BEM solutions to a class of elliptic BVPs for anisotropic trigonometrically graded media

St. N. Jabir\(^1\), M. I. Azis\(^2,\ast\), Z. Djafar\(^3\), B. Nurwahyu\(^2\)

\(^1\)Department of Electrical Engineering, ATI Polytechnic, Makassar, Indonesia
\(^2\)Department of Mathematics, Hasanuddin University, Makassar, Indonesia
\(^3\)Department of Mechanical Engineering, Hasanuddin University, Makassar, Indonesia

E-mail: mohivanazis@yahoo.co.id (\*Corresponding author)

Abstract. Boundary value problems (BVPs) governed by a class of elliptic equations for anisotropic trigonometrically graded media are solved using Boundary Element Method (BEM). The variable coefficient governing equation is transformed to a constant coefficient equation which is then transformed to a boundary integral equation. The results show the convergence, consistency, and accuracy of the BEM solutions.

1. Introduction

Several types of constant coefficient equations have been solved using BEM (see for examples [1, 2, 3, 4]. But in general this is not the case for variable coefficient equation. There is some progress in using BEM to solve several types of variable coefficient governing equations (see for examples [5, 6, 7, 8, 9, 10, 11, 12, 13]).

The governing equation considered by Salam et. al in [11] takes the form

\[
\frac{\partial}{\partial x_1} \left[ \lambda_{ij} (x_1, x_2) \frac{\partial \varsigma (x_1, x_2)}{\partial x_j} \right] = 0
\]

This paper is intended to extend the work by Salam et. al [11] for problems with governing equation (1) to for 2D boundary value problems governed by another type of (dimensionless) elliptic equation of the form

\[
\frac{\partial}{\partial x_i} \left[ \lambda_{ij} (x_1, x_2) \frac{\partial \varsigma (x_1, x_2)}{\partial x_j} \right] + \beta (x_1, x_2) \varsigma (x_1, x_2) = 0
\]

where the coefficients \(\lambda_{ij}\) depend on \(x_1\) and \(x_2\) and the repeated summation convention (summing from 1 to 2) is employed.

The matrix of coefficients \([\lambda_{ij}]\) is a real symmetric positive definite matrix so that equation (2) is a second order elliptic partial differential equation and may be written explicitly as

\[
\frac{\partial}{\partial x_1} \left( \lambda_{11} \frac{\partial \varsigma}{\partial x_1} \right) + 2 \frac{\partial}{\partial x_1} \left( \lambda_{12} \frac{\partial \varsigma}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( \lambda_{22} \frac{\partial \varsigma}{\partial x_2} \right) + \beta \varsigma = 0
\]

Further, the coefficients \(\lambda_{ij}\) and \(\beta\) are required to be twice differentiable functions of the two independent variables \(x_1\) and \(x_2\). The analysis here is specially relevant to an anisotropic medium.
but it equally applies to isotropic media. For isotropy, the coefficients in (2) take the form \( \lambda_{11} = \lambda_{22} \) and \( \lambda_{12} = 0 \) and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

Steady infiltration problems (when \( \beta < 0 \), see for examples [14, 15]), acoustic problems (when \( \beta > 0 \), see for examples [16, 17]), and antiplane strain in elastostatics and plane thermostatic problems (when \( \beta = 0 \)) are the areas for which the governing equation is of the type (2).

The technique of transforming (2) to a constant coefficient equation will again be used for obtaining a boundary integral equation for the solution of (2).

2. The boundary value problem

Referred to a Cartesian frame \( O_{x_1, x_2} \) a solution to (2) is sought which is valid in a region \( \Omega \) in \( \mathbb{R}^2 \) with boundary \( \partial \Omega \) which consists of a finite number of piecewise smooth closed curves. On \( \partial \Omega_1 \) the dependent variable \( \varsigma(x) \) (\( x = (x_1, x_2) \)) is specified and on \( \partial \Omega_2 \)

\[
P(x) = \lambda_{ij} \left( \partial \varsigma / \partial x_j \right) n_i
\]

is specified where \( \partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \) and \( n = (n_1, n_2) \) denotes the outward pointing normal to \( \partial \Omega \).

3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (2) to a constant coefficient equation. The coefficients \( \lambda_{ij} \) and \( \beta \) are required to take the form

\[
\lambda_{ij}(x) = \overline{\lambda}_{ij} g(x)
\]

\[
\beta(x) = \overline{\beta} g(x)
\]

where the \( \overline{\lambda}_{ij} \) and \( \overline{\beta} \) are constants and \( g \) is a differentiable function of \( x \). Use of (4) and (5) and in (2) yields

\[
\overline{\lambda}_{ij} \frac{\partial}{\partial x_i} \left( g \frac{\partial \varsigma}{\partial x_j} \right) + \overline{\beta} g \varsigma = 0
\]

Let

\[
\varsigma(x) = g^{-1/2}(x) \psi(x)
\]

so that (6) may be written in the form

\[
\overline{\lambda}_{ij} \frac{\partial}{\partial x_i} \left[ g \left( g^{-1/2} \psi \right) \frac{\partial \psi}{\partial x_j} \right] + \overline{\beta} g^{1/2} \psi = 0
\]

That is

\[
\overline{\lambda}_{ij} \left[ \left( \frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j} \right) \psi + g^{1/2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] + \overline{\beta} g^{1/2} \psi = 0
\]

Use of the identity

\[
\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j}
\]

permits (8) to be written in the form

\[
g^{1/2} \overline{\lambda}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \overline{\lambda}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + \overline{\beta} g^{1/2} \psi = 0
\]

If we further restrict the function \( g(x) \) to take the exponential form
\[ g(x) = \{A[\cos(\alpha_mx_m) + \sin(\alpha_mx_m)]\}^2 \quad \bar{\beta} = -\bar{\lambda}_{ij}\alpha_i\alpha_j < 0 \] (10)

where \(\alpha_m\) are constant, then

\[ \bar{\lambda}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \bar{\beta}g^{1/2} = 0 \] (11)

Substitution (11) into (9) implies a constant coefficients equation

\[ \bar{\lambda}_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0 \] (12)

Also, substitution of (4) and (7) into (3) gives

\[ P = -P_g \psi + P_g^{1/2} \] (13)

where

\[ P_g(x) = \bar{\lambda}_{ij}\frac{\partial g^{1/2}}{\partial x_j} n_i \quad P_\psi(x) = \bar{\lambda}_{ij}\frac{\partial \psi}{\partial x_j} n_i \]

A boundary integral equation for the solution of (12) is given in the form

\[ \kappa(x_0) \psi(x_0) = \int_{\partial \Omega} [\Gamma(x,x_0) \psi(x) - \Phi(x,x_0) P_\psi(x)] ds(x) \] (14)

where \(x_0 = (a,b)\), \(\kappa = 0\) if \((a,b) \notin \Omega \cup \partial \Omega\), \(\kappa = 1\) if \((a,b) \in \Omega\), \(\kappa = \frac{1}{2}\) if \((a,b) \in \partial \Omega\) and \(\partial \Omega\) has a continuously turning tangent at \((a,b)\).

The so called fundamental solution \(\Phi\) in (14) is any solution of the equation

\[ \bar{\lambda}_{ij}\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \delta(x - x_0) \]

and the \(\Gamma\) is given by

\[ \Gamma(x,x_0) = \bar{\lambda}_{ij}\frac{\partial \Phi(x,x_0)}{\partial x_j} n_i \]

where \(\delta\) is the Dirac delta function. Following Azis in [18], for two-dimensional problems \(\Phi\) and \(\Gamma\) are given by

\[ \Phi(x,x_0) = \frac{K}{2\pi} \ln R \]
\[ \Gamma(x,x_0) = \frac{K}{2\pi} \bar{\lambda}_{ij}\frac{\partial R}{\partial x_j} n_i \] (15)

where

\[ K = \ddot{\tau}/\zeta \]
\[ \zeta = \left[\bar{\lambda}_{11} + 2\bar{\lambda}_{12} \dot{\tau} + \bar{\lambda}_{22} (\dot{\tau}^2 + \ddot{\tau}^2)\right]/2 \]
\[ R = \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2} \]
\[ \dot{x}_1 = x_1 + \dot{\tau} x_2 \]
\[ \dot{a} = a + \dot{\tau} b \]
\[ \dot{x}_2 = \dot{\tau} x_2 \]
\[ \dot{b} = \ddot{\tau} b \]
where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root $\tau$ of the quadratic

$$\lambda_{11} + 2\lambda_{12}\tau + \lambda_{22}\tau^2 = 0$$

The derivatives $\frac{\partial R}{\partial x_j}$ needed for the calculation of the $\Gamma$ in (15) are given by

$$\frac{\partial R}{\partial x_1} = \frac{1}{R}\left(\dot{x}_1 - \dot{a}\right)$$

$$\frac{\partial R}{\partial x_2} = \dot{\tau}\left[\frac{1}{R}\left(\dot{x}_1 - \dot{a}\right)\right] + \ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2 - \dot{b}\right)\right]$$

Use of (7) and (13) in (14) yields

$$\kappa(x_0) g^{1/2}(x_0) \varsigma(x_0) = \int_{\partial \Omega} \left\{\left[g^{1/2}(x) \Gamma(x, x_0) - P_g(x) \Phi(x, x_0)\right] \varsigma(x) - \left[g^{-1/2}(x) \Phi(x, x_0)\right] P(x)\right\} ds(x) \quad (16)$$

This equation provides a boundary integral equation for determining $\varsigma$ and $P$ at all points of $\Omega$.

4. Numerical examples

In order to show the appropriateness of the BEM and the validity of the analysis used above for deriving the boundary integral equation (16), some particular boundary value problems will be solved. The integrals in equation (16) are evaluated numerically using the Bode’s quadrature (see Abramowitz and Stegun [19]).

4.1. Examples with analytical solutions

In order to see the convergence and accuracy of the BEM we will consider some examples of problems with analytical solutions. The parameters for the trigonometrical inhomogeneity function $g(x)$ are $A = 3, \alpha_1 = 0.15, \alpha_2 = 0.55$. Plot of $g(x)$ is shown in Figure 1.

![Figure 1](image)

**Figure 1.** A trigonometrical inhomogeneity function $g(x) = \{3[\cos(0.15x_1 + 0.55x_2) + \sin(0.15x_1 + 0.55x_2)]\}^2$

The geometry of the region $\Omega$ and the boundary conditions are as depicted in Figure 2.
The values of the constant coefficients $\lambda_{ij}$ for the governing equation (2) are

$$\lambda_{11} = 0.25, \lambda_{12} = 0.25, \lambda_{22} = 1$$

Therefore from (9)

$$\beta = -\lambda_{ij} \alpha_i \alpha_j = -0.349375$$

The function $g(x)$ satisfies (9). Therefore equation (12) has to be the corresponding constant coefficient equation for $\psi(x)$. For test problems, we will take $\psi(x)$ as a linear function satisfying (12)

$$\psi(x) = B (\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2)$$

4.1.1. Problem 4.1.1 We choose analytical solution with parameters for $\psi(x)$

$$B = 2, \gamma_0 = 1, \gamma_1 = 0.85, \gamma_2 = 0.25$$

so that from (7)

$$\varsigma(x) = 2 (1 + 0.85 x_1 + 0.25 x_2) / \{3 [\cos (0.15 x_1 + 0.55 x_2) + \sin (0.15 x_1 + 0.55 x_2)]\}$$

Table 1 shows the results of the analytical and BEM solutions with 20, 40 and 80 elements of equal length. The BEM solution converges to the analytical solution as the number of elements increases.
Table 1. BEM and analytical solutions for Problem 4.1.1

| (x₁, x₂) | ζ | ∂ζ/∂x₁ | ∂ζ/∂x₂ | ζ | ∂ζ/∂x₁ | ∂ζ/∂x₂ |
|----------|---|---------|---------|---|---------|---------|
| (.1,.5)  | .6501 | .3993 | -.0567 | .6489 | .4012 | -.0569 |
| (.3,.5)  | .7286 | .3896 | -.0636 | .7282 | .3921 | -.0666 |
| (.5,.5)  | .8060 | .3850 | -.0707 | .8059 | .3854 | -.0736 |
| (.7,.5)  | .8823 | .3797 | -.0753 | .8824 | .3795 | -.0777 |
| (.9,.5)  | .9582 | .3835 | -.0795 | .9580 | .3773 | -.0798 |
| (1.1,.5) | .8655 | -.0026 | -.2967 | .8607 | .3935 | -.1999 |
| (.5,.3)  | .8262 | .3973 | -.1351 | .8266 | .3933 | -.1353 |
| (.5,.7)  | .7970 | .3861 | -.0221 | .7965 | .3822 | -.0211 |
| (.5,.9)  | .7999 | .6441 | .1031 | .7976 | .3931 | .0378 |

| BEM 80 elements | Analytical |
|----------------|------------|
| (.1,.5)  | .6486 | .4020 | -.0576 | .6483 | .4029 | -.0587 |
| (.3,.5)  | .7280 | .3931 | -.0679 | .7280 | .3935 | -.0692 |
| (.5,.5)  | .8059 | .3856 | -.0749 | .8059 | .3857 | -.0762 |
| (.7,.5)  | .8824 | .3794 | -.0789 | .8823 | .3792 | -.0800 |
| (.9,.5)  | .9578 | .3756 | -.0803 | .9576 | .3741 | -.0809 |
| (1.1,.5) | .8616 | .4044 | -.2137 | .8622 | .4060 | -.2159 |
| (.5,.3)  | .8269 | .3934 | -.1367 | .8271 | .3934 | -.1382 |
| (.5,.7)  | .7963 | .3825 | -.0222 | .7960 | .3826 | -.0233 |
| (.5,.9)  | .7967 | .3823 | .0262 | .7962 | .3840 | .0248 |

4.1.2. Problem 4.1.2 We choose analytical solution with parameters

\[ B = 4, \gamma_0 = 1, \gamma_1 = 0.85, \gamma_2 = 0.25 \]

so that

\[ \zeta(x) = 4(1 + 0.85x_1 + 0.25x_2) / \{3[\cos(0.15x_1 + 0.55x_2) + \sin(0.15x_1 + 0.55x_2)]\} \]

Table 2 shows the results of the analytical and BEM solutions with 20, 40 and 80 elements of equal length. The BEM solution converges to the analytical solution as the number of elements increases.
Table 2. BEM and analytical solutions for Problem 4.1.2

| $(x_1, x_2)$ | $\varsigma$ | $\partial \varsigma / \partial x_1$ | $\partial \varsigma / \partial x_2$ | $\varsigma$ | $\partial \varsigma / \partial x_1$ | $\partial \varsigma / \partial x_2$ |
|-------------|-------------|-------------------------------|-------------------------------|-------------|-------------------------------|-------------------------------|
| BEM 20 elements | BEM 40 elements |
| (.1,.5)     | 1.3003 | .7986 | -.1134 | 1.2978 | .8024 | -.1138 |
| (.3,.5)     | 1.4572 | .7793 | -.1272 | 1.4563 | .7843 | -.1331 |
| (.5,.5)     | 1.6119 | .7701 | -.1415 | 1.6118 | .7708 | -.1473 |
| (.7,.5)     | 1.7646 | .7595 | -.1505 | 1.7648 | .7591 | -.1555 |
| (.9,.5)     | 1.9164 | .7671 | -.1591 | 1.9159 | .7546 | -.1597 |
| (.5,.1)     | 1.7309 | -.0053 | -.5933 | 1.7213 | .7870 | -.3998 |
| (.5,.3)     | 1.6524 | .7946 | -.2702 | 1.6532 | .7866 | -.2705 |
| (.5,.7)     | 1.5939 | .7722 | -.0442 | 1.5931 | .7645 | -.0423 |
| (.5,.9)     | 1.5997 | 1.2882 | .2062 | 1.5952 | .7861 | .0756 |

| BEM 80 elements | Analytical |
|-------------|-------------|-------------------------------|-------------------------------|-------------|-------------------------------|-------------------------------|
| (.1,.5)     | 1.2971 | .8040 | -.1152 | 1.2967 | .8058 | -.1174 |
| (.3,.5)     | 1.4561 | .7861 | -.1358 | 1.4559 | .7871 | -.1384 |
| (.5,.5)     | 1.6118 | .7712 | -.1499 | 1.6117 | .7714 | -.1524 |
| (.7,.5)     | 1.7647 | .7589 | -.1578 | 1.7647 | .7585 | -.1600 |
| (.9,.5)     | 1.9156 | .7511 | -.1607 | 1.9153 | .7482 | -.1618 |
| (.5,.1)     | 1.7232 | .8087 | -.4273 | 1.7243 | .8120 | -.4318 |
| (.5,.3)     | 1.6537 | .7868 | -.2735 | 1.6542 | .7868 | -.2764 |
| (.5,.7)     | 1.5926 | .7650 | -.0444 | 1.5920 | .7652 | -.0467 |
| (.5,.9)     | 1.5934 | .7646 | .0525 | 1.5924 | .7681 | .0497 |

4.2. Examples without analytical solutions

In this section we will consider some examples of problems without simple analytical solutions. We setup some problems for a homogeneous isotropic material by taking $\lambda_{11} = \lambda_{22} = 1, \lambda_{12} = 0$ and with symmetrical boundary conditions. The aim is to see the consistency of the BEM of whether it produces symmetrical solutions.

4.2.1. Problem 4.2.1 For this problem we take $g(\mathbf{x}) = \{A [\cos(\alpha_i x_i) + \sin(\alpha_i x_i)]\}^2 = 1$ with $A = 1, \alpha_1 = 0, \alpha_2 = 0$ so that $\beta = \lambda_{ij} \alpha_i \alpha_j = 0$. And the symmetrical boundary conditions are as shown in Figure 3.

![Figure 3. The geometry of Problem 4.2.1](image-url)
Table 3 shows the results of the BEM solution using 40, 80, 160 and 320 elements of equal length. As expected, the results converge as the number of elements increases and also they are symmetrical about the axes $x_2 = 0.5$.

| $(x_1, x_2)$ | $\zeta$ | $\partial \zeta / \partial x_1$ | $\partial \zeta / \partial x_2$ | $\zeta$ | $\partial \zeta / \partial x_1$ | $\partial \zeta / \partial x_2$ |
|-------------|---------|-------------------------------|-------------------------------|---------|-------------------------------|-------------------------------|
|              | BEM 40 elements | BEM 80 elements | BEM 160 elements | BEM 320 elements |
| (.1,.5)     | .8924   | -.9973                        | .0000                         | .8967   | -.9987                        | .0000                         |
| (.3,.5)     | .6932   | -.9952                        | .0000                         | .6971   | -.9980                        | .0000                         |
| (.5,.5)     | .4942   | -.9943                        | .0000                         | .4975   | -.9977                        | .0000                         |
| (.7,.5)     | .2955   | -.9932                        | .0000                         | .2980   | -.9973                        | .0000                         |
| (.9,.5)     | .0970   | -.9919                        | .0000                         | .0986   | -.9968                        | .0000                         |
| (.5,.1)     | .4940   | -.9975                        | .0005                         | .4974   | -.9982                        | .0002                         |
| (.5,.3)     | .4942   | -.9945                        | .0007                         | .4975   | -.9978                        | .0002                         |
| (.5,.7)     | .4942   | -.9945                        | .0007                         | .4975   | -.9978                        | .0002                         |
| (.5,.9)     | .4940   | -.9975                        | .0005                         | .4974   | -.9982                        | .0002                         |

4.2.2. Problem 4.2.2 Now we consider a problem with $g(x) = 1$ again and the symmetrical boundary conditions are as shown in Figure 4. Table 4 shows the results of the BEM solution using 40, 80, 160 and 320 elements of equal length. The results converge as the number of elements increases and also they are symmetrical about the axes $x_1 = 0.5$.

![Figure 4. The geometry of Problem 4.2.2](image-url)
Table 4. BEM solution for Problem 4.2.2

| \((x_1, x_2)\) | \(\varsigma\) | \(\partial \varsigma / \partial x_1\) | \(\partial \varsigma / \partial x_2\) | \(\varsigma\) | \(\partial \varsigma / \partial x_1\) | \(\partial \varsigma / \partial x_2\) |
|-----------------|-------------|-----------------|-----------------|-------------|-----------------|-----------------|
| \((.1,.5)\)     | .5003      | .0031           | 1.0020          | .5001      | .0014           | 1.0001          |
| \((.3,.5)\)     | .5008      | .0016           | 1.0004          | .5004      | .0007           | 1.0003          |
| \((.5,.5)\)     | .5010      | -.0000          | 1.0005          | .5004      | .0000           | 1.0004          |
| \((.7,.5)\)     | .5008      | -.0016          | 1.0004          | .5004      | -.0007          | 1.0003          |
| \((.9,.5)\)     | .5003      | -.0031          | 1.0020          | .5001      | -.0014          | 1.0001          |
| \((.5,.1)\)     | .1012      | -.0000          | .9987           | .1005      | -.0000          | .9993           |
| \((.5,.3)\)     | .3010      | -.0000          | .9992           | .3004      | -.0000          | .9998           |
| \((.5,.7)\)     | .7012      | -.0000          | 1.0022          | .7006      | .0000           | 1.0012          |
| \((.5,.9)\)     | .9019      | -.0000          | 1.0037          | .9009      | -.0000          | 1.0019          |

| \((.1,.5)\)     | .5001      | .0006           | 1.0001          | .5000      | .0003           | 1.0000          |
| \((.3,.5)\)     | .5002      | .0004           | 1.0002          | .5001      | .0002           | 1.0001          |
| \((.5,.5)\)     | .5002      | .0000           | 1.0002          | .5001      | .0000           | 1.0001          |
| \((.7,.5)\)     | .5002      | -.0004          | 1.0002          | .5001      | -.0002          | 1.0001          |
| \((.9,.5)\)     | .5001      | -.0006          | 1.0001          | .5000      | -.0003          | 1.0000          |
| \((.5,.1)\)     | .1002      | .0000           | .9996           | .1001      | -.0000          | .9998           |
| \((.5,.3)\)     | .3002      | .0000           | .9999           | .3001      | -.0000          | .9999           |
| \((.5,.7)\)     | .7003      | .0000           | 1.0006          | .7001      | -.0000          | 1.0003          |
| \((.5,.9)\)     | .9005      | -.0000          | 1.0010          | .9002      | -.0000          | 1.0005          |

5. Conclusion

The scalar elliptic governing equation (2) is used for modeling physical problems such as steady infiltration problems (when \(\beta < 0\)), acoustic problems (when \(\beta > 0\)), and antiplane strain in elastostatics and plane thermostatic problems (when \(\beta = 0\). The boundary integral equation (16) was derived from this governing equation (2) and straight from (16) a BEM was then constructed for calculation of numerical solutions to the problems for anisotropic trigonometrically graded media. The results show the convergence, consistency, and accuracy of the BEM solutions. Together with its ease in implementation, it may be concluded that BEM is a good numerical method for solving such kind of problems.

Acknowledgements

The author would like to thank The Hasanuddin University and The Ministry of Research, Technology and Higher Education of the Republic of Indonesia for the provided support.

References

[1] Azis M I, Kasbawati, Haddade A and Thamrin S A 2018 On some examples of pollutant transport problems solved numerically using the boundary element method Journal of Physics: Conference Series 979(1)
[2] Azis M I, Asrul L, Khaeruddin and Paharuddin 2018 BEM solutions for unsteady transport problems in anisotropic media JP Journal of Heat and Mass Transfer 15(4) 915
[3] Haddade A, Salam N, Khaeruddin and Azis M I 2017 A boundary element method for 2D diffusion-convection problems in anisotropic media Far East Journal of Mathematical Sciences 102(8) 1593
[4] Azis M I 2019 Numerical solutions for the Helmholtz boundary value problems of anisotropic media Journal of Computational Physics 381 42
[5] Cheng A H-D 1984 Darcy’s Flow with Variable Permeability: A Boundary Integral Solution Water Resources Research 20 980
[6] Clements D L and Azis M I 2000 A Note on a Boundary Element Method for the Numerical Solution of Boundary Value Problems in Isotropic Inhomogeneous Elasticity Journal of the Chinese Institute of Engineers 23(3) 261
[7] Azis M I and Clements D L 2014 On some problems concerning deformations of functionally graded anisotropic elastic materials Far East Journal of Mathematical Sciences 87(2) 173
[8] Azis M I, Tosha S, Bahri M and Ilyas N 2018 A boundary element method with analytical integration for deformation of inhomogeneous elastic materials Journal of Physics: Conference Series 979(1) 012072
[9] Azis M I and Clements D L 2014 A Boundary Element Method for Transient Heat Conduction Problem of Nonhomogeneous Anisotropic Materials Far East Journal of Mathematical Sciences, 89(1) 51
[10] Azis M I and Clements D L 2008 Nonlinear transient heat conduction problems for a class of inhomogeneous anisotropic materials by BEM Engineering Analysis with Boundary Elements 32(12) 1054
[11] Salam N, Haddade A, Clements D L and Azis M I 2017 A boundary element method for a class of elliptic boundary value problems of functionally graded media Engineering Analysis with Boundary Elements 84(3) 186
[12] Azis M I 2019 Numerical solutions to a class of scalar elliptic BVPs for anisotropic exponentially graded media Journal of Physics: Conference Series 1218 012001
[13] Azis M I 2019 Standard-BEM solutions to two types of anisotropic-diffusion convection reaction equations with variable coefficients Engineering Analysis with Boundary Elements 105 87
[14] Clements D L and Lobo M 2010 A BEM for time dependent infiltration from an irrigation channel Engineering Analysis with Boundary Elements 34 1100
[15] Solekhudin I and Ang K-C 2012 A DRBEM with a predictor-corrector scheme for steady infiltration from periodic channels with root-water uptake Engineering Analysis with Boundary Elements 36 1199
[16] Barucq H, Bendali A, Fares M, Mattesi V and Tordeux S 2017 A symmetric Trefftz-DG formulation based on a local boundary element method for the solution of the Helmholtz equation Journal of Computational Physics 330 1069
[17] Loeffler C F, Mansur W J, Barcelos H D M and Bulcão A 2015 Solving Helmholtz problems with the boundary element method using direct radial basis function interpolation Engineering Analysis with Boundary Elements 61 218
[18] Azis M I 2017 Fundamental solutions to two types of 2D boundary value problems of anisotropic materials Far East Journal of Mathematical Sciences 101(11) 2405
[19] Abramowitz M and Stegun I A 1972 Handbook of mathematical functions: with formulas, graphs and mathematical tables, Dover Publications, Washington