QCD Sum Rules and the $\Pi(1300)$ Resonance

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Abstract

Global fits to the shape of the first QCD Laplace sum rule exhibiting sensitivity to pion-resonance [$\Pi(1300)$] parameters are performed, leading to predictions for the pion-resonance mass and decay constant. Two scenarios are considered which differ only in their treatment of the dimension-six quark condensate $\langle \mathcal{O}_6 \rangle$. The first scenario assumes an effective scale for $\langle \mathcal{O}_6 \rangle$ from other sum-rule applications which is assumed to be independent of the physical value of the quark mass, while the second scenario requires self-consistency between the value of $\langle \mathcal{O}_6 \rangle$ and the current algebra constraint $2m_\pi^2 = -f_\pi^2 m_\pi^2$. Predictions of the pion-resonance mass $M_\pi$ and decay constant $F_\pi$ are obtained in these two scenarios. A byproduct of this analysis is a prediction of the renormalization-group invariant quark mass $(\hat{m}_u + \hat{m}_d)/2$.

QCD sum-rule treatments of the pseudoscalar mesons have long been known to be successful in predicting a near-massless pion with a decay constant $f_\pi$ that upholds current algebra (GMOR) constraints relating the pion parameters $m_\pi$ and $f_\pi$ to $\langle m\bar{q}q \rangle$. In this approach, the correlation function of charged axial vector currents $J_5^\mu(x) = \bar{u}(x)\gamma^\mu\gamma^5d(x)$ is considered:

$$\Pi_{\mu\nu}(q) = i\int d^4xe^{iq\cdot x}\langle O|T\left(J_5^\mu(x)\bar{J}_\nu(0)^*\right)|O\rangle$$

$$= \left(g_{\mu\nu} - q_{\mu}q_{\nu}/q^2\right)\Pi^T(q^2) + \frac{q_{\mu}q_{\nu}}{q^2}\Pi^L(q^2).$$  \hspace{1cm} (1)

The longitudinal part $\Pi^L(q^2)$ of this correlation function is related to the pseudoscalar resonances with the quantum numbers of the pion.

In the work presented here, we use the QCD Laplace sum-rules for $\Pi^L(q^2)$ to make explicit predictions concerning the mass $M_\pi$ and the decay constant $F_\pi$ of the first excited (pion-resonance) state. To leading order in the quark mass, the QCD Laplace sum rules for $\Pi^L(s)$ and their relation to the QCD continuum and the pseudoscalar resonances in the narrow width approximation are given by...
\[ \frac{1}{\pi} \int_{0}^{\infty} ds \, \text{Im} \Pi^L(s) e^{-s\tau} = \mathcal{R}_0(\tau) \]  
(2)

\[ -4m\langle \bar{q}q \rangle + O(m^2) = 2f_\pi^2m_\pi^2e^{-m_\pi^2\tau} + 2F_\pi^2M_\pi^2e^{-M_\pi^2\tau} + \ldots \]  
(3)

\[ \frac{1}{\pi} \int_{0}^{\infty} ds \, \text{Im} \Pi^L(s)s e^{-s\tau} = \mathcal{R}_1(\tau) \]  
(4)

\[ 2f_\pi^2m_\pi^4e^{-m_\pi^2\tau} + 2F_\pi^2M_\pi^4e^{-M_\pi^2\tau} + m^2c_1(\tau, s_0) = \]  
\[ m^2 \left( \frac{3}{2\pi^2\tau^2} \left[ 1 + \frac{17\alpha}{3\pi} \right] - \frac{3}{\pi^2\tau^2} \left[ 1 - \gamma_E \right] - 4\langle \bar{q}q \rangle + \frac{1}{2\pi}(\alpha G^2) \right. \]  
\[ + \pi\langle O_6 \rangle \tau + \frac{3\rho_0^2}{2\pi^2\tau^3}e^{-\frac{\rho_0^2}{2\pi\tau}} \left[ K_0 \left( \frac{\rho_0^2}{2\pi\tau} \right) + K_1 \left( \frac{\rho_0^2}{2\pi\tau} \right) \right] \right) + O(m^3) \]  
(5)

\[ c_1(\tau, s_0) = \frac{3}{2\pi^2\tau^2} \left[ 1 + \frac{17\alpha}{3\pi} \right] \left[ 1 + s_0\tau \right] e^{-s_0\tau} \]  
\[ -2\frac{\alpha}{\pi} \left[ e^{-s_0\tau} + E_1(s_0\tau) + (1 + s_0\tau)e^{-s_0\tau}\log(s_0\tau) \right] \]  
(6)

where \( m = (m_u + m_d)/2, \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle \), and the functions \( E_1(x) \) and \( K_n(x) \) are respectively the exponential integral and modified Bessel functions [7]. The quantity \( m^2c_1(\tau, s_0) \) represents the phenomenological contribution of the QCD continuum above the continuum threshold \( s_0 \) for which perturbative-QCD contributions to the longitudinal component of the axial vector current correlator are dual to the phenomenological hadronic contributions \( s_0 > M_\pi^2 \) [8]. We have not included in the hadronic contributions to \( \mathcal{R}_0 \) and \( \mathcal{R}_1 \) the explicit contribution of the three-pion continuum of states when \( s \) is between \( 9m_\pi^2 \) and \( s_0 \), because this contribution is expected to be small compared to the single pion pole. For sufficiently small \( s \), the contributions of the pion pole and 3\( \pi \) continuum to the correlation function \( \Pi^L \) can be extracted from their relative contributions to the pseudoscalar correlation function [9]

\[ \frac{1}{\pi} \text{Im} \left[ \Pi^L(s) \right] \bigg|_{\pi+3\pi} = 2f_\pi^2m_\pi^4 \left[ \delta(s - m_\pi^2) + \frac{8s}{3(8\pi f_\pi)^4} \left( s - 9m_\pi^2 \right) \right] \]  
(7)

The relative size of the 3\( \pi \) continuum contribution to \( \mathcal{R}_0 \) is easily seen to be negligible, as is necessary to ensure the sum-rule validity of the GMOR relation \( f_\pi^2m_\pi^2 = -2m\langle \bar{q}q \rangle \) [2]. To ascertain roughly the relative size of these two hadronic contributions to \( \mathcal{R}_1 \), we can substitute (7) into the integrand (4) and integrate up to the continuum threshold \( s_0 \) [for \( s > s_0 \), the 3\( \pi \) continuum and all other hadronic effects are accounted for in \( m^2c_1(\tau, s_0) \), the QCD continuum term]. We find that the 3\( \pi \) continuum contribution to \( \mathcal{R}_1 \) is less than 7\% of the pion-pole’s contribution for a representative Borel-parameter choice of \( \tau = 1 \) GeV\(^{-2} \), assuming \( s_0 = 4 \) GeV\(^2 \), and that this percentage is substantially decreased if \( s_0 \) is chosen to be smaller (e.g. 4\% if \( s_0 = 2 \) GeV\(^2 \)). Even if we allow \( \tau \) to be as small as 0.4 GeV\(^{-2} \) (the minimum value utilized in our fit), and \( s_0 \) to be as large as 4 GeV\(^2 \), we still find that the 3\( \pi \) continuum contribution to \( \mathcal{R}_1 \) is less than 24\% of the pion-pole’s contribution, which itself will be seen to be of secondary importance compared to the pole-contribution of the first pion excitation state for such smaller values of \( \tau \).

Finally, the quantity \( \langle O_6 \rangle \) in (3) denotes the dimension-six quark condensates

\[ \langle O_6 \rangle \equiv \alpha_s \left[ 2\langle \bar{u}\sigma_{\mu\nu}\gamma_5T^a\bar{u}\bar{u}\sigma^{\mu\nu}\gamma_5T^a\bar{u} \rangle + u \to d \right] - 4\langle \bar{u}\sigma_{\mu\nu}\gamma_5T^a\bar{u}\sigma^{\mu\nu}\gamma_5T^a\bar{d} \rangle \]
perturbative term and direct instanton effects has not previously been performed. The SU(2) breaking effects in (5, 6) proportional to $m_u - m_d$ are subleading in the quark mass since they are proportional to $(m_u - m_d)^2(m_u + m_d)^2$ (8). As first noted in (3), effects other than perturbative and power-law terms can be present in the $\mathcal{R}_1$ sum-rule. Direct instanton contributions to the $\mathcal{R}_1$ sum-rule in the instanton liquid model (5) are scaled by the parameter $\rho_c$, and can be excised simply by going to the $\rho_c \to \infty$ limit. It should be noted that a sum-rule analysis containing both the rather large two-loop perturbative term and direct instanton effects has not previously been performed.

The above sum-rules satisfy a renormalization group (RG) equation in $\tau$ which implies that $m$ and $\alpha$ are the running mass and coupling constant at the energy scale $\tau$ (10) which to next-to-leading order are (11)

$$\alpha(\tau) = \alpha^{(2)}(\tau) \left[ 1 - \frac{\beta_2}{\beta_1 \pi} \alpha^{(2)}(\tau) \log \left( - \log(\tau \Lambda^2) \right) \right]$$

(9)

$$\alpha^{(2)}(\tau) = \frac{2\pi}{\beta_1 \log(\tau \Lambda^2)}$$

(10)

$$m(\tau) \equiv \hat{m} w(\tau)$$

(11)

$$w(\tau) = \frac{1}{\left( -\frac{1}{2} \log(\tau \Lambda^2) \right)^{-\gamma_1/\beta_1} \left[ 1 - \frac{\gamma_2 - \frac{\gamma_2}{\beta_1}}{\beta_1^2 \frac{1}{2} \log(\tau \Lambda^2)} + \frac{\gamma_1 \beta_2 \log \left( - \log(\tau \Lambda^2) \right)}{\beta_1^4 \frac{1}{2} \log(\tau \Lambda^2)} \right]}$$

(12)

$$\beta_1 = -\frac{9}{2}, \quad \beta_2 = -8, \quad \gamma_1 = 2, \quad \gamma_2 = \frac{91}{12}$$

(13)

The quantity $\hat{m}$ is RG invariant and is thus a fundamental quark mass parameter in QCD. The leading-order versions of (12) and (11) are used for the (leading order) power-law and instanton corrections in (11).

The qualitative behaviour of the sum-rules $\mathcal{R}_0(\tau)$ and $\mathcal{R}_1(\tau)$ provides significant information about the first pseudoscalar resonance. First, if $\hat{m}$ is reasonably small, then $\mathcal{R}_0(\tau)$ is essentially independent of $\tau$, since it is dominated by $\langle m \bar{q} q \rangle$. This then implies that the phenomenological side of the sum-rule is dominated by a light pseudogoldstone boson, since $\exp(-m^2/2\tau) \approx 1$ for appropriate mass scales [recall $\tau = 1/M^2$ and note that $\Lambda < M < \sqrt{s_0}$]. Thus $\mathcal{R}_0(\tau)$ mainly contains the information that the quark condensate must balance the phenomenological contribution of the pseudogoldstone pion, resulting in the GMR (2) relation $4m(\bar{q}q) = -2f^2 \pi m^2_\pi$, with $f_\pi = 93$ MeV, as noted above.

Although the pion dominates $\mathcal{R}_0(\tau)$, the excited state $M_\pi$ in $\mathcal{R}_1(\tau)$ is enhanced relative to the pion by an additional factor of $M^2_\pi/m^2_\pi$ relative to its contribution to $\mathcal{R}_0(\tau)$. For the $\Pi(1300)$ resonance ($M^2_\pi/m^2_\pi \approx 100$), we see that even a 1% contribution from the excited state in (3) corresponds to the excited state’s domination of (8). In such a case, the excited state would be too strong to be absorbed into the QCD continuum, a point which will be discussed in more detail below.

To obtain some quantitative understanding of this excited pion resonance state, it is necessary to reduce the dependence on the relatively uncertain value of the quark mass (13). We utilize the explicit RG dependence in (11) to obtain the following expression from (3):

$$\frac{\mathcal{R}_1(\tau)}{\hat{m}^2} = w(\tau)^2 \left[ \frac{3}{2\pi^2 \tau^2} \left[ 1 + \frac{17}{3} \frac{\alpha}{\pi} \right] - \frac{3}{\pi^2 \tau^2} \frac{\alpha}{\pi} \left[ 1 - \gamma_E \right] - 4 \langle m \bar{q} q \rangle + \frac{1}{2\pi} \langle \alpha G^2 \rangle \right]$$

(8)
\[ +\pi \langle O_6 \rangle \tau + \frac{3\rho_c^2}{2\pi^2\tau^3} e^{-\rho_c^2} \left[ K_0 \left( \frac{\rho_c^2}{2\tau} \right) + K_1 \left( \frac{\rho_c^2}{2\tau} \right) \right] \]  

(14)

Any implicit mass dependence in the above expression occurs through the value of the dimension-six quark condensate. There are two points of view that can be taken for the value of \( \langle \bar{q}q \rangle \).

1. The scale of \( \langle O_6 \rangle \) is an effective (chiral-limiting) value set in numerous sum-rule applications which is not contingent upon a particular value of the quark mass. In this case vacuum saturation and an effective \( \langle \bar{q}q \rangle \) scale \( \bar{m} \) are used to find

\[ \langle O_6 \rangle = f_{vs} \frac{448}{27} \alpha \langle \bar{q}qq \rangle = f_{vs} 3 \times 10^{-3} \text{GeV}^6 \equiv \langle O_6^{(1)} \rangle \]

where \( f_{vs} = 1 \) for exact vacuum saturation. Larger values of effective dimension-six operators found in \( [14] \) imply that \( f_{vs} \) could be as large as \( f_{vs} = 2 \).

2. In a self-consistent approach, vacuum saturation is imposed at a characteristic 1 GeV scale to give

\[ \langle O_6 \rangle = \frac{448}{27} \alpha (1) \langle \bar{q}q \rangle (1)^2 = \frac{112\alpha(1) f^2_{\pi} m_\pi^4 \left[ -\frac{1}{2} \log(\Lambda^2) \right]^{8/9}}{27\dot{m}^2} \equiv \langle O_6^{(2)} \rangle \]

where the GMOR relation \( 2m \langle \bar{q}q \rangle = -f^2_{\pi} \dot{m}^2 \) has been imposed to make the value of the quark condensate consistent with the quark mass.

By comparing the values of \( \langle O_6 \rangle \) in the two scenarios we see that they are identical if the following identification is made:

\[ f_{vs} = \frac{2.6 \times 10^{-5} \text{GeV}^2}{\dot{m}^2} \]

(17)

Hence a wide enough variation in the parameter \( f_{vs} \) in the first scenario can account for any inconsistency between the value of \( \langle O_6 \rangle \), the quark mass, and the GMOR relation. Furthermore, Figure 1 shows that the actual numerical effect of these two values of \( \langle O_6 \rangle \) on the sum-rule \( R_1/\dot{m}^2 \) is small enough to be accommodated within the theoretical uncertainties associated with \( R_1 \) discussed below. For now it suffices to observe that a wide range of smooth functions lying between the extreme solid curves in the figure occur within our error model, accommodating the small difference between the curves obtained for the two scenarios, including variations in \( \dot{m} \) for scenario 2. Although we could accommodate scenario 2 within the theoretical uncertainties of scenario 1, we will present explicit results for the two scenarios distinguished by the values \( \langle O_6^{(1)} \rangle \) and \( \langle O_6^{(2)} \rangle \), and it will be seen that the two are self-consistent when theoretical uncertainties in the predicted parameters are considered.

In the first scenario, we thus find the following relation between the QCD sum-rule and phenomenology by using \( (12) \) explicitly, and by dividing both sides of \( (3) \) by \( \dot{m}^2 \).

\[ w(\tau)^2 \left( \frac{3}{2\pi^2\tau^2} \left[ 1 + \frac{17\alpha}{3\pi} - \frac{3\alpha}{\pi^2\tau^2} \left[ 1 - \gamma_E \right] - \frac{1}{2\pi} \langle \alpha G \rangle \right] \right. \]

\[ \left. + f_{vs} 3 \times 10^{-3} \text{GeV}^6 \tau + \frac{3\rho_c^2}{2\pi^2\tau^3} e^{-\rho_c^2} \left[ K_0 \left( \frac{\rho_c^2}{2\tau} \right) + K_1 \left( \frac{\rho_c^2}{2\tau} \right) \right] \right) \]

\[ = 2 \frac{f^2_{\pi} m_\pi^4}{\dot{m}^2} \left[ 1 + \frac{F^2_2 M_\pi^4}{f^2_{\pi} m_\pi^4} e^{-M_\pi^2\tau} \right] + w^2(\tau) c_1(\tau, s_0) \]

(18)
In the first scenario, all \( \hat{m} \) dependence occurs on the right-hand side of (18). Input of the QCD parameters permits a least-squares fit of the \( \tau \)-dependence of \( R_1(\tau)/\hat{m}^2 \) to the form

\[
\frac{2f_\pi^2m_\pi^4}{\hat{m}^2} \left[ 1 + re^{-M_\pi^2\tau} \right] + w^2(\tau)c_1(\tau, s_0)
\]

with the fitted parameters \( \hat{m}, r, M_\pi \) and \( s_0 \) in correspondence with the right-hand side of (18): \( r = F_\pi^2M_\pi^4/(f_\pi^2m_\pi^4) \).

In the second scenario, we find a similar relation between the QCD sum-rule and phenomenology after making use of (16).

\[
w(\tau)^2 \left[ \frac{3}{2\pi^2\tau^2} \left[ 1 + \frac{17}{3} \alpha - \frac{3}{\pi^2\tau^2} \alpha \right] [1 - \gamma_\epsilon] - 4 \langle m\bar{q}q \rangle + \frac{1}{2\pi} \alpha G^2 \right] \\
+ \frac{3\rho_c^2}{2\pi^2\tau^3} e^{-\frac{m_\pi^2}{\tau^2}} \left[ K_0 \left( \frac{\rho_c^2}{2\tau} \right) + K_1 \left( \frac{\rho_c^2}{2\tau} \right) \right] \\
= \frac{112\alpha(1)F_\pi^2m_\pi^4}{27} \left[ -\frac{4}{3} \log(\Lambda^2) \right]^{8/9} \left[ w(\tau)^2 \frac{1}{\hat{m}^2} + \frac{2f_\pi^2m_\pi^4}{\hat{m}^2} \left[ 1 + \frac{F_\pi^2M_\pi^4}{f_\pi^2m_\pi^4} e^{-M_\pi^2\tau} \right] + w^2(\tau)c_1(\tau, s_0) \right]
\]

All the \( \hat{m} \) dependence is again arranged to be on the right-hand side, and the parameters \( \hat{m}, r, M_\pi \) and \( s_0 \) can again be obtained from a least-squares fit.

To obtain these resonance and quark mass parameters from a fit to the above shape dependence, we use the standard set of values \( \langle \alpha G^2 \rangle = 0.045 \text{ GeV}^4 \), \( \Lambda = 0.15 \text{ GeV} \), and \( \rho_c = 1/600 \text{ MeV} \), along with the GMOR relation \( \langle m\bar{q}q \rangle = -f_\pi^2m_\pi^2/2 \) and physical values for \( m_\pi \) and \( f_\pi \). The optimum value of the parameters are then obtained via a fit to the \( \tau \)-dependence of the left hand side of (18) (scenario 1), or the left-hand side of (20) (scenario 2) which leads to the smallest value of a weighted \( \chi^2 \). The weights for the minimum \( \chi^2 \) are obtained from a 50% uncertainty for power-law corrections\(^1\) and a 30% uncertainty for the continuum contributions. The magnitude of the relative uncertainty in the power-law, perturbative and continuum contributions is shown in Figure 2 for a typical value of \( s_0 \). The \( \tau \) region chosen for the \( \chi^2 \) minimization is the range for which the relative uncertainty reaches the 20% level, in this case \( 0.4 \text{ GeV}^{-2} < \tau < 2.5 \text{ GeV}^{-2} \).

It is interesting to observe that the minimum \( \chi^2 \) increases by an order of magnitude when the pion-resonance is excluded from the phenomenological model on the right hand side of (18). This increase in \( \chi^2 \) occurs for a variety of scenarios where the \( 3\pi \) continuum is also included with the QCD continuum and pion pole, a clear indication that the excited state is too strong to be absorbed into continuum effects. The sharp fall-off of \( R_1 \) with increasing \( \tau \) [Fig. 1] also provides strong evidence for a substantial pion-resonance contribution, which is expected from (19) to fall off exponentially with \( \tau \) compared with the constant contribution anticipated from the pion itself. Comparison of the form of (19) to Fig. 1 indicates that the pion-resonance contribution dominates the small \( \tau \) region, whereas the constant contribution from the pion itself is evident in the flattening out of \( R_1 \) at large \( \tau \).

Uncertainties in the fitted parameters are obtained from a Monte-Carlo simulation based upon a 15% variation in \( \rho_c \), and a simulation of the previously described power-law and continuum uncertainties. In scenario 1, the parameter \( f_{es} \) is allowed to vary in the range \( 0 < f_{es} < 2 \) to accommodate any

\(^{1}\)The 50% uncertainty is actually larger than the SVZ criterion of the (dimensionless) square of the power law corrections.
inconsistency between $\langle O_6^{(1)} \rangle$ and the fitted value of $\hat{m}$. Figure 3 shows the effect of the 15% variation in $\rho_c$ on the instanton contributions to $R_1$. As evident from the figure, this variation can easily accommodate any uncertainty associated with the zero-mode approximation in Shuryak’s instanton liquid model [6]. Figure 4 shows that the power-law and continuum uncertainties are well described by the empirical formula

$$\delta R_1(\tau)/\hat{m}^2 + w^2 \delta c_1(\tau, s_0) = \frac{\sigma_1}{\tau^3} + \sigma_2 \sqrt{\tau}$$

(21)

where $\sigma_1 = 0.007$ and $\sigma_2 = 0.006$ in GeV units. It is now possible to simulate the uncertainties of the minimum $\chi^2$ parameters by performing a Monte Carlo simulation with random variations in the parameter set $\rho_c, \sigma_1, \sigma_2$, and $f_{us}$ (note that $f_{us}$ only occurs in scenario 1).

The results for the parameters and their error estimates in each scenario are summarized in Table 1. The predictions from the two scenarios exhibit an overlap within their uncertainties, so we conclude that the two scenarios for the operator $\langle O_6 \rangle$ are consistent. The excellent quality of the minimum $\chi^2$ fits is illustrated in Figure 5. While the decay constant $F_\pi$ (and hence $r$) has not been measured, the PDG estimate of the $\Pi(1300)$ mass is $1300 \pm 100$ MeV [13]. Our results tend to favour a pion resonance mass at the lower end of the experimental range. As is evident from Table 1, the results we obtain are quite insensitive to the fitted value of $s_0$—both scenarios remain within 90% confidence levels for values of $s_0$ between 1.7 GeV$^2$ and 4.4 GeV$^2$. This entire range, however, is consistent with the methodological constraint that $s_0$ be greater than $M_\pi^2 + M_\pi \Gamma$—i.e. the requirement that the first pion excitation resonance be entirely below the QCD continuum threshold for the resonance not to be absorbed in the QCD continuum contribution. Note that the Particle Data Guide [13] estimates $\Gamma$ to be between 200 and 600 MeV, suggesting values of $s_0$ in excess of 2 GeV$^2$. Our quark mass estimates for $m(1\text{GeV}) = [m_u(1\text{GeV}) + m_d(1\text{GeV})]/2$ are also consistent with [13] values.

The role of direct instanton contributions to the sum-rule in the instanton liquid model can be understood by comparing to a fit in which such contributions are absent (the $\rho_c \to \infty$ limit). Corresponding parameter values for scenario 1 in the $\rho_c \to \infty$ limit are $M_\pi = 1.34 \pm 0.16$ GeV, $r = 8.88 \pm 3.2$, $\hat{m} = 12.15 \pm 2.0$ MeV ($m(1\text{GeV}) = 9.14 \pm 1.5$ MeV), and $s_0 = 3.44 \pm 1.4$ GeV. Thus, the effect of direct instanton contributions is to lower both the pion-resonance mass and its decay constant.

It is also possible to gain some insight into the instanton size $\rho_c$ by adding $\rho_c$ to the set of fit parameters. We then find an optimum value of $\rho = 1.47$ GeV$^{-1}$, $\hat{m} = 10.2$ MeV, $r = 6.68$, $M_\pi = 1.13$ GeV and $s_0 = 3.8$ GeV$^2$. A Monte Carlo simulation of errors for this case is beyond our present computational capacity.

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2 Random variations in $\sigma_1$ and $\sigma_2$ are confined to the region $-0.007 < \sigma_1 < 0.007$ and $-0.006 < \sigma_2 < 0.006$ so that the simulation does not exceed the error estimate $\delta R_1$. 
Table 1: Pion resonance and quark mass parameters obtained from minimization of the weighted $\chi^2$. The two scenarios correspond to the two possibilities for the value of $\langle O_6 \rangle$. All uncertainties are at the 90% confidence level.

| Scenario | 1                  | 2                  |
|----------|--------------------|--------------------|
| $M_\pi$  | 0.996 ± 0.25 (GeV) | 0.95 ± 0.24 (GeV)  |
| $r$      | 5.40 ± 3.8         | 7.28 ± 5.2         |
| $\bar{m}$ | 10.88 ± 2.9 (MeV) | 13.95 ± 3.8 (MeV)  |
| $m(1\text{GeV})$ | 8.03 ± 2.2 (MeV) | 10.30 ± 2.8 (MeV)  |
| $s_0$    | 3.21 ± 1.5 (GeV$^2$) | 3.02 ± 1.4 (GeV$^2$) |

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**Figure Captions**

**Figure 1:** Upper and lower curves show extremes of the error model in scenario 1. The middle solid curves show the dependence of the sum-rule $\mathcal{R}_1(\tau)$ on $\langle O_6 \rangle$ for both scenarios, including quark mass dependence in scenario 2 for $5 \text{ MeV} < \hat{m} < 20 \text{ MeV}$.

**Figure 2:** Relative uncertainty in the power-law, perturbative, and continuum contributions to $\mathcal{R}_1(\tau)/\hat{m}^2 - w^2(\tau)c_1(\tau, s_0)$ for $s_0 = 3 \text{ GeV}^2$.

**Figure 3:** Instanton contributions to $\mathcal{R}_1/\hat{m}^2$ for $\rho_c = 1/600 \text{ MeV}$ are shown in the central curve. The upper (lower) curves are the instanton contributions for $\rho$ 20% smaller (larger) than the central value of $\rho_c = 1/600 \text{ MeV}$.

**Figure 4:** Error model from (21) (upper curve) and uncertainty $\delta \mathcal{R}_1(\tau)/\hat{m}^2 + w^2(\tau)\delta c_1(\tau, s_0)$ for $s_0 = 3 \text{ GeV}^2$ (lower curve).

**Figure 5:** Ratio of the left and right-hand sides of (18) for the fitted parameters leading to a minimum weighted $\chi^2$. 
Figure 1

\[ \tau (\text{GeV}^{-2}) \]

\[ R_i / m^2 (\text{GeV}^4) \]
Figure 2

\[ \tau \text{ (GeV}^{-2}) \]

\[ \frac{(\delta R_1 + m^2 \delta c_1)}{R_1} \]

\[ 0.06 \quad 0.08 \quad 0.10 \quad 0.12 \quad 0.14 \quad 0.16 \quad 0.18 \quad 0.20 \quad 0.22 \quad 0.24 \quad 0.26 \quad 0.28 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \]
Figure 3

\[ R_1 / m^2 \text{ (GeV}^4) \]

\[ \tau \text{ (GeV}^{-2}) \]
Figure 4

\[ \frac{\delta R_1}{m^2 + w^2 \delta c_1} \text{ (GeV}^4) \]

vs.

\[ \tau \text{ (GeV}^{-2}) \]
Figure 5

\[ \left[ a \left( 1 + r e^{-M_{\tau}^2} \right) + w^2 c_1 \right] m^2/R_1 \]

\[ \tau \text{ (GeV}^{-2}) \]