THE RICCI FLOW ON SOLVMANIFOLDS OF REAL TYPE

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Abstract. We show that for any solvable Lie group of real type, any homogeneous Ricci flow solution converges in Cheeger-Gromov topology to a unique non-flat solvsoliton, which is independent of the initial left-invariant metric. As an application, we obtain results on the isometry groups of non-flat solvsoliton metrics and Einstein solvmanifolds.

1. Introduction

A solvmanifold is a simply-connected solvable Lie group $S$ endowed with a left-invariant Riemannian metric. It is called of real type, if $S$ is non-abelian and if for each element of its Lie algebra the corresponding adjoint map is either nilpotent or has an eigenvalue with non-zero real part. Nilmanifolds are of real type, and so are –up to isometry– all homogeneous manifolds with negative sectional curvature, see [Hei74] and [Jab15b], but flat solvmanifolds are not [Mil76]. More interestingly, by the deep structure results of [Heb98], [Lau10] and [Lau11b], examples of solvmanifolds of real type include all non-flat Einstein solvmanifolds and solvsolitons. Recall that a solvsoliton is a solvmanifold which is also an expanding Ricci soliton, and whose corresponding Ricci flow evolution is driven by diffeomorphisms which are Lie group automorphisms. Thus, up to isometry, solvmanifolds of real type contain all known examples of non-compact, non-flat homogeneous Ricci solitons.

Since the homogeneous Ricci flow solution starting at a solvmanifold exists for all positive times [Laf15b], any sequence of blow-downs subconverges to a homogeneous limit Ricci soliton $\bar{S}$, $\bar{g}_{sol}$. Our first main result addresses the question of uniqueness of such limits:

Theorem A. On a simply-connected solvable Lie group $S$ of real type, any scalar-curvature-normalized homogeneous Ricci flow solution converges in Cheeger-Gromov topology to a non-flat solvsoliton $(\bar{S}, \bar{g}_{sol})$, which does not depend on the initial metric.

The Lie group $S$ is of course called of real type, if it satisfies the above condition. Notice, that the limit transitive group $\bar{S}$ may be non-isomorphic to $S$, but must still be of real type: see Remark 2.2 and Remark 2.6. Theorem A was known for $\dim S = 3$ and partially for $\dim S = 4$ [Lot07], for nilpotent Lie groups [Lau11a], [Jab11], and for unimodular, almost-abelian Lie groups [Arr13]. In the compact case, such a uniqueness result does not hold in general, since most compact Lie groups admit non-isometric Einstein metrics.

A first immediate consequence of Theorem A is

Corollary B. Let $(S, g_{sol})$ be a non-flat solvsoliton. Then, any scalar-curvature-normalized homogeneous Ricci flow solution on $S$ converges in Cheeger-Gromov topology to $(S, g_{sol})$.

We not only recover that solvsolitons on such $S$ are unique up to isometry, proved in [Lau11b], but show also that homogeneous Ricci solitons on a solvable Lie group of real type are pairwise equivariantly isometric: see Corollary 4.3.

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Recall that along a homogeneous Ricci flow solution the full isometry group remains unchanged \cite{Kot10}. However, this does not imply in general that this group will be a subgroup of the full isometry group of the limit: see \cite[Example 1.6]{GJ17}. The main reason for this is that, in the non-Einstein solvsoliton case, we only have convergence in Cheeger-Gromov topology: see Remark 7.1. Still, Theorem \ref{A} yields

\textbf{Corollary C.} For a non-flat solvsoliton \((S, g_{sol})\), the dimension of its isometry group \(\text{Iso}(S, g_{sol})\) is maximal among all left-invariant metrics on \(S\).

In the Einstein case, the convergence can be improved as follows:

\textbf{Theorem D.} Let \((S, g_E)\) be a non-flat Einstein solvmanifold. Then, any scalar-curvature-normalized homogeneous Ricci flow solution on \(S\) converges in \(C^\infty\) topology to \(\psi^* g_E\), for some \(\psi \in \text{Aut}(S)\).

Notice that Theorem \ref{D} gives in particular a dynamical proof of the recent result of Gordon and Jablonski on the maximal symmetry of Einstein solvmanifolds \cite{GJ17}.

There exist already in the literature several results on the stability of certain non-compact homogeneous Einstein metrics and Ricci solitons, see for instance \cite{SSS11}, \cite{Bam15}, \cite{JPW16}, \cite{WW16}. Even though more general, non-homogeneous variations are considered in these articles, none of them implies Theorem \ref{D} since two different homogeneous metrics on \(\mathbb{R}^n\) are not within bounded distance to each other.

Our last main result provides a geometric characterization of solvmanifolds of real type, in terms of the behavior of homogeneous Ricci flow solutions. More precisely, recall that given a Cheeger-Gromov-convergent sequence \((M^n_k, g_k)_{k \in \mathbb{N}}\) of homogeneous manifolds, each of which having an \(N\)-dimensional transitive group of isometries \(G_k\), there is a natural way of making sense of an \(N\)-dimensional limit group of isometries \(\bar{G}\), by taking limits of appropriately rescaled Killing fields (see \cite[§9]{BL17a}). The sequence is then called \emph{algebraically non-collapsed} if the action of \(\bar{G}\) is transitive on the limit space \((\bar{M}^n, \bar{g})\), and collapsed otherwise. A homogeneous Ricci flow solution is algebraically non-collapsed if any convergent sequence of parabolic blow-downs has that property.

\textbf{Theorem E.} On a simply-connected, non-abelian, solvable Lie group \(S\), a homogeneous Ricci flow solution is algebraically non-collapsed if and only if \(S\) is of real type.

A first consequence of Theorem \ref{E} is that for Ricci flow solutions on a simply-connected, non-abelian solvable Lie group which are \textit{not} of real type, the dimension of the isometry group must always jump in the limit: see Section \ref{B}. But even more importantly, studying Ricci flow solutions on such Lie groups would in general involve the understanding of algebraic collapse, which cannot be achieved by using the moving brackets framework only.

We turn to the content of the paper and the proofs of the above results. In Section \ref{B} we discuss algebraic properties of solvmanifolds of real type. In Section \ref{C} we recall the GIT stratification of the space of brackets and state in Theorem \ref{3.3} a uniqueness result for critical points of the energy map. This is the main ingredient in the proof Theorem \ref{4.2} a uniqueness result for solvsoliton brackets lying in the intersection of the closure of an orbit and the stratum containing that orbit. Finally, using the equivalence of the Ricci flow and the gauged bracket flow, we show in Section \ref{D} that on a solvmanifold of real type any scalar curvature normalized bracket flow converges to a unique solvsoliton bracket in the closure of the corresponding orbit. The proof uses essentially a Lyapunov function for the bracket flow, described in \cite{BL17a}. Then Theorem \ref{A} follows immediately. The proof of Theorem \ref{E} is given in Section \ref{C}. Finally, in Section \ref{D} we prove Theorem \ref{D} using the computations of the linearization of the gauged bracket flow at an Einstein bracket given in Section \ref{E}.
2. Solvable Lie groups of real type

In this section we discuss algebraic properties of solvable Lie groups or real type. Let \( (\mathfrak{s}, \mu) \) be the Lie algebra of a solvable Lie group \( \mathcal{S} \). The Lie bracket \( \mu \) is a skew-symmetric bilinear map, i.e. an element of the vector space of brackets

\[
\mathcal{V}(\mathfrak{s}) := \Lambda^2 \mathfrak{s}^* \otimes \mathfrak{s},
\]

that satisfies the Jacobi identity. We denote by \( \text{ad}_\mu(X)(Y) := \mu(X,Y) \in \mathfrak{s}, X,Y \in \mathfrak{s} \), the corresponding adjoint maps. For convenience, we also fix a scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{s} \).

**Definition 2.1.** A non-abelian solvable Lie algebra \( (\mathfrak{s}, \mu) \) is called

(i) of **imaginary type**, if \( \text{sp}(\text{ad}_\mu(\mathfrak{s})) \subset i \mathbb{R} \) for all \( X \in \mathfrak{s} \);

(ii) of **real type**, if all its subalgebras of imaginary type are nilpotent.

A solvable Lie group is called of imaginary or real type if its Lie algebra is of that type.

In the above definition, \( \text{sp}(E) \) denotes the spectrum of an endomorphism \( E \) of \( \mathfrak{s} \). Notice that \( \mathfrak{s} \) is of real type if and only if all for all \( X \in \mathfrak{s} \) the adjoint map \( \text{ad}_\mu(X) \) is either nilpotent or has an eigenvalue \( \lambda \notin i\mathbb{R} \). Recall also that there exist non-abelian solvable Lie groups admitting flat left-invariant metrics. By \( \text{[Mil76]} \), they are all of imaginary type and have abelian nilradical; we call the corresponding Lie brackets **flat brackets**.

**Remark 2.2.** In \( \text{[Jab15b]} \), Lie algebras of real type are called **almost completely solvable**. Completely solvable Lie algebras, characterized by \( \text{sp}(\text{ad}(\mathfrak{s})) \subset \mathbb{R} \) for all \( X \in \mathfrak{s} \), are of course of real type. Moreover, by \( \text{[Jab15a] Prop.8.4} \), any solvable Lie group \( \mathcal{S} \) admitting a solvmanifold metric is of real type.

Considering \( \mathfrak{s} \) as a real vector space, there is a natural ‘change of basis’ linear action of the group \( \text{GL}(\mathfrak{s}) \) on \( \mathcal{V}(\mathfrak{s}) \), given by

\[
(h \cdot \mu)(\langle \cdot, \cdot \rangle) := h \mu(h^{-1} \cdot, h^{-1} \cdot), \quad h \in \text{GL}(\mathfrak{s}), \quad \mu \in \mathcal{V}(\mathfrak{s}).
\]

The orbit \( \text{GL}(\mathfrak{s}) \cdot \mu \) is precisely the set of brackets on \( \mathfrak{s} \) such that the corresponding Lie algebra is isomorphic to \( (\mathfrak{s}, \mu) \). Since by \( \text{[2]} \) we have

\[
\text{ad}_{h \cdot \mu}(X) = h \text{ad}_\mu(h^{-1}X)h^{-1},
\]

the type of a Lie bracket is constant on each \( \text{GL}(\mathfrak{s}) \)-orbit. Next, for \( \mu \in \mathcal{V}(\mathfrak{s}) \), \( X \in \mathfrak{s} \) we set

\[
\varphi(\mu,X) = \max \{ |\text{Re}(\lambda)| : \lambda \in \text{sp}(\text{ad}_\mu(X)) \}, \quad \psi(\mu,X) = \max \{ |\lambda| : \lambda \in \text{sp}(\text{ad}_\mu(X)) \}. \]

**Lemma 2.3.** Let \( (\mathfrak{s}, \mu_0) \) be a solvable Lie algebra with nilradical \( \mathfrak{n} \), and let \( \mathfrak{a} \) be the orthogonal complement of \( \mathfrak{n} \) in \( \mathfrak{s} \). If \( \mu = h \cdot \mu_0 \) for \( h \in \text{GL}(\mathfrak{s}) \), then for all \( X \in \mathfrak{s} \) we have

\[
\varphi(h \cdot \mu_0, X) = \varphi(\mu_0, (h^{-1}X)_a) \quad \text{and} \quad \psi(h \cdot \mu_0, X) = \psi(\mu_0, (h^{-1}X)_a),
\]

where the \( a \)-subscript denotes orthogonal projection onto \( \mathfrak{a} \). Moreover, if \( \mathfrak{n} \neq \mathfrak{s} \) then \( \mu_0 \) is of real type if and only if \( \sigma_\mathfrak{a}(\mu_0) := \min_{X \in \mathfrak{a}, \|X\|=1} \varphi(\mu_0, X) > 0 \).

**Proof.** By \( \text{[3]} \) we have \( \varphi(h \cdot \mu_0, X) = \varphi(\mu_0, h^{-1} \cdot X) \) and we obtain the first two claims, since for any \( X \in \mathfrak{s} \) and \( Y \in \mathfrak{n} \) the adjoint maps \( \text{ad}_{\mu_0}(X+Y) \) and \( \text{ad}_{\mu_0}(X) \) share the same eigenvalues. This follows from \( \text{[Var84]} \) Theorem 3.7.3], since there is a basis for the complexified Lie algebra \( \mathfrak{s}^c \) such that all the adjoint maps are upper triangular, and strictly upper triangular for \( Y \in \mathfrak{n} \).

Regarding the second claim, \( \varphi \) is continuous in \( X \) and \( \mu \), because the eigenvalues of linear maps vary continuously. Since for \( X \in \mathfrak{a} \setminus \{0\} \) the endomorphism \( \text{ad}_{\mu_0}(X) \) is not nilpotent, it immediately follows that \( \mu_0 \) is not of real type if and only if \( \sigma_\mathfrak{a}(\mu_0) = 0 \).
Lemma 2.4. Let $\mu_0 \in V(\mathfrak{s})$ be a solvable Lie bracket of real type. Then, there are no non-zero flat brackets in $\mathbf{GL}(\mathfrak{s}) \cdot \mu_0$.

Proof. Since $(\mathfrak{s}, \mu_0)$ is of real type, $\varphi(\mu_0, Y) > 0$ for all $Y \in \mathfrak{a}\{0\}$. Therefore, using that the continuous maps $(\mu, X) \mapsto \varphi(\mu, X)$, $\psi(\mu, X)$ are homogeneous of degree one in $X$, we see that there exists a constant $C_{\mu_0} > 0$ such that $\psi(\mu_0, Y) \leq C_{\mu_0} \cdot \varphi(\mu_0, Y)$ for all $Y \in \mathfrak{a}$.

Suppose now that $h_k \cdot \mu_0$ converges for $k \to \infty$ to a flat bracket $\bar{\mu} \in V(\mathfrak{s})$. Recall, that by [Mil76] the bracket $\bar{\mu}$ is of imaginary type, that is $\varphi(\bar{\mu}, \cdot) \equiv 0$. For any fixed $X \in \mathfrak{s}$ we then obtain by Lemma 2.3

$$\psi(h_k \cdot \mu_0, X) = \psi(\mu_0, (h_k^{-1}X)_a) \leq C_{\mu_0} \cdot \varphi(\mu_0, (h_k^{-1}X)_a) = C_{\mu_0} \cdot \varphi(h_k \cdot \mu_0, X) \xrightarrow{k \to \infty} 0.$$ 

Therefore, $\psi(\bar{\mu}, \cdot) \equiv 0$ and $\bar{\mu}$ is nilpotent by Engel’s theorem. Thus $\bar{\mu} = 0$ by [Mil76]. □

Recall, that the rank of a solvable Lie algebra is the codimension of its nilradical.

Lemma 2.5. Among solvable Lie brackets of fixed rank, those of real type form an open set.

Proof. If the rank is zero, the claim is obvious. Let now $(\mu_k)_{k \in \mathbb{N}} \in V(\mathfrak{s})$ be a sequence of rank $a \in \mathbb{N}$ solvable Lie brackets which are not of real type, converging to $\mu_0$ as $k \to \infty$, also of rank $a$. After acting with suitable orthogonal maps on each $\mu_k$ and passing to a convergent subsequence, we may assume that the nilradicals of $\mu_k, \mu_0$ are a fixed subspace $n \neq \mathfrak{s}$. The claim now follows by Lemma 2.3 since $0 = \lim_{k \to \infty} \sigma_a(\mu_k) = \sigma_a(\mu_0)$. □

Remark 2.6. There is only one 2-dimensional non-abelian solvable Lie algebra, and it is of real type.

Table 1 contains up to isomorphism all non-abelian, solvable Lie algebras of dimension 3, according to [ABDO05]. After fixing an orthonormal basis $\{e_1, e_2, e_3\}$, they are described by $(\text{ad } e_1)|_n \in \mathbf{gl}_2(\mathbb{R})$, where $n = \text{span}_{\mathbb{R}}\{e_2, e_3\}$ is an abelian ideal. Since $\mu^{(2)} \in \mathbf{GL}_3(\mathbb{R}) \cdot \mu^{(2)}$, real type is not an open condition in the space of brackets. Here $\mathfrak{e}(2)$ denotes the Lie algebra of rigid motions of the Euclidean plane and $\mathfrak{h}_3$ the 3-dimensional Heisenberg Lie algebra.

On $\mathbf{S}_3$, the simply-connected solvable Lie group with Lie algebra $\mathfrak{s}_3$, any homogeneous Ricci flow solution converges to the soliton on $\mathbf{S}_3$ in Cheeger-Gromov topology. This follows from Remark 5.6 since the Lie brackets $\mu^{(3)}$ and $\mu^{(2, 3, 1)}$ lie in the same stratum, $\mu^{(2, 3, 1)} \in \mathbf{GL}_3(\mathbb{R}) \cdot \mu^{(3)}$, and $\mathbf{S}_3$ admits a solvsoliton (isometric to the hyperbolic 3-space).

In dimension 4, solvable Lie algebras have rank at most 2. The Lie algebra $\mathfrak{aff}(\mathbb{C})$ is the unique example of rank 2 which is not of real type. From Lemma 2.3 and the fact that the rank is lower semi-continuous on the space of brackets, it follows that there is no sequence of brackets of real type converging to a bracket of type $\mathfrak{aff}(\mathbb{C})$. Thus, the set of solvable Lie brackets of real type is not dense in the space of all solvable Lie brackets.

| $(\text{ad } e_1)|_n$ | constraints | real type | flat |
|---------------------|-------------|-----------|------|
| $\mathfrak{h}_3$    | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | -         | ✓    | -    |
| $\mathfrak{s}_3$    | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | -         | ✓    | -    |
| $\mathfrak{s}_3,\lambda$ | $\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$ | $-1 \leq \lambda \leq 1$ | ✓ | - |
| $\mathfrak{s}_3',\lambda$ | $\begin{bmatrix} \lambda & 1 \\ 0 & 1 \end{bmatrix}$ | $0 < \lambda$ | ✓ | - |
| $\mathfrak{e}(2)$ | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ | -         | -    | ✓    |

Table 1. 3-dimensional solvable Lie algebras
3. Stratification and uniqueness results for the moment map

We review in this section the GIT stratification of the space of brackets $V(s)$ with respect to the linear action $\mathcal{A}$ of the real reductive Lie group $\text{GL}(s)$. We also recall a uniqueness result for critical points of the moment map which will play a key role in the proof of Theorem \[A\] The reader is referred to \[BL17b, BL17c\] for a more thorough presentation.

Let us fix a Euclidean vector space $(s, \langle \cdot, \cdot \rangle)$. Denote also by $\langle \cdot, \cdot \rangle$ the induced scalar products on $\text{gl}(s) \simeq s^* \otimes s$ and $V(s)$. If $O(s)$ denotes the orthogonal group of $(s, \langle \cdot, \cdot \rangle)$, $\text{so}(s)$ its Lie algebra, and $p := \text{Sym}(s, \langle \cdot, \cdot \rangle) \subset \text{gl}(s)$ the subspace of $\langle \cdot, \cdot \rangle$-symmetric endomorphisms, then there is a Cartan decomposition

\[
\text{gl}(s) = O(s) \exp(p), \quad \text{gl}(s) = \text{so}(s) \oplus p.
\]

The maximal compact subgroup $O(s)$ of $\text{GL}(s)$ acts orthogonally on $V(s)$ via $\mathcal{A}$, and the elements in $\exp(p)$ act by symmetric maps. At the linear level, there is a corresponding $\text{gl}(s)$-representation $\pi : \text{gl}(s) \to \text{End}(V(s))$ given by

\[
\pi(A) \mu (\cdot, \cdot) := A\mu(\cdot, \cdot) - \mu(A, \cdot) - \mu(\cdot, A), \quad A \in \text{gl}(s), \quad \mu \in V(s),
\]

which yields $\pi(\text{so}(s)) \subset \text{so}(V(s))$, $\pi(p) \subset \text{Sym}(V(s), \langle \cdot, \cdot \rangle)$. Thus, $\text{GL}(s)$ is a real reductive Lie group in the sense of \[BL17b\], and one can study its linear action on $V(s)$ using real geometric invariant theory.

The moment map $m : V(s) \setminus \{0\} \to p$ and its energy $F : V(s) \setminus \{0\} \to \mathbb{R}$ are respectively defined by

\[
m(\mu) = \frac{1}{\|\mu\|^2} \cdot \langle \pi(A) \mu, \mu \rangle,
\]

for all $A \in p$, $\mu \in V(s) \setminus \{0\}$. Notice that the moment map is $O(s)$-equivariant:

\[
m(k \cdot \mu) = k \cdot m(\mu) k^{-1}, \quad k \in O(s), \quad \mu \in V(s) \setminus \{0\}.
\]

The energy $F$ is therefore $O(s)$-invariant. The following theorem shows how $F$ determines a $\text{GL}(s)$-invariant, “Morse-type” stratification of $V(s) \setminus \{0\}$:

**Theorem 3.1.** \[BL17b\] There exists a finite subset $\mathcal{B} \subset p$ and a collection of smooth, $\text{GL}(s)$-invariant submanifolds $\{S_\beta\}_{\beta \in \mathcal{B}}$ of $V(s)$, with the following properties:

(i) We have $V(s) \setminus \{0\} = \bigcup_{\beta \in \mathcal{B}} S_\beta$ and $S_\beta \cap S_{\beta'} = \emptyset$ for $\beta \neq \beta'$.

(ii) We have $\overline{S_\beta} \subset \bigcup_{\beta \in \mathcal{B}, \|\beta\| > \|\beta'\|} S_{\beta'}$ (the closure taken in $V(s) \setminus \{0\}$).

(iii) A bracket $\mu$ is contained in $S_\beta$ if and only if the negative gradient flow of $F$ starting at $\mu$ converges to a critical point $\mu_C$ of $F$ with $m(\mu_C) \in O(s) \cdot \beta$.

We now describe the strata in more detail. For $\beta \in p$ we set

\[
\beta^+ := \beta + \|\beta\|^2 \text{Id}_s.
\]

Denote by $V^\gamma_{\beta^+} \subset V(s)$ the eigenspace of $\pi(\beta^+) = \pi(\beta) - \|\beta\|^2 \text{Id}_{V(s)}$ corresponding to the eigenvalue $r \in \mathbb{R}$ (recall that $\pi(\text{Id}_s) = -\text{Id}_{V(s)}$), and consider the subspace

\[
V_{\beta^+}^{\geq 0} := \bigoplus_{r \geq 0} V^{\gamma}_{\beta^+}.
\]

There exist subgroups of $\text{GL}(s)$ adapted to these subspaces. In order to describe them, since the linear map $\text{ad}(\beta) : \text{gl}(s) \to \text{gl}(s)$, $A \mapsto [\beta, A]$ is self-adjoint, we may decompose $\text{gl}(s) = \bigoplus_{r \in \mathbb{R}} \text{gl}(s)_r$, as a sum of $\text{ad}(\beta)$-eigenspaces, and set accordingly

\[
\text{g}_\beta := \text{gl}(s)_0 = \ker(\text{ad}(\beta)), \quad \text{u}_\beta := \bigoplus_{r > 0} \text{gl}(s)_r, \quad \text{q}_\beta := \text{g}_\beta \oplus \text{u}_\beta.
\]
We then denote by
\[ G_\beta := \{ g \in \text{GL}(s) : g \beta g^{-1} = \beta \}, \quad U_\beta := \exp(u_\beta), \quad Q_\beta := G_\beta U_\beta, \]
the centralizer of \( \beta \) in \( \text{GL}(s) \), the unipotent subgroup associated with \( \beta \), and the parabolic subgroup associated with \( \beta \), respectively. Set \( K_\beta := \text{O}(s) \cap G_\beta \) and consider also
\[ H_\beta := K_\beta \exp(p \cap h_\beta), \quad h_\beta := \{ A \in g_\beta : \langle A, \beta \rangle = 0 \}, \]
a codimension-one reductive subgroup (resp. subalgebra) of \( G_\beta \) (resp. \( g_\beta \)).

We then denote by \( \mu \) that it is large enough so that we have
\[ \text{GL}(s) = \text{O}(s)Q_\beta. \]
For a critical point \( \mu_C \) of \( F \) we set \( \beta := \text{m}(\mu_C) \) and define
\[ U^0_{\beta+} := \{ \mu \in V^0_{\beta^+} : 0 \notin H_\beta : \mu \} \quad U^0_{\beta+} := p^{-1}_\beta(U^0_{\beta^+}), \]
where \( p_\beta : V^0_{\beta^+} \to V^0_{\beta^+} \) denotes the orthogonal projection. The set \( U^0_{\beta+} \) (resp. \( U^0_{\beta^+} \)) is open and dense in \( V^0_{\beta^+} \) (resp. in \( V^0_{\beta^+} \)). The proof of Theorem \[ \ref{thm:unique} \] in fact shows that
\[ S_\beta = \text{O}(s) \cdot U^0_{\beta^+}. \]
In particular, for any \( \mu \in S_\beta \) one can always find \( k \in \text{O}(s) \) such that \( k \cdot \mu \in V^0_{\beta^+} \).

**Definition 3.2.** We say a bracket \( \mu \in S_\beta \subset V(s) \) is gauged correctly w.r.t. \( \beta \), if \( \mu \in U^0_{\beta^+} \).

Later on, we will fix a stratum label \( \beta \), and then exploit (11) to gauge everything, in order to work on the set \( U^0_{\beta^+} \) which is better adapted to \( \beta \).

Finally, we recall the following uniqueness result for critical points of \( F \). We think of it as a generalization of the Kempf-Ness theorem, which gives ‘uniqueness’ of minimal vectors (zeros of the moment map).

**Theorem 3.3.** \[ \text{BL17b} \] For \( \mu \in S_\beta \), the set of critical points of \( F \) contained in \( \text{GL}(s) \cdot \mu \cap S_\beta \) equals \( \mathbb{R}_{>0} \cdot \text{O}(s) \cdot \mu_C \), where \( \mu_C \in S_\beta \) is a critical point of \( F \) with \( \text{m}(\mu_C) = \beta \).

4. **Uniqueness of solvsolitons**

The main result of this section is Theorem \[ \ref{thm:uniqueness} \] a ‘solvsoliton analogue’ of Theorem \[ \ref{thm:unique} \].

Before turning to that, we recall the correspondence between left-invariant metrics on a Lie group and Lie brackets.

Let \( S \) be a simply-connected Lie group whose Lie algebra \( s \) is endowed with a fixed background scalar product \( \langle \cdot, \cdot \rangle \). Denote by \( \mu^s \in V(s) \) the Lie bracket of \( s \). The set \( \mathcal{M}_{\text{left}}(S) \) of left-invariant Riemannian metrics on \( S \) can be parameterized by the orbit \( \text{GL}(s) \cdot \mu^s \subset V(s) \) as follows: any \( g' \in \mathcal{M}_{\text{left}}(S) \) is determined by a scalar product \( \langle \cdot, \cdot \rangle' \) on \( s \), which may be written as \( \langle \cdot, \cdot \rangle' = \langle h \cdot, h \cdot \rangle \) for some \( h \in \text{GL}(s) \). We then associate to \( g' \) the bracket \( h \cdot \mu^s \).

Recall that \( \text{GL}(s) \) acts on \( V(s) \) via (2).

Conversely, to a bracket \( h \cdot \mu^s \in \text{GL}(s) \cdot \mu^s \) we associate the left-invariant metric on \( S \) determined by the scalar product \( \langle h \cdot, h \cdot \rangle \) on \( s \). Notice that in both directions the map \( h \)
is not unique, thus this correspondence is one to one only when we take into account the action of the groups Aut(s, μ²) ≃ Aut(S) and O(s):

\[ \mathcal{M}_{\text{left}}(S) / \text{Aut}(S) \sim \text{GL}(s) \cdot \mu^2 / O(s). \]

Here Aut(S) acts by pull-back on \( \mathcal{M}_{\text{left}}(S) \), each orbit consisting of pairwise isometric metrics. To every Lie bracket \( \mu \in V(s) \) there corresponds a Riemannian manifold \( (S_\mu, g_\mu) \), where \( S_\mu \) is the simply-connected Lie group with Lie algebra \( (s, \mu) \), and the metric \( g_\mu \in \mathcal{M}_{\text{left}}(S_\mu) \) is determined by \( \langle \cdot, \cdot \rangle \) on \( s \simeq T_e S_\mu. \)

**Definition 4.1.** A solvsoliton \( (S, g_{\text{sol}}) \) is a solvmanifold for which

\[ \text{Ric}_{g_{\text{sol}}} = c \cdot \text{Id}_s + D, \quad c \in \mathbb{R}, \quad D \in \text{Der}(s), \]

where \( \text{Ric}_{g_{\text{sol}}} \) denotes the Ricci endomorphism at \( e \in S \) and \( \text{Der}(s) \) is the Lie algebra of derivations of \( s \). The corresponding Lie bracket \( \mu \in \text{GL}(s) \cdot \mu^2 \) is also called a solvsoliton.

We now turn to the main result of this section.

**Theorem 4.2.** Let \( (s, \mu) \) be a solvable Lie algebra of real type with \( \mu \in S_\beta \). Then, up to scaling the set of solvsolitons in \( \text{GL}(s) \cdot \mu \cap S_\beta \) is contained in a unique \( O(s) \)-orbit.

This immediately implies

**Corollary 4.3.** Let \( (S, g_{\text{sol}}) \) be a non-flat solvmanifold. Then, any other left-invariant Ricci soliton metric on \( S \) is of the form \( \alpha \cdot y_g_{\text{sol}}, \) for some \( \alpha > 0, y \in \text{Aut}(S) \).

**Proof.** The group \( S \) is of real type by Remark 2.2. Hence by [Jab15a, Prop. 8.4] any left-invariant Ricci soliton \( g \) on \( S \) is a solvmanifold. After rescaling \( g \), by Theorem 4.2 we may assume that \( g_{\text{sol}} \) and \( g \) have associated solvmanifold brackets \( \mu_{\text{sol}}, \mu \in V(s) \) with \( \mu_{\text{sol}} = k \cdot \mu \) for some \( k \in O(s) \). Thus, by (12) the metrics \( g_{\text{sol}} \) and \( g \) are isometric via an automorphism. □

The uniqueness of solvsolitons up to equivariant isometry was known for completely solvable groups: see [Hel98] for the Einstein case and [Lau11b] for solvsolitons.

We now work towards a proof of Theorem 4.2. The idea is that starting with a solvmanifold bracket one can explicitly construct a bracket on the same \( \text{GL}(s) \)-orbit which is a critical point of the energy map \( F \), and then Theorem 3.3 can be applied.

Let us recall the following formula for the Ricci endomorphism \( \text{Ric}_\mu \in \text{Sym}(s) \) of a Lie bracket \( \mu \in V(s) \):

\[ \text{Ric}_\mu = M_\mu - \frac{1}{2} B_\mu - \frac{1}{2} (\text{ad}_\mu H_\mu + (\text{ad}_\mu H_\mu)^t). \]

Here, \( M_\mu = \frac{1}{2} \cdot m(\mu) \cdot \| \mu \|^2 \) is a multiple of the moment map \( m(\mu) \) defined in (9) and \( \langle B_\mu X, Y \rangle = \text{tr}(\text{ad}_\mu X \text{ad}_\mu Y) \) is the endomorphism associated to the Killing form. The mean curvature vector \( H_\mu \) is implicitly defined by \( \langle H_\mu, X \rangle = \text{tr} \text{ad}_\mu X \) for all \( X \in s \), and \( (\cdot)^t \) denotes the transpose with respect to \( \langle \cdot, \cdot \rangle \). The modified Ricci curvature is defined by

\[ \text{Ric}^*_\mu = M_\mu - \frac{1}{2} B_\mu. \]

Moreover, we set \( \text{scal}^*(\mu) \) := \text{tr} \text{Ric}^*_\mu. Notice, that for non-flat solvmanifolds \( \text{scal}^*(\mu) < 0 \) by Lemmas 3.5 and 3.6 in [BL17a].

In terms of the stratification from Section 3 by [BL17a] Thm. 6.4 & Cor. C.3] we have

**Proposition 4.4.** [BL17a] Let \( \mu_{\text{sol}} \in U_{\beta^+}^\infty \subset S_\beta \) be a solvmanifold with scal*(\mu_{\text{sol}}) = -1. Then,

\[ \text{Ric}^*_\mu_{\text{sol}} = \beta = c \cdot \text{Id}_s + D, \quad c = -\| \beta \|^2, \quad D = \beta^+ \in \text{Der}(s, \mu_{\text{sol}}). \]

Moreover, \( \beta^+ = \beta + \| \beta \|^2 \cdot \text{Id}_s \geq 0 \), and its image is the nilradical of \( (s, \mu_{\text{sol}}) \).
Next, let us briefly review some of the structural results for solvsolitons from [Lau13].

Given a solvsoliton bracket $\mu \in V(\mathfrak{s})$ with nilradical $\mathfrak{n}$, consider the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$. We have that $\mu(\mathfrak{a}, \mathfrak{a}) = 0$, and for all $Y \in \mathfrak{a}, X \in \mathfrak{n}$, it holds that

$$[\text{ad}_\mu Y, (\text{ad}_\mu Y)^t] = 0, \quad \text{tr} \left( (\text{ad}_\mu Y) (\text{ad}_\mu X)^t \right) = 0.$$  \hfill (16)

Moreover, the symmetric endomorphism $M_\mu$ defined after equation \[\text{(14)}\] satisfies

$$\langle M_\mu Y, Y \rangle = -\frac{1}{2} \|\text{ad}_\mu Y\|^2, \quad \langle M_\mu Y, X \rangle = 0, \quad \langle M_\mu X, X \rangle = (M_\nu X, X),$$  \hfill (17)

for all $Y \in \mathfrak{a}, X \in \mathfrak{n}$. Here $\nu: \mathfrak{n} \cap \mathfrak{n} \to \mathfrak{n}$ denotes the restriction of $\mu$ to $\mathfrak{n}$ (see [LL14] Prop. 4.13]). This yields in particular $M_\mu |_\mathfrak{a} < 0$, since $\text{ad}_\mu Y = 0$ implies $Y \in \mathfrak{n}$.

The next lemma shows how to modify a solvsoliton to obtain a critical point of $F$.

**Lemma 4.5.** Let $\mu \in S_\beta$ be a solvsoliton Lie bracket with $\text{Ric}_\mu^* = \beta$, and let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ be the orthogonal decomposition where $\mathfrak{n}$ is the nilradical of $\mu$.

(i) If $h = \begin{bmatrix} h_\mathfrak{a} & 0 \\ 0 & \text{Id}_\mathfrak{n} \end{bmatrix} \in \text{GL}(\mathfrak{s})$, then

$$M_{h,\mu} = (h^{-1})^t M_\mu h^{-1}, \quad \text{Ric}_{h,\mu} = (h^{-1})^t \text{Ric}_\mu h^{-1}.$$  \hfill (i)

(ii) There exists $h = \begin{bmatrix} h_\mathfrak{a} & 0 \\ 0 & \text{Id}_\mathfrak{n} \end{bmatrix} \in \text{GL}(\mathfrak{s})$ such that $h \cdot \mu$ is a critical point of the norm squared of the moment map $F$, with $m(h \cdot \mu) = \beta$.

**Proof.** Set $\tilde{\mu} := h \cdot \mu$. Notice that $\tilde{\mu}(\mathfrak{a}, \mathfrak{a}) = 0$ and $\tilde{\mu}|_{\mathfrak{n} \cap \mathfrak{n}} = \nu$. By [LL14] Lemma 4.4], if $\{Y_i\}$ is an orthonormal basis of $\mathfrak{a}$ then for $Y \in \mathfrak{a}, X \in \mathfrak{n}$ we have that

$$\langle M_{\tilde{\mu}} Y, Y \rangle = -\frac{1}{2} \text{tr} \text{ad}_{\tilde{\mu}} Y (\text{ad}_{\tilde{\mu}} Y)^t,$$

$$\langle M_{\tilde{\mu}} X, X \rangle = \langle M_\nu X, X \rangle + \frac{1}{2} \sum \langle \text{ad}_{\tilde{\mu}} Y_i, (\text{ad}_{\tilde{\mu}} Y_i)^t \rangle X, X,$$

$$\langle M_{\tilde{\mu}} Y, X \rangle = -\frac{1}{2} \text{tr} \text{ad}_{\tilde{\mu}} Y (\text{ad}_{\tilde{\mu}} X)^t.$$  \hfill (ii)

Since $(\text{ad}_\mu Y) (\mathfrak{n}) \subset \mathfrak{n}$ and $\text{ad}_{\tilde{\mu}} Y |_{\mathfrak{a}} = 0$, \[\text{(3)}\] implies that $\text{ad}_{\tilde{\mu}} Y = \text{ad}_\mu (h^{-1}Y)$ for any $Y \in \mathfrak{a}$. Using that and \[\text{(10)}, \text{(17)}\] one can easily verify the formula for $M_{\tilde{\mu}}$. Since $B_{\tilde{\mu}} = (h^{-1})^t B_\mu h^{-1}$ (see [Lau13] Lemma 3.7]), the formula for the modified Ricci curvature follows.

To prove (ii), we look for a map $h = \begin{bmatrix} h_\mathfrak{a} & 0 \\ 0 & \text{Id}_\mathfrak{n} \end{bmatrix} \in \text{GL}(\mathfrak{s})$ such that $M_{h,\mu} = \beta$. It will then follow that $h \cdot \mu$ is a critical point of the energy map $F$. Indeed, $M_{h,\mu} = \frac{1}{2} m(h \cdot \mu) \|h \cdot \mu\|^2$, and $\text{tr} m(h \cdot \mu) = -1 = \text{tr} \beta$ by [Lau13] Lemma 3.7], from which we deduce $m(h \cdot \mu) = \beta$. But $F \geq \|\beta\|^2$ on $S_\beta$ by Theorem \[\text{(3)}\] (iii).

To that end, let $h = \begin{bmatrix} h_\mathfrak{a} & 0 \\ 0 & \text{Id}_\mathfrak{n} \end{bmatrix} \in \text{GL}(\mathfrak{s})$ satisfying

$$h^t h = \text{Id}_\mathfrak{a} - \frac{1}{2\|\beta\|^2} \cdot B_\mu.$$  \hfill (iii)

Such an $h$ exists if and only if the right-hand-side is positive definite. But $B_\mu |_{\mathfrak{n}} = 0$, thus on $\mathfrak{n}$ we have $\text{Id}_\mathfrak{n}$. And the restriction to $\mathfrak{a}$ equals $-\|\beta\|^2 M_\mu |_{\mathfrak{a}} > 0$, by the fact that $\text{Ric}_\mu^* = \beta$ and Proposition \[\text{(4.4)}\]. Hence there is at least one such $h$. Using (i) we obtain

$$M_{h,\mu} = (h^{-1})^t M_\mu h^{-1} = (h^{-1})^t \left( \text{Ric}_\mu^* + \frac{1}{2} \cdot B_\mu \right) h^{-1}$$

$$= (h^{-1})^t \left( \beta^+ - \|\beta\|^2 \cdot \text{Id}_\mathfrak{a} + \frac{1}{2} \cdot B_\mu \right) h^{-1}$$

$$= (h^{-1})^t \left( \beta^+ - \|\beta\|^2 \cdot h^t h \right) h^{-1} = \beta,$$

where in the last step we are using the identity $(h^{-1})^t \beta^+ h^{-1} = \beta^+$, which follows at once from the special form of $h$ and the properties of $\beta^+$ stated in Proposition \[\text{(4.4)}\]. \hfill \Box
Proof of Theorem 4.2. Let $\mu_1, \mu_2 \in \mathop{GL}(\mathfrak{s}) \cdot \mu^0 \cap \mathcal{S}_\beta$ be two solv-soliton brackets. By (11) and Proposition 4.4, after acting with $\mathcal{O}(\mathfrak{s})$, we may assume that $\operatorname{Ric}\mu^*_\mu = \operatorname{Ric}\mu^*_\mu = \beta$ and that the nilradicals of $\mu_1$ and $\mu_2$ equal to $\mathfrak{n}$. Setting $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, by Lemma 4.5 (ii) there exist maps $h_i = \left[ \begin{matrix} (h_i)_1 & 0 \\ 0 & \mathop{Id}_n \end{matrix} \right] \in \mathop{GL}(\mathfrak{s})$, $i = 1, 2$, such that $h_i \cdot \mu_i \in \mathop{GL}(\mathfrak{s}) \cdot \mu^0$ are critical points of $F$ with $\mathfrak{m}(h_i \cdot \mu_i) = \beta$. Theorem 5.3 then yields $h_1 \cdot \mu_2 = (kh_2) \cdot \mu_2$ for some $k \in \mathcal{O}(\mathfrak{s})$. This implies that $\beta = m(h_1 \cdot \mu_1) = k m(h_2 \cdot \mu_2) k^{-1} = k \beta k^{-1}$ by (17), and hence $k$ commutes with $\beta$ and $\beta^+$, thus $k = \left[ \begin{matrix} k_a & 0 \\ 0 & k_n \end{matrix} \right]$. After acting on $\mu_2$ with $\left[ \begin{matrix} \mathop{Id}_n & 0 \\ 0 & k_n^{-1} \end{matrix} \right]$ we may assume $k_n = \mathop{Id}_n$.

Hence, $\mu_1 = h \cdot \mu_2$ for $h = \left[ \begin{matrix} h_a & 0 \\ 0 & \mathop{Id}_n \end{matrix} \right] \in \mathop{GL}(\mathfrak{s})$. Finally, using $\operatorname{Ric}\mu^*_\mu = \operatorname{Ric}\mu^*_\mu = \beta$, the fact that $\beta|_\mathfrak{a} = c \cdot \mathop{Id}_\mathfrak{a}$, see Proposition 4.4 and Lemma 4.5 (i), we get $h' = h = \mathop{Id}_\mathfrak{a}$, from which it follows that $h \in \mathcal{O}(\mathfrak{s})$.

\[ \square \]

5. Proof of Theorem A

Before turning to the proof of Theorem A, we discuss here one of our main tools, an ODE on the space of brackets which is equivalent to the Ricci flow of left-invariant metrics.

It was shown in [Lau13] that each Ricci flow solution of left-invariant metrics on $\mathcal{S}$ corresponds to a curve of brackets in $V(\mathfrak{s})$ solving the bracket flow $\mu' = -\pi(\operatorname{Ric}\mu) \mu$. Since the right-hand-side is always tangent to the $\mathop{GL}(\mathfrak{s})$-orbit, the flow preserves orbits. Hence if the initial bracket $\mu(0)$ belongs to a stratum $\mathcal{S}_\beta$, then also $\mu(t) \in \mathcal{S}_\beta$ for all $t$ for which the solution exists, since $\mathcal{S}_\beta$ is $\mathop{GL}(\mathfrak{s})$-invariant. Moreover, the flow can be ‘gauged’ so that it consists of brackets which are gauged correctly w.r.t. $\beta$ (Def. 3.2). To that end, consider the subspace $\mathfrak{t}_{\mathfrak{u}_\beta} = \{ A - A^t : A \in \mathfrak{u}_\beta \} \subset \mathfrak{so}(\mathfrak{s})$, a direct complement for $\mathfrak{q}_\beta$:

\[ (18) \quad \mathfrak{gl}(\mathfrak{s}) = \mathfrak{q}_\beta \oplus \mathfrak{t}_{\mathfrak{u}_\beta}. \]

Denote by $(\cdot)_{\mathfrak{q}_\beta}$ the corresponding linear projection onto $\mathfrak{q}_\beta$.

Remark 5.1. In general [18], [33] is not an orthogonal decomposition. We do have an orthogonal decomposition $\mathfrak{gl}(\mathfrak{s}) = \mathfrak{u}_\beta \oplus \mathfrak{g}_\beta \oplus \mathfrak{u}_\beta^t$, and for $A = A_{\mathfrak{u}_\beta} + A_{\mathfrak{g}_\beta} + A_{\mathfrak{u}_\beta^t} \in \mathfrak{gl}(\mathfrak{s})$ the projection according to (18) is given by $A_{\mathfrak{q}_\beta} = A_{\mathfrak{g}_\beta} + A_{\mathfrak{u}_\beta} + (A_{\mathfrak{u}_\beta^t})^t$. In particular, for $A \in \mathfrak{Sym}(\mathfrak{s})$ we have $A_{\mathfrak{q}_\beta} = A_{\mathfrak{g}_\beta} + 2 \cdot A_{\mathfrak{u}_\beta}$ and

\[ \|A\| \leq \|A_{\mathfrak{q}_\beta}\| \leq 2 \cdot \|A\|. \]

Given a Ricci flow solution $(g(t)) \subset \mathcal{M}_{\text{left}}(\mathcal{S})$ with $g(0) = g_0$, we write $g_0$ at the point $e \in \mathcal{S}$ with respect to the fixed scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}$, that is $(g_0)_e(\cdot, \cdot) = \langle g_0 \cdot, \cdot \rangle$, and we may assume that $h_0 \in \mathcal{Q}_\beta$ by (10). We call the following ODE the gauged bracket flow:

\[ (19) \quad \frac{d\mu}{dt} = -\pi(\operatorname{Ric}\mu^*_\mu) \mu, \quad \mu(0) = h_0 \cdot \mu^0 \in \mathcal{Q}_\beta \cdot \mu^0. \]

Theorem 5.2. [Lau13] [BL17a] Let $(\mathcal{S}, g_0)$ be a non-abelian, simply-connected Lie group with left-invariant metric $g_0$ and Lie algebra $(\mathfrak{s}, \mu^0)$, and consider a fixed scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}$. Then, the solution $(g(t))_{t \in [0, \infty)}$ to the Ricci flow starting at $g_0$ and the solution $(\mu(t))_{t \in [0, \infty)}$ to the gauged bracket flow (19) starting at $h_0 \cdot \mu^0$ differ only by pull-back by time-dependent diffeomorphisms. Here, $h_0 \in \mathcal{Q}_\beta$ is such that $(g_0)_e(\cdot, \cdot) = \langle h_0 \cdot, h_0 \cdot \rangle$.

The proof shows the existence of Lie group isomorphisms $\varphi_t : \mathcal{S} \to \mathcal{S}_{\mu(t)}$ such that $g(t) = \varphi_t^* g_{\mu(t)}$. They can be obtained from the corresponding Lie algebra isomorphisms $h(t) : (\mathfrak{s}, \mu^0) \to (\mathfrak{s}, \mu(t))$, which satisfy the ODE

\[ (20) \quad h' = -(\operatorname{Ric}\mu^*_\mu)_{\mathfrak{q}_\beta} \cdot h, \quad h(0) = h_0. \]

The gauging is chosen so that for $\mu(0) \in \mathcal{U}^\infty_{\beta^+}$ we have $\mu(t) \in \mathcal{Q}_\beta \cdot \mu^0 \subset \mathcal{U}^\infty_{\beta^+}$ for all $t \in [0, \infty)$. 

Remark 5.3. Unlike the bracket flow, the gauged bracket flow is not $O(s)$-equivariant. However, it is still $K_\beta$-equivariant: for $k \in K_\beta$ one has that
\[
k \cdot (-\pi((\text{Ric}^*_\nu)_{q_\beta}) \mu) = -\pi(k(\text{Ric}^*_\nu)_{q_\beta} k^{-1})(k \cdot \mu) = -\pi((k \text{Ric}^*_\mu k^{-1})_{q_\beta})(k \cdot \mu) = -\pi((\text{Ric}^*_k \mu)_{q_\beta})(k \cdot \mu).
\]
The second identity follows from Remark 5.1 since conjugation by $k \in K_\beta$ preserves $g_\beta$ and $u_\beta$, thus for $A \in \text{Sym}(s)$ we have that $k A_{q_\beta} k^{-1} = (k A k^{-1})_{q_\beta}$.

Next, we recall a scale-invariant Lyapunov function, which is monotone along immortal solutions to [19]: see [BL17a, §7]. Consider the following codimension-one subgroup of $Q_\beta$,
\[
\text{SL}_\beta := H_\beta U_\beta \subset Q_\beta,
\]
and let $\mathfrak{s}_\beta$ be its Lie algebra. Assume without loss of generality that $\mu^s \in U_{\beta_0}^\beta$. Since $Q_\beta \cdot \mu^s$ is a cone over $\text{SL}_\beta \cdot \mu^s$ by [BL17a], for every $\mu \in Q_\beta \cdot \mu^s$ there exists a unique scalar $v_\beta(\mu) \in \mathbb{R}_{>0}$ such that
\[
v_\beta(\mu) \in \text{SL}_\beta \cdot \mu^s.
\]
We call $v_\beta$ the $\beta$-volume functional; notice that it depends on the ‘base bracket’ $\mu^s$. It has the property that for some constant $C_{\mu^s} > 0$ and for all $\mu \in Q_\beta \cdot \mu^s$ we have
\[
v_\beta(\mu) \geq C_{\mu^s} \cdot ||\mu||^{-1}.
\]

Theorem 5.4. [BL17a] Let $(\mu(t))_{t \in [0, \infty)}$ be a solution to [19] with $\mu(0) \in Q_\beta \cdot \mu^s$. Then,
\[
F_\beta : Q_\beta \cdot \mu^s \to \mathbb{R}, \quad \mu \mapsto v_\beta(\mu)^2 \cdot \text{scal}^*(\mu),
\]
is scale-invariant and evolves along [19] by
\[
\frac{d}{dt} F_\beta(\mu) = 2 \cdot v_\beta(\mu)^2 \cdot \left( ||\text{Ric}^*_\mu ||^2 + \text{scal}^*(\mu) \cdot (\text{Ric}^*_\mu, \beta) \right) \geq 0.
\]
Equality holds for some $t > 0$ if and only if $\mu(0)$ is a solvsoliton.

The monotonicity of $F_\beta$ follows from Cauchy-Schwarz and the estimate
\[
(\text{Ric}^*_\mu, \beta) \geq ||\text{scal}^*_\mu|| \cdot ||\beta||^2,
\]
which holds on $Q_\beta \cdot \mu^s$. Moreover, even though $F_\beta$ is defined only on one orbit, the rigidity statement also holds for all potential limits:

Proposition 5.5. [BL17a] Let $\bar{\mu} \in \overline{Q_\beta \cdot \mu^s}$ with $\text{scal}^*(\bar{\mu}) = -1$. Then,
\[
||\text{Ric}^*_\bar{\mu}||^2 - (\text{Ric}^*_\bar{\mu}, \beta) = 0
\]
if and only if $\bar{\mu}$ is a solvsoliton with $\text{Ric}^*_\bar{\mu} = \beta$ and $\bar{\mu} \in S_\beta$.

We are now in a position to prove the main result of the article.

Proof of Theorem 4. Let $(S, g_0)$ be a solvmanifold of real type, with Lie algebra $(s, \mu^s)$. By Theorem 5.2 since $\mu^s \neq 0$, to the Ricci flow solution $g(t)$ with $g(0) = g_0$ there corresponds a solution $\mu(t)$ to the gauged bracket flow [19]. Recall that $\mu(t) \in Q_\beta \cdot \mu^s$ for all $t \geq 0$, and that $Q_\beta \cdot \mu^s \subset S_\beta$ for some $\beta$.

For non-compact homogeneous spaces there exists a normalized bracket flow keeping the modified scalar curvature $\text{scal}^*$ constant. More precisely, since $\text{scal}^*(\mu(t)) < 0$ for all $t \geq 0$, as explained in [Lau13] §3.3, after an appropriate time reparameterization the corresponding $\text{scal}^*$-normalized family $\nu(t) := |\text{scal}^*(\mu(t))|^{-1/2} \cdot \mu(t)$ solves
\[
\frac{d\nu}{dt} = -\pi((\text{Ric}^*_\nu)_{q_\beta} + ||\text{Ric}^*_\nu||^2 \cdot \text{Id}_s) \nu, \quad \nu(0) = \nu_0 \in Q_\beta \cdot \mu^s.
\]
To this end, recall that by [Lau13] we have \( d \scal^* |_{\mu}(\pi(A)\nu) = -2 \cdot (\Ric^*_s, A) \). Thus \( \scal^*(\nu(t)) \equiv -1 \), since \( (\Ric^*_s)_{t\nu} = \Ric^*_s - (Ric^*_s)_{t\nu} \) and \( \Ric^*_s \perp (Ric^*_s)_{t\nu} \), see (18).

We first show that there exist \( c_{\mu_0}, C_{\mu_0} > 0 \) such that for all \( t \geq 0 \) it holds

\[
0 < C_{\mu_0} \leq \|\nu(t)\| \leq C_{\mu_0}.
\]

The existence of \( c_{\mu_0} \) is clear since \( \scal^*(0) = 0 \). On the other hand, if some subsequence \( \nu(t_k) \) is unbounded, then the sequence \( \nu_k := \nu(t_k)/\|\nu(t_k)\| \) satisfies \( \scal^*(\nu_k) \to 0 \) as \( k \to \infty \). A subsequential limit would then contradict the real type hypothesis: see Lemma 2.4.

Next, we claim that the \( \omega \)-limit of the solution \( (\nu(t))_{t \in [0, \infty)} \) consists entirely of solvsoliton brackets lying in the same stratum \( S_\beta \). This will follow from Proposition 5.5 once we show that the non-negative function \( f(t) := \|\Ric^*_s(t)\|^2 - \langle \Ric^*_s(t), \beta \rangle \) tends to 0 as \( t \to \infty \). To see that, notice first that by scale-invariance of the Lyapunov function \( F_\beta \) (Theorem 5.4) we have that \( (\grad \nu)_t \perp \mathbb{R} \nu \). Hence, along the normalized bracket flow \( F_\beta \) satisfies the same evolution equation (22). Together with (25) and (21) this implies

\[
\frac{d}{dt} F_\beta(t) \geq C_{\mu_0} \cdot f(t) \geq 0,
\]

for some constant \( C_{\mu_0} > 0 \). Since \( F_\beta \) is monotone non-decreasing and \( F_\beta(\nu(t)) < 0 \) for all \( t \geq 0 \), it follows that \( \int_0^\infty f(t) dt < \infty \). On the other hand, notice that with respect to a fixed orthonormal basis of \( V(\mu) \) the entries of \( \frac{d}{dt} \nu \) are polynomials in the entries of \( \nu \). By using again the upper bound in (25) we deduce that \( f'(t) \leq D_{\mu_0} \) for all \( t \geq 0 \). It is now clear that \( \lim_{t \to \infty} f(t) = 0 \), which proves our claim.

Applying Theorem 4.2 we conclude that the \( \omega \)-limit is contained in a single \( O(\mu) \cdot \nu_\sol \). From this we deduce that it is equivalent to normalize the scalar curvature. Indeed, by \( O(\mu) \)-invariance of \( \scal \), we have \( 0 > s_\infty = \lim_{t \to \infty} \scal(\nu(t)) \).

Hence, the \( \omega \)-limit of the scalar-normalized bracket family

\[
(\|\scal(\mu(t))\|^{-1/2} \mu(t))_{t \in [0, \infty)} = (\|\scal(\nu(t))\|^{-1/2} \nu(t))_{t \in [0, \infty)}
\]

is contained in \( O(\mu) \cdot \nu_\sol \), where \( \nu_\sol := \|s_\infty\|^{-1/2} \nu_\sol \).

The bracket \( \nu_\sol \) corresponds to a solvmanifold \( (\tilde{S}, \tilde{g}_\sol) \), which by Theorem 4.2 does not depend (up to isometry) on the initial metric \( g_0 \). By [Lau12, Corollary 6.20], bracket convergence implies Cheeger-Gromov subconvergence to a space locally isometric to \( (\tilde{S}, \tilde{g}_\sol) \). If we set \( \tilde{g}(t) := |\scal(t)| \cdot g(t) \), this says that any sequence \( (\tilde{S}, \tilde{g}(t_k))_{k \in \mathbb{N}} \), \( t_k \to \infty \), has a subsequence converging in Cheeger-Gromov topology to a Riemannian manifold locally isometric to \( (\tilde{S}, \tilde{g}_\sol) \). By Theorem D.2 in [BL17a], the limit is in fact simply-connected, and hence equal to \( (\tilde{S}, \tilde{g}_\sol) \), as claimed. \( \square \)

Remark 5.6. The above proof shows that in fact the scal-normalized Ricci flow converges to the unique soliton whose bracket lies in \( GL(\mu) \cdot \mu \cap S_\beta \), see Theorem 4.2.

6. No algebraic collapsing

Let \( (M^n, g_k)_{k \in \mathbb{N}} \) be a sequence of Riemannian metrics converging in pointed Cheeger-Gromov topology to a limit space \( (M^n, \tilde{g}) \). Assume that for each \( k \in \mathbb{N} \) there is a connected, \( n \)-dimensional Lie group \( G_k \) of \( g_k \)-isometries acting transitively on \( M^n \). A natural way of obtaining an isometric group action in the limit is by arguing at the infinitesimal level, as follows: for each \( k \in \mathbb{N} \) one considers \( n \) linearly independent \( g_k \)-Killing fields, which after a suitable normalization subconverge in \( C^1 \)-topology to \( n \) linearly independent \( \tilde{g} \)-Killing fields on \( M^n \); see [Heb89, §6.2] and [BL17a, §9]. The sequence \( (M^n, g_k)_{k \in \mathbb{N}} \) is called algebraically non-collapsed, if the \( n \) limit Killing fields span the tangent space at each point of \( M^n \). Notice that if this is the case, then after lifting them to the universal cover \( (X^n, \tilde{g}) \) of \( (M^n, \tilde{g}) \), they
can be ‘integrated’ to a simply-transitive, $\bar{g}$-isometric action of a simply-connected Lie group $G$ on $X^n$ [KN96, Ch.VI, Thm.3.4].

**Definition 6.1.** An immortal homogeneous Ricci flow solution $(M^n, g(t))_{t \in [0, \infty)}$ is called 

*algebraically non-collapsed*, if any Cheeger-Gromov-convergent sequence of parabolic blow-downs

$$g_{s_k}(t) := \frac{1}{s_k} \cdot g(s_k t), \quad s_k \to \infty,$$

is algebraically non-collapsed in the above sense.

We now work towards a proof of Theorem 6.2. Let $(\bar{S}, g(t))_{t \in [0, \infty)}$ be an immortal homogeneous Ricci flow solution of left-invariant metrics on a simply-connected solvable Lie group $\bar{S}$. Recall that $\bar{S}$ is diffeomorphic to $\mathbb{R}^n$. Consider the associated bracket flow solution $(\mu(t))_{t \in [0, \infty)}$, $\mu(0) = \mu_0$ and recall that for a blown-down solution $\frac{1}{t} \cdot g(st)$, $s > 0$, the corresponding brackets $\mu_s(t)$ scale like

$$\|\mu_s(1)\| = \sqrt{s} \cdot \|\mu(s)\|. \tag{27}$$

By [Boh15] the solution is Type-III, and we have injectivity radius estimates by [BL17a, Thm. 8.2]. Hence, Hamilton’s compactness theorem [Ham95] implies that any sequence of blow-downs $(\bar{S}, (g_{s_k}(t))_{t \in [1, \infty)})_{k \in \mathbb{N}}$ subconverges to a limit Ricci flow solution in Cheeger-Gromov topology, uniformly over compact subsets of $\bar{S} \times [1, \infty)$. We claim that this limit Ricci flow solution may be written as $(\bar{S}, \bar{g}(t))_{t \in [1, \infty)}$, where $\bar{S}$ is a simply-connected solvable Lie group, in general not isomorphic to $\bar{S}$.

To that end, notice that by Theorem D.2 in [BL17a] the limit is simply connected. Thus we are in a position to apply the results in [Heb98, §6] and conclude that there is a solvable Lie group of isometries acting transitively on $(M^n, \bar{g})$. More precisely, one may use items (i), (ii) of Step 1 in the proof of Theorem 6.6 from that paper. A quick inspection of the proof shows that the Einstein hypothesis is not used at all for these items, and indeed all that is needed is that the limit space is simply-connected. By [GWSS, Lemma 1.2], there is also a simply-transitive solvable group of isometries, hence the limit is a solvmanifold.

**Lemma 6.2.** Suppose that the solution $(g(t))_{t \in [0, \infty)}$ is algebraically non-collapsed. Then, there exists $0 < c_{\mu_0} < C_{\mu_0}$ such that $c_{\mu_0} \leq t \cdot \|\mu(t)\|^2 \leq C_{\mu_0}$ for all $t \geq 1$.

Proof. To prove the upper bound, assume on the contrary that $s_k \cdot \|\mu(s_k)\|^2 \to +\infty$ for some sequence $s_k \to \infty$. After extracting a convergent subsequence of blow-downs and using the algebraic non-collapsedness, we may apply [BL17a, Thm. 9.2] and conclude that the corresponding brackets are bounded. This is a contradiction: see (27).

The lower bound holds even without the non-collapsedness assumption. To see that, notice that the vector field defining the bracket flow, $\mu \mapsto -\pi(A_{\mu})\mu$, can be extended to a smooth vector field on $V(\mathfrak{s})$, which is homogeneous of degree 3. By compactness of the sphere in $V(\mathfrak{s})$ we conclude that there is a uniform bound $\|\pi(A_{\mu})\mu\| \leq C \cdot \|\mu\|^3$ for some $C > 0$, see also [Laf15a, §3]. This implies that $\frac{4}{3} \|\mu\|^2 = 2 \langle \mu, \frac{4}{3} \mu \rangle \geq -2 C \cdot \|\mu\|^4$, which by integrating yields $\|\mu(t)\|^2 \geq 1/2 C t + \|\mu_0\|^2$ for all $t \geq 0$. The desired lower bound for $t \geq 1$ now follows. \[ \square \]

Let $C_\alpha$ denote the norm of the linear map $\pi : \mathfrak{gl}(\mathfrak{s}) \to \text{End}(V(\mathfrak{s}))$ defined in (3). The next lemma says that the Ricci curvature cannot be too small for a very long time in the algebraically non-collapsed case.

**Lemma 6.3.** Suppose that the solution $(g(t))_{t \in [1, \infty)}$ is algebraically non-collapsed. Then, there exists $\alpha_{\mu_0} > 0$ such that if $t \cdot \|\text{Ric}_{\mu(t)}\| \leq \frac{1}{8C_\alpha}$ holds for all $t \in [t_1, t_2]$, then $t_2 \leq \alpha_{\mu_0} \cdot t_1$. 

Proof. By Lemma 6.2 the function \( \varphi : [1, \infty) \to \mathbb{R} : t \mapsto t \cdot \| \mu(t) \|^2 \) is bounded. Moreover, if \( A_\mu := (\text{Ric}_\mu)^{\alpha_0} \), then by Cauchy-Schwarz and Remark 5.1 we have

\[
\langle \mu, \pi(A_\mu)\mu \rangle \leq C_n \cdot t \cdot \| \mu \|^2 \leq 2C_n \cdot t \cdot \| \text{Ric}_\mu \| \| \mu \|^2 \leq 2C_n \cdot t \cdot \| \text{Ric}_\mu \| \| \mu \|^2.
\]

Using that, for \( t \in [t_1, t_2] \) we obtain

\[
\frac{d}{dt} \varphi = \| \mu \|^2 + 2t \cdot \langle \mu, \frac{d}{dt} \mu \rangle = \| \mu \|^2 - 2t \cdot \langle \mu, \pi(A_\mu)\mu \rangle \\
\geq \| \mu \|^2 - 4C_n \cdot t \cdot \| \text{Ric}_\mu \| \| \mu \|^2 \geq \frac{1}{2} \| \mu \|^2 = \frac{1}{2t} \cdot \varphi.
\]

Integrating on \([t_1, t_2]\) one gets \( \varphi(t_2)/\varphi(t_1) \geq \sqrt{t_2/t_1} \), and the lemma follows. \( \square \)

We now show that the blow-down limits cannot be flat.

Lemma 6.4. Suppose that the solution \((g(t))_{t \in [1, \infty)}\) is algebraically non-collapsed. Then, there exists \( \delta_{\mu_0} > 0 \) such that \( t \cdot \| \text{Ric}_{\mu(t)} \| \geq \delta_{\mu_0} \) for all \( t \geq 1 \).

Proof. Assume that this is not the case and let \( s_k \to \infty \) be a sequence of times with \( s_k \cdot \| \text{Ric}_{\mu(s_k)} \| \to 0 \) as \( k \to \infty \). Any convergent subsequence of the corresponding sequence of blow-downs \( g_{s_k}(t) = \frac{1}{s_k} g(s_k, t) \) must have a Ricci-flat limit. After passing to such a subsequence, it follows that there exists \( k_0 \) such that for all \( k \geq k_0 \) and all \( t \in [1, 1+\alpha_{\mu_0}] \), we have \( \| \text{Ric}(g_{s_k}(t)) \| \leq \frac{1}{8C_n(1+\alpha_{\mu_0})} \). This yields

\[
(\langle k \rangle t \cdot \| \text{Ric}(g_{s_k}(t)) \| = t \cdot \| \text{Ric}(g_{s_k}(t)) \| \leq \frac{1}{8C_n}
\]

for all \( t \in [1, 1+\alpha_{\mu_0}] \), thus \( \hat{t} \cdot \| \text{Ric}_{\mu(t)} \| \leq \frac{1}{8C_n} \) for all \( \hat{t} \in [s_k, (1+\alpha_{\mu_0})s_k] \). But this contradicts Lemma 6.3. \( \square \)

We are finally in a position to prove Theorem 3.

Proof of Theorem 3. Let \((S, g(t))_{t \in [0, \infty)}\) be an algebraically non-collapsed Ricci flow solution of left-invariant metrics, and let \( 0 \neq \mu^* \in S_\beta \) correspond to the initial metric \( g(0) \).

As in the proof of Theorem 1 let \( \mu(t) \) be the corresponding solution to the gauged bracket flow and \( \nu(t) := (\text{scal}^*(\mu(t))^{-1/2} \cdot \mu(t) \) the \( \text{scal}^* \)-normalized solution, which after a time reparameterization solves (2).

Assume that \( \| \nu(t_k) \| \to \infty \) for some sequence \( t_k \to \infty \), and let \( \hat{\nu}_k := \nu(t_k)/\| \nu(t_k) \| = \mu(t_k)/\| \mu(t_k) \| \). Then any subsequential limit \( \hat{\nu} \) is a solvable Lie bracket with \( \text{scal}^*(\nu) = 0 \), hence flat (see [Heb98] Rmk. 3.2(b)). On the other hand, \( \| \mu(t) \| \sim 1/\sqrt{t} \) by Lemma 6.2 thus

\[
\| \text{Ric}_{\mu/\| \mu \| \| \mu \| \geq \delta_{\mu_0} > 0,
\]

thanks to Lemma 6.3. This implies that \( \hat{\nu} \) cannot be flat, a contradiction.

Precisely as in the proof of Theorem 1 it follows that for some subsequence \( s_k \to \infty \) we have \( \nu(s_k) \to \nu_{\text{real}} \in S_\beta \), a solvsoliton. By Remark 2.2 \( \nu_{\text{real}} \) is of real type. And since the nilradical of solvable brackets in \( S_\beta \) is of constant dimension (equal to \( \text{rank}(\beta^+)) \), for \( k \) large enough \( \nu(s_k) \) is of real type by Lemma 2.3. Since \( \mu^* \) and \( \nu(t) \) are isomorphic for all \( t \), it follows that \( S \) is of real type.

Conversely, assume that \( S \) is of real type and let \( \mu(t) \) be a bracket flow solution corresponding to a Ricci flow of left-invariant metrics on \( S \). By Corollary 9.13 in [BL17a] it suffices to show that \( t \cdot \| \mu(t) \|^2 \leq C_{\mu_0} \) for some constant \( C_{\mu_0} > 0 \). But this follows immediately from (26) and the Type-III behavior of homogeneous Ricci flow solutions. \( \square \)
7. THE EINSTEIN CASE

In this section we prove Theorem [12] an improvement of the above convergence results in the Einstein case, made possible by the linearization computations from Section 8.

Proof of Theorem [12] Let \((\nu^*(t))_{t \in [0, \infty)}\) denote a solution to the \(\text{scal}^*\)-normalized gauged bracket flow \([23]\) keeping \(\text{scal}^*(\nu^*(t)) \equiv -1\), with \(\nu^*(0) \in Q_\beta \cdot \mu^g\). By Theorem [12] we may assume that for a large time we are as close to an Einstein bracket \(\mu_E\) as we like. The set of Einstein brackets in \(Q_\beta \cdot \mu^g\) with \(\text{scal}^* \equiv -1\) equals \(K_\beta \cdot \mu_E\) by Theorem [12] and [BL17b, Cor. 8.4]. Moreover, by Theorem 8.1 for such an Einstein bracket \(\mu_E\) the tangent space to its tangent \(\nu\)-orbit may be decomposed as \(T_{\mu_E}(\SL_\beta \cdot \mu_E) = T_{\mu_E}(K_\beta \cdot \mu_E) \oplus V_{\mu_E}\), where \(V_{\mu_E}\) denotes the sum of the eigenspaces of the linearization of \([24]\) with negative eigenvalues. This decomposition is \(K_\beta\)-equivariant, since the gauged bracket flow is so by Remark 5.3. Using the normal exponential map of the compact orbit \(K_\beta \cdot \mu_E\) in \(\SL_\beta \cdot \mu_E\) in direction of \(V_{\mu_E}\), we can find coordinates \((x, y) \in U := (1,3)^k \times (-1,1)^j\) of the orbit \(\SL_\beta \cdot \mu_E\) close to \(\mu_E\), where \((x,0)\) parametrizes to \(K_\beta\)-orbit of \(\mu_E\) locally and \((0,y)\) the transversal slice given by \(V_{\mu_E}\). In these coordinates the differential equation \([24]\) reads as \((x,y)' = F(x,y) = (F_1(x,y), F_2(x,y))\) with \(F(x,0) = 0\) and

\[
(dF)_{(x,0)} = \begin{pmatrix} 0 & \frac{\partial F_1}{\partial y} \\ 0 & \frac{\partial F_2}{\partial y} \end{pmatrix},
\]

where \((\frac{\partial F_j}{\partial y})_{(x,0)}\) has only eigenvalues with negative real part, say bounded from the above by \(-\epsilon < 0\). It is easy to see that choosing \(y(0)\) small enough one can conclude that

\[
\nu^*(t) \underset{t \to \infty}{\longrightarrow} \mu_E, \quad \text{exponentially fast.}
\]

Next, consider the solution \((\nu(t))_{t \in [0, \infty)}\) to the \(\text{scal}^*\)-normalized bracket flow

\[
\frac{d\nu}{dt} = -\pi(\text{Ric}_\nu + \|\text{Ric}^*_\nu\|^2 \cdot \text{Id}_s)\nu, \quad \nu(0) = \nu^*(0) \in Q_\beta \cdot \mu^g.
\]

Since \(\nu^*(t)\) is obtained by ‘gaugauging’ \(\nu(t)\), by [BL17a, §3] there exists a smooth family of orthogonal maps \((k(t)) \in O(s)\) such that

\[
\nu(t) = k(t) \cdot \nu^*(t), \quad \forall \ t \in [0, \infty).
\]

It might be the case that the \(\omega\)-limit of \((\nu(t))_{t \in [0, \infty)}\) is not a single bracket. However, by [25] and compactness of \(O(s)\), it must be contained in \(O(s) \cdot \mu_E\). In particular, since the function \(\mu \mapsto \|\text{Ric}^*_\mu\|^2\) is \(O(s)\)-invariant, there exists a limit \(\|\text{Ric}^*_\mu(t)\|^2 \to c_1\) as \(t \to \infty\). And also by [25], and using that the entries of \(\text{Ric}_\mu\) are quadratic in the entries of \(\mu\), we have that \(\text{Ric}_\nu(t) \to c \cdot \text{Id}_s\) exponentially fast, since \(\mu_E\) is Einstein. Hence, from [24] and the \(O(s)\)-equivariance of \(\mu \mapsto \text{Ric}_\mu\), we deduce that \(\text{Ric}_\nu(t) \to c_2 \cdot \text{Id}_s\) as \(t \to \infty\), exponentially fast. We thus get exponentially fast convergence

\[
\text{Ric}_\nu(t) + \|\text{Ric}^*_\nu(t)\|^2 \cdot \text{Id}_s \underset{t \to \infty}{\longrightarrow} \alpha \cdot \text{Id}_s.
\]

Taking scalar products against \(\text{Ric}^*_\nu(t)\) we get \(\alpha = 0\), since \(\text{scal}^* \equiv -1\) and \(\langle \text{Ric}^*_\mu, \text{Ric}_\mu \rangle = \|\text{Ric}^*_\mu\|^2\) (see [Laf15a, Lemma 2.1]).

Recall now that by Theorem [5.3] for a curve \((h(t)) \in GL(s)\) solving the linear equation

\[
h'(t) = -\left( \text{Ric}_\nu(t) + \|\text{Ric}^*_\nu(t)\|^2 \cdot \text{Id}_s \right) \cdot h(t), \quad h(0) = \text{Id}_s,
\]
one can recover the corresponding scal*-normalized Ricci flow solution. By \(31\), the differential equation \(31\) can be rewritten as

\[
h'(t) = \delta(t) \cdot h(t), \quad h(0) = \text{Id}_a,
\]

where \(\delta(t) \in \text{Sym}(s)\) converges exponentially fast to 0 for \(t \to \infty\). We denote by \(\sigma(t) \geq 0\) the maximum between 0 and the largest eigenvalue of \(\delta(t)\), and by \(f(t) := \text{tr}(h(t) \cdot h(t)^T) = \|h(t)\|^2\). Since \(\sigma(t)\) is integrable over \([0, \infty)\), from

\[
\text{tr}(h'(t)h(t)^T) = \text{tr}(\delta(t) \cdot h(t) \cdot h(t)^T),
\]

we get a differential inequality \(f'(t) \leq 2\sigma(t)f(t)\), thus \(f(t)\) is bounded above on \([0, \infty)\).

The function \(g(t) := \text{det}(h(t))\) satisfies an equation \(g'(t) = g(t) \cdot s(t)\), where \(|s(t)|\) is again integrable over \([0, \infty)\). Thus, there exists a limit \(\lim_{t \to \infty} g(t) > 0\). Consequently, we can find a subsequence \((t_i)_{i \in \mathbb{N}}\) of times converging to infinity, such that \(\lim_{t \to \infty} h(t_i) \to h_\infty \in \text{GL}(s)\).

From \(32\) it follows that \(\|h'(t)\|\) is integrable over \([0, \infty)\), hence the curve \(h : [0, \infty) \to \text{GL}(s)\) has finite length and we must have \(\lim_{t \to \infty} h(t) = h_\infty\).

After knowing that the scal*-normalized Ricci flow has a non-flat limit, the same is also true for the scalar-curvature-normalized solution, as they differ only by scaling. The theorem now follows using the uniqueness of Einstein metrics stated in Corollary \(43\). \(\square\)

**Remark 7.1.** If the limit bracket is not Einstein but a non-trivial solvsoliton, then the endomorphism \(\text{Ric}_{\nu(t)} + \|	ext{Ric}_{\nu(t)}^*\|^2 \cdot \text{Id}_a\) converges exponentially fast to a derivation \(D \neq 0\) of the limit bracket \(\mu_{\text{sol}}\). Thus equation \(32\) becomes

\[
h'(t) = -(\delta(t) + D) \cdot h(t), \quad h(0) = \text{Id}_a,
\]

with \(\delta(t) \to 0\). It follows that the solution \(h(t)\) does not converge.

### 8. The Linearization of the Bracket Flow at a Solvsoliton

We finally compute the linearization of the scal*-normalized gauged bracket flow

\[
\frac{d\nu}{dt} = -\pi \left( (\text{Ric}_{\nu}^*)_{q_\beta} + r_\nu \cdot \text{Id}_a \right) \nu, \quad \nu(0) = \nu_0 = q_\beta \cdot \mu^s,
\]

at a solvsoliton bracket \(\mu\) which is gauged correctly w.r.t. \(\beta\). Here, \(r_\nu = \|\text{Ric}_{\nu}^*\|^2\). Recall that for such a bracket \(\bar{\mu} \in \mathcal{U}_{\beta^+}^\geq 0 \subset S_\beta\) we have \(\text{Ric}_{\bar{\mu}}^* = \beta\) and

\[
(\text{Ric}_{\bar{\mu}}^*)_{q_\beta} + r_{\bar{\mu}} \cdot \text{Id}_a = \beta + \|\beta\|^2 \cdot \text{Id}_a = \beta^+ \in \text{Der}(\bar{\mu}),
\]

thanks to Proposition \(4.3\). Thus, \(\bar{\mu}\) is a fixed point of \(33\).

The evolution equations for \(\text{Ric}^*_{\bar{\mu}}\) stated in \([\text{Lau13}, \text{pp. 390}]\), applied to \(33\), imply that

\[
T_{\bar{\mu}} \left( (q_\beta \cdot \bar{\mu}) \cap \{\text{scal}^* = -1\} \right) = \left\{ \pi(A)\bar{\mu} : A \in q_\beta, \langle A, \text{Ric}^*_{\bar{\mu}} \rangle = 0 \right\}.
\]

In particular if \(\bar{\mu}\) is a solvsoliton with \(\text{Ric}^*_{\bar{\mu}} = \beta\) then

\[
T_{\bar{\mu}} \left( (q_\beta \cdot \bar{\mu}) \cap \{\text{scal}^* = -1\} \right) = T_{\bar{\mu}} (\text{SL}_\beta \cdot \bar{\mu}).
\]

**Theorem 8.1.** Let \(\bar{\mu} \in \mathcal{U}_{\beta^+}^\geq 0 \subset S_\beta\) be a solvsoliton bracket with \(\text{Ric}^*_{\bar{\mu}} = \beta\). Then, the linearization of the scal*-normalized gauged bracket flow \(33\) at \(\bar{\mu}\),

\[
L_{\bar{\mu}} : T_{\bar{\mu}} (\text{SL}_\beta \cdot \bar{\mu}) \to T_{\bar{\mu}} (\text{SL}_\beta \cdot \bar{\mu}),
\]

has kernel given by \(\pi(K_\beta)\bar{\mu}\), and its non-zero eigenvalues are negative.
Proof. We apply the formula for $L_{\bar{\mu}}$ given in Lemma \ref{lem:1}. Lemma \ref{lem:2} implies that its kernel of $L_{\bar{\mu}}$ is contained in $\pi(\xi_\beta)\bar{\mu}$. By Lemmas \ref{lem:3} and \ref{lem:4} we may now choose $A \in \mathfrak{s}l_\beta$ an eigenvector of both $P_{\bar{\mu}}$ and $\text{ad}(\beta^+)$, with eigenvalues adding up to $c > 0$. Then,

$$L_{\bar{\mu}}(\pi(A)\bar{\mu}) = -\pi(P_{\bar{\mu}}(A) + [\beta^+, A])\bar{\mu} = -c \cdot \pi(A)\bar{\mu},$$

hence $\pi(A)\bar{\mu}$ is an eigenvector of $L_{\bar{\mu}}$ with negative eigenvalue. The theorem follows. \hfill \Box

In the rest of this section $\bar{\mu}$ will denote a solvsoliton bracket as in Theorem \ref{thm:1}.

Lemma 8.2. If $A \in \mathfrak{s}l_\beta$ then $L_{\bar{\mu}}(\pi(A)\bar{\mu}) = -\pi(P_{\bar{\mu}}(A) + [\beta^+, A])\bar{\mu}$, where

\begin{equation}
P_{\bar{\mu}} : \mathfrak{s}l_\beta \rightarrow \mathfrak{s}l_\beta; \quad A \mapsto \left( d \text{Ric}^*|_{\bar{\mu}}(\pi(A)\bar{\mu}) \right)_{\bar{q}_\beta}.
\end{equation}

Proof. Since $(\text{Ric}^*_{\bar{\mu}})_{\bar{q}_\beta} + r_{\bar{\mu}} \cdot \text{Id}_{\bar{\beta}} = \beta^+$ by \ref{eq:22}, a direct computation yields

\begin{equation}
L_{\bar{\mu}}(\pi(A)\bar{\mu}) = -\pi(P_{\bar{\mu}}(A))\bar{\mu} - \pi \left( (\text{dric}^*_{\bar{\mu}}(\pi(A)\bar{\mu})) \cdot \text{Id}_{\bar{\beta}} \right)\bar{\mu} - \pi(\beta^+)\pi(A)\bar{\mu}.
\end{equation}

Using that $\beta^+ \in \text{Der}(\bar{\mu})$, we obtain

$$\pi([\beta^+, A])\bar{\mu} = \pi(\beta^+)\pi(A)\bar{\mu} - \pi(A)\pi(\beta^+)\bar{\mu} = \pi(\beta^+)\pi(A)\bar{\mu}.$$ 

On the other hand, by \ref{eq:22} we know that $|\text{Ric}^*_{\bar{\mu}}| \geq |\text{scal}^*_{\bar{\mu}}| \cdot |\beta^+|$ for all $\mu \in Q_{\beta} \cdot \bar{\mu}$, and at $\bar{\mu}$ equality holds. Thus the first variation of $|\text{Ric}^*_{\bar{\mu}}|$ at $\bar{\mu}$ along directions tangent to the subset of brackets with $\text{scal}^* = -1$ must vanish, and this amounts to saying that $\bar{\mu}$ is a critical point for $r_{\bar{\mu}}$ restricted to $\mathbf{SL}_\beta \cdot \bar{\mu}$. Therefore, the second term in \ref{eq:37} vanishes.

Finally note, that the image of $P_{\bar{\mu}}$ is contained in $q_{\beta}$. But since by \ref{eq:23} $\bar{\mu}$ is also a minimum for the functional $\mu \mapsto \langle \text{Ric}^*_{\bar{\mu}}, \beta \rangle$ restricted to $\mathbf{SL}_\beta \cdot \bar{\mu}$, it follows that the image is contained in the subalgebra $\mathfrak{s}l_\beta$ by its very definition. \hfill \Box

Recall from \ref{eq:3} that $\text{ad}(\beta) = \text{ad}(\beta^+) : \mathfrak{g}l(\mathfrak{s}) \rightarrow \mathfrak{g}l(\mathfrak{s})$ is a symmetric map, and if $(\mathfrak{g}l(\mathfrak{s})_r)_{r \in \mathbb{R}}$ denote its pairwise orthogonal eigenspaces, then $\mathfrak{h}_\beta \subset \mathfrak{g}_\beta = \mathfrak{g}l(\mathfrak{s})_0$ and $\mathfrak{u}_\beta = \bigoplus_{r > 0} \mathfrak{g}l(\mathfrak{s})_r$.

Lemma 8.3. For $A \in \mathfrak{g}l(\mathfrak{s})_r$ we have that $\pi(A)\bar{\mu} \in V_{\beta^+}^\ast$ (see paragraph before \ref{eq:5}).

Proof. Since $\bar{\mu} \in V_{\beta^+}^\ast$, we have $\pi(\beta^+)\pi(A)\bar{\mu} = \pi([\beta^+, A])\bar{\mu} + \pi(A)\pi(\beta^+)\bar{\mu} = r \cdot \pi(A)\bar{\mu}$. \hfill \Box

Lemma 8.4. The linear maps $P_{\bar{\mu}} : \mathfrak{s}l_\beta \rightarrow \mathfrak{s}l_\beta$ commute. In particular, $P_{\bar{\mu}}$ preserves $\mathfrak{h}_\beta$ and $\mathfrak{u}_\beta$, and it satisfies

$$P_{\bar{\mu}}(A) = \begin{cases}
\frac{1}{2} \cdot (S \circ \delta_{\bar{\mu}}^\perp \pi(A) + A^t B_{\bar{\mu}} + B_{\bar{\mu}} A), & A \in \mathfrak{h}_\beta; \\
\frac{1}{2} \cdot \delta_{\bar{\mu}}^\perp \pi(A), & A \in \mathfrak{u}_\beta.
\end{cases}$$

Here, $S(A) := \frac{1}{2}(A + A^t)$, and

$$\delta_{\bar{\mu}} : \mathfrak{g}l(\mathfrak{s}) \rightarrow \mathcal{V}(\mathfrak{s}) : A \mapsto -\pi(A)\bar{\mu},$$

and $\delta_{\bar{\mu}}^\perp : (\mathcal{V}(\mathfrak{s})(\langle \cdot , \cdot \rangle)) \rightarrow (\mathfrak{g}l(\mathfrak{s}), \langle \cdot , \cdot \rangle)$ is the usual adjoint map.

Proof. Using the formula for $d \text{Ric}^*_{\bar{\mu}}(\pi(A)\bar{\mu})$ given in \ref{eq:36} and \ref{eq:37} in \ref{eq:13} we have

$$P_{\bar{\mu}}(A) = \frac{1}{2}(S \circ \delta_{\bar{\mu}}^\perp \pi(A))_{\bar{q}_\beta} + \frac{1}{2}(A^t B_{\bar{\mu}} + B_{\bar{\mu}} A)_{\bar{q}_\beta}.$$

We show that $P_{\bar{\mu}}$ preserves the eigenspaces of $\text{ad}(\beta^+)$, recalling that by Lemma \ref{lem:1} $P_{\bar{\mu}}$ preserves $\mathfrak{s}l_\beta$.

First, we claim that the linear map $A \mapsto \delta_{\bar{\mu}}^\perp \delta_{\bar{\mu}}(A)$ preserves the eigenspaces of $\text{ad}(\beta^+)$. Indeed, if $A_1, A_2 \in \mathfrak{s}l_\beta$ are eigenvectors of $\text{ad}(\beta^+)$ with eigenvalues $r_1 \neq r_2$, then

$$\langle \delta_{\bar{\mu}}^\perp \delta_{\bar{\mu}}(A_1), A_2 \rangle = \langle \pi(A_1)\bar{\mu}, \pi(A_2)\bar{\mu} \rangle = 0.$$
by Lemma 8.3, since two different eigenspaces of $\pi^{\beta+}$ are orthogonal. For $A \in \mathfrak{g}_\beta$ this implies that $S \circ \delta^{\mu}_\beta(A) \in \mathfrak{g}_\beta$, and the projection $\langle \cdot \rangle_{\mathfrak{g}_\beta}$ is the identity when restricted to $\mathfrak{g}_\beta$ (see Remark 5.1). For $A \in \mathfrak{gl}(s)_r \subset \mathfrak{u}_\beta$, $r > 0$, we have that $\delta^{\mu}_\beta(A) \in \mathfrak{gl}(s)_r$ as well, and the map $\langle S(\cdot) \rangle_{\mathfrak{g}_\beta}$ is the identity on $\mathfrak{u}_\beta$ (see again Remark 5.1). The statement for the first summand thus follows.

Regarding the second term, the decomposition $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ induces natural inclusions $\mathfrak{gl}(\mathfrak{a}), \mathfrak{gl}(\mathfrak{n}) \subset \mathfrak{gl}(\mathfrak{s})$. By Proposition 4.3 we have that $\mathfrak{gl}(\mathfrak{a}) \subset \mathfrak{g}_\beta$. Since the Killing form is trivial on the nilradical, we also have $\mathfrak{B}_\mu \in \mathfrak{gl}(\mathfrak{a})$, or informally, $\mathfrak{B}_\mu = [0, 0]$. On the other hand, any $A \in \mathfrak{u}_\beta$ is of the form $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, thus the map $A \mapsto (A^t \mathfrak{B}_\mu + \mathfrak{B}_\mu A)$ vanishes on $\mathfrak{u}_\beta$. It clearly preserves $\mathfrak{g}_\beta$.

Finally, the formula stated for $P_\mu$ now follows from the previous discussion. \qed

**Lemma 8.5.** The linear map $P_\mu : \mathfrak{sl}_3 \rightarrow \mathfrak{sl}_3$ defined in (36) is symmetric, positive semi-definite, and its kernel is given by $\text{Der}(\mu) + \mathfrak{t}_\beta$.

**Proof.** For $A \in \mathfrak{t}_\beta$, by $\mathcal{O}(\mathfrak{s})$-equivariance of $\mu \mapsto \text{Ric}^*_\mu$, we have that

$$\text{Ric}^{\exp(sA)}_\mu = \exp(sA) \text{Ric}^*_\mu \exp(-sA) = \exp(sA)\beta \exp(-sA) = \beta,$$

thus $P_\mu(\mathfrak{t}_\beta) = 0$. Recall also that since $\mu \in U_{\mathfrak{g}_{\beta \beta}}$, by [BL17a, Cor. 4.11] we have $\text{Der}(\mu) \subset \mathfrak{sl}_3$.

It remains to show that on the orthogonal complement of $\text{Der}(\mu) + \mathfrak{t}_\beta$ in $\mathfrak{sl}_3$ the map $P_\mu$ is symmetric and positive definite. By Lemma 8.4 we may argue on $\mathfrak{h}_{\beta}$ and $\mathfrak{u}_\beta$ separately. The formula given in that lemma for $P_\mu|_{\mathfrak{u}_\beta}$ immediately implies the claim in this case (recall that $\ker \delta_{\beta} = \text{Der}(\mu)$).

Regarding the restriction to $\mathfrak{h}_{\beta}$, using that $P_\mu(\mathfrak{t}_\beta) = 0$ and the formula from Lemma 8.4 we need to worry only about the restriction $P_\mu : \mathfrak{p}_{\beta} \rightarrow \mathfrak{p}_{\beta}$, where $\mathfrak{p}_{\beta} = \mathfrak{g}_{\beta} \cap \mathfrak{Sym}(\mathfrak{s})$. Using $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, by Proposition 4.4 $\mathfrak{p}_\beta$ decomposes as

$$\mathfrak{p}_\beta = \mathfrak{Sym}(\mathfrak{a}) \oplus \mathfrak{p}_\beta^n, \quad \mathfrak{p}_\beta^n := \mathfrak{p}_\beta \cap \mathfrak{gl}(\mathfrak{n}).$$

Let us first see that $P_\mu$ maps these two subspaces onto orthogonal subspaces. For the Killing form term this is clear, since $\mathfrak{B}_\mu \in \mathfrak{Sym}(\mathfrak{a})$. We must thus show that for symmetric maps $A_{\mathfrak{a}} \in \mathfrak{gl}(\mathfrak{a})$ and $A_{\mathfrak{n}} \in \mathfrak{g}_\beta^n$ we have $\langle \pi(A_{\mathfrak{a}})\mu, \pi(A_{\mathfrak{n}})\mu \rangle = 0$. By linearity we may assume that the rank of $A_{\mathfrak{n}}$ is one, and that $A_{\mathfrak{n}} e_1 = e_1$ for some vector $e_1 \in \mathfrak{a}$ of norm one. Then,

$$\langle \pi(A_{\mathfrak{a}})\mu, \pi(A_{\mathfrak{n}})\mu \rangle = 2 \cdot \langle \text{ad}_{\pi(A_{\mathfrak{a}})\mu} e_1, \text{ad}_{\pi(A_{\mathfrak{n}})\mu} e_1 \rangle = -2 \cdot \langle \text{ad}_{\mu} e_1, [A_{\mathfrak{n}}, \text{ad}_{\mu} e_1] \rangle = -2 \text{tr} A_{\mathfrak{n}} [\text{ad}_{\mu} e_1, (\text{ad}_{\mu} e_1)^t],$$

and the last expression vanishes since for a solvsoliton $\mu$ we have that $\text{ad}_{\mu} e_1$ is a normal operator, by [Lau11b, Theorem 4.8].

Having this at hand, we may prove the statement of the lemma separately for $\mathfrak{Sym}(\mathfrak{a})$ and $\mathfrak{p}_\beta^n$. On $A_{\mathfrak{n}} \in \mathfrak{p}_\beta^n$ we have that $P_\mu = \frac{1}{2} S \circ \delta^{\mu}_\beta$, and the claim is clear. Finally, for $A_{\mathfrak{a}} \in \mathfrak{Sym}(\mathfrak{a})$, we may apply Lemma 8.4 (i) to obtain

$$P_\mu(A_{\mathfrak{a}}) = \frac{d}{ds} \bigg|_{s=0} \text{Ric}^{\exp(sA_{\mathfrak{a}})\mu} = \frac{d}{ds} \bigg|_{s=0} \exp(-sA_{\mathfrak{a}}) \text{Ric}^{\mu} \exp(-sA_{\mathfrak{a}}) = -A_{\mathfrak{a}} \beta - \beta A_{\mathfrak{a}}.$$

Thus, $P_\mu(A_{\mathfrak{a}}) = 2 \cdot \|\beta\|^2 A_{\mathfrak{a}}$, since $\beta^\perp|_{\mathfrak{a}} = 0$, and $A_{\mathfrak{a}}$ is an eigenvector. \qed

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