LOCALIZATION AND PROJECTIONS ON BI–PARAMETER BMO

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Abstract. We prove that for any operator $T$ on bi–parameter $BMO(\delta^2)$ the identity factors through $T$ or $Id - T$. As a consequence, $BMO(\delta^2)$ is a primary Banach space. Bourgain’s localization method provides the conceptual framework of our proof. It consists in replacing the factorization problem on the non–separable Banach space $BMO(\delta^2)$ by its localized, finite dimensional counterpart on $BMO_n(\delta^2)$. We solve the resulting finite dimensional factorization problems on $BMO_n(\delta^2)$ by combinatorics of colored dyadic rectangles.

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1. Introduction
The dyadic intervals $\mathcal{D}$ on the unit interval are given by
$$\mathcal{D} = \{ [2^{-j}k, 2^{-j}(k + 1)] : j, k \in \mathbb{N}_0, k \leq 2^j - 1 \},$$
and the dyadic rectangles $\mathcal{R}$ on the unit square by $\mathcal{R} = \mathcal{D} \times \mathcal{D}$. For any given dyadic interval $I \in \mathcal{D}$ we define the $L^\infty$ normalized Haar function $h_I$, to be $+1$ on the left half of $I$ and $-1$ on the right half of $I$. Given two dyadic intervals $I, J$ we have
$$h_{I \times J}(s, t) = h_I(s) h_J(t), \quad s, t \in [0, 1].$$
We define the bi–parameter space $H^1(\delta^2)$ to be the completion of
$$\text{span}\{ h_{I \times J} : I \times J \in \mathcal{R} \}$$
under the norm
$$\| f \|_{H^1(\delta^2)} = \int_0^1 \int_0^1 \left( \sum a_{I \times J}^2 h_{I \times J}^2 \right)^{1/2} \, ds \, dt,$$
(1.1)
where is the finite linear combination $f = \sum a_{I \times J} h_{I \times J}$. For basic information and background we refer to [1], [2], [6], [8], [9], [11] and [14].
In this paper we are primarily concerned with isomorphic invariant stability properties of the finite dimensional building blocks of bi-parameter $H^1(\delta^2)$ given by

$$H^1_n(\delta^2) = \text{span}\{h_{I \times J} : |I|, |J| \geq 2^{-n}\} \cap H^1(\delta^2).$$

In [16] we proved the following stability result. Given $\delta : \mathcal{N} \to \{0, 1\}$ let

$$T_\delta h_{I \times J} = m(I \times J) h_{I \times J}$$

be the associated Haar multiplier operator. Then, the identity on $H^1(\delta^2)$ factors through $H = T_\delta$ or $H = \text{Id} - T_\delta$, that is

$$\text{Id}_{H^1(\delta^2)} = P H_m E$$  \hspace{1cm} (1.2)

where $E, P$ are bounded linear operators on $H^1(\delta^2)$. Observe that $T_\delta$ is an orthogonal projection since $m$ takes values in $\{0, 1\}$. As noted in [16] the factorization theorem (1.3) for the multiplier operators $T_\delta$ can be used to show that $H^1(\delta^2)$ is a primary Banach space. For the method of proof see also [7]. Recall that a Banach space $X$ is primary if for any projection $P : X \rightarrow X$ one of the spaces $P(X)$ or $(\text{Id}_X - P)(X)$ is isomorphic to $X$. For definitions and background on this classical isomorphic invariant concept we refer to [13, 17, 22].

In this article we prove that $\text{BMO}(\delta^2)$, the dual of $H^1(\delta^2)$, is a primary Banach space. More generally, we show that for any operator

$$S : \text{BMO}(\delta^2) \rightarrow \text{BMO}(\delta^2)$$

the identity on $\text{BMO}(\delta^2)$ factors through $H = S$ or $H = \text{Id} - S$, that is

$$\begin{array}{c|c|c}
\text{BMO}(\delta^2) & \text{Id} & \text{BMO}(\delta^2) \\
\hline
E & P & H \\
\end{array}$$

where $E, P$ are bounded linear operators. Our approach is the localization method introduced by J. Bourgain in [3]. See also [4, 5] and [20] for one of the first papers in this direction. Bourgain’s method is particularly useful for treating factorization problems on non-separable Banach spaces such as $\text{BMO}(\delta^2)$. It aims at replacing (1.3) by its localized, finite dimensional counterpart, and in our context it consists of three basic steps.

(i) The starting point is Wojtaszczyk’s isomorphism [21], that is

$$\text{BMO}(\delta^2) \sim \left( \sum_n \text{BMO}_n(\delta^2) \right)_\infty,$$

where $\text{BMO}_n(\delta^2) = \text{span}\{h_{I \times J} : I \times J \in \mathcal{N} \} \cap \text{BMO}(\delta^2)$.

(ii) Reduction to diagonal operators on $\left( \sum_n \text{BMO}_n(\delta^2) \right)_\infty$.

(iii) Verification of the following finite dimensional and quantitative factorization problem: For any $n \in \mathbb{N}$ there exists $N = N(n)$ such that for any norm one operator $T : \text{BMO}_N(\delta^2) \rightarrow \text{BMO}_N(\delta^2)$ we have that $H = T$ or $H = \text{Id} - T$ satisfies

$$\begin{array}{c|c|c|c}
\text{BMO}_n(\delta^2) & \text{Id} & \text{BMO}_n(\delta^2) \\
\hline
E & P & \|E\|\|P\| \leq C, \\
\end{array}$$

where $C > 0$ is some universal constant.
The most challenging aspect in connection with the localization method of Bourgain consists in proving the finite dimensional factorization problem (1.4) while simultaneously controlling $N$ in terms of $n$. In one-parameter $BMO(\delta)$-spaces the factorization problems analogous to (1.3) and (1.4) were solved in [15]. See also [17, 18]. The one-parameter factorization is both the model case and also a special case of our present problem.

2. Preliminaries

Basic notation.

Here we collect basic notation and definitions. We refer to [17] for reference. The dyadic intervals $\mathcal{D}$ on the unit interval are given by

$$\mathcal{D} = \{2^{-j}k, 2^{-j}(k + 1) : j, k \in \mathbb{N}_0, k \leq 2^j - 1\},$$

and the dyadic rectangles $\mathcal{R}$ on the unit square by

$$\mathcal{R} = \mathcal{D} \times \mathcal{D}.$$

Let $\pi : \mathcal{D} \setminus \{[0, 1)\} \to \mathcal{D}$ denote the dyadic predecessor map, that is $\pi(I) = \bigcap\{J \in \mathcal{D} : J \supseteq I\}$. The level $\text{lev}(I)$ of a dyadic interval $I \in \mathcal{D}$ is defined as $\text{lev}(I) = -\log_2(|I|)$, its position $\text{pos}(I)$ is given by $\text{pos}(I) = \inf I / |I|$. The collection $\mathcal{D}_j$ of dyadic intervals at level $j$ is given by

$$\mathcal{D}_j = \{I \in \mathcal{D} : \text{lev}(I) = j\}$$

and we set $\mathcal{D}_n = \bigcup_{j \leq n} \mathcal{D}_j$. For $n \in \mathbb{N}$ we define

$$\mathcal{R}_n = \{I \times J \in \mathcal{R} : I, J \in \mathcal{D}_n\}.$$

Given a collection of sets $\mathcal{C}$ we define

$$\mathcal{C}^* = \bigcup\{C : C \in \mathcal{C}\}.$$

If $A$ is some set, then $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$. The Carleson constant $[\mathcal{A}]$ of a collection $\mathcal{A} \subset \mathcal{D}$ is given by

$$[\mathcal{A}] = \sup_{I \in \mathcal{A}} \sum_{J \in \mathcal{A} \cap I} |J| / |I|.$$ 

Note that $[\mathcal{A} \cup \mathcal{B}] \leq [\mathcal{A}] + [\mathcal{B}]$ for any two collections $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$.

For any given dyadic interval $I \in \mathcal{D}$ we define $h_I = 1_{I_0} - 1_{I_1}$, where $1_A$ denotes the characteristic function of a set $A$, $I_0 = [\inf I, (\inf I + \sup I)/2]$ and $I_1 = [(\inf I + \sup I)/2, \sup I]$. The one parameter hardy space $H^1(\delta)$ is the completion of

$$\text{span}\{h_I : I \in \mathcal{D}\}$$

under the square function norm

$$\|f\|_{H^1(\delta)} = \int_0^1 \left(\sum a_I^2 h_I^2\right)^{1/2} dt,$$

where $f = \sum a_I h_I$. We set

$$H^1_n(\delta) = \text{span}\{h_I : I \in \mathcal{D}_n\} \cap H^1(\delta),$$

$$BMO_n(\delta) = \text{span}\{h_I : I \in \mathcal{D}_n\} \cap BMO(\delta).$$
The Gamlen–Gaudet factorization.

We recall the relation between large Carleson constants and factorization, see [17]. Let \( A \) be a collection of dyadic intervals satisfying \( \llbracket A \rrbracket \geq N \) and define

\[
X_A = \text{span}\{h_I : I \in A\} \cap H^1(\delta).
\]

If \( N = n4^n \), then there exist linear operators \( E \) and \( P \) so that

\[
\|E\| \|P\| \leq C. \tag{2.1}
\]

The bi–parameter analogues of these finite dimensional building blocks are:

\[
H^1_n(\delta^2) = \text{span}\{h_{I \times J} : I \times J \in \mathcal{R}_n\} \cap H^1(\delta^2),
\]

\[
BMO_n(\delta^2) = \text{span}\{h_{I \times J} : I \times J \in \mathcal{R}_n\} \cap BMO(\delta^2).
\]

In the bi–parameter context, factorization and large Carleson constants are related for collections of rectangles satisfying tensor product structure. Given collections of dyadic intervals \( A, B \subset \mathcal{D} \) we define the tensor product space \( X_{A \times B} \) by

\[
X_{A \times B} = \text{span}\{h_{I \times J} : I \times J \in A \times B\} \cap H^1(\delta^2).
\]

If \( N = n4^n \) and \( \llbracket A \rrbracket \geq N, \llbracket B \rrbracket \geq N \), then there exist linear operators \( E \) and \( P \) such that

\[
\|E\| \|P\| \leq C. \tag{2.2}
\]

Due to tensor product structure of \( X_{A \times B} \), the bi–parameter factorization \((2.2)\) results directly from its one–parameter predecessor \((2.1)\). In the next paragraph we discuss Ramsey’s theorem for colored dyadic rectangles. Its relevance for the constructions of this paper comes from the fact that for any two–coloring of \( \mathcal{R}_n \), Ramsey’s theorem detects a large monochromatic collection of the form \( A \times B \).

**Ramsey theorem for colored dyadic rectangles.**

Ramsey’s theorem asserts that for any two–coloring of the dyadic rectangles

\[
\mathcal{R}_n = \{I \times J : |I| \geq 2^{-n}, |J| \geq 2^{-n}\}
\]

there exist collections \( A, B \) of dyadic intervals, each of which has large Carleson constant and, moreover,

\[ A \times B \text{ is monochromatic in } \mathcal{R}_n. \]

Specifically, given \( n_0 \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that for any collection \( C \subset \mathcal{R}_n \) one finds \( A, B \subset \mathcal{D} \) satisfying

(i) \( A \times B \subset C \) or \( A \times B \subset \mathcal{R}_n \setminus C \),

(ii) \( \llbracket A \rrbracket \geq n_0 \) and \( \llbracket B \rrbracket \geq n_0 \).

One can choose \( n = n_0 2^{4^n} \). For the above formulation of Ramsey’s theorem we refer to [16].
Block bases and Projections in $H^1(\delta^2)$. 

We introduce next some frequently used terminology and record a boundedness criterion for projections on $H^1(\delta^2)$. We say that a sequence $\{z_{I \times J} : I \times J \in P\}$ in a Banach space $E$ is equivalent to the 2D Haar basis $\{h_{I \times J} : I \times J \in P\}$ in $H^1(\delta^2)$ if the following holds: The map

$$T : \sum a_{I \times J} h_{I \times J} \to \sum a_{I \times J} z_{I \times J}$$

defined initially on finite linear combinations of 2D Haar functions and extended by density to $H^1(\delta^2)$ satisfies

$$C^{-1} \|T(x)\|_E \leq \|T(x)\|_{H^1(\delta^2)}, \quad x \in H^1(\delta^2).$$

Let $\{\mathcal{A}_{I \times J} : I \times J \in P\}$ be pairwise disjoint collections of dyadic rectangles and let $A_{I \times J} = \bigcup_{R \in \mathcal{A}_{I \times J}} R$ be the point-set covered by the collection $\mathcal{A}_{I \times J}$. We denote by

$$z_{I \times J} = \sum_{R \in \mathcal{A}_{I \times J}} h_R$$

the block-basis generated by $\mathcal{A}_{I \times J}$. We assume throughout, that $\|z_{I \times J}\|^2 = |A_{I \times J}|$ or equivalently that $\mathcal{A}_{I \times J}$ consists of pairwise disjoint dyadic rectangles. We formulate conditions on the collections $\{\mathcal{A}_{I \times J}\}$ so that the block basis $\{z_{I \times J}\}$ is equivalent to the 2D Haar system. The sets $\{A_{I \times J} : I \times J \in P\}$ satisfy the bi-tree condition if the following two conditions hold. First, there exists $C > 0$ so that for each $I \times J \in P$

$$C^{-1}|I \times J| \leq |A_{I \times J}| \leq C|I \times J|.$$

Second, given $(I_0 \times J_0), (I_1 \times J_1) \in P$ such that $I = I_0 = I_1$ we have $J = J_0 = J_1$

$$A_{I_0 \times J_0} \cap A_{I_1 \times J_1} = \emptyset, \quad A_{I_0 \times J_0} \cup A_{I_1 \times J_1} \subset A_{I \times J},$$

$$A_{I_0 \times J_0} \cap A_{I_1 \times J_1} = \emptyset, \quad A_{I_0 \times J_0} \cup A_{I_1 \times J_1} \subset A_{I \times J}.$$ (2.3)

(2.4)

Under the above conditions it follows that the block bases $\{z_{I \times J}\}$ is equivalent to the 2D Haar system in and on $H^1(\delta^2)$. The following proposition is a basic tool that allows to project onto the span of the block bases $\{z_{I \times J} : I \times J \in P\}$. It was instrumental in proving that $H^1(\delta^2)$ is a primary space, see [7] and [19].

**Proposition 2.1.** Let $\mathcal{A}_{I \times J}, I \times J \in P$ be pairwise disjoint collections consisting of disjoint dyadic rectangles. Let $A_{I \times J} = \mathcal{A}_{I \times J}$. Assume that $\{A_{I \times J} : I, J \in P\}$ is a bi-tree, then the following hold

(i) The block basis $\{z_{I \times J} : I \times J \in P\}$ is equivalent to the 2D-Haar basis in $H^1(\delta^2)$.

(ii) If there exists $C > 0$ so that for each $R, R_0 \in P$ with $R \supset R_0$ and for every $K \times L \in \mathcal{A}_R$ we have

$$C^{-1}\frac{|A_{R_0}|}{|A_R|} \leq \frac{|(K \times L) \cap A_{R_0}|}{|K \times L|} \leq C\frac{|A_{R_0}|}{|A_R|},$$

then the orthogonal projection

$$Pf = \sum \langle f, z_{I \times J} \rangle \frac{z_{I \times J}}{|z_{I \times J}|^2}$$

defines a bounded operator on $H^1(\delta^2)$ with norm only depending on $C$.

The corresponding criterion for the 1D Haar system and orthogonal projections in $L^p$ and $H^1$ is a theorem of P. W. Jones [12]. It is important to realize that the boundedness of the projection $P$ can be verified by checking the criterion (2.3), which involves only testing dyadic rectangles and not arbitrary open sets. Related to this is Fefferman’s theorem [10] which determines the boundedness of singular
integral operators by testing atoms supported on rectangles. By contrast, the space $H^1(\delta^2)$ itself is not determined by atoms supported on dyadic rectangles. The corresponding counterexample is due to Carleson, see \cite{9}.

Rademacher type functions in $H^1(\delta^2)$ and $BMO(\delta^2)$. We define the following Rademacher type system as block basis of the Haar system. Given $r \geq k_0$ and $K_0 \times L_0 \in \mathcal{R}$ with $|K_0| = 2^{-k_0}$ we specify the following functions. First, for any choice of signs we set

$$d_i = \sum_{K \in D_i \cap K_0} \pm h_K, \quad i \geq r.$$  

Then it is easy to see that if we define

$$g_i(s,t) = d_i(s) h_{L_0}(t), \quad s,t \in [0,1]$$

for each dyadic interval $L_0$, then by (1.1) and duality we have

$$\| \sum_{i=r}^{r+k-1} g_i \|_{H^1(\delta^2)} = \sqrt{r} |L_0| \quad \text{and} \quad \| \sum_{i=r}^{r+k-1} g_i \|_{BMO(\delta)} = \sqrt{r}.$$

(2.6)

3. Results

The main result of this paper is the following quantitative factorization theorem 3.1.

**Theorem 3.1.** For $n \in \mathbb{N}$ there exists $N = N(n)$ so that the following holds: For any operator $T : H^1_N(\delta^2) \to H^1_N(\delta^2)$ the identity on $H^1_n(\delta^2)$ well–factors through $H = T$ or $H = \text{Id} - T$. That is

$$H^1_n(\delta^2) \xrightarrow{\text{Id}} H^1_n(\delta^2) \xrightarrow{E} H^1_n(\delta^2) \xrightarrow{\text{Id}} H^1_n(\delta^2) \xrightarrow{P} H^1_n(\delta^2)$$

where $C > 0$ is a universal constant.

The proof is based on a Ramsey type theorem for colored dyadic rectangles as well as a reduction argument to multiplier operators on the Haar system.

Reduction to multiplier operators – quasi-diagonalization.

The following theorem asserts that the factorization problem is solved as soon as we are able to prove it for the very special class of operators – the multipliers of the Haar system.

**Theorem 3.2.** Let $n \in \mathbb{N}$ and $\{\varepsilon_{I \times J} : I \times J \in \mathcal{R}_n\}$ be a given sequence of small positive scalars. Let $N$ be given by

$$\log_2 \log_2 (N \min \{\varepsilon_{I \times J}\}) = C_1 n$$

for some universal constant $C_1 > 0$. Let $T : H^1_N(\delta^2) \to H^1_N(\delta^2)$ linear with $\|T\| = 1$. Then there exist disjoint collections $\delta_{I \times J}$, indexed by $I \times J \in \mathcal{R}_n$, consisting of pairwise disjoint dyadic rectangles defining the functions

$$b_{I \times J} = \sum_{K \times L \in \delta_{I \times J}} h_{K \times L},$$

which satisfy the following conditions:

(i) $\delta_{I \times J} \subset \mathcal{R}_N$ and $|b_{I \times J}| \leq 1$ for all $I \times J \in \mathcal{R}_n$. 

constant $C > \frac{1}{2}$, we now show that Theorem 3.1 can be deduced from the Ramsey theorem for colored dyadic rectangles (see Section 2) and the reduction theorem 3.2.

Proof of Theorem 3.1.

**(i)** The orthogonal projection

$$Q(f) = \sum_{I \times J \in \mathcal{R}_n} \langle f, \frac{b_{I \times J}}{\|b_{I \times J}\|_2} \rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2}$$

is a bounded operator on $H^1(\delta^2)$ with $Q(H^1(\delta^2)) = \text{span}\{b_{I \times J}\}$ satisfying

$$\|Q : H^1(\delta^2) \to H^1(\delta^2)\| \leq C_2,$$

for some universal constant $C_2 > 0$.

**(ii)** The map $\mathcal{S} : H^1_n(\delta^2) \to \text{span}\{b_{I \times J}\} \cap H^1_N(\delta^2)$ defined as the linear extension of $b_{I \times J} \mapsto b_{I \times J}$ is an isomorphism with

$$\|\mathcal{S}\| \|\mathcal{S}^{-1}\| \leq C_3,$$

for some universal constant $C_3 > 0$.

**(iii)** We have the estimate

$$\sum_{K \times L \neq I \times J} |(Tb_{K \times L}, b_{I \times J})| \leq \varepsilon_{I \times J}\|b_{I \times J}\|_2^2,$$

for all $I \times J \in \mathcal{R}_n$.

We now show that Theorem 3.1 can be deduced from the Ramsey theorem for colored dyadic rectangles (see Section 2) and the reduction theorem 3.2.

**Proof of Theorem 3.1.**

Let $n \in \mathbb{N}$. We define $N$ by the chain of the following conditions:

$$\log_2 \log_2(N \min \{\varepsilon_{I \times J}\}) = C_1N_1, \quad N_1 = N_2 2^{4N_2}, \quad N_2 = n4^n.$$  \hspace{1cm} (3.3)

We select $\{\varepsilon_{I \times J} : I \times J \in \mathcal{R}_{N_1}\}$ such that

$$\sum_{I \times J \in \mathcal{R}_{N_1}} \varepsilon_{I \times J} \leq \frac{1}{4}. \hspace{1cm} (3.4)$$

For instance, we could take $\varepsilon_{I \times J} = (164N_i)^{-1}$. Let $T : H^1_N(\delta^2) \to H^1_N(\delta^2)$ be an operator such that $\|T\| = 1$. Then, applying Theorem 3.2 with $n = N_1$ to $T$ yields the block basis $\{b_{I \times J} : I \times J \in \mathcal{R}_{N_1}\}$ satisfying the conclusion of Theorem 3.2.

The Ramsey theorem for colored dyadic rectangles applied to

$$\mathcal{C} = \{I \times J \in \mathcal{R}_{N_1} : |(Tb_{I \times J}, b_{I \times J})| \geq \|b_{I \times J}\|_2^2/2\}$$

yields collections $\mathcal{A}, \mathcal{B} \subset \mathcal{P}_{N_1}$, with Carleson constants $\|\mathcal{A}\| \geq N_2$ and $\|\mathcal{B}\| \geq N_2$, such that $\mathcal{A} \cap \mathcal{B} \subset \mathcal{C}$ or $\mathcal{A} \times \mathcal{B} \subset \mathcal{R}_{N_1} \setminus \mathcal{C}$. We choose $H = T$ if $\mathcal{A} \times \mathcal{B} \subset \mathcal{C}$ and $H = \text{Id} - T$ if $\mathcal{A} \times \mathcal{B} \subset \mathcal{R}_{N_1} \setminus \mathcal{C}$.

The following lower estimate will be essential below:

$$|\langle (Hb_{I \times J}, b_{I \times J}) \rangle| \geq \|b_{I \times J}\|_2^2/2, \quad I \times J \in \mathcal{A} \times \mathcal{B}. \hspace{1cm} (3.5)$$

We define the tensor product space $X_{\mathcal{A} \times \mathcal{B}}$ by

$$X_{\mathcal{A} \times \mathcal{B}} = \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\} \cap H^1(\delta^2).$$

Since $\|\mathcal{A}\| \geq N_2$, $\|\mathcal{B}\| \geq N_2$, we know from (2.2) that there exists a universal constant $C > 0$ so that

$$H^1_n(\delta^2) \xrightarrow{\text{Id}} H^1(n(\delta^2))$$

$$E_0 \xrightarrow{p_0} \|E_0\| \|P_0\| \leq C.$$
We claim that Theorem 3.2 and the choices we made in (3.3), (3.4) and (3.5) imply that there exist linear operators $S_1$ and $P_1$ such that

$$X_{\mathcal{A} \times \mathcal{B}} \xrightarrow{\text{Id}} X_{\mathcal{A} \times \mathcal{B}} \xrightarrow{S_1} \xrightarrow{P_1} \|E_1\| \|P_1\| \leq C.$$ 

For the verification of the claim we remark that the method lined out in [17, 288–290] is directly applicable: The isomorphic embedding

$$S_1 : X_{\mathcal{A} \times \mathcal{B}} \to \text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$$

is defined as the linear extension of the map

$$h_{I\times J} \mapsto b_{I\times J}.$$ 

For the norm estimate of $S_1$ we refer to (3.1). Next, define

$$\tilde{P}_1 : H^1_N(\delta^2) \to \text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$$

by the formula

$$f \mapsto \sum_{I \times J \in \mathcal{A} \times \mathcal{B}} \langle f, b_{I\times J} \rangle b_{I\times J} (Hb_{I\times J}, b_{I\times J})^{-1}.$$ 

We observe that for $g \in \text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$ we have

$$\tilde{P}_1 H g = g + G g,$$

where the error term $G g$ is controlled via $\|G\| : \text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\} \to \text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\} \|_{H^1(\delta^2)} \leq \frac{1}{2}$. Hence, we may invert $\text{Id} + G$ on $\text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$ so that

$$(\text{Id} + G)^{-1} \tilde{P}_1 H g = g, \quad g \in \text{span}\{b_{I\times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}.$$ 

This defines $P_1$ as follows:

$$P_1 f = S_1^{-1} (\text{Id} + G)^{-1} \tilde{P}_1 H S_1 f, \quad f \in X_{\mathcal{A} \times \mathcal{B}}.$$ 

We should emphasize that $S_1^{-1}$ is well defined on the range of $(\text{Id} + G)^{-1}$ and furthermore $(\text{Id} + G)^{-1}$ is well defined on the range of $\tilde{P}_1$.

Finally, it remains to merge the diagrams yielding the following factorization:

$$H^1_N(\delta^2) \xrightarrow{\text{Id}} H^1_N(\delta^2) \xrightarrow{E} \xrightarrow{P} \|E\| \|P\| \leq C.$$ 

□
4. Quantitative Quasi-diagonalization

In this section we give the proof of Theorem 5.2. Our argument is inductive. We use induction within the collection of dyadic rectangles. It is therefore important that we introduce a suitable linear ordering relation on the collection of dyadic rectangles. Below we specifically construct the linear ordering relation $\prec$ so that the bijective index function $\mathcal{O}_3 : \mathcal{R} \to \mathbb{N}$, which is defined by

$$\mathcal{O}_3(R_0) < \mathcal{O}_3(R_1) \iff R_0 \prec R_1, \quad R_0, R_1 \in \mathcal{R},$$

has the following properties (4.1) and (4.2). For a picture of the index function $\mathcal{O}_3$, see Figure 1. The geometry of a dyadic rectangle and its position within our linear ordering $\prec$ are linked by the inequalities

$$(2^k - 1)^2 < \mathcal{O}_3(I \times J) \leq (2^{k+1} - 1)^2, \quad \text{whenever } \min(|I|, |J|) = 2^{-k},$$

as well as

$$4|I_1 \times J_1| \leq \frac{|I_0 \times J_0|}{\min(|I_1|, |J_1|)}, \quad \text{whenever } I_0 \times J_0 \prec I_1 \times J_1.$$  

Any linear orderings on the dyadic rectangles for which (4.1) and (4.2) hold may serve as basis for our induction argument in the proof of Theorem 5.2.

4.1. Constructing the linear ordering relation $\prec$ on $\mathcal{R}$.

First, we define the rectangles of fixed side lengths $2^{-m}$ and $2^{-n}$ by setting

$$\mathcal{B}_{m,n} = \{I \times J \in \mathcal{R} : |I| = 2^{-m}, |J| = 2^{-n} \}, \quad m, n \geq 0.$$  

(4.3)

Second, we will define the ordering relation $\prec_\ell$ on each of the blocks $\mathcal{B}_{m,n}$. Given two dyadic rectangles $I_0 \times J_0, I_1 \times J_1 \in \mathcal{B}_{m,n}$ we set

$$I_0 \times J_0 \prec_\ell I_1 \times J_1 : \iff (\inf I_0, \inf J_0) <_\ell (\inf I_1, \inf J_1),$$

where $<_\ell$ denotes the lexicographic ordering on $\mathbb{R}^2$. Third, we shall collect the blocks $\mathcal{B}_{m,n}$ in the collections

$$\mathcal{J}_k = \{\mathcal{B}_{m,n} : \max(m, n) = k \}, \quad k \geq 0.$$  

Third, we need to bring the blocks $\mathcal{B}_{m,n}$ in order. To this end, we consider

$$w : \{\mathcal{B}_{m,n} : m, n \geq 0\} \to \mathbb{N}_0$$

such that the following conditions hold for all $k \geq 1$:

(i) $w|_{\mathcal{J}_k} : \mathcal{J}_k \to \{k^2, \ldots, (k + 1)^2 - 1\}$ is bijective.

(ii) we set $w(\mathcal{B}_{0,0}) = k^2$ and moreover

$$w(\mathcal{B}_{m_0,n_0}) < w(\mathcal{B}_{m_1,n_1}) \iff \begin{cases} m_0 > n_0 & \text{and } m_1 \leq n_1, \\
_0 > m_0 & \text{and } n_0 > n_1 \text{ and } n_0 < n_1, \\
_0 < m_0 & \text{and } m_0 \leq n_0 \text{ and } m_0 < n_1, \\
_0 < m_0 & \text{and } m_0 \leq n_1 \text{ and } m_0 < n_1, \end{cases}$$

for all $\mathcal{B}_{m_0,n_0}, \mathcal{B}_{m_1,n_1} \in \mathcal{J}_k \setminus \{\mathcal{B}_{0,0}\}$.

Finally, we use the function $w$ and its properties as well as the properties of $<_\ell$ to define our linear ordering relation $\prec$ on the dyadic rectangles $\mathcal{R}$. If $I_0 \times J_0, I_1 \times J_1 \in \mathcal{R}$ we set

$$(I_0 \times J_0) \prec (I_1 \times J_1) : \iff \begin{cases} w(\mathcal{B}_{\ell I_0,\ell J_0}) < w(\mathcal{B}_{\ell I_1,\ell J_1}) \text{ or} \\
w(\mathcal{B}_{\ell I_0,\ell J_0}) = w(\mathcal{B}_{\ell I_1,\ell J_1}) \text{ and } (I_0, J_0) <_\ell (I_1, J_1). \end{cases}$$

Since our ordering relation $\prec$ is linear, we may well define the bijective index function $\mathcal{O}_3 : \mathcal{R} \to \mathbb{N}$ by the following property:

$$\mathcal{O}_3(R_0) < \mathcal{O}_3(R_1) \iff R_0 \prec R_1, \quad R_0, R_1 \in \mathcal{R}.$$
Figure 1. Index function $O_q(I \times J)$ for all $I \times J \subseteq \mathcal{R}_2$.

Observe that the crucial relations between the geometry of a dyadic rectangle and its position within our linear ordering (4.1) and (4.2) are satisfied by design.

4.2. Combinatorial Lemma.

Let $\{r_i\}$ denote the sequence of independent Rademacher functions which are given by

$$r_i(t) = \text{sign}(\sin(2\pi 2^i t)), \quad t \in [0, 1], \ i \in \mathbb{N}.$$  

We consider the tensor product $r_{i,j}$ of the standard Rademacher system defined as

$$r_{ij}(s, t) = r_i(s) r_j(t), \quad (s, t) \in [0, 1]^2$$  

It is well known and easy to verify that in both spaces, $H^1(\delta^2)$ and $BMO(\delta^2)$, the system $\{r_{ij}\}$ is equivalent to the unit vector basis of $\ell^2$. Specifically, there exists constants $c_0, C_0$ so that for any sequence of scalars $\{a_{ij}\}$ the following inequalities hold.

$$\| \sum a_{ij} r_{ij} \|_{H^1(\delta^2)} = \sum a_{ij}^2$$

and

$$c_0 \sum a_{ij}^2 \leq \| \sum a_{ij} r_{ij} \|_{BMO(\delta^2)}^2 \leq C_0 \sum a_{ij}^2.$$  

Hence, $\{r_{ij}\}$ is a weak null sequence in both spaces $H^1(\delta^2)$ and $BMO(\delta^2)$,

$$r_{ij} \to 0 \quad \text{weakly in } H^1(\delta^2), \ i \to \infty \text{ or } j \to \infty$$

and

$$r_{ij} \to 0 \quad \text{weakly in } BMO(\delta^2), \ i \to \infty \text{ or } j \to \infty.$$  

For the purpose of our present work we need a quantitative strengthening of these considerations. This is done in the following combinatorial lemma. Our combinatorial argument is controlled by the local frequency weight

$$f(K \times L) = |\langle x, h_{K \times L} \rangle| + |\langle y, h_{K \times L} \rangle|, \quad K \times L \subseteq K_0 \times L_0$$

where $x \in BMO(\delta^2)$ and $y \in H^1(\delta^2)$ are fixed functions and $K_0 \times L_0 \subseteq \mathcal{R}$. For us, it will be extremely important that the collection

$$\{K \times L : f(K \times L) \leq \tau |K \times L|\}$$
contains almost complete and well-structured coverings of $K_0 \times L_0$ of the form

$$\{K_0 \times L : L \in \mathcal{D}_k \cap L_0\} \quad \text{and} \quad \{K \times L_0 : K \in \mathcal{D}_k \cap K_0\},$$

with $k$ and $\ell$ well under control in terms of $\tau$.

**Lemma 4.1.** Let $i \in \mathbb{N}$, $K_0, L_0 \in \mathcal{D}$, $x_j \in BMO(\delta^2)$, $y_j \in H^1(\delta^2)$, $1 \leq j \leq i$, such that

$$\sum_{j=1}^{i} ||x_j||_{BMO(\delta^2)} \leq 1 \quad \text{and} \quad \sum_{j=1}^{i} ||y_j||_{H^1(\delta^2)} \leq |K_0 \times L_0|. \quad (4.4)$$

Let $\tau > 0$, $r \in \mathbb{N}_0$, $K \times L \in \mathcal{D}$ and define the local frequency weight

$$f_i(K \times L) = \sum_{j=1}^{i} ||x_j, h_{K \times L}|| + ||y_j, h_{K \times L}|| \quad (4.5)$$
as well as the collections

$$\mathcal{K}_k(K_0 \times L_0) = \{K \times L_0 : K \subset K_0, |K| \leq 2^{-r}|K_0|, f_i(K \times L_0) \leq \tau|K \times L_0|\},$$

$$\mathcal{L}_k(K_0 \times L_0) = \{K \times L_0 : L \subset L_0, |L| \leq 2^{-r}|L_0|, f_i(K \times L_0) \leq \tau|K \times L_0|\}.$$  

For all integers $k, \ell$ the collections $\mathcal{K}_k(K_0 \times L_0)$ and $\mathcal{L}_k(K_0 \times L_0)$ are given by

$$\mathcal{K}_k(K_0 \times L_0) = \mathcal{K}(K_0 \times L_0) \cap ([K \in \mathcal{D} : |K| = 2^{-k}|K_0|] \times \mathcal{D}),$$

$$\mathcal{L}_\ell(K_0 \times L_0) = \mathcal{L}(K_0 \times L_0) \cap (\mathcal{D} \times \{L \in \mathcal{D} : |L| = 2^{-\ell}|L_0|\}).$$

Let $\delta > 0$. Then there exist integers $k, \ell$ with

$$r \leq k, \ell \leq \left\lfloor \frac{i^2}{\delta + \tau^2} \right\rfloor + r \quad (4.6)$$
such that

$$|\mathcal{K}_k(K_0 \times L_0)| \geq (1 - \delta)|K_0 \times L_0| \quad \text{and} \quad |\mathcal{L}_\ell(K_0 \times L_0)| \leq (1 - \delta)|K_0 \times L_0|. \quad (4.7)$$

**Proof.** Define $\mathcal{B} = (\mathcal{D} \cap K_0 \times L_0) \setminus \mathcal{K}(K_0 \times L_0)$ and

$$\mathcal{B}_k = \mathcal{B} \cap ([K \in \mathcal{D} : |K| = 2^{-k}|K_0|] \times \mathcal{D}).$$

Define

$$A = \left\lfloor \frac{i^2}{\delta + \tau^2} \right\rfloor + r.$$  

Assume that

$$|\mathcal{B}_k| \geq \delta|K_0 \times L_0|, \quad r \leq k \leq A.$$

Summing these estimates yields

$$\sum_{k=r}^{A} |\mathcal{B}_k| \geq (A - r + 1) \delta|K_0 \times L_0|, \quad (4.8)$$

Observe that

$$\tau \cdot \sum_{k=r}^{A} |\mathcal{B}_k| \leq \sum_{j=1}^{i} \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{B}_k} ||x_j, h_{K \times L_0}|| + ||y_j, h_{K \times L_0}||$$

$$= \sum_{j=1}^{i} \left( \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \right) + \left( \sum_{j=1}^{i} \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \right),$$
By (2.10) we have
\[ \left\| \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{R}_k} \pm h_{K \times L_0} \right\|_{H^1(\delta^2)} = \sqrt{A - r + 1} |K_0 \times L_0|, \]
\[ \left\| \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{R}_k} \pm h_{K \times L_0} \right\|_{BMO(\delta^2)} = \sqrt{A - r + 1}, \]
thus, by duality and (4.3) we obtain
\[ \tau \cdot \sum_{k=r}^{A} |\mathcal{R}_k^*| \leq i \sqrt{A - r + 1} |K_0 \times L_0|. \] (4.9)
Combining (4.8) and (4.9) we conclude
\[ A \leq \frac{i^2}{\delta^2 \tau^2} + r - 1, \]
which contradicts the definition of $A$. The same proof in the other variable can be used to show the estimate for $C_t^*$.

**4.3. Proof of Theorem 3.2**

Theorem 3.2 asserts that we are able to construct a large block basis $\{b_{I \times J}\}$ in $H^n_1(\delta^2)$ which are almost eigenvectors for $T$. Moreover, the block basis is such that it spans a well complemented copy of $H^n_1(\delta^2)$ in $H^n_1(\delta^2)$. It is of equal importance that the relation between the dimensions $N$, $n$ and the precision $\{\varepsilon_{I \times J}\}$ is given quantitatively by $\log_2 \log_2 (N \min \{\varepsilon_{I \times J}\}) = C_1 n$.

It is here where we will exploit our linear order $\prec$ introduced on the collection of dyadic rectangles $\mathcal{R}$. The proof described below is by mathematical induction executed along the linear order given by $\mathcal{O}_4$. To make the transition from standard indexing by dyadic rectangles to indexing by natural numbers we employ the following convention. Given a dyadic rectangle $I \times J$ with $\mathcal{O}_4 (I \times J) = i$ we will systematically relabel the collections $\delta_{I \times J}$, the functions $b_{I \times J}$ and the constants $\delta_{I \times J}$, $\tau_{I \times J}$ by $\delta_i$, $b_i$ and $\delta_i$, $\tau_i$, respectively.

Before we begin with our construction we define the following constants:
\[ \delta_i = 2^{-i} / (8n) \quad \text{and} \quad \tau_i = \frac{2^{-i}}{4(i-1)} \min_{j \leq i} \varepsilon_j |\mathcal{O}_4^{-1}(i)|. \] (4.10)

**Inductive Construction.**

*First stage of the induction.* We begin the induction by setting $\delta_1 := \delta_{[0,1] \times [0,1]} := \{[0,1] \times [0,1]\}$ and $b_1 := b_{[0,1] \times [0,1]} := b_{[0,1] \times [0,1]}$.

*At stage $i$ of the induction.* We assume that we have already defined the disjoint collections of dyadic rectangles $\delta_j$ for all $1 \leq j \leq i - 1$. Now, we will construct $\delta_i$. The construction of $\delta_i$ depends crucially on the value of $i$. We will distinguish between two principal cases, where the second one is divided again into two sub cases.

- **Case 1:** The stage ordinal $i$ is given by $i = \mathcal{O}_4 ([0,1] \times J)$.
- **Case 2:** The stage ordinal $i$ is given by $i = \mathcal{O}_4 (I \times J)$, where $I \neq [0,1]$.
  - **Case 2.a:** The second component $J$ satisfies $J = [0,1]$.
  - **Case 2.b:** The second component $J$ satisfies $J \neq [0,1]$. 

\[ \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{R}_k} \pm h_{K \times L_0} \]
\[ \sum_{k=r}^{A} \sum_{K \times L_0 \in \mathcal{R}_k} \pm h_{K \times L_0} \]
Case 1: \( I = [0, 1] \). The stage ordinal \( i \) is given by \( i = \mathcal{O}_d([0, 1] \times J) \). Case 1 is applicable to the light rectangles. The collections \( \mathcal{E}_{K \times L_0} \) indexed by the dark rectangles \( L_0 \times J_0 \) are already well defined at this stage. The white ones are ignored.

Recall that
\[
b_j = \sum_{K \times L \in \mathcal{E}_j} h_{K \times L}, \quad 1 \leq j \leq i - 1.
\]

Since the collection \( \mathcal{E}_j \) consists of pairwise disjoint rectangles we have by (4.1) and duality that
\[
\|b_j\|_{BMO(\mathbb{R}^2)} = 1 \quad \text{and} \quad \|b_j\|_{H^1(\mathbb{R}^2)} = |\mathcal{E}_j^*|.
\]

Now define
\[
x_j := \frac{1}{i - 1} T^* b_j, \quad y_j := \frac{[0, 1] \times j}{(i - 1)|\mathcal{E}_j|} T b_j, \quad 1 \leq j \leq i - 1. \tag{4.11}
\]

and observe
\[
\sum_{j=1}^{i-1} \|x_j\|_{BMO(\mathbb{R}^2)} \leq 1 \quad \text{and} \quad \sum_{j=1}^{i-1} \|y_j\|_{H^1(\mathbb{R}^2)} \leq |[0, 1] \times J|.
\]

Let \( \tilde{J} \) denotes the unique dyadic interval satisfying \( \tilde{J} \supset J \) and \( |\tilde{J}| = 2|J| \). By definition of our linear ordering we have \( \mathcal{O}_d([0, 1] \times \tilde{J}) \leq i - 1 \). Hence, \( \mathcal{E}_{[0, 1] \times \tilde{J}} \) is already defined. Given \( L_0 \) we remark that by our previous choices we have the following convenient implication:
\[
K \times L_0 \in \mathcal{E}_{[0, 1] \times \tilde{J}} \quad \text{implies} \quad K = [0, 1]. \tag{4.12}
\]

For all \( L_0 \) such that \([0, 1] \times L_0 \in \mathcal{E}_{[0, 1] \times \tilde{J}} \) we define the collection of dyadic rectangles
\[
\mathcal{L}([0, 1] \times L_0) = \{ [0, 1] \times L : L \subseteq L_0, f_{i-1}([0, 1] \times L) \leq \tau |L| \},
\]

where the local frequency weight \( f_{i-1} \) is specified in (4.15). Applying Lemma 4.1 to \( \mathcal{L}([0, 1] \times L_0) \) yields an integer \( \ell = \ell([0, 1] \times L_0) \) so that
\[
1 \leq \ell([0, 1] \times L_0) < \frac{(i - 1)^2}{\delta_l^2 \tau_l^2} + 1 \tag{4.13}
\]

such that the collection of disjoint dyadic rectangles
\[
\mathcal{L}([0, 1] \times L_0) = \{ [0, 1] \times L \in \mathcal{L}([0, 1] \times L_0) : |L| = 2^{-\ell([0, 1] \times L_0)} |L_0| \}
\]
satisfies the estimate
\[
(1 - \delta_l)[0, 1] \times L_0 \leq |\mathcal{L}([0, 1] \times L_0)| \leq |[0, 1] \times L_0|. \tag{4.14}
\]

Note that in Lemma 4.1 \( \mathcal{L}([0, 1] \times L_0) \) was denoted \( \mathcal{L}_{\ell}(0, 1] \times L_0) \). Now we take the union and define
\[
\mathcal{L}_{[0, 1] \times J} = \bigcup \{ \mathcal{L}([0, 1] \times L_0) : \ [0, 1] \times L_0 \in \mathcal{E}_{[0, 1] \times \tilde{J}} \}.
\]

Since \( \mathcal{L}_{[0, 1] \times J}([0, 1] \times L_0) \subset \mathcal{L}([0, 1] \times L_0) \), we know
\[
f_{i-1}([0, 1] \times L) \leq \tau |L|, \quad \text{for } [0, 1] \times L \in \mathcal{L}_{[0, 1] \times J}. \tag{4.15}
\]

Following Gamlen-Gaudet, we define
\[
\mathcal{E}_{[0, 1] \times J} = \{ [0, 1] \times L \in \mathcal{L}_{[0, 1] \times J} : \text{pos}(L) = \text{pos}(J) \mod 2 \}. \tag{4.16}
\]

Recall that \( i = \mathcal{O}_d([0, 1] \times J) \) and \( \delta_l = \delta_{[0, 1] \times J} \). An immediate consequence of the Gamlen-Gaudet construction and (4.14) is the estimate
\[
\frac{1}{2} (1 - \delta_{[0, 1] \times J}) |[0, 1] \times L| \leq |([0, 1] \times L) \cap \mathcal{E}_{[0, 1] \times J}^*| \leq \frac{1}{2} |[0, 1] \times L|, \tag{4.17}
\]

where \( \mathcal{E}_{[0, 1] \times J}^* \) is the collection
\[
\mathcal{E}_{[0, 1] \times J}^* = \{ [0, 1] \times L \in \mathcal{L}_{[0, 1] \times J} : \text{pos}(L) = \text{pos}(J) \mod 2 \}.
\]
for all \([0, 1] \times L \in \mathcal{E}_{[0,1] \times J}\). Note that all the rectangles in \(\mathcal{E}_{[0,1] \times J}\) are of the form \([0, 1] \times L\), see \([4.12]\).

**Case 2: \(I \neq [0, 1]\).** The figure on the right depicts the transition from Case 1 to Case 2. Here, the stage ordinal \(i\) is given by \(i = 0_{s}(I \times J)\) with \(I \neq [0, 1]\). The rectangle \(I \times J\) is one of the light rectangles. The light rectangles fall into two separate cases, see below. Up to \([4.2]\) both cases are treated in tandem.

Recall that

\[
b_j = \sum_{K \times L \in \mathcal{E}_j} h_{K \times L}, \quad 1 \leq j \leq i - 1.
\]

Since the collection \(\mathcal{E}_j\) consists of pairwise disjoint rectangles we have by \([4.1]\) and duality that

\[
\|b_j\|_{BMO(\mathbb{R}^2)} = 1 \quad \text{and} \quad \|b_j\|_{H^1(\mathbb{R}^2)} = |\mathcal{E}_j^*|.
\]

Now define

\[
x_j := \frac{1}{i - 1} T^* b_j, \quad y_j := \frac{|I \times J|}{(i - 1)|\mathcal{E}_j^*|} Tb_j, \quad 1 \leq j \leq i - 1. \tag{4.18}
\]

Observe

\[
\sum_{j=1}^{i-1} \|x_j\|_{BMO(\mathbb{R}^2)} \leq 1 \quad \text{and} \quad \sum_{j=1}^{i-1} \|y_j\|_{H^1(\mathbb{R}^2)} \leq |I \times J|.
\]

We will now construct the collections \(\mathcal{Y}_{I \times J}\) of \(y\)-frequencies and depending on each \(y\)-frequency \(L_0 \in \mathcal{Y}_{I \times J}\) the collection \(\mathcal{X}_{I \times J}(L_0)\) of \(x\)-frequencies. Those frequencies will be our building blocks for \(\mathcal{E}_{I \times J}\). Let us define the collection \(\mathcal{Y}_{I \times J}\) by

\[
\mathcal{Y}_{I \times J} = \{L_0 : [0, 1] \times L_0 \in \mathcal{E}_{[0,1] \times J}\}.
\]

We remark that \(K \times L_0 \in \mathcal{E}_{[0,1] \times J}\) implies \(K \neq [0, 1]\).

We now turn to the construction of \(\mathcal{X}_{I \times J}(L_0)\), \(L_0 \in \mathcal{Y}_{I \times J}\). This will be more involved, and in particular the construction relies on the combinatorial Lemma \([4.1]\).

First, let \(\mathcal{D}\) denote the previous dyadic rectangle indices that are not located in the same macro block \(\mathcal{B}_{m,n}\) as \(I \times J\), see \([4.3]\). That is

\[
\mathcal{D} = \{I_0 \times J_0 : I_0 \times J_0 \cap I \times J, (|I_0|, |J_0|) \neq (|I|, |J|)\}
\]

Let \(\mathcal{A}\) denote collection of index strips \(\mathcal{D}_m \times \{J_0\}\) given by

\[
\mathcal{A} = \{\mathcal{D}_m \times \{J_0\} : m \in \mathbb{N}_0, I_0 \times J_0 \in \mathcal{D}, I_0 \in \mathcal{B}_m\}.
\]

Recall that \(\mathcal{D}_m = \{I \in \mathcal{D} : |I| = 2^{-m}\}\). Given \(L_0 \in \mathcal{Y}_{I \times J}\) and \(\mathcal{F} \in \mathcal{A}\) we define

\[
W_{I \times J}(\mathcal{F}, L_0) = \{x \in [0, 1] : \exists y \in [0, 1], (x, y) \in \bigcup_{I_0 \times J_0 \in \mathcal{F}} \mathcal{E}_{I_0 \times J_0} \cap ([0, 1] \times L_0)\}.
\]

Note that if \(I_0 \times J_0 \in \mathcal{F}\), then \(\mathcal{F} = \mathcal{D}_m \times \{J_0\}\) for \(2^{-m} = |I_0|\). Furthermore, let \(L_0 \in \mathcal{Y}_{I \times J}\), then \(\mathcal{E}_{I_0 \times J_0} \cap ([0, 1] \times L_0) \neq \emptyset\) if and only if \(J \cap J_0 \neq \emptyset\). To see this we proceed as follows. First, observe that \(\mathcal{E}_{I_0 \times J_0} \cap ([0, 1] \times L_0) \neq \emptyset\) is equivalent (!) to \(\mathcal{E}_{[0,1] \times J_0} \cap [0, 1] \times L_0 \neq \emptyset\). We remark that as a consequence of the Gaunet-Gaudet construction used in Case 1 the collection \(\{\mathcal{E}_{[0,1] \times J} : J \in \mathcal{D}\}\) is a nested collection of sets for which \(J \cap J_0 \neq \emptyset\) is equivalent to \(\mathcal{E}_{[0,1] \times J} \cap \mathcal{E}_{[0,1] \times J_0} \neq \emptyset\). With that in mind we define

\[
W_{I \times J}(L_0) = \bigcap_{\mathcal{D}_m \times \{J_0\} \in \mathcal{A}} W_{I \times J}(\mathcal{D}_m \times \{J_0\}, L_0).
\]
The point-set $W_{I \times J}(L_0)$ is the smallest common $x$–support of all previous collections $\mathcal{E}_{I \times J}$, $I_0 \times J_0 \in \mathcal{P}$, which are located in $y$–space around $L_0$. Define $\eta_i$ to be half the size of the smallest $x$–frequency previously used, thus

$$\eta_i = \frac{1}{2} \min\{|K| : \exists L, K \times L \in \bigcup_{I_0 \times J_0 \in \mathcal{P}} \mathcal{E}_{I_0 \times J_0}\}.$$  

Finally let $\mathcal{W}_{I \times J}(L_0)$ denote the high–frequency cover of $W_{I \times J}(L_0)$ given by

$$\mathcal{W}_{I \times J}(L_0) = \{ K \in \mathcal{P} : |K| = \eta_i, K \subset W_{I \times J}(L_0) \}.$$  

For each $L_0 \in \mathcal{W}_{I \times J}(L_0)$ and $K_0 \in \mathcal{W}_{I \times J}(L_0)$ define the collection

$$\mathcal{X}(K_0 \times L_0) = \{ K \times L : K \subseteq K_0, f_{i-1}(K \times L_0) \leq \tau_i[K \times L_0] \},$$

where the local frequency weight $f_{i-1}$ is specified in (4.21). Applying Lemma 4.1 to $\mathcal{X}(K_0 \times L_0)$ yields an integer $k = k(K_0 \times L_0)$ such that

$$1 \leq k(K_0 \times L_0) < \left( \frac{i-1}{\delta_i^2 \tau_i^2} \right) + 1$$

(4.19)

and so that the following condition holds. If we form the collection $\mathcal{Z}_{I \times J}(K_0 \times L_0)$ by

$$\mathcal{Z}_{I \times J}(K_0 \times L_0) = \{ K \times L \in \mathcal{X}(K_0 \times L_0) : |K| = 2^{-k(K_0 \times L_0)}|K_0| \}$$

we have the estimate

$$(1 - \delta_i)|K_0 \times L_0| \leq |\mathcal{Z}_{I \times J}(K_0 \times L_0)| \leq |K_0 \times L_0|. \quad (4.20)$$

Note that $K \times L \in \mathcal{Z}_{I \times J}(K_0 \times L_0)$ implies $L = L_0$. For each $y$–frequency $L_0 \in \mathcal{W}_{I \times J}$ we define the collection $\mathcal{Z}_{I \times J}(L_0)$ of $x$–frequencies by

$$\mathcal{Z}_{I \times J}(L_0) = \{ K : \exists K_0 \in \mathcal{W}_{I \times J}(L_0), K \times L_0 \in \mathcal{Z}_{I \times J}(K_0 \times L_0) \}. \quad (4.21)$$

Note that the $K_0$ in (4.21) is uniquely determined.

Now, we define the building blocks $\mathcal{Z}_{I \times J}$ of the collection $\mathcal{E}_{I \times J}$ as the local product

$$\mathcal{Z}_{I \times J} = \bigcup \{ \mathcal{Z}_{I \times J}(L_0) : L_0 \in \mathcal{W}_{I \times J} \}. \quad (4.22)$$

Observe that the following identity holds:

$$\mathcal{Z}_{I \times J} = \bigcup \{ \mathcal{Z}_{I \times J}(L_0) \times \{ L_0 \} : L_0 \in \mathcal{W}_{I \times J}, K_0 \in \mathcal{W}_{I \times J}(L_0) \}. \quad (4.23)$$

Since $\mathcal{Z}_{I \times J}(K_0 \times L_0) \subset \mathcal{X}(K_0 \times L_0)$, we have the estimate

$$f_{i-1}(K \times L_0) \leq \tau_i|K \times L_0|, \quad \text{for all } K \times L_0 \in \mathcal{Z}_{I \times J}. \quad (4.24)$$

Up to this point, the construction for Case 2.a and Case 2.b are identical. Now is the time to distinguish between the cases $J = [0, 1]$ and $J \neq [0, 1]$.

**Case 2.a:** $I \neq [0, 1], J = [0, 1]$. The light rectangles $I \times J$ are the ones to which Case 2.a is applicable. The collection $\mathcal{E}_{I \times J}$ is defined in (4.24a). The collections $\mathcal{E}_{I_0 \times J_0}$ indexed by the dark rectangles $I_0 \times J_0$ are already well defined. The white ones are ignored.

$$\mathcal{E}_{I \times J} = \{ K \times L_0 \in \mathcal{Z}_{I \times J} : K \times L_0 \subset \mathcal{E}_{I \times J}^*, \text{pos}(K) = \text{pos}(I) \text{ mod } 2 \} \quad (4.24a)$$

**Case 2.b:** $I \neq [0, 1], J \neq [0, 1]$. The figure on the right depicts the transition from Case 2.a to Case 2.b. The light rectangles $I \times J$ are the ones covered by Case 2.b. The collection $\mathcal{E}_{I \times J}$ is defined in (4.24b). The collections $\mathcal{E}_{I_0 \times J_0}$ indexed by the dark rectangles $I_0 \times J_0$ are well defined before the first light rectangle is treated.

$$\mathcal{E}_{I \times J} = \{ K \times L_0 \in \mathcal{Z}_{I \times J} : K \times L_0 \subset \mathcal{E}_{I \times J}^* \} \quad (4.24b)$$
Now we have completed the construction part and we turn to verifying our first crucial measure estimate \( (4.25) \) which asserts that the division by which we produced \( \delta_{I \times J} \) is scale invariant: it acts locally on each rectangle the same as globally. We claim that

\[
\frac{1}{2} \prod_{I_0 \times J_0 \subseteq I \times J} (1 - \delta_{I_0 \times J_0})^2 |K \times L| \leq |(K \times L) \cap \delta_{I \times J}^*| \leq \frac{1}{2} |K \times L|,
\]

(4.25a)

for all \( K \times L \in \delta_{I \times J} \) as well as

\[
\frac{1}{2} \prod_{I_0 \times J_0 \subseteq I \times J} (1 - \delta_{I_0 \times J_0})^2 |K \times L| \leq |(K \times L) \cap \delta_{I \times J}^*| \leq \frac{1}{2} |K \times L|,
\]

(4.25b)

for all \( K \times L \in \delta_{I \times J}^* \), if \( J \neq [0,1] \). Indeed, we only have to verify the left hand side estimates. First, let \( K \times L \in \delta_{I \times J} \). Observe that since \( \mathcal{F}_{I \times J} = \mathcal{F}_{I \times J} \) and \( |\delta_{I \times J}^* \cap ([0,1] \times L)| = \frac{1}{2} |\mathcal{F}_{I \times J}^*(L) \times L| \) for all \( L \in \mathcal{F}_{I \times J} \), we have

\[
|(K \times L) \cap \delta_{I \times J}^*| = \frac{1}{2} |(K \times L) \cap (\mathcal{F}_{I \times J}^*(L) \times L)|.
\]

(4.26)

Obviously, by \( (4.20) \), the right hand side is larger than

\[
\frac{1}{2} |(K \times L) \cap W_{I \times J}(L)| |L|.
\]

(4.27)

We go back over the course by which we have come and see that

\[
|K \cap W_{I \times J}(L)| \geq \prod_{I \times J \supseteq I_0 \times J_0 \subseteq I \times J} (1 - \delta_{I_0 \times J_0}) |K|.
\]

(4.28)

Combining \( (4.26) \), with \( (4.27) \) and \( (4.28) \) yields \( (4.25a) \). Second, let \( K \times L \in \delta_{I \times J}^* \) and \( J \neq [0,1] \). By the definition of \( \delta_{I \times J}^* \) and \( (4.22) \) we have

\[
|(K \times L) \cap \delta_{I \times J}^*| = \sum_{L_0 \in \mathcal{F}_{I \times J}} |(K \times L) \cap (\mathcal{F}_{I \times J}^*(L_0) \times L_0)|.
\]

(4.29)

For each summand note the identity

\[
|(K \times L) \cap (\mathcal{F}_{I \times J}^*(L_0) \times L_0)| = |K \cap \mathcal{F}_{I \times J}^*(L_0)| |L \cap L_0|.
\]

(4.30)

As before, we have

\[
|K \cap \mathcal{F}_{I \times J}^*(L_0)| \geq (1 - \delta_{I \times J}) |K \cap W_{I \times J}(L_0)|
\]

(4.31)

and

\[
|K \cap W_{I \times J}(L_0)| \geq \prod_{I \times J \supseteq I_0 \times J_0 \subseteq I \times J} (1 - \delta_{I_0 \times J_0}) |K|.
\]

(4.32)

Next, we observe that by \( (4.30) \), \( (4.31) \) and \( (4.32) \), the sum in the right hand side of \( (4.29) \) is larger than

\[
\left( \prod_{I \times J \supseteq I_0 \times J_0 \subseteq I \times J} (1 - \delta_{I_0 \times J_0}) \right) |K| \sum_{L_0 \in \mathcal{F}_{I \times J}} |L \cap L_0|.
\]

(4.33)

Taking into account that \( J \subset J \), the Gamlen-Gaudet construction of Case 1 gives

\[
\sum_{L_0 \in \mathcal{F}_{I \times J}} |L \cap L_0| \geq \frac{1}{2} |(1 - \delta_{[0,1] \times J})| L|.
\]

(4.34)

Finally, combining \( (4.33) \) and \( (4.34) \) with \( (4.29) \) yields \( (4.25b) \).

**Essential properties of our construction.**
Here, we exploit our choice of the constants $\delta$ and (4.24) imply such that $K \cap I \subset I$ and $|I_0 \times J_0| = 2|I_1 \times J_1|$ we have

$$\frac{1}{2} \prod_{I \times J \subseteq I_1 \times J_1} (1 - \delta_{I \times J})^2 |K \times L| \leq |(K \times L) \cap \mathcal{E}_{I_1 \times J_1}^+| \leq \frac{1}{2} |K \times L|, \quad (4.35)$$

for all $K \times L \in \mathcal{E}_{I_0 \times J_0}$. Second, (4.11), (4.15) and (4.16) as well as (4.18), (4.23) and (4.24) imply

$$\sum_{j=1}^{i-1} |(T^*b_j, h_{K \times L})| + |(Tb_j, h_{K \times L})| \leq \frac{(i - 1) \tau_j}{|I \times J|} |K \times L|, \quad (4.36)$$

for all $i \in \mathbb{N}$ and $K \times L \in \mathcal{E}_i$. Recall that $\mathcal{E}_i = \mathcal{E}_{I \times J}$ provided $i = O_\omega(I \times J)$.

Bi-tree property.

The collection $\{\mathcal{E}_{I \times J} : I \times J \in \mathcal{D}_n\}$ forms a bi-tree. The bi-tree constant is determined by the local product structure (4.38) verified below. In particular

$$\frac{1}{2} |I \times J| \leq |\mathcal{E}_{I \times J}^+| \leq |I \times J|. \quad (4.37)$$

The local product structure of $\mathcal{E}_{I \times J}$.

Here, we exploit our choice of the constants $\delta_{I \times J}$, see (4.10). We carry over (4.35) to each pair of nested dyadic rectangles. Let $I_0 \times J_0, I_1 \times J_1 \in \mathcal{D}$ such that $I_0 = \pi^i(I_1)$ and $J_0 = \pi^j(J_1)$ for some $i, j \in \mathbb{N}_0$. Then, iterating (4.35) yields

$$\frac{1}{2} |K \times L| \leq 2^{i+j} |K \times L) \cap \mathcal{E}_{I_1 \times J_1}^+| \leq |K \times L|, \quad (4.38)$$

for all $K \times L \in \mathcal{E}_{I_0 \times J_0}$. Our construction with its inherent complications permits us now verify the crucial estimate (4.38). We present only the proof for the lower estimate since the verification of the upper estimate follows the same line of reasoning. Let $I_0 \times J_0$ and $I_1 \times J_1$ be a nested pair of dyadic rectangles as specified above. We now define a path $p(I_0 \times J_0, I_1 \times J_1)$ of nested rectangles $I^{(n)} \times J^{(n)}$ connecting $I_1 \times J_1$ to $I_0 \times J_0$ as follows. We define $I^{(0)} = I_0, J^{(0)} = J_0$ and $J^{(i)} = J_1, J^{(i+1)} = J_0$ as well as

$$f^{(m+1)} = \tilde{f}^{(m)} \quad \text{and} \quad J^{(m+1)} = J^{(m)}, \quad \text{if} \quad 0 \leq m \leq i - 1, \quad (4.35)$$

$$f^{(m+1)} = f^{(m)} \quad \text{and} \quad J^{(m+1)} = \tilde{J}^{(m)}, \quad \text{if} \quad i \leq m \leq i + j - 1.$$

Iterating the local property (4.35) along the path $p = p(I_0 \times J_0, I_1 \times J_1)$ we obtain

$$|K \times L) \cap \mathcal{E}_{I_1 \times J_1}^+| \geq 2^{-(i+j)} \prod_{I \times J \in p} (1 - \alpha_{I \times J}) |K \times L|,$$

where we put

$$1 - \alpha_{I \times J} = \prod_{k \leq \Omega(I \times J)} (1 - \delta_k)^2.$$

Since the length of the path $p$ is at most $2n$, we obtain

$$|(K \times L) \cap \mathcal{E}_{I_1 \times J_1}^+| \geq 2^{-(i+j)} |K \times L| \left(1 - 4n \sum_{k=1}^{\infty} \delta_k\right).$$

As we specified $\delta_k = 2^{-k}/(8n)$ in (4.10) we see that (4.38) holds.
The boundedness of the orthogonal projection $Q$.

The collections of dyadic rectangles $\mathcal{E}_{I \times J}$ give rise to the block basis

$$b_{I \times J} = \sum_{K \times L \in \mathcal{E}_{I \times J}} h_{K \times L}$$

and the orthogonal projection

$$Q(f) = \sum_{I \times J \in \mathcal{R}_n} \langle f, \frac{b_{I \times J}}{\|b_{I \times J}\|_2} \rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2}.$$ 

Feeding the estimate (4.38) into Proposition 2.1 we obtain that

$$\|Q : H^1(\delta^2) \to H^1(\delta^2)\| \leq C_2,$$

for some universal constant $C_2 > 0$.

The basis $\{b_i\}$ are almost eigenvectors for $T$.

Here we exploit our choice of the constants $\tau_i$ defined in (4.10); recall that was

$$\tau_i = \frac{2^{-i}}{4(i-1) \min_{j \leq i} \epsilon_i |O_0^{-1}(i)|}.$$  

(4.39)

We show that we have

$$T b_i = \frac{|\langle T b_i, b_i \rangle|}{\|b_i\|_2^2} b_i + \text{tiny error}.$$ 

(4.40)

We now calculate the size of the error terms. We claim that

$$\sum_{j: j \neq i} |\langle T b_j, b_i \rangle| \leq \epsilon_i \|b_i\|_2^2,$$

for all $i$.  

(4.41)

Summing (4.36) over $K \times L \in \mathcal{E}_i$ we obtain

$$\sum_{j=1}^{i-1} |\langle T^* b_j, b_i \rangle| + |\langle T b_j, b_i \rangle| \leq \frac{(i-1) \tau_i}{\|O_0^{-1}(j)\|_2^2} \|b_i\|_2^2.$$

Thus, in view of (4.37) we have

$$\sum_{j=1}^{i-1} |\langle T^* b_j, b_i \rangle| + |\langle T b_j, b_i \rangle| \leq 2(i-1) \tau_i.$$  

(4.42)

From this estimate we obtain

$$\sum_{j=1}^{i-1} |\langle T b_j, b_i \rangle| \leq 2(i-1) \tau_i.$$  

(4.43)

Replacing $i$ by $j$ in (4.41) it is easy to see that we obtain the following inequalities.

$$|\langle T b_j, b_i \rangle| = |\langle b_j, T^* b_i \rangle| \leq 2(j-1) \tau_j, \quad j \geq i + 1.$$  

(4.44)

Taking the sum in (4.43) and adding (4.42) gives

$$\sum_{j: j \neq i} |\langle T b_j, b_i \rangle| \leq 2 \sum_{j \geq i} (j-1) \tau_j.$$  

(4.45)

Invoking (4.39), (4.44) and using (4.39) we obtain therefore

$$\sum_{j: j \neq i} |\langle T b_j, b_i \rangle| \leq 2^{-i} \min_{k \leq i} \epsilon_k |O_0^{-1}(k)| \leq 2^{-i+1} \epsilon_i \|b_i\|_2^2,$$

which certainly implies the estimate (4.40).
Estimating $N$ in Terms of $n$ and $\{\varepsilon_i\}$.

Here we exploit the quantitative constraints (4.6) in the combinatorial orthogonality lemma (4.1). We give an upper bound for the size of the rectangle collection $\mathcal{R}_N$ which ensures that we can carry out the inductive construction over $n$ levels and precision $\{\varepsilon_i\}$ so that (4.38) and (4.40) hold.

We denote by $m_i$ the highest frequency used in all the building blocks of $b_1, \ldots, b_i$, that is

$$2^{-m_i} = \min\{|K|, |L| : K \times L \in \bigcup_{j \leq i} \mathcal{E}_j\}.$$ 

Recall that the combinatorial Lemma (4.1) provided bounds for the size of the building blocks of $b_i$ in each step, see (4.13) and (4.19). Consequently, we obtain the recursive estimates

$$m_{i+1} \leq m_i + 1 + \frac{i^2}{\delta_{i+1} \tau_{i+1}}, \quad i \geq 1.$$ 

Considering the definitions of $\delta_i$ and $\tau_i$, see (4.10), we get

$$m_i \leq A 2^{8i} / \min_{j \leq i} \varepsilon_j,$$

where $A$ is some absolute constant, which, for convenience, we assume to be 1. Thus, since $N \geq m_{(2^{n+1} - 1)^2}$, we have

$$\mathcal{E}_i \subset \mathcal{R}_N, \quad i \geq 1.$$ 

5. Localization in Bi–parameter BMO

In this section we prove that $\text{BMO}(\delta^2)$ is primary. Since

$$\text{BMO}(\delta^2) \sim \left(\sum_n \text{BMO}_n(\delta^2)\right)_\infty,$$

this results from the Pelczynski decomposition method [19] and the following theorem on the factorization of the identity operator on $\text{BMO}(\delta^2)$.

**Theorem 5.1.** For any operator

$$T : \text{BMO}(\delta^2) \rightarrow \text{BMO}(\delta^2)$$

the identity on $\text{BMO}(\delta^2)$ factors through $H = T$ or $H = \text{Id} - T$, that is

$$\text{BMO}(\delta^2) \xrightarrow{\text{Id}} \text{BMO}(\delta^2) \xrightarrow{E} \text{BMO}(\delta^2) \xrightarrow{P} \text{BMO}(\delta^2) \xrightarrow{H} \text{BMO}(\delta^2) \xrightarrow{H} \text{BMO}(\delta^2),$$

with $\|E\|, \|P\| \leq C$. (5.1)

The structure of the proof given below follows the general localization method introduced by Bourgain [3] to treat factorization in non–separable Banach spaces:

(i) We exploit Wojtaszczyk’s isomorphism asserting that

$$\text{BMO}(\delta^2) \sim \left(\sum_n \text{BMO}_n(\delta^2)\right)_\infty.$$ 

(ii) We reduce the factorization problem to the case where the operator $T$ is a diagonal operator on $\left(\sum_n \text{BMO}_n(\delta^2)\right)_\infty$.

(iii) We invoke our finite dimensional factorization Theorem (3.1) to see that in fact Theorem 5.1 holds true for diagonal operators.
We say an operator \( D : (\sum_n \text{BMO}_n(\delta^2))^\infty \to (\sum_n \text{BMO}_n(\delta^2))^\infty \) is a diagonal operator if there exists a sequence of operators \( A_n : \text{BMO}_n(\delta^2) \to \text{BMO}_n(\delta^2) \) such that

\[
D(f_1, f_2, \ldots, f_n, \ldots) = (A_1 f_1, A_2 f_2, \ldots, A_n f_n, \ldots) .
\]

The following theorem provides the reduction to diagonal operators.

**Theorem 5.2.** For any linear operator \( T \) on \((\sum_n \text{BMO}_n(\delta^2))^\infty \) there exists a diagonal operator

\[
D : (\sum_n \text{BMO}_n(\delta^2))^\infty \to (\sum_n \text{BMO}_n(\delta^2))^\infty
\]

such that

\[
D = RTE \quad \text{and} \quad \text{Id} - D = R(\text{Id} - T)E,
\]

for some bounded linear operators \( R, E \) on \((\sum_n \text{BMO}_n(\delta^2))^\infty \).

**Proof.** The proof of Theorem 5.2 is quantitative and finite dimensional in nature. The estimates are provided by Theorem 5.3. In all other aspects, the reduction procedure is analogous to the one–parameter case in \([17, 292–294]\).

We now turn to the main analytic ingredient in the proof of Theorem 5.2.

**Theorem 5.3.** Let \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), then there exists an \( N = N(n, \varepsilon) \) so that the following holds. For any \( n \)-dimensional subspace \( F \subset \text{BMO}_N(\delta^2) \), there exists a block–basis \( \{b_{1 \times j}\} \in \text{BMO}_N(\delta^2) \) satisfying the following conditions.

(i) The map \( S : \text{BMO}_n(\delta^2) \to \text{BMO}_N(\delta^2) \) defined as the linear extension of \( h_{1 \times j} \mapsto b_{1 \times j} \) satisfies

\[
\frac{1}{C} \|f\| \leq \|Sf\| \leq C\|f\|,
\]

with universal constant \( C \).

(ii) The orthogonal projection \( Q : \text{BMO}_N(\delta^2) \to \text{BMO}_N(\delta^2) \), given by

\[
Q(f) = \sum_{I \times J \in \mathbb{S}_n} \langle f, b_{I \times J} \rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2} \frac{b_{I \times J}}{\|b_{I \times J}\|_2}
\]

is bounded by

\[
\|Qf\|_{\text{BMO}_N(\delta^2)} \leq C\|f\|_{\text{BMO}_N(\delta^2)}, \quad f \in \text{BMO}_N(\delta^2),
\]

for some universal constant \( C \) and almost annihilates the space \( F \),

\[
\|Qf\|_{\text{BMO}_N(\delta^2)} \leq \varepsilon\|f\|_{\text{BMO}_N(\delta^2)} \quad f \in F. \tag{5.2}
\]

**Proof.** The proof of Theorem 5.3 is a repetition of the quasi–diagonalization argument in the proof of Theorem 5.2, where condition (5.2) is simpler to realize than (3.2). The situation is once more analogous to the one–parameter case treated in \([17, 290–291]\).

**Theorem 5.4.** Let \( D \) be a diagonal operator on \((\sum_n \text{BMO}_n(\delta^2))^\infty \). Then the identity factors through \( H \) or \( H = \text{Id} - D \), that is

\[
\begin{array}{c}
(\sum_n \text{BMO}_n(\delta^2))^\infty \xrightarrow{E} (\sum_n \text{BMO}_n(\delta^2))^\infty \\
\xrightarrow{P} \|E\|\|P\| \leq C\|D\|.
\end{array}
\]
Proof. Let $A_n : \text{BMO}_n(\delta^2) \to \text{BMO}_n(\delta^2)$ be the linear map defining the diagonal operator $D$, that is
$$D(f_1, f_2, \ldots, f_n, \ldots) = (A_1f_1, A_2f_2, \ldots, A_nf_n, \ldots).$$
By Theorem 5.1, the identity on $\text{BMO}_n(\delta^2)$ factors through $H_n = A_{N(n)}$ or $H_n = \text{Id} - A_{N(n)}$, that is
$$\text{BMO}_n(\delta^2) \xrightarrow{\text{Id}} \text{BMO}_n(\delta^2)$$
$$\xrightarrow{E_n} \text{BMO}_n(\delta^2) \xrightarrow{p_n} \text{BMO}_n(\delta^2) , 
\|E_n\|\|P_n\| \leq C\|D\|.$$ 
If there exists an infinite sequence $\{k(n)\}$ so that $H_{k(n)} = A_{N(k(n))}$, then the identity on $(\sum_n \text{BMO}_n(\delta^2))_\infty$ factors through $D$. If $H_{k(n)} = \text{Id} - A_{N(k(n))}$, then the identity factors through $\text{Id} - D$.

Proof of Theorem 5.7. By Wojtaszczyk’s isomorphism, see [21], the Banach space $\text{BMO}((\delta^2))$ is isomorphic to the infinite sum of its finite dimensional building blocks $(\sum_n \text{BMO}_n(\delta^2))_\infty$. Hence, in Theorem 5.1 we replace operators on $\text{BMO}(\delta^2)$ by operators on $(\sum_n \text{BMO}_n(\delta^2))_\infty$. Moreover, by Theorem 5.2 it suffices to consider only diagonal operators on $(\sum_n \text{BMO}_n(\delta^2))_\infty$. In Theorem 5.4 we proved that for any diagonal operator $D$ on $(\sum_n \text{BMO}_n(\delta^2))_\infty$ the identity factors through $H = D$ or $H = \text{Id} - D$, that is
$$\xrightarrow{\text{Id}} \xrightarrow{E} \xrightarrow{p} \xrightarrow{H} \xrightarrow{\text{Id}} , 
\|E\|\|P\| \leq C\|D\|.$$ 

\[\square\]

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