Displaced and Squeezed Number States

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ABSTRACT

After beginning with a short historical review of the concept of displaced (coherent) and squeezed states, we discuss previous (often forgotten) work on displaced and squeezed number states. Next, we obtain the most general displaced and squeezed number states. We do this in both the functional and operator (Fock) formalisms, thereby demonstrating the necessary equivalence. We then obtain the time-dependent expectation values, uncertainties, wave-functions, and probability densities. In conclusion, there is a discussion on the possibility of experimentally observing these states.

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1 Background

The coherent states were discovered in 1926 by Schrödinger \[1\], as an example of how his wave functions could mimic classical particles. They consisted of Gaussians in an harmonic oscillator potential. The widths of these Gaussians were those of the ground-state Gaussian. In modern notation, they were

\[
\psi = \pi^{-1/4} \exp \left[ -\frac{(x-x_0)^2}{2} + ip_0 x \right].
\]

(1)

It is to be noted that, this being before Born’s probability interpretation of $\psi^* \psi$, Schrödinger was concerned by the complex nature of the wave function, wondering if it was the real part that was significant.

Lorentz was the person who had been especially bothered by the lack of classical properties of eigenstate wave functions. Schrödinger and Lorentz exchanged many letters on this subject, and their correspondence is published \[2\].

Soon after this, Kennard \[3\] published a paper on quantum motion. There he described what are, in modern parlance, squeezed states. They follow the classical motion, they are Gaussians whose widths are not that of the ground state, and the widths and uncertainty products oscillate with time. However, relatively little notice was given to this paper in later times \[4\].

In the 1960’s the emergence of quantum optics was a fertile background for the modern development of coherent states by Glauber, Klauder, and Sudarshan \[5\]-\[8\]. Using boson operator calculus, these states could be defined as either the eigenstates of the destruction operator,

\[
a|\alpha\rangle = \alpha|\alpha\rangle,
\]

(2)

or equivalently as the state obtained by operating on the ground state by the displacement operator:

\[
D(\alpha)|0\rangle = |\alpha\rangle,
\]

(3)

\[
D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] = \exp[-|\alpha|^2/2] \exp[\alpha a^\dagger] \exp[-\alpha^* a],
\]

(4)
the last equality coming from a Baker-Campbell-Hausdorff (BCH) calculation. These states are

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$  \hspace{1cm} (5)

Use of a generating formula for Hermite polynomials shows that this yields the same wave function as Eq. (1), with the identifications

$$\alpha = \alpha_1 + i\alpha_2 = \frac{x_0 + ip_0}{\sqrt{2}} .$$  \hspace{1cm} (6)

Squeezed states were rediscovered and elucidated by a number of people [9]-[12]. (Ref. [11] was where the term “squeeze” was invented. See Ref. [13] for a discussion of this history.) In operator form, these states are created by

$$D(\alpha)S(z)|0\rangle = |\alpha, z\rangle ,$$  \hspace{1cm} (7)

where the squeeze operator is

$$S(z) = \exp \left[ \frac{1}{2} za^\dagger a^\dagger - \frac{1}{2} z^* aa \right] , \hspace{1cm} z = re^{i\phi} = z_1 + iz_2 .$$  \hspace{1cm} (8)

By BCH relations the squeeze operator can be written as

$$S(z) = \exp \left[ \frac{1}{2} e^{i\phi}(\tanh r) a^\dagger a^\dagger \right] \left( \frac{1}{\cosh r} \right)^{(\frac{1}{2} + a^\dagger a)} \exp \left[ -\frac{1}{2} e^{-i\phi}(\tanh r) aa \right]$$  \hspace{1cm} (9)

$$= \exp \left[ \frac{1}{2} e^{i\phi}(\tanh r) a^\dagger a^\dagger \right] (\cosh r)^{-1/2} \sum_{n=0}^{\infty} \frac{(\text{sech} r - 1)^n}{n!} (a^\dagger)^n (a)^n$$  \hspace{1cm} (10)

$$\times \exp \left[ -\frac{1}{2} e^{-i\phi}(\tanh r) aa \right] .$$

The functional forms of $D$ and $S$ are [14]

$$D(\alpha) = \exp[-ix_0 p_0/2] \exp[ip_0 x] \exp[-x_0 \partial] ,$$  \hspace{1cm} (11)

$$S = S^{-1/2} \exp \left[ \frac{i z_2 \sinh r}{2r} \frac{r S}{S} (x^2) \right] \exp[-(\ln S)(x \partial)] \exp \left[ \frac{i z_2 \sinh r}{2r} \frac{S}{S} (\partial^2) \right] ,$$  \hspace{1cm} (12)

where

$$S = \cosh r + \frac{z_1}{r} \sinh r = \cosh r + \cos \phi \sinh r = e^r \cos^2 \frac{\phi}{2} + e^{-r} \sin^2 \frac{\phi}{2} ,$$  \hspace{1cm} (13)
and one should recall the operator definitions on a function $h$:

\begin{align}
\exp[c\partial]h(x) &= h(x + c) \\
\exp[\tau(x\partial)]h(x) &= h(xe^\tau) \\
\exp[c(\partial^2)]h(x) &= \frac{1}{[4\pi c]^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(y-x)^2}{4c}\right]h(y)dy .
\end{align}

Using these functional forms for $D$ and $S$ the most general squeezed wave function is \cite{14}

$$\psi_{ss} = D(\alpha)S(z)\psi_0$$

$$= \frac{1}{\pi^{1/4}} \frac{\exp[-ix_0p_0/2]}{|S(1+i2\kappa)|^{1/2}} \exp\left[-(x-x_0)^2 \left(\frac{1}{2S^2(1+i2\kappa)} - i\kappa\right) + ip_0x\right],$$

where

$$\kappa \equiv \frac{z_2 \sinh r}{2rS} .$$

Setting $z$ to be real and positive, yields the most commonly studied example:

$$\psi_{ss} = [\pi^{1/2}s]^{-1/2} \exp\left[-\frac{(x-x_0)^2}{2s^2} - ip_0x\right], \quad s = e^r .$$

## 2 The Forgotten Displaced Number States

In the 1950’s, before the modern work on coherent states started, there was a short flurry of activity by four authors, which has essentially been forgotten. These papers were by Senitzky \cite{15} (perhaps the best known), Plebanski \cite{16}, Husimi \cite{17}, and Epstein \cite{18}. These authors investigated whether there are other wave packets, besides the coherent-state packets, which keep their shapes and follow the classical motion. They found that there are. In our modern parlance, they found that any displaced number state follows the classical motion and keeps its shape.

In particular, given that the number-state wave functions are

$$\psi_n = \exp\left[-\frac{x^2}{2} \right] \frac{H_n(x)}{\pi^{1/4}[2^n n!]^{1/2}} ,$$
the displaced number states are found by applying the functional form of the displacement operator onto the number wave functions:

\[ \psi_{cs(n)} = D(\alpha)\psi_n = \frac{\exp[-ix_0p_0/2] \exp[ip_0x]}{\pi^{1/4}} \exp \left[ -\frac{(x - x_0)^2}{2} \right] \frac{H_n(x - x_0)}{[2^n n!]^{1/2}}. \] (21)

An equivalent result can be obtained in Fock notation by writing

\[ D(\alpha)|n\rangle = \exp[-\alpha^2/2] \exp[\alpha a^\dagger] \exp[-\alpha^* a]|n\rangle \] (22)

\[ = \exp[-\alpha^2/2] \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \sum_{j=0}^{n} \frac{(-\alpha^*)^j}{j!} \left[ (n - j + k)!n! \right]^{1/2} |n - j + k\rangle. \] (23)

Then by changing from the Fock state basis to the wave function basis, interchanging the order of the summations, and using the sum 49.4.2 of Ref. [19], one has

\[ D(\alpha)|n\rangle \rightarrow \pi^{-1/4} \exp[-(x - \sqrt{2}\alpha_1)^2/2 + i\sqrt{2}\alpha_2 x - i\alpha_1\alpha_2] \]

\[ \frac{(n!)^{1/2}}{2^{n/2}} \sum_{j=0}^{n} \frac{(-\alpha^*)^j 2^{j/2}}{j!(n - j)!} H_{n-j}(x - \alpha/\sqrt{2}). \] (24)

Changing the sum over \( j \) to a sum over \( k = n - j \), the sum is in the form of the seventh equality on page 255 of Ref. [20]. Using this one obtains the same answer as Eq. (21) above.

Time-evolution can be shown to be described by [14]

\[ x_0 \rightarrow x_0(t) = (x_0 \cos t + p_0 \sin t), \quad p_0 \rightarrow p_0(t) = (p_0 \cos t - x_0 \sin t). \] (25)

The wave packets have the properties

\[ [\Delta x(t)]^2 = [\Delta p(t)]^2 = (n + 1/2), \] (26)

\[ \langle H \rangle = (n + 1/2) + |\alpha|^2. \] (27)

In the later excitement over ordinary coherent and squeezed states, the work of the earlier authors [15]-[18] was, unfortunately, mainly forgotten. In 1973 one paper appeared applying the operator-formalism displacement operator on the Fock number state [21]. Then, in the 1980’s, papers started appearing studying these states [22]-[24], one of which [22] noted the early work by Senitzky and others [15]-[18].
3 Squeezed Number States

Plebanski [10], in wave-function form, also looked a little into what we would call squeezed number states. By using the functional forms in Eqs. (11) and (12) of D and S [14], we find that the most general squeezed number states are

\[ \psi_{ss(n)} = D(\alpha)S(z)\psi_n = \frac{\exp[-i x_0 p_0/2]}{\pi^{1/4}[F_1]^{1/2}} \exp \left[ -\frac{(x-x_0)^2}{2} F_2 + ip_0 x \right] \]

\[ \left[ (F_3)^{n/2} \frac{1}{[2^n n!]^{1/2}} H_n \left( \frac{x-x_0}{F_4} \right) \right], \]

(28)

where

\[ F_1 = S(1+i2\kappa) = \cosh r + e^{i\phi} \sinh r, \]

(29)

\[ F_2 = \left( \frac{1}{S^2(1+i2\kappa)} - i2\kappa \right) = \left( \frac{1}{S^2(1+4\kappa^2)} \right) - i2\kappa \left[ 1 + \left( \frac{1}{S^2(1+4\kappa^2)} \right) \right] \]

\[ = \frac{1 - i \sin \phi \sinh r (\cosh r + e^{i\phi} \sinh r)}{(\cosh r + \cos \phi \sinh r)(\cosh r + e^{i\phi} \sinh r)}. \]

(30)

\[ F_3 = \frac{1 - i2\kappa}{1 + i2\kappa} = \left( \frac{\cosh r + e^{-i\phi} \sin \phi \sinh r}{\cosh r + e^{i\phi} \sin \phi \sinh r} \right) = e^{-i2 \tan^{-1}(2\kappa)}, \]

(31)

\[ F_4 = S(1+4\kappa^2)^{1/2} = [\cosh^2 r + \sinh^2 r + 2 \cos \phi \cosh r \sinh r]^{1/2}. \]

(32)

By using properties of the \(F\)'s. such as that \(F_3\) is a phase, \([F_2 + F_2^*] = 2/F_4^2\), and \(F_1^* F_1 = F_4^2\), one can verify that

\[ 1 = \int_{-\infty}^{\infty} dx \, \psi_{ss(n)}^*(x) \psi_{ss(n)}(x). \]

(33)

Note that when \(n = 0\), the final large square bracket of Eq. (28) is unity and the wave function reduces to the normal squeezed states.

Satyanarayana [23] defined the problem in Fock notation. He obtained that the squeezed number states are

\[ |\alpha, z, n\rangle = D(\alpha)S(z)|n\rangle \]
\[
\exp\left[-|\alpha|^2/2\right] \sum_{m,l} G_{mn}(z) \left(\frac{m!}{l!}\right)^{1/2} L_{(m)}^{(l-m)}(|\alpha|^2) \alpha^{l-m} |l\rangle,
\]

where the \(G_{mn}(z)\) are given by infinite sums. Later discussions followed. One emphasized number states having the displacement operator and squeeze operator being applied separately (rather than in combination) \cite{23}. Another was interested in these states from the point of view of the parity operator \cite{26}. A third still addressed the topic from the viewpoint of molecular wave packets \cite{27}.

Now we will show how to sum the Fock problem in closed form, thereby reproducing the result of Eq. (28). Start with a number state and apply the squeeze operator. If

\[
d = \frac{1}{2} e^{i\phi} \tanh r,
\]

then

\[
S(z)|n\rangle = e^{da^\dagger a^\dagger} \left(\frac{1}{\cosh r}\right)^{1/2} e^{-d^\dagger a a} |n\rangle
\]

\[
= e^{da^\dagger a^\dagger} \left(\frac{1}{\cosh r}\right)^{1/2} \sum_{j=0}^{[n/2]} \left[\frac{n!}{(n-2j)!}\right]^{1/2} \frac{(-d^\dagger)^j}{j!} |n-2j\rangle
\]

\[
= \left(\frac{1}{\cosh r}\right)^{n+1/2} (n!)^{1/2} \sum_{j=0}^{[n/2]} \frac{(-d^\dagger)^j (\cosh r)^{2j}}{(n-2j)!j!} \sum_{k=0}^{\infty} \frac{d^k[(n-2j+2k)!]^{1/2}}{k!} |n-2j+2k\rangle,
\]

where, in the upper limit of the sums, \([n/2]\) is the greatest integer function.

With the identification

\[
|n-2j+2k\rangle \rightarrow \frac{e^{-x^2/2} H_{n-2j+2k}(x)}{\pi^{1/4} [2^{n-2j+2k} (n-2j+2k)!]^{1/2}},
\]

and using the known infinite sum 49.4.4 of Ref. \cite{19},

\[
\sum_{k=0}^{\infty} \frac{v^k}{k!} H_{2k+m}(x) = (1 + 4v)^{-m/2-1/2} \exp\left[\frac{4vx^2}{1 + 4v}\right] H_{m}\left(\frac{x}{[1 + 4v]^{1/2}}\right),
\]

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one finds

$$S(z)|n\rangle \rightarrow \frac{\exp[-x^2\mathcal{F}_2/2](n!)^{1/2}}{\pi^{1/4}\mathcal{F}_1^{1/2}} \left[ \cosh r (\cosh r + e^{i\phi} \sinh r) \right]^{n/2} R,$$

where

$$R = \sum_{j=0}^{[\frac{n}{2}]} \tau^j \left( \frac{n - 2l}{(n - 2l)!} \right) H_{n-2j}(xy),$$

and

$$\tau = -e^{-i\phi} \sinh r (\cosh r + e^{i\phi} \sinh r), \quad y = \frac{\cosh r}{\mathcal{F}_1^{1/2}}.$$

To do the sum in Eq. (42), we start with the $n$ being even case, $[n/2] = n/2$. By changing the variable of the sum to $k = n/2 - j$ and doing a little algebra, one finds

$$R_E = \tau^{n/2} \left( \frac{n}{(n/2)!} \sum_{k=0}^{n/2} \left( -\frac{n}{2} \right)_k \left( \frac{-1}{\tau} \right)^k H_{2k}(xy).$$

But this sum is of the form of equation 49.4.12 of Ref. [19]:

$$\sum_{k=0}^{n} \left( -\frac{n}{2} \right)_k \frac{v^k}{(2k)!} H_{2k}(x) = \frac{n!}{(2n)!} (-1 - v)^n H_{2n} \left[ x(1 + 1/v)^{-1/2} \right].$$

Using this and a fair amount of further algebra one finally obtains for the $n$ being even case ($n = n_E$),

$$S(z)|n = n_E\rangle \rightarrow \frac{1}{\pi^{1/4}\mathcal{F}_1^{1/2}} \exp \left[ -\frac{x^2}{2} \mathcal{F}_2 \right] \left[ \frac{(\mathcal{F}_3)^{n/2}}{[2^n n!]^{1/2}} H_n \left( \frac{x}{\mathcal{F}_4} \right) \right].$$

But this is exactly the form of Eq. (28) for the special case of no displacement, $x_0 = p_0 = 0$. Therefore, now applying the displacement operator trivially yields the general result of Eq. (28), as it should.

The $n$ being odd case $n = n_O$ is obtained in the same manner. The only significant difference is that instead of using the sum of Eq. (45), one uses Eq. 49.4.14 of Ref. [19]:

$$\sum_{k=0}^{n} \frac{(-n)_k}{(2k + 1)!} v^k H_{2k+1}(x) = \frac{n!}{(2n + 1)!} (-v)^n (1 + 1/v)^{n+1/2} H_{2n+1} \left[ x(1 + 1/v)^{-1/2} \right].$$
The identical functional form is obtained and so we have demonstrated, as it must be, that the functional and operator formalisms yield the identical squeezed number states.

4 Time-Dependent Uncertainties

Calculating the time-dependent uncertainties is most easily done with the boson-operator formalism. This is because we know that

\[
D^\dagger(\alpha)aD(\alpha) = a - \alpha ,
\]

\[
S^\dagger(z)aS(z) = (\cosh r)a + e^{i\phi}(\sinh r)a^\dagger ,
\]

\[
T^\dagger(t)aT(t) = ae^{-it} ,
\]

where \( T \) is the time-evolution operator

\[
T(t) = \exp[-iHt] = \exp[-i(a^\dagger a + 1/2)t] = \exp[-i(-\partial^2 + x^2)t/2] .
\]

With this it is straight-forward to calculate the uncertainties, as a function of time, for the squeezed number states.

To begin,

\[
\langle x_s(t) \rangle_n = \langle n|S^\dagger D^\dagger T^\dagger xTDS|n \rangle = x_0 \cos t + p_0 \sin t .
\]

Similarly,

\[
\langle p_s(t) \rangle_n = p_0 \cos t - x_0 \sin t ,
\]

\[
\langle x_s^2(t) \rangle_n = (2n + 1)[(\cosh r)^2 + (\sinh r)^2 + 2(\cosh r)(\sinh r) \cos(2t - \phi)]
\]

\[+ \langle x_s(t) \rangle_n^2 , \]

\[
\langle p_s^2(t) \rangle_n = (2n + 1)[(\cosh r)^2 + (\sinh r)^2 - 2(\cosh r)(\sinh r) \cos(2t - \phi)]
\]

\[+ \langle p_s(t) \rangle_n^2 . \]
Therefore, the uncertainty-product as a function of time is
\[
\frac{[\Delta x(t)]^2[\Delta p(t)]^2}{(n + 1/2)^2} = 1 + 4(\cosh r)^2(\sinh r)^2 \sin^2(2t - \phi) \quad (56)
\]
\[
= 1 + \frac{1}{4} \left( s^2 - \frac{1}{s^2} \right)^2 \sin^2(2t - \phi) . \quad (57)
\]
Eq. (56) agrees with the results of Ref. [26] for \( t = 0 \) and Eq. (57) agrees with the standard squeezed-state result when \( n = 0 \) [28].

5 Time-Evolution of the States

Taking the functional definition of the time-displacement operator, \( T \), in Eq. (51), and using BCH relations it has been shown [14] that the harmonic-oscillator time-displacement operator can be written as
\[
T = \cos t - \frac{1}{2} \exp \left[ -i \frac{2}{\tan t} (x^2) \right] \exp[ - (\ln \cos t) (x \partial)] \exp \left[ i \frac{2}{\tan t}(\partial^2) \right] . \quad (58)
\]
Using this result, along with the functional definitions in Eqs. (14-16), and combining it with the squeezed state wave function, \( \psi_{ss(n)} \) of Eq. (28), one has
\[
\Psi_{ss(n)}(x,t) = T\psi_{ss(n)}(x) = e^{-i\tan t x^2/2}e^{-ix_0p_0/2} F_n^{3/2} I, \quad (59)
\]
where
\[
I = \int_{-\infty}^{\infty} dy \exp \left[ -\frac{(y - x/\cos t)^2}{i2 \tan t} - \frac{1}{2} (y - x_0)^2 F_2 + ip_0y \right] H_n \left( \frac{y - x_0}{F_4} \right) . \quad (60)
\]
This integral can be evaluated with the aid of 7.374.8 of Ref. [29], yielding
\[
\Psi_{ss(n)}(x,t) = \left[ \frac{1}{\pi^{1/4}[BF_1]^{1/2}} \left[ \frac{F_3^2 A^n}{2^n n!} \right] \right]^{1/2} H_n \left( \frac{X(t)}{F_4 B[A]^{1/2}} \right) \exp \left[ -\frac{x^2 F_2 \cos t + \sin t}{2B} + \frac{x (x_0 F_2 + ip_0) \cos t}{B} - \frac{x_0^2 F_2 \cos t}{2B} \right] \exp \left[ -\frac{p_0^2 (i \sin t)}{2B} - x_0 p_0 \left( \frac{F_2 \sin t}{B} + \frac{i}{2} \right) \right] , \quad (61)
\]
where the \( F_i \) are given in Eqs. (29) - (32), and

\[
A = \left( 1 - \frac{i2 \sin t}{F_i^2 B} \right) \left( \frac{B - i2 \sin t/F_i^2}{B} \right), \tag{62}
\]

\[
B = \cos t + iF_2 \sin t, \tag{63}
\]

\[
X(t) = x - (x_0 \cos t + p_0 \sin t). \tag{64}
\]

Appropriate limits can be checked. For the case \( t = 0 \) one finds \( \Psi_{ss(n)}(x, t = 0) = \psi_{ss(n)}(x) \), the function given in Eq. (28). Also, for the non-squeezed case of \( z = 0 \) or \( F_i = 1 \), one finds the appropriate time-evolution of displaced number states:

\[
\Psi_{ss(n)}(x, t, F_i = 1) = e^{-i(n+1/2)t} \frac{1}{\pi^{1/4} [2^n n!]^{1/2}} H_n(X(t)) \exp \left[ -\frac{X^2(t)}{2} \right] \exp \left[ i \left\{ x - (x_0 \cos t + p_0 \sin t)/2 \right\} (p_0 \cos t - x_0 \sin t) \right]. \tag{65}
\]

The probability density is given by

\[
\rho_{ss(n)}(x, t) = \Psi_{ss(n)}^*(x, t) \Psi_{ss(n)}(x, t) = \frac{[AA^*]^{n/2}}{2^n (n!)^{1/2}} \frac{1}{\pi B B^*[A^*]^{1/2}} H_n \left( \frac{X(t)}{F_4 B^* [A^*]^{1/2}} \right) \frac{X(t)}{F_4 B [A]^{1/2}} \exp \left[ -\frac{X^2(t)}{F_4^2 B B^*} \right]. \tag{66}
\]

(67)

With the aid of integral 2.20.16.2 of Ref. [30],

\[
\int_0^\infty dx \ e^{-nx^2} H_n(bx) H_n(cx) = \frac{2^{n-1} n! \sqrt{\pi}}{p^{(n+1)/2}} (b^2 + c^3 - p)^{n/2} P_n \left( \frac{bc}{\sqrt{p(b^2 + c^2 - p)}} \right), \tag{68}
\]

where the \( P_n \) are the Legendre functions, it can in principle be demonstrated that the probability density is properly normalized:

\[
1 = \int_{-\infty}^{\infty} dx \ \rho_{ss(n)}(x, t). \tag{69}
\]

However, for small \( n \) it is easier to verify the normalization on a case by case basis. The first three probability densities are

\[
\rho_{ss(0)}(x, t) = \frac{1}{\pi B B^* [A^*]^{1/2}} \exp \left[ -\frac{X^2}{BB^* F_4^2} \right], \tag{70}
\]

11
\[ \rho_{ss(1)}(x,t) = \frac{2}{\pi BB^* F_4^{1/2}} \exp \left[ -\frac{X^2}{BB^* F_4^2} \right] \left( \frac{X^2}{BB^* F_4^2} \right), \]  
(71)

\[ \rho_{ss(2)}(x,t) = \frac{AA^*}{2\pi BB^* F_4^{1/2}} \exp \left[ -\frac{X^2}{BB^* F_4^2} \right] \left( \frac{2X^2}{B^2 A F_4^2} - 1 \right) \left( \frac{2X^2}{B^2 A F_4^2} - 1 \right). \]  
(72)

\( \rho_{ss(0)}(x,t) \) is the probability density for the ordinary squeezed state. \( \rho_{ss(1)}(x,t) \) and \( \rho_{ss(2)}(x,t) \) are \( n = 1 \) and \( n = 2 \) squeezed number states, having two and three humps, respectively, in their wave packets. In Figs. 1-4 we show three-dimensional plots of \( \rho_{ss(1)} \) vs. \( x \) and \( t \) for various \( n = 1 \) squeezed states.

Fig. 1 looks at the case \( x_0 = 8, p_0 = 0, \) and \( z = \ln 2. \) The picket starts with no velocity at maximum displacement. It is at its broadest when \( t = 0, \pi, \) and \( 2\pi \) for \( x = 8, -8, \) and \( 8. \) In between the locations where the wave packet is broadest, the two humps are closer together, narrower, and higher in amplitude. In Fig 2, we show the case again having \( x_0 = 8 \) and \( p_0 = 0, \) but this time \( z = -\ln 2. \) This basically changes the phase by \( \pi/2. \) The narrow, peaked wave packets are when \( t = 0, \pi, \) and \( 2\pi \) for \( x = 8, -8, \) and \( 8. \) The broader and shorter conditions are in between.

Fig. 3 is again an example with \( x_0 = 8 \) and \( p_0 = 0, \) but now \( z \) is imaginary: \( z = (\ln 2) \exp[i\pi/2] = i \ln 2. \) One can see from the \( t = 2\pi \) case that now we have a phase shift such that neither extreme configuration is located at the \( t = \) an integral number times \( \pi \) positions. Finally, in Fig. 4, we change to a central position and maximum velocity to the right at \( t = 0: x_0 = 0 \) and \( p_0 = 8, \) with \( z = \ln 2. \) Now the broadest wave packets are centered at \( x = 0 \) for \( t = 0, \pi, \) and \( 2\pi. \) The narrow peaked packets are when \( t = \pi/2 \) and \( 3\pi/2. \)

6 Discussion

There is hope that, in the not too distant future, displaced and squeezed number states can be observed. This optimism is based on the recent work of Wineland’s
group with trapped $Be^+$ ions [31, 32].

With laser cooling and a series of laser pulses to entangle the electronic and motional states they produced (displaced) even and odd coherent states [31, 33]. They also have been able to produce a squeezed (but not displaced) ground state and number states [32]. If all these techniques can be combined then in principle displaced and squeezed number states perhaps can be produced. However, this would be no easy feat since there are very complicated heating problems involved.

When these states are produced, they should mimic the features displayed in our Figs. 1 to 4.

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References

[1] E. Schrödinger, Naturwiss. 14 (1926) 664.

[2] E. Przibram, *Letters on Wave Mechanics*, translated into English by M. J. Klein (Philosophical Library, NY, 1967).

[3] E. H. Kennard, Zeit. Phys. 44 (1927) 326. See Sec. 4C.

[4] D. S. Saxon, *Elementary Quantum Mechanics, Preliminary Edition*, (Holden-Day, San Francisco,1964), p. 162.

[5] R. J. Glauber, Phys. Rev. 130 (1963) 2529.
[6] J. R. Klauder, Annals of Physics 11 (1960) 123; J. Math. Phys. 4 (1963) 1055; J. Math. Phys. 4 (1963) 1058.

[7] E. C. G. Sudarshan, Phys. Rev. Lett. 10 (1963) 227.

[8] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, NY, 1968).

[9] H. Takahasi, Ad. Communication Systems 1 (1965) 227.

[10] H. P. Yuen, Phys. Rev. A 13 (1976) 2226.

[11] J. N. Hollenhorst, Phys. Rev. D 19 (1979) 1669.

[12] C. M. Caves, J. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmerman, Rev. Mod. Phys. 52 (1980) 341.

[13] M. M. Nieto, in: *Frontiers of Nonequilibrium Statistical Physics*, eds. G. T. Moore and M. O. Scully (Plenum, New York, 1986) p. 287.

[14] M. M. Nieto, Quantum Semiclass. Optics 8 (1996) 1061.

[15] I. R. Senitzky, Phys. Rev. 95 (1954) 1115.

[16] J. Plebanski, Phys. Rev. 101 (1956) 1825; Bull. Acad. Polon. 11 (1954) 213; ibid. 14 (1955) 275.

[17] K. Husimi, Prog. Theort. Phys. 9 (1953) 381.

[18] S. T. Epstein, Am. J. Phys. 27 (1959) 291.

[19] E. R. Hansen, *A Table of Series and Products*, (Prentice-Hall, Englewood Cliffs, N.J., 1975).

[20] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, (Springer, New York, 1966).
[21] M. Boiteux and A. Levelut, J. Phys. A 6 (1973) 589.

[22] S. M. Roy and V. Singh, Phys. Rev. D 25 (1982) 3413.

[23] M. V. Satyanarayana, Phys. Rev. D 32 (1985) 400.

[24] F. A. M. de Oliveira, M. S. Kim, P. L. Knight, and V. Bužek, Phys. Rev. A 41 (1990) 2645.

[25] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Opt. Comm. 12 (1989) 99; Phys. Rev. A 40 (1989) 2494; J. Mod. Opt. 37 (1990) 659.

[26] R. F. Bishop and A. Vourdas, Phys. Rev. A 50 (1994) 4488.

[27] J. P. Dowling, W. P. Schleich, and J. A. Wheeler, Ann. Phys. (Leipzig) 48 (1991) 423.

[28] M. M. Nieto, Quantum Optics 6 (1994) 9. The quantity in the large round brackets of Eq. (5) should be squared.

[29] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products, (Academic Press, N.Y., 1965).

[30] A. P. Prudnikov, Yu. A. Brychkov, and O. J. Marichev, Integrals and Series, Vol. 2, (Gordon and Breach, New York, 1986).

[31] C. Monroe, D. M. Meekhof, B. E. King, and D. J. Wineland, Science 272 (1996) 1131.

[32] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. 76 (1996) 1796.

[33] Also see the analyses of these states in R. L. de Matos Filho and W. Vogel, Phys. Rev. Lett. 76 (1996) 608; M. M. Nieto, Phys. Lett A 219 (1966) 180.
Figure Captions

Figure 1. A three-dimensional plot of the $n = 1$ squeezed-state probability density $\rho_{ss(1)}$ as a function of position, $x$, and time, $t$, for $x_0 = 8$, $p_0 = 0$, and $z = \ln 2$. The spikes near the maxima of $\rho_{ss(1)}$ are an artifact of the numerical routine. The true, smooth maxima follow the tops of the spikes.

Figure 2. A three-dimensional plot of the $n = 1$ squeezed-state probability density $\rho_{ss(1)}$ as a function of position, $x$, and time, $t$, for $x_0 = 8$, $p_0 = 0$, and $z = -\ln 2$.

Figure 3. A three-dimensional plot of the $n = 1$ squeezed-state probability density $\rho_{ss(1)}$ as a function of position, $x$, and time, $t$, for $x_0 = 8$, $p_0 = 0$, and $z = (\ln 2) \exp[i\pi/2] = i \ln 2$. The spikes near the maxima of $\rho_{ss(1)}$ are an artifact of the numerical routine. The true, smooth maxima follow the tops of the spikes.

Figure 4. A three-dimensional plot of the $n = 1$ squeezed-state probability density $\rho_{ss(1)}$ as a function of position, $x$, and time, $t$, for $x_0 = 0$, $p_0 = 8$, and $z = \ln 2$. 

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