BRAID GROUP ACTIONS OF QUANTUM BORCHERDS-BOZEC ALGEBRAS

ZHAOBING FAN AND BOLUN TONG

Abstract. In this paper, we construct the Lusztig symmetries for quantum Borcherds-Bozec algebra $U_q(g)$ and its weight module $M \in \mathcal{O}$, on which the generators with real indices of $U_q(g)$ act nilpotently. We show that these symmetries satisfy the defining relations of the braid group, associated to the Weyl group $W$ of $U_q(g)$, which gives a braid group action.

Introduction

Lusztig considered in [L90] certain perverse sheaves on representation varieties of a quiver of type A,D,E, and gave a geometric approach to the half parts of corresponding quantum groups. The canonical basis theory arose in this setting, which is given by simple perverse sheaves. Moreover, the algebraic and coalgebraic structures of the quantum groups are given by induction functor and restriction functor, respectively. These results were generalized by Lusztig in [L91] to Kac-Moody cases. Later on, Lusztig consider more general cases, namely arbitrary quivers, possibly carrying loops in [L93]. The obtained algebra is denoted by $U^-$. In [L93], Lusztig proposed a question that if $U^-$ is generated by the elementary simple perverse sheaves $F_i^{(n)}$ with all vertices $i$ and $n \in \mathbb{N}$ as an algebra. The question is answered by himself in the case of the quiver with one vertex and multiple loops, by a quadratic form criterion for a monomial to be tight or semi-tight. Based on Kang and Schiffmann’s work for quantum generalized Kac-Moody algebras (cf. [KS06]), Li and Lin [LL09] answered the question when the quiver has at least two loops on each imaginary vertex. In [B15], Bozec solved the Lusztig’s question completely. As a bialgebra, the resulting $U^-$ is so called quantum Borcherds-Bozec algebra, which is the main object we studied in the current paper.

2010 Mathematics Subject Classification. 17B37, 17B67, 16G20.
Key words and phrases. quantum Borcherds-Bozec algebra, braid group action, PBW-basis, primitive generator.
On algebraic side, the quantum Borcherds-Bozec algebras can be treated as a further generalization of quantum generalized Kac-Moody algebras [K95]. More precisely, a quantum Borcherds-Bozec algebra \( U_q(\mathfrak{g}) \) has infinitely many generators \( e_{il}, f_{il} \) \( (l \in \mathbb{Z}_{>0}) \) for each imaginary index \( i \in \mathcal{I}_{im} \), and their degrees are \( l \) multiples of \( \alpha_i \) and \( -\alpha_i \), respectively. The commutation relations between these generators are rather complicated and are higher order in some sense (cf. [FKKT20]). Thanks to Bozec, there exists a set of primitive generators \( a_{il}, b_{il} \) \( ((i, l) \in \mathcal{I}^\infty) \) with better properties and simpler commutation relations. Using these generators, Bozec constructed the Kashiwara operators. He then developed the crystal basis theory for quantum Borcherds-Bozec algebras and their irreducible highest weight modules [B16]. In [FKKT21], the authors and Kang and Kim constructed the Global bases for the quantum Borcherds-Bozec algebras.

Lusztig defined in [L10] the symmetries \( T_{i,e}^l, T_{i,e}^m \) \( (e = \pm 1) \) for the quantum group \( U \) of Kac-Moody type and its integrable weight modules. He proved that these automorphisms \( T_{i,e}^l, T_{i,e}^m \) satisfy the braid group relations both on \( U \) and integrable \( U \)-modules. As an important application, he gave some linearly independent subsets of \( U \) associated to reduced expressions of elements in the Weyl group \( W \). In the case of finite types, if one choose the longest element \( \omega_0 \in W \), the independent set considered form the PBW-basis. He also shown how to extend as much as possible this construction to arbitrary Cartan data, especially for affine cases.

In this paper, we shall define the Lusztig symmetries for quantum Borcherds-Bozec algebra \( U_q(\mathfrak{g}) \). Note that, the Weyl group of \( U_q(\mathfrak{g}) \) is generated by the reflections associated to real indexes, it is not enough to construct the PBW-type basis as Lusztig did. But since the primitive generators \( a_{il} \) and \( b_{il} \) satisfy the ‘higher order’ quantum Serre relations for \( i \in \mathcal{I}_{im} \), we could define the Lusztig symmetries in a natural way. In the case where \( I \) consists of exact two real indexes \( i \neq j \) and \( a_{ij}a_{ji} \) is finite, Lusztig proved that there is a braid group action on the integral modules (cf. [L10]) through the symmetries. In Theorem 2.6, we prove that it can be generalized to an arbitrary Borcherds-Cartan datum by using our constructed symmetries, which give the braid group actions on \( U_q(\mathfrak{g}) \) and \( U_q(\mathfrak{g}) \)-module \( M \) in a certain category \( \mathcal{O} \).

In Lusztig’s construction for PBW-basis, a crucial result is that the inner product of \( U^+ \) is \( T_{i,j}^{\pm \alpha_i} \)-invariant on the subalgebra \( U^+[i] \), which is generated by a set of elements of the forms similar to the Serre-type relations but allows for smaller degrees. To verify this in quantum Borcherds-Bozec algebra case (Theorem 3.5), we follow the framework given in [J95, Chapter 8A] rather that Section 38.2 in [L10], since we have a more general setting for the values of our bilinear form. We make a notice here that, as a consequence of Theorem 3.5, one could get a lot of linearly independent subsets of \( U_q^+(\mathfrak{g}) \) as in [L10, 38.2.2] by a similar argument.
This paper is organized as follows. In Section 1, we review the definition of quantum Borcherds-Bozec algebras and the notion of the primitive generators. In Section 2, we define the Lusztig symmetries $L'_{i,e}, L''_{i,e}$ ($e = \pm 1$) for these algebras and their weight module $M \in \mathcal{O}$, and prove the braid group actions on them. In Section 3, we investigate the relations between the symmetries and the bilinear form $\{ , \}$ on $U_q^+(g) \times U_q^-(g)$.

Acknowledgements.

Z. Fan was partially supported by the NSF of China grant 11671108, the NSF of Heilongjiang Province grant JQ2020A001, and the Fundamental Research Funds for the central universities. We would like to express our sincere gratitude to Professor Seok-Jin Kang for his helpful discussions.

1. Quantum Borcherds-Bozec Algebras

Let $I$ be a finite or countably infinite index set. An integer-valued matrix $A = (a_{ij})_{i,j \in I}$ is called an even symmetrizable Borcherds-Cartan matrix if it satisfies the following conditions:

(i) $a_{ii} = 2, 0, -2, -4, \ldots$,
(ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
(iii) there is a diagonal matrix $D = \text{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $DA$ is symmetric.

Let $I^r = \{ i \in I \mid a_{ii} = 2 \}$ be the set of real indices. Let $I^{\text{im}} = \{ i \in I \mid a_{ii} \leq 0 \}$ and $I^{\text{iso}} = \{ i \in I \mid a_{ii} = 0 \}$ be the set of imaginary indices and isotropic indices, respectively.

A Borcherds-Cartan datum consists of

(a) an even symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$,
(b) a free abelian group $P^\vee = (\bigoplus_{i \in I} \mathbb{Z}h_i) \oplus (\bigoplus_{i \in I} \mathbb{Z}d_i)$, the dual weight lattice,
(c) $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$, the Cartan subalgebra,
(d) $P = \{ \lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subseteq \mathbb{Z} \}$, the weight lattice,
(e) $\Pi^\vee = \{ h_i \in P^\vee \mid i \in I \}$, the set of simple coroots,
(f) $\Pi = \{ \alpha_i \in P \mid i \in I \}$, the set of simple roots, which is linearly independent over $\mathbb{Q}$ and satisfies

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_i) = \delta_{ij} \quad \text{for all } i, j \in I,$$

(g) for each $i \in I$, there is an element $\Lambda_i \in P$, called the fundamental weight, defined by

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d_j) = 0 \quad \text{for all } i, j \in I.$$

We denote by $P^+$ the set $\{ \lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \}$ of dominant integral weights. The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}h_i$ is called the root lattice. Set $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $Q_- = -Q_+$. For $\beta = \sum_{i \in I} k_i \alpha_i \in Q_+$, we define its height to be $|\beta| = \sum_{i \in I} k_i$. 


There is a non-degenerate symmetric bilinear form \((\ , \ )\) on \(\mathfrak{h}^*\) satisfying
\[
(\alpha_i, \lambda) = s_i \lambda(h_i), \ (\Lambda_i, \lambda) = s_i \lambda(d_i) \quad \text{for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I,
\]
and therefore we have
\[
(\alpha_i, \alpha_j) = s_i a_{ij} = s_j a_{ji} \quad \text{for all } i, j \in I.
\]

For \(i \in I^e\), we define the simple reflection \(r_i \in GL(\mathfrak{h}^*)\) by
\[
r_i(\lambda) = \lambda - \lambda(h_i) \alpha_i \quad \text{for } \lambda \in \mathfrak{h}^*.
\]
The subgroup \(W\) of \(GL(\mathfrak{h}^*)\) generated by \(r_i \ (i \in I^e)\) is called the Weyl group of the Borcherds-Cartan datum. Note that the symmetric bilinear form \((\ , \ )\) is \(W\)-invariant.

\(W\) is a coxeter group generated by \(r_i \ (i \in I^e)\) with defining relations \(r_i^2 = 1 \ (i \in I^e)\) and \((r_i r_j)^{m_{ij}} = 1 \ (i \neq j)\), where \(m_{ij}\) is the order of \(r_i r_j\) and is related to \(a_{ij}\) as follows:

| \(a_{ij} a_{ji}\) | 0 | 1 | 2 | 3 | 4 \(\geq 4\)
|---|---|---|---|---|---|
| \(m_{ij}\) | 2 | 3 | 4 | 6 | \(\infty\)

We also use \(r_i \ (i \in I^e)\) to denote the automorphism of \(P^\vee\) given by
\[
r_i(h) = h - \alpha_i(h) h_i \quad \text{for all } h \in P^\vee.
\]
Let \(i = (i_1, i_2, \ldots, i_N)\) be a sequence in \(I^e\). Note that for any \(\lambda \in \mathfrak{h}^*\) and \(h \in P^\vee\), we have
\[
\lambda(r_{i_1} r_{i_2} \cdots r_{i_N}(h)) = (r_{i_N} r_{i_{N-1}} \cdots r_{i_1}(\lambda))(h).
\]

Let \(I^\infty = (I^e \times \{1\}) \cup (I^m \times \mathbb{Z}_{>0})\). For simplicity, we will often write \(i\) instead of \((i, 1)\) when \(i \in I^e\). Let \(q\) be an indeterminate, and set for each \(i \in I\)
\[
q_i = q^{s_i}, \ q(i) = q^{\frac{(\alpha_i, \alpha_i)}{2}}.
\]
For \(i \in I^e\) and \(n \in \mathbb{Z}_{\geq 0}\), we define
\[
[n]_i = \frac{q^n_i - q_i^{-n}}{q^i - q^{-i}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \binom{n}{k}_i = \frac{[n]_i!}{[k]_i! [n-k]_i!}.
\]

Let \(\mathcal{E} = \mathbb{Q}(q) \langle e_{il} \mid (i, l) \in I^\infty\rangle\) be the free associative algebra over \(\mathbb{Q}(q)\) generated by the symbols \(e_{il}\) for \((i, l) \in I^\infty\). By setting \(\deg e_{il} = l \alpha_i\), \(\mathcal{E}\) becomes a \(Q_+\)-graded algebra. For a homogeneous element \(u\) in \(\mathcal{E}\), we denote by \(|u|\) the degree of \(u\), and for any \(A \subseteq \mathbb{Q}_+\), set \(\mathcal{E}_A = \{ x \in \mathcal{E} \mid |x| \in A\}\).

We define a twisted multiplication on \(\mathcal{E} \otimes \mathcal{E}\) by
\[
(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{(|x_2|, |y_1|)} x_1 y_1 \otimes x_2 y_2
\]
for all homogeneous elements $x_1, x_2, y_1, y_2 \in \mathcal{E}$, and equip $\mathcal{E}$ with a comultiplication $\rho$ defined by

$$\rho(e_{il}) = \sum_{m+n=l} q_{il}^{mn} e_{im} \otimes e_{in} \text{ for } (i, l) \in I^\infty.$$  

Here, we set $e_{i0} = 1$, and $e_{il} = 0$ for $l < 0$.

**Proposition 1.1.** [B15, B16] For a family $\nu = (\nu_{il})_{(i,l) \in I^\infty}$ of non-zero elements in $\mathbb{Q}(q)$, there exists a symmetric bilinear form $\{ , \} : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{Q}(q)$ such that

(a) $\{x, y\} = 0$ if $|x| \neq |y|$, 
(b) $\{1, 1\} = 1$, 
(c) $\{e_{il}, e_{il'}\} = \nu_{il}$ for all $(i, l) \in I^\infty$, 
(d) $\{x, yz\} = \{\rho(x), y \otimes z\}$ for all $x, y, z \in \mathcal{E}$.

Here, $\{x_1 \otimes x_2, y_1 \otimes y_2\} = \{x_1, y_1\}\{x_2, y_2\}$ for any $x_1, x_2, y_1, y_2 \in \mathcal{E}$.

Let $\mathcal{C}_n$ be the set of compositions $c$ of $n$, and $e_{i,c} = e_{ic_1} \cdots e_{ic_m}$ for each $i \in I^m$, $c = (c_1, \ldots, c_m) \in \mathcal{C}_n$. It is clear that $\{e_{i,c} \mid c \in \mathcal{C}_n\}$ form a basis of $\mathcal{E}_{\alpha_i}$.

Assume that $i, j \in I^e$, $j \in I$ and $i \neq j$. Let $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 0}$ with $m > -a_{ij} n$, then for any $c \in \mathcal{C}_n$, the following element of $\mathcal{E}'$ belongs to the radical $\mathcal{J}$ of the form $\{ , \}$

$$F_{i,j,n,m,c,\pm 1} = \sum_{r+s=m} (-1)^r q_i^{\pm r(-a_{ij} n - m + 1)} e_i^{(r)} e_j c_{i,e}^{(s)}$$

Here, if $j \in I^e$, we set $e_{j,c} = e_j^{(n)}$ as the divided power of $e_j$.

Moreover, if $(i, k), (j, l) \in I^\infty$ such that $a_{ij} = 0$, one can show that the element $e_{ik} e_{jl} - e_{jl} e_{ik}$ belongs to $\mathcal{J}$.

From now on, we assume that

$$\nu_{il} \in 1 + q^{-1} \mathbb{Z}_{\geq 0}[[q^{-1}]] \text{ for all } (i, l) \in I^\infty.$$  

Under this assumption, the bilinear form $\{ , \}$ is non-degenerate on $\mathcal{E}(i) = \bigoplus_{l \geq 1} \mathcal{E}_{\alpha_i}$ for $i \in I^m \setminus I^{iso}$. Moreover, its radical is generated by a simpler set consisting of

$$\sum_{r+s=1-l a_{ij}} (-1)^r e_i^{(r)} e_{jl} c_{i,e}^{(s)} \text{ for } i \in I^e, (j, l) \in I^\infty \text{ and } i \neq (j, l),$$

and $e_{ik} e_{jl} - e_{jl} e_{ik}$ for all $(i, k), (j, l) \in I^\infty$ with $a_{ij} = 0$ (cf. [B15, Proposition 14]).

**Definition 1.2.** Given a Borcherds-Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$, the quantum Borcherds-Bozec algebra $U_q(\mathfrak{g})$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by the elements
$q^h$ ($h \in P^\vee$) and $e_{il}, f_{il}$ ($(i, l) \in I^\infty$), subjecting to

\begin{align*}
q^0 &= 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee, \\
q^h e_{jl} q^{-h} &= q^{\lambda_j(h)} e_{jl}, \quad q^h f_{jl} q^{-h} = q^{-\lambda_j(h)} f_{jl} \quad \text{for } h \in P^\vee, (j, l) \in I^\infty, \\
\sum_{r+s=1-\lambda_{ij}} (-1)^r e_i^{(r)} e_j^{(s)} &= 0 \quad \text{for } i \in I^e, (j, l) \in I^\infty \text{ and } i \neq (j, l), \\
\sum_{r+s=1-\lambda_{ij}} (-1)^r f_i^{(r)} f_j^{(s)} &= 0 \quad \text{for } i \in I^e, (j, l) \in I^\infty \text{ and } i \neq (j, l), \\
e_{ik} e_{jl} - e_{jl} e_{ik} &= f_{ik} f_{jl} - f_{jl} f_{ik} = 0 \quad \text{for } a_{ij} = 0, \\
e_{ik} f_{jl} &= f_{jl} e_{ik} \quad \text{for } (i, k), (j, l) \in I^\infty \text{ and } i \neq j, \\
\sum_{m+n=k \atop n+s=l} q^{n(m-s)}_{i(i)} \nu_{im} K^{-n}_i f_{im} e_{is} &= \sum_{m+n=k \atop n+s=l} q^{n(s-m)}_{i(i)} \nu_{im} K^n_i e_{im} f_{is} \quad \text{for } (i, k), (i, l) \in I^\infty.
\end{align*}

Here, $K_i = q^h_i$ for all $i \in I$. We extend the grading by setting $|q^h| = 0$ and $|f_{il}| = -\lambda_i$. For each $\beta = \sum n_i \alpha_i \in Q$, we set $K_\beta = \prod_i K_i^{n_i}$.

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_{il}$ (resp. $f_{il}$) for $(i, l) \in I^\infty$, and $U_q^0(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $q^h$ for $h \in P^\vee$. Then the quantum Borcherds-Boec algebra $U_q(\mathfrak{g})$ has the triangular decomposition

$$U_q(\mathfrak{g}) \cong U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).$$

We shall denote by $U$ (resp. $U^+$ and $U^-$) for $U_q(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$) for simplicity.

The algebra $U$ is endowed with a comultiplication $\Delta : U \to U \otimes U$ given by

\begin{align}
\Delta(q^h) &= q^h \otimes q^h, \\
\Delta(e_{il}) &= \sum_{m+n=l} q^{mn}_{i(i)} e_{im} K^n_i \otimes e_{in}, \\
\Delta(f_{il}) &= \sum_{m+n=l} q^{-mn}_{i(i)} f_{im} \otimes K^{-m}_i f_{im}.
\end{align}

We shall give the relations between $\rho : U^+ \to U^+ \otimes U^+$ and $\Delta$. Let $x \in U^+$ be a homogeneous element such that $\rho(x) = \sum x_1 \otimes x_2$, then we have

\begin{align}
\Delta(x) = \sum x_1 K_{|x_2|} \otimes x_2.
\end{align}
If \( y \in U^- \) is a homogeneous element with \( \rho(\omega(y)) = \sum y_1 \otimes y_2 \), where \( \omega \) is the involution of \( U \) such that \( \omega(q^h) = q^{-h} \), \( \omega(e_{il}) = f_{il} \), \( \omega(f_{il}) = e_{il} \) for \( h \in P^\vee \) and \((i,l) \in I^\infty \). We have
\[
\Delta(y) = \sum q^{-(|y_1|+|y_2|)} \omega(y_2) \otimes K_{-|y_2|} \omega(y_1).
\]

**Proposition 1.3.** [B15, B16] For any \( i \in I^\rm{im} \) and \( l \geq 1 \), there exist unique elements \( a_{il} \in U^+_{l\lambda_i} \) and \( b_{il} = \omega(a_{il}) \) such that
\begin{enumerate}
\item \( Q(q) \langle e_{il} | l \geq 1 \rangle = Q(q) \langle a_{il} | l \geq 1 \rangle \) and \( Q(q) \langle f_{il} | l \geq 1 \rangle = Q(q) \langle b_{il} | l \geq 1 \rangle \),
\item \( (a_{il}, z)_L = 0 \) for all \( z \in Q(q) \langle e_{i1}, \cdots, e_{il-1} \rangle \), \( (b_{il}, z)_L = 0 \) for all \( z \in Q(q) \langle f_{i1}, \cdots, f_{il-1} \rangle \),
\item \( a_{il} - e_{il} \in Q(q) \langle e_{ik} | k < l \rangle \) and \( b_{il} - f_{il} \in Q(q) \langle f_{ik} | k < l \rangle \),
\item \( \bar{a}_{il} = a_{il}, \bar{b}_{il} = b_{il} \),
\item \( \rho(a_{il}) = a_{il} \otimes 1 + 1 \otimes a_{il}, \rho(b_{il}) = b_{il} \otimes 1 + 1 \otimes b_{il} \).
\end{enumerate}
Here, \( - : U^\pm \to U^\pm \) is the \( \mathbb{Q} \)-algebra homomorphism defined by \( \tau_{il} = e_{il}, \bar{\tau}_{il} = f_{il} \) and \( \bar{q} = q^{-1} \).

Set \( \tau_{il} = \{a_{il}, a_{il}\} = \{b_{il}, b_{il}\} \). We have the following commutation relations in \( U \) derived from the Drinfeld double process,
\[
a_{il}b_{jk} - b_{jk}a_{il} = \delta_{ij}\delta_{lk}(K_i^{-l} - K_i^l).
\]

The \( a_{il} \)'s and \( b_{il} \)'s are called the **primitive generators** of \( U_q(g) \).

Let \( C_l \) (resp. \( P_l \)) be the set of compositions (resp. partitions) of \( l \). For \( i \in I^\rm{im} \), we define
\[
\mathcal{D}_{i,l} = \begin{cases} 
C_l & \text{if } i \in I^\rm{im} \setminus I^\rm{iso}, \\
P_l & \text{if } i \in I^\rm{iso}.
\end{cases}
\]
and \( \mathcal{D}_i = \bigcup_{j \geq 0} \mathcal{D}_{i,l} \). Let \( c = (c_1, \cdots, c_l) \in \mathcal{D}_{i,l} \), we set
\[
a_{i,c} = a_{i,c_1} \cdots a_{i,c_l}, \quad b_{i,c} = b_{i,c_1} \cdots b_{i,c_l} \quad \text{and} \quad \tau_{i,c} = \tau_{i,c_1} \cdots \tau_{i,c_l}.
\]
Note that \( \{a_{i,c} | c \in \mathcal{D}_{i,l}\} \) forms a basis of \( U^+_{l\lambda_i} \). For each \( i \in I^\vee \), we set \( a_{i1} = e_{i1}, b_{i1} = f_{i1} \), and write \( a_i \) (resp. \( b_i \)) instead of \( a_{i1} \) (resp. \( b_{i1} \)) in this case for simplicity.

**Example 1.4.** \( \lambda \in P_l \) can be written as the form \( \lambda = \lambda_1 2^{\lambda_2} \cdots l^{\lambda_l} \), where \( \lambda_k \) are non-negative integers such that \( \lambda_1 + 2\lambda_2 + \cdots + l\lambda_l = l \). For \( i \in I^\rm{iso} \), we have
\[
a_{il} = e_{il} - \sum_{\lambda \in P_l \setminus \{l\}} \frac{1}{\prod_{i=1}^l \lambda_i!} a_{i,\lambda}.
\]
Applying the involution we also have \( \{ q \in \mathbb{C} \} \). In particular, we see that \( \{ a_i, c_i, c'_i \} = \{ b_i, c_i, c'_i \} = 0 \).

**Example 1.6.** Let \( c = (c_1, c_2, \cdots, c_r) \) be a composition of \( l \), we denote by \( \tilde{c} \) the reverse of \( c \), i.e. \( \tilde{c} = (c_r, c_{r-1}, \cdots, c_1) \). Assume that \( c, c' \) are two compositions of \( l \), which determine the same partition, say it is \( p = (p_1, p_2, \cdots, p_r) \). For \( i \in I^\text{im}\backslash I^\text{iso} \), if \( \{ a_i, c_i, c'_i \} = \gamma \tau_i, p \) for some \( \gamma \in \mathbb{Q}(q) \), then we have

\[
\{ a_i, c_i, c'_i \} = q^{m} \tau_i, p.
\]

Here, \( m = 2 \sum_{r<s} c_r c_s = 2 \sum_{r<s} c_r' c_s' \) and \( - \) be the involution of \( \mathbb{Q}(q) \) mapping \( q \) to \( q^{-1} \). In particular, we see that \( \{ a_i, c_i, c'_i \} = \{ a_i, c_i, c'_i \} \).

**Definition 1.7.** For every \((i, l) \in I^\infty\), we define the linear maps \( \delta_{i,l}, \delta^{i,l} : U^+ \to U^+ \) by

\[
\delta_{i,l}(1) = 0, \quad \delta_{i,l}(a_{jk}) = \delta_{ij} \delta_{lk} \quad \text{and} \quad \delta_{i,l}(xy) = q^{l(y|\alpha_i)} \delta_{i,l}(x) y + x \delta_{i,l}(y),
\]

\[
\delta^{i,l}(1) = 0, \quad \delta^{i,l}(a_{jk}) = \delta_{ij} \delta_{lk} \quad \text{and} \quad \delta^{i,l}(xy) = \delta^{i,l}(x) y + q^{l(x|\alpha_i)} x \delta^{i,l}(y),
\]

for any homogeneous elements \( x, y \) in \( U^+ \).

Let \((i, l) \in I^\infty \) and \( z \in U^+ \), one deduce from the Drinfeld double process that (cf. [B16, Proposition 3.10])

\[
[a_{il}, \omega(z)] = \tau_{il} \{ \omega(\delta_{i,l}(z)) K_i^{l} - K_i^{l} \omega(\delta^{i,l}(z)) \}. \tag{1.7}
\]

Applying the involution \( \omega \) to both sides of (1.7), we obtain the following commutation relation

\[
[b_{il}, z] = \tau_{il} \{ \delta_{i,l}(z) K_i^{l} - K_i^{l} \delta^{i,l}(z) \}. \tag{1.8}
\]

2. Lusztig symmetries and the braid group actions

Let \( U_q(\mathfrak{g}) \) be a quantum Borcherds-Bozec algebra, we denote by \( \mathcal{O} \) the category of \( U_q(\mathfrak{g}) \)-module \( M \) satisfying the following two conditions

(i) \( M \) has a weight space decomposition

\[
M = \bigoplus_{\mu \in P} M_\mu, \quad \text{where} \quad M_\mu = \{ z \in M \mid q^h z = q^{\mu(h)} z \},
\]

(ii) \( a_i \) and \( b_i \) act locally nilpotent on \( M \) for all \( i \in I^\infty \).
For \( i \in I^e \), we set \( B_i = b_i/\tau_i(q_i^{-1} - q_i) \), then the commutation relation between \( a_i \) and \( B_i \) is given by

\[
a_i B_i - B_i a_i = (K_i - K_i^{-1})/(q_i - q_i^{-1}).
\]

Since every element \( M \) in \( O \) is a weight module by (i), so it has a direct sum decomposition as follow

\[
M = \bigoplus_{n \in \mathbb{Z}} M^n, \text{ where } M^n = \{ z \in M \mid K_i z = q_i^nz \}.
\]

We see that \( a_i(M^n) \subseteq M^{n+2}, B_i(M^n) \subseteq M^{n-2} \) for all \( n \), and \( a_iB_i - B_i a_i \) acts on \( M^n \) is the multiplication by \( [n]_q \) for all \( n \), these properties yield \( M \) becomes an object of the category \( C_i' \) defined in [L10, 5.1.1]. Hence the standard argument in [L10, Chapter 5] can be applied to our case. We define the \( \mathbb{Q}(q) \)-linear maps \( L'_{i,e}, L''_{i,e} : M \to M \) for \( i \in I^e \) and \( M \in O \) by

\[
L'_{i,e}(z) = \sum_{a,b,c:a-b+c=n} (-1)^b q_i^e(-ac+b)B_i^a a_i^{(b)} B_i^{(c)} z,
\]

\[
L''_{i,e}(z) = \sum_{a,b,c:a-b+c=n} (-1)^b q_i^e(-ac+b) B_i^{(a)} a_i^{(b)} B_i^{(c)} z,
\]

where \( z \in M^n \) and \( e = \pm 1 \). \( L'_{i,e}, L''_{i,e} \) are called symmetries on \( M \). According to [L10, 5.2.3], we have \( L'_{i,e} L''_{i,-e} = L''_{i,-e} L'_{i,e} = \text{id} : M \to M \).

Given \( i \in I^e \), \( j \in I \) and \( i \neq j \). Let \( m, n \in \mathbb{Z}_{\geq 0} \). Along the notations in (1.1), we set

\[
\mathcal{F}_{i,j,n,m,e} = \sum_{r+s=m} (-1)^r q_i^e(-a_{ij}n - m + 1) a_i^{(r)} a_{jn} a_i^{(s)} \in U^+.
\]

Here \( a_{jn} = 1 \) if \( n = 0 \). Otherwise, \( a_{jn} \) are the generators of \( U^+ \) corresponding to \( (j, n) \in I^\infty \) when \( j \in I^m \), and \( a_{jn} = a_j^{(n)} \) when \( j \in I^e \).

We shall denote \( \mathcal{F}_{i,j,n,m,e} \) by \( \mathcal{F}_{n,m,e} \) for simplicity if there is no risk of confusion, and set \( \beta = -a_{ij} \), \( \beta' = -a_{ji} \).

We shall give several equations about \( \mathcal{F}_{n,m,e} \) that can be proved by similar inductive processes in [L10, Chapter 7].

**Lemma 2.1.**

(i) \(-q_i^{(e(\beta n - 2m))} a_i \mathcal{F}_{n,m,e} + \mathcal{F}_{n,m,e} a_i = [m + 1]_q \mathcal{F}_{n,m+1,e} \),

(ii) \( a_i^{(p)} \mathcal{F}_{n,m,e} = \sum_{p'=0}^{p} (-1)^{p'} q_i^{e(2pm - \beta pm + pp' - p')} [m + p']_i \mathcal{F}_{n,m+1,p',e} a_i^{(p-p')} \),

(iii) \(-B_i \mathcal{F}_{n,m,e} + \mathcal{F}_{n,m,e} B_i = [\beta n - m + 1]_q K_{-ei} \mathcal{F}_{n,m-1,e} \).
Lemma 2.2.

(i) If \( j \in I^{re} \) and \( m = 1 + \beta n \), then

\[
B_j F_{n,m,e} = K_j \frac{q_j^{n-1}}{q_j - q_j^{-1}} F_{n-1,m,1} - K_j \frac{q_j^{1-n}}{q_j - q_j^{-1}} F_{n-1,m,-1}.
\]

(ii) If \( j \in I^{im}, (j, l) \in I^\infty \) and \( m = 1 + \beta n \), then

\[
b_{jl} F_{n,m,e} - F_{n,m,e} b_{jl} = 0.
\]

Proof. In the case of \( j \in I^{re} \), the proof is the same as [L10, Lemma 7.1.4]. We now prove (ii). Note that

\[
b_{jl} F_{n,m,e} - F_{n,m,e} b_{jl} = \sum_{r+s=m} (-1)^r q_i^{(\beta n-m+1)} a_i^{(r)} (b_{jl} a_{jn} - a_{jn} b_{jl}) a_i^{(s)}
\]

\[
= \delta_{ln} q_j^n (\beta n-m+1) a_i^{(r)} (b_{jl} a_{jn} - a_{jn} b_{jl}) a_i^{(s)}
\]

\[
= \delta_{ln} q_j^n (\beta n-m+1) (q_i^{nr \beta} K_j^n - q_i^{-nr \beta} K_j^{n-1}) a_i^{(r)} a_i^{(s)}.
\]

Since \( m = 1 + \beta n \), we see that the right hand side of the above equality equals to

\[
\delta_{ln} q_j^n (\beta n-m+1) a_i^{(r)} a_i^{(s)}
\]

and our assertion follows by the identity \( \sum_{r=0}^{m} (-1)^r q_i^{r(1-m)} \left[ \begin{array}{c} m \\ r \end{array} \right] = 0 \) for all \( m > 0 \). 

The following lemma is an analogue of [L10, Lemma 3.5.4]. One just need to note the fact that if \( i \in I^{re}, (j, l) \in I^\infty \) and \( i \neq j \), then

\[
a_i^m a_{jl} \in \sum_{k=0}^{-l a_{ij}} Q(q) a_k^l a_{jl} a_i^{m-k}
\]

for all \( m \geq 1 - l a_{ij} \).

Lemma 2.3. Assume that \( I^{re} \neq \emptyset \). Let \( u \) be an element of \( U \) such that \( u \) annihilates all \( M \in O \). Then \( u = 0 \).
Given an $i \in I^e$, we define the symmetries $L'_{i,e} : U \to U$ ($e = \pm 1$) on the generators of $U$ as follows

$$L'_{i,e}(a_i) = -K_e B_i, \quad L'_{i,e}(B_i) = -a_i K_{-ei};$$

$$L'_{i,e}(a_{jl}) = \sum_{r+s=-la_{ij}} (-1)^r q^{er} a_i^{(s)} a_{jl} a_i^{(r)} \quad \text{for } (j,l) \in I^\infty \text{ and } i \neq (j,l);$$

$$L'_{i,e}(b_{jl}) = \sum_{r+s=-la_{ij}} (-1)^r q_i^{er} B_i^{(s)} b_{jl} B_i^{(r)} \quad \text{for } (j,l) \in I^\infty \text{ and } i \neq (j,l);$$

$$L'_{i,e}(q^h) = q^n(h) = q^{h-a_i(h)} h_i,$$

and define the symmetries $L''_{i,-e}$ ($e = \pm 1$) by

$$L''_{i,-e}(a_i) = -B_i K_{-ei}, \quad L''_{i,-e}(B_i) = -K_{ei} a_i;$$

$$L''_{i,-e}(a_{jl}) = \sum_{r+s=-la_{ij}} (-1)^r q^{er} a_i^{(s)} a_{jl} a_i^{(r)} \quad \text{for } (j,l) \in I^\infty \text{ and } i \neq (j,l);$$

$$L''_{i,-e}(b_{jl}) = \sum_{r+s=-la_{ij}} (-1)^r q_i^{er} B_i^{(s)} b_{jl} B_i^{(r)} \quad \text{for } (j,l) \in I^\infty \text{ and } i \neq (j,l);$$

$$L''_{i,-e}(q^h) = q^{h-a_i(h)} h_i.$$

Let $j \in I$ with $j \neq i$, and let $m, n \in \mathbb{Z}_{\geq 0}$. We set $\mathcal{F}_{i,j,n,m,e} = \mathcal{F}_{n,m,e}$ as in (2.1) and

$$\mathcal{F}'_{i,j,n,m,e} = \mathcal{F}'_{n,m,e} = \sum_{r+s=m} (-1)^r q^{er(\beta n-m+1)} a_i^{(s)} a_{jn} a_i^{(r)};$$

$$\mathcal{G}_{i,j,n,m,e} = \mathcal{G}_{n,m,e} = \sum_{r+s=m} (-1)^r q_i^{-er(\beta n-m+1)} b_i^{(s)} b_{jn} B_i^{(r)};$$

$$\mathcal{G}'_{i,j,n,m,e} = \mathcal{G}'_{n,m,e} = \sum_{r+s=m} (-1)^r q_i^{-er(\beta n-m+1)} b_i^{(r)} b_{jn} B_i^{(s)}.$$

Here, $b_{jn}$ has the same interpretation as $a_{jn}$ and $\beta = -a_{ij}$.

Using Lemma 2.1 and Lemma 2.3, we can verify the following statements step by step according to [L10, Chapter 37], we leave it to readers. Precisely, thanks to Lemma 2.1, one can treat the generator $a_{jn}$ as the $a_i^{(n)}$ in form when we work with (i) in the following proposition, which is a counterpart of Lemma 37.2.2 in *loc.cit.* Then the proof of (ii) is entirely similar to Lemma 37.2.3 in [L10] with the help of (i) and Lemma 2.3.

**Proposition 2.4.**
Lemma 2.5. Let $U$ be the anti-automorphism of $U$ such that $U_{\alpha} = \alpha^{-1} U_{\alpha}$ for any $\alpha \in U$. Moreover, if $U_{\alpha} e = e U_{\alpha}$, then $U_{\alpha}$ commutes with $L^i_{\alpha}$ and $L^i_{\alpha^{-1}}$ for $i = 1, 2, 3$. Furthermore, the operators $L^i_{\alpha}$ and $L^i_{\alpha^{-1}}$ of $U$ are uniquely determined by these properties.

(iii) For any $m \in \mathbb{Z}$, we have

$$L^i_{m,e} (F_{n,m,e}) = F_{n,m,m,e}$$

and

$$L^i_{m,e} (G_{n,m,e}) = G_{n,m,m,e}.$$  

It is more convenient to twist the involution $\omega$. We shall denote by $\varpi$ the automorphism of $U$ such that

$$\varpi(a_i) = b_i, \quad \varpi(b_i) = a_i, \quad \varpi(q^h) = q^{-h}, \quad \varpi(a_{jl}) = b_{jl} \quad \text{and} \quad \varpi(b_{jl}) = a_{jl} \quad \text{for} \quad (j,l) \neq i.$$  

Let $*$ be the anti-automorphism of $U$ given by

$$*(q^h) = q^{-h}, \quad *(a_{jl}) = a_{jl} \quad \text{and} \quad *(b_{jl}) = b_{jl} \quad \text{for} \quad h \in P^\vee, \quad (i,l) \in I^\infty.$$  

Checking for the generators of $U$, we obtain $\varpi L^i_{m,e} = L^i_{m,e} \varpi : U \to U$ and $* L^i_{m,e} = L^i_{m,e} * : U \to U$. Moreover, if $u \in U$ such that $K_i u K_i^{-1} = q^m u$, we have

$$L^i_{m,e}(u) = (-1)^m q_i^m L^i_{m,e}(u).$$  

Lemma 2.5. [L10, Lemma 39.4.1] Assume that $I = I^\infty$ consists of exact two elements $i \neq j$ such that $m_{ij} = 2, 3, 4$ or 6, and assume $M \in \mathcal{O}$ in this case, we have

$$L^i_{m,e} L^j_{m,e} L^i_{m,e} \cdots = L^j_{m,e} L^i_{m,e} L^i_{m,e} \cdots : M \to M$$  

with both sides have $m_{ij}$ terms.

We shall generalize it to an arbitrary Borcherds-Cartan datum by the method in [L10, Lemma 39.4.3].

Theorem 2.6. Given a Borcherds-Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ with $\Pi^\vee \neq \emptyset$, let $U$ be the associated quantum Borcherds-Bozec algebra. For any $i \neq j$ such that $m_{ij} < \infty$, we have the following equalities of automorphisms of $U$ and any $U$-module $M$ in $\mathcal{O}$

$$L^i_{m,e} L^j_{m,e} L^i_{m,e} \cdots = L^j_{m,e} L^i_{m,e} L^i_{m,e} \cdots,$$

$$L^i_{m,e} L^j_{m,e} L^i_{m,e} \cdots = L^j_{m,e} L^i_{m,e} L^j_{m,e} \cdots.$$
where all the products have $m_{ij}$ factors.

Proof. By the preceding argument, equality (2.5) for $M \in \mathcal{O}$ can be shown by considering $M$ as a $U'$-module, where $U'$ is the subalgebra of $U$ generated by $a_i, b_i, a_j, b_j$ and $U^0$.

Let $u \in U$, we set $u_1 = (L''_{l_{i,j}^{-1}}L''_{l_{j,i}^{-1}}L''_{l_{i,j}^{-1}})(u)$ and $u_2 = (L''_{l_{j,i}^{-1}}L''_{l_{i,j}^{-1}}L''_{l_{j,i}^{-1}})(u)$. For any $M \in \mathcal{O}$ and $z \in M$, we have by using Proposition 2.4 (ii) that

$$u_1((L''_{l_{i,j}^{-1}}L''_{l_{j,i}^{-1}}L''_{l_{i,j}^{-1}})z) = u_1((L''_{l_{i,j}^{-1}}L''_{l_{j,i}^{-1}}L''_{l_{i,j}^{-1}})(uz)) = u_2((L''_{l_{j,i}^{-1}}L''_{l_{i,j}^{-1}}L''_{l_{j,i}^{-1}})(uz)).$$

It follows that $u_1 - u_2$ annihilates all $M \in \mathcal{O}$. Hence we get $u_1 = u_2$ by Lemma 2.3, this proves (2.5) for $U$. Finally, the equality for $L'$ follows by taking inverse. 

The braid group associated to a Borcherds-Cartan datum is the group generated by $r_i$ ($i \in I^n$) with defining relations

$$r_ir_jr_i \cdots = r_jr_ir_j \cdots \quad \text{for} \quad i \neq j \quad \text{and} \quad m_{ij} < \infty,$$

where both sides have $m_{ij}$ factors. Theorem 2.6 yields the braid group actions on $U$ and $U$-module $M \in \mathcal{O}$.

If $r_{i_1}r_{i_2} \cdots r_{i_N}$ and $r_{i'_1}r_{i'_2} \cdots r_{i'_N}$ are two reduced expressions of $r \in W$, then we have the equality $r_{i_1}r_{i_2} \cdots r_{i_N} = r_{i'_1}r_{i'_2} \cdots r_{i'_N}$ in the braid group. Hence the following definition is valid

$$L''_{i_{r_{i_1}r_{i_2} \cdots r_{i_N}}e} = L''_{i_{r_{i_1}r_{i_2} \cdots r_{i_N}}e} = L''_{i_{r_{i'_1}r_{i'_2} \cdots r_{i'_N}}e} = L''_{i_{r_{i'_1}r_{i'_2} \cdots r_{i'_N}}e},$$

where $r = r_{i_1}r_{i_2} \cdots r_{i_N}$ is a reduced expression of $r \in W$. From the definition, we have

$$L''_{r'r_{i_{r_{i_1}r_{i_2} \cdots r_{i_N}}e}} = L''_{r'r_{i_{r_{i_1}r_{i_2} \cdots r_{i_N}}e}} = L''_{r'r_{i_{r_{i'_1}r_{i'_2} \cdots r_{i'_N}}e}} = L''_{r'r_{i_{r_{i'_1}r_{i'_2} \cdots r_{i'_N}}e}}$$

if $r, r' \in W$ such that $l(rr') = l(r) + l(r')$. The almost same calculate in [L10, Chapter 40] gives the following lemma.

**Lemma 2.7.** Let $r \in W$ and let $i \in I^n$ with $l(rr_i) = l(r) + 1$, we have $L''_{r,e}(a_i) \in U^+$ and $L'_{r,e}(a_i) \in U^+$ for $e = \pm 1$.

**Corollary 2.8.** Let $r_{i_1}r_{i_2} \cdots r_{i_N}$ be a reduced expression for some $r \in W$. Then

(i) $L''_{i_{r_{i_1}r_{i_2} \cdots r_{i_{N-1}}e}(a_{i_N})} \in U^+$ and $L'_{i_{r_{i_1}r_{i_2} \cdots r_{i_{N-1}}e}(a_{i_N})} \in U^+$.

(ii) For any $j \in I^m$, we have $L''_{r,e}(a_{i_j}) \in U^+$ and $L'_{r,e}(a_{i_j}) \in U^+$.

**Proof.** The first assertion follows from the previous lemma directly. We prove (ii) by induction on $l(r)$. Let $r' = r_{i_1}r_{i_2} \cdots r_{i_{N-1}}$ and $i = i_N$, we have $r = r'r_1$. Since $l(r') =
and where elements write \( f \). Hence by (i), we deduce that 
\[
L_{r,e}^u(a_{jl}) = L_{r,e}^u(a_{jl}) = L_{r,e}^u(\sum_{r+s=-la_{ij}} (-1)^r q_i^{d_{er}} a_i^{(s)} a_j a_i^{(r)})
\]
\[
= \sum_{r+s=-la_{ij}} (-1)^r q_i^{d_{er}} L_{r,e}^u(a_i^{(s)}) L_{r,e}^u(a_{jl}) L_{r,e}^u(a_i^{(r)}).
\]
Hence by (i), we deduce that \( L_{r,e}^u(a_{jl}) \in U^+ \). Similarly, we have \( L_{r,e}^u(a_{jl}) \in U^+ \). 

3. Link with the bilinear form on \( U \)

Fix \( i \in I^e \). For any \((j, l) \in I^\infty\) with \( i \neq (j, l)\) and any \( m \in \mathbb{Z}\), we set

(i) \( f(i, (j, l), m) = \sum_{k=0}^{\infty} q_i^k (-1)^k a_i^{(m-k)} a_j a_i^{(k)} \); 

(ii) \( f'(i, (j, l), m) = \sum_{k=0}^{\infty} q_i^k (-1)^k a_i^{(m-k)} a_j a_i^{(k)} \); 

(iii) \( g(i, (j, l), m) = \sum_{k=0}^{\infty} q_i^k (-1)^k a_i^{(m-k)} b_j b_i^{(k)} \); 

(iv) \( g'(i, (j, l), m) = \sum_{k=0}^{\infty} q_i^k (-1)^k a_i^{(m-k)} b_j b_i^{(k)} \). 

Note that \( f(i, (j, l), m) = F_{i,j,i,m,1}, f'(i, (j, l), m) = F'_{i,j,i,m,1}, g(i, (j, l), m) = G_{i,j,i,m,1} \) and \( g'(i, (j, l), m) = G'_{i,j,i,m,1} \) by using the notations in (2.2). Sometimes, we will simply write \( f_m, f'_m, g_m, g'_m \) for convenience.

Let \( U^+[i] \) (resp. \( U^+[i], U^-[i] \) and \( U^-[i] \)) be the subalgebra of \( U \) generated by the elements \( f(i, (j, l), m) \) (resp. \( f'(i, (j, l), m), g(i, (j, l), m) \) and \( g'(i, (j, l), m) \)) for all \((j, l) \neq i\) and \( m \in \mathbb{Z}\). We have \(*\{U^+[i]\} = U^+[i] \) and \(*\{U^-[i]\} = U^-[i] \).

According to Proposition 2.4(iii) and equality (2.3), for any \( m \in \mathbb{Z}\), we have
\[
L''_{i,1}(f_m) = (-1)^n q_i^m L'_{i,1}(f_m) = f_{i,\beta-m};
\]
where \( n = 2m - l\beta \), and we obtain by using \(*\) that
\[
L'_{i,-1}(f'_m) = (-1)^n q_i^m L''_{i,-1}(f'_m) = f_{i,\beta-m}.
\]

Note that
\[
\omega f_m = (-1)^m q_i^m (la_{ij} + m-1) g_m, \quad \omega f'_m = (-1)^m q_i^m (la_{ij} + m-1) g'_m;
\]
so we obtain
\[ L''_{i,1}(g_m) = (-1)^m q_i^{-m(l\beta-m+1)} L''_{i,1}(\varpi f_m) \]
\[ = (-1)^m q_i^{-m(l\beta-m+1)} \varpi L'_{i,1}(f_m) \]
\[ = (-1)^{m+n} q_i^{-m(l\beta-m+1)-n} \varpi f'_{i\beta-m} = g'_{i\beta-m}, \]
and \( L'_{i,-1}(g'_m) = g_{i\beta-m} \) by taking inverse. Hence we have
\[ \varpi U^+[i] = U^-[i], \quad \varpi U^+_*[i] = U^*_-[i], \]
and
\[ L''_{i,1} U^+[i] = U^+_*[i]; \quad L''_{i,1} U^-[i] = U^*_-[i]; \]
\[ L'_{i,-1} U^+_*[i] = U^+[i]; \quad L'_{i,-1} U^*_-[i] = U^-_[i]. \]

**Lemma 3.1.** We have

(i) \( U^+ = \bigoplus_{t \geq 0} a_i^t U^+[i] = \bigoplus_{t \geq 0} U^+[i] a_i^t; \)
(ii) \( U^+ = \bigoplus_{t \geq 0} a_i^t U^+_*[i] = \bigoplus_{t \geq 0} U^*_+[i] a_i^t. \)

**Proof.** Note that (ii) follows from (i) by applying \( \ast \). To prove (i), it suffices to show that \( U^+[i] = \ker \delta^t \). By Lemma 2.1(i), we see that \( a_i \) and \( f_m \) interchange by the following formula
\[ -q_i^{2m-l\beta} a_i f(i, (j, l), m) + f(i, (j, l), m) a_i = [m + 1] f(i, (j, l), m + 1). \]

Note that \( U^+ \) is generated by \( a_i \) and \( U^+[i] \), we have the following decomposition
\[ U^+ = \sum_{t \geq 0} a_i^t U^+[i]. \]

Since \( L''_{i,1} U^+[i] \subseteq U^+ \), one can show by the exposition in [L10, Lemma 38.1.4] that \( U^+[i] \) is contained in \( \ker \delta^t \). Let \( x \in \ker \delta^t \), we write \( x \) into the form \( x = \sum_{t \geq 0} a_i^t x_t \), where \( x_t \in U^+[i] \subseteq \ker \delta^t \). Note that \( U^+ = \bigoplus_{t \geq 0} a_i^t \ker \delta^t \), we have \( x = x_0 \in U^+[i] \). This completes the proof. \( \square \)

Using \( \varpi \), we have the decompositions for \( U^- \)
\[ U^- = \bigoplus_{t \geq 0} B_i^t U^-_[i] = \bigoplus_{t \geq 0} U^-_[i] B_i^t = \bigoplus_{t \geq 0} B_i^t U^-_[i] = \bigoplus_{t \geq 0} U^-_[i] B_i^t. \]

We now assume that \( x \in U^+ \) and \( L''_{i,1}(x) \in U^+ \), and write \( x = \sum_{\alpha \in Q^+_+} x_\alpha \). Since \( L''_{i,1} \)
sends the different root spaces into the different root spaces, we have \( L''_{i,1}(x_\alpha) \in U^+ \) for
each $\alpha \in Q^+$. Note that $L''_{i,1}(x_\alpha) = (-1)^{\alpha(h)}q_i^{\alpha(h)}L'_{i,1}(x_\alpha)$, we deduce that $L'_{i,1}(x_\alpha) \in U^+$ and therefore $L''_{i,1}(x) \in U^+$. Thus, the following four subspaces of $U^+$ coincide

$$U^*[i]; \ker \delta^i; \{x \in U^+ \mid L''_{i,1}(x) \in U^+\}; \text{ and } \{x \in U^+ \mid L'_{i,1}(x) \in U^+\}.$$ 

By using $\ast$, we have the following equal subspaces

$$U^*[i]; \ker \delta^i; \{x \in U^+ \mid L''_{i,-1}(x) \in U^+\}; \text{ and } \{x \in U^+ \mid L'_{i,-1}(x) \in U^+\},$$

and by using $\varpi$, we obtain

$$U^[-i] = \{x \in U^- \mid L''_{i,1}(x) \in U^-\} = \{x \in U^- \mid L'_{i,1}(x) \in U^-\},$$

$$U^[-i] = \{x \in U^- \mid L''_{i,-1}(x) \in U^-\} = \{x \in U^- \mid L'_{i,-1}(x) \in U^-\}.$$ 

Let $f_m = f(i,(j,l),m)$ and $f'_m = f'(i,(j,l),m)$. By direct calculation, we can get the following equations in $U^+$ easily as in [L10, Lemma 38.1.7],

$$\rho(f_m) = 1 \otimes f_m + \sum_{t=0}^{m} \prod_{h=0}^{m-t-1} (1 - q_i^{2m-2h-2l+2}q_i^{t(m-t)} f_t \otimes a_i^{(m-t)},$$

(3.1) $$\rho(f'_m) = f'_m \otimes 1 + \sum_{t=0}^{m} \prod_{h=0}^{m-t-1} (1 - q_i^{2m-2h-2l+2}q_i^{t(m-t)} a_i^{(m-t)} \otimes f'_t.$$ 

Hence we have $\delta^i(f_m) = 0, \delta^i(f'_m) = (1 - q_i^{2m-2l+2})q_i^{m-1} f_{m-1}$ and

$$\delta^{i,l}(f_m) = \prod_{h=0}^{m-1} (1 - q_i^{2m-2h-2l+2}a_i^{(m)} = \prod_{h=1}^{m} (1 - q_i^{-2l+2})a_i^{(m)}.$$ 

Since $\ast \delta^k,s = \delta_k,s \ast$ for any $(k,s) \in I^\infty$, so we obtain

$$\delta^i(f'_m) = 0, \delta^i(f'_m) = (1 - q_i^{2m-2l+2})q_i^{m-1} f'_{m-1},$$

and

$$\delta_{j,l}(f'_m) = \prod_{h=1}^{m} (1 - q_i^{-2l+2})a_i^{(m)}.$$ 

Lemma 3.2. Let $f_m = f(i,(j,l),m)$ and $g_m = g(i,(j,l),m)$, we have

$$\{f_m, g_m\} = \{L''_{i,1}f_m, L''_{i,1}g_m\}.$$
Proof. Note that \( \{B_i U^-, f_m\} = 0 \), so we have by (3.2) that
\[
\{g_m, f_m\} = (-1)^m q_i^{m(\beta - m + 1)} \{b_j B_i^{(m)}, f_m\} = (-1)^m q_i^{m(\beta - m + 1)} \tau_{jl} \prod_{h=1}^m (1 - q_i^{-2(\beta + 1 - n)}) \{B_i^{(m)}, a_i^{(m)}\}.
\]
Since \( \{a_i^{(m)}, a_i^{(m)}\} = \tau_i q_i^{m(m-1)/2}[m]_i^{-1} \), we have
\[
\{B_i^{(m)}, a_i^{(m)}\} = q_i^{m(m-1)/2}(q_i^{-1} - q_i)^{-m}[m]_i^{-1}.
\]
Hence
\[
\{g_m, f_m\} = \tau_{jl} \left[ \frac{l^\beta}{m} \right] \{g_{\beta - m}, f_{\beta - m}\} = \{g'_{\beta - m}, f'_{\beta - m}\} = \{L''_{i,1} g_m, L''_{i,1} f_m\}.
\]
The lemma is proved.

Let \( n > 0 \) and \( i \in I^\infty \), we define the linear maps \( \delta_{ni}, \delta^{ni} : U^+ \rightarrow U^+ \) by
\[
\rho(x) = x \otimes 1 + \sum_{n>0} \delta_{ni}(x) \otimes a_i^{(n)} \text{ terms of bidegree not in } Q_+ \times N \alpha_i,
\]
\[
\rho(x) = 1 \otimes x + \sum_{n>0} a_i^{(n)} \otimes \delta^{ni}(x) \text{ terms of bidegree not in } N \alpha_i \times Q_+,
\]
where \( x \) is a homogeneous element of \( U^+ \). Let \( |x| = \mu \) and let \( (j, l) \in I^\infty \) with \( i \neq (j, l) \), we have
\[
\delta_{ni}(a_i x) = a_i \delta_{ni}(x) + [n_i]q^{(\alpha_i, \mu - (n-1)\alpha_i)} \delta_{(n-1)i}(x),
\]
\[
\delta^{ni}(x a_i) = \delta^{ni}(x)a_i + [n_i]q^{(\alpha_i, \mu - (n-1)\alpha_i)} \delta^{(n-1)i}(x),
\]
and
\[
\delta_{ni}(a_{jl} x) = a_{jl} \delta_{ni}(x), \quad \delta^{ni}(x a_{jl}) = \delta^{ni}(x)a_{jl}.
\]
Note that, we have by induction on \( n \) that
\[
(\delta_i)^n(a_i x) = a_i(\delta_i)^n(x) + [n_i]q^{(\alpha_i, \mu)} q_i^{-(n-1)}(\delta_i)^{n-1}(x),
\]
\[
(\delta^i)^n(x a_i) = (\delta^i)^n(x)a_i + [n_i]q^{(\alpha_i, \mu)} q_i^{-(n-1)}(\delta^i)^{n-1}(x),
\]
and
\[
(\delta_i)^n(a_{jl} x) = a_{jl}(\delta_i)^n(x), \quad (\delta^i)^n(x a_{jl}) = (\delta^i)^n(x)a_{jl}.
\]
It follows that
\[
(\delta_i)^n = q_i^{n(n-1)/2} \delta_{ni} \text{ and } (\delta^i)^n = q_i^{n(n-1)/2} \delta^{ni}.
\]
Fix $i \in I^e$, let $(j, l) \in I^\infty$ with $(j, l) \neq i$. Note that, the following set of elements are linearly independent in $U^+$

$$\{a_i^{(n)}, a_i^{(r)}a_{jl}a_i^{(s)} | n, r, s \geq 0 \text{ and } r + s \leq -la_{ij}\}.$$ 

It can be extended to be a basis of $U^+$, by adding a set $\{u_{w}\}_{w \in S}$ of monomials in $U^+$.

Hence for any $x \in U^+$, we can write $\rho(x)$ into

$$\rho(x) = 1 \otimes x + \sum_{n>0} a_i^{(n)} \otimes \delta^{ni}(x) + \sum_{r=0}^{l_{\beta}} \sum_{s=0}^{l_{\beta} - r} a_i^{(r)} a_{jl} a_i^{(s)} \otimes \delta^{ri;(j,l);si}(x) + \sum_{w \in S} u_w \otimes x_w.\quad (3.4)$$

Here $\beta = -a_{ij}$ as usual.

**Lemma 3.3.** Let $x \in U^+$ and $y \in U^-$. For any $m \in \mathbb{Z}$, we have

(i) $\{g_m y, x\} = \{g_m, f_m\} \{y, \delta^{(j,l);mi}(x)\}$.
(ii) $\{g'_m y, x\} = \{g'_m, f'_m\} \{y, \delta^{mi;(j,l)}(x)\}$.

**Proof.** We shall prove (i) first. Since $f_m = g_m = 0$ when $m > l_{\beta}$ or $m < 0$, we may assume $0 \leq m \leq l_{\beta}$. Let $u$ be a monomial in $U_{m\alpha_i + la_j}^+$. If $u$ contains no $a_{jl}$, then we can write $u = a_l'^{a_{jk}} u'$ for some $t \geq 0$, $k < l$ and a monomial $u'$ of $U_{(m-t)\alpha_i + (l-k)\alpha_j}^+$. Note that

$$\rho(a_i^{(r)} a_{jl} a_i^{(m-r)}) = (\sum_{a+b=r} q_{i}^{ab} a_i^{(a)} \otimes a_i^{(b)}) (a_{jl} \otimes 1 + 1 \otimes a_{jl}) (\sum_{a'+b'=m-r} q_{i}^{a'b'} (a') \otimes (b')),$$

and the right hand side is in

$$\sum_{a+b=r \atop a'+b'=m-r} Q(q) a_i^{(a)} a_{jl} a_i^{(a')} \otimes a_i^{b+b'} + \sum_{a+b=r \atop a'+b'=m-r} Q(q) a_i^{a+a'} \otimes a_i^{(b)} a_{jl} a_i^{(b')}.$$

Hence we obtain

$$\{a_i^{(r)} a_{jl} a_i^{(m-r)}, u\} = \{\rho(a_i^{(r)} a_{jl} a_i^{(m-r)}), a_i^{a_{jk}} \otimes u'\} = 0,$$

and therefore $\{g_m, u\} = 0$ for such $u$.

Since $\{g_m, a_i U^+\} = 0$, we get $\{g_m, f_m\} = \{g_m, a_i^{(m)}\}$. On the other hand, by the above argument, we have $\{g_m, u_w\} = 0$ for any $w \in S$, which implies

$$\{g_m y, x\} = \{g_m \otimes y, a_i^{(m)} \otimes \delta^{(j,l);mi}(x)\} = \{g_m, f_m\} \{y, \delta^{(j,l);mi}(x)\}.$$

The proof of (ii) is similar. ☐
Let $x \in U^{+}_{\mu}$ and $x' \in U^{+}_{\mu}$. Let $0 \leq n \leq l \beta$. The expression (3.4) yields
\[
\delta^{(j,l);ni}(xx') = \delta^{(j,l);ni}(x)x' + q^{(\mu,la_{j}+na_{i})}x\delta^{(j,l);ni}(x')
\]
\[\text{(3.5)}\]
\[
+ \sum_{t=0}^{n-1} q^{(\mu-la_{j}-ta_{i}(n-t)\alpha)} \left[ \frac{n}{t} \right] \delta^{(j,l);ti}(x)\delta^{(n-t)i}(x'),
\]
and
\[
\delta^{ni;(j,l)}(xx') = \delta^{ni;(j,l)}(x)x' + q^{(\mu,na_{i}+la_{j})}x\delta^{ni;(j,l)}(x')
\]
\[\text{(3.6)}\]
\[
+ \sum_{t=0}^{n-1} q^{(\mu-(n-t)\alpha_{i}+la_{j})} \left[ \frac{n}{t} \right] \delta^{(n-t)i}(x)\delta^{ti;(j,l)}(x').
\]
In particular, if $x' \in U^{+}[i]$, we have $\delta^{ni}(x') = 0$ for all $n > 0$, which implies
\[
\delta^{(j,l);ni}(xx') = \delta^{(j,l);ni}(x)x' + q^{(\mu,la_{j}+na_{i})}x\delta^{(j,l);ni}(x').
\]
\[\text{(3.7)}\]
\[
\text{Let } f_{m} = f(i,(j,l),m) \text{ and } f'_{m} = f'(i,(j,l),m). \text{ By (3.1), we have}
\]
\[
\delta^{(j,l);mi}(f_{m}) = \begin{cases} 
\gamma_{mn} a_{i}^{(m-n)} 
& \text{if } n \leq m, \\
0 
& \text{if } n > m,
\end{cases}
\text{and } \delta^{ni;(j,l)}(f'_{m}) = \begin{cases} 
1 
& \text{if } n = m, \\
0 
& \text{otherwise},
\end{cases}
\]
where
\[
\gamma_{mn} = \prod_{h=0}^{m-n-1} (1 - q_{i}^{2m-2t-2l\beta-2}) q_{i}^{n(m-n)}.
\]

Denote by $P_{i} : U^{+} \to U^{+}[i]$ the projection obeys the decomposition $U^{+} = a_{i}U^{+} \oplus U^{+}[i]$ given in Lemma 3.1. We have by the definition
\[
P_{i}(xx') = P_{i}(P_{i}(x)x') \text{ for any } x, x' \in U^{+}.
\]
In particular, we have a more simpler formula $P_{i}(xx') = P_{i}(x)x'$ when $x' \in U^{+}[i].$

**Lemma 3.4.** Let $x \in U^{+}[i]$ with $|x| = \mu$ and let $n > 0$, we have
\[
P_{i}(xa_{i}^{n}) = \frac{q^{n(\mu,\alpha_{i})}q_{i}^{n(n+1)}}{(q_{i} - q_{i}^{-1})^{n}} L_{i,-1}^{n}(\delta_{i}^{n}) L_{i,1}^{n}(x).
\]

**Proof.** We use induction on $n$. Assume that $n = 1$. Since $L_{i,1}^{n}(x) \in U^{+}[i]$, we have $\delta_{i}(L_{i,1}^{n}(x)) = 0$ and so
\[
L_{i,1}^{n}(x)B_{i} - B_{i}L_{i,1}^{n}(x) = -K_{i}^{-1} \delta_{i}(L_{i,1}^{n}(x))/(q_{i} - q_{i}^{-1}).
\]
Applying $L'_{i-1}$ to both sides, we get

$$-xa_iK_i + a_iK_i x = -K_i L'_{i-1} \delta^i L''_{i,1}(x)/(q_i - q_i^{-1}),$$

and by applying $K^{-1}$, we obtain

$$-q^{-(\mu + \alpha_i \alpha_i)}ax_i + q^{-(\alpha_i \alpha_i)}a_i x = -L'_{i-1} \delta^i L''_{i,1}(x)/(q_i - q_i^{-1}).$$

It follows that

$$xa_i = q^{(\mu + \alpha_i \alpha_i)}a_i x + q^{(\mu + \alpha_i \alpha_i)}(q_i - q_i^{-1})^{-1}L'_{i-1} \delta^i L''_{i,1}(x).$$

Note that, according to the second formula in (3.1), we have $\rho(U^+_*[i]) \subseteq U^+ \otimes U^+_*[i]$, which implies $\delta^i L''_{i,1}(x) \in U^+_*[i]$. Hence

$$P_i(xa_i) = \frac{q^{(\mu + \alpha_i \alpha_i)}}{q_i - q_i^{-1}}L'_{i-1} \delta^i L''_{i,1}(x).$$

This proves our assertion for $n = 1$. For the induction step, assume the lemma is true for $n$, then we have

$$P_i(xa_i^{n+1}) = P_i(P_i(xa_i^n)a_i) = \frac{q^{(\mu + (n+1) \alpha_i \alpha_i)}}{q_i - q_i^{-1}}L'_{i-1} \delta^i L''_{i,1}(P_i(xa_i^n)) = \frac{q^{(n+1)(\mu \alpha_i)}}{q_i - q_i^{-1}}(n+1) \delta^i L''_{i,1}(x)$$

as desired. \qed

Now, we shall prove our main theorem by using the method given in [J95, Chapter 8A]. Note that there is no special restriction on the values of $\tau_i$’s for $i \in \mathbb{I}^e$, but only ask them to take values in $1 + q^{-1}\mathbb{Z}_{\geq 0}[q^{-1}]$, hence the argument in [L10, Lemma 38.2.1] is not available for our case.

**Theorem 3.5.** For any $x \in U^+[i], y \in U^-[\bar{i}]$, we have

$$\{L''_{i,1}(x), L''_{i,1}(y)\} = \{x, y\}.$$

**Proof.** Assume that our assertion holds for a given $y$ in $U^-[\bar{i}]$ and arbitrary $x$ in $U^+[i]$. By Lemma 3.3(ii), we have

$$\{L''_{i,1}(y), L''_{i,1}(x)\} = \{g_{\ell \beta - m} L''_{i,1}(y), L''_{i,1}(x)\} = \{g_{\ell \beta - m}, f_{\ell \beta - m} \{L''_{i,1}(y), L''(\ell \beta - m)_{i,1} L''_{i,1}(x)\} = \{g_{\ell \beta - m}, f_{\ell \beta - m} \{y, L''_{i,1} \delta^i (\ell \beta - m)_{i,1} L''_{i,1}(x)\},$$
where the last equality follows from the fact that \( \delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x) \in U^+_i \). On the other hand, by Lemma 3.2 and Lemma 3.3(i), we have

\[
\{g_m y, x\} = \{g_m, f_m\} \{y, \delta^{(j,l)mi}(x)\} = \{g_m, f_m\} \{y, \delta^{(j,l)mi}(x)\}.
\]

Since \( \{y, a_i U^+\} = 0 \). To show \( \{L''_{i,1}(g_m y), L_{i,1}''(x)\} = \{g_m, x\} \), it is enough to show

\[
L'_{i,-1} \delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x) \equiv \delta^{(j,l)mi}(x) \mod a_i U^+
\]

for all \( x \in U^+_{ij} \). This is equivalent to

\[
\delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x) = L''_{i,1} P_i \delta^{(j,l)mi}(x) (x').
\]

On the other hand, by (3.6), we have

\[
\delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x) = \delta^{(l-\beta-m)i;j,l} (x) L''_{i,1}(x'),
\]

hence (3.8) holds for \( x' \) in this case.

If \( (j', l') = (j, l) \), then \( x' = f_s = f_i, (j, l), s \). By using (3.7), we obtain

\[
\delta^{(j,l)mi}(x f_s) = \delta^{(j,l)mi}(x) f_s + X,
\]

where \( X = 0 \) when \( m > s \), and \( X = q^{(\mu, l\alpha_j + m\alpha_i)} \gamma_{sm} x a_i^{(s-m)} \) when \( m \leq s \).

By (3.6), we have

\[
\delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x f_s) = \delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x) f'_{l-\beta-s} + Y,
\]

where \( Y = 0 \) when \( m > s \), and \( m \leq s \),

\[
Y = q^{(r, \mu -(s-m)\alpha_j, l(\beta-s)\alpha_j + l\alpha_j)} \left[ \frac{[l(\beta-s)]}{[l-\beta-s]} \right] \delta^{(s-m)i} L''_{i,1}(x).
\]

Here we understand \( \delta^0(L''_{i,1}(x)) = L''_{i,1}(x) \). According to our assumption, we have

\[
L''_{i,1} \left( P_i \delta^{(j,l)mi}(x) f_s \right) = \delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x) f'_{l-\beta-s}.
\]

Hence in order to show

\[
\delta^{(l-\beta-m)i;(j,l)} L''_{i,1}(x f_s) = L''_{i,1} P_i \delta^{(j,l)mi}(x f_s),
\]
it suffices to prove that $L''_{i,1}P_1(X) = Y$ in the case of $m \leq s$. In this case, using the equality given Lemma 3.4 and in (3.3), we get

$$\begin{align*}
L''_{i,1}P_1(X) &= q^{(m, \alpha_i + \alpha_j)}\gamma_{sm}L''_{i,1}P_1(xa_x^{(s-m)}) \\
&= \frac{1}{[s-m]!}q^{(s-m)(m, \alpha_i)}q^{(m, \alpha_i + \alpha_j)}(s-m)(3(s-m)+1)/2\gamma_{sm}\delta^{(s-m)}L''_{i,1}(x).
\end{align*}$$

Note that, we have

$$\gamma_{sm} = q_i^{m(s-m)+(s-\ell)(s-m)-(s-m)(s-m+1)/2} \prod_{h=1}^{s-m} (q_i^{\ell-s} - q_i^{-\ell+s-h}).$$

Thus,

$$L''_{i,1}P_1(X) = q^{(m, \alpha_i + \alpha_j)}q^{(2s-\ell)(s-m)}\delta^{(s-m)}L''_{i,1}(x) = Y.$$ 

The theorem is proved. \qed 

Let $P'_i : U^+ \to U^+ [i]$ be the projection along the decomposition $U^+ = U^+_*[i] \oplus U^+ a_i$. We have the following result as a direct corollary of the previous theorem.

**Corollary 3.6.** Let $x \in U^+[i]$, we have

$$P'_i \otimes \text{id} \circ \rho \circ L''_{i,1}(x) = (L''_{i,1} \otimes L''_{i,1}) \circ \text{id} \otimes P_i \circ \rho(x).$$

**Proof.** Note that both sides in the equation (3.9) belong to $U^+_*[i] \otimes U^+_*[i]$. Let $y, y' \in U^- [i]$, we have by the definition

$$\{yy', x\} = \{y \otimes y', \rho(x)\} = \{y \otimes y', (\text{id} \otimes P_i) \circ \rho(x)\}$$

$$= \{L''_{i,1}(y) \otimes L''_{i,1}(y'), (L''_{i,1} \otimes L''_{i,1}) \circ \text{id} \otimes P_i \circ \rho(x)\},$$

where the last equality follows from Theorem 3.5. On the other hand, since $yy' \in U^- [i]$, we obtain by using Theorem 3.5 directly

$$\{yy', x\} = \{L''_{i,1}(y)\} = \{L''_{i,1}(y')\},$$

Let $u = (P'_i \otimes \text{id}) \circ \rho \circ L''_{i,1}(x) - (L''_{i,1} \otimes L''_{i,1}) \circ (\text{id} \otimes P_i) \circ \rho(x)$. The above argument implies that

$$\{U^+_*[i] \otimes U^-*[i], u\} = 0.$$

Since $U^+_1 = U^+_*[i] \oplus U^- B_1$ and $\{U^- B_1, u\} = 0$, thus we deduce that $\{U^- \otimes U^-, u\} = 0$. Note that the bilinear form $\{\, , \}$ is non-degenerated on $U^+ \otimes U^+$, hence $u = 0$ and our assertion follows. \qed
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