I. INTRODUCTION

Tens of gravitational wave (GW) detections in the frequency range tens to hundreds of hertz were reported by the Laser Interferometer Gravitational-Wave Observatory (LIGO) Scientific Collaboration and the Virgo Collaboration [1–13]. It is expected that the proposed space-based observatories such as Laser Interferometer Space Antenna (LISA) [14, 15], Taiji [16] and TianQin [17] will detect GWs in the frequency regimes $10^{-4}$–$10^{-1}$ Hz which are undetectable by the ground-based GW observatories due to the seismic noise and gravity gradient noise.

There are two different configurations and designs for the space-based GW observatory. One configuration has the geocentric orbit with three spacecrafts orbiting the Earth and further rotating around the Sun together with the Earth. TianQin takes such design with the normal vector of the detector plane pointing to the source RX J0806.3+1527 [17]. LISA [14, 15] and Taiji [16] adopt the design with the three spacecrafts moving in the heliocentric orbit ahead or behind the Earth by about 20°. LISA and Taiji constellations have an inclination angle of 60° with respect to the ecliptic plane. The inclination angle ensures the three spacecrafts to keep the geometry of an equilateral triangle throughout the mission. The normal vector of the detector plane rotates around the normal vector of the ecliptic plane and forms a cone with a half opening angle of 60° in one year.

Each measured GW signal shaped by a set of parameters carries information about the source: its location in the sky, the inclination angle between the binary’s orbital angular momentum and the line of sight, the polarization angle, the amplitude, the initial phase and the frequency which can be accurately determined by the orbital period. The physical parameters of the binary source extracted from the GW signal are strongly correlated. The ability to locate a source in the sky is important in GW observations and parameter estimations, therefore localizing the sky position of the GW source is a key scientific goal for GW observations. The accuracy of the source localization is essential for the follow-up observations of counterparts and the statistical identification of the host galaxy when no counterpart is present. With the information about counterparts or the host galaxy, the redshift $z$ of the source can be determined and unprecedented information about its environment may be uncovered. Combining the redshift $z$ with the luminosity distance $d_L$ to the source determined from the GW waveform, we can probe the thermal history of the Universe and measure fundamental cosmological parameters. In particular, the independent measurements of $d_L$ and $z$ from GWs can be used as standard sirens [18, 19] to understand the problem of Hubble tension [20].

Except for different frequency bands between space-based and ground-based detectors, another important difference is that space-based GW observatories can observe binary sources for months to years before the final coalescence. Thus for space-based observations, the periodic Doppler shift imposed on the signal by the translational motion of the detector around the Sun results in amplitude and phase modulations of the detector output which encode information about both the detector position and the angular position of the source. Hence, the periodic Doppler shift provides us a method of identifying the angular position of the source in the sky with one detector. The extraction of parameters from merging compact binaries for LIGO and Virgo network was first discussed in [21]. The accuracy of the angular resolution
of LISA for a monochromatic GW was first investigated in [22, 23] with the simplified detector response and the assumption that all other parameters are known a priori. Cutler then estimated the angular resolution of LISA and all other parameters including the frequency simultaneously for galactic and extragalactic sources by using a more realistic detector response which accounts for the rotation of the LISA constellation in the long-wavelength approximation [24, 25]. More accurate waveforms were also included in the context of coalescing massive black hole (MBH) binaries and captures of small compact objects by MBHs [26–29]. For equal mass black hole (BH) binary system with the total mass $10^5 M_\odot$ at the redshift $z = 1$, the LISA-Taiji network can improve the accuracy of the sky localization by two orders of magnitude than each individual detector for one year’s joint observation, and an optimal configuration angle of 40° was suggested for the LISA-Taiji network [30, 31]. By simulating coalescing GWs from MBH binary systems with masses $M_1 = 10^7 M_\odot$, $10^6 M_\odot$, and $10^5 M_\odot$ respectively for the primary BH and the mass ratio $q = 1/3$ at the redshift $z = 2$, and 30 days observation until the merger, it was found that the LISA-Taiji network could improve the angular resolution for various time-delay interferometry channels by more than 10 times than each individual LISA or Taiji detector [32]. The improvement from the LISA-Taiji network is relatively moderate for monochromatic GWs at 3 mHz and 10 mHz with one year observation [32]. The precisions of the parameter estimation and the sky localization of equal mass MBH binary systems with masses in the range $10^{-7} – 10^{-5} M_\odot$ for TianQin were discussed in [33]. The LISA-TianQin network may improve the sky localization of Galactic double white dwarf binaries up to 3 orders of magnitude [34] comparing with single TianQin detector. The effects on the sky localization due to the different configuration between LISA/Taiji and TianQin and their sky maps of the sky localization for monochromatic GWs were analyzed in [35]. For more discussion on the accuracy of sky localization, please see Refs. [36–46].

In this paper, we consider GW signals from two compact objects such as white dwarfs, neutron stars or black holes. During the operation time of the space-borne GW observatory, there is nearly no frequency evolution in the early inspiral of these systems. The GWs emitted by these sources can be treated as monochromatic. The detection of these sources is anticipated to provide us with new insights into the formation and the evolution of relativistic objects and the physics in the early universe. The aim of the present paper is to investigate the accuracy of parameter estimations for TianQin and LISA. The analysis can be easily applied to Taiji because of its similarity with LISA. Estimation errors of sky positions of sources and the other parameters are mainly dependent on the amplitude modulation due to the changing orientation of the detector plane and the Doppler effect due to the translational motion of the center of the detector around the Sun. The amplitude of the detector response is modulated by the annual rotation of the LISA array which improves the measurement of the sky position. For TianQin, there exists no such modulation and its sky localization is expected to be different from LISA. Apart from the amplitude modulation, the phase of the detector response modulated by the Doppler effect for TianQin and LISA also improves the accuracy of parameter estimation. Therefore, the correlation between the amplitude modulation and Doppler modulation makes the analysis of sky localization quite different between TianQin and LISA.

We present detailed analysis of the estimation of the parameters with TianQin and LISA for monochromatic GWs. To understand the result, the exact detector signal and its long-wavelength approximation are considered. In the long-wavelength regime we give compact formulas for the errors as functions of the frequency. With these analytical formulas, it is easy to understand the frequency dependence of the parameter estimation errors in the low and medium frequency regimes. In the high frequency domain we numerically calculate errors as functions of the frequency for the single Michelson observable. The organization of the paper is as follows. In Sec. II, we review the Fisher Information Matrix (FIM) method of signal analysis. In Sec. III, we apply the FIM method to TianQin and LISA. We derive the analytical formulas of the parameter estimation errors in the long-wavelength domain and we put the detailed analyses in the Appendices A, B and C; in high frequency regime, we give the numerical results of the parameter estimation errors. For comparison, we consider the parameter estimation errors for four parameters in Appendix D and the Bayesian analysis for LISA in Appendix E. We present our conclusion and discussion in the last section.

II. THE FISHER INFORMATION MATRIX METHOD

It is convenient to describe GWs and the motion of detectors in the heliocentric right-handed orthogonal reference frame with the constant basis vectors $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ [47]. For GWs propagating in the direction $\hat{w}$, we introduce a set of unit vectors $\{\hat{w}, \hat{\theta}, \hat{\phi}\}$ which are perpendicular to each other as

$$\hat{\theta} = \cos(\theta) \cos(\phi) \hat{e}_x + \cos(\theta) \sin(\phi) \hat{e}_y - \sin(\theta) \hat{e}_z,$$
$$\hat{\phi} = -\sin(\phi) \hat{e}_x + \cos(\phi) \hat{e}_y,$$
$$\hat{w} = -\sin(\theta) \cos(\phi) \hat{e}_x - \sin(\theta) \sin(\phi) \hat{e}_y - \cos(\theta) \hat{e}_z,$$

where the angles $\theta$ and $\phi$ are the angular coordinates of the source. To describe GW signals, the polarization angle $\psi$ is introduced to form polarization tensors

$$e_{ij}^p = \hat{p}_i \hat{p}_j - \hat{q}_i \hat{q}_j, \quad e_{ij}^q = \hat{p}_i \hat{q}_j + \hat{q}_i \hat{p}_j,$$
$$\hat{p} = \cos \psi \hat{\theta} + \sin \psi \hat{\phi}, \quad \hat{q} = -\sin \psi \hat{\theta} + \cos \psi \hat{\phi},$$

where $\hat{p}$ and $\hat{q}$ represent the directions of the two polarization axes of the gravitational radiation. For this
particular choice of \( \{\hat{w}, \hat{p}, \hat{q}\} \), GWs have the form
\[
h_{ij}(t) = \sum_A e^{iA} h_A(t),
\]  
where \( A = +, \times \) stands for the plus and cross polarization states,
\[
\begin{align*}
  h_+ &= \mathcal{A} \left[ 1 + \cos^2(\iota) \right] \exp(2\pi if t + i\phi_0), \\
  h_\times &= 2i\mathcal{A} \cos(\iota) \exp(2\pi if t + i\phi_0),
\end{align*}
\]  
\( \mathcal{A} \) and \( f \) are the amplitude and the frequency of GWs respectively, \( \iota \) is the inclination angle and \( \phi_0 \) is the initial phase.

For a monochromatic GW with the frequency \( f \) propagating along the direction \( \hat{w} \), the output of an equal arm space-based interferometric detector with a single round trip of light travel is
\[
H(t) = \sum_A F^A h_A(t) e^{i\phi(t)},
\]  
where \( F^A = \sum_{i,j} D^{ij} e^{iA} \), the Doppler phase \( \phi_D(t) \) is
\[
\phi_D(t) = \frac{2\pi f R}{c} \sin \theta \cos \left( \frac{2\pi f t}{T} - \phi - \phi_a \right),
\]  
\( \phi_a \) is the ecliptic longitude of the detector \( \alpha \) at \( t = 0 \), \( c \) is the speed of light, the rotational period \( T \) is 1 year and the radius \( R \) of the orbit is 1 AU. The detector tensor \( D^{ij} \) is
\[
D^{ij} = \frac{1}{2} \left[ \hat{w}^i \hat{w}^j T(f, \hat{u} \cdot \hat{w}) - \hat{v}^i \hat{v}^j T(f, \hat{v} \cdot \hat{w}) \right],
\]  
where \( \hat{u} \) and \( \hat{v} \) are the unit vectors along the arms of the detector and \( T(f, \hat{u} \cdot \hat{w}) \) are [48, 49]
\[
T(f, \hat{u} \cdot \hat{w}) = \frac{1}{2} \left\{ \sin \left( \frac{f(1 - \hat{u} \cdot \hat{w})}{2f^*} \right) \exp \left( \frac{f(3 + \hat{u} \cdot \hat{w})}{2if^*} \right) \right. \\
\left. + \sin \left( \frac{f(1 + \hat{u} \cdot \hat{w})}{2f^*} \right) \exp \left( \frac{f(1 + \hat{u} \cdot \hat{w})}{2if^*} \right) \right\},
\]  
\( \sin(x) = \sin x / x, f^* = c / (2\pi L) \) is the transfer frequency of the detector and \( L \) is the arm length of the detector.

The signal \( H(t) \) depends on the source parameters which are to be estimated. To estimate the accuracy of the parameters by a single detector, we introduce the FIM in the frequency domain
\[
\Gamma_{ij} = \left( \frac{\partial H}{\partial \xi_i} \frac{\partial H}{\partial \xi_j} \right) \equiv 4 \Re \int_0^{T_{\text{obs}}} \frac{\partial_i H(f) \partial_j H^*(f)}{S_n(f)} df,
\]  
where \( \Re \) means the real part, \( \partial_i H = \partial H / \partial \xi_i \) and \( \xi_i \) is the \( i \)-th parameter. For monochromatic GW sources there is almost no frequency evolution, the FIM can be simplified in the time domain as
\[
\Gamma_{ij} = \frac{2}{S_n(f)} \mathcal{R} \int_0^{T_{\text{obs}}} \partial_i H(t) \partial_j H^*(t) dt,
\]  
where \( T_{\text{obs}} = 1 \) yr is the observational time. We focus mainly on the following parameters of the monochromatic GW signal considered in Eq. (10),
\[
\xi = \{ \theta, \phi, \ln A, \iota, t, \psi, \phi_0 \}.
\]  

For space-based interferometers, the noise power spectral density \( S_n(f) \) is [15]
\[
S_n(f) = \frac{S_a}{L^2} + \frac{4S_a}{(2\pi f)^4 L^2} \left( 1 + \frac{10^{-4} \text{Hz}}{f} \right).
\]  

For LISA, the acceleration noise is \( \sqrt{S_a} = 3 \times 10^{-15} \text{ m s}^{-2} / \text{Hz}^{1/2} \), the displacement noise is \( \sqrt{S_a} = 15 \text{ pm/Hz}^{1/2} \), the arm length is \( L_a = 2.5 \times 10^6 \text{ km} \) [15], and its transfer frequency is \( f_s^* = 0.02 \text{ Hz} \). For TianQin, the acceleration noise is \( \sqrt{S_a} = 10^{-15} \text{ m s}^{-2} / \text{Hz}^{1/2} \), the displacement noise is \( \sqrt{S_a} = 1 \text{ pm/Hz}^{1/2} \), the arm length is \( L_t = 1.7 \times 10^5 \text{ km} \) [17], and its transfer frequency is \( f_t^* = 0.28 \text{ Hz} \).

The FIM can be split into three parts. \( \Gamma_{ij}^m \) denotes the total amplitude modulation and \( \Gamma_{ij}^{dm} \) denotes the Doppler phase modulation and \( \Gamma_{ij}^{ad} \) denotes the interaction of the total amplitude and Doppler phase modulation
\[
\Gamma_{ij} = \Gamma_{ij}^m + \Gamma_{ij}^{dm} + \Gamma_{ij}^{ad},
\]  
where
\[
\begin{align*}
\Gamma_{ij}^m &= \left( \frac{\partial (\sum_A F^A h_A)}{\partial \xi_i} \right) \left( \frac{\partial (\sum_A F^A h_A)}{\partial \xi_j} \right), \\
\Gamma_{ij}^{dm} &= \left( \sum_A F^A h_A \frac{\partial (e^{i\phi_D(t)})}{\partial \xi_i} \right) \left( \sum_A F^A h_A \frac{\partial (e^{i\phi_D(t)})}{\partial \xi_j} \right), \\
\Gamma_{ij}^{ad} &= \left( \frac{\partial (\sum_A F^A h_A)}{\partial \xi_i} \right) \left( \frac{\partial (\sum_A F^A h_A)}{\partial \xi_j} \right) e^{i\phi_D(t)} + \left( \sum_A F^A h_A \frac{\partial (e^{i\phi_D(t)})}{\partial \xi_i} \right) \left( \frac{\partial (\sum_A F^A h_A)}{\partial \xi_j} \right) e^{i\phi_D(t)}.
\end{align*}
\]  
The covariance matrix of the parameter errors is
\[
\sigma_{ij} = \langle \Delta \xi_i \Delta \xi_j \rangle \approx (\Gamma^{-1})_{ij}.
\]  
The root mean square errors of the parameters are given by
\[
\sigma_i = \sqrt{\sigma_{ii}} = \sqrt{(\Gamma^{-1})_{ii}}.
\]  
The angular uncertainty of the sky localization is evaluated as [24]
\[
\Delta \Omega = 2\pi \sin \theta \sqrt{\sigma_{\theta \theta} \sigma_{\phi \phi} - \sigma_{\theta \phi}^2}.
\]
III. PARAMETER ESTIMATION ERRORS

In this section, we analyze the errors of the six parameters discussed in the previous section. If the parameters \( \psi \) and \( \phi_0 \) are known, then the six parameters are reduced to four parameters, we discuss the results of the four parameters in Appendix D.

A. The long-wavelength approximation

In this subsection we derive analytical formulas for the parameter estimation errors in the long-wavelength (LW) approximation for LISA and TianQin. We also use numerical method to confirm the analytical behaviours. Note that in the LW approximation, \( f \ll f^* \), and \( T(f, \hat{u} \cdot \hat{w}) \to 1 \) in Eq. (8).

TianQin is an equilateral triangle constellation with sides of \( 1.73 \times 10^9 \) km designed to orbit the Earth with the period of 3.65 days and further rotate around the Sun together with the Earth [17]. In the heliocentric coordinate system, the normal vector of the detector plane points to the direction of RX J0806.3+1527 with the latitude \( \beta = 94.7^\circ \) and the longitude \( \alpha = 120.5^\circ \). The orbits of the unit vectors of detector arms (two arms only) for TianQin are [50]

\[
\hat{u}_x = \cos(\omega_s t) \cos(\alpha) \sin(\beta) - \sin(\omega_s t) \cos(\beta), \\
\hat{u}_y = \cos(\omega_s t) \sin(\alpha) + \cos(\omega_s t) \cos(\beta) \sin(\alpha), \\
\hat{u}_z = - \cos(\omega_s t) \sin(\beta),
\]

\[
\hat{v}_x = \cos(\omega_s t + \frac{\pi}{3}) \cos(\alpha) \cos(\beta) - \sin(\omega_s t + \frac{\pi}{3}) \sin(\alpha), \\
\hat{v}_y = \cos(\omega_s t + \frac{\pi}{3}) \sin(\alpha) + \cos(\omega_s t + \frac{\pi}{3}) \cos(\beta) \sin(\alpha), \\
\hat{v}_z = - \cos(\omega_s t + \frac{\pi}{3}) \sin(\beta),
\]

where the rotation frequency \( \omega_s = 2\pi/(3.65 \text{ days}) \). The FIM in the LW approximation can be written as a sum in a compact form (the matrix elements are presented in Appendix A)

\[
\Phi^{\text{LW}} = \phi^{am} + \phi^{dm} + \phi^{ad},
\]

where

\[
\phi^{am} = \frac{A^2 T_{\text{obs}}}{S_n(f)} M^{am}, \\
\phi^{dm} = \frac{A^2 T_{\text{obs}}}{S_n(f)} M^{dm}, \\
\phi^{ad} = \frac{A^2 T_{\text{obs}}}{S_n(f)} M^{ad},
\]

\[
M^{am} \text{ is singular and independent of the frequency},
\]

\[
M^{dm} = \left( \frac{2\pi f R}{c} \right)^2 \begin{pmatrix} D_{2 \times 2} & 0_{2 \times 4} \\ 0_{4 \times 2} & 0_{4 \times 4} \end{pmatrix}, \quad M^{ad} = 0.
\]

Note that only the submatrix \( D \) is nonzero, the matrix \( M^{dm} \) for the Doppler phase modulation depends on the frequency as \( f^2 \) and the matrix \( M^{am} \) is independent of the frequency.

LISA mission was proposed as an equilateral triangle constellation with sides of \( 2.5 \times 10^6 \) km [14, 15]. The constellation has an inclination angle of 60° with respect to the ecliptic plane and trails the Earth by about 20°. In the heliocentric coordinate system, the orbits of the unit vectors of detector arms (two arms only) for LISA are [25]

\[
\hat{u}_x = - \sin(\omega_s t) \cos(\omega_s t) \cos(\omega_s t) \sin(\omega_s t)/2, \\
\hat{u}_y = \cos(\omega_s t) \cos(\omega_s t) + \sin(\omega_s t) \sin(\omega_s t)/2, \\
\hat{u}_z = \sin(\pi/3) \sin(\omega_s t), \\
\hat{v}_x = - \sin(\omega_s t) \cos(\omega_s t - \alpha) \cos(\omega_s t - \alpha) \sin(\omega_s t - \alpha)/2, \\
\hat{v}_y = \cos(\omega_s t) \cos(\omega_s t - \alpha) + \sin(\omega_s t) \sin(\omega_s t - \alpha)/2, \\
\hat{v}_z = \sin(\pi/3) \sin(\omega_s t - \alpha),
\]

where the rotation frequency \( \omega_s = 2\pi/(365 \text{ days}) \) and \( \alpha = \pi/3 \). The FIM in the LW approximation can also be written in the compact forms (17) and (18) with

\[
M^{dm} = \left( \frac{2\pi f R}{c} \right)^2 \begin{pmatrix} A_{2 \times 2} & 0_{2 \times 4} \\ 0_{4 \times 2} & 0_{4 \times 4} \end{pmatrix}, \quad M^{ad} = \left( \frac{2\pi f R}{c} \right)^2 \begin{pmatrix} B_{2 \times 2} & C_{2 \times 4} \\ C_{4 \times 2} & C_{4 \times 4} \end{pmatrix},
\]

where the matrix elements of \( M^{am} \), \( M^{dm} \) and \( M^{ad} \) are presented in Appendix B. The matrix \( M^{am} \) is non-singular and independent of the frequency. In the matrix \( M^{dm} \), only the submatrix \( A \) is nonzero.

From Eqs. (10) and (18), we see that the FIM is proportional to \( A^2/S_n(f) \), so \( \Delta \Omega \propto S_n(f)/A^2 \) and the parameter estimation errors are \( \sigma_i \propto \sqrt{S_n(f)/A} \). For convenience, we write the parameter estimation errors as

\[
\Delta \Omega = \frac{S_n(f)}{A^2} \Delta \Omega, \quad \sigma_i = \frac{S_n(f)}{A} \sigma_i,
\]

(21)

to factor out the effect of the noise curve \( S_n(f) \) because it is just a number for a monochromatic source.

Using the FIM derived above, we get the normalized angular resolutions \( \Delta \Omega \) of LISA and TianQin as shown in Fig. 1. In Fig. 1, we compare the normalized angular resolutions \( \Delta \Omega \) of LISA and TianQin with and without the LW approximation. From Fig. 1, we see that the approximate result of \( \Delta \Omega \) derived with the LW approximation is almost the same as the exact result derived without the LW approximation when the GW frequency is less than the transfer frequency, i.e., \( f \leq 0.02 \text{ Hz} \) for LISA and \( f \leq 0.28 \text{ Hz} \) for TianQin, respectively. At higher frequencies, \( f > f^* \), the transfer function \( T(f, \hat{u} \cdot \hat{w}) \) in Eq. (8) is no longer constant and becomes frequency dependent. In other words, the arm length of the detector is shorter than the GW’s wavelength and the LW approximation breaks down when the frequency \( f > f^* \). In summary, the LW approximation works quite well when the frequencies are below \( 10^{-2} \text{ Hz} \) for LISA and TianQin,
so we can use the LW approximation to estimate the errors of parameters in the low ($\sim 10^{-4}$ Hz) and medium ($\sim 10^{-2}$ Hz) frequency regimes.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{The normalized angular resolutions $\Delta \tilde{\Omega}$ of LISA and TianQin as functions of frequency for the source with $(\theta = \pi/5, \phi = 4\pi/3, t = \pi/4, \psi = \pi/4, \phi_0 = 0)$. The dashed curve represents the exact $\Delta \tilde{\Omega}$ without the LW approximation, the solid curve represents the $\Delta \tilde{\Omega}$ calculated with the LW approximation.}
\end{figure}

As seen from Fig. 1, for LISA, the normalized angular resolution $\Delta \tilde{\Omega}$ is almost independent of the frequency below sub-nHz and depends on the frequency in a power-law form till $f \sim f^*$. However, the normalized angular resolution $\Delta \tilde{\Omega}$ of TianQin depends on the frequency in a power-law form when $f < f^*$. To understand these results, we use the inverse of the FIM, $(\Gamma^{LW})^{-1}$ to estimate the parameter errors. In general, it is not easy to obtain the analytical expression for $(\Gamma^{LW})^{-1}$. In Appendix C, we give the frequency dependent behaviour of $(\Gamma^{LW})^{-1}$ in the low and high frequency limits so that we can understand how the amplitude and Doppler phase modulations contribute to the parameter error estimations. Note that the frequency limit (especially the high frequency limit) is relative and it still satisfies the LW condition. The main results obtained in Appendix C are as follows. For TianQin in the low frequency limit $f \to 0$, the errors $\tilde{\sigma}_i$ fall off as $1/f$ because $M^{am}$ is singular and the contribution is from the Doppler phase modulation only. In the high frequency limit $f \to \infty$, the errors $\tilde{\sigma}_i$ fall off as $1/f$ for $i = (\theta, \phi)$ and approach a constant for $i = (\ln A, \tau, \psi, \phi_0)$. For LISA in the low frequency limit $f \to 0$, the errors $\tilde{\sigma}_i$ approach a constant because $M^{am}$ is non-singular and the amplitude modulation dominates over the Doppler phase modulation. In the high frequency limit $f \to \infty$, the errors $\tilde{\sigma}_i$ fall off as $1/f$ for $i = (\theta, \phi)$ and approach a constant for $i = (\ln A, \tau, \psi, \phi_0)$. These results are consistent with Fig. 1.

To show the frequency dependent behavior is general and independent of the particular choice of the source, we simulate 2000 sources with parameters $(\cos \theta, \phi, \cos \tau, \psi, \phi_0)$ uniformly distributed. The medians of parameter estimation errors are shown in Figs. 2 and 3. Fig. 2 shows the normalized angular resolutions $\Delta \tilde{\Omega}$ of LISA and TianQin. As expected, they behave differently in the low frequency regimes. The angular resolution $\Delta \tilde{\Omega}$ of LISA depends on the frequency as $S_n(f)$ but the angular resolution $\Delta \tilde{\Omega}$ of TianQin depends on the frequency as $S_n(f)/f^2$. For LISA, the matrix $M^{am}$ is non-singular and independent of the frequency, so it contributes to the parameter estimation and it helps the localization of the source due to the changing orientation of the detector plane. As the frequency decreases, both the Doppler phase modulation $\Gamma^{dm}$ which depends on $f^2$ and $\Gamma^{ad}$ which depends on $f$ tend to be $0$ compared with the total amplitude modulation $\Gamma^{am}$ which is independent of $f$, so the normalized angular resolution $\Delta \tilde{\Omega}$ of LISA approaches a constant. For TianQin, the total amplitude modulation $\Gamma^{am}$ is singular, so its ability of localization comes only from the Doppler phase modulation $\Gamma^{dm}$ which depends on the frequency as $f^2$. This means that even though the total amplitude modulation $\Gamma^{am}$ is singular, but the full Fisher matrix $\Gamma^{LW}$ including the Doppler phase modulation $\Gamma^{dm}$ is non-singular, so TianQin uses the Doppler phase modulation only to locate the source. Therefore, the angular resolution $\Delta \tilde{\Omega}$ of TianQin depends on the frequency as $S_n(f)/f^2$. As the frequency $f$ increases, the main contribution to sky localizations for LISA and TianQin comes from the Doppler phase modulation effect $\Gamma^{dm}$ and their angular resolutions $\Delta \tilde{\Omega}$ have the same behavior of $S_n(f)/f^2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig2.png}
\caption{The medians of the normalized angular resolutions $\Delta \tilde{\Omega}$ of LISA and TianQin as functions of frequency for monochromatic sources.}
\end{figure}

Fig. 3 shows the parameter estimation errors $\sigma_\xi$ for LISA and TianQin. At low frequency regime (below sub-mHz) $\sigma_\xi$ is proportional to $\sqrt{S_n(f)/f}$ for TianQin but $\sigma_\xi$ approaches $\sqrt{S_n(f)}$ for LISA. For TianQin, the Doppler
phase modulation $\Gamma^{dm}$ dominates and it improves both the sky localization and the parameter estimation as the frequency increases. For LISA, as analyzed above, the amplitude modulation $\Gamma^{am}$ dominates, so the parameter estimation errors $\hat{\sigma}_\xi$ are independent of frequency. In the median frequency regime, the errors of the parameters $(\theta, \phi)$ are proportional to $\sqrt{S_n(f)/f}$ and the errors of the parameters $(\ln A, \i, \psi, \phi_0)$ approach to $\sqrt{S_n(f)}$ for both TianQin and LISA. The difference between the parameters $(\ln A, \i, \psi, \phi_0)$ and the parameters $(\theta, \phi)$ is because the $f^2$ dependent Doppler modulation $\Gamma^{dm}$ contributes only to the $(\theta, \phi)$ components in the FIM and it dominates the total matrix $\Gamma^{LW}$ in the medium frequency regime. For the parameters $(\ln A, \i, \psi, \phi_0)$, the errors approach to constants because they are determined by the $4 \times 4$ submatrix in the total amplitude modulation $\Gamma^{am}$.

If the parameters $\psi$ and $\phi_0$ are known, then we are left with four parameters $\xi = \{\theta, \phi, \ln A, \i\}$. In Appendix D, we give the estimation errors of the four parameters. In this case, the matrices $\Gamma^{am}$ for LISA and TianQin are both non-singular and independent of frequency, so the behaviours of the parameter errors for LISA and TianQin are similar except that the parameter errors for LISA are much smaller in the low frequency regime. We also simulate the signal in LISA and use Bayesian analysis to estimate the parameter errors and the results are presented in Appendix E. The results show that the parameter estimation errors with the Bayesian analysis are $3-6$ times larger than those with the FIM.

**B. The high frequency regime**

In the high frequency regime from $10^{-2}$ Hz to $10^{-1}$ Hz, the LW approximation breaks down for LISA. We numerically calculate the parameter errors for LISA. Fig. 4 shows the results of $\hat{\sigma}_\xi$ for LISA and TianQin at high frequencies. When the frequency $f > 0.02$ Hz, we see from Fig. 4 that the errors of parameters become larger for LISA. The reason is that the transfer function $T(f, \hat{u}, \hat{w})$ in Eq. (8) becomes less than 1 and decreases as the frequency $f$ increases. However, at the same frequency the transfer function of TianQin is much bigger than that of LISA, so the estimation errors of parameters for TianQin are smaller than LISA.

**IV. CONCLUSION**

After splitting the FIM into three parts, we separately analyze the effects on the parameter estimation errors by the amplitude modulation due to the changing orientation of the detector plane and the Doppler phase modulation due to the transential motion of the center of the detector around the Sun. In the low and medium frequency regimes, we take the long-wavelength approximation to give analytical formulas for the parameter estimation errors. We find that in the low frequency regime, the angular resolution falls off as $S_n(f)/f^2$ for TianQin but $S_n(f)$ for LISA, and the estimation errors of the other parameters fall off as $\sqrt{S_n(f)/f}$ for TianQin but $\sqrt{S_n(f)}$ for LISA. The different pattern between LISA and TianQin is because the total amplitude modulation is missing in TianQin. For LISA the total amplitude modulation dominates over the Doppler phase modulation, so the parameter error $\hat{\sigma}_i$ is independent of frequency; For TianQin only the Doppler phase modulation contributes to the parameter estimation, so the parameter error $\hat{\sigma}_i$ falls off as $1/f$. In the medium frequency regime, the Doppler phase modulation dominates and it affects on the angular resolutions only, but its effect on the other parameters is negligible. Because of this, for both TianQin and LISA, the angular resolutions fall off as $S_n(f)/f^2$, and the estimation errors of the parameters $(\ln A, \i, \psi, \phi_0)$ fall off as $\sqrt{S_n(f)}$ because they are determined by the frequency independent $4 \times 4$ submatrix in the total amplitude modulation $\Gamma^{am}$. These results are also confirmed by numerical method. In the high frequency regime where the long-wavelength approximation fails, we numerically calculate the parameter estimation errors for LISA and TianQin and find that because of different transfer frequency due to the difference in arm lengths, the parameter estimation errors measured by TianQin are smaller than LISA. Although the parameter estimations with the Bayesian analysis are more reliable and the estimation errors are $3-6$ times larger than those with the FIM, the FIM helps us understand the effects on the parameter estimations by the total amplitude modulation and the Doppler phase modulation. The results are useful for understanding the parameter estimation errors measured by LISA and TianQin due to the difference in the constellation.

**ACKNOWLEDGMENTS**

This research is supported in part by the National Key Research and Development Program of China under Grant No. 2020YFC2201504; the National Natural Science Foundation of China under Grant Nos. 11875136 and 12075202; and the Major Program of the National Natural Science Foundation of China under Grant No. 11690021.

**Appendix A: The FIM for TianQin**

This appendix provides the nonzero matrix elements of FIM for TianQin. The nonzero elements of the matrix
$M_{14}^m$ for TianQin in the detector coordinate system are

$$M_{11}^m = \frac{3}{32} \sin^2(\theta_d) \left( -16 \sin^2(\theta_d) \sin^4(\iota) \cos(4\psi_d) 
+ (\cos(2\theta_d) + 3)(28 \cos(2\iota) + \cos(4\iota) + 35) \right),$$

$$M_{12}^m = -\frac{3}{2} \sin^3(\theta_d) \sin^4(\iota) \sin(4\psi),$$

$$M_{13}^m = -\frac{3}{128} \sin(2\theta_d) \left( 16 \sin^2(\theta_d) \sin^4(\iota) \cos(4\psi_d) 
- (\cos(2\theta_d) + 3)(28 \cos(2\iota) + \cos(4\iota) + 35) \right),$$

$$M_{14}^m = \frac{3}{32} \sin(2\theta_d) \sin(2\iota) \left( 4 \sin^2(\theta_d) \sin^2(\iota) \cos(4\psi_d) 
+ (\cos(2\theta_d) + 3)(28 \cos(2\iota) + \cos(4\iota) + 35) \right),$$

$$M_{15}^m = -\frac{3}{2} \sin^3(\theta_d) \cos(\theta_d) \sin^4(\iota) \sin(4\psi_d),$$

$$M_{22}^m = \frac{3}{128} \left( 64 \sin^4(\theta_d) \sin^4(\iota) \cos(4\psi_d) 
+ (28 \cos(2\theta_d) + \cos(4\theta_d) + 35) \times (28 \cos(2\iota) + \cos(4\iota) + 35) \right),$$

$$M_{25}^m = \frac{3}{16} \left( 7 \cos(\theta_d) + \cos(3\theta_d) \right) \times (28 \cos(2\iota) + \cos(4\iota) + 35),$$

$$M_{26}^m = -\frac{3}{4} \left( 7 \cos(\theta_d) + \cos(3\theta_d) \right) \left( 7 \cos(\iota) + \cos(3\iota) \right),$$

$$M_{33}^m = \frac{3}{512} \left( 64 \sin^4(\theta_d) \sin^4(\iota) \cos(4\psi_d) 
+ (28 \cos(2\theta_d) + \cos(4\theta_d) + 35) \times (28 \cos(2\iota) + \cos(4\iota) + 35) \right),$$

$$M_{34}^m = -\frac{3}{512} \sin(2\iota) \left( -64 \sin^4(\theta_d) \sin^2(\iota) \cos(4\psi_d) 
+ 4(28 \cos(2\theta_d) + \cos(4\theta_d) + 35)(\cos(2\iota) + 7) \right),$$

$$M_{35}^m = -\frac{3}{4} \sin^4(\theta_d) \sin^4(\iota) \sin(4\psi_d),$$

FIG. 3. The medians of parameter estimation errors of LISA and TianQin as functions of frequency for monochromatic sources in the low and medium frequency regimes.
FIG. 4. The medians of parameter estimation errors of LISA and TianQin as functions of frequency for monochromatic sources in the high frequency regimes.

\[
M_{44}^{am} = \frac{3}{32} \sin^2(\iota) \left(-16 \sin^4(\theta_d) \sin^2(\iota) \cos(4\psi_d)
+ (28 \cos(2\theta_d) + \cos(4\theta_d) + 35)(\cos(2\iota) + 3)\right),
\]

\[
M_{45}^{am} = -\frac{3}{2} \sin^4(\theta_d) \sin^3(\iota) \cos(\iota) \sin(4\psi_d),
\]

\[
M_{46}^{am} = \frac{3}{4} \sin^4(\theta_d) \sin^3(\iota) \sin(4\psi_d),
\]

\[
M_{55}^{am} = \frac{3}{128} \left(-64 \sin^4(\theta_d) \sin^4(\iota) \cos(4\psi_d)
+ (28 \cos(2\theta_d) + \cos(4\theta_d) + 35)
\times (28 \cos(2\iota) + \cos(4\iota) + 35)\right),
\]

\[
M_{56}^{am} = -\frac{3}{32} (28 \cos(2\theta_d) + \cos(4\theta_d) + 35)
\times (7 \cos(\iota) + \cos(3\iota)),
\]

\[
M_{66}^{am} = \frac{3}{512} (64 \sin^4(\theta_d) \sin^4(\iota) \cos(4\psi_d)
+ (28 \cos(2\theta_d) + \cos(4\theta_d) + 35)
\times (28 \cos(2\iota) + \cos(4\iota) + 35)).
\]

It is straightforward to check that \(\det M^{am} = 0\). For the same source in the sky, the parameters \((\theta_d, \phi_d, \psi_d)\) in the detector coordinate system are related with the parameters \((\theta, \phi, \psi)\) in the heliocentric coordinate system as

\[
\theta_d = \arccos(\sin(\alpha) \sin(\beta) \sin(\phi) \sin(\theta)
+ \cos(\alpha) \sin(\beta) \cos(\phi) \sin(\theta) + \cos(\beta) \cos(\theta))
\]

\[
\phi_d = \arctan\left(\frac{\cos(\alpha) \sin(\phi) \sin(\theta) - \sin(\alpha) \cos(\phi) \sin(\theta)}{\cos(\alpha) \cos(\beta) \sin(\phi) \sin(\theta)
+ \sin(\alpha) \cos(\beta) \sin(\phi) \sin(\theta) - \sin(\beta) \cos(\theta)}\right),
\]

\[
\psi_d = \arctan\left((\cos(\beta) \sin(\theta) \sin(\psi) + \sin(\beta) \cos(\alpha) \times
((\sin(\phi) (\cos(\psi) - \cos(\phi) \cos(\theta) \sin(\psi))
+ \tan(\alpha) (\cos(\phi) \cos(\psi) - \sin(\phi) \cos(\theta) \sin(\psi)))
\times (\sin(\beta) \sin(\phi) \sin(\theta) \cos(\psi) + \cos(\phi) \sin(\psi))
+ \cos(\alpha) \sin(\beta) (\cos(\phi) \cos(\theta) \cos(\psi) - \sin(\phi) \sin(\psi))
- \cos(\beta) \sin(\theta) \cos(\psi)))\right).
\]
The matrix $M_{ij}^m$ for TianQin in the heliocentric coordinate system becomes

$$M_{ij}^m \rightarrow F^{di} M_{ik}^m F^{kj},$$

where $F^{ij} = \partial \xi_i / \partial \xi_j$ and $\xi_i = (\theta_d, \phi_d, \ln A_\iota, \psi_d, \phi_0)$.

The nonzero elements of the matrix $M_{ij}^m$ for TianQin in the heliocentric coordinate system are

$$M_{11}^{dm} = \frac{2\pi^2 \rho^2 f^2 R^2}{c^2} \sin^2(\theta),$$

$$M_{22}^{dm} = \frac{2\pi^2 \rho^2 f^2 R^2}{c^2} \sin^2(\theta),$$

where

$$\rho^2 = \frac{3}{512} \left(64 \sin^4(\theta_d) \cos(4\psi_d) \sin^4(\iota) + (28 \cos(2\theta_d) + \cos(4\theta_d) + 35) \times (28 \cos(2\iota) + \cos(4\iota) + 35)\right).$$

### Appendix B: The FIM for LISA

This appendix provides the nonzero matrix elements of FIM for LISA in the heliocentric coordinate system.
\[ M_{m}^{22} = \frac{\sin^2(\alpha)}{16384} \left( 8 \sin^4(\iota) \cos(4\psi) (162 \cos(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(4\phi) \right) \\
+ 10368(7 \cos(\theta) + \cos(3\theta)) \sin^4(\iota) \sin(4\psi) \sin(2\alpha - 4\phi) \\
+ 4(28 \cos(2\iota) + \cos(4\iota) + 35)(1735 - 19 \cos(4\theta)) \\
+ 4(28 \cos(2\iota) + \cos(4\iota) + 35)(324 \sin^4(\theta) \cos(2\alpha - 4\phi) + 908 \cos(2\theta) \\
+ 8 \sin^4(\iota) \cos(4\psi) (162 \sin(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \sin(4\phi) - 608 \sin^4(\theta)) \right), \\
\]
\[ M_{m}^{23} = \frac{81 \sin^2(\alpha)}{2048} \left( 8 \sin^4(\iota)(7 \cos(\theta) + \cos(3\theta)) \sin(4\psi) \cos(2\alpha - 4\phi) \\
- \sin^4(\theta)(28 \cos(2\iota) + \cos(4\iota) + 35) \sin(2\alpha - 4\phi) \\
- \sin^4(\iota)(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(4\psi) \sin(2\alpha - 4\phi) \right), \\
\]
\[ M_{m}^{24} = \frac{81 \sin^2(\alpha) \sin(\iota) \cos(\iota)}{1024} \left( 8(7 \cos(\theta) + \cos(3\theta)) \sin^2(\iota) \sin(4\psi) \cos(2\alpha - 4\phi) \\
+ 4 \sin(2\alpha - 4\phi) \sin^4(\iota) \cos(4\psi)(2\cos(2\theta) + \cos(4\theta) + 7) \\
- \sin(2\alpha - 4\phi) \sin^2(\iota) \cos(4\psi)(28 \cos(2\theta) + \cos(4\theta) + 35) \right), \\
\]
\[ M_{m}^{25} = \frac{\sin^2(\alpha)}{1024} \left( 648(7 \cos(\theta) + \cos(3\theta)) \sin^4(\iota) \cos(4\psi) \cos(2\alpha - 4\phi) \\
+ 2(347 \cos(\theta) - 19 \cos(3\theta))(28 \cos(2\iota) + \cos(4\iota) + 35) \\
+ 81(28 \cos(2\theta) + \cos(4\theta) + 35) \sin^4(\iota) \sin(4\psi) \sin(2\alpha - 4\phi) \right), \\
\]
\[ M_{m}^{26} = \frac{1}{128} \sin^2(\alpha)(19 \cos(3\theta) - 347 \cos(\theta))(7 \cos(\iota) + \cos(3\iota)), \\
\]
\[ M_{m}^{33} = \frac{\sin^2(\alpha)}{65536} \left( -16 \sin^4(\iota) \cos(4\psi) (81 \cos(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(4\phi) \\
- 4(28 \cos(2\iota) + \cos(4\iota) + 35)(37 \cos(4\theta) - 3121) \\
- 10368(7 \cos(\theta) + \cos(3\theta)) \sin^4(\iota) \sin(4\psi) \sin(2\alpha - 4\phi) \\
- 4(28 \cos(2\iota) + \cos(4\iota) + 35)(324 \sin^4(\theta) \cos(2\alpha - 4\phi) + 460 \cos(2\theta)) \\
- 16 \sin^4(\iota) \cos(4\psi) (81 \sin(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \sin(4\phi) + 592 \sin^4(\theta)) \right), \\
\]
\[ M_{m}^{34} = \frac{\sin^2(\alpha)}{32768} \left( 8 \sin^3(\iota) \cos(\iota) (-1296(7 \cos(\theta) + \cos(3\theta)) \sin(4\psi) \sin(2\alpha - 4\phi) \\
+ 4(14 \sin(2\iota) + \sin(4\iota)) (324 \sin^4(\theta) \cos(2\alpha - 4\phi) + 460 \cos(2\theta)) \\
- 16 \sin^3(\iota) \cos(\iota) \sin(4\psi) (81(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(2\alpha - 4\phi) \\
- 9472 \sin^3(\iota) \cos(\iota) \sin(4\psi) \sin^4(\theta) + 4(14 \sin(2\iota) + \sin(4\iota))(37 \cos(4\theta) - 3121) \right), \\
\]
\[ M_{m}^{35} = \frac{\sin^2(\alpha) \sin^4(\iota)}{2048} \left( \sin(4\psi) (81(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(2\alpha - 4\phi) \\
- 648(7 \cos(\theta) + \cos(3\theta)) \cos(4\psi) \sin(2\alpha - 4\phi) + 592 \sin(4\psi) \sin^4(\theta) \right), \\
\]
\[ M_{m}^{44} = \frac{\sin^2(\alpha) \sin^2(\iota)}{4096} \left( 2592(7 \cos(\theta) + \cos(3\theta)) \sin^2(\iota) \sin(4\psi) \sin(2\alpha - 4\phi) \\
+ 4 \sin^2(\iota) \cos(4\psi) (81 \cos(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(4\phi) \\
+ 81 \sin(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \sin(4\phi) + 592 \sin^4(\theta) \right) \\
- 4(\cos(2\iota) + 3) (324 \sin^4(\theta) \cos(2\alpha - 4\phi) + 460 \cos(2\theta) + 37 \cos(4\theta) - 3121) \right), \\
\]
\[ M_{m}^{45} = \frac{\sin^2(\alpha) \sin^3(\iota) \cos(\iota)}{1024} \left( -648(7 \cos(\theta) + \cos(3\theta)) \cos(4\psi) \sin(2\alpha - 4\phi) \\
+ \sin(4\psi) (81(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(2\alpha - 4\phi) + 592 \sin^4(\theta)) \right), \\
\]
\[ M_{m}^{46} = \frac{\sin^2(\alpha) \sin^3(\iota)}{2048} \left( 648(7 \cos(\theta) + \cos(3\theta)) \cos(4\psi) \sin(2\alpha - 4\phi) \\
- \sin(4\psi) (81(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(2\alpha - 4\phi) + 592 \sin^4(\theta)) \right), \\
\]
\[ M_{55}^{m} = \frac{\sin^2(\alpha)}{16384} \left( 16 \sin^4(\psi) \cos(4\psi) \left( 81 \cos(2\alpha) \right) (28 \cos(2\theta) + \cos(4\theta) + 35) \cos(4\phi) \\
- 4(28 \cos(2\theta) + \cos(4\theta) + 35)(37 \cos(4\theta) - 3121) \\
+ 10368(7 \cos(\theta) + \cos(3\theta)) \sin^4(\psi) \sin(2\alpha - 4\phi) \\
+ 81 \sin(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \sin(4\phi) + 592 \sin^4(\theta) \\
- 4(28 \cos(2\theta) + \cos(4\theta) + 35) \left( 324 \sin^4(\theta) \cos(2\alpha - 4\phi) + 460 \cos(2\theta) \right) \right), \]

\[ M_{66}^{m} = \frac{\sin^2(\alpha)}{1024} \left( 7 \cos(\psi) \left( 324 \sin^4(\theta) \cos(2\alpha - 4\phi) \right) + 460 \cos(2\theta) + 37 \cos(4\theta) - 3121 \right), \]

\[ M_{56}^{m} = \frac{\sin^2(\alpha)}{65536} \left( -16 \sin^4(\psi) \cos(4\psi) \left( 81 \cos(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \cos(4\phi) \\
- 4(28 \cos(2\theta) + \cos(4\theta) + 35)(37 \cos(4\theta) - 3121) \\
- 10368(7 \cos(\theta) + \cos(3\theta)) \sin^4(\psi) \sin(2\alpha - 4\phi) \\
+ 81 \sin(2\alpha)(28 \cos(2\theta) + \cos(4\theta) + 35) \sin(4\phi) + 592 \sin^4(\theta) \\
- 4(28 \cos(2\theta) + \cos(4\theta) + 35) \left( 324 \sin^4(\theta) \cos(2\alpha - 4\phi) + 460 \cos(2\theta) \right) \right). \]

\[ M_{11}^{dn} = \frac{\pi^2 f^2 R^2 \sin^2(\alpha) \cos^2(\theta)}{32768c^2} \left( 8 \sin^4(\psi) \cos(4\psi) (16 \sin^2(\theta)(55 \cos(2\theta) + 17) \\
+ 27648 \cos(\theta)(28 \cos(2\theta) - 4) \sin^4(\psi) \sin(2\alpha - 4\phi) \\
+ 8 \sin^4(\psi) \cos(4\psi)(54 \cos(2\alpha)(-28 \cos(2\theta) + 11 \cos(4\theta) - 175) \cos(4\theta)) \\
+ 432 \sin^4(\psi) \cos(4\psi) \sin(2\alpha)(-28 \cos(2\theta) + 11 \cos(4\theta) - 175) \sin(4\phi) \\
+ 4(28 \cos(2\theta) + \cos(4\theta) + 35)(-1108 \cos(2\theta) - 55 \cos(4\theta) + 3787) \\
- 216(28 \cos(2\theta) + \cos(4\theta) + 35) \sin^2(\theta)(11 \cos(2\theta) + 13) \cos(2\alpha - 4\phi) \right), \]

\[ M_{12}^{dn} = \frac{9\pi^2 f^2 R^2 \sin^2(\alpha) \sin^3(\theta)}{4096c^2} \left( 64 \cos^2(\theta) \sin^4(\psi)(21 \cos(2\alpha - 4\phi) + 2) \\
- 168(7 \cos(\theta) + \cos(3\theta)) \sin^4(\psi) \cos(4\psi) \sin(2\alpha - 4\phi) \\
- 3(17 \cos(\theta) + 7 \cos(3\theta))(28 \cos(2\theta) + \cos(4\theta) + 35) \sin(2\alpha - 4\phi) \right), \]

\[ M_{22}^{dn} = \frac{\pi^2 f^2 R^2 \sin^2(\alpha) \sin^2(\theta)}{32768c^2} \left( -8 \sin^4(\psi) \cos(4\psi) (54 \sin(2\alpha)(140 \cos(2\theta)) \\
- 6912(7 \cos(\theta) + 5 \cos(3\theta)) \sin^4(\psi) \sin(2\alpha - 4\phi) \\
- 8 \sin^4(\psi) \cos(4\psi)(54 \cos(2\alpha)(140 \cos(2\theta) + 17 \cos(4\theta) + 35) \cos(4\phi)) \\
- 8 \sin^4(\psi) \cos(4\psi)(+17 \cos(4\theta) + 35) \sin(4\phi) + 16 \sin^2(\theta)(91 - 19 \cos(2\theta)) \\
+ 216(28 \cos(2\theta) + \cos(4\theta) + 35) \sin^2(\theta)(17 \cos(2\theta) + 7) \cos(2\alpha - 4\phi) \\
+ 4(28 \cos(2\theta) + \cos(4\theta) + 35)(188 \cos(2\theta) - 19 \cos(4\theta) + 2455) \right). \]
\[ M'_{11} = \frac{81\sqrt{3}\pi fR \sin^2(\alpha) \sin^2(2\theta)(7 \cos(\iota) + \cos(3\iota)) \sin(2\alpha - 4\phi)}{256c} \]
\[ M'_{12} = \frac{\sqrt{3}\pi fR \sin^2(\alpha) \sin(2\theta)(7 \cos(\iota) + \cos(3\iota))}{128c} (-27 \sin^2(\theta) \cos(2\alpha - 4\phi) + 47 \cos(2\theta) + 161), \]
\[ M'_{14} = \frac{\sqrt{3}\pi fR \sin^2(\alpha) \sin^3(\iota)}{256c} (-27(10 \sin(2\theta) + 3 \sin(4\theta)) \cos(4\psi) \sin(2\alpha - 4\phi) + 2 \sin(\theta) \cos^2(\theta) \sin(4\psi) (54 \cos(2\alpha)(\cos(2\theta) + 7) \cos(4\phi) + 54 \sin(2\alpha)(\cos(2\theta) + 7) \sin(4\phi) - 40 \sin^2(\theta)) \} , \]
\[ M'_{15} = \frac{\sqrt{3}\pi fR \sin^2(\alpha) \sin(\theta) \cos^2(\theta)(7 \cos(\iota) + \cos(3\iota))}{32c} (27 \sin^2(\theta) \cos(2\alpha - 4\phi) - 5 \cos(2\theta) + 109), \]
\[ M'_{16} = \frac{\sqrt{3}\pi fR \sin^2(\alpha) \sin^4(\iota) \sin(4\psi) \sin(2\alpha - 4\phi)}{2048c} + 4 \sin(\theta) \cos^2(\theta) \sin(4\psi) (2 \sin^4(\iota) \sin(4\psi) (54 \cos(2\alpha)(\cos(2\theta) + 7) \cos(4\phi) + 54 \sin(2\alpha)(\cos(2\theta) + 7) \sin(4\phi) - 40 \sin^2(\theta)) \} , \]
\[ M'_{22} = \frac{27\sqrt{3}\pi fR \sin^2(\alpha) \sin^4(\theta)(7 \cos(\iota) + \cos(3\iota)) \sin(2\alpha - 4\phi)}{64c}, \]
\[ M'_{24} = \frac{\sqrt{3}\pi fR \sin^2(\alpha) \sin^2(\theta) \sin^3(\iota)}{128c} (2 \cos(4\psi) (27 \cos(2\theta) + 5) \cos(2\alpha - 4\phi) + 27(15 \cos(\theta) + \cos(3\theta)) \sin(4\psi) \sin(2\alpha - 4\phi) + 40 \cos(4\psi) \sin^2(\theta)), \]
\[ M'_{25} = \frac{27\sqrt{3}\pi fR \sin^2(\alpha) \sin^4(\theta) \cos(3\iota)}{32c} \sin(2\alpha - 4\phi), \]
\[ M'_{26} = \frac{\sqrt{3}\pi fR \sin^2(\alpha) \sin^2(\theta)}{512c} (54(15 \cos(\theta) + \cos(3\theta)) \sin^4(\iota) \cos(4\psi) \sin(2\alpha - 4\phi) - 108 \sin^4(\iota) \sin(4\psi) \cos(2\alpha)(3 \cos(2\theta) + 5) \cos(4\phi) - 27 \sin^2(\theta) \cos(\theta)(28 \cos(2\iota) + \cos(4\iota) + 35) \sin(2\alpha - 4\phi) - 2 \sin^4(\iota) \sin(4\psi) (54 \sin(2\alpha)(3 \cos(2\theta) + 5) \sin(4\phi) + 40 \sin^2(\theta) ) \}, \]

---

**Appendix C: The frequency dependence of the inverse of FIM**

1. TianQin

We can write the FIM as \( \Gamma = H + f^2 N \), where \( H \) and \( N \) are independent of the frequency. In the low frequency limit \( f \to 0 \),

\[
\det \Gamma = \det (H + f^2 N) = \det H + f^2 \text{Tr} [\text{adj} (H) N] + O(f^4),
\]

where \( \text{adj} (H) \) is the adjugate of the matrix \( H \) and \( \text{Tr} \) denotes the trace of the matrix, so the inverse of the FIM is

\[
\Gamma^{-1} = \frac{1}{\det \Gamma} \text{adj} (\Gamma) = \frac{\text{adj} (H + f^2 N)}{\det H + f^2 \text{Tr} [\text{adj} (H) N]},
\]

If the matrix \( H \) is non-singular then \( \Gamma^{-1} \) can be expanded as

\[
\Gamma^{-1} = H^{-1} + O(f^2).
\]

If the matrix \( H \) is singular then \( \Gamma^{-1} \) can be expanded as

\[
\Gamma^{-1} = \frac{\text{adj} (H)}{f^2 \text{Tr} [\text{adj} (H) N]} + O(f^6).
\]

In order to discuss the high frequency limit, we explicitly write the matrices \( H \) and \( N \) as block matrices, then the FIM becomes

\[
\Gamma = \begin{pmatrix} Q_{2 \times 2} & W_{2 \times 4} \\ W_{4 \times 2} & X_{4 \times 4} \end{pmatrix} + f^2 \begin{pmatrix} Y_{2 \times 2} & 0_{2 \times 4} \\ 0_{4 \times 2} & 0_{4 \times 4} \end{pmatrix},
\]

and the inverse of the FIM is

\[
\Gamma^{-1} = \begin{pmatrix} V^{-1} & * \\ * & Z^{-1} \end{pmatrix},
\]
where \( V = Q + f^2 Y - W X^{-1} W^T \), \( Z = X - W^T (Q + f^2 Y)^{-1} W \), the off diagonal elements are not specified because they are irrelevant for our discussion. In the high frequency limit \( f \to \infty \), we get
\[
\Gamma^{-1} = \begin{pmatrix} f^{-2} Y^{-1} & 0 \\ 0 & X^{-1} \end{pmatrix}, \tag{C7}
\]

2. LISA

We can write the FIM as \( \Gamma = H + fK + f^2 N \), where \( H, K \) and \( N \) are independent of the frequency. In the low frequency limit \( f \to 0 \), the matrix \( H \) is non-singular and \( \Gamma^{-1} \) can be expanded as
\[
\Gamma^{-1} = H^{-1} + O(f). \tag{C8}
\]

In order to discuss the high frequency limit, we explicitly write the matrices \( H, K \) and \( N \) as block matrices, then the FIM becomes
\[
\Gamma = \begin{pmatrix} Q & W \\ W^T & X \end{pmatrix} + \begin{pmatrix} fB + f^2 Y & fC \\ fC^T & 0 \end{pmatrix}, \tag{C9}
\]
and the inverse of the FIM is
\[
\Gamma^{-1} = \begin{pmatrix} V^{-1} & 0 \\ 0 & Z^{-1} \end{pmatrix}, \tag{C10}
\]
where \( V = Q + fB + f^2 Y - (W + fC)X^{-1}(W + fC)^T \), \( Z = X - (W + fC)^T(Q + fB + f^2 Y)^{-1}(W + fC) \), the off diagonal elements are not specified because they are irrelevant for our discussion. In the high frequency limit \( f \to \infty \), we get
\[
\Gamma^{-1} = \begin{pmatrix} f^{-2}(Y - CX^{-1}C^T)^{-1} & 0 \\ 0 & (X - C^T Y^{-1} C)^{-1} \end{pmatrix}.
\]

Appendix D: The FIM with four parameters

If we consider only four parameters of the monochromatic GW signal
\[
\xi = \{\theta, \phi, \ln A, \iota\}, \tag{D1}
\]
then the FIM in the LW approximation can be written in a compact form as
\[
\Gamma^{LW} = \Gamma^{am} + \Gamma^{dm}, \tag{D2}
\]
where
\[
\Gamma^{am} = \frac{A^2 T_{\text{obs}}}{S_n(f)} F^{am}, \quad \Gamma^{dm} = \frac{A^2 T_{\text{obs}}}{S_n(f)} F^{dm}, \tag{D3}
\]
\( F^{am} \) is a 4 by 4 submatrix of \( M^{am} \) and independent of the frequency,
\[
F^{dm} = \left( \frac{2\pi f R}{c} \right)^2 \begin{pmatrix} D_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} \tag{D4}
\]

Appendix E: The Bayesian analysis

The likelihood \( \mathcal{L} = p (d | \xi) \) for a given gravitational wave signal takes the form
\[
\ln \mathcal{L} = -\frac{1}{2} \left( |H(\xi) - d| |H(\xi) - d| \right), \tag{E1}
\]

FIG. 5. The medians of the normalized angular resolutions \( \Delta \Omega \) of LISA and TianQin as functions of frequency for monochromatic sources with four parameters.
where the Hermitian inner product \((A|B)\) is defined in Eq. (9), the data \(d = H(\xi_0) + n\) is the superposition of the gravitational wave signal \(H(\xi_0)\) for the true parameters \(\xi_0\) and the noise \(n\) in the detector. The posterior distribution for the parameters \(\xi\) is

\[
p(\xi|d) = \frac{p(d|\xi)p(\xi)}{p(d)},
\]

(E2)

where \(p(\xi)\) is the prior on the parameters and \(p(d)\) is the evidence. In our analysis, we simulate signals in LISA for the source with parameters \((A = 10^{-20}, \theta = \pi/5, \phi = 4\pi/3, \iota = \pi/4, \psi = \pi/4, \phi_0 = 0, f = 10^{-3})\) and perform Bayesian analyses with Eq. (E1) and (E2) to obtain the posteriors of the physical parameter \(\xi\). The signal to noise ratio for the source is \(\text{SNR}=955\). The results on the parameter estimations are shown in Fig. 8 and the comparison of the Bayesian analysis with the FIM is shown in Table I. The results show that the parameter estimation errors with the Bayesian analysis are 3 – 6 times larger than those with the FIM.

---

[1] B. P. Abbott et al. (LIGO Scientific, Virgo), GW150914: The Advanced LIGO Detectors in the Era of First Discoveries, Phys. Rev. Lett. 116, 131103 (2016).

[2] B. P. Abbott et al. (LIGO Scientific, Virgo), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, 061102 (2016).
FIG. 7. The medians of parameter estimation errors of LISA and TianQin as functions of frequency for monochromatic sources with four parameters in the high frequency regimes.

| Method | $\sigma_\theta(10^{-3})$ | $\sigma_\phi(10^{-3})$ | $\sigma_{\log A}(10^{-3})$ | $\sigma_\iota(10^{-3})$ |
|--------|-----------------|-----------------|-----------------|-----------------|
| Bayesian | 1.0             | 0.9             | 5.5             | 18              |
| FIM     | 0.30            | 0.35            | 1.3             | 3.4             |

TABLE I. The parameter estimation errors using the FIM and the Bayesian analysis for the fiducial source with ($A = 10^{-20}$, $\theta = \pi/5$, $\phi = 4\pi/3$, $\iota = \pi/4$, $\psi = \pi/4$, $\phi_0 = 0$, $f = 10^{-3}$).

[3] B. P. Abbott et al. (LIGO Scientific, Virgo), GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. 116, 241103 (2016).
[4] B. P. Abbott et al. (LIGO Scientific, VIRGO), GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett. 118, 221101 (2017), [Erratum: Phys.Rev.Lett. 121, 129901 (2018)].
[5] B. P. Abbott et al. (LIGO Scientific, Virgo), GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, Phys. Rev. Lett. 119, 141101 (2017).
[6] B. P. Abbott et al. (LIGO Scientific, Virgo), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. 119, 161101 (2017).
[7] B. P. Abbott et al. (LIGO Scientific, Virgo), GW170608: Observation of a 19-solar-mass Binary Black Hole Coalescence, Astrophys. J. Lett. 851, L35 (2017).
[8] B. P. Abbott et al. (LIGO Scientific, Virgo), GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs, Phys. Rev. X 9, 031040 (2019).
[9] B. P. Abbott et al. (LIGO Scientific, Virgo), GW190425: Observation of a Compact Binary Coalescence with Total Mass $\sim 3.4M_\odot$, Astrophys. J. Lett. 892, L3 (2020).
[10] R. Abbott et al. (LIGO Scientific, Virgo), GW190412: Observation of a Binary-Black-Hole Coalescence with Asymmetric Masses, Phys. Rev. D 102, 043015 (2020).
FIG. 8. The parameter estimation errors using the Bayesian analysis for the simulated source with $(A = 10^{-20}, \theta = \pi/5, \phi = 4\pi/3, \iota = \pi/4, \psi = \pi/4, \phi_0 = 0, f = 10^{-5})$.

[11] R. Abbott et al. (LIGO Scientific, Virgo), GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object, Astrophys. J. Lett. 896, L44 (2020).
[12] R. Abbott et al. (LIGO Scientific, Virgo), GW190521: A Binary Black Hole Merger with a Total Mass of 150$M_\odot$, Phys. Rev. Lett. 125, 101102 (2020).
[13] R. Abbott et al. (LIGO Scientific, Virgo), GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run, arXiv:2010.14527 [gr-qc].
[14] K. Danzmann, LISA: An ESA cornerstone mission for a gravitational wave observatory, Class. Quant. Grav. 14, 1399 (1997).
[15] P. Amaro-Seoane et al. (LISA), Laser Interferometer Space Antenna, arXiv:1702.00786 [astro-ph.IM].
[16] W.-R. Hu and Y.-L. Wu, The Taiji Program in Space for gravitational wave physics and the nature of gravity, Natl. Sci. Rev. 4, 685 (2017).
[17] J. Luo et al. (TianQin), TianQin: a space-borne gravitational wave detector, Class. Quant. Grav. 33, 035010 (2016).
[18] B. F. Schutz, Determining the Hubble Constant from Gravitational Wave Observations, Nature 323, 310 (1986).
[19] D. E. Holz and S. A. Hughes, Using gravitational-wave standard sirens, Astrophys. J. 629, 15 (2005).
[20] A. G. Riess, S. Casertano, W. Yuan, L. M. Macri, and D. Scogninc, Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM, Astrophys. J. 876, 85 (2019).
[21] C. Cutler and E. E. Flanagan, Gravitational waves from merging compact binaries: How accurately can one extract the binary’s parameters from the inspiral wave form?, Phys. Rev. D 49, 2658 (1994).
[22] M. Peterseim, O. Jennrich, and K. Danzmann, Accuracy of parameter estimation of gravitational waves with LISA, Class. Quant. Grav. 13, A279 (1996).
[23] M. Peterseim, O. Jennrich, K. Danzmann, and B. F. Schutz, Angular resolution of LISA, Class. Quant. Grav. 14, 1507 (1997).
[24] C. Cutler, Angular resolution of the LISA gravitational wave detector, Phys. Rev. D 57, 7089 (1998).
[25] C. Cutler and A. Vecchio, LISA’s angular resolution for monochromatic sources, AIP Conf. Proc. 456, 95 (1998).
[26] T. A. Moore and R. W. Hellings, The Angular resolution of space based gravitational wave detectors, AIP Conf. Proc. 523, 255 (2000).
[27] L. Barack and C. Cutler, LISA capture sources: Approximate waveforms, signal-to-noise ratios, and parameter estimation accuracy, Phys. Rev. D 69, 082005 (2004).
[28] E. K. Porter and N. J. Cornish, The Effect of Higher Harmonic Corrections on the Detection of massive black hole binaries with LISA, Phys. Rev. D 78, 064005 (2008).
[29] A. Blaut, Accuracy of estimation of parameters with LISA, Phys. Rev. D 83, 083006 (2011).
[30] W.-H. Ruan, C. Liu, Z.-K. Guo, Y.-L. Wu, and R.-G. Cai, The LISA-Taiji network: precision localization of massive black hole binaries, arXiv:1909.07104 [gr-qc].
[31] W.-H. Ruan, C. Liu, Z.-K. Guo, Y.-L. Wu, and R.-G. Cai, The LISA-Taiji network, Nature Astron. 4, 108 (2020).
[32] G. Wang, W.-T. Ni, W.-B. Han, S.-C. Yang, and X.-Y. Zhong, Numerical simulation of sky localization for LISA-TAIJI joint observation, Phys. Rev. D 102, 024089 (2020).
[33] W.-F. Feng, H.-T. Wang, X.-C. Hu, Y.-M. Hu, and Y. Wang, Preliminary study on parameter estimation accuracy of supermassive black hole binary inspirals for TianQin, Phys. Rev. D 99, 123002 (2019).
[34] S.-J. Huang, Y.-M. Hu, V. Korol, P.-C. Li, Z.-C. Liang, Y. Lu, H.-T. Wang, S. Yu, and J. Mei, Science with the TianQin Observatory: Preliminary results on Galactic double white dwarf binaries, Phys. Rev. D 102, 063021 (2020).
[35] C. Zhang, Y. Gong, H. Liu, B. Wang, and C. Zhang, Sky localization of space-based gravitational wave detectors, arXiv:2009.03476 [astro-ph.IM].
[36] M. Vallisneri, Use and abuse of the Fisher information matrix in the assessment of gravitational-wave parameter-estimation prospects, Phys. Rev. D 77, 042001 (2008).
[37] L. Wen and Y. Chen, Geometrical Expression for the Angular Resolution of a Network of Gravitational-Wave Detectors, Phys. Rev. D 81, 082001 (2010).
[38] B. P. Abbott et al. (KAGRA, LIGO Scientific, VIRGO), Prospects for Observing and Localizing Gravitational-Wave Transients with Advanced LIGO, Advanced Virgo and KAGRA, Living Rev. Rel. 21, 3 (2018).
[39] K. Grover, S. Fairhurst, B. F. Farr, I. Mandel, C. Rodriguez, T. Sidery, and A.Vecchio, Comparison of Gravitational Wave Detector Network Sky Localization Approximations, Phys. Rev. D 89, 042004 (2014).
[40] C. P. L. Berry et al., Parameter estimation for binary neutron-star coalescences with realistic noise during the Advanced LIGO era, Astrophys. J. 804, 114 (2015).
[41] L. P. Singer and L. R. Price, Rapid Bayesian position reconstruction for gravitational-wave transients, Phys.
[42] B. Bécsey, P. Raffai, N. J. Cornish, R. Essick, J. Kanner, E. Katsavounidis, T. B. Littenberg, M. Millhouse, and S. Vitale, Parameter estimation for gravitational-wave bursts with the BayesWave pipeline, Astrophys. J. **839**, 15 (2017).

[43] W. Zhao and L. Wen, Localization accuracy of compact binary coalescences detected by the third-generation gravitational-wave detectors and implication for cosmology, Phys. Rev. D **97**, 064031 (2018).

[44] C. Mills, V. Tiwari, and S. Fairhurst, Localization of binary neutron star mergers with second and third generation gravitational-wave detectors, Phys. Rev. D **97**, 104064 (2018).

[45] S. Fairhurst, Localization of transient gravitational wave sources: beyond triangulation, Class. Quant. Grav. **35**, 105002 (2018).

[46] Y. Fujii, T. Adams, F. Marion, and R. Flaminio, Fast localization of coalescing binaries with a heterogeneous network of advanced gravitational wave detectors, Astropart. Phys. **113**, 1 (2019).

[47] L. J. Rubbo, N. J. Cornish, and O. Poujade, Forward modeling of space borne gravitational wave detectors, Phys. Rev. D **69**, 082003 (2004).

[48] N. J. Cornish and S. L. Larson, Space missions to detect the cosmic gravitational wave background, Class. Quant. Grav. **18**, 3473 (2001).

[49] F. B. Estabrook and H. D. Wahlquist, Response of Doppler spacecraft tracking to gravitational radiation, Gen. Relat. Gravit. **6**, 439 (1975).

[50] X.-C. Hu, X.-H. Li, Y. Wang, W.-F. Feng, M.-Y. Zhou, Y.-M. Hu, S.-C. Hu, J.-W. Mei, and C.-G. Shao, Fundamentals of the orbit and response for TianQin, Class. Quant. Grav. **35**, 095008 (2018).