The SO(32) Heterotic and Type IIB Membranes

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Abstract

A two dimensional anomaly cancellation argument is used to construct the
SO(32) heterotic and type IIB membranes. By imposing different boundary
conditions at the two boundaries of a membrane, we shift all of the two dimen-
sional anomaly to one of the boundaries. The topology of these membranes is
that of a 2-dimensional cone propagating in the 11-dimensional target space.
Dimensional reduction of these membranes yields the SO(32) heterotic and type
IIB strings.

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The current membrane actions could be named the type IIA \cite{1,2} and $E_8$ \cite{3,4} membranes since their dimensional reduction yields the type IIA and $E_8 \times E_8$ strings, respectively. Recall that the type IIA membrane \cite{1,2} has the topology of a torus, while the $E_8$ membrane \cite{3,4} has the topology of a cylinder. To construct the latter membrane, one needs to invoke anomaly cancellation arguments to justify the coupling of the membrane to an $E_8$ gauge group at each of its ends. When the boundary conditions on the membrane are NN (Neumann boundary conditions at both ends), there is a symmetry between both dynamical boundaries. Each boundary contributes half of the two dimensional anomaly present at the boundaries.

Here we use similar anomaly arguments to construct the SO(32) heterotic and type IIB membranes. By imposing DN boundary conditions (Dirichlet boundary conditions at one boundary and Neumann boundary conditions at the other boundary), we shift all of the two dimensional anomaly to the N boundary. Indeed, the observer at the D boundary will see no dynamics at all. Therefore he will be unable to construct any diagram which contributes to the anomaly. On the other hand, the observer at the N boundary sees all the relevant string degrees of freedom. The observer at the N boundary will thus be able to construct a diagram which spoils the two dimensional coordinate invariance.

The topology of the SO(32) heterotic and type IIB membranes turns out to be that of a two dimensional cone propagating in the 11-dimensional target space. That is, one of its boundaries is a string, while the other boundary is a point in target space.

To fix our notation and conventions, we first give a short review of the $\kappa$-symmetric action for the supermembrane on the Minkowski background \cite{5,6}. The action is

$$S = -\frac{1}{2} \int d^3 \zeta \{ \sqrt{-g} (g^{ij} \Pi_i^\mu \Pi_j^\nu \eta_{\mu\nu} - 1) - \dot{\epsilon}^{ijk} \Pi_j^\mu \chi^i_{\dot{k}}(\dot{\theta} \Gamma^\mu \theta_{\dot{k}}) + \frac{1}{3} \epsilon^{ijk} (\dot{\theta} \Gamma^\mu \theta_{\dot{j}})(\dot{\theta} \Gamma^\nu \theta_{\dot{k}})(\dot{\theta} \Gamma_{\mu\nu} \theta_{\dot{i}}) \},$$

(1)
where,

\[ \Pi^\mu_i = X^\mu_i - i\bar{\theta}\Gamma^\mu\theta_{i}. \] (2)

Here \( X^\mu = (X^0, X^1, \ldots, X^{10}) \) and the world-volume coordinates are \( \zeta^i = (\tau, \sigma, z) \). The range of these coordinates will be specified later.

The equations of motion corresponding to the action are

\[ g_{ij} = \Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu}, \] (3)

\[ (\sqrt{-g}g^{ij}\Pi^\mu_j)_i + \epsilon^{ijk}\Pi^\nu_i(\bar{\theta}\Gamma^\nu_\theta_{j,k}) = 0, \] (4)

\[ g^{ij}\Pi^\mu_i(\Gamma^\mu_\theta_{j,k}) = \frac{\epsilon^{ijk}}{2\sqrt{-g}}\left\{ (X^\nu_k - \frac{2i}{3}\bar{\theta}\Gamma^\mu\theta_{j,k})(\bar{\theta}\Gamma^\mu_\theta_{i,j})(\Gamma^\mu_\theta_{j,k}) - (\Pi^\mu_i X^\nu_k - \frac{1}{3}(\bar{\theta}\Gamma^\mu_\theta_{j,k})(\bar{\theta}\Gamma^\nu_\theta_{j,k})(\Gamma^\mu_\theta_{j,k}) \right\}. \] (5)

Notice that eq.(5) can be written more compactly, see for instance Refs.\[5, 6\].

The boundary terms, that must vanish to obtain the equations of motion (3)-(5), are

\[ -\int d^3\zeta (\mathcal{P}^i_\mu \delta X^\mu + S^i \delta \theta)_i, \] (6)

where,

\[ \mathcal{P}^i_\mu = \sqrt{-g}g^{ij}\Pi^\mu_j + \epsilon^{ijk}(\bar{\theta}\Gamma^\nu_\theta_{j,k})(\Pi^\nu_k + \frac{i}{2}\bar{\theta}\Gamma^\nu_\theta_{j,k}), \] (7)

\[ S^i = -i\sqrt{-g}g^{ij}\Pi^\mu_j(\bar{\theta}\Gamma^\mu_{i}) - \frac{i}{2}\epsilon^{ijk}(X^\nu_k - \frac{2i}{3}\bar{\theta}\Gamma^\mu_\theta_{j,k})(\bar{\theta}\Gamma^\nu_\theta_{j,k})(\bar{\theta}\Gamma^\mu_{i}) \]

\[ -\frac{1}{2}\epsilon^{ijk}(\Pi^\mu_k X^\nu_{j,k} - \frac{1}{3}(\bar{\theta}\Gamma^\mu_\theta_{j,k})(\bar{\theta}\Gamma^\nu_\theta_{j,k})(\bar{\theta}\Gamma^\mu_{i,k}). \] (8)

It is also convenient to introduce the following notation for a vector \( V^\mu \)

\[ V^\pm = \frac{1}{\sqrt{2}}(V^0 \pm V^9), \quad V^M = (V^1, V^2, \ldots, V^8, V^{10}), \] (9)

so that

\[ V^\mu W_\mu = V^M W^M - V^+ W^- - V^- W^+, \quad V^M = V_M, \quad V^+ = -V_-, \quad V^- = -V_+ \] (10)
We follow Refs. [5, 6] and make the "light-cone" gauge choice

\[ X^+ = p^+ \tau, \quad \Gamma^+ \theta = 0 \]

\[ g_{a\tau} = 0, \quad a = (\sigma, z) \]

\[ g_{\tau\tau} = -\det(g_{ab}) \equiv -\det(h_{ab}) \equiv -h \]

Notice that the condition \( \Gamma^+ \theta = 0 \) effectively reduces to 16 the number of components of the spinor \( \theta \)

\[ \theta = \begin{pmatrix} S_1 \\ S_2 \\ S_1 \\ S_2 \end{pmatrix}, \quad (12) \]

where \((S_1, S_2)\) are 8-component spinors (for our conventions for the gamma matrices, see the appendix).

In the gauge (11), the equations of motion reduce to

\[ g_{ab} = X_a^M X_b^M \equiv h_{ab}, \quad (13) \]

\[ \ddot{X}^M - (hh^{ab} X_b^M)_a + p^+ \epsilon^{ab} \bar{\theta}_a \Gamma^{-M} \theta_b = 0, \quad (14) \]

\[ \Gamma^{-} \dot{\theta} + i \epsilon^{ab} \Gamma^{-M} \theta_a X_b^M = 0, \quad (15) \]

where we introduced also \( \epsilon^{ab} = \epsilon^{a\tau b} \). To obtain these equations of motion, various properties of the gamma matrices have been used. Our conventions for the gamma matrices are given in the appendix.

The equations of motion must be supplemented by the gauge constraints

\[ p^+ \Pi_a^- = \dot{X}^M X_a^M, \quad 2p^+ \Pi_\tau^- = \dot{X}^M \dot{X}^M + h, \quad (16) \]

which imply, among other things, the relation [7]

\[ \epsilon^{ab} \dot{X}^M X_a^M X_b^M + ip^+ \epsilon^{ab} \bar{\theta}_a \Gamma^{-} \theta_b = 0. \quad (17) \]

These conditions can be used to effectively eliminate three bosonic degrees of freedom, namely \( X^+, X^- \) and one more.
Now consider a boundary of the membrane defined by \( z = \text{const} \). The corresponding normal vector is

\[
n_i = \frac{1}{\sqrt{h_{zz}}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |n| = 1.
\]  

(18)

Then the boundary terms (6)-(8) reduce to

\[
- \oint_B (\mathcal{P}_\mu^i \delta X^\mu + S^i \delta \theta) d\Sigma_i = - \oint_B (\mathcal{P}_\mu^z \delta X^\mu + S^z \delta \theta) d\Sigma_z, \tag{19}
\]

where we used that \( d\Sigma_i \propto n_i \), and the boundary corresponds to \( z = \text{const} \). Now,

\[
\mathcal{P}_M^z = hh^a X_{\alpha}^M - p^+ \bar{\theta} \Gamma^M - \theta_\sigma
\]

\[
\mathcal{P}_-^z = 0
\]

\[
\mathcal{P}_+^z = -hh^a \Pi_a - \epsilon^{ijk} \bar{\theta} \Gamma^M - \theta_\sigma X_k^M
\]

such that

\[
\mathcal{P}_\mu^z \delta X^\mu = (hh^a X_{\alpha}^M - p^+ \bar{\theta} \Gamma^M - \theta_\sigma) \delta X^M - (hh^a \Pi_a - \epsilon^{ijk} \bar{\theta} \Gamma^M - \theta_\sigma X_k^M) \delta X^+.
\]  

(21)

On the other hand

\[
S^z \delta \theta = p^+ X_{\alpha}^M \bar{\theta} \Gamma^M \delta \theta. \tag{22}
\]

We shall be interested in a membrane with two boundaries at \( z = z_0 \) and \( z = 0 \), respectively. The other spatial world-volume coordinate \( \sigma \) parametrizes an angular direction, \( \sigma \in [0, 2\pi] \).

Consider first the boundary \( z = z_0 \). We use the notation \( M = (I, 10), \quad I = (1, 2, ..., 8) \), and enforce the boundary conditions

\[
X^{10} = \text{const}., \quad X^I_z = 0, \quad S_1 = S_2 \equiv S, \quad (z = z_0) \tag{23}
\]

It is straightforward to show that these conditions in fact kill the boundary terms (19)-(22). We shall call this boundary of the membrane the N boundary, since the
conditions on the $X^I$ coordinates are of Neumann type. Concerning the conditions on the spinor, we notice that the choice $S_1 = S_2$ determines the chirality of the surviving fermions at the boundary. Indeed, for $S_1 = S_2$ we have

$$ (I - \Gamma^{11})\theta = 0. \quad (24) $$

To ensure that the Neumann boundary conditions are compatible with supersymmetry, we enforce the additional boundary condition

$$ S_{,z} = 0, \quad (z = z_0) \quad (25) $$

At the boundary $z = z_0$, the target space coordinates $X^\mu$ are functions of (at most) $(\tau, \sigma)$. It means that this boundary of the membrane is a closed string in target space.

We now study this string in more detail.

Using the gauge choice (11), the action (1) takes the form

$$ S_{lc} = -\frac{1}{2} \int d^3\zeta \{ h - \dot{X}^M \dot{X}^M - 2ip^+ \bar{\theta} \Gamma^- \dot{\theta} + 2p^+ \epsilon^{ab} \bar{\theta} \Gamma^a \theta X^M_{,b} \}, \quad (26) $$

where we also used (13) to put $h_{ab}$ on-shell. From (23), (25) follows that the fields have expansions near $z = z_0$

$$ X^{10}(\tau, \sigma, z) = \text{const} + (z - z_0) + \mathcal{O}((z - z_0)^2) $$

$$ X^I(\tau, \sigma, z) = X^I(\tau, \sigma) + \mathcal{O}((z - z_0)^2) \quad (27) $$

$$ \theta(\tau, \sigma, z) = \theta(\tau, \sigma) + \mathcal{O}((z - z_0)^2) $$

where $\theta(\tau, \sigma) = (S, S, S, S), \quad S = S(\tau, \sigma)$. Notice that we used the remaining freedom mentioned after eq.(17) to fix to 1 the coefficient of the linear term in $X^{10}$.

The action (26) can be written

$$ S_{lc} = -\frac{1}{2} \int_0^{z_0} dz \, S_{lc}(z), \quad (28) $$

where,

$$ S_{lc}(z) \equiv \int d\tau d\sigma \{ h - \dot{X}^M \dot{X}^M - 2ip^+ \bar{\theta} \Gamma^- \dot{\theta} + 2p^+ \epsilon^{ab} \bar{\theta} \Gamma^a \theta X^M_{,b} \}. \quad (29) $$
Using (27), it follows that

\[ S_{lc}(z_0) = \int d\tau d\sigma \{ X^I_\sigma X^I_\sigma - X^I_\tau X^I_\tau + iS^T(S_\sigma + S_\tau) \}. \] (30)

after a constant redefinition of the spinor \( S \).

Thus we can interpret the action (30) as the action seen by the observer living at the boundary \( z = z_0 \). This action is nothing but the spacetime part of the heterotic string action [9, 10] (i.e. without the SO(32) or \( E_8 \times E_8 \) current algebra). It must be stressed, that to obtain this string action, we have enforced particular boundary conditions on the membrane. However, all the relevant string degrees of freedom are still present.

We now turn to the boundary \( z = 0 \). At this boundary, we take the following boundary conditions

\[ X^M = \text{const}, \quad (z = 0) \] (31)

This is in fact sufficient to kill all the boundary terms (19)-(22) at the boundary \( z = 0 \). We shall call this boundary of the membrane the D boundary, since the conditions on the \( X^M \) coordinates are of Dirichlet type. Notice that at this boundary, the target space coordinates are at most functions of \( \tau \). In fact, \( X^+ \) is proportional to \( \tau \), while all the other target space coordinates are constant. This means that this boundary of the string is just a point in target space, that is, a stationary point with momentum only in the \( X^+ \) direction.

As in the previous case, the condition (31) is supplemented by an additional condition

\[ \theta = \text{const}, \quad (z = 0) \] (32)

Then the fields have expansions near \( z = 0 \)

\[ X^{10}(\tau, \sigma, z) = \text{const} + \mathcal{O}(z) \]
\[ X^I(\tau, \sigma, z) = \text{const} + \mathcal{O}(z) \] (33)
\[ \theta(\tau, \sigma, z) = \text{const} + \mathcal{O}(z) \]
It follows that the action (29) vanishes identically at \( z = 0 \)

\[
S_{lc}(0) = 0. \tag{34}
\]

Thus the boundary conditions (23), (31) describe a membrane whose spatial topology is a two-cone, that is, one boundary is a string, while the other is a point in target space. Moreover, we obtained that the dynamics near the string boundary \( z = z_0 \) is the spacetime content of the heterotic string, while there is no dynamics at all (except for the momentum \( p^+ \)) near the point boundary \( z = 0 \). This concludes the discussion of the boundary conditions.

The action at the boundary \( z = 0 \) has no gravitational anomaly because it is not \( 4k + 2 \)-dimensional at that boundary: the topology of the two-cone there is just a point particle from a worldvolume point of view. In addition there is no dynamics there and thus it is impossible to construct any diagram. From the target space point of view there is also no anomaly since the spacetime is 9-dimensional. This follows from the fact that the topology of the 11-dimensional targetspace is that of 9-dimensional Minkowski space times a two-cone, necessary for topological stability of the membranes.

The action at the boundary \( z = z_0 \) clearly has a two dimensional gravitational anomaly. In order to cancel such anomaly we must introduce additional massless fields. The anomaly can be cancelled at \( z = z_0 \) for three different situations. Adding an \( SO(32) \) or \( E_8 \times E_8 \) current algebra, or adding additional left moving fermions. The resulting theories couple a string to the end of the open membrane. The respective strings are \( SO(32) \) and \( E_8 \times E_8 \) heterotic strings and type IIB strings. Notice that the type IIB string is consistent with the boundary conditions. This follows from the fact that half the worldvolume spinors were ”projected out” of the action after satisfying the boundary conditions. But the boundary action (such as the one needed to obtain the type IIB string) does not itself need to satisfy any boundary conditions. Therefore is it acceptable to have the additional field content needed to obtain a type IIB string at the boundary.
Thus, at $z = z_0$ we have
\begin{equation}
S'_{lc}(z_0) = S_{lc}(z_0) + S_{an}(z_0)
\end{equation}
where $S_{an}$ is one of the following actions
\begin{align}
S_{an}(z_0) &= \int d\tau d\sigma \{X^K_\sigma X^K_\sigma - X^K_\tau X^K_\tau\}, \\
S_{an}(z_0) &= \int d\tau d\sigma i \tilde{S}^T (\partial_\sigma \tilde{S} + \partial_\tau \tilde{S}).
\end{align}
In (36), the bosons $X^K$ will generate the $SO(32)$ or the $E_8 \times E_8$ current algebra required to cancel the gravitational anomaly. In (37) the fermions $\tilde{S}$ have the same chirality as the fermions $S$, and are thus also able to cancel the gravitational anomaly.

The dimensional reduction of these membranes leads to the expected string theories. Dimensional reduction in the presence of a boundary has certain peculiarities. Moreover, in the present case we cannot use the standard double dimensional reduction ansatz since it is incompatible with our boundary conditions. However, consider the membrane in the limit that its world-volume becomes a world-sheet
\begin{equation}
z \in [0, z_0] \rightarrow 0 \cup z_0
\end{equation}
Then, in that limit, the two boundaries overlap and the integral over $z$ in (28) collapses to a sum of two terms
\begin{equation}
\int_0^{z_0} dz \ S_{lc}(z) \rightarrow S_{lc}(0) + S_{lc}(z_0).
\end{equation}
The first term vanishes, thus, using also (35), the end-result in the limit (38) is
\begin{equation}
\int_0^{z_0} dz \ S_{lc}(z) \rightarrow S'_{lc}(z_0)
\end{equation}
which reflects the dynamics of the $SO(32)$ and $E_8 \times E_8$ heterotic strings and the type IIB string, depending on the type of action $S_{an}$ chosen to cancel the two dimensional anomaly (see (36)-(37)).
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A The Gamma Matrices

In this appendix we give our conventions for the gamma matrices. We follow closely the conventions of [11], however some relabeling of the coordinates will be required.

The $32 \times 32$ gamma matrices are in the Majorana representation and are purely imaginary. They are

$$
\Gamma^0 = \tau_2 \times I_{16} \\
\Gamma^I = i\tau_1 \times \gamma^I, \quad I = 1, \ldots 8 \\
\Gamma^9 = i\tau_3 \times I_{16} \\
\Gamma^{10} = i\tau_1 \times \gamma^9
$$

(1)

where $\tau_i$ are the Pauli matrices, $I_x$ are $x \times x$ identity matrices and the $16 \times 16$ real matrices $\gamma^I$ satisfy

$$
\{\gamma^I, \gamma^J\} = 2\delta^{IJ}, \quad I, J = 1, \ldots 8.
$$

(2)

and

$$
\gamma^9 = \prod_{I=1}^{8} \gamma^I.
$$

(3)

This ensures that

$$
\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}.
$$

(4)

We now construct the $spin(8)$ Clifford algebra. The matrices $\gamma^I$ take the form

$$
\gamma^I = \begin{pmatrix}
0 & \tilde{\gamma}^I \\
-\tilde{\gamma}^I & 0 \\
\end{pmatrix}, \quad \hat{I} = 1, \ldots 7, \\
\gamma^8 = \begin{pmatrix}
I_8 & 0 \\
0 & -I_8 \\
\end{pmatrix}
$$

(5)

1 This construction is that presented in Appendix 5.B of Ref.[12]
where the $8 \times 8$ matrices $\tilde{\gamma}^I$ are antisymmetric and explicitly given by

\begin{align*}
\tilde{\gamma}^1 &= -i\tau_2 \times \tau_2 \\
\tilde{\gamma}^2 &= i1_2 \times \tau_1 \times \tau_2 \\
\tilde{\gamma}^3 &= i1_2 \times \tau_3 \times \tau_2 \\
\tilde{\gamma}^4 &= i\tau_1 \times \tau_2 \times 1_2 \\
\tilde{\gamma}^5 &= i\tau_3 \times \tau_2 \times 1_2 \\
\tilde{\gamma}^6 &= i\tau_2 \times 1_2 \times \tau_1 \\
\tilde{\gamma}^7 &= i\tau_2 \times 1_2 \times \tau_3 
\end{align*}

It follows that $\gamma^9$ is given by

\begin{equation}
\gamma^9 = \begin{pmatrix} 0 & -I_8 \\ -I_8 & 0 \end{pmatrix}.
\end{equation}

Furthermore

\begin{equation}
\Gamma^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ i & -i \end{pmatrix} \times I_{16}, \quad \Gamma^- = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -i \\ i & i \end{pmatrix} \times I_{16},
\end{equation}

such that

\begin{equation}
(\Gamma^+)^2 = (\Gamma^-)^2 = 1, \quad \{\Gamma^+, \Gamma^-\} = 2.
\end{equation}

Then it is straightforward to show that the condition $\Gamma^+ \theta = 0$ leads to

\begin{equation}
\theta = \begin{pmatrix} S_1 \\ S_2 \\ S_1 \\ S_2 \end{pmatrix}.
\end{equation}

Moreover, it follows that

\begin{align*}
\bar{\theta} \Gamma^\mu \partial \theta &= 0 \quad \text{unless } \mu = - \\
\bar{\theta} \Gamma^{\mu \nu} \partial \theta &= 0 \quad \text{unless } \mu \nu = -M
\end{align*}

where $\bar{\theta} = \theta^T \Gamma_0 = \theta^T \Gamma_0$ ($\theta$ is real). Finally notice that

\begin{equation}
(\Gamma^\mu)^\dagger = \Gamma^0 \Gamma^\mu \Gamma^0, \quad \Gamma^{11} = \prod_{\mu=0}^{10} \Gamma^\mu = i\Gamma^{10}.
\end{equation}
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