NORMAL ALMOST CONTACT STRUCTURES AND
NON-KÄHLER COMPACT COMPLEX MANIFOLDS

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ABSTRACT. We construct some families of complex structures on compact manifolds by means of normal almost contact structures (nacs) so that each complex manifold in the family has a non-singular holomorphic flow. These families include as particular cases the Hopf and Calabi-Eckmann manifolds and the complex structures on the product of two normal almost contact manifolds constructed by Morimoto. We prove that every compact Kähler manifold admitting a non-vanishing holomorphic vector field belongs to one of these families and is a complexification of a normal almost contact manifold. Finally we show that if a complex manifold obtained by our constructions is Kählerian the Euler class of the nacs (a cohomological invariant associated to the structure) is zero. Under extra hypothesis we give necessary and sufficient conditions for the complex manifolds so obtained to be Kählerian.

1. INTRODUCTION

Most of the known examples of complex manifolds, in particular all projective manifolds, are of Kähler type. Nevertheless, the existence of a Kähler metric imposes strong topological restrictions on the manifold, for instance its odd Betti numbers are even. Riemann surfaces are always Kählerian and compact complex surfaces if and only its first Betti number is even (c.f. [6],[17]). For higher dimensions there is not a simple characterization of Kähler manifolds, however one would expect that they are rather the exception than the rule. For instance a corollary of a result by Taubes (c.f [26]) implies that every finite presentation group is the fundamental group of a non-Kähler compact complex 3-manifold. Historically the first examples of non-Kähler manifolds were constructed by H. Hopf as a quocient of \( \mathbb{C}^n - \{0\} \) for \( n > 1 \) by a contracting holomorphism of \( \mathbb{C}^n \) which fixes the origin. Later, E. Calabi and B. Eckmann described a class of non-Kähler complex structures on the product \( S^{2n+1} \times S^{2m+1} \) for \( n, m \geq 0 \) such that the corresponding complex manifold is the total space of a holomorphic elliptic principal bundle over \( \mathbb{P}^n \times \mathbb{P}^m \). In [20] J.J. Loeb and M. Nicolau generalized Calabi-Eckmann and Hopf structures by the construction of a class of complex structures on the product \( S^{2n+1} \times S^{2m+1} \) that contains the precedents. Similar techniques have been used by S.López de Medrano and A.Verjovsky in [21] to construct another family of non-Kählerian compact manifolds and later generalized by L.Meersseman in [22].

The second section of the paper is devoted to describe a general procedure to obtain complex manifolds by means of elementary geometrical constructions. We
departure from odd-dimensional manifolds admitting a normal almost contact structure (nacs for shortness), i.e. a CR-structure of maximal dimension and a transverse CR-action of \( \mathbb{R} \). More precisely we consider three cases: (A) products of two real manifolds endowed with a nacs, (B) \( S^1 \)-principal bundles over a manifold with a nacs (with an extra restriction on the bundle) and (C) suspensions of a manifold with a nacs by a suitable automorphism. In particular we generalize Morimoto’s construction of a complex structure on a product of two normal almost contact manifolds (c.f. [23]). The constructions of cases A and C produce a compact complexification of the original normal almost contact manifold \( M \), i.e. a compact complex manifold \( X \) such that \( M \) is a real submanifold of \( X \) so that its CR-structure is compatible with the complex structure of \( X \) and a holomorphic vector field on \( X \) whose real part is the vector field of the CR-action on \( M \). Moreover we prove that given a compact Kähler manifold admitting a holomorphic vector field without zeros its complex structure can be recovered by the construction of case C. Therefore, every compact Kähler manifold admitting a non-vanishing holomorphic vector field is a compact complexification of a nacs. We also show that double suspensions of compact complex manifolds by two commuting automorphisms (see p.8), which can be obtained by means of the construction of case C, present a remarkable property. Namely, every compact Kähler manifold admitting a non-vanishing holomorphic vector field can be endowed with a complex structure on the underlying smooth manifold arbitrarily close to the original one which turns it into a double suspension.

In the third section we study criteria to determine when the complex manifolds of the above three families are Kählerian in terms of properties of the departing nacs. The common feature of all the complex manifolds that we construct is the existence of a holomorphic vector field without zeros. Using this fact we will show that for these complex structures to be Kählerian the Euler class of the nacs must be zero. When the flows associated to the nacs are isometric we prove that a compact complex manifold obtained by the constructions of cases A or B is Kähler if and only if the Euler class (or classes) is zero and the flows (or flows) is transversely Kählerian. For suspensions (case C) of a normal almost contact manifold \( M \) by an automorphism \( f \) we give a complete characterization when the CR-structure is Levi-flat and \( f^* = id \) acting on \( H^1(M, \mathbb{C}) \). Finally, we prove that a double suspension is Kählerian if and only if the departing manifold is Kählerian and the two automorphisms preserve a Kähler class.

Throughout the paper all manifolds are supposed to be smooth and connected and all differentiable objects to be of class \( C^\infty \).

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2. Compact complex structures defined from nacs

2.1. Normal almost contact structures (nacs). Recall that a complex subbundle \( \Phi^{1,0} \) of dimension \( m \) of the complexified tangent bundle \( T^CM = TM \otimes \mathbb{C} \) of a manifold \( M \) is called a CR-structure on \( M \) of dimension \( m \) (cf. [16] or [3]) if:
(i) $\Phi^{1,0} \cap \overline{\Phi^{1,0}} = \{0\};$

(ii) $\Phi^{1,0}$ is involutive, i.e. $[\Phi^{1,0}, \Phi^{1,0}] \subset \Phi^{1,0}.$

The complex bundle $\Phi^{1,0}$ induces a real subbundle $D = TM \cap (\Phi^{1,0} \oplus \overline{\Phi^{1,0}})$ of $TM.$ We define an endomorphism $J : D \to D$ imposing that $v - iJu \in \Phi^{1,0}$ for every $v \in D.$ Note that we can determine the CR-structure by giving $(M, D, J).$ Setting $\Phi^{0,1} = \overline{\Phi^{1,0}}$ we have a decomposition $D \cong \Phi^{1,0} \oplus \Phi^{0,1}$ where $\Phi^{1,0}$ and $\Phi^{0,1}$ are the eigenspaces of $J$ (extended by complex linearity to $D \otimes \mathbb{C}$) of eigenvalue $i$ and $-i$ respectively. We denote by $\text{Aut}_{CR}(M)$ the subset ofDiff$(M)$ of maps $f$ such that $df$ preserves $D$ and commutes with $J.$ Let $\{\varphi_t : t \in \mathbb{R}\}$ be the flow induced by a smooth $\mathbb{R}$-action on $M.$ We say that $\{\varphi_t\}$ defines a CR-action if $\varphi_t \in \text{Aut}_{CR}(M)$ for each $t.$ When $	ext{dim}_M M = 2m + 1$ we call the action transverse to the CR-structure if the smooth vector field $T = (d\varphi_t/dt)|_{t=0}$ is everywhere transverse to $D,$ i.e. $(T) \oplus D$ has real dimension $2m + 1$ at every point.

**Definition 2.1.1.** A normal almost contact structure (or nacs) on a manifold $M$ of odd-dimension is a pair $(\Phi^{1,0}, \varphi_t)$ where $\Phi^{1,0}$ is a CR-structure of maximal dimension and $\{\varphi_t\}$ a flow induced by a smooth $\mathbb{R}$-action defining a transverse CR-action. Given a nacs $(\Phi^{1,0}, \varphi_t)$ we define its characteristic 1-form $\omega$ by the conditions $\omega(T) = 1$ and $\ker \omega = D$ (where $T = (d\varphi_t/dt)|_{t=0}$ and $D = TM \cap (\Phi^{1,0} \oplus \overline{\Phi^{1,0}})$) and its associated flow $F$ as the flow induced by $\{\varphi_t\},$ which is transversely holomorphic.

Alternatively a nacs can be determined by an endomorphism $\varphi$ on the tangent space, a vector field $T$ and a 1-form $\omega.$ The tercet $(\varphi, T, \omega)$ is called an almost contact structure on $M$ if: (1) $\omega(T) = 1,$ (2) rank $\varphi = 2n,$ (3) $\varphi(T) = 0,$ (4) $\omega(\varphi(X)) = 0$ and $\varphi^2(X) = -X + \omega(X)T$ for every tangent vector field $X$ on $M.$ There is an almost contact structure on $\mathbb{R}$ given by $(\frac{\partial}{\partial t}, dt).$ If $M_1$ has an almost contact structure $(\varphi_1, T_1, \omega_1)$ then there is an almost complex structure $K$ on $M_1 \times \mathbb{R}$ defined by $K(X_1, a\frac{\partial}{\partial t}) = (\varphi_1(X_1) - aT_1, \omega_1(X_1)\frac{\partial}{\partial t}).$ The almost contact structure on $M_1$ is called normal if $K$ is integrable (cf. [1]). It is not difficult to see that the two definitions are equivalent.

Recall that a form $\omega \in \Omega^1(M)$ is called basic with respect to a foliation $F$ if $i_S \omega = i_S d\omega = 0$ for every vector field $S$ tangent to the leaves of $F.$

**Lemma 2.1.2.** Let $F$ be a transversely holomorphic flow on a compact manifold $M$ generated by a real vector field $T$ without zeros and let $\omega$ be a 1-form such that $\omega(T) = 1.$ Set $D = \ker \omega$ and let $J$ be the almost-complex structure on $D$ induced by $F.$ Then $(D, J, T)$ is a nacs if and only if $i_T d\omega = 0$ and the basic form $d\omega$ is of type $(1,1)$ with respect to the holomorphic structure transverse to $F.$

**Proof.** Let $\Phi^{1,0}$ the vectors in $D^C$ of type $(1,0)$ with respect to $J.$ The vector field $T$ preserves $D,$ i.e. $[T, D] \subset D$ if and only if $i_T d\omega = 0.$ In this case $\Phi^{1,0}$ defines a CR-structure, i.e. $[\Phi^{1,0}, \Phi^{1,0}] \subset \Phi^{1,0},$ if and only if $d\omega$ is of type $(1,1)$ with respect to the complex structure transverse to $F.$ It is clear that then $T$ defines a transverse CR-action.

Let $M$ be a compact manifold and $F$ a transversely holomorphic isometric flow on $M$ defined by a Killing vector field $T.$ If there exists a characteristic 1-form $\omega$ (which verifies $\omega(T) = 1$ and $i_T d\omega = 0$) such that $d\omega$ is of type $(1,1)$ then $M$ admits a nacs. Analogously a $S^1$-principal bundle over a compact complex manifold admits a nacs provided that we can choose a connection 1-form such that its curvature form is of
type \((1, 1)\). It is well known that a \(S^1\)-principal bundle admits such a connection form if and only if it is the unit bundle associated to a hermitian metric on a holomorphic line bundle. More generally, compact Seifert fibrations over a complex orbifold also provide examples of transversely holomorphic isometric flows, therefore they admit a nacs provided that there exists a suitable characteristic 1-form. Notice that if \(\dim \mathbb{R} M = 3\) the last condition of the lemma is always fulfilled since every 2-form on a compact Riemann surface is of type \((1, 1)\).

It is also known that a compact connected Lie group \(K\) of odd dimension greater than one always admits a non left-invariant nacs (cf. [19]).

Let now \((D, J)\) be a CR-structure on \(M\) and suppose that the distribution \(D\) is a contact structure, i.e. the characteristic 1-form \(\omega\) verifies \(\omega \wedge (d\omega)^n \neq 0\). Then \((D, J)\) is called a strictly pseudo-convex CR-structure on \(M\). The couple of a strictly pseudo-convex CR-structure of maximal dimension and a transverse CR-action of \(\mathbb{R}\) on an odd-dimensional manifold is also known as a normal contact structure. For compact connected 3-manifolds it is known that if they admit a normal contact structure the vector field defining the CR-action is Killing (c.f. [1]). The opposite situation to a strictly pseudo-convex CR-structure from the point of view of the real integrability of the distribution \(D\) is Levi-flatness, that is, the condition \(\beta \wedge d\beta \equiv 0\) where \(\beta\) is a 1-form such that \(\ker \beta = D\) or equivalently \(d\omega = 0\) where \(\omega\) is the characteristic 1-form of the nacs. In this case we can easily construct examples of nacs such that its associated flow is not isometric. Recall that the suspension of a compact manifold \(N\) by \(g \in \text{Diff}(N)\) is the compact manifold \(N \times \mathbb{R}\) given by \(N \times \mathbb{R}/ \sim\) where \((z, s) \sim (g(z), s + 1)\). When \(N\) is a compact complex manifold and \(g \in \text{Aut}_{\mathbb{C}}(N)\) the suspension \(N \times \mathbb{R}\) carries a natural Levi-flat CR-structure defined by \(TN\) and a transverse CR-action induced by \(\frac{\partial}{\partial s}\). If we choose \(g\) such that it is not an isometry for any metric on \(N\), for instance \(N = \mathbb{C}P^1\) and \(g(z) = \lambda z\) with \(\lambda \in \mathbb{C}\) such that \(|\lambda| \neq 1\), the flow \(F\) generated by \(\frac{\partial}{\partial s}\) is clearly not isometric for any Riemannian metric on \(N \times \mathbb{R}\).

2.2. The Euler Class. We introduce here the notion of Euler class, which generalizes the classical notion of Euler class of an isometric flow.

**Definition 2.2.1.** Let \(M\) be a compact manifold endowed with non-vanishing vector field \(T\) and a 1-form \(\omega\) such that \(\omega(T) = 1\) and \(i_T d\omega = 0\). We denote by \(\mathcal{F}\) the flow induced by \(T\). We define the Euler class of the pair \((M, T)\) as the basic cohomology class given by

\[
e_{\mathcal{F}}(M) = [d\omega] \in H^2(M/\mathcal{F}).
\]

Recall that the basic cohomology \(H^*(M/\mathcal{F})\) is defined as the cohomology of the differential complex \((\Omega^*(M/\mathcal{F}), d)\) of basic forms for \(\mathcal{F}\). Note that the class \(e_{\mathcal{F}}(M)\) depends on the vector field \(T\) but not on the 1-form, provided that it verifies the above conditions. In particular we can consider the Euler class of a nacs. As the vanishing of the Euler class will be a necessary condition for the complex manifolds that we construct to be Kählerian we will discuss some criteria to determine when it is zero.

**Lemma 2.2.2.** Let \(M\) be a compact manifold endowed with non-vanishing vector field \(T\) and a 1-form \(\omega\) such that \(\omega(T) = 1\) and \(i_T d\omega = 0\). We denote by \(\mathcal{F}\) the flow induced by \(T\). Then the following conditions are equivalent:
There exists a closed 1-form $\chi$ on $M$ such that $\chi(T) = 1$.

Proof. To prove (a) $\Rightarrow$ (b) note that if $e_F(M) = 0$ then there exists a basic 1-form $\alpha$ such that $d\alpha = d\omega$. Then $\chi = \omega - \alpha$ is a closed 1-form such that $\chi(T) = 1$. To see that (b) $\Rightarrow$ (c) it is enough to remark that the distribution is given by $\ker \chi$ and since $L_T \chi = 0$ the form $\chi$ is invariant by the flow. To prove (c) $\Rightarrow$ (a) one defines the 1-form $\chi$ imposing that it vanishes on the distribution and $\chi(T) = 1$. As $d\chi = 0$ we have $e_F(M) = 0$. \hfill $\square$

Corollary 2.2.3. Let $M$ be a compact manifold endowed with a nacs with Levi-flat CR-structure. Then $e_F(M) = 0$.

Proposition 2.2.4. Let $M$ be a compact manifold endowed with a nacs. If $e_F(M) = 0$ then $M$ is a fiber bundle over $S^1$. In particular $b_1(M) \neq 0$ and $M$ is not simply connected.

Proof. The first statement is a consequence of a theorem by D. Tischler [27] that states that if $M$ is a compact manifold admitting a non-vanishing closed 1-form then $M$ is a fibre bundle over $S^1$. The second statement is an immediate consequence of the homotopy exact sequence associated to a fibration. \hfill $\square$

Proposition 2.2.5. Let $M$ be a compact manifold endowed with a nacs and $T$ the vector field inducing the CR-action. If there exists a contact form $\chi$ on $M$ such that $\chi(T) = 1$ and $i_T d\chi = 0$ then $e_F(M) \neq 0$.

The proof is analogous to the one of the corresponding statement for isometry flows due to Saralegui (cf. [25]).

Corollary 2.2.6. Let $M$ be a compact manifold endowed with a normal contact structure. Then $e_F(M) \neq 0$.

2.3. Geometrical constructions of complex structures.

Proposition 2.3.1. (Case A) Let $M_1$ and $M_2$ be two manifolds endowed with a nacs. There exists a 1-parametric family of complex structures $K_\tau$ on the product $M_1 \times M_2$ for $\tau \in \mathbb{C} \setminus \mathbb{R}$ so that the complex manifold $M_1 \times M_2$ admits a non-vanishing holomorphic vector field $v$.

Proof. Let us denote by $(T_1, \omega_1)$ and $(T_2, \omega_2)$ the vector fields and the characteristic 1-forms of the nacs on $M_1$ and $M_2$ respectively. The 2-foliation $\mathcal{F}$ on $M_1 \times M_2$ generated by $T_1$ and $T_2$ is transversely holomorphic and $[T_1, T_2] = 0$. We set a distribution $\mathcal{D}$ given by $\ker \omega_1 \oplus \ker \omega_2$ and we define a complex-valued 1-form $\chi = \frac{1}{2\Im \tau} (\bar{\tau} \omega_1 + \omega_2)$. The 2-form $d\chi$ is basic with respect to $\mathcal{F}$ and $d\chi$ is of type $(1, 1)$ with respect to the transverse holomorphic structure of $\mathcal{F}$. We define an almost complex structure $K_\tau$ on $M_1 \times M_2$ imposing that $K_\tau$ is compatible with the transverse holomorphic structure of $\mathcal{F}$ and that $\chi$ is of type $(1, 0)$. Using Newlander-Nirenberg theorem one can check that $K_\tau$ is integrable, i.e. a complex structure, if and only if $d\chi^{0.2} = 0$, which holds in our case. Moreover the vector field $v = T_1 - \tau T_2$ is of type $(1, 0)$, therefore it is holomorphic if and only if $[v, Q^{0,1}] \subset Q^{0,1}$ where $Q^{0,1}$ denotes the subbundle of $T(M_1 \times M_2)^\mathbb{C}$ of vector fields of type $(0, 1)$ with respect to $K_\tau$, which holds as a consequence of the equality $i_v d\chi = 0$. \hfill $\square$
Example 2.3.2. Let $K$ be a compact connected real Lie group of odd dimension. Since $K$ admits a nacs the previous proposition describes a complex structure on the product $K \times S^1$. As $K \times S^1$ is also a Lie group this can be seen as a particular case of a result by Samelson (cf. [24]) which states that every compact connected Lie group of even dimension admits a left-invariant complex structure. Nevertheless one obtains more complex structures by this construction. Indeed, when $\dim(K) > 1$ we can assume that the nacs is non-invariant. Moreover, by topological reasons $K \times S^1$ cannot be Kählerian except when $K$ is a real torus.

Proposition 2.3.3. (Case B) Let $M$ be a manifold endowed with a nacs and $\mathcal{F}$ its associated flow. Let $\pi : X \to M$ be a $S^1$-principal bundle over $M$ with Chern class $[\beta]$, where $\beta$ is a 1-form on $X$ such that $d\beta$ is the pull-back of a closed $(1,1)$-form on $M$ basic for the flow $\mathcal{F}$. Then there exists a 1-parametric family of complex structures $K_\tau$ on $X$ for $\tau \in \mathbb{C}\setminus\mathbb{R}$ so that the complex manifold $X$ admits a non-vanishing holomorphic vector field $v$.

Proof. Let $\omega$ be the characteristic 1-form of the nacs and let $T$ be the vector field inducing the CR-action. We denote by $\overline{T}$ the vector field on $X$ contained in $\ker \beta$ such that $\pi_* (\overline{T}) = T$ and we define the 1-form $\overline{\omega} = \pi^* \omega$. Let $R$ denote the fundamental vector field of the action corresponding to the $S^1$-principal bundle $\pi : X \to M$ such that $\beta(R) = 1$. We apply now the same arguments as in proposition 2.3.1 using the vector fields $\overline{T}$ and $R$, the distribution $D = \ker \beta \cap \ker \overline{\omega}$ and the transverse holomorphic structure for the flow $\overline{\mathcal{F}} = \langle \overline{T}, R \rangle$ induced by the CR-structure of $M$. We define a complex-valued 1-form $\chi$ by imposing $\ker \chi = D$, $\chi(v) = 1$ and $\chi(\overline{T}) = 0$. The holomorphic vector field $v$ is $\overline{T} - \tau R$ for $\tau \in \mathbb{C}\setminus\mathbb{R}$. The hypothesis $d\beta \in \pi^* \Omega^{1,1}(M/\mathcal{F})$ and $d\omega \in \Omega^{1,1}(M/\mathcal{F})$ imply that $d\chi$ is of type $(1,1)$, thus the complex structure is integrable.

Definition 2.3.4. Let $M$ be a compact manifold with a nacs $(\Phi^{1,0}, T)$. We define

$$\text{Aut}_\tau(M) = \{ f \in \text{Aut}_{\text{CR}}(M) : f_* T = T \}.$$ 

Proposition 2.3.5. (Case C) Let $M$ be a manifold endowed with a nacs $(\Phi^{1,0}, T)$. Given $f \in \text{Aut}_\tau(M)$ the suspension $X$ of $M$ by $f$, i.e. $X = M \times \mathbb{R}/ \sim$ where $(x, s) \sim (f(x), s + 1)$, admits a 1-parametric family of complex structures $K_\tau$ for $\tau \in \mathbb{C}\setminus\mathbb{R}$ so that the complex manifold $X$ admits a non-vanishing holomorphic vector field $v$ induced by $T - \tau \frac{\partial}{\partial s}$.

Proof. The proof is straightforward using the same arguments as in the previous cases, the distribution $D$ is induced by the CR-structure $\Phi^{1,0}$ on $M$.

Definition 2.3.6. Let $M$ be a manifold endowed with a nacs, $T$ the vector field defining the CR-action and $\mathcal{F}$ its associated flow. We say that the pair $(X, v)$ of a compact complex manifold $X$ and a non-singular holomorphic vector field $v$ is a compact complexification of the pair $(M, T)$ if:

(i) $M$ is a real submanifold of $X$.

(ii) The CR-structure of $M$ is compatible with the complex structure of $X$.

(iii) There exists $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda v) = T$.

Both the constructions of case A and case C produce complex manifolds that are compact complexifications of the departing normal almost contact manifolds. Indeed, if $v = T_1 - \tau T_2$ for $\tau \in \mathbb{C}\setminus\mathbb{R}$ then $T_1 = \text{Re} \left( \frac{i}{\text{Im} \tau} v \right)$ and $T_2 = \text{Re} \left( \frac{1}{\text{Im} \tau} v \right)$. 

Theorem 2.3.7. Every compact Kähler manifold admitting a non-vanishing holomorphic vector field can be obtained as a suspension by of proposition 2.3.3 (case C). In particular it is a compact complexification of a compact manifold endowed with a nacs.

Proof. Let $X$ be a compact Kähler manifold admitting a non-vanishing holomorphic vector field $v$. By a result by Carell-Lieberman (c.f. [7]) there exists a holomorphic 1-form $\chi$ over $X$ such that $\chi(v) \neq 0$. As $X$ is compact we can assume $\chi(v) = 1$ and as $X$ is Kählerian we have $d\chi = 0$. Assume $b_1(X) = 2k$. Let $\gamma_1, ..., \gamma_{2k}$ be closed paths giving a basis of $H_1(X, \mathbb{Z})$ modulus torsion and let $\xi_1, ..., \xi_k$ be the dual basis of closed 1-forms. Fix a basis $\omega_1, ..., \omega_k$ of $H^0(X, \Omega^1)$. By Hodge’s decomposition theorem we have

$$\eta = a_1^1 \omega_1 + ... + a_k^k \omega_k + d_1^1 \omega_1 + ... + d_k^k \omega_k$$

where $f_i$ is a differentiable function, for $i = 1, ..., 2k$, and $a_i, d_i \in \mathbb{C}$. By Stokes theorem the two sets of 1-forms $\{\xi_i\}$ and $\{\eta_i\}$ have the same periods. In particular $\{\eta_1, ..., \eta_{2k}\}$ is a basis of $H^1(X, \mathbb{C})$ dual of $\{\gamma_1, ..., \gamma_{2k}\}$. Since $\mathbb{Q} + i\mathbb{Q}$ is a dense subset in $\mathbb{C}$ we can choose $a_i \in \mathbb{C}$ for $i = 1, ..., 2k$ arbitrarily small so that $\eta = \chi + \sum a_i \eta_i$ is a closed 1-form and $\int_{\gamma_i} \eta \in \mathbb{Q} + i\mathbb{Q}$ for $j = 1, ..., 2k$. Moreover by construction the 1-form $\eta$ is of the form:

$$\eta = c_1 \omega_1 + ... + c_k \omega_k + d_1^1 \omega_1 + ... + d_k^k \omega_k$$

with $c_i, d_i \in \mathbb{C}$ for $i = 1, ..., k$. It follows that $\eta(v)$ is constant and close to 1 by construction, set $\eta(v) = \delta$. In an analogous way $\eta(\overline{v})$ is constant and close to 0, set $\eta(\overline{v}) = \epsilon$. Therefore $\Gamma = \{ \int_{\gamma_i} \eta : \gamma \in H_1(X, \mathbb{Z}) \}$ is finitely generated and it is contained in $\mathbb{Q} + i\mathbb{Q}$, thus $\Gamma \cong \mathbb{Z} + i\mathbb{Z}$. Fixing a base point $p_0$ the differentiable map

$$\pi_1 : X \to \mathbb{C}/\Gamma$$

$$p \mapsto \int_{p_0}^p \eta \mod \Gamma$$

over the elliptic curve $\mathbb{C}/\Gamma$ is well defined. Furthermore $\pi_1$ is a proper submersion and thus a fibration. The real vector fields $v + \overline{v}$ and $i(v - \overline{v})$ are transverse to the fibres of $\pi_1$ and preserve the fibration (because $\eta(v + \overline{v}) = \epsilon + \delta, \eta(i(v - \overline{v})) = i(\delta - \epsilon)$ are constants close to 1 and $i$ respectively and $\eta$ is closed). Since $\eta(v + \overline{v}) = \delta + \epsilon \sim 1$ we can find a linear map $h : \mathbb{C} \to \mathbb{R}$ such that $h(\eta(v + \overline{v})) > 0$ and $h(\Gamma) \subset \mathbb{Z}$. Let $\tilde{h} : \mathbb{C}/\Gamma \to \mathbb{R}/\mathbb{Z}$ be the induced fibration. The composition $\pi_2 = \tilde{h} \circ \pi_1 : X \to \mathbb{R}/\mathbb{Z}$ is a fibration over the circle. The fibres of $\pi_2$, denoted by $M = \pi_2^{-1}(p)$, admit a CR-structure induced by the complex structure on $X$. There exists $a \in \mathbb{C} \setminus \mathbb{R}$ such that the real vector field $v_1 = \text{Re}(av)$ is tangent to $M$. As the flow associated to $v$ is holomorphic the vector field $v_1$ preserves the CR-structure of $M$ and induces a transverse CR-action. On the other hand there exists $b \in \mathbb{R}^+$ such that the vector field $v_2 = \text{Re}(bv)$ projects over the vector field $\frac{\partial}{\partial \theta}$ on $S^1$. The flow of $v_2$ preserves the CR-structure over $M$ and clearly $[v_1, v_2] = 0$. Finally setting $\tau = a \cdot b^{-1}$ we obtain $v_1 - \tau v_2 = \text{Re}(av) - \tau \text{Re}(bv) = \mu \cdot v$, where $\mu \in \mathbb{C}$. Taking the automorphism $f$ over $M$ induced by the flow of $v_2$ for time 1 the compact complexification of case C for the preceding $\tau$ gives rise to the original complex structure. □

We have shown the suspension of a complex manifold $N$ by an automorphism $g \in \text{Aut}_\mathbb{C}(N)$ admits a nacs. Applying the construction of case C to such a manifold is equivalent to consider the quotient $X$ of $N \times \mathbb{C}$ by the subgroup generated by
Theorem 2.3.8. Let $X$ be a compact Kähler manifold admitting a holomorphic vector field $v$ without zeros and $\mathcal{F}$ the flow induced by $v$. In the previous theorem we have seen that as a consequence of Carrell-Lieberman theorem (c.f. [7]) there exists a closed holomorphic 1-form $\chi$ on $X$ such that $\chi(v) = 1$. Then the complex structure on $X$ is the only one compatible with the transverse holomorphic structure of $\mathcal{F}$ such that $\chi$ is of type $(1,0)$.

**Proof.** We proceed as in theorem 2.3.7 to obtain from $\chi$ a closed 1-form $\eta$ with group of periods $\Gamma \cong \mathbb{Z} + i\mathbb{Z}$ and a smooth fibration $\pi_1 : \mathcal{M} \to \mathbb{C}/\Gamma$ given by $x \mapsto \int_{x_0}^x \eta$. Every fibre $N$ of $\pi_1$ is transverse to the foliation $\mathcal{F}_\eta$ generated by $v$. Therefore $N$ admits a complex structure. Note that $\eta = \pi_1^* (dz)$. Consider the universal covering $p : \mathcal{C} \to \mathbb{C}/\Gamma$ and the pullback $\pi_2 : N \times \mathcal{C} \to \mathbb{C}$ of the fibration $\pi_1$ by the map $p$. There exists a map $q : N \times \mathcal{C} \to X$ such that $\pi_1 \circ q = p \circ \pi_2$. The holomorphic vector field $v$ is transverse to the leaves of $\pi_1$ and it preserves the complex structure on $N$. We recall that $\eta(v) = \delta \sim 1$ and that $\eta(\bar{v}) = \epsilon \sim 0$. Fix $\tau \in \mathbb{C}\setminus \mathbb{R}$. We decompose $\delta^{-1}v = v_1 - \tau v_2$, where $v_1$ and $v_2$ are real vector fields. Then $v_1$ and $v_2$ are transverse to the fibers of $\pi_1$, they preserve the fibration and the complex structure on $N$ and $\langle v_1, v_2 \rangle = 0$ (for $\delta^{-1}v$ is holomorphic). Finally, they project over $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ on $\mathbb{C}/\Gamma$ respectively where $z = 2 \text{Im} \tau (ds + \bar{\tau} dt)$. We set $f, g \in \text{Aut}_\mathbb{C}(N)$ as the flows $v_1$ and $v_2$ for time 1 and $-1$ respectively. Thus $X$ is diffeomorphic to the double suspension $\mathcal{N} \times \mathbb{C}/(F,G)$ where $F(x, z) = (f(x), z + 1)$, $G(x, z) = (g(x), z + \tau)$ and $f \circ g = g \circ f$. Moreover the complex structure on $X$ induced by it which is arbitrarily close to the original one. Note that with the new complex structure on $X$ the fibration $\pi_1 : X \to \mathbb{C}/\Gamma$ is holomorphic and the fibres $N$ are analytic submanifolds. As we can choose $\eta$ arbitrarily close to the starting holomorphic 1-form $\chi$ we can conclude. \[ \]

There is a natural generalization to a suspension of a compact complex manifold $N$ by a commutative subgroup $\Gamma = \langle f_1, ..., f_s, g_1, ..., g_s \rangle$ of $\text{Aut}_\mathbb{C}(N)$. The resulting manifold has dimension $\dim_\mathbb{C} N + s$ and fibers over the torus $\mathbb{T}^s$. Then one can prove, with the same arguments as in theorem 2.3.8, the result below.

**Theorem 2.3.9.** Let $X$ be a compact Kähler manifold such that $\mathfrak{h}$ admits an abelian subalgebra $\mathfrak{h}$ of holomorphic vector fields without zeros such that $\dim_\mathbb{C} \mathfrak{h} = s > 0$. The underlying smooth manifold $X$ admits a complex structure arbitrarily close to the original one, obtained as a suspension over the complex torus $\mathbb{T}^s$. \[ \]
We denote by $\mathfrak{h}$ the Lie algebra of holomorphic vector fields on $X$ and by $\mathfrak{h}_0$ the Lie algebra of holomorphic vector field with zeros. Recall that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}_0$. Therefore if $\dim_{\mathbb{C}} \mathfrak{h} = s > 0$ and $\mathfrak{h}_0 = 0$ the hypothesis of the above theorem hold. The limit case, i.e. when $\dim_{\mathbb{C}} \mathfrak{h} = \dim_{\mathbb{C}} X$, is a classical result by Wang’s:

**Corollary 2.3.10.** Let $X$ be a complex parallisable compact Kähler manifold, then $X$ is a complex torus.

### 2.4. Nacs on compact 3-manifolds.

When $M$ is a compact manifold of dimension 3 there is a classification due to H. Geiges of the manifolds admitting a nacs based on the classification of compact complex surfaces (see [12]). Using the classification of transversely holomorphic flows on a compact connected 3-manifold (see [5] and [13]) together with the condition of the existence of a CR-structure and a transverse CR-action we give an alternative proof of the classification. The main interest of this point of view is that it determines not only the 3-manifold but also the flow of the CR-action.

**Proposition 2.4.1.** Let $M$ be a compact connected 3-manifold endowed with a nacs. Then, up to diffeomorphism, the manifold $M$ and the vector field inducing the CR-action belong to the following list:

1. **Seifert fibrations over a Riemann surface** such that the isometric flow of the $S^1$-action admits a characteristic 1-form $\omega$ such that $d\omega$ is of type $(1, 1)$.
2. **Linear vector fields in** $T^3$.
3. **Flows on** $S^3$ **induced by a singularity of a holomorphic vector field in** $\mathbb{C}^2$ in the Poincaré domain and their finite quotients, i.e. flows on lens spaces.
4. **Suspensions of a holomorphic automorphism of** $\mathbb{P}^1$ **with a vector field tangent to the flow associated to the suspension.**

Moreover, all the previous manifolds admit a nacs such that the CR-action is the one induced by the corresponding vector field.

**Proof.** Since the last statement of the proposition is clear it is enough to depart from the classification of transversely holomorphic flows and rule out the two cases that do not admit a nacs: strong stable foliations associated to suspensions of hyperbolic diffeomorphisms of $\mathbb{T}^2$ and $\mathbb{C} \times \mathbb{R}\{(0,0)\}/\sim$ where $(z, t) \sim (\lambda z, 2t)$ for $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ with the flow induced by the vertical vector field $\frac{\partial}{\partial t}$. The first ones are are examples of non-isometric Riemannian flows (see [8]), if they admitted an invariant CR-structure together with a transverse CR-action they would be isometric (for the flow of a nacs admits a transverse invariant distribution). Let now $M$ be $\mathbb{C} \times \mathbb{R}\{(0,0)\}/\sim$ where $(z, t) \sim (\lambda z, 2t)$ for $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ with the flow $F$ induced by the vertical vector field $\frac{\partial}{\partial t}$. Suppose that $M$ admits a CR-structure $\Phi^{1,0}$ transverse to $F$ and a vector field $T$ tangent to $F$ inducing a CR-action. Then $M^3 \times S^1$ is a compact complex surface that admits a holomorphic non-vanishing vector field. Therefore it belongs to the following list (c.f. [9, 10]):

1. **Complex tori.**
2. **Principal Seifert fibre bundles over a Riemann surface of genus $g \geq 1$ with fiber an elliptic curve.**
3. **Ruled surfaces over an elliptic curve.**
4. **Almost-homogeneous Hopf-surfaces.**

Since $X$ is homeomorphic to $S^2 \times S^1 \times S^1$ the complex surface $X$ must be a ruled surface over an elliptic curve. However we will see that this is a contradiction.
Recall that the universal covering of a ruled surface over an elliptic curve is either \( \mathbb{D} \times \mathbb{P}^1 \) or \( \mathbb{C} \times \mathbb{P}^1 \). By construction of the complex structure on \( X = M \times S^1 \) we have that \( t + \tau s \) is a holomorphic coordinate for some \( \tau \in \mathbb{C} \setminus \mathbb{R} \), therefore the analytic universal covering \( \tilde{X} = \mathbb{C} \times \mathbb{R} \setminus \{(0,0)\} \times \mathbb{R} \) of \( X \) admits a holomorphic projection \( p : \tilde{X} \to \mathbb{C} \) defined by \( p(z,t,s) = z \) with fiber an open subset of \( \mathbb{C} \). As \( \mathbb{P}^1 \) is compact by the maximum principle it is immersed in the fibers of \( p \), which is a contradiction. \( \square \)

3. Kähler criteria

**Theorem 3.1.1.** Let \( X \) be a compact complex manifold with a non-vanishing holomorphic vector field \( v \). For every \( \tau \in \mathbb{C} \setminus \mathbb{R} \) let \( T_1 \) and \( T_2 \) be the two real vector fields defined by \( v = T_1 - \tau T_2 \) and \( \mathcal{F}_1, \mathcal{F}_2 \) the flows defined by \( T_1, T_2 \) respectively. If \( X \) is Kählerian then \( e_{\mathcal{F}_1}(X) = e_{\mathcal{F}_2}(X) = 0 \).

**Proof.** Since \( X \) is a compact Kähler manifold with a holomorphic vector field \( v \) without zeros by Carrell-Liebermann’s theorem there exists a closed holomorphic 1-form \( \alpha \) such that \( \alpha(v) = 1 \). We decompose
\[
\alpha = \frac{i}{2 \text{Im}(\tau)} (\alpha_1 + \tau \alpha_2)
\]
where \( \alpha_1 \) and \( \alpha_2 \) are real closed 1-forms. Using that \( \alpha(v) = 1 \) and \( \alpha(\overline{\tau}) = 0 \), a direct computation shows that \( \alpha_i(T_j) = \delta_{ij} \) for \( i, j = 1, 2 \). Thus \( e_{\mathcal{F}_1}(X) = e_{\mathcal{F}_2}(X) = 0 \). \( \square \)

**Proposition 3.1.2.** Let \( M \) be a compact manifold endowed with a nacs, let \( T \) be the vector field of the CR-action and \( \mathcal{F} \) its associated flow. Assume that the compact complex manifold \( X \) together with a holomorphic vector field \( v \) are a compact complexification of \((M,T)\). If \( X \) is Kählerian then \( e_{\mathcal{F}}(M) = 0 \).

**Proof.** We denote by \( v \) the holomorphic vector field on \( X \) and by \( \lambda \) the complex number such that \( T = \text{Re}(\lambda v) \) on \( M \). Since \( X \) is a compact Kähler manifold with a non-singular vector field \( v \) there exists a holomorphic closed 1-form \( \alpha \) such that \( \alpha(v) = 1 \). We can decompose \( \lambda \cdot v \)\( |_M = T - iS \) where \( S \) is a real vector field. Set \( \alpha = \frac{1}{2i}(\alpha_1 + \alpha_2) \) where \( \alpha_1, \alpha_2 \) are real 1-forms on \( X \). Then \( \alpha_1 \) and \( \alpha_2 \) are closed and \( \alpha_1(T^2) = 1 \) on \( M \). The closed real 1-form \( \omega := \alpha_1|_M \) verifies \( \omega(T) = 1 \) (because \( \alpha(v) = 1 \) and \( \alpha(\overline{\tau}) = 0 \), therefore \( e_{\mathcal{F}}(M) = 0 \). \( \square \)

**Corollary 3.1.3.** Let \( M \) be a compact manifold endowed with a nacs. If \( e_{\mathcal{F}}(M) \neq 0 \) then no compact complexification obtained as a product by proposition 2.3.1 (case A) or as a suspension by proposition 2.3.7 (case C) is Kählerian.

**Corollary 3.1.4.** Let \( M \) be a compact manifold endowed with a nacs, let \( T \) be the vector field of the CR-action and \( \mathcal{F} \) its associated flow. If \( M \) admits a normal contact structure compatible with the CR-action induced by \( T \) then \( (M,T) \) admits no Kähler compact complexification.

It follows from corollary 2.2.6.

**Corollary 3.1.5.** Let \( M \) be a compact manifold endowed with a nacs. If \( b_1(M) = 0 \), in particular if \( M \) is simply connected, then \( (M,T) \) admits no Kähler compact complexification.

It is a consequence of proposition 2.2.4.
Corollary 3.1.6. Let $M$ be a compact connected semisimple real Lie group of odd dimension endowed with a nacs and $T$ the vector field of the CR-action. Then $(M, T)$ admits no Kähler compact complexification.

It is enough to recall that for any such group $b_1(M) = 0$.

Proposition 3.1.7. Let $M$ be a compact manifold endowed with a nacs and let $X$ be a compact complex manifold obtained as a $S^1$-principal bundle $\pi : X \to M$ over $M$ by proposition 2.3.6 (case B). If $X$ is Kählerian then $c_X(M) = 0$ and the $S^1$-principal bundle is flat. In particular, if $X$ is Kählerian and $H^2(M, \mathbb{Z})$ has no torsion then the $S^1$-principal bundle is topologically trivial. Moreover, if $\alpha$ is a connection 1-form on $X$ such that $da \in \pi^*\Omega^{1,1}(M/F)$ then $[d\alpha] = 0$ in $H^2(M/F)$.

Proof. If $\nu$ is the holomorphic vector field of the complexification there exists a closed holomorphic 1-form $\alpha$ on $X$ such that $\alpha(v) = 1$. The connected group $S^1$ acts holomorphically on $X$ (as the group of the action of the $S^1$-principal bundle), therefore the forms $\alpha$ and $\tilde{\alpha}$ are invariant by the action of $S^1$. Notice that $\nu = \tilde{T} - \tau R$ where $R$ is the vector field of the action and, $\tilde{T}$ is the vector field contained in $\ker \beta$ such that $\pi_*(\tilde{T}) = T$ and $\tau \in C \backslash \mathbb{R}$. We decompose $\alpha = \frac{i}{2 \im \tau}(\alpha_2 + i \alpha_1)$ where $\alpha_1, \alpha_2$ are real 1-forms. Then $\alpha_1$ and $\alpha_2$ are closed 1-forms invariant by the action of $S^1$ (for they are a linear combination of $\alpha$, $\tilde{\alpha}$ with constant coefficients) such that $\alpha_1(\tilde{T}) = \alpha_2(R) = 1$ and $\alpha_1(R) = \alpha_2(\tilde{T}) = 0$ (because $\alpha(v) = 1$ and $\alpha(\tau) = 0$). Since $\alpha_1$ is a closed real basic $S^1$-invariant 1-form it induces a closed 1-form $\omega$ on $M$ such that $\omega(T) = 1$, thus $c_X(M) = 0$. Finally, $\alpha_2$ is a closed connection 1-form for the $S^1$-principal bundle $\pi : X \to M$, so it is flat. When $H^2(M, \mathbb{Z})$ has no torsion flat bundles are topologically trivial. Moreover, if $\alpha$ is a connection 1-form on $X$ such that $da \in \pi^*\Omega^{1,1}(M/F)$ then $[d\alpha] = [d\alpha_2] = 0$ in $H^2(M/F)$.

Finally for the suspension (case C) of a compact connected Lie group endowed with a nacs an elementary computation of the second cohomology group shows that the resulting complex manifold cannot be Kählerian:

Proposition 3.1.8. Let $K$ be a non-abelian compact connected real Lie group of odd dimension endowed with a nacs, $f \in \text{Aut}_r(K)$ and $X = K \times f \mathbb{R}$ endowed with the complex structure obtained as a suspension by proposition 2.3.6 (case C). Then X is not Kählerian.

Proof. The suspension $X = K \times f \mathbb{R}$ admits a finite covering $\tilde{X} = M \times f \mathbb{R}$ such that $M \cong K' \times (S^1)^r$ where $K'$ is a compact connected semisimple real Lie group, $0 \leq r < \dim_R K$ and $f$ the lift of $f$ to $\tilde{X}$. Since $b_1(K') = b_2(K') = 0$ we conclude that $H^2(M) \cong H^2((S^1)^r)$ and $H^1(M) \cong H^1((S^1)^r)$. Then using Mayer-Vietoris sequence for the De Rham cohomology groups (c.f. [4]) one proves that $H^2(\tilde{X})$ is isomorphic to

$$\{[\sigma] \in H^2(M) : f^*[\sigma] = [\sigma]\} \oplus \bigg\{(\frac{H^1(M)}{[\sigma - f^*\sigma] : [\sigma] \in H^1(M)}\bigg\) \wedge [ds].$$

It is not difficult to see that $\tilde{X}$ cannot be Kählerian and it follows that $X$ is not Kählerian. □
3.2. Criteria for isometric flows.

**Theorem 3.2.1.** Let $X$ be a compact complex manifold with a non-vanishing holomorphic vector field $v$. For every $\tau \in \mathbb{C} \setminus \mathbb{R}$ let $T_1$ and $T_2$ be the two real vector fields defined by $v = T_1 - \tau T_2$ and $F_1, F_2$ the flows defined by $T_1, T_2$ respectively. Assume that the flows $F_1$ and $F_2$ are isometric. Then the manifold $X$ is Kählerian if and only if $e_{F_1}(X) = e_{F_2}(X) = 0$ and the real foliation $\mathcal{F} = \langle T_1, T_2 \rangle$ is transversely Kählerian.

**Proof.** $\Rightarrow$ : The same argument as in theorem 3.1.1 shows that there are two closed real 1-forms $\alpha_1$ and $\alpha_2$ on $M$ such that $\alpha_i(T_j) = \delta_{ij}$ for $i, j = 1, 2$. Then $\alpha_1$ and $\alpha_2$ are characteristic forms for $F_1$ and $F_2$ respectively and by $H$ the closure in Isom$(X)$ of the abelian group generated by $(\varphi_1)_t$ and $(\varphi_2)_t$. If $\Phi$ is the Kähler form on $X$ then the transverse part $\Psi(\cdot, \cdot)$ with respect to $F$ of

$$\int_H \Phi(\sigma_* \cdot, \sigma_* \cdot),$$

where we integrate with respect to the Haar measure on $H$, is a transverse Kähler form.

$\Leftarrow$ : As $F_1$ and $F_2$ are isometric we can assume that there exists an invariant transverse distribution $D$ of maximal dimension for the real foliation $\mathcal{F} = \langle T_1, T_2 \rangle$. We denote by $\omega_1$ and $\omega_2$ the 1-forms on $X$ defined by $\omega_i(T_j) = \delta_{ij}$ and $\omega_i|D = 0$ for $i, j = 1, 2$. Since $e_{F_1}(X) = e_{F_2}(X) = 0$ there exist $\beta_1, \beta_2 \in \Omega^1(X/F)$ such that $d\beta_i = d\omega_i$ for $i = 1, 2$. We denote by $\beta$ the basic form $\beta = \frac{1}{2i} \omega_1 \wedge \omega_2$. It follows that $d\beta = d\chi$. We begin by showing that it is enough to find $\alpha \in \Omega^1(X/F, \mathbb{C})$ of type $(1, 0)$ such that $d\alpha = d\chi$. Indeed, if $\alpha$ exists the form $\Phi = (\chi - \alpha)^\wedge (\chi - \alpha)$ is closed and of type $(1, 1)$. Adding to $\Phi$ a positive multiple of the transverse Kähler form of $\mathcal{F}$ we obtain a Kähler form on $X$ and the proof is complete. We will now show that such a form exists. Since $d\beta^{0,2} = d\chi^{0,2} = 0$ we have $d\beta = 0\beta^{1,0} + \partial(\beta^{0,1})$, i.e. $\partial(\beta^{0,1}) = 0$, and

$$d(\partial \beta^{0,1}) = (\partial + \overline{\partial})(\partial \beta^{0,1}) = -\partial \overline{\partial} \beta^{0,1} = 0$$

so $\partial(\beta^{0,1})$ is a $(1, 0)$-form which is $\partial$-exact as a basic form and $\partial$-closed. Applying the basic $\partial \overline{\partial}$-lemma (c.f. [11]) to $\partial(\beta^{0,1})$ we obtain a basic function $f$ such that $\partial(\beta^{0,1}) = \partial \overline{\partial} f = \overline{\partial}(-f)$.

Then $\partial f$ is a basic form of type $(1, 0)$ such that $d(-\partial f) = -\overline{\partial} \partial f = \partial(\beta^{0,1})$. The form $\alpha = \beta^{1,0} - \partial f$ is basic, of type $(1, 0)$ and $d\alpha = d\beta = d\chi$ so the conclusion follows.

Every Riemannian holomorphic flow $\mathcal{F}$ in a compact complex surface $S$ is transversely Kählerian. Therefore with the above hypothesis when $\dim_{\mathbb{C}} X = 2$ the complex manifold $X$ is Kählerian if and only if $e_{F_1}(X) = e_{F_2}(X) = 0$.

**Corollary 3.2.2.** Let $X$ be a complex manifold obtained as a product by proposition 3.3.1 (case A) from two manifolds $M_1$ and $M_2$ endowed with a nacs such that the flows $F_1$ and $F_2$ in $M_1$ and $M_2$ respectively associated to the nacs are isometric. Then $X$ is Kählerian if and only if $e_{F_1}(M_1) = e_{F_2}(M_2) = 0$ and the flows $F_1$ and $F_2$ are transversely Kählerian.
Corollary 3.2.3. Let $M$ be a compact manifold endowed with a nacs such that its associated flow $\mathcal{F}$ is isometric. Let $X$ be a complex manifold obtained as a $S^1$-principal bundle over $M$ by proposition 2.3.3 (case B). Then $X$ is Kähler if and only if the $S^1$-principal bundle $\pi : X \to M$ is flat, the flow $\mathcal{F}$ is transversely Kählerian on $M$ and $e_\mathcal{F}(M) = 0$.

Example 3.2.4. Let $S$ be a compact complex surface obtained as a $S^1$-principal bundle over a 3-manifold $M$ with a nacs by means of the construction of case B. From the last result together with the classification of compact complex surfaces one concludes the following. With the notation of section §2.4 for each of the possibilities for $M$ the corresponding surface $S$ is:

1) an elliptic Seifert principal fibre bundle,
2) a complex torus or non-Kählerian elliptic Seifert principal fibre bundle,
3) a Hopf surface and
4) either a Hopf surface or a ruled surface over an elliptic curve.

Proposition 3.2.5. Let $X$ be a complex manifold obtained as a suspension by proposition 2.3.5 (case C) from a manifold $M$ endowed with a nacs such that its associated flow $\mathcal{F}$ is isometric. If $X$ is Kählerian then the flow $\mathcal{F}$ is transversely Kählerian and there exists a basic Kähler form $\Phi$ such that $[f^*\Phi] = [\Phi] \in H^{1,1}(M/\mathcal{F})$.

Proof. We choose a Kähler form $\Psi_0$ on $X$ and define $\Psi$ as the pullback to $M \times \mathbb{R}$. We define $\Phi_0 = \Psi(x, s_0)|_M$ so that $[f_\mathcal{F}\Phi_0] = [\Phi_0]$ and $\Phi_0$ is a closed form of type $(1,1)$ with respect to the holomorphic transverse structure. We denote by $\varphi_t$ the 1-parameter group on $M$ associated to $T$ and by $H$ the closure in Isom($M$) of the abelian group generated by $\varphi_t$. The form $\Phi$ defined as the basic part of

$$\int_H \Phi_0(\sigma_*^t, \sigma_*),$$

where we integrate with respect to the Haar measure on $H$, is a basic Kähler form on $M$. Moreover since $f^*\Phi_0 = \Phi_0 + d\alpha$ where $\alpha$ is a 1-form on $M$, the form $f^*\Phi$ is the basic part of

$$\int_H f^*(\Phi_0(\sigma_*^t, \sigma_*)) = \int_H (f^*\Phi_0)(\sigma_*^t, \sigma_*) = \int_H \Phi_0(\sigma_*^t, \sigma_*) + \int_H d\alpha(\sigma_*^t, \sigma_*)$$

$$= \int_H \Phi_0(\sigma_*^t, \sigma_*) + d \int_H \alpha(\sigma_*)$$

(see [14] and [15] and note that $f_*T = T$ so $f_*\sigma = \sigma \circ f$). Therefore $[f^*\Phi] = [\Phi]$. $\square$

3.3. Suspensions of Levi-flat nacs. In this section we find necessary and sufficient conditions for a complex manifold obtained as a suspension by the construction of case C from a manifold endowed with a Levi-flat nacs to be Kählerian.

Let $M$ be a compact manifold endowed with a nacs and let $\omega$ be its characteristic 1-form. Assume that the CR-structure is Levi-flat, i.e. $d\omega = 0$. The suspension $M$ of a compact complex manifold $N$ by $g \in Aut_C(N)$ with the natural nacs is an example of this situation. Conversely, one has the following result:

Proposition 3.3.1. Let $M$ be a manifold endowed with a nacs with Levi-flat CR-structure. If its characteristic 1-form $\omega$ has group of periods $\Gamma_\omega \cong \mathbb{Z}$ then $M$ and the given nacs can be obtained as a suspension of a compact complex manifold $V$ by $g \in Aut_C(V)$. 

Proof. Since $\omega$ has periods group $\Gamma_\omega \cong \mathbb{Z}$ there is a well defined fibration
\[ \pi : M \to S^1 \cong \mathbb{R}/\Gamma_\omega, \quad x \mapsto \int_{x_0}^x \omega \]
with compact fibers isomorphic to $V$ which are the leaves of the foliation induced by $\ker \omega$. The compact leaf $V$ carries a complex structure induced by the CR-structure $\Phi^{1,0}$ on $M$. The automorphism $g \in \text{Aut}(V)$ corresponding to the suspension giving rise to $M$ is induced by $\varphi_{t_0}$, where $t_0$ verifies $\varphi_t(x_0) \notin V$ for $0 < t < t_0$ and $\varphi_{t_0}(x_0) \in V$ for every $x_0 \in V$. The vector field of the CR-action $T$ is transverse to the leaves of the fibration and $\omega = \pi^*(dt)$, where $t$ denotes the real coordinate on $S^1$. The choice of $g$ and the hypothesis $\omega(T) = 1$ imply that $T$ is the vector field induced by $\frac{\partial}{\partial t}$. \hfill \Box

**Theorem 3.3.2.** Let $X$ be a complex manifold obtained as a suspension by proposition 2.3.3 (case C) from a manifold $M$ endowed with a nacs with Levi-flat CR-structure and $f \in \text{Aut}_T(M)$. Assume that its associated flow $F$ is isometric and that $f^* = \text{id}$ acting on $H^1(M, \mathbb{C})$. Then $X$ is Kählerian if and only there exists a basic Kähler form $\Phi$ for the flow $F$ such that $[f^* \Phi] = [\Phi] \in H^{1,1}(M/F)$.

**Lemma 3.3.3.** Let $M$ a compact manifold with a transversely Kählerian flow $F$ depending smoothly on a real parameter $s \in \mathbb{R}$. Let $\omega$ be an exact basic form on $M$ of type $(p,q)$ such that $\omega = d_M \alpha$ for a basic form $\alpha$ depending smoothly on $s$. Then there exists a basic $(p-1,q)$-form $\mu$ and a basic $(p,q-1)$-form $\nu$ that depend smoothly on $s$ and such that $\omega = d_M \mu = d_M \nu$.

**Lemma 3.3.4.** Let $M$ a compact manifold with a transversely Kählerian flow $F$ depending smoothly on a real parameter $s \in \mathbb{R}$. Let $\omega$ be an exact basic form on $M$ such that $\omega = d_M \alpha$ for a basic form $\alpha$ depending smoothly on $s$. Assume that $\omega = \omega_1 + \omega_2$ where $\omega_1$ is of type $(p+1,q)$ and $\omega_2$ is of type $(p,q+1)$, both basic and depending smoothly on $s$. There exists a basic $(p,q)$-form $\nu$ such that $\omega = d_M \nu$ and it depends smoothly on $s$.

The proofs of the previous lemmas are analogous to the ones in [2], pages 185-186, using the fact that on a transversely Kählerian isometric flow there exists a transversal Hodge theory regarding basic forms and in particular a basic $\overline{\partial}$-lemma (c.f. [11]). Note that locally every suspension is a product, thus we can consider the exterior derivative $d_s$ with respect to the local coordinate $s$ and then $d = d_M + d_s$.

**Proof.** $\implies$ : It follows from proposition 3.2.5.

$\Longleftarrow$: We define $\alpha$ as the $(1,0)$-form induced by \( \frac{i}{2 \pi \Im} (ds + \bar{\tau} \omega) \), which is a closed holomorphic 1-form on $X$ such that $\alpha(v) = 1$. Besides, from Gysin’s exact sequence we conclude that
\[ H^1(M) \cong H^1(M/F) \oplus \langle [\omega] \rangle. \]

From the hypothesis $f^* = \text{id}$ acting on $H^1(X)$ and a short computation using Mayer-Vietoris sequence for the De Rham cohomology groups (c.f. [4]), it follows that $H^1(X) \cong H^1(M) \oplus \langle [ds] \rangle$. Since the flow $F$ is transversely Kählerian on $M$ by Hodge theory
\[ H^1(M/F) \cong H^{1,0}(M/F) \oplus H^{0,1}(M/F) \]
and there exists a basis $\alpha_1, \ldots, \alpha_k$ of $H^{1,0}(M/F)$ of closed $(1,0)$-forms (and therefore holomorphic) on $M$ such that $f^* \alpha_j = \alpha_j$ for $j = 1, \ldots, k$ (for $f^* = \text{id}$ on $H^1(M)$ and the representatives of cohomology classes of type $(1,0)$ are unique as a consequence.
of the transversal Hodge theory). Note also that $H^{1,0}(X/F_0) \cong H^{1,0}(M/F)$ since a basic 1-form on $X$ cannot depend on $s$ and therefore lives on $M$. Fix the transverse Kähler form $\Phi$ on $M$. We can choose an open covering $\{U_i\}$ of $S^1$ so that the fibration $p : X \rightarrow S^1$ is trivial over $U_i$. Then $\Phi$ induces a well-defined form $\Phi_i$ on $p^{-1}(U_i) \cong U_i \times M$. Let $\{\rho_i\}$ be a unit partition associated to $\{U_i\}$. Then $\Phi_0 = \sum \rho_i(s)\Phi_i$ is a real global $(1,1)$-form on $X$ so that $\Phi_{0|M}$ is a transverse Kähler form on $M$ representing a fixed cohomology class, since $[f^*\Phi] = [\Phi_L]$. Moreover $d\Phi_0 = d_s\Phi_0$. Now we want to obtain a closed real valued $(1,1)$-form $\Phi$ on $X$ such that $\Phi_{0|M} = \Phi_0$. We search $\Phi$ of the type:

$$\tilde{\Phi} = \Phi_0 + H \wedge \bar{\alpha} + \bar{\bar{\alpha}} \wedge \alpha + iF \alpha \wedge \bar{\alpha},$$

where $H$ is a basic $(1,0)$-form on $X$ and $F$ a real-valued function on $X$. The hypothesis $d\Phi = 0$ is then equivalent to the following equations:

$$\begin{cases} 
    d_s\Phi_0 - \frac{1}{2i\text{Im}\tau}(-d_M H \wedge ds + d_M \bar{\alpha} \wedge ds) &= 0 \quad (I) \\
    -\tau d_M H \wedge \omega + \bar{\tau} d_M \bar{\alpha} \wedge \omega &= 0 \quad (II) \\
    \frac{1}{2i\text{Im}\tau}(-\tau d_s H \wedge \omega + \bar{\tau} d_s \bar{\alpha} \wedge \omega) - idF \alpha \wedge \bar{\alpha} &= 0 \quad (III).
\end{cases}$$

Note that

$$idF \alpha \wedge \bar{\alpha} = -\frac{dF}{2\text{Im}\tau} ds \wedge \omega.$$ 

Obtaining the form $\tilde{\Phi}$ is equivalent to finding $H$ and $F$ that solve the previous system. Roughly speaking, we will first solve $(I)$ and $(II)$ so that we determine $H$ and then define $F$ as the solution of $(III)$. The equation $(I)$ is equivalent to

$$2i\text{Im}\tau i_\pi ds \Phi_0 = d_M \bar{\alpha} - d_M H = d_M(\bar{\bar{\alpha}} - H)$$

which is fulfilled if

$$d_M H = \bar{\tau} i_\pi ds \Phi_0 = (IV).$$

The form $\gamma := \bar{\tau} i_\pi ds \Phi_0$ can be seen locally as a basic form on $M$ of type $(1,1)$ that depend smoothly on the real parameter $s$. To apply lemma 3.3.3 to obtain local basic $(1,0)$-forms $H_i$ on $p^{-1}(U_i) \cong U_i \times M$ which solve (IV) we must check that $\gamma$ is $d_M$-exact. Note that

$$d_M \gamma = \bar{\tau} i_\pi ds d_M \Phi_0 = 0.$$ 

To prove the exactness it is enough to see that

$$\int_C \gamma = \int_C \bar{\tau} i_\pi ds \int_C \Phi_0 = 0$$

for every cycle $C$ on $M$. This holds because $\Phi_0$ represents the same cohomology class on every fibre so $\int_C \Phi_0$ does not depend on $s$. Therefore there exist 1-forms $\{\beta_i\}$ on $M$ depending smoothly on $s$ such that $\gamma = d_M \beta_j$. We denote by $\varphi$ the 1-parameter group associated to $T$ and by $L$ the closure of the abelian group generated by $\varphi_t$. Define $\beta_j$ as the transverse part of $\int_L \beta_j(\sigma) \, (\text{integrating with respect to the Haar measure on } L)$. Then $\beta_j$ is a transverse 1-form on $X$ which depends smoothly on $s$ and such that

$$d_M \tilde{\beta}_j = d_M \int_L \sigma^* \beta_j = \int_L \sigma^* d_M \beta_j = \int_L \sigma^* \gamma = \gamma$$

(recall that $\gamma$ is basic, so $\int_L \sigma^* \gamma = \gamma$). Therefore there exist basic local solutions $\{H_i\}$ to (IV) which depend smoothly on $s$. Using a unit partition as above we obtain a global solution $H_0 = \sum \rho(s)H_i$. Note that by construction of $H$ it verifies
\[ \tau d_M H = \bar{\tau} d_M \bar{H}, \] 
therefore it is also a solution of (II). Now, to proceed with our plan we should define \( F \) as the solution of (III) for the previous \( H_0 \). Instead of finding a solution of (III) we will solve the following equation:

\[ d_M F = i \left( \tau i \frac{\partial}{\partial s} d_s H - \bar{\tau} i \frac{\partial}{\partial s} d_s \bar{H} \right) = \nu \quad (V). \]

Note that the term on the right \( \nu \) is a basic real form on \( M \) depending smoothly on \( s \) which is the sum of a form of type \((1, 0)\) and a form of type \((0, 1)\), therefore we can try to apply lemma 3.3.4. Moreover \( \nu \) is \( d_M \)-closed, for

\[ d_M \nu = i \cdot i \frac{\partial}{\partial s} d_s (\tau d_M H - \bar{\tau} d_M \bar{H}) = 0 \]

Nevertheless, here we encounter a difficulty, since we cannot prove that \( \nu_0 \), for the previous solution \( H_0 \), is \( d_M \)-exact. To overcome it we will modify \( H_0 \) to obtain a new solution of (IV) for which the corresponding \( \nu \) in \( (V) \) is \( d_M \)-exact. Consider the basis \( \{ \alpha_1, \ldots, \alpha_k, \bar{\alpha}_1, \ldots, \bar{\alpha}_k \} \) of \( H^{1,0}(M/F) \oplus H^{0,1}(M/F) \) of forms fixed by \( f \) so that they are well-defined closed forms on \( X \) (note that \( \alpha_1, \ldots, \alpha_k \) are holomorphic). Let \( \{ \gamma_1, \ldots, \gamma_{k+1}, \bar{\gamma}_1, \ldots, \bar{\gamma}_{k+1} \} \) be the basis of \( H_1(M, \mathbb{C}) \) dual to \( \{ \alpha_1, \ldots, \alpha_k, \bar{\alpha}_1, \ldots, \bar{\alpha}_k \} \). We define

\[ u_j = i \int_{\gamma_j} (\tau d_s H - \bar{\tau} d_s \bar{H}) = i \cdot d_s \int_{\gamma_j} (\tau H - \bar{\tau} \bar{H}) \]

for \( 1 \leq j \leq k \). Notice that

\[ \bar{u}_j = i \cdot d_s \int_{\bar{\gamma}_j} (\tau H - \bar{\tau} \bar{H}) \]

It is not difficult to see that they are \( d_s \)-closed and exact real 1-forms on \( S^1 \) (the base of the natural fibration). There exists a family \( \{ v_1, \ldots, v_k \} \) of real functions on \( S^1 \) (that we extend by pullback to \( X \)) such that \( d_s v_j = u_j \), namely

\[ v_j = i \int_{\gamma_j} (\tau H - \bar{\tau} \bar{H}). \]

We can define now a new solution \( H \) of (IV), and therefore a solution of (I) and (II), by the formula

\[ H = H_0 + i \cdot \tau^{-1} \sum_{j=1}^{k} \alpha_j \cdot v_j \]

so that

\[ \nu ds = i (\tau d_s H_0 - \bar{\tau} d_s \bar{H}_0) + \sum_{j=1}^{k} u_j \land \alpha_j - \sum_{j=1}^{k} \bar{u}_j \land \bar{\alpha}_j. \]

We will next verify that the integral of \( \nu ds = i (\tau d_s H - \bar{\tau} d_s \bar{H}) \) is zero for any cycle \( C \) on \( M \), for it is equivalent to \( \int_C \nu = 0 \). Note that since we saw that \( \nu \) is \( d_M \)-closed when \( H \) is a solution of (IV) it is enough to check that \( \int_{\gamma_j} \nu ds = \int_{\bar{\gamma}_j} \nu ds = 0 \) (note that if \( \nu = d_M G \) the function \( G \) must be basic). Indeed,

\[ \int_{\gamma_j} \nu ds = \int_{\gamma_j} \nu_0 ds - d_s v_j = u_j - u_j = 0 \]

\[ \int_{\bar{\gamma}_j} \nu ds = \int_{\bar{\gamma}_j} \nu_0 ds - d_s \bar{v}_j = \bar{u}_j - \bar{u}_j = 0. \]
Therefore, we are left to solve $d_M F = \nu$ with $\nu$ satisfying all the hypothesis in lemma 3.3.3 to obtain local real functions $F_i$ such that $d_M F_i = \nu$. We define thus $F$ by means of a unit partition, $F = \sum \rho_i F_i$. To finish the proof it is enough to add a positive constant to $F$ so that we obtain a positive closed $(1,1)$-form on $X$. \hfill \Box

3.4. **Double suspensions.** We will finally see that in the case of a double suspension we have a more explicit version of theorem 3.3.2 to decide when the resulting complex manifold is Kählerian. If $N$ is a compact complex manifold we denote by $\text{Aut}_0(N)$ the connected component of the identity in the group of holomorphic transformations $\text{Aut}_C(N)$ of $N$.

**Theorem 3.4.1.** Let $N$ be a compact complex manifold and $f, g \in \text{Aut}_C(N)$ such that $f \circ g = g \circ f$. Let $X$ be the suspension $N \times C/(F, G)$ where $F(x, z) = (f(x), z+1)$, $G(x, z) = (g(x), z+\tau)$ and $\tau \in C \setminus \mathbb{R}$. Then the following conditions are equivalent:

(i) $X$ is Kählerian.

(ii) There is a Kähler form $\omega$ on $N$ such that $[f^* \omega] = [g^* \omega] = [\omega]$.

(iii) $N$ is Kählerian and there are integers $n, m > 0$ such that $f^n, g^m \in \text{Aut}_0(N)$.

**Proof.** (i) $\implies$ (ii): Let $\Psi$ be a Kähler form on $X$. Its pull-back $\Phi = \pi^* \Psi$ by the covering map $\pi : N \times C \to X$ is a Kähler form on $N \times C$. Let us choose $z_0 \in C$. If we denote by $N_z = N \times \{z\}$ and by $\omega_z(x) = \Phi(x, z)|_{N_z}$ then $\omega_{z_0}$ is a Kähler form on $N_{z_0}$. It suffices to show that $[f^* \omega_{z_0}] = [\omega_{z_0}]$ (for $g$ the argument is analogous). By construction of $\Phi$ we know that $F^* \Phi = \Phi$ where $F(x, z) = (f(x), z+1)$, therefore $f^* \omega_{z_0} = \omega_{z_0-1}$. Recall that if $[\Phi] = [\Phi'] \in H^2(N \times C, \mathbb{R})$ then $[\Phi(x, z_0)|_{N}] = [\Phi'(x, z_0)|_{N}]$ in $H^2(N, \mathbb{R})$. Therefore it is enough to see that $[\Phi(x, z)] = [\Phi(x, z+a_0)]$ in $H^2(N \times C, \mathbb{R})$ for all $a_0 \in \mathbb{C}$. Since the map $(x, z) \mapsto (x, z + a_0)$ is homotopic to the identity on $N \times C$ this condition is verified and we can conclude.

(ii) $\implies$ (iii): It is an immediate consequence of a result by D.Lieberman (c.f. [12]) that asserts the following: if $M$ is a compact Kähler manifold, $\omega$ a Kähler form and we denote by $\text{Aut}_C(M)$ the group of automorphisms of $M$ preserving the Kähler class $[\omega]$ then $\text{Aut}_0(M)/\text{Aut}_0(M)$ is a finite group. Indeed, if we consider $\{f, f^2, \ldots\}$ there must exist $n_1 > n_2 > 0$ such that $f^{n_1} = f^{n_2} \cdot h$ with $h \in \text{Aut}_0(N)$. Therefore for $n = n_1 - n_2$ we have $f^n \in \text{Aut}_0(N)$ and we would proceed identically for $g$.

(iii) $\implies$ (i): It is enough to show that a double suspension of a compact Kähler manifold $N$ by $f, g \in \text{Aut}_0(N)$ is Kählerian. Indeed, a finite covering $\tilde{X}$ of a compact complex manifold $X$ is Kählerian if and only if $X$ is Kählerian and the suspension $N \times C/(F, G)$, for $\tilde{F}(x, z) = (f^n(x), z+1)$ and $\tilde{G}(x, z) = (g^m(x), z + \frac{m}{n} \tau)$, is a finite covering of $X$. The statement follows now from a theorem by A.Blanchard (cf. [2], p.192) which states the total space $X$ of a fibred space with base $B$ and fibre $F$ such that $\pi_1(B)$ acts trivially on $H^1(F, \mathbb{R})$ is Kählerian if and only if there is a Kähler form on $F$ which represents a cohomology class invariant by $\pi_1(B)$, $B$ is Kählerian and $b_1(X) = b_1(B) + b_1(F)$. Note that since $f, g \in \text{Aut}_0(N)$ the fundamental group $\pi_1(E)$ acts trivially on $H^1(N, \mathbb{R})$ and consequently $b_1(X) = b_1(N) + 2$. \hfill \Box

In particular if $N$ is a compact Kähler manifold and $\text{Aut}_C(N)$ is finite then any complex manifold $X$ thus obtained from $N$ must be Kählerian.

**Corollary 3.4.2.** Let $N$ be a compact Kähler manifold, if $b_2(N) = 1$ any compact complex manifold $X$ obtained by a double suspension from $N$ is Kählerian.
Theorem 3.4.3. Let $\omega$ be a Kähler form on $N$ and $f \in \text{Aut}_C(N)$ we have $[f^*\omega] = [\lambda \omega]$, where $\lambda \in \mathbb{R}^+$. Then $f^*\omega = \lambda \omega + d\alpha$ where $\alpha$ is a 1-form. Moreover we know that $[\lambda \omega^n] = f^*[\omega^n] = [\omega^n]$, therefore $\lambda = \pm 1$. Assume that $\lambda = -1$, then the Kähler form $\omega + f^*\omega = d\alpha$ is exact, which is a contradiction. Thus $\lambda = 1$ and we conclude that the hypothesis (ii) in the theorem 3.4.1 are trivially fulfilled. 

Proof. Applying Hodge decomposition theorem to $N$ one concludes that $H^2(N, \mathbb{C}) \cong H^{1,1}(N, \mathbb{C}) \cong \mathbb{C}$. Then if $\omega$ is a Kähler form on $N$ and $f \in \text{Aut}_C(N)$ we have $[f^*\omega] = [\lambda \omega]$, where $\lambda \in \mathbb{R}^+$. Then $f^*\omega = \lambda \omega + d\alpha$ where $\alpha$ is a 1-form. Moreover we know that $[\lambda \omega^n] = f^*[\omega^n] = [\omega^n]$, therefore $\lambda = \pm 1$. Assume that $\lambda = -1$, then the Kähler form $\omega + f^*\omega = d\alpha$ is exact, which is a contradiction. Thus $\lambda = 1$ and we conclude that the hypothesis (ii) in the theorem 3.4.1 are trivially fulfilled. 

Although it escapes the scope of this paper note that the same arguments as in the above theorem yield the following theorem. We start by recalling the general definition of suspension. Let $N$ and $B$ be two complex manifolds, let $\tilde{B}$ be the universal cover of $B$ and $\Gamma = \pi_1(B)$ (acting by the right on $\tilde{B}$). If $h : \Gamma \to \text{Aut}_C(N)$ is a group homomorphism we define the suspension of $N$ by $B$ and $h$ as the quotient $\Gamma \backslash N \times B$ where the action by $\Gamma$ is

$$\Gamma \times (N \times \tilde{B}) \to (N \times B)$$

$$(\gamma, (x, z)) \mapsto (h(\gamma)(x), z \cdot \gamma^{-1}).$$

Theorem 3.4.3. Let $N$ and $B$ be two compact connected complex manifolds and $h : \Gamma = \pi_1(B) \to \text{Aut}_C(N)$ a group homomorphism. Then the following conditions are equivalent:

(i) The suspension of $N$ by $B$ and $h$ is Kählerian.

(ii) $N$ is Kählerian and there is a Kähler form $\omega$ on $N$ such that $[f^*\omega] = [\omega]$ for every $f \in h(\Gamma)$.

(iii) $N$ and $B$ are Kählerian and there exists a subgroup $\Gamma'$ of $\Gamma$ such that $h(\Gamma') \subset \text{Aut}_0(N)$ and $h(\Gamma') / h(\Gamma')$ is finite.

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