Some Ropelength-Critical Clasps

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We describe several configurations of clasped ropes which are balanced and thus critical for the Gehring ropelength problem.

1. INTRODUCTION

The ropelength of a link is given by the ratio of its length to its thickness. There are many ways to measure the thickness of a space curve, but for links one particularly simple notion, the Gehring thickness, is simply the minimum distance between different components. With Cantarella, Fu and Kusner we introduced a theory of criticality [2] for the Gehring ropelength problem. Our necessary and sufficient conditions for ropelength criticality take the form of a balance criterion which says that the tension force trying to reduce the length of the curves must be balanced by contact forces acting at points achieving the minimum distance. One simple example, with surprising intricacy for its solution [2], is the clasp. This is a generalized link whose components are not closed curves but instead have constrained endpoints. In the clasp, one rope whose ends are attached to the ceiling is looped around another whose ends are attached to the floor. The ropelength critical clasp we described is presumably the minimizer for ropelength, and is surprising in several ways: the tips of the two components are 6% further apart than they need to be—leaving a small gap between the ropes—and the curvature of the core curves blows up at the tips.

Here, we describe critical configurations of several generalized clasps with two or more components. Our new examples include clasps with one or both curves doubled, and connect sums of clasps with Hopf links. All of our examples are proven critical by the balance criterion. Many of them we expect are global minimizers (though we know no way to show this), but others are clearly not minimizing and are presumably unstable equilibria.

2. BACKGROUND

Here we recall the necessary definitions and theorems from our work [2] with Cantarella, Fu and Kusner. These will show the balance criterion for Gehring ropelength in the form in which we apply it to our new examples.

Definition. A generalized link \( L \) is a curve \( L \) (with disjoint components) together with obstacles and endpoint constraints.

In particular, each endpoint \( x \in \partial L \) is constrained to stay on some affine subspace \( M_x \subset \mathbb{R}^3 \). Furthermore, there is a finite collection of obstacles for the link, each obstacle

\[
\{ p \in \mathbb{R}^3 : g_j(p) < 0 \}
\]

being given in terms of a \( C^1 \) function \( g_j \) having 0 as a regular value. By calling these sets obstacles, we mean that \( L \) is constrained to stay in the region where \( \min g_j \geq 0 \).

Definition. The Gehring thickness \( GThi(L) \) of a curve \( L \) is the minimal distance between points on different components of \( L \). This is the supremal \( \tau \) for which the \((\tau/2)\)-neighborhoods of the components of \( L \) are disjoint.

We formulate the ropelength problem as minimizing the length of a generalized link subject to the constraint that its Gehring thickness remains at least 1. The contact points of the different components of the link are of primary importance, and are called struts.

Definition. An (unordered) pair of points \( x \) and \( y \) on different components of \( L \) is a Gehring strut if \( |y-x| = GThi(L) \). The set of all Gehring struts of \( L \) is denoted \( GStrut(L) \).

Given a generalized link \( L \), only variations preserving the endpoint constraints should be allowed. A continuous vectorfield \( \xi \) along \( L \) is said to be compatible with these constraints if it is tangent to \( M_x \) at each endpoint \( x \in \partial L \). We write \( VF_c(L) \) for the space of all compatible vectorfields.

Given a set of obstacles \( g_j < 0 \) for a link \( L \), we write

\[
O(L) := \min_j \min_{x \in L} g_j(x).
\]

Then \( L \) avoids the obstacles \( g_j \) if and only if \( O(L) \geq 0 \). We define the wall struts of \( L \) by

\[
\text{Wall}_j(L) := L \cap \{ g_j = 0 \}, \quad \text{Wall}(L) = \bigcup_j \text{Wall}_j(L).
\]

This incorporates those parts of \( L \) which are on the boundary of the obstacles.

Usually we will be minimizing the length \( \text{Len}(L) \) of a link while constraining \( GThi(L) \) to be at least 1. Sometimes, however, we wish to consider a slightly more general objective functional, a weighted sum \( \text{Len}^w(L) := \sum w_i \text{Len}(L_i) \) of the lengths of the different components. Here \( w_i \geq 0 \) can be viewed as the elastic tension within component \( L_i \).

It is known [2] that critical links for this (weighted) Gehring ropelength problem are curves of finite total curvature. (See [3] for an expository account of such curves.) This
means that the first variation of length under a compatible \( \xi \) is
given by \( \delta_\xi \text{Len}(L) = - \int_L (\xi, \kappa) \), where \( \kappa \) is a vector-valued
Radon measure along \( L \) (what we call a force along \( L \)) called
the curvature force. For a \( C^2 \) link, we have \( \kappa = N \kappa \) ds in
terms of the Frenet frame. For weighted length, it follows that
\( \delta \text{Len}^w = -\kappa^w \) where \( \kappa^w = w_i \kappa \) along component \( L_i \).

A variation of \( L \) will change the length of the struts and
change the values of the obstacle functions \( g_j \). We
collect these changes into the rigidity operator \( A \). If \( L \) is varied
with initial velocity \( \xi \in \mathcal{VF}(L) \), then \( A(\xi) \) is by definition a con-
tinuous function on \( \mathcal{GStrut} \cap \mathcal{Wall}(L) \); its value on a strut
\( \{x, y\} \) of length \( 1 \) is \( \delta_\xi [x-y] = (\xi_x - \xi_y)(x-y) \), and its value
on a wall strut \( x \) (where \( g_j(x) = 0 \)) is \( \delta_\xi g_j(x) = \xi - \nabla g_j \).

The (one-sided) first variation \( \delta^+_\xi \text{GT} \text{Hi}(L) \) of Gehring
thickness is (by Clarke’s differentiation theorem for min-
functions, see [2]) the minimum of \( A(\xi) \) over all struts; simi-
larly the first variation \( \delta^+_\xi O(L) \) of the obstacle function is the
minimum over all wall struts.

Intuitively, we expect a ropelength-critical configuration to
be one whose length cannot be reduced without also reducing
thickness. For technical reasons (see [2]) we define strong
criticality to require this reduction to happen at a definite rate:

**Definition.** We say that a generalized link \( L \) is strongly criti-
cal for minimizing weighted length when constrained by
\( \text{GT} \text{Hi} \) if there is an \( \varepsilon > 0 \) such that, for all compatible \( \xi \) with
\( \delta^+_\xi \text{Len}^w = -1 \), we have

\[
\min \delta^+_\xi \text{GT} \text{Hi}(L), \delta^+_\xi O(L) \leq -\varepsilon.
\]

The adjoint \( A^* \) of the rigidity operator takes a Radon mea-
sure on \( \mathcal{GStrut} \cap \mathcal{Wall} \) and gives a force along \( L \). This adjoint
is what appears in our balance criterion [2], which in turn
allows us to explicitly solve for the shapes of critical configura-
tions of various links.

**Theorem 2.1.** A generalized link \( L \) is strongly critical for
weighted Gehring ropelength if and only if there is a pos-
tive Radon measure \( \mu \) on \( \mathcal{GStrut} \cap \mathcal{Wall}(L) \) such that
\( -\kappa^w = A^* \mu \) as linear functionals on \( \mathcal{VF}(L) \). That is, \( -\kappa^w \)
and \( A^* \mu \) agree as forces along \( L \) except at endpoints \( x \in \partial L \),
where they may differ by an atomic force in the direction
normal to \( M_L \).

Most of the configurations we care about here consist—as
did the tight simple clasp [2]—of convex curves in perpen-
dicular planes. Let \( P_i \) (\( i = 1, 2 \)) be the \( x_i z \)-coordinate plane in
\( \mathbb{R}^3 = \{(x_1, x_2, z)\} \). If \( \gamma_1 \subset P_1 \) are two components of our
generalized link, then we parametrize each curve \( \gamma_i \) by the \( z \)-
component, \( u_i \), of its unit tangent vector. If there is a strut
\( \{p_1, p_2\} \) of length 1 connecting these two components \( \gamma_i \),
then the curves are both perpendicular to this strut, so elementary
trigonometry gives the following lemma from [2]:

**Lemma 2.2.** Let \( \gamma_i \) and \( \gamma_2 \) be two components of a link \( L \), ly-
ing in perpendicular planes. Suppose there is a strut \( \{p_1, p_2\} \)
of length 1 connecting these components. Then in the notation
of the previous paragraph, the parameters \( x_i \) and \( u_i \) for
the points \( p_i \) satisfy \( 0 \leq x_i \leq u_i \leq 1 \), and any two of the numbers
\( x_1, x_2, u_1, u_2 \) determine the other two (up to sign) according
to the formulas

\[
x_i^2 = 1 - x_j^2 = \frac{u_i^2(1 - u_j^2)}{1 - u_i^2 u_j^2},
\]

\[
u_i^2 = 1 - x_j^2 = \frac{u_i^2}{1 - u_j^2},
\]

where \( j \neq i \).

Note that the lemma above says nothing about balancing of
forces, but is merely a geometric fact about curves in perpen-
dicular planes that stay distance 1 apart. To balance symmet-
ric planar curves, we make use of the following lemma, again from [2]:

**Lemma 2.3.** Suppose \( \gamma_i \in P_i \) is symmetric across the \( z \)-axis,
and parametrized by \( u_i \) as above. Consider the net curvature
force of a mirror-image pair of infinitesimal arcs of \( \gamma \). This net
force acts in the vertical direction with magnitude \( 2|\text{d}u_i| \).

**3. WEIGHTED CLASPS**

The \( \tau \)-clasp \( C(\tau) \) is a generalized link consisting of two
clasped ropes arranged according to the following descrip-
tion. We fix four planes, in a tetrahedral pattern as shown in
Figure 1, each making angle \( \arcsin \tau \) with the vertical. The \( \tau \)-clasp
consists of two unknotted arcs \( \alpha_1 \) and \( \alpha_2 \) whose end-
points are constrained to the four planes; the complement of
the tetrahedron also serves as an obstacle for the link. The
isotopy class of the link is specified so that closing each arc
within the planes of its endpoints would produce a Hopf link.

If the two components \( \alpha_1 \) and \( \alpha_2 \) of a clasp have differ-
et tensions, the configuration can only be balanced if the
opening angles also differ. For \( \tau_1, \tau_2 \in (0, 1] \), we now de-
define the weighted clasp \( C(\tau_1, \tau_2) \). It is just like \( C(\tau) \) except
that its component \( \alpha_i \) is attached to planes at angle \( \arcsin \tau_i \).
For this generalized link, we will describe a critical configuration for the weighted Gehring ropelength problem, where the weights are \( w_i = 1/\tau_i \) on the two components. These tensions are chosen to ensure a net balance of vertical forces \( 2w_1\tau_1 = 2 = 2w_2\tau_2 \) at the ends of the clasp.

It follows from Lemma 3.1 that the clamped contact between \( \alpha_1 \) and \( \alpha_2 \) is determined by the balance equation \( u_1/\tau_1 + w_2/\tau_2 = 1 \). Plugging this into Lemma 3.2 gives us equations for the shapes. Namely, we get an explicit formula for the \( x_i \)-coordinate of \( \alpha_i \):

\[
x_i(u_i) = \frac{u_i^2(1 - u_j^2)}{1 - u_i^2u_j^2} = \frac{u_i^2(1 - \tau_i^2(1 - u_i/\tau_i)^2)}{1 - u_i^2\tau_j^2(1 - u_i/\tau_j)^2},
\]

where \( j \neq i \). As for the critical \( \tau \)-clasp [2], the \( z \)-coordinate is then determined as a hyperelliptic integral, through the relation \( dz/dz_i = u_i/\sqrt{1 - u_i^2} \) that defines \( u_i \). This integral, and the similar one for the total arclength of the curve, can easily be computed numerically.

Note that when \( \tau_1 = \tau_2 \), the curves arising here are exactly the symmetric \( \tau \)-clasp curves of [2]. But when \( \tau_1 \neq \tau_2 \), the two touching curves have shapes different from each other and from any symmetric clasp.

The calculations above, combined with Theorem 3.1, serve to prove:

**Theorem 3.1.** The configuration of \( C(\tau_1, \tau_2) \) described above is critical for weighted ropelength. \( \Box \)

We expect these configurations are in fact the ropelength minimizers.

4. CLASPS WITH PARALLELS

We next describe a family of examples based on the connect sum of the \( \tau \)-clasp with the Hopf link. We describe configurations which we show are balanced and thus critical points, but they are not minimizers and are presumably very unstable equilibria. Although we use unweighted length as our objective functional here, some of the shapes that appear are those of the weighted clasp curves we described above.

We first consider a configuration \( C_{1,1}(\tau) \) which can be defined as the connect sum of the \( \tau \)-clasp \( C(\tau) \) with a Hopf link. Letting \( \beta \) and \( \gamma \) denote the two components of \( C(\tau) \), we add a third component \( \alpha \), which is linked to \( \gamma \) but not to \( \beta \). We guess that in the ropelength minimizer for this link, the components \( \beta \) and \( \gamma \) retain the shapes they have in the minimizing clasp, and \( \alpha \) is a round circle around a point on a straight end of \( \gamma \), far from \( \beta \). Here, however, we will be interested in describing a critical configuration with more symmetry, where \( \alpha \) lies in the plane of \( \beta \).

**Definition.** The generalized link \( C_{p,q}^{\alpha,\beta}(\tau) \) is defined from \( C_{1,1}(\tau) \) by replacing \( \alpha \) by \( k \) parallel copies, \( \beta \) by \( l \) parallel copies, and \( \gamma \) by \( m \) parallel copies. We also adjust the angles of the bounding planes: those containing the ends of the \( l \) copies of \( \beta \) now lie at angle \( \arcsin(\tau/l) \) from the vertical, and those with \( \gamma \) at angle \( \arcsin(\tau/m) \).

In the configurations we describe, the copies of \( \alpha \) and \( \beta \) all lie nested with one another in a single vertical plane \( P_1 \), and the copies of \( \gamma \) are nested in the perpendicular vertical plane \( P_2 \). (Figure 2 sketches what such a configuration for \( C_{2,2}^{1,1}(\tau) \) might look like.) In particular, the shapes of all components are determined by those of the innermost \( \alpha \) and \( \gamma \); the other copies of \( \alpha \) are successive outer parallels at distance \( 1 \) and the copies of \( \beta \) are further outer parallels except that they peel off at the angle corresponding to \( u_1 = \pm \tau/l \), to proceed straight out and down to meet their bounding planes perpendicularly. Similarly, the copies of \( \gamma \) are determined as successive outer parallels at distance \( 1 \) to the innermost copy; all of them have straight segments out and up to the bounding planes at \( u_1 = \pm \tau/m \) and have a curved arc parametrized by \( u_2 \in [\tau/m, \tau/m] \). Here \( \tau \) can be any nonnegative number not exceeding \( \min(m,n) \). The angles are again determined by an overall balance of vertical forces: the \( l \) strands of \( \beta \) exert a vertical force of \( 2|u_1| = 2\tau/l \) each, while the \( m \) strands of \( \gamma \) exert a vertical force of \( 2|u_2| = 2\tau/m \) each.

Our entire configuration, like the \( \tau \)-clasp, has mirror symmetry across the planes \( P_1 \) and \( P_2 \), but unless we have \( k = 0 \) and \( l = m \), there is no longer the extra symmetry interchanging top and bottom.

It remains to describe the shapes of the innermost copies \( \alpha \) and \( \gamma \) exactly, and to show the resulting configuration is balanced. (See Figure 3 which shows \( C_{2,2}^{1,1}(1) \).) We will find it useful to use the abbreviation \( n := k+l \) for the total number of curves in the plane \( P_1 \) containing \( \alpha \) and \( \beta \) and their parallels. Note that the \( m \) copies of \( \gamma \) intersect the plane \( P_1 \) in a series of \( m \) points \( q_i \) spaced at distance \( 1 \) along the \( z \)-axis. The inner copy \( \alpha \) would naively be a stadium curve looping around these points at distance \( 1 \). Our configuration is close to this, changing only part of the upper semicircle. That is, \( \alpha \) consists of a semicircle around the bottommost \( q_i \), joined to vertical segments of length \( m \), and to an upper arc parametrized by \( u_1 \in [\tau/n, \tau/n] \). For \( |u_1| \geq \tau/n \) this upper arc is also part of the unit circle around the topmost \( q_i \), but for \( u_1 \in [\tau/n, \tau/n] \).
it is an analytic arc determined below. Note that the parallel copies of $\beta$ include outer parallels of this analytic arc in the range $|u_1| \leq \tau/n$, but also outer parallels to the circle (that is, larger circular arcs of integer radius around the topmost $q_i$) in the range $|u_1| \in [\tau/n, \tau/l]$, before peeling off straight at $|u_1| = \tau/l$.

The innermost $\gamma$ consists more simply of an analytic arc parametrized by $|u_2| \leq \tau/m$ joined to the straight segments out to the bounding planes. It now remains merely to describe the analytic arcs of the innermost $\alpha$ and $\gamma$; these form a weighted clasp $C(\tau/n, \tau/m)$ as in Theorem 3.1.

The important observation is that in our situation we have a convex planar curve with nested outer parallels. Each of the parallels has struts only to the next ones inward and outward, and these touch at points with equal direction $u_i$. Thus the innermost curve behaves like a curve with increased tension, proportional to the total number of parallel curves. If this increased force is balanced by struts to the other innermost curve, then the struts between the parallels distribute this balancing force outwards to balance the equal tension on each of the parallel strands.

**Theorem 4.1.** The configuration of $C^m_{k,l}(\tau)$ described above is critical for ropelength.

**Proof.** The lower semicircle of $\alpha$ and its parallels exert a total force $2k$ upwards on the bottommost $q_i$. This is transmitted upwards to the topmost $q_i$ by atomic strut forces in the struts connecting the $q_i$. The remaining circular pieces around this topmost $q_i$, namely $k$ circles for $|u_1| \in [\tau/l, 1]$ and a total of $n$ for $|u_1| \in [\tau/n, \tau/l]$, exert a balancing downwards force of $2k(1 - \tau/l) + 2n(\tau/l - \tau/n) = 2k$ on this $q_i$. The analytic arcs of the innermost $\alpha$ and $\gamma$ stay at constant distance 1 from each other; since they are the curves of the critical weighted clasp $C(\tau/n, \tau/m)$, they balance each other with weights $n$ and $m$, or equivalently, with unit weights on each of them and their respective $n$ and $m$ parallels.

We now consider in more detail the specific example of $C^m_{1,1}(1)$. This is the ordinary simple clasp $C(1)$—whose endpoints are attached to horizontal planes—with the addition of a closed component $\alpha$. The lower half of $\alpha$ is a semicircle of radius 1 centered at the tip of $\gamma$. The upper half of $\alpha$ consists of three parts: two circular arcs of angle $30^\circ$ and an arc clasped to $\gamma$. The curve $\beta$ consists of two vertical segments from the floor up to the height of the tip of $\gamma$, connected by an arc that is an outer parallel to the upper half of $\alpha$. The curve $\gamma$ includes no circular arcs, but only the analytic arc determined above and straight segments up to the ceiling.

In general, note that our critical configuration of $C^m_{k,l}(\tau)$ includes circular arcs in the $\beta$ components when $k > 0$, but not when $k = 0$. For $k = 0$ all the force balancing happens between the analytic arcs described above.

5. CONJECTURED MINIMIZERS FOR TWO CASES

For the two cases $k = 0$, $l = 2$, $m = 1, 2$, we now describe a different critical configuration $L^m(\tau)$ for the generalized link $C^m_{0,2}(\tau)$. In both cases, we conjecture that this configuration (unlike the one described above) is the minimizer for Gehring ropelength. The drawings in Figure 4 indicate the relative positions of the components in these configurations, without showing the exact geometric features described below.
The struts here close into hexagons. The horizontal components of force to be properly balanced.

Again, new horizontal struts connecting the copies of 
struts are pulled apart from each other. They carry exactly the
same balancing forces as before, but the horizontal compo-
nents of those forces need to be balanced by further horizontal
edges. In our new configuration, the quadrilateral of struts is split apart only in one direction.

The case \( m = 1 \) is similar but even simpler. In this case the quadrilateral of struts is split apart only in one direction. Again, new horizontal struts connecting the copies of \( \beta \) allow the horizontal components of force to be properly balanced. The struts here close into hexagons.

Figure 4: The configurations \( L^m(\tau) \) are presumably the minimizers for the clasp problems \( C^m_{0,2}(\tau) \) for \( m = 1 \) (left) and \( m = 2 \) (right). They involve congruent curves in parallel planes. The exact geometry of the critical configurations—including unit-length horizontal segments at the tips of most components, and the fact that the lower ends on the left must proceed out within 30° of the horizontal—is not shown in these sketches.

For \( C^1_{0,2}(\tau) \) the configuration we described above consists of noncongruent curves \( \beta_0 \) and \( \gamma_0 \), together with an outer parallel to \( \beta_0 \). In our new configuration \( L^1(\tau) \), both \( \beta \) curves are congruent to \( \beta_0 \), but translated out to lie in parallel planes \( x_2 = \pm \frac{1}{2} \). The \( \gamma \) curve is a copy of \( \gamma_0 \), but split apart at the tip, with a unit-length straight segment inserted from \( x_2 = -\frac{1}{2} \) to \( x_2 = \frac{1}{2} \).

For \( C^2_{0,2}(\tau) \) the configuration we described above consists of two ordinary \( \tau \)-clasp curves together with their outer parallels. In our new configuration \( L^2(\tau) \), all four curves are congruent, and each looks like the ordinary \( \tau \)-clasp but split at the tip, with a unit-length segment inserted. The two copies of \( \beta \) lie in the planes \( x_2 = \pm \frac{1}{2} \), and are translates of each other perpendicular to these planes. Similarly the two copies of \( \gamma \) lie in the planes \( x_1 = \pm \frac{1}{2} \). Note that for \( \tau = 1 \), the curvature of each curve is unbounded near the tip; \( L^2(\tau) \) gives an example of a Gehring-critical configuration in which curvature approaches infinity but then immediately jumps to zero.

**Theorem 5.1.** The configurations \( L^1(\tau) \) and \( L^2(\tau) \) described above are critical for Gehring ropelength.

**Proof.** Consider the case \( m = 2 \). In the usual \( \tau \)-clasp, there is 2–2 strut contact, with the four struts in any given set lying over the four quadrants of the \( x_1x_2 \)-plane. Here, those struts are pulled apart from each other. They carry exactly the same balancing forces as before, but the horizontal components of those forces need to be balanced by further horizontal struts. These connect corresponding points on the two copies of \( \beta \), and similarly for \( \gamma \). Each quadrilateral of struts has now been replaced by an octagon (with four translated copies of the original diagonal edges plus four new horizontal edges). But the forces still balance within each such octagon.

The case \( m = 1 \) is similar but even simpler. In this case the quadrilateral of struts is split apart only in one direction. Again, new horizontal struts connecting the copies of \( \beta \) allow the horizontal components of force to be properly balanced. The struts here close into hexagons.

Figure 5: Two clasps, when joined end to end, form a generalized link with three components. For the case \( \tau = 1 \) of vertical ends, the middle closed component \( \alpha \) has varying length in our family of critical configurations.

6. A CHAINED CLASP

Our next example is that of two clasps joined end to end, as in Figure 5. That is, we have three components: the open arcs \( \beta \) and \( \gamma \) are attached to the floor and ceiling, respectively, and are unlinked with each other; each, however, is linked to the closed component \( \alpha \). The configurations we consider have reflection symmetry across a horizontal plane, interchanging \( \beta \) and \( \gamma \) while preserving \( \alpha \).

If we make such a configuration with \( \tau < 1 \), the junctions (where \( \alpha \) clasps to \( \beta \) and to \( \gamma \)) will move towards each other, until the tips of \( \beta \) and \( \gamma \) touch at distance 1 producing an isolated strut. Here \( \beta \) and \( \gamma \) are congruent \( \tau \)-clasp curves. The curve \( \alpha \) is like a stadium curve, but its tips (for \( |u| \leq \tau \)) are \( \tau \)-clasps. These are followed by circular arcs around the tips of \( \beta \) and \( \gamma \), which are finally joined by unit-length vertical segments. Assuming the top/bottom symmetry, this critical
configuration is uniquely determined.

On the other hand, when $\tau = 1$, the critical configuration has one simple clasp at the junction of $\alpha$ with $\beta$, and another symmetric one between $\alpha$ and $\gamma$. These clasps can be close to each other (with the tips of $\beta$ and $\gamma$ as close as distance 1) or can move farther apart. There is a one-parameter family of equal-length symmetric critical configurations. In this family, the length of the closed component $\alpha$ varies.

**Theorem 6.1.** The configurations of the chained clasps described above are critical for Gehring ropelength.

*Proof.* For $\tau = 1$ the balancing is just that for the $\alpha, \beta$ and $\alpha, \gamma$ clasps separately.

For $\tau < 1$ we need to combine the balancing for the clasp with that for the simple closed chain described in [1, 2]. The curves $\beta$ and $\gamma$ are ordinary $\tau$-clasp curves, positioned so their tips are unit distance apart. These are balanced by clasp arcs of $\alpha$. The circular arcs of $\alpha$ focus net force $2 - 2\tau$ downwards on the tip of $\gamma$ and upwards on the tips of $\beta$. These forces are balanced by a force on the isolated strut connecting these two tips.

Note that we could build similar configurations with several chained components $\alpha_i$ in between $\beta$ and $\gamma$. For $\tau < 1$ the $\alpha_i$ are all congruent to the curve $\alpha$ described above. For $\tau = 1$, each would be a stadium curve, but they could have differing lengths. Again, we expect that all of these are ropelength minimizers.

### 7. THE GRANNY CLASP

Our final example generalizes the simple clasp not by introducing extra components, but by clasping the two components in a more intricate way. The connect sum of two trefoil knots of the same handedness is called the granny knot. We define the granny clasp $G_1$ to be the generalized link shown in Figure 6(left), a clasp of two ropes based on this granny knot.

It seems clear that if this configuration of the granny clasp were tied tight in rope, each component would contact itself as well as the other, as suggested in Figure 6(center). But our constraint on the Gehring thickness $GThi \geq 1$ does not see self-contact of a single component. It is thus important to remember that the natural setting for Gehring ropelength problems [2] is Milnor’s link homotopy. Two configurations are link-homotopic if there is a homotopy between them where the components stay disjoint but self-intersections of any given component are allowed.

In the critical configuration we describe for the granny clasp $G_1$, each component does have a point of self-intersection, like those shown in Figure 6(right). More precisely, if $\beta$ denotes the component attached to the floor, it—like the curves in all our previous examples—lies in a vertical plane and has mirror symmetry across a vertical line $\ell$ in that plane. In Figure 7 we see (as the solid line) one symmetric half of $\beta$, consisting of four analytic pieces joined in a $C^1$ fashion: first a vertical segment up from the floor, then a $\tau = 1$ clasp arc leading to the point $p$ of self-intersection, then continuing on the other side of $\ell$ with two more clasp arcs leading to the tip $t$ of $\beta$. We expect that this configuration is the minimizer, but as usual will prove only that it is critical.

**Theorem 7.1.** The (nonembedded) configuration of the generalized link-homotopy class $G_1$ built from clasp arcs as described above is critical for the Gehring ropelength problem.
Proof. The configuration can be balanced as follows: Cut space with horizontal planes at the heights of $t'$, $p$, $p'$ and $t$. Below $t'$ and above $t$ we have only vertical straight segments. In each of the three intermediate slabs we see exactly a $\tau = 1$ clasp. (Even though the symmetric halves of $\beta$ do not connect to each other through $p$ as in a clasp, they still balance the other component in the same way.)

We generalize this example as follows: let $G_n$ be the generalized link of two components obtained from the connect sum of two $(2,2n+1)$–torus knots of the same handedness. That is, each chain of three half-twists in our first picture of $G_1$ is replaced by $2n+1$. The link $G_2$ is shown in Figure 8 (left); note that $G_0$ is the ordinary clasp $C(1)$.

Again we can describe a critical configuration for $G_n$—as shown in Figure 8 (right)—which we expect is the minimizer. We obtain this configuration from the initial one by twisting the pair of endpoints in the ceiling around each other $n$ full turns (relative to the pair in the floor) and then letting the $n$ points of self-contact of each component become self-intersections. These $n$ self-intersections of each component occur where it crosses its plane of symmetry. Each half of each component is built from $2n+1$ clasp arcs plus a straight vertical segment.

A final generalization $G_n(\tau)$ would allow the ends of the clasp to be attached to slanted planes. Here the critical configuration would be built from $\tau$-clasp arcs (for $|u| \leq \tau$) near each self-intersection point, and arcs of circles (for $|u| \geq \tau$) centered at the self-intersections of the other component.

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