ELLiptic equations with transmIssIon and wentzell boundary conditions anD an application to steady water waves in the presence of wind

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Abstract. In this paper, we present results about the existence and uniqueness of solutions of elliptic equations with transmission and Wentzell boundary conditions. We provide Schauder estimates and existence results in Hölder spaces. As an application, we develop an existence theory for small-amplitude two-dimensional traveling waves in an air-water system with surface tension. The water region is assumed to be irrotational and of finite depth, and we permit a general distribution of vorticity in the atmosphere.

1. Introduction

1.1. Elliptic theory. Let $\Omega \subset \mathbb{R}^n$ be a connected bounded $C^{2,\beta}$ domain for $n > 1$ and $\beta \in (0,1)$. Suppose that there exists a $C^{2,\beta}$ hypersurface $\Gamma$ that divides $\Omega$ into two connected regions such that

$$\Omega = \Omega_1 \cup \Gamma \cup \Omega_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \partial \Omega_1 \cap \partial \Omega_2 = \Gamma,$$

and denote by $S := \partial \Omega$. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the normal vector field on the interface $\Gamma$ pointing outward from $\Omega_1$. We define the co-normal derivative operator on $\Gamma$

$$\partial_N := \sum_{i,j=1}^n a^{ij} \nu_i \partial_{x_j},$$

and the tangential differential operator along $\Gamma$

$$D_s := \sum_{t=1}^n w^{st} \partial_{x_t}, \quad 1 \leq s \leq n,$$

where $w := I_n - \nu \otimes \nu$, and $I_n$ is the $n \times n$ identity matrix.

Our main object of study is the following transmission problem with a Wentzell boundary condition

$$\begin{cases}
Lu &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } S, \\
[u] &= 0 \quad \text{on } \Gamma, \\
Bu &= g \quad \text{on } \Gamma,
\end{cases} \quad (1.1)$$
where
\[ Lu := - \sum_{i,j=1}^{n} \partial_{x_i}(a^{ij}(x)\partial_{x_j}u) + \sum_{i=1}^{n} b^i(x)\partial_{x_i}u + c(x)u, \quad (1.2) \]
\[ Bu := - \sum_{s,t=1}^{n} \mathcal{D}_s(a^{st}(x)\mathcal{D}_t u) + \alpha \, [\partial_N u] + \sum_{s=1}^{n} b^s(x)\mathcal{D}_s u + c(x)u, \quad \alpha = \pm 1. \quad (1.3) \]
Here, we are using \([\cdot] := (\cdot)|_{\Omega_1} - (\cdot)|_{\Omega_2}\) to denote the jump operator across \(\Gamma\). We think of \(\alpha = +1\) as favorable and \(\alpha = -1\) as unfavorable. We shall assume uniform ellipticity condition on the operators \(L\) and \(B\); that is, there exist constants \(\lambda, \mu > 0\) such that
\[ \lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n, \quad (1.4) \]
and
\[ \mu|\xi|^2 \leq \sum_{s,t=1}^{n} a^{st}(x)\xi_s\xi_t \quad \text{for all } x \in \Gamma \text{ and } \xi \in \mathbb{R}^n \text{ such that } \xi \cdot \nu(x) = 0. \quad (1.5) \]
The coefficients \(a^{ij}\) and \(a^{st}\) satisfy \(a^{ij} = a^{ji}, a^{st} = a^{ts}\) for all \(i,j,s,t = 1, \ldots, n\). We also assume that \(a^{ij}, b^i, c\) are in \(L^\infty(\Omega)\), and \(a^{st}, b^s, c\) are in \(L^\infty(\Gamma)\).

Note that \(B\) contains second-order tangential derivatives of \(u\). This is characteristic of so-called Wentzell-type boundary conditions, whose study was initiated by Wentzell in [43]. They arise, for example, in stochastic equations [22] or as an asymptotic model for roughness of the boundary or other more complex geometrical effects [9]. They also appear in water waves and continuum mechanics, which is our principal interest here. For instance, the Young–Laplace Law states that at the interface between two immiscible fluids, the pressure experiences a jump proportional to the curvature. In a free boundary problem where the interface is given as the graph of an unknown function, this naturally leads to quasilinear versions of Wentzell-type conditions. More generally, the curvature of a hyperplane is the first variation of its surface area. Thus, these types of conditions are frequently encountered in free boundary problems where the shape of the interface contributes to the energy.

Transmission conditions refers to the jump operator in \(B\). They are commonly found in multiphase problems, where physically their purpose is to enforce continuity of the normal stress across a material interface. Many researchers alternatively call these diffraction problems (see, for example, [26, 5]).

We first have an a priori estimate for classical solutions in Hölder spaces.

**Theorem 1.1** (Schauder estimate). Assume that \(a^{ij} \in C^{1,\beta}(\Omega_1) \cap C^{1,\beta}(\Omega_2)\), \(b^i, c \in C^{0,\beta}(\Omega_1) \cap C^{0,\beta}(\Omega_2)\), and \(a^{st} \in C^{1,\beta}(\Gamma)\), \(b^s, c \in C^{0,\beta}(\Gamma)\); and suppose that
\[ \|a^{ij}\|_{C^{1,\beta}(\Omega_k)}, \|b^i\|_{C^{0,\beta}(\Omega_k)}, \|c\|_{C^{0,\beta}(\Omega_k)}, \|a^{st}\|_{C^{1,\beta}(\Gamma)}, \|b^s\|_{C^{0,\beta}(\Gamma)}, \|c\|_{C^{0,\beta}(\Gamma)} < \Lambda_2 \]
for some constant $\Lambda_2 > 0$, for all $i,j,s,t = 1,\ldots,n$, and $k = 1,2$. Suppose that $u \in C^0(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega}_1) \cap C^{2,\beta}(\overline{\Omega}_2)$ solves equation (1.1) with $\alpha = \pm 1$. Then for any $\Omega' \subset \subset \Omega \setminus (\Gamma \cap S)$, if $f \in C^0,\beta(\overline{\Omega}_1) \cap C^{0,\beta}(\overline{\Omega}_2)$ and $g \in C^{0,\beta}(\Gamma)$, the following estimate holds:

$$
\|u_1\|_{C^{2,\beta}(\Omega')^2} + \|u_2\|_{C^{2,\beta}(\Omega'_2)} \leq C \left( \|u\|_{C^0(\Omega)} + \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)} \right)
$$

for some constant $C = C(n,\beta,\Lambda_2,\lambda,\mu,\Omega_1',\Omega_2') > 0$, where $\Omega'_k := \Omega' \cap \Omega_k$ and $u_k := u|_{\Omega_k}$.

Note that the estimates in the theorem hold on subsets that are positively separated from $\Gamma \cap S$ where the boundary may not be smooth. The next theorems will assume that $\Omega \subset T^{n-1} \times \mathbb{R}$, where $T$ is a torus and $S \cap \Gamma = \emptyset$.

**Theorem 1.2** (Existence and uniqueness of solutions in $C^{2,\beta}$). Suppose the coefficients $a^{ij}, b^i, c, a^{st}, b^s, \mathbf{e}$ exhibit the same regularity as in Theorem 1.1 and assume in addition that $c \geq 0$ and $\epsilon > 0$. Then for all $f \in C^{0,\beta}(\overline{\Omega}_1) \cap C^{2,\beta}(\overline{\Omega}_2)$ and $g \in C^{0,\beta}(\Gamma)$, the problem (1.1) with $\alpha = 1$ has a unique $C^{2,\beta}(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega}_1) \cap C^{2,\beta}(\overline{\Omega}_2)$ solution.

We emphasize that this theorem only holds if we assume $c \geq 0$, $\epsilon > 0$, and $\alpha$ has a favorable sign. However, for a general $c$, $\epsilon$, and $\alpha$, we are still able to assert the Fredholm solvability of the problem. Letting

$$
X := C^{2,\beta}(\overline{\Omega}_1) \cap C^{2,\beta}(\overline{\Omega}_2) \cap \{ u|_S = 0 \} \cap C^{0,\beta}(\overline{\Omega}),
$$

which is a Banach space with respect to the norm

$$
\|u\|_X := \|u\|_{C^{2,\beta}(\Omega_1)} + \|u\|_{C^{2,\beta}(\Omega_2)} + \|u\|_{C^{0,\beta}(\Omega)},
$$

we have the following theorem.

**Theorem 1.3** (Fredholm solvability). Suppose the coefficients $a^{ij}, b^i, c, a^{st}, b^s, \mathbf{e}$ exhibit the same regularity as in Theorem 1.1 and $\alpha = \pm 1$. Then either

- a) the homogeneous problem (1.1) with $f = g = 0$ has nontrivial solutions that form a finite dimensional subspace of $X$, or
- b) the homogeneous problem has only the trivial solution in which case the inhomogeneous problem has a unique solution in $X$ for all $f \in C^{0,\beta}(\overline{\Omega}_1) \cap C^{0,\beta}(\overline{\Omega}_2)$ and $g \in C^{0,\beta}(\Gamma)$.

A great deal of research has been devoted to studying elliptic problems with linear and nonlinear Wentzell boundary conditions, but they remain comparatively less well-understood. One of the earliest works to consider Wentzell conditions was Korman [25] who, like us, was interested in their connection to a problem in water waves. Specifically, he investigated a model describing three-dimensional periodic capillary-gravity waves where the gravity pointed upward. A Schauder theory was later provided by Luo and Trudinger [29] for the linear case. In the quasilinear setting, Luo [28] gave a priori estimates for uniformly elliptic Wentzell conditions, while later Luo and Trudinger [30] studied the degenerate case. More recently, Nazarov and Paletsikikh [31] derived local Hölder estimates
in the spirit of De Giorgi for divergence form elliptic equations with measurable coefficients and a Wentzell condition imposed on a portion of the boundary. See also the survey by Apushkinskaya and Nazarov [3] for a summary of the progress made on the nonlinear problem.

Transmission boundary conditions are of great importance to physics and other applied sciences. They are also of interest from a purely mathematical perspective as they arise naturally in the weak formulation of PDEs with discontinuous coefficients. The study of transmission problems dates back to the 1950s and 1960s. Schechter [37] and Šeftel’ [42] investigated even-order elliptic equations on a smooth and bounded domain with smooth coefficients. Schechter obtained estimates and provided an existence for weak solutions. His strategy involved transforming the transmission problem into a mixed boundary value problem for a system of equations. On the other hand, Šeftel’ found a priori $L^p$-estimates. Oleı́nik [36] also studied transmission problems for second-order elliptic equations with smooth coefficients; approximating equations were used to derive results for weak solutions. One of the most foundational work was done by Ladyzhenskaya and Ural’tseva [26], who considered second-order elliptic equations on a bounded domain and then obtained estimates for weak and classical solutions in Sobolev and Hölder spaces, respectively. In contrast to Schechter’s approach, Ladyzhenskaya and Ural’tseva exploited cleverly chosen test functions to deduce their a priori estimates. More recently, Borsuk [10, 11, 12] has treated linear and quasilinear transmission problems on non-smooth domains.

Apushkinskaya and Nazarov [4] considered Sobolev and Hölder solutions of linear elliptic and parabolic equations for two-phase systems. However, they only examined the problem with a favorable sign $\alpha = +1$ of the transmission term, and did not study Fredholm property. Note that in water wave applications, the sign is typically unfavorable. With that in mind, in this paper we make the effort to also include Schauder estimates and Fredholm solvability for $\alpha = -1$ as well; see also Remarks 2.1, 2.5, and 2.7. Our approach is to view the Wentzell boundary condition as a non-local $(n-1)$-dimensional elliptic equation, treating the jump in the co-normal derivative term as forcing that can be controlled using techniques from the literature on transmission problems.

1.2. Steady wind-driven capillary-gravity water waves. Our second set of results considers an application of the above elliptic theory to a problem in water waves. In particular, we will prove the existence of small amplitude periodic wind-driven capillary-gravity waves in a two-phase air-water system. One of the main novelties here is that we also allow for a general distribution of vorticity in the air region. For simplicity we take the flow in the water to be irrotational. When discussing these results, we adopt notational conventions common in studies of steady water waves which occasionally conflict with our notations in the elliptic theory part.

Let us now formulate the problem more precisely. Fix a Cartesian coordinate system $(X, Y) \in \mathbb{R}^2$ so that the $X$-axis points in the direction of wave propagation and the $Y$-axis is vertical. The ocean bed is assumed to be flat and at the depth $Y = -d$, while the interface between the water and the atmosphere is a free surface given as the graph of $\eta = \eta(X, t)$. We then normalize $\eta$ so that the free surface is oscillating around the line
Air \( \Omega_1(t) \)

\[ \quad Y = \ell \]

\[ \phantom{Y = \ell} \]

Water \( \Omega_2(t) \)

\[ \quad Y = \eta(X,t) \]

\[ \phantom{Y = \eta(X,t)} \]

\[ \quad Y = -d \]

**Figure 1.** The air-water system

\( Y = 0 \). The atmospheric domain is assumed to be bounded in \( Y \); that is, the air region lies below \( Y = \ell \) for some fixed \( \ell > 0 \). At a given time \( t \), the fluid domain is

\[ \Omega(t) = \Omega_1(t) \cup \Omega_2(t), \]

where \( \Omega_1 \) is the air region,

\[ \Omega_1(t) := \{(X,Y) \in \mathbb{R}^2 : \eta(X,t) < Y < \ell \}, \]

and \( \Omega_2 \) is the water region,

\[ \Omega_2(t) := \{(X,Y) \in \mathbb{R}^2 : -d < Y < \eta(X,t) \}. \]

We also denote \( \mathcal{I}(t) := \partial \Omega_1(t) \cap \partial \Omega_2(t) \). Here we think of \( \mathcal{I}(t) \) as playing the role of \( \Gamma \) in the notation of the previous subsection.

Let \( u = u(X,Y,t) \) and \( v = v(X,Y,t) \) be the horizontal and vertical fluid velocities, respectively, and denote by \( P = P(X,Y,t) \) the pressure. We say that this is a *traveling wave* provided that there exists a wave speed \( c > 0 \) such that the change of variables

\[ (X,Y) \mapsto (x,y) := (X - ct, Y) \]

eliminates time dependence. The velocity field is assumed to be incompressible and, in the moving frame, \((u,v,\eta,P)\) are taken to be \(2\pi\)-periodic in \( x \).

For water waves, the governing equations are the incompressible steady Euler system:

\[
\begin{aligned}
  u_x + v_y &= 0, \\
  \varrho(u - c)u_x + \varrho vu_y &= -P_x, \quad \text{in } \Omega \\
  \varrho(u - c)v_x + \varrho vv_y &= -P_y - g \varrho,
\end{aligned}
\]

where \( g > 0 \) is the gravitational constant, and \( \varrho = \varrho_1\mathbb{I}_{\Omega_1} + \varrho_2\mathbb{I}_{\Omega_2} \) with \( \varrho_1 \) and \( \varrho_2 \) assumed to be constant densities of \( \Omega_1 \) and \( \Omega_2 \), respectively. We assume \( \varrho_1 < \varrho_2 \). The symbol \( \mathbb{I}_{\Omega_i} \) stands for the characteristic function on \( \Omega_i \). As above, \([\cdot]\) denotes the jump over \( \mathcal{I} \), that is, \([\cdot]\) = \((\cdot)|_{\Omega_1} - (\cdot)|_{\Omega_2}\).
The kinematic and dynamic boundary conditions for the lidded atmosphere problem with surface tension $\sigma$ are
\[
\begin{align*}
&v = 0 \quad \text{on } y = \ell, \\
&v = 0 \quad \text{on } y = -d, \\
&v = (u - c)\eta_x \quad \text{on } y = \eta(x), \\
\|P\| = -\sigma \frac{\eta_{xx}}{(1 + (\eta_x)^2)^{3/2}} \quad \text{on } y = \eta(x).
\end{align*}
\] (1.9)

Note that the last condition will give rise to nonlinear Wentzell and transmission terms. In particular, the right hand side can be viewed as a second-order elliptic operator acting on $\eta$, while the jump in the pressure will relate to a jump in $(u - c)^2 + v^2$ via Bernoulli’s theorem that we discuss below.

We consider waves without (horizontal) stagnation, that is, we will always assume
\[ u - c < 0 \quad \text{in } \Omega. \] (1.10)
As $(u, v)$ is divergence free according to (1.8), we can define the pseudostream function $\psi = \psi(x, y)$ for the flow by
\[
\begin{align*}
&\psi_x = \sqrt{\varrho} v, \\
&\psi_y = \sqrt{\varrho}(u - c)
\end{align*}
\] in $\Omega$. (1.11)

The level sets of $\psi$ are called streamlines. Without stagnation (1.10), we have $\psi_y < 0$, which implies that each streamline is given as the graph of a function of $x$ via a simple Implicit Function Theorem argument. The boundary conditions in (1.9) show that the air-water interface, bed, and lid are each level sets of $\psi$. We will take $\psi = 0$ on the upper lid so that $\psi = -p_0$ on $y = -d$, where $p_0$ is defined by
\[
p_0 := \int_{-d}^{\eta(x)} \sqrt{\varrho(x, y)}(u(x, y) - c) \, dy.
\]

It can be shown that $p_0$ does not depend on $x$ (see, for example, [46]). Bernoulli’s theorem states that
\[ E := P + \frac{\varrho}{2}(u - c)^2 + v^2 + g\varrho y \]
is constant along streamlines. Evaluating the jump of $E$ on the interface gives
\[
\left[ [\nabla \psi] \right] + 2g \left[ \varrho \right] (\eta + d) + \sigma \kappa = Q \quad \text{on } y = \eta(x),
\]
where $\kappa$ is the signed curvature of the air-water interface and $Q := 2 \left[ E + g\varrho d \right]$.

Recall that in two dimensions, the vorticity $\omega$ is defined to be
\[ \omega := v_x - u_y. \]

If there is no stagnation (1.10), there exists a function $\gamma$, called the vorticity strength function, such that
\[ \omega(x, y) = \gamma(\psi(x, y)) \quad \text{for all } (x, y) \in \Omega. \]
The vorticity plays a key role in the wind generation of water waves as we will discuss below. Mathematically, it substantially complicates the analysis.
Finally, we will use the following notational conventions. For any integer $k \geq 0$, $\alpha \in (0, 1)$, and an open region $R \subset \mathbb{R}^n$, we define the space $C^{k+\alpha}_{\text{per}}(R)$ to be the set of $C^{k+\alpha}(R)$ functions that are $2\pi$-periodic in their first argument.

Our main theorem is an existence result for traveling capillary-gravity water waves in the presence of wind.

**Theorem 1.4** (Existence of small amplitude wind-driven water waves). Fix $d, \ell, c > 0$, and $p_0 < p_1 < 0$. For any vorticity function $\gamma \in C^{0,\alpha}([p_1,0])$ and $\sigma > 0$ sufficiently large, there exists a $C^1$ curve

$$C'_{\text{loc}} := \{(u(s), v(s), \eta(s), Q(s)) : s \in (-\epsilon, \epsilon)\}$$

of traveling wave solutions to the capillary-gravity water wave problem (1.8) – (1.10) such that

- a) Each $(u, v, \eta, Q) \in C'_{\text{loc}}$ is of class

$$(u(s), v(s), \eta(s), Q(s)) \in \left(C^{\alpha}_{\text{per}}(\Omega) \cap C^{1+\alpha}_{\text{per}}(\Omega(s) \setminus \mathcal{I}(s))\right)^2 \times C^{2+\alpha}_{\text{per}}(\mathbb{R}) \times \mathbb{R} : = \mathscr{A},$$

where $u(s)$ and $v(s)$ are even and odd in the first coordinate, respectively, $\eta(s)$ is even in $x$, and $\Omega(s)$ is the domain corresponding to $\eta(s)$;

- b) $(u(0), v(0), \eta(0), Q(0)) = (U_*(y), 0, 0, Q_*)$, where $(U_*, Q_*)$ is laminar solution.

We prove this theorem using a local bifurcation theoretic strategy that draws on the ideas of Constantin and Strauss [17], who studied rotational periodic gravity water waves in a single fluid. Indeed, following the publication of [17], traveling water waves with vorticity have been an extremely active area of research (see, for example, the surveys in [38, 16]).

Our most direct influence is the work of Bühler, Shatah, and Walsh [13] on the existence of steady gravity waves in the presence of wind. These authors studied exactly the system (1.8) – (1.10) taking $\sigma = 0$. One of the main objectives of that paper was to construct waves that were dynamically accessible from an initial state where the flow is laminar and the horizontal velocity experiences a jump over the interface. More specifically, this meant that the circulation along each streamline was prescribed in order to ensure that its values in the air and water regions were distinct (see Remark 3.3). We also adopt this approach in the present paper, though the addition of surface tension necessitate many nontrivial adaptations.

1.3. History of the problem. Steady capillary and capillary-gravity waves have been the subject of extensive research. Because we are particularly interested in the role of vorticity, we will restrict our discussion to rotational waves. In this setting, progress is much more recent and begins with the work of Wahlén [44, 45], who proved the existence of small-amplitude periodic capillary and capillary-gravity waves in two-dimensions for a single fluid system. As in [17], this was done for a general vorticity function $\gamma$. Contrary to the gravity wave case, Wahlén showed that with surface tension there can be double bifurcation points; this is a rotational analogue of the famous Wilton ripples [49]. Later, Walsh considered two-dimensional periodic capillary-gravity waves with density stratification [47, 48].
Recently, Martin and B-V Matioc proved the existence of steady small-amplitude capillary-
gravity water waves with piecewise constant vorticity [31]. While they consider a one-layer
model, the analysis has a similar flavor to that in the present paper. A-V Matioc and B-V
Matioc also constructed weak solutions for steady capillary-gravity water waves in a single
fluid [32].

The waves we construct can also be viewed as internal waves moving along the interface
between two immiscible fluid layers confined in a channel. Versions of this problem have
been investigated by many authors. For instance, Amick–Turner [1] and Sun [39, 40] con-
sidered the existence of solitary waves in a channel where the flow is irrotational at infinity.
Amick–Turner built their solitary waves as limits of periodic waves with the period tending
to infinity. Sun, on the other hand, exploited the fact that the leading-order form of the
wave is given by the Benjamin–Ono equation, and then used singular integral operator
estimates to control the remainder. The existence of continuously stratified channel flows
has also been verified in a number of regimes. Note that these are rotational, since hetero-
genesis in the density produces vorticity. Specifically, Turner [41] and Kirchgässner [24]
investigated small-amplitude continuously stratified waves using a variational scheme and
a center manifold reduction method, respectively. A large-amplitude existence theory was
also provided by Bona, Bose, and Turner [8], Lankers and Friesecke [27], and Amick [2].

As mentioned above, steady water waves in the presence of wind was studied by Bühler,
Shatah, and Walsh in [13]. Our main contribution relative to that work is to account for
capillary effects on the air-water interface. It is known that surface tension is important in
the formation of wind-driven waves. Indeed, high frequency and small-amplitude capillary-
gravity waves are the first to form when wind blows over a quiescent body of water.

One of the most successful explanations for the mechanism behind the wind generation
of water waves was given by Miles [33]. His main observation was that vorticity in the air
region can create a certain resonance phenomenon that destabilizes the system. Impor-
tantly, this so-called critical layer instability can occur even when the horizontal velocity
is continuous — or nearly continuous — over the interface, and therefore does not re-
quire exceedingly strong wind speeds like the Kelvin–Helmholtz model. The mathematical
ideas underlying Miles’s theory were recently reexamined and rigorously proved by Bühler,
Shatah, Walsh, and Zeng [14]. In that work, the authors also allowed surface tension.

This is somewhat important as the interface Euler problem itself is ill-posed when there
is a jump in the tangential velocity and there is no surface tension (see, for example, [6]).
In a forthcoming work, the author intends to study the stability of the family of waves
constructed in Theorem 1. This will serve as a model for wind generation of water waves
in the spirit of Miles, but with an initial state that is not purely laminar.
1.4. **Plan of the article.** We now briefly discuss the strategies we use to derive these results. The elliptic theory is proved in Section 2. Our approach is based on the work of Luo and Trudinger [29], who gave Schauder estimates for elliptic equations with Wentzell boundary conditions.

In Section 3, we construct capillary-gravity water waves where the air region is rotational. Following Bühler, Shatah, and Walsh [13], the first step in this procedure is to reformulate the interface Euler system as a quasilinear elliptic equation on a fixed domain. Due to surface tension, there is now a nonlinear Wentzell condition on the image of the interface in these new coordinates. We construct the non-laminar waves using local bifurcation theory. This entails studying the spectrum of the linearized equation at a laminar flow, and here we make essential use of the elliptic theory developed in Section 2. One major difficulty that arises is that this linearized problem is of Sturm–Liouville type, but associated to an indefinite inner product. Consequently, to successfully determine the spectral behavior, we must work in Pontryagin spaces. A similar issue was encountered by Wahlén in [44, 45]. Finally, we apply the Crandall–Rabinowitz local bifurcation theorem to obtain Theorem 1.4.

2. **Elliptic Theory**

To simplify subsequent calculations, it is convenient to first change variables. Fix a point \( x^0 \in \Gamma \). Then by the assumption on \( \Omega \), there is a neighborhood \( U \) of \( x^0 \) and a \( C^{2,\beta} \) diffeomorphism that maps \( U \) to some ball \( B \subset \mathbb{R}^n \) so that \( \Gamma \) maps to \( \{ x_n = 0 \} \), \( \Omega_1 \) to \( B \cap \{ x_n > 0 \} \), and \( \Omega_2 \) to \( B \cap \{ x_n < 0 \} \) (see, for example, [29]). Then it suffices to assume that \( \Gamma \) is the hyper-plane \( \{ x_n = 0 \} \), and consequently, \( \Omega_1 \) and \( \Omega_2 \) lie inside the upper-half and lower-half planes respectively.

In this case, the co-normal derivative operator simplifies to

\[
\partial_N u = - \sum_{j=1}^n a^{nj} \partial_{x_j} u,
\]

and the Wentzell and transmission condition on \( \Gamma \) becomes

\[
Bu = - \sum_{s,t=1}^{n-1} \partial_{x_s}(a^{st} \partial_{x_t} u) + \alpha [\partial_N u] + \sum_{s=1}^{n-1} b^s \partial_{x_s} u + cu.
\]

We also denote by \( \nabla' \) the tangential gradient on \( \Gamma \) in this case.

2.1. **Classical solutions.** First, we prove our theorem on Schauder estimates for solutions in Hölder spaces. This relies on the observation that one can apply \((n - 1)\)-dimensional elliptic estimates for \( B \) on \( \Gamma \) with transmission boundary condition being lower ordered.

**Proof of Theorem 1.1.** Using the above change of variables, we rewrite the condition on \( \Gamma \)

\[
- \sum_{s,t=1}^{n-1} a^{st} \partial_{x_s} \partial_{x_t} u - \sum_{s,t=1}^{n-1} (\partial_{x_s} a^{st}) (\partial_{x_t} u) - \alpha \sum_{j=1}^n a^{nj} \partial_{x_j} u_1 + \alpha \sum_{j=1}^n a^{nj} \partial_{x_j} u_2 + \sum_{s=1}^{n-1} b^s \partial_{x_s} u + cu = g.
\]
We then cover $\Gamma$ by a finite number of spheres in which the estimate in [21, Theorem 6.2] for $B$ on $\Gamma$ can be applied. This ensures the existence of a positive constant $C = C(n, \beta, L, \mu)$ such that

$$\|u_1\|_{C^{2, \beta}(\Gamma')} \leq C(\|u_1\|_{C^0(\Gamma)} + \|u_2\|_{C^{1, \beta}(\Gamma')} + \|g\|_{C^0, \beta}(\Gamma)).$$

(2.1)

Similarly, we have

$$\|u_2\|_{C^{2, \beta}(\Gamma')} \leq C(\|u_2\|_{C^0(\Gamma)} + \|u_1\|_{C^{1, \beta}(\Gamma')} + \|g\|_{C^0, \beta}(\Gamma)).$$

(2.2)

Next, we use a basic elliptic estimate for the Dirichlet problem in $\Omega_k'$ with boundary condition $u_k|_{\Gamma}$ (see, for example, [21, Theorem 6.6]), to obtain

$$\|u_k\|_{C^{2, \beta}(\Omega_k')} \leq C \left( \|u_k\|_{C^0(\Omega_k)} + \|u_k\|_{C^{2, \beta}(\Gamma')} + \|f\|_{C^0, \beta}(\Omega_k') \right).$$

(2.3)

Moreover, we have the following interpolation

$$\|u_k\|_{C^{1, \beta}(\Gamma')} \leq C_\epsilon \|u_k\|_{C^0(\Gamma)} + \epsilon \|u_k\|_{C^{2, \beta}(\Gamma')}$$

(2.4)

for some $\epsilon > 0$. Finally, evaluating (2.3) with $k = 1, 2$ and summing, using $\|u\|_{\Gamma} = 0$ and the estimates (2.1), (2.2), (2.4) and choosing appropriate $\epsilon$ give

$$\|u_1\|_{C^{2, \beta}(\Omega_1')} + \|u_2\|_{C^{2, \beta}(\Omega_2')} \leq C \left( \|u\|_{C^0(\Omega)} + \|f\|_{C^0, \beta}(\Omega_1) + \|f\|_{C^0, \beta}(\Omega_2) + \|g\|_{C^0, \beta}(\Gamma') \right). \quad \Box$$

Remark 2.1. A version of Theorem 1.1 was stated in [3] Theorem 2.3* without proof under the assumption that $\alpha = +1$ in the boundary operator (1.13). However, according to the above proof, this theorem holds regardless of the sign of $[\partial_N u]$.

Next, in preparation for proving the existence and uniqueness result, we first establish a maximum principle. Apushkinskaya and Nazarov state a similar result in [3] Theorem 3.1. Using our notations, we have the lemma.

Lemma 2.2 (Maximum Principle). Suppose the coefficients $a^{ij}, b^i, c, a^{st}, b^s, c$ exhibit the same regularity as in Theorem 1.1 and assume in addition that $c \geq 0$ and $c > 0$. Let $\alpha = 1$ and suppose that $u \in C^0(\overline{\Omega}) \cap C^2(\Omega_1) \cap C^2(\Omega_2)$ satisfies

$$Lu \leq f \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ S, \quad Bu \leq g \quad \text{on} \ \Gamma.$$

Then we have the estimate

$$\sup_{\Omega} u \leq \sup_{\Gamma} \left| \frac{g}{c} \right| + C \sup_{\overline{\Omega}} \left| \frac{f}{\lambda} \right|$$

(2.5)

for some positive constant $C = C(\text{diam} \ \Omega, \lambda, \|\partial_x a^{ij\ell}\|_{L^\infty}, \|b^{i\ell}\|_{L^\infty})$, where the $L^\infty$ norms are taken over $\Omega_1$ and $\Omega_2$.

Proof. We will follow very closely the classical arguments when proving this maximum principle in the interior. Rewriting $L$ in non-divergence form gives

$$Lu = - \sum_{i,j=1}^n a^{ij} \partial_{x_i} \partial_{x_j} u + \sum_{i=1}^n b^i \partial_{x_i} u + cu,$$
where \( \tilde{b} := b^i - \partial_x^i a^{ij} \). Setting
\[
\tau := \frac{\|\tilde{b}^i\|_{L^\infty(\Omega_1)} + \|\tilde{b}^i\|_{L^\infty(\Omega_2)}}{\lambda},
\]
choosing \( \sigma \geq 1 \) large enough so that \( \sigma^2 - \tau \sigma \geq 1 \), and without loss of generality, because of the boundedness of \( \Omega \), assuming \( \Omega \) lies between \( \{x_1 = 0\} \) and \( \{x_1 = d\} \), let
\[
v := \sup_{\partial\Omega_k} u^+ + \left( e^{\sigma d} - e^{\sigma x_1} \right) \sup_{\Omega_k} \frac{f^+}{\lambda},
\]
where \( u^+ := \max(u, 0) \) and \( d := \text{diam} \Omega \). Then
\[
Le^{\sigma x_1} = (-a^{11} \sigma^2 + \tilde{b}^1 \sigma + c)e^{\sigma x_1} \leq -\lambda(\sigma^2 - \tau \sigma)e^{\sigma x_1} + ce^{\sigma x_1} \leq -\lambda + ce^{\sigma x_1}.
\]
Then since \( c \geq 0 \),
\[
Lv \geq c \sup_{\partial\Omega_k} u^+ + ce^{\sigma d} \sup_{\Omega_k} \frac{f^+}{\lambda} - (-\lambda + ce^{\sigma x_1}) \sup_{\Omega_k} \frac{f^+}{\lambda} \geq c(e^{\sigma d} - e^{\sigma x_1}) \sup_{\Omega_k} \frac{f^+}{\lambda} + \sup_{\Omega_k} f^+ \geq \sup_{\Omega_k} f^+,
\]
so we have \( L(u - v) \leq 0 \) in \( \Omega_k \). On the other hand, by construction \( u - v \leq 0 \) on \( \partial\Omega_k \).
Therefore, the maximum principle implies \( u \leq v \) in \( \Omega_k \), and that there exists a positive constant \( C = C(d, \lambda, \|\partial_x^i a^{ij}\|_{L^\infty}, \|b^i\|_{L^\infty}) \) such that
\[
\sup_{\Omega_k} u \leq \sup_{\partial\Omega_k} u^+ + C \sup_{\Omega_k} \frac{f^+}{\lambda} \text{ for } k = 1, 2.
\]
Next, since \( u|_S = 0 \), if \( u|_\Gamma \leq 0 \) for all \( x \in \Gamma \), then
\[
\sup_{\partial\Omega_k} u^+ = 0 \leq \sup_{\Gamma} \left| \frac{g}{\epsilon} \right|,
\]
If we suppose that \( u \) attains its local maximum at some point \( x_0 \in \Gamma \) and \( u(x_0) > 0 \), then by the positive definiteness of the matrix \( (a^{st}) \),
\[
\nabla' u(x_0) = 0 \quad \text{and} \quad \sum_{s,t=1}^{n-1} (a^{st} \partial_{x_s} \partial_{x_t} u)(x_0) \leq 0.
\]
By the positive-definiteness of the matrix \( (a^{ij}) \), we have
\[
\partial_N u_1(x_0) = -a^{nn} \partial_{x_n} u_1(x_0) \geq 0
\]
\[
\partial_N u_2(x_0) = -(a^{nn} \partial_{x_n} u_2)(x_0) \leq 0,
\]
and hence \( \|\partial_N u(x_0)\| \geq 0 \). Then the condition on \( \Gamma \) gives
\[
(cu)(x_0) \leq g(x_0) + \sum_{s,t=1}^{n-1} (a^{st} \partial_{x_s} \partial_{x_t} u)(x_0) - \|\partial_N u(x_0)\| \leq g(x_0),
\]
so since \( c > 0 \) for all \( x \in \Gamma \), we obtain
\[
\sup_{\Gamma} u = u(x_0) \leq \frac{g(x_0)}{c(x_0)} \leq \sup_{\Gamma} \frac{g}{c}.
\]
Therefore,
\[
\sup_{\partial \Omega_k} u^+ = \sup_{\Gamma} u \leq \sup_{\Gamma} \frac{g}{c},
\]
and hence we obtain the desired estimate (2.5) by using \( \sup_{\Omega} u = \max(\sup_{\Omega_1} u, \sup_{\Omega_2} u) \). \( \square \)

**Remark 2.3.** Note that if \( \Omega \) is periodic in one variable, the lemma still holds by modifying the proof to assume that \( \Omega \) lies between two hyperplanes parallel to the periodic direction.

Using the notation of a Hölder seminorm, we have the following simple lemma whose proof will be omitted:

**Lemma 2.4.** Suppose \( u \in C^0(\Omega) \cap C^{0,\beta}(\Omega_1) \cap C^{0,\beta}(\Omega_2) \). Then \( [u]_{0,\beta,\Omega} \) is finite, and
\[
\|u\|_{C^{0,\beta}(\Omega)} \leq C \left( \|u_1\|_{C^{0,\beta}(\Omega_1)} + \|u_2\|_{C^{0,\beta}(\Omega_2)} \right).
\]

Now we can derive the existence and uniqueness of solution in Hölder spaces.

**Proof of Theorem 1.2.** Consider the family of problems indexed by \( \theta \in [0, 1] \):
\[
\begin{cases}
Lu = f & \text{in } \Omega, \\
u = 0 & \text{on } S, \\
[u] = 0 & \text{on } \Gamma, \\
B_\theta u = g & \text{on } \Gamma,
\end{cases}
\tag{2.6}
\]
where \( B_\theta u = \theta Bu + (1 - \theta)B' u \) and
\[
B' u := -\sum_{s=1}^{n-1} \partial_{x_s}^2 u + u.
\]
We note that \( B_1 = B, B_0 = B' \), and that
\[
B_\theta u = -\sum_{s,t=1}^{n-1} \tilde{a}^{st} \partial_{x_s} \partial_{x_t} u + \sum_{s=1}^{n-1} \tilde{b}^s \partial_{x_s} u + \theta [\partial_{\mathcal{N}} u] + \tilde{c} u,
\]
where all of the coefficients \( \tilde{a}^{st}, \tilde{b}^s, \tilde{c} \) of \( B_\theta \) are bounded in \( C^{0,\beta}(\Gamma) \) independently of \( \theta \) with \( \tilde{c} > 0 \) and
\[
\min(1, \mu)|\xi|^2 =: \mu_0|\xi|^2 \leq \tilde{a}^{st} \xi_s \xi_t \quad \text{for all } x \in \Gamma, \xi \in \mathbb{R}^{n-1}.
\]
Consider any solution \( u \in C^0(\Omega) \cap C^{2,\beta}(\Omega_1) \cap C^{2,\beta}(\Omega_2) \) of (2.6). Then by estimates (1.6) and (2.5), the following inequality holds
\[
\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)} \leq C \left( \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)} \right), \tag{2.7}
\]
where the constant \( C \) is independent of \( \theta \). Note that the above estimate is valid for \( \Omega_k \) with \( k = 1, 2 \) since \( S \cap \Gamma = \emptyset \).
Next, recalling the definition of $X$ as in (1.7), let $Y = Y_1 \times Y_2$ where
\[ Y_1 = C^{0,\beta}(\Omega_1) \cap C^{0,\beta}(\Omega_2), \quad \text{and} \quad Y_2 = C^{0,\beta}(\Gamma). \]
Then $Y$ is a Banach space with respect to the norm
\[ \|(f,g)\|_Y := \|f\|_{Y_1} + \|g\|_{Y_2} := \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)}. \]
Thus, problem (2.6) can be written as
\[ L\theta u := (Lu, B\theta u) = (f,g), \]
where $L\theta : X \to Y$, so the solvability of the problem (2.6) for arbitrary $f \in C^{0,\beta}(\Omega_1) \cap C^{0,\beta}(\Omega_2)$ and $g \in C^{0,\beta}(\Gamma)$ is then equivalent to the invertibility of the mapping $L\theta$. We note that $L_0$ and $L_1$ are bounded operators.

On the other hand, by Lemma 2.4, Lemma 2.2, and estimate (2.7), we have
\[ \|u\|_{C^{0,\beta}(\Omega)} \leq C \left( \|u\|_{C^{0,\beta}(\Omega_1)} + \|u\|_{C^{0,\beta}(\Omega_2)} \right) \]
for some $\epsilon > 0$, and hence
\[ \|u\|_X = \|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)} + \|u\|_{C^{0,\beta}(\Omega)} \]
\[ \leq C \left( \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)} \right) \]
\[ = C (\|f\|_{Y_1} + \|g\|_{Y_2}) = C\|L\theta u\|_Y, \]
where the constant $C$ does not depend on $\theta$. Thus, by the method of continuity (see, for example, [21, Theorem 5.2]), the surjectivity of $L_1$, which we are investigating, is equivalent to that of $L_0$ which is the problem
\[
\begin{align*}
L u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } S, \\
[Bu] &= 0 \quad \text{on } \Gamma, \\
B' &u = g \quad \text{on } \Gamma.
\end{align*}
\]
Finally, we recall that
\[ B' u = -\sum_{s=1}^{n-1} \partial_{x_s}^2 u + u \]
is invertible on $\Gamma$. If $\varphi \in C^{2,\beta}(\Gamma)$ is the unique solution to $B' \varphi = g$ on $\Gamma$ for a given $g \in C^{0,\beta}(\Gamma)$, then by [21, Lemma 6.38] we can make an extension to have $\varphi \in C^{2,\beta}(\Omega_1) \cap C^{2,\beta}(\Omega_2)$. Now we have a Dirichlet problem
\[ Lu_k = f \quad \text{in } \Omega_k, \quad u_k = 0 \quad \text{on } S, \quad u_k = \varphi \quad \text{on } \Gamma, \]
which has a unique solution \( u_k \in C^{2,\beta}(\Omega_k) \) by [21, Theorem 6.14]. Therefore, by Lemma 2.3, we conclude that there is a unique solution in \( C^{0,\beta}(\Omega) \cap C^{2,\beta}(\Omega_1) \cap C^{2,\beta}(\Omega_2) \) to the system (1.1). □

Remark 2.5. As a consequence of Theorem 1.2, we see that \( L_1 = (L, B) \) is a Fredholm operator of index 0 despite the sign of the transmission term. Indeed, for \( \theta \in [0,1] \), consider the following linear operator

\[
\tilde{L}_\theta := (Lu, (1 - \theta)B + \theta \tilde{B}),
\]

where \( \tilde{L}_\theta : X \rightarrow Y \) and

\[
\tilde{B}u = - \sum_{s,t=1}^{n-1} \partial_{x_s} (a^{st} \partial_{x_s} u) - \|\partial_N u\| + \sum_{s=1}^{n-1} b^s \partial_{x_s} u + cu
\]

with coefficients \( a^{st}, b^s, \) and \( c \) satisfying the hypotheses of Theorem 1.2. Note that the sign of the transmission term is unfavorable. It is clear that the map \( \theta \mapsto \tilde{L}_\theta \in L(X,Y) \) is continuous. Then Schauder estimate from Theorem 1.1 and Remark 2.1 give

\[
\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)} + \|u\|_{C^{0,\beta}(\Omega)} \leq C\|\tilde{L}_\theta u\|_Y + C_{\epsilon} \|u\|_{C^{0,\beta}(\Omega)} + \epsilon \left(\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)}\right)
\]

for some small \( \epsilon > 0 \), so

\[
(1 - \epsilon) \left(\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)}\right) + \|u\|_{C^{0,\beta}(\Omega)} \leq C_{\epsilon} \|u\|_{C^{0,\beta}(\Omega)} + C\|\tilde{L}_\theta u\|_Y.
\]

Choosing \( \epsilon > 0 \) small, we have

\[
\|u\|_X \leq C \left(\|u\|_{C^{0,\beta}(\Omega)} + \|\tilde{L}_\theta u\|_Y\right)
\]

for some constant \( C > 0 \) independent of \( \theta \), which implies that \( \tilde{L}_\theta \) has finite dimensional null space and closed range. Thus, \( \tilde{L}_\theta \) is semi-Fredholm. If \( \theta < \frac{1}{2} \), the map \( \tilde{L}_\theta \) is invertible by Theorem 1.2 and hence has index 0. By the continuity of the index, it also holds for \( \theta \geq \frac{1}{2} \), which means that we have Fredholm index 0 regardless of the sign of the transmission term.

2.2. Fredholm property. In light of Remark 2.5, it suffices to take \( \alpha = +1 \). To simplify our notation, we write \( L \) for \( L_1 \), which is the problem we are considering. If \( L \) and \( B \) do not satisfy the conditions \( c \geq 0 \) and \( c > 0 \), it is still possible to assert a Fredholm alternative, which we formulate as in Theorem 1.3.

Proof of Theorem 1.3. For all \( \sigma, \tau \in \mathbb{R} \), notice that for \( u \in X \), \((f, g) \in Y\),

\[
\mathcal{L}u = (f, g)
\]

is equivalent to

\[
\mathcal{L}_{\sigma, \tau}u = (f + \sigma u, g + \tau u),
\]
where $\mathcal{L}_{\sigma,\tau}u := ((L + \sigma)u, (B + \tau)u)$. From Theorem 1.2, the mapping $\mathcal{L}_{\sigma,\tau} : X \rightarrow Y$ is invertible for $\sigma$ and $\tau$ sufficiently large. Now, applying $\mathcal{L}_{\sigma,\tau}^{-1}$ to both sides, we obtain
\[ u = \mathcal{L}_{\sigma,\tau}^{-1}(f + \sigma u, g + \tau u|_\Gamma) \]
which can be written as
\[ u - \mathcal{L}_{\sigma,\tau}^{-1}(\sigma u, \tau u|_\Gamma) = \mathcal{L}_{\sigma,\tau}^{-1}(f, g). \]
Letting $Ku : u \in X \subset Y \mapsto \mathcal{L}_{\sigma,\tau}^{-1}(\sigma u, \tau u|_\Gamma) \in Y_1$, and $h := \mathcal{L}_{\sigma,\tau}^{-1}(f, g)$, the equation becomes
\[ (I - K)u = h. \] (2.9)
We claim that $K$ is a compact operator. Let $\{(f_m, g_m)\} \subset Y$ be bounded, and define $u_m := K(f_m, g_m) \in Y_1$. We want to show that $\{u_m\}$ has a convergent subsequence in $Y_1$. By definition of $u_m$ and $K$, we have
\[ \begin{cases} Lu_m + \sigma u_m = f_m & \text{in } \Omega \\ Bu_m + \tau u_m = g_m & \text{on } \Gamma, \end{cases} \]
where $u_m \in X$, $f_m \in Y_1$, $g_m \in Y_2$. Thus, by Theorem 1.1, there exists a positive constant $C = C(n, \beta, L, B, \lambda, \mu)$ such that
\[ \|u_m\|_{C^{2,\beta}(\Omega_1)} + \|u_m\|_{C^{2,\beta}(\Omega_2)} \leq C(\|u_m\|_{C^0(\Omega)} + \|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}). \] (2.10)
Note that the estimate holds for $\Omega_k$, since $S \cap \Gamma = \emptyset$. Since $C^{0,\beta}(\Omega_k) \subset C^0(\Omega_k)$ and $C^{2,\beta}(\Omega_k) \subset C^{0,\beta}(\Omega_k)$, $k = 1, 2$, using estimates as in the proof of Theorem 1.2, we find that
\[ \|u_m\|_{C^0(\Omega)} \leq C\|u_m\|_{C^{0,\beta}(\Omega)} \leq C(\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}). \]
Then the inequality (2.10) becomes
\[ \|u_m\|_{C^{2,\beta}(\Omega_1)} + \|u_m\|_{C^{2,\beta}(\Omega_2)} \leq C(\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}), \]
or we can write this to be
\[ \|u_m\|_{C^{2,\beta}(\Omega_1)} + \|u_m\|_{C^{2,\beta}(\Omega_2)} + \|u_m\|_{C^{0,\beta}(\Omega)} \leq C(\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}), \]
which is equivalent to
\[ \|u_m\|_X \leq C\|(f_m, g_m)\|_Y, \]
so $\|u_m\|_X$ is bounded in $X$. Since $X \subset Y_1$, we conclude that $\{u_m\}$ contains a subsequence $\{u_{m_k}\}$ such that $u_{m_k} \rightarrow u$ in $Y_1$, which proves the claim that $K$ is a compact operator. Applying the Fredholm Alternative, equation (2.9) always has a solution $u \in X$ provided the homogeneous equation $(I - K)u = 0$ has only the trivial solution $u = 0$. When this condition is not satisfied, the kernel of $I - K$ is a finite dimensional subspace of $Y_1$. Since the solutions of (2.9) are in one-to-one correspondence to the solutions of (1.1), we therefore can conclude the alternative stated in the theorem. \[ \square \]

Finally, the last result in this subsection gives Hölder continuity for a classical solution provided sufficient smoothness of the data and coefficients.
Proposition 2.6. Suppose the coefficients $a^{ij}, b^i, c, a^{st}, b^s, c$ exhibit the same regularity as in Theorem 1.1. If $u \in C^0(\Omega) \cap C^2(\Omega_1) \cap C^2(\Omega_2)$ is a solution to equation (1.1) with $\alpha = \pm 1$ for $f \in C^{0,\beta}(\Omega_1) \cap C^{0,\beta}(\Omega_2)$ and $g \in C^{0,\beta}(\Gamma)$, then $u \in C^{0,\beta}(\Omega) \cap C^{2,\beta}(\Omega_1) \cap C^{2,\beta}(\Omega_2)$.

Proof. By the hypothesis, we have $u_k \in C^2(\Gamma)$, and hence, $\partial_N u_k \in C^1(\Gamma) \subset C^{0,\beta}(\Gamma)$ for $k = 1, 2$. Thus, the boundary condition $Bu = g$ can be re-expressed as

$$- \sum_{s,t=1}^{n-1} \partial_{x_s} (a^{st} \partial_{x_t} u) + \sum_{s=1}^{n-1} b^s \partial_{x_s} u + cu = h \quad \text{on} \, \Gamma,$$

where $h := g - \alpha \, [\partial_N u] \in C^{0,\beta}(\Gamma)$. By standard elliptic regularity theory, $u_{|\Gamma} \in C^{2,\beta}(\Gamma)$.

Now the Dirichlet problem

$$Lu_k = f \quad \text{in} \, \Omega_k, \quad u_k = 0 \quad \text{on} \, S, \quad u_k = u_{|\Gamma} \quad \text{on} \, \Gamma$$

has a unique solution $u \in C^{2,\beta}(\Omega_1) \cap C^{2,\beta}(\Omega_2)$. Using the fact that $C^{2,\beta}(\Omega_k) \subset C^{0,\beta}(\Omega_k)$ and Lemma 2.4, we conclude that $u \in C^{2,\beta}(\Omega_1) \cap C^{2,\beta}(\Omega_2) \cap C^{0,\beta}(\Omega)$. \hfill $\Box$

Remark 2.7. If we change the boundary term $B$ to

$$\hat{B}u := \sum_{s,t=1}^{n-1} \partial_{x_s} (a^{st} \partial_{x_t} u) + \alpha \, [\partial_N u] + \sum_{s=1}^{n-1} b^s \partial_{x_s} u + cu,$$

where the signs of the second-order term is switched, we obtain the same results as in Theorems 1.1, 1.3 and Proposition 2.6. For Lemma 2.2 and Theorem 1.2 to be valid, we have to assume in addition that $\alpha = -1$ and $c < 0$, which means the signs of the second-order term and zeroth-order term must be opposite.

3. Steady capillary-gravity waves in the presence of wind

In this section, we will apply the results found above to investigate the existence of steady wind-driven water waves. There exists a well-known change of variables due to Dubreil-Jacotin that maps $\Omega$ to a strip (see [20]). We change variables $(x, y) \in \Omega \mapsto (x, -\psi) =: (g, p) \in D$. We recall that $\psi$ is the (relative) pseudostream function for the flow defined by (1.11), along with the boundary conditions $\psi = 0$ on the upper lid, $\psi = -p_0$ at the bed, and $\psi \in C^{0,\alpha}(\Omega) \cap C^{2,\alpha}(\Omega_1) \cap C^{2,\alpha}(\Omega_2)$ for a fixed $\alpha \in (0, 1)$. Thus, the problem is now posed in a union of rectangles $D = D_1 \cup D_2 \subset \mathbb{R}^2$, where the air region is mapped to

$$D_1 := \{(q, p) \in D : 0 < q < 2\pi, p_1 < p < 0\},$$

and the water region is mapped to

$$D_2 := \{(q, p) \in D : 0 < q < 2\pi, p_0 < p < p_1\}.$$
Under this change of coordinates, the Euler problem \((1.8)-(1.10)\) becomes the following height equation

\[
\begin{cases}
(1 + h_q^2)h_{pp} + h_{qq} h_p^2 - 2h_p h_q h_{pq} = -\gamma(-p)h_p^3 & \text{in } D_1, \\
(1 + h_q^2)h_{pp} + h_{qq} h_p^2 - 2h_p h_q h_{pq} = 0 & \text{in } D_2, \\
\left[\frac{1 + h_q^2}{h_p^2}\right] + 2g[\rho] h - Q + \sigma \frac{h_{qq}}{(1 + h_q^2)^{3/2}} = 0 & \text{on } p = p_1, \\
[h] = 0 & \text{on } p = p_1, \\
h = 0 & \text{on } p = p_0, \\
h = \ell + d(h) & \text{on } p = 0,
\end{cases}
\tag{3.1}
\]

where \(h(q, p)\) is the height above the bed of the point \((x, y)\), where \(x = q\) and \((x, y)\) lies on \(\{-\psi = p\}\), and the depth operator \(d\) is defined to be

\[
d(h) := \frac{1}{2\pi} \int_{-\pi}^{\pi} h(q, p_1) \, dq.
\]

Note that \(\rho\) in the above equation is for \((q, p)\)-coordinates after the transformation. The equivalence of \((3.1)\) to the original system \((1.8)-(1.10)\) can be proved following \([15\, \text{Lemma } A.2]\).

Our objective is to find solutions \((h, Q) \in \mathcal{S}'\), where

\[
\mathcal{S}' := (C^{2,\alpha}_{\text{per}}(D_1) \cap C^{2,\alpha}_{\text{per}}(D_2) \cap C^{0,\alpha}_{\text{per}}(\overline{D})) \times \mathbb{R}
\]

and \(h_p > 0\) in \(\overline{D}\) because of no stagnation condition \((1.10)\). Recall that the space \(C^{k,\alpha}_{\text{per}}(\mathbb{R})\) is the set of \(C^{k,\alpha}(\mathbb{R})\) functions that are \(2\pi\)-periodic and even in their first coordinate. The presence of surface tension \(\sigma\) is manifested as the nonlinear second-order term in the boundary condition.

We will prove the following theorem stated in the Dubreil-Jacotin variables, which implies Theorem \([14\, \text{Theorem } 3.3]\).

**Theorem 3.1** (existence). Let \(p_1 < 0, \ell > 0, \) and atmospheric vorticity function \(\gamma \in C^{0,\alpha}((p_1, 0))\) be given. Then there exists \(\sigma_0 \geq 0\) such that for each \(\sigma > \sigma_0\), there is a continuous curve \(C_{\text{loc}} \subset \mathcal{S}'\) of solution to \((3.1)\) with the following properties:

1. \(C_{\text{loc}} := \{(h(\lambda), Q(\lambda)) : \lambda \in (\lambda^* + \epsilon, \lambda^* - \epsilon)\}, \quad \text{where } \lambda \in (\lambda^* + \epsilon, \lambda^* - \epsilon) \mapsto (h(\lambda), Q(\lambda)) \in \mathcal{S}'\) is \(C^1\).
2. \((h(\lambda^*), Q(\lambda^*)) = (H(\lambda^*), Q(\lambda^*))\) is a laminar solution.
3. \(h(\lambda)\) is non-laminar for \(\lambda \neq \lambda^*\).

**Remark 3.2.** In fact, there is a necessary and sufficient condition that we call local bifurcation condition \((LBC)\), which will be given explicit in Lemma \([3.9]\). In particular, \((LBC)\) always holds for \(\sigma\) sufficiently large. When \(\sigma\) is small, a local bifurcation argument can still be carried out, but the eigenvalue of the linearized problem may not be simple. In this case, a more sophisticated analysis is required (see, for example, \([45\, [44\, [47]\).
3.1. Laminar solutions. We first consider laminar flows which are solutions of the height equation (3.1) that are independent of \( q \). Physically, this entails a wave where all of the streamlines are parallel to the bed. These will serve as the trivial solution curve when we apply the Crandall-Rabinowitz theorem to obtain Theorem 3.1.

Let us define \( \Gamma_{\text{rel}} \) by

\[
\partial_p(\Gamma_{\text{rel}}(p)^2) = 2\gamma(-p), \quad \ell = \int_{p_1}^{0} \frac{dp}{\Gamma_{\text{rel}}(p)},
\]

(3.2)

Remark 3.3. \( \Gamma_{\text{rel}} \) is called the (pseudo) relative circulation and is given by

\[
\Gamma_{\text{rel}}(p) = \frac{1}{2\pi} \int_{\{\psi = -p\}} |\nabla \psi| dH^1,
\]

where \( H^1 \) denotes one-dimensional Hausdorff measure. Note that circulation around a closed loop is conserved for the time-dependent problem by Kelvin’s circulation law. For periodic domains, this includes the circulation along the streamlines \( \{\psi = -p\} \). If the waves we construct are to be viewed as generated dynamically by the wind, the circulation along each streamline must agree with the initial configuration.

For laminar flows, since \( h \) does not depend on \( q \), we can write \( h = H(p) \), where \( H \) satisfies the following ODE:

\[
\begin{cases}
H_{pp} = -\gamma(-p)H_p^3 & \text{in } p_1 < p < 0, \\
H_{pp} = 0 & \text{in } p_0 < p < p_1, \\
[H_p^{-2}] + 2g[H] H - Q = 0 & \text{on } p = p_1, \\
H = 0 & \text{on } p = p_0, \\
H = \ell + d(H) & \text{on } p = 0.
\end{cases}
\]

(3.3)

Note that \( d(H) = H(p_1) \). The above equation can be solved explicitly, but we still need some compatibility conditions to ensure continuity across the interface.

Lemma 3.4 (laminar flow). If the compatibility condition (3.2) is satisfied, then there exists a one-parameter family of solutions \( \{(H(\cdot ; \lambda), Q(\lambda)) : \lambda > 0\} \) to the laminar flow equation (3.3) with \( H_p > 0 \). Each member of the family has the explicit form

\[
H(p; \lambda) = \begin{cases}
\int_{p_1}^{p} \frac{ds}{\Gamma_{\text{rel}}(s)} + \frac{p_1 - p_0}{\lambda}, & p_1 < p < 0, \\
\frac{p - p_0}{\lambda}, & p_0 < p < p_1,
\end{cases}
\]

(3.4)

and

\[
Q(\lambda) = \frac{2g[H](p_1 - p_0)}{\lambda} + \Gamma_{\text{rel}}(p_1)^2 - \lambda^2.
\]

(3.5)

Moreover, the depth of the fluid at parameter value \( \lambda \) is

\[
d(H(\cdot ; \lambda)) = \frac{p_1 - p_0}{\lambda}.
\]

(3.6)
Since the laminar flow is independent of the surface tension $\sigma$, the proof of Lemma 3.4 can be obtained by similar arguments as in [13, Lemma 4.2], which we will omit. Note that differentiating (3.5) with respect to $\lambda$ gives

$$Q'(\lambda) = -\frac{2g [\rho]}{\lambda^2} (p_1 - p_0) - 2\lambda$$

and

$$Q''(\lambda) = \frac{4g [\rho]}{\lambda^3} (p_1 - p_0) - 2 < 0,$$

so $\lambda \mapsto Q(\lambda)$ is concave and has a unique maximum at $\lambda_0$ satisfying

$$\lambda_0^3 = -\frac{g [\rho]}{\lambda} (p_1 - p_0).$$

### 3.2. Linearized problem

Next, let us consider the linearization of the height equation (3.1) at one of the laminar solutions $(H(\cdot; \lambda), Q(\lambda))$ constructed in Lemma 3.4:

\[
\begin{cases}
(a^3 m_p)_p + (am_q)_q = 0 & \text{in } D_1 \cup D_2, \\
-2 [a^3 m_p] + 2g [\rho] m + \sigma m_{qq} = 0 & \text{on } p = p_1, \\
m = 0 & \text{on } p = p_0, \\
m - d(m) = 0 & \text{on } p = 0,
\end{cases}
\]  

(3.8)

where

$$a := a(p; \lambda) = H_p(p; \lambda)^{-1} = \begin{cases} 
\Gamma_{rel}(p), & p_1 < p < 0, \\
\lambda, & p_0 < p < p_1.
\end{cases}$$

Since we seek solutions that are $2\pi$-periodic and even in $q$, we first consider $m$ of the form $m(q, p) = M(p) \cos(nq)$, for some $n \geq 0$. If $n = 0$, $m$ does not depend on $q$ and the linearized problem (3.8) becomes

\[
\begin{cases}
(a^3 M_p)_p = 0 & \text{in } p_1 < p < 0, \\
M_{pp} = 0 & \text{in } p_0 < p < p_1, \\
- [a^3 M_p] + g [\rho] M = 0 & \text{on } p = p_1, \\
M = 0 & \text{on } p = p_0, \\
M - d(M) = 0 & \text{on } p = 0.
\end{cases}
\]

This equation can be solved explicitly. Using the boundary condition at $p = p_0$ and the continuity of $M$ across the interface, we find that in the water region

$$M^{(2)}(p) = \frac{p - p_0}{p_1 - p_0} M(p_1), \quad \text{in } p_0 < p < p_1.$$

For the air region, we first observe that

$$M(0) = d(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} M(p_1) \, dq = M(p_1),$$
and hence, \( M_p \) must vanish at least once inside \((p_1, 0)\) by Rolle’s Theorem. We also have, from the above ODE, that \( a^3 M_p \) is constant, so we conclude that \( M_p \equiv 0 \) in \((p_1, 0)\). Finally, the jump condition gives

\[
-\frac{\lambda^3}{p_1 - p_0} M(p_1) = g \left[ \rho \right] M(p_1),
\]

which implies that there can be a zero-mode solution if and only if \( \lambda = \lambda_0 \), where \( \lambda_0 \) is defined according to (3.7).

On the other hand, if \( n > 0 \), the linearized problem (3.8) becomes

\[
\begin{cases}
-(a^3 M_p)_p = -n^2 aM & \text{in } (p_0, p_1) \cup (p_1, 0), \\
2 \left[ a^3 M_p \right] - 2g \left[ \rho \right] M = -n^2 \sigma M & \text{on } p = p_1, \\
M = 0 & \text{on } p = p_0, \\
M = 0 & \text{on } p = 0.
\end{cases}
\] (3.9)

To investigate the ODE (3.9), we consider the following general eigenvalue problem

\[
\begin{cases}
-\frac{1}{a} (a^3 u')' = \mu u & \text{in } (p_0, p_1) \cup (p_1, 0), \\
2 \left[ a^3 u' \right] - 2g \left[ \rho \right] u = \mu \sigma u & \text{on } p = p_1, \\
u = 0 & \text{on } p = p_0, \\
u = 0 & \text{on } p = 0.
\end{cases}
\] (P\(\mu\))

In particular, we are interested in the case \( \mu = -n^2 \).

The eigenvalue problem (P\(\mu\)) closely resembles a Sturm–Liouville equation, but the eigenvalue occurs both in the interior and boundary conditions. Moreover, the associated inner product defining the relation between eigenfunctions is indefinite. For that reason, it is natural to reformulate it in a Pontryagin space. Here we follow the general approach of Wahlén [44, 45] and Walsh [47].

With that in mind, we introduce the complex Pontryagin space (see [7, 23])

\[
\mathbb{H} := \{ \tilde{u} = (u, b) \in L^2([p_0, 0]) \times \mathbb{C} \}
\]

with the indefinite inner product

\[
[\tilde{u}_1, \tilde{u}_2] := \langle au_1, u_2 \rangle_{L^2} - \frac{1}{2\sigma} \overline{b_1 b_2}.
\]

We understand that the \( L^2 \)-inner product is taken over \((p_0, 0)\). On \( \mathbb{H} \), there is also an associated Hilbert space inner product, given by \( \langle \tilde{u}, \tilde{v} \rangle_{\mathbb{H}} = [J\tilde{u}, \tilde{v}] \), where

\[
J = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}.
\]

**Proposition 3.5.** \( \mathbb{H} \) is \( \pi_1 \)-space, that is \( \mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_- \), where

\[
\mathbb{H}_+ \subset \{ x \in \mathbb{H} : [x, x] > 0, \text{ or } x = 0 \}, \quad \mathbb{H}_- \subset \{ x \in \mathbb{H} : [x, x] < 0, \text{ or } x = 0 \}
\]

are complete subspaces with \( \dim \mathbb{H}_+ = 1 \) or \( \dim \mathbb{H}_- = 1 \).
We omit the proof of this proposition, as it is elementary. In fact, we have explicitly that
\[ H = H_+ \oplus H_. \]
where
\[ H_+ := L^2((p_0,0)) \times \{0\}, \]
\[ H_- := \{0 \in L^2((p_0,0))\} \times \mathbb{C}. \]

Next, define the linear operator \( K : D(K) \subset \mathbb{H} \rightarrow \mathbb{H} \) by
\[ K\tilde{u} := \left( -\frac{1}{a} (a^3 u')', 2 \left[ a^3 u' \right] - 2g \left[ \rho \right] u(p_1) \right), \]
where
\[ D(K) := \{ \tilde{u} = (u,b) \in \left( H^2((p_0,p_1)) \cap H^2((p_1,0)) \cap C^0((p_0,0)) \right) \times \mathbb{C} : u(p_0) = u(0) = 0, \sigma u(p_1) = b \}. \]

Thus, there exists a nontrivial solution of \([K\tilde{u}, \tilde{u}] \geq 0\) if and only if \( \mu \) is an eigenvalue of \( K \). Moreover, it is clear that \( D(K) \) is dense in \( \mathbb{H} \), and the operator \( K \) is closed. Recalling the convention \([u] = u|_{p_1^+} - u|_{p_1^-}\) and using integration by parts, we can show
\[ [K\tilde{u}, \tilde{u}] = \left\langle a^3 u', u' \right\rangle_{L^2} + g[\rho] |u(p_1)|^2 = \tilde{u}, K\tilde{u} \in \mathbb{R}, \quad (3.10) \]
which implies that \( K \) is symmetric, and in fact, self-adjoint. The next proposition provides a condition under which the operator \( K \) is positive, that is, \([K\tilde{u}, \tilde{u}] > 0\) for all non-zero \( \tilde{u} \in D(K) \).

**Proposition 3.6.** \( K \) is self-adjoint with simple eigenvalues. Moreover, it has a maximal negative semidefinite subspace invariant under \( K \) that has dimension one. For \( \lambda > \lambda_0 \), the operator \( K \) is positive with a unique negative eigenvalue.

**Proof.** It follows from the above discussion that \( K \) is self-adjoint. Since \( \mathbb{H} \) is a \( \pi_1 \)-space, \([23, \text{Theorem 12.1}]\) implies that \( K \) has a maximal negative semidefinite subspace invariant under \( K \) that has dimension one. By an argument similar to \([44, \text{Lemma 3.8}]\) and \([45, \text{Lemma 2}]\), we see that \( K \) has discrete spectrum and its eigenvalues are geometrically simple.

Next, since \( K \) has a maximal invariant negative semidefinite subspace which is of dimension one, it has at least one eigenvalue of negative-semidefinite type. By this, we mean the restriction of \([\cdot, \cdot]\) to the eigenspace corresponding to an eigenvalue is a negative semidefinite inner product. We caution that this does not say anything about the sign of the eigenvalue itself. Let \( \mu \) be a general eigenvalue of \( K \) with corresponding non-zero eigenvector \( \tilde{u} \). Taking the complex conjugate of the equation \( K\tilde{u} = \mu \tilde{u} \), we see that \( \bar{\mu} \) is also an eigenvalue of \( K \) (note that the coefficients of the operator \( K \) are real). Thus, either \( \mu \) is real, or both \( \mu \) and its complex conjugate are eigenvalues. For the latter case,
the corresponding eigenvector $\tilde{u}$ must be neutral, that is, $[\tilde{u}, \tilde{u}] = 0$. This follows from the observation that

$$
\mu[\tilde{u}, \tilde{u}] = [K\tilde{u}, \tilde{u}] = [\tilde{u}, K\tilde{u}] = \mu[\tilde{u}, \tilde{u}].
$$

On the other hand, if $\mu \in \mathbb{R}$ is an eigenvalue with corresponding eigenvector $\tilde{u}$ such that $[\tilde{u}, \tilde{u}] \neq 0$, then letting

$$
\mathcal{N} := \text{span } \tilde{u} = \ker(K - \mu I) \quad \text{and} \quad \mathcal{N}^\perp := \{ \tilde{v} \in \mathbb{H} : [\tilde{v}, \mathcal{N}] = 0 \}.
$$

we have $\mathbb{H} = \mathcal{N} \mathcal{N}^\perp$ as an orthogonal direct sum. Note that we are using $\mathcal{N}^\perp$ and $\mathcal{N}^\perp$ to emphasize that the orthogonality is with respect to the indefinite inner product $\langle \cdot, \cdot \rangle$. If $\tilde{w}$ is in the range of $K - \mu I$, then $\tilde{w} = (K - \mu I)\tilde{v}$ for some $\tilde{v} \in \mathbb{H}$, and hence

$$
[\tilde{w}, \tilde{u}] = [K\tilde{v}, \tilde{u}] - \mu[\tilde{v}, \tilde{u}] = [\tilde{v}, K\tilde{u}] - [\tilde{v}, \mu\tilde{u}] = 0,
$$

which implies that the range of $K - \mu I$ is in $\mathcal{N}^\perp$. Thus, $\mu$ is algebraically simple if it is real.

Finally, suppose $\lambda > \lambda_0$. By the Cauchy–Schwarz inequality,

$$
|u(p_1)|^2 = \left| \int_{p_0}^{p_1} u'(p) \, dp \right|^2 \leq \int_{p_0}^{p_1} a^3 |u'|^2 \, dp \int_{p_0}^{p_1} a^{-3} \, dp < -\frac{1}{g[\rho]} \int_{p_0}^{p_1} a^3 |u'|^2 \, dp,
$$

which gives

$$
g[\rho]|u(p_1)|^2 + \int_{p_0}^{0} a^3 |u'|^2 \, dp > 0.
$$

Thus, $[K\tilde{u}, \tilde{u}] > 0$, that is, $K$ is positive. Then $[\tilde{u}, \tilde{u}] \neq 0$, so $\tilde{u}$ is non-neutral. Hence all eigenvalues are real when $\lambda > \lambda_0$.

If $\mu$ is a negative semidefinite eigenvalue with the corresponding eigenvector $\tilde{u}$, then

$$
\mu[\tilde{u}, \tilde{u}] = [K\tilde{u}, \tilde{u}] > 0,
$$

and hence, it follows that $\mu < 0$. This means that any real negative semidefinite eigenvalue of $K$ must be negative. In fact, there is only one such eigenvalue. Indeed, if $\nu$ is another eigenvalue with corresponding eigenvector $\tilde{v}$, then

$$
[\tilde{u}, \tilde{v}] = \frac{1}{\mu}[\mu\tilde{u}, \tilde{v}] = \frac{1}{\mu}[K\tilde{u}, \tilde{v}] = \frac{1}{\mu}[\tilde{u}, K\tilde{v}] = \frac{\nu}{\mu}[\tilde{u}, \tilde{v}],
$$

which implies $\mu = \nu$ or $[\tilde{u}, \tilde{v}] = 0$. Since any maximal invariant semidefinite subspace of $K$ is one dimensional, we must have $\mu = \nu$. We have therefore shown $K$ has a unique negative eigenvalue. \hfill \qed

Define the Rayleigh quotient $\mathcal{R}$ corresponding to $(P\mu)$ by

$$
\mathcal{R}(\varphi; \lambda) := \frac{\int_{p_0}^{0} a^3 \varphi^2 \, dp + g[\rho] \varphi(p_1)^2}{\int_{p_0}^{0} a \varphi^2 \, dp - g[\rho] \varphi(p_1)^2}, \quad \lambda > \lambda_0, \varphi \in \mathcal{A},
$$
where the admissible set is defined by

\[ \mathcal{A} := \{ \varphi \in H^2((p_0, p_1)) \cap H^2((p_1, 0)) \cap C((p_0, 0)) : \]

\[ \varphi(p_0) = \varphi(0) = 0 \quad \text{and} \quad \int_{p_0}^{0} a \varphi^2 \, dp - \frac{\sigma}{2} \varphi(p_1)^2 < 0 \}. \]

Note that we are considering \( \varphi \) only in the negative definite subspace of \( K \) because of the condition

\[ \int_{p_0}^{0} a \varphi^2 \, dp - \frac{\sigma}{2} \varphi(p_1)^2 < 0. \]  \hspace{1cm} (3.11)

Simple arguments can show that if for a fixed \( \lambda > \lambda_0 \), \( \varphi \) is a critical point of \( R(\cdot; \lambda) \), then \( \varphi \) solves \( (P_{\mu}) \) for \( \mu = R(\varphi; \lambda) \).

Next, let us define

\[ \nu(\lambda) := \sup_{\varphi \in \mathcal{A}, \varphi \neq 0} R(\varphi; \lambda). \]

First, we want to show that \(-1\) is in the range of \( \nu \). This is because we want our solutions to be \( 2\pi \)-periodic in \( q \), and the null space of \( F_m(\lambda^*, 0) \) is spanned by \( \varphi_1(p) \cos(q) \) (see Lemma 3.12), which is the case where \( n = 1 \) and hence \( \mu = -n^2 = -1 \) in (3.9).

**Lemma 3.7.** Let \( a_{\min} := \min_{[p_1, 0]} a \) (which does not depend on \( \lambda \)). Then for each \( n \geq 1 \), \( \nu(\lambda) < -n^2 \) when \( \lambda \) satisfies

\[ \lambda^2 > -a_{\min}^2 - \frac{g [\rho]}{n} + \frac{\sigma n^2}{2}. \]

**Proof.** Let \( \varphi \in \mathcal{A} \) be given and fix any \( \lambda \) as in the hypothesis. Then

\[ \int_{p_1}^{0} (a^3 \varphi_p^2 + n^2 a \varphi^2) \, dp \geq a_{\min} \int_{p_1}^{0} (a_{\min}^2 \varphi_p^2 + n^2 \varphi^2) \, dp \]

\[ \geq -2n a_{\min} \int_{p_1}^{0} \varphi_p \varphi \, dp \]

\[ = -n a_{\min} \int_{p_1}^{0} (\varphi^2)_p \, dp = n a_{\min}^2 \varphi(p_1)^2. \]

On the other hand, since \( a^{(2)} = \lambda \),

\[ \int_{p_0}^{p_1} (a^3 \varphi_p^2 + n^2 a \varphi^2) \, dp = \lambda \int_{p_0}^{p_1} (a^2 \varphi_p^2 + n^2 \varphi^2) \, dp \geq 2n \lambda^2 \int_{p_0}^{p_1} \varphi_p \varphi \, dp = n \lambda^2 \varphi(p_1)^2. \]

Summing these together and using the hypothesis for \( \lambda \), we find

\[ \int_{p_0}^{0} (a^3 \varphi_p^2 + n^2 a \varphi^2) \, dp \geq (n \lambda^2 + n a_{\min}^2) \varphi(p_1)^2 \]

\[ > \left( -g [\rho] + \frac{\sigma n^2}{2} \right) \varphi(p_1)^2, \]
which implies
\[
\int_{p_0}^{p_1} a^3 \varphi_0^2 \, dp + g \left[ \rho \right] \varphi(p_1)^2 > -n^2 \left( \int_{p_0}^{p_1} a \varphi_0^2 \, dp - \frac{\sigma}{2} \varphi(p_1)^2 \right),
\]
so \( R(\varphi; \lambda) < -n^2 \). Thus, \( \nu(\lambda) < -n^2 \). \( \square \)

Next, we need to verify that \( \nu(\lambda) > -1 \) for some \( \lambda > \lambda_0 \). Since this is not true in general, we will have it as one of our hypotheses.

**Definition 3.8.** We say that the local bifurcation condition is satisfied provided that
\[
\sup_{\lambda > \lambda_0} \nu(\lambda) > -1.
\] (LBC)

This is necessary and sufficient for our main result Theorem 3.1 to hold. An explicit but not sharp condition is the following:

**Lemma 3.9** (size condition). For
\[
\sigma > \frac{2\lambda_0(p_1 - p_0)}{3} + \frac{2}{p_1^3} \int_{p_1}^{p_0} \left( \Gamma_{\text{rel}}^3 + p^2 \Gamma_{\text{rel}} \right) \, dp,
\] (3.12)
where \( \lambda_0 \) is defined as in (3.7), (LBC) holds.

**Proof.** Let
\[
\varphi(p) := \begin{cases} 
\frac{p}{p_1}, & p_1 < p < 0, \\
\frac{p - p_0}{p_1 - p_0}, & p_0 < p < p_1.
\end{cases}
\]
We first check if \( \varphi \) is in the admissible set \( \mathcal{A} \). We see that
\[
\int_{p_0}^{p_1} a \varphi_0^2 \, dp - \frac{\sigma}{2} \varphi(p_1)^2 = \int_{p_0}^{p_1} \lambda \left( \frac{p - p_0}{p_1 - p_0} \right)^2 \, dp + \int_{p_1}^{p_0} \Gamma_{\text{rel}} \left( \frac{p}{p_1} \right)^2 \, dp - \frac{\sigma}{2}
\]
\[
= \frac{\lambda(p_1 - p_0)}{3} + \frac{1}{p_1} \int_{p_1}^{p_0} p^2 \Gamma_{\text{rel}} \, dp - \frac{\sigma}{2},
\]
but from the hypothesis (3.12),
\[
\frac{\lambda_0(p_1 - p_0)}{3} + \frac{1}{p_1^2} \int_{p_1}^{p_0} p^2 \Gamma_{\text{rel}} \, dp - \frac{\sigma}{2} < 0.
\]
Thus, for \(|\lambda - \lambda_0|\) small, \(\varphi \in \mathcal{A}\). With this particular \(\varphi\), we then compute

\[
\mathcal{R}(\varphi; \lambda) = \frac{\int_{p_0}^{p_1} \frac{\lambda^3}{(p_1 - p_0)^2} \, dp + \int_{p_1}^0 \frac{\Gamma_{rel}}{p_1} \, dp + g [\rho]}{\int_{p_0}^{p_1} \frac{\lambda}{(p_1 - p_0)} \, dp + \int_{p_1}^0 \frac{\Gamma_{rel}}{p_1} \, dp - \frac{\sigma}{2}}.
\]

Rewriting the hypothesis (3.12) gives

\[
\frac{\sigma}{2} > \frac{\lambda^3_0}{p_1 - p_0} + \frac{\lambda_0 (p_1 - p_0)}{3} + \frac{1}{p_1^2} \int_{p_1}^0 (\Gamma_{rel}^3 + p^2 \Gamma_{rel}) \, dp + g [\rho].
\]

Then for \(|\lambda - \lambda_0|\) small, we have

\[
\frac{\sigma}{2} > \frac{\lambda^3}{p_1 - p_0} + \frac{\lambda (p_1 - p_0)}{3} + \frac{1}{p_1^2} \int_{p_1}^0 (\Gamma_{rel}^3 + p^2 \Gamma_{rel}) \, dp + g [\rho].
\]

Recalling that \(\varphi\) satisfies inequality (3.11), the above estimate implies that \(\mathcal{R}(\varphi; \lambda) > -1\), so \((LBC)\) holds.

We note that the above proof can be further refined following arguments of [18, Theorem 4] to find a smaller lower bound on \(\sigma\) guaranteeing \((LBC)\) than that in (3.12).

**Lemma 3.10** (monotonicity of \(\nu\)). If \(\nu(\lambda) < 0\), then \(\nu(\lambda)\) is decreasing in \(\lambda\).

**Proof.** Denoting derivatives with respect to \(\lambda\) by a dot, differentiating the eigenvalue problem \(P\mu\) with \(u = \varphi \in \mathcal{A}\) gives

\[
\begin{cases}
- (3a^2 \dot{\varphi}_p + a^3 \ddot{\varphi}_p)_p = (\dot{\varphi} + \nu a \varphi + \nu a \dot{\varphi}) & \text{in } (p_0, p_1) \cup (p_1, 0), \\
2 \left[ 3a^2 \ddot{\varphi}_p + a^3 \dot{\varphi}_p \right] - 2g [\rho] \dot{\varphi} = a \dot{\varphi} + \sigma \nu \dot{\varphi} & \text{on } p = p_1, \\
\dot{\varphi} = 0 & \text{on } p = p_0, \\
\ddot{\varphi} = 0 & \text{on } p = 0.
\end{cases}
\]

Multiplying \((P\mu)\) by \(\dot{\varphi}\) and integrating yields

\[
\int_{p_0}^0 a^3 \dot{\varphi}_p \dot{\varphi}_p \, dp + g [\rho] \varphi(p_1) \dot{\varphi}(p_1) + \frac{\sigma}{2} \nu \varphi(p_1) \dot{\varphi}(p_1) = \int_{p_0}^0 a \dot{\varphi} \dot{\varphi} \, dp. \quad (3.13)
\]
On the other hand, multiplying \( \{P_{\mu}\} \) by \( \varphi \) and integrating gives
\[
\int_0^p 3a^2 \dot{a} \varphi_p^2 dp + g [\rho] \dot{\varphi}(p_1) \varphi(p_1) + \frac{\sigma}{2} \dot{\nu} \varphi(p_1)^2 + \frac{\sigma}{2} \nu \ddot{\varphi}(p_1) \varphi(p_1)
\]
(3.14)
\[
+ \int_0^p a^3 \dddot{\varphi} \dot{\varphi}_p dp = \int_0^p (\dot{\nu} \varphi^2 + \nu \dot{\varphi}^2 + \nu a \dot{\varphi} \varphi) dp.
\]
Subtracting (3.13) from (3.14), we have the following Green’s identity
\[
\int_0^p 3a^2 \dot{a} \varphi_p^2 dp = \int_0^p (\dot{\nu} a \varphi^2 + \nu \dot{\varphi}^2) dp.
\]
Since \( \dot{a} = \frac{1}{BD(p_0, p_1)} \), we can simplify this to find
\[
\int_{p_0}^{p_1} 3a^2 \dot{a} \varphi_p^2 dp - \int_{p_0}^{p_1} \nu \varphi^2 dp = \left( \int_{p_0}^{p_1} a \varphi^2 dp - \sigma \frac{p^2}{2} \varphi(p_1)^2 \right) \dot{\nu}.
\]
Therefore, since \( \nu < 0 \) by assumption and the quantity in parenthesis is negative, we must have \( \dot{\nu} < 0 \).

\[\square\]

**Lemma 3.11.** Suppose that the [LBC] holds. Then there exists a unique value \( \lambda^* > 0 \) such that \( \nu(\lambda^*) = -1 \). Equivalently, there exists a unique value of \( \lambda \) for which there is a nontrivial solution to the linearized problem (3.8) with the ansatz \( m(q, p) = M(p) \cos(q) \). Moreover, \( Q \) is an invertible function of \( \lambda \) in a neighborhood of \( \lambda^* \).

**Proof.** From Lemma 3.7, we have \( \nu(\lambda) < -1 \) for \( \lambda \) sufficiently large, and \( \nu(\lambda) > -1 \) for some \( \lambda \) by [LBC]. By continuity, there exists \( \lambda^* \) such that \( \nu(\lambda^*) = -1 \). Moreover, Lemma 3.10 tells us that \( \nu \) is a decreasing function when \( \nu < 0 \), so \( \lambda^* \) is unique.

Next, as noted at the end of Lemma 3.4, \( Q \) is a concave function of \( \lambda \) according to (3.5), so we only need to show that \( \lambda^* \neq \lambda_0 \), where \( \lambda_0 \) is defined in (3.7) to be the critical point of \( Q \). But \( \lambda^* > \lambda \) by [LBC].

\[\square\]

### 3.3. Proof of local bifurcation.

We are now prepared to prove Theorem 3.1. As stated above, our approach is based on the classical theory of Crandall–Rabinowitz on local bifurcation from simple (generalized) eigenvalues. Specifically, we will treat the family of laminar flows as our trivial solutions. Suppose the solution to the height equation (3.1) can be decomposed as \( h(q, p) = H(p; \lambda) + m(q, p) \) and \( Q = Q(\lambda) \). Then substituting it into the equation gives
\[
F(\lambda, m) = 0,
\]
where \( F = (F_1, F_2, F_3, F_4) : \Lambda \times O \to Y \) with \( \Lambda \subset \mathbb{R} \) to be a neighborhood of \( \lambda^* \), and
\[
O := \{ m \in X : \inf(m_p + H_p) > 0 \text{ in } D \text{ for all } \lambda \in \Lambda \},
\]
\[ \mathcal{F}_1(\lambda, m) := (1 + (m_{pq}^{(1)})^2)(m_{pp}^{(1)} + H_{pp}) + m_{pp}^{(1)}(m_{pq}^{(1)} + H_p)^2 \\
- 2m_{pq}^{(1)}(H_p + m_{pq}^{(1)})m_{pq}^{(1)} + \gamma(-p)(H_p + m_{pq}^{(1)})^3, \]
\[ \mathcal{F}_2(\lambda, m) := (1 + (m_{pq}^{(1)})^2)(m_{pp}^{(2)} + H_{pp}) + m_{pp}^{(2)}(m_{pq}^{(2)} + H_p)^2 \\
- 2m_{pq}^{(2)}(H_p + m_{pq}^{(2)})m_{pq}^{(2)}, \]
\[ \mathcal{F}_3(\lambda, m) := - \left[ \frac{1 + m_{pq}^2}{(H_p + m_{pq})^2} \right] - 2g [\rho] (m + H) + Q - \frac{\sigma m_{pq}}{(1 + m_{pq}^2)^{3/2}}, \]
\[ \mathcal{F}_4(\lambda, m) := \left( m + H - \ell - d(m) - d(H) \right) \right|_T. \]

The Banach spaces \( X \) and \( Y = Y_1 \times Y_2 \times Y_3 \times Y_4 \) are defined by
\[ X := \{ h \in C^{2, \alpha}(\overline{D_1}) \cap C^{2, \alpha}(\overline{D_2}) \cap C^{0, \alpha}_{\text{per}}(\overline{T}) : h(p_0) = 0 \}, \]
\[ Y_1 := C^{0, \alpha}_{\text{per}}(\overline{D_1}), \quad Y_2 := C^{0, \alpha}_{\text{per}}(\overline{D_2}), \quad Y_3 := C^{\alpha}_{\text{per}}(I), \quad Y_4 := C^{2, \alpha}_{\text{per}}(T). \]

It is clear that \( \mathcal{F}(\lambda, 0) = 0 \) for all \( \lambda > 0 \). Let us record the Fréchet derivative of \( \mathcal{F} \) with respect to \( m \) at \( (\lambda^*, 0) \).
\[ \mathcal{F}_{1m}(\lambda^*, 0)\varphi = (\partial_p^2 + H_{pp}^2\partial_q^2 + 3\gamma H_{pp}^2\partial_p) \varphi^{(1)}, \]
\[ \mathcal{F}_{2m}(\lambda^*, 0)\varphi = (\partial_p^2 + H_{pp}^2\partial_q^2) \varphi^{(2)}, \]
\[ \mathcal{F}_{3m}(\lambda^*, 0)\varphi = 2 [H_p^{-3}\varphi_p] - 2g [\rho] \varphi - \sigma \varphi_{qq}, \]
\[ \mathcal{F}_{4m}(\lambda^*, 0)\varphi = \left( \varphi - d(\varphi) \right) \right|_T. \]

Note that in \( D_1 \), from (3.3), we have \( \gamma = -H_{pp}/H_{pp}^3 \), so we can write
\[ \mathcal{F}_{1m}(\lambda^*, 0)\varphi = \varphi_{pp} + a^{-2}\varphi_{qq} + 3H_{pp}\partial_p \left( H_{pp}^{-1} \right) \varphi_p \]
\[ = \varphi_{pp} + a^{-2}\varphi_{qq} + a^{-3}a^2(\partial_p \varphi) \varphi_p \]
\[ = a^{-3} \left( a^3 \varphi_p \right) + a^{-2}\varphi_{qq}, \]
which is the same quantity as in \( D_2 \). Thus, the first expression can be written as
\[ \mathcal{F}_{im}(\lambda^*, 0)\varphi = a^{-3} \partial_p \left( a^3 \partial_q \varphi \right) + a^{-2}\partial_q^2 \varphi \right|_{i=1,2} \]

**Lemma 3.12** (null space). The null space of \( \mathcal{F}_m(\lambda^*, 0) \) is one-dimensional and spanned by \( \varphi^*(q, p) := \varphi_1(p) \cos(q) \).

**Proof.** Let \( \varphi \) be in the null space of \( \mathcal{F}_m(\lambda^*, 0) \). Since \( \varphi \) is even and \( C^{0, \alpha} \), we can express it via a cosine series
\[ \varphi(q, p) = \sum_{n=0}^{\infty} \varphi_n(p) \cos(nq). \]
Clearly, \( \mathcal{F}_m(\lambda^*, 0) (\varphi_n(p) \cos(nq)) = 0 \) for every \( n \geq 0 \) meaning that \( \varphi_n \) must solve (8.34) for \( n \). Since \( \lambda^* > \lambda_0 \), we can apply Proposition 3.6 to conclude that the null space of \( \mathcal{F}_m(\lambda^*, 0) \) is one-dimensional. In particular, it is generated by \( \varphi^*(q, p) := \varphi_1(p) \cos(q) \), where \( \varphi_1 \) is the unique solution to equation (3.3) for \( n = 1 \). □
Our next lemma characterizes the range of $F_m(\lambda^*, 0)$.

**Lemma 3.13** (range). $A = (A_1, A_2, A_3, A_4) \in Y$ is in the range of $F_m(\lambda^*, 0)$ if and only if it satisfies the following orthogonality condition:

$$
\iint_{D_1} a^3 A_1 \varphi^* \, dq \, dp + \iint_{D_2} a^3 A_2 \varphi^* \, dq \, dp + \frac{1}{2} \int_I A_3 \varphi^* \, dq + \int_T a^3 A_4 \varphi^* \, dq = 0, \quad (3.16)
$$

where $\varphi^*$ generates the null space of $F_m(\lambda^*, 0)$.

**Proof.** Denoting by $\mathcal{R}(F_m(\lambda^*, 0))$ the range of $F_m(\lambda^*, 0)$, we first suppose that $A \in \mathcal{R}(F_m(\lambda^*, 0))$. Let $\varphi$ be given such that $F_m(\lambda^*, 0) \varphi = \mathcal{A}$. Using integration by parts and the PDE for $\varphi$ and $\varphi^*$ ($\varphi^*$ satisfies equation (3.9) with $n = 1$), we can compute

$$
\langle a^3 \varphi^*, A_1 \rangle_{L^2(D_1)} + \langle a^3 \varphi^*, A_2 \rangle_{L^2(D_2)}
= \int_{D_1 \cup D_2} \left( (a^3 \varphi_p)_p + a \varphi_{qq} \right) \varphi^* \, dq \, dp
= - \int_{D_1 \cup D_2} a^3 \varphi_p \varphi^*_p \, dq \, dp - \int_I \left[ a^3 \varphi_p \varphi^* \right] \, dq + \int_{D_1 \cup D_2} a \varphi^*_q \varphi \, dq \, dp
= \int_{D_1 \cup D_2} (a^3 \varphi_{pp} + a \varphi_{qq}) \varphi \, dq \, dp + \int_I (\left[ a^3 \varphi_p \varphi^* \right] - \left[ a^3 \varphi_p \varphi^* \right]) \, dq - \int_T a^3 \varphi_p \, dq.
$$

Using the jump condition, the fact that $\varphi$ and $\varphi^*$ are continuous across the interface, and integration by parts, we obtain

$$
\langle a^3 \varphi^*, A_1 \rangle_{L^2(D_1)} + \langle a^3 \varphi^*, A_2 \rangle_{L^2(D_2)}
= \int_I \left( g [\rho] \varphi^* - \frac{\sigma}{2} \varphi^* \right) \varphi \, dq - \int_I \left( g [\rho] \varphi + \frac{\sigma}{2} \varphi_{qq} + \frac{1}{2} A_3 \right) \varphi^* \, dq
- \int_T a^3 A_4 \varphi_p^* \, dq - d(\varphi) \int_T a^3 \varphi_p^* \, dq
= \int_I \left( \frac{\sigma}{2} \varphi^* \varphi - \frac{\sigma}{2} \varphi_{qq} \varphi - \frac{1}{2} A_3 \varphi^* \right) \, dq - \int_T a^3 A_4 \varphi_p^* \, dq.
$$

Recalling that $\varphi^* = \varphi_1(p) \cos(q)$, the last equality comes from the observation that

$$
\int_T a^3 \varphi_p^* \, dq = a^3 \varphi_1(0) \int_0^{2\pi} \cos(q) \, dq = 0.
$$

Finally, we have

$$
\langle a^3 \varphi^*, A_1 \rangle_{L^2(D_1)} + \langle a^3 \varphi^*, A_2 \rangle_{L^2(D_2)} = -\frac{1}{2} \int_I A_3 \varphi^* \, dq - \int_T a^3 A_4 \varphi_p^* \, dq,
$$

which can be rearranged to yield the identity (3.16).
Next, we prove the orthogonality condition (3.16) is sufficient. By similar arguments as in [4, Lemma 3.6], we define the following inner product for each $\lambda$:

$$<(U_1, U_2, U_3, U_4), (V_1, V_2, V_3, V_4)>_Y :=<a^3 U_1, V_1>_{L^2(D_1)} + <a^3 U_2, V_2>_{L^2(D_2)} + \frac{1}{2} <U_3, V_3>_{L^2(I)} + <a^3 U_4, V_4>_{L^2(T)}.$$ 

Note that the null space $\tilde{N}$ of $\mathcal{F}_m(\lambda^*, 0)$ can be identified with the subspace $\tilde{N} := \{(V_1, V_2, V_3, V_4) \in Y : V_1 = V|_{D_1}, V_2 = V|_{D_2}, V_3 = V|_I, V_4 = V|_T \text{ for some } V \in \tilde{N}\}$. Then the necessary condition, which is shown above, implies that if $A$ is in the range of $\mathcal{F}_m(\lambda^*, 0)$, then

$$<(\varphi^*[D_1], \varphi^*[D_2], \varphi^*[I], \varphi^*[T]), A>_Y = 0,$$

so $\mathcal{R}(\mathcal{F}_m(\lambda^*, 0)) \subset \tilde{N}^\perp$. By Remark 2.5, $\mathcal{F}_m(\lambda^*, 0)$ has Fredholm index 0, and hence

$$\text{codim} \mathcal{R}(\mathcal{F}_m(\lambda^*, 0)) = \dim \tilde{N} = \dim \tilde{N} - \text{codim} \tilde{N}^\perp < \infty,$$

which means that $\mathcal{R}(\mathcal{F}_m(\lambda^*, 0)) = \tilde{N}^\perp$. This concludes the proof of the lemma.

**Lemma 3.14** (transversality). The following transversality condition holds

$$\mathcal{F}_{\lambda m}(\lambda^*, 0)\varphi^* \notin \mathcal{R}(\mathcal{F}_m(\lambda^*, 0)),$$

(3.17)

where $\varphi^*$ generates the null space of $\mathcal{F}_m(\lambda^*, 0)$ solves equation (3.9) for $n = 1$.

**Proof.** Using Lemma 3.13, it suffices to show that $A := \mathcal{F}_{\lambda m}(\lambda^*, 0)\varphi^*$ does not satisfy the orthogonality condition (3.16). We must confirm that

$$\Xi := \int_{D_1} a^3 A_1 \varphi^* \, dq \, dp + \int_{D_2} a^3 A_2 \varphi^* \, dq \, dp + \frac{1}{2} \int_I A_3 \varphi^* \, dq + \int_T a^3 A_4 \varphi^* \, dq \neq 0.$$

Computing the derivatives gives

$$\mathcal{F}_{1\lambda m}(\lambda^*, 0)\varphi^* = 0,$$

$$\mathcal{F}_{2\lambda m}(\lambda^*, 0)\varphi^* = -\frac{2}{(\lambda^*)^3} \varphi^*qq,$$

$$\mathcal{F}_{3\lambda m}(\lambda^*, 0)\varphi^* = \left(6(\lambda^*)^2 (\varphi^*_p)^2(2)\right)_I,$$

$$\mathcal{F}_{4\lambda m}(\lambda^*, 0)\varphi^* = 0,$$

so we have

$$\int_{D_1} a^3 A_1 \varphi^* \, dq \, dp = \int_T a^3 A_4 \varphi^*_p \, dq = 0.$$

Using (3.9) with $n = 1$ to have $\varphi^* = (\lambda^*)^2 \varphi^*_p$, we can derive

$$(\lambda^*)^2 (\varphi^*_p \varphi^*_p)_p = (\lambda^*)^2 (\varphi^*_p)^2 + (\lambda^*)^2 \varphi^*_p \varphi^*_p = (\lambda^*)^2 (\varphi^*_p)^2 + (\varphi^*)^2 \text{ in } D_2$$
so that using integration by parts and the fact that $\varphi_{qq}^* = -\varphi^*$ yields

$$\int\int_{D_2} a^3 A_2 \varphi^* dq dp = \int\int_{D_2} (\varphi^*)^2 dq dp$$

and

$$\frac{1}{2} \int_I A_3 \varphi^* dq = -3 (\lambda^*)^2 \int_I (\varphi_p^*)^2 dq$$

$$= -3 (\lambda^*)^2 \int_{D_2} (\varphi_p^*)^2 dq dp - 3 \int_{D_2} (\varphi^*)^2 dq dp.$$ 

Finally, combining all terms gives

$$\Xi = -\int\int_{D_2} (\varphi^*)^2 dq dp - 3 (\lambda^*)^2 \int_{D_2} (\varphi_p^*)^2 dq dp < 0.$$

Now we are ready to prove our main theorem.

**Proof of Theorem 3.1.** Suppose conditions (3.2) and (LBC) are satisfied. Then $F(\lambda,0) = 0$ for all $\lambda > 0$ and $F_m$, $F_\lambda$, $F_{\lambda m}$ exist and are continuous, which means parts (i) and (ii) are confirmed. Moreover, Lemma 3.12 and Lemma 3.13 give dimension 1 for the null space of $F_m(\lambda^*,0)$ and co-dimension 1 for the range of $F_m(\lambda^*,0)$. Thus, $F_m(\lambda^*,0)$ has Fredholm index 0, and hence part (iii) is justified. Lastly, the transversality condition in Lemma 3.14 fulfills part (iv). Therefore, the local bifurcation result follows directly from Theorem A.1.

Finally, back to our objective, we note that the existence of a solution of class $S'$ to the height equation (3.1) is equivalent to the existence of a solution of class $S$ to the Euler system (1.8)–(1.10) (see, for example, [17, Lemma 2.1] or [46, Lemma 2.1]).

**Acknowledgements**

The author would like to express his sincere gratitude to Samuel Walsh for the continuous support and insightful comments. The author is also grateful to Erik Wahlén for assisting with the Pontryagin space material. Finally, the author is deeply indebted to the referee for many helpful comments and suggestions which significantly improved the paper.

This work was supported in part by the National Science Foundation through DMS-1514910.

**Appendix.**

**Theorem A.1** (Crandall and Rabinowitz [19]). Let $X$ and $Y$ be Banach spaces and $I \subset \mathbb{R}$ be an open interval with $\lambda^* \in I$. Suppose that $F : I \times X \to Y$ is a continuous map with the following properties:

(i) $F(\lambda,0) = 0$ for all $\lambda \in I$;

(ii) $D_1 F$, $D_2 F$, and $D_1 D_2 F$ exist and are continuous, where $D_i$ denotes the Fréchet derivative with respect to the $i$-th coordinate;
(iii) $D_2 F(\lambda^*, 0)$ is a Fredholm operator of index 0. In particular, the null space is one-dimensional and spanned by some element $w^*$.

(iv) $D_1 D_2 F(\lambda^*, 0)w^* \notin \mathcal{R}(D_2 F(\lambda^*, 0))$.

There exists a continuous local bifurcation curve $\{(\lambda(s), w(s)) \in \mathbb{R} \times X : |s| < \epsilon\}$ with $\epsilon > 0$ sufficiently small such that $(\lambda(0), w(0)) = (\lambda^*, w^*)$, and

$$\{(\lambda, w) \in \mathcal{U} : w \neq 0, F(\lambda, w) = 0\} = \{(\lambda(s), w(s)) \in \mathbb{R} \times Y : |s| < \epsilon\}$$

for some neighborhood $\mathcal{U}$ of $(\lambda^*, 0)$ in $\mathbb{R} \times X$. Moreover, we have

$$w(s) = sw^* + o(s) \quad \text{in } X, \quad |s| < \epsilon.$$

If $D_2^2$ exists and is continuous, then the curve is of class $C^1$.

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