On the stability problem in relativistic thermodynamics: implications of the Chapman-Enskog formalism.

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Abstract

Extended theories are widely used in the literature to describe relativistic fluids. The motivation for this is mostly due to the causality issues allegedly present in the first order in the gradients theories. However, the decay of fluctuations in the system is also at stake when first order theories that couple heat with acceleration are used. This paper shows that although the introduction of the Maxwell-Cattaneo equation in the description of a simple relativistic fluid formally eliminates the generic instabilities identified by Hiscock and Lindblom in 1985, the hypothesis on the order of magnitude of the corresponding relaxation term contradicts the basic ordering in Knudsen's parameter present in the kinetic approach to hydrodynamics. It is shown that the time derivative, stabilizing term is of second order in such parameter and thus does not belong to the Navier-Stokes regime where the so-called instability arises.
I. INTRODUCTION

The transport equations for relativistic fluids have been well established within the framework of irreversible thermodynamics \[1\]–\[3\]. However, it was shown that Eckart’s constitutive equation, as a closure relation for the heat flux including an acceleration term as driving force \[4\], renders the system unstable \[5\] and seems to allow propagation of signals at arbitrary velocities \[6\]. This instability and acausality features of the relativistic gas lead to the formulation and use of higher order in the gradients, or extended theories \[7\]–\[10\].

The definite proof of the violation of Onsager’s regression of fluctuations hypothesis was firstly identified by Hiscock and Lindblom even though they interpreted this contradiction with the tenets of irreversible thermodynamics as an early onset of instabilities \[3\]. In their paper it is shown that fluctuations in the linearized system of equations grow exponentially with a very small characteristic time. Meanwhile, buried deep in the fundamental hypothesis of non-equilibrium thermodynamics, Onsager’s regression assumption asserts that spontaneous fluctuations of microscopic origin should decay following the linearized equations for the state variables. It is this assumption that is violated when Eckart’s equation for the heat flux is used, as is shown in Ref. \[11\]. As a result, Hiscock and Lindblom suggested in their work that, due to the unphysical behavior predicted by Eckart’s formalism, “standard” first order in the gradients theories should be discarded in favor of the extended type ones developed by Israel.

In this work we revisit the stability analysis for fluctuations within the linearized system using a Maxwell-Cattaneo type constitutive equation for the heat flux. This calculation sheds light on the “stabilization” mechanism by showing that the dynamics of the fluctuations is dominated by two competing drives: Eckart’s acceleration term and Cattaneo’s relaxation. By assuming the characteristic time in Maxwell-Cattaneo’s equation is large enough, the relaxation term dominates leading to an exponential decay of density fluctuations.

On the other hand an equation describing the evolution of the heat flux can be obtained using kinetic theory. In this work we perform this task including explicitly an ordering parameter by means of the Chapman-Enskog method in order to avoid the fundamental inconsistencies previously reported in the literature \[12\], \[13\]. As a consequence, an evolution equation for the heat flux is obtained from Boltzmann’s equation in which the ordering of the terms can be clearly identified. It is then shown that the time derivative term is of
second order in the Knudsen parameter. Moreover the relaxation time $\tau$, of microscopic scale, present in kinetic theory must be identified in this scheme with the characteristic time of Maxwell-Cattaneo’s equation. It is then concluded that kinetic theory does not provide a microscopic basis for the presence of the relaxation time required to solve the stability problem of a dissipative relativistic fluid in the Maxwell-Cattaneo approach in the Navier-Stokes regime.

The rest of this paper is divided as follows. In section II we review the balance equations used in the description of the evolution of density fluctuations in a dissipative relativistic monoatomic fluid. In section III we show how in the linearized system the generic instability previously reported is formally suppressed using the Maxwell-Cattaneo evolution equation for the heat flux. In section IV a Maxwell-Cattaneo type equation is obtained within the Chapman Enskog method and the different orders in powers of the Knudsen’s parameter are identified. Discussion and final remarks are included in section V.

II. BASIC FORMALISM: RELATIVISTIC FLUIDS AND GENERIC INSTABILITIES

The transport equations for a simple relativistic fluid are obtained from the conservation laws for the particle and momentum-energy fluxes, that is

$$N_{;\mu}^\mu = 0$$

and

$$T_{;\nu}^{\mu\nu} = 0$$

where $N^\mu$ is the particle four flux given by

$$N^\mu = nU^\mu$$

and

$$T^{\mu\nu} = \frac{n\varepsilon}{c^2} U^\mu U^\nu + p h^{\mu\nu} + \Pi^{\mu\nu} + \frac{1}{c^2} q^\nu U^\mu + \frac{1}{c^2} U^\mu q^\nu$$

is the stress-energy tensor. Here $n$ is the particle number density, $U^\mu$ the hydrodynamic four-velocity, $c$ the speed of light, $\varepsilon$ the internal energy density per particle and $p$ the hydrostatic pressure. The dissipative fluxes are the heat flux $q^\nu$ and the Navier tensor $\Pi^\mu_\nu$. The spatial
projector $h_{\mu}^\nu$ for a $+++-$ signature is given by

\[ h_{\mu}^\nu = \delta_{\mu}^\nu + \frac{U_{\mu}U_{\nu}}{c^2} \]  

which satisfies $h_{\mu}^\nu U_{\nu} = 0$ for $\mu = 1, \ldots, 4$. Also, the orthogonality conditions implied in this 3+1 representation are given by

\[ U_{\mu}\Pi_{\nu}^\mu = 0, \quad q_{\mu}U_{\mu} = 0. \]  

Substituting Eq. (3) in Eq. (1) yields the relativistic continuity equation, i.e.

\[ \dot{n} + n\theta = 0 \]  

where $\theta = U_{\nu}^\nu$, a comma indicates a partial derivative and a semicolon a covariant one. Also, a dot denotes a proper time derivative, that is $\dot{\phi} = U_{\nu}\phi_{,\nu}$ and $\dot{A}_\mu = U_{\nu}A_{\mu\nu}$ for scalar and first rank tensors respectively. Both energy and momentum balances are extracted from Eq. (2), which reads:

\[ \left( \frac{n\varepsilon}{c^2} + \frac{p}{c^2} \right) U_{\nu}^\nu + \left( \frac{n\varepsilon}{c^2} + \frac{p}{c^2}\theta \right) U_{\nu}^\nu + p_{,\mu} h_{\mu\nu} + \Pi_{\mu\nu}^\nu + \frac{1}{c^2} \left( q_{\mu}U_{\nu}^\nu + q_{,\mu}U_{\nu}^\nu + \theta q_{,\nu} + U_{\nu}^\mu q_{,\mu} \right) = 0. \]  

For $\nu = 1, 2, 3$ one obtains the momentum balance, and $\nu = 4$ leads to a total energy balance. A shortcut leading directly to the internal energy equation is to consider the projection $U_{\nu}^\nu T_{\nu\mu} = 0$ which leads to

\[ n\dot{\varepsilon} + p\dot{\theta} + U_{\mu\nu}\Pi_{\nu}^\mu + q_{,\mu}^\mu + \frac{1}{c^2} U_{\mu\nu}U_{\nu}^\mu q_{,\nu} = 0. \]  

The relation $\dot{\varepsilon} = \left( \frac{\partial \varepsilon}{\partial n} \right)_T \dot{n} + \left( \frac{\partial \varepsilon}{\partial T} \right)_n \dot{T}$ together with the ideal gas law $p = nkT$, which holds in the relativistic case, is introduced in order to obtain an evolution equation for $T$, i.e.

\[ nC_n \dot{T} + p\dot{\theta} + U_{\mu\nu}\Pi_{\nu}^\mu + q_{,\mu}^\mu + \frac{1}{c^2} U_{\mu\nu}q_{,\nu} = 0 \]  

where $C_n$ is the specific heat capacity per particle. The set of hydrodynamic equations for the relativistic fluid is thus given by Eqs. (7), (8) and (10). The set is not complete and constitutive equations must be introduced in order to express the dissipative fluxes in terms of the state variables. In 1940, Eckart proposed, from a purely phenomenological approach by enforcing consistency with the second law of thermodynamics, a coupling of heat with hydrodynamic acceleration additional to the Fourier term. Various authors have shown...
that the system of equations leads to unphysical results \cite{5, 11, 14, 15}. It is important to remark that no generic instabilities arise from viscous effects \cite{11} so that we shall only concentrate in the constitutive equation for the heat flux.

III. THE MAXWELL-CATTANEO EQUATION AND GENERIC INSTABILITIES

Following Hiscock and Lindblom’s result, we here study the behavior of the linearized system by introducing a Maxwell-Cattaneo constitutive relation for the heat flux. Such equation reads:

\[ \tau q'' + q'' = -\kappa h^{\nu\mu} \left( T_{,\mu} + \frac{T}{c^2} \dot{U}_\mu \right) \] (11)

where \( \tau \) is a relaxation time. Notice that Eq. (11) becomes Eckart’s constitutive equation if \( \tau = 0 \) \cite{4}.

The relativistic version of Maxwell-Cattaneo’s equation, Eq. (11), can be motivated by means of a phenomenological entropy balance using extended theories of irreversible thermodynamics \cite{16}. No specific arguments regarding the relativistic generalization are required to sustain the general structure of Eq. (11) since our key argument here corresponds to the role of the first term of this equation in solving the stability problem.

In order to study the behavior of thermal fluctuations in the set of relativistic hydrodynamic equations, a linearized set is obtained by considering

\[ F = F_0 + \delta F \] (12)

for the state variables, in this case \( n, \dot{U}_\mu \) and \( T \). Introducing this hypothesis one obtains, to first order in fluctuations, linearized continuity, momentum and heat equations given by

\[ \delta \dot{n} + n_0 \delta \theta = 0 \] (13)

\[ \frac{1}{c^2} (n_0 \varepsilon_0 + p_0) \delta \dot{U}_\nu + kT_0 \delta n_{,\nu} + n_0 k \delta T_{,\nu} + \delta q_{\nu} = 0 \] (14)

\[ nC_n \delta \dot{T} + n_0 kT_0 \delta \theta + \delta q''_{\nu} = 0. \] (15)

where \( \delta q_{\nu} \) is understood as the heat flux calculated to first order in fluctuations of state variables. Here, and for the rest of this work, we consider an inviscid fluid for the sake of
simplicity. The stability problem pertains heat conduction exclusively and thus the inclusion of viscous effects shall not alter the arguments here presented.

The relativistic version of Maxwell-Cattaneo’s equation, in terms of fluctuations, is given by

$$\tau \delta q'' + \delta q'' = -\kappa h'' \left( \delta T_{,\mu} + \frac{T_0}{c^2} \delta U_{,\mu} \right).$$  (16)

Since equations (13) and (15) only depend on the velocity through $\delta \theta$, it is convenient to separate the longitudinal and transverse velocity gradient fluctuations, by calculating the curl and the divergence of Eq. (14). Following this standard procedure, the transverse component of the momentum balance is uncoupled from the system, as shown in Ref. [11]. On the other hand the longitudinal component, obtained by calculating the divergence of Eq. (14), reads:

$$\frac{1}{c^2} \left( n_0 \varepsilon_0 + p_0 \right) \delta \dot{\theta} + kT_0 \left( h'' \delta n_{,\nu} ; \alpha \right) + n_0 k \left( h'' \delta T_{,\nu} ; \alpha \right) + \frac{1}{c^2} \delta \dot{q}_{\mu}'' = 0.$$  (17)

Thus, considering the divergence of the linearized Maxwell-Cattaneo equation, Eq. (16), reduces the system to a set of 4 scalar equations for the unknowns $\delta n$, $\delta \theta$, $\delta T$ and $\delta q_{\nu}''$, which in Fourier-Laplace space leads to the following dispersion relation

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$  (18)

where the coefficients $a_0$, $a_1$, $a_2$, $a_3$ and $a_4$ are given by the following expressions

$$a_0 = \frac{kT_0 \kappa}{C_n mn_0} q^4$$  (19)

$$a_1 = \frac{kT_0}{m} \left( 1 + \frac{k}{C_n} \right) q^2$$  (20)

$$a_2 = q^2 \left[ \frac{\kappa}{C_n n_0} \left( 1 - \frac{2kT_0}{mc^2} \right) + \frac{kT_0}{m} \left( 1 + \frac{k}{C_n} \right) \tau \right]$$  (21)

$$a_3 = 1$$  (22)

$$a_4 = \tau - \frac{T_0 \kappa}{c^2 mn_0}$$  (23)
where \( s \) and \( q \) are the Laplace and Fourier parameters respectively. The roots of such an equation, in the non-relativistic case, lead to the well known Rayleigh-Brillouin spectrum \[17\].

In order to determine accurate enough solutions for the fourth order dispersion relation given in Eq. \[18\], we follow the same ideas as in Ref. \[18\]. Since \( a_4 \) is a very small quantity compared to the other coefficients, we assume that the roots of the cubic equation obtained by neglecting \( a_4 \) are approximate roots of Eq. \[18\]. Using Mountain’s method \[19\], for the equation

\[
a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0
\]

(24)
yields a real root given by

\[
s_1 = -\frac{D_T}{\gamma} q^2
\]

(25)
and the two conjugate solutions

\[
s_{2,3} = -\left[ \frac{D_T}{5} + z \left( \frac{5}{6} c^2 \tau - D_T \right) \right] q^2 \pm i \sqrt{\frac{5 kT}{3 m}} q.
\]

(26)

In Eq. \[26\] use has been made of the fact that for an ideal gas, with \( z = \frac{kT}{mc^2} \ll 1 \), \( C_n = \frac{3}{2} k \) and \( \gamma = \frac{5}{3} \). Also, the thermal diffusivity \( D_T = \frac{\kappa}{n_0 C_n} \) has been introduced. These roots give rise to the usual behavior of density fluctuations, that is, a decaying mode with a characteristic time given by \( s_1 \), and two oscillating ones with frequencies given by the imaginary parts of \( s_2 \) and \( s_3 \). Notice that the only relativistic corrections to this behavior corresponds to the Stokes-Kirchoff coefficient, the term in parenthesis in Eq. \[26\]. Clearly, since the characteristic time \( \tau \) is of the order of the mean collision time \( \sim 10^{-6} s \) and typical thermal diffusivities are in the \((10^{-5}, 10^{-7}) m^2 s^{-1} \) range \[20\], the first term in the parenthesis dominates. This correction yields only a slight increase in the real part of the conjugate roots (width of the Brillouin peaks), which vanishes as \( z \to 0 \), in the non-relativistic limit. It is also important to point out the fact that the relativistic heat diffusion term, \( z D_T \), opposes this effect.

To approximate the fourth root, we use the information of the previous solutions by assuming the three known solutions of the cubic equation are still approximate roots of the complete equation. This assumption is plausible given the smallness of \( a_4 \). We also use the fact that the sum of all roots in an \( n \)-th order polynomial is equal to the ratio of the
coefficients of the \( n - 1 \) and the \( n \)-th power \((a_4/a_3)\), so that:

\[
s_4 = -\left(\tau - \frac{3}{2} D_T \alpha^2 \right)^{-1} \left[ 1 + \frac{q^2}{mn_0 c^4} \left( \frac{2 T_0 \kappa^2}{3 kn_0} - \frac{4}{3} \frac{T_0^3 \kappa^2}{c^2 mn_0} \right)
- \frac{2 c^4 m \kappa \tau}{3 k} + \frac{4}{3} c^2 T_0 \kappa \tau + \frac{5 k T_0^2 \kappa \tau}{3 m} + \frac{5}{3} \frac{c^4 kn_0 T_0^2 \kappa \tau}{m} \right].
\]  
(27)

Notice that, by taking \( \tau = 0 \), the fourth root is positive and thus corresponds to a growing mode. In Ref. [18], the existence of this positive large solution was used to question the validity of Eckart’s constitutive equation, in particular the coupling of heat with acceleration. On the other hand, for \( \tau \neq 0 \), one can approximate

\[
s_4 \sim -\frac{1}{\tau} + q^2 \left( D_T + \frac{5 k T_0}{3 m} \tau \right).
\]  
(28)

Now, Eq. (28) sheds more light on this issue since, by assuming \( \tau \gg \frac{T_0 \rho \tau}{c^4} \), the fourth root corresponds, instead of to a fast growing mode, to a fast decaying one that does not interfere with the formation of the spectrum and thus stabilizes the system. Moreover, Eq. (28) clearly shows that stabilization takes place as the relaxation term dominates over the exponential growth caused by Eckart’s coupling. In other words, the time derivative term in Maxwell-Cattaneo’s constitutive equations yields a modified time scale for the evolution of the fluctuations and, more importantly, changes the overall physical behavior because of the different signs of the terms in Eq. (28). The corresponding root found in Ref. [18] for Eckart’s formalism reads:

\[
s_4 \sim \frac{c^4 m n_0 \kappa T_0}{\kappa T_0}
\]  
(29)

which yields an exponential growth in the structure factor. On the other hand, the result in the current calculation

\[
s_4 \sim -\frac{1}{\tau}
\]  
(30)
corresponds to a finite spectrum which could eventually be measured although is very difficult to be observed [16].

It is important to mention at this point that a general mathematical formalism including relaxation times for both heat conduction and viscous effects has recently been performed in Ref. [21] comparing the behavior of eigenmodes within the moment method. However, the procedure here presented is focused on the particular damping mechanism of the instability triggered by the acceleration term in Eckart’s relation.
IV. THE KINETIC THEORY APPROACH: KNUDSEN’S PARAMETER ORDERING

The Maxwell-Cattaneo equation introduced in Sect. III as an evolution equation for the heat flux arises from Grad’s method of solution of the Boltzmann equation [20]. In such a procedure, the solution is written in terms of a combination of the chosen moments of the distribution function where the coefficients are given in terms of Hermite tensor polynomials. As recognized in Refs. [12] and [13], this method of solution may lead to inconsistencies in the equations for high order moments. These apparent contradictions arise primarily from the fact that there is no explicit ordering of the terms in the expansion and can be eliminated by introducing an ordering parameter, such as Knudsen’s parameter, and matching terms of equal order. On the other hand, a similar scheme is introduced right at the first step in the Chapman-Enskog’s method. The arguments in Refs. [12, 13], pertain the expression of the heat flux when calculated using two different moment definitions for it. In this section, we show that a more important issue arises when establishing the evolution equation for the heat flux.

In order to show in a transparent fashion the ordering problem regarding the time derivative term in Maxwell-Cattaneo equation, we start by considering the non-relativistic Boltzmann’s equation with a simplified, BGK type, collision operator.

\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f = -\frac{1}{\tau} (f - f^{(0)})
\]  
(31)

where \( f \) is the distribution function giving the number of particles per unit volume in a cell of phase space and \( \vec{v} \) is the molecular velocity, that is the velocity of a particle measured from an arbitrary reference frame. This approximation will enable us to obtain an evolution equation for the heat flux and analyze the relative order of the terms without introducing further complications. The details of the collisions are included in this model in the parameter \( \tau \) which is a relaxation time for the distribution function to attain the corresponding equilibrium distribution given by \( f^{(0)} \). Multiplying both sides of Eq. (31) and integrating over velocity space leads to a general transport equation given by

\[
\tau \frac{\partial n \langle \psi \rangle}{\partial t} + n \langle \psi \rangle + \tau \nabla \cdot n \langle \vec{v} \psi \rangle - \tau \left[ n \langle \frac{\partial \psi}{\partial t} \rangle + n \langle \vec{v} \cdot \nabla \psi \rangle \right] = \hat{\psi} f^{(0)} d^3v
\]  
(32)

where \( \psi \) is a general vector or tensor quantity which can depend on space, time and molecular or chaotic velocity. The angle brackets denote a distribution function weighted average given
by

\[ \langle \psi \rangle = \frac{1}{n} \int \psi f d^3v. \]  (33)

In order to obtain an evolution equation for the heat flux, one considers its definition, given by

\[ \vec{q} = \int \frac{m c^2}{2} \vec{c} f d^3v = n \langle \frac{m}{2} \vec{c}^2 \rangle \]  (34)

where \( \vec{c} = \vec{u} - \vec{v} \) is the chaotic or peculiar velocity, being \( \vec{u} \) the hydrodynamic velocity: \( \vec{u} = \langle \vec{v} \rangle \). Thus, with \( \psi = \frac{m}{2} \vec{c}^2 \) in Eq. (32) and considering for simplicity, a frame comoving with the fluid, i.e. \( \vec{u} = 0 \) one obtains

\[ \tau \frac{\partial \vec{q}}{\partial t} + \vec{q} + \tau \frac{m}{2} \nabla \cdot n \langle \vec{c} \vec{c} \rangle = 0 \]  (35)

where use has been made of the fact that the chaotic velocity, time and position vector are here independent variables. In order to analyze the relative magnitude of the terms in Eq. (35), we introduce the Chapman-Enskog assumption

\[ f = f^{(0)} (1 + \epsilon \Phi) \]  (36)

where \( \epsilon \) is Knudsen’s parameter quantifying the relative magnitude of microscopic and macroscopic scales and \( \Phi \) a function of the chaotic velocity and the state variables such that \( \epsilon \Phi \) is a correction to first order in their gradients. As mentioned above, this parameter is key for keeping track of the ordering of the terms in the transport equations both in Chapman-Enskog and Grad’s schemes [12, 13]. Also, we introduce the dimensionless variables

\[ f^{(0)} = \frac{1}{L^3 v^{(0)}_{th}} \hat{f}^{(0)} \]  (37)

\[ \vec{r} = L \hat{r}, \quad t = \frac{L}{v^{(0)}_{th}} t, \quad \vec{v} = v^{(0)}_{th} \hat{v} \]  (38)

where \( v_{th} \) is the thermal velocity and \( L \) a characteristic macroscopic scale for the system. Such procedure yields

\[ \tau \frac{v_{th}}{L} \frac{\partial \vec{q}}{\partial t} + \vec{q} + \tau \frac{v_{th}}{L} \nabla \cdot \int \frac{c^2}{2} \vec{c} \vec{c} f^{(0)} d^3v = 0. \]  (39)

Here use has been made of the fact that, since \( f^{(0)} \) and \( \Phi \) have even and odd parities in \( c \) respectively,

\[ \int \frac{c^2}{2} \vec{c} \vec{c} f^{(0)} d^3v = \int \frac{c^2}{2} \vec{c} \vec{c} f^{(0)} d^3v \]  (40)
and the order $\epsilon$ term vanishes. On the other hand, for the heat flux one has

$$\vec{q} = \epsilon \int \frac{mc^2}{2} \vec{c} f^{(0)} \Phi d^3v$$

(41)

and thus we can introduce a dimensionless flux $\hat{q}$ as follows

$$\vec{q} = \epsilon \frac{mv^3_{th}}{L^3} \int \frac{mc^2}{2} \vec{c} f^{(0)} \Phi d^3\hat{v} = \epsilon \frac{mv^3_{th}}{L^3} \hat{q}.$$  

(42)

Recalling that $\tau$ in the relaxation approximation corresponds to a microscopic time $\tau \sim \frac{\ell}{v_{th}}$, being $\ell$ a microscopic length, for instance a mean free path, one has that the ordering of the terms in Eq. (42) is a follows

$$\tau \frac{\partial \vec{q}}{\partial t} + \vec{q} + \tau \frac{m}{2} \nabla \cdot n \langle c^2 \vec{c} \vec{c} \rangle = 0.$$  

$$\downarrow \downarrow \downarrow$$

$$\sim \epsilon^2 \sim \epsilon \sim \epsilon$$

(43)

The time derivative term is thus of higher order than the rest of the terms in the equation, in particular the second one. Such a term belongs in a separate equation with other terms of second order in $\epsilon$, related to Burnett’s regime. One can thus clearly conclude that including a Maxwell-Cattaneo type equation in order to avoid the pathological behavior of fluctuations in the relativistic irreversible thermodynamics is inconsistent with the Navier-Stokes regime, and such inconsistency is directly inherited from the lack of an ordering parameter in the method of solution of Boltzmann’s equation.

V. FINAL REMARKS

It is widely accepted to this point that the kinetic theory of gases serves, additional to a methodology to estimate transport coefficients, a solid formalism to sustain the phenomenology in both the relativistic and non-relativistic scenarios in the case of diluted, non-degenerate gases. Even though this microscopic theory yields a relativistic heat flux proportional to the gradients of state variables, most works in this field favor a term proportional to the hydrodynamic acceleration which can only be sustained from a phenomenological point of view. Moreover, the so-called generic instabilities are exclusively triggered by the presence of the acceleration in the corresponding constitutive equation for the heat flux. This suggested the addition of a relaxation term, within a Maxwell-Cattaneo type extended formalism, in order to suppress the generic instabilities.
In this work we accomplished two important tasks. Firstly, in Section III we analyzed the specific mechanism by means of which the stabilization of the system of relativistic hydrodynamic occurs. A dispersion relation for such system was obtained using a Maxwell-Cattaneo heat evolution equation. Such procedure produced a fourth order equation for the Fourier variable $s$ for a given wavenumber $q$. This result appears to be similar to the one obtained in Ref. [18] but, as was emphatically pointed out, the overall behavior of the system changes radically. This can be clearly seen by comparing the approximate roots in Eqs. (29) and (30). The former yields an exponential growth in the corresponding structure factor which destroys, theoretically, the spectrum. This fact is clearly unacceptable. However, the latter predicts a decay in time of the fluctuations with a characteristic time determined by $\tau$, the relaxation parameter introduced by Cattaneo. Thus, the relaxation term introduces a term that, when large enough, overrides the one leading to the instability.

Secondly, based on the fact that not including an explicit ordering parameter in the solution method for Boltzmann’s equation that involve considering perturbations around the equilibrium distribution function, we have used Chapman-Enskog hypothesis in order to analyze the relative magnitude of the time derivative term in the evolution equation for the heat flux and the heat flux itself. We found that the former is an order higher than the latter in the Knudsen parameter and thus shall be considered in the next order equation for consistency. This term is precisely the key in the stabilization mechanism and without its consideration the acceleration term leads to the exponential growth of fluctuations. From the previous argument, one can thus clearly conclude that including a Maxwell-Cattaneo type equation in order to avoid the pathological behavior of fluctuations in the relativistic irreversible thermodynamics is inconsistent, and such inconsistency is directly inherited from the lack of an ordering parameter in the method of solution of Boltzmann’s equation. In view of these facts, the authors consider that the basic idea of describing relativistic fluids through a first order in the gradients theory obtained from microscopic grounds remains sound and attractive.

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