NON-NORMAL EDGE RINGS SATISFYING \((S_2)\)-CONDITION

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Abstract. Let \(G\) be a finite simple connected graph on the vertex set \(V(G) = [d] = \{1, \ldots, d\}\), with edge set \(E(G) = \{e_1, \ldots, e_n\}\). Let \(K[t] = K[t_1, \ldots, t_d]\) be the polynomial ring in \(d\) variables over a field \(K\). The edge ring of \(G\) is the semigroup ring \(K[G]\) generated by monomials \(t^e := t_1^{e_1} \cdots t_d^{e_d}\), for \(e = \{i, j\} \in E(G)\). In this paper, we will prove that, given integers \(d\) and \(n\), where \(d \geq 7\) and \(d + 1 \leq n \leq \frac{d^2 - 7d + 24}{2}\), there exists a finite simple connected graph \(G\) with \(|V(G)| = d\) and \(|E(G)| = n\), such that \(K[G]\) is non-normal and satisfies \((S_2)\)-condition.

1. Introduction

Let \(G\) be a finite simple connected graph on the vertex set \(V(G) = [d]\) and let \(E(G) = \{e_1, \ldots, e_n\}\) be the edge set of \(G\). Let us consider, \(K[t] = K[t_1, \ldots, t_d]\) to be the polynomial ring in \(d\) variables over a field \(K\). For an edge \(e = \{i, j\} \in E(G)\), we define \(t^e := t_i t_j\). The subring of \(K[t]\) generated by \(t^{e_1}, \ldots, t^{e_n}\) is called the edge ring of \(G\), denoted by \(K[G]\). Let \(e_1, \ldots, e_d\) be the canonical unit coordinate vectors of \(\mathbb{R}^d\) and for each \(e = \{i, j\} \in E(G)\), we define \(\rho(e) := e_i + e_j\). Let \(S_G\) be the affine semigroup ring of \(S\). Then, the edge ring \(K[G]\) is the subring of \(S_G\).

For an affine semigroup \(S \subset \mathbb{N}^d\), let \(K[S]\) be the affine semigroup ring of \(S\). While studying for a characterization of the Cohen-Macaulay affine semigroup ring, Goto and Watanabe [2] have defined an extension \(S'\) of \(S\) and claimed that the condition, \(S' = S\) is the necessary and sufficient condition for \(K[S]\) to be Cohen-Macaulay. Trung and Hoa [3] presented a counterexample and also demonstrated that \(S' = S\) is insufficient to establish the Cohen-Macaulayness of \(K[S]\). They have also provided an additional topological condition on \(C_S\), the convex rational polyhedral cone spanned by \(S\) in \(\mathbb{Q}^d\) and characterized the Cohen-Macaulayness of \(K[S]\). Schäfer and Schenzel [7] Theorem 6.3 claimed that the condition \(S' = S\) corresponds to the Serre’s condition \((S_2)\). For the introduction of Serre’s condition \((S_2)\), readers may refer to [1] Section 2.

The Cohen-Macaulayness of the edge ring \(K[G]\) in terms of the corresponding graph \(G\) has been a subject of extensive research. Given that the edge ring \(K[G]\) is an affine semigroup ring, it is known from [5] Theorem 1 that, if \(K[G]\) is normal then \(K[G]\) is Cohen-Macaulay. Ohsugi and Hibi [6] have characterized the normality of an edge ring in terms of its graph. At about the same time, Simis-Vasconcelos-Villarreal independently came to the same conclusion and reported it in [8]. Recall from [1] Theorem 2.2.22] that, the edge ring \(K[G]\) is normal if and only if \(K[G]\) satisfies Serre’s conditions \((R_1)\) and \((S_2)\). In [3] Theorem 2.1, Hibi and Kathân
have characterized the edge rings satisfying $(R_1)$-condition. Note that Serre’s condition $(S_2)$ is a necessary condition for $K[G]$ to be Cohen-Macaulay. Based on these insights, Higashitani and Kimura \cite{4} have provided the necessary condition for an edge ring to satisfy the $(S_2)$-condition.

The main theorem that we will prove in this paper is as follows:

**Theorem 1.1.** Given integers $d$ and $n$ such that, $d \geq 7$ and $d + 1 \leq n \leq \frac{d^2 - 7d + 24}{2}$, there exists a finite simple connected graph $G$ with $|V(G)| = d$ and $|E(G)| = n$ such that, the edge ring $K[G]$ is non-normal and satisfies $(S_2)$-condition.

A detailed explanation of why the quadratic expression $\frac{d^2 - 7d + 24}{2}$ appears in the main theorem is provided in Section 6.

This paper is organized as follows. In Section 2, we revisit some basic prerequisite definitions and results that will be encountered throughout this article. Section 3 deals with introduction of the graph $G_{a,b}$, whose edge ring $K[G_{a,b}]$ is non-normal. In this section, we further prove that the edge ring $K[G_{a,b}]$ satisfies $(S_2)$-condition. Section 4 focuses on the step-wise removal of edges from $G_{a,b}$, such that each new graph obtained per step also satisfies both non-normality and $(S_2)$-condition. In Section 5, we prove that any addition of new edges to the graph $G_{a,b}$, either affects the non-normality of the edge ring or leads to the violation of $(S_2)$-condition. We conclude the article with Section 6, which deals with supporting evidence for Theorem 1.1 and the conclusions.

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2. Preliminaries

Let $G$ be a finite simple connected graph on the vertex set $V(G) = [d]$, with edge set $E(G) = \{e_1, \ldots, e_n\}$. Let us consider $K[t] = K[t_1, \ldots, t_d]$ to be the polynomial ring in $d$ variables over a field $K$. For an edge $e = \{i, j\} \in E(G)$, we define $t^e := t_it_j$. The subring of $K[t]$ generated by $t^{e_1}, \ldots, t^{e_n}$ is called the edge ring of $G$ and let it be denoted as $K[G]$.

We consider $e_1, \ldots, e_d$ to be the canonical unit coordinate vectors of $\mathbb{R}^d$. For some edge $e = \{i, j\} \in E(G)$, define $\rho(e) := e_i + e_j$. Let $A_G := \{\rho(e) : e \in E(G)\}$ and let $S_G$ be the affine semigroup generated by $\rho(e_1), \ldots, \rho(e_n)$. We can express, $S_G := \mathbb{Z}_{\geq 0}A_G$. Thus, the edge ring $K[G]$ is the affine semigroup ring of $S_G$.

Consider $C_G$ to be the convex rational polyhedral cone spanned by $S_G$ in $\mathbb{Q}^d$. We may assume that $C_G$ is of dimension $d$ and let $\mathcal{F}(G)$ be the set of all facets of $C_G$. We define, $\overline{S}_G := \mathbb{Q}_{\geq 0}A_G \cap \mathbb{Z}A_G$. For any facet $F \in \mathcal{F}(G)$, we define

$$S_F := S_G - S_G \cap F = \{x \in \mathbb{Z}A_G : \exists y \in S_G \cap F \text{ such that } x + y \in S_G\},$$

and $S_G' := \bigcap_{F \in \mathcal{F}(G)} S_F$. By definition, $A_G$ is said to be normal when we have

$$\mathbb{Z}_{\geq 0}A_G = \mathbb{Z}A_G \cap \mathbb{Q}_{\geq 0}A_G,$$

that is, when $\overline{S}_G = S_G$. 
A cycle is said to be minimal in $G$ if there exists no chord in it. Let $C$ and $C'$ be two minimal cycles of $G$ with $V(C) \cap V(C') = \emptyset$, if we have $i \in V(C)$ and $j \in V(C')$, then $e = \{i, j\} \in E(G)$ is called a bridge between $C$ and $C'$.

A pair of odd cycles $(C, C')$ is called exceptional if $C$ and $C'$ are minimal odd cycles in $G$ such that $V(C) \cap V(C') = \emptyset$ and there exists no bridge connecting them. We say that a graph $G$ satisfies odd cycle condition, if for any two odd cycles $C$ and $C'$ of $G$, either $V(C) \cap V(C') \neq \emptyset$ or there exists a bridge between $C$ and $C'$. In other words, the graph $G$ has no exceptional pairs.

**Theorem 2.1** (From [6, Theorem 2.2] and [8, Theorem 1.1]). Let $G$ be a finite simple graph. Then, the edge ring $K[G]$ is normal if and only if $G$ satisfies the odd cycle condition.

From all of the above observations, we have:

$$S_G = \overline{S_G} \iff K[G] \text{ is normal } \iff G \text{ satisfies odd cycle condition.}$$

Let us consider $U \subset V(G)$, and we define $G_U$ as the induced subgraph of $G$ with $V(G_U) = U$ and $E(G_U) = \{e \in E(G): e \subset U\}$. Let $i \in V(G)$ and we denote $G \setminus i$ as the induced subgraph of $G$ on the vertex set $V(G \setminus i) = V(G) \setminus \{i\}$. Consider a subset $T \subset V(G)$ and we define:

$$N(G; T) := \{v \in V(G): \{v, w\} \in E(G) \text{ for some } w \in T\}.$$ 

A subset $T \subset V(G)$ is called independent if $\{t_i, t_j\} \notin E(G)$ for any $t_i, t_j \in T$. For an independent set $T \subset V(G)$, we define a bipartite graph induced by $T$ as the graph on vertex set $T \cup N(G; T)$ with edge set $\{\{v, w\} \in E(G): v \in T, w \in N(G; T)\}$.

Now, we will be looking at the facets of $C_G$. For that, let us look at some important definitions and theorems that have been discussed in [6].

**Definition 2.2.** Let $G$ be a finite connected simple graph with vertex set $V(G)$. A vertex $v \in V(G)$ is said to be regular in $G$ if every connected component of $G \setminus \{v\}$ contains at least one odd cycle. A non-empty set $T \subset V(G)$ is said to be fundamental in $G$ if all the conditions below are satisfied by $T$:

1. $T$ is an independent set;
2. the bipartite graph induced by $T$ is connected;
3. either $T \cup N(G; T) = V(G)$ or every connected component of the graph $G_{V(G) \setminus T \cup N(G; T)}$ contains at least one odd cycle.

Facets of $C_G$ are given by the intersection of the half-spaces defined by the supporting hyperplanes of $C_G$, and was investigated by Ohsugi and Hibi [6].

**Theorem 2.3** (From [6, Theorem 1.7]). Let $G$ be a finite connected simple graph on the vertex set $[d]$, containing at least one odd cycle. Then, all the supporting hyperplanes of $C_G$ are as follows:

1. $\mathcal{H}_v = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d: x_v = 0\}$, where $v$ is a regular vertex in $G$.
2. $\mathcal{H}_T = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d: \sum_{i \in T} x_i = \sum_{j \in N(G; T)} x_j\}$, where $T$ is a fundamental set in $G$.

In this paper, we denote $F_v$ and $F_T$ as the facets of $C_G$ corresponding to the hyperplanes $\mathcal{H}_v$ and $\mathcal{H}_T$ respectively.
With reference to the work of Ohsugi and Hibi [6, Theorem 2.2], normalization of the edge ring $K[G]$ can be expressed as

$$S_G = S_G + \mathbb{Z}_{\geq 0}\{\mathbb{E}_C + \mathbb{E}_{C'} : (C, C') \text{ is exceptional in } G\},$$

where for any odd cycle $C$, we define $\mathbb{E}_C := \sum_{i \in V(C)} e_i$. We observe that,

$$2(\mathbb{E}_C + \mathbb{E}_{C'}) = \left(\sum_{e \in E(C)} \rho(e) + \sum_{e' \in E(C')} \rho(e')\right) \in S_G.$$

Through the work of Goto and Watanabe [2] and the investigation by Trung and Hoa [9], the necessary and sufficient conditions for $K[G]$ to be Cohen-Macaulay were established. Followed by this, Schäfer and Schenzel [7, Theorem 6.3] stated that, the condition $S'_G = S_G$ corresponds to the Serre’s condition $(S_2)$. In general, $S_G \subseteq S'_G \subseteq S_G$. Therefore, in order to prove that the edge ring $K[G]$ satisfies $(S_2)$-condition, it is enough to show that for any $\alpha \in S_G \setminus S_G$, the element $\alpha \notin S'_G$. This implies that $S'_G \subseteq S_G$ and therefore $S'_G = S_G$.

In [4], Higashitani and Kimura have provided the necessary condition that a graph $G$ has to hold in order to satisfy $(S_2)$-condition.

**Theorem 2.4** (From [4, Theorem 4.1]). Let $G$ be a finite simple connected graph. Suppose that, there exists an exceptional pair $(C, C')$ satisfying the following two conditions:

1. for each regular vertex $v \in V(G) \setminus [V(C) \cup V(C')]$ in $G$, both $C$ and $C'$ belong to the same connected component of the graph $G \setminus v$;
2. for each fundamental set $T \in G$ with $[V(C) \cup V(C')] \cap [T \cup N(G; T)] = \emptyset$, both $C$ and $C'$ belong to the same connected component of $G_{V(G) \setminus (T \cup N(G; T))}$.

Then, $\mathbb{E}_C + \mathbb{E}_{C'} \in S'_G$. In particular, $S_G \neq S'_G$.

3. The Graph $G_{a,b}$

This section explicitly deals with the construction and study of a special graph $G_{a,b}$, whose edge ring $K[G_{a,b}]$ is non-normal. We conclude the section by stating and proving a proposition that, the edge ring $K[G_{a,b}]$ satisfies $(S_2)$-condition.

Let $G_{a,b}$ be a simple finite connected graph with $|V(G_{a,b})| = d = a + b + 1$, where $3 \leq a \leq b$. We construct the graph $G_{a,b}$ (Figure 1) such that, it is formed by the union of two complete graphs $K_{a+1}$ and $K_{b+1}$ with exactly one common vertex. Let us consider the vertex set $V(G_{a,b}) = V(K_{a+1}) \cup V(K_{b+1})$, such that $V(K_{a+1}) = \{u_1, \ldots, u_a, w\}$ and $V(K_{b+1}) = \{v_1, \ldots, v_b, w\}$.

**Figure 1.** The graph $G_{a,b}$
Any vertex \( v \in V(G_{a,b}) \) is regular in \( G_{a,b} \). We observe that the fundamental sets in \( G_{a,b} \) are:
\[
\{i\}, \forall \ i \in V(G_{a,b}) \text{ and } \{i,j\}, \forall \ {i,j} \notin E(G_{a,b}).
\]

For any fundamental set \( T \subset V(G_{a,b}) \), let \( H_T \) be the bipartite graph induced by \( T \). We consider \( K_a \) to be the induced subgraph of \( G_{a,b} \) on vertex set \( V(K_{a+1}) \setminus \{w\} \), that is, the complete graph on \( V(K_a) = \{u_1,\ldots,u_a\} \). Similarly, \( K_b \) is the induced subgraph of \( G_{a,b} \) on the vertex set \( V(K_{b+1}) \setminus \{w\} \), which is the complete graph on \( V(K_b) = \{v_1,\ldots,v_b\} \).

Let us denote \( A_{G_{a,b}} := \{\rho(e) : e \in E(G_{a,b})\} \), \( S := S_{G_{a,b}} \) and \( \overline{S} := \overline{S_{G_{a,b}}} \). We consider \( C_S \) to be the convex rational polyhedral cone spanned by \( A_{G_{a,b}} \) in \( \mathbb{Q}^d \), i.e., \( C_S := C_{G_{a,b}} \). Let \( F(G_{a,b}) \) be the set of all facets of \( C_S \) and for any \( F_i \in F(G_{a,b}) \),
\[
S_i := S - S \cap F_i = \{x \in \mathbb{Z}A_{G_{a,b}} : \exists y \in S \cap F_i \text{ such that } x + y \in S\},
\]
\[
S' := \bigcap_{F_i \in F(G_{a,b})} S_i.
\]

Now, let us look at the facets of \( C_S \) in detail. We observe that, for any regular vertex \( v \in V(G_{a,b}) \),
\[
S \cap F_v = \mathbb{Z}_{\geq 0}A_{G_{a,b}} v.
\]

Corresponding to each fundamental set in \( G_{a,b} \), we have
\[
S \cap F_{\{w\}} = \mathbb{Z}_{\geq 0}A_{H_{\{w\}}} v,
\]
\[
S \cap F_{\{u_i\}} = \mathbb{Z}_{\geq 0}A_{H_{\{u_i\}}} \cup K_b,
\]
\[
S \cap F_{\{v_j\}} = \mathbb{Z}_{\geq 0}A_{K_a \cup H_{\{v_j\}}} v,
\]
\[
S \cap F_{\{u_i,v_j\}} = \mathbb{Z}_{\geq 0}A_{H_{\{u_i,v_j\}}} v,
\]
where \( 1 \leq i \leq a \) and \( 1 \leq j \leq b \). Note that, throughout this investigation, we are only concerned about the description of \( S \cap F_w \), for a regular vertex \( w \).

**Lemma 3.1.** Let the pair of odd cycles \((C,C')\) be exceptional in \( G_{a,b} \). Consider vertices \( v, w \in V(G_{a,b}) \), where \( w \) is the common vertex of \( K_{a+1} \) and \( K_{b+1} \). Let \( e_v \) and \( e_w \) be the canonical unit coordinate vectors of \( \mathbb{R}^d \) corresponding to vertices \( v \) and \( w \) respectively. Then,
\[
E_C + E_{C'} + e_v + e_w \in S.
\]

**Proof.** We consider an exceptional pair \((C,C')\) in \( G_{a,b} \). Without loss of generality, let \( C = \{u_{i_1},u_{i_2},u_{i_3}\} \) be a minimal odd cycle in \( K_{a+1} \) and \( C' = \{v_{j_1},v_{j_2},v_{j_3}\} \) be a minimal odd cycle in \( K_{b+1} \).

Since \( K_{a+1} \) and \( K_{b+1} \) are complete graphs with common vertex \( w \) and \((C,C')\) is exceptional in \( G_{a,b} \), we have \( V(C) \cap V(C') = \emptyset \) and \( w \notin V(C) \cup V(C') \).

Without loss of generality, we may assume \( v \in V(K_{a+1}) \). Given that \( K_{a+1} \) and \( K_{b+1} \) are complete graphs, for any \( u_{i_k} \in V(C) \) that is distinct from \( v \), we have \( \{v,u_{i_k}\} \in E(G_{a,b}) \) and for any \( v_{j_k} \in V(C') \), we have \( \{w,v_{j_k}\} \in E(G_{a,b}) \). Suppose, we choose \( u_{i_1} \neq v \). Then we can express
\[
E_C + E_{C'} + e_v + e_w = \sum_{k=1}^{3} e_{u_{i_k}} + \sum_{k=1}^{3} e_{v_{j_k}} + e_v + e_w
\]
\[
= (e_v + e_{u_{i_1}}) + \sum_{k=2}^{3} e_{u_{i_k}} + \sum_{k=1}^{2} e_{v_{j_k}} + (e_{v_{j_3}} + e_w).
\]
This can be written as,
\[
\mathbb{E}_C + \mathbb{E}_{C'} + e_v + e_w = \rho(\{v, u_1\}) + \rho(\{u_1, u_2\}) + \rho(\{v_1, v_2\}) + \rho(\{v_3, w\}).
\]

As we can see, the expression \(\mathbb{E}_C + \mathbb{E}_{C'} + e_v + e_w\) can be written as a linear combination of some \(\rho(e)\), where \(e \in E(G_{a,b})\). Therefore, for any \(v, w \in V(G_{a,b})\), we have \((\mathbb{E}_C + \mathbb{E}_{C'} + e_v + e_w) \in S\).

Let us consider \(x = (x_{u_1}, \ldots, x_{u_a}, x_{w}, x_{v_1}, \ldots, x_{v_b}) \in \mathbb{Z}_{\geq 0}^d\). We define a set,
\[
A := \left\{ x : x_w = 0, \sum_{u \in \{u_1, \ldots, u_a\}} x_u \text{ is odd, } \sum_{v \in \{v_1, \ldots, v_b\}} x_v \text{ is odd} \right\}.
\]

For the simple finite connected graph \(G_{a,b}\) (Figure 1) and the set \(A\) as defined above, we can state the following lemma.

**Lemma 3.2.** \(\overline{S} \subset S \cup A\).

**Proof.** Let \(\alpha\) be an arbitrary element in \(\overline{S}\). As we have seen in Section 2 the normalization of the edge ring \(K[G_{a,b}]\) can be expressed as
\[
\overline{S} = S + \mathbb{Z}_{\geq 0}\{\mathbb{E}_C + \mathbb{E}_{C'} : (C, C') \text{ is exceptional in } G_{a,b}\}.
\]

Therefore, any \(\alpha \in \overline{S}\) can be expressed as \(\alpha = \beta + \gamma\), where we have \(\beta \in S\) and \(\gamma \in \mathbb{Z}_{\geq 0}\{\mathbb{E}_C + \mathbb{E}_{C'} : (C, C') \text{ is exceptional in } G_{a,b}\}\). If \(\gamma = 0\), then \(\alpha \in S\). So, let us consider the non-trivial case where \(\gamma \neq 0\). Let \(\alpha_i, \beta_i\) and \(\gamma_i\) represent the \(i^{th}\) coordinates of \(\alpha, \beta\) and \(\gamma\) respectively.

Since for any two (possibly identical) exceptional pairs \((C, C'), (\overline{C}, \overline{C'})\), it follows from the completeness of the graphs \(K_{a+1}\) and \(K_{b+1}\) that \(\mathbb{E}_C + \mathbb{E}_{C'} + \mathbb{E}_{C'} + \mathbb{E}_{C} \in S\). Therefore, without loss of generality, for an exceptional pair \((C, C')\) in \(G_{a,b}\), we may assume that \(\gamma = \mathbb{E}_C + \mathbb{E}_{C'}\).

**Case 1.** Let \(\alpha_w = 0\).

We have \(\alpha_w = 0\) and \(\gamma \neq 0\) with \(\gamma_w = 0\). Therefore \(\beta_w = 0\), that is, we are not considering any edge adjacent to the common vertex \(w\). This assures that, both \(\sum_{u \in V(K_a)} \alpha_u\) and \(\sum_{v \in V(K_b)} \beta_v\) have to be even. Hence, both \(\sum_{u \in V(K_a)} \alpha_u\) and \(\sum_{v \in V(K_b)} \beta_v\) will be odd. Thus we have \(\alpha \in A\) and therefore, \(\alpha \in S \cup A\).

**Case 2.** Let \(\alpha_w > 0\).

Consider an exceptional pair \((C, C')\) in \(G_{a,b}\). We have \(\gamma = \mathbb{E}_C + \mathbb{E}_{C'}\). The condition \(\alpha_w > 0\) implies \(\beta_w > 0\). This indicates that there must be at least one edge adjacent to the common vertex \(w\), say \(\{v, w\}\). For any exceptional pair \((C, C')\) in \(G_{a,b}\), by Lemma 3.1, we have \(\mathbb{E}_C + \mathbb{E}_{C'} + e_v + e_w \in S\). Hence, \(\alpha = \beta + \gamma \in S\). Thus, it proves that \(\alpha \in S \cup A\).

All of the observations we have made so far lead us to the conclusion that,
\[
S \subset S' \subset \overline{S} \subset S \cup A.
\]

**Proposition 3.3.** Let \(G_{a,b}\) be a simple finite connected graph with \(3 \leq a \leq b\) and \(|V(G_{a,b})| = a + b + 1 = d\), such that \(G_{a,b}\) (Figure 1) consists of two complete graphs \(K_{a+1}\) and \(K_{b+1}\) joined at a common vertex \(w\). Let \(K[G_{a,b}]\) be the edge ring of the graph \(G_{a,b}\). Then, \(K[G_{a,b}]\) is non-normal and satisfies \((S_2)\)-condition.
Proof. Since $G_{a,b}$ is the union of two complete graphs with a common vertex $w$, it is assured that all the pairs of odd cycles of the form $\{(u_i, u_{i+1}, u_{i+2}), \{v_j, v_{j+1}, v_{j+2}\}\}$ where $1 \leq i \leq a - 2$ and $1 \leq j \leq b - 2$ are exceptional. Therefore, the graph $G_{a,b}$ does not satisfy the odd cycle condition and by Theorem 2.1, we conclude that the edge ring $K[G_{a,b}]$ is non-normal.  

Let us consider an element from the set $A$, say $\alpha = (x_{u_1}, \ldots, x_{u_a}, 0, x_{v_1}, \ldots, x_{v_b})$ such that $\alpha \in \overline{S} \setminus S$. We have seen that the common vertex $w$ is regular in $G_{a,b}$ and corresponding to this regular vertex we have $S \cap F_w = \mathbb{Z}_{\geq 0}A_{G_{a,b}\setminus w}$. For any $\beta \in S \cap F_w$, let $\beta_i$ represent the $i^{th}$ coordinate of $\beta$. We observe that $\beta_w = 0$ and both $\sum_{i=1}^{a} \beta_{u_i}$ and $\sum_{j=1}^{b} \beta_{v_j}$ are even. Therefore, for all $\beta \in S \cap F_w$, we have $\alpha + \beta \in A$ and not in $S$. Thus, there exists no $\beta \in S \cap F_w$, such that $\alpha + \beta \in S$. Therefore, $\alpha \notin S_w$. Hence, 

$$\alpha \notin \bigcap_{F_i \in F(G_{a,b})} S_i = S'.$$

This implies that, for any $\alpha \in \overline{S} \setminus S$, we have $\alpha \notin S'$. Hence, $(\overline{S} \setminus S) \cap S' = \emptyset$ and $S' \subset S$. We know that $S \subset S'$. Therefore, $S' = S$. □

We have proved that $K[G_{a,b}]$ is non-normal and satisfies $(S_2)$-condition. Now, we are interested in modifying the graph $G_{a,b}$, to see how the behaviour of the corresponding edge ring varies.

4. REMOVING EDGES OF $G_{a,b}$ AND $(S_2)$-CONDITION

In Section 3, we have studied the graph $G_{a,b}$ in detail. In this section, we will investigate whether we can remove edges of $G_{a,b}$ to obtain $\tilde{G}$, a subgraph of $G_{a,b}$, with $V(\tilde{G}) = V(G_{a,b})$ and $|E(\tilde{G})| = d + 1$ such that the edge ring $K[\tilde{G}]$ is non-normal and satisfies $(S_2)$-condition. We will modify a method of eliminating edges of $G_{a,b}$ so that the common vertex $w$ remains regular in any graph created using this method. By the end of this section, we prove that $K[\tilde{G}]$ is non-normal and satisfies $(S_2)$-condition.

By eliminating one edge from the graph $G_{a,b}$ per step, we will gradually build up $\tilde{G}$. First of all, we will be removing edges of the graph $K_{a+1}$ and will not alter the graph $K_{b+1}$. Let us denote $u_0 := w$ and $G_{0}^{u_i} := G_{a,b}$. We construct a new subgraph of $G_{a,b}$ through an edge removal process such that for each $1 \leq i \leq a - 3$,

- we remove the edge $\{u_0, u_i\}$ from $G_{0}^{u_i}$ to obtain $G_{0}^{u_i}$, and
- remove the edge $\{u_i, u_j\}$ from $G_{j-1}^{u_i}$ to obtain $G_{j}^{u_i}$, $\forall i + 1 \leq j \leq a - 1$.

Let us denote $G_{0}^{u_i} := G_{a-3}^{u_i}$, $\forall 2 \leq i \leq a - 2$. By construction, we observe that

$$E(G_{0}^{u_i}) = E(G_{a,b}) \setminus \bigcup_{p=1}^{i-1} \{u_p, u_q\} : q \neq p, 0 \leq q \leq a - 1, \forall 2 \leq i \leq a - 2.$$

For $i = a - 2$, the construction of the subgraph $G_{j}^{u_i}$, $a - 2 \leq j \leq a - 1$ is as follows:

- the subgraph $G_{a-2}^{u_{a-2}}$ is constructed by removing the edge $\{u_0, u_{a-2}\}$ from $G_{0}^{u_{a-2}}$;
- the edge $\{u_0, u_{a-1}\}$ is removed from $G_{a-2}^{u_{a-2}}$, to obtain the subgraph $G_{a-1}^{u_{a-2}}$. 
Remark 4.1. For $1 \leq i \leq a - 2$, $i \leq j \leq a - 1$ and $(i, j) \neq (a - 2, a - 1)$, we have

$$E(G_{j}^{u}) = E(G_{a,b}) \backslash \{u_p, u_q\} : q \neq p, (p, q) \in [a - 1] \times [i - 1] \cup \{(0, 1), (0, 2), \ldots, (0, i), (i, i + 1), \ldots, (i, j)\}.$$

In particular, if $1 \leq p < q \leq a$, then $\{u_p, u_q\} \in E(G_{j}^{u})$ if and only if one of the following cases happens:

(i) $p \geq i + 1$;
(ii) $p = i$, $j + 1 \leq q \leq a$;
(iii) $p \leq i - 1$, $q = a$.

If $1 \leq p < q < r \leq a$, then $u_p, u_q, u_r$ form a triangle in $G_{j}^{u}$ if and only if either of the following cases happens:

(a) $p = i$, $j + 1 \leq q$;
(b) $p \geq i + 1$.

Moreover, for $1 \leq t \leq a$, we have $\{u_t, w\} \in E(G_{j}^{u})$ if and only if $t \geq i + 1$.

Remark 4.2. For $(i, j) = (a - 2, a - 1)$, we have

$$E(G_{a-1}^{u_{a-2}}) = E(G_{a,b}) \backslash \{u_p, u_q\} : q \neq p, (p, q) \in [a - 1] \times [a - 3] \cup \{(0, 1), (0, 2), \ldots, (0, a - 1)\}.$$

In particular, if $1 \leq p < q$, then $\{u_p, u_q\} \in E(G_{a-1}^{u_{a-2}})$ if and only if one of the following cases happens:

(i) $p \leq a - 1$, $q = a$;
(ii) $(p, q) = (a - 2, a - 1)$.

If $1 \leq p < q < r \leq a$, then $u_p, u_q, u_r$ form a triangle in $G_{a-1}^{u_{a-2}}$ if and only if $(p, q, r) = (a - 2, a - 1, a)$. Moreover, $\{u_t, w\} \in E(G_{a-1}^{u_{a-2}})$ if and only if $t = a$.

An example of the sequence of subgraphs $G_{j}^{u}$; $1 \leq i \leq 2$ and $i \leq j \leq 3$ that is constructed from the graph $G_{4,3}$ using the edge removal process defined above is illustrated in Figure 2

$G_{4,3} =: G_{0}^{u_{1}} \rightarrow G_{1}^{u_{1}} \rightarrow G_{2}^{u_{1}} \rightarrow G_{3}^{u_{1}} =: G_{0}^{2}$

$G_{2}^{u_{2}} \rightarrow G_{3}^{u_{2}} \rightarrow G_{3}^{u_{2}}$  \(\{u_{0}, u_{2}\}\)

**Figure 2.** A sequence of subgraphs constructed from $G_{4,3}$

According to our construction, $V(G_{j}^{u}) = V(G_{a,b})$, $\forall 1 \leq i \leq a - 2$, $i \leq j \leq a - 1$. By the end of this entire process, we construct the subgraph of $G_{a,b}$, as shown in Figure 3

Let $C = \{u_{i_{1}}, u_{i_{2}}, \ldots, u_{i_{2l+1}}\}$ be an odd cycle in $G_{j}^{u}$, such that $2l + 1 \geq 5$ and $1 \leq i_{1} < i_{2} < \cdots < i_{2l+1}$. We have $\{u_{i_{1}}, u_{i_{2}}\} \in E(G_{j}^{u})$ and $\{u_{i_{1}}, u_{i_{2l+1}}\} \in E(G_{j}^{u})$. Thus according to our construction, $\{u_{i_{k}}, u_{k}\} \in E(G_{j}^{u})$, for all $i_{k} < k \leq i_{2l+1}$. Hence, for all $1 \leq i \leq a - 2$; $i \leq j \leq a - 1$, the minimal odd cycles of $G_{j}^{u}$ are cycles of length three.
Remark 4.3. Let $1 \leq i \leq a - 2$, $i \leq j \leq a - 1$ be integers. Let $C = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ be a triangle in $G^a_{i_2}$, where $1 \leq i_1 < i_2 < i_3$. Then from Remarks 4.1 and 4.2, $w$ is always adjacent to $u_{i_3}$. Indeed, if $(i, j) \neq (a - 2, a - 1)$ then $w$ is even adjacent to both $u_{i_2}$ and $u_{i_3}$. If $(i, j) = (a - 2, a - 1)$ then $(i_1, i_2, i_3) = (a - 2, a - 1, a)$ and $w$ is adjacent to $u_{i_3} = u_a$.

Remark 4.4. Let $1 \leq i \leq a - 2$, $i \leq j \leq a - 1$ be integers. Let $C$ and $\overline{C}$ be two (possibly identical) triangles in $G^a_{i_2}$. Then by Remarks 4.1 and 4.2, there is an edge connecting two vertices of $C$ and $\overline{C}$. Indeed, let $C = \{u_{i_1}, u_{i_2}, u_{i_3}\}$ and $\overline{C} = \{u_{j_1}, u_{j_2}, u_{j_3}\}$ where $1 \leq i_1 < i_2 < i_3$, $1 \leq j_1 < j_2 < j_3$. Then $i_2, j_2 \geq i + 1$, so $u_{i_2}$ is adjacent to either $u_{j_2}$ or $u_{j_3}$.

Lemma 4.5. Let $1 \leq i \leq a - 2$, $i \leq j \leq a - 1$ be integers. Let $(C, C')$ be an exceptional pair in $G^a_{i_2}$. If $\{w, v\} \in E(G^a_{i_2})$, then

$$E_C + E_{C'} + e_w + e_v \in S_{G^a_{i_2}}.$$

Proof. Since $(C, C')$ is an exceptional pair, we have $V(C) \cap V(C') = \emptyset$ and $w \notin V(C) \cup V(C')$. By Remark 4.3, we may assume that $V(C) = \{u_{i_1}, u_{i_2}, u_{i_3}\} \subset V(K_a)$ and $V(C') = \{v_{j_1}, v_{j_2}, v_{j_3}\} \subset V(K_b)$, where $1 \leq i_1 < i_2 < i_3$, $1 \leq j_1 < j_2 < j_3$.

Case 1. Let $v = u_k \in V(K_a)$. We claim that $E_C + e_v, E_{C'} + e_w \in S_{G^a_{i_2}}$. By Remark 4.3, $w$ is adjacent to $v_{j_3}$, so

$$E_C + e_w = \rho(\{w, v_{j_3}\}) + \rho(\{v_{j_1}, v_{j_2}\}) \in S_{G^a_{i_2}}.$$

Since $\{u_{i_1}, u_{i_2}\}, \{u_{i_1}, u_{i_3}\}$ are edges and $i_1 < i_2 < i_3$, by Remarks 4.4 and 4.2, $i_1 \geq i$ and hence $i_2 \geq i + 1$. Since $\{w, u_k\}$ is an edge, by the same results, $k \geq i + 1$. Hence Remarks 4.4 and 4.2 imply that $u_k$ is adjacent to either $u_{i_2}$ or $u_{i_3}$. This implies $E_C + e_v \in S_{G^a_{i_2}}$.

Case 2. Let $v \in V(K_b)$. We claim that $E_C + e_w, E_{C'} + e_v \in S_{G^a_{i_2}}$. Since $K_b$ is complete, $E_{C'} + e_v \in S_{G^a_{i_2}}$. By Remark 4.3, $w$ is adjacent to $u_{i_3}$. Hence $E_C + e_w \in S_{G^a_{i_2}}$.

In both cases, we get the desired containment. \hfill \square

For the graph $G^a_{i_2}$, where $1 \leq i \leq a - 2$, $i \leq j \leq a - 1$ and the set $A$ defined in Section 3, we have the following lemma.

Lemma 4.6. $\overline{S_{G^a_{i_2}}} \subset S_{G^a_{i_2}} \cup A$, for all $1 \leq i \leq a - 2$ and $i \leq j \leq a - 1$. 
Proof. Let $\alpha$ be an arbitrary element in $\overline{S_{G_j}}$, where $1 \leq i \leq a - 2$, $i \leq j \leq a - 1$. The normalization of the semigroup $S_{G_j}$ can be expressed as

$$\overline{S_{G_j}} = S_{G_j} + \mathbb{Z}_{\geq 0}\{E_C + E_{C'} : (C, C') \text{ is exceptional in } G_j\}.$$ 

Therefore, any $\alpha \in \overline{S_{G_j}}$ can be expressed as $\alpha = \beta + \gamma$, where $\beta \in S_{G_j}$ and $\gamma \in \mathbb{Z}_{\geq 0}\{E_C + E_{C'} : (C, C') \text{ is exceptional in } G_j\}$. If $\gamma = 0$, then $\alpha \in S_{G_j}$. So, let us consider the non-trivial case where $\gamma \neq 0$. Let $\alpha_k$, $\beta_k$ and $\gamma_k$ represent the $k$th coordinates of $\alpha$, $\beta$ and $\gamma$ respectively.

For any two (possibly identical) exceptional pairs $(C, C')$, $(C, C')$, using Remark 4.4 and from the completeness of $K_{b+1}$, we have $E_C + E_{C'} + E_{C''} + E_{C'''} \in S_{G_j}$, for all $1 \leq i \leq a - 2$ and $i \leq j \leq a - 1$. Therefore, without loss of generality, for an exceptional pair $(C, C')$ in $G_j$, we may assume that $\gamma = E_C + E_{C'}$.

**Case 1.** Let $\alpha_w = 0$.

We have $\gamma_w = 0$ and $\alpha_w = 0$. Therefore $\beta_w = 0$, that is, we are not considering any edge adjacent to the common vertex $w$. This assures that, both $\sum_{u \in V(K_u)} \beta_u$ and $\sum_{v \in V(K_v)} \beta_v$ have to be even. Hence, both $\sum_{u \in V(K_u)} \alpha_u$ and $\sum_{v \in V(K_v)} \alpha_v$ will be odd. Thus we have $\alpha \in A$.

**Case 2.** Let $\alpha_w > 0$.

The condition $\alpha_w > 0$ implies $\beta_w > 0$. This indicates that among the edges defining the vector $\beta$, there must be at least one edge adjacent to $w$, say $\{w, v\}$. For any exceptional pair $(C, C')$ in $G_j$, by Lemma 4.5, $E_C + E_{C'} + e_w + e_v \in S_{G_j}$, and thus $\alpha = \beta + \gamma \in S_{G_j}$.

Therefore, for all $1 \leq i \leq a - 2$, and $i \leq j \leq a - 1$, we can observe that,

$$S_{G_j} \subset S_{G_j} \subset \overline{S_{G_j}} \subset S_{G_j} \cup A.$$ 

**Proposition 4.7.** The edge ring $K[G_j]$ of the graph $G_j$ is non-normal and satisfies $(S_2)$-condition, for all $1 \leq i \leq a - 2$ and $i \leq j \leq a - 1$.

Proof. For any $1 \leq k \leq b - 2$, the pair $\{(u_{a-2}, u_{a-1}, u_a), \{v_k, v_{k+1}, v_{k+2}\}\}$ is always exceptional in $G_j$. Hence, $K[G_j]$ is always non-normal.

Let us consider an element $\alpha \in \overline{S_{G_j}} \setminus S_{G_j}$. By Lemma 4.5, we have $\alpha \in A$ and $\alpha_w = 0$. We observe that, the common vertex $w$ is regular in $G_j$. Hence, corresponding to $w$, we have $S_{G_j} \cap F_w := \mathbb{Z}_{\geq 0}\{e_w + e_v : v \in V(K_v)\}$. For any $\beta \in S_{G_j} \cap F_w$, let $\beta_k$ be the $k$th coordinate of $\beta$. We observe that $\beta_k = 0$ and both $\sum_{i=1}^a \beta_{ui}$, and $\sum_{j=1}^b \beta_{vj}$ are even. Therefore, for all $\beta \in S_{G_j} \cap F_w$, we have $\alpha + \beta \in A$ and not in $S_{G_j}$. Thus, there exists no $\beta \in S_{G_j} \cap F_w$, such that $\alpha + \beta \in S_{G_j}$, and this implies that, $\alpha \notin S_{G_j}$. As a result, we have $(\overline{S_{G_j}} \setminus S_{G_j}) \cap S_{G_j} = \emptyset$ and $S_{G_j} \subset S_{G_j}$. Therefore, $S'_{G_j} = S_{G_j}$.

Now, let us continue a similar edge removal process on $K_{b+1}$ and remove the maximum number of edges from $K_{b+1}$ resulting in the formation of the graph $\overline{G}$, as
per our requirement. Let \( v_0 := w \) and \( \widetilde{G}^{v_0}_{0} := G^{a_{-2}} \). We construct a new subgraph of \( G_{a,b} \) through an edge removal process such that for each \( 1 \leq i \leq b - 3 \),

- we remove the edge \( \{v_b, v_i\} \) from \( G^{v_0}_{0} \) to obtain \( \widetilde{G}^{v_0}_{i} \), and
- remove the edge \( \{v_i, v_j\} \) from \( \widetilde{G}^{v_0}_{j-1} \) to obtain \( \widetilde{G}^{v_0}_{j} \), \( \forall i + 1 \leq j \leq b - 1 \).

Let us denote \( \overline{G}^{v_0}_{0} := \overline{G}^{v_{i-1}}_{b-1} \), \( \forall 2 \leq i \leq b - 2 \). By construction, we observe that

\[
E(\overline{G}^{v_0}_{0}) = E(\overline{G}^{v_{i-1}}_{0}) \setminus \bigcup_{p=1}^{i-1} \{v_p, v_q\} : q \neq p, 0 \leq q \leq b - 1 \}, \ \forall 2 \leq i \leq b - 2.
\]

For \( i = b - 2 \), the construction of the subgraph \( \overline{G}^{v_{b-2}}_{b} \), \( b - 2 \leq j \leq b - 1 \) is as follows:

- the subgraph \( \overline{G}^{v_{b-2}}_{b} \) is constructed by removing the edge \( \{v_0, v_{b-2}\} \) from \( \overline{G}^{v_{b-2}}_{0} \), and
- the edge \( \{v_0, v_{b-1}\} \) is removed from \( \overline{G}^{v_{b-2}}_{b-2} \), to obtain the subgraph \( \overline{G}^{v_{b-2}}_{b-1} \).

As per construction, \( V(\overline{G}^{v_i}_{j}) = V(G_{a,b}) \), \( \forall 1 \leq i \leq b - 2 \), \( i \leq j \leq b - 1 \) and by the end of this removal procedure, we construct the graph depicted in Figure 4.

![Figure 4: The graph \( \overline{G} \)](image)

**Remark 4.8.** For \( 1 \leq i \leq b - 2 \), \( i \leq j \leq b - 1 \) and \( (i, j) \neq (b - 2, b - 1) \), we have

\[
E(\overline{G}^{v_0}_{i}) = E(\overline{G}^{v_{i-1}}_{0}) \setminus \bigcup_{p=1}^{i-1} \{v_p, v_q\} : q \neq p, (p, q) \in [b - 1] \times [i - 1] \\
\cup \{(0, 1), (0, 2), \ldots, (0, i), (i, i + 1), \ldots, (i, j)\}.
\]

In particular, if \( 1 \leq p < q \leq b \), then \( \{v_p, v_q\} \in E(\overline{G}^{v_i}_{j}) \) if and only if one of the following cases happens:

(i) \( p \geq i + 1 \);
(ii) \( p = i, j + 1 \leq q \leq b \);
(iii) \( p \leq i - 1, q = b \).

If \( 1 \leq p < q < r \leq b \), then \( v_p, v_q, v_r \) form a triangle in \( \overline{G}^{v_i}_{j} \) if and only if either of the following cases happens:

(a) \( p = i, j + 1 \leq q \);
(b) \( p \geq i + 1 \).

Moreover, for \( 1 \leq t \leq b \), we have \( \{v_t, w\} \in E(\overline{G}^{v_i}_{j}) \) if and only if \( t \geq i + 1 \).

**Remark 4.9.** For \( (i, j) = (b - 2, b - 1) \), we have

\[
E(\overline{G}^{v_{b-2}}_{b-1}) = E(\overline{G}^{v_{b-2}}_{0}) \setminus \bigcup_{p=1}^{b-3} \{v_p, v_q\} : q \neq p, (p, q) \in [b - 1] \times [b - 3] \\
\cup \{(0, 1), (0, 2), \ldots, (0, b - 1)\}.
\]
In particular, if $1 \leq p < q$, then $\{v_p, v_q\} \in E(\tilde{G}_b^{t-2})$ if and only if one of the following cases happens:

(i) $p \leq b - 1$, $q = b$;

(ii) $(p, q) = (b - 2, b - 1)$.

If $1 \leq p < q < r \leq b$, then $v_p, v_q, v_r$ form a triangle in $\tilde{G}_b^{t-2}$ if and only if $(p, q, r) = (b - 2, b - 1, b)$. Moreover, $\{v_i, w\} \in E(\tilde{G}_b^{t-2})$ if and only if $t = b$.

**Remark 4.10.** From Remarks 4.8 and 4.9 we see that the minimal odd cycles of $\tilde{G}^v_i$ are triangles. Let $C$ be a triangle of $\tilde{G}^v_i$, we claim that a vertex of $C$ is adjacent to $w$. If $V(C) \subseteq V(K_b)$, as $\tilde{G}^v_i$ is a subgraph of $\tilde{G}^a_i = G^a_{a-2}$, we must have $C = \{u_{a-2}, u_a, a\}$. In this case, $w$ adjacent to $u_a$.

If $C$ is a subgraph of $K_b$, let its vertices be $v_{j_1}, v_{j_2}, v_{j_3}$ where $1 \leq j_1 < j_2 < j_3 \leq b$. Then $w$ is adjacent to $v_{j_1}$, as Remarks 4.8 and 4.9 implies that $j_1 \geq i$ and $i + 1 \leq j_3$. In both cases, a vertex of $C$ is adjacent to $w$.

Moreover, for any two (possibly identical) triangles $C$ and $C'$ of $\tilde{G}^v_i$, whose vertices are inside $K_b$, there is an edge of $\tilde{G}^v_i$ connecting a vertex of $C$ to a vertex of $C'$.

**Lemma 4.11.** Let us consider an exceptional pair $(C, C')$ in $\tilde{G}^v_i$, where $1 \leq i \leq b - 2$ and $i \leq j \leq b - 1$. If $\{w, v\} \in E(\tilde{G}^v_i)$, then we have

$$E_C + E_{C'} + e_w + e_v \in S_{\tilde{G}^v_i}.$$ 

**Proof.** By Remark 4.10, we may assume that $V(C) \subseteq V(K_b)$ and $V(C') \subseteq V(K_b)$. The same remark implies that $C = \{u_{a-2}, u_{a-1}, u_a\}$. Let the vertices of $C'$ be $v_{j_1}, v_{j_2}, v_{j_3}$ where $1 \leq j_1 < j_2 < j_3 \leq b$.

**Case 1.** $v \in V(K_b)$. We claim that $E_C + e_v, E_{C'} + e_w \in S_{\tilde{G}^v_i}$. Given $\{w, v\} \in E(\tilde{G}^v_i)$, as per the construction of $\tilde{G}^v_i$, $v = u_a$. Since $C = \{u_{a-2}, u_{a-1}, u_a\}$, we see that $v$ is adjacent to both $u_{a-2}$ and $u_{a-1}$, so $E_C + e_v \in S_{\tilde{G}^v_i}$. By Remark 4.10 $w$ is adjacent to a vertex of $C'$, hence $E_{C'} + e_w \in S_{\tilde{G}^v_i}$.

**Case 2.** $v = v_k \in V(K_b)$. We claim that $E_C + e_w, E_{C'} + e_v \in S_{\tilde{G}^v_i}$. Since $w$ is adjacent to $u_{a-2}$ and $u_{a-1}$, so $E_C + e_w \in S_{\tilde{G}^v_i}$. By Remarks 4.8 and 4.9 $k \geq i + 1$. The same remarks imply that $j_1 \geq i$, $j_2 \geq i + 1$. Hence $v_k$ is adjacent to either $v_{j_2}$ or $v_{j_3}$. This yields $E_{C'} + e_v \in S_{\tilde{G}^v_i}$.

In both cases, we get the desired containment. 

For the set $A$ as defined in Section 3 and the graph $\tilde{G}^v_i$, where $1 \leq i \leq b - 2$, and $1 \leq j \leq b - 1$, we have the following lemma.

**Lemma 4.12.** $\overline{S_{\tilde{G}^v_i}} \subset S_{\tilde{G}^v_i} \cup A$, for all $1 \leq i \leq b - 2$ and $1 \leq j \leq b - 1$.

**Proof.** Let $\alpha$ be an arbitrary element in $\overline{S_{\tilde{G}^v_i}}$. Due to the similar edge removal process, the proof is similar to that of Lemma 1.6. By similar arguments as in the proof of Lemma 4.6 we reduce to the case $\alpha = \beta + \gamma$, where $\beta \in S_{\tilde{G}^v_i}$, $\gamma = E_C + E_{C'}$ for an exceptional pair $(C, C')$ of $\tilde{G}^v_i$. Furthermore, we also get that $\alpha \in A$ if $\alpha_w = 0$. Assume that $\alpha_w > 0$, then so is $\beta_w$. Hence among the edges defining the vector $\beta$, there is at least one edge of the form $\{w, v\}$. Using Lemma 4.11 we get that the semigroup $S_{\tilde{G}^v_i}$ contains $E_C + E_{C'} + e_w + e_v$, hence it also contains $\alpha$. 


From the above observations, we have \( S_{G_{j}'} \subseteq S'_{G_{j}'} \subseteq \overline{S_{G_{j}'}}, \subseteq S'_{G_{j}'} \cup A \), for all \( 1 \leq i \leq b - 2 \), and \( i \leq j \leq b - 1 \).

**Proposition 4.13.** The edge ring \( K[\overline{G_{j}''}] \) of the graph \( \overline{G_{j}''} \) is non-normal and satisfies \((S_2)\)-condition, for all \( 1 \leq i \leq b - 2 \) and \( i \leq j \leq b - 1 \).

**Proof.** As per our construction, the pair \( \{\{u_{a-2}, u_{a-1}, u_{a}\}, \{v_{b-2}, v_{b-1}, v_{b}\}\} \) is contained in every \( \overline{G_{j}''} \) and is exceptional, for all \( 1 \leq i \leq b - 2 \), and \( i \leq j \leq b - 1 \). Hence, \( K[\overline{G_{j}''}] \) is always non-normal.

Let us consider an element \( \alpha \in A \) such that \( \alpha \in S_{G_{j}'} \setminus S'_{G_{j}'} \). The common vertex \( w \) is regular in \( \overline{G_{j}''} \), and corresponding to this regular vertex, we have \( S_{G_{j}''} \cap F_{w} := \mathbb{Z}_{\geq 0}A_{G_{j}''} \setminus w \). By a similar proof as that of Proposition 4.13, we can demonstrate that there exists no \( \beta \in S_{G_{j}''} \cap F_{w} \), such that \( \alpha + \beta \in S'_{G_{j}'} \), and hence \( S'_{G_{j}'} = S_{G_{j}''} \). \( \square \)

Let the graph \( \overline{G_{b-1}'} := \overline{G} \) (Figure 5). We observe that, \( \overline{G} \) is a subgraph of \( G_{a,b} \) with \( |V(\overline{G})| = |V(G_{a,b})| \) and \( |E(\overline{G})| = a + b + 2 = d + 1 \). By Proposition 4.13, we know that the edge ring \( K[\overline{G}] \) is non-normal and also satisfies \((S_2)\)-condition. Therefore, we observe that \( \overline{G} \) is the graph on \( d \) vertices with the least number of edges, \( d + 1 \) edges, such that the edge ring is non-normal and meets \((S_2)\)-condition. This completes the proof of a part of the statement of Theorem 1.1.

5. Addition of Edges to \( G_{a,b} \) breaks Non-normality or \((S_2)\)-condition

In this section, we prove that any addition of (one or more) new edges to \( G_{a,b} \) either breaks the non-normality of the edge ring or violates the \((S_2)\)-condition.

Let us construct a new graph \( G' \) on the vertex set \( V(G') = V(G_{a,b}) \), by introducing one or more edges to \( G_{a,b} \). Since \( K_{a+1} \) and \( K_{b+1} \) are complete graphs, each of the new edges will be of the form \( \{u_i, v_j\} \), for some \( 1 \leq i \leq a \) and \( 1 \leq j \leq b \).

For instance, addition of a single edge \( \{u_2, v_3\} \) to the graph \( G_{a,b} \) is illustrated in Figure 5.

![Figure 5. The graph \( G' \) obtained by adding edge \( \{u_2, v_3\} \) to \( G_{a,b} \)](image)

We can observe that for any \( G' \), all of its vertices are regular and the fundamental sets are: \( \{i\} \), for some \( i \in V(G') \) and \( \{i, j\} \), \( \forall \{i, j\} \notin E(G') \).

**Proposition 5.1.** Let \( G' \) be a graph on the vertex set \( V(G') = V(G_{a,b}) \), such that \( G' \) is constructed by adding one or more new edges to \( G_{a,b} \). Then for any \( G' \), the edge ring \( K[G'] \) is either normal or it does not satisfy \((S_2)\)-condition.
Proof. Suppose we construct a graph \( G' \) by adding at least minimal number of edges \( \{u_i, v_j\} \), where \( 1 \leq i \leq a \) and \( 1 \leq j \leq b \), such that we connect all the exceptional pairs of \( G_{a,b} \). Thus, \( G' \) satisfies the odd-cycle condition and therefore the corresponding edge ring \( K[G'] \) is normal.

Now, let us consider the case where we construct a graph \( G' \) such that \( K[G'] \) is non-normal. Then, we prove that for any such \( G' \), the edge ring \( K[G'] \) will not satisfy \((S_2)\)-condition.

Suppose, we construct \( G' \) by adding new edges \( \{u_i, v_j\} \) to the graph \( G_{a,b} \), for some \( 1 \leq i \leq a \) and \( 1 \leq j \leq b \), such that \( G' \) consists of at least one pair of 3-cycles, \( \{(u_{i_1}, u_{i_2}, u_{i_3}), (v_{j_1}, v_{j_2}, v_{j_3})\} \) with either \( i_k \neq i \) or \( j_k \neq j \) for any \( 1 \leq k \leq 3 \). This pair will be exceptional in \( G' \) and thus, the corresponding edge ring \( K[G'] \) is non-normal.

Now, we consider any exceptional pair \((C, C')\) of the graph \( G' \). For any regular vertex \( v \in V(G') \) such that \( v \neq w \), we observe that \( G' \setminus v \) is a connected graph with the common vertex \( w \). Let us consider the regular vertex \( w \in V(G') \) such that \( w \neq v \). As per our construction, the graph \( G' \) contains edges of the type \( \{u_i, v_j\} \), for some \( 1 \leq i \leq a \) and \( 1 \leq j \leq b \). The existence of such edges in \( G' \) ensures the connectedness of the graph \( G' \setminus w \).

Thus for any regular vertex \( v \in V(G') \), we observe that the graph \( G' \setminus v \) is always a connected graph. Hence both \( C \) and \( C' \) belong to the same connected components of \( G' \setminus v \).

Since both \( K_{a+1} \) and \( K_{b+1} \) are complete graphs, any vertex in \( V(K_{a+1}) \) or \( V(K_{b+1}) \) is adjacent to all the other vertices of \( K_{a+1} \) and \( K_{b+1} \) respectively. Hence for all \( v \in V(G') \), we have \( |V(C) \cup V(C')| \cap \{v\} \cup N(G'; \{v\})| \neq \emptyset \). Let us consider the fundamental set of the form \( \{u_i, v_j\} \), such that \( \{u_i, v_j\} \notin E(G') \). By the completeness of \( K_{a+1} \) and \( K_{b+1} \), \( \{u_i, v_j\} \cup N(G'; \{u_i, v_j\}) = V(G') \). Hence for any fundamental set \( T \) of \( G' \), we have

\[
\left| V(C) \cup V(C') \right| \cap \left[ T \cup N(G'; T) \right] \neq \emptyset.
\]

Therefore, by Theorem 2.2, \( E_C + E_{C'} \in S_{G'}. \) In particular, \( S_{G'} \neq S_{G'}'. \) Hence, the edge ring \( K[G'] \) does not satisfy \((S_2)\)-condition.

6. Conclusions

A simple connected finite graph on \( d \) vertices with a non-normal edge ring must contain at least one exceptional pair of odd cycles. Thus the minimal graph on \( d \) vertices satisfying the above condition must be a graph consisting of two minimal odd cycles and a path (of at least length 2) connecting the two cycles. This minimal graph will have exactly \( d + 1 \) number of edges. In Section 4, we proved the existence of such a minimal graph \( \tilde{G} \), which satisfies the main theorem (Theorem 1.1).

We have examined the graph \( G_{a,b} \) in detail. From Section 5, we can conclude that any addition of (one or more) new edges to \( G_{a,b} \) either breaks the non-normality of the edge ring or violates \((S_2)\)-condition. Thus, we may conclude that \( G_{a,b} \) is the graph on \( d \) vertices with the maximum number of edges such that, the corresponding edge ring is non-normal and satisfies \((S_2)\)-condition. For the graph \( G_{a,b} \), we have \(|V(G_{a,b})| = d = a + b + 1\) and \( 3 \leq a \leq b \). Therefore, in order to maximize the number of edges in \( G_{a,b} \), we have to consider \( a = 3 \) and \( b = d - 4 \). That is,

\[
|E(G_{a,b})| \leq \frac{4}{2} + \left( \frac{d - 3}{2} \right) = \frac{d^2 - 7d + 24}{2}.
\]
This provides us very strong supporting evidence that \( \frac{d^2 - 7d + 24}{2} \) could be the maximal number of edges possible for a graph on \( d \) vertices such that, its edge ring is non-normal and satisfies \( (S_2)\)-condition.

Proof of Theorem 1.1 Let us consider the graph \( G_{3,b} \) on \( d \) vertices such that \( d \geq 7 \). We have \( |E(G_{3,b})| = \frac{d^2 - 7d + 24}{2} \) and by Proposition 3.3, the edge ring \( K[G_{3,b}] \) is non-normal and satisfies \( (S_2)\)-condition.

Through the edge removal processes discussed in Section 4, by eliminating one edge from the graph \( G_{3,b} \) per step, we can gradually build up a graph on \( d \) vertices with \( d + 1 \) edges such that, its edge ring is non-normal and satisfies \( (S_2)\)-condition. Proposition 4.7 and Proposition 4.13 guarantee that the edge ring of each of the graphs obtained after each removal step is always non-normal and will satisfy \( (S_2)\)-condition.

Therefore we prove that for any given integers \( d \) and \( n \) such that, \( d \geq 7 \) and \( d + 1 \leq n \leq \frac{d^2 - 7d + 24}{2} \), we can always construct a finite simple connected graph on \( d \) vertices and having \( n \) edges such that, the edge ring of the graph is non-normal and satisfies \( (S_2)\)-condition. \( \square \)

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