CONFORMAL AND INVARIANT MEASURE OF A
HYPERBOLIC ENTIRE TRANSCENDENTAL MAP FROM
THE VIEWPOINT OF SYMBOLIC DYNAMICS

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Abstract. We prove that there is a hyperbolic transcendental entire
map that generates a class of potentials which is different from the ones
studied by Mayer and Urbanski (2010). Moreover, a metric on a non-
compact subset of the full shift with a countable alphabet is given, which
is not necessarily compatible with the natural shift metric. This subset
encodes the dynamics of a subset of the Julia set of hyperbolic tran-
scedental maps, which is non-compact and non-Markov, so we study
the existence of a conformal and an invariant probability measure of the
shift on this subset and a class of weakly Hölder continuous potentials.

1. Introduction

Over the last decades, a great deal of attention has been paid to the study
of the iteration of transcendental entire and meromorphic func-
tions [De89, De89a, DT86, Er89, Er92, Er84, GK86, St90, Be90, Be91, Be92, Be93, Be95,
St91, RS99, RS05, BK90, BK91, BK91a, RS09, Re03]. In particular
the dynamics of the exponential family $f_\lambda(z) = \lambda \exp(z)$ for a large class of
parameters has been described in great detail, see for example [Mi81, De84,
De85, De86, De87, Er89a, De91, De94, DG87, DK84, DT86, Er89, Er92,
AO93] and references therein. On the ergodic theory of certain transcen-
dental entire functions, see [McMS77, St91a, Ka99, UZ03, UZ04, IS06, Ba07,
BK07, CS07, MU10].

In [MU10], the authors give a complete description of the thermody-
namic formalism for a very general class of hyperbolic meromorphic maps of finite
order, satisfying some additional properties, and for a class of tame potentials.

In this paper, for a given hyperbolic transcendental entire map we study
a class of potentials and we prove that for a particular entire transcendental
map the class of potentials is different from the ones studied in [MU10]. Indeed,
this class of potentials have an intersection with the class of potentials studied in [MU10],
but the difference is not empty. Compare § 2.

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measures, Entire transcendental maps, Meromorphic functions.
Most of difficulties when dealing with transcendental entire functions are two fold, firstly the set where the interesting dynamics is concentrated, namely the Julia set is never compact, secondly the system restricted to the Julia set is not Markov, so unfortunately the symbolic spaces we need to model Julia sets of these maps fall out of the framework developed by Sarig in [Sa99] and Mauldin-Urbański [MU01], who developed the thermodynamic formalism for topologically mixing Markov shift spaces with infinitely many symbols.

We study the shift map acting on some non-compact and invariant subset of the full shift space with a countably infinite number of symbols, equipped with a metric which is not necessarily equivalent to the natural shift metric, and for a weakly Hölder continuous potentials we prove the existence of a conformal and absolutely continuous invariant probability measure. Our approach is to use an approximation argument as stated in [UZ93], [UZ94], [DU91], we consider restrictions of the map to subsets of the Julia set that can be encoded by a full shift on $2N+1$ symbols, with $N$ as large as desired, since the space is compact and Markov a conformal measure $\nu_N$ exists. The next step is to show that the sequence $\{\nu_N\}_N$ is tight and therefore this sequence has an accumulation point $\nu$. Then, we prove that this measure $\nu$ is a conformal measure for the original map.

Given a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$, the Fatou set $F(f)$ is the subset of $\mathbb{C}$ where the iterates $f^n$ of $f$ form a normal family and its complement is namely called the Julia set, which is denoted by $J(f)$.

Denote by $\text{Sing}(f^{-1})$ the set of finite singularities of the inverse function $f^{-1}$, which is the set of critical values (images of critical points) and asymptotic values of $f$ together with their finite limit points.

The post-singular set $\mathcal{PS}(f)$ of $f$ is defined as,

$$
\mathcal{PS}(f) := \bigcup_{n=0}^{\infty} f^n(\text{Sing}(f^{-1}))
$$

and $\rho_f := \limsup_{z \to \infty} \frac{\log \log |f(z)|}{\log |z|}$ is namely called the order of $f$.

**Definition 1.** Define $\mathcal{F}$ as a class of transcendental entire functions $f$ satisfying the following,

1. It is of finite order;
2. satisfies the rapid derivative growth condition: There are $\alpha_2 > 0$ $\alpha_1 > \alpha_2$ and $\kappa > 0$ such that for every $z \in J(f) \setminus f^{-1}(\infty)$ we have $|f'(z)| \geq \kappa^{-1}|z|^{\alpha_1}|f(z)|^{\alpha_2}$;
3. the set $\text{Sing}(f^{-1})$ is contained in a compact subset of the immediate basin $B = B(z_0)$ of an attracting fixed point $z_0 \in \mathbb{C}$.

Note that each $f \in \mathcal{F}$ belongs to the Eremenko-Lyubich class

$$
\mathcal{B} := \{ f : \mathbb{C} \to \mathbb{C} : \text{Sing}(f^{-1}) \text{ is bounded} \}.
$$
HYPERBOLIC ENTIRE TRANSCENDENTAL MAPS

It was proved in [Er92] that for \( f \in B \) all the Fatou components of \( f \) are simply connected. Hence the immediate basin \( B \) is simply connected. Moreover each \( f \in F \) is hyperbolic in the sense that the closure of \( \overline{PS(f)} \) is disjoint from the Julia set and \( \overline{PS(f)} \) is compact. From [Er89, Er92, GK86, BHKMT93], we have \( f \) has no wandering and Baker domains. So \( B \) is the only Fatou component of \( f \). Transcendental entire functions may have wandering domains, see [Ba76, Ba84, Ba85, Ba87]. Examples in the class \( F \) include the family \( \lambda \exp(z) \) for \( \lambda \in (0, 1/e) \), the family of maps \( \lambda \sin(z) \) for \( \lambda \in (0, 1) \), and \( \lambda g(z) \), where \( \lambda \in \mathbb{C}\{0\} \) and \( g \) is an arbitrary map of finite order such that \( \text{Sing}(g^{-1}) \) is bounded and \( |\lambda| \) is enough small, other examples are the expanding entire maps \( \sum_{p+qj=0} a_j e^{(j-p)z} \), \( p, q > 0, a_j \in \mathbb{C} \), studied early in [CS07].

1.1. Symbolic representation. We fix \( f \in F \), since the immediate attraction basin \( B = B(z_0) \) of an attracting fixed point \( z_0 \) is simply connected, there exists a bounded simply connected domain \( D \subset \mathbb{C} \), such that its closure \( \overline{D} \subset B \), and boundary \( \partial D \) is an analytic Jordan curve. Moreover, \( \text{Sing}(f^{-1}) \subset D \) and \( f(D) \subset D \), for more details see [BK07, Lemma 3.1]. Following [BK07], the pre-images of \( \mathbb{C}\setminus D \) by \( f \) consists of countably many unbounded connected components called tracts of \( f \). We denote the collection of all these tracts by \( \mathcal{R} \).

Since the closure of each tract is simply connected, there exists an open simple arc \( \alpha : (0, \infty) \to \mathbb{C}\setminus \overline{D} \), which is disjoint from the union of the closures of all tracts and such that \( \alpha(t) \) tends to a point of \( \partial D \) as \( t \) tends to \( 0^+ \), and \( \alpha(t) \) tends to \( +\infty \) as \( t \) tends to \( +\infty \). We use this curve to define the fundamental domains on each tract as follows: since for every \( T \in \mathcal{R} \) the map \( f|T \) is a cover of \( \mathbb{C}\setminus \overline{D} \), we have \( T\setminus f^{-1}(\alpha) \) is the union of infinitely many disjoint simply connected domains \( S \) such that the function \( f|S : S \to \mathbb{C}\setminus (\overline{D} \cup \alpha) \) is bijective. Given \( T \in \mathcal{R} \), we denote by \( S_T \) the collection of connected components of \( T\setminus f^{-1}(\alpha) \). The elements of

\[
S := \bigcup_{T \in \mathcal{R}} S_T
\]

are namely called fundamental domains.

For each \( S \in S \), we have that the restriction \( f|S \) is univalent, so we denote its inverse branch by \( g_S := (f|S)^{-1} : \mathbb{C}\setminus (\overline{D} \cup \alpha) \to S \). For \( n \geq 1 \) and each \( j \in \{0, 1, \cdots, n\} \) denote by \( S_j \) an element of \( S \) and put \( g_{S_0 \cdots S_n} = g_{S_0} \circ \cdots \circ g_{S_n} \). Then,

\[
(2) \quad g_{S_0 \cdots S_n}(\mathbb{C}\setminus (\overline{D} \cup \alpha)) = \{ z \in \mathbb{C} : f^j(z) \in S_j, \text{ for every } j = 0, \cdots, n \}.
\]
For each sequence \( S = (S_0S_1 \cdots) \in S^\mathbb{N} \), let \( K_S := \bigcap_{n=0}^{\infty} g_{S_0S_1 \cdots S_n}(C \setminus (\mathcal{D} \cup \alpha)) \).

Then, the Julia set of \( f \) is given by the disjoint union of \( K_S \), that means

\[
J(f) = \bigcup_{S \in S^\mathbb{N}} K_S.
\]

The following theorem proved by Barański [Ba07], states that for each \( S \in S^\mathbb{N} \) such that \( K_S \neq \emptyset \), the set \( K_S \) is a curve tending to infinity (hair) attached to the unique point accessible from \( B \), called endpoint of the hair. This result generalizes previous results of Devaney and Goldberg [DG87] for the exponential map having an attracting fixed point.

**Theorem 1** (Barański [Ba07]). Let \( f \in \mathcal{F} \). Then, either \( K_S \) is empty or there is a homeomorphism \( h_S : [0, +\infty) \to K_S \) such that \( \lim_{t \to +\infty} h_S(t) = \infty \), and such that for every \( t > 0 \) we have \( \lim_{n \to +\infty} f^n(h_S(t)) = \infty \). In the latter case \( z_S := h_S(0) \) is the only point of \( K_S \) accessible from the immediate basin \( B \).

Let us define \( \Sigma_Z := \{ s = (s_0s_1 \ldots) : s_j \in \mathbb{Z}, \text{ for all } j \geq 0 \} \) be the full shift space. The natural shift metric, is defined for some \( \theta \in (0, 1) \) by

\[
d(s, t) = \theta^{\inf\{k : s_k \neq t_k\} \cup \{\infty\}}.\tag{3}
\]

Let \( n \) be a positive integer number. We denote a finite word \( s_0 \cdots s_{n-1} \) simply by \( s^* \). The set

\[
[s^*] = \{ w \in \Sigma_Z : w_i = s_i, 0 \leq i \leq n-1 \}
\]

is namely called cylinder, in particular for \( s \in \mathbb{Z} \),

\[
[s] = \{ w \in \Sigma_Z : w_0 = s \}.
\]

The left-sided shift map \( \sigma : \Sigma_Z \longrightarrow \Sigma_Z \) is defined by

\[
\sigma(s_0s_1 \cdots) = (s_1s_2 \cdots).
\]

Observe that by definition the set \( S \) is countably infinite, so we identify \( S \) with \( \mathbb{Z} \) and put

\[
X := \{ S \in S^\mathbb{N} : K_S \neq \emptyset \} \subseteq \Sigma_Z,
\]

and

\[
Z = \bigcup_{S \in X} K_S.
\]

From (2), we have for each \( S \in S^\mathbb{N} \), \( f(K_S) = K_{\sigma(S)} \), then the function \( f \) on the Julia set \( J(f) \) is semi-conjugate to \( \sigma \) on \( X \). However, if we consider the set of endpoint of hairs,

\[
\mathcal{EP} := \{ z_S = h_S(0) : S \in X \},
\]

then \( f|_{\mathcal{EP}} \) is conjugate to \( \sigma|_X \). Hence the set \( X \) is completely \( \sigma \)-invariant.

---

1 If \( U \) is simply connected domain in the Riemann sphere \( \overline{\mathbb{C}} \), we say that a point \( z \in \partial U \) is **accessible** from \( U \) if there exists a curve \( v : [0, \infty) \to U \) such that \( \lim_{t \to \infty} v(t) = z \).
The set $\mathcal{E}\mathcal{P}$ is the set of accessible points from $B$. It is totally disconnected, however $\mathcal{E}\mathcal{P} \cup \{\infty\}$ is connected, see [BK07]. Moreover its Hausdorff dimension is equal to two, see [Ba08], this result generalizes previous results of Karpińska [Ka99] for the exponential map $E_{\lambda}(z) = \lambda e^z$ for parameters $\lambda \in (0, 1/e)$. The hyperbolic exponential map is probably the best known example in the family $\mathcal{F}$, its Julia set is a Cantor bouquet and the set of endpoints is modelled by the symbolic space of all allowable sequences, see [DK84] and [DG87].

1.2. Induced metric on $X$.

Let $H : \Sigma_\mathbb{Z} \times [0, \infty) \to J(f)$, and $H|_{X \times \{0\}} : X \times \{0\} \to \mathcal{E}\mathcal{P}$ defined by $H(s, 0) = h_s(0)$, we have that $H$ induces a metric $\varrho$ on $X$

$$\varrho(s, w) := |h_s(0) - h_w(0)|.$$ 

Indeed, this metric which is induced by the euclidean metric on $J(f)$ which is not compatible with the topology generated by the natural shift metric. The shift map $\sigma|_X$ is continuous respect to $\varrho$, moreover it can be continuously extended to its completion $Z = \bigcup_{s \in X} K_s$ of $X$ respect to $\varrho$, which, symbolically represents the Julia set $J(f)$.

For more details over the properties of $(X, \varrho)$, see Section § 2.

1.3. A different class of potentials.

For $f \in \mathcal{F}$, let $\rho_f$ be the order of $f$ and $\alpha_1, \alpha_2 > 0$ be the corresponding constants of the rapid derivative growth condition of $f$.

Fix $\tau \in (0, \alpha_2)$ and let $\gamma : \mathbb{C}\setminus\{0\} \to \mathbb{R} \cup \{\infty\}$ defined by $\gamma(z) = \frac{1}{|z|^\tau}$. Let $\theta$ be the Riemannian metric on $\mathbb{C}\setminus\{0\}$ defined by

$$d\theta(z) = \gamma(z)|dz|,$$

and we derive $f$ with respect to $\theta$.

$$|f'(z)|_\theta = |f'(z)| \frac{\gamma \circ f(z)}{\gamma(z)} = |f'(z)| \frac{|z|^\tau}{|f(z)|^\tau}.$$ 

Denote by $\mathcal{C}$ be the set of functions $\psi$ from $\bigcup_{S \in \mathcal{S}} S$ to $\mathbb{R}^+$ that are bounded from above and are constant on each element of $\mathcal{S}$.

$$\mathcal{C} := \left\{ \psi : \bigcup_{S \in \mathcal{S}} S \to \mathbb{R}^+ : \psi \text{ is bounded from above and constant over each } S \in \mathcal{S} \right\}$$

Consider the following class of potentials for $f$:

$$\mathcal{P}_f = \left\{ \varphi_{\psi, t}(z) = \log\psi(z) - t \log|f'(z)|_\theta, \psi \in \mathcal{C}, t > \frac{\rho_f}{\alpha_1 + \tau} \right\}.$$ 

Observe that this class contains potentials $-t \log|f'|_\theta$, which from (4) are cohomologous to $-t \log|f'|$. 
For each \( f \in \mathcal{F} \) we denote by \( \mathcal{H}_f \) the general class of tame potentials considered in [MU10] defined by

\[
\mathcal{H}_f := \left\{ \varphi = h - t \log |f'|_{\theta}; \ h \text{ is bounded weakly Hölder function, } t > \frac{\theta'}{\alpha_1 + \alpha_2} \right\}
\]

Although the class \( \mathcal{F} \) does not include all the maps considered in [MU10], the class \( \mathcal{P}_f \) determined for some function \( f \in \mathcal{F} \) has non-empty intersection with the class of potentials considered in [MU10], but the difference with respect to \( \mathcal{P}_f \) is non-empty.

**Proposition 1.** There exists \( f \in \mathcal{F} \) such that \( \mathcal{P}_f \cap \mathcal{H}_f \neq \emptyset \) and \( \mathcal{P}_f \setminus \mathcal{H}_f \neq \emptyset \).

**Proof.** Consider the exponential family \( \{E_\lambda(z) = \lambda e^z, \lambda \in (0, 1/e)\} \). Each \( E_\lambda \) belongs to \( \mathcal{F} \), because has order equal to 1, satisfies the rapid derivative growth condition with \( \alpha_1 = 0 \) and \( \alpha_2 = 1 \), and since 0 is the only singular value of \( E_\lambda \), so this set is hyperbolic. Moreover the potentials \(-t \log |z| = -t \log |f'_\lambda(z)| + \log \gamma_1 - \log \gamma_1 \circ E_\lambda \), where \( \gamma_1 = |z|^{-t} \), are tame potentials and also belong to the class \( \mathcal{P}_{E_\lambda} \).

On the other hand, the difference between these two classes of potentials with respect to \( \mathcal{P}_{f_\lambda} \) is non-empty. In fact, let \( \mathbb{D} \) be the open unit disk in \( \mathbb{C} \), then we have \( E_\lambda \left( \left\{ z : \text{Re} \ z < \ln \left( \frac{1}{\lambda} \right) \right\} \right) = \mathbb{D} \setminus \{0\} \), and since \( 1 < \ln \left( \frac{1}{\lambda} \right) \) we have \( \overline{E_\lambda(\mathbb{D})} \subset \mathbb{D} \). Moreover since the immediate basin \( B \) of the attracting fixed point is the only Fatou component of \( E_\lambda \) we have \( \overline{\mathbb{D}} \subset B \).

Since \( f_\lambda^{-1}(\mathbb{C} \setminus \mathbb{D}) = \left\{ z : \text{Re} \ z > \ln \left( \frac{1}{\lambda} \right) \right\} \), then the only tract of \( E_\lambda \) is the half plane \( T = \left\{ z : \text{Re} \ z > \ln \left( \frac{1}{\lambda} \right) \right\} \). Let us consider the ray \( \alpha : (0, \infty) \to \mathbb{C} \setminus \mathbb{D} \) defined by \( \alpha(t) = -(1 + t) \), then

\[
E_\lambda^{-1}(\alpha(0, \infty)) = \bigcup_{k \in \mathbb{Z}} \left\{ x + (2k - 1)\pi i : x > \ln \left( \frac{1}{\lambda} \right) \right\},
\]

for each \( k \in \mathbb{Z} \), put \( S_k := \left\{ z : \text{Re} \ z > \ln \left( \frac{1}{\lambda} \right), (2k - 1)\pi < \text{Im} \ z < (2k + 1)\pi \right\} \). Then \( T \setminus E_\lambda^{-1}(\alpha(t)) \) is the disjoint union of the fundamental domains \( S_k \).

Following [IS06], let \( c : J(E_\lambda) \to \mathbb{R}^+ \) be a function such that for each \( k \in \mathbb{Z} \), is constant on \( J(E_\lambda) \cap (S_{-k} \cup S_k) \) and we denote by \( c_k \) its value on this set. Furthermore we assume that the sequence \( (c_k)_{k \in \mathbb{Z}} \) of positive numbers satisfies

\[
(6) \lim_{k \to \infty} \frac{\log c_k}{\log k} = -\infty.
\]

Define \( \varphi(z) := \log \left( c(z) |z|^{-t} \right) \), where \( t > 0 \), \( c(z) = c_k \) if \( z \in S_{-k} \cup S_k \) and the sequence \( (c_k)_{k \in \mathbb{Z}} \) satisfies \([9]\). Observe that any potential as above \( \varphi(z) = \log(c(z)|z|^{-t}) \) satisfies \( \lim_{k \to \infty} c_k = 0 \), so, \( \varphi \) is not a tame potential however this potential belongs to the class \( \mathcal{P}_{E_\lambda} \) because the function \( c \) is bounded on each \( S_k \) and \( |E_\lambda'(z)|_{\theta} = |z| \).

\( \square \)
1.4. Potentials defined on $X$.

Given $\mathbf{s} \in \Sigma_{\mathbb{Z}}$ and $w^{*} \in \mathbb{Z}^{n}$ let us write $w^{*}\mathbf{s} = (w_{0}\cdots w_{n-1}s_{0}s_{1}\cdots)$. For a set $A \subset \Sigma_{\mathbb{Z}}$ let us write $w^{*}A = \{w^{*}\mathbf{s} : \mathbf{s} \in A\}$. For $\mathbf{s} \in X$ and $\delta > 0$ we define

$$B(\mathbf{s}, \delta) := \{w \in X : \varphi(\mathbf{s}, w) < \delta\} = \{w \in X : |h_{\mathbf{s}}(0) - h_{w}(0)| < \delta\},$$

$$\overline{B}(0, \delta) := \{\mathbf{s} \in X : \varphi(\mathbf{s}, 0) \leq \delta\}.$$

For every $n \geq 1$ and $\mathbf{s} \in X$ define

$$\mathbb{B}_{n}(\mathbf{s}, \delta) := \{w \in X : \sigma^{j}(w) \in \mathbb{B}_{0}(\sigma^{j}(\mathbf{s}), \delta), \text{ for all } j = 0, 1, \cdots, n\}.$$

**Definition 2.** Given any $\delta_{1} > 0$ and $\alpha \in (0,1]$, we say that a function $\varphi : X \to \mathbb{R}$ is uniformly $\delta_{1}$-locally $\alpha$-Hölder, if there exists $L \geq 0$ such that for every $\mathbf{s}, \mathbf{t}, w \in X$ satisfying $\mathbf{s}, \mathbf{t} \in \mathbb{B}_{0}(w, \delta_{1})$, we have

$$|\varphi(\mathbf{s}) - \varphi(\mathbf{t})| \leq L\varphi(\mathbf{s}, \mathbf{t})^{\alpha}.$$

For a bounded continuous function $\varphi$, define

$$v_{\alpha, \delta_{1}}(\varphi) := \inf \{L \geq 0 : \text{ for all } \mathbf{s}, \mathbf{t} \in X, \text{ if } \varphi(\mathbf{s}, \mathbf{t}) \leq \delta_{1} \text{ then } |\varphi(\mathbf{s}) - \varphi(\mathbf{t})| \leq L\varphi(\mathbf{s}, \mathbf{t})^{\alpha}\}.$$

The vector space $H_{\alpha, \delta_{1}} := \{\varphi \text{ is bounded continuous : } ||\varphi||_{\alpha, \delta_{1}} < \infty\}$ with the norm $||\varphi||_{\alpha, \delta_{1}} := v_{\alpha, \delta_{1}}(\varphi) + ||\varphi||_{\infty}$, is a Banach space.

Given $\delta > 0$, and $r \in (0,1)$.

**Definition 3.** We say that a continuous potential $\varphi : X \to \mathbb{R}$ is $(\delta, r)$-weakly Hölder continuous if there exist $\tilde{C} > 0$ such that for every $n \geq 0$ we have

$$\text{Var}_{n}(\varphi) := \sup_{\mathbf{s} \in X} \sup_{w \in \mathbb{B}_{n}(\mathbf{s}, \delta)} |\varphi(\mathbf{t}) - \varphi(w)| \leq \tilde{C}r^{n}.$$

Moreover we say that $\varphi$ is weakly Hölder continuous if there are $\delta > 0$ and $r \in (0,1)$ such that $\varphi$ is $(\delta, r)$-weakly Hölder continuous function

We will see that on $(X, \varrho)$, and for an appropriate $\delta$, every uniformly $\delta$-locally $\alpha$-Hölder potential is weakly Hölder, compare Lemma $\mathbb{X}$

If we fix $\delta > 0$ and $r \in (0,1)$, we denote by $\text{BV}_{r}(X, \mathbb{R})$ the space of functions $\varphi : X \to \mathbb{R}$, which are $(\delta, r)$-weakly Hölder continuous.

Observe that for every $\varphi_{c,t} \in \mathcal{P}_{f}$, we have $\varphi_{c,t} \circ H$ and $\exp \varphi_{c,t} \circ H$ are weakly Hölder continuous. It is due to $\varrho$ is induced by the euclidean metric on $J(f)$.

**Proposition 2.** Let $\varphi_{c,t} \in \mathcal{P}_{f}$, then $\varphi_{c,t} \circ H$ and $\exp \varphi_{c,t} \circ H$ are weakly Hölder continuous.
Proof. Follows from [MU10, Lemma 4.7, Lemma 4.10]. □

Definition 4. A potential $\varphi : X \to \mathbb{R}$ is summable if

$$\sup_{s \in X} \sum_{u \in \mathbb{Z}} e^{\varphi(u s)} < \infty,$$

Defintion 5. The function $\varphi$ is bounded on balls if for every $R > 0$ we have

$$\sup_{B(0,R)} |\varphi| < \infty.$$

Definition 6. Given $R > 0$ and $k \in \mathbb{Z}$, we say that a potential $\varphi$ is rapidly decreasing if satisfies

$$\lim_{R \to \infty} \sum_{u \in \mathbb{Z}} \exp \left( \sup_{s \in X \setminus B(0,R) \cap [u]} \varphi(s) \right) = 0.$$

Let us define

$$X_{\text{rad}} := \{ \underline{s} \in \mathbb{Z} : \omega(\underline{s}) \cap X \neq \emptyset \},$$

where $\omega(\underline{s})$ is the omega limit of $\underline{s}$. We have $X_{\text{rad}} \subset X$.

Theorem A. Let $\sigma : X \to X$ be the shift map on

$$X = \{ \underline{s} \in \Sigma \mathbb{Z} : K_{\underline{s}} \neq \emptyset \}$$

and let $\varphi$ be a weakly Hölder continuous potential satisfying conditions (7), (8), (9). Then there exists a conformal measure $m$ and absolutely continuous invariant measure $\nu$ equivalent to $m$ supported on $X_{\text{rad}}$.

Corollary 1. Let $f \in F$, Then for every potential $\varphi \in P_f$. Then there exists a conformal measure $\nu_{\varphi}$ of $f$ and there exists an invariant probability measure $\mu_{\varphi}$, both measures are supported on the radial Julia set

$$J_{\text{rad}}(f) := \left\{ z \in J(f) : \lim_{n \to \infty} f^n(z) < \infty \right\}.$$

1.5. Organization. We gather several definitions, properties of the space $(X, g)$ and potentials in § 3. After these considerations Theorem A is proved in § 4.

2. Preliminaries

2.1. Definitions.

Definition 7. (Conformal measure) Let $T : Y \to Y$ be a measurable map on a measurable space. If $g : Y \to \mathbb{R}$ is a non-negative measurable function. A probability measure $\mu$ on $Y$ is called $g$-conformal for $T$, if for every measurable set $A$ which $T(A)$ is measurable and $T|_A$ is injective we have

$$\mu(T(A)) = \int_A g d\mu.$$
We give the definition of Gibbs measures of potentials adapted for the entire transcendental functions.

**Definition 8.** (Gibbs measure) A probability measure $\eta$ on $X$ is called a Gibbs measure for the potential $\varphi$ if there exist $\mathcal{P} \in \mathbb{R}$, $D \geq 1$ such that for every $s \in X$, there exists $R = R(s)$ such that for all $n \geq 1$ and $u^* \in \mathbb{Z}^n$ we have
\begin{equation}
D^{-1} R(s) \leq \frac{\eta(u^*B_0(s, \delta))}{\exp(S_n(\varphi)(u^*s) - n\mathcal{P})} \leq D.
\end{equation}
If additionally $\eta$ is $\sigma$-invariant, we call $\eta$ an invariant Gibbs measure.

**Definition 9.** (Tightness) A family of probability measures $\{\nu_N\}$ is tight if for every $\epsilon$ there exists a compact set $K \subset Y$ such that for every $N$ we have $\nu_N(Y \setminus K) < \epsilon$.

**Theorem 2.** (Prokhorov’s theorem) If $Y$ is a Polish space, and $M(Y)$ is the space of all Borel probability measures in $Y$, then every tight family measures from $M(Y)$ is precompact subset of $M(Y)$, see [Bi99].

2.2. Properties of the metric space $(X, \varrho)$.

Consider
\begin{equation}
X = \{ \underline{s} \in \mathcal{S} : K_{\underline{s}} \neq \emptyset \} \subseteq \Sigma_{\mathbb{Z}},
\end{equation}
equipped with the metric $\varrho$, and let
\[ Z = \bigcup_{\underline{s} \in X} K_{\underline{s}}. \]
The set $X$ is non-compact, however we have that there exists a family of compact subsets $(\Sigma_N)_{N \geq 1}$, each $\Sigma_N$ is $\sigma$-invariant. Moreover for each $N \geq 1$, we have $\Sigma_N \subset X$, $\Sigma_N \subset \Sigma_{N+1}$ and $\bigcup_{N \geq 1} \Sigma_N$ is dense in $X$.

For each $N \geq 1$, define
\[ \Sigma_N := \{ \underline{s} = (s_0 s_1 \cdots) \in X : \text{ for } j \geq 0 \text{ we have } s_j \in \{-N, \cdots, N\} \}. \]

**Proposition 3.** The following properties hold
\begin{enumerate}
\item For each positive integer number $N$, we have $\Sigma_N$ is compact with respect to $\varrho$ and invariant by $\sigma$.
\item For each compact subset $\Lambda$ of $X$ with respect to the metric $\varrho$, so that $\sigma(\Lambda) \subset \Lambda$, we have, there exists $N_0 \geq 1$, such that $\Lambda \subset \Sigma_{N_0}$.
\end{enumerate}

**Proof.** By Ahlfors Spiral Theorem [Ha89], transcendental entire functions of finite order have only a finite numbers of tracts, then we assume without loss of generality that for $f \in \mathcal{F}$ there is only one tract $T$.

Let $N \geq 1$ and denote by $S_N$ the union of $2N + 1$ fundamental domains in $T$ and define
\[ K_N := \{ z \in S_N : \text{ for every } j \geq 0, f^j(z) \in S_N \}. \]
Let \((x^m)_{m \geq 1}\) be a sequence in \(\Sigma_N\). By taking a subsequence if necessary, we can assume that for some \(R\) being chosen large enough, the subsequence \((x^m)_{m \geq 1}\) is contained in \(\Sigma_N \cap B(0, R)\). So for every \(m \geq 1\) there is an endpoint \(x^m = h_{x^m}(0) \in K_N \cap B(z^m, R)\). Since \(K_N \cap B(z^m, R)\) is bounded and \(K_N\) is closed in \(J(f)\) there is a subsequence converging to some point \(z \in K_N\). Let \(s\) be the allowable itinerary sequence associated to \(z\), then \(s \in \Sigma_N^+\) and

\[\varrho(s^{m_j}, s) = |x^{m_j} - z| \to 0, \text{ as } j \to \infty.\]

Moreover, \(\Sigma_N\) is invariant by \(\sigma\).

On the other hand, let \(\Lambda\) be a compact subset of \(X\) with respect to the metric \(\varrho\) such that \(\sigma(\Lambda) \subset \Lambda\). Let \(s \in \Lambda\) and let \(z \in H(\Lambda)\) with allowable itinerary sequence \(s\), then since the compact subset \(H(\Lambda)\) intersects only a finite numbers of tracts (see [BK07, Lemma 3.2]), there exists \(N_0 \geq 1\) such that for every \(j \geq 0\) we have \(|s_j| \leq N_0\). Therefore \(\Lambda \subset \Sigma_{N_0}\). \(\square\)

This metric has the property of exponentially shrinking of preimages.

**Proposition 4.** There exists \(\delta_0\) such that the following condition holds:

There exist \(C_E > 0\) and \(\lambda_E > 1\) such that for every \(n \in \mathbb{N}\) and \(\bar{s}, \bar{t} \in X\) and \(u^* \in \mathbb{Z}^n\), if \(\varrho(\bar{s}, \bar{t}) < \delta_0\) then we have

\[\varrho(u^* \bar{s}, u^* \bar{t}) \leq C_E \lambda_E^{-n} \varrho(\bar{s}, \bar{t}).\]

**Proof.** Follows from the derivative grown condition and the uniformly expanding property of \(f\), compare [MU10] Proposition 4.4. \(\square\)

We note that the natural shift metric \(d\) in (3) satisfies in particular this condition, compare \((35)\) in §4

The following properties follow immediately from the definitions, and will be used several times. For every \(\delta > 0\), \(\bar{s} \in X\), \(n, m \geq 0\) and \(u^* \in \mathbb{Z}^n\) we have

\[\sigma^n(\mathbb{B}_{m+n}(\bar{s}, \delta)) \subseteq \mathbb{B}_m(\sigma^n \bar{s}, \delta),\]

\[\mathbb{B}_{m+n}(u^* \bar{s}, \delta) \subseteq u^* \mathbb{B}_m(\bar{s}, \delta).\]

For the rest of this section we fix \(\delta_0\) from Proposition 4. So, the following lemmas and propositions hold.

**Lemma 1.** Fix \(n_0 \geq 0\) such that \(C_E \lambda_E^{-n_0} \leq \min\{1, \frac{1}{C_E}\}\). Then for every \(\delta \in (0, \delta_0]\), \(n, m \geq 0\), \(u^* \in \mathbb{Z}^n\) and \(\bar{s} \in X\), we have

1. \(\mathbb{B}_{m+n}(u^* \bar{s}, \delta) \subseteq u^* \mathbb{B}_m(\bar{s}, \delta) \subseteq \mathbb{B}_m(u^* \bar{s}, C_E \lambda_E^{-n} \delta).\)
   In particular taking \(n = n_0\), we have
   \[\mathbb{B}_{m+n_0}(u^* \bar{s}, \delta) \subseteq u^* \mathbb{B}_m(\bar{s}, \min\{\delta, \delta/C_E\}).\]
2. \(\mathbb{B}_{m+n}(u^* \bar{s}, \min\{\delta, \delta/C_E\}) \subseteq u^* \mathbb{B}_m(\bar{s}, \min\{\delta, \delta/C_E\}) \subseteq \mathbb{B}_{m+n}(u^* \bar{s}, \delta).\)
Proof. 1. The first inclusion is (14). The second inclusion follows since for \( \bar{t} \in u^* B_m(\underline{s}, \delta) \) there is \( \bar{t}' \in B_m(\underline{s}, \delta) \) such that \( \bar{t} = u^* \bar{t}' \) and for each \( j = 0, \cdots, m \), we have
\[
\varrho(\sigma^j \underline{s}, \sigma^j \underline{t}') < \delta \quad \text{and} \quad \underline{x}_j = \underline{t}_j.
\]
So, from Proposition 4 implies that for such \( j \) we have
\[
\varrho(\sigma^j (u^* \underline{s}), \sigma^j (u^* \bar{t}')) < C E^{\lambda_n^j \varrho(\underline{t}', \underline{s})} < C \delta' \leq \delta.
\]

2. Set \( \delta' = \min \{ \delta, \delta / C_E \} \). The first inclusion is (14) with \( \delta' \) replaced by \( \delta' \).
In order to prove the second inclusion let \( \bar{t} \in u^* B_m(\underline{s}, \delta') \). Then there exists \( \bar{t}' \in B_m(\underline{s}, \delta') \) such that \( b = u^* \bar{t}' \), hence for every \( j = 0, 1, \cdots, m \) we have
\[
\varrho(\sigma^j \bar{t}', \sigma^j \underline{s}) < \delta' \quad \text{and} \quad \underline{t}_j = \underline{s}_j.
\]
Moreover from Proposition 3 for each \( j = 0, 1, \cdots, n \) we have
\[
\varrho(\sigma^j (u^* \bar{t}'), \sigma^j (u^* \underline{s})) 
\leq C E^{\lambda_n^j \varrho(\underline{t}', \underline{s})} < C \delta' \leq \delta.
\]

\( \square \)

**Proposition 5.** For every \( R > 0 \) there exists \( n \geq 1 \) such that for every \( \underline{s} \in B(0, R) \), we have \( \sigma^n(B(\underline{s}, \delta_0)) \supset B(0, R) \).

**Corollary 2.** The system \((X, \sigma)\) is topologically mixing

**Proof.** Let \( \underline{s}, \underline{t} \in X \) and \( U = B(\underline{s}, \delta_1), \quad V = B(\underline{t}, \delta_2), \quad \delta_1, \delta_2 > 0 \). Then from Proposition 3 we have there is \( n_1 > 0 \) such \( \sigma^{n_1}(U) \supset B(\sigma^{n_1} \underline{s}, \delta_0) \).

Let \( R > 0 \) such that \( C E^{\lambda_{n_1} \delta_0} \). Then, there is \( N_1 \) such that
\[
\sigma^{N_1}(B(\sigma^{n_1} \underline{s}, \delta_0)) \supset B(0, R).
\]

So, \( \sigma^{N_1\lambda_{n_1}}(U) \supset \sigma^{N_1}(B(\sigma^{n_1} \underline{s}, \delta_0)) \supset B(0, R) \). Taking \( m > N_1 + n_1 \), we have for every \( k \geq m \), \( \sigma^k(U) \cap V \neq \emptyset \).

\( \square \)

**Proposition 6.** The set \( \bigcup_{N \geq 1} \Sigma_N \) is dense in \( X \).

**Proof.** Let \( \underline{s} = (s_0 s_1 \cdots) \in X \) and \( \epsilon > 0 \). Then, there exists \( n_1 > 0 \) such that
\[
B(\sigma^{n_1} \underline{s}, \delta_0) \subseteq \sigma^{n_1} B(\underline{s}, \epsilon).
\]

It is enough to take \( n_1 > 0 \) such that \( C E^{\lambda_{n_1} \delta_0} \leq \epsilon \). So, we have \( s_0 s_1 \cdots s_{n_1 - 1} B(\sigma^{n_1} \underline{s}, \delta_0) \subseteq B(\underline{s}, C E^{\lambda_{n_1} \delta_0}) \subseteq B(\underline{s}, \epsilon) \).

Hence, \( B(\sigma^{n_1} \underline{s}, \delta_0) \subseteq \sigma^{n_1} B(\underline{s}, \epsilon) \).

On the other hand, since \( X = \bigcup_{R > 0} B(0, R) \), then for some \( R > 0 \) we have \( \sigma^{n_1} \underline{s} \in B(0, R) \). Moreover from Proposition 5 there is \( n_2 > 0 \) such that \( B(0, R) \subseteq \sigma^{n_2} B(\sigma^{n_1} \underline{s}, \delta_0) \).

Therefore, for \( n = n_1 + n_2 \) we follow that \( B(0, R) \subseteq \sigma^n B(\underline{s}, \epsilon) \). Hence, the set \( \sigma^n B(\underline{s}, \epsilon) \) contains the sequence \( \underline{0} = 00 \cdots \). Let \( w^* = w_0 \cdots w_{n_1 - 1} \in \mathbb{Z}^n \) such that \( w^* B(0, R) \subset B(\underline{s}, \epsilon) \), then \( w_0 \cdots w_{n_1 - 1} \underline{0} \in B(\underline{s}, \epsilon) \). Then, taking \( N := \max\{|w_0|, \cdots, |w_{n_1 - 1}|\} \) we conclude \( w_0 w_1 \cdots w_{n_1} \underline{0} \in \Sigma_N \).

\( \square \)
2.3. Properties of potentials. From the dynamics of \((J(f), f_J(f))\) we have the following properties on \((X, \varrho)\).

**Proposition 7.** Let \(\delta \in (0, \delta_0] \text{ and } \delta' = \min\{\delta, \frac{\delta}{C_E}\}.\) There exists \(\ell \geq 1\) such that for every \(s \in X\) and for every \(t, w \in B_0(s, \delta)\), there exists a sequence
\[
\ell = s_0, \ldots, s_\ell = w,
\]
such that for each \(j \geq 0\) we have
\[
\varrho(s_j, s_{j+1}) < \delta'.
\]

**Proposition 8.** The following properties hold
a) The map \(\sigma : X \to X\) can be continuously extended to its completion \(\bar{\sigma} : Z \to Z\).

b) For every \(R > 0\), the set \(\bar{B}(0, R)\) is compact in \(Z\).

c) For every \(R > 0\) and for every \(t \in Z \setminus X\) there exists \(N \geq 1\) such that for \(n \geq N\), we have \(\bar{\sigma}^n(t) \in Z \setminus B(0, R)\).

For every \(s, t \in X\) and \(k \geq 1\), we define
\[
\varrho_k(s, t) := \max_{0 \leq j \leq k} \varrho(\sigma^j s, \sigma^j t).
\]

**Lemma 2.** Let \(\delta \in (0, \delta_0] \text{ and } \delta' = \min\{\delta, \delta/C_E\}.\) For every \(k \geq 1, w \in X\) and \(s, t \in B_k(w, \delta)\) we have that there exists a sequence \(s = u_0, \ldots, u_\ell = t\) such that for each \(j \in \{0, \ldots, \ell - 1\}\),
\[
\varrho_k(u_j, u_{j+1}) < \delta'.
\]

Proof. We will proceed by induction on \(k\). From Proposition 7, the desired assertion is satisfied for \(k = 1\). Suppose that this is true for \(k > 1\). For \(s, t \in B_{k+1}(w, \delta)\) we have \(\sigma(s), \sigma(t) \in B_k(\sigma(w), \delta)\), then there exists a sequence \(\sigma(s) = u_0, \ldots, u_\ell = \sigma(t)\) such that for every \(j \in \{0, \ldots, \ell - 1\}\) we have
\[
\varrho_k(u_j, u_{j+1}) < \delta'.
\]

Then, let \(u^*\) the first word from \(w\) such that by item 1 of Lemma 1, we have \(u^*B_0(\sigma(w), \delta') \subset B_0(u^* \sigma(w), \delta)\) and for every \(j = 0, \ldots, \ell - 1\),
\[
\varrho(u^* u_j, u^* u_{j+1}) < \delta.
\]

Then for each \(j = 0, \ldots, \ell - 1\) there is a sequence \(u^* u_j = w_0^j, w_1^j, \ldots, w_\ell^j = u^* u_{j+1}\), such that for every \(0 \leq h \leq \ell - 1\) we have \(\varrho(w_h^j, w_{h+1}^j) < \delta'.\) So there is a path
\[
s = w_0^0, \ldots, w_0^\ell, w_0^1, \ldots, w_\ell^1, \ldots, w_{\ell - 1}^1, \ldots, w_\ell^{\ell - 1} = t,
\]
such that for each \(0 \leq j \leq \ell - 1\) and \(0 \leq h \leq \ell - 1\) we have
\[
\varrho_{k+1}(w_h^j, w_{h+1}^j) < \delta'.
\]
Lemma 3. Let $\alpha \in (0, 1)$ and $\delta \in (0, \delta_0]$. Then every uniformly $\delta$-locally $\alpha$-Hölder continuous potential is $(\delta, r)$-weakly Hölder continuous with constant $r = \lambda_E^{-\alpha}$, where $\lambda_E$ comes from Proposition 4.

Proof. By part 1 of Lemma 1 with $u^s$ replaced by $s$ and with $m = 0$, we obtain $B_n(s, \delta) \subseteq B_0(s, C_E \lambda_E^{-n} \delta)$. So, if $\varphi$ is uniformly $\delta$-locally $\alpha$-Hölder, then we have that there exists $L \geq 0$ such that for every $n \geq 0$, $s \in X$ and $t, u \in B_n(s, \delta)$ we have

$$|\varphi(t) - \varphi(u)| \leq L \varphi(t, u)^{\alpha} \leq L(C_E \lambda_E^{-n})^{\alpha} \delta^{\alpha} = LC_E^{\alpha}(\lambda_E^{-\alpha})^n. \tag*{□}$$

Fix $r \in (0, 1)$ and let $\varphi \in BV_r(X, \mathbb{R})$, we define

$$W_r(\varphi) := \sup \left\{ \frac{\text{Var}_n(\varphi)}{r^n} : n \geq 1 \right\}.$$

So, $\|\varphi\|_r = \|\varphi\|_\infty + W_r(\varphi)$ defines a norm on $BV_r(X, \mathbb{R})$, which makes it a Banach space.

Put $\delta' = \min\{\delta, \delta/C_E\}$.

Proposition 9. If $\varphi$ is $(\delta', r)$-weakly Hölder continuous then $\varphi$ is $(\delta, r)$-weakly Hölder continuous.

Proof. Let $\varphi$ be a $(\delta', r)$-weakly Hölder potential. Let $w \in X$, $m \geq 0$ and $s, t \in B_m(w, \delta)$. From part 1 of Lemma 1 we have that for some $n_0$ satisfying $C_E \lambda_E^{n_0} \leq \min\{1, 1/r\}$ and for every $n \geq 0$,

$$B_{n+n_0}(w, \delta) \subseteq B_n(w, \delta').$$

If $m \geq n_0$ then $B_m(w, \delta) \subseteq B_{m-n_0}(w, \delta')$. Thus we have

$$|\varphi(s) - \varphi(t)| \leq W_r(\varphi)^{m-n_0} = (W_r(\varphi)^r)^{m-n_0}. \tag*{□}$$

If $1 < m < n_0$, then from Proposition 7, we have that Lemma 3 is satisfied. So, for every $s, t \in B_m(w, \delta)$, there is a sequence $s = u_0, \ldots, u_m = t$ such that for every $j = 0, \ldots, \ell^m - 1$ we have $\varphi_m(u_j, u_{j+1}) \leq \delta'$. Hence

$$|\varphi(s) - \varphi(t)| \leq \sum_{j=0}^{\ell^m-1} |\varphi(u_j) - \varphi(u_{j+1})| \leq (\ell^mW_r(\varphi))r^m. \tag*{□}$$

For every $n \geq 1$, we denote

$$S_n(\varphi) := \sum_{k=0}^{n-1} \varphi \circ \sigma^k.$$ 

If $n = 0$, then put $S_0(\varphi) \equiv 0$.

Lemma 4. Let $\varphi \in BV_r(X, \mathbb{R})$. Then for each $n \geq 0$ we have the following assertions.
1. For every $m \geq 0$ we have

$$V_{m+n}(S_n(\varphi)) \leq \left( W_r(\varphi) \frac{r^n}{1-r} \right)^{m+n}. $$

2. For every $u^* \in \mathbb{Z}^n$ and $\mathbf{s} \in X$, let $\varphi_{n,u^*}(\mathbf{s}) := S_n(\varphi)(u^* \mathbf{s})$, then $\varphi_{n,u^*} \in BV_r(X,\mathbb{R})$.

Proof. 1. Let $\mathbf{w} \in X$ and $m \geq 0$. Using (13) we have that for each $j = 0, 1, \cdots, n-1$,

$$\sigma^j \mathbb{B}_{m+n}(\mathbf{w}, \delta) \subseteq \mathbb{B}_{m+n-j}(\sigma^j \mathbf{w}, \delta).$$

Then, for each $\mathbf{s}, \mathbf{t} \in \mathbb{B}_{m+n}(\mathbf{w}, \delta)$ we have

$$|S_n(\varphi)(\mathbf{s}) - S_n(\varphi)(\mathbf{t})| \leq \sum_{j=0}^{n-1} |\varphi(\sigma^j \mathbf{s}) - \varphi(\sigma^j \mathbf{t})|$$

$$\leq \sum_{j=0}^{n-1} V_{m+n-j}(\varphi) \leq W_r(\varphi) \sum_{j=0}^{n-1} j^{m+n-j} \leq W_r(\varphi)r^{m+n} \frac{1}{1-r} = \left( W_r(\varphi) \frac{r^n}{1-r} \right)^{m+n}.$$

2. Let $n \geq 0$, $u^* \in \mathbb{Z}^n$ and $m \geq 0$. We have to prove that $\varphi_{n,u^*}$ is $(\delta, r)$-weakly Hölder, with $\delta' = \min \left\{ \delta, \frac{\delta}{C_E} \right\}$. From Proposition 9 it is enough to show that $\varphi_{n,u^*}$ is $(\delta', r)$-weakly Hölder. In fact, let $\mathbf{w} \in X$ and $\mathbf{s}, \mathbf{t} \in \mathbb{B}(\mathbf{w}, \delta')$. By part 2 of Lemma 1 we have

$$u^* \mathbb{B}_n(\mathbf{w}, \delta') \subseteq \mathbb{B}_{m+n}(u^* \mathbf{w}, \delta).$$

Hence, by using (15) of part 1 of this lemma we get, for $u^* \mathbf{s}, u^* \mathbf{t} \in \mathbb{B}_{m+n}(u^* \mathbf{w}, \delta)$

$$|\varphi_{n,u^*}(\mathbf{s}) - \varphi_{n,u^*}(\mathbf{t})| = |S_n(\varphi)(u^* \mathbf{s}) - S_n(\varphi)(u^* \mathbf{t})| \leq \left( W_r(\varphi) \frac{r^n}{1-r} \right)^{m+n}.$$

Therefore, the function $\varphi_{n,u^*}$ is $(\delta, r)$-weakly Hölder continuous.

Lemma 5. Let $\varphi \in BV_r(X,\mathbb{R})$. Then for every $n \geq 1$, $u^* \in \mathbb{Z}^n$, $m \geq 1$, $\mathbf{w} \in X$ and $\mathbf{s}, \mathbf{t} \in \mathbb{B}_m(\mathbf{w}, \delta)$, if we put $K := W_r(\varphi_{n,u^*})$, then

$$|e^{S_n(\varphi)(u^* \mathbf{s})} - e^{S_n(\varphi)(u^* \mathbf{t})}| \leq e^K e^{S_n(\varphi)(u^* \mathbf{t})}.$$

Proof. Let $m \geq 1$, $\mathbf{w} \in X$ and $\mathbf{s}, \mathbf{t} \in \mathbb{B}_m(\mathbf{w}, \delta)$. Since for every $n \geq 1$ and $u^* \in \mathbb{Z}^n$ the function $\varphi_{n,u^*} \in BV_r(X,\mathbb{R})$, we have

$$|\varphi_{n,u^*}(\mathbf{s}) - \varphi_{n,u^*}(\mathbf{t})| \leq K r^m \leq K,$$

with $K = W_r(\varphi_{n,u^*})$.

Then,

$$e^{\varphi_{n,u^*}(\mathbf{s})} - \varphi_{n,u^*}(\mathbf{t}) - 1 \leq e^K - 1,$$

and

$$1 - e^{\varphi_{n,u^*}(\mathbf{s})} - \varphi_{n,u^*}(\mathbf{t}) \leq 1 - e^{-K} \leq e^K - 1.$$
Therefore
\[ |e^{S_n(\varphi)(u^*\mathbf{1}_s)} - S_n(\varphi)(u^*\mathbf{1}_t) - 1| \leq e^K - 1. \]
\[ \square \]

For \( \varphi : X \to \mathbb{R} \) be a weakly Hölder continuous potential satisfying summability condition (7), define the transfer operator of \( \varphi \) acting on any \( g : X \to \mathbb{R} \), be a bounded continuous function,
\[ \mathcal{L}_\varphi(g)(\mathbf{s}) = \sum_{u \in \mathbb{Z}} e^{S_n(\varphi)(u^*\mathbf{s})} g(u^*\mathbf{s}). \]
For every \( n \geq 1 \) and \( \mathbf{s} \in X \), let
\[ \mathcal{L}_\varphi^n(g)(\mathbf{s}) = \sum_{u^* \in \mathbb{Z}^n} e^{S_n(\varphi)(u^*\mathbf{s})} g(u^*\mathbf{s}). \]

Define \( \mathcal{K}(X, \sigma) \) the set of all \( \sigma \)-invariant and compact subset of \( X \) with respect to \( \varrho \). From Proposition 3 we have that \( \mathcal{K}(X, \sigma) \neq \emptyset \).

Define
\[ P(\sigma, \varphi) := \sup_{K \in \mathcal{K}(X, \sigma)} P_K(\sigma, \varphi), \]
where,
\[ P_K(\sigma, \varphi) := \sup \left\{ h_\mu + \int \varphi d\mu, \mu \text{ is invariant probability measure on } K \right\}. \]
Moreover, since for each set \( K \in \mathcal{K}(X, \sigma) \), there is some subset \( \Sigma_N \) of \( X \), such that \( K \subset \Sigma_N \), we have
\[ P(\sigma, \varphi) = \sup_{N \geq 1} P_{\Sigma_N}(\sigma, \varphi). \]

On the other hand, since for each \( N \geq 1 \), \( \Sigma_N \subset \Sigma_{N+1} \), then \( P_{\Lambda_N}(\sigma, \varphi) \leq P_{\Lambda_{N+1}}(\sigma, \varphi) \), then the limit
\[ \lim_{N \to \infty} e^{P_{\Lambda_N}(\sigma, \varphi)} \]
exists and it is equal to \( e^{P(\sigma, \varphi)} \).

We will denote \( e^{P(\sigma, \varphi)} \) simply by \( \vartheta \).

**Lemma 6.** Let \( \varphi \in \text{BV}_r(X, \mathbb{R}) \), summable. For every \( n \geq 0 \) and \( m \geq 0 \) there exists a constant \( K > 0 \) such that for every \( \mathbf{w} \in X \) and \( \mathbf{s}, \mathbf{t} \in \mathbb{B}_m(\mathbf{w}, \delta) \) we have
\[ \left| \mathcal{L}^n_{\varphi}(\mathbf{s}) - \mathcal{L}^n_{\varphi}(\mathbf{t}) \right| \leq e^K \mathcal{L}^n_{\varphi}(\mathbf{s}). \]
Lemma 7. Let $\varphi \in BV_{\ell}(X, \mathbb{R})$ summable and bounded on balls. Then for every $R > 0$ there exists $L_{\varphi,R} > 0$ such that for every $n \geq 0$, $\mathbf{s}, \mathbf{t} \in B(0, R)$ we have

$$L_{\varphi,R}^{-1} \leq L_{\varphi,R} \leq K L_{\varphi,R}^{n}.$$ 

Proof. Let $n \geq 0$, $u^* \in \mathbb{Z}^n$ and $K = \mathcal{W}_2(\varphi_{n,u^*})$. Then for $\mathbf{s}, \mathbf{t} \in \mathbb{B}_n(\mathbf{w}, \delta)$ we have $|e^{S_n(\varphi)(\mathbf{s})} - e^{S_n(\varphi)(\mathbf{t})}| \leq e^K e^{S_n(\varphi)(\mathbf{s}^*)}$. Therefore,

$$\left| L_{\varphi,R}^{-1}(\mathbf{s}) - L_{\varphi,R}^{-1}(\mathbf{t}) \right| \leq \sum_{u^* \in \mathbb{Z}^n} e^{S_n(\varphi)(\mathbf{s}^*)} - e^{S_n(\varphi)(\mathbf{t}^*)} \leq \sum_{u^* \in \mathbb{Z}^n} |e^{S_n(\varphi)(\mathbf{s}^*)} - e^{S_n(\varphi)(\mathbf{t}^*)}| \leq e^K \sum_{u^* \in \mathbb{Z}^n} e^{S_n(\varphi)(\mathbf{s}^*)} \leq e^K L_{\varphi,R}^{-1}(\mathbf{t}).$$

Observe that

$$L_{\varphi,R}^{-1}(\mathbf{s}) = \sum_{\mathbf{w}: \sigma^{n+\eta}(\mathbf{w}) = \mathbf{s}} e^{S_n(\varphi)(\mathbf{w})} e^{S_{n+\eta}(\varphi)(\mathbf{w})} = \sum_{\mathbf{w}: \sigma^{n+\eta}(\mathbf{w}) = \mathbf{s}} e^{S_n(\varphi)(\mathbf{w})} e^{S_{n+\eta}(\varphi)(\mathbf{w})} \geq \sum_{\mathbf{w}: \sigma^{n+\eta}(\mathbf{w}) = \mathbf{s}} e^{S_n(\varphi)(\mathbf{w})} e^{S_{n+\eta}(\varphi)(\mathbf{w})} = e^{S_{n+\eta}(\varphi)(\mathbf{w})} L_{\varphi,R}^{-1}(\mathbf{w}).$$

From Lemma 5 we have, for every $n \geq 0$,

$$L_{\varphi,R}^{-1}(\mathbf{s}) \leq (1 + e^K) L_{\varphi,R}^{-1}(\mathbf{w}).$$
Therefore,
\[ L^n_{\varphi}(s) \leq (1 + e^K)e^{-S_{n_0}(\varphi)(\underline{u})}L_{\varphi}^{n+n_0}(t) \]
\[ \leq (1 + e^K)e^{-S_{n_0}(\varphi)(\underline{u})}\|L_{\varphi}\|_\infty^n L_{\varphi}(t) \]
\[ \leq (1 + e^K)L_{\varphi,R}\|L_{\varphi}\|_\infty^n L_{\varphi}(t), \]
where \( L_{\varphi,R} = \sup_{u \in B(0,R+\delta_0)} e^{-S_{n_0}(\varphi)(u)}, \) which is bounded by hypothesis. □

Corollary 3. The limit \( \limsup_{n \to \infty} \frac{1}{n} \log L^n_{\varphi}(s) \) is independent of \( s \in X. \)

**Proof.** Let \( s, t \in X, \) and set \( R = \max\{\varrho(0,s), \varrho(0,t)\} + 1, \) and let \( L_{\varphi,R} \) such as in Lemma 7. Then for every \( n \geq 0 \) we have
\[ L^n_{\varphi}(s) \leq L_{\varphi,R}L^n_{\varphi}(t). \]
Hence, \( \limsup_{n \to \infty} \frac{1}{n} \log L^n_{\varphi}(s) \leq \limsup_{n \to \infty} \frac{1}{n} \log L^n_{\varphi}(t). \) So, the corollary follows immediately. □

In the next section we will show that the sequence \( \{\frac{1}{n} \log L^n_{\varphi}(s)\} \) with \( s \in X \) actually converges and its limit is precisely \( P(\sigma, \varphi). \)

3. Proof of Theorem A

**Theorem A.** Let \( \sigma : X \to X \) be the shift map on
\[ X = \{s \in \Sigma_Z : K_s \neq \emptyset\} \]
and let \( \varphi \) be a weakly Hölder continuous potential satisfying conditions (7), (8) and (9). Then there exists a conformal measure \( m \) and absolutely continuous invariant measure \( \nu \) equivalent to \( m \) supported on \( X_{\text{rad}}. \)

**Proof.** For each \( k \in \mathbb{Z} \) and \( R > 0, \) we define
\[ [k,R] := [k] \cap (X \setminus B(0,R)), \]
and for \( N \geq 1 \) we define
\[ [k,R]_N := [k,R] \cap \Sigma_N \]
Each \( \Sigma_N \) is invariant and compact with respect to \( \rho, \) so that for each \( N \geq 1 \) there exists a \( e^{P_{\Sigma_N}(\sigma,\varphi)\|_{\Sigma_N}} \)-conformal measure for \( \sigma|_{\Sigma_N} \) (compare [Bo08] and [PU10]). That means that for each \( N \geq 1 \) such that \( \sigma|_{[k,R]_N} \) is invertible, we have
\[ \nu_N(\sigma([k,R]_N)) = e^{P_{\Sigma_N}(\sigma,\varphi)} \int_{[k,R]_N} e^{-\varphi} d\nu_N. \]
We prove that the sequence of measure \( \{\nu_N\}_{N \geq 1} \) is tight.

In fact, since for each \( N \geq 1 \) we have
\[ \nu_N(\sigma([k,R])) = e^{P_{\Sigma_N}(\sigma,\varphi)} \int_{[k,R]} e^{-\varphi} d\nu_N(s) \geq e^{P_{\Sigma_N}(\sigma,\varphi)} \nu_N([k,R]) e^{-\sup \varphi([k,R])}. \]
Hence $\nu_N([k, R]) \leq e^{-P_{\Sigma_N}(\sigma, \varphi)} \nu_N(\sigma([k, R])) e^{\sup \varphi|_{[k, R]}} \leq e^{-P_{\Sigma_N}(\sigma, \varphi)} e^{\sup \varphi|_{[k, R]}}$.

Furthermore observe

\begin{equation}
\nu_N(X \setminus \overline{B}(0, R)) = \nu_N \left( \bigcup_{k \in \mathbb{Z}} [k, R] \right) \leq \sum_{k \in \mathbb{Z}} \nu_N([k, R]) \leq e^{-P_{\Sigma_N}(\sigma, \varphi)} \sum_{k \in \mathbb{Z}} e^{\sup \varphi|_{[k, R]}}.
\end{equation}

From (20) we have that the last term tends to zero when $R$ tends to infinity. Therefore since $\overline{B}(0, R)$ is compact in $\mathbb{Z}$, the tightness of $\{\nu_N\}_{N \geq 1}$ is proved.

From Prokhorov’s theorem there exists a subsequence $\{\nu_{N_i}\}_{i \geq 1}$ which converges in the weak topology to some probability measure $\nu$. It follows in particular that for every Borel set $A$ such that $\nu(\partial A) = 0$ and for every bounded continuous function $g$ with bounded support we have

\begin{equation}
\lim_{i \to \infty} \int_A g d\nu_{N_i} = \int_A g d\nu.
\end{equation}

For every $N \geq 1$, let $\overline{B}(\mathbb{0}, N)$ be the closed ball in $\mathbb{Z}$ and for $k$ an integer number let $[k]$ be the closure of the cylinder $[k]$. Thus we consider a sequence of $A_N$, where

$$A_N := \overline{B}(\mathbb{0}, N) \cap \bigcup_{k \in \mathbb{Z}, [k] \subset \Sigma_N} [k].$$

Note that for every $N \geq 1$, $A_N \subset A_{N+1}$ and $\bigcup_{N=1}^\infty A_N = \mathbb{Z}$. Moreover note that for each $N \geq 1$, the measure $\nu_N$ is $e^{P_{\Sigma_N}(\sigma, \varphi)}$-conformal for $\sigma|_{\Sigma_N}$ but is not for $\sigma$ defined on $\mathbb{Z}$. However, if $N$ is large enough, we prove that for every $A \subset A_N$ such that $\sigma|A$ is one-to-one,

$$\nu_N(\sigma(A)) = e^{P_{\Sigma_N}(\sigma, \varphi)} \int_A e^{-\varphi} d\nu_N.$$

In order to show this, we first prove the following equality.

\begin{equation}
\sigma(A) \cap \Sigma_N = \sigma(A \cap \Sigma_N).
\end{equation}

Observe that $\sigma(A \cap \Sigma_N) \subseteq \sigma(A) \cap \sigma(\Sigma_N) \subseteq \sigma(A) \cap \Sigma_N$. To get the contrary inclusion, let $\underline{s} \in \sigma(A) \cap \Sigma_N$, then there exists $\underline{t} \in A$ such that $\underline{s} = \sigma(\underline{t})$. We will prove that $\underline{t} \in \Sigma_N$. In fact, let $c \in \mathbb{Z}$ a word such that $c\underline{s} = \underline{t}$. Since $\underline{t} \in A$ and $A \subset A_N$ we have that there is $k \in \mathbb{Z}$ with $[k] \subset \Sigma_N$ such that $c\underline{s} \in [k]$. Then

$$B_1(c\underline{s}, \delta) \cap [k] \neq \emptyset.$$

By part 1 of Lemma 1 we have that $B_1(c\underline{s}, \delta) \subseteq cB_0(\underline{s}, \delta)$, hence

$$cB_0(\underline{s}, \delta) \cap [k] \neq \emptyset.$$

Hence $c = k$, thus $\underline{t} = c\underline{s} \in [k] \subset \Sigma_N$. 

}\end{proof}
Using (22) we can write
\[
\nu_N(\sigma(A)) = \nu_N(\sigma(A) \cap \Sigma_N) = \nu_N(\sigma(A \cap \Sigma_N))
\]
\[
= e^{\mathcal{P}_{\Sigma_N}(\sigma, \varphi)} \int_{A \cap \Sigma_N} e^{-\varphi} d\nu_N = e^{\mathcal{P}_{\Sigma_N}(\sigma, \varphi)} \int_A e^{-\varphi} d\nu_N
\]
Since the sequence \(\{\nu_{N_i}\}\) converges weakly to \(\nu\), we have for every Borel set \(A\) such that \(\nu(\partial A) = 0\) satisfies \(\nu_{N_i}(A) \to \nu(A)\). In particular this holds for every bounded Borel \(A\) such that \(\nu(\partial A) = 0\) and \(\nu(\partial(A)) = 0\). Then,
\[
\nu(\sigma(A)) = \lim_{i \to \infty} \nu_{N_i}(\sigma(A)) = \lim_{i \to \infty} e^{\mathcal{P}_{\Sigma_N}(\sigma, \varphi)} \int_A e^{-\varphi} d\nu_{N_i} = \vartheta \int_A e^{-\varphi} d\nu.
\]
Take an arbitrary bounded Borel set \(A\) such that \(\sigma|_A\) is injective. We have that \(\text{Sing}(\sigma : Z \to Z) = \emptyset\) and \(\nu(Z) = 1\). Hence we obtain that \(\nu\) is a conformal measure (compare [DU91]).

In order to prove that the measure \(\nu\) is supported on \(X_{\text{rad}}\), observe that
by the inequality (6), there exists \(R > 0\) such that
\[
(23) \quad \vartheta^{-1} \sum_{k \in \mathbb{Z}} e^{\varphi[k, k]} < 1/2.
\]
Let \(Z(R, n) := \{z \in Z : (\sigma)^k(z) \in Z \setminus B(0, R)\text{ for }k = 0, \ldots, n - 1\}\). Then
\[
\nu(Z(R, n)) \geq \nu(\sigma(Z(R, n + 1) \cap [k])) = \vartheta \int_{Z(R, n + 1) \cap [k]} e^{-\varphi} d\nu
\]
\[
\geq \vartheta \nu(Z(R, n + 1) \cap [k]) e^{-\sup_{z \in Z(R, n + 1) \cap [k]} \varphi(z)}.
\]
Hence, by using (23) we have
\[
\nu(Z(R, n)) \leq \vartheta^{-1} \sum_{k \in \mathbb{Z}} e^{\sup_{z \in Z(R, n + 1) \cap [k]} \varphi(z)} \nu(Z(R, n - 1)).
\]
Then, \(\nu(Z(R, n)) \leq (1/2)^n\) and hence \(\nu(X_{\text{rad}}) = 1\).
Therefore, the measure \(\nu\) is \(\vartheta e^{-\varphi}\)-conformal for \(\sigma : \overline{X}^0 \to \overline{X}^0\) and \(\nu(X_{\text{rad}}) = 1\).

**Proposition 10.** Let \(\varphi \in \text{BV}_r(X, \mathbb{R})\) satisfying (7), (8) and (9). The sequence \(\left\{\frac{1}{n} \log \mathcal{L}_\varphi^n 1(z)\right\}\) with \(z \in X\) converges to \(P(\sigma, \varphi)\).

**Proof.** Consider the normalized transfer operator \(\widehat{\mathcal{L}}_\varphi = \vartheta^{-1} \mathcal{L}_\varphi\). It is enough to prove that there exists \(\Xi > 0\) so that for every \(R > 0\) there exists \(\xi_R > 0\) such that for every \(n \geq 1\) and \(z \in B(0, R)\), we have
\[
(24) \quad \xi_R \leq \widehat{\mathcal{L}}_\varphi^n 1(z) \leq \Xi.
\]
First, we prove the right hand side inequality. Since \(\varphi\) satisfies (9) we have that there exists sufficiently large \(R_0 > 0\) that
\[
(25) \quad \sup \left\{\widehat{\mathcal{L}}_\varphi 1(z) : z \in \overline{X}^0 \setminus B(0, R_0)\right\} \leq 1.
\]
We will show by induction, for every \(n \geq 0\) we have
\[
(26) \quad \|\widehat{\mathcal{L}}_\varphi^n 1\|_\infty \leq \frac{\mathcal{L}_\varphi R_0}{\nu(B(0, R_0))} := \Xi,
\]
where $L_{\varphi, R_0}$ is the constant coming from Lemma 7 with $\mathcal{L}_\varphi$ replaced by the operator $\hat{\mathcal{L}}_\varphi$. Observe that for each $n \geq 0$,

$$
\|\hat{\mathcal{L}}_\varphi^n 1\|_\infty \leq \vartheta^{-n} \|\mathcal{L}_\varphi 1\|_\infty^n.
$$

(27)

So, for $n = 0$ is clear, since $\|\hat{\mathcal{L}}_\varphi^n 1\|_\infty \leq 1 \leq L_{\varphi, R_0} \leq \frac{L_{\varphi, R_0}}{\nu(B(0, R_0))}$.

Suppose that $\|\hat{\mathcal{L}}_\varphi^n 1\|_\infty \leq \frac{L_{\varphi, R_0}}{\nu(B(0, R_0))}$. We have that there exists $t \in \bar{X}^0$ such that $\|\hat{\mathcal{L}}_\varphi^n 1\|_\infty = \hat{\mathcal{L}}_\varphi^n 1(t)$. If $t \in \bar{X}^0 \setminus B(0, R_0)$, then by combining (25) and (26) we have

$$
\|\hat{\mathcal{L}}_\varphi^n 1\|_\infty = \hat{\mathcal{L}}_\varphi^n 1(t) = \hat{\mathcal{L}}_\varphi(\hat{\mathcal{L}}_\varphi^n 1)(t) \leq \vartheta^{-n} \sum_{\sigma(\bar{u}) = t} \exp(\nu) \hat{\mathcal{L}}_\varphi^n 1(\bar{u})
$$

$$
\leq \|\hat{\mathcal{L}}_\varphi^n 1\|_\infty \hat{\mathcal{L}}_\varphi 1(t) \leq \frac{L_{\varphi, R_0}}{\nu(B(0, R_0))}.
$$

Otherwise, if $t \in B(0, R_0)$, then from Lemma 7 and the fact that for every $n \geq 0$, $\hat{\mathcal{L}}_\varphi^n \nu = \nu$ we follow

$$
1 = \int d\nu = \int \hat{\mathcal{L}}_\varphi^n 1 d\nu \geq \int_{B(0, R_0)} \hat{\mathcal{L}}_\varphi^n 1 d\nu \geq L_{\varphi, R_0}^{-1} \nu(B(0, R_0)) \|\hat{\mathcal{L}}_\varphi^n 1\|_\infty.
$$

Therefore the right hand inequality is proved.

In order to prove the other inequality in (24), let $R_1 > R_0$ be such that $\nu(\bar{X}^0 \setminus B(0, R_1)) \leq 1/4\Xi$. Let $R > R_1$ then we have

$$
1 = \int \hat{\mathcal{L}}_\varphi^n 1 d\nu \leq \int_{B(0, R)} \hat{\mathcal{L}}_\varphi^n 1 d\nu + \int_{\bar{X}^0 \setminus B(0, R)} \hat{\mathcal{L}}_\varphi^n 1 d\nu
$$

$$
\leq \int_{B(0, R)} \hat{\mathcal{L}}_\varphi^n 1 d\nu + \Xi \nu(\bar{X}^0 \setminus B(0, R)) \leq \int_{B(0, R)} \hat{\mathcal{L}}_\varphi^n 1 d\nu + 1/4.
$$

Hence for any $n \geq 0$ there is $\bar{u}_n \in B(0, R_0)$ such that $\hat{\mathcal{L}}_\varphi^n 1(\bar{u}_n) \geq 3/4$. If $\bar{u} \in B(0, R)$ is any other point, then from Lemma 7 we have

$$
3/4 \leq \|\hat{\mathcal{L}}_\varphi^n 1(\bar{u}_n) \leq L_{\varphi, R} \|\hat{\mathcal{L}}_\varphi^n 1(\bar{u})\|.
$$

Therefore, for each $R \geq R_1$ the inequality holds with $\xi_R = 3M_{\varphi, R}/4$. If $R < R_1$ then we just put $\xi_R := \xi_{R_1}$.

The rest of this section is devoted to prove the following: for a weakly Hölder potential there is an invariant measure $\mu$ which is absolutely continuous with respect to the conformal measure $\nu$, and $\mu$ is a Gibbs measure.

We first need the following Lemma.

**Lemma 8.** Let $\nu$ be the conformal measure for $\varphi$ as stated in 3 then for every $\bar{s} \in X$ and for every $\varepsilon > 0$, we have $\nu(B(\bar{s}, \varepsilon)) > 0$. 

Proof. Let $s \in X$ and $\varepsilon > 0$. Since $X = \bigcup_{R \geq 0} B(0, R)$ we have there exists $R > 0$ such that $\nu(B(0, R)) > 0$. From Proposition [5] there exists $n_0 > 0$ such that

$$B(\sigma^{n_0} s, \delta_0) \subseteq \sigma^{n_0}(B(s, \varepsilon)).$$

Take $R_1 > 0$ such that $R_1 \geq R$ and $\sigma^{n_0} s \in B(0, R_1)$. Moreover, there exists $n_1$ such that

$$B(0, R_1) \subseteq \sigma^{n_1}(B(\sigma^{n_0} s, \delta_0)) \subseteq \sigma^{n_1+n_0}(B(s, \varepsilon)).$$

Thus for $n = n_0 + n_1$ we have $B(0, R) \subseteq \sigma^n(B(s, \varepsilon))$.

Since $B(s, \varepsilon) = \bigcup_{u^* \in \mathbb{Z}^n} B(s, \varepsilon) \cap [u^*]$. Then

$$\sigma^n \left( \bigcup_{u^* \in \mathbb{Z}^n} B(s, \varepsilon) \cap [u^*] \right) \supseteq B(0, R).$$

Then there exists $u^* \in \mathbb{Z}^n$ such that $\nu(\sigma^n(B(s, \varepsilon) \cap [u^*])) > 0$. Therefore, we have

$$0 < \nu(\sigma^n(B(s, \varepsilon) \cap [u^*])) = \int_{B(s, \varepsilon) \cap [u^*]} \partial^n e^{-S_n(\varphi)} d\nu$$

$$\leq \partial^n e^{-\inf B(s, \varepsilon) \cap [u^*] \nu(B(s, \varepsilon) \cap [u^*])}$$

$$\leq \partial^n e^{-S_n(\varphi)\mathcal{L}} C\nu(B(s, \varepsilon)),$$

where $C'$ is the positive constant $\exp\left(\frac{\nu(\varphi)}{1-r}\right)$. Therefore,

$$\nu(B(s, \varepsilon)) \geq \nu(\sigma^n B(s, \varepsilon) \cap [u^*]) \partial^n e^{-S_n(\varphi)\mathcal{L}} C' > 0.$$

Let $A$ be a Borel set such that $A \subset B(s, \delta) \subset B(0, R)$, then

$$\nu(A) = \int_{u^* A} \partial^n e^{-S_n(\varphi)} d\nu,$$

From Lemma [7], we have

$$\sum_{u^* \in \mathbb{Z}^n} \partial^n e^{-S_n(\varphi)(u^*)} \geq L_{\varphi,R}^{-1} \int_{B(0, R)} \mathcal{L}^{n-1} d\nu = L_{\varphi,R}^{-1} \nu(B(0, R)).$$

Then, from (15) and (28), we have

$$\nu(A) \geq \partial^n e^{-\sup B(s, \delta_{\varphi})} \nu(u^* A) \geq \partial^n e^{-S_n(\varphi)(u^*)} C' \nu(u^* A)$$

and

$$\nu(A) \leq \partial^n e^{-\inf B(s, \delta_{\varphi})} \nu(u^* A) \leq \partial^n e^{-S_n(\varphi)(u^*)} C' \nu(u^* A).$$

In particular

$$\partial^n e^{-S_n(\varphi)(u^*)} \leq \frac{C' \nu(u^* B(s, \delta))}{\nu(B(s, \delta))}.$$
Then from (30), we have
\[
\nu(\sigma^{-n}(A)) = \sum_{u^* \in \mathbb{Z}^n} \nu(u^* A) \leq \nu(A) C' \sum_{u^* \in \mathbb{Z}^n} \vartheta^{-n} e^{S_n(\varphi)(u^* x)} \leq \frac{C'^2}{\nu(B(s, \delta))},
\]
using (31) and (29),
\[
\nu(\sigma^{-n}(A)) = \sum_{u^* \in \mathbb{Z}^n} \nu(u^* A) \geq \nu(A) C'^{-1} \sum_{u^* \in \mathbb{Z}^n} \vartheta^{-n} e^{S_n(\varphi)(u^* x)} \geq \nu(A)(C'L_{\varphi,R})^{-1} \nu(B(\emptyset, R)).
\]
Therefore,
\[
(32) \quad \nu(A)(C'L_{\varphi,R})^{-1} \nu(B(\emptyset, R)) \leq \nu(\sigma^{-n}(A)) \leq \nu(A) \frac{C'^2}{\nu(B(s, \delta))}.
\]
In the Banach space of all bounded sequences of real numbers we consider a continuous linear functional \( \mathfrak{L} \) called a Banach limit \( \mathfrak{L} : \ell^\infty \to \mathbb{R} \) (compare [BKS5]) such that \( \mathfrak{L}(\{s_n\}_{n \geq 1}) = \mathfrak{L}(\{s_{n+1}\}_{n \geq 1}) \),
\[
\liminf_{n \to \infty} a_n \leq \mathfrak{L}(\{a_n\}) \leq \limsup_{n \to \infty} a_n,
\]
and if \( \{a_n\} \) converges then \( \lim_{n \to \infty} a_n = \mathfrak{L}(\{a_n\}) \). Then let \( \nu \) be a measure defined by formula
\[
\mu(A) = \mathfrak{L}(\{\nu \circ \sigma^{-n}(A)\}_{n = 0}^\infty).
\]
Since \( \mathfrak{L}(\{s_n\}_{n \geq 1}) = \mathfrak{L}(\{s_{n+1}\}_{n \geq 1}) \), the measure is invariant. Moreover, since \( \liminf_{n \to \infty} a_n \leq \mathfrak{L}(\{a_n\}) \leq \limsup_{n \to \infty} a_n \), and from (32) we obtain, for any Borel set \( A \subset B(s, \delta) \subset B(\emptyset, R) \),
\[
(33) \quad \nu(A)(C'L_{\varphi,R})^{-1} \nu(B(\emptyset, R)) \leq \mu(A) \leq \nu(A) \frac{C'^2}{\nu(B(s, \delta))}.
\]
Hence the measure \( \mu \) is equivalent to the measure \( \nu \).

**Corollary 4.** We have that \( \nu \) and \( \mu \) are Gibbs measures.

**Proof.** From (30) and (31) we have
\[
(34) \quad \mathfrak{D}^{-1} \nu(B(s, \delta)) \leq \frac{\nu(u^* B(s, \delta))}{\exp(S_n(\varphi)(u^* s))} \vartheta^{-n} \leq \mathfrak{D}.
\]
Using Lemma [S] follows that for every \( s \in X \), \( \nu(B(s, \delta)) > 0 \). Therefore we get that \( \nu \) is a Gibbs measure with \( P = \ln \vartheta \), \( \mathfrak{D} = C' \) and \( \mathfrak{R}(a) = \nu(B(s, \delta_0)) \).

We also have that from (33) and (31)
\[
\mathfrak{D}^{-2} L_{\varphi,R}^{-1} \mathfrak{R}(s)^2 \leq \frac{\mu(u^* B(s, \delta))}{\exp(S_n(\varphi)(u^* s))} \vartheta^{-n} \leq \mathfrak{D}^3.
\]
Therefore \( \mu \) is a Gibbs measure.
4. Comments

It would be optimal to study the thermodynamic formalism and Multifractal analysis of a general subset \( X \) of \( \Sigma \), which \( X \) would be equipped with a metric \( \varrho \) which is not necessarily compatible with the metric, 
\[
d(\underline{s},\underline{t}) = \varrho(\inf\{k:s_k \neq t_k\} \cup \{\infty\}) ,
\]
and the dynamics \( \sigma|_X : X \to x \) satisfying some regular conditions

1. \( X \) is not compact subset of \( \Sigma \)
2. \( X \) is completely invariant, that means \( \sigma(X) = X = \sigma^{-1}(X) \)
3. There exists a nested family of compact subsets of \( X \), \( \{\Lambda_N\}_{N \geq 1} \) such that \( \sigma(\Lambda_N) \subset \Lambda_N \), such that \( \bigcup_{N \geq 1} \Lambda_N \) is dense in \( X \).
4. If \( \Lambda \) is a subset of \( X \), which is compact and forward invariant, then, there exists \( N \geq 1 \) such that \( \Lambda \subset \Lambda_N \).

Moreover, assume that there exists \( \delta_0 > 0 \) such that the following properties with respect to \( \varrho \) hold.

\( (B1) \) There exist \( C_E \) and \( \lambda_E > 1 \) such that, for all \( n \in \mathbb{N} \), \( u^* \in \mathbb{Z}^n \), 
\( \underline{s}, \underline{t} \in X \), if \( \varrho(\underline{s}, \underline{t}) < \delta_0 \), then
\[
\varrho(u^* \underline{s}, u^* \underline{t}) \leq C_E \lambda_E^{-n} \varrho(\underline{s}, \underline{t}).
\]

\( (B2) \) \( \sigma : (X, \varrho) \to (X, \varrho) \) is topologically mixing.

Let \( \delta \in (0, \delta_0] \) and \( \delta' = \min\{\delta, \delta/C_E\} \).

\( (B3) \) There exists \( \ell \geq 1 \) such that for every \( \underline{s} \in X \) and for every \( \underline{t}, \underline{u} \in \mathbb{B}_0(\underline{s}, \delta) \), there exists a sequence
\[
\underline{t} = \underline{s}_0, \underline{s}_1, \cdots, \underline{s}_\ell = \underline{u},
\]
such that for each \( j \geq 0 \) we have \( \varrho(\underline{s}_j, \underline{s}_{j+1}) < \delta' \).

Note that the natural shift metric \( d \) satisfies condition \( (B1) \) provided \( \theta \in (0, 1) \). If we put \( \delta_0 = 1 \), \( C = 1 \), \( \lambda_E = 1/\theta \), then for every \( \underline{s}, \underline{t} \in \mathbb{Z}^n \), \( u^* \in \mathbb{Z}^n \), we have
\[
d(\underline{s}, \underline{t}) \leq 1, d(\underline{s}, \underline{t}) = \theta^s \text{ with } s = \inf\{k : a_k \neq b_k\}, \text{ so}
\]
\[
d(u^* \underline{s}, u^* \underline{t}) \leq \theta^{n+s} = \theta^n d(\underline{s}, \underline{t}).
\]

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