QCD chiral Lagrangian on the lattice, strong coupling expansion and Ward identities with Wilson fermions

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Abstract
We discuss a general strategy to compute the coefficients of the QCD chiral Lagrangian using lattice QCD with Wilson fermions. This procedure requires the introduction of a lattice chiral Lagrangian as an intermediate step in the calculation. The QCD chiral Lagrangian is then obtained by expanding the lattice effective theory in increasing powers of the lattice spacing and the external momenta. In order to investigate the consequences of the chiral symmetry breaking induced by the Wilson term, we study the lattice chiral Lagrangian at the leading order of the strong coupling and large $N$ expansion. We show that the effects of the Wilson term can be conveniently taken into account, in the lattice effective theory, by a suitable renormalization procedure. In particular, we show that, at the leading order of the strong coupling and large $N$ expansion, the chiral symmetry is exactly recovered on the lattice provided that the bare quark mass and the lattice operators are properly renormalized.

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1 Introduction

The QCD Chiral Lagrangian, originally introduced by Weinberg [1] as a convenient way to reproduce the predictions of PCAC and current algebra, has been subsequently promoted to the role of a consistently renormalizable effective theory [2, 3], which provides a powerful tool for describing the phenomenology of low-energy QCD. A recent review of the phenomenological results obtained in this framework can be found in refs. [4]-[6].

The basic observation of the chiral Lagrangian approach is that, at very low energy, the light pseudoscalar mesons are the only hadronic particles which can be close to the mass shell in the strong interaction processes, and can therefore contribute poles to the analytic structure of the amplitudes. Once these contributions are accounted for by the introduction of a suitable set of fields, an expansion in powers of momenta becomes possible. The QCD chiral Lagrangian is then the most general effective theory, constructed in terms of pseudoscalar fields and a set of external sources, which is consistent with QCD chiral symmetry and its assumed spontaneous dynamical breaking.

Another important feature of low-energy strong interactions is that, in the limit of massless light quarks \((u, d, s)\), chiral symmetry prevents the interactions among the pseudoscalar Goldstone bosons at zero external momenta. Therefore, the effective theory can be systematically organized as an expansion in increasing powers of the quark masses and external momenta:

\[
L_{\text{eff}} = L_2 + L_4 + \ldots
\] (1)

The most general form of the chiral Lagrangian is constrained by the symmetries of the fundamental QCD theory. This fixes completely the form of the interactions among the pseudo-Goldstone bosons and the correlation functions of the external currents in the low-energy limit. However, a set of numerical coefficients, which define the strength of these couplings, cannot be fixed by symmetry requirements alone.

At the lowest order \(p^2\), the effective Lagrangian \(L_2\) can be expressed in terms of only two arbitrary parameters which represent, in the chiral limit, the pseudoscalar decay constant \(F_\pi\) and the quark condensate. One finds then that ten additional chiral couplings are necessary to describe the low-energy QCD phenomenology at \(O(p^4)\) [3]. They are usually denoted by the symbols \(L_1, \ldots, L_{10}\). The main purpose of this paper is to discuss a possible first principle lattice calculation of these coefficients.
At present, our knowledge of the coefficients of QCD chiral Lagrangian mainly comes from the experimental data. The couplings are fixed by comparing the predictions of the effective theory with a set of physical amplitudes measured in the experiments. Further constraints on the coefficients can be derived by general theoretical considerations, relying, for instance, on the large $N$ expansion. Such a determination is typically affected by large uncertainties, and some of these coefficients are still known with a relative error larger than 100% [7].

From a theoretical point of view, the coefficients of chiral Lagrangian ought to be calculable from the QCD Lagrangian. They are functions of the fundamental QCD scale, $\Lambda_{QCD}$, and the heavy quark masses. A rough estimate of their size can be obtained by requiring that the tree-level amplitudes from $L_4$ are of the same order of magnitude of the loop corrections from $L_2$ [8]. This leads to the prediction $L_i \sim (F_\pi^2/4)/\Lambda_{\chi}^2 \sim 2 \times 10^{-3}$, where $\Lambda_{\chi} \simeq 4\pi F_\pi$ is the scale of chiral symmetry breaking. Such an estimate reproduce in fact the correct order of magnitude of the chiral coefficients renormalized at the rho mass scale, indicating a good convergence of the momentum expansion below the resonance region.

A more quantitative picture can be obtained by exploiting the role of meson resonances in the low-energy effective theory [9]. The basic idea consists in writing down the most general low-energy effective Lagrangian for effective vector, axial, scalar, pseudoscalar and tensor fields. The standard chiral Lagrangian is then obtained by integrating out the mesonic degrees of freedom but for the pseudoscalar ones. In this way, one ends up with an explicit expression for the chiral coefficients in terms of low-energy mesonic parameters, masses and decay constants. This approach provides values of the chiral couplings in good agreement with the experimental determinations, showing that these couplings are almost completely saturated by the mesonic resonances contributions. However, in order to determine their actual values, one must still rely on the experimental measurements.

In principle, the QCD chiral Lagrangian can be obtained by integrating out the non effective quark and gluonic degrees of freedom from the original fundamental theory. Indeed, being the two theories mathematically equivalent, the partition function $Z$ can be expressed as:

$$Z = \int (dA d\psi d\overline{\psi}) \exp \left\{ i \int d^4x L_{QCD} \right\} = \int (dU) \exp \left\{ i \int d^4x L_{eff} \right\}$$  \hspace{1cm} (2)
However, although the integral over the fermionic fields in the above equation can be performed explicitly, we do not know, in the general case, how to perform analytically the remaining integration over the gluonic fields.

A more feasible theoretical approach would consist in performing a matching between the effective and the fundamental theory. Specifically, one could compute a sufficient number of physical amplitudes, both in the effective and the original theory, and derive the values of the chiral coefficients from a comparison of the results. The calculation in the full theory, being non perturbative, would then require the implementation of a numerical lattice simulation.

However, such an approach would be affected by the following technical difficulty: in order to perform the matching between the effective and the full theory, one should consider a set physical amplitudes defined in the region of low external momenta, $p < m_\rho$, where the predictions of the chiral Lagrangian can be reliably obtained. On the other hand, in the numerical calculation, an intrinsic infra-red cut-off is introduced by the finite size of the lattice. The minimum value of momentum that can be considered is $p_{\text{min}} = 2\pi/La$, where $L$ is the lattice size and $a$ the lattice spacing. In current lattice calculations, $p_{\text{min}}$ is typically of the order of the rho mass and thus lies at the border of the kinematical region accessible to the calculations in the effective theory. In order to overcome this difficulty, one should consider either larger values of the lattice spacing, thus increasing the finite cut-off effects on the lattice, or larger lattices, which soon becomes computationally prohibitive.

An alternative approach to the calculation of the coefficients of the chiral Lagrangian has been suggested in ref. [10]. The basic observation there is that the separation between effective and non-effective degrees of freedom, which occurs in the continuum QCD theory, must be mirrored by an equivalent distinction in theory regularized on the lattice. Specifically, one can consider an effective Lagrangian, defined on the lattice in terms of an effective pseudoscalar field and external sources, which is equivalent to the fundamental QCD theory regularized on the lattice, for any value of the lattice spacing or bare coupling constant.

This effective lattice theory can be then introduced as an intermediate step in the calculation of the continuum chiral Lagrangian. In order to derive the effective lattice Lagrangian, one can assume a sufficiently large set of couplings and fix the corresponding coefficients through a matching with an overcomplete set of expectation values,
computed both in the effective and the fundamental lattice theory. The several interactions, allowed in the lattice effective Lagrangian, can be organized as an expansion in terms of the distance of couplings, rather than in powers of the external momenta as in the continuum effective theory. Therefore, the lattice effective theory is not specifically a low-energy effective theory. This means that, in performing the matching between the two lattice theories, the existence of an infrared cut-off in the numerical simulation should not represent a problem any longer.

A second advantage of this procedure is that the matching is performed between theories which are defined in the same regularization scheme, the four-dimensional lattice grid. For this reason, the finite ultra-violet cut-off effects, which affect the results of the numerical simulation performed with the fundamental lattice theory, can be kept better under control, because these effects are in principle exactly predicted and reproduced by the corresponding lattice effective theory.

An important observation of ref. [10] is that the lattice effective theory explicitly contains the collective fields which are responsible for the long distance behavior of the corresponding fundamental theory. For this reason, only short distance couplings, typically on the scale length of the order of the inverse rho mass, are expected to play a significant role in the effective theory. This observation then provides a criterion to select a finite number of possible interactions to be considered in the lattice effective theory. For the same reason, one expects that the determination of the corresponding coefficients should be feasible on a lattice of moderate size, thus allowing to achieve a better numerical accuracy in the calculation.

Once the matching has been performed, and the lattice effective theory has been derived, one can consider the infinite volume limit of this theory and expand in increasing powers of the external momenta. The result of this expansion is the QCD chiral Lagrangian. In this way, the chiral coefficients can be calculated.

A useful feature in this approach is that, in the limit of strong coupling and large number of colors $N$, the integration (2) of the non effective degrees of freedom in the QCD Lagrangian can be performed analytically on the lattice [11, 12]. In this way, at the leading order of the strong coupling and large $N$ expansion, the lattice effective theory can be exactly computed.

This procedure has been followed by the authors of ref. [11] in order to compute the coefficients of the QCD chiral Lagrangian in the strong coupling and large $N$ limit. They
have considered a body centered hypercubical lattice, whose greater symmetry implies that invariance under lattice transformations carries over to Euclidean invariance up to the terms of $O(p^4)$ in the continuum. Despite the strong coupling and large $N$ approximation, and an additional simplifying assumption of complete decoupling of the mesonic resonances, the final results for the chiral coefficients are in remarkable qualitative agreement with the experimental values. These results strongly encourage the attempt of a numerical calculation performed in the region of intermediate couplings, which is relevant for continuum QCD calculations.

The feasibility of such a calculation is further investigated in this paper. With respect to ref. [10], we consider the calculation on a standard hypercubical lattice, whose symmetry properties are well understood and four-dimensional Euclidean symmetry is known to be properly recovered in the continuum limit. Moreover, in this paper we study a fermionic action with the Wilson term [13], which serves to prevent the appearance of the undesired doubler fermionic species. A well known effect of the Wilson term on the lattice is the introduction of an additional source of chiral symmetry breaking, which persists even in the limit of vanishing bare quark masses. Main goal of this paper is a study of the effects of such a symmetry breaking from the specific point of view of the calculation of the coefficients of chiral Lagrangian, along the lines discussed above.

In order to derive the low-energy effective theory for QCD, it is convenient to introduce a given set of external source in the fundamental Lagrangian [3]. In this way, the original global chiral symmetry of QCD can be formally promoted to an exact local invariance. On the lattice, because of the presence of the Wilson term in the fermionic action, it is necessary to introduce an additional set of external sources, which have no direct continuum correspondence. This step is discussed in sec. 2, where we define all the external sources added to the action and the corresponding lattice operators which these sources are coupled with.

In sec. 3, we derive the corresponding lattice effective Lagrangian, at the leading order in the strong coupling and large $N$ expansion, by following the procedure of refs. [10]-[12]. The main purpose of this calculation is to investigate the general structure of the lattice effective theory, in the presence of the Wilson term and the external sources. This general structure is a pure consequence of chiral invariance and discrete space-time symmetries, and must persist even in the region of intermediate lattice couplings.
Therefore, the lattice effective Lagrangian, as derived in the strong coupling limit, can be assumed as a guideline to define the corresponding theory in the region of intermediate couplings.

A primary effect of the presence of the Wilson term in the fermionic action is the appearance, in the lattice effective theory, of additional terms, which do not have any direct correspondence in the continuum chiral Lagrangian. The role of these terms is mainly to reproduce the effects of chiral symmetry breaking induced by the Wilson term. These effects are first investigated in sec. 4, where some typical physical quantities, like the pseudoscalar mass, the decay constant and the kaon $B_K$-parameter, are computed directly from the strong coupling lattice effective Lagrangian.

The effects of chiral symmetry breaking are then further investigated in sec. 5 where we study the axial and vector Ward identities in the lattice theory. For the weak coupling region, these identities have been studied in ref. [14]. In that paper it was shown that, once the chiral limit of the theory is correctly identified and the lattice operators are properly renormalized, the predictions of continuum PCAC and current algebra are reproduced on the lattice up to vanishing cut-off effects, unaffected by the chiral breaking introduced by regularization. In this paper, we show that the same conclusion also applies to the lattice theory at the leading order of the strong coupling and large $N$ expansion. In this limit, the effects of the Wilson terms in the effective Lagrangian can be taken into account through a proper renormalization of the quark mass and the lattice operators. The correlation functions of these renormalized operators then satisfy, even in the strong coupling limit, the Ward identities of the continuum theory, as predicted by recovered chiral symmetry.

In the effective continuum limit, that is in the limit of vanishing lattice bare coupling constant, the effects of chiral symmetry breaking induced by the Wilson term are expected to become completely negligible. However, it is well known that for typical values of couplings considered in current numerical simulations these effects are still quite relevant. For this reason, the role of the “Wilson terms”, in the lattice effective Lagrangian, cannot be neglected. On the other hand, the presence of these terms introduces additional difficulties. They significantly increase the number of couplings in the effective theory, they do not have a direct continuum limit in the QCD chiral Lagrangian and finally, because of their presence, the external sources in the lattice theory do not reduce straightforwardly to their direct continuum counterparts.
A procedure which allows to completely discard the Wilson terms in the lattice effective theory is then described in sec. 6. In this way, the general strategy discussed in this paper to derive the QCD chiral Lagrangian should be significantly simplified. The basic idea is to consider a new “renormalized” effective lattice Lagrangian, which reproduce the correlation functions of properly renormalized lattice operators in the fundamental theory. Because these correlation functions satisfy all the Ward identities predicted by continuum current algebra, the renormalized effective Lagrangian does not contain the Wilson terms at all. The way in which this mechanism works is illustrated through the discussion of few significative examples in the framework of the strong coupling effective theory, which exhibits the same chiral structure expected in the weak coupling limit.

Finally, in sec. 7 we briefly discuss some aspects of the numerical calculation to be performed in the region of intermediate couplings of lattice QCD and present our conclusions.

2 The chiral invariant Wilson action with external sources

According to Gasser and Leutwyler [3], a convenient preliminary step in the derivation of the QCD chiral Lagrangian consists in adding to the original QCD action a proper set of external sources. These sources are introduced in such a way that the resulting action becomes invariant with respect to combined local chiral transformations of the external sources and fundamental fields. In this section, we perform this step for QCD regularized on the lattice. But for the introduction of the Wilson term in the lattice action, our definitions of the external sources closely follow those of ref. [10].

We consider the Wilson formulation of lattice QCD [13]. The total action is:

$$S = S_g + S_\psi$$  \hspace{1cm} (3)

where $S_g$ is the pure gauge action:

$$S_g = -\frac{1}{g^2} \sum_P \text{Tr} \left( U_P + U_P^\dagger \right)$$  \hspace{1cm} (4)
$U_P$ being the plaquette variable and the sum over $P$ running over all the oriented plaquettes on the lattice, and $S_\psi$ is the fermionic action:

$$S_\psi = -\frac{1}{2} \sum_{x,\mu} \left[ \overline{\psi}(x) \left( r - \gamma_\mu \right) U_\mu(x) \psi(x + \hat{\mu}) + \overline{\psi}(x + \hat{\mu}) \left( r + \gamma_\mu \right) U_\mu^\dagger(x) \psi(x) \right] +$$

$$+ \left( m + 4r \right) \sum_x \left[ \overline{\psi}(x) \psi(x) \right]$$

(5)

The quark fields, $\psi$ and $\overline{\psi}$, are $n$–component vectors in flavour space; $U_\mu$ are the lattice gauge variables, defined on the links, and $m$ is the bare quark mass. For simplicity, we assume in this paper that the $n$ flavours of quarks are degenerate in mass.

In the limit of zero quark mass and vanishing Wilson term ($m = r = 0$), the lattice action is invariant with respect to global chiral transformations of the quark fields: $\psi_L(x) \rightarrow L \psi_L(x)$ and $\psi_R(x) \rightarrow R \psi_R(x)$, with $L$ and $R$ constant unitary matrices in flavour space. Formally, by adding to the fermionic action a proper set of external sources, this global invariance can be promoted to a local invariance, even in the presence of quark masses and Wilson term. We consider a pair of “link-type” external sources, $W_\mu(x)$ and $Z_\mu(x)$, defined on the link between $x$ and $x + \hat{\mu}$, and a pair of local sources, $\xi(x)$ and $\chi(x)$, defined on the site $x$. The sources $W_\mu(x)$ and $Z_\mu(x)$ are introduced to perform the parallel transport of the right- and left-handed quark fields respectively. The sources $\xi(x)$ and $\chi(x)$, which include the Wilson and mass term respectively, and are introduced to achieve chiral invariance of these terms as well. The lattice fermionic action, including the external sources, can be written as:

$$S_{\psi j} = -\frac{1}{2} \sum_{x,\mu} \left\{ \overline{\psi}(x) \left[ rK(x) - \gamma_\mu \right] U_\mu(x) Y_\mu(x) \psi(x + \hat{\mu}) \right\} +$$

$$+ \overline{\psi}(x + \hat{\mu}) \left[ rK(x + \hat{\mu}) + \gamma_\mu \right] U_\mu^\dagger(x) Y_\mu^\dagger(x) \psi(x) \right\} +$$

$$+ \sum_x \left[ \overline{\psi}(x) J(x) \psi(x) \right]$$

(6)

where the external sources, $Y_\mu(x)$, $K(x)$ and $J(x)$ are given by:

$$Y_\mu(x) = \frac{1}{2} \left[ W_\mu(x) \left( 1 + \gamma_5 \right) + Z_\mu(x) \left( 1 - \gamma_5 \right) \right]$$

$$K(x) = \frac{1}{2} \left[ \xi^\dagger(x) \left( 1 + \gamma_5 \right) + \xi(x) \left( 1 - \gamma_5 \right) \right]$$

$$J(x) = \frac{1}{2} \left[ \chi^\dagger(x) \left( 1 + \gamma_5 \right) + \chi(x) \left( 1 - \gamma_5 \right) \right]$$

(7)
$W_\mu(x)$, $Z_\mu(x)$, $\xi(x)$ and $\chi(x)$ are matrices in the pure flavour space.

It is easy to verify that the total action, $S_g + S_\psi$, is now invariant with respect to local chiral transformations of the quark fields:

$$\psi_L(x) \rightarrow L(x) \psi_L(x), \quad \psi_R(x) \rightarrow R(x) \psi_R(x)$$

with $L(x)$ and $R(x)$ unitary matrices, provided the external sources are defined to transform according to:

$$W_\mu(x) \rightarrow R(x)W_\mu(x)R^\dagger(x+\hat{\mu})$$
$$Z_\mu(x) \rightarrow L(x)Z_\mu(x)L^\dagger(x+\hat{\mu})$$
$$\xi(x) \rightarrow R(x)\xi(x)L^\dagger(x)$$
$$\chi(x) \rightarrow R(x)\chi(x)L^\dagger(x)$$

The Wilson standard fermionic action (5) is recovered from $S_\psi$ in eq. (6) by considering the limit:

$$W_\mu(x), \quad Z_\mu(x), \quad \xi, \quad \xi^\dagger, \quad \chi, \quad \chi^\dagger \rightarrow 1$$
$$\chi, \quad \chi^\dagger \rightarrow m + 4r$$

In the following, we will refer to this limit as the limit of vanishing external sources. This limit will be also indicated with the symbol $(\ldots)_0$.

By following ref. [10], we express, in a non linear way, the external fields $W_\mu(x)$ and $Z_\mu(x)$ in terms of traceless hermitian vector and axial sources:

$$W_\mu(x) = \exp \{-i[v_\mu(x) + a_\mu(x)]\}$$
$$Z_\mu(x) = \exp \{-i[v_\mu(x) - a_\mu(x)]\}$$

where:

$$v_\mu(x) = v_\mu^k(x) \ t^k, \quad a_\mu(x) = a_\mu^k(x) \ t^k$$

and $t^k$ are the generators of the $SU(n)$ flavour group ($k = 1, \ldots , n^2 - 1$). In the fermionic action (3), the sources $v_\mu$ and $a_\mu$ are directly coupled to non-singlet vector and axial lattice currents. In the limit of vanishing external sources, the functional derivatives of the action with respect to $v_\mu$ and $a_\mu$ are given by:

$$\left( \frac{i \delta S}{\delta v_\mu^k(x)} \right)_0 = V_\mu^k(x)$$
$$\left( \frac{i \delta S}{\delta a_\mu^k(x)} \right)_0 = A_\mu^k(x)$$
where
\[ V^k_\mu(x) = \frac{1}{2} \left[ \overline{\psi}(x) U_\mu(x) \left( \gamma_\mu - r \right) t^k \psi(x + \hat{\mu}) + \right. \]
\[ \left. + \overline{\psi}(x + \hat{\mu}) U_\mu^\dagger(x) \left( \gamma_\mu + r \right) t^k \psi(x) \right] \]
(\ref{14})

and
\[ A^k_\mu(x) = \frac{1}{2} \left[ \overline{\psi}(x) U_\mu(x) \left( \gamma_\mu - r \right) \gamma_5 t^k \psi(x + \hat{\mu}) + \right. \]
\[ \left. + \overline{\psi}(x + \hat{\mu}) U_\mu^\dagger(x) \left( \gamma_\mu + r \right) \gamma_5 t^k \psi(x) \right] \]
(\ref{15})

\( V^k_\mu(x) \) is the extended lattice vector current, conserved for degenerate quark masses, and \( A^k_\mu(x) \) is the corresponding (non conserved) axial current.

Similarly, we decompose the external fields \( \xi(x) \) and \( \chi(x) \) in terms of hermitian scalar and pseudoscalar sources:
\[ \xi(x) = \sigma(x) + i\pi(x) \quad , \quad \chi(x) = s(x) + ip(x) \]
(\ref{16})

Derivatives of the action with respect to these sources are given by:
\[ \left( \frac{\delta S}{\delta s^\alpha(x)} \right)_0 = S^\alpha(x) \quad , \quad \left( \frac{i \delta S}{\delta p^\alpha(x)} \right)_0 = P^\alpha(x) \]
(\ref{17})

\[ \left( \frac{\delta S}{\delta \sigma^\alpha(x)} \right)_0 = -4 r \Sigma^\alpha(x) \quad , \quad \left( \frac{i \delta S}{\delta \pi^\alpha(x)} \right)_0 = -4 r \Pi^\alpha(x) \]

where \( S(x) \) and \( P(x) \) are the local scalar and pseudoscalar densities:
\[ S^\alpha(x) = \overline{\psi}(x) t^\alpha \psi(x) \quad , \quad P^\alpha(x) = \overline{\psi}(x) t^\alpha \gamma_5 \psi(x) \]
(\ref{18})

\((\alpha = 0, 1, \ldots, n^2 - 1 \text{ and we use the notation } t^0 = 1)\). The operators \( \Sigma(x) \) and \( \Pi(x) \) are defined as:
\[ \Sigma^\alpha(x) = \frac{1}{8} \sum_\mu \left[ \overline{\psi}(x) U_\mu(x) t^\alpha \psi(x + \hat{\mu}) + \overline{\psi}(x + \hat{\mu}) U_\mu^\dagger(x) t^\alpha \psi(x - \hat{\mu}) \right] \]
\[ \Pi^\alpha(x) = \frac{1}{8} \sum_\mu \left[ \overline{\psi}(x) U_\mu(x) t^\alpha \gamma_5 \psi(x + \hat{\mu}) + \overline{\psi}(x + \hat{\mu}) U_\mu^\dagger(x) t^\alpha \gamma_5 \psi(x - \hat{\mu}) \right] \]
(\ref{19})

and are extended bilinears that, in the continuum limit, reduce to \( S(x) \) and \( P(x) \) respectively.

Beside providing the formal invariance of the action with respect to local chiral transformations, the introduction of the external sources also offers a powerful tool
to compute the correlation functions of the lattice operators. Indeed, the partition function of the theory:

$$Z(j) = \int (dUd\psi d\bar{\psi}) \exp \left\{ -S(U, \psi, \bar{\psi}; j) \right\}$$

(20)

is an explicit functional of the external sources \((j = v_\mu, a_\mu, s, p, \sigma, \pi)\). We can then use eqs. (13) and (17) to compute, with standard techniques, the correlation functions of \(V_\mu, A_\mu, S\) and \(P\) as functional derivatives of \(Z(j)\) with respect to the proper external sources. Eventually, this approach becomes of practical utility once the partition function is expressed in terms of an effective Lagrangian. In the following, we discuss how this effective Lagrangian can, in fact, be calculated.

### 3 The effective lattice Lagrangian in the strong coupling and large \(N\) limit

In the limit of strong coupling and large number of colors \(N\), the high-energy degrees of freedom of lattice QCD can be analytically integrated out \([11, 12]\). After this integration, the partition function turns out to be expressed in terms of an effective action, containing only mesonic fields and external sources. The purpose of this paper is to investigate the feasibility of performing the same integration, numerically, in the region of intermediate couplings relevant for simulations of lattice QCD. Still, the study of the strong coupling limit is an important preliminary step. The effective lattice Lagrangian, obtained in this limit, has a general structure, in terms of effective fields and external sources, that only follows from chiral invariance, charge conjugation and discrete lattice space-time symmetries. Consequently, the same structure must be preserved in the region of intermediate gauge coupling. In this region, one can consider an effective lattice Lagrangian of this form and determine its free parameters through a numerical matching with the full QCD lattice theory.

The procedure to integrate out the high-energy degrees of freedom of lattice QCD, in the strong coupling and large \(N\) limit, has been developed in refs. \([11, 12]\). In ref. \([10]\) the method has been implemented in the formalism of modern chiral Lagrangian introduced

\(^1\)Contrary to what happens in the continuum formalism, however, here one has to be aware of the fact that the action \(S_{\psi j}\) is not simply linear in the external sources. Thus, the derivatives of the partition function might differ, from the corresponding correlation functions, for the contributions of additional contact terms.
by Gasser and Leutwyler [3]. In this section, we apply the results of these papers to obtain the effective Lagrangian for lattice QCD with Wilson fermions and the set of external sources introduced in the previous section.

At the leading order in the strong coupling and large $N$ expansion, the generating functional of eq. (20) can be expressed in terms of an effective action [11, 12]:

$$Z(j) = \int (d\mathcal{M}) \exp \{-S_{\text{eff}}(\mathcal{M}; j)\}$$ (21)

where $S_{\text{eff}}(\mathcal{M}; j)$ is given by:

$$S_{\text{eff}}(\mathcal{M}; j) = -N \left\{ \sum_{x,\mu} \text{Tr} \left[ F(\lambda_{\mu}(x)) \right] - \sum_x \text{Tr} \left[ \ln \mathcal{M}(x) \right] + \sum_x \text{Tr} \left[ J(x) \mathcal{M}(x) \right] \right\}$$ (22)

and $\mathcal{M}(x)$ is a matrix in the spin and flavour space representing an effective bosonic field. The function $F(\lambda)$ in eq. (22) is defined as:

$$F(\lambda) = 1 - \sqrt{1 - \lambda} + \ln \left( \frac{1 + \sqrt{1 - \lambda}}{2} \right)$$ (23)

and the matrix $\lambda_{\mu}(x)$ is a function of the effective field and the external sources:

$$\lambda_{\mu}(x) = -\mathcal{M}(x) \left[ rK(x) - \gamma_{\mu} \right] Y_{\mu}(x) \mathcal{M}(x + \hat{\mu}) \left[ rK(x + \hat{\mu}) + \gamma_{\mu} \right] Y_{\mu}^{\dagger}(x)$$ (24)

The functional integration in eq. (21) is over a generalized contour in the complex matrix space, which can be parameterized by writing $\mathcal{M}$ in polar form: $\mathcal{M} = RU$, with $R$ and $U$ hermitian and unitary matrices respectively. The functional measure in eq. (21) is $(d\mathcal{M}) = (dU)$, where $(dU)$ is the Haar measure on the unitary group of $U$. The functional integral $Z(j)$ is independent on the choice of $R$.

In order to evaluate the partition function in perturbation theory, following refs. [11], [12] we look for a translationally invariant stationary point of the action $S_{\text{eff}}(\mathcal{M})$, taken in the the limit of vanishing external sources. The expansion of $\mathcal{M}$ around its vacuum expectation value will then generate the standard perturbative expansion and, for large $N$, this also corresponds to evaluate the partition function $Z(j)$ by the saddle-point method.
The vacuum expectation value of $\mathcal{M}$ is found by choosing:

$$\mathcal{M}_0(x) = u_0 I$$

and considering the effective action in the limit of vanishing external sources. In this limit, the matrix $\lambda_\mu(x)$ of eq. (24) reduces to:

$$\lambda_0 = (1 - r^2) u_0^2$$

The stationary point of $S(\mathcal{M})$ is then given by $u_0$ satisfying the equation:

$$\frac{4 (1 - r^2) u_0^2}{1 + \sqrt{1 - (1 - r^2) u_0^2}} + (m + 4r) u_0 = 1$$

whose positive solution is:

$$u_0(m, r) = \frac{-3 (m + 4r) + 4 \sqrt{(m + 4r)^2 + 7 (1 - r^2)}}{(m + 4r)^2 + 16 (1 - r^2)}$$

For $r = 1$, one simply has:

$$u_0(m, 1) = \frac{1}{m + 4}$$

The matrix $\mathcal{M}$ can be now expressed in terms of fields with vanishing vacuum expectation value. A convenient general form is:

$$\mathcal{M}(x) = u_0 \exp [i S(x) + i \mathcal{P}(x) \gamma_5 + i \mathcal{V}_\mu(x) \gamma_\mu + i \mathcal{A}_\mu(x) \gamma_\mu \gamma_5 + i \mathcal{T}_{\mu\nu}(x) \sigma_{\mu\nu}]$$

where $S$, $\mathcal{P}$, $\mathcal{V}_\mu$, $\mathcal{A}_\mu$ and $\mathcal{T}_{\mu\nu}$ are matrices in the pure flavour space. In the lattice effective theory these fields, which carry integer spin, represent the mesonic excitations with the corresponding quantum numbers.

The lattice effective action of eq. (22) differs from the standard continuum chiral Lagrangian in that the latter is only expressed in terms of external sources and pseudoscalar fields. Thus, in order to derive an effective lattice action corresponding to this continuum Lagrangian, the contributions of the scalar, vector, axial and tensor components of $\mathcal{M}$, in eq. (30), should be integrated out. In principle, this could be done explicitly. However, in the context of the strong coupling approximation considered so far we are more interested in deriving the general structure of the effective Lagrangian.
than in computing its specific expression. Therefore, also in the attempt to keep discussion simple, we will not perform such an integration. Rather, we assume a complete decoupling of the higher resonances (corresponding to the infinite mass limit) by simply neglecting their contributions. That is, we constrain $\mathcal{M}(x)$ to the form:

$$\mathcal{M}(x) = u_0 \exp \left[ i \mathcal{P}(x) \gamma_5 \right] = u_0 \left[ U(x) \frac{1 + \gamma_5}{2} + U^\dagger(x) \frac{1 - \gamma_5}{2} \right]$$  \hspace{1cm} (31)

where $U(x) = \exp \left[ i \mathcal{P}(x) \right]$ is a unitary matrix in the flavour space. In the continuum limit, $U(x)$ would correspond to the standard effective field entering in the QCD chiral Lagrangian.

An explicit expression for the effective action, in terms of the field $U$, is obtained by Taylor expanding the function $F(\lambda)$ of eq. (22) around the stationary point $\lambda = \lambda_0$. The resulting action has then the form of an infinite series:

$$S_{\text{eff}}(U) = \sum_{k=1}^{\infty} S_k(U) + S_m(U)$$  \hspace{1cm} (32)

where:

$$S_k(U) = -\frac{N C_k}{4 u_0^{2k}} \sum_{x,\mu} \text{Tr} \left[ \lambda_\mu(x) - \lambda_0 \right]^k$$  \hspace{1cm} (33)

and

$$S_m(U) = -\frac{N C_m}{2} \sum_x \text{Tr} \left[ \chi^\dagger(x) U(x) + \chi(x) U^\dagger(x) \right]$$  \hspace{1cm} (34)

The coefficients $C_m$ and $C_k (k = 1, 2, \ldots)$ are given by:

$$C_m = 4 u_0 \quad , \quad C_k = 4 u_0^{2k} \frac{F(k)(\lambda_0)}{k!}$$  \hspace{1cm} (35)

and are of order one in the large $N$ expansion. For practical calculations, the spin-flavour matrix $\lambda_\mu(x) - \lambda_0$ can be conveniently written in the form:

$$\lambda_\mu(x) - \lambda_0 = -\frac{u_0^2}{2} \left[ -U(x) (\Delta_\mu U(x))^\dagger (1 + \gamma_5) - U^\dagger(x) (\Delta_\mu U(x)) (1 - \gamma_5) + + r^2 U(x) \left( \Omega^A_\mu U(x) \right)^\dagger (1 + \gamma_5) + r^2 U^\dagger(x) \left( \Omega^B_\mu U(x) \right) (1 - \gamma_5) + + r U(x) \left( \Sigma^A_\mu U(x) \right)^\dagger (1 + \gamma_5) \gamma_\mu + r U^\dagger(x) \left( \Sigma^B_\mu U(x) \right) (1 - \gamma_5) \gamma_\mu \right]$$  \hspace{1cm} (36)
where $\Delta_\mu U$ is:

$$
\Delta_\mu U(x) = W_\mu(x)U(x + \hat{\mu})Z^*_\mu(x) - U(x) \tag{37}
$$

and the quantities $\Omega^{A,B}_\mu U$ and $\Sigma^{A,B}_\mu U$ are defined as:

$$
\begin{align*}
\Omega^A_\mu U(x) &= W_\mu(x)\xi(x + \mu)U^\dagger(x + \hat{\mu})W^*_\mu(x)\xi(x) - U(x) \\
\Omega^B_\mu U(x) &= \xi(x)Z_\mu(x)U^\dagger(x + \hat{\mu})\xi(x + \mu)Z^*_\mu(x) - U(x) \\
\Sigma^A_\mu U(x) &= Z_\mu(x)U^\dagger(x + \hat{\mu})W^*_\mu(x)\xi(x) - Z_\mu(x)\xi^\dagger(x + \mu)U(x + \hat{\mu})Z^*_\mu(x) \\
\Sigma^B_\mu U(x) &= \xi(x)Z_\mu(x)U^\dagger(x + \hat{\mu})W^*_\mu(x) - W_\mu(x)U(x + \hat{\mu})\xi^\dagger(x + \mu)W^*_\mu(x) \tag{38}
\end{align*}
$$

Equation (32) is the effective chiral Lagrangian for lattice QCD with Wilson fermions at the leading order in the strong coupling and large $N$ expansion. In the next section we will study the predictions of this Lagrangian in some more detail. Here we conclude with some important remarks.

i) As the original fermionic theory, the effective Lagrangian is invariant with respect to local chiral transformations. This is easily shown to be the case when we assume that the field $U(x)$ transforms according to:

$$
U(x) \rightarrow R(x)U(x)L^\dagger(x) \tag{39}
$$

and the external sources $W_\mu$, $Z_\mu$, $\xi$ and $\chi$ have the transformation properties defined in eq. (3). Notice that the quantity $\Delta_\mu U(x)$, introduced in eq. (37), transforms as the matrix $U(x)$ and, in the limit of vanishing external sources, it reduces to the standard discrete derivative of $U$:

$$
(\Delta_\mu U(x))_0 = \nabla_\mu U(x) \equiv U(x + \hat{\mu}) - U(x) \tag{40}
$$

Thus, $\Delta_\mu U$ represents a possible definition of the lattice chiral covariant derivative of $U$ and, in the continuum limit, it reduces in fact to the continuum covariant derivative $D_\mu U$.

ii) In the limit of vanishing Wilson term ($r = 0$), the quantity $(\lambda_\mu - \lambda_0)$ in eq. (30) is proportional to the covariant derivative $\Delta_\mu U(x)$. Therefore, it vanishes for vanishing external sources and for constant field $U(x)$. In this limit, the series expansion of the effective action in eq. (32) corresponds to an expansion in powers of the external momenta $p$, each term $S_k$ being at least of order $p^k$. As in the continuum case, this
feature is a consequence of chiral symmetry, and is what makes the continuum QCD chiral Lagrangian a useful tool for the study of low-energy strong interactions. However, due to the chiral breaking of the Wilson term, this feature is lost on the lattice. In particular, contributions of the order of $p^0$ enter in each of the infinite terms $S_k$ of the effective action. Thus in this case, even in the low-energy limit, each of these terms should be in principle considered to perform calculations. Nevertheless, as we will show in the next section, in the large $N$ limit one can use the standard perturbative technique, so that the lattice effective action of eq. (32) is still useful in practice.

iii) The effective lattice action of eq. (32) is not specifically a low-energy effective theory. Indeed, but for our deliberate neglecting of the contributions of the higher resonances, no low-energy approximation has been introduced to derive it. Rather, eq. (32) defines an effective theory describing, in the strong coupling and large $N$ limit, the strong interactions among pseudoscalar mesons at arbitrary energy. In the intermediate coupling region, a corresponding effective Lagrangian, with the same symmetry properties, can also be considered. The free parameters entering in this Lagrangian, including those coming from integrating out the higher resonances, can be eventually computed by matching with a numerical simulation within the full lattice theory. In this approach, the infra-red cut-off introduced by the finite lattice size in numerical simulations does not represent a problem any longer, since no low-energy expansion has been performed in deriving the effective theory.

iv) The most evident effect of the Wilson term in the lattice effective action is the appearance of additional couplings. For $r = 0$, the effective action of eq. (32) reduces to the form:

$$S_D(U) = \frac{1}{2} N \sum_{x,\mu} \text{Tr} \left\{ (C_1 + 2C_2) \left[ (\Delta_\mu U)^\dagger (\Delta_\mu U) \right] - C_2 \left[ (\Delta_\mu U)^\dagger (\Delta_\mu U) \right]^2 \right\} - \frac{1}{2} NCm \sum_x \text{Tr} \left( \chi^\dagger U + U^\dagger \chi \right) + \ldots$$

which, but for the obvious effects of discretization and lattice breaking of Lorentz invariance, has a direct correspondence with the continuum chiral Lagrangian.

The presence of the Wilson term in the original lattice action gives rise to additional interactions in the effective theory, of the kind:

$$S_W(U) = N \tau^2 \sum_{x,\mu} \text{Tr} \left\{ \frac{1}{2} C_1 \left[ U \left( \Omega_\mu^A U \right)^\dagger + U^\dagger \left( \Omega_\mu^B U \right) \right] + C_2 \left[ U (\Delta_\mu U)^\dagger U^\dagger (\Omega_\mu^A U) + U^\dagger (\Delta_\mu U) U^\dagger (\Omega_\mu^B U) - U \left( \Sigma_\mu^A U \right)^\dagger U^\dagger \left( \Sigma_\mu^B U \right) \right] + \ldots \right\}$$

(42)
These terms are explicit sources of chiral symmetry breaking, since in the limit of van-
ishing Wilson sources, $\xi \rightarrow 1$, they reduce to interactions of the form $U^2$, $(\Delta_{\mu}U)(\Delta_{\mu}U)$, .... In the continuum limit ($g \rightarrow 0$), the coefficients of these terms are expected to vanish. However, in the region of intermediate couplings relevant for current lattice simulations, their effects are still not negligible. These terms significantly increase the number of allowed couplings in the lattice effective theory and, furthermore, they do not have a direct correspondence in the continuum QCD chiral Lagrangian. In the following of this paper we will discuss the way to overcome this problem. It will be shown that an effective lattice theory, without the Wilson terms, can be in fact considered, and this theory reproduces the values of properly renormalized lattice correlation functions.

4 Correlation functions and matrix elements from the effective Lagrangian

In this section we use the effective lattice Lagrangian, derived in eq. (32), to calculate explicitly some simple physical quantities. From a phenomenological point of view, the results are significantly affected by the approximations introduced so far: strong coupling limit, large $N$ expansion and assumption of complete decoupling of the higher resonance. On the other hand, these calculations achieve a twofold goal: first, they offer us the opportunity to illustrate the perturbative technique, as applied to the lattice effective Lagrangian. Possibly, the same technique can be used for determining the coefficients of the effective Lagrangian in the intermediate coupling region. In addition, the analysis which follows, combined with the study of chiral Ward identities in the next section, will point out some important effects of the several Wilson terms entering in the lattice effective theory.

As discussed in sec. 2, the correlation functions of the lattice operators can be calculated by differentiating the partition function $Z(j)$ with respect to the external sources. The first convenient step in order to do that, consists in expressing the effective field $U(x)$ in terms of $n^2 - 1$ independent real fields, representing the true degrees of freedom of the effective theory. We choose the following parameterization:

$$U(x) = \exp \left\{2i\phi(x)/F_0 \right\}$$ (43)

where $\phi(x) = \phi_k(x)t_k$ and $F_0$ is a coefficient introduced to conventionally normalize the pseudoscalar field.
Since the derivatives of the partition functions must be eventually computed in the limit of vanishing external sources, it is also convenient to perform, from the beginning, the following change of variables for the external scalar sources:

\[ s_0(x) = (m + 4r) + s'_0(x) \quad , \quad \sigma_0(x) = 1 + \sigma'_0(x) \]  \tag{44}

In this way, the new sources \( s'_0(x) \) and \( \sigma'_0(x) \) (we will omit the primes in the following) do actually vanish in the limit of vanishing external sources.

The next step consists in expanding the effective action (32) in increasing powers of the pseudoscalar field \( \phi \) and the external sources. Since \( \lambda_0 \) represents the vacuum expectation value of the matrix \( \lambda_\mu(x) \), the difference \( \lambda_\mu(x) - \lambda_0 \) is, by construction, of the order of the field \( \phi \) and the external sources. Thus, although the action (32) has the form of an infinite series in powers of \( (\lambda_\mu(x) - \lambda_0) \), the above expansion can be systematically performed. For convenience, we fix the parameter \( F_0 \) of eq. (43) in such a way that the the kinetic term of the effective Lagrangian for the real scalar field has the standard normalization, i.e. \( (1/2) (\nabla_\mu \phi)^2 \). In this way we obtain:

\[ F_0^2 = 2N \left( 1 + r^2 \right) \left[ C_1 + 2C_2 \left( 1 - r^2 \right) \right] \]  \tag{45}

where \( C_1 \) and \( C_2 \) are the coefficients defined in eq. (35). The effective action \( S_\phi \), as a function of the field \( \phi \) and the external sources, can be written in the form:

\[ S_\phi = S^{(0)}_\phi + S^{(1)}_\phi + S^{(2)}_\phi + \ldots \]  \tag{46}

where the generic term \( S^{(k)}_\phi \) contains the external sources at the power \( k \). The action \( S^{(0)}_\phi \) represents therefore the effective lattice action in the limit of vanishing external sources. It is given by:

\[ S^{(0)}_\phi = \sum_x \left\{ \frac{1}{2} (\nabla_\mu \phi)^2 + \frac{1}{2} M_\sigma^2 \phi^2 + d_{41} K (\nabla_\mu \phi)^4 + d_{42} K (\nabla_\mu \phi^2)^2 + d_{43} K (\nabla_\mu \phi^3) (\nabla_\mu \phi) + d_{44} K \phi^4 + \ldots \right\} \]  \tag{47}

For simplicity, all the flavour indices in the above expression have been omitted. Thus, for instance, \( (\nabla_\mu \phi)^2 \) stands for \( (\nabla_\mu \phi_i) (\nabla_\mu \phi_i) \) and \( K \phi^4 \) for \( K_{ijkl} \phi_i \phi_j \phi_k \phi_l \). The tensor \( K_{ijkl} \) is given by:

\[ K_{ijkl} = \frac{1}{n} \delta_{ij} \delta_{kl} + \frac{1}{2} d_{ijkl} \]  \tag{48}
and the structure constants $d_{ijk}$, as well as the constants $f_{ijk}$ to be introduced below, are defined by the algebra:

$$\text{Tr}(t_i t_j) = \frac{1}{2} \delta_{ij}, \quad [t_i, t_j] = i f_{ijk} t_k, \quad \{t_i, t_j\} = \frac{1}{n} \delta_{ij} + d_{ijk} t_k$$

(49)

The coefficient $M_\pi^2$ of the $\phi^2$ term in eq. (47) (we are anticipating that this coefficient represents in fact the pseudoscalar meson mass) is given by:

$$M_\pi^2 = \frac{2NC_m}{F_0^2} (m + 4r) - \frac{16r^2}{(1 + r^2)}$$

(50)

and the couplings $d_{ij}$ of the several $\phi^4$ terms are:

$$d_{41} = -\frac{(1 + 6r^2 + r^4) A_2}{2N (1 + r^2)^2 A_1^2}, \quad d_{42} = \frac{(1 - r^2) A_1 - 64r^2 A_2}{8N (1 + r^2)^2 A_1^2},$$

$$d_{43} = -\frac{(1 + r^2) A_1 - 24r^2 (3 + r^2) A_2}{6N (1 + r^2)^2 A_1^2}, \quad d_{44} = \frac{8r^2 (A_1 - 12r^2 A_2)}{3N (1 + r^2)^2 A_1^2}$$

(51)

where:

$$A_1 = C_1 + 2C_2 (1 - r^2)$$

$$A_2 = C_2 + 3C_3 (1 - r^2) + 2C_4 (1 - r^2)^2$$

(52)

Since, at the leading order, the effective action $S_{\text{eff}}(U)$ is proportional to the number of colors $N$, the perturbative expansion in terms of the scalar field $\phi$ corresponds in fact to an expansion in increasing powers of $1/N$. The couplings of the $\phi^2$ terms are by construction of order 1 and the couplings of the terms $\phi^{2k}$ are of order $1/N^{k-1}$ (see eq. (51)). Thus, in the large $N$ limit, the lattice effective Lagrangian can be treated in perturbation theory. Furthermore, it is important to remember that, in this context, only the lowest (non trivial) perturbative order should be considered in each particular calculation, since contributions suppressed by additional powers of $1/N$ have been already neglected, from the very beginning, in the calculation of the effective Lagrangian itself.

At the leading order in perturbation theory, the propagator of the scalar field $\phi$ satisfies the equation:

$$\left(\nabla_x^2 - M_\pi^2\right) \Delta(x, y) = -\delta_{x,y}$$

(53)
where $\nabla^2$ represents the (symmetric) discretized version of the four-dimensional Laplacian operator:

$$\nabla^2 f(x) \equiv \sum_{\mu} \left[ f(x + \hat{\mu}) - 2f(x) + f(x - \hat{\mu}) \right]$$

(54)

The solution of eq. (53) is the standard lattice propagator of the scalar field:

$$\Delta(x, y) = \frac{1}{V} \sum_p \frac{e^{-ip(x-y)}}{4 \sum_{\mu} \left[ \sin^2\left(\frac{p_{\mu}}{2}\right) \right] + M_\pi^2}$$

(55)

where $V = L_x L_y L_z L_t$ is the lattice volume and the sum is extended over all the lattice momenta, $p_i = 2\pi k_i / L_i$. For later use we also give the expression of the propagator as a function of the time $t$ and the spatial momentum $\vec{p}$. In the infinite volume limit, and for $t > 0$, one finds:

$$\Delta(t, \vec{p}) = \sum_{\vec{x}} \Delta(t, \vec{x}; 0) e^{-i\vec{p} \cdot \vec{x}} = e^{-\tilde{E}_\pi t} \frac{e^{-\tilde{E}_\pi t}}{2 \sinh \tilde{E}_\pi}$$

(56)

where $\tilde{E}_\pi$ is given by:

$$\tilde{E}_\pi = 2 \text{arcsinh} \left[ \sin^2 \left( \frac{p_i}{2} \right) + \sinh^2 \left( \frac{M_\pi}{2} \right) \right]^{1/2}$$

(57)

and:

$$\tilde{M}_\pi = 2 \text{arcsinh} \left( \frac{M_\pi}{2} \right)$$

(58)

In the continuum limit, $\tilde{M}_\pi \rightarrow M_\pi$ and $\tilde{E}_\pi \rightarrow \sqrt{p_i^2 + M_\pi^2}$.

The next term, $S_\phi^{(1)}$, of the lattice effective action in eq. (46) is the one linear in the external sources. This term can be conveniently written in the form:

$$S_\phi^{(1)} = \sum_x \left\{ s_0(x) \tilde{S}_0(x) + \sigma_0(x) \tilde{\Sigma}_0(x) + p_k(x) \tilde{P}_k(x) + \pi_k(x) \tilde{\Pi}_k(x) + a^{k}_{\mu}(x) \tilde{A}^{k}_{\mu}(x) + v^{k}_{\mu}(x) \tilde{V}^{k}_{\mu}(x) + \ldots \right\}$$

(59)

where the “effective” operators $\tilde{O}_i$ are given by:

$$\begin{align*}
\tilde{S}_0(x) &= d_s + d_{s1} \phi_k^2(x) + \ldots \\
\tilde{\Sigma}_0(x) &= d_\sigma + d_{\sigma1} \phi_k^2(x) + \ldots \\
\tilde{P}_k(x) &= d_p \phi_k(x) + \ldots \\
\tilde{\Pi}_k(x) &= d_{\pi1} \phi_k(x) + d_{\pi2} \nabla^2 \phi_k(x) + \ldots \\
\tilde{A}^k_{\mu}(x) &= d_a \nabla_\mu \phi_k(x) + \ldots \\
\tilde{V}^k_{\mu}(x) &= d_v f_{ijk} \phi_i(x) \left( \nabla_\mu \phi_j(x) \right) + \ldots
\end{align*}$$

(60)
with coefficients:

\[
\begin{align*}
    d_s &= -nNC_m, \quad d_{s1} = NC_m/F_0^2 \\
    d_p &= -NC_m/F_0, \quad d_a = F_0/(1 + r^2), \quad d_v = 1 \\
    d_\sigma &= 8nNC_1r^2, \quad d_{\sigma 1} = -8r^2/(1 + r^2) \\
    d_{\pi 1} &= 8r^2F_0/(1 + r^2), \quad d_{\pi 2} = r^2F_0/2(1 + r^2)
\end{align*}
\]

(61)

In the effective theory, the operators \( \tilde{O}_i \) can be used to calculate directly the correlation functions of the corresponding operators in the original theory.

When one needs to compute the correlation functions of higher powers of the lattice operators (like for instance \( A_\mu^2 \)) then additional contributions might come from terms containing higher powers of the external sources (like \( a_\mu^2 \) and \( a_\mu^2 \phi^2 \)). Examples of such terms, that will be considered in the study of the lattice Ward identities, are found in the next term of the effective action (46):

\[
S^{(2)}_\phi = \sum_x \left\{ d_{\pi 1} \pi_k^2(x) + d_{\pi 2} (\nabla_\mu \pi_k(x))^2 + d_{aa} a_\mu^k(x)^2 + \\
+ d_{av} f_{ijk} [(\nabla_\mu \phi_i(x)) \pi_j(x) - \phi_i(x) (\nabla_\mu \pi_j(x))] v_\mu^k(x) + \\
+ d_{av} f_{ijk} a_\mu^i(x)v_\mu^j(x) [\phi_k(x + \hat{\mu}) + \phi_k(x)] + \\
+ d_{vv} f_{ijk} f_{lmk} v_\mu^i(x)v_\mu^j(x)\phi_j(x)\phi_m(x + \hat{\mu}) + \ldots \right\}
\]

(62)

where:

\[
\begin{align*}
    d_{\pi 1} &= 2NC_1r^2 - 2r^2F_0^2/(1 + r^2), \quad d_{\pi 2} = r^2F_0^2/8(1 + r^2) \\
    d_{aa} &= -d_{av} = F_0^2/2(1 + r^2), \quad d_{vv} = r^2F_0/2(1 + r^2) \\
    d_{vv} &= 1/2
\end{align*}
\]

(63)

In the effective theory, the correlation functions are computed by expressing the partition function \( Z(j) \) in terms of a functional integral on the fields \( \phi \). Up to higher terms in \( 1/N \), the functional measure simply reduces to \( (d\phi) \):

\[
Z(j) = \int (d\phi) \exp \{-S_\phi(\phi; j)\}
\]

(64)

All the correlation functions of interest can be then obtained by subsequent derivatives of \( Z(j) \) with respect to the external sources. We conclude this section by discussing a few interesting examples.
The quark condensate

The simplest relevant case is the calculation of the quark condensate $\langle \bar{\psi}\psi \rangle$. This quantity vanishes to all orders in standard perturbation theory and a value of $\langle \bar{\psi}\psi \rangle$ different from zero is expected to be the signal of spontaneous chiral symmetry breaking. In the effective theory, at the leading order of the strong coupling and large $N$ expansion, the absolute value of the condensate for a single flavour of quarks is given by:

$$\langle \bar{\psi}\psi \rangle = \frac{1}{n} \left( \frac{\delta Z}{Z \delta s^0(x)} \right)_0 = \frac{d_s}{n} = 4N u_0 \quad (65)$$

Thus $u_0$, the vacuum expectation value of the effective field $M(x)$, basically represents the quark condensate.

The pion mass and the critical quark mass

In the effective theory, the correlation function of two pseudoscalar densities is given by:

$$\langle P^j(x)P^{k\dagger}(y) \rangle = \left( \frac{1}{Z} \frac{\delta^2 Z}{\delta p^j(x)\delta p^{k}(y)} \right)_0 = d^2 \delta^{jk} \Delta(x,y) \quad (66)$$

where $\Delta(x,y)$ is the propagator of the $\phi$-field of eq. (55). At this order, the pseudoscalar correlation function only receives contribution from the propagation of a single pseudoscalar meson. Thus, $\Delta(x,y)$ is just the pion propagator and the pole of this function, $M^2_\pi$, defines the pion mass. From eq. (60) we can write this mass in the form:

$$M^2_\pi = \frac{2NC_m}{F_0^2} (m - \bar{m}(m,r)) \quad (67)$$

where $\bar{m}(m,r)$ is the function:

$$\bar{m}(m,r) = \frac{8 r^2 F_0^2}{(1 + r^2) NC_m} - 4r \quad (68)$$

The vanishing of the pion mass identifies the chiral limit of the theory. The corresponding critical value of the quark mass, $m_c$, is therefore defined by the equation:

$$m_c = \bar{m}(m_c,r) \quad (69)$$

In the limit $r = 0$, the function $\bar{m}$ identically vanishes and the critical quark mass is $m_c = 0$. For finite values of $r$, the Wilson term induces an additive renormalization of
the quark mass. In particular, for \( r = 1 \) one finds from eqs. (68) and (69) that the pion mass vanishes for \( m_c = -2 \), corresponding to the critical value \( k_c = 1/4 \) of the Wilson hopping parameter. This is indeed a well known result of strong coupling lattice QCD\(^2\).

In numerical simulations, it is often convenient to compute the correlation functions as a function of the time \( t \) and the spatial momentum \( \vec{p} \). In the effective theory (in the infinite volume limit) these functions can be calculated by using eq. (56). Thus, for instance, for the pseudoscalar correlation function we find:

\[
C_{pp}^{jk}(t, \vec{p}) \equiv \sum_{\vec{x}} \langle P_j(x) P^k(0) \rangle e^{-i\vec{p} \cdot \vec{x}} = \delta^{jk} \frac{d_p}{2 \sinh E_{\pi}} e^{-\tilde{E}_{\pi} t} \\
(70)
\]

with \( \tilde{E}_{\pi} \) given by eq. (57). It is interesting to notice that the energy dependence of the correlation function predicted by the effective Lagrangian differs, by lattice discretization effects, from the corresponding continuum form:

\[
C_{pp}^{jk}(t, \vec{p}) \sim e^{-E_{\pi} t} \\
(71)
\]

where \( E_{\pi} = (\vec{p}^2 + M_{\pi}^2)^{1/2} \). A behaviour as given in eq. (70) has been observed in numerical QCD simulations [13]. Therefore, this is an example of pure lattice artifacts, that are actually found in numerical calculations, and whose existence is predicted by the lattice effective Lagrangian, even in the strong coupling approximation considered so far. It is likely that, in the numerical determination of the coefficients of the effective Lagrangian in the intermediate coupling region, the possibility of taking into account such lattice artifacts will help in better controlling the associated systematic errors. This is another advantage of introducing the effective Lagrangian on the lattice as an intermediate step in the calculations.

**The pseudoscalar decay constant**

The decay constant of pseudoscalar mesons can be computed from the following correlation function:

\[
C_{A_0 p}^{jk}(t, \vec{p}) \equiv \sum_{\vec{x}} \langle A_0^j(x) P^{k}(0) \rangle e^{-i\vec{p} \cdot \vec{x}} = -\delta^{jk} \frac{d_p d_a}{2 \sinh E_{\pi}} \left( 1 - e^{-E_{\pi}} \right) e^{-\tilde{E}_{\pi} t} \\
(72)
\]

\(^2\)It can be proved, in fact, that the result at \( r = 1 \) is true for any \( N \), and applies equally well to \( N = 3 \) [12].

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In general, two-point correlation functions in momentum space can be used to extract the values of the matrix elements of lattice operators between the vacuum and a single particle state. For instance, the correlation function of any two operators $O_A$ and $O_B$, carrying the quantum number of the pion field, are expected to have in the large time limit the following behaviour:

$$\lim_{t \to \infty} C_{AB}(t, \vec{p}) = \frac{\langle 0\mid O_A(0)\mid \pi \rangle \langle \pi \mid O_B^\dagger(0) \mid 0 \rangle}{2 \sinh \tilde{E}_\pi} e^{-\tilde{E}_\pi t}$$  \hspace{1cm} (75)

At the lowest order in the strong coupling effective theory the large time limit is not a necessary requirement, since the correlation functions are always of the form (75). Thus, from eqs. (70) and (72), we can obtain (up to irrelevant complex phases) the values of the following matrix elements:

$$\langle 0\mid P^i(0)\mid \pi^k \rangle = \delta^{jk} d_p, \quad \langle 0\mid A^i_0(0)\mid \pi^k \rangle = -\delta^{jk} d_a \left(1 - e^{-\tilde{E}_\pi}\right)$$  \hspace{1cm} (76)

The last equality provides the value of the pseudoscalar decay constant $F_\pi$ in the strong coupling limit. By considering the continuum definition $\langle 0\mid A^0_0(0)\mid \pi^k \rangle = \delta^{jk} F_\pi E_\pi$, we find that, up to lattice artifacts:

$$F_\pi = -d_a = F_0/(1 + r^2)$$  \hspace{1cm} (77)

From eqs. (63), (67) and (77), we can also derive a relation between $F_\pi$ and the quark condensate:

$$\left(1 + r^2\right)^2 F_\pi^2 M_\pi^2 = 2 \left(m - \hat{m}\right) \langle \bar{\psi} \psi \rangle$$  \hspace{1cm} (78)

In the next section, the same relation will be derived directly from the axial Ward identity of the lattice theory. In that context the meaning of the factors $(1 + r^2)$, entering in eqs. (77) and (78) because of the Wilson term, will be explicitly clarified.

---

3For convenience we choose here, for the one-particle states, the normalization:

$$\langle M(\vec{p})\mid M(\vec{p}') \rangle = 2 \sinh \tilde{E}_M \delta_{\vec{p}\vec{p}'}$$  \hspace{1cm} (73)

instead of the standard covariant normalization:

$$\langle M(\vec{p})\mid M(\vec{p}') \rangle = 2 E_M \delta_{\vec{p}\vec{p}'}$$  \hspace{1cm} (74)
The form factor of semileptonic pion decay

In the limit of exact isotopic invariance, the semileptonic decays of charged pions, \( \pi^+ \to \pi^0 e^+ \nu_e \), are described in terms of a single invariant form factor, \( f_+(q^2) \). This form factor parameterizes the matrix element of the weak vector current between external pion states:

\[
\langle \pi^0(p_2)|\bar{u}\gamma_\mu d|\pi^+(p_1)\rangle = \sqrt{2} f_+(q^2) (p_1 + p_2)_\mu
\]

(79)

\( q_\mu = (p_1 - p_2)_\mu \) is the momentum transferred in the decay and the form factor is normalized at \( q^2 = 0 \) by current conservation: \( f_+(0) = 1 \).

The matrix element of eq. (79) can be computed by considering the following three-point correlation function:

\[
C_{jkl}^{\mu}(t_x,t_y;\vec{p}_1,\vec{p}_2) \equiv \sum_{\vec{x},\vec{y}} \langle P_j(y)V_{\mu}^k(0)P_l^\dagger(x) \rangle e^{i\vec{p}_1 \cdot \vec{x} - i\vec{p}_2 \cdot \vec{y}} =
\]

(80)

\[
= \langle 0|P_j(0)|\pi^r(\vec{p}_2)\rangle \langle \pi^r(\vec{p}_2)|V_{\mu}^k(0)|\pi^s(\vec{p}_1)\rangle \langle \pi^s(\vec{p}_1)|P_l^\dagger(0)|0\rangle \cdot e^{-E_2 t_y + E_1 t_x} \cdot \frac{4 \sinh E_1 \sinh E_2}{4 \sinh E_1 \sinh E_2}
\]

where the last equality holds for \( t_x < 0 < t_y \). An explicit calculation of this function in the effective theory shows that the matrix element of the weak current between two external pion states has the form:

\[
\langle \pi^j(\vec{p}_2)|V_{\mu}^k(0)|\pi^l(\vec{p}_1)\rangle = -i d_v f_{jkl} \left( e^{i\vec{p}_1 \cdot \vec{x}} - e^{-i\vec{p}_2 \cdot \vec{y}} \right)
\]

(81)

Thus, up to lattice artifacts, we find that the vector form factor for pion semileptonic decays is just given by:

\[
f_+(q^2) = d_v = 1
\]

(82)

This result has likely a simple explanation. In reality, the form factor \( f_+(q^2) \) turns out to be well described by a phenomenological vector meson dominance model, where it is expected to have the form: \( f_+(q^2) = m_\rho / (m_\rho - q^2) \), with \( m_\rho \) the rho meson mass. In this case, in the limit of complete decoupling of the rho meson as the one considered so far, the form factor becomes completely independent on \( q^2 \) and reduces to 1 for current conservation. In a typical lattice simulation, the value of \( q^2 / m_\rho^2 \) is roughly of the order of 1. Thus, in that context, the assumption of complete decoupling of the rho
meson is certainly unreliable. What we expect to happen is that the effect of the rho and other resonances will manifest itself through the structure of the couplings in the effective lattice Lagrangian, which, rather than being limited to the nearest neighbors as in the strong coupling approximation, will fall off with a range \( \approx m_\rho^{-1} \). It should be also noticed that the equality \( d_v = 1 \), obtained for the coefficient of the effective theory, is a consequence of the lattice vector current conservation. In the next section we will find in fact that the same equality can be also obtained directly from the vector Ward identities of the lattice theory.

### The \( B_K \)-parameter

The calculation of the \( B_K \)-parameter in the effective theory shows that some chiral symmetry breaking effects, induced on the lattice by the Wilson term \([14, 16]\) and actually observed in numerical simulations (see e.g. ref. \([17]\)), do not appear in the strong coupling and large \( N \) limit. This seems to be a quite general property of the theory in this limit: that is, the chiral symmetry is not broken to all the extents it could possibly be. The same feature will be illustrated by the study of the lattice Ward identities in the next section.

The parameter \( B_K \) parameterizes the matrix element \( \langle K^0 \mid O^{\Delta S=2} \mid K^0 \rangle \) relevant for \( K^0 - \bar{K}^0 \) mixing in weak interactions. \( O^{\Delta S=2} \) is the four-fermion operator \((\bar{s}\gamma_\mu (1 - \gamma_5) d)^2\) entering in the effective \( \Delta S = 2 \) weak Hamiltonian. For external states of kaons at rest, the parameter \( B_K \) is defined as:

\[
\langle K^0 \mid O^{\Delta S=2} \mid K^0 \rangle = \frac{16}{3} B_K F_K^2 M_K^2
\]

This definition is such that in the vacuum saturation approximation:

\[
B_K = \frac{3}{4} \left( 1 + \frac{1}{N} \right)
\]

so that, in this limit, \( B_K = 1 \) for \( N = 3 \).

On the lattice, with Wilson fermions, the matrix element between external states of arbitrary momenta can be conveniently parameterized in the form:

\[
\langle K^0(\mathbf{p}_2) \mid O^{\Delta S=2} \mid K^0(\mathbf{p}_1) \rangle = \alpha + \beta M_K^2 + \gamma (p_1 \cdot p_2) + \ldots
\]

where the dots stand for higher order terms in \( M_K^2 \) and \( (p_1 \cdot p_2) \). The parameters \( \alpha \) and \( \beta \) are lattice artifacts that should vanish in the true continuum limit, \( g \to 0 \). The origin
of these terms is the mixing of the lattice operator \( O^{\Delta S=2} \) with operators of different chirality\(^2\), and it is only due to the presence of the Wilson term in the lattice action.

The calculation of the matrix element (85) in the effective theory proceeds through the evaluation of the corresponding three-point correlation function, by using the technique discussed above. In the large \( N \) limit, one finds that the matrix element only receives contribution from the \( A^2 \) part of the \( (V - A)^2 \) lattice operator. The matrix elements of the \( VA \) and \( AV \) operators vanish identically for parity invariance and the remaining contribution, \( V^2 \), is suppressed by an additional power of \( 1/N \). Technically, this follows from the fact that the relevant coefficients in the effective Lagrangian are \( d_a \sim \sqrt{N} \) and \( d_v = 1 \) (see eq. (61)). In addition, we find that in the large \( N \) limit the matrix element is saturated by the vacuum insertion and it is given by:

\[
\langle K_0(\vec{p}_2)|O^{\Delta S=2}|K_0(\vec{p}_1)\rangle = 4 d_a^2 (e^{ip_1\mu} - 1) (e^{-ip_2\mu} - 1) \tag{86}
\]

Up to discretization effects, eq. (86) corresponds to eq. (85) with \( \alpha = \beta = 0 \), as would be predicted by unbroken chiral symmetry. Therefore, at the leading order of strong coupling and large \( N \) expansion, the mixing of \( O^{\Delta S=2} \) with operators of different chirality does not occur. We also find in this limit \( \gamma = 4 d_a^2 = 4 F_K^2 \), corresponding to:

\[
B_K = \frac{3}{4} \tag{87}
\]

According to eq. (84), this is the result predicted by the vacuum insertion approximation in the large \( N \) limit.

Although the \( 1/N \) corrections to eq. (87) cannot be consistently computed within the approximations considered so far, it may be useful to observe that \( 1/N \) suppressed contributions to the parameter \( \alpha \) in eq. (85) in fact exist. They come for instance from the Wilson term:

\[
-NC_1 r^2 \sum_{x,\mu} \text{Tr} [a_\mu(x)\phi(x + \hat{\mu})a_\mu(x)\phi(x) + (a_\mu \rightarrow v_\mu)] \tag{88}
\]

which can be obtained by expanding the effective action \( S_1 \) of eq. (32). Other \( 1/N \) suppressed contributions to \( \alpha \) are also found at the next order in perturbation theory, coming from the pure \( \phi^4 \) term of the effective action (see eq. (17)). Since the coefficient \( d_{44} \) of this term is proportional to \( r^2 \), it represents again a pure effect of the Wilson chiral symmetry breaking.

\(^2\)The same combination (88) is also found in the symmetric term \( (\Delta_\mu U)^\dagger (\Delta_\mu U) \) of the effective action, but now with a relative minus sign between the vector and axial part. Thus, in this case, the contributions of the two terms cancel in the final result.
5 Chiral invariance and Ward identities

The chiral invariance of the Wilson action, in the presence of external sources, implies the existence of a full set of vector and axial Ward identities (WI). These identities have been studied in ref. [14] to show that, in the continuum limit, they reduce to the known WI's of QCD. Precisely, once the chiral limit of the theory is correctly identified and the lattice operators are properly renormalized, the WI's on the lattice reproduce the relations of continuum current algebra.

In this section, we will first verify that the lattice WI's are indeed satisfied, by explicitly computing the relevant correlation functions in the effective theory. This is nothing more than a useful check of the calculations performed so far. Then, however, by using these results, we will show that the relations of continuum current algebra are exactly reproduced by the lattice theory even in the (leading order) strong coupling and large $N$ expansion considered so far. In a sense, this is quite a unexpected result. According to ref. [14], the current algebra relations are expected to be recovered only in the continuum limit, when the contribution of higher dimensional operators to the correlation functions of the WI's can be exactly neglected. In contrast, the strong coupling approximation represents rather the opposite limit, in which such contributions, if any, should appear as genuine and finite corrections.

For the purpose of this paper, the possibility of recovering exact chiral symmetry with the Wilson action in the strong coupling approximation is a very useful result. In fact we can define, even in the strong coupling limit, “renormalized” operators that satisfy all the WI's of the continuum theory. Then in the next section, by considering some specific examples, we will be able to show, analytically, that the correlation functions of these renormalized operators are completely reproduced by a lattice effective Lagrangian which does not contain the Wilson terms at all. Thus, each term of this Lagrangian, but for discretization effects, has a well defined correspondence in the continuum effective theory. The same result must be valid in the scaling region of couplings relevant for QCD simulations. In that case, this effective lattice Lagrangian can be eventually expanded in powers of the lattice spacing and external momenta to compute the coefficients of the continuum QCD chiral Lagrangian.

In order to study the lattice WI’s in the framework of the effective theory, it is convenient to derive them in the form of equalities among derivatives of the partition function with respect to the external sources (see footnote 1).
Let us consider the set of infinitesimal vector and axial transformations generated, from eqs. (8) and (9), in the limit in which:

\[ R(x) \simeq 1 + i [\alpha_V(x) + \alpha_A(x)] \]
\[ L(x) \simeq 1 + i [\alpha_V(x) - \alpha_A(x)] \]  

(89)

where \( \alpha_{V,A}(x) = \alpha_{V,A}^k t^k \) are infinitesimal quantities. In this limit, the chiral transformations of the quark fields in eq. (8) reduce to:

\[ \psi(x) \rightarrow \psi'(x) = \psi(x) + i [\alpha_V(x) + \alpha_A(x) \gamma_5] \psi(x) \]
\[ \bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \bar{\psi}(x) - i \bar{\psi}(x) [\alpha_V(x) - \gamma_5 \alpha_A(x)] \]  

(90)

Similarly, from eq. (9), we can derive the transformations of the external sources. For the vector and axial sources they have the form:

\[ \delta v_\mu(x) = \nabla_\mu \alpha_V(x) + \frac{i}{2} [\alpha_V(x + \hat{\mu}) + \alpha_V(x), v_\mu(x)] + \]
\[ + \frac{i}{2} [\alpha_A(x + \hat{\mu}) + \alpha_A(x), a_\mu(x)] + \ldots \]  

\[ \delta a_\mu(x) = \nabla_\mu \alpha_A(x) + \frac{i}{2} [\alpha_V(x + \hat{\mu}) + \alpha_V(x), a_\mu(x)] + \]
\[ + \frac{i}{2} [\alpha_A(x + \hat{\mu}) + \alpha_A(x), v_\mu(x)] + \ldots \]  

(91)

where the dots indicate terms of higher order in the external sources. For the scalar and pseudoscalar sources these transformations, by being local, have the same form expected in the continuum theory:

\[ \delta \sigma(x) = i [\alpha_V(x), \sigma(x)] - \{\alpha_A(x), \pi(x)\} \]
\[ \delta \pi(x) = i [\alpha_V(x), \pi(x)] + \{\alpha_A(x), \sigma(x)\} \]
\[ \delta s(x) = i [\alpha_V(x), s(x)] - \{\alpha_A(x), p(x)\} \]
\[ \delta p(x) = i [\alpha_V(x), p(x)] + \{\alpha_A(x), s(x)\} \]  

(92)

Let us then perform in the partition function \( Z(j) \) of the full lattice theory the change of variables defined by eq. (90). With respect to this change of variables the measure in the functional integral is invariant\(^5\). On the other hand, by definition, a change of

\(^5\)In the more general case of \( U(n) \otimes U(n) \) flavour transformations the invariance of the measure is lost and the consequent variation generates the quantum anomaly.
variables does not affect the value of the integral. Thus, by considering that the lattice action is invariant with respect to combined transformations of the quark fields and external sources, it is easy to proof that the partition function itself is invariant with respect to transformations of the external sources only. Indeed, by denoting with a prime the transformed quantities, one has:

\[
Z(j) = \int (dU d\psi d\bar{\psi}) \exp \{-S(U, \psi, \bar{\psi}; j)\} = \int (dU d\psi d\bar{\psi}) \exp \{-S(U, \psi', \bar{\psi}'; j')\} = \int (dU d\psi' d\bar{\psi}') \exp \{-S(U, \psi', \bar{\psi}'; j')\} = Z(j')
\]

(93)

Formally, we can express this invariance in infinitesimal form:

\[
\delta Z(j) = \sum_a \sum_x \left( \frac{\delta Z(j)}{\delta j_a(x)} \right) \delta j_a(x) = 0
\]

(94)

where \( \delta j(x) \) are the variations of the sources given in eqs. (91) and (92). The whole set of vector and axial WI’s are finally obtained by further differentiating the above equation with respect to the external sources and taking the limit of vanishing external sources. For example, for the two- and three-point correlations functions, the generic WI’s have the form:

\[
\sum_a \sum_x \left[ \left( \frac{\delta^2 Z(j)}{\delta j_a(x) \delta j_b(y)} \right) \delta j_a(x) + \left( \frac{\delta Z(j)}{\delta j_a(x)} \right) \left( \frac{\delta j_a(x)}{\delta j_b(y)} \right) \right] = 0
\]

(95)

and

\[
\sum_a \sum_x \left[ \left( \frac{\delta^3 Z(j)}{\delta j_a(x) \delta j_b(y) \delta j_c(z)} \right) \delta j_a(x) + \left( \frac{\delta^2 Z(j)}{\delta j_a(x) \delta j_b(y)} \right) \left( \frac{\delta j_a(x)}{\delta j_c(z)} \right) + \left( \frac{\delta^2 Z(j)}{\delta j_a(x) \delta j_c(z)} \right) \left( \frac{\delta j_a(x)}{\delta j_b(y)} \right) \right] = 0
\]

(96)

respectively.

To begin the analysis of the WI’s in the effective theory, let us consider the simple case of the two-point axial-pseudoscalar correlation function. The corresponding WI is obtained from eq. (93) by choosing \( j^b(y) = p^j(y) \). The result reads:

\[
\nabla^L \mu \left( \frac{1}{Z} \frac{\delta^2 Z}{\delta a^*_k(x) \delta p^j(y)} \right)_0 = 2 \left( m + 4r \right) \left( \frac{1}{Z} \frac{\delta^2 Z}{\delta p^k(x) \delta p^j(y)} \right)_0 + 2 \left( \frac{1}{Z} \frac{\delta^2 Z}{\delta \pi^k(x) \delta p^j(y)} \right)_0 - \frac{1}{n} \delta^{kj} \delta_{x,y} \left( \frac{1}{Z} \frac{\delta Z}{\delta s^0(x)} \right)_0
\]

(97)
where $\nabla^L_\mu$ denotes the left discrete derivative:

$$\nabla^L_\mu f(x) \equiv f(x) - f(x - \mu)$$  \hfill (98)

and, we recall, $n$ in eq. (97) is the number of flavours.

In the effective theory, the derivatives of the partition function with respect to the external sources can be easily calculated. For the terms entering in eq. (97) we find, at the lowest order:

$$\nabla^{Lx}_\mu \left( \frac{1}{Z} \frac{\delta^2 Z}{\delta a^k_\mu(x) \delta p^j(y)} \right)_0 = d_a d_p M^2 \pi \delta^{kj} \Delta(x,y) - d_a d_p \delta^{kj} \delta_{x,y} \Delta(x,y)$$

$$\left( \frac{1}{Z} \frac{\delta^2 Z}{\delta \pi^k(x) \delta p^j(y)} \right)_0 = d_p (d_{\pi^1} + d_{\pi^2} M^2 \pi) \delta^{kj} \Delta(x,y) - d_p d_{\pi^2} \delta^{kj} \delta_{x,y}$$

$$\left( \frac{1}{Z} \frac{\delta^2 Z}{\delta p^k(x) \delta p^j(y)} \right)_0 = d_{\pi^1}^2 \delta^{kj} \Delta(x,y) \quad , \quad \left( \frac{1}{Z} \frac{\delta Z}{\delta \delta^{ij}(x)} \right)_0 = -d_s$$  \hfill (99)

Now, for the WI (97) to be satisfied, the terms proportional to $\Delta(x,y)$ and those proportional to the $\delta_{x,y}$ must be equal on both sides of the equation. From these requirements we find two relations among the several coefficients $d_i$:

$$(d_a - 2d_{\pi^2}) M^2 \pi = 2 (m + 4r) d_p + 2d_{\pi^1}$$  \hfill (100)

and

$$(d_a - 2d_{\pi^2}) d_p = -d_s / n$$  \hfill (101)

By using the explicit values of the coefficients, (eqs. (51) and (61)), it is easy to verify that both the relations (100) and (101) are indeed satisfied. Thus, in the effective theory, the axial WI (97) is identically satisfied as well.

To extend this check, we have considered a larger set of both vector and axial WI’s, for two- and three-point correlation functions. In each case, one or more relations among the coefficients $d_i$ are found, each of the kind of eq. (101). Examples of such relations, as derived from the axial WI’s for the correlation functions $\langle AP \rangle$, $\langle AA \rangle$, $\langle ASP \rangle$ and $\langle AA V \rangle$, are:

$$(d_a - 2d_{\pi^2}) d_{\pi^1} = -d_s / n + 4d_{\pi^1} \quad , \quad (d_a - 2d_{\pi^2}) d_{\pi^2} = -4d_{\pi^2}$$

$$(d_a - 2d_{\pi^2}) d_a = 2d_{aa} \quad , \quad (d_a - 2d_{\pi^2}) d_{s^1} = d_p$$

$$(d_a - 2d_{\pi^2}) d_v = 2d_{av} - 2d_{\pi v} \quad , \quad (d_a - 2d_{\pi^2}) d_v = d_a - 2d_{\pi v}$$  \hfill (102)
Similarly, by considering the vector WI’s for \langle VV \rangle, \langle VPP \rangle, \langle VP\Pi \rangle and \langle VAA \rangle, we find:

\[ d_v = 1 , \quad d_v^2 = 2d_{vv} , \quad d_{\pi v} = d_{\pi 2} , \quad d_{av} = d_a/2 \]  \hspace{1cm} (103)

All the relations in eqs. (102) and (103) are satisfied regardless the specific values of the coefficients \( C_m, C_1, C_2, \ldots \) entering in the effective Lagrangian. In other words, eqs. (102) and (103) are a pure consequence of chiral symmetry and the fact that these relations are satisfied is basically just a useful check of the calculations performed so far.

It is useful to rewrite eq. (100) in the following form:

\[ (d_a - 2d_{\pi 2}) M_{\pi}^2 = 2 \left[ m - \hat{m}(m, r) \right] d_p \] \hspace{1cm} (104)

where we have defined:

\[ \hat{m}(m, r) = \frac{-d_{\pi 1}}{d_p} - 4r \] \hspace{1cm} (105)

We easily recognize that the identity (104) is just eq. (67), that fixes the ratio between the square of the pion mass and the quark mass. In particular, the function \( \hat{m} \), defined in eq. (103), is exactly the same quantity previously introduced in eq. (68). In addition, the identities (104) and (101) can be combined together to obtain:

\[ (d_a - 2d_{\pi 2})^2 M_{\pi}^2 = -2 \left[ m - \hat{m}(m, r) \right] d_s/n \] \hspace{1cm} (106)

By observing that the ratio \((d_a - 2d_{\pi 2})/d_a\) is equal to \((1 + r^2)\), we can also express eq. (106) in terms of the pion decay constant \( F_\pi = -d_a \) and the quark condensate \( \langle \bar{\psi} \psi \rangle = d_s/n \). In this way we obtain:

\[ \left( 1 + r^2 \right)^2 F_\pi^2 M_{\pi}^2 = 2 \left( m - \hat{m} \right) \langle \bar{\psi} \psi \rangle \] \hspace{1cm} (107)

that is again eq. (78) derived before. Thus this equation, as its continuum analogous, is a direct consequence of the axial WI.

We now show how the continuum current algebra relations and the partial conservation of the axial current are reproduced in the lattice theory at the leading order of the strong coupling and large \( N \) expansion. Technically, the mechanism turns out to be exactly the same discussed in ref. [14] for the weak coupling regime.

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In the continuum theory, the axial WI, expressed in terms of renormalized operators $O_R(x)$ and the renormalized quark mass $m_R$ has the form:

$$\partial^\mu \langle A^k_{R\mu}(x)P^j_R(y) \rangle = 2 m_R \langle P^k_R(x)P^j_R(y) \rangle - \frac{1}{n} \delta^{kj} \delta(x-y) \langle S^0_R(x) \rangle$$

(108)

On the lattice, since the fermionic action is simply linear in the external source $p(x)$, the derivatives of the partition function entering in the WI (97) are proportional to the corresponding correlation functions. Thus, eq. (97) can be also written in the form:

$$\nabla^\mu L^x \langle A^k_{\mu}(x)P^j(y) \rangle = 2 (m + 4r) \langle P^k(x)P^j(y) \rangle - 8r \langle \Pi^k(x)P^j(y) \rangle - \frac{1}{n} \delta^{kj} \delta(x,y) \langle S^0(x) \rangle$$

(109)

Formally, this differs from the continuum expression (108) for the presence of terms proportional to $r$, originated by the Wilson term in the lattice action. We now observe that these contributions can be expressed in terms of correlation functions of the two lattice operators $\nabla^\mu L^x A_\mu(x)$ and $P(x)$ respectively. Explicitly, from eq. (99), we find:

$$8r \langle \left[ P^k(x) - \Pi^k(x) \right] P^j(y) \rangle =$$

$$= (1 - Z_A) \nabla^\mu L^x \langle A^k_{\mu}(x)P^j(y) \rangle - 2 \tilde{m} \langle P^k(x)P^j(y) \rangle$$

(110)

where:

$$Z_A = 1 - 2 \frac{d_\pi^2}{d_a} = 1 + r^2$$

(111)

Therefore, by substituting eq. (110) in eq. (109), we obtain for the lattice WI the following expression:

$$Z_A \nabla^\mu L^x \langle A^k_{\mu}(x)P^j(y) \rangle = 2 (m - \tilde{m}) \langle P^k(x)P^j(y) \rangle - \frac{1}{n} \delta^{kj} \delta(x,y) \langle S^0(x) \rangle$$

(112)

Despite the strong coupling limit we are considering, this result has exactly the form (108) required by continuum current algebra, provided we identify the following renormalized operators and quark mass:

$$A^k_{R\mu}(x) = Z_A A^k_{\mu}(x) \quad , \quad P^k_R(x) = Z_P P^k(x)$$

$$S^0_R(x) = Z_P S^0(x) \quad , \quad m_R = Z_m (m - \tilde{m})$$

(113)

6In the weak coupling region this is also true, but for the presence of additional Schwinger terms. This follows from the fact that $\nabla^\mu A_\mu(x)$ and $P(x)$ are the only operators of dimension four or less with which the Wilson terms can mix [14].
with $Z_A$ given by eq. (111) and $Z_m = 1/Z_P$. In particular, the vanishing of the renormalized quark mass $m_R$ identifies the proper chiral limit, in which the axial current becomes exactly conserved and the pseudoscalar meson mass vanishes.

To verify the extent to which chiral symmetry is in fact recovered also in the strong coupling limit, we can study other examples of lattice WI's, like for instance the axial WI for the $\langle ASP \rangle$ correlation function. In this case, current algebra requires:

$$\nabla_\mu L^x \langle A_\mu^k(x) S^j(y) P^l(z) \rangle = 2 (m - \bar{m}) \frac{Z_m Z_P}{Z_A} \langle P^k(x) S^j(y) P^l(z) \rangle -$$

$$- d^{kjr} \delta_{x,y} \frac{Z_P}{Z_S Z_A} \langle P^r(x) P^l(z) \rangle - d^{klr} \delta_{x,z} \frac{Z_S}{Z_P Z_A} \langle S^r(x) S^j(y) \rangle$$

An explicit calculation of the correlation functions in the effective theory shows that again the WI has the form prescribed by eq. (114). $Z_A$ is still given by eq. (111) and $Z_m = 1/Z_P$. In addition, we also find the ratio:

$$\frac{Z_S}{Z_P} = 1$$

As a last example, we wish to discuss a vector WI. As far as the vector current is concerned, we know that, for any value of the gauge coupling, this current is conserved. Thus, its renormalization constant must be exactly equal to 1. This can be verified by considering, for instance, the vector WI for the $\langle VPP \rangle$ correlation function. In this case, in the limit of degenerate quark masses, the prescription of current algebra is:

$$Z_V \nabla_\mu L^x \langle V_\mu^k(x) P^j(y) P^l(z) \rangle = i f^{kjr} \delta_{x,y} \langle P^r(x) P^l(z) \rangle +$$

$$+ i f^{klr} \delta_{x,z} \langle P^r(x) P^j(y) \rangle$$

This is exactly the identity we obtain in the effective theory, where, as expected, we also find:

$$Z_V = 1$$

To conclude this section, we wish to notice that above results provide a clear interpretation of eq. (107). Since the combination $Z_m Z_P$ is equal to 1, the right-hand side of eq. (107) is renormalization invariant. In contrast, the pion decay constant $F_\pi$ on the left-hand side of the equation must be renormalized, the proper renormalization constant being $Z_A = 1 + r^2$. Thus, once expressed in terms of all renormalized quantities, with $F_{\pi R} = Z_A F_\pi = F_0$, eq. (107) becomes:

$$F_{\pi R}^2 M_{\pi R}^2 = 2 m_R \langle \overline{\psi} \psi \rangle_R$$
that reproduces the well known continuum relation.

Thus, all the results obtained in this section are consistent with the following picture: at the leading order in the strong coupling and large $N$ expansion, the lattice theory reproduces correctly the relations of continuum current algebra. This allows one to define, even in the strong coupling limit, renormalized lattice operators that corresponds to their continuum counterpart. The lattice vector current, being conserved, does not require renormalization. In contrast, the axial current does. The resulting renormalized operators then respect all requirements of the PCAC hypothesis. The way in which this scenario sets up is exactly the one outlined in ref. [14] for the weak coupling regime of lattice QCD.

6 The “renormalized” effective Lagrangian

The discussion of the two previous sections has been mainly devoted to investigate the role of the Wilson terms (eq. (42)) in the lattice effective Lagrangian.

The original motivation, which led to the introduction of the Wilson term in the lattice action, was the removing of the doubler fermion species [13]. In the naively discretized version of the Dirac theory, these fermions would originate additional pseudo-Goldstone bosons. Clearly, were these particles present in the spectrum of the theory, they could not have been integrated out from the effective Lagrangian. Thus in this sense, by having considered only a multiplet of pseudo-Goldstone bosons, this primary effect of the Wilson term has been taken into account, form the very beginning, in our lattice effective theory.

A well known consequence of removing the doubler species is the appearance on the lattice of the axial anomaly [18]. However, in the effective theory considered in this paper, we have not introduced external sources coupled to flavour-singlet anomalous axial currents. Thus, the Wilson terms in our effective action do not play any role in reproducing the correct anomaly. From this particular point of view, they could have been completely neglected.

Beside that, most of the other effects of the Wilson term on the lattice are usually undesired, in that they contribute to further differentiate the lattice theory from the corresponding continuum one. By being formally vanishing in the continuum limit, the Wilson term in the original lattice action can only induce physical effects through
the loops of the quantum theory. In the weak coupling region, these effects can be cured by considering a proper set of renormalization prescriptions, which allows one to relate the lattice quantities to the corresponding continuum counterparts. Examples of such prescriptions, already encountered in this paper, are the renormalization of the quark mass and the multiplicative renormalization of the bilinear quark operators. The several Wilson terms in the effective lattice Lagrangian enter to exactly reproduce these “undesired” effects of the Wilson term in the correlation functions of the lattice operators.

In this context, it appears natural to consider instead the following approach: on one hand, one can simply discard the Wilson terms in the lattice effective theory, thus ending up with an effective Lagrangian with a reduced number of terms each of those has a direct correspondence in the continuum chiral Lagrangian. On the other hand, in comparing the predictions of this effective theory with the results of a numerical simulation, one must consider, in the latter, only the correlation functions of properly renormalized operators.

In the following, we will show how this approach works in details by considering two specific examples. In doing that, we will take advantage of the fact that, as discussed in the previous sections, from the particular point of view of renormalization, the lattice theory at the leading order in the strong coupling and large \( N \) limit shows basically the same features expected in the weak coupling region. Thus, we will be able to continue our discussion by dealing with the effective lattice Lagrangian derived in this approximation.

**The quark mass renormalization**

In the limit of vanishing bare quark mass and Wilson term, the generation of a quark mass from pure quantum corrections in the lattice theory is protected by chiral symmetry. In contrast, in the presence of the Wilson term, a physical quark mass can arise even in the limit in which the mass term of the lattice action is set to zero. This is a well known consequence of the Wilson breaking of chiral symmetry.

In this respect, the local external source \( \chi(x) \) of the lattice Lagrangian cannot be identified, directly, with the corresponding source of the continuum theory. In fact, when in the latter this source is removed, the pseudoscalar Goldstone bosons remain
exactly massless. On the lattice, instead, when we consider the limit \( \chi(x) = 0 \), corresponding to \( m = -4r \), we still find a finite value of the pion mass (precisely, from eq. (50), \( M_{\pi}^2 = -16r^2/(1 + r^2) \)).

In order to define an external source that is proportional, in the continuum limit, to the corresponding source of the continuum Lagrangian, we can perform in the effective action the following change of variable:

\[
\chi(x) = (\hat{m} + 4r) \xi(x) + \delta \chi(x)
\]

where \( \hat{m} \) is the quantity defined in eq. (105) and the Wilson source \( \xi(x) \) is introduced in eq. (119) to preserve chiral covariance. The new field \( \delta \chi(x) \) is defined in such a way that, in the limit of vanishing external sources, it reduces to the renormalized quark mass:

\[
(\delta \chi)_0 = \delta m = m - \hat{m}
\]

For simplicity, we are only considering here the additive renormalization of the quark mass, thus neglecting the existence of an additional multiplicative renormalization.

By construction, after this change of variable the whole pion mass in the effective theory is generated by the new source \( \delta \chi \). Precisely, as a consequence of the substitution (119), a new term appears in the lattice effective action, of the form:

\[
- \frac{1}{2} NC_m (\hat{m} + 4r) \sum_x \text{Tr} \left[ \xi^\dagger(x) U(x) + \xi(x) U^\dagger(x) \right]
\]

The contribution to the pion mass coming from this term is exactly canceled by an the opposite contribution coming from the Wilson terms. As a result, the pion mass in the effective theory is only generated by the term \( S_m \) of eq. (114), now containing the new source \( \delta \chi \).

If in a specific case we were only interested in the calculation of the pion mass, it is clear that, after the change of variables (119), we could completely forget about all the Wilson terms in the effective action. In addition it is worth to notice that, but for the formal substitution \( \chi(x) \to \delta \chi(x) \), the mass term \( S_m \) in the effective action is left unchanged. Thus in this respect, we could have also considered, from the very beginning, a completely different point of view. That is, we could have discarded all the Wilson terms in the lattice effective action and simply interpreted the source \( \chi(x) \) as the “renormalized” source associated with the renormalized quark mass \( \delta m \).
Technically, there is a difference between the two particular points of view. This is due to the fact that, after the substitution (119), the Wilson sources \( \sigma(x) \) and \( \pi(x) \) are coupled in the action to different operators, namely \( \Sigma'(x) \) and \( \Pi'(x) \) given by:

\[
\begin{align*}
\Sigma'(x) &= \Sigma(x) + (\bar{m} + 4r) S(x) \\
\Pi'(x) &= \Pi(x) + (\bar{m} + 4r) P(x)
\end{align*}
\]  

(122)

However, as far as we are not interested in calculating the correlation functions of these particular operators, the above difference becomes completely immaterial.

Other interesting consequences of the change of variable defined by eq. (119), and in particular those related to the lattice chiral WI’s, will be discussed at the end of this section. Before that, we want to consider another significative example.

**The axial current renormalization**

In the previous example we have shown that some of the Wilson terms in the effective Lagrangian can be removed by considering an external source which is directly coupled to a renormalized quantity, in that specific case the quark mass. Similarly, we can argue that other Wilson terms would be also removed by considering for instance a new axial source, \( a'_\mu(x) \), directly coupled to the renormalized axial current, \( A_{R\mu}(x) = Z_A A_{\mu}(x) \). Specifically, we would like to consider a change of variable leading to the introduction of the following set of external sources:

\[
\begin{align*}
W'_\mu(x) &= \exp \left\{ -i \left[ v'_\mu(x) + a'_\mu(x) \right] \right\} \\
Z'_\mu(x) &= \exp \left\{ -i \left[ v'_\mu(x) - a'_\mu(x) \right] \right\}
\end{align*}
\]  

(123)

Since the vector current is conserved, one might look for the new vector and axial sources in the form:

\[
\begin{align*}
v_\mu(x) &= v'_\mu(x) \\
a_\mu(x) &= Z_A a'_\mu(x)
\end{align*}
\]  

(124)

with \( Z_A \) = given by eq. \((111)\) in the strong coupling limit.

This would imply a relation between the old and new \( W_\mu \) and \( Z_\mu \) external sources, which takes the form of an infinite series:

\[
\begin{align*}
W_\mu(x) &= W'_\mu(x) + \frac{r^2}{2} \left[ W''_\mu(x) - Z''_\mu(x) \right] + \ldots \\
Z_\mu(x) &= Z'_\mu(x) + \frac{r^2}{2} \left[ Z''_\mu(x) - W''_\mu(x) \right] + \ldots
\end{align*}
\]  

(125)
where the dots represent terms which are at least quadratic in the external sources \( v'_\mu \) and \( a'_\mu \). However, the changes of variables of eqs. (124) and (125) cannot be implemented as such, not being consistent with chiral covariance. Requiring the validity of eq. (124) is too restrictive, since adding to the right-hand sides of that equation any term linear in the Wilson external sources, \( \sigma(x) \) and \( \pi(x) \), would not change the desired result. Indeed, even with this addition, the sources \( v'_\mu(x) \) and \( a'_\mu(x) \) would still be coupled, in the lattice action, to the currents \( V_\mu(x) \) and \( Z_A A_\mu(x) \) respectively. Similarly, we could also add any other term which is at least quadratic in any of the defined external sources.

Taking advantage of this freedom one can obtain chirally covariant relations between old and new sources, satisfying all the desired requirements, by modifying eq. (125) in the following manner:

\[
W_\mu(x) = W'_\mu(x) + \frac{r^2}{4} \left[ \xi(x)\xi^\dagger(x) W'_\mu(x) + W'_\mu(x)\xi(x + \bar{\mu})\xi^\dagger(x + \bar{\mu}) - 2\xi(x) Z'_\mu(x)\xi^\dagger(x + \bar{\mu}) \right] + \ldots
\]

\[
Z_\mu(x) = Z'_\mu(x) + \frac{r^2}{4} \left[ \xi(x)^\dagger\xi(x) Z'_\mu(x) + Z'_\mu(x)\xi^\dagger(x + \bar{\mu})\xi(x + \bar{\mu}) - \right. \\
\left. 2\xi^\dagger(x) W'_\mu(x)\xi(x + \bar{\mu}) \right] + \ldots
\]

(126)

In this expressions we have included on the right-hand side all the terms which are at most linear in the \( v'_\mu \), \( a'_\mu \), \( \sigma \) and \( \pi \) external sources, whereas the dots represent an infinite series of terms at least quadratic in these sources. The missing terms can be determined, order by order, by requiring the unitarity of the new fields \( W'_\mu(x) \) and \( Z'_\mu(x) \). By expanding eq. (126) in terms of vector, axial, scalar and pseudoscalar sources, we find, up to higher orders:

\[
v_\mu(x) = v'_\mu(x) + O(j^2)
\]

\[
a_\mu(x) = (1 + r^2) a'_\mu(x) - \frac{r^2}{2} \nabla_\mu \pi(x) + O(j^2)
\]

(127)

which generalize eq. (124).

It is worth to discuss an interesting consequence of the combined change of variables (119) and (127). Once both these changes are performed, the terms in the effective
action which are linear in the external sources can be still written in the form (59) and
(60), but now with new coefficients $d_i$ given by:

$$
\begin{align*}
  d_s &= -nNC_m, \quad d_{s1} = NC_m/F_0^2 \\
  d_p &= -NC_m/F_0, \quad d_a = -F_0, \quad d_v = 1 \\
  d_\sigma &= 8m \pi^2 [NC_1 - F_0^2/(1 + \pi^2)] \quad , \quad d_{\sigma 1} = 0 \\
  d_{\pi 1} &= 0 \quad , \quad d_{\pi 2} = 0
\end{align*}
$$

(128)

In particular, because of the last two equalities ($d_{\pi 1} = d_{\pi 2} = 0$), the operator:

$$
\Pi'(x) = \Pi(x) + (\bar{m} + 4r) P(x) + \frac{\pi^2}{2} \nabla^L_{\mu} A_\mu(x)
$$

(129)

that is now coupled to the source $\pi(x)$ in the lattice action, has always vanishing on-shell matrix elements, and can only contribute to the lattice correlation functions through localized contact terms. In some cases it does not contribute at all. For instance, the lattice chiral WI for the $\langle AP \rangle$ correlation function can be now directly derived in the form (112), without any explicit additional contribution from the Wilson terms.

**The continuum-like effective Lagrangian**

The above discussion naturally leads to the following conclusion: since it is possible, both in the weak and strong coupling limit, to renormalize the lattice correlation functions in such a way that they satisfy all the relations of continuum current algebra, it must be also possible to describe the physics of this renormalized lattice theory in terms of a “renormalized”, continuum-like, effective Lagrangian. Specifically, the only sources of chiral symmetry breaking in this Lagrangian must be represented by the quark mass terms, whereas Wilson terms like those in eq. (12) should not enter at all.

The two examples discussed in this section were aimed to support the above argument. An evident feature of eq. (128) is that, after the external sources renormalization, many of the coefficients related to the Wilson operators $\Sigma(x)$ and $\Pi(x)$ do actually vanish. In addition, the Wilson parameter $r$ does not explicitly enter anymore in the expressions of the remaining coefficients, related to the lattice operators $V_\mu, Z_A A_\mu, S$ and $P$. In fact, one can easily show that after the change of variables (113) and (126)
many of the correlation functions we have considered so far are simply reproduced by the following effective action:

\[ S(U) = \sum_x \text{Tr} \left[ \frac{1}{4} F_\mu U \Delta U \right] - \frac{1}{2} NC m \left( \chi U + U \chi \right) \]  

(130)

provided the external sources \( W_\mu(x), Z_\mu(x) \) and \( \chi(x) \) entering in it are identified with the renormalized sources of eqs. (119) and (123). Equation (130) exactly corresponds, up to discretization effects, to the continuum chiral Lagrangian at order \( p^2 \) [3].

On the other hand, it should be pointed out that a continuum-like effective action, like that of eq. (130), cannot be obtained from the original one by simply renormalizing the external sources. For instance, the correlation functions of the operator \( A_\mu^2(x) \) obtained from eq. (130) are not equal to the corresponding correlation functions given by the lattice effective Lagrangian after the renormalization of the axial source. The reason is that the lattice the operator \( A_\mu^2(x) \) does not renormalize with renormalization constant \( Z_\mu^2 \), and in fact it does not even renormalize in a simple multiplicative way.

Only some of the effects of the Wilson terms in the lattice effective theory can be accounted for by a renormalization of the external sources, because these effects rather imply the renormalization of an infinite number of lattice operators. The important point here is the following: with a proper renormalization of the lattice operators, their correlation functions will be correctly reproduced, up to pure discretization effects, by a continuum-like effective lattice Lagrangian which does not contain the Wilson terms at all. The reason is that these correlation functions satisfy all the WI’s predicted by continuum current algebra, and these identities can be only reproduced by a continuum-like effective Lagrangian.

7 A numerical calculation of the lattice effective Lagrangian in the scaling region

In the region of intermediate couplings, which is relevant for numerical simulations of continuum QCD, the non-effective degrees of freedom of the lattice theory cannot be analytically integrated out. Therefore, in this case we would not be able to derive the exact form of the resulting effective theory. However, such an integration can be performed numerically, at least in an approximate way.
The idea consists in defining an effective lattice Lagrangian of the general form derived in this paper for the strong coupling and large $N$ limit. The basic fields entering in this Lagrangian are the effective field $U(x)$ and the set of external sources $W_\mu(x)$, $Z_\mu(x)$ and $\chi(x)$. On the basis of the results derived in this paper, we expect that the correlation functions of properly renormalized lattice operators can be reproduced by an effective lattice Lagrangian which does not necessitate Wilson terms. Furthermore, the external sources of this effective Lagrangian have a direct and well defined correspondence with the external sources entering in the continuum QCD chiral Lagrangian.

The most general form of the effective lattice Lagrangian is dictated by local chiral invariance, lattice symmetries, parity and charge conjugation. The chiral parallel transport between two nearest neighbor lattice sites is performed by the left- and right-handed external sources $Z_\mu(x)$ and $W_\mu(x)$ respectively. They can enter in the Lagrangian in the form of chiral plaquettes or through the lattice covariant derivative $\Delta_\mu U(x)$ defined in eq. $\text{(37)}$. The local source $\chi(x)$, which reverses chirality, allows to introduce in the theory the effects of chiral symmetry breaking induced by the light quark mass term.

The general strategy to calculate the effective lattice Lagrangian has been outlined in ref. [10]. The idea is to assume a sufficiently large set of effective couplings, with strength determined by unknown numerical coefficients. These coefficients can be then fixed through the matching of an overcomplete set of expectation values, computed both in the effective and the full theory.

The calculation in the full theory, being highly non perturbative, requires the implementation of a numerical lattice simulation. An open question is whether or not the lattice effective Lagrangian, in the region of intermediate values of the lattice QCD coupling constant, can be treated in perturbation theory. This is possible in the strong coupling and large $N$ limit, where the coefficients of the higher order terms ($\phi^4, \phi^6, \ldots$) are suppressed by increasing powers of $1/N$. In addition, even for the actual value $N = 3$, these coefficients (see e.g. eq. $\text{(51)}$) are found to lie in the perturbative regime. Certainly, such a feature could be lost as one approaches the weak coupling region. Even if this should be the case, however, the difficulty could be more technical than conceptual, in the sense that also the effective theory could be treated by numerical techniques.

A crucial observation is that, since the lattice effective Lagrangian contains explicitly the collective fields responsible for the long distance behaviour of the fundamental
lattice theory, then only short distance couplings are expected to play a relevant role in the effective theory. Therefore, in defining a criterion to truncate the infinite number of allowed couplings in the effective Lagrangian one can limit oneself to considering local or quasi-local interactions (e.g. only local, nearest and next-to-nearest neighbors couplings). For the same reason, the numerical simulation in the fundamental theory, which is necessary to perform the matching, should be feasible on lattices of moderate sizes, which would allow one to achieve a higher numerical accuracy.

In the strong coupling limit, the coefficients $C_k$ of the effective Lagrangian are found to be explicit functions of the light quark mass $m$ and the Wilson parameter $r$. Both these parameters enter in the coefficients through the vacuum expectation value $u_0$. Such a dependence persists even after the process of renormalization, but there are reasons to believe that it is an artifact of the strong coupling expansion. In the weak coupling limit, the matrix elements of the lattice renormalized operators must be independent on the value of $r$. Therefore, the coefficients of the renormalized effective Lagrangian cannot depend on the Wilson parameter. Furthermore, there is clear phenomenological evidence that the effects of the light quark masses, in the physical amplitudes, are well reproduced in the chiral QCD Lagrangian by the terms containing the external source $\chi(x)$, without any additional mass dependence entering in the coefficients. For these reasons, in defining an effective lattice Lagrangian for the weak coupling region one should assume the coefficients independent on $m$ and $r$.

Once one has been able to determine an expression for the lattice effective Lagrangian which accurately reproduces the Green’s functions of the fundamental lattice theory at moderate distances, one can compute its large volume limit and expand in increasing powers of the external momenta. The result of such an expansion, to any given order in $p$, corresponds to the continuum low-energy QCD effective theory. In such a way, the procedure discussed in this paper would allow a first principle theoretical calculation of the coefficients of the QCD chiral Lagrangian.

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