\end{align*}
\]

where \( 1 \leq j \leq \mu_i - 2, \mu_i \leq 1, \mu_i + 1 + \mu_i \geq 1, \mu_i + 2 \geq 3 \), the functions \( p = p(w), \bar{w}_2^{j+2} \) and \( q = q(w^{p_i-2k+2}, w^{p_i-2k+2}) \) are such that \( \bar{w}_1^{j+2} \neq 0 \) and verify additional regularity conditions such that \( k \) is as in Assumption (A2).

Moreover, the functions \( (\bar{z}_1, \bar{z}_2) \) for NF1, and \((\bar{w}_1, \bar{w}_2) \) for NF2, are flat outputs of differential weight \( n + 2 + 2 = n + 4 \).

The functions \( b \) and \( d \) of NF1 (resp., \( p \) and \( q \) of NF2) are briefly discussed in Section 3.1.1 below.

Flatness described by Theorem 1 (treating the case \( k \geq 2 \)) is local around \( x_0 \) but, like for flat systems of differential weight \( n + m \) or \( n + m + 1 \) (with \( k \geq 1 \)), is global with respect to the control \( u \). This changes if \( k = 1 \) or \( k = 0 \), in which cases we have to consider flatness at \((x_0, u_0)\) as described by Theorem 2. Observe that we face a similar situation for flatness of differential weight \( n + m + 1 \) when \( k = 0 \), Nicolau and Rondepé [2017, 2019]. Below, \( \bar{u}_{10} \) stands for the first component of \( \bar{u}_0 \) given by (4).

**Theorem 2.** Assume \( k = 0 \) or \( k = 1 \). The following are equivalent:

(i) \( \Sigma \) is flat at \((x_0, u_0)\) of differential weight \( n + 4 \);
(ii) $\Sigma$ is locally, around $x_0$, static feedback equivalent in a neighborhood of $z_0 \in \mathbb{R}^n$ to:

$$\begin{cases}
\frac{\partial^2 z}{\partial x^2}(z_0, u_0) \neq 0, & \frac{\partial B}{\partial \Sigma^{2+2}}(z_0, u_0) \neq 0, \\
- \text{either } k = 0 \text{ and then} \quad A(z, \tilde{u}_1) = a(z, u_1) + z_2^{p+1} u_1, \\
= 0, & \text{or } k = 1 \text{ and then} \quad \text{either } A(z, \tilde{u}_1) = a(z, u_1) + z_2^{p+1} u_1, \\
= 0, & \text{or } A(z, \tilde{u}_1) = a(z, u_1) + z_2^{p+1} u_1, \\
\text{and } B(z, \tilde{u}_1) = d(z) + d(z) u_1, & \text{where the function } b \text{ of the second case of (8)} \quad \text{satisfy some regularity conditions assuring that} \\
\text{satisfy some regularity conditions assuring that} & \text{is as in Assumption (A2).} \\
\text{is as in Assumption (A2).} & \text{is as in Assumption (A2).} \\
\text{is as in Assumption (A2).} & \text{is as in Assumption (A2).}
\end{cases}
$$

Moreover, for NF3 and NF4, $(\varphi_1, \varphi_2) = (z_1, z_2)$ is a flat output of differential weight $n + 4$.

3.1.1 Nonlinearities and invariants of NF1 and NF2. For NF1, the value of $k$ is encoded when looking more precisely at the functions $b$ and $d$, which depend on $k$ and $d_2$. They are defined as $b(z, u_0)$ and $d_2(z)$, respectively. The conditions $\rho_i > 1$ for $i \geq 1$, impose on $b$ and $d$ three more conditions (which can be computed by a straightforward calculation) that we do not present here. Moreover, by Assumption (A2), the distribution $D^k_0$ is noninvolutive and $\text{cork}(D^k_0) = 2$. Hence, see Proposition 2, $rk D^k = 2k + 2$. The integers $\rho_i$ for $i > 1$, the integers $\mu_i$ for $i > 2$, and the functions $b$ and $d$ of NF1, resp., $\rho$ and $q$ of NF2, should also satisfy some regularity conditions assuring $rk D^k = 2k + 4$. One may also distinguish the subcase when the vector field $D^k_0$ is $2k + 2, 2k + 3, 2k + 4$ from that when the vector field $D^k$ is $(2k + 2, 2k + 4)$. The only difference with NF2 is that the control $u_1$ replaces the variable $w_1^{(i+1)}$ in the only nonlinear equation $w_2 - p(w_1^{(i+1)}, w_2) = q(w_1^{(i+1)}, w_2)$.

3.2 Discussion of the normal forms. All normal forms are valid around $z_0 \in \mathbb{R}^n$, which may be zero or not. Thus all forms can be used around any point (equilibrium or not). All forms and the minimal $x$-flat outputs are compatible, that is, for a given flat system $\Sigma$ of differential weight $n + 4$, we can always simultaneously normalize $\Sigma$ and a priori given minimal flat output $\varphi$, as asserted by:

**Proposition 3.** Let $\Sigma$ be flat at $x_0$ (at $(x_0, u_0)$, if $k = 0$ or $k = 1$) and $\varphi$ a minimal flat output of differential weight $n + 4$ of $\Sigma$. Then $\Sigma$ is locally around $z_0$ static feedback equivalent to NF1 or NF2, if $\rho_i > 2$ (resp. to NF3 or NF4, if $k = 0$ or $k = 1$, where $\varphi = (z_1, z_2)$) for NF1 (resp. for NF3 and NF4) and $\varphi = (w_1, w_2)$ for NF2.

Normal forms NF1, NF3 and NF4 always contain a linear chain $\Sigma = \tilde{u}_1$, called $z_1$-chain whose control has to be prolonged twice. For each form there are (at most) two nonlinearities of different possible forms (see Table 1) associated to the $z_2$-chain (which is called nonlinear). Observe that the normal forms for $k > 1$ may actually present only one nonlinear function (see Example 2): this happens only if the function $d$ involved in the expression of $z_2^{p+1}$ is identically zero. On the other hand, the normal forms for $k = 0$ always involve two nonlinearities. The number of possible nonlinearities of the forms presented in this paper, is due to the fact that the first noninvolutive distribution $D^k_0$ is actually squeezed between two involutive ones, namely $D^{k-1} \subset D^k \subset \overline{D}^k$ and both inclusions are of corank two, see Nicolau and Respondek [2016a].

NF2 also contains a linear $w_1$-subsystem that is a chain of pure $\rho_1$-fold integrator $\frac{d_1}{d_1} w_1 = \tilde{u}_1$ (we actually have $\rho_1 = \rho_1 + k$). The $w_2$-chain has two nonlinearities $p(w_1^{(i+1)}, w_2)$ and $q(w_1^{(i+1)}, w_2)$ defining the only nonlinear component $w_2^{(i+1)} = p(w_1^{(i+1)}, w_2) + q(w_1^{(i+1)}, w_2)$. In NF2, the integer $k$ appears explicitly, so the noninvolutive distribution $D^k$ is easier to be analyzed with the help of NF2. From NF2, it is obvious that in the case $k < 2$, the flat outputs provide a parametrization of system’s trajectories that is global with respect to controls. If $k = 1$, then the system is static feedback equivalent to a form that reminds NF2, namely:

$$\begin{cases}
\Sigma = \tilde{u}_1, & \mu_1 = 1, \\
\Sigma = \tilde{u}_1 + \tilde{u}_2, & \mu_1 = 1, \\
\Sigma = \tilde{u}_1 + \tilde{u}_2, & \mu_1 = 1, \\
\Sigma = \tilde{u}_1 + \tilde{u}_2, & \mu_1 = 1, \\
\Sigma = \tilde{u}_1 + \tilde{u}_2, & \mu_1 = 1, \\
\Sigma = \tilde{u}_1 + \tilde{u}_2, & \mu_1 = 1.
\end{cases}
$$

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3.2.1 Comparison with flat systems of differential weight \(n + 2\) and \(n + 3\). For two-input flat control systems of differential weight \(n + m + 1 = n + 3\) (or equivalently, linearizable via a one-fold prolongation), we proposed in Nicolau and Respondek [2019], without constant rank assumption, four normal forms that are analogous to those presented here. Each of them contains only one nonlinear function and they differ notably by the role played by the first distribution destroying feedback linearizability (\(k\) being defined as the smallest integer such that either involutivity or constant rank is not satisfied for \(\mathcal{D}^k\)). Therefore the normal forms \(\text{NF1}, \text{NF3}\) and \(\text{NF4}\) as well as those for differential weight \(n + 2\) are always normalize at least two of them. Table 1 (where \(d.w.\) stands for differential weight) presents all possible cases. Moreover, \((\varphi_1, \varphi_2) = (z_1, z_2)\) is a minimal flat output of differential weight at most \(n + 4\).

Normal form \(\text{NF0}\) presents four nonlinearities, but we can always normalize it to at least two of them. Table 1 (where \(d.w.\) stands for differential weight) presents all possible cases.

| \(d.w.\) | Nonlinearities |
|---|---|
| \(n + 2\) (Br) | \(a = z_2^{p_2 + 2}\), \(b \equiv 0\) |
| \(c = z_2^{p_2 + 2}\) | \(d \equiv 0\) |
| \(n + 3\) All forms | \(a = z_2^{p_2 + 2}\), \(b \equiv 0\) |
| \(c = z_2^{p_2 + 2}\) | \(d(\bar{z}_1, z_2^{p_2 + 1})\) any |
| \(\text{NF1}, k \geq 1\) | \(c + \bar{u}_{10}d = z_2^{p_2 + 2}\) |
| \(\text{NF3}, k = 0\) | \(c(z)\) any |
| \(\text{NF4}, k = 0\) | \(d(z)\) any |

Moreover,

| \(\varphi_1, \varphi_2\) | \(\bar{z}_1, z_2\) |
|---|---|
| \(\text{NF0}\) | \(\varphi_1 = a_{11}(z, u)\), \(\varphi_2 = a_{12}(z, u)\) |
| \(\text{NF1}, k \geq 1\) | \(a_{11}(z, u)\) any |
| \(\text{NF3}, k = 0\) | \(a_{11}(z, u)\) any |
| \(\text{NF4}, k = 0\) | \(a_{11}(z, u)\) any |

Under the constant rank assumption, only the first form shows up.

3.3 Identifying singularities in the control space. When \(k = 0\) or \(k = 1\), the system exhibits flatness singularities in the control space and we will explain that, according to Theorem 2, there are two ways to deal with them (reminding very much the case of flatness of differential weight \(n + m + 1\) with \(k = 0\), Nicolau and Respondek [2019]). Normal forms \(\text{NF3}\) and \(\text{NF4}\) are local, around \(z_0\), but global with respect to \(\bar{u} = (\bar{u}_1, \bar{u}_2) \in \mathbb{R}^2\) and thus allow to identify all points \((z, \bar{u})\) at which the system is \(x\)-flat and distinguish them from \((\hat{z}, \bar{u})\) at which it is not. Let \(u_0\) be a nominal control. Its value is involved in \(\text{NF3}\) in such a way that \(\text{NF3}\) is flat around \((x_0, u_0)\). On the other hand, \(\text{NF4}\) does not use the knowledge of \(u_0\) and in order to verify that \(\text{NF4}\) is flat around \((x_0, u_0)\) one needs to check conditions (5). More precisely, the value of \(\bar{u}_{10}\) appears explicitly in \(\text{NF3}\) and it yields, at the nominal point \((z_0, \bar{u}_0)\), a zero multiplying the functions \(b \) and \(d\) and allowing us to normalize \(a(z) + b(z)\bar{u}_0\) as \(z_2^{p_2 + 1}\) and \(c(z) + d(z)\bar{u}_{10}\) as \(z_2^{p_2 + 2}\). The value of \(\bar{u}_{10}\) does not appear in \(\text{NF4}\) and the normalization of the nonlinearities is different. In the case \(k = 0\), due to noninvolutivity of \(\mathcal{D}^0\) whose grow vector is \((2, 3, 4)\), we can always normalize \(b\) and \(d\) to \(b = z_2^{p_2 + 1}\) and \(d = z_2^{p_2 + 2}\), resp. If \(k = 1\), the function \(d\) does not depend on \(z_2^{p_2 + 2}\) (so we can always normalize \(c\) to \(c = z_2^{p_2 + 2}\) and from \(\partial(a + \bar{u}_{10}b)/(\partial z_2^{p_2 + 1})\neq 0\), it follows that we can normalize either \(a\) or \(b\). Forms \(\text{NF3}\) and \(\text{NF4}\) hold on \(O \times \mathbb{R}^2\), where \(O\) is a neighborhood of \(z_0\). The identification of all points at which the system is not flat can be performed as follows. Define by \(a(z) + b(z)\bar{u}_1\) and \(c(z) + d(z)\bar{u}_1\) the expressions for \(z_2^{p_2 + 1}\) and \(z_2^{p_2 + 2}\), resp., for both \(\text{NF3}\) and \(\text{NF4}\), independently of the normalization. Set \(S_1(z, \bar{u}_1) = \partial(a(z) + \bar{u}_1b)/(\partial z_2^{p_2 + 1})\) and \(S_2(z, \bar{u}_1) = \partial(c(z) + \bar{u}_1d)/(\partial z_2^{p_2 + 2})\), which depend (in an affine way) on \(\bar{u}_1\), and fix \(z \in O\). For \(k = 1\), flatness singularities are \((z, \bar{u}^*(z)) \in O \times \mathbb{R}^2\), where \(\bar{u}^*(z) = (\bar{u}_1(z), \bar{u}_2(z))\), with \(\bar{u}_2(z)\) being the unique root of \(S_1(z, \bar{u}_1) = 0\) and any \(\bar{u}_1(z)\). Those singularities always exist and form, for a fixed \(z\), one line in the control space \(\mathbb{R}^2\). Similarly, if \(k = 0\), flatness singularities are \((z, \bar{u}^*(z)) \in O \times \mathbb{R}^2\), where \(\bar{u}_1^*(z)\) is a root of the product \(S_1(z, \bar{u}_1) \cdot S_2(z, \bar{u}_1) = 0\). Notice that, for each fixed \(z \in O\), the above product admits either one or two distinct real roots. Therefore for a given \(z \in O\), the values of the singular controls form, resp., two lines or one line in \(\mathbb{R}^2\).

4. EXAMPLES

Example 1. Flatness of differential weight \(n + 4\) for four-dimensional control systems. The simplest two-input control system that may satisfy the assumptions under which we work (that is \(\text{cork}(\mathcal{D}^k \subset \mathcal{D}^{k+1}) = 2\)) are those in dimension four. The problem of flatness for four-dimensional control systems with two inputs has been solved by Pomet [1997] whose results can be interpreted in terms of dynamic linearizability via a \(p\)-fold prolongation of a suitably chosen control with \(p \leq 3\) (or equivalently in terms of flatness of differential weight \(n + m + p = 6 + p\)). While the cases \(p \leq 2\) correspond to \(x\)-flatness (which is consistent with Proposition 1 and Nicolau and Respondek [2017]), the last case \(p = 3\) describes \((x, u)\)-flatness (i.e., all possible flat outputs depend explicitly on \(u\)). We will focus on the case \(p = 2\) (which is the subject of this paper). For four-dimensional two-input control systems that satisfy Assumption (A2), the first noninvolutive distribution is necessarily \(\mathcal{D}^0\), i.e., \(k = 0\), and as we have already seen, under the hypotheses \(\text{cork}(\mathcal{D}^0 \subset \mathcal{D}^1) = 2\), only the grow vector \((2, 3, 4)\) for \(\mathcal{D}^0\) is possible (and in particular, \(\mathcal{D}^0 = TX\)). In fact, if a system with four states and two controls satisfying \(\text{cork}(\mathcal{D}^0 \subset \mathcal{D}^1) = 1\) is flat, then it is necessarily dynamically linearizable via a one-fold prolongation (and thus of differential weight \(n + 3 = 7\)). Therefore the
condition \( c(D^0 \subset \overline{D}^0) = 2 \) is actually necessary for the differential weight \( n + 4 = 8 \) and, according to Theorem 2, all systems of differential weight 8 admit, around \((x_0, u_0)\), one normal form that can be taken as either
\[
NF_{3n=4} : \begin{cases}
\frac{\partial}{\partial x_1}(x_0) \neq 0 \text{ and } \frac{\partial}{\partial x_2}(x_0) \neq 0, \text{ or} \\
\frac{\partial}{\partial z_1}(x_0) = \frac{\partial}{\partial z_2}(x_0) = 0,
\end{cases}
\]
where \( \frac{\partial}{\partial z_1}(x_0) \neq 0 \) and \( \frac{\partial}{\partial z_2}(x_0) \neq 0 \), or
\[
NF_{4n=4} : \begin{cases}
\frac{\partial}{\partial x_1}(x_0) \neq 0 \text{ and } \frac{\partial}{\partial x_2}(x_0) \neq 0, \text{ or} \\
\frac{\partial}{\partial z_1}(x_0) = \frac{\partial}{\partial z_2}(x_0) = 0,
\end{cases}
\]
where \( \frac{\partial}{\partial z_1}(x_0) \neq 0 \) and \( \frac{\partial}{\partial z_2}(x_0) \neq 0 \). Form \( NF_{4n=4} \) agrees with that of Pomet [1997] concerning to linearizability via a two-fold prolongation and our regularity conditions coincide with one case of Pomet [1997] while the other case of Pomet [1997] is excluded by the constant rank Assumption (A1).

Example 2. The PVTOL aircraft. The following model of a planar vertical take off and landing (PVTOL) aircraft was introduced in Hauser et al. [1992] and has attracted a lot of attention in the last years (see, e.g., Martin et al. [1996], Lozano et al. [2004]). The configuration of the system is \( (\theta, x_1, y_1) \), with \( \theta \) the angle the aircraft makes with the horizontal axis and \( (x_1, y_1) \) the position of its center of mass. After normalisation of \( n \) and \( J \), the dynamics of the PVTOL aircraft is given by:
\[
\dot{\theta} = \omega \\
\dot{x}_1 = x_2 \\
\dot{x}_2 = -u_1 \sin \theta \\
\dot{y}_1 = y_2 \\
\dot{y}_2 = -a_0 + u_1 \cos \theta + c_1 u_2 \sin \theta,
\]
where \( u_1 \) and \( u_2 \) correspond, resp., to the body vertical force (minus the gravity) and to forces on the tips of the wings, \( a_0 \) is the gravity acceleration and \( \epsilon \neq 0 \) is a fixed constant related to the geometry of the aircraft. The PVTOL aircraft has been shown to be locally flat with \( \varphi = (x_1 - \epsilon \sin \theta, y_1 + \epsilon \cos \theta) \) a flat output of differential weight \( n + 4 \), see Martin et al. [1996]. By a direct calculation we get \( k = 1 \) and \( \overline{D} = D^1 + [D^1, D^1] = TX \). We show that the PVTOL model can be brought into NF4. Suppose that we work around a nominal point such that \( \sin \theta_0 \neq 0 \). By introducing the local coordinates \( \bar{x}_1 = x_1 - \epsilon \sin \theta, \bar{x}_2 = L_f \bar{x}_1, \bar{y}_1 = y_1 + \epsilon \cos \theta, \bar{y}_2 = L_f \bar{y}_1, \theta = -\cot \theta, \omega = L_f \theta \), followed by an suitable invertible feedback transformation (where \( \bar{u}_1 = c_2 \epsilon \sin \theta - u_1 \sin \theta \)), we get
\[
\bar{x}_1 = \bar{x}_2 \\
\bar{y}_1 = \bar{y}_2 \\
\bar{y}_2 = -a_0 + \bar{u}_1 \\
\bar{\theta} = \bar{\omega}
\]
This is normal form NF4 for \( k = 1 \) for which, with respect to the Brunovsky canonical form, only one component (the third from the bottom) of the second chain is modified. In these coordinates, we have \( \varphi = (\bar{x}_1, \bar{y}_1) \) and in order to express all states and controls, we need to differentiate \( \bar{u}_1 \) twice (thus obtaining the differential weight \( n + 4 \)). The input \( \bar{u}_1 \) is also the control that has to be prolonged twice in order to obtain a static feedback linearizable prolonged system. Finally, notice that \( \bar{u}_{10} = 0 \) is a singular control for flatness of differential weight \( n + 4 \).

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