ON THE AJ CONJECTURE FOR KNOTS

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ABSTRACT. We confirm the AJ conjecture [Ga2] that relates the $A$-polynomial and the colored Jones polynomial for those hyperbolic knots satisfying certain conditions. In particular, we show that the conjecture holds true for some classes of two-bridge knots and pretzel knots. This extends the result of the first author in [Le1] where he established the AJ conjecture for a large class of two-bridge knots, including all twist knots. Along the way, we explicitly calculate the universal character ring of the knot group of the $(-2,3,2n+1)$-pretzel knot and show that it is reduced for all integers $n$.

0. Introduction

0.1. The AJ conjecture. For a knot $K$ in $S^3$, let $J_K(n) \in \mathbb{Z}[t^\pm 1]$ be the colored Jones polynomial of $K$ colored by the (unique) $n$-dimensional simple representation of $sl_2$ [Jo, RT], normalized so that for the unknot $U$,

$$J_U(n) = [n] := \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$ 

The color $n$ can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, $J_K(0) = 0$. It is known that $J_K(1) = 1$, and $J_K(2)$ is the ordinary Jones polynomial.

Define two linear operators $L, M$ acting on the set of discrete functions $f : \mathbb{Z} \to \mathcal{R} := \mathbb{C}[t^\pm 1]$ by

$$(Lf)(n) = f(n + 1), \quad (Mf)(n) = t^{2n}f(n).$$

It is easy to see that $LM = t^2 ML$. The inverse operators $L^{-1}, M^{-1}$ are well-defined. One can consider $L, M$ as elements of the quantum torus

$$\mathcal{T} = \mathcal{R}(L^\pm 1, M^\pm 1)/(LM - t^2 ML),$$

which is not commutative, but almost commutative.

Let

$$\mathcal{A}_K = \{ P \in \mathcal{T} \mid PJ_K = 0 \}.$$ 

It is a left ideal of $\mathcal{T}$, called the recurrence ideal of $K$. It was proved in [GL] that for every knot $K$, the recurrence ideal $\mathcal{A}_K$ is non-zero. Partial results were obtained earlier by Frohman, Gelca, and Lofaro through their theory of non-commutative A-ideal [FGL, Ge]. An element in $\mathcal{A}_K$ is called a recurrence relation for the colored Jones polynomial of $K$.

The ring $\mathcal{T}$ is not a principal left-ideal domain, i.e. not every left-ideal of $\mathcal{T}$ is generated by one element. By adding the inverses of polynomials in $t, M$ to $\mathcal{T}$ one gets a principal

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left-ideal domain $\mathcal{T}$, cf. [Ga2]. Denote the generator of the extension $\tilde{A}_K = A_K \cdot \mathcal{T}$ by $\alpha_K$. The element $\alpha_K$ can be presented in the form

$$\alpha_K(t; M, L) = \sum_{j=0}^{d} \alpha_{K,j}(t, M) L^j,$$

where the degree in $L$ is assumed to be minimal and all the coefficients $\alpha_{K,j}(t, M) \in \mathbb{Z}[t^{\pm 1}, M]$ are assumed to be co-prime. The polynomial $\alpha_K$ is defined up to a polynomial in $\mathbb{Z}[t^{\pm 1}, M]$. Moreover, one can choose $\alpha_K \in A_K$, i.e. it is a recurrence relation for the colored Jones polynomial. We call $\alpha_K$ the recurrence polynomial of $K$.

Garoufalidis [Ga2] formulated the following conjecture (see also [FGL, Ge]).

**Conjecture 1.** (AJ conjecture) For every knot $K$, $\alpha_K|_{t=-1}$ is equal to the $A$-polynomial, up to a factor depending on $M$ only.

In the definition of the $A$-polynomial [CCGLS], we also allow the abelian component of the character variety, see Section 2.

0.2. **Main results.** Conjecture 1 was established for a large class of two-bridge knots, including all twist knots, by the first author [Le1] using skein theory. In this paper we generalize his results as follows.

**Theorem 1.** Suppose $K$ is a knot satisfying all the following conditions:

(i) $K$ is hyperbolic,
(ii) The $SL_2$-character variety of $\pi_1(S^3 \setminus K)$ consists of 2 irreducible components (one abelian and one non-abelian),
(iii) The universal $SL_2$-character ring of $\pi_1(S^3 \setminus K)$ is reduced,
(iv) The recurrence polynomial of $K$ has $L$-degree greater than 1.

Then the AJ conjecture holds true for $K$.

**Theorem 2.** The following knots satisfy all the conditions (i)–(iv) of Theorem 1 and hence the AJ conjecture holds true for them.

(a) All pretzel knots of type $(-2, 3, 6n \pm 1)$, $n \in \mathbb{Z}$.
(b) All two-bridge knots for which the character variety has exactly 2 irreducible components; these include all twist knots, double twist knots of the form $J(k, l)$ with $k \neq l$ in the notation of [HS], all two-bridge knots $b(p, m)$ with $p$ prime or $m = 3$. Here we use the notation $b(p, m)$ from [BZ].

**Remark 0.1.** Besides the infinitely many cases of two-bridge knots listed in Theorem 2, explicit calculation seems to confirm that ”most two-bridge knots” satisfy the conditions of Theorem 1 and hence AJ conjecture holds for them. In fact, among 155 $b(p, m)$ with $p < 45$, only 9 knots do not satisfy the conditions of Theorem 1.

0.3. **Other results.** In our proof of Theorem 2 it is important to know whether the universal character ring of a knot group is reduced, i.e. whether its nil-radical is 0. Although it is difficult to construct a group whose universal character ring is not reduced (see [LM]), so far there are a few groups for which the universal character ring is known to be reduced: free groups [Si], surface groups [CM, Si], two-bridge knot groups [PS], torus knot groups [Mar], two-bridge link groups [LT]. In the present paper, we show that the universal character ring of the $(-2, 3, 2n + 1)$-pretzel knot is reduced for all integers $n$. 
0.4. **Plan of the paper.** We review skein modules and their relation with the colored Jones polynomial in Section 1. In Section 2 we prove some properties of the character variety and the $A$-polynomial a knot. We discuss the role of the quantum peripheral polynomial in the AJ conjecture and give the proof of Theorem 1 and Theorem 2 in Section 3. In Section 4, we prove the reducedness of the universal character ring of the $(-2,3,2n+1)$-pretzel knot for all integers $n$. Finally we study the irreducibility of the non-abelian character varieties of two-bridge knots in the appendix.

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1. **Skein Modules and the colored Jones polynomial**

In this section we will review skein modules and their relation with the colored Jones polynomial. The theory of Kauffman bracket skein module (KBSM) was introduced by Przytycki [Pr] and Turaev [Tu] as a generalization of the Kauffman bracket [Ka] in $S^3$ to an arbitrary 3-manifold. The KBSM of a knot complement contains a lot, if not all, of information about its colored Jones polynomials.

1.1. **Skein modules.** Recall that $\mathcal{R} = \mathbb{C}[t^\pm 1]$. A framed link in an oriented 3-manifold $Y$ is a disjoint union of embedded circles, equipped with a non-zero normal vector field. Framed links are considered up to isotopy. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold $Y$, including the empty link. Consider the free $\mathcal{R}$-module with basis $\mathcal{L}$, and factor it by the smallest submodule containing all expressions of the form $\langle -t \rangle \langle -t^{-1} \rangle$ (and $\bigcirc + (t^2 + t^{-2})\emptyset$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $\mathcal{S}(Y)$ and is called the Kauffman bracket skein module, or just skein module, of $Y$.

For an oriented surface $\Sigma$ we define $\mathcal{S}(\Sigma) = \mathcal{S}(Y)$, where $Y = \Sigma \times [0,1]$, the cylinder over $\Sigma$. The skein module $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over $\partial Y$ to $Y$ induces a $\mathcal{S}(\partial Y)$-left module structure on $\mathcal{S}(Y)$.

1.2. **The skein module of $S^3$ and the colored Jones polynomial.** When $Y = S^3$, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. Thus if $\ell$ is a framed link in $S^3$, then its value in the skein module $\mathcal{S}(S^3)$ is $\langle \ell \rangle$ times the empty link, where $\langle \ell \rangle \in \mathcal{R}$ is the Kauffman bracket of $\ell$ $[Ka]$ which is the Jones polynomial of the framed link $\ell$ in a suitable normalization.

Let $S_n(z)$ be the Chebychev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_{n+1}(z) = zS_n(z) - S_{n-1}(z)$ for all $n \in \mathbb{Z}$. For a framed knot $K$ in $S^3$ and an integer $n \geq 0$, we define the $n$-th power $K^n$ as the link consisting of $n$ parallel copies of $K$ (this is a 0-framing cabling operation). Using these powers of a knot, $S_n(K)$ is defined as an element of $\mathcal{S}(S^3)$. We then define the colored Jones polynomial $J_K(n)$ by the equation

$$J_K(n+1) := (-1)^n \times \langle S_n(K) \rangle.$$  

The $(-1)^n$ sign is added so that for the unknot $U$, $J_U(n) = [n]$. Then $J_K(1) = 1$, $J_K(2) = -\langle K \rangle$. We extend the definition for all integers $n$ by $J_K(-n) = -J_K(n)$ and $J_K(0) = 0$. In the framework of quantum invariants, $J_K(n)$ is the $sl_2$-quantum invariant of $K$ colored by the $n$-dimensional simple representation of $sl_2$. 

1.3. The skein module of the torus. Let $\mathbb{T}^2$ be the torus with a fixed pair $(\mu, \lambda)$ of simple closed curves intersecting at exactly 1 point. For co-prime $k$ and $l$, let $\lambda_{k,l}$ be a simple closed curve on the torus homologically equal to $k\mu + l\lambda$. It is not difficult to show that the skein algebra $S(\mathbb{T}^2)$ of the torus is generated, as an $\mathcal{R}$-algebra, by all $\lambda_{k,l}$. In fact, Bullock and Przytycki [BP] showed that $S(\mathbb{T}^2)$ is generated over $\mathcal{R}$ by 3 elements $\mu, \lambda$ and $\lambda_{1,1}$, subject to some explicit relations.

Recall that $\mathcal{T} = \mathcal{R}(M^{\pm 1}, L^{\pm 1})/(LM - t^2ML)$ is the quantum torus. Let $\sigma : \mathcal{T} \to \mathcal{T}$ be the involution defined by $\sigma(M^kL^l) = M^{-k}L^{-l}$. Frohman and Gelca [FG] showed that there is an algebra isomorphism $\Upsilon : S(\mathbb{T}^2) \to \mathcal{T}^\sigma$ given by

$$\Upsilon(\lambda_{k,l}) = (-1)^{k+l}t^{kl}(M^kL^l + M^{-k}L^{-l}).$$

The fact that $S(\mathbb{T}^2)$ and $\mathcal{T}^\sigma$ are isomorphic algebras was also proved by Sallenave [Sa].

1.4. The orthogonal and peripheral ideals. Let $N(K)$ be a tubular neighborhood of an oriented knot $K$ in $S^3$, and $X$ the closure of $S^3 \setminus N(K)$. Then $\partial(N(K)) = \partial(X) = \mathbb{T}^2$. There is a standard choice of a meridian $\mu$ and a longitude $\lambda$ on $\mathbb{T}^2$ such that the linking number between the longitude and the knot is 0. We use this pair $(\mu, \lambda)$ and the map $\Upsilon$ in the previous subsection to identify $S(\mathbb{T}^2)$ with $\mathcal{T}^\sigma$.

The torus $\mathbb{T}^2 = \partial(N(K))$ cut $S^3$ into two parts: $N(K)$ and $X$. We can consider $S(X)$ as a left $S(\mathbb{T}^2)$-module and $S(N(K))$ as a right $S(\mathbb{T}^2)$-module. There is a bilinear bracket

$$\langle \cdot, \cdot \rangle : S(N(K)) \otimes_{S(\mathbb{T}^2)} S(X) \to S(S^3) \equiv \mathcal{R}$$

given by $\langle \ell', \ell'' \rangle = \langle \ell' \cup \ell'' \rangle$, where $\ell'$ and $\ell''$ are links in respectively $N(K)$ and $X$. Note that if $\ell \in S(\mathbb{T}^2)$, then

$$\langle \ell' \cdot \ell, \ell'' \rangle = \langle \ell', \ell \cdot \ell'' \rangle.$$

In general $S(X)$ does not have an algebra structure, but it has the identity element—the empty link. The map

$$\Theta : S(\mathbb{T}^2) \to S(X), \quad \Theta(\ell) = \ell \cdot \emptyset$$

is $S(\mathbb{T}^2)$-linear. Its kernel $\mathcal{P} = \ker \Theta$ is called the quantum peripheral ideal, first introduced in [FGL]. In [FGL Ge], it was proved that every element in $\mathcal{P}$ gives rise to a recurrence relation for the colored Jones polynomial.

The orthogonal ideal $\mathcal{O}$ in [FGL] is defined by

$$\mathcal{O} := \{ \ell \in S(\partial X) \mid \langle \ell', \Theta(\ell) \rangle = 0 \quad \text{for every } \ell' \in S(N(K)) \}.$$ 

It is clear that $\mathcal{O}$ is a left ideal of $S(\partial X) \equiv \mathcal{T}^\sigma$ and $\mathcal{P} \subset \mathcal{O}$. In [FGL], $\mathcal{O}$ was called the formal ideal. According to [Le], if $\mathcal{P} = \mathcal{O}$ for all knots then the colored Jones polynomial distinguishes the unknot from other knots.

1.5. Relation between the recurrence ideal and the orthogonal ideal. As mentioned above, the skein algebra of the torus $S(\mathbb{T}^2)$ can be identified with $\mathcal{T}^\sigma$ via the $\mathcal{R}$-algebra isomorphism $\Upsilon$ sending $\mu, \lambda$ and $\lambda_{1,1}$ respectively $-(M + M^{-1}), -(L + L^{-1})$ and $t(ML + M^{-1}L^{-1})$.

**Proposition 1.1.** One has

$$(-1)^n \langle S_{n-1}(\lambda), \Theta(\ell) \rangle = \Upsilon(\ell)J_K(n)$$

for all $\ell \in S(\mathbb{T}^2)$. 

Proof. We know from the properties of the Jones-Wenzl idempotent (see e.g. [Oh]) that

\[
\langle S_{n-1}(\lambda) \cdot \mu, \emptyset \rangle = (t^{2n} + t^{-2n}) \langle S_{n-1}(\lambda), \emptyset \rangle \\
\langle S_{n-1}(\lambda) \cdot \lambda, \emptyset \rangle = \langle S_n(\lambda) + S_{n-2}(\lambda), \emptyset \rangle \\
\langle S_{n-1}(\lambda) \cdot \lambda_{1,1}, \emptyset \rangle = -\langle t^{2n+1} S_n(\lambda) + t^{-2n+1} S_{n-2}(\lambda), \emptyset \rangle.
\]

By definition \( J_K(n) = (-1)^{n-1} \langle S_{n-1}(\lambda), \emptyset \rangle \) and \((MJ_K)(n) = t^{2n} J_K(n), (LJ_K)(n) = J_K(n + 1) \). Hence the above equations can be rewritten as

\[
(-1)^n \langle S_{n-1}(\lambda), \Theta(\mu) \rangle = -(M + M^{-1}) J_K(n) = \Upsilon(\mu) J_K(n), \\
(-1)^n \langle S_{n-1}(\lambda), \Theta(\lambda) \rangle = -(L + L^{-1}) J_K(n) = \Upsilon(\lambda) J_K(n), \\
(-1)^n \langle S_{n-1}(\lambda), \Theta(\lambda_{1,1}) \rangle = t(ML + M^{-1}L^{-1}) J_K(n) = \Upsilon(\lambda_{1,1}) J(n).
\]

Since \( S(\mathbb{T}^2) \) is generated by \( \mu, \lambda \) and \( \lambda_{1,1} \), we conclude that

\[
(-1)^n \langle S_{n-1}(\lambda), \Theta(\ell) \rangle = \Upsilon(\ell) J_K(n)
\]
for all \( \ell \in S(\mathbb{T}^2) \). \( \square \)

Corollary 1.2. One has \( \mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma \).

Proof. Since \( \{S_n(\lambda)\}_n \) generates the skein module \( S(N(K)) \), Proposition [11] implies that

\[
\mathcal{O} = \{ \ell \in S(\partial X) \mid \langle \ell, \Theta(\ell) \rangle = 0 \text{ for every } \ell' \in S(N(K)) \} \\
= \{ \ell \in S(\partial X) \mid \langle S_n(\lambda), \Theta(\ell) \rangle = 0 \text{ for all integers } n \} \\
= \{ \ell \in S(\partial X) \mid \Upsilon(\ell) J_K(n) = 0 \text{ for all integers } n \}.
\]

Hence \( \mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma \). \( \square \)

Remark 1.3. Corollary [1,2] was already obtained in [Ga1] by another method. Our proof here uses the properties of the Jones-Wenzl idempotent only.

2. Character varieties and the A-polynomial

For non-zero \( f, g \in \mathbb{C}[M, L] \), we say that \( f \) is M-essentially equal to \( g \), and write \( f \equiv M \), if the quotient \( f / g \) does not depend on \( L \). We say that \( f \) is M-essentially divisible by \( g \) if \( f \) is M-essentially equal to a polynomial divisible by \( g \).

2.1. The character variety of a group. The set of representations of a finitely presented group \( G \) into \( SL_2(\mathbb{C}) \) is an algebraic set defined over \( \mathbb{C} \), on which \( SL_2(\mathbb{C}) \) acts by conjugation. The set-theoretic quotient of the representation space by that action does not have good topological properties, because two representations with the same character may belong to different orbits of that action. A better quotient, the algebraico-geometric quotient denoted by \( \chi(G) \) (see [CS, LM]), has the structure of an algebraic set. There is a bijection between \( \chi(G) \) and the set of all characters of representations of \( G \) into \( SL_2(\mathbb{C}) \), hence \( \chi(G) \) is usually called the character variety of \( G \). For a manifold \( Y \) we use \( \chi(Y) \) also to denote \( \chi(\pi_1(Y)) \).

Suppose \( G = \mathbb{Z}^2 \), the free abelian group with 2 generators. Every pair of generators \( \lambda, \mu \) will define an isomorphism between \( \chi(G) \) and \( (\mathbb{C}^*)^2 / \tau \), where \( (\mathbb{C}^*)^2 \) is the set of non-zero complex pairs \( (L, M) \) and \( \tau \) is the involution \( \tau(M, L) = (M^{-1}, L^{-1}) \), as follows: Every representation is conjugate to an upper diagonal one, with \( L \) and \( M \) being the upper left entry of \( \lambda \) and \( \mu \) respectively. The isomorphism does not change if one replaces \( (\lambda, \mu) \) with \( (\lambda^{-1}, \mu^{-1}) \).
2.2. The universal character ring. For a finitely presented group $G$, the character variety $\chi(G)$ is determined by the traces of some fixed elements $g_1, \ldots, g_k$ in $G$. More precisely, one can find $g_1, \ldots, g_k$ in $G$ such that for every element $g \in G$ there exists a polynomial $P_g$ in $k$ variables such that for any representation $r : G \to SL_2(\mathbb{C})$ we have $\text{tr}(r(g)) = P_g(x_1, \ldots, x_k)$ where $x_j := \text{tr}(r(g_j))$. The universal character ring of $G$ is then defined to be the quotient of the ring $\mathbb{C}[x_1, \ldots, x_k]$ by the ideal generated by all expressions of the form $\text{tr}(r(v)) - \text{tr}(r(w))$, where $v$ and $w$ are any two words in $g_1, \ldots, g_k$ which are equal in $G$. The universal character ring of $G$ is actually independent of the choice of $g_1, \ldots, g_k$. The quotient of the universal character ring of $G$ by its nil-radical is equal to the ring of regular functions on the character variety of $G$.

The universal character ring defined here is the sklein algebra of $G$ of [PS], where it is proved that it is $TH(G)$ of Brumfiel-Hilden’s book [BH]. They prove that it is the universal character ring, which is defined as the coefficient algebra of the universal representation.

2.3. The $A$-polynomial. Let $X$ be the closure of $S^3$ minus a tubular neighborhood $N(K)$ of a knot $K$. The boundary of $X$ is a torus whose fundamental group is free abelian of rank 2. An orientation of $K$ will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is 0, as in Subsection 1.4. The pair provides an identification of $\chi(\partial X)$ and $(\mathbb{C}^*)^2/\tau$ which actually does not depend on the orientation of $K$.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\rho : \chi(X) \longrightarrow \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau$$

Let $Z$ be the image of $\rho$ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of $Z$ under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the 1-dimension components is defined by a single polynomial $A_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Note that $A_K$ is defined up to $\pm 1$. We call $A_K$ the $A$-polynomial of $K$. By definition, $A_K$ does not have repeated factors. It is known that $A_K$ is always divisible by $L - 1$. The $A$-polynomial here is actually equal to $L - 1$ times the A-polynomial defined in [CCGLS].

2.4. The $B$-polynomial. It is also instructive and convenient to see the dual picture in the construction of the $A$-polynomial. For an algebraic set $V$ (over $\mathbb{C}$) let $\mathbb{C}[V]$ denote the ring of regular functions on $V$. For example, $\mathbb{C}[[\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the $\sigma$-invariant subspace of $\mathfrak{t} := \mathbb{C}[\mathbb{Z}^+L^\sigma]$, where $\sigma(M^k L^l) = M^{-k} L^{-l}$.

The map $\rho$ in the previous subsection induces an algebra homomorphism

$$\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)].$$

We will call the kernel $\mathfrak{p}$ of $\theta$ the classical peripheral ideal; it is an ideal of $\mathfrak{t}^\sigma$. Let $\hat{\mathfrak{p}} := \mathfrak{t}\mathfrak{p}$ be the ideal extension of $\mathfrak{p}$ in $\mathfrak{t}$. The set of zero points of $\hat{\mathfrak{p}}$ is the closure of $\hat{Z}$ in $\mathbb{C}^2$.

The ring $\mathfrak{t} = \mathbb{C}[\mathbb{Z}^+L^\pm 1]$ embeds naturally into the principal ideal domain $\mathfrak{i} := \mathbb{C}(M)[L^\pm 1]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. The ideal extension of $\hat{\mathfrak{p}}$ in $\mathfrak{i}$, which is $\mathfrak{i}\hat{\mathfrak{p}} = \mathfrak{i}\mathfrak{p}$, is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^k$ with $k \in \mathbb{Z}$. Again $B_K$ can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$. We will call $B_K$ the $B$-polynomial of $K$. 
2.5. Relation between the A-polynomial and the B-polynomial. From the definitions one has immediately that the polynomial $B_K$ is $M$-essentially divisible by $A_K$. Moreover, their zero sets \{ $B_K = 0$ \} and \{ $A_K = 0$ \} are equal, up to some lines parallel to the $L$-axis in the $LM$-plane.

**Lemma 2.1.** One has $\tilde{t} = t^\sigma \otimes C[M^{\pm 1}]^\sigma C(M)$.

**Proof.** The extension from $C[M^{\pm 1}]^\sigma$ to $C(M)$ can be done in two steps: The first one is from $C[M^{\pm 1}]^\sigma$ to $C[M^{\pm 1}]$ (note that $C[M^{\pm 1}]$ is free over $C[M^{\pm 1}]^\sigma$ since $C[M^{\pm 1}] = C[M^{\pm 1}]^\sigma \oplus MC[M^{\pm 1}]^\sigma$); the second step is from $C[M^{\pm 1}]$ to its field of fractions $C(M)$. Each step is a flat extension, hence $C(M)$ is flat over $C[M^{\pm 1}]^\sigma$.

Look at the first extension. Let us show that the $C[M^{\pm 1}]$-linear map

$$\psi : t^\sigma \otimes C[M^{\pm 1}]^\sigma C[M^{\pm 1}] \to t, \quad \psi(x \otimes y) = xy,$$

is injective. Since $C[M^{\pm 1}] = C[M^{\pm 1}]^\sigma \oplus M C[M^{\pm 1}]^\sigma$, the map

$$\psi_1 : t^\sigma \oplus t^\sigma \to t^\sigma \otimes C[M^{\pm 1}]^\sigma C[M^{\pm 1}], \quad \psi_1(x, y) = x \otimes 1 + y \otimes M$$

is an isomorphism of $C[M^{\pm 1}]^\sigma$-modules.

In $t$ we have $t^\sigma \cap Mt^\sigma = \{0\}$. This means $t^\sigma + Mt^\sigma$ is a direct sum, or the map

$$\psi_2 : t^\sigma \oplus t^\sigma \to t, \quad \psi_2(x, y) = x + My$$

is injective.

Note that $\psi_2 = \psi \circ \psi_1$. Since $\psi_1$ is bijective, and $\psi_2$ is injective, $\psi$ must also be injective. Now consider the second extension. Tensoring (2.1) with $C(M)$, we get an algebra homomorphism

$$\tilde{\psi} : t^\sigma \otimes C[M^{\pm 1}]^\sigma C(M) \to t \otimes C[M^{\pm 1}] C(M) = \tilde{t}, \quad \tilde{\psi}(x \otimes y) = xy,$$

which is still injective, since $C(M)$ is flat over $C[M^{\pm 1}]$. Let us show that $\tilde{\psi}$ is surjective. Note that

$L = \tilde{\psi}((ML + M^{-1}L^{-1}) \otimes \frac{1}{M - M^{-1}}) - \tilde{\psi}((L + L^{-1}) \otimes \frac{M^{-1}}{M - M^{-1}}),$

$L^{-1} = \tilde{\psi}((L + L^{-1}) \otimes \frac{M}{M - M^{-1}}) - \tilde{\psi}((ML + M^{-1}L^{-1}) \otimes \frac{1}{M - M^{-1}})$

are in the image of $\tilde{\psi}$. Since $L, L^{-1}$ generate $\tilde{t} = C(M)[L^{\pm 1}]$, the map $\tilde{\psi}$ is surjective. Therefore $\tilde{\psi}$ is an isomorphism. \qed

**Proposition 2.2.** The B-polynomial $B_K$ does not have repeated factors.

**Proof.** We first note that the ring $C[\chi(X)]$ has a $t^\sigma$-module structure via the algebra homomorphism $\theta : C[\chi(\partial X)] \equiv t^\sigma \to C[\chi(X)]$, hence a $C[M^{\pm 1}]^\sigma$-module structure since $C[M^{\pm 1}]^\sigma$ is a subring of $t^\sigma$.

By Lemma 2.1 $\tilde{t} = t^\sigma \otimes C[M^{\pm 1}]^\sigma C(M)$. This implies that $\tilde{p} = p \otimes C[M^{\pm 1}]^\sigma C(M)$. We want to show that $\tilde{p}$ is radical, i.e. $\sqrt{\tilde{p}} = \tilde{p}$. Here $\sqrt{\tilde{p}}$ denotes the radical of $\tilde{p}$.

Let

$$\tilde{t} = t^\sigma \otimes C[M^{\pm 1}]^\sigma C(x), \quad \tilde{p} = p \otimes C[M^{\pm 1}]^\sigma C(x).$$

Note that $p$, the kernel of $\theta : t^\sigma \to C[\chi(X)]$, is radical since the ring $C[\chi(X)]$ is reduced. We claim that $\tilde{p}$ is also radical. Indeed, suppose $\gamma \in \tilde{t}$ and $\gamma^2 \in \tilde{p}$. Then $\gamma^2 = \delta/f$ for some $\delta \in \tilde{p}$ and $f \in C[x]$. It implies that $(f\gamma)^2 = f\delta$ is in $\tilde{p}$. Hence $f\gamma \in \sqrt{\tilde{p}} = \tilde{p}$ which means $\gamma \in \tilde{p}$. 

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Since $\mathfrak{I} = \mathbb{C}(x)[L^{\pm 1}]$ is a principal ideal domain, the radical ideal $\bar{\mathfrak{p}}$ can be generated by one element, say $\gamma(L) \in \mathbb{C}(x)[L^{\pm 1}]$, which does not have repeated factors. Note that the polynomial $\gamma(L)$ and $\delta(L) := \gamma'(L)$, the derivative of $\gamma(L)$ with respect to $L$, are co-prime. Since $\mathbb{C}(x)[L^{\pm 1}]$ is an Euclidean domain, there are $f, g \in \mathbb{C}(x)$ such that $f\gamma + g\delta = 1$.

It follows that $\gamma(L)$ and $\delta(L)$ are also co-prime in $\mathbb{C}(M)[L^{\pm 1}]$. Hence the ideal $\tilde{\mathfrak{p}} = \mathfrak{p} \otimes_{\mathbb{C}(x)} \mathbb{C}(M)$ in $\mathbb{C}(M)[L^{\pm 1}]$ is radical. This means that the $B$-polynomial $B_K$ does not have repeated factors.

Corollary 2.3. For every knot $K$ one has

$$B_K = \frac{A_K}{\text{its } M\text{-factor}}.$$ 

Here the $M$-factor of $A_K$ is the maximal factor of $A_K$ depending on $M$ only; it is defined up to a non-zero complex number.

2.6. Small knots. A knot $K$ is called small if its complement $X$ does not contain closed essential surfaces. It is known that all 2-bridge knots and all 3-tangle pretzel knots are small [HT, Oe].

Proposition 2.4. Suppose $K$ is a small knot. Then the $A$-polynomial $A_K$ has trivial $M$-factor. Hence the $A$-polynomial and $B$-polynomial of a small knot are equal.

Proof. The $A$-polynomial $A_K$ always contains the factor $L - 1$ coming from characters of abelian representations [CCGLS]. Hence we write $A_K = (L - 1)A_{\text{ab}}$ where $A_{\text{ab}}$ is a polynomial in $\mathbb{C}[M, L]$.

Suppose the polynomial $A_{\text{ab}}$ of a knot has non-trivial $M$-factor, then the Newton polygon of $A_{\text{ab}}$ has the slope infinity. It is known that every slope of the Newton polygon of $A_{\text{ab}}$ is a boundary slope of the knot complement [CCGLS]. Hence the knot complement has boundary slope infinity. The complement of a small knot in $S^3$ does not have boundary slope infinity (this fact follows easily from [CGLS, Thm. 2.0.3]), hence its polynomial $A_{\text{ab}}$ has trivial $M$-factor. □

Remark 2.5. By [IMS], according to a calculation by Culler, there exists a non-small knot whose $A$-polynomial has non-trivial $M$-factor; it is the knot $9_{38}$ in the Rolfsen table.

3. Skein modules and the AJ conjecture

Our proof of the main theorems is more or less based on the ideology that the KBSM is a quantization of the $SL_2(\mathbb{C})$-character variety [Bul, PS] which has been exploited in the work of Frohman, Gelca, and Lofaro [FGL] where they defined the non-commutative $A$-ideal. In this section we will discuss the quantum peripheral polynomial and its role in our approach to the AJ conjecture, and then prove Theorems 1 and 2.

3.1. Skein modules as quantizations of character varieties. Let $\varepsilon$ be the map reducing $t = -1$. An important result [Bul, PS] in the theory of skein modules is that $\varepsilon(S(Y))$ has a natural $\mathbb{C}$-algebra structure and is isomorphic to the universal $SL_2$-character algebra of the fundamental group of $Y$. The product of 2 links in $\varepsilon(S(Y))$ is their disjoint union. Using the skein relation with $t = -1$, it is easy to see that the product is well-defined, and that the value of a knot in the skein module depends only on the homotopy class of
the knot in $Y$. The isomorphism between $\varepsilon(S(Y))$ and the universal $SL_2$-character algebra of $\pi_1(Y)$ is given by $K(r) = -\text{tr } r(K)$, where $K$ is a homotopy class of a knot in $Y$, represented by an element, also denoted by $K$, of $\pi_1(Y)$, and $r : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$. The quotient of $\varepsilon(S(Y))$ by its nilradical is canonically isomorphic to $\mathbb{C}[\chi(Y)]$, the ring of regular functions on the $SL_2$-character variety of $\pi_1(Y)$.

In many cases $\varepsilon(S(Y))$ is reduced, i.e. its nilradical is 0, and hence $\varepsilon(S(Y))$ is exactly equal to the ring of regular functions on the character variety of $\pi_1(Y)$. For example, this is the case when $Y$ is a torus, or when $Y$ is the complement of a two-bridge knot/link $\text{[Le1, PS, LT]}$, or when $Y$ is the complement of the $(-2,3,2n+1)$-pretzel knot for any integer $n$ (see Section 4 below). We conjecture that

**Conjecture 2.** For every knot $K$ the universal character ring is reduced.

### 3.2. The peripheral polynomial and its role in the AJ conjecture.

Suppose for a knot $K$, the nilradical of $\varepsilon(S(X))$ is trivial. One has the following commutative diagram

$$
\begin{array}{ccc}
S(\partial X) & \xrightarrow{\varepsilon} & S(X) \\
\downarrow & & \downarrow \\
\mathbb{C}[\chi(\partial X)] & \xrightarrow{\varepsilon} & \mathbb{C}[\chi(X)].
\end{array}
$$

Recall that the classical peripheral ideal $p$ is the kernel of $\theta$; it is an ideal of $t^\sigma$. The ring $t^\sigma$ embeds into the principal ideal domain $\tilde{\mathfrak{i}} = \mathbb{C}(M)[L^{\pm 1}]$. The ideal extension $\tilde{p} = \tilde{t}p$ of $p$ in $\tilde{\mathfrak{i}}$ is generated by the $B$-polynomial $B_K$.

Let us now adapt the construction of the polynomial $B_K$ to the quantum setting. By definition the quantum peripheral ideal $\mathcal{P}$ is the kernel of $\Theta$; it is a left-ideal of $T^\sigma$. The ring $T^\sigma$ embeds into the principal left-ideal domain $\tilde{T}$ which can be formally defined as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{T}$ be the set of all Laurent polynomials in the variable $L$ with coefficients in $\mathcal{R}(M)$:

$$
\tilde{T} = \left\{ \sum_{j \in \mathbb{Z}} f_j(M) L^j \mid f_j(M) \in \mathcal{R}(M), \ f_j = 0 \quad \text{almost everywhere} \right\},
$$

and define the product in $\tilde{T}$ by $f(M)L^k \cdot g(M)L^l = f(M)g(t^{2k} M)L^{k+l}$.

The left-ideal extension $\tilde{\mathcal{P}} := \tilde{T}\mathcal{P}$ of $\mathcal{P}$ in $\tilde{T}$ is then generated by a polynomial

$$
\beta_K(t; M, L) = \sum_{j=0}^{d'} \beta_K(t, M, L^j),
$$

where $d'$ is assumed to be minimum and all the coefficients $\beta_K(t, M) \in \mathbb{Z}[t^{\pm 1}, M]$ are co-prime. Note that the polynomial $\beta_K$ is defined up to $\pm t^k M^l$ with $k, l \in \mathbb{Z}$. We will call $\beta_K$ the peripheral polynomial of $K$. The polynomial $\beta_K$ was introduced in $\text{[Le1]}$.

**Proposition 3.1.** a) $\varepsilon(\beta_K)$ is $M$-essentially divisible by $B_K$, and hence is $M$-essentially divisible by $A_K$.

b) $\beta_K$ is divisible by the recurrence polynomial $\alpha_K$ in the sense that there are polynomials $g(t, M) \in \mathbb{Z}[t, M]$ and $\gamma(t, M, L) \in T$ such that

$$
(3.1) \quad \beta_K(t, M, L) = \frac{1}{g(t, M)} \gamma(t, M, L) \alpha_K(t, M, L).
$$

Moreover $g(t, M)$ and $\gamma(t, M, L)$ can be chosen so that $\varepsilon(g) \neq 0$. 
Proof. See [Le1]. Note that the first part follows from the fact that $\varepsilon(\mathcal{P}) \subset \mathfrak{p}$ which can be easily deduced from the above commutative diagram, while the second fact follows the fact that the peripheral ideal is contained in the recurrence ideal by Proposition [1.2]. □

From Proposition 3.1, we see that both $\varepsilon(\alpha_K)$ and $A_K$ divide $\varepsilon(\beta_K)$ for a knot $K$. This observation gives us some important information for studying the AJ conjecture for $K$. Indeed, we have the following.

**Proposition 3.2.** Suppose $K$ is a knot satisfying all the following conditions:

(i) The $L$-degrees of the polynomials $A_K$ and $\varepsilon(\beta_K)$ are equal,

(ii) The $A$-polynomial has exactly two irreducible factors,

(iii) The recurrence polynomial $\alpha_K$ has $L$-degree greater than 1.

Then the AJ conjecture holds for $K$.

**Proof.** Since $\varepsilon(\beta_K)$ is $M$-essentially divisible by $A_K$, condition (i) implies that $\varepsilon(\beta_K) = A_K$. Combining this with (3.1), we get

$$\varepsilon(\gamma)\varepsilon(\alpha_K) = A_K.$$  

(3.2)

It is known that $A_K$ always contains the factor $L - 1$ coming from characters of abelian representations [CCGLS]), and $\varepsilon(\alpha_K)$ is also divisible by $L - 1$ [Le1, Proposition 2.3]. Hence we can rewrite (3.2) as follows:

$$\varepsilon(\gamma)\varepsilon(\alpha_K) = \frac{A_K}{L - 1}.$$  

(3.3)

By (ii), $A_K$ has exactly two irreducible factors. One of them is $L - 1$, hence the other factor $\frac{A_K}{L - 1}$ is irreducible. Equation (3.3) then implies that

$$\frac{\varepsilon(\alpha_K)}{L - 1} = A_K.$$  

(3.4)

If $\frac{\varepsilon(\alpha_K)}{L - 1} = 1$, then, by Lemma 3.3 below, the recurrence polynomial $\alpha_K$ has $L$-degree 1. This contradicts (iii), hence we must have $\frac{\varepsilon(\alpha_K)}{L - 1} = \frac{A_K}{L - 1}$. In other words, the AJ conjecture holds true for $K$. □

**Lemma 3.3.** The polynomial $\varepsilon(\alpha_K)$ is $M$-essentially equal to $L - 1$ if and only if the $L$-degree of the recurrence polynomial $\alpha_K$ is 1.

**Proof.** The backward direction is obvious since $\varepsilon(\alpha_K)$ is always divisible by $L - 1$.

Now suppose the polynomial $\varepsilon(\alpha_K)$ is $M$-essentially equal to $L - 1$, i.e. $\varepsilon(\alpha_K) = g(M)(L - 1)$ for some non-zero $g(M) \in \mathbb{C}[M^{\pm 1}]$. Then we have

$$\alpha_K = g(M)(L - 1) + (1 + t) \sum_{j=0}^{d} f_j(M)L^j$$  

(3.4)

where $f_j(M)$'s are Laurent polynomials in $\mathbb{R}[M^{\pm 1}]$ and $d$ is the $L$-degree of $\alpha_K$.

By a result in [Ga1], the recurrence ideal $\mathcal{A}_K$ is invariant under the involution $\sigma$. Hence $\sigma(\alpha_K)$ is contained in $\mathcal{A}_K$. Since $\alpha_K$ is the generator, it follows that $\alpha_K = h(M)\sigma(\alpha_K)L^d$ for some $h(M) \in \mathbb{R}(M)$. Equation (3.4) implies that

$$g(M)(L - 1) + (1 + t) \sum_{j=0}^{d} f_j(M)L^j$$
Dunfield showed that the map \( \rho \) representations. Then the restriction map components (one abelian and one non-abelian containing the characters of discrete faithful representation. By a result of Thurston, the 3.3. The \( L \)-degree of the \( A \)-polynomial. Suppose \( K \) is a hyperbolic knot. Then it has discrete faithful \( SL_2(\mathbb{C}) \)-representations. Let \( \chi \) denote an irreducible component of \( \chi(X) \) containing the character of a discrete faithful representation. By a result of Thurston, \( \chi_0 \) has dimension 1 since \( X \) has one boundary component.

Recall that the inclusion \( \partial X \hookrightarrow X \) induces the restriction map \( \rho: \chi(X) \to \chi(\partial X) \). Dunfield showed that the map \( \rho|_{\chi_0}: \chi_0 \to \chi(\partial X) \) is a birational isomorphism onto its image. Suppose \( K \) is hyperbolic and the character variety \( \chi(X) \) consists of 2 irreducible components (one abelian and one non-abelian containing the characters of discrete faithful representations). Then the restriction map \( \rho: \chi(X) \to \chi(\partial X) \) is a birational isomorphism onto its image. i.e. \( \chi(X) \) and its image are equal up to adding a finite number of points, since \( \chi(X) \) has dimension 1. It implies that the map \( \tilde{\theta}: \mathbb{C}[\chi(\partial X)] \otimes_{\mathbb{C}[M\pm1]^\sigma} \mathbb{C}(M) \to \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M\pm1]^\sigma} \mathbb{C}(M), \) induced by \( \rho \), is surjective. Note that

\[
\mathbb{C}[\chi(\partial X)] \otimes_{\mathbb{C}[M\pm1]^\sigma} \mathbb{C}(M) = \mathbb{C}[M^{\pm1}, L^{\pm1}]^\sigma \otimes_{\mathbb{C}[M\pm1]^\sigma} \mathbb{C}(M) = \mathbb{C}(M)[L^{\pm1}]
\]

by Lemma 2.1. Hence the polynomial \( B_K \), the generator of the kernel of \( \tilde{\theta} \), has \( L \)-degree equal to the dimension of the \( \mathbb{C}(M) \)-vector space \( \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M\pm1]^\sigma} \mathbb{C}(M) \), which is equal to the rank of the \( \mathbb{C}[M+M^{-1}] \)-module \( \mathbb{C}[\chi(X)] \). Since the \( A \)-polynomial is \( M \)-essentially equal to the \( B \)-polynomial by Corollary 2.3, we have

**Proposition 3.4.** Suppose \( K \) is a knot satisfying all the following conditions:

(i) \( K \) is hyperbolic,

(ii) The \( SL_2 \)-character variety of \( \pi_1(S^3 \setminus K) \) consists of 2 irreducible components (one abelian and one non-abelian).

Then the \( L \)-degree of the \( A \)-polynomial of \( K \) is equal to the rank of the \( \mathbb{C}[M+M^{-1}] \)-module \( \mathbb{C}[\chi(X)] \).

3.4. Localization and the \( L \)-degree of the peripheral polynomial.

3.4.1. Localization. Recall that \( T \) is the quantum torus and \( \varepsilon \) is the map reducing \( t = -1 \). Let \( D := \mathcal{R}[M^{\pm1}] \) and consider the localization of \( D \) at the ideal \( (1 + t) \):

\[
\overline{D} := \{ \frac{f}{g} \mid f, g \in D, \varepsilon(g) \neq 0 \}.
\]

Note that \( \overline{D} \), being a localization of \( D \), is flat over \( D \). The ring \( D = \mathcal{R}[M^{\pm1}] \) is flat over \( \mathcal{R}[M^{\pm1}]^\sigma \), since it is free over \( \mathcal{R}[M^{\pm1}]^\sigma \):

\[
\mathcal{R}[M^{\pm1}] = \mathcal{R}[M^{\pm1}]^\sigma \oplus M \mathcal{R}[M^{\pm1}]^\sigma.
\]
Let \( \mathcal{T} := \{ \sum_{j \in \mathbb{Z}} f_j(M) L^j \mid f_j(M) \in \mathcal{D}, ~ f_j = 0 \text{ almost everywhere} \} \)

with commutation rule: \( f(M) L^k \cdot g(M) L^l = f(M) g(t^{2k} M) L^{k+l} \). Then \( \mathcal{T} = \mathcal{T} \otimes_D \mathcal{D} \). Note that if in the definition of \( \mathcal{T} \) we allow \( f_j(M) \) to be in the fractional field \( \mathcal{R}(M) \) of \( D \) then we get \( \tilde{\mathcal{T}} \).

For a left-ideal \( I \) of \( \mathcal{T} \) (or \( \mathcal{T}^\sigma \)) let \( I \) and \( \tilde{I} \) be its extensions in \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \) respectively.

### 3.4.2. The \( L \)-degree of the peripheral polynomial.

**Proposition 3.5.** Suppose for a knot \( K \), \( \varepsilon(S(X)) \) is a finite rank \( \mathbb{C}[M + M^{-1}] \)-module. Then the \( L \)-degree of the peripheral polynomial \( \beta_K \) is less than or equal the rank of the \( \mathbb{C}[M + M^{-1}] \)-module \( \varepsilon(S(X)) \).

**Proof.** The involution \( \sigma \) acts on \( D \). Let \( D^\sigma \) denote the \( \sigma \)-invariant part of \( D \). There is a natural \( D^\sigma = \mathcal{R}([\bar{x}] \) module structure on \( S(X) \), here \( \bar{x} \equiv -(M + M^{-1}) \) is a meridian and it does belong to the boundary of \( X \). Hence by tensoring \( S(X) \) with \( \mathcal{D} \), \( \mathcal{S}(X) := S(X) \otimes_{D^\sigma} \mathcal{D} \) is an \( \mathcal{D} \)-module. Note that \( \mathcal{S}(\mathbb{T}^2) := S(\mathbb{T}^2) \otimes_{D^\sigma} \mathcal{D} \) is a free \( \mathcal{D} \)-module with basis \( \{ L^j : j \in \mathbb{Z} \} \).

Consider the following exact sequence of \( D^\sigma \)-modules

\[
0 \rightarrow \mathcal{P} \hookrightarrow \mathcal{T}^\sigma \xrightarrow{\Theta} S(X)
\]

Since \( \mathcal{D} \) is a flat over \( D^\sigma \), the following sequence is also exact

\[
0 \rightarrow \mathcal{D} \otimes_{D^\sigma} \mathcal{P} \hookrightarrow \mathcal{D} \otimes_{D^\sigma} \mathcal{T}^\sigma \xrightarrow{id \otimes \Theta} \mathcal{D} \otimes_{D^\sigma} S(X)
\]

i.e. the sequence of \( \mathcal{D} \)-modules

\[
(3.6) \quad 0 \rightarrow \mathcal{P} \hookrightarrow \mathcal{T} \xrightarrow{i} \mathcal{S}(X)
\]

is exact. Note that \( \mathcal{S}(X) \) is a module over the principal ideal domain \( \mathcal{D} \).

Suppose \( \mathcal{S}(X) \otimes_{\mathcal{D}} \mathbb{C}(M) \) is a finite dimensional vector space over \( \mathbb{C}(M) \). Consider the exact sequence \((3.6)\). The third module \( \mathcal{S}(X) \) is a \( \mathcal{D} \)-module of finite rank and hence

\[
\mathcal{S}(X) = \mathcal{D}^k \bigoplus_{j=1}^l \mathcal{D}/(f_j).
\]

Here each \( f_j \) is a power of \((1 + t)\). The middle module \( \mathcal{T} \) of \((3.6)\) is free \( \mathcal{D} \)-module with basis \( \{ L^j : j \in \mathbb{Z} \} \). Hence the image of \( k + l + 1 \) elements \( 1, L, L^2, \ldots, L^{k+l} \) are linearly dependent. Hence there must be a non-trivial element \( \delta \) in the kernel \( \mathcal{P} \) of \( L \)-degree less than or equal to \( k + l \). This element \( \delta \) is in \( \tilde{\mathcal{P}} \), hence it divides \( \beta_K \), the generator of \( \tilde{\mathcal{P}} \). From this, we conclude that \( \beta_K \) is non-trivial and has \( L \)-degree less that or equal to \( k + l \), which is the dimension of the vector space \( \mathcal{S}(X) \otimes_{\mathcal{D}} \mathbb{C}(M) \) over \( \mathbb{C}(M) \) since \( \mathcal{S}(X) \otimes_{\mathcal{D}} \mathbb{C}(M) = \mathbb{C}(M)^{k+l} \).

It is easy to check that \( \mathcal{S}(X) \otimes_{\mathcal{D}} \mathbb{C}(M) = \varepsilon(S(X)) \otimes_{\mathbb{C}[M \pm 1]^\sigma} \mathbb{C}(M) \). Since the dimension of the \( \mathbb{C}(M) \)-vector space \( \varepsilon(S(X)) \otimes_{\mathbb{C}[M \pm 1]^\sigma} \mathbb{C}(M) \) is equal to the rank of the \( \mathbb{C}[M + M^{-1}] \)-module \( \varepsilon(S(X)) \), the proposition follows. \( \square \)

### 3.5. Proof of Theorems [1] and [2]
3.5.1. **Proof of Theorem 1.** Suppose the knot $K$ satisfies all the conditions of the theorem. By assumptions (i), (ii) and Proposition 3.4, the $L$-degree of the $A$-polynomial $A_K$ is equal to the rank of the $\mathbb{C}[M+M^{-1}]$-module $\mathbb{C}[\chi(X)]$. Since the universal $SL_2$-character variety is reduced, $\varepsilon(S(X)) = \mathbb{C}[\chi(X)]$. Hence the $L$-degree of the $A$-polynomial $A_K$ is also equal to the the rank of the $\mathbb{C}[M+M^{-1}]$-module $\varepsilon(S(X))$. This, together with Proposition 3.5, implies that the $L$-degree of $\beta_K$ is less than or equal to that of $A_K$. From this, one can easily check that the knot $K$ satisfies all the conditions of Proposition 3.2 and hence the AJ conjecture holds true for $K$.

3.5.2. **Proof of Theorem 2.** It is known that two-bridge knots and $(-2, 3, 2n+1)$-pretzel knots, excluding torus knots, are hyperbolic. (Note that the AJ conjecture holds true for torus knots by [Hi, Tr]). Their universal character rings are reduced by [Le1] and Theorem 4.6 below, respectively. The $L$-degrees of their recurrence polynomials are greater than $1$ according to the results in [Le1 Proposition 2.2] and [Ga3 Section 4.7] respectively. Double twist knots of the form $J(k,l)$ with $k \neq l$, two-bridge knots of the form $b(p,m)$ with $p$ prime or $m = 3$, and $(-2, 3, 6n \pm 1)$-pretzel knots satisfy assumption (ii) of Theorem 1 by [MPL], Theorem A.1 and [Bur], and [Mat] respectively. Hence the theorem follows.

4. The universal character ring of $(-2, 3, 2n + 1)$-pretzel knots

In this section we explicitly calculate the universal character ring of the $(-2, 3, 2n + 1)$-pretzel knot and prove its reducedness for all integers $n$.

4.1. **The character variety.** For the $(-2, 3, 2n + 1)$-pretzel knot $K_{2n+1}$, we have

$$\pi_1(X) = \langle a, b, c \mid cacb = acba, \ ba(cb)^n = a(cb)^n c \rangle,$$

where $X = S^3 \setminus K_{2n+1}$ and $a, b, c$ are meridians depicted in Figure 1.

![Figure 1. The $(-2, 3, 2n+1)$-pretzel knot](image)

Let $w = cb$ then the first relation of $\pi_1(X)$ becomes $caw = awa$. It implies that $c = awaw^{-1}a^{-1}$ and $b = c^{-1}w = awa^{-1}w^{-1}a^{-1}w$. The second relation then has the form

$$awa^{-1}w^{-1}a^{-1}waw^n = awaw^{-1}a^{-1}$$

i.e.

$$w^na^{-1}w^{-1}a^{-1} = a^{-1}w^{-1}awaw^{-1}w^n.$$

Hence we obtain a presentation of $\pi_1(X)$ with two generators and one relation

$$\pi_1(X) = \langle a, w \mid w^nE = Fw^n \rangle$$
where $E := a w a^{-1} w^{-1} a^{-1}$ and $F := a^{-1} w^{-1} a w a w^{-1}$.

The characteristic variety of the free group $F_2 = \langle a, w \rangle$ in 2 letters $a$ and $w$ is isomorphic to $\mathbb{C}^3$ by the Fricke-Klein-Vogt theorem, see [LM]. For every element $\omega \in F_2$ there is a unique polynomial $P_\omega$ in 3 variables such that for any representation $r : F_2 \to SL_2(\mathbb{C})$ we have $\text{tr}(r(\omega)) = P_\omega(x, y, z)$ where $x := \text{tr}(r(a))$, $y := \text{tr}(r(w))$ and $z := \text{tr}(r(aw))$. The polynomial $P_\omega$ can be calculated inductively using the following identities for traces of matrices $A, B \in SL_2(\mathbb{C})$:

\begin{align}
(4.1) & \quad \text{tr}(A) = \text{tr}(A^{-1}), \quad \text{tr}(AB) + \text{tr}(AB^{-1}) = \text{tr}(A) \text{tr}(B).
\end{align}

Thus for every representation $r : \pi_1(X) \to SL_2(\mathbb{C})$, we consider $x, y$, and $z$ as functions of $r$. The character variety of $\pi_1(X)$ is the zero locus of an ideal in $\mathbb{C}[x, y, z]$, which we describe explicitly in the next theorem.

**Theorem 4.1.** The character variety of the pretzel knot $K_{2n+1}$ is the zero locus of 2 polynomials $P := P_E - P_F$ and $Q_n := P_{w^nEa} - P_{Fw^n a}$. Explicitly,

\begin{align}
(4.2) & \quad P = x - xy + (-3 + x^2 + y^2)z - xyz^2 + z^3, \\
(4.3) & \quad Q_n = S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)x^2 \\
& \quad \quad + (S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y))xz - (S_{n-2}(y) + S_{n-3}(y))z^2
\end{align}

where $S_n(y)$ are the Chebychev polynomials defined by $S_0(y) = 1$, $S_1(y) = y$ and $S_{n+1}(y) = yS_n(y) - S_{n-1}(y)$ for all integer $n$.

**Proof.** The explicit formulas (4.2) and (4.3) follow from an easy calculation of the trace polynomials using (4.1).

Because $E$ and $F$ are conjugate (by $w^n$) and $w^nEa = Fw^n a$ in $\pi_1(X)$, we have $P = Q_n = 0$ for every representation $r : \pi_1(X) \to SL_2(\mathbb{C})$.

We will prove the converse: fix a solution $(x, y, z)$ of $P = Q_n = 0$, we will find a representation $r : \pi_1(X) \to SL_2(\mathbb{C})$ such that $x = \text{tr}(r(a))$, $y = \text{tr}(r(w))$ and $z = \text{tr}(r(aw))$.

We consider the following 3 cases:

**Case 1:** $y^2 \neq 4$. Then there exist $s, u, v \in \mathbb{C}$ such that $s + s^{-1} = y$, $u + v = x$, $su + s^{-1}v = z$. Since $S_k(y) = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}}$ for all integer $k$, we have

\begin{align*}
P &= s^{-3}(s - 1)P', \\
Q_n &= s^{-3-n}((s^2u - sv)P' - (1 + s)(-1 + uv)Q'_n),
\end{align*}

where

\begin{align*}
P' &= s^3u - s^4u - s^5u + v + sv - s^2v - s^2u^2v - s^3u^2v + s^4u^2v + s^5u^2v \\
&\quad - uv^2 - s^2uv^2 + s^2uv^2, \\
Q'_n &= s^5 + s^{2n} - s^{2+2n}u^2 + s^{4+2n}u^2 + s^{3}uv - s^5uv - s^{2n}uv + s^{2+2n}uv + sv^2 - s^3v^2.
\end{align*}

Since $s \neq \pm1$, $P = Q_n = 0$ is equivalent to $P' = (-1 + uv)Q'_n = 0$. We consider the following 2 subcases:

**Subcase 1.1:** $Q'_n = 0$. Choose $r(a) = \begin{pmatrix} u & 1 \\ uv - 1 & v \end{pmatrix}$ and $r(w) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$. It is easy to check $x = \text{tr}(r(a))$, $y = \text{tr}(r(w))$, $z = \text{tr}(r(aw))$ and the calculations in the following 2 lemmas.
Lemma 4.2. One has
\[ r(E) = \begin{pmatrix} s^{-2}H_{11} & -s^{-2}H_{12} \\ s^{-2}(-1 + uv)H_{21} & -s^{-2}H_{22} \end{pmatrix}, \quad r(F) = \begin{pmatrix} -s^{-3}H_{22} & -s^{-1}H_{21} \\ s^{-3}(-1 + uv)H_{12} & -s^{-1}H_{11} \end{pmatrix} \]
where
\[ H_{11} = s^2u - s^4u + v - s^2u^2v + s^4u^2v - uv^2 + s^2uv^2, \]
\[ H_{12} = 1 - s^2u^2 + s^4u^2 - uv + s^2uv, \]
\[ H_{21} = -s^4 - s^2uv + s^4uv - v^2 + s^2v^2, \]
\[ H_{22} = -s^4u + v - s^2u^2v + s^4u^2v - uv^2 + s^2uv^2. \]

Lemma 4.3. One has
\[ r(w^nE - Fw^n) = \begin{pmatrix} s^{-3+n}P' & -s^{-2-n}Q'_n \\ -s^{-3-n}(-1 + uv)Q'_n & -s^{-2-n}P' \end{pmatrix}. \]

Since \( P' = Q'_n = 0 \), Lemma 4.3 implies that \( r(w^nE - Fw^n) = 0 \), i.e. \( r(w^nE) = r(Fw^n) \).

Subcase 1.2: \(-1 + uv = 0\) then \( v = u^{-1} \). In this case the equation \( P' = 0 \) becomes \( s^2u^{-1}(s - u^2) = 0 \) i.e. \( s = u^2 \). Let
\[ r(a) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad r(w) = \begin{pmatrix} u^2 & 0 \\ 0 & u^{-2} \end{pmatrix}. \]
Then it is easy to check that \( x = \text{tr}(r(a)), \ y = \text{tr}(r(w)), \ z = \text{tr}(r(aw)) \) and \( r(Ew^n) = r(w^nF) \). (Note that \( r(a) \) and \( r(w) \) commute in this case).

Case 2: \( y = 2 \). Then \( S_k(y) = k \) for all integer \( k \). Hence
\[ P = (x - z)(-1 + xz - z^2), \]
\[ Q_n = 4 - (n - 1)x^2 + (3n - 5)xz - (2n - 3)z^2. \]
Hence \((x, z) = (-2, -2), (2, 2)\) or \((x = z + z^{-1} \) and \( 1 - n + (1 + n)z^2 - z^4 = 0 \).
If \( x = z = 2 \) we choose
\[ r(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]
If \( x = z = -2 \) we choose
\[ r(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]
If \( x = z + z^{-1} \) and \( 1 - n + (1 + n)z^2 - z^4 = 0 \) we choose
\[ r(a) = \begin{pmatrix} z & 0 \\ -z^{-1} & z^{-1} \end{pmatrix}, \quad r(w) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Lemma 4.4. One has
\[ r(w^nE - Fw^n) = \begin{pmatrix} 0 & z^{-1}(-1 + n - (1 + n)z^2 + z^4) \\ 0 & 0 \end{pmatrix} \]
Proof. By direct calculations we have \( r(E) = \begin{pmatrix} z & -2z + z^3 \\ 0 & z^{-1} - z \end{pmatrix} \), \( r(F) = \begin{pmatrix} z & z^{-1} - z \\ 0 & z^{-1} \end{pmatrix} \) and \( r(w^n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \). The lemma follows. \( \square \)

Hence \( x = \text{tr}(r(a)) \), \( y = \text{tr}(r(w)) \), \( z = \text{tr}(r(aw)) \) and \( r(w^nE) = r(Fw^n) \).

Case 3: \( y = -2 \). Then \( S_k(y) = (-1)^k k \) for all integer \( k \). Hence
\[
\begin{align*}
P &= 3x + z + x^2z + 2xz^2 + z^3, \\
Q_n &= (-1)^n (xP - (x + z)Q_n')/2,
\end{align*}
\]
where \( Q_n'' = x + 2nx + 2z + x^2z + xz^2 \).

Hence the system \( P = Q_n = 0 \) is equivalent to \( P = (x + z)Q_n'' = 0 \). We consider the following 2 subcases:

Subcase 3.1: \( x + z = 0 \). Then it is easy to see that \( P = 0 \) is equivalent to \( x = z = 0 \). In this case we choose
\[
r(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad r(w) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]
where \( i \) is the imaginary number.

Subcase 3.2: \( x + z \neq 0 \). Then \( Q_n'' = 0 \). Choose
\[
r(a) = \begin{pmatrix} x/2 & (1 - x^2/4)/(x + z) \\ -x - z & x/2 \end{pmatrix}, \quad r(w) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.
\]

Lemma 4.5. One has
\[
r(w^nE - Fw^n) = (-1)^n \begin{pmatrix} nP - Q_n''/2 & (n - 1)P + Q_n'' \\ 0 & -(n - 1)P + Q_n'' \end{pmatrix}.
\]

Proof. By direct calculations, we have \( r(w^n) = (-1)^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \) and
\[
\begin{align*}
r(E) &= \begin{pmatrix} -(x + 2z + x^2z + xz^2)/2 & -4 + 3x^2 + 4xz + x^3z + x^2z^2 \\ (x + z)(1 + xz + z^2) & 4(x + z) \end{pmatrix}, \\
r(F) &= \begin{pmatrix} (x + 2z + x^2z + xz^2)/2 & -4 + 5x^2 + 10xz + 3x^3z + 4x^2z + 5xz^2 + 2xz^3 \\ (x + z)(1 + xz + z^2) & 4(x + z) \end{pmatrix}.
\end{align*}
\]
The lemma follows. \( \square \)

Hence \( x = \text{tr}(r(a)) \), \( y = \text{tr}(r(w)) \), \( z = \text{tr}(r(aw)) \) and \( r(w^nE) = r(Fw^n) \) in all cases. It implies that the character variety of the pretzel knot \( K_{2n+1} \) is exactly equal to the algebraic set \( \{P = Q_n = 0\} \). \( \square \)

4.2. The universal character ring. In this subsection, we will prove the following theorem.

Theorem 4.6. The universal character ring of \( K_{2n+1} \) is reduced and is equal to the ring \( \mathbb{C}[x, y, z]/(P, Q_n) \).
Proof. Suppose we have shown that the ring $\mathbb{C}[x, y, z]/(P, Q_n)$ is reduced, then it is exactly the character ring $\mathbb{C}[\chi(x)]$ of $K_{2n+1}$.  
Recall that $\pi_1(X) = \langle a, w \mid u^mE = Fw^a \rangle$ where $F_2 = \langle a, w \rangle$ is the free group on two generators $a, w$. It is known that the universal character ring of $F_2$ is the ring $\mathbb{C}[x, y, z]$ where $x = \operatorname{tr}(r(a))$, $y = \operatorname{tr}(r(w))$ and $z = \operatorname{tr}(r_aw)$ as above. The quotient map $h : F_2 \to \pi_1(X)$ induces the epimorphism $h_* : \mathbb{C}[x, y, z] \to \mathbb{C}[\chi(X)]$. Since $P, Q_n$ come from traces, they are contained in ker $h_*$.  
Since $\mathbb{C}[\chi(X)]$ is the quotient of $\mathbb{C}(S(X))$ by its nilradical, we have the quotient homomorphism $\phi : \mathbb{C}(S(X)) \to \mathbb{C}[\chi(X)] = \mathbb{C}[x, y, z]/(P, Q_n)$. Then  
$$\phi \circ h_* : \mathbb{C}[x, y, z] \to \mathbb{C}(S(X)) \to \mathbb{C}[\chi(\pi)] = \mathbb{C}[x, y, z]/(P, Q_n)$$  
is a homomorphism. It follows that ker $h_* \subseteq (P, Q_n)$. Hence we must have ker $h_* = (P, Q_n)$, which implies $\mathbb{C}(S(X)) \cong \mathbb{C}[x, y, z]/(P, Q_n) \cong \mathbb{C}[\chi(X)]$.  
In the remaining part of this section we will show that the ring $\mathbb{C}[x, y, z]/(P, Q_n)$ is reduced, i.e. the ideal $I_n := (P, Q_n)$ is radical. The proof of this fact will be divided into several steps.  

4.2.1. $\mathbb{C}[x, y, z]/I_n$ is free over $\mathbb{C}[x]$.  

**Lemma 4.7.** For every $x_0 \neq 0, \pm 2$, the polynomial $P |_{x=x_0}$ is irreducible in $\mathbb{C}[y, z]$.  

**Proof.** Assume that $P |_{x=x_0}$ can be decomposed as  
$$z^3 - x_0y^2 + (y^2 + x_0^2 - 3)z + x_0(1 - y) = (z + f_1)(z^2 - (x_0y + f_1)z + f_2),$$  
where $f_j \in \mathbb{C}[y]$. Equation (4.4) implies that $f_2 - f_1(x_0y + f_1) = y^2 + x_0^2 - 3$ and $f_1f_2 = x_0(1 - y)$.  
If $f_1$ is a constant then $f_2 = x_0(1 - y)/f_1$ has $y$-degree 1. Hence $f_2 - f_1(x_0y + f_1)$ has $y$-degree 1 also. It implies that $f_2 - f_1(x_0y + f_1) \neq y^2 + x_0^2 - 3$.  
If $f_2$ is a constant then $f_1 = x_0(1 - y)/f_2$. Hence  
$$f_2 - f_1(x_0y + f_1) = f_2 - \left(\frac{x_0}{f_2} \cdot \frac{y}{f_2} \cdot \frac{x_0}{f_2} - \frac{x_0}{f_2} \cdot \frac{x_0y + x_0y}{f_2} \right) = \frac{x_0}{f_2} (1 - \frac{y}{f_2})^2 - \frac{x_0^2}{f_2} (1 - \frac{2}{f_2})y + (f_2 - \frac{x_0^2}{f_2}).$$  
Then since $f_2 - f_1(x_0y + f_1) = y^2 + x_0^2 - 3$, we have $\frac{x_0^2}{f_2} (1 - \frac{1}{f_2}) = 1, \frac{x_0^2}{f_2} (1 - \frac{2}{f_2}) = 0$, and $f_2 - \frac{x_0^2}{f_2} = x_0^2 - 3$. This implies $x_0 = 0$ or $x_0 = \pm 2$.  

**Lemma 4.8.** For every $x_0$, the polynomials $P |_{x=x_0}$ and $Q_n |_{x=x_0}$ are co-prime in $\mathbb{C}[y, z]$.  

**Proof.** If $x_0 \neq 0, \pm 2$ then, by Lemma 4.7, $P |_{x=x_0}$ is irreducible in $\mathbb{C}[y, z]$. Lemma 4.8 then follows since $P |_{x=x_0}$ and $Q_n |_{x=x_0}$ have $z$-degrees 3 and 2 respectively.  
At $x_0 = 0$, we have $P = z(-3 + y^2 + z^2)$ and $Q_n = a_n + b_nz^2$ where  
$$a_n = S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y),$$  
$$b_n = -S_{n-2}(y) - S_{n-3}(y).$$  
In this case, it suffices to show that $Q_n |_{z^2=3-y^2}= a_n + b_n(3 - y^2) \neq 0$. This is true by Lemma 4.13 below.
At \( x_0 = 2 \), we have \( P = (z + 1 - y)(z^2 - (1 + y)z + 2) \) and \( Q_n = a_n' + b_n'z + c_n'z^2 \) where \( a_n', b_n', c_n' \in \mathbb{C}[y] \). When \( z = y - 1 \), we have \( Q_0 = 1 \) and \( Q_1 = y - 1 \) and \( Q_{n+1} = yQ_n - Q_{n-1} \).

It implies that \( Q_n \mid_{z = y - 1} = S_n(y) - S_{n-1}(y) \) is a polynomial of \( y \)-degree \( n \) if \( n \geq 0 \) and \( -(n + 1) \) if \( n \leq -1 \), with leading coefficient \( 1 \).

Hence \( Q_n \mid_{z = y - 1} \) is not identically 0. It remains to show that \( Q_n = a_n' + b_n'z + c_n'z^2 \neq c_n'(z^2 - (1 + y)z + 2) \). It suffices to show that \( b_n' \mid_{y = -1} \neq 0 \). Indeed, when \( x_0 = 2 \) and \( y = -1 \) we have \( b_n' = 2(S_{n-1}(-1) + S_{n-3}(-1) + S_{n-4}(-1)) \). It is easy to check that \( S_k(-1) = 1 \) if \( k = 0 \) (mod 3), \( S_k(-1) = -1 \) if \( k = 1 \) (mod 3) and \( S_k(-1) = 0 \) otherwise. Hence \( b_n' = 2(S_{n-1}(-1) + S_{n-3}(-1) + S_{n-4}(-1)) \neq 0 \).

The case \( x_0 = -2 \) is similar. \( \square \)

**Proposition 4.9.** \( \mathbb{C}[x, y, z]/I_n \) is a torsion-free \( \mathbb{C}[x] \)-module.

**Proof.** Suppose \( S \in \mathbb{C}[x, y, z] \) and \( (x - x_0)S \in I_n \) for some \( x_0 \in \mathbb{C} \). We will show that \( S \in I_n \). Indeed, we have \( (x - x_0)S = fP - gQ_n \) for some \( f, g \in \mathbb{C}[x, y, z] \). Hence \( (fP) \mid_{x = x_0} = (gQ_n) \mid_{x = x_0} \) which implies that \( f \mid_{x = x_0} \) is divisible by \( Q_n \mid_{x = x_0} \), since \( P \mid_{x = x_0} \) and \( Q_n \mid_{x = x_0} \) are co-prime in the UFD \( \mathbb{C}[y, z] \) by Lemma 4.1. Hence \( f_{x = x_0} = h Q_n \mid_{x = x_0} \) for some \( h \in \mathbb{C}[y, z] \). From this, we may write \( f = hQ_n + (x - x_0)Q \) for some \( Q \in \mathbb{C}[x, y, z] \). Then we have

\( (x - x_0)S = fP - gQ_n = (hQ_n + (x - x_0)Q)P - gQ_n = (x - x_0)QP + (hP - g)Qn \)

which implies that \( hP - g \) is divisible by \( x - x_0 \) and \( S = QP + \frac{hP - g}{x - x_0}Q_n \in I_n \). \( \square \)

**Proposition 4.10.** \( \mathbb{C}[x, y, z]/I_n \) is a finitely generated \( \mathbb{C}[x] \)-module.

**Proof.** We want to show that \( y \) and \( z \), considered as elements of \( \mathbb{C}[x, y, z]/I_n \), are integral over \( \mathbb{C}[x] \). Indeed, the resultant of \( P \) and \( Q_n \) w.r.t. \( z \) is

\[
\begin{vmatrix}
0 & P_0 & P_1 & P_2 & P_3 & 0 \\
0 & P_0 & P_1 & P_2 & P_3 & 0 \\
Q_{n,0} & Q_{n,1} & Q_{n,2} & 0 & 0 & 0 \\
0 & Q_{n,0} & Q_{n,1} & Q_{n,2} & 0 & 0 \\
0 & 0 & Q_{n,0} & Q_{n,1} & Q_{n,2} & 0 \\
\end{vmatrix}
\]

where \( P_0 = x - xy, \ P_1 = -3 + x^2 + y^2, \ P_2 = -xy, \ P_3 = 1 \) and

\[
\begin{align*}
Q_{n,0} &= S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y) - S_{n-2}(y)x^2, \\
Q_{n,1} &= (S_{n-1}(y) + S_{n-3}(y) + S_{n-4}(y))x, \\
Q_{n,2} &= -(S_{n-2}(y) + S_{n-3}(y)).
\end{align*}
\]

Write \( y = s + s^{-1} \) then \( S_k(y) = \frac{s^{k+1} - s^{-k-1}}{s - s^{-1}} \) for all integers \( k \). By a direct calculation

\[
\begin{align*}
Res &= \frac{t + t^{-1} + 2 - x^2}{(t + t^{-1} + 2)}(t^{3n} + t^{-3n} + 3t^{3n-1} + 3t^{1-3n} + 3t^{3n-2} + 3t^{2-3n} \\
&+ t^{3n-3} + t^{-3n+3} + t^{n+5} + t^{-n+5} + 3t^{n+4} + 3t^{n-4} + 3t^{n+3} + 3t^{n-3} + t^{n+2} + t^{-n-2} \\
&- 2t^{-n-1} - 2t^{1-n} - 6t^{n-2} - 6t^{-n-2} - 6t^{n-3} - 6t^{3-n} - 6t^{3-n} - 2t^{n-4} + 2t^{n+1} - 2t^{n+1} - 2t^{n-3} - 3t^{n+2} - 3t^{-n-2} \\
&- t^{n+1} - t^{n-1} - 5t^{n} - 5t^{-n} - 2t^{n+1} - 2t^{n+1} - 8t^{n-2} + 8t^{n-2} + 6t^{n-3} + 6t^{3-n} \\
&+ t^{n-4} + t^{4-n}) + x^4(t^{n+1} + t^{n-1} + 2t^{n} + 2t^{n} - 2t^{n-2} - 2t^{n-2} - t^{n-4} - 2t^{n-4} - t^{n-4} - 3).
\end{align*}
\]
Let $T_k(y) = t^k + t^{-k}$. Then we have

\[
\text{Res} = \frac{y + 2 - x^2}{(y^2 - 4)(y + 2)}(T_{3n}(y) + 3T_{3n-1}(y) + 3T_{3n-2}(y) + T_{3n-3}(y) + T_{n+5}(y) + 3T_n+4(y)
+ 3T_{n+3}(y) + T_{n+2}(y) - 2T_{n-1}(y) - 6T_{n-2}(y) - 6T_{n-3}(y) - 2T_{n-4}(y)
+ x^2(-T_{3n-1}(y) - T_{3n-2}(y) - 2T_{n+3}(y) - 3T_{n+2}(y) - T_{n+1}(y) - 5T_n(y) - 2T_{n-1}(y)
+ 8T_{n-2}(y) + 6T_{n-3}(y) + T_{n-4}(y)) + x^4(T_{n+1}(y) + 2T_n(y) - 2T_{n-2}(y) - T_{n-3}(y)))
\]

Note that $T_k(y)$ has $y$-degree $|k|$ with leading coefficient 1. If $n \geq 4$ then it is easy to see that $\text{Res}$ has $y$-degree $3n - 2$; moreover their coefficient of $y^{3n-2}$ is 1. Similarly, if $n \leq -5$ then $\text{Res}$ has $y$-degree $1 - 3n$; moreover the coefficient of $y^{1-3n}$ is 1. If $-4 \leq n \leq 3$ then by direct calculations, we can check that the coefficient of the highest power of $y$ in $\text{Res}$ is 1. Hence the coefficient of the highest power of $y$ in $\text{Res}$ is 1 for all integers $n$. It implies that $y$, considered as an element of $\mathbb{C}[x, y, z]/I_n$, is integral over $\mathbb{C}[x]$. Since $z$, considered as an element of $\mathbb{C}[x, y, z]/I_n$, satisfies the equation $P = x - xy + (-3 + x^2 + y^2)z^2 - xyz^2 + z^3 = 0$ with 1 being the coefficient of the highest power of $z$, it is also integral over $\mathbb{C}[x]$. Therefore $\mathbb{C}[x, y, z]/I_n$ is a finitely generated $\mathbb{C}[x]$-module.

Since $\mathbb{C}[x]$ is a PID, Propositions 4.9 and 4.10 imply that

**Proposition 4.11.** $\mathbb{C}[x, y, z]/I_n$ is a free $\mathbb{C}[x]$-module.

4.2.2. Reduction to a special case. For a $\mathbb{C}[x]$-module $J$, let $J |_{x=x_0} := J \otimes_{\mathbb{C}[x]} \mathbb{C}$, where $\mathbb{C}$ is considered as an $\mathbb{C}[x]$-module by reducing $x = x_0$.

**Proposition 4.12.** $I_n$ is radical if $I_n |_{x=x_0}$ is radical for some $x_0 \in \mathbb{C}$.

*Proof.* Let $R = \mathbb{C}[x, y, z]$. Consider the exact sequence of $\mathbb{C}[x]$-modules

\[0 \to \sqrt{I_n}/I_n \to R/\sqrt{I_n} \to R/I_n \to 0.\]

By Proposition 4.11, $R/I_n$ is free, hence the sequence splits and $\sqrt{T_n}/I_n$ is projective. Since $\mathbb{C}[x]$ is a PID, $\sqrt{T_n}/I_n$ is free. Let $k$ be the rank of the $\mathbb{C}[x]$-module $\sqrt{T_n}/I_n$ then the rank of the $\mathbb{C}$-module $(\sqrt{T_n}/I_n) |_{x=x_0}$ is always $k$ for every $x_0 \in \mathbb{C}$. Hence if $I_n |_{x=x_0}$ is radical for some $x_0 \in \mathbb{C}$ then $k = 0$ which implies that $\sqrt{T_n} = I_n$. \hfill \Box

4.2.3. $I_n |_{x=0}$ is radical. By Lemma 4.8, $P |_{x=0}$ and $Q_n |_{x=0}$ are co-prime. This means $I_n |_{x=0}$ is a zero-dimensional ideal of $\mathbb{C}[y, z]$. By Seidenberg’s lemma (see [KL, Proposition 3.7.15]), if there exist two non-zero free-square polynomials in $I_n |_{x=0} \cap \mathbb{C}[y]$ and $I_n |_{x=0} \cap \mathbb{C}[z]$ respectively, then $I_n |_{x=0}$ is radical.

From now on we fix $x = 0$. Then $P = z(-3 + y^2 + z^2)$ and $Q_n = a_n + b_n z^2$ where

\[
a_n = S_{n-2}(y) + S_{n-3}(y) - S_{n-4}(y) - S_{n-5}(y),
\]

\[
b_n = -S_{n-3}(y) - S_{n-3}(y).
\]

Let $U_n = a_n + b_n (3 - y^2)$. Then $U_0 = 1, U_1 = y + 1$ and $U_{n+1} = y U_n - U_{n-1}$. Hence

\[U_n = S_n(y) + S_{n-1}(y).
\]

We first consider the case $n \geq 3$. Then $U_n$ and $a_n$ have $y$-degrees $n$ and $n-2$ respectively; moreover their leading coefficients are equal to 1.
Lemma 4.13. One has

\[ U_n = \prod_{j=1}^{n} (y - 2 \cos \frac{j2\pi}{2n+1}). \]

Proof. It is easy to see that \( U_n \) is a polynomial of degree \( n \) in \( y \). Note that if \( y = s + s^{-1} \neq \pm 2 \) then \( S_k(y) = \frac{s^{k+1} + s^{-k-1}}{s-s^{-1}} \). We now take \( y = e^{j2\pi/2n+1} + e^{-j2\pi/2n+1} = 2 \cos \frac{j2\pi}{2n+1} \) where \( 1 \leq j \leq n \). Then

\[ S_n(y) = \frac{\sin((n+1)\frac{j2\pi}{2n+1})}{\sin(\frac{j2\pi}{2n+1})} = -\frac{\sin(n\frac{j2\pi}{2n+1})}{\sin(\frac{j2\pi}{2n+1})} = -S_{n-1}(y). \]

The lemma follows. \( \square \)

Lemma 4.14. One has

\[ a_n = \prod_{k=0}^{n-3} (y - 2 \cos \frac{(2k+1)\pi}{2n-5}). \]

Proof. The proof is similar to that of the previous lemma. \( \square \)

Note that

\[ b_n^2 z P = b_n z^2 (-3 + y^2)b_n + b_n z^2) = (Q_n - a_n)(Q_n - U_n). \]

Hence \( a_n U_n = a_n Q_n - Q_n^2 + Q_n U_n + b_n^2 z P \) is contained in \( I_n |_{x=0} \). But \( a_n U_n \) is a polynomial in \( y \), hence it is actually contained in \( I_n |_{x=0} \cap \mathbb{C}[y] \). It is easy to see that \( a_n U_n \) is square-free, i.e. does not have repeated factors.

Let

\[ V_n = z \prod_{j=1}^{n} \left(-3 + 4 \cos^2 \frac{j2\pi}{2n+1} + z^2\right) \prod_{k=0}^{n-3} \left(-3 + 4 \cos^2 \frac{(2k+1)\pi}{2n-5} + z^2\right). \]

Then it is easy to show that \( V_n \in \mathbb{C}[z] \) is square-free. Moreover, since

\[ V_n \equiv z \prod_{j=1}^{n} \left(-3 + y^2 + z^2 + (4 \cos^2 \frac{j2\pi}{2n+1} - y^2)\right) \times \prod_{k=0}^{n-3} \left(-3 + y^2 + z^2 + (4 \cos^2 \frac{(2k+1)\pi}{2n-5} - y^2)\right) \]

\[ \equiv z \prod_{j=1}^{n} \left(4 \cos^2 \frac{j2\pi}{2n+1} - y^2\right) \prod_{k=0}^{n-3} \left(4 \cos^2 \frac{(2k+1)\pi}{2n-5} - y^2\right) \pmod{P}, \]

it is contained in \( I_n |_{x=0} \). Hence \( V_n \) is in \( I_n |_{x=0} \cap \mathbb{C}[z] \) and is square-free.

Since both \( a_n U_n \in I_n |_{x=0} \cap \mathbb{C}[y] \) and \( V_n \in I_n |_{x=0} \cap \mathbb{C}[z] \) are square-free, \( I_n |_{x=0} \) is a radical ideal by Seidenberg's lemma. Hence by Proposition 4.12, \( I_n \) is also radical. It implies that \( R/I_n \) is reduced. Hence the ring \( \mathbb{C}[x, y, z]/(P, Q_n) \) is reduced and is equal to the universal character ring of \( K_{2n+1} \). This proves Theorem 4.6 for the case \( n \geq 3 \). The case \( n \leq -1 \) is similar (in this case \( U_n \) and \( a_n \) have \( y \)-degrees \( 3 - n \) and \( -(n+1) \) respectively; moreover their leading coefficients are equal to 1 and \( -1 \) respectively). If \( 0 \leq n \leq 2 \) then by direct calculations we can check that \( I_n |_{x=0} \) is reduced. This completes the proof of Theorem 4.6 for all integers \( n \). \( \square \)
Appendix A. The character variety of two-bridge knots

In this appendix we prove the following result.

Theorem A.1. Suppose $K = b(p, m)$ is a two-bridge knot. Then the character variety of $K$ has two (irreducible) components if $p$ is prime.

We first review the description of the character variety of two-bridge knots from [Le2]. Suppose $K = b(p, m)$ is a two-bridge knot. Let $X = S^3 \setminus K$. Then

$$\pi_1(X) = \langle a, b \mid wa = bw \rangle,$$

where both $a$ and $b$ are meridians. The word $w$ has the form $a^{\varepsilon_1}b^{\varepsilon_2} \cdots a^{\varepsilon_{2d-1}}b^{2d}$, where $d = (p - 1)/2$ and $\varepsilon_j = (-1)^{\lfloor jm/p \rfloor}$. In particular, if we read $w$ from right to left and interchange $a$ and $b$ then we get $w$ again. For example, $b(2d + 1, 1)$ is the torus knot $T(2, 2d + 1)$, and in this case $w = (ab)^d$.

For any representation $r : \pi_1(X) \to SL_2(\mathbb{C})$ let $x = tr(r(a)) = tr(r(b))$ and $z = tr(r(ab))$. It was shown in [Le2] that the non-abelian character variety, i.e. the characters of non-abelian representations, of $\pi_1(X)$ is the zero set of the polynomial

$$\Phi_{(p,m)}(x, z) = tr(r(w)) - tr(r(w')) + \cdots + (-1)^{n-1} tr(r(w^{(n-1)})) + (-1)^n,$$

here if $w$ is a word then $w'$ denotes the word obtained from $w$ by deleting the two letters at the two ends.

Since we want to know the number of components of the character variety of $\pi_1(X)$, considered as an algebraic set over $\mathbb{Q}$, we need to understand the irreducibility of the polynomial $\Phi(x, z)$ in $\mathbb{Q}[x, z]$.

Let $\Phi_d(x, z) = \Phi_{(p,1)}(x, z)$, where $d = (p - 1)/2$. It was shown in [Le2] Proposition 4.3.1 (also see below) that $\Phi_d(x, z)$ does not depend on $x$. We claim that

Proposition A.2. The polynomial $\Phi_d(z) \in \mathbb{Q}[z]$ is irreducible if and only if $p = 2d + 1$ is prime.

Proof. It is immediate from [Le2] Proposition 4.3.1 that $\Phi_d(2z) = (T_{d+1}(z) + T_d(z))/(z + 1)$, where $T_n(z)$ are the Chebyshev polynomials defined by $T_{n+1}(z) = 2z T_n(z) - T_{n-1}(z)$ and $T_0(z) = 1, T_1(z) = z$.

Let $\Phi_d(z) = \Phi_d(2z)$. It is well-known that by letting $z = \cos \zeta$, we can write $T_d(z) = \cos(d\zeta)$, and so $\Phi_d(z) = \cos ((2d + 1)\frac{\zeta}{2})/\cos(\frac{\zeta}{2})$. It also follows that $\Phi_d(z)$ is an integer polynomial of degree $d$ with exactly $d$ roots given by $z = \cos \left( \frac{2j+1}{2d+1} \pi \right)$, $0 \leq j \leq d-1$. Fix $\zeta = \pi/p$. It is easy to see that $\Phi_d$ is irreducible, and so is $\Phi_d$, if and only if the extension field degree $[\mathbb{Q}(\cos \zeta) : \mathbb{Q}]$ is exactly the degree of $\Phi_d$.

Note that $\cos \zeta = (e^{i\zeta} + e^{-i\zeta})/2$. Hence we need to study the extension field $\mathbb{Q}(e^{i\zeta})/\mathbb{Q}$. It is well-known [La] p. 276 that the irreducible polynomial of $e^{i\zeta}$ is the cyclotomic polynomial

$$C_{2p}(t) = \prod_{1 \leq j \leq 2p, \gcd(j, 2p) = 1} (t - e^{j\pi i/p}).$$

This is an integer polynomial whose degree is $\varphi(2p) = \varphi(p)$, where $\varphi$ is the Euler totient function. Thus the degree of the extension field is $[\mathbb{Q}(e^{i\zeta}) : \mathbb{Q}] = \varphi(p)$. From the identity $(t - e^{\zeta})(t - e^{-\zeta}) = t^2 - 2(\cos \zeta)t + 1$, we see that $[\mathbb{Q}(e^{i\zeta}) : \mathbb{Q}(\cos \zeta)] = 2$, thus $[\mathbb{Q}(\cos \zeta) : \mathbb{Q}] = \varphi(p)/2$. Therefore $\Phi_d$ is irreducible if and only if $\varphi(p) = p - 1$, which happens if and only if $p$ is prime. □
Proposition A.3. We have $\Phi_{(p,m)}(0, z) = \Phi_{(p,1)}(z)$. Hence if $\Phi_{(p,1)}(z)$ is irreducible in $\mathbb{Q}[z]$ then $\Phi_{(p,m)}(x, z)$ is irreducible in $\mathbb{Q}[x, z]$.

Proof. If $x = \text{tr}(r(a)) = \text{tr}(r(b)) = 0$ then it is immediate that $r(a)^{-1} = -r(a)$ and $r(b)^{-1} = -r(b)$ (this follows from the Cayley-Hamilton theorem applying for $SL_2(\mathbb{C})$).

Recall that $\Phi_{(p,m)}(x, z) = \text{tr}(r(w)) - \text{tr}(r(w')) + \cdots + (-1)^{n-1} \text{tr}(r(w^{(n-1)})) + (-1)^n z$. From the definition of the word $w$, it is easy to see that $a^{-1}$ and $b^{-1}$ appear in pairs in $w$. This is also true for $a^{-1}$ and $b^{-1}$ in each word $w^{(j)}$, $0 \leq j \leq d - 1$, hence $r(w^{(j)})$ does not change if we replace $a^{-1}$ by $a$ and $b^{-1}$ by $b$. Thus $r(w^{(j)}) = r((ab)^{d-j})$. Note that for the torus knot $b(2d+1, 1)$ we have $w = (ab)^d$, hence the first claim of the proposition follows.

The second claim is easy to deduce from the first claim and the fact from [Le2] that the polynomial $\Phi_{(p,m)}(x, z)$ is monic in $z$ and has $z$-degree $d = (p - 1)/2$. \qed

Theorem A.1 follows directly from Propositions A.2 and A.3.

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