A contemporary look at Hermann Hankel’s 1861 pioneering work on Lagrangian fluid dynamics

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The present paper is a companion to the paper by Villone and Rampf (2017), titled “Hermann Hankel’s On the general theory of motion of fluids, an essay including an English translation of the complete Preisschrift from 1861” together with connected documents. Here we give a critical assessment of Hankel’s work, which covers many important aspects of fluid dynamics considered from a Lagrangian-coordinates point of view: variational formulation in the spirit of Hamilton for elastic (barotropic) fluids, transport (we would now say Lie transport) of vorticity, the Lagrangian significance of Clebsch variables, etc. Hankel’s work is also put in the perspective of previous and future work. Hence, the action spans about two centuries: from Lagrange’s 1760–1761 Turin paper on variational approaches to mechanics and fluid mechanics problems to Arnold’s 1966 founding paper on the geometrical/variational formulation of incompressible flow. The 22-year old Hankel—who was to die 12 years later—emerges as a highly innovative master of mathematical fluid dynamics, fully deserving Riemann’s assessment that his Preisschrift contains “all manner of good things.”

I. INTRODUCTION

It has been known for over two centuries that there are two ways for describing the motion of a fluid. In the Eulerian approach, the velocity, the density and the pressure are expressed in terms of the coordinates $x = (x, y, z)$ of the point of measurement and of the time $t$, whereas in the Lagrangian approach they are expressed in terms of the coordinates $a = (a, b, c)$ of the initial position of the fluid particle and of the time elapsed. In modernized notation, the two systems are related by the Lagrangian map $a \mapsto x(a, t)$, which satisfies the characteristic equation $\dot{x} = u(x, t)$, where $u$ is the velocity and the dot denotes the Lagrangian (or material) time derivative. Before the Eulerian/Lagrangian terminology was in use, in his famous 1788 treatise “Méchanique analytique”, Lagrange warned his readers that “The [Eulerian] equations are much simpler than the [Lagrangian] ones; thus, in the theory of fluids, one should prefer the former over the latter.” Even a couple of centuries after “Lagrange’s curse,” Lagrangian coordinates have been occasionally viewed as an “agony”.

Nevertheless, in the 20th century, Lagrangian coordinates have been used extensively by mathematicians, in analyzing the well-posedness of ideal incompressible fluid flow, governed by the Euler equations. The goal was to prove theorems regarding the existence, regularity and uniqueness of the solution to the initial-value problem, where one prescribes the flow at time $t = 0$. In the 20s and the 30s, the works of Günther, Lichtenstein and Wolibner were the first to show that ideal incompressible fluid flow with suitable initial regularity, stays so for at least a finite time (in three dimensions) and for all times in two dimensions (in a bounded domain). These results have barely been improved since and the situation is not much better for viscous flow, at least in three dimensions.

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1 For Lagrangian coordinates, see Euler, 1760; Lagrange, 1788. For Lagrange’s curse, see Lagrange, 1788:453 and also p. 450. For the “agony”, see Price, 2006:31.
2 Günther, 1926. Lichtenstein, 1927. Wolibner, 1933.
A major reason why Lagrangian approaches work better than Eulerian ones for such questions has been identified several decades later by Ebin and Marsden, using an (infinite-dimensional) geometrical approach to the dynamics of ideal incompressible fluids, pioneered by Arnold. The key is that the application of Lagrangian time derivatives do not lose spatial derivatives, whereas Eulerian time derivatives do lose spatial derivatives. What this means, is that if the initial velocity field has spatial derivatives up to order \( p \geq 2 \), then initial Eulerian time derivatives exist only up to the same order \( p \), whereas Lagrangian time derivatives exist up to any order. Roughly, the reason is that in an Eulerian formulation a time derivative \( \partial_t \) is always accompanied by a space derivative \( u \cdot \nabla \), whereas in a Lagrangian formulation these two terms combine to form the Lagrangian time derivative. \(^3\)

We mention that there has been for several decades a strong interest for Lagrangian approaches in geophysical fluid dynamics, cosmology and magnetohydrodynamics. \(^4\)

In the 19th century there was relatively little use of Lagrangian coordinates, not so much because of Lagrange’s curse but because there was a more objective reason to stay away from Lagrangian coordinates: After the work on the Navier–Stokes equations, it was realized that for most realistic flows one cannot ignore the role of viscosity and to achieve this, it is indeed easier to use Eulerian coordinates.

Some key theoretical results obtained during the 19th century were actually formulated as Lagrangian conservation theorems, but their derivations were given in Eulerian coordinates. This includes Helmholtz’s result on the conservation of vorticity flux and Thomson/Kelvin’s result on the conservation of circulation. \(^5\)

But there are also 19th century studies that make end-to-end use of Lagrangian variables. Two stand out: Cauchy’s derivation of the three-dimensional invariants, now called the Cauchy invariants and the results by Hankel presented here and in the companion paper. Both Cauchy’s and Hankel’s were prized contributions on a prescribed theme. For Cauchy it was “the problem of waves on the surface of a liquid of arbitrary depth”, so that his results on Lagrangian invariants were somewhat incidental and occupied only a small fraction of his long memoir. For Hankel the situation was quite different: he was being asked to write a memoir on the theory of fluids centered on the use of Lagrangian coordinates, and that is basically what he did. \(^6\)

Around the end of the 1850s, among the mathematicians and physicists at the University of Göttingen, such as Weber, Dirichlet and Riemann, all of whom were interested in fluid mechanics, awareness of the importance of Lagrangian coordinates started rising. Following the death of Dirichlet in 1859, an extraordinary prize was set up, which was won in 1861 by Hermann Hankel, a student in Mathematics on leave from Leipzig University. We shall come back to the detailed circumstances around that prize in Section II, but we wish to explain briefly why two companion papers in this issue of EPJ H are dedicated to Hankel’s Preisschrift (Prize manuscript), the present one and another one containing the full translation of the Preisschrift with related documents. \(^7\)

A detailed study of Hankel’s Preisschrift reveals indeed a deep understanding of Lagrangian fluid mechanics, with variational methods and differential geometry present throughout. This did not escape Riemann’s attention who, in the official assessment for the special prize committee set up by the faculty, wrote that the manuscript contained ‘all manner of good things’. The style of writing of Hankel may occasionally appear awkward, but is actually fully consistent with mid-19th century German scientific writing tradition. \(^8\)

Later Hankel did make important contributions to mathematics until he died at age 34. His work on hydrodynamics did not achieve much visibility. In the 19th century, Hankel’s hydrodynamics work is cited several times by Beltrami, and Auerbach covers the work rather

\(^3\) Ebin and Marsden, 1970. Arnold, 1966; Arnold & Khesin, 1998.
\(^4\) Frisch and Villone, 2014 and references therein.
\(^5\) Helmholtz, 1858; Thomson (Lord Kelvin), 1869
\(^6\) For Cauchy and how his work on invariants was mostly forgotten in the 19th century, see Cauchy, 1815/1827; Frisch and Villone, 2014 §§ 2–4.
\(^7\) For the Preisschrift, see Hankel, 1861. For the companion paper, see Villone and Rampf, 2017.
\(^8\) For exchanges between Riemann and Weber relating to Hankel’s application for the prize and for the official assessment, see the companion paper by Villone & Rampf, 2017.
extensively. In addition, there are references concerned exclusively with Hankel’s proof of the Stokes theorem, namely Lipschitz and Lamb.\(^9\)

In the 20th century, we have to wait for Truesdell to find extensive coverage of Hankel’s hydrodynamics, which is also mentioned by Darrigol.\(^10\)

The paper is organized as follows. Section II describes the context of the mathematical and fluid dynamical research in Germany and particularly in Göttingen, at the time of Hankel’s Preisschrift.

Section III describes the tools and methods available at the time and also their relation to more modern methods used currently, e.g., in differential geometry.

Section IV is about Hankel’s first variational approach to elastic (barotropic) fluid dynamics in the spirit of Hamilton’s least action principle.

Section V shows how Hankel derived the Helmholtz theorem for the conservation of vorticity flux by using Cauchy’s 1815 Lagrangian reformulation of the incompressible Euler equations. The conservation of circulation plays there a role as an intermediate step.\(^11\)

Section VI is about Hankel’s rederivation of the Clebsch-variable approach to the Euler equations using clever switches between Lagrangian and Eulerian coordinates, now called pullbacks and pushforwards. This is followed by two subsections: Section VI.A briefly describes how Clebsch introduced the variables known by his name two years before Hankel’s work; Section VI.B is about a topological pitfall of Clebsch variables, understood only in the last fifty years.

Concluding remarks are presented in Section VII, in particular about the difficult birth and rebirth of variational methods.

II. CONTEXT OF HANKEL’S 1861 PRIZE

Three important scientists connected with the Georg-August university of Göttingen around the middle of the 19th century played a major role in the establishment of the prize that was attributed to Hankel. They are the physicist Wilhelm Eduard Weber (1804–1891), particularly known for his work on electricity and magnetism and closely associated to one of the brightest stars of Göttingen, Carl Friedrich Gauss (1777–1855), and two mathematicians, Gustav Peter Lejeune-Dirichlet (1805–1859) and Bernhard Riemann (1826–1866), who successively held the chair of Gauss (Riemann had also been assistant to Weber).\(^12\)

Around 1850 the Georg-August university was quickly turning into a major center for mathematics and physics. Gauss was of course still an important driving element but until the political liberalisation that followed the 1848 upheavals in many European countries, Georg-August suffered from serious lack of freedom for academics, which in turn led to a dramatic drop in student enrollment. For example, Weber was one of the seven professors (including the Brothers Grimm) who were summarily dismissed in 1837, although he was not banned from Göttingen and was eventually reinstated in 1849, becoming one of its most influential professors. Weber played a leading role in establishing the prize discussed here. Dirichlet had been teaching for many years in Berlin, where he and Carl Gustav Jacob Jacobi (1804–1851) attracted many bright students, among whom Riemann. After the death of Gauss in 1855, with the recommendation of Weber, Dirichlet was appointed to Gauss’s chair in Göttingen. He died however in 1859 and the chair went to Riemann, who in December 1859 presented to the Academy of Göttingen an unpublished work of Dirichlet on hydrodynamics and another unpublished work of his own on hydrodynamics.\(^13\)

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\(^9\) Beltrami, 1871. Auerbach, 1881. Lipschitz, 1870-1871. Lamb, 1895.

\(^10\) Truesdell, 1954a; Truesdell and Toupin, 1960. Darrigol, 2005.

\(^11\) Cauchy, 1815/1827.

\(^12\) Biographies of Weber, Dirichlet and Riemann may be found in the Dictionary of Scientific Biography, 1970-1980. Biographies of Weber, Dirichlet and Riemann: these are by Woodruff, 1976, Ore, 1859 and Freudenthal, 1975, respectively.

\(^13\) For the Göttingen seven and its impact on student enrollment, see Hunger, 2002. For Weber’s role in the hiring of Dirichlet, see Minkowski, 1905. For the presentation of Dirichlet’s hydrodynamics unpublished work, see Nachrichten, 1857, 1859 and for the posthumous published work, see Dirichlet, 1860. For Riemann’s 1859 presentation of his own hydrodynamics work, see Nachrichten, 1859 and the published
The death of Dirichlet and his posthumous paper “On a problem of hydrodynamics,” edited by Richard Dedekind (1831–1916), which discusses the advantages and disadvantages of using Eulerian or Lagrangian coordinates, are highlighted in the official announcement as one of the motivations of the prize for which Hankel applied. For over one century, establishing and studying the equations of hydrodynamics had been considered a major challenge by mathematicians. In 1858 Hermann Helmholtz (1821–1894) had discovered the basic laws of vortex dynamics and developed an interesting analogy with the nascent electromagnetic theory: velocity is expressible in terms of vorticity using the same Biot–Savart law that relates magnetic fields to currents. Helmholtz’s theorem on the conservation of vorticity flux through a surface element following the flow, did probably strike Riemann as a potentially Lagrangian result, although Helmholtz gave an Eulerian derivation. Riemann had himself been interested in the Eulerian vs. Lagrangian approaches to hydrodynamics and observed that they both originated with Euler. It is thus not surprising that the extraordinary prize which was established after these events requested a fully Lagrangian derivation of Helmholtz’ result.

Here is the official statement of the Georg–August university about the prize: The most useful equations for determining fluid motion may be presented in two ways, one of which is Eulerian, the other one is Lagrangian. The illustrious Dirichlet pointed out in the posthumous unpublished paper “On a problem of hydrodynamics” the almost completely overlooked advantages of the Lagrangian way, but he was prevented from unfolding this way further by a fatal illness. So, this institution asks for a theory of fluid motion based on the equations of Lagrange, yielding, at least, the laws of vortex motion already derived in another way by the illustrious Helmholtz. The prize was set up on 4th June 1860 and had a deadline of end of March 1861.

Enters Hermann Hankel, who arrived in the spring 1860 in Göttingen, as a 21-year old mathematics student, coming from the University of Leipzig. In Göttingen, Hankel worked on the prize, which he won as single contestant. This was a common situation for such thematic prizes, but the prize was not necessarily awarded. According to Zahn, while in Göttingen, Hankel attended the lectures of Riemann during the 1860 summer term and the 1860–1861 winter term on such topics as: “The mathematical theory of gravity, electricity and magnetism” and “The theory of partial differential equations with applications to physical problems.” Hankel stayed in Göttingen until the Fall of 1861, defending his doctoral dissertation in Leipzig during the same year. Then he left for Berlin to extend his mathematical training, attending lectures by Weierstrass and others.

III. STATE OF THE ART NEAR THE MIDDLE OF THE 19TH CENTURY, AS USED BY HANKEL

The most popular way of formulating the equations of fluid dynamics and, more generally of continuous media, from the time of Lagrange to the middle of the 19th century and beyond was the method of virtual work (also known as “virtual displacements” or “virtual velocities”), applied to the various external forces and the inertial force. A considerable extension and synthesis of previous work by Johann (John) Bernoulli (Principle of virtual velocities)
and of D’Alembert’s Principle, Lagrange presented it as a founding principle of mechanics.\footnote{For the virtual velocities Principle, see Bernouilli, Johann (John), 1717; for D’Alembert’s Principle and its discussion, see D’Alembert, 1743; Lagrange, 1788 and Truesdell, 1960:188–190.}

In modernized notation this amounts roughly to the following. One avoids Euler’s, traditional way of writing Newton’s equation for each infinitesimal element (“molecule”), namely $F = ma$, where $F$, $m$ and $a$ are the force, the mass and the acceleration, respectively. (Henceforth we use modernized notation, mostly boldface vector notation.) Instead, two changes are performed. First, one lumps together into a total external force $F_{\text{ext}}(x,t)$ all the external forces acting on the molecule near $x$ at time $t$, including the “inertial force” $-ma$. Second, one gives a weak formulation (with space derivatives taken in the sense of distributions) of $F_{\text{ext}}(x,t) = 0$, by equating to zero, at any given time, the total work $\int F_{\text{ext}}(x,t) \cdot \delta x(x,t) d^3x$ of these forces under assumed infinitesimal virtual displacements $\delta x(x,t)$. An instance is (1) of §4 of HT.\footnote{Henceforth, HT designates the translation of Hankel’s Preisschrift in the companion paper by Villone and Rampf, 2017. By (n) §p we understand the equation in §p that was given the number (n) by Hankel (such numbers are preserved in the translation). Since many of Hankel’s equations are unnumbered, in the translation all the equations have been numbered, using the format [p,n], which means the nth equation of §p.}

As was common throughout the second half of the 18th century and the 19th, Hankel systematically considers both the case of incompressible fluids and that of elastic fluids having a well-defined functional relation between the pressure and the density. Nowadays, the latter would be called barotropic. In the incompressible case, the infinitesimal virtual displacements are chosen to preserve the volume, i.e., $\nabla \cdot \delta x = 0$.

When reading Hankel, it must be kept in mind that he always starts from the virtual work formulation and then derives what we generally call Euler’s equations. So, actually he goes from a weak to a strong formulation. Since no analytic convergence issues were considered at the time of Hankel, this does not matter.

Interestingly, Hankel did not shy off the now standard notation of Euler, avoiding virtual displacements. However it was probably important for the young man, still a student, to demonstrate that he was fully familiar with the notation most common at that time.

A second important point when reading Hankel’s Preisschrift, is the central role played by the duality of approaches mentioned in the theme of the prize, namely the use of either Eulerian coordinates. referred to by Hankel as “first manner/method” or Lagrangian coordinates, referred to by Hankel as “second manner/method”.

A central feature of Hankel’s Preisschrift is the constant juggling with Eulerian and Lagrangian coordinates. Hankel performs changes of variables from Eulerian to Lagrangian coordinates (or vice-versa) on both scalar functions and on differential forms (in modern geometrical parlance these are called pullbacks and pushforwards). As long as he works with scalar functions, these are just changes of coordinates. As was common at the time, he does not change the name of the function so transformed; neither will we. When he performs pushbacks and pullforwards on differential forms (or on gradients), the changes in the differentials are controlled by the Jacobian matrix $\partial x_i/\partial a_j$. Because of the heavy use of virtual work (weak) formalism, Hankel frequently needs the volume element. Denoting the mass density by $\rho(x,t)$ and the initial density by $\rho_0(a)$, one has of course

$$\rho(x,t) d^3x = \rho_0(a) d^3a. \quad (1)$$

Although he makes frequent indirect use of this mass conservation relation, the closest he comes to writing it, is in HT [2.9] which, in modernized form, reads

$$\det \left( \frac{\partial x_i}{\partial a_j} \right) = \frac{\rho_0}{\rho}. \quad (2)$$

Hankel stresses in §1 that, according to his advisor Bernhard Riemann, Euler is the originator of both approaches, the Eulerian and the Lagrangian ones. This issue has been revisited by Truesdell. Of course, Hankel had to make central use of Lagrangian coordinates...
in the Preisschrift. Actually, he succeeded amazingly well in this, and particularly so because he became fully aware of Cauchy’s 1815 work on Lagrangian coordinates.\(^{20}\)

Finally, Hankel and most theoretical mechanicians of his time were quite familiar with developments of variational methods that started about one century earlier with the work of Euler and Lagrange. It is not our purpose here to review the history of variational methods. Lagrange, a few years after he started discussing with Euler his ideas on analytical methods for variational problems, tried applying such methods to various mechanical problems, including fluid dynamics. This attempt was not very successful, in particular because Lagrange used the Maupertuis form of the action, based just on the kinetic energy; furthermore, he used “Lagrangian” coordinates only to monitor the evolution of volume elements.\(^21\)

A crucial change of perspective occurred in the 1830th when Hamilton realized that for most mechanical systems one should include in the action not only a kinetic but also a potential contribution and that it can be advantageous to switch to so-called canonical variables. This line of research was pursued and disseminated widely all over Europe by Jacobi and many others.\(^22\)

However such development concerned chiefly solid mechanics. Fluid mechanical developments had to wait until around 1860 with the work of Helmholtz and Clebsch and, of course, Hankel’s Preisschrift, as we shall now see.\(^23\)

### IV. HANKEL’S FIRST VARIATIONAL FORMULATION FOR ELASTIC FLUIDS

Hankel’s derivation of a variational formulation for the dynamics of elastic/barotropic flow is presented at the beginning of his §5, where it takes barely more than one page. The corresponding Eulerian equations (HT §4 (4) [4.5]) are written in his previous section. In modernized form they read (after pulling out a factor \(\rho\) in the first equation):

\[
\rho \left[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{X} + \frac{1}{\rho} \nabla p \right] = 0, \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{3}
\]

where \(\mathbf{u}\) is the velocity, \(p\) the pressure, \(\rho\) the density, \(\nabla\) the Eulerian spatial gradient and \(\mathbf{X}\) the external force per unit volume, assumed by Hankel to derive from a potential \((\mathbf{X} = \nabla V)\), as is for example the case for gravitation. This equation has an equivalent virtual-work (weak) form also derived in Hankel’s previous section (HT §4 (1) [4.1]):

\[
\int d^3 \mathbf{x} \left\{ \rho \left[ (\mathbf{X} - \ddot{x}) \cdot \delta \mathbf{x} \right] + p \nabla \cdot \delta \mathbf{x} \right\} = 0, \tag{4}
\]

where \(\ddot{x}\) denotes the acceleration of the fluid particle, that is the Lagrangian/material time derivative of the velocity \(\dot{x}\).

From (4), using \(\mathbf{X} = \nabla V\) and integrating by parts the term involving the pressure, Hankel obtains, after a global change of sign, [5.2]

\[
\int d^3 \mathbf{x} \rho \left[ \ddot{x} - \nabla V + \frac{1}{\rho} \nabla p \right] \cdot \delta \mathbf{x} = 0. \tag{5}
\]

Then, using the barotropic relation \(\rho = \phi(p)\) and introducing the function \(f(p) = \int dp/\rho(p)\), Hankel observes that \(\frac{1}{\rho} \nabla p = \nabla f(p)\). Next come two important steps, not particularly underlined by Hankel. First, introducing the new function \(\Omega = V - f(p)\) he makes use of \(\delta \Omega = \nabla \Omega \cdot \delta \mathbf{x}\). Second, he performs a pullback to Lagrangian coordinates, using (1), to obtain [5.6]:

\[
\int d^3 \mathbf{a} \rho_0(\mathbf{a}) \left[ \ddot{\mathbf{x}} - \delta \mathbf{x} - \delta \Omega \right] = 0. \tag{6}
\]

\(^{20}\) Truesdell 1954b. Cauchy, 1815/1827. 
\(^{21}\) For the history of variational methods, see Goldstine, 1980; Fraser, 2003. For the early variational work, see Lagrange 1760–1761; for “Lagrangian” coordinates, see his §XLIV, p. 276. 
\(^{22}\) Hamilton, 1834; 1835. Jacobi, 1837; 1866. 
\(^{23}\) Helmholtz, 1858. Clebsch, 1857; 1859.
It is now obvious — both for us and for a mechanician of Hankel’s time — that the variational equation (6), which should hold at any time in, say, the interval \([0, T]\), is equivalent, after a time integration by parts, to the vanishing of the variation of an action, namely (HT §5 ((1)) [5.9], essentially)

\[\delta \int d^3a \rho_0(a) \int_0^T dt \left[ \frac{1}{2} |\dot{\mathbf{x}}|^2 + \Omega \right] = 0, \tag{7}\]

with the usual constraint that the infinitesimal variations \(\delta \mathbf{x}\) vanish at time 0 and \(T\). Here we have modernized Hankel’s \((\frac{ds}{dt})^2\) (where \(s\) is the arclength) into \(|\dot{\mathbf{x}}|^2\). Actually, the form with \(ds^2\) is quite useful, as Hankel will explain a little further.

Hankel has thus given a variational formulation for the elastic/barotropic fluid. It involves an action that is the space-time integral of the Lagrangian density function \(\rho_0(a) \left[ \frac{1}{2} |\dot{\mathbf{x}}|^2 + \Omega \right]\), namely the difference of the kinetic energy \((1/2)\rho_0(a)|\dot{\mathbf{x}}|^2\) and of the potential energy \(-\rho_0(a)\Omega\), the latter being itself the sum of the potential energy \(-\rho_0(a)V\) due to external forces and of the elastic potential \(\rho_0(a)f(p)\).

Here Hankel pauses to comment on the nature of his result. He qualifies it as a theorem possessing some analogy to the principle of least action. Of course, the latter was historically based on the Maupertuis action, involving just the kinetic energy. Only with Hamilton and Jacobi did mechanicians realize that the simplest correct variational formulation for conservative systems must use an action involving both the kinetic and the potential energy. Such modern variants of the least action principle were commonplace for discrete mechanical system but not yet for fluids. One possible exception is the work of Clebsch on incompressible flow, to which we shall return in Section VI.\(^{24}\)

At this point, Hankel makes an important observation about the use of his variational formulation. One essential ingredient in the action appearing in (7) is the square of the line element \(ds^2\). In Cartesian coordinates we have \(ds^2 = dx^2 + dy^2 + dz^2\). In another coordinate system, involving arbitrary curvilinear coordinates, denoted \(\rho_1, \rho_2\) and \(\rho_3\), the squared line element will be some other quadratic form of the \(d\rho_i\) \((i = 1, 2, 3)\) and the volume element can be easily evaluated in arbitrary coordinates. Hankel observes that it is much easier to perform the change of variables on the action integral than doing it directly on the equation of motion. He actually devotes 13 out of the 53 pages of the Preisschrift to deriving such equations in a number of different coordinate systems. He then points out that there is no need to stay with three variables and that the same process can be carried in an \(n\)-dimensional space. Here, he was probably influenced by Riemann’s teaching, who in the same year 1861 submitted (unsuccessfully) to the French Academy a prize essay on the conduction of heat using the same notation for curvilinear coordinates. Nevertheless, Hankel does of course not try leaving ordinary Euclidean space in favor of hydrodynamics in curved Riemann spaces.\(^{25}\)

V. FROM CAUCHY TO HELMHOLTZ (VIA THE CIRCULATION THEOREM)

The §§6–9 of Hankel present a fully Lagrangian-coordinates-based proof of one of Helmholtz’s main results, that the flux of the vorticity vector \(\nabla \times \mathbf{u}\) through an infinitesimal surface element remains unchanged as we follow the surface element in its Lagrangian motion. Helmholtz proved this for a three-dimensional incompressible flow subject to a potential external force. Hankel’s proof is not just Lagrangian, as required by the rules of the prize, instead of incompressibility, he assumes that the flow is elastic/barotropic. Yet, as he points out, the essence of his proof applies equally well to an incompressible flow. The starting point in §6 is the pullback to Lagrangian coordinates of the elastic/barotropic Euler

\(^{24}\) For the Maupertuis action, see Maupertuis, 1740, 1744; Euler, 1744; 1750. Hamilton, 1834; 1835. Jacobi, 1866. Clebsch, 1857; 1859.

\(^{25}\) Riemann, 1861a.
momentum equation (3) (HT [6.1])

\[ \ddot{x}_k \frac{\partial x_k}{\partial a_i} - \frac{\partial \Omega}{\partial a_i} = 0, \]  

(8)

where \( \Omega \equiv V - f(p) \) as before and repeated indices are summed upon. This is obtained as follows: the momentum equation (3) may be rewritten either as \( \ddot{x} - \nabla \Omega = 0 \) or, in the language of differential geometry, as \( \ddot{x} \cdot dx - d\Omega = 0 \). The Eulerian gradient \( \nabla \Omega \) is then converted into a Lagrangian gradient \( \nabla_L \Omega \) through multiplication by the Jacobian matrix \( \nabla_L x \) of the Lagrangian map. In the language of differential geometry, this is the pullback of the vanishing 1-form \( \ddot{x} \cdot dx - d\Omega \) from Eulerian to Lagrangian coordinates. Henceforth we shall rewrite equations such as (8) in more compact notation as

\[ \ddot{x}_k \nabla^L x_k = \nabla^L \Omega. \]  

(9)

So far, Hankel has been essentially following Lagrange’s own rewriting of the hydrodynamical equations in “Lagrangian coordinates”. The next step will be to follow Cauchy in his 1815 Prize work. Hankel takes a Lagrangian curl of (9) to kill the gradient on the r.h.s. and obtain (HT [6.2])

\[ \nabla^L \times (\ddot{x}_k \nabla^L x_k) = 0. \]  

(10)

Still, following exactly Cauchy, Hankel observes that this equation can be integrated in time to yield

\[ \nabla^L \dot{x}_k \times \nabla^L x_k = \omega_0, \]  

(11)

where \( \omega_0 \equiv \nabla^L \times u_0 \) is the initial vorticity, which obviously does not depend on time. Hence, the three components of the l.h.s. are time-independent. They are nowadays called the Cauchy invariants.26

The next step is Hankel’s own: he rewrites (11) as

\[ \nabla^L \times (\dot{x}_k \nabla^L x_k) = \omega_0, \]  

(12)

thus pulling out a curl operator. This is crucial because it will allow him to use what is now called the Stokes theorem, which as is known, relates the line integral along a closed contour of a vector field to the flux of its curl across a surface bounded by the contour. The proof is presented in Hankel’s §7 and constitutes the first published proof of the theorem. This theorem was actually known much before Hankel’s 1861 work, as explained by Katz. Furthermore, Hankel benefited from close association with Riemann who had proved a two-dimensional version of the theorem, itself related to an earlier result of Cauchy. It is interesting that Riemann’s work discussed the topological aspects of the theorem, in particular issues of simply- and multiply-connected domains, whereas Hankel assumed not only a simply-connected domain but also that the surface can be resolved uniquely for the \( z \)-coordinate in terms of the \( x \)- and \( y \)-coordinates. These simplifications somewhat reduce the scope of Hankel’s proved theorem, but did not matter, since eventually, Hankel only needed to work with an infinitesimal surface, as assumed by Helmholtz.27

The derivation of the Helmholtz theorem, in §§8–9, proceeds as follows. In the Lagrangian space, Hankel considers a connected (\( \text{zusammenhängende} \)) surface, here denoted by \( S_0 \), whose boundary is a curve, here denoted by \( C_0 \), and applies the Stokes theorem to (12).

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26 For the pullback to Lagrangian coordinates, see Lagrange, 1788:446. For the Cauchy invariants, see Cauchy, 1815/1827; Frisch and Villone, 2014. It is noteworthy that Cauchy’s derivation of equations (10)-(11) has been apparently rediscovered (in words only) by Dedekind in his own footnote 1 to Dirichlet’s posthumous paper (Lejeune-Dirichlet, 1860), where Dedekind also states that Dirichlet cannot have failed to notice this, as well as the consequence, nowadays called the Cauchy vorticity formula.

27 Katz, 1979. Cauchy, 1846. Riemann, 1857; Roy, 2011:362.
Hankel then infers that the flux through $S_0$ of the r.h.s. of (12), namely the initial vorticity, is given by the circulation along $C_0$ of the initial velocity $u_0$ (HT [8.5] and also § 8 (2) [8.6]):

$$\int_{C_0} u_0 \cdot da = \int_{S_0} \omega_0 \cdot n_0 \, d\sigma_0,$$

(13)

where $n_0$ denotes the local unit normal to $S_0$ and $d\sigma_0$ the surface element. Then, he similarly handles the l.h.s. of (12) and first notices that (HT [9.3])

$$u_k \nabla^L x_k \cdot da = u \cdot dx.$$

(14)

He thus obtains the Eulerian circulation, an integral over the curve $C$ where are presently located the fluid particles initially on $C_0$ (HT [9.5]):

$$\int_C u \cdot dx = \int_{S_0} \omega_0 \cdot n_0 \, d\sigma_0.$$

(15)

Eq. (15), together with (13), although they do not appear in the same numbered paragraph of Hankel, are clearly a statement of the standard circulation theorem, mostly associated to the name of William Thomson (Kelvin). However, as already stated, Hankel’s formulation of the Stokes theorem is restricted to simply-connected domains, which is enough to recover Helmholtz’s flux-invariance result. Hence Hankel’s 1861 circulation result is less general than Kelvin’s 1869 result in which multiply-connected domains are taken into consideration. Of course, Hankel was aware of Riemann’s emphasis on connectedness, since he cited Riemann’s 1857 founding paper on the subject.28

At this point, it is clear that Hankel also proved the constancy in time of the flux of the vorticity through any finite surface moving with the fluid. Letting this surface shrink to an infinitesimal element, he obtains Helmholtz’s theorem.29

Hankel has thus obtained both global and local forms of the Helmholtz theorem and, incidentally, of the circulation theorem.

VI. CLEBSCH VARIABLES IN HANKEL’S LAGRANGIAN FORMULATION

Alfred Clebsch (1833–1872) entered Königsberg University in 1850, attended lectures by Jacobi’s disciples, Otto Hesse, Friedrich Richelot and Franz Neumann. He graduated in 1854, moved to Berlin where he became Privat Dozent in 1858. He went on to Karlsruhe in 1858, to Gießen in 1863 and finally became Professor in Göttingen in 1868, where he occupied the chair previously held by Gauss, Lejeune Dirichlet and Riemann, until his premature death in 1872. His early research work (1857–1859), focussed on hydrodynamics and the calculation of variations and was directly inspired by the methodology developed by Jacobi and Hesse. (Eventually, Clebsch became one of the editors of the complete works of Jacobi.) Around 1860 his research interests shifted towards the theory of algebraic invariants, where he used the analytical view of projective geometry implemented by Plücker (later, Clebsch and Felix Klein edited Plücker’s posthumous works). After 1863 in Gießen, he worked on Abelian functions with Gordan, following the path opened by Riemann. After 1868, he became the chief organizer of mathematical research in Göttingen, creating the journal Matematische Annalen and developing contacts with young French mathematicians, such as Jordan or Darboux, and with the English school of Sylvester and Cayley. He interacted with Felix Klein and Sophus Lie. Here, we are only interested in the very early period of Clebsch’s life before Hankel’s 1861 Preisschrift. More detailed biographies of Clebsch are of course available.30

28 Thomson (Lord Kelvin), 1869. Riemann, 1857.
29 Helmholtz, 1858.
30 For detailed biographical information on Clebsch, see Burau, 1970–1980 and Various Authors, 1873a; 1873b.
The Clebsch-variable formulation of three-dimensional incompressible flow is nowadays usually written as follows. There are two three-dimensional time-dependent scalar fields $\phi(x, t)$ and $\psi(x, t)$ that are material invariants, that is, they satisfy

$$(\partial_t + u \cdot \nabla)\phi = 0 \quad \text{and} \quad (\partial_t + u \cdot \nabla)\psi = 0.$$ (16)

Furthermore, there is a scalar field $F(x, t)$, such that the velocity field $u$ is coupled back to the fields $\phi$ and $\psi$ by

$$u = \nabla F + \phi \nabla \psi \quad \text{and} \quad \nabla \cdot u = 0.$$ (17)

Actually, Clebsch had a somewhat complicated proof that a velocity field thus constructed satisfies the incompressible Euler equations. We shall come back to this a little further, in Section VI.A.31

In §10–11 of the Preisschrift, Hankel gives a very short proof of this result, while stressing that the equations just given are written in Eulerian coordinates. Hankel’s proof uses his modified form of the Cauchy invariants equation (12) and performs a reversion from Lagrangian to Eulerian coordinates, which one now calls a pushforward.

First, he assumes that the initial vorticity $\omega_0(a) \equiv \omega(a, 0)$ has the following Pfaff-type representation, obtained by taking at $t = 0$ the curl of (17) (HT §10 (2) [10.4]):

$$\omega_0 = \nabla^L \times u_0 = \nabla^L \phi_0 \times \nabla^L \psi_0 = \nabla^L \times (\phi_0 \nabla^L \psi_0),$$ (18)

where we recall that $\nabla^L$ denotes the Lagrangian gradient. We will come back to the validity of this representation at the end of Section VI.A.32

The functions here denoted by $\phi_0$ and $\psi_0$ were written by Hankel without the subscript zero but he stressed that this assumption is made only at time $t = 0$. Hankel then substitutes (18) in the r.h.s. of the Cauchy invariants equation (12). He notices that there are now Lagrangian curls on both sides, which he can remove provided he adds the Lagrangian gradient of a suitable function $F$. He thus obtains:

$$\dot{x}_k \nabla^L x_k = \nabla^L \cdot F + \phi_0 \nabla^L \psi_0.$$ (19)

Now comes a crucial step. Hankel reverts to Eulerian coordinates and also switches from Lagrangian gradients (of $F$ and $\psi_0$) to Eulerian ones. For the latter step he must use what is now called the inverse Jacobian matrix of the Lagrangian map, that is $\nabla a$. On the l.h.s. there is already a direct Jacobian matrix $\nabla^L x$. In modern language these two matrices, inverses of each other, multiply to the identity, so that after this multiplication the l.h.s of (19) becomes just $\dot{x} = u$ and (19) becomes exactly (17), hence concluding Hankel’s derivation.

These manipulations, described here in matrix language, are essentially what Hankel does when invoking his equations §6 (2) (HT [6.4]) and §2 (3) (HT [2.14]). Furthermore, since he performs a change of variables, he might have changed the name of his functions. Instead, following the usage of the time, he resorts to warning language: “in so far as time $t$ appears explicitly or implicitly through $x, y, z$.” We avoid this problem by switching from $\phi_0$ and $\psi_0$ to $\phi$ and $\psi$, although they are actually the same functions in Lagrangian coordinates, being material invariants.

It is of interest to point out that, shortly after the beginning of this clever and simple proof, Hankel makes a digression concerning his geometrical Helmholtzian vision of Clebsch variables. Knowing, from Helmholtz, that vortex lines are material, he observes that under the assumption that $\omega_0 = \nabla^L \phi_0 \times \nabla^L \psi_0$ a simple way to ensure the materiality of vortex lines is to take the surfaces $\phi = \text{Const.}$ and $\psi = \text{Const.}$ to be material, namely to have $\dot{\phi} = 0$ and $\dot{\psi} = 0$. Hankel also adds, at the end of §11, after completion of the proof, that Clebsch himself did not bring out the meaning of his variables. Probably Hankel meant the geometrical meaning. Connected to this, there is also an important footnote [A.26] to which we shall come back after having summarized the Clebsch approach.

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31 Clebsch, 1859.
32 Pfaff, 1814.
A. The Clebsch 1859 approach

Our purpose here is to find out how the 1859 work of Clebsch may have influenced the subsequent work of Hankel, who was fully aware of it and cites it several times. For the convenience of the reader we shall use Hankel’s notation (except for a certain Hamiltonian function present only in Clebsch). Since Clebsch was including the pressure $p$ in the potential of external forces, we shall use $V - p$ where Clebsch uses $V$. For simplicity we assume that the constant and uniform density is unity. Furthermore we shall specialize to the three-dimensional incompressible case, commenting afterwards on the generalization by Clebsch. Here, the presentation does not aim at analyzing Clebsch’s hydrodynamical work exhaustively. It would be of interest to do so and also to translate his 1857 and 1859 papers.\(^{33}\)

The starting point are the Euler equations

$$\nabla(V - p) = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}; \quad \nabla \cdot \mathbf{u} = 0, \quad (20)$$

the first of which is written as a 1-form differential relation, with the infinitesimal element denoted $\delta x$:

$$\delta(V - p) = \delta x \cdot (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}). \quad (21)$$

Clebsch assumes that, at any time and not just initially, the velocity has the (Clebsch) representation (17). After various algebraic manipulations, he arrives at the following relation

$$\delta \{ V - p - (1/2)|\mathbf{u}|^2 - \partial_t F - \phi \partial_t \psi \} = \dot{\phi} \delta \psi - \dot{\psi} \delta \phi, \quad (22)$$

where $\dot{\phi} \equiv (\partial_t + \mathbf{u} \cdot \nabla) \phi$ and $\dot{\psi} \equiv (\partial_t + \mathbf{u} \cdot \nabla) \psi$.

Clebsch then notes that the l.h.s., being the variation of a scalar function, so must the r.h.s. By counting the number of independent variations he argues that the r.h.s. is of the form $\delta \Pi$ where $\Pi$ can only depend on $\phi$, $\psi$ and the time $t$. From there, immediately follows that we have a pair of canonical Hamilton equations

$$\dot{\phi} = -\frac{\partial \Pi}{\partial \psi}, \quad \dot{\psi} = \frac{\partial \Pi}{\partial \phi}. \quad (23)$$

However, Clebsch shows that by performing a suitable canonical transformation, this Hamiltonian $\Pi$ morphs into a vanishing one. The details of this transformation, which actually originated with Jacobi are now standard.\(^{34}\) As a consequence, there is a pair of new Clebsch variables, which we shall here still denote by $\phi$ and $\psi$ whose material time derivatives vanish, so that they satisfy (16).

Clebsch does not comment on the vortical and Lagrangian interpretation of his variables, but his work appeared only shortly after the Helmholtz 1858 vorticity paper, which is not cited. Hankel’s subsequent work is of course much simpler, mostly because he makes use of Lagrangian coordinates, as required for the Prize.\(^{35}\)

Here it is appropriate to mention Hankel’s own assessment of some of the 1859 work of Clebsch, as it appears in the first part of the footnote [A.26] at the end of §11.\(^{36}\) Hankel stresses that, on the one hand, the Clebsch derivation (probably of (22)) can be further simplified and, on the other hand, that his direct proof — that the variables $\phi$ and $\psi$ can be chosen to be material — may be adapted to a purely Eulerian proof. Eventually, such a purely Eulerian derivation of the material character of the Clebsch variables will be given.\(^{37}\)

Of course there are also modern derivations of such results, using the tools of differential geometry and in particular the concept of Lie-advection invariance. For example, proceeding

\(^{33}\) Clebsch, 1857; 1859.

\(^{34}\) Jacobi, 1836-1837/1890: 393; Lanczos, 1970: Chap. 8, Sec. 2, p. 238; Landau and Lifshitz, 1976:148.

\(^{35}\) Helmholtz, 1858.

\(^{36}\) Footnote 1 on p. 45 of the German original.

\(^{37}\) For Eulerian derivations, see Duhem, 1901 and references therein.
as in Besse and Frisch, one can take $\phi$ and $\chi$ to be two Lie-advection-invariant 0-forms (modern parlance for stating that they are material-invariant scalars). Then, the exterior derivative $d\psi$ and the product $\phi d\psi$ will be Lie-advection-invariant 1-forms. The latter’s exterior derivative $d\phi \wedge d\psi$ is then a Lie-advection-invariant 2-form. If initially this 2-form is taken to be the vorticity 2-form, it will stay so. Thus the vorticity vector (the Hodge dual of the vorticity 2-form) can be written at all time as $\nabla \phi \times \nabla \chi$. Except for the modernized language, this is essentially the proof given by Hankel in §§10-11. Hankel here makes use of concepts and calculation procedures, which will take their full significance only within the framework of Lie’s theory.\(^{38}\)

A variational formulation of the incompressible Euler equations, in the spirit of Hamilton and Jacobi, can then be given as follows. One observes that, $\phi$ and $\psi$ being now material invariants, (22), after integration over space and time (over an interval $[0,T]$ at the ends of which the variations are taken to vanish) and after an integration by parts in time, becomes

$$\delta \int_0^T dt \int d^3x \{ \psi \partial_t \phi - \frac{1}{2} |\mathbf{u}|^2 + V - p \} = 0.$$  \(^{(24)}\)

Eq. (24) constitutes a variational formulation for the canonically conjugate variables $\psi$ and $\phi$ and a Hamiltonian $H = \frac{1}{2} |\mathbf{u}|^2 - V + p$, provided $V$ and $p$ are somehow reexpressed functionally in terms of $\psi$, $\phi$ and time.

We should also mention that Clebsch did not restrict his work to three dimensions. He was working in dimension $2n + 1$ where $n$ is an arbitrary positive integer and he had not one but $n$ pairs of variables.

### B. Clebsch variables and global topology

There is a topological issue that could not be perceived at the time of Clebsch, Hankel and all their contemporaries. Indeed, about a century later, with the work on Kolmogorov-Arnold-Moser (KAM) theory and the work relating helicity to vortex line knottedness, it became clear that writing the vorticity in the Pfaffian way, namely as a vector product of two gradients, as in (18), may not hold globally in space. Hadamard already had pointed out that the Clebsch representation is generally only local.

In the three-dimensional case, it was shown by Darboux that a Pfaff representation can be made only locally, near a point of non-vanishing vorticity. To understand why such a representation can generically not be extended globally, we note that the vortex lines are at the intersections of the surfaces of constant $\phi$ and constant $\psi$, as observed by Hankel. Such intersections constitute typically closed lines. However the most general topology of the integral lines of a three-dimensional vector field of vanishing divergence is given by the KAM theorem and has chaotic topology. One well-known instance are the ABC flows. A global Clebsch representation holds only in special cases, one instance being the Taylor–Green flow. This difficulty with the usual Clebsch variables can however be bypassed by using multi-Clebsch variables, that is more than one pair of Clebsch variables, even in three dimensions, where Clebsch had just one pair.\(^{39}\)

There is however another mechanism that can prevent a global Clebsch representation: if the vortex line are closed but knotted. Such flows are said to possess helicity.\(^{40}\)

\(^{38}\)Besse and Frisch, 2017.

\(^{39}\)For the Pfaff problem, see Pfaff, 1814. Clebsch, 1862. Darboux, 1882. Hadamard, 1903:79–81. For the history of the Pfaff problem, see Hawkins, 2013:155–204. For the KAM theory, see Kolmogorov, 1954; Arnold, 1963; Moser, 1962. For the ABC flow, see Dombre, Frisch, Greene, and Hénon, 1986 and references therein. For the Taylor–Green flow, see Taylor and Green, 1937; Brachet, Meiron, Orszag, Nickel, Morf and Frisch, 1983. For multi-Clebsch variables, see Zakharov, L’vov and Falkovich, 1992:28.

\(^{40}\)Moreau, 1961; Moffatt, 1969; Bretherton, 1970; Kotiuga, 1991.
VII. CONCLUSIONS

Before attempting to give a modern assessment of Hankel’s Preisschrift, it is of interest to look at Hankel’s own assessment of his work, as it appears in his 4-page summary published in Fortschritte der Physik (Progress in Physics). He first presents his variational formulation and observes that it is a very convenient way for studying hydrodynamics in an arbitrary system of coordinates by just changing the expression of the element of length $ds^2$. This shows his high regard for Riemann. He then turns to Cauchy’s Lagrangian formulation and applications to the Helmholtz vortical motion. He also states Riemann’s view that the Cauchy invariants are an expression of conservation of rotation. Hankel also points out that the exploitation of Cauchy’s result requires the use of what we now call the Stokes theorem, which Hankel had just rediscovered independently and of which it is not clear that Helmholtz was aware in 1858. Hankel then gives the essence of the derivation of the Helmholtz theorem from the Cauchy invariants. Finally, Hankel gives his own view on Clebsch variables, in which their material invariance is established directly by using Lagrangian coordinates. More than 150 years later, we do not feel that this summary has aged much. We would just add Hankel’s establishment of the conservation of circulation, albeit he never emphasizes this result, now considered by many as one of the most important of ideal fluid dynamics.

We have left for these concluding remarks the thorny issue of how the variational formulation of the incompressible Euler equations has appeared. Given the important role it has played in the work of the last fifty years (initiated by Arnold’s 1966 SDiff formulation), we have felt that this is a worthwhile question.

Already in his founding paper on three-dimensional incompressible flow, Euler speculated that the equation may have a least action formulation in the sense of Maupertuis. The Maupertuisian action is a sum or integral of the velocity $v$ times the particle infinitesimal displacement $ds$. Since $ds = vdt$, this action reduces — except for an overall factor two — to the kinetic energy contribution of the modern (Hamiltonian) action, leaving out the potential energy contribution. The peculiarity of an incompressible fluid is that there is no genuine potential energy: everything is subsumed in the geometrical constraint of volume conservation. Hence, a Maupertuisian attack can be successful, provided incompressibility is correctly taken into account. Lagrange has precisely introduced the appropriate tool, the multipliers known by his name. One would thus expect to find the first variational/least-action formulation of the incompressible flow in Lagrange’s *opus magnum* of 1788. It is actually there, but only in an indirect way: Lagrange points out that for a rather general class of mechanical systems, to which the principle of live forces is applicable, a least action formulation leads to the same equations as his virtual velocity formulation. Here the principle of live forces is essentially equivalent to the modern statement of total energy conservation.

Of course, as is well known, around the middle of the 1760s, Lagrange stopped being very enthusiastic with variational approaches and switched to virtual velocity/virtual power approaches, which he viewed as more general. Before this methodological transition, he published in 1760–1761 a fairly long and rather difficult paper, where variational methods were applied to a number of different mechanical problems, including hydrodynamics. He gave a Maupertuisian least action derivation of the elastic Euler equations, using what we would now call the conservation of total energy. Doing a better job required the development of Hamilton’s version of the least action principle in which both the kinetic and the potential energy appear in the Hamiltonian and the Lagrangian. In the case of elastic hydrodynamics, this project was carried out by Hankel in a most elegant and simple manner using Lagrangian coordinates (see Section IV).

41 Hankel, 1863. This is signed with a somewhat cryptic “Hf”, which is however explained in the list of authors at the end of the volume.
42 Arnold, 1966.
43 Lagrange, 1788:188–189.
44 For Lagrange’s transition to virtual velocities, see Lagrange, 1764; Dahan-Dalmenico, 1992:39–42. For Lagrange’s early work on least action methods applied to mechanics, see Lagrange, 1760–1761; Fraser, 1983.
Why does Hankel not also discuss the case of incompressible flow? Actually he touches on
this issue in the second half of the footnote [A.26] at the end of § 11. He notes that in his 1859
paper Clebsch gives an Eulerian variational formulation for three-dimensional incompressible
flow and states that “In another form, this is the theorem I presented in § 5 [discussed in our
Section IV], deduced immediately from the principle of virtual velocities.” Hankel may not
have desired or found the time to discuss an issue already touched by Clebsch. Furthermore,
Hankel may have thought, as we do, that Lagrange essentially was aware of the variational
formulation for incompressible flow, nearly one century before Hankel’s work.

It is interesting that it took yet another century before Arnold’s 1966 variational for-
mulation of ideal incompressible 3D flow in terms of SDiff, the space of volume-preserving
diffeomorphisms.

Arnold’s 1966 work was however much more than just a variational formulation. La-
grange, as he stresses in the Avertissement (Warning) at the beginning of his 1788 Analytic
Mechanics, engaged in a much-needed degeometrization of mechanics, which brought
considerably more rigor in the presentation. Eventually — but this took two centuries —
Arnold became the father of the regeometrization of fluid mechanics. The central
quantity in the geometrical approach of Arnold, of Ebin and Marsden and of others, already
present in the work of Cauchy and of Hankel, is the Lagrangian map. Dealing with such
maps requires an entirely novel type of geometry, capable of handling infinite-dimensional
Riemannian manifolds and their geodesics, concepts that where very far from ripe at the
time of Riemann and Hankel.

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