CANONICAL INTEGRAL OPERATORS ON THE FOCK SPACE

XINGTANG DONG AND KEHE ZHU

ABSTRACT. In this paper we introduce and study a two-parameter family of integral operators on the Fock space $F^2(\mathbb{C})$. We determine exactly when these operators are bounded and when they are unitary. We show that, under the Bargmann transform, these operators include the classical linear canonical transforms as special cases. As an application, we obtain a new unitary projective representation for the special linear group $SL(2, \mathbb{R})$ on the Fock space.

1. INTRODUCTION

Let $\mathbb{R}$ denote the real line and $\mathbb{C}$ denote the complex plane. We are going to study operator theory on three particular Hilbert spaces: the Lebesgue space $L^2(\mathbb{R}, dx)$, the weighted Lebesgue space $L^2(\mathbb{C}, d\lambda)$, and the Fock space

$$F^2 = L^2(\mathbb{C}, d\lambda) \cap H(\mathbb{C}),$$

where $H(\mathbb{C})$ is the space of all entire functions on $\mathbb{C}$ and

$$d\lambda(z) = \frac{1}{\pi} e^{-|z|^2} dA(z)$$

is the Gaussian measure on $\mathbb{C}$. Here $dA$ is ordinary area measure. The inner product on $F^2$ is inherited from $L^2(\mathbb{C}, d\lambda)$.

The Fock space has its origin in mathematical physics. The mathematical theory of Fock spaces has also experienced a rapid development over the past few decades. In particular, starting from the 1980s, Berger and Coburn systematically studied Toeplitz and Hankel operators on the Fock space; see [5, 6, 7] for example. On the other hand, motivated by problems from engineering, Seip and Wallsten completely characterized the so-called interpolating and sampling sequences for Fock spaces in [15, 16]. See [19] for a recent survey of the mathematical theory of Fock spaces.

Although all separable infinite-dimensional Hilbert spaces are isomorphic, there is a particular unitary transformation $B$ from the Lebesgue space $L^2(\mathbb{R}, dx)$ onto the Fock space $F^2$ that is very natural and extremely useful. This is the Bargmann transform, which enables us to translate operators on $L^2(\mathbb{R}, dx)$ to operators on $F^2$, and vice versa. A prominent example is the Fourier transform $\mathcal{F}$ as a bounded

2010 Mathematics Subject Classification. Primary 30H20; Secondary 47G10.

Key words and phrases. Fock space, linear canonical transforms, Bargmann transform, general linear group, special linear group.

Research of Dong was supported in part by the Natural Science Foundation of Tianjin City of China (Grant No. 19JCQNJC14700). Research of Zhu was supported by NNSF of China (Grant No. 11720101003).
operator on $L^2(\mathbb{R}, dx)$, which, under the Bargmann transform, becomes the operator $f(z) \mapsto f(i z)$ on $F^2$. It is clear that the latter form of the Fourier transform on $F^2$ has a much better structure (at least on surface) than the original form on $L^2(\mathbb{R}, dx)$. This example illustrates the enormous potential of the Bargmann transform in operator theory.

Another important example is the Hilbert transform $H$ as a bounded operator on $L^2(\mathbb{R}, dx)$. Under the Bargmann transform, it becomes the following integral operator on $F^2$:

$$Hf(z) = \int_{\mathbb{C}} e^{z \overline{w}} \varphi(z - \overline{w}) f(w) d\lambda(w),$$

where $\varphi$ is a particular function in $F^2$ which can be written down explicitly. Motivated by this example, the second author here raised the following question in [20]: characterize all functions $\varphi \in F^2$ such that the integral operator above is bounded on $F^2$. A beautiful answer to this question was given in [8], and more related work was done in [17]. See [21] for many other examples of operators on $L^2(\mathbb{R}, dx)$ and their counterparts, under the Bargmann transform, on $F^2$.

In this paper, we introduce and study a two-parameter family of integral operators on $F^2$. More specifically, for any $(s, t) \in \mathbb{C}^2$ with $s \neq 0$, we define

$$T^{(s,t)} f(z) = \int_{\mathbb{C}} K^{(s,t)}(z, w) f(w) d\lambda(w),$$

where

$$K^{(s,t)}(z, w) = \frac{1}{\sqrt{s}} \exp \left\{ \frac{t z^2 - \bar{t} w^2 + 2 z \overline{w}}{2s} \right\}.$$

Here and throughout the paper, the complex square root is defined as follows:

$$\sqrt{z} = \sqrt{|z|} e^{i\theta/2}, \quad z = |z| e^{i\theta}, \quad \theta \in (-\pi, \pi].$$

It is clear that $T^{(s,t)} f$, whenever well-defined, is an entire function. Therefore, when $T^{(s,t)}$ is a bounded operator on $L^2(\mathbb{C}, d\lambda)$, it will also be a bounded operator on $F^2$. This simple observation will be used many times later without being explicitly mentioned again.

If $f$ is continuous on $\mathbb{C}$ with compact support, it is easy to see that $T^{(s,t)} f$ is a well-defined entire function for any $s \neq 0$. Therefore, $T^{(s,t)} : L^2(\mathbb{C}, d\lambda) \to H(\mathbb{C})$ is densely defined for any $s \neq 0$. However, when $|t| \geq 2|s|$, it is not clear if there is any nonzero function $f \in F^2$ such that the integral defining $T^{(s,t)} f$ is convergent. In particular, for $|t| \geq 2|s|$, the integral for $T^{(s,t)} f$ is divergent for all nonzero polynomials $f$ and for all finite linear combinations of (ordinary) kernel functions. Therefore, when considering $T^{(s,t)}$ as operators on $F^2$, we will make the natural assumption that $|t| < 2|s|$.

Our main results are Theorems A, B, C, D, and E below.

**Theorem A.** The operators $T^{(s,t)}$ on $F^2$, where $|t| < 2|s|$, have the following properties.

(i) $T^{(s,t)}$ is bounded if and only if $|s|^2 \geq |t|^2 + 1$.

(ii) $T^{(s,t)}$ is unitary if and only if $|s|^2 = |t|^2 + 1$. 
When \( |s|^2 > |t|^2 + 1 \), the operator \( T(s,t) \) belongs to the Hilbert-Schmidt class \( S_2 \) with

\[
\|T(s,t)\|^2_{S_2} = \frac{|s|}{\sqrt{|s|^2 - |t|^2 - 1}}.
\]

Our next result concerns the general and special real linear groups of order 2. Let us write

\[
GL(\mathbb{C} \times \mathbb{C}) = \{(s, t) \in \mathbb{C}^2 : |s| \neq |t|\}
\]
and

\[
SL(\mathbb{C} \times \mathbb{C}) = \{(s, t) \in \mathbb{C}^2 : |s|^2 = |t|^2 + 1\}.
\]

We show that these are the complex versions of the general and special real linear groups, respectively. We also show that the special linear group of order 2 can be represented as unitary operators on the Fock space.

**Theorem B.** Let \( GL(2, \mathbb{R}) \) denote the general real linear group and \( SL(2, \mathbb{R}) \) the special real linear group of order 2.

(i) The set \( GL(\mathbb{C} \times \mathbb{C}) \) is a group with the operation

\[
(s_1, t_1) \cdot (s_2, t_2) = (s_1 s_2 + \overline{t_1} t_2, t_1 s_2 + \overline{s_1} t_2).
\]

Moreover, the set \( SL(\mathbb{C} \times \mathbb{C}) \) is a subgroup of \( GL(\mathbb{C} \times \mathbb{C}) \).

(ii) The mapping \( \varphi : GL(2, \mathbb{R}) \to GL(\mathbb{C} \times \mathbb{C}) \) defined by

\[
\varphi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left( \frac{a + ib + d - ic}{2}, \frac{a + ib - d + ic}{2} \right)
\]

is a group isomorphism. Moreover, \( \varphi \) maps \( SL(2, \mathbb{R}) \) onto \( SL(\mathbb{C} \times \mathbb{C}) \).

(iii) The mapping \( (s, t) \mapsto T(s,t) \) is a unitary projective representation of the group \( SL(\mathbb{C} \times \mathbb{C}) \) on \( F^2 \).

Recall from mathematical physics that, for every matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) in \( SL(2, \mathbb{R}) \), the linear canonical transform

\[
\mathcal{F}^A : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)
\]

is defined by

\[
\mathcal{F}^A f(x) = \frac{1}{\sqrt{1 + b}} e^{i dx^2 / b} \int_{\mathbb{R}} e^{-i(2xt - at^2) / b} f(t) dt
\]
when \( b \neq 0 \). If \( b = 0 \), then \( \mathcal{F}^A \) is defined by

\[
\mathcal{F}^A f(x) = \sqrt{d} e^{i dx^2} f(dx).
\]

Note that \( d \) here is a real number but not the differential. Also, the parameter \( c \) does not appear in the definition of \( \mathcal{F}^A f \) for \( b \neq 0 \) above because the four parameters are subject to the condition \( ad - bc = 1 \), so \( c = (ad - 1) / b \) in this case.

The linear canonical transforms are widely used in science, engineering, and mathematical physics. Special forms of them include fractional Fourier transforms, scaling (or dilation), the Fresnel transform, and chirp multiplication. See [4][11][13][18] for more information about linear canonical transforms.
Theorem C. Under the Bargmann transform, the linear canonical transforms become the operators $T^{(s,t)}$ on $F^2$, where $(s, t) \in SL(\mathbb{C} \times \mathbb{C})$.

Recall that the Fourier transform $F : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$ becomes the operator $F^2 \to F^2$, where $F(f)(z) = f(iz)$. Using this form and with the help of the canonical monomial orthonormal basis for $F^2$, it is easy to show that the spectrum of the unitary operator $F$ on $L^2(\mathbb{R}, dx)$ or on $F^2$ consists of four points on the unit circle: $i^k$, $k = 0, 1, 2, 3$. Each point is an eigenvalue and the corresponding eigenspace can be described in terms of the Hermite polynomials in the case of $L^2(\mathbb{R}, dx)$ and in terms of Taylor coefficients in the case of $F^2$. See [21] again. The following result shows that much of this can be generalized to linear canonical transforms.

Theorem D. Let $(s, t) \in SL(\mathbb{C} \times \mathbb{C})$ with $|\text{Re } s| < 1$. Then there exists a number $\gamma$ in the open unit disk such that for each $n = 0, 1, 2, \ldots$ the number

$$\lambda_n = \frac{1}{\sqrt{s}} \sqrt{\frac{1}{s + \overline{t}\gamma} \frac{1}{(s + \overline{t}\gamma)^n}}$$

is an eigenvalue of $T^{(s,t)}$, and a corresponding eigenvector is the function

$$f_n(z) = e^{\gamma z^2/2} \int_{\mathbb{R}} H_n \left( \frac{x}{\rho} \right) \exp \left[ -\frac{2}{1 + \gamma} \left( x - \frac{1 + \gamma}{2} z \right)^2 \right] dx,$$

where $H_n$ is the classical Hermite polynomial of degree $n$ and $\rho$ is a certain positive number.

In order to prove Theorem D, we will first obtain some properties of Hermite polynomials, which appear to be new (to the best of our knowledge). In particular, we find out that each Hermite polynomial is a solution of a certain integral equation, which enables us to obtain the classical closed-form formula for Hermite polynomials. More specifically, we have the following.

Theorem E. Let $\mu, \nu, a, b \in \mathbb{C}$ with $a \neq 0$, $b \neq 0$, $\text{Re } \mu > 0$, $\text{Re } \nu > 0$, and $\nu b^2 \neq \mu a^2$. Suppose

$$P_n(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n, \quad c_0 \neq 0.$$

If $a^k \neq b^k$ for $1 \leq k \leq n$, then the following conditions are equivalent:

(a) $P_n(x)$ is a constant multiple of the Hermite polynomial $H_n(x)$.

(b) $P_n(x)$ is a solution of the integral equation

$$\int_{\mathbb{R}} P_n(x) e^{-\mu(x-az)^2} dx = C_n \int_{\mathbb{R}} P_n(x) e^{-\nu(x-bz)^2} dx,$$

where

$$C_n = \frac{\sqrt{\nu}}{\sqrt{\mu}} \left( \frac{a}{b} \right)^n, \quad \frac{(b^2 - a^2)\mu \nu}{\nu b^2 - \mu a^2} = 1. \quad (1)$$

(c) $c_{2k+1} = 0$ whenever $2k + 1 \leq n$, and

$$(n - 2k)! c_{2k} = -2(2k + 2)(n - 2k - 2)! c_{2k+2}$$
Lemma 2. For $k = 1, 2$ let $(s_k, t_k) \in \mathbb{C}^2$ with $|s_k| > |t_k|$. Then
\[
\int_{\mathbb{C}} K^{(s_1, t_1)}(z, w) K^{(s_2, t_2)}(w, u) d\lambda(w) =
\]
\[
cK^{(s, t)}(z, u) \exp \left[ \frac{t_2(|t_2|^2 + 1 - |s_1|^2) z^2}{2s_2(s_1 s_2 + t_1 t_2)} - \frac{t_1(|t_2|^2 + 1 - |s_2|^2)}{2s_2(s_1 s_2 + t_1 t_2)} \right],
\]
where $(s, t) = (s_1 s_2 + \bar{t}_1 t_2, t_1 s_2 + \bar{s}_1 t_2), |s| > |t|$, and
\[
c = \sqrt{\frac{s_1 s_2 + t_1 t_2}{\sqrt{s_1} \sqrt{s_2}}} \sqrt{\frac{s_1 s_2}{s_1 s_2 + t_1 t_2}} = \pm 1.
\]
\textbf{Proof.} It follows from Lemma 1 and Hölder's inequality that the integral in question exists for all \( z \) and \( u \) in \( \mathbb{C} \). The desired integral formula then follows from the elementary identity (see (1.18) of [2] for example),

\[
\int_{\mathbb{C}} \exp \left[ \frac{\gamma}{2} w^2 + aw + \frac{\delta}{2} \overline{w}^2 + \overline{b} \overline{w} \right] d\lambda(w) = \frac{1}{\sqrt{1 - \gamma \delta}} \exp \left[ \frac{\delta a^2 + \gamma b^2 + 2a\overline{b}}{2(1 - \gamma \delta)} \right]
\]

under the assumption \(| \gamma + \delta |^2 < 4 \), which implies that

\[
\text{Re}(1 - \gamma \overline{\delta}) = 1 - \frac{1}{4} | \gamma + \delta |^2 + | \gamma - \delta |^2 > 0.
\]

It is easy to check that

\[
|s|^2 - |t|^2 = (|s_1|^2 - |t_1|^2)(|s_2|^2 - |t_2|^2) > 0.
\]

This proves the lemma. \( \Box \)

\textbf{Corollary 3.} If \((s, t) \in \mathbb{C}^2 \) with \(|s| > |t|\), then

\[
\left\| K_w^{(s, t)} \right\| = \frac{1}{\sqrt{|s|^2 - |t|^2}} \exp \left[ \frac{|w|^2}{2(|s|^2 - |t|^2)} \right] \exp \left[ \frac{t(|t|^2 + 1 - |s|^2)}{2\pi(|s|^2 - |t|^2)} w^2 \right].
\]

\textbf{Proof.} This is a direct consequence of Lemmas 1 and 2. We leave the elementary details to the interested reader. \( \Box \)

We now begin to study the two-parameter family of operators \( T^{(s, t)} \). First note that for \(|s| > |t|\) the function \( T^{(s, t)} f \) is well-defined and entire for every \( f \in L^2(\mathbb{C}, d\lambda) \). In fact, by Hölder's inequality and part (b) of Lemma 1 we have

\[
\left| T^{(s, t)} f(z) \right| \leq \frac{1}{\sqrt{|s|}} \| f\| \left[ \int_{\mathbb{C}} \left| e^{\frac{1}{2}z^2 - \frac{\gamma}{2\pi}z^2 + \frac{a\overline{w}}{2\pi}} \right|^2 d\lambda(w) \right]^{\frac{1}{2}} = \| f\| \| K^{(s, t)} \| < \infty.
\]

For any \( w \in \mathbb{C} \) let us write

\[
K_w(z) = K(z, w) = K^{(1, 0)}(z, w) = e^{z\overline{w}}, \quad z \in \mathbb{C},
\]

which is the well known reproducing kernel of \( F^2 \) at \( w \).

\textbf{Lemma 4.} Let \((s, t) \in \mathbb{C}^2 \) with \(|s| > |t|\) and \( u \in \mathbb{C} \). Then the operator \( T^{(s, t)} \) has the following properties.

(a) \( T^{(s, t)} K_u = K_u^{(s, t)} \).

(b) If \(|s|^2 = |t|^2 + 1\), then \( T^{(s, t)} K_u^{(s, t)} = cK_u \), where \( c = \pm 1 \) as in Lemma 1.

\textbf{Proof.} It follows from Lemma 2 that

\[
T^{(s, t)} K_u(z) = \int_{\mathbb{C}} K^{(s, t)}(z, w) K^{(1, 0)}(w, u) d\lambda(w) = K_u^{(s, t)}(z).
\]
If $|s|^2 = |t|^2 + 1$, Lemma 2 again gives
\[
T^{(s,t)}K_u^{(s,-t)}(z) = \int_C K^{(s,t)}(z,w)K^{(s,-t)}(w,u)\,d\lambda(w) = cK_u^{(1,0)}(z),
\]
where $c = |s|/(\sqrt{s}\sqrt{s})$. This proves the desired result. \qed

We can now prove the first main result of the paper, Theorem A, which is stated in a slightly different way as follows.

**Theorem 5.** The integral operators $T^{(s,t)}$ on $F^2$, where $|t| < 2|s|$, have the following properties.

(a) If $|s|^2 - |t|^2 < 1$, then $T^{(s,t)}$ is unbounded on $F^2$.
(b) If $|s|^2 - |t|^2 > 1$, then $T^{(s,t)}$ is not only bounded on $F^2$ but also in the Hilbert-Schmidt class $S_2$ with
\[
\|T^{(s,t)}\|_{S_2} = \frac{\sqrt{|s|}}{|s|^2 - |t|^2 - 1},
\]
(c) If $|s|^2 - |t|^2 = 1$, then $T^{(s,t)}$ is a unitary operator on $F^2$ with
\[
[T^{(s,t)}]^{-1} = [T^{(s,t)}]^* = cT^{(s,-t)},
\]
where $c = \pm 1$ as in Lemma 7.

**Proof.** If $|s| \leq |t| < 2|s|$, then for the constant function $f = 1$ in $F^2$ we have
\[
T^{(s,t)}f(z) = \sqrt{s} e^{t \bar{z}^2/(2s)},
\]
which does not belong to $F^2$, because it is a nonzero function of order 2 and type greater than or equal to 1/2. See Theorem 2.12 of [19]. This shows that $T^{(s,t)}$ is unbounded on $F^2$ when $|s| \leq |t| < 2|s|$.

Next we assume $0 < |s|^2 - |t|^2 < 1$. By Lemma 4 and Corollary 3, we have
\[
\|T^{(s,t)}K_u\| = \|K_u^{(s,t)}\| = \frac{1}{\sqrt{|s|^2 - |t|^2}} \exp \left[ \frac{|u|^2}{2(|s|^2 - |t|^2)} \right] \exp \left[ \frac{t(|t|^2 + 1 - |s|^2)}{2\pi(|s|^2 - |t|^2)}u^2 \right].
\]
If $T^{(s,t)}$ is bounded on $F^2$, then there would exist a constant $C > 0$ such that
\[
\|T^{(s,t)}K_u\| \leq C\|K_u\| = Ce^{u^2/2}, \quad u \in \mathbb{C}.
\]
With the change of variables $u = \sqrt{2(|s|^2 - |t|^2)}/z$, this would give
\[
|e^{-t^2/2}z| \leq C \sqrt{|s|^2 - |t|^2} e^{-|z|^2}, \quad z \in \mathbb{C},
\]
which is clearly impossible. This together with the conclusion in the previous paragraph proves (a).

If $|s|^2 - |t|^2 > 1$, it follows from Corollary 3 that
\[
\int_C \int_C |K^{(s,t)}(z,w)|^2 \,d\lambda(z) \,d\lambda(w)
\]
variables and using (2) again, we get
\[
T \left( \frac{t}{|t|} \right) = \int_{C} \left( \frac{t}{|t|} \right) d\lambda(w).
\]

With the change of variables \( u = \sqrt{|s^2 - |t|^2 - 1} - w \) and the help of (2), we obtain
\[
\int_{C} \int_{C} |K(s, u)(z, w)|^2 d\lambda(z) d\lambda(w) = \frac{\sqrt{|s^2 - |t|^2}}{|s^2 - |t|^2 - 1}} \int_{C} \| e^{-tu^2/(2\pi)} \|^2 d\lambda(u) \leq \frac{|s|}{|s^2 - |t|^2 - 1|} < \infty.
\]

Therefore, \( T(s, t) \) belongs to the Hilbert-Schmidt class \( S_2 \) and part (b) is proved.

To prove (c) we assume \( |s|^2 - |t|^2 = 1 \). We will actually prove the boundedness of \( T(s, t) \) on \( L^2(C, d\lambda) \). We do this with the help of Schur’s test; see [19] Lemma 2.14.

The boundedness of \( T(s, t) \) on \( L^2(C, d\lambda) \) will follow if we can show that the following operator, whose kernel is positive (which is necessary for Schur’s test), is bounded on \( L^2(C, d\lambda) \):
\[
Q(s, t) f(z) = \int_{C} |K(s, t)(z, w)| f(w) d\lambda(w).
\]

To this end, we consider the positive function \( h(z) = e^{(|z|^2/4)} \). Making a change of variables and using (2) again, we get
\[
\int_{C} |K(s, t)(z, w)| h(w)^2 d\lambda(w) = \frac{1}{\sqrt{|s|}} \int_{C} \left| \exp \left[ \frac{t}{4s} z^2 - \frac{7}{4s} \bar{w}^2 + \frac{\bar{w}}{2s} \right] \right|^2 e^{\frac{|w|^2}{2}} d\lambda(w) = 2 \sqrt{|s|} \exp \left( \frac{t}{4s} z^2 \right)^2 \exp \left[ -\frac{7}{4s} \bar{w}^2 - \frac{t}{4s} z^2 + \frac{|z|^2}{2} \right] = 2 \sqrt{|s|} h(z)^2.
\]

Similarly,
\[
\int_{C} |K(s, t)(z, w)| h(z)^2 d\lambda(z) = \frac{1}{\sqrt{|s|}} \int_{C} \left| \exp \left[ \frac{t}{4s} z^2 - \frac{7}{4s} \bar{w}^2 + \frac{\bar{w}}{2s} \right] \right|^2 e^{\frac{|w|^2}{2}} d\lambda(z) = 2 \sqrt{|s|} \exp \left( -\frac{7}{4s} \bar{w}^2 \right)^2 \int_{C} \left| \exp \left[ \frac{t}{2s} u^2 + \frac{\bar{w} u}{\sqrt{2s}} \right] \right|^2 d\lambda(u) = 2 \sqrt{|s|} h(w)^2.
\]
Therefore, Schur’s test tells us that $Q(s,t)$ is bounded on $L^2(\mathbb{C}, d\lambda)$ with its norm not exceeding $2\sqrt{|s|}$.

We next focus on the case of $F^2$. It follows from Lemma 3 that

$$T(s,t) cT(\overline{s},-t) K_u = cT(s,t) K_{u}^{T(\overline{s},-t)} = K_u$$

and

$$cT(\overline{s},-t) T(s,t) K_u = cT(\overline{s},-t) K_{u}^{T(s,t)} = K_u.$$ 

Since the set of finite linear combinations of kernel functions is dense in $F^2$, we conclude that

$$T(s,t) cT(\overline{s},-t) = cT(\overline{s},-t) T(s,t) = I.$$ 

By Lemma 1, the adjoint operator of $T(s,t)$ is given by

$$\left[T(s,t)^*\right] f (z) = \int_{\mathbb{C}} K(s,t)(w,z)f(w) d\lambda(w) = cT(\overline{s},-t)f(z).$$

Thus $T(s,t)$ is unitary on $F^2$. This completes the proof of the theorem. □

3. Eigenvalues and eigenvectors of $T(s,t)$

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disk in the complex plane and let $T = \partial \mathbb{D}$ be the unit circle. For any fixed $(s, t) \in \mathbb{C}^2$ with $|s|^2 = 1 + |t|^2$, the operator $T(s,t)$ is unitary, so its spectrum $\sigma(T(s,t))$ is contained in $T$. When $|\text{Re } s| < 1$, we will find a sequence of eigenvalues for $T(s,t)$ together with certain corresponding eigenfunctions. In some special cases, this will imply that $\sigma(T(s,t)) = T$.

We first observe that for $f(z) = e^{\gamma z^2/2}$, where $\gamma \in \mathbb{D}$, we have

$$T(s,t) f (z) = \frac{1}{\sqrt{s}} \int_{\mathbb{C}} \exp \left[ \frac{\gamma w^2}{2s} + \frac{t}{2s} z^2 - \frac{7}{2s} \overline{w}^2 + \frac{z\overline{w}}{s} \right] d\lambda(w)$$

$$= \frac{1}{\sqrt{s}} \sqrt{\frac{s}{s + \gamma t}} \exp \left[ \frac{t}{2s} z^2 + \frac{\gamma z^2}{2s(s + \gamma t)} \right].$$

Therefore, if $\gamma$ is a solution of the equation

$$st\gamma^2 + (s^2 - 1 - |t|^2) \gamma - st = 0,$$  

then the unitary operator $T(s,t)$ on $F^2$ has an eigenvalue

$$\lambda_0 = \frac{1}{\sqrt{s}} \sqrt{\frac{s}{s + \gamma t}}$$

with the function $e^{\gamma z^2/2}$ as a corresponding eigenvector.

**Lemma 6.** Let $(s, t) \in \mathbb{C}^2$ with $|s|^2 = 1 + |t|^2$. Then the equation (3) has a unique solution $\gamma$ in $\mathbb{D}$ if and only if $|\text{Re } s| < 1$.

**Proof:** If $t = 0$, then $|s| = 1$ and the equation (3) becomes $(s^2 - 1) \gamma = 0$, which has a unique solution $\gamma = 0 \in \mathbb{D}$ if and only if $s^2 - 1 \neq 0$. It is clear that, under the assumption $|s| = 1$, the condition $s^2 \neq 1$ is equivalent to $|\text{Re } s| < 1$. 

CFK!
If \( t \neq 0 \), then by the quadratic formula, the solutions of (3) are given by

\[
\gamma = \frac{-(s - \overline{s}) \pm \sqrt{(s + \overline{s})^2 - 4}}{2t}.
\]

Writing \( s = x + iy \), we have

\[
\gamma = \frac{-iy \pm \sqrt{x^2 - 1}}{t}, \quad |\gamma|^2 = \frac{|-iy \pm \sqrt{x^2 - 1}|^2}{|t|^2},
\]

and

\[
s + \gamma \overline{t} = x \pm \sqrt{x^2 - 1}.
\]

If \(|\text{Re } s| \geq 1\), then both solutions of (3) satisfy

\[
|\gamma|^2 = \frac{|s|^2 - 1}{|t|^2} = 1,
\]

so (3) does not have a solution in the unit disk. We now consider the case when \(|\text{Re } s| < 1\). Since \(|s|^2 = x^2 + y^2 > 1\), it is easy to check that the solution

\[
\gamma_1 = \frac{-y + \text{sgn}(y) \sqrt{1 - x^2}}{t} i
\]

satisfies

\[
|\gamma_1|^2 = \frac{y^2 + 1 - x^2 - 2|y| \sqrt{1 - x^2}}{x^2 + y^2 - 1} < 1.
\]

Also,

\[
s + \gamma_1 \overline{t} = x + i \text{sgn}(y) \sqrt{1 - x^2}, \quad |s + \gamma_1 \overline{t}| = 1.
\]

It is also easy to check that the other solution

\[
\gamma_2 = \frac{-y - \text{sgn}(y) \sqrt{1 - x^2}}{t} i
\]

satisfies \(|\gamma_2| \geq 1\). This completes the proof of the lemma. \(\Box\)

When \(|s|^2 = 1 + |t|^2\) with \(|\text{Re } s| < 1\), we will modify \(\lambda_0\) above to obtain additional eigenvalues for the unitary operator \(T(s,t)\) on \(F^2\). We will also obtain a corresponding eigenvector for each such eigenvalue. To this end, we will need a new characterization of the classical Hermite polynomials.

**Lemma 7.** Let \(\mu \in \mathbb{C}\) with \(\text{Re } \mu > 0\) and let \(n\) be a nonnegative integer. Then

\[
\int_{\mathbb{R}} x^n e^{-\mu(x+z)^2} dx = c_0 z^n + c_1 z^{n-1} + \cdots + c_n
\]

for all \(z \in \mathbb{C}\), where \(c_k = 0\) when \(k\) is odd and

\[
c_k = (-1)^{n-k} \frac{n! \Gamma \left( \frac{k+1}{2} \right)}{k! (n-k)! (\sqrt{\mu})^{k+1}}
\]

when \(k\) is even.
Proof. Consider the integral
\[ I(z) = \int_{\mathbb{R}} x^n e^{-u(x+z)^2} \, dx, \quad z \in \mathbb{C}. \]
Since \( \text{Re} \mu > 0 \), it is clear that \( I(z) \) is an entire function. Moreover, differentiating under the integral sign, we obtain
\[
I'(z) = \int_{\mathbb{R}} x^n \frac{\partial}{\partial z} e^{-u(x+z)^2} \, dx
= \int_{\mathbb{R}} x^n \frac{\partial}{\partial x} e^{-u(x+z)^2} \, dx
= -n \int_{\mathbb{R}} x^{n-1} e^{-u(x+z)^2} \, dx.
\]
Repeat this process, we will then get
\[
I^{(j)}(z) = n(n-1) \cdots (n-j+1)(-1)^j \int_{\mathbb{R}} x^{n-j} e^{-u(x+z)^2} \, dx
\]
for \( j = 2, 3, \cdots, n \). Combining this with Lemma 2 of [9], we obtain
\[
I^{(n)}(z) = (-1)^n n! \int_{\mathbb{R}} e^{-u(x+z)^2} \, dx = (-1)^n n! \frac{\sqrt{\pi}}{\sqrt{u}}
\]
for all \( z \in \mathbb{C} \). From this, we deduce that
\[
I(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_n
\]
with
\[
c_k = \frac{I^{(n-k)}(0)}{(n-k)!} = (-1)^{n-k} \frac{n!}{k!(n-k)!} \int_{\mathbb{R}} x^k e^{-ux^2} \, dx
\]
for any \( k = 0, \cdots, n \). This proves the desired result. \( \square \)

We are now ready to prove the following interesting result which is more general than Theorem E.

**Theorem 8.** Let \( \mu, \nu, a, b \in \mathbb{C} \) with \( a \neq 0, b \neq 0, \text{Re} \mu > 0, \text{Re} \nu > 0, \) and \( \nu b^2 \neq \mu a^2 \). Suppose
\[
\delta = \frac{(b^2 - a^2) \mu \nu}{\nu b^2 - \mu a^2}
\]
and
\[
P_n(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n, \quad c_0 \neq 0.
\]
If \( a^k \neq b^k \) for \( 1 \leq k \leq n \), then the following conditions are equivalent:
(a) \( P_n(x) \) is the result of the recursion
\[
P_k(x) = 2x P_{k-1}(x) - \frac{1}{\delta} P'_{k-1}(x), \quad 1 \leq k \leq n,
\]
where \( P_0(x) = C \) is a nonzero constant.
(b) $P_n(x)$ is a solution of the integral equation

$$\int_{\mathbb{R}} P_n(x) e^{-\mu(x-az)^2} \, dx = C_n \int_{\mathbb{R}} P_n(x) e^{-\nu(x-bz)^2} \, dx$$

for some constant $C_n$.

(c) $c_{2k+1} = 0$ whenever $2k + 1 \leq n$, and

$$(n - 2k)! c_{2k} = -2\delta(2k + 2)(n - 2k - 2)! c_{2k+2}$$

whenever $2k + 2 \leq n$.

Furthermore, in the cases above, we have

$$C_n = \frac{\sqrt{\nu}}{\sqrt{\mu}} \left(\frac{a}{b}\right)^n$$

and

$$P_n(x) = C \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k 2^n n!}{4^k k! (n - 2k)! \delta^k} x^{n-2k}$$

with $C = c_0 2^{-n} \neq 0$.

**Proof.** We begin with some elementary calculations. Differentiating under the integral sign and using integration by parts, we obtain

$$\frac{d}{dz} \left( \int_{\mathbb{R}} P_n(x) e^{-\mu(x-az)^2} \, dx \right) = 2\mu a \int_{\mathbb{R}} (x - az) P_n(x) e^{-\mu(x-az)^2} \, dx$$

$$= a \int_{\mathbb{R}} P_n'(x) e^{-\mu(x-az)^2} \, dx.$$  \hfill (8)

Continuing this process $k \leq n$ times, we obtain

$$\frac{d^k}{dz^k} \left( \int_{\mathbb{R}} P_n(x) e^{-\mu(x-az)^2} \, dx \right) = a^k \int_{\mathbb{R}} P_n^{(k)}(x) e^{-\mu(x-az)^2} \, dx.$$  \hfill (9)

It follows from (8) that

$$\int_{\mathbb{R}} 2x P_n(x) e^{-\mu(x-az)^2} \, dx$$

$$= \frac{1}{\mu} \int_{\mathbb{R}} P_n'(x) e^{-\mu(x-az)^2} \, dx + 2az \int_{\mathbb{R}} P_n(x) e^{-\mu(x-az)^2} \, dx,$$

which implies that

$$\int_{\mathbb{R}} \left[ 2x P_n(x) - \frac{1}{\delta} P_n'(x) \right] e^{-\mu(x-az)^2} \, dx$$

$$= \frac{\delta - \mu}{\mu \delta} \int_{\mathbb{R}} P_n'(x) e^{-\mu(x-az)^2} \, dx + 2az \int_{\mathbb{R}} P_n(x) e^{-\mu(x-az)^2} \, dx$$  \hfill (10)

whenever $\delta \neq 0$.

To show that (a) implies (b), we use induction on $n$. When $n = 0$, it is clear from Lemma 7 that $P_0(x) = C$, where $C$ is any nonzero constant, is a solution of
equation (6) with $C_0 = \sqrt{\nu}/\sqrt{\mu}$. Similarly, $P_1(x) = 2Cx$ is a solution of equation (6) with $C_1 = \sqrt{\nu/a}/\sqrt{\mu/b}$.

So fix $n \geq 1$ and assume that the $n$th degree polynomial $P_n(x)$ is a solution of equation (6). Then we consider the $(n+1)$st degree polynomial

$$P_{n+1}(x) = 2xP_n(x) - \frac{1}{\delta}P_n'(x).$$

Differentiate both sides of (6) and use (8) to obtain

$$\int_{\mathbb{R}} P_n'(x)e^{-\mu(x-az)^2} dx = \frac{bC_n}{a} \int_{\mathbb{R}} P_n'(x)e^{-\nu(x-bz)^2} dx.$$

Combining the above identity with (6) and (10), we obtain

$$\int_{\mathbb{R}} P_{n+1}(x)e^{-\mu(x-az)^2} dx = \frac{bC_n(\delta - \mu)}{\mu \delta} \int_{\mathbb{R}} P_n'(x)e^{-\nu(x-bz)^2} dx + 2aC_n z \int_{\mathbb{R}} P_n(x)e^{-\nu(x-bz)^2} dx.$$

By an argument similar to that given for (10) we get

$$\int_{\mathbb{R}} P_{n+1}(x)e^{-\nu(x-bz)^2} dx = \frac{\delta - \nu}{\nu \delta} \int_{\mathbb{R}} P_n'(x)e^{-\nu(x-bz)^2} dx + 2bz \int_{\mathbb{R}} P_n(x)e^{-\nu(x-bz)^2} dx.$$

Clearly, if we define $C_{n+1}$ by $aC_n = C_{n+1}b$, or equivalently,

$$C_{n+1} = \left(\frac{\nu}{\mu}\right)^{n+1},$$

by induction, then it follows from (5) that

$$\frac{bC_n(\mu - \delta)}{a \mu} = \frac{C_{n+1}(\nu - \delta)}{\nu}.$$

Therefore,

$$\int_{\mathbb{R}} P_{n+1}(x)e^{-\mu(x-az)^2} dx = C_{n+1} \int_{\mathbb{R}} P_{n+1}(x)e^{-\nu(x-bz)^2} dx.$$

This completes the induction argument and proves that condition (a) implies (b).

To prove that (b) implies (c), we differentiate both sides of (6) $n$ times. It follows from (9) and Lemma 7 that

$$a^n n! c_0 \frac{\sqrt{\pi}}{\sqrt{\mu}} = b^n n! c_0 \frac{\sqrt{\pi}}{\sqrt{\nu}},$$

which gives

$$C_n = \left(\frac{\nu}{\mu}\right)^{n} \left(\frac{a}{b}\right)^n.$$

(11)
Therefore, if \( n = 0 \), there is nothing to prove. For \( n \geq 1 \), we differentiate both sides of (6) \( n - 1 \) times and use (9) to obtain

\[
a^{n-1} \int_{\mathbb{R}} \left[ n! c_0 x + (n - 1)! c_1 \right] e^{-\mu(x-az)^2} \, dx = b^{n-1} C_n \int_{\mathbb{R}} \left[ n! c_0 x + (n - 1)! c_1 \right] e^{-\nu(x-bz)^2} \, dx.
\]

Lemma 7 shows that both sides of the above equation are first degree polynomials of \( z \). If we compare the constant terms of these two linear polynomials, it follows from the assumption \( a \neq b \) and Lemma 7 that

\[
c_1 = 0. \tag{12}
\]

Thus condition (c) holds for \( n = 1 \). For \( n \geq 2 \) we use (12) to write

\[
P_n(x) = c_0 x^n + c_2 x^{n-2} + \cdots + c_n.
\]

Differentiating both sides of (6) \( n - 2 \) times and using (9), we obtain

\[
a^{n-2} \int_{\mathbb{R}} \left[ \frac{n! c_0}{2} x^2 + (n - 2)! c_2 \right] e^{-\mu(x-az)^2} \, dx = b^{n-2} C_n \int_{\mathbb{R}} \left[ \frac{n! c_0}{2} x^2 + (n - 2)! c_2 \right] e^{-\nu(x-bz)^2} \, dx.
\]

Comparing the constant terms on both sides of the above equation and using Lemma 7 and (11), we arrive at

\[
\frac{n! c_0}{4\mu} + (n - 2)! c_2 = \frac{a^2}{b^2} \left( \frac{n! c_0}{4\nu} + (n - 2)! c_2 \right).
\]

This together with (5) and the assumption \( a^2 \neq b^2 \) yields

\[
n! c_0 = -4\delta(n - 2)! c_2.
\]

Therefore, condition (c) holds for \( n = 2 \) as well.

When \( n \geq 3 \), we will prove (c) by induction on \( k \). Thus we assume

\[
c_{2k-1} = 0 \quad \text{and} \quad (n - 2k + 2)! c_{2k-2} = -4\delta k(n - 2k)! c_{2k}
\]

whenever \( 2k < n \). Then we have

\[
P_n(x) = \sum_{j=0}^{k} c_{2j} x^{n-2j} + c_{2k+1} x^{n-2k-1} + \cdots + c_n.
\]

Differentiating both sides of (6) \( n - 2k - 1 \) times and using (9), we obtain

\[
a^{n-2k-1} \int_{\mathbb{R}} \left[ \sum_{j=0}^{k} \frac{(n - 2j)! c_{2j} x^{2k-2j+1}}{(2k-2j+1)!} + (n - 2k - 1)! c_{2k+1} \right] e^{-\mu(x-az)^2} \, dx = b^{n-2k-1} C_n \int_{\mathbb{R}} \left[ \sum_{j=0}^{k} \frac{(n - 2j)! c_{2j} x^{2k-2j+1}}{(2k-2j+1)!} + (n - 2k - 1)! c_{2k+1} \right] e^{-\nu(x-bz)^2} \, dx.
\]
Note that \( 2k - 2j + 1 \) is odd for all \( 0 \leq j \leq k \). Comparing the constant terms on both sides of the above equation and using Lemma 7 again, we arrive at

\[
b^{2k+1}(n - 2k - 1)! c_{2k+1} = a^{2k+1}(n - 2k - 1)! c_{2k+1}.
\]

Since \( a^{2k+1} \neq b^{2k+1} \), we must have \( c_{2k+1} = 0 \). Thus

\[
P_n(x) = \sum_{j=0}^{k} c_{2j} x^{n-2j} + c_{2k+2} x^{n-2k-2} + \cdots + c_n.
\]

If \( 2k+1 < n \), we continue this process by differentiating both sides of (13) \( n-2k-2 \) times to obtain

\[
a^{n-2k-2} \int_{\mathbb{R}} \left[ \sum_{j=0}^{k} \frac{(n-2j)! c_{2j} x^{2k-2j+2}}{(2k-2j+2)!} \right] e^{-\mu(x-az)^2} dx
\]

\[
= b^{n-2k-2} C_n \int_{\mathbb{R}} \left[ \sum_{j=0}^{k} \frac{(n-2j)! c_{2j} x^{2k-2j+2}}{(2k-2j+2)!} \right] e^{-\nu(x-bz)^2} dx.
\]

Again, comparing the constant terms on both sides of the above equation, we get

\[
b^{2k+2} \left[ \sum_{j=0}^{k} \frac{(n-2j)! c_{2j} (2k-2j+1)!}{(2k-2j+2)! 2k-j+1 \mu^{k-j+1}} + (n-2k-2)! c_{2k+2} \right]
\]

\[
= a^{2k+2} \left[ \sum_{j=0}^{k} \frac{(n-2j)! c_{2j} (2k-2j+1)!}{(2k-2j+2)! 2k-j+1 \mu^{k-j+1}} + (n-2k-2)! c_{2k+2} \right].
\]

From the induction hypothesis we deduce that

\[
\frac{(n-2j+2)! c_{2j-2}}{(2k-2j+4)! 2k-j+2 \mu^{k-j+2}} = -\frac{j\delta}{(k-j+2)\mu} \left[ \frac{(n-2j)! c_{2j} (2k-2j+1)!}{(2k-2j+2)! 2k-j+1 \mu^{k-j+1}} \right]
\]

for any \( j = 1, \cdots, k \). Consequently,

\[
\sum_{j=0}^{k} \frac{(n-2j)! c_{2j} (2k-2j+1)!}{(2k-2j+2)! 2k-j+1 \mu^{k-j+1}} - \frac{(n-2k)! c_{2k}}{2(2k+2)\delta}
\]

\[
= -\frac{(n-2k)! c_{2k}}{2(2k+2)\delta} \left[ 1 - \frac{(k+1)\delta}{\mu} + \frac{(k+1)k\delta^2}{2\mu^2} + \cdots + (-1)^{k+1} \frac{(k+1)k \cdots 1 \delta^{k+1}}{1 \cdot 2 \cdots (k+1) \mu^{k+1}} \right]
\]

\[
= -\frac{(n-2k)! c_{2k}}{2(2k+2)\delta} \left( 1 - \frac{\delta}{\mu} \right)^{k+1}.
\]

By (5), we have

\[
\left( 1 - \frac{\delta}{\nu} \right) b^2 = \left( 1 - \frac{\delta}{\nu} \right) a^2.
\]

Combining this with (13), we get
Since this completes the induction argument.

and if

This proves that condition (c) implies (a).

To finish the proof, we assume that condition (c) holds. Recall that the leading term of the polynomial \( P_n(x) \) is \( c_0 x^n \). If we let \( C = c_0 2^{-n} \neq 0 \), then it is easy to see that

\[
P_n(x) = C \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k 2^n n!}{4^k k!(n-2k)!} x^{n-2k}.
\]

A straightforward calculation shows that

\[
\frac{1}{C} \left[ 2x P_n(x) - \frac{1}{\delta} P'_n(x) \right] - (2x)^{n+1}
= \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k 2^{n+1} n!}{4^k k!(n-2k)!} x^{n-2k+1} = \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor+1} \frac{(-1)^{k-1} 2^n n!}{4^{k-1}(k-1)!} x^{n-2k+1}
= \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k 2^{n+1} (n+1)!}{4^k k!(n-2k+1)!} x^{n-2k+1} = \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} 2^n n!}{4^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{n+1}{2} \right)!} x^{n-2\left\lfloor \frac{n}{2} \right\rfloor+1}.
\]

Consequently, if \( n \) is even, then

\[
2x P_n(x) - \frac{1}{\delta} P'_n(x) = C \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k 2^{n+1} (n+1)!}{4^k k!(n-2k+1)!} x^{n-2k+1}
= P_{n+1}(x),
\]

and if \( n \) is odd, then

\[
2x P_n(x) - \frac{1}{\delta} P'_n(x) = C \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k 2^{n+1} (n+1)!}{4^k k!(n-2k+1)!} x^{n-2k+1} = \frac{C(-1)^{\frac{n-1}{2}} 2^n n!}{4^{\frac{n-1}{2}} (\frac{n-1}{2})!} x^{n-2\left\lfloor \frac{n}{2} \right\rfloor+1}
= C \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \frac{(-1)^k 2^{n+1} (n+1)!}{4^k k!(n-2k+1)!} x^{n-2k+1}
= P_{n+1}(x).
\]

This proves that condition (c) implies (a). \( \square \)

Recall that for any nonnegative integer \( n \), the \( n \)th Hermite polynomial \( H_n(x) \) is defined by

\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.
\]
In general, it is easy to check that each $H_n(x)$ has degree $n$ and

$$H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x), \quad n \geq 1,$$

which can be used to compute $H_n(x)$ inductively. Obviously, to show that Theorem E holds, simply take $\delta = 1$ and apply Theorem 8. Furthermore, as a by-product of the proof of Theorem 8, we will also obtain the explicit formula for the integral that appeared in (6).

**Corollary 9.** Let $\mu, a \in \mathbb{C}$ with $\Re \mu > 0$, and let $P_n(x)$ be the $n$ degree polynomial defined in (7). Then

$$\int_{\mathbb{R}} P_n(x)e^{-\mu(x-az)^2} \, dx = C \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^n n! a^{n-2k} \sqrt{\pi}}{4^k k! (n-2k)! \delta^k \sqrt{\mu}} \left(1 - \frac{\delta}{\mu}\right)^k z^{n-2k}$$

for all $z \in \mathbb{C}$.

**Proof.** Write

$$Q_n(z) = \int_{\mathbb{R}} P_n(x)e^{-\mu(x-az)^2} \, dx.$$

Since $c_{2k+1} = 0$ whenever $2k + 1 \leq n$, it follows from Lemma 7 that

$$Q_n(z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{Q_{n-2k}^{(0)}}{(n-2k)!} z^{n-2k}.$$

Furthermore, by the proof of Theorem 8 we have

$$Q_{n-2k}^{(0)} = \frac{d^{n-2k}}{dx^{n-2k}} \left( \int_{\mathbb{R}} P_n(x)e^{-\mu(x-az)^2} \, dx \right) \bigg|_{z=0}$$

$$= a^{n-2k} \int_{\mathbb{R}} P_n^{(n-2k)}(x)e^{-\mu(x-az)^2} \, dx \bigg|_{z=0}$$

$$= -a^{n-2k} (n-2k+2)! c_{2k-2} \sqrt{\pi} \left(1 - \frac{\delta}{\mu}\right)^k$$

$$= Ca^{n-2k} \frac{(-1)^k 2^n n! \sqrt{\pi}}{4^k k! \delta^k \sqrt{\mu}} \left(1 - \frac{\delta}{\mu}\right)^k,$$

whenever $k \in \mathbb{N}$ with $2k \leq n$. This proves the desired result. \hfill \Box

We return to the calculation of eigenvalues and eigenvectors of the operators $T^{(s,t)}$. Recall from the analysis at the beginning of this section that if $|\Re s| < 1$ and $|s|^2 = 1 + |t|^2$ then $T^{(s,t)}$ has an eigenvalue $\lambda_0$ with corresponding eigenvector $e^{\gamma z^2/2}$, where $\lambda_0 = \frac{1}{\sqrt{s}} \sqrt{\frac{s}{1+|t|^2}}$ for a certain choice of $\gamma$ in the unit disk that is guaranteed by Lemma 6. With the help of Theorem 8 we are going to obtain additional eigenvalues and corresponding eigenvectors for $T^{(s,t)}$ in the following result.
Theorem 10. Suppose \((s, t) \in \mathbb{C}^2\) with \(|s|^2 = |t|^2 + 1\) and \(\text{Re } s < 1\). Let \(\gamma\) be the unique number in \(\mathbb{D}\) from Lemma 6. Then for each nonnegative integer \(n\) the complex number
\[
\lambda_n = \frac{1}{\sqrt{s}} \sqrt{\frac{s}{s + t\gamma}} \frac{1}{(s + t\gamma)^n} \in \mathbb{T}
\]
is an eigenvalue of \(T(s, t)\) and the function \(Q_n(z) e^{\gamma z^2/2}\) is a corresponding eigenvector, where
\[
Q_n(z) = \int_{\mathbb{R}} H_n \left( \frac{x}{\rho} \right) \exp \left[ -\frac{2}{1 + \gamma} \left( x - \frac{1 + \gamma}{2} z \right)^2 \right] dx
\]
is a polynomial of degree \(n\). Here \(H_n(x)\) is the \(n\)th Hermite polynomial and \(\rho\) is a positive number such that
\[
\rho^2 = \frac{(1 + \gamma) \left[ (s - \bar{t})(s + t\gamma) - 1 \right]}{2 \left( (s + t\gamma)^2 - 1 \right)}.
\]

Proof. Since \(|\gamma| < 1\), we use Fubini’s theorem and (2) to obtain
\[
T(s, t) \left[ Q_n(z) e^{\frac{z^2}{2}} \right](z)
\]
\[
= \frac{1}{\sqrt{s}} \int_{\mathbb{C}} Q_n(w) \exp \left[ \frac{\gamma}{2} w^2 + \frac{t}{2s} z^2 - \frac{7}{2s} w^2 + \frac{z\bar{w}}{s} \right] d\lambda(w)
\]
\[
= \frac{e^{\frac{z^2}{2}}}{\sqrt{s}} \int_{\mathbb{R}} H_n \left( \frac{x}{\rho} \right) e^{-\frac{2(1 + \gamma)^2}{s - t} x^2} dx \int_{\mathbb{C}} \exp \left[ -\frac{w^2}{2} + 2xw - \frac{7}{2s} w^2 + \frac{z\bar{w}}{s} \right] d\lambda(w)
\]
\[
= C e^{\frac{(1 + \gamma)^2 z^2}{2(s - t)^2}} \int_{\mathbb{R}} H_n \left( \frac{x}{\rho} \right) \exp \left[ -\frac{2(s + t\gamma)}{(1 + \gamma)(s - t)} \left( x - \frac{1 + \gamma}{2(s + t\gamma)} z \right)^2 \right] dx,
\]
where
\[
C = \frac{1}{\sqrt{s}} \sqrt{\frac{s}{s - t}}.
\]
It follows from \(|s|^2 = 1 + |t|^2\) that the equation in (3) can be rewritten as
\[
\gamma(s + t\gamma) = t + \bar{t}\gamma,
\]
which implies
\[
e^{\frac{2(t + t\gamma)^2}{s - t}} = e^{\gamma^2}.
\]
So it suffices for us to prove that
\[
\frac{1}{\sqrt{s}} \sqrt{\frac{s}{s - t}} \int_{\mathbb{R}} H_n \left( \frac{x}{\rho} \right) \exp \left[ -\frac{2(s + t\gamma)}{(1 + \gamma)(s - t)} \left( x - \frac{1 + \gamma}{2(s + t\gamma)} z \right)^2 \right] dx
\]
\[
= \lambda_n \int_{\mathbb{R}} H_n \left( \frac{x}{\rho} \right) \exp \left[ -\frac{2}{1 + \gamma} \left( x - \frac{1 + \gamma}{2} z \right)^2 \right] dx. \quad (15)
\]
It follows from $|\text{Re } s| < 1$ and $|s|^2 = 1 + |t|^2$ that $|\text{Im } t| < |\text{Im } s|$. This together with (14) and (4) implies that

$$\rho^2 = \frac{(s - \bar{t})(s + \bar{\gamma}t) + (s - \bar{t})(t + \bar{\gamma} - 1 - \gamma)}{2|s(s + \bar{\gamma}t) + \bar{t}(t + \bar{\gamma}) - 1|}$$

$$= \frac{(s + \bar{\gamma}t)(s + t - \bar{\gamma} - \bar{t})}{2|(s + \bar{\gamma})(s + \bar{\gamma}t) - 2|}$$

$$= \frac{(s + t - \bar{\gamma} - \bar{t})}{2(s + \bar{\gamma} - \frac{2}{s + \bar{\gamma}t})} > 0.$$  

It is easy to check that

$$\text{Re } \left( \frac{s}{s - \bar{t}} \right) > 0 \quad \text{and} \quad \text{Re } \left( \frac{s}{s - \bar{\gamma}t} \right) > 0.$$ 

Thus (15) is equivalent to

$$\int_{\mathbb{R}} H_n(x) \exp \left[ -\frac{2(s + \bar{\gamma}t)^2}{(1 + \gamma)(s - \bar{t})} \left( x - \frac{1 + \gamma}{2(s + \bar{\gamma}t)\rho} \right)^2 \right] dx$$

$$= \sqrt{\frac{s - \bar{t}}{s + \bar{\gamma}(s + \bar{\gamma}t)}} \int_{\mathbb{R}} H_n(x) \exp \left[ -\frac{2\rho^2}{1 + \gamma} (x - \frac{1 + \gamma}{2\rho}z)^2 \right] dx.$$  (16)

Write

$$\mu = \frac{2(s + \bar{\gamma}t)^2}{(1 + \gamma)(s - \bar{t})}, \quad \nu = \frac{2\rho^2}{1 + \gamma}, \quad a = \frac{1 + \gamma}{2(s + \bar{\gamma}t)\rho}, \quad b = \frac{1 + \gamma}{2\rho}.$$ 

Then a straightforward calculation shows that $\text{Re } \nu > 0$ and

$$\text{Re } \mu = \frac{2\rho^2(|s|^2 - \text{Re}(st) + \text{Re}(\gamma) + \text{Re}(st)|\gamma|^2 - |t|^2|\gamma|^2)}{|1 + \gamma|^2|s - \bar{t}|^2}$$

$$= \frac{\rho^2 \left[ (1 + |s - \bar{t}|^2) + 2\text{Re}(\gamma) + (1 - |s - \bar{t}|^2)|\gamma|^2 \right]}{|1 + \gamma|^2|s - \bar{t}|^2}$$

$$> \frac{\rho^2}{|s - \bar{t}|^2} > 0.$$ 

Recall from the proof of Lemma 6 that $(s + \bar{\gamma}t)^2 \neq 1$. Since

$$\frac{(1 + \gamma)(s - \bar{t})(s + \bar{\gamma}t)}{2\rho^2} - (s + \bar{\gamma}t)^2 = \frac{1 + \gamma}{2\rho^2} - 1,$$

it follows that

$$\nu b^2 \neq \mu a^2 \quad \text{and} \quad \frac{(b^2 - a^2)\mu}{\nu b^2 - \mu a^2} = 1.$$ 

Then by the proof of Theorem 8, the $n$th Hermite polynomial $H_n(x)$ satisfies (16). This complete the proof of the theorem. $\square$
Some special cases are worth mentioning here. If \((s + \bar{t} \gamma)^k = 1\) for some \(2 < k \leq n\), then by the proof of Theorem 8, each Hermite polynomial \(H_{n+kN}(x)\), \(N = 0, 1, \cdots\), satisfies (16), which implies that all functions \(Q_{n+kN}(z)e^{\gamma z^2/2}\) are eigenvectors for the operator \(T(s,t)\) corresponding to the eigenvalue \(\lambda_n\). Conversely, if \((s + \bar{t} \gamma)^k \neq 1\) whenever \(1 \leq k \leq n\), then \(H_n(x)\) is the unique polynomial of degree less than or equal to \(n\) satisfying (16). Furthermore, if \((s + \bar{t} \gamma)^k \neq 1\) for any positive number \(k\), then we can completely determine the spectrum of the operator \(T(s,t)\). More specifically, we have the following.

**Corollary 11.** Suppose \((s, t) \in \mathbb{C}^2\) with \(|s|^2 = 1 + |t|^2\) and \(|\text{Re } s| < 1\). Let \(\gamma\) be the number from Lemma 6 and \(1/(s + \bar{t} \gamma) = e^{i\theta}, \theta \in (-\pi, \pi]\). If \(\theta\) is not a rational multiple of \(\pi\), then the spectrum of the unitary operator \(T(s,t)\) is the full unit circle.

**Proof.** If \(\theta\) is not a rational multiple of \(\pi\), it is well known that the sequence \(\{e^{in\theta} : n = 0, 1, 2, \cdots\}\) is dense in the unit circle \(\mathbb{T}\). Since the spectrum \(\sigma(T(s,t))\) of the unitary operator \(T(s,t)\) is a closed subset of \(\mathbb{T}\), it follows from Theorem 10 that \(\sigma(T(s,t)) = \mathbb{T}\).

\[\square\]

4. **Linear Canonical Transforms**

Recall that the general linear group of order \(n\), denoted by \(GL(n, \mathbb{R})\), is the group of all invertible \(n \times n\) matrices with real entries. The special linear group of order \(n\), denoted by \(SL(n, \mathbb{R})\), is the subgroup of \(GL(n, \mathbb{R})\) consisting of matrices \(A\) with \(\det(A) = 1\).

We will be mostly interested in the case when \(n = 2\). Instead of using four real entries to describe an element in \(GL(2, \mathbb{R})\), it will be more convenient for us to think of a matrix in \(GL(2, \mathbb{C})\) as a pair of complex numbers.

**Lemma 12.** Let \(GL(\mathbb{C} \times \mathbb{C}) = \{(s, t) \in \mathbb{C} \times \mathbb{C} : |s| \neq |t|\}\). Then \(GL(\mathbb{C} \times \mathbb{C})\) is a group with the following operation:

\[(s_1, t_1) \cdot (s_2, t_2) = (s_1 s_2 + \bar{t}_1 t_2, t_1 s_2 + \bar{s}_1 t_2).\]

Furthermore, \(SL(\mathbb{C} \times \mathbb{C}) = \{(s, t) \in \mathbb{C} \times \mathbb{C} : |s|^2 - |t|^2 = 1\}\) is a subgroup of \(GL(\mathbb{C} \times \mathbb{C})\).

**Proof.** The proof follows easily from the definitions. We omit the routine details. This will also follow from Theorem 13 and its proof below.

\[\square\]

It is clear that the group \(GL(\mathbb{C} \times \mathbb{C})\) has unit \((1, 0)\) and it is easy to check that the inverse of \((s, t)\) is given by

\[\varphi(A) = \left(\frac{s}{|s|^2 - |t|^2}, -\frac{t}{|s|^2 - |t|^2}\right).\]

**Theorem 13.** Let

\[\varphi(A) = \left(\frac{a + bi + d - ci}{2}, \frac{a + bi - d + ci}{2}\right)\]
for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$. Then $\varphi : GL(2, \mathbb{R}) \to GL(\mathbb{C} \times \mathbb{C})$ is a group isomorphism, and it maps $SL(2, \mathbb{R})$ onto $SL(\mathbb{C} \times \mathbb{C})$.

**Proof.** Since

$$\frac{|a + bi + d - ci|^2}{2} - \frac{|a + bi - d + ci|^2}{2} = ad - bc,$$

we conclude that $\varphi(A) \in GL(\mathbb{C} \times \mathbb{C})$ for any $A \in GL(2, \mathbb{R})$.

It is clear that $\varphi$ is injective. Moreover, for any $(s, t) \in GL(\mathbb{C} \times \mathbb{C})$, if we let

$$A = \begin{bmatrix} \text{Re}(s + t) & \text{Im}(s + t) \\ -\text{Im}(s - t) & \text{Re}(s - t) \end{bmatrix},$$

then $A \in GL(2, \mathbb{R})$ and $\varphi(A) = (s, t)$. Thus $\varphi$ is also surjective.

A direct calculation shows that

$$\varphi(A_1 A_2) = \left( \frac{a_1 a_2 + b_1 c_2 + (a_1 b_2 + b_1 d_2)i + c_1 b_2 + d_1 d_2 - (c_1 a_2 + d_1 c_2)i}{2}, \right.$$

$$\left. \frac{a_1 a_2 + b_1 c_2 + (a_1 b_2 + b_1 d_2)i - (c_1 b_2 + d_1 d_2) + (c_1 a_2 + d_1 c_2)i}{2} \right)$$

$$= \left( \frac{a_1 + b_1 i + d_1 - c_1 i}{2}, \frac{a_1 + b_1 i - d_1 + c_1 i}{2} \right)$$

$$\cdot \left( \frac{a_2 + b_2 i + d_2 - c_2 i}{2}, \frac{a_2 + b_2 i - d_2 + c_2 i}{2} \right)$$

$$= \varphi(A_1) \cdot \varphi(A_2),$$

which shows that $\varphi$ preserves group operations in $GL(2, \mathbb{R})$ and $GL(\mathbb{C} \times \mathbb{C})$. Thus $\varphi$ is an isomorphism from $GL(2, \mathbb{R})$ onto $GL(\mathbb{C} \times \mathbb{C})$. It follows from (17) that $\varphi$ maps $SL(2, \mathbb{R})$ onto $SL(\mathbb{C} \times \mathbb{C})$. \(\square\)

The following result gives a new unitary projective representation of $SL(\mathbb{C} \times \mathbb{C})$, and hence $SL(2, \mathbb{R})$, on the Fock space. Recall that a mapping $\phi : G \to B(H)$ from a group $G$ to the algebra $B(H)$ of all bounded linear operators on a Hilbert space $H$ is called a unitary projective (or ray) representation if

(i) for every $x \in G$ the operator $\phi(x)$ is unitary, and

(ii) for any $x_1, x_2 \in G$ there exists a unimodular constant $\lambda$ such that $\phi(x_1 x_2) = \lambda \phi(x_1) \phi(x_2)$.

See [1] [12] for the theory of unitary projective or ray representations.

**Theorem 14.** The mapping $(s, t) \mapsto T^{(s, t)}$ is a unitary projective representation of the group $SL(\mathbb{C} \times \mathbb{C})$ on $F^2$.

**Proof.** For $k = 1, 2$ let $(s_k, t_k) \in SL(\mathbb{C} \times \mathbb{C})$, so $|s_k|^2 = |t_k|^2 + 1$. It follows from Lemma[2] that

$$\int_{\mathbb{C}} K^{(s_1, t_1)}(z, w) K^{(s_2, t_2)}(w, u) d\lambda(w) = C_{(s, t)} K^{(s_1, t_1) \cdot (s_2, t_2)}(z, u),$$
where
\[ C(s,t) = \sqrt{\frac{s_1 s_2 + t_1 t_2}{s_1 s_2}} \sqrt{\frac{s_1 s_2}{s_1 s_2 + t_1 t_2}} = \pm 1. \]

Therefore,
\[
T^{(s_1,t_1)}T^{(s_2,t_2)}f(z) = \int_C f(u) d\lambda(u) \int_C K^{(s_1,t_1)}(z,w)K^{(s_2,t_2)}(w,u) d\lambda(w)
\]
\[ = C(s,t) \int_C f(u) K^{(s_1,t_1)-(s_2,t_2)}(z,u) d\lambda(u) \]
\[ = C(s,t)T^{(s_1,t_1)-(s_2,t_2)}f(z). \]

By Theorem 5, each \( T^{(s,t)} \) is a unitary operator on \( F^2 \), so \( s, t \mapsto T^{(s,t)} \) is a unitary projective representation of \( SL(\mathbb{C} \times \mathbb{C}) \) on \( F^2 \). \( \square \)

Next we will consider the Hilbert space \( L^2(\mathbb{R}) = L^2(\mathbb{R}, dx) \) and a family of unitary operators on it, namely, the so-called linear canonical transforms that were traditionally studied in Hamiltonian mechanics. These transforms include the fractional Fourier transforms, the Fresnel transform, as well as many other classical transforms on \( L^2(\mathbb{R}) \). The books [10, 11, 13, 18] and survey paper [4] are excellent sources of information for these operators.

The linear canonical transforms can be defined in several ways. One way is to define them as parameterized by the special linear group \( SL(2, \mathbb{R}) \). More specifically, for any real matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( \det(A) = ad - bc = 1 \), we define an operator \( F^A \) on \( L^2(\mathbb{R}) \) by
\[
F^A(f)(x) = \frac{1}{\sqrt{2\pi b}} e^{i\alpha \cot \alpha} \int_\mathbb{R} e^{-i(2xt \csc \alpha - t^2 \cot \alpha)} f(t) dt
\]
for \( b \neq 0 \). When \( b = 0 \), we define
\[
F^A(f)(x) = \sqrt{d} e^{i\alpha \cot \alpha} f(x)
\]
which is the limit in (18) as \( b \to 0 \). Note that the symbol \( d \) here is a number but not the differential!

The linear canonical transforms include several prominent classical transforms as special cases. First, for any angle \( \alpha \in [-\pi, \pi] \), the matrix
\[
A_\alpha = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}
\]
is clearly a member of \( SL(2, \mathbb{R}) \). The corresponding linear canonical transform is given by
\[
F^{A_\alpha}(f)(x) = \frac{1}{\sqrt{2\pi \sin \alpha}} e^{i2x \cot \alpha} \int_\mathbb{R} e^{-i(2xt \csc \alpha - t^2 \cot \alpha)} f(t) dt
\]
when \( \sin \alpha \neq 0 \), and
\[
F^{A_\alpha}(f)(x) = \sqrt{\cos \alpha} e^{-i2x \sin \alpha \cos \alpha} f(x \cos \alpha) = \sqrt{\pm 1} f(\pm x)\]
when \( \sin \alpha = 0 \). Up to a unimodular constant, these are the classical fractional Fourier transforms. In fact, since

\[
\frac{e^{i\alpha/2}}{\sqrt{1 - i \cot \alpha}} = \frac{\sqrt{1 - i \cot \alpha}}{\sqrt{\pi}},
\]

the classical \( \alpha \)th order fractional Fourier transform \( \mathcal{F}^\alpha \) (see [4, 13] for example) can be written as \( \mathcal{F}^\alpha = e^{i\alpha/2} \mathcal{F}_A^\alpha \) for any \( \alpha \in [-\pi, \pi) \). When \( \alpha = \pi/2 \), we obtain the ordinary Fourier transform. The inverse Fourier transform is obtained when \( \alpha = -\pi/2 \).

For any \( \sigma > 0 \), the matrix

\[
A = \begin{bmatrix}
1/\sigma & 0 \\
0 & \sigma
\end{bmatrix}
\]

belongs to \( SL(2, \mathbb{R}) \) and the corresponding linear canonical transform is given by

\[
\mathcal{F}_A^\sigma(f)(x) = \sqrt{\sigma} f(\sigma x).
\]

This is “scaling” or “dilation” in \( L^2(\mathbb{R}) \).

Let \( b = \lambda l/\pi > 0 \) (where \( l \) and \( \lambda \) represent distance and wavelength, respectively, in mechanics). The linear canonical transform corresponding to the matrix

\[
A = \begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix}
\]

is given by

\[
\mathcal{F}_A^b(f)(x) = \frac{1}{\sqrt{i\pi b}} \int_{\mathbb{R}} e^{i(x-t)^2/b} f(t) \, dt.
\]

This is basically the classical Fresnel transform \( \mathcal{F}_l^A \), which corresponds to shearing in continuum mechanics. More precisely, we have

\[
\mathcal{F}_l^A(f)(x) = e^{i\pi l/\lambda} \mathcal{F}_A^b(f)(x).
\]

Finally, for the matrix

\[
A = \begin{bmatrix}
1 & 0 \\
\tau & 1
\end{bmatrix}
\]

in \( SL(2, \mathbb{R}) \), the associated linear canonical transform is given by

\[
\mathcal{F}_A^\tau(x) = e^{i\tau x^2} f(x),
\]

which is traditionally called the “chirp multiplication” in optics.

Our next goal is to show that every linear canonical transform is unitarily equivalent to some operator \( T^{(s,t)} \) on \( F^2 \). To this end, recall that the Bargmann transform \( B \) is the operator from \( L^2(\mathbb{R}) \to F^2 \) defined by

\[
Bf(z) = C \int_{\mathbb{R}} f(x) e^{2xz-x^2-(z^2/2)} \, dx,
\]

where \( C = (2/\pi)^{1/4} \). It is well known that \( B \) is a unitary operator from \( L^2(\mathbb{R}) \) onto \( F^2 \). Furthermore, the inverse of \( B \) is also an integral operator, namely,

\[
B^{-1}f(x) = C \int_{\mathbb{C}} f(z) e^{2z\pi-x^2-(\pi^2/2)} \, d\lambda(z).
\]
We consider the action of the Bargmann transform on linear canonical transforms. In other words, we are going to compute the equivalent form, under the Bargmann transform, of the linear canonical transforms on the Fock space.

**Theorem 15.** Suppose

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is a matrix in \( SL(2, \mathbb{R}) \). Then the operator \( T^A = B F^A B^{-1} : F^2 \to F^2 \) is given by \( T^A = C_A T(s, t) \), where

\[
\begin{align*}
    s &= \frac{a + bi + d - ci}{2}, \\
    t &= \frac{a + bi - d + ci}{2}
\end{align*}
\]

with \((s, t) \in SL(\mathbb{C} \times \mathbb{C})\), and

\[
C_A = \pm 1 = \begin{cases} 
\sqrt{s} & \text{if } b \neq 0, \\
\sqrt{s + t - \frac{s}{2}} & \text{if } b = 0.
\end{cases}
\]

**Proof.** First we assume \( b \neq 0 \). For the purpose of applying Fubini’s theorem in the calculations below, we assume that \( f \) is any polynomial (recall that the polynomials are dense in \( F^2 \), and under the inverse Bargmann transform, they become the Hermite polynomials times the Gauss function, which have very good integrability properties on the real line). Recall that \( C = (2/\pi)^{1/4} \) and let us write \( C' = \frac{1}{\sqrt{b \pi}} \) for simplicity. Then we have

\[
F^A(B^{-1}f)(x) = CC' e^{idx^2/b} \int_{\mathbb{R}} e^{-i(2xt - at^2)/b} dt \int_{\mathbb{C}} f(w) e^{2i\overline{w} - t^2 - (\overline{w}/2)} d\lambda(w)
\]

\[
= CC' e^{idx^2/b} \int_{\mathbb{C}} f(w) \exp \left[ -\frac{\overline{w}^2}{2} + \frac{(b\overline{w} - ix)^2}{b(b - ia)} \right] d\lambda(w)
\]

\[
\cdot \int_{\mathbb{R}} \exp \left[ -\left(1 - \frac{a}{b}\right) \left(t - \frac{b\overline{w} - ix}{b - ia}\right)^2 \right] dt.
\]

It follows from Lemma 7 that

\[
F^A(B^{-1}f)(x)
\]

\[
= CC'' e^{idx^2/b} \int_{\mathbb{C}} f(w) \exp \left[ -\frac{\overline{w}^2}{2} + \frac{(b\overline{w} - ix)^2}{b(b - ia)} \right] d\lambda(w)
\]

\[
= CC'' \exp \left[ \left(\frac{id}{b} - \frac{1}{b(b - ia)}\right)x^2 \right] \int_{\mathbb{C}} f(w) \exp \left[ \frac{b + ia}{2(b - ia)} \overline{w}^2 - \frac{2i}{b - ia} x \overline{w} \right] d\lambda(w),
\]

where \( C'' = CC' \sqrt{\frac{b\pi}{b - ia}} \). Since \( ad - bc = 1 \), a direct calculation shows that

\[
(b - id)(b - ia) + 1 = b(b - ia - c - id).
\]

Therefore,

\[
BF^A B^{-1} f(z)
\]
\[ CC'' \int_{\mathbb{R}} \exp \left[ \frac{2xz - z^2}{2} - \frac{(b-id)(b-ia) + 1}{b(b-ia)} \right] dx \]
\[ \cdot \int_{\mathbb{C}} f(w) \exp \left[ \frac{b+ia}{2(b-ia)} \bar{w}^2 - \frac{2i}{b-ia} \bar{w} \right] d\lambda(w) \]
\[ = CC'' e^{-z^2/2} \int_{\mathbb{C}} f(w) \exp \left[ \frac{b+ia}{2(b-ia)} \bar{w}^2 + \frac{[(b-ia)z - i\bar{w}]^2}{(b-ia)(b-ia-c-id)} \right] d\lambda(w) \]
\[ \cdot \int_{\mathbb{R}} \exp \left[ - \frac{b-ia-c-id}{b-ia} \left( x - \frac{(b-ia)z - i\bar{w}}{b-ia-c-id} \right)^2 \right] dx. \]

Observe that
\[ \text{Re} \frac{b-ia-c-id}{b-ia} = \frac{b^2 + a^2 + 1}{b^2 + a^2} > 0. \]

Using Lemma 7 again, we obtain
\[ BF^A B^{-1} f(z) = C_A \frac{\sqrt{2}}{\sqrt{a+bi-ci+d}} e^{-z^2/2} \]
\[ \cdot \int_{\mathbb{C}} f(w) \exp \left[ \frac{b+ia}{2(b-ia)} \bar{w}^2 + \frac{[(b-ia)z - i\bar{w}]^2}{(b-ia)(b-ia-c-id)} \right] d\lambda(w) \]

where
\[ C_A = \sqrt{\frac{a+bi-ci+d}{\sqrt{ib}}} \sqrt{\frac{ib}{a+bi+d-ci}}. \]

Then a few lines of elementary calculations show that
\[ BF^A B^{-1} f(z) = C_A \frac{\sqrt{2}}{\sqrt{a+bi-ci+d}} \]
\[ \cdot \int_{\mathbb{C}} f(w) \exp \left[ \frac{(a+bi+ci-d)z^2 - (a-bi-ci-d)\bar{w}^2 + 4z\bar{w}}{2(a+ib-ci+d)} \right] d\lambda(w) \]
\[ = \frac{C_A}{s} \int_{\mathbb{C}} f(w) e^{\frac{1}{2}z^2 - \frac{7}{2}z\bar{w}^2 + \frac{\Delta}{s}} d\lambda(w), \]
where
\[ s = \frac{a+bi+d-ci}{2}, \quad t = \frac{a+bi-d+ci}{2}. \]

It is easy to check that \(|s|^2 = |t|^2 + 1\), so \((s, t) \in SL(\mathbb{C} \times \mathbb{C})\).

Next we assume \(b = 0\). Note that
\[ d = \frac{1}{a} = \frac{1}{s+t} \]

The arguments used above can be simplified to show that
\[ T^A f(z) = \sqrt{s} \sqrt{\frac{1}{s+t}} \sqrt{\frac{s+t}{s}} T^{(s,t)} f(z). \]

This completes the proof of the theorem. \(\square\)
As a consequence of Theorem 15 we immediately derive a number of basic properties for the operators \( F^A \). In particular, we obtain an alternative proof of the unitarity and the composition formula of linear canonical transforms.

**Corollary 16.** Let \( A, A_1, A_2 \in \text{SL}(2, \mathbb{R}) \). Then the following statements hold.

(a) \( F^A \) is a unitary operator on \( L^2(\mathbb{R}) \).
(b) \( F^{A_1} F^{A_2} = C F^{A_1 A_2} \) for some \( C = \pm 1 \).
(c) \( (F^A)^{-1} = C F^{A^{-1}} \), where \( C = -1 \) whenever \( a < 0 \) and \( b = 0 \), and \( C = 1 \) for other cases. Here \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{R}) \).

**Proof.** Condition (a) clearly follows from Theorems 8 and 15.

Recall from Theorem 13 that \( \text{SL}(2, \mathbb{R}) \) is isomorphic to \( \text{SL}(\mathbb{C} \times \mathbb{C}) \). Therefore, it follows from Theorems 14 and 15 that

\[
B F^{A_1} F^{A_2} B^{-1} f(z) = C A_1 C_2 T^{(s_1, t_1)} T^{(s_2, t_2)} f(z) = C A_1 C_2 C A_2 C^{(s, t)} f(z) = C B F^{A_1 A_2} B^{-1} f(z)
\]

for any \( f \in F^2 \), where \( C = C A_1 C_2 C A_2 C^{(s, t)} = \pm 1 \). This proves (b).

A special case of assertion (b) is that

\[
B F^{A^*} F^{-1} B^{-1} f(z) = C T^{(1, 0)} f(z) = C \int_{\mathbb{C}} f(u) e^{uz} d\lambda(u) = C f(z),
\]

where

\[
C = \frac{|s|}{\sqrt{s + t - \overline{s} - \overline{t}}} \sqrt{\frac{s + t - \overline{s} - \overline{t}}{s}} \sqrt{\frac{1}{s - t + \overline{s}}} \sqrt{\frac{1}{s - t + \overline{s}}} \sqrt{\frac{\overline{s} - t - s + \overline{t}}{\overline{s}}}
\]

when the real number \(-i(s + t - \overline{s} - \overline{t}) \neq 0 \), and

\[
C = |s| \sqrt{\frac{1}{s + t}} \sqrt{\frac{s + t}{s}} \sqrt{\frac{1}{s - t}} \sqrt{\frac{\overline{s} - t}{\overline{s}}}
\]

when \(-i(s + t - \overline{s} - \overline{t}) = 0 \).

Suppose \(-i(s + t - \overline{s} - \overline{t}) \neq 0 \). Then \((s + t - \overline{s} - \overline{t})/s < 0 \) if and only if

\[
s = -\overline{s} \quad \text{and} \quad [-i(s + t - \overline{s} - \overline{t})](-is) < 0.
\]

However, this condition means that

\[
|s|^2 = | -is|^2 < |\text{Im} t| \leq |t|^2,
\]

which contradicts with the fact that \((s, t) \in \text{SL}(\mathbb{C} \times \mathbb{C}) \). Therefore,

\[
\sqrt{\frac{s + t - \overline{s} - \overline{t}}{s}} \sqrt{\frac{\overline{s} - t - s + \overline{t}}{\overline{s}}} = |s + t - \overline{s} - \overline{t}|/|s|.
\]

It is clear that \( s + t - \overline{s} - \overline{t} \) is not a negative number as well. So we have \( C = 1 \).

Now we consider the remaining case \(-i(s + t - \overline{s} - \overline{t}) = 0 \). Since \((s, t) \in \text{SL}(\mathbb{C} \times \mathbb{C}) \), it is clear that both \( s + t \) and \( \overline{s} - t \) are nonzero real numbers, and

\[
(s + t)(\overline{s} - t) = 1 > 0.
\]
Note that
\[ \text{Re} \left( \frac{s + t}{s} \right) > 0 \quad \text{and} \quad \text{Re} \left( \frac{\bar{s} - t}{\bar{s}} \right) > 0. \]
So we have
\[ C = \sqrt{\frac{1}{s + t}} \sqrt{\frac{1}{\bar{s} - t}} = \text{sgn}(s + t). \]
This proves (c) and completes the proof of the corollary. \(\square\)

We encountered the possibility of an additional minus sign in the unitary representation (sometimes called the composition formula in the literature) of the special linear group \(\text{SL}(\mathbb{C} \times \mathbb{C})\). This problem is known as the metaplectic sign problem and is carefully studied in [18, Sect. 9.1.4].

Recall that for each fixed \((s, t) \in \text{SL}(\mathbb{C} \times \mathbb{C})\) with \(|\text{Re} s| < 1\) we found a sequence \(\{\lambda_n\}\) of eigenvalues for the unitary operator \(T(s, t) : F^2 \rightarrow F^2\) with the corresponding eigenfunctions
\[ f_n(z) = Q_n(z)e^{\frac{z^2}{2}}, n = 0, 1, 2, \ldots. \]
Under the inverse Bargmann transform, these eigenfunctions change as follows:
\[ B^{-1}(f_n)(x) = C \int_{\mathbb{C}} \exp \left[ \frac{2ix\tau - x^2 - \frac{\tau^2}{2}}{2} \right] d\lambda(z) \cdot \int_{\mathbb{R}} H_n(t/\rho) \exp \left[ -\frac{1}{1 + \gamma} t^2 - \frac{1}{2} z^2 + 2tz - t^2 \right] dt \]
\[ = CH_n(x/\rho) \exp \left[ -\frac{1}{1 + \gamma} x^2 \right], \]
where \(H_n\) are the Hermite polynomials and \(C = (2/\pi)^{1/4}\). In view of (20), we have \(|\text{Re} s| < 1\) if and only if \(|a + d| < 2\). See [14] for related work on linear canonical transforms.

As another consequence of Theorem 15, we obtain the equivalent form, via the Bargmann transform, of several linear canonical transforms on \(F^2\).

First consider the fractional Fourier transform, which is one of the most valuable and powerful tools in mathematics, quantum mechanics, optics and signal processing. There are several well-accepted normalizations for the fractional Fourier transform. We will chose the following definition. For \(\alpha \in [-\pi, \pi]\) (or for any real number \(\alpha\)), we define the \(\alpha\)th order fractional Fourier transform by
\[ F^\alpha(f)(x) = \frac{1 - i \cot \alpha}{\sqrt{\pi}} e^{ix^2 \cot \alpha} \int_{\mathbb{R}} e^{-i(2xt \csc \alpha - t^2 \cot \alpha)} f(t) dt. \]
Several special cases are worth mentioning. First, the case \(\alpha = 0\) needs to be interpreted as a limit, and as such, it is just the identity operator (this is partial justification for the particular normalization above). When \(\alpha = \pm \pi\), we also need to interpret the corresponding transform as a limit, and an elementary calculation shows that \(F^{\pm \pi}(f)(x) = f(-x)\), which is also called the parity transform. The most prominent special cases are when \(\alpha = \pm \pi/2\): \(F^{\pi/2}\) is the classical Fourier
transform and $\mathcal{F}^{-\pi/2}$ is the inverse Fourier transform. See [4, 13] for more information about fractional Fourier transforms.

In the framework of linear canonical transforms, we have

$$\mathcal{F}^\alpha = e^{i\alpha/2} \mathcal{F} A_\alpha, \quad \alpha \in [-\pi, \pi),$$

(21)

where

$$A_\alpha = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}.$$ 

This leads to the following result, which can be found in [9].

**Corollary 17.** For any $\alpha \in [-\pi, \pi]$ the operator

$$T = B \mathcal{F}^\alpha B^{-1} : F^2 \to F^2$$

is given by $T f(z) = f(e^{-i\alpha} z)$ for all $f \in F^2$. Consequently, the operator

$$T^{-1} = B (\mathcal{F}^\alpha)^{-1} B^{-1} : F^2 \to F^2,$$

where $(\mathcal{F}^\alpha)^{-1}$ is the inverse fractional Fourier transform, is given by $T^{-1} f(z) = f(e^{i\alpha} z)$ for all $f \in F^2$.

**Proof.** First notice that $\varphi(A) = (e^{i\alpha}, 0)$ for the real rotation matrix $A$ above in view of (20). So it follows from (19), (21) and Theorem 15 that

$$B \mathcal{F}^\alpha B^{-1} f(z) = \int_C f(w) \exp \left[ e^{-i\alpha} z \overline{w} \right] d\lambda(w) = f(e^{-i\alpha} z).$$

It is then clear that $B (\mathcal{F}^\alpha)^{-1} B^{-1} f(z) = f(e^{i\alpha} z)$. \hfill \Box

Recall that for any positive $r$ the (weighted) dilation operator $D_r : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by $D_r f(x) = \sqrt{r} f(rx)$. We have $D_r = \mathcal{F} A_r$ with

$$A_r = \begin{bmatrix} 1/r & 0 \\ 0 & r \end{bmatrix},$$

and the linear canonical transform $\mathcal{F} A_r$ in this case is also called scaling.

**Corollary 18.** For any positive $r$ the Bargmann transform takes the operator $D_r$ to the following unitarily equivalent form on the Fock space:

$$T_r f(z) = B D_r B^{-1} f(z)$$

$$= \sqrt{\frac{2r}{1 + r^2}} \int_C f(w) \exp \left[ \frac{2r}{1 + r^2} z \overline{w} + \frac{1 - r^2}{2(1 + r^2)} (z^2 - \overline{w}^2) \right] d\lambda(w).$$

**Proof.** This follows from Theorem 15 and the observation that

$$\varphi(A_r) = \left( \frac{1 + r^2}{2r}, \frac{1 - r^2}{2r} \right).$$

For a positive parameter $b$, the matrix

$$A_b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$
in $SL(2, \mathbb{R})$ gives rise to the following linear canonical transform:

$$F^{A_b}(f)(x) = \frac{1}{\sqrt{2\pi b}} \int_{\mathbb{R}} e^{i(x-t)^2/2b} f(t) \, dt.$$ 

This is called the Fresnel transform or the Gauss-Weierstrass transform. It is also called the chirp convolution.

**Corollary 19.** With the above notation, the Bargmann transform takes the operator $F^{A_b}$ on $L^2(\mathbb{R})$ to the following unitarily equivalent form on $F^2$:

$$T_b f(z) = B F^{A_b} B^{-1} f(z)$$

$$= \sqrt{\frac{2}{2+ib}} \int_{\mathbb{C}} f(w) \exp \left[ z\overline{w} + \frac{bi}{2(2+bi)}(z - \overline{w})^2 \right] d\lambda(w).$$

**Proof.** This follows from Theorem 15 and the fact that

$$\varphi(A_b) = \left( \frac{2+ib}{2}, \frac{ib}{2} \right).$$

Finally in this section, recall that the chirp multiplication (or multiplication by a Gaussian) is the linear canonical transform $F^{A_c}$ on $L^2(\mathbb{R})$ corresponding to

$$A_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}.$$ 

More specifically, $F^{A_c}(f)(x) = e^{icx^2} f(x)$.

**Corollary 20.** The Bargmann transform sends the chirp multiplication $F^{A_c}$ on $L^2(\mathbb{R})$ above to the following unitarily equivalent form on the Fock space $F^2$:

$$T_c f(z) = B F^{A_c} B^{-1} f(z)$$

$$= \sqrt{\frac{2}{2-ic}} \int_{\mathbb{C}} f(w) \exp \left[ z\overline{w} + \frac{ic}{2(2-ic)}(z + \overline{w})^2 \right] d\lambda(w).$$

**Proof.** This is a consequence of Theorem 15 and the identity

$$\varphi(A_c) = \left( \frac{2-ic}{2}, \frac{ic}{2} \right).$$

In the case of fractional Fourier transforms, the Bargmann transform takes a very complicated integral transform on $L^2(\mathbb{R})$ to an extremely simple operator on $F^2$. On the other hand, the Bargmann transform takes the very simple dilation and chirp multiplication on $L^2(\mathbb{R})$ to relatively complicated integral operators on $F^2$. This is a good illustration of the usefulness of the Bargmann transform (and its inverse): it can sometimes be used to simplify operators on $L^2(\mathbb{R})$ and it can sometimes also be used to simplify operators on $F^2$. 


REFERENCES

[1] V. Bargmann, On unitary ray representations of continuous groups, *Ann. Math.* **59** (1954), 1-46.
[2] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform I, *Comm. Pure Appl. Math.* **14** (1961), 187-214.
[3] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform II, *Comm. Pure Appl. Math.* **20** (1967), 1-101.
[4] A. Bultheel and H. Martínez, Recent developments in the theory of the fractional Fourier transforms and linear canonical transforms. *Bull. Belg. Math. Soc. Simon Stevin* **13** (2007), 971-1005.
[5] C. Berger and L. Coburn, Toeplitz operators and quantum mechanics, *J. Funct. Anal.* **68** (1986), 273-299.
[6] C. Berger and L. Coburn, Toeplitz operators on the Segal-Bargmann space, *Trans. Amer. Math. Soc.* **301** (1987), 813-829.
[7] C. Berger and L. Coburn, Heat flow and Berezin-Toeplitz estimates, *Amer. J. Math.* **116** (1994), 563-590.
[8] G. Cao, J. Li, M. Shen, B.D. Wick and L. Yan, A Boundedness Criterion for Singular Integral Operators of convolution type on the Fock Space. *Adv. Math.* **363** (2020), 107001.
[9] X. Dong and K. Zhu, The Fourier and Hilbert transforms under the Bargmann transform, *Complex Variables and Elliptic Equations*, **63** (2018), 517-531.
[10] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
[11] J. Healy, M. Kutay, H. Ozaktas, and J. Sheridan, Linear Canonical Transforms Theory and Applications, Springer, New York, 2016.
[12] G. Mackey, Unitary representations of group extensions, *Acta Math.* **99** (1958), 265-311.
[13] H. Ozaktas, Z. Zalevsky, and M.A. Kutay, The fractional Fourier transform, Wiley, Chichester, 2001.
[14] S. Pei and J. Ding. Eigenfunctions of linear canonical transform. *IEEE Trans. Sig. Proc.*, **50** (2002), 11-26.
[15] K. Seip, Density theorems for sampling and interpolation in the Bargmann-Fock space I, *J. Reine Angew. Math.* **429** (1992), 91-106.
[16] K. Seip and R. Wallsten, Density theorems for sampling and interpolation in the Bargmann-Fock space, *J. Reine Angew. Math.* **429** (1992), 107-113.
[17] B. Wick and S. Wu, Integral Operators on Fock-Sobolev Spaces via Multipliers on Gauss-Sobolev Spaces, *Integr. Equat. Oper. Theory* **94** (2022), no. 2, Paper No. 22, 24 pp.
[18] K.B. Wolf, Integral Transforms in Science and Engineering, Plenum Press, New York, 1979.
[19] K. Zhu, Analysis on Fock Spaces, Springer, New York, 2012.
[20] K. Zhu, Singular integral operators on the Fock space, *Integr. Equat. Oper. Theory*, **81** (2015), 451-454.
[21] K. Zhu, Towards a dictionary for the Bargmann transform, in: *Handbook of Analytic Operator Theory*, Chapman and Hall/CRC, 2019.