Abstract

In this short note we describe an alternative global version of the twisting procedure used by Dolgushev to prove formality theorems. This allows us to describe the maps of Fedosov resolutions, which are key factors of the formality morphisms, in terms of a twist of the fiberwise quasi-isomorphisms induced by the local formality theorems proved by Kontsevich and Shoikhet. The key point consists in considering $L_\infty$-resolutions of the Fedosov resolutions obtained by Dolgushev and an adapted notion of Maurer–Cartan element. This allows us to perform the twisting of the quasi-isomorphism intertwining them in a global manner.
1 Introduction

One of the most important results of deformation quantization is the so-called formality theorem, due to Kontsevich [9], proving the existence and classification of formal star products. More precisely, the formality theorem provides an $L_\infty$-quasi-isomorphism from polyvectorfields to polydifferential operators on Euclidean space. In [4, 5] Dolgushev proves the theorem for general manifolds $M$ by using Fedosov’s formal geometric methods [8], Kontsevich’s quasi-isomorphism [9] and the twisting procedure inspired by Quillen [10]. These techniques have also been used to prove formality for Lie algebroids [1] and formality for chains [6].

The main goal of this paper is to give an alternative version of the twisting procedure used by Dolgushev. In [9] Kontsevich proved that there exists an $L_\infty$-quasi-isomorphism between dgla’s

$$\mathcal{H} : T_{\text{poly}}(\mathbb{R}^d) \longrightarrow D_{\text{poly}}(\mathbb{R}^d).$$

(1.1)

For a generic manifold $M$ the quasi-isomorphism $\mathcal{H}$ induces a fiberwise quasi-isomorphism $\mathcal{U}$ between the Fedosov resolutions of $T_{\text{poly}}(M)$ and $D_{\text{poly}}(M)$. Twisting procedures are used by Dolgushev to obtain a formality quasi-isomorphism by twisting the fiberwise quasi-isomorphism induced by $\mathcal{U}$. However, Dolgushev twists locally and checks consistency on overlapping charts, this means that the global quasi-isomorphism is not described, a priori, as a twist of another morphism. We are interested in presenting the quasi-isomorphism of Fedosov resolutions as a twist of the fiberwise map globally. This is possible if, first of all, one allows for curvature in the definition of an $L_\infty$-algebra and secondly one uses the notion of Maurer–Cartan elements adapted to a resolution of $L_\infty$-modules. This allows us to present the fiberwise morphism $\mathcal{U}$ as a morphism of curved Lie algebras and obtain the global map as a twist of $\mathcal{U}$ directly. In complete analogy, starting with the quasi-isomorphism $\mathcal{S}$ of dgla-modules proved by Shoikhet in [11] we are able to obtain the relevant map for chains as a twist of $\mathcal{S}$. The generalization to chains doesn’t cost anything since we already phrase the case of cochains completely in terms of $L_\infty$-modules (and their resolutions as modules).

In Dolgushev’s approach the (global) $L_\infty$-algebra which one can twist is (globally) curved. Thus it makes no sense to speak of twisting a quasi-isomorphism. Our strategy consists in resolving the Fedosov resolutions obtained by Dolgushev as $L_\infty$-modules. Then, $\mathcal{U}$ induces an $L_\infty$-morphism of resolutions and we prove that the element by which one twists satisfies certain conditions. Locally we establish that the morphism of curved $L_\infty$-algebras is actually the twist of a quasi-isomorphism of flat $L_\infty$-algebras. Thus the use of a resolution allows us to consider whether $L_\infty$-morphisms of curved algebras are “quasi-isomorphisms”, at least as far as twisting is concerned.

The paper is organized as follows. In Section 2 we recall the language of $L_\infty$-algebras and in particular we present the notions of $L_\infty$-algebra and $L_\infty$-morphisms in the presence of curvature. In this setting, we recall the twisting procedure and the effects that it has on $L_\infty$-algebras and $L_\infty$-morphisms. Section 3 contains the main result of this paper. First, we introduce the concept of Maurer–Cartan elements compatible with resolutions of $L_\infty$-modules and we prove the theorem stated above. As a second step, we apply this result to prove formality theorems for Hochschild cochains and chains.

Acknowledgments

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2 Preliminaries

In this section we recall the notions of $L_\infty$-algebras, $L_\infty$-modules, $L_\infty$-morphisms and their twists by Maurer–Cartan elements. The idea of such twisting procedures comes from Quillen’s seminal
work [10]. Proofs and details can be found in [4, 5, 7].

Given a graded vector space $V^\bullet$ over $\mathbb{K}$ we shall define the \textit{shifted} vector space, denoted by $V[k]^\bullet$, by

$$V[k]^\ell = V^{\ell+k}$$

We shall fix a field $\mathbb{K}$ of characteristic 0. Recall that a degree $+1$ coderivation $Q$ on the counital conilpotent cocommutative coalgebra $S^c(\mathfrak{L})$ cofreely cogenerated by the graded vector space $\mathfrak{L}[1]^\bullet$ over $\mathbb{K}$ is called an $L_\infty$-\textit{structure} on the graded vector space $\mathfrak{L}$ if $Q^2 = 0$.

We recall that $S^c(\mathfrak{L})$ can be realized by the symmetrized deconcatenation product on the space $\bigoplus_{n \geq 0} \mathfrak{L}^n$, see e.g. [7]. Here $\mathfrak{L}^n$ is the space of coinvariants for the usual action of $S_n$ (the symmetric group in $n$ letters) on $\bigotimes^n \mathfrak{L}[1]$ (keeping the grading in mind). Any degree $+1$ coderivation $Q$ on $S^c(\mathfrak{L})$ is uniquely determined by the components

$$Q_n : \bigvee^n(\mathfrak{L}[1]) \rightarrow \mathfrak{L}[2]$$

through the formula

$$(2.1) \quad Q(\gamma_1 \vee \ldots \vee \gamma_n) = \sum_{k=0}^n \sum_{\sigma \in \text{Sh}(k,n-k)} \varepsilon(\sigma) Q_k(\gamma_{\sigma(1)} \vee \ldots \vee \gamma_{\sigma(k)}) \vee \gamma_{\sigma(k+1)} \vee \ldots \vee \gamma_{\sigma(n)}.$$  

where $\text{Sh}(k,n-k)$ denotes the set of $(k,n-k)$ shuffles in $S_n$, $\varepsilon(\sigma) = \varepsilon(\sigma, \gamma_1, \ldots, \gamma_n)$ is a sign given by the rule

$$(2.3) \quad \gamma_{\sigma(1)} \vee \ldots \vee \gamma_{\sigma(n)} = \varepsilon(\sigma) \gamma_1 \vee \ldots \vee \gamma_n$$

and we use the conventions that $\text{Sh}(n,0) = \text{Sh}(0,n) = \{\text{id}\}$ and that the empty product equals the unit. Note that $Q_0(1)$ is of degree 1 in $\mathfrak{L}[1]$ (thus of degree 2 in $\mathfrak{L}$). The $Q^2 = 0$ condition can now be expressed in terms of a quadratic equation in the $Q_n$.

\textbf{Example 2.1 (Curved Lie algebra)} Our main example of an $L_\infty$-algebra is that of a (curved) Lie algebra $(\mathfrak{L}, R, d, \{\cdot, \cdot\})$ by setting $Q_0(1) = -R$, $Q_1 = -d$, $Q_2(\gamma \vee \mu) = -(-1)^{|\gamma||\mu|}[\gamma, \mu]$ and $Q_i = 0$ for all $i \geq 3$.

In the following we call the $L_\infty$-algebras with $Q_0 = 0$ \textit{flat} $L_\infty$-algebras. For our purposes, we need to consider $L_\infty$-algebras $\mathfrak{L}$ that are equipped with a decreasing filtration

$$\mathfrak{L} = \mathfrak{F}^0 \mathfrak{L} \supset \mathfrak{F}^1 \mathfrak{L} \supset \ldots \supset \mathfrak{F}^k \mathfrak{L} \supset \ldots ,$$

that respects the $L_\infty$-structure and which is moreover \textit{complete}, i.e.

$$\bigcap_k \mathfrak{F}^k \mathfrak{L} = \{0\}.$$  

This yields an associated complete metric topology and we consider convergence of infinite sums in terms of this topology.

Suppose $(\mathfrak{L}, Q)$ and $(\widetilde{\mathfrak{L}}, \widetilde{Q})$ are two $L_\infty$-algebras. A degree 0, filtration respecting, counital coalgebra morphism

$$F : S^c(\mathfrak{L}) \rightarrow S^c(\widetilde{\mathfrak{L}})$$

such that $FQ = \widetilde{Q}F$ is said to be an $L_\infty$-\textit{morphism}. A coalgebra morphism $F$ from $S^c(\mathfrak{L})$ to $S^c(\widetilde{\mathfrak{L}})$ is uniquely determined by its components (also called \textit{Taylor coefficients})

$$(2.3) \quad F_n : \bigvee^n(\mathfrak{L}[1]) \rightarrow \widetilde{\mathfrak{L}}[1],$$
where \( n \geq 1 \). Namely, we set \( F(1) = 1 \) and use the formula

\[
F(\gamma_1 \lor \ldots \lor \gamma_n) = \sum_{p \geq 1} \sum_{k_1, \ldots, k_p \geq 1} \sum_{\sigma \in \text{Sh}(k_1, \ldots, k_p)} \frac{\varepsilon(\sigma)}{p!} F_{k_1}(\gamma_{\sigma(1)} \lor \ldots \lor \gamma_{\sigma(k_1)}) \lor \ldots \lor F_{k_p}(\gamma_{\sigma(n-k_p+1)} \lor \ldots \lor \gamma_{\sigma(n)}), \tag{2.4}
\]

where \( \text{Sh}(k_1, \ldots, k_p) \) denotes the set of \( (k_1, \ldots, k_p) \)-shuffles in \( S_n \) (again we set \( \text{Sh}(n) = \{ \text{id} \} \)). Given an \( L_\infty \)-morphism of flat \( L_\infty \)-algebras \( \mathfrak{L} \) and \( \tilde{\mathfrak{L}} \), we obtain the map of complexes

\[
F_1: (\mathfrak{L}, Q_1) \rightarrow (\tilde{\mathfrak{L}}, \tilde{Q}_1).
\]

The \( L_\infty \)-morphism \( F \) is called an \( L_\infty \)-quasi-isomorphism if this map \( F_1 \) is a quasi-isomorphism of complexes.

Let \( \mathfrak{L} \) be an \( L_\infty \)-algebra. Then an \( L_\infty \)-module over \( \mathfrak{L} \) is a graded vector space \( \mathfrak{M} \) equipped with a square-zero degree +1 coderivation \( \varphi \) on the cofree \( S^c(\mathfrak{L}) \)-comodule \( S^c(\mathfrak{L}) \otimes \mathfrak{M} \) cogenerated by \( \mathfrak{M} \).

Note that as for a coderivation on \( S^c(\mathfrak{L}) \) a coderivation \( \varphi \) on \( S^c(\mathfrak{L}) \otimes \mathfrak{M} \) is given by the components

\[
\varphi_n: \bigwedge^n \mathfrak{L}[1] \otimes \mathfrak{M} \rightarrow \mathfrak{M}[1] \tag{2.5}
\]

through the formula

\[
\varphi(\gamma_1 \lor \ldots \lor \gamma_n \otimes m) = Q(\gamma_1 \lor \ldots \lor \gamma_n) \otimes m + \sum_{k=0}^{n} \sum_{\sigma \in \text{Sh}(k,n-k)} \epsilon'(\sigma) \epsilon(\gamma_{\sigma(1)} \lor \ldots \lor \gamma_{\sigma(k)} \otimes \varphi_{n-k}(\gamma_{\sigma(k+1)} \lor \ldots \lor \gamma_{\sigma(n)} \otimes m), \tag{2.6}
\]

where \( \epsilon'(\sigma) = \epsilon'(\sigma, \gamma_1 \lor \ldots \lor \gamma_n) = (-1)^{\sum_{i=1}^{k} |\gamma_{\sigma(i)}|} \). So for instance we find that \( \varphi(\gamma \otimes m) \) is given by \( Q(\gamma) \otimes m + 1 \otimes \varphi_1(\gamma \otimes m) + (-1)^{|\gamma|} \gamma \otimes \varphi_0(1 \otimes m) \). The square-zero condition yields conditions quadratic in the \( \varphi_n \) and \( Q_n \), for example

\[
\varphi_0(1 \otimes \varphi_0(1 \otimes m)) + \varphi_1(Q_0(1) \otimes m) = 0
\]

\[
\varphi_0(1 \otimes \varphi_1(\gamma \otimes m)) + (-1)^{|\gamma|} \varphi_1(\gamma \otimes \varphi_0(1 \otimes m)) + \varphi_1(Q_1(\gamma) \otimes m) + \varphi_2(Q_0(1) \lor \gamma \otimes m) = 0.
\]

Note that if \( Q_0(1) = 0 \) then by identifying \( \bigwedge^n \mathfrak{L}[1] \otimes \mathfrak{M} \) with \( \mathfrak{M} \) we obtain that \( \varphi_0 \) is a differential on \( \mathfrak{M} \).

**Example 2.2** The second most basic example of an \( L_\infty \)-module consists of a dg module \( (\mathfrak{M}, b, \rho) \) over a dgla \( (\mathfrak{L}, d, [\cdot, \cdot]) \). In this case we have

\[
\varphi_0(v) = -bv \\
\varphi_1(\gamma \lor v) = -(-1)^{|\gamma|} \rho(\gamma)v \quad \text{and}
\]

\[
\varphi_k = 0 \quad \text{for all } k \geq 2, \tag{2.7}
\]

for \( v \in \mathfrak{M} \) and \( \gamma \in \mathfrak{L} \), where \( \rho \) is the action of \( \mathfrak{L} \) on \( \mathfrak{M} \).

**Example 2.3** (Morphism of \( L_\infty \)-algebras) Suppose \( F: \mathfrak{L} \rightarrow \mathfrak{R} \) is an \( L_\infty \)-morphism. This induces the structure of \( L_\infty \)-module over \( \mathfrak{L} \) on \( \mathfrak{R} \). Namely, we consider the module structure with components \( \varphi_k \) given by

\[
\varphi_k(\gamma_1 \lor \ldots \lor \gamma_k \otimes m) = \text{pr}_\mathfrak{R}(P_{\mathfrak{R}}(F(\gamma_1 \lor \ldots \lor \gamma_k) \lor m)) \tag{2.8}
\]
Let \( \mathfrak{L} \) be an \( L_\infty \)-algebra and \( (\mathfrak{M}, \varphi), (\widehat{\mathfrak{M}}, \hat{\varphi}) \) be \( L_\infty \)-modules over \( \mathfrak{L} \). Then a morphism \( F \) from the comodule \( S^c(\mathfrak{L}) \otimes \mathfrak{M} \) to the comodule \( S^c(\mathfrak{L}) \otimes \widehat{\mathfrak{M}} \) is said to be an \( L_\infty \)-morphism if it satisfies the condition:

\[
F \varphi = \hat{\varphi} F.
\]

As before a degree 0 comodule morphism \( F : S^c(\mathfrak{L}) \otimes \mathfrak{M} \to S^c(\mathfrak{L}) \otimes \widehat{\mathfrak{M}} \) is given by components \( F_n : \bigwedge^n \mathfrak{L}[1] \otimes \mathfrak{M} \to \widehat{\mathfrak{M}} \) through the formula

\[
F(\bigwedge^n \gamma_1 \wedge \cdots \wedge \gamma_n \otimes m) = \sum_{k=0}^{n} \sum_{\sigma \in \text{Sh}(k,n-k)} \epsilon(\sigma) \gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(k)} \otimes F_{n-k}(\gamma_{\sigma(k+1)} \wedge \cdots \wedge \gamma_{\sigma(n)} \otimes m)
\]

(2.9)

In particular, in the case that \( \mathfrak{L} \) is flat, \( F \) is a \textit{quis} of \( \mathfrak{L}_\infty \)-modules if the zero-th component \( F_0 \) is a \textit{quis} of complexes.

**Example 2.4 (Morphism of \( \mathfrak{L}_\infty \)-algebras)**

Suppose

\[
\mathfrak{L} \xrightarrow{G} \mathfrak{G} \xleftarrow{F} \mathfrak{H}
\]

(2.10)

is a commuting diagram of morphisms of \( \mathfrak{L}_\infty \)-algebras. Then we may equip \( \mathfrak{H} \) and \( \mathfrak{G} \) with the \( \mathfrak{L} \)-module structure as in Example 2.3 and we find that the map \( \mathcal{F} : \mathfrak{G} \to \mathfrak{H} \) given by

\[
\mathcal{F}_n(\bigwedge^n \gamma_1 \wedge \cdots \wedge \gamma_n \otimes m) = \text{pr}_\mathfrak{G}(F(H(\bigwedge^n \gamma_1 \wedge \cdots \wedge \gamma_n \otimes m)))
\]

(2.11)

is a morphism of \( \mathfrak{L}_\infty \)-modules. Note in particular that \( \mathcal{F}_0(1 \otimes m) = F_1(m) \) so, if \( \mathfrak{L}, \mathfrak{G} \) and \( \mathfrak{H} \) are flat, then \( \mathcal{F} \) is a quasi-isomorphism if and only if \( F \) is a quasi-isomorphism.

Let \( \pi \in \mathfrak{F}^1 \mathfrak{L}[1]^0 \), by direct computations we find that the element

\[
\exp(\pi) := \sum_{n=0}^{\infty} \frac{\pi^n}{n!} \in S^c(\mathfrak{L})
\]

is well-defined, invertible and group-like. As a consequence one can prove the following claim.

**Lemma 2.5** The pair \( \mathfrak{L}^\pi = (\mathfrak{L}, Q^\pi) \) with \( Q^\pi(a) := \exp(-\pi) \vee Q(\exp(\pi) \vee a) \) is still an \( \mathfrak{L}_\infty \)-algebra.

**Example 2.6 (Twisted curved Lie algebra)** In the case of a curved Lie algebra \( (\mathfrak{L}, R, d, [\cdot, \cdot]) \) we find the twisted curved Lie algebra \( (\mathfrak{L}, R^\pi, d + [\pi, \cdot], [\cdot, \cdot]) \), where

\[
R^\pi := R + d\pi + \frac{1}{2} [\pi, \pi].
\]

Given an \( \mathfrak{L}_\infty \)-algebra \( (\mathfrak{L}, Q) \), an element \( \pi \in \mathfrak{F}^1 \mathfrak{L}[1]^0 \) is called a Maurer-Cartan (MC) element if it satisfies the following equation

\[
\sum_{n=0}^{\infty} \frac{Q_n(\pi^n)}{n!} = 0.
\]

(2.12)
Note that this is equivalent to $Q_0^2 = 0$, and it is equivalent to $Q(\exp(\pi)) = 0$. For a dgla $(\mathfrak{L}, d, [\cdot, \cdot])$ the definition above boils down to the usual Maurer–Cartan equation. If we have similarly a curved Lie algebra with curvature $-R$ it comes down to the non-homogeneous equation

$$d\pi + \frac{1}{2} [\pi, \pi] = R.$$  

**Lemma 2.7** Given an $L_\infty$-morphism $F$ from $\mathfrak{L}$ to $\tilde{\mathfrak{L}}$ and an element $\pi \in \mathfrak{L}^1\mathfrak{L}[1]^0$, we define the $F$-associated element $\pi_F \in \mathfrak{L}^1\mathfrak{L}[1]^0$ by the formula

$$\pi_F := \sum_{n=1}^{\infty} \frac{F_n(\pi^n)}{n!}.$$  

Then:

i.) we have

$$F(\exp(\pi)) = \exp(\pi_F);$$

ii.) if $\pi$ is an MC element, then $\pi_F$ is also an MC element;

iii.) $\pi$-twist $F^\pi : \mathfrak{L}^\pi \rightarrow \tilde{\mathfrak{L}}^{\pi F}$ of $F$ defined by

$$F^\pi(a) := \exp(-\pi_F) \vee F(\exp(\pi)) \vee a$$

is an $L_\infty$-morphism;

iv.) if $F$ is an $L_\infty$-morphism such that the induced morphisms

$$F|_{\mathfrak{L}^k} : \mathfrak{L}^k \rightarrow \tilde{\mathfrak{L}}^k$$

are $L_\infty$-quasi-isomorphisms for all $k$ and $\pi \in \mathfrak{L}^1\mathfrak{L}[1]^0$ is an MC element, then the $\pi$-twist $F^\pi$ of $F$ is also a quasi-isomorphism.

Here i.)-iii.) are proved through simple computations and for iv.) we refer to [5, Prop. 1].

**Remark 2.8** Note that, given $L_\infty$-morphisms $F$ and $G$ from $\mathfrak{L}$ to $\mathfrak{L}'$ and $\mathfrak{L}'$ to $\mathfrak{L}''$ respectively and the elements $\pi, B \in \mathfrak{L}^1\mathfrak{L}[1]^0$, we have that

$$\left(\pi^B\right)^B = \pi^{B_+ B} = (Q^B)^\pi, \quad (F^B)^B = F^{B_+ B} = (F^B)^\pi,$$

$$\pi_F + B_{F^\pi} = (\pi + B)_F = B_F + \pi_{F^B} \quad \text{and} \quad \left(\pi_F\right)_G = \pi_{G \circ F}.$$  

The results recalled in Lemma 2.7 can be also obtained in the setting of $L_\infty$-modules. More precisely, let $(\mathfrak{M}, \varphi)$ be an $L_\infty$-module over the $L_\infty$-algebra $(\mathfrak{L}, Q)$. The same graded (filtered) vector space $S^\even(\mathfrak{L}) \otimes \mathfrak{M}$ that forms the cofree comodule cogenerated by $\mathfrak{M}$ is simultaneously the free module generated by $\mathfrak{M}$. Thus, since $S^\even(\mathfrak{L})$ is commutative, every element $a \in S^\even(\mathfrak{L})$ defines a module morphisms of $S^\even(\mathfrak{L}) \otimes \mathfrak{M}$ that we will denote by concatenation. Then it is easy to see that given $\pi \in \mathfrak{L}^1\mathfrak{L}[1]^0$ we obtain the twisted $L_\infty$-module $(\mathfrak{M}_\pi, \varphi^\pi)$ over the twisted $L_\infty$-algebra $\mathfrak{L}_\pi$ by setting $\mathfrak{M}_\pi := \mathfrak{M}$ and

$$\varphi^\pi(X) = e^{-\pi} \varphi(e^\pi X). \quad (2.13)$$

Similarly if $F : (\mathfrak{M}, \varphi) \rightarrow (\widehat{\mathfrak{M}}, \widehat{\varphi})$ is an $L_\infty$-morphism we obtain the $L_\infty$-morphism $F^\pi : \mathfrak{M}_\pi \rightarrow \widehat{\mathfrak{M}}_\pi$ given by

$$F^\pi(X) = e^{-\pi} F(e^\pi X). \quad (2.14)$$

As before we find that if $F$ is an $L_\infty$-quas (respecting filtrations) then $F^\pi$ is also an $L_\infty$-quas.
Example 2.9 (Morphism of $L_\infty$-algebras) Consider Example 2.3, in this case the above twisting leads a priori to two different modules. Namely we can either twist the module $\varphi$ obtained from the $L_\infty$-algebra morphism $F$ or we can first twist $F$ to obtain a module structure. A straightforward check shows that these two modules coincide.

Similarly we may consider Example 2.4. In this case we can either twist the morphism $F$ or note that $F^\pi_H \circ H^\pi = (F \circ H)^\pi = G^\pi$ to obtain the morphism induced by $F^\pi_H$ directly. By the previous paragraph these are two morphisms between the same modules. In fact they coincide.

3 A global approach to twisting procedure

In [5, 6] Dolgushev uses the twisting method to obtain a certain quasi-isomorphism. This method proceeds roughly as follows. One starts with two flat $L_\infty$-algebras $\mathfrak{R}$ and $\mathfrak{L}$ and one wants to find an $L_\infty$-quasi-isomorphism $F$ between them. It is often too hard to construct such a morphism directly, but one can make use of the twisting procedure of $L_\infty$-algebras.

Let us recall the result of this procedure in order to make this point explicit. The result of the twisting procedure is that the elements $\pi \in \mathfrak{F}\mathfrak{L}\mathfrak{L}[1]^0$ parametrize a family of $L_\infty$-algebras $\mathfrak{L}^\pi$ and the subset of Maurer–Cartan elements $MC(\mathfrak{L})$ parametrize a subfamily characterized by flatness. The same is of course true for $\mathfrak{R}$. Given a morphism $G: \mathfrak{L}^\pi \to \mathfrak{R}^\pi'$ from an algebra in one family to an algebra in the other we obtain morphisms from each algebra in family parametrized by $\mathfrak{F}\mathfrak{L}\mathfrak{L}[1]^0$ to an algebra in the family parametrized by $\mathfrak{F}\mathfrak{L}\mathfrak{L}[1]^0$. Thus each such morphism also induces a family of morphisms parametrized by $\mathfrak{F}\mathfrak{L}\mathfrak{L}[1]^0$. Moreover, if we start with $\pi \in MC(\mathfrak{L})$ and $G$ is a quasi-isomorphism (respecting filtrations) then the subfamily parametrized by $MC(\mathfrak{L})$ consists of quasi-isomorphisms. Note that this is also of significance to the rest of the morphisms in the family; in a sense these are “quasi-isomorphisms” of curved $L_\infty$-algebras.

The idea used by Dolgushev, when $\mathfrak{L}$ and $\mathfrak{R}$ are given by the Fedosov resolutions of polyvector fields and polydifferential operators respectively (see section 3.1), is to look for a quasi-isomorphism in the family parametrized by $\mathfrak{F}\mathfrak{L}\mathfrak{L}[1]$ that gets twisted into a map from $\mathfrak{L}$ to $\mathfrak{R}$, thus showing that these algebras are quasi-isomorphic. Such a quasi-isomorphism is made readily available in his case by considering Kontsevich’s map from Theorem 3.4 applied fiberwise. A problem that arises is that the $L_\infty$-algebras between which Kontsevich’s map operates are not flat.

Dolgushev resolves this issue by first working locally and showing that these local solutions glue appropriately. This is not preferable since the resulting quasi-isomorphism is not explicitly realized as a twist. In this section we construct the tools needed to perform the twisting in an explicitly global manner and apply the method to establish formality of polydifferential operators on an arbitrary manifold given the formal formality of Theorem 3.4. The basic idea is to replace the “glueing” argument of Dolgushev by a resolution of $L_\infty$-modules given by a cover. We thus show that the fiberwise application of Kontsevich’s map actually yields one of the “quasi-isomorphisms” of curved $L_\infty$-algebras mentioned above. This method has the added benefit of working mutatis mutandis for the case of chains, which we will also exemplify.

Suppose $\mathfrak{L}$ is an $L_\infty$-algebra and $\mathfrak{M}$ is an $L_\infty$-module over $\mathfrak{L}$. Consider a resolution

\[
0 \to (\mathfrak{M}, \varphi) \xrightarrow{F} (\mathfrak{M}^0, \varphi^0) \xrightarrow{\partial^0} (\mathfrak{M}^1, \varphi^1) \xrightarrow{\partial^1} \ldots
\]

of $\mathfrak{M}$, which we denote by $F: \mathfrak{M} \to (\mathfrak{M}^\bullet, \partial^\bullet)$ or simply $F$. Note that this means that the graded vector spaces $\mathfrak{M}^i$ are all $L_\infty$-modules over $\mathfrak{L}$ and the maps $\partial^i$ are $L_\infty$-morphisms.

Definition 3.1 (Resolution adapted MC elements) The resolution adapted MC elements are those MC elements $\pi$ of $\mathfrak{L}$ that have the property that the induced complex

\[
0 \to H(\mathfrak{M}, \varphi_0^\pi) \xrightarrow{F^\pi_0} H(\mathfrak{M}^0, (\varphi_0^\pi)_0) \xrightarrow{\partial^0_0} H(\mathfrak{M}^1, (\varphi_0^\pi)_1) \xrightarrow{\partial^1_0} \ldots
\]

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is acyclic. The set of resolution adapted MC elements is denoted by $MC(F)$.

**Definition 3.2 (Morphisms of Resolutions)** Given resolutions $F: \mathcal{M} \to \mathcal{M}^\bullet$ and $G: \mathcal{N} \to \mathcal{N}^\bullet$ of $L_\infty$-modules over $\mathcal{L}$, a series of $L_\infty$-morphisms $U: \mathcal{M} \to \mathcal{N}$ and $U^\bullet: \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ is an $L_\infty$-morphism from $F$ to $G$ if the following diagram

\[
\begin{array}{cccccccc}
0 & \to & \mathcal{M} & \xrightarrow{F} & \mathcal{M}^0 & \xrightarrow{\partial_F^0} & \mathcal{M}^1 & \xrightarrow{\partial_F^1} & \cdots \\
& & V & \downarrow & V^0 & \downarrow & V^1 & & \\
0 & \to & \mathcal{M} & \xrightarrow{G} & \mathcal{N}^0 & \xrightarrow{\partial_G^0} & \mathcal{N}^1 & \xrightarrow{\partial_G^1} & \cdots \\
\end{array}
\]

(3.3)

commutes.

**Proposition 3.3** Suppose $U: F \to G$ is an $L_\infty$-morphism of resolutions from a resolution of the $L_\infty$-module $\mathcal{M}$ to a resolution of the $L_\infty$-module $\mathcal{N}$. Suppose further that $\pi \in MC(F) \cap MC(G)$ and $(U^n)_{\pi}$ is a quasi-isomorphism for all $n \geq 0$. Then $U_{\pi}$ is an $L_\infty$-quasi-isomorphism.

**Proof:** Since $\pi$ is adapted to both $F$ and $G$ we find that taking the cohomology with respect to the differentials on $(\mathcal{M})_{\pi}^\bullet$ and $(\mathcal{N})_{\pi}^\bullet$ yields the commutative diagram

\[
\begin{array}{cccccccc}
0 & \to & H(\mathcal{M}) & \xrightarrow{HF_{\pi}} & H(\mathcal{M}^0) & \xrightarrow{H(\partial_F^0)_{\pi}} & H(\mathcal{M}^1) & \xrightarrow{H(\partial_F^1)_{\pi}} & \cdots \\
& & H(\mathcal{N}) & \xrightarrow{HG_{\pi}} & H(\mathcal{N}^0) & \xrightarrow{H(\partial_G^0)_{\pi}} & H(\mathcal{N}^1) & \xrightarrow{H(\partial_G^1)_{\pi}} & \cdots \\
\end{array}
\]

(3.4)

with exact rows and such that the the vertical arrows to right of $H_{\mathcal{N}}_{\pi}$ are all isomorphisms. Thus $H_{\mathcal{N}}_{\pi}$ is also an isomorphism and $U_{\pi}$ is a quasi-isomorphism.

3.1 Applications: formality theorems

In the following section we apply the above result to obtain a proof of formality of Hochschild cochains and chains. The main point consists in showing that the Fedosov resolutions obtained by Dolgushev in [4,6] form resolutions of $L_\infty$-modules. Thus, formality maps can be obtained as a twisted morphism of resolutions via Prop. 3.3.

3.1.1 Formality for Hochschild cochains

As a first step we here need to recall the resolutions obtained in [4,5] of the dgla’s of poly-vector fields and poly-differential operators on a generic manifold. A more detailed discussion can also be found in [7].

Let us denote the formal neighborhood at $0 \in \mathbb{R}^d$ by $\mathbb{R}^d_{\text{formal}}$. The smooth functions $\mathcal{C}^\infty(\mathbb{R}^d_{\text{formal}})$ on $\mathbb{R}^d_{\text{formal}}$ are given by the algebra

$$\mathcal{C}^\infty(\mathbb{R}^d_{\text{formal}}) := \lim_{k \to \infty} \mathcal{C}^\infty(\mathbb{R}^d)/\mathcal{J}_0^k,$$

where $\mathcal{J}_0$ denotes the ideal of functions vanishing at $0 \in \mathbb{R}^d$. Note that $\mathcal{C}^\infty(\mathbb{R}^d_{\text{formal}})$ comes equipped with the complete decreasing filtration

$$\mathcal{C}^\infty(\mathbb{R}^d_{\text{formal}}) \supset \mathcal{J}_0 \supset \mathcal{J}_0^2 \supset \cdots$$
and corresponding (metric) topology. The Lie algebra of continuous derivations of \( \mathcal{C}^{\infty}(\mathbb{R}^d_{\text{formal}}) \) is denoted by \( T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \). Setting \( T^{-1}_{\text{poly}} := \mathcal{C}^{\infty}(\mathbb{R}^d_{\text{formal}}) \) we obtain the Lie-Rinehart pair \( (T^{-1}_{\text{poly}}, T^0_{\text{poly}}) \) and the graded vector space

\[
T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) := \bigoplus_{k \geq -1} T^k_{\text{poly}}(\mathbb{R}^d_{\text{formal}}),
\]

where \( T^k_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) := \Lambda^{k+1} T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) for \( k \geq 0 \). Here the tensor product is understood to be over \( T^{-1}_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and completed. The Lie bracket \([\cdot, \cdot]\) on \( T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) extends to a graded Lie algebra structure on \( T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \).

The universal enveloping algebra of the Lie-Rinehart pair \( (T^{-1}_{\text{poly}}(\mathbb{R}^d_{\text{formal}}), T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}})) \) is denoted by \( D^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \). We extend the algebra structure in the obvious (componentwise) way to

\[
D^k_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) := \bigoplus_{k \geq -1} D^k_{\text{poly}}(\mathbb{R}^d_{\text{formal}}),
\]

where \( D^1_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) := T^1_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and \( D^k_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) := (D^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}))^\otimes k+1 \). Here, as before, the tensor product is understood to be over \( D^1_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and completed. This allows us to define the Gerstenhaber bracket \([\cdot, \cdot]_G\) which endows \( D^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) with a graded Lie algebra structure.

**Theorem 3.4 (Kontsevich [9])** There exists an \( L_\infty \)-quasi-isomorphism between dgla’s

\[
\mathcal{K} : (T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}), 0, [\cdot, \cdot]) \rightarrow (D_{\text{poly}}(\mathbb{R}^d_{\text{formal}}), \partial, [\cdot, \cdot]_G)
\]

where \( \partial = [\mu, \cdot]_G \) for \( \mu = 1 \otimes 1 \in D^1_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \). Moreover

i.) \( \mathcal{K} \) is \( \text{GL}(d, \mathbb{R}) \) invariant;

ii.) \( \mathcal{K}(X_1 \vee \ldots \vee X_n) = 0 \) for all \( X_i \in T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and \( n > 1 \);

iii.) \( \mathcal{K}(X \vee Y_2 \vee \ldots \vee Y_n) = 0 \) for all \( Y_i \in T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and \( n \geq 2 \) whenever \( X \in T^0_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \)

is induced by the action of \( \text{gl}(d, \mathbb{R}) \).

The definitions of the dgla’s \( T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and \( D_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) go through mutatis mutandis to define the dgla’s \( T_{\text{poly}}(M) \) and \( D_{\text{poly}}(M) \) on a generic manifold \( M \), starting from the Lie-Rinehart pair \( (\mathcal{C}^{\infty}(M), \Gamma^{\infty}(TM)) \). Note that the resulting spaces \( D^k_{\text{poly}}(M) \) can be identified with the vector space of polydifferential operators of order \( k+1 \).

The bundle \( T_{\text{poly}} \) of formal fiberwise polyvector fields is the bundle over \( M \) with fiber \( T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) associated with the principal bundle of general linear frames in \( TM \). Similarly, for the bundle \( D_{\text{poly}} \) of formal fiberwise polydifferential operators. The differential forms with values in these bundles form the dgla’s \( (\Omega(M; T_{\text{poly}}), 0, [\cdot, \cdot]) \) and \( (\Omega(M; D_{\text{poly}}), \partial, [\cdot, \cdot]_G) \) respectively. The dgla structure may be induced from the dgla structures on the fibers since it is compatible with the general linear action.

Note that \( T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \rightarrow D_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) by the usual anti-symmetrization map and thus also \( \Omega(M; T_{\text{poly}}) \rightarrow \Omega(M; D_{\text{poly}}) \). So, any element \( A \in \Omega^k(M; T^k_{\text{poly}}) \) defines an operator \([A, \cdot]\) of degree \( k+\ell \) on \( \Omega(M; T^\ell_{\text{poly}}) \) and \( \Omega(M; D^0_{\text{poly}}) \). Let us consider the \( T^\ell_{\text{poly}} \) analog of the fundamental one-form used in the Fedosov construction [8, Def. 1.3.1], which we denote by \( A_{-1} \) and set \( \delta = [A_{-1}, \cdot]_G \). In local coordinates \((x^1, \ldots, x^n)\) we have \( A_{-1} = \sum_{i=1}^n \partial_i \otimes dx^i \) and thus \( [A_{-1}, A_{-1}] = 0 \) which implies that \( \delta^2 = 0 \) and yields the dgla’s \((\Omega(M; T^\ell_{\text{poly}}), -\delta, [\cdot, \cdot]) \) and \((\Omega(M; D^0_{\text{poly}}), \partial - \delta, [\cdot, \cdot]_G) \).

Following the idea of Fedosov, one changes the differential \( \delta \) by adding terms of higher degree in the fiberwise grading. This way the cohomology remains the same, but with the correct dgla
structure. Since $\delta$ is of fiberwise degree $-1$ we start by adding a linear connection $\nabla$ as a degree 0 term. It can be checked that $\nabla A_{-1}$ coincides with the $\mathcal{T}_{\text{poly}}$ equivalent of the torsion 2-form of $\nabla$. Thus by picking a torsion-free connection $\nabla$ we find that $(-\delta + \nabla)^2 = \nabla^2$. Since there is no reason to assume that we can find $\nabla$ such that $\nabla^2 = 0$ we correct $-\delta + \nabla$ by an inner derivation and make the ansatz

$$D := -\delta + \nabla + [A, \cdot]$$

(3.6)

with $A \in \Omega^1(M; \mathcal{T}_{\text{poly}}^0)$ (of fiberwise degree greater than 1). It can be proved that one may always find $A$ such that $D^2 = 0$. In fact such $A$ is unique if one adds the normalization $\delta^{-1} A = 0$. Such $A$ also yields the maps from $T_{\text{poly}}(M)$ and $D_{\text{poly}}(M)$ to $\Omega(M; \mathcal{T}_{\text{poly}})$ and $\Omega(M; \mathcal{D}_{\text{poly}})$ respectively that yield the following theorem.

**Theorem 3.5 (Fedosov Resolutions, Dolgushev [4, 5])** There exist dgla quasi-isomorphisms

$$\lambda_D : (D_{\text{poly}}(M), \partial) \to (\Omega(M; \mathcal{D}_{\text{poly}}), \partial + D) \quad \text{and} \quad \lambda_T : (T_{\text{poly}}(M), 0) \to (\Omega(M; \mathcal{T}_{\text{poly}}), D).$$

**Example 3.6 (Formality for $\mathbb{R}^d$)** In this example we generalize the result of Theorem 3.4 from $\mathbb{R}^d_{\text{formal}}$ to $\mathbb{R}^d$ by using the twisting procedure discussed in Section 2. Note that we are looking for an $L_\infty$-quasi-isomorphism

$$\mathcal{U}^{A_{-1}} : (\Omega(\mathbb{R}^d; \mathcal{T}_{\text{poly}}), D) \to (\Omega(\mathbb{R}^d; \mathcal{D}_{\text{poly}}), \partial + D),$$

since this would complete the diagram

$$\begin{array}{ccc}
(T_{\text{poly}}(\mathbb{R}^d), 0) & \xrightarrow{\lambda_T} & (\Omega(\mathbb{R}^d; \mathcal{T}_{\text{poly}}), D) \\
\xrightarrow{\mathcal{U}^{A_{-1}}} & & \xrightarrow{\lambda_D} (\Omega(\mathbb{R}^d; \mathcal{D}_{\text{poly}}), \partial + D) \\
\end{array}$$

(3.7)

of $L_\infty$-quasi-isomorphisms. Also, note that $D := -\delta + d$ follows from the natural choice $\nabla = d$. We obtain this map $\mathcal{U}^{A_{-1}}$ as follows. First we note that, by applying the map $\mathcal{U}$ from Theorem 3.4 fiberwise, we obtain the $L_\infty$-morphism

$$\mathcal{U} : (\Omega(\mathbb{R}^d; \mathcal{T}_{\text{poly}}), d) \to (\Omega(\mathbb{R}^d; \mathcal{D}_{\text{poly}}), \partial + d).$$

By considering the filtrations by exterior degree on both these algebras we construct spectral sequences which show that $\mathcal{U}$ is a quasi-isomorphism. Using this same filtration we may consider the MC element $A_{-1} \in \mathcal{F}^1 \Omega(\mathbb{R}^d; \mathcal{T}_{\text{poly}})$. Now note that $\Omega(\mathbb{R}^d; \mathcal{T}_{\text{poly}})^{A_{-1}}$ is exactly $(\Omega(\mathbb{R}^d; \mathcal{T}_{\text{poly}}), D)$ and $\Omega(\mathbb{R}^d; \mathcal{D}_{\text{poly}})^{A_{-1} \mathcal{U}}$ is exactly $(\Omega(\mathbb{R}^d; \mathcal{D}_{\text{poly}}), D)$, since $A_{-1} \mathcal{U} = A_{-1}$ by point (ii) of Theorem 3.4. So we obtain the diagram (3.7). Finally, to obtain the quasi-isomorphism

$$\mathcal{U} : T_{\text{poly}}(\mathbb{R}^d) \to D_{\text{poly}}(\mathbb{R}^d)$$

we need to invert the final arrow in the diagram (3.7). To do this we note that this arrow is actually an identification (by dgla-morphism) with the kernel of $D$ in exterior degree 0. Thus it can be inverted if we can guarantee that the map $\mathcal{U}^{A_{-1}} \circ \lambda_T$ maps $T_{\text{poly}}(\mathbb{R}^d)$ into this kernel. In [5] Dolgushev demonstrates a procedure to construct an $L_\infty$-morphism $\mathcal{V}$ homotopic to $\mathcal{U}^{A_{-1}} \circ \lambda_T$ that has this property.

The next step consists in the globalization, i.e. the generalization of the above result to any manifold $M$. The only delicate part is now the twisting procedure. Dolgushev presents the twisting locally in a way that is compatible on pairwise intersections of coordinate charts. This means that the global quasi-isomorphism is not described, a priori, as a twist of another morphism. In the following we prove that the quasi-isomorphism of Fedosov resolutions is given by a twist of the fiberwise map $\mathcal{U} : \Omega(M; \mathcal{T}_{\text{poly}}) \to \Omega(M; \mathcal{D}_{\text{poly}})$, by showing that it induces a morphism of resolutions and using
Prop. 3.3. First, we need to find suitable resolutions of $\Omega(M; T_{\text{poly}})$ and $\Omega(M; D_{\text{poly}})$. Let us fix a good cover $(U_i)_{i \in I}$ of $M$ by coordinate neighborhoods. By abuse of notation we shall denote the set of $k$-tuples $(i_1, \ldots, i_k)$ in $I$ such that $U_{i_1} \cap \cdots \cap U_{i_k} \neq \emptyset$ by $\mathcal{J}^k$. For $(i_1, \ldots, i_k) \in \mathcal{J}^k$ we shall denote $U_{i_1,\ldots,i_k} := U_{i_1} \cap \cdots \cap U_{i_k}$. As discussed in Example 2.1, $(\Omega(M; T_{\text{poly}}), \nabla - \delta([\cdot, \cdot]))$ and $(\Omega(M; D_{\text{poly}}), \partial + \nabla - \delta([\cdot, \cdot], \l))$, being curved Lie algebras, have the corresponding structure of $L_\infty$-algebras. Note that for each $a \in \mathcal{J}^k$ we obtain the curved Lie algebras $(\Omega(U_a; T_{\text{poly}}), Q^a)$ and $(\Omega(U_a; D_{\text{poly}}), P^a)$ by simply restricting the structures on $\Omega(M; T_{\text{poly}})$ and $\Omega(M; D_{\text{poly}})$ respectively. In order to obtain an $L_\infty$-algebra structure on $\check{C}^i(\mathcal{J}, \Omega(M; T_{\text{poly}})) = \prod_{a \in \mathcal{J}^i} \Omega(U_a; T_{\text{poly}})$ we need to introduce the notion of product of $L_\infty$-algebras.

Let $\{(\mathcal{L}_i, Q^i)\}_{i \in I}$ be a collection of $L_\infty$-algebras indexed over $I$. Set

$$\prod_{i \in I} \mathcal{L}_i := \left\{ f \in \text{Hom}_{\text{Set}} \left( I, \prod_{i \in I} \mathcal{L}_i \right) \mid f(i) \in \mathcal{L}_i \right\}.$$ 

Note that $\prod_{i \in I} \mathcal{L}_i$ is a vector space with $(f + g)(i) = f(i) + g(i)$, $0(i) = 0$ and $(\lambda f)(i) = \lambda f(i)$ for all $i \in I$ and $\lambda$ scalar. Moreover, $\prod_{i \in I} \mathcal{L}_i$ inherits a $\mathbb{Z}$-grading where $f$ is homogeneous of degree $n$ if $f(i)$ is homogeneous of degree $n$ for all $i \in I$. Furthermore, we obtain the projections $\prod_{i \in I} \mathcal{L}_i \to \mathcal{L}_i$ as evaluation at $i \in I$.

**Definition 3.7 (Product of $L_\infty$-algebras)** A graded vector space $\prod_{i \in I} \mathcal{L}_i$ is called product of $L_\infty$-algebras if equipped with the $L_\infty$-structure $Q$ given by the components $Q_0(1) = (i \mapsto Q_0^i(1))$ and $Q_k(f_1 \vee \cdots \vee f_k) = (i \mapsto Q_k^i(f_1(i) \vee \cdots \vee f_k(i)))$ for all $k \geq 1$. We denote the product of $(\mathcal{L}_i, Q^i)_{i \in I}$ by $\prod_{i \in I}(\mathcal{L}_i, Q^i)$.

**Example 3.8 ($\check{\text{Čech}}$ complex)** By the above definition, we immediately obtain an $L_\infty$-algebra structure on $\check{C}^i(\mathcal{J}, \Omega(M; T_{\text{poly}})) = \prod_{a \in \mathcal{J}^i} \Omega(U_a; T_{\text{poly}})$ for all $i \geq 0$ and similarly for $D_{\text{poly}}$. We denote these structures by $\check{Q}^i$ and $\check{\mathcal{J}}^i$ respectively.

As an immediate consequence of Definition 3.7 we obtain the following lemma.

**Lemma 3.9** The limit of the discrete diagram of $L_\infty$-algebras $(\mathcal{L}_i, Q^i)$ exists and is given by the product $\prod_{i \in I}(\mathcal{L}_i, Q^i)$. The relevant projection maps $\text{pr}^j : \prod_{i \in I}(\mathcal{L}_i, Q^i) \to (\mathcal{L}_i, Q^i)$ are given by the components $\text{pr}^j_0(f) = f(i)$ and $\text{pr}^j_k = 0$ for all $k > 1$.

Let $\{(\mathcal{L}_i, Q^i)\}_{i \in I}$ and $\{(\mathcal{R}_i, P^i)\}_{i \in I}$ be two collections of $L_\infty$-algebras and $F^i$ be morphisms from $\mathcal{L}_i$ to $\mathcal{R}_i$. Then we obtain the morphisms

$$F^j \circ \text{pr}^j : \prod_{i \in I}(\mathcal{L}_i, Q^i) \to (\mathcal{R}_j, P^j)$$

for all $j \in I$. By the universal property of the product we thus obtain the product morphism

$$F : \prod_{i \in I}(\mathcal{L}_i, Q^i) \to \prod_{i \in I}(\mathcal{R}_i, P^i).$$  \tag{3.8}

**Lemma 3.10** Given a collection of elements $\pi_i \in \mathcal{J}^i \mathcal{L}_i[1]^0$ and the corresponding $\pi \in \mathcal{J}^1 \prod_{i \in I} \mathcal{L}_i[1]^0$ given by $(i \mapsto \pi_i)$, we have the following properties:

i.) The $L_\infty$-algebra $\prod_{i \in I}(\mathcal{L}_i, Q^i)$ is naturally $L_\infty$-isomorphic to $\prod_{i \in I}(\mathcal{L}_i, (Q^i)^{\pi_i})$.

ii.) Given a collection of morphisms $F^i$ between $\mathcal{L}^i$ and $\mathcal{R}^i$, the collection $(F^i)^{\pi_i}$ induces the $L_\infty$-morphism $F^\pi$ using the notation of (3.8).
iii.) If all the elements $\pi_i$ are Maurer–Cartan, then $\pi$ is Maurer–Cartan.

PROOF: The first claim follows easily from the formula

$$Q_k^\pi(\gamma_1, \ldots, \gamma_k) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} Q_{k+\ell}(\pi^\ell \vee \gamma_1 \vee \ldots \vee \gamma_k)$$

for $Q^\pi$. The second claim follows from the similar expression

$$F_k^\pi(\gamma_1 \vee \ldots \vee \gamma_k) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} F_{k+\ell}(\pi^\ell \vee \gamma_1 \vee \ldots \vee \gamma_k)$$

for the twist of an $L_\infty$-morphism. Finally by the first point we have $Q^\pi(1)i = (Q^\pi)^{\pi}(1)$ which shows the third claim.

Recall the $L_\infty$-structures on the components of the Čech complex $\check{C}^\bullet(\beta; \Omega(M; T_{\text{poly}}))$ from Example 3.8. For $i \geq 0$ and each $a \in \beta^i$ the restriction map from $\Omega(M; T_{\text{poly}})$ to $\Omega(U_a; T_{\text{poly}})$ induces the $L_\infty$-morphism $R^a$ where $R^a_i = 0$ for all $i > 1$ and where $R^a_1$ is the restriction map. By the universal property of the product these maps combine into the $L_\infty$-morphisms

$$\mathcal{R}^i: \Omega(M; T_{\text{poly}}) \to \check{C}^i(\beta; \Omega(M; T_{\text{poly}})). \quad (3.9)$$

We equip the components $\check{C}^i(\beta; \Omega(M; T_{\text{poly}}))$ of the Čech complex with the structures $\varphi^i$ of $L_\infty$-modules over $\Omega(M; T_{\text{poly}})$ induced from the maps $\mathcal{R}^i$ as in Example 2.3. In the case of $\Omega(M; \mathcal{D}_{\text{poly}})$ we denote the restriction maps by $A^i$. Note that, since $\nabla$ is a linear connection and by Theorem 3.4 iii.), the fiberwise application of $\nabla$ yields the $L_\infty$-morphism $\nabla$ from $\Omega(M; T_{\text{poly}})$ to $\Omega(M; \mathcal{D}_{\text{poly}})$ with the $L_\infty$-structures with first Taylor coefficients $\nabla - \delta$ and $\delta + \nabla - \delta$ respectively. Thus we may equip the components $\check{C}^i(\beta; \Omega(M; \mathcal{D}_{\text{poly}}))$ of the Čech complex of $\Omega(M; \mathcal{D}_{\text{poly}})$ with the $\Omega(M; T_{\text{poly}})$-module structures $\psi^i$ obtained through example 2.3 by the maps $A^i \circ \nabla$. Of course we may also consider $\Omega(M; \mathcal{D}_{\text{poly}})$ itself as an $\Omega(M; T_{\text{poly}})$ module by applying example 2.3 to the map $\nabla$.

Lemma 3.11 The sequence

$$0 \to \Omega(M; T_{\text{poly}}) \xrightarrow{\mathcal{R}^0} \check{C}^0(\beta; \Omega(M; T_{\text{poly}})) \xrightarrow{\partial_0} \check{C}^1(\beta; \Omega(M; T_{\text{poly}})) \xrightarrow{\partial_1} \ldots \quad (3.10)$$

forms a resolution of $\Omega(M; T_{\text{poly}})$ and there is a similar resolution $A^0$ of $\mathcal{D}_{\text{poly}}$.

PROOF: The maps $\partial_i$ are given by the components $(\partial_i)_j = 0$ for all $j > 0$ and $(\partial_i)_0$ simply the Čech differential. Thus it is clear that $\partial_0 \circ \mathcal{R}^0 = 0$ and $\partial_{i+1} \circ A_i = 0$ for all $i \geq 0$ and we only need to show that the maps $\partial_i$ and $\mathcal{R}^0$ indeed define $L_\infty$-module morphisms.

Note that $\mathcal{R}^0$ is automatically a map of $L_\infty$-modules by Example 2.4, but the same is not true of the $\partial_j$ since they do not preserve the $L_\infty$-algebra structure. Note that by definition of the maps $\partial_j$ we have that $\partial_j(\gamma_1 \vee \ldots \vee \gamma_n \otimes m) = \gamma_1 \vee \ldots \vee \gamma_n \otimes \partial_j m$. Thus the fact that the $\partial_j$ are maps of $L_\infty$-modules follows from the fact that

$$\partial_j \varphi^i_k(\gamma_1 \vee \ldots \vee \gamma_n \otimes m) = \varphi^i_k(\gamma_1 \vee \ldots \vee \gamma_n \otimes \partial_j m). \quad (3.11)$$

This is established by simply writing out the definitions of both sides.

Similarly the map $A^0$ yields a map of $L_\infty$-modules by applying Example 2.4 and Eq. (3.11) holds with $\varphi$ replaced by $\psi$. As with $\Omega(M; T_{\text{poly}})$ we have $\partial_0 \circ A^0 = 0$ and $\partial_{j+1} \circ A_j = 0$ for all $j \geq 0$. ⊳
Replacing $M$ by $U_a$ with $a \in \mathcal{I}^k$ and $k \geq 0$ in the discussion of $\mathcal{U}$ preceding Lemma 3.11 and inducing maps to the product (as done in the discussion after Lemma 3.10), we obtain the $L_\infty$-morphisms

$$\mathcal{U}^k : \check{C}^k(\mathcal{F}, \Omega(\pi; \mathcal{I})) \to \check{C}^k(\mathcal{F}, \Omega(M; \mathcal{T}_{\text{poly}}))$$

(3.12)

for all $k \geq 0$. Note that for each $k \geq 0$ this yields the commuting diagram

$$
\begin{array}{ccc}
\Omega(M; \mathcal{T}_{\text{poly}}) & \xrightarrow{\mathcal{U}^k} & \check{C}^k(\mathcal{F}, \Omega(M; \mathcal{T}_{\text{poly}})) \\
\downarrow & & \downarrow \\
\Omega(M; \mathcal{D}_{\text{poly}}) & \xrightarrow{\mathcal{A}^0} & \check{C}^k(\mathcal{F}, \Omega(M; \mathcal{D}_{\text{poly}}))
\end{array}
$$

(3.13)

which allows us to realize the maps $\mathcal{U}^k$ as maps of $L_\infty$-modules over $\Omega(M; \mathcal{T}_{\text{poly}})$ by using Example 2.4. We denote these maps by $\mathcal{U}^k$ again since it should not cause any confusion. As an immediate consequence we can prove the following lemma.

**Lemma 3.12** The maps $\mathcal{U}$ and $\mathcal{U}^\bullet$ form a morphism from the resolution $\mathcal{R}^0$ to the resolution $\mathcal{A}^0$.

The next step consists in constructing a resolution adapted Maurer–Cartan element. Let us consider the one-forms $A_{-1}$ and $A$ in $\Omega(M; \mathcal{T}_{\text{poly}})$ introduced above in order to construct the Fedosov differential as in (3.6).

**Lemma 3.13** The element $\Gamma := A_{-1} + A \in \Omega^1(M; \mathcal{T}_{\text{poly}})$ is a Maurer–Cartan element adapted to both $\mathcal{R}^0$ and $\mathcal{A}^0$.

**Proof:** First of all we note that the filtration appearing in the definition of a Maurer–Cartan element is given here by the exterior degree. Consider the Eq. (3.6) and note that the condition $D^2 = 0$ says that

$$\nabla^2 + [\nabla \Gamma, \cdot ] + \frac{1}{2}[\Gamma, \Gamma] = 0,$$

which (together with $\delta^{-1}A = 0$) implies that

$$R + \nabla \Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0,$$

where we have denoted the curvature of $\nabla$ by $R$, i.e. $R \in \Omega^2(M; \mathcal{T}_{\text{poly}})$ is defined by $\nabla^2 \alpha = [R, \alpha]$. This shows that $\Gamma$ is an MC-element and it is left to show that it is adapted to both $\mathcal{R}^0$ and $\mathcal{A}^0$.

Suppose we have that $\partial_j^\pi = \partial_j$ for all $j \geq 0$. Then $\pi$ would obviously be adapted to both $\mathcal{R}^0$ and $\mathcal{A}^0$, since the twisted resolution would simply be the Čech complex of polyvector fields and polydifferential operators respectively. These two sheaves are fine, thus the corresponding Čech complexes on a good cover are acyclic. Now simply note that indeed $\partial_j^\pi = \partial_j$ for all $j \geq 0$ since the Taylor coefficients of these maps vanish except in the lowest order and we may use the formula from the proof of Lemma 3.10.

Finally, using the techniques introduced above we can give another proof of the formality theorem that we state below.

**Theorem 3.14** The dgla’s $\mathcal{T}_{\text{poly}}(M)$ and $\mathcal{D}_{\text{poly}}(M)$ are quasi-isomorphic.

**Proof:** We prove that the $L_\infty$-morphisms

$$
\begin{array}{ccc}
(\mathcal{T}_{\text{poly}}(M), 0) & \xrightarrow{\lambda^\mathcal{T}} & (\Omega(M; \mathcal{T}_{\text{poly}}), D) \\
& \xrightarrow{\mathcal{U}^\Gamma} & (\Omega(M; \mathcal{D}_{\text{poly}}), \partial + D) \\
& \xrightarrow{\lambda^\mathcal{D}} & (\mathcal{D}_{\text{poly}}(M), \partial)
\end{array}
$$

(3.14)
are all quasi-isomorphisms. Then, in order to obtain the quasi-isomorphism \( T_{\text{poly}}(M) \rightarrow D_{\text{poly}}(M) \) we only need to invert the final arrow in (3.14). This is done in the same way as in Example 3.6 (see [4, section 4.2]). From Prop 3.12, \( \mathcal{W} \) is an \( L_{\infty} \)-morphism of resolutions. Thus, using Lemma 3.13, Proposition 3.3, Lemma 3.10 and Example 2.4 it is enough to show that \( \mathcal{W}^\Gamma \) is a quasi-isomorphism on the \( U_a \) for \( a \in \mathcal{J}^k \) and \( k \geq 1 \). To show this we note first that

\[
\mathcal{W} : (\Omega(U_a; T_{\text{poly}}), d, [\cdot, \cdot]) \rightarrow (\Omega(U_a; D_{\text{poly}}), d, [\cdot, \cdot]_G)
\]

is a well-defined \( L_{\infty} \)-quasi-isomorphism. On \( U_a \) we have the decomposition \( \nabla = d + [B_a, \cdot] \) for some \( B_a \in \Omega(U_a; T_{\text{poly}}) \). By Theorem 3.4 it follows that

\[
\mathcal{W}^{B_a + \Gamma} = \mathcal{W}^\Gamma : (\Omega(U_a; T_{\text{poly}}), D, [\cdot, \cdot]) \rightarrow (\Omega(U_a; D_{\text{poly}}), D, [\cdot, \cdot]_G).
\]

Now since \( \mathcal{W} \) was a quasi-isomorphism this proves that \( \mathcal{W}^\Gamma \) is a quasi-isomorphism, since \( B_a + \Gamma \) is an MC-element (see [5]).

**Remark 3.15 (Formality for Lie algebroids)** Formality for Lie algebroids has been proved in [1] and also uses the twisting procedure. We here remark that the techniques discussed above also apply to formality for Lie algebroids. Similarly, the authors conjecture that this observation immediately extend to the result presented in [3].

### 3.1.2 Formality for Hochschild chains

Formality for Hochschild chains has been conjectured in [13] and proved by Dolgushev in [6] by using the globalization techniques proposed in [4, 5] and the local formality for Hochschild chains proved by Shoikhet in [11]. Here we briefly recall the Fedosov resolutions proved in [6, section 4]. In the previous we were concerned with the analog \( D_{\text{poly}}(M) \) of Hochschild cochains on \( C^\infty(M) \). Similarly we consider the analog \( C^\text{poly}(M) \) of Hochschild chains on \( M \) given by

\[
C_{-n}^\text{poly}(M) = C_{n+1}^\infty(M), \quad C_0^\text{poly}(M) = C_1^\infty(M),
\]

where \( M^{n+1} \) denotes the \( n + 1 \)-fold Cartesian product of \( M \) with itself. The space \( C^\text{poly}(M) \) can be naturally endowed with a structure of graded module over the Lie algebra \( D_{\text{poly}}(M) \) and we denote the corresponding action by \( \rho \). The multiplication \( \mu \) in the algebra \( C^\infty(M) \) induces a differential \( b \) on \( C^\text{poly}(M) \) by

\[
b := \rho_\mu : C_n^\text{poly}(M) \rightarrow C_{n+1}^\text{poly}(M).
\]

It is easy to see that \( (C^\text{poly}(M), b) \) is a dg module over the dgla \( D_{\text{poly}}(M) \) (see [6, section 3]). Its cohomology is isomorphic, as a vector space, to the space \( A^\bullet(M) \) of forms on \( M \) with an inverted grading, as proved by Teleman in [12]. The dg \( D_{\text{poly}}(M) \)-module structure on \( C^\text{poly}(M) \) induces a dg \( T_{\text{poly}}(M) \)-module structure on \( A^\bullet(M) \), with action denoted by \( \lambda \) (defined by the action of a polyvector field on exterior forms via the Lie derivative). The first step to find Fedosov resolutions for \( C^\text{poly}(M) \) and \( A^\bullet(M) \) consists in a local statement. Note that composing the quis \( \mathcal{W} \) discussed in Example 3.6 with the action \( \rho \) we obtain an \( L_{\infty} \)-module structure \( \psi \) on \( C^\text{poly}(\mathbb{R}^d) \) over the dgla \( T_{\text{poly}}(\mathbb{R}^d) \).

**Theorem 3.16 (Shoikhet, [11])** There exists a quis

\[
\mathcal{J} : (C^\text{poly}(\mathbb{R}^d), b) \rightarrow (A^\bullet(\mathbb{R}^d), 0)
\]

of \( L_{\infty} \)-modules over \( T_{\text{poly}}(\mathbb{R}^d) \), with actions given by \( \psi \) and \( \lambda \) respectively and where \( \mathcal{J}_0 \) is given by Teleman’s theorem and satisfying the same properties of Theorem 3.4.
Let us denote the bundle of formal fiberwise Hochschild chains whose fibers are dg $D_{\text{poly}}(\mathbb{R}^d)$-modules by $\mathcal{E}_{\text{poly}}$. Similarly, we consider the bundle $\mathcal{E}$ of formal fiberwise exterior forms, i.e., exterior forms with values in the bundle of the formally completed symmetric algebra $S\mathcal{M}$ of $T^*\mathcal{M}$. Clearly $(\Omega(M; \mathcal{E}), 0)$ and $(\Omega(M; \mathcal{E}_{\text{poly}}), b)$ are fiberwise dglas over $\Omega(M; T_{\text{poly}}^\infty)$ and $\Omega(M; D_{\text{poly}}^\infty)$, respectively. We denote the fiberwise Lie derivative on $\Omega(M; \mathcal{E})$ and the fiberwise action of $\Omega(M; \mathcal{E}_{\text{poly}})$ again by $\lambda$ and $\rho$, resp. Also, the differential on $\Omega(M; \mathcal{E}_{\text{poly}})$ can be written as $b = \rho_\mu$ with $\mu \in D_{\text{poly}}^1$. Finally, in complete analogy with the above discussion, one obtains the following statement.

**Theorem 3.17 (Fedosov Resolutions, Dolgushev [6])** There exist quasi-isomorphisms of dglas

$$\lambda_A: (\Lambda^*(M), 0) \longrightarrow (\Omega(M; \mathcal{E}), D) \quad \text{and} \quad \lambda_C: (\mathcal{C}_{\text{poly}}(M), \rho_\mu) \longrightarrow (\Omega(M; \mathcal{E}_{\text{poly}}), D + \rho_\mu),$$

where $D$ denotes the Fedosov differential.

Composing the fiberwise quis $\mathcal{Y}: (\Omega(M; T_{\text{poly}}^\infty), 0) \longrightarrow (\Omega(M; D_{\text{poly}}^\infty), \partial)$ with the fiberwise action $\rho$ of $\Omega(M; D_{\text{poly}}^\infty)$ on $\Omega(M, \mathcal{E}_{\text{poly}})$ we obtain an $L_\infty$-module structure on $\Omega(M, \mathcal{E}_{\text{poly}})$ over $\Omega(M; T_{\text{poly}}^\infty)$, also denoted by $\psi$. Moreover, from Theorem 3.16 we obtain a fiberwise quis, denoted by $\mathcal{Y}$ between the modules $(\Omega(M, \mathcal{E}_{\text{poly}}), b, \psi)$ and $(\Omega(M; \mathcal{E}), 0, \lambda)$. Thus we find that $\mathcal{Y}$ is a morphism of the $L_\infty$-modules $(\Omega(M, \mathcal{E}_{\text{poly}}), b)$ and $(\Omega(M; \mathcal{E}), 0)$ over $(\Omega(M; T_{\text{poly}}^\infty), 0, [\cdot, \cdot])$. It is easy to observe that, in analogy with last section, the sequence

$$0 \longrightarrow \Omega(M; \mathcal{E}_{\text{poly}}) \overset{\mathcal{Y}_0}{\longrightarrow} \hat{C}^0(\Omega(M; \mathcal{E}_{\text{poly}})) \overset{\partial_1}{\longrightarrow} \hat{C}^1(\Omega(M; \mathcal{E}_{\text{poly}})) \overset{\partial_1}{\longrightarrow} \ldots \quad (3.16)$$

forms a resolution of $\Omega(M; \mathcal{E}_{\text{poly}})$. Similarly, there is a resolution of $\Omega(M; \mathcal{E})$. Using the same argument as for the morphism $\mathcal{Y}$ we obtain:

**Lemma 3.18** $\mathcal{Y}: \Omega(M, \mathcal{E}_{\text{poly}}) \longrightarrow \Omega(M; \mathcal{E})$ is a morphism of resolutions.

Finally, as an immediate consequence of the above lemma we can prove formality for Hochschild chains as follows.

**Theorem 3.19** The dg modules $(T_{\text{poly}}^*(M), A^*(M))$ and $(D_{\text{poly}}^*(M), C_{\text{poly}}^*(M))$ are quasi-isomorphic.

**Proof:** Twisting the resolution morphism $\mathcal{Y}$ by $\Gamma := A_{-1} + A \in \Omega^1(M; T^\infty_{\text{poly}})$ and using Prop. 3.3 we obtain the $L_\infty$-quasi-isomorphism $\mathcal{Y}_\Gamma: (\Omega(M, \mathcal{E}_{\text{poly}}), D + \rho_\mu) \longrightarrow (\Omega(M; \mathcal{E}), D)$ as in theorem 3.14. In fact, given the $L_\infty$-quasi-isomorphism $\mathcal{Y}_\Gamma$ it is not hard to show that the dglas $\Omega(M; \mathcal{E})$ and $\Omega(M, \mathcal{E}_{\text{poly}})$ over $(\Omega(M; T_{\text{poly}}^\infty), D)$ and $(\Omega(M; D_{\text{poly}}^\infty), D + \partial)$, resp. obtained by twisting via $\Gamma$ coincide with those defined by the fiberwise structures $\lambda$ and $\rho$. This concludes the proof. $\heartsuit$

**Remark 3.20 (Formality for chains in the Lie algebroid setting)** The same techniques can be used to prove formality for Hochschild chains in the Lie algebroid setting, whose original proof can be found in [2].

**References**

[1] Calaque, D.: *Formality for Lie algebroids*. Comm. Math. Phys. 257,3 (2005).

[2] Calaque, D., Dolgushev, V. A., Halbout, G.: *Formality theorems for Hochschild cochains in the Lie algebroid setting*. Crelle’s J. reine angew. Math. 612 (2007).

[3] Chevla, S.: *Formality theorem with coefficients in a module*. Transf. Groups 13 (2008).
[4] Dolgushev, V. A.: Covariant and equivariant formality theorems. Adv. Math. 191.1 (2005), 147–177.

[5] Dolgushev, V. A.: A Proof of Tsygan’s Formality Conjecture for an Arbitrary Smooth Manifold. PhD thesis, MIT, 2005.

[6] Dolgushev, V. A.: A Formality Theorem for Hochschild Chains. Adv. Math. 200.1 (2006), 51–101.

[7] Esposito, C., de Kleijn, N.: Universal Deformation Formula, Formality and Actions. arXiv:1704.07054, 2017.

[8] Fedosov, B.: Deformation quantization and Index theory. Akademie-Verlag, 1996.

[9] Kontsevich, M.: Deformation Quantization of Poisson Manifolds. Lett. Math. Phys. 66 (2003), 157–216.

[10] Quillen, D.: Rational homotopy theory. Ann. of Math. 90.2 (1969), 205–295.

[11] Shoikhet, B.: A proof of the Tsygan formality conjecture for chains. Adv. Math. 179.1 (2003), 7–37.

[12] Teleman, N.: Microlocalisation de l’homologie de Hochschild. C.R. Acad. Sci.Paris 326 (1998), 1261–1264.

[13] Tsygan, B.: Formality conjecture for chains. arXiv:math/9904132, 1999.