Does General Relativity Require a Metric

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The nexus between the gravitational field and the space-time metric was an essential element in Einstein’s development of General Relativity and led him to his discovery of the field equations for the gravitational field/metric. We will argue here that the metric is in fact an inessential element of this theory and can be dispensed with entirely. Its sole function in the theory was to describe the space-time measurements made by ideal clocks and rods. However, the behavior of model clocks and measuring rods can be derived directly from the field equations of general relativity using the Einstein-Infeld-Hoffmann (EIH) approximation procedure. Therefore one does not need to introduce these ideal clocks and rods and hence has no need of a metric.

I. INTRODUCTION

Both the Newtonian and special relativistic descriptions of dynamical systems require the introduction of absolute objects for their formulation. Newtonian laws use planes of absolute simultaneity and straight lines while special relativistic laws employ light cones and straight lines. These elements are absolute in the sense that they are unaffected by the presence or behavior of any kind of physical system. They also characterize the geometry of Newtonian and special relativistic space-time. In the case of special relativity, a specification of the light cone structure and the time-like straight lines is equivalent to the introduction of a flat Lorentzian metric onto the four dimensional space-time manifold. And finally, these objects are the source of all inertial effects such as Coriolis and centripetal forces.

Einstein’s famous elevator gedanken experiment led him to his Equivalence Principle which asserts the local equivalence of inertial and gravitational effects. This equivalence led him in turn to associate the gravitational field with the space-time metric. His great achievement was to realize that, since the gravitational field is a dynamical object, the metric must therefore also be dynamical. His search for field equations for this gravitational field/metric took a number of twists and turns but in 1915 he published the details of his General Theory of Relativity which included his now well-known field equations. Today these equations are almost universally accepted and in the last few years their experimental testing has left little doubt concerning their validity in the macroscopic realm. Since Einstein first introduced his theory there have been numerous attempts at alternate theories of gravity, none of which have survived. At the same time, the success of Einstein’s ”geometrization” of gravity led him and many other physicists including Weyl, Schrödinger, Kalutza and Klein to attempt a geometrical unification of gravity and electromagnetism. Most of these attempts survive now as relics of a bygone, simpler age when there were only two known fundamental forces.

As originally conceived, general relativity consisted of several disjoint parts. There were the field equations for the gravitational field/metric tensor $g_{\mu\nu}$ together with prescriptions on how to couple this field to other fields, e.g., the electromagnetic field. In addition it was assumed that otherwise free particles followed time-like geodesics determined by the $g_{\mu\nu}$ while light rays followed null geodesics. And finally the metric character of $g_{\mu\nu}$ was used to determine the results of space-time measurements made with so-called ideal rods and clocks through the introduction of the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$$  
(1)

(Here and in what follows Greek indices take the values 0,1,2,3, Latin indices take the values 1,2,3 and I use the Einstein summation convention.) In particular, ideal clocks were assumed to measure the integral of $ds$ along their world line.

At this point several comments are in order. First, the Einstein field equations do not depend on a metric interpretation of $g_{\mu\nu}$: if one requires that these equations be linear in second derivatives of the components of $g_{\mu\nu}$, that they contain no absolute objects, are generally covariant and follow from a variational principle then they are unique modulo the so-called cosmological term. Second, the geodesic equations, even when applied to test masses, can only be approximate since any finite mass will in general radiate in the course of its motion through a gravitational field. There is also the question of what field to use to calculate the geodesics. The whole procedure only makes sense if one uses the background gravitational field/metric in which the mass moves. But because the field equations are highly
non-linear such a separation between an external field and the field of the mass can only be accomplished by the use of some approximation procedure. Furthermore, the geodesic hypothesis can say nothing about the motion of two comparable masses. And finally, there is a problem with the proper time hypothesis - how does one identify an ideal clock. Any real physical clock will be subject to tidal forces which, if they are sufficiently strong, can disrupt or at least perturb its workings and hence will cease to be an ideal clock. Thus any real clock is at best an approximation to an ideal clock.

These of course are not new problems - they have been around as long as general relativity itself. However, what is not well known is that in 1949 Einstein, together with Infeld and Hoffman and later with Infeld (collectively EIH) laid the groundwork for their resolution. For what these authors showed was that the motion of compact sources of the gravitational field followed from the field equations themselves without the need for additional assumptions such as the form of a force law such as one must make in electrodynamics. Even the motion of electrically charged particles was shown to follow from the combined Einstein-Maxwell field equation without the need to postulate the Lorentz force law. In particular it is unnecessary to make the geodesic hypothesis since an approximate form of the geodesic equations follows directly from the field equations. Furthermore, it is possible to construct simple clock models, essentially two compact masses or charges moving around each other in circular orbits, whose dynamics in an external field can again be determined directly from the field equations. As a consequence one does not need to use the metric interpretation of $g_{\mu\nu}$ for any purpose and hence the notion of a metric can be dispensed with entirely in general relativity.

While this view may appear to be quite radical I would characterize it rather as conservative since it sets forth the minimum number of elements needed to formulate general relativity. Furthermore, it is important in understanding a theory, particularly a fundamental theory, to know what is essential to that theory and what is not. This is not to say that the geometrical interpretation is of no use as it is often of heuristic value in affording a picture of what is going on in a particular situation. And of course many of the concepts of differential geometry such as Killing vectors, Lie derivatives etc. are extremely useful in the analysis of particular gravitational fields. And finally, it does not detract in any way from Einstein’s great achievement in constructing the general theory of relativity.

In what follows I will outline briefly the main ideas behind the EIH procedure and how one can use it to determine the behavior of simple clock models in an external gravitational field. Since most of this material is already in the literature I will try to avoid unnecessary details as much as possible.

II. EQUATIONS OF MOTION

Perhaps Einstein’s least well known achievement, but arguably one of his most important, is the work he did with L. Infeld and B. Hoffmann on the equations of motion of sources of the gravitational field. General relativity is unique among field theories in that one does not have to postulate separately the equations of motion of the sources of the fields they produce. In classical electrodynamics one must postulate not only the Lorentz force but the equations of motion in which it appears as well. In general relativity neither of these types of postulates are necessary. Furthermore, in deriving these equations one does not encounter self energy infinities found in other derivations.

The equations of motion can most easily be derived from a form of the field equations given by Landau and Lifshitz which are

$$U_{\mu\nu\rho} = \Theta^{\mu\nu},$$

where $U_{\mu\nu\rho}$ is a so-called superpotential, a function of $g_{\mu\nu}$ and linear in its first derivatives and

$$\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}).$$

In this latter equation $g = \text{det}(g_{\mu\nu})$, $t_{LL}^{\mu\nu}$ is the Landau-Lifshitz energy-stress pseudotensor and $T^{\mu\nu}$ is energy-stress tensor due to the presence of other fields. A contribution to this latter tensor due to the sources of these and the gravitational field is not included because these sources are assumed to be compact and vanish on the closed surface integrals used in the EIH derivation. Because of the antisymmetry of $U_{\mu\nu\rho}$ in $\nu$ and $\rho$, $U_{\mu\nu\rho}$ is a curl whose integral over any closed spatial 2-surface vanishes identically. As a consequence, integration of Eq. (2.1) over such a surface in a $t = \text{constant}$ hypersurface gives

$$\oint (U_{\mu\nu\rho \rho} - \Theta^{\mu\nu}) n_r dS = 0,$$

where $n_r$ is a unit surface normal. It is this last equation that yields the equations of motion of a source when the surface encloses it.
The actual details of deriving these equations requires the use of a number of approximation procedures and the identification of the small parameters associated with the system under consideration. For the types of systems to which one can apply the EIH method, one of these parameters, \( \epsilon_S \), is a characteristic time scale \( T_S \) of the system such as its period, divided by the light travel time \( T_L \) across the system and is equivalent to a characteristic velocity of the system divided by the speed of light. Since this ratio must be small compared to one, the approximation is referred to as a slow-motion approximation. I will not attempt here to give the details of the derivation since they are rather lengthy and refer the interested reader to the references cited below. If one perturbs off a "flat" gravitational field \( (g_{\mu\nu} = \text{diag} \{ 1, -1, -1, -1 \} ) \) as was done in the original EIH papers one obtains, in first order or Newtonian approximation, the equations of motion, including the inertial force term, of compact sources interacting via the \( 1/r^2 \) Newtonian force law. Since there is only one mass parameter per source in these equations one also derives the equivalence of inertial and gravitational mass by these means.

In ref. I the EIH method to derive the approximate laws of motion of compact masses, both charged and uncharged, in a Einstein-deSitter gravitational field given by

\[
g_{\mu\nu} = \text{diag} \{ 1, -R^2(t), -R^2(t), -R^2(t) \}
\]

where \( R(t) = (t/t_0)^{2/3} \) and \( t \) is the cosmic time. In this case a second small parameter \( \epsilon_H \ll \epsilon_S \), the ratio of \( T_S \) divided by the Hubble time \( T_H = R(t)/\dot{R}(t) \) is used in a multi-time approximation scheme. In this scheme the source coordinates are assumed to depend both on the cosmic time \( t \) and a fast time \( \tau = t/\epsilon \) where \( \epsilon = \epsilon_H/\epsilon_S \). The resulting equations of motion in lowest order of approximation in \( \epsilon_S \) and \( \epsilon_H \) take the form

\[
m_A \dot{x}_A + 2 \epsilon m_A R \ddot{x}_A + 2 \epsilon m_A R \dot{x}_A = \frac{1}{R^3} \sum_{B \neq A} \frac{m_A m_B}{r_{AB}^3} r_{AB}
\]

where \( r_{AB} = x_A - x_B \). For a two-body system moving in circular orbits about each other one finds that

\[
\omega = \text{constant} \quad \text{and} \quad rR(t) = \text{constant}
\]

where \( \omega \) is the angular frequency of the motion and \( r \) is the coordinate radius of the orbit. We see that such a system can serve as a clock and such a clock measures the cosmic time \( t \). This result is valid both for charged and uncharged clocks so, in the level of approximation, there is no difference in the time measured by a "gravitational" and an "electrical" clock. However, this will not be true in higher orders of approximation. Unfortunately, the next corrections in \( \epsilon_H \) are of order \( \epsilon_H^2 \) and so are probably not observable at the present time. Note that, as \( R(t) \) increases in time, the coordinate radius of a clock decreases. Depending on whether we choose to take the coordinate radius or the coordinate radius times the scale factor to be a measure of the size of the clock we can say either that the size of the clocks, viewed now as a measuring rod, is decreasing while the size of the universe remains fixed or else that the size of the clocks stays fixed while the size of the universe increases. There is no way to distinguish physically between the two interpretations. With either interpretation however the number of such rods needed to measure the distance to a distant galaxy will increase in time. We also see that small systems such as the solar system or atoms with short time scales compared to the Hubble time will effectively not see the effects of expansion while big ones for with \( \epsilon_H \sim 1 \) will.

**III. CONCLUSIONS**

In this paper I have argued that a metric interpretation is not needed in general relativity and that the purposes for which it was originally introduced, i.e., temporal and spatial measurements and the determination of geodesic paths, can be all be derived from the field equations of this theory by means of the EIH approximation scheme. As a consequence, the only ab initio space-time concept that is required is that of the blank space-time manifold. In this view what general relativity really succeeded in doing was to eliminate geometry from physics. The gravitational field is, again in this view, just another field on the space-time manifold. It is however a very special field since it is needed in order to formulate the field equations for, what other fields are present and hence couples universally with all other fields. It is hoped that this identification of unnecessary concepts in general relativity will help in the discussion of a number of fundamental issues, chief among which is that of construction a quantum theory of gravity.
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