Abstract

For controller implementation and due to practical considerations a reduced order model is usually needed. So in this work, the problem of order reduction of nonlinear multi time scales systems in the framework of singular perturbations is addressed. A successive separation algebraic procedure is proposed to derive a polynomial reduced model by using the Kronecker product based state representation and the properties of the tensor algebra.

Keywords: Kronecker Product, Polynomial Reduced Order Model, Nonlinear Large-scale Systems, Singular Perturbation Theory.

1. Introduction

Due to not only theoretical interest, but also to the relevance of this topic to control engineering applications, the singular perturbations techniques have received a great deal of attention over the past two decades. Several results are developed in the literature to cope with order reduction of linear systems and even for particular classes of nonlinear systems using the singular perturbations theory. But no simple and easy implemented approach for order reduction of multi-time-scale interconnected nonlinear systems are provided.

Indeed, the complexity encountered in solving nonlinear differential equations and the lack of rigorous analytical reduced model have motivated the development of the proposed algebraic approach based on the description of the studied systems by an analytical control affine state space equation with polynomial vector fields, based on the use of Kronecker product. The properties of the tensor algebra allow to overcome many difficulties.

The layout of the paper is as follows. In the first section, the theory background of the singular perturbation theory is state briefly. In the next section, the description of the studied systems and the problem statement are formulated. A proposed procedure is also given where the different steps of dynamic separation and construction of the reduced model is detailed. The numerical simulations demonstrate the efficiency of the proposed approach and the accuracy of the approximate reduced model. Finally, some conclusions are drawn.
2. Materials and Methods

2.1 Singular Perturbation Theory

Let us consider the general nonlinear singularly perturbed systems described by the following state equations:

\[
\begin{align*}
    \dot{x} &= F(x,z,t), \quad x(t_0) = x_0, x \in \mathbb{R}^n \\
    \varepsilon \dot{z} &= G(x,z,t), \quad z(t_0) = z_0, z \in \mathbb{R}^m
\end{align*}
\]  

where the state variables are divided into \(n_1\) slow states \(x\) and \(n_2\) fast states \(z\) and \(\varepsilon\) is a positive small singular perturbed parameter such that \(0 < \varepsilon << 1\), which quantifies the speed ratio of the slow versus the fast dynamical phenomena of the process, \(t \in \mathbb{R}^+\).

One of the delicate problems is to explicit not defined at fortiori the parameter \(\varepsilon\) depending on the adopted representation of the studied system.

Let us introduce the following assumptions\(^8,11\).

**Assumption 1**

The Jacobian \(\frac{\partial G(x,z,t)}{\partial z}\) at \(\varepsilon = 0\) is non singular for all \(x, z \in D\), where \(D\) is a region of interest for the approximate analysis\(^8\).

**Assumption 2**

For all \(x, z \in D\), real parts of the eigenvalues of \(\frac{\partial G(x,z,t)}{\partial z}\) at \(\varepsilon = 0\) are smaller than a fixed negative number:

\[
\text{Re} \left\{ \frac{\partial G}{\partial z} \right\} \bigg|_{\varepsilon=0} < c < 0
\]

When the Assumption 1 is satisfied, the slow reduced model is obtained by formally setting \(\varepsilon = 0\) in equation (1) as:

\[
\begin{align*}
    \dot{\bar{x}} &= F(\bar{x},\bar{z},t) \quad \text{(a)} \\
    0 &= G(\bar{x},\bar{z},t) \quad \text{(b)}
\end{align*}
\]

Solving for \(\bar{z}\), the equation (2-b)

\[
\bar{z} = \psi(\bar{x})
\]

and substituting \(\bar{z}\) into (2) lead to the following model:

\[
\bar{x} = F(\bar{x},\psi(\bar{x}),t) = \bar{F}(\bar{x},t), \quad \bar{x}(0) = x_0
\]

It neglects the fast phenomenon, considering that it has instantaneously reached its equilibrium. The fast reduced model is obtained by introducing the fast time scale

\[
\tau = \frac{t-t_0}{\varepsilon}
\]

and the fast variables \(\tilde{x}(\tau)\) and \(\tilde{z}(\tau)\) such that:

\[
\begin{align*}
    x(t) &= \bar{x}(t) + \tilde{x}(\tau) \\
    z(t) &= \bar{z}(t) + \tilde{z}(\tau)
\end{align*}
\]

Expressing the derivatives of \(x\) and \(z\) with respect to \(t\) and \(\tau\), substituting them into (14) and letting \(\varepsilon \to 0\), we obtain:

\[
\begin{align*}
    \frac{d\bar{x}}{d\tau} &= 0; \quad \bar{x}(0) = x(0) - \bar{x}(0) = 0 \\
    \frac{d\bar{z}}{d\tau} &= G(\bar{x}(\tau) \tilde{z}(\tau) + \tilde{z}(\tau)); \quad \tilde{z}(0) = \bar{z}(0) - \tilde{z}(0)
\end{align*}
\]
where the fast part $\tilde{z}(\tau)$ is the state of the fast reduced model (7), called also the solution of the boundary layer. Its equilibrium $\tilde{z}(\tau) = 0$ is asymptotically stable if assumption 2 is satisfied. If in addition $\tilde{z}(0)$ is in the region of attraction of this equilibrium, then the following two time scales approximation is valid, [8]:

$$
\begin{align*}
    x(t) &= \tilde{x}(t) + O(\epsilon) \\
    z(t) &= \tilde{z}(t) + \tilde{z}(\tau) + O(\epsilon)
\end{align*}
$$

(8)

It is shown here that the reduced model will be determined by solving for $\tilde{z}$ the equation (2-b). This appears to be easy in the case of linear systems but seems to be a difficult task in the general case of nonlinear systems. Several results are obtained when considering linear systems and particular class of nonlinear systems and some others in the case of nonlinear systems described by state equations with no direct transmission matrix $L(x, z)$ (9).

$$
\begin{align*}
    \dot{x} &= F(x, z) + G(x, z)U \\
    \dot{\epsilon} z &= H(x, z) + L(x, z)U
\end{align*}
$$

(9)

So the problem we are addressing in this paper can be stated as follows; given a high order multi-time scale system with nonlinear vector functions $F(\cdot), G(\cdot), H(\cdot)$ and $L(\cdot)$, how can we exploit the singular perturbation theory and the tensors representation to derive in a relatively simple way a simplified model and provide a computational procedure for this aim.

The following sections will show that good results are obtained. The principal key to our work is the use of a powerful mathematical tool in the description of the nonlinear studied systems: the Kronecker product.

### 2.2 Description of the Studied Systems and Problem Statement

We will focus on realistic models of $q$ time-scales systems which involve interacting dynamic phenomena of widely different speeds and described by the following state separable form:

$$
\begin{align*}
    \epsilon_i \dot{X}_i &= F_i(X_i, \cdots, X_j, \cdots, X_e) + G_i(X_i, \cdots, X_j, \cdots, X_e) U \\
    \vdots \\
    \epsilon_q \dot{X}_q &= F_q(X_i, \cdots, X_j, \cdots, X_e) + G_q(X_i, \cdots, X_j, \cdots, X_e) U
\end{align*}
$$

(10)

where $X_i \in \mathbb{R}^n, i = 1, \ldots, q$ are the state vectors, $U \in \mathbb{R}^n$ is the control vector and $F_i(\cdot), G_i(\cdot) \mid_{t \to \infty}$, the vector functions, continuously differentiable, are supposed analytic. Moreover, $\epsilon_i$ denotes small non-negative singular perturbation parameters such that $\epsilon_i < \epsilon_j$ for $k > i$, making the state variables to be ordered accordingly to their increasing speed. The system (10) can be studied like a multi-parameter system $\epsilon_i$ appears here for the aim of symmetry but it is going to be considered equal to 1.

The simplification of the studied system (10) can be done gradually when posing successively $\epsilon_q = 0, \epsilon_{q-1} = 0, \ldots$

It will appear a recurrence that makes possible to derive, going from the fastest to the slowest subsystem, the degenerate slow model. In each stage of reduction, we apply an analytical identification approach which is
are introduced in this paper to calculate all the matrices appearing in the description of each reduced order model.

### 3. Results and Discussions

#### 3.1 Algebraic Approach for Order Reduction

The separation of time scales starts with the following model obtained by rewriting the system (10) in a compact form:

\[
\begin{align*}
    \dot{X}_{s1} &= F_s(X_{s1}, X_q) + G_s(X_{s1}, X_q) U \\
    \varepsilon_s \dot{X}_q &= H_q(X_{s1}, X_q) + L_q(X_{s1}, X_q) U
\end{align*}
\]  

(11)

where \(X_{s1} = [X_1^T \quad X_2^T \quad \cdots \quad X_{s1}^T]^T \in \mathbb{R}^{n_{s1}}\) is predominantly slow and \(X_q\) contains the \(n_q\) fastest state variables. \(n\) denotes the order of the full studied system defined by \(n = \sum_{j=1}^{r} n_j\). \(F_s(\cdot), G_s(\cdot), H_s(\cdot)\) and \(L_s(\cdot)\), are expressed from the functions \(F_s(\cdot)|_{s=0, q}\) and \(G_s(\cdot)|_{s=1, q}\) (see previous work).

They admit a generalized Taylor series developments using the Kronecker product and power notations as follows

\[
\begin{align*}
    F_s(X_{s1}, X_q) &= \sum_{i=0}^{\lfloor r \rfloor} \sum_{j=0}^{\lfloor r \rfloor} F_{s,ij}(X_{s1}^{[i+1-j]} \otimes X_q^{[j-1]}) \\
    G_s(X_{s1}, X_q) &= \sum_{i=0}^{\lfloor r \rfloor} \sum_{j=0}^{\lfloor r \rfloor} G_{s,ij} (I_s \otimes X_{s1}^{[i+1-j]} \otimes X_q^{[j-1]}) \\
    H_s(X_{s1}, X_q) &= \sum_{i=0}^{\lfloor r \rfloor} \sum_{j=0}^{\lfloor r \rfloor} H_{s,ij} (X_{s1}^{[i+1-j]} \otimes X_q^{[j-1]}) \\
    L_s(X_{s1}, X_q) &= \sum_{i=0}^{\lfloor r \rfloor} \sum_{j=0}^{\lfloor r \rfloor} L_{s,ij} (I_s \otimes X_{s1}^{[i+1-j]} \otimes X_q^{[j-1]})
\end{align*}
\]

(12)

where \(X_{s1}^{[i]}\) and \(X_q^{[i]}\) are the \(i\)-th redundant Kronecker power of a \(n_{s1}\) dimensional vector \(n_q\) dimensional vector. \(F_{s,ij}, G_{s,ij}, H_{s,ij}, L_{s,ij}\) are constant matrices with appropriate dimensions.

Based on\(^1^{14}\), it is proved that the previous development (12) can be truncated at an arbitrary order \(r\). In practice, the choice of the third order polynomial description \((r = 3)\) is considered enough to model many physical processes\(^1^{15,16}\). So, all the analytical functions (12) can be approximated by the following vector polynomials:

\[
\begin{align*}
    F_s(X_{s1}, X_q) &= F_{s,00}(X_{s1}) + F_{s,01}(X_{s1}) X_q + F_{s,10}(X_{s1}) + F_{s,11}(X_{s1}) X_q^2 \\
    &+ F_{s,20}(X_{s1}) X_q^3 \\
    G_s(X_{s1}, X_q) &= G_{s,00}(X_{s1}) + G_{s,01}(X_{s1}) X_q + G_{s,10}(X_{s1}) + G_{s,11}(X_{s1}) X_q^2 \\
    &+ G_{s,20}(X_{s1}) X_q^3 \\
    H_s(X_{s1}, X_q) &= H_{s,00}(X_{s1}) + H_{s,01}(X_{s1}) X_q + H_{s,10}(X_{s1}) + H_{s,11}(X_{s1}) X_q^2 \\
    &+ H_{s,20}(X_{s1}) X_q^3 \\
    L_s(X_{s1}, X_q) &= L_{s,00}(X_{s1}) + L_{s,01}(X_{s1}) X_q + L_{s,10}(X_{s1}) + L_{s,11}(X_{s1}) X_q^2 \\
    &+ L_{s,20}(X_{s1}) X_q^3 \\
    L_{s,30}(X_{s1}) &= L_{s,30}(X_{s1}) + L_{s,31}(X_{s1}) X_q + L_{s,32}(X_{s1}) X_q^2 + L_{s,33}(X_{s1}) X_q^3
\end{align*}
\]
The terms $O(X_{st}^{[1]}, X_{s}^{[1]})$, $O_{2}(X_{st}^{[1]}, X_{s}^{[1]})$, $O_{3}(X_{st}^{[1]}, X_{s}^{[1]})$, $O_{4}(X_{st}^{[1]}, X_{s}^{[1]})$ are the fourth order error of the previous developments.

- First step of reduction

Setting $\varepsilon = 0$ and assuming that it exists a unique root solution of (13-b) in the interesting region which is considered here as a neighborhoods of the equilibrium taken to be the origin, the first generated subsystem verifying the assumption 1 and the assumption 2 is obtained by solving:

$$\begin{cases}
\dot{\hat{X}}_{st} = F_{s}(\hat{X}_{st}, \hat{X}_{s}) + G_{s}(\hat{X}_{st}, \hat{X}_{s}) U & (a) \\
0 = H_{s}(\hat{X}_{st}, \hat{X}_{s}) + L_{s}(\hat{X}_{st}, \hat{X}_{s}) U & (b)
\end{cases}$$

(13)

After the first step of reduction, the first obtained reduced model is considered to be described by a polynomial representation to preserve the same description of the studied system

$$\hat{X}_{st} = \sum_{i=1}^{\hat{\hat{A}}_{st}} A_{st} \hat{X}_{st}^{[i]} + \sum_{i=1}^{\hat{\hat{B}}_{st}} B_{st}(I_{n_{w}} \otimes \hat{X}_{st}^{[i]}) U + O(\hat{X}_{st}^{[i]})$$

(14)

where $O(\hat{X}_{st}^{[i]})$ is the fourth order error of (14).

The constant matrices $A_{st}$ and $B_{st}$ are of appropriate dimensions and are calculated by applying the identification approach presented below and:

To get the property of full rank matrices, we first rewrite the equation (14) in the non-redundant form as:

$$\hat{X}_{st} = \sum_{i=1}^{\hat{\hat{A}}_{st}} A_{st} \hat{X}_{st}^{[i]} + \sum_{i=1}^{\hat{\hat{B}}_{st}} B_{st}(I_{n_{w}} \otimes \hat{\hat{X}}_{st}^{[i]}) U$$

(15)

where $\hat{\hat{A}}_{st} = A_{st} T_{t}$ and $\hat{\hat{B}}_{st} = B_{st}(I_{n_{w}} \otimes T_{t})$

(16)

$\hat{\hat{X}}_{st}$ is defined by (A.6) (resp.(A.7)) and $T_{t}^{+}$ is the pseudo-inverse of $T_{t}$ expressed in (A.8), (see Appendix).

- Considering the above notations and applying the vec function defined in (A.9) and the corresponding properties (A.10), (A.11), (A.12), the equation (15) is rewritten as:

$$\text{vec}(\hat{X}_{st}) = \text{vec}(\hat{X}_{st}) = \sum_{i=1}^{\hat{\hat{A}}_{st}} \text{vec}(\hat{A}_{st} \hat{X}_{st}^{[i]}) + \sum_{i=1}^{\hat{\hat{B}}_{st}} \text{vec}(B_{st}(I_{n_{w}} \otimes \hat{\hat{X}}_{st}^{[i]}) U)$$

(17)

and one can write

$$\text{vec}(\hat{A}_{st} \hat{\hat{X}}) = \left(\hat{\hat{X}} \otimes I_{n_{w}}\right) \text{vec}(\hat{A}_{st})$$

(18)

$$\text{vec}(\hat{\hat{B}}_{st}(I_{n_{w}} \otimes \hat{\hat{X}}_{st}^{[i]}) U) = \left((I_{n_{w}} \otimes \hat{\hat{X}}_{st}^{[i]}) U \otimes I_{n_{w}}\right) \text{vec}(\hat{\hat{B}}_{st})$$

(19)
\[
\tilde{X}_{s1} = \left[ X_s^T \otimes I_m ... \tilde{X}_{s1}^{(v)} \otimes I_m ... \right] \begin{bmatrix}
\text{vec}(A_{s1}) \\
\vdots \\
\text{vec}(A_{si}) \\
\vdots \\
\text{vec}(\hat{A}_{si}) \\
\vdots \\
\text{vec}(\hat{B}_{si}) \\
\vdots \\
\text{vec}(\hat{B}_{sj}) \\
\end{bmatrix}
\]

which can be written as a vectorial equation:
\[
\hat{X}_{s1} = \theta_{s1}^T \vartheta_{s1}
\]

where
\[
\vartheta_{s1} = \left[ X_s^T \otimes I_m ... \tilde{X}_{s1}^{(v)} \otimes I_m ... (u^T \otimes I_m)(I_m \otimes \tilde{X}_{s1}^{(v)} \otimes I_m) \right]
\]

\[
\vartheta_{s1} = \begin{bmatrix}
\text{vec}(A_{s1}) \\
\vdots \\
\text{vec}(A_{si}) \\
\vdots \\
\text{vec}(\hat{A}_{si}) \\
\vdots \\
\text{vec}(\hat{B}_{si}) \\
\vdots \\
\text{vec}(\hat{B}_{sj}) \\
\end{bmatrix}
\]

\[
\theta_{s1} = \left[ \theta_{s1}^T \theta_{s1} \right]^{-1} \theta_{s1}
\]  

(24)

The solution of the vectorial equation (24) is then equal to:
\[
\vartheta_{s1} = \theta_{s1}^T \hat{X}_{s1}
\]  

(25)

Applying a random input to the system (15), the proposed approach provides the determination of the matrices \( A_{s1}, B_{s1} \), \( i = 1, \ldots, r \) and so on the matrices \( A_{s1}, B_{s1} \) from (16).

Starting from the obtained reduced model (14) which contains \((q-1)\) state vectors of different speeds, the gradually application of the proposed approach of system reduction derives when going from the fastest to the slowest subsystem a reduced order slow subsystem model.

- After \((K+1)\) stage of reduction

It appears from the above developments a recurrence. So, assumption 1 and 2 being verified at each stage, the successive separation procedure of order reduction leads after the kth stage to the following reduced model described by the polynomial state equations:
\[
\hat{X}_{sk} = \sum_{i=1}^{r} A_{sk,i} \hat{X}_{sk}^{[i]} + \sum_{j=1}^{r} B_{sk,i} (I_m \otimes \hat{X}_{sk}^{[i]}) U
\]

(26)

where
\[
\hat{X}_{sk} = [\hat{X}_{sk,1}^{r} \hat{X}_{sk,2}^{r} ... \hat{X}_{sk,r}^{r} ... \hat{X}_{sk,q-k}^{r}]^T \in \mathbb{R}^{n_k}
\]

and \( \hat{X}_{sk,i} \in \mathbb{R}^{n_j} \) with \( n_k = \sum_{j=1}^{q-k} n_j \).
After the calculus of the matrices $\hat{A}_{i,k}$ and $\hat{B}_{i,k}$ the next step is to extract again the fastest state variables given by $\bar{X}_{sk,q-k+1}$ and to derive the reduced model that approximate the slowest state variables of the full studied systems where the original variables are finally approximated by:

$$\hat{X}_{sk+1} = [\bar{X}_{sk+1,1}^\tau, \bar{X}_{sk+1,2}^\tau, \ldots, \bar{X}_{sk+1,q-1}^\tau] \in \mathbb{R}^{n(x_{st})}$$

For this aim, it is necessary to develop the terms $\bar{X}_{sk}^{[l]}$ and $(I_m \otimes \bar{X}_{sk}^{[l]})$ (26) in order to make to appear the slow state variables $\bar{X}_{sk+1}$ and the fastest ones $\bar{X}_{sk,q-1}$.

Consequently, the system (26) can be formulated (appendix, [15]) by a similar description as (11) and (12).

It comes out that all the constant matrices $F_{q,k}, G_{q,k}, L_{q,k}$ and $H_{q,k}$ are then defined and we can derive the finally degenerate slow model:

$$\hat{X}_{sk+1} = \sum_{j=1}^{q} A_{j,k+1} \bar{X}_{sk+1}^{[j]} + \sum_{j=1}^{q} B_{j,k+1} (I_m \otimes \bar{X}_{sk+1}^{[j]})$$

(27)

where $A_{j,k} = \hat{A}_{j,k} T'$ and $B_{j,k} = \hat{B}_{j,k} (I_m \otimes T')$

$\hat{A}_{j,k+1}$ and $\hat{B}_{j,k+1}$ are the solutions of the following equation: $\theta_{sk+1} = \theta_{\tau x+1} \hat{X}_{sk+1}$

(28)

and $\bar{X}_{sk+1}$ is the solution of the following equations:

$$\begin{cases}
\bar{X}_{sk+1} = F_{sk+1} (\bar{X}_{sk+1}) + G_{sk+1} (\bar{X}_{sk+1}) U \\
0 = H_{sk+1} (\bar{X}_{sk+1}) + L_{sk+1} (\bar{X}_{sk+1}) U
\end{cases}$$

(29)

when computing, all the factor scales $c_i$ are incorporated in the corresponding matrices.

### 3.2 Illustrative Example

Let’s consider a 7th order polynomial nonlinear system with four time-scales:

$$\begin{cases}
\epsilon_1 \dot{x}_1 = 1.6x_1 - 1.3x_1 + 0.3x_1 + 1.2x_2 - 0.15x_2 + 0.5x_3x_5 \\
+ 0.5x_1^2 + 0.1x_1 + (1 - 0.5)x_1 + 0.6x_2 + 0.5x_4 - 0.7x_2 u(t) \\
\epsilon_2 \dot{x}_2 = -1.15x_1 - 0.95x_2 - 0.55x_3 + 0.15x_4 + 0.2x_4 - 0.2x_3 - 0.1x_3x_4 \\
+ 0.15x_1^2x_4 + 0.5x_1^2 - 0.2x_2x_4 + (0.5 + 0.4)x_5 + 0.7x_4 - 0.35x_4 - 0.2x_4 u(t) \\
\epsilon_3 \dot{x}_3 = -0.39x_1 - 2.4x_2 - 0.14x_3 - 0.3x_4 - 0.09x_5 - 0.045x_1^2 + 0.3x_1x_5 \\
+ (0.15 + 0.225x_4 + 0.06x_2^2 - 0.15x_4)x_5 u(t) \\
\epsilon_4 \dot{x}_4 = 0.11x_1 - 0.12x_2 - 0.16x_3 + 2.83x_4 - 0.06x_4 + 0.15x_6 \\
+ 0.3x_1 - x_4 + x_4^2 + (0.03 - 0.09)x_4 + 0.12x_4 - 0.18x_4x_4 u(t) \\
\epsilon_5 \dot{x}_5 = -0.185x_1 - 0.1x_1 - 0.15x_2 + 0.26x_3 - 2.78x_4 - 0.16x_5 + x_5^2 + \\
+ 0.15 - 0.075x_5 - 0.03x_5 u(t) \\
\epsilon_6 \dot{x}_6 = -0.15x_1 - 0.165x_2 - 0.045x_3 - 0.3x_3 - 2.73x_4 - 0.345x_5 + 0.15x_8x_4 \\
- 0.45x_8^2 + (-0.009 + 0.04x_8 - 0.045) x_5 u(t) \\
\epsilon_7 \dot{x}_7 = -0.12x_1 - 0.198x_2 - 0.057x_3 - 0.3x_4 + 0.3x_4 - 2.7x_5 - 0.3x_8x_5 \\
- 0.4x_8^2 + (-0.045 - 0.027) x_5 - 0.018x_8^2 + 0.009x_8 u(t)
\end{cases}$$

3.2 Illustrative Example

Let’s consider a 7th order polynomial nonlinear system with four time-scales:
After a three successive separation of dynamics, the proposed procedure provides the following slow reduced subsystem with second order described by

$$\bar{X}_{s3} = A_{s3} \bar{X}_{s3} + A_{s3} \bar{X}_{s3} + A_{s3} \bar{X}_{s3} + (B_{s3} + B_{s3} (I_u \otimes \bar{X}_{s3})) + B_{s3} (I_u \otimes \bar{X}_{s3}) U$$

where $\bar{X}_{s3} = [\bar{X}_{s3,1}, \bar{X}_{s3,2}]^T$.

When considering the random input signal as defined in Figure 1, all the characteristic matrices $A_{s,i}$ and $B_{s,i}$ are calculated.

The obtained simulation results, Figure 2 to Figure 4, show the effectiveness of the recursive order reduction procedure and the quality of the reduced model as illustrated in the Figure 4.

| Figure 1. A random input signal $u(t)$. |
Figure 2. Simulation curve of state variables $X_i$ and obtained after the first step of reduction (___).
Figure 3. Simulation curve of state variables $X_i$ and $X_j$ obtained after the second stage of reduction (___).
Good results, figure 4, are obtained when comparing the state variables evolution of the initial system and those of the constructed reduced model showing the effectiveness of the proposed procedure.

4. Conclusion

The proposed approach appears to be a very powerful alternative to the general problem of modeling...
complex systems and seems to correspond very well to industrial applications and especially to those interconnected with multi-time-scale. It relies on the use of the Kronecker product, polynomial representation and algebraic manipulations for the estimation of the matrices of the polynomial reduced order model. It must be pointed out that the algebraic formulation of the solutions of the nonlinear differential equations allows an easy implementation.

5. Appendix

The Kronecker product of \( A \) and \( B \) denoted by \( A \otimes B \) is defined by:

\[
A \otimes B = \begin{bmatrix}
a_1 B & \ldots & a_n B \\
a_1 B & \ldots & a_n B \\
\vdots & \ddots & \vdots \\
a_1 B & \ldots & a_n B
\end{bmatrix}
\]

\( A \otimes B \) is then a \((pr \times qs)\) matrix.

Among the main properties of this product, let us consider the following useful ones:

- Mixed product rules:
  \[(A \otimes B)(C \otimes D) = (AC) \otimes (BD)\]

- Transposition rule:
  \[(A \otimes B)^T = A^T \otimes B^T\]

The Kronecker power of order \( i \), noted \( X^{[i]} \), of the vector \( X \in \mathbb{R}^n \) is defined by:

\[
\begin{cases}
X^{[1]} = 1 \\
X^{[i]} = X^{[i-1]} \otimes X = X \otimes X^{[i-1]} \text{ pour } i \geq 1
\end{cases}
\]

The non-redundant \( i \)-power \( \hat{X}^{[i]} \) of \( X \) is defined as:

\[
\hat{X}^{[i]} = X^{[i]} = X
\]

\[
\forall i \geq 2; \quad \hat{X}^{[i]} = \begin{bmatrix} x_1^i, x_1^{i-1} x_2, \ldots, x_1^{i-1} x_n, x_1^i x_2^2, \ldots, x_1^{i-2} x_2 x_3, \ldots, x_1^{i-3} x_2 x_3^2, \ldots, x_1^{i-3} x_2 x_3^3, \ldots, x_1^{i-3} x_2 x_3^4 \end{bmatrix}^T
\]

(A.5)

It corresponds to the previous power where have been removed the repeated components.

Then we have:

\[
\forall i \in \mathbb{R}; \exists ! T_i \in \mathbb{R}^{n \times n}, \quad \hat{X}^{[i]} = T_i \hat{X}^{[i]}
\]

(A.6)

Thus, one possible solution for the inversion can be written as:

\[
\forall i \in \mathbb{R}; \exists ! T_i^+ \in \mathbb{R}^{n \times n}; \quad \hat{X}^{[i]} = T_i^+ X^{[i]}
\]

(A.7)

where \( T_i^+ \) is the Moore-Penrose pseudo-inverse of \( T_i \) given by:

\[
T_i^+ = (T_i^T T_i)^{-1} T_i^T
\]

(A.8)

An important vector valued function of a matrix was defined as follows:

\[
\forall i \in \{1, \ldots, q\}, \quad A_i \in \mathbb{R}^r, \quad A = \begin{bmatrix} A_1 & A_2 & \cdots & A_q \end{bmatrix};
\]

\[
\text{vec}(A) = \begin{bmatrix} A_1 \\
A_2 \\
\vdots \\
A_q \end{bmatrix}
\]

(A.9)
We recall the following needed properties

\[
\text{vec}(EAC) = (C^T \otimes E) \text{vec}(A) \tag{A.10}
\]

\[
\text{vec}(A^T) = U_{pq} \text{vec}(A) \tag{A.11}
\]

\[
\text{vec}(AC) = (C^T \otimes I_p) \text{vec}(A) = (I_q \otimes A) \text{vec}(C) \tag{A.12}
\]

where

\[
U_{pq} \text{ is a square permutation matrix ( } pq \times pq \text{) defined by:}
\]

\[
U_{pq} = \sum_{i=1}^{p} \sum_{j=1}^{q} E_{ij}^{pq} \otimes E_{ij}^{pq} \tag{A.13}
\]

with \( E_{ij}^{pq} = e_i^p \otimes e_j^q \), where \( e_i^p \) is a unit p-dimensional vector which is ‘1’ in the i-th column and zero elsewhere, and has precisely a single ‘1’ in each row and in each column.

6. References

1. Esteban S. and Rivas D. Singular perturbation control of the longitudinal flight dynamics of an UAV. In International Conference on Control, 2012; p.310–315. Crossref.

2. Zhang Y, Naidu DS, Cai C and Zou Y. Singulars perturbations and time scales in control theories and applications: an overview 2002-2012. International Journal of Information and Systems Sciences Computing and Information. 2014; 9(1):1–36.

3. Kokotovic PV, O’malley RE, Sannuti P. Singular perturbations and order reduction in control theory–An overview. Introduction to Sociology American on 6th IFAC World Congress, part 1c. Pittsburgh; 1975.

4. Chen Y and Liu YQ. Summary of singular perturbation modeling of multi-time scale power systems. IEEE/PES Transmission and Distribution Conference and Exhibition: Asia and Pacific. 2005; p.1–4.

5. Winkelman JR, Chow JH, Allemong JJ, Kokotovic P. Multi-time-scale analysis of a power system. Automatica. 1980; 16:35–43. Crossref.

6. Singh NP, Singh YP, Ahson SI. New Design of Composite Controllers for Synchronous Machines via Separation of Timescales. IEEE Proceedings.1988 May. 135(3):182–8.

7. Kokotovic PV, Allemong J, Winkelman JR and Chow JH. Singular perturbation and iterative separation of time scales. Automatica. 1980; 16:23–33. Crossref.

8. Tihonov AN. Systems of differential equations containing a small parameter multiplying the derivatives. Matematicheskii Sbornik. 1952; 31(73):575–86.

9. Brewer JW. Kronecker product and matrix calculus in system theory. Transactions on Circuits and Systems. 1978; 25(9):772–81. Crossref.

10. Rotella F, Dauphin-Tanguy G. Non-linear systems: identification and optimal control. International Journal of Control. 1988; 48(2):525–44.

11. Peponides G. Kokotovic PV, and Chow JH. Singular perturbations and time scales in nonlinear models of power systems. IEEE, 1982; 29(11):758–67.

12. Bouzaouache H. On Guaranteed Global Exponential Stability of Polynomial Singularly Perturbed Control Systems. International Journal of Computers, Communications and Control. 2006; 1(4):21–34. Crossref.

13. Bouzaouache. On the stability analysis of nonlinear systems using polynomial Lyapunov functions. Mathematics and Computers in Simulation. 2008; 76(5):316–29.

14. Bouzaouache H. The global stability analysis of nonlinear large scale interconnected systems. Large Scale Complex Systems Theory and Applications. 2007; 11(1):386–91.

15. Bouzaouache H. The Tensor Product Based Analytic Solutions of Nonlinear Optimal Control Problem. Applied Mechanics and Materials. 2014; 511-512:1063–7.

16. Saeedi H, Moghadam MM. A Wavelet Operational Matrix Approach for Solving a Nonlinear Mixed Type Fractional Integro- Differential Equation. Journal of Computer Science and Computational Mathematics. 2014; 4(3):33–40. Crossref.

17. Kimio T, Natarajan G, Hideki A, Taichi K and Nanao K. Higher involvement of subtelomere regions for chromosome rearrangements in leukemia and lymphoma and in irradiated leukemic cell line. Indian Journal of Science and Technology. 2012 April; 5(1):1801–11.
18. Cunningham CH. A laboratory guide in virology. 6th edition. Burgess Publication Company: Minnesota, 1973.

19. Kumar E, Rajan M. Microbiology of Indian desert. In: Ecology and vegetation of Indian desert. DN Sen (editor), Agro Botanical Publisher: India; 1990, p. 83–105.

20. Rajan M, Rao BS, Anjaria KB, Unny VKP, Thyagarajan S. Radiotoxicity of sulfur-35. Proceedings of 10th NSRP, India; 1993, p. 257–8.

21. Article title. http://www.indjst.org/index.php/vision. Date accessed: 01/01/2015.