BEHAVIOR NEAR THE ORIGIN OF \( f'(u^*) \) IN RADIAL SINGULAR EXTREMAL SOLUTIONS

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Abstract. Consider the semilinear elliptic equation \(-\Delta u = \lambda f(u)\) in the unit ball \( B_1 \subset \mathbb{R}^N \), with Dirichlet data \( u|_{\partial B_1} = 0 \), where \( \lambda \geq 0 \) is a real parameter and \( f \) is a \( C^1 \) positive, nondecreasing and convex function in \([0, \infty)\) such that \( f(s)/s \to \infty \) as \( s \to \infty \). In this paper we study the behavior of \( f'(u^*) \) near the origin when \( u^* \), the extremal solution of the previous problem associated to \( \lambda = \lambda^* \), is singular. This answers to an open problems posed by Brezis and Vázquez [2, Open problem 5].

1. Introduction and main results

Consider the following semilinear elliptic equation, which has been extensively studied:

\[
\begin{cases}
-\Delta u = \lambda f(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

\((P_\lambda)\)

where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, \( N \geq 1 \), \( \lambda \geq 0 \) is a real parameter and the nonlinearity \( f : [0, \infty) \to \mathbb{R} \) satisfies

\[ f \text{ is } C^1, \text{ nondecreasing and convex, } f(0) > 0, \text{ and } \lim_{u \to +\infty} \frac{f(t)}{t} = +\infty. \]

It is well known that there exists a finite positive extremal parameter \( \lambda^* \) such that \((P_\lambda)\) has a minimal classical solution \( u_\lambda \in C^0(\overline{\Omega}) \cap C^2(\Omega) \) if \( 0 < \lambda < \lambda^* \), while no solution exists, even in the weak sense, for \( \lambda > \lambda^* \). The set \( \{ u_\lambda : 0 < \lambda < \lambda^* \} \) forms a branch of classical solutions increasing in \( \lambda \). Its increasing pointwise limit \( u^*(x) := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x) \) is a weak solution of \((P_\lambda)\) for \( \lambda = \lambda^* \), which is called the extremal solution of \((P_\lambda)\) (see [1 2]).

The regularity and properties of extremal solutions depend strongly on the dimension \( N \), domain \( \Omega \) and nonlinearity \( f \). When \( f(u) = e^u \), it was proven that \( u^* \in L^\infty(\Omega) \) if \( N < 10 \) (for every \( \Omega \)) (see [8 11]), while \( u^*(x) = -2 \log |x| \) and \( \lambda^* = 2(N - 2) \) if \( N \geq 10 \) and \( \Omega = B_1 \) (see [10]). There is an analogous result for \( f(u) = (1 + u)^p \) with \( p > 1 \) (see [2]). Brezis

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and Vázquez [2] raised the question of determining the boundedness of $u^*$, depending only on the dimension $N$, for general smooth bounded domains $\Omega \subset \mathbb{R}^N$ and nonlinearities $f$ satisfying (1.1). This was proven by Nedev [12] when $N \leq 3$; by Cabré and Capella [4] when $\Omega = B_1$ and $N \leq 9$; by Cabré [3] when $N = 4$ and $\Omega$ is convex; by the author [13] when $N = 4$; by Cabré and Ros-Oton [6] when $N \leq 7$ and $\Omega$ is a convex domain of double revolution; by Cabré, Sanchón, and Spruck [7] when $N = 5$ and \( \limsup_{t \to \infty} f'(t)/f(t)^{1+\epsilon} < +\infty \) for every $\epsilon > 0$. Finally, in a recent paper Cabré, Figalli, Ros-Oton and Serra [5] solved completely this question by proving that $u^*$ is bounded if $N \leq 9$.

Another question posed by Brezis and Vázquez [2] Open problem 5 for singular extremal solutions is the following: What is the behavior of $u$ near the singularities? Does it look like $C/r^2$?

This question is motivated by the fact that in the explicit examples $\Omega = B_1$ and $f(u) = (1 + u)^p$, $p > 1$ or $f(u) = e^u$ it is always $f'(u^*(r)) = C/r^2$ for certain positive constant $C$, when the extremal solution $u^*$ is singular.

In this paper we give a negative answer to this question, by showing that, in the case in which $\Omega = B_1$ and $u^*$ is singular, we always have $\limsup_{r \to 0} r^2 f'(u^*(r)) \in (0, +\infty)$. However, it is possible to give examples of $f \in C^\infty([0, +\infty))$ satisfying (1.1) for which $u^*$ is singular and $\liminf_{r \to 0} r^2 f'(u^*(r)) = 0$. In fact, we exhibit a large family of functions $f \in C^\infty([0, +\infty))$ satisfying (1.1) for which $u^*$ is singular and $f'(u^*)$ can have a very oscillating behavior.

**Theorem 1.1.** Assume that $\Omega = B_1$, $N \geq 10$, and that $f$ satisfies (1.1). Suppose that the extremal solution $u^*$ of $(P_\lambda)$ is unbounded. Then $\limsup_{r \to 0} r^2 f'(u^*(r)) \in (0, +\infty)$. Moreover

\[
\frac{2(N - 2)}{\lambda^*} \leq \limsup_{r \to 0} r^2 f'(u^*(r)) \leq \frac{\lambda_1}{\lambda^*},
\]

where $\lambda_1$ denotes the first eigenvalue of the linear problem $-\Delta v = \lambda v$ in $B_1 \subset \mathbb{R}^N$ with Dirichlet conditions $v = 0$ on $\partial B_1$.

**Theorem 1.2.** Assume that $\Omega = B_1$, $N \geq 10$, and that $\varphi : (0, 1) \to \mathbb{R}^+$ satisfies $\lim_{r \to 0} \varphi(r) = +\infty$. Then there exists $f \in C^\infty([0, +\infty))$ satisfying (1.1) such that the extremal solution $u^*$ of $(P_\lambda)$ is unbounded and

\[
\liminf_{r \to 0} \frac{f'(u^*(r))}{\varphi(r)} = 0.
\]

Note that in the case $\varphi(r) = 1/r^2$, we would obtain $\liminf_{r \to 0} r^2 f'(u^*(r)) = 0$. This answers negatively to [2 Open problem 5]. In fact, $r^2 f'(u^*(r))$ could be very oscillating, as the next result shows.

**Theorem 1.3.** Assume that $\Omega = B_1$, $N \geq 10$, and let $0 \leq C_1 \leq C_2$, where $C_2 \in [2(N - 2), (N - 2)^2/4]$. Then there exists $f \in C^\infty([0, +\infty))$ satisfying (1.1) such that the extremal solution $u^*$ of $(P_\lambda)$ is unbounded, $\lambda^* = 1$ and
\[
\lim_{r \to 0} r^2 f'(u^*(r)) = C_1,
\]
\[
\limsup_{r \to 0} r^2 f'(u^*(r)) = C_2.
\]

Note that if \(C_1 = C_2\), then the interval \([2(N - 2), (N - 2)^2/4] \) is optimal: \(C_2 \geq 2(N - 2)^2/4\) by Hardy’s inequality.

**Theorem 1.4.** Assume that \(\Omega = B_1\), \(N \geq 11\), and that \(\Psi \in C(B_1 \setminus \{0\})\) is a radially symmetric decreasing function satisfying

\[
\frac{2(N - 2)}{r^2} \leq \Psi(r) \leq \frac{(N - 2)^2}{4r^2}, \quad \text{for every } 0 < r \leq 1.
\]

Then there exist \(f \in C^1([0, +\infty))\) satisfying (1.1) such that \(\lambda^* = 1\) and

\[
f'(u^*(x)) = \Psi(x), \quad \text{for every } x \in \overline{B_1 \setminus \{0\}}.
\]

Moreover, this function \(f\) is unique up to a multiplicative constant. That is, if \(g\) is a function with the above properties, then there exists \(\alpha > 0\) such that \(g = \alpha f(\cdot/\alpha)\) (whose extremal solution is \(\alpha u^*\)).

## 2. Proof of the main results

First of all, if \(\Omega = B_1\), and \(f\) satisfies (1.1), it is easily seen by the Gidas-Ni-Nirenberg symmetry result that \(u_\lambda\), the solution of \((P_\lambda)\), is radially decreasing for \(0 < \lambda < \lambda^*\). Hence, its limit \(u^*\) is also radially decreasing. In fact \(u^*_r(r) < 0\) for all \(r \in (0, 1]\), where \(u_r\) denotes the radial derivative of a radial function \(u\). Moreover, it is immediate that the minimality of \(u_\lambda\) implies its stability. Clearly, we can pass to the limit and obtain that \(u^*\) is also stable, which means

\[
(2.1) \quad \int_{B_1} |\nabla \xi|^2 \, dx \geq \int_{B_1} \lambda^* f'(u^*) \xi^2 \, dx
\]

for every \(\xi \in C^\infty(B_1)\) with compact support in \(B_1\).

On the other hand, differentiating \(-\Delta u^* = \lambda^* f(u^*)\) with respect to \(r\), we have

\[
(2.2) \quad -\Delta u^*_r = \left(\lambda^* f'(u^*) - \frac{N - 1}{r^2}\right) u^*_r, \quad \text{for all } r \in (0, 1).
\]

**Proposition 2.1.** Let \(N \geq 3\) and \(\Psi : \overline{B_1 \setminus \{0\}} \to \mathbb{R}\) be a radially symmetric function satisfying that there exists \(C > 0\) such that \(|\Psi(r)|/r^2 \leq C\), for every \(0 < r \leq 1\), and

\[
(2.3) \quad \int_{B_1} |\nabla \xi|^2 \, dx \geq \int_{B_1} \Psi \xi^2 \, dx
\]

for every \(\xi \in C^\infty(B_1)\) with compact support in \(B_1\).

Then
i) The problem

\[
\begin{cases}
-\Delta \omega(x) = \left( \Psi(x) - \frac{N-1}{|x|^2} \right) \omega(x) & \text{in } B_1, \\
\omega(x) = 1 & \text{on } \partial B_1,
\end{cases}
\]

has an unique solution \( \omega \in W^{1,2}(B_1) \). Moreover \( \omega \) is radial and strictly positive in \( B_1 \setminus \{0\} \).

ii) If \( \Psi_1 \leq \Psi_2 \) in \( B_1 \setminus \{0\} \) satisfy the above hypotheses and \( \omega_i \) (\( i = 1, 2 \)) are the solutions of the problems \( (P_{\Psi_i}) \) then \( \omega_1 \leq \omega_2 \) in \( B_1 \setminus \{0\} \).

Proof. i) By Hardy’s inequality

\[
\int_{B_1} |\nabla \xi|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{B_1} \frac{\xi^2}{|x|^2} \, dx,
\]

for every \( \xi \in C^\infty(B_1) \) with compact support in \( B_1 \), we can define the functional \( I : X \to \mathbb{R} \) by

\[
I(\omega) := \frac{1}{2} \int_{B_1} |\nabla \omega|^2 \, dx - \frac{1}{2} \int_{B_1} \left( \Psi - \frac{N-1}{|x|^2} \right) \omega^2 \, dx,
\]

for every \( \omega \in X \), where \( X = \left\{ \omega : B_1 \to \mathbb{R} \text{ such that } \omega - 1 \in W^{1,2}_0(B_1) \right\} \).

It is immediate that

\[
I'(\omega)(v) = \int_{B_1} \nabla \omega \nabla v \, dx - \int_{B_1} \left( \Psi - \frac{N-1}{|x|^2} \right) \omega v \, dx; \quad \omega \in X, v \in W^{1,2}_0(B_1).
\]

Therefore to prove the existence of a solution of \( (P_{\Psi}) \) it is sufficient to show that \( I \) has a global minimum in \( X \). To do this, we first prove that \( I \) is bounded from below in \( X \). Taking \( v = \omega - 1 \) in (2.3) and applying Cauchy-Schwarz inequality we obtain

\[
I(\omega) \geq \frac{1}{2} \int_{B_1} \Psi(\omega - 1)^2 \, dx - \frac{1}{2} \int_{B_1} \left( \Psi - \frac{N-1}{|x|^2} \right) \omega^2 \, dx =
\]

\[
= \frac{1}{2} \int_{B_1} \Psi(-2\omega + 1) \, dx + \frac{1}{2} \int_{B_1} \frac{N-1}{|x|^2} \omega^2 \, dx
\]

\[
\geq \frac{1}{2} \int_{B_1} \frac{-C(2|\omega| + 1) + (N-1)\omega^2}{|x|^2} \, dx \geq \frac{1}{2} \int_{B_1} \frac{-C - C^2}{|x|^2} \, dx.
\]

Hence \( I \) is bounded from below in \( X \). Take \( \{w_n\} \subset X \) such that \( I(\omega_n) \to \inf I \). Let us show that \( \{w_n\} \) is bounded in \( W^{1,2} \). To this end, taking into account the above inequalities and that \( -C(2|s| + 1) + (N-1)s^2 \geq -C(2|s| + 1) + 2s^2 \geq s^2 - C - C^2 \) for every \( N \geq 3 \) and \( s \in \mathbb{R} \), we have
From the definition of $\Psi$ it clearly implies that $\{\omega_n\}_{n=1}^\infty$, which implies that $\limsup_{n \to \infty} \omega_n = 0$ in $B_1 \setminus \{0\}$, and the boundary condition of the problem.

Finally, to prove that the solution $\omega$ of $\Psi(x) = (N - 1)/|x|^2$ and the boundary condition of the problem.

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0 = I'(\omega)(v) = \int_{B_{r_0}} |\nabla \omega|^2 dx - \int_{B_{r_0}} \Psi \omega^2 dx + \int_{B_{r_0}} \frac{(N-1)\omega^2}{|x|^2} dx
\geq \int_{B_{r_0}} \frac{(N-1)\omega^2}{|x|^2} dx.

Therefore, \omega = 0 in \ B_{r_0}. In particular, \omega(r_0) = \omega'(r_0) = 0 (with radial notation), which implies, by the uniqueness of the corresponding Cauchy problem, that \omega = 0 in (0, 1]. This contradicts \omega(1) = 1.

ii) Consider the function \( v = (\omega_1 - \omega_2)^+ = \max(0, \omega_1 - \omega_2) \in W^{1,2}_0(B_1) \) in the weak formulation of problem (P\( \Psi_1 \)). We have

\[
0 = \int_{B_1} \left( \nabla \omega_1 \nabla (\omega_1 - \omega_2)^+ - \Psi_1 \omega_1 (\omega_1 - \omega_2)^+ + \frac{(N-1)\omega_1 (\omega_1 - \omega_2)^+}{|x|^2} \right) dx
\]

Consider the same function \( v = (\omega_1 - \omega_2)^+ \) in the weak formulation of problem (P\( \Psi_2 \)). Taking into account that \( \Psi_1 \leq \Psi_2 \) and \( \omega_2 \geq 0 \) we obtain

\[
0 = \int_{B_1} \left( \nabla \omega_2 \nabla (\omega_1 - \omega_2)^+ - \Psi_2 \omega_2 (\omega_1 - \omega_2)^+ + \frac{(N-1)\omega_2 (\omega_1 - \omega_2)^+}{|x|^2} \right) dx
\leq \int_{B_1} \left( \nabla \omega_2 \nabla (\omega_1 - \omega_2)^+ - \Psi_1 \omega_2 (\omega_1 - \omega_2)^+ + \frac{(N-1)\omega_2 (\omega_1 - \omega_2)^+}{|x|^2} \right) dx
\]

Subtracting the above two expressions it is follows that

\[
0 \geq \int_{B_1} |\nabla (\omega_1 - \omega_2)^+|^2 dx - \int_{B_1} \Psi_1 (\omega_1 - \omega_2)^+ dx + \int_{B_1} \frac{(N-1)(\omega_1 - \omega_2)^+}{|x|^2} dx
\geq \int_{B_1} \frac{(N-1)(\omega_1 - \omega_2)^+}{|x|^2} dx.
\]

This implies \( (\omega_1 - \omega_2)^+ = 0 \). Hence \( \omega_1 \leq \omega_2 \), which is our claim. \( \square \)

**Proof of Theorem 1.1** We first prove that \( \lambda^* f'(u^*(r)) \leq \lambda_1/r^2 \) for every \( r \in (0, 1] \). To see this, let \( 0 < \varphi_1 \) be the first eigenfunction of the linear problem \(-\Delta v = \lambda v \) in \( B_1 \subset \mathbb{R}^N \) with Dirichlet conditions \( v = 0 \) on \( \partial B_1 \). Then \( \int_{B_1} |\nabla \varphi_1|^2 = \lambda_1 \int_{B_1} \varphi_1^2 \). By density, for arbitrary \( 0 < r \leq 1 \), we could take in (2.1) the radial function \( \xi = \varphi_1 (\cdot / r) \) in \( B_r \) and \( \xi = 0 \) in \( B_1 \setminus B_r \).

Since \( f' \) is nondecreasing and \( u^* \) is radially decreasing, then \( f'(u^*) \) is radially decreasing. An easy computation shows that

\[
\int_{B_1} |\nabla \xi|^2 = \int_{B_r} |\nabla \xi|^2 = r^{N-2} \int_{B_1} |\nabla \varphi_1|^2 = \lambda_1 r^{N-2} \int_{B_1} \varphi_1^2,
\]

\[
\int_{B_1} \lambda^* f'(u^*) \xi^2 = \int_{B_r} \lambda^* f'(u^*) \xi^2 \geq \lambda^* f'(u^*) \int_{B_r} \xi^2 \int_{B_r} \xi^2 = \lambda^* f'(u^*) r^N \int_{B_1} \varphi_1^2.
\]
Combining this with (2.1) we obtain the desired conclusion. Consequently
\[
\limsup_{r \to 0} r^2 f'(u^*(r)) \leq \lambda_1/\lambda^*.
\]
We now prove that \(\limsup_{r \to 0} r^2 f'(u^*(r)) \geq 2(N - 2)/\lambda^*\). To obtain a contradiction, suppose that there exists \(r_0 \in (0, 1)\) and \(\varepsilon > 0\) such that

\[
\lambda^* f'(u^*(r)) \leq \frac{2(N - 2) - \varepsilon}{r^2},
\]
for every \(r \in (0, r_0]\). Consider now the radial function \(\omega(r) := u^*_r(r_0 r)/u^*_r(r_0)\), defined in \(\overline{B_1}\backslash\{0\}\). Applying (2.2), an easy computation shows that \(\omega(1) = 1\) and

\[
-\Delta \omega(r) = \frac{1}{u^*_r(r_0)} r^2_0 (\Delta (u^*_r(r_0)))
= \frac{1}{u^*_r(r_0)} r^2_0 \left( \lambda^* f'(u^*(r_0)) - \frac{N - 1}{r^2} u^*_r(r_0) \right) = \left( \Psi(r) - \frac{N - 1}{r^2} \right) \omega(r),
\]
for every \(r \in (0, 1)\), where \(\Psi(r) := r^2_0 \lambda^* f'(u^*(r_0))\). From (2.4) we obtain \(\Psi(r) \leq \Psi_2(r) := \frac{2(N - 2) - \varepsilon}{r^2}\) for every \(r \in (0, 1]\). It is easy to check that the solution \(\omega_2\) of the problem \((P_{\Psi_2})\) is given by \(w_2(r) = r^\alpha\) \((0 < r \leq 1)\) where

\[
\alpha = \frac{2 - N + \sqrt{(N - 4)^2 + 4\varepsilon}}{2}.
\]
Therefore, applying Proposition 2.1 we can assert that \(0 < \omega(r) \leq r^\alpha\) for every \(r \in (0, 1]\). It is clear that \(\alpha > -1\). Hence \(\omega \in L^1(0, 1]\). This gives \(u^*_r \in L^1(0, r_0)\), which contradicts the unboundedness of \(u^*\).

\[\square\]

\textbf{Lemma 2.2.} Let \(N \geq 10\) and \(0 < A < B \leq 1\). Define the radial function

\[
\Psi_{A, B} : \overline{B_1} \backslash \{0\} \to \mathbb{R}
\]

by

\[
\Psi_{A, B}(r) := \begin{cases} 
0 & \text{if } 0 < r < A \\
\frac{2(N - 2)}{r^2} & \text{if } A \leq r \leq B, \\
0 & \text{if } B < r \leq 1.
\end{cases}
\]

Let \(\omega[A, B]\) be the unique radial solution of \((P_{\Psi_{A, B}})\). Then

\[
\lim_{s \to 0} \int_0^1 \omega[se^{-1/s^3}, s](r)dr = +\infty.
\]

\textbf{Proof.} We first observe that since \(N \geq 10\) we have \(2(N - 2) \leq (N - 2)^2/4\). Hence \(0 \leq \Psi_{A, B} \leq (N - 2)^2/(4r^2)\) for every \(0 < r \leq 1\). Thus, by Hardy’s inequality, \(\Psi_{A, B}\) satisfies (2.3) and we can apply Proposition 2.1.

We check at once that

[Details of the proof are omitted for brevity.]
\[
\omega[A, B](r) = \begin{cases} 
\frac{N(N-4)B^{N-2}A^{-2} r}{(N-2)^2B^{N-4} - 4A^{N-4} + 2(N-2)BN(B^{N-4} - A^{N-4})} & \text{if } 0 \leq r < A, \\
\frac{N(N-2)B^{N-2} r^{-1} - 2NA^{N-4}B^{N-2}A^{-N} r^{-1}}{(N-2)^2B^{N-4} - 4A^{N-4} + 2(N-2)BN(B^{N-4} - A^{N-4})} & \text{if } A \leq r \leq B, \\
\frac{(N(N-2)B^{N-4} - 4A^{N-4}) r + 2(N-2)BN(B^{N-4} - A^{N-4}) r^{-1-N}}{(N-2)^2B^{N-4} - 4A^{N-4} + 2(N-2)BN(B^{N-4} - A^{N-4})} & \text{if } B < r \leq 1.
\end{cases}
\]

To see that \( \omega[A, B] \) is the solution of \((P_{\Psi_{A,B}})\) it suffices to observe that \( \omega[A, B] \in C^1(\overline{B}_1 \setminus \{0\}) \cap W^{1,2}(B_1) \) satisfies pointwise \((P_{\Psi_{A,B}})\) if \(|x| \neq A, B\).

On the other hand, taking into account that \( r^{3-N} \leq A^{4-N} r^{-1} \) if \( A \leq r \leq B \), we have that

\[
\omega[A, B](r) \geq \frac{N(N-2)B^{N-2} r^{-1} - 2NA^{N-4}B^{N-2}A^{4-N} r^{-1}}{(N-2)^2B^{N-4} - 4A^{N-4} + 2(N-2)BN(B^{N-4} - A^{N-4})} \\
\geq \frac{N(N-2)B^{N-2} r^{-1} - 2NA^{N-4}B^{N-2}A^{4-N} r^{-1}}{(N-2)^2B^{N-4} + 2(N-2)BNB^{N-4}} \\
= \frac{N(N-4)B^{2} r^{-1}}{(N-2)^2 + 2(N-2)BN}, \quad \text{if } A \leq r \leq B.
\]

From this and the positiveness of \( \omega[A, B] \) it follows that

\[
\int_0^1 \omega[A, B](r) \geq \int_A^B \omega[A, B](r)dr \geq \int_A^B \frac{N(N-4)B^{2} r^{-1}}{(N-2)^2 + 2(N-2)BN}dr \\
= \frac{N(N-4)B^{2} \log(B/A)}{(N-2)^2 + 2(N-2)BN}.
\]

Taking in this inequality \( A = se^{-1/s^3}, B = s \) (for arbitrary \( 0 < s \leq 1 \)), it may be concluded that

\[
\int_0^1 \omega[se^{-1/s^3}, s](r)dr \geq \frac{N(N-4)}{s ((N-2)^2 + 2(N-2)s^N)}
\]

and the lemma follows. \( \square \)

**Proposition 2.3.** Let \( N \geq 10 \) and \( \varphi : (0, 1) \rightarrow \mathbb{R}^+ \) such that \( \lim_{r \to 0} \varphi(r) = +\infty \). Then there exists \( \Psi \in C^\infty(\overline{B}_1 \setminus \{0\}) \) an unbounded radially symmetric decreasing function satisfying

i) \( 0 < \Psi(r) \leq \frac{2(N-2)}{r^2} \) and \( \Psi'(r) < 0 \) for every \( 0 < r \leq 1 \).

ii) \( \liminf_{r \to 0} \frac{\varphi(r)}{\Psi(r)} = 0, \limsup_{r \to 0} r^2\Psi(r) = 2(N-2) \).
iii) \( \int_0^1 \omega(r)dr = +\infty \), where \( \omega \) is the radial solution of \((P\psi)\).

**Proof.** Without loss of generality we can assume that \( \varphi(r) \leq 2(N-2)/r^2 \) for \( r \in (0,1] \), since otherwise we can replace \( \varphi \) with \( \overline{\varphi} = \min \{\varphi, 2(N-2)/r^2\} \). It is immediate that \( \lim_{r \to 0} \varphi(r) = +\infty \) implies \( \lim_{r \to 0} \overline{\varphi}(r) = +\infty \) and that \( 0 \leq \liminf_{r \to 0} \Psi(r)/\overline{\varphi}(r) \).

We begin by constructing by induction two sequence \( \{x_n\}, \{y_n\} \subset (0,1] \) in the following way: \( x_1 = 1 \) and, knowing the value of \( x_n \) (\( n \geq 1 \)), take \( y_n \) and \( x_{n+1} \) such that

\[
x_{n+1} < y_n < x_n e^{-1/x_n^3} < x_n,
\]

where \( y_n \in (0, x_n e^{-1/x_n^3}) \) is chosen such that

\[
\varphi(y_n) > (n+1) \frac{2(N-2)}{(x_n e^{-1/x_n^3})^2},
\]

which is also possible since \( \lim_{r \to 0} \varphi(r) = +\infty \). The inequality \( x_{n+1} < x_n e^{-1/x_n^3} \) for every integer \( n \geq 1 \) implies that \( \{x_n\} \) is a decreasing sequence tending to zero as \( n \) goes to infinity. For this reason, to construct the radial function \( \Psi \) in \( B_1 \setminus \{0\} \), it suffices to define \( \Psi \) in every interval \( [x_{n+1}, x_n) = [x_{n+1}, y_n) \cup [y_n, x_n e^{-1/x_n^3}] \cup (x_n e^{-1/x_n^3}, x_n) \).

First, we define

\[
\Psi(r) := \frac{2(N-2)}{r^2}, \quad \text{if} \quad x_n e^{-1/x_n^3} < r < x_n,
\]

\[
\Psi(y_n) := \frac{\varphi(y_n)}{n+1}.
\]

By the definition of \( y_n \) we have that

\[
\Psi(y_n) = \frac{\varphi(y_n)}{n+1} > \frac{2(N-2)}{(x_n e^{-1/x_n^3})^2} \quad \text{and} \quad \Psi(y_n) < \varphi(y_n) \leq \frac{2(N-2)}{y_n^2}.
\]

Thus, it is a simple matter to see that it is possible to take a decreasing function \( \Psi \) in \( (y_n, x_n e^{-1/x_n^3}] \) such that \( \Psi(r) < 2(N-2)/r^2 \) and \( \Psi'(r) < 0 \) for \( r \in (y_n, x_n e^{-1/x_n^3}] \) and \( \Psi \in C^\infty([y_n, x_n]) \).

Finally, we will define similarly \( \Psi \) in \( [x_{n+1}, y_n) \). Taking into account that

\[
\Psi(y_n) < \varphi(y_n) \leq \frac{2(N-2)}{y_n^2} < \frac{2(N-2)}{x_{n+1}^2},
\]

we see at once that it is possible to take a decreasing function \( \Psi \) in \( [x_{n+1}, y_n) \) such that

\[
\Psi(x_{n+1}) = \frac{2(N-2)}{x_{n+1}^2},
\]
\[ \partial^{(k)}_r \Psi(x_{n+1}) = \partial^{(k)}_r (2(N - 2)/r^2) (x_{n+1}), \quad \text{for every } k \geq 1, \]

\[ \Psi(r) < 2(N - 2)/r^2 \quad \text{and} \quad \Psi'(r) < 0 \quad \text{for } r \in (x_{n+1}, y_n), \]

\[ \Psi \in C^\infty([x_{n+1}, x_n]). \]

Once we have constructed the radial function \( \Psi \) it is evident that \( \Psi \in C^\infty(B_1 \setminus \{0\}) \) an unbounded radially symmetric decreasing function satisfying i).

To prove ii) it is sufficient to observe that the sequences \( \{x_n\}, \{y_n\} \) tend to zero and satisfy \( x_n^2 \Psi(x_n) = 2(N - 2) \) and \( \Psi(y_n)/\varphi(y_n) = 1/(n + 1) \) for every integer \( n \geq 1 \).

It remains to prove iii). To this end consider an arbitrary \( K > 0 \). Since \( \{x_n\} \) tends to zero, applying Lemma 2.2 we can assert that there exists a natural number \( m \) such that

\[ \int_0^1 \omega[x_m e^{-1/x_m^3}, x_m](r)dr \geq K. \]

Observe that \( \Psi \geq \Psi x_m e^{-1/x_m^3, x_m} \). By Proposition 2.1 it follows that \( \omega \geq \omega[x_m e^{-1/x_m^3}, x_m] \). Thus

\[ \int_0^1 \omega(r)dr \geq \int_0^1 \omega[x_m e^{-1/x_m^3}, x_m](r)dr \geq K. \]

Since \( K > 0 \) is arbitrary we conclude \( \int_0^1 \omega(r)dr = +\infty \). \( \square \)

**Proof of Theorem 1.2** Consider the function \( \Psi \) of Proposition 2.3 and let \( \omega \) be the radial solution of \( (P_\Psi) \). Since \( \Psi \in C^\infty(B_1 \setminus \{0\}) \) we obtain \( \omega \in C^\infty(B_1 \setminus \{0\}) \cap W^{1,2}(B_1) \). Define the radial function \( u \) by

\[ u(r) := \int_r^1 \omega(t)dt, \quad 0 < r \leq 1. \]

It is obvious that \( u \in C^\infty(B_1 \setminus \{0\}) \). Since \( u' = -\omega \) (with radial notation), we have \( u \in W^{2,2}(B_1) \subset W^{1,2}(B_1) \). Moreover, from \( \int_0^1 \omega(r)dr = +\infty \) we see that \( u \) is unbounded.

On the other hand, since \( u' = -\omega < 0 \) in \( (0, 1] \) (by Proposition 2.1), it follows that \( u \) is a decreasing \( C^\infty \) diffeomorphism between \( (0, 1] \) and \( [0, +\infty) \). Therefore we can define \( f \in C^\infty([0, +\infty)) \) by

\[ f := (-\Delta u) \circ u^{-1}. \]

We conclude that \( u \in W^{1,2}_0(B_1) \) is an unbounded solution of \( (P_\lambda) \) for \( \lambda = 1 \).
Now, substituting $u_r$ by $-\omega$ in (2.2) it follows that
\[ -\Delta (-\omega) + f'(u)(-\omega) = \frac{N-1}{r^2}(-\omega) \quad \text{for } 0 < r \leq 1 \]

Hence, since $\omega$ is a solution of $(P_\Psi)$ we obtain $f'(u)\omega = \Psi \omega$ in $(0,1]$. From $\omega > 0$ in $(0,1]$ we conclude that
\[ f'(u(x)) = \Psi(x) \quad \text{for every } x \in \overline{B_r} \setminus \{0\}. \]

We now prove that $f$ satisfies (1.1). To do this, we first claim that $\omega'(1) \geq -1$. Since $\Psi \leq 2(N-2)/r^2$, applying Proposition 2.1 with $\Psi_1 = \Psi$ and $\Psi_2 = 2(N-2)/r^2$, we deduce $\omega_1 \leq \omega_2$, where $\omega_1 = \omega$ and $\omega_2 = r^{-1}$, as is easy to check. Since $\omega_1(1) = \omega_2(1)$ it follows $\omega_1'(1) \geq \omega_2'(1) = -1$, as claimed.

Thus
\[ f(0) = f(u(1)) = -\Delta u(1) = -u''(1) - (N-1)u'(1) \geq \omega'(1) + (N-1)\omega(1) \geq (-1) + (N-1) > 0. \]

On the other hand, since $f'(u(r)) = \Psi(r) > 0$ for every $r \in (0,1]$ it follows $f' > 0$ in $[0, +\infty)$. Moreover $\lim_{s \to +\infty} f'(s) = \lim_{r \to 0} f'(u(r)) = \lim_{r \to 0} \Psi(r) = +\infty$, and the superlinearity of $f$ is proven. Finally, to show the convexity of $f$, it suffices to differentiate the expression $f'(u) = \Psi$ with respect to $r$ (with radial notation), obtaining $u'(r)f''(u(r)) = \Psi'(r)$ in $(0,1]$. Since $u' < 0$ and $\Psi' < 0$ we obtain $f''(u(r)) > 0$ in $(0,1]$, which gives the convexity of $f$ in $[0, +\infty)$.

Finally, we show that $u$ is a stable solution of $(P_\lambda)$ for $\lambda = 1$. Since $N \geq 10$ then $2(N-2) \leq (N-2)^2/4$, hence
\[ f'(u(r)) = \Psi(r) \leq \frac{2(N-2)}{r^2} \leq \frac{(N-2)^2}{4r^2} \quad \text{for every } 0 < r \leq 1. \]

Thus, by Hardy’s inequality, we conclude that $u$ is a stable solution of $(P_\lambda)$ for $\lambda = 1$.

On the other hand, in [2, Th. 3.1] it is proved that if $f$ satisfies (1.1) and $u \in W_0^{1,2}(\Omega)$ is an unbounded stable weak solution of $(P_\lambda)$ for some $\lambda > 0$, then $u = u^*$ and $\lambda = \lambda^*$. Therefore we conclude that $\lambda^* = 1$, $u^* = u$ and
\[ \liminf_{r \to 0} \frac{f'(u^*(r))}{\varphi(r)} = \liminf_{r \to 0} \frac{\Psi(r)}{\varphi(r)} = 0. \]

\[ \square \]

**Proof of Theorem 1.3** Take $\varphi(r) = 1/r^2$, $0 < r \leq 1$, and consider the function $\Psi$ of Proposition 2.3 Define
\[ \Phi(r) := \frac{C_2 - C_1}{2(N-2)} \Psi(r) + \frac{C_1}{r^2}, \]
for every $0 < r \leq 1$. Then it follows easily that $\Phi \in C^{\infty}(\overline{B_1} \setminus \{0\})$ is an unbounded radially symmetric decreasing function satisfying

1. $\Psi(r) \leq \Phi(r) \leq \frac{(N-2)^2}{4r^2}$ and $\Phi'(r) < 0$ for every $0 < r \leq 1$.
2. $\liminf_{r \to 0} r^2\Phi(r) = C_1$, $\limsup_{r \to 0} r^2\Phi(r) = C_2$.
3. $\int_0^1 \varpi(r)dr = +\infty$, where $\varpi$ is the radial solution of $(P_\Psi)$.

Note that iii) follows from Proposition 2.1, Proposition 2.3 and the fact that $\varpi \geq \omega$, being $\omega$ the radial solution of $(P_\Psi)$.

The rest of the proof runs as before.

Proof of Theorem 1.4. Since $0 < \Psi \leq (N-2)^2/(4r^2)$ we have that $\Psi$ satisfies the hypothesis of Proposition 2.1. Thus we can consider the solution $\omega$ of the problem $(P_\Psi)$. From $\Psi \in C(\overline{B_1} \setminus \{0\})$ it follows that $\omega \in C^2(\overline{B_1} \setminus \{0\}) \cap W^{1,2}(B_1)$. On the other hand, since $\Psi(r) \geq \Psi_1(r) := 2(N-2)/r^2$ for $0 < r \leq 1$, we have that $\omega(r) \geq \omega_1(r) := r^{-1}$ for $0 < r \leq 1$, where have used that $\omega_1$ is the solution of $(P_{\Psi_1})$ and we have applied Proposition 2.1. Define the radial function $u$ by

$$u(r) := \int_r^1 \varpi(t)dt, \quad 0 < r \leq 1.$$ 

Analysis similar to that in the proof of Theorem 1.2 shows that $u \in W^{2,2}$ is a decreasing $C^\infty$ diffeomorphism between $(0, 1]$ and $[0, +\infty)$. Defining again $f := (-\Delta) \circ u^{-1}$, it is obtained that $f \in C^{\infty}((0, +\infty))$. Thus $u \in W^{1,2}(B_1)$ is an unbounded solution of $(P_\lambda)$ for $\lambda = 1$. It remains to prove that $f$ satisfies (1.1). At this point, the only difference with respect to the proof of Theorem 1.2 is that $\Phi(r) \leq \Psi_2(r) := (N-2)^2/(4r^2)$ implies that $\varpi \leq \omega_2$, being $\omega_2(r) = r^{-N/2+\sqrt{N-1}+1}$ the solution of the problem $(P_{\Psi_2})$. Hence $\varpi'(1) \geq \omega_2'(1) = -N/2 + \sqrt{N-1} + 1$. Therefore

$$f(0) = f(u(1)) = -\Delta u(1) = -u''(1) - (N-1)u'(1) = \varpi'(1) + (N-1)\varpi(1) \geq (-N/2 + \sqrt{N-1} + 1) + (N-1) > 0.$$ 

The rest of the proof runs as before. 

Proof of Theorem 1.4. Since $0 < \Psi \leq (N-2)^2/(4r^2)$ we have that $\Psi$ satisfies the hypothesis of Proposition 2.1. Thus we can consider the solution $\omega$ of the problem $(P_\Psi)$. From $\Psi \in C(\overline{B_1} \setminus \{0\})$ it follow that $\omega \in C^2(\overline{B_1} \setminus \{0\}) \cap W^{1,2}(B_1)$. On the other hand, since $\Psi(r) \geq \Psi_1(r) := 2(N-2)/r^2$ for $0 < r \leq 1$, we have that $\omega(r) \geq \omega_1(r) := r^{-1}$ for $0 < r \leq 1$, where have used that $\omega_1$ is the solution of $(P_{\Psi_1})$ and we have applied Proposition 2.1. Define the radial function $u$ by

$$u(r) := \int_r^1 \omega(t)dt, \quad 0 < r \leq 1.$$ 

Therefore $u(r) \geq |\log r|$ for $0 < r \leq 1$. In particular, $u$ is unbounded. From been proved, it follows that $u \in C^3(\overline{B_1} \setminus \{0\}) \cap W^{2,2}(B_1)$. Hence (with radial notation) we have that $u$ is a decreasing $C^3$ diffeomorphism between $(0, 1]$ and $[0, +\infty)$. Thus we can define $f \in C^1((0, +\infty))$ by
BEHAVIOR NEAR THE ORIGIN OF $f'(u^*)$ IN EXTREMAL SOLUTIONS

$$f := (-\Delta u) \circ u^{-1}.$$  

Analysis similar to that in the proof of Theorems 1.2 and 1.3 shows that $f$ satisfies (1.1), $\lambda^* = 1$ and $u = u^*$.

Finally, to prove that $f$ is unique up to a multiplicative constant, suppose that $g$ is a function satisfying (1.1), $\lambda^* = 1$ and $g'(v^*(x)) = \Psi(x)$, for every $x \in \overline{B_1} \setminus \{0\}$, where $v^*$ is the extremal solution associated to $g$. From (2.2) we see that

$$-\Delta v^*_r = \left(g'(v^*) - \frac{N - 1}{r^2}\right) v^*_r, \quad \text{for all } r \in (0, 1].$$

It follows immediately that $v^*_r(1)/v^*_r(1)$ is the solution of the problem $(P_\Psi)$. Since this problem has an unique solution we deduce that $v^*_r(1)/v^*_r(1) = \omega(r) = -u^*_r(r)$, for every $r \in (0, 1]$. Thus $v^*_r = \alpha u^*$ for some $\alpha > 0$, which implies, since $v^*_r(1) = u^*_r(1) = 0$, that $v^* = \alpha u^*$. The proof is completed by showing that

$$g(v^*(x)) = -\Delta v^*(x) = \alpha(-\Delta u^*(x)) = \alpha f(u^*(x)) = \alpha f(v^*(x)/\alpha),$$

for every $x \in \overline{B_1} \setminus \{0\}$ and taking into account that $v^*(\overline{B_1} \setminus \{0\}) = [0, +\infty)$.

□

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