Some remarks on a formula for Sobolev norms due to Brezis, Van Schaftingen and Yung

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Abstract

We provide answers to some questions raised in a recent work by H. Brezis, J. Van Schaftingen and Po-Lam Yung [2, 3] concerning the Gagliardo semi-norm $|u|_{W^{s,q}}$ computed at $s = 1$, when the strong $L^q$ is replaced by weak $L^q$. In particular, we address generalization of the results in [2, 3] for a general domain and non-smooth functions.

1 Introduction

The following two remarkable theorems were proved by H. Brezis, J. Van Schaftingen and Po-Lam Yung in [2, 3]:

Theorem 1.1. Let $q \geq 1$. Then, for every dimension $N \geq 1$ there exist constants $c_N, C_N > 0$ such that, for every $u \in C^\infty_0(\mathbb{R}^N)$ we have

$$c_N^q \int_{\mathbb{R}^N} |\nabla u(x)|^q dx \leq$$

$$\sup_{s \in (0, +\infty)} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}$$

$$\leq C_N \int_{\mathbb{R}^N} |\nabla u(x)|^q dx. \quad (1.1)$$

Theorem 1.2. Let $q \geq 1$. Then, for every $u \in C^\infty_0(\mathbb{R}^N)$ we have

$$\lim_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}$$

$$= \int_{S^{N-1}} \frac{|z_1|^q d\mathcal{H}^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u(x)|^q dx. \quad (1.2)$$

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These results shed light on what happened when one replaces the strong $L^q$ by weak $L^q$ in the expression for the Gagliardo semi-norm $|u|_{W^{s,q}}$, computed at $s = 1$. Several interesting open problems, related to Theorems 1.1 and 1.2 were raised in [3]:

(i) If $u \in L^q$ for some $q \geq 1$ satisfies

$$\sup_{s \in (0, +\infty)} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^q} > s \right\} \right) \right\} < +\infty,$$

does it imply that $u \in W^{1,q}$ (when $q > 1$) or $u \in BV$ (when $q = 1$)?

(ii) Does Theorem 1.1 hold in the cases $u \in W^{1,q}$ (when $q > 1$) and $u \in BV$ (when $q = 1$)?

(iii) Same question as in (ii), but for Theorem 1.2.

(iv) Given $u \in L^q$, does

$$\lim_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^q} > s \right\} \right) \right\} = 0,$$

imply that $u$ necessarily equals a constant a.e. in $\mathbb{R}^N$?

(v) For $r \in (0, q)$ characterize the class of functions $u \in L^q$, satisfying

$$\sup_{s \in (0, \infty)} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q + N}} > s \right\} \right) \right\} < +\infty. \quad (1.3)$$

In particular, determine how this class is related to an appropriate Besov space.

In the current paper we give full affirmative answers to questions (i), (ii) and (iv), see Theorem 1.3 and Corollary 1.1 bellow. Moreover, we give a partial answer to questions (iii), see Corollary 1.2 bellow (in particular, we completely resolve this question in the case $q > 1$). Concerning question (v), we give only some partial information about the quantity appearing in (1.3) that could be obtained by combining Theorem 1.4, treating the quantities

$$\limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{r + N}} > s \right\} \right) \right\}$$

and

$$\liminf_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{r + N}} > s \right\} \right) \right\}, \quad (1.4)$$

for general $r$, together with Proposition 1.1.

Our first main result, answering Questions (i) and (ii), is:

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^N$ be an open domain with Lipschitz boundary and let $q \geq 1$. Then there exist constants $C_\Omega > 0$ and $\tilde{C}_N > 0$ satisfying $C_\Omega = 1$ if $\Omega = \mathbb{R}^N$, such that for every $u \in L^q(\Omega, \mathbb{R}^m)$ we have:
(i) When \( q > 1 \),
\[
\int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z) \frac{1}{(N+q)} \int_\Omega |\nabla u(x)|^q dx \\
\leq \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \\
\leq \sup_{s \in (0, +\infty)} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq C_q \tilde{C}_N \int_\Omega |\nabla u(x)|^q dx,
\]
with the convention that \( \int_\Omega |\nabla u(x)|^q dx = +\infty \) if \( u \notin W^{1,q}(\Omega, \mathbb{R}^m) \).

(ii) When \( q = 1 \),
\[
\int_{S^{N-1}} |z_1| d\mathcal{H}^{N-1}(z) \frac{1}{(N+1)} \|Du\|((\Omega) \leq \\
\limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|^{1+N}} > s \right\} \right) \right\} \\
\leq \sup_{s \in (0, +\infty)} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|^{1+N}} > s \right\} \right) \right\} \leq C_1 \tilde{C}_N \|Du\|((\Omega),
\]
with the convention \( \|Du\|((\Omega) = +\infty \) if \( u \notin BV(\Omega, \mathbb{R}^m) \).

Remark 1.1. Setting \( \tilde{C}_N = \frac{\int_{S^{N-1}} |z_1| d\mathcal{H}^{N-1}(z)}{(N+1) \left( \mathcal{H}^{N-1}(S^{N-1}) + 1 \right)} \) it is easy to deduce by Holder inequality that
\[
\tilde{C}_N^q \leq \frac{\int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z)}{(N+q)}
\]
Therefore, the lower-bound in inequality (1.5) can be also written, analogously to (1.1), as
\[
\tilde{C}_N^q \int_\Omega |\nabla u(x)|^q dx \leq \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}.
\]

The proof of Theorem 1.3 is given in Section 3 below. From Theorem 1.3 we deduce the next corollary, that provides a positive answer to Question (iv) (see [3, Open Problem 1]):

Corollary 1.1. Let \( \Omega \subset \mathbb{R}^N \) be an open domain, \( q \geq 1 \) and \( u \in L^q(\Omega, \mathbb{R}^m) \). If
\[
\lim_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} = 0,
\]
then \( u(x) \) necessarily equals a constant a.e. in \( \Omega \).

Moreover, by combining Theorem 1.3 with Theorem 1.2 we obtain the following corollary, that provides a full positive answer to Question (iii) for \( q > 1 \) and a positive answer to this Question in the case \( q = 1 \) and \( u \in W^{1,1} \not\subset BV \):
Corollary 1.2. Let \( q \geq 1 \). Then, for every \( u \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m) \) we have

\[
\lim_{s \to +\infty} \left\{ s L^{2N} \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+1}} > s \right\} \right\} = \frac{\int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u(x)|^q dx.
\] (1.8)

The proof of Corollary 1.2 is given in Section 4 below.

Remark 1.2. Only the lower bound in Theorem 1.3 requires a non-trivial proof. Indeed, the upper bound in this Theorem follows quite easily from Theorem 1.1, by an extension of a formula due to J. Bourgain, H. Brezis, P. Mironescu \[1\] for \( q > 1 \). For \( q = 1 \) the formula in the general case of BV functions is due to J. Dávila \[4\]. This formula states, in particular, that given an open domain with Lipschitz boundary \( \Omega \subset \mathbb{R}^N \), a family of radial mollifiers \( \rho_\varepsilon(|z|) : \mathbb{R}^N \to [0, +\infty) \), satisfying \( \int_{\mathbb{R}^N} \rho_\varepsilon(|z|) dz = 1 \) and such that for every \( r > 0 \) there exists \( \delta := \delta_r > 0 \), satisfying \( \text{supp}(\rho_\varepsilon) \subset B_r(0) \) for every \( \varepsilon \in (0, \delta_r) \), the following holds true:

(i) For any \( q > 1 \) and any \( u \in L^q(\Omega, \mathbb{R}^m) \) we have

\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^q}{|x - y|^q} \rho_\varepsilon(|x - y|) \, dx \, dy = K_{q,N} \int_{\Omega} |\nabla u(x)|^q \, dx,
\] (1.9)

with the convention that \( \int_{\Omega} |\nabla u(x)|^q \, dx = +\infty \) if \( u \notin W^{1,q}(\Omega, \mathbb{R}^m) \) and with \( K_{q,N} \) given by

\[
K_{q,N} := \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z) \quad \forall q \geq 1.
\] (1.10)

(ii) In the case \( q = 1 \), for any \( u \in L^1(\Omega, \mathbb{R}^m) \) we have

\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|) \, dx \, dy = K_{1,N} \|Du\|(\Omega),
\] (1.11)

with the convention that \( \|Du\|(\Omega) = +\infty \) if \( u \notin BV(\Omega, \mathbb{R}^m) \).

In particular, taking

\[
\rho_\varepsilon(|z|) := \frac{1}{2\sigma_\varepsilon \mathcal{H}^{N-1}(S^{N-1}) |z|^{N-1}} \chi_{[\sigma_\varepsilon, \varepsilon + \sigma_\varepsilon]}(|z|) \quad \forall z \in \mathbb{R}^N
\]

with sufficiently small \( 0 < \sigma_\varepsilon \ll \varepsilon \), we deduce the following variant of the “BBM formula”:

\[
\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \lim_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\varepsilon(x + \varepsilon n) \|u(x + \varepsilon n) - u(x)|^q \, dx \right) d\mathcal{H}^{N-1}(n) = K_{q,N} \int_{\Omega} |\nabla u(x)|^q \, dx \quad \text{for} \quad q > 1,
\] (1.12)
and
\[
\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \lim_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|}{\varepsilon} \, dx \, d\mathcal{H}^{N-1}(n) \right)
\]
\[
= K_{1,N} \| Du \|(\Omega) \quad \text{for} \ q = 1, \quad (1.13)
\]
where we denote
\[
\chi_\Omega(z) := \begin{cases} 
1 & z \in \Omega, \\
0 & z \in \mathbb{R}^N \setminus \Omega. 
\end{cases} \quad (1.14)
\]
In the spirit of (1.12) and (1.13) we prove the following Theorem. The special case \( r = q \) provides the key ingredient in the proof of Theorem 1.3.

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain, \( q \geq 1 \), \( r \geq 0 \) and \( u \in L^\infty(\Omega, \mathbb{R}^m) \). Then,
\[
\liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} \, dx \, d\mathcal{H}^{N-1}(n) \right)
\]
\[
\leq (N + r) \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{r+N}} > s \right\} \right) \right\}, \quad (1.15)
\]
and
\[
\limsup_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} \, dx \, d\mathcal{H}^{N-1}(n) \right)
\]
\[
\geq N \liminf_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{r+N}} > s \right\} \right) \right\}. \quad (1.16)
\]

We refer the reader to Lemma A.1 in Appendix for the significance of the quantity
\[
\lim_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} \, dx \, d\mathcal{H}^{N-1}(n) \right),
\]
appearing in Theorem 1.4 for general \( r \).

**Remark 1.3.** Although we stated Theorem 1.4 for every \( r \geq 0 \), it is useful only for \( r \in (0, q] \), since in the case \( r > q \) we have
\[
\liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} \, dx \, d\mathcal{H}^{N-1}(n) \right) < +\infty \quad \text{implies}
\]
\[
\liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \, dx \, d\mathcal{H}^{N-1}(n) \right) = 0, \quad (1.17)
\]
and thus, by the “BBM formula”, \( u \) must be a constant. On the other hand, in the case \( r = 0 \) we obviously have
\[
\lim_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) |u(x + \varepsilon n) - u(x)|^q \, dx \, d\mathcal{H}^{N-1}(n) \right) = 0. \quad (1.18)
\]
Next we recall the definition of the Besov Spaces $B_{q,\infty}^s$ with $s \in (0, 1)$:

**Definition 1.1.** Given $q \geq 1$ and $s \in (0, 1)$, we say that $u \in L^q(\mathbb{R}^N, \mathbb{R}^m)$ belongs to the Besov space $B_{q,\infty}^s(\mathbb{R}^N, \mathbb{R}^m)$ if

$$\sup_{\rho \in (0, \infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \frac{|u(x + h) - u(x)|^q}{\rho^q} \, dx \right) < +\infty. \quad (1.19)$$

Moreover, for every open $\Omega \subset \mathbb{R}^N$ we say that $u \in L^q_{loc}(\Omega, \mathbb{R}^m)$ belongs to Besov space $(B_{q,\infty}^s)_{loc}(\Omega, \mathbb{R}^m)$ if for every compact $K \subset \subset \Omega$ there exists $u_K \in B_{q,\infty}^s(\mathbb{R}^N, \mathbb{R}^d)$ such that $u_K(x) = u(x)$ for every $x \in K$.

The following technical proposition makes the connection between Besov spaces and the quantities appearing in the statement of Theorem 1.4. This proposition is a direct consequence of Corollary A.3 and Lemma A.5; see in the Appendix, whose proofs are based on similar arguments to those used in [5].

**Proposition 1.1.** If $q \geq 1$, $r \in (0, q)$ and $u \in L^q(\mathbb{R}^N, \mathbb{R}^m)$, then, $u \in \left( B_{q,\infty}^{r/q}(\mathbb{R}^N, \mathbb{R}^m) \right)$ if and only if we have

$$\limsup_{\varepsilon \to 0^+} \left\{ \int_{S^{N-1}} \int_{\mathbb{R}^N} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} \, dx \, d\mathcal{H}^{N-1}(n) \right\} < +\infty. \quad (1.20)$$

Moreover, if $\Omega \subset \mathbb{R}^N$ be an open set, $q \geq 1$, $r \in (0, q)$ and $u \in L^q_{loc}(\Omega, \mathbb{R}^m)$, then, $u \in \left( B_{q,\infty}^{r/q}(\Omega, \mathbb{R}^m) \right)$ if and only if for every open set $G \subset \subset \Omega$ we have

$$\limsup_{\varepsilon \to 0^+} \left\{ \int_{S^{N-1}} \int_G \chi_G(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} \, dx \, d\mathcal{H}^{N-1}(n) \right\} < +\infty. \quad (1.21)$$

Combining Theorem 1.4 in the case $r \in (0, q)$ with Proposition 1.1 might be a first step towards an answer to open question (v).

Our last result links the quantities in (1.4) for $r = 1, q > 1$ and $u \in BV \cap L^\infty$ with the “jump in the power $q$” of $u$:

**Theorem 1.5.** Let $\Omega$ be an open set with bounded Lipschitz boundary, $q > 1$ and $u \in BV(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$. Then,

$$N \liminf_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{N+1}} > s \right\} \right) \right\} \leq \left( \int_{S^{N-1}} |z_1| \, d\mathcal{H}^{N-1}(z) \right) \left( \int_{J_u \cap \Omega} \left| u^+(x) - u^-(x) \right|^q \, d\mathcal{H}^{N-1}(x) \right) \leq (N + r) \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{N+1}} > s \right\} \right) \right\}. \quad (1.22)$$

Here $J_u$ denotes the jump set of $u \in BV$ and $u^+, u^-$ are the approximate one-side limits of $u$. 

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To conclude, we list some interesting open problems for future research:

(a) Does a complete version of Theorem 1.4 hold, where we replace lim inf by lim sup in (1.15) and (1.16). In particular, following Proposition 1.1, this would provide a full answer to Question (v) in the case \( u \in L^\infty \).

(b) In the spirit of Theorem 1.2, does (1.15) in Theorem 1.4 hold with the constant \( N \), instead of \( (N + r) \)?

(c) Does Corollary 1.2 hold for \( q = 1 \) and \( u \in BV \setminus W^{1,1} \)?

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# 2 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Its special case \( r = q \) is essential for the proof of the main results Theorem 1.3 and Corollary 1.1.

**Proof of Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( u \in L^\infty(\Omega, \mathbb{R}^m) \). Given \( \alpha > 0 \), consider

\[
\eta_\varepsilon(t) := \frac{1}{\alpha |\ln \varepsilon| t^N} \chi_{[\varepsilon^{1+\alpha}, \varepsilon]}(t) \quad \forall t \in \mathbb{R}.
\]

In particular, we have

\[
\eta_\varepsilon(t) t^N = \frac{1}{\alpha |\ln \varepsilon|} \chi_{[\varepsilon^{1+\alpha}, \varepsilon]}(t). \tag{2.2}
\]

Then, for every \( z \in \mathbb{R}^N \) every \( h \geq 0 \) and every \( s \geq 0 \) considering

\[
K_{\varepsilon, u}(z, s, h) := L^N \left( \left\{ \chi_{\Omega}(x + z) \frac{|u(x + z) - u(x)|^q}{|z|^h} > s \right\} \right)
\]

\[
= K_{\varepsilon, u} \left( z, \frac{s}{|z|^h}, h + l \right) \quad \forall l \geq 0, \tag{2.3}
\]
by Fubini Theorem we deduce:

\[
\int \int_{\Omega} \eta_{\varepsilon}(|y-x|) \frac{|u(y) - u(x)|^q}{|y-x|^r} dydx = \int \int_{\mathbb{R}^N} \eta_{\varepsilon}(|z|) \chi_{\Omega}(x+z) \frac{|u(x+z) - u(x)|^q}{|z|^r} dx dz
\]

\[
= \int \int_{\mathbb{R}^+ \times \mathbb{R}^N} \eta_{\varepsilon}(z) K_{\varepsilon,u}(z,s,r) dz ds = \int \int \int_{S^{N-1} \times \mathbb{R}^+} \eta_{\varepsilon}(t) t^{N-1} K_{\varepsilon,u}(t n, s t N + r + N) dt ds dH^{N-1}(n)
\]

\[
= \int \int \int_{S^{N-1} \times \mathbb{R}^+} \eta_{\varepsilon}(t) t^N t^{N-1} K_{\varepsilon,u}(t n, s + r + N) dt ds dH^{N-1}(n). \tag{2.4}
\]

Thus, by (2.4) and (2.2) we have

\[
\int \int_{\Omega} \eta_{\varepsilon}(|y-x|) \frac{|u(y) - u(x)|^q}{|y-x|^r} dydx = I_{\varepsilon,u,\alpha}([0, +\infty]). \tag{2.5}
\]

where, for every \(0 \leq a \leq b \leq +\infty\) we denote:

\[
I_{\varepsilon,u,\alpha}([a, b]) := \int \int_{\mathbb{R} \times S^{N-1}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon^{1+\alpha}, \varepsilon]}(t) \chi_{[a,b]}(s) t^{N-1} s K_{\varepsilon,u}(t n, s + N) dH^{N-1}(n) dt ds
\]

\[
= \int \int \int_{\mathbb{R} \times S^{N-1}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon^{1+\alpha}, \varepsilon]}(t) \chi_{[a,b]}(s) \times
\]

\[
\times t^N t^{N-1} s \mathcal{L}^N \left( \left\{ x \in \Omega : \frac{\chi_{\Omega}(x + t n) |u(x + t n) - u(x)|^q}{|t|^{r+N}} > s \right\} \right) dH^{N-1}(n) dt ds. \tag{2.6}
\]

So, for every \(d > 0\) and \(\gamma > 0\) we have

\[
\int \int_{\Omega} \eta_{\varepsilon}(|y-x|) \frac{|u(y) - u(x)|^q}{|y-x|^r} dydx
\]

\[
= I_{\varepsilon,u,\alpha} \left( \left[ \frac{1}{\varepsilon^d}, \frac{1}{\varepsilon^{d+\gamma}} \right] \right) + I_{\varepsilon,u,\alpha} \left( \left[ 0, \frac{1}{\varepsilon^d} \right] \right) + I_{\varepsilon,u,\alpha} \left( \left[ \frac{1}{\varepsilon^{d+\gamma}}, +\infty \right] \right). \tag{2.7}
\]

Furthermore, since \(\Omega\) is bonded, by (2.6) we can obtain

\[
I_{\varepsilon,u,\alpha} \left( \left[ 0, \frac{1}{\varepsilon^d} \right] \right) \leq \frac{C}{\alpha \ln \varepsilon} \varepsilon^{N-d}, \tag{2.8}
\]
and

\[
I_{\varepsilon,u,\alpha}\left(\frac{1}{\varepsilon^{d+\gamma}}, +\infty\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{S^{N-1}} \frac{1}{\alpha \ln \varepsilon} \frac{1}{t^{r+1}} \chi_{[\varepsilon^{1+\alpha},\varepsilon]}(t) \chi_{[0,1]}(s) \times
\]

\[
\times t^{r+N} K_{\varepsilon,u}(tn, st^{r+N}, 0) dH^{N-1}(n) dt ds
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{S^{N-1}} \frac{1}{\alpha \ln \varepsilon} \frac{1}{t^{r+1}} \chi_{[\varepsilon^{1+\alpha},\varepsilon]}(t) \chi_{[\varepsilon^{r+N}/\varepsilon^{d+\gamma}, +\infty]}(\tau) K_{\varepsilon,u}(tn, \tau, 0) dH^{N-1}(n) dt d\tau
\]

\[
\leq \int_{\varepsilon^{(d+\gamma)-(1+\alpha)(r+N)}}^{+\infty} \int_{\mathbb{R}} \int_{S^{N-1}} \frac{1}{\alpha \ln \varepsilon} \frac{1}{t^{r+1}} \chi_{[\varepsilon^{1+\alpha},\varepsilon]}(t) K_{\varepsilon,u}(tn, \varepsilon^{(d+\gamma)-(1+\alpha)(r+N)}, 0) dH^{N-1}(n) dt d\tau.
\]

Thus, since \( u \in L^\infty \), in the case \( d \leq N \) and \( \gamma > (1 + \alpha)(r + N) - d \) for sufficiently small \( \varepsilon > 0 \) we have

\[
K_{\varepsilon,u}(tn, \varepsilon^{(d+\gamma)-(1+\alpha)(r+N)}, 0) := \left\{ x \in \Omega : \chi_{\Omega}(x + tn) |u(x + tn) - u(x)|^q > 1/\varepsilon^{(d+\gamma)-(1+\alpha)(r+N)} \right\} = \emptyset,
\]

and thus by (2.8) and (2.9) we rewrite (2.7), in the case \( d \leq N \), \( \gamma > (1 + \alpha)(r + N) - d \) and sufficiently small \( \varepsilon > 0 \), as:

\[
\int_{\Omega} \int_{\Omega} \eta_\varepsilon (|y - x|) \left| \frac{u(y) - u(x)}{|y - x|^r} \right|^q dy dx = O\left(\frac{1}{\alpha \ln \varepsilon}\right) + I_{\varepsilon,u,\alpha}\left(\frac{1}{\varepsilon^{d+\gamma}}, +\infty\right)
\]

\[
\leq O\left(\frac{1}{\alpha \ln \varepsilon}\right) + \int_{\mathbb{R}} \int_{S^{N-1}} \frac{1}{\alpha \ln \varepsilon} \chi_{[0,1]}(s) \chi_{[\varepsilon^{1+\alpha},\varepsilon]}(s) \times
\]

\[
\times t^{r+N} sL^N\left(\left\{ x \in \Omega : \chi_{\Omega}(x + tn) \left| \frac{u(x + tn) - u(x)}{|t|^{r+N}} \right|^q > s \right\} \right) dH^{N-1}(n) dt ds =
\]

\[
O\left(\frac{1}{\alpha \ln \varepsilon}\right) + \int_{\mathbb{R}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon^{1+\alpha},\varepsilon^{d+\gamma}]}(s) sL^N\left(\left\{ (x, y) \in \Omega \times \Omega : \left| \frac{u(y) - u(x)}{|y - x|^r} \right|^q > s \right\} \right) ds.
\]

(2.11)
On the other hand, we have (2.12), then we deduce:

\[
\int_\Omega \int_\Omega \eta_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} \, dy \, dx = \int_\Omega \int_\Omega \eta_\varepsilon(|z|) \chi_\Omega(x + z) \frac{|u(x + z) - u(x)|^q}{|z|^r} \, dx \, dz
\]

\[
= \int \int \int_{S^{N-1} \times \Omega} t^{N-1} \eta_\varepsilon(t) \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, dt \, d\mathcal{H}^{N-1}(\mathbf{n})
\]

\[
= \int_{\varepsilon^{1+\alpha}}^\varepsilon \frac{1}{\alpha |\ln \varepsilon|^t} \left( \int_{S^{N-1} \times \Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, dt \right) d\mathcal{H}^{N-1}(\mathbf{n}) dt.
\] (2.12)

Thus, by (2.12) and (2.11) for \( d = N, \gamma > r + \alpha(N + r) \) and sufficiently small \( \varepsilon > 0 \) we have

\[
\int_{\varepsilon^{1+\alpha}}^\varepsilon \frac{1}{\alpha |\ln \varepsilon|^t} \left( \int_{S^{N-1} \times \Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, dt \right) d\mathcal{H}^{N-1}(\mathbf{n}) dt
\]

\[
\leq \int_\Omega \int_\Omega \eta_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} \, dy \, dx
\]

\[
= \int_{\varepsilon^{1+\alpha}}^\varepsilon \frac{1}{\alpha |\ln \varepsilon|^t} \left( \int_{S^{N-1} \times \Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, dt \right) d\mathcal{H}^{N-1}(\mathbf{n}) dt
\]

\[
\leq \sup_{t \in (0, \varepsilon)} \left( \int_{S^{N-1} \times \Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, dt \right),
\] (2.14)

and

\[
\inf_{s > (1/\varepsilon)^N} \left\{ s L^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{r+N}} > s \right\} \right) \right\} \leq
\]

\[
\int_{\gamma |\ln \varepsilon|^s} \chi_{[1/\varepsilon^N, 1/\varepsilon^N+\gamma]}(s) s L^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{r+N}} > s \right\} \right) ds
\]

\[
\leq \sup_{s > (1/\varepsilon)^N} \left\{ s L^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{r+N}} > s \right\} \right) \right\}. (2.15)
\]
Therefore, inserting these into (2.13) gives that for \( d = N, \gamma > r + \alpha(N + r) \) and sufficiently small \( \varepsilon > 0 \) we have

\[
\inf_{t \in (0, \varepsilon)} \left( \int \int_{S^{N-1}} \chi_{(x + t n)} \left| \frac{u(x + t n) - u(x)}{t^r} \right|^{q} \, dx d\mathcal{H}^{N-1}(n) \right) \leq O \left( \frac{1}{\alpha \ln \varepsilon} \right) + \gamma \sup_{s > (1/\varepsilon N)} \left\{ s L^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \left| \frac{u(y) - u(x)}{|y - x|^{r+N}} > s \right\} \right) \right\}. \tag{2.16}
\]

Thus, letting \( \varepsilon \to 0^+ \) in (2.16) gives that for \( d = N \) and \( \gamma > r + \alpha(N + r) \) we have

\[
\lim_{\varepsilon \to 0^+} \inf_{t \in (0, \varepsilon)} \left( \int \int_{S^{N-1}} \chi_{(x + \varepsilon n)} \left| \frac{u(x + \varepsilon n) - u(x)}{\varepsilon^{r}} \right|^{q} \, dx d\mathcal{H}^{N-1}(n) \right) \leq \frac{\gamma}{\alpha} \lim_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \left| \frac{u(y) - u(x)}{|y - x|^{r+N}} > s \right\} \right) \right\}. \tag{2.17}
\]

Therefore, letting \( \gamma \to (r + \alpha(N + r))^+ \) in (2.17) we deduce for \( d = N \) and \( \gamma = r + \alpha(N + r) \):

\[
\lim_{\varepsilon \to 0^+} \inf_{t \in (0, \varepsilon)} \left( \int \int_{S^{N-1}} \chi_{(x + \varepsilon n)} \left| \frac{u(x + \varepsilon n) - u(x)}{\varepsilon^{r}} \right|^{q} \, dx d\mathcal{H}^{N-1}(n) \right) \leq \left( \frac{r}{\alpha} + (N + r) \right) \lim_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \left| \frac{u(y) - u(x)}{|y - x|^{r+N}} > s \right\} \right) \right\}. \tag{2.18}
\]

Letting \( \alpha \to +\infty \) in (2.18) we finally deduce (1.15).

Next, by (2.7) for every \( d > 0 \) and \( \gamma > 0 \) we have

\[
\int_{\Omega} \int_{\Omega} \eta_{\varepsilon}(|y - x|) \left| \frac{u(y) - u(x)}{|y - x|^{r}} \right|^{q} \, dy dx \geq I_{\varepsilon,u,\alpha} \left( \left[ \frac{1}{\varepsilon^{d}}, \frac{1}{\varepsilon^{d+\gamma}} \right] \right). \tag{2.19}
\]

Furthermore, by (2.3) we have

\[
\int \int_{S^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^{d}, 1/\varepsilon^{d+\gamma}]}(s) t^{N-1} s K_{\varepsilon,u}(tn, s, r + N) \, d\mathcal{H}^{N-1}(n) \, dt ds = \int \int_{S^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^{d}, 1/\varepsilon^{d+\gamma}]}(s) \frac{1}{t^{r+1}} t^{N+r} K_{\varepsilon,u}(tn, st^{r+N}, 0) \, d\mathcal{H}^{N-1}(n) \, dt ds = \int \int_{S^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^{d-(r+N)}, 1/\varepsilon^{d+\gamma}]}(t) \frac{1}{t^{r+1}} K_{\varepsilon,u}(tn, \tau, 0) \, d\mathcal{H}^{N-1}(n) \, dt d\tau \leq \int \int_{S^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^{d-(r+N)}, 1/\varepsilon^{d+\gamma}]}(t) \frac{1}{t^{r+1}} K_{\varepsilon,u}(tn, \tau, 0) \, d\mathcal{H}^{N-1}(n) \, dt d\tau \leq \int \int_{S^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\alpha \ln \varepsilon} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^{d-(r+N)}, 1/\varepsilon^{d+\gamma}]}(t) \frac{1}{t^{r+1}} K_{\varepsilon,u}(tn, \tau, 0) \, d\mathcal{H}^{N-1}(n) \, dt d\tau. \tag{2.20}
\]
On the other hand, since \( u \in L^\infty \), in the case \( d > (N + r) \) for sufficiently small \( \varepsilon > 0 \) we have

\[
K_{\varepsilon,u}(t, n; \frac{1}{\varepsilon^{d-(r+N)}}, 0) = \left\{ x \in \Omega : \chi_\Omega(x + tn)|u(x + tn) - u(x)|^q > 1/\varepsilon^{d-(r+N)} \right\} = \emptyset, \quad (2.21)
\]

and thus by (2.20), in the case \( d > (N + r) \) for sufficiently small \( \varepsilon > 0 \) we have

\[
\int \int \int _{\mathbb{R}^2} \frac{1}{|\ln \varepsilon|} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^d, 1/\varepsilon^{d+\gamma}]}(s) \times
\]

\[
\times t^{N-1} s^{L^N} \left( \left\{ x \in \Omega : \frac{\chi_\Omega(x + tn)|u(x + tn) - u(x)|^q}{|t|^{r+N}} > s \right\} \right) d\mathcal{H}^{N-1}(n) dt ds = 0. \quad (2.22)
\]

In particular, by (2.19) and (2.22) in the case \( d > (N + r) \) for sufficiently small \( \varepsilon > 0 \) we have

\[
\int \int _{\Omega} \eta_\varepsilon \left( |y - x| \right) \frac{|u(y) - u(x)|^q}{|y - x|^r} dy dx 
\]

\[
\leq \int \int \int _{\mathbb{R}^2} \frac{1}{|\ln \varepsilon|} \chi_{[\varepsilon, +\infty)}(t) \chi_{[1/\varepsilon^d, 1/\varepsilon^{d+\gamma}]}(s) \times
\]

\[
\times t^{N-1} s^{L^N} \left( \left\{ x \in \Omega : \frac{\chi_\Omega(x + tn)|u(x + tn) - u(x)|^q}{|t|^{r+N}} > s \right\} \right) d\mathcal{H}^{N-1}(n) dt ds \quad (2.23)
\]

On the other hand, since \( \Omega \) is bonded, we have

\[
\int \int \int _{\mathbb{R}^2} \frac{1}{|\ln \varepsilon|} \chi_{[0, \varepsilon^{1+\alpha}]}(t) \chi_{[1/\varepsilon^d, 1/\varepsilon^{d+\gamma}]}(s) \times
\]

\[
\times t^{N-1} s^{L^N} \left( \left\{ x \in \Omega : \frac{\chi_\Omega(x + tn)|u(x + tn) - u(x)|^q}{|t|^{r+N}} > s \right\} \right) d\mathcal{H}^{N-1}(n) dt ds
\]

\[
\leq \int \int \int _{\mathbb{R}^2} \frac{1}{|\ln \varepsilon|} t^{N-1} \chi_{[0, \varepsilon^{1+\alpha}]}(t) \chi_{[1/\varepsilon^d, 1/\varepsilon^{d+\gamma}]}(s) C dt ds \leq C \frac{\varepsilon^{N(1+\alpha)-(d+\gamma)}}{N \alpha |\ln \varepsilon|}. \quad (2.24)
\]

Moreover, by (2.2) we deduce:

\[
\int \int _{\Omega} \eta_\varepsilon \left( |y - x| \right) \frac{|u(y) - u(x)|^q}{|y - x|^r} dy dx = \int \int _{\mathbb{R}^N} \eta_\varepsilon \left( |z| \right) \chi_\Omega(x + z) \frac{|u(x + z) - u(x)|^q}{|z|^r} dz dx
\]

\[
= \int \int \int _{\mathbb{R}^N} \eta_\varepsilon(t) \chi_\Omega(x + tn) \frac{|u(x + tn) - u(x)|^q}{t^r} dt dx d\mathcal{H}^{N-1}(n)
\]

\[
= \int \frac{1}{\alpha |\ln \varepsilon|} \left( \int \int _{\mathbb{R}^N} \chi_\Omega(x + tn) \frac{|u(x + tn) - u(x)|^q}{t^r} dxdH^{N-1}(n) \right) dt. \quad (2.25)
\]

Thus, by (2.23), (2.24) and (2.25), in the case \( d > (N + r) \) and \( N(1+\alpha)-(d+\gamma) > 0 \) for
sufficiently small $\varepsilon > 0$ we deduce

$$\int_{\varepsilon^{1+\alpha}}^{\varepsilon} \frac{1}{\alpha |\ln \varepsilon|} t \left( \int_{S^{N-1}} \int_{\Omega} \chi_{\Omega}(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, d\mathcal{H}^{N-1}(\mathbf{n}) \right) dt$$

$$= \int \int_{\Omega} \eta_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} \, dy \, dx \geq O \left( \frac{1}{\alpha |\ln \varepsilon|} \right) + \int \int_{S^{N-1}} \int_{\Omega} \frac{\gamma}{\alpha} \frac{1}{\varepsilon |\ln \varepsilon|} \chi_{[1/\varepsilon^{d},1/\varepsilon^{d+\gamma}]}(t) \chi_{[1/\varepsilon^{d},1/\varepsilon^{d+\gamma}]}(s) t^{N-1} sK_{\varepsilon,u}(t\mathbf{n},s,r + N) \, d\mathcal{H}^{N-1}(\mathbf{n}) \, dt \, ds = O \left( \frac{1}{\alpha |\ln \varepsilon|} \right) + \int \int_{\Omega} \frac{\gamma}{\alpha} \frac{1}{\varepsilon |\ln \varepsilon|} \chi_{[1/\varepsilon^{d},1/\varepsilon^{d+\gamma}]}(t) \chi_{[1/\varepsilon^{d},1/\varepsilon^{d+\gamma}]}(s) \, d\mathcal{L}^{2N} \left( \left\{ (x,z) \in \Omega \times \mathbb{R}^N : \frac{\chi_{\Omega}(x + z)|u(x + z) - u(x)|^q}{|z|^{r+N}} > s \right\} \right) ds. \quad (2.26)$$

Then, by (2.26) we infer

$$\int_{\varepsilon^{1+\alpha}}^{\varepsilon} \frac{1}{\alpha |\ln \varepsilon|} t \left( \int_{S^{N-1}} \int_{\Omega} \chi_{\Omega}(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, d\mathcal{H}^{N-1}(\mathbf{n}) \right) dt$$

$$= \int \int_{\Omega} \eta_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} \, dy \, dx \geq O \left( \frac{1}{\alpha |\ln \varepsilon|} \right) + \int \int_{S^{N-1}} \int_{\Omega} \frac{\gamma}{\alpha} \frac{1}{\varepsilon |\ln \varepsilon|} \chi_{[1/\varepsilon^{d},1/\varepsilon^{d+\gamma}]}(t) s^{-1} \mathcal{L}^{2N} \left( \left\{ (x,y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{r+N}} > s \right\} \right) ds. \quad (2.27)$$

On the other hand, we have

$$\inf_{t \in (0,\varepsilon)} \left( \int_{S^{N-1}} \int_{\Omega} \chi_{\Omega}(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, d\mathcal{H}^{N-1}(\mathbf{n}) \right)$$

$$\leq \int \int_{\Omega} \eta_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} \, dy \, dx = \int_{\varepsilon^{1+\alpha}}^{\varepsilon} \frac{1}{\alpha |\ln \varepsilon|} t \left( \int_{S^{N-1}} \int_{\Omega} \chi_{\Omega}(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, d\mathcal{H}^{N-1}(\mathbf{n}) \right) dt$$

$$\leq \sup_{t \in (0,\varepsilon)} \left( \int_{S^{N-1}} \int_{\Omega} \chi_{\Omega}(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} \, dx \, d\mathcal{H}^{N-1}(\mathbf{n}) \right). \quad (2.28)$$
Lemma 3.1 after proving some intermediate results. We start with the following Proposition:

This section is devoted to the proof of Theorem 1.3, that will follow from Corollary 3.2 and Lemma 3.1 after proving some intermediate results. We start with the following Proposition:
Proposition 3.1. Let $\Omega \subset \mathbb{R}^N$ be an open set, $q \geq 1$ and $u \in L^q(\Omega, \mathbb{R}^m)$. Furthermore, let $G \subset \Omega$ be an open subset, such that $\mathcal{L}^N(\partial G) = 0$ and either $G$ is convex or $G \subset \subset \Omega$. Then,

$$
\limsup_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{G} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \right)
$$

$$
\leq \sup_{\varepsilon \in (0, h)} \left( \int_{S^{N-1}} \int_{G} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \right)
$$

$$
\leq (N + q) \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}, \quad (3.1)
$$

where

$$
h := \begin{cases} 
+\infty & \text{if } G \text{ is convex}, \\
\text{dist}(G, \mathbb{R}^N \setminus \Omega) & \text{otherwise}.
\end{cases} \quad (3.2)
$$

Proof. In the case of bounded $\Omega$ and $u \in L^\infty(\Omega, \mathbb{R}^m)$, (3.1) follow from Theorem 1.4 together with either Corollary A.1 or Corollary A.2 from the Appendix. So it remains to prove (3.1) in the case of unbounded $\Omega$ and/or $u \notin L^\infty(\Omega, \mathbb{R}^m)$. Thus for every $k \in \mathbb{N}$ consider a bounded open sets $G_k \subset \Omega_k \subset \Omega$ with $\mathcal{L}^N(\partial G_k) = 0$, defined by

$$
G_k := G \cap B_k(0) \quad \text{and} \quad \Omega_k := \Omega \cap B_k(0), \quad (3.3)
$$

and consider $u^{(k)}(x) := (u^{(k)}_1(x), \ldots, u^{(k)}_m(x)) \in L^\infty(\Omega, \mathbb{R}^m) \cap L^q(\Omega, \mathbb{R}^m)$, defined by

$$
u^{(k)}_j(x) := \begin{cases} 
-k & \text{if } u(x) \leq -k, \\
u(x) & \text{if } u(x) \in [-k, k], \\
k & \text{if } u(x) \geq k,
\end{cases} \quad \forall x \in \Omega \quad \forall j \in \{1, \ldots, m\}. \quad (3.4)
$$

Then obviously,

$$
|u^{(k)}(y) - u^{(k)}(x)|^q \leq |u(y) - u(x)|^q \quad \forall (x, y) \in \Omega \times \Omega, \quad \forall k \in \mathbb{N}. \quad (3.5)
$$

and

$$
\lim_{k \to +\infty} u^{(k)}(x) = u(x) \quad \forall x \in \Omega. \quad (3.6)
$$

Moreover, if $G$ is convex then $G_k$ is also convex. Otherwise, $G_k = G$ for sufficiently large $k$ and $\text{dist}(G_k, \mathbb{R}^N \setminus \Omega_k) = \text{dist}(G, \mathbb{R}^N \setminus \Omega)$ for sufficiently large $k$. Thus, by (3.1) with $\Omega_k$ instead of $\Omega$ and $u^{(k)}$ instead of $u$, for sufficiently large $k$ we have

$$
\sup_{\varepsilon \in (0, h)} \left( \int_{S^{N-1}} \int_{G_k} \frac{|u^{(k)}(x + \varepsilon n) - u^{(k)}(x)|^q}{\varepsilon^q} \chi_{G_k}(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \right)
$$

$$
\leq (N + q) \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega_k \times \Omega_k : \frac{|u^{(k)}(y) - u^{(k)}(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \quad \forall k \in \mathbb{N}. \quad (3.7)
$$
Thus, since $\Omega_k \subset \Omega$ by (3.5) we deduce from (3.7) that, for sufficiently large $k$ we have

$$\int_{S^{N-1} \setminus G_k} \frac{|u^{(k)}(x + \varepsilon n) - u^{(k)}(x)|^q}{\varepsilon^q} \chi_{G_k}(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \leq$$

$$(N+q) \limsup_{s \to +\infty} \left\{ \mathcal{L}^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \quad \forall \varepsilon \in (0, h), \quad \forall k \in \mathbb{N}. \quad (3.8)$$

Then, letting $k \to +\infty$ in (3.8), using (3.5), (3.6) and the Dominated Convergence Theorem, gives

$$\int_{S^{N-1} \setminus G} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \leq$$

$$(N+q) \limsup_{s \to +\infty} \left\{ \mathcal{L}^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \quad \forall \varepsilon \in (0, h). \quad (3.9)$$

In particular,

$$\sup_{\varepsilon \in (0,h)} \left( \int_{S^{N-1} \setminus G} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \right)$$

$$\leq (N+q) \limsup_{s \to +\infty} \left\{ \mathcal{L}^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}. \quad (3.10)$$

Thus, by (3.10) we finally deduce (3.1). \[\square\]

**Corollary 3.1.** Let $\Omega \subset \mathbb{R}^N$ be a convex open domain, such that $\mathcal{L}^N(\partial \Omega) = 0$, $q \geq 1$ and $u \in L^q(\Omega, \mathbb{R}^m)$. Then,

$$\sup_{\varepsilon \in (0, +\infty)} \left( \int_{S^{N-1} \setminus \Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_{\Omega}(x + \varepsilon n) d\mathcal{H}^{N-1}(n) \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( \int_{S^{N-1} \setminus \Omega} \chi_{\Omega}(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} d\mathcal{H}^{N-1}(n) \right)$$

$$\leq (N+q) \limsup_{s \to +\infty} \left\{ \mathcal{L}^2 \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}. \quad (3.11)$$

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Moreover, in the case of bounded \( \Omega \) and \( u \in L^\infty(\Omega, \mathbb{R}^m) \) we also have

\[
\sup_{\varepsilon \in (0, +\infty)} \left( \int_{S^{n-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_\Omega(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right) \\
= \lim_{\varepsilon \to 0^+} \left( \int_{S^{n-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} dx d\mathcal{H}^{N-1}(n) \right) \\
\geq N \liminf_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}.
\]

(3.12)

**Proof.** In the case of bounded \( \Omega \) and \( u \in L^\infty(\Omega, \mathbb{R}^m) \), (3.11) and (3.12) follow from Theorem 1.4 together with Corollary A.2 from the Appendix. On the other hand, in the case of unbounded \( \Omega \) and/or \( u \notin L^\infty(\Omega, \mathbb{R}^m) \), in order to prove (3.11) we use Proposition 3.1 with \( G = \Omega \), together with Corollary A.2 from the Appendix. \( \square \)

**Corollary 3.2.** Let \( \Omega \subset \mathbb{R}^N \) be an open domain, \( q \geq 1 \) and \( u \in L^q(\Omega, \mathbb{R}^m) \). Then, in the case \( q > 1 \) we have

\[
\frac{\int_{S^{n-1}} |z| q d\mathcal{H}^{N-1}(z)}{(N + q)} \int_{\Omega} |\nabla u(x)|^q dx \leq \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}. \]

(3.13)

with the convention that \( \int_{\Omega} |\nabla u(x)|^q dx = +\infty \) if \( u \notin W^{1,q}(\Omega, \mathbb{R}^m) \). On the other hand, in the case \( q = 1 \) we have:

\[
\frac{\int_{S^{n-1}} |z| d\mathcal{H}^{N-1}(z)}{(N + 1)} \|Du\|(\Omega) \leq \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|^{1+N}} > s \right\} \right) \right\}. \]

(3.14)

with the convention \( \|Du\|(\Omega) = +\infty \) if \( u \notin BV(\Omega, \mathbb{R}^m) \).

**Proof.** Let \( \rho_\varepsilon(|z|) : \mathbb{R}^N \to [0, +\infty) \) be radial mollifiers, so that \( \int_{\mathbb{R}^N} \rho_\varepsilon(|z|) dz = 1 \) and for every \( r > 0 \) there exits \( \delta := \delta_r > 0 \), such that \( \text{supp}(\rho_\varepsilon) \subset B_r(0) \) for every \( \varepsilon \in (0, \delta_r) \). Next fix an open subset \( G \subset \subset \Omega \) with Lipschitz boundary. Since, by Proposition 3.1 we have

\[
\limsup_{\varepsilon \to 0^+} \left( \int_{S^{n-1}} \int_G \chi_G(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} dx d\mathcal{H}^{N-1}(n) \right) \\
\leq (N + q) \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}. \]

(3.15)
and at the same time by Lemma \[A.1\] we have:

\[
\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \limsup_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{G} \chi_{G}(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \rho_{\varepsilon} |y - x|^q dxd\mathcal{H}^{N-1}(n) \right) \\
\geq \limsup_{\varepsilon \to 0^+} \int_{G} \int \rho_{\varepsilon} \left| \frac{u(y) - u(x)}{|y - x|^q} \right| dydx,
\]

we deduce from (3.15) and (3.16) that

\[
\limsup_{\varepsilon \to 0^+} \int_{G} \int \rho_{\varepsilon} \left| \frac{u(y) - u(x)}{|y - x|^q} \right| dydx \\
\leq \frac{(N + q)}{\mathcal{H}^{N-1}(S^{N-1})} \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}.
\]

On the other hand, since bounded $G \subset \mathbb{R}^N$ has a Lipschitz boundary, the so called “BBM formula” (see [1] and [4] for the details) states that for $q > 1$ we have

\[
\lim_{\varepsilon \to 0^+} \int_{G} \int \frac{|u(x) - u(y)|^q}{|x - y|^q} \rho_{\varepsilon}(|x - y|) \, dx \, dy = K_{q,N} \int_{G} \left| \nabla u(x) \right|^q \, dx,
\]

with the convention that $\int_{G} \left| \nabla u(x) \right|^q \, dx = +\infty$ if $u \notin W^{1,q}(G, \mathbb{R}^m)$ and with $K_{q,N}$ given by

\[
K_{q,N} := \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \int_{S^{N-1}} |z|^q d\mathcal{H}^{N-1}(z) \quad \forall q \geq 1,
\]

where we denote $z := (z_1, \ldots, z_N) \in \mathbb{R}^N$. Moreover, for $q = 1$ we have

\[
\lim_{\varepsilon \to 0^+} \int_{G} \int \frac{|u(x) - u(y)|}{|x - y|} \rho_{\varepsilon}(|x - y|) \, dx \, dy = K_{1,N} \|Du\|(G),
\]

with the convention $\|Du\|(G) = +\infty$ if $u \notin BV(G, \mathbb{R}^m)$. Inserting it into (3.17) gives for $q > 1$:

\[
\int_{S^{N-1}} |z|^q d\mathcal{H}^{N-1}(z) \\
\frac{(N + q)}{\mathcal{H}^{N-1}(S^{N-1})} \int_{G} \left| \nabla u(x) \right|^q \, dx = K_{q,N} \frac{\mathcal{H}^{N-1}(S^{N-1})}{(N + q)} \int_{G} \left| \nabla u(x) \right|^q \, dx \\
\leq \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\},
\]

and for $q = 1$:

\[
\int_{S^{N-1}} |z| \, d\mathcal{H}^{N-1}(z) \\
\frac{(N + 1)}{\mathcal{H}^{N-1}(S^{N-1})} \|Du\|(G) = K_{1,N} \frac{\mathcal{H}^{N-1}(S^{N-1})}{(N + 1)} \|Du\|(G) \\
\leq \limsup_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|^{1+N}} > s \right\} \right) \right\}.
\]

Thus, taking the supremum of the left hand side of (3.21) over all open $G \subset \subset \Omega$ with Lipschitz boundary, we deduce (3.13) and taking the supremum of the left hand side of (3.22) over all open $G \subset \subset \Omega$ with Lipschitz boundary, we deduce (3.14). \[\square\]
Lemma 3.1. Let \( q \geq 1 \) and let \( \Omega \subset \mathbb{R}^N \) be a domain with Lipschitz boundary. Then there exist constants \( C := C_\Omega > 0 \) and \( \tilde{C}_N > 0 \), satisfying \( C_\Omega = 1 \) if \( \Omega = \mathbb{R}^N \), such that, in the case \( q > 1 \), for every \( u \in W^{1,q}(\Omega, \mathbb{R}^m) \) we have

\[
\sup_{s \in (0, +\infty)} \left\{ s L^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq C_\Omega \tilde{C}_N \int_\Omega |\nabla u(x)|^q dx , \tag{3.23}
\]

and, in the case \( q = 1 \), for every \( u \in BV(\Omega, \mathbb{R}^m) \) we have:

\[
\sup_{s \in (0, +\infty)} \left\{ s L^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|^{1+N}} > s \right\} \right) \right\} \leq C_\Omega \tilde{C}_N \|Du\|(\Omega) . \tag{3.24}
\]

Proof. By Extension Theorem for Sobolev and \( BV \) functions there exist a constant \( C := C_\Omega > 0 \) such that, in the case \( q > 1 \) for every \( u \in W^{1,q}(\Omega, \mathbb{R}^m) \) there exists its extension \( \tilde{u} \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m) \) with the property

\[
\begin{cases}
\tilde{u}(x) = u(x) & \forall x \in \Omega , \\
\int_{\mathbb{R}^N} |\nabla \tilde{u}(x)|^q dx \leq C_\Omega^q \int_\Omega |\nabla u(x)|^q dx ,
\end{cases}
\tag{3.25}
\]

and in the case \( q = 1 \) for every \( u \in BV(\Omega, \mathbb{R}^m) \) there exists its extension \( \tilde{u} \in BV(\mathbb{R}^N, \mathbb{R}^m) \) with the property

\[
\begin{cases}
\tilde{u}(x) = u(x) & \forall x \in \Omega , \\
\|Du\|(\mathbb{R}^N) \leq C_\Omega \|Du\|(\Omega) .
\end{cases}
\tag{3.26}
\]

Moreover, in the trivial case \( \Omega = \mathbb{R}^N \) we obviously can consider \( C_\Omega = 1 \). Next, by the standard properties of the Sobolev and the \( BV \) functions, there exist a sequence \( \{\varphi_n(x)\}_{n=1}^{+\infty} \subset C_c^\infty(\mathbb{R}^N, \mathbb{R}^m) \) such that in the case \( q > 1 \) we have

\[
\begin{cases}
\varphi_n \to \tilde{u} \quad \text{strongly in} \quad L^q(\mathbb{R}^N, \mathbb{R}^m) , \\
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^q dx = \int_{\mathbb{R}^N} |\nabla \tilde{u}(x)|^q dx ,
\end{cases}
\tag{3.27}
\]

and in the case \( q = 1 \) we have

\[
\begin{cases}
\varphi_n \to \tilde{u} \quad \text{strongly in} \quad L^1(\mathbb{R}^N, \mathbb{R}^m) , \\
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |\nabla \varphi_n(x)| dx = \|Du\|(\mathbb{R}^N) .
\end{cases}
\tag{3.28}
\]

On the other hand, H. Brezis, J. Van Schaftingen and Po-Lam Yung in [2] or [3] proved that for every \( q \geq 1 \) there exists a constant \( \tilde{C} := \tilde{C}_N > 0 \) such that for every \( \varphi(x) \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^m) \) we have

\[
\sup_{s \in (0, +\infty)} \left\{ s L^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|\varphi(y) - \varphi(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \tilde{C}_N \int_{\mathbb{R}^N} |\nabla \varphi(x)|^q dx . \tag{3.29}
\]
In particular, for every \( s > 0 \) and every \( n \in \mathbb{N} \) we have:

\[
s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{\varphi_n(y) - \varphi_n(x)}{|y - x|^{q+N}} > s \right\} \right) \leq \tilde{C}_N \int_{\mathbb{R}^N} |\nabla \varphi_n(x)|^q dx. \tag{3.30}
\]

Thus letting \( n \to +\infty \) in (3.30) and using either (3.27) (for \( q > 1 \)) or (3.28) (for \( q = 1 \)), in the case \( q > 1 \) for every \( s > 0 \) we deduce

\[
s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{\bar{u}(y) - \bar{u}(x)}{|y - x|^{q+N}} > s \right\} \right) \leq \tilde{C}_N \int_{\mathbb{R}^N} |\nabla \bar{u}(x)|^q dx. \tag{3.31}
\]

and in the case \( q = 1 \) for every \( s > 0 \) we deduce

\[
s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|}{|y - x|} > s \right\} \right) \leq \tilde{C}_N \|D\bar{u}\|_R^q. \tag{3.32}
\]

Thus, by (3.33) and (3.32), in the case \( q > 1 \) for every \( s > 0 \) we deduce

\[
s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|} > s \right\} \right) \leq \tilde{C}_N \|
abla u\|_{\Omega}. \tag{3.33}
\]

and by (3.26) and (3.32) in the case \( q = 1 \) for every \( s > 0 \) we deduce

\[
s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(y) - u(x)|}{|y - x|} > s \right\} \right) \leq \tilde{C}_N \|
abla u\|_{\Omega}. \tag{3.34}
\]

Finally, taking the supremum of either (3.33) or (3.34) over the set \( s \in (0, +\infty) \) completes the proof.

\[\square\]

**Proof of Theorem 1.3.** It suffices to combine Corollary 3.2 with Lemma 3.1. \[\square\]

### 4 Proof of Corollary 1.2

**Proof of Corollary 1.2.** Let \( \{u_n\}_{n=1}^{+\infty} \subset C^\infty_c(\mathbb{R}^N, \mathbb{R}^m) \) be such that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \left( |\nabla u_n(x) - \nabla u(x)|^q + |u_n(x) - u(x)|^q \right) = 0. \tag{4.1}
\]

Then, by Theorem 1.2, for every fixed \( n \in \mathbb{N} \) we have

\[
\lim_{\tau \to +\infty} \left\{ \tau^q \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u_n(y) - u_n(x)|}{|y - x|^{q+N}} > \tau \right\} \right) \right\} = \lim_{s \to +\infty} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u_n(y) - u_n(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} = \int_{\mathbb{R}^N} |\nabla u_n(x)|^q dx. \tag{4.2}
\]
On the other hand, given arbitrary $v \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m)$, $w \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m)$, $\tau > 0$ and $\alpha > 1$, denoting the sets:

$$A_1 := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(v(y) + w(y)) - (v(x) + w(x))|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \tau \right\}$$

$$A_2 := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|v(y) - v(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} + \frac{|w(y) - w(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \tau \right\}$$

$$B_1 := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|v(y) - v(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \frac{\tau}{\alpha} \right\}$$

$$B_2 := \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|w(y) - w(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \frac{(\alpha - 1)\tau}{\alpha} \right\},$$

by triangle inequality we obviously deduce:

$$A_1 \subset A_2 \subset B_1 \cup B_2,$$

and so,

$$\mathcal{L}^{2N}(A_1) \leq \mathcal{L}^{2N}(B_1) + \mathcal{L}^{2N}(B_2). \quad (4.3)$$

i.e. for every $v \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m)$ and $w \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m)$, every $\tau > 0$ and every $\alpha > 1$ we have

$$\mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(v(y) + w(y)) - (v(x) + w(x))|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \tau \right\} \right) \leq$$

$$\mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|v(y) - v(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \frac{\tau}{\alpha} \right\} \right) +$$

$$\mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|w(y) - w(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \frac{(\alpha - 1)\tau}{\alpha} \right\} \right), \quad (4.5)$$

and thus,

$$\tau^q \mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(v(y) + w(y)) - (v(x) + w(x))|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \tau \right\} \right) \leq$$

$$\alpha^q \left( \frac{\tau}{\alpha} \right)^q \mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|v(y) - v(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \frac{\tau}{\alpha} \right\} \right) +$$

$$\left( \frac{\alpha}{\alpha - 1} \right)^q \left( \frac{\alpha - 1}{\alpha} \right)^q \mathcal{L}^{2N}\left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|w(y) - w(x)|}{|y - x|} \frac{1}{|y - x|^\frac{N}{q}} > \frac{(\alpha - 1)\tau}{\alpha} \right\} \right). \quad (4.6)$$
Then, by (4.6) we deduce that for every $v \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m)$, $w \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^m)$ and every $\alpha > 1$ we have

\[
\lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|v(y) - v(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \alpha^q \lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|w(y) - w(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} + \left( \frac{\alpha}{\alpha - 1} \right)^q \lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(v(y) + w(y)) - (v(x) + w(x))|^q}{|y - x|^{q+N}} > s \right\} \right) \right\},
\]

(4.7)

In particular, taking firstly $v = u$ and $w = -u_n$ and secondly $v = u_n$ and $w = -u$ in (4.7), for every $\alpha > 1$ we deduce:

\[
\lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \alpha^q \lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u_n(y) - u_n(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} + \left( \frac{\alpha}{\alpha - 1} \right)^q \lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(u_n(y) - u_n(x)) - (u_n(x) - u_n(x))|^q}{|y - x|^{q+N}} > s \right\} \right) \right\},
\]

(4.8)

and

\[
\lim_{s \to +\infty} \inf_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u_n(y) - u_n(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \alpha^q \lim_{s \to +\infty} \inf_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} + \left( \frac{\alpha}{\alpha - 1} \right)^q \lim_{s \to +\infty} \inf_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(u_n(y) - u_n(x)) - (u_n(x) - u_n(x))|^q}{|y - x|^{q+N}} > s \right\} \right) \right\},
\]

(4.9)

Thus by combining (4.2) with (4.8) we deduce

\[
\lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \alpha^q \int_{\mathbb{R}^N} |z_1|^q d\mathcal{H}^{N-1}(z) \int_{\mathbb{R}^N} |\nabla u_n(x)|^q dx + \left( \frac{\alpha}{\alpha - 1} \right)^q \lim_{s \to +\infty} \sup_{s} \left\{ s \mathcal{L}^{2N} \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|(u_n(y) - u_n(x)) - (u_n(x) - u_n(x))|^q}{|y - x|^{q+N}} > s \right\} \right) \right\},
\]

(4.10)
and by combining (4.12) with (4.9) we deduce

\[
\frac{\int_{S^{N-1}} |z_1|^q dH^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u_n(x)|^q \, dx \leq \\
\alpha^q \liminf_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} + \\
\left( \frac{\alpha}{\alpha - 1} \right)^q \limsup_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}.
\]  

(4.11)

Then, using Theorem 1.3 by (4.10) for every fixed \( n \in \mathbb{N} \) and every \( \alpha > 1 \) we have

\[
\limsup_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \\
\alpha^q \frac{\int_{S^{N-1}} |z_1|^q dH^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u_n(x)|^q \, dx \\
+ \left( \frac{\alpha}{\alpha - 1} \right)^q \tilde{C}_N \int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^q \, dx. \tag{4.12}
\]

and using Theorem 1.3 by (4.11) for every fixed \( n \in \mathbb{N} \) and every \( \alpha > 1 \) we have

\[
\frac{\int_{S^{N-1}} |z_1|^q dH^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u_n(x)|^q \, dx \leq \\
\alpha^q \liminf_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} + \\
\left( \frac{\alpha}{\alpha - 1} \right)^q \tilde{C}_N \int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^q \, dx. \tag{4.13}
\]

Therefore, letting \( n \to +\infty \) in (4.12) and (4.13) and using (4.1) we deduce:

\[
\limsup_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \\
\alpha^q \frac{\int_{S^{N-1}} |z_1|^q dH^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u(x)|^q \, dx, \tag{4.14}
\]

and

\[
\frac{\int_{S^{N-1}} |z_1|^q dH^{N-1}(z)}{N} \int_{\mathbb{R}^N} |\nabla u(x)|^q \, dx \leq \\
\alpha^q \liminf_{s \to +\infty} \left\{ s L^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\}. \tag{4.15}
\]
Finally, letting $\alpha \to 1^+$ in (4.14) and (4.15) we infer
\[
\int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z) \int_{\mathbb{R}^N} |\nabla u(x)|^q dx \leq \\
\liminf_{s \to +\infty} \left\{ s \mathcal{L}^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq \\
\limsup_{s \to +\infty} \left\{ s \mathcal{L}^2 \left( \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : \frac{|u(y) - u(x)|^q}{|y - x|^{q+N}} > s \right\} \right) \right\} \leq 
\int_{S^{N-1}} |z_1|^q d\mathcal{H}^{N-1}(z) \int_{\mathbb{R}^N} |\nabla u(x)|^q dx,
\] (4.16)
and we obtain (1.8).

5 Proof of Theorem 1.5

The next Proposition is proved exactly as a similar statement in [5]; the proof is postponed to the Appendix; in both cases the key ingredient is Proposition A.1 which is part of [5, Proposition 2.4]).

Proposition 5.1. Let $\Omega$ be an open set with bounded Lipschitz boundary, $q > 1$ and $u \in BV(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$. Then,
\[
\lim_{\varepsilon \to 0^+} \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon} \chi_\Omega(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) = \\
\left( \int_{S^{N-1}} |z_1| d\mathcal{H}^{N-1}(z) \right) \left( \int_{\Omega \cap \Omega} |u^+(x) - u^-(x)|^q d\mathcal{H}^{N-1}(x) \right).
\] (5.1)
Proof of Theorem 1.5. The result is a direct consequence of Theorem 1.4 and Proposition 5.1.

A Appendix

Lemma A.1. Let $\Omega \subset \mathbb{R}^N$ be a domain $q \geq 1$, $r \geq 0$ and $u \in L^p(\Omega, \mathbb{R}^m)$. Next let $\rho_\varepsilon(|z|) : \mathbb{R}^N \to [0, +\infty)$ be radial mollifiers so that $\int_{\mathbb{R}^N} \rho_\varepsilon(|z|) dz = 1$ and for every $r > 0$ there exits $\delta := \delta_r > 0$, such that $\text{supp}(\rho_\varepsilon) \subset B_r(0)$ for every $\varepsilon \in (0, \delta_r)$. Then
\[
\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} dx d\mathcal{H}^{N-1}(n) \right) \\
\leq \liminf_{\varepsilon \to 0^+} \int_{\Omega} \rho_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} dy dx \leq \limsup_{\varepsilon \to 0^+} \int_{\Omega} \rho_\varepsilon(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} dy dx \\
\leq \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \limsup_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + \varepsilon n) \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^r} dx d\mathcal{H}^{N-1}(n) \right).
\] (A.1)
Proof. Obviously, we have

\[
\int_{\Omega} \int_{\Omega} \rho_r(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} dy dx = \int_{\mathbb{R}^N} \int_{\Omega} \rho_r(|z|) \chi_\Omega(x + z) \frac{|u(x + z) - u(x)|^q}{|z|^r} dz dx
\]

\[
= \int_{S^{N-1}} \int_{\mathbb{R}^+} t^{N-1} \rho_r(t) \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} dx dt d\mathcal{H}^{N-1}(\mathbf{n})
\]

\[
= \int_0^{+\infty} \left\{ t^{N-1} \rho_r(t) \left( \int_{S^{N-1}} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} dx \right) \right\} dt. \quad (A.2)
\]

Therefore, if \( \varepsilon \in (0, \delta_r) \) by (A.2) we deduce

\[
\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \inf_{t \in (0, r)} \left\{ \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} dx d\mathcal{H}^{N-1}(\mathbf{n}) \right\}
\]

\[
= \left( \int_0^{+\infty} t^{N-1} \rho_r(t) dt \right) \inf_{t \in (0, r)} \left\{ \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} dx d\mathcal{H}^{N-1}(\mathbf{n}) \right\}
\]

\[
\leq \int_{\Omega} \int_{\Omega} \rho_r(|y - x|) \frac{|u(y) - u(x)|^q}{|y - x|^r} dy dx
\]

\[
\leq \left( \int_0^{+\infty} t^{N-1} \rho_r(t) dt \right) \sup_{t \in (0, r)} \left\{ \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} dx d\mathcal{H}^{N-1}(\mathbf{n}) \right\}
\]

\[
= \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \sup_{t \in (0, r)} \left\{ \int_{S^{N-1}} \int_{\Omega} \chi_\Omega(x + t\mathbf{n}) \frac{|u(x + t\mathbf{n}) - u(x)|^q}{t^r} dx d\mathcal{H}^{N-1}(\mathbf{n}) \right\}. \quad (A.3)
\]

Thus, letting \( r \to 0^+ \) in (A.3) we easily deduce (A.1). \( \square \)

### A.1 The case \( r=q \)

#### Lemma A.2.

Let \( \Omega \subset \mathbb{R}^N \) be an open set, \( q \geq 1, u \in L^q(\Omega, \mathbb{R}^m) \) and \( t_1, t_2 > 0 \). Furthermore, let \( G \subset \Omega \) be an open subset such that either \( G \) is convex or \( t_1 < \text{dist}(G, \mathbb{R}^N \setminus \Omega) \). Then, we have

\[
\int_{S^{N-1}} \int_{G} \frac{1}{(t_1 + t_2)^q} |u(x + (t_1 + t_2)\mathbf{n}) - u(x)|^q \chi_G(x + (t_1 + t_2)\mathbf{n}) dx d\mathcal{H}^{N-1}(\mathbf{n}) \leq \int_{S^{N-1}} \int_{\Omega} \frac{t_2}{(t_1 + t_2)^q} |u(x + t_2\mathbf{n}) - u(x)|^q \chi_{\Omega}(x + t_2\mathbf{n}) dx d\mathcal{H}^{N-1}(\mathbf{n})
\]

\[
+ \frac{t_1}{(t_1 + t_2)^q} \int_{S^{N-1}} \int_{\Omega} |u(x + t_1\mathbf{n}) - u(x)|^q \chi_{\Omega}(x + t_1\mathbf{n}) dx d\mathcal{H}^{N-1}(\mathbf{n}). \quad (A.4)
\]
In particular,
\[ \int_{S^{N-1}} \int_{G} \frac{1}{(t_1 + t_2)^q} |u(x + (t_1 + t_2)n) - u(x)|^q \chi_G(x + (t_1 + t_2)n) dx d\mathcal{H}^{N-1}(n) \leq \]
\[ \max \left\{ \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + t_2n) - u(x)|^q}{t_2^q} \chi_{\Omega}(x + t_2n) dx d\mathcal{H}^{N-1}(n), \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + t_1n) - u(x)|^q}{t_1^q} \chi_{\Omega}(x + t_1n) dx d\mathcal{H}^{N-1}(n) \right\}. \] (A.5)

Proof. By the triangle inequality, for every \( n \in S^{N-1} \) and every \( h_1, h_2 > 0 \) we have
\[ \int_{G} \frac{1}{(h_1 + h_2)^q} |u(x + (h_1 + h_2)n) - u(x)|^q \chi_G(x + (h_1 + h_2)n) dx = \]
\[ \int_{\mathbb{R}^N} \frac{1}{(h_1 + h_2)^q} |u(x + (h_1 + h_2)n) - u(x)|^q \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx = \]
\[ \int_{\mathbb{R}^N} \frac{1}{(h_1 + h_2)^q} \left( |u(x + (h_1 + h_2)n) - u(x + h_1n) + u(x + h_1n) - u(x)|^q \right) \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx \leq \]
\[ \int_{\mathbb{R}^N} \frac{1}{(h_1 + h_2)^q} \left( |u(x + (h_1 + h_2)n) - u(x + h_1n)| + |u(x + h_1n) - u(x)| \right) \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx \]
\[ = \int_{\mathbb{R}^N} \left( \frac{h_2}{h_1 + h_2} \frac{|u(x + (h_1 + h_2)n) - u(x + h_1n)|}{h_2} \right)^q \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx. \] (A.6)

Thus, by (A.6) and convexity of \( g(s) := |s|^q \), for every \( n \in S^{N-1} \) and every \( h_1, h_2 > 0 \) we deduce
\[ \int_{G} \frac{1}{(h_1 + h_2)^q} |u(x + (h_1 + h_2)n) - u(x)|^q \chi_G(x + (h_1 + h_2)n) dx \leq \]
\[ \int_{\mathbb{R}^N} \left( \frac{h_2}{h_1 + h_2} \left( \frac{|u(x + (h_1 + h_2)n) - u(x + h_1n)|}{h_2} \right)^q \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx = \]
\[ \frac{h_2}{h_1 + h_2} \int_{\mathbb{R}^N} \frac{h_2^q}{h_1 + h_2} \left( \frac{|u(x + (h_1 + h_2)n) - u(x + h_1n)|}{h_2} \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx \right. \]
\[ + \frac{h_1}{h_1 + h_2} \int_{\mathbb{R}^N} \left. \frac{|u(x + nh_1) - u(x)|^q}{h_1^q} \chi_G(x) \chi_G(x + (h_1 + h_2)n) dx. \] (A.7)
However, if \( G \subset \Omega \) is convex then \( x \in G \) and \( x + (h_1 + h_2)\mathbf{n} \in G \) implies \( x + h_1\mathbf{n} \in G \) and then
\[
\chi_G(x)\chi_G(x + (h_1 + h_2)\mathbf{n}) = \chi_G(x)\chi_G(x + h_1\mathbf{n})\chi_G(x + (h_1 + h_2)\mathbf{n}) \\
\leq \chi_\Omega(x)\chi_\Omega(x + h_1\mathbf{n}) \chi_\Omega(x + (h_1 + h_2)\mathbf{n}). \tag{A.8}
\]

On the other hand, if \( h_1 < \text{dist}(G, \mathbb{R}^N \setminus \Omega) \), then \( x \in G \) implies \( x + h_1\mathbf{n} \in \Omega \) and so we also deduce (A.8) in that case. Thus, inserting (A.8) into (A.7), in both cases we have
\[
\int_G \frac{1}{(h_1 + h_2)^q} \left| u(x + (h_1 + h_2)\mathbf{n}) - u(x) \right|^q \chi_G(x + (h_1 + h_2)\mathbf{n}) \, dx \leq \\
\int_{\mathbb{R}^N} \frac{h_2}{(h_1 + h_2)} \left| u(x + (h_1 + h_2)\mathbf{n}) - u(x + h_1\mathbf{n}) \right|^q \chi_\Omega(x) \chi_\Omega(x + h_1\mathbf{n}) \chi_\Omega(x + (h_1 + h_2)\mathbf{n}) \, dx \\
+ \frac{h_1}{(h_1 + h_2)} \int_{\mathbb{R}^N} \left| u(x + h_1\mathbf{n}) - u(x) \right|^q \chi_\Omega(x) \chi_\Omega(x + h_1\mathbf{n}) \chi_\Omega(x + (h_1 + h_2)\mathbf{n}) \, dx. \tag{A.9}
\]

Therefore, since \( \chi_\Omega \leq 1 \) by (A.9) we infer
\[
\int_G \frac{1}{(h_1 + h_2)^q} \left| u(x + (h_1 + h_2)\mathbf{n}) - u(x) \right|^q \chi_G(x + (h_1 + h_2)\mathbf{n}) \, dx \leq \\
\frac{h_2}{(h_1 + h_2)} \int_{\mathbb{R}^N} \left| u(x + (h_1 + h_2)\mathbf{n}) - u(x + h_1\mathbf{n}) \right|^q \chi_\Omega(x + h_1\mathbf{n}) \chi_\Omega(x + (h_1 + h_2)\mathbf{n}) \, dx \\
+ \frac{h_1}{(h_1 + h_2)} \int_{\mathbb{R}^N} \left| u(x + h_1\mathbf{n}) - u(x) \right|^q \chi_\Omega(x) \chi_\Omega(x + h_1\mathbf{n}) \chi_\Omega(x + (h_1 + h_2)\mathbf{n}) \, dx \\
= \frac{h_2}{(h_1 + h_2)} \int_{\mathbb{R}^N} \left| u(x + h_2\mathbf{n}) - u(x) \right|^q \chi_\Omega(x) \chi_\Omega(x + h_2\mathbf{n}) \, dx \\
+ \frac{h_1}{(h_1 + h_2)} \int_{\mathbb{R}^N} \left| u(x + h_1\mathbf{n}) - u(x) \right|^q \chi_\Omega(x) \chi_\Omega(x + h_1\mathbf{n}) \, dx. \tag{A.10}
\]

So, we deduce (A.4). In particular, by (A.4) we finally obtain (A.5). \( \Box \)

**Lemma A.3.** Let \( \Omega \subset \mathbb{R}^N \) be an open set, \( q \geq 1 \), \( u \in L^q(\Omega, \mathbb{R}^m) \) and \( t > 0 \). Furthermore, let \( G \subset \Omega \) be an open subset, such that \( \mathcal{L}^N(\partial G) = 0 \) and, either \( t < \text{dist}(G, \mathbb{R}^N \setminus \Omega) \) or \( G \) is convex. Then, we have
\[
\int_{S^{N-1}} \int_G \frac{1}{t^q} \left| u(x + tn) - u(x) \right|^q \chi_G(x + tn) \, dx \, d\mathcal{H}^{N-1}(n) \\
\leq \liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \frac{1}{\varepsilon^q} \left| u(x + \varepsilon n) - u(x) \right|^q \chi_\Omega(x + \varepsilon n) \, dx \, d\mathcal{H}^{N-1}(n) \right). \tag{A.11}
\]

**Proof.** First of all, in the case of convex \( G \), for every \( s > 0 \) and every \( j = 1, 2, \ldots \), taking \( t_1 = s \)
and $t_2 = js$ in (A.5), with $G$ instead of $\Omega$, gives

$$
\int_{S^{N-1}} \int_G \frac{1}{(j+1)^q} \left| u(x + (j+1)sn) - u(x) \right|^q \chi_G(x + (j+1)sn) dxdH^{N-1}(n) \leq
$$

$$
\max \left\{ \int_{S^{N-1}} \int_G \frac{1}{s^q} \left| u(x + sn) - u(x) \right|^q \chi_G(x + sn) dxdH^{N-1}(n), \right. $$

$$
\int_{S^{N-1}} \int_G \frac{1}{(js)^q} \left| u(x + jsn) - u(x) \right|^q \chi_G(x + jsn) dxdH^{N-1}(n) \right\}. \quad (A.12)
$$

Therefore, using (A.12), by induction, in the case of convex $G$ we prove

$$
\int_{S^{N-1}} \int_G \frac{1}{(js)^q} \left| u(x + jsn) - u(x) \right|^q \chi_G(x + jsn) dxdH^{N-1}(n) \leq
$$

$$
\max \left\{ \int_{S^{N-1}} \int_G \frac{1}{s^q} \left| u(x + sn) - u(x) \right|^q \chi_G(x + sn) dxdH^{N-1}(n), \right. $$

$$
\int_{S^{N-1}} \int_\Omega \frac{1}{s^q} \left| u(x + sn) - u(x) \right|^q \chi_\Omega(x + sn) dxdH^{N-1}(n) \right\} \quad \forall j = 1, 2, \ldots \quad (A.13)
$$

On the other hand, for every $s > 0$, every $j = 1, 2, \ldots$ and every open $G_{j+1} \subset G_j \subset \Omega$, such that $s < \text{dist}(G_{j+1}, \mathbb{R}^N \setminus G_j)$, taking $t_1 = s$ and $t_2 = js$ in (A.5) gives

$$
\int_{S^{N-1}} \int_{G_{j+1}} \frac{1}{(j+1)^q} \left| u(x + (j+1)sn) - u(x) \right|^q \chi_{G_{j+1}}(x + (j+1)sn) dxdH^{N-1}(n) \leq
$$

$$
\max \left\{ \int_{S^{N-1}} \int_{G_j} \frac{1}{s^q} \left| u(x + sn) - u(x) \right|^q \chi_{G_j}(x + sn) dxdH^{N-1}(n), \right. $$

$$
\int_{S^{N-1}} \int_{G_j} \frac{1}{(js)^q} \left| u(x + jsn) - u(x) \right|^q \chi_{G_j}(x + jsn) dxdH^{N-1}(n) \right\} \leq
$$

$$
\max \left\{ \int_{S^{N-1}} \int_\Omega \frac{1}{s^q} \left| u(x + sn) - u(x) \right|^q \chi_\Omega(x + sn) dxdH^{N-1}(n), \right. $$

$$
\int_{S^{N-1}} \int_{G_j} \frac{1}{(js)^q} \left| u(x + jsn) - u(x) \right|^q \chi_{G_j}(x + jsn) dxdH^{N-1}(n) \right\}. \quad (A.14)
$$

Therefore, using (A.14), by induction we prove that, for every $s > 0$, every $j = 1, 2, \ldots$ and
every \( G_j \subset \Omega \), such that \( js < \text{dist}(G_j, \mathbb{R}^N \setminus \Omega) \), we have

\[
\int \int_{S^{N-1}} \frac{1}{(js)^q} \left| u(x + (js)n) - u(x) \right|^q \chi_{G_j}(x + (js)n) dx d\mathcal{H}^{N-1}(n) \leq \\
\int \int_{S^{N-1}} \frac{1}{s^q} \left| u(x + sn) - u(x) \right|^q \chi_{\Omega}(x + sn) dx d\mathcal{H}^{N-1}(n) \quad \forall \ j = 1, 2, \ldots. \quad (A.15)
\]

Next, assume that a sequence \( \{\varepsilon_k\}_{k=1}^{+\infty} \) satisfies \( \varepsilon_k \downarrow 0 \) and

\[
\lim_{k \to +\infty} \left( \int \int_{S^{N-1}} \frac{1}{\varepsilon_k^q} \left| u(x + \varepsilon_k n) - u(x) \right|^q \chi_{\Omega}(x + \varepsilon_k n) dx d\mathcal{H}^{N-1}(n) \right) = \\
\lim_{\varepsilon \to 0^+} \inf \left( \int \int_{S^{N-1}} \frac{1}{\varepsilon^q} \left| u(x + \varepsilon n) - u(x) \right|^q \chi_{\Omega}(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right). \quad (A.16)
\]

Then, given \( t > 0 \), for every \( k \in \mathbb{N} \) consider \( j_k \in \mathbb{N} \) and \( r_k \in [0, 1) \) such that

\[
\frac{t}{\varepsilon_k} = j_k + r_k \quad \forall \ k \in \mathbb{N}, \quad (A.17)
\]

so that

\[
t = (j_k + r_k)\varepsilon_k = j_k \varepsilon_k + r_k \varepsilon_k \quad \forall \ k \in \mathbb{N}. \quad (A.18)
\]

In particular, since \( \varepsilon_k \downarrow 0 \) and \( r_k \in [0, 1) \) we deduce

\[
\lim_{k \to +\infty} j_k \varepsilon_k = t. \quad (A.19)
\]

Then, by (A.19), obviously we have

\[
\lim_{k \to +\infty} \int \int_{S^{N-1}} \frac{1}{(j_k \varepsilon_k)^q} \left| u\left(x + (j_k \varepsilon_k)n\right) - u(x) \right|^q \chi_{G}(x + (j_k \varepsilon_k)n) dx d\mathcal{H}^{N-1}(n) = \\
\lim_{k \to +\infty} \int \int_{S^{N-1}} \frac{1}{t^q} \left| u\left(x + (j_k \varepsilon_k)n\right) - u(x) \right|^q \chi_{G}(x + (j_k \varepsilon_k)n) dx d\mathcal{H}^{N-1}(n) = \\
\int \int_{S^{N-1}} \frac{1}{t^q} \left| u\left(x + tn\right) - u(x) \right|^q \chi_{G}(x + tn) dx d\mathcal{H}^{N-1}(n), \quad (A.20)
\]

where we include the fact \( \mathcal{L}^N(\partial G) = 0 \) in the proof of the last equation. However, by either (A.13) with \( s = \varepsilon_k \) and \( j = j_k \), in the case of convex \( G \), together with (A.16), or by (A.15), with \( s = \varepsilon_k, j = j_k \) and \( G_j = G \), in the case of \( t < \text{dist}(G, \mathbb{R}^N \setminus \Omega) \), together with (A.16) and
(A.19), we infer,

\[
\lim_{k \to +\infty} \int_{S^{N-1}} \int_{G} \frac{1}{(j_k \varepsilon_k)^q} \left| u\left( x + (j_k \varepsilon_k) n \right) - u(x) \right|^q \chi_G \left( x + (j_k \varepsilon_k) n \right) dx d\mathcal{H}^{N-1}(n)
\]

\[
\leq \lim_{k \to +\infty} \left( \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right) = \liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right). \tag{A.21}
\]

Therefore, by inserting (A.21) into (A.20) we finally obtain (A.11).

**Corollary A.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set, \( q \geq 1 \) and \( u \in L^q(\Omega, \mathbb{R}^m) \). Furthermore, let \( G \subset \Omega \) be an open subset, such that \( \mathcal{L}^N(\partial G) = 0 \) and \( h := \text{dist}(G, \mathbb{R}^N \setminus \Omega) > 0 \). Then, we have

\[
\sup_{\varepsilon \in (0, h)} \left( \int_{S^{N-1}} \int_{G} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right) = \limsup_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right). \tag{A.22}
\]

**Corollary A.2.** Let \( \Omega \subset \mathbb{R}^N \) be a convex open domain, such that \( \mathcal{L}^N(\partial \Omega) = 0 \), \( q \geq 1 \) and \( u \in L^q(\Omega, \mathbb{R}^m) \). Then, we have

\[
\sup_{\varepsilon \in (0, +\infty)} \left( \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right) = \limsup_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right) = \liminf_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon^q} \chi_G(x + \varepsilon n) dx d\mathcal{H}^{N-1}(n) \right). \tag{A.23}
\]

In particular, for that case, if \( \rho_\varepsilon(|z|) : \mathbb{R}^N \to [0, +\infty) \) are radial mollifiers, so that \( \int_{\mathbb{R}^N} \rho_\varepsilon(|z|)dz = 1 \) and for every \( r > 0 \) there exits \( \delta := \delta_r > 0 \), such that \( \text{supp} \ (\rho_\varepsilon) \subset B_r(0) \) for every \( \varepsilon \in (0, \delta_r) \), then by Lemma A.1, we have:

\[
\frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \lim_{\varepsilon \to 0^+} \left( \int_{S^{N-1}} \int_{\Omega} \chi_G(x + \varepsilon n) \left| u(x + \varepsilon n) - u(x) \right|^q dx d\mathcal{H}^{N-1}(n) \right) = \lim_{\varepsilon \to 0^+} \int_{\Omega} \rho_\varepsilon(|y - x|) \left| u(y) - u(x) \right|^q dy dx
\]

\[
= \frac{1}{\mathcal{H}^{N-1}(S^{N-1})} \sup_{\varepsilon \in (0, +\infty)} \left( \int_{S^{N-1}} \int_{\Omega} \chi_G(x + \varepsilon n) \left| u(x + \varepsilon n) - u(x) \right|^q dx d\mathcal{H}^{N-1}(n) \right). \tag{A.24}
\]

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A.2 The case \( r \in (0, q) \)

The following Lemma is a part of the statement, that was proven in [5]:

**Lemma A.4.** Let \( \Omega \subset \mathbb{R}^N \) be an open set, \( q \geq 1 \) and let \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^m) \). Then, for every open \( \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega \), \( k \in S^{N-1} \) and \( \varepsilon \) satisfying

\[
0 < \varepsilon < \min \{ \text{dist}(\Omega_1, \mathbb{R}^N \setminus \Omega_2), \text{dist}(\Omega_2, \mathbb{R}^N \setminus \Omega) \},
\]

we have

\[
\int_{S^{N-1}} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx \leq \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{\Omega_2} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx \leq \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{\Omega_2} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx.
\]

**Proof.** In the case \( \Omega = \mathbb{R}^N \), we easily deduce from (A.26) that for every \( \varepsilon > 0 \) we have

\[
\int_{\mathbb{R}^N} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx \leq \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx \leq \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx.
\]

Then, by either (A.26) for every \( \varepsilon \) satisfying (A.25) or by (A.28) for every \( \varepsilon > 0 \) we have

\[
\sup_{k \in S^{N-1}} \left\{ \int_{S^{N-1}} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx \right\} \leq \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{S^{N-1}} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx ds d\mathcal{H}^{N-1}(n)
\]

\[
= \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{\mathbb{T}^2} s^{N-1} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx ds d\mathcal{H}^{N-1}(n)
\]

\[
= \frac{2^{N+q}}{L^N(B_1(0))} \int_{B_1(0)} \int_{\mathbb{T}^2} t^{N-1} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx dt d\mathcal{H}^{N-1}(n)
\]

\[
\leq \frac{2^{N+q}}{L^N(B_1(0))} \left( \sup_{\varepsilon \in (0, \varepsilon)} \left\{ \int_{S^{N-1}} \int_{\mathbb{T}^2} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx ds d\mathcal{H}^{N-1}(n) \right\} \right) \int_{0}^{\varepsilon} \frac{\tau^{N-2+r}}{\varepsilon^{N-1+r}} d\tau
\]

\[
= \frac{2^{N+q}}{(N-1+r)L^N(B_1(0))} \left( \sup_{\varepsilon \in (0, \varepsilon)} \left\{ \int_{S^{N-1}} \int_{\mathbb{T}^2} \frac{1}{\varepsilon^ disc(\Omega_1, \mathbb{R}^N \setminus \Omega)} dx ds d\mathcal{H}^{N-1}(n) \right\} \right) \int_{0}^{\varepsilon} \frac{\tau^{N-2+r}}{\varepsilon^{N-1+r}} d\tau.
\]

In particular, by (A.29) we deduce (A.27).
Definition A.1. Given a compact set $\mathcal{U} \subset \Omega$ let
\[
B_{u,q,r}(\mathcal{U}) := \limsup_{\varepsilon \to 0^+} \sup_{k \in S^{N-1}} \int_{\mathcal{U} \cap \varepsilon^r} \frac{1}{\varepsilon^r} |u(x + \varepsilon k) - u(x)|^q \, dx.
\] (A.30)

Next, given an open set $\Omega \subset \mathbb{R}^N$ define
\[
B_{u,q,r}(\Omega) := \sup_{K \subset \subset \Omega} B_{u,q,r}(K).
\] (A.31)

Finally, set
\[
\hat{B}_{u,q,r}(\mathbb{R}^N) := \limsup_{\varepsilon \to 0^+} \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^r} |u(x + \varepsilon k) - u(x)|^q \, dx.
\] (A.32)

The following result is known; for the convenience of a reader we will give its proof.

Lemma A.5. For any $q \geq 1$ and $r \in (0, q)$, a function $u \in L^q(\mathbb{R}^N, \mathbb{R}^m)$ belongs to $B_{q,\infty}^{r/q}(\mathbb{R}^N, \mathbb{R}^m)$ if and only if $\hat{B}_{u,q,r}(\mathbb{R}^N) < +\infty$. Moreover, for any open $\Omega \subset \mathbb{R}^N$, a function $u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^m)$ belongs to $(B_{q,\infty}^{r/q})_{\text{loc}}(\Omega, \mathbb{R}^m)$ if and only if for every compact $K \subset \subset \Omega$ we have $B_{u,q,r}(K) < +\infty$.

Proof. We have,
\[
\sup_{\rho \in (0, \infty)} \left( \sup_{|h| = \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^r} |u(x+h) - u(x)| \right)^q \, dx \right) \leq \sup_{\rho \in (0, \infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^r} |u(x+h) - u(x)| \right)^q \, dx \right)
\]
\[
= \sup_{\rho \in (0, \infty)} \sup_{t \in (0, \rho)} \left( \sup_{|h| = t} \int_{\mathbb{R}^N} \left( \frac{1}{t^r} |u(x+h) - u(x)| \right)^q \, dx \right)
\]
\[
\leq \sup_{\rho \in (0, \infty)} \left( \sup_{|h| = \rho} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^r} |u(x+h) - u(x)| \right)^q \, dx \right)
\]
\[
= \sup_{\rho \in (0, \infty)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \left( \frac{1}{\rho^r} |u(x+\rho k) - u(x)| \right)^q \, dx \right). \tag{A.33}
\]

On the other hand by the triangle inequality and the convexity of $g(s) := |s|^q$ for every $\delta > 0$ we have,
\[
\sup_{\rho \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho^r} |u(x+\rho k) - u(x)|^q \, dx \right) \leq \sup_{\rho \in (0, \infty)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho^r} |u(x+\rho k) - u(x)|^q \, dx \right) \leq \sup_{\rho \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho^r} |u(x+\rho k) - u(x)|^q \, dx \right)
\]
\[
+ \frac{2^{q-1}}{\delta^r} \sup_{\rho \in (\delta, \infty)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \left( |u(x+\rho k)|^q + |u(x)|^q \right) \, dx \right)
\]
\[
= \sup_{\rho \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\rho^r} |u(x+\rho k) - u(x)|^q \, dx \right) + \frac{2^q}{\delta^r} \|u\|_{L^q(\mathbb{R}^N, \mathbb{R}^d)}^q \tag{A.34}
\]
Therefore, by (A.33) and (A.34) we have:

\[
\sup_{\varepsilon \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^r} |u(x + \varepsilon k) - u(x)|^q \, dx \right) \leq \sup_{\rho \in (0, \infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \frac{1}{\rho^{r/q}} |u(x + h) - u(x)|^q \, dx \right)
\]

\[
\leq \sup_{\varepsilon \in (0, \delta)} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^r} |u(x + \varepsilon k) - u(x)|^q \, dx \right) + \frac{2q}{\delta^r} \|u\|_{L^q(\mathbb{R}^N, \mathbb{R}^d)}^q. \tag{A.35}
\]

Thus by (A.33) we clearly obtain that if \( u \in L^q(\mathbb{R}^N, \mathbb{R}^m) \) then

\[
\sup_{\rho \in (0, \infty)} \left( \sup_{|h| \leq \rho} \int_{\mathbb{R}^N} \frac{1}{\rho^{r/q}} |u(x + h) - u(x)|^q \, dx \right) < +\infty \text{ if and only if } \limsup_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^r} |u(x + \varepsilon k) - u(x)|^q \, dx \right) < +\infty. \tag{A.36}
\]

So we proved that \( u \in L^q(\mathbb{R}^N, \mathbb{R}^m) \) belongs to \( B^{r/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^m) \) if and only if we have \( \hat{B}_{u,q,r}(\mathbb{R}^N) < +\infty \).

Next, given open \( \Omega \subset \mathbb{R}^N \) let \( u \in L^q_{\text{loc}}(\Omega, \mathbb{R}^m) \) and \( \bar{K} \subset \subset \Omega \) be a compact set. Moreover, consider an open set \( U \subset \mathbb{R}^N \) such that we have the following compact embedding:

\[
\bar{K} \subset \subset U \subset \overline{U} \subset \subset \Omega.
\]

Then, assuming \( u \in (B^{r/q}_{q,\infty})_{\text{loc}}(\Omega, \mathbb{R}^m) \) implies existence of \( \hat{u} \in B^{r/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^m) \) such that \( \hat{u}(x) = u(x) \) for every \( x \in \overline{U} \), that gives

\[
B_{u,q,r}(K) = B_{\hat{u},q,r}(K) \leq \hat{B}_{\hat{u},q,r}(\mathbb{R}^N) < +\infty.
\]

On the other hand, if we assume

\[
B_{u,q,r}(U) < +\infty, \tag{A.37}
\]

then define

\[
\hat{u}(x) = \begin{cases} 
\eta(x)u(x) & \forall x \in U \\
0 & \forall x \in \mathbb{R}^N \setminus U,
\end{cases} \tag{A.38}
\]

where \( \eta(x) \in C^\infty_c(U, [0, 1]) \) is some cut-off function such that \( \eta(x) = 1 \) for every \( x \in K \). Thus in particular \( \hat{u}(x) = u(x) \) for every \( x \in K \) and so, in order to complete the proof, we need just to show that \( \hat{u} \in B^{r/q}_{q,\infty}(\mathbb{R}^N, \mathbb{R}^m) \). Thus by (A.36) it is sufficient to show:

\[
\limsup_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\mathbb{R}^N} \frac{1}{\varepsilon^r} |\hat{u}(x + \varepsilon k) - \hat{u}(x)|^q \, dx \right) < +\infty. \tag{A.39}
\]
However, since $|\eta| \leq 1$, $\text{supp} \eta \subset U$ and $\eta$ is smooth we have:

\[
\lim_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\Omega} \frac{1}{\varepsilon^q} |\hat{u}(x + \varepsilon k) - \hat{u}(x)|^q dx \right) = \\
\lim_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\Omega} \frac{1}{\varepsilon^q} |\eta(x + \varepsilon k)u(x + \varepsilon k) - \eta(x)u(x)|^q dx \right) = \\
\lim_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\Omega} \frac{1}{\varepsilon^q} \left| \eta(x + \varepsilon k)(u(x + \varepsilon k) - u(x)) + (\eta(x + \varepsilon k) - \eta(x))u(x) \right|^q dx \right) \\
\leq \lim_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\Omega} \frac{2^{q-1}}{\varepsilon^q} \left( \left| \eta(x + \varepsilon k)(u(x + \varepsilon k) - u(x)) \right|^q + \left| (\eta(x + \varepsilon k) - \eta(x))u(x) \right|^q \right) dx \right) \\
\leq 2^{q-1} \lim_{\varepsilon \to 0^+} \left( \sup_{k \in S^{N-1}} \int_{\Omega} \frac{1}{\varepsilon^q} \left( |u(x + \varepsilon k) - u(x)|^q \right) dx \right) + \\
2^{q-1} \lim_{\varepsilon \to 0^+} \sup_{k \in S^{N-1}} \int_{\Omega} \left| \frac{\eta(x + \varepsilon k) - \eta(x)}{\varepsilon} \right|^q |u(x)|^q dx \\
\leq 2^{q-1} \mathcal{B}_{u,q,r}(\Omega) + 2^{q-1} \left( \int_{\Omega} |u(x)|^q dx \right) \|\nabla \eta\|_{L^\infty} \left( \lim_{\varepsilon \to 0^+} \varepsilon^{q-r} \right) = 2^{q-1} \mathcal{B}_{u,q,r}(\Omega) < +\infty.
\]

(A.40)

The next Proposition is part of [5, Proposition 2.4], see [5] for the proof.

**Proposition A.1.** Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in BV_{loc}(\Omega, \mathbb{R}^m) \cap L^\infty_{loc}(\Omega, \mathbb{R}^m)$. Then, for every compact set $K \subset \Omega$ such that $\|Du\|(\partial K) = 0$, any $q > 1$ and any vector $k \subset S^{N-1}$ we have

\[
\lim_{\varepsilon \to 0^+} \int_{K} \frac{1}{\varepsilon^q} |u(x + \varepsilon k) - u(x)|^q dx = \int_{J_u \cap K} |u^+(x) - u^-(x)|^q |k \cdot \mathbf{v}(x)| d\mathcal{H}^{N-1}(x). \quad (A.41)
\]

**Proof of Proposition A.1.** For any compact set $K \subset \Omega$, we can choose $\Omega_1 \subset \subset \Omega$, such that $K \subset \subset \Omega_1$ and then, for every small $\varepsilon > 0$ and any vector $k \subset S^{N-1}$ we clearly have

\[
0 \leq \frac{1}{\varepsilon} \int_{K} |u(x + \varepsilon k) - u(x)|^q dx \leq 2^{q-1} \|u\|_{L^\infty(\Omega_1)} \int_{K} \frac{1}{\varepsilon^q} |u(x + \varepsilon k) - u(x)| dx \\
\leq 2^{q-1} \|u\|_{L^\infty(\Omega_1)} \|u\|_{BV(\Omega_1)}. \quad (A.42)
\]

Thus by dominated convergence, by (A.41) in Proposition A.1 we get

\[
\lim_{\varepsilon \to 0^+} \left\{ \int_{S^{N-1}} \int_{K} \frac{1}{\varepsilon^q} |u(x + \varepsilon n) - u(x)|^q dx d\mathcal{H}^{N-1}(n) \right\} \\
= \int_{S^{N-1}} \left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q |n \cdot \mathbf{v}(x)| d\mathcal{H}^{N-1}(x) \right) d\mathcal{H}^{N-1}(n) = \\
\left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q \left( \int_{S^{N-1}} |n \cdot \mathbf{v}(x)| d\mathcal{H}^{N-1}(n) \right) d\mathcal{H}^{N-1}(x) \right) = \\
\left( \int_{S^{N-1}} |z_1 d\mathcal{H}^{N-1}(z)\right) \left( \int_{J_u \cap K} |u^+(x) - u^-(x)|^q d\mathcal{H}^{N-1}(x) \right). \quad (A.43)
\]
Next, if $\Omega$ is an open set with bounded Lipschitz boundary and $u \in BV(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$, then we can extend the function $u(x)$ to all of $\mathbb{R}^N$ in such a way that $u \in BV(\mathbb{R}^N, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^m)$ and $\|Du\|((\partial \Omega)) = 0$. Next, consider an increasing sequence of compact sets $K_n \subset \Omega$ such that $K_n \subset K_{n+1}$, dist$(K_n, \mathbb{R}^N \setminus K_{n+1}) > 0$ and $\bigcup_{n=1}^{+\infty} K_n = \Omega$. Moreover, consider a decreasing sequence of open sets $V_n$ such that $\overline{\Omega} \subset V_n$, $V_{n+1} \subset V_n$, dist$(V_{n+1}, \mathbb{R}^N \setminus V_n) > 0$ and $\bigcap_{n=1}^{+\infty} V_n = \overline{\Omega}$, so that, denoting an open set $U_n := V_n \setminus K_n$, we have $U_{n+1} \subset U_n$ dist$(U_{n+1}, \mathbb{R}^N \setminus U_n) > 0$ and $\bigcap_{n=1}^{+\infty} U_n = \partial \Omega$. Then, by (A.43) for every $n$ we have

\[
\left( \int_{S^{n-1}} |z_1| d\mathcal{H}^{N-1}(z) \right) \left( \int_{J_n \cap K_n} \left| u^+(x) - u^-(x) \right|^q d\mathcal{H}^{N-1}(x) \right) = \lim_{\varepsilon \to 0^+} \left\{ \int_{S^{n-1}} \int_{K_n} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon} dx d\mathcal{H}^{N-1}(n) \right\} \leq \limsup_{\varepsilon \to 0^+} \left\{ \int_{S^{n-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon} dx d\mathcal{H}^{N-1}(n) \right\} \leq \limsup_{\varepsilon \to 0^+} \left\{ \int_{S^{n-1}} \int_{\Omega} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon} dx d\mathcal{H}^{N-1}(n) \right\}.
\] (A.44)

On the other hand, as in (A.42), for every small $\varepsilon > 0$ and any vector $k \subset S^{n-1}$ we clearly have

\[
0 \leq \frac{1}{\varepsilon} \int_{\Omega \setminus K_n} |u(x + \varepsilon k) - u(x)|^q dx \leq 2^{q-1} \|u\|_{L^\infty(\mathbb{R}^N)}^{q-1} \int_{\Omega \setminus K_n} \frac{1}{\varepsilon} |u(x + \varepsilon k) - u(x)| dx \leq 2^{q-1} \|u\|_{L^\infty(\mathbb{R}^N)}^{q-1} \|u\|_{BV(U_n)}.
\] (A.45)

In particular,

\[
0 \leq \limsup_{\varepsilon \to 0^+} \left\{ \int_{S^{n-1}} \int_{\Omega \setminus K_n} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon} dx d\mathcal{H}^{N-1}(n) \right\} \leq 2^{q-1} \left( \mathcal{H}^{N-1}(S^{n-1}) \right) \|u\|_{L^\infty(\mathbb{R}^N)}^{q-1} \|u\|_{BV(U_n)}.
\] (A.46)

Thus, since $\bigcap_{n=1}^{+\infty} U_n = \partial \Omega$, letting $n \to +\infty$ in (A.46) and using the fact $\|Du\|((\partial \Omega)) = 0$ gives

\[
0 \leq \limsup_{n \to +\infty} \left( \limsup_{\varepsilon \to 0^+} \left\{ \int_{S^{n-1}} \int_{\Omega \setminus K_n} \frac{|u(x + \varepsilon n) - u(x)|^q}{\varepsilon} dx d\mathcal{H}^{N-1}(n) \right\} \right) \leq 2^{q-1} \left( \mathcal{H}^{N-1}(S^{n-1}) \right) \|u\|_{L^\infty(\mathbb{R}^N)}^{q-1} \|Du\|((\partial \Omega)) = 0.
\] (A.47)

Thus, inserting (A.47) into (A.44) we finally deduce (5.1). \(\Box\)

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