Solving parallel transport equations in the higher-dimensional Kerr-NUT-(A)dS spacetimes

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We obtain and study the equations describing the parallel transport of orthonormal frames along geodesics in a spacetime admitting a non-degenerate principal conformal Killing–Yano tensor $h$. We demonstrate that the operator $F$, obtained by a projection of $h$ to a subspace orthogonal to the velocity, has in a generic case eigenspaces of dimension not greater than 2. Each of these eigenspaces are independently parallel-propagated. This allows one to reduce the parallel transport equations to a set of the first order ordinary differential equations for the angles of rotation in the 2D eigenspaces. General analysis is illustrated by studying the equations of the parallel transport in the Kerr-NUT-(A)dS metrics. Examples of three, four, and five dimensional Kerr-NUT-(A)dS are considered and it is shown that the obtained first order equations can be solved by a separation of variables.

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I. INTRODUCTION

Higher-dimensional black hole solutions of Einstein equations attract a lot of interest. The most general known solution for an isolated rotating higher-dimensional black hole in an asymptotically (anti) de Sitter background, the Kerr-NUT-(A)dS metric, was obtained recently by Chen, Lü, and Pope [1]. In many respects this spacetime is similar to the 4D Kerr-NUT-(A)dS solution. In particular, it possesses hidden symmetries generated by the Killing–Yano (KY) and Killing tensors. Namely, it admits a principal conformal Killing-Yano (CKY) tensor [2, 3] which generates a tower of the Killing–Yano and Killing tensors [4, 5]. The existence of the complete set of irreducible Killing tensors and corresponding integrals of motion makes the particle geodesic equation completely integrable [4, 6]. For the same reason the Hamilton–Jacobi, Klein–Gordon, and Dirac equations allow a separation of variables [7, 8]. For more details on these results see recent reviews [9, 10] and references therein.

One of the additional remarkable properties of the 4D Kerr metric, discovered by Marck in 1983 [11, 12], is that the equations of parallel transport can be integrated. This result allows a generalization: A parallel-propagated frame along a geodesic can be constructed explicitly in an arbitrary 4D spacetime admitting the rank-2 Killing–Yano tensor [13].

Solving the parallel transport equations is useful for many problems whose study is the behavior of extended objects moving in the Kerr and more general geometries. In particular, it facilitated the study of tidal forces acting on a moving body, for example a star, in the background of a massive black hole (see, e.g., [14, 15, 16, 17, 18, 19]). In the special case of null geodesics (rays) a solution to the parallel transport equations is required for studying the propagation of light polarization [20, 21, 22]. In quantum physics the parallel transport of frames is an important technical element of the point splitting method which is used for calculation of renormalized values of local observables (such as vacuum expectation values of currents, stress-energy tensor etc.) in a curved spacetime. Solving of the parallel transport equations is useful especially when fields with spin are considered (see, e.g., [23]).

The purpose of the present paper is to generalize the results [11] to the case of spacetimes with arbitrary number of dimensions admitting a non-degenerate principal CKY tensor. It was demonstrated in [24] that the existence of such a CKY tensor obeying the additional condition implies complete integrability of geodesic motion. This ‘special’ CKY tensor also distinguishes uniquely the Kerr-NUT-(A)dS spacetimes among all the Einstein spaces [25]. However, in the first part of the paper we do not limit our analysis to these metrics and consider a more general situation of a spacetime with a closed 2-form of the non-degenerate principal CKY tensor $h$ and an integrable particle geodesic motion.

Let us outline the main idea of our construction. Any 2-form determines what is called a Darboux basis, that is a basis in which it has a simple standard canonical form. We call the Darboux basis of $h$ a principal basis. For a strictly non-degenerate...
Consider now a timelike geodesic describing the motion of a particle with velocity $u$. We focus our attention on a special 2-form $F$—obtained by a projection of the principal CKY tensor $h$ to a subspace orthogonal to the velocity $u$. $F$ has its own Darboux basis, which we call comoving. For any chosen geodesic the comoving basis is determined along its trajectory. It can be easily shown that $F$ is parallel-transported along the geodesic. In particular, it means that its eigenvalues and its Darboux subspaces, which we call the eigenspaces of $F$, are parallel-transported. We shall show that for generic geodesics the eigenspaces of $F$ are at most 2-dimensional. In fact, the eigenspaces with non-zero eigenvalues are 2-dimensional, and the zero-value eigenspace is 1-dimensional for odd number of spacetime dimensions and 2-dimensional for even. So, the comoving basis is defined up to rotations in each of the 2D eigenspaces. The parallel-transported basis is a special comoving basis. It can be found by solving a set of the first order ordinary differential equations for the angles of rotation in the 2D eigenspaces.

For special geodesic trajectories the 2-form $F$ may become degenerate, that is at least one of its eigenspaces will have more than 2 dimensions. We shall demonstrate that the eigenspaces with non-vanishing eigenvalues in such a degenerate case may be 4-dimensional. In the odd number of spacetime dimensions one may also have a 3-dimensional eigenspace with a zero eigenvalue. Nevertheless, in these degenerate cases one can also obtain the parallel-transported basis by (now rather more complicated) time dependent rotations of the comoving basis.

The paper is organized as follows. In Section II we introduce the form $F$ and construct its Darboux basis. In Section III we use this comoving basis to derive the equations for the parallel transport of frames. In Section IV we adopt general results to the case of the Kerr-NUT-(A)dS metrics. Concrete examples of parallel-transported frames in $D = 3, 4, 5$ Kerr-NUT-(A)dS spacetimes are described in Section V. Section VI contains discussion.

II. HIDDEN SYMMETRIES AND COMOVING BASIS

A. Operator $F$

Consider a $D$-dimensional spacetime $M^D$ with the metric

$$g = g_{ab} dx^a dx^b.$$  \hspace{1cm} (1)

We assume that this metric has the signature $(-, +, +, +)$. We also assume that the spacetime possesses a closed CKY 2-form $h$:

$$h = \frac{1}{2} h_{ab} dx^a \wedge dx^b, \hspace{1cm} (2)$$

$$\nabla_X h = -\frac{1}{D-1} X^b \wedge \delta h.$$  \hspace{1cm} (3)

Here $X$ is an arbitrary vector, $X^a$ is the corresponding form which has components $(X^a)_a = g_{ab} X^b$. An inverse to $\delta$ operation is denoted by $\xi$. Namely if $\alpha$ is a 1-form then $\alpha^\sharp$ denotes a vector with components $(\alpha^\sharp)^a = g^{ab} \alpha_b$. $\delta$ denotes the co-derivative. For a $p$-form $\alpha_p$ one has $\delta \alpha_p = \epsilon \ast d \ast \alpha_p$, where $\epsilon = (-1)^{(d-p)+p-1}$, and $\ast$ denotes the Hodge star operator. In usual tensor notations the definition (3) of the closed CKY tensor $h$ reads

$$\nabla_c h_{ab} = 2 g_{[a} \xi_b]. \hspace{1cm} (4)$$

We call $h$ a principal conformal Killing-Yano tensor. Its dual tensor $\ast h$ is the Killing–Yano tensor ($(D-2)$-form).

Our aim is to construct a parallel-propagated frame along geodesic in a spacetime with such a principal CKY tensor. First of all, we construct a comoving basis determined by the principal CKY tensor and a time-like unit vector (vector of velocity)$^2$. At this stage our construction is local. We consider a tangent space $T$ at a chosen point $p_0$. We denote by $u$ a chosen time-like unit velocity vector ($u^a u_a = -1$), by $U$ a 1-dimensional space generated by $u$, and by $V$ the $(D-1)$-dimensional

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1 In an odd number of spacetime dimensions there exists an additional one-dimensional zero-eigenvalue Darboux subspace of $h$.

2 The construction we describe can be easily adapted for the parallel transport of frames along spacelike geodesics.
subspace orthogonal to \( u \). Thus \( T \) is a direct sum of two orthogonal subspaces, \( U \) and \( V \),

\[ T = U \oplus V. \]

Let \( F \) be a 2-form defined by the relation (cf. \([4, 6]\))

\[ F = h + u^b \wedge s, \quad s = u_a h, \quad s_b = u^a h_{ab}, \]

or in components

\[ F_{ab} = h_{ab} + u_a u^c h_{cb} + h_{ac} u^c u_b = P^a_a P^d_b h_{cd}. \]

Here \( P^a_a = \delta^a_a + u^a u_a \) is the projector to \( V \). \( F_{ab} = g_{ab} + u_a u_b \) can be also understood as a positive definite metric in \( V \) induced by its embedding into \( T \). \( F_{ab} \) and \( F^a_a = g^{ac} F_{cb} \) can be considered as a 2-form and an operator, respectively, either in the subspace \( V \) or in the complete tangent space \( T \). Since \( F^a_a u^b = 0 \), the vector \( u \in T \) is an eigenvector of \( F \) with a zero eigenvalue.

**B. Comoving basis**

We demonstrate now that there exists such an orthonormal basis in \( V \) in which the operator \( F \) has the (matrix) form (see e.g. \([26]\))

\[ \text{diag}(0, \ldots, 0, \Lambda_1, \ldots, \Lambda_p), \]

where \( \Lambda_\mu \) are matrices of the form

\[ \Lambda_\mu = \begin{pmatrix} 0 & \lambda_\mu \ I_\mu \\ -\lambda_\mu \ I_\mu & 0 \end{pmatrix}, \]

and \( I_\mu \) are unit matrices. Such a basis, known as the Darboux basis, can be constructed for any 2-form by a modified version of the Gram–Schmidt process. We recall this procedure, mainly, in order to fix the notations we shall use later in the paper.

To construct the Darboux basis let us define an operator \( F^2 \) with the components

\[ (F^2)^a_b = F^a_c F^c_b = P^a_d h_{c e} P^c_f h^{k l} P^k_b. \]

It annihilates the vector \( u \) and hence it can be considered as an operator in \( V \). Consider the following eigenvalue problem

\[ F^2 v = -\lambda^2 v. \]

Then we find

\[ \lambda^2 v^2 = -(v, F^2 v) = (Fv)^2 \geq 0. \]

Equation (13) implies that \( \lambda^2 \) is non-negative. For non-vanishing eigenvalues it is convenient to choose all \( \lambda \) to be positive. We enumerate \( \lambda \) by index \( \mu = 0, \ldots, p \) and order them as follows

\[ 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_p. \]

If \( F^2 \) does not have a zero eigenvalue, the first term \( \lambda_0 \) in (14) is omitted. We denote by \( V_\mu \) a subspace spanned by the eigenvectors of \( F^2 \) corresponding to \( \lambda_\mu \). These (Darboux) subspaces \( V_\mu \) possess the property

\[ FV_\mu = \lambda_\mu V_\mu. \]

We call them the eigenspaces of \( F \). Similarly we call \( \lambda_\mu \) the ‘eigenvalues’ of \( F \). The eigenspaces with different eigenvalues are mutually orthogonal and their direct sum forms the space \( V \):

\[ V = V_0 \oplus V_1 \oplus \ldots \oplus V_p. \]

If \( \lambda = 0 \) and the corresponding subspace \( V_0 \) has \( q_0 \) dimensions, we denote an orthonormal basis in \( V_0 \) by

\[ \{n_{01}, \ldots, n_{0q_0}\}. \]

Let \( \lambda \neq 0 \). Consider a unit vector \( n \) from \( V_\lambda \) and denote \( \hat{n} = -\lambda^{-1} F^1 n \). One has

\[ (n, \hat{n}) = -\lambda^{-1} (n, F^1 n) = 0, \]

\[ F^1 \hat{n} = -\lambda^{-1} F^2 \hat{n} = \lambda^1 n, \]

\[ (\hat{n}, \hat{n}) = -\lambda^{-1} (\hat{n}, F^1 \hat{n}) = (n, \hat{n}) = 1 \]

If \( V_\lambda \) has 2 dimensions, \( \{n, \hat{n}\} \) is an orthonormal basis in it. If \( V_\lambda \) has more than 2 dimensions we choose another unit vector \( \hat{n} \in V_\lambda \) orthogonal to both \( n \) and \( \hat{n} \) and denote \( \tilde{n} = -\lambda^{-1} F^1 \hat{n} \). Evidently, \( \tilde{n} \) is orthogonal to \( \hat{n} \) and has a unit norm. One also has

\[ (\tilde{n}, \hat{n}) = \lambda^{-1} (\tilde{n}, F^1 \hat{n}) = -(\tilde{n}, n) = 0, \]

\[ (\hat{\tilde{n}}, \tilde{n}) = \lambda^{-1} (\hat{\tilde{n}}, F^1 \tilde{n}) = (n, \hat{n}) = 1. \]

Thus the vectors \( \{n, \hat{n}, \tilde{n}\} \) can be added to the set \( \{n, \hat{n}\} \). If the dimension of \( V_\lambda \) is 4 we already got an orthonormal basis in it. If the dimension is greater than 4 the procedure must be repeated until finally a complete basis is constructed. For the eigenvalue \( \lambda = \lambda_\mu \) such a basis in \( V_\mu \) is

\[ \{n_\mu, \hat{n}_\mu, \ldots, n_{q_\mu}, \hat{n}_{q_\mu}\}. \]

\(^3\) We put \( \hat{\cdot} \) marks to indicate that, strictly speaking, \( \lambda_\mu \) are eigenvalues not of \( F \) but of \( F^2 \).
It is evident that each $V_q$ is an even dimensional space; we denote its dimension by $2q$. We further denote by

$$\{ \xi^\hat{0}, \ldots, \xi^\hat{q} \}, \quad \{ \xi^{\mu}, \xi^{\hat{\mu}}, \ldots, \xi^{\hat{q}} \xi^\hat{0}, \xi^\hat{q} \},$$

bases of forms dual to the constructed orthonormal vector bases $\{ \hat{0}, 2q \}$. These forms give bases in the cotangent spaces $V^\ast_q$ and $V^\ast_{q}$. We combine the bases $\{ \hat{0}, 2q \}$ with $\mu = 0, \ldots, p$ to obtain a complete orthonormal basis of vectors (forms) in the space $V (V^\ast)$. The duality conditions read

$$\xi^\mu (\xi^{\mu'}) = \delta^{\mu}_{\mu'} \delta^{s}_{s'},$$

$$\xi^{\hat{\mu}} (\xi^{\hat{\mu'}}) = \delta_{\mu}^{\mu'} \delta^{s}_{s'} = 0.$$  

Here for a given $\mu = 0, \ldots, p$ index $s$ takes the values $s = 1, \ldots, q_{\mu}$. It is evident from the orthonormality of the constructed basis that we also have

$$(\xi^{\mu})^b = \xi^{\hat{\mu}}, \quad (\xi^{\hat{\mu}})^b = \xi^{\hat{\mu}},$$

$$\xi^{\hat{\mu}} = (\xi^{\hat{\mu}})^b = (\xi^{\hat{\mu}})^b = (\xi^{\hat{\mu}})^b.$$  

In this basis the antisymmetric operator $F$ takes the form $[6]$. If we consider $F$ as an operator in the complete tangent space $T$, the corresponding Darboux basis is enlarged by adding the vector $u$ to it. In this enlarged basis the operator $F$ has the same form $[9]$, with the only difference that now the total number of zeros is not $q_0$, but $q_0 + 1$. To remind that the constructed Darboux basis depends on the velocity $u$ of a particle and $u$ is one of its elements we call this basis comoving. The characteristic property of the comoving frame is that all spatial components of the velocity vanish.

Although our construction was local, we can naturally extend the comoving basis along the whole geodesic trajectory. In a general case, however, the constructed comoving frame is not parallel-propagated. The parallel-propagated frame can be obtained by performing additional rotations in each of the parallel-propagated eigenspaces of $F$. The equations for the corresponding rotation angles will be derived in the next section. Before we do that we demonstrate that for a strictly non-degenerate principal CKY tensor $h$ the structure of the eigenspaces of $F$, and hence the comoving basis, significantly simplifies.

### C. Eigenspaces of $F$

To treat both cases of the even and odd dimensional spacetime $M^D$ simultaneously we denote

$$D = 2n + \varepsilon,$$  

where $\varepsilon = 0$ and $\varepsilon = 1$ for even and odd number of dimensions, respectively.

In the comoving frame constructed above the form $F$ reads:

$$F = \sum_{\mu=1}^{p} \lambda^{\mu} \sum_{j=1}^{q_{\mu}} \xi^{\mu} \wedge \xi^{\mu}.$$  

It is convenient to call the ordered eigenvalues $\lambda^{\mu}$ together with their degeneracies a 'spectrum' of $F$ (cf. Eq. (12)). We denote it by

$$S(F) = \{ 0, \ldots, 0, \lambda_1, \ldots, \lambda_1, \ldots, \lambda_p, \ldots, \lambda_p \}.$$  

The first zero eigenvalue corresponds to the 1-dimensional subspace $U$ spanned by $u$. One also has

$$D = 1 + q_0 + 2k, \quad k = \sum_{\mu=1}^{p} q_{\mu}.$$  

### 1. Structure of $V_0$

Let us now impose the condition that $h$ is non-degenerate, that is its (matrix) rank is $2n$. Then one has

$$q_0 = \begin{cases} 1, & \text{for } \varepsilon = 0, \\ 0 \text{ or } 2, & \text{for } \varepsilon = 1. \end{cases}$$  

Let us prove this assertion. For an arbitrary form $\alpha$ we denote

$$\alpha^{\wedge m} = \alpha \wedge \ldots \wedge \alpha,$$  

where $\alpha^{\wedge m}$ is the total of $m$ factors $\alpha$. From the definition (6) of $F$ we find

$$h^{\wedge m} = F^{\wedge m} - m F^{\wedge (m-1)} \wedge u^b \wedge s,$$  

where we have used the property of the external product $\alpha_p \wedge \alpha_q = (-1)^{pq} \alpha_q \wedge \alpha_p$. It is obvious from (25) that the (matrix) rank of $F$ is $2k$, that is $F^{\wedge (k+1)} = 0$. So, using (30) we have $h^{\wedge (k+2)} = 0$. It means that for a non-degenerate (matrix rank $2n$) $h$ we have $k + 2 \geq n + 1$. Employing this is equivalent to $q_0 \leq 1 + \varepsilon$. This, together with the fact that $q_0$ has to be even for $D$ odd and vice versa, proves (28).

Let us now consider a nontrivial $V_0$, that is $V_0$ with $q_0 = 1 + \varepsilon$, $n = 1$. The vectors spanning it can be found as the eigenvectors of the operator $F^2$ with zero eigenvalue, not belonging to $U$. There is, however, a more direct way which was already used
by Marck in 4D. Let us consider a Killing–Yano $(2 + \varepsilon)$-form\(^4\)

\[ f = *h^\lambda_{\kappa}, \] \hspace{1cm} (31)

and use it to define a $(1 + \varepsilon)$-form

\[ z = u \wedge f. \] \hspace{1cm} (32)

Using the relation

\[ X \wedge \ast \alpha = * (\alpha \wedge X^b), \] \hspace{1cm} (33)

and the equation \((30)\) one obtains

\[ z = u \wedge \ast h^\lambda_{\kappa} = * (h^\lambda_{\kappa} \wedge u^b) = *(F^\lambda_{\kappa} \wedge u^b). \] \hspace{1cm} (34)

Employing \((25)\) we have

\[ F^\lambda_{\kappa} = B \hat{s}^1 \wedge \hat{s}^2 \wedge \ldots \wedge \hat{s}^{2p}, \quad B = k! \prod_{\mu=1}^{p} \lambda_{\mu}. \] \hspace{1cm} (35)

This means that \(z\) spans \(V_0^*\). In the even number of spacetime dimensions the space \(V_0^*\) is 1-dimensional and \(\hat{s}^0 = z/|z|\). Hence, using \((22)\),

\[ n_0 = z^2/|z| \] spans \(V_0\). In the odd number of spacetime dimensions

\[ z = \text{const} \hat{s}^0 \wedge \hat{s}^0. \] \hspace{1cm} (36)

Hence, the 2-form \(z\) determines the orthonormal basis \(\{n_0, \hat{n}_0\}\) in \(V_0\) up to a 2D rotation.

Let us finally consider the odd dimensional case in more detail. Expanding the characteristic equation for the operator \(F\) one has

\[ 0 = \det(F - \lambda I) = a(u) + b(u)\lambda^2 + \ldots \] \hspace{1cm} (37)

The condition that \(q_0 = 2\) implies that \(a(u) = \det(F) = 0\). This imposes a constraint on \(u\). It means that \(q_0 = 2\) is a degenerate case which happens only for special trajectories \(u\). For a generic (not special) \(u\) one has trivial \(V_0\) with \(q_0 = 0\).

2. Eigenspaces \(V_\mu\)

We call \(h\) strictly non-degenerate if it is non-degenerate and its eigenvalues are different, that is each of the corresponding Darboux subspaces has not more than 2 dimensions. It is possible to show (see Appendix A) that for a strictly non-degenerate \(h\) the dimensionalities of the eigenspaces of \(F\) with non-zero eigenvalues obey the inequalities \(q_\mu \leq 2\). The case of \(q_\mu = 2\) is possible only in a degenerate case when the vector \(u\) obeys a special condition.

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\(^4\) It was demonstrated in [7] (see also [3, 10]) that an exterior product of closed CKY tensors is again a closed CKY tensor. The Hodge dual of a closed CKY tensor is a Killing–Yano tensor.
The parallel-propagated basis can be obtained from the comoving basis by time dependent rotations in the eigenspaces of $F$. We denote the corresponding matrix of rotations by $O(\tau)$. Similar to $F$ it has the following structure

$$O = \text{diag}(O_0, O_1, \ldots, O_6).$$

For $\lambda_\mu > 0$, $O_\mu$ are $2g_\mu \times 2g_\mu$ orthogonal matrices. Let $\{p_\mu, \bar{p}_\mu\}$ be a parallel-propagated basis in the eigenspace $V_\mu$ and $\{n_\mu, \bar{n}_\mu\}$ be the 'original' comoving basis then

$$\left( {s'p}_\mu \over {s'\bar{p}_\mu} \right) = \sum_{s'=1}^{q_\mu} O_\mu s' s' \left( {s'n}_\mu \over {s'\bar{n}_\mu} \right).$$

Here, for fixed values $\{\mu, s, s'\}$, $O_\mu s' s'$ are $2 \times 2$ matrices. Differentiating along the geodesic and using the fact that $\{p_\mu, \bar{p}_\mu\}$ are parallel-propagated one gets

$$\sum_{s'=1}^{q_\mu} \frac{\dot{O}_\mu s' s'}{s' n_\mu} = - \sum_{s'=1}^{q_\mu} O_\mu s' s' \left( {s'n}_\mu \over {s'\bar{n}_\mu} \right).$$

This gives the following set of the first order differential equations for $O_\mu s'$,

$$\dot{O}_\mu s' = \sum_{s'=1}^{q_\mu} O_\mu s' s' N_\mu s'',$n
t

where

$$N_\mu s'' = \left( \left( {s'n}_\mu, {s''n}_\mu \right), \left( {s'\bar{n}_\mu}, {s''\bar{n}_\mu} \right) \right).$$

For generic geodesics the parallel transport equations are greatly simplified. In this case each of the eigenspaces $V_\mu$ is two-dimensional. The equations take the form

$$p_\mu = \cos \beta_\mu n_\mu - \sin \beta_\mu \bar{n}_\mu,$n
t

$$\bar{p}_\mu = \sin \beta_\mu \bar{n}_\mu + \cos \beta_\mu n_\mu.$$n
t

It is easy to check that the equations reduce to the following first order equations

$$\dot{\beta}_\mu = \left( \bar{n}_\mu, n_\mu \right) = - \left( n_\mu, \bar{n}_\mu \right).$$

If at the initial point $\tau = 0$ bases $\{p\}$ and $\{n\}$ coincide, the initial conditions for the equations are

$$\beta_\mu(\tau = 0) = 0.$$n
t

For $\lambda_0 = 0$, $O_0$ is a $g_0 \times g_0$ matrix. In even number of spacetime dimensions, $g_0 = 1$, and $V_0$ is spanned by $n_0$ which is already parallel-propagated. Therefore we have $O_0 = 1$. For odd number of spacetime dimensions, $O_0$ is present only in the degenerate case, $g_0 = 2$, that is when $V_0$ is spanned by $\{n_0, \bar{n}_0\}$. The parallel-propagated vectors $\{p_0, \bar{p}_0\}$ are then given by the analogue of the equations.

**IV. PARALLEL TRANSPORT IN KERR-NUT-(A)dS SPACETIMES**

**A. Kerr-NUT-(A)dS spacetimes**

In a Kerr-NUT-(A)dS spacetime of arbitrary dimension ($D > 2$) the metric can be written in the form

$$g = \sum_{\mu=1}^{n-1} (\omega^\mu \omega^\mu + \bar{\omega}^\mu \bar{\omega}^\mu) + \omega^\mu \omega^\mu - \bar{\omega}^\mu \bar{\omega}^\mu + \epsilon \omega^\mu \omega^\mu,$n
t

where the basis 1-forms are

$$\omega^\mu = \frac{dr}{\sqrt{Q_n}}, \quad \omega^\mu = \frac{d\theta^\mu}{\sqrt{Q_n}}, \quad \mu = 1, \ldots, n - 1,$n
t

$$\bar{\omega}^\mu = \frac{d\psi^\mu}{\sqrt{Q_n}}, \quad \mu = 1, \ldots, n,$n
t

$$\omega^\mu = \sqrt{-c} \frac{d\tau}{A^{(n)}} \sum_{j=0}^{n} A^{(j)} d\psi_j.$$n
t

Notice that we enumerate the basis $\{\omega\}$ so that $\bar{\omega}$ is (the only one) timelike 1-form. Here,

$$A^{(j)} = \sum_{\nu_1 < \cdots < \nu_j} x_{\nu_1}^2 \cdots x_{\nu_j}^2, \quad A^{(j)} = \sum_{\nu_1 < \cdots < \nu_j} x_{\nu_1}^2 \cdots x_{\nu_j}^2,$n
t

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^{n} (x_{\nu}^2 - x_{\mu}^2), \quad x_n^2 = - r^2,$n
t

$$X_n = - \sum_{k=\epsilon}^{n} c_k (-r^2)^k - 2m - \epsilon c \frac{r^2}{x_{\mu}^2},$$n
t

$$X_\mu = \sum_{k=\epsilon}^{n} c_k x_{\mu}^{2k} - 2b_{\mu} x_{\mu}^{1-\epsilon} + \epsilon c \frac{x_{\mu}^2}{x_{\mu}^2}.$$n
t

Time is denoted by $\psi_0$, azimuthal coordinates by $\psi_j$, $j = 1, \ldots, D - n - 1$, $r$ is the Boyer-Lindquist type radial coordinate, and $x_\mu$, $\mu = 1, \ldots, n - 1$, stand for latitude coordinates. The parameter $c_n$ is proportional to the cosmological constant.

$$R_{ab} = (-1)^n (D - 1) c_n g_{ab},$$n
t

and the remaining constants $c_k$, $c > 0$, and $b_\mu$ are related to rotation parameters, mass, and NUT parameters.

More generally, it is possible to consider a broader class of metrics where $X_n(r), X_\mu(x_\mu)$, are arbitrary functions. To stress that such metrics do not necessarily satisfy the Einstein equations we call them off-shell metrics. All the statements and formulas formulated for the Kerr-NUT-(A)dS solutions below are also valid off-shell.
The principal CKY tensor reads
\[ h = \sum_{\mu=1}^{n-1} x_\mu \omega^\mu \wedge \bar{\omega}^\mu - r \omega^\hat{n} \wedge \bar{\omega}^\hat{n}. \] (56)

This means that basis \{\omega\} is a principal one. This principal basis has an additional nice property; namely that many of the Ricci coefficients of rotation vanish \[27\]. We call this special principal basis canonical. The second-rank irreducible Killing tensors are \[3, 2, 8\] (\(j = 1, \ldots, D-n-1\))
\[ K^{(j)} = \sum_{\mu=1}^{n-1} A^{(j)}_{\mu} (\omega^\mu \omega^\mu + \bar{\omega}^\mu \bar{\omega}^\mu) \]
\[ + A^{(j)}_{\mu} (\omega^\hat{n} \omega^\hat{n} - \bar{\omega}^\hat{n} \bar{\omega}^\hat{n}) + \varepsilon A^{(j)} \omega^\hat{\mu} \omega^\hat{\mu}. \] (57)

The geodesic motion of a particle in the Kerr-NUT-(A)dS spacetime is completely integrable \[4, 6\] and its velocity reads \[3, 10\]:
\[ u^\mu = \sum_{\mu=1}^{n} (u_\mu \omega^\mu + \bar{u}_\mu \bar{\omega}^\mu) + \varepsilon u_\zeta \omega^\zeta, \] (58)
where the vielbein components of the velocity are
\[ u_\mu = \frac{\sigma_n}{(X_n U^i)_{1/2}} (W_n^2 - X_n V_n)_{1/2}, \]
\[ u_\mu = \frac{\sigma_\mu}{(X_\mu U^i)_{1/2}} (X_\mu V_\mu - W_\mu^2)_{1/2}, \]
\[ \bar{u}_\mu = \frac{W_n}{(X_n U^i)_{1/2}}, \bar{u}_\mu = \frac{W_\mu}{(X_\mu U^i)_{1/2}}, \]
\[ u_\zeta = \frac{\Psi_n}{\sqrt{-cA(n)}}. \] (59)

Here constants \(\sigma_\mu = \pm 1 (\mu = 1, \ldots, n)\) are independent of one another and we have defined
\[ V_n = -\sum_{j=0}^{m} r^{2(n-j)} \kappa_j, \quad V_\mu = \sum_{j=0}^{m} (-x_\mu^2)^{n-j} \kappa_j, \]
\[ W_n = -\sum_{j=0}^{m} r^{2(n-j)} \Psi_j, \quad W_\mu = \sum_{j=0}^{m} (-x_\mu^2)^{n-j} \Psi_j. \] (60)

The quantities \(\Psi_j\) and \(\kappa_j\) are conserved and connected with the Killing vectors and the Killing tensors, respectively. The constant \(\kappa_0\) denotes the normalization of the velocity \(\kappa_n = u^\mu u_\mu = -1\) and
\[ \kappa_n = -\frac{\Psi_n^2}{c}. \] (61)

We shall construct a parallel-propagated frame for geodesic motion in 3 steps. At first we use the freedom of local rotations in the 2D Darboux spaces of \(h\) to introduce the velocity adapted principal basis in which \(n\) components of the velocity vanish. As the second step, by studying the eigenvalue problem for the operator \(F^2\) we find a transformation connecting the velocity adapted basis to a comoving basis. And finally, we derive the equations for the rotation angles in the eigenspaces of \(F\) which transform the obtained comoving basis into the parallel-propagated one.

**B. Velocity adapted principal basis**

To construct the velocity adapted principal basis we perform the boost transformation in the \{\(\omega^\mu, \omega^\hat{n}\)\} 2-plane and the rotation transformations in each of the \{\(\omega^\mu, \omega^\hat{n}\), \(\mu < n\), 2-planes:
\[ \bar{\sigma}^\mu = \cosh \alpha_\mu \omega^\mu + \sinh \alpha_\mu \omega^\hat{n}, \]
\[ \bar{\sigma}^\hat{n} = \sinh \alpha_\mu \omega^\mu + \cosh \alpha_\mu \omega^\hat{n}, \]
\[ \bar{\sigma}^\mu = \cos \alpha_\mu \omega^\mu + \sin \alpha_\mu \omega^\hat{n}, \]
\[ \bar{\sigma}^\hat{n} = -\sin \alpha_\mu \omega^\mu + \cos \alpha_\mu \omega^\hat{n}, \]
\[ \bar{\sigma}^\zeta = \omega^\zeta. \] (62)

For arbitrary angles \(\alpha_\mu (\mu = 1, \ldots, n)\) this transformation preserves the form of the metric and of the principal CKY tensor:
\[ g = \sum_{\mu=1}^{n-1} (\bar{\sigma}^\mu \bar{\sigma}^\mu + \bar{\sigma}^\hat{n} \bar{\sigma}^\hat{n}) + \bar{\sigma}^\mu \bar{\sigma}^\hat{n} - \bar{\sigma}^\hat{n} \bar{\sigma}^\hat{n} + \varepsilon \bar{\sigma}^\zeta \bar{\sigma}^\zeta, \]
\[ h = \sum_{\mu=1}^{n-1} x_\mu \bar{\sigma}^\mu \wedge \bar{\sigma}^\mu - r \bar{\sigma}^\hat{n} \wedge \bar{\sigma}^\hat{n}. \] (63)

Let us define
\[ \bar{v}_\mu = -\sqrt{u_\mu^2 - u_\mu^2} = \sqrt{V_\mu}, \]
\[ \bar{v}_\mu = \sqrt{\bar{u}_\mu^2 + \bar{u}_\mu^2} = \sqrt{W_\mu}. \] (64)

Then, specifying the values of \(\alpha_\mu\) to be
\[ \cosh \alpha_\mu = \frac{\bar{v}_\mu}{\bar{v}_\mu}, \quad \sinh \alpha_\mu = \frac{\bar{u}_\mu}{\bar{v}_\mu}, \]
\[ \cos \alpha_\mu = \frac{\bar{u}_\mu}{\bar{v}_\mu}, \quad \sin \alpha_\mu = \frac{\bar{u}_\mu}{\bar{v}_\mu}, \] (65)

one obtains the following form of the velocity
\[ u^\mu = \sum_{\mu=1}^{n} \bar{v}_\mu \bar{\sigma}^\mu + \varepsilon u_\zeta \bar{\sigma}^\zeta. \] (66)

It means that after this transformation the velocity vector \(u\) has only \((n + \varepsilon)\) non-vanishing components. This simplifies considerably the construction of the comoving and the parallel-propagated
bases. Notice also that the boost in the \(\{\hat{\omega}^\mu, \omega^\mu\}\) 2-plane is function of \(r\) only and the rotation in each \(\{\hat{\omega}^\mu, \omega^\mu\}\) 2-plane is function of \(x_\mu\) only. The components of the velocity in the adapted basis \(\{\mathbf{o}\}\) depend on constants \(\kappa_j\) only; constants \(\Psi_j\) and \(\sigma_\mu\) are absorbed in the definition of the new frame.

\[ \hat{\beta}_\mu = \frac{f^{(\mu)}(r)}{U_n} + \sum_{\nu=1}^{n-1} \frac{f^{(\mu)}(x_\nu)}{U_\nu}, \]  

C. Parallel-propagated frame

It is obvious from the expression \(\{66\}\) that at a generic point of the manifold \(\{62\}\) the principal CKY tensor \(h\) is strictly non-degenerate and we may use the theory described in Sections II and III. In particular, for generic geodesics the operator \(F^2\) possesses twice degenerate non-zero eigenvalues, and the nontrivial eigenspace \(V_0\), which is present only in even dimensions, is 1-dimensional space determined by the properly normalized \(z^i\). Therefore the problem of finding the parallel-propagated frame in Kerr-NUT-(A)dS spacetimes reduces to finding the eigenvectors \(\{n_\mu, \bar{n}_\mu\}\) spanning the 2-plane eigenspaces \(V_\mu\) and subsequent 2D rotations \(\{69\}\) in these spaces.

A degenerate case which requires a special consideration arises when initially different elements of the spectrum \(S(F)\), \(\{66\}\), coincide one with another. It happens for special values of the integrals of motion characterizing the geodesic trajectories. The larger is the number of spacetime dimensions the larger is the number of different degenerate cases. Some of them will be discussed in the next section.

In our setup it is somewhat more natural to construct, instead of the vector basis \(\{p\}\), the parallel-propagated basis of forms \(\{\pi\}\). In the generic case it consists of

\[ \{u^i, z, \pi^1, \bar{\pi}^1, \ldots, \pi^n, \bar{\pi}^n\}. \]  

(The element \(z\) is present only in even dimensions.) If \(\{\hat{\pi}^i, \hat{\sigma}^i\}\) are comoving basis forms spanning \(V_\mu^*\), then (cf. Eq. \(\{69\}\))

\[ \begin{align*} 
\pi^\mu &= \hat{\pi}^\mu \cos \beta_\mu - \hat{\sigma}^\mu \sin \beta_\mu, \\
\bar{\pi}^\mu &= \hat{\pi}^\mu \sin \beta_\mu + \hat{\sigma}^\mu \cos \beta_\mu, 
\end{align*} \]  

where

\[ \dot{\beta}_\mu = (\hat{\pi}^i, \hat{\sigma}^i) = - (\hat{\sigma}^i, \hat{\pi}^i), \]  

with the initial condition \(\beta_\mu(\tau = 0) = 0\).

The rotation angles \(\beta_\mu\) as given by \(\{69\}\) are functions of \(r\) and \(x_\mu\). In the case when \(\beta_\mu\) can be brought into the form

\[ \begin{align*} 
\beta_\mu &= \int \frac{f^{(\mu)}(r)}{U_n} dr + \sum_{\nu=1}^{n-1} \int \frac{f^{(\mu)}(x_\nu)}{U_\nu} dx_\nu, 
\end{align*} \]  

the problem \(\{69\}\) is separable and the particular solution is given by (see Appendix B)

\[ \beta_\mu = \int \frac{\sigma_\mu f^{(\mu)}(r)}{\sqrt{W_n^2 - X_n V_n}} - \sum_{\nu=1}^{n-1} \int \frac{\sigma_\nu f^{(\nu)}(x_\nu)}{\sqrt{X_\nu V_\nu - W_\nu}}. \]  

V. EXAMPLES

We shall now illustrate the above described formalism by considering \(D = 3, 4, 5\) Kerr-NUT-(A)dS spacetimes. We take the normalization of the velocity \(\kappa_0 = -1\) and normalize other vectors of the parallel-transported frame to +1. In the derivation of the equations for \(\beta_\mu\) we used the Maple program.

A. 3D spacetime: BTZ black holes

1. Generic case

As the first example we consider the case when \(D = 3\), that is when the metric \(\{62\}\) describes a BTZ black hole \(\{28\}\). We first discuss the generic case, \(q_0 = 0\), and then briefly mention what happens for the degenerate geodesics with \(q_0 = 2\). Since in three dimensions \(n = 1\) we drop everywhere index \(\mu\).

So, we have the metric

\[ g = -\omega^2 + \omega^2 + \omega^2 \omega^i, \]  

where

\[ \omega = \sqrt{X} d\psi_0, \quad \omega = \frac{dr}{\sqrt{X}}, \quad \omega^i = \sqrt{\frac{c}{r}} (d\psi_0 - r^2 d\psi_1), \quad X = c_1 r^2 - 2m + \frac{c}{r^2}. \]  

The parameter \(c_1\) is proportional to the cosmological constant and parameters \(m\) and \(c > 0\) are related to mass and rotation parameter.

The principal CKY tensor and the Killing tensor are:

\[ h = -r \omega^i \omega^i, \quad K = -r^2 \omega^i \omega^i. \]  

The velocity

\[ u^i = \tilde{u} \omega + u \omega + u_\xi \omega^\xi, \]  

has the components

\[ \tilde{u} = \frac{W}{\sqrt{X}}, \quad u = \frac{\sigma}{\sqrt{W^2 X + V}}, \quad u_\xi = \frac{\Psi_1}{\sqrt{cr}}. \]
where
\[ W = -\Psi_0 - \frac{\Psi_1}{r^2}, \quad V = 1 + \frac{\Psi_1^2}{cr^2}. \]

In the velocity adapted frame \( \{ \bar{o}, \bar{o}, \bar{o} \} \) given by (62) we have
\[ u^\xi = \bar{v}\bar{o} + u_0\bar{o}^\xi, \quad \bar{v} = -\sqrt{V}, \quad F = ru_0\bar{o} \wedge (u_0\bar{o} + \bar{v}\bar{o}^\xi). \]

The spectrum (20) of \( F \) is
\[ S(F) = \{ 0, \lambda, \lambda \}, \quad \lambda = \frac{|\Psi_1|}{\sqrt{c}}. \]

The zero eigenvalue corresponds to the space \( U^* \) spanned by \( u^\xi \). In the non-degenerate case, that is when \( \Psi_1 \neq 0 \), the eigenspace \( V_0^* \) is trivial. The orthonormal forms spanning \( V_0^* \) are:
\[ \zeta = o^\xi, \quad \bar{\zeta} = u_0\bar{o} + \bar{v}\bar{o}^\xi. \]

Using (69) one finds
\[ \dot{\beta} = \frac{M}{r^2 + \lambda^2}, \quad M = \frac{c - \Psi_0\Psi_1}{\sqrt{c}}. \]

The parallel-transported forms \( (\pi, \bar{\pi}) \) are given by (68), where
\[ \beta = \frac{\sqrt{M}dr}{(r^2 + \lambda^2)\sqrt{W^2 - XV}}. \]

2. Degenerate case

Let us now consider special geodesic trajectories with \( \Psi_1 = 0 \) for which \( \varphi_0 = 2 \). For such trajectories one has
\[ \bar{u} = -\frac{\Psi_0}{\sqrt{X}}, \quad u = \sigma\sqrt{\frac{\Psi_0^2}{X} - 1}, \quad u_i = 0. \]

In the adapted basis the velocity is \( u^\xi = -\bar{\omega} \). Operators \( F \) and \( F^2 \) become trivial. The space \( V_0^* \) is spanned by \( \{ \bar{o}^\xi, \bar{o}^0 \} \), where
\[ \bar{o}^0 = o^\xi, \quad \bar{o}^0 = -\bar{\omega}^\xi. \]

Similar to (68) and (69) parallel-transported forms can be written as follows
\[ \pi = \bar{o}^\xi \cos \beta - \bar{o}^0 \sin \beta, \]
\[ \bar{\pi} = \bar{o}^0 \sin \beta + \bar{o}^\xi \cos \beta, \]
\[ \dot{\beta} = (\bar{o}^\xi, \bar{o}^0) = -\langle \bar{o}^0, \bar{o}^\xi \rangle, \]
with the initial condition \( \beta(\tau = 0) = 0 \). Using these equations we find \( \dot{\beta} = \sqrt{c}/r^2 \) and hence
\[ \beta = \int \frac{\sigma \sqrt{c}dr}{r^2 \sqrt{\Psi_0^2 - X}}. \]

Notice that this relation can be obtained from (88) by taking the limit \( \Psi_1 \to 0 \).

To conclude, the parallel-propagated orthonormal frame around a BTZ black hole is \( \{ u^\xi, \pi, \bar{\pi} \} \). This frame remains parallel-propagated also off-shell, when \( X \) given by (73) becomes an arbitrary function of \( r \).

### B. 4D spacetime: Carter’s family of solutions

Let us now consider the case of \( D = 4 \). We have
\[ g = -\bar{\omega}^2 + \omega^2 \bar{\omega}^2 + \bar{\omega}^1 \omega^1 + \omega^1 \bar{\omega}^1, \]
where
\[ \bar{\omega}^2 = \sqrt{\frac{X_2}{U_2}}(d\psi_0 + x_1d\psi_1), \quad \omega^2 = \sqrt{\frac{U_2}{X_2}}dr, \]
\[ \bar{\omega}^1 = \sqrt{\frac{X_1}{U_1}}(d\psi_0 - r^2d\psi_1), \quad \omega^1 = \sqrt{\frac{U_1}{X_1}}dx_1. \]

Here, \( U_2 = -U_1 = x_1^2 + r^2 \), and we shall not be specifying functions \( X_1(x_1), X_2(r) \) at this point.

The principal CKY tensor and the Killing tensor are:
\[ h = x_1\omega^1 \wedge \bar{\omega}^1 - r\omega^2 \wedge \bar{\omega}^2, \]
\[ K = x_1^2(\omega^2 \bar{\omega}^2 - \bar{\omega}^2 \omega^2) - r^2(\bar{\omega}^2 \omega^1 + \omega^2 \bar{\omega}^1). \]

The components of the velocity are
\[ \bar{u}_2 = \frac{W_2}{\sqrt{X_2U_2}}, \quad u_2 = \frac{\sigma_2}{\sqrt{X_2U_2}}\sqrt{\bar{\omega}^2 - X_2V_2}, \]
\[ \bar{u}_1 = \frac{W_1}{\sqrt{X_1U_1}}, \quad u_1 = \frac{\sigma_1}{\sqrt{X_1U_1}}\sqrt{X_1V_1 - W_1^2}, \]
where
\[ W_2 = -r^2\Psi_0 - \Psi_1, \quad V_2 = r^2 - \kappa_1, \]
\[ W_1 = -x_1^2\Psi_0 + \Psi_1, \quad V_1 = x_1^2 + \kappa_1. \]

The constants of geodesic motion \( \Psi_0 \) and \( \Psi_1 \) are associated with isometries and \( \kappa_1 < 0 \) corresponds to the Killing tensor (91). In the velocity adapted frame \( \{ \bar{\omega}^2, \omega^2, \bar{\omega}^1, \omega^1 \} \) given by (62) we have
\[ u^\xi = \bar{v}_2\bar{\omega}^2 + \bar{v}_1\bar{\omega}^1, \quad \bar{v}_2 = -\frac{W_2}{U_2}, \quad \bar{v}_1 = \sqrt{\frac{V_1}{U_1}} \]
\[ F = (r\bar{v}_1\bar{\omega}^2 + x_1\bar{v}_2\bar{\omega}^1) \wedge (\bar{v}_1\bar{\omega}^2 + \bar{v}_2\bar{\omega}^1). \]
The spectrum \( \{20\} \) of \( F \) is
\[
S(F) = \{0, 0, \lambda, \lambda\}, \quad \lambda = \sqrt{-\kappa_1}. \quad (96)
\]
The first zero eigenvalue corresponds to \( U^* \), while the second one corresponds to the eigenspace \( V^*_A \), spanned by 1-form \( z \) \( \{52\} \). When normalized \( z \) reads:
\[
z = \lambda^{-1}(-x_1 \bar{v}_2 \sigma^2 + r \bar{v}_1 \sigma^1). \quad (97)
\]
The orthonormal forms spanning \( V^*_A \) are:
\[
\zeta = \bar{v}_1 \sigma^2 + \bar{v}_2 \sigma^1, \quad \bar{\zeta} = \lambda^{-1}(r \bar{v}_1 \sigma^2 + x_1 \bar{v}_2 \sigma^1). \quad (98)
\]
Using \( \{69\} \) one finds
\[
\beta = \frac{M}{(x_1^2 - \lambda^2)(r^2 + \lambda^2)} = \frac{f_1}{U_1} + \frac{f_2}{U_2},
\]
\[
f_1 = -\frac{M}{x_1^2 - \lambda^2}, \quad f_2 = \frac{M}{r^2 + \lambda^2}, \quad (99)
\]
where \( M = \lambda(\Psi_1 - \lambda^2 \Psi_0) \). Therefore, \( \beta \) allows a separation of variables and can be written as
\[
\beta = \int \frac{\sigma_2 f_2 dx}{\sqrt{W_2 - X_2 Y_2}} - \int \frac{\sigma_1 f_1 dx_1}{\sqrt{X_1 Y_1 - W_1}}, \quad (100)
\]
where functions \( f_1, f_2 \) are defined in \( \{69\} \). The parallel-transported forms \( \{\pi, \bar{\pi}\} \) are given by \( \{68\} \).

To summarize, the parallel-propagated orthonormal frame in the spacetime \( \{88, 89\} \) is \( \{u^b, \zeta, \pi, \bar{\pi}\} \). This parallel-propagated basis is constructed for arbitrary functions \( X_1(x_1), X_2(r) \), and in particular for the Carter’s class of solutions \( \{29, 30\} \)—describing among others a rotating charged black hole in the cosmological background. So we have re-derived the results obtained earlier in \( \{11, 13\} \).

### C. 5D Kerr-NUT-(A)dS spacetime

#### 1. Generic case

As the last example we consider the 5D off-shell Kerr-NUT-(A)dS spacetime. Similar to the 3D case we shall first obtain the parallel-propagated frame for generic geodesics and then briefly discuss what happens for the special trajectories characterized by \( q_0 = 2 \), or \( q_1 = 2 \). The metric reads
\[
g = -\omega^2 \omega^2 + \omega^2 \omega^2 + \omega^2 \omega^1 + \omega^1 \omega^1 + \omega^1 \omega^1 + \omega^1 \omega^1, \quad (101)
\]
where
\[
\omega^2 = \sqrt{\frac{X_2}{U_2}(d\psi_0 + x_1 d\psi_1)}, \quad \omega^2 = \sqrt{\frac{U_2}{X_2}} dr,
\]
\[
\omega^1 = \sqrt{\frac{x_1}{U_1}(d\psi_0 - r^2 d\psi_1)}, \quad \omega^1 = \sqrt{\frac{U_1}{X_1}} dx_1,
\]
\[
\omega^2 = \sqrt{\frac{r}{x_1}} [d\psi_0 + (x_1 - r^2) d\psi_1 - x_1 r^2 d\psi_2], \quad (102)
\]
and \( U_2 = -U_1 = x_1^2 + r^2 \). The principal CKY tensor and the Killing tensor for this metric are:
\[
h = x_1 \omega^1 \wedge \omega^1 - r \omega^2 \wedge \omega^2, \quad (103)
\]
\[
K = x_1^2(-\omega^2 \omega^2 + \omega^2 \omega^2 + \omega^1 \omega^1 - \omega^1 \omega^1) - r^2(\omega^1 \omega^1 + \omega^1 \omega^1 + \omega^2 \omega^2). \quad (104)
\]
The components of the velocity are
\[
\bar{u}_2 = \frac{W_2}{\sqrt{X_2 U_2}}, \quad u_2 = \frac{\sigma_2}{\sqrt{X_2 U_2}} \sqrt{W_2^2 - X_2 V_2},
\]
\[
\bar{u}_1 = \frac{W_1}{\sqrt{X_1 U_1}}, \quad u_1 = \frac{\sigma_1}{\sqrt{X_1 U_1}} \sqrt{X_1 V_1 - W_1^2},
\]
\[
u_\bar{c} = \frac{\Psi_2}{\sqrt{X_1^2 r^2}}, \quad (105)
\]
where
\[
W_1 = -x_1^2 \Psi_0 - \Psi_1 \frac{\Psi_2}{x_1^2}, \quad V_1 = x_1^2 + \kappa_1 + \Psi_2 \frac{\Psi_2}{x_1^2},
\]
\[
W_2 = -r^2 \Psi_0 - \Psi_1 - \frac{\Psi_2}{r^2}, \quad V_2 = r^2 + \kappa_1 + \Psi_2 \frac{\Psi_2}{r^2}. \quad (106)
\]
In the velocity adapted frame \( \{\bar{a}\} \) given by \( \{62\} \) we have
\[
\bar{u}^b = \tilde{v}_2 \sigma^2 + \tilde{v}_1 \sigma^1 + u_2 \sigma^2,
\]
\[
\tilde{v}_2 = \frac{\sqrt{V_2}}{U_2}, \quad \tilde{v}_1 = \frac{V_1}{U_1}. \quad (107)
\]
The form \( F \) is
\[
F = (r \tilde{v}_1 \sigma^2 + x_1 \tilde{v}_2 \sigma^2) \wedge (\tilde{v}_1 \sigma^2 + \tilde{v}_2 \sigma^1)
\]
\[
+ ru_2 \sigma^2 \wedge (\tilde{v}_2 \sigma^2 + u_2 \sigma^2)
\]
\[
+ x_1 u_2 \sigma^1 \wedge (\tilde{v}_1 \sigma^2 - u_1 \sigma^1). \quad (108)
\]
The 2-form \( z \) \( \{32\} \) reads
\[
z = \sigma^2 \wedge (r \tilde{v}_1 \sigma^2 - x_1 \tilde{v}_2 \sigma^2) + u_2(x_1 \sigma^2 \wedge \sigma^2 + r \sigma^2 \wedge \sigma^2). \quad (109)
\]
The spectrum \( \{20\} \) of \( F \) is
\[
S(F) = \{0, \lambda_1, \lambda_2, \lambda_2\}. \quad (110)
\]
The parallel-propagated forms \( \{ \pi^1, \pi^2 \} \) are given by rotation \((88)\), where
\[
\beta_2 = \int \frac{\sigma_2 \hat{f}_2^{(2)}}{\sqrt{W_2^2 - X_2 V_2}} - \int \frac{\sigma_1 \hat{f}_1^{(2)}}{\sqrt{X_1 V_1 - W_1^2}}. \tag{119}
\]

2. Degenerate case

In \( D = 5 \) two different degenerate cases are possible. One can have either a 2-dimensional \( V_0 \) \((q_0 = 2)\) which happens for the special geodesics characterized by \( \Psi_2 = 0 \), or a 4-dimensional \( V_\lambda \) \((q_1 = 2)\) which happens when \( \kappa_1^2 = 4 \Psi_2^2/c \). The latter case is more complicated and the general formulas \((115)\)–\((118)\) have to be used. We shall not do this here and rather concentrate on the first degeneracy which has an interesting consequence.

So we consider the special geodesics characterized by \( \Psi_2 = 0 \). It can be checked by direct calculations that in this case the results can be obtained by taking the limit \( \Psi_2 \to 0 \) of previous formulas. In particular one has \( u_2 = 0 \),
\[
W_1 = - x_2^2 \Psi_0 + \Psi_1, \quad V_1 = x_2^2 + \kappa_1, \\
W_2 = - r^2 \Psi_0 - \Psi_1, \quad V_2 = r^2 - \kappa_1.
\]

The velocity \( u \) becomes effectively 4-dimensional:
\[
u^\flat = \bar{v}_2 \bar{o}^2 + \bar{v}_1 \bar{o}^1. \tag{121}
\]

The form \( F \), \((108)\), becomes degenerate and takes the form
\[
F = (r \bar{v}_1 \bar{o}^2 + x_1 \bar{v}_2 \bar{o}^1) \wedge (\bar{v}_1 \bar{o}^2 + \bar{v}_2 \bar{o}^1). \tag{122}
\]

The 2-form \( z \) \((109)\) reduces to
\[
z = \bar{o}^1 \wedge (r \bar{v}_1 \bar{o}^2 - x_1 \bar{v}_2 \bar{o}^1). \tag{123}
\]

The spectrum is
\[
S(F) = \{ 0, 0, 0, \lambda, \lambda \}, \quad \lambda = \sqrt{-\kappa_1}. \tag{124}
\]

The eigenspace \( V_0^* \) is spread by \( \{ \mathbf{s}^0, \mathbf{s}^0 \} \), where
\[
\mathbf{s}^0 = \bar{o}^1, \quad \mathbf{s}^0 = \lambda^{-1}(r \bar{v}_1 \bar{o}^1 - x_1 \bar{v}_2 \bar{o}^2). \tag{125}
\]

(Note that \( \mathbf{s}^0 \) is identical to the normalized 4-dimensional 1-form \( z \) given by \((97)\).) The angle of rotation in the \( \mathbf{s}^0, \mathbf{s}^0 \) 2-plane obeys the equation
\[
\hat{\beta}_1 = \frac{M^{(1)}}{x_1^2} = \frac{f_1^{(1)}}{U_1} + \frac{f_1^{(2)}}{U_2}, \quad M^{(3)} = \lambda \sqrt{c}, \\
\hat{f}_1^{(1)} = - \frac{M^{(1)}}{x_1^2}, \quad \hat{f}_2^{(1)} = \frac{M^{(1)}}{r^2}. \tag{126}
\]
Thus $\beta_1$ is given by $\{11\}$ with functions $f_1^{(1)}, f_2^{(1)}$ defined in $\{12\}$. The eigenspace $V_\lambda^+$ spread by
\[ \zeta = \bar{v}_1 \sigma^2 + \bar{v}_2 \sigma^1, \quad \zeta = \lambda^{-1} (r \bar{v}_1 \sigma^2 + x_1 \bar{v}_2 \sigma^1), \quad (127) \]
is identical to the $V_\lambda^+$ subspace in the 4D case. Thus the rotation angle $\beta_2$ coincides with $\beta$ given by $\{11\}$.

3. Summary of 5D

To summarize, we have demonstrated that also in $D = 5$ Kerr-NUT-(A)dS spacetime the rotation angles in 2D eigenspaces can be separated and the parallel-transported frame $\{\pi\}$ explicitly constructed. This result is again valid off-shell, that is for arbitrary functions $X_2(r)$, $X_1(x_1)$.

The special degenerate case characterized by $\Psi_2 = 0$ has the following interesting feature. The zero-value eigenspace is spanned by the 4-dimensional $\nu^\nu$, by the 4-dimensional 1-form $\nu^\nu$, and $\nu^\nu$. The structure of $V_\lambda^+$ is identical to the 4D case and the equation of parallel transport in this plane is identical to the equation in 4D. Therefore this 5D degenerate problem effectively reduces to the generic 4D problem plus the rotation in the 2D $\{\nu^\nu, \nu^\nu\}$ plane.

This indicates that a similar reduction might be valid also in higher dimensions. Namely, one may expect that the degenerate odd dimensional problem, $\Psi_2 = 0$, effectively reduces to the problem in a spacetime of one dimension lower plus the rotation in the 2D $\{\nu^\nu, \nu^\nu\}$ plane. If this is so, one can use this odd dimensional degenerate case to generate the solution for the generic (one dimension lower) even dimensional problem.

VI. CONCLUSIONS

In this paper we have described the construction of a parallel-transported frame in a spacetime admitting the principal CKY tensor $h$. This tensor determines a principal (Darboux) basis at each point of the spacetime. The geodesic motion of a particle in such a space can be characterized by the components of its velocity $\nu^\nu$ with respect to this basis. For a moving particle it is also natural to introduce a comoving basis, which is just a Darboux basis of $F$, where $F$ is a projection of $h$ along the velocity $\nu^\nu$. Since $F$ is parallel-propagated along $\nu^\nu$, its eigenvalues are constant along the geodesic and its eigenspaces are parallel-propagated. We have demonstrated that for a generic motion the parallel-propagated basis can be obtained from the comoving basis by simple two-dimensional rotations in the 2D eigenspaces of $F$.

To illustrate the general theory we have considered the parallel transport in the Kerr-NUT-(A)dS spacetimes. Namely, we have newly constructed the parallel-propagated frames in three and five dimensions and re-derived the results $\{11\}$ in 4D. One of the interesting features of the 4D construction, observed already by Marck, is that the equation for the rotation angle allows a separation of variables. Remarkably, we have shown that also in five dimensions equations for the rotation angles can be solved by a separation of variables. Moreover, the 4D result can be understood as a special degenerate case of the 5D construction. Is this a general property valid in the Kerr-NUT-(A)dS spacetime with any number of dimensions? What underlines the separability of the rotation angles? These are interesting open questions.

The analysis of the present paper was restricted to the problem of parallel transport along timelike geodesics. As we already mentioned, the generalization to the case of spatial geodesics is straightforward. The case of null geodesics is a special one and requires additional consideration (see $\{12\}$ for the 4D case). The parallel transport along null geodesics might be of interest for the study of the polarization propagation of massless fields with spin in the geometric optics approximation.

To conclude the paper we would like to mention that described in the paper the possibility of solving the parallel transport equations in the Kerr-NUT-(A)dS spacetime is one more evidence of the miraculous properties of these metrics connected with their hidden symmetries.

APPENDIX A: REMARKS ON THE DEGENERACY OF EIGENVALUES OF $F$

Consider a strictly non-degenerate operator $h$. Let us discuss what restrictions this condition imposes on the operator $F$. If $h$ is strictly non-degenerate, then
\[ \Delta(\lambda) = \det(h^\mu_\nu - \lambda \delta^\mu_\nu) = (-\lambda)^5 \prod_{k=1}^n (\lambda^2 + \nu_k^2), \]
where all $\nu_k$ are different.

Let us re-calculate this determinant in terms of $F$ and compare the results. For this calculation we use the Darboux basis of $F$ and corresponding matrix form of the objects. In particular, we have
the expression (9) for $F$, and

\[ s = (s_0, s_\ldots, s_\mu), \quad s_0 = (s_0, \ldots, q_0 s_0), \]

\[ s_\mu = (s_\mu, s_\ldots, q_\mu s_\mu, q_\mu s_\mu), \quad (A2) \]

for the 1-form $s$. Using (8), (9), and (A2), one can rewrite (A1) as

\[ \Delta(\lambda) = \det(F^\mu_b - u^\mu s_b + s^\mu u_b - \lambda_\delta^\mu_b) = \begin{vmatrix} A & B \\ C & E \end{vmatrix}, \quad (A3) \]

where $A = -\lambda, B = -s, C = -s^T, \text{and } E$ is the $(D - 1)$-dimensional matrix of the form

\[ E = \text{diag}(-\lambda I_{q_0}, 1, \ldots, 1, \ldots, p), \quad (A4) \]

\[ \mu Z = \begin{pmatrix} -\lambda I_\mu & \lambda I_\mu \\ -\lambda I_\mu & -\lambda I_\mu \end{pmatrix}. \]

(A5)

Here $I_{q_0}$ is a unit $q_0 \times q_0$ matrix, and we use $X^T$ to denote a matrix transposed to $X$. It is easy to check that

\[ E^{-1} = \text{diag}(-\lambda^{-1} I_{q_0}, 1, \ldots, 1, \ldots, p^{-1}), \quad (A6) \]

\[ \mu Z^{-1} = Q_\mu^{-1} \mu Z^{-T}, \quad Q_\mu = (\lambda^2 + \mu^2), \quad \det(\mu Z) = Q_\mu. \]

One has the following relation for the determinant of a block matrix (see, e.g., 31)

\[ \begin{vmatrix} A & B \\ C & E \end{vmatrix} = A |E|, \quad A = |A - B E^{-1} C|. \]

Using (A4) and (A5) one finds

\[ \det(E) = (-\lambda)^{q_0} \prod_{\mu = 1}^p Q_\mu^{q_\mu}, \quad A = -\lambda - s E^{-1} s^T. \]

(A7)

Combining all these relations one obtains

\[ \Delta(\lambda) = (-\lambda)^{q_0 - 1} \prod_{\mu = 1}^p Q_\mu^{q_\mu} \times \left[ \lambda^2 \left( 1 - s_\mu^2 \frac{Q_\mu}{Q_\mu} \right) - s_0^2 \right], \quad (A8) \]

\[ s_0^2 = \sum_{i = 1}^{q_0} i^2, \quad s_\mu^2 = \sum_{i = 1}^{q_\mu} (i^2 + s_\mu^2). \]

Let us now compare (A1) and (A8). First of all, let us compare the powers of $(-\lambda)$. For $s^2 \neq 0$ we have match for $q_0 - 1 = 3$, whereas the case $s^2 = 0$ may happen only in odd dimensions and one must have $q_0 = 0$ (cf. 23). Another result of the comparison is that $q_\mu \leq 2$. Really, if $q_\mu > 2$, then at least 2 roots of $\Delta(\lambda)$ in (A8) coincide. This contradicts the assumption previously stated, since for a strictly non-degenerate operator $h$ the characteristic polynomial has only single roots $\lambda^2 = -\nu^2$. The case when $q_\mu = 2$ is degenerate. It is valid only for a special value of the velocity $\mathbf{u}$. Really, in this case one of the eigenvalues, say $\nu_\mu$, of $h$ coincides with one of the eigenvalues of $F$ so that one has $\det(F - \nu I) = 0$. The latter is an equation restricting the value of $\mathbf{u}$.

**APPENDIX B: SEPARABILITY OF ROTATION ANGLES IN KERR-NUT-(A)DS SPACETIMES**

The right hand sides of the equations (69) for rotation angles $\dot{\beta}_\mu$ are in general complicated functions of $r$ and $x_\mu$. However, it turns out that, at least in four and five dimensions, one can find a particular solution for the rotation angles which allows the additive separation of variables. Let us probe this possibility in more detail. The separability means, that we seek the solution in the form

\[ \beta_\mu = S_\mu(r) - \sum_{\nu = 1}^{n - 1} S_\nu(x_\nu). \]

Using (69) and (69) one finds

\[ \dot{\beta}_\mu = \left( S_\mu(r) \right)' \sum_{\nu = 1}^{n - 1} (S_\nu(x_\nu))' \frac{x_\nu}{U_\mu} \frac{\sqrt{X_\nu U_\nu}}{U_\nu}. \]

Prime denotes the derivative with respect to a single argument. For each $\nu$ the numerator of the latter expression is function of one variable only. If $\dot{\beta}_\mu$ given by (69) can be brought into the form

\[ \dot{\beta}_\mu = \frac{f_\mu(r)}{U_\mu} + \sum_{\nu = 1}^{n - 1} \frac{g_\nu(x_\nu)}{U_\mu}, \]

the problem is separable. By comparing (B2) and (B3) we arrive at the relation

\[ \frac{g_\nu(x_\nu)}{U_\mu} + \sum_{\nu = 1}^{n - 1} \frac{g_\nu(x_\nu)}{U_\mu} = 0, \]

where $(\nu = 1, \ldots, n)$

\[ g_\nu(x_\nu) = (S_\nu(x_\nu))' \frac{x_\nu}{U_\mu} \frac{\sqrt{X_\nu U_\nu}}{U_\nu} - f_\nu(\nu). \]

The general solution of (B4) is (see, e.g., 32)

\[ g_\nu(\nu) = \sum_{k = 1}^{n - 1} C_k(\nu)^{-n - 1 - k}, \quad \nu = 1, \ldots, n. \]
However, what we need is a particular solution for which we may choose all constants \( c_k^{(\mu)} = 0 \). Such a particular solution is

\[
\beta_\mu = \int \frac{\sigma_n f_n^{(\mu)}(r)}{\sqrt{W_n^2 - X_n V_n}} - \sum_{\nu=1}^{n-1} \frac{\sigma_\nu f_\nu^{(\mu)}(r)}{\sqrt{X_\nu V_\nu - W_\nu^2}}.
\] (B7)

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