FAMILIES OF IRREDUCIBLE SINGULAR GELFAND-TSETLIN MODULES OF $\mathfrak{gl}(n)$

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ABSTRACT. We prove a conjecture for the irreducibility of singular Gelfand-
Tsetlin modules announced in [9]. We describe explicitly the irreducible sub-
quotients of certain classes of singular Gelfand-Tsetlin modules.

1. Introduction

In 1950, I. Gelfand and M. Tsetlin [11] constructed a basis for any irreducible
finite-dimensional $\mathfrak{gl}(n)$-module. These bases are parameterized by the so-called
Gelfand-Tsetlin tableaux, whose entries satisfy some integer relations. Observing
that the coefficients in the Gelfand-Tsetlin formulas are rational functions on the
entries of the tableaux, it is natural to extend the Gelfand-Tsetlin construction to
more general modules. For a generic tableau $T(L)$ (no integer relations between
elements of the same row) the corresponding module was constructed in [2] and ex-
licit bases for their irreducible subquotients were given in [7] providing explicitly
new irreducible modules for $\mathfrak{gl}(n)$. The first attempt of generalization is the case
when there is only one row and a unique pair of entries in this row with integ-
condition (1-singularity). This cases was treated in [9]. In this construction be-
sides regular tableaux appear new tableaux which are called derivatives tableaux.
The vector space $V(T(\bar{v}))$ generated by regulars and derivatives tableaux has a
$\mathfrak{gl}(n)$-module structure. In [9], V. Futorny, D. Grantcharov and L.E. Ramirez gave
sufficient condition for irreducibility of the module $V(T(\bar{v}))$ and conjectured that
this same condition is necessary. In particular, for $n = 3$ this was shown in [7].

Conjecture: Let $n \geq 2$, $V(T(\bar{v}))$ the 1-singular $\mathfrak{gl}(n)$-module. If $V(T(\bar{v}))$ is
irreducible then the differences between elements of neighboring rows of $T(\bar{v})$ are
not integers.

In the current paper we give a positive answer to this conjecture and describe
the irreducible subquotients for certain families of 1-singular modules $V(T(\bar{v}))$.

The paper is organized as follows. In Section 2 we introduce the necessary def-
itions and notations used through the paper. In Section 3 we recall the definition
of Gelfand-Tsetlin subalgebras and Gelfand-Tsetlin modules. In the same section
we recall the Gelfand-Tsetlin theorem about realizations of irreducible finite di-
dimensional modules via Gelfand-Tsetlin tableaux. Also, we recall the construction

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of generic Gelfand-Tsetlin modules presented in [2]. In Section 4 we discuss 1-
singular Gelfand-Tsetlin modules constructed in [9]. In the following
section we define a preorder relation in the set of all tableaux (regular end deriva-
tives) and establish important properties of this relation that will be used in the
next section. Finally, in Sections 6 and 7 we establish main results in this paper;
the first result gives us an explicit basis for an irreducible subquotient of the 1-
singular module $V(T(\bar{v}))$ that contains a given tableau, and the second result gives
a positive answer for the Conjecture.

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2. Conventions and notation

The ground field will be $\mathbb{C}$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers $m$
such that $m \geq a$. We fix an integer $n \geq 2$. By $\mathfrak{gl}(n)$ we denote the general linear
Lie algebra consisting of all $n \times n$ complex matrices, and by $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ -
the standard basis of $\mathfrak{gl}(n)$ of elementary matrices. We fix the standard triangular
decomposition and the corresponding basis of simple roots of $\mathfrak{gl}(n)$. The weights of
$\mathfrak{gl}(n)$ will be written as $n$-tuples $(\lambda_1, ..., \lambda_n)$.

For a Lie algebra $a$ by $U(a)$ we denote the universal enveloping algebra of $a$.
Throughout this paper $U = U(\mathfrak{gl}(n))$. For a commutative ring $R$, by $\text{Specm } R$
we denote the set of maximal ideals of $R$.

We will write the vectors in $\mathbb{C}^{\mathbb{Z}_{n+1}}$ in the following form:

$$v = (v_{n1}, ..., v_{nn}, v_{n-1,1}, ..., v_{n-1,n-1}, \cdots, v_{21}, v_{22}, v_{11}).$$

For $1 \leq j \leq n$, $\delta^{ij} \in \mathbb{Z}^n_{\geq i}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{m\ell}$ are
zero. For $i > 0$ by $S_i$ we denote the $i$th symmetric group. By $(m, \ell)$ we denote the
transposition of $S_i$ switching $m$ and $\ell$.

3. Gelfand-Tsetlin modules

3.1. Definitions. Recall that $U = U(\mathfrak{gl}(n))$. Let for $m \leq n$, let $\mathfrak{gl}_m$
be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \mid i, j = 1, \ldots, m\}$. We have the following chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n,$$

which induces the chain $U_1 \subset U_2 \subset \ldots \subset U_n$ of the universal enveloping algebras
$U_m = U(\mathfrak{gl}_m), 1 \leq m \leq n$. Let $Z_m$ be the center of $U_m$. Then $Z_m$ is the polynomial
algebra in the $m$ variables $\{c_{mk} \mid k = 1, \ldots, m\}$,

$$c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} E_{i_1i_2}E_{i_2i_3} \cdots E_{i_ki_1}.$$  

(1)

Following [2], we call the subalgebra of $U$ generated by $\{Z_m \mid m = 1, \ldots, n\}$ the
(standard) Gelfand-Tsetlin subalgebra of $U$ and will be denoted by $\Gamma$. In fact, $\Gamma$
is the polynomial algebra in the $\frac{n(n+1)}{2}$ variables $\{c_{ij} \mid 1 \leq j \leq i \leq n\}$ ([21]). Let $\Lambda$
be the polynomial algebra in the variables $\{\lambda_{ij} \mid 1 \leq j \leq i \leq n\}$.
Let \( \iota : \Gamma \to \Lambda \) be the embedding defined by \( \iota(c_{mk}) = \gamma_{mk}(\lambda) \), where

\[
\gamma_{mk}(\lambda) := \sum_{i=1}^{m} (\lambda_{mi} + m - 1)^{k} \prod_{j \neq i} \left( 1 - \frac{1}{\lambda_{mi} - \lambda_{mj}} \right).
\]

The image of \( \iota \) coincides with the subalgebra of \( \mathcal{G} \)-invariant polynomials in \( \Lambda \), where \( \mathcal{G} := S_{n} \times \cdots \times S_{1} \) which we identify with \( \Gamma \).

**Definition 3.1.** A finitely generated \( \mathcal{U} \)-module \( M \) is called a Gelfand-Tsetlin module (with respect to \( \Gamma \)) if \( M \) splits into a direct sum of \( \Gamma \)-modules:

\[
M = \bigoplus_{m \in \Spec \mathcal{G}} M(m),
\]

where

\[
M(m) = \{ v \in M \mid m^{k}v = 0 \text{ for some } k \geq 0 \}.
\]

Identifying \( m \) with the homomorphism \( \chi : \Gamma \to \mathbb{C} \) with \( \text{Ker} \chi = m \), we will call \( m \) a Gelfand-Tsetlin character of \( M \) if \( M(m) \neq 0 \), and \( \dim M(m) \) - the Gelfand-Tsetlin multiplicity of \( m \). The Gelfand-Tsetlin support of a Gelfand-Tsetlin module \( M \) is the set of all Gelfand-Tsetlin characters of \( M \).

**Remark 3.2.** Note that any irreducible Gelfand-Tsetlin module over \( \mathfrak{gl}(n) \) is a weight module with respect to the standard Cartan subalgebra \( \mathfrak{h} \) spanned by \( E_{ii}, i = 1, \ldots, n \). In particular, every highest weight module or, more generally, every module from the category \( \mathcal{O} \) is a Gelfand-Tsetlin module.

### 3.2. Finite dimensional modules for \( \mathfrak{gl}(n) \)

In this section we recall a classical result of I. Gelfand and M. Tsetlin which provides an explicit basis for every irreducible finite dimensional \( \mathfrak{gl}(n) \)-module.

**Definition 3.3.** The following array \( T(v) \) of complex numbers \( \{ v_{ij} \mid 1 \leq j \leq i \leq n \} \)

\[
\begin{array}{ccccccc}
& v_{n1} & v_{n2} & \cdots & v_{n,n-1} & v_{nn} \\
v_{n-1,1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
v_{21} & v_{22} \\
v_{11} & \\
\end{array}
\]

is called a Gelfand-Tsetlin tableau.

A Gelfand-Tsetlin tableau \( T(v) \) is called standard if:

\[
v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{> 0}, \quad \text{for all } 1 \leq i \leq k \leq n.
\]

**Theorem 3.4 ([11]).** Let \( L(\lambda) \) be the finite dimensional irreducible module over \( \mathfrak{gl}(n) \) of highest weight \( \lambda = (\lambda_{1}, \ldots, \lambda_{n}) \). Then there exists a basis of \( L(\lambda) \) consisting of all standard tableaux \( T(v) \) with fixed top row \( v_{n1} = \lambda_{1}, v_{n2} = \lambda_{2} - 1, \ldots, v_{nn} = \lambda_{n} \).
\[ (\lambda_n - n + 1). \] Moreover, the action of the generators of \( \mathfrak{gl}(n) \) on \( L(\lambda) \) is given by the Gelfand-Tsetlin formulas:

\[
E_{k,k+1}(T(v)) = -\sum_{i=1}^{k} \left( \prod_{j=1}^{k+1} (v_{ki} - v_{k+1,j}) \right) T(v + \delta_{ki}),
\]

(3)

\[
E_{k+1,k}(T(v)) = \sum_{i=1}^{k} \left( \prod_{j=1}^{k-1} (v_{ki} - v_{k-1,j}) \right) T(v - \delta_{ki}),
\]

\[
E_{kk}(T(v)) = \left( k - 1 + \sum_{i=1}^{k} v_{ki} - \sum_{i=1}^{k-1} v_{k-1,i} \right) T(v).
\]

If the new tableau \( T(v \pm \delta_{ki}) \) is not standard, then the corresponding summand of \( E_{k,k+1}(T(v)) \) or \( E_{k+1,k}(T(v)) \) is zero by definition. Furthermore, the action of generators \( c_{rs} \) of \( \Gamma \) defined by (1) is given by,

\[
c_{rs}(T(v)) = \gamma_{rs}(v) T(v),
\]

where \( \gamma_{rs} \) are defined in (2).

3.3. Generic Gelfand-Tsetlin modules. In the case when all denominators are nonintegers, one can use the same formulas and define a new class of infinite dimensional \( \mathfrak{gl}(n) \)-modules: generic Gelfand-Tsetlin modules (cf. [2], Section 2.3).

Definition 3.5. A Gelfand-Tsetlin tableau \( T(v) \) is called generic if \( v_{rs} - v_{ru} \notin \mathbb{Z} \) for each \( 1 \leq s < u \leq r \leq n - 1 \).

Theorem 3.6 ([2] and Theorem 2 in [21]). Let \( T(v) \) be a generic Gelfand-Tsetlin tableau. Denote by \( V(T(v)) \) the vector space with basis consisting of all Gelfand-Tsetlin tableaux \( T(w) \) satisfying \( w_{nj} = v_{nj}, w_{ij} - v_{ij} \in \mathbb{Z} \) for \( 1 \leq j \leq i \leq n - 1 \).

(i) The vector space \( V(T(v)) \) has a structure of a \( \mathfrak{gl}(n) \)-module with action of the generators of \( \mathfrak{gl}(n) \) given by the Gelfand-Tsetlin formulas (3). The module \( V(T(v)) \) has finite length.

(ii) The action of the generators of \( \Gamma \) on the basis elements of \( V(T(v)) \) is given by (4).

(iii) The module defined in (i) is a Gelfand-Tsetlin module all of whose Gelfand-Tsetlin multiplicities are 1.

(iv) The action of the generators of \( \Gamma \) is given by (4).

We will denote the module constructed in Theorem 3.6 by \( V(T(v)) \). Note that \( V(T(v)) \) need not to be irreducible. Because \( \Gamma \) has simple spectrum on \( V(T(v)) \) for \( T(w) \) in \( V(T(v)) \) we may define the irreducible \( \mathfrak{gl}(n) \)-module in \( V(T(v)) \) containing \( T(w) \) to be the subquotient of \( V(T(v)) \) containing \( T(w) \) (see Theorem 3.6(i)). A basis for the irreducible subquotients of \( V(T(v)) \) can be described in terms of the following set.

\[
\Omega^+(T(w)) := \{(r, s, u) \mid w_{rs} - w_{r-1,u} \in \mathbb{Z}_{\geq 0}\}.
\]

Theorem 3.7 (Theorem 6.14, in [8]). Let \( T(v) \) be a generic tableau and let \( T(w) \) be a tableau in \( V(T(v)) \). Then the following hold.

(i) The submodule of \( V(T(v)) \) generated by \( T(w) \) has basis

\[
\mathcal{N}(T(w)) := \{T(w') \in V(T(v)) \mid \Omega^+(T(w)) \subseteq \Omega^+(T(w'))\};
\]
(ii) The irreducible \( \mathfrak{gl}(n) \)-module in \( V(T(v)) \) containing \( T(w) \) has basis
\[
\mathcal{I}(T(w)) := \{ T(w') \in V(T(v)) \mid \Omega^+(T(w)) = \Omega^+(T(w')) \}.
\]
The action of \( \mathfrak{gl}(n) \) on both \( \mathcal{N}(T(w)) \) and \( \mathcal{I}(T(w)) \) is given by the Gelfand-Tsetlin formulas.

3.4. Gelfand-Tsetlin formulas in terms of permutations. In this subsection we rewrite and generalize the Gelfand-Tsetlin formulas in Theorem 3.4 in convenient for us terms.

Recall the convention that for a vector \( v = (v_{n1}, ..., v_{11}) \) in \( \mathbb{C}^{n(n+1)/2} \), by \( T(v) \), we denote the corresponding to \( v \) Gelfand-Tsetlin tableau. Let us call \( v \) in \( \mathbb{C}^{n(n+1)/2} \) generic if \( T(v) \) is a generic Gelfand-Tsetlin tableau, and denote by \( \mathbb{C}^{n(n+1)/2}_\text{gen} \) the set of all generic vectors in \( \mathbb{C}^{n(n+1)/2} \).

Remember that \( G = S_n \times \cdots \times S_1 \). Let \( \mathcal{S}_m \) denotes the subset of \( S_m \) consisting of the transpositions \( (1, i) \), \( i = 1, ..., m \). Also, we consider every \( \sigma \in \mathcal{S}_m \) as an element of \( G \) by letting \( \sigma[t] = \text{Id} \) whenever \( t \neq m \).

**Definition 3.8.** Let \( 1 \leq r \leq n-1 \). Set
\[
\varepsilon_{r,r+1} := \delta^{r-1} \in \mathbb{Z}^{n(n+1)/2}.
\]
Furthermore, define \( \varepsilon_{rr} = 0 \) and \( \varepsilon_{r+1,r} = -\delta^{r-1} \).

**Definition 3.9.** For each generic vector \( w \) and any \( 1 \leq r, s \leq n \) we define
\[
e_{r,s+1}(w) := \prod_{j=1}^{r+1} (w_{r_{1,j}} - w_{r_{r+1,j}}) / \prod_{j=1}^{r} (w_{r_{1,j}} - w_{r_{r,j}}),
\]
\[
e_{r+1,r}(w) := \prod_{j=1}^{r-1} (w_{r_{1,j}} - w_{r_{k-1,j}}) / \prod_{j=1}^{r} (w_{r_{1,j}} - w_{r_{r,j}}),
\]
\[
e_{rr}(w) := r - 1 + \sum_{i=1}^{r} w_{r_{1,i}} - \sum_{i=1}^{r-1} w_{r_{1,i}}.
\]

By using permutations we can rewrite the Gelfand-Tsetlin formulas as follows:

**Proposition 3.10.** Let \( v \in \mathbb{C}^{n(n+1)/2}_\text{gen} \). The Gelfand-Tsetlin formulas for the generic Gelfand Tsetlin \( \mathfrak{gl}(n) \)-module \( V(T(v)) \) can be written as follows:
\[
E_{\ell m}(T(v + z)) = \sum_{\sigma \in \mathcal{S}_r} e_{\ell m}(\sigma(v + z)) T(v + z + \sigma(\varepsilon_{\ell m})),
\]
where \( (\ell, m) \in \{(r, r+1), (r+1, r), (r, r)\} \) and \( z \in \mathbb{Z}^{n(n+1)/2} \) has top row zero.

4. Singular Gelfand-Tsetlin modules

In this section we will remember the construction of singular Gelfand-Tsetlin modules given in [9].

**Definition 4.1.** A vector \( v \in \mathbb{C}^{n(n+1)/2}_\text{gen} \) will be called singular if there exist \( 1 \leq s < t \leq n \) such that \( v_{rs} - v_{rt} \in \mathbb{Z} \). The vector \( v \) will be called 1-singular if there exist \( i, j, k \) with \( 1 \leq i < j \leq k \leq n-1 \) such that \( v_{ki} - v_{kj} \in \mathbb{Z} \) and \( v_{rs} - v_{rt} \notin \mathbb{Z} \) for all \((r, s, t) \neq (k, i, j)\).
From now on we fix \((i, j, k)\) such that \(1 \leq i < j \leq k \leq n - 1\). In [9], associated with any 1-singular tableau \(T(\bar{v})\) is constructed a Gelfand-Tsetlin module \(V(T(\bar{v}))\) and explicit formulas for the action of the generators of \(\mathfrak{gl}(n)\) and the generators of \(\Gamma\) is given, in this section we will remember that construction.

Let us fix a 1-singular vector \(\bar{v}\) such that \(\bar{v}_{ki} - \bar{v}_{kj} = 0\). From now on by \(\tau\) we denote the element \((\tau[n], \ldots, \tau[2], \tau[1])\) in \(G\) such that \(\tau[k]\) is the transposition \((i, j)\) and all other \(\tau[l]\) are \(\text{Id}\). We formally introduce new tableaux \(DT(\bar{v} + w)\) for every \(w \in \mathbb{Z}_{\frac{n(n-1)}{2}}\) subject to the relations \(DT(\bar{v} + w) + DT(\bar{v} + \tau(w)) = 0\). We call \(DT(\bar{v} + w)\) the derivative Gelfand-Tsetlin tableau associated with \(w\).

**Definition 4.2.** We set \(V(T(\bar{v}))\) to be the vector space spanned by the set of tableaux \(\{T(\bar{v} + w), DT(\bar{v} + w) \mid w \in \mathbb{Z}_{\frac{n(n-1)}{2}}\}\), subject to the relations \(T(\bar{v} + w) = T(\bar{v} + \tau(w))\) and \(DT(\bar{v} + w) + DT(\bar{v} + \tau(w)) = 0\). We also fix a basis of \(V(T(\bar{v}))\) to be the set \(\{\text{Tab}(w) : w \in \mathbb{Z}_{\frac{n(n-1)}{2}}\}\), where

\[
\text{Tab}(w) := \begin{cases} T(\bar{v} + w), & \text{if } w_{ki} - w_{kj} \leq 0 \\ DT(\bar{v} + w), & \text{if } w_{ki} - w_{kj} > 0 \end{cases}
\]

For a variables vector \(v\) and \(f\) is a rational function on variables \(v_{rs}\) which is smooth on the hyperplane \(v_{ki} - v_{kj} = 0\) we can define the linear map

\[
D^v(fT(v + z)) = D^v(f)T(\bar{v} + z) + f(\bar{v})DT(\bar{v} + z),
\]

where \(D^v(f) = \frac{1}{2}(\frac{\partial f}{\partial v_{ki}} - \frac{\partial f}{\partial v_{kj}})(\bar{v})\). The following lemma will be useful in order to do some computations.

**Lemma 4.3.** Let \(f\) be a rational function on variables \(v_{rs}\) smooth on the hyperplane \(v_{ki} - v_{kj} = 0\). Then,

(i) \(D^v((v_{ki} - v_{kj})f) = f(\bar{v})\).

(ii) If \(f\) is symmetric with respect to \(v_{ki}\) and \(v_{kj}\) then, \(D^v(f) = 0\).

**Theorem 4.4 (9) Theorems 4.11 and 4.12.** \(V(T(\bar{v}))\) is an 1-singular Gelfand-Tsetlin \(\mathfrak{gl}(n)\)-module, with action of the generators of \(\mathfrak{gl}(n)\) given by

\[
E_{rs}(T(\bar{v} + z)) = D^v((v_{ki} - v_{kj})E_{rs}(T(v + z)))
\]

\[
E_{rs}(DT(\bar{v} + z')) = D^v(E_{rs}(T(v + z'))),
\]

and the action of the generators of \(\Gamma\) can be written explicitly as follows:

\[
c_{rs}(T(\bar{v} + z)) = \gamma_{rs}(\bar{v} + z)T(\bar{v} + z)
\]

\[
c_{rs}(DT(\bar{v} + z')) = \gamma_{rs}(\bar{v} + z')DT(\bar{v} + z') + D^\gamma(\gamma_{rs}(v + z'))T(\bar{v} + z')
\]

for any \(z, z' \in \mathbb{Z}_{\frac{n(n-1)}{2}}\) with \(z' \neq \tau(z')\).

For some of the generators of \(\mathfrak{gl}(n)\) the action on \(V(T(\bar{v}))\) coincide with the classical Gelfand-Tsetlin formulas. The following corollary gives some sufficient conditions in order to have this property.

**Corollary 4.5.** Let \(z\) be any element of \(\mathbb{Z}_{\frac{n(n-1)}{2}}\).

(i) For any \(\ell, m\) such that \(k < \min\{\ell, m\}\) or \(\max\{\ell, m\} \leq k\) we have

\[
E_{\ell m}(T(\bar{v} + z)) = \sum_{\sigma \in \Phi_{\ell m}} e_{\ell m}(\sigma(\bar{v} + z))\bar{T}(\bar{v} + z + \sigma(\bar{e}_{\ell m}))
\]
Remark 4.6. Note that \( c \).\n
Lemma 5.1. As the polynomials \( \gamma \) we have \( \gamma(T(\bar{v}))) = 0 \). Now, by (7) we have the equality.

(ii) Under these conditions on \( r, s \) the function \( e_{rs}(\sigma(v + z)) \) is symmetric with respect to \( v_{ki} \) and \( v_{kj} \) for any \( \sigma \in \Phi_{rs} \). Therefore, by Lemma 4.3, \( D^v e_{rs}(\sigma(v + z)) = 0 \) and then, by (8) we have the desired equality.

Remark 4.7. As the polynomials \( \{ \gamma_{rs}(v) \}_{1 \leq s \leq r \leq n} \) are symmetric in the entries of \( v \) and generated an symmetric polynomials, we have \( \gamma_{rs}(v) = \gamma_{rs}(v') \) for any \( r, s \) if only if, \( v = \sigma(v') \) for some \( \sigma \in G \). In particular, for the 1-singular vector \( \bar{v} \) we have \( \gamma_{rs}(\bar{v} + z) = \gamma_{rs}(\bar{v} + w) \) for any \( 1 \leq s \leq r \leq n \) if, and only if, \( w = z \) or \( w = \tau(z) \).

Lemma 4.8. Assume \( \tau(z) \neq z \). Then:

(i) \((c_k - \gamma_k(\bar{v} + z))T(\bar{v} + z) = 0\).

(ii) \((c_{k2} - \gamma_{k2}(\bar{v} + z))DT(\bar{v} + z) = D^v(\gamma_{k2}(\bar{v} + z))T(\bar{v} + z) \) with \( D^v(\gamma_{k2}(\bar{v} + z)) \neq 0 \).

(iii) \((c_k - \gamma_k(\bar{v} + z))^2DT(\bar{v} + z) = 0\).

5. \( \Gamma \) separates tableaux in \( V(T(\bar{v})) \)

One essential property of generic Gelfand-Tsetlin modules described in Theorem 3.6 is that for any different tableaux in \( V(T(\bar{v})) \) there exists an element \( \gamma \) of \( \Gamma \) that separates those tableaux (i.e. the action of \( \gamma \) has different eigenvalues on this two tableaux). In this section we will give a detailed prove of this fact for any 1-singular module \( V(T(\bar{v})) \). Remember that we fix a basis \( \{ Tab(z) \mid z \in \mathbb{Z}^{n(n-1)} \} \) for \( V(T(\bar{v})) \).

For any \( 1 \leq s \leq r \leq n \) and \( z \in \mathbb{Z}^{n(n-1)} \) we will denote by \( C_{rs}(z) \) the element \( c_{rs} - \gamma_{rs}(\bar{v} + z) \) of \( \Gamma \).

Lemma 5.1. For any \( 1 \leq s \leq r \leq n \), any \( z \in \mathbb{Z}^{n(n-1)} \) and \( \chi \in \text{Supp}_{GT}(V(T(\bar{v}))) \), the subspace \( V(T(\bar{v}))(\chi) \) is \( C_{rs}(z) \)-invariant.
Proof. By Remark 4.7, any Gelfand-Tsetlin subspace $V(T(\bar{v}))(\chi)$ is generated by $T(\bar{v} + w)$ and $DT(\bar{v} + w)$ for some $w \in \mathbb{Z}^{(n+1)/(2)}$. Now,

$$C_{rs}(z)T(\bar{v} + w) = (\gamma_{rs}(\bar{v} + w) - \gamma_{rs}(\bar{v} + z))T(\bar{v} + w)$$

$$C_{rs}(z)DT(\bar{v} + w) = D^\psi(\gamma_{rs}(\bar{v} + z))T(\bar{v} + w) + (\gamma_{rs}(\bar{v} + w) - \gamma_{rs}(\bar{v} + z))DT(\bar{v} + w).$$

\[\square\]

Definition 5.2. Given $z, w \in \mathbb{Z}^{(n+1)/(2)}$, we write $\text{Tab}(z) \prec \text{Tab}(w)$ if, and only if, there exists $u \in U(\mathfrak{gl}(n))$ such that $\text{Tab}(w)$ appears with nonzero coefficient in the decomposition of $u \cdot \text{Tab}(z)$ as a linear combination of tableaux.

Lemma 5.3. Let $w \in \mathbb{Z}^{(n+1)/(2)}$ be such that $w \neq \tau(w)$, then $DT(\bar{v} + w) \prec T(\bar{v} + w)$.

Proof. By Lemma 5.2(ii), we have $C_{rs}(w)DT(\bar{v} + w) = D^\psi(\gamma_{rs}(v + w))T(\bar{v} + w)$. As $w \neq \tau(w)$, $D^\psi(\gamma_{rs}(v + w)) \neq 0$ and then $DT(\bar{v} + w) \prec T(\bar{v} + w)$.

Lemma 5.4. Let $z, w \in \mathbb{Z}^{(n+1)/(2)}$ be such that $w \neq z$ and $w \neq \tau(z)$. There exists $\gamma_z^w \in \Gamma$ such that

$$\gamma_z^w \cdot T(\bar{v} + z) = \gamma_z^w \cdot DT(\bar{v} + z) = 0$$

and $\gamma_z^w \cdot \text{Tab}(w) = \text{Tab}(w)$

Proof. Let us fix $r$ and $s$ such that $\gamma_{rs}(\bar{v} + w) \neq \gamma_{rs}(\bar{v} + z)$ (such $r, s$ exist because of Remark 4.7). Set $a := \gamma_{rs}(\bar{v} + w) - \gamma_{rs}(\bar{v} + z) \neq 0$, by a direct computation we have the following identities:

(i) $C_{rs}(z)T(\bar{v} + z) = 0 = C^2_{rs}(z)DT(\bar{v} + z)$.

(ii) $C_{rs}(z)DT(\bar{v} + z) = D^\psi(\gamma_{rs}(v + z))T(\bar{v} + z)$.

(iii) $C_{rs}(z)DT(\bar{v} + w) = aDT(\bar{v} + w)$

(iv) $C_{rs}(z)DT(\bar{v} + w) = D^\psi(\gamma_{rs}(v + w))T(\bar{v} + w) + aDT(\bar{v} + w)$.

Now, we have two cases:

Case 1. Suppose $\text{Tab}(w) = T(\bar{v} + w)$. In this case we take $\gamma_z^w = \frac{1}{a}C^2_{rs}(z)$.

Case 2. Suppose that $\text{Tab}(w) = DT(\bar{v} + w)$. In this case we have two possibilities, namely:

(i) $D^\psi(\gamma_{rs}(v + w)) = 0$. In this case, from (v) we have $C^2_{rs}(z) \cdot DT(\bar{v} + w) = a^2DT(\bar{v} + w)$. So, we can take $\gamma_z^w = \frac{1}{a}C^2_{rs}(z)$.

(ii) $D^\psi(\gamma_{rs}(v + w)) \neq 0$. By Lemma 4.8(ii), $D^\psi(\gamma_{k2}(v + w)) \neq 0$ and

$$\begin{cases} C_{k2}(w)T(\bar{v} + w) = 0, \\ C_{k2}(w)DT(\bar{v} + w) = D^\psi(\gamma_{k2}(v + w))T(\bar{v} + w). \end{cases}$$

So, applying $C_{k2}(w)$ to the equality (vi), we have:

$$C_{k2}(w)C^2_{rs}(z)DT(\bar{v} + w) = a^2D^\psi(\gamma_{k2}(v + w))T(\bar{v} + w)$$

Now, replacing Equality (i) in (vi), we have:

$$\begin{equation} (1 - \frac{2C_{k2}(w)}{aD^\psi(\gamma_{k2}(v + w))}) C^2_{rs}(z)DT(\bar{v} + w) = a^2DT(\bar{v} + w). \end{equation}$$

Therefore, in this case we can consider $\gamma_z^w = \frac{1}{a^2} \left(1 - \frac{2C_{k2}(w)}{aD^\psi(\gamma_{k2}(v + w))}\right) C^2_{rs}(z)$. 

\[\square\]
Summarizing, we have:

$$\gamma^w_z = \begin{cases} \frac{1}{a}C^2_{rs}(z), & \text{if } Tab(w) = T(\bar{v} + w) \text{ or } D^v(\gamma_{rs}(v + w)) = 0 \\ \frac{1}{a} \left(1 - \frac{2Cz^2(w)}{aD^v(\gamma_{rz}(v + w))}\right) C^2_{rs}(z), & \text{if } D^v(\gamma_{rs}(v + w)) \neq 0 \end{cases}$$

\[\square\]

**Remark 5.5.** Note that each $\gamma^w_z$ on the previous lemma is a combination of products of elements of $\Gamma$ of the form $C_{lm}(w')$. So, by Lemma 5.4, the Gelfand-Tsetlin subspaces $V(T(\bar{v}))((\chi))$ are $\gamma^w_z$-invariant.

**Lemma 5.6.** Let us consider $T := aT(\bar{v} + w) + bDT(\bar{v} + \tau(w))$, where $w \in \mathbb{Z}^{n(n-1)/2}$ is such that $w \notin \tau(w)$ and $a, b \in \mathbb{C}$. Then,

(i) If $a \neq 0$, then $\gamma_1 \cdot T = T(\bar{v} + w)$ for some $\gamma_1 \in \Gamma$.

(ii) If $b \neq 0$, then $\gamma_2 \cdot T = DT(\bar{v} + \tau(w))$ for some $\gamma_2 \in \Gamma$.

**Proof.** Let $\gamma_1, \gamma_2 \in \Gamma$ defined by

$$\gamma_1 = \begin{cases} \frac{1}{a}, & \text{if } b = 0 \\ \frac{1}{b}, & \text{if } b \neq 0 \end{cases}, \quad \gamma_2 = \begin{cases} \frac{1}{b}, & \text{if } a = 0 \\ \frac{1}{b} \left(1 - \frac{2Cz^2(w)}{aD^v(\gamma_{rz}(v + w))}\right), & \text{if } a \neq 0 \end{cases}$$

First we note that by Lemma 1.3, the denominators of $\gamma_1, \gamma_2$ are not zero. The rest of the proof is a straightforward verification \[\square\]

**Theorem 5.7.** If $z, w \in \mathbb{Z}^{n(n-1)/2}$ are such that $Tab(z) \prec Tab(w)$ then, there exist $u \in U(\mathfrak{gl}(n))$ such that $u \cdot Tab(z) = Tab(w)$.

**Proof.** As $Tab(z) \prec Tab(w)$, there exist $u^0 \in U(\mathfrak{gl}(n))$ such that $u^0 \cdot Tab(z)$ can be written as follows

$$\sum_{i=0}^s a_i T(\bar{v} + w_i) + b_i DT(\bar{v} + \tau(w_i)) \in V(T(\bar{v}))(\chi_0) \cdots \oplus V(T(\bar{v}))(\chi_s)$$

where $w_i \neq w_j, \tau(w_i)$ for any $i \neq j$ and $w_0 = w$ or $w_0 = \tau(w)$, $a_0 \neq 0$ or $b_0 \neq 0$, and $\chi_l$ the Gelfand-Tsetlin character associated with $w_l$. By Lemma 5.4, for each $j \in \{1, 2 \cdots, s\}$, there exist $\gamma^w_{w_j} \in \Gamma$ such that

$$\gamma^w_{w_j} T(\bar{v} + w_j) = \gamma^w_{w_j} DT(\bar{v} + w_j) = 0 \text{ and } \gamma^w_{w_j} Tab(w_0) = Tab(w_0)$$

Then, by Remark 5.5, applying $\gamma := \gamma^w_{w_0} \cdots \gamma^w_{w_1}$ to $u^0 \cdot Tab(z)$, we have

$$\gamma u^0 \cdot Tab(z) = \gamma^w_{w_0} \cdots \gamma^w_{w_1} \left(\sum_{i=1}^s a_i T(\bar{v} + w_i) + b_i DT(\bar{v} + \tau(w_i))\right) \in V(T(\bar{v}))(\chi_0)$$

So, $\gamma u \cdot Tab(z) = aT(\bar{v} + w) + bDT(\bar{v} + \tau(w))$ for some $a, b \in \mathbb{C}$. Let us see the relation between the coefficients $a$, $b$ and $a_0$, $b_0$.

**Case 1.** Suppose $Tab(w) = T(\bar{v} + w)$. In this case, for any $j = 1, \ldots, s$, we have $\gamma^w_{w_j} T(\bar{v} + w) = T(\bar{v} + w)$ (by construction of $\gamma^w_{w_j}$) and by Remark 5.5 $\gamma^w_{w_j} DT(\bar{v} + w) = DT(\bar{v} + w)$. Therefore,

$$\gamma u \cdot Tab(z) = a_0 T(\bar{v} + w) + b_0 DT(\bar{v} + \tau(w)).$$

Now, as $a_0 \neq 0$, by Lemma 5.6 there exist $\gamma_1 \in \Gamma$ such that

$$\gamma_1 \gamma u \cdot Tab(z) = \gamma_1 (a_0 T(\bar{v} + w) + b_0 DT(\bar{v} + \tau(w))) = T(\bar{v} + w).$$
Case 2. Suppose $\text{Tab}(w) = DT(\bar{v} + w)$. In this case, for any $j = 1, \ldots, s$, we have $\gamma^{w}_{w_{j}}DT(\bar{v} + w) = DT(\bar{v} + w)$ (by construction of $\gamma^{w}_{w_{j}}$ and $\gamma^{w}_{w_{j}}T(\bar{v} + w) = \alpha_{j}T(\bar{v} + w)$ for some $\alpha_{j} \in \mathbb{C}$. Therefore,

$$\gamma u \cdot \text{Tab}(z) = aT(\bar{v} + w) + b_{0}DT(\bar{v} + \tau(w)),$$

with $a = \alpha_{s} \cdots \alpha_{1}a_{0}$ and $b_{0} \neq 0$. By Lemma 5.6 there exist $\gamma_{2} \in \Gamma$ such that $\gamma_{2}\gamma u \cdot \text{Tab}(z) = \gamma_{2}(aT(\bar{v} + w) + b_{0}DT(\bar{v} + \tau(w))) = DT(\bar{v} + w)$.

**Corollary 5.8.** The relation \(\prec\) define a preorder on the set of tableaux $B(T(\bar{v}))$ (i.e. \(\prec\) is a reflexive and transitive relation).

**Proof.** Reflexivity is clear from the definition of \(\prec\). For transitivity, assume that $\text{Tab}(w_{1}) \prec \text{Tab}(w_{2})$ and $\text{Tab}(w_{2}) \prec \text{Tab}(w_{3})$ for some $w_{1}, w_{2}, w_{3} \in \mathbb{Z}^{n(n-1)}$. By Theorem 5.7 there exists $u_{1}, u_{2} \in U(g(l(n)))$ such that $u_{1}\text{Tab}(w_{1}) = \text{Tab}(w_{2})$ and $u_{2}\text{Tab}(w_{2}) = \text{Tab}(w_{3})$. Therefore, $u_{2}u_{1}\text{Tab}(w_{1}) = \text{Tab}(w_{3})$. That is, $\text{Tab}(w_{1}) \prec \text{Tab}(w_{3})$. \(\square\)

### 6. Irreducible Subquotients in $V(T(\bar{v}))$

The Theorem 5.7 provides an explicit basis for an irreducible submodule that contains a given tableau for generic case. In this section we will present a similar result for 1-singular case and this will lead us an alternative proof for Theorem 4.14 in \(9\).

**Definition 6.1.** Given $z, w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, define the distance between the tableaux, $\text{Tab}(z)$ and $\text{Tab}(w)$ by

$$d(z, w) = \sum_{1 \leq s \leq t \leq n} |z_{rs} - w_{rs}|.$$

The Theorem 4.14 in \(9\) states that if $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$, for every $(r, s, t)$, then the module $V(T(\bar{v}))$ is irreducible. Now consider a tableau such that the condition $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$, for every $(r, s, t)$ is satisfied for any $r \geq k + 1$. We will show how to construct a basis for an irreducible subquotient of $V(T(\bar{v}))$ that contains a given tableau. For this we need of some definitions.

**Definition 6.2.** For $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$, we define

$$\Omega_{k}(\text{Tab}(w)) = \{(r, s, t) \mid r \leq k, (\bar{v}_{rs} + w_{rs}) - (\bar{v}_{r-1,t} + w_{r-1,t}) \in \mathbb{Z}\}$$

$$\Omega_{k}^{-}(\text{Tab}(w)) = \{(r, s, t) \mid r \leq k, (\bar{v}_{rs} + w_{rs}) - (\bar{v}_{r-1,t} + w_{r-1,t}) \in \mathbb{Z}_{\geq 0}\}$$

$$\mathcal{I}_{k}(\text{Tab}(w)) = \{\text{Tab}(w') \in V(T(\bar{v})) \mid \Omega_{k}^{+}(\text{Tab}(w)) = \Omega_{k}^{-}(\text{Tab}(w'))\}$$

**Lemma 6.3.** Let $\text{Tab}(w'), \text{Tab}(w') \in \mathcal{I}_{k}(\text{Tab}(w))$ be tableaux such that $\text{Tab}(w') \not\prec \text{Tab}(w')$, then there exists $i, j$ with $k + 1 \leq j \leq i < n$ such that $w'_{ij} \neq w'_{ij}$.

**Proof.** The tableaux $\text{Tab}(w'), \text{Tab}(w') \in \mathcal{I}_{k}(\text{Tab}(w))$ can be separated in two parts: the top part, i.e. the part from $(k + 1)$-th row to $n$-th row and the bottom part from row 1 to row $k$. Now, suppose that $w'_{ij} = w'_{ij}$ for any $k + 1 \leq j \leq i < n$. In this case we have $\text{Tab}(w') \not\prec \text{Tab}(w')$, because $\text{Tab}(w'), \text{Tab}(w') \in \mathcal{I}_{k}(\text{Tab}(w))$ implies (as in the generic case for $g(l(k))$) that bottom parts of $\text{Tab}(w')$ and $\text{Tab}(w')$ (that we denote by $\text{Tab}_{k}(w'), \text{Tab}_{k}(w')$) are such that $\text{Tab}_{k}(w') \not\prec \text{Tab}_{k}(w')$. \(\square\)

**Theorem 6.4.** The set $\mathcal{I}_{k}(\text{Tab}(w))$ is a basis for irreducible subquotient of $V(T(\bar{v}))$ that contains the tableau $\text{Tab}(w)$.
Proof. Let $N = \text{Span}_C \mathcal{I}_k(Tab(w))$ be the submodule of $V(T(\bar{v}))$ generated by the set $\mathcal{I}_k(Tab(w))$. If $N$ is not irreducible. By Corollary 5.8, there exist tableaux $Tab(z')$ and $Tab(w)$ such that $Tab(z') \neq Tab(w)$. Now, fix $w$ and choose $z \in \mathbb{Z}^{n(n+1)/2}$ such that $d(z, w)$ is minimal. Set $d := d(z, w)$ then, if $z' \in \mathbb{Z}^{n(n+1)/2}$ is such that $d(z', w) < d$, we should have $Tab(z') \prec Tab(w)$. As $Tab(z) \not\prec Tab(w)$, we have $d \geq 1$, that is, $z_{rs} \neq w_{rs}$ for some $r, s$ and by Theorem 6.3 we have that is possible for $r \geq k + 1$. Now fix the position $r, s$ and assume without lose of generality that $z_{rs} < w_{rs}$. The case $w_{rs} < z_{rs}$ will be analogous.

As $z_{rs} < w_{rs}$, we have $d(z + \delta^{rs}, w) = d - 1 < d$. Then, $Tab(z + \delta^{rs}) \prec Tab(w)$. Therefore, as by Corollary 5.8, $\prec$ is transitive, we should have $Tab(z) \not\prec Tab(z + \delta^{rs})$. So, if $Tab(z + \delta^{rs})$ appear in the decomposition of $u \cdot Tab(z)$ for some $u \in U$, the corresponding coefficient should be zero.

As the formulas that define the action of $\mathfrak{gl}(n)$ on $V(T(\bar{v}))$ depend of the type of tableau (derivative tableaux or regular tableau), we will consider four cases as follows:

| Case  | $Tab(z)$          | $Tab(z + \delta^{rs})$          |
|-------|-------------------|---------------------------------|
| Case 1 | $T(\bar{v} + z)$ | $T(\bar{v} + z + \delta^{rs})$ |
| Case 2 | $DT(\bar{v} + z)$ | $DT(\bar{v} + z + \delta^{rs})$ |
| Case 3 | $T(\bar{v} + z)$ | $DT(\bar{v} + z + \delta^{rs})$ |
| Case 4 | $DT(\bar{v} + z)$ | $T(\bar{v} + z + \delta^{rs})$ |

**Case 1.** The coefficient of $T(\bar{v} + z + \delta^{rs})$ in decomposition $E_{r,r+1}T(\bar{v} + z)$ is given by $D^\bar{v}((v_{ki} - v_{kj})e_{r,r+1}(\sigma_s(v + z)))$ where $\sigma_s$ is the transposition $(1, s)$ on row $r$ and identity on the other rows. But, $D^\bar{v}((v_{ki} - v_{kj})e_{r,r+1}(\sigma_s(v + z))) = e_{r,r+1}(\sigma_s(\bar{v} + z))$, because in this case the function $e_{r,r+1}(\sigma_s(v + z))$ is smooth. As the numerator of $e_{r,r+1}(\sigma_s(\bar{v} + z))$ is a product of differences of type $(\bar{v}_r + z_r)(\bar{v}_{r+1, t} + z_{r+1, t})$. We necessarily have $(\bar{v}_r + z_r)(\bar{v}_{r+1, t} + z_{r+1, t}) = 0$ for some $t$, that is: $\bar{v}_{r+1, t} - \bar{v}_r, s \in \mathbb{Z}$, with $r \geq k + 1$, that is a contradiction.

**Case 2.** The coefficient of $DT(\bar{v} + z + \delta^{rs})$ in decomposition $E_{r,r+1}DT(\bar{v} + z)$ is given by $e_{r,r+1}(\sigma_s(\bar{v} + z))$. Analogously the first case, we obtain that $\bar{v}_{r+1, t} - \bar{v}_r, s \in \mathbb{Z}$ for some $t$, with $r \geq k + 1$, that is a contradiction.

**Case 3.** The only possibility in order to have $Tab(z) = T(\bar{v} + z)$ and $Tab(z + \delta^{rs}) = DT(\bar{v} + z + \delta^{rs})$ is $z_{ki} = z_{kj}$ and $(r, s) \in \{(k, i), (k, j)\}$. As the coefficient of $DT(\bar{v} + z + \delta^{ks})$ is $ev(\bar{v})(v_{ki} - v_{kj})e_{k,k+1}(\sigma_s(v + z))$ and $e_{k,k+1}(\sigma_s(v + z))$ has singularity at $v_{ki} - v_{kj} = 0$, $ev(\bar{v})(v_{ki} - v_{kj})e_{k,k+1}(\sigma_s(v + z)) = 0$ if, and only if \[
\prod_{j=1}^{k+1}((\bar{v} + z)_{ks} - (\bar{v} + z)_{k+1,j}) = 0
\]
which implies that some difference $\bar{v}_{ks} - \bar{v}_{k+1, t} \in \mathbb{Z}$, with $r \geq k + 1$, that is a contradiction.

**Case 4.** In this case we have:
\[
E_{r,r+1}(DT(\bar{v} + z)) = \sum_{\sigma \in \Phi_{r,r+1}} D^\bar{v}(e_{r,r+1}(\sigma(v + z)))T(\bar{v} + z + \sigma(\varepsilon_{r,r+1})) + \sum_{\sigma \in \Phi_{r,r+1}} e_{r,r+1}(\sigma(\bar{v} + z))DT(\bar{v} + z + \sigma(\varepsilon_{r,r+1}))
\]
and the coefficient of all possible tableaux $T_{ab}$

**Definition 7.1.** Let $V$ be the irreducibility of the module $\Omega_{\ell}(\text{Tab}(w))$. We will analyze the action of generators of $\mathfrak{gl}(n)$ to have the irreducibility of $\Omega_{\ell}(\text{Tab}(w))$.

**Proposition 7.3.** Let $\text{Tab}(z)$ and $\text{Tab}(w)$ be tableaux in $\mathcal{B}(T(\bar{v}))$ such that $\text{Tab}(w) \in U \cdot \text{Tab}(z)$, then $|\Omega^+(\text{Tab}(w))| \geq |\Omega^+(\text{Tab}(z))| - 1$.

To prove this proposition we will prove first that the lemma is true when $\text{Tab}(z) \prec_g \text{Tab}(w)$ for some $g \in \mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$.

**Lemma 7.4.** Let $\text{Tab}(z)$ and $\text{Tab}(w)$ be tableaux in $\mathcal{B}(T(\bar{v}))$ such that $\text{Tab}(z) \prec_g \text{Tab}(w)$ for some $g \in \mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ with $1 \leq r \leq n-1$, then $|\Omega^+(\text{Tab}(w))| \geq |\Omega^+(\text{Tab}(z))| - 1$.

**Proof.** We will analyze the action of generators of $\mathfrak{gl}(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ in all tableaux $\text{Tab}(z) \in V(T(\bar{v}))$ such that $|\Omega^+(\text{Tab}(w))| \leq |\Omega^+(\text{Tab}(z))| - 1$ and $\text{Tab}(z) \prec \text{Tab}(w)$. A case by case verification we have that, the coefficient of tableau $\text{Tab}(w)$ is equal to zero whenever $|\Omega^+(\text{Tab}(w))| \leq |\Omega^+(\text{Tab}(z))| - 2$ and the list of all possible tableaux $\text{Tab}(z)$ and $\text{Tab}(w)$ such that $|\Omega^+(\text{Tab}(w))| = |\Omega^+(\text{Tab}(z))| - 1$ and the coefficient of $\text{Tab}(w)$ is not zero is the following:

1. $D \left( \begin{array}{cc} x & x + a \\ x & x + 1 \end{array} \right) \prec_{E_{k-1,k}} \left( \begin{array}{cc} x & x + a \\ x + 1 & x + 1 \end{array} \right)$, $a \in \mathbb{Z}_{<0}$.
2. $D \left( \begin{array}{cc} x & x + a \\ x & x + 1 \end{array} \right) \prec_{E_{k-1,k}} \left( \begin{array}{cc} x + a & x \\ x & x + 1 \end{array} \right)$, $a \in \mathbb{Z}_{<0}$.
Where configurations above represent the part of the tableaux around row \( k \). First of all, by Corollary 4.5 it is enough to consider \( r \in \{ k, k-1 \} \), in fact, for the other cases the action is given by the classical Gelfand-Tsetlin formulas, as in the generic case. So we have \( |\Omega^+(\text{Tab}(w))| \geq |\Omega^+(\text{Tab}(z))| \) (see Theorem 3.7(1)).

Assume first that \( g = E_{r,r+1} \). The action of \( E_{r,r+1} \) on basis elements of \( V(T(\bar{v})) \) is given by:

\[
E_{r,r+1}T(\bar{v} + w) = \sum_{\sigma} \mathcal{D}^\sigma \left( (v_{r_i} - v_{r_j})e_{r,r+1}(\sigma(v + w)) T(\bar{v} + w + \sigma(\delta^r)) \right) + \\
\sum_{\sigma} ((v_{r_i} - v_{r_j})e_{r,r+1}(\sigma(v + w))) (\bar{v})DT(\bar{v} + w + \sigma(\delta^r)),
\]

\[
E_{r,r+1}DT(\bar{v} + w) = \sum_{\sigma} \mathcal{D}^\sigma \left( e_{r,r+1}(\sigma(v + w)) T(\bar{v} + w + \sigma(\delta^r)) \right) + \\
\sum_{\sigma} e_{r,r+1}(\sigma(\bar{v} + w))DT(\bar{v} + w + \sigma(\delta^r)).
\]

Depending on the \( \text{Tab}(z) \) and \( \text{Tab}(z + \delta^r) \) being regular tableau or derivative tableau, we have to look at different coefficients as shows the following table:

| Type | \( \text{Tab}(z) \) | \( \text{Tab}(z + \delta^r) \) | Coefficient of \( \text{Tab}(z + \delta^r) \) |
|------|----------------|----------------|---------------------------------|
| (a)  | \( T(\bar{v} + z) \) | \( T(\bar{v} + z + \delta^r) \) | \( \mathcal{D}^\sigma ((v_{k_i} - v_{k_j})e_{r,r+1}(\bar{v} + z)) \) |
| (b)  | \( DT(\bar{v} + z) \) | \( DT(\bar{v} + z + \delta^r) \) | \( e_{r,r+1}(\bar{v} + z) \) |
| (c)  | \( T(\bar{v} + z) \) | \( DT(\bar{v} + z + \delta^r) \) | \( ((v_{k_i} - v_{k_j})e_{r,r+1}(\bar{v} + z))(\bar{v}) \) |
| (d)  | \( DT(\bar{v} + z) \) | \( T(\bar{v} + z + \delta^r) \) | \( \mathcal{D}^\sigma (e_{r,r+1}(\bar{v} + z)) \) |

(i) Consider a tableau \( \text{Tab}(z) \) such that \( (\bar{v} + z)_{k_i} = (\bar{v} + z)_{k-1,t} = x \) and \( (\bar{v} + z)_{k_j} = x + a \), with \( a \in \mathbb{Z} \). We will represent this part of tableau \( \text{Tab}(z) \) by (the row where appear two variable equal to \( x \) is the \( k \)-th row of the tableau \( \text{Tab}(z) \)).

\[
\text{Tab}(z) = \begin{pmatrix} x & x + a \\ x & \end{pmatrix}
\]

This tableau can be regular (if \( a \geq 0 \)) or derivative (if \( a < 0 \)). We will analyse this two cases. When \( E_{k-1,k} \) acts in this tableau we obtain the following tableau:

\[
\text{Tab}(z + \delta^{k-1,t}) = \begin{pmatrix} x & x + a \\ x + 1 & \end{pmatrix}
\]

Note that

\[
|\Omega^+(\text{Tab}(z + \delta^{k-1,t}))| = \begin{cases} 
|\Omega^+(\text{Tab}(z))| - 2, & \text{if } a = 0 \\
|\Omega^+(\text{Tab}(z))| - 1, & \text{otherwise.}
\end{cases}
\]
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For \( a \geq 0 \) the tableaux \( \text{Tab}(z) \) is regular. In this case we will analyse the coefficients of types (a) and (c). Recall that,

\[
e_{k-1,k}(v + z) = \frac{\prod_{t=1}^{k-1}((v + z)_{k-1,t} - (v + z)_{k-1,t})}{\prod_{t \neq 1}^{k-1}((v + z)_{k-1,t} - (v + z)_{k-1,t})}
\]

In this case, \( e_{k-1,k}(v + z) \) is smooth function (because the singularity is in \( k \)-th row and in the denominator appear differences between elements of the \( (k-1) \)-th row). Thus, by Lemma [4.3] follows that:

- \( D^\beta((v_{ri} - v_{rj})e_{k-1,k}(v + z)) = e_{k-1,k}(\bar{v} + z) \), and, as in this case we have the relation \( (\bar{v} + z)_{kj} = (v + z)_{k-1,t} \), therefore \( e_{k-1,k}(\bar{v} + z) = 0 \).
- \( (v_{ki} - v_{kj})e_{k-1,k}(v + z))(\bar{v}) = 0 \).

If \( a < 0 \), the tableau \( \text{Tab}(z) \) is derivative tableau. In this case, we need analyze the coefficients of types (b) and (d). Now the coefficients of tableau \( \text{Tab}(z + \delta^{k-1,t}) \) are \( e_{k-1,k}(\bar{v} + z) \) and \( D^\beta(e_{k-1,k}(v + z)) \). In this case, we have that

- \( e_{k-1,k}(\bar{v} + z) = 0 \), because in the numerator of this rational function appear a difference \( (\bar{v} + z)_{kj} - (v + z)_{k-1,t} \) that is equal to zero, in this case.
- For the coefficient \( D^\beta(e_{k-1,k}(v + z)) \), we have the following:

\[
D^\beta(e_{k-1,k}(v + z)) = D^\beta(-(v_{k-1,t} - (v_{ki} + a))(v_{k-1,t} - v_{ki})\varphi(v)) = -\frac{1}{2}\varphi(\bar{v})a
\]

where \( \varphi(v) \) is a rational function of the entries of the vector \( v \) that not depends of the entries \( v_{ki} \) end \( v_{kj} \), moreover \( \varphi(\bar{v}) \neq 0 \). As \( a \neq 0 \), in this case we have \( D^\beta(e_{k-1,k}(v + z)) \neq 0 \).

(ii) Now we will consider a tableau \( \text{Tab}(z) \) such that \( k \neq n - 1 \), \( (\bar{v} + z)_{k+1,t} = (v + z)_k = x \) and \( (\bar{v} + z)_{kj} = x + a \), with \( a \in \mathbb{Z} \). A representation of part this tableau is

\[
\text{Tab}(z) = \begin{pmatrix} x & x + a \\ x + 1 & x + a \end{pmatrix}
\]

This tableau should be regular or derivative depending on the value of \( a \).

We will analyze this two cases. When \( E_{k,k+1} \) acts in this tableau we obtain the tableau

\[
\text{Tab}(z + \delta^{k,t}) = \begin{pmatrix} x & x + a \\ x + 1 & x + a \end{pmatrix}
\]

In this case, we have that \( |\Omega^+\text{Tab}(z + \delta^{k,t})| = |\Omega^+\text{Tab}(z)| - 1 \). Initially we will assume that this tableau is regular tableau \( (a \geq 0) \) and we will analyze the coefficients of type (a) and (c). For this, recall that:

\[
e_{k,k+1}(v + z) = -\frac{\prod_{t=1}^{k}((v + z)_{k,t} - (v + z)_{k+1,t})}{\prod_{t \neq 1}^{k}((v + z)_{k,t} - (v + z)_{kt})}
\]

If \( a > 0 \), in this case that \( e_{k,k+1}(v + z) \) is a smooth function, then by Lemma [4.3] follows that:

- \( D^\beta((v_{ri} - v_{rj})e_{k,k+1}(v + z)) = e_{k,k+1}(\bar{v} + z) \), and, as we have the relation \( (\bar{v} + z)_{k+1,t} = (v + z)_{k-1,t} \) then, \( e_{k,k+1}(\bar{v} + z) = 0 \).
- \( (v_{ki} - v_{kj})e_{k-1,k}(v + z))(\bar{v}) = 0 \).
On the other hand, if $a = 0$, we have

$$((v_{k_1} - v_{k_2})e_{k-1,k}(v + z))((\bar{v})) = \left(\prod_{i=1}^{k} \left(\prod_{t\neq 1, j} \left((v + z)_{k-1,1} - (v + z)_{k-1,i}\right)\right) \right)(\bar{v}) = 0$$

$$D^{\bar{v}} \left((v_{k_1} - v_{k_2})e_{k-1,k}(v + z)\right) = D^{\bar{v}} \left(\prod_{i=1}^{k} \left(\prod_{t\neq 1, j} \left((v + z)_{k-1,1} - (v + z)_{k-1,i}\right)\right) \right) = 0$$

Finally, if $a < 0$ the tableau $Tab(\bar{v} + z)$ is a derivative tableau. In this case we will analyze the coefficients of type (b) and (d). In this case, using the formula (17), follows that:

- $e_{k-1,k}(\bar{v} + z) = 0$, because in the numerator of this rational function appear the difference $(\bar{v} + z)_{k1} - (\bar{v} + z)_{k+1,1}$, that is equal to zero in this case.

- More one time using the formula (17), we have that $D^{\bar{v}} (e_{k-1,k}(\bar{v} + z)) \neq 0$.

Continuing this analysis, case by case, we can identify all tableaux $Tab(w) \in U \cdot Tab(z)$ such that the correspondent coefficient is no zero is a tableau $Tab(w)$ such that $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$, that was described in begin the proof of Lemma 7.4 (cases (I)-(V)). Moreover, for all tableaux $Tab(w) \in V(T(\bar{V}))$ such that $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(z))| - 2$ have respective coefficients equal to zero.

\[\square\]

The following lemma shows that, for each of the cases described in Lemma 7.4, where $Tab(z) \prec Tab(w)$ and $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$, (cases (I)-(V)), the action of the basis elements of $gl(n)$ on $Tab(w)$ will generate tableaux $Tab(w')$ such that $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(w'))| \leq |\Omega^+(Tab(w'))|$. (i.e. $Tab(w) \prec_g Tab(w')$ for some $g \in gl(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ implies $|\Omega^+(Tab(w'))| \leq |\Omega^+(Tab(w'))|$).

**Lemma 7.5.** Let $Tab(w)$ be a tableau such that $Tab(z) \prec Tab(w)$ and $|\Omega^+(Tab(w))| = |\Omega^+(Tab(z))| - 1$ for some $Tab(z)$. If $Tab(w) \prec_g Tab(w')$ for some $g \in gl(n)$ of the form $E_{r,r+1}$ or $E_{r+1,r}$ then $|\Omega^+(Tab(w))| \leq |\Omega^+(Tab(w'))|$.

**Proof.** By Lemma 7.4 the set of all possible tableaux $Tab(w)$ satisfying the condition is given by:

1. $\begin{pmatrix} x & x + 1 & x + a \\ x + 1 & x & x + 1 \\ x + a & x + 1 & x + a \end{pmatrix}$
   with $a \in \mathbb{Z}_{<0}$.
2. $\begin{pmatrix} x & x + 1 \\ x & x + 1 \\ x \end{pmatrix}$.

Now, for each $Tab(w)$ as before we will construct subsets $W(Tab(w))$ and $W^*(Tab(w))$, of $\mathcal{B}(T(\bar{v}))$ such that:

(i) $Tab(w) \in W(Tab(w)) \cup W^*(Tab(w))$ and $Tab(w') \in W(Tab(w)) \cup W^*(Tab(w))$ implies $|\Omega^+(Tab(w'))| \geq |\Omega^+(Tab(w))|$.

(ii) $W(Tab(w)) \cup W^*(Tab(w))$ is $gl(n)$-invariant.
The construction before will be enough to finish the proof. In fact, if $\text{Tab}(w) \prec_g \text{Tab}(w')$ with $g \in \mathfrak{gl}(n)$, then $\text{Tab}(w') \in W(\text{Tab}(w)) \cup W^+(\text{Tab}(w))$ (because of condition (ii)) and, then $|\Omega^+(\text{Tab}(w'))| \geq |\Omega^+(\text{Tab}(w))|$ (because of condition (i)).

For instance, consider $\text{Tab}(w)$ to be of the form $\begin{pmatrix} x & x+1 & x+a \end{pmatrix}$. Define the following subsets of $B(T(\vec{v}))$:

$W(\text{Tab}(w)) = \left\{ \begin{pmatrix} x+b & x+a+c \end{pmatrix} \mid b - 1 - d < 0 \text{ and } b + a - 1 - d < 0 \right\}$

$W^+(\text{Tab}(w)) = \left\{ \begin{pmatrix} x+b & x+a+c \end{pmatrix} \mid b - 1 - d \geq 0 \text{ or } b + a - 1 - d \geq 0 \right\}$.

Note that, the tableaux in $W(\text{Tab}(w))$ or $W^+(\text{Tab}(w))$ can be regular or derivative. It is easy to check that $W(\text{Tab}(w)) \cup W^+(\text{Tab}(w))$ is $\mathfrak{gl}(n)$-invariant. Also, $\text{Tab}(w') \in W(\text{Tab}(w)) \cup W^+(\text{Tab}(w))$ satisfies $|\Omega^+(\text{Tab}(w'))| \geq |\Omega^+(\text{Tab}(w))|$. In fact, if $\text{Tab}(w') \in W(\text{Tab}(w))$ follows that $|\Omega^+(\text{Tab}(w'))| = |\Omega^+(\text{Tab}(w))|$ and, by on the other hand, if $\text{Tab}(w') \in W^+(\text{Tab}(w))$ follows that $|\Omega^+(\text{Tab}(w'))| > |\Omega^+(\text{Tab}(w))|$.

The construction of $W(\text{Tab}(w))$ and $W^+(\text{Tab}(w))$ for the other cases is analogous.

\[ \square \]

**Proof of Lemma 7.3** We will show that if $\text{Tab}(w) \in E_1 \cdots E_t \cdot \text{Tab}(z)$, then

\[ |\Omega^+(\text{Tab}(w))| \geq |\Omega^+(\text{Tab}(z))| - 1, \]

and by linearity the result follows for every element of $u \in U(\mathfrak{gl}(n))$.

**Proof.** Let $\text{Tab}(z) \in V(T(\vec{v}))$ and $E_1 \cdots E_t \in U$. We will use induction over $t$. Indeed, if $t = 1$, this result is true by Lemma 7.4. Suppose that the lemma is true for every $s$ such that $1 \leq s \leq t - 1$. We will proof that the result is true for $t$. Indeed, as

\[ E_1 E_2 \cdots E_{t-1} E_t \cdot \text{Tab}(z) = E_1 E_2 \cdots E_{t-1} (E_t \cdot \text{Tab}(z)) \]

we have to consider two cases:

Case 1. If $\text{Tab}(z)$ is not a tableau described in Lemma 7.4 (cases (I)-(V)), follows that $E_t \cdot \text{Tab}(z) = \sum_i \alpha_i \text{Tab}(w_i)$, where $|\Omega^+(\text{Tab}(w_i))| \geq |\Omega^+(\text{Tab}(z))|$. In this case, by induction hypothesis, the action of $E_1 E_2 \cdots E_{t-1}$ over $(E_t \cdot \text{Tab}(w_i))$ we obtain a linear combination of tableaux $\text{Tab}(w''_i)$ such that $|\Omega^+(\text{Tab}(w''_i))| \geq |\Omega^+(\text{Tab}(z))| - 1$ which proves the desired result in this case.

Case 2. Now we will assume that $\text{Tab}(z)$ is a tableau described in Lemma 7.4 (cases (I)-(V)). In this case follows that

\[ E_t \cdot \text{Tab}(z) = \sum_i \alpha_i \text{Tab}(w_i) + \sum_i \beta_i \text{Tab}(w'_i) \]

where $|\Omega^+(\text{Tab}(w_i))| \geq |\Omega^+(\text{Tab}(z))|$ and $|\Omega^+(\text{Tab}(w'_i))| = |\Omega^+(\text{Tab}(z))| - 1$. Thus, by induction hypothesis, follows that when $E_1 E_2 \cdots E_{t-1}$ acts in tableaux $\text{Tab}(w_i)$ we obtain a linear combination of new tableaux $\text{Tab}(w''_i)$ such that $|\Omega^+(\text{Tab}(w''_i))| \geq |\Omega^+(\text{Tab}(z))| - 1$. By other hand, when $E_1 \cdots E_{t-1}$ acts in tableaux $\text{Tab}(w'_i)$, we will get a linear combination of tableaux belong the set $W(\text{Tab}(w)) \cup W^+(\text{Tab}(w))$ (that
was described in Lemma 7.3. Thus, in this case we have that in decomposition $E_1 \cdots E_{i-1} \cdot Tab(w_i)$ will appear tableaux $Tab(w_i)$ such that

$$|\Omega^+(Tab(w_i))| = |\Omega^+(Tab(z))| - 1$$

which prove the result in this second case. \qed

**Corollary 7.6.** If $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ is such that $|\Omega^+(Tab(z))| \geq 1$ then, $U \cdot Tab(z)$ is a proper submodule of $V(T(\bar{v}))$.

**Proof.** Indeed, if the module $U \cdot Tab(z)$ is not a proper submodule of $V(T(\bar{v}))$, follows that for every $Tab(w) \in V(T(\bar{v}))$ we have $Tab(z) \prec Tab(w)$. But by Lemma 7.3, follows that $|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1$. We have two cases to consider:

**Case 1.** If $|\Omega^+(Tab(z))| \geq 2$, follows that

$$|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| - 1 \geq 2 - 1 = 1$$

but, this is a contradiction, we always have $T(w) \in V(T(\bar{v}))$ such that $|\Omega^+(Tab(w))| = 0$.

**Case 2.** If $|\Omega^+(Tab(z))| = 1$, follows that the unique integer difference $\bar{v}_{rs} - \bar{v}_{r-1,t}$ is not close to critical line, because if this integer difference is close to critical line we would $|\Omega^+(Tab(z))| \geq 2$. But away from the critical line we have the generic case, where we have the following inequality

$$|\Omega^+(Tab(w))| \geq |\Omega^+(Tab(z))| = 1$$

As in the last case this is a contradiction. \qed

Hence, if $|\Omega^+(Tab(z))| \geq 1$ the module $U \cdot Tab(z)$ could not be the full module $V(T(\bar{v}))$, thus in this case, $U \cdot Tab(z)$ is a proper submodule. \qed

**Theorem 7.7.** If $V(T(\bar{v}))$ is irreducible then, $\bar{v}_{rs} - \bar{v}_{r-1,t} \notin \mathbb{Z}$ for any $1 \leq s \leq r \leq n$ and $1 \leq t \leq r - 1$.

**Proof.** Suppose that $\bar{v}_{rs} - \bar{v}_{r-1,t} \in \mathbb{Z}$ for some $1 \leq t < r \leq n$, $1 \leq s \leq r$, choose $w \in \mathbb{Z}^{\frac{n(n-3)}{2}}$ such that $|\Omega^+(Tab(w))|$ is maximal ($|\Omega^+(Tab(w))| \geq 1$) and consider the submodule $N$ of $V(T(\bar{v}))$ generate by $Tab(w)$. Thus by 7.6 follows that the submodule $N$ is proper. \qed

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