Conflict Between Classical Mechanics and Electromagnetism: The Harmonic Oscillator in Equilibrium with a Bath

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Abstract

It is pointed out that an electric charge oscillating in a one-dimensional purely-harmonic potential is in detailed balance at its harmonics with a radiation bath whose energy $U_{\text{rad}}$ per normal mode is linear in frequency $\omega$, $U_{\text{rad}} = \text{const} \times \omega$, and hence is Lorentz invariant, as seems appropriate for relativistic electromagnetism. The oscillating charge is not in equilibrium with the Rayleigh-Jeans spectrum which arises from energy-sharing equipartition ideas which are valid only in nonrelativistic mechanics. Here we explore the contrasting behavior of harmonic oscillators connected to baths in classical mechanics and electromagnetism. It is emphasized that modern physics text are in error in suggesting that the Rayleigh-Jeans spectrum corresponds to the equilibrium spectrum of random classical radiation, and in ignoring Lorentz-invariant classical zero-point radiation which is indeed a classical equilibrium spectrum.
I. INTRODUCTION

A. Harmonic Oscillators as Illustrative of the Mechanics-Electromagnetism Conflict

Ask a physicist to name the spectrum of random classical radiation with which an oscillating classical charge in a one-dimensional purely-harmonic potential will come to equilibrium at its harmonics, and, influenced by current modern physics textbooks, he will invariably give the wrong answer. He will suggest “the Rayleigh-Jeans spectrum.” But this suggestion is untrue. The correct answer is “a Lorentz-invariant spectrum.” The Rayleigh-Jeans spectrum is associated with the energy-sharing equipartition ideas which are valid only in nonrelativistic classical mechanics. On the other hand, electromagnetism is a relativistic theory, and a classical charge oscillating in one dimension is in equilibrium at its harmonics with a relativistically-invariant spectrum of random radiation. In this article, we explore the conflict between Newtonian mechanics and electromagnetism in connection with harmonic oscillators and equilibrium baths.

The mismatch between Newtonian mechanics and classical electromagnetism goes unappreciated by many physics students and by their instructors. The mismatch is fundamental since Newtonian mechanics satisfies Galilean invariance while electrodynamics is relativistically invariant. Yet the mismatch is usually ignored. The mismatch is usually ignored in our classes because relativity is treated as a specialty subject covered at the beginning of a class on modern physics; only point collisions between particles are considered for relativistic interactions, and relativity is regarded as needed only in nuclear and elementary-particle physics. It is usually assumed that relativity becomes significant only for situations out of our ordinary experience when particles are traveling at velocities close to the speed of light. Here we show that relativity is central to striking contrasts in harmonic oscillator systems involving low particle velocities in conjunction with ambient baths. The simple analysis exposes the profound conflict between nonrelativistic mechanical systems and relativistic electromagnetic systems.

The essential contrast is as follows. A frictionless harmonic oscillator system in Newtonian mechanics can oscillate forever with a constant amplitude and energy. There is no need for any bath for the oscillator. However, if the mechanical oscillator is connected by point
collisions to a bath of particles, the total system energy is shared between the oscillator and the particle bath so as to give energy equipartition. In contradiction with this mechanical situation, an electromagnetic oscillator involving the oscillation of a charged particle cannot exist on its own; the charged particle must be coupled to radiation. Within classical electrodynamics, there is no such thing as an oscillating charge which exists without radiation. It turns out that the spectrum of radiation which must provide the equilibrium bath for an oscillating charge in a one-dimensional purely-harmonic potential is uniquely defined by electromagnetic theory, and turns out to correspond to a Lorentz-invariant spectrum of classical electromagnetic radiation.

This last statement will come as a surprise to the many physicists who are aware of the historical situation involving Planck’s linear oscillator taken in dipole approximation which will match the energy of the radiation bath at the oscillator frequency, but will not determine the radiation spectrum. Such an oscillator appears in many modern physics texts in connection with the blackbody radiation spectrum.[1] However, any electrodynamic oscillator must involve an oscillating charge with a finite non-zero amplitude of motion in the harmonic potential. The moment the amplitude of oscillation is non-zero, the charge radiates not only at the fundamental frequency (as given by the dipole approximation), but also at all the harmonics. The radiation at the harmonics determines the spectrum of the ambient radiation whose absorption is needed to balance the emitted radiation. Thus by considering the harmonics, one arrives at a unique spectrum for the equilibrium bath for the oscillating charge. A recent relativistic calculation shows[2] that classical electromagnetic theory for the one-dimensional electromagnetic harmonic oscillator uniquely determines the radiation spectrum with which the oscillator is in equilibrium, and the spectrum is that of Lorentz-invariance where the energy of a radiation normal mode is proportional to the frequency of the mode, \( U_{rad} = \text{const} \times \omega \). The well-known radiation spectrum \( U_{rad}(\omega) = (1/2)\hbar\omega \) of classical electromagnetic zero-point radiation fits this Lorentz-invariance requirement.

B. Outline of the Article

The outline of the presentation is as follows. First we describe the assumed mechanical and electromagnetic oscillators and their baths. Then we consider the equilibrium situations for the oscillators. For the electromagnetic oscillator, we first describe the dipole harmonic
approximation which is of historical importance in suggesting the Rayleigh-Jeans spectrum, and also the nonlinear oscillators taken in the dipole approximation which seem to confirm the Rayleigh-Jeans conclusion. Then we sketch the extension of the oscillator analysis to quadrupole radiation at double the frequency of the oscillator and note that equilibrium requires a linear (and hence relativistic) radiation spectrum for equilibrium. Next we turn to adiabatic changes of the oscillator’s fundamental frequency, and see how these changes fit with thermodynamic ideas of the equilibrium temperature. The final analysis notes the limitations in the use of a one-dimensional purely-harmonic oscillator. Finally, we give some concluding remarks.

II. THE OSCILLATORS

A. Mechanical Oscillator

Here we discuss the conflict between Newtonian mechanics and electromagnetism in connection with equilibrium for small harmonic oscillators in a bath. In the case of classical mechanics, the oscillator consists of a particle attached to a spring along the $x$-axis in the absence of any friction. The particle of mass $M$ is attached to a spring of constant $\kappa$ so that Newton’s second law gives the equation of motion due to the linear restoring force $F = -\hat{x}\kappa x$ as

$$M\ddot{x} = -\kappa x,$$

with the solution

$$x(t) = x_0 \cos [\omega_0 t + \phi]$$

where the angular frequency is $\omega_0 = \sqrt{\kappa/M}$ and $\phi$ is a constant phase angle.

B. Electromagnetic Oscillator

In the case of the electromagnetic oscillator, we are considering only a one-dimensional oscillator consisting of a charge $e$ with mass $M$ confined along the $x$-axis by two charges $q$ of the same sign as $e$. Thus we consider two charges $q$ placed along the $x$-axis at $x_{q\pm} = \pm a$ with the charge $e$ in unstable equilibrium at the origin of coordinates. If the charge $e$ is
displaced from the origin to the point $x$, then the force on the charge is

$$\mathbf{F} = \hat{x}q e \left[ \frac{1}{(-a - x)^2} - \frac{1}{(a - x)^2} \right] \approx -\frac{4q e}{a^3} \hat{x}. \quad (3)$$

Thus for small displacements $x$, there is a linear restoring force on the charge $e$. If the system were treated as a purely mechanical system, the (angular) frequency of oscillation would be $\omega_0 = \sqrt{\frac{4q e}{(a^3 M)}}$. Now electromagnetism is a relativistic theory. Our electromagnetic oscillator can be regarded as relativistic to any degree of approximation provided that the amplitude $x_0$ of oscillation and hence the maximum velocity $x_0 \omega_0$ of the oscillation is taken as sufficiently small, $x_0 \omega_0 \ll c$.

### III. EQUILIBRIUM OF OSCILLATORS IN BATHS

#### A. Optional Mechanical Bath

Nonrelativistic mechanics deals only with massive particles, and accordingly, the bath with which the mechanical oscillator can be regarded as interacting will be taken as composed of particles of mass $m$ moving in one dimension. The oscillator particle of mass $M$ is connected by point collisions to a bath of non-interacting particles of mass $m$ confined to a large one-dimensional box of length $L$ with an elastically reflecting wall. The distant end of the box provides the elastic-rebound wall, and the oscillator particle $M$ (attached to the spring and wall) provides the other end of the one-dimensional box. Then the point collisions of the bath particles with the oscillator of mass $M$ provides a means of transfer of energy and momentum between the bath particles and the oscillator system.

Point collisions of massive particles lead to “energy sharing” among all the interacting particles of the system. The work on kinetic theory carried out in the 19th century introduced the idea of kinetic energy equipartition among the particles. Furthermore, the mechanical harmonic oscillator of mass $M$ shares energy equally between its kinetic energy and its potential energy. Thus the “energy-sharing” idea extends to all the modes of the nonrelativistic mechanical oscillator coupled by point collisions to the bath of non-interacting particles. This energy-sharing idea is carried over into classical statistical mechanics and provides the basis for the Boltzmann distribution on phase space. The system can be described satisfactorily by classical statistical mechanics with the resulting Boltzmann probability distribution. We note that the energy-sharing concept has no particular role for
the frequency of any oscillator nor any limit on the (finite) number of parameters which
may enter the nonrelativistic mechanical system interacting with a finite number-density of
particles. Equilibrium will involve the preferred inertial frame of the confining box and the
average kinetic energy of a particle.

B. Required Electromagnetic Bath

For the electromagnetic oscillator, the bath is totally different from that of the non
relativistic mechanical situation. The mechanical oscillator can be imagined to oscillate
without friction and without any mechanical bath of particles. In contrast, the charge $e$ of
the electromagnetic oscillator may oscillate harmonically without mechanical friction, but
the oscillating charge must accelerate. And the accelerating oscillating charge emits radia-
tion at its natural frequency of oscillation $\omega_0$. According to classical theory, the oscillating
electromagnetic particle is always coupled to radiation. Thus within classical theory, equi-
librium for an electromagnetic oscillator requires the presence of a bath of electromagnetic
radiation.

Furthermore, electromagnetism is a relativistic theory. Although a bath of mechanical
particles with finite particle density can be described in terms of a finite number of wave
modes, the number of normal modes for relativistic waves is infinite. Thus an electromagnetic
oscillator is necessarily connected to electromagnetic radiation involving an infinite
number of wave modes. It is natural to ask, “What is the spectrum of the radiation
bath which is tied to a small (relativistic) motion of the charge $e$ of the one-dimensional
electromagnetic oscillator?”

IV. REVIEW OF THE ELECTROMAGNETIC RADIATION BATH IN DIPOLE
APPROXIMATION

A. Random Classical Radiation

An oscillator in a closed container with perfectly reflecting walls might be in equilibrium
with coherent radiation. The far more usual case is for the oscillator to come to equilibrium
with random radiation. The problem of an electromagnetic oscillator interacting with
random radiation is an old problem going back to Planck’s work at the end of the 19th century. Here we will first review the historical dipole approximation calculation before turning to the new aspects which are unfamiliar to most physicists.

Random classical radiation can be described as Planck described it at then end of the 19th century in terms of plane waves with random phases. If we consider a very large cubic box with sides of length $a$, then the random radiation can be written as

$$E(r, t) = \sum_{k, \lambda} \hat{\epsilon}(k, \lambda) \left( \frac{8\pi U_{\text{rad}}(\omega)}{a^3} \right)^{1/2} \{\exp [i k \cdot r - i\omega t + i\theta(k, \lambda)] + cc \}$$

(4)

$$B(r, t) = \sum_{k, \lambda} \hat{k} \times \hat{\epsilon}(k, \lambda) \left( \frac{8\pi U_{\text{rad}}(\omega)}{a^3} \right)^{1/2} \{\exp [i k \cdot r - i\omega t + i\theta(k, \lambda)] + cc \}$$

(5)

where the sum over the wave vectors $k = \hat{x}2\pi l/a + \hat{y}2\pi m/a + \hat{z}2\pi n/a$ involves integers $l, m, n = 0, \pm 1, \pm 2, ...$ running over all positive and negative values, there are two polarizations $\lambda = 1, 2$, and the random phases $\theta(k, \lambda)$ are distributed uniformly over the interval $(0, 2\pi)$, independently for each wave vector $k$ and polarization $\lambda$. The energy per normal mode at radiation frequency $\omega$ is given by $U_{\text{rad}}(\omega)$, and we have assumed that the radiation spectrum is isotropic.

**B. Oscillator Equation of Motion**

The one-dimensional electromagnetic oscillator is located at the origin and oscillates along the $x$-axis. For small displacements $x$, the oscillator motion satisfies Newton’s second law

$$M \ddot{x} = -M \omega_0 x + M \tau \dot{x} + e E_x(x, 0, 0, t),$$

(6)

where the time $\tau = 2e^2/(3Mc^3)$, and the $x$-component of the electric field follows from Eq. (4). In the electric dipole approximation corresponding to a point oscillator, the electric field $E_x$ is approximated as that located at the origin (center of the oscillator), $E_x(x, 0, 0, t) \approx E_x(0, 0, 0, t)$. Substituting from Eq. (4), we have a linear stochastic differential equation with the steady-state solution

$$x(t) = \frac{e}{M} \sum_{k, \lambda} \epsilon_x(k, \lambda) \left( \frac{8\pi U_{\text{rad}}(\omega)}{a^3} \right)^{1/2} \frac{1}{2} \left\{ \exp \left\{ i [k \cdot r - \omega t + \theta(k, \lambda)] \right\} \frac{1}{-\omega^2 + \omega_0^2 + i\tau\omega_0^3} + cc \right\}$$

(7)
The time derivative $\dot{x}(t)$ follows from Eq. (7). The average values can be obtained by averaging over the random phases as
\[
\langle\exp[i\theta(k, \lambda)]\exp[-i\theta(k', \lambda')]\rangle = \delta_{k,k'}\delta_{\lambda,\lambda'}.
\] (8)
Thus averaging over the random phases and then summing over the Kronecker delta, the mean-square displacement is
\[
\langle x^2(t) \rangle = \sum_{k,\lambda} \epsilon_x^2(k, \lambda) \left( \frac{8\pi U_{\text{rad}}(\omega)}{a^3} \right) \frac{e^2}{2m^2 \left[ (-\omega^2 + \omega_0^2)^2 + (\tau \omega^3)^2 \right]}.
\] (9)
Assuming that the box for the periodic boundary conditions becomes very large, the sum over the discrete values of $k$ can be replaced by an integral, $\sum_k \rightarrow \int d^3 k [a/(2\pi)]^3 = \int_0^\infty k^2 dk \int d\Omega [a/(2\pi)]^3$. The only angular dependence appears in the polarization vectors $\epsilon_x^2(k, \lambda)$, so that, on angular integration, each polarization will contribute a value of $1/3$.

We are left with the integration over $k$. If we assume that the charge $e$ is small so that the damping is small and the integrand is sharply peaked at $\omega_0$, then we may extend the lower limit of the integral to $-\infty$, and replace all frequencies $\omega$ by $\omega_0$, except where the combination $\omega - \omega_0$ appears. The remaining integral is of the form
\[
\int_{-\infty}^{\infty} \frac{du}{a^2 u^2 + b^2} = \frac{\pi}{ab}.
\] (10)
The mean-square displacement is
\[
\langle x^2 \rangle = \frac{U_{\text{rad}}(\omega_0)}{m\omega_0^2}.
\] (11)
An analogous procedure can be followed for the evaluation of $\langle \dot{x}^2 \rangle$. Then the average energy of the oscillator follows as
\[
U_{\text{osc}} = \left\langle \frac{1}{2} M \dot{x}^2 + \frac{1}{2} M\omega_0^2 x^2 \right\rangle = U_{\text{rad}}(\omega_0),
\] (12)
so that in equilibrium (for the small-charge approximation) the point dipole oscillator has the same average energy as the radiation modes at the same frequency as the oscillator. [4]

C. Phase Space Distribution for the Electromagnetic Oscillator

One can also obtain the averages for all products involving an arbitrary number of factors of $x(t)$ and $\dot{x}(t)$ by repeated use of the average in Eq. (8). Indeed, the averages
correspond to a probability distribution $P(x, p)$ for the displacement $x(t)$ and momentum $p(t) = M \dot{x}(t)$, for $-\infty < x < \infty$, $-\infty < p < \infty$, given by

\begin{equation}
P_{osc}(x, p) dx dp = \frac{1}{2\pi U_{osc}} \sqrt{\frac{\kappa}{M}} \exp \left[ -\frac{(\kappa x^2 + p^2/M)}{2U_{osc}} \right] dx dp. \quad (13)
\end{equation}

It is interesting to rewrite the probability distribution for the oscillator in random radiation in terms of action-angle variables using

\begin{align*}
x(t) &= \sqrt{\frac{2J}{m\omega_0}} \cos w \\
p(t) &= \sqrt{2mJ\omega_0} \cos w
\end{align*}

Then the phase space distribution for the electromagnetic oscillator, for $0 \leq w \leq 2\pi$, $0 \leq J < \infty$, is given by

\begin{equation}
P_{osc}(w, J) dwdJ = \frac{1}{2\pi} \frac{U_{osc}}{\omega_0} \exp \left[ -\frac{J\omega_0}{U_{osc}} \right] dwdJ. \quad (15)
\end{equation}

As shown in earlier work, the scattering of the random radiation by the point electric dipole oscillator does not change the frequency spectrum or the isotropic nature of the random radiation. We emphasize that the energy $U_{osc}$ of the point electric dipole oscillator matches the radiation energy $U_{rad}$ in the normal modes of the radiation field at the oscillator natural frequency $\omega_0$. However, this is merely a connection at one point $\omega = \omega_0$ of the spectral function $U_{rad}(\omega)$. There is nothing in this calculation which suggests any preferred spectrum for the random radiation in equilibrium with the electromagnetic oscillator in dipole approximation.

D. Historical Use of the Electromagnetic Oscillator in Dipole Approximation

It was Planck who introduced the electromagnetic oscillator taken in the dipole approximation in connection with the problem of blackbody radiation at the end of the 19th century. The oscillator was regarded as a connection between the thermodynamics of matter and the corresponding electromagnetic thermal radiation. Initially, Planck hoped that the scattering of radiation by the oscillator itself would force electromagnetic radiation into equilibrium and so reveal the blackbody radiation spectrum. However, Boltzmann pointed out that electromagnetic theory is invariant under time reversal so that Planck’s hope was empty. It was only when Planck became convinced that the dipole oscillator
would not change the frequencies of scattered radiation and so would not push a spectrum of random radiation towards equilibrium that Planck turned to statistical mechanical ideas for particles. Only then did Planck seek to obtain the thermodynamic behavior of the electromagnetic oscillator in order to determine the equilibrium spectrum of the associated thermal radiation.\[8\][9]

The historical use of the electromagnetic dipole oscillator still appears in the textbooks of modern physics.\[1\] Using the ideas of nonrelativistic classical statistical mechanics for the oscillator, the texts claim that a classical electromagnetic oscillator must assume the equipartition energy $U_{osc} = k_B T$ of a Newtonian mechanical oscillator, and therefore, by Planck’s calculation showing the equality of energy between the electromagnetic dipole oscillator and the radiation normal modes, the spectrum of thermal radiation $U_{\text{rad}}$ must be the Rayleigh-Jeans spectrum

$$U_{\text{rad}}(\omega, T) = U_{\text{RJ}}(\omega, T) = k_B T.$$

(16)

Although energy equipartition is strictly a result of nonrelativistic mechanics and has nothing to do with relativistic classical electromagnetism, it is claimed that this Rayleigh-Jeans result is the prediction of “classical physics.” And this prediction is made without any allowance for the conflict between nonrelativistic mechanics and relativistic electromagnetism.

This same result that nonrelativistic physics leads to the Rayleigh-Jeans radiation spectrum has been obtained by other researchers without using classical statistical mechanics but by using nonrelativistic nonlinear scattering systems which are connected to electromagnetic radiation by the dipole radiation approximation for the mechanical systems.\[10\]

Whereas the analysis of the modern physics texts uses nonrelativistic classical statistical mechanics, all of these scattering calculation use nonrelativistic nonlinear potential systems for the radiation scatterers. All of these derivations give a false result because (despite any claims to being relativistic calculations) they all exclude Coulomb potentials and involve a nonrelativistic basis for the attainment of equilibrium.
V. RELATIVISTIC RADIATION EQUILIBRIUM

A. Few Relativistic Scattering Systems

The use of nonrelativistic mechanical systems in connection with electromagnetic radiation equilibrium persists because there are very few relativistic scattering systems in nature which are familiar to physicists and which allow tractable analytic calculations. Indeed, only the system of a point charge in a Coulomb potential when coupled to electromagnetic radiation can be regarded as fully relativistic. And analytic treatment of the Coulomb-potential system seems extraordinarily difficult. To date, only numerical simulations have provided some insight into the Coulomb system in radiation, and then only in the nonrelativistic approximation for the particle motion. The one other system which can be regarded as approximately relativistic is that of a point charge in a one-dimensional harmonic potential when the amplitude of oscillation is taken as very small so that the particle velocity is very small, \( v << c \).

B. Analysis of the Electromagnetic Oscillator Beyond the Dipole Approximation

Although the classical electromagnetic calculation for the point electric dipole oscillator in random radiation has been presented repeatedly for a century, the possibility of going beyond the dipole approximation was emphasized only recently in work by Huang and Batelaan who considered the absorption of a radiation pulse by an oscillator of non-zero amplitude. We will not consider the absorption of a radiation pulse, but, inspired by the work of Huang and Batelaan, we wish to continue the analysis of the equilibrium between an electromagnetic harmonic oscillator and a radiation bath by considering the higher harmonics. We note that by taking the amplitude of oscillation as small enough, the motion of the oscillating charge can be regarded as relativistic to any degree of approximation, \( v << c \), and yet, provided that the amplitude of motion is non-zero, there is always non-zero radiation at harmonics.

Thus if the amplitude of oscillation of the electromagnetic oscillator is not zero, then the harmonic oscillator of natural frequency \( \omega_0 \) must be emitting quadrupole radiation at frequency \( 2\omega_0 \). But here indeed is an opportunity to determine the equilibrium spectrum of a very small electromagnetic oscillator. There must be energy in the random radiation spectrum at frequency \( 2\omega_0 \) which is absorbed by the oscillator and balances the energy radi-
ated as quadrupole radiation at $2\omega_0$. And indeed, the argument can be repeated for all the higher radiation multipoles of the oscillating charge $e$. The situation here is totally different from the earlier scattering calculations [10], all of which involved nonrelativistic non-linear oscillators scattering electromagnetic radiation in the dipole approximation, and all of which led to the Rayleigh-Jeans spectrum as the equilibrium spectrum of random radiation. Here for an electromagnetic oscillator in a purely harmonic potential, the equilibrium spectrum for the electromagnetic oscillator is not determined by any assumption about the mechanical structure of the oscillator, but rather the spectrum is determined by purely electromagnetic considerations. This is the first scattering calculation for electromagnetic radiation which can be regarded as fully relativistic [2].

The change from the earlier work involves the equation of motion for the oscillator given in Eq. (6), but now not taking the dipole approximation $E_x(x,0,0,t) \approx E_x(0,0,0,t)$, but rather, for the quadrupole term, making the next approximation

$$E_x(x,0,0,t) \approx E_x(0,0,0,t) + x(t) \left( \frac{d}{dx'} E_x(x',0,0,t) \right)_{x'=0}$$

where on the right-hand side we will insert for $x(t)$ the dipole-approximation result appearing in Eq. (7). Indeed, the whole idea is to make successive approximations in the small amplitude of oscillator motion. Thus now for the quadrupole radiation approximation, we write the equation of motion for the oscillating charge $e$ as

$$M\ddot{x} \approx -M\omega_0^2 x^2 + M\tau \ddot{x} + eE_x(0,0,0,t) + x^{(0)}(t) \left( \frac{d}{dx'} E_x(x',0,0,t) \right)_{x'=0}$$

where $x^{(0)}(t)$ is the dipole approximation appearing in Eq. (7). We notice that whereas the equation of motion for the electromagnetic oscillator in the dipole approximation involved one factor of $\exp[-i\omega t + i\theta(k,\lambda)]$ on the right-hand side there are now two factors of this form in Eq. (18).

One can solve [2] the equation of motion (18), and one finds that the scattered radiation preserves the frequency spectrum and the angular distribution of isotropic radiation provided that the spectrum of random radiation has twice as much energy per normal mode at frequency $2\omega_0$ compared to the energy per normal mode at frequency $\omega_0$. It is easy to see where this analysis is going. The electromagnetic oscillator in one dimension is in equilibrium only with a radiation bath $U_{rad}(n\omega_0) = \text{const} \times n\omega_0$ for all multiples $n$ of the fundamental oscillator frequency $\omega_0$. Thus detailed balance for radiation energy holds
provided that the spectrum of random radiation is linear in frequency

\[ U_{\text{rad}}(\omega) = \text{const} \times \omega. \]  

This situation corresponds to Lorentz invariance for the spectrum. Indeed, it seems natural that the Lorentz-invariant theory of classical electrodynamics should pick out a Lorentz-invariant spectrum of random radiation for the electromagnetic oscillator which has no structure other than the characteristic frequency \( \omega_0 \).

VI. ADIABATIC CHANGE AND TEMPERATURE

A. Adiabatic Change of the Mechanical Spring Constant

Adiabatic changes in the oscillator frequency for a Newtonian mechanical oscillator in equilibrium through point collisions with a particle bath show that the temperature of the bath can never be zero unless the total system energy vanishes. If the spring constant \( \kappa \) of the oscillator is readjusted very slowly, then the external agent which provides the readjustment is carrying out an adiabatic change of the system. If we imagine that the collision interactions between the oscillator and the bath particles are removed during the adiabatic change of the oscillator, then the ratio of the energy \( U_{\text{osc}} \) to the oscillator (angular) frequency \( \omega_0 \) is an adiabatic invariant, \( U_{\text{osc}}/\omega_0 = J_{\text{osc}} \); where \( J_{\text{osc}} \) is the action variable in the action-angle treatment of the mechanics of the oscillator, with \( U_{\text{osc}} = J_{\text{osc}}\omega_0 \). Thus during the adiabatic readjustment of the spring constant, the probability distribution of the action variable \( P(J_{\text{osc}}) \) is unchanged, but the average energy of the oscillator does indeed change since \( U_{\text{osc}} = J_{\text{osc}}\omega_0 \), and \( \omega_0 \) has changed.

If the mechanical oscillator is now reconnected to the bath of particles, the oscillator will no longer be in equilibrium with the bath because its average energy per normal mode has been changed. Rather, energy will be exchanged between the oscillator and the bath by means of point collisions, and a new equilibrium will be established with a different total energy for the combined system of the oscillator and the bath. Since the adiabatic change is not the same as a isothermal change, the temperature of the oscillator and its particle bath must be regarded as non-zero. Thus provided that the mechanical motion of the oscillator and its surrounding bath particles has not actually ceased, the oscillator-bath system must be regarded as at equilibrium at some positive temperature determined by the
average kinetic energy of any of the particles. There is no such thing as particle motion at the absolute zero of temperature in nonrelativistic classical mechanics.

B. Adiabatic Change of the Frequency of the Electromagnetic Oscillator

In contrast with the Newtonian mechanical oscillator, an adiabatic change in the oscillator frequency of a one-dimensional electromagnetic oscillator in equilibrium with random radiation shows that the one-dimensional-electromagnetic-oscillator-radiation system has zero temperature.

Electromagnetism depends crucially on frequency-dependent resonances whereas point collisions do not have any associated frequency. Thus the frequency of the Newtonian mechanical oscillator is largely irrelevant in connection with the particle bath. However, the frequency of the electromagnetic oscillator determines the radiation frequencies with which the electromagnetic oscillator will interact. If we imagine the electromagnetic oscillator temporarily uncoupled from its radiation and the separation \( a \) between the confining charges \( q \) as changed, then the adiabatic change of the oscillation frequency \( \omega_0 \) appearing in Eq. (3) leads to a change in energy \( \Delta U_{osc} = J_{osc} \Delta \omega_0 \) of the oscillator while keeping the value of the action variable \( J_{osc} \) fixed. For an ensemble of oscillators, the phase space distribution involving \( J_{osc} \) is unchanged under the adiabatic change in frequency, but the average oscillator energy is still linear in frequency with the same phase space distribution as given in Eq. (15) since the ratio of the oscillator energy to the oscillator frequency has not changed. But now when the electromagnetic oscillator is reconnected with the initial Lorentz-invariant spectrum of random radiation, there is no transfer of average energy between the oscillator and the bath of electromagnetic radiation. Thus an adiabatic change is the same as an isothermal change for this situation, and we would describe the one-dimensional electromagnetic oscillator as in equilibrium with its radiation bath at the absolute zero of temperature, \( T = 0 \).

We emphasize that the energy introduced in the adiabatic change went into the energy of the oscillator, and (on average) no energy was transferred to the energy-divergent radiation field. There is “apparent decoupling” between the electromagnetic oscillator and its equilibrium radiation bath. Thus in a sense the electromagnetic oscillator acts as though it were decoupled from the radiation field even though the radiation field is required for the stability
of the oscillating charge in classical electromagnetism. In this connection, it is noteworthy that textbooks of thermodynamics and statistical mechanics will often introduce oscillators and electromagnetic systems without worrying about any divergent radiation bath which would be required for their stability within classical physics.

VII. CLASSICAL ELECTROMAGNETIC ZERO-POINT RADIATION

A. Unique Lorentz-Invariant Spectrum

In nature, oscillating charges can not each have their own Lorentz-invariant spectrum of random radiation. Thus classical electromagnetic theory demands that in order to have oscillating electric charges, there must be one Lorentz-invariant spectrum of random classical electromagnetic radiation. This radiation spectrum is usually termed classical electromagnetic zero-point radiation. And within a purely classical electromagnetic description of nature, there is good experimental evidence for classical electromagnetic zero-point radiation with an energy per radiation normal mode

$$U_{\text{rad}}(\omega) = \frac{1}{2} \hbar \omega,$$

(20)

where the constant \( \hbar \) takes the same value as is given for Planck’s constant \( \hbar = h/(2\pi) \).

The existence of a spectrum of classical electromagnetic zero-point radiation provides the basis for classical explanations of Casimir forces, van der Waals forces, the specific heats of solids, diamagnetism, and the absence of atomic collapse.

B. Phase Space of Classical Zero-Point Radiation

An electromagnetic oscillator in zero-point radiation takes on the same phase space distribution as is found for the radiation normal modes. Both the electromagnetic oscillator and all the modes of the electromagnetic radiation field have the distribution on phase space following from Eq. (15) for \( 0 \leq \omega \leq 2\pi, 0 \leq J < \infty \),

$$P(\omega, J) \, d\omega \, dJ = \frac{1}{2\pi} \frac{\hbar}{2} \exp \left[ -\frac{J}{\hbar/2} \right] \, d\omega \, dJ,$$

(21)

which does not depend upon the frequency of the oscillator or of the radiation mode. On adiabatic change of the frequency of a purely-harmonic oscillator, the phase space distribution is unchanged and remains in equilibrium with the zero-point radiation spectrum.
C. Spectrum of “Least Possible Information”

The Lorentz-invariant spectrum of zero-point radiation is the “radiation spectrum of least possible information in a relativistic theory.” The zero-point radiation spectrum involves one overall constant $\hbar$ and takes the same isotropic form in every inertial frame. On the other hand, the Rayleigh-Jeans spectrum contains more information than the Lorentz-invariant zero-point radiation spectrum. Thus in addition to one over-all parameter $k_B T$, the Rayleigh-Jeans spectrum has a preferred inertial frame in which the simple Rayleigh-Jeans form in Eq. (16) appears; any other inertial frame moving with constant velocity relative to the preferred frame will find a non-isotropic spectrum of radiation. Nonrelativistic nonlinear oscillators scatter radiation so as to impose their own inertial frame as the preferred inertial frame of the random radiation spectrum to which they are attached. In contrast, the calculation showing that the one-dimensional electromagnetic oscillator in a purely-harmonic potential is in equilibrium with a Lorentz-invariant radiation spectrum indeed suggests that the purely-harmonic oscillator is not imposing its own inertial frame on the equilibrium radiation spectrum upon which it depends. Indeed, the electromagnetic oscillator based upon the electrostatic consideration leading up to Eq. (3) will have relativistic energy transformation properties between inertial frames provided the the particle velocity is small in the oscillator frame. On the other hand, the potential energy of a nonrelativistic nonlinear oscillator will not transform in a relativistic fashion.

In addition to providing a Lorentz-invariant radiation spectrum in Minkowski spacetime, zero-point radiation allows incorporation into general relativity where it again corresponds to the “radiation spectrum of least possible information in a relativistic theory.” In the general relativistic situation, the correlation functions for zero-point radiation between spacetime points depend upon only the geodesic separations between the spacetime points.

VIII. LIMITATIONS OF THE ANALYSIS

The one-dimensional electromagnetic oscillator in a purely-harmonic potential scatters radiation so as to push the radiation toward its equilibrium spectrum which is a Lorentz-invariant spectrum. This is indeed consistent with classical electrodynamics which is a Lorentz-invariant theory. However, zero-point radiation must be interpreted as corre-
sponding to a temperature of absolute zero, and the scattering analysis does not suggest any equilibrium at a thermal spectrum with non-zero temperature. Also, we note that the one-dimensional oscillator shows no velocity-dependent damping, which must exist for thermal radiation at non-zero temperature. Indeed the one-dimensional analysis allows no role for the magnetic field of the random radiation which would cause forces perpendicular to the direction of oscillation which are eliminated by the constraint giving one-dimensional motion. Finally, the one-dimensional oscillator is not a purely electromagnetic system because of the non-electromagnetic constraint which gives one-dimensional motion. Indeed, Earnshaw’s theory tells us that a stable purely-harmonic potential cannot be obtained from interacting point charges.

It is conjectured that all of these considerations are related. It is suggested that velocity-dependent damping in random radiation is possible only if the charged particle can oscillate (in a sort of zitterbewegung) in the direction perpendicular to the direction of damping. Indeed, the Einstein-Hopf analysis\[20\] for the motion of a classical particle in thermal radiation involves a dipole oscillator, internal to the particle, which oscillates in a direction perpendicular to the direction of the particle's constrained one-dimension motion. Thus it is conjectured that the complete understanding of the thermal radiation spectrum in terms of scattering by a purely classical system must wait until there is a classical understanding of something like the classical hydrogen atom. While awaiting such a classical scattering analysis, one is encouraged by many points of view that suggest that thermal radiation can indeed be understood in terms of classical physics.\[9\]

IX. CONCLUDING COMMENTS

The biggest shock to undergraduate physics students is their encounter with the unusual ideas of quantum theory. However, they are often not confronted with just how different the ideas of classical electrodynamics are from the every-day experiences of Newtonian mechanics. Like the physicists of the early 20th century, contemporary physicists still do not appreciated the conflict between Newtonian mechanics and electromagnetism. Furthermore, the textbooks of modern physics continue to mislead physics students regarding this matter by insisting that the energy-sharing ideas of nonrelativistic classical mechanics are to be carried over to the relativistic theory of classical electromagnetic radiation equilibrium.
Newtonian mechanics and classical electrodynamics are vastly different in their assumptions and in their descriptions of nature. A nonrelativistic mechanical harmonic oscillator can be assumed to have no friction, and so will oscillate indefinitely with no need for any bath associated with the equilibrium situation. In addition, nonrelativistic mechanical systems of particles come to equilibrium by energy-sharing throughout the mechanical system. This energy-sharing idea appears in the equipartition theorem and in the Boltzmann distribution which are incorporated into nonrelativistic classical statistical mechanics.

In contrast with the mechanical situation, an electric charge is always connected to electromagnetic fields. If the electric charge oscillates, then the charge must exchange energy with the radiation field. There is no such thing as a classical electromagnetic oscillator which is not coupled to electromagnetic radiation. Furthermore, a relativistic field theory will involve an infinite number of radiation normal modes. Thus the nonrelativistic idea of energy-sharing across the entire system has no place in electromagnetism. Rather, resonance phenomena associated with frequencies appear crucially in electromagnetism, whereas such frequency-dependent resonances are of no significance in nonrelativistic mechanical equilibrium.

It is very hard to find relativistic electromagnetic systems allowing easy calculations involving radiation equilibrium. In this article, we have considered the simplest relativistic classical electromagnetic system which will lead to a preferred equilibrium radiation spectrum. The oscillation of an electric charge in a one-dimensional purely-harmonic potential leads to a required equilibrium radiation spectrum, not through its mechanical motion which is always simply harmonic, but through the electromagnetic aspects of the multipole radiation which arise in classical electromagnetism. One finds that the one-dimensional electromagnetic oscillator is in equilibrium with a spectrum whose normal mode energy $U_{rad}$ is linear in frequency $\omega$, $U_{rad} = \text{const} \times \omega$, corresponding to a Lorentz-invariant radiation spectrum. Indeed, the relativistic nature of the equilibrium spectrum fits with the relativistic nature of classical electromagnetism.

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[6] See, for example, H. Goldstein, Classical Mechanics 2nd edn, (Addison-Wesley, Reading, MA 1981), p. 462. We have rewritten the expressions by factors of $2\pi$, corresponding to the transition from $\hbar$ over to $\hbar$.

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