THE MODULI SPACE OF SHEAVES AND THE GENERALIZATION OF MACMAHON’S FORMULA

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Abstract. Recently M. Vuletic found a two-parameter generalization of the MacMahon’s formula. In this note we show that certain ingredients of her formula have a clear interpretation in terms of the geometry of the moduli space of sheaves on the projective plane.

1. Introduction

A plane partition is a Young diagram filled with positive integers that form nonincreasing rows and columns. For a plane partition \( \pi \) one defines the weight \( |\pi| \) to be the sum of all entries. Denote by \( \mathcal{P} \) the set of all plane partitions.

A generating function for the number of plane partitions is given by the famous MacMahon’s formula (see e.g. [9]):

\[
\sum_{\pi \in \mathcal{P}} s^{\pi} = \prod_{n=1}^{\infty} \frac{1}{(1 - s^n)^n}.
\]

There are several generalizations of MacMahon’s formula, see e.g. [4, 10]. In this paper we investigate the generalization of M. Vuletic from [10]. For each plane partition \( \pi \) she defined a rational function \( F_\pi(q, t) \) and proved that

\[
\sum_{\pi \in \mathcal{P}} F_\pi(q, t) s^{\pi} = \prod_{n=1}^{\infty} \prod_{k=0}^{\infty} \left( \frac{1 - ts^n q^k}{1 - s^n q^k} \right)^n.
\]

Her proof was inspired by [8] and [11].

Let \( \mathcal{M}_{r,n} \) be the framed moduli space of torsion free sheaves on \( \mathbb{P}^2 \) with rank \( r \) and \( c_2 = n \). This is a smooth irreducible quasi-projective variety of dimension \( 2rn \). In the case \( r = 1 \) it is isomorphic to the Hilbert scheme of \( n \) points on the plane. The moduli space \( \mathcal{M}_{r,n} \) has a simple quiver description and we recall it in Section 2.1. There is a natural action of the two-dimensional torus \( T = (\mathbb{C}^*)^2 \) on \( \mathcal{M}_{r,n} \). We refer the reader to the book [5] for a more detailed discussion of the moduli space \( \mathcal{M}_{r,n} \).

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We denote by $K_0(\nu_C)$ the Grothendieck ring of complex quasi-projective varieties.

In this note we show that the coefficients $F_{\pi}(q, 0)$ give the formulas for the classes in $K_0(\nu_C)$ of the irreducible components of the fixed point set $\mathcal{M}_{r,n}^T$.

We also show how to use the $T$-action on $\mathcal{M}_{r,n}$ to get a combinatorial identity, which is close to (1).

We refer the reader to [3] for results about the Hilbert scheme of $n$ points on the plane close to this work.

1.1. Definition of $F_{\pi}(q, t)$. For nonnegative integers $n$ and $m$ let

$$f(n, m) = \begin{cases} \prod_{i=0}^{n-1} \frac{1-q^i t^{m+1}}{1-q^{i+1} t^m}, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Let $\pi \in \mathcal{P}$ be a plane partition and let $(i, j)$ be a box in its support (where the entries are nonzero). Let $\lambda, \mu$ and $\nu$ be the ordinary partitions defined by

$$\lambda = (\pi_{i,j}, \pi_{i+1,j+1}, \ldots),$$
$$\mu = (\pi_{i+1,j}, \pi_{i+2,j+1}, \ldots),$$
$$\nu = (\pi_{i,j+1}, \pi_{i+1,j+2}, \ldots).$$

For a box $(i, j)$ of $\pi$ let

$$F_{\pi}(i, j)(q, t) = \prod_{m=0}^{\infty} \frac{f(\lambda_1 - \mu_{m+1}, m) f(\lambda_1 - \nu_{m+1}, m)}{f(\lambda_1 - \lambda_{m+1}, m) f(\lambda_1 - \lambda_{m+2}, m)}.$$ 

An example is on Figure 1.

1.2. Grothendieck ring of quasi-projective varieties. Here we recall a definition of the Grothendieck ring $K_0(\nu_C)$ of complex quasi-projective varieties. It is an abelian group generated by the classes $[X]$ of all complex quasi-projective varieties $X$ modulo the relations:

(1) if varieties $X$ and $Y$ are isomorphic, then $[X] = [Y]$;
(2) if $Y$ is a Zariski closed subvariety of $X$, then $[X] = [Y] + [X \setminus Y]$.

\[
\begin{array}{cccc}
1 & & & \\
3 & 2 & 1 & \\
3 & 2 & 2 & \\
4 & 4 & 3 & 1 & 1 \\
\end{array}
\]

\[
F_{\pi}(0, 0)(q, t) = \frac{1-q^2}{1-qt}, \quad \frac{1-q^3}{1-q^2 t^2}, \quad \frac{1-q^4 t^3}{1-q^3 t^2}
\]

\textbf{Figure 1.}

For a plane partition $\pi$ the rational function $F_{\pi}(q, t)$ is defined by

$$F_{\pi}(q, t) = \prod_{(i, j) \in \pi} F_{\pi}(i, j)(q, t).$$

\[\]
1.3. **Moduli space of sheaves on \( \mathbb{P}^2 \).** In this section we show a geometric meaning of the functions \( F_\nu(q,0) \).

The moduli space \( \mathcal{M}_{r,n} \) is defined by

\[
\mathcal{M}_{r,n} = \left\{ (E, \Phi) \mid E: \text{a torsion free sheaf on } \mathbb{P}^2 \right. \\
\left. \text{rank}(E) = r, \text{deg}(E) = n \right\}/ \text{isomorphism},
\]

where \( l_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2 \} \subset \mathbb{P}^2 \) is the line at infinity.

The torus \( T = (\mathbb{C}^*)^2 \) acts on \( \mathbb{C}^2 \) by scaling the coordinates, \((t_1, t_2)(x, y) = (t_1x, t_2y)\). This action lifts to the \( T \)-action on the moduli space \( \mathcal{M}_{r,n} \).

We will prove that the irreducible components of the variety \( \mathcal{M}_{r,n} \) are enumerated by plane partitions \( \pi \) such that \(|\pi| = n\) and \( \pi_{0,0} \leq r \). We denote by \( \mathcal{M}_{r,n}^T(\pi) \) the corresponding irreducible components. We use the notation \( [N]!_q = \prod_{i=1}^{N} (1 - q^i) \). We will prove the following statement.

**Theorem 1.1.** Let \( \pi \) be a plane partition such that \(|\pi| = n\) and \( \pi_{0,0} \leq r \), then

\[
\left[ \mathcal{M}_{r,n}^T(\pi) \right] = \frac{[r]!_L}{[r - \pi_{0,0}]!_L} \prod_{i,j \geq 0} \frac{[\pi_{i,j} - \pi_{i+1,j+1}]!_L}{[\pi_{i,j} - \pi_{i+1,j}]!_L [\pi_{i,j} - \pi_{i,j+1}]!_L}.
\]

Consider the map \( \psi: \mathcal{M}_{r,n} \to \mathcal{M}_{r+1,n} \) defined by \( E \mapsto E \oplus \mathcal{O}_{\mathbb{P}^2} \), where \( E \) is a sheaf. The map \( \psi \) is an embedding of \( \mathcal{M}_{r,n} \) into \( \mathcal{M}_{r+1,n} \).

This embedding induces an embedding of \( \mathcal{M}_{r,n}^T(\pi) \) into \( \mathcal{M}_{r+1,n}^T(\pi) \). We denote by \( \mathcal{M}_{\infty,n}^T(\pi) \) the limit space. The space \( \mathcal{M}_{\infty,n}^T(\pi) \) has infinite dimension, but using a generalization of the ring \( K_0(\mathcal{O}_C) \) the class \( [\mathcal{M}_{\infty,n}^T(\pi)] \) can be defined. The class \( [\mathcal{M}_{\infty,n}^T(\pi)] \) is an infinite series in \( L \) equal to \( \lim_{r \to \infty} [\mathcal{M}_{r,n}^T(\pi)] \). From Theorem 1.1 it follows that

\[
[\mathcal{M}_{\infty,n}^T(\pi)] = \prod_{i,j \geq 0} \frac{[\pi_{i,j} - \pi_{i+1,j+1}]!_L}{[\pi_{i,j} - \pi_{i+1,j}]!_L [\pi_{i,j} - \pi_{i,j+1}]!_L}.
\]

The following statement shows a geometric interpretation of the series \( F_\nu(q,0) \).

**Theorem 1.2.** \( F_\nu(L,0) = [\mathcal{M}_{\infty,n}^T(\pi)] \).

**Proof.** Direct computation. \( \square \)

Using the result of M. Vuletic we obtain the following corollary.

**Corollary 1.3.**

\[
\sum_{n \geq 0} [\mathcal{M}_{\infty,n}^T] t^n = \prod_{i,j \geq 0, j \geq 1} \frac{1}{(1 - L^i t^j)^j}.
\]
1.4. Combinatorial identity. Here we give an application of Theorem 1.1. In Section 3 we use the $T$-action on $\mathcal{M}_{r,n}$ to get a decomposition of $\mathcal{M}_{r,n}$ into locally closed subvarieties. These subvarieties are locally trivial bundles over the varieties $\mathcal{M}_{r,n}^T(\pi)$. Then Theorem 1.1 can be applied to obtain the following statement.

Theorem 1.4.

$$\sum_{\pi \in \mathcal{P}} t^{|\pi|} \frac{[r]!}{[r-n,0]^!} q^{|\chi(\pi)|} F(\pi,0) = \prod_{n \geq 1} \frac{1}{1-q^{n+1}}$$

where $\chi(\pi) = \sum_{i,j \geq 0} \pi_{i,j} (\pi_{i,j} - \pi_{i,j+1})$. In particular

$$\sum_{\pi \in \mathcal{P}} t^{|\pi|} q^{|\chi(\pi)|} F_{\pi}(q,0) = \prod_{m,n \geq 1} \frac{1}{1-q^{m+n}}.$$ 

1.5. Organization of the paper. In Section 2 we recall the quiver description of the moduli space $\mathcal{M}_{r,n}$. Then we use it to describe the irreducible components of the variety $\mathcal{M}_{r,n}^T$. Finally, we prove Theorem 1.1. Section 3 contains the proof of Theorem 1.4.

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2. Moduli space of sheaves on $\mathbb{P}^2$

2.1. Quiver description of $\mathcal{M}_{r,n}$. The variety $\mathcal{M}_{r,n}$ has the following quiver description (see e.g. [5]).

$$\mathcal{M}_{r,n} \cong \left\{ (B_1, B_2, i, j) \left| \begin{array}{c} 1) B_1, B_2 ] + i j = 0 \\ 2) (\text{stability}) \quad \text{There is no subspace} \\ S \subseteq \mathbb{C}^n \text{ such that } B_\alpha(S) \subseteq S (\alpha = 1, 2) \\ \text{and } \text{im}(i) \subseteq S \end{array} \right. \right\} / GL_n(\mathbb{C}),$$

where $B_1, B_2 \in \text{End}(\mathbb{C}^n), i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and $j \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ with the action given by

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, gj^{-1})$$

for $g \in GL_n(\mathbb{C})$.

In the quiver description the map $\psi: \mathcal{M}_{r,n} \to \mathcal{M}_{r+1,n}$ is induced by the coordinate embedding of $\mathbb{C}^r$ into $\mathbb{C}^{r+1}$.

2.2. Irreducible components of $\mathcal{M}_{r,n}^T$. In terms of Section 2.1 the $T$-action on $\mathcal{M}_{r,n}$ is given by (see e.g. [6])

$$(t_1, t_2) \cdot [(B_1, B_2, i, j)] = [(t_1B_1, t_2B_2, i, t_1t_2j)].$$

By definition, $[(B_1, B_2, i, j)] \in \mathcal{M}_{r,n}$ is a fixed point if and only if there exists a homomorphism $\lambda: T \to GL_n(\mathbb{C})$ satisfying the following
We see that the numbers \( \dim \mathcal{M} \) conditions:
\[
\begin{align*}
  t_1 B_1 &= \lambda(t)^{-1} B_1 \lambda(t), \\
  t_2 B_2 &= \lambda(t)^{-1} B_2 \lambda(t), \\
  i &= \lambda(t)^{-1} i, \\
  t_1 t_2 j &= j \lambda(t).
\end{align*}
\]
(2)

Suppose that \([(B_1, B_2, i, j)]\) is a fixed point. Then we have the weight decomposition of \( \mathbb{C}^n \) with respect to \( \lambda(t) \), i.e. \( \mathbb{C}^n = \bigoplus_{k,l} V_{k,l} \), where \( V_{k,l} = \{ v \in \mathbb{C}^n | \lambda(t) \cdot v = t_1^k t_2^l v \} \). From the conditions (2) it follows that the only components of \( B_1, B_2, i \) and \( j \) which might survive are
\[
\begin{align*}
  B_1 &: V_{k,l} \rightarrow V_{k-1,l}, \\
  B_2 &: V_{k,l} \rightarrow V_{k,l-1}, \\
  i &: \mathbb{C}^r \rightarrow V_{0,0}, \\
  j &: V_{1,1} \rightarrow \mathbb{C}^r.
\end{align*}
\]

From the stability condition it follows that
\[
\begin{align*}
  V_{k,l} &= 0, \text{ if } k > 0 \text{ or } l > 0, \\
  j &= 0, \\
  \dim V_{0,0} &\leq r, \\
  \dim V_{k,l} &\geq \dim V_{k-1,l}, \\
  \dim V_{k,l} &\geq \dim V_{k,l-1}.
\end{align*}
\]

We see that the numbers \( (\dim V_{k-1,l})_{k,l \geq 0} \) form a plane partition.

Let \( (\pi_{i,j})_{i,j \geq 0} \) be a plane partition such that \( |\pi| = n \) and \( \pi_{0,0} \leq r \). Let \( \mathcal{M}_{r,n}^T(\pi) \) be the subset of points from \( \mathcal{M}_{r,n}^T \) such that \( \dim V_{k-1,l} = \pi_{k,l} \). It is easy to see that \( \mathcal{M}_{r,n}^T(\pi) \) is a closed subvariety of \( \mathcal{M}_{r,n}^T \) and
\[
\mathcal{M}_{r,n}^T = \bigoplus_{\pi \in \mathcal{P}} \mathcal{M}_{r,n}^T(\pi).
\]

We see that the variety \( \mathcal{M}_{r,n}^T(\pi) \) has the following quiver description. Let \( V_{-k,-l} = \mathbb{C}^{\pi_{k,l}} \). Then
\[
\mathcal{M}_{r,n}^T(\pi) \cong \left\{ \left. (B_1,k,l \geq 0, (B_2,k,l)_{k,l \geq 0}, i) \right| \begin{aligned}
  &1) B_{1,k,l+1} B_{2,k,l} = B_{2,k+1,l} B_{1,k,l}, \\
  &2) B_{1,k,l}, B_{2,k,l}, i \text{ are surjective} \end{aligned} \right\} / \prod_{k,l} GL_{\pi_{k,l}},
\]
where \( B_{1,k,l} \in Hom(V_{-k,-l}, V_{-k-1,l}), B_{2,k,l} \in Hom(V_{-k,-l}, V_{-k-1,l-1}) \) and \( i \in Hom(\mathbb{C}^r, V_{0,0}) \) (see Figure 2).

From Theorem 1.1 it follows that the class of \( \mathcal{M}_{r,n}^T(\pi) \) is a polynomial in \( \mathbb{L} \). The coefficient of \( \mathbb{L}^i \) in this polynomial is equal to the \( 2i \)-th Betti number of \( \mathcal{M}_{r,n}^T(\pi) \). By Theorem 1.1 the coefficient of \( \mathbb{L}^0 \) is equal to 1, hence the variety \( \mathcal{M}_{r,n}^T(\pi) \) is irreducible.
Let and that the variety \( \varphi \) be the Grassmanian of \( 1 \leq i \leq k-1 \), where \( h_i \in Hom(\mathbb{C}^{\mu_i}, \mathbb{C}^{\nu_i+1}) \) are surjective homomorphisms. We consider the variety \( N_{\mu,\nu,h} \) defined by

\[
N_{\mu,\nu,h} = \left\{ (f_i)_{1 \leq i \leq k-1}, (g_i)_{1 \leq i \leq k} \mid \prod_{i=1}^{k-1} (f_i)_{1 \leq i \leq k} \right\},
\]

where \( f_i \in Hom(\mathbb{C}^{\mu_i}, \mathbb{C}^{\mu_{i+1}}) \) and \( g_i \in Hom(\mathbb{C}^{\nu_i}, \mathbb{C}^{\nu_i}) \). It is easy to see that the variety \( N_{\mu,\nu,h} \) is smooth.

**Lemma 2.1.**

\[
[N_{\mu,\nu,h}] = \frac{[\mu_1]! L[GL_{\mu_1}]}{[\nu_1]! [\mu_1 - \nu_1]! L \prod_{i=1}^{k-1} [\mu_i - \nu_{i+1}]!} \frac{[\mu_i]! L[GL_{\mu_{i+1}}]}{[\nu_i]! [\mu_i - \nu_{i+1}]! L \prod_{i=1}^{k-1} [\mu_i - \nu_{i+1}]!}
\]

**Proof.** The proof is by induction on \( k \). Suppose \( k = 1 \), then \( N_{\mu,\nu,h} \) is the variety of \( \mu_1 \times \nu_1 \)-matrices of the maximal rank. The class of this variety is equal to \( \frac{[\mu_1]! L[GL_{\mu_1}]}{[\nu_1]! [\mu_1 - \nu_1]! L} \).

Suppose \( k \geq 2 \). Obviously, we have \( Ker(f_{k-1}) \subset Ker(h_{k-1}g_{k-1}) \).

Let \( Gr_M(V) \) be the Grassmanian of \( M \)-dimensional vector subspaces in a vector space \( V \). Let \( h' = (h_1, h_2, \ldots, h_{k-2}), \mu' = (\mu_1, \mu_2, \ldots, \mu_{k-1}) \) and \( \nu' = (\nu_1, \nu_2, \ldots, \nu_{k-1}) \). Let

\[
N'_{\mu',\nu',h'} = \left\{ (f', g') \mid (f', g') \in N_{\mu',\nu',h'}, v \in Gr_{\mu_{k-1} - \mu_k}(Ker(h_{k-1}g_{k-1})) \right\}.
\]

We define the map \( \phi: N_{\mu,\nu,h} \to N'_{\mu',\nu',h'} \) by the following formula

\[
((f_i)_{1 \leq i \leq k-1}, (g_i)_{1 \leq i \leq k}) \mapsto ((f_i)_{1 \leq i \leq k-2}, (g_i)_{1 \leq i \leq k-1}, Ker(f_{k-1})).
\]

It is easy to see that the map \( \phi \) is a locally trivial bundle with the variety \( GL_{\mu_k} \) as the fiber. Therefore, we have

\[
[N_{\mu,\nu,h}] = [N'_{\mu',\nu',h'} \cdot GL_{\mu_k}] = [N'_{\mu',\nu',h'}] [Gr_{\mu_{k-1} - \mu_k}(\mathbb{C}^{\mu_{k-1} - \nu_k})] [GL_{\mu_k}] =
\]

\[
= [N'_{\mu',\nu',h'}] \frac{[\mu_{k-1} - \nu_k]! L [GL_{\mu_k}]}{[\mu_{k-1} - \mu_k]! [\mu_k - \nu_k]! L}.
\]

**Figure 2.** The quivers description of \( M_{r,n}^T(\pi) \)
By the induction hypothesis this is equal to the right-hand side of (3). \(\square\)

It is easy to see that the varieties \(N_{\mu,\nu,h}\) are isomorphic for different choices of the maps \(h_i\). Let

\[
N(\pi) = \left\{ ((B_{1,i,j})_{i,j\geq 0}, (B_{2,i,j})_{i,j\geq 0}) \mid \begin{array}{l}
1)B_{1,i,j+1}B_{2,i,j} = B_{2,i,j+1}B_{1,i,j}\\
2)B_{2,i,j} \text{ are surjective}
\end{array} \right\},
\]

where \(B_{1,i,j} \in \text{Hom}(V_{i-j,j}, V_{i+1-j,j})\), \(B_{2,i,j} \in \text{Hom}(V_{i-j,j}, V_{i-1-j,j})\).

Theorem 1.1 is equivalent to the following equation

\[
(4) \quad [N(\pi)] = \frac{[\pi_{0,0}]!_L}{[GL_{\pi_{0,0}}]} \prod_{i,j \geq 0} \frac{[\pi_{i,j} - \pi_{i+1,j+1}]!_L [GL_{\pi_{i,j}}]}{[\pi_{i,j} - \pi_{i+1,j}]!_L [\pi_{i,j} - \pi_{i,j+1}]!_L}.
\]

Let \(l\) be the number of rows in the support of \(\pi\). The proof of (4) is by induction on \(l\). Suppose \(l = 0\), then (4) is obvious. Suppose \(l \geq 1\). Let \(\tilde{\pi}\) be the plane partition defined by \(\tilde{\pi}_{i,j} = \pi_{i,j+1}\). Consider a point \(((B_{1,i,j}), (B_{2,i,j})) \in N(\tilde{\pi})\). If we forget the maps \(B_{1,i,0}\) and \(B_{2,i,0}\), then we obtain a point from \(N(\tilde{\pi})\). This defines the map \(\rho: N(\tilde{\pi}) \to N(\pi)\). It is easy to see that the map \(\rho\) is a locally trivial bundle. The fiber over a point \(((B_{1,i,j}), (B_{2,i,j})) \in N(\tilde{\pi})\) is the variety \(N_{\mu,\nu,h}\), where \(\mu_i = \pi_{i,0}, \nu_i = \pi_{i,1}\) and \(h_i = B_{1,i,0}\). Using the induction hypothesis and Lemma 2.1, we get

\[
[N(\pi)] = \left( \frac{[\pi_{0,1}]!_L}{[GL_{\pi_{0,1}}]} \prod_{i,j \geq 1} \frac{[\pi_{i,j} - \pi_{i+1,j+1}]!_L [GL_{\pi_{i,j}}]}{[\pi_{i,j} - \pi_{i+1,j}]!_L [\pi_{i,j} - \pi_{i,j+1}]!_L} \right) \times
\]

\[
\times \frac{[\pi_{0,0}]!_L [GL_{\pi_{0,1}}]}{[\pi_{0,1}]!_L [\pi_{0,0} - \pi_{0,1}]!_L} \prod_{i \geq 0} \frac{[\pi_{i,0} - \pi_{i+1,0}]!_L [GL_{\pi_{i+1,0}}]}{[\pi_{i,0} - \pi_{i+1,1}]!_L [\pi_{i+1,0} - \pi_{i+1,1}]!_L} =
\]

\[
= \frac{[\pi_{0,0}]!_L}{[GL_{\pi_{0,0}}]} \prod_{i,j \geq 0} \frac{[\pi_{i,j} - \pi_{i+1,j+1}]!_L [GL_{\pi_{i,j}}]}{[\pi_{i,j} - \pi_{i+1,j}]!_L [\pi_{i,j} - \pi_{i,j+1}]!_L}.
\]

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.4

Let \(T_{a,b} = \{(t^a, t^b) \in T | t \in \mathbb{C}^*\}\), be a one dimensional subtorus of \(T\). Consider the \(T_{1,\alpha}\)-action on \(\mathcal{M}_{r,n}\), where \(\alpha\) is a positive integer. Suppose \(\alpha\) is big enough. Then the set of fixed points of the \(T_{1,\alpha}\)-action coincides with the set of fixed points of the \(T\)-action on \(\mathcal{M}_{r,n}\). We define the map \(\rho: \mathcal{M}_{r,n} \to \mathcal{M}_{r,n}^T\) by the following formula \(\rho(p) = \lim_{t \to 0} t \cdot p\), where \(p \in \mathcal{M}_{r,n}\) and \(t \in T_{1,\alpha}\). Therefore, we have

\[
\mathcal{M}_{r,n} = \bigsqcup_{\pi_0,0 \leq r} \rho^{-1}(\mathcal{M}_{r,n}^T(\pi)).
\]

From [1, 2] it follows that \(\rho^{-1}(\mathcal{M}_{r,n}^T(\pi))\) is a locally closed subvariety such that the map \(\rho^{-1}: (\mathcal{M}_{r,n}^T(\pi)) \to \mathcal{M}_{r,n}^T(\pi)\) is a locally trivial bundle.
with an affine space as the fiber. We denote by $d^+_{i, \alpha}(\pi)$ the dimension of the fiber. Therefore, we have

$$[\mathcal{M}_{r,n}] = \sum_{\pi \in P_{\pi_{0,0} \leq r}} [\mathcal{M}^T_{r,n}(\pi)] \mathcal{L}^{d^+_{i, \alpha}(\pi)}.$$  \hspace{1cm} (5)

Let us prove that

$$d^+_{i, \alpha}(\pi) = rn + \sum_{i,j \geq 0} \pi_{i,j}(\pi_{i,j} - \pi_{i,j+1}).$$ \hspace{1cm} (6)

The $T$-action on $\mathcal{M}_{r,n}$ is the part of the action of the $(r + 2)$-dimensional torus $T \times (\mathbb{C}^*)^r$. In terms of Section 2.1 this action is given by (see e.g. [6])

$$(t_1, t_2, e_1, e_2, \ldots, e_r) \cdot [(B_1, B_2, i, j)] = [(t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e j)],$$

where $e = \text{diag}(e_1, e_2, \ldots, e_r)$ is the diagonal $r \times r$-matrix.

The set of fixed points of the $T \times (\mathbb{C}^*)^r$-action is finite and is parametrized by the set of $r$-tuples $D = (D_1, D_2, \ldots, D_r)$ of Young diagrams $D_i$, such that $\sum_{i=1}^r |D_i| = n$ (see e.g. [3]). It is easy to see that the fixed point corresponding to an $r$-tuple $D$ belongs to $\mathcal{M}^T_{r,n}(\pi)$, where $\pi_{i,j} = |\{\alpha \mid (i, j) \in D \alpha\}|$.

For a Young diagram $Y$ let

$$r_l(Y) = |\{(i, j) \in D \mid j = l\}|,$$
$$c_l(Y) = |\{(i, j) \in D \mid i = l\}|.$$

For a point $s = (i, j) \in \mathbb{Z}_{\geq 0}^2$ let

$$l_Y(s) = r_j(Y) - i - 1,$$
$$a_Y(s) = c_i(Y) - j - 1,$$

see Figure 3. Note that $l_Y(s)$ and $a_Y(s)$ are negative if $s \notin Y$.

![Figure 3](image)

Let $p$ be the fixed point of the $T \times (\mathbb{C}^*)^r$-action corresponding to an $r$-tuple $D$. Let $R(T \times (\mathbb{C}^*)^r) = \mathbb{Z}[t_1, t_2, e_1, e_2, \ldots, e_r]$ be the representation ring of $T \times (\mathbb{C}^*)^r$. Then the weight decomposition of $T_p(\mathcal{M}_{r,n})$ is given
by (see e.g. [6])

\[ T_p(\mathcal{M}_{r,n}) = \sum_{i,j=1}^{r} e_{j} e_{i}^{-1} \left( \sum_{s \in D_i} l_{1}^{-l_{D_i}(s)} l_{2}^{a_{D_i}(s)+1} + \sum_{s \in D_j} l_{1}^{l_{D_j}(s)+1} l_{2}^{-a_{D_j}(s)} \right). \]

Let \( p \) be an arbitrary fixed point of the \( T \times (\mathbb{C}^*)^r \)-action on \( \mathcal{M}^T_{r,n}(\pi) \).

Let \( D \) be the corresponding \( r \)-tuple of Young diagrams. From (7) it follows that

\[
d_{i,a}(\pi) = \sum_{i,j=1}^{r} \left( |\{ s \in D_i | \alpha(a_{D_i}(s) + 1) - l_{D_j}(s) > 0 \}| + |\{ s \in D_j | - \alpha a_{D_j}(s) + l_{D_i}(s) + 1 > 0 \}| \right).
\]

Since \( \alpha \) is big, the right-hand side is equal to

\[n + \sum_{i,j=1}^{r} |\{ s \in D_j | a_{D_j}(s) = 0, s \in D_i \}| = rn + \sum_{i,j \geq 0} \pi_{i,j}(\pi_{i,j} - \pi_{i,j+1}).
\]

Thus, we have proved (5).

It is well known that

\[
\sum_{n \geq 0} [\mathcal{M}_{r,n}] t^n = \prod_{m=1}^{r} \prod_{n \geq 1} \frac{1}{1 - L^{r_m + m} t^n},
\]

see e.g. [7]. If we combine Theorem 1.1 and the equations (5), (6) and (8), we obtain the proof of Theorem 1.4.

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