THE GIBBONS-HAWKING ANSATZ OVER A WEDGE

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ABSTRACT. We discuss the Ricci-flat ‘model metrics’ on $\mathbb{C}^2$ with cone singularities along the conic $\{zw = 1\}$ constructed by Donaldson -[3], Section 5- using the Gibbons-Hawking ansatz over wedges in $\mathbb{R}^3$. In particular we describe their asymptotic behavior at infinity and compute their energies.

1. Introduction

Fix $0 < \beta < 1$. Let $W$ be a wedge in $\mathbb{R}^3$ of angle $2\pi\beta$ delimited by two planes. Let $p$ be a point in the interior of the wedge which is equidistant from the two faces and is located at distance 1 from the edge of $W$. Take $f$ to be the associated Green’s function for the Laplacian with pole at $p$ and zero normal derivative at the boundary of $W$. The Gibbons-Hawking ansatz produces an hyperkähler 4-manifold with boundary $P$ endowed with a circle action which preserves all the structure and has the closure of $W$ as its space of orbits. The rotation that takes one face of $W$ to the other is lifted to an isometry $F$ of $P$, fixing the points over the edge of $W$. We identify points on the boundary of $P$ which correspond under $F$ to obtain a smooth manifold $P$ without boundary, endowed with a metric $g_{RF}$ which has cone angle $2\pi\beta$ in transverse directions to the points fixed by $F$. The upshot is that the direction of the edge of $W$ defines a global complex structure $I$ on $P$ with respect to which $g_{RF}$ is Kähler; and the complex manifold $(P,I)$ is indeed a very familiar one.

Theorem 1. $g_{RF}$ defines a Ricci-flat Kähler metric on $\mathbb{C}^2$; it is invariant under the $S^1$-action $e^{i\theta}(z,w) = (e^{i\theta}z, e^{-i\theta}w)$, it has cone angle $2\pi\beta$ along the conic $C = \{zw = 1\}$ and its volume form is

$$\text{Vol}(g_{RF}) = (\beta^2/2)|1 - zw|^{2\beta - 2}\Omega \wedge \overline{\Omega},$$

where $\Omega = (1/\sqrt{2})dzdw$. Moreover

1. At points on $C$ it has cone singularities in a $C^\alpha$ sense -as defined in [3]- with Hölder exponent $\alpha = 1$ if $0 < \beta \leq 1/2$ and $\alpha = (1/\beta) - 1$ if $1/2 < \beta < 1$.
2. It is asymptotic to the Riemannian cone $C_{\beta} \times C_{\beta}$ at rate $-4$ if $0 < \beta \leq 1/2$ and $-2/\beta$ if $1/2 < \beta < 1$.
3. Its energy is finite and is given by

$$E(g_{RF}) = 1 - \beta^2$$

We collect some background material on the Green’s function of a wedge and the Gibbons-Hawking ansatz in Section 2. The proof of Theorem 1 is done in Section 3. The identification of $(P, I)$ with $\mathbb{C}^2$ is already in [3], we repeat the argument filling-in small details. The original content of this article rests on the three items 1, 2 and 3, which are proved in 3.1, 3.2 and 3.3 respectively. Finally, in Section 4 we discuss the sectional curvature of the metrics $g_{RF}$ and the limits when $\beta \to 0$.

The interest in Theorem 1 comes from the blow-up analysis of the Kähler-Einstein (KE) equations in the context of solutions with cone singularities. In the case of smooth KE metrics on complex surfaces the solutions can only degenerate in the non-collapsed regime by developing isolated orbifold points, and the blow-up limits at these are the well-known ALE spaces. In the conical case a new feature arises when the curves along which the metrics have singularities degenerate. In this setting, the $g_{RF}$ furnish a model for blow-up limits at a point where a sequence of smooth curves develops an ordinary double point. Models for blow-up limits of sequences in which the curves develop an ordinary $d$-tuple point are constructed in [2].

The energy of a Riemannian manifold $(M, g)$ is defined as

$$E(g) = \frac{1}{8\pi^2} \int_M |\text{Rm}(g)|^2dV_g,$$
where $\text{Rm}(g)$ denotes the curvature operator of the metric. In our case $g_{RF}$ is smooth on $C^2 \setminus C$ and we integrate on this region. Following next we clarify the meaning of the first two items in Theorem \[1\] but first let us introduce some notation.

We write $C_\beta$ for the complex numbers endowed with the singular metric $\beta^2|\xi|^{2\beta-2}|d\xi|^2$. We recognize it as the standard cone of total angle $2\pi\beta$. Indeed, if we introduce the ‘cone coordinates’

$$\xi = r^{1/\beta}e^{i\theta}$$

then $\beta^2|\xi|^{2\beta-2}|d\xi|^2 = dr^2 + \beta^2r^2d\theta^2$. There are two flat model metrics which are relevant to us. The first is $C_\beta \times C$, which captures the local behavior of $g_{RF}$ at points on the conic. In complex coordinates $g_{loc} = \beta^2|z_1|^{2\beta-2}|dz_1|^2 + |dz_2|^2$. Secondly is $C_\beta \times \overline{C}_\beta$, which models the asymptotic behavior of $g_{RF}$ at infinity. In complex coordinates $g_F = \beta^2|u|^{2\beta-2}|du|^2 + \beta^2|v|^{2\beta-2}|dv|^2$. We introduce ‘spherical coordinates’ by setting

$$\rho^2 = |u|^{2\beta} + |v|^{2\beta}.$$ 

The function $\rho$ measures the intrinsic distance to 0 and it is easy to check to that $g_F = d\rho^2 + \rho^2\overline{g}$ where $\overline{g}$ is a metric on the 3-sphere with cone angle $2\pi\beta$ along the Hopf circles determined by the complex lines \{u = 0\} and \{v = 0\}. We proceed with the explanation of Theorem \[1\]

- **Item 1.** Let $p \in C$ and $(z_1, z_2)$ be complex coordinates centered at $p$ such that $C = \{z_1 = 0\}$. Let $z_1 = r_1^{1/\beta}e^{i\theta_1}$. Set $\epsilon_1 = dr_1 + i\beta r_1 d\theta_1$ and $\epsilon_2 = dw$. The Kähler from associated to $g_{RF}$ writes as

$$\omega_{RF} = i \sum_{j,k} a_{jk} \overline{x_j} \overline{r_k}$$

for smooth functions $a_{jk}$ on the complement of $\{z_1 = 0\}$. We say that $g_{RF}$ is $C^\alpha$ if, for every $p \in C$ and holomorphic coordinates centered at $p$ as above, the $a_{jk}$ extend to $\{z_1 = 0\}$ as $C^\alpha$ functions in the cone coordinates $(r_1e^{i\theta_1}, z_2)$. We also require the matrix $(a_{jk}(p))$ to be positive definite and that $a_{11} = 0$ when $z_1 = 0$. In particular these conditions imply that $A^{-1}g_{loc} \leq g_{RF} \leq A g_{loc}$ for some $A > 0$.

- **Item 2.** We say that $g_{RF}$ is asymptotic to $g_F$ at rate $-\mu$, for some $\mu > 0$, if there is a closed ball $B \subset C^2$ and a map $\Phi: C^2 \setminus B \rightarrow C^2$ which is a diffeomorphism onto its image and with the property that

$$|\Phi^*g_{RF} - g_F|_{g_F} \leq A\rho^{-\mu}, \quad |\Phi^*I - I|_{g_F} \leq A\rho^{-\mu}$$

for some constant $A > 0$; where $I$ denotes the standard complex structure of $C^2$. We write $(z, w) = \Phi(u, v)$, so that necessarily $\Psi(\{uv = 0\}) \subset \{zw = 1\}$.

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2. **Background**

2.1. **Potential theory on a wedge.** Consider the wedge

$$W = \{(re^{i\theta}, s) \in \mathbb{R}^3 : \text{s.t. } -\pi\beta < \theta < \pi\beta\}.$$ 

Write $S = \{0\} \times \mathbb{R}$ for the edge of $W$ and let $p = (1, 0, 0)$. It is a fact that there is a unique continuous and positive function $\tilde{\Gamma}_p: \overline{W} \setminus \{p\} \rightarrow \mathbb{R}_{>0}$ which tends to 0 at infinity and solves the boundary value problem

$$\begin{cases}
\Delta \tilde{\Gamma}_p = \delta_p & \text{on } W, \\
\frac{\partial \tilde{\Gamma}_p}{\partial v} = 0 & \text{on } \partial W.
\end{cases}$$ 

The first equation is meant to be interpreted in the sense of distributions, $\Delta$ is the standard Euclidean Laplacian and $\delta_p$ is the Dirac delta at $p$. In the second equation $\nu$ denotes the outward unit normal vector,
which is well defined on the complement of the edge. In other words, $\tilde{\Gamma}_p$ is the Green’s function for the Laplace operator with pole at $p$ associated to the Neumann boundary value problem over $W$.

Standard elliptic regularity theory -Weyl’s lemma- implies that $\tilde{\Gamma}_p$ is smooth on $W \setminus \{p\}$. Indeed, on $W$ we can write

$$\tilde{\Gamma}_p(x) = \frac{1}{4\pi|x-p|} + F$$

for some smooth harmonic function $F$. The behavior of $\tilde{\Gamma}_p$ around points on the edge is more subtle.

We use a rotation to identify the faces of $W$. Write $\theta = \beta \theta$. We are led with a metric on $\mathbb{R}^3$ with cone angle $2\pi \beta$ along $S = \{0\} \times \mathbb{R}$,

$$g_\beta = dv^2 + \beta^2 v^2 d\theta^2 + ds^2.$$ 

Write $\Delta_\beta$ for its Laplacian. We let $\Gamma_p(re^{i\theta}, s) = \hat{\Gamma}_p(re^{i\theta}, s)$. The function $\Gamma_p$ is continuous on $\mathbb{R}^3 \setminus \{p\}$, smooth on the complement of $S \cup \{p\}$ and solves the distributional equation

$$\Delta_\beta \Gamma_p = \delta_p.$$ 

It is shown in [3] that $\Gamma_p$ is $\beta$-smooth at points of $S$; which means that it is a smooth function of the variables $r^{1/\beta}e^{i\theta}, r^2, s$. Moreover, we have a ‘polyhomogeneous’ expansion

$$\Gamma_p = \sum_{j,k \geq 0} a_{j,k}(s)r^{(k/\beta)+j} \cos(k\theta)$$

with $a_{j,k}$ smooth functions of $s$ and which converges uniformly when $r \leq 1/4$.

Allowing the point $p$ to vary we obtain, in the usual way, the function $G(p, q) = \Gamma_p(q)$ which provides an inverse for $\Delta_\beta \psi = \phi$ by letting $\psi(x) = \int G(x, y)\phi(y)dv_\beta(y)$. The symmetries and dilations of $g_\beta$ are reflected in that

$$G(T_l p, T_l q) = G(p, q), \quad G(R_\gamma p, R_\gamma q) = G(p, q), \quad G(m_\lambda p, m_\lambda q) = \lambda^{-1}G(p, q)$$

where $T_l(r, \theta, s) = (r, \theta, s + l)$, $R_\gamma(r, \theta, s) = (r, \theta + \gamma, s)$ and $m_\lambda(r, \theta, s) = (\lambda r, \theta, \lambda s)$ for $\lambda > 0$. There is also the symmetric property $G(p, q) = G(q, p)$. It follows from the $\beta$-smoothness that there is $\kappa > 0$ such that

1. $|G(0, p)| \leq \kappa$ for every $p$ with $|p| = 1$
2. $|G(x_1, p) - G(x_2, p)| \leq \kappa |x_1 - x_2|^{1/\beta}$ whenever $|p| = 1$ and $|x_1|, |x_2| \leq 1/2$

It is easy to write the Green’s function with the pole located at $S$,

$$G(0, x) = \frac{1}{4\pi \beta|x|}.$$ 

By homogeneity, if $|x| \geq 2|p|$

$$|G(x, p) - G(x, 0)| = |x|^{-1}|G(|p||x|^{-1}, x|x|^{-1}) - G(0, |x|^{-1}x)| \leq \kappa |x|^{-1-1/\beta}.$$ 

In particular we see that $\Gamma_p$ decays as $|x|^{-1}$.

We include an observation regarding formula (2.1) which will be useful for us later on.

**Lemma 1.** $F > 0$

**Proof.** Since $F$ is harmonic it is enough to show that it is positive on $\partial(B \cap W)$ for any sufficiently large ball $B$. Since $\tilde{\Gamma}_p$ is asymptotic to $1/4\pi \beta|x|$ it follows that $F > 0$ on $\partial B \cap W$. Note that $F$ restricted to the edge is equal to $(\beta^{-1} - 1)/4\pi|x-p| > 0$. The fact that the normal derivative of $\tilde{\Gamma}_p$ is zero at the boundary of $W$ implies that $F$ has no critical points when restricted to these planes, it then follows that $F > 0$ on $\partial W \cap B$.

To finish this section and for the sake of completeness we comment a bit more on the expansion (2.3). The coefficients $a_{j,k}$ are given in terms of Bessel’s functions and we want to indicate how these arise. The technique is separation of variables. We write

$$G(r, \theta, s; r', \theta', s') = \sum_{k=0}^{\infty} G_k(r, r', R) \cos k(\theta - \theta')$$

where $G_k(r, r', R) = \sum_{j=0}^{\infty} (\beta^{-1} - 1)j^{1/\beta} (j + \beta^{-1} - 1) B_j(r, r', a_k)$. The function $B_j(r, r', a_k)$ is the Bessel’s function of first kind of order $j$ and argument $a_k$. The integral representation of $B_j(r, r', a_k)$ is

$$B_j(r, r', a_k) = \frac{1}{2\pi i} \int_{C_{\lambda k}} \left( \frac{\gamma_k}{\lambda_k} \right)^{1/2} e^{-\pi \lambda_k / 2} d\lambda_k,$$ 

where $\gamma_k$ is the gamma function, $\lambda_k$ is the eigenvalue of the Laplace operator with pole at $p$, and $C_{\lambda k}$ is a contour in the complex plane enclosing the eigenvalues of the Laplace operator with pole at $p$. The contour $C_{\lambda k}$ is chosen so that it does not intersect the real axis and it encloses the eigenvalues of the Laplace operator with pole at $p$.
with \( R = |s - s'| \). We decompose
\[
\Delta_\beta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{\beta^2 r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial s^2} = L + \frac{\partial^2}{\partial s^2}.
\]

The point is that -for any integer \( k \geq 0 \) and \( \lambda \geq 0 \)- the function \( \phi = J_\nu(\lambda r)e^{ik\theta} \) is an eigenfunction for \( L \) with \( L\phi = -\lambda^2 \phi \); where \( \nu = k/\beta \) and
\[
J_\nu = \sum_{j=0}^{\infty} (-1)^j \frac{(\nu/2)^{\nu+j}}{j!(\nu+j)!}
\]
is Bessel’s function, which solves \( f'' + \frac{1}{r} f' + (1 - \nu^2 r^2)f = 0 \). This leads to a formula for the heat kernel associated to \( \Delta_\beta \)
\[
(4\pi t)^{-1/2}e^{-R^2/4t} \sum_{k=0}^{\infty} \left( \pi^{-1} \int_0^\infty e^{-\lambda^2 t} J_\nu(\lambda r)J_\nu(\lambda r')d\lambda \right) \cos(\theta - \theta')
\]
The Green’s function is obtained by integration of the heat kernel with respect to the time parameter and this gives us
\[
G_k(r, r', R) = \int_0^\infty \int_0^\infty (4\pi t)^{-1/2}e^{-\lambda^2 t - R^2/4t} J_\nu(\lambda r)J_\nu(\lambda r')d\lambda dt.
\]
We fix \((r', \theta', s') = (1, 0, 0)\), replace \(2.7\) into \(2.5\) expand the Bessel’s functions into power series \(2.6\) and exchange the integral with the summation; to obtain a formal polyhomogeneous expansion as in \(2.3\). The validity of the expression is guaranteed provided we check uniform convergence. In order to do this the integral \(2.7\) has to be properly manipulated, suitable bounds must be derived and some expertise with Bessel’s functions is required -see \(3\)-.

2.2. The Gibbons-Hawking ansatz. This well-known construction provides a ‘local’ correspondence between positive harmonic functions on domains in \( \mathbb{R}^3 \) and hyperkähler structures with \( S^1 \) symmetry. More precisely, let \( x_1, x_2, x_3 \) be standard coordinates on \( \mathbb{R}^3 \) and let \( f \) be a positive harmonic function on \( \Omega \). Consider an \( S^1 \)-bundle over \( \Omega \) equipped with a connection\(^1\) \( \alpha \) which satisfies the Bogomolony equation
\[
(2.8) \quad d\alpha = -\ast df.
\]
The hyperkähler structure is then defined by means of the three 2-forms
\[
(2.9) \quad \omega_i = \alpha dx_i + f dx_j dx_k,
\]
here in and the rest of the article we use the notation of the indices \((i, j, k)\) varying over the cyclic permutations of \((1, 2, 3)\). The Bogomolony equation \(2.8\) is indeed equivalent to the \( \omega_i \) being closed. The bundle projection is then characterized as the hyperkähler moment map for the \( S^1 \)-action.

**Example 1.** A basic case is that of the Euclidean metric on \( \mathbb{R}^4 \cong \mathbb{C}^2 \) equipped with the \( S^1 \)-action \( e^{i\theta}(z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2) \) which preserves its standard hyperkähler structure. The hyperkähler moment map agrees with the Hopf map
\[
(2.10) \quad H(z_1, z_2) = \left( z_1 z_2, \frac{|z_1|^2 - |z_2|^2}{2} \right).
\]
Removing 0 gives an \( S^1 \)-bundle with first Chern class equal to \(-1\); and it is straightforward to check that
\[
(2.11) \quad f = \frac{1}{2|x|} \quad \alpha = \text{Re} \left( \frac{i \overline{z}_2 dz_2 - \overline{z}_1 dz_1}{|z_1|^2 + |z_2|^2} \right).
\]

\(^1\)By a connection we mean an \( S^1 \)-invariant 1-form on the total space which gives 1 when contracted with the derivative of the \( S^1 \)-action. Its curvature is \( da \) and it is a general fact that it is the pull-back by the bundle projection of a closed 2-form on the base whose de Rham cohomology class represents \(-2\pi c_1\). We shall often suppress the pull-back by the bundle projection in our formulas.
A unit vector $v$ in $\mathbb{R}^3$ determines a parallel complex structure on the hyperkähler manifold, by sending the horizontal lift of the constant vector field $v$ to the derivative of the $S^1$-action. We now review an explicit construction of holomorphic functions for these complex structures on a particular case which will be relevant for us, our reference is Section 5.3 in [3].

Assume that $\Omega$ is the product of a simply connected domain $U \subset \mathbb{R}^2$ with the $x_3$ axis. We will consider the complex structure determined by the $x_3$ direction. We trivialize the bundle and denote the circle coordinate with $e^{i\alpha}$, so that

$$\alpha = dt + \sum_{j=1}^{3} a_j dx_j.$$  

We can change gauge by $\tilde{t} = t - \int_{0}^{x_3} a_3(x_1, x_2, q) dq$ - or in other words parallel translate in the $x_3$ direction- and assume that $a_3 = 0$. The Bogomolony equation (2.8) amounts to

$$\frac{\partial f_1}{\partial x_3} = f_1, \quad \frac{\partial f_2}{\partial x_3} = -f_2, \quad \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} = f_3;$$

where $f_i$ means $\partial f / \partial x_i$. We further assume that $f_3 = 0$ on $\{x_3 = 0\}$, in other words this is to say that $\alpha$ restricts to a flat connection over $U$. Since $U$ is simply connected we can perform a gauge transformation $\tilde{t} = t - \phi(x_1, x_2)$ and assume that $a_1 = a_2 = 0$ on the slice $\{x_3 = 0\}$.

The horizontal lifts of the coordinate vectors $\partial / \partial x_i$ are given by

$$\hat{\partial} \bigg|_{x_1} = \frac{\partial}{\partial x_1} - ai \frac{\partial}{\partial t}, \quad \hat{\partial} \bigg|_{x_2} = \frac{\partial}{\partial x_2} - a_2 \frac{\partial}{\partial t}, \quad \hat{\partial} \bigg|_{x_3} = \frac{\partial}{\partial x_3};$$

and the complex structure is defined as

$$I \frac{\hat{\partial}}{\partial x_1} = \frac{\hat{\partial}}{\partial x_2}, \quad I \frac{\hat{\partial}}{\partial x_3} = -\frac{\hat{\partial}}{\partial t}.$$  

The Cauchy-Riemann equations for a function $h$ to be holomorphic w.r.t. $I$ are then given by

$$\frac{\partial h}{\partial x_1} + i \frac{\partial h}{\partial x_2} = (a_1 + ia_2) \frac{\partial h}{\partial t}, \quad \frac{\partial h}{\partial x_3} = i f \frac{\partial h}{\partial t}.$$  

We look for a function $h$ which has weight one for the circle action, so that $\partial h / \partial t = i h$. We use separation of variables and write $h = \tilde{h} e^{it}$ with $\tilde{h} = \tilde{h}(x_1, x_2, x_3)$. The second equation in (2.14) gives us $\partial \tilde{h} / \partial x_3 = - f \tilde{h}$; so that

$$\tilde{h} = h_0 e^{-u e^{it}}, \quad u = \int_{0}^{x_3} f(x_1, x_2, q) dq, \quad h_0 = h_0(x_1, x_2)$$

Recall that $a_1 = a_2 = 0$ on the slice $\{x_3 = 0\}$. Let $h_0$ be any solution of the equation

$$\frac{\partial h_0}{\partial x_1} + i \frac{\partial h_0}{\partial x_2} = 0,$$

in other words an holomorphic function of $x_1 + i x_2$.

**Lemma 2.** The function $h$ defined by (2.15) and (2.16) solves 2.14.

**Proof.** The Cauchy-Riemann equations 2.14 is an over-determined system. It follows from the definition of $h$ that it solves the second equation in 2.14 and that it solves the first equation in 2.14 only in the slice $\{x_1 = 0\}$. The point is that $h$ is a solution thanks to the integrability condition provided by the Bogomolony equation 2.8 as the following computation shows

$$\frac{\partial h}{\partial x_1} + i \frac{\partial h}{\partial x_2} = e^{-u} e^{it} \left( \frac{\partial h_0}{\partial x_1} + i \frac{\partial h_0}{\partial x_2} - h_0 \frac{\partial u}{\partial x_1} - h_0 \frac{\partial u}{\partial x_2} \right) = ih \left( - \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_1} \right)$$

and it follows from (2.12) that

$$\frac{\partial u}{\partial x_1} = a_2, \quad \frac{\partial u}{\partial x_2} = -a_1.$$
Similarly, \( h_0 e^{i\psi} \) defines a holomorphic function with weight \(-1\) for the circle action. Given any holomorphic function of the complex variable \( x_1 + ix_2 \), \( h_0 \) on \( U \), the pair

\[
(2.17) \quad z = h_0 e^{i\psi}, \quad w = h_0 e^{i\psi}
\]
defines an \( S^1 \)-equivariant holomorphic map from \( (\Omega \times S^1, I) \) to a domain in \( \mathbb{C}^2 \) equipped with the circle action \( e^{it}(z, w) = (e^{it}z, e^{-it}w) \).

**Example 2. Taub-Nut metric.** Let \( c > 0 \) and consider the harmonic function

\[
f = 2c + \frac{1}{2|z|}.
\]

Let \( \alpha \) be the connection on the Hopf bundle \( H : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\} \) given in Example 1.\footnote{The restriction of the Hopf connection to the punctured plane \( \{x_3 = 0\} \setminus \{(0,0)\} \) is a flat connection with non-trivial holonomy, this forces us to introduce a cut in the plane in order to define the desired trivialization with \( a_1 = a_2 = 0 \).} so that \( d\alpha = -\ast df \).

We look at complex structure \( I \) on \( \mathbb{R}^4 \) determined by the \( x_3 \)-axis.\footnote{On \( \mathbb{R}^4 \) we have standard coordinates \( s_1, s_2, s_3, s_4 \). For notational convenience we write \( z_1 = s_1 + is_2 \) and \( z_2 = s_3 + is_4 \), but \( z_1, z_2 \) are not necessarily complex coordinates for \((\mathbb{R}^4, I)\).}

We want a suitable trivialization of the bundle which fits us in the context of Lemma 3.\footnote{We apply Lemma 2 with \( h_0 = \sqrt{x} \).} Note that it follows from 2.10 that

\[
2|x| = |z_1|^2 + |z_2|^2,
\]

so \( |z_1| = (|x| - x_3)^{1/2} \) and \( |z_2| = (|x| + x_3)^{1/2} \). Write \( \xi = x_1 + ix_2 = |\xi|e^{i\theta} \) and let \( U \) be the complement of the negative real axis in the \( \xi \)-plane so that \(-\pi < \theta < \pi\). Over \( \Omega \) we have the following trivialization

\[
\Phi(x, e^{it}) = (z_1, z_2) = \left( (|x| - x_3)^{1/2} e^{i\theta/2} e^{it}, (|x| + x_3)^{1/2} e^{i\theta/2} e^{-it} \right)
\]

and it is easy to check that

\[
\alpha = dt - \frac{x_3}{2|x|} d\theta,
\]

which clearly satisfies \( a_3 \equiv 0 \) and \( a_1 = a_2 = 0 \) when \( x_3 = 0 \).\footnote{Indeed, \( \omega = \alpha dx_3 + f dx_1 dx_2 \) and it is easy to check that \( \omega = i\sqrt{x}(|x| + c(x_2 + x_1^2 + 2x_3^2)) \). Since \( |x| = (1/2)(|z_1|^2 + |z_2|^2) \) and \( x_1^2 + x_2^2 + 2x_3^2 = (1/2)(|z_1|^4 + |z_2|^4) \) we obtain

\[
\omega = \frac{i}{2} \partial \bar{\partial} \left( |z_1|^2 + |z_2|^2 + c(|z_1|^4 + |z_2|^4) \right),
\]

with \( |z_1|, |z_2| \) determined implicitly in terms of \( z, w \) by means of 2.18.}
2.3. 4-dimensional Riemannian geometry. Let \((M, g)\) be an oriented Riemannian four-manifold and let \(\text{Rm}(g) : \Lambda^2 \to \Lambda^2\) be its curvature operator. The decomposition \(\Lambda = \Lambda^+ \oplus \Lambda^-\) of the 2-forms into self and anti-self-dual given by the Hodge star operator of \(g\) determines the well-known decomposition of \(\text{Rm}(g)\) into four three by three blocks

\[
\begin{pmatrix}
\frac{s}{r^2} + W^+ \\
\frac{\dot{r}}{r} \\
\frac{s}{r^2} + W^- 
\end{pmatrix}.
\]

These blocks can also be interpreted in terms of the curvature \(F_\nabla\) of the Levi-Civita connection on the bundles \(\Lambda^+, \Lambda^-\). Indeed, if \(\theta_1, \theta_2, \theta_3\) is an orthonormal triple of anti-self-dual forms we can write

\[
F_\nabla(\theta_i) = F_j \otimes \theta_k - F_k \otimes \theta_j
\]

for some 2-forms \(F_1, F_2, F_3\). We write the anti-self-dual parts as \(F_i^- = \sum_{j=1}^{3} c_{ij} \theta_j\). The fact is that \((c_{ij})_{1 \leq i,j \leq 3}\) agrees with the block \(s/12 + W^-\); and similarly for the other blocks. The curvature 2-forms are given by \(F_i = T_i - T_j \wedge T_k\) where \(T_1, T_2, T_3\) are the connection 1-forms \(\nabla(\theta_i) = T_j \otimes \theta_k - T_k \otimes \theta_j\).

The torsion-free property gives us

\[
d\theta_i = T_j \wedge \theta_k - T_k \wedge \theta_j, \tag{2.19}
\]

A simple algebraic fact -see Proposition 2.3 in [4]- is that the system of equations 2.19 for \(T_1, T_2, T_3\) has the unique solution

\[
2T_i = \ast \psi_i + \ast(\ast \psi_j \wedge \theta_k) - \ast(\ast \psi_k \wedge \theta_j), \tag{2.20}
\]

where \(\psi_i = d\theta_i\) and \(\ast\) denotes the Hodge operator of \(g\). This fact can be interpreted as a characterization of the Levi-Civita connection on \(\Lambda^-\) as the unique which is metric and torsion-free; somewhat analogous to the Cartan’s lemma.

We use the previous discussion to compute the energy distribution \(|\text{Rm}(g)|^2\) of a metric \(g = f dx^2 + f^{-1} \alpha^2\) given by the Gibbons-Hawking ansatz. There is an orthonormal frame of self-dual 2-forms \(\omega_i = \alpha dx_i + f dx_j dx_k\); since \(d\omega_i = 0\) the only non-vanishing part of the curvature operator is \(W^-\), the anti-self-dual Weyl curvature tensor and it is a general fact that \(W^-\) is symmetric and trace-free. We consider the orthonormal frame of \(\Lambda^-\) given by \(\theta_i = \alpha dx_i - f dx_j dx_k\); so \(d\theta_i = -2f dx_1 dx_2 dx_3\) and 2.20 gives us

\[
T_i = \frac{f_i}{f^2} \alpha - \frac{f_j dx_k - f_k dx_j}{f}. \tag{2.21}
\]

We compute the curvature forms and express their anti-self-dual components with respect to the \(\theta_i\) frame to obtain

\[
c_{ij} = \frac{f_{ij}}{f^2} - \frac{3 f_i f_j}{f^3} + \delta_{ij} \frac{|Df|^2}{f^3}. \tag{2.22}
\]

We can write this more succinctly as

\[
W^- = (-f/2)\text{Hess}(f^{-2}),
\]

where \(\text{Hess}\) denotes the standard Euclidean trace-free Hessian.

In order to achieve our goal we proceed by straightforward computation,

\[
|\text{Rm}(g)|^2 = \sum_{i,j} c_{ij}^2 = 6f^{-6} |Df|^4 + f^{-4} \sum_i f_{ij}^2 - 6f^{-5} \sum_{i,j} f_{ij} f_i f_j.
\]

Let \(\Delta\) be the Euclidean Laplacian, so that \(\Delta f = 0\). It follows easily that

\[
\Delta f^{-1} = -2f^{-3} |Df|^2, \quad (1/2) \Delta \Delta f^{-1} = 12f^{-5} |Df|^4 + 2f^{-3} \sum_i f_{ij}^2 - 12f^{-4} \sum_{i,j} f_{ij} f_i f_j.
\]

Comparing we obtain the formula -see Remark 2.4 in [5]-

\[
|\text{Rm}(g)|^2 = \frac{1}{4f} \Delta \Delta f^{-1}. \tag{2.22}
\]
3. Proof of Theorem

We go back to the construction of the metric $g_{RF}$ mentioned in the Introduction. It is convenient to perform the gluing right in the beginning. Equivalently, we start with

\[(3.1) \quad g_{\beta} = dr^2 + \beta^2 r^2 d\theta^2 + ds^2.\]

Write $*_{\beta}$ for its Hodge operator, which acts one 1-forms as

\[*_{\beta}dr = \beta r d\theta ds, \quad *_{\beta}r d\theta = ds dr, \quad *_{\beta}ds = \beta r dr d\theta.\]

Let $p = (1,0,0)$ and let $\Gamma_p$ be the Green’s function for $\Delta_{\beta}$ with pole at $p$ - see Subsection 2.1. We take $f = 2\pi \Gamma_p$, so that $- *_{\beta} df$ integrates $2\pi$ over spheres centered at $p$. Note that

\[d *_{\beta} df = (\Delta_{\beta} f) V_{\beta} = 0\]

where $dV_{\beta} = \beta r dr d\theta ds$ denotes the volume form. Let $\Pi : P_0 \to \mathbb{R}^3 \setminus (S \cup \{p\})$ be the $S^1$-bundle with $c_1(P_0) = -1$. We shall show that there is a connection $\alpha$ on $P_0$ with curvature $- *_{\beta} df$ and with trivial holonomy along small loops that shrink to $S$; note that these two conditions determine $\alpha$ uniquely up to gauge equivalence.

We consider the metric

\[(3.2) \quad g_{RF} = fg_{\beta} + f^{-1}\alpha^2.\]

It is clear that $g_{RF}$ is locally hyperkähler. On the other hand we can extend $P_0$ to $P = P_0 \cup (\mathbb{R} \times S^1) \cup \{\hat{p}\}$ so that the $S^1$-action extends smoothly to $P$, acting freely on $P \setminus \{\hat{p}\}$ and fixing $\hat{p}$. The map $\Pi$ also extends smoothly as the orbit projection $\Pi : P \to \mathbb{R}^3$ with $\Pi(\hat{p}) = p$. We shall see that $g_{RF}$ extends smoothly over $\hat{p}$ and as a metric with cone singularities along $\Pi^{-1}(S) \cong \mathbb{R} \times S^1$.

3.1. Complex structure. Let $\mathbb{R}^2_* \times \mathbb{R}$ be the $re^{i\theta}$-plane with the point $(1,0)$ removed. Define the 1-form on $\mathbb{R}^2_* \times \mathbb{R}$

\[(3.3) \quad \alpha_0 = a_1 dr + a_2 \beta r d\theta\]

where

\[(3.4) \quad a_1 = \frac{1}{\beta r} \int_0^s \frac{\partial f}{\partial \theta}(r, \theta, q) dq, \quad a_2 = \int_0^s \frac{\partial f}{\partial r}(r, \theta, q) dq.\]

It is clear that $\alpha_0$ is smooth on $(\mathbb{R}^2_* \times \mathbb{R}) \setminus S$ and the functions $a_1, a_2$ extend continuously by 0 over $S$ as $C^\alpha$ functions for $\alpha = \beta^{-1} - 1$. The computations that follow are done over the complement of $S$.

Claim 1.

\[(3.5) \quad d\alpha_0 = - *_{\beta} df\]

Proof. We have that

\[d\alpha_0 = - \frac{\partial a_2}{\partial s} \beta r dr ds dr + \frac{\partial a_1}{\partial s} ds dr + \left( \frac{\partial a_2}{\partial r} + \frac{1}{r} \frac{a_2}{\beta r} \frac{\partial a_1}{\partial \theta} \right) \beta r dr d\theta\]

and

\[*_{\beta} df = \frac{\partial f}{\partial r} \beta r dr d\theta + \frac{1}{\beta r} \frac{\partial f}{\partial \theta} ds dr + \frac{\partial f}{\partial s} \beta r dr d\theta\]

It is clear from \[3.4\] that $\partial a_2 / \partial s = \partial f / \partial r$ and $\partial a_1 / \partial s = -(1/\beta r) \partial f / \partial \theta$.

It follows by symmetry that $\partial f / \partial s = 0$ when $s = 0$, so that $\partial f / \partial s = \int_0^s \frac{\partial^2 f}{\partial s^2}(\cdot, t) dt$. Using that $\Delta_{\beta} f = 0$ we get

\[\frac{\partial a_2}{\partial r} + \frac{1}{r} a_2 - \frac{1}{\beta r} \frac{\partial a_1}{\partial \theta} = \int_0^s \left( \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{\beta^2 r^2} \frac{\partial^2 f}{\partial \theta^2} \right) (r, \theta, q) dq = - \int_0^s \frac{\partial^2 f}{\partial s^2}(r, \theta, q) dq = - \frac{\partial f}{\partial s}\]
Consider the product $\mathbb{R}^2 \times \mathbb{R} \times S^1$ and write points in the circle factor as $e^{it}$. Define the connection 1-form $\alpha = dt + \alpha_0$ and the metric

$$g_{RF} = fg_\beta + f^{-1} \alpha^2$$

The horizontal lifts of $\partial/\partial r, \partial/\partial \theta, \partial/\partial s$ are

$$\tilde{\partial}/\partial r = \partial/\partial r - a_1 \partial/\partial t, \quad \tilde{\partial}/\partial \theta = \partial/\partial \theta - a_2 \beta r \partial/\partial t, \quad \tilde{\partial}/\partial s = \partial/\partial s$$

We consider the complex structure determined by the $s$-axis

$$I \tilde{\partial}/\partial r = 1/\beta r \tilde{\partial}/\partial \theta, \quad I \tilde{\partial}/\partial s = -f \tilde{\partial}/\partial t$$

The associated 2-form is

$$\omega_{RF} = g_{RF}(I,\cdot) = \alpha ds + f \beta r drd\theta$$

Claim 2. $(g_{RF}, \omega_{RF}, I)$ defines a Kähler structure on $\mathbb{R}^2 \times \mathbb{R} \times S^1$

Proof. The equation $d\alpha = -\ast \beta df$ implies that $d\omega = d\alpha ds = -\partial f/\partial s \beta r dr d\theta ds$ and this gives $d\omega = 0$. To prove that $I$ is integrable one can check that

$$\left[ \tilde{\partial}/\partial r + i \frac{1}{\beta r} \tilde{\partial}/\partial \theta, \partial/\partial s - if \partial/\partial t \right] = 0$$

Strictly speaking we don’t need to do this since we are going to find complex coordinates in what follows.

The Cauchy-Riemann equations for a function $h$ to be holomorphic w.r.t. $I$ are given by

$$\frac{\partial h}{\partial r} + i \frac{1}{\beta r} \frac{\partial h}{\partial \theta} = (a_1 + ia_2) \frac{\partial h}{\partial t} + \frac{\partial h}{\partial s} = if \frac{\partial h}{\partial t}$$

We look for a function $h$ which has weight one for the circle action, this goes as in Lemma 2. Let $h_0$ be any solution of the equation

$$\frac{\partial h_0}{\partial r} + i \frac{1}{\beta r} \frac{\partial h_0}{\partial \theta} = 0,$$

that is $h_0$ is a holomorphic function of the variable $r^{1/\beta} e^{it}$. Set

$$h = h_0 e^{-u} e^{it}, \quad u = \int_0^s f(r, \theta, q) dq$$

Claim 3. The function $h$ defined by (3.11) and (3.10) solves (3.9). Similarly, holomorphic functions with weight $-1$ for the circle action are given by $h_0 e^u e^{-it}$.

Proof. We check

$$\frac{\partial h}{\partial r} + i \frac{1}{\beta r} \frac{\partial h}{\partial \theta} = e^{-u} e^{it} \left( \frac{\partial h_0}{\partial r} + i \frac{1}{\beta r} \frac{\partial h_0}{\partial \theta} - h_0 \frac{\partial u}{\partial r} - h_0 i \frac{\partial u}{\partial \theta} \right) = ih \left( - \frac{1}{\beta r} \frac{\partial u}{\partial \theta} + i \frac{\partial u}{\partial r} \right)$$

and $-\frac{1}{\beta r} \frac{\partial u}{\partial \theta} + i \frac{\partial u}{\partial r} = a_1 + ia_2$.
Consider the segment in the \(re^{i\theta}\)-plane given by the points in the real line which are \(\geq 1\), let \(U\) be the complement of that segment and \(U^* = U \setminus \{0\}\). Write \(c = \beta^{-1}\). The function \(1 - re^{i\theta}\) maps \(U\) to the complement of the negative real axis, so

\[
(3.12) \quad h_0 = (1 - re^{i\theta})^{1/2}
\]

is a well defined function on \(U\) which satisfies \(3.10\). From now on we set \(h_0\) to be given by \(3.12\) and define

\[
(3.13) \quad z = h_0 e^{-u e^{i\psi}}, \quad w = h_0 e^{u e^{-i\psi}}
\]

Let \(V \subset \mathbb{C}^2\) be the open set of points \((z, w)\) such that \(zw \notin \mathbb{R}_{\leq 0}\). Write \(C = \{zw = 1\}\).

**Claim 4.** The map \(H = (z, w)\) gives a biholomorphism between \((U^* \times \mathbb{R} \times S^1, I)\) and \(W \setminus C\). Moreover \(H\) extends as an homeomorphism between \((U \times \mathbb{R} \times S^1, I)\) with \(H(\{0\} \times \mathbb{R} \times S^1) = C\).

**Proof.** First we provide an inverse for \(H\), showing the homeomorphism part. The pair \((r, \theta)\) is determined by \(r e^{i\theta} = 1 - zw\). The function \(u(s) = \int_0^s f(r, \theta, q) dq\) is increasing \(u' = f > 0\) and \(\lim_{s \to \pm \infty} u(s) = \pm \infty\); so \((s, e^t)\) is given by \(e^{-u}e^t = h_0^{-1}z\).

Next we compute the Jacobian of the map \(H\). Let \(\eta_1 = dr + i\beta rd\theta\) and \(\eta_2 = ds - i\beta^{-1} s\alpha\), so that \(\{\eta_1, \eta_2\}\) is a basis of the \((1,0)\)-forms on \(U^* \times \mathbb{R} \times S^1\). It is straightforward to check that

\[
dz = z \left(h_0^{-1} \frac{\partial h_0}{\partial r} - a_2 - ia_1\right) \eta_1 - z f \eta_2
\]

\[
dw = w \left(h_0^{-1} \frac{\partial h_0}{\partial r} + a_2 + ia_1\right) \eta_1 + w f \eta_2.
\]

The determinant of the linear map that takes \(\{\eta_1, \eta_2\}\) to \(\{dz, dw\}\) is \(-fc r^{-1} e^{i\theta}\) and is non-zero on \(U^* \times \mathbb{R} \times S^1\).

We can compose the inverse of \(H\) with the projection of the trivial \(S^1\)-bundle \(pr(r e^{i\theta}, s, e^t) = (re^{i\theta}, s)\) to obtain the map \(\Pi = pr \circ H^{-1} : V \to \mathbb{R}^3\). We want to show that \(\Pi\) extends to all of \(C^2\), as an orbit map for the \(S^1\)-action \(e^{it}(z, w) = (e^{it} z, e^{-it} w)\). Clearly the \(r, \theta\) coordinates of \(\Pi\) extend, since \(r e^{i\theta} = 1 - zw\). The key step is to extend the function \(s\).

**Claim 5.** \(s\) extends to \(C^2\), smoothly on the complement of \(C\). The map \(\Pi : C^2 \to \mathbb{R}^3\) is an orbit projection for the \(S^1\)-action \(e^{it}(z, w) = (e^{it} z, e^{-it} w)\) with \(\Pi(0) = p\) and \(\Pi(C) = \{0\} \times \mathbb{R}\).

**Proof.** The coordinate \(s\) is determined by \(e^{-u} e^t = h_0^{-1} z\), since \(|h_0| = |z|^{1/2} |w|^{1/2}\) we obtain \(e^{-u} = |z|^{1/2} |w|^{-1/2}\) and taking logarithms

\[
(3.14) \quad \int_0^s f(re^{i\theta}, q) dq = \frac{1}{2} \log \left(\frac{|z|}{|w|}\right).
\]

It is then clear that in the complement of \(\{zw = 0\}\) the map \(\Pi\) extends with the desired properties. We assume that \(|zw| < \epsilon\) for some small \(\epsilon\). Since \(re^{i\theta} = 1 - zw\), we can suppose that \(-\pi < \theta < \pi\). Let \(\tilde{\theta} = \beta\theta\), so that

\[
(3.15) \quad f(re^{i\theta}, q) = \frac{1}{2\sqrt{|re^{i\theta} - 1|^2 + q^2}} + \frac{F}{2}
\]

for some smooth harmonic positive function \(F = F(re^{i\theta}, q)\)-see Lemma \[1\]. Write \(\xi = re^{i\theta} - 1 = (1 - zw)^{\beta} - 1 = \beta zw\psi\) for some \(\psi\) holomorphic function of \(zw\) with \(\psi(0) = 1\). We plug \(3.15\) into \(3.14\) to obtain

\[
\log \left(\frac{s + \sqrt{s^2 + |\xi|^2}}{|\xi|}\right) + \int_0^s F = \log \left(\frac{|w|}{|z|}\right).
\]

We exponentiate and re-arrange terms to obtain

\[
(3.16) \quad 2s = \beta |w|^2 \psi e^{-\int_0^s F} - \beta |z|^2 \psi e^{\int_0^s F};
\]

note that in the standard case of \(\beta = 1\), we have \(F \equiv 0, \psi \equiv 1\) and therefore \(2s = |w|^2 - |z|^2\)-see \(2.10\). The claim follows from \(3.16\). The map \(\Pi\) sends the \(\{z = 0\}\) complex line to the ray \(\{(1, 0, s) : s > 0\}\) via
2 \pi e^{i q} F(1,0,q) dq = \beta |w|^2, the fact that \( F > 0 \) implies that \( s \to \int_0^s F \) is a diffeomorphism of \([0, \infty)\). Similarly, \( \Pi \) sends \( \{ w = 0 \} \) to \( \{ (1,0,s) : s < 0 \} \) via

\[
\int_0^s F(1,0,q) dq = -\beta |z|^2.
\]

\( \square \)

\( I \) is the standard complex structure in the \( z,w \) coordinates. Since \( I ds = f^{-1} \alpha \), it follows that \( \alpha \) extends as a connection 1-form on \( \mathbb{C}^2 \setminus \{ 0 \} \) which is smooth on the complement of the conic. It is then standard to show -see for example [1]- that \( g_{RF} \) extends smoothly over 0, since \( g \) is Euclidean in a neighborhood of \( p \), \( f \) differs from the Newtonian potential by a smooth harmonic function and \( \alpha \) is a smooth connection in a punctured neighborhood of 0. The function \( s \) is a moment map for the circle action, in the sense that

\[
\omega_{RF}(\vec{Y}, \cdot) = ds
\]

where \( \vec{Y} = 2 \text{Im} (w \partial/\partial w - z \partial/\partial z) \). \( s > 0 \) when \( |w| > |z| \) and \( s < 0 \) when \( |z| > |w| \).

The first item in Theorem 1 then follows from

\[
h_0^{-1} \partial h_0 / \partial r = (1/zw) cr^{c-1} e^{i \theta} \text{ and the formula } 3.4 \text{ for } a_1, a_2.
\]

To conclude this first part we compute the volume form of \( g_{RF} \). We write \( \Omega = (1/\sqrt{2}) dz dw \).

\textbf{Claim 6.}

\begin{equation}
\omega_{RF}^2 = \beta^2 |1 - wz|^2 \Omega \wedge \overline{\Omega}
\end{equation}

\textbf{Proof.} It is immediate from the previous computation of the Jacobian of the map \( H = (z,w) \) that

\[
\Omega \wedge \overline{\Omega} = (1/2) f^2 c^2 r^{2c-2} \eta_1 \eta_2 \eta_1 \eta_2.
\]

We use that

\[
\eta_1 \eta_1 = -2i \beta r dr d\theta, \quad \eta_2 \eta_2 = 2i f^{-1} ds \alpha, \quad \omega^2 = 2 f \beta r dr ds d\theta
\]
and \( r = |1 - zw|^β \) to conclude the claim. \( \square \)

### 3.2. Asymptotics

There is a simple explanation, in terms of the complex curve \( C \), for the exponent \(-2/β\) in Item 2 of Theorem 1. Among the diffeomorphisms \( F \) of \( C \) which, outside a compact set, take the conic \( C = \{zw=1\} \) to its asymptotic lines \( \{zw=0\} \), the ones which are closest to being holomorphic satisfy \( |∂F(x)| = O(|x|^{-2}) \). On the other hand \( ρ^2 = |z|^{2β} + |w|^{2β} \) and therefore \( |∂F(x)| = O(ρ^{-2/β}) \). Our proof of Item 2 is based on the simple observation that applying the Gibbons-Hawking ansatz to \( 2πΓ₀ \) gives rise to \( C_β \times C_β \). The asymptotic behavior of \( gRF \) then follows from the fact that \( Γ_p \) is asymptotic to \( Γ₀ \). The ‘matching map’ \( Φ \) is as a suitable bundle map.

We begin by writing \( g_F = β^3|u|^{2β−2}|du|^2 + β^3|v|^{2β−2}|dv|^2 \) as a Gibbons-Hawking metric. We use the cone coordinates \( u = ρ_1^{1/β}e^{iψ_1} \) and \( v = ρ_2^{1/β}e^{iψ_2} \) so that

\[
\Pi_0(u,v) = \left( βρ_1ρ_2e^{i(ψ_1+ψ_2)}, β\frac{ρ_2^2−ρ_1^2}{2} \right).
\]

This is an orbit map for the \( S^1 \)-action \( e^{it}(u,v) = (e^{it}u, e^{-it}v) \). If we let \( x = Π_0(u,v) \) then \( |x|^2 = (β^2/4)(ρ_1^2 + ρ_2^2)^2 \), equivalently

\[
βρ^2 = 2|x|.
\]

The derivative of the action is \( Y = \frac{∂}{∂ψ_1} − \frac{∂}{∂ψ_2} \) and \( |Y|^2_{g_F} = β^2ρ^2 = 1/(2β|x|) \). We let \( α_0 = |Y|^2_{g_F}g_F(Y,·) = ρ^{-2}(ρ_1^2ψ_1 − ρ_2^2ψ_2) \). It requires a simple computation to check that

\[
g_F = f_0g + f_0^{-1}α_0^2, \quad \text{with} \quad f_0 = \frac{1}{2β|x|}.
\]

Note that \( f_0 = 2πΓ₀ \).

Now let \((C^2,gRF)\) together with \( Π : C^2 \rightarrow \mathbb{R}^3 \). We let \( B \subset \mathbb{R}^3 \) a closed ball of radius 2, say, so that \( p \in B \). The bundles \( Π₀ \) and \( Π \) are isomorphic on the complement of \( B \), so there is an \( S^1 \)-equivariant diffeomorphism \( Φ : Π₀ \rightarrow Π^{-1}(\mathbb{R}^3 \setminus B) \) which induces the identity on \( \mathbb{R}^3 \setminus B \), in particular note that \( Φ(\{w=0\}) \subset \{zw=1\} \).

\( Φ_β^α \) is a connection on \( Π₀ \), therefore \( Φ_β^α − α_0 = η \) with \( η \) a 1-form on \( \mathbb{R}^3 \setminus B \). Moreover

\[
dη = −2π∗β d(Γ_p − Γ₀).
\]

On the other hand, since \( |Γ_p − Γ₀| = O(|x|^{-1−1/β}) \), we can assume -after changing gauge if necessary- that \( |α − α_0| = O(|x|^{-1−1/β}) \), we evaluate \( gRF \) in the orthonormal basis of \( g_F \) given by \( v_1 = f_0^{1/2}Y \) and \( v_2, v_3, v_4 \) the horizontal lifts of \( f_0^{-1/2}\frac{∂}{∂r}, f_0^{-1/2}(βr)^{-1}\frac{∂}{∂θ}, f_0^{-1/2}\frac{∂}{∂s} \) in our goal is to show that \( gRF(v_i,v_j) = δ_{ij} + O(|x|^{-1/β}) \) which is the same -due to 3.18- as \( |gRF − g_F|_{gF} = O(ρ^{-2/β}) \). Note that \( α(v_1) = f_0^{1/2} \) and \( α(v_j) = f_0^{-1/2}O(|x|^{-1−1/β}) \) for \( j = 2, 3, 4 \). This follows from straightforward computation, but before doing that we state a simple observation.

**Lemma 3.** Let \( f = O(|x|^{-a}) \) and \( f − g = O(|x|^{-b}) \) with \( 0 < a < b \) and \( f > 0 \). Then \( g/f = 1 + O(|x|^{−(b−a)}) \).

Indeed \( g/f = f/f + (g − f)/f \).

In particular, we conclude that \( f/f₀ = 1 + O(|x|^{-1/β}) \).

We proceed with our proof, we let \( 2 ≤ j, k ≤ 4 \) and \( j ≠ k \)

\[
gRF(v_1,v_1) = f^{-1}f₀ = 1 + O(|x|^{-1/β}), \quad gRF(v_j,v_j) = f₀^{-1}f + O(|x|^{-2/β}) = 1 + O(|x|^{-1/β})
\]

\[
 gRF(v_1,v_j) = f^{-1}O(|x|^{-1−1/β}) = O(|x|^{-1/β}), \quad gRF(v_j,v_k) = O(|x|^{-2/β}).
\]

Similarly, we can show that \( |Φ^∗ωRF − ωF|_{gF} = O(ρ^{-2/β}) \) and therefore \( |Φ^∗I − I|_{gF} = O(ρ^{-2/β}) \).

**Remark 1.** We can include derivatives in the statement of Item 2 as \( \nabla_X(Φ^∗gRF − gF)|_{gF} = O(ρ^{-2/β−1}) \) and so on; but care must be taken in not to differentiate in transverse directions to the cone singularities more than once.
3.3. Energy. It follows from \(\text{(2.21)}\) that the curvature operator of \(g_{RF}\) is given, up to the \(-f/2\) factor, by the trace free part of the Hessian of \(f^{-2}\) with respect to the \(g_\beta\) metric. In particular, close to the conic the curvature behaves as \(r^{1/\beta - 2}\) and this is unbounded when \(\beta > 1/2\). The norm-square of the curvature operator is \(O(r^{2/\beta - 4})\). Comparison with the integral \(\int_0^1 r^{2/\beta - 3} \, dr < \infty\) shows that \(|\text{Rm}(g_{RF})|^2\) is locally integrable.

According to our formula \(\text{(2.22)}\)
\[|\text{Rm}(g_{RF})|^2 = \frac{1}{4f} \Delta_\beta \Delta_\beta f^{-1}.\]
We want to compute \(\int_{C_\epsilon} |\text{Rm}(g_{RF})|^2\). We note that \(\Pi : (C^2, g_{RF}) \to (\mathbb{R}^3, f \cdot g)\) is a Riemannian submersion whose fiber over \(x\) is a circle of length \(2\pi f^{-1/2}(x)\). The volume form of \(f \cdot g\) is \(f^{3/2} dV_\beta\) and it is easy to conclude that
\[\|\text{Rm}(g)\|^2_{L^2} = (\pi/2) \int_{\mathbb{R}^3} \Delta_\beta \Delta_\beta f^{-1} \, dV_\beta.\]
In order to compute this quantity we use Stokes’ theorem
\[\int_{\Omega} \Delta_\beta \Delta_\beta f^{-1} \, dV_\beta = \int_{\partial \Omega} \langle D \Delta_\beta f^{-1}, \nu \rangle \, dA_\beta\]
for an increasing sequence of domains \(\Omega\).

There are two key lemmas

**Lemma 4.** Let \(C_r\) be a bounded cylinder consisting of points which are at distance \(r\) from the singular set \(S = \{0\} \times \mathbb{R}\). Then
\[\lim_{r \to 0} \int_{C_r} \langle D \Delta_\beta f^{-1}, \nu \rangle \, dA_\beta = 0\]
*Proof.* The lemma is a consequence of the \(\beta\)-smoothness of \(f^{-1}\) together with the fact that \(\Delta_\beta f = 0\). Indeed, since \(f\) is harmonic, \(\Delta_\beta f^{-1} = 2f^{-3} |Df|^2\). The \(\beta\)-smoothness then gives us \(|D \Delta_\beta f^{-1}| = O(r^{2\beta - 1})\) and therefore \(\int_{C_r} |D \Delta_\beta f^{-1}, \nu \rangle \, dA_\beta = O(r^{2\beta - 1})\).

**Lemma 5.** Let \(S_R\) denote the sphere of points which are at distance \(R\) from \(0\). Then
\[\lim_{R \to \infty} \int_{S_R} \langle D \Delta_\beta (f^{-1} - f_0^{-1}), \nu \rangle \, dA_\beta = 0\]
*Proof.*

\[\Delta_\beta (f^{-1} - f_0^{-1}) = f^{-3} |Df|^2 - f_0^{-3} |Df_0|^2 = f_0^{-3} |Df_0|^2 \left(1 - \frac{|Df|^2}{|Df_0|^2} \right)\]
\[= f_0^{-3} |Df_0|^2 = O(|x|^{-1})\] and \(f/f_0 = 1 + O(|x|^{-1/\beta})\). On the other hand, \(|Df|^2/|Df_0|^2 = O(|x|^{-1/\beta})\). We conclude that \(\Delta_\beta (f^{-1} - f_0^{-1}) = O(|x|^{-1/\beta})\). Note that \(\nu\) is tangential to \(S\), so that \(\langle D \Delta_\beta (f^{-1} - f_0^{-1}), \nu \rangle = O(|x|^{-1/\beta})\). We deduce that the integral is \(O(|x|^{-1/\beta})\).

It follows easily from these results that
\[\int_{\mathbb{R}^3} \Delta_\beta \Delta_\beta f^{-1} \, dV_\beta = \lim_{R \to \infty} \int_{S_R(0)} \langle D \Delta_\beta f_0^{-1}, \nu \rangle \, dA_\beta - \lim_{\epsilon \to 0} \int_{S_\epsilon(p)} \langle D \Delta_\beta f^{-1}, \nu \rangle \, dA_\beta.\]
Finally,
\[\lim_{\epsilon \to 0} \int_{S_\epsilon(p)} \langle D \Delta_\beta f^{-1}, \nu \rangle \, dA_\beta = -16\pi.\]
Indeed, \(g\) is isometric to the Euclidean metric in a neighborhood of \(p\) and we reduce to the standard situation where \(f = 1/2|x|\). We compute in spherical coordinates to obtain \(\Delta |x| = 2|x|^{-3}\) and \(\int_S \langle D(|x|), \nu \rangle \, dA = -4\pi\), for any sphere \(S\) centered at \(0\).
well-defined notion of a normal direction to the curve $C$. The extra factor $\beta^2$ comes from $f_0^{-1} = 2\beta|x|$ and $dA_\beta = \beta dA$.

This goes along the same lines as in the previous item, replacing the Euclidean metric with $g$. We can also ask in which directions the curvature blows-up. We can be more precise with the concept of direction at points $p$ and evaluate at $p$ to obtain $g_{1\Gamma,1\Gamma} + g_{2\Gamma,2\Gamma} = 0$; similarly differentiating with respect to $z_2$, we obtain $g_{1\Gamma,2\Gamma} + g_{2\Gamma,2\Gamma} = 0$. It follows that $g_{1\Gamma,1\Gamma} = g_{2\Gamma,2\Gamma}$; which is to say that the sectional curvature of the $\partial/\partial z_1$ and $\partial/\partial z_2$ planes at $p$ agree.

We go back to our setting, $g_{RF}$ is smooth in tangent directions to $C$ and it induces on it the metric of a rotationally symmetric negatively curved cylinder. The sectional curvature of $g_{RF}$ remains bounded as we approach the curve in either tangential or normal directions. The upshot is that if $p \in C$ and $P_p \cong \mathbb{C}P^1$ with $\perp$ the standard $\xi \rightarrow -1/\xi$ and the tangent and normal directions to the curve corresponding to the North and South poles; then the sectional curvature is invariant under $\perp$, bounded around the poles and unbounded around the equator.

**Quotients and limits when $\beta \rightarrow 0$.**

We study the case when $\beta = 1/n$ with $n$ a positive integer $\geq 2$. The Green’s function is explicit in this case, given by the Neumann reflection trick

$$\Gamma_p(x) = \frac{1}{4\pi} \sum_{j=0}^{n-1} \frac{1}{|x - p_j|},$$

with $x = (re^{i\theta}, s)$ and $p_j = (e^{2\pi i(j/n)}, 0)$.

Let $X$ be the standard $A_{n-1}$-ALE space determined by $p_0, \ldots, p_{n-1}$. The orthogonal transformation that fixes the $s$-axis and rotates by $\pi/n$ has a lift as an isometry of $X$, this lift is unique if we require that it fixes the points over the axis. This isometry generates a $\mathbb{Z}_n$-action and the quotient space is then identified with $(\mathbb{C}^2, g_{RF})$. When $n = 2$, $X$ is the Eguchi-Hanson space and there is an explicit Kähler potential for $g_{RF}$ which -up to a constant factor- is given by $\phi = (|z|^2 + |w|^2 + |1 - zw| + 1)^{1/2}$.

We let $g_n$ denote the metric $g_{RF}$ with $\beta = 1/n$. We want to know the possible Gromov-Hausdorff limits of the sequence $\{g_n\}$ as $n \rightarrow \infty$. First of all we note that $g_n$ are complete, in the sense that Cauchy sequences with respect to the induced distance converge. Indeed the standard proof for Gibbons-Hawking spaces applies in our case -see [1]-, the point being that the bundle projection is a Riemannian submersion onto a complete space. Since we are dealing with non-compact spaces we must choose points and talk about pointed Gromov-Hausdorff limits. We choose the points to be the ones fixed by the $S^1$-action. As we shall

\[\lim_{R \rightarrow \infty} \int_{S_n(0)} \langle D\Delta f_0^{-1}, \nu \rangle dA_\beta = -\beta^2 16\pi.\]

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see the curvature of $g_n$ blows-up at this point and if we re-scale in order to keep it bounded we obtain the Taub-Nut metric in the limit, in symbols $(C^2, \lambda_n g_n, 0) \to (C^2, g_{TN}, 0)$ with $\lambda_n \approx |\text{Rm}(g_n)|$ and

$$0 < \lim_{n \to \infty} \frac{|\text{Rm}(g_n)|(0)}{n \log n} < \infty.$$ 

We consider a unit circle in the plane with $n$ points equally separated. If we fix one of these points and consider the sum of the inverse distances to the others, then -up to a constant factor- the sum is $n(1 + 1/2 + \ldots + 1/n)$. We go back to the sequence $g_n$, with the marked points mapping to 0 $\in \mathbb{R}^3$ and conclude that in a small ball we can write the harmonic functions as $1/2|x| + (n \log n)F_n$ with $F_n$ converging uniformly to a positive constant. It is then easy to derive the claims made in the previous paragraph.

It is worth to point out that in the previous limit we are magnifying a neighborhood of 0 and pushing-off the cone singularities to infinity. On the other hand

$$\lim_{n \to \infty} \|\text{Rm}(g_n)\|^2_{L^2} = \lim_{n \to \infty} 8\pi^2 (1 - 1/n^2) = 8\pi^2 = \|\text{Rm}(g_{TN})\|^2_{L^2}.$$ 

So the metrics $g_n$ become nearly flat, as $n \to \infty$, around the conic. It is also tempting -by approximating large circles with a line- to compare the metrics $g_n$ for $n$ large with ‘the’ Ooguri-Vafa metric [3], obtained from the Gibbons-Hawking ansatz applied to the potential of infinitely many charges lying on a line and equally separated. However, as a word of caution, it must be said that the Ooguri-Vafa metric is not complete. The Ooguri-Vafa metric is indeed a one parameter family of metrics, parametrized by the distance between the charges. Scaling the metrics when the parameter tends to zero at the fixed point of the $S^1$-action recovers the Taub-Nut metric in the limit; we can use the triangle inequality to relate this sequence to $\lambda_n g_n$. However one can ask for a more precise correspondence, relating their associated harmonic functions -both admitting asymptotic expansions in terms of Bessel functions-.

We can also scale the metrics $g_{RF}$ so that their volume forms are $|1 - zw|^{2\beta - 2}\Omega \wedge \overline{\Omega}$ and then take the point-wise limit of these tensors as $\beta \to 0$, proceeding like this results into a degenerate limit $g_{RF} \to 0$. It is not known whether there is a Kähler metric on the complement of the conic with volume form $|1 - zw|^{-2}\Omega \wedge \overline{\Omega}$. Note that such a metric would be necessarily complete Ricci-flat and that $\pi_1(C^2 \setminus C) \cong \mathbb{Z}$.

**Variants.** As mentioned in [3], there are many variants of the construction. Finite sums of Green’s functions $\Gamma_p$ at different points give rise to Ricci-flat metrics with cone singularities on $A_n$ manifolds. It is also possible to consider several parallel wedges of an obtain metrics on $C^2$ with cone singularities along disjoint conics. Another variant is to add a positive constant term to the Green’s function to obtain analogs of (multi)-Taub-Nut spaces.

More interesting is the case of a curve $C \subset C^2$ which is invariant under an $S^1$-action different from the one we considered. For example $\{ w = z^2 \}$ and $\{ wz^2 = 1 \}$ are invariant under $(e^{it}z, e^{2it}w)$ and $(e^{it}z, e^{-2it}w)$ respectively. We can ask for $S^1$-invariant Ricci-flat metrics with cone singularities along the curve; but a suitable extension of the Gibbons-Hawking ansatz to the context of Seifert fibrations or $S^1$-actions which rotate the complex volume form seems not to be available.

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