Extension of holomorphic canonical forms on complete $d$-bounded 
Kähler manifolds

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Abstract. In this paper we study the extension of holomorphic canonical forms on 
complete $d$-bounded Kähler manifolds by using $L^2$ analytic methods and $L^2$ Hodge 
theory, which generalizes some classical results to noncompact cases.

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1. Introduction

Following the recent papers [10, 12] we will present some results in this paper 
about the extension of holomorphic canonical forms on complete $d$-bounded Kähler manifolds, by using $L^2$ analytic methods [1, 2, 4, 7] and $L^2$ Hodge theory [5, 8, 12, 20], 
which is related to some extent to Siu’s conjecture of the invariance of plurigenera 
for any compact Kähler manifolds [16, 17]. Part of our results generalizes some 
important results obtained in [10, 12] from the compact to noncompact cases. Recall 
that a differential form $\alpha$ on a Riemannian manifold $(X, g)$ is said to be bounded with 
respect to the Riemannian metric $g$ if the $L^\infty$-norm of $\alpha$ is finite, that is,

$$||\alpha||_{L^\infty(X)} := \sup_{x \in X} ||\alpha||_g < \infty.$$ 

Following Gromov[4] we say that $\alpha$ is $d$-bounded if $\alpha$ is the exterior differential of a 
bounded form $\beta$, that is,

$$\alpha = d\beta \text{ with } ||\beta||_{L^\infty(X)} < \infty.$$ 

In particular, we need to introduce the following definition.

Definition 1.1 ([4]). A Kähler manifold $(X, \omega)$ of dimension $n$ is called a $d$-bounded 
Kähler manifold, if the Kähler form $\omega$ is $d$-bounded, that is, there exists a bounded 
1-form $\theta$ on $X$ with respect to $g$ such that $\omega = d\theta$, where $g$ is the Riemannian metric 
induced by $\omega$. 

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For any Kähler manifold \((X, \omega)\) of dimension \(n\), we denote by \(L^{p,q}_{(2)}(X, \omega)\) the space of \((p,q)\) forms \(\psi\) on \(X\) with \(0 \leq p, q \leq n\) such that \(\psi\) is \(L^2\) global integrable with respect to \(\omega\) in this paper. For the details of \(L^2\) space and \(L^2\) method we recommend the reader see [1, 2, 3, 7, 13]. By 1.4. A. Theorem in [4] we have

**Theorem 1.2.** If \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold, then

\[ \overline{Im \partial} = Im \overline{\partial}, \quad \overline{Im \partial^*} = Im \overline{\partial^*}. \]

Moreover, \((X, \omega)\) admits the following fundamental \(L^2\) Hodge theory:

1. \(L^2\) Hodge orthogonal decomposition: \(L^{p,q}_{(2)}(X, \omega) = H^{p,q}_{(2)}(X, \omega) \oplus Im \overline{\partial} \oplus Im \overline{\partial^*}\)
2. \(L^2\) Hodge isomorphism: \(H^{p,q}_{(2)}(X, \omega) \cong H^{p,q}_{(2)}(X, \omega)\)
3. denoting \(\mathbb{H}\) to be the projection operator \(\mathbb{H} : L^{p,q}_{(2)}(X, \omega) \to H^{p,q}_{(2)}(X, \omega)\) then the Green operator \(G = (\Delta_{\mathbb{H}}|_{H^{p,q}_{(2)}(X, \omega)})^{-1}(I - \mathbb{H})\) is well-defined and bounded.

And, we have the following identity

\[ \Delta_{\mathbb{H}} G = G \Delta_{\mathbb{H}} = I - \mathbb{H}, \mathbb{H} G = G \mathbb{H} = 0 \]

From Theorem 1.2 we know that the \(L^2\) Hodge theory holds for any complete \(d\)-bounded Kähler manifold \((X, \omega)\). This is very important for us since it provides us a potential Hodge method as used in [10, 12] to solve the extension \(\partial\)-equation (2.13) from the deformation theory of Kodaira-Spencer-Kuranishi with suitable \(L^2\) estimates on \((X, \omega)\). A complete \(d\)-bounded Kähler manifold \((X, \omega)\) is not necessarily compact, which constitutes the main difficulty in our research and can makes almost everything different from the classical compact cases. In fact, to study the extension \(\overline{\partial}\)-equation (2.13) on the complete \(d\)-bounded Kähler manifold \((X, \omega)\), we need to prove some basic \(L^2\) estimates on \((X, \omega)\) by applying the Hopf-Rinow lemma in terms of the operators as presented in the \(L^2\) Hodge theory. In particular, we obtain a quasi-isometry formula in \(L^2\)-norm with respect to the operator \(\overline{\partial}^* G \partial\).

**Theorem 1.3.** Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\). Then for any \(g \in L^{p-1,q}_{(2)}(X, \omega)\) with \(\partial g \in L^{p,q}_{(2)}(X, \omega)\), we have

\[ \|\overline{\partial}^* G \partial g\|^2 \leq \|g\|^2. \]

We recall some basic concepts on complex structures and Beltrami differentials. Let \(M\) be a complex manifold of complex dimension \(\dim M = n\). A Beltrami differential \(\varphi\) is by definition a tangent bundle valued \((0, 1)\)-form in \(A^{0,1}(M, T^{1,0}M)\). If the Beltrami differential \(\varphi\) is integrable in the sense that

\[ \overline{\partial} \varphi = \frac{1}{2} [\varphi, \varphi] \]

then \(\varphi\) determines a new complex structure on \(M\), which is denoted by \(M_\varphi\) in this paper. By applying Theorem 1.3 we will study in detail the \(L^2\) extension equation of holomorphic canonical forms on complete \(d\)-bounded Kähler manifolds in Section 2.
2. Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\), \(A^{p,q}_{(2)}(X, \omega)\) be the space of smooth \((p, q)\) forms \(g\) on \(X\) such that \(g\) is \(L^2\) global integrable with respect to \(\omega\) and \(\varphi \in A^{0,1}(X, T^{1,0}_X)\) be an integral Beltrami differential on \(X\) such that its \(L_\infty\)-norm \(\|\varphi\|_{\omega, \infty}\) with respect to \(\omega\) is less than 1, that is, \(\|\varphi\|_{\omega, \infty} < 1\). Then for any \(g \in A^{p,q}_{(2)}(X, \omega)\) we have

\[
\|\varphi \cdot g\|_\omega \leq \|\varphi\|_{\omega, \infty}\|g\|_\omega \leq \|g\|_{\omega} < \infty.
\]

Following the paper \cite{12} we consider the operator

\[
T : L^{p, q}_{(2)}(X, \omega) \to L^{p+1, q-1}_{(2)}(X, \omega)
\]

defined by

\[
T = \overline{\partial} \mathcal{G} \partial.
\]

By Theorem 1.3 we will show that the operator \(I + T\varphi\) is injective. It follows that the deformation operator

\[
\rho_{\omega, \varphi} : A^{n,0}_{(2)}(X, \omega, \varphi) \to A^{n,0}(X, \varphi)
\]

defined by

\[
\rho_{\omega, \varphi}(\Omega) = e^{\varphi}(I + T\varphi)^{-1}\Omega, \ \Omega \in A^{n,0}_{(2)}(X, \omega, \varphi)
\]

is well-defined, where

\[
A^{n,0}_{(2)}(X, \omega, \varphi) := \operatorname{Im}(I + T\varphi) \subset A^{n,0}_{(2)}(X, \omega).
\]

Finally, by using our \(L^2\) estimates on the complete \(d\)-bounded Kähler manifold \((X, \omega)\) and some analysis of the \(L^2\) extension equation (2.14), see the following section, we have the following main result about extensions of holomorphic canonical forms from the complex manifold \(X\) to complex manifold \(X, \varphi\).

**Theorem 1.4.** Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\) and \(\varphi \in A^{0,1}(X, T^{1,0}_X)\) be an integral Beltrami differential on \(X\) such that \(\|\varphi\|_{\omega, \infty} < 1\). Then for any holomorphic \((n, 0)\)-form \(\Omega\) in \(A^{n,0}_{(2)}(X, \omega, \varphi)\), the expression \(\rho_{\omega, \varphi}(\Omega)\) defines a holomorphic \((n, 0)\)-form on \(X, \varphi\) with \(\rho_{\omega, 0}(\Omega) = \Omega\).

We say that the canonical holomorphic \((n, 0)\)-form \(\rho_{\omega, \varphi}(\Omega)\) on \(X, \varphi\) is a holomorphic extension of the canonical holomorphic \((n, 0)\)-form \(\Omega\) on \(X\) in this paper. Theorem 1.4 tell us that, for any complete \(d\)-bounded Kähler manifold \((X, \omega)\) of dimension \(n\), there always exists a subspace \(A^{n,0}_{(2)}(X, \omega, \varphi)\) of \(A^{n,0}_{(2)}(X, \omega)\) such that every holomorphic \((n, 0)\)-form \(\Omega\) in \(A^{n,0}_{(2)}(X, \omega, \varphi)\) admits a holomorphic extension \(\rho_{\omega, \varphi}(\Omega)\) from \(X\) to \(X, \varphi\) with \(\rho_{\omega, 0}(\Omega) = \Omega\). This result generalizes Theorem 1.1 in \cite{12} from the compact to noncompact cases, which is closely related to a famous conjecture due to Siu \cite{16, 17}, about the invariance of plurigenera for compact Kähler manifolds.

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2. Proof of Theorem 1.4

In this section we give a detailed proof of Theorem 1.4. First we need the following estimate from [4] which plays a key role in building the $L^2$-Hodge theory on complete $d$-bounded Kähler manifolds.

**Lemma 2.1** (1.4.A. Theorem[4]). Let $(X,\omega)$ be a complete $d$-bounded Kähler manifold of dimension $n = 2m$ and $\omega = d\eta$ where $\eta$ is a bounded 1-form on $X$. Then every $L^2$-form $\psi$ on $X$ of degree $p \neq m$ satisfies the inequality
\[
\langle\langle \psi, \Delta \psi \rangle\rangle \geq \lambda^2 \langle\langle \psi, \psi \rangle\rangle
\]
where $\lambda$ is strictly positive constant which depends only on $n = \dim X$ and the bound on $\eta$. Furthermore, inequality in the (2.1) is satisfied by the $L^2$-forms of degree $m$ which are orthogonal to the harmonic $m$-forms. Here every term in (2.1) is allowed to be infinity.

For any Kähler manifold $(X,\omega)$ we denote by $L^{p,q}_{(2)}(X,\omega)$ the space of $(p,q)$ forms $\psi$ on $X$ such that $\psi$ is $L^2$ global integrable with respect to the metric $\omega$. As an important consequence of Lemma 2.1 we obtain

**Theorem 2.2** (=Theorem 1.2). If $(X,\omega)$ be a complete $d$-bounded Kähler manifold, then we have
\[
\overline{\text{Im} \partial} = \text{Im} \overline{\partial}, \quad \overline{\text{Im} \partial^*} = \text{Im} \overline{\partial^*}.
\]
Moreover, $(X,\omega)$ admits the following fundamental $L^2$ Hodge theory:
1. $L^2$ Hodge orthogonal decomposition : $L^{p,q}_{(2)}(X,\omega) = \mathcal{H}^{p,q}_{(2)}(X,\omega) \oplus \text{Im} \overline{\partial} \oplus \text{Im} \overline{\partial^*}$,
2. $L^2$ Hodge isomorphism : $\mathcal{H}^{p,q}_{(2)}(X,\omega) \cong \mathcal{H}^{p,q}_{(2)}(X,\omega)$.
3. denoting $\mathbb{H}$ to be the projection operator $\mathbb{H} : L^{p,q}_{(2)}(X,\omega) \to \mathcal{H}^{p,q}_{(2)}(X,\omega)$ then the Green operator $G = (\Delta_{\overline{\partial}}|_{\mathcal{H}^{p,q}_{(2)}(X,\omega)})^{-1}(I - \mathbb{H})$ is well-defined and bounded. And, we have the following identity
\[
\Delta_{\overline{\partial}} G = G \Delta_{\overline{\partial}} = I - \mathbb{H}, \quad \mathbb{H} G = G \mathbb{H} = 0.
\]

**Proof.** From the arguments in the proof of l.l.B.Lemma [4] we have
\[
\langle\langle \psi, \Delta \psi \rangle\rangle = \langle\langle \overline{\partial} \psi, \overline{\partial} \psi \rangle\rangle + \langle\langle \overline{\partial}^* \psi, \overline{\partial}^* \psi \rangle\rangle = \| \overline{\partial} \psi \|^2 + \| \overline{\partial}^* \psi \|^2
\]
for any $\psi \in \text{Dom} \overline{\partial} \cap \text{Dom} \overline{\partial}^* \cap (\text{Ker} \overline{\partial} \cap \text{Ker} \overline{\partial}^*)^\perp$. It follows from Lemma 2.1 that
\[
\lambda^2 \| \psi \|^2 = \lambda^2 \langle\langle \psi, \psi \rangle\rangle \\
\leq \langle\langle \psi, \Delta \psi \rangle\rangle \\
\leq \| \overline{\partial} \psi \|^2 + \| \overline{\partial}^* \psi \|^2 \\
\leq (\| \overline{\partial} \psi \| + \| \overline{\partial}^* \psi \|)^2
\]
which yields that
\[
\| \psi \| \leq \lambda^{-1}(\| \overline{\partial} \psi \| + \| \overline{\partial}^* \psi \|)\]
for any \( \psi \in \text{Dom}\overline{\partial} \cap \text{Dom}\partial^* \cap (\text{Ker}\overline{\partial} \cap \text{Ker}\partial^*)^\perp \). Then Theorem 2.2 follows from Theorem 2.1 and Theorem 2.2 in Chapter 2.1.2 of [15], pages 42-43.

Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\). We will prove some basic \(L^2\) estimates on \((X, \omega)\) in terms of the operators as in Theorem 2.2, in particular a quasi-isometry formula, that is, Theorem 2.6 as below, in \(L^2\)-norm with respect to the operator \(\overline{\partial}G\partial\). As one application, we will see that these estimates give a rather simple and explicit way to solve some \(\partial\)-equations with suitable \(L^2\)-estimates on \((X, \omega)\), see Theorem 2.5. As another application of our estimates, we will apply these estimates to studying the \(L^2\) extension equation of holomorphic canonical forms on the complete \(d\)-bounded Kähler manifold \((X, \omega)\) as follows. To begin, we need the following Hopf-Rinow lemma, which is very effective when we deal with \(L^2\) estimates on complete manifolds.

**Lemma 2.3** (cf. page 366 in [2]). The following properties are equivalent:

(a) \((M, g)\) is complete;

(b) there exists an exhaustive sequence \(\{K_\nu\}_{\nu \in \mathbb{N}}\) of compact subsets of \(M\) and functions \(\psi_\nu \in C^\infty(M, \mathbb{R})\) such that \(\psi_\nu = 1\) in a neighborhood of \(K_\nu\), \(\text{Supp} \psi_\nu \subset K_{\nu+1}\), \(0 \leq \psi_\nu \leq 1\) and \(|d\psi_\nu|_g \leq 2^{-\nu}\).

By applying Lemma 2.3 we obtain

**Lemma 2.4.** Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\). Then for any \(g \in L^{p, q}_2(X, \omega)\) with \(q \geq 1\)

\[
\|\overline{\partial}^* Gg\|^2 \leq \langle\langle g, Gg\rangle\rangle \leq \|G\| \cdot \|g\|^2.
\]

**Proof.** Since \(\|\overline{\partial}^* Gg\|^2 = \lim_{\nu \to \infty} \|\psi_\nu \overline{\partial}^* Gg\|^2\) and

\[
\|\psi_\nu \overline{\partial}^* Gg\|^2 = \lim_{\nu \to \infty} \langle\langle \psi_\nu^2 \overline{\partial}^* Gg, \overline{\partial}^* Gg\rangle\rangle = \lim_{\nu \to \infty} \langle\langle \psi_\nu^2 \overline{\partial}^* Gg, Gg\rangle\rangle
\]

\[
= \lim_{\nu \to \infty} \langle\langle 2\psi_\nu \overline{\partial} \psi_\nu \wedge \overline{\partial}^* Gg, Gg\rangle\rangle + \lim_{\nu \to \infty} \langle\langle \psi_\nu^2 \overline{\partial} \partial^* Gg, Gg\rangle\rangle
\]

it follows that

(2.2) \[
\lim_{\nu \to \infty} \|\psi_\nu \overline{\partial}^* Gg\|^2 = \lim_{\nu \to \infty} \langle\langle \psi_\nu^2 \overline{\partial}^* Gg, Gg\rangle\rangle.
\]

Here we used the fact that

\[
\lim_{\nu \to \infty} \langle\langle 2\psi_\nu \overline{\partial} \psi_\nu \wedge \overline{\partial}^* Gg, Gg\rangle\rangle \leq \lim_{\nu \to \infty} \|\overline{\partial} \psi_\nu\| \|2\psi_\nu \overline{\partial}^* Gg\| \|Gg\| \leq \lim_{\nu \to \infty} \|\overline{\partial} \psi_\nu\| \|2\overline{\partial}^* Gg\| \|Gg\| = 0.
\]
We compute

\begin{equation}
\lim_{\nu \to \infty} \langle \psi^2 \partial^* \mathcal{G} g, \mathcal{G} g \rangle = \lim_{\nu \to \infty} \langle \psi^2 (\Delta_{\mathcal{G}} - \partial^* \mathcal{G}) g, \mathcal{G} g \rangle
\end{equation}

\begin{equation}
= \lim_{\nu \to \infty} \langle \psi^2 (I - \mathbb{H} - \partial^* \mathcal{G}) g, \mathcal{G} g \rangle
\end{equation}

\begin{equation}
= \langle \mathcal{O} g, \mathcal{G} g \rangle - \langle \mathbb{H} g, \mathcal{G} g \rangle - \lim_{\nu \to \infty} \langle \psi^2 \partial^* \mathcal{G} g, \mathcal{G} g \rangle
\end{equation}

Note that \( \langle \mathbb{H} g, \mathcal{G} g \rangle = \langle \mathcal{G} \mathbb{H} g, g \rangle = 0 \) since the Green operator is self-adjoint and zero on the kernel of Laplacian by definition. Moreover,

\begin{equation}
\langle \psi^2 \partial^* \mathcal{G} g, \mathcal{G} g \rangle = \langle \partial^* \mathcal{G} g, \psi^2 \mathcal{G} g \rangle = \langle \partial \mathcal{G} g, \partial^* (\psi^2 \mathcal{G} g) \rangle
\end{equation}

But

\begin{equation}
\| \langle \partial^* \mathcal{G} g, \partial^* (\psi^2 \mathcal{G} g) \rangle \| \leq \| \mathcal{G} \| \| \partial \mathcal{G} g \| \| \psi^2 \mathcal{G} g \|
\end{equation}

\begin{equation}
\leq \| \mathcal{O} \| \| \mathcal{G} \mathcal{G} g \| \| \mathcal{G} g \|
\end{equation}

\begin{equation}
\leq 2^{-v} \| \mathcal{G} \mathcal{G} g \| \| \mathcal{G} g \| \to 0.
\end{equation}

Therefore

\begin{equation}
\lim_{\nu \to \infty} \langle \psi^2 \partial^* \mathcal{G} g, \mathcal{G} g \rangle = \| \mathcal{G} \mathcal{G} g \|^2.
\end{equation}

It follows from (2.2) and (2.3) that

\begin{equation}
\| \mathcal{G} \mathcal{G} g \|^2 = \lim_{\nu \to \infty} \| \psi^2 \mathcal{G} \mathcal{G} g \|^2 = \lim_{\nu \to \infty} \langle \psi^2 \partial^* \mathcal{G} g, \mathcal{G} g \rangle
\end{equation}

\begin{equation}
= \langle \mathcal{O} g, \mathcal{G} g \rangle - \| \mathcal{G} \mathcal{G} g \|^2 \leq \langle \mathcal{O} g, \mathcal{G} g \rangle
\end{equation}

This completes the proof. \( \Box \)

As an easy application of Theorem 2.2 and Lemma 2.4, we have

**Theorem 2.5.** Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\). Then for any \(g \in L^{p-1,q}(X, \omega)\) with \(\partial g \in L^{p,q}(X, \omega)\) and \(q \geq 1\), the differential form

\[ s = \partial^* \mathcal{G} \partial g \in L^{p,q-1}(X, \omega) \]

gives a solution to the equation

\begin{equation}
\partial s = \partial g
\end{equation}

with \(\partial \partial g = 0\) and

\[ \| s \|^2 \leq \langle \partial g, \mathcal{G} \partial g \rangle \leq \| \mathcal{G} \| \| \partial g \|^2. \]

This solution is unique if we require \(\mathbb{H}(s) = 0\) and \(\mathcal{T} s = 0\).
Proof. By the $L^2$ Hodge decomposition, see Theorem 2.2, we have
\[ \overline{\partial} s = \overline{\partial} \partial^* G \partial g = \partial g - \partial g - \overline{\partial} \partial^* G \partial g = \partial g - 2 \partial g = \partial g, \]
where we used the identity $\partial g = 0$. The estimate in Theorem 2.5 follows from Lemma 2.4. The uniqueness of this solution is obvious. In fact, if $s_1$ and $s_2$ are two solutions to $\overline{\partial} s = \partial g$ with $H(s_1) = H(s_2) = 0$ and $\overline{\partial} s_1 = \overline{\partial} s_2 = 0$, by setting $\eta = s_1 - s_2$, we see $\overline{\partial} \eta = 0$, $H(\eta) = 0$ and $\overline{\partial} \eta = 0$. Therefore,
\[ \eta = H(\eta) + \Delta_G G(\eta) = H(\eta) + (\overline{\partial} \partial^* + \overline{\partial} \overline{\partial} G(\eta) = 0, \]
as desired. \qed

We remark that such kind of $\overline{\partial}$-equation as in formula (2.4) is very important in the study of the holomorphic deformation theory of Kodaira-Spencer-Kuranishi since its solution can be used to construct holomorphic deformations of complex structures (cf. [6, 14, 11, 18, 19]). Theorem 2.5 gives a rather simple and explicit way to solve this kind of $\overline{\partial}$-equation with suitable $L^2$-estimates on complete Kähler manifolds, which generalizes Proposition 2.3 in [10].

Next by using Lemma 2.3 we have the following quasi-isometry formula in $L^2$-norm with respect to the operator $\overline{\partial} G \partial$. We will apply it to studying the $L^2$ extension equation, from the deformation theory of Kodaira-Spencer-Kuranishi, of holomorphic canonical forms on complete $d$-bounded Kähler manifolds.

**Theorem 2.6** (=Theorem 1.3). Let $(X, \omega)$ be a complete $d$-bounded Kähler manifold of dimension $n$. Then for any $g \in L^{p-1,q}(X, \omega)$ with $\partial g \in L^{p,q}(X, \omega)$, we have
\[ \|\overline{\partial} G \partial g\|^2 \leq \|g\|^2. \]

**Proof.** First, for any $g \in L^{p-1,q}(X, \omega)$ with $\partial g \in L^{p,q}(X, \omega)$, we find $\Delta_G G \partial g = \partial g - \partial g \in L^p(X, \omega)$, in particular,
\[ \mathbb{G} \partial g \in \text{Dom} \Delta_G \subset \text{Dom} \overline{\partial} \cap \text{Dom} \overline{\partial}^* \]
and
\[ \mathbb{G} \partial g \in \text{Dom} \Delta_G = \text{Dom} \overline{\partial} \subset \text{Dom} \overline{\partial} \cap \text{Dom} \overline{\partial}^*. \]

Next, we notice that
\[ \|\mathbb{G} \partial g\|^2 = \langle \mathbb{G} \partial g, \mathbb{G} \partial g \rangle \]
\[ = \langle \overline{\partial} \mathbb{G} \partial g, \mathbb{G} \partial g \rangle \]
\[ = \langle 2 \mathbb{G} \mathbb{G} \mathbb{G} ^{\partial} \mathbb{G} \partial g, \mathbb{G} \partial g \rangle + \langle \mathbb{G} \overline{\partial} \mathbb{G} \partial g, \mathbb{G} \partial g \rangle \]
\[ \leq 2^{-\nu} \|\mathbb{G} \partial g\| \|\mathbb{G} \partial g\| + \|\mathbb{G} \partial g\|^2 \]
\[ \leq 2^{-\nu} \|\mathbb{G} \partial g\|^2 + \|\mathbb{G} \partial g\|^2 + \langle \mathbb{G} \partial g, \mathbb{G} \partial g \rangle \]
\[ \leq 2^{-\nu} \|\mathbb{G} \partial g\|^2 + 2^{-\nu} \|\mathbb{G} \partial g\|^2 + \langle \mathbb{G} \partial g, \mathbb{G} \partial g \rangle. \]
Thus we obtain

\begin{equation}
\|\psi_\nu\overline{\partial}^*G\partial g\|^2 \leq \frac{1}{1 - 2^{-\nu}} \left( 2^{-\nu}\|G\partial g\|^2 + \left\langle \psi_\nu^2\overline{\partial}^*G\partial g, G\partial g \right\rangle \right).
\end{equation}

In the following we give a detailed estimate of the term \(\left\langle \psi_\nu^2\overline{\partial}^*G\partial g, G\partial g \right\rangle\) in the above inequality by using the \(L^2\) Hodge decomposition. First we compute

\[\left\langle \psi_\nu^2\overline{\partial}^*G\partial g, G\partial g \right\rangle = \left\langle \psi_\nu^2(\Delta^G - \overline{\partial}^*\overline{\partial}G)\partial g, G\partial g \right\rangle = \left\langle \psi_\nu^2(\mathbb{I} - \mathbb{H} - \overline{\partial}^*\overline{\partial}G)\partial g, G\partial g \right\rangle = \left\langle \psi_\nu^2(\partial g - \overline{\partial}^*\overline{\partial}G\partial g), G\partial g \right\rangle = \left\langle \psi_\nu^2\partial g, G\partial g \right\rangle - \left\langle \psi_\nu^2\overline{\partial}^*G\partial g, G\partial g \right\rangle.\]

On one hand,

\[\left\langle \psi_\nu^2\partial g, G\partial g \right\rangle = \left\langle (\partial(\psi_\nu^2 g) - 2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle = \left\langle (\psi_\nu^2 g, \partial^*G\partial g) \right\rangle - \left\langle (2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle = \left\langle (\psi_\nu^2 g, g - \mathbb{H}g - \partial\partial^*Gg) \right\rangle - \left\langle (2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle = \left\langle (\psi_\nu^2 g, g - \mathbb{H}g) \right\rangle - \left\langle (\psi_\nu^2 g, \partial\partial^*Gg) \right\rangle - \left\langle (2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle \]

from which we see that

\[\lim_{\nu \rightarrow \infty} \left\langle \psi_\nu^2\partial g, G\partial g \right\rangle = \left\langle (g, g) \right\rangle - \left\langle (g, \mathbb{H}g) \right\rangle - \left\langle (g, \partial\partial^*Gg) \right\rangle - \lim_{\nu \rightarrow \infty} \left\langle (2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle.
\]

We notice that

\[\left\langle (g, \partial^*Gg) \right\rangle = \lim_{k \rightarrow \infty} \left\langle (g_k, \partial^*Gg) \right\rangle = \lim_{k \rightarrow \infty} \left\langle (\partial^*g_k, \partial^*Gg) \right\rangle = \left\langle (\partial^*g, \partial^*Gg) \right\rangle \geq 0\]

for some sequence \(\{g_k\}_{k=1}^\infty\) with compact support in \(X\) such that \(\partial^*g_k \rightarrow \partial^*g\) with respect to the weak \(L^2\)-topology thanks to the completeness of \(X\). We also notice that

\[\left\langle (2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle \leq \|\partial\psi_\nu\|\|g\|\|2\psi_\nu\overline{\partial}^*G\partial g\| \leq \|\partial\psi_\nu\|\|g\|^2 + \|\psi_\nu G\partial g\|^2 \leq 2^{-\nu}\|g\|^2 + \|\psi_\nu G\partial g\|^2 \leq 2^{-\nu}\|g\|^2 + \|G\partial g\|^2 \rightarrow 0.\]

Thus we get

\begin{equation}
\lim_{\nu \rightarrow \infty} \left\langle \psi_\nu^2\partial g, G\partial g \right\rangle \leq \|g\|^2 - \lim_{\nu \rightarrow \infty} \left\langle (2\psi_\nu\partial\psi_\nu \wedge g, G\partial g) \right\rangle = \|g\|^2.
\end{equation}
On the other hand,
\[
\langle\langle \psi^2 \bar{\omega} \bar{G} \partial g, G \partial g \rangle \rangle = \langle\langle \bar{G} \partial g, \bar{\psi}^2 \bar{G} \partial g \rangle \rangle = \langle\langle \bar{G} \partial g, 2\psi \bar{\partial} \psi \wedge G \partial g \rangle \rangle + \langle\langle \bar{G} \partial g, \psi^2 \partial^* G \partial g \rangle \rangle.
\]
But note that
\[
|\langle\langle \partial^* G \partial g, 2\psi \partial \psi \wedge G \partial g \rangle \rangle| \leq \|\partial^* \psi\| \|2\psi \partial G \partial g\| \|G \partial g\|
\leq \|\partial^* \psi\| \|2\partial^* G \partial g\| \|G \partial g\|
\leq 2^{-\nu} \|2\partial G \partial g\| \|G \partial g\| \to 0.
\]
Therefore
\[
(2.10) \quad \lim_{\nu \to \infty} \langle\langle \psi^2 \bar{\omega} \bar{G} \partial g, G \partial g \rangle \rangle = \|\partial G \partial g\|^2.
\]
Now, from (2.7), (2.8), (2.9) and (2.10) we see that
\[
\|\partial^* G \partial g\|^2 = \lim_{\nu \to \infty} \|\psi^2 \bar{\omega} \bar{G} \partial g\|^2
\leq \lim_{\nu \to \infty} \frac{1}{1 - 2^{-\nu}} \left(2^{-\nu}\|G \partial g\|^2 + \langle\langle \psi^2 \partial^* G \partial g, G \partial g \rangle \rangle \right)
\leq \lim_{\nu \to \infty} \langle\langle \psi^2 \partial^* G \partial g, G \partial g \rangle \rangle
\leq \lim_{\nu \to \infty} \langle\langle \psi^2 G \partial g, G \partial g \rangle \rangle - \lim_{\nu \to \infty} \langle\langle \psi^2 \partial G \partial g, G \partial g \rangle \rangle
\leq \|g\|^2 - \|\partial G \partial g\|^2 \leq \|g\|^2
\]
where we used \(\lim_{\nu \to \infty} \frac{1}{1 - 2^{-\nu}} 2^{-\nu} \|G \partial g\|^2 = 0\). This completes the proof. \(\square\)

Let \((X, \omega)\) be a complete \(d\)-bounded Kähler manifold of dimension \(n\). We want to use Theorem 2.6 to study the problem of extension of holomorphic canonical forms on the complete \(d\)-bounded Kähler manifold \((X, \omega)\) in the following. For this purpose, we let \(A^{p,q}_{(2)}(X, \omega)\) be the space of smooth \((p,q)\) forms \(g\) on \(X\) such that \(g\) is \(L^2\) global integrable with respect to \(\omega\) and let \(\varphi \in A^{0,1}(X, T^1_X)\) be an integral Beltrami differential on \(X\) such that its \(L^\infty\)-norm \(\|\varphi\|_{\omega, \infty}\) with respect to \(\omega\) is less than 1, that is,
\[
\|\varphi\|_{\omega, \infty} < 1.
\]
Then for any \(g \in A^{p,q}_{(2)}(X, \omega)\) we have
\[
\|\varphi \cdot g\|_{\omega} \leq \|\varphi\|_{\omega, \infty} \|g\|_{\omega} \leq \|g\|_{\omega} < \infty.
\]
Following the paper [12] we consider the operator
\[
T : L^{p,q}_{(2)}(X, \omega) \to L^{p+1,q-1}_{(2)}(X, \omega)
\]
defined by
\[
T = \partial^* G \partial.
\]
The domain $\text{Dom} T$ of $T$ by definition is
\[ \text{Dom} T = \{ g \in L^{p,q}_{(2)}(X) : Tg \in L^{p+1,q-1}_{(2)}(X) \}, \]
which, obviously, coincides with the domain $\text{Dom}\partial$, that is,
\[ \text{(2.11)} \quad \text{Dom} T = \text{Dom} \partial = \{ g \in L^{p,q}_{(2)}(X) : \partial g \in L^{p+1,q-1}_{(2)}(X) \}. \]
For the details of (2.11) see formulas (2.5) and (2.6). By Theorem 2.6 we have that $T$ is an operator of norm less than or equal to 1 in the Hilbert space of $L^2$ forms, that is, for any $g \in L^{p-1,q}_{(2)}(X)$ with $\partial g \in L^{p,q}_{(2)}(X)$,
\[ \|Tg\|_{\omega} = \|T^*G\partial g\|_{\omega} \leq \|g\|_{\omega}. \]
In particular, we have

**Lemma 2.7.** Let $(X,\omega)$ be a complete $d$-bounded Kähler manifold of dimension $n$ and $\varphi \in A^{0,1}(X,T_{X}^{1,0})$ be an integral Beltrami differential on $X$ such that $\|\varphi\|_{\omega,\infty} < 1$. Then the operator $I + T\varphi$ is injective on the domain $\text{Dom} T\varphi \cap A^{p,q}_{(2)}(X,\omega)$.

**Proof.** For any nonzero $x \in \text{Dom} T\varphi \cap A^{p,q}_{(2)}(X,\omega)$, we have
\[ (2.12) \quad \|T\varphi x\| \leq \|\varphi x\| \leq \|\varphi\|_{\omega,\infty} \|x\| < \|x\| \]
by Theorem 2.6. Let $(I + T\varphi)x = 0$, where $x \in \text{Dom} T\varphi \cap A^{p,q}_{(2)}(X,\omega)$. If $x \neq 0$ then we find
\[ 0 = \|(I + T\varphi)x\| \geq \|x\| - \|T\varphi\| > \|x\| - \|x\| = 0, \]
a contradiction. Therefore the equation $(I + T\varphi)x = 0$ has only zero solution, which gives the injectivity of $I + T\varphi$. $\square$

In order to study the extension of holomorphic canonical forms on the complete $d$-bounded Kähler manifold $(X,\omega)$, we need to restrict the operator $I + T\varphi$ to the domain $\text{Dom} T\varphi \cap A^{p,q}_{(2)}(X,\omega)$. Thus, in the following, we only consider the case that the operator $I + T\varphi$ is defined by
\[ I + T\varphi : \text{Dom} T\varphi \cap A^{p,q}_{(2)}(X,\omega) \rightarrow A^{p,q}_{(2)}(X,\omega). \]
Then the operator $I + T\varphi$ is injective by Lemma 2.7 and
\[ \text{Im}(I + T\varphi) = (I + T\varphi)(\text{Dom} T\varphi \cap A^{p,q}_{(2)}(X,\omega)). \]
Here we need some basic results from classical deformation theory. For the details, we recommend the reader to see [6, 9, 10, 12].

**Lemma 2.8** ([10, 12]). Let $M$ be a complex manifold of dimension $n$ and let $\varphi \in A^{0,1}(M,T_{M}^{1,0})$ be an integral Beltrami differential. Then for any smooth $(n,0)$-form $\Omega \in A^{n,0}(X)$, the corresponding $(n,0)$-form $\rho_{\varphi}(\Omega) = e^{\varphi} \cdot \Omega$ on $M_{\varphi}$ is holomorphic if and only if
\[ (2.13) \quad \overline{\partial}\Omega = -\partial(\varphi,\Omega). \]
Remark 2.9. Equation (2.13) is called the extension equation since its solution can be used to construct the extensions of holomorphic $(n, 0)$-forms from complex manifold $M$ to complex manifold $M'$. 

Let $(X, \omega)$ be a complete $d$-bounded Kähler manifold of dimension $n$. Then the complete Kähler manifold $(X, \omega)$ admits $L^2$ Hodge theory, see Theorem 2.2 as above. Let $\varphi \in A^{0,1}(X, T_X^{1,0})$ be an integral Beltrami differential on $X$. We want to apply the $L^2$ Hodge theory on $X$ to solving the equation (2.13). Thus we consider naturally the extension equation in $L^2$ sense, that is, we want to find $\Omega \in A^{n,0}(X, \omega)$ such that

\begin{equation}
\overline{\partial}\Omega = -\partial(\varphi, \Omega), \quad \text{with } \Omega \in \text{Dom}\overline{\partial} \text{ and } \varphi, \Omega \in \text{Dom}\partial.
\end{equation}

By Lemma 2.8 the solution of equation (2.14) can be used to construct extensions of holomorphic $(n, 0)$-forms from $X$ to $X'$. Here we should point out that since the complex manifold $X$ we considered in this paper may not be compact we cannot directly use Hodge theory as in [10, 12] to solve the equation (2.14). Indeed, in general, the classical Hodge theory does not hold when the underlying complex manifold is not compact. However, for a complete $d$-bounded Kähler manifold $(X, \omega)$, by Theorem 2.2 we know that $(X, \omega)$ admits the $L^2$ Hodge theory, which provides a possibility to solve the equation (2.14) by the same methods as presented in [10, 12].

Lemma 2.10. Let $(X, \omega)$ be a complete $d$-bounded Kähler manifold of dimension $n$ and $\varphi \in A^{0,1}(X, T_X^{1,0})$ be an integral Beltrami differential on $X$ such that $\|\varphi\|_{\omega, \infty} < 1$. If $\Omega$ is a smooth $(n, 0)$-form in $A^{n,0}(X, \omega)$ such that

\begin{equation}
\overline{\partial}(I + T\varphi)\Omega = 0, \quad \text{with } \Omega \in \text{Dom}T\varphi \cap A^{n,0}(X, \omega)
\end{equation}

then $\Omega$ gives a solution of the $L^2$ extension equation (2.14).

Proof. First we note that for any $\Omega \in \text{Dom}T\varphi \cap A^{n,0}(X, \omega)$ we have

$$\varphi, \Omega \in \text{Dom}T = \text{Dom}\partial.$$

Let $\Omega_0 = (I + T\varphi)\Omega \in \text{Im}(I + T\varphi) \subset A^{n,0}(X, \omega)$. Then

$$\Omega = \Omega_0 - T\varphi\Omega = \Omega_0 - \overline{T}^*G\partial(\varphi, \Omega).$$

It follows that

$$\Omega \in \text{Dom}\overline{\partial}$$

since $\overline{\partial}\Omega_0 = \overline{\partial}(I + T\varphi)\Omega = 0$ and $\overline{T}^*G\partial(\varphi, \Omega) \in \text{Dom}\overline{\partial}$. By the injectivity of $I + T\varphi$ (see Lemma 2.7) we have the equation

$$\Omega = (I + T\varphi)^{-1}\Omega_0$$

in $\text{Dom}T\varphi \cap A^{n,0}(X, \omega)$. We need to show

$$\overline{\partial}\Omega = -\partial(\varphi, \Omega).$$
Indeed, from the $L^2$ Hodge theory on $(X,\omega)$, it follows that
\[
\overline{\partial}\Omega = -\overline{\partial}^* G \partial (\varphi, \Omega) \\
= (\overline{\partial} \overline{\partial} - \Delta_{\overline{\partial}}) G \partial (\varphi, \Omega) \\
= (\overline{\partial} \overline{\partial} G - I + H) \partial (\varphi, \Omega) \\
= -\partial (\varphi, \Omega) + \overline{\partial}^* G \partial (\varphi, \Omega). 
\tag{2.16}
\]

Let $\Phi = \overline{\partial}\Omega + \partial (\varphi, \Omega)$. Then it suffices to show $\Phi = 0$. We compute
\[
\Phi = \overline{\partial}\Omega + \partial (\varphi, \Omega) \\
= \overline{\partial}^* G \partial (\varphi, \Omega) \\
= -\overline{\partial} G \partial (\varphi, \Omega) \\
= -\overline{\partial}^* G \partial ((\overline{\partial} \varphi), \Omega + \varphi, \overline{\partial} \Omega) \\
= -\overline{\partial}^* G \partial (\frac{1}{2} [\varphi, \varphi], \Omega + \varphi, (\Phi - \partial (\varphi, \Omega))) \\
= -\overline{\partial}^* G \partial (\varphi, \Phi) 
\tag{2.17}
\]
where in the last equality, we have used $\partial \Omega = 0$, $\partial^2 = 0$ and the formula
\[
[\varphi, \varphi], \Omega = 2\varphi, \varphi, \Omega - \partial (\varphi, \varphi, \Omega) - \varphi, \varphi, \partial \Omega.
\]

If $\Phi \neq 0$ then by Theorem 2.6 and the condition $\|\varphi\|_{\omega, \infty} < 1$ we obtain
\[
\|\Phi\| \leq \|\varphi, \Phi\| \leq \|\varphi\|_{\omega, \infty} \|\Phi\| < \|\Phi\|,
\]
a contradiction. Thus we conclude that $\Phi = 0$ and
\[
\overline{\partial}\Omega = -\partial (\varphi, \Omega),
\]
as desired. \qed

Conversely, we have

**Lemma 2.11.** Let $(X,\omega)$ be a complete d-bounded Kähler manifold of dimension $n$ and $\varphi \in A^{0,1}(X, T^{1,0})$ be an integral Beltrami differential on $X$ such that $\|\varphi\|_{\omega, \infty} < 1$. If the $(n,0)$-form $\Omega \in A^{n,0}_{(2)}(X, \omega)$ satisfies the equation (2.14), then $\Omega$ satisfies the equation (2.15).

**Proof.** Assume that $\Omega \in A^{n,0}_{(2)}(X, \omega)$ satisfies the equation (2.14). Then we have $\Omega \in \text{Dom} T \varphi \cap A^{n,0}_{(2)}(X, \omega)$ since $\varphi, \Omega \in \text{Dom} \partial = \text{Dom} T$. Applying the operator $\overline{\partial}^* G$ to (2.14), we find
\[
\overline{\partial}^* G \overline{\partial} \Omega = -\overline{\partial}^* G \partial (\varphi, \Omega). 
\tag{2.18}
\]

From the basic properties of $G$ and $H$, we have
\[
\overline{\partial}^* G \overline{\partial} \Omega = \overline{\partial}^* \overline{\partial} \Omega = \Delta_{\overline{\partial}} G = \Omega - \overline{\partial} \Omega. 
\tag{2.19}
\]

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Then combining the equations (2.18) and (2.19) we obtain

\[(2.20) \quad \mathbb{H}\Omega = \Omega + \bar{\partial}G\partial(\varphi, \Omega) = (I + T\varphi)\Omega.\]

Note that $\mathbb{H}\Omega$ is a harmonic $(n,0)$-form on $X$. Thus it follows that

$$\bar{\partial}(I + T\varphi)\Omega = \bar{\partial}\mathbb{H}\Omega = 0$$

as desired. $\square$

Summarizing Lemma 2.10 and Lemma 2.11, we obtain

**Theorem 2.12.** Let $(X, \omega)$ be a complete $d$-bounded Kähler manifold of dimension $n$ and $\varphi \in A^{0,1}(X, T_X^{1,0})$ be an integral Beltrami differential on $X$ such that $\|\varphi\|_{\omega, \infty} < 1$. Then an $(n,0)$-form $\Omega \in A_{(2)}^{n,0}(X,\omega)$ satisfies the equation (2.14), that is,

$$\bar{\partial}\Omega = -\partial(\varphi, \Omega), \quad \text{with } \Omega \in \text{Dom} \bar{\partial} \text{ and } \varphi, \Omega \in \text{Dom} \partial,$$

if and only if it satisfies the equation (2.15), that is,

$$\bar{\partial}(I + T\varphi)\Omega = 0, \quad \text{with } \Omega \in \text{Dom} T\varphi \cap A_{(2)}^{n,0}(X,\omega).$$

Denote

$$A_{(2)}^{n,0}(X,\omega,\varphi) := \text{Im}(I + T\varphi) \subset A_{(2)}^{n,0}(X,\omega).$$

Following the papers [10, 12] we define the deformation operator $\rho_{\omega,\varphi} : A_{(2)}^{n,0}(X,\omega,\varphi) \to A_{(2)}^{n,0}(X,\omega)$ by

$$\rho_{\omega,\varphi}(\Omega) = e^{\varphi, \omega}(I + T\varphi)^{-1}\Omega.$$

By Lemma 2.7 the deformation operator is well-defined. Then we have the following main result about extensions of holomorphic canonical forms from $X$ to $X\varphi$.

**Theorem 2.13** (=Theorem 1.4). Let $(X, \omega)$ be a complete $d$-bounded Kähler manifold of dimension $n$ and $\varphi \in A^{0,1}(X, T_X^{1,0})$ be an integral Beltrami differential on $X$ such that $\|\varphi\|_{\omega, \infty} < 1$. Then for any holomorphic $(n,0)$-form $\Omega$ in $A_{(2)}^{n,0}(X,\omega,\varphi)$, the expression $\rho_{\omega,\varphi}(\Omega)$ defines a holomorphic $(n,0)$-form on $X\varphi$ with $\rho_{\omega,\varphi}(\Omega) = \Omega$.

**Proof.** By the definition of $A_{(2)}^{n,0}(X,\omega,\varphi)$ we know that for any smooth $(n,0)$-form $\Omega$ in $A_{(2)}^{n,0}(X,\omega,\varphi)$ there exists an $(n,0)$-form $\alpha \in \text{Dom} T\varphi \cap A_{(2)}^{n,0}(X,\omega)$ such that

$$\Omega = (I + T\varphi)\alpha.$$

By the injectivity of $I + T\varphi$, see Lemma 2.7, we know that this $\alpha$ is exactly $(I + T\varphi)^{-1}\Omega$, that is,

$$\alpha = (I + T\varphi)^{-1}\Omega.$$

Moreover, if $\Omega$ is holomorphic, then

$$\bar{\partial}(I + T\varphi)\alpha = \bar{\partial}\Omega = 0.$$

By Theorem 2.12 and Lemma 2.8 the equation

$$\rho_{\omega,\varphi}(\Omega) = e^{\varphi, \omega}(I + T\varphi)^{-1}\Omega = e^{\varphi, \omega}\alpha$$
defines a holomorphic \((n,0)\)-form on \(X\) with \(\rho_{\omega,0}(\Omega) = \Omega\), as desired. \(\Box\)

We remark that theorem 2.13 generalizes Theorem 1.1 in [12] from the compact to noncompact cases, which is closely related to a famous conjecture due to Siu [16, 17], about the invariance of plurigenera for compact Kähler manifolds.

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