Heteroclinic and homoclinic solutions for nonlinear second-order coupled systems with $\phi$-Laplacians

Robert de Sousa $^{1,2}$ · Feliz Minhós $^3$

Received: 2 October 2020 / Accepted: 4 June 2021 / Published online: 16 June 2021
© SBMAC - Sociedade Brasileira de Matemática Aplicada e Computacional 2021

Abstract
In this paper, we present sufficient conditions for the existence of heteroclinic or homoclinic solutions for second-order coupled systems of differential equations on the real line. We point out that it is required only conditions on the homeomorphisms and no growth or asymptotic conditions are assumed on the nonlinearities. The arguments make use of the fixed point theory, $L^1$-Carathéodory functions and Schauder’s fixed point theorem. An application to a family of second-order nonlinear coupled systems of two degrees of freedom, shows the applicability of the main theorem.

Keywords  Heteroclinic and homoclinic solutions · Coupled systems · $L^1$-Carathéodory functions · Homeomorphism · Schauder’s fixed-point theorem · Operator theory

Mathematics Subject Classification  47H10 · 34K25 · 34B27 · 34L30

Robert de Sousa
robert.sousa@docente.unicv.edu.cv
Feliz Minhós
fminhos@uevora.pt

1 Faculdade de Ciências e Tecnologia, Núcleo de Matemática e Aplicações (NUMAT), Centro de Investigação em Ciências Exatas (CiCE), Universidade de Cabo Verde, Campus de Palmarejo, 279, Praia, Cabo Verde

2 Centro de Investigação em Matemática e Aplicações (CIMA), Instituto de Investigação e Formação Avançada, Universidade de Évora Rua Romão, Ramalho 59, 7000-671 Évora, Portugal

3 Departamento de Matemática, Escola de Ciências e Tecnologia, Centro de Investigação em Matemática e Aplicações (CIMA), Instituto de Investigação e Formação Avançada, Universidade de Évora Rua Romão, Ramalho 59, 7000-671 Évora, Portugal
1 Introduction

In this paper, we consider the second-order coupled system on the real line

\[
\begin{align*}
\left\{ \begin{array}{l}
(a(t)\phi(u'(t)))' = f(t, u(t), v(t), u'(t), v'(t)), \\
(b(t)\psi(v'(t)))' = h(t, u(t), v(t), u'(t), v'(t)),
\end{array} \right. \\
t \in \mathbb{R},
\end{align*}
\]

(1)

with \(\phi\) and \(\psi\) increasing homeomorphisms verifying some adequate relations on their inverses, \(a, b : \mathbb{R} \rightarrow (0, +\infty]\) are continuous functions, \(f, h : \mathbb{R}^5 \rightarrow \mathbb{R}\) are \(L^1\)-Carathéodory functions, together with asymptotic conditions

\[u(-\infty) = A, \quad u'(+\infty) = 0, \quad v(-\infty) = B, \quad v'(+\infty) = 0\]

(2)

for \(A, B \in \mathbb{R}\).

Heteroclinic trajectories play an important role in geometrical analysis of dynamical systems, connecting unstable and stable equilibria having two or more equilibrium points, Izhikevich (2007). In fact, the homoclinic or heteroclinic orbits are a kind of spiral structures, which are general phenomena in nature (Zuntao et al. 2005). Graphical illustrations and a very complete explanation on homoclinics and heteroclinics bifurcations can be seen in Homburg and Sandstede (2010). A planar homoclinic theorem and heteroclinic orbits, to analyze fluid models, is studied in Bertozzi (1988). Applications of dynamic systems techniques to the problem of heteroclinic connections and resonance transitions are treated in Koon et al. (2000), on planar circular domains. To prove the existence of heteroclinic solutions, for a class of non-autonomous second-order equations, see Ellero and Zanolin (2013) and Marcelli and Papalini (2007). Topological, variational and minimization methods to find heteroclinic connections can be found in Zelati and Rabinowitz (2001).

On heteroclinic coupled systems, among many published works, we highlight some of them: in Aguiar et al. (2011), Aguiar et al. consider the dynamics of small networks of coupled cells, with one of the points, analyzed as invariant subsets, can support robust heteroclinic attractors. In Ashwin and Karabacak (2011), Ashwin and Karabacak study coupled phase oscillators and discuss heteroclinic cycles and networks between partially synchronized states and in Karabacak and Ashwin (2010), they analyze coupled phase oscillators, highlighting a dynamic mechanism, nothing more than a heteroclinic network. Moreover, in Feng and Hu (2003), the authors present some applications, results, methods and problems that have been recently reported and, in addition, they suggest some possible research directions, and some problems for further studies on homoclinics and heteroclinics.

Cabada and Cid (2009) study the following boundary value problem on the real line:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\phi(u'(t)))' = f(t, u(t), u'(t)), \quad \text{on } \mathbb{R}, \\
u(-\infty) = -1, \quad u(+\infty) = 1,
\end{array} \right.
\end{align*}
\]

with a singular \(\phi\)-Laplacian operator, where \(f\) is a continuous function that satisfies suitable symmetric conditions. In Calamai (2011), Calamai discusses the solvability of the following strongly nonlinear problem:

\[
\begin{align*}
\left\{ \begin{array}{l}
(a(x(t))\phi(x'(t)))' = f(t, x(t), x'(t)), \quad t \in \mathbb{R}, \\
x(-\infty) = \alpha, \quad x(+\infty) = \beta,
\end{array} \right.
\end{align*}
\]

where \(\alpha < \beta\), \(\phi : (-r, r) \rightarrow \mathbb{R}\) is a general increasing homeomorphism with bounded domain (singular \(\phi\)-Laplacian), \(a\) is a positive, continuous function and \(f\) is a Carathéodory function.
nonlinear function. Recently, in Kajiwara (2017), Kajiwara proved the existence of a heteroclinic solution of the FitzHugh–Nagumo type reaction–diffusion system, under certain conditions on the heterogeneity.

Motivated by these works and applying the techniques suggested in Liu and Chen (2014), Minhós and Carrasco (2016), Minhós (2017) and Minhós and de Sousa (2018), we apply the fixed point theory, to obtain sufficient conditions for the existence of heteroclinic solutions of the coupled system (1), (2), assuming some adequate conditions on $\phi^{-1}$, $\psi^{-1}$.

We emphasize that it is the first time where heteroclinic solutions for second-order coupled differential systems are considered for systems with full nonlinearities depending on both unknown functions and their first derivatives. An example illustrates the potentialities of our main result, and an application to coupled nonlinear systems of two degrees of freedom (2-DOF) shows the applicability of the main theorem.

This paper is organized as it follows: Sect. 2 contains some preliminary results. In Sect. 3, we present the main theorem: an existence result of, at least, a pair of heteroclinic solutions. An application to a family of coupled 2-DOF nonlinear systems is presented in the last section.

2 Notations and preliminary results

Consider the following spaces:

$$X := \left\{ x \in C^1(\mathbb{R}) : \lim_{t \to \pm \infty} x^{(i)}(t) \in \mathbb{R}, \; i = 0, 1 \right\}$$

equipped with the norm

$$\|x\|_X = \max \left\{ \|x\|_\infty, \|x'\|_\infty \right\},$$

where

$$\|x\|_\infty := \sup_{t \in \mathbb{R}} |x(t)|,$$

and $X^2 := X \times X$ with

$$\|(u, v)\|_{X^2} = \max \left\{ \|u\|_X, \|v\|_X \right\}.$$

It can be proved that $(X, \|\cdot\|_X)$ and $(X^2, \|\cdot\|_{X^2})$ are Banach spaces.

**Remark 1** If $w \in X$ then $w'(\pm \infty) = 0$.

By solution of problem (1), (2), we mean a pair $(u, v) \in X^2$ such that

$$a(t)\phi(u'(t)) \in W^{1,1}(\mathbb{R}) \quad \text{and} \quad b(t)\psi(v'(t)) \in W^{1,1}(\mathbb{R}),$$

verifying (1), (2).

For the reader’s convenience, we consider the definition of $L^1$–Carathéodory functions:

**Definition 2** A function $g : \mathbb{R}^5 \to \mathbb{R}$ is $L^1$–Carathéodory if

(i) for each $(x, y, z, w) \in \mathbb{R}^4$, $t \mapsto g(t, x, y, z, w)$ is measurable on $\mathbb{R}$;

(ii) for a.e. $t \in \mathbb{R}$, $(x, y, z, w) \mapsto g(t, x, y, z, w)$ is continuous on $\mathbb{R}^4$;

(iii) for each $\rho > 0$, there exists a positive function $\vartheta_\rho \in L^1(\mathbb{R})$ such that, whenever $x, y, z, w \in [-\rho, \rho]$, then

$$|g(t, x, y, z, w)| \leq \vartheta_\rho(t), \; \text{a.e.} \; t \in \mathbb{R}. \quad (3)$$
Along this chapter, we assume that

\((H1)\) \(\phi, \psi : \mathbb{R} \to \mathbb{R}\) are increasing homeomorphisms such that

(a) \(\phi(\mathbb{R}) = \mathbb{R}\), \(\phi(0) = 0\), \(\psi(\mathbb{R}) = \mathbb{R}\), \(\psi(0) = 0\);  
(b) \(|\phi^{-1}(x)| \leq |\psi^{-1}(|x|)|\), \(|\psi^{-1}(x)| \leq |\phi^{-1}(|x|)|\).

\((H2)\) \(a, b : \mathbb{R} \to (0, +\infty)\) are positive continuous functions such that

\[
\lim_{t \to \pm \infty} \frac{1}{a(t)} \in \mathbb{R} \text{ and } \lim_{t \to \pm \infty} \frac{1}{b(t)} \in \mathbb{R}.
\]

A convenient criterion for the compacity of the operators is given by next theorem:

**Theorem 3** (Minhós and Carrasco 2016, Theorem 2.3) A set \(M \subset X\) is relatively compact if the following conditions hold:

(i) both \(t \mapsto x(t) : x \in M\) and \(t \mapsto x'(t) : x \in M\) are uniformly bounded;  
(ii) both \(t \mapsto x(t) : x \in M\) and \(t \mapsto x'(t) : x \in M\) are equicontinuous on any compact interval of \(\mathbb{R}\);  
(iii) both \(t \mapsto x(t) : x \in M\) and \(t \mapsto x'(t) : x \in M\) are equiconvergent at \(\pm \infty\), that is, for any given \(\epsilon > 0\), there exists \(t_\epsilon > 0\) such that

\[
|f(t) - f(\pm \infty)| < \epsilon, \quad |f'(t) - f'(\pm \infty)| < \epsilon, \forall |t| > t_\epsilon, \ f \in M.
\]

The existence tool will be given by Schauder’s fixed point theorem:

**Theorem 4** (Zeidler 1986) Let \(Y\) be a nonempty, closed, bounded and convex subset of a Banach space \(X\), and suppose that \(P : Y \to Y\) is a compact operator. Then \(P\) has at least one fixed point in \(Y\).

## 3 Existence of heteroclinics

In this section, we prove the existence for a pair of heteroclinic solutions to the coupled system (1), (2), for some constants \(A, B \in \mathbb{R}\).

**Theorem 5** Let \(\phi, \psi : \mathbb{R} \to \mathbb{R}\) be increasing homeomorphisms and \(a, b : \mathbb{R} \to (0, +\infty)\) continuous functions satisfying \((H1)\) and \((H2)\). Assume that \(f, h : \mathbb{R}^2 \to \mathbb{R}\) are \(L^1\)–Carathéodory functions and there is \(R > 0\) and \(\vartheta_R, \theta_R \in L^1(\mathbb{R})\) such that

\[
\int_{-\infty}^{+\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \vartheta_R(r) dr}{a(s)} \right) ds < +\infty,
\]

\[
\int_{-\infty}^{+\infty} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \theta_R(r) dr}{b(s)} \right) ds < +\infty,
\]

with

\[
\sup_{t \in \mathbb{R}} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \vartheta_R(r) dr}{a(t)} \right) ds < +\infty, \quad \sup_{t \in \mathbb{R}} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \theta_R(r) dr}{b(t)} \right) ds < +\infty,
\]

\[
|f(t, x, y, z, w)| \leq \vartheta_R(t), \quad |h(t, x, y, z, w)| \leq \theta_R(t),
\]

\(\vartheta_R, \theta_R \in L^1(\mathbb{R})\) such that

\[
\int_{-\infty}^{+\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \vartheta_R(r) dr}{a(s)} \right) ds < +\infty, \quad \int_{-\infty}^{+\infty} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \theta_R(r) dr}{b(s)} \right) ds < +\infty,
\]
whenever $x, y, z, w \in [-R, R]$.

Then for given $A, B \in \mathbb{R}$, problem (1), (2) has, at least, a pair of heteroclinic solutions $(u, v) \in X^2$.

Proof Define the operators $T_1 : X^2 \to X$, $T_2 : X^2 \to X$ and $T : X^2 \to X^2$ by

$$T (u, v) = (T_1 (u, v), T_2 (u, v)),$$

with

$$(T_1 (u, v)) (t) = \int_{-\infty}^{t} \phi^{-1} \left( \frac{\int_{-\infty}^{s} f (r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds + A,$$

$$(T_2 (u, v)) (t) = \int_{-\infty}^{t} \psi^{-1} \left( \frac{\int_{-\infty}^{s} h (r, u(r), v(r), u'(r), v'(r)) dr}{b(s)} \right) ds + B,$$

with $A$ and $B$ given by (2).

To apply Theorem 4, we shall prove that $T$ is compact and has a fixed point and to simplify the proof, we detail the arguments only for $T_1 (u, v)$, as for the operator $T_2 (u, v)$ the technique is similar. To be clear, we divide the proof into claims (i)–(v).

(i) $T$ is well defined and continuous in $X^2$

Let $(u, v) \in X^2$ and take $\rho > 0$ such that $\| (u, v) \|_{X^2} < \rho$. As $f$ is a $L^1$–Carathéodory function, there exists a positive function $\varphi_{\rho} \in L^1(\mathbb{R})$ verifying (6). Moreover, $T_1 \in C^1(\mathbb{R})$, as

$$\int_{-\infty}^{t} |f(r, u(r), v(r), u'(r), v'(r))| dr \leq \int_{-\infty}^{+\infty} \varphi_{\rho}(t) dt < +\infty$$

and

$$(T_1 (u, v))' (t) = \phi^{-1} \left( \frac{\int_{-\infty}^{t} f (r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right).$$

By (2), (4), (6) and (H2),

$$\lim_{t \to -\infty} T_1 (u, v) (t) = \lim_{t \to -\infty} \int_{-\infty}^{t} \phi^{-1} \left( \frac{\int_{-\infty}^{s} f (r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds + A = A,$$

$$\lim_{t \to +\infty} T_1 (u, v) (t) = \int_{-\infty}^{+\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{s} f (r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds + A < +\infty,$$

and

$$\lim_{t \to \pm\infty} (T_1 (u, v) (t))' = \lim_{t \to \pm\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{t} f (r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \leq \lim_{t \to \pm\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \varphi_{\rho}(r) dr}{a(t)} \right) < +\infty.$$

Therefore, $T_1 (u, v) \in X$, and, by the same arguments, $T_2 (u, v) \in X$. So, $T (u, v) \in X^2$. 

 Springer
(ii) $TM$ is uniformly bounded on $M \subseteq X^2$, for some bounded $M$

Let $M$ be a bounded set of $X^2$, defined by

$$M := \{(u, v) \in X^2 : \max \{\|u\|_{\infty}, \|u'\|_{\infty}, \|v\|_{\infty}, \|v'\|_{\infty}\} \leq \rho_1\},$$  \hspace{1cm} (9)

for some $\rho_1 > 0$.

By (4), (6), (H1) and (H2), we have

$$\|T_1(u, v)(t)\|_{\infty} = \sup_{t \in \mathbb{R}} \left|\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right)ds + A\right|$$

$$\leq \sup_{t \in \mathbb{R}} \left|\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right)ds + |A|\right|$$

$$\leq \sup_{t \in \mathbb{R}} \left|\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right)ds + |A|\right|$$

$$\leq \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \vartheta \rho_1(r)dr}{a(s)}\right)ds + |A| < +\infty,$$

and

$$\| (T_1(u, v))'(t) \|_{\infty} = \sup_{t \in \mathbb{R}} \left|\phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r))dr}{a(t)}\right)\right|$$

$$\leq \sup_{t \in \mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r))dr}{a(t)}\right)$$

$$\leq \sup_{t \in \mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \vartheta \rho_1(r)dr}{a(t)}\right) < +\infty.$$

So, $\|T_1(u, v)(t)\|_{X} < +\infty$, that is, $T_1M$ is uniformly bounded on $X$ and by similar arguments, $T_2$ is uniformly bounded on $X$. Therefore, $TM$ is uniformly bounded on $X^2$.

(iii) $TM$ is equicontinuous on $X^2$

Let $t_1, t_2 \in [-K, K] \subseteq \mathbb{R}$ for some $K > 0$, and suppose, without loss of generality, that $t_1 \leq t_2$. Thus, by (4), (6) and (H1),

$$|T_1(u, v)(t_1) - T_1(u, v)(t_2)| = \left|\int_{-\infty}^{t_1} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r))dr}{a(s)}\right)dsight.$$
uniformly for \((u, v) \in M\), as \(t_1 \to t_2\), and

\[
\left| (T_1 (u, v))' (t_1) - (T_1 (u, v))' (t_2) \right| = \left| \phi^{-1} \left( \frac{\int_{t_1}^{t_2} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \right|
\]

\[
- \phi^{-1} \left( \frac{\int_{t_1}^{t_2} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \to 0,
\]

uniformly for \((u, v) \in M\), as \(t_1 \to t_2\).

Therefore, \(T_1 M\) is equicontinuous on \(X\). Analogously, it can be proved that \(T_2 M\) is equicontinuous on \(X\). So, \(TM\) is equicontinuous on \(X^2\).

(iv) \(TM\) is equiconvergent at \(t = \pm \infty\)

Let \((u, v) \in M\). For the operator \(T_1\), we have, by (4), (6) and (H1),

\[
\left| T_1 (u, v) (t) - \lim_{t \to -\infty} T_1 (u, v) (t) \right| = \left| \int_{-\infty}^{t} \phi^{-1} \left( \frac{\int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds \right|
\]

\[
\leq \int_{-\infty}^{t} \phi^{-1} \left( \frac{\int_{-\infty}^{s} \vartheta \rho_1 (r) dr}{a(s)} \right) ds \to 0,
\]

uniformly in \((u, v) \in M\), as \(t \to -\infty\), and,

\[
\left| T_1 (u, v) (t) - \lim_{t \to +\infty} T_1 (u, v) (t) \right| = \left| \int_{t}^{+\infty} \phi^{-1} \left( \frac{\int_{t}^{s} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds \right|
\]

\[
- \int_{-\infty}^{t} \phi^{-1} \left( \frac{\int_{-\infty}^{s} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds \left| \right|
\]

\[
= \left| \int_{t}^{+\infty} \phi^{-1} \left( \frac{\int_{t}^{s} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds \right|
\]

\[
\leq \int_{t}^{+\infty} \phi^{-1} \left( \frac{\int_{t}^{s} \vartheta \rho_1 (r) dr}{a(s)} \right) ds \to 0,
\]

uniformly in \((u, v) \in M\), as \(t \to +\infty\).

For the derivative, it follows that

\[
\left| (T_1 (u, v))' (t) - \lim_{t \to +\infty} (T_1 (u, v))' (t) \right| = \left| \phi^{-1} \left( \frac{\int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \right|
\]

\[
- \phi^{-1} \left( \lim_{t \to +\infty} \frac{\int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \left| \right| \to 0
\]

uniformly in \((u, v) \in M\), as \(t \to +\infty\), and

\[
\left| (T_1 (u, v))' (t) - \lim_{t \to -\infty} (T_1 (u, v))' (t) \right| = \left| \phi^{-1} \left( \frac{\int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \right|
\]

\[
\leq \phi^{-1} \left( \frac{\int_{-\infty}^{t} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(t)} \right) \left| \right|
\]

\[
\leq \phi^{-1} \left( \frac{\int_{-\infty}^{t} \vartheta \rho_1 (r) dr}{|a(t)|} \right) \to 0
\]
uniformly in \((u, v) \in M\), as \(t \to -\infty\).

Therefore, \(T_1 M\) is equiconvergent at \(\pm \infty\) and, following a similar technique, we can prove that \(T_2 M\) is equiconvergent at \(\pm \infty\), too. So, \(TM\) is equiconvergent at \(\pm \infty\) and by Theorem 3, \(TM\) is relatively compact.

(v) \(T : X \to X\) has a fixed point

To apply Schauder’s fixed point theorem for operator \(T (u, v)\), we need to prove that \(TD \subset D\), for some closed, bounded and convex \(D \subset X^2\).

Consider

\[
D := \{ (u, v) \in X^2 : \| (u, v) \|_X^2 \leq \rho_2 \},
\]

with \(\rho_2 > 0\) such that

\[
\rho_2 \geq \max \left\{ \rho_1, \int_{-\infty}^{+\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \varphi(t) dt}{a(t)} \right) ds + |A|, \right. \\
\int_{-\infty}^{+\infty} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \chi(t) dt}{b(t)} \right) ds + |B|, \\
\left. \sup_{t \in \mathbb{R}} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \varphi(t) dt}{a(t)} \right), \sup_{t \in \mathbb{R}} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \chi(t) dt}{b(t)} \right) \right\}
\]

with \(\rho_1\) given by (9).

Following similar arguments as in (ii), we have, for \((u, v) \in D\),

\[
\| T(u, v) \|_X^2 = \| (T_1(u, v), T_2(u, v)) \|_X^2 \\
= \max \{ \| T_1(u, v) \|_X, \| T_2(u, v) \|_X \} \\
= \max \{ \| T_1(u, v) \|_\infty, \| T_2(u, v) \|_\infty \} \leq \rho_2,
\]

and \(TD \subset D\).

By Theorem 4, the operator \(T (u, v) = (T_1(u, v), T_2(u, v))\) has a fixed point \((u, v) \in X^2\) and following standard arguments, it can be proved that this fixed point defines a pair of heteroclinic or homoclinic solutions of problem (1), (2).

\(\square\)

Remark 6 If

\[
\int_{-\infty}^{+\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} f(r, u(r), v(r), u'(r), v'(r)) dr}{a(s)} \right) ds = 0
\]

and

\[
\int_{-\infty}^{+\infty} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} h(r, u(r), v(r), u'(r), v'(r)) dr}{b(s)} \right) ds = 0,
\]

the solutions \((u, v) \in X^2\) of problem (1), (2) will be a pair of homoclinic solutions.

4 Application to coupled systems of nonlinear 2-DOF model

Generic nonlinear coupled systems of two degrees of freedom (2-DOF), are especially important in Physics and Mechanics. For example in Mikhlin et al. (2008), the authors use this
type of system to investigate the transient in a system containing a linear oscillator, linearly coupled to an essentially nonlinear attachment with a comparatively small mass. The family of coupled non-linear systems of 2-DOF is used to study the global bifurcations in the motion of an externally forced coupled nonlinear oscillatory system or for the nonlinear vibration absorber subjected to periodic excitation, see Malhotra and Sri Namachchivaya (1995). Moreover, in Ariaratnam et al. (1991), the authors deal with the stochastic moment stability of such systems.

Motivated by these works, in this section, we consider an application of system (1), (2), to a family of coupled non-linear systems of 2-DOF model, given by the nonlinear coupled system (see Malhotra and Sri Namachchivaya 1995)

\[
\begin{align*}
(1 + t^4) (q'_1(t))^3 & = \frac{t^4}{(1 + r^6)^2} \left[ 2\zeta \omega_0 q_1(t) + \omega_0^2 q_1(t) + \gamma (q_1(t))^3 \right] - 3d^2 q_1(t)q_2(t) + \cos(t), \\
(1 + t^4) (q'_2(t))^3 & = \frac{t^4}{(1 + r^6)^2} \left[ 2\zeta \omega_0 q_2(t) + \omega_0^2 q_2(t) + \gamma (d^2 q_2(t))^3 \right] - 3 (q_1(t))^2 q_2(t), \\
\end{align*}
\]

where

- \(q_1(t)\) and \(q_2(t)\) represent the generalized coordinates;
- \(d, \tau, \gamma\) are positive constant coefficients which depend on the characteristics of the physical or mechanical system under consideration;
- \(\cos(t)\) is related to the type of excitation of the system under consideration;
- \(\zeta, \omega_0\), are the damping coefficient and the frequency, respectively.

As the asymptotic conditions, we consider

\[
q_1(-\infty) = A, \quad q'_1(+\infty) = 0, \quad q_2(-\infty) = B, \quad q'_2(+\infty) = 0,
\]

with \(A, B \in \mathbb{R}\), and, moreover, assume that the real coefficients \(\zeta, \omega_0, \gamma, d, r\) are such that the integrals

\[
\int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \right) \right) ds \quad \text{(12)}
\]

and

\[
\int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \right) \right) ds \quad \text{(13)}
\]

are finite.

It is clear that (10) is a particular case of (1) with

\[
\phi(z) = \psi(z) = z^3, \quad a(t) = b(t) = 1 + t^4,
\]
\[ f(t, x, y, z, w) = \frac{t^4}{(1 + t^6)^2} (2\xi \omega_0 z^3 + \omega_0^2 x + \gamma x^3 - 3d^2 x y + \cos(t)) \]
\[ \leq \frac{t^4}{(t^6 + 1)^2} (2 |\xi \omega_0| \rho^3 + \omega_0^2 \rho + \gamma \rho^3 + 3d^2 \rho^2 + 1) \]
\[ = \delta_\rho(t) \]
\[ h(t, x, y, z, w) = \frac{t^4}{\tau^2(1 + t^6)^2} (2\xi \omega_0 w^3 + \omega_0^2 y + \gamma d^2 y^3 - 3x^2 y) \]
\[ \leq \frac{t^4}{\tau^2(t^6 + 1)^2} (2 |\xi \omega_0| \rho^3 + \omega_0^2 \rho + \gamma d^2 \rho^3 + 3\rho^3) \]
\[ = \epsilon_\rho(t), \]

where \( \delta_\rho(t) \) and \( \epsilon_\rho(t) \) are functions in \( L^1(\mathbb{R}) \), for \( \rho > 0 \) such that
\[ \rho := \max \{|x|, |y|, |z|, |w|\}. \tag{14} \]

Moreover, conditions (H1) and (H2) hold as
- \( \phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R} \) and \( \phi(0) = \psi(0) = 0; \)
- \( |\phi^{-1}(z)| = |\sqrt{z}| = \phi^{-1}(|z|) = \sqrt{|z|} \) and \( \psi^{-1}(w) = |\sqrt{w}| = \psi^{-1}(|w|) = \sqrt{|w|}; \)
- \( \lim_{t \to \pm \infty} \frac{1}{\frac{1}{a(t)}} = \lim_{t \to \pm \infty} \frac{1}{1 + t^4} = \lim_{t \to \pm \infty} \frac{1}{b(t)} = 0. \)

For \( \rho > 0 \) such that
\[
\int_{-\infty}^{+\infty} \phi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \delta_\rho(r) dr}{a(s)} \right) ds = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} r^4 \frac{2|\xi \omega_0| \rho^3 + \omega_0^2 \rho + \gamma \rho^3 + 3d^2 \rho^2 + 1}{(1 + t^6)^2} \frac{dr}{1 + s^4} \right) ds < \rho \tag{15}
\]
and
\[
\int_{-\infty}^{+\infty} \psi^{-1} \left( \frac{\int_{-\infty}^{+\infty} \epsilon_\rho(r) dr}{b(s)} \right) ds = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} r^4 \frac{2|\xi \omega_0| \rho^3 + \omega_0^2 \rho + \gamma d^2 \rho^3 + 3\rho^3}{\tau^2(1 + t^6)^2} \frac{dr}{1 + s^4} \right) ds < \rho, \tag{16}
\]

by Theorem 5, system (10) together with the asymptotic conditions (11) has at least a pair \((q_1, q_2) \in X^2\) of heteroclinic solutions since the integrals (12) and (13) are finite. As example, in particular, for
\[ |\xi| = \frac{1}{2\sqrt{1000}}, |\omega_0| = \frac{1}{\sqrt{1000}}, \gamma = \frac{1}{1000}, d^2 = \frac{1}{3000}, \tau = 23, \]
inequalities (15) and (16) take the following form:

$$
\int_{-\infty}^{+\infty} \left( \sqrt{\frac{1}{1 + s^4}} \left[ \frac{1}{2\sqrt{1000}} \frac{1}{\sqrt{1000}} \rho^3 + \left( \frac{1}{\sqrt{1000}} \right)^2 \rho + \frac{1}{1000} \rho^3 + \frac{1}{3000} \rho^2 + 1 \right] \right) \, ds < \rho
$$
\[
\int_{-\infty}^{+\infty} \left[ \frac{3}{4} \int_{-\infty}^{+\infty} r^4 \left( 2 \frac{1}{2\sqrt{1000}} \frac{1}{\sqrt{1000}} \left( \rho^3 + \frac{1}{\sqrt{1000}} \frac{1}{\sqrt{1000}} \rho + \frac{1}{\sqrt{1000}} \frac{1}{\sqrt{1000}} \rho^3 + 3\rho^3 \right) \right) dr \right] ds < \rho,
\]

that with the help of the Maple software, we can see that both relations hold for \( \rho > 6.3542 \).

For the values of the above parameters, \( A = 10 \) and \( B = 8 \), the heteroclinics solutions \( q_1 \) and \( q_2 \) have the graphs given in Fig. 1. In Fig. 2, we present the real shape of the \( q_1 \) trajectory, which is not detailed in Fig. 1 due to the scale range.

Remark that, if integrals (12) and (13) are null, then system (10) has a pair of homoclinic solutions \((q_1, q_2) \in X^2 \).

References

Aguiar M, Ashwin P, Dias A, Field M (2011) Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation. J Nonlinear Sci 21(2):271–323

Ashwin P, Karabacak Ö (2011) Robust heteroclinic behaviour, synchronisation, and ratcheting of coupled oscillators. Dyn Games Sci II:125–140

Ariaratnam S, Tan D, Xie W-C (1991) Lyapunov exponents of two-degrees-of-freedom linear stochastic systems, stochastic structural dynamics 1, new theoretical developments. Springer, Berlin, Heidelberg

Bertozzi A (1988) Heteroclinic orbits and chaotic dynamics in planar fluid flows. SIAM J Math Anal 19(6):1271–1294

Cabada A, Cid J (2009) Heteroclinic solutions for non-autonomous boundary value problems with singular \( \Phi \)-Laplacian operators. Discret and continuous dynamical systems supplement, pp 118–122

Calamai A (2011) Heteroclinic solutions of boundary value problems on the real line involving singular \( \Phi \)-Laplacian operators. J Mat. Anal Appl 378:667–679

Ellero E, Zanolli F (2013) Homoclinic and heteroclinic solutions for a class of second-order non-autonomous ordinary differential equations: multiplicity results for stepeweight potentials. Bound Value Probl 2013:167. https://doi.org/10.1186/1687-2770-2013-167

Feng B, Hu R (2003) A survey on homoclinic and heteroclinic orbits. Appl Math E Notes 3:16–37

Fu ZT, Liu SD, Liu SK, Liang FM, Xin GJ (2005) Homoclinic (heteroclinic) orbit of complex dynamical system and spiral structure. Commun Theor Phys 43(4):601–603

Homburg A, Sandstede B (2010) Homoclinic and heteroclinic bifurcations in vector fields. Handb Dyn Syst 3:379–524. https://doi.org/10.1016/S1874-575X(10)000316-4

Izhikevich E (2007) Dynamical systems in neuroscience: the geometry of excitability and bursting. Computational Neuroscience. The MIT Press, Cambridge

Kajiwara T (2017) A heteroclinic solution to a variational problem corresponding to FitzHugh–Nagumo type reaction-diffusion system with heterogeneity. Commun Pure Appl Anal 16(6):2133–2156

Karabacak Ö, Ashwin P (2010) Heteroclinic Ratchets in networks of coupled oscillators. J Nonlinear Sci 20:105–129

Koon W, Lo M, Marsden J, Ross S (2000) Heteroclinic orbits and chaotic dynamics in planar fluid flows. Chaos 10(2):427–469

Liu Y, Chen S (2014) Existence of bounded solutions of integral boundary value problems for singular differential equations on whole lines. Int J Math 25(8):1450078 (28 pages)

Malhotra N, Sri Namachchivaya N (1995) Global bifurcations in externally excited two-degree-of-freedom nonlinear systems. Nonlinear Dyn 8(1):85–109

Marcelli C, Papalini F (2007) Heteroclinic connections for fully non-linear non-autonomous second-order differential equations. J Differ Equations 241:160–183

Mikhlin Y, Bunakova T, Rudneva G, Perepelkin N (2008) Transient in 2-DOF nonlinear systems. ENOC 2008, Saint Petersburg (June, 30-July, 4)

Minhós F, Carrasco H (2016) Existence of homoclinic solutions for nonlinear second-order problems. Mediterr J Math 13:3849–3861. https://doi.org/10.1007/s00009-016-0718-4

Minhós F (2017) Sufficient conditions for the existence of heteroclinic solutions for \( \varphi \)-Laplacian differential equations. Complex Var Elliptic Equations 62(1):123–134
Minhós F, de Sousa R (2018) Existence of homoclinic solutions for nonlinear second-order coupled systems. J Differ Equations. https://doi.org/10.1016/j.jde.2018.07.072
Zeidler E (1986) Nonlinear functional analysis and its applications: fixed-point theorems. Springer, New York
Zelati V, Rabinowitz P (2001) Heteroclinic solutions between stationary points at different energy levels. Topological methods in nonlinear analysis. J Juliusz Schauder Center 17:1–21

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.