Josephson junctions in thin and narrow rectangular superconducting strips

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I consider a Josephson junction crossing the middle of a thin rectangular superconducting strip of length \( L \) and width \( W \) subjected to a perpendicular magnetic induction \( B \). I calculate the spatial dependence of the gauge-invariant phase difference across the junction and the resulting \( B \) dependence of the critical current \( I_c(B) \).

I. INTRODUCTION

Grain-boundary Josephson junctions play an important role in thin films of \( \text{YBa}_2\text{Cu}_3\text{O}_{7-\delta} \) (YBCO). Various theoretical approaches have been taken to understand the physics of Josephson junctions in thin films. When the film thickness \( d \) is less than the London penetration depth \( \lambda \), the current density \( j \) is practically uniform across the thickness, and the characteristic length describing the spatial variation of the gauge-invariant phase across the junction is (in SI units)

\[
\Lambda = 2\lambda^2/d.
\]

However, various studies have shown that there is a non-local relationship between the Josephson-current distribution in the vicinity of a Josephson vortex core and the magnetic field these currents generate, and the characteristic length governing the spatial distribution of the magnetic field distribution is the Pearl length, \( \ell = \phi_0/4\pi\mu_0\lambda^2 j_c \),

where \( \phi_0 = h/2e \) is the superconducting flux quantum and \( j_c \) (assumed to be independent of position) is the maximum Josephson current density that can flow as a supercurrent through the junction.

The integral equations relating the gauge-invariant phase difference across the junction to the magnetic field generated by a vortex in a long Josephson junction in a thin \((d < \lambda)\) film of lateral dimensions large by comparison with \( \Lambda \) and \( \ell \) were examined analytically and solved numerically in Ref. 6 for arbitrary ratios of \( \ell/\Lambda \). The case of a short Josephson junction bisecting a long superconducting strip of width \( W \) was studied in Ref. 7 under the assumptions that \( W \ll \Lambda \) and \( W \ll \ell \).

In this paper I revisit the latter problem by considering a thin rectangular uniform superconducting strip of length \( L \), width \( W \), and thickness \( d \) \((d < \lambda)\) divided into two halves by a Josephson junction at \( x = 0 \), as shown in Fig. 1. An applied magnetic induction \( B = \hat{z}B \) induces screening currents in the film. However, I consider here only the simplest case for which \( \Lambda = 2\lambda^2/d \) is much larger than the smaller of \( L \) and \( W \), such that the self-field generated by the screening currents can be neglected. The purpose of this paper is to calculate how the screening currents induced in response to \( B \) affect the \( B \) dependence of the maximum Josephson critical current \( I_c(B) \).

In Sec. II I give a brief discussion of the derivation of the basic equation for the gauge-invariant phase difference \( \Delta\gamma(y) \) across the junction, in Sec. III I present the solutions for \( \Delta\gamma(y) \) for arbitrary ratios of \( L/W \), and in Sec. IV I briefly summarize the results.

II. GAUGE-ININVARIANT PHASE DIFFERENCE

In the context of the Ginzburg-Landau (GL) theory, the superconducting order parameter can be expressed as \( \Psi = \Psi_0 fe^{i\gamma} \), where \( \Psi_0 \) is the magnitude of the order parameter in equilibrium, \( f = |\Psi|/\Psi_0 \) is the reduced order parameter, and \( \gamma \) is the phase. The second GL equation (in SI units) is

\[
\frac{f^2}{\mu_0\lambda^2} (A + \frac{\phi_0}{2\pi} \nabla \gamma),
\]

where \( A \) is the vector potential and \( B = \nabla \times A \) is the magnetic induction. Since \( j \) is a gauge-invariant quantity, so is the quantity within the parentheses on the
right-hand side. Different choices for the gauge of the vector potential $A$ result in different expressions for $\gamma$.

With a sinusoidal current-phase relation, the Josephson current density in the $x$ direction across the junction of width $d_i$ at $x = 0$ is $j_x(y) = j_x \sin \Delta \gamma(y)$, where $j_x$ is the maximum Josephson current density and $\Delta \gamma(y)$ is the gauge-invariant phase difference between the left ($a$) and right ($b$) superconductors,

$$\Delta \gamma(y) = \gamma_a(-\frac{d_i}{2}, y) - \gamma_b(\frac{d_i}{2}, y) - \frac{2\pi}{\phi_0} \int_{-d_i/2}^{d_i/2} A_x(x, y) dx. \tag{4}$$

I assume here that the induced or applied current densities $j_a$ and $j_b$ on the left- and right-hand sides of the junction are so weak that the suppression of the magnitude of the superconducting order parameter is negligible, such that $f = 1$. A simple relation between these current densities and the gauge-invariant phase difference can be obtained by integrating the vector potential $A$ around a very narrow rectangular loop of width $d_i$ in the $xy$ plane that just encloses the junction (with the bottom end at the origin and the top end at $y$), neglecting the magnetic flux up through the contour, making use of Eq. (3) with $f = 1$ for those portions of the integration along the sides of the junction, and noting that, by symmetry, $j_{xy}(0, y) = -j_{by}(0, y)$:

$${\Delta \gamma(y) = \Delta \gamma_0 + \frac{4\pi \mu_0 \lambda^2}{\phi_0} \int_0^y j_{by}(0, y') dy',} \tag{5}$$

where $\Delta \gamma_0 = \Delta \gamma(0)$, such that

$${d\Delta \gamma(y) / dy = (4\pi \mu_0 \lambda^2 / \phi_0) j_{by}(0, y).} \tag{6}$$

III. SOLVING FOR $\Delta \gamma$

I next assume that the Josephson coupling is so weak that the currents $j_a$ and $j_b$ on the left- and right-hand sides of the junction induced in response to the applied magnetic induction $B$ are far larger than the Josephson current density. This is equivalent to the assumption that $W \ll \ell$. Since $j_a$ easily can be obtained by symmetry from $j_b$, I calculate only $j_b$ in the region $x > 0$ and suppress the subscript $b$.

With the gauge choice $A = -\hat{x} By$, since $\nabla \cdot j = 0$ [see Eq. (3)], $\nabla^2 \gamma = 0$ must be solved subject to the boundary conditions following from $j_x(0, y) = j_x(L/2, y) = 0$ and $j_y(x, \pm W/2) = 0$,

$$\gamma_x(0, y) = \gamma_x(L/2, y) = 2\pi By / \phi_0, \quad \gamma_y(x, \pm W/2) = 0, \tag{7}$$

$$\gamma_x = \partial \gamma / \partial x$$ and $\gamma_y = \partial \gamma / \partial y$. The solution, obtained by the method of separation of variables, is (up to a constant)

$$\gamma(x, y) = \frac{8\pi B}{\phi_0 W} \sum_{n=0}^{\infty} \frac{(-1)^n \sinh[k_n(x - L/4)] \sin(k_ny)}{k_n^3 \cosh(k_nL/4)}, \tag{9}$$

where $k_n = (n + 1/2)2\pi / W$.

The current density $j_x(x, y)$ now can be obtained from Eq. (3). Since $\nabla \cdot j = 0$, we also can write $j = \nabla \times S$, where $S = \hat{z} S$, and $S(x, y) = (B/2\mu_0 \lambda^2) s(x, y)$ is the stream function given by

$$s(x, y) = y^2 + \frac{8}{W} \sum_{n=0}^{\infty} \frac{(-1)^n \cosh[k_n(x - L/4)] \cos(k_ny)}{k_n^3 \cosh(k_nL/4)}. \tag{10}$$

The series converges rapidly, and Fig. 2 shows the result for $x > 0$ obtained by summing over $n$ from 0 to 10 when $L/W = 2$.

Since Eq. (3) yields $j_y(0, y) = -(\phi_0 / 2\pi \mu_0 \lambda^2) \gamma_y(0, y)$, Eq. (6) can be integrated to obtain (assuming the additive constant $\Delta \gamma_0 = 0$)

$$\Delta \gamma(y) = \frac{16\pi B}{\phi_0 W} \sum_{n=0}^{\infty} \frac{(-1)^n \tanh(k_nL/4) \sin(k_ny)}{k_n^3}. \tag{11}$$

As shown in the Appendix, when $L \to \infty$, the sum can be expressed in terms of the Lerch transcendent $\Phi(z, s, a)$, and in this limit $\Delta \gamma(y)$ also can be expressed as $\Delta \gamma(y) = (4BW^2 / \phi_0) \varphi_0(\pi y / W)$, where $\varphi_0(\mu)$ is a function defined in Ref. 7. This is the explanation of why $\varphi_0(\mu)$ was found to be a material-independent universal function when $\Lambda \gg W$ and $\ell \gg W$.

For all ratios of $L/W$, the maximum value of $\Delta \gamma(y)$ occurs at $y = W/2$, where

$$\Delta \gamma(W/2) = \frac{14c(3)BW^2}{\pi^2 \phi_0} = 1.705 \frac{BW^2}{\phi_0}, \quad L \to \infty, \tag{12}$$

$$\Delta \gamma(W/2) = \frac{\pi BWL}{2\phi_0} = 1.571 \frac{BWL}{\phi_0}, \quad L < W. \tag{13}$$
The solid curve in Fig. 3 shows $\Delta \gamma(W/2)$ normalized to $BW^2/\phi_0$ as a function of $L/W$ along with the limiting behaviors of Eqs. (12) (dot-dashed) and (13) (dotted). The dashed curve shows the interpolating function,

$$\Delta \gamma(W/2) = \frac{14\zeta(3)BW^2}{\pi^2\phi_0} \tanh\left[\frac{\pi^3L}{28\zeta(3)W}\right]$$

where $\zeta(3) = 1.20206$ is the Riemann zeta function.

The plots of $\Delta \gamma(y)/\Delta \gamma(W/2)$ vs $y/(W/2)$ in Fig. 4 show how the gauge-invariant phase difference depends upon the ratio $L/W$. For $L/W \to \infty$, the curve lies below $\sin(\pi y/W)$, shown as the long-dashed curve, and for $L \ll W$, $\Delta \gamma(y)/\Delta \gamma(W/2) = y/(W/2)$, a straight line.

The maximum Josephson current $I_c(B)$, the maximum integral of $j_y \sin \Delta \gamma(y)$ over $y$ from $-W/2$ to $W/2$, occurs when $\Delta \gamma_0 = \pm \pi/2$, such that

$$\frac{I_c(B)}{I_c(0)} = \frac{2}{W} \int_{-W/2}^{W/2} \cos[\Delta \gamma(y)]dy$$

where $\Delta \gamma(y)$ is given in Eq. (11). Figure 5 shows plots of $I_c(B)/I_c(0)$ vs $BW^2/\phi_0$ for $L/W = \infty$, 1, and 1/2. The stretching out of the pattern along the horizontal axis as $L$ decreases is easily understood with the help of Fig. 3.

Let us define $\Delta B_1$ as the value of $B$ at which $I_c(B)$ has its first zero, $\Delta B_2$ as the difference of the values at which $I_c(B)$ has its second and first zeros, and $\Delta B_n$ as the difference of the values at which $I_c(B)$ has its $n$th and $(n-1)$th zeros. For all finite values of $L/W$, the $\Delta B_n$ are smaller for small $n$ than for large $n$. However, for large $n$, the $\Delta B_n$ approach the limiting value

$$\Delta B = \left(\frac{\pi^3\phi_0}{16W^2}\right)\sum_{n=0}^{\infty} \frac{\tanh[(2n + 1)\pi L/4W]}{(2n + 1)^3}$$

To illustrate this, if we approximate $\Delta \gamma(y)$ for $L \to \infty$ by $\Delta \gamma(W/2) \sin(\pi y/W)$ (see the long-dashed curve in Fig. 4), as in Ref. 7, then the integral in Eq. 15 can be evaluated in terms of the Bessel function $J_0$ with the result

$$\frac{I_c(B)}{I_c(0)} = \left|J_0\left(\frac{14\zeta(3)BW^2}{\pi^2\phi_0}\right)\right|.$$  (17)

For $L/W \to \infty$, the sum in Eq. (16) can be evaluated as

$$\Delta B = \left[\frac{\pi^3}{14\zeta(3)}\right]\phi_0/W^2 = 1.842\phi_0/W^2,$$  (18)

as pointed out in Ref. 15. Using Eq. (17) and the well-known zeros of $J_0(x)$, we find the following values for $n = 1, 2, 3, 4,$ and 5: $\Delta B_n/\Delta B = 0.7655, 0.9916, 0.9975, 0.9988$, and 0.9993. However, to evaluate the zeros of $I_c(B)/I_c(0)$ without using the Bessel-function approximation, we must numerically evaluate Eq. (15). This yields the following more accurate values for $n = 1, 2, 3, 4,$ and $5$: $\Delta B_n/\Delta B = 0.8173, 0.9866, 0.9946, 0.9968$, and 0.9979.

When $d_y \ll L \ll W$, the gauge-invariant phase difference $\Delta \gamma(y)$ of Eq. (11) becomes linear in $y$, and in this case we have the familiar Fraunhofer-like pattern,

$$\frac{I_c(B)}{I_c(0)} = \left|\frac{\sin(\pi BWL/2\phi_0)}{\pi BWL/2\phi_0}\right|.$$  (19)
such that all the $\Delta B_n$ are the same and equal to

$$\Delta B = 2\phi_0/4L.\quad (20)$$

The magnitude of this $\Delta B$ agrees with that in sandwich-type Josephson junctions with thickness $d \gg \lambda$ along the $z$ direction only in the limit $d \ll L < \lambda$.16-18

The solid curve in Fig. 6 shows $\Delta B$, the large-$n$ limit of $\Delta B_n$, calculated via Eq. (16) as a function of $L/W$, along with the expressions for $\Delta B$ in the limits $L/W \to \infty$ (dot-dashed) and $L \ll W$ (dotted). The dashed curve shows the approximate interpolating function obtained from Eq. (14),

$$\Delta B = \left(\frac{\pi^3\phi_0}{14\zeta(3)W^2}\right)/\tanh\left[\frac{\pi^3L}{28\zeta(3)W}\right].\quad (21)$$

IV. SUMMARY

In this paper I have considered a Josephson junction bisecting a rectangular superconducting thin film of large Pearl length $\Lambda = 2\lambda^2/d$ subjected to a perpendicular magnetic induction $B$. I calculated the gauge-invariant phase difference and used it to determine the $B$ dependence of the Josephson critical current density $I_c(B)$.

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Appendix A: The limit $L \to \infty$

In the limit $L \to \infty$, the gauge-invariant phase difference $\Delta \gamma(y)$ given in Eq. (11) can be expressed as

$$\Delta \gamma(y) = (16BW^2/\pi^2\phi_0)\sigma(\pi y/W),$$

where

$$\sigma(\psi) = \sum_{n=0}^{\infty} (-1)^n \sin[(2n+1)\psi]/(2n+1)^3\quad (A1)$$

and

$$\Phi(z,s,a) = \sum_{k=0}^{\infty} z^k/(k+a)^s$$

is the Lerch transcendent.19 Note that $\sigma(0) = 0$ and $\sigma(\pi/2) = 7\zeta(3)/6 = 1.0518$.

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