ON THE SECOND AND THIRD EXTERIOR POWER OF TANGENT BUNDLES OF FANO MANIFOLDS WITH BIRATIONAL CONTRACTIONS

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ABSTRACT. In this paper, we classify Fano manifolds with elementary contractions of birational type such that the second or third exterior power of tangent bundles are numerically effective.

INTRODUCTION

In the paper [1], F. Campana and T. Peternell classified smooth projective threefolds \( X \) such that the second exterior power of tangent bundle \( \bigwedge^2 T_X \) are numerically effective (nef, for short). Among such threefolds, the blowing-up of three-dimensional projective space at a point is the only manifold having an elementary contraction of birational type. In [11], the author classified Fano fourfolds with the same condition and see that the blowing-up of four-dimensional projective space at a point is the only manifold having an elementary contraction of birational type. In this paper, we show that the same statement also holds in arbitrary dimension.

**Main Theorem 1.** Let \( X \) be an \( n \)-dimensional Fano manifold with at least one elementary contraction of birational type such that the second exterior power of tangent bundle \( \bigwedge^2 T_X \) is nef where \( n \geq 3 \). Then \( X \) is isomorphic to the blowing-up of the \( n \)-dimensional projective space \( \mathbb{P}^n \) at a point.

Furthermore, we also study the case where the third exterior power of tangent bundle \( \bigwedge^3 T_X \) is nef and obtain the following theorem.

**Main Theorem 2.** Let \( X \) be an \( n \)-dimensional Fano manifold with at least one elementary contraction of birational type such that the third exterior power of tangent bundle \( \bigwedge^3 T_X \) is nef where \( n \geq 4 \). Then \( X \) is isomorphic to one of the following,

1. blowing-up of \( \mathbb{P}^n \) at a point,
2. \( \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-2)) \),
3. \( \mathbb{P}_{\mathcal{Q}^{n-1}}(\mathcal{O}_{\mathcal{Q}^{n-1}} \oplus \mathcal{O}_{\mathcal{Q}^{n-1}}(-1)) \) where \( \mathcal{Q}^{n-1} \) is a \((n-1)\)-dimensional smooth quadric hypersurface in \( \mathbb{P}^n \).

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(4) blowing-up of $\mathbb{P}^n$ along a line,
(5) product of $\mathbb{P}^1$ and the blowing-up of $\mathbb{P}^{n-1}$ at a point,
(6) blowing-up of $Q^1$ along a line,
(7) blowing-up of $Q^4$ along a conic not on a plane contained in $Q^4$.

There is a problem about the nefness of $\wedge^q T_X$ posed by F. Campana and T. Peternell.

Problem 0.1 ([4], Problem 6.4). Let $X$ be a Fano manifold. Assume that $\wedge^q T_X$ is nef on every extremal rational curve. Is then $\wedge^q T_X$ already nef?

In the proof of above theorems, we can see that the following theorem also holds.

Theorem 0.2. Let $X$ be an $n$-dimensional Fano manifold with at least one elementary contraction of birational type where $n \geq 4$. Assume that $\wedge^2 T_X$ (resp. $\wedge^3 T_X$) is nef on every extremal rational curves in $X$. Then $\wedge^2 T_X$ (resp. $\wedge^3 T_X$) is nef.

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Notation

Throughout this paper we work over the complex number field $\mathbb{C}$. For a projective manifold $X$ and a vector bundle $\mathcal{E}$ on $X$, let $\mathbb{P}_X(\mathcal{E})$ be the projectivization of $\mathcal{E}$ in the sense of Grothendieck and $\xi_\mathcal{E}$ be the tautological divisor. We usually denote the projection by $\pi : \mathbb{P}_X(\mathcal{E}) \to X$. We denote the blowing-up of smooth projective manifold $M$ along a smooth subvariety $Z$ by $Bl_Z(M)$.

1. Examples

In this section, we confirm that manifolds appeared in Main Theorem 1 (resp. Main Theorem 2) satisfy the condition that $\wedge^2 T_X$ (resp. $\wedge^3 T_X$) are nef.

Example 1.1. Let $X = Bl_{pt}(\mathbb{P}^n)$ be the blowing-up of $\mathbb{P}^n$ at a point where $n \geq 4$. Then, by Lemma 2.1 in [11], we know that $\wedge^2 T_X$ is nef.

Example 1.2. Set $X = \mathbb{P}_{\mathbb{P}^{n-1}}(O_{\mathbb{P}^{n-1}}(2) \oplus O_{\mathbb{P}^{n-1}}) \cong \mathbb{P}_{\mathbb{P}^{n-1}}(O_{\mathbb{P}^{n-1}} \oplus O_{\mathbb{P}^{n-1}}(-2))$ where $n \geq 4$. Put $\xi$ the tautological divisor on $\mathbb{P}_{\mathbb{P}^{n-1}}(O_{\mathbb{P}^{n-1}}(2) \oplus O_{\mathbb{P}^{n-1}})$ and $\pi : \mathbb{P}_{\mathbb{P}^{n-1}}(O_{\mathbb{P}^{n-1}}(2) \oplus O_{\mathbb{P}^{n-1}}) \to \mathbb{P}^{n-1}$ the natural projection. We have the following exact sequence:

$$0 \to \mathcal{T}_\pi \to \mathcal{T}_X \to \pi^* T_{\mathbb{P}^{n-1}} \to 0.$$  

From this exact sequence, we obtain the following exact sequence:

$$0 \to \mathcal{T}_\pi \otimes \pi^*(\wedge^2 T_{\mathbb{P}^{n-1}}) \to \wedge^3 T_X \to \pi^*(\wedge^3 T_{\mathbb{P}^{n-1}}) \to 0.$$
The subbundle $\mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^{n-1}}) \cong \mathcal{O}_X(2\xi) \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^2}(-2))$ is nef. Therefore, we have $\wedge^3 \mathcal{T}_X$ is nef since $\pi^*(\wedge^3 \mathcal{T}_{\mathbb{P}^{n-1}})$ is nef.

**Example 1.3.** Set $X = \mathbb{P}_{Q^{n-1}}(O_{Q^{n-1}}(1) \oplus O_{Q^{n-1}})$ where $n \geq 4$. Put $\xi$ the tautological divisor on $\mathbb{P}_{Q^{n-1}}(O_{Q^{n-1}}(1) \oplus O_{Q^{n-1}})$ and $\pi : \mathbb{P}_{Q^{n-1}}(O_{Q^{n-1}}(1) \oplus O_{Q^{n-1}}) \to Q^{n-1}$ the natural projection. We have the following exact sequence:

$$0 \to \mathcal{T}_\pi \to \mathcal{T}_X \to \pi^*\mathcal{T}_{Q^{n-1}} \to 0.$$ 

From this exact sequence, we obtain the following exact sequence:

$$0 \to \mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{Q^{n-1}}) \to \wedge^3 \mathcal{T}_X \to \pi^*(\wedge^3 \mathcal{T}_{Q^{n-1}}) \to 0.$$ 

We can show that the subbundle $\mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{Q^{n-1}}) \cong \mathcal{O}_X(2\xi) \otimes \pi^*(\wedge^2 \mathcal{T}_{Q^{n-1}}(-1))$ is nef by the surjection from nef bundle $\wedge^3 \mathcal{T}_{\mathbb{P}^{n}}(-3)|_{Q^{n-1}} \to \wedge^2 \mathcal{T}_{Q^{n-1}}(-1) \to 0$. Hence, we have $\wedge^3 \mathcal{T}_X$ is nef since $\pi^*(\wedge^3 \mathcal{T}_{Q^{n-1}})$ is nef.

**Example 1.4.** Let $X = B_l(\mathbb{P}^n)$ be the blowing-up of $\mathbb{P}^n$ along a line $l$ where $n \geq 4$. Then, it is known that $X$ has a $\mathbb{P}^2$-bundle structure $X \cong \mathbb{P}_{\mathbb{P}^{n-2}}(O_{\mathbb{P}^{n-2}} \oplus O_{\mathbb{P}^{n-2}}(1))$. Put $\xi$ the tautological divisor and $\pi : \mathbb{P}_{\mathbb{P}^{n-2}}(O_{\mathbb{P}^{n-2}} \oplus O_{\mathbb{P}^{n-2}}(1)) \to \mathbb{P}^{n-2}$ the natural projection. We have the following two exact sequences:

$$0 \to \mathcal{T}_\pi \to \mathcal{T}_X \to \pi^*\mathcal{T}_{\mathbb{P}^{n-2}} \to 0$$

where $\mathcal{T}_\pi$ is the relative tangent bundle and

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\xi) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^{n-2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(-1)) \to \mathcal{T}_\pi \to 0.$$ 

If $n = 4$, we have the following exact sequence from the first exact sequence:

$$0 \to \wedge^2 \mathcal{T}_\pi \otimes \pi^*\mathcal{T}_{\mathbb{P}^2} \to \wedge^3 \mathcal{T}_X \to \mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^2}) \to 0.$$ 

The subbundle $\wedge^2 \mathcal{T}_\pi \otimes \pi^*\mathcal{T}_{\mathbb{P}^2} \cong \mathcal{O}_X(3\xi) \otimes \pi^*\mathcal{T}_{\mathbb{P}^2}(-1)$ is nef. The quotient bundle is also nef by the surjection from the ample vector bundle $\mathcal{O}_X(\xi) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^{n-2}}(3) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(-2)) \to \mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^2}) \to 0$. Therefore, $\wedge^3 \mathcal{T}_X$ is nef.

If $n \geq 5$, we have the following two exact sequences:

$$0 \to \mathcal{F} \to \wedge^3 \mathcal{T}_X \to \pi^*(\wedge^3 \mathcal{T}_{\mathbb{P}^{n-2}}) \to 0$$

and

$$0 \to \wedge^2 \mathcal{T}_\pi \otimes \pi^*\mathcal{T}_{\mathbb{P}^{n-2}} \to \mathcal{F} \to \mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^{n-2}}) \to 0.$$ 

We can easily show that bundles $\pi^*(\wedge^3 \mathcal{T}_{\mathbb{P}^{n-2}})$ and $\wedge^2 \mathcal{T}_\pi \otimes \pi^*\mathcal{T}_{\mathbb{P}^{n-2}}$ are nef. We also see that $\mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^{n-2}})$ is nef by the surjection from the ample vector bundle $\mathcal{O}_X(\xi) \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^{n-2}} \oplus \wedge^2 \mathcal{T}_{\mathbb{P}^{n-2}}(-1)) \to \mathcal{T}_\pi \otimes \pi^*(\wedge^2 \mathcal{T}_{\mathbb{P}^{n-2}}) \to 0$. Therefore, $\wedge^3 \mathcal{T}_X$ is nef.
Example 1.5. Let $X = \mathbb{P}^1 \times Y$ be the product of $\mathbb{P}^1$ and the blowing-up of $\mathbb{P}^{n-1}$ at a point $Y = Bl_{pt}(\mathbb{P}^{n-1})$ where $n \geq 4$. Set $p : X \to \mathbb{P}^1$ the first projection and $q : X \to Bl_{pt}(\mathbb{P}^{n-1})$ the second projection. Then, we have $\mathcal{T}_X \cong p^*\mathcal{T}_{\mathbb{P}^1} \oplus q^*\mathcal{T}_Y$. Hence, $\wedge^3\mathcal{T}_X \cong p^*\mathcal{T}_{\mathbb{P}^1} \otimes q^*(\wedge^2\mathcal{T}_Y) \oplus q^*(\wedge^3\mathcal{T}_Y)$ is nef because we can easily check that $\wedge^3\mathcal{T}_Y$ is nef.

Example 1.6. Let $X = Bl_l(Q^n)$ be the blowing-up of $Q^n$ along a line $l$ where $n \geq 4$. At first, we show that $\wedge^{n-1}\mathcal{T}_X$ is nef. Let $Y = Bl_l(\mathbb{P}^{n+1})$ be the blowing-up of $\mathbb{P}^{n+1}$ containing $Q^n$ along a line $l$. Then, $X$ is a closed subvariety of $Y$ of codimension one. $Y$ has a $\mathbb{P}^2$-bundle structure $Y \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}^2 \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. Put $\xi$ the tautological divisor and $H$ a divisor associated with the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ where $\pi : \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}^2 \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)) \to \mathbb{P}^{n-1}$ is the natural projection. We note that fibers of $\pi$ correspond to planes in $\mathbb{P}^{n+1}$ containing $l$. From this fact, we can see that $\pi^{-1}(z) \cap X \cong \mathbb{P}^1$ for a general $z \in \mathbb{P}^{n-1}$.

We show that $X \sim \xi + H$ in $Y$. We set that $X \sim a\xi + bH$ for some integers $a$ and $b$. Let $C$ be the strict transform of a line in $P^{n+1}$ not contained in $Q^n$ such that it does not intersect with $l$. Then, we have that $X.C = 2$, $\xi.C = 1$ and $H.C = 1$. Let $C'$ be the strict transform of a line in $P^{n+1}$ not contained in $Q^n$ such that it meets $l$ at one point. Then, we have that $X.C' = 2 - 1 = 1$, $\xi.C' = 1$ and $H.C' = 1 - 1 = 0$. Hence, we obtain that $a = b = 1$. From this, we know that

$$\mathcal{N}_{X/Y} = \xi + H|_X.$$  

By the exact sequence $0 \to \mathcal{T}_X \to \mathcal{T}_Y|_X \to \mathcal{N}_{X/Y} \to 0$, we get the surjection of vector bundles on $X$, $\wedge^n\mathcal{T}_Y|_X \otimes \mathcal{N}_{X/Y} \to \wedge^{n-1}\mathcal{T}_X \to 0$. We confirm that $\wedge^n\mathcal{T}_Y(\xi - H)|_X \cong \wedge^n\mathcal{T}_Y|_X \otimes \mathcal{N}_{X/Y}$ is nef. From the exact sequence

$$0 \to \mathcal{T}_\pi \to \mathcal{T}_Y \to \pi^*\mathcal{T}_{\mathbb{P}^{n-1}} \to 0,$$

we have the following exact sequence

$$0 \to \wedge^2\mathcal{T}_\pi(\xi - H) \otimes \pi^*(\wedge^{n-2}\mathcal{T}_{\mathbb{P}^{n-1}}) \to \wedge^n\mathcal{T}_Y(\xi - H) \to \mathcal{T}_\pi(\xi - H) \otimes \pi^*(\wedge^{n-1}\mathcal{T}_{\mathbb{P}^{n-1}}) \to 0.$$

The subbundle $\wedge^2\mathcal{T}_\pi(\xi - H) \otimes \pi^*(\wedge^{n-2}\mathcal{T}_{\mathbb{P}^{n-1}}) \cong \mathcal{O}_Y(2\xi) \otimes \pi^*(\wedge^{n-2}\mathcal{T}_{\mathbb{P}^{n-1}}(-2))$ is nef. To show the nefness of the quotient bundle, we use the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(\xi) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^{n-1}}^2 \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)) \to \mathcal{T}_\pi \to 0.$$

We can check that the quotient bundle is also nef by the surjection from the nef vector bundle

$$\pi^*(\wedge^{n-1}\mathcal{T}_{\mathbb{P}^{n-1}}(-1)) \oplus \pi^*(\wedge^{n-1}\mathcal{T}_{\mathbb{P}^{n-1}}(-2)) \to \mathcal{T}_\pi(\xi - H) \otimes \pi^*(\wedge^{n-1}\mathcal{T}_{\mathbb{P}^{n-1}}) \to 0.$$

Therefore, $\wedge^n\mathcal{T}_Y(\xi - H)$ is nef. Hence, we have that $\wedge^{n-1}\mathcal{T}_X$ is nef.

Next, we confirm that $\wedge^i\mathcal{T}_X$ is not nef for $i \leq n - 2$. We suppose that $\wedge^i\mathcal{T}_X$ is nef for $i \leq n - 2$. We have $X \sim \xi + H$ in $Y$. Sections of $\mathcal{O}_Y(\xi + H)$ correspond to sections of $\mathcal{O}_{\mathbb{P}^{n-1}}^2(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(2)$. Since $c_3(\mathcal{O}_{\mathbb{P}^{n-1}}^2(1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(2)) = 2 \neq 0$, we know that any section of $\mathcal{O}_Y(\xi + H)$ is zero on some fiber of $\pi$. Hence, the elementary contraction

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\[ \pi|_X : X \to \mathbb{P}^{n-1} \] has a two-dimensional fiber and corresponding extremal rational curve \( C \) satisfies \(-K_X.C = 2\). This contradicts to Lemma 1.8.

**Lemma 1.7.** Let \( X \) be an \( n \)-dimensional smooth projective manifold with nef vector bundle \( \wedge^i T_X \) for \( i \leq n - 1 \). Put \( C \) be a rational curve in \( X \) and \( \nu : \mathbb{P}^1 \to C \) the normalization. Then \(-K_X.C \geq 2\). We further assume that \( i \leq n - 2 \) and \(-K_X.C = 2\), then \( \nu^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-1} \).

**Proof.** Set \( \nu^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \) where \( a_1 \geq a_2 \geq \cdots \geq a_n \). We know that \( a_1 \geq 2 \) and \( a_{n-i+1} + \cdots + a_n \geq 0 \). Therefore we have \(-K_X.C \geq 2\). We assume that \( i \leq n - 2 \) and \(-K_X.C = 2\). If \( a_n < 0 \), then we get \(-K_X.C = a_1 + a_2 + \cdots + a_n \geq n - i + 1 \geq 3\). Therefore, we have the assertion. \( \square \)

**Lemma 1.8.** Let \( X \) be an \( n \)-dimensional smooth projective manifold with nef vector bundle \( \wedge^i T_X \) for \( i \leq n - 2 \). Let \( C \) be an rational curve in \( X \) such that \(-K_X.C = 2\). Then \( C \) is in some extremal ray and corresponding elementary contraction \( \varphi : X \to Y \) is a \( \mathbb{P}^1 \)-bundle over a smooth projective manifold \( Y \).

**Proof.** The proof is based on the argument in the proof of Theorem 8 in [8]. Let \( \nu : \mathbb{P}^1 \to C \subset X \) be the normalization. Let \( H \) be an irreducible component of the Hom scheme \( \text{Hom}(\mathbb{P}^1, X) \) containing the morphism \( \nu \). In this case \( h^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-1} \) for all \( h \in H \) by Lemma 1.7. Then we see that \( \dim H = n + 2 \) and \( H \) is smooth by Proposition 3 in [8]. Let \( G \) be the \( \text{Aut}(\mathbb{P}^1) \). We can construct an \((n-1)\)-dimensional smooth projective manifold \( Y \) (resp. \( n \)-dimensional smooth projective manifold \( Z \)) which is the geometric quotient of \( H \) (resp. \( H \times \mathbb{P}^1 \)) by \( G \) from the argument in the proof of Theorem 8 in [8]. Moreover, we have a morphisms \( q : Z \to Y \) which is \( \mathbb{P}^1 \)-bundle in the étale topology and the evaluation morphism \( p : Z \to X \). We can show that \( p \) is a smooth surjective morphism from \( n \)-dimensional smooth manifold to \( n \)-dimensional Fano manifold. Since any Fano manifold is simply connected, \( p \) is isomorphism and we finish the proof. \( \square \)

**Example 1.9.** Let \( X = Bl_C(Q^n) \) be the blowing-up of \( Q^n \subset \mathbb{P}^{n+1} \) along a conic \( C \) not on a plain contained in \( Q^n \) where \( n \geq 4 \). At first, we show that \( \wedge^{n-1} T_X \) is nef. Take a plain \( S \cong \mathbb{P}^2 \) which contains \( C \). Let \( Y = Bl_S(\mathbb{P}^{n+1}) \) be the blowing-up of \( \mathbb{P}^{n+1} \) containing \( Q^n \) along a plain \( S \). Then, \( X \) is a closed subvariety of \( Y \) of codimension one. \( Y \) has a \( \mathbb{P}^3 \)-bundle structure \( Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-2}}(3) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)) \). Put \( \xi \) the tautological divisor and \( H \) a divisor associated with the line bundle \( \pi^* \mathcal{O}_{\mathbb{P}^{n-2}}(1) \) where \( \pi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-2}}(3) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)) \to \mathbb{P}^{n-2} \) is the natural projection. We note that fibers of \( \pi \) correspond to 3-planes in \( \mathbb{P}^{n+1} \) containing \( S \). From this fact, we can see that \( \pi^{-1}(z) \cap X \) is an (possibly singular) irreducible and reduced quadric surface for any \( z \in \mathbb{P}^{n+1} \).

We show that \( X \cong 2\xi \) in \( Y \). We set that \( X \cong a\xi + bH \) for some integers \( a \) and \( b \). Let \( C' \) be the strict transform of a line in \( P^{n+1} \) not contained in \( Q^n \) such that it does not intersect with \( C \). Then, we have that \( X.C' = 2, \xi.C = 1 \) and \( H.C = 1 \). Let \( C'' \) be the
strict transform of a line in $P^{n+1}$ not contained in $Q^n$ such that it meets $C$ at one point. Then, we have that $X.C'' = 2$, $\xi.C'' = 1$ and $H.C'' = 0$. Hence, we obtain that $a = 2$ and $b = 0$. From this, we know that

$$N_{X/Y} = 2\xi|_X.$$  

By the exact sequence $0 \to T_X \to T_Y|_X \to N_{X/Y} \to 0$, we get the surjection of vector bundles on $X$, $\wedge^n T_Y|_X \otimes N^{\vee}_{X/Y} \to \wedge^{n-1} T_X \to 0$. We confirm that $\wedge^n T_Y(-2\xi)|_X \cong \wedge^n T_Y|_X \otimes N^{\vee}_{X/Y}$ is nef. From the exact sequence

$$0 \to \pi^* \to \pi^* \to 0,$$

we have the following exact sequence

$$0 \to (\wedge^3 T)(-2\xi) \otimes \pi^* (\wedge^{n-3} T_{P^{n-2}}) \to \wedge^n T_Y(-2\xi) \to (\wedge^2 T)(-2\xi) \otimes \pi^* (\wedge^{n-2} T_{P^{n-2}}) \to 0.$$

The subbundle $\wedge^3 T(-2\xi) \otimes \pi^* (\wedge^{n-3} T_{P^{n-2}}) \cong O_Y(2\xi) \otimes \pi^* (\wedge^{n-3} T_{P^{n-1}}(-1))$ is nef. To show the nefness of the quotient bundle, we use the exact sequence,

$$0 \to O_Y \to O_Y(\xi) \otimes \pi^* (O_{P^{n-2}}^3 \oplus O_{P^{n-2}}(-1)) \to T_\pi \to 0.$$

We can check that the quotient bundle is also nef by the surjection from the nef vector bundle,

$$\pi^* (\wedge^{n-2} T_{P^{n-2}})^{\otimes 3} \oplus \pi^* (\wedge^{n-2} T_{P^{n-2}}(-1))^{\otimes 3} \to (\wedge^2 T)(-2\xi) \otimes \pi^* (\wedge^{n-2} T_{P^{n-2}}) \to 0.$$

Therefore, $\wedge^n T_Y(-2\xi)$ is nef. Hence, we have that $\wedge^{n-1} T_X$ is nef.

Next, we confirm that $\wedge^i T_X$ is not nef for $i \leq n - 2$. We suppose that $\wedge^i T_X$ is nef for $i \leq n - 2$. We have the elementary contraction $\pi|_X : X \to \mathbb{P}^{n-2}$ such that it has a two dimensional fiber and corresponding extremal rational curve $C'$ satisfies $-K_X.C' = 2$. This contradicts to Lemma 1.8.

2. Proof of Main Theorem 1

In this section, we give a proof of Main Theorem 1.

**Theorem 2.1.** Let $X$ be an $n$-dimensional Fano manifold with at least one elementary contraction of birational type such that the second exterior power of tangent bundle $\wedge^2 T_X$ is nef where $n \geq 3$. Then $X$ is isomorphic to the blowing-up of the $n$-dimensional projective space $\mathbb{P}^n$ at a point.

If part is proved in Example 1.1. Therefore, we prove only if part.

Let $X$ be an $n$-dimensional Fano manifold with an elementary contraction of birational type $\tau : X \to Y$ such that the second exterior power of tangent bundle $\wedge^2 T_X$ is nef where $n \geq 3$.

**Claim 2.2.** There is an extremal rational curve contracted by $\tau$ such that $n + 1 \geq -K_X.C \geq n - 1$. 

PROOF. By Bend and Break Lemma we have a rational curve $C$ contracted by $\tau$ such that $n + 1 \geq -K_X.C$ and $T_X|_C$ is not nef since $\tau$ is of birational type. Put $\nu : \mathbb{P}^1 \to C$ the normalization. Set $\nu^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n$. We know that $a_1 \geq 2$, $a_{n-1} + a_n \geq 0$ and $a_n < 0$. In particular, $a_{n-1} \geq 1$. Therefore we get $-K_X.C = a_1 + a_2 + \cdots + a_n \geq n - 1$.

Set $E$ an irreducible component of the exceptional locus of $\tau$. Then we know that $(\dim E, \dim \tau(E)) = (n - 1, 0)$ by the following proposition proved in [11]. Im particular, $\tau$ is a divisorial contraction.

**Proposition 2.3.** Let $X$ be a smooth projective manifold of dimension $n$ and $R$ an extremal ray on $X$ which corresponds to the contraction $\tau : X \to Y$. Set $l(R) := \min\{-K_X.C[C] \in R\}$. Let $E$ be any irreducible component of the exceptional locus of $\tau$ and $F$ any irreducible component of any nontrivial fiber of $\tau$. Then

\[
\dim E + \dim F \geq n + l(R) - 1.
\]

Let $E$ be the exceptional divisor of $\tau$. Then, we can find a rational curve $C$ of minimal degree in some extremal ray $R$ such that $E.C > 0$ by Cone theorem since $X$ is Fano and $E$ is effective and nontrivial. Let $\phi : X \to Z$ be the elementary contraction associated with $R$. Then, any fiber of $\phi$ is of dimension $\leq 1$. Since $-K_X.C' > 1$ for any rational curve $C'$ on $X$ we have that $\phi$ is $\mathbb{P}^1$-bundle by Lemma 2.1 of [5]. Moreover we know that $E$ is the section of $\phi$ by Claim 2.3 in [5]. Hence $E \cong Z$ is smooth and there is a rank 2 vector bundle $\mathcal{E}$ on $Z$ such that $X \cong \mathbb{P}_Z(\mathcal{E})$. Since $n \geq -K_E.C = -K_X|_E.C - N_{E/X}.C \geq (n-1)+1 = n$, we have that $C$ is a line in $E$ and $-K_E = nL$ for some ample line bundle $L \in \text{Pic}(E)$ by the following theorem.

**Theorem 2.4 ([1], Theorem 2.1).** Let $X$ be a smooth projective manifold of dimension $n$. Let $f : X \to Y$ be an extremal contraction and $E$ the exceptional locus of $f$. Assume that $\dim E = n - 1$. Let $F$ be a general fiber of $f_E : E \to f(E)$. Then there exists a Cartier divisor $L$ on $X$ such that

1. $\text{Im}(\text{Pic}X \to \text{Pic}F) = \mathbb{Z}[L]_F$ and $L|_F$ is ample on $F$;
2. $\mathcal{O}_F(-K_X) \cong \mathcal{O}_F(pL)$ and $\mathcal{O}_F(-E) \cong \mathcal{O}_F(qL)$ for some $p, q \in \mathbb{N}$.

Therefore, we have $E \cong \mathbb{P}^{n-1}$ by Kobayashi-Ochiai’s characterization [6] and $\mathcal{N}_{E/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Taking the push-forward of the exact sequence

\[
0 \to \mathcal{O}_X \to \mathcal{O}_X(E) \to \mathcal{N}_{E/X} \to 0
\]

by $\phi$, we have that $X \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ which is the blowing up of $\mathbb{P}^n$ at a point. Hence we complete the proof.
3. Proof of Main Theorem 2

In this section, we give a proof of Main Theorem 2.

**Theorem 3.1.** Let $X$ be an $n$-dimensional Fano manifold with at least one elementary contraction of birational type such that the third exterior power of tangent bundle $\wedge^3 T_X$ is nef where $n \geq 4$. Assume that $\wedge^2 T_X$ is not nef. Then $X$ is isomorphic to one of the following,

1. $\mathbb{P}^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-2))$,
2. $\mathbb{P}^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ where $\mathbb{P}^{n-1}$ is a $(n - 1)$-dimensional smooth quadric hypersurface in $\mathbb{P}^n$,
3. blowing-up of $\mathbb{P}^n$ along a line,
4. product of $\mathbb{P}^1$ and the blowing-up of $\mathbb{P}^{n-1}$ at a point,
5. blowing-up of $\mathbb{P}^1$ along a line,
6. blowing-up of $\mathbb{P}^4$ along a conic not on a plane contained in $\mathbb{P}^4$,

If part is proved in Example 1.2, 1.3, 1.4, 1.5, 1.6 and 1.9. Therefore, we prove only if part.

Let $X$ be an $n$-dimensional Fano manifold with an elementary contraction of birational type $\tau : X \to Y$ such that the third exterior power of tangent bundle $\wedge^3 T_X$ is nef where $n \geq 4$. Furthermore we suppose that $\wedge^2 T_X$ is not nef. Put $C$ a rational curve of minimal degree in an extremal ray which corresponds to the given birational contraction $\tau$.

**Claim 3.2.** There is an extremal rational curve contracted by $\tau$ such that $n + 1 \geq -K_X.C \geq n - 2$.

**Proof.** By Bend and Break Lemma we have an extremal rational curve $C$ contracted by $\tau$ such that $n + 1 \geq -K_X.C$ and $T_X|_C$ is not nef since $\tau$ is of birational type. Put $\nu : \mathbb{P}^1 \to C$ the normalization. Set $\nu^* T_X \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$ where $a_1 \geq a_2 \geq \cdots \geq a_n$. We know that $a_1 \geq 2$, $a_{n-2} + a_{n-1} + a_n \geq 0$ and $a_n < 0$. In particular, $a_{n-2} \geq 1$. Therefore we get $-K_X.C = a_1 + a_2 + \cdots + a_n \geq n - 2$. \[\square\]

Set $E$ an irreducible component of the exceptional locus of $\tau$. Then we know that $(\dim E, \dim \tau(E)) = (n - 1, 0)$ or $(n - 1, 1)$ by Proposition 2.3. In particular, $\tau$ is a divisorial contraction.

Let $E$ be the exceptional divisor of $\tau$. Then, we can find a rational curve $C$ of minimal degree in some extremal ray $R$ such that $E.C > 0$ by Cone theorem since $X$ is Fano and $E$ is effective and nontrivial. Let $\phi : X \to Z$ be the elementary contraction associated with $R$. Then, any fiber of $\phi$ is of dimension $\leq 2$.

We consider the case where $(\dim E, \dim \tau(E)) = (n - 1, 0)$. Then, any fiber of $\phi$ is of dimension $\leq 1$. Therefore we have $\phi$ is $\mathbb{P}^1$-bundle, $Z$ is a Fano manifold and $E$ is the section of $\phi$ by the same argument in the proof of Theorem 1.1. Hence $E \cong Z$ is smooth...
and there is a rank 2 vector bundle $E$ on $Z$ such that $X \cong \mathbb{P}(\mathcal{E})$. Since $n \geq -K_{E}.C = -K_{X}|_{E}.C - N_{E/X}.C \geq (n - 2) + 1 = n - 1$, we have $(-K_{E}.C, -K_{X}|_{E}.C, -N_{E/X}.C) = (n, n - 1, 1), (n - 1, n - 2, 1)$ or $(n, n - 2, 2)$.

If $(-K_{E}.C, -K_{X}|_{E}.C, -N_{E/X}.C) = (n, n - 1, 1)$, then we have $X \cong Bl_{\text{point}}\mathbb{P}^{n}$ by the proof of Main theorem 1. In this case, we know $\Lambda^{2}\mathcal{T}_{X}$ is nef by Example 1.1. This is a contradiction.

If $(-K_{E}.C, -K_{X}|_{E}.C, -N_{E/X}.C) = (n - 1, n - 2, 1)$, we have that $C$ is a line in $E$ and $-K_{E} = (n - 1)L$ for some ample line bundle $L \in \text{Pic}(E)$ by Theorem 2.3. Therefore, we have $E \cong Q^{n-1}$ by Kobayashi-Ochiai’s characterization [8] and $N_{E/X} \cong \mathcal{O}_{Q^{n-1}}(-1)$.

Taking the push-forward of the exact sequence

$$0 \to \mathcal{O}_{X} \to \mathcal{O}_{X}(E) \to N_{E/X} \to 0$$

denoted by $\phi$, we have that $X \cong \mathbb{P}_{Q^{n-1}}(\mathcal{O}_{Q^{n-1}} \oplus \mathcal{O}_{Q^{n-1}}(-1))$.

We consider the case where $(-K_{E}.C, -K_{X}|_{E}.C, -N_{E/X}.C) = (n, n - 2, 2)$. We have $-K_{E}.C' \geq n = \dim E + 1$ for any rational curve in $E$ because $C'$ is numerically equivalent to $aC$ for some positive integer $a$. From the characterization of projective spaces due to Cho–Miyaoka–Shepherd-Barron [8], we know $E \cong \mathbb{P}^{n-1}$ and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$. Taking the push-forward of the exact sequence

$$0 \to \mathcal{O}_{X} \to \mathcal{O}_{X}(E) \to N_{E/X} \to 0$$

denoted by $\phi$, we have that $X \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-2))$.

We have the assertion if $\dim E, \dim \tau(E) = (n - 1, 0)$.

Secondary, we consider the case where $(\dim E, \dim \tau(E)) = (n - 1, 1)$. In this case, $Y$ and $W := \tau(E)$ are smooth and $\tau$ is the blowing-up of $Y$ along a smooth curve $W$ by Theorem 5.1 in [2] and Claim 3.2 By Lemma 1.7, we know that the pseudo-index $i(X) := \min\{-K_{X}.C| C \text{ is a rational curve in } X\} \geq 2$.

From the classification of Tsukioka [9], we have that $X$ is isomorphic to one of the following,

1. blowing-up of $\mathbb{P}^{n}$ along a line,
2. product of $\mathbb{P}^{1}$ and the blowing-up of $\mathbb{P}^{n-1}$ at a point,
3. blowing-up of $Q^{n}$ along a line,
4. blowing-up of $Q^{n}$ along a conic not on a plain contained in $Q^{n}$.

Note that we use the fact that if $X$ is a blowing-up of a smooth $n$-dimensional manifold along a smooth subvariety of codimension 2, then $i(X) = 1$. By Example 1.4, 1.5, 1.6 and 1.9 we have the assertion.

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