Bounding the number of characters in a block of a finite group

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Abstract

We present a strong upper bound on the number \( k(B) \) of irreducible characters of a \( p \)-block \( B \) of a finite group \( G \) in terms of local invariants. More precisely, the bound depends on a chosen major \( B \)-subsection \((u, b)\), its normalizer \( N_G(\langle u \rangle, b) \) in the fusion system and a weighted sum of the Cartan invariants of \( b \). In this way we strengthen and unify previous bounds given by Brauer, Wada, Külshammer–Wada, Héthelyi–Külshammer–Sambale and the present author.

Keywords: number of characters in a block, Cartan matrix, Brauer’s \( k(B) \)-Conjecture

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1 Introduction

Let \( B \) be a \( p \)-block of a finite group \( G \) with defect \( d \). Since Richard Brauer [4] conjectured that the number of irreducible characters \( k(B) \) in \( B \) is at most \( p^d \), there has been great interest in bounding \( k(B) \) in terms of local invariants. Brauer and Feit [6] used some properties of the Cartan matrix \( C = (c_{ij}) \in \mathbb{Z}^{l(B) \times l(B)} \) of \( B \) to prove their celebrated bound \( k(B) \leq p^{2d} \) (here and in the following \( l(B) \) denotes the number of irreducible Brauer characters of \( B \)). In the present paper we investigate stronger bounds by making use of further local invariants. By elementary facts on decomposition numbers, it is easy to see that

\[ k(B) \leq \text{tr}(C) \]

where \( \text{tr}(C) \) denotes the trace of \( C \). However, it is not true in general that \( \text{tr}(C) \leq p^d \). In fact, there are examples with \( \text{tr}(C) > l(B)p^d \) (see [11]) although Brauer already knew that \( k(B) \leq l(B)p^d \) (see Corollary 15 below) and this was subsequently improved by Olsson [12, Theorem 4]. For this reason, some authors strengthened (1) in a number of ways. Brandt [3, Proposition 4.2] proved

\[ k(B) \leq \text{tr}(C) - l(B) + 1 \]

and this was generalized by the present author in [17, Proposition 8] to

\[ k(B) \leq \sum_{i=1}^m \det(C_i) - m + 1 \]

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where \( S_1, \ldots, S_m \) is a partition of \( \{1, \ldots, l(B)\} \) and \( C_i := (c_{st})_{s,t \in S_i} \). Using different methods, Wada [20] observed that

\[
k(B) \leq \text{tr}(C) - \sum_{i=1}^{l(B)-1} c_{i,i+1}.
\]  

(2)

In Külshammer–Wada [10], the authors noted that (2) is a special case of

\[
k(B) \leq \sum_{1 \leq i \leq j \leq l(B)} q_{ij}c_{ij}
\]  

(3)

where \( q(x) = \sum_{1 \leq i \leq j \leq l(B)} q_{ij}x_ix_j \) is a (weakly) positive definite integral quadratic form.

Since \( C \) is often harder to compute than \( k(B) \), it is desirable to replace \( C \) by the Cartan matrix of a Brauer correspondent of \( B \) in a proper subgroup. For this purpose let \( D \) be a defect group of \( B \) and choose \( u \in \mathbb{Z}(D) \). Then a Brauer correspondent \( b \) of \( B \) in \( C_G(u) \) has defect group \( D \) as well. The present author replaced \( c_{ij} \) in [3] by the corresponding entries of the Cartan matrix \( C_u \) of \( b \) (see [15, Lemma 1]).

In Héthelyi–Külshammer–Sambale [9] Theorem 2.4] we have invoked Galois actions to obtain stronger bounds although only in the special cases \( p = 2 \) and \( l(b) = 1 \) (see [9, Theorems 3.1 and 4.10]). More precisely, in the latter case we proved

\[
k(B) \leq \sum_{i=1}^{\infty} p^{2i}k_i(B) \leq \left(n + \frac{|\langle u \rangle| - 1}{n}\right) p^d \leq p^d = \text{tr}(C_u)
\]  

(4)

where \( n := |N_G(\langle u \rangle, b) : C_G(\langle u \rangle)| \) and \( k_i(B) \) is the number of irreducible characters of height \( i \geq 0 \) in \( B \). This is a refinement of a result of Robinson [13, Theorem 3.4.3]. In [18, Theorem 2.6], the present author relaxed the condition \( l(b) = 1 \) to the weaker requirement that \( \mathcal{N} := N_G(\langle u \rangle, b)/C_G(\langle u \rangle) \) acts trivially on the set \( \text{IBr}(b) \) of irreducible Brauer characters of \( b \).

In this paper we replace integral quadratic forms by real matrices \( W \) describing weighted sums of Cartan invariants. This allows us to drop all restrictions imposed above. We prove the following general result which incorporates the previous special cases (see Section 3 for details).

**Theorem A.** Let \( B \) be a block of a finite group \( G \) with defect group \( D \). Let \( u \in \mathbb{Z}(D) \) and let \( b \) be a Brauer correspondent of \( B \) in \( C_G(u) \). Let \( \mathcal{N} := N_G(\langle u \rangle, b)/C_G(\langle u \rangle) \) and let \( C \) be the Cartan matrix of the block \( \bar{b} \) of \( C_G(u)/\langle u \rangle \) dominated by \( b \). Let \( W \in \mathbb{R}^{l(b) \times l(b)} \) such that \( xWx^t \geq 1 \) for every \( x \in \mathbb{Z}^{l(b)} \setminus \{0\} \). Then

\[
k(B) \leq \left|\mathcal{N}\right| + \frac{|\langle u \rangle| - 1}{|\mathcal{N}|} \text{tr}(WC) \leq |\langle u \rangle| \text{tr}(WC).
\]

The first inequality is strict if \( \mathcal{N} \) acts non-trivially on \( \text{IBr}(b) \) and the second inequality is strict if and only if \( 1 < |\mathcal{N}| < |\langle u \rangle| - 1 \).

In contrast to [4], we cannot replace \( k(B) \) by \( \sum p^{2i}k_i(B) \) in Theorem A (the principal 2-block of \( \text{SL}(2,3) \) is a counterexample with \( u = 1 \)). By a classical fusion argument of Burnside, the automorphism group \( \mathcal{N} \) of \( \langle u \rangle \) in Theorem A is the restriction of the inertial quotient \( N_G(D, bD)/DC_G(D) \leq \text{Aut}(D) \) where \( bD \) is a Brauer correspondent of \( B \) in \( C_G(D) \) (see [11, Corollary 4.18]). In particular, \( \mathcal{N} \) is a \( p^l \)-group and \( |\mathcal{N}| \) divides \( p - 1 \).

As noted in previous papers, if \( u \in D \setminus \mathbb{Z}(D) \), one still gets upper bounds on the number of height 0 characters and this is of interest with respect to Olsson’s Conjecture \( k_0(B) \leq |D : D'| \) where \( D' \) denotes the commutator subgroup of \( D \). In fact, we will deduce Theorem A from our second main theorem:
Theorem B. Let $B$ be a block of a finite group $G$ with defect group $D$. Let $u \in D$ and let $b$ be a Brauer correspondent of $B$ in $C_G(u)$. Let $\mathcal{N} := N_G(\langle u \rangle)/C_G(u)$ and let $C$ be the Cartan matrix of the block $\tilde{b}$ of $C_G(u)/\langle u \rangle$ dominated by $b$. Let $W \in \mathbb{R}^{l(b) \times l(b)}$ such that $xWx^t \geq 1$ for every $x \in \mathbb{Z}^{l(b)} \setminus \{0\}$. Then

$$k_0(B) \leq k_0(\langle u \rangle \times \mathcal{N}) \text{tr}(WC) \leq |\langle u \rangle| \text{tr}(WC).$$

The first inequality is strict if $\mathcal{N}$ acts non-trivially on $\text{IBr}(b)$.

In the situation of Theorem B we may assume, after conjugation, that $N_D(\langle u \rangle)/C_D(u)$ is a Sylow $p$-subgroup of $\mathcal{N}$ (see [2] Proposition 2.5]). In particular, $\mathcal{N} = N_D(\langle u \rangle)/C_D(u)$ whenever $p = 2$.

If $\mathcal{N}$ acts trivially on $\text{IBr}(b)$, then our bounds cannot be improved in general. To see this, let $\langle u \rangle$ be any cyclic $p$-group, and let $\mathcal{N} \leq \text{Aut}(\langle u \rangle)$. Then $G := \langle u \rangle \times \mathcal{N}$ has only one $p$-block $B$. In this situation $l(b) = 1$ and $C = (1)$. Hence, $k_0(B) = k_0(G) = k_0(\langle u \rangle \times \mathcal{N}) \text{tr}(WC)$ for $W = (1)$. Similarly, if $\mathcal{N}$ is a $p'$-group, then $k(B) = k(G) = |\mathcal{N}| + |\langle u \rangle|^{-1}$.

It is known that the ordinary character table of $C_G(u)/\langle u \rangle$ determines $C$ up to basic sets, i.e. up to transformations of the form $S^tCS$ where $S \in \text{GL}(l(b), \mathbb{Z})$ and $S^t$ is the transpose of $S$. Then $\tilde{W} := S^{-1}WS^{-1}$ still satisfies $x\tilde{W}x^t \geq 1$ for every $x \in \mathbb{Z}^{l(b)} \setminus \{0\}$ and

$$\text{tr}(\tilde{W}S^tCS) = \text{tr}(S^{-1}WCS) = \text{tr}(WC).$$

Hence, our results do not depend on the chosen basic set.

2 Proofs

First we outline the proof of Theorem B: For sake of simplicity suppose first that $u = 1$. Then every row $d_\chi$ of the decomposition matrix $Q$ of $B$ is non-zero and $Q^tQ = C$. Hence,

$$k(B) \leq \sum_{\chi \in \text{Irr}(B)} d_\chi W d_\chi^t = \text{tr}(QWQ^t) = \text{tr}(WQ^tQ) = \text{tr}(WC).$$

In the general case we replace $Q$ by the generalized decomposition matrix with respect to the subsection $\langle u, b \rangle$. Then $Q$ consists of algebraic integers in the cyclotomic field of degree $q := |\langle u \rangle|$. We apply a discrete Fourier transformation to turn $Q$ into an integral matrix with the same number of rows, but with more columns. At the same time we need to blow up $W$ to a larger matrix with similar properties. Afterwards we use the fact that the rows of $Q$ corresponding to height 0 characters are non-zero and fulfill a certain $p$-adic valuation. For $p = 2$ the proof can be completed directly, while for $p > 2$ we argue by induction on $q$. Additional arguments are required to handle the case where $|\mathcal{N}|$ is divisible by $p$. These calculations make use of sophisticated matrix analysis.

We fix the following matrix notation. For $n \in \mathbb{N}$ let $1_n$ be the identity matrix of size $n \times n$ and similarly let $0_n$ be the zero matrix of the same size. Moreover, let

$$U_n := \frac{1}{2} \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix} \in \mathbb{Q}^{n \times n}.$$

For $d \in \mathbb{N}$ let $d^{n \times n}$ be the $n \times n$ matrix which has every entry equal to $d$. For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$ we construct the direct sum $A \oplus B \in \mathbb{R}^{(n+m) \times (n+m)}$ and the Kronecker product $A \otimes B \in \mathbb{R}^{nm \times nm}$.
in the usual manner. Note that \( \text{tr}(A \oplus B) = \text{tr}(A) + \text{tr}(B) \) and \( \text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B) \). Finally, let \( \delta_{ij} \) be the Kronecker delta. We assume that every positive (semi)definite matrix is symmetric. Moreover, we call a symmetric matrix \( A \in \mathbb{R}^{n \times n} \) integral positive definite, if \( xAx^t \geq 1 \) for every \( x \in \mathbb{Z}^n \setminus \{0\} \).

The proof of Theorem B is deduced from a series of lemmas and propositions.

**Lemma 1.** Every integral positive definite matrix is positive definite.

*Proof.* Let \( W \in \mathbb{R}^{n \times n} \) be integral positive definite. By way of contradiction, suppose that there exists an eigenvector \( v \in \mathbb{R}^n \) of \( W \) with eigenvalue \( \lambda \leq 0 \) and (euclidean) norm 1. If \( \lambda < 0 \), choose \( x \in \mathbb{Q}^n \) such that \( \|x\| \leq \|v\| = 1 \) and \( \|x - v\| < -\frac{\lambda}{\|W\|} \) where \( \|W\| \) denotes the Frobenius matrix norm of \( W \). Then
\[
xWx^t = (x - v)W(x + v)^t + vWv^t \leq \|x - v\|\|W\|\|x + v\| + \lambda < 0.
\]

However, there exists \( m \in \mathbb{N} \) such that \( mx \in \mathbb{Z}^n \) and \( (mx)W(mx)^t < 0 \). This contradiction implies \( \lambda = 0 \). By Dirichlet’s approximation theorem (see [8] Theorem 200) there exist infinitely many integers \( m \) and \( x \in \mathbb{Z}^n \) such that
\[
\|x - mv\| < \frac{\sqrt{n}}{\sqrt{m}}.
\]

It follows that
\[
xWx^t = (x - mv)W(x - mv)^t \leq \|x - mv\|^2\|W\| < 1
\]
if \( m \) is sufficiently large. Again we have a contradiction. \( \square \)

Conversely, every positive definite matrix can be scaled to an integral positive definite matrix. The next lemma is a key argument when dealing with non-trivial actions of \( \mathcal{N} \) on \( \text{IBr}(b) \).

**Lemma 2.** Let \( A, B \in \mathbb{R}^{n \times n} \) positive semidefinite matrices such that \( A \) commutes with a permutation matrix \( P \in \mathbb{R}^{n \times n} \). Then \( \text{tr}(ABP) \leq \text{tr}(AB) \). If \( A \) and \( B \) are positive definite, then \( \text{tr}(ABP) = \text{tr}(AB) \) if and only if \( P = 1_n \).

*Proof.* By the spectral theorem, \( A \) and \( P \) are diagonalizable. Since they commute, they are simultaneously diagonalizable. Since \( A \) has real, non-negative eigenvalues, there exists a positive semidefinite matrix \( A^{1/2} \in \mathbb{R}^{n \times n} \) such that \( A^{1/2}A^{1/2} = A \) and \( A^{1/2}P = PA^{1/2} \). Then \( M := (m_{ij}) = A^{1/2}BA^{1/2} \) is also positive semidefinite. In particular \( m_{ij} \leq (m_{ii} + m_{jj})/2 \) for \( i, j \in \{1, \ldots, n\} \). If \( \sigma \) denotes the permutation corresponding to \( P \), then we obtain
\[
\text{tr}(ABP) = \text{tr}(A^{1/2}BPA^{1/2}) = \text{tr}(MP) = \sum_{i=1}^n m_{\sigma(i)} \leq \sum_{i=1}^n \frac{m_{ii} + m_{\sigma(i)\sigma(i)}}{2} = \text{tr}(M) = \text{tr}(AB).
\]
If \( A \) and \( B \) are positive definite, then so is \( M \) and we have \( m_{ij} < (m_{ii} + m_{jj})/2 \) whenever \( i \neq j \). This implies the last claim. \( \square \)

**Lemma 3.** Let \( W \in \mathbb{R}^{n \times n} \) be integral positive definite and suppose that \( W \) commutes with a permutation matrix \( P \). Let
\[
W_m := \frac{1}{2}
\begin{pmatrix}
2W & -PW & 0 \\
-P^tW & \ddots & \ddots \\
0 & \ddots & -PW \\
& & & 2W
\end{pmatrix}
\in \mathbb{R}^{mn \times mn}.
\]

Then \( W_m \) is integral positive definite. In particular, \( U_m \otimes W \) is integral positive definite.
Proof. Let \( x = (x_1, \ldots, x_m) \) with \( x_i \in \mathbb{Z}^n \). Since \( WP = PW \) we have

\[
xW_m x^t = \sum_{i=1}^{m} x_i W x_i^t - \sum_{i=1}^{m-1} x_i WP x_{i+1}^t
\]

\[
= \frac{1}{2} x_1 W x_1^t + \frac{1}{2} x_m W x_m^t + \frac{1}{2} \sum_{i=1}^{m-1} (x_i P - x_{i+1}) W (x_i P - x_{i+1})^t.
\]

We may assume that \( x_i \neq 0 \) for some \( i \in \{1, \ldots, m\} \). If \( i = 1 \), then \( x_m \neq 0 \) or \( x_j P \neq x_{j+1} \) for some \( j \). In any case \( xW_m x^t \geq 1 \). If \( i > 1 \), then the claim can be seen in a similar fashion. The last claim follows with \( P = 1_n \).

Now assume the notation of Theorem B. In addition, let \( p \) be the characteristic of \( B \) such that \( q := |\langle u \rangle| \) is a power of \( p \). Let \( k := k(B), l := l(b) \) and \( \zeta := e^{2\pi i / q} \in \mathbb{C} \). Then the generalized decomposition matrix \( Q = (d_xu)^{\gamma} \) of \( B \) with respect to the subsection \( (u, b) \) has size \( k \times l \) and entries in \( \mathbb{Z}[\zeta] \) (see \cite[Definition 1.19]{[16]} for instance). By the orthogonality relations of generalized decomposition numbers, we have \( Q^t \overline{Q} = qC \) where \( qC \) is the Cartan matrix of \( b \) (see \cite[Theorems 1.14 and 1.22]{[16]}). Recall that \( C \) is positive definite and has non-negative integer entries.

The first part of the next lemma is a result of Broué \cite{[17]} while the second part was known to Brauer \cite[(5H)]{[5]}.

Lemma 4 \cite[Proposition 1.36]{[16]}]. Let \( d^u_{\chi} \) be a row of \( Q \) corresponding to a character \( \chi \in \text{Irr}(B) \) of height 0. Let \( d \) be the defect of \( b \) and let \( \overline{C} := p^d C^{-1} \in \mathbb{Z}^{l \times l} \). Then the \( p \)-adic valuation of \( d_xu \overline{C} d_x^{\gamma} \) is 0. In particular, \( d_x \neq 0 \). Now assume that \( u \in \mathbb{Z}(D) \) and \( \chi \in \text{Irr}(B) \) is arbitrary. Then \( d_x \neq 0 \).

We identify the Galois group \( \mathcal{G} := \text{Gal}(\mathbb{Q}(\zeta)|\mathbb{Q}) \) with \( \text{Aut}((u)) \cong (\mathbb{Z}/q\mathbb{Z})^x \) such that \( \gamma(\zeta) = \zeta^\gamma \) for \( \gamma \in \mathcal{G} \). In this way we regard \( N \) as a subgroup of \( \mathcal{G} \). Let \( n := |N| \). For any \( \gamma \in \mathcal{G}, \gamma(Q) \) is the generalized decomposition matrix with respect to \((u^\gamma, b)\). If the subsections \((u, b)\) and \((u^\gamma, b)\) are not conjugate in \( G \), then \( \gamma \notin N \) and \( \gamma(Q)^t \overline{Q} = 0 \). On the other hand, if they are conjugate, then \( \gamma \in N \) and

\[
\gamma(d_xu)^{\gamma} = d_xu^\gamma = d_xu^{\gamma \gamma}(5)
\]

for \( \chi \in \text{Irr}(B) \) and \( \varphi \in \text{IBr}(b) \). Hence, in this case, \( \gamma \) acts on the columns of \( Q \) and there exists a permutation matrix \( P_\gamma \) such that \( \gamma(Q) = QP_\gamma \). Recall that permutation matrices are orthogonal, i.e. \( P_{\gamma^{-1}} = P_{\gamma} \). Since \( \mathcal{G} \) is abelian, we obtain

\[
CP_\gamma = Q^t \gamma(\overline{Q}) = \gamma^{-1}(Q)^t \overline{Q} = P_{\gamma^{-1}} C = P_\gamma C
\]

for every \( \gamma \in N \) and

\[
\gamma(Q)^t \delta(Q) = \begin{cases} CP_{\gamma^{-1}} \delta & \text{if } \gamma \equiv \delta \pmod{N} \\ 0 & \text{otherwise} \end{cases}
\]

for \( \gamma, \delta \in \mathcal{G} \). For any subset \( S \subseteq N \) we write \( P_S := \sum_{\delta \in S} P_\delta \).

Lemma 5. In the situation of Theorem B we may assume that \( W \) is (integral) positive definite and commutes with \( P_\gamma \) for every \( \gamma \in N \).

Proof. Let

\[
W := \frac{1}{2n} \sum_{\delta \in N} P_\delta (W + W^t) P_\delta^t.
\]
Then $W$ is symmetric and commutes with $P_\delta$ for every $\delta \in \mathcal{N}$. Moreover, $W$ is integral positive definite and by Lemma 6, $W$ is positive definite. Finally,

$$\text{tr}(WC) = \frac{1}{2n} \sum_{\delta \in \mathcal{N}} \text{tr}(P_\delta WCP_\delta) + \text{tr}(P_\delta W^{i}CP_\delta) = \text{tr}(WC),$$

since $P_\delta$ commutes with $C$. Hence, we may replace $W$ by $W$. \hfill \square

In the following we revisit some arguments from [16, Section 5.2]. Write $Q = \sum_{i=1}^{\varphi(q)} A_i \zeta^i$ where $A_i \in \mathbb{Z}^{k \times l}$ for $i = 1, \ldots, \varphi(q)$ and $\varphi(q) = q - q/p$ is Euler’s function. Let

$$A_q = (A_i : i = 1, \ldots, \varphi(q)) \in \mathbb{Z}^{k \times \varphi(q)}.$$

**Lemma 6.** The matrix $A_q$ has rank $l\varphi(q)/n$.

**Proof.** It is well-known that the Vandermonde matrix $V := (\zeta^{i\gamma} : 1 \leq i \leq \varphi(q), \gamma \in \mathcal{G})$ is invertible. Since $Q$ has full rank, the facts stated above show that $(\gamma(Q) : \gamma \in \mathcal{G})$ has rank $l|\mathcal{G} : \mathcal{N}| = l\varphi(q)/n$. Then also $A_q = (\gamma(Q) : \gamma \in \mathcal{G})(V \otimes 1_l)^{-1}$ has rank $l\varphi(q)/n$. \hfill \square

Let $T_q$ be the trace of $Q(\zeta)$ with respect to $Q$. Recall that

$$T_q(\zeta^i) = \begin{cases} \varphi(q) & \text{if } q \mid i, \\ -q/p & \text{if } q \nmid i \text{ and } q/p \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$T_q(Q\zeta^{-i}) = \sum_{j=1}^{\varphi(q)} A_j T_q(\zeta^{j-i}) = \frac{q}{p} \left( pA_i - \sum_{j \equiv i \pmod{q/p}} A_j \right).$$

**Definition 7.** For $1 \leq i \leq \varphi(q)$ let $i'$ be the unique integer such that $0 \leq i' < q/p$ and $i' \equiv -i \pmod{q/p}$.

Then $q/p \leq i + i' \leq \varphi(q)$ and $\sum_{j \equiv i \pmod{q/p}} \zeta^{-j} = -\zeta^{i'}$ where we consider only those summands with $1 \leq j \leq \varphi(q)$. With this convention we obtain

$$T_q(Q(\zeta^{-i} - \zeta^{-i'})) = \frac{q}{p} \left( pA_i - \sum_{j \equiv i \pmod{q/p}} A_j + \sum_{j \equiv i \pmod{q/p}} (pA_j - \sum_{s \equiv j \pmod{q/p}} A_s) \right)$$

$$= \frac{q}{p} \left( pA_i - (p-1) \sum_{j \equiv i \pmod{q/p}} A_j + (p-1) \sum_{s \equiv i \pmod{q/p}} A_s \right) = qA_i$$

and (7) yields

$$q^2 A_i A_j = \sum_{\gamma, \delta \in \mathcal{G}} (\zeta^{-i\gamma} - \zeta^{-i'\gamma})(\zeta^{j\delta} - \zeta^{-j'\delta})\gamma(Q)^i \delta(\overline{Q})$$

$$= \sum_{\delta \in \mathcal{N}} \sum_{\gamma \in \mathcal{G}} (\zeta^{-i\gamma} - \zeta^{-i'\gamma})(\zeta^{j\gamma} - \zeta^{-j'\delta})qCP_\delta$$

$$= qC \sum_{\delta \in \mathcal{N}} P_\delta T_q(\zeta^{j\delta-i'} - \zeta^{j'\delta-i'} - \zeta^{-j\delta-i'} + \zeta^{-j'\delta+i'})$$

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for $1 \leq i, j \leq \varphi(q)$. Note that
\[ j\delta - i \equiv j\delta + i' \equiv -j'\delta - i \equiv -j'\delta + i' \pmod{q/p}. \]
Moreover, if $j\delta - i \equiv 0 \pmod{q}$, then $j\delta + i' \neq 0 \pmod{q}$. In this case $T_q(zj\delta - i - zj\delta + i') = \varphi(q) + q/p = q$. In a similar way we obtain
\[ A_1^t A_j = C \sum_{\delta \in \mathcal{N}} P_\delta ([j\delta \equiv i] - [j\delta \equiv -i] + [j'\delta \equiv i'] - [j'\delta \equiv -i]) \] (8)
where all congruences are modulo $q$ and $[. \equiv .]$ evaluates to 1 if the congruence is fulfilled and to 0 otherwise.

By Lemma 4, $A_q$ has non-zero rows $a_1, \ldots, a_{k_0(B)}$. If $\mathcal{W} \in \mathbb{R}^{l\varphi(q) \times l\varphi(q)}$ is integral positive definite, then
\[ k_0(B) \leq \sum_{i=1}^{k_0(B)} a_i \mathcal{W} a_i^t \leq \text{tr}(A_q \mathcal{W} A_q^t) = \text{tr}(A_q \mathcal{W} A_q^t) \]
and this is what we are going to show. We need to discuss the case $p = 2$ separately.

Proposition 8. Theorem B holds for $p = 2$.

Proof. If $q \leq 2$, then $Q = A_1 = A_q$, $n = 1$ and
\[ k_0(B) \leq \text{tr}(WQ^tQ) = \text{tr}(WqC) = q \text{tr}(WC) = k_0(\langle u \rangle \rtimes \mathcal{N}) \text{tr}(WC). \]

Hence, we will assume for the remainder of the proof that $q \geq 4$. Then $i' = q/2 - i$ for every $1 \leq i \leq \varphi(q) = q/2$. Hence, (8) simplifies to
\[ A_1^t A_j = 2C \sum_{\delta \in \mathcal{N}} P_\delta ([j\delta \equiv i] - [j\delta \equiv i + q/2]). \] (9)

It is well-known that
\[ G = (-1 + q\mathbb{Z}) \times (5 + q\mathbb{Z}) \cong C_2 \times C_{q/4}. \]
In particular, $\mathcal{N}$ is a 2-group and so is $U := \langle u \rangle \rtimes \mathcal{N}$. Therefore, $k_0(U) = |U : U'|$ where $U'$ denotes the commutator subgroup of $U$.

Case 1: $\mathcal{N} = \langle 5^{2m} + q\mathbb{Z} \rangle$ for some $m \geq 0$.

Then $q = |\mathcal{N}|^{2m+2} = n^{2m+2}$ and $U'$ is generated by $u^{5^{2m} - 1}$. Since $5^{2m} - 1 \equiv 2^{m+2} \pmod{2^{m+3}}$, we conclude that $|U'| = n$ and $k_0(U) = |U : U'| = q$.

For any given $\delta \in \mathcal{N} \setminus \{1\}$ both congruences $i\delta \equiv i \pmod{q}$ and $i\delta \equiv i + q/2 \pmod{q}$ have solutions $i \in \{1, \ldots, q/2\}$. Moreover, the number of solutions is the same, since they both form residue classes modulo a common integer. On the other hand, $i\delta \equiv i + q/2 \pmod{q}$ has no solution for $\delta = 1$. An application of (9) yields
\[ \sum_{i=1}^{q/2} A_1^t A_i = 2C \sum_{\delta \in \mathcal{N}} P_\delta \sum_{i=1}^{q/2} [i\delta \equiv i] - [i\delta \equiv i + q/2] = qCP_1 = qC. \]

The matrix $\mathcal{W} := 1_{q/2} \otimes \mathcal{W}$ is certainly integral positive definite. Moreover,
\[ k_0(B) \leq \text{tr}(\mathcal{W} A_q^t A_q) = q \text{tr}(WC) = k_0(U) \text{tr}(WC). \] (10)
It remains to check when this bound is sharp. If \( k_0(B) = \text{tr}(WA_q^tA_q) \), then every row of \( A_q \) vanishes in all but (possibly) one \( A_i \). Moreover, characters of positive height vanish completely in \( A_q \). By way of contradiction, suppose that \( \mathcal{N} \) acts non-trivially on \( \text{IBr}(b) \). Using [5], it follows that there exists a character \( \chi \in \text{Irr}(B) \) of height 0 such that the corresponding row \( d_\chi = a_\chi^{-1} \) of \( Q \) satisfies \( aP_\delta = -a \) for some \( \delta \in \mathcal{N} \). We write \( a = (\alpha_1, \ldots, \alpha_s, -\alpha_1, \ldots, -\alpha_s, 0, \ldots, 0) \) with non-zero \( \alpha_1, \ldots, \alpha_s \in \mathbb{Z} \). With the notation of Lemma 4 let \( C = (c_{ij}) \). By [6], we have \( P_3C = CP_3 \). Now Lemma 4 leads to the contradiction

\[
0 \neq d_\chi \tilde{C}d_\chi^{-1} = a\tilde{C}a^{-1} = \sum_{i=1}^s 2a_i^2\tilde{c}_{ii} \equiv 0 \pmod{2},
\]

since the diagonal of \( \tilde{C} \) is constant on the orbits of \( \mathcal{N} \). Therefore, equality in (10) can only hold if \( \mathcal{N} \) acts trivially on \( \text{IBr}(b) \).

**Case 2:** \( \delta := -5^m + q \mathbb{Z} \in \mathcal{N} \setminus \{1\} \) for some \( m \geq 0 \).

Since \( 1 + 5^m \equiv 2 \pmod{4} \), we have \( U' = \langle u^{1+5^m} \rangle = \langle u^2 \rangle \) and \( k_0(U) = |U : U'| = 2n \). We show that every row of \( A_{q/2} \) corresponding to a height 0 character \( \chi \in \text{Irr}(B) \) is non-zero. Let \( d_\chi = \sum_{i=1}^{q/2} a_i\zeta_i^j \) be the corresponding row of \( Q \) where \( a_i \) is a row of \( A_i \). Let \( \nu \) be the \( p \)-adic valuation. By Lemma 4

\[
0 = \nu(d_\chi \tilde{C}d_\chi^{-1}) = \nu\left( \sum_{1 \leq i, j \leq q/2} a_i\tilde{C}a_j^{\nu-1} \right) = \nu\left( \sum_{i=1}^{q/2} a_i\tilde{C}a_i^{\nu} \right),
\]

i.e.

\[
\sum_{i=1}^{q/2} a_i\tilde{C}a_i^{\nu} \equiv 1 \pmod{2}. \tag{11}
\]

On the other hand,

\[
\sum_{i=1}^{q/2} a_iP_\delta \zeta_i^j = d_\chi P_\delta = \delta(d_\chi) = \sum_{i=1}^{q/2} a_i\zeta_i^j \delta.
\]

Now \( i\delta \equiv i \pmod{q} \) implies \(-5^m \equiv \delta \equiv 1 \pmod{q} \) and \( i \equiv q/2 \). Similarly \( i\delta \equiv i + q/2 \pmod{q} \) implies \( i = q/4 \). Then \( A_{q/4}P_\delta = -A_{q/4} \). As in Case 1, it follows that \( a_{q/4} \tilde{C}a_{q/4} \equiv 0 \pmod{2} \). For \( i \notin \{q/2, q/4\} \) we have \( A_iP_\delta = \pm A_j \) for some \( j \in \{1, \ldots, q/2\} \) \( \setminus \{i\} \). Then, using (6),

\[
a_j\tilde{C}a_j = a_iP_3P_3a_i = a_i\tilde{C}a_i.
\]

Now (11) yields \( a_{q/2} \tilde{C}a_{q/2} \equiv 1 \pmod{2} \) and \( a_{q/2} \neq 0 \). Therefore, \( A_{q/2} \) has non-zero rows for height 0 characters.

By [9], \( A_{q/2}A_{q/2} = 2CP_N \) and Lemma 2 implies

\[
k_0(B) \leq \text{tr}(WA_{q/2}A_{q/2}) = 2\text{tr}(CP_N) = 2\sum_{\gamma \in \mathcal{N}} \text{tr}(WCP_{\gamma}) \leq 2n \text{tr}(WC) = k_0(U) \text{tr}(WC)
\]

with strict inequality if \( \mathcal{N} \) acts non-trivially on \( \text{IBr}(b) \).

We are left with the case \( p > 2 \). Here \( G \) is cyclic and \( \mathcal{N} \) is uniquely determined by \( n \). Let \( n_p \) be the \( p \)-part of \( n \) and \( n_{p'} \) the \( p' \)-part. Then \( n_p \mid \frac{2}{p} \) and \( n_{p'} \mid p - 1 \).

**Lemma 9.** We have \( k_0(\langle u \rangle \times \mathcal{N}) = n + \frac{q-n_p}{n_{p'}} \) for \( p > 2 \).
Finally, Lemma 2 implies

\[ k_0(U) = n + np \frac{q/n_p - 1}{n_p'} = n + \frac{q - np}{n_p'}. \]

The following settles Theorem B in the special case \( n_p = 1 \) (use Lemma 9).

**Proposition 10.** Let \( p > 2 \) and \( n_p = 1 \). With the notation above there exists an integral positive definite matrix \( W \in \mathbb{R}^{\varphi(q) \times \varphi(q)} \) such that

\[ \text{tr}(WA_q^t A_q) \leq (n + \frac{q - 1}{n}) \text{tr}(W) \]

with equality if and only if \( N \) acts trivially on \( \text{IBr}(b) \).

**Proof.** We argue by induction on \( q \). If \( q = 1 \), then \( A_q = A_1 = Q \), \( n = 1 \) and the claim holds with \( W = W \) [Lemma 5]. The next case requires special treatment as well.

**Case 1:** \( q = p \).

Then \( i' = 0 \) for all \( i \) and \( \mathbf{[8]} \) simplifies to

\[ A_i^t A_j = C \sum_{\delta \in N} P_\delta ([j \delta \equiv i] + [0 \delta \equiv 0]) = \begin{cases} CP_N & \text{if } i \neq j \pmod{N}, \\ C(P_N + P_{j-i}) & \text{if } i \equiv j \pmod{N}. \end{cases} \]

After permuting the columns of \( A_q \) if necessary, we obtain

\[ A_q^t A_q = 1^{\varphi(q) \times \varphi(q)} \otimes P_N C + 1_{n'} \otimes (P_{\gamma^{-i}C})_{\gamma, \delta \in N} \]

where \( n' := (p-1)/n \). We fix a generator \( \rho \) of \( N \). Then we may write \( (P_{\gamma^{-i}C})_{\gamma, \delta \in N} = (P_{\rho^i}^j C)_{i, j = 1}^n \).

By [Lemma 5], we may assume that \( W \) is (integer) positive definite and commutes with \( P_\rho \). Let \( W_n \) as in Lemma 3 where we use \( P_\rho \) instead of \( P \). A repeated application of that lemma shows that the matrix \( W := U_{n'} \otimes W_n \) is integral positive definite. Moreover, since \( P_N P_\rho = P_N = P_N^i P_\rho^i \), we have

\[ \text{tr}(WA_q^t A_q) = \text{tr}((U_{n'} \otimes W_n)(1^{\varphi(q) \times \varphi(q)} \otimes P_N C)) + \text{tr}((U_{n'} \otimes W_n)(1_{n'} \otimes (P_{\rho^{-i}C}^j))) \]

\[ = \text{tr}((U_{n'} \otimes W_n)(1^{n \times n'} \otimes 1_{n \times n} \otimes P_N C)) + \text{tr}(U_{n'} \otimes W_n(P_{\rho^{-i}C}^j)) \]

\[ = \text{tr}(W_n(1^{n \times n} \otimes P_N C)) + n' \text{tr}(W_n(P_{\rho^{-i}C}^j)) \]

\[ = \sum_{i=1}^n \text{tr}(WCP_N) - \sum_{i=1}^{n-1} \text{tr}(WCP_N P_\rho) + n' \left( \sum_{i=1}^n \text{tr}(W) - \sum_{i=1}^{n-1} \text{tr}(WCP_N P_\rho^i) \right) \]

\[ = \text{tr}(WCP_N) + n' \text{tr}(W). \]

Finally, Lemma 2 implies

\[ \text{tr}(WCP_N) = \sum_{\delta \in N} \text{tr}(WCP_\delta) \leq n \text{tr}(W). \]
with equality if and only if $N$ acts trivially on $\text{IBr}(b)$. This completes the proof in the case $q = p$.

**Case 2:** $q > p$. Let

$$I_p := \{1 \leq i \leq \varphi(q) : p \mid i\}, \quad I_{p'} := \{1 \leq i \leq \varphi(q) : p \nmid i\}.$$ 

Then $|I_p| = \varphi(q)/p = \varphi(q/p)$ and $|I_{p'}| = \varphi(q) - \varphi(q/p) = \varphi(q/p)(p - 1)$. If $i \in I_p$ and $j \in I_{p'}$, then $j\delta - i \not\equiv 0 \pmod{q/p}$ for every $\delta \in N$ and $A_q^i A_j = 0$ by [8]. Hence, after relabeling the columns of $A_q$, we obtain

$$A_q^i A_q = \begin{pmatrix} \Delta_p & 0 \\ 0 & \Delta_{p'} \end{pmatrix}$$

where $\Delta_p$ corresponds to the indices in $I_p$. Since $n \mid p - 1$, we may regard $N$ as a subgroup of $\text{Gal}(\mathbb{Q}(\zeta^p)/\mathbb{Q})$. For $i \in I_p$ let $j = i/p$. Then $j' \equiv -i \pmod{q/p}$ implies $j'/p \equiv -j \pmod{q/p^2}$ and $0 \leq j'/p < q/p^2$. Hence, $j' = j'/p$ where the left hand side refers to $q/p$. It follows from [8] that $\Delta_p = A_{q/p}^i A_{q/p}$. By induction on $q$ there exists an integral positive definite $W_p$ such that

$$\text{tr}(W_p \Delta_p) \leq \left(n + \frac{q/p - 1}{n}\right) \text{tr}(WC)$$

with equality if and only if $N$ acts trivially on $\text{IBr}(b)$.

It remains to consider $\Delta_{p'}$. By Lemma 6, $A_q^i/p$ and $\Delta_p$ have rank $l\varphi(q/p)/n$ and therefore $\Delta_{p'}$ has rank

$$l(\varphi(q) - \varphi(q/p))/n = l\varphi(q/p)(p - 1)/n.$$ 

We define a subset $J \subseteq I_{p'}$ such that $|J| = \varphi(q/p)(p - 1)/n$ and the matrix $(A_i : i \in J)$ has full rank. Let $R$ be a set of representatives for the orbits of $\{i \in I_{p'} : 1 \leq i \leq q/p\}$ under the multiplication action of $N$ modulo $q/p$. Note that every orbit has size $n$. For $r \in R$ let

$$J_r := \{r + jq/p : j = 0, \ldots, p - 2\} \subseteq I_{p'}$$

and $J := \bigcup_{r \in R} J_r$. Since $J_r \cap J_s = \emptyset$ for $r \neq s$, we have $|J| = \varphi(q/p)(p - 1)/n$. If $i \in J_r$ and $j \in J_s$ with $r \neq s$, then $j\delta \equiv i \pmod{q/p}$ for every $\delta \in N$. Consequently, $A_q^i A_j = 0$. Now let $i, j \in J_r$. Then [8] implies

$$A_q^i A_j = C(1 + \delta_{ij}).$$

After relabeling we obtain

$$(A_i : i \in J)^t (A_i : i \in J) = 1_{\varphi(q/p)/n} \otimes (1 + \delta_{ij})^{p - 1}_{i,j=1} \otimes C.$$ 

In particular, $(A_i : i \in J)$ has full rank. Since $\Delta_{p'}$ has the same rank, there exists an integral matrix $S \in \text{GL}(\varphi(q/p)(p - 1), \mathbb{Q})$ such that

$$S^t \Delta_{p'} S = 1_{\varphi(q/p)/n} \otimes (1 + \delta_{ij}) \otimes C \oplus 0_s$$

where $s := l\varphi(q/p)(p - 1)(n - 1)/n$. Let

$$W_{p'} := S(1_{\varphi(q/p)/n} \otimes U_{p - 1} \otimes W \oplus 1_s) S^t.$$ 

Then $W_{p'}$ is integral positive definite by Lemma 3. Moreover,

$$\text{tr}(W_{p'} \Delta_{p'}) = \text{tr}((1_{\varphi(q/p)/n} \otimes U_{p - 1} \otimes W)(1_{\varphi(q/p)/n} \otimes (1 + \delta_{ij}) \otimes C)) + \text{tr}(1_{0_s})$$

$$= \frac{\varphi(q/p)}{n} \text{tr}(U_{p - 1}(1 + \delta_{ij})) \text{tr}(WC) = \frac{\varphi(q/p)p}{n} \text{tr}(WC) = \frac{\varphi(q)}{n} \text{tr}(WC).$$
Finally, we set $\mathcal{W} := \mathcal{W}_p \oplus \mathcal{W}_{p'}$. Then $\mathcal{W}$ is integral positive definite and
\[
\text{tr}(\mathcal{W} A_q^t A_q) = \text{tr}(\mathcal{W}_p \Delta_p) + \text{tr}(\mathcal{W}_{p'} \Delta_{p'}) \leq \left(n + \frac{q/p - 1}{n} \right) \text{tr}(WC) + \frac{\varphi(q)}{n} \text{tr}(WC)
\]
\[
= \left(n + \frac{q - 1}{n} \right) \text{tr}(WC)
\]
with equality if and only if $\mathcal{N}$ acts trivially on $\text{IBr}(b)$.

To complete the proof of Theorem B it remains to show the following.

**Proposition 11.** *Theorem B holds in the case $p > 2$ and $n_p > 1$.*

**Proof.** Let
\[
I_1 := \{1 \leq i \leq \varphi(q) : n_p \mid i\}, \quad I_2 := \{1 \leq i \leq \varphi(q) : n_p \nmid i\}.
\]
As in the proof of Proposition 10 we have
\[
A_q^t A_q = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}
\]
where $\Delta_1$ corresponds to the indices in $I_1$. Let $\mathcal{N} = \mathcal{N}_p \times \mathcal{N}_{p'}$ where $\mathcal{N}_p := \langle 1 + q/n_p + q\mathbb{Z} \rangle$ is the unique Sylow $p$-subgroup of $\mathcal{N}$. Then $\delta i \equiv i \pmod{q}$ for $\delta \in \mathcal{N}_p$ and $i \in I_1$. Hence, for $i, j \in I_1$ we have
\[
A_q^t A_j = C \sum_{\delta \in \mathcal{N}} P_\delta([j\delta \equiv i] - [j\delta \equiv -i]) + [j'\delta \equiv i'] - [j'\delta \equiv -i])
\]
\[
= C P_{\mathcal{N}_p} \sum_{\delta \in \mathcal{N}_{p'}} P_\delta([j\delta \equiv i] - [j\delta \equiv -i]) + [j'\delta \equiv i'] - [j'\delta \equiv -i])
\]
\[
\text{For } i \in I_1 \text{ it is easy to see that } i'_{n_p} = (i/n_p)' \text{ when the right hand side is considered with respect to } q/n_p \text{ (see proof of Proposition 10). It follows that}
\]
\[
\Delta_1 = (1_{\varphi(q/n_p)} \otimes P_{\mathcal{N}_p}) A_q^t A_q/n_p
\]
where we consider $A_q/n_p$ with respect to the $p'$-group $\mathcal{N}_{p'}$. By Proposition 10 there exists an integral positive definite $\mathcal{W}_1$ such that
\[
\text{tr}(\mathcal{W}_1 A_q^t A_q/n_p) \leq \left(n_{p'} + \frac{q/n_p - 1}{n_{p'}} \right) \text{tr}(WC).
\]
(12)

Moreover, equality holds if and only if $\mathcal{N}_{p'}$ acts trivially on $\text{IBr}(b)$. By construction, $A_q^t A_q/n_p$ is positive semidefinite. By [6] and [8], $A_q^t A_q/n_p$ commutes with $1_{\varphi(q/n_p)} \otimes P_\delta$ for $\delta \in \mathcal{N}_p$. Hence, Lemma 2 implies
\[
\text{tr}(\mathcal{W}_1 \Delta_1) = \text{tr}(\mathcal{W}_1 (1_{\varphi(q/n_p)} \otimes P_{\mathcal{N}_p}) A_q^t A_q/n_p)
\]
\[
\leq n_p \text{tr}(\mathcal{W}_1 A_q^t A_q/n_p) \leq \left(n + \frac{q - n_p}{n_{p'}} \right) \text{tr}(WC).
\]
(13)

Suppose that $\text{tr}(\mathcal{W}_1 \Delta_1) = \left(n + \frac{q - n_p}{n_{p'}} \right) \text{tr}(WC)$. Then, by (12), $\mathcal{N}_{p'}$ acts trivially on $\text{IBr}(b)$ and the matrices $A_q^t A_j$ with $i, j \in I_1$ are scalar multiples of $C P_{\mathcal{N}_p}$. We write $A_q^t A_q/n_p = (A_{ij})$ such that

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\( A^t_{in_p} A_{jn_p} = P_{N_p} \delta_{ij} \). Note that \( A_{11} = 2C \) is positive definite. As in the proof of Lemma 2, we construct a positive semidefinite matrix \( M = (m_{ij}) := A^{1/2} W_i A^{1/2} \) where \( A^{1/2} A^{1/2} = (A_{ij})_{ij} \). By way of contradiction, suppose that \( P_{\delta} \neq I_1 \) for some \( \delta \in N_p \). Let \( 1 \leq i \leq l \) such that \( \delta(i) \neq i \), and let \( x = (x_i) \in \mathbb{Z}[q/n]l \) with \( x_i = x_{\delta(i)} = 1 \) and zero elsewhere. Then \( x(A_{ij})x^t > 0 \) since \( A_{11} \) is positive definite. Thus, \( A^{1/2} x^t \neq 0 \). Since \( W_i \) is positive definite (Lemma 1), it follows that \( xMx^t > 0 \) and \( m_{i\delta(i)} < (m_{ii} + m_{\delta(i)\delta(i)})/2 \). Hence, the proof of Lemma 2 leads to

\[
\text{tr}(W_1(1_{\varphi(q/n_p)} \otimes P_{\delta})A^t_{q/n_p} A_{q/n_p}) = \text{tr}(A^{1/2} W_1(1_{\varphi(q/n_p)} \otimes P_{\delta})A^{1/2})
= \text{tr}(M(1_{\varphi(q/n_p)} \otimes P_{\delta})) < \text{tr}(M) = \text{tr}(W_1 A^t_{q/n_p} A_{q/n_p})
\]

and we derive the contradiction \( \text{tr}(W_1 \Delta_1) < n_p \text{tr}(W_1 A^t_{q/n_p} A_{q/n_p}) \). Thus, we have shown that equality in (13) can only hold if \( N \) acts trivially on \( \text{IBr}(b) \).

Now we use the argument from Proposition 8 to deal with \( \Delta_2 \). Let \( \chi \in \text{Irr}(B) \) of height 0, and let \( d_{\chi} = \sum_{i=1}^{\varphi(q)} a_i \zeta^i \) be the corresponding row of \( Q \). By Lemma 4 we have

\[
0 = \nu(d_{\chi} \tilde{C} a_{\chi}^t) = \nu\left( \sum_{i,j=1}^{\varphi(q)} a_i \tilde{C} a_j^t \right)
\]

where \( \nu \) is the \( p \)-adic valuation. In order to show that \( a_i \neq 0 \) for some \( i \in I_1 \), it suffices to show that

\[
\sum_{i,j \in I_2} a_i \tilde{C} a_j^t \equiv 0 \pmod{p}. \tag{14}
\]

For any \( \delta \in N_p \), we have

\[
\sum_{i=1}^{\varphi(q)} A_i P_{\delta} \zeta^i = QP_{\delta} = \delta(Q) = \sum_{i=1}^{\varphi(q)} A_i \zeta^i_{\delta}.
\]

Restricting to the indices \( i \in I_2 \) and taking the valuation yields

\[
\sum_{i \in I_2} A_i P_{\delta} = \sum_{i \in I_2} A_i \pmod{p}.
\]

Let \( i \in I_2 \) be arbitrary and choose \( \delta \in N_p \) such that \( \gcd(q,i)p = |(\delta)| \). Let

\[
\{i_1, \ldots, i_{p-1}\} = \{j \in I_2 : j \equiv i \pmod{q/p} \}.
\]

We may assume that \( i_1 \delta \equiv -i' \pmod{q} \) and \( i_j \delta \equiv i_{j-1} \pmod{q} \) for \( j = 2, \ldots, p-1 \). Since \( \zeta^{-i'} = -\zeta^{i_1} - \ldots - \zeta^{i_{p-1}} \), we obtain \( A_{p-1} P_{\delta} = A_{i_1} \) and \( A_i P_{\delta} = A_{i_{j+1}} - A_{i_1} \) for \( j = 1, \ldots, p-2 \). Hence,

\[
\left( \sum_{j \in I_2} a_j \tilde{C} a_{i_1}^t \right) = \left( \sum_{j \in I_2} a_j \right) \tilde{C} P_{\delta} \tilde{C} a_{i_1}^t = \left( \sum_{j \in I_2} a_j \right) \tilde{C} (a_{i_2} - a_{i_1})^t = \left( \sum_{j \in I_2} a_j \right) \tilde{C} (a_{i_3} - a_{i_2})\]

\[
= \ldots = \left( \sum_{j \in I_2} a_j \right) \tilde{C} (a_{i_{p-1}} - a_{i_{p-2}})^t = -\left( \sum_{j \in I_2} a_j \right) \tilde{C} a_{i_{p-1}}^t \pmod{p}.
\]

Now it is easy to see that

\[
\left( \sum_{j \in I_2} a_j \right) \tilde{C} (a_{i_1} + \ldots + a_{i_{p-1}})^t = \frac{p(p-1)}{2} \left( \sum_{j \in I_2} a_j \right) \tilde{C} a_{i_1}^t \equiv 0 \pmod{p}
\]

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and (14) follows. Thus, we have shown that every height 0 character has a non-vanishing part in $A_i$ for some $i \in I_1$. Hence by (13),

$$k_0(B) \leq \text{tr}(W_1 \Delta_1) \leq \left( n + \frac{q - n_{p'}}{n_{p'}} \right) \text{tr}(W_C)$$

with strict inequality if $\mathcal{N}$ acts non-trivially on $\text{IBr}(b)$. By Lemma 9 the proof is complete. 

Now it is time to derive Theorem A from Theorem B. For the convenience of the reader we restate it as follows.

**Proposition 12.** If $u \in \mathbb{Z}(D)$ in the situation above, then

$$k(B) \leq \left( n + \frac{q - 1}{n} \right) \text{tr}(W_C) \leq q \text{tr}(W_C).$$

The first inequality is strict if $\mathcal{N}$ acts non-trivially on $\text{IBr}(b)$ and the second inequality is strict if and only if $1 < n < q - 1$.

**Proof.** As mentioned in the introduction, the inertial quotient $N_G(D,b_D)/D\text{C}_G(D)$ restricts to $\mathcal{N}$ and therefore $\mathcal{N}$ is a $p'$-group. As a subgroup of $\text{Aut}(\langle u \rangle)$, its order $n$ must divide $p - 1$. For $p = 2$ we obtain $n = 1$ and $k_0(\langle u \rangle \times \mathcal{N}) = q$. For $p > 2$, Lemma 9 gives $k_0(\langle u \rangle \times \mathcal{N}) = n + \frac{q - 1}{n}$. By Lemma 4, all rows of $Q$ are non-zero. Hence, the proofs of Propositions 8 and 10 actually show that $k(B) \leq k_0(\langle u \rangle \times \mathcal{N}) \text{tr}(W_C)$ with strict inequality if $\mathcal{N}$ acts non-trivially on $\text{IBr}(b)$ (note that only Case 1 in the proof of Proposition 8 is relevant). This implies the first two claims. The last claim follows, since $n + \frac{q - 1}{n}$ is a convex function in $n$ and $1 \leq n \leq q - 1$.

If the action of $\mathcal{N}$ on $\text{IBr}(b)$ is known, a careful analysis of the proofs above leads to even stronger estimates. For instance, in Proposition 10 we have actually shown that

$$k_0(B) \leq \text{tr}(W_C P_N) + \frac{q - 1}{n} \text{tr}(W_C)$$

for $p > 2$ and $n_{p'} = 1$. If $b$ has cyclic defect groups, then $P_N$ is a direct sum of equal blocks of the form $d^{n/d \times n/d}$ (see [18, Proposition 3.2]). This can be used to give a simpler proof of [18, Theorem 3.1].

### 3 Consequences

In this section we deduce some of the results stated in the introduction.

**Corollary 13** (Sambale [14, Lemma 1]). Let $C = (c_{ij})_{i,j=1}^l$ be the Cartan matrix of a Brauer correspondent of $B$ in $C_G(u)$ where $u \in \mathbb{Z}(D)$. Then for every positive definite, integral quadratic form $q(x_1, \ldots, x_l) = \sum_{1 \leq i \leq j \leq l} q_{ij} x_i x_j$ we have

$$k(B) \leq \sum_{1 \leq i \leq j \leq l} q_{ij} c_{ij}.$$
Proof. Let $t := |\langle u \rangle|$. Then $t^{-1}C$ is the Cartan matrix of the block $\overline{b}$ in Theorem A (see [16, Theorem 1.22]). Taking $W := \frac{1}{2}(q_{ij}(1 + \delta_{ij}))$ with $q_{ij} = q_{ji}$ we obtain
\[
xWx^t = \frac{1}{2} \sum_{1 \leq i,j \leq l} q_{ij}(1 + \delta_{ij})x_ix_j = \sum_{1 \leq i,j \leq l} q_{ij}x_ix_j = q(x) \geq 1
\]
for every $x = (x_1, \ldots, x_l) \in \mathbb{Z}^l \setminus \{0\}$ and
\[
k(B) \leq t \text{ tr}(Wt^{-1}C) = \text{ tr}(WC) = \sum_{1 \leq i,j \leq l} q_{ij}c_{ij}.
\]
Wada’s inequality \cite{2} follows from Corollary 13 with $q(x) = \sum_{i=1}^{l} x_i^2 - \sum_{i=1}^{l-1} x_ix_{i+1}$ (or $W = U_l$ in Theorem A).

Corollary 14 (Héthelyi–Külshammer–Sambale \cite{9} Theorem 4.10]). Suppose $p > 2$. Let $b$ be a Brauer correspondent of $B$ in $C_G(u)$ where $u \in D$ and $l(b) = 1$. Let $|N_G(\langle u \rangle, b) : C_G(u)| = p^s \cdot r$ with $s \geq 0$ and $p \nmid r$. Then
\[
k(B) \leq \frac{|\langle u \rangle| + p^s(r^2 - 1)}{|\langle u \rangle|} \cdot p^d
\]
where $d$ is the defect of $b$.

Proof. Setting $q := |\langle u \rangle|$ we obtain $C = (p^d/q)$ in the situation of Theorem B. By Lemma 9, $k_0(\langle u \rangle \rtimes N) = (q + p^s(r^2 - 1))/r$ and the claim follows with $W = (1)$. \hfill \Box

The following result of Brauer cannot be seen in the framework of integral quadratic forms. It was a crucial ingredient in the proof of the $k(GV)$-Problem (see [19, Theorem 2.5d]).

Corollary 15 (Brauer \cite{5} 5D]). Let $B$ be a $p$-block with defect $d$, and let $C$ be the Cartan matrix of a Brauer correspondent $b$ of $B$ in $C_G(u)$ where $u \in \mathbb{Z}(D)$. Then $k(B) \leq l(b)/m \leq l(b)p^d$ where
\[
m := \min\{xC^{-1}x^t : x \in \mathbb{Z}^{l(b)} \setminus \{0\}\}.
\]

Proof. By the definition of $m$, the matrix $W := \frac{1}{m}C^{-1}$ is integral positive definite. Theorem A gives $k(B) \leq \text{ tr}(WC) = l(b)/m$. For the second inequality we recall that the elementary divisors of $C$ divide $p^d$. Hence, $p^dc^{-1}$ has integral entries and $m \geq p^{-d}$. \hfill \Box

In [16], we referred to the Cartan method and the inverse Cartan method when applying Corollary 13 and Corollary 15 respectively. Now we know that both methods are special cases of a single theorem. In fact, the following examples show that Theorem A is stronger than Corollary 13 and Corollary 15.

(i) Let $B$ be the principal 2-block of the affine semilinear group $G = AGL(1,8)$, and let $u = 1$. Then
\[
C = \begin{pmatrix}
2 & . & 1 & 1 \\
. & 2 & 1 & 1 \\
1 & 1 & 1 & 4 & 3 \\
1 & 1 & 1 & 3 & 4
\end{pmatrix}
\]
and $m = \frac{1}{2}$ with the notation of Corollary 15. This implies $k(B) \leq 10$. On the other hand, $q(x_1, \ldots, x_5) = x_1^2 + \ldots + x_5^2 + x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5$ in Corollary 13 gives $k(B) \leq 8$ and in fact equality holds (cf. \cite{10} p. 84)].
(ii) Let $B$ be the principal 2-block of $G = A_4 \times A_4$ where $A_4$ denotes the alternating group of degree 4. Let $u = 1$. Then

$$C = (1 + \delta_{ij})^3_{i,j=1} \otimes (1 + \delta_{ij})^3_{i,j=1}$$

and $m = 9/16$ with the notation of Corollary 15. Hence, we obtain $k(B) \leq 16$ and equality holds. On the other hand, it has been shown in [15, Section 3] that there is no positive definite, integral quadratic form $q$ such that $k(B) \leq 16$ in Corollary 13.

We give a final application where the Cartan matrix $C$ is known up to basic sets. It reveals an interesting symmetry in the formula.

**Proposition 16.** Let $B$ be a block of a finite group with abelian defect group $D$ and inertial quotient $E \leq \text{Aut}(D)$. Suppose that $u \in D$ such that $D/\langle u \rangle$ is cyclic. Then

$$k(B) \leq \left( |N_E(\langle u \rangle)/C_E(u)| + \frac{|\langle u \rangle| - 1}{|N_E(\langle u \rangle)/C_E(u)|} \right) \left( |C_E(u)| + \frac{|D/\langle u \rangle| - 1}{|C_E(u)|} \right) \leq |D|.$$ 

**Proof.** With the notation of Theorem A we have $N = N_E(\langle u \rangle)/C_E(u)$. Moreover, $\overline{b}$ has defect group $D/\langle u \rangle$ and inertial quotient $C_E(u)$. By Dade’s theorem on blocks with cyclic defect groups, $l(b) = |C_E(u)|$ and $C = (m + \delta_{ij})$ up to basic sets where $m := (|D/\langle u \rangle| - 1)/l(b)$ (see [16, Theorem 8.6]). With $W = U_{l(b)}$ we obtain

$$k(B) \leq \left( |N| + \frac{|\langle u \rangle| - 1}{|N|} \right) \text{tr}(WC)$$

$$= \left( |N_E(\langle u \rangle)/C_E(u)| + \frac{|\langle u \rangle| - 1}{|N_E(\langle u \rangle)/C_E(u)|} \right) \left( |C_E(u)| + \frac{|D/\langle u \rangle| - 1}{|C_E(u)|} \right).$$

The first factor is at most $|\langle u \rangle|$ and the second factor is bounded by $|D/\langle u \rangle|$. This implies the second inequality.

Coming back to the initial motivation of this paper, we remark that Theorem A implies Brauer’s Conjecture $k(B) \leq |D|$ in all examples we have considered.

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