Universality of Uhrig dynamical decoupling for suppressing qubit pure dephasing and relaxation

Wen Yang and Ren-Bao Liu

Department of Physics, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong, China

The optimal $N$-pulse dynamical decoupling discovered by Uhrig for a spin-boson model [Phys. Rev. Lett, 98, 100504 (2007)] is proved to be universal in suppressing to $O(T^{N+1})$ the pure dephasing or the longitudinal relaxation of a qubit (or spin-1/2) coupled to a generic bath in a short-time evolution of duration $T$. It is also found that for the purpose of suppressing the longitudinal relaxation, an ideal Uhrig $\pi$-pulse sequence can be generalized to a sequence consisting of the ideal one superimposed with finite-duration pulses satisfying certain symmetry requirements.

PACS numbers: 03.65.Yz, 03.67.Pp, 76.20.+q, 33.25.+k

Introduction – A central topic in spin resonance spectroscopy [1] is the decoherence of spins due to coupling to environments, including the longitudinal relaxation of the population and the transverse relaxation of the phase correlation (i.e., dephasing) in the basis quantized along an external magnetic field [2,3]. Also, the decoherence of a qubit, which can be modeled by a spin-1/2, is the main obstacle in implementing scalable quantum computing [4]. To deal with the spin or qubit decoherence, various strategies have been developed, including quantum error correction [5,6,7,8], decoherence-free subspace [9,10], dynamical decoupling (DD) or bang-bang control [11,12,13,14,15,16,17,18,19,20,21,22,23,24], and dynamical control by pulse spectrum or shape engineering [23,26,27]. In particular, the DD suppresses the decoherence by eliminating the qubit-bath coupling through stroboscopic rotation of the qubit. An especially interesting DD scheme is the concatenated DD [18,19,20,21,22,23] which applies recursively a lower-order periodic pulse sequence as the building block of the next higher order sequence. For an evolution of a short duration $T$, an $N$th order concatenated DD eliminates the qubit-bath coupling up to $O(T^{N+1})$. The number of pulses in concatenated DD, however, increases exponentially with increasing order $N$. Since errors are inherently introduced by the controlling pulses, it is desirable to have DD sequences with the minimum number of controlling pulses.

An optimal DD scheme was first discovered by Uhrig for a pure dephasing spin-boson model [24], which uses $N \pi$-pulses applied at

$$T_j = T \sin^2 \frac{j \pi}{2(N+1)}, \quad \text{for } j=1,2,\ldots,N, \quad (1)$$

to eliminate the dephasing up to $O(T^{N+1})$. Optimal pulse sequences for $N \leq 5$ have also been noticed by Dhar et al. earlier for controlling the Zeno effect [28]. Lee, Witzel and Das Sarma conjectured that the Uhrig dynamical decoupling (UDD) may work for a generic pure-dephasing model with an analytical verification up to $N=9$ [29]. Later computer-assisted algebra was used to verify the conjecture up to $N=14$ [30]. Aiming at a general proof of the conjecture, Cardy and Dhar [31] have given a very inspiring though unsuccessful attempt by formulating the problem in a time-dependent perturbation theory.

In this Letter, we shall complete the proof of the universality of the UDD in suppressing the pure dephasing or the longitudinal relaxation of a qubit (or spin-1/2) coupled to a generic bath. The proof is based on the observation that to preserve the spin coherence up to a given order, one does not have to eliminate all terms of the effective qubit-bath coupling to the given order as in a generic concatenated DD but just needs to eliminate the terms relevant to the decoherence. An extension of the proof is that an ideal UDD sequence, for countering the longitudinal spin relaxation, can be replaced with a more general UDD sequence consisting of the same ideal $\pi$-pulses and some extra finite-duration pulses as long as certain symmetry requirements are fulfilled.

Ideal UDD for a generic pure dephasing Hamiltonian – Let us first consider the ideal UDD pulse sequences for a Hamiltonian of the form

$$\hat{H} = \hat{\mathcal{C}} + \hat{\sigma}_z \otimes \hat{Z}, \quad (2)$$

where $\hat{\sigma}_z$ is the qubit Pauli matrix along the $z$-direction, and $\hat{\mathcal{C}}$ and $\hat{Z}$ are bath operators. This Hamiltonian describes a pure dephasing model for it contains no qubit flip processes and therefore leads to no longitudinal relaxation but only transverse dephasing. Pure dephasing models are of special interest in quantum computing since very often the qubit flip terms can be essentially eliminated by applying a strong static magnetic field along the $z$-direction. A specific example is the spin-boson model in which $\hat{\mathcal{C}} = \sum_i \omega_i \hat{b}_i^\dagger \hat{b}_i$ and $\hat{Z} = \sum_i (\lambda_i/2) (\hat{b}_i^\dagger + \hat{b}_i)$ with $\hat{b}_i$ being a boson annihilation operator. It is for this spin-boson model that Uhrig [24] has discovered the optimal pulse sequences with timing given in Eq. (1).

Now we shall prove that the UDD applies for arbitrary $\hat{\mathcal{C}}$ and $\hat{Z}$. To overcome the pure dephasing, the ideal UDD sequences consist of $\hat{\sigma}_z$-pulse $\pi$-rotations about a transverse axis (say, the $x$-axis) [24]. The qubit dephasing is characterized by the decay of the expectation value of the raising or lowering operator $\hat{\sigma}_\pm \equiv \hat{\sigma}_x \pm i \hat{\sigma}_y$,

$$L_{\pm}(T) \equiv |\langle \hat{\sigma}_\pm(T) \rangle| = \left| \left\langle \hat{\sigma}_- \hat{\mathcal{C}}_1^{\dagger} \hat{\mathcal{C}}_2^{\dagger} \right\rangle \right|, \quad (3)$$

where the qubit state-dependent bath propagators under the
The key feature of the Fourier expansion to be exploited is that it contains only odd harmonics of \(\sin[(N + 1)\theta]\). With the Fourier expansion, we just need to show that

\[ \int_0^n dt_0 \cdots \int_0^{t_1} dt_2 \int_0^{t_1} \prod_{j=1}^n \cos (r_j \theta_j + q_j \theta_j) = 0, \tag{12} \]

for \(n\) being odd, \(r_j\) being an odd multiple of \((N + 1)\), and \(|\{q_j\}| \leq N\). With the product-to-sum trigonometric formula repeatedly used, it can be shown by induction that after an even number of variables \(\theta_1, \theta_2, \ldots, \theta_{2k}\) have been integrated over, the resultant integrand as a function of \(\theta_{2k+1}\) can be written as the sum of cosine functions of the form

\[ \cos (R_{2k+1} \theta_{2k+1} + Q_{2k+1} \theta_{2k+1}), \tag{13} \]

with \(R_{2k+1}\) being an odd multiple of \((N + 1)\) and \(|Q_{2k+1}| \leq \Sigma_{j=1}^n |q_j|\). In particular, the last step is

\[ \int_0^n \cos (R_n \theta_n + Q_n \theta_n) d\theta_n. \tag{14} \]

Since \(R_n\) is an odd (non-zero, of course) multiple of \((N + 1)\), and \(|Q| \leq \Sigma_{j=1}^n |q_j| \leq N\), we have \(R_n + Q_n \neq 0\) and the integral above must be zero. Thus Eq. (11) holds. The proof is done.

**Ideal UDD for suppressing longitudinal spin relaxation** – Now we consider the most generic qubit-bath Hamiltonian

\[ \hat{H} = \hat{C} \otimes \hat{X} + \hat{\sigma}_x \otimes \hat{Y} + \hat{\sigma}_z \otimes \hat{Z}, \tag{15} \]

where \(\hat{\sigma}_i\) are the Pauli matrices of the qubit and \(\hat{C}, \hat{X}, \hat{Y}, \text{and} \hat{Z}\) are bath operators. Without loss of generality, we assume the \(z\)-axis as the rotation axis for qubit control. We aim to show that the spin polarization along the rotation axis \(\langle \hat{\sigma}_z(T) \rangle\) is preserved up to \(O(T^{N+1})\) under the control of the \(N\)th order UDD. The spin polarization is

\[ \langle \hat{\sigma}_z(T) \rangle = \langle \hat{\sigma}_z^{(N)} \rangle, \tag{16} \]
where the propagator $\hat{U}$ can be written as
\[
\hat{U}^{(N)} = e^{-i(\hat{C}^* + (-1)^N \hat{D})(T - T_N)} \cdots e^{-i(\hat{C}^* - \hat{D})(T_2 - T_1)} e^{-i(\hat{C} + \hat{D})T_1},
\]
(17)
in which the Hamiltonian has been separated into $\hat{C}' \equiv \hat{C} + \hat{\sigma}_z \otimes \hat{Z}$ and $\hat{D} \equiv \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y}$. With the definition $\hat{D}_t(t) \equiv e^{i\hat{C}t}D e^{-i\hat{C}t}$, the propagator can be formally expressed as
\[
\hat{U}^{(N)} = e^{-i\hat{C}_T \hat{\sigma}} e^{-i\int_0^T F_{\hat{S}}(t) D_t(t) dt},
\]
(18)
which has the same form as Eq. (4). Following the same procedure as for proving Eq. (4), we find that up to $O\left(T^{N+1}\right)$, the expansion of the propagator contains only terms consisting of even power of $\hat{D}$. Since $\hat{D}$ contains only the Pauli matrices $\hat{\sigma}_x$ and $\hat{\sigma}_y$, and an even power of the two Pauli matrices $\hat{\sigma}_x^k \hat{\sigma}_y^k$ (with $n_x + n_y$ being even) is either unity or $i\hat{\sigma}_z$, the propagator
\[
\hat{U}^{(N)} = e^{-i\hat{H}_{\text{eff}} T} O(T^{N+1}),
\]
(19)
where the effective Hamiltonian $\hat{H}_{\text{eff}}$ commutes with $\hat{\sigma}_z$. Thus the $N$-pulse UDD eliminates the longitudinal qubit relaxation up to $O\left(T^{N+1}\right)$.

**UDD with non-ideal pulses:** Longitudinal relaxation –
With the help of Eq. (12), we realize that Eq. (10) holds for more general modulation functions $F_{\hat{N}}(t)$ as long as the scaled modulation function $f_N(t) \equiv F_{\hat{N}} \left(T \sin^2(\theta/2)\right)$ contains only odd harmonics of $\sin[(N+1)\theta]$ as in Eq. (11), i.e.,
\[
f_N(t) = \sum_{k=0}^{\infty} A_k \sin \left[(2k+1)(N+1)t\right],
\]
(20)
with arbitrary coefficients $A_k$. Motivated by this observation, we try to generalize the UDD to the case of non-ideal pulses.

Consider the control of the qubit by an arbitrary time-dependent magnetic field $B(t)$ applied along the $z$-direction, the general qubit-bath Hamiltonian is
\[
\hat{H}(t) = \hat{\mathcal{C}} + \hat{\sigma}_z \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{\gamma} + \hat{\sigma}_z \otimes \hat{Z} + \frac{1}{2} \hat{\sigma}_z B(t).
\]
(21)
In the rotating reference frame following the qubit precession under the magnetic field, the Hamiltonian becomes
\[
\hat{\mathcal{H}}_R(t) = \hat{\mathcal{C}}' + \cos[\phi(t)]\hat{D}^* + \sin[\phi(t)]\hat{D}^-,\n\]
(22)
where the precession angle $\phi(t) = \int_0^t B(t')\, dt'$, $\hat{C}' \equiv \hat{\mathcal{C}} + \hat{\sigma}_z \otimes \hat{Z}$, $\hat{D}^* \equiv \hat{\sigma}_x \otimes \hat{X} + \hat{\sigma}_y \otimes \hat{Y}$, and $\hat{D}^- \equiv \hat{\sigma}_x \otimes \hat{Y} - \hat{\sigma}_y \otimes \hat{X}$. In the rotating reference frame the propagator is
\[
\hat{U} = e^{-i\hat{\mathcal{C}}_T \hat{\mathcal{S}}^-} \exp \left\{ -i \int_0^T \sum_{k=\pm} F_{\hat{N}}(t) D^k(t) dt \right\},
\]
(23)
with $F_{\hat{N}}(t) \equiv \cos[\phi(t)]$, $F_{\hat{N}}(t) \equiv \sin[\phi(t)]$, and $D^k(t) \equiv e^{i\hat{\mathcal{C}}t}D^k e^{-i\hat{\mathcal{C}}t}$. To consider the qubit relaxation, we just need to examine the odd power of $D^k$ in the Taylor expansion of the propagator. The same way as derive Eq. (9), we find that for the $n$th power of $\hat{D}^k$, the expansion in $T$ has coefficients as long as the scaled modulation functions $f_{N}(\theta)$ have the following symmetries:

1. periodic with period of $2\pi/(N+1)$;
2. anti-symmetric with respect to $\theta = j\pi/(N+1)$;
3. symmetric with respect to $\theta = (j+1/2)\pi/(N+1)$.

The anti-symmetry condition requires $f_{N}(\theta)$ to be either zero or discontinuous at $\theta = j\pi/(N+1)$. But $f_{N}(\theta)$ and $f_{N}(\theta)$ cannot be simultaneously zero since they have to satisfy the normalization condition
\[
[f_{N}(\theta)]^2 + [f_{N}(\theta)]^2 = 1,
\]
(24)
according to the definition of $F_{\hat{N}}(t)$. So there must be sudden jumps at least in one of two modulation functions at $\theta = j\pi/(N+1)$, which means the controlling magnetic field $B(t)$ has to contain a $\delta$-pulse for $\pi$-rotation at $t = T_j$. With the initial conditions $f_{N}(0) = 1$ and $f_{N}(0) = 0$, one can choose the field such that $f_{N}(\theta)$ is continuous while $f_{N}(\theta)$ has sudden
jumps between +1 and −1 at $\theta = j\pi/(N + 1)$. Thus, a generalized UDD sequence can be chosen the following way: For $\theta \in [0, \pi/(2N + 2)]$, $f_N^\theta(t)$ can be arbitrary but sudden jumps from −1 to +1 at $\theta = 0$ and from +1 to −1 at $\pi/(2N + 2)$, and $f_N^\theta(t)$ is determined from the normalization condition as

$$f_N^\theta(t) = \pm \sqrt{1 - \left[f_N^\theta(t)\right]^2}.$$ At other regions, $f_N^\theta(t)$ are determined by the symmetry requirements. The pulse amplitude $B(t)$ for the generalized UDD is

$$B(t) = \frac{1}{F_N^\theta(t)} \frac{d}{dt} F_N^\theta(t) = \sum_{j=1}^{N} \pi \delta(t - T_j) + B_{\text{extra}}(t), \quad (25)$$

which is a superposition of the ideal UDD pulses and an extra component $B_{\text{extra}}(t)$ being arbitrary but subject to the symmetry requirements. The demand of $\delta$-pulses in the generalized UDD is consistent with the previous finding in Ref. $[32]$ that the effect of an ideal $\pi$-pulse on the evolution of a qubit coupled to a bath cannot be exactly reproduced by a pulse with a finite magnitude. An example of the scaled modulation functions and the corresponding magnetic field for the generalized 3rd order UDD control are shown in Fig. 1. Notice that due to the variable transformation from $\theta$ to $t$, the magnetic field $B(t)$ does not have the symmetries as the scaled modulation functions $f_N^\theta(t)$. For example, $B(t)$ is not periodic and the pulse at different time has different width.

Summary – To summarize, we have proven that with $N$ ideal $\delta$-pulses for $\pi$-rotations, the Uhrig dynamical decoupling can suppress the pure dephasing or the longitudinal relaxation of a qubit (or spin-1/2) coupled to an arbitrary bath, up to $O(T^{N+1})$. The qubit-bath coupling is not eliminated to the same order as in generic concatenated dynamical decoupling. But the remaining coupling would not induce the qubit decoherence under consideration as it commutes with the relevant observable of the qubit. As an extension of the proof, we also put forward a design of generalized UDD sequences which are the ideal UDD $\delta$-pulse sequences superimposed with arbitrary finite-duration pulses satisfying certain symmetry requirements. It should be pointed out that the present proof of the UDD applies either to pure dephasing or longitudinal relaxation and is limited to spin-1/2. It would be very interesting if the UDD can be generalized for simultaneous suppression of transverse and longitudinal relaxation and for higher spins.

This work was supported by Hong Kong RGC Project 2160285. We would like to thank Jian-Liang Hu for discussions.

* rbliu@phy.cuhk.edu.hk

[1] C. P. Slichter, Principles of Magnetic Resonance (Springer-Verlag, Berlin, 1992), 3rd ed.
[2] D. Pines and C. P. Slichter, Phys. Rev. 100, 1014 (1955).
[3] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, Decoherence and the Appearance of a Classical World in Quantum Theory (Springer, New York, 2003), 2nd ed.
[4] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University, Cambridge, 2000).
[5] P. W. Shor, Phys. Rev. A 52, R2493 (1995).
[6] A. M. Steane, Phys. Rev. Lett. 77, 793 (1996).
[7] E. Knill and R. Laflamme, Phys. Rev. A 55, 900 (1997).
[8] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997).
[9] L. M. Duan and G. C. Guo, Phys. Rev. Lett. 79, 1953 (1997).
[10] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. 81, 2594 (1998).
[11] M. Mehring, Principles of High Resolution NMR in Solids (Springer-Verlag, Berlin, 1983), 2nd ed.
[12] W.-K. Rhim, A. Pines, and J. S. Waugh, Phys. Rev. Lett. 25, 218 (1970).
[13] U. Haeberlen, High resolution NMR in solids: selective averaging (Academic Press, New York, 1976).
[14] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998).
[15] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999).
[16] L. Viola and E. Knill, Phys. Rev. Lett. 94, 060502 (2005).
[17] O. Kern and G. Alber, Phys. Rev. Lett. 95, 250501 (2005).
[18] K. Khodjasteh and D. A. Lidar, Phys. Rev. Lett. 95, 180501 (2005).
[19] K. Khodjasteh and D. A. Lidar, Phys. Rev. A 75, 062310 (2007).
[20] L. F. Santos and L. Viola, Phys. Rev. Lett. 97, 150501 (2006).
[21] W. Yao, R. B. Liu, and L. J. Sham, Phys. Rev. Lett. 98, 077602 (2007).
[22] W. M. Witzel and S. Das Sarma, Phys. Rev. B 76, 241303(R) (2007).
[23] W. X. Zhang, V. V. Dobrovitski, L. F. Santos, L. Viola, and B. N. Harmon, Phys. Rev. B 75, 201302(R) (2007).
[24] G. S. Uhrig, Phys. Rev. Lett. 98, 100504 (2007).
[25] G. S. Agarwal, Phys. Rev. A 61, 013809 (1999).
[26] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. 87, 270405 (2001).
[27] G. Gordon, G. Kurizki, and D. A. Lidar, Phys. Rev. Lett. 101, 010403 (2008).
[28] D. Dhar, L. K. Grover, and S. M. Roy, Phys. Rev. Lett. 96, 100405 (2006).
[29] B. Lee, W. M. Witzel, and S. Das Sarma, Phys. Rev. Lett. 100, 160505 (2008).
[30] G. S. Uhrig, arXiv:0803.1427 (2008).
[31] J. Cardy and D. Dhar, arXiv:0805.0970 (2008).
[32] S. Pasini, T. Fischer, P. Karbach, and G. S. Uhrig, Phys. Rev. A 77, 032315 (2008).