Microcanonical functional integral and entropy for eternal black holes

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Abstract

The microcanonical functional integral for an eternal black hole system is considered. This requires computing the microcanonical action for a spatially bounded spacetime region when its two disconnected timelike boundary surfaces are located in different wedges of the Kruskal diagram. The path integral is a sum over Lorentzian geometries and is evaluated semiclassically when its boundary data are chosen such that the system is approximated by any Lorentzian, stationary eternal black hole. This approach opens the possibility of including explicitly the internal degrees of freedom of a physical black hole in path integral descriptions of its thermodynamical properties. If the functional integral is interpreted as the density of states of the system, the corresponding entropy equals \( S = A_H/4 - A_H/4 = 0 \) in the semiclassical approximation, where \( A_H \) is the area of the black hole horizon. The functional integral reflects the properties of a pure state. The description of the black hole density of states in terms of the eternal black hole functional integral is also discussed.

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I. INTRODUCTION

Despite considerable progress in the path integral description of gravitational systems [1–6], the statistical mechanical origin of black hole entropy remains unclear in this approach. The dynamical origin of entropy has been recently studied with the help of different methods (see, for example, Refs. [7–9]). Given these developments, it would be interesting to include explicitly the internal degrees of freedom of a black hole in the functional integral and study their contribution to black hole entropy. An attempt in this direction, which we pursue in this paper, consists in investigating the microcanonical functional integral when applied to an eternal black hole statistical system which effectively contains information about the internal degrees of freedom of a physical black hole.

A proposal for the density of states of a gravitational system obtained as the trace of a microcanonical density matrix has been suggested recently in Refs. [1,2]. The density of quantum states for a self-gravitating system spatially bounded by a timelike three-dimensional surface \(B\) is given by the functional integral

\[
\nu[\varepsilon, j, \sigma] = \sum_{M} \int D \mathcal{H} \exp(iS_m/\hbar). \tag{1.1}
\]

The phase of the functional integral is proportional to the so-called microcanonical action \(S_m\) which describes the dynamics of a gravitational system whose surface energy density \(\varepsilon\), surface momentum density \(j_a\), and size (specified by the two-dimensional metric \(\sigma_{ab}\)) are fixed at the spatial boundary. The quantities \(\varepsilon\), \(j_a\), and \(\sigma_{ab}\) are constructed from the dynamical phase space variables that include the three-metric \(h_{ij}\) of an initial spacelike hypersurface \(\Sigma\) and its conjugate momentum \(P^{ij}\). The density of states is defined as a formal integral over Lorentzian metrics that satisfy the boundary conditions and is a functional of the quantities \(\varepsilon\), \(j_a\), and \(\sigma_{ab}\). The sum over \(M\) in (1.1) refers to a sum over manifolds of different topologies which are periodic in the time-like direction and whose three-dimensional boundary has topology \(S \times S^1\), where the two-dimensional surface \(S\) is the intersection of the boundary \(B\) and the slice \(\Sigma\). The symbol \(D \mathcal{H}\) in (1.1) denotes a formal measure in the
space of these manifolds. The black hole density of states $\nu_\ast$ is obtained from the functional integral (1.1) when the latter is approximated semiclassically by using a complex metric whose boundary data at its single boundary surface coincide with the boundary data of a Lorentzian, stationary, axisymmetric black hole. The density of states defined accordingly equals the exponential of one fourth of the area of the black hole horizon.

The proposal (1.1) opens the possibility of determining the thermodynamical properties of black hole systems starting from a sum over real Lorentzian geometries. However, several problems remain in this approach. First, a spacelike hypersurface $\Sigma$ that describes the initial data of a Lorentzian black hole has to cross necessarily the event horizon and eventually intersect the interior singularity. This implies that additional information has to be provided on $\Sigma$ in order to describe the properties of the singularity [10]. Second, the microcanonical functional integral and action used in [1] to calculate the black hole entropy are appropriate when the spacetime has a single timelike boundary surface. However, as already noted in [1], a Lorentzian, stationary, axisymmetric black hole is not a extremum of this action since it cannot be placed on a manifold with a single timelike boundary. In particular, this implies that the black hole density of states $\nu_\ast$ whose boundary data correspond to the boundary data of a Lorentzian, stationary, axisymmetric black hole cannot be approximated semiclassically by using the same Lorentzian metric that motivates its boundary conditions.

These difficulties do not prevent the evaluation of the black hole density of states in the semiclassical approximation [1]. As already mentioned, there exists a related complex metric which satisfies the boundary data and which can be used to calculate the Lorentzian functional integral in a steepest descents approximation by distorting its contours of integration [6,1]. This approximation yields the correct result for the black hole entropy but conceals its origin. As with other complexification schemes previously used in calculations of black hole partition functions [11–13], the interior of the Lorentzian black hole literally disappears by virtue of this procedure, leaving effectively only a periodically identified Euclidean version of the “right” wedge region of a Kruskal diagram. The properties of the black hole interior become encoded in a set of conditions at the so-called “bolt” of the complex geometry.
In this approach, as in other formulations of gravitational thermodynamics in terms of path integrals, the statistical origin of entropy and its relationship to the internal degrees of freedom of a black hole remain obscure.

We believe that the problems mentioned above and the role of internal degrees of freedom in functional integral descriptions of black hole thermodynamics can be addressed by explicitly considering the eternal version of a black hole. The description of states of a physical black hole formed from gravitational collapse in terms of the states of its eternal version has been proposed in Ref. [8]. The late time geometry of a physical black hole can be analytically continued into the spacetime of an eternal black hole if the latter configuration possesses the same macroscopic parameters as the former one. The excitations of the physical black hole can be associated with the deformations of an initial global Cauchy surface $\Sigma$ of the eternal black hole plus initial data for the non-gravitational fields defined on such a distorted surface [8,14]. In general, the spatial slices $\Sigma$ that foliate an eternal black hole are (deformed) Einstein-Rosen bridges with wormhole topology $R^1 \times S^2$. The spacetime is composed of two wedges $M_+$ and $M_-$ located in the right ($R_+$) and left ($R_-$) sectors of a Kruskal diagram [10]. Internal and external degrees of freedom of the black hole can be easily identified in this approach since the hypersurfaces $\Sigma$ are naturally divided in two parts $\Sigma_+$ and $\Sigma_-$ by a bifurcation two-surface $S_0$. While the “external” degrees of freedom of the original black hole are naturally given by the initial data at $\Sigma_+$, its “internal” degrees of freedom can be identified with initial data defined at $\Sigma_-$. The importance of finite size systems in gravitational thermodynamics has been stressed repeatedly [14]. Finite spacetime regions are required in thermodynamical applications since a gravitational system in thermal equilibrium with a radiation bath is not described by an asymptotically flat spacetime. In particular, rotating black holes can be in thermal equilibrium only if contained inside a spatially finite boundary [12,14]. Other advantages of bounded systems include the possibility of describing thermally or mechanically stable configurations under gravitational collapse. However, a single three-dimensional boundary does not confine a finite spacetime region of an eternal black hole. In order to describe black
hole thermodynamics starting from eternal black hole systems, it is necessary to consider two
three-dimensional timelike boundary surfaces \( B_+ \) and \( B_- \) located in the right \( M_+ \) and left
\( M_- \) regions of the spacetime. This has been noted in Refs. [10,17], where the Hamiltonian,
quasilocal energy, and angular momentum for a finite region of a (distorted) eternal black
hole have been constructed from the gravitational action. In particular, the Hamiltonian
for an eternal black hole is of the general form \( H = H_+ - H_- \), where \( H_+ \) and \( H_- \) are the
Hamiltonian functions for the two separate wedges \( M_+ \) and \( M_- \).

The aim of this paper is to generalize the microcanonical functional integral (1.1) to
quantum self-gravitating systems that include spacetimes whose topology and boundary
conditions coincide with the ones of (either distorted or Kerr-Newman) eternal black holes. This naturally requires the construction of the microcanonical action (appropriate for fixed
energy systems) when the two boundaries \( B_+ \) and \( B_- \) are located in the regions \( M_+ \) and
\( M_- \) of an eternal black hole geometry. The evaluation of the functional integral as well as
its thermodynamical consequences are discussed. It turns out that if the microcanonical
sum over geometries for an eternal black hole system is interpreted as its density of states,
the total entropy of the system equals zero in the semiclassical approximation. This result
applies to the gravitational field itself of any type of eternal black holes (not only of the
Kerr-Newman form) for which the geometry is regular at the bifurcation surface. Since in
a microcanonical description it seems natural to relate the external and internal degrees of
freedom of a black hole with the boundary data at the surfaces \( B_+ \) and \( B_- \) respectively [10],
we believe that the microcanonical functional integral for an eternal black hole system opens
the possibility of extending the path integral formulation of gravitational thermodynamics
to situations when internal degrees of freedom are present and allows the formulation of
black hole thermodynamics in terms of a single pure state.

The paper is organized as follows. We review in Section II the relevant kinematical prop-
erties of a finite spacetime region generated by the so-called “tilted foliation” introduced in
Ref. [10] and compute its microcanonical action. The results are applied to the particular
case of a (distorted) Lorentzian eternal black hole. The microcanonical sum over geometries
for a quantum gravitational system whose boundary conditions equal the boundary conditions of a physical Lorentzian, stationary, axisymmetric eternal black hole is presented in Section III. The path integral is evaluated semiclassically by using the Lorentzian eternal black hole metric that motivates its boundary conditions as well as a complex saddle point of the microcanonical action. The latter approximation allows one to understand the relationship between the functional integral for eternal black holes and the black hole density of states computed in Ref. [1]. We conclude in Section IV with general remarks concerning the construction of the density of states for the “exterior” region $M_+$ in terms of the functional integral for the complete spacetime, and the relevance of the results in a thermofield dynamics interpretation of black hole thermodynamics.

II. MICROCANONICAL ACTION

Consider a spacelike hypersurface $\Sigma$ with Einstein-Rosen bridge topology $R^1 \times S^2$ whose intrinsic geometry and time derivatives are chosen to satisfy the gravitational constraint equations. The evolution of these data is presumed to define a regular spacetime region to the future and past of the slice $\Sigma$ [10]. We assume that there exist two different spacelike hypersurfaces $\Sigma'$ and $\Sigma''$ which intersect each other at a two-dimensional, topologically spherical spacelike surface $S_0$. The “bifurcation” surface $S_0$ divides the slice $\Sigma$ in two parts denoted by $\Sigma_+$ and $\Sigma_-$. The sequence of slices (generically denoted by the symbol $\Sigma$ in what follows) which intersect at the same bifurcation surface $S_0$ is called a “tilted foliation” [10]. The spacetime region $M$ lying between the two spacelike Cauchy surfaces $\Sigma'$ and $\Sigma''$ consists therefore of two regions $M_+$ and $M_-$ (foliated by $\Sigma_+$ and $\Sigma_-$ respectively) that join at $S_0$. The region $M$ we consider is bounded not only by the slices $\Sigma'$ and $\Sigma''$ but also by a three-dimensional timelike boundary $B$ that consists of two disconnected parts $B_+$ and $B_-$. For a general eternal black hole geometry the boundaries $B_+$ and $B_-$ are located in $M_+$ and $M_-$ respectively. The intersections of the boundaries $B_+$ and $B_-$ with $\Sigma$ are topologically spherical two-dimensional surfaces denoted by $S_+$ and $S_-$ respectively. The topology of the
slices Σ is therefore $I \times S^2$ (the interval $I$ referring to a finite spatial distance), while the topology of the boundary surfaces $B_\pm$ is $I \times S_\pm$ (the interval $I$ referring to a finite time-like distance).

The line element of $M$ is of the general form

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt),$$

(2.1)

where $N$ is the corresponding lapse function and the spacelike surfaces $\Sigma$ are chosen to coincide with surfaces of constant values of $t$, so that the time coordinate $t$ is the scalar function that labels the foliation. In particular, $\Sigma' = \Sigma_{\nu}$ and $\Sigma'' = \Sigma_{\nu''}$. The four-velocity vector $u^\mu$ is the timelike unit vector normal to the slices $\Sigma$ and is defined by $u_\mu = -N \partial_\mu t$.

Following [19], greek indices are used for tensors in $M$ while latin indices are used for tensors defined in either $\Sigma$ of $B_\pm$. The lapse function $N$ is defined so that $u \cdot u = -1$. The vector $t^\mu$ that connects points with the same spatial coordinates is

$$t^\mu = Nu^\mu + V^\mu,$$

(2.2)

so that $V^i = h_0^i = -Nu^i$ is the shift vector. For the “tilted foliations” considered here the slices corresponding to different values of the parameter $t$ join at the bifurcation surface where the lapse function $N$ vanishes. The vector $u^\mu$ is chosen to be future oriented in $M$ and the lapse $N$ is positive at $\Sigma_+$ and negative at $\Sigma_-$. The spacelike normal $n^\mu$ to the three-dimensional boundaries $B_\pm$ is defined to be outward pointing at $B_+$, inward pointing at $B_-$, and normalized so that $n \cdot n = +1$. We shall assume that the foliation is further restricted by the conditions $(u \cdot n)|_{B_\pm} = 0$ [10].

As argued in Ref. [10], it is convenient to define a set of “standard” coordinates $(t, x^i)$ for the “tilted” foliation. These coordinates are in a one-to-one correspondence with the “standard” coordinates $(t, y, \theta, \phi)$ of a “tilted” foliation in a Schwarzschild-Kruskal space-time. The spatial coordinates $x^i$ have the same space orientation in both $R_+$ and $R_-$, but the time coordinate $t$ has opposite orientations in $R_+$ and $R_-.$

The metric and extrinsic curvature of $\Sigma$ as a surface embedded in $M$ are denoted by $h_{ij}$ and $K_{ij} = -h_i^k \nabla_k u_j$ respectively, while the metric and extrinsic curvature of the boundaries
$B_\pm$ as surfaces embedded in $M$ are $\gamma_{ij}$ and $\Theta_{ij} = -\gamma_i^k \nabla_k n_j$ [17,18]. Covariant differentiation with respect to the metrics $g_{\mu\nu}$ and $h_{ij}$ is denoted by $\nabla$ and $D$ respectively. The induced metric and extrinsic curvature of the boundaries $S_\pm$ as surfaces embedded on $\Sigma$ are $\sigma_{ab}$ and $k_{ab} = -\sigma_a^k D_k n_b$ respectively ($a, b = 2, 3$). The normal vector $n^\mu$ to $B_\pm$ is also the normal vector to $S_\pm$. The extrinsic curvature tensors for the different surfaces are defined so that

$$\Theta^\mu_\nu = k^\mu_\nu + u^\mu u^\nu n_\alpha a^\alpha + 2\sigma^{\alpha(\mu} u_{\nu)} n^\beta K_{\alpha\beta} ,$$

while the traces $\Theta$ and $k$ of the tensors $\Theta^\mu_\nu$ and $k^\mu_\nu$ obey the relation

$$\Theta = k - n_\beta a^\beta ,$$

where the acceleration $a^\mu$ of the timelike unit normal $u^\mu$ to the hypersurfaces $\Sigma$ is $a^\mu = u^\alpha \nabla_\alpha u^\mu = (D^\mu N)/N$. Finally, the determinants of the metric tensors are related by

$$\sqrt{-g} = |N|\sqrt{h} ,$$

$$\sqrt{-\gamma} = |N|\sqrt{\sigma} .$$

As an illustration of a “tilted” foliation, consider the simple case of a static, spherically symmetric eternal black hole whose line element is [10]

$$ds^2 = -N^2(y)dt^2 + dy^2 + r^2(y)d\Omega^2 .$$

The set $(t, y, \theta, \phi)$ has the same spatial orientation but differing time orientation in $R_+$ and $R_-$. The coordinate $y$ represents the proper geodesic distance from the “throat” of the Einstein-Rosen bridge at $S_0$. The Hamiltonian constraint equation implies that

$$dy = \pm \frac{dr}{\sqrt{1 - r_+/r}}$$

in $M_\pm$. It is convenient to choose $y$ positive in $\Sigma_+$, negative in $\Sigma_-$, and zero at $S_0$ [so that $r(y = 0) \equiv r_+$]. The solution is regular at the surface $S_0$. The behavior of the gradient $r_y$ exemplifies an important property of eternal black holes: the area of two-dimensional
surfaces $S_+ (S_-)$ in $\Sigma_+ (\Sigma_-)$ increases (decreases) as the proper coordinate $y$ increases. The lapse function in the Schwarzschild-Kruskal spacetime is $N = \pm (1 - r_+ / r)^{1/2}$ at $\Sigma_\pm$. Observe that the gradient $D_i N = N_g \delta_i^y = r_+ / 2 r^2 \delta_i^y$, so that $n^i D_i N = r_+ / 2 r^2$ for both regions $M_+$ and $M_-$. 

We turn now to consider the microcanonical action $S_m$ for a "tilted" foliation. The action $S_m$ is the action appropriate to a variational principle in which the fixed boundary conditions at the timelike boundaries $B_+$ and $B_-$ are not the spacetime three-geometry (that is, the metric components $N, V^i$, and $\sigma_{ab}$) but the surface energy density $\varepsilon$, surface momentum density $j_a$, and boundary metric $\sigma_{ab}$ [6,19]. The action $S_m$ has been constructed for spacetimes with a single timelike boundary in Refs. [1,20] by adding the appropriate boundary terms to the ordinary gravitational action. The surface energy density $\varepsilon$ and momentum density $j_a$ for a slice $\Sigma = \Sigma_+ \cup \Sigma_-$ of an eternal black hole spacetime has been calculated in [10] when the two-dimensional boundary surfaces $S_+$ and $S_-$ are located in either (1) the same space (either $\Sigma_+$ or $\Sigma_-$), or (2) the two separate spaces $\Sigma_+$ and $\Sigma_-$ respectively. The energy density $\varepsilon$ is the value (per unit boundary area) of the Hamiltonian that generates unit time translations orthogonal to the boundaries $S_+$ and $S_-$. At each one of these surfaces the energy and momentum densities are defined by 

$$\varepsilon = (k / \kappa) , \quad j_i = -2 \sigma_{ij} n_k P^{jk} / \sqrt{h} ,$$

(2.8)

where contributions due to functionals of the three-metrics at $B_+$ or $B_-$ have been neglected. The signs of the extrinsic curvatures $k$ of the surfaces $S_+$ and $S_-$ depend on the location of these surfaces for a chosen orientation of the normal $n^\mu$. The quantities $\varepsilon$ and $j_i$, as well as their associated integrated quantities, namely, the quasilocal energy $E_\pm$ and angular momentum $J_\pm$ for an eternal black hole, have been discussed in [10].

The covariant form of the microcanonical action for a general spacetime $M$ generated by a “tilted” foliation and whose respective three-dimensional timelike surfaces $B_+$ and $B_-$
are located in $M_+$ and $M_-$ can be written as

$$
S_m = \frac{1}{2\kappa} \int_{M_+} d^4x \sqrt{-g} \mathcal{R} + \frac{1}{\kappa} \int_{t''} d^3x \sqrt{h} K - \frac{1}{\kappa} \int_{B_+} d^3x \sqrt{-\gamma} t_\mu \Theta^{\mu\nu} \partial_\nu t
- \frac{1}{2\kappa} \int_{M_-} d^4x \sqrt{-g} \mathcal{R} + \frac{1}{\kappa} \int_{t''} d^3x \sqrt{h} K - \frac{1}{\kappa} \int_{B_-} d^3x \sqrt{-\gamma} t_\mu \Theta^{\mu\nu} \partial_\nu t,
$$

(2.9)

where $\mathcal{R}$ denotes the four-dimensional scalar curvature, and $\kappa \equiv 8\pi$. (We follow the conventions of Ref. [21] and units are chosen so that $G = \hbar = c = 1$.) The notation $\int_{(\pm)\nu}$ represents an integral over the three-boundary $\Sigma_{\pm}$ at $t''$ minus an integral over the three-boundary $\Sigma_{\pm}$ at $t'$. The integrations are taken over coordinates $x^\mu$ which possess the same orientation as the “standard” coordinates $(t, x^i)$ of the “tilted” foliation. The differing signs in the integrations over $M_+$ and $M_-$ reflect the fact that the coordinates have different time orientations in $M_+$ and $M_-$. The action (2.9) is independent of functionals of the three-metric at the timelike boundaries $B_+$ and $B_-$ (“subtraction terms”), and reduces to the microcanonical action introduced in Ref. [1] when the spacetime region is bounded by a single timelike surface $B_+$.

The Hamiltonian form of the microcanonical action is easily obtained under a $3 + 1$ spacetime split by recognizing that there exists a direction of time at the boundaries $B_+$ and $B_-$ inherited by the time vector $t^\mu$. The four-dimensional scalar curvature is

$$
\mathcal{R} = R + K^{\mu\nu} K_{\mu\nu} - (K)^2 - 2\nabla_\mu (K u^\mu + a^\mu),
$$

(2.10)

where $R$ is the curvature scalar on $\Sigma$. By using Gauss’ theorem and the conditions [11]

$$
u \cdot n|_{B_\pm} = 0, u \cdot a = 0, u \cdot u = -1, n \cdot n = 1,
$$

(2.11)

as well as Eqns. (2.2) and (2.3), the action (2.9) can be written as

$$
S_m = \frac{1}{2\kappa} \int_{M_+} d^4x \sqrt{-g} [R + K_{\mu\nu} K^{\mu\nu} - (K)^2] + \frac{1}{2\kappa} \int_{B_+} d^3x \sqrt{\sigma} n_i V_j (K h^{ij} - K^{ij})
- \frac{1}{2\kappa} \int_{M_-} d^4x \sqrt{-g} [R + K_{\mu\nu} K^{\mu\nu} - (K)^2] - \frac{1}{2\kappa} \int_{B_-} d^3x \sqrt{\sigma} n_i V_j (K h^{ij} - K^{ij}).
$$

(2.12)

In the most general case, there would be contributions to the action (2.12) associated with the “corners” $B''_\pm = \Sigma'' \cap B_\pm$ and $B'_\pm = \Sigma' \cap B_\pm$, as well as with the cusp-like part $S_0$ of the
These contributions are related to the angles between the unit normal $u^\mu$ of $\Sigma$ and the spacelike normal $n^\mu$. For simplicity, we consider here only the case when $u \cdot n = 0$ at the boundaries $B_{\pm}$. For a “tilted” foliation the contributions at $S_0$ connected with the region $M_+$ and $M_-$ have opposite signs and cancel identically due to the regularity of the geometry at the bifurcation surface $S_0$ [10], and no extra contributions appear in (2.12).

The momentum $P^{ij}$ conjugate to the three-metric $h_{ij}$ of $\Sigma$ for the “tilted” foliation can be defined as [10]

$$P^{ij} = \frac{1}{2\kappa} \sqrt{h} \left( K h^{ij} - K^{ij} \right).$$  (2.13)

Since the sphere $S_0$ consists of points which remain fixed under the change of the parameter $t$, the time derivative of the three-metric must vanish at $S_0$. The behaviour of the canonical variables in the vicinity of the fixed sphere $S_0$ has been discussed in Ref. [24]. Upon integration of the kinetic part of the volume integrals in (2.12) the action becomes

$$S_m = \int_M d^4x \left[ P^{ij} \dot{h}_{ij} - N \mathcal{H} - V^i \mathcal{H}_i \right],$$  (2.14)

where the dot denotes differentiation with respect to the global time $t$ and the gravitational contribution to the Hamiltonian and momentum constraints are given by the usual expressions

$$\mathcal{H} = (2\kappa)G_{ijk\ell} P^{ij} P^{k\ell} - \sqrt{h} \ R/(2\kappa),$$

$$\mathcal{H}_i = -2D_j P_i^j,$$  (2.15)

with $G_{ijk\ell} = (h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell})/(2\sqrt{h})$.

The microcanonical action (2.14) applies to any smooth Lorentzian geometry generated by a “tilted” foliation when $B_+$ and $B_-$ are located in the regions $M_+$ and $M_-$. It has the same form as the ordinary canonical action with no explicit boundary terms. In particular, the action (2.14) vanishes identically for stationary solutions of the vacuum Einstein equations describing stationary eternal black holes (with no extra assumptions required about their symmetry). In this case $\dot{h}_{ij} = 0$, the constraint equations are satisfied, and no boundary terms remain in the action. This situation may of course be different in the presence of
matter fields. For example, matter distributions at the horizon could alter the regularity of the geometry there and give extra contributions to the action.

The ordinary gravitational action $S$ for the “tilted” foliation can be constructed from the microcanonical action (2.14) by adding boundary terms that change the boundary conditions from fixed surface energy density $\varepsilon$, surface momentum density $j_a$ and boundary metric $\sigma_{ab}$ at $B_\pm$ to fixed metric components $N$, $V^i$, and $\sigma_{ab}$ at $B_\pm$. Two of these boundary terms are needed. The action $S$ is

$$S = S_m - \int_{B_+} d^3x \sqrt{\sigma} [N\varepsilon - V^i j_i] + \int_{B_-} d^3x \sqrt{\sigma} [N\varepsilon - V^i j_i]$$

$$= \int_M d^4x [P^{ij} h_{ij} - NH - V^i H_i] - \int_{B_+} d^3x \sqrt{\sigma} [N\varepsilon - V^i j_i] + \int_{B_-} d^3x \sqrt{\sigma} [N\varepsilon - V^i j_i].$$

(2.16)

This form for the action $S$ and its consequences in the description of eternal black holes have been discussed in Ref. [10].

Consider finally, as an illustration, the microcanonical action for a spacetime region generated by the standard “untilted” foliation when both timelike boundaries $B_+$ and $B_-$ are located in the “right” wedge $M_+$ of an eternal black hole. The foliation is regular everywhere in the region between the initial $\Sigma'$ and final $\Sigma''$ slices. The global time parameter $t$ labels the foliation and the four-velocity vector is $u^\mu = -N \delta^\mu_t$, with the lapse function being positive everywhere in $M_+$. In this case the microcanonical action is

$$S_m = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g} R + \frac{1}{\kappa} \int_{\Sigma'} d^3x \sqrt{h} K - \frac{1}{\kappa} \int_{B_+} d^3x \sqrt{-g} \left( t^\mu \Theta_{\mu\nu} \partial_\nu t \right)$$

$$+ \frac{1}{\kappa} \int_{B_-} d^3x \sqrt{-g} \left( t^\mu \Theta_{\mu\nu} \partial_\nu t \right).$$

(2.17)

It is easy to show that the Hamiltonian version of this action is also given by Eqn. (2.14). The difference between the microcanonical action for “tilted” and “untilted” foliations manifests itself in their boundary data. (For instance, since the sign of the surface energy density $\varepsilon_-$ is connected with the sign of extrinsic curvature of the surface $S_-$ for a chosen orientation of the normal $n^\mu$, the sign of $\varepsilon_-$ when $S_-$ is located in $M_+$ for the “untilted” foliation is opposite to the sign of $\varepsilon_-$ when $S_-$ is located in $M_-$ for the “tilted” foliation.) The Hamiltonian form
of the microcanonical action for “untitled” foliations has been used in Refs. [23] when the two three-dimensional boundaries of the spacetime are located in the single complex sector of an ordinary black hole and the internal boundary approaches the black hole horizon. We would like to emphasize that, even if the microcanonical actions for “tilted” and “untitled” foliations reduce to similar Hamiltonian forms, the former applies to spacetimes whose two regions intersect at a fixed surface $S_0$. The action $(2.14)$ is the necessary action to describe the dynamics of finite regions of a distorted eternal black hole and will play an important role in the sum over geometries for eternal black hole systems presented below.

III. FUNCTIONAL INTEGRAL

We consider in this section a microcanonical functional integral for a physical system whose boundary conditions correspond to the ones of an eternal version of a black hole. Consider first the functional integral for a microcanonical gravitational system for which two timelike boundary surfaces $B_+^+$ and $B_-$ are needed in order to contain a finite spacetime region. The functional integral takes the form

$$\tilde{\nu}[\varepsilon^+, j^+; \varepsilon^-, j^-] = \sum_M \int DH \exp(iS_m),$$

and is a functional of the energy density $\varepsilon$, momentum density $j_a$, and two-metric $\sigma_{ab}$ at the boundaries $B_+$ and $B_-$. For simplicity the notation $j_{\pm}$ indicates that the quantity $j_a$ is specified at the surface $B_{\pm}$. The sum over $M$ refers to a sum over manifolds of different topologies whose boundaries have topologies $B_+ = S_+ \times S^1 = S^2 \times S^1$ and $B_- = S_- \times S^1 = S^2 \times S^1$. The element $S^1$ is due to the periodic identification in the global time direction at the boundaries when the initial and final hypersurfaces are identified. The integral is a sum over periodic Lorentzian metrics that satisfy the boundary conditions at $B_+$ and $B_-$. The action appearing in $(3.1)$ is the microcanonical action $S_m$ discussed in Section II, but with the boundary terms corresponding to $\Sigma'$ and $\Sigma''$ dropped because the manifolds summed over possess only two boundary elements, namely, $B_+$ and $B_-$. 

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As with the density of states (1.1) [2], the functional integral (3.1) can be considered as the result of tracing over initial and final configurations in a microcanonical density matrix $\rho_m$ of the form:

$$\tilde{\nu}[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] = \int \mathcal{D}h \rho_m[h, h; \alpha''_\pm, \alpha'_\pm; \varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] ,$$  \hspace{1cm} (3.2)

where the angles $\alpha''_\pm$ and $\alpha'_\pm$ at the corners $B''_\pm$ and $B'_\pm$ are required to satisfy the condition $\alpha''_\pm + \alpha'_\pm = \pi$ to guarantee the smoothness of the boundaries $B_+$ and $B_-.$

Consider now the functional integral (3.1) in the case when the boundary surfaces $B_+$ and $B_-$ are located in separate regions $M_+$ and $M_-$ and the fixed boundary data $(\varepsilon_+, j_+, \sigma_+)$ and $(\varepsilon_-, j_-, \sigma_-)$ correspond to the boundary data of a general Lorentzian, stationary, axisymmetric eternal black hole. This spacetime is a solution of Einstein equations whose line element is of the form (2.1):

$$ds^2 = -\tilde{N}^2 dt^2 + \tilde{h}_{ij}(dx^i + \tilde{V}^i dt)(dx^j + \tilde{V}^j dt) ,$$  \hspace{1cm} (3.3)

where the lapse $\tilde{N}$, shift vector $\tilde{V}^i$, and three-metric $\tilde{h}_{ij}$ are particular functions of the spatial coordinates $x^i (i = 1, 2, 3).$ For convenience, the spatial coordinates can be chosen to be co-rotating with the horizon [24,3], so that $\tilde{V}^i / \tilde{N} = 0$ at the horizon. In this spacetime the spacelike slices $\Sigma$ are constant stationary time surfaces that contain the closed orbits of the axial Killing vector field. The two-dimensional boundaries $S_+$ and $S_-$ of $\Sigma$ also contain the orbits of the axial Killing field. The boundary data $(\varepsilon_+, j_+, \sigma_+)$ and $(\varepsilon_-, j_-, \sigma_-)$ of this solution can be determined at $S_+$ and $S_-$ for each slice $\Sigma.$ By virtue of the gravitational constraint equations, these data determine uniquely the size of the black hole horizon [20] and are such that the two-metric $\tilde{\sigma}_{ab}$ is continuous at this horizon. We will assume that both boundaries $S_+$ and $S_-$ of the rotating solution used to generate the boundary data are not located beyond the speed-of-light surfaces surrounding the black hole [3,10]. The eternal black hole functional integral $\tilde{\nu}_*$ is given by expression (3.1) when the boundary data at $B_+$ and $B_-$ of the geometries summed over coincide with the data of the classical Lorentzian eternal black hole. The topology of each one of these spacetimes is arbitrary but each boundary $B_\pm$ is required to have the boundary topology $S_\pm \times S^1.$
We evaluate now the functional integral in the semiclassical approximation. This requires finding a four-metric that extremizes the action $S_m$ and satisfies the boundary conditions $(\varepsilon_+, j_+, \sigma_+)$ at $S_+$ and $(\varepsilon_-, j_-, \sigma_-)$ at $S_-$. Fortunately, the Lorentzian eternal black hole metric (3.3) can be periodically identified in the global time direction and placed on a manifold whose two spatial boundaries have the desired topologies $S_\pm \times S^1$. The periodic identification alters neither the constraint equations nor the boundary data and the resulting metric can be used to approximate the path integral. As observed in [1], if the physical system can be approximated by a single classical configuration, this configuration will be the real spacetime (3.3) that induced the boundary data. In the semiclassical approximation the functional integral $\tilde{\nu}_*$ becomes

$$\tilde{\nu}_*[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp \left(i S_m[\tilde{N}, \tilde{V}, \tilde{h}]\right), \quad (3.4)$$

where the action $S_m[\tilde{N}, \tilde{V}, \tilde{h}]$ is the microcanonical action evaluated at the periodic manifold (3.3).

The action $S_m[\tilde{N}, \tilde{V}, \tilde{h}]$ is obtained from (2.9) by dropping the integrals at $t'$ and $t''$, and its Hamiltonian form is given by Eqn. (2.14). This action vanishes identically: the volume term equals zero because $P^{ij}\dot{h}_{ij}$ is zero by stationarity and the gravitational constraints are satisfied. The functional integral is therefore

$$\tilde{\nu}_*[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp (0) = 1 \quad (3.5)$$

in the semiclassical approximation.

It is illustrative to consider now a complex four-metric which also extremizes the microcanonical action for eternal black hole boundary conditions and which can be used to reevaluate the path integral (3.1) in a steepest descent approximation. This alternative approximation of the quantity $\tilde{\nu}_*$ is useful in understanding the relationship of the result (3.5) with the density of states for an ordinary (that is, non-eternal) black hole computed in Ref. [1]. The complex metric can be obtained from the Lorentzian eternal black hole metric (3.3) by replacing the stationary time $t$ with imaginary time, namely, $t \rightarrow -it$, with $t$ real. Its line element is
\[ ds^2 = -(-i\tilde{N})^2 dt^2 + \tilde{h}_{ij}(dx^i - i\tilde{V}^i dt)(dx^j - i\tilde{V}^j dt), \] (3.6)

with \( \tilde{N}, \tilde{V}^i, \) and \( \tilde{h}_{ij} \) real. The complex metric has \( -i\tilde{N} \) as its lapse function and \( -i\tilde{V}^i \) as its shift vector, with \( \tilde{N} \) being real and positive in \( M_+ \) and real and negative in \( M_- \). (The metric becomes Euclidean if \( \tilde{V}^i = 0 \).) The complexification map \( \Psi \) defined by \( \Psi(N) = -iN \), \( \Psi(V^i) = -i\tilde{V}^i \) is equivalent to transforming the global vector \( t^\mu \) so that \( t^\mu \rightarrow \exp(i\vartheta)t^\mu \), with \( \vartheta = -\pi/2 \). In particular, \( \Psi(|N|) = -i|N| \). Under the map \( \Psi \) and the periodic identification in the time-like direction, the “right” and “left” wedges of a Lorentzian eternal black hole are mapped into two complex sectors (which we denote \( \tilde{M}_+ \) and \( \tilde{M}_- \) for simplicity).

The complexification map \( \Psi \) preserves the reflection symmetry and the canonical variables \( h_{ij} \) and \( P^{ij} \) of the Lorentzian eternal black hole solution. This implies that the microcanonical boundary data (constructed uniquely from those canonical variables) that characterize the real Lorentzian solution and the functional integral are also the boundary data of the complex metric (3.6). As pointed out in Ref. [2], this property guarantees that the sum over geometries extremized by the complex eternal black hole metric will indeed describe the physical properties of a real Lorentzian eternal black hole in the semiclassical approximation. The complexification map \( \Psi \) is in fact the only complexification map that preserves the boundary data of the Lorentzian solution. Complexifications of the type \( N \rightarrow -iN \) for \( \tilde{M}_+ \) and \( N \rightarrow iN \) for \( \tilde{M}_- \) would produce complex metrics whose boundary surface energy densities do not coincide with the boundary surface energy densities of the Lorentzian eternal black hole. This can be checked by using the explicit expressions presented in [10] for the quasilocal energy of the latter solution.

\(^1\)The complexification \( \Psi \) maps the “right” and “left” Lorentzian wedges of an eternal black hole into distinct complex sectors. This can be seen by considering a finite matter distribution located at a finite distance in one of the regions of a static Lorentzian black hole. Because of the presence of matter, the complexification \( \Psi \) produces two complex sectors that cannot be identified.
The complex geometry consists of two complex sectors \( \bar{M}_+ \) and \( \bar{M}_- \) which join at the locus of points at which \( \tilde{N} = 0 \). For each sector the two-surface at which \( \tilde{N} = 0 \) is called a “bolt” [4]. The geometric structure of each of these sectors resembles the structure of the single black hole complex sector used in Refs. [3,4] to approximate black hole functional integrals. Since the Lorentzian metric is a solution of Einstein equations, the complex metric (3.6) is also a solution of Einstein equations with the exception of the locus \( \tilde{N} = 0 \). Einstein equations are not satisfied at the “bolt” if a conical singularity exists there for every \( \Sigma \). Each sector \( \bar{M}_+ \) and \( \bar{M}_- \) has consequently the topology of a “punctured” disk \( \times S^2 \) because the two-space defined by the plane generated by the unit normals \( u^\mu \) and \( n^\mu \) has the topology of a “punctured” disk [1]. The outer three-dimensional boundaries of the sectors \( \bar{M}_+ \) and \( \bar{M}_- \) are \( B_+ \) and \( B_- \), while their inner three-dimensional boundaries are denoted by \( ^3H_+ \) and \( ^3H_- \) respectively. The boundary data \( (\varepsilon_+, j_+, \sigma_+) \) and \( (\varepsilon_-, j_-, \sigma_-) \) are specified at \( B_+ \) and \( B_- \). The outer boundaries \( B_\pm \) of \( M_\pm \) have topologies \( S_\pm \times S^1 \) while the inner boundaries \( ^3H_\pm \) of \( M_\pm \) have topologies \( ^2H_\pm \times S^1 \), where \( ^2H_+ \) and \( ^2H_- \) denote respectively the intersection of the slices \( \Sigma_+ \) and \( \Sigma_- \) with the black hole horizon for the Lorentzian metric. Each one of the slices \( \Sigma_+ \) and \( \Sigma_- \) of the complex metric has the topology \( I \times S^2 \) due to the openings at \( ^3H_+ \) and \( ^3H_- \).

To satisfy the vacuum Einstein equations and assure the smoothness of the complex geometry it is necessary to impose regularity conditions in the submanifolds that contain the unit normals \( n^i \) to the “bolt” for each surface \( t = \text{const.} \) [3,4] and to require the two-metric \( \sigma_{ab} \) to be continuous at \( ^2H_+ \) and \( ^2H_- \). As one approaches the “bolt” from both \( \bar{M}_+ \) and \( \bar{M}_- \) the metric becomes Euclidean

\[
ds^2 \approx \tilde{N}^2 dt^2 + \tilde{h}_{ij} dx^i dx^j.
\]

The regularity is enforced if, for each sector \( \bar{M}_+ \) and \( \bar{M}_- \), the proper circumference of circles surrounding the “bolt” equals \( 2\pi \) times their proper distance to the “bolt”. The proper circumference is given by \( P|\tilde{N}| \) in both \( \bar{M}_+ \) and \( \bar{M}_- \), where \( P \) denotes the period of the geometry in coordinate (stationary) time \( t \). The complexification map \( \Psi \) guarantees that
the unit normals $\tilde{n}^i$ to the “bolt” for each surface $\Sigma$ are continuously defined. Because of this, the regularity conditions at the “bolt” as approached from either region $\tilde{M}_+$ and $\tilde{M}_-$ take the form

$$P = \frac{2\pi}{\tilde{n}^i D_i \tilde{N}}.$$  \hspace{1cm} (3.8)

As mentioned in Section II, the quantity $\tilde{n}^i D_i \tilde{N}$ (defined in terms of the “standard” coordinates) has the same relative signs in both regions $\tilde{M}_+$ and $\tilde{M}_-$ of an eternal black hole. Condition (3.8) holds at each point on the bolt $[1]$. The regularity conditions (3.8) and the requirement that $\tilde{N}$ at $3H_+$ and $3H_-$ assure the smoothness of the complex geometries by sealing the openings at $3H_+$ and $3H_-$ with no conical singularities. They effectively guarantee the absence of inner boundaries for either sector $\tilde{M}_+$ or $\tilde{M}_-$ and imply that the plane generated by the normals $u^\mu$ and $n^\mu$ becomes a smooth disk with $R^2$ topology. The topology of each sector $\tilde{M}_+$ and $\tilde{M}_-$ becomes $R^2 \times S^2$. In this way the conditions mentioned above amount to the absence of inner boundary information $[2]$ at either $3H_+$ or $3H_-$. However, each element $3H_+$ and $3H_-$ does contribute a term to the microcanonical action for the complex geometry (3.6). For an ordinary black hole the contribution from the single inner element $3H_+$ to the action is indeed responsible for the black hole entropy. In the present case, two such contributions to the action arise at $3H_+$ and $3H_-$, and it becomes important to determine whether they either add or cancel each other.

The complex metric periodically identified with a coordinate period satisfying (3.8) is an extremum of the action $S_m$ and satisfies the desired boundary conditions. It is not included in the sum over Lorentzian geometries $\tilde{\nu}_s$ in (3.1) but it can be used to approximate it by distorting the contours of integration for both the lapse $\tilde{N}$ and the shift $\tilde{V}^i$ into the complex plane $[1]$. In this approximation the functional integral becomes

$$\tilde{\nu}_s[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp \left( i S_m [-i \tilde{N}, -i \tilde{V}, \tilde{h}] \right).$$  \hspace{1cm} (3.9)

The action $S_m [-i \tilde{N}, -i \tilde{V}, \tilde{h}]$ is the microcanonical action (2.9) for a “tilted” foliation evaluated at the complex metric (3.6) when the smoothness of the geometries at $3H_+$ and $3H_-$ is
inforced. As before, no integrals at \( t' \) and \( t'' \) are present because of the periodic identification of \( \Sigma' \) and \( \Sigma'' \) in the complex manifold. However, we cannot use Eqn. (2.14) directly to evaluate the action for the complex metric since the latter action must include terms at both \( 3H_+ \) and \( 3H_- \). If one repeats the Hamiltonian decomposition (2.10)-(2.13) of the Lorentzian action when two internal “boundary” elements exist at \( 3H_+ \) and \( 3H_- \) one obtains

\[
S_m = \int_M d^4x [P^{ij} \dot{h}_{ij} - N H - V^i H_i] + \int_{H_+} d^3x \sqrt{\sigma} (|N| n^\mu a_\mu / \kappa + 2 n_i V_j P^{ij} / \sqrt{h})
\]

\[
+ \int_{H_-} d^3x \sqrt{\sigma} (|N| n^\mu a_\mu / \kappa - 2 n_i V_j P^{ij} / \sqrt{h}).
\]  

(3.10)

This action can now be used to evaluate the action \( S_m[-i \tilde{N}, -i \tilde{V}, \tilde{h}] \). The volume term in the latter action vanishes due to stationarity and to the Hamiltonian and momentum constraints being satisfied by the complex metric (3.6). Since the shift vector \( \tilde{V}^i \) vanishes at both \( 3H_+ \) and \( 3H_- \), only the terms involving the acceleration of the unit normal \( u^\mu \) remain to be evaluated. By using the regularity conditions (3.8) and the expression \( \tilde{a}_i = (D_i \tilde{N})/\tilde{N} \), the action at the complex metric becomes

\[
S_m[-i \tilde{N}, -i \tilde{V}, \tilde{h}] = -\frac{i}{\kappa} \int_{H_+} d^3x \sqrt{\sigma} |\tilde{N}| \tilde{n}^\mu \tilde{a}_\mu - \frac{i}{\kappa} \int_{H_-} d^3x \sqrt{\sigma} |\tilde{N}| \tilde{n}^\mu \tilde{a}_\mu
\]

\[
= -\frac{i}{\kappa} \int_{H_+} d^2x \sqrt{\sigma} P \tilde{n}^i D_i \tilde{N} + \frac{i}{\kappa} \int_{H_-} d^2x \sqrt{\sigma} P \tilde{n}^i D_i \tilde{N}
\]

\[
= -\frac{2\pi i}{\kappa} \int_{H_+} d^2x \sqrt{\sigma} + \frac{2\pi i}{\kappa} \int_{H_-} d^2x \sqrt{\sigma}
\]

\[
= -\frac{i}{4} A_+ + \frac{i}{4} A_- ,
\]  

(3.11)

where \( A_+ \) and \( A_- \) denote the surface area of the horizon elements \( 3H_+ \) and \( 3H_- \). The gravitational constraint equations imply that \( A_+ \) and \( A_- \) are functions of the boundary data \( (\varepsilon_+, j_+, \sigma_+) \) and \( (\varepsilon_-, j_-, \sigma_-) \) respectively. The functional integral (3.9) is therefore

\[
\bar{\nu}_s[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp \left( \frac{1}{4} A_+ - \frac{1}{4} A_- \right).
\]  

(3.12)

Recall that the data \( (\varepsilon_+, j_+, \sigma_+) \) and \( (\varepsilon_-, j_-, \sigma_-) \) correspond to the boundary data of the classical Lorentzian eternal black hole solution (3.3). As such, they are not an arbitrary set of boundary data but a set that guarantees that the two-metric is continuous at the horizon.
of the Lorentzian black hole. Since the periodic identification and the complexification $\Psi$ do not alter these boundary data nor the gravitational constraint equations, the area $A_+$ of $2H_+$ coincides with the area $A_-$ of $2H_-: A_+(\varepsilon_+, j_+, \sigma_+) = A_-(\varepsilon_-, j_-, \sigma_-) \equiv A_H$. This implies that, in agreement with (3.5), the eternal black hole functional integral is

$$\bar{\nu}_*[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp \left( \frac{1}{4}A_H - \frac{1}{4}A_H \right) = 1$$

(3.13)
in the “zero-loop” approximation.

If the microcanonical functional integral (3.1) is interpreted as the density of states of the statistical system, it is possible to express $\bar{\nu}_*$ approximately as

$$\bar{\nu}_*[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-] \approx \exp(S[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-]),$$

(3.14)

where $S$ represents the total entropy of the system. The result (3.13) implies that the entropy for the system is

$$S \approx \frac{1}{4}A_H - \frac{1}{4}A_H = 0$$

(3.15)
in the semiclassical approximation. Notice that the total entropy is given formally by the subtraction $S = S_+[\varepsilon_+, j_+, \sigma_+] - S_-[\varepsilon_-, j_-, \sigma_-]$, where both $S_+$ and $S_-$ equal one fourth of the area of the horizon in this approximation and can be interpreted as the semiclassical entropies associated with the external ($M_+$) and internal ($M_-$) regions respectively of the eternal black hole system.

IV. CONCLUDING REMARKS

The functional integral (3.1) and (3.13) refers to a quantum-statistical system which is classically approximated by a general stationary, axisymmetric, eternal black hole solution of Einstein equations within a region bounded by two timelike surfaces $B_+$ and $B_-$. If the functional integral is interpreted as the density of states of the system, the entropy of the latter in the semiclassical approximation equals $S = A_H/4 - A_H/4 = 0$, where $A_H$ is the area of the horizon of the physical eternal black hole solution that classically approximates
the system. This result is a consequence of the choice of boundary data, the gravitational constraint equations, and the vanishing of the microcanonical action for the four-geometries that satisfy the boundary conditions and approximate the path integral.

Although the result (3.15) for the entropy can be expected on physical grounds, it is important to stress its generality. Since the spacetime is not necessarily asymptotically flat outside the boundaries $B_+$ and $B_-$, the physical eternal black hole that approximates the quantum system is in general a distorted black hole not necessarily of the Kerr-Newman form. Expression (3.15) applies to any of these configurations in the strong gravity regime (even in the case when gravitational perturbations are not small) since the functional integral refers to the gravitational field itself of any type of spacetime whose geometry is regular at the bifurcation surface and which satisfies eternal black hole boundary conditions. As is the case for the ordinary black hole entropy computed in [1], the entropy (3.15) does not seem to depend on axisymmetry. These results indicate that a pure state (of zero entropy) can be defined not only for matter fields perturbations propagating in the spacetime of an eternal black hole but also for the gravitational field itself. This is physically appealing: the initial data for the eternal black hole specified at the spacelike hypersurface $\Sigma$ contain all the information required for the evolution of both the exterior and interior parts of a physical black hole. The entropy associated with $\Sigma$ must therefore equal zero.

These conclusions are in complete agreement with thermofield dynamics descriptions of quantum processes [26] and, in particular, with the application of this approach to black hole thermodynamics developed originally by Israel [27] for small perturbations (see also Refs. [28]). In the original formulation of thermofield dynamics an extended Fock space $\mathcal{F} \otimes \tilde{\mathcal{F}}$ is obtained by augmenting the physical Fock space $\mathcal{F}$ by a “fictitious” Fock space $\tilde{\mathcal{F}}$. A pure vacuum state in the extended Fock space $\mathcal{F} \otimes \tilde{\mathcal{F}}$ corresponds to a mixed state in the physical Fock space $\mathcal{F}$. In the application of this approach to black hole processes the Boulware states of particles in the two causally disconnected regions $R_+$ and $R_-$ of an eternal black hole can be identified with the spaces $\mathcal{F}$ and $\tilde{\mathcal{F}}$ respectively, and the space $\mathcal{F} \otimes \tilde{\mathcal{F}}$ describes states for the complete system. The results of Ref. [10] regarding the gravitational
Hamiltonian $H = H_+ - H_-$ for a spatially bounded region of an eternal black hole and the thermodynamical functional integral for eternal black holes presented in this paper strongly indicate that the thermofield dynamics description of quantum field processes in a curved background can be extended beyond small perturbations to the gravitational field itself of distorted eternal black holes.

The microcanonical functional integral (3.9) reflects the properties of a pure state of zero entropy. It would be specially interesting to recover the density of states and entropy for “mixed” states in the “exterior region” $M_+$ of an eternal black hole by explicitly tracing out in (3.9) the internal degrees of freedom of the black hole itself. This operation must yield the density of states $\nu_*$ for a black hole computed in [1] (with a corresponding entropy given by one fourth of the horizon area) in the semiclassical approximation. It is not yet clear how to perform this “tracing” operation satisfactorily beginning with the functional integral (3.9). There are several ways in which one could proceed. For example, it has been suggested [10] that the internal degrees of freedom of a black hole can be identified with the set of boundary data specified at the boundary $B_-$. One could formally construct a functional integral $\nu_*$ on $M_+$ by integrating over these boundary data in the form

$$\nu_*[\varepsilon_+, j_+, \sigma_+] \approx \int \mathcal{D}\mu[\varepsilon_-, j_-, \sigma_-] \nu_*[\varepsilon_+, j_+, \sigma_+; \varepsilon_-, j_-, \sigma_-],$$

(4.1)

where $\mathcal{D}\mu[\varepsilon_-, j_-, \sigma_-]$ denotes some measure in the space of boundary data at $B_-$. The definition of this measure is delicate. Since the initial data $(\varepsilon_-, j_-, \sigma_-)$ at $B_-$ uniquely determine the horizon area $A_-$ in a microcanonical description (see, for example, Ref. [20]), the measure $\mathcal{D}\mu$ may be tentatively regarded as proportional to the differential $dS_-$ of the entropy $S_-$ in a first approximation. (Although in a different context, this measure has been previously considered in Ref. [13].) If (3.12) is substituted directly in (4.1) the integral would become

$$\nu_*[\varepsilon_+, j_+, \sigma_+] \approx \int_0^\infty dS_- \exp(S_+ - S_-) \approx \exp(S_+).$$

(4.2)

While this is the desired result for the quantity $\nu_*$, the approach has several obvious conceptual difficulties. For a given value of $A_+$, the integration (4.2) implies a sum over the
whole range of areas $A_\pm$. This “decoupling” of the “degrees of freedom” $A_+$ and $A_-$ is not a semiclassical effect because the boundary data $(\varepsilon_{\pm}, j_{\pm}, \sigma_{\pm})$ at $B_\pm$ for a classical eternal black hole are such that $A_+ = A_- = A_H$ in the absence of matter at the horizon. The integral (4.2) therefore represents a sum over quantum spacetimes which satisfy the boundary data at $B_\pm$ but whose two-metric is not regular at the bifurcation surface $S_0$. However, it is not clear whether the expression $\bar{\nu}_\ast \approx \exp(\mathcal{S}_+ - \mathcal{S}_-)$ is appropriate for non-smooth geometries. The contribution of these geometries to the functional integral (3.1) could perhaps be calculated using the approaches developed in Refs. [22,23,29].

Another approach to recover the black hole density of states $\nu_\ast$ from the eternal black hole functional integral $\bar{\nu}_\ast$ is the following. The quantity $\nu_\ast$ computed in [2] is obtained as the trace of a density matrix when a special set of conditions (which include $\tilde{N} = 0$, $\tilde{V}^i = 0$, and the regularity conditions) is imposed at the “bolt” of a complex geometry. These conditions imply that the complex sector has no inner boundary. Similarly, the eternal black hole functional integral $\bar{\nu}_\ast$ computed in Section III is obtained as the trace (3.2) of a density matrix when similar conditions are imposed at $^3H_+$ and $^3H_-$. However, it is not difficult to see that $\bar{\nu}_\ast$ would equal $\nu_\ast$ if the above conditions are only imposed at $^3H_+$ while microcanonical boundary conditions are imposed at $^3H_-$. Tracing out internal degrees of freedom would seem to be equivalent to imposing microcanonical boundary conditions at $^3H_-$ in the functional integral (3.1). If this procedure is physically sensible, the geometries summed over in the tracing operation will not be smooth at the “bolt”. It might be interesting to study the relationship between this approach and the proposals for black hole entropy presented in Refs. [1,3], and to reproduce the thermodynamical results presented in this paper by using the Hamiltonian methods developed in Refs. [3,30,10].

Finally, the relationship between vacuum states in the left and right wedges of the Kruskal diagram to the Hartle-Hawking vacuum for quantum fields defined on the maximally extended black hole is well known [27,31,22,8]. Recently, the Hartle-Hawking vacuum state for linearized field perturbations for all fields has been constructed by using a no-boundary wave function proposal for a black hole [8]. The essential properties defining a general Hartle-
Hawking state have been described in Ref. [32]. It would be interesting to understand the significance of the thermodynamical functional integral presented here in the construction of the Hartle-Hawking vacuum state (within properly defined boundary surfaces that do not exceed the speed-of-light surfaces) for stationary, axisymmetric black holes in the strong gravity regime when the perturbations of the gravitational field are not necessarily small.

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REFERENCES

[1] J. D. Brown and J. W. York, Jr., Phys. Rev. D 47, 1420 (1993).

[2] J. D. Brown and J. W. York, Jr., “The path integral formulation of gravitational thermodynamics”, preprint IFP-UNC-491, TAR-UNC-043, CTMP/007/NCSU, gr-qc/9405024.

[3] C. Teitelboim, “Action and entropy for extremal and non-extremal black holes”, preprint, hep/9410103, (1994).

[4] S. Carlip and C. Teitelboim, “The off-shell black hole”, preprint, IASSNS-HEP-93/84, UCD-93-34, gr-qc/9312002.

[5] S. W. Hawking, G. T. Horowitz, and S. F. Ross, “Entropy, area, and black hole pairs”, preprint, DAMTP-R-94/26, UCSBTH-94-25, gr-qc/9409013, (1994).

[6] J. D. Brown, E. A. Martinez, and J. W. York, Jr., Phys. Rev. Lett., 66, 2281 (1991); in Nonlinear Problems in Relativity and Cosmology, edited by J. R. Buchler, S. L. Detweiler, and J.R. Ipser (New York Academy of Sciences, New York, 1991).

[7] V. P. Frolov and I. Novikov, Phys. Rev. D 48, 4545 (1993).

[8] A. O. Barvinsky, V. Frolov, and A. Zelnikov, “Wavefunction of a black hole and the dynamical origin of entropy”, preprint, Alberta-Thy-13-94, gr-qc/9404036, (1994).

[9] J. Louko and B. F. Whiting, “Hamiltonian thermodynamics of the Schwarzschild black hole”, preprint, UF-RAP-94-13, WISC-MILW-94-TH-24, gr-qc/9411017, (1994).

[10] V. Frolov and E. A. Martinez, “Action and Hamiltonian for eternal black holes”, preprint, Alberta-Thy-32-94, gr-qc/9411001, (1994).

[11] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2752 (1977); S. W. Hawking, in General Relativity, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

[12] J. W. York, Jr., Phys. Rev. D 33, 2092 (1986).
[13] B. F. Whiting and J. W. York, Jr., Phys. Rev. Lett., 61, 1336 (1988).

[14] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. 66, 291 (1979).

[15] W. Israel and J. M. Stewart, in General Relativity and Gravitation. II, edited by A. Held (Plenum Press, New York, 1980).

[16] V. Frolov and K. S. Thorne, Phys. Rev. D 31, 2125 (1989).

[17] V. Frolov and E. A. Martinez, “Eternal black holes and quasilocal energy”, Alberta-Thy-19-94, gr-qc/9405041, in Proceedings of the Lake Louise Winter Institute on Particle Physics and Cosmology, edited by B. Campbell and F. Khana, World Scientific, 1994.

[18] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current Research, edited by L. Witten (Wiley, New York, 1962).

[19] J. D. Brown and J. W. York, Jr., Phys. Rev. D 47, 1407 (1993).

[20] J. D. Brown, G. L. Comer, E. A. Martinez, J. Melmed, B. F. Whiting, and J. W. York, Class. Quantum Grav. 7, 1433 (1990).

[21] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation, W. H. Freeman, San Francisco, 1973.

[22] G. Hayward, Phys. Rev. D 47, 3275, (1993); G. Hayward and K. Wong, Phys. Rev. D 46, 620, (1992); Phys. Rev. D 47, 4778, (1993).

[23] D. Brill and G. Hayward, Phys. Rev. D 50, 4914 (1994).

[24] K. V. Kuchař, J. Math. Phys. 17, 777 (1976); 17, 792 (1976); 17, 801 (1976).

[25] J. M. Bardeen, in Black Holes, edited by C. DeWitt and B. S. DeWitt, Gordon and Breach Science Publishers, New York, 1973.

[26] H. Umezawa and Y. Takahashi, Collective Phenomena 2, 55 (1975); H. Umezawa, Advanced Field Theory, American Institute of Physics, New York, 1993.
[27] W. Israel, Phys. Lett. **57A**, 107 (1976).

[28] R. Laflamme, Nucl. Phys. **B324**, 233 (1989); N. Sanchez and B. F. Whiting, Nucl. Phys. **B283**, 605 (1987); B. F. Whiting, in *Proceedings of the Workshop on Thermal Field Theories and their applications*, edited by K. L. Kowalski, N. P. Landsman, and Ch. G. van Weert, Physica A, North Holland (1988).

[29] G. Hayward and J. Louko, Phys. Rev. **D 42**, 4032, (1990).

[30] K. V. Kuchař, Phys. Rev. **D 50**, 3961 (1994).

[31] J. B. Hartle and S. W. Hawking, Phys. Rev. **D 13**, 2188 (1976).

[32] T. Jacobson, “A note on Hartle-Hawking vacua”, preprint, [gr-qc/9407022](http://arxiv.org/abs/gr-qc/9407022), (1994).