Second homology of generalized periplectic Lie superalgebras

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Abstract

Let \((R, -)\) be an arbitrary unital associative superalgebra with superinvolution over a commutative ring \(k\) with 2 invertible. The second homology of the generalized periplectic Lie superalgebra \(p_m(R, -)\) for \(m \geq 3\) has been completely determined via an explicit construction of its universal central extension. In particular, this second homology could be identified with the first \(\mathbb{Z}/2\mathbb{Z}\)-graded dihedral homology of \(R\) with certain superinvolution whenever \(m \geq 5\).

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1 Introduction

It is well known that the second homology of a Lie (super)algebra \(\mathfrak{g}\) is identified with the kernel of its universal central extension, and thus classifies all central extension of \(\mathfrak{g}\) up to isomorphism (c.f. \([15, 17]\)). They play crucial rules in the theory of Lie (super)algebras.

A remarkable work about the second homology of a Lie algebra is the nice connection between the second homology of a matrix Lie algebra and the first cyclic homology of its coordinates associative algebra established in \([10]\). Concretely, let \(A\) be a unital associative algebra over a commutative ring with 2 invertible, one denotes \(\mathfrak{gl}_n(A)\) the Lie algebra of all \(n \times n\)-matrices with entries in \(A\) under commutator operation and \(\mathfrak{sl}_n(A)\) the derived Lie subalgebra of \(\mathfrak{gl}_n(A)\). It is shown in \([10]\) that the second homology \(H_2(\mathfrak{sl}_n(A))\) with \(n \geq 2\) is isomorphic to the first cyclic homology \(HC_1(A)\).

Such an isomorphism has been extended to many other classes of Lie (super)algebras. For instance, Y. Gao showed in \([6]\) that the second homology of elementary unitary Lie algebra \(\mathfrak{eu}_n(R, -)\) with \(n \geq 5\) is identified with the first skew-dihedral homology of \((R, -)\) that is a unital associative algebra with anti-involution. The super analogue of C. Kassel and J. L. Loday’s work was obtained in \([3, 4]\). The isomorphism from the second homology of the Lie superalgebra \(\mathfrak{sl}_{m|n}(S)\) coordinated by a unital associative superalgebra \(S\) with \(m + n \geq 5\) to the first \(\mathbb{Z}/2\mathbb{Z}\)-graded cyclic homology \(HC_1(S)\) was established. Recent investigation \([2]\) further gave the identification between the second homology of the orthosymplectic Lie superalgebra \(\mathfrak{osp}_{m|2n}(R, -)\) with positive integer pair \((m, n) \neq (1, 1)\) or \((2, 1)\) and the first \(\mathbb{Z}/2\mathbb{Z}\)-grade skew-dihedral homology of \((R, -)\), where \((R, -)\) is a unital associative superalgebra with superinvolution (see \((2.3)\) for definition). A series of deep investigations on the relationship between the homology theory of Lie algebras and the homology theory of associative algebras have been made in \([13, 12]\).

Inspired by above developments, we aim to establish an isomorphism that is analogous to C. Kassel and J. L. Loday’s isomorphism for the generalized periplectic Lie superalgebra \(p_m(R, -)\) coordinatized by a unital associative superalgebra \((R, -)\) with superinvolution. As in Section 2,
a generalized periplectic Lie superalgebra is defined as the derived sub-superalgebra of the Lie superalgebra of all skew-symmetric matrices with respect to certain superinvolution. It could be understood as a super analogue of a unitary Lie algebra introduced in [1]. This family of Lie superalgebras provides us with a realization of an arbitrary generalized root graded Lie superalgebra of type $P(m-1)$ for $m \neq 4$ up to central isogeneous (c.f. [5]), which is a complement to the realization of a root graded Lie superalgebra of type $P(m-1)$ given in [14].

A primary result of this paper is Theorem 5.5 which states that the second homology of the Lie superalgebra $p_m(R, -)$ with $m \geq 5$ is isomorphic to the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, - \circ \rho)$, where $- \circ \rho$ is the superinvolution on $R$ obtained by twisting the superinvolution $-$ with the sign map $\rho$ (see (2.5) in Section 2). In the special case where $R$ is super-commutative, the isomorphism indicates that the second homology of $p_m(k) \otimes_k R$ for a super-commutative superalgebra $R$ is trivial, which was obtained by K. Iohara and Y. Koga in [7, 8]. While the isomorphism also reveals that the second homology of $p_m(R, -)$ is not necessarily trivial if $R$ is not super-commutative.

The methods used in this paper non-suprisingly involve explicitly construction of the universal central extension of $p_m(R, -)$, which will be achieved via introducing the notion of Steinberg periplectic Lie superalgebra $stp_m(R, -)$ in Section 3.

The isomorphism between the second homology of $p_m(R, -)$ and the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, - \circ \rho)$ fails when $m = 3$ or 4. Nonetheless, the second homology of $p_4(R, -)$ and $p_3(R, -)$ will also been explicitly computed in Theorems 6.3 and 7.3.

## 2 Basics on generalized periplectic Lie superalgebras

We briefly review the definition of a generalized periplectic Lie superalgebra and prove a few properties in this section.

Throughout the paper, we always assume $k$ is a commutative base ring with 2 invertible. All modules, associative superalgebras and Lie superalgebras are assumed to be over $k$. Let $R$ be a unital associative superalgebra, in which the parity of $a \in R$ is denoted by $|a|$. Then the associative superalgebra $M_{m|m}(R)$ of all $2m \times 2m$-matrices is also equipped with a $\mathbb{Z}/2\mathbb{Z}$-gradation by setting

$$|e_{ij}(a)| := |i| + |j| + |a|, \quad a \in R, \quad 1 \leq i, j \leq 2m,$$

where $e_{ij}(a)$ is the matrix unit with $a$ at the $(i, j)$-position and 0 elsewhere,

$$|i| = \begin{cases} 0, & \text{if } i \leq m, \\ 1, & \text{if } i > m. \end{cases}$$

This makes $M_{m|m}(R)$ an associative superalgebra.

We assume in addition that $R$ is equipped with a superinvolution\(^1\) $- : R \to R$ that is a $k$-linear map satisfying

$$\overline{ab} = (-1)^{|a||b|} \overline{b} \overline{a}, \text{ and } \overline{a} = a,$$

for homogeneous $a, b \in R$. This further gives rise to a periplectic superinvolution on the associative superalgebra $M_{m|m}(R)$ defined by

$$(A \ B)_{\text{p}}p = \begin{pmatrix} D & -\rho(B) \rho(C) \\ \rho(C) & A \end{pmatrix},$$

where $A, B, C, D$ are $m \times m$-matrices with entries in $R$, $\rho : R \to R$ is the $k$-linear map defined by

$$\rho(a) = (-1)^{|a|}a,$$

\(^1\)The superinvolution on the matrix superalgebra $M_{m|m}(k)$ over a field $k$ of characteristic not 2 were classified in [16]. A superinvolution on $M_{m|m}(k)$ may not exist. Whenever it exists, a superinvolution on $M_{m|m}(k)$ is equivalent to either a periplectic superinvolution or an orthosymplectic superinvolution. This motivates us to define the periplectic superinvolution on $M_{m|m}(R)$ here.
for homogeneous $a \in R$, $\rho(A)$ denotes the matrix $(\rho(a_{ij}))$ and $\overline{A} = (\overline{a_{ij}})$ for $A = (a_{ij})$. In this situation, one defines a Lie superalgebra

$$\hat{p}_m(R, -) := \{X \in M_{m|m}(R)|X^{\operatorname{top}} = -X\},$$

with the standard super-commutator as the super-bracket. Its derived Lie sub-superalgebra

$$p_m(R, -) := [\hat{p}_m(R, -), \hat{p}_m(R, -)]$$

is called the generalized periplectic Lie superalgebra coordinatized by the associative superalgebra with the super-commutator as the super-bracket. Its derived Lie sub-superalgebra is super-commutative. In order to discuss the universal central extension of the generalized periplectic Lie superalgebra $(R, -)$ with superinvolution.

As an example, we consider $R = k$ on which the identity map is a superinvolution. The Lie superalgebra $p_m(k, \text{id})$ coincides with the simple Lie superalgebra of type $P(m-1)$ as defined in \cite{9}. We simply write $p_m(k) := p_m(k, \text{id})$.

If $R$ is super-commutative, there is a natural Lie superalgebra structure on $p_m(k) \otimes_k R$ with the super-bracket

$$[x \otimes a, y \otimes b] = (-1)^{|a||b|}[x, y] \otimes ab,$$

for homogeneous $x, y \in p_m(k)$ and $a, b \in R$. The Lie superalgebra $p_m(k) \otimes_k R$ is actually isomorphic to the generalized periplectic Lie superalgebra $p_m(R, \rho)$, where the $k$-linear map $\rho : R \to R$ is a superinvolution on $R$ when $R$ is super-commutative.

Before going into the discussion on the properties of generalized periplectic Lie superalgebras, we exhibit another example here:

**Example 2.1.** Let $S$ be an arbitrary unital associative superalgebra and $S^{\text{op}}$ denote its opposite superalgebra with the multiplication

$$a^{\text{op}} b = (-1)^{|a||b|}b \cdot a,$$

for homogeneous $a, b \in S$. Then the $k$-linear map

$$\operatorname{ex} : S \oplus S^{\text{op}} \to S \oplus S^{\text{op}}, \quad a \oplus b \mapsto b \oplus a.$$

is a superinvolution on $S \oplus S^{\text{op}}$. In this situation, we have an isomorphism of Lie superalgebras

$$p_m(S \oplus S^{\text{op}}, \operatorname{ex}) \cong \mathfrak{gl}_{m|m}(S) := [\mathfrak{gl}_{m|m}(S), \mathfrak{gl}_{m|m}(S)], \quad m \geq 1,$$

where $\mathfrak{gl}_{m|m}(S)$ is the Lie superalgebra of $2m \times 2m$-matrices with entries in $S$.

**Proof.** In fact, the Lie superalgebra $\hat{p}_m(S \oplus S^{\text{op}}, \operatorname{ex})$ is isomorphic to the Lie superalgebra $\mathfrak{gl}_{m|m}(S)$, where an explicit isomorphism $\mathfrak{gl}_{m|m}(S) \to \hat{p}_m(S \oplus S^{\text{op}}, \operatorname{ex})$ is given as follows:

$$e_{i,j}(a) \mapsto e_{i,j}(a) + e_{m+i,j}(0 \oplus a),$$

$$e_{i,m+j}(a) \mapsto e_{i,m+j}(a) + e_{j,m+i}(0 \oplus \rho(a)),$$

$$e_{m+i,j}(a) \mapsto e_{m+i,j}(a) - e_{i,j}(0 \oplus \rho(a)),$$

$$e_{m+i,m+j}(a) \mapsto -e_{j,i}(0 \oplus a) + e_{m+i,m+j}(a \oplus 0),$$

for $a \in S$ and $1 \leq i, j \leq m$. Taking their derived Lie sub-superalgebras, we conclude that the Lie superalgebra $p_m(S \oplus S^{\text{op}}, \operatorname{ex})$ is isomorphic to the Lie superalgebra $\mathfrak{gl}_{m|m}(S)$.

It is known from \cite{15} that a Lie superalgebra admits a universal central extension if and only if it is perfect. In order to discuss the universal central extension of the generalized periplectic Lie superalgebra $p_m(R, -)$, we explore the perfectness of $p_m(R, -)$. We will use the following notation:

$$l_{ij}(a) := e_{ij}(a) - e_{m+j,m+i}(\bar{a}),$$

$$f_{ij}(a) := e_{i,m+j}(a) + e_{j,m+i}(\rho(\bar{a})),$$

$$g_{ij}(a) := e_{m+i,j}(a) - e_{m+j,i}(\rho(\bar{a})).$$

We always denote $R_{(k)} := \{a \in R|\bar{a} = \pm \rho(a)\}$. 

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Lemma 2.2. For $m \geq 2$, every element of $x \in \tilde{p}_m(R, -)$ is written as
\[
x = t_{11}(a) + \sum_{i=2}^{m} (t_{ii}(a_i) - t_{11}(a_i)) + \sum_{1 \leq i < j \leq m} t_{ij}(a_{ij}) + \sum_{i=1}^{m} (e_{i,m+1}(b_i) + e_{m+1,i}(c_i)) + \sum_{1 \leq i < j \leq m} (f_{ij}(b_{ij}) + g_{ij}(c_{ij})),
\]
(2.15)
where $a, a_i, a_{ij}, b_j, c_i, j \in R, b \in R_+$ and $c_i \in R_{(-)}$ are uniquely determined by $x$. Moreover, such an element $x$ is contained in $p_m(R, -) = [\tilde{p}_m(R, -), \tilde{p}_m(R, -)]$ if and only if $a \in [R, R] + R_{(-)}$.

Proof. The first statement follows from the definition of $\tilde{p}_m(R, -)$. We show that $x \in p_m(R, -)$ if and only if $a \in [R, R] + R_{(-)}$.

We observe that each term on the right hand side of (2.15) except $t_{11}(a)$ is a super-commutator of two elements in $\tilde{p}_m(R, -)$, i.e., they are contained in $p_m(R, -) = [\tilde{p}_m(R, -), \tilde{p}_m(R, -)]$. Hence, it suffices to show that $t_{11}(a) \in p_m(R, -)$ if and only if $a \in [R, R] + R_{(-)}$.

If $a \in [R, R]$, we write $a = \sum [a_i', a_i'']$ with $a_i', a_i'' \in R$, then
\[
t_{11}(a) = \sum [t_{11}(a'), t_{11}(a'')] \in p_m(R, -).
\]
While an element $a \in R_{(-)}$ satisfies that
\[
t_{11}(a) + t_{22}(a) = t_{11}(a) - t_{22}(\rho(a)) = [f_{12}(a), g_{21}(1)] \in p_m(R, -).
\]
Combining with $t_{22}(a) - t_{11}(a) \in p_m(R, -)$ and $\frac{1}{2} \in k$, we conclude that $t_{11}(a) \in p_m(R, -)$. This shows that $t_{11}(a) \in p_m(R, -)$ if $a \in [R, R] + R_{(-)}$.

For the inverse implication, we observe that every element
\[
\left( \begin{array}{cc} A & B \\ C & -A \end{array} \right) \in p_m(R, -) = [\tilde{p}_m(R, -), \tilde{p}_m(R, -)]
\]
satisfies $\text{Tr}(A) \in [R, R] + R_{(-)}$. Hence, $a \in [R, R] + R_{(-)}$ if $t_{11}(a) \in p_m(R, -)$. \qed

Proposition 2.3. Let $(R, -)$ be a unital associative superalgebra with superinvolution and $m \geq 2$.

(i) There is an exact sequence of Lie superalgebras
\[
0 \to p_m(R, -) \to \tilde{p}_m(R, -) \to \frac{R}{[R, R] + R_{(-)}} \to 0.
\]

(ii) The Lie superalgebra $p_m(R, -)$ is generated by $t_{ij}(a), f_{ij}(a), g_{ij}(a)$ for $a \in R$, $1 \leq i \neq j \leq m$.

(iii) If $m \geq 3$, then the Lie superalgebra $p_m(R, -)$ is perfect, i.e.,
\[
p_m(R, -) = [p_m(R, -), p_m(R, -)].
\]

Proof. (i) We define a surjective $k$–linear map
\[
\eta : \tilde{p}_m(R, -) \to \frac{R}{[R, R] + R_{(-)}}, \left( \begin{array}{cc} A & B \\ C & -A \end{array} \right) \mapsto \text{Tr}(A) + ([R, R] + R_{(-)}).
\]
By Lemma 2.2, $\ker \eta = p_m(R, -)$. Hence, we obtain an exact sequence of $k$–modules:
\[
0 \to p_m(R, -) \to \tilde{p}_m(R, -) \to \frac{R}{[R, R] + R_{(-)}} \to 0.
\]
Note that $R/([R, R] + R_{(-)})$ is a super-commutative Lie superalgebra, we obtain that all $k$–linear maps appearing in this exact sequence are homomorphisms of Lie superalgebras.

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(ii) By Lemma 2.2, it suffices to show \( t_{11}(a) \) with \( a \in [R, R] + R(-) \), \( t_{ii}(a) - t_{11}(a) \) with \( a \in R, 2 \leq i \leq m \), \( e_{i,m+i}(a) \) with \( a \in R_{(i+1)}, 1 \leq i \leq m \), and \( e_{m+i,i}(a) \) with \( a \in R, 1 \leq i \leq m \) can be generated by \( t_{ij}(b), f_{ij}(b) \) and \( g_{ij}(b) \) for \( b \in R \) and \( 1 \leq i \neq j \leq m \). Indeed,

\[
t_{ii}(a) - t_{11}(a) = [t_{ii}(a), t_{11}(1)], \quad e_{i,m+i}(a) = \frac{1}{2}[t_{ij}(1), f_{ji}(a)], \quad \text{and} \quad e_{m+i,i}(a) = \frac{1}{2}[g_{ij}(a), t_{ji}(1)],
\]

where \( 1 \leq j \leq m \) is chosen such that \( i \neq j \). Furthermore, for \( a \in [R, R] + R(-) \), we also have

\[
t_{11}(a') a'' = \frac{1}{2}[t_{12}(1), t_{21}(a'')] a'' - (-1)^{i''[a'']} [t_{12}(1), t_{21}(a'') a'], \quad a', \ a'' \in R
\]

This proves (ii).

(iii) By (ii), \( p_m(R, -) \) is generated by \( t_{ij}(a), f_{ij}(a), g_{ij}(a) \) with \( a \in R \) and \( 1 \leq i \neq j \leq m \). We shall show that these elements are also contained in \( p_m(R, -), p_m(R, -) \). Note that \( m \geq 3 \), for \( 1 \leq i \neq j \leq m \), we may choose \( 1 \leq k \leq m \) such that \( i, j, k \) are distinct. Then the equalities

\[
t_{ij}(a) = [t_{ik}(1), t_{kj}(a)], \quad f_{ij}(a) = [t_{ik}(1), f_{kj}(a)], \quad g_{ij}(a) = [g_{ik}(a), t_{kj}(1)]
\]

imply that \( t_{ij}(a), f_{ij}(a), g_{ij}(a) \in [p_m(R, -), p_m(R, -)] \). Hence, \( p_m(R, -) \) is perfect for \( m \geq 3 \).

Remark 2.4. The Lie superalgebra \( p_1(R, -) \) is not necessarily perfect. For instance, if \( R \) is supercommutative, then

\[
\tilde{p}_1(R, \rho) := \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \mid a, b, c \in R \right\}, \quad \text{and} \quad p_1(R, \rho) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid a, b, c \in R \right\}.
\]

The Lie superalgebra \( p_1(R, \rho) \) is not perfect since \( [p_1(R, \rho), p_1(R, \rho)] = 0 \). In general, the condition for the perfectness of \( p_1(R, -) \) is unknown yet.

Similarly, the Lie superalgebra \( p_2(R, -) \) is also not necessarily perfect. Hence, the existence of a universal central extension of \( p_1(R, -) \) or \( p_2(R, -) \) is not ensured. We only consider the second homology of \( p_m(R, -) \) for \( m \geq 3 \).

### 3 Steinberg periplectic Lie superalgebras

In the previous section, we have shown that the Lie superalgebra \( p_m(R, -) \) is perfect for \( m \geq 3 \). The perfectness allows us to further study its universal central extension, whose kernel will finally provides us with the second homology of \( p_m(R, -) \).

In this section, we will introduce the notion of Steinberg periplectic Lie superalgebra \( \mathfrak{sp}_m(R, -) \) and prove that it is a central extension of \( p_m(R, -) \). Its universality will be discussed in the sequent sections.

**Definition 3.1.** Let \((R, -)\) be a unital associative superalgebra with superinvolution and \( m \geq 3 \). The Steinberg periplectic Lie superalgebra coordinatized by \((R, -)\), denoted by \( \mathfrak{sp}_m(R, -) \), is defined to be the abstract Lie superalgebra generated by homogenous elements \( t_{ij}(a), f_{ij}(a), g_{ij}(a) \) with parity \(|a|, |a| + 1, |a| + 1 \) respectively, for homogeneous \( a \in R \) and \( 1 \leq i \neq j \leq m \), subjecting to the relations:

\[
\begin{align*}
t_{ij}, f_{ij}, g_{ij} & \text{ are all } k\text{-linear,} \\
f_{ij}(a) &= f_{ij}(\rho(a)), \\
g_{ij}(a) &= -g_{ij}(\rho(a)), \\
[t_{ij}(a), t_{jk}(b)] &= t_{ik}(ab), \\
[t_{ij}(a), t_{kl}(b)] &= 0,
\end{align*}
\]

for \( i \neq j \), \( i \neq j \), \( i \neq j \), \( \text{for distinct } i, j, k \), and \( i \neq j \neq k \neq l \neq i \).
\[ [t_{ij}(a), f_{jk}(b)] = f_{ik}(ab), \quad \text{for distinct } i, j, k, \quad \text{(STP05)} \]
\[ [t_{ij}(a), t_{lk}(b)] = 0, \quad \text{for } i \neq j \neq k \neq l \neq j, \quad \text{(STP06)} \]
\[ [g_{ij}(a), t_{jk}(b)] = g_{ik}(ab), \quad \text{for distinct } i, j, k, \quad \text{(STP07)} \]
\[ [g_{ij}(a), t_{kl}(b)] = 0, \quad \text{for } l \neq k \neq j \neq i \neq k, \quad \text{(STP08)} \]
\[ [f_{ij}(a), f_{lk}(b)] = 0, \quad \text{for } i \neq j, \text{ and } k \neq l, \quad \text{(STP09)} \]
\[ [g_{ij}(a), g_{kl}(b)] = 0, \quad \text{for } i \neq j, \text{ and } k \neq l, \quad \text{(STP10)} \]
\[ [f_{ij}(a), g_{jk}(b)] = t_{ik}(ab), \quad \text{for distinct } i, j, k, \quad \text{(STP11)} \]
\[ [f_{ij}(a), g_{kl}(b)] = 0, \quad \text{for distinct } i, j, k, \quad \text{(STP12)} \]

where \( a, b \in R \) and \( 1 \leq i, j, k, l \leq m \).

Recall Proposition 2.3 that \( p_m(R, -) \) is generated by \( t_{ij}(a), f_{ij}(a) \) and \( g_{ij}(a) \) for \( a \in R \) and \( 1 \leq i \neq j \leq m \). These generators satisfy all relations (STP00)-(STP12). Hence, there is a canonical homomorphism of Lie superalgebras:

\[ \psi : stp_m(R, -) \rightarrow p_m(R, -), \quad \text{(3.1)} \]

such that \( \psi(t_{ij}(a)) = t_{ij}(a), \psi(f_{ij}(a)) = f_{ij}(a) \) and \( \psi(g_{ij}(a)) = g_{ij}(a) \), which will be demonstrated to be a central extension, i.e., the kernel of \( \psi \) is included in the center of \( stp_m(R, -) \).

It is easy to observe that all diagonal, upper triangular and lower triangular matrices in \( p_m(R, -) \) form three Lie sub-superalgebras of \( p_m(R, -) \), respectively. Their direct sum gives a decomposition of \( p_m(R, -) \). We first show that the Steinberg periplectic Lie superalgebra \( stp_m(R, -) \) also possesses a similar decomposition.

**Lemma 3.2.** In the Lie superalgebra \( stp_m(R, -) \), the following equalities hold:

\[ [t_{ij}(a), f_{jk}(b)] = [t_{ik}(a), f_{ki}(b)], \quad \text{and } [g_{ij}(a), t_{ji}(b)] = [g_{ik}(a), t_{ki}(b)], \]

for \( a, b \in R \) and \( 1 \leq i, j, k \leq m \) with \( i \neq j, k \).

**Proof.** We assume \( i, j, k \) are distinct and deduce from (STP03), (STP05) and (STP06) that

\[ [t_{ik}(a), f_{ki}(b)] = [[t_{ij}(a), t_{jk}(1)], f_{ki}(b)] = [t_{ij}(a), [t_{jk}(1), f_{ki}(b)]] = 0 + [t_{ij}(a), f_{ji}(b)] = [t_{ij}(a), f_{ji}(b)]. \]

Similarly, \( [g_{ij}(a), t_{ji}(b)] = [g_{ik}(a), t_{ki}(b)] \) follows from (STP03), (STP07) and (STP08). □

**Lemma 3.2** permits us to introduce the following well-defined elements of \( stp_m(R, -) \):

\[ f_i(a) := [t_{ij}(1), f_{ji}(a)], \quad \text{for some } j \neq i, \quad \text{(3.2)} \]
\[ g_i(a) := [g_{ij}(a), t_{ji}(1)], \quad \text{for some } j \neq i, \quad \text{(3.3)} \]
\[ h_{ij}(a, b) := [f_j(a), g_{ji}(b)], \quad \text{for } i \neq j, \quad \text{(3.4)} \]

where \( a, b \in R \) and \( 1 \leq i, j \leq m \). One easily deduce that

\[ f_i(\rho(a)) = f_i(a), \quad \text{and } g_i(\rho(a)) = -g_i(a), \quad \text{(3.5)} \]

for \( 1 \leq i \leq m \) and \( a \in R \).

**Proposition 3.3.** The Lie superalgebra \( stp_m(R, -) \) is decomposed as a direct sum of \( k \)-modules:

\[ stp_m(R, -) = stp_m^0(R, -) \oplus stp_m^0(R, -) \oplus stp_m^+(R, -), \quad \text{(3.6)} \]
where

\[
\begin{align*}
stp_0(R^-) &= \text{span}_k \{ h_{ij}(a, b) \mid a, b \in R, 1 \leq i \neq j \leq m \}, \\
stp_+^0(R^-) &= \text{span}_k \{ f_{ij}(a), f_i(a) \mid a \in R, 1 \leq i, j \leq m \text{ and } i < j \}, \\
stp^-_m(R^-) &= \text{span}_k \{ t_{ij}(a), g_{ij}(a), g_k(a) \mid a \in R, 1 \leq i, j, k \leq m \text{ and } i > j \},
\end{align*}
\]

are all Lie sub-superalgebras of \( \stp_m(R^-) \) and \( [\stp_0^0(R^-), \stp_+^0(R^-), \stp^-_m(R^-)] \subseteq \stp_m^+(R^-) \).

**Proof.** We first deduce from (STP00)-(STP12) that \( \stp^0_0(R^-), \stp^-_m(R^-) \) and \( \stp^+_m(R^-) \) are all Lie sub-superalgebras of \( \stp_m(R^-) \) and

\[
[\stp_0^0(R^-), \stp_+^0(R^-)] \subseteq \stp_m^+(R^-).
\]

Next, we denote \( g := \stp_m^0(R^-) + \stp_0^0(R^-) + \stp_+^0(R^-) \) and show that \( \stp_m(R^-) = g \). The \( k \)-module \( g \) is invariant under \( \text{ad}(t_{ij}(a)), \text{ad}(f_{ij}(a)), \text{ad}(g_{ij}(a)) \). Note that \( t_{ij}(a), f_{ij}(a), \text{and } g_{ij}(a) \) with \( a \in R \) and \( 1 \leq i \neq j \leq m \) generate the Lie superalgebra \( \stp_m(R^-) \), we obtain that \( g \) is an ideal of the Lie superalgebra \( \stp_m(R^-) \). It follows that \( \stp_m(R^-) = g \) since \( g \) contains a complete family of generators of \( \stp_m(R^-) \).

Finally, we prove that the summation in the decomposition (3.6) is a direct sum. We claim that the restriction \( \psi|\stp_m^+(R^-) \) of the canonical homomorphism (3.1) is injective. Suppose that \( x^+ \in \stp_m^+(R^-) \) satisfying \( \psi(x^+) = 0 \). Write

\[
x^+ = \sum_{1 \leq i < j \leq m} t_{ij}(a_{ij}) + \sum_{1 \leq i < j \leq m} f_{ij}(b_{ij}) + \sum_i f_i(c_i),
\]

where \( a_{ij}, b_{ij}, c_i \in R \). Applying \( \psi \), we obtain

\[
0 = \psi(x^+) = \sum_{1 \leq i < j \leq m} (t_{ij}(a_{ij}) + f_{ij}(b_{ij})) + \sum_{i=1}^m c_{i,m+i}(c_i + \rho(c_i)) \in p_m(R^-).
\]

It follows that \( a_{ij} = b_{ij} = 0 \) for \( 1 \leq i < j \leq m \) and \( c_i + \rho(c_i) = 0 \) for \( i = 1, \ldots, m \). Now,

\[
x^+ = \sum_{1 \leq i < j \leq m} t_{ij}(a_{ij}) + \sum_{1 \leq i < j \leq m} f_{ij}(b_{ij}) + \sum_i f_i(c_i)
\]

\[
= \sum_{1 \leq i < j \leq m} t_{ij}(a_{ij}) + \sum_{1 \leq i < j \leq m} f_{ij}(b_{ij}) + \frac{1}{2} \sum_i (f_i(c_i) + f_i(\rho(c_i))) = 0,
\]

since \( f_i(\bar{c}) = f_i(\rho(c)) \). Hence, \( \psi|\stp^+_m(R^-) \) is injective. Similarly, \( \psi|\stp^-_m(R^-) \) is injective. Now, if \( 0 = x^- + x^0 + x^+ \) for \( x^0 \in \stp^0_0(R^-) \) and \( x^+ \in \stp^+_m(R^-) \), then

\[
0 = \psi(x^-) + \psi(x^0) + \psi(x^+) \in p_m(R^-),
\]

in \( p_m(R^-) \), where \( \psi(x^-) \) (resp. \( \psi(x^0) \) or \( \psi(x^+) \)) is a lower-triangular (resp. diagonal or upper-triangular) matrix. It follows that \( \psi(x^-) = \psi(x^0) = \psi(x^+) = 0 \) and yields \( x^- = x^+ = 0 \) since \( \psi|\stp^+_m(R^-) \) is injective. Hence, the summation (3.6) is a direct sum. \( \square \)

**Proposition 3.4.** Let \( (R, \cdot) \) be a unital associative superalgebra with superinvolution and \( m \geq 3 \). Then \( \psi : \stp_m(R^-) \to p_m(R^-) \) is a central extension and \( \ker \psi \subseteq \stp^0_m(R^-) \).

**Proof.** Let \( x \in \ker \psi \). We write \( x = x^- + x^0 + x^+ \) with respect to the decomposition (3.6). Then

\[
0 = \psi(x) = \psi(x^-) + \psi(x^0) + \psi(x^+) \in p_m(R^-),
\]

where \( \psi(x^-) \) (resp. \( \psi(x^0) \) or \( \psi(x^+) \)) is a lower-triangular (resp. diagonal or upper-triangular) matrix. Hence, \( \psi(x^-) = \psi(x^0) = \psi(x^+) = 0 \). Recall from the proof of Proposition 3.3 that \( \psi|\stp^+_m(R^-) \) are injective. It follows that \( x^+ = x^- = 0 \). Hence, \( x = x^0 \in \stp^0_0(R^-) \).
It remains to show that every element \( x \in \ker \psi \) commutes with the generators \( t_{ij}(a), f_{ij}(a) \) and \( g_{ij}(a) \). For \( x \in \ker \psi \), we have
\[
\psi([x, t_{ij}(a)]) = \psi([x, f_{ij}(a)]) = \psi([x, g_{ij}(a)]) = 0.
\]
Note that \( x \in \ker \psi \subseteq \mathfrak{stp}_m^0(R, -) \), it follows from Proposition 3.3 that \([x, t_{ij}(a)], [x, f_{ij}(a)]\) and \([x, g_{ij}(a)]\) are all contained in either \( \mathfrak{stp}_m^0(R, -) \) or \( \mathfrak{stp}_m^+(R, -) \). Hence,
\[
[x, t_{ij}(a)] = [x, f_{ij}(a)] = [x, g_{ij}(a)] = 0
\]
since \( \psi|_{\mathfrak{stp}_m^0(R, -)} \) are injective.

In the special case where \((R, -) = (S \oplus S^{op}, ex)\) for a unital associative superalgebra \( S \). Example 2.1 implies that the Lie superalgebra \( \mathfrak{p}_m(S \oplus S^{op}, ex) \) is isomorphic to \( \mathfrak{st}_{m|m}(S) \). According to Proposition 3.4, a central extension of \( \mathfrak{p}_m(S \oplus S^{op}, ex) \) is given by \( \mathfrak{stp}_m(S \oplus S^{op}, ex) \), which is isomorphic to the Steinberg Lie superalgebra \( \mathfrak{st}_{m|m}(S) \) defined in [4]:

**Proposition 3.5.** Let \( S \) be an arbitrary unital associative superalgebra and \( m \geq 3 \). Then
\[
\mathfrak{stp}_m(S \oplus S^{op}, ex) \cong \mathfrak{st}_{m|m}(S)
\]
as Lie superalgebras over \( \mathbb{k} \).

**Proof.** The Steinberg Lie superalgebra \( \mathfrak{st}_{m|m}(S) \) as defined in [4] is the abstract Lie superalgebra generated by homogeneous elements \( e_{ij}(a) \) of degree \(|i| + |j| + |a|\) for \( a \in R \) and \( 1 \leq i \neq j \leq m + n \), subjecting to the relations:
\[
a \mapsto e_{ij}(a) \text{ is } \mathbb{k}-linear, \tag{ST0}
\]
\[
[e_{ij}(a), e_{jk}(b)] = e_{ik}(ab), \quad \text{for distinct } i, j, k, \tag{ST1}
\]
\[
[e_{ij}(a), e_{kl}(b)] = 0, \quad \text{for } i \neq j \neq k \neq l \neq i, \tag{ST2}
\]
where \( a, b \in R \) and \( 1 \leq i, j, k, l \leq m + n \).

According to the relations \((STP00)-(STP12)\), there is a homomorphism of Lie superalgebras \( \phi : \mathfrak{st}_{m|m}(S) \to \mathfrak{stp}_m(S \oplus S^{op}, ex) \) such that
\[
\phi(e_{ij}(a)) = t_{ij}(a \oplus 0), \quad \phi(e_{i,m+j}(a)) = f_{ij}(a \oplus 0),
\]
\[
\phi(e_{m+i,j}(a)) = g_{ij}(a \oplus 0), \quad \phi(e_{m+i,m+j}(a)) = -t_{ji}(0 \oplus a),
\]
\[
\phi(e_{i,m+j}(a)) : = [t_{ij}(1 \oplus 0), f_{ji}(a \oplus 0)], \quad \phi(e_{m+i,m+j}(a)) : = [g_{ij}(a \oplus 0), t_{ji}(1 \oplus 0)],
\]
for \( a \in S \) and \( 1 \leq i \neq j \leq m \). It has an inverse \( \tilde{\phi} : \mathfrak{stp}_m(S \oplus S^{op}, ex) \to \mathfrak{st}_{m|m}(S) \) given by
\[
\tilde{\phi}(t_{ij}(a \oplus b)) = e_{ij}(a) - e_{m+j,m+i}(b),
\]
\[
\tilde{\phi}(f_{ij}(a \oplus b)) = e_{i,m+j}(a) + e_{j,m+i}(b),
\]
\[
\tilde{\phi}(g_{ij}(a \oplus b)) = e_{m+i,j}(a) - e_{m+j,i}(b),
\]
for \( a, b \in S \) and \( 1 \leq i \neq j \leq m \). Hence, we obtain the desired isomorphism.

**4 Characterization of the kernel**

We have shown that the canonical epimorphism \( \psi : \mathfrak{stp}_m(R, -) \to \mathfrak{p}_m(R, -) \) is a central extension. This section is devoted to explicitly characterizing the kernel of \( \psi \). Here, we need the notion of the first \( \mathbb{Z}/2\mathbb{Z} \)-graded dihedral homology \( + \text{HD}_1(R, -) \) for a unital associative superalgebra \( (R, -) \) with superinvolution.
The $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, -)$ is a natural $\mathbb{Z}/2\mathbb{Z}$-graded analogue of the dihedral homology of a unital associative algebra with anti-involution. It can be defined through the coinvariant complex of the Hochschild complex under certain action of the dihedral group as in [11]. For the use in this paper, we only describe its degree one term here:

Let $I$ be the $k$-submodule of $R \otimes_k R$ spanned by

$$a \otimes b + (-1)^{|a||b|} b \otimes a,$$

for homogenous $a, b, c \in R$. Let $(R, R) := (R \otimes_k R)/I$ and $(a, b) = a \otimes b + I$, then the first $\mathbb{Z}/2\mathbb{Z}$-graded dihedral homology of $(R, -)$ is

$$-\text{HD}_1(R, -) := \left\{ \sum_i (a_i, b_i) \left| \sum_i (a_i, b_i) = - \sum_i [a_i, b_i] \right. \right\}.$$

**Proposition 4.1.** Let $(R, -)$ be a unital associative superalgebra with superinvolution, $m \geq 3$, and $\psi : \text{stp}_m(R, -) \to \text{p}_m(R, -)$ the canonical epimorphism (3.4). Then

$$\ker \psi \cong -\text{HD}_1(R, - \circ \rho)$$

as $k$-modules, where $\rho$ is the $k$-linear map (2.5) and $- \circ \rho$ is also a superinvolution on $R$.

In order to prove this proposition, we need a few lemmas:

**Lemma 4.2.** The elements $h_{ij}(a, b) = [f_{ij}(a), g_{ij}(b)] \in \text{stp}_m(R, -)$ satisfy

$$h_{11}(a, b) - (-1)^{|a||b|} h_{1k}(1, ba) = h_{1k}(a, b) - (-1)^{|a||b|} h_{1k}(1, ba),$$

$$h_{11}(1, a) + h_{1j}(1, a) - h_{ij}(1, a) = h_{ki}(1, a) + h_{1j}(1, a) - h_{ki}(1, a).$$

for homogenous $a, b \in R$ and $2 \leq i, j, k, l \leq m$ with $i \neq j$ and $k \neq l$.

**Proof.** Observing that the equality (4.3) is trivial when $i = k$, we assume that $2 \leq i \neq k \leq m$.

$$h_{1k}(a, b) - (-1)^{|a||b|} h_{1k}(1, ba) = [f_{ik}(a), g_{ik}(b)] - (-1)^{|a||b|} [f_{ik}(1), g_{ik}(1) ]$$

$$= -[[f_{1k}(a), t_{ki}(1)], g_{ik}(b)] + (-1)^{|a||b|}[[f_{1k}(1), t_{ki}(1)], g_{ik}(ba) ]$$

$$= -[[f_{1k}(a), g_{ik}(b)], t_{ki}(1)] - [f_{1k}(a), [t_{ki}(1), g_{ik}(b)]]$$

$$+ (-1)^{|a||b|}[[f_{1k}(1), g_{ik}(ba)], t_{ki}(1)] + (-1)^{|a||b|}[[f_{1k}(1), t_{ki}(1), g_{ik}(ba)]]$$

$$= (-1)^{|a||b|}[t_{ik}(\bar{a}b), t_{ki}(1)] + [f_{1k}(a), g_{ik}(b) ]$$

$$- (-1)^{|a||b|}[t_{ik}(\bar{a}b), t_{ki}(1)] - (-1)^{|a||b|}[[f_{1k}(1), g_{ik}(ba)]]$$

$$= [f_{1k}(a), g_{ik}(b)] - (-1)^{|a||b|}[[f_{1k}(1), g_{ik}(ba)]]$$

$$= h_{i1}(a, b) - (-1)^{|a||b|} h_{i1}(1, ba).$$

It yields the equality (4.3).

For the equality (4.4), we first consider the case where $2 \leq i \neq k \leq m$. We have

$$[t_{1k}(1), t_{i1}(a)] = [t_{1k}(1), t_{ki}(1)], t_{i1}(a) = [t_{ki}(1), t_{ki}(a)] + [t_{ki}(1), t_{ki}(a)].$$

On the other hand, it follows from $t_{ij}(a) = [f_{ik}(a), g_{kj}(1)]$ for distinct $i, j, k$ that

$$[t_{ij}(a), t_{ji}(b)] = h_{ik}(a, b) - (-1)^{|a||b|} h_{ik}(1, ba).$$

Hence,

$$h_{ij}(1, a) - h_{ij}(1, a) = h_{ki}(1, a) - h_{1i}(1, a) + h_{1j}(1, a) - h_{ki}(1, a),$$

i.e., (4.4) holds when $2 \leq i \neq k \leq m$.

For $i = k$, the equality (4.4) is reduced to

$$h_{ij}(1, a) - h_{ij}(1, a) = h_{1i}(1, a) - h_{ij}(1, a),$$

for distinct $i, j, l \in \{2, \ldots, m\}$, whose both sides are equal to $[t_{1i}(1), t_{i1}(a)]$ by (4.5).
Lemma 4.2 ensures that
\[
\lambda(a, b) := h_{1i}(a, b) - (-1)^{|a||b|} h_{1i}(1, ba) \in \text{stp}_m(R, -)
\] (4.6)
is independent of \(2 \leq i \leq m\), and
\[
\mu(a) := h_{1i}(1, a) + h_{1j}(1, a) - h_{ij}(1, a) \in \text{stp}_m(R, -)
\] (4.7)
is independent of \(2 \leq i \neq j \leq m\). Moreover, they satisfy the following properties:

**Lemma 4.3.** For homogeneous \(a, b, c \in R\), we have

(i) \((-1)^{|a||c|} \lambda(ab, c) + (-1)^{|b||a|} \lambda(bc, a) + (-1)^{|c||b|} \lambda(ca, b) = 0, \)

(ii) \(\lambda(a, 1) = \lambda(1, b) = 0, \)

(iii) \(\lambda(a, b) = -(1)^{|a||b|} \lambda(b, a), \)

(iv) \(\mu(a) = -\mu(\rho(a)). \)

**Proof.** We claim that
\[
\lambda(a, b) = [t_{1j}(a), t_{j1}(b)] - (-1)^{|a||b|} [t_{1j}(1), t_{j1}(ba)],
\] (4.8)
for \(a, b \in R\) and \(j \neq 1\). Indeed, we deduce from (4.5) that
\[
[t_{1j}(a), t_{j1}(b)] = h_{jk}(a, b) - (-1)^{|a||b|} h_{jk}(1, ba),
\]
\[
[t_{1j}(1), t_{j1}(ba)] = h_{1k}(1, ba) - h_{jk}(1, ba),
\]
for some \(k \neq 1, j\). Hence, (4.8) holds.

Secondly, (STP03) and the Jacobi identity yield that
\[
(-1)^{|a||c|} [t_{1j}(ab), t_{j1}(c)] + (-1)^{|a||b|} [t_{ki}(bc), t_{ik}(a)] + (-1)^{|b||c|} [t_{jk}(ca), t_{kj}(b)] = 0,
\] (4.9)
for distinct \(i, j, k\).

(i) We deduce from (4.8) and (4.9) that
\[
(-1)^{|c||a|} \lambda(ab, c)
\]
\[
= (-1)^{|c||a|} [t_{1j}(ab), t_{j1}(c)] - (-1)^{|b||c|} [t_{1j}(1), t_{j1}(cab)]
\]
\[
= (-1)^{|c||a|} [t_{1j}(ab), t_{j1}(c)] - (-1)^{|b||c|} [t_{1j}(1), [t_{j1}(ca), t_{1i}(b)]]
\]
\[
= (-1)^{|c||a|} [t_{1j}(ab), t_{j1}(c)] + (-1)^{|b||c|} [t_{j1}(ca), t_{ij}(b)] - (-1)^{|b||c|} [t_{1i}(ca), t_{1j}(b)]
\]
\[
= (-1)^{|a||b|} [t_{1i}(bc), t_{1j}(a)] - (-1)^{|b||c|} [t_{1i}(ca), t_{1j}(b)].
\]

On the other hand, we also compute that
\[
\lambda(ab, c) = [t_{1j}(ab), t_{j1}(c)] - (-1)^{|a||c|+|b|} [t_{1j}(1), t_{j1}(cab)]
\]
\[
= [t_{1j}(ab), t_{j1}(c)] - (-1)^{|a||c|+|b|} [t_{1j}(1), [t_{j1}(ca), t_{1i}(ab)]]
\]
\[
= [t_{1j}(ab), t_{j1}(c)] + [t_{1i}(ab), t_{1i}(c)] - [t_{1j}(ab), t_{j1}(c)].
\]

It follows from (4.9) again that
\[
(-1)^{|a||c|} \lambda(ab, c) + (-1)^{|a||b|} \lambda(bc, a)
\]
\[
= -(-1)^{|a||b|} [t_{1i}(bc), t_{1j}(a)] - (-1)^{|b||c|} [t_{1i}(ca), t_{1j}(b)]
\]
\[
= -(-1)^{|b||c|} [t_{j1}(ca), t_{1j}(b)] - (-1)^{|c||a|} [t_{j1}(ab), t_{1j}(c)]
\]
\[
= (-1)^{|b||c|} [t_{j1}(ca), t_{1j}(b)] - [t_{1i}(ca), t_{1j}(b)] - [t_{j1}(ca), t_{1j}(b)].
\]
\[ = -(−1)^{|b||c|} \lambda(c, a, b). \]

This proves (i).

(ii) \( \lambda(1, b) = 0 \) is obvious. Taking \( b = c = 1 \) in (i), we obtain

\[ \lambda(a, 1) + \lambda(1, a) + \lambda(a, 1) = 0 \]

which implies \( \lambda(a, 1) = 0 \) since \( \frac{1}{k} \in \mathbb{k} \) and \( \lambda(1, a) = 0 \).

(iii) follows from (i) by taking \( c = 1 \).

(iv) follows from the equality \( h_{ij}(a, b) = -h_{ji}(\rho(\bar{a}), \rho(\bar{b})). \)

Lemma 4.4. Every element \( x \in \text{stp}_m^0(R, -) \) can be written as

\[ x = \sum_{i \in I_x} \lambda(a_i, b_i) + \mu(c) + \sum_{j=2}^m h_{ij}(1, d_j), \quad (4.10) \]

where \( I_x \) is a finite index set, \( a_i, b_i, c, d_j \in R \) for \( i \in I_x \) and \( j = 2, \ldots, m \). Moreover,

\[ \mu([a, b]) = \lambda(a, b) + \lambda(\rho(\bar{a}), \rho(\bar{b})) \quad (4.11) \]

for homogeneous \( a, b \in R \).

Proof. Recall that \( \text{stp}_m^0(R, -) \) is spanned by \( h_{ij}(a, b) \) for homogeneous \( a, b \in R \) and \( 1 \leq i \neq j \leq m \). It suffices to show that every \( h_{ij}(a, b) \) can be written in the form of (4.10).

We first observe that

\[ -h_{i1}(\rho(\bar{a}), \rho(\bar{b})) = h_{1i}(a, b) = h(a, b) + (-1)^{|a||b|} h_{i1}(1, ba) \]

for \( a, b \in R \) and \( i = 2, \ldots, m \).

If \( 2 \leq i \neq j \leq m \), then

\[ h_{ij}(a, b) = [f_{ij}(a), g_{ji}(b)] \]
\[ = [[t_{i1}(1), f_{i1}(a)], g_{ji}(b)] \]
\[ = [f_{i1}(a), g_{ji}(b)] + [t_{i1}(1), t_{i1}(ab)] \]
\[ = h_{ij}(a, b) + [t_{i1}(1), t_{i1}(ab)] \]
\[ = h_{ij}(a, b) + h_{ij}(1, ab) - h_{1j}(1, ab) \]
\[ = h_{ij}(a, b) - \mu(ab) + h_{i1}(1, ab), \]

which is of the form (4.10) since \( h_{1j}(a, b) \) and \( h_{i1}(1, ab) \) have already been of the form (4.10).

Now, we prove the equality (4.11). For \( 2 \leq i \neq j \leq m \), we have already obtained that

\[ \mu(ab) = h_{ij}(a, b) + h_{i1}(1, ab) - h_{ij}(a, b) \]
\[ = h_{ij}(a, b) - (-1)^{|a||b|} h_{i1}(1, \rho(\bar{b})\rho(\bar{a})) - h_{ij}(a, b). \]

It follows from Lemma 4.3 that

\[ \mu(ba) = -(-1)^{|a||b|}\mu(\rho(\bar{a}), \rho(\bar{b})) \]
\[ = -(-1)^{|a||b|} h_{i1}(\rho(\bar{a}), \rho(\bar{b})) - (-1)^{|a||b|} h_{ij}(1, ba) - h_{ij}(\rho(\bar{a}), \rho(\bar{b}))) \]
\[ = -(-1)^{|a||b|} h_{i1}(\rho(\bar{a}), \rho(\bar{b})) + h_{ij}(1, ba) - (-1)^{|a||b|} h_{ij}(a, b). \]

Hence,

\[ \mu([a, b]) = \mu(ab) - (-1)^{|a||b|}\mu(ba) \]
\[ \begin{align*}
&= h_{ij} (a, b) - (-1)^{|a||b|} h_{ij} (1, ba) \\
&\quad + h_{ij} (\rho (\bar{a}), \rho (\bar{b})) - (-1)^{|a||b|} h_{ij} (1, \rho (\bar{b}) \rho (\bar{a})) \\
&= \lambda (a, b) + \lambda (\rho (\bar{a}), \rho (\bar{b})).
\end{align*} \]

This completes the proof. \hfill \square

Now, we may proceed to prove Proposition 4.1.

**Proof of Proposition 4.1.** Recall (4.1) that

\[ H(D^1 (R, -) \circ \rho) = \left\{ \sum_i \langle a_i, b_i \rangle \in \langle R, R \rangle \mid \sum_i [a_i, b_i] = - \sum_i [\rho (a_i), \rho (b_i)] \right\}, \]

where \( \langle R, R \rangle = (R \otimes_k R) / I \) and I is the \( \mathbb{k} \)-submodule of \( R \otimes_k R \) spanned by \( a \otimes b + (-1)^{|a||b|} b \otimes a, a \otimes b + \rho (\bar{a}) \otimes \rho (\bar{b}) \) and \( (-1)^{|a||c|} ab \otimes c + (-1)^{|a||b|} bc \otimes a + (-1)^{|c||b|} ca \otimes b \) for homogeneous \( a, b, c \in R \).

By Lemmas 4.3 and 4.4, there exists a well-defined \( \mathbb{k} \)-linear map

\[ \eta : (R, R) \to \text{stp}_m (R, -), \]

\[ \langle a, b \rangle \mapsto \lambda (a, b) - \frac{1}{2} \mu ([a, b]) = \frac{1}{2} (\lambda (a, b) - \lambda (\rho (\bar{a}), \rho (\bar{b}))). \]

We will prove that its restriction on \( H(D^1 (R, -) \circ \rho) \) is an isomorphism of \( \mathbb{k} \)-modules onto \( \ker \psi \).

We claim that \( \eta (H(D^1 (R, -) \circ \rho)) \subseteq \ker \psi \). For \( \sum_i \langle a_i, b_i \rangle \in H(D^1 (R, -) \circ \rho) \), we have

\[ \sum_i [a_i, b_i] = - \sum_i [\rho (a_i), \rho (b_i)]. \]

Hence,

\[ \psi (\eta (\sum_i \langle a_i, b_i \rangle)) = \frac{1}{2} \sum_i \psi (\lambda (a_i, b_i) - \lambda (\rho (\bar{a}_i), \rho (\bar{b}_i))) = \frac{1}{2} \sum_i e_{11} ([a_i, b_i] - [\rho (\bar{a}_i), \rho (\bar{b}_i)]) = 0. \]

Conversely, let \( x \in \ker \psi \subseteq \text{stp}_m^0 (R, -) \) (see Proposition 3.4). It follows from Lemma 4.4 that

\[ x = \sum_{i \in I_x} \lambda (a_i, b_i) + \mu (c) + \sum_{j=2}^m h_{ij} (1, d_j), \]

where \( I_x \) is a finite index set, \( a_i, b_i, c, d_j \in R \) for \( i \in I_x \) and \( j = 2, \ldots, m \), and hence,

\[ 0 = \psi (x) = \sum_{i \in I_x} e_{11} ([a_i, b_i]) + e_{11} (c_{\cdot \cdot}) + \sum_{j=2}^m (e_{11} (d_j) - e_{ij} (\bar{d}_j)), \]

which implies that \( d_j = 0 \) for \( j = 2, \ldots, m \) and

\[ \sum_{i \in I_x} [a_i, b_i] = - c_{\cdot \cdot} \in R(-). \]

Since \( \frac{1}{2} \in \mathbb{k} \) and \( \mu (\bar{a}) = - \mu (\rho (a)) \) for homogeneous \( a \in R \), we have

\[ \mu (c) = \frac{1}{2} \mu (c_{\cdot \cdot}) = - \frac{1}{2} \sum_{i \in I_x} \mu ([a_i, b_i]). \]

Hence, we conclude that

\[ x = \sum_{i \in I_x} (\lambda (a_i, b_i) - \frac{1}{2} \mu ([a_i, b_i])) = \sum_{i \in I_x} \eta (\langle a_i, b_i \rangle), \]

\[ x = \sum_{i \in I_x} h_{ij} (\rho (\bar{a}_i), \rho (\bar{b}_i)). \]
and \( \sum_{i \in I} \langle a_i, b_i \rangle \in + \text{HD}_1(R, - \circ \rho) \).

It remains to show the injectivity of \( \eta \). Define a \( k \)-bilinear map

\[ \alpha : \mathfrak{gl}_{m|n}(R) \times \mathfrak{gl}_{m|n}(R) \to \langle R, R \rangle \]

by

\[ \alpha(e_{ij}(a), e_{kl}(b)) = \delta_{jk} \delta_{il}(-1)^{\langle i \rangle + \langle a \rangle + \langle b \rangle}(a, b) \]

for homogeneous \( a, b \in R \) and \( 1 \leq i, j \leq 2m \), where \( \langle i \rangle \) is the parity of \( i \) given by (2.2). It is verified that \( \alpha \) is a 2-cocycle on the Lie superalgebra \( \mathfrak{gl}_{m|n}(R) \).

Now, the restriction of \( \alpha \) on \( \mathfrak{p}_m(R, -) \times \mathfrak{p}_m(R, -) \) is a 2-cocycle on \( \mathfrak{p}_m(R, -) \subseteq \mathfrak{gl}_{m|n}(R) \). Hence, there is a Lie superalgebra structure on \( \mathfrak{p}_m(R, -) \oplus \langle R, R \rangle \):

\[ [x \oplus c, y \oplus c'] = [x, y] \oplus \alpha(x, y), \quad x, y \in \mathfrak{p}_m(R, -) \text{ and } c, c' \in \langle R, R \rangle. \]

Observing that \( t_{ij}(a) \oplus 0 \), \( f_{ij}(a) \oplus 0 \) and \( g_{ij}(a) \oplus 0 \in \mathfrak{p}_m(R, -) \oplus \langle R, R \rangle \) satisfy all relations (STP00)-(STP12), there is a canonical homomorphism of Lie superalgebras

\[ \phi : \text{stp}_m(R, -) \to \mathfrak{p}_m(R, -) \oplus \langle R, R \rangle \]

such that

\[ \phi(t_{ij}(a)) = t_{ij}(a) \oplus 0, \quad \phi(f_{ij}(a)) = f_{ij}(a) \oplus 0, \quad \phi(g_{ij}(a)) = g_{ij}(a) \oplus 0, \]

for \( a \in R \) and \( 1 \leq i \neq j \leq m \). We now compute that

\[
\phi(h_{ij}(a, b)) = \phi(f_{ij}(a), g_{ji}(b)) = [f_{ij}(a) \oplus 0, g_{ji}(b) \oplus 0] = [t_{ij}(ab) - t_{jj}(\rho(\bar{a})\rho(\bar{b})) + (\langle a, b \rangle - \langle \rho(\bar{a}), \rho(\bar{b}) \rangle) + 2\langle a, b \rangle, \]

which implies that

\[
\phi(\lambda(a, b)) = \phi(h_{11}(a, b) - (-1)^{|a||b|}h_{11}(1, ba)) = t_{11}([a, b]) \oplus 2\langle a, b \rangle - 2(-1)^{|a||b|}(1, ba)).
\]

Since \( \langle 1, a \rangle = -\langle a, 1 \rangle \) and \( \langle a, 1 \rangle + \langle a, 1 \rangle + \langle 1, a \rangle = 0 \), we obtain that \( \langle 1, a \rangle = 0 \). Hence,

\[ \phi(\lambda(a, b)) = t_{11}(\langle [a, b] \rangle) \oplus 2\langle a, b \rangle. \]

Since \( \frac{1}{2} \in k \),

\[
\phi(\eta(\langle a, b \rangle)) = \frac{1}{2}(\phi(\lambda(a, b)) - \phi(\lambda(\rho(\bar{a}), \rho(\bar{b})))) = \frac{1}{2}(t_{11}(\langle [a, b] \rangle) \oplus 2\langle a, b \rangle^{(+)} - t_{11}(\langle [\rho(\bar{a}), \rho(\bar{b})] \rangle) \oplus 2\langle \rho(\bar{a}), \rho(\bar{b}) \rangle^{(+)}) = \frac{1}{2}(t_{11}(\langle [a, b] \rangle - [\rho(\bar{a}), \rho(\bar{b})]) \oplus (\langle a, b \rangle^{(+)}) - \langle \rho(\bar{a}), \rho(\bar{b}) \rangle^{(+)}) \]

which shows that \( \eta \) is injective and completes the proof. \( \square \)
5 The universality of the central extension \( \psi \)

It is shown in Section 3 that the canonical homomorphism \( \psi : \mathfrak{st}p_m(R, \mathbb{R}) \to p_m(R, \mathbb{R}) \) is a central extension, whose kernel has been explicitly characterized in Section 4. In this section, we will prove that the central extension \( \psi \) is universal for \( m \geq 3 \) and thus obtain a precise description of the second homology \( p_m(R, \mathbb{R}) \).

A necessary condition for the universality of \( \psi \) is the perfectness \( \mathfrak{st}p_m(R, \mathbb{R}) \), which can be easily observed from the defining relations (STP03), (STP05) and (STP07). Next, we proceed to prove the universality of \( \psi \).

Let \( \varphi : \mathfrak{e} \to p_m(R, \mathbb{R}) \) be an arbitrary central extension of \( p_m(R, \mathbb{R}) \) with \( m \geq 3 \). For \( a \in R \) and \( 1 \leq i \neq j \leq m \), we pick

\[
\hat{t}_{ij}(a) \in \varphi^{-1}(t_{ij}(a)), \quad \hat{f}_{ij}(a) \in \varphi^{-1}(f_{ij}(a)), \quad \text{and } \hat{g}_{ij}(a) \in \varphi^{-1}(g_{ij}(a)).
\]

Obviously, the element \( [\hat{x}, \hat{y}] \in \mathfrak{e} \) is independent the choice of the representatives \( \hat{x} \in \varphi^{-1}(x) \) and \( \hat{y} \in \varphi^{-1}(y) \) for \( x, y \in p_m(R, \mathbb{R}) \). Moreover, we have the following lemma:

**Lemma 5.1.** In the Lie superalgebra \( \mathfrak{e} \), the following equalities hold:

\[
\begin{align*}
(i) & \quad [\hat{f}_{ik}(a), \hat{g}_{kj}(b)] = [\hat{f}_{il}(a), \hat{g}_{lj}(b)], \\
(ii) & \quad [\hat{t}_{ik}(a), \hat{f}_{kj}(b)] = [\hat{t}_{il}(a), \hat{f}_{lj}(b)], \\
(iii) & \quad [\hat{g}_{ik}(a), \hat{t}_{kj}(b)] = [\hat{g}_{il}(a), \hat{t}_{lj}(b)].
\end{align*}
\]

for \( a, b \in R \) and distinct \( i, j, k, l \).

**Proof.** (i) Since \([\hat{f}_{ik}(a), \hat{t}_{ik}(1)] + \hat{t}_{il}(a) \in \ker \varphi \) that is included in the center of \( \mathfrak{e} \), we deduce

\[
[\hat{f}_{ik}(a), \hat{g}_{kj}(b)] = -[[\hat{f}_{ik}(a), \hat{t}_{ik}(1)], \hat{g}_{kj}(b)]
= -[[\hat{f}_{ik}(a), \hat{g}_{kj}(b)], \hat{t}_{ik}(1)] - [\hat{f}_{ik}(a), [\hat{t}_{ik}(1), \hat{g}_{kj}(b)]]
= 0 + [\hat{f}_{ik}(a), \hat{g}_{kj}(b)],
\]

which shows (i). The equalities (ii) and (iii) follows similarly. \( \square \)

According to Lemma 5.1, we define for each pair \((i, j)\) with \( 1 \leq i \neq j \leq m \) that:

\[
\hat{t}_{ij}(a) := [\hat{f}_{ik}(a), \hat{g}_{kj}(1)], \quad \hat{f}_{ij}(a) := [\hat{t}_{ik}(a), \hat{f}_{kj}(a)], \quad \text{and } \hat{g}_{ij}(a) := [\hat{g}_{ik}(a), \hat{t}_{kj}(1)],
\]

(5.1)

where \( a \in R \) and \( 1 \leq k \leq m \) is an arbitrary integer such that \( k \neq i, j \).

**Lemma 5.2.** Suppose \( m \geq 3 \). Let \( \hat{t}_{ij}(a), \hat{f}_{ij}(a) \) and \( \hat{g}_{ij}(a) \) be the elements of \( \mathfrak{e} \) given in (5.1), where \( a \in R \) and \( 1 \leq i \neq j \leq m \). Then they satisfy all relations (STP00)-(STP12) except (STP04) and (STP10). Moreover, for \( a, b \in R \), we have

\[
\begin{align*}
[\hat{t}_{ik}(a), \hat{t}_{jk}(b)] &= 0, \quad \text{if } i, j, k \text{ are distinct}, \quad \text{ (STP04a)} \\
[\hat{t}_{ij}(a), \hat{g}_{kl}(b)] &= 0, \quad \text{if } i, j, k, l \text{ are distinct}, \quad \text{ (STP04b)} \\
[\hat{g}_{ij}(a), \hat{g}_{ij}(b)] &= 0, \quad \text{if } i \neq j, \quad \text{ (STP10a)} \\
[\hat{g}_{ij}(a), \hat{g}_{jk}(b)] &= 0, \quad \text{if } i, j, k \text{ are distinct.} \quad \text{ (STP10b)}
\end{align*}
\]

**Proof.** The \( k \)-linearity of \( \hat{t}_{ij} \) is obvious since both \( \hat{f}_{ik}(ca) \) and \( c\hat{f}_{ik}(a) \) are contained in \( \varphi^{-1}(f_{ik}(a)) \) for \( a \in R \) and \( c \in k \). Similarly, we have the \( k \)-linearity of \( \hat{f}_{ij} \) and \( \hat{g}_{ij} \), which shows (STP00).

For (STP01), we aim to show \( \hat{f}_{ij}(a) = \hat{f}_{ji}(\rho(a)) \) for \( a \in R \). Note that \( m \geq 3 \), we choose \( 1 \leq k \leq m \) such that \( i, j, k \) are distinct and set

\[
\hat{h}_{ik} = [\hat{f}_{ik}(1), \hat{g}_{ki}(1)],
\]

(5.2)
Let \( \phi \in \mathcal{E} \) be a universal central extension of \( \mathcal{E} \). Then \( (\phi) \) is contained in the center of \( \mathcal{E} \) (Lemma 5.2). This shows \( \tilde{\phi} \) to be a universal central extension of \( \mathcal{E} \). Applying \( \tilde{\phi} \) to \( f_{ij}(a) \), we obtain that

\[
0 = [\tilde{h}_{ij}, \tilde{f}_{ij}(a)] - [\tilde{f}_{ij}(a), \tilde{h}_{ij}].
\]

This shows \( \tilde{f}_{ij}(a) \) satisfies the relation (STP01). All other relations can be verified similarly. \( \square \)

**Remark 5.3.** The relation (STP04) states that

\[
[t_{ij}(a), t_{kl}(b)] = 0, \quad \text{if } i \neq j \neq k \neq l \neq i,
\]

which is equivalent to (STP04a), (STP04b) and

\[
[t_{ij}(a), t_{ik}(b)] = 0, \quad \text{if } i, j, k \text{ are distinct}. \tag{STP04c}
\]

By Lemma 5.2, the elements \( f_{ij}(a) \)'s in an arbitrary central extension \( \mathcal{E} \) of \( \mathcal{E} \) always satisfy (STP04a) and (STP04b), but do not necessarily satisfy (STP04c). Such examples will appear in central extensions of \( \mathcal{E} \). The similar phenomenon also occurs for the relation (STP10).

**Proposition 5.4.** Let \( (R, -) \) be a unital associative superalgebra with superinvolution and \( m \geq 5 \). Then \( \psi : \text{stp}_m(R, -) \rightarrow \mathcal{E}_m(R, -) \) is a universal central extension.

**Proof.** Let \( \phi : \mathcal{E} \rightarrow \mathcal{E}_m(R, -) \) be an arbitrary central extension. Take \( f_{ij}(a), \tilde{f}_{ij}(a), \tilde{g}_{ij}(a) \in \mathcal{E} \) for \( a \in R \) and \( 1 \leq i \neq j \leq m \) as in (5.1). Then we have already known from Lemma 5.2 that they satisfy all relations (STP00)-(STP12) except (STP04) and (STP10). Now, under the additional assumption that \( m \geq 5 \), we will show that these elements also satisfy (STP04) and (STP10).

For (STP04), since (STP04a) and (STP04b) have already been verified in Lemma 5.2, it suffices to show

\[
[t_{ij}(a), \tilde{t}_{ik}(b)] = 0, \quad \text{if } i, j, k \text{ are distinct}. \tag{STP04c}
\]

Indeed, we observe that \( [\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] \in \ker \phi \). Since \( m \geq 5 \), we are allowed to choose \( 1 \leq l \leq m \) such that \( l \neq i, j, k \). Applying \( h_{ij} \) defined in (5.2), we obtain that

\[
0 = [\tilde{h}_{ij}, [\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)]]
\]

\[
= [[\tilde{h}_{ij}, \tilde{t}_{ij}(a)], \tilde{t}_{ik}(b)] + [\tilde{t}_{ij}(a), [\tilde{h}_{ij}, \tilde{t}_{ik}(b)]]
\]

\[
= [\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)].
\]

Then (STP04c) follows.

For (STP10), we have obtained (STP10a) and (STP10b) in Lemma 5.2. It suffices to show

\[
[\tilde{g}_{ij}(a), \tilde{g}_{kl}(b)] = 0, \quad \text{if } i, j, k, l \text{ are distinct}. \tag{STP10c}
\]

Since \( m \geq 5 \), we are permitted to choose \( k' \) such \( k' \neq i, j, k, l \). Hence,

\[
[\tilde{g}_{ij}(a), \tilde{g}_{kl}(b)] = [\tilde{g}_{ij}(a), [\tilde{g}_{kk'}(b), \tilde{t}_{k'l'}(1)]]
\]

\[
= [[\tilde{g}_{ij}(a), \tilde{g}_{kk'}(b)], \tilde{t}_{k'l'}(1)] + (-1)^{(1+|a|)(1+|b|)}[\tilde{g}_{kk'}(b), [\tilde{g}_{ij}(a), \tilde{t}_{k'l'}(1)]] = 0.
\]
This proves (STP10).

In summary, we have shown that the elements \( \tilde{t}_{ij}(a), \tilde{f}_{ij}(a), \tilde{g}_{ij}(a) \in \mathfrak{E} \) with \( a \in \mathbb{R} \) and \( 1 \leq i \neq j \leq m \) satisfy all relations (STP00)-(STP12). Hence, there is a homomorphism of Lie superalgebras

\[
\varphi' : \text{stp}_m(R, -) \to \mathfrak{E}
\]

such that

\[
\varphi'(t_{ij}(a)) = \tilde{t}_{ij}(a), \quad \varphi'(f_{ij}(a)) = \tilde{f}_{ij}(a), \quad \varphi'(g_{ij}(a)) = \tilde{g}_{ij}(a),
\]

for \( a \in \mathbb{R} \) and \( 1 \leq i \neq j \leq m \), i.e., \( \varphi \circ \varphi' = \psi \).

To show the uniqueness of \( \varphi' \), let \( \tilde{\varphi}' : \text{stp}_m(R, -) \to \mathfrak{E} \) be another homomorphism of Lie superalgebras such that \( \varphi \circ \tilde{\varphi}' = \psi \). Then

\[
\tilde{\varphi}'(t_{ij}(a)) \in \varphi^{-1}(t_{ij}(a)), \quad \tilde{\varphi}'(f_{ij}(a)) \in \varphi^{-1}(f_{ij}(a)), \quad \tilde{\varphi}'(g_{ij}(a)) \in \varphi^{-1}(g_{ij}(a)),
\]

for \( 1 \leq i \neq j \leq m \). Note that

\[
t_{ij}(a) = [t_{ik}(a), t_{kj}(1)], \quad f_{ij}(a) = [t_{ik}(1), g_{kj}(a)], \quad g_{ij}(a) = [g_{ik}(a), t_{kj}(1)],
\]

for \( a \in \mathbb{R} \) and distinct \( i, j, k \), we deduce that

\[
\tilde{\varphi}'(t_{ij}(a)) = \tilde{t}_{ij}(a), \quad \tilde{\varphi}'(f_{ij}(a)) = \tilde{f}_{ij}(a), \quad \tilde{\varphi}'(g_{ij}(a)) = \tilde{g}_{ij}(a).
\]

It yields that \( \tilde{\varphi}' = \varphi' \) since \( \text{stp}_m(R, -) \) is generated by \( t_{ij}(a), f_{ij}(a), g_{ij}(a) \) with \( a \in \mathbb{R} \) and \( 1 \leq i \neq j \leq m \). Thus, there is a unique homomorphism \( \varphi' : \text{stp}_m(R, -) \to \mathfrak{E} \) such that \( \varphi \circ \varphi' = \psi \).

Therefore, the central extension \( \psi : \text{stp}_m(R, -) \to \mathfrak{p}_m(R, -) \) is universal. \( \square \)

**Theorem 5.5.** Let \((R, -)\) be a unital associative superalgebra with superinvolution and \( m \geq 5 \). Then

\[
H_2(\mathfrak{p}_m(R, -)) = \vphantom{\vdash} \text{H}_2(\mathfrak{p}_m(R, -) \circ \rho)
\]

where \( \rho \) is the \( k \)-linear map given in (2.5).

**Proof.** The second homology of \( \mathfrak{p}_m(R, -) \) can be identified with the kernel of its universal central extension \( \psi \), which has been shown in Proposition 4.1 to be \( \vphantom{\vdash} \text{H}_2(\mathfrak{p}_m(R, -) \circ \rho) \).

**Remark 5.6.** If \( R \) is super-commutative, then one deduce from the definition that \( \vphantom{\vdash} \text{H}_2(\mathfrak{p}_m(R, \text{id})) = 0 \). Hence,

\[
H_2(\mathfrak{p}_m(\mathbb{k} \otimes_\mathbb{K} R) \cong H_2(\mathfrak{p}_m(R, \rho)) \cong \vphantom{\vdash} \text{H}_2(\mathfrak{p}_m(R, \text{id})) = 0.
\]

This recovers the results about the second homology of \( \mathfrak{p}_m(\mathbb{k} \otimes_\mathbb{K} R \text{ given in } [8] \text{ and } [14] \).

In the special case where \((R, -) = (S \oplus S^{op}, ex)\) for a unital associative superalgebra \( S \), we have

**Corollary 5.7.** Let \( S \) be an arbitrary unital associative superalgebra and \( m \geq 5 \). Then

\[
H_2(\mathfrak{sl}_m(S)) \cong \vphantom{\vdash} \text{H}_2(S \oplus S^{op}, \text{ex } \circ \rho) \cong \text{H}_2(S),
\]

where \( \text{H}_2(S) \) is the first \( \mathbb{Z}/2\mathbb{Z} \)-graded cyclic homology of \( S \) as defined in [4].

**Proof.** By Example 2.1, the Lie superalgebra \( \mathfrak{sl}_m(S) \) is isomorphic to \( \mathfrak{p}_m(S \oplus S^{op}, \text{ex}) \). Theorem 5.5 insures that

\[
H_2(\mathfrak{sl}_m(S)) \cong H_2(\mathfrak{p}_m(S \oplus S^{op}, \text{ex})) \cong \vphantom{\vdash} \text{H}_2(S \oplus S^{op}, \text{ex } \circ \rho),
\]

for \( m \geq 5 \).

Next, we identify \( \vphantom{\vdash} \text{H}_2(S \oplus S^{op}, \text{ex } \circ \rho) \) with the first \( \mathbb{Z}/2\mathbb{Z} \)-graded cyclic homology \( \text{H}_2(S) \) defined in [4]. In the \( \mathbb{K} \)-module \((S \oplus S^{op}, S \oplus S^{op}) \) defined in (4.1), we have

\[
(a \oplus b) = ((a \oplus 0)(1 \oplus 0), 0 \oplus b) + 0 + 0
\]
Then, we define a
for
This shows that

\[ \langle a_1 \oplus a_2, b_1 \oplus b_2 \rangle = \langle a_1 \oplus 0, b_1 \oplus 0 \rangle + \langle a_2 \oplus 0, b_2 \oplus 0 \rangle, \]

for \(a_1, a_2, b_1, b_2 \in S\).

Let \( I_c \) be the \( k \)-submodule of \( S \) generated by \( a \otimes b - (-1)^{|a||b|} b \otimes a \) and \( (-1)^{|a||c|} ab \otimes c + (-1)^{|b||a|} bc \otimes a + (-1)^{|c||b|} ca \otimes b \) for homogeneous \( a, b, c \in S \) and \( \langle S, S \rangle_c = (S \otimes S)/I_c \). Then one observes that

\[ \langle a_1 \oplus a_2, b_1 \oplus b_2 \rangle \mapsto \langle a_1, b_1 \rangle_c + \langle a_2, b_2 \rangle_c \]
defines an isomorphism \( \langle S \oplus S^{\text{op}}, S \oplus S^{\text{op}} \rangle \to \langle S, S \rangle_c \). Its restriction on \( \partial \text{HD}_1(S \oplus S^{\text{op}}, \text{ex} \circ \rho) \) gives an isomorphism onto

\[ \text{HC}_1(S) := \left\{ \sum_i \langle a_i, b_i \rangle_c \in \langle S, S \rangle_c \left| \sum_i [a_i, b_i] = 0 \right. \right\}. \]

This completes the proof.

Remark 5.8. The above corollary recovers the second homology of \( \mathfrak{sl}_m(m) \) for \( m \geq 5 \) obtained in [4]. As a byproduct, we obtain the isomorphism

\[ \partial \text{HD}_1(S \oplus S^{\text{op}}, \text{ex} \circ \rho) \cong \text{HC}_1(S), \]

which indicates that the first \( \mathbb{Z}/2\mathbb{Z} \)-graded cyclic homology can be regarded as a special case of the first \( \mathbb{Z}/2\mathbb{Z} \)-graded dihedral homology. However, it is unknown yet whether such an isomorphism exists for higher degree cyclic homology and higher degree dihedral homology.

6 The second homology of \( p_4(R, -) \)

The second homology of \( p_m(R, -) \) for \( m \geq 5 \) have been explicitly characterized in the previous section. However, the central extension \( \psi : \text{stp}_4(R, -) \to p_4(R, -) \) is not necessarily universal. This section is devoted to explicitly constructing a universal central extension of \( \text{stp}_4(R, -) \), which can be accomplished by creating a 2-cocycle on \( \text{stp}_4(R, -) \).

Such a 2-cocycle takes values in the \( k \)-module \( R/(R_{(-)} \cdot R) \), where \( R_{(-)} \cdot R \) is the right ideal of \( R \) generated by \( \tilde{a} - \rho(a) \) for \( a \in R \). In fact, the \( k \)-module \( R/(R_{(-)} \cdot R) \) is a super-commutative \( k \)-superalgebra since \( [R, R] \cdot R \subseteq R_{(-)} \cdot R \). We denote \( \pi : R \to R/(R_{(-)} \cdot R) \) the canonical quotient map of \( k \)-modules. It satisfies

\[ \pi(ab) = \pi(\rho(a)b), \quad a, b \in R. \tag{6.1} \]

Similar to Proposition 3.3, \( \text{stp}_4(R, -) \) is decomposed as a direct sum of Lie sub-superalgebras:

\[ \text{stp}_4(R, -) = a \oplus b \]

where

\[ a := \text{span}_k \{ h_{ij}(a, b), t_{ij}(a), f_i(a), f_{ij}(a) | a, b \in R, 1 \leq i \neq j \leq 4 \}, \]

\[ b := \text{span}_k \{ g_i(a), g_{ij}(a) | a \in R, 1 \leq i \neq j \leq 4 \}. \]

Then, we define a \( k \)-linear map \( \beta_0 : b \times b \to R/(R_{(-)} \cdot R) \) by

\[ \beta_0(g_{ij}(a), g_{kl}(b)) = \epsilon(ijkl)\pi(a \cdot \rho(b)), \]

\[ \beta_0(g_i(a), b) = \beta_0(b, g_i(a)) = 0, \]

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for \(a, b \in R, 1 \leq i \neq j \leq 4 \) and \(1 \leq k \neq l \leq 4\), where \(\epsilon(ijkl)\) denotes the sign of the permutation \((ijkl)\) if \((ijkl)\) is a permutation of \(\{1, 2, 3, 4\}\) and denotes 0 if \((ijkl)\) is not a permutation. Such a \(k\)-linear map \(\beta_0\) is well-defined since \(g_{ij}(a) = -g_{ij}(\rho(a))\) while \(\pi(ab) = \pi(\rho(a)b)\).

Furthermore, the \(k\)-bilinear map \(\beta_0 : b \times b \to R/(R_{(-)} \cdot R)\) is extended to a \(k\)-bilinear map

\[
\beta : stp_4(R, -) \times stp_4(R, -) \to R/(R_{(-)} \cdot R)
\]
such that \(a\) lies in the radical of \(\beta\), i.e.,

\[
\beta(a, stp_4(R, -)) = \beta(stp_4(R, -), a) = 0.
\]

Now, we can show that:

**Lemma 6.1.** The \(k\)-bilinear map \(\beta\) is a 2-cocycle on \(stp_4(R, -)\) with values in \(R/(R_{(-)} \cdot R)\).

**Proof.** We have to show \(\beta\) satisfies

\[
\beta(x, y) = -(-1)^{|x||y|}\beta(y, x),
\]

(6.2)

\[
(1-|x|)|y| \beta([x, y], z) + (1-|y|)|x| \beta([y, z], x) + (-1)^{|x||y|} \beta([z, x], y) = 0,
\]

(6.3)

for homogeneous \(x, y, z \in stp_4(R, -)\).

For (6.2), it suffices to show

\[
\beta(g_{ij}(a), g_{kl}(b)) = -(-1)^{|a||b|+|a||l|+|b||k|} \beta(g_{kl}(b), g_{ij}(a)),
\]

for homogeneous \(a, b \in R, i \neq j\) and \(k \neq l\). Note that \(\pi(ab) = (-1)^{|a||b|} \pi(ba)\), we deduce that

\[
\beta(g_{ij}(a), g_{kl}(b)) = \epsilon(ijkl) \pi(a \cdot \rho(b)) = \epsilon(klij) (-1)^{|a||b|} \pi(\rho(b) \cdot a)
\]

\[
= (-1)^{|a||b|} \beta(g_{kl}(\rho(b)), g_{ij}(\rho(a)))
\]

\[
= (-1)^{|a|+|b|} \beta(g_{kl}(b), g_{ij}(a)).
\]

Next, we show (6.3). Observing that \(a\) is a Lie sub-superalgebra of \(stp_4(R, -)\) included in the radical of \(\beta\) and \([b, b] = 0\), we deduce that \(\beta([x, y], z) = \beta([y, z], x) = \beta([z, x], y) = 0\) if \((x, y, z)\) is contained in one of the subspaces \(a \times a \times b, a \times b \times b, b \times b \times b\). Note also that (6.3) is symmetric with respect to all permutations on \(\{x, y, z\}\), the proof is reduced to verify (6.3) for \(x \in a\) and \(y, z \in b\). In this situation, \([y, z] = 0\) and (6.3) is equivalent to

\[
\beta([y, x], z) = -(-1)^{|y||x|+|y||z|+|z||x|} \beta([z, x], y)
\]

(6.4)

If \(x = f_{ij}(a)\) or \(x = f_{ij}(a)\), then \([x, b] \subseteq a\) is included in the radical of \(\beta\). It yields that both sides of (6.4) are zero, and hence, we may assume that \(x = h_{ij}(a)\) or \(x = t_{ij}(a)\). In this situation, it is also obvious that both sides of (6.4) are zero if \(y = g_{ij}(a)\) or \(z = g_{ij}(a)\). Now, it remains to verify the following two equalities

\[
\beta([g_{ij}(a), t_{rs}(c)], g_{kl}(b)) = -(-1)^{|a|+|b|+|c|+|d|} \beta([g_{kl}(b), t_{rs}(c)], g_{ij}(a)),
\]

(6.5)

\[
\beta([g_{ij}(a), h_{rs}(c, c')], g_{kl}(b)) = -(-1)^{|a|+|b|+|c|+|c'|+|d|} \beta([g_{kl}(b), h_{rs}(c, c')], g_{ij}(a)),
\]

(6.6)

for homogeneous \(a, b, c, c' \in R\) and \(i \neq j, k \neq l\) and \(r \neq s\).

For (6.5), we compute that

\[
\beta([g_{ij}(a), t_{rs}(c)], g_{kl}(b)) = \delta_{jr} \beta(g_{is}(ac), g_{kl}(b)) - \delta_{ir} \beta(g_{js}(\rho(a)c), g_{kl}(b))
\]

\[
= \delta_{jr} \epsilon(iskl) \pi(acp(b)) - \delta_{ir} \epsilon(jskl) \pi(\rho(a)c)p(b))
\]

\[
= (\delta_{jr} \epsilon(iskl) - \delta_{ir} \epsilon(jskl)) \pi(acp(b)).
\]

Then, (6.5) follows from the facts that

\[
\pi(abc) = (-1)^{|a||b|+|b||c|+|c||a|} \pi(abc)
\]

\[
\delta_{jr} \epsilon(iskl) - \delta_{ir} \epsilon(jskl) = \delta_{lt} \epsilon(ksij) - \delta_{kt} \epsilon(lsiij)
\]

for all \(1 \leq i, j, k, l \leq 4\). The equality (6.6) follows similarly.
The 2-cocycle $\beta : \mathfrak{stp}_4(R, -) \times \mathfrak{stp}_4(R, -) \rightarrow R/(R_{(-)} \cdot R)$ gives rise to a new Lie superalgebra

$$\widehat{\mathfrak{stp}}_4(R, -) := \mathfrak{stp}_4(R, -) \oplus (R/(R_{(-)} \cdot R)),$$

under the super-bracket

$$[x \oplus c, y \oplus d] := [x, y] \oplus \beta(x, y)$$

for $x, y \in \mathfrak{stp}_4(R, -)$ and $c, d \in R/(R_{(-)} \cdot R)$, which is a central extension with the canonical projection $\psi'_4 : \widehat{\mathfrak{stp}}_4(R, -) \rightarrow \mathfrak{stp}_4(R, -)$. Furthermore, we may show that

**Proposition 6.2.** Let $(R, -)$ be a unital associative superalgebra with superinvolution. Then the central extension $\psi' : \mathfrak{stp}_4(R, -) \rightarrow \widehat{\mathfrak{stp}}_4(R, -)$ is universal.

**Proof.** We have already known from Proposition 3.4 that $\psi : \mathfrak{stp}_4(R, -) \rightarrow \mathfrak{p}_4(R, -)$ is a central extension. Hence, $\psi \circ \psi' : \mathfrak{stp}_4(R, -) \rightarrow \mathfrak{p}_4(R, -)$ is a central extension. It suffices to show that $\psi \circ \psi'$ is universal.

Let $\varphi : \mathcal{E} \rightarrow \mathfrak{p}_4(R, -)$ be an arbitrary central extension of $\mathfrak{p}_4(R, -)$. We also take $\hat{t}_{ij}(a)$, $\hat{f}_{ij}(a)$ and $\hat{g}_{ij}(a) \in \mathcal{E}$ as in (5.1). By Lemma 5.2, these elements satisfy (STP00)-(STP12) except (STP04) and (STP10). While the same argument as in Theorem 5.4 also shows that (STP04) holds.

For $a \in R$, we define

$$\hat{\pi}(a) := [\hat{g}_{12}(a), \hat{g}_{34}(1)] \in \mathcal{E},$$

which is contained in the center of $\mathcal{E}$ since $\varphi(\hat{\pi}(a)) = 0$. We next prove that $\hat{\pi}(R_{(-)} \cdot R) = 0$.

Let $a, b \in R$ be homogeneous. We compute that

$$\hat{\pi}(ab) = [\hat{g}_{12}(ab), \hat{g}_{34}(1)] = [[\hat{g}_{13}(a), \hat{t}_{12}(b)], \hat{g}_{34}(1)]$$

$$= [\hat{g}_{13}(a), [\hat{t}_{12}(b), \hat{g}_{34}(1)]] = -[\hat{g}_{13}(a), \hat{g}_{24}(b)]$$

$$= -[[\hat{g}_{14}(1), \hat{t}_{13}(a)], \hat{g}_{24}(b)] = -[\hat{g}_{14}(1), [\hat{t}_{13}(a), \hat{g}_{24}(b)]]$$

$$= (-1)^{|a||1+b|}[\hat{g}_{14}(1), \hat{g}_{23}(ba)] = (-1)^{|a||1+b|}[\hat{g}_{14}(1), \hat{g}_{21}(ba), \hat{t}_{13}(1)]$$

$$= (-1)^{|a||1+b|}(-1)^{|a|+|b|}[\hat{g}_{21}(ba), [\hat{g}_{14}(1), \hat{t}_{13}(1)]] = -(-1)^{|b|+|a|}[\hat{g}_{21}(ba), \hat{g}_{34}(1)]$$

$$= (-1)^{|a|}[[\hat{g}_{12}(ab), \hat{g}_{34}(1)] = (-1)^{|a|} \hat{\pi}(ab).$$

It follows that $(a - (-1)^{|a|}a)b \in \ker \hat{\pi}$ for homogeneous $a, b \in R$. Hence, $\hat{\pi}(R_{(-)} \cdot R) = 0$. We obtain a $k$-linear map

$$R/(R_{(-)} \cdot R) \rightarrow \ker \varphi, \quad \pi(a) \mapsto \hat{\pi}(a).$$

Since $\hat{\pi}(a) = [\hat{g}_{12}(a), \hat{g}_{34}(1)]$ and $\hat{\pi}(ab) = \hat{\pi}(\rho(a)b)$, we deduce that

$$[\hat{g}_{ij}(a), \hat{g}_{kl}(b)] = \epsilon(ijkl)\hat{\pi}(a \cdot \rho(b)),$$

for distinct $i, j, k, l$. Combining with (STP10a) and (STP10b), we have

$$[\hat{g}_{ij}(a), \hat{g}_{kl}(b)] = \epsilon(ijkl)\hat{\pi}(a \cdot \rho(b)),$$

for $i \neq j$ and $k \neq l$.

Hence, there is a homomorphism $\varphi' : \widehat{\mathfrak{stp}}_4(R, -) \rightarrow \mathcal{E}$ such that

$$\varphi'(t_{ij}(a) \oplus 0) = \hat{t}_{ij}(a), \quad \varphi'(f_{ij}(a) \oplus 0) = \hat{f}_{ij}(a), \quad \varphi'(g_{ij}(a) \oplus 0) = \hat{g}_{ij}(a), \quad \varphi'(0 \oplus \pi(a)) = \hat{\pi}(a),$$

where $a \in R$ and $1 \leq i \neq j \leq 4$, i.e., $\varphi \circ \varphi' = \varphi \circ \varphi'$. The uniqueness of $\varphi'$ follows from a similar argument as the proof of Proposition 5.4. Hence, we conclude that $\psi \circ \psi' : \widehat{\mathfrak{stp}}_4(R, -) \rightarrow \mathfrak{p}_4(R, -)$ is a universal central extension. \[\square\]

Using Propositions 3.4 and 6.2, we conclude that
Lemma 7.1. Let \( (R, -) \) be a unital associative superalgebra with superinvolution. Then
\[
H_2(\mathfrak{p}_4(R, -)) = +\mathbf{HD}_1(R, - \circ \rho) \oplus R/(R(-) \cdot R). \]
\[\square\]

Remark 6.4. If \( R \) is super-commutative, then \( R(-) = 0 \). In this situation,
\[
H_2(\mathfrak{p}_4(k) \otimes_k R) \cong H_2(\mathfrak{p}_4(R, \rho)) \cong +\mathbf{HD}_1(R, \text{id}) \oplus R \cong R,
\]
which recovers the second homology of \( \mathfrak{p}_4(k) \otimes_k R \) obtained in [8].

In the special case where \( (R, -) = (S \oplus S^{op}, \text{ex}) \), Theorem 6.3 recovers the result about the universal central extension of \( \mathfrak{sl}_4(S) \) given in [4].

Corollary 6.5. Let \( S \) be an arbitrary unital associative superalgebra. Then
\[
H_2(\mathfrak{sl}_4(S)) = HC_1(S).
\]

Proof. Recall from Example 2.1 that the Lie superalgebra \( \mathfrak{sl}_4(S) \) is isomorphic to \( \mathfrak{p}_4(S \oplus S^{op}, \text{ex}) \).
Hence,
\[
H_2(\mathfrak{sl}_4(S)) \cong H_2(\mathfrak{p}_4(S \oplus S^{op}, \text{ex})) \cong +\mathbf{HD}_1(S \oplus S^{op}, \text{ex} \circ \rho) \oplus R/(R(-) \cdot R),
\]
where \( (R, -) = (S \oplus S^{op}, \text{ex}) \). Now, \( R(-) \) contains a unit element \( 1 \oplus (-1) \), which yields that \( R/(R(-) \cdot R) = 0 \). Hence,
\[
H_2(\mathfrak{sl}_4(S)) \cong +\mathbf{HD}_1(S \oplus S^{op}, \text{ex} \circ \rho) \cong HC_1(S),
\]
where the last isomorphism follows from Corollary 5.7. \[\square\]

7 The second homology of \( \mathfrak{p}_3(R, -) \)

Analogous to Section 6, we will calculate the second homology of \( \mathfrak{p}_3(R, -) \) via explicitly creating the universal central extension of \( \mathfrak{stp}_3(R, -) \). This will be accomplished by introducing a 2-cocycle on \( \mathfrak{stp}_3(R, -) \) with values in the \( k \)-module:
\[
\mathfrak{z} := \frac{R}{3R + R(-) \cdot R} \oplus \frac{R}{3R + R(-) \cdot R} \oplus \frac{R}{3R + R(-) \cdot R},
\]
where \( R(-) \cdot R \) is the right-ideal of \( R \) generated by \( \bar{a} - \rho(a) \) for \( a \in R \).

Let \( \pi_i(a), i = 1, 2, 3 \) denote the canonical image of \( a \) in one of the three direct summands, respectively. For distinct \( i, j, k \), we will also use \( e(ij,k) \) to denote the sign of the permutation \( (ijk) \).

Recall from Lemma 3.3 that \( \mathfrak{stp}_3(R, -) \) is spanned as a \( k \)-module by
\[
\mathfrak{B} := \{ h_{ij}(a, b), t_{ij}(a), f_{ij}(a, b), g_{ij}(a), f_k(a), g_k(a) | a, b \in R, 1 \leq i, j, k \leq 3 \text{ with } i \neq j \}.
\]
We define a \( k \)-bilinear map \( \beta : \mathfrak{stp}_3(R, -) \times \mathfrak{stp}_3(R, -) \rightarrow \mathfrak{z} \) as follows:
\[
\beta(t_{ij}(a), t_{ik}(b)) = e(ij,k)\pi_1(ab), \quad \beta(f_i(a), g_{jk}(b)) = (-1)^{(i+|a|)(j+|b|)}\beta(g_{jk}(b), f_i(a)) = e(ij,k)\pi_1(ab),
\]
where \( a, b \in R \) are homogeneous and \( \{i, j, k\} = \{1, 2, 3\} \). For other pairs \( (x, y) \in \mathfrak{B} \times \mathfrak{B} \), we set \( \beta(x, y) = 0 \). The \( k \)-bilinear map \( \beta \) is well-defined since
\[
f_i(\bar{a}) = f_i(\rho(a)), \quad f_{ij}(\bar{a}) = f_{ij}(\rho(a)), \quad g_i(\bar{a}) = -g_i(\rho(a)), \quad g_{ij}(\bar{a}) = -g_{ij}(\rho(a)),
\]
while \( \pi_1(\bar{ab}) = \pi_1(\rho(a)b) \).

Lemma 7.1. The \( k \)-bilinear map \( \beta \) is a 2-cocycle on \( \mathfrak{stp}_3(R, -) \) with values in \( \mathfrak{z} \).
Proof. Since $[R, R] \cdot R \subseteq R_{(−)} \cdot R$ implies that $\pi_i(ab) = (-1)^{|a||b|} \pi_i(ba)$, the $\mathbb{k}$-bilinear map $\beta$ satisfies
\[
\beta(x, y) = (-1)^{|x||y|} \beta(y, x),
\]
for homogeneous elements $x, y \in stp_3(R, -)$. It suffices to show
\[
J(x, y, z) := (-1)^{|x||z|} \beta([x, y], z) + (-1)^{|y||z|} \beta([y, z], x) + (-1)^{|z||x|} \beta([z, x], y) = 0,
\]
for $x, y, z \in stp_3(R, -)$. Since (7.2) is symmetric under all permutations of $\{x, y, z\}$, we may assume $\beta([x, y], z) \neq 0$, which only occurs when $z = t_{ik}(a)$, $z = g_{jk}(a)$, or $z = f_i(a)$.

If $z = t_{ik}(a)$ for $a \in R$ and $1 \leq i \neq k \leq 3$, we pick $j$ to be the unique element of $\{1, 2, 3\}$ such that $\{i, j, k\} = \{1, 2, 3\}$. Then we directly verified that $J(x, y, z) = 0$ for all possible choices of $(x, y) \in \mathfrak{B} \times \mathfrak{B}$ such that $\beta([x, y], z) \neq 0$. The pair $(x, y)$ might be one of the following pairs
\[
(h_{ij}(a, a'), t_{ij}(b)), (h_{ik}(a, a'), t_{ij}(b)), (t_{ik}(a), t_{kj}(b)), (f_{ik}(a), g_{kj}(b)), (f_i(a), g_{ij}(b)), (f_{ij}(a), g_j(b)),
\]
for homogeneous $a, a', b \in R$. Similarly, $J(x, y, z) = 0$ when $z = g_{jk}(a)$ or $z = f_i(a)$.

The 2-cocycle $\beta : stp_3(R, -) \times stp_3(R, -) \to \mathfrak{z}$ determines a central extension
\[
\psi' \colon stp_3(R, -) \oplus \mathfrak{z} \to stp_3(R, -),
\]
where $\psi'$ is the canonical projection and the super-bracket on $stp_3(R, -) \oplus \mathfrak{z}$ is given by
\[
[x \oplus c, y \oplus c'] = [x, y] \oplus \beta(x, y), \quad x, y \in stp_3(R, -), \text{ and } c, c' \in \mathfrak{z}.
\]

Proposition 7.2. The central extension $\psi' \colon stp_3(R, -) \oplus \mathfrak{z} \to stp_3(R, -)$ is universal.

Proof. It suffices to show the central extension $\psi \circ \psi' \colon stp_3(R, -) \oplus \mathfrak{z} \to p_3(R, -)$ is universal.

Let $\varphi : \mathfrak{c} \to p_3(R, -)$ be an arbitrary central extension of $p_3(R, -)$. Pick elements $\tilde{h}_{ij}(a), \tilde{f}_{ij}(a), \tilde{g}_{ij}(a) \in \mathfrak{c}$ with $a \in R$ and $1 \leq i \neq j \leq 3$ as in (5.1). By Lemma 5.2, they satisfying all relations (STP00)-(STP11) except (STP04) and (STP10). Moreover, since $m = 3$, there are no four distinct indices $1 \leq i, j, k, l \leq 3$. Hence, (STP10a) and (STP10b) imply (STP10).

For $i \in \{1, 2, 3\}$, there are unique $j$ and $k$ such that $(i, j, k)$ is an even permutation of $\{1, 2, 3\}$. We define
\[
\tilde{\pi}_i(a) := [\tilde{h}_{ij}(1), \tilde{f}_{ik}(a)] \in \ker \varphi,
\]
for $i = 1, 2, 3$ and $a \in R$.

We next show that $\tilde{\pi}_i(3R + R_{(−)} \cdot R) = 0$ for $i = 1, 2, 3$. First, we take $\tilde{h}_{ij} = [\tilde{f}_{ij}(1), \tilde{g}_{ij}(1)]$ and deduce that
\[
0 = [\tilde{h}_{ij}, \tilde{\pi}_i(a)] = [\tilde{h}_{ij}, [\tilde{f}_{ij}(1), \tilde{f}_{ik}(a)]] = [[\tilde{h}_{ij}, \tilde{f}_{ij}(1)], \tilde{f}_{ik}(a)] + [\tilde{h}_{ij}, [\tilde{f}_{ij}(1), \tilde{f}_{ik}(a)]] = 2[\tilde{f}_{ij}(1), \tilde{f}_{ik}(a)] + [\tilde{h}_{ij}(1), \tilde{f}_{ik}(a)] = \tilde{\pi}_i(3a),
\]
where $a \in R$ and $\{i, j, k\} = \{1, 2, 3\}$ are chosen as in (7.3). It follows that $\tilde{\pi}_i(3R) = 0$.

Now, we claim that $\tilde{\pi}_i(ab) = (-1)^{|a||b|} \tilde{\pi}_i(ba)$ for $i = 1, 2, 3$ and homogeneous $a, b \in R$. Indeed, for $i \in \{1, 2, 3\}$, we may pick $j, k$ as in (7.3). Then
\[
\tilde{\pi}_i(ab) = [\tilde{f}_{ij}(1), \tilde{g}_{ij}(ab)] = [\tilde{f}_{ij}(1), [\tilde{f}_{ij}(a), \tilde{g}_{jk}(b)]] = [[\tilde{f}_{ij}(1), \tilde{f}_{ij}(a)], \tilde{g}_{jk}(b)] + [\tilde{f}_{ij}(1), [\tilde{f}_{ij}(a), \tilde{g}_{jk}(b)]]
\]
= (-1)^{a|}[\tilde{t}_{ij}(1), \tilde{f}_{ji}(a), \tilde{g}_{jk}(b)].

Since \([\tilde{t}_{ij}(1), \tilde{f}_{ji}(a)] = [\tilde{f}_{ik}(1), \tilde{f}_{kj}(a)] + c\) for some \(c \in \ker \varphi\), we further deduce that

\[
\tilde{\pi}_i(ab) = (-1)^{a|}[\tilde{f}_{ik}(1), \tilde{f}_{kj}(a), \tilde{g}_{jk}(b)]
= (-1)^{a + [b]}[\tilde{f}_{ik}(1), \tilde{g}_{jk}(b)] + \tilde{f}_{ki}(a) + (1^{a}[\tilde{f}_{ik}(1), [\tilde{f}_{kj}(a), \tilde{g}_{jk}(b)]
= 0 - (-1)^{b}[\tilde{f}_{ik}(1), \tilde{f}_{ij}(ab)]
= (-1)^{b}[\tilde{f}_{ij}(1), \tilde{f}_{jk}(ab)] - (1)^{b}[\tilde{f}_{ij}(1), \tilde{f}_{jk}(ab)]
= (1)^{b}[\tilde{f}_{ij}(1), \tilde{f}_{jk}(ab)]
= (-1)^{b}\tilde{\pi}_i(ab).
\]

It follows that \(\tilde{\pi}_i(b) = (-1)^{b}\tilde{\pi}_i(ab)\) and

\[
\tilde{\pi}_i(ab) = (-1)^{a + [b]}\tilde{\pi}_i(ab) = (-1)^{a + [b]}\tilde{\pi}_i(ab) = (-1)^{b}\tilde{\pi}_i(ba)
= (-1)^{a + [b]}\tilde{\pi}_i(ba) = (-1)^{a}\tilde{\pi}_i(ab).
\]

Therefore, we conclude that \(\tilde{\pi}_i(3R + R_{(-)} : R) = 0\) for \(i = 1, 2, 3\).

Next, we show that \([\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = \epsilon_{ijk}\tilde{\pi}_i(ab)\) for \(\{i, j, k\} = \{1, 2, 3\}\). We first assume that the permutation taking 1 to \(i\), 2 to \(j\), and 3 to \(k\) has positive sign. Then

\[
[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = \tilde{t}_{ij}(a), [\tilde{t}_{ij}(a), \tilde{t}_{jk}(b)]
= [\tilde{t}_{ij}(a), \tilde{t}_{ij}(1), \tilde{t}_{jk}(b)] + [\tilde{t}_{ij}(1), \tilde{t}_{ij}(a), \tilde{t}_{jk}(b)]
= [\tilde{t}_{ij}(1), \tilde{t}_{ik}(ab)]
= \tilde{\pi}_i(ab).
\]

If the permutation \((ijk)\) has negative sign, then \((ikj)\) has positive sign. We have

\[
[\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = -(-1)^{[a]}[\tilde{t}_{ik}(b), \tilde{t}_{ij}(a)] = (-1)^{b}\tilde{\pi}_i(ba) = -\tilde{\pi}_i(ab).
\]

Hence, we conclude that \([\tilde{t}_{ij}(a), \tilde{t}_{ik}(b)] = \epsilon_{ijk}\tilde{\pi}_i(ab)\).

Therefore, there is a homomorphism of Lie superalgebras

\[
\varphi' : \mathfrak{so}_3(\bar{R}, -) \to \mathfrak{e}
\]

such that

\[
\varphi'(t_{ij}(a)) = \tilde{t}_{ij}(a), \quad \varphi'(f_{ij}(a)) = \tilde{f}_{ij}(a), \quad \varphi' (g_{ij}(a)) = \tilde{g}_{ij}(a),
\]

for \(1 \leq i \neq j \leq 3\) and \(a \in \bar{R}\). Hence, \(\varphi \circ \varphi' = \psi \circ \psi'\). The uniqueness of \(\varphi'\) follows from the same argument as in the proof of Proposition 5.4.

Now, we conclude from Propositions 3.4 and 7.2 that:

**Theorem 7.3.** Let \((\bar{R}, -)\) be a unital associative superalgebra with superinvolution. Then

\[
H_2(p_3(R, -)) = +HD_1(R, - \circ \rho) \oplus \frac{R}{3R + R_{(-)} : R} \oplus \frac{R}{3R + R_{(-)} : R} \oplus \frac{R}{3R + R_{(-)} : R}.
\]

**Remark 7.4.** If \(R\) is super-commutative, then \(p_3(R, \rho) \cong p_3(k) \otimes_k R, +HD_1(R, \rho \circ \rho) = 0\), and \(R_{(-)} = 0\). Hence,

\[
H_2(p_3(k) \otimes_k R) = (R/3R) \oplus (R/3R) \oplus (R/3R),
\]

which equals 0 whenever 3 is invertible in \(R\). When \(k\) is a field of characteristic zero, this coincides with the second homology group of \(p_3(k) \otimes_k R\) given in [8].
In the special case where \((R, -) = (S \oplus S^{op}, \text{ex})\), Theorem 7.3 recovers the second homology of \(\mathfrak{sl}_3(S)\) obtained in [4].

**Corollary 7.5.** Let \(S\) be an arbitrary unital associative superalgebra. Then

\[
H_2(\mathfrak{sl}_3(S)) \cong H^1(C(S)).
\]

**Proof.** It is known from Example 2.1 that \(\mathfrak{sl}_3(S)\) is isomorphic to \(p_3(S \oplus S^{op}, \text{ex})\). On the other hand, \(\mathfrak{z} = 0\) since \(1 \oplus (-1)\) is invertible and is contained in the right-ideal of \(S \oplus S^{op}\) generated by \((S \oplus S^{op})(-)\). Hence, \(H_2(\mathfrak{sl}_3(S)) \cong \text{HD}_1(S \oplus S^{op}, \text{ex}) \cong H^1(C(S))\). \(\square\)

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