Softener than normal, but not as soft as one might think: Spontaneous flux lattices in ferromagnetic spin-triplet superconductors

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A theory is developed for the spontaneous vortex lattice that is expected to occur in the ferromagnetic superconductors ZrZn\textsubscript{2}, UGe\textsubscript{2}, and URhGe, where the superconductivity is likely of spin-triplet nature. The long-wavelength fluctuations of this spontaneous flux lattice are predicted to be huge compared to those of a conventional flux lattice, and to be the same as those for spin-singlet ferromagnetic superconductors. It is shown that these fluctuations lead to unambiguous experimental signatures which may provide the easiest way to observe the spontaneous flux lattice.

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Conventional wisdom holds that ferromagnetism and superconductivity cannot coexist \cite{1, 2}. However, recent experimental studies of UGe\textsubscript{2} \cite{3, 4}, URhGe \cite{5}, and ZrZn\textsubscript{2} \cite{6}, backed by band structure calculations \cite{7, 8}, have demonstrated coexistence of both types of order in the same electron band in these materials. Theoretical considerations suggest that the superconducting order parameter is the spin-triplet non-unitary type \cite{9}.

A state which displays coexistence of ferromagnetism and superconductivity is expected to have many unusual features, among them a spontaneous vortex or flux lattice. The internal magnetic field generated by the spontaneous magnetization makes topological excitations, viz. vortices, in the superconducting order parameter energetically favorable \cite{10}. Unlike the well-known Abrikosov flux lattice \cite{11}, this spontaneous flux lattice state requires no external magnetic field. For spin-singlet superconductors, such spontaneous flux lattices have been proposed and theoretically studied previously \cite{12, 13, 14}.

This Letter addresses the heretofore unstudied problem of spontaneous flux lattices in spin-triplet p-wave superconductors. One might expect this problem to be much more complicated than the spin-singlet case, since the order parameter is much more complicated, which should lead to many more soft modes. One of our chief conclusions is that, surprisingly, this is not the case. Rather, the low-energy elastic properties of any hexagonal \cite{15} spontaneous spin-triplet vortex lattice, regardless of the precise superconducting order parameter symmetry, map onto those of the corresponding spin-singlet problem. The reason is that the more complicated order parameter, while allowing for more modes, also allows for additional couplings among them, which renders the additional modes massive. Our second main conclusion is that the very unusual elastic properties of the spontaneous vortex lattice have easily observable consequences. Specifically, we predict that in ultraclean samples, the magnetic induction $B$ depends nonanalytically on an external magnetic field $H$, namely, $B(H) = \mu H + c H^{3/2}$, with $\mu$ the (linear) magnetic permeability, and $c$ a constant. The $H^{3/2}$ term is the leading nonanalyticity. In samples with quenched disorder strong enough to dominate the elastic properties of the flux lattice, but not strong enough to destroy either the superconductivity or the flux lattice, the nonanalyticity is the leading term as $H \to 0$, and given by $B(H) \propto H^\alpha$ with $\alpha \approx 0.72$. This is the same result as in the spin-singlet case \cite{14}. For very small magnetic fields either nonanalyticity is cut off by the lattice, which breaks the spatial rotational invariance, and the disorder-induced nonanalyticity is cut off at high fields by a field scale, or a corresponding length scale, that bounds the nonlinear elasticity regime. These results imply the $dB/dH$ versus $H$ curves shown in Fig.1.

To derive these results, we start from a Landau-Ginzburg-Wilson (LGW) functional that allows for both ferromagnetic and spin-triplet superconducting order. The superconducting order parameter is a matrix in spin space \cite{16}, $\Delta_{\alpha\beta}(k) = \sum_{\mu=1}^{3} d_{\mu}(k)(\sigma_{\mu}\sigma_{2})_{\alpha\beta}$. Here $\alpha$, $\beta$ are spin indices, $k$ is the wave vector, and $\sigma_{1,2,3}$ are

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Schematic representation of the predicted nonanalytic behavior of $B(H)$ for clean (a) and disordered (b) systems. $H_{cf}$ is the crystal-field scale that cuts of the nonanalyticity at small fields, and $H_{nl}$ is an upper cutoff scale determined by the disorder. See the text for additional information.}
\end{figure}
the Pauli matrices. This relation expresses the isomorphism between the complex symmetric matrix $\Delta$ and the complex 3-vector $d(k)$. We will consider p-wave symmetry, which implies $d_{\mu}(k) = \sum_{j=1}^{3} d_{\mu j} k_j$. The tensor field $d_{\mu j}(x)$ is the appropriate order parameter for either He 3 or a p-wave superconductor. $d_{\mu j}$ is characterized by 18 real numbers, which explains the rich phenomenology of order parameter textures observed in He 3\cite{16}. The superconducting part of the LGW functional contains all scalars that can be constructed from $d_{\mu j}$ and the covariant derivative $D_{\mu} = \partial_{\mu} - ig A_{\mu}(x)$ ($g = 2e/\hbar c$), where the $A_{\mu}$ are the components of the fluctuating vector potential $A$, and indices in spin space and real space, respectively, must be contracted among themselves\cite{16}. Up to bi-quadratic order in covariant gradients and tensor fields\cite{17} one finds (with summation over repeated indices implied)

$$S_{\text{sc}} = \int dx \left[ c_d^{(1)} (D_{\mu} d_{\mu})^2 + c_d^{(2)} (D_{\mu} d_{\mu})^3 + c_d^{(3)} (D_{\mu} d_{\mu})^4 + u_d d_{\mu j} d_{\nu j} d_{\mu i} d_{\nu i} + u_d d_{\mu j} d_{\nu j} d_{\mu i} d_{\nu i} + u_d d_{\mu j} d_{\nu j} d_{\mu i} d_{\nu i} + u_d d_{\mu j} d_{\nu j} d_{\mu i} d_{\nu i} \right]. \quad (1a)$$

The $c_d$, $t_d$, and $u_d$ are the coefficients of the LGW theory. To describe a ferromagnetic superconductor we need, in addition to $d_{\mu j}(x)$, a real vector field in spin space, $M(x)$, which serves as the magnetic order parameter. The magnetic part of the LGW functional takes the form of a $\varphi^4$-theory for the fluctuating magnetization, and terms for the magnetic field energy and the coupling between $M$ and $A$\cite{18},

$$S_{\text{fm}} = \int dx \left[ c_m (\nabla M)^2 + t_m M^2 + u_m (M^2)^2 \right], \quad (1b)$$

Finally, there is a direct coupling between the magnetic and superconducting order parameters (in addition to the indirect one via the vector potential). Considerations analogous to those that lead to Eq.\cite{13} show that up to quartic order in the fields there are two such terms,

$$S_{\text{fm-sc}} = \int dx \left[ -ig_1 \epsilon_{\mu \nu \lambda} M_{\mu} d_{\nu i} d_{\lambda i} + g_2 M_{\mu} M_{\nu} d_{\nu i} d_{\mu i} \right], \quad (1c)$$

where $g_1$ and $g_2$ are two additional real coupling constants and $\epsilon_{\mu \nu \lambda}$ is the Levi-Civita tensor. The complete LGW action is the sum of the terms in Eqs.\cite{11}.

This action is much more complicated than that for an s-wave superconductor. Nevertheless, the number of soft modes, and their long-wavelength effective action, are identical to those of the spin-singlet case. In particular, the spontaneous flux lattice has the same long-wavelength properties. This surprising result, which is the basis for all of the predictions in this Letter, is independent of the exact nature of the spin-triplet superconducting phase. In what follows, we first give a heuristic argument, and then a formal symmetry argument.

To make the heuristic argument more transparent, let us consider a superconducting order parameter for the simplest non-unitary state, the so-called $\beta$-state\cite{16}. (Analogous arguments can be made for other states.) It is given by a tensor product $d = \psi \otimes \phi$ of a complex vector $\psi$ in spin space, and a real unit vector $\phi$ in orbital space, and the ground-state order parameter is given by $\psi = \Delta_0 (1, i, 0), \phi = (0, 0, 1)$. In terms of $\psi$ and $\phi$, the parts $S_{\text{sc}}$ and $S_{\text{fm-sc}}$ of the action read

$$S_{\text{sc}} = \int dx \left[ t_{\psi} |\psi|^2 + u_{\psi} |\psi|^4 + v_{\psi} |\psi \times \psi|^2 + c_{\psi} |D\psi|^2 \right], \quad (2a)$$

$$S_{\text{fm-sc}} = \int dx \left[ -ig_1 M (\psi \times \psi^*) + g_2 |M \cdot \psi|^2 \right]. \quad (2b)$$

The coefficients are simply related to those in Eqs.\cite{11}.

Compared to a spin-singlet s-wave superconductor, with a complex scalar order parameter field, we have more order parameter components, and also additional coupling terms. It turns out that the effects of these two complications effectively cancel each other, since the additional couplings generate masses for the extra degrees of freedom, leaving the number of massless degrees of freedom, and their long-wavelength Hamiltonian, identical to that for the spontaneous flux lattice in the spin-singlet case.

This can be seen as follows. We consider a region in parameter space where the ground state of the LGW functional, Eqs.\cite{2, 13} is a vortex lattice\cite{10}, i.e., a lattice of parallel lines which each represent a topological “winding” singularity of an overall phase associated with the complex vector $\psi$. As in spin-singlet superconductors, the gauge couplings implicit in the covariant derivatives in Eqs.\cite{2} force $B = \nabla \times A$ to run along these vortex lines. Physically, this is the Meissner effect. Fluctuations in the direction of $B$ are therefore massive. The $B \cdot M$
coupling in Eq. \(1d\) then fixes the direction of \(M\) to also be parallel to the flux lines, up to massive fluctuations, and the \(M \cdot (\psi \times \psi^*)\) term in Eq. \(2a\) fixes the direction and phases of \(\psi\), up to one free (Goldstone) phase whose vortex singularities are the flux lines. Finally, the direction of \(\phi\) is fixed by the gauge couplings in Eq. \(2a\), which force \(\phi\) to also run along the flux lines. Hence, the entire order parameter structure is determined (up to massive fluctuations) by the positions of the vortex lines. The only Goldstone modes in the system are thus the two-component positional fluctuations \(u\) of the flux lines relative to some perfect reference lattice of parallel lines with an arbitrary orientation.

A more general conclusion is reached by formal symmetry arguments. The superconducting part of the action, Eq. \(1a\), is invariant under a group \(\text{SO}(3) \times \text{SO}(3) \times U(1)\) of rotations in spin space, rotations in real space, and gauge transformations \(1c\). The couplings in Eq. \(1c\) are invariant under \(\text{SO}(3)\) co-rotations of the spin part of the tensor \(d_{ij}\) and the vector \(M\), which expresses the fact that both are elements of the same spin space. The magnetic part of the action, Eq. \(1b\), is invariant only under co-rotations in spin space and in real space, so the entire action is invariant under a group \(\text{SO}(3) \times U(1)\). In a ferromagnetic superconducting state this symmetry is spontaneously broken to \(SO(2)\), and the number of Goldstone modes is given by \(\text{dim}(\text{SO}(3) \times U(1)/\text{SO}(2)) = 3\) \(19\). These are, two spin-wave-like modes due to the broken spin rotation symmetry, and one Anderson-Bogoliubov mode due to the broken gauge symmetry. The latter is rendered massless by means of the Higgs mechanism, and the former can be eliminated in favor of vortex-lattice degrees of freedom as shown in Ref. \(14\). Notice that, although the triplet superconducting action is invariant under a much larger symmetry group than its singlet analog, the symmetry properties of the full action, and hence the number of Goldstone modes, are the same as in the singlet case, in agreement with the heuristic arguments given above. This symmetry argument is not tied to a particular superconducting ground state.

We have corroborated the general arguments given above by expanding the action to Gaussian order about both the \(\beta\)-state, and an order parameter appropriate for the \(A_1\)-phase in \(\text{He}\ 3\), which is another nonunitary state. The number and nature of the soft modes found is consistent with the general arguments given above.

The final effective action for any Heisenberg ferromagnetic \(p\)-wave superconductor \(19\) is therefore the same as for a ferromagnetic \(s\)-wave spin-singlet superconductor. It describes an Anderson-Bogoliubov mode by means of an \(O(2)\) nonlinear sigma model for a phase \(\theta\), generalized spin waves by means of an \(O(3)\) nonlinear sigma model for a unit \(3\)-vector \(\varphi\), and a coupling between the two by means of the gauge field \(A\). With LGW coefficients \(c\) and \(\tilde{a}\) one thus has

\[
S_{\text{eff}} = \int dx \, \left[ \frac{-\lambda_0^2}{2} \left( \nabla \theta(x) - q A(x) \right)^2 + \frac{\tilde{a} M_0^2}{2} \left( \nabla \varphi(x) \right)^2 \right. \\
\left. + \frac{1}{8\pi} \left( \nabla \times A(x) \right)^2 - M_0 \varphi(x) \cdot \left( \nabla \times A(x) \right) \right].
\]

(3)

with \(\Delta_0\) and \(M_0\) the average amplitudes of the superconducting and magnetic order parameters, respectively.

We now look for saddle-point solutions of this action that take the form of vortices \(\mathbf{10}\), i.e., where the superfluid velocity \(\mathbf{v} = \nabla \theta\) obeys the condition

\[
\frac{\nabla \times \mathbf{v}(x)}{2\pi} = \sum_n \int d\tau \, \frac{dr_n}{d\tau} \delta(x - r_n(\tau)) \equiv \mathbf{t}(x).
\]

(4)

Here \(r_n(\tau)\) is a parameterized line in \(\mathbb{R}^3\) representing the \(n\)th vortex line. By minimizing the action with respect to \(\theta(x)\) and \(A(x)\) we can express the saddle-point action in terms of the vortex line degrees of freedom \(t\) coupled to \(\varphi\) \(14\ 20\).

\[
S_{\text{eff}}^{(0)} = \frac{\pi}{2q^2} \int dx \, dy \, V(x - y) \mathbf{t}(x) \cdot \mathbf{t}(y) \\
- \frac{2\pi M_0}{q} \int dx \, dy \, V(x - y) \mathbf{t}(x) \cdot \varphi(y) \\
+ \frac{a M_0^2}{2} \int dx \left( \nabla \varphi(x) \right)^2 + 2 \pi M_0^2 \lambda^2 \int dx \left( \nabla \cdot \varphi(x) \right)^2.
\]

(5)

Here \(\lambda = 1/\sqrt{4\pi cq^2 \Delta_0^2}\) is the London penetration length, \(a = \tilde{a} - 4\pi \lambda^2\), and \(V(x) = (1/4\pi \lambda^2 |x|) \exp(-|x|/\lambda)\) is a screened Coulomb potential.

The action given by Eq. \(5\) can be analyzed as explained in Ref. \(14\). The equilibrium state is a hexagonal vortex lattice \(13\) described by two-dimensional lattice vectors \(R_n = (X_n, Y_n)\). Fluctuations of the vortex lattice are described by a two-dimensional displacement field \(u(R_n, z)\) such that the vortex lines are given by

\[
\mathbf{r}_n(z) = (X_n + u_x(R_n, z), Y_n + u_y(R_n, z), z), \tag{6}
\]

where we use \(z\) as the parameter of the line. After integrating out the generalized spin waves, the fluctuation action to second order in the strain tensor \(21\)

\[
u_{ij}(x) = \frac{1}{2} \left[ \partial_i u_j(x) + \partial_j u_i(x) - \partial_i u_i(x) \partial_j u_j(x) \right]
\]

reads

\[
S_{\text{fluc}} = \frac{1}{2} \int dx \left[ \kappa \left( \partial_i u_i \right)^2 + 2 \mu u_{ij} u_{ij} + \lambda (u_{ii})^2 \right], \tag{8}
\]

where the elastic constants \(\kappa, \mu\) (not to be confused with the magnetic permeability in Fig. \(1\)), and \(\lambda\) (not to be confused with the penetration depth) can be expressed in terms of the coefficients of the LGW action, Eq. \(4\).

Unlike the flux lines in a conventional Abrikosov flux lattice, which are induced by an external magnetic field, the
flux lines in this system are spontaneously generated, and therefore do not have a preferred direction. As described for the singlet case in Ref. [14] this rotational invariance leads to an additional softness in the elasticity of the spontaneous flux lattice, due to the absence of the usual “tilt energy” term proportional to $(\partial_z u)^2$. This feature leads to anisotropic Gaussian $u$-propagators, where $k_x^2$ scales as $k_1^2$, with $k = (k_1, k_2)$ the wave vector. Power counting shows that the Gaussian theory is stable for all dimensions $d > 5/2$. This argument neglects rotational symmetry breaking by crystal fields, which provide a long-wavelength cutoff to the applicability of our theory, as illustrated in Fig. 1.

The leading corrections to the Gaussian action are terms of the structure $\partial_\perp u (\partial_\perp u)^2$ and $(\partial_z u)^4$, respectively. They lead to a least irrelevant operator with scale dimension $-(d-5/2)$, or to a wave vector dependence of the elastic constant $\mu(k_1 = 0, k_2) = \mu(1 + \text{const} \times k^{2d-5})$. By arguments analogous to those given in Ref. [14] this leads to a nonanalytic strain-stress relation, and finally to the nonanalytic dependence of $B$ on $H$ discussed above in the context of Fig. 1(a).

Even more unusual elastic properties result from the presence of quenched disorder. It was shown in Ref. [14] that ordinary impurities lead to a random-field term in $S_{\text{Bac}}$ that couples linearly to $\partial_z u$. As a result of the strong random-field effects, the Gaussian theory becomes unstable for all dimensions $d < 7/2$. A renormalization-group analysis showed that the elastic constants, as well as the variance of the random field, become singular functions of the wave number, and the corresponding exponents have been calculated to first order in an expansion in powers of $\epsilon = 7/2 - d$. The resulting non-Hookian, or nonlinear, elastic properties of the vortex lattice extend up to a length scale $\xi_\text{al} = k^2/\nu$, where $\nu$ is the variance of the disorder distribution. They lead to the predictions shown in Fig. 1(b), which provide a way to experimentally observe the spontaneous flux lattice.

An important question is whether there is a parameter range where the disorder is strong enough to lead to observable anomalous elastic effects, but not so strong as to destroy the $p$-wave superconductivity, which is very disorder sensitive. A necessary condition is $\xi \ll \ell \ll a$, with $\xi$ the superconducting coherence length, $\ell$ the mean-free path, and $a$ the flux lattice constant. Let us consider ZrZn$_2$ [22], where $\xi \approx 290\AA$. From the normal-state residual resistivity, $\rho = 0.62\mu\Omega\text{cm}$, we estimate $\ell \approx 600 - 1000\AA \gg \xi$. The system is thus sufficiently clean to sustain $p$-wave superconductivity. On the other hand, from the value of the ordered moment, $\mu_s = 0.17\mu_B$ per formula unit [22], which gives rise to a relatively small spontaneous magnetic induction, $B \approx 0.03\ T$, one estimates $a \approx \sqrt{\phi_0/B} \approx 2,500\AA$. Here, $\phi_0$ is the flux-quantum. The above condition is thus fulfilled in ZrZn$_2$.

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[17] It is easy to verify that keeping terms of higher than quartic power in either order parameter in Eqs. [2] does not affect our conclusions.
[18] If one is interested in ferromagnetism by itself, the fluctuating vector potential has no important effects and is usually omitted from the action. For the properties of a ferromagnetic superconductor, on the other hand, the coupling of the vector potential to the magnetization is crucial, as we will see.
[19] We consider only magnetic states which are rotationally invariant in the plane perpendicular to the direction of the spontaneous magnetization.
[20] To compare with Ref. [14] notice that these authors localized the potential in the second term and omitted both the last term and the subtraction leading to $a$ instead of $a$ in the next-to-last term. These approximations do not affect the leading elastic properties of the vortex lattice.
[21] Note that this is the so-called left strain tensor, as opposed to the more commonly used right strain tensor [23]. A discussion of why the left strain tensor is appropriate for the situation at hand can be found in Ref. [23].
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