Holonomies of gauge fields in twistor space 4: functional MHV rules and one-loop amplitudes

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Abstract

We consider generalization of the Cachazo-Svrcek-Witten (CSW) rules to one-loop amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory in a recently developed holonomy formalism in twistor space. We first reconsider off-shell continuation of the Lorentz-invariant Nair measure for the incorporation of loop integrals. We then formulate an S-matrix functional for general amplitudes such that it implements the CSW rules at quantum level. For one-loop MHV amplitudes, the S-matrix functional correctly reproduces the analytic expressions obtained in the Brandhuber-Spence-Travaglini (BST) method. Motivated by this result, we propose a novel regularization scheme by use of an iterated-integral representation of polylogarithms and obtain a set of new analytic expressions for one-loop NMHV and $\mathcal{N}^2$MHV amplitudes in a conjectural form. We also briefly sketch how the extension to one-loop non-MHV amplitudes in general can be carried out.
1 Introduction

Recent years witness a lot of progress in the study of loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory. At a relatively early stage, the progress is initiated by the discovery of the so-called Cachazo-Svrcek-Witten (CSW) rules for tree-level gluon amplitudes [1] (for related works, see, e.g., [2]-[5]) and the applicability of the CSW rules to one-loop MHV (Maximally Helicity Violating) amplitudes in $\mathcal{N} = 4$ and less supersymmetric theories [6, 7, 8]. A particularly important set of results in regard to the applicability are the so-called Brandhuber-Spence-Travaglini (BST) method [9]-[12]; see also [13, 14] and a recent review [15]. In the original BST paper [9], it is analytically shown that the CSW-based loop calculation, together with the use of an off-shell continuation of the Nair measure (i.e., the Lorentz-invariant measure in terms of spinor momenta, first given in [16]), leads to the previously known results on the one-loop MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory which Bern, Dixon, Dunbar and Kosower (BDDK) have obtained many years before, utilizing a unitary-cut method [17, 18]. Although the success of the BST method is somewhat restricted to the MHV amplitudes, the importance of the CSW rules in gauge theories has been well-recognized at this stage and it has motivated further developments in the unitary-cut method for various types of one-loop amplitudes [19]-[23].

At a later stage, the progress in loop calculation integrates higher-loop planar amplitudes of the $\mathcal{N} = 4$ theory. There are a series of important studies which guide this advance, including a simple proportional relation between four-point one- and two-loop planar amplitudes [24], a conjectured iterative relation (in loop order) among MHV loop amplitudes [24, 25], and a duality between gluon amplitudes and light-like Wilson loops in strong coupling region of planar $\mathcal{N} = 4$ super Yang-Mills theory [34] (see [35, 36] for a review of this duality). A particularly important result in this context is the discovery of dual superconformal symmetry [37]. There are many investigations on this new symmetry in the loop amplitudes; see, e.g., [38]-[41] and a recent review [42]. The discovery of Yangian symmetry in the $\mathcal{N} = 4$ amplitudes is also reported in this context [43]; for the study of Yangian symmetry in loop amplitudes, see, e.g., [44, 45] and a review [46]. More recently, largely motivated by the above progress, a powerful computational framework for the planar amplitudes is proposed [47, 48, 49]. This is carried out by use of a so-called momentum twistor space and is followed by a number of intensive works; some of them can be traced in [50]-[59]. Along with these developments, we also notice that many new results are obtained in regard to the unitary-cut method for higher-loop calculations; see, e.g., [60]-[63] and a series of recent reviews [64]-[67].

In the present paper, we introduce a new CSW- or BST-based calculation for one-loop amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory in the framework of a recently developed holonomy formalism in twistor space [68, 69]. As shown in [68], an S-matrix functional for tree gluon amplitudes can be obtained from a holonomy operator in supertwistor space. The S-matrix functional contains a Wick-like contraction operator that implements the CSW rules. Thus, in terms of functional formulation, it would and should be straightforward to

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1These relations, known as the ABDK/BDS relations, do not hold at higher-loop level in general. An analytic expression for such a discrepancy is called a reminder function. Attentive studies of the reminder function at two-loop level are carried out recently. For relevant works, see, e.g., [26]-[33].
generalize the holonomy formalism to loop amplitudes. We shall pursue this possibility in the present paper.

In carrying out practical calculations at loop level, there are subtleties in regard to an off-shell property of the loop momentum. Remember that the original CSW rules give a prescription for the non-MHV tree amplitudes in terms of a combination of MHV vertices which are connected one another by a scalar propagator having an off-shell momentum transfer. Thus the issue of off-shell continuation in the CSW rules exists even at tree level. For tree amplitudes, this can be circumvented by considering in a momentum-space representation. In fact, it turns out that the momentum-space tree amplitudes can be determined up to the choice of the so-called reference spinors. Starting at one-loop level, however, we need to seriously consider the off-shell continuation of the Nair measure. This is analogous to the calculation of quantum anomalies since, in either case, quantum effects arise from an analysis of the integral measures which behave nontrivially at loop level.

The first step toward such an analysis for one-loop MHV amplitudes is taken in the BST method [9]. In the present paper, we shall follow their step in spirit and consider natural generalization to the other (non-MHV) one-loop amplitudes. It should be noticed that there exist no analytic confirmations of the BST method to one-loop non-MHV amplitudes although it is widely recognized in the literature that the CSW generalization is applicable to any types of loop amplitudes. In fact, quite interestingly, an S-matrix functional which realizes the CSW generalization to multi-loop, non-MHV and non-planar amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory is recently proposed by Sever and Vieira in [70]. Motivated by these considerations, in the present paper, we reconsider the off-shell continuation of the Nair measure and then introduce a novel realization of the CSW rules in the holonomy formalism such that the CSW generalization to loop amplitudes becomes straightforward. To be more specific, we shall reproduce one-loop MHV amplitudes by reformulating the S-matrix functional in a coordinate-space representation such that off-shell Nair measures are naturally incorporated into the holonomy operator. We then obtain a set of new analytic expressions for one-loop NMHV amplitudes in a conventional functional method. We also consider the generalization to other non-MHV amplitudes. The results suggest that general one-loop amplitudes can be expressed in terms of a set of polylogarithms, $\text{Li}_k (k \leq m + 2)$ for $n$-gluon one-loop $\mathcal{N}^m$MHV amplitudes ($m = 0, 1, 2, \ldots, n - 4$). The main purpose of this paper is to show that such expressions can systematically be obtained in the holonomy formalism.

This paper is organized as follows. In section 2, we review the holonomy formalism at tree level. We discuss how the CSW rules are implemented by use of a contraction operator. As a simple example, we present explicit calculations of the six-point NMHV tree amplitudes in a momentum-space representation. In section 3, we first consider the off-shell continuation of the Nair measure. We then reformulate the S-matrix functional in a coordinate-space representation such that off-shell Nair measures are naturally incorporated into the holonomy operator. We discuss that an application of such a formulation correctly reproduces the

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2 A main purpose of ref. [70] is to show superconformal invariance of the proposed S-matrix (which, by the way, is structurally similar to the S-matrix of the holonomy formalism) at loop level and, hence, no analytic expressions for any loop amplitudes are provided.
BST representation of one-loop MHV amplitudes. In section 4, we present an alternative derivation of the BST representation, making explicit use of the off-shell Nair measure. This enable us to propose a new efficient regularization scheme for the one-loop amplitudes. In section 5, we apply the alternative method to NMHV amplitudes and obtain a new analytic expression for them. As an example, we present explicit calculations of the six-point one-loop NMHV amplitudes in the momentum-space representation in section 6. In section 7, we further consider the application to one-loop N\(^2\)MHV amplitudes and obtain new analytic expressions for them in a conjectural form. We also briefly sketch how the application to one-loop non-MHV amplitudes in general can be carried out. Lastly, we present some concluding remarks.

2 Tree amplitudes

In this section, we review how the original Cachazo-Svrcek-Witten (CSW) rules for tree amplitudes [1] are implemented in the holonomy formalism [68]. We first review the CSW rules briefly and then explicitly write down an S-matrix functional for the tree amplitudes of gluons in terms of a holonomy operator that is suitably defined in superwistor space. As an example, we show how to calculate the six-point NMHV tree amplitude in our formulation. We also discuss the color structure of the tree amplitudes in some detail.

MHV amplitudes and spinor momenta

In the spinor-helicity formalism, the simplest way of describing gluon amplitudes is to factorize the amplitudes into a set of MHV amplitudes. The MHV amplitudes are the scattering amplitudes of \((n - 2)\) positive-helicity gluons and 2 negative-helicity gluons or the other way around. In a momentum-space representation, the MHV tree amplitudes of gluons are expressed as

\[
A^{(1,2_3 \ldots a_\ldots b_\ldots n_+)}_{\text{MHV}(0)}(u, \bar{u}) = A^{(a\ldots b\ldots)}_{\text{MHV}(0)}(u, \bar{u})
\]

\[
= ig^{n-2} (2\pi)^4 \delta(4) \left( \sum_{i=1}^{n} p_i \right) \hat{A}_{\text{MHV}(0)}^{(a\ldots b\ldots)}(u) \tag{2.1}
\]

\[
\hat{A}_{\text{MHV}(0)}^{(a\ldots b\ldots)}(u) = \sum_{\sigma \in S_{n-1}} \text{Tr}(t^{c_1}t^{c_2}t^{c_3} \ldots t^{c_n}) \hat{C}_{\text{MHV}(0)}^{(a\ldots b\ldots)}(u; \sigma) \tag{2.2}
\]

\[
\hat{C}_{\text{MHV}(0)}^{(a\ldots b\ldots)}(u; \sigma) = \frac{(u_a u_b)^4}{(u_1 u_{\sigma_2})(u_{\sigma_2} u_{\sigma_3}) \cdots (u_{\sigma_n} u_1)} \tag{2.3}
\]

where \(u_i\) denotes a two-component spinor momentum of the \(i\)-th gluon \((i = 1, 2, \ldots, n)\) and \(\bar{u}_i\) denotes its complex conjugate. In (2.1), \(a\) and \(b\) label the numbering indices of the two negative-helicity gluons, \(g\) denotes the Yang-Mills coupling constant, and \(t^{c_i}\)'s denote the generators of a gauge group in the fundamental representation. In the present paper, we consider an \(SU(N)\) gauge theory, with \(t^{c_i}\)'s given by the fundamental representation of the \(SU(N)\) algebra. In (2.2), the sum is taken over the permutations labeled by

\[
\sigma = \begin{pmatrix} 2 & 3 & \cdots & n \\ \sigma_2 \sigma_3 \cdots \sigma_n \end{pmatrix} \tag{2.4}
\]
Notice that a set of the permutations forms a rank-\((n - 1)\) symmetric group \(S_{n-1}\). This is why we denote the sum by that of \(\sigma \in S_{n-1}\) in (2.2).

The product of the spinor momenta \((u_i u_j)\) and its conjugate are defined as

\[
 u_i \cdot u_j \equiv (u_i u_j) = \epsilon_{AB} u_i^A u_j^B, \quad \bar{u}_i \cdot \bar{u}_j \equiv [\bar{u}_i \bar{u}_j] = \epsilon^{AB} \bar{u}_i^A \bar{u}_j^B
\]

where \(\epsilon_{AB} \epsilon^{AB}\) are the rank-2 Levi-Civita tensors, with the upper-case indices taking the value of \((1, 2)\). These tensors can also be used to raise or lower the indices, e.g., \(u_B = \epsilon_{AB} u^A\) and \(\bar{u}_B = \epsilon^{AB} \bar{u}_A\). In terms of the spinor momenta, the four-momentum \(p^{A\dot{A}}\) of a massless gluon is parametrized as

\[
 p^{A\dot{A}} = p^\mu (\tau_\mu)^{A\dot{A}} = u^A \bar{u}^{\dot{A}}
\]

where \(\mu (= 0, 1, 2, 3)\) is the ordinary Minkowski index and \(\tau_\mu\) is given by \(\tau_\mu = (1, \bar{\tau})\), with \(1\) and \(\bar{\tau}\) being the \((2 \times 2)\) identity matrix and the Pauli matrices, respectively. The spinor momentum \(u^A\) or \(\bar{u}^{\dot{A}}\) can be determined by the null momentum \(p^\mu\), with \(p^2 = 0\), up to a phase factor. An explicit choice of \(u^A\) can be given by

\[
 u^A = \frac{1}{\sqrt{p_0 - p_3}} \left( p_1 - i p_2 \right)
\]

The Lorentz symmetry of \(u^A\)’s is given by an \(SL(2, \mathbb{C})\) group. The product \((u_i u_j)\) in (2.5) is invariant under the Lorentz transformations. Similarly, the Lorentz group of \(\bar{u}^{\dot{A}}\) is given by another \(SL(2, \mathbb{C})\). The four-dimensional Lorentz symmetry is then given by a combination of these, i.e., by \(SL(2, \mathbb{C}) \times SL(2, \mathbb{C})\).

Non-MHV amplitudes, reference spinors and the CSW rules

The non-MHV gluon amplitudes, or the general gluon amplitudes, can be expressed in terms of the MHV amplitudes \(\hat{A}^{(\ldots)}_{\text{MHV}(0)}(u)\). Prescription for such expressions is called the CSW rules [1]. For the next-to-MHV (NMHV) amplitudes, which contain three negative-helicity gluons and \((n - 3)\) positive-helicity gluons, the CSW rules can be expressed as

\[
 \hat{A}^{(a \ldots, b \ldots, c \ldots)}_{\text{NMHV}(0)}(u) = \sum_{(i,j)} \hat{A}^{(i_a \ldots, j_b \ldots, i_c \ldots)}_{\text{MHV}(0)}(u) \frac{1}{q_{ij}^2} \hat{A}^{((-i) \ldots, (j+1) \ldots, c \ldots, (i-1) \ldots)}_{\text{MHV}(0)}(u)
\]

where the sum is taken over all possible choices for \((i,j)\) that satisfy the ordering \(i \leq a < b \leq j < c \, (\text{mod} \, n)\) such that the number of indices for each of the MHV amplitudes is more than or equal to three. The numbering indices for the negative-helicity gluons are now given by \(a, b\) and \(c\). The momentum transfer \(q_{ij}\) between the two MHV vertices can be expressed by a partial sum of the gluon four-momenta:

\[
 q_{ij}^{A\dot{A}} = p_i^{A\dot{A}} + p_{i+1}^{A\dot{A}} + \cdots + p_a^{A\dot{A}} + \cdots + p_b^{A\dot{A}} + \cdots + p_j^{A\dot{A}}.
\]

The non-MHV amplitudes are then obtained by iterative use of the relation (2.8), given that the spinor momentum for the index \(l\) or \(-l\) is defined by

\[
 u^A_l \equiv q_{ij}^{A\dot{A}} \bar{\eta}_{\dot{A}} \quad (u^A_{-l} = -u^A_l)
\]
where $u^A_{ij}$ can be treated as an on-shell spinor momenta corresponding to the off-shell momentum $q_{ij}$ or the four-momentum of the virtual gluon. $\bar{\eta}_A$ is an arbitrarily fixed two-component spinor which behaves like $\bar{u}$'s. Once fixed, this so-called reference spinor should be identical for any $q_{ij}$'s in the expression (2.8). We shall call this requirement the single reference-spinor principle in the tree-level CSW rules. Notice that the parametrization (2.10) is always possible, which can be understood directly by decomposing $q_{ij}^{AA}$ as

$$q_{ij}^{AA} = u^A_{ij} \bar{u}^A_{ij} + w \eta^A \bar{\eta}^A$$

(2.11)

where $w$ is a real number. From this decomposition, we can easily find the relation $u^A_{ij} = q_{ij}^A \bar{\eta}_A / [\bar{u}_{ij} \bar{\eta}]$. Thus, thanks to the scale invariance of the spinor momenta, this relation naturally reduces to the definition (2.10).

**Single-trace color structure in non-MHV amplitudes**

Now we assign a $U(1)$ direction of the $SU(N)$ gauge group to the color factor of the propagator such that the full algebra of the amplitudes becomes the $U(N)$ algebra. Notice that the $U(1)$ part gives an auxiliary degree of freedom here since the external legs are all assigned by the $SU(N)$ generators in the fundamental representation. The completeness relation for the generators of the $U(N)$ group is conventionally given by

$$(t^\dot{c})_{ij}(t^\dot{c})_{kl} = \frac{1}{2} \delta_{il} \delta_{jk}$$

(2.12)

where we take a sum over the $U(N)$ color index $\dot{c} = 1, 2, \cdots, N^2$. We here distinguish the $U(N)$ index $\dot{c}$ from the $SU(N)$ index $c = 1, 2, \cdots, N^2 - 1$. The normalization factor $\frac{1}{2}$ can be absorbed into the definition of the reference spinor. Alternatively, we can redefine $q_{ij}^2$ in (2.8) as a square of $q_{ij}^{AA}$ rather than that of $q_{ij}^\mu$. Since the latter is twice as large as the former, the factor $\frac{1}{2}$ cancels in the expression (2.8). Notice that the product of four-vectors, say, $v^A$ and $v'^A$, is related to that of $v^\mu$ and $v'^\mu$ by

$$v^A v'^A = 2(v_0 v'_0 - v_1 v'_1 - v_2 v'_2 - v_3 v'_3) = 2v^\mu v'^\mu$$

(2.13)

in general regardless whether the four-vectors are null or not. In any respect, the completeness condition (2.12) guarantees the single-trace property of the tree amplitudes. A naive extension of the CSW rules to loop amplitudes also preserves this property. We shall take this feature for granted throughout the present paper.

Assigning the $U(1)$ color index to the internal numbering indices $l$ and $-l$ in the CSW rules (2.8), we can in principle express $\tilde{N}^k$ MHV tree amplitudes ($k = 0, 1, 2, \cdots, n - 4$) in general as

$$A_{N^k_{MHV}(0)}^{(1h_1 2h_2 \cdots n_h_{n})}(u, \bar{u}) = i g^{-2} (2\pi)^{4} \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \tilde{A}_{N^k_{MHV}(0)}^{(1h_1 2h_2 \cdots n_h_{n})}(u)$$

(2.14)

$$\tilde{A}_{N^k_{MHV}(0)}^{(1h_1 2h_2 \cdots n_h_{n})}(u) = \sum_{\sigma \in \delta_{n-1}} \text{Tr}(t^{a_1} t^{a_2} t^{a_3} \cdots t^{c_{s_n}}) \tilde{C}_{N^k_{MHV}(0)}^{(1h_1 2h_2 \cdots n_h_{n})}(u; \sigma)$$

(2.15)

where $h_i = \pm$ denotes the helicity of the $i$-th gluon, with the total number of negative helicities being $k + 2$. $\tilde{C}_{N^k_{MHV}(0)}^{(1h_1 2h_2 \cdots n_h_{n})}(u; \sigma)$ denotes a function of the Lorentz-invariant scalar
products \((u_i u_j)\). The simplest form of this function is given by the MHV case (2.3). By use of the CSW rules, we can obtain \(\hat{C}(u; \sigma)\)'s of non-MHV helicity configurations in terms of (2.3) and a set of off-shell momenta defined in (2.9). Accordingly, the permutations of the non-MHV amplitudes decreases, which means that many of \(\hat{C}(u; \sigma)\)'s vanish for particular choices of \(\sigma\). This is related to the well-known redundancy of the expression (2.15) due to the so-called \(U(1)\) decoupling identities. For example, the NMHV tree amplitudes (2.8) can alternatively be written as

\[
\tilde{A}_{NMHV(0)}^{(a \ldots c)}(u) = \sum_{i=1}^{n} \sum_{r=1}^{n-3} \tilde{A}_{MHV(0)}^{(i \ldots a \ldots b \ldots (i+r)_{+}^1)}(u) \frac{1}{q_{i+r}} \tilde{A}_{MHV(0)}^{((-l)_{-} \ (i+r+1)_{+} \ldots (i-1)_{+})}(u)
\]

where we replace index \(j\) in (2.8) by \(i + r\) with \(r = 1, 2, \ldots, n - 3\). Here the \(SU(N)\) generators \(t^{a_{\tau}}\)'s are abbreviated by \(t^{a_{\tau}}\)'s. To avoid unnecessary complexity, we shall use this notation from here on. The expression (2.16) explicitly shows that the color structure of the NMHV tree amplitudes is decomposed into the two sums over the permutations \(\sigma^{(1)}\) and \(\sigma^{(2)}\), respectively. This is essentially the same as the sum over the so-called cyclicly ordered permutations (COP's) used in the unitary-cut method [71, 72], except that the single-trace color structure is preserved here thanks to our abelian choice of internal degrees of freedom.

Lastly, we would like to emphasize that there are ambiguities in the expression (2.14) due to the dependence on the reference spinors. Once a single reference spinor is fixed, it should be identical for all propagators. In other words, the non-MHV amplitudes are determined up to the choice of the reference spinor or equivalently the reference null-vector \(\eta^{A\bar{A}} = \eta^{A} \bar{\eta}^{\bar{A}}\). This fact is referred to as the single reference-spinor principle in the present paper. As shown in [1], if we choose \(\bar{\eta}^{\bar{A}}\) to be equal to one of the \(\bar{u}_{i}^{A}\)'s, then we can obtain manifestly Lorentz covariant expressions for the non-MHV amplitudes.

A holonomy operator in supertwistor space

From here on, we briefly review the construction of a holonomy operator in supertwistor space as proposed in [68]. An S-matrix functional for tree amplitudes can be obtained in terms of this holonomy operator. The holonomy operator is defined by

\[
\Theta^{(A)}_{R, \gamma}(u; x, \theta) = \text{Tr}_{R, \gamma} \text{P exp} \left[ \sum_{m \geq 2} \oint_{\gamma} A \wedge A \wedge \ldots \wedge A \right]
\]

(2.17)

where \(A\) is what we call a comprehensive gauge one-form and is defined by the following set of equations.

\[
A = g \sum_{1 \leq i < j \leq n} A_{ij} \omega_{ij}
\]

(2.18)

\[
\omega_{ij} = d \log(u_i u_j) = \frac{d(u_i u_j)}{(u_i u_j)}
\]

(2.19)
\[ A_{ij} = \sum_{\hat{h}_i} a^{(\hat{h}_i)}_i(x, \theta) \otimes a^{(0)}_j \]  
\[ a^{(\hat{h}_i)}_i(x, \theta) = \int d\mu(p_i) a^{(\hat{h}_i)}_i(\xi^i) e^{ixu_i\mu} \bigg|_{\xi^i = \theta^\alpha A^A} \]  
\[ d\mu(p_i) \equiv \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2p_{0i}} = \frac{1}{4} \left[ \frac{u_i \cdot du_i}{2\pi i} \frac{d^2 \bar{u}_i}{(2\pi)^2} - \frac{\bar{u}_i \cdot d\bar{u}_i}{2\pi i} \frac{d^2 u_i}{(2\pi)^2} \right] \]  

\( a^{(\hat{h}_i)}_i(\xi^i)'s \) are physical operators that are defined in a four-dimensional \( \mathcal{N} = 4 \) chiral superspace \((x, \theta)\) where \( x_{AA} \) denotes coordinates of four-dimensional spacetime and \( \theta^\alpha_A (A = 1, 2; \alpha = 1, 2, 3, 4) \) denotes their chiral superpartners with \( \mathcal{N} = 4 \) extended supersymmetry. As explicitly discussed in [68], these coordinates emerges from homogeneous coordinates of the supertwistor space \( \mathbb{CP}^{3|4} \), denoted by \((u^A, v_A, \xi^\alpha)\), that satisfy the so-called supertwistor conditions

\[ v_A = x_{AA} u^A, \quad \xi^\alpha = \theta^\alpha_A u^A. \]  

Notice that for the ordinary twistor space its homogeneous coordinate is given by \( Z_I = (u^A, v_A) \) \((I = 1, 2, 3, 4)\) where \( u^A \) and \( v_A \) are two-component complex spinors, with \( Z_I \) satisfying the scale invariance. The spacetime coordinates emerges from the first condition in (2.23). As in (2.6), the Minkowski coordinates \( x_\mu \) can easily be related to \( x_{AA} \) via \( x_{AA} = x_\mu(\tau^\mu)_{AA} \). In the spinor-helicity formalism in supertwistor space, we identify \( u^A \) as the spinor momentum defined in (2.7) so that we can essentially describe four-dimensional physics in terms of \( u^A \)'s with an imposition of the supertwistor conditions (2.23).

The physical operators \( a^{(\hat{h}_i)}_i(\xi^i) \) are relevant to creations of gluons and their superpartners, having the helicity \( \hat{h}_i = (0, \pm \frac{1}{2}, \pm 1) \). In the two-component indices, we can properly define the Pauli-Lubanski spin vector for the \( i \)-th gluon as

\[ S^{AA}_i = p^{AA}_i \left( 1 - \frac{1}{2} u^B_i \frac{\partial}{\partial u^B_i} \right). \]  

This means that the helicity \( \hat{h}_i \) of the \( i \)-th particle or supermultiplet can be determined by

\[ \hat{h}_i = 1 - \frac{1}{2} u^A_i \frac{\partial}{\partial u^A_i}. \]  

In other words, the helicity is essentially given by the degree of homogeneity in \( \xi^{\alpha_i} \)'s for each of the physical operators \( a^{(\hat{h}_i)}_i(\xi^i) \). Thus we can write down explicit forms of \( a^{(\hat{h}_i)}_i(\xi^i) \)'s as below.

\[ a^{(+)}_i(\xi^i) = a^{(+)}_i \]  
\[ a^{(+ \frac{1}{2})}_i(\xi^i) = \xi^\alpha_i a^{(+ \frac{1}{2})}_i \]  
\[ a^{(0)}_i(\xi^i) = \frac{1}{2} \xi^\alpha_i \xi^\beta_i a^{(0)}_i \]  
\[ a^{(- \frac{1}{2})}_i(\xi^i) = \frac{1}{3！} \xi^\alpha_i \xi^\beta_i \xi^\gamma_i \xi_{\alpha \beta \gamma \delta} a^{(- \frac{1}{2}) \delta}_i \]  
\[ a^{(-)}_i(\xi^i) = \xi^1_i \xi^2_i \xi^3_i \xi^4_i a^{(-)}_i \]
The color factor can be attached to each of the physical operators \( a_i^{(h_i)} \):
\[
a_i^{(h_i)} = t^{c_i} a_i^{(h_i)c_i}
\] (2.27)
where, as in the case of (2.2), \( t^{c_i} \)'s are given by the generators of the \( SU(N) \) gauge group in the fundamental representation. The color part of the symbol \( R \) in (2.17) then refers to the fundamental representation of the \( SU(N) \) algebra in this paper.\(^3\)

The physical Hilbert space of the holonomy formalism is given by \( V^\otimes n = V_1 \otimes V_2 \otimes \cdots \otimes V_n \) where \( V_i \) \((i = 1, 2, \cdots, n)\) denotes a Fock space that creation operators of the \( i \)-th particle with helicity \( \pm \) act on. The creation operators of gluons and their scalar partner form an \( SL(2, \mathbb{C}) \) algebra:
\[
[a_i^{(+)}, a_j^{(-)}] = 2a_i^{(0)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(+)}] = a_i^{(+)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(-)}] = -a_i^{(-)} \delta_{ij}
\] (2.28)
where Kronecker’s deltas show that the non-zero commutators are obtained only when \( i = j \). The remaining of commutators, those expressed otherwise, all vanish.

Non-supersymmetric part of the holonomy operator

In the following, we consider the gluonic part of the holonomy operator in (2.17); as we shall discuss later, contributions from gluinos and scalar partners to the MHV vertices vanish upon Grassmann integrals. To distinguish the gluon operators from the super multiplets, we denote the former by \( a_i^{(h_i)} \) with \( h_i = \pm 1 = \pm \) instead of \( a_i^{(\hat{h}_i)} \) with \( \hat{h}_i = (0, \pm \frac{1}{2}, \pm 1) \). The physical configuration space for gluons can be defined in terms of a set of \( \mathbb{C}P^1 \)'s on which the gluonic spinor momenta are defined. Since gluons are bosons, these \( \mathbb{C}P^1 \)'s are symmetric under permutations. The physical configuration space is therefore given by \( \mathcal{C} = \mathbb{C}^n / \mathbb{S}_n \). The fundamental homotopy group of \( \mathcal{C} \) is given by the braid group, \( \Pi_1(\mathcal{C}) = \mathcal{B}_n \). The symbol \( \gamma \) in (2.17) denotes a closed path defined in \( \mathcal{C} \). Linear transformations of the holonomy operator depends on the homotopy class of the closed path \( \gamma \) on \( \mathcal{C} = \mathbb{C}^n / \mathbb{S}_n \). Thus the holonomy operator gives a linear representation of the braid group \( \mathcal{B}_n \).

The symbol \( P \) in (2.17) denotes a “path” ordering of the numbering indices along \( \gamma \). The meaning of \( P \) acting on the exponent of (2.17) can explicitly be written as
\[
P \sum_{m \geq 2} \oint_{\gamma} A \wedge A \wedge \cdots \wedge A = \sum_{m \geq 2} \oint_{\gamma} A_{12} A_{23} \cdots A_{m1} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_m
\]
\[
= \sum_{m \geq 2} 2^{m+1} \sum_{(h_1, h_2, \cdots, h_m)} (-1)^{h_1+h_2+\cdots+h_m} \times a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_m^{(h_m)} \oint_{\gamma} \omega_1 \wedge \cdots \wedge \omega_m
\] (2.29)
where we consider only the gluonic part so that \( h_i = \pm = \pm 1 \) \((i = 1, 2, \cdots, m)\) denotes the helicity of the \( i \)-th gluon. In obtaining the above expression, we use an ordinary definition.

\(^3\)Notice that the symbol \( R \) also refers to an irreducible representation of the Iwahori-Hecke algebra in the holonomy formalism [69]. As shown in (2.32), this can be materialized by a sum over permutations over the numbering indices. In this sense, we can practically regard the symbol \( R \) as the fundamental representation of the gauge group. For the emergence of the Iwahori-Hecke algebra as an irreducible representation of the braid group, one may refer to [73, 74].
of commutators for bialgebraic operators. For example, using the commutation relations (2.28), we can calculate $[A_{12}, A_{23}]$ as

$$[A_{12}, A_{23}] = a_1^{(+)} \otimes a_2^{(+)} \otimes a_3^{(0)} - a_1^{(+)} \otimes a_2^{(-)} \otimes a_3^{(0)} + a_1^{(-)} \otimes a_2^{(+)} \otimes a_3^{(0)} - a_1^{(-)} \otimes a_2^{(-)} \otimes a_3^{(0)}.$$  

(2.30)

In (2.29), we also define $a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_m^{(h_m)} \otimes a_1^{(0)}$ as

$$\frac{1}{2} [a_1^{(0)}, a_1^{(\pm)}] \otimes a_2^{(h_2)} \otimes \cdots \otimes a_m^{(h_m)} = \pm \frac{1}{2} a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_m^{(h_m)}$$  

(2.31)

where we implicitly use an antisymmetric property for the indices due to the wedge products in (2.29).

The trace $\text{Tr}_{R,\gamma}$ in the definition (2.17) means a trace over the color factors of the gluons. As discussed in [69], this trace includes not only a trace over the $SU(N)$ generators but also that of braid generators. The latter, a so-called braid trace, is realized by a sum over permutations of the numbering indices. Thus the trace $\text{Tr}_{R,\gamma}$ over the non-supersymmetric exponent of (2.17) can be expressed as

$$\text{Tr}_{R,\gamma} = \sum_{m \geq 2} \sum_{\sigma \in S_{m-1}} \oint A_1 \wedge A_2 \wedge \cdots \wedge A_m = \sum_{m \geq 2} \sum_{\sigma \in S_{m-1}} \oint A_{1\sigma_2} A_{\sigma_2 \sigma_3} \cdots A_{\sigma_m \omega_{\sigma_1} \wedge \omega_{\sigma_2} \wedge \cdots \wedge \omega_{\sigma_m}}$$  

(2.32)

where the sum of $\sigma \in S_{m-1}$ is taken over the permutations $\sigma = \begin{pmatrix} 2 & 3 & \cdots & m \\ \sigma_2 & \sigma_3 & \cdots & \sigma_m \end{pmatrix}$.

From (2.29) and (2.32), we find that the holonomy operator can be described in terms of the logarithmic one-forms $\omega_{i,i+1}$ and the permutations of the numbering indices. This fact suggests the dual conformal invariance [37] of the holonomy operator. As mentioned above, the holonomy operator gives a linear representation of the braid group $\mathcal{B}_q$. Thus, by use of the Kohno-Drinfel’d monodromy theorem [75], one can argue that the holonomy operator and hence the tree amplitudes in general preserve the Yangian symmetry [43]. Interested readers may refer to [68] for details on this topic.

**An S-matrix functional for tree amplitudes**

In terms of the supersymmetric holonomy operator (2.17), we can construct an S-matrix functional for the MHV tree amplitudes as

$$\mathcal{F}_{MHV} \left[ a^{(h)c} \right] = \exp \left[ \frac{i}{g^2} \int d^4 x d^8 \theta \, \Theta^{(A)}_{R,\gamma}(u;x,\theta) \right]$$  

(2.33)

where $a^{(h)c}$ denotes a generic expression for $a_i^{(h)c}$. In terms of $\mathcal{F}_{MHV} \left[ a^{(h)c} \right]$, the MHV tree amplitudes (2.2) are generated as

$$\frac{\delta}{\delta a_1^{(+)} c_1(x_1)} \otimes \cdots \otimes \frac{\delta}{\delta a_a^{(-)} c_a(x_a)} \otimes \cdots$$
\[ \cdots \otimes \frac{\delta}{\delta a_b^{(-)i}(x_b)} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(+)}(x_n)} \mathcal{F}_{\text{MHV}} \left[ a^{(h)c}(x) \right]_{a^{(h)c}(x)=0} = ig^{n-2} \widetilde{A}_{\text{MHV}(0)}^{(a\ldots b\ldots)}(u) \]

where \( a^{(h)c}(x) \)'s denote x-space representation of the gluon creation operators:

\[ a_i^{(h)}(x) = \int d\mu(p_i) a_i^{(h)}(x) e^{ix\mu p_i^c} \]

where \( d\mu(p_i) \) is the Nair measure \((2.22)\).

The expression \text{(2.34)} can easily be checked with the following two relations. First, we choose the normalization of the spinor momenta as

\[ \oint \gamma d(u_1u_2) \wedge d(u_2u_3) \wedge \cdots \wedge d(u_mu_1) = 2^{m+1} \]  

\text{(2.36)}

Under a permutation of the numbering indices, a sign factor arises in the above expression. We omit this sign factor as well as the factor \((-1)^{h_1+h_2+\cdots+h_n}\) in \text{(2.29)} since physical quantities are given by the squares of the amplitudes. Secondly, the Grassmann integral over \( \theta \)'s vanishes unless we have the following integrand:

\[ \int d^8\theta \xi^1_1 \xi^2_2 \xi^3_3 \xi^4_4 \left| \mathcal{F}_{\text{i}i} \right|_{\xi^{\alpha}_{i} = 0} = u_r^4 u_s^4. \]

\text{(2.37)}

This relation guarantees that only the MHV amplitudes are picked up upon the execution of the Grassmann integral.

We can therefore naturally express the MHV S-matrix functional \( \mathcal{F}_{\text{MHV}} \left[ a^{(h)c} \right] \) in terms of the supersymmetric holonomy operator \( \Theta_{R,\gamma}(u,x,\theta) \). We can also express an S-matrix functional for non-MHV tree amplitudes in general by use of \( \mathcal{F}_{\text{MHV}} \left[ a^{(h)c} \right] \) if we manage to realize the CSW rules in a functional language. Such a general S-matrix functional, denoted by \( \mathcal{F} \left[ a^{(h)c} \right] \), can be expressed as

\[ \mathcal{F} \left[ a^{(h)c} \right] = \tilde{\mathcal{W}}(A) \mathcal{F}_{\text{MHV}} \left[ a^{(h)c} \right], \]  

\text{(2.38)}

\[ \tilde{\mathcal{W}}(A) = \exp \left[ -i \int d^4x d^4y \frac{1}{q^2} \delta \frac{\delta}{\delta a_1^{(+)}(x)} \otimes \delta \frac{\delta}{\delta a_1^{(-)}(y)} \right]. \]

\text{(2.39)}

In terms of \( \mathcal{F} \left[ a^{(h)c} \right] \), the general tree amplitudes \( \widehat{A}_{N^k\text{MHV}(0)}^{(1_{h_1}2_{h_2}\ldots n_{h_n})}(u) \) in \text{(2.15)} can be generated as

\[ \delta \frac{\delta}{\delta a_1^{(h_1)c_1}(x_1)} \otimes \delta \frac{\delta}{\delta a_2^{(h_2)c_2}(x_2)} \otimes \cdots \otimes \delta \frac{\delta}{\delta a_n^{(h_n)c_n}(x_n)} \mathcal{F} \left[ a^{(h)c} \right]_{a^{(h)c}(x)=0} = ig^{n-2} \widehat{A}_{N^k\text{MHV}(0)}^{(1_{h_1}2_{h_2}\ldots n_{h_n})}(u) \]  

\text{(2.40)}

where a set of \( h_i = \pm (i = 1, 2, \ldots, n) \) are arbitrarily chosen on condition that they contain \( k + 2 \) negative helicities in total. The condition \( a^{(h)}(x) = 0 \) means that the remaining
operators (or, to be precise, source functions) should be evaluated as zero in the end of the calculation.

The Wick-like contraction operator \( \bar{W}^{(A)} \) in (2.39) is introduced so that the CSW rules are realized in a functional method. The functional derivatives in (2.40), with a help of the Grassmann integral (2.37) automatically lead to the summation over \((i, j)\) in (2.8); this is why we denote the momentum transfer by \(q\), without the \((i, j)\) indices, in (2.39).

The Grassmann integral (2.37) guarantees that gluon amplitudes vanish unless the helicity configuration can be factorized into the MHV helicity configurations. Thus the CSW rules are automatically satisfied by the Grassmann integral. In this respect, we can specify the helicity index by \(h_i = (+, -)\), rather than the supersymmetric version \(\bar{h}_i = (0, \pm \frac{1}{2}, \pm)\). In the functional formulation of the tree amplitudes, the essence of the CSW rules therefore lies in the use of Grassmann integral (2.37). This feature would and should be held in loop calculations as well, since the formula (2.40) can naturally apply to loop amplitudes, with an appropriate understanding of loop-integral measures. We shall consider this issue in the next section.

**Six-point NMHV tree amplitudes**

As an example of our formulation, we now calculate the six-point NMHV tree amplitudes, the simplest non-MHV tree amplitudes. All six-point NMHV amplitudes can be obtained by applying cyclic permutations and reflections to the three distinct helicity configurations \(1_{-2}3_{-4}4_{+}5_{+}6_{+}\), \((1_{-2}+3_{-4}+4_{-5}+5_{-6}+6_{-})\) and \((1_{-2}+3_{-4}+4_{-5}+6_{-})\). Using the expression (2.40) for the first configuration, we can explicitly carry out the functional derivatives as

\[
\begin{align*}
\frac{\delta}{\delta a_1^{(-)}(x_1)} \otimes \frac{\delta}{\delta a_2^{(-)}(x_2)} \otimes \frac{\delta}{\delta a_3^{(-)}(x_3)} \\
\otimes \frac{\delta}{\delta a_4^{(+)}(x_4)} \otimes \frac{\delta}{\delta a_5^{(+)}(x_5)} \otimes \frac{\delta}{\delta a_6^{(+)}(x_6)} \\
\left. \exp \left[ \frac{i}{g^2} \int_{x,y} \chi (u; x, \theta) \right] \right|_{a^{(+)}c(x) = 0}
\end{align*}
\]

\[
\begin{align*}
= i g^4 & \left[ \tilde{A}_{MHV(0)}^{(2\cdot 3 \cdot L_4)}(u) \frac{1}{q_{23}} \tilde{A}_{MHV(0)}^{((-) \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 1)}(u) + \tilde{A}_{MHV(0)}^{(2 \cdot 3 \cdot 4 \cdot L_4)}(u) \frac{1}{q_{34}} \tilde{A}_{MHV(0)}^{((-) \cdot 5 \cdot 6 \cdot 1)}(u) \\
& + \tilde{A}_{MHV(0)}^{(2 \cdot 3 \cdot 4 \cdot 5 \cdot L_4)}(u) \frac{1}{q_{24}} \tilde{A}_{MHV(0)}^{((-) \cdot 6 \cdot 1)}(u) + \tilde{A}_{MHV(0)}^{(5 \cdot 6 \cdot 1 \cdot 2 \cdot L_4)}(u) \frac{1}{q_{52}} \tilde{A}_{MHV(0)}^{((-) \cdot 3 \cdot 4)}(u) \\
& + \tilde{A}_{MHV(0)}^{(6 \cdot 1 \cdot 2 \cdot L_4)}(u) \frac{1}{q_{62}} \tilde{A}_{MHV(0)}^{((-) \cdot 3 \cdot 4 \cdot 5)}(u) + \tilde{A}_{MHV(0)}^{(1 \cdot 2 \cdot L_4)}(u) \frac{1}{q_{12}} \tilde{A}_{MHV(0)}^{((-) \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 1)}(u) \right]
\end{align*}
\]

where in the second lines we use an abbreviated notation for the integral measures over the spacetime coordinates and the Grassmann variables. In the third lines, we omit the supplemental conditions \(l = q_{ij} \tilde{n}_i\), which can easily read off form each term. Those terms
that contribute to $\hat{A}^{(1-2,3-4,5+6+)}_{NMHV}(u)$ are diagrammatically shown in Figure 1.

\[\hat{A}^{(1-2,3-4,5+6+)}_{NMHV}(u) = \hat{A}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-4+5-6+)}_{MHV}(u) + \hat{A}^{(1-2,3-4,5+)}_{MHV}(0) \left( \frac{1}{q_{t4}} \hat{A}^{((-1)-5-6+)}_{MHV}(u) \right) + \hat{A}^{(0,1-2,3-4,5+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-4-6+)}_{MHV}(u) \right) + \hat{A}^{(1-2,3-4,6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-5-6+)}_{MHV}(u) \right) + \hat{A}^{(0,1-2,3-4,6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) + \hat{A}^{(1-2,3-4,6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) + \hat{A}^{(2+3,4-5-6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) + \hat{A}^{(2+3,4-5-6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) \right) \]

Figure 1: Diagrams contributing to the six-gluon NMHV tree amplitude $\hat{A}^{(1-2,3-4,5+6+)}_{NMHV}(u)$

Similarly, for the remaining helicity configurations we can calculate the six-point tree amplitudes as

\[\hat{A}^{(1-2,3-4,5+6+)}_{NMHV}(u) = \hat{A}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-4+5-6+)}_{MHV}(u) + \hat{A}^{(1-2,3-4,5+)}_{MHV}(0) \left( \frac{1}{q_{t4}} \hat{A}^{((-1)-5-6+)}_{MHV}(u) \right) + \hat{A}^{(0,1-2,3-4,5+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-4-6+)}_{MHV}(u) \right) + \hat{A}^{(1-2,3-4,6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-5-6+)}_{MHV}(u) \right) + \hat{A}^{(0,1-2,3-4,6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) + \hat{A}^{(1-2,3-4,6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) + \hat{A}^{(2+3,4-5-6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) + \hat{A}^{(2+3,4-5-6+)}_{MHV}(0) \left( \frac{1}{q_{t3}} \hat{A}^{((-1)-3-6+)}_{MHV}(u) \right) \right) \]

(2.42)

where, as in the case of (2.41), the spinor momenta $u_t$, $u_{-t}$ are defined through the corresponding $q_{ij}$ as (2.9).
From Figure 1, we can easily find that the number of the contractions due to the operator $\hat{W}^{(A)}$ corresponds to the number of negative-helicity states minus two. Notice that the power of $g$ is independent of the helicity configuration; it depends only on the total number of scattering gluons. This feature is inherent in all tree amplitudes since iterative use of contraction operator $\hat{W}^{(A)}$ does not alter the power of $g$. The above example (2.41) illustrates that the functional formula (2.40) elegantly materializes the CSW rules by use of the Grassmann integral (2.37).

Lastly, we would like to emphasize the characteristic feature of tree amplitudes in our formulation of the CSW rules, namely, the single-trace color structure of the amplitudes furnished with sums over distinct permutations of the numbering indices. The sums arise from the braid trace that is inherent in the definition of the holonomy operator (2.17) and, hence, in the S-matrix functional (2.38) for general gluon amplitudes. The number of such sums corresponds to the number of negative-helicity gluons minus one. The single-trace color structure is guaranteed by assigning the $U(1)$ color factor to the internal propagators that connect two MHV vertices. As we discuss in the following sections, this feature is pertinent to one-loop amplitudes as well. Notice that there are similarity and difference between our functional formulation and the known unitary-cut method [71, 72]. The similarity is that the sums over permutations in the former is basically the same as the sum over what is called the cyclicly ordered permutations (COP’s) in the latter, while the difference is that the former has a single-trace property but the latter has multi-trace decomposition in terms of the color structure of one-loop amplitudes. These are essential similarity and difference between a CSW-based method and a non-CSW method in the calculation of gluon amplitudes. In this context, our formulation is qualitatively different from the unitary-cut method. Since the unitary-cut method borrows a computational technique from string theory (or, to be more precise, four-dimensional heterotic string theory), we can alternatively state that our resultant amplitudes are qualitatively different from bosonic open-string amplitudes in structure.

3 One-loop MHV amplitudes: functional derivation

In this section, we generalize the above formulation to one-loop MHV amplitudes. We first consider an off-shell continuation of the Nair measure $d\mu(p)$ defined in (2.22). We then reformulate the S-matrix functional in an $x$-space representation such that it incorporates the off-shell Nair measures. We find that such reformulation naturally leads to the Brandhuber-Spence-Travaglini (BST) representation of one-loop MHV amplitudes [9] in a functional method.

Off-shell continuation of the Nair measure

The spinor momentum $u^A$ in (2.7) can be regarded as a homogeneous coordinate of the complex projective space $\mathbb{CP}^1$. Thus it can be parametrized as

$$u^A = \frac{1}{\sqrt{p_0 - p_3}} \left( \begin{array}{c} p_1 - ip_2 \\ p_0 - p_3 \end{array} \right) = \alpha \left( \begin{array}{c} 1 \\ z \end{array} \right), \quad \alpha \in \mathbb{C} - \{0\}$$

(3.1)
where $z$ represents a local complex coordinate of $\mathbb{CP}^1$, with $\alpha$ being a non-zero complex number. In terms of $z$ and $\alpha$, the Nair measure (2.22) can be expressed as

$$d\mu(p) = \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} = \frac{1}{(2\pi)^3} \frac{(\tilde{\alpha}\alpha)d(\tilde{\alpha}\alpha)}{2} \frac{dzd\bar{z}}{(-2i)} \quad (3.2)$$

where we omit the numbering index for simplicity. In terms of the null momentum components, $z$ and $\bar{\alpha}\alpha$ are expressed as

$$z = \frac{p_1 + ip_2}{p_0 + p_3}, \quad \bar{\alpha}\alpha = p_0 + p_3. \quad (3.3)$$

This means that the on-shell Nair measure can alternatively be written as

$$d\mu(p) = \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \frac{1}{4\pi} \frac{d(p_0 + p_3)}{(p_0 + p_3)}. \quad (3.4)$$

Upon suitable normalization along $p_1$ and $p_2$ directions, this implies that the Nair measure is essentially encoded by $d\log(\bar{\alpha}\alpha)$, i.e.,

$$d\mu(p) \approx \frac{1}{4\pi} d\log(\bar{\alpha}\alpha) \quad (3.5)$$

where $\approx$ denotes that we use a conventional normalization for spatial area. Notice that this can also be interpreted as a projection of single-particle trajectories onto a certain spacial direction.

We now consider an off-shell continuation of the form (3.4). Let $L_\mu$ and $l_\mu$ be off-shell and on-shell four-momenta, respectively. Following the notation (2.11), we relate these to each other by

$$L_\mu = l_\mu + w\eta_\mu \quad (3.6)$$

where $\eta_\mu$ is a reference null-vector, satisfying $\eta^2 = 0$, and $w$ is a real number. Since both $\eta_\mu$ and $w$ can arbitrarily be chosen, we can in fact fix the scaling freedom for either $\eta_\mu$ or $w$.

According to the CSW rules, the physics and hence the off-shell measure should be independent of the reference null-vector. This implies that such off-shell measures are parametrized in terms of $l_\mu$ and $w$. Thus, in order to obtain one realization of the off-shell measures, we can fix $\eta_\mu$ to a suitable null vector. We here fix the reference vector by $\eta_\mu = (1, 0, 0, -1)$. We can then calculate the off-shell Nair measure as

$$d\mu(L) = \frac{dL_1}{2\pi} \frac{dL_2}{2\pi} \frac{1}{4\pi} \frac{d(L_0 + L_3)}{(L_0 + L_3)}$$

$$= \frac{dl_1}{2\pi} \frac{dl_2}{2\pi} \frac{1}{4\pi} \left[ \frac{d(l_0 + l_3)}{(l_0 + l_3)} + \frac{dw^2}{w^2} \right] \quad (3.7)$$

where we have used the off-shell condition

$$L^2 = L_0^2 - L_1^2 - L_2^2 - L_3^2 = 2w(L_0 + L_3) \quad (3.8)$$
to calculate the particular factor \( d(L_0 + L_3) \). Using the on-shell notation \( l_0 + l_3 = \bar{\alpha}\alpha \) as before, we can also express \( d\mu(L) \) as

\[
d\mu(L) \approx \frac{1}{4\pi} \left[ d\log(\bar{\alpha}\alpha) + d\log w^2 \right] \\
\approx \frac{1}{4\pi} d\log(\bar{\alpha}\alpha) w^2 \\
\approx d\mu(l) + \frac{1}{4\pi} dw^2.
\]  

(3.9)

This is an off-shell analog of the expression (3.5) and is a very interesting result in terms of the application of the spinor-helicity formalism to massive theories. The second-line expression, in particular, shows that the off-shell continuation can simply be implemented by the scaling of \( \alpha \to w\alpha \) and \( \bar{\alpha} \to w\bar{\alpha} \) \((w \in \mathbb{R} - \{0\})\). This suggests that in the holonomy formalism massive particles can effectively be treated as massless ones by means of the scaling \( \alpha \to w\alpha \) and the use of the off-shell Nair measures. The off-shell momentum \( L_\mu \) is parametrized as (3.6) which is, of course, not proportional to the on-shell momentum \( l_\mu \). In this sense, the above prescription is merely effective and its usage should be clarified. We shall relate it to the CSW prescription in a moment but clarification on this matter is still missing. We shall investigate this point in a future paper.

Notice that the \( w \)-dependence of the resultant off-shell measure comes in only by the factor of \( d\log w^2 \). Thus there is a scaling ambiguity in defining \( w^2 \). This ambiguity can in fact be absorbed into the definition of the reference null-vector. Therefore, without losing generality, we can limit the range of \( w^2 \) by

\[
0 < w^2 < 1.
\]  

(3.10)

If \( w^2 \) is more than one, we can always redefine \( \eta_\mu \) such that the integral part of \( w^2 \) is absorbed into the reference null-vector.

**An x-space representation, off-shell Nair measure and the CSW rules**

The CSW rules, which are realized by the S-matrix functional in (2.38), do lead to the correct tree amplitudes, however, the appearance of \( 1/q^2 \) in the contraction operator (2.39) is rather abrupt. Conventionally, this factor is interpreted as a contribution from a massless scalar propagator that connects MHV vertices. This interpretation is convenient and correct in writing down the momentum-space tree amplitudes which do not require the knowledge of off-shell measures. As discussed above, however, by use of the off-shell Nair measure we can deal with massive fields somewhat analogous to massless fields. This suggests that we may find a more natural understanding of the factor \( 1/q^2 \) by introducing the off-shell Nair measure to the S-matrix functional (or, more concretely, to the contraction operator \( \hat{W}^{(A)} \)) in an x-space representation. We shall consider this possibility in what follows.

Notice that the x-space representation of the amplitudes is given by

\[
\mathcal{A}_{N^kMHV(0)}^{(1_{h_1}2_{h_2}...n_{h_n})}(x) = \prod_{i=1}^{n} \int d\mu(p_i) \mathcal{A}_{N^kMHV(0)}^{(1_{h_1}2_{h_2}...n_{h_n})}(u, \bar{u})
\]  

(3.11)
where $A_{N^k_{MHV(0)}}^{(1h_1,2h_2,...,nh_n)}(u, \bar{u})$ is defined in (2.14). In terms of the MHV S-matrix functional $F_{MHV}^{(a(h)c)}$ in (2.33), the $x$-space MHV amplitudes at tree level can be expressed as

$$
\delta \frac{\delta}{\delta a_1^{(+)}c_1} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(-)}c_n} F_{MHV}^{(a(h)c)} \bigg|_{a(h)c=0} = A_{MHV(0)}^{(a-b-)}(x) \quad (3.12)
$$

where $a^{(h)c}$'s denote the gluon creation operators in the momentum-space representation, which are treated as source functions here. As in the previous section, the notation $(0)$ specifies that the amplitudes are at tree level.

Generalization of this expression or the $x$-space analog of (2.40), not necessarily limited to the case of tree amplitudes, can be written as

$$
\frac{\delta}{\delta a_1^{(h)c_1}} \otimes \frac{\delta}{\delta a_2^{(h)c_2}} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(h)c_n}} F^{(a(h)c)} \bigg|_{a(h)c=0} = A_{N^k_{MHV}}^{(1h_1,2h_2,...,nh_n)}(x) \quad (3.13)
$$

where the S-matrix functional is now represented with an $x$-space contraction operator $W^{(A)}(x)$:

$$
F^{(a(h)c)} = W^{(A)}(x) F_{MHV}^{(a(h)c)}
$$

$$
\tilde{W}^{(A)}(x) = \exp \left[ - \int d\mu(p) \left( \frac{\delta}{\delta a_p^{(+)}} \otimes \frac{\delta}{\delta a_p^{(-)}} \right) e^{-ip(x-y)} \right] y \to x \quad (3.15)
$$

where $p$ denotes the on-shell partner of the momentum transfer $q$, corresponding to the one in (2.9). For simplicity, we here express the relation (2.11) as

$$
q_{\mu} = p_{\mu} + w_{\eta_{\mu}} \quad (3.16)
$$

where $\eta_{\mu}$ denotes the reference null-vector. As in the expression (3.9), the off-shell Nair measure for $q$ is given by

$$
d\mu(q) \approx d\mu(p) + \frac{1}{4\pi} \frac{dw^2}{w^2}. \quad (3.17)
$$

The factor $e^{-ip(x-y)}$ in (3.15) is necessary to guarantee the energy-momentum conservation for $x - y \neq 0$. At the end of calculation, we eventually take the limit $y \to x$. We assume that the limit is taken such that the time ordering $x^0 > y^0$ is preserved. In other words, we shall take the limit $y \to x$ with $x^0 - y^0 \to 0_+$. Since we take this limit, the null property of $p$ is not necessary in performing the integral and in assuring the energy-momentum conservation. Therefore we may also express the contraction operator $\tilde{W}^{(A)}(x)$ in (3.15) as

$$
\tilde{W}^{(A)}(x) = \exp \left[ - \int d\mu(q) \left( \frac{\delta}{\delta a^{(+)}} \otimes \frac{\delta}{\delta a^{(-)}} \right) e^{-iq(x-y)} \right] y \to x
$$

$$
= \exp \left[ - \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2} \left( \frac{\delta}{\delta a^{(+)}} \otimes \frac{\delta}{\delta a^{(-)}} \right) e^{-iq(x-y)} \right] y \to x \quad (3.18)
$$

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where we use a well-known identity in the calculation of the Feynman propagator
\[
\int d\mu(p) \left[ \theta(x^0 - y^0)e^{-ip(x-y)} + \theta(y^0 - x^0)e^{ip(x-y)} \right] = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 + i\epsilon} e^{-ip(x-y)} \tag{3.19}
\]
\(p\) is a null momentum and \(\epsilon\) is a positive infinitesimal. Notice that there appears no propagator mass for \(q\) in (3.18). This is a consequence of the aforementioned interpretation of the off-shell momentum \(q\) (as an effective null-momentum with an incorporation of the scaling factor \(w\)) and essentially embodies the CSW prescription in our formulation.

In the holonomy formalism, the CSW rules are encoded in the definition of \(\hat{W}^{(A)}(x)\). The CSW rules in our formulation are then stated as follows.

1. There is no propagator mass for the virtual gluon represented by the off-shell momentum transfer \(q\).

2. The factor \(e^{-iq(x-y)}\) can be replaced by \(e^{-ip(x-y)}\) as we eventually take the limit \(y \to x\), keeping the time ordering \(x^0 > y^0\).

3. We do not define creation operators for the virtual gluon, \(a^{(+)}_q\) or \(a^{(-)}_{-q}\), since these are irrelevant, if related, to the creation operators of gluons in terms of which the MHV S-matrix functional \(F_{MHV}\) is constructed.

These rules are materialized by the following contraction operator.
\[
\hat{W}^{(A)}(x) = \exp \left[ -\int d\mu(q) \left( \frac{\delta}{\delta a^{(+)}_p} \otimes \frac{\delta}{\delta a^{(-)}_{-p}} \right) e^{-ip(x-y)} \right]_{y \to x} \approx \exp \left[ -\int d\mu(p) \left( \frac{\delta}{\delta a^{(+)}_p} \otimes \frac{\delta}{\delta a^{(-)}_{-p}} \right) e^{-ip(x-y)} \right] \times \exp \left[ \frac{1}{4\pi} \int \frac{dw^2}{w^2} \left( \frac{\delta}{\delta a^{(+)}_p} \otimes \frac{\delta}{\delta a^{(-)}_{-p}} \right) e^{-ip(x-y)} \right] \bigg|_{y \to x} \tag{3.20}
\]
where we use (3.17) and (3.18) with a replacement of the factor \(e^{-iq(x-y)}\) by \(e^{-ip(x-y)}\).

Apparently, the expressions (3.15) and (3.20) contradict each other. Quantum theoretically, however, the two (and the other expressions in (3.18)) can be regarded as equivalent, with the former being identified as a regularized form of the latter. The exponent of the second factor in (3.20) is proportional to
\[
\int_0^1 \frac{dw^2}{w^2} = \infty \tag{3.21}
\]
where we use the condition (3.10). Thus we can indeed obtain the expression (3.15) from (3.20) by regularizing the factor involving the above log divergence.

As we have seen in the previous section, the above understanding of regularization on the contraction operators is valid through tree amplitudes where the log divergence arises
at each propagator. As long as we resort to the functional derivation of gluon amplitudes, this interpretation should be applicable to loop amplitudes in general. In the following, we consider a derivation of one-loop MHV amplitudes in particular, utilizing the \(x\)-space S-matrix functional (3.14).

Reproduction of the BST results in the holonomy formalism

In terms of the S-matrix functional (3.14), the MHV amplitude in the \(x\)-space representation is generated as a loop expansion:

\[
\frac{\delta}{\delta a_1^{(+)c_1}} \otimes \cdots \otimes \frac{\delta}{\delta a_{n}^{(-)c_n}} \otimes \cdots \otimes \frac{\delta}{\delta a_{b}^{(+)(-c_{b})}} \otimes \cdots \otimes \frac{\delta}{\delta a_{a}^{(+)c_a}} F[a^{(h)c}]_{a^{(h)c}=0}
\]

\[
= \mathcal{A}_{MHV}^{(\cdots)}(x)
\]

\[
= \mathcal{A}_{MHV(0)}^{(\cdots)}(x) + \mathcal{A}_{MHV(1)}^{(\cdots)}(x) + \mathcal{A}_{MHV(2)}^{(\cdots)}(x) + \cdots.
\]

(3.22)

The number of contractions by \(\hat{W}^{(A)}(x)\) corresponds to the loop order of the \(L\)-loop amplitude \(\mathcal{A}_{MHV(L)}^{(\cdots)}(x)\). In terms of the coupling constant \(g\), the \(L\)-loop amplitude can be expressed as

\[
\mathcal{A}_{MHV(L)}^{(\cdots)}(x) \sim g^{n-2+2L}.
\]

(3.23)

Picking up the \(L = 1\) term in (3.22), we can calculate the one-loop MHV amplitudes as

\[
\mathcal{A}_{MHV(1)}^{(\cdots)}(x) = \left[ -\int d\mu(L_1) \left( \frac{\delta}{\delta a_{l_1}^{(+)c_{l_1}}} \right) \right] \left[ -\int d\mu(L_2) \left( \frac{\delta}{\delta a_{l_2}^{(+)c_{l_2}}} \right) \right]
\]

\[
\times \left[ \frac{\delta}{\delta a_{1}^{(+)c_1}} \otimes \cdots \otimes \frac{\delta}{\delta a_{n}^{(+)(-c_n)}} F[a^{(h)c}]_{a^{(h)c}=0} \right]
\]

\[
= \prod_{i=1}^{n} \int d\mu(p_i) \mathcal{A}_{MHV(1)}^{(\cdots)}(u, \bar{u})
\]

(3.24)

\[
\mathcal{A}_{MHV(1)}^{(\cdots)}(u, \bar{u}) = ig^{n-2} (2\pi)^4 \delta(4) \left( \sum_{i=1}^{n} p_i \right) \hat{\mathcal{A}}_{MHV(1)}^{(\cdots)}(u)
\]

(3.25)

where \(L_1\) and \(L_2\) denote loop momenta. \(l_1\) and \(l_2\) are corresponding on-shell vectors, respectively. A typical one-loop MHV diagram is shown in Figure 2.

The holomorphic part of the one-loop amplitudes \(\hat{\mathcal{A}}_{MHV(1)}^{(\cdots)}(u)\) are then calculated as

\[
\hat{\mathcal{A}}_{MHV(1)}^{(\cdots)}(u) = ig^2 \sum_{i=1}^{n} \sum_{r=1}^{\left[ \frac{n}{2} \right]-1} \left( 1 - \frac{1}{2} \delta_{2r+1}^{\frac{n}{2}} \right) \int d\mu(L_1) d\mu(L_2)
\]

\[
\times \hat{\mathcal{A}}_{MHV(0)}^{((-l_1_{1}+a_{i} \cdots (l_1+r)+l_2+(u) \hat{\mathcal{A}}_{MHV(0)}^{((-l_2_{1}+a_{(i+r)+1}+b_{i} \cdots (i-1)+l_1+(u)}(u)
\]

(3.26)

where \(\left[ \frac{n}{2} \right]\) denotes the largest integer less than or equal to \(\frac{n}{2}\). Since \(r\) is a positive integer, \(\delta_{2r+1}^{\frac{n}{2}}\) vanishes unless \(n = 2k\) \((k = 2, 3, \cdots)\). The factor \(\left( 1 - \frac{1}{2} \delta_{2r+1}^{\frac{n}{2}} \right)\) arises in order to compensate the double counting due to the left-right reflection symmetry. There is a freedom
Figure 2: One-loop MHV diagram — The internal off-shell momenta are labeled by $L_i = l_i + w_i \eta_i$ ($i = 1, 2$), with $l_i$ and $\eta_i$ being null four-vectors. $w_i$ are real variables. The diagram corresponds to the case where the negative-helicity indices $(a_+ b_-)$ are split into the left and the right MHV vertices. If the both indices are on the same side, which always occurs as we rotate the index $i$, we need to flip the direction of one of the propagators accordingly. When the left and the right vertices have the same number of external legs, we have reflection symmetry. This explains the factor of $\left(1 - \frac{1}{2} \delta_{\frac{n}{2} r + 1} \right)$ in the equation (3.26).

of choice for the positions of the internal indices $\{l_1, -l_1, l_2, -l_2\}$. Diagrammatically, this means that we can arbitrarily choose the indices $i$ ($= 1, 2, \cdots, n$) and $r$ ($= 1, 2, \cdots, \lfloor n/2 \rfloor - 1$) in Figure 2. We have reflection symmetry when the left and the right MHV vertices have the same number of external legs. In the functional language, this means that $x$ and $y$ are exchangeable in carrying out the contraction with (3.20). Since we preserve the time ordering $x^0 > y^0$ in taking the $y \rightarrow x$ limit, there are only half contributions from the diagrams with reflection symmetry. Alternatively, we can also understand this fact from the Taylor expansion of the MHV S-matrix $F_{MHV} \left[a^{(h) \ell}c\right]$ in (3.14), whose explicit form is defined by (2.33), since in the symmetric case we can treat $x$ and $y$ identical to each other. Thus a factor $\frac{1}{2!}$ automatically arises in the functional method. This naturally explains the factor $\left(1 - \frac{1}{2} \delta_{\frac{n}{2} r + 1} \right)$ in (3.26).

Notice also that the use of the functional derivatives in (3.22) and the Grassmann integral in (2.37) automatically pick a correct choice of the internal helicity configuration for given $i$ and $r$. Thus, as far as formalism is concerned, our functional formulation of the one-loop amplitudes is more efficient and natural than a diagrammatic method.

The single-trace color structure of the one-loop MHV amplitudes arises from the product of the holomorphic tree MHV amplitudes in (3.26) and it can be calculated as

$$
\hat{A}_{MHV(0)}^{((-l_1) \cdots (-i) \cdots (i+r) \cdots l_2)}(u) \hat{A}_{MHV(0)}^{((-l_2) \cdots (-i) \cdots (i+r+1) \cdots l_1)}(u) = \sum_{\sigma^{(1)} \in S_{n+1}} \sum_{\sigma^{(2)} \in S_{n-r-1}} \text{Tr}(t^{a_1}_{\sigma^{(1)}} \cdots t^{a_r}_{\sigma^{(1)}} t^{a_{r+1}}_{\sigma^{(2)}} \cdots t^{a_{n-r-1}}_{\sigma^{(2)}})
\times \hat{C}_{MHV(0)}((l_1) \cdots (-i) \cdots (i+r) \cdots l_2)(u; \sigma^{(1)}) \hat{C}_{MHV(0)}((-l_2) \cdots (-i) \cdots (i+r+1) \cdots l_1)(u; \sigma^{(2)})
$$
a limit of the scalar products being defined in (2.5). Notice that the factor \( R_y \) (\( N \) as discussed below (2.22), the holonomy operator is defined in the framework the relation (3.31) is naturally required, and so is the relation (3.29). \( \sigma_y, \theta \) should be understood as taking the limit of \( (1) \) 

where \( \mathcal{P}(i \cdots i + r'| i + r + 1 \cdots i - 1) \) denotes the terms obtained by the double permutations of \( \sigma^{(1)} \) and \( \sigma^{(2)} \). \( 1^{(1)} \) and \( 1^{(2)} \) denote the identity transpositions for \( \sigma^{(1)} \) and \( \sigma^{(2)} \), respectively:

\[
1^{(1)} = \left( \begin{array}{c} i \cdots i + r \\ i \cdots i + r \end{array} \right), \quad 1^{(2)} = \left( \begin{array}{c} i + r + 1 \cdots i - 1 \\ i + r + 1 \cdots i - 1 \end{array} \right).
\] (3.28)

That the functional method has generality in the position of negative-helicity indices \((a_- b_-)\) means that the product of the colorless factors in (3.27) can be replaced by the following product:

\[
\begin{align*}
&\hat{C}_{MHV(0)}^{((-l_1)_- \cdots (-l_{i+r})_--(i+r)_+)b_+} (u; 1^{(1)}) \hat{C}_{MHV(0)}^{((-l_2)_- \cdots (i+r+1)_+ \cdots (-l_1)_+)b_+} (u; 1^{(2)}) \\
&\rightarrow \hat{C}_{MHV(0)}^{((-l_1)_- \cdots (-l_{i+r})_--(i+r)_+)b_+} (u; 1^{(1)}) \hat{C}_{MHV(0)}^{((-l_2)_- \cdots (i+r+1)_+ \cdots (-l_1)_+)b_+} (u; 1^{(2)}) \\
&= \mathcal{R}^{(l_1 l_2)}_{n;r;i} \hat{C}_{MHV(0)}^{(i \cdots i - b_- \cdots (i-1)_+)b_+} (u; 1^{(1)} \otimes 1^{(2)})
\end{align*}
\] (3.29)

where \( \mathcal{R}^{(l_1 l_2)}_{n;r;i} \) is defined by

\[
\mathcal{R}^{(l_1 l_2)}_{n;r;i} = \frac{(i + r + 1)(l_1 l_2)}{(i + r l_2)(-l_1 i)} \frac{(i - 1 i)(l_2 l_1)}{(i - 1 l_1)(-l_2 i + r + 1)}.
\] (3.30)

Here, for simplicity, the holomorphic scalar products of spinor momenta are represented by the numbering indices, e.g., \((i + r + 1) = (u_{i+r} u_{i+r+1})\), \((l_1 l_2) = (u_{l_1} u_{l_2})\) and so on, with the scalar products being defined in (2.5). Notice that the factor \( \mathcal{R}^{(l_1 l_2)}_{n;r;i} \) is quadratic in either \( l_1 \) or \( l_2 \). Thus it is independent of the signs of \( l_1, l_2 \). The arrow in (3.29) means that we take a limit of \( \theta' \rightarrow \theta \) where \( \theta' \) and \( \theta \) are the \( \mathcal{N} = 4 \) chiral superpartners of the four-dimensional space-time coordinates \( x \) and \( \mathcal{R} \). In this limit, we have the relation

\[
\begin{align*}
&(-l_1 a)^4(b - l_2)^4 \\
&= \int d^8 \theta \xi^1 \xi^2 \xi^3 \xi^4 \xi^1 \xi^2 \xi^3 \xi^4 \xi_{i}^{a_5} u_{i}^{a_4} \int d^8 \theta' \eta_1^{a_5} \eta_2^{a_4} \eta_3^{a_3} \eta_4^{a_2} \eta_{-l_2}^{a_1} \eta_{-l_2}^{a_0} \eta_{-l_2}^{a_1} \eta_{-l_2}^{a_0} \eta_{-l_2}^{a_1} \eta_{-l_2}^{a_0} \xi_{i}^{a_5} = \theta_5^{a_5} u_{i}^{a_4} \\
&\rightarrow (a b)^4(l_1 l_2)^4 \quad (\theta' \rightarrow \theta).
\end{align*}
\] (3.31)

As discussed below (2.22), the holonomy operator is defined in \( \mathcal{N} = 4 \) chiral superspace \((x, \theta) = (x_{AA}, \theta^a_\alpha)\). This means that taking the limit of \( y \rightarrow x \) in the definition of \( \hat{W}^{(a)}(x) \) should be understood as taking the limit of \( (y, \theta') \rightarrow (x, \theta) \). Thus, in the supertwistor framework the relation (3.31) is naturally required, and so is the relation (3.29).

The holomorphic one-loon MHV amplitudes (3.26) are then calculated as

\[
\hat{A}_{MHV(1)}^{(a_- b_-)} (u) = i g^2 \sum_{i=1}^{n} \sum_{r=1}^{n-i} \left( 1 - \frac{1}{2} \delta_{-r+1}^{n} \right) \\
\times \sum_{\sigma^{(1)} \in \mathcal{S}_{r+1}} \sum_{\sigma^{(2)} \in \mathcal{S}_{n-r-1}} \text{Tr}(t_{\sigma^{(1)}}^{x_i} \cdots t_{\sigma^{(1)}}^{x_{i+r}} t_{\sigma^{(2)}}^{x_{i+r+1}} \cdots t_{\sigma^{(2)}}^{x_{i}})
\]
by use of the notation $R$ can attribute the calculation of (3.32) to that of where $\hat{\sigma}$ Here we abbreviate $\sigma$ i.e. any values here, one-loop MHV amplitudes by rewriting the off-shell measures as

The BST representation of one-loop MHV amplitudes

\[ \mathcal{L}^{(l_1,l_2)}_{n:r;\sigma_i}(u;\sigma^{(1)} \otimes \sigma^{(2)}) = \int d\mu(L_1)d\mu(L_2) \mathcal{R}^{(l_1,l_2)}_{n:r;\sigma_i}(u;\sigma^{(1)} \otimes \sigma^{(2)}) \]  

(3.32)

\[ \mathcal{L}^{(l_1,l_2)}_{n:r;\sigma_i} = \int d\mu(L_1)d\mu(L_2) \mathcal{R}^{(l_1,l_2)}_{n:r;\sigma_i} \]  

(3.33)

\[ \mathcal{R}^{(l_1,l_2)}_{n:r;\sigma_i} = \frac{(\sigma_{i+r}^{(2)} \sigma_{i+r+1}^{(1)})(l_1 l_2)}{(\sigma_{i+1}^{(1)} l_2)(-l_1 \sigma_i^{(1)})(l_2 l_1)} \]  

(3.34)

where an explicit form of $\tilde{\mathcal{C}}_{MHV(0)}^{(a-b-)}(u;\sigma^{(1)} \otimes \sigma^{(2)})$ is given by

\[ \tilde{\mathcal{C}}_{MHV(0)}^{(a-b-)}(u;\sigma^{(1)} \otimes \sigma^{(2)}) = \frac{(a b)^4}{(\sigma_{i+1}^{(1)} \sigma_{i+2}^{(1)})(\sigma_{i+1}^{(2)} \sigma_{i+2}^{(2)})\cdots(\sigma_{i+r}^{(2)} \sigma_{i+r+1}^{(2)})\cdots(\sigma_{i-1}^{(2)} \sigma_i^{(1)})} \]  

(3.35)

Here we abbreviate $\sigma^{(1)} \otimes \sigma^{(2)}$ by $\sigma^{(1\otimes2)}$. Notice that the negative-helicity indices can take any values here, i.e., $a, b \in \{i, i+1 \cdots, i-1\} = \{1, 2, \cdots, n\}$. As in the expression (3.27), by use of the notation $\mathcal{P}(i\cdots i+r|i+r+1\cdots i-1)$ for the double-permutation terms, we can attribute the calculation of (3.32) to that of

\[ \mathcal{L}^{(l_1,l_2)}_{n:r;i} = \int d\mu(L_1)d\mu(L_2) \mathcal{R}^{(l_1,l_2)}_{n:r;i} \]  

(3.36)

where $\mathcal{R}^{(l_1,l_2)}_{n:r;i}$ is defined by (3.30). Now, using the Schouten identities

\[ (i+r i+r+1)(l_1 l_2) = (i+r l_2)(l_1 i+r+1 + (i+r l_1)(i+r+1 l_2), \]

\[ (i-1 i)(l_2 l_1) = (i-1 l_1)(l_2 i) + (i-1 l_2)(i l_1), \]

we can express $\mathcal{R}^{(l_1,l_2)}_{n:r;i}$ as

\[ \mathcal{R}^{(l_1,l_2)}_{n:r;i} = \hat{R}^{(l_1,l_2)}_{(i,i+r+1)} - \hat{R}^{(l_1,l_2)}_{(i,i+r)} - \hat{R}^{(l_1,l_2)}_{(i-1,i+r+1)} + \hat{R}^{(l_1,l_2)}_{(i-1,i+r)} \]  

(3.37)

where $\hat{R}^{(l_1,l_2)}_{(i,j)}$ is defined by

\[ \hat{R}^{(l_1,l_2)}_{(i,j)} = \frac{(l_2 i)(l_1 j)}{(l_1 i)(l_2 j)}, \]  

(3.38)

We here suppress the total number suffix $n$ for simplicity.

It is explicitly shown in [9, 12] that the expression (3.36) leads to the BST results for one-loop MHV amplitudes by rewriting the off-shell measures as

\[ d\mu(L_i) \longrightarrow \frac{d^4L_i}{L_i^2} \quad (\text{for } i = 1, 2) \]  

(3.39)

up to a certain normalization factor. In our formulation, this prescription can naturally be understood from the expression (3.18). This means that the BST representation of one-loop MHV amplitudes, which we show in a moment, can be reproduced from the S-matrix functional (3.14) with a suitable choice of normalization for the off-shell measures.

The BST representation of one-loop MHV amplitudes

For the completion of our discussion, in the following we briefly review the results of the BST method for one-loop MHV amplitudes [9]. The essence of the BST results is that the
calculation of the integral (3.36) with the prescription (3.39) correctly leads to the previously known unitary-cut results obtained by Bern, Dixon, Dunbar and Kosower (BDDK) [17, 18].

The BDDK representation of one-loop MHV amplitudes is given in terms of color-stripped holomorphic amplitudes. In our notation, this can be expressed in terms of the colorless known unitary-cut results obtained by Bern, Dixon, Dunbar and Kosower (BDDK) [17, 18].

\[
\hat{C}_{MHV(0)}(u) = C_{MHV(0)}(u; 1),
\]

\[
\hat{C}_{MHV(1)}(u) = C_{MHV(1)}(u; 1^{(1)} \otimes 1^{(2)})
\]

where \( \hat{C}_{MHV(0)}(u; \sigma) \) is defined by (2.3) and \( \hat{C}_{MHV(1)}(u; \sigma^{(1)} \otimes \sigma^{(2)}) \) is defined by the expression

\[
\hat{A}_{MHV(1)}^{(a_{-b_{-}})}(u) = \sum_{\sigma^{(1)} \in \mathcal{S}_{n+1}} \sum_{\sigma^{(2)} \in \mathcal{S}_{n-r-1}} \text{Tr}(t^{\sigma^{(1)}}_1 \cdots t^{\sigma^{(1)}}_{i+r+1} t^{\sigma^{(2)}}_{i+r+2} \cdots t^{\sigma^{(2)}}_{i-1}) \hat{C}_{MHV(1)}^{(a_{-b_{-}})}(u; \sigma^{(1)} \otimes \sigma^{(2)}).
\]

In terms of (3.40), the BDDK representation is then expressed as [17]

\[
\hat{C}_{MHV(1)}^{(a_{-b_{-}})}(u) = ig^2 \mathcal{C}_T V_n \hat{C}_{MHV(0)}^{(a_{-b_{-}})}(u)
\]

\[
\mathcal{C}_T = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \to \frac{1}{(4\pi)^2} \quad (\epsilon \to 0)
\]

where \( \epsilon = (4 - D)/2 \) is the dimensional regularization parameter. \( V_n \) is independent of the negative-helicity indices \((a_{-b_{-}})\) and is given by

\[
V_n = \sum_n \left( \sum_{r=1}^n \left( 1 - \frac{1}{2} \delta_{\frac{n}{2}, r+1} \right) F_{n;r;i}^{2m,e} \right).
\]

\( F_{n;r;i}^{2m,e} \) is known as the (easy two-mass scalar) box function and is given by

\[
F_{n;r;i}^{2m,e} \equiv F(s, t, P^2, Q^2)
\]

\[
= -\frac{1}{\epsilon^2} \left[ (-s)^{-\epsilon} + (-t)^{-\epsilon} - ((-P)^2)^{-\epsilon} - ((-Q)^2)^{-\epsilon} \right]
\]

\[ + \text{Li}_2 \left( 1 - \frac{P^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{P^2}{t} \right) + \text{Li}_2 \left( 1 - \frac{Q^2}{s} \right) + \text{Li}_2 \left( 1 - \frac{Q^2}{t} \right)
\]

\[ - \text{Li}_2 \left( 1 - \frac{Q^2}{st} \right) + \frac{1}{2} \log^2 \left( \frac{s}{t} \right)
\]

in terms of the following parametrization

\[
P = q_{i+r-1} = p_i + p_{i+1} + \cdots p_{i+r-1}
\]

\[
Q = q_{i+r+1} = p_{i+r} + p_{i+r+1} + \cdots p_{i+2}
\]

\[
s = (p_{i-1} + P)^2 = (p_{i+r} + Q)^2
\]

\[
t = (p_{i+r} + P)^2 = (p_{i-1} + Q)^2
\]

with \( p_{i-1} + P + p_{i+r} + Q = 0 \) where \( p_i \)'s denote the \( i \)-th out-going gluon momenta \((i = 1, 2, \cdots, n)\). In (3.46) and (3.47) we use the notation (2.9), expressing \( P \) and \( Q \) as \( P = q_{i+i+r-1} \).
and \( Q = q_{i+r+1}i-2 \), respectively. Diagrammatic relation of these variables can be seen in Figure 2.

Comparing the expressions (3.32) and (3.41)-(3.44), we can easily find that the box function (3.45) corresponds to the integral \( \mathcal{L}_{n;r;i}^{(l_1,l_2)} \) in (3.36). In the BST paper [9], this correspondence is analytically confirmed, including the \( \epsilon \)-dependent part, by careful evaluation of the integral. The upshot of the BST results is given by the BST representation of the box function:

\[
F(s, t, P^2, Q^2) = -\frac{1}{\epsilon^2} \left[ (-s)^{-\epsilon} + (-t)^{-\epsilon} - ((-P)^2)^{-\epsilon} - ((-Q)^2)^{-\epsilon} \right] + B(s, t, P^2, Q^2) \tag{3.50}
\]

\[
B(s, t, P^2, Q^2) = \text{Li}_2(1 - c_{r;i}P^2) + \text{Li}_2(1 - c_{r;i}Q^2) - \text{Li}_2(1 - c_{r;i}s) - \text{Li}_2(1 - c_{r;i}t) \tag{3.51}
\]

where

\[
c_{r;i} = \frac{(P^2 + Q^2) - (s + t)}{P^2Q^2 - st} = \frac{(p_{i-1} + p_{i+r})^2}{P^2Q^2 - st}. \tag{3.52}
\]

At the limit of \( \epsilon \to 0 \), the finite part of the box function \( F(s, t, P^2, Q^2) \) is given by \( B(s, t, P^2, Q^2) \).

### 4 One-loop MHV amplitudes: polylog regularization

In this section, we propose an alternative derivation of the BST representation from the expression (3.36) in a rather intuitive way.

Our strategy to obtain the one-loop amplitudes is to use the off-shell continuation of the contraction operator (3.20) such that the finite part of the amplitude arises from the \( w \)-part of the off-shell Nair measure, namely, contributions form the second factor in (3.20).

Instead of relying on the BST prescription (3.39), we now explicitly use the off-shell Nair measure of the form (3.17). The integral measure of \( \mathcal{L}_{n;r;i}^{(l_1,l_2)} \) in (3.36) is then expanded as

\[
d\mu(L_1)d\mu(L_2) = d\mu(l_1)d\mu(l_2) + d\mu(l_1) \frac{1}{4\pi} \frac{dW_2}{W_2} + \frac{1}{4\pi} \frac{dW_1}{W_1} d\mu(l_2) + \frac{1}{(4\pi)^2} \frac{dW_1}{W_1} \frac{dW_2}{W_2} \tag{4.1}
\]

where \( W_i \equiv w_i^2 \) \( (i = 1, 2) \).

Since \( \mathcal{R}_{n;r;i}^{(l_1,l_2)} \) in (3.36) is a dimensionless quantity, the integral over \( d\mu(l_1) \) or \( d\mu(l_2) \) leads to quadratic divergence. In order to obtain finite quantities from the measures involving these, we need some regularization scheme. In the unitary-cut method, dimensional regularization is utilized to express the one-loop MHV amplitudes in terms of the box functions. Similarly, we may calculate the integral over \( d\mu(l_1)d\mu(l_2) \), the first term in (4.1), by use of dimensional regularization. The resultant amplitudes would correspond to a certain collinear sector of the one-loop MHV amplitudes. The collinear sector can be specified by assuming that external legs for one of the MHV vertices, say, the left-hand side MHV vertex, are all collinear.
The second and the third terms in (4.1) also include the quadratically divergent $d\mu(l_i)$ measure. Thus, for the same reasons above, we need to resort to some regularization scheme unless the $W_i$-measure vanishes to cancel the divergence. Such a cancelation, however, never happens because of the following. The vanishing of the $W_i$ measure means that $W_i$ is fixed. But what does fixed $W_i$ or $w_i$ mean in the off-shell definition of $L_i = l_i + w_i\eta_i$ where $\eta_i$ is an arbitrary null vector? Since $w_i$ couples to $\eta_i$, we can always shift $w_i$ unless it is zero. In this context, the fixed $w_i$ literally means that we fix $w_i$ to zero, which of course contradicts the off-shell definition. Thus the $W_i$ measure never vanishes and the second and third terms in (4.1) lead to infinity.

Another interesting argument for the divergence can be made as follows. In terms of the integral measure per se, the single-$W_i$ measure is equivalent to the one that appears in the tree amplitudes in (3.21). As discussed there, the integral has log divergence unless we change the range of $W_i$ from $0 < W_i < 1$. As we shall discuss later, such an alternation may occur for multiple $W_i$’s, particularly for the double-$W_i$ measure in (4.1), but for the single-$W_i$ case that will not happen since the single-$W_i$ measure is independent of the other $W_i$’s and can effectively be treated as a tree-level integral measure. Thus, as in the tree-level case, we can interpret the integrals over the second and third terms in (4.1) as unphysical.

Iterated integral representation of polylogarithms

In the rest of this section, we argue that finite physical quantities of the one-loop MHV amplitudes, i.e., the function $B(s, t, P^2, Q^2)$ in (3.51), can be obtained from an integral over the last term in (4.1). We shall carry out the analysis by paying attention to the range of $W_i$’s. For this purpose, it is convenient to consider the double-$W_i$ measure in terms of differential forms as we have defined the logarithmic one-form $\omega_{ij} = d\log(u_i u_j)$ in (2.19). Using the local coordinate parametrization (3.1), the one-form $\omega_{ij}$ is expressed as

$$\omega_{ij} = d\log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j} \tag{4.2}$$

where $i, j$ denote the numbering indices, satisfying $1 \leq i < j \leq n$. Motivated by this form, we define the following one-forms

$$\omega_1^{(1)} \equiv -d\log(W^{(1)} - W_1) = \frac{dW_1}{1 - W_1}, \tag{4.3}$$

$$\omega_2^{(0)} \equiv d\log(W^{(0)} - W_2) = \frac{dW_2}{W_2} \tag{4.4}$$

where $W_1$ and $W_2$ are positive real variables and we set $W^{(0)}$ and $W^{(1)}$ by

$$W^{(0)} \equiv 0, \quad W^{(1)} \equiv 1. \tag{4.5}$$

It is well-known (see, e.g., [73]) that the dilogarithm function can be represented as an iterated integral:

$$\text{Li}_2(x) = \int_0^x \omega^{(1)} \omega^{(0)} \tag{4.6}$$

$$\omega^{(0)} \equiv \frac{dt}{t}, \quad \omega^{(1)} \equiv \frac{dt}{1 - t} \tag{4.7}$$
where \( t \in [0, x] \) with \( x \in \mathbb{R} - \{0, 1\} \). To be more rigorous, it is known that \( x \) can be analytically continued to \( x \in \mathbb{C} - \{0, 1\} \) but, for our discussion, it is enough to restrict \( t \) (and \( x \)) to be real. Generalization of this expression provides a neat set of representations for the polylogarithm functions:

\[
\text{Li}_k(x) = \int_0^x \frac{\omega^{(1)}(0) \omega^{(0)}(0) \cdots \omega^{(0)}(k-1)}{t} \, dt \tag{4.8}
\]

for \( k = 2, 3, \cdots \). This can be understood from the iterative relation

\[
\text{Li}_k(x) = \int_0^x \frac{\text{Li}_{k-1}(t)}{t} \, dt \tag{4.9}
\]

With a suitable ordering of \( W_i \)'s, say,

\[
0 < W_1 \leq W_2 < x, \quad x \in \mathbb{R} - \{0, 1\}, \tag{4.10}
\]

one can then express the dilogarithm function as an iterated integral over (4.3) and (4.4):

\[
\int_0^x \omega^{(1)}(0) \omega^{(0)}(0) = \text{Li}_2(x). \tag{4.11}
\]

An analog of (4.8) is then written as

\[
\text{Li}_k(x) = \int_0^x \omega^{(1)}(0) \omega^{(0)}(0) \cdots \omega^{(0)}(k) \tag{4.12}
\]

where, as in the case of (4.4), \( \omega^{(0)}(0) \) is defined by

\[
\omega^{(0)}(0) \equiv d \log(W^{(0)} - W_k) = \frac{dW_k}{W_k} \tag{4.13}
\]

for \( k = 2, 3, \cdots \). In this general case, the range of \( W_i \)'s is given by

\[
0 < W_1 \leq W_2 \leq \cdots \leq W_k < x, \quad x \in \mathbb{R} - \{0, 1\}. \tag{4.14}
\]

For details of these iterated integral expressions, one may also refer to [76].

In what follows, we identify the integral over \( \frac{dW_1}{W_1} \frac{dW_2}{W_2} \) as the iterated integral (4.11). With such an identification, the problem of integral can be reduced to that of determining the range of \( W_i \)'s which corresponds to the value of \( x \in \mathbb{R} - \{0, 1\} \). We shall consider this problem in the following.

**Prescription for the BST representation**

In terms of the iterated integral, the integral \( \mathcal{L}_{n,r;i}^{(l_1,l_2)} \) in (3.36) can be expressed as

\[
\int \omega^{(1)}(0) \omega^{(0)}(0) R_{n,r;i}^{(l_1,l_2)} = \hat{R}^{(l_1,l_2)}_{(i,i+r+1)} \int \omega^{(1)}(0) \omega^{(0)}(0) - \hat{R}^{(l_1,l_2)}_{(i,i+r)} \int \omega^{(1)}(0) \omega^{(0)}(0)
\]

\[
- \hat{R}^{(l_1,l_2)}_{(i-1,i+r+1)} \int \omega^{(1)}(0) \omega^{(0)}(0) + \hat{R}^{(l_1,l_2)}_{(i-1,i+r)} \int \omega^{(1)}(0) \omega^{(0)}(0) \tag{4.15}
\]
where we use the expansion of $\mathcal{R}_{n;r,i}^{(l_1,l_2)}$ in (3.37). Notice that $\mathcal{R}_{n;r,i}^{(l_1,l_2)}$ is not a function of $(W_1,W_2)$ but that of $(l_1,l_2)$. The paths of iterated integrals are to be determined. The expression (4.15) lead to the BST box function (3.51), if we impose the following prescription:

$$
\tilde{R}_{(i,j)}^{(l_1,l_2)} \int \omega_1^{(1)} \omega_2^{(0)} \rightarrow \int_0^{1-\Delta_{(i,j)}} \omega_1^{(1)} \omega_2^{(0)} = \text{Li}_2(1 - \Delta_{(i,j)})
$$

(4.16)

where $\Delta_{(i,i+r+1)}$, $\Delta_{(i,i+r)}$, $\Delta_{(i-1,i+r+1)}$ and $\Delta_{(i-1,i+r)}$ are defined by

\begin{align*}
\Delta_{(i,i+r+1)} & \equiv c_{r;i} Q^2 = c_{r;i} (P + p_{i-1} + p_{i+r})^2 \\
\Delta_{(i,i+r)} & \equiv c_{r;i} s = c_{r;i} (P + p_{i-1})^2 \\
\Delta_{(i-1,i+r+1)} & \equiv c_{r;i} t = c_{r;i} (P + p_{i+r})^2 \\
\Delta_{(i-1,i+r)} & \equiv c_{r;i} P^2 
\end{align*}

(4.17)-(4.20)

with $c_{r;i}$ given in (3.52). In terms of $P = p_i + p_{i+1} + \cdots + p_{i+r-1}$, $p_{i-1}$ and $p_{i+r}$, the factor $c_{r;i}$ can be written as

$$
c_{r;i} = \frac{(p_{i-1} + p_{i+r})^2}{P^2(P + p_{i-1} + p_{i+r})^2 - (P + p_{i-1})^2(P + p_{i+r})^2}.
$$

(4.21)

As pointed out in [9], this is invariant under the transformations

$$
P \rightarrow P + \alpha p_{i-1} + \beta p_{i+r} \equiv P_{(\alpha,\beta)}
$$

(4.22)

where $\alpha$ and $\beta$ are real numbers. Using $P_{(\alpha,\beta)}$, we can uniformly express the definitions (4.17)-(4.20) as

$$
\Delta_{(i-1+i\alpha,i+r+\beta)} = c_{r;i} P^2_{(\alpha,\beta)}
$$

(4.23)

for $(\alpha,\beta) = (1,1), (1,0), (0,1), (0,0)$. Notice that the BST prescription (4.16) is symmetric under the exchange of $l_1$ and $l_2$ indices since the original integration measure is given by (4.1). This means that the action of $\tilde{R}_{(i,j)}^{(l_1,l_2)} = \tilde{R}_{(i,j)}^{(l_2,l_1)-1}$ is effectively the same as that of $\tilde{R}_{(i,j)}^{(l_1,l_2)}$ in terms of the resultant dilogarithms.

At the preset, it is not clear how to derive the prescription (4.16). What is interesting here is, however, that the same analytic results as the BST method can be obtained from the $W$-part integrals more directly. In other words, by use of the iterated-integral representation of the polylogarithm functions, we can extract finite physical quantities of one-loop MHV amplitudes more efficiently than the BST method. In this sense, it is reasonable to interpret our prescription (4.16)-(4.20) as a new regularization scheme for one-loop calculation. In what follows, we call this scheme “polylog regularization” for simplicity.

Use of the iterated-integral representation of polylogarithms in (4.12) suggests that the prescription, if generalized to non-MHV amplitudes, would lead to a systematic production of polylogarithms in the computation of one-loop amplitudes. Polylogarithm contributions

\footnote{We have tried to figure out the derivation considerably in preparing the present paper. However, many approaches (including an idea of zero-modes in the holonomy formalism [76]) turn out to be unsuccessful. We expect that some clues will be given by the invariance of $c_{r;i}$ under the transformations (4.22) but can not make a convincing argument yet.}
are generally observed in higher-loop calculations and corresponding reminder functions. Thus we do not think that the polylog regularization is just a computational trick which is applicable to one-loop MHV amplitudes. Bearing in mind that there are no CSW- or BST-based analytic expressions for one-loop non-MHV amplitudes, we find it worth trying how far we can generalize the polylog regularization scheme to the one-loop calculations; this is exactly what we shall carry out in the rest of the present paper.

5 One-loop NMHV amplitudes: formalism

In the following sections, we apply the above formulation to one-loop $N^m$MHV amplitudes ($m = 1, 2, \cdots n - 4$) where the number of negative-helicity gluons is $(m + 2)$, with $n$ being the total number of scattering gluons. We do this in a deductive fashion, focusing on on the case of $m = 1$ in the present section.

The $x$-space representation

As in the MHV amplitudes (3.22), the NMHV amplitudes in the $x$-space representation are generated as

\[
\frac{\delta}{\delta a_{1}^{(+)c_1}} \otimes \cdots \otimes \frac{\delta}{\delta a_{a}^{(-)c_a}} \otimes \cdots \otimes \frac{\delta}{\delta a_{b}^{(-)c_b}} \otimes \cdots \otimes \frac{\delta}{\delta a_{c}^{(-)c_c}} \otimes \cdots \otimes \frac{\delta}{\delta a_{n}^{(+)c_n}} \mathcal{F} \left[ a^{(h)c} \right] \bigg|_{a^{(h)c}=0}
\]

\[
= \mathcal{A}_{NMHV}^{(a\_b\_c\_\ldots\_c)}(x)
\]

\[
= \mathcal{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(x) + \mathcal{A}_{NMHV(1)}^{(a\_b\_c\_\ldots\_c)}(x) + \mathcal{A}_{NMHV(2)}^{(a\_b\_c\_\ldots\_c)}(x) + \cdots.
\]

(5.1)

For tree NMHV amplitudes $\mathcal{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(x)$, there is a single contraction by the operator $\tilde{W}^{(A)}(x)$ in (3.20). Thus, to this order, the amplitudes are expressed as

\[
\mathcal{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(u, \overset{\_}{u}) = \frac{\delta}{\delta a_{1}^{(+)c_1}} \otimes \cdots \otimes \frac{\delta}{\delta a_{a}^{(-)c_a}} \otimes \cdots \otimes \frac{\delta}{\delta a_{b}^{(-)c_b}} \otimes \cdots \otimes \frac{\delta}{\delta a_{c}^{(-)c_c}} \otimes \cdots \otimes \frac{\delta}{\delta a_{n}^{(+)c_n}} \int d\mu(q) \left( \frac{\delta}{\delta a_{I}^{(+)c_I}} \otimes \frac{\delta}{\delta a_{-I}^{(-)c_{-I}}} \right) \mathcal{F} \left[ a^{(h)c} \right] \bigg|_{a^{(h)c}=0}
\]

\[
= \prod_{i=1}^{n} \int d\mu(p_i) \mathcal{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(u, \overset{\_}{u})
\]

(5.2)

\[
\mathcal{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(u, \overset{\_}{u}) = ig n^{-2} (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \tilde{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(u)
\]

(5.3)

\[
\tilde{A}_{NMHV(0)}^{(a\_b\_c\_\ldots\_c)}(u)
\]

\[
= -i \sum_{i=1}^{n} \sum_{r=1}^{n-3} \int d\mu(q_{i+r}) \tilde{A}_{MHV(0)}^{(i_+\_a\_b\_c\_\ldots\_c)(i_+\_a\_b\_c\_\ldots\_c)}(u) \tilde{A}_{MHV(0)}^{(i\_a\_b\_c\_\ldots\_c)(i\_a\_b\_c\_\ldots\_c)}(u) \bigg|_{l=q_{i+r}l}
\]

28
\[
\sum_{i=1}^{n} \sum_{r=1}^{n-3} \int \frac{d^4 q_{i+r}}{(2\pi)^4} \tilde{A}_{\text{MHV}(0)}^{(i, \ldots, a, \ldots, b, \ldots, (i+r)+)}(u) \frac{1}{q_{i+r}^3} \tilde{A}_{\text{MHV}(0)}^{((-l)+ \ldots (i+r)+ \ldots (i-1)+)}(u) \bigg|_{l=q_{i+r}} (5.4)
\]

where the expression (5.4) is identical to the tree-level CSW-results (2.16) in section 2 except that the off-shell integral measure \( \frac{d^4 q_{i+r}}{(2\pi)^4} \) is inserted here. The difference arises simply because we here consider the amplitudes in the coordinate-space representation while in section 2 we have formulated the CSW rules in the momentum-space representation. As discussed in (3.21), the off-shell Nair measure \( d\mu(q_{i+r}) = d\mu(l) + \frac{1}{4\pi} dw^2 \) (with \( q_{i+r} = l + w\eta \)) may be replaced by the on-shell Nair measure \( d\mu(l) \) upon regularizing the log divergence.

Similarly, one-loop NMHV amplitudes \( A_{\text{NMHV}(1)}^{(a,b,c)}(x) \) can be generated as

\[
A_{\text{NMHV}(1)}^{(a,b,c)}(x) = \delta \left( \frac{\delta}{\delta a_1^{(+)}c_1} \otimes \cdots \otimes \frac{\delta}{\delta a_a^{(+)}c_a} \otimes \cdots \otimes \frac{\delta}{\delta a_b^{(-)}c_b} \otimes \cdots \otimes \frac{\delta}{\delta a_c^{(-)}c_c} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(+)}c_n} \right) \\
\times \left[ - \int d\mu(L_1) \frac{\delta}{\delta a_{l_1}^{(+)}c_{l_1}} \otimes \frac{\delta}{\delta a_{l_1}^{(-)}c_{l_1}} e^{-il_1(x-y_1)} \right]_{y_1 \rightarrow x} \\
\times \left[ - \int d\mu(L_2) \frac{\delta}{\delta a_{l_2}^{(+)}c_{l_2}} \otimes \frac{\delta}{\delta a_{l_2}^{(-)}c_{l_2}} e^{-il_2(x-y_2)} \right]_{y_2 \rightarrow x} \\
\times \left[ - \int d\mu(L_3) \frac{\delta}{\delta a_{l_3}^{(+)}c_{l_3}} \otimes \frac{\delta}{\delta a_{l_3}^{(-)}c_{l_3}} e^{-il_3(x-y_3)} \right]_{y_3 \rightarrow x} \\
\times \mathcal{F} \left[ a^{(h)c} \right]_{a^{(h)c}=0} (5.5)
\]

where we make the limit \( y \rightarrow x \) explicit for each of the contractions. Up to the cyclicity of the indices, there are basically two ways of taking these limits:

\text{Type-I: } y_1 \rightarrow y_2, \ y_2 \rightarrow y_1, \ y_3 \rightarrow y_2, \quad (5.6)
\text{Type-II: } y_1 \rightarrow y_2, \ y_2 \rightarrow y_3, \ y_3 \rightarrow y_1 \quad (5.7)

which lead to the one-loop amplitudes.

Figure 3: MHV diagrams, Type-I (left) and Type-II (right), contributing to one-loop NMHV amplitudes — Each of the MHV clusters has more than or equal to two external legs.

\textbf{Type-I case: dilogarithm contributions}
In the type-I case, (5.5) can be calculated as

\[ A_{NMHV}^{(a,b,c)}(u) = \frac{1}{2} \int d\mu(p_1) A_{NMHV}^{(a,b,c)}(u, \bar{u}) \cdot n \int d\mu(p_i) A_{NMHV}^{(a,b,c)}(u, \bar{u}) \cdot \]

The expression in the second line, inside the squared parentheses, is the same as the tree NMHV amplitudes (5.2). Thus, using (5.3) and (5.4), we can explicitly write down the momentum-space amplitudes \( A_{NMHV}^{(a,b,c)}(u, \bar{u}) \) as

\[ A_{NMHV}^{(a,b,c)}(u) = ig^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} P_i \right) A_{NMHV}^{(a,b,c)}(u) \]

\[ \hat{A}_{NMHV}^{(a,b,c)}(u) = \sum_{i=1}^{n} \sum_{r=1}^{n-3} \int d^4q_{i+r} \hat{A}_{MHV}^{(a+b-c-d)(i+r)+l_3+}(u) \]

\[ \times \frac{1}{q_{i+r}} \hat{A}_{MHV}^{(a,b,c)}(u) \]

where the one-loop MHV amplitudes \( \hat{A}_{MHV}^{(i+j+k+l_3+)}(u) \) can be obtained from (3.32) by replacing \( n \) with \( r+1 \). We have an extra internal index \( l_3 \) here. Since we assign a \( U(1) \) color direction to the internal index, this is an auxiliary index in practical calculation. Notice that the internal index has positive helicity, otherwise one of the MHV clusters become no more MHV but NMHV due to the freedom of the choice for the signs of internal null-momenta \( l_1 \) and \( l_2 \). The amplitudes \( \hat{A}_{MHV}^{(a+b-c-d)(i+r)+l_3+}(u) \) can then be written explicitly as

\[ \hat{A}_{MHV}^{(i+j+k+l_3+)}(u) = ig^{i+r} \sum_{j=i}^{i+r-1} \sum_{t=1}^{i+r-t} \left( 1 - \frac{1}{2} \delta_{i,t+1} \right) \sum_{\sigma^{(1)} \in S_{i+1}} \sum_{\sigma^{(2)} \in S_{i-t}} Tr(t^{\sigma^{(1)}_1} \cdots t^{\sigma^{(1)}_{i+r}} \cdots t^{\sigma^{(2)}_i} t^{\sigma^{(2)}_{i+r}} t^{\sigma^{(2)}_{l+1}}) \]

\[ \mathcal{L}_{r+1:1}^{(i,j)} C_{MHV}^{(j+k+l_3+)}(u; \sigma^{(1)} \otimes \sigma^{(2)}) \]

\[ = ig^{i+r} \sum_{j=i}^{i+r-1} \sum_{t=1}^{i+r-t} \left( 1 - \frac{1}{2} \delta_{i,t+1} \right) \left[ Tr(t^{j+1} t^{j+2} \cdots t^{j+r} t^{j+r+t_1} \cdots t^{j+r+t_3}) \mathcal{L}_{r+1:1}^{(i,j)} \right. \]

\[ \times C_{MHV}^{(j+k+l_3+)}(u; \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}) + \mathcal{P}(j \cdots j + t \cdots t + 1 \cdots j + r) \]
The expression (5.11) suggests that the one-loop MHV diagram can be treated as a unit cluster in a broad sense. Namely, the position of the internal index \( l_3 \) in the one-loop MHV diagram can be chosen arbitrarily; only the fact that the one-loop MHV diagram is connected to the other MHV cluster is essential in drawing the MHV diagram of interest; see the type-I diagram in Figure 3. In application of the polylog regularization, \( \mathcal{L}_{r+1:t:j}^{(l_1, l_2)} \) is given by

\[
\mathcal{L}_{r+1:t:j}^{(l_1, l_2)} = \frac{1}{(4\pi)^2} \int \omega_1^{(1)} \omega_2^{(0)} \mathcal{R}_{r+1:t:j}^{(l_1, l_2)}
\]

\[
= \frac{1}{(4\pi)^2} \left[ \text{Li}_2(1 - \Delta_{(j,j+t+1)}) - \text{Li}_2(1 - \Delta_{(j,j+t)}) - \text{Li}_2(1 - \Delta_{(j+r,j+t+1)}) + \text{Li}_2(1 - \Delta_{(j+r,j+t)}) \right]
\]

(5.13)

where \( \Delta \)'s are defined by

\[
\Delta_{(j,j+t+1)} = c_{t:j} (P + p_{j+r} + p_{j+t})^2
\]

\[
\Delta_{(j,j+t)} = c_{t:j} (P + p_{j+r})^2
\]

\[
\Delta_{(j+r,j+t+1)} = c_{t:j} (P + p_{j+t})^2
\]

\[
\Delta_{(j+r,j+t)} = c_{t:j} P^2
\]

(5.14)

with \( P = q_{j+r,t-1} = p_j + p_{j+1} + \cdots + p_{j+t-1} \), and \( c_{t:j} \) being defined as

\[
c_{t:j} = \frac{(p_{j+r} + p_{j+t})^2}{P^2(P + p_{j+r} + p_{j+t})^2 - (P + p_{j+r})^2(P + p_{j+t})^2}.
\]

(5.15)

The color factor of \( \tilde{A}_{N_{\text{MHV}}(1; L_i^2)}^{(a, b, c)}(u) \) becomes single-trace, with the internal indices \( \pm l_3 \) contracted by each other. Explicitly, we can write down the full amplitudes as

\[
\tilde{A}_{N_{\text{MHV}}(1; L_i^2)}^{(a, b, c)}(u)
\]

\[
= ig^2 \sum_{i=1}^{n} \sum_{r=1}^{n-1+i} \sum_{j=1}^{r} \left( \frac{1}{2\delta_{i+1,t+1}} \right) \sum_{\sigma(1) \in S_{i+1}} \sum_{\sigma(2) \in S_{r-1}} \sum_{\sigma(3) \in S_{-r-1}} \text{Tr}(t^{(1)} \cdots t^{(1)} t^{(1)} (\cdots t^{(2)} t^{(2)} t^{(2)} t^{(3)} \cdots) \cdots \cdot)
\]

\[
 \int \frac{d^4q_{i+r+t}}{(2\pi)^4} \mathcal{L}_{r+1:t:j}^{(l_1, l_2)}(u; \sigma^{(1)} \otimes \sigma^{(2)})
\]

\[
\times \frac{1}{q_{i+r+t}^2} \tilde{A}_{N_{\text{MHV}}(0)}^{(l_1, l_2)}(u; \sigma^{(3)}) \bigg|_{l_3 = q_{i+r+t}^2}
\]

(5.16)

where \( \sigma^{(3)} \) is the transposition corresponding to the subamplitude \( \tilde{A}_{N_{\text{MHV}}(0)}^{(l_1, l_2)}(u; \sigma^{(3)}) \) in (5.10):

\[
\sigma^{(3)} = \left( \begin{array}{ccc}
  i + r + 1 & \cdots & i - 1 \\
  \sigma^{(3)}_{i+r+1} & \cdots & \sigma^{(3)}_{i-1}
\end{array} \right).
\]

(5.17)

Since the amplitudes \( \tilde{A}_{N_{\text{MHV}}(1; L_i^2)}^{(a, b, c)}(u) \) include a single-line propagator, as in the case of tree NMHV amplitudes, these are determined up to the choice of reference spinors. This fact
is reflected in the \( \pm l_3 \)-dependence in the expression (5.16). Notice that \( \pm l_3 \) enter only in \( \hat{C}_{\text{MHV}(0)} \)'s. Thus the sign of \( l_3 \) is irrelevant in the final results as expected from our functional derivation.

**Type-II case: trilogarithm contributions**

In the type-II case (5.7), on the other hand, we can no more utilize the results of the previous sections. This is because the three contraction operators in (5.5) are now treated equally so that we need to consider the three as one set of operators. The situation can readily be perceived from the MHV diagrams in Figure 3 contributing to one-loop NMHV amplitudes.

In the type-II case, the functional derivatives in (5.5) can be calculated as

\[
\mathcal{A}_{\text{NMHV}(1; Li_3)}^{(a, b, c,-)}(x) = \left[ -\int d\mu(L_1) \left( \frac{\delta}{\delta a_{l_1}^{(+)}(+)} \otimes \frac{\delta}{\delta a_{l_1}^{(-)}} \right) \right] \left[ -\int d\mu(L_2) \left( \frac{\delta}{\delta a_{l_2}^{(+)}(+)} \otimes \frac{\delta}{\delta a_{l_2}^{(-)}} \right) \right] \\
\times \left[ -\int d\mu(L_3) \left( \frac{\delta}{\delta a_{l_3}^{(+)}(+)} \otimes \frac{\delta}{\delta a_{l_3}^{(-)}} \right) \right] \frac{\delta}{\delta a_n^{(+)}c_1} \ldots \frac{\delta}{\delta a_n^{(+)}c_n} \mathcal{F} [a^{(h)c}]_{i_q^{(h)c}=0}.
\]

where we label the numbering indices of the type-II MHV diagram as shown in Figure 4.

\[
\hat{A}^{(a, b, c,-)}_{\text{NMHV}(1; Li_3)}(u, \bar{u}) = ig^{n-2} (2\pi)^4 \delta(4) \left( \sum_{i=1}^{n} p_i \right) A^{(a, b, c,-)}_{\text{NMHV}(1; Li_3)}(u)
\]

Figure 4: The type-II MHV diagram that contributes to one-loop NMHV amplitudes

The amplitudes of interest, \( \hat{A}^{(a, b, c,-)}_{\text{NMHV}(1; Li_3)}(u, \bar{u}) \) in (5.18), are then straightforwardly calculated as

\[
\mathcal{A}_{\text{NMHV}(1; Li_3)}^{(a, b, c,-)}(u, \bar{u}) = g^2 \sum_{i=1}^{n} \sum_{r_1=1}^{[\frac{n}{3}]-1} \sum_{r_2=1}^{[\frac{n}{3}]-1} \left( 1 - \frac{2}{3} \delta_{r_1+1}^{a_i} \delta_{r_2+1}^{a_j} \right) \int d\mu(L_1)d\mu(L_2)d\mu(L_3) \hat{A}_{\text{MHV}(0)}^{((-l_1) \ldots (i+r_1+1) \ldots (i+r_2+1) \ldots l_3)}(u) \hat{A}_{\text{MHV}(0)}^{((-l_3) \ldots (i+r_1+2) \ldots c_i \ldots (i-1) \ldots l_1)}(u)
\]

\[
\hat{A}_{\text{MHV}(0)}^{((-l_1) \ldots (i+r_1+1) \ldots b_i \ldots (i+r_1+2) \ldots l_3)}(u)
\]

(5.20)
where the factor \( 1 - \frac{2}{3} \delta^\pm_{r_1+1} \delta^\pm_{r_2+1} \) arises to compensate the redundant counting when the type-II MHV diagram preserves a cyclic symmetry. The cyclic symmetry is an analog of the reflection symmetry considered in the one-loop MHV diagram. As in the MHV case, this factor is related to a factor of \( \frac{1}{3!} \) coming from the Taylor expansion of the MHV S-matrix functional (2.34) under the cyclic symmetry. In the above expression, however, the calculation is made in a rotational fashion in terms of the numbering indices. Thus the “reflection” part of the factor \( \frac{1}{3!} \), i.e., the factor of \( \frac{1}{2} \), is irrelevant and only the factor of \( \frac{1}{3} \) survives to take care of the redundancy in (5.20). The negative-helicity indices \((a, b, c)\) satisfy the ordering

\[
i \leq a \leq i + r_1 < b \leq i + r_1 + r_2 + 1 < c \pmod{n}.
\]  

(5.21)

This choice is the same as the tree NMHV amplitudes in (2.8), with an identification of \( j = i + r_1 + r_2 + 1 \). In the expression (5.20), we specify the positions and the signs of \((\pm l_1, \pm l_2, \pm l_3)\) in accord with the arrows in Figure 4. This specification, however, is superficial because, as in the one-loop MHV amplitudes, these positions can arbitrarily be chosen (thanks to the functional derivatives) and the holomorphic part of the amplitudes (in terms of the spinor momenta) is independent of the signs of these indices. This feature is already discussed in (3.29)-(3.31) and is used to show that the one-loop MHV amplitudes are proportional to the color-stripped tree MHV amplitudes; see (3.32). Analogously, in the present case, we expect that the type-II one-loop NMHV amplitudes are proportional to some holomorphic tree amplitudes which are independent of the internal indices \((l_1, l_2, l_3)\). This can be carried out as follows.

For simplicity, we first denote the key indices as

\[
a_1 = i, \quad a_2 = i + r_1, \\
b_1 = i + r_1 + 1, \quad b_2 = i + r_1 + r_2 + 1, \\
c_1 = i + r_1 + r_2 + 2, \quad c_2 = i - 1.
\]

(5.22)

The tree-part of the amplitude in (5.20) is then expressed as

\[
\hat{A}_{\text{MHV}(0)}^{((l_1), \ldots, l_2+1)}(u) \hat{A}_{\text{MHV}(0)}^{((l_2), \ldots, l_2+3)}(u) \hat{A}_{\text{MHV}(0)}^{((l_3), \ldots, c_2+1)}(u) \\
= \sum_{\sigma(1) \in \mathcal{S}_{r_1+1}} \sum_{\sigma(2) \in \mathcal{S}_{r_2+1}} \sum_{\sigma(3) \in \mathcal{S}_{r_3+1}} \text{Tr}(t^{a_1} t^{b_1} t^{c_1} \ldots t^{a_2} t^{b_2} t^{c_2}) \\
\times \hat{C}_{\text{MHV}(0)}^{((l_1), \ldots, l_2+1)}(u; \sigma(1)) \hat{C}_{\text{MHV}(0)}^{((l_2), \ldots, l_2+3)}(u; \sigma(2)) \hat{C}_{\text{MHV}(0)}^{((l_3), \ldots, c_2+1)}(u; \sigma(3)) \\
= \text{Tr}(t^{a_1} t^{a_2} t^{b_1} \ldots t^{b_2} t^{c_1} \ldots t^{c_2}) \hat{C}_{\text{MHV}(0)}^{((l_1), \ldots, l_2+1)}(u; 1(1)) \\
\times \hat{C}_{\text{MHV}(0)}^{((l_2), \ldots, l_2+3)}(u; 1(2)) \hat{C}_{\text{MHV}(0)}^{((l_3), \ldots, c_2+1)}(u; 1(3)) \\
+ \mathcal{P}(a_1 \ldots a_2 b_1 \ldots b_2 | c_1 \ldots c_2)
\]

(5.23)

where we introduce an auxiliary parameter \( r_3 = n - (r_1 + r_2 + 3) \), \( \mathcal{P}(a_1 \ldots a_2 b_1 \ldots b_2 | c_1 \ldots c_2) \) denotes the terms obtained by the triple permutations of \( \sigma(1), \sigma(2) \) and \( \sigma(3) \). The above expressions is the type-II NMHV version of the MHV expression (3.27). Using the relation
From the definition (3.30), we can readily write down (3.29), we then find the following relation:

\[
\mathcal{C}^{((-l_1)\ldots a_1 \ldots a_2)_{l_2} (u; 1^{(1)}) \mathcal{C}^{((-l_2)\ldots b_1 \ldots b_{2+} l_3)_{l_3} (u; 1^{(2)})} = \frac{\mathcal{C}^{((-l_1)\ldots a_1 \ldots a_2)_{l_2} (u; 1^{(1)}) \mathcal{C}^{((-l_2)\ldots b_1 \ldots b_{2+} l_3)_{l_3} (u; 1^{(2)}) (b_2 l_3)(l_3 (-l_2))}{(b_2 l_3)(l_3 (-l_2))} \]

(5.24)

where the total number of indices involved in \( \mathcal{R}_{r_1+r_2+2:r_1:a_1} \) is now \((r_1 + r_2 + 2)\). In terms of (5.22), it is written as

\[
\mathcal{R}_{r_1+r_2+2:r_1:a_1}^{(l_1,l_2)} = \frac{(a_2 b_1)(l_1 l_2)(b_2 a_1)(l_2 l_1)}{(a_2 l_2)(-l_1 a_1)(b_2 l_1)(-l_2 b_1)} \frac{(a_1 b_2)(b_1 a_2)(l_1 l_2)^2}{(l_1 a_1)(l_1 b_2)(l_2 a_2)(l_2 b_1)}. \]

(5.25)

Similarly, we find

\[
\mathcal{R}_{r_2+r_3+2:r_2:b_1}^{(l_2,l_3)} \mathcal{R}_{r_3+r_1+2:r_3:c_1}^{(l_3,l_1)}
\]

(5.26)

From the definition (3.30), we can readily write down \( \mathcal{R}_{r_2+r_3+2:r_2:b_1}^{(l_2,l_3)} \) and \( \mathcal{R}_{r_3+r_1+2:r_3:c_1}^{(l_3,l_1)} \) as

\[
\mathcal{R}_{r_2+r_3+2:r_2:b_1}^{(l_2,l_3)} = \frac{(b_1 c_2)(c_1 b_2)(l_2 l_3)^2}{(l_2 b_1)(l_2 c_2)(l_3 b_2)(l_3 c_1)}, \]

(5.28)

\[
\mathcal{R}_{r_3+r_1+2:r_3:c_1}^{(l_3,l_1)} = \frac{(c_1 a_2)(a_1 c_2)(l_3 l_1)^2}{(l_3 c_1)(l_3 a_2)(l_1 c_2)(l_1 a_1)}.
\]

(5.29)

(5.29)

From these relations, we can express the square of the holomorphic factor in (5.23) as

\[
\left( \mathcal{C}^{((-l_1)\ldots a_1 \ldots a_2)_{l_2} (u; 1^{(1)}) \mathcal{C}^{((-l_2)\ldots b_1 \ldots b_{2+} l_3)_{l_3} (u; 1^{(2)})} \mathcal{C}^{((-l_3)\ldots c_1 \ldots c_2)_{l_4} (u; 1^{(3)})} \right)^2
\]

(5.30)

where \( \mathcal{R}_{(a_2,b_2,c_2)}^{(l_1,l_2,l_3)} \) is defined by

\[
\mathcal{R}_{(a_2,b_2,c_2)}^{(l_1,l_2,l_3)} = \frac{(l_3 b_2)(l_1 c_2)(l_2 a_2)}{(l_1 b_2)(l_2 c_2)(l_3 a_2)}. \]

(5.31)

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From (5.20), (5.23) and (5.30), we find that the amplitudes of interest can be expressed as

\[\hat{A}_{N\text{MHV}(1;L_i)}(u)\]

\[= i g^2 \sum_{n} \sum_{n=1}^{2} \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} \left(1 - \frac{2}{3} \delta_{+1} \delta_{+2} + \frac{1}{3} \delta_{+3} \right) \int d\mu(L_1)d\mu(L_2)d\mu(L_3)\]

\[\text{Tr}(t^{a_1} t^{a_2} t^{b_1} t^{b_2} \cdots t^{c_2}) \sqrt{-R_{1+1} R_{2+2} R_{3+3}} \times \sqrt{C_{\text{MHV}(0)}(u; \mathbf{1}_{(1\otimes 2)}) \hat{C}_{\text{MHV}(0)}(u; \mathbf{1}_{(2\otimes 3)}) \hat{C}_{\text{MHV}(0)}(u; \mathbf{1}_{(3\otimes 1)})} + \mathcal{P}(a_1 \cdots a_2 | b_1 \cdots b_2 | c_1 \cdots c_2)\]  

(5.32)

where we abbreviate \(\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}\) by \(\mathbf{1}^{(1\otimes 2)}\), and so on. Precisely speaking, there exists a sign ambiguity in defining the above amplitudes which arises from taking the square root of the quantity (5.30). This ambiguity always exists in the holonomy formalism as discussed below (2.36). The physical quantities are, however, independent of this ambiguity so that, in this sense, we can fix the sign of (5.32) uniquely. The indices \(a_1, a_2, b_1, b_2, c_1, c_2\) depend on the summing indices \(i, r_1\) and \(r_2\) as specified in (5.22). The holomorphic part of the amplitudes are given by the square root in the forth line of (5.32). Notice that this factor is in accord with the conventional relation (2.25) between the helicity configurations and the degrees of homogeneity in spinor momenta. The square root in the third line can explicitly be calculated as

\[\sqrt{-R_{1+1} R_{2+2} R_{3+3}} \times \sqrt{C_{\text{MHV}(0)}(u; \mathbf{1}_{(1\otimes 2)}) \hat{C}_{\text{MHV}(0)}(u; \mathbf{1}_{(2\otimes 3)}) \hat{C}_{\text{MHV}(0)}(u; \mathbf{1}_{(3\otimes 1)})} + \mathcal{P}(a_1 \cdots a_2 | b_1 \cdots b_2 | c_1 \cdots c_2)\]  

(5.33)

We now simplify the expression (5.33), using the single reference-spinor principle in the CSW rules. Namely, we shall impose the condition

\[\eta_1 = \eta_2 = \eta_3.\]  

(5.34)

In Figure 4, this implies that at least one of the external legs for each MHV cluster is collinear to one another. For example, we can impose the collinearity

\[u_{a_1} \parallel u_{b_1} \parallel u_{c_1}.\]  

(5.35)

Notice that this condition is a natural consequence of the CSW generalization rather than a convenient assumption. By use of the Schouten identities, we can then simplify (5.33) as

\[\sqrt{-R_{1+1} R_{2+2} R_{3+3}} \times \sqrt{C_{\text{MHV}(0)}(u; \mathbf{1}_{(1\otimes 2)}) \hat{C}_{\text{MHV}(0)}(u; \mathbf{1}_{(2\otimes 3)}) \hat{C}_{\text{MHV}(0)}(u; \mathbf{1}_{(3\otimes 1)})} + \mathcal{P}(a_1 \cdots a_2 | b_1 \cdots b_2 | c_1 \cdots c_2)\]  

(5.36)
where we make the condition (5.35) explicit by the abbreviated notation \( a_1 \parallel b_1 \parallel c_1 \). Note that \( \hat{R} \)’s are defined as before, e.g.,

\[
\hat{R}_{(a_1, a_2)} = \frac{(l_2 a_1)(l_1 a_2)}{(l_1 a_1)(l_2 a_2)}. \tag{5.37}
\]

Under the collinearity condition (5.35), the indices \( a_1, b_1 \) and \( c_1 \) can arbitrarily be chosen in the suffix of \( \hat{R} \)’s. In the above expression, we make such choices that are convenient for later purposes.

In the present type-II limit (5.7), we can further impose the cyclicity of the \((l_1, l_2, l_3)\)-indices. Since these indices enter only in the factor of (5.36), this condition can be realized by rewriting (5.36) as

\[
\mathcal{R}_{n: r_1, r_2; i}^{(l_1, l_2, l_3)} \equiv \sqrt{-\mathcal{R}_{r_1, a_1}^{(l_1, l_2, l_3)} \mathcal{R}_{r_2, b_1}^{(l_2, l_3, l_1)} \mathcal{R}_{r_3, c_1}^{(l_3, l_1, l_2)} \mathcal{R}_{(a_2, b_2, c_2)}^{(l_1, l_2, l_3)}}_{a_1 \parallel b_1 \parallel c_1, \text{cycl}}
\]

\[
= -\hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \hat{R}_{(c_2, c_1)}^{(l_2, l_3)} \hat{R}_{(a_2, a_1)}^{(l_3, l_1)} + \hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \hat{R}_{(c_2, c_1)}^{(l_2, l_3)} \hat{R}_{(a_2, a_1)}^{(l_3, l_1)} - \hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \hat{R}_{(c_2, c_1)}^{(l_2, l_3)} \hat{R}_{(a_2, a_1)}^{(l_3, l_1)}
\]

\[
+ \hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \hat{R}_{(c_2, c_1)}^{(l_2, l_3)} \hat{R}_{(a_2, a_1)}^{(l_3, l_1)} - \hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \hat{R}_{(c_2, c_1)}^{(l_2, l_3)} \hat{R}_{(a_2, a_1)}^{(l_3, l_1)}
\] \tag{5.38}

where we make \( \hat{R} \)’s in appropriate forms such that we can relate them to those that appear in (3.37) with the cyclicity of indices.

**Polylog regularization: emergence of trilogarithms**

Applying the results in section 4, we can then express the integral in (5.32) as an iterated integral:

\[
\mathcal{L}_{n: r_1, r_2; i}^{(l_1, l_2, l_3)} \equiv \int d\mu(L_1)d\mu(L_2)d\mu(L_3) \mathcal{R}_{n: r_1, r_2; i}^{(l_1, l_2, l_3)} \equiv \frac{1}{(4\pi)^3} \int \omega_1^{(1)} \omega_2^{(0)} \omega_3^{(0)} \mathcal{R}_{r_1, r_2; i}^{(l_1, l_2, l_3)} \tag{5.39}
\]

where \( \omega_1^{(1)} \) and \( \omega_k^{(0)} \) \((k = 2, 3)\) are defined by (4.3) and (4.13), respectively. As mentioned below (4.23), the action of \( \hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \) leads to the same result as that of \( \hat{R}_{(b_2, b_1)}^{(l_1, l_2)} \) in implementing the BST prescription (4.16). Thus, applying the BST prescription to the integral (5.39), with the knowledge of the iterated-integral representation of polylogarithms (4.12), we find that the following NMHV analog of the BST prescription:

\[
\int \omega_1^{(1)} \omega_2^{(0)} \omega_3^{(0)} \mathcal{R}_{r_1, r_2; i}^{(l_1, l_2, l_3)} \rightarrow \text{Li}_3 \left( 1 - \Delta_{(b_2, a_2)}^{(1)} - \Delta_{(c_1, c_2)}^{(3)} \right) - \text{Li}_3 \left( 1 - \Delta_{(b_2, a_2)}^{(1)} - \Delta_{(c_2, c_1)}^{(3)} \right)
\]

\[
+ \text{Li}_3 \left( 1 - \Delta_{(c_2, b_2)}^{(2)} - \Delta_{(a_1, a_2)}^{(1)} \right) - \text{Li}_3 \left( 1 - \Delta_{(c_2, b_2)}^{(2)} - \Delta_{(a_2, a_1)}^{(3)} \right)
\]

\[
+ \text{Li}_3 \left( 1 - \Delta_{(a_2, c_2)}^{(3)} - \Delta_{(b_1, b_2)}^{(2)} \right) - \text{Li}_3 \left( 1 - \Delta_{(a_2, c_2)}^{(3)} - \Delta_{(b_2, b_1)}^{(1)} \right) \tag{5.40}
\]
where $\Delta^{(1)}$, $\Delta^{(2)}$'s and $\Delta^{(3)}$'s are defined as follows.

\[
\begin{align*}
\Delta^{(1)}_{(a_1, b_1)} &= c^{(1)}_{r_1:a_1} (P(a) + p_{b_2} + p_{a_2})^2 \\
\Delta^{(1)}_{(a_1, a_2)} &= c^{(1)}_{r_1:a_2} (P(a) + p_{b_2})^2 \\
\Delta^{(1)}_{(b_2, b_1)} &= c^{(1)}_{r_1:b_1} (P(a) + p_{a_2})^2 \\
\Delta^{(1)}_{(b_2, a_2)} &= c^{(1)}_{r_1:a_2} P(a)^2 \\
\Delta^{(3)}_{(c_1, a_1)} &= c^{(3)}_{r_3:c_1} (P(c) + p_{a_2} + p_{c_2})^2 \\
\Delta^{(3)}_{(c_1, c_2)} &= c^{(3)}_{r_3:c_2} (P(c) + p_{a_2})^2 \\
\Delta^{(3)}_{(a_2, a_1)} &= c^{(3)}_{r_3:a_1} (P(c) + p_{c_2})^2 \\
\Delta^{(3)}_{(a_2, c_2)} &= c^{(3)}_{r_3:c_2} P(c)^2 \\
\end{align*}
\]

\[
\begin{align*}
\frac{c^{(1)}_{r_1:a_1}}{P(a)^2} &= \frac{(p_{b_2} + p_{a_2})^2}{(P(a) + p_{b_2} + p_{a_2})^2 - (P(a) + p_{b_2})^2 (P(a) + p_{a_2})^2} \\
\frac{c^{(2)}_{r_2:b_1}}{P(b)^2} &= \frac{(p_{c_2} + p_{b_2})^2}{(P(b) + p_{c_2} + p_{b_2})^2 - (P(b) + p_{c_2})^2 (P(b) + p_{b_2})^2} \\
\frac{c^{(3)}_{r_3:c_1}}{P(c)^2} &= \frac{(p_{a_2} + p_{c_2})^2}{(P(c) + p_{a_2} + p_{c_2})^2 - (P(c) + p_{a_2})^2 (P(c) + p_{c_2})^2} \\
\end{align*}
\]

Here we include $\Delta^{(1)}_{(a_1, b_1)}$, $\Delta^{(2)}_{(b_1, c_1)}$ and $\Delta^{(3)}_{(c_1, a_1)}$, which do not appear in (5.40), simply for the completion of the arguments.

There is one caveat about an ambiguity of the expression (5.38). Namely, the second term in (5.38), $R_{(b_2, b_1)}^{(l_2, l_3)} R_{(c_2, c_1)}^{(l_2, l_3)} R_{(a_2, a_1)}^{(l_1, l_1)}$, can alternatively be written as $R_{(b_1, b_2)}^{(l_1, l_3)} R_{(c_1, c_2)}^{(l_1, l_1)} R_{(a_1, a_2)}^{(l_1, l_2)}$, using the cyclicity of the indices. A naive application of the NMHV prescription (5.40) leads to different results, $\Delta^{(1)}_{(b_2, b_1)} + \Delta^{(2)}_{(c_2, c_1)} + \Delta^{(3)}_{(a_2, a_1)}$ and $\Delta^{(1)}_{(a_1, a_2)} + \Delta^{(2)}_{(b_1, b_2)} + \Delta^{(3)}_{(c_1, c_2)}$ in terms of the arguments of trigonometric functions. Apparently, this causes discrepancy in our prescription but, as far as the amplitudes (5.32) are concerned, the two expressions lead to the same result thanks to the split-permutations of the numbering indices. To be more specific, in (5.32) we can change the ordering of the indices from $(a_1 \ldots a_2 b_1 \ldots b_2 c_1 \ldots c_2)$ to $(a_2 a_1 \ldots (a_2 - 1) b_2 b_1 \ldots (b_2 - 1) c_2 c_1 \ldots (c_2 - 1))$ thanks to the sum over split-permutations represented by $P(a_1 \ldots a_2 | b_1 \ldots b_2 | c_1 \ldots c_2)$. In the MHV diagram, this corresponds to the replacements of $a_2$ with $b_2$, $b_2$ with $c_2$, and $c_2$ with $a_2$. This means that we have transformations $P(a) + p_{a_2} \rightarrow P(a) + p_{b_2}$, $P(b) + p_{b_2} \rightarrow P(b) + p_{c_2}$, and $P(c) + p_{c_2} \rightarrow P(c) + p_{a_2}$, respectively. As mentioned in (4.22), $c^{(1)}_{r_1:a_1}$, $c^{(2)}_{r_2:b_1}$ and $c^{(3)}_{r_3:c_1}$ are invariant under these transformations. Thus, under the above change of index orderings we can replace $\Delta^{(1)}_{(b_2, b_1)} + \Delta^{(2)}_{(c_2, c_1)} + \Delta^{(3)}_{(a_2, a_1)}$ by $\Delta^{(1)}_{(a_1, a_2)} + \Delta^{(2)}_{(b_1, b_2)} + \Delta^{(3)}_{(c_1, c_2)}$ so that the resultant amplitudes are identical regardless the choice of the expression $R_{(b_2, b_1)}^{(l_2, l_3)} R_{(c_2, c_1)}^{(l_2, l_3)} R_{(a_2, a_1)}^{(l_1, l_1)}$ or $R_{(b_1, b_2)}^{(l_1, l_3)} R_{(c_1, c_2)}^{(l_1, l_1)} R_{(a_1, a_2)}^{(l_1, l_2)}$ in (5.38). Eventually, however, this term cancels out with the first term in (5.38). Thus the above argument on the solution of ambiguity is in fact digressive.

The NMHV prescription (5.40) is not derived rigorously but is obtained somewhat in a conjectural fashion by use of the BST prescription (4.16) which is, on the other hand,
confirmed analytically thanks to the BST method. In obtaining the above expressions, we therefore keep the structure of the BST prescription as much as possible, using the proper cyclicity and the single reference-spinor principle of the CSW rules. One may wonder why the arguments of $\text{Li}_3$’s are additive rather than multiplicative. This is a reasonable question but can be answered in a few interrelated ways. First, the MHV clusters that involve a one-loop diagram have always two and only two internal lines. This suggests that the deformation of the integral path can be analyzed by an iterative use of the BST prescription. Secondly, the original path for the iterated integral can be defined by a real line segment $[0, 1]$ regardless the number of $w_i$’s. Thus the way that the integral paths deform is independent of the number of MHV clusters in a one-loop diagram. In other words, the BST prescription can be applied, in an iterative fashion, to the type-II one-loop NMHV amplitudes as long as the deformations of integral paths are concerned. Lastly, consider for example the cases of $r_1 = 1$, $r_2 > 1$ and $r_3 > 1$ where $\Delta^{(1)}_{(b_3, a_2)} = 0$ but $\Delta^{(2)}$’s and $\Delta^{(3)}$’s are not zero. If the arguments of $\text{Li}_3$’s were to be multiplicative, we would have trivial arguments (identities) for such cases regardless the choice of $r_2$ and $r_3$. This is not consistent with the above picture of iterative contributions by the MHV clusters. Thus, although we have only circumstantial evidences, it is natural to require the arguments of $\text{Li}_3$’s to be additive.

Summary

To recapitulate the results of this section, we now write down the complete form of the one-loop NMHV amplitudes in the momentum-space representation:

$$A^{(a,b,c,d)}_{\text{NMHV}(1)}(u, \bar{u}) = i g^{n-2} (2\pi)^4 \delta^{(4)}(\sum_{i=1}^{n} p_i) \tilde{A}^{(a,b,c,d)}_{\text{NMHV}(1)}(u),$$

$$\tilde{A}^{(a,b,c,d)}_{\text{NMHV}(1)}(u) = \tilde{A}^{(a,b,c,d)}_{\text{NMHV}(1;\text{Li}_2)}(u) + \tilde{A}^{(a,b,c,d)}_{\text{NMHV}(1;\text{Li}_3)}(u).$$

In the momentum-space, the type-I and type-II subamplitudes are written as

$$\tilde{A}^{(a,b,c,d)}_{\text{NMHV}(1;\text{Li}_2)}(u)$$

$$= i g^2 \sum_{i=1}^{n} \sum_{r=1}^{n-3} \sum_{j=i}^{r} \sum_{t=1}^{r+1} \left(1 - \frac{1}{2} \delta_{t+1, t+1} \right) \frac{1}{q_{i+r}^2} \text{Tr}(t^{(1)} \ldots t^{(1)} \ldots t^{(2)} \ldots t^{(2)} \ldots t^{(3)} \ldots t^{(3)}) L^{(l_1, l_2)}_{r+1; t; \sigma_f}$$

$$\tilde{C}^{(a,b,c,d)}_{\text{NMHV}(0)}(u; \sigma^{(1)\otimes 2}) \left|q_{i+r}^2\right| \tilde{C}^{(a,b,c,d)}_{\text{NMHV}(0)}(u; \sigma^{(3)}) \right|_{l_3 = q_{i+r}^2}$$

where $\sigma^{(1)}$, $\sigma^{(2)}$, and $\sigma^{(3)}$ are defined in (5.12) and (5.17) and $L^{(l_1, l_2)}_{r+1; t; \sigma_f}$ can be defined through (5.13)-(5.15).

$$\tilde{A}^{(a,b,c,d)}_{\text{NMHV}(1;\text{Li}_3)}(u).$$

Notice that we can also derive the BST result (3.37) in a squared form as in (5.38), using the fact that the identity can be replaced by $\tilde{R}^{(l_1, l_2)}_{(a_1, b_1)}$ under the condition of $u_{a_1} \parallel u_{b_1}$.
\[\begin{align*}
&= i g^2 \sum_{i=1}^{n} \sum_{r_1=1}^{\frac{n}{2}} \sum_{r_2=1}^{\frac{n}{2}} \left( 1 - \frac{2}{3} \delta_{i,r_1+1} \delta_{i,r_2+1} \right) \\
&\quad \sum_{\sigma(1)\in S_{r_1+1}} \sum_{\sigma(2)\in S_{r_2+1}} \sum_{\sigma(3)\in S_{r_3+1}} \text{Tr}(t^{\sigma_{a_1}} \ldots t^{\sigma_{a_2}} t^{\sigma_{b_1}} \ldots t^{\sigma_{b_2}} t^{\sigma_{c_1}} \ldots t^{\sigma_{c_2}}) L_{n;r_1,r_2;\sigma_i}^l \end{align*}\]

where the indices \(\{a_1, a_2, b_1, b_2, c_1, c_2\}\) are defined by (5.22) and \(L_{n;r_1,r_2;\sigma_i}^l\) can be defined through (5.39)-(5.43).

As discussed in the previous section, the type-I or dilogarithm contributions are a direct consequence of the BST results for one-loop MHV amplitudes so that the validity of the expression (5.46) is well confirmed, while the type-II or trilogarithm contributions in (5.47) are obtained by use of the somewhat conjectural polylog regularization. The latter contributions, however, provide what we consider the most natural generalization of the BST representation to one-loop NMHV amplitudes.

6 One-loop NMHV amplitudes: examples

As a simple example of our formulation, we calculate the six-point one-loop NMHV amplitudes in this section. The MHV diagrams contributing to the six-point one-loop NMHV amplitudes are shown in Figure 5.

Figure 5: The type-I and type-II MHV diagrams that contribute to the six-point one-loop NMHV amplitude

As discussed earlier, there are three distinct helicity configurations \((1_2^{+}3_4^{+}5_6^{+}), (1_2^{+}3_4^{-}5_6^{+})\) and \((1_2^{+}3_4^{+}5_6^{-})\) for six-point amplitudes. The corresponding tree amplitudes are obtained in (2.41)-(2.43). Applying the formulae of the previous section, we can then easily write down the amplitudes of interest in the momentum-space representation. The results are listed below.
Six-point type-I amplitudes

The type-I part of the \((1\ 2\ 3\ 4\ 5\ 6)\) amplitudes can be expanded as

\[
\hat{A}_{N_{MHV}(1;L_2)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u)
= \hat{A}_{MHV(1)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u) \frac{1}{q_{24}^2} \hat{A}_{MHV(0)}^{((-l_2)\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) + \hat{A}_{MHV(1)}^{(5\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) \frac{1}{q_{52}^2} \hat{A}_{MHV(0)}^{((-l_2)\ 3\ 4\ 5\ 6)}(u)
\]

where

\[
\hat{A}_{MHV(1)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u) = \frac{ig^2}{(4\pi)^2} \left\{ \text{Tr}(t^{2\ 4\ 5} t^{3\ 6}) \right\} \left[ \text{Li}_2 \left( 1 + \frac{(p_5 \cdot p_3)}{(p_2 \cdot p_3)} + \frac{(p_5 \cdot p_3)}{(p_2 \cdot p_5)} + \frac{(p_5 \cdot p_3)^2}{(p_2 \cdot p_5)(p_2 \cdot p_5)} \right)
- \text{Li}_2 \left( 1 + \frac{(p_5 \cdot p_3)}{(p_2 \cdot p_5)} \right)
- \text{Li}_2 \left( 1 + \frac{(p_5 \cdot p_3)}{(p_2 \cdot p_5)} \right) + \frac{\pi^2}{6} \right] C_{MHV(0)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u; 1) + \mathcal{P}(23|45) \right) \right.+ \left( \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \end{array} \right) \leftrightarrow \left( \begin{array}{c} 3 \\ 4 \\ 5 \\ 2 \end{array} \right),
\]

\[
\hat{A}_{MHV(1)}^{(5\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) = \frac{ig^2}{(4\pi)^2} \left\{ \text{Tr}(t^{5\ 6\ 1\ 2\ 3\ 4\ 5\ 6}) \right\} \left[ \text{Li}_2 \left( 1 + \frac{(p_2 \cdot p_6)}{(p_5 \cdot p_6)} + \frac{(p_2 \cdot p_6)}{(p_5 \cdot p_2)} + \frac{(p_2 \cdot p_6)^2}{(p_5 \cdot p_2)(p_5 \cdot p_6)} \right)
- \text{Li}_2 \left( 1 + \frac{(p_2 \cdot p_6)}{(p_5 \cdot p_2)} \right)
- \text{Li}_2 \left( 1 + \frac{(p_2 \cdot p_6)}{(p_5 \cdot p_2)} \right) + \frac{\pi^2}{6} \right] C_{MHV(0)}^{(5\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u; 1) + \mathcal{P}(56|12) \right) \right.+ \left( \begin{array}{c} 5 \\ 6 \\ 1 \\ 2 \end{array} \right) \leftrightarrow \left( \begin{array}{c} 6 \\ 1 \\ 2 \\ 5 \end{array} \right).
\]

The tree subamplitudes in (6.1) can be written as

\[
\hat{A}_{N_{MHV}(1;L_2)}^{((-l_2)\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) = \text{Tr}(t^{(-l_2)\ 6\ 1\ 2\ 3\ 4\ 5\ 6}) \hat{C}_{MHV(0)}^{((-l_2)\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u; 1) + \mathcal{P}(61),
\]

\[
\hat{A}_{N_{MHV}(1;L_2)}^{((-l_2)\ 3\ 4\ 5\ 6)}(u) = \text{Tr}(t^{(-l_2)\ 3\ 4\ 5\ 6}) \hat{C}_{MHV(0)}^{((-l_2)\ 3\ 4\ 5\ 6)}(u; 1) + \mathcal{P}(34).
\]

Thus, upon the contraction of color indices, the amplitudes \(\hat{A}_{N_{MHV}(1;L_2)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u)\) become single-trace amplitudes.

Similarly, for the helicity configurations \((1\ 2\ 3\ 4\ 5\ 6)\) and \((1\ 2\ 3\ 4\ 5\ 6)\) the type-I amplitudes are given by

\[
\hat{A}_{N_{MHV}(1;L_2)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u)
= \hat{A}_{MHV(1)}^{(1\ 2\ 3\ 4\ 5\ 6)}(u) \frac{1}{q_{14}^2} \hat{A}_{MHV(0)}^{((-l_3)\ 5\ 6)}(u) + \hat{A}_{MHV(1)}^{(6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) \frac{1}{q_{63}^2} \hat{A}_{MHV(0)}^{((-l_3)\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u)
+ \hat{A}_{MHV(1)}^{(5\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) \frac{1}{q_{52}^2} \hat{A}_{MHV(0)}^{((-l_2)\ 3\ 4\ 5\ 6)}(u) + \hat{A}_{MHV(1)}^{(4\ 5\ 6\ 1\ 2\ 3\ 4\ 5\ 6)}(u) \frac{1}{q_{41}^2} \hat{A}_{MHV(0)}^{((-l_3)\ 2\ 3\ 4\ 5\ 6)}(u).
\]
follows.

Therefore the type-II part of the (1

\[ A_{\text{MHV}(1)}^{(3,-4,5,-6,13+)}(u) \frac{1}{q_{36}^2} A_{\text{MHV}(0)}^{(-l_3)-1-2+}(u) + A_{\text{MHV}(1)}^{(2,3,-4,5,-13+)}(u) \frac{1}{q_{25}^2} A_{\text{MHV}(0)}^{(-l_3)-6+1-}(u), \] 

(6.5)

\[ A_{\text{MHV}(1)}^{(1,-2,3,4,-5,6+)}(u) \]

\[ = \frac{i g^2}{(4\pi)^2} \left\{ \text{Tr}(t^1 t^2 t^3 t^4 t^5 t^6) \left[ \text{Li}_3 \left( 1 + \frac{p_4 \cdot p_2}{p_1 \cdot p_4} + \frac{p_4 \cdot p_2}{p_1 \cdot p_4} + \frac{(p_4 \cdot p_2)^2}{p_1 \cdot p_4 (p_1 \cdot p_4)} \right) \right] - \text{Li}_3 \left( 1 + \frac{p_4 \cdot p_2}{p_1 \cdot p_4} + \frac{p_4 \cdot p_2}{p_1 \cdot p_4} + \frac{\pi^2}{6} \right) C_{\text{MHV}(0)}^{(1,-2,3,4,5,13+)}(u; 1) + \mathcal{P}(12|34) \right\} \]

(6.7)

Six-point type-II amplitudes

We first consider the case of (1_2_3_4_5_6+). In applying the formula (5.47) in this helicity configuration, we find that we can not have three \( C_{\text{MHV}(0)} \)'s, that is, one of them can not have two negative-helicity states in any way; it has only a single negative-helicity state. Therefore the type-II part of the (1_2_3_4_5_6+) amplitudes vanish:

\[ A_{\text{N\!MHV}(1;1,13)}^{(1,-2,3,4,5,6+)}(u) = 0. \] 

(6.8)

For the other cases, the type-II amplitudes are nontrivial. We can calculate them as follows.

\[ \sqrt{\widehat{C}_{\text{MHV}(0)}^{(1,-2,3,4,5,6+)}(u; 1) \widehat{C}_{\text{MHV}(0)}^{(3,-4,5,6+)}(u; 1) \widehat{C}_{\text{MHV}(0)}^{(5,-6+1-2+)}(u; 1) + \mathcal{P}(12|34|56)} \]

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momentum-space representation, the combined amplitudes can be calculated as

\[ \hat{A}_\text{N}^{(1, 2, 3, 4, 5, 6, +)} (u) \]

It is true that the amplitudes have the single-trace holomorphic structure but, as discussed in section 2, this feature is a direct consequence of the CSW generalization. It is also true that the amplitudes are encoded in the single-trace color structure. Thus, if necessary to tell the planarity, it is natural to interpret our results as non-planar amplitudes.

These results are qualitatively different from the unitary-cut results [18, 37] in many ways. First, the Li3-dependence does not appear in the unitary-cut method. Second, the non-holomorphicity arises from rather involved so-called spin factors. Lastly and most significantly, our amplitudes are not necessarily planar while the unitary-cut results are. It is true that the amplitudes have the single-trace structure but, as discussed in section 2, this feature is a direct consequence of the CSW generalization and is completely independent of the conventional large N analysis. In a large-N independent nonperturbative analysis, the single-trace color structure does not necessarily mean the planarity of the amplitudes. Thus, if necessary to tell the planarity, it is natural to interpret our results as non-planar amplitudes.

7 One-loop N2MHV amplitudes and beyond

The MHV diagrams that contribute to one-loop N2MHV amplitudes (with four negative-helicity gluons and \((n - 4)\) positive-helicity gluons) are shown in Figure 6.

Li2- and Li3-type contributions

As in the NMHV case, the Li2- and Li3-type amplitudes contributing the one-loop N2MHV amplitudes can be calculated by a direct application of the CSW rules. In the momentum-space representation, the combined amplitudes can be calculated as

\[ \hat{A}_\text{N}^{(a, b, c, d, \ldots)} (u) = \sum_{i=1}^{n-3} \sum_{r=1}^{n-3} \hat{A}_\text{N}^{(i, a, b, c, \ldots; (i+r) + l_4)} (u) \frac{1}{q^{i+r}} \hat{A}_\text{N}^{((-i_4) \ldots; (i-1) \ldots)} (u) \right|_{l_4=q^{i+r}} \]
Figure 6: MHV diagrams contributing to one-loop $N^2$MHV amplitudes

can be expressed as

$$
\hat{A}_{N^{MHV}(1;L_{i2})}^{(i_1 \cdots a \cdots c \cdots (i+r)_{+}l_{4+})}(u) = \hat{A}_{N^{MHV}(1;L_{i2})}^{(i_1 \cdots a \cdots c \cdots (i+r)_{+}l_{4+})}(u) + \hat{A}_{N^{MHV}(1;L_{i3})}^{(i_1 \cdots a \cdots c \cdots (i+r)_{+}l_{4+})}(u)
$$

(7.2)

where the $L_{i2}$-type amplitudes are given by

$$
\hat{A}_{N^{MHV}(1;L_{i2})}^{(i_1 \cdots a \cdots c \cdots (i+r)_{+}l_{4+})}(u) = ig^{2} \sum_{j=1}^{i+r-2} \sum_{t_1=1}^{j+1} \sum_{t_2=1}^{j+1} \left[ 1 - \frac{1}{2} \delta_{t_1+1,t_2+1} \right]

\sum_{\sigma(1) \in S_{t_1+1} \sigma(2) \in S_{t_1-t_2} \sigma(3) \in S_{r-t_1}} \text{Tr}(t^{(1)}_{\sigma} \cdots t^{(1)}_{k+t_2} t^{(2)}_{k+t_2+1} \cdots t^{(2)}_{k+t_1} t^{(3)}_{j+t_1+1} \cdots t^{(3)}_{j+r+1} l_{4+}) C^{(l_{1:t},l_{2:k})}_{MHV(0)}

\left[ \hat{C}_{MHV(0)}^{((k_{+1} \cdots a \cdots b_{-(k+t_{1})}+l_{3+})}(u; \sigma^{(1 \otimes 2)}) \frac{1}{q_{k,k+t_{1}}} C_{MHV(0)}^{((-(k_{-1})_{+}(j+t_{1}+1)_{+} \cdots (i+r)_{+}l_{4+})}(u; \sigma^{(3)})

+ \left( 1 - \frac{1}{2} \delta_{t-1,n-r-1} \right) \hat{C}_{MHV(0)}^{((k_{+1} \cdots a \cdots b_{-(k+t_{1})}+l_{3+})}(u; \sigma^{(1 \otimes 2)})

\times \frac{1}{q_{k,k+t_{1}}} C_{MHV(0)}^{((-(k_{-1})_{+}(j+t_{1}+1)_{+} \cdots (j+r)_{+}l_{4+})}(u; \sigma^{(3)})\right]

\right].

(7.3)

As indicated in Figure 6, there are asymmetric and symmetric MHV diagrams for the $L_{i2}$-type amplitudes. These are reflected in the two terms inside the square bracket of (7.3). In
the second term, which corresponds to the symmetric MHV diagram, we take care of the double counting due to the reflection symmetry. Upon polylog regularization, we can treat the one-loop MHV diagram as a unit of MHV diagrams. Thus the symmetric MHV diagram double counting due to the reflection symmetry. Upon polylog regularization, we can treat the second term, which corresponds to the symmetric MHV diagram, we take care of the

\[ L_{\sigma_1 \sigma_2 \sigma_3} \left[ \hat{C}^{(j+1 \rightarrow b \cdots c \rightarrow (j+r))} (u; \sigma^{(1 \otimes 2)}) \hat{C}^{(j+1 \rightarrow b \cdots c \rightarrow (j+r))} (u; \sigma^{(1 \otimes 3)}) \right] \frac{1}{2}. \]

As in the case of (5.11), the position of \( l_4 \) can be chosen arbitrarily. In the above expression, we place the internal index \( l_4 \) adjacent to \((j + r)\) and \( j \).

**Li\(_4\)-type contributions**

In analogy to (5.47), we can calculate the Li\(_4\)-type amplitudes in Figure 6 as follows.

\[ \tilde{A}^{(a \rightarrow b \cdots c \rightarrow d)}_{N^2MHV(1:1l_4)} (u) \]

\[ = -ig^2 \sum_{i=1}^{n} \sum_{r_1=1}^{1} \sum_{r_2=1}^{1} \sum_{r_3=1}^{1} \left( 1 - \frac{3}{4} \delta_{\hat{a}r_1+1} \delta_{\hat{a}r_2+1} \delta_{\hat{a}r_3+1} \right) \]

\[ \sum_{\sigma^{(1)} \in S_{r_1+1}} \sum_{\sigma^{(2)} \in S_{r_2+1}} \sum_{\sigma^{(3)} \in S_{r_3+1}} \sum_{\sigma^{(4)} \in S_{r_4+1}} \sum_{\sigma \in S_{r_4+1}} \text{Tr}(t^{\sigma_1} \cdots t^{\sigma_2} t^{\sigma_3} \cdots t^{\sigma_4}) \]

\[ \hat{C}_{MHV(0)}^{(c_{1+1 \rightarrow \cdots a \cdots d}_d)} (u; \sigma^{(3 \otimes 4)}) \hat{C}_{MHV(0)}^{(d_{1+1 \rightarrow \cdots a \cdots d}_a)} (u; \sigma^{(4 \otimes 1)}) \frac{1}{2}. \]

Figure 7: Symmetric Li\(_2\)-type MHV diagrams contributing to one-loop N\(^2\)MHV amplitudes
where the indices $a_1, a_2, \ldots, d_2$ are labeled by

$$a_1 = i, \quad a_2 = i + r_1, \quad b_1 = i + r_1 + 1, \quad b_2 = i + r_1 + r_2 + 1, \quad c_1 = i + r_1 + r_2 + 2, \quad c_2 = i + r_1 + r_2 + r_3 + 2, \quad d_1 = i + r_1 + r_2 + r_3 + 3, \quad d_2 = i - 1.$$  

(7.6)

The symbol $\mathcal{L}_{n=1,r_2,r_3|\sigma_i}^{(l_3, l_3, l_4, l_4)}$ is given by

$$\mathcal{L}_{n=1,r_2,r_3|\sigma_i}^{(l_3, l_3, l_4, l_4)} = \frac{1}{(4\pi)^4} \int \omega_1^{(1)} \omega_2^{(0)} \omega_3^{(0)} \omega_4^{(0)} \mathcal{R}_{n=1,r_2,r_3|\sigma_i}^{(l_3, l_3, l_4, l_4)},$$  

(7.7)

where $\mathcal{R}_{n=1,r_2,r_3|\sigma_i}^{(l_3, l_3, l_4, l_4)}$ can be obtained in the same way as $\mathcal{R}_{n=1,r_2|\sigma_i}^{(l_4, l_4)}$ in (5.33)-(5.38). Namely, we can obtain it from

$$\mathcal{R}_{n=1,r_2,r_3|\sigma_i}^{(l_3, l_3, l_4, l_4)} = \frac{(l_1 l_2)(l_2 l_3)(l_3 l_4)(l_4 l_1)(a_1 a_2)(b_1 b_2)(c_1 c_2)(d_1 d_2)}{(l_1 a_1)(l_2 b_1)(l_3 c_1)(l_4 d_1)(l_1 b_2)(l_2 c_2)(l_3 d_2)(l_4 a_2)} |_{a_1 \mid b_1 \mid c_1 \mid d_1, cyc.}$$  

(7.8)

with replacements of $a_1$ and $a_2$ by $\sigma_i^{(1)}$ and $\sigma_i^{(1)}$, and so on.

As in (3.36), we can make a tedious but straightforward calculation for the raw data:

$$\begin{align*}
(l_1 l_2)(l_2 l_3)(l_3 l_4)(l_4 l_1)(a_1 a_2)(b_1 b_2)(c_1 c_2)(d_1 d_2) \\
(l_1 a_1)(l_2 b_1)(l_3 c_1)(l_4 d_1)(l_1 b_2)(l_2 c_2)(l_3 d_2)(l_4 a_2) &\mid_{a_1 \mid b_1 \mid c_1 \mid d_1} \\
&= \left[ R^{(l_1 l_2)}_{(b_2 b_1)} R^{(l_2 l_3)}_{(c_2 c_1)} R^{(l_3 l_4)}_{(d_2 d_1)} R^{(l_4 l_1)}_{(e_2 e_1)} - R^{(l_1 l_2)}_{(e_2 e_1)} R^{(l_2 l_3)}_{(c_2 c_1)} R^{(l_3 l_4)}_{(d_2 d_1)} R^{(l_4 l_1)}_{(b_2 b_1)} \right. \\
&- R^{(l_1 l_2)}_{(b_2 b_1)} R^{(l_2 l_3)}_{(c_2 c_1)} R^{(l_3 l_4)}_{(d_2 d_1)} R^{(l_4 l_1)}_{(b_2 b_1)} + R^{(l_1 l_2)}_{(d_2 d_1)} R^{(l_2 l_3)}_{(c_2 c_1)} R^{(l_3 l_4)}_{(b_2 b_1)} R^{(l_4 l_1)}_{(b_2 b_1)} \]
\end{align*}$$

(7.9)

Taking account of the cyclic property of the $\{l_1, l_2, l_3, l_4\}$ indices, we can carry out the polylog regularization in analogy to (5.40) as

$$\frac{1}{(4\pi)^4} \int \omega_1^{(1)} \omega_2^{(0)} \omega_3^{(0)} \omega_4^{(0)} \mathcal{R}_{n=1,r_2,r_3|\sigma_i}^{(l_3, l_3, l_4, l_4)},$$

$$\rightarrow \quad \text{Li}_4 \left( 1 - \Delta_{(b_2, b_1)}^{(12)} - \Delta_{(c_2, c_1)}^{(23)} - \Delta_{(d_2, d_1)}^{(34)} - \Delta_{(e_2, e_1)}^{(41)} \right) - \text{Li}_4 \left( 1 - \Delta_{(a_1, a_2)}^{(12)} - \Delta_{(c_2, b_2)}^{(23)} - \Delta_{(d_2, c_2)}^{(42)} \right)$$

$$- \text{Li}_4 \left( 1 - \Delta_{(c_1, b_2)}^{(12)} - \Delta_{(a_2, c_2)}^{(23)} - \Delta_{(d_1, d_2)}^{(41)} \right) + \text{Li}_4 \left( 1 - \Delta_{(c_2, b_1)}^{(12)} - \Delta_{(e_2, c_2)}^{(31)} - \Delta_{(d_2, a_2)}^{(42)} \right)$$

$$- \text{Li}_4 \left( 1 - \Delta_{(c_1, c_2)}^{(12)} - \Delta_{(a_2, d_2)}^{(23)} - \Delta_{(d_1, d_2)}^{(34)} \right) + \text{Li}_4 \left( 1 - \Delta_{(c_1, b_2)}^{(12)} - \Delta_{(e_2, c_2)}^{(31)} - \Delta_{(a_1, a_2)}^{(42)} \right).$$
\[ \Delta^{(31)}_{(c_2,c_1)} - \Delta^{(41)}_{(a_2,a_1)} - \Delta^{(42)}_{(b_2,d_2)} - \Delta^{(12)}_{(b_2,a_2)} = \Delta^{(12)}_{(b_2,a_2)} - \Delta^{(23)}_{(a_2,a_1)} - \Delta^{(41)}_{(c_2,c_1)} - \Delta^{(42)}_{(b_2,d_2)} \]

where \( \Delta^{(12)} \), \( \Delta^{(23)} \), \( \Delta^{(34)} \) and \( \Delta^{(41)} \) are defined by

\[ \Delta^{(12)}_{(b_2,c_1)} = c^{(1)}_{r_1,a_1} (p^{(a)} + p_{a_2})^2, \quad \Delta^{(12)}_{(a_1,a_2)} = c^{(1)}_{r_1,a_1} (p^{(a)} + p_{b_2})^2, \quad \Delta^{(12)}_{(b_2,a_2)} = c^{(1)}_{r_1,a_1} (p^{(a)} + p_{b_2})^2, \]

\[ \Delta^{(23)}_{(c_2,c_1)} = c^{(2)}_{r_2,b_1} (p^{(b)} + p_{c_2})^2, \quad \Delta^{(23)}_{(b_2,b_1)} = c^{(2)}_{r_2,b_1} (p^{(b)} + p_{c_2})^2, \quad \Delta^{(23)}_{(c_2,c_1)} = c^{(2)}_{r_2,b_1} (p^{(b)} + p_{c_2})^2, \]

\[ \Delta^{(34)}_{(d_2,d_1)} = c^{(3)}_{r_3,c_1} (p^{(c)} + p_{d_2})^2, \quad \Delta^{(34)}_{(c_1,c_2)} = c^{(3)}_{r_3,c_1} (p^{(c)} + p_{d_2})^2, \quad \Delta^{(34)}_{(d_2,d_1)} = c^{(3)}_{r_3,c_1} (p^{(c)} + p_{d_2})^2, \]

\[ \Delta^{(41)}_{(a_2,a_1)} = c^{(4)}_{r_4,d_1} (p^{(d)} + p_{d_2})^2, \quad \Delta^{(41)}_{(d_1,d_2)} = c^{(4)}_{r_4,d_1} (p^{(d)} + p_{d_2})^2, \quad \Delta^{(41)}_{(a_2,a_2)} = c^{(4)}_{r_4,d_1} (p^{(d)} + p_{d_2})^2, \]

\[ \Delta^{(12)}_{(d_2,a_2)} = c^{(1)}_{r_1,a_1} (p^{(a)} + p_{a_2})^2, \quad \Delta^{(34)}_{(a_2,c_2)} = c^{(3)}_{r_3,c_1} (p^{(c)} + p_{d_2})^2. \]

Explicit forms of these terms can be obtained as in the cases of (5.41)-(5.43). Strictly speaking, however, the last two entries \( \Delta^{(12)}_{(a_2,a_2)} \) and \( \Delta^{(34)}_{(a_2,a_2)} \) can not be defined directly from the previous cases. Namely, these are expressions that do not appear in the BST-type polylog regularization (4.16). Thus, if we apply the BST method strictly, we can not define these terms. We here simply regard the undefined indices (say, \( d_2 \) of \( \Delta^{(12)}_{(a_2,d_2)} \)) as dummy indices to define these terms. Similarly, we can define the rest of \( \Delta \)'s in (7.10) as below.

\[ \Delta^{(42)}_{(d_2,c_2)} = c^{(4)}_{r_4,d_1} (p^{(d)} + p_{d_2})^2, \quad \Delta^{(42)}_{(d_1,d_2)} = c^{(4)}_{r_4,d_1} (p^{(d)} + p_{d_2})^2 \]

\[ \Delta^{(31)}_{(a_2,c_2)} = c^{(3)}_{r_3,c_1} (p^{(c)} + p_{a_2})^2 \]

These may be considered as natural extension of the BST prescription but it should be emphasized that these are not obtained solely by the BST prescription; there are ambiguities in these definitions. For example, \( \Delta^{(42)}_{(b_2,d_2)} \) can also be defined as \( c^{(4)}_{r_4,d_1} (p^{(d)} + p_{d_2})^2 \) if we treat the index \( b_2 \) as a dummy index. At present, we do not know how to fix such an ambiguity. This indicates the limitation of our analysis. We expect that some solution to this problem will be provided by seeking a proof of the BST prescription in section 4.

**Generalization**

For the completion of our analysis, we now sketch how to obtain one-loop N\( ^m \)MHV amplitudes \( (m = 1, 2, \ldots, n - 4) \) in general. The amplitudes can be obtained in a recursive manner. Let \( \hat{A}^{(a_1, a_2, \ldots, a_{m+1})}_{N^m MHV(1)}(u) \) be the \( n \)-point one-loop N\( ^m \)MHV amplitudes in the momentum-space representation. Suppose that we know the form of \( \hat{A}^{(a_1, a_2, \ldots, a_{m+1})}_{N^{m-1} MHV(1)}(u) \). Then the one-loop...
N\textsuperscript{m}MHV amplitudes can be decomposed as
\[
\hat{A}_{N^m MHV}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})}(u) = \hat{A}_{N^m MHV(1: Li_1, \ldots, m+1)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})}(u) + \hat{A}_{N^m MHV(1: Li_{m+2})}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})}(u). \tag{7.13}
\]
As in the N\textsuperscript{2}MHV case (7.1), the first term can be expressed by direct use of the CSW rules:
\[
\hat{A}_{N^m MHV(1: Li_1, \ldots, m+1)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})}(u) = \sum_{l=1}^{m+1} \hat{A}_{N^{m-1} MHV(1)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+1}^{(m+1)}; l)}(u) \frac{1}{q_{l+i}^2} \hat{A}_{MHV(0)}^{(-a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)}; l)}(u) \tag{7.14}
\]
where the one-loop N\textsuperscript{m-1}MHV subamplitudes are given by a sum of \(N_k\)-type contributions (\(k = 2, 3, \ldots, m+1\)). Explicitly, these are written as
\[
\hat{A}_{N^{m-1} MHV(1)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+1}^{(m+1)}; l)}(u) = \sum_{k=2}^{m+1} \hat{A}_{N^{m-1} MHV(1: Li_k)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+1}^{(m+1)}; l)}(u). \tag{7.15}
\]
As in the case of (7.3), we need to take account of symmetries of MHV diagrams in order to obtain explicit forms of the amplitudes (7.14).

The characteristic feature of the one-loop N\textsuperscript{m}MHV amplitude (7.13) arises from its second term. In analogy to (7.5) and (5.47), we can compute this term as follows.
\[
\hat{A}_{N^m MHV(1: Li_{m+2})}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})}(u) = (-1)^{\frac{m}{2}} i g^2 \sum_{a_1^{(1)} \in S_{m+1}} \sum_{a_2^{(2)} \in S_{m+2}} \cdots \sum_{a_{m+2}^{(m+2)} \in S_{m+2}} \left( \sum_{\sigma_1^{(1)} \in S_{m+1}} \sum_{\sigma_2^{(2)} \in S_{m+2}} \cdots \sum_{\sigma_{m+2}^{(m+2)} \in S_{m+2}} \right) \text{Tr} \left( t^{a_1^{(1)}}_1 \cdots t^{a_2^{(2)}}_2 \cdots t^{a_1^{(1)}}_{m+1} \right) \right.
\]
\[
\left. \left. \text{L}_{m; r_1, r_2, \ldots, r_{m+1}; \sigma_1} \left[ \hat{C}_{MHV(0)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})} (u; \sigma^{(1)} \otimes 2) \hat{C}_{MHV(0)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})} (u; \sigma^{(2)} \otimes 3) \times \cdots \right. \right.
\]
\[
\left. \left. \cdots \times \hat{C}_{MHV(0)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})} (u; \sigma^{(m+1) \otimes m+2}) \hat{C}_{MHV(0)}^{(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)})} (u; \sigma^{(m+2) \otimes 1}) \right) \right]^{\frac{1}{2}} \right) \tag{7.16}
\]
where the indices \(a_1^{(1)}, a_2^{(2)}, \ldots, a_{m+2}^{(m+2)}\) are labeled by
\[
\begin{align*}
a_1^{(1)} &= i, & a_2^{(1)} &= i + r_1, \\
a_1^{(2)} &= i + r_1 + 1, & a_2^{(2)} &= i + r_1 + r_2 + 1, \\
& \vdots \\
a_1^{(m+1)} &= i + r_1 + r_2 + \cdots + r_m + m, & a_2^{(m+1)} &= i + r_1 + r_2 + \cdots + r_{m+1} + m, \\
a_1^{(m+2)} &= i + r_1 + r_2 + \cdots + r_{m+1} + m + 1, & a_2^{(m+2)} &= i - 1. \tag{7.17}
\end{align*}
\]
The auxiliary index \( r_{m+2} \) is given by \( r_{m+2} = n - (r_1 + r_2 + \cdots + r_{m+1} + m + 2) \). In terms of the indices \( (7.17) \), the set of permutations are explicitly denoted as

\[
\sigma^{(1)} = \begin{pmatrix} a_1^{(1)} & \cdots & a_2^{(1)} \\ a_1^{(1)} & \cdots & a_2^{(1)} \end{pmatrix}, \quad \sigma^{(2)} = \begin{pmatrix} a_1^{(2)} & \cdots & a_2^{(2)} \\ a_1^{(2)} & \cdots & a_2^{(2)} \end{pmatrix}, \ldots, \quad \sigma^{(m+2)} = \begin{pmatrix} a_1^{(m+2)} & \cdots & a_2^{(m+2)} \\ a_1^{(m+2)} & \cdots & a_2^{(m+2)} \end{pmatrix}.
\] (7.18)

The polylog regularization is to be realized in the symbol \( \mathcal{L}_{n;r_1,r_2,\cdots,r_{m+1};\sigma_i}^{l_1,l_2,\cdots,l_{m+2}} \) which can be expressed as

\[
\mathcal{L}_{n;r_1,r_2,\cdots,r_{m+1};\sigma_i}^{l_1,l_2,\cdots,l_{m+2}} = \frac{1}{(4\pi)^{m+2}} \int \omega_1^{(1)} \omega_2^{(0)} \omega_3^{(0)} \cdots \omega_{m+2}^{(0)} R_{n;r_1,r_2,\cdots,r_{m+1};\sigma_i}^{l_1,l_2,\cdots,l_{m+2}} (7.19)
\]

where as in (7.8) the integrand can be written as

\[
R_{n;r_1,r_2,\cdots,r_{m+1};\sigma_i}^{l_1,l_2,\cdots,l_{m+2}} = \frac{(l_1 l_2 l_3) \cdots (l_{m+2} l_1)}{\left( l_1 \sigma^{(1)}_{a_1^{(1)}} l_2 \sigma^{(2)}_{a_2^{(2)}} \cdots (l_{m+2} \sigma^{(m+2)}_{a_{(m+2)}^{(m+2)}}) \right) \left( \sigma^{(1)}_{a_1^{(1)}} \sigma^{(1)}_{a_2^{(2)}} \cdots \sigma^{(m+2)}_{a_{(m+2)}^{(m+2)}} \right) \left( l_1 \sigma^{(2)}_{a_1^{(1)}} l_2 \sigma^{(3)}_{a_2^{(2)}} \cdots (l_{m+2} \sigma^{(1)}_{a_{(m+2)}^{(m+2)}}) \right) \sigma^{(1)}_{a_1^{(1)}} \sigma^{(2)}_{a_2^{(2)}} \cdots \sigma^{(m+2)}_{a_{(m+2)}^{(m+2)}}}.
\] (7.20)

Using the Schouten identities, we can in principle factorize the above quantity into a sum of the products of the \( R \)'s as shown in (7.9). Its final form is not clear at the present state although we find that it involves at most \( 2^{m+2} \) such terms. The fact that such a factorization is possible, however, guarantees that we can utilize the polylog regularization in simplifying the expression (7.19). To be more concrete, by use of the polylog regularization, we can naturally express \( \mathcal{L}_{n;r_1,r_2,\cdots,r_{m+1};\sigma_i}^{l_1,l_2,\cdots,l_{m+2}} \) in terms of a set of \( \text{Li}_{m+2} \) contributions. For \( m = 0 \), this is analytically confirmed by the BST method. In the previous sections, we have developed the BST method in the holonomy formalism so as to show that this picture can also be applicable to the cases of \( m = 1, 2 \) in a systematic way. It is therefore natural to consider that we can generalize this picture to arbitrary \( m \). The above expressions (7.19), (7.20) firmly suggest the possibility of such a generalization in a concrete fashion.

8 Concluding remarks

In the present paper, we carry out a first serious analysis on quantum aspects of the holonomy formalism introduced in the recent paper [68]. As mentioned in the concluding section of [68], we can extend the holonomy formalism to obtain one-loop amplitudes without any modifications. In this paper, we basically confirm this proposition by (a) considering the off-shell continuation of the Nair measure which is necessary to incorporate the loop integrals into the formalism and (b) considering the one-loop amplitudes in the coordinate-space representation, rather than the conventional momentum-space representation.
To be more specific in part (b), we define the $x$-space contraction operator (3.20). It is this operator that embodies the CSW rules in the functional derivation of gluon amplitudes in terms of the $x$-space S-matrix functional (3.14). The basic result of this paper is that we can systematically obtain one-loop gluon amplitudes from this S-matrix functional by a conventional functional method, presenting what we consider the most natural generalization of the CSW rules or the BST method to one-loop amplitudes in the framework of the holonomy formalism.

As mentioned at the end of section 2, the CSW-based calculation of one-loop amplitudes are qualitatively different from the previously known unitary-cut method (or the open-string calculation). The former leads to single-trace color structure while the latter has multi-trace color decomposition in describing the one-loop amplitudes. Notice that the the single-trace property does not necessarily mean the planarity of the amplitudes in the present case since this property is a direct consequence of the CSW generalization to one-loop amplitudes and is completely independent of the conventional large $N$ analysis. In the literature, as far as the author notices, there exist no analytic expressions for the CSW-based one-loop non-MHV amplitudes. In this sense, our new analytic results for one-loop NMHV and $N^2$MHV amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory are conjectural in nature.

Lastly, we comment on the polylog regularization introduced in section 4. The regularization can be executed by use of what we call the BST prescription (4.16). In this paper we could not provide a proof of this prescription. Thus the polylog regularization itself is to some extent speculative. It does, however, lead to the correct BST representation of one-loop MHV amplitudes in (3.41)-(3.49). Furthermore, with the knowledge of the iterated-integral representation of polylogarithm functions (4.12), we can naturally generalize the polylog regularization scheme to one-loop non-MHV amplitudes. It is widely recognized in the literature that the CSW rules should apply not only to one-loop MHV amplitudes but also to any type of loop amplitudes, and besides, the recent developments indicate that reminder functions of higher-loop amplitudes can be described systematically in terms of polylogarithms. Given these facts, partly in search of a new perspective to one-loop amplitudes, we find it of some significance to pursue the application of the polylog regularization to one-loop non-MHV amplitudes despite of its conjectural nature. The latter half of this paper has been developed along these lines of reasonings.

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