A radiometer for stochastic gravitational waves

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Abstract

The LIGO Scientific Collaboration recently reported a new upper limit on an isotropic stochastic background of gravitational waves obtained based on the data from the third LIGO science run (S3). Here I present a new method for obtaining directional upper limits on stochastic gravitational waves that essentially implements a gravitational wave radiometer. The LIGO Scientific Collaboration intends to use this method for future LIGO science runs.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The LIGO Scientific Collaboration reported a new 90%-confidence Bayesian upper limit on an isotropic background of gravitational waves of \( \Omega_{gw}(f) < 8.4 \times 10^{-4} \) [1] using data from the third LIGO science run (S3), where \( \Omega_{gw}(f) \) is the energy density per logarithmic frequency interval associated with gravitational waves normalized by the critical energy density \( \rho_c \) required to close the universe. However, it is possible that the dominant source of stochastic gravitational waves in the LIGO frequency band comes from an ensemble of astrophysical sources (e.g., [3, 4]). If such an ensemble turns out to be dominated by its strongest members then the assumption of isotropy is no longer valid. Instead one should look for anisotropies in the stochastic gravitational wave background. This was addressed in [5, 6], but they focused on getting optimal estimates for each spherical harmonic. If the stochastic gravitational wave background is indeed dominated by individual sources, one can get a better signal-to-noise ratio by obtaining optimal filters for small patches of the sky.

I present a directional method that essentially implements a gravitational wave radiometer. The algorithm has been implemented and will be used to analyse future LIGO science runs starting with S4.
2. Search for an isotropic background

The data from the first three LIGO science runs were analysed with a method described in detail in [1, 2, 7]. The data streams from a pair of detectors were cross-correlated with a cross-correlation kernel $Q$ chosen to be optimal for an assumed strain power spectrum $H(f) = H(|f|)$ and angular distribution $P(\Omega) = 1$ (isotropic distribution). Specifically, with $S_1(f)$ and $S_2(f)$ representing the Fourier transforms of the strain outputs of two detectors, this cross-correlation is computed in the frequency domain segment by segment as

$$ Y = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \delta_T(f - f') S_1^*(f) Q(f') S_2(f'), \quad (1) $$

where $\delta_T$ is a finite-time approximation to the Dirac delta function. The optimal filter $Q$ has the form:

$$ Q(f) = \lambda \frac{\gamma_{iso}(f) H(f)}{P_1(f) P_2(f)}, \quad (2) $$

where $\lambda$ is a normalization factor, $P_1$ and $P_2$ are the strain noise power spectra of the two detectors, $H(f)$ is the strain power spectrum of the stochastic background being searched for ($=S_{gw}(f)$ in [1, 2]) and the factor $\gamma_{iso}$ takes into account the cancellation of an isotropic omni-directional signal ($P(\hat{\omega}/Omega_1) = 1$) at higher frequencies due to the detector separation. $\gamma_{iso}$ is called the overlap reduction function [8] and is given by (the normalization is such that $\gamma_{iso}(f = 0) = 1$ for aligned and collocated detectors)

$$ \gamma_{iso}(f) = \frac{5}{8\pi} \sum_A \int_{S^2} d\Omega e^{2\pi f\hat{\Omega}} \bar{\Omega} F_i^{A}(\hat{\Omega}) F_j^{A}(\hat{\Omega}), \quad (3) $$

where $\Delta x = \hat{x}_2 - \hat{x}_1$ is the detector separation vector, $\hat{\Omega}$ is the unit vector specifying the sky position and

$$ F_i^{A}(\hat{\Omega}) = e^{iA}(\hat{\Omega}) \frac{1}{2} \left( \hat{X}_i^{a} \hat{X}_i^{b} - \hat{Y}_i^{a} \hat{Y}_i^{b} \right) \quad (4) $$

is the response of detector $i$ to a zero frequency, unit amplitude, $A = +, \times$ polarized gravitational wave. $e^{iA}(\hat{\Omega})$ is the spin-2 polarization tensor for polarization $A$, and $\hat{X}_i^{a}$ and $\hat{Y}_i^{a}$ are unit vectors pointing in the directions of the detector arms (see [7] for details).

The optimal filter $Q$ is derived assuming that the intrinsic detector noise is Gaussian and stationary over the measurement time, uncorrelated between detectors, and uncorrelated with and much greater in power than the stochastic gravitational wave signal. Under these assumptions the expected variance, $\sigma_Y^2$, of the cross-correlation is dominated by the noise in the individual detectors, whereas the expected value of the cross-correlation $Y$ depends on the stochastic background power spectrum:

$$ \sigma_Y^2 \equiv \langle Y^2 \rangle - \langle Y \rangle^2 \approx \frac{T}{4} \langle Q, Q \rangle, \quad \langle Y \rangle = T \left( \frac{\gamma_{iso} H}{P_1 P_2} \right). \quad (5) $$

Here the scalar product $(\cdot, \cdot)$ is defined as $(A, B) = \int_{-\infty}^{\infty} A^*(f) B(f) P_1(f) P_2(f) df$ and $T$ is the duration of the measurement.

In order to address the long-term non-stationarity of the detector noise, the data set from a given interferometer pair is divided into equal-length intervals, and the cross-correlation $Y$ and theoretical $\sigma_Y$ are calculated for each interval, yielding a set $\{Y_i, \sigma_{Y_i}\}$ of such values,
with $I$ labelling the intervals. The interval length can be chosen such that the detector noise is relatively stationary over one interval. In [1, 2], the interval length was chosen to be 60 s. The cross-correlation values are combined to produce a final cross-correlation estimator, $Y_{\text{opt}}$, that maximizes the signal-to-noise ratio, and has variance $\sigma_{\text{opt}}^2$:

$$Y_{\text{opt}} = \sum_I \sigma_{Y_I}^{-2} Y_I / \sigma_{\text{opt}}^{-2}, \quad \sigma_{\text{opt}}^{-2} = \sum_I \sigma_{Y_I}^{-2}.$$  \hspace{1cm} (6)

Since the LIGO Hanford and Livingston sites are separated by 3000 km, the overlap reduction function for this pair has already dropped below 5% around each interferometer’s sweet spot of 150 Hz. One unfortunate drawback of this analysis thus is the limited use it makes of the individual interferometer’s most sensitive frequency region. Moreover, if the dominant gravitational wave background were of astrophysical origin, the assumption of an isotropic background is not well justified. If for example the signal is dominated by a few strong sources, a directed search can achieve a better signal-to-noise ratio.

3. Directional search: a gravitational wave radiometer

A natural generalization of the method described above can be achieved by finding the optimal filter for an angular power distribution $P(\hat{\Omega})$. In this case the second part of equation (5) generalizes to

$$\langle Y \rangle = T \left( Q, \int \frac{d\Omega \gamma_{\hat{\Omega}} P(\hat{\Omega}) H}{P_1 P_2} \right),$$  \hspace{1cm} (7)

where $\gamma_{\hat{\Omega}}$ is now just the integrand of $\gamma_{\text{iso}}$, i.e.

$$\gamma_{\hat{\Omega}} = \frac{1}{2} \sum_A e^{i2\pi f_{\hat{\Omega}} \cdot \Delta x} F_1^A(\hat{\Omega}) F_2^A(\hat{\Omega})$$  \hspace{1cm} (8)

and $H(f)$ is the strain power spectrum of an unpolarized point source, summed over both polarizations. Note that $\gamma_{\hat{\Omega}}$ also becomes sidereal time dependent both through $\Delta x$ and $F_1^A(\hat{\Omega})$.

Equation (7) was used in [5] as a starting point to derive optimal filters for each spherical harmonic. However, if one wants to optimize the method for well-located astrophysical sources, it seems more natural to use a $P(\hat{\Omega})$ that only covers a localized patch in the sky. Indeed, for most reasonable choices of $H(f)$, the resulting maximal resolution of this method will be bigger than several tens of square degrees such that most astrophysical sources would not be resolved. Therefore, it makes sense to optimize the method for true point sources, i.e. $P(\hat{\Omega}) = \delta^2(\hat{\Omega}, \hat{\Omega}')$.

With this choice of $P(\hat{\Omega})$, the optimal filter $Q_{\hat{\Omega}'}$ for the sky direction $\hat{\Omega}'$ becomes

$$Q_{\hat{\Omega}'}(f) = \gamma_{\hat{\Omega}'}(f) H(f) \frac{P_1(f) P_2(f)}{P_1(f) P_2(f)},$$  \hspace{1cm} (9)

and the expected cross-correlation $Y_{\hat{\Omega}'}$ and its expected variance $\sigma_{Y_{\hat{\Omega}'}^2}$ are

$$\sigma_{Y_{\hat{\Omega}'}^2} = \langle Y_{\hat{\Omega}'}^2 \rangle - \langle Y_{\hat{\Omega}'} \rangle^2 \approx \frac{T}{4} \langle Q_{\hat{\Omega}'} \cdot Q_{\hat{\Omega}'} \rangle, \quad \langle Y_{\hat{\Omega}'} \rangle = T \left( Q_{\hat{\Omega}'} \gamma_{\hat{\Omega}'} H \frac{P_1 P_2}{P_1 P_2} \right).$$  \hspace{1cm} (10)
3.1. Integration over sidereal time

Just as in the isotropic case, the long-term non-stationarity of the detector noise can be addressed by processing the data on a segment by segment basis. However, \( \gamma \) changes with sidereal time. By setting it to its mid-segment value one can get rid of the first-order error, but a second-order error remains and is of the order

\[
Y_{\mathrm{er}(T_{\mathrm{seg}})/Y} = \frac{T_{\mathrm{seg}}^{2}}{24} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{Y_{\Omega}^{2} H}{P_{1} P_{2}} S_{1}^{*} S_{2} \, df \, \frac{1}{\gamma_{\Omega}^{2}} \left| \frac{\mu_{\Omega}}{\mu_{\Omega}} \right| d\phi = O \left( \frac{2\pi f d T_{\mathrm{seg}}}{c} \right)^{2}
\]

with \( f \) the typical frequency and \( d \) the detector separation. For \( T_{\mathrm{seg}} = 60 \, \text{s} \), \( f = 2 \, \text{kHz} \) and \( d = 3000 \, \text{km} \) this error is less than 1%.

Thus the integration over sidereal time for each \( \hat{\Omega} \) again reduces to the optimal combination of the set \( \{ Y_{i}, \sigma_{y_i} \}_{\hat{\Omega}} \) given by equation (6). The only difference from the isotropic \( P(\hat{\Omega}) = 1 \) case is that the optimal filter \( Q_{\hat{\Omega}} \) is different for each interval \( I \) and each sky direction \( \hat{\Omega} \).

3.2. Numerical aspects

To implement this method one thus has to calculate

\[
Y_{\hat{\Omega}} = \lambda \Omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{Y_{\Omega}^{2} H}{P_{1} P_{2}} S_{1}^{*} S_{2}, \quad \sigma_{\hat{\Omega}}^{2} = \frac{\lambda^{2} T}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|Y_{\Omega}|^{2} H^{2}}{P_{1} P_{2}}
\]

for each sky direction \( \hat{\Omega} \) and each segment \( I \). This can be done very efficiently by realizing that \( Y_{\Omega} \) splits into a DC part \( \frac{1}{2} \sum F_{A}(\hat{\Omega}) F_{A}^{*}(\hat{\Omega}) \) and a phasor \( \exp\{i2\pi f \hat{\Omega} \cdot \Delta \hat{x} / c \} \). For both integrals, the DC part can be taken out of the frequency integration, leaving all the directional information of the integrands in the phasor. Thus, with \( N \) the number of sky directions \( \hat{\Omega} \), the computational load per segment is reduced from \( 2N \) integrations to the one fast Fourier transform, one integral and \( N \) readouts of the cross-correlation \( Y_{\hat{\Omega}} \) at the time shifts \( \tau = \hat{\Omega} \cdot \Delta \hat{x} / c \).

Since the fast Fourier transform of \( S_{1}^{*} S_{2} H/(P_{1} P_{2}) \) is sampled at \( f_{\text{sample}} = 2 f_{\text{Nyquist}} \), it is necessary to interpolate to get the cross-correlation \( Y_{\hat{\Omega}} \) at the time shift \( \tau \). However, by choosing a high enough Nyquist frequency and zero-padding the unused bandwidth this interpolation error can be kept small while the overall efficiency is still improved.

3.3. Comparison to the isotropic case

It is interesting to look at the potential signal-to-noise ratio improvement of this directional method compared to the isotropic method if indeed all correlated signal were to come from one point \( \hat{\Omega} \), i.e. \( \langle S_{1}^{*} S_{2} \rangle = Y_{\hat{\Omega}} H \). The ratio between the two signal-to-noise ratios works out to

\[
\frac{\text{SNR}_{\text{iso}}}{\text{SNR}_{\hat{\Omega}}} = \frac{\langle Y_{1}^{\text{opt}} \rangle / \sigma_{1}^{\text{opt}}}{\langle Y_{\hat{\Omega}}^{\text{opt}} \rangle / \sigma_{\hat{\Omega}}^{\text{opt}}} = \left[ \frac{\nu_{1}}{\nu_{\hat{\Omega}}} \right]^{1/2}
\]

with \( [A, B] = \sum_{i} (A_{i} H/(P_{1,i} P_{2,i}), B_{i} H/(P_{1,i} P_{2,i})) \) and \( i \) the index summing over sidereal time. This ratio is bounded between \( -1 \) and \( 1 \), i.e. the directional search not only performs better in this case but also, for a point source at an unfortunate position, the isotropic search can even yield negative or zero correlation.
It is also possible to recover the isotropic result as an integral over the sky. The definitions of $\gamma_{iso}$ and $\gamma_{\hat{\Omega}}$ (equations (3) and (8)) imply

$$ Y_{iso}^{opt} = \int d\hat{\Omega} \frac{5}{4\pi} \sigma_{iso}^{opt}, \quad \sigma_{iso}^{opt} = \left(\frac{5}{4\pi}\right)^2 \int d\hat{\Omega} \int d\hat{\Omega}' \sigma_{\hat{\Omega},\hat{\Omega}'}^{opt}. \quad (14) $$

Here $\sigma_{\hat{\Omega},\hat{\Omega}'}^2$ is the covariance of the two sky directions $\hat{\Omega}$ and $\hat{\Omega}'$. It is given by

$$ \sigma_{\hat{\Omega},\hat{\Omega}'}^2 \approx \frac{T}{4} (Q\hat{\Omega}, Q\hat{\Omega}') \quad (15) $$

and describes the antenna lobe of the gravitational wave radiometer.

### 3.4. Achievable sensitivity

The $1-\sigma$ sensitivity of this method is given by

$$ H_{sens, \hat{\Omega}}(f) = \frac{\sigma_{\hat{\Omega}}}{T} H(f) = \frac{H(f)}{2\sqrt{T} \int_{-\infty}^{\infty} |\gamma_{\hat{\Omega}}^{opt} H_j| \frac{d f}{\text{sidereal day}}} \quad (16) $$

$H_{sens}$ is somewhat dependent on the declination and in theory independent of right ascension. In practice though an uneven coverage of the sidereal day due to downtime and time-of-day dependent sensitivity will break this symmetry, leaving only an antipodal symmetry.

For the initial LIGO Hanford 4 km–Livingston 4 km pair (H1–L1), both at design sensitivity, and a flat source power spectrum $H(f) = \text{const}$ this works out to

$$ H_{sens}^{H1-L1} \approx 1.5 \times 10^{-50} \text{ Hz}^{-1} \left(\frac{1 \text{ yr}}{T}\right)^\frac{1}{2} \quad (17) $$

with a 35% variation depending on the declination. This corresponds to a gravitational wave energy flux density of

$$ F_{gw} df \approx 5 \times 10^{-11} \text{ Watt m}^2 \text{ Hz} \cdot \left(\frac{f}{100 \text{ Hz}}\right)^2 \cdot \left(\frac{1 \text{ yr}}{T}\right)^\frac{1}{2} df. \quad (18) $$

### 3.5. Results from simulated data

The described algorithm was implemented such that it is ready to run on real LIGO data. However, in order to test the code, the real data were blanked out and simulated Gaussian noise uncorrelated between the two detectors and with a power spectrum shape equal to the LIGO design sensitivity was added. To take into account the non-uniform day coverage real lock segment start and stop times were used. To get a shorter turn-around time during testing the code was only run on 1.7 days of integrated simulated data. The signal power spectrum was assumed to be flat, $H(f) = \text{const}$.

The algorithm was run on a 360 $\times$ 181 point grid covering the whole sky. While this clearly over-samples the intrinsic resolution—for the $H(f) = \text{const}$ case the antenna lobe has a FWHM area of 50–100 deg$^2$, depending on the declination—it produces nicer pictures as final product as shown in figure 1.

For figure 2, additional coherent noise with a flat power spectrum $H(f)$ and a sidereal time-dependent time shift and amplitude modulation appropriate for a true point source was added. The data were produced by splicing together short pieces of data with fixed time shift. The injected source has a signal-to-noise ratio of 14 and is clearly recovered. This figure
4. Conclusion

The presented gravitational wave radiometer method aims for the optimal detection of localized stochastic gravitational wave sources with a given power spectrum $H(f)$ and significant uptime. It produces a map of point estimates and corresponding variances. The point estimate map corresponds to the true power distribution of gravitational waves $P(\tilde{\Omega})$ convolved with the antenna lobe of the radiometer and an uncertainty determined by the detector noise. This antenna lobe in turn depends on the assumed source power spectrum $H(f)$.

The method is well suited for searching for a stochastic gravitational wave background of astrophysical origin and the LIGO Scientific Collaboration intends to use it on future LIGO science runs starting with S4.

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