DEFORMED YANGIANS
AND INTEGRABLE MODELS

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ABSTRACT
Twisted Hopf algebra $sl_\xi(2)$ gives rise to a deformation of the Yangian $Y(sl(2))$. The corresponding deformations of the integrable $XXX$-spin chain and the Gaudin model are discussed.
1 Introduction

The development of the quantum inverse scattering method (QISM) gave rise to the theory of quantum groups [1–3]: the latter influenced the QISM as well (cf. [4]). Taking into account recent results on the further deformation of the Yangian algebra $\mathcal{Y}$ [5], and applications of the Yangian symmetries to a variety of integrable models [6], the simplest integrable $XXX$-model related to the Yangian $Y(gl(2))$ is discussed in this paper. The deformation related to a simple twist of the Lie algebra $sl(2)$: $\Delta_\xi = F \Delta F^{-1}$, with an appropriate element $F \in U(sl(2)) \otimes U(sl(2))$ preserves the regularity property [7] of the corresponding $R$-matrix [8]: $R(0) = P$, where $P$ is the permutation map in $C^2 \otimes C^2$. This observation enables us to write down a deformed Hamiltonian for the Heisenberg chain of length $N$ ($XXX_\xi$-model) [8],

$$H = \sum_n (\sigma^x_n \sigma^x_{n+1} + \sigma^y_n \sigma^y_{n+1} + \sigma^z_n \sigma^z_{n+1} + \xi^2 \sigma^z_n \sigma^{-z}_{n+1} + \xi (\sigma^-_n - \sigma^-_{n+1})),$$

where $\xi$ is a deformation parameter, $\sigma^x_n$, $\sigma^y_n$, and $\sigma^z_n$ are Pauli sigma-matrices acting in $C^2_n$ related to the $n$th site of the chain and $\sigma^-_n = \frac{1}{2}(\sigma^x_n - i\sigma^y_n)$. According to the general scheme of the QISM one can construct integrable models for other values of spin $s = 1, \frac{3}{2}, 2, \ldots$ as well.

However, it turns out that this Hamiltonian is not Hermitian, and thus creates extra difficulties in constructing the algebraic Bethe Ansatz (ABA) for this model. Due to the triangular structure of the twist, we demonstrate that the spectrum of the transfer matrix and the Bethe equations of the deformed $XXX$-model coincide with the usual ones.

The connection of the $XXX$-spin chain with the representation theory of the $sl(2)$ algebra in the framework of the QISM was explained in [9]. The deformed $XX\hat{X}_\xi$-spin chain has a similar connection with the representation theory of the twisted $sl(2)$.

Let us point out that for the periodic boundary condition the obtained Hamiltonian coincides with appropriate limit of the seven-vertex model Hamiltonian [10]. An interesting application of the Drinfeld twist depending on the spectral parameter, to the original $XXX$-model is discussed in the recent paper [11]. The Gaudin model directly related to the Yangian, was studied also in the framework of the QISM [12].

2 Yang–Baxter algebra

Let us consider the Yang–Baxter algebra (see e.g. [7, 13]),

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v),$$

for the entries of the $2 \times 2$ matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$
given by twisting the Yang solution,

\[ R(u-v) = F_{21} \left( I - \frac{\eta}{u-v} P \right) F_{12}^{-1} = R_\xi - \frac{\eta}{u-v} P, \tag{1} \]

with \( P \) as the permutation matrix and

\[
F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\xi & 1 \end{pmatrix}, \quad R_\xi = F_{21} F_{12}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix}.
\]

This is a reduction to the fundamental representation \( \rho \) in \( \mathfrak{sl}^2 \) of the universal twist element \( F \), \( F_{12} = (\rho \otimes \rho) F \) and the universal \( R \)-matrix \( R \) of the twisted Yangian \( Y_\xi(\mathfrak{sl}(2)) \) \([5]\). Using the generators \( h, e \) of the \( \mathfrak{sl}(2) \): \( [h, e] = -2e \), the universal twist \( F \) is

\[
F = 1 + \xi h \otimes e + \frac{1}{2} \xi^2 h(h+2) \otimes e^2 + \ldots.
\]

There are quite a few papers on the non-standard quantum algebra \( \mathfrak{sl}_\xi(2) \) (see e.g. \([14, 15]\) and Refs therein). Introducing the variable \( \sigma : 1 - 2\xi e = \exp(-\sigma) \) this twist is \( F = \exp(h \otimes \sigma/2) \).

To study the corresponding Bethe Ansatz related to \( T(u) = \{ t_{ij}(u) \} \), we need all 16 commutation relations (CR) of the form

\[
\sum t_{ij}(u) t_{kl}(v) = \sum t_{mn}(v) t_{pq}(u) \ldots.
\]

Constructing \( T(u) = L_N(u) \ldots L_2(u) L_1(u) \) from the local \( L \)-operators given by the \( R \)-matrix (1) itself according to the QISM \([7, 13]\), we write down at the beginning the CR of \( A(u) \) and \( D(u) \) with \( C(v) \), using notations for the \( R \)-matrix entries:

\[
\alpha(u, v) = 1 + \beta(u, v) = 1 - \frac{\eta}{u-v},
\]

\[
A(u)C(v) = \alpha(u, v) C(v) A(u) - (\beta(u, v) C(u) - \xi A(u)) A(v) - \xi D(v) A(u) + (\xi C(v) + \xi^2 D(v)) B(u),
\]

\[
D(u)C(v) = (\alpha(u, v) C(v) - \xi A(v)) D(u) - (\beta(u, v) C(u) - \xi D(u)) D(v) + (\xi C(v) + \xi^2 A(v)) B(u),
\]

\[
B(u)C(v) = (C(v) + \xi D(v)) B(u) + (\xi B(u) - \beta(u, v) D(u)) A(v) + (\beta(u, v) D(u)) A(u) + \xi (u, v) C(v) A(v) - \xi D(v) C(u) A(v) + (\xi C(v) + \xi^2 D(v)) D(u) + (-\xi C(u) + \xi^2 A(u)) A(v) + \xi A(u) C(v),
\]

\[
A(u, v) A(u) = (\alpha(u, v) A(v) - \xi B(v)) A(u) + (\xi A(v) + \xi^2 B(v)) B(u),
\]

\[
A(u, v) B(v) = B(v) A(u) + (\beta(u, v) A(v) - \xi B(v)) B(u),
\]

\[
B(u) B(v) = B(v) B(u),
\]
\( A(u)D(v) = D(v)A(u) + (\xi A(u) - \beta(u, v)C(u))B(v) + (\beta(u, v)C(v) - \xi D(v))B(u), \)
\( \alpha(v, u)D(u)B(v) = B(v)D(u) + (\beta(v, u)D(v) - \xi B(v))B(u), \)
\( (C(u) + \xi A(u))A(v) = (\alpha(u, v)A(v) - \xi B(v))C(u) + \xi A(v) + \xi^2 B(v))D(u) - \beta(u, v)A(u)C(v), \)
\( B(u)C(v) = (C(v) + \xi A(v))B(u) + \xi B(u)B(v) + \beta(v, u)(A(v)D(u) - A(u)D(v)), \)
\( \beta(u, v)B(u)C(v) = \beta(u, v)B(v)C(u) + (A(v) + \xi B(v))D(u) - (D(u) + \xi B(u))A(v), \)
\[ A(u)\Omega = a(u)\Omega, \quad D(u)\Omega = d(u)\Omega, \quad B(u)\Omega = 0. \]

For the twisted XXX-model of spin 1/2 (there are the corresponding states for the XXX\(_\xi\)-models of other spin values too)

\[ \Omega = \bigotimes_{k=1}^{N} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad a(u) = 1, \quad d(u) = (\alpha(u))^N. \quad (2) \]

Let us study the conditions under which the state vector \( C(v)\Omega \) is an eigenvector of the transfer matrix \( t(u) \):

\[ t(u) = \text{tr} T(u) = A(u) + D(u), \quad t(u)\Omega = (1 + d(u))\Omega. \]

Using the CR to intertwine \( (A(u) + D(u)) \) over \( C(v) \) and the properties of the bare vacuum \( \Omega \) (2) one gets

\[ t(u)C(v)\Omega = (1 \cdot \alpha(u, v) + d(u)\alpha(v, u))C(v)\Omega - (\beta(u, v) + \beta(v, u)d(v))C(u)\Omega + \xi(1 - d(u))(1 - d(v))\Omega. \]

Therefore we can conclude that \( C(v)\Omega \) is an eigenvector of the transfer matrix \( t(u) \) if \( (\alpha(v))^N = 1 \). This is the Bethe equation for the one magnon sector. The action of \( t(u) = A(u) + D(u) \) on the vector \( C(v_1)C(v_2)\Omega \) results in the following combination of vectors \( C \cdot C \cdot \Omega, C\Omega \) and \( \Omega : \)

\[ (\alpha(u, v_1)\alpha(u, v_2) + d(u)\alpha(v_1, u)\alpha(v_2, u))C(v_1)C(v_2)\Omega + "\text{unwanted terms}". \]
Hence, the eigenvalue \( \Lambda(u, \{v_j\}) \) of the transfer matrix \( t(u) \) is the same as for the XXX-model [7, 13], although the structure of the “unwanted terms” is more complicated and the standard Bethe equations, which read
\[
\left( \frac{v_j - \eta}{v_j} \right)^N = \prod_k \frac{v_k - v_j + \eta}{v_k - v_j - \eta},
\]
are not enough to have
\[
\Psi(v_1, \ldots, v_M) = C(v_1) \ldots C(v_M)\Omega
\]
as an eigenvector of the transfer matrix: \( t(u)\Psi = \Lambda(u, \{v_j\})\Psi \). This fact is related with triangularity of the perturbation of the XXX-model Hamiltonian [10]. One can see this phenomenon by looking even at the Clebsch-Gordan coefficients of the twisted \( sl(2) \) algebra in the original basis \( h, e, f \). Due to the connection of the twisted coproduct with the standard one by the similarity transformation \( \Delta_\xi = F \Delta F^{-1} \), the CG coefficients of the \( sl(2) \xi \) are linear combinations of the usual ones with the elements of triangular matrix \( F_{m_1,m_2}^{\pi,\rho}(\pi \otimes \rho) \) of the twist \( (\pi \otimes \rho)F \) in the tensor product of two irreducible representations \( V_\pi \otimes V_\rho \) (cf.[15]).

3 Symmetry algebra \( sl_\xi(2) \)

The local structure of the monodromy matrix \( T(u) \) leads to the following asymptotic expansion,
\[
T_N(u) = L_N(u) \ldots L_1(u) = \prod_{k=1}^N R_{ak}(\xi) + \frac{1}{u} \sum_{k=1}^N M_k P_{ak} M_{N-k-1} + O\left( \frac{1}{u^2} \right),
\]
where \( M_k = \prod_{m=1}^{k-1} R_{am}(\xi) = (id \otimes \Delta^{(k-1)}) R(\xi) \). Using for the constant term the notation
\[
T_0 = \begin{pmatrix} E & 0 \\ G & E^{-1} \end{pmatrix} = \prod_{k=1}^N R_{ak}(\xi) = (id \otimes \Delta^{(N-1)}) R(\xi),
\]
one gets the symmetry algebra \( sl_\xi(2) \) of the quantum scattering data:
\[
\begin{align*}
E G &= GE - \xi (1 - E^2), \\
E A(u) &= A(u)E - \xi B(u)E, \\
E D(u) &= D(u)E + \xi E B(u), \\
E B(u) &= B(u)E, \\
E C(u) &= C(u)E + \xi E A(u) - \xi D(u)E = C(u)E + \xi (A(u) - D(u))E - \xi^2 B(u)E;
\end{align*}
\]
\[ GB(u) = B(u)G - \xi(EB(u) + B(u)E^{-1}), \]
\[ GA(u) = A(u)G - \xi(EA(u) - A(u)E^{-1} + B(u)G) + \xi^2 B(u)E^{-1}, \]
\[ GD(u) = D(u)G + \xi(ED(u) - D(u)E^{-1} - GB(u)) - \xi^2 B(u)E. \]
\[ GC(u) = C(u)G + \xi(EC(u) + C(u)E^{-1} - GA(u) - DA(u)) + \xi^2 (D(u)E^{-1} - EA(u)). \]

The element \( E \) is the group-like one, and it commutes with the transfer matrix \( t(u) \). The elements \( E \) and \( G \) have the following simple behaviour under the coproduct map:
\[ \Delta(E) = E \otimes E, \quad \Delta(G) = G \otimes E + E^{-1} \otimes G. \]

However, to extract from \( T(u) \) the third generator of the twisted Hopf algebra \( sl_\xi(2) \) one has to consider the \( 1/u \) term in the asymptotic expansion of \( T(u) \).

The twist structure of the \( R \)-matrix (1) leads to the similar structure of the monodromy matrix \( T(u) \). Hence, the quantum scattering data of the \( XXX_\xi \)-model can be expressed as linear combinations of the quantum scattering data of the \( XXX \)-model with coefficients constructed from the twist \( F \).

The twisted \( R \)-matrix has the same spectral decomposition as the original one:
\[ PR = F_{12}PF_{12}^{-1} = P_+(\xi) - P_-(\xi). \]

The same is true for the \( R \)-matrix which depends on the spectral parameter. Hence, applying the fusion procedure [7] to the monodromy matrix one gets relations among the transfer matrices in different representations of the \( sl(2) \) for the auxiliary space
\[ t^{(l+1)}(u) = t^{(l)}(u)t^{(1)}(u - \eta) + (u - \eta)^N t^{(l-1)}(u). \]

Another important ingredient of the QISM is the Baxter difference equation
\[ \Lambda(u)Q(u) = Q(u - \eta) + d(u)Q(u + \eta). \]

The appropriate quasiclassical limit \( \eta \approx \xi \rightarrow 0 \) gives rise to a deformation of the Gaudin model, analysed also in the framework of the QISM [12]. One can apply to the proposed \( XXX_\xi \)-model the functional Bethe Ansatz [16] as well.

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