Theoretical and Practical Limits of
Kolmogorov-Zurbenko Periodograms with DiRienzo-Zurbenko Algorithm Smoothing
in the Spectral Analysis of Time Series Data

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ABSTRACT

This investigation establishes the theoretical and practical limits of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing with respect to sensitivity (i.e., ability to detect weak signals), accuracy (i.e., ability to correctly identify signal frequencies), resolution (i.e., ability to separate signals with close frequencies), and robustness (i.e., sensitivity, accuracy, and resolution despite high levels of missing data). Compared to standard periodograms that utilize static smoothing with a fixed window width, Kolmogorov-Zurbenko periodograms with DiRienzo-Zurbenko algorithm smoothing utilize dynamic smoothing with a variable window width. This article begins with a summary of its statistical derivation and development followed by instructions for accessing and utilizing this approach within the R statistical program platform. Brief definitions, importance, statistical bases, theoretical and practical limits, and demonstrations are provided for its sensitivity, accuracy, resolution, and robustness. Next using a simulated time series in which two signals close in frequency are embedded in a significant level of random noise, the predictive power of this approach is compared to an autoregressive integral moving average (ARIMA), with support also garnered for its being robust even in the face of a high level of missing data. The article concludes with brief descriptions of studies across a range of scientific disciplines that have capitalized on the power of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing.

KEYWORDS

Time-series analysis, spectral analysis, Kolmogorov-Zurbenko periodogram, DiRienzo-Zurbenko algorithm smoothing, signal sensitivity, frequency accuracy, frequency resolution

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Theoretical and Practical Limits of Kolmogorov-Zurbenko Periodograms with DiRienzo-Zurbenko Algorithm Smoothing in the Spectral Analysis of Time Series Data

In sharp contrast to traditional periodograms that use static smoothing with a fixed window width, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing is a dynamic method for conducting spectral analysis of time series data that uses a variable window width. This investigation establishes its theoretical and practical limits for sensitivity, accuracy, resolution, and robustness with missing data and demonstrates its performance with respect to these limits using simulated datasets.

Simply put, a time series tracks the value of a given variable over time and time series analyses focus on how that variable changes with respect to time. Some time series analyses assume a linear model and its most common algorithm is the autoregressive integrated moving average (ARIMA). However, the linear model underlying these algorithms limits their predictive power when periodicity is present because the data are better described as signals that are sinusoidal in nature. For a full discussion, see the texts of Koopmans (1995), Montgomery, Jennings, and Kulahci (2015), and Wei (2006).

Time series analyses based on sinusoidal models have two aspects. The first is the temporal dimension and it examines regularity in patterns with which the given variable changes over time. To extract this pattern, these analyses use band pass filters that typically utilize a moving average with a fixed width (i.e., the number of observations averaged) to extract the true signal from an otherwise noisy background of random fluctuations in the variable (Zurbenko, 1986). However, a better filter is the Kolmogorov-Zurbenko adaptive filter (Yang & Zurbenko, 2010; Zurbenko, 1991). While this filter also utilizes a moving average, it does so by varying the width of the filter’s window such that it more accurately eliminates random noise and the actual variation in the variable over time (i.e., the signal) can be more clearly seen.

The second aspect of time series analyses based on sinusoidal models is the spectral dimension and it searches for signals present in a given time series. To find these signals, such analyses utilize periodograms that are smoothed using a moving average with a fixed window width to identify specific signal frequencies present across the range of all possible frequencies. However, a better periodogram is the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing (DiRienzo & Zurbenko, 1999). This periodogram, likewise, uses a moving average but with a window width that is dynamic and varies across increments of the frequency spectrum based on the proportion of total variance present such that it can precisely pinpoint the frequency of each signal present in the time series.

This article begins with a definition of time series as a stochastic process, followed by a brief exploration of the statistical foundations of a standard spectral analysis along with its subsequent refinement with the Kolmogorov-Zurbenko periodogram in concert with DiRienzo-Zurbenko algorithm smoothing. Instructions for accessing and utilizing these algorithms are also provided. To establish its sensitivity, accuracy, resolution, and robustness, we provide brief definitions, importance, statistical bases, theoretical and practical limits, and demonstrations for each. Next using a simulated time series dataset with relatively weak signals embedded in significant levels
of noise, the article presents a further demonstration of the sensitivity, accuracy, resolution, and robustness of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing, in addition to showcasing its predictive power in contrast to the ARIMA model. The article concludes with a brief section describing several substantive studies from across a range of scientific disciplines that have utilized the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing.
1. Development and Logic of the Kolmogorov-Zurbenko Periodogram with DiRienzo-Zurbenko Algorithm Smoothing

This section provides a brief summary of the mathematical development and underlying logic of spectral analysis via the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing as explicated in an article published by its creators (DiRienzo & Zurbenko, 1999). In approaching the problem of finding the spectral density and related smoothed periodogram, DiRienzo and Zurbenko began with a ‘worst case scenario’ by assuming a nonstationary time series process with a mixed spectrum. Defined in trigonometric form, it is:

\[ X_t = \sum_{j=1}^{I} \rho_j \cos(\omega_j t + \phi_j) + \varepsilon_t, \quad t = 0, \pm 1, \pm 2, \ldots, \] (1.1)

where \( \omega_j \) is a constant frequency, \( \rho_j \) is a constant amplitude, \( \phi_j \) is a uniform phase shift on the interval \(( -\pi, \pi )\), and \( \varepsilon_t \) is random noise that is stationary and bounded with continuous spectral density \( f_{\varepsilon}(\cdot) \). Consequently, the spectral density of \( X_t \), that is \( f_X(\cdot) \), is superimposed on the spectral density of \( \varepsilon_t \), that is \( f_{\varepsilon}(\cdot) \). However, the sample spectral density (i.e., periodogram) often has a considerable amount of variation, leading to a spectral estimate that is unstable and, thus, jagged, especially as sample size increases. To correct this, variation in the sample spectrum can be reduced by smoothing (Wei, 2006).

For an ordinary stationary process \( X_t \), the smoothed spectral density is a function of a Fourier transform of the time series dataset, a selected spectral window form, and a selected spectral window bandwidth. One such basic estimate of the spectral density and its corresponding smoothed periodogram is the Grenander-Rosenblatt spectral estimate (Grenander, 1984), which is given by:

\[ \hat{f}_N(\lambda_k) = \sum_{j=-N/2+1}^{N/2} \Phi_N(\lambda_j - \lambda_k) I_N(\lambda_j) \] (1.2)

with periodogram

\[ I_N(\lambda_k) = \frac{1}{2\pi N} |\sum_{t=0}^{N-1} X_t \exp(-i\lambda_k t)|^2 \] (1.3)

calculated at the Fourier frequencies \( \lambda_k = \frac{2\pi k}{N} \), where \( k = -N/2 + 1, ..., N/2, \) and with \( \Phi_N(\cdot) \) being a spectral window such that

\[ \Phi_N(x) = A_N G(A Nx) \left\{ \int_{-A N \pi}^{A N \pi} G(\tilde{x}) d\tilde{x} \right\}^{-1}, \quad -\pi \leq x \leq \pi; \quad 1 \ll A_N \ll N \] (1.4)

where \( G(\cdot) \) is the spectral window form and \( A_N \) is the inverse of the spectral window bandwidth. Spectral window forms include uniform, Bartlett, Gaussian, Parzen, and Tukey-Hamming (Priestley, 1980; Zurbenko, 1986) and the spectral window width is assumed to be a selected constant.

However, the spectral estimate \( \hat{f}_N(\lambda_k) \), can have a large bias, a large variance, or both; such a situation typically occurs when the underlying spectral density has a large amount of variability in its values or order of smoothness. Unfortunately, regular statistical estimations of spectra use
fixed spectral window smoothing across the spectrum and this use of a selected fixed spectral window frequently smooths out some of the spectral lines; thus, the analyst is “... forced to compromise between variance reduction and bias” (Wei, 2006, p. 302)

To address this shortcoming, DiRienzo and Zurbenko developed a dynamic approach. First, they recognized that because the mean square error of a periodogram includes both bias and variance, it can be used to assess the quality of a spectral estimate, thus serving as the criterion for selecting the best spectral window and, further, that the mean square error can be used to assess the local quality of a spectral estimate (i.e., the quality around a given frequency). They also noted that optimal spectral estimation requires that both the window form and the bandwidth vary with frequency, but that bandwidth has a substantially greater impact on the asymptotic mean square error than does the window form. Consequently, DiRienzo and Zurbenko’s dynamic approach ensures the optimal local quality of a spectrum estimate by focusing on selection of optimal local bandwidths.

To this end, they first proved two theorems. Their first theorem states that the asymptote of the local squared variation of \( f(\lambda) \) is similar to the asymptote of the inverse of the optimal bandwidth, \( A_N(\lambda) \), as defined in Equation (1.4). That is

\[
\frac{T_\Delta(f)}{C_{\lambda \Delta}} \approx N^{1/(1+2\alpha_\lambda)}(1 - \epsilon) \tag{1.5}
\]

where:
- \( T_\Delta(f) \) = largest possible local squared variation;
- \( N^{1/(1+2\alpha_\lambda)}(1 - \epsilon) \) = the optimal inverse bandwidth, \( A_N(\lambda) \); and
- \( C_{\lambda \Delta} \) = constant under Hölder’s condition.

Their second theorem states that the local squared variation of \( f(\lambda) \) within the optimal bandwidth, \( A_N^{-1}(\lambda) \), is virtually constant independent of sample size. That is:

\[
T_{\Delta N^{-1}}(f) = K_\Delta (1 - \epsilon) \tag{1.6}
\]

where:
- \( T_{\Delta N^{-1}}(f) \) = local squared variance for spectral window bandwidth \( A_N^{-1}(x) \);
- \( A_N^{-1}(\lambda) \) = the optimal spectral window bandwidth, with \( \Delta > A_N^{-1} > 0 \); and
- \( K_{\lambda} = \frac{C_\lambda^2 (2\pi)^{2\alpha_\lambda}}{C_{\lambda \Delta}^{opt} \pi} \), where \( 2\alpha_\lambda \) is the window width, \( C_\lambda \) is a spectral constant under Hölder’s condition, and \( C_{\lambda \Delta}^{opt} \) is the optimal spectral constant under Hölder’s condition.

Based on these two theorems, they created an algorithm which permits a dynamic (variable) bandwidth that depends on the local characteristics of the underlying spectral density.
The logic behind the DiRienzo-Zurbenko algorithm is as follows. For each frequency, $\lambda_k$, in Equation (1.2), the width of the spectral window is increased until the local squared variation of the periodogram, $I_N(\cdot)$, reaches a pre-selected constant value; this constant value is selected by the analyst and is a set proportion of the total variance in the periodogram (e.g., 0.05, 0.01), designated as the DiRienzo-Zurbenko proportion of smoothness. Consequently, the resulting smoothed periodogram is formed using a dynamic window – when a signal is present, local variance is high, the width of the window becomes narrow (i.e., zooms in), and the smoothed periodogram spikes up; conversely when no signal is present, local variance is low, the width of the window becomes wide (i.e., zooms out), and the smoothed periodogram becomes relatively flat. As a result, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing more clearly displays a spectral estimate that is sensitive to weak signals, accurate in identifying frequencies correctly, and can resolve signals that are close to one another.

However, time series analysts are frequently more comfortable framing spectral analyses in terms of information (i.e., entropy) by taking the log of a periodogram because it allows comparisons in terms of percentages and proportions rather than the magnitudes of variances. Using the log-periodogram to cast the spectral domain as information has a distinct advantage over the use of standard periodograms: comparisons across the log-periodogram are independent of scaling units because differences are in percentages (or proportions) rather than the magnitudes of variance used in standard periodograms. For this reason, Kolmogorov-Zurbenko periodograms are by default log-periodograms and, without loss of generality with respect to spectral estimates, DiRienzo-Zurbenko smoothing can also be used in conjunction with Kolmogorov-Zurbenko periodograms.

It is important to note an alternative algorithm that adaptively smooths log-periodograms has also been developed. Based on Kolmogorov’s proof that the error of prediction for a stationary sequence is constant (Kolmogorov, 1941, 1992), Kolmogorov’s algorithmic approach to quantifying information (Kolmogorov, 1965), and Wu’s casting the spectral domain as information (Wu, 1997), Zurbenko (2001) developed an alternate algorithm that adaptively smooths log periodograms in a different way: local changes in the log-periodogram are done in terms of linear rates of change which, in turn, can enhance the sensitivity of the periodogram in its ability to detect weak signals.

Zurbenko’s alternative approach was subsequently refined by Neagu and Zurbenko to form the Neagu-Zurbenko algorithm (Neagu & Zurbenko, 2003). Like the DiRienzo-Zurbenko algorithm, it adaptively smooths the log-periodogram using a dynamic window, but this alternate approach first computes a linear approximation to the local information in the process, then uses the dynamic window to minimize the dissimilarity between the estimate and the log of the observed spectrum.

As a result, the Kolmogorov-Zurbenko periodogram with Neagu-Zurbenko smoothing utilizes a dynamic window that expands and contracts based on a set proportion of the total departure from linearity found in the given spectrum, with that total departure from linearity being a fixed value. Thus, it ‘zooms in’ when local information spikes relative to the linear approximation, indicating the presence of a signal, and it ‘zooms out’ when local information is low and commensurate
with the linear estimate, indicating that no signal is present. A preliminary comparison of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing to one using Neagu-Zurbenko algorithm smoothing can be found in Neagu and Zurbenko (2003) and an in-depth comparison will be conducted in subsequent work.
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2. Access and Utilization of Software
While DiRienzo and Zurbenko have fully described the motivation, development, and basic principles underlying their algorithm, there is little in the way of how one can access and utilize it. This section details where the DiRienzo-Zurbenko algorithm can be found and how to use it with a given dataset.

2.1 Access
The DiRienzo-Zurbenko algorithm is available in the R platform for data analysis and is an option in the \texttt{kzp()} function within the \texttt{kza} package at https://CRAN.R-project.org/package=kza (Close et al., 2020). Because errors can occur when downloading the \texttt{kza} package using R’s \texttt{install.packages()} command, the analyst should download the \texttt{kza} Package Source folder using the \texttt{kza_4.1.0.tar.gz} link; within the \texttt{kza} folder is an R sub-folder that includes three R script files: \texttt{rlv.R}, \texttt{kza.R}, and \texttt{kzft.R}. The \texttt{kzp()} function is in the \texttt{kzft} script file. However, before conducting a time series analysis, the analyst should delete an incorrect version of the \texttt{plot.kzp()} function in the \texttt{kza.R} script, thus retaining the correct version of the \texttt{plot.kzp()} function placed in the \texttt{kzft.R} script. Next, the analyst should run the \texttt{kza.R} script, the \texttt{kzft.R} script, and the \texttt{rlv.R} script, in that order, to ensure any dependencies among and between scripts are addressed. To summarize:

Go to: https://CRAN.R-project.org/package=kza
Download the \texttt{kza} folder by clicking the \texttt{kza_4.1.0.tar.gz} link
Open the \texttt{kza} folder
Open the R sub-folder
Delete the incorrect version of the \texttt{plot.kzp()} function in the \texttt{kza.R} script
Run the edited \texttt{kza.R} script file, the \texttt{kzft.R} script file, and the \texttt{rlv.R} script file, in that order.

As discussed below, the DiRienzo-Zurbenko algorithm is an option in the \texttt{kzp()} function found in the \texttt{kzft.R} script file. Thus, the \texttt{kzp()} function is used to construct the Kolmogorov-Zurbenko log periodogram for a given time series dataset and results in a spectral density estimate using the Kolmogorov-Zurbenko Fourier transform (KZFT). In turn, the DiRienzo-Zurbenko algorithm is an optional method that is accessed when running the \texttt{kzp()} function.

2.2 Utilization
To construct a Kolmogorov-Zurbenko periodogram utilizing DiRienzo-Zurbenko algorithm smoothing for a given time series dataset, a researcher must specify the raw time series dataset to be analyzed, the initial width of the filtering window, the number of iterations of the KZFT, the smoothness level of the periodogram, the DiRienzo-Zurbenko algorithm as the method for smoothing, and the number of top signal frequencies and periods to be identified. With respect to the number of iterations, the first iteration applies the KZFT to the time series dataset, the second iteration applies the KZFT to the result of the first iteration, and so on, such that the transfer function for the selected number of iterations is a product of the selected number of iterations of the KZFT transfer function. With a starting frequency of 0, one iteration results in a standard periodogram while two or more iterations approach a Gaussian distribution, thereby eliminating false low-level frequency spikes.
In its basic form, generating a Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing utilizes four functions:

\[
\text{totalSignal} \leftarrow \text{kzp}(y, m, k) \\
\text{kzPeriodogram} \leftarrow \text{smooth.kzp}(\text{totalSignal}, \log=\text{TRUE}, \text{smooth.level}, \text{method}) \\
\text{kzpPlot} \leftarrow \text{plot.kzp}(\text{kzPeriodogram}, \text{type} = \text{"1"}) \\
\text{kzpSummary} \leftarrow \text{summary.kzp}(\text{kzPeriodogram}, \text{digits}, \text{top})
\]

where the respective arguments are:

- **y**: raw time series dataset (a time ordered vector of values)
- **m**: initial window width
- **k**: number of iterations of KZFT
- **smooth_level**: proportion of total variance (for method = “DZ”) or proportion of total departure from linearity (for method = “NZ”) (a.k.a proportion of smoothness)
- **method**: method for smoothing:
  - “DZ” for DiRienzo-Zurbenko, “NZ” for Neagu-Zurbenko
- **digits**: number of significant digits
- **top**: number of top frequencies and periods to be reported

As a result, the \text{kzp()} function uses the Kolmogorov-Zurbenko Fourier transform to generate the raw periodogram; \text{smooth.kzp()} returns the smoothed periodogram ordinates and the smoothing method; \text{plot.kzp()} returns a plot of the smoothed Kolmogorov-Zurbenko periodogram that is mean-centered (i.e., the mean of all ordinate values); and \text{summary.kzp()} returns the top frequencies and top periods of interest.

In this basic form, the \text{kzp()} function estimates the spectral density across the range of frequencies through use of a moving average. The initial width of the window used to compute the moving average is selected by the researcher (i.e., m=m). When the \text{smooth.kzp()} function is invoked, the DiRienzo-Zurbenko algorithm adapts the width in such a way as to ‘zoom in’ where values of the spectral density spike up and ‘zoom out’ where values of the spectral density are relatively flat. It does this by utilizing the preselected smoothness level; that is, the selected smoothness level is a proportion of total variance in the periodogram for the DiRienzo-Zurbenko algorithm. With the smoothness level set, the width of the window used in computing moving averages across the frequency spectrum varies so that the proportion of variance contained in the window remains constant (i.e., the set proportion); thus, the window shrinks in size when the spectral density spikes up (i.e., local variance is relatively high) and expands in size when spectral density is relatively flat (i.e., local variance is relatively low). In turn, the \text{plot.kzp()} function generates a plot of the smooth Kolmogorov-Zurbenko periodogram and the \text{summary.kzp()} function returns the top frequencies and periods in the time series dataset.

Finally, for time series analysts who prefer to conduct spectral analyses in terms of linearity rather than in terms of variance, the Neagu-Zurbenko algorithm is also included as a method of smoothing in the \text{kzp} program found within the \text{kza} package in R (i.e., set \text{method} = “NZ”). Consequently, the \text{kzp} program adapts to the spectral analyst, depending on his or her preference for working in variances or in linearity.
3. Theoretical and Practical Limits

As noted above, the purpose of this study is to establish the theoretical and practical limits of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing with respect to sensitivity, accuracy, resolution, and robustness, and to compare its predictive power to that of a standard ARIMA model. Theoretical and practical limits are established in this Section, while its predictive power is presented in Section 4.

Establishing limits of Kolmogorov-Zurbenko periodograms with DiRienzo-Zurbenko algorithm smoothing utilized four criteria: (1) sensitivity, (2) accuracy, (3) resolution, and (4) robustness with respect to missing data. For each criterion, this section provides a brief definition, importance, statistical basis, theoretical limits where available, practical limits where applicable, and a demonstration the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing performance through analyses across a series of simulated datasets.

All demonstrations used signals at Fourier frequencies consistent with selected algorithm settings. Initial window widths were set to \( m = 500 \) and the number of iterations of the KZFT were set to \( k = 3 \). For each criterion, two different numbers of observations were examined, \( N = 5000 \) and \( N = 1000 \), and two different levels for the DiRienzo-Zurbenko proportion of smoothing were utilized, \( DZ = 0.05 \) and \( DZ = 0.01 \); taken in combination, this resulted in four possible scenarios for each of the four criteria (i.e., sixteen series of analyses).

3.1 Sensitivity

Definition and Importance. The first criterion in assessing the performance of a given periodogram algorithm is sensitivity. In spectral analysis, sensitivity is the ability to detect a signal embedded in random noise. Thus, a smoothing algorithm is assessed with respect to the signal-to-noise ratio present in a given time series dataset. As such, the analyst must know how strong a given signal must be relative to the strength of the surrounding noise in order for it to be detected. To this end, a good smoothing algorithm will be able to detect a weak signal in the presence of high levels of noise. Without this quality, an algorithm will be virtually useless with all but the strongest of signals relative to noise.

Statistical Basis and Limits. With regard to the theoretical limit for sensitivity using the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing, the signal-to-noise ratio must be greater than or equal to the proportion of DiRienzo-Zurbenko smoothing set by the analyst. The total variance of a Kolmogorov-Zurbenko periodogram consists of variance due to one or more signals, \( s^2 \), and variance due to noise, \( n^2 \). In order for a signal to be detected using the DiRienzo-Zurbenko algorithm, the variance of a signal must be greater than or equal to the variance due to noise in the locale of that signal frequency. When using the DiRienzo-Zurbenko algorithm, the amount of total variance in the locale of a signal frequency is restricted to the proportion of total variance, also known as the proportion of smoothing, \( DZ \), as set by the analyst. Thus, the variance of a signal at a given frequency must be greater than or equal to the proportion of variance of noise in the locale of that frequency. That is:

\[
s^2 \geq DZ \cdot n^2
\]  

where:
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\[ s^2 = \text{variance of signal}; \]
\[ n^2 = \text{total variance of noise}; \text{ and} \]
\[ DZ = \text{DiRienzo-Zurbenko proportion of smoothness} \]

Indeed, stated differently, this becomes:

\[ \frac{s^2}{n^2} \geq DZ \]  \hspace{1cm} (3.1.2)

Because \( \frac{s^2}{n^2} \) is none other than the signal-to-noise ratio, the limit for detecting a signal is a signal-to-noise ratio equal to the DiRienzo-Zurbenko proportion of smoothness. Typical values for the DiRienzo-Zurbenko proportion of smoothness are 0.05 and 0.01. Thus, the theoretical limit of the signal-to-noise ratio for detecting a signal is quite small. Consequently, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing has the potential to detect a very weak signal in the midst of high levels of noise.

**Demonstration.** To demonstrate sensitivity, the amplitude of noise was set at 16 and the signal-to-noise ratio was made to vary by adjusting the amplitude of the respective signal across the four scenarios. For scenarios in which the DiRienzo-Zurbenko proportion of smoothness was set at 0.05, the signal-to-noise ratio ranged from 0.057 to 0.050 in increments of 0.001. Likewise, for scenarios in which the DiRienzo-Zurbenko proportion of smoothness was set at 0.01, the signal-to-noise ratio ranged from 0.017 to 0.010, in increments of 0.001. The ranges of signal-to-noise ratio for the two levels of smoothness were different because the theoretical limit of sensitivity for a given scenario is determined by the respective level of the DiRienzo-Zurbenko proportion of smoothness.

In scenarios where the number of observations was set at \( N = 5000 \), the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing reached the respective theoretical limits in terms of signal-to-noise ratio, detecting the signal frequency \( \lambda = 0.040 \), regardless of proportion of smoothness. When the number of observations was set to \( N = 1000 \) and proportion of smoothness set at \( DZ = 0.05 \), a signal frequency of 0.044, rather than 0.040, was detected across its entire range of signal-to-noise ratios. However, with \( N = 1000 \) and proportion of smoothness set at \( DZ = 0.01 \), the theoretical limit with respect to signal-to-noise ratio was approached: the signal frequency of 0.040 was detected up to a signal-to-noise ratio of 0.013, but the signal was lost when moving from a signal-to-noise ratio of 0.013 to one of 0.012. Thus, it appears the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing using relatively large proportions of smoothness (e.g., 0.05) becomes less sensitive as number of observations decreases and the sensitivity of Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing using relatively small proportions of smoothness (e.g., 0.01) approaches but does not achieve its theoretical limit as the number of observations decreases (see Table 3.1.1.). Given these findings, subsequent demonstrations for accuracy, resolution, and robustness set the respective signal-to-noise ratio at 0.10. While this value is above the theoretical limit for both \( DZ = 0.05 \) and \( DZ = 0.01 \), it is well below the accepted lower bound of 9 using the Rose criterion (Rose, 1973).
Table 3.1.1. Observed Top Frequencies in Kolmogorov-Zurbenko Periodograms using DiRienzo-Zurbenko Algorithm Smoothing across Varying Signal-To-Noise (S/N) Ratios for Signal Frequency = 0.040 and Noise Amplitude = 16, with Number of Observations $N = 5000$ or $N = 1000$.

NB. Initial Window Width $m = 500$, Number of Iterations $k = 3$, and Proportions of Smoothness $DZ = 0.05$ or $DZ = 0.01$. 

| S/N Ratio | Observed Top Frequency | S/N Ratio | Observed Top Frequency | S/N Ratio | Observed Top Frequency | S/N Ratio | Observed Top Frequency |
|-----------|------------------------|-----------|------------------------|-----------|------------------------|-----------|------------------------|
| 0.057     | 0.040                  | 0.017     | 0.040                  | 0.057     | 0.044                  | 0.017     | 0.040                  |
| 0.056     | 0.040                  | 0.016     | 0.040                  | 0.056     | 0.044                  | 0.016     | 0.040                  |
| 0.055     | 0.040                  | 0.015     | 0.040                  | 0.055     | 0.044                  | 0.015     | 0.040                  |
| 0.054     | 0.040                  | 0.014     | 0.040                  | 0.054     | 0.044                  | 0.014     | 0.040                  |
| 0.053     | 0.040                  | 0.013     | 0.040                  | 0.053     | 0.044                  | 0.013     | 0.040                  |
| 0.052     | 0.040                  | 0.012     | 0.040                  | 0.052     | 0.044                  | 0.012     | 0.160                  |
| 0.051     | 0.040                  | 0.011     | 0.040                  | 0.051     | 0.044                  | 0.011     | 0.160                  |
| 0.050     | 0.040                  | 0.010     | 0.040                  | 0.050     | 0.044                  | 0.010     | 0.160                  |
3.2 Accuracy

Definition and Importance. The second criterion in assessing the performance of a given periodogram algorithm is accuracy. In spectral analysis, accuracy is the ability to precisely identify the frequency of a given signal. Combined with sensitivity, accuracy ensures a periodogram can pinpoint the exact frequency of a weak signal embedded in high levels of noise. Without accuracy, predicted values of the variable will quickly diverge from the true values as one moves forward in time; conversely, with high levels of accuracy, predicted values of the variable will remain close to the true values into perpetuity for all intents and purposes.

Statistical Basis and Limits. The theoretical limit for accuracy using the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing is $1/N$. The Kolmogorov-Zurbenko periodogram is determined by using the Kolmogorov-Zurbenko Fourier transform (KZFT). The KZFT works by reconstructing the original signal in a raw time series dataset using a weighted sum of all possible empirical sinusoidal waves that could be present in the dataset. Because there are $N$ observations in a given dataset, there are $N-1$ possible sinusoidal waves, with any two consecutive waves separated by $1/N$. Because deviation of a signal by more than $1/2N$ above or below that signal frequency would be attributed to the next possible sinusoidal wave for the given number of observations, the theoretical limit of accuracy for the frequency of any given signal is $1/N$.

However, the DiRienzo-Zurbenko algorithm identifies signal frequencies by examining the local quality of a spectral estimate by using a dynamic window with an initial width of $m$. Thus, the practical limit of accuracy for a Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing is $1/m$. It should be noted that the maximum initial window width is $N$, so both the theoretical limit and the smallest possible practical limit of accuracy is $1/N$.

Demonstration. To demonstrate accuracy, the amplitude of noise was set at 16 and the signal-to-noise ratio was set at 0.10, resulting in a signal amplitude of 5.06. Actual signal frequencies were set at 0.400, 0.440, or 0.444, in an effort to demonstrate accuracy in tenths, hundredths, and thousands. As with demonstration of sensitivity, two number of observations were considered, $N = 5000$ and $N = 1000$, as were two levels of proportion of smoothing, $DZ = 0.05$ and $DZ = 0.01$. The initial window width was set at $m = 500$, with a practical limit for accuracy of 0.002.

Signal frequency estimates were accurate to the thousands across all four scenarios: $N = 5000, DZ = 0.05$; $N = 5000, DZ = 0.01$; $N = 1000, DZ = 0.05$; and $N = 1000, DZ = 0.01$. Thus, accuracy was maintained regardless of whether the number of observations was relatively large or small and regardless of whether the proportion of smoothness was relatively high or low. (See Table 3.2.1.)
### ACCURACY: OBSERVED TOP SIGNAL FREQUENCY

Amplitude of Signal = 5.06
Amplitude of Noise = 16
S/N = 0.10

| Actual Signal Frequency | $N = 5000$ | $N = 1000$ |
|-------------------------|------------|------------|
| $DZ = 0.05$             | $DZ = 0.01$| $DZ = 0.05$| $DZ = 0.01$|
| 0.400                   | 0.400      | 0.400      | 0.400       |
| 0.440                   | 0.440      | 0.440      | 0.440       |
| 0.444                   | 0.444      | 0.444      | 0.444       |

Table 3.2.1. Observed Top Frequencies in Kolmogorov-Zurbenko Periodograms using DiRienzo-Zurbenko Algorithm Smoothing across Varying Actual Signal Frequencies, with Signal-to-Noise Ratio = 0.10 and Number of Observations $N = 5000$ or $N = 1000$. NB. Initial Window Width $m = 500$, Number of Iterations $k = 3$, and Proportion of Smoothness $DZ = 0.05$ and $DZ = 0.01$. 
### 3.3 Resolution

*Definition and Importance.* The third criterion in assessing the performance of a given periodogram algorithm is resolution. In spectral analysis, *resolution* is the ability to sensitively detect and accurately identify the frequencies of two signals that are close together. In this situation, it is important to know how close two frequencies can be before a smoothing algorithm is no longer able to separate them. Without the ability to resolve close signals, a periodogram will merge the signals into a single peak, making accurate prediction of future values of a variable all but impossible. Thus, despite having a common starting point, two signals with two different frequencies will have a much different effect on the predicted value of a variable over time compared to one signal of a single frequency. Consequently, a smoothing algorithm must be able to not only identify each signal with accuracy but must also have the ability to resolve two such signals when they are close together.

*Statistical Basis and Limits.* The theoretical limit of resolution for the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing is $2/N$. As stated above, the Kolmogorov-Zurbenko periodogram is determined by using the KZFT which reconstructs an original raw dataset using a weighted sum of all possible empirical sinusoidal waves that could be present in a time series dataset. Given $N$ observations in such a dataset, there are $N-1$ possible sinusoidal waves and any two consecutive sinusoidal waves have a separation of $1/N$. Clearly, the spike in spectral density of two consecutive empirical frequencies would overlap and could not be distinguished from one another. However, if the distance between two close frequencies is $2 \cdot (1/N)$ or $2/N$, then resolution of the frequencies becomes possible. Thus, the theoretical limit for resolution of two signals that are close to one another is a separation in frequency of $2/N$.

As with accuracy, the DiRienzo-Zurbenko algorithm identifies signal frequencies by examining the local quality of the spectral estimate by using dynamic windows with an initial window width of $m$. Thus, the practical limit of resolution for Kolmogorov-Zurbenko periodograms with DiRienzo-Zurbenko algorithm smoothing is $2/m$. Again, similar to accuracy, it should be noted that the maximum initial window width is $N$, so both the theoretical limit and the smallest possible practical limit of resolution is $2/N$.

*Demonstration.* To demonstrate resolution, the amplitude of noise was set at 16 and the signal-to-noise ratio was set at 0.10, resulting in signal amplitudes of 3.58 for each of two signals. The initial window width was set at $m=500$, with a practical limit for accuracy of 0.002 and a practical limit for resolution of 0.004. Because this was a demonstration of how well the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing could resolve two frequencies that are close to one another, one frequency was fixed at $\lambda_1 = 0.040$, while a second signal frequency increased so as to approach the first signal frequency up to the practical limit for resolution of 0.004, with $\lambda_2 = 0.030, 0.032, 0.034, \text{ and } 0.036$. Again, and as with the demonstration of sensitivity and accuracy, the number of observations were $N = 5000$ and $N = 1000$ and DiRienzo-Zurbenko algorithm proportions of smoothness were $DZ = 0.05$ and $DZ = 0.01$.

With regard to resolution, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing successfully resolved signal pairs across scenarios with $N = 5000$. 



regardless of whether the proportion of smoothness was $DZ = 0.05$ or $DZ = 0.01$. In these scenarios, top observed frequencies were within the practical limit for accuracy (0.002) and, thus, the separate frequencies were accurate and clearly resolved. When the number of observations was reduced to $N = 1000$ with $DZ = 0.05$, performance of the DiRienzo-Zurbenko algorithm was poor: top signal frequencies were within 0.002 of each other and, thus, Fourier signal frequencies were consecutive and could not be resolved. However, when the number of observations was $N = 1000$ and the proportion of smoothness was reduced to $DZ = 0.01$, signal frequencies were within the practical limit for accuracy (0.002) and, thus, the separate frequencies were accurate and clearly resolved. Thus, it appears that the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing using relatively high proportions of smoothness loses its ability to resolve signal frequencies as the number of observations decreases but maintains its ability to do so when using relatively low proportions of smoothness. (See Table 3.3.1.)
### RESOLUTION: OBSERVED TOP SIGNAL FREQUENCIES

Amplitude of Signals = 3.58  
Amplitude of Noise = 16 
S/N = 0.10

| Actual Signal Frequencies | $N = 5000$ | $N = 1000$ |
|---------------------------|------------|------------|
|                           | $DZ = 0.05$ | $DZ = 0.01$ | $DZ = 0.05$ | $DZ = 0.01$ |
| $\lambda_1$              | 0.040      | 0.040      | 0.040       | 0.036       |
| $\lambda_2$              | 0.030      | 0.030      | 0.030       | 0.036       |
| $\tilde{\lambda}_1$      | 0.040      | 0.040      | 0.040       | 0.038       |
| $\tilde{\lambda}_2$      | 0.030      | 0.030      | 0.030       | 0.036       |
| $\tilde{\lambda}_1$      | 0.040      | 0.040      | 0.040       | 0.040       |
| $\tilde{\lambda}_2$      | 0.036      | 0.036      | 0.036       | 0.040       |

Table 3.3.1. Observed Top Frequencies in Kolmogorov-Zurbenko Periodograms using DiRienzo-Zurbenko Algorithm Smoothing across Varying Actual Signal Frequencies, with Signal-to-Noise Ratio = 0.10 and Number of Observations $N = 5000$ or $N = 1000$.  
NB. Initial Window Width $m = 500$, Number of Iterations $k = 3$, and Proportion of Smoothness $DZ = 0.05$ or $DZ = 0.01$
3.4 Robustness with Respect to Missing Data

**Definition and Importance.** The fourth criterion used to assess periodogram performance is robustness with respect to missing data. Here, a periodogram algorithm can be viewed as robust with respect to missingness if it can detect, identify, and separate signal frequencies with a high percentage of missing data. This criterion is important because time series analysts frequently encounter situations where data is missing (e.g., instrument malfunction, human error in data collection).

**Statistical Basis and Limits.** With regard to missingness, there is no statistical basis to establish a theoretical limit for the percent missing. However, it is possible to demonstrate a periodogram algorithm’s performance under increasing percentages of missingness with respect to its theoretical and practical limits of sensitivity, accuracy, and resolution. In other words, one must discern how high a level of missingness is tolerable while still approaching established practical limits for sensitivity, accuracy, and resolution.

**Demonstration.** To demonstrate the robustness of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing, an analysis was done on a simulated time series dataset in which signal frequencies were set at $\lambda_1 = 0.040$ and $\lambda_2 = 0.036$, amplitude of noise set at 16, and signal-to-noise ratio set at 0.10, resulting in amplitudes of 3.58 for both signals. As with demonstrations of sensitivity, accuracy, and resolution, two numbers of observations were considered: $N = 5000$ and $N = 1000$; likewise, DiRienzo-Zurbenko algorithm proportions of smoothness were $DZ = 0.05$ and $DZ = 0.01$.

Once a simulated dataset was generated for a given scenario, missingness was introduced utilizing random draws from a binomial distribution (i.e., 0 = not missing, 1 = missing) such that data were missing completely at random and the percentage missing ranged from 0 percent to 70 percent, in increments of 10 percent. Setting window width at $m = 500$ and the number of iterations at $k = 3$ in each scenario and at each percentage of missingness, it was possible to assess sensitivity at a signal-to-noise ratio of 0.10, accuracy at the practical limit of 0.002, and resolution at the practical limit of 0.004, by determining whether or not the top identified frequencies were the same as the true frequencies of 0.040 and 0.036.

With respect to robustness in the context of missing data, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing demonstrated respectable performance. When the number of observations was $N = 5000$, both signals were detected (sensitivity) and observed frequencies were exact (accuracy) and separated (resolution) with up to 50 percent missing with $DZ = 0.05$ and up to 70% missing with $DZ = 0.01$. When the number of observations was $N = 1000$ and the proportion of smoothness was set at $DZ = 0.05$, performance of the DiRienzo-Zurbenko algorithm was poor: with up to 60 percent missing, Fourier frequencies were consecutive, so signals could not be resolved, and frequencies were inaccurate with 70 percent missing. However, when the number of observations remained at $N = 1000$ but the proportion of smoothness decreased to $DZ = 0.01$, signals were again detected (sensitivity) and observed frequencies were again exact (accuracy) and separated (resolution) with up to 50 percent of the data missing. Given the decrement in performance observed when the proportion of smoothness of $DZ = 0.05$ moved from $N = 5000$ to $N = 1000$ and the continued sensitivity, accuracy, and resolution observed when the proportion of smoothness of $DZ = 0.01$ moved from $N = 5000$ to N
= 1000, it appears robustness for higher proportions of smoothness wanes as the number of observations decreases, but this is corrected using lower proportions of smoothness. With this caveat in mind, the overall findings are quite good given the relatively low signal-to-noise ratio used across the respective scenarios.
### ROBUSTNESS: OBSERVED TOP FREQUENCIES

Signal Frequencies: $\lambda_1 = 0.040$ and $\lambda_2 = 0.036$
Amplitude of Signals = 3.58
Amplitude of Noise = 16
S/N = 0.10

| Percent of Missing Data | $N = 5000$ | $N = 1000$ |
|-------------------------|------------|------------|
|                         | $DZ = 0.05$ | $DZ = 0.01$ | $DZ = 0.05$ | $DZ = 0.01$ |
| $\hat{\lambda}_1$     | 0.040      | 0.040      | 0.038      | 0.038      |
| $\hat{\lambda}_2$     | 0.036      | 0.036      | 0.040      | 0.040      |

Table 3.4.1. Observed Top Frequencies in the Kolmogorov-Zurbenko Periodogram using DiRienzo-Zurbenko Algorithm Smoothing across Varying Percentages of Missing Data, with Signal-to-Noise Ratio = 0.10 and Number of Observations $N = 5000$ or $N = 1000$.

NB. Initial Window Width $m = 500$, Number of Iterations $k = 3$, and Proportion of Smoothness $DZ = 0.05$ or $DZ = 0.01$. 
4. Simulated Example
The Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing used in conjunction with the Kolmogorov-Zurbenko Fourier transform is a state-of-the-art method for time series analysis. To further demonstrate its sensitivity, accuracy, and resolution as well as its predictive power, a simulated time series dataset was constructed with known frequencies and a known signal-to-noise ratio. The Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing was used to identify signal frequencies and the Kolmogorov-Zurbenko Fourier transform was used to reconstruct the signal. The predictive power of the reconstructed signal was assessed by computing the proportion of variance in the original signal accounted for by the reconstructed signal; this result, in turn was compared to the predictive power of an autoregression model, another standard method used by time series analysts. The example concludes with a re-analysis of the timeseries dataset with 50% of the data missing, again demonstrating the sensitivity, accuracy, and resolution as well as its predictive power with the reconstructed signal.

4.1 Construction of the Time Series Dataset
A time series dataset of variable $X$ at time $t$ was constructed as:

$$X_t = a_1 \cdot \sin(2\pi \lambda_1 t) + a_2 \cdot \sin(2\pi \lambda_2 t) + N(\mu, \sigma^2) \quad (4.1.1)$$

setting frequency 1 at $\lambda_1 = 0.084$, with amplitude $a_1 = 1$, frequency 2 at $\lambda_2 = 0.098$, with amplitude $a_2 = 1.5$, and random noise at $N(\mu, \sigma^2) = N(0, 16)$. Thus, the signal-to-noise ratio was 0.203, frequencies were to the thousands place, and frequency separation was 0.014. In generating the dataset, time went from $t = 1$ to $t = 5000$, thus producing 5000 observed values of $X$. An interval of this time series, from $t = 30$ to $t = 80$, is displayed in Figure 4.1.1. Here, one can see its rather chaotic nature, with no apparent pattern visible, because two regular signals close in frequency and in amplitude were combined with a high level of random noise.

4.2 Identification of Signals
A standard periodogram algorithm (i.e., R command `periodogram()` ) is also available in the `kzft.R` script file and was used to identify signals in the time series dataset. The result of this algorithm is displayed in Figure 4.2.1(a). One can see that there are two prominent signal spikes, one at $\lambda_1 = 0.084$ and the other at $\lambda_2 = 0.098$. However, there are a multitude of minor spikes and it is not clear whether these are actual signals or random fluctuations in the periodogram. To gain a clearer notion of what signals are actually present, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing, with $DZ = 0.01$, $m = 500$, and $k = 3$, was constructed and is displayed in Figure 4.2.1(b). One can see that the multitude of minor spikes have dissolved and those remaining are all but muted.

To conclude, the standard periodogram is somewhat ambiguous with respect to the presence of weaker signals, but the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing demonstrates its sensitivity, accuracy, and resolution, clearly identifying $\lambda_1 = 0.084$ and $\lambda_2 = 0.098$ with less ambiguity in regard to other possible signals. Given the obvious strength of the two identified signals, it appeared possible to reconstruct of the original signal.
Figure 4.1. Time Series Plot between $t=30$ and $t=80$ for Complete Data where:

$$X(t) = 1 \cdot \sin(2\pi(0.084)t) + 1.5 \cdot \sin(2\pi(0.098)t) + N(0,16).$$
Figure 4.2.1. Spectral Analyses using Complete Data
(a) Standard Periodogram
(b) The Kolmogorov-Zurbenko Periodogram with DiRienzo-Zurbenko Smoothing
4.3 Reconstruction of Original Signal
In reconstructing the original time series signal, the goal is to correctly predict the values of variable $X$ over time, using the identified signals from the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing. To reconstruct the signal, the Kolmogorov-Zurbenko Fourier transform was used to construct two signals, one with $\lambda_1 = 0.084$ and the other with $\lambda_2 = 0.098$. These two signals were added together, then the real-valued component extracted and multiplied by 2 to form the reconstructed signal. It must be remembered that the Kolmogorov-Zurbenko Fourier transform generates complex numbers; thus, the real part must be extracted and multiplied by 2 to obtain the estimated value of $X$ for each value of $t$.

For the time series interval running from $t = 30$ to $t = 80$, Figure 4.3.1 displays the original data (i.e., signal plus noise), the original signal (i.e., signal only), and the reconstructed signal. Despite the fact that the original signal cannot be easily identified in the plot of the original data (signal plus noise), it can be seen that the reconstructed signal runs a very close approximation to the original signal (signal only). Further, the predictive power of the reconstructed signal is strong. Across the interval from $t = 4$ to $t = 5000$, the correlation between the original signal and the reconstructed signal is $r = 0.977$, thus accounting for 95.4% of the variance in the original signal. It should be noted that the first three observations are lost due to averaging within the algorithm.

4.4 Comparison to Autoregression
Autoregression models are used extensively in time series analysis. However, the approach using the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing followed with signal reconstruction using the Kolmogorov-Zurbenko Fourier transform is superior to autoregression modeling in relation to prediction with a sinusoidal time series dataset. To demonstrate this, autoregression analysis was done on the time series dataset. The standard autoregression algorithm in R is `ar()` and can be found in the `stats` package. It identified an autoregression model of order 12, meaning that the values of $X$ at time $t$ correlated with subsequent values of $X$ up to 12-time units ahead. However, correlations were extremely weak, with magnitudes ranging from 0.021 at $t+8$ to 0.125 at $t+6$. Figure 4.4.1 presents the related correlogram.

The autoregression analysis indicated that the variance that could not be accounted for in the model was 16.578, while the total variance of the time series dataset was 17.403. As a result, the autoregression analysis could not account for 95.3% of the total variance. Thus, the autoregression model accounted for only 4.7% of the total variance in the time series dataset, while the reconstructed signal based on the approach using the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing in conjunction with the Kolmogorov-Zurbenko Fourier transform accounted for 95.4% of the variance in the actual signal. The autoregression model for the signal provided above resembles mostly heavy noise with no practical opportunity to predict the periodic signal buried within that noise. This situation is common in practice and can be resolved by utilizing the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing to identify hidden signal frequencies, followed by use of the Kolmogorov-Zurbenko Fourier transform to reconstruct the hidden signal, as was demonstrated above.
Figure 4.3.1. Plots of signal+noise, signal, and reconstructed signal using the Kolmogorov-Zurbenko Fourier transform ($m=500$ and $k=3$).
Series SignalAndNoise

Autocorrelation Order = 12

Figure 4.4.1. Correlogram of Autoregression Analysis for Simulated Time Series for Complete Data
4.5 Robustness with Respect to Missing Data
Identification of signal frequencies and reconstruction of the original signal was repeated, but with 50% of the datapoints missing. As in the earlier demonstration of robustness in the context of missing data, this began with the complete time series dataset and missingness was introduced using random draws from a binomial distribution (i.e., 0 = not missing, 1 = missing) so that data were missing completely at random with the percentage set at 50%. Next, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing was constructed and is displayed in Figure 4.5.1(a). Despite a missingness rate of 50%, sensitivity, accuracy, and resolution were maintained as, again, two signal spikes were evident, one at $\lambda_1 = 0.084$ and the other at $\lambda_2 = 0.098$.

To reconstruct the original signal, the Kolmogorov-Zurbenko Fourier transform was again used to construct two signals, one at $\lambda_1 = 0.084$ and the other at $\lambda_2 = 0.098$, which were, in turn, added together with the real-valued component extracted and multiplied by 2. Figure 4.5.1(b) displays the original data (signal plus noise), original signal (signal only), and the reconstructed signal, with the former two plots notably missing values of $X_t$. Again, one can see that the reconstructed signal closely approximates the original signal.

The predictive power of the reconstructed signal was also strong. Across the available data, the correlation of the reconstructed signal with the original data was 0.954, thus accounting for 91.0% of the variance in the original signal.

4.6 Summary of Simulated Example
This section demonstrated the ease of use and predictive power of an approach to time series analysis that utilizes the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko smoothing to detect, identify, and separate signals and the Kolmogorov-Zurbenko Fourier transform to reconstruct the original time series. The utility and power of this approach is apparent, even when signals are weak relative to random noise (i.e., low signal-to-noise ratio), even when signals are close in frequency and amplitude, and even when there is a substantial amount of missing data.

When periodicity is present in a time series dataset, the predictive power of this approach was quite strong compared to another standard of time series analysis, the autoregression model. Not only is this spectral analytic approach good in predicting future values of an underlying set of signals, but it is also able to do so into perpetuity, assuming no catastrophic event impacts the time series process.
Figure 4.5.1. Time Series Analysis and Signal Reconstruction with 50% Missing Data.
(a) The Kolmogorov-Zurbenko periodogram using DiRienzo-Zurbenko smoothing
\((m=500, k=3, \text{ and } DZ=0.01)\)
(b) Plots of signal+noise, signal, and estimated signal using the Kolmogorov-Zurbenko Fourier transform \((m=500 \text{ and } k=3)\)
5. Examples in Current Research
There are a range of studies, in fields as diverse as earth and atmospheric sciences as well as public health, that have utilized the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing to identify heretofore hidden periodicities in time series data. For example just as oceans have oceanic tides, the atmosphere has its own tides and Zurbenko and Potrzeba (2010, 2013) have used this approach to identify important patterns in atmospheric tides and atmospheric tidal waves, highlighting critical patterns (signals) that can attenuate or can exacerbate the destructive power of hurricanes. In regard to climate, these same researchers utilized the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing to identify patterns related to climate change: they found that cycles in the planetary orbits have an impact on sunspot activity (Potrzeba-Macrina & Zurbenko, 2019) and, in turn, were able to assess the relative impact of sunspot activity, solar irradiation, and human factors on temperature (Potrzeba-Macrina & Zurbenko, 2020; Zurbenko & Potrzeba-Macrina, 2019b, 2019a). In the field of public health, both Close and Zurbenko (2008) as well as Zurbenko and Pera (2020) have used the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing to identify seasonal, weekly, and daily patterns in traffic fatalities, with the latter pair of researchers finding increased fatalities related to use of daylight savings time. The reader is encouraged to review these original works to gain a better understanding of how this approach has been utilized as well as the power of the methods themselves.
6. Conclusions
This article began with a brief description of the development of and logic underlying the Kolmogorov-Zurbenko algorithm periodogram with DiRienzo-Zurbenko algorithm smoothing. Because log periodograms have the advantage of allowing comparisons in terms of percentage and proportion of variance rather than the magnitude of variance, the Kolmogorov-Zurbenko periodogram is, by default, a log periodogram. While other methods used to construct a smoothed periodogram utilize a static spectral window with a fixed width, the DiRienzo-Zurbenko algorithm is dynamic in its ability to zoom in (decrease the width of the filtering window) when a signal frequency is present and zoom out (increase the width of the filtering window) when signal frequencies are not present. As a result, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing more clearly identifies hidden signals and represents a significant improvement in the construction of periodograms.

The discussion then turned to accessing the Kolmogorov-Zurbenko periodogram function within the statistical platform R, along with basic instructions in its use. It was noted that spectral analysts who prefer working in terms of proportion of total variance can invoke DiRienzo-Zurbenko algorithm smoothing, while those who prefer to work in terms of proportion of total departure from linearity can choose Neagu-Zurbenko algorithm smoothing.

The theoretical and practical limits of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing were established with respect to sensitivity (ability to detect weak signals), accuracy (ability to correctly identify signal frequencies), resolution (ability to resolve signals with close frequencies), and robustness (sensitivity, accuracy, and resolution when data is missing), with each including demonstrations using simulated datasets. For given number of observations, $N$, signal-to-noise ratio, $s^2/n^2$, initial window width, $m$, and DiRienzo-Zurbenko proportion of smoothness, $DZ$, the theoretical limit of signal sensitivity is $s^2/n^2 \geq DZ$; the theoretical limit of frequency accuracy is $\pm 1/2N$, with a practical limit of $\pm 1/2m$; and the theoretical limit of frequency resolution is $\pm 1/N$, with a practical limit of $\pm 1/m$. With respect to the practical limits of robustness, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing demonstrated sensitivity, accuracy, and resolution with up to 50% of data missing, except when the number of observations was relatively small ($N = 1000$) and the proportion of smoothness was relatively high ($DZ = 0.05$).

In a series of demonstrations, the performance of the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing was quite good across all criteria, with one exception: the poorest performance occurred for sensitivity, resolution, and robustness when the number of observations was relatively small ($N = 1000$ vs $N = 5000$) and proportion of smoothness was relatively high ($DZ = 0.05$ vs $DZ = 0.01$). This implies that when the number of observations is relatively large, a higher proportion of smoothness can be used; however when the number of observations is relatively small, the analyst should use a lower proportion of smoothness to better detect, identify, and resolve signal frequencies in the context of moderate amounts of noise because the lower proportion of smoothing results in a narrower smoothing window width in the presence of a signal.

Finally, an example was presented using a simulated time series dataset in which two signals close in frequency and amplitude were embedded in a significant level of random noise. The
Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing was used to identify signal frequencies and the Kolmogorov-Zurbenko Fourier transform was used to recreate the original signal. The reconstructed signal had a high level of predictive power, accounting for 95.4% of the variance in the original signal using the complete dataset and accounting for 91.0% of the variance in the original signal when 50% of the data were missing. By comparison, an autoregression model of order 12 only accounted for 4.7% of the variance present in the original dataset (signal-plus-noise). Thus, an approach using the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing to identify hidden signal frequencies followed by signal reconstruction using the Kolmogorov-Zurbenko Fourier transform represents a state-of-the-art approach that is straightforward in its use and produces results that have a considerably high level of predictive power when periodicity is present in a time series dataset.

Just as the current study focused on the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing, our next investigation will establish the theoretical and practical limits of the Kolmogorov-Zurbenko periodogram with Neagu-Zurbenko algorithm smoothing with respect to sensitivity, accuracy, resolution, and robustness. As noted earlier, the Kolmogorov-Zurbenko periodogram with DiRienzo-Zurbenko algorithm smoothing adapts according to the proportion of total variance, while the Kolmogorov-Zurbenko periodogram with Neagu-Zurbenko algorithm smoothing adapt window size according to the proportion of total departure from linearity. Consequently, this next investigation will also include an in-depth comparison of the DiRienzo-Zurbenko algorithm to the Neagu-Zurbenko algorithm across all criteria.
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