Subalgebras of FA-presentable algebras

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ABSTRACT

Automatic presentations, also called FA-presentations, were introduced to extend finite model theory to infinite structures whilst retaining the solubility of fundamental decision problems. This paper studies FA-presentable algebras. First, an example is given to show that the class of finitely generated FA-presentable algebras is not closed under forming finitely generated subalgebras, even within the class of algebras with only unary operations. However, it is proven that a finitely generated subalgebra of an FA-presentable algebra with a single unary operation is itself FA-presentable. Furthermore, it is proven that the class of unary FA-presentable algebras is closed under forming finitely generated subalgebras, and that the membership problem for such subalgebras is decidable.

1 INTRODUCTION

Automatic presentations, also known as FA-presentations, were introduced by Khoussainov & Nerode [KN95] to fulfill a need to extend finite model theory to infinite structures while retaining the solubility of interesting decision problems. Informally, an FA-presentation for a relational structure consists of a regular language of abstract representatives for elements of the structure such that the relations of the structure can be recognized by syn-

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chronous finite automata. FA-presentations have been considered for structures such as orders [KRS03, KRS05, Del04], graphs [KLM09], and groups, semigroups, and rings [OT05, CORT09, NT08].

This paper studies subalgebras of FA-presentable algebras. In the particular case of groups, it was already known that there exists an FA-presentation for the group $\mathbb{Z} \times \mathbb{Z}$ under which the sublanguage of representatives for elements of any non-trivial cyclic subgroups is not regular [NS09, §6]. However, such subgroups, like all abelian groups [OT05, Theorem 3], are FA-presentable with a different language of representatives. We construct an example of a finitely generated FA-presentable algebra that contains a non-FA-presentable finitely generated subalgebra (Example 3.3). This shows that the class of FA-presentable algebras is not closed under taking finitely generated subalgebras. Furthermore, this non-closure holds even within the class of algebras equipped with only unary operations (Remark 3.7). However, the class of FA-presentable algebras with a single unary operation is closed under forming finitely generated subalgebras (Proposition 3.2).

On the other hand, we prove that the class of algebras that admit unary FA-presentations (that is, FA-presentations over a one-letter alphabet) is closed under forming finitely generated subalgebras (Theorem 5.2). The proof depends on the sublanguage of representatives for elements of the subalgebra being regular and effectively constructible (Theorem 5.1), which also implies that the membership problem is decidable for such subalgebras (Theorem 5.3). We also prove that finitely generated unary FA-presentable algebras have growth level bounded by a linear function (Proposition 5.4).

These results for unary FA-presentations are proved using a new diagrammatic representation, developed in §4. This representation allows us to visualize and manipulate elements of a unary FA-presentable relational structure in a way that is more accessible than the corresponding arguments using languages and automata. In a forthcoming paper [CR], we deploy this representation in an analysis of unary FA-presentable binary relations. This representation is thus potentially a unifying framework in which to reason about unary FA-presentable algebraic and relational structures.

2 PRELIMINARIES

The reader is assumed to be familiar with the theory of finite automata and regular languages; see [HU79, Chs 2–3] for background reading. The empty word (over any alphabet) is denoted $\varepsilon$.

**Definition 2.1.** Let $L$ be a regular language over a finite alphabet $A$. Define, for $n \in \mathbb{N}$,

$$L^n = \{(w_1, \ldots, w_n) : w_i \in L \text{ for } i = 1, \ldots, n\}.$$  

Let $\$ be a new symbol not in $A$. The mapping $\text{conv} : (A^*)^n \to ((A \cup \{\$\})^n)^*$ is defined as follows. Suppose

$$w_1 = w_{1,1}w_{1,2} \cdots w_{1,m_1},$$

$$w_2 = w_{2,1}w_{2,2} \cdots w_{2,m_2},$$

$$\vdots$$

$$w_n = w_{n,1}w_{n,2} \cdots w_{n,m_n},$$
where \( w_{i,j} \in A \). Then \( \text{conv}(w_1, \ldots, w_n) \) is defined to be

\[
(w_{1,1}, w_{2,1}, \ldots, w_{n,1})(w_{1,2}, w_{2,2}, \ldots, w_{n,2}) \cdots (w_{1,m}, w_{2,m}, \ldots, w_{n,m}),
\]

where \( m = \max\{m_i : i = 1, \ldots, n\} \) and with \( w_{i,j} = s \) whenever \( j > m_i \).

Observe that the mapping \( \text{conv} \) maps an \( n \)-tuple of words to a word of \( n \)-tuples.

**Definition 2.2.** Let \( A \) be a finite alphabet, and let \( R \subseteq (A^*)^n \) be a relation on \( A^* \). Then the relation \( R \) is said to be *regular* if

\[
\text{conv}R = \{ \text{conv}(w_1, \ldots, w_n) : (w_1, \ldots, w_n) \in R \}
\]

is a regular language over \( (A \cup \{s\})^n \).

**Definition 2.3.** Let \( S = (S, R_1, \ldots, R_n) \) be a relational structure. Let \( L \) be a regular language over a finite alphabet \( A \), and let \( \phi : L \rightarrow S \) be a surjective mapping. Then \( (L, \phi) \) is an *automatic presentation* or an *FA-presentation* for \( S \) if, for all relations \( R_i \in \{=, R_1, \ldots, R_n\} \) of relation

\[
\Lambda(R, \phi) = \{(w_1, w_2, \ldots, w_{r_i}) \in L^{r_i} : R(w_1 \phi, \ldots, w_{r} \phi)\},
\]

where \( r \) is the arity of \( R_i \) is regular.

If \( S \) admits an FA-presentation, it is said to be *FA-presentable*.

If \( (L, \phi) \) is a FA-presentation for \( S \) and the mapping \( \phi \) is injective (so that every element of the structure has exactly one representative in \( L \)), then \( (L, \phi) \) is said to be *injective*.

If \( (L, \phi) \) is an FA-presentation for \( S \) and \( L \) is a language over a one-letter alphabet, then \( (L, \phi) \) is a *unary* FA-presentation for \( S \), and \( S \) is said to be *unary FA-presentable*.

Every FA-presentable structure admits an injective binary FA-presentation; that is, where the language of representatives is over a two-letter alphabet; see [KN95, Corollary 4.3] and [Blu99, Lemma 3.3]. Therefore the class of binary FA-presentable structures is simply the class of FA-presentable structures. However, there are many structures that admit FA-presentations but not unary FA-presentations: for instance, any finitely generated virtually abelian group is FA-presentable [OT05, Theorem 8], but unary FA-presentable groups must be finite [Blu99, Theorem 7.19]. Thus there is a fundamental difference between unary FA-presentable structures and all other FA-presentable structures.

The fact that a tuple of elements \( \{s_1, \ldots, s_n\} \) of a structure \( S \) satisfies a first-order formula \( \theta(x_1, \ldots, x_n) \) is denoted \( S \models \theta(s_1, \ldots, s_n) \).

**Proposition 2.4 ([KN95, Theorem 4.4]).** Let \( S \) be a structure with an FA-presentation \( (L, \phi) \). For every first-order formula \( \theta(x_1, \ldots, x_n) \) over the structure, the relation

\[
\Lambda(\theta, \phi) = \{(w_1, \ldots, w_n) \in L^n : S \models \theta(w_1 \phi, \ldots, w_n \phi)\}
\]

is regular, and an automaton recognizing it can be effectively constructed.

Proposition 2.4 is fundamental to the theory of FA-presentations and will be used without explicit reference throughout the paper.

The following important result shows that in the case of unary FA-presentations for infinite structures, we can assume that the language of representatives is the language of all words over a one letter alphabet:

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THEOREM 2.5 ([CRT12, Theorem 3.1]). Let $S$ be an infinite relational structure that admits a unary FA-presentation. Then $S$ has an injective unary FA-presentation $(a^*, \psi)$.

3 SUBALGEBRAS OF FA-PRESENTABLE ALGEBRAS

This section presents various contrasting results for finitely generated subalgebras of FA-presentable subalgebras. First, a positive result is proven: the class of FA-presentable algebras whose signature consists of a single unary operation is closed under forming finitely generated subalgebras (Proposition 3.2). Then we exhibit an example of a finitely generated FA-presentable algebra that admits a non-FA-presentable finitely generated subalgebra (Example 3.3). Although this example algebra is equipped with a binary operation, we note afterwards how it can be modified into an algebra with only unary operations (Remark 3.7), thus contrasting our previous positive result.

Although this section shows that the class of FA-presentable algebras is not closed under forming finitely generated subalgebras. However, closure under forming finitely generated subalgebras may hold within classes of FA-presentable algebras of a particular type. For instance, the following result holds:

PROPOSITION 3.1. Every finitely generated subgroup of an FA-presentable group is FA-presentable.

Proof of 3.1. A finitely generated subgroup of an FA-presentable group is virtually abelian by [NT08, Theorem 10(i)] and hence FA-presentable by [OT05, Theorem 3].

PROPOSITION 3.2. Let $S = (S, \sigma)$ be an algebra, where $\sigma$ is a unary operation. Then every finitely generated subalgebra of $S$ is FA-presentable.

Proof of 3.2. Let $X = \{x_1, \ldots, x_n\}$ be a finite subset of $S$ and let $T$ be the subalgebra of $S$ that $X$ generates. We will inductively define FA-presentations $(L_i, \phi_i)$ for the subalgebras $T_i$ of $S$ generated by $X_i = \{x_1, \ldots, x_i\}$. To avoid having to treat the case $i = 1$ separately, formally define $L_0$ to be the empty language, $\sigma_0$ to be the empty map and $\Lambda(\sigma, \phi_0)$ to be the empty relation.

So suppose we have an FA-presentation $(L_{i-1}, \phi_{i-1})$ for the subalgebra $T_{i-1}$ generated by $X_{i-1}$, where $i \in \{1, \ldots, n\}$. Notice that the subalgebra generated by $X_i$ consists of elements in the set $L_{i-1}\phi_{i-1} \cup \{x_i^{\sigma^k} : k \in \mathbb{N}^0\}$. There are three cases to consider:

1. For some $k \in \mathbb{N}^0$, the element $x_i^{\sigma^k}$ lies in $L_{i-1}\phi_{i-1}$, with $u\phi_{i-1} = x_i^{\sigma^k}$. In this case, let

$$L_i = L_{i-1} \cup \{p_{i,j} : j \in \{0, \ldots, k-1\}\};$$

then $L_i$ is regular. Define $\phi_i$ by $\phi_i|_{L_{i-1}} = \phi_{i-1}$ and $p_{i,j}\phi_i = x_i^{\sigma^j}$ for $j \in \{0, \ldots, k-1\}$. Then

$$\Lambda(\sigma, \phi_i) = \Lambda(L_{i-1}, \phi_{i-1})$$

$$\cup \{(p_{i,j}, p_{i,j+1}) : j \in \{0, \ldots, k-2\}\} \cup \{(p_{i,k-1}, u)\}$$

is a union of regular relations and is thus regular.
2. The element $x_i\sigma^k$ does not lie in $L_{i-1}\phi_{i-1}$ for any $k \in \mathbb{N}^0$, but $x_i\sigma^k = x_i\sigma^{k+m}$ for some $k \in \mathbb{N}^0$ and $m \in \mathbb{N}$. In this case, let

$$L_i = L_{i-1} \cup \{p_{i,j} : j \in \{0, \ldots, k + m - 1\}\};$$

then $L_i$ is regular. Define $\phi_i$ by $\phi_i|_{L_{i-1}} = \phi_{i-1}$ and $p_{i,j}\phi_i = x_i\sigma^j$ for $j \in \{0, \ldots, k + m - 1\}$. Then

$$\Lambda(\sigma, \phi_i) = \Lambda(L_{i-1}, \phi_{i-1}) \cup \{(p_{i,j}, p_{i,j+1}) : j \in \{0, \ldots, k + m - 2\}\} \cup \{(p_{i,k+m-1}, p_{i,k})\}$$

is a union of regular relations and is thus regular.

3. The element $x_i\sigma^k$ does not lie in $L_{i-1}\phi_{i-1}$ for any $k \in \mathbb{N}^0$, and all $x_i\sigma^k$ are distinct. In this case, let

$$L_i = L_{i-1} \cup i\alpha^*,$$

where $i$ is treated as a symbol $\alpha$ is a new symbol; then $L_i$ is regular. Define $\phi_i$ by $\phi_i|_{L_{i-1}} = \phi_{i-1}$ and $i\alpha^j\phi_i = x_i\sigma^j$ for $j \in \mathbb{N}^0$. Then

$$\Lambda(\sigma, \phi_i) = \Lambda(L_{i-1}, \phi_{i-1}) \cup \{(i\alpha^j, i\alpha^{j+1}) : j \in \mathbb{N}^0\}$$

is a union of regular relations and is thus regular.

In any case, $(L_{i+1}, \phi_{i+1})$ is an FA-presentation for the subalgebra $T_i$. In particular, $T = T_n$ is FA-presentable.

Note, however, that the proof of Proposition 3.2 does not yield an effective algorithm for constructing the subalgebra. If such an algorithm existed, reachability in the configuration graphs of deterministic Turing machines would be soluble.

Example 3.3. The example algebra $X$ will consist of the disjoint union of a semilattice and two copies of the configuration graph of a Turing machine, augmented by extra unary operations.

For each $i \in \mathbb{N}$, let $M_i$ be a chain of $2^i$ elements. Let $S$ be the zero-union of all the $M_i$; the zero of $S$ is denoted $0_S$. Notice that $S$ is a semilattice and can either be viewed as a partially-ordered set or a semigroup where the multiplication is the meet operation.

Let $\Sigma$ be a deterministic Turing machine that generates sequences of symbols $a_j^i$, where $j \in \mathbb{N}$. More precisely, $\Sigma$ starts with an empty tape, performs some computation and arrives in a distinguished state $q_\square$ with its tape contents being $a_1^1$, then computes again and reaches state $q_\square$ with its contents being $a_2^2$. In general at various points during its computation $\Sigma$ has tape contents $a_j^i$ for every $j \in \mathbb{N}$, and $\Sigma$ enters state $q_\square$ exactly when its tape contents are $a_j^i$ for some $j \in \mathbb{N}$. Notice that $\Sigma$ runs forever without halting. Suppose $Q$ is the state set and $B$ the tape alphabet of $\Sigma$.

Recall that an instantaneous description, or configuration, of $\Sigma$ consists of its state, its tape contents, and the position of its read/write head on its tape. The configuration graph of $\Sigma$ is an infinite graph whose vertices are all conceivable configurations of $\Sigma$, with a directed edge from $g$ to $g'$ precisely if $\Sigma$, when in configuration $g$, can make a single computation step and reach configuration $g'$. Note that in general not all configurations are reachable from the initial configuration.
Let \( G_T \) and \( G_\perp \) be two copies of the configuration graph of \( \Sigma \). The carrier set for the algebra \( \mathcal{X} \) will be \( \mathcal{X} = S \cup G_T \cup G_\perp \). The semilattice \( S \) is already equipped with a multiplication \( \circ \); extend this multiplication to \( \mathcal{X} \) by defining \( g \circ g' = g \) and \( g \circ s = s \circ g = g \) for \( g, g' \in G_T \cup G_\perp \) and \( s \in S \). The configuration graph \( G_\perp \) is equipped with a directed edge relation \( \delta \). Since \( \Sigma \) is deterministic, each vertex of the graph has outdegree 1, and so the relation \( \delta \) can be viewed as a unary operation. Extend \( \delta \) to \( \mathcal{X} \) by \( x \delta = x \) for all \( x \in S \cup G_T \cup G_\perp \). We emphasize that \( \delta \) acts like a computation step by \( \Sigma \) in the configuration graph \( G_\perp \), but acts like the identity map on the configuration graph \( G_T \).

Now define three new unary operations. First, \( \alpha \) sends each configuration in \( G_T \) to the corresponding configuration in \( G_\perp \), and otherwise (for all elements of \( S \cup G_\perp \)) acts like the identity map. Second, \( \beta \) sends each element of a chain \( M_i \) to the element immediately below it in that same \( M_i \), sends the minimum element of each \( M_i \) to \( 0_S \), and otherwise (for all elements of \( \{0_S\} \cup G_T \cup G_\perp \)) acts like the identity map. Third, \( \gamma \) maps configurations in \( G_\perp \) with state \( q \) and tape contents \( a^k \) to the maximum element of the chain \( M_k \), and otherwise (for all other elements of \( G_\perp \) and all elements of \( \{0_S\} \cup G_T \cup G_\perp \)) acts like the identity map. We will shortly define yet another unary operation \( \zeta \), but we must first set up the FA-presentation. The algebra \( \mathcal{X} \) will be
Let $I$ be the language $\{ z \} \cup \{ 0,1 \}^* \cup \{ T, \bot \} B^* QB^*$, where $z$ is a new symbol not in $B$ or $Q$. Define $\phi : I \to X$ as follows:

- $z \phi = 0_S$.
- If $u \in \{ 0,1 \}^k$, then $u \phi$ is the $u$-th element (interpreting $u$ as a binary number) from the bottom in $M_k$. (Notice that since $M_k$ contains exactly $2^k$ elements, $\phi$ restricts to a bijection between $\{ 0,1 \}^k$ and $M_k$.)
- If $t \in \{ T, \bot \}$, $u,v \in B^*$ and $q \in Q$, then $(tuqv) \phi$ is the configuration in $G_t$ where the state is $q$, the tape contains $uv$, and the head points to the first symbol in $v$.

Let us first show that the definition of FA-presentability is satisfied for the operations $\circ, \alpha, \beta, \gamma, \delta$. 

To see that $\Lambda(\circ, \phi)$ is regular, it is simplest to notice that the $\Lambda(\leq, \phi)$ is regular, where $\leq$ is the order on the semilattice $S$, since an automaton recognizing $\Lambda(\leq, \phi)$ must simply compare the lengths of two strings over $\{ 0,1 \}^*$ and compare them as binary numbers, and also always accept if the left-hand input word is $z$ and the right lies in $\{ z \} \cup \{ 0,1 \}^*$. Then, since $\circ$ is first-order definable in terms of $\leq$, it follows that $\Lambda(\circ, \phi)$ is regular.

Next, 

$$
\Lambda(\alpha, \phi) = \{ (T uq, v, \bot) : u, v \in B^*, q \in Q \}
\cup \{ (w, w) : w \in \{ z \} \cup \{ 0,1 \}^* \cup \bot B^* QB^* \}
$$

is clearly regular.

An automaton recognizing $\Lambda(\beta, \phi)$ need only decrement a binary number by 1, recognize $\{ 0^k, z \}$, and recognize the identity relation on $\{ z \} \cup \{ 0,1 \}^* B^* QB^*$.

Now,

$$
\Lambda(\gamma, \phi) = \{ (\bot a^k q a^l, 1^{k+1}) : k, l \in \mathbb{N}^0 \} \cup \{ (u, u) : u \in L - \bot a^* Q a^* \},
$$

which is easily seen to be regular.

The relation $\Lambda(\delta, \phi)$ is easily seen to be regular, since each computation of a Turing machine makes only a small localized change to the configuration as represented by words in $B^* QB^*$; see [KN95, p. 374].

We can now define our last operation $\zeta$. Let $\sqsubseteq$ be the length-plus-lexicographic ordering of words in $\top B^* QB^*$ induced by some order on $\{ T \} \cup B \cup Q$. For any element $g \in G_\bot$, define $g \zeta$ as follows. Let $u$ be the unique word in $\top B^* QB^*$ with $u \phi = g$. Let $u'$ be the word in $\top B^* QB^*$ that succeeds $u$ in the $\sqsubseteq$ ordering. Then $g \zeta$ is defined to be $u' \phi$. For all $x \in S \cup G_\bot$, define $x \zeta = x$. Notice that 

$$
\Lambda(\zeta, \phi) = \{ (u, u') : u \in \top B^* QB^* \wedge (u \sqsubseteq u') \\
\wedge (\forall v \in \top B^* QB^* ) (u \sqsubseteq v \implies u' \sqsubseteq v) \}
\cup \{ (w, w) : w \in L - \top B^* QB^* \}
$$

is regular since an automaton can recognize the $\sqsubseteq$ relation.

Thus $(I, \phi)$ is an FA-presentation for the algebra $X = (X, \circ, \alpha, \beta, \gamma, \delta, \zeta)$.

By Lemma 3.4 below, the algebra $X$ is finitely generated. Let $g_0$ be the initial configuration of $\Sigma$ in the configuration graph $G_\bot$. Let $Y$ be the subalgebra generated by $g_0$. Then $Y$ is not FA-presentable by Lemma 3.5 below.
Lemma 3.4. The algebra $X$ is finitely generated.

Proof of 3.4. Let $u$ be the $\sqsubseteq$-minimal word in $\top \mathcal{B}^* \mathcal{Q} \mathcal{B}^*$. Let $T$ be the set of elements in the subalgebra generated by $u \phi \in G_T$; the aim is to show that $T = X$.

By repeated application of the operation $\zeta$ to $u \phi$, all elements of $G_T$ lie in $T$. By applying $\alpha$ to elements of $G_T$, all elements of $G_L$ lie in $T$. By applying $\gamma$ to those configurations in $G_L$ where the state is $q_\square$ and the tape contains $a^k$ for some $k \in \mathbb{N}$, the maximum elements of each chain $M_i$ lie in $T$. By repeatedly applying $\beta$ to these maximum elements, all elements of the chains $M_i$ lie in $T$, as does $0_S$. Hence all elements of $X$ lie in $T$ and so $X = T$.

Lemma 3.5. The subalgebra $Y$ is not FA-presentable.

Proof of 3.5. The first step is to show that the subalgebra $Y$ contains the chains $M_{12}$ and no other chains $M_i$.

Recall that $Y$ is generated by $g_0$, the initial configuration of $X$ in $G_{\perp}$. The operation $\delta$ applied repeatedly to $g_0$ yields every element reachable from $g_0$ in the configuration graph $G_{\perp}$. Let $H$ be the set of these reachable elements. By the definition of $X$, the set $H$ includes configurations with state $q_\square$ and tape contents $a^j$ for all $j \in \mathbb{N}$. Furthermore, the definition of $X$ ensures that $H$ includes no other configuration with state $q_\square$. The operation $\gamma$ applied to $H$ yields the maximum element of every $M_{12}$ (where $j \in \mathbb{N}$). The operation $\beta$ now yields all elements of each $M_{12}$ and also yields $0_S$. So $Y$ contains the set $Y = \{0_S\} \cup H \cup \bigcup_{j \in \mathbb{N}} M_{12}$. It is easy to see that $Y$ is closed under every operation. So the domain of $Y$ is $Y$. In particular, $Y$ contains the chains $M_{12}$ and no other chains $M_i$.

Now suppose, with the aim of obtaining a contradiction, that $Y$ admits an injective automatic presentation $(L, \phi)$. Let

$$Y_1 = \{ y \in Y : y \circ 0_S = 0_S \};$$

notice that $Y_1 = \{0_S\} \cup \bigcup_{i \in \mathbb{N}} M_{12}$. Notice further that $Y_1$ is defined by a first order formula and so $L_1 = Y_1 \phi^{-1}$ is regular. Observe that the order relation $\leq$ on the subsemilattice $Y_1$ is first-order definable in terms of $\circ$. Let

$$K_1 = \{ u \in L_1 : (\forall v) ( (u \phi \leq v \phi) \implies (u \phi = v \phi) ) \}.$$  \hspace{1cm} (3.2)

Then $K_1$ consists of representatives in $L$ of the maximum elements in the various sub-chains $M_{12}$ of $Y_1$. Since it is defined by a first-order formula, $K_1$ is regular. Let

$$K_2 = \{ u \in K_1 : (\forall v \in L_1) ((v \in K_1 \land |v| = |u|) \implies (v \sqsubseteq u)) \};$$

then $K_2$ consists of length-plus-lexicographically minimal words of each length in $K_1$. The language $K_2$ is regular. The relation

$$R_1 = \{ (u, v) : (u \in K_2) \land (u \phi \geq v \phi) \land (v \phi \neq 0_S) \};$$

is regular. Notice that $R_1$ relates a word $u \in K_2$, which represents the maximum element of some chain $M_{12}$, to all the words $v$ representing elements of that chain. Let $n$ be the number of states in an automaton recognizing $\text{conv}(R_1)$. 

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If \((u, v) \in R_1\), then \(|v| \leq |u| + n\), for otherwise one could pump the subword of \(v\) that extends beyond \(u\) to obtain infinitely many words representing elements of a single \(M_{j_2}\), which would entail infinitely many distinct elements of \(M_{j_2}\) (since \(\phi\) is injective), which is a contradiction.

Let 
\[ R_2 = \{(u^\#^n, v) : (u, v) \in R_1\}, \]
where \# is a new symbol. By the observation in the last paragraph, if \((u, v) \in R_2\), then \(|u| \geq |v|\). Furthermore, if \((u, v), (u', v') \in R_2\) and \(|u| = |u'|\), then \(u = u'\) by the definition of \(R_2\) and \(R_2\). Moreover, no word in \(\text{conv}(R_2)\) contains a letter whose left-hand component is \$. Therefore the number of words of length \(k\) in \(\text{conv}(R_2)\) is either \(\emptyset\) or, if there is a word \(u \in K_2\) of length \(k - n\), the number of possible words \(v\) such that \((u^\#^n, v)\) lies in \(R_2\), which is in turn the number of elements of the chain \(M_{j_2}\) in which \(u\phi\) lies, which is \(2^{12}\).

Let \(z_k\) be the number of words in \(\text{conv}(R_2)\) of length \(k\). By the observation in the last paragraph, whenever \(z_k\) is non-zero, it is the number of elements in some chain \(M_{j_2}\). Since \(\text{conv}(R_2)\) is a regular language, the generating function
\[ f(x) = \sum_{k=0}^{\infty} z_k x^k \]
is a rational function with no singularity at \(0\). Thus the radius of convergence of its power series expansion must be strictly greater than zero. The aim is to obtain a contradiction by showing that this power series has radius of convergence zero.

By the pumping lemma for regular languages, there are constants \(p, q\) such that \(z_{p+kq}\) is non-zero for all \(k \in \mathbb{N}^0\). So for every \(k \in \mathbb{N}^0\), there exists \(k\theta \in \mathbb{N}^0\) such that \(z_{p+kq} = 2^{(k\theta)^2}\). This defines an injection \(\theta : \mathbb{N}^0 \to \mathbb{N}^0\). By Lemma 3.6 below, \(k \leq k\theta\) for infinitely many values of \(k \in \mathbb{N}^0\). So by choosing \(k\) to be large enough and also satisfying \(k \leq k\theta\), the value
\[ |z_{p+kq}|^{1/(p+kq)} = 2^{(k\theta)^2/(p+kq)} = 2^{(k\theta)^2/(p+kq)} \]
can be made arbitrarily large. Therefore
\[ \lim sup_{k \to \infty} |z_{p+kq}|^{1/(p+kq)} = \infty, \]
and hence \(\lim sup_{k \to \infty} |z_k|^{1/k} = \infty\), from which it follows that the radius of convergence of the power series \(\sum_{k=0}^{\infty} z_k x^k\) is zero.

**Lemma 3.6.** Let \(\theta : \mathbb{N}^0 \to \mathbb{N}^0\) be an injection. Then there are infinitely many \(i \in \mathbb{N}\) such that \(i \leq i\theta\).

**Proof of 3.6.** Suppose, with the aim of obtaining a contradiction, that there are only finitely many \(i \in \mathbb{N}^0\) such that \(i \leq i\theta\). Let \(I = \{i \in \mathbb{N} : i \leq i\theta\}\); by supposition, \(I\) is finite. Let \(m = \max(I)\) and \(n = \max(I\theta)\). Then \(m \leq m\theta\) and \(m\theta \leq n\), so \(m \leq n\). Furthermore, \(i\theta < i\) for \(i \notin I\), and \(i\theta \leq n\) for \(i \in I\). Hence \(i\theta \leq n\) for all \(i \leq n\). Since \(m = \max(I)\) and \(m \leq n\), it follows that \(n + 1 \notin I\) and so \((n + 1)\theta < n + 1\). Putting the last two sentences together shows that \(\{0, \ldots, n + 1\} \cap \{0, \ldots, n\}\), which contradicts \(\theta\) being an injection. Thus there are infinitely many \(i \in \mathbb{N}^0\) such that \(i\theta \geq i\).
Remark 3.7. In Example 3.3, the algebra $\mathcal{X}$ has exactly one binary operation, namely the multiplication $\circ$. However, this is not used for the finite generation of $\mathcal{X}$ or $\mathcal{Y}$, and is used in only three places in the proof of Lemma 3.5, namely (3.1), (3.2), and (3.3). (In the latter two, it is hidden within the first-order definition of the order $\preceq$ on $S$.)

However, we can modify $\mathcal{X}$ by removing $\circ$ and adding two new unary operations $\lambda$ and $\mu$ to obtain a new algebra $\mathcal{X}'$, where the subalgebra $\mathcal{Y}'$ generated by $g_0$ has the same domain $Y$ as $\mathcal{Y}$ and where the $\mathcal{Y}'$ can be proved to be non-FA-presentable in the same way. Hence, even the class of FA-presentable algebras with only unary operations is not closed under forming finitely generated subalgebras.

The first operation $\lambda$ sends every element of each chain $M_i$ to the maximum element of that chain, and acts like the identity map elsewhere (that is, on $\{0_S\} \cup G_T \cup G_\perp$). The second operation $\mu$ sends every element of $S$ to $0_S$ and acts like the identity map elsewhere (on $G_T \cup G_\perp$). Notice that

$$\Lambda(\lambda, \phi) = \{(u, 1^{|u|}) : u \in \{0, 1\}^* \} \cup \{(u, u) : u \in \{z\} \cup \{\top, \perp\}^*B^*QB^*\}$$

$$\Lambda(\mu, \phi) = \{(u, z) : u \in \{z\} \cup \{0, 1\}^* \} \cup \{(u, u) : u \in \{\top, \perp\}^*B^*QB^*\};$$

both relations are clearly regular. Thus $\mathcal{X}' = (X, \alpha, \beta, \gamma, \delta, \zeta, \lambda, \mu)$ is FA-presentable and has only unary operations.

To prove that the subalgebra $\mathcal{Y}'$ (generated by $g_0$) is not FA-presentable, follow the proof of Lemma 3.5, with the following definitions for $\mathcal{Y}'$, $K_1$, and $R_1$ replacing (3.1), (3.2), and (3.3):

$$Y_1 = \{y \in Y : y\mu = 0_S\},$$

$$K_1 = \{u \in L_1 : ((u\phi)\lambda = u\phi) \wedge (u\phi \neq 0_S)\},$$

$$R_1 = \{(u, v) : (u \in K_2) \wedge (u\phi = v\phi)\lambda\}. $$

Note that these are first order definitions in terms of the new signature in which $\lambda$ and $\mu$ replace $\circ$.

4 Diagrams for Unary FA-Presentations

This section develops a diagrammatic representation for unary FA-presentations. In the following section, we apply this representation to prove results about subalgebras of unary FA-presentable algebras.

Let $(a^*, \phi)$ be an injective unary FA-presentation for a relational structure $S$ with relations $R_1, \ldots, R_n$; such an FA-presentation exists by Theorem 2.5. For each $i \in \{1, \ldots, n\}$, let $A_i$ be a deterministic $r_i$-tape automaton recognizing $\Lambda(R_i, \phi)$, where $r_i$ is the arity of $R_i$. Let us examine the structure of the automata $A_i$. For ease of explanation, view $A_i$ as a directed graph with no failure states: $A_i$ fails if it is in a state and reads a symbol that does not label any outgoing edge from that state.

Define a partial order $\preceq$ on elements of $(a, \$)^{r_i}$ by $(x_1, \ldots, x_{r_i}) \preceq (x'_1, \ldots, x'_{r_i})$ if and only if $x'_i = \$ \implies x_i = \$ for all $i$. Since $A_i$ recognizes words in $\text{conv}((a^*)^{r_i})$, it will only successfully read words consisting of a $\preceq$-decreasing sequence of tuples in $(a, \$)^{r_i}$. Thus an edge labelled by a tuple $b$ leads to a state all of with outgoing edges are labelled by $\preceq$-preceding tuples.

Since $A_i$ is deterministic, while it reads letters of a fixed tuple $b \in (a, \$)^{r_i}$, it follows a fixed path which, if the string of letters $b$ is long enough, will form
Figure 2. Example automaton recognizing $\Lambda(R_1,\phi)$ where $r_1$ is 2. Edges labelled $(a,a)$ form a path that leads into a uniquely determined loop. From this path and loop paths labelled by $(a,$ or $(\$, a) branch off. (Notice that $(a,$, $(\$, a) $\prec$ $(a,a)$.)

Consider what this means in terms of the tuple $\bar{p} = (a^{p_1}, \ldots, a^{p_n})$ such that $\text{conv}(\bar{p}) = uvw$. Since $v' \in b^*$, it follows that

$$uv'(v'')^{m+1}v'''w = \text{conv}(a^{p_1+q_1}, \ldots, a^{p_n+q_n}),$$

where

$$q_i = \begin{cases} 0 & \text{if } p_j \leq |u| \\ D & \text{if } p_j \geq |uw| \end{cases}.$$  

(Note that either $p_j \leq |u|$ or $p_j \geq |uw|$ since $v \in b^*$ for a fixed $b \in \{A \cup \{\}\}^r$.) Therefore we have the following:

**PUMPING RULE 1.** If the components of a tuple in $\Lambda(R_1,\phi)$ can be partitioned into those that are of length at most $l \in \mathbb{N}$ and those that have length at least $l + D$, then [the word encoding] this tuple can be pumped so as to increase by $D$ the lengths of those components that are at least $l + D$ letters long and yield another [word encoding a] tuple in $\Lambda(R_1,\phi)$.

(Notice that this also applies when all components have length at least $D$; in this case, set $l = 0$.)
With the same setup as above, suppose $|v| \geq 2D$. Then $A_i$ must follow the loop labelled by $v''$ starting at $q'$ at least $m = D/|v''|$ times. That is, $v$ factorizes as $v'(v'')^m \hat{v}''$. By the pumping lemma, $uvv'' \in \text{conv}(L(A_i))$ and $|v''| = |v| - D$. Therefore, we also have the following:

**Pumping rule 2.** If the components of a tuple in $\Lambda(R_i, \phi)$ can be divided into those that are of length less than $1 \in \mathbb{N}$ and those that have length at least $l + 2D$, then [the word encoding] this tuple can be pumped so as to decrease by $D$ the length of those components that are at least $l + 2D$ letters long and yield another [word encoding a] tuple in $\Lambda(R_i, \phi)$.

This ability to pump so as to increase or decrease lengths of components by a constant $D$ lends itself to a very useful diagrammatic representation of the unary FA-presentation $\{a^*, \phi\}$. Consider a grid of $D$ rows and infinitely many columns. The rows, from bottom to top, are $B[0], \ldots, B[D - 1]$. The columns, starting from the left, are $C[0], C[1], \ldots$. The point in column $C[x]$ and row $B[y]$ corresponds to the word $a^x a^{D+y}$. For example, in the following diagram, the distinguished point is in column $C[3]$ and row $B[2]$ and so corresponds to $a^{3D+2}$.

![Diagram](image)

The power of such diagrams is due to a natural correspondence between pumping as in **Pumping rules 1 and 2** and certain simple manipulations of tuples of points in the diagram. Before describing this correspondence, we must set up some notation. We will not distinguish between a point in the grid and the word to which it corresponds. The columns are ordered in the obvious way, with $C[x] < C[x']$ if and only if $x < x'$. Extend the notation for intervals on $\mathbb{N}$ to intervals of contiguous columns. For example, for $x, x' \in \mathbb{N}$ with $x \leq x'$, let $C[x, x')$ denotes the set of elements in columns $C[x], \ldots, C[x' - 1]$, and $C(x, \infty)$ denotes the set of elements in columns $C[x + 1], C[x + 2], \ldots$. For any element $u \in a^*$, let $c(u)$ be the index of the column containing $u$.

Consider the components of an $r_1$-tuple $\vec{p}$ in $\Lambda(R_i, \phi)$, viewed as an $r_1$-tuple of points in the diagram. If there is a column $C[x]$ that contains none of the components of $\vec{p}$, then all the components that lie in $C[0, x)$ are at least $D$ shorter than those lying in $C(x, \infty)$. Hence the word encoding the tuple $\vec{p}$ can be pumped between these two sets of components in accordance with **Pumping rule 1**. This corresponds to shifting all those components that lie in $C(x, \infty)$ rightwards by one column. The tuple that results after this rightward shift of some components also lies in $\Lambda(R_i, \phi)$. [Notice in particular that if column $C[0]$ contains none of the components of $\vec{p}$, then every component can be shifted right by one column, giving a new tuple that also lies in $\Lambda(R_i, \phi)$.] This rightward shifting of components can be iterated arbitrarily many times to yield new tuples. Thus we have the following diagrammatic version of **Pumping rule 1**:
**Shift rule 1.** Consider the components of an \( \tau_1 \)-tuple \( \vec{p} \) in \( \Lambda(R_1, \phi) \), viewed as an \( \tau_1 \)-tuple of points in the diagram. If there is a column \( C[x] \) that contains none of the components of \( \vec{p} \), then for any \( k \in \mathbb{N} \), shifting the components in \( C(x, \infty) \) to the right by \( k \) columns yields a tuple that also lies in \( \Lambda(R_1, \phi) \).

Similarly, if there are two adjacent columns \( C[x] \) and \( C[x+1] \) that contain none of the components of \( \vec{p} \), then every component in \( C[0, x] \) is at least 2D shorter than every component in \( C(x+1, \infty) \). Therefore the word encoding this tuple can be pumped between these sets of components in accordance with **Pumping rule 2.** This corresponds to shifting all components in \( C(x+1, \infty) \) leftwards by one column. The tuple that results after this leftward shift of some components also lies in \( \Lambda(R_1, \phi) \). [Notice in particular that if columns \( C[0] \) and \( C[1] \) contain none of the components of \( \vec{p} \), then every component can be shifted left by one column, giving a new tuple that also lies in \( \Lambda(R_1, \phi) \).]

This leftward shifting of components can be iterated to yield new tuples for as long as the two columns \( C[x] \) and \( C[x+1] \) do not contain any elements of the latest tuple. Thus we have the following diagrammatic version of **Pumping rule 2.**

**Shift rule 2.** Consider the components of an \( \tau_1 \)-tuple \( \vec{p} \) in \( \Lambda(R_1, \phi) \), viewed as an \( \tau_1 \)-tuple of points in the diagram. If the columns in \( C[x, x+h] \) contain none of the components of \( \vec{p} \), then for any \( k \) with \( 0 < k \leq h \), shifting the components in \( C(x+h, \infty) \) to the left by \( k \) columns yields a tuple that also lies in \( \Lambda(R_1, \phi) \).

For convenience, define for every \( n \in \mathbb{Z} \) a partial map \( \tau_n : a^* \rightarrow a^* \), where \( a^k \tau_n \) is defined to be \( a^{k+nD} \) if \( k+nD \geq 0 \) and is otherwise undefined. Notice that if \( n \geq 0 \), the map \( \tau_n \) is defined everywhere. In terms of the diagram, \( a^k \tau_n \) is the element obtained by shifting \( a^k \) to the right by \( n \) columns if \( n \geq 0 \) and to the left by \( -n \) columns if \( n < 0 \). The values of \( k \) and \( n < 0 \) for which \( a^k \tau_n \) are undefined are precisely those where shifting \( a^k \) to the left by \( -n \) columns would carry it beyond the left-hand edge of the diagram.

**Example 4.1.** In order to illustrate Shift rules 1 and 2, consider a 4-tuple \( \vec{p} = (a^{2D+1}, a^{D+2}, a^{5D+3}, a^{7D+2}) \). This corresponds to the following four points in the diagram.

\[
\begin{array}{c}
\text{B}[-1] \\
\text{B}[4] \\
\text{B}[3] \\
\text{B}[2] \\
\text{B}[1] \\
\text{B}[0] \\
\text{C}[0] \\
\text{C}[1] \\
\text{C}[2] \\
\text{C}[3] \\
\text{C}[4] \\
\text{C}[5] \\
\text{C}[6] \\
\text{C}[7] \\
\text{C}[8] \\
\end{array}
\]

1. The column \( C[0] \) contains no components of \( \vec{p} \), so, by **Shift rule 1**, for any \( k \in \mathbb{N} \), all components can be shifted rightward by \( k \) columns, yielding the tuple \( (a^{2+2k}D+1, a^{1+k}D+2, a^{5+k}D+3, a^{7+k}D+2) \).

2. The columns \( C[3] \) and \( C[4] \) contain no components of \( \vec{p} \), so, by **Shift rule 1**, for any \( k \in \mathbb{N} \), the third and fourth components can be shifted rightwards by \( k \) columns, yielding the tuple \( (a^{2D+1}, a^{D+2}, a^{5+k}D+3, a^{7+k}D+2) \).
3. The column $C[6]$ contains no components of $\vec{p}$, so, by Shift rule 1, for any $k \in \mathbb{N}$, the fourth component of $\vec{p}$ can be shifted rightwards by $k$ columns, yielding the tuple $(a^{2D+1}, a^{D+2}, a^{5D+3}, a^{(7+k)D+2})$.

Shift rule 2 (or the corresponding Pumping rule 2) can be applied in only one way here: columns $C[3]$ and $C[4]$ contain no component of $\vec{p}$, so the third and fourth components can be shifted leftwards by one column, yielding the tuple $(a^{2D+1}, a^{D+2}, a^{4D+3}, a^{6D+2})$.

5 UNARY FA-PRESENTABLE ALGEBRAS

This section studies finitely generated subalgebras of unary FA-presentable algebras. The key result, Theorem 5.1, shows that the language representing elements of such a subalgebra is regular and that there is an algorithm that effectively constructs this language. From this it follows that the class of unary FA-presentable algebras is closed under taking finitely generated subalgebras (Theorem 5.2) and that the membership problem for finitely generated subalgebras is decidable (Theorem 5.3).

**Theorem 5.1.** Let $S$ be an algebra that admits an injective unary FA-presentation $(a^*, \phi)$ and let $T$ be a finitely generated subalgebra of $S$. Let $L$ be the sublanguage of $a^*$ consisting of representatives of elements of $T$. Then $L$ is regular, and an automaton recognizing $L$ can be constructed effectively from a finite set of words representing a generating set for $T$.

**Proof of 5.1.** We will first of all show $L$ is regular and then show how it can be constructed effectively.

**Regularity.** Let $r$ be the maximum arity of any of the operations in the signature of $S$. (That is, $r$ is the maximum of their arities qua operations, not qua relations.)

For $x, y \in \mathbb{N}^0$ with $x \leq y$, define new notation $L[x] = C[x] \cap L$ and $L[x, y] = C[x, y] \cap L$. Since the subalgebra $T$ is finitely generated, it is generated by the elements in $L[0, m] \phi$ for some $m \in \mathbb{N}^0$.

Suppose that $h, h' \in \mathbb{N} \cup \{0\}$, where $h' > h > m$, are such that

$$L[h, h + r] \phi [h'-h] \subseteq L[h', h' + r].$$

(5.1)

The immediate aim is to prove that

$$L[h + r + 1] \phi [h'-h] \subseteq L[h' + r + 1].$$

(5.2)

Because $h + r > m$, the elements of $L[0, h + r] \phi$ generate the elements of $L[h + r + 1] \phi$. That is, by applying the operations of $S$ to elements of $L[0, h + r] \phi$, one can obtain a finite sequence of points $p_1, \ldots, p_n \in L[h + r + 1, \infty)$ such that each $p_i \phi$ is obtained by a single application of some operation to elements from $\{L[0, h + r] \cup \{p_1, \ldots, p_{i-1}\}\} \phi$, and such that $L[h + r + 1] \phi \subseteq \{p_1, \ldots, p_n\} \phi$.

[It may be necessary for some $p_i$ to lie in columns to the right of $C[h + r + 1]$, in order to later generate the elements of $L[h + r + 1] \phi$.]

For each $i \in \{1, \ldots, n\}$, let $p'_i = p_i \phi [h'-h]$. The aim is to prove by induction on $i$ that $p'_i \in L$. We will show that, just as $p_i \phi$ is obtained by an application of some operation to elements from $\{L[0, h + r] \cup \{p_1, \ldots, p_{i-1}\}\} \phi$, so $p'_i \phi$ can be obtained by an application of the same operation to elements of

$$(L[0, h + r] \cup L[h', h' + r] \cup \{p'_1, \ldots, p'_{i-1}\} \phi [h'-h]) \phi.$$
Figure 3. The indices $h$ and $h'$ are such that an elements of $L[h,h+r]|_{\tau_{h'-h}} \subseteq L[h',h'+r]$. The point $p'_i$ is defined to be $p_i|_{\tau_{h'-h}}$. The solid lines indicate how $p_i\phi$ is obtained by an application of some operation to the elements $x\phi$, $y\phi$, $z\phi$, and $p_j\phi$ (where $j < i$).

Figure 4. The dotted lines indicate how $p'_i$ is obtained by an application of the same operation to the elements $x, y, z|_{\tau_{h'-h}}$, and $p'_j = p_j|_{\tau_{h'-h}}$.

So suppose that $p'_1, \ldots, p'_{l-1} \in L$. Suppose $p_i\phi = (x_1\phi, \ldots, x_k\phi)f$, where $f$ is an operation of arity $k \leq r$ and $x_1, \ldots, x_k \in L[0, h + r] \cup \{p_1, \ldots, p_{l-1}\}$. (See Figure 3.) Without loss of generality, assume that $c(x_1) \leq c(x_{j+1})$ for all $j \in \{1, \ldots, k-1\}$. Since $k \leq r$, there is at least one column $C'$ in $C[h], \ldots, C[h+r]$ that does not contain any point $x_1, \ldots, x_k$. Let $x_1, \ldots, x_k$ be the points lying to the left of this column, and $x_{j+1}, \ldots, x_k$ be those lying to the right.

For $l \in \{j + 1, \ldots, k\}$, let $x'_1 = x_1|_{\tau_{h'-h}}$. Recall that $p'_1 = p_1|_{\tau_{h'-h}}$. Now, since $x_1 \in L[h, h+r]$, it follows that $x'_1 \in L[h', h'+r]$ by (5.1). On the other hand, if $x'_1$ is one of the points $p'_1, \ldots, p'_{l-1}$, then it lies in $L$ by the induction hypothesis. The application of the operation $f$ to the elements $x_1\phi, \ldots, x_k\phi$ gives $p_1\phi$. Let $\bar{p} = (x_1, \ldots, x_k, p_1) \in \Lambda(f, \phi)$. Then by Shift rule 1, the tuple

$$\bar{q} = (x_1, \ldots, x_j, x'_1, x_{j+1}, \ldots, x_k, p'_1),$$

obtained by shifting rightwards the components $x_{j+1}, \ldots, x_k, p_L$, also lies in $\Lambda(f, \phi)$. Since all of $x_1\phi, x_j\phi$ and $x'_1\phi, \ldots, x'_k\phi$ lie in the subalgebra $\tau$, so does $p'_1\phi$. (See Figure 4.) Hence $p'_1 \in L$.

Therefore, by induction, all the points $p'_i$ lie in $L$, and hence condition (5.2) holds. Thus condition (5.1) entails condition (5.2).

Since each of the sets $L[h, h+r]$ contains at most $(r+1)D$ elements, there must exist $h, h' \in \mathbb{N} \cup \{0\}$ with $h' > h$ such that (5.1) holds. Fix two such values $h$ and $h'$. Then it follows by induction on $i$ that $L[i]|_{\tau_{h'-h}} \subseteq L[i+h'-h]$ for all $i \geq h$. Since the size of the sets $L[i]$ is bounded above by $D$, there exists $g \in \mathbb{N}$ such that $L[i]|_{\tau_{h'-h}} = L[i+h'-h]$ for all $i \geq g$. Thus

$$L = L[0, g-1] \cup (a^{D}(h'-h))^{*}L[g, g+h'-h-1]$$

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and so is regular.

Effective construction. Let \( L_0 \) be a finite set of words representing a generating set for the subalgebra \( \mathcal{T} \). We will inductively construct a sequence of regular sublanguages \( L_i \) of \( L \) for \( i \in \mathbb{N}^0 \). From some point onwards, every language in this sequence will be \( L \) itself. We will be able to detect when \( L_i = L \), but we cannot bound in advance the number of terms we must compute before obtaining \( L \). For all \( i, x, y \in \mathbb{N}^0 \) with \( x \leq y \), let \( L_i[x] = C[x] \cap L_i \) and \( L_i[x, y] = C[x, y] \cap L_i \).

Inductively define the language \( L_{i+1} \) as follows: find the minimal \( h \) such that there exists \( h' \) such that

\[
L_i[h, h + r] \tau_{h' - h} \subseteq L_i[h', h' + r].
\]

(Notice that this is (5.1) restated with \( L_i \) in place of \( L \).) Let \( h_i \) be \( h \) and let \( h_i' \) be minimal among corresponding such \( h' \), and let

\[
L_{i+1} = L_i \cup \{(a^{D(h' - h_i)} \ast L_i[h_i, h_i' - 1]) \cup K_{i+1}\}
\]

where

\[
K_{i+1} = \{(s_1, \ldots, s_{r_j}, x)f_j : j \in \{1, \ldots, k\}, s_1, \ldots, s_{r_j} \in L_i \phi, f_j \text{ has arity } r_j\} \phi^{-1}.
\]

[Notice that \( h_i \) and \( h_i' \) always exist since because the sets \( L_i[h_i, h_i' + r] \) are all of bounded size. Notice further that when \( L_i \) is finite, \( L_i[h_i, h_i' + r] \) may be empty. Observe that \( h_i \) and \( h_i' \) can be found simply by enumerating sets \( L_i[h_i, h_i + r] \).]

Let us prove by induction that \( L_i \subseteq L \). Clearly \( L_0 \subseteq L \). Suppose that \( L_i \subseteq L \). By the reasoning in used in the proof of regularity above, each element of \( (a^{D(h' - h_i)} \ast L_i[h_i, h_i' - 1]) \) lies in \( L \). The language \( K_{i+1} \) consists of representatives of elements obtained by applying the operations of \( \mathcal{S} \) to elements of \( L_i \phi \). Since \( \mathcal{T} \) is a subalgebra, every element of the language \( K_{i+1} \) thus lies in \( L \). Hence \( L_{i+1} \subseteq L \).

Furthermore, the language \( K_{i+1} \) consists of representatives of elements satisfying a first-order formula. Hence, if \( L_i \) is regular and given by a finite automaton, a finite automaton recognizing \( L_{i+1} \) can be effectively constructed. Since \( L_0 \) is finite, it follows by induction that every \( L_i \) is regular, and that for any \( i \in \mathbb{N}^0 \) an automaton recognizing \( L_i \) can be effectively constructed.

Notice further that \( L_i \subseteq L_{i+1} \) and that for any \( u \in L \), there exists some \( L_i \) such that \( u \in L_i \).

By the reasoning in the proof of regularity above, there exist \( g, h, h' \in \mathbb{N} \) be such that \( L_i[\tau_{h' - h}] = L_i[h + h' - h] \) for all \( i \geq g \). Note that \( L = L[0, g - 1] \cup (a^{D(h' - h)} \ast L[g, g + h' - h]) \). Let \( n \) be such that \( L[0, g + h' - h] \subseteq L_n \). Then, by definition, \( L_{n+1} \) contains \( L_n \) and \( (a^{D(h' - h)} \ast L[g, g + h' - h]) \). Hence \( L \subseteq L_{n+1} \).

Therefore, the algorithm constructing the various \( L_i \) will at some point construct \( L_{n+1} = L \). Furthermore, the algorithm can check whether \( L_i = L \) simply by checking whether \( L_i = L_{i+1} \), for if this holds, then \( L_i \phi \) is closed under all the operations of \( \mathcal{S} \) and hence must be the domain of the subalgebra \( \mathcal{T} \). Thus there is an effective procedure that constructs \( L \).

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**Theorem 5.2.** The class of unary FA-presentable algebras is closed under taking finitely presented subalgebras.
Theorem 5.2. Let $S$ be an algebra that admits an injective unary FA-presentation $(a^*,\phi)$ and let $T$ be a finitely generated subalgebra of $S$. Let $L$ be the sublanguage of $a^*$ consisting of representatives of elements of $T$. By Theorem 5.1, $L$ is regular, whence

$$\Lambda(R,\phi|_L) = \Lambda(R,\phi) \cap (L \times L \times \ldots \times L)^k$$

for any $k$-ary relation (or operation) $R$ of $S$, which shows that $(L,\phi|_L)$ is a unary FA-presentation for $T$.

Note that Theorem 5.2 does not hold without the hypothesis of finite generation: an arbitrary subsemigroup of a unary FA-presentable semigroup may not even be FA-presentable [CRT12, Example 9.4].

The following theorem deals with the membership problem for finitely generated subalgebras of unary FA-presentable algebras. This problem is not decidable for general FA-presentable algebras, because reachability in the configuration graph of a Turing machine is undecidable.

**Theorem 5.3.** There is an algorithm that takes a unary FA-presentation $(a^*,\phi)$ for an algebra $S$, a finite set $X$ of words in $a^*$, and a word $w \in a^*$, and decides whether $w\phi$ lies in the subalgebra generated by $X\phi$.

**Proof of 5.3.** By Theorem 5.1, there is an algorithm that takes the finite set of words $X$ and constructs the sublanguage $L$ of $a^*$ consisting of representatives of elements of the subalgebra $T$ generated by $X\phi$. To decide whether $w\phi$ lies in $T$, it remains to check whether $w$ lies in $L$.

Let $S$ be a finitely generated algebra. Let $f_1,\ldots,f_k$ be the operations of $S$. Let $G_0$ be a finite generating set for $S$. Inductively define the following finite sets for all $i \in \mathbb{N}$:

$$G_i = G_{i-1} \cup \{(s_1,\ldots,s_r)f_j : j \in [1,\ldots,k], s_1,\ldots,s_r \in G_{i-1}, f_j \text{ has arity } r_j\}.$$  

Define $g : \mathbb{N}^0 \to \mathbb{N}^0$ by $n \mapsto |G_n|$. The function $g$ is called the growth level of $S$ with respect to the generating set $G_0$. (This definition is taken from [KN95, §4].)

If $S$ is FA-presentable, then there exist constants $s,a,b \in \mathbb{N}$ such that $g(n) \leq s^{a+1+bn}$ for all $n \in \mathbb{N}^0$ [KN95, Lemma 4.5].

**Proposition 5.4.** If $S$ is a unary FA-presentable algebra, then there exist constants $a,b \in \mathbb{N}$ such that $g(n) \leq a + bn$ for all $n \in \mathbb{N}^0$.

**Proof of 5.4.** Let $(a^*,\phi)$ be an injective unary FA-presentation for $A$. Let $x \in \mathbb{N}^0$ be such that $G_0 \subseteq C[0,x]\phi$. The first aim is to prove, by induction, that $G_n \subseteq C[0,x+n]\phi$ for all $n \in \mathbb{N}^0$. This clearly holds for $n = 0$.

Let $f_j$ be an operation of $S$ whose arity is $r_j$. Let $u_1,\ldots,u_{r_j} \in G_1\phi^{-1} \subseteq C[0,x+n]$. Let $v \in a^*$ be such that $(u_1\phi_1,\ldots,u_{r_j}\phi)f_j = v\phi$. Suppose for reductio ad absurdum that $c(v) > x + n + 1$. Then no component of the tuple $(u_1,\ldots,u_{r_j},v) \in \Lambda(f_j,\phi)$ lies in $C[x+n+1]$ and $v$ is the only component lying in $C[x+n+2,\infty)$ and so, by Shift rule 1, $(u_1,\ldots,u_{r_j},v\tau_1) \in \Lambda(f_j,\phi)$. Hence $v\phi = (u_1\phi_1,\ldots,u_{r_j}\phi)f_j = (v\tau_1)\phi$, which contradicts the injectivity of $\phi$. Therefore $v \in C[0,x+n+1]$. Since $v\phi$ is the result of applying an arbitrary operation $f_j$ of $S$ to arbitrary elements of $G_n$, it follows that $G_{n+1} \subseteq C[0,x+n+1]\phi$.
Let \( a = (x + 1)D \); then \(|C[0,x]| = a\). Furthermore, \(|C[0,x+n+1]| = |C[0,x+n]| + D\). Let \( b = D \); then \(|C[0,x+n+1]| = a + bn\). Since \( G_n \subseteq C[0,x+n+1]\), it follows that \( g(n) = |G_n| \leq a + bn\).

The contrast between the growth levels of finitely generated FA-presentable algebras (bounded by an exponential function) and finitely generated unary FA-presentable algebras (bounded by a linear function) resembles the contrast between the growth of finitely generated FA-presentable semigroups (polynomial growth [CORT09, Theorem 7.4]) and finitely generated unary FA-presentable semigroups (sublinear growth, which implies finiteness [CRT12, Proof of Theorem 13]). Note, however, the difference between the two types of growth: the growth level of an algebra counts elements of a given term complexity, and the growth of a semigroup or group counts elements of given word length.

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