Nonlinear Simulation of an Embedded Crack in the Presence of Holes and Inclusions by XFEM

A. S. Shedbale, I. V. Singh*, B. K. Mishra

Department of Mechanical and Industrial Engineering, Indian Institute of Technology Roorkee, Roorkee, U.K, India.

Abstract

In this work, nonlinear analysis of an embedded cracked plate is performed by XFEM in presence of multiple discontinuities under plane stress condition. Von-Mises yield criterion is used to model the elasto-plastic behavior of the plate assuming isotropic strain hardening. The stress-strain response of the plate is modeled using Ramberg-Osgood material model. XFEM is used as a numerical tool to simulate the plate containing a major centre crack, holes and inclusions. A domain based J-integral approach is used to evaluate the stress intensity factors. The generalized Paris law is used to estimate the fatigue life of the plate under mode-I cyclic loading condition. The effect of inclusions and holes on the fatigue life of the component for linear and nonlinear material behavior is discussed in detail.

© 2013 The Authors. Published by Elsevier Ltd.
Selection and peer-review under responsibility of the organizing and review committee of IConDM 2013

Keywords: fatigue life; XFEM; Von-Mises yield criterion; multiple discontinuities; SIF

Nomenclature

| Symbol | Definition |
|--------|------------|
| $\sigma$ | Cauchy stress tensor |
| $b$ | Body force per unit volume |
| $d\lambda$ | Plastic multiplier |
| $f$ | Yield function |
| $\varepsilon$ | Strain tensor |
| $D_{ep}$ | Elasto-plastic constitutive matrix |

* Corresponding author. Tel.: +91-1332-285888; fax: +91-1332-285665.
E-mail address: ivsingh@gmail.com
1. Introduction

At present, the prediction of crack growth behavior becomes a major issue to ensure the reliability of the structures. In many cases (aerospace structures, engines and many other industrial applications), the fatigue crack growth leads to the catastrophic failure of the structures/components.

Thus, there is need to analyze such structures/components accurately to get better service throughout their lifespan. Nowadays, numerous methods are available for crack growth modeling such as boundary element method [1-3], finite element method [4], and meshfree methods [5-7]. Though, the FEM has been widely used for the crack growth modeling it has some drawbacks. It requires a conformal mesh to model the crack and remeshing at each stage of crack propagation which is a quite cumbersome process. Also, to handle the asymptotic tip stress field at the crack tip it needs some special elements. So, in order to overcome these difficulties, a novel approach, known as extended finite element method (XFEM) [8-11] has been developed to simulate the fracture mechanics problems. By this technique, it is possible to model the crack mesh independently and crack growth without remeshing. In XFEM, modeling of crack growth and other arbitrary discontinuities (holes and inclusions) is done by enriching the standard FE approximation with some additional functions [9]. The level set approach [10-11] has been widely used to model the crack growth.

Generally, it has been observed that the fatigue life of the components/structures is affected due to the presence of flaws (holes and inclusions). The fatigue life simulation of plate is already carried out by [12] in presence of multiple discontinuities with the linear analysis. Therefore, this paper presents the fatigue life simulation of a centre cracked plate in the presence of multiple discontinuities (holes and inclusions) for nonlinear material behavior. Under cyclic loading, some plane crack problems are solved in the presence of randomly distributed discontinuities of arbitrary size. A domain based J-integral approach is used to extract the stress intensity factor (SIF) values from XFEM solution. The maximum principal stress criterion [7, 12] is used to determine the crack growth direction. The generalized Paris law is used to determine the fatigue life from computed SIF values and crack extension. Under plane stress condition, some problems are solved by XFEM for linear and nonlinear material behavior. Finally, the fatigue life obtained using linear and nonlinear material behavior is compared.

2. Numerical Formulation

2.1. Governing Equations

The equilibrium equation [4] can be defined as

\[ \nabla \cdot \mathbf{\sigma} + \mathbf{b} = 0 \quad \text{in} \quad \Omega \]  

(1)

where, \( \mathbf{\sigma} \) is Cauchy stress tensor and \( \mathbf{b} \) is body force per unit volume. The mathematical theory of plasticity [13] is used to obtain the stress-strain relationship for a material which exhibits an elasto-plastic response. By applying Von-Mises yield criterion with isotropic hardening, the complete stress-strain relation is obtained. The change of strain is assumed to be divisible into elastic and plastic strain components and it is given as
The flow rule applied in this work is associated flow rule i.e. the plastic potential function is equivalent to yield function and which is given as

\[ (d\varepsilon^p_{ij}) = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \]  

(3)

where, \( d\lambda \) is plastic multiplier and \( f \) is yield function. An elasto-plastic incremental stress-strain relation can be written as

\[ d\sigma = D_{ep} d\varepsilon \]  

(4)

where \( D_{ep} \) is the elasto-plastic constitutive matrix. The stress-strain relation is obtained based on associated theory [14] of plasticity, thus the matrix \( D_{ep} \) becomes symmetric. However, when yield function is different than plastic potential function (non-associative flow rule), the matrix \( D_{ep} \) becomes un-symmetric.

2.2. Weak Formulation

The weak form of governing equations can be written as [9]

\[ \int_{\Omega} \sigma(u) : \varepsilon(v) d\Omega = \int_{\Omega} b \cdot v d\Omega + \int_{\Gamma} \bar{t} \cdot v d\Gamma \]  

(5)

On substituting the trial and test functions and using the arbitrariness of nodal variations, the following discrete system of equations are obtained

\[ [K]\{d\} = \{f\} \]  

(6)

where, \( K \) is the global stiffness matrix, \( d \) is the vector of nodal unknowns and \( f \) is the external force vector.

2.2.1. Displacement Approximation for Crack, Inclusions and Holes

The displacement approximation for 2-D body having multiple discontinuities such as cracks, inclusions and holes can be written as [11-12, 15]

\[ \mathbf{u}^h(x) = \sum_{i=1}^{n} N_i(x) \left[ \mathbf{u}_i + \sum_{\alpha=1}^{d} \left[ \xi_{\alpha}(x) - \xi_{\alpha}(x_i) \right] \mathbf{b}_{\alpha} + \sum_{i \in n_t} \left[ \phi(x) + \phi(x_i) \right] \mathbf{c}_i + \sum_{i \in n_h} \left[ \psi(x) - \psi(x_i) \right] \mathbf{d}_i \right] \]  

(7)

where,
\( \mathbf{u}_i \) = nodal displacement vector associated with the continuous part of the FE solution;
\( n \) = set of all nodes in the mesh;
\( n_r \) = set of nodes belonging to those elements which are completely cut by the crack;
\( n_d \) = set of nodes belonging to those elements which are partially cut by the crack;
\( H(x) = \) the discontinuous Heaviside function, defined for those elements, which are completely cut by the crack; it takes the value +1 on one side and -1 on other side of the crack.

\( a_i = \) nodal enriched degrees of freedom associated with \( H(x) \);

\( \xi_a(x) = \) the asymptotic crack tip enrichment functions, which are defined as

\[
\xi_a(x) = \{ \xi_1, \xi_2, \xi_3, \xi_4 \} = \begin{bmatrix}
  r^\gamma \cos \theta / 2, & r^\gamma \sin \theta / 2, & r^\gamma \cos \theta / 2 \sin \theta, & r^\gamma \sin \theta / 2 \sin \theta
\end{bmatrix}
\] (8)

where, \( (r, \theta) \) is a polar coordinate system with its origin defined at the crack tip. For linear analysis, the value of exponent \( \gamma = 0.5 \), and for nonlinear analysis, the value of exponent \( \gamma = \frac{1}{n+\bar{n}} \), where \( \bar{n} \) is the hardening exponent which depends on the material.

\( i_a = \) nodal enriched degrees of freedom vector associated with \( \xi_a(x) \);

\( n_i = \) set of nodes belonging to those elements which are cut by inclusions;

\( n_h = \) set of nodes belonging to those elements which are cut by holes;

\( c_i = \) nodal enriched degrees of freedom associated with \( \phi(x) \); where \( \phi(x) \) is level set function which can be defined as

\[
\phi(x) = \pm \min \left\| x - x_r \right\|
\] (9)

where, \( x_r \) is nearest point on the interface from the point \( x \).

\( b_i = \) nodal enriched degrees of freedom associated with \( \psi(x) \); where \( \psi(x) \) is the Heaviside function, which takes the value +1 for the nodes lying outside the hole and 0 for the nodes lying inside the hole.

The elemental matrices \( K \) and \( f \) in "Eq. (6)" are obtained using the approximation function defined in "Eq. (7)."

\[
K^e_{ij} = \begin{bmatrix}
  K_{uu}^{ia} & K_{ua}^{ia} & K_{ub}^{ia} & K_{uc}^{ia} & K_{ud}^{ia} \\
  K_{au}^{ia} & K_{aa}^{ia} & K_{ab}^{ia} & K_{ac}^{ia} & K_{ad}^{ia} \\
  K_{bu}^{ia} & K_{ba}^{ia} & K_{bb}^{ia} & K_{bc}^{ia} & K_{bd}^{ia} \\
  K_{cu}^{ia} & K_{ca}^{ia} & K_{cb}^{ia} & K_{cc}^{ia} & K_{cd}^{ia} \\
  K_{du}^{ia} & K_{da}^{ia} & K_{db}^{ia} & K_{dc}^{ia} & K_{dd}^{ia}
\end{bmatrix}
\] (10)

\[
f^h = \{ f^u_i, f^a_i, f^{b1}_i, f^{b2}_i, f^{b3}_i, f^{b4}_i, f^c_i, f^d_i \}^T
\] (11)

The expressions of force vectors for crack are given by [12], so, in similar way the force vector associated with holes and inclusions can be written as

\[
K^{rs}_{ij} = \int_{\alpha_r} (B^r_i)^T C B^s_j \, h d \Omega \quad \text{where} \quad r, s = u, a, b, c, d
\] (12)

\[
f^c = \int_{\alpha_r} N_i \phi(x) b \, \Omega + \int_{\Gamma_i} N_i \phi(x) \bar{t} d \Gamma
\] (13)
where, \( N_i \) are finite element shape function, \( B_i^\phi, B_i^\psi, B_i^\phi, B_i^{\phi\alpha}, B_i^\psi \) and \( B_i^{\psi\alpha} \) are the matrices of shape function derivatives. The expressions of \( B_i^\phi, B_i^\psi, B_i^\phi \) and \( B_i^{\psi\alpha} \) matrices are given by [12], thus, in similar way the matrices of shape function derivatives associated with the holes and inclusions can be written as

\[
B_i^\phi = \begin{bmatrix}
\left(N_i \phi(x)\right)_x & 0 \\
0 & \left(N_i \phi(x)\right)_y \\
\left(N_i \phi(x)\right)_y & \left(N_i \phi(x)\right)_x
\end{bmatrix}_{3\times8}
\]

\[
B_i^\psi = \begin{bmatrix}
\left(N_i \left(\psi(x) - \psi(x_i)\right)\right)_x & 0 \\
0 & \left(N_i \left(\psi(x) - \psi(x_i)\right)\right)_y \\
\left(N_i \left(\psi(x) - \psi(x_i)\right)\right)_y & \left(N_i \left(\psi(x) - \psi(x_i)\right)\right)_x
\end{bmatrix}_{3\times8}
\]

2.3. Computation of Stress Intensity Factor

The interaction integral is an effective tool for extracting the mixed-mode stress intensity factors [12, 16-18]. For two independent equilibrium states of a cracked body, the interaction integral is given as

\[
M^{(1,2)} = \int_A \left[ \sigma^{(1)}_{ij} \frac{\partial u^{(2)}_i}{\partial x_j} + \sigma^{(2)}_{ij} \frac{\partial u^{(1)}_i}{\partial x_j} - W^{(1,2)} \delta_{ij} \right] \frac{\partial q}{\partial x_j} dA
\]

where, \( W^{(1,2)} \) is the mutual strain energy and is given as

\[
W^{(1,2)} = \frac{1}{2} \left( \sigma^{(1)}_{ij} \varepsilon^{(2)}_{ij} + \sigma^{(2)}_{ij} \varepsilon^{(1)}_{ij} \right) = \sigma^{(1)}_{ij} \varepsilon^{(2)}_{ij} = \sigma^{(2)}_{ij} \varepsilon^{(1)}_{ij}
\]

2.4. Fatigue Life Calculation

The crack growth direction is determined by using the maximum principal stress criterion which postulates that the crack growth occurs in a direction perpendicular to the maximum principal stress. Thus, at each crack tip, the local direction of crack growth \( \theta_c \) is determined by the condition that the local shear stress should be zero, i.e.

\[
K_j \sin \theta_c + K_n \left(3 \cos \theta_c - 1\right) = 0
\]

The solution of above equation gives
According to this criterion, the equivalent mode-I SIF is

\[ K_{eq} = K_I \cos^3 \left( \frac{\theta_c}{2} \right) - 3K_{II} \cos^2 \left( \frac{\theta_c}{2} \right) \sin \left( \frac{\theta_c}{2} \right) \]  

(21)

The generalized Paris law is used to calculate the fatigue life [7], which is defined as

\[ \frac{da}{dN} = C (\Delta K_{eq})^m \]  

(22)

where, \( a \) is crack length, \( N \) is number of loading cycles, \( C \) and \( m \) are material properties and \( \Delta K_{eq} \) can be obtained with \( \Delta K_I \) and \( \Delta K_{II} \) from "Eq. (21)". The crack becomes unstable whenever \( K_{eq} \) exceeds \( K_{IC} \) where \( K_{IC} \) is the fracture toughness of the material.

3. Numerical Results and Discussion

In this work, the fatigue life is evaluated for a centre cracked plate in presence of multiple discontinuities such as holes and inclusions. The discontinuities are distributed randomly in the material. The holes and inclusions are circular in shape and their radii vary from 3.5 to 4 mm. The generalized material model of Ramberg-Osgood is used to model the elasto-plastic material behavior which is given as [19]

\[ \frac{1}{n} \frac{E}{H} \]  

(23)

where, \( H \) is strength coefficient and \( n \) is strain hardening exponent. The value of \( n = 0.0946 \) is used for all simulation problems. It should be noted that, the SIF values are computed based on the same criteria i.e. for same value of strain hardening exponent \( n = 1 \) in linear and nonlinear analysis but stress computation methodology is different in these cases.

The material (Aluminum 7075 T-6) properties [12, 20-21] used for the simulations are

Elastic Modulus \( E = 71.7 \) GPa
Elastic Modulus for Inclusions \( E_f = 20 \) GPa
Poisson Ratio for Material \( \nu = 0.33 \)
Poisson Ratio for Inclusions \( \nu_f = 0.3 \)
Fracture Toughness \( K_{IC} = 29 \) MPa√m
Paris Exponent \( m = 3.21 \)
Paris Constant \( C = 6.85 \times 10^{-8} \)

The plane stress condition is assumed in the simulations. A centre cracked rectangular specimen of size \( 100 \) mm × \( 200 \) mm is used for two simulation problems. The specimen subjected to a cyclic tensile load which is applied at the top edge while the bottom edge is kept constrained. The domain is discretized into four noded Lagrangian type quadrilateral elements. The higher order Gauss quadrature is used in the region near the discontinuity. The effect of various discontinuities on the fatigue life of component is analyzed. A comparison of the results obtained from linear and nonlinear analysis is discussed. The effect of yield stress on the fatigue life is also analyzed.
3.1. Plate with a Centre Crack and Holes

A specimen of size $100 \text{ mm} \times 200 \text{ mm}$ with a centre crack of initial length $a = 20 \text{ mm}$ containing randomly distributed thirty holes of radii vary from 3.5 mm to 4 mm is taken for simulation shown in Fig 1a. A cyclic tensile load of $\sigma_{\text{min}} = 0 \text{ N/mm}$ and $\sigma_{\text{max}} = 60 \text{ N/mm}$ is applied at the top edge of the specimen while the bottom edge is constrained in $y$-direction. A regular mesh of 60 by 120 nodes is used for the simulation.

![Fig. 1. (a) Centre cracked plate with multiple holes; (b) Stress contour plot ($\sigma_{yy}$) for centre crack plate in the presence of multiple holes](image)

An equal crack increment of 2 mm is given at each tip for each step and crack is propagated until its final failure i.e. when $K_{\text{eq}}$ exceeds $K_{IC}$. The fatigue life is evaluated using Paris law from SIF values computed at each stage of crack growth. The stress contour of $\sigma_{yy}$ is shown in Fig 1b. Fig 2 shows the variation in the plastic zone size due to the difference in yield stresses. The fatigue life obtained by XFEM is plotted with crack extension as shown in Fig 3a. The SIF variation against crack extension is shown in Fig 3b. In case of linear analysis, the fatigue life is found as 157260 cycles whereas in case of nonlinear analysis, the fatigue life is found as 149300 and 147230 cycles corresponding to the yield strength of 520 MPa and 200 MPa respectively. Thus, in case of nonlinear analysis, the fatigue life is reduced by 5.06% and 6.38% corresponding to the yield strength of 520 MPa and 200 MPa respectively.

![Fig. 2. Plastic zone with (a) $S_y = 520 \text{ MPa}$; (b) $S_y = 200 \text{ MPa}$ for the centre crack plate in the presence of multiple holes](image)
3.2. Plate with a Centre Crack and Inclusions

A centre cracked specimen along with randomly distributed inclusions is modeled for fatigue crack propagation. A specimen of size $100 \text{ mm} \times 200 \text{ mm}$ with a centre crack of initial length $a = 20 \text{ mm}$ containing thirty inclusions of radii vary from 3.5 mm to 4 mm is taken for simulation shown in Fig 4a. All the modeling conditions i.e. loading, boundary conditions, mesh size and crack increment are kept similar to problem 3.1. The fatigue life is evaluated using Paris law from SIF values computed at each stage of crack growth. The stress distribution of $\sigma_{yy}$ is shown in the Fig 4b. Fig 5 shows the variation in the plastic zone size due to the difference in yield stresses. The fatigue life obtained from XFEM solution is plotted with crack extension as shown in Fig 6a. Fig 6b shows the SIF variation against crack extension. The fatigue life is found as 147580 cycles in case of linear analysis whereas in case of nonlinear analysis, the fatigue life is found as 147210 cycles and 146280 cycles corresponding to the yield strength of 520 MPa and 200 MPa respectively.
Thus, in case of nonlinear analysis, the reduction of 0.25% and 0.88% is observed in fatigue life corresponding to the yield strength of 520 MPa and 200 MPa respectively.

4. Conclusion

The fatigue life of a centre cracked rectangular plate is carried out in the presence of multiple discontinuities using XFEM. The plasticity modeling is performed using Von-Mises yield criterion and isotropic hardening. The generalized Ramberg-Osgood material model is used to simulate the problems under plane stress condition. All the simulations are performed under constant amplitude cyclic loading. A domain based \( J \)-integral approach is used to compute SIF values from XFEM solution. The fatigue life is evaluated using generalized Paris law at each stage of crack growth. A negligible difference is observed in results obtained using linear and nonlinear analysis at higher yield stress as compared to lower yield stress. It is also observed that the presence of holes significantly affects the fatigue life of the plate as compared to inclusions. Moreover, it is seen that XFEM can be easily extended to fatigue crack growth modeling in the presence of discontinuities.
References

[1] Leitao, V. M. A., Aliabadi, M. H., Rooke, D. P., 1995. Elastoplastic simulation of fatigue crack growth: Dual boundary element formulation, International Journal of Fatigue, 17, pp. 353-363.
[2] Yan, X., 2006. A boundary element modeling of fatigue crack growth in a plane elastic plate, Mechanics Research Communications, 33, pp. 470-481.
[3] Leonel, E. D., Chateauneuf, A., Venturini, W. S., 2012. Probabilistic crack growth analysis using a boundary elemental model: Applications in linear elastic fracture and fatigue problems, Engineering Analysis with Boundary Elements, 36, pp. 944-959.
[4] Moes, N., Dolbow, J., Belytschko, T., 1999. A finite element method for crack growth without remeshing, International Journal for Numerical Methods in Engineering, 46, pp. 131-150.
[5] Belytschko, T., Gu, L., Lu, Y. Y., 1994. Fracture and crack growth by element-free Galerkin methods, Modelling and Simulation in Material Science and Engineering, 2, pp. 519-534.
[6] Belytschko, T., Lu, Y. Y., Gu, L., 1995. Crack propagation by element-free Galerkin methods, Engineering Fracture Mechanics, 51, pp. 295-315.
[7] Duflot, M., Dang, H. N., 2004. Fatigue crack growth analysis by an enriched meshless method, Journal of Computational and Applied Mathematics, 168, pp. 155-164.
[8] Melenk, J. M., Babuska, I., 1996. The partition of unity finite element method: basic theory and applications, Computer Methods in Applied Mechanics and Engineering, 139, pp. 289-314.
[9] Belytschko, T., Black, T., 1999. Elastic crack growth in finite elements with minimal remeshing, International Journal for Numerical Methods in Engineering, 45, pp. 601-620.
[10] Stolarska, M., Chopp, D., Moes, N., Belytschko, T., 2001. Modeling crack growth by level sets in the extended finite element method, International Journal for Numerical Methods in Engineering, 51, pp. 943-960.
[11] Sukumar, N., Chopp, D. L., Moes, N., Belytschko, T., 2001. Modeling of holes and inclusions by level sets in the extended finite element method, Computer Methods in Applied Mechanics and Engineering, 190, pp. 6183-6200.
[12] Singh, I. V., Mishra, B. K., Bhattacharya, S., Patil, R. U., 2012. The numerical simulation of fatigue crack growth using extended finite element method, International Journal of Fatigue, 36, pp. 109-119.
[13] Hinton, E., Owen, D.R.J., 1980. Finite Element in Plasticity, Pineridge Press Ltd, Swansea, U. K.
[14] Bland, D. R., 1957. The associated flow rule of plasticity, Journal of Mechanics and Physics of Solids, 6, pp. 71-78.
[15] Mohammadi, S., 2008. Extended finite element method for fracture analysis of structures, Blackwell Publishing Ltd.
[16] Daux, C., Moes, N., Dolbow, J., Sukumar, N., Belytschko, T., 2000. Arbitrary branched and intersecting cracks with the extended finite element method, International Journal for Numerical Methods in Engineering, 48, pp. 1741-1760.
[17] Rao, B. N., Rahman, S., 2003. An interaction integral method for analysis of cracks in orthotropic functionally graded materials, Computational Mechanics, 32, pp. 40-51
[18] Gdoutos, E. E., 2005. ‘Fracture Mechanics: An Introduction’, Second Edition, Springer.
[19] Long, K. S., 1996. Experimental study of inelastic stress concentration around a circular notch, M.Sc. Thesis, Naval Postgraduate School Monterey, California.
[20] Xiaoping, H., Moan, T., Weicheng, C., 2008. An engineering model of fatigue crack growth under variable amplitude loading, International Journal of Fatigue, 30, pp. 2-10.
[21] Aluminum 7075 T-6-ASM Material data sheet. (asm.matweb.com)