New gauge conditions in general relativity: what can we learn from them?

Giampiero Esposito * and Cosimo Stornaiolo †
Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d’Oltremare Padiglione 20, 80125 Napoli, Italy
Dipartimento di Scienze Fisiche, Università degli Studi di Napoli Federico II, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio G, 80126 Napoli, Italy

Abstract

The construction of conformally invariant gauge conditions for Maxwell and Einstein theories on a manifold $M$ is found to involve two basic ingredients. First, covariant derivatives of a linear gauge (e.g. Lorenz or de Donder), completely contracted with the tensor field representing the metric on the vector bundle of the theory. Second, the addition of a compensating term, obtained by covariant differentiation of a suitable tensor field built from the geometric data of the problem. If the manifold $M$ is endowed with a $m$-dimensional positive-definite metric $g$, the existence theorem for such a gauge in gravitational theory can be proved. If the metric $g$ is Lorentzian, which corresponds to general relativity, some technical steps are harder, but one has again to solve integral equations on curved space-time to be able to impose such gauges.

*Electronic address: giampiero.esposito@na.infn.it
†Electronic address: cosmo@na.infn.it
Our recent work on gauge conditions in (linearized) general relativity has been motivated by the Eastwood–Singer [1] derivation of conformally invariant gauge conditions for vacuum Maxwell theory. We are now going to describe the key steps of our construction.

For vacuum Maxwell theory in four dimensions in the absence of sources, the operator acting on the potential \( A^b \) is well known to be

\[
P_a^b = -\delta_a^b \Box + R_a^b + \nabla_a \nabla^b,
\]

(1)

where \( \nabla \) is the Levi–Civita connection on space-time, \( \Box \equiv g^{ab} \nabla_a \nabla_b \), and \( R_a^b \) is the Ricci tensor. Thus, the supplementary (or gauge) condition of the Lorenz type, i.e.

\[
\nabla^b A_b = 0,
\]

(2)

is of crucial importance to obtain a wave equation for \( A_b \). The drawback of Eq. (2), however, is that it is not preserved under conformal rescalings of the metric:

\[
\hat{g}_{ab} = \Omega^2 g_{ab}, \quad \hat{g}^{ab} = \Omega^{-2} g^{ab},
\]

(3)

whereas the Maxwell equations

\[
\nabla^b F_{ab} = 0
\]

(4)

are invariant under the rescalings (3). This remark was the starting point of the investigation by Eastwood and Singer [1], who found that a conformally invariant supplementary condition may be imposed, i.e.

\[
\nabla^b \left[ (\nabla^b \nabla^a - 2 R^b^a + \frac{2}{3} R g^{ba}) A_a \right] = 0.
\]

(5)

As is clear from Eq. (5), conformal invariance is achieved at the price of introducing third-order derivatives of the potential. In flat backgrounds, such a condition reduces to

\[
\Box \nabla^b A_b = 0.
\]

(6)

Of course, all solutions of the Lorenz gauge are also solutions of Eq. (6), whereas the converse does not hold.

Leaving aside the severe technical problems resulting from the attempt of quantizing in the Eastwood–Singer gauge [2], we are now interested in understanding the key features of the counterpart for Einstein’s theory of general relativity. In other words, although the vacuum Einstein equations

\[
R_{ab} - \frac{1}{2} g_{ab} R = 0
\]

(7)

are not invariant under the conformal rescalings (3), we would like to see whether the geometric structures leading to Eq. (5) admit a non-trivial generalization to Einstein’s theory, so that a conformally invariant supplementary condition with a higher order operator may be found as well. For this purpose, we re-express Eqs. (2) and (5) in the form

\[
g^{ab} \nabla_a A_b = 0,
\]

(8)
\[ g^{ab} \nabla_a \nabla_b \nabla^c A_c + \nabla_b \left[ \left( -2R^{ba} + \frac{2}{3}Rg^{ba} \right) A_a \right] = 0. \] (9)

Equation (8) involves the space-time metric in its contravariant form, which is also the metric on the bundle of 1-forms on \( M \). In Einstein’s theory, one deals instead with the vector bundle of symmetric rank-2 tensors on space-time with DeWitt supermetric

\[ E^{ab cd} \equiv \frac{1}{2} \left( g^{ac} g^{bd} + g^{ad} g^{bc} + \alpha g^{ab} g^{cd} \right), \] (10)

\( \alpha \) being a real parameter different from \(-\frac{2}{m}\), where \( m \) is the dimension of space-time (this restriction on \( \alpha \) is necessary to make sure that the metric \( E^{ab cd} \) has an inverse). One is thus led to replace Eq. (8) with the de Donder gauge

\[ W^a \equiv E^{ab cd} \nabla_b h_{cd} = 0. \] (11)

Hereafter, \( h_{ab} \) denotes metric perturbations in linearized general relativity. The supplementary condition (11) is not invariant under conformal rescalings, but the expression of the Eastwood–Singer gauge in the form (9) suggests considering as a “candidate” for a conformally invariant gauge involving a higher-order operator the equation

\[ E^{ab cd} \nabla_a \nabla_b \nabla_c \nabla_d W^e \left[ \left( \nabla_p \tilde{T}^pebc \right) + T^pebc \nabla_p \right] h_{bc} = 0. \] (12)

More precisely, Eq. (12) is obtained from Eq. (9) by applying the replacement prescriptions

\[ g^{ab} \rightarrow E^{ab cd}, \]
\[ A_b \rightarrow h_{ab}, \]
\[ \nabla^b A_b \rightarrow W^e, \]

with \( T^pebc \) a rank-4 tensor field obtained from the Riemann tensor, the Ricci tensor, the trace of Ricci and the metric. In other words, \( T^pebc \) is expected to involve all possible contributions of the kind \( R^pebc, R^pe g^{bc}, Rg^pe g^{bc} \), assuming that it should be linear in the curvature. The analysis of the full theory, however, shows that \( T^pebc \) is even more involved.

Indeed, if \( \gamma \) is a metric solving the full Einstein equations in vacuum, and \( g \) is a background metric, a gauge condition linear in \( \gamma \) which reduces to Eq. (12) in the linearized approximation may be written in the form [3]

\[ S^e(\gamma) \equiv E^{ab cd}(g) \nabla_a \nabla_b \nabla_c \nabla_d W^e(\gamma) + \nabla_p \tilde{T}^pe(\gamma) = 0, \] (13)

where

\[ W^e(\gamma) \equiv E^{ep qr}(g) \nabla_p \gamma_{qr}, \] (14)

and the connection \( \nabla \) annihilates \( g \) but not \( \gamma \). \( \tilde{T}^pe(\gamma) \) is a rank-2 tensor field to be determined (see below). We now study the behaviour of \( S^e(\gamma) \) under conformal rescalings of the physical metric \( \gamma \), since it is \( \gamma \) which solves the vacuum Einstein equations. It is then convenient to denote by \( Q^e(\gamma) \) the first term on the right-hand side of Eq. (13). On the one hand, the invariance of \( S^e \) under conformal rescalings of \( \gamma \) means that
\[ S^e(\Omega^2 \gamma) = \Omega^2 \left( Q^e(\gamma) + \nabla_p \tilde{T}^{pe}(\gamma) \right). \] (15)

On the other hand, the explicit calculation shows that
\[ S^e(\Omega^2 \gamma) = \Omega^2 Q^e(\gamma) + U^e + \nabla_p \tilde{T}^{pe}(\Omega^2 \gamma), \] (16)

where \( U^e \) depends on \( \Omega, \gamma \) and their covariant derivatives up to the fourth order. Its lengthy expression can be found in Sec. 5 of [3]. By virtue of (15) and (16), and of the identity
\[ \nabla_p \left( \Omega^2 \tilde{T}^{pe}(\Omega^2 \gamma) \right) = 2\Omega \Omega_p \tilde{T}^{pe}(\gamma) + \Omega^2 \nabla_p \tilde{T}^{pe}(\gamma), \] (17)

the desired equation for \( \tilde{T}^{pe}(\gamma) \) reads
\[ \nabla_p \left[ \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right] + 2\Omega \Omega_p \tilde{T}^{pe}(\gamma) = -U^e. \] (18)

Since \( \tilde{T}^{pe}(\gamma) \) can be arbitrarily chosen, it is sufficient to show that a particular class of such tensors exists for which Eq. (18) (and hence Eq. (15)) is satisfied. For this purpose, we assume that
\[ \nabla_q \left[ \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right] + 2\Omega \Omega_q \tilde{T}^{pe}(\gamma) = -\frac{1}{m} \delta_q^p U^e. \] (19)

Both sides of Eq. (19) can be contracted with a vector \( f^q \) on \((M, g)\). After defining the operator
\[ \mathcal{D} \equiv f^q \nabla_q, \] (20)

together with (the symbol \( \otimes_s \) represents symmetrized tensor product)
\[ \tilde{T}_(\cdot) \equiv \tilde{T}_{pe} dx^p \otimes_s dx^e = \tilde{T}_{(pe)} dx^p \otimes_s dx^e; \] (21)
\[ \tilde{T}_\wedge \equiv \tilde{T}_{pe} dx^p \wedge dx^e = \tilde{T}_{[pe]} dx^p \wedge dx^e; \] (22)
\[ (fU)_\cdot(\cdot) \equiv f_p U_e dx^p \otimes_s dx^e = f_{(p} U_{e)} dx^p \otimes_s dx^e; \] (23)
\[ (fU)_\wedge \equiv f_p U_e dx^p \wedge dx^e = f_{[p} U_{e]} dx^p \wedge dx^e; \] (24)

and introducing the kernel (with our notation, \( G_{\mathcal{D}}(x, y) \) is the Green kernel of the operator \( \mathcal{D} \))
\[ K_\Omega(x, y) \equiv 2G_{\mathcal{D}}(x, y)[\Omega \mathcal{D} \Omega](y) - \delta(x, y)\Omega^2(y), \] (25)

we eventually derive from Eq. (18) the integral equation
\[ [\tilde{T}_\circ(\Omega^2 \gamma)](x) + \int_M K_\Omega(x, y)[\tilde{T}_\circ(\gamma)](y) dV(y) = -\frac{1}{m} \int_M G_{\mathcal{D}}(x, y)(fU)\circ(y) dV(y), \] (26)
where the symbol ⋄ is a concise notation for the subscript ( ) or ∧ used in (21)–(24). The right-hand side of Eq. (26) is completely known for a given choice of the vector \( f^p \) and of the dimension \( m \) of \( M \).

If the metric \( \gamma \) is positive-definite, and if the operator \( \mathcal{D} \) defined in (20) is symmetric and elliptic on a compact Riemannian manifold \( M \) without boundary, with \( f^p \) so chosen that \( \mathcal{D} \) has no zero-modes, the solution of Eq. (26) can be reduced to the task of solving an infinite system of algebraic equations (see Eq. (5.40) of [3]).

In our analysis of conformal invariance of gauge conditions, it is crucial to consider conformal rescalings of the physical metric \( \gamma_{ab} \), while the background metric \( g_{ab} \) is kept fixed. We have done so because it is \( \gamma_{ab} \) which solves the Einstein equations, which are not conformally invariant. The consideration of general mathematical structures seems to suggest that a key ingredient is the addition of a “compensating term” \( \nabla_p \tilde{T}^{pe}(\gamma) \) to the higher-order covariant derivatives of the original gauge condition (see Eqs. (9) and (13)). Unlike the case of Maxwell theory in curved backgrounds, where conformal rescalings of the background metric are considered, we have therefore studied conformal rescalings of the physical metric only in general relativity. Still, it remains of interest for further research to consider conformal rescalings of both background and physical metric.

It also remains to be seen how to extend our results [3], which hold for positive-definite metrics, to the case of Lorentzian metrics, which are of course the object of interest in general relativity. We can however point out that the integral equation resulting from Eq. (18) does not depend on the signature of the metric, and hence the construction of \( \tilde{T}^{pe}(\gamma) \) remains non-local also in the Lorentzian case [3]. The Green kernels that one may want to use will be distinguished by various boundary conditions, and hence the Lorentzian framework will be actually richer in this respect.

It therefore seems that new perspectives are in sight in the investigation of gauge conditions in general relativity. They might be applied both in classical theory (linearized equations in gravitational wave theory, symmetry principles), and in the attempts of quantizing the gravitational field [3].

ACKNOWLEDGMENTS

This work has been partially supported by PRIN97 “Sintesi”.

5
REFERENCES

[1] M. Eastwood and I. M. Singer, Phys. Lett. A107 (1985) 73.
[2] G. Esposito, Phys. Rev. D56 (1997) 2442.
[3] G. Esposito and C. Stornaiolo, “A New Family of Gauges in Linearized General Relativity” (gr-qc 9812044).