Nonsmooth stabilization and its computational aspects

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Abstract: This work has the goal of briefly surveying some key stabilization techniques for general nonlinear systems, for which, as it is well known, a smooth control Lyapunov function may fail to exist. A general overview of the situation with smooth and nonsmooth stabilization is provided, followed by a concise summary of basic tools and techniques, including general stabilization, sliding-mode control and nonsmooth backstepping. Their presentation is accompanied with examples. The survey is concluded with some remarks on computational aspects related to determination of sampling times and control actions.

Keywords: Stabilization, nonsmooth analysis, control Lyapunov function

1. INTRODUCTION

Nonsmooth tools play central role in nonlinear system stabilization theory as the smooth ones are prone to limitations posed by the celebrated work of Brockett (1983). He demonstrated that even such simple systems, as a three-wheel robot, do not admit a smooth control Lyapunov function (CLF). A particular consequence of this fact is that there can be no continuous control law, which depends only on the system’s state and which parks the robot into the desired position. The alternatives are either to consider a time-varying (or dynamical) control law or to give up the continuity condition. The former approach received great attention in the 80s and 90s (Aeyels, 1985; Kawski, 1989; Coron and d’Andrea Novel, 1991; Samson, 1991; Pomet, 1992; Coron, 1992; Coron and Pomet, 1993; Coron and Rosier, 1994; Coron, 1995; Khaneja and Brockett, 1999; Morin et al., 1999). Although the design of time-varying control laws may happen to be somewhat involved, the great advantage of this approach is that it requires the usual analysis tools and the closed-loop system trajectory is a classical Carathéodory solution which enjoys uniqueness properties. Contrary to this approach, if the continuity condition of the control law is omitted, care must be taken when defining what a system trajectory actually is. That is where alternative solutions to the respective initial value problem with discontinuous right-hand side come into play. One of the most well-known is the Filippov solution (Filippov, 1988). In brief, it is an absolutely continuous function that satisfies the said initial value problem with the right-hand side interpreted in the sense of a differential inclusion of the kind \( \dot{x} \in F(x,t) \), where \( F \) is a set-valued map. Sliding-mode control (SMC) makes wide use of Filippov solutions (Slotine and Li, 1991; Young et al., 1996; Fridman and Levant, 1996; Fridland and Levant, 1999; Perruquetti and Barbot, 2002). A particular drawback of this kind of a solution in the context of stabilization is that the very same Brockett’s conditions apply just as if one were to limit to the classical solutions (Coron and Rosier, 1994; Ryan, 1994). Further solutions include the ones in the sense of Hermes, Krasovskii, Sentis etc. whose good overview was done by Cortes (2008).

Of particular interest for this survey is the setting of sample-and-hold (S&H) solutions which are very simple to interpret. For the original nonlinear system

\[
\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad \text{(Sys)}
\]

if one designs a discontinuous static control law \( \kappa : \mathbb{R}^n \to \mathbb{R}^m \), instead of applying \( \kappa \) literally, one can consider its digital implementation with a sampling time \( \delta > 0 \) in the form

\[
\dot{x} = f(x,u_k), \quad t \in [k\delta, (k+1)\delta], \quad u_k \equiv \kappa(x(k\delta)), \quad k \in \mathbb{N}. \quad \text{(Sys-SH)}
\]

In this case, if \( f \) is locally Lipschitz w.r.t. \( x \), classical trajectories of (Sys-SH) exist (at least locally) and are unique, since \( f \) is also measurable in \( t \), precisely due to the sample nature of the control. The only property one has to give up in general when stabilizing (Sys) in the S&H mode, is asymptotic stabilizability of the closed loop. Instead, one has practical stabilizability in the sense of Definition 1. (Practical stabilizability). A control law \( \kappa \) is said to practically stabilize (Sys) in the sense of S&H, i.e., (Sys-SH) if, for any data \( R > r > 0 \), there is a sufficiently small \( \delta > 0 \) such that any closed-loop trajectory \( x(t), t \geq 0, x(0) \in B_R \) is bounded and enters and stays in the ball \( B_r \) within a time \( T \) depending uniformly on \( R, r \).

So, by properly selecting the sampling time, one can achieve any desirable precision of stabilization. Thus, S&H scenario of stabilization can be justified from both the implementation and usefulness sides.

A brief summary of the discussed stabilization methods can be found in Table 1. The focus of this survey is set to discontinuous static control laws in their S&H realization due to the aforementioned usefulness and practicability. In general, as said earlier, a nonlinear system does not admit a smooth CLF, so one has to widen the perspective to include nonsmooth tools, such as nonsmooth CLFs, gener-
alized derivatives, subgradients etc., whose brief overview is given in the next section. Particular stabilization techniques are surveyed in Section 3. In Section 4, we overview a very powerful technique of nonsmooth backstepping with application to dynamically actuated three-wheel robot and dynamical Artstein’s circles. A brief survey of SMC in the S&H setting is given in Section 5. The work is concluded with a discussion on computational aspects of stabilization techniques and a related case study in Section 6.

In the following, $\|x\|$ describes the Euclidean norm of $x$ and $\overline{co}(X)$ is defined as the closure of the convex hull of a set $X$. Furthermore, $B_R(x)$ denotes a ball with radius $R$ at $x$, i.e., $B_R(x) := \{x \in \mathbb{R}^n : \|x\| \leq R\}$ and $B_0$ means the same with $x = 0$. Finally, $\Rightarrow$ denotes a set-valued mapping.

## 2. BASIC NONSMOOTH TOOLS

Perhaps, the most central difference between the machinery of continuous and discontinuous stabilization lies at the level of a CLF. As mentioned earlier, a general nonlinear system does not admit a smooth CLF, so one is forced to consider nonsmooth alternatives. For this sake, let us first consider

**Definition 2.** (Lower directional generalized derivative). For a locally Lipschitz function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, the function $D_V(x) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as

$$D_V(x) := \lim_{\mu \rightarrow 0^+} \inf_{\mu \rightarrow 0^+} \frac{V(x + \mu \theta) - V(x)}{\mu}.$$  

This is called the lower directional generalized derivative (further, just “LDGD”).

**Example 1.** Consider a function

$$V(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x < 0, \\ 0, & x = 0, \\ 2x^2 \sin \frac{1}{x}, & x > 0, \end{cases}$$

LDGD at zero along one is $-2$.

Fig. 1. Graph of the derivative of $V$

Using the introduced LDGD, we can now consider one type of a nonsmooth CLF:

**Definition 3.** (CLF in LDGD sense). For the system (Sys), a locally Lipschitz, proper and positive-definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **CLF in LDGD sense** if there exists a continuous function $w : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \neq 0 \implies w(x) > 0$ satisfying a decay condition: for any compact set $X \subseteq \mathbb{R}^n$, there exists a compact set $U \subseteq \mathbb{R}^n$ such that

$$\forall x \in X \inf_{\sigma \in \text{cl}(\{x, U\})} D_\sigma V(x) \leq -w(x). \quad \text{(Dec)}$$

The function $w$ is also called decay function.

The condition (Dec) effectively means that $V$ is an upper minimax solution of the Hamilton-Jacobi equation

$$\inf_{u \in U_{x_i}} \langle \nabla V, f(x, u) \rangle + w(x) = 0 \quad \text{(HJ)}$$

on the respective domain (Camilli et al., 2008; Subbotin, 2013). There exist several techniques of practical stabilization of (Sys) using an LDGD CLF in the sense of Definition 3, some of which are reviewed in Section 3. Here, it has to be clarified what “$\nabla V$” means in the nonsmooth setting. There are several substitutes for gradients with different contexts, some of which are overviewed here. First of all, $\nabla V$ for a smooth function $V$ is a unique vector at each point. If $V$ is nonsmooth, there is no unique vector which describes a descent direction of $V$, so one speaks of a set of those, summarizing them in a subdifferential. Here is the first such subdifferential, which is useful in practical stabilization (Clarke et al., 1997):

**Definition 4.** (Proximal subdifferential). A vector $\zeta \in \mathbb{R}^n$ is called **proximal subgradient of $V$ at $x \in \mathbb{R}^n$** if there exists a ball $B_\epsilon(x)$ and $\sigma > 0$ s.t.

$$\forall y \in B_{\epsilon}(x) \quad V(y) \geq V(x) + \langle \zeta, y-x \rangle - \sigma \|y-x\|^2. \quad \text{(Prox)}$$

The set of all such vectors is called proximal subdifferential and is denoted by $\partial_P V(x)$.

It follows straight from the definition of an LDGD, that for any vector $\mu$ and any proximal subgradient $\zeta$, it holds that

$$\langle \zeta, \mu \rangle \leq D_\mu V(x). \quad \text{(1)}$$

Therefore, a decay condition in the spirit of (Dec) can be formulated as

$$\forall \zeta \in \partial_P V(x) \quad \inf_{u \in U_{x_i}} \langle \zeta, f(x, u) \rangle \leq -w(x). \quad \text{(PDec)}$$

The condition (PDec) means that $V$ is a proximal supersolution of (HJ) (Clarke et al., 1995), or, equivalently, a viscosity supersolution thereof (Crandall and Lions, 1983).

Other subdifferentials exist, for instance:

**Definition 5.** (Limiting subdifferential). Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. The set

$$\partial_L V(x) := \{\zeta \in \partial_P V(x_i) : \zeta = \text{w-lim} \zeta_i, \ z_i \rightarrow V(x_i) \rightarrow V(x)\} \quad \text{(2)}$$

is called limiting subdifferential of $V$ at $x$, where w-lim is the weak limit.

Here, a sequence $\{\zeta_i\}_{i=1,2,...} \subseteq \mathbb{R}^n$ is said to converge weakly to $\zeta \in \mathbb{R}^n$, if $\langle \zeta_i, \theta \rangle \rightarrow \langle \zeta, \theta \rangle$ for all $\theta \in \mathbb{R}^n$. Such limiting constructions are used, e.g., in the nonsmooth practical stability analysis of SMC (Clarke and Vinter, 2009). An overview is presented in Section 5.

**Definition 6.** (Clarke subdifferential). Let $\partial_C V(x)$ be the limiting subdifferential of $V$ at $x$. Then, the Clarke subdifferential $\partial_C V(x)$ is defined as the closed convex hull of (2), i.e.

$$\partial_C V(x) = \text{co} (\partial_L V(x)). \quad \text{(3)}$$
Example 2. Consider \( V(x) = \begin{cases} x^2, & \text{if } x < 1 \\ (x-1)^2 + 1, & \text{else} \end{cases} \).

The subdifferentials defined in Definitions 4, 5 and 6 are given as

\[
\partial_p V(1) = \emptyset, \quad \partial_L V(1) = \left\{ \left( \frac{1}{0}, \frac{1}{2} \right) \right\}, \\
\partial_C V(1) = \overline{co} \left( \left\{ \left( \frac{1}{0}, \frac{1}{2} \right) \right\} \right). 
\]

\[\text{Fig. 2. Graph of } V \text{ and the three subdifferentials}\]

An important property of many practical stabilizing techniques is *semiconcavity*. In fact, semiconcavity is a ubiquitous property of CLFs (Clarke, 2011; Cannarsa and Sinestrari, 2004).

**Definition 7.** (Locally semiconcave function). A function \( V : \mathbb{R} \to \mathbb{R} \) with \( \mathbb{X} \subseteq \mathbb{R}^n \) being convex, is called *locally semiconcave with linear modulus*, if it is continuous in \( \mathbb{X} \) and there exists \( C \geq 0 \) such that the following inequality holds for all \( x, y \in \mathbb{X} \):

\[ V(x) + V(y) - 2V \left( \frac{x+y}{2} \right) \leq C \left\| x - y \right\|^2. \] *(SemiConc)*

The following theorem states that any locally semiconcave function can be represented as the infimum of a family of \( C^2 \) functions. The proof can be found in (Cannarsa and Sinestrari, 2004).

**Theorem 1.** Let \( V : \mathbb{X} \to \mathbb{R}, \mathbb{X} \subseteq \mathbb{R}^n \) be a locally semiconcave function with linear modulus according to Definition 7. Then, for each compact subset \( \mathbb{K} \subset \mathbb{X} \), there exists a compact set \( \Theta \subset \mathbb{R}^{2n} \) and a continuous function \( F : \mathbb{X} \times \Theta \to \mathbb{R} \), s.t. \( F(\bullet, \theta) \in C^2 \) for any \( \theta \in \Theta \), the gradients \( \nabla_x F(\bullet, \theta) \) are equicontinuous, and \( V \) can be expressed as

\[ V(x) = \min_{\theta \in \Theta} F(x; \theta), \] *(4)*

for all \( x \in \mathbb{X} \). If there exists such a representation of \( V \), then \( V \) is called *marginal function* (Cannarsa and Sinestrari, 2004).

Theorem 1 will be useful in presenting a particular technique of nonsmooth backstepping in Section 4. Marginal functions are also used to define

**Definition 8.** *(F-disassembled subdifferential).* Let \( V : \mathbb{X} \to \mathbb{R}, \mathbb{X} \subseteq \mathbb{R}^n \) be a locally semiconcave function and let \( F : \mathbb{X} \times \Theta \to \mathbb{R} \), where \( \mathbb{K} \subset \mathbb{R}^n \) and \( \Theta \subset \mathbb{R}^{2n} \) are compact sets. The set-valued map \( \partial_D^F V : \mathbb{X} \rightrightarrows \mathbb{R}^n \) defined as

\[ \partial_D^F V(x) := \left\{ \frac{\partial F(x; \theta)}{\partial x} : \theta \in \arg \min_{\theta \in \Theta} F(x; \theta) \right\} \] *(5)*

is called *F-disassembled subdifferential*. A single element of \( \partial_D^F V(x) \) is called *F-disassembled subgradient*.

Such a subdifferential was used in (Cannarsa and Sinestrari, 2004) and was named in (Nakamura et al., 2013). Using disassembled subdifferentials, the following type of a nonsmooth CLF can be introduced (Nakamura et al., 2013)

**Definition 9.** *(F-disassembled CLF).* A proper, positive-definite, locally semiconcave function \( V : \mathbb{R}^n \to \mathbb{R} \) is called *F-disassembled CLF* for \((\text{Sys}), \) if there exists a continuous, positive definite function \( w : \mathbb{R}^n \to \mathbb{R} \) satisfying the following decay condition: for any compact subset \( \mathbb{X} \subseteq \mathbb{R}^n \), there exists a compact set \( U_X \subseteq U \), such that

\[ \forall x \in \mathbb{X}, \exists \zeta \in \partial_D^F V(x) : \min_{u \in U_X} \langle \zeta, f(x, u) \rangle \leq -w(x). \]

*(DisDec)*

An immediate relation between \( F\)-disassembled and proximal subdifferentials can be stated in the following lemma.

**Lemma 2.** Let \( X \subseteq \mathbb{R}^n \) be open and let \( \Theta \subset \mathbb{R}^{2n} \) be compact. Let \( V \) be a locally semiconcave function given as \((4)\) for \( F \) continuous in \( X \times \Theta \). If \( \zeta \) is a proximal subgradient of \( V \) at \( x \), then it is also an \( F\)-disassembled subgradient of \( V \) at \( x \), i.e.,

\[ \partial_P V(x) \subseteq \partial_L V(x) \subseteq \partial_D^F V(x). \]

*(6)*

**Proof.** The first inclusion follows from Definition 5. The second one is a part of the proof of Theorem 3.4.4 in (Cannarsa and Sinestrari, 2004).

The following lemma shows the relation between the two different kinds of CLFs.

**Lemma 3.** Consider \((\text{Sys})\) and a related \( F\)-disassembled CLF \( V : \mathbb{R}^n \to \mathbb{R} \) with an arbitrary \( F \). Then, \( V \) is also a CLF in LDGD sense.

**Proof.** Choose a function \( F : \mathbb{X} \times \Theta \to \mathbb{R}, \mathbb{F} \in C^2(\mathbb{X} \times \Theta) \) satisfying \((4)\) and let \( \partial_D^F V(x) \) be the corresponding \( F\)-disassembled differential. Define for all \( x \in \mathbb{X} \)

\[ \Phi(x, \vartheta, \varepsilon; \theta) := F(x + \varepsilon \vartheta; \theta) - F(x; \theta) \]

Obviously,

\[ \lim_{\varepsilon \to 0} \Phi(x, \vartheta, \varepsilon; \theta) := \left\{ \frac{\partial F(x; \theta)}{\partial x} : \theta \in \arg \min_{\theta \in \Theta} F(x; \theta) \right\}, \] *(7)*

holds for all \( x \in \mathbb{X} \), which follows from the definition of \( \lim_{\varepsilon \to 0} \). Since \( \Phi \) is differentiable w.r.t. \( x \) and \( \theta \), the equality in \((7)\) holds for all \( \theta \in \Theta \). In particular, for \( \theta^* \in \arg \min_{\theta \in \Theta} F(x; \theta) \) it holds that

\[ \lim_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta^*) = \left\{ \frac{\partial F(x; \theta^*)}{\partial x} : \theta = \theta^* \right\} = \left( \zeta, \theta^* \right), \] *(8)*

for some \( \zeta \in \partial_D^F V(x) \) corresponding to \( \theta^* \) at \( x \). In turn, one obtains using \((1)\)

\[ \lim_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta^*) = \left( \zeta, \theta^* \right) \leq D_\theta V(x). \] *(9)*

Since \( \lim_{\varepsilon \to 0^+} \Phi(x, \vartheta, \varepsilon; \theta^*) \) is independent of the choice of \( F \), \((\text{DisDec}) \implies (\text{Dec})\) holds with \((9)\). And since \( F \) was chosen arbitrary, \((9)\) holds for all \( F \) that satisfy \((4)\).
Before proceeding to concrete stabilizing techniques, the inf-convolution (InfC) should be recalled (Clarke et al., 2008):

**Definition 10. (Inf-convolution).** Let \( V : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \subseteq \mathbb{R}^n \). For \( \alpha \in (0, 1) \), the inf-convolution of \( V \) at \( x \) is defined by

\[
V_{\alpha}(x) = \inf_{y \in \mathbb{R}^n} \left\{ V(y) + \frac{1}{2\alpha^2} \| y - x \|^2 \right\}.
\]

(InfC)

In classical convex analysis, \( V_{\alpha} \) is known as Yoreau-Mosida regularization of a (convex) function \( V \). Furthermore, if \( V \) is a lower semicontinuous function and bounded from below, then \( V_{\alpha} \) is locally Lipschitz and an approximation of \( V \) in the sense of \( \lim_{\alpha \to 0} V_{\alpha}(x) = V(x) \) (Clarke et al., 1997).

### 3. STABILIZATION TECHNIQUES

In the following section, some stabilization techniques are presented. Some of them are also described in (Braun et al., 2017).

#### 3.1 Steepest descent

A steepest descent control law \( \kappa : \mathbb{R}^n \to \mathbb{U} \subseteq \mathbb{R}^m \) at \( x \neq 0 \) is computed via

\[
\kappa(x) \in \arg \min_{u \in \mathbb{U}} \mathcal{D}_{f(x,u)}V(x).
\]

(10)

It is shown in (Braun et al., 2017) that \( \kappa(x) \), computed by (10) with a semiconcave LDGD CLF \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \), practically asymptotically stabilizes the origin of (Sys). The semiconcavity of \( V \) is crucial to guarantee practical stabilizability by steepest descent (Braun et al., 2017).

#### 3.2 Dini Aiming

A control law at the state \( x \in \mathbb{R}^n \) by Dini Aiming (Kellett and Teel, 2000) is computed in two steps based on a nondecreasing continuous function \( \sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \).

1. Identify a direction \( \theta^* \) by means of minimizing the LDGD CLF \( V \) over a neighborhood of \( x \) for a given \( r > 0 \), i.e.,

\[
\theta^* \in \arg \min_{s \in \mathcal{B}_r(x)} V(s).
\]

(11)

2. Compute an admissible control \( \kappa(x) \in \mathbb{U} \cap \mathcal{B}_{\sigma(\|x\| + r)} \), \( \mathbb{U} \cap \mathcal{B}_{\sigma(\|x\| + r)} = \{ u \in \mathbb{U} : \| u \| \leq \sigma(\|x\| + r) \} \) via

\[
\kappa(x) = \arg \min_{u \in \mathbb{U} \cap \mathcal{B}_{\sigma(\|x\| + r)}} \frac{\| x - \theta^* - f(x, u) \|}{\| x - \theta^* \|}.
\]

(12)

If the sampling time \( \delta \) is chosen small enough (in accordance with \( \sigma \) and \( r \)), the control computed in (12) practically asymptotically stabilizes the origin of (Sys) (Kellett et al., 2004; Braun et al., 2017). Such a function \( \sigma \) always exists, since \( V \) is an LDGD CLF. Furthermore, \( \|x\| \to 0 \Rightarrow \|u\| \to 0 \) has to hold. In contrast to the steepest descent, \( V \) does not necessarily need to be semiconcave.

#### 3.3 Optimization-based control

In optimization-based control, the given LDGD is directly minimized over the set of admissible constant inputs (Braun et al., 2017), i.e.,

\[
\min_{u \in \mathcal{U}} \int_0^\delta \mathcal{D}_{f(x,u)}V(\varphi(t, x, u)) \, dt
\]

or, equivalently,

\[
\min_{u \in \mathcal{U}} |V(\varphi(\delta, x, u)) - V(x)|.
\]

At every step, \( \varphi(\delta, x, u) \), as a solution of (Sys), is computed over the sampling time period \([0, \delta]\). The one step optimization-based control can be defined as

\[
\kappa_1(x) \in \arg \min_{u \in \mathcal{U}} V(\varphi(\delta, x, u)).
\]

(15)

This method combines the two steps (11) and (12) in one single optimization problem. In comparison to other techniques, \( \kappa_1(x) \) computed in (15), explicitly depends on \( \delta \).

#### 3.4 Inf-convolution-based stabilization

In this technique, a minimizer of (InfC), i.e.,

\[
y_\alpha(x) \in \arg \min_{y \in \mathbb{R}^n} \left\{ V(y) + \frac{1}{2\alpha^2} \| y - x \|^2 \right\}
\]

(16)

is computed to define a proximal subgradient \( \zeta_\alpha(x) := \frac{\partial y_\alpha(x)}{\partial x} \) (Clarke et al., 1997). A control law is obtained by

\[
\kappa(x) = \arg \inf_{u \in \mathcal{U}} \left\{ \zeta_\alpha(x) \cdot f(x, u), y_\alpha(x, u) \right\}
\]

(17)

where \( \mathcal{Y} \) is a compact set which contains \( y_\alpha(x) \). Numerical studies with the above described methods can be found in (Braun et al., 2017; Osinenko et al., 2018a). The next section discusses specifically the use of F-disassembled subdifferentials and CLFs for nonsmooth backstepping.

### 4. NONSMOOTH BACKSTEPPING

In this section, a variant of nonsmooth backstepping on the example of three-wheel robot with dynamical actuators and dynamical Artstein’s circles is presented based on F-disassembled subdifferentials and CLFs.

#### 4.1 Three-wheel robot with dynamical actuators

A three-wheel robot with dynamical control of the driving and steering torques can be described as follows:

\[
\begin{align*}
\dot{x}_1 &= \eta_1, \\
\dot{x}_2 &= \eta_2, \\
\dot{\eta}_2 &= u_2. 
\end{align*}
\]

(ENDI)

The system (ENDI), \( \frac{d}{dt} [x \eta] = f_{\text{ENDI}}(x, \eta, u) \), is also called extended nonholonomic dynamical integrator (ENDI) (Ab-basi et al., 2017; Sankaranarayanan and Mahindaraka, 2009; Pascoal and Aguiar, 2002). It is essentially the Brockett’s nonholonomic integrator (NI) with additional integrators before the control inputs. The former reads, accordingly, as

\[
\dot{x} = f_{\text{NI}}(x, u) = \begin{pmatrix} 1 & 0 \\ -x_2 & 1 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ x_1 \end{pmatrix} u_2.
\]

(NI)

The following functions are nonsmooth LDGD CLFs for (NI) (Braun et al., 2017; Clarke, 2011):

\[
V_1(x) = x_1^2 + x_2^2 + 2x_3^2 - 2|x_3| \sqrt{x_1^2 + x_2^2},
\]

\[
V_2(x) = x_1^2 + x_2^2 + 2x_3^2 + |x_3| (10 - 2(|x_1| + |x_2|)).
\]
For the ENDI, nonsmooth backstepping based on (Matsumoto et al., 2015) may be utilized. To this end, consider the following function (Kimura et al., 2015):

\[ F(x; \theta) := x_1^4 + x_2^4 + \left[ \frac{|x_3|^3}{x_1 \cos(\theta) + x_2 \sin(\theta) + \sqrt{|x_3|^3}} \right]^2. \]

(18)

So, \( V(x) := \min_{\theta \in \Theta} F(x; \theta) \) is an F-disassembled CLF for (NI). Choose a minimizing \( \theta^* \in \Theta^* := \arg \min_{\theta \in \Theta} F(x; \theta) \) and compute \( \zeta(x; \theta) = \nabla_x F(x; \theta) \). Then, \( \zeta(x; \theta^*) \in \partial_D^2 V(x) = \nabla_x F(x; \theta) |_{\theta = \theta^*} \), holds.

A corresponding stabilizing controller for (NI) can be chosen using the Sontag’s formula as (Nakamura et al., 2013)

\[ \kappa(x; \theta) = - \left( \begin{array}{c} \langle \zeta(x; \theta), g_1(x) \rangle \\ \langle \zeta(x; \theta), g_2(x) \rangle \end{array} \right). \]

(19)

Then, for all \( x \in \mathbb{X} \) and \( \theta \in \Theta \), a decay of the CLF is ensured due to

\[ \langle \zeta(x; \theta), g_1(x) \rangle u_1 + \langle \zeta(x; \theta), g_2(x) \rangle u_2 \]

(20)

where \( \Theta^* \subseteq \Theta^* \) is obtained by choosing \( \theta^* \in \Theta^* \) in (26). Thus, the origin of (ENDI) is practically stabilized by the described control law. As a recap of Section 4.1, first compute the set \( \Theta^* \) by minimizing \( F(x; \theta) \). Second, compute a control law for the NI-subsystem according to (19). Third, the F-disassembled CLF \( F(x; \theta) \) is augmented with \( z = -\kappa(x; \theta) \) to a new function \( F(x; \theta) + \frac{1}{2} \| z \|^2 \). The resulting control law for the whole system is given either via (19), or, respectively (26), depending on \( z \).

4.2 Dynamical Artstein’s circles

The same method as described in the previous section can be applied to the following system, a dynamic extension of the Artstein’s Circle:

\[ \dot{x}_1 = -x_1^3 + x_2^3, \quad \dot{x}_2 = -2x_1x_2x_3, \quad \dot{x}_3 = u, \]

(AC)

which can be written as

\[ \dot{v} = g(v)w \]

(28)

with \( v := (x_1, x_2, x_3)^T \in \mathbb{R}^3 \) and \( w := x_3 \in \mathbb{R} \).

A nonsmooth LDGD CLF for \( \dot{v} = g(v)u \) is given as (Braun et al., 2018):

\[ V(v) = \sqrt{3x_1^2 + 4x_2^2 - |x_1|}. \]

(29)

Referring to (18), the following marginal function gives rise to an F-disassembled CLF:

\[ F(v; \theta) := \sqrt{3x_1 \cos(\theta) + 2x_2 \sin(\theta) - |x_1|}. \]

(30)

Obviously, for (29) and (30), (4) holds with \( \Theta = [0, 2\pi] \).

After computing the F-disassembled differential \( \partial_D^2 V \mid_{\theta = \theta^*} \), choose \( \zeta(v; \theta^*) \in \partial_D^2 V \mid_{\theta = \theta^*} \), where \( \theta^* \in \Theta^* := \arg \min_{\theta \in \Theta} F(v; \theta) \). Then, with \( \kappa(v; \theta^*) = -\langle \zeta(v; \theta), g(v) \rangle \) as a feedback, it follows

\[ -\langle \zeta(v; \theta), g(v) \rangle \mid_{\theta = \theta^*}. \]

(31)

which holds also for all \( \theta \in \Theta \) and all \( v \in \mathbb{R}^2 \setminus \{0\} \), and practically stabilizes \( \dot{v} = g(v)w \). Now, (30) can be extended via backstepping to

\[ V(v, w) = \min_{\theta \in \Theta^*} \left\{ F(v; \theta) + \frac{1}{2} \| w - \kappa(v; \theta) \|^2 \right\}. \]

(32)

Define \( z := w - \kappa(v; \theta) \). Since the disassembled subdifferential is given similarly to (23), one can show, that for \( \zeta(v, w; \theta^*) \in \partial \partial_D^2 V \mid_{\theta = \theta^*} \) with \( \theta^* \in \Theta^*, \Theta^* \) defined as in Section 4.1, the following equality holds:

\[ \langle \zeta(v, w; \theta^*), g(v) \rangle = \langle \zeta(v; \theta^*), g(v) \rangle + \langle \zeta(v; \theta^*), z \rangle. \]

(33)
The corresponding control law can be derived in a similar manner as in Section 4.1.

5. SLIDING MODE
Consider a system
\[
\begin{align*}
\dot{x} &= f(x, u), x \in \mathbb{R}^n, \\
u &= \kappa(x) + s(x)\sigma(x),
\end{align*}
\]
(34)
where \(\kappa, s\) are continuous and
\[
\sigma(x) = \begin{cases} 
1, & \chi(x) > 0, \\
-1, & \chi(x) < 0 
\end{cases}
\]
(35)
describes the discontinuous part of the controller around a sliding surface \(\Sigma = \{x : \chi(x) = 0\}\). To process (34), unlike in the usual case, two Lyapunov-like functions \(V_1, V_2\) are employed that satisfy, for some \(w_1 > 0\),
\[
V_1 = 0, V_2 > 0, \\
\dot{V}_1(x) \leq -w_1, x \in \mathbb{R}^n \setminus \Sigma; \\
\dot{V}_2(x) = 0, x \in \Sigma \setminus \{0\} \implies V_2(x) > 0, w_2(x) > 0,
\]
\[
\sup_{x \in \Omega_2} |\nabla V_2(x, v)| \leq -w_2(x), x \in \Sigma \setminus \{0\},
\]
where \(\Omega_2 := \{x, \, s(x) > 0, \, s(x) > 0, \, s(x) > 0, \, s(x) > 0\}\), respectively. The initial condition \(x(0) = x_0\) and \(\dot{\theta}(0) := f(x_0, \kappa(0) + s(x_0)\sigma(0))\) describes the limiting behavior of possible system velocities (cf. the construction in Definition 5). Let 
\[
f_1(x) := f(x, \kappa(x) + s(x)), \\
f_2(x) := f(x, \kappa(x)), \\
X_1 := \{x: \chi(x) > 0\}, \\
X_2 := \{x: \chi(x) < 0\},
\]
and
\[
\chi(x) = \begin{cases} 
\chi(x), & x \in X_1, \\
(1 - \alpha) f_1(x) + \alpha f_2(x), & x \in \Sigma, \\
f_2(x), & x \in X_2,
\end{cases}
\]
(36)
This is a particular example of Filippov regularization (Cortes, 2008) and, in this case, is upper semi-continuous, has closed and convex images and thus admits a Filippov solution (Zabczyk, 2009), which is an absolutely continuous function whose derivative satisfies the differential inclusion almost everywhere. A Filippov solution is an idealized construction describing a perfect sliding mode in this context. In practice, since the control is usually realized digitally, S&H analysis of SMC comes in handy (Clarke and Vinter, 2009). The general idea thereof is to chose a sampling time bound small enough that the system approach a specially chosen vicinity of \(\Sigma\) where \(V_2\) still retains some of it decay rate \(w_2\) (by a continuity argument). The attraction to this vicinity is in turn ensured by retaining some of the decay rate \(w_1\). Combining the two decay properties together ensures practical stabilization of (34) in the S&H mode (Clarke and Vinter, 2009).

6. COMPUTATIONAL ASPECTS
The described practical stabilization methods of the previous sections rely on computation of primarily two things: (1) allowed sampling time for the desired stabilization precision; (2) control actions. General techniques of Section 3 rely heavily on various optimizations. It was shown that optimization accuracy greatly influences stabilization precision (Osinenko et al., 2018a). Moreover, the involved LDGD CLF must satisfy certain regularity properties. In particular, the following: For all compact sets \(\mathbb{V}, \mathbb{F} \subset \mathbb{R}^n\) and for all \(\nu, \chi > 0\) there exist \(\mathbb{Y} \subseteq \mathbb{V}, \mu \geq 0\) such that:

1. For each \(\tilde{y} \in \tilde{Y}, \theta \in \mathbb{F}\) and \(\forall \mu \in (0, \mu]\) it holds that
\[
\frac{V(\tilde{y} + \mu \theta) - V(\tilde{y})}{\mu} - D_\theta V(\tilde{y}) \leq \nu; \\
\text{(hom)}
\]
2. For each \(y \in \mathbb{Y}\) there exists \(\tilde{y} \in \tilde{Y}\) such that
\[
\|y - \tilde{y}\| \leq \chi. \\
\text{(npt)}
\]
Under these conditions, one can find bounds on the optimization accuracy of the inf-convolution-based method so as to achieve the desired stabilization precision. Furthermore, it was shown in (Osinenko et al., 2018b) what regularity properties of the involved Lyapunov-like functions have to be satisfied to effectively compute required bounds on the sampling time. The respective machinery was addressed on the example of SMC. In contrast to (Clarke and Vinter, 2009), an actual numerical example of practical SMC stabilization for vehicle slip control was shown. The computed sampling time bounds were satisfactory for the considered application.

Here, we give a short case study that demonstrates the effect of the described computational uncertainty. Namely, for (ENDI) and its CLF (22), a control law was computed in the S&H framework using (16) and (17) with optimization accuracy \(\varepsilon\) and \(\eta\), respectively. The initial condition is set to \(x_0 = (-1, 0.5, 0.01, 0.05, 0.075)^T\) and the set of admissible controls is given as \(U = [-3, 3]\). Furthermore, we set \(\alpha\) in (InIC) to \(\alpha = 0.1\) and the sampling time \(\delta = 0.005\).

The influence of the optimization accuracy on the state and CLF behavior can be seen in Fig. 3 for different values of \(\varepsilon\) and \(\eta\), namely \(\varepsilon = \eta \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}, 10^{-8}\}\).

It can be observed that insufficient accuracy (\(\varepsilon = \eta = 10^{-3}\) and \(\varepsilon = \eta = 10^{-2}\)) leads instability. Higher accuracies lead to ever smaller vicinities of the origin that the state converges into. This clearly demonstrates that computational uncertainty must be taken into account in practical stabilization.

7. CONCLUSION
This work surveyed, in a brief form, some of the key modern nonsmooth stabilization tools and techniques. These include general practical stabilization, sliding-mode control, nonsmooth backstepping. Examples were provided. In addition, this work briefly discussed some computational aspects of practical stabilization one should be concerned about in applications.

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