THE CHOW RING OF THE STACK OF HYPERELLPTIC CURVES OF ODD GENUS

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Abstract. We find a new presentation of the stack of hyperelliptic curves of odd genus as a quotient stack and we use it to compute its integral Chow ring by means of equivariant intersection theory.

Contents
Introduction 1
1. Preliminaries on PGL_2-schemes 3
2. A new presentation of \( H_g \) as a quotient stack 7
3. Intersection theory of \( P(V_n) \) 17
4. The Chow ring of \( H_g \): generators and first relations 21
5. Other generators of im(\( i_\ast \)) 26
6. The Chow ring of \( H_g \): end of the computation 31
References 33

Introduction
There is a well defined intersection theory with integral coefficients for quotient stacks, first developed in [EG98], generalizing some ideas contained in [Tot98]. In [EG98] the authors defined the integral Chow ring \( A^\ast(\mathcal{X}) \) of a smooth quotient stack \( \mathcal{X} = [U/G] \), and they also showed that if \( \mathcal{X} \) is Deligne-Mumford, then the ring \( A^\ast(\mathcal{X}) \otimes \mathbb{Q} \) coincides with the rational Chow ring of Deligne-Mumford stacks, whose notion had already been introduced in [Gil84, Mum83, Vis89].

Since then, some explicit computations of integral Chow rings of interesting algebraic stacks have been carried on: in [EG98] the authors computed \( A^\ast(\mathcal{M}_{1,1}) \), the integral Chow ring of the compactified moduli stack of elliptic curves, and Vistoli in the appendix [Vis98] computed \( A^\ast(\mathcal{M}_2) \), the integral Chow ring of the moduli stack of curves of genus 2. Furthermore, among moduli stack of curves, the integral Chow ring of the stack of at most 1-nodal rational curves had been computed in [EF08] and the integral Chow ring of the stack of hyperelliptic curves of even genus had been determined explicitly in [EF09].

The main goal of this paper is to compute the integral Chow ring of \( H_g \), the moduli stack of hyperelliptic curves of genus \( g \), when \( g \geq 3 \) is an odd number. Our main result is the following:

Theorem. \( A^\ast(H_g) = \mathbb{Z}[\tau,c_2,c_3]/(4(2g+1)\tau,8\tau^2-2g(g+1)c_2,2c_3) \)

We also provide a geometrical interpretation of the generators of this ring.

The content of the theorem above had already been presented in the paper [FV11], but recently R.Pirisi pointed out a mistake in the proof of [FV11, lemma 5.6] which is crucial in order to complete the computation (for a more detailed
analysis of this, see lemma 5.8 and the following remark). Actually, the content of corollary 5.3 implies that the proof of [FV11, lemma 5.6] cannot be fixed, because its consequences, in particular [FV11, lemma 5.3], are wrong.

The main difference between our methods and the ones in [FV11] consists of the presentation of $H_g$ as a quotient stack that is used in order to carry on the equivariant computations. Indeed, in [FV11] the authors exploit a presentation that had been first obtained in [AV04], which involves the algebraic group $\text{PGL}_2 \times \mathbb{G}_m$. The group $\text{PGL}_2$ is a non-special group, i.e. there exist $\text{PGL}_2$-torsors over certain base schemes that are not Zariski-locally trivial but only étale-locally trivial. A consequence of this fact, which can be interpreted in numerous distinct ways, is that in general equivariant computations involving $\text{PGL}_2$ may be hard to carry on. For instance, not every projective space endowed with an action of $\text{PGL}_2$ can be seen as the projectivization of a $\text{PGL}_2$-representation: this makes the computation of the $\text{PGL}_2$-equivariant Chow ring of $\mathbb{P}^1$ a non trivial challenge.

On the other hand, equivariant computations involving a special group, i.e. a group $G$ such that every $G$-torsor can be trivialized Zariski-locally, are more approachable. An important example of special group is the general linear group $\text{GL}_n$.

A key result of the present work is the following theorem, where a presentation of $H_g$ as a quotient stack with respect to the action of a special group is explicitly obtained:

**Theorem.** There exists a scheme $U'$ such that $H_g = [U'/\text{GL}_3 \times \mathbb{G}_m]$

To obtain the presentation above, we introduce the notions of $\text{GL}_3$-counterpart of a $\text{PGL}_2$-scheme and of $\text{GL}_3 \times \mathbb{G}_m$-counterpart of a $\text{PGL}_2 \times \mathbb{G}_m$-scheme. More precisely, we give the following definitions:

**Definition.** Let $k$ be a field, and let $X$ be a scheme of finite type over $\text{Spec}(k)$ endowed with a $\text{PGL}_2$-action. Then the $\text{GL}_3$-counterpart of $X$ is a scheme $Y$ endowed with a $\text{GL}_3$-action such that $[Y/\text{GL}_3] \simeq [X/\text{PGL}_2]$.

**Definition.** Let $k$ be a field, and let $X$ be a scheme of finite type over $\text{Spec}(k)$ endowed with a $\text{PGL}_2 \times \mathbb{G}_m$-action. Then the $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $X$ is a scheme $Y$ endowed with a $\text{GL}_3 \times \mathbb{G}_m$-action such that $[Y/\text{GL}_3 \times \mathbb{G}_m] \simeq [X/\text{PGL}_2 \times \mathbb{G}_m]$.

We show that every $\text{PGL}_2$-scheme (resp. $\text{PGL}_2 \times \mathbb{G}_m$-scheme) admits a $\text{GL}_3$-counterpart (resp. $\text{GL}_3 \times \mathbb{G}_m$-counterpart), by explicit construction. These results are then applied to produce a new description of $H_g$ as a quotient stack: indeed, the presentation contained in [AV04] is of the form

$$H_g \simeq [(A(1,2g+2) \setminus \Delta')/\text{PGL}_2 \times \mathbb{G}_m]$$

where $A(1,2g+2)$ denotes the affine space of binary forms of degree $2g+2$, the closed subscheme $\Delta'$ is the hypersurface parametrising binary forms with multiple roots and the action is defined as

$$(A, \lambda) \cdot (f(x,y)) = \lambda^{-2} \det(A)^{g+1} f(A^{-1}(x,y))$$

By describing explicitly the $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $A(1,2g+2) \setminus \Delta'$, we obtain the new presentation.

The fact that $\text{GL}_3 \times \mathbb{G}_m$ is special enables us to use a set of new tools that were not available using the presentation of [AV04], and these new tools allows us to complete the computation of the integral Chow ring of the moduli stack of hyperelliptic curves of odd genus.
Description of contents. The whole paper is ideally divided in two parts, the first one that goes from section 1 to section 2 and is more stack-theoretical, the second one that goes from section 3 to the end which is more computational.

In section 1 we introduce the notion of GL₃-counterpart of a PGL₂-scheme, we prove the existence of a GL₃-counterpart for every PGL₂-scheme and some related results on equivariant Chow groups. In section 2 we give a new presentation of $H_g$, for $g \geq 3$ an odd number. We also introduce a class of vector bundles, denoted $V_n$, which will play a central role in the remainder of the notes. In section 3 we study some intersection theoretical properties of the projective bundles $P(V_n)$. In section 4 we begin the computation of the Chow ring of $H_g$, obtaining the generators and some relations. Section 5 is the technical core of the paper: here is where the new presentation of $H_g$ obtained in the second section will prove to be particularly useful in order to find other relations for the Chow ring of $H_g$. The computation of $A^*(H_g)$ is completed in section 6, where we also provide a geometrical interpretation of the generating cycles of $A^*(H_g)$. For the convenience of the reader, a more detailed description of the contents can be found at the beginning of every section.

We assume the knowledge of the basic tools of equivariant intersection theory. An excellent survey of the techniques used in this paper may be found in [FV, section 2]. We adopt a notation which is slightly different from the one adopted in [FV]: for us the ring $A^*_G(Spec(k))$ is generated by the cycles $\lambda_i$, for $i = 1, ..., n$, and the Chern classes $c_i$ are the elementary symmetric polynomials in $\lambda_1, ..., \lambda_n$ of degree $i$. In particular, this means that the GL₃-equivariant ring of a point is generated by $c_1, c_2$ and $c_3$. We will use this notation throughout the present work.

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1. Preliminaries on PGL₂-schemes

Fix a base field $k$. We begin with the following definitions:

**Definition 1.1.** Let $X$ be a scheme of finite type over $Spec(k)$ endowed with a PGL₂-action. Then a GL₃-counterpart of $X$ is a scheme $Y$ endowed with a GL₃-action such that $[Y/GL_3] \simeq [X/PGL_2]$.

**Definition 1.2.** Let $X$ and $X'$ be two schemes of finite type over $Spec(k)$ endowed with a PGL₂-action, and let $f : X \to X'$ be a proper PGL₂-equivariant morphism. Then a GL₃-counterpart of $f$ is a proper GL₃-equivariant morphism $g : Y \to Y'$ between two schemes endowed with a GL₃-action such that:

1. The scheme $Y$ (resp. $Y'$) is a GL₃-counterpart of $X$ (resp. $X'$).
2. The following diagram commutes:

$$
\begin{array}{ccc}
[X/PGL_2] & \xrightarrow{f} & [X'/PGL_2] \\
\downarrow \cong & & \downarrow \cong \\
[Y/GL_3] & \xrightarrow{g} & [Y'/GL_3]
\end{array}
$$

The existence of a GL₃-counterpart of a PGL₂-scheme has some consequences on equivariant Chow groups. Recall from [EG98] that if $X$ is a scheme of finite type over $Spec(k)$ on which an algebraic group $G$ acts, we can form the equivariant Chow groups $A^0_G(X)$, which can be shown to only depend on the quotient stack $[X/G]$, and thus can be thought as the integral Chow groups of $[X/G]$. 


Moreover, if $X \to X'$ is a proper $G$-equivariant morphism between two schemes both endowed with a $G$-action, there is an induced pushforward morphism between $A^G_i(X)$ and $A^G_i(X')$, that seen as a morphism between the Chow groups of the quotient stacks $A_i([X/PGL_2])$ and $A_i([X'/PGL_2])$ coincides with the pushforward morphism induced by the representable morphism $[X/PGL_2] \to [X'/PGL_2]$.

From this we deduce the following result:

**Proposition 1.3.** Let $f : X \to X'$ be a $PGL_2$-equivariant proper morphism between two schemes of finite type over $\text{Spec}(k)$ both endowed with a $PGL_2$-action, and let $g : Y \to Y'$ be its $GL_3$-equivariant counterpart. Then we have a commutative diagram of equivariant Chow groups of the form

$$
\begin{array}{ccc}
A^i_{PGL_2}(X) & \xrightarrow{f^*} & A^i_{PGL_2}(X') \\
\downarrow{\cong} & & \downarrow{\cong} \\
A^i_{GL_3}(Y) & \xrightarrow{g^*} & A^i_{GL_3}(Y')
\end{array}
$$

The following proposition assures us that definitions 1.1 and 1.2 are not useless:

**Proposition 1.4.** Let $f : X \to X'$ be a proper $PGL_2$-equivariant morphism between $PGL_2$-schemes. Then the morphism $f$ always admits a $GL_3$-counterpart.

In particular, the proposition above tells us that given a $PGL_2$-scheme $X$, we can always find a $GL_3$-counterpart $Y$. The remainder of this section is devoted to the proof of this statement.

Recall the definition of the moduli stack $M_0$ of smooth curves of genus 0, which is

$$
M_0(S) = \{(C \to S)\}
$$

where $S$ is a scheme over $\text{Spec}(k)$ and $C \to S$ is a smooth and proper morphism whose fibres are curves of genus 0. From now on, the relative scheme $C \to S$ will be called a *family of rational curves*.

It is well known that $M_0$ is an algebraic stack isomorphic to the classifying stack $BPGL_2$, and thus isomorphic to the quotient stack $[\text{Spec}(k)/PGL_2]$, where the action of the group on the point is the trivial one. In other words, the point $\text{Spec}(k)$ is a $PGL_2$-torsor over $M_0$.

The next proposition gives us another presentation of $M_0$ as a quotient stack, but before its statement we need to introduce some additional notation: let $\mathbb{A}(2,2)$ be the affine space parametrising trinary forms of degree 2, and let $S \subset \mathbb{A}(2,2)$ be the open subscheme of $\mathbb{A}(2,2)$ parametrising smooth trinary forms of degree 2. Then we have:

**Proposition 1.5.** The scheme $S$ is a $GL_3$-torsor over $M_0$ with action defined as $A \cdot q(X) := \det(A)q(A^{-1}X)$, where $X = (X_0, X_1, X_2)$. In particular, there is an isomorphism $[S/GL_3] \cong M_0$.

The proof of this proposition is postponed to the end of the section, as for the moment we prefer to show how it can be applied to prove proposition 1.4

**Proof of prop. 1.4.** Let $X$ be a $PGL_2$-scheme. Then we can form the quotient stack $[X/PGL_2]$, which fits into the cartesian diagram

$$
\begin{array}{ccc}
X & \to & [X/PGL_2] \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \to & M_0
\end{array}
$$
It is easy to check that the morphism of stacks \([X/PGL_2] \to \mathcal{M}_0\) is representable. We can form another cartesian diagram of the form

\[
\begin{array}{ccc}
Y & \rightarrow & [X/PGL_2] \\
\downarrow & & \downarrow \\
S & \rightarrow & \mathcal{M}_0
\end{array}
\]

and the representability of the right vertical morphism implies that \(Y\) is a scheme. Moreover, we see that \(Y\) must be a \(GL_3\)-torsor over the stack \([X/PGL_2]\). In other terms, there is an isomorphism \([Y/GL_3] \simeq [X/PGL_2]\).

If \(X \to X'\) is a proper \(PGL_2\)-equivariant morphism between two schemes both endowed with a \(PGL_2\)-action, this induces a proper morphism of quotient stacks \([X/PGL_2] \to [X'/PGL_2]\). We can actually pull back this proper morphism along \(S \to \mathcal{M}_0\), obtaining in this way a proper \(GL_3\)-equivariant morphism \(Y \to Y'\), where \(Y'\) is the \(GL_3\)-counterpart of \(X'\). In particular, the induced morphism \([Y/GL_3] \to [Y'/GL_3]\) coincides with the morphism \([X/PGL_2] \to [X'/PGL_2]\).

**Remark 1.6.** There is another way to obtain the \(GL_3\)-counterpart of a \(PGL_2\)-scheme. Recall that if we have a \(G\)-torsor \(X \to Z\) and a morphism of algebraic groups \(\varphi : G \to H\), we can construct the associated \(H\)-torsor of \(X\) as \(X \times^GH = X \times H/G\), where the (right) action of \(G\) is

\[
g \cdot (x, h) = (xg, \varphi(g)^{-1}h)
\]

Consider now the morphism of algebraic groups \(PGL_2 \to GL_3\) induced by the adjoint representation of \(PGL_2\). Then this morphism permits us to produce from the \(PGL_2\)-torsor \(X \to [X/PGL_2]\) a \(GL_3\)-torsor

\[
Y = X \times^{PGL_2} GL_3 \longrightarrow [X/PGL_2]
\]

and it can be checked that \(Y\) is the \(GL_3\)-counterpart of \(X\).

Now we give a proof of proposition \(\mathcal{M}_0\) stated at the beginning. We start with two technical lemmas:

**Lemma 1.7.** Let \(L\) be an invertible sheaf on a scheme \(\pi : X \to S\) such that \(\pi_*L\) is a globally generated locally free sheaf of rank \(n + 1\). Then giving an isomorphism \(\pi_*L \simeq \mathcal{O}_S^{n+1}\) induces a morphism \(f : X \to \mathbb{P}_S^n\) and an isomorphism \(f^*\mathcal{O}(1) \simeq L\), and vice versa.

**Proof.** The proof is standard, and basically follows from the canonical isomorphism \(\text{pr}_2^*\mathcal{O}_{\mathbb{P}_S^n \times S}(1) \simeq \mathcal{O}_S^{n+1}\). \(\square\)

**Lemma 1.8.** Let \(\pi : C \to S\) be a family of rational curves. Then the sheaf \(\pi_*\omega_{C/S}^{-1}\) is a locally free sheaf on \(S\) of rank \(3\) which satisfies the base change property. The morphism \(\pi_*\omega_{C/S}^{-1} \to \omega_{C/S}^{-1}\) is surjective and induces a closed immersion \(C \hookrightarrow \mathbb{P}(\pi_*\omega_{C/S}^{-1})\).

**Proof.** Follows from the base change theorem in cohomology applied to \(\pi_*\omega_{C/S}^{-1}\). \(\square\)

Consider the prestack in groupoids over the category \(\text{Sch}/k\) of schemes

\[
\mathcal{E}(S) = \left\{ (\pi : C \to S, \alpha : \pi_*\omega_{C/S}^{-1} \simeq \mathcal{O}_S^{\oplus 3}) \right\}
\]

where \(C \to S\) is a family of rational curves, and the morphisms

\[
(C \to S, \alpha : \pi_*\omega_{C/S}^{-1} \simeq \mathcal{O}_S^{\oplus 3}) \to (C' \to S', \alpha' : \pi'_*\omega_{C'/S'}^{-1} \simeq \mathcal{O}_{S'}^{\oplus 3})
\]
are given by triples \((\varphi : S' \to S, \psi : C' \simeq \varphi^* C, \phi : \pi'_* \omega_{C/S'}^{-1} \simeq \varphi^* \pi_\ast \omega_{C/S}^{-1})\), where \(\phi\) must commute with \(\alpha\) and \(\alpha'\). It can be easily checked that this prestack is equivalent to a sheaf.

Observe that there is a free and transitive action of \(\text{GL}_3\) on \(E\), which turns \(E\) into a \(\text{GL}_3\)-torsor sheaf over \(\mathcal{M}_0\). Consider also the auxiliary prestack
\[
E'(S) = \left\{ (D), \beta : i^* \mathcal{O}(1) \simeq \omega_{C/S}^{-1} \right\}
\]
where \((D)\) is a commutative diagram of the form
\[
\begin{array}{ccc}
C & \xrightarrow{i} & \mathbb{P}^2_S \\
\downarrow & & \downarrow \\
S & \xrightarrow{\alpha} & \mathbb{P}^2_S
\end{array}
\]
with \(C \to S\) a family of rational curves, and \(i\) a closed immersion. Recall that \(S = \mathbb{A}(2,2)_{\text{sm}}\) is the scheme parametrising smooth forms of degree 2 in three variables.

**Lemma 1.9.** There are isomorphisms \(E \simeq E' \simeq S\).

**Proof.** The first isomorphism follows from lemma 1.7. Suppose to have a commutative triangle
\[
\begin{array}{ccc}
C & \xrightarrow{i} & \mathbb{P}^2_S \\
\downarrow & & \downarrow \\
S & \xrightarrow{\alpha} & \mathbb{P}^2_S
\end{array}
\]
and an isomorphism \(\varphi : i^* \mathcal{O}(1) \simeq \omega_{C/S}^{-1}\). This one can be seen as a non-zero section of \(H^0(C, i^* \mathcal{O}(1) \otimes \omega_{C/S})\). Now we have the following chain of isomorphisms:
\[
\begin{align*}
H^0(C, i^* \mathcal{O}(1) \otimes \omega_{C/S}) &= H^0(\mathbb{P}^2_S, i_* (i^* \mathcal{O}(1) \otimes \omega_{C/S})) \\
&= H^0(\mathbb{P}^2_S, i_*(i^* (\mathcal{O}(1) \otimes \omega_{\mathbb{P}^2_S} \otimes \mathcal{I}^{-1}))) \\
&= H^0(\mathbb{P}^2_S, \mathcal{O}(-2) \otimes \mathcal{I}^{-1} \otimes i_* \omega_C)
\end{align*}
\]
where \(\mathcal{I}\) denotes the ideal sheaf of \(i(C) \subset \mathbb{P}^2_S\) and in the last line we used the projection formula and the canonical isomorphism \(\omega_{\mathbb{P}^2_S} \simeq \mathcal{O}(-3)\). If \(L := \mathcal{O}(-2) \otimes \mathcal{I}^{-1}\), then by twisting the exact sequence
\[
0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^2_S} \to i_* \omega_C \to 0
\]
by \(L\) and by taking the associated long exact sequence in cohomology, we easily deduce the isomorphism
\[
H^0(\mathbb{P}^2_S, \mathcal{O}(-2) \otimes \mathcal{I}^{-1} \otimes i_* \omega_C) \simeq H^0(\mathbb{P}^2_S, L)
\]
Now observe that a non-zero global section of \(L\) induces an isomorphism \(\mathcal{I} \simeq \mathcal{O}(-2)\), and vice versa.

Thus, by dualizing the injective morphism of sheaves \(\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^2_S}\) and by applying the isomorphism above, we obtain a morphism \(\mathcal{O}_{\mathbb{P}^2_S} \to \mathcal{O}(2)\), which is equivalent to choosing a global section \(q\) of \(\mathcal{O}(2)\), that will be smooth because of the hypotheses on \(C\).

It is easy to check that the induced morphism
\[
E' \to S, \quad (D, \varphi) \mapsto q
\]
is an isomorphism, whose inverse is given by sending \( q \) to the object \( ((D), \varphi) \), where \((D)\) is the commutative triangle

\[
\begin{array}{c}
Q \\
\downarrow \\
S
\end{array}
\]

and the isomorphism \( \varphi : i^*\mathcal{O}(1) \simeq \omega_{Q/S}^{-1} \) is induced by

\[
\mathcal{I}_Q \simeq \omega_{\mathbb{P}^2_S/(1)} \simeq \mathcal{O}(-2)
\]

where

**Proof of prop. 1.5.** From lemma 1.9 we readily deduce proposition 1.5. We only have to check that the action of \( \text{GL}_3 \) on \( S \) is the correct one, but this immediately follows from the isomorphism \( I \simeq \omega_{\mathbb{P}^2_S}(1) \) seen in the proof of lemma 1.9. \( \square \)

Now we give other two definitions that are useful for our purposes:

**Definition 1.10.** Let \( X \) be a scheme of finite type over \( \text{Spec}(k) \) endowed with a \( \text{PGL}_2 \times \mathbb{G}_m \)-action. Then a \( \text{GL}_3 \times \mathbb{G}_m \)-counterpart of \( X \) is a scheme \( Y \) endowed with a \( \text{GL}_3 \times \mathbb{G}_m \)-action such that \( [Y/\text{GL}_3 \times \mathbb{G}_m] \simeq [X/\text{PGL}_2 \times \mathbb{G}_m] \).

**Definition 1.11.** Let \( X \) and \( X' \) be two schemes of finite type over \( \text{Spec}(k) \) endowed with a \( \text{PGL}_2 \times \mathbb{G}_m \)-action, and let \( f : X \to X' \) be a proper \( \text{PGL}_2 \times \mathbb{G}_m \)-equivariant morphism. Then a \( \text{GL}_3 \times \mathbb{G}_m \)-counterpart of \( f \) is a proper \( \text{GL}_3 \times \mathbb{G}_m \)-equivariant morphism \( g : Y \to Y' \) between two schemes endowed with a \( \text{GL}_3 \times \mathbb{G}_m \)-action such that:

1. The scheme \( Y \) (resp. \( Y' \)) is a \( \text{GL}_3 \times \mathbb{G}_m \)-counterpart of \( X \) (resp. \( X' \)).
2. The following diagram commutes:

\[
\begin{array}{ccc}
[X/\text{PGL}_2 \times \mathbb{G}_m] & \xrightarrow{f} & [X'/\text{PGL}_2 \times \mathbb{G}_m] \\
\cong & & \cong \\
[Y/\text{GL}_3 \times \mathbb{G}_m] & \xrightarrow{g} & [Y'/\text{GL}_3 \times \mathbb{G}_m]
\end{array}
\]

Then, just as in the previous case, we have:

**Proposition 1.12.** Let \( f : X \to X' \) be a proper \( \text{PGL}_2 \times \mathbb{G}_m \)-equivariant morphism between \( \text{PGL}_2 \times \mathbb{G}_m \)-schemes. Then it always admits a \( \text{GL}_3 \times \mathbb{G}_m \)-counterpart.

**Proof.** The proof of the proposition above works exactly in the same way as the proof of proposition 1.3 one has only to take into account the action of \( \mathbb{G}_m \), but is immediate to check that \( A(2,2)_m \) is a \( \text{GL}_3 \times \mathbb{G}_m \)-torsor over \( B(\text{PGL}_2 \times \mathbb{G}_m) \), where the action of \( \mathbb{G}_m \) is the trivial one. From this the proposition easily follows. \( \square \)

2. A NEW PRESENTATION OF \( \mathcal{H}_g \) AS A QUOTIENT STACK

Fix an base field \( k \) of characteristic different from 2 and an odd integer \( g \geq 3 \). Let us stress the fact that \( g \) will always be odd, as this is a key property in most of the constructions presented here. Recall that by a *family of rational curves over \( S \) we mean a proper and smooth scheme over a \( k \)-base scheme \( S \) such that every fiber is a connected curve of genus 0. Then a *family of hyperelliptic curves of genus \( g \) over \( S \) is defined as a pair \((C \to S, \iota)\) where \( C \to S \) is a proper and smooth scheme over a base \( k \)-scheme \( S \) such that every fiber is a connected curve of genus \( g \), and \( \iota \in \text{Aut}(C) \) is an involution such that \( C/\langle \iota \rangle \to S \) is a family of rational curves.
Let $\mathcal{H}_g$ be the moduli stack of smooth hyperelliptic curves of genus $g$, so that
$$\mathcal{H}_g(S) = \{(C \to S, \iota)\}$$
where $(C \to S, \iota)$ is a family of hyperelliptic curves, and the morphisms are the isomorphisms over $S$ (the condition of commuting with the involutions is automatically satisfied). The goal of this section is to give a presentation of this stack as a quotient stack $[U'/\mathrm{GL}_2 \times \mathbb{G}_m]$, where $U'$ is an an certain scheme that will be defined later. This is done in theorem 2.8.

2.1. Properties of hyperelliptic curves. Now we briefly recall some basic facts about hyperelliptic curves (for an extensive treatment see [KK79]). Let $C \to S$ be a family of hyperelliptic curves of genus $g$. By definition there exists a global involution $\iota$ which induces the hyperelliptic involution on every geometric fiber.

There exists also a canonical, finite, surjective $S$-morphism $f : C \to C'$ of degree 2 that on each geometric fiber corresponds to taking the quotient w.r.t. the hyperelliptic involution. The scheme $C' \to S$ is a smooth family of rational curves. The morphism $f$ can also be described as the canonical morphism $f : C \to \mathbb{P}(\pi_*\omega_{C/S})$ whose image is $C'$.

Families of hyperelliptic curves have a canonical subscheme $W_{C/S}$, called the Weierstrass subscheme, that is the ramification divisor of $f$ endowed with the scheme structure given by the zeroth Fitting ideal of $\Omega^1_{C/S}$. It is finite and étale over $S$ of degree $2g + 2$, and its associated line bundle, when seen as an effective Cartier divisor, is the dualizing sheaf $\omega_f$ relative to the finite morphism $f$. Clearly, $f$ induces an isomorphism between $W_{C/S}$ and the branch divisor $D$ on $C'$.

2.2. Preliminaries on $\mathcal{H}_g$. Recall (see for instance [Par91]) that giving a family of hyperelliptic curves $C \to S$ of genus $g$ is the same as giving a family of rational curves $C' \to S$, a line bundle $L$ over $C'$ of degree $-g - 1$ and a global section $\sigma$ of $L^{-g^2}$ such that the zero locus of $\sigma$ is étale over $S$. This can be rephrased by saying that the stack $\mathcal{H}_g$ above is isomorphic to the following one:
$$\mathcal{H}_g^-(S) = \{([C' \to S, L, \sigma])\}$$
with morphisms between $([C' \to S, L, \sigma])$ and $([C'' \to S, M, \tau])$ given by isomorphisms $f : C' \simeq C''$ and $g : L \simeq f^* M$ which induce $\sigma \simeq f^* \tau$.

In [AV04] the authors exploited this isomorphism of stacks to produce a presentation of $\mathcal{H}_g$ as a quotient stack. Let us briefly recall what is their result, and how it is obtained.

Let $\mathbb{A}(1, 2g + 2)_{\text{sm}}$ be the scheme parametrising smooth binary forms of degree $2g + 2$. This scheme can be described as a stack in sets (i.e. a sheaf) over the category of schemes $\text{Sch}/k$ as follows: the objects of $\mathbb{A}(1, 2g + 2)_{\text{sm}}$ are pairs $(S, \sigma)$ where $S$ is a scheme and $\sigma$ is an element of $H^0(\mathbb{P}_S^1, O(2g + 2))$ whose zero locus is étale over $S$, and the only morphisms are identities.

Let us consider the prestack $\mathbb{A}(1, 2g + 2)_{\text{sm}}'$ over $\text{Sch}/k$ whose objects are
$$\mathbb{A}(1, 2g + 2)_{\text{sm}}'(S) = \{((\pi : C' \to S, L, \sigma, \phi, \psi))\}$$
where:
- $\pi : C' \to S$ is a family of rational curves.
- $L$ is a line bundle over $C'$ of degree $-g - 1$.
- $\sigma$ is a global section of $L^{-g^2}$.
- $\phi : C' \simeq \mathbb{P}_S^1$.
- $\psi : \pi_* (L \otimes O(g + 1)) \simeq O_S$.

The morphism $(\pi_1, L_1, \sigma_1, \phi_1, \psi_1) \to (\pi_2, L_2, \sigma_2, \phi_2, \psi_2)$ in $\mathbb{A}(1, 2g + 2)'$ is given by isomorphisms $f : C_1 \simeq C_2$ and $g : L_1 \simeq f^* L_2$ which induce $\sigma_1 \simeq f^* \sigma_2$ and are compatible with the isomorphisms $\phi_i, \psi_i$ for $i = 1, 2$. 

There is an obvious action of $\text{PGL}_2 \times \mathbb{G}_m$ over $\mathbb{A}((1,2g+2)_{\text{sm}}$: an element $(A, \lambda)$ of $\text{PGL}_2(S) \times \mathbb{G}_m(S)$ acts by multiplication on $(\phi, \psi)$, that is

$$(A, \lambda) \cdot (\phi, \psi) \mapsto (A \circ \phi, \lambda \cdot \psi)$$

It is immediate to verify that this makes $\mathbb{A}((1,2g+2)_{\text{sm}}$ into a $\text{PGL}_2 \times \mathbb{G}_m$-torsor over $\mathcal{H}_g$. It is also easy to see that $\mathbb{A}((1,2g+2)_{\text{sm}}$ is equivalent to the prestack whose objects are:

$$(\mathbb{P}^1_S \to S, \mathcal{O}(-g-1), \sigma)$$

where $\sigma$ is an element of $H^0(\mathbb{P}^1_S, \mathcal{O}(2g+2))$ whose zero locus is étale over $S$. Indeed, we are fixing an isomorphism $\mathbb{P}^1_S \simeq C'$ using $\phi$, and moreover $\psi$ induces an isomorphism $L \simeq \mathcal{O}(-g-1)$, from which our claim follows.

In other terms, $\mathbb{A}((1,2g+2)_{\text{sm}} \simeq \mathbb{A}((1,2g+2)_{\text{sm}}$ and the isomorphism is $\text{PGL}_2 \times \mathbb{G}_m$-equivariant, thus

$$\mathbb{A}((1,2g+2)_{\text{sm}}/\text{PGL}_2 \times \mathbb{G}_m \simeq \mathcal{H}_g^- \simeq \mathcal{H}_g$$

where the action of $\text{PGL}_2 \times \mathbb{G}_m$ is

$$(A, \lambda) \cdot (f(x,y)) = \lambda^{-2} \det(A)^{g+1} f(A^{-1}(x,y))$$

Therefore, a new presentation of $\mathcal{H}_g$ as a quotient stack with respect to the action of $\text{GL}_3 \times \mathbb{G}_m$ can be obtained by finding a $\text{GL}_3 \times \mathbb{G}_m$-counterpart of the $\text{PGL}_2 \times \mathbb{G}_m$-scheme $\mathbb{A}((1,2g+2)$.

Actually, in this section we will also study some $\text{GL}_3$-counterparts and $\text{GL}_3 \times \mathbb{G}_m$-counterparts of other schemes that will be relevant for our purposes.

### 2.3. Computation of $\text{GL}_3$-counterparts.

Let $\mathbb{A}((1,2n)$ be the affine space of the homogeneous polynomials of degree $2n$ in two variables. There is an action of $\text{PGL}_2$ on this scheme given by:

$$A \cdot f(x,y) := \det(A)^n f(A^{-1}(x,y))$$

We want to find a $\text{GL}_3$-counterpart of $\mathbb{A}((1,2n)$.

Let $\mathbb{A}((n,d)$ be the affine space parametrising homogeneous polynomials (forms) of degree $d$ in $n+1$ variables. This scheme represents the (free) sheaf

$$\mathbb{A}((n,d) : S \mapsto H^0(\mathbb{P}^n_S, \mathcal{O}(d))$$

The open subscheme parametrising smooth forms is denoted $\mathbb{A}((n,d)_{\text{sm}}$. Moreover, from now on, if $f$ is a form in three variables, its zero locus inside $\mathbb{P}^2_S$ will be denoted $F$. In other terms, with the capital letter we indicate the zero locus, whereas the lowercase letter will stand for the polynomial. For $n \geq 2$ we can define an injective morphism of free sheaves over $\mathbb{A}((2,2) \setminus \{0\}$ as follows:

$$\mathbb{A}((2,2) \setminus \{0\} \times \mathbb{A}((2, n-2) \mapsto \mathbb{A}((2,2) \setminus \{0\} \times \mathbb{A}((2, n), (q,f) \mapsto (q,qf)$$

The quotient is a locally free sheaf on $\mathbb{A}((2,2) \setminus \{0\)$, that we will call $V'_n$

Moreover, we define $V'_1$ to be the locally free sheaf $\mathbb{A}((2,1)$ over $\mathbb{A}((2,2) \setminus \{0\$. Let us present another characterization of these locally free sheaves. Observe that by definition they are the sheafification of the presheaves over $\mathbb{A}((2,2) \setminus \{0\)$ defined as

$$V'_n : (S \mapsto \mathbb{A}((2,2) \setminus \{0\) \mapsto H^0(\mathbb{P}^n_S, \mathcal{O}(n))/q \cdot H^0(\mathbb{P}^2_S, \mathcal{O}(n-2))$$

where $q$ is the non-zero form of degree 2 associated to the morphism $S \to \mathbb{A}((2,2) \setminus \{0\). Consider the short exact sequence of sheaves on $\mathbb{P}^2_S$ given by

$$0 \to \mathcal{O}(-2) \to \mathcal{O} \to i_*\mathcal{O}_Q \to 0$$

where the first non-trivial arrow is given by multiplication by $q$ and the last non-trivial sheaf is the pushforward of the structure sheaf of the smooth conic $Q$, defined as the zero locus of $q$, along its closed immersion in $\mathbb{P}^2_S$. After twisting by $\mathcal{O}(n)$ the
sequence remains exact, and the first terms of the associated long exact sequence in cohomology are

\[ 0 \rightarrow H^0(P^2 \mathbb{P}^2, \mathcal{O}(n-2)) \rightarrow H^0(P^2, \mathcal{O}(n)) \rightarrow H^0(Q, \mathcal{O}_Q(n)) \rightarrow 0 \]

This implies that the sheaf \( V'_n \) can also be characterized as the sheafification of

\[ V'_n : (S \xrightarrow{\psi} \mathbb{A}(2, 2) \setminus \{0\}) \rightarrow H^0(Q, \mathcal{O}_Q(n)) \]

We can also consider the projectivization \( \mathbb{P}(V'_n) \) of the vector bundle \( V'_n \), that as a sheaf over \( \mathbb{A}(2, 2) \setminus \{0\} \) coincides with the sheafification of

\[ \mathbb{P}(V'_n) : (S \xrightarrow{\psi} \mathbb{A}(2, 2) \setminus \{0\}) \rightarrow (H^0(Q, \mathcal{O}_Q(n)) \setminus \{0\}) / G_m(S) \]

where \( G_m(S) \) acts by multiplication. Finally, we define the action of \( \text{GL}_3 \) over \( \mathbb{A}(2, 2) \setminus \{0\} \times \mathbb{A}(2, n) \) as follows:

\[ A \cdot (q(X), f(X)) := (\det(A)q(A^{-1}X), f(A^{-1}X)) \]

where \( X = (X_0, X_1, X_2) \). This action passes to \( V'_n \) and \( \mathbb{P}(V'_n) \). From now on we will focus on the open subscheme \( S := \mathbb{A}(2, 2)_{\text{sm}} \)

\textbf{Definition 2.1.} Let \( V'_n \) be the vector bundles over \( \mathbb{A}(2, 2) \setminus \{0\} \) defined as the cokernel of

\[ (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{A}(2, n-2) \hookrightarrow (\mathbb{A}(2, 2) \setminus \{0\}) \times \mathbb{A}(2, n), \quad (q, f) \mapsto (q, qf) \]

Then we define the vector bundles \( V_n \) as the restrictions of \( V'_n \) to \( S := \mathbb{A}(2, 2)_{\text{sm}} \).

Before going on, let us describe the scheme \( V_n \) as a stack in sets over the category \( \text{Sch}/k \) of schemes:

\[ V_n(S) = \{(q, f)\} \]

where:

- \( q \) is a global section of \( \mathcal{O}_{\mathbb{P}^2}(2) \) whose zero locus \( Q \subset \mathbb{P}^2 \) is smooth over \( S \).
- \( f \) is a global section of \( \mathcal{O}_Q(n) \).

This description of \( V_n \) as a stack in sets will be frequently used in this section.

We are ready to give explicit descriptions of some \( \text{GL}_3 \)-counterparts. We begin with the following:

\textbf{Proposition 2.2.} The vector bundle \( V_n \) is a \( \text{GL}_3 \)-counterpart of \( \mathbb{A}(1, 2n) \).

\textbf{Proof.} The scheme \( \mathbb{A}(1, 2n) \) is equivalent to the stack in sets \( \mathbb{A}(1, 2n)^{\sim} \) whose objects are:

\[ \mathbb{A}(1, 2n)^{\sim}(S) = \{(\pi : C' \rightarrow S, \sigma, \phi)\} \]

where:

- \( \pi : C' \rightarrow S \) is a family of rational curves.
- \( \sigma \) is a global section of \( T^\otimes_{C'/S} \).
- \( \phi : \mathbb{P}^1 S \simeq C' \)

Let us make a few comments: once we fixed an isomorphism \( \phi : \mathbb{P}^1 S \simeq C' \), we have a canonical isomorphism of the tangent bundle \( \phi^*T_{C'/S} \) with \( \mathcal{O}(2) \), thus a canonical isomorphism \( \psi_{\text{can}} : \mathcal{O}(2n) \simeq \phi^*T^\otimes_{C'/S} \).

From this follows that we have a morphism

\[ \mathbb{A}(1, 2n)^{\sim} \rightarrow \mathbb{A}(1, 2n), \quad (\pi, \sigma, \phi) \mapsto \psi^*_{\text{can}} \phi^* \sigma \]

It is not hard to show that this morphism is actually a \( \text{PGL}_2 \)-equivariant equivalence, where \( \text{PGL}_2 \) acts on \( \phi \) by multiplication. Observe that this action is free, thus we have:

\[ \mathbb{A}(1, 2n)^{\sim}/\text{PGL}_2(S) = \{(\pi : C' \rightarrow S, \sigma)\} \]
where $\sigma$ is a global section of $T_{C/S}^\otimes n$ with the usual properties, and the morphisms $(C', \tau) \to (C', \sigma)$ are given by isomorphisms $f : C' \simeq C$ such that $f^* \sigma = \tau$. There is an obvious morphism $[\mathbb{A}(1, 2n)^-/\text{PGL}_2] \to \mathcal{M}_0$ which forgets the section $\sigma$.

Identifying the quotient stack $[\mathbb{A}(1, 2n)/\text{PGL}_2]$ with $[\mathbb{A}(1, 2n)^-/\text{PGL}_2]$, by construction of the $\text{GL}_3$-counterpart (see proposition 1.5), all we have to do if we want to give an explicit description of a $\text{GL}_3$-counterpart of $\mathbb{A}(1, 2n)$ is to understand what is the pullback of $[\mathbb{A}(1, 2n)^-/\text{PGL}_2]$ along the $\text{GL}_3$-torsor $\mathbb{A}(2, 2)_{\text{sm}} =: S \to \mathcal{M}_0$.

This is not a hard task: recall that the objects of $S$, seen as a stack in sets, are pairs $((D), \varphi)$ where $(D)$ is a commutative diagram of the form

$$
\begin{array}{ccc}
C \to & \mathbb{P}^3_S \\
\downarrow & \downarrow \\
S & \\
\end{array}
$$

with $C \to S$ a family of rational curves, a closed immersion $i$ and an isomorphism $\varphi : i^* \mathcal{O}(1) \simeq T_{C/S}$. Thus the objects of $[\mathbb{A}(1, 2n)^-/\text{PGL}_2] \times_{\mathcal{M}_0} S$ are

$$
((D), \varphi, \sigma)
$$

where $(D)$ and $\varphi$ are as above, and $\sigma$ is a global section of $T_{C/S}^\otimes n$. Thanks to the fact that we are fixing now an isomorphism of $T_{C/S}$ with $i^* \mathcal{O}(1)$, we can identify $\sigma$ with a global section $f$ of $i^* \mathcal{O}(n) \simeq \mathcal{O}_Q(n)$, where $Q = i(C')$.

Recall that $V_n$ admits a description as stack in sets with objects $(S, q, f)$, where $q$ is a global section of $\mathcal{O}_\mathbb{P}_S^2(2)$ whose zero locus $Q$ is smooth over $S$, and $f$ is a global section over $Q$ of $\mathcal{O}_Q(n)$.

In this way we can define a morphism

$$
V_n \longrightarrow [\mathbb{A}(1, 2n)^-/\text{PGL}_2] \times_{\mathcal{M}_0} S
$$

that sends an object $(S, q, f)$ of $V_n$ to $(Q \subset \mathbb{P}^3_S, i^* \mathcal{O}(1) \simeq T_{Q/S}, f)$.

Using the same arguments of the proof of lemma 1.14 we deduce that this morphism is actually an isomorphism, which concludes the proof of the proposition. □

**Proposition 2.3.** Let $\mathbb{G}_m$ acts on $\mathbb{A}(1, 2n)$ by multiplication for $\lambda$. Then a $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}(1, 2n)$ is $V_n$, where $\mathbb{G}_m$ acts as

$$
\lambda \cdot (q, f) := (q, \lambda f)
$$

**Proof.** We will argue as in the proof of proposition 2.2. Again, we have that $\mathbb{A}(1, 2n)$ can be described as the stack in sets $\mathbb{A}(1, 2n)^-$ whose objects are

$$
\mathbb{A}(1, 2n)^-(S) = \{ (\pi : C' \to S, \sigma, \phi) \}
$$

where:

- The relative scheme $\pi : C' \to S$ is a family of rational curves.
- The section $\sigma$ is a global section of $T_{C/S}^\otimes n$.
- $\phi : \mathbb{P}^3_S \simeq C'$

and $\text{PGL}_2$ acts by multiplication on $\phi$ and $\mathbb{G}_m$ acts by multiplication on $\sigma$.

Actually, this stack in sets is equivalent to the stack $\mathbb{A}(1, 2n)''$ whose objects are

$$
\mathbb{A}(1, 2n)''(S) = \{ (\pi : C' \to S, M, \sigma, \phi, \psi) \}
$$

where:

- $\pi : C' \to S$ is a family of rational curves.
- $M$ is a line bundle over $C'$ of degree $n$.
- $\sigma$ is a global section of $M$.
- $\phi : \mathbb{P}^3_S \simeq C'$
and whose morphisms are given by the data of

- an isomorphism \( f : C'_1 \simeq C'_2 \) which commutes with \( \phi_1 \) and \( \phi_2 \).
- an isomorphism \( g : M_1 \simeq f^*M_2 \) which commutes with \( \psi_1 \) and \( f^*\psi_2 \), and such that \( g^*f^*\sigma_2 = \sigma_1 \).

and \( G_m \) acts on \( \psi \) this time, by sending \( \psi \) to \( \lambda\psi \). There is an obvious morphism of stack \( \mathcal{A}(1,2n)^- \to \mathcal{A}(1,2n)^\nu \) defined as
\[
(\pi : C' \to S, \sigma \in H^0(C', \mathcal{O}_{C'/S}), \phi) \mapsto (\pi : C' \to S, \mathcal{T}_{C'/S}^{\otimes n}, \sigma, \phi, \text{id})
\]
which induces an equivalence \( \mathcal{A}(1,2n)^- \simeq \mathcal{A}(1,2n)^\nu \).

Observe that, instead of asking for an isomorphism \( \psi : M \simeq \mathcal{T}_{C'/S}^{\otimes n} \), we can equivalently ask for an isomorphism \( \psi' : \pi_*(M^{-1} \otimes \mathcal{T}_{C'/S}^{\otimes n}) \simeq \mathcal{O}_S \). In other terms, we can think of \( \mathcal{A}(1,2n)^\nu \) as the stack whose objects are
\[
\mathcal{A}(1,2n)^\nu(S) = \{ (\pi : C' \to S, M, \sigma, \phi, \psi') \}
\]
and \( G_m \) acts \( \psi' \) by multiplication. Observe that \( \text{PGL}_2 \times G_m \) acts freely.

We deduce then that the quotient of \( \mathcal{A}(1,2n)^\nu \) with respect to the action of \( \text{PGL}_2 \times G_m \) is the stack \( [\mathcal{A}(1,2n)^\nu/\text{PGL}_2 \times G_m] \) whose objects are
\[
(\pi : C' \to S, M, \sigma, \pi_*(M^{-1} \otimes \mathcal{T}_{C'/S}^{\otimes n}))
\]
that is, we are forgetting the isomorphisms \( \phi \) and \( \psi' \).

Recall again that \( S := \mathcal{A}(2,2)_{\text{sm}} \) is equivalent to the stack whose objects are pairs \((D, \varphi)\) where \( D \) is a commutative diagram of the form
\[
\begin{array}{ccc}
C' & \xrightarrow{\varphi} & \mathbb{P}^2_S \\
\downarrow & & \downarrow \\
S & \searrow & \\
& &
\end{array}
\]
with \( C' \to S \) a family of rational curves, a closed immersion \( i \) and an isomorphism \( \varphi : i^*\mathcal{O}(1) \simeq \mathcal{T}_{C'/S} \). Moreover, \( \text{GL}_3 \) acts by multiplication on \( \varphi \) and \( G_m \) acts trivially.

So we have the following diagram:
\[
[\mathcal{A}(1,2n)^\nu/\text{PGL}_2 \times G_m] \xrightarrow{\mathcal{M}_0 \times \mathcal{B}G_m} S
\]
The morphism \( [\mathcal{A}(1,2n)^\nu/\text{PGL}_2 \times G_m] \to \mathcal{M}_0 \times \mathcal{B}G_m \) is by definition:
\[
(\pi : C' \to S, M, \sigma, \pi_*(M^{-1} \otimes \mathcal{T}_{C'/S}^{\otimes n})) \mapsto (\pi : C' \to S, \pi_*(M^{-1} \otimes \mathcal{T}_{C'/S}^{\otimes n}))
\]
The morphism \( S \to \mathcal{M}_0 \times \mathcal{B}G_m \) is by definition:
\[
((D), \varphi) \mapsto (\pi : C' \to S, \mathcal{O}_S)
\]
From this we deduce that the objects of
\[
[\mathcal{A}(1,2n)^\nu/\text{PGL}_2 \times G_m] \times_{\mathcal{M}_0 \times \mathcal{B}G_m} S
\]
are:
\[
((D), \varphi, M, \sigma, \alpha : \pi_*(M^{-1} \otimes \mathcal{T}_{C'/S}^{\otimes n}) \simeq \mathcal{O}_S)
\]
where the action of \( G_m \) is by multiplication on \( \alpha \).

Now, the isomorphism \( \varphi : i^*\mathcal{O}(1) \simeq \mathcal{T}_{C'/S} \) allows us to identify \( \mathcal{T}_{C'/S}^{\otimes n} \) with \( i^*\mathcal{O}(n) \), and the isomorphism \( \alpha \) induces (and is actually equivalent to) fixing an isomorphism \( M \simeq \mathcal{T}_{C'/S}^{\otimes n} \). Combining these two data, we see that we are fixing an
isomorphism $M \simeq i^*\mathcal{O}(n)$. Therefore, we can think of $\sigma$, via all these identifications, as a global section of $i^*\mathcal{O}(n)$, on which $G_m$ acts by multiplication.

From this we see that we have morphism
\[
V_n \to [\mathbb{A}(1, 2n)/\text{PGL}_2 \times G_m] \times_{\mathbb{A}_0 \times \text{BG}_m} S
\]
defined as we did in the end of the proof of proposition 2.2. Moreover, this morphism is $G_m$-equivariant with respect to the action of $G_m$ on $V_n$ defined as:
\[
\lambda \cdot (q, f) = (q, \lambda f)
\]
This implies that $V_n$ with this action is also a $GL_3 \times G_m$-counterpart of $\mathbb{A}(1, 2n)$.

Obviously, the quotient $(\mathbb{A}(1, 2n) \setminus \{0\})/G_m$ is $\mathbb{P}(1, 2n)$, the projective space that parametrises binary forms of degree $2n$. Moreover, the actions of $\text{PGL}_2$ and $G_m$ on $\mathbb{A}(1, 2n)$ commutes, and a $GL_3$-counterpart of $\{0\} \subset \mathbb{A}(1, 2n)$ is exactly the zero section $\sigma_0$ inside $V_n$. From these simple observations and proposition 2.2, we obtain that a $GL_3 \times G_m$-counterpart of $\mathbb{A}(1, 2n) \setminus \{0\}$ is $V_n \setminus \sigma_0$, where $GL_3$ acts as before and $G_m$ acts as follows:
\[
\lambda \cdot (q, f) = (q, \lambda f)
\]
By taking the quotient with respect to the $G_m$-action on both $\mathbb{A}(1, 2n) \setminus \{0\}$ and $V_n \setminus \sigma_0$, we immediately deduce the following result:

**Proposition 2.4.** The projective bundle $\mathbb{P}(V_n)$ is a $GL_3 \times G_m$-counterpart of $\mathbb{P}(1, 2n)$.

**Remark 2.5.** The scheme $\mathbb{P}(1, 2n)$ can be thought as the Hilbert scheme $\text{Hilb}_{2n}^{2n}$ of $2n$ points on $\mathbb{P}^1$. Its quotient $[\mathbb{P}(1, 2n)/\text{PGL}_2]$ can be identified with the Hilbert stack $\text{Hilb}_{2n}^{2n}/\text{BGGL}_2$ of $2n$ points relative to the universal torsor $P$ over the classifying stack $\text{BGGL}_2$. Equivalently, we can think of this stack as the Hilbert stack $\text{Hilb}_{2n}^{2n}/\mathcal{M}_0$ of $2n$ points relative to the universal rational curve $C$ over $\mathcal{M}_0$.

So proposition 2.2 gives us the following presentation of this stack as a quotient stack:
\[
\text{Hilb}_{2n}^{2n}/\mathcal{M}_0 \simeq [\mathbb{P}(V_n)/GL_3]
\]
Observe that the projective bundle $\mathbb{P}(V_n)$ itself can be thought as the Hilbert scheme $\text{Hilb}_{2n}^{2n}/\mathcal{Q}_S$ of $2n$ points relative to the universal quadric $\mathcal{Q}$ over $S = \mathbb{A}(2, 2)\text{sm}$.

An interesting feature of this new presentation is that provides us with a natural way to partially extend the Hilbert stack $\text{Hilb}_{2n}^{2n}/\mathcal{M}_0$, which is a stack over $\mathcal{M}_0$, to a stack over the stack of genus 0 and at most 1 nodal curves $\mathcal{M}_0^{1,1}$. Indeed, instead of taking $\mathbb{P}(V_n)$ we can take the projectivization of the vector bundle $V_n^{2n,1}$ defined over $\mathbb{A}(2, 2)\leq_{1}$, the scheme parametrising quadrics in three variables of rank strictly greater than 1. Then the quotient stack $[\mathbb{P}(V_n^{2,1})/GL_3]$ gives a natural enlargement of the Hilbert stack $\text{Hilb}_{2n}^{2n}/\mathcal{M}_0$ to a stack over $\mathcal{M}_0^{1,1}$.

Let $\Delta' \subset \mathbb{A}(1, 2n)$ be the $\text{PGL}_2$-invariant, closed subscheme parametrising singular binary forms of degree $2n$. In other terms, the points of $\Delta'$ corresponds to global sections $\sigma$ of $\mathcal{O}_{\mathbb{P}_2}(2n)$ with multiple roots. We want to find its $GL_3$-counterpart.

Consider the set $D'$ inside $V_{2,1}^{+}$ defined as follows:
\[
D' := \{ (q, f) \text{ such that } V_{+}(q, f) \subset \mathbb{P}^2 \text{ is singular} \}
\]
Let us show how to put a scheme structure on this set: consider the closed sub-scheme of $S \times \mathbb{A}(2, n) \times \mathbb{P}^2$ defined as
\[
D'' := \{ (q, f, u) \text{ such that } u \text{ is a singular point of } V_{+}(q, f) \}
\]
Then $D''$ is a scheme, because it can be defined as the locus where
\[
qu(u) = f(u) = 0, \quad \text{rk}(J(q, f)(u)) \text{ is not maximal}
\]
Here \( J(q,f) \) is the Jacobian matrix of \( q \) and \( f \). Then the image of \( D' \) via the proper morphism
\[
pr : S \times \mathbb{A}(2,n) \times \mathbb{P}^2 \longrightarrow S \times \mathbb{A}(2,n)
\]
holds a scheme structure, and projecting again \( pr(D'') \) along the quotient morphism
\[
S \times \mathbb{A}(2,n) \longrightarrow V_n
\]
we obtain exactly \( D' \), which in this way inherits a scheme structure.

It is then almost immediate, using exactly the same arguments used to prove proposition 2.2 to deduce the following result:

**Proposition 2.6.** We have:

1. \( D' \) is a \( GL_3 \)-counterpart and a \( GL_3 \times \mathbb{G}_m \)-counterpart of \( \Delta' \).
2. \( V_n \setminus D' \) is a \( GL_3 \)-counterpart and the \( GL_3 \times \mathbb{G}_m \)-counterpart of \( \mathbb{A}(1,2n) \setminus \Delta' \).

Let \( \Delta \subset \mathbb{P}(1,2n) \) denotes the image of \( \Delta' \) via the projection \( \mathbb{A}(1,2n) \setminus \{0\} \to \mathbb{P}(1,2n) \), and let \( D \subset \mathbb{P}(V_n) \) be the projection of \( D' \) along \( V_n \setminus \sigma_0 \to \mathbb{P}(V_n) \).

**Corollary 2.7.** We have:

1. \( D \) is a \( GL_3 \)-counterpart of \( \Delta \).
2. \( V_n \setminus D \) is a \( GL_3 \)-counterpart of \( \mathbb{P}(1,2n) \setminus \Delta \).

**2.4. Some comparison results.** In the previous subsection, we found that \( \mathbb{P}(V_n) \) is a \( GL_3 \)-counterpart (see definition 1.2) of the \( \mathbb{P}(1,2n) \). We want now to apply proposition 2.2 to this particular case. There are two relevant classes of morphisms that we want to consider, which are

\[
\psi_n : \mathbb{P}(1,2n) \longrightarrow \mathbb{P}(1,4n), \quad f \longmapsto f^2
\]
\[
\psi_{n,m} : \mathbb{P}(1,2n) \times \mathbb{P}(1,2n) \longrightarrow \mathbb{P}(1,2n + 2m), \quad (f,g) \longmapsto fg
\]

All these maps are \( PGL_2 \)-equivariant and it is immediate to verify that \( GL_3 \)-counterparts of these morphisms (see definition 1.2) are

\[
\psi'_n : \mathbb{P}(V_n) \longrightarrow \mathbb{P}(V_{2n}), \quad (q,f) \longmapsto (q,f^2)
\]
\[
\psi'_{n,m} : \mathbb{P}(V_n) \times \mathbb{P}(V_m) \longrightarrow \mathbb{P}(V_{n+m}), \quad (q,f,g) \longmapsto (q,fg)
\]

All the morphisms involved are proper, and proposition 1.3 gives us the following commutative diagrams of equivariant Chow rings:

\[
\begin{array}{ccc}
A^*_{GL_3} (\mathbb{P}(V_n)) & \cong & A^*_{GL_3} (\mathbb{P}(V_{2n})) \\
\downarrow \psi'_{n,*} & & \downarrow \psi'_{n,*} \\
A^*_{PGL_2} (\mathbb{P}(1,2n)) & \cong & A^*_{PGL_2} (\mathbb{P}(1,4n))
\end{array}
\]

\[
\begin{array}{ccc}
A^*_{GL_3} (\mathbb{P}(V_n)) \otimes A^*_{GL_3} (\mathbb{P}(V_m)) & \cong & A^*_{GL_3} (\mathbb{P}(V_{n+m})) \\
\downarrow \psi'_{n,m,*} & & \downarrow \psi'_{n,m,*} \\
A^*_{PGL_2} (\mathbb{P}(1,2n)) \otimes A^*_{PGL_2} (\mathbb{P}(1,2m)) & \cong & A^*_{PGL_2} (\mathbb{P}(1,2n + 2m))
\end{array}
\]

Every morphism obtained composing \( \psi_{n,m} \) and \( \psi_n \) induces a commutative diagram as the ones above. This will be one of the key tools used to compute the Chow ring of \( \mathcal{H}_g \).
2.5. **The main result.** We are ready to give a new presentation of the stack $\mathcal{H}_g$ as a quotient stack. Recall the presentation of $\mathcal{H}_g$ as a quotient stack. Recall the presentation of $\mathcal{H}_g$ as a quotient stack.

\[ \mathcal{H}_g \simeq \mathcal{A}(1, 2g + 2) \setminus \Delta'/\text{PGL}_2 \times \mathbb{G}_m \]

with action defined as

\[ (A, \lambda) \cdot (x, y) := \lambda^{-2} \det(A)^{g+1} f(A^{-1}(x, y)) \]

Proposition 2.6 (2) tells us that a $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathcal{A}(1, 2g + 2) \setminus \Delta'$, with $\mathbb{G}_m$ acting by simple multiplication, is $V_{g+1} \setminus D'$, where the action of $\text{GL}_3 \times \mathbb{G}_m$ is:

\[ (A, \lambda) \cdot (q, f) := (\det(A)q(A^{-1}(x, y)), \lambda f(A^{-1}(x, y, z)) \]

It is then natural to expect that a $\text{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathcal{A}(1, 2g + 2) \setminus \Delta'$ with $\mathbb{G}_m$ acting by $\lambda^{-2}$ is $V_{g+1}$, with $\mathbb{G}_m$ acting by multiplication for $\lambda^{-2}$. This is indeed the case:

**Theorem 2.8.** Let $U' := V_{g+1} \setminus D'$. Then we have an isomorphism

\[ \mathcal{H}_g \simeq [U'/'\text{GL}_3 \times \mathbb{G}_m] \]

where the action on $U'$ is given by the formula

\[ (A, \lambda) \cdot (q, f) = (\det(A) q(A^{-1} X), \lambda^{-2} f(A^{-1} X)) \]

**Proof.** We argue as in the proof of proposition 2.3 with only one subtle difference, which we now explain. In the proof of proposition 2.3 we defined a stack $\mathcal{A}(1, 2n)'$ which was equivalent to $\mathcal{A}(1, 2n)$, and such that the equivalence was $\mathbb{G}_m$-equivariant.

Let us recall how $\mathcal{A}(1, 2n)'$ is defined: its objects are

\[ (\pi : C' \to S, M, \sigma, \phi, \psi) \]

where

- $\pi : C' \to S$ is a family of rational curves.
- $M$ is a line bundle over $C'$ of degree $2n$.
- $\sigma$ is a global section of $M$.
- $\phi : \mathbb{P}^1_S \simeq C'$.
- $\psi : M \simeq T^\otimes n_{C'/S}$

and its morphisms are given by the data of

- an isomorphism $f : C'_1 \simeq C'_2$ which commutes with $\phi_1$ and $\phi_2$.
- an isomorphism $g : M_1 \simeq f^* M_2$ which commutes with $\psi_1$ and $f^* \psi_2$, and such that $g^* f^* \sigma_2 = \sigma_1$.

and $\mathbb{G}_m$ acts on $\psi$, by sending $\psi$ to $\lambda \psi$. The equivalence $\mathcal{A}(1, 2n) \to \mathcal{A}(1, 2n)'$ is defined as

\[ f \in H^0(\mathbb{P}^1_S, \mathcal{O}(2n)) \mapsto (\pi : \mathbb{P}^1_S \to S, \mathcal{O}(2n), f, \text{id}, \mathcal{O}(2n) \simeq T^\otimes n_{\mathbb{P}^1_S/S}) \]

where the last isomorphism is the canonical one.

It can be checked that this equivalence is $\mathbb{G}_m$-equivariant if we let $\mathbb{G}_m$ acts on $\mathcal{A}(1, 2n)$ by multiplication for $\lambda$, but it is no more $\mathbb{G}_m$-equivariant if $\mathbb{G}_m$ acts by multiplication for $\lambda^{-2}$.

To solve this issue, we consider the stack $\mathcal{A}(1, 2n)'$ such that $\mathcal{A}(1, 2n) \simeq \mathcal{A}(1, 2n)'$, this isomorphism is $\mathbb{G}_m$-equivariant and we can apply to $\mathcal{A}(1, 2n)'$ the same arguments that we used to prove proposition 2.3.

Let $n$ be even (this is the case when $n = g + 1$) and define $\mathcal{A}(1, 2n)'$ as the stack whose objects are

\[ \mathcal{A}(1, 2n)'(S) = \{ (\pi : C' \to S, L, \sigma, \phi, \psi) \} \]

where:
• \( \pi : C' \rightarrow S \) is a family of rational curves.
• \( L \) is a line bundle over \( C' \) of degree \(-n/2\).
• \( \sigma \) is a global section of \( L^{-\otimes 2} \).
• \( \phi : \mathbb{P}^1_S \cong C' \)
• \( \psi : L \simeq \mathcal{T}_{C'/S} \)

The key point here is that now \( G_m \) acts by standard multiplication for \( \lambda \) on \( \psi \), and thus it acts on the global section \( \sigma \) by multiplication for \( \lambda^{-2} \).

There is an obvious morphism \( \mathbb{A}(1, 2n) \rightarrow \mathbb{A}(1, 2n)' \) defined as

\[
\begin{align*}
\phi & \in H^0(\mathbb{P}^1_S, \mathcal{O}(2n)) \mapsto (\mathbb{P}^1_S \rightarrow S, \mathcal{O}(-n), f, \text{id}, \psi_{\text{can}})
\end{align*}
\]

where the last isomorphism is the canonical one. If we let \( G_m \) acts by multiplication for \( \lambda^{-2} \) on \( \mathbb{A}(1, 2n) \), we see that this morphism is a \( G_m \)-equivariant isomorphism.

To conclude that \( V_n \), with \( G_m \) acting by multiplication for \( \lambda^{-2} \), is a \( G_m \times G_m \)-torsor over \( \mathbb{A}(1, 2n) \), in the same way it can be proved that a \( G_m \times G_m \)-equivariant morphism from \( \mathbb{A}(1, 2n) \) to \( \mathbb{A}(1, 2n)' \) is exactly \( V_n \setminus D' \).

Then the theorem follows by simply substituting \( n \) with \( g + 1 \).

The theorem above can be rephrased by saying that \( V_{g+1} \setminus D' \) is a \( G_m \times G_m \)-torsor over \( \mathcal{H}_g \). It is well known that to every \( G_m \times G_m \)-equivariant isogeny \( \mathcal{E} \oplus \mathcal{L} \), where \( \mathcal{E} \) is a rank 3 vector bundle and \( \mathcal{L} \) is a line bundle, and viceversa. We want to find out what is the vector bundle over \( \mathcal{H}_g \) associated to \( V_{g+1} \setminus D' \).

Recall that \( V_{g+1} \setminus D' \), seen as a stack in sets, has as objects the triples \((S, q, f)\) where:

• \( S \) is a scheme.
• \( q \) is a global section of \( \mathcal{O}_{\mathbb{P}^2_S}(2) \) whose zero locus \( Q \subset \mathbb{P}^2_S \) is smooth over \( S \).
• \( f \) is a global section of \( \mathcal{O}_Q(g + 1) \) over \( Q \).

and \( GL_3 \times G_m \) acts as described in theorem \( \ref{theo:GmEquivariant} \). In the proof of theorem \( \ref{theo:GmEquivariant} \) we actually showed that this stack is isomorphic to the stack \( \mathcal{P} \) whose objects are

\[
((D), \varphi, L, \sigma, \alpha)
\]

where:

1. \((D)\) is a commutative diagram of the form

\[
\begin{array}{ccc}
C' & \xrightarrow{i} & \mathbb{P}^2_S \\
\downarrow & & \downarrow \pi \\
S & \xrightarrow{\varphi} & \\
\end{array}
\]

with \( C' \rightarrow S \) a family of rational curves and \( i \) a closed immersion.
2. \( \varphi : i^* \mathcal{O}(1) \cong T_{C'/S} \).
3. \( L \) is a line bundle over \( C' \) of degree \(-(g + 1)/2\).
4. \( \sigma \) is a global section of \( L^{-\otimes 2} \).
5. \( \alpha : \pi_*(L^{-\otimes 2} \otimes T_{C'/S}^{-\otimes (g+1)/2}) \simeq \mathcal{O}_S \).

The elements (1) and (2) above induce by lemma \( \ref{lem:GmEquivariant} \) an isomorphism \( \beta : \pi_* T_{C'/S} \cong \mathcal{O}_{\mathbb{P}^3_S} \), and vice versa. Therefore, it is easy to prove that the stack \( \mathcal{P} \) is equivalent to the stack \( \mathcal{P}' \) whose objects are

\[
(\pi : C' \rightarrow S, L, \sigma, \alpha, \beta)
\]

where:

• \( \pi : C' \rightarrow S \) is a family of rational curves.
• \( L \) is a line bundle of degree \(-(g + 1)/2\) over \( C' \).
• $\sigma$ is a global section of $L^{-\otimes 2}$.
• $\alpha : \pi_* (L^{-1} \otimes T_{C/S}^{-\otimes (q+1)/2}) \simeq \mathcal{O}_S$.
• $\beta : \pi_* T_{C/S} \simeq \mathcal{O}_S^{3}$.

From this we see that there is a morphism $\mathcal{P}' \to \mathcal{H}_g^\sim$ defined as
$$(\pi : C' \to S, L, \sigma, \alpha, \beta) \mapsto (\pi : C' \to S, L, \sigma)$$
that realizes $\mathcal{P}'$ as a $GL_3 \times \mathbb{G}_m$-torsor over $\mathcal{H}_g^\sim$, because $GL_3$ acts by multiplication on $\beta$ and $\mathbb{G}_m$ acts by multiplication on $\alpha$.

This description of $\mathcal{P}' \to \mathcal{H}_g^\sim$ allows us to determine the associated rank 4 vector bundle: it coincides with $E \oplus L$, where $E$ is the rank 3 vector bundle over $\mathcal{H}_g^\sim$ functorially defined as
$$E : (\pi : C' \to S, L, \sigma) \mapsto \pi_* T_{C/S}$$
and $L$ is the line bundle over $\mathcal{H}_g^\sim$ functorially defined as
$$L : (\pi : C' \to S, L, \sigma) \mapsto \pi_* (L^{-1} \otimes T_{C/S}^{-\otimes (q+1)/2})$$

Recall that there is an isomorphism of $\mathcal{H}_g^\sim \simeq \mathcal{H}_g$. We may ask for a description of the vector bundles $E$ and $L$ as vector bundles over $\mathcal{H}_g$. This can be easily deduced from the description we gave before: indeed, if $C \to S$ is a family of hyperelliptic curves of genus $g$ which is a double cover of $C' \to S$ via the morphism $\eta : C \to C'$, and if $W$ is the associated Weierstrass divisor, then
1. $\eta^* T_{C/S} \simeq \omega_{C/S}^{-1} \otimes \mathcal{O}(W)$.
2. $\eta^* L \simeq \mathcal{O}(-\frac{2g}{2}W)$.

From the formulas above it can be easily deduced that the vector bundle $E$, seen as a vector bundle over $\mathcal{H}_g$, is functorially defined as
$$E((\pi : C \to S, i)) = \pi_* \omega_{C/S}^{-1} (W)$$
whereas $L$, seen as a line bundle over $\mathcal{H}_g$, is functorially defined as
$$L((\pi : C \to S, i)) = \pi_* \omega_{C/S}^\otimes \left(\frac{1-g}{2}W\right)$$

These considerations will be used at the end of the paper in order to provide a geometrical description of the generators of the Chow ring of $\mathcal{H}_g$.

3. Intersection theory of $\mathbb{P}(V_n)$

The aim of this section is to study the vector bundles $\mathbb{P}(V_n)$ on $\mathcal{S}$ that were introduced in the previous section (see definition 2.4). In particular, in the first subsection we concentrate on the geometry of $\mathbb{P}(V_n)$, and we show that over certain particular open subschemes of $\mathcal{S}$ the bundles $V_n$ become trivial (lemma 3.1). We also study some interesting morphisms between $\mathbb{P}(V_n)$ for different $n$. In the second subsection we do some computations in the $T$-equivariant Chow ring of $\mathbb{P}(V_n)$, where $T \subset GL_3$ is the subgroup of diagonal matrices, focusing on the cycle classes of some specific $T$-invariant subvarieties (lemmas 3.6, 3.7, and 3.8). All these results will be needed for computing the Chow ring of $\mathcal{H}_g$, which is done in the last three sections.

3.1. Properties of $\mathbb{P}(V_n)$. We will use the following notational shorthand: an underlined letter $\underline{i}$ will indicate a triple $(i_0, i_1, i_2)$, and the expression $\underline{X}$ will indicate the monomial $X_{i_0}^{i_1} X_{i_2}$. A form $f$ of degree $n$ in three variables with coefficients in a ring $R$ can then be expressed as $f = \sum b_i \underline{X}$, where the $b_i$ are elements of $R$ and the sum is taken over the triples $\underline{i}$ such that $|\underline{i}| := i_0 + i_1 + i_2 = n$. The coefficients $b_i$ give us coordinates in $\mathbb{A}(2, n)$, thus homogeneous coordinates in $\mathbb{P}(2, n)$. 

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The symbols $a_i$ will be used only for the coefficients of quadrics, or equivalently
for the coordinates of $\mathbb{A}(2,2)$. Finally, we say that $i \leq j$ iff $i_\alpha \leq j_\alpha$ for $\alpha = 0, 1, 2$.
This is equivalent to the condition $X_i \leq X_j$.

Let $S_i$ be the open subscheme of $S = \mathbb{A}(2,2)_{sm}$ where the coordinate $a_i$ is
not zero, and let $Y_i$ be its complement. It can be easily checked that the open
subschemas $S_{(0,2,0)}, S_{(0,1,1)}$ and $S_{(0,0,2)}$ constitute an open covering of $S$: indeed, a
point not in the union of these three open subschemes will necessarily parametrise
a quadric divisible by $X_0$, thus not smooth. The open subschemas $S_i$ share another
property, expressed in the following lemma:

**Lemma 3.1.** The projective bundles $\mathbb{P}(V_i)$ are trival over the opens $S_i$.

The proof of the lemma above relies on a lemma of linear algebra concerning the
vector spaces of forms in three variables of fixed degree.

**Lemma 3.2.** Let $i$ be a triple such that $|i| = 2$ and let $B_i$ be the set of monomials
of degree $n$ in three variables not divisible by $X_i$. Fix a quadric $q$ in three variables
with non-zero coefficient $a_i$. Define $B_i'$ to be the set of polynomials obtained by
multiplying $q$ with a monomial of degree $n-2$ in three variables. Then the two sets $B_i$ and $B_i'$ are disjoint and $B_i \cup B_i'$ is a base for the vector space of homogeneous
polynomials of degree $n$ in three variables with coefficients in a field $k$.

**Proof.** The fact that $B_i$ and $B_i'$ are disjoint is obvious, because the polynomials
in $B_i'$ are all sums of monomials divisible by $X_i$, thanks to the fact that $a_i$ is
not zero. The monomials form a base for the vector space of homogeneous polynomials
of degree $n$ in three variables. Let $M$ be the matrix representing the unique linear
transformation that sends the base of monomials to the set $B_i \cup B_i'$ in the following way: monomials not divisible by $X_i$ are sent to themselves, and monomials of the
form $X_i f$ are sent to $q f$. It can be proved that the determinant of $M$ is invertible,
thus $B_i' \cup B_i$ is a base. \qed

From lemma 3.2 we see that over $S_i$ the coordinates $b_k$, for $|k| = n$ and $i \not\subseteq k$,
trivialize the vector bundle $V_i|_{S_i}$, thus proving lemma 3.1. We now define some
morphisms that will play an important role in the remainder of the paper: these morphisms are

$$\pi_{n,m} : \mathbb{P}(1,2n) \times \mathbb{P}(1,2m) \longrightarrow \mathbb{P}(1,4n + 2m), \quad (f, g) \longmapsto f^2 g$$

whose $\text{GL}_3$-counterparts are

$$\pi_{n,m} : (V_i) \times_S \mathbb{P}(V_m) \longrightarrow \mathbb{P}(V_{2n+m}), \quad (q, f, g) \longmapsto (q, f^2 g)$$

Applying proposition 1.3 we deduce the following commutative diagrams of equivariant Chow rings:

$$
\begin{array}{ccc}
A_{\text{GL}_3}^*(\mathbb{P}(V_i)) \otimes A_{\text{GL}_3}^*(S) & A_{\text{GL}_3}^*(\mathbb{P}(V_m)) & \pi_{n,m}^* \longrightarrow \pi_{n+m}^* \longrightarrow A_{\text{GL}_3}^*(\mathbb{P}(V_{2n+m})) \\
\cong & & \cong \\
A_{\text{PGL_2}}^*(\mathbb{P}(1,2n)) \otimes A_{\text{PGL_2}}^*(\mathbb{P}(1,2m)) & A_{\text{PGL_2}}^*(\mathbb{P}(1,4n + 2m)) & \\
\end{array}
$$

Observe that we also have the following class of closed linear immersions of projective bundles:

$$j_{n,r,l} : \mathbb{P}(V_{n-r-l}) \hookrightarrow \mathbb{P}(V_n), \quad (q, f) \longmapsto (q, X_0^r X_1^l f)$$

These are not $\text{GL}_3$-equivariant. But if we restrict to the induced action of the maximal subtorus $T \subset \text{GL}_3$ of diagonal matrices, we see that $j_{n,r,l}$ is indeed $T$-equivariant.
Definition 3.3. The $T$-invariant, closed subscheme $W_{n,r,t} \subset \mathbb{P}(V_n)$ is defined as the image of $j_{n,r,t}$.

3.2. Computations in the $T$-equivariant Chow ring of $\mathbb{P}(V_n)$. Let us introduce another little piece of notation: with $\lambda$ we mean the triple $(\lambda_1, \lambda_2, \lambda_3)$. If $\hat{i}$ is another triple (most of the times, we will have $\hat{i} = (i_0, i_1, i_2)$), we indicate with $\hat{i} \cdot \lambda$ their scalar product.

As already observed, on $\mathbb{P}(V_n)$ there is a well defined action of $GL_3$, which induces an action of its maximal subtorus $T$ of diagonal matrices. Thus we can consider the $T$-equivariant Chow ring $A^*_T(\mathbb{P}(V_n))$. If $Z \subset \mathbb{P}(V_n)$ is a $T$-invariant subvariety, its $T$-equivariant cycle class will be denoted $[Z]$.

Recall that, as stated in the introduction, when dealing with $T$-equivariant Chow groups we denote $\lambda_i$, for $i = 1, 2, 3$, the generators of $A^2(\text{Spec}(k))$ and we denote $c_i$ the elementary symmetric polynomials in $\lambda_1, \lambda_2, \lambda_3$.

Lemma 3.4. We have $A^*_T(S) = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3]/(c_1, 2c_3)$.

Proof. Observe that $S$ is an open subscheme of the representation $A(2, 2)$ of $T$. This plus the localization exact sequence implies that $A^*_T(S)$ is generated by $\lambda_1, \lambda_2$ and $\lambda_3$. We need to find the relations among the generators. Let $W$ be the closed subscheme inside $\mathbb{P}(2, 2) \times \mathbb{P}^2$ whose points correspond to pairs $(q, p)$ with $p$ a singular point of $\{q = 0\}$. We have $[W] = pr_1^*\xi_3 + pr_2^*\xi_2 + pr_3^*\xi_1^2$, where $t$ denotes the pullback to $\mathbb{P}(2, 2) \times \mathbb{P}^2$ of the hyperplane section of $\mathbb{P}^2$. Using the same arguments of [FV, theorem 5.5] we deduce that the image of

$$i_* : A^*_T(\mathbb{P}(2, 2)_{\text{sing}}) \longrightarrow A^*_T(\mathbb{P}(2, 2))$$

is exactly $(\xi_1, \xi_2, \xi_3)$. Arguing as in [Vi98, pg.638], we have that the pullback map

$$A^*_T(\mathbb{P}(S)) \rightarrow A^*_T(S)$$

is surjective with kernel $(c_1 - h)$, where $h$ denotes the hyperplane section of $\mathbb{P}(S)$. This implies that the ideal of relations of $A^*_T(S)$ is generated by the elements $\xi_1, \xi_2, \xi_3, c_1 - h$ and $f(h)$, where $f$ denotes the monic polynomial of degree 6 satisfied by $h$ in $A^*_T(\mathbb{P}(2, 2))$. After having computed explicitly $[W]$, we see that the actual generators of the ideal of relations are $c_1$ and $2c_3$. \(\square\)

We know that $A^*_T(\mathbb{P}(V_n))$ is generated as $A^*_T(S)$-algebra by the hyperplane section $h_n$, so that we have

$$A^*_T(\mathbb{P}(V_n)) \simeq \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, h_n]/(c_1, 2c_3, p_n(h_n))$$

where $p_n(h_n)$ is a monic polynomial of degree $2n + 1$. Recall now that in the previous subsection we defined the open subscheme $\mathcal{S}_n$ of $S$ as the subscheme whose points are smooth quadratic forms with coefficient $a_i$ not zero. The complement of $\mathcal{S}_n$ was denoted $Y_n$. The $T$-equivariant Chow ring of $\mathbb{P}(V_n)|_{\mathcal{S}_n}$ may be easily computed.

Lemma 3.5. We have $A^*_T(\mathbb{P}(V_n)|_{\mathcal{S}_n}) \simeq A^*_T(\mathbb{P}(V_n))/\langle \hat{i} \cdot \lambda \rangle$.

Proof. From the localization exact sequence

$$A^*_T(\mathbb{P}(V_n)|_{Y_n}) \xrightarrow{i_*} A^*_T(\mathbb{P}(V_n)) \xrightarrow{j^*} A^*_T(\mathbb{P}(V_n)|_{\mathcal{S}_n}) \rightarrow 0$$

we see that what we need to prove is that $\text{im}(i_*) = \langle \hat{i} \cdot \lambda \rangle$. Let $t$ be the hyperplane section of $\mathbb{P}(V_n)|_{Y_n}$ so that the $T$-equivariant Chow ring of $\mathbb{P}(V_n)|_{Y_n}$ is generated, as abelian group, by elements of the form $p^* \cdot t^d$ for $d \leq 2n$, where $p^*$ is the projection map to $Y_n$. This implies that $\text{im}(i_*)$ is generated by the elements $i_* (p^* \cdot t^d)$.
Observe that \( i^* h_n = t \). From the cartesian square

\[
\begin{array}{ccc}
\mathbb{P}(V_n)|_{Y_2} & \xrightarrow{\ i\ } & \mathbb{P}(V_n) \\
\downarrow p' & & \downarrow p \\
Y_2 & \xrightarrow{\ i'\ } & S
\end{array}
\]

we deduce the following chain of equivalences:

\[
\begin{align*}
i_* (p' \cdot \alpha \cdot t^d) &= i_* (p' \cdot \alpha \cdot i^* h_n^d) \\
&= i_* p' \cdot \alpha \cdot h_n^d = p' \cdot i' \cdot \alpha \cdot h_n^d
\end{align*}
\]

This means that \( \text{im}(i_*) = p'(\text{im}(i'_*)) \). Now observe that \( Y_2 \) is an open subscheme of a representation of \( T \), namely the vector subspace of forms of degree 2 with the coefficient \( a_2 \) equal to zero. This implies that \( A_T^*(Y_2) \) is a quotient of \( A_T^* \), from which we deduce that \( \text{im}(i'_*) \) is generated, as an ideal, by \( [Y_2] \), so that we only have to compute this class in \( A_T^*(\mathcal{S}) \). Because of the fact that \( Y_2 \) is defined as the zero locus of the coordinate \( a_2 \), the class of \( Y_2 \) corresponds to \( -c_1(\chi) \); here \( \chi \) is the character such that if \( r \) is an element of \( T \) then \( a_2(r^{-1} \cdot x) = \chi(r) a_2 \). In this case, with the action that we defined before, we obtain \( c_1(\chi) = c_1 - \frac{1}{2} \lambda \). Using the relations that we already had in \( A_T^*(\mathcal{S}) \), we deduce that \( [Y_2] = -\frac{1}{2} \lambda \). \( \square \)

Recall that we previously defined \( T \)-invariant closed subschemes \( W_{n,r,l} \subset \mathbb{P}(V_n) \) (see definition 3.3). The cycle classes \( [W_{n,r,l}] \) have degree \( 2r + 2l \). We can pull back them via the open immersion

\[
j : \mathbb{P}(V_n)|_{\mathcal{S}(0,0,2)} \simeq \mathbb{P}^{2n} \times \mathcal{S}(0,0,2) \hookrightarrow \mathbb{P}(V_n)
\]

so that they can be actually computed. Indeed here we have homogeneous coordinates given by the coefficients \( b_k \), for \( |k| = n \) and \( (0, 0, 2) \not\in k \) and we see that

\[
j^{-1} W_{n,r,l} = \{b_k = 0 \text{ for } k_0 < r \text{ or } k_1 < l\}
\]

from which we deduce that they are complete intersection, and consequently we obtain

\[
j^* [W_{n,r,l}] = \prod (h_n - k \cdot \lambda) \text{ for } k \text{ s.t. } |k| = n, k_2 < 2, k_0 < r \text{ or } k_1 < l
\]

Applying lemma 3.5 we deduce:

**Lemma 3.6.** We have

\[
[W_{n,r,l}] = \prod (h_n - k \cdot \lambda) + 2\lambda \xi \text{ for } k \text{ s.t. } |k| = n, k_2 < 2, k_0 < r \text{ or } k_1 < l
\]

where \( \xi \) is an element of \( A_T^*(\mathbb{P}(V_n)) \).

In particular, all these classes are monic in \( h_n \). Another useful property is the following: the set-theoretic intersection of \( W_{n,r,0} \) and \( W_{n,0,l} \) is exactly \( W_{n,r,l} \). This is also true at the level of Chow rings: indeed \( W_{n,r,l} \) is the only component of \( W_{n,r,0} \cap W_{n,0,l} \), all the varieties involved are smooth and it is easy to check that the intersection is transversal, so that:

**Lemma 3.7.** We have \( [W_{n,r,0}] \cdot [W_{n,0,l}] = [W_{n,r,l}] \).

Recall that we have defined the morphism

\[
\pi_{n,m} : \mathbb{P}(V_n) \times S \mathbb{P}(V_m) \longrightarrow \mathbb{P}(V_{2n+m})
\]
as \( \pi_{n,m}^I(q, f, g) = (q, f^2 g) \). If we restrict this morphism to \( W_{n,1,n-1} \times S \mathbb{P}(V_m) \) we obtain

\[
\pi_{n,m}^I : W_{n,1,n-1} \times S \mathbb{P}(V_m) \longrightarrow W_{2n+m, 2n-2}
\]

and from this we easily deduce:
Lemma 3.8. We have $\pi_{n,m}^*(\{W_{n,1,n-1} \times S \mathbb{P}(V_m)\}) = [W_{2n+m,2,n-2}]$.

4. The Chow ring of $\mathcal{H}_g$: generators and first relations

The goal of this section is to do the first steps in the computation of the Chow ring of $\mathcal{H}_g$, finding the generators (first subsection) and some relations (second subsection). The intermediate result that we find is the content of corollary 4.11.

4.1. Setup. Theorem 2.8 tells us that the Chow ring $A^*(\mathcal{H}_g)$ is isomorphic to the equivariant Chow ring $A^*_{\text{GL}_3 \times \mathbb{G}_m}(U')$, where $U'$ is the open subscheme of $V_{g+1}$ consisting of all the pairs $(q, f)$ such that the intersection $Q \cap F$ is smooth, i.e. the intersection consists of $2g + 2$ distinct points.

Let $D'$ be the complement of $U'$ in $V_{g+1}$. It is easy to see that $D'$ is a closed subscheme of codimension 1. In this section, we will always assume $n = g + 1$. The localization exact sequence in this case is

$$A^*_{\text{GL}_3 \times \mathbb{G}_m}(D') \xrightarrow{(\cdot)^*} A^*_{\text{GL}_3 \times \mathbb{G}_m}(V_n) \xrightarrow{\cdot} A^*_{\text{GL}_3 \times \mathbb{G}_m}(U') \rightarrow 0$$

Observe that $V_n$ is a $\text{GL}_3 \times \mathbb{G}_m$-equivariant vector bundle over $S$, from which we deduce

$$(1) \quad A^*_{\text{GL}_3 \times \mathbb{G}_m}(V_n) \simeq A^*_{\text{GL}_3 \times \mathbb{G}_m}(S) \simeq \mathbb{Z}[c_2, c_3]/(2c_3)$$

where $\tau$ is the first Chern class of the standard $\mathbb{G}_m$-representation and $c_2, c_3$ are respectively the second and the third Chern class of the standard $\text{GL}_3$-representation. The last isomorphism can be deduced from Lemma 3.4 using [FV] Lemma 2.1.

We have found a set of generators for $A^*(\mathcal{H}_g)$. Now we have to find the relations among the generators, which is the same as computing the generators of the ideal $\text{im}(i_v)$ inside the equivariant Chow ring of $V_n$.

Consider the projective bundle $\mathbb{P}(V_n)$ and its open subscheme $U$, whose preimage in $V_n \setminus \sigma_0$ is exactly $U'$ (recall that $\sigma_0 : S \rightarrow V_n$ is the zero section). Observe that $V_n \setminus \sigma_0$ is equivariantly isomorphic to the $\mathbb{G}_m$-torsor over $\mathbb{P}(V_n)$ associated to the line bundle $\mathcal{O}(\mathcal{O}_{\mathbb{P}(V_n)}) \oplus \mathcal{V}^{-\otimes 2}$, where $\mathcal{V}$ is the standard representation of $\mathbb{G}_m$ pulled back to $\mathbb{P}(V_n)$. Clearly, we can say the same thing for $U'$ over $U$. This implies, arguing as in [Vis98, pg.638], that we have a surjective morphism

$$A^*_{\text{GL}_3 \times \mathbb{G}_m}(U) \twoheadrightarrow A^*_{\text{GL}_3 \times \mathbb{G}_m}(U')$$

whose kernel is given by $c_1(\mathcal{O}_U(-1) \otimes (\mathcal{V}^{-\otimes 2}) = -h_n - 2\tau$.

This means that, if $p(h_n)$ is a relation in $A^*_{\text{GL}_3 \times \mathbb{G}_m}(U)$, then $p(-2\tau)$ is a relation in $A^*_{\text{GL}_3 \times \mathbb{G}_m}(U')$, and all the relations in this last ring are obtained from relations in the Chow ring of $U'$ in this way.

Clearly, after passing to $\mathbb{P}(V_n)$ the action of $\mathbb{G}_m$ becomes trivial, so that we can restrict ourselves to consider only the $\text{GL}_3$-action. Is then enough, in order to determine the equivariant Chow ring of $U'$, to compute the $\text{GL}_3$-equivariant Chow ring of $U$.

Again, we have the localization exact sequence

$$A^*_{\text{GL}_3}(D) \xrightarrow{(\cdot)^*} A^*_{\text{GL}_3}(\mathbb{P}(V_n)) \xrightarrow{\cdot} A^*_{\text{GL}_3}(U) \rightarrow 0$$

The ring in the middle is isomorphic to

$$(2) \quad A^*_{\text{GL}_3}(S)[h_n]/(p_n(h_n)) \simeq \mathbb{Z}[c_2, c_3, h_n]/(2c_3, p_n(h_n))$$

where $p_n(h_n)$ is monic of degree $2n + 1$. Thus to find the relations we have to compute the generators of the ideal $\text{im}(i_v)$.

Observe that the closed subscheme $D$ admits a stratification

$$D_n \subset D_{n-1} \subset ... \subset D_1 = D$$
where $D_n$ is the locus of pairs $(q, f)$ such that $Q \cap F = 2E + E'$, with $\deg(E) = m$.

All these sets are clearly $GL_3$-invariant. Observe moreover that $D_{2s}$ coincides with the image of the equivariant, proper morphism

$$\pi_{2s}^*: P(V_s) \times_S P(V_{n-2s}) \to P(V_n), \quad (q, f, g) \mapsto (q, f^2g)$$

which coincides with the morphism $\pi'_{s,n-2s}$ that we have defined in subsection 3.1. This induces a scheme structure on $D_{2s}$.

Consider also the $GL_3$-invariant closed subscheme

$$\mathcal{Y}_1 = \{(q, l) \text{ such that } L \text{ is tangent to } Q \} \subset P(V_1)$$

and let us define the closed subschemes $\mathcal{Y}_{2s+1}$ as the image of the morphisms

$$\phi: \mathcal{Y}_1 \times_S P(V_s) \to P(V_{2s+1}), \quad (q, l, f) \mapsto (q, lf^2)$$

We can think of $\mathcal{Y}_{2s+1}$ as the locus of quadrics plus a divisor of the form $2E$, with $\deg(E) = 2s + 1$. Restricting the morphism $\psi'_{2s+1,n-2s-1}$ defined in subsection 2.4 to this closed subscheme we obtain a proper morphism

$$\pi'_{2s+1}: \mathcal{Y}_{2s+1} \times_S P(V_{n-2s-1}) \to P(V_n)$$

whose image is $D_{2s+1}$. This induces the scheme structure on $D_{2s+1}$.

The stratification defined above resembles the stratification

$$\Delta_n \subset \ldots \subset \Delta_1 = \Delta \subset P(1, 2n)$$

that has been introduced in [FV11]. Indeed, we have that $D_s$ is the $GL_3$-counterpart of $\Delta_s$. Furthermore, it is easy to see that the $GL_3$-counterparts of the morphisms

$$\pi_{2s}: P(1, 2s) \times P(1, 2n - 4s) \to P(1, 2n), \quad (f, g) \mapsto f^2g$$

are exactly the morphisms $\pi'_{2s}: P(V_s) \times_S P(V_{n-2s}) \to P(V_n)$. Applying again proposition 1.3 we immediately obtain the commutative diagram

$$A^*_PGL_2(P(1, 2s) \times P(1, 2n - 4s)) \xrightarrow{\sim} A^*_GL_3(P(V_s) \times_S P(V_{n-2s}))$$

$$A^*_PGL_2(P(1, 2n)) \xrightarrow{\sim} A^*_GL_3(P(V_n))$$

We also have that $\mathcal{Y}_1$ is the $GL_3$-counterpart of $P^1$, as we can think of $\mathcal{Y}_1$ as the tautological conic over $S$, and in general the closed subschemes $\mathcal{Y}_{2s+1} \subset P(V_{2s+1})$ are the $GL_3$-counterpart of $P(1, 2s+1)$ sitting inside $P(1, 4s + 2)$ via the square map, so that we still have

$$A^*_PGL_2(P(1, 2s + 1) \times P(1, 2n - 4s - 2)) \xrightarrow{\sim} A^*_GL_3(\mathcal{Y}_{2s+1} \times_S P(V_{n-2s-1}))$$

$$A^*_PGL_2(P(1, 4s + 2) \times P(1, 2n - 4s - 2)) \xrightarrow{\sim} A^*_GL_3(P(V_{2s+1}) \times_S P(V_{n-2s-1}))$$

$$A^*_PGL_2(P(1, 2n)) \xrightarrow{\sim} A^*_GL_3(P(V_n))$$

Combining [FV11] lemma 3.1 with the diagrams above we obtain:

**Lemma 4.1.** The ideal $\text{im}(i_s)$ is the sum of the ideals $\text{im}(\pi'_{s,n-2s})$. 


4.2. Computation of \(\text{im}(\pi'_1)\). The goal of this subsection is to prove the

**Proposition 4.2.** We have \(\text{im}(\pi'_1) = (2h^2_n - 2n(n-1)c_2, 4(n-2)h_n)\).

Consider the closed subscheme \(Z \subset \mathbb{P}(V_n) \times \mathbb{P}^2\) which is defined as

\[ Z = \{(q, f, p) \text{ such that } Q \text{ and } F \text{ intersect non transversely at } p\} \]

Observe that it is \(GL_3\)-invariant, where \(GL_3\) acts on \(\mathbb{P}^2\) in the standard way (we can think of \(\mathbb{P}^2\) as the projectivization of the standard representation of \(GL_3\)). The image of \(Z\) via the projection on \(\mathbb{P}(V_n)\) is clearly \(D\), and moreover \(pr_1 : Z \to D\) is injective over \(D_1 \setminus D_2\). Before going on, let us pause a moment to study the geometry of \(Z\). We begin with a well known technical lemma:

**Lemma 4.3.** Let \(p \in \mathbb{P}^2\) be a point of \(\mathbb{P}^2\) and let \(Q, F\) be the plane projective curves defined by the homogeneous polynomials \(q\) and \(f\). Let \(J(q, f)\) be the \(2 \times 3\)-jacobian matrix, and suppose that \(Q\) and \(F\) intersect in \(p\). Then their intersection is transversal iff there exists one \(2 \times 2\)-minor of \(J(q, f)\) such that its determinant does not vanish in \(p\).

Let us denote the determinant of the minor of \(J(q, f)\) obtained by removing the column with the partial derivatives w.r.t. \(X_0\) (resp. \(X_1\) and \(X_2\)) as \(\det_0 J(q, f)\) (resp. \(\det_1 J(q, f)\) and \(\det_2 J(q, f)\)). Then we have the following equational characterization of \(Z\), which directly follows from the lemma above:

**Lemma 4.4.** Consider \(Z\) restricted to \(\mathbb{P}(V_n)|_{S^1} \times \mathbb{P}^2\), where \(\underline{1} = (1, 0, 0)\) (resp. \((0, 1, 0)\) and \((0, 0, 1)\)). Then \(Z\) is defined by the following equations in \(p\) and in the coefficients \(a_{ij}, b_{ij}\) for \(\underline{1} \neq \underline{2}\) of \(q\) and \(f\):

- \(q(p) = 0\)
- \(f(p) = 0\)
- \(\det(J(q, f))(p) = 0\), for \(i = 0\) (resp. \(i = 1\) and \(i = 2\))

A first step in the proof of proposition 4.2 is the following, which enables us to work with the morphism \(pr_1 : Z \to \mathbb{P}(V_n)\) rather than \(\pi'_1 : Y_1 \times_S \mathbb{P}(V_{n-1}) \to \mathbb{P}(V_n)\).

**Lemma 4.5.** We have \(Y_1 \times_S \mathbb{P}(V_{n-1}) \simeq Z\) and the isomorphism commutes with the morphisms to \(\mathbb{P}(V_n)\).

**Proof.** Consider the morphism \(Y_1 \times_S \mathbb{P}(V_{n-1}) \to \mathbb{P}(V_n) \times \mathbb{P}^2\) which sends a triple \((q, l, f, p)\) to \((q, l, f, p)\) where \(p\) is the point of tangency of \(Q\) and \(L\).

By construction, the image of this morphism is \(Z\). We want to define an inverse \(Z \to Y_1 \times_S \mathbb{P}(V_{n-1})\): this is done by sending a triple \((q, f, p)\) of \(Z\), \((q, l, f, p)\) to \((q, l, f^{-1})\), where \(L\) is the only line tangent to \(Q\) in \(p\). Details are omitted. \(\square\)

**Corollary 4.6.** We have \(\text{im}(\pi'_1) = \text{im}(pr_{1 \ast})\).

In order to prove proposition 4.2, i.e. then equivalent compute \(\text{im}(pr_{1 \ast})\). Let us call \(t\) the hyperplane section of \(\mathbb{P}^2\), so that the equivariant Chow ring \(A^*_GL_3(\mathbb{P}(V_n) \times \mathbb{P}^2)\) is generated as \(A^*_GL_3(S)\)-algebra by \(pr_{1 \ast} h_n\) and \(pr_{2 \ast} t\): with a little abuse of notation we will keep calling these cycles \(h_n\) and \(t\).

The class \([Z]\) can then be written as a polynomial in \(t\) of degree 3 with coefficients in \(A^*_GL_3(\mathbb{P}(V_n))\): indeed, the dimension of \(Y_1 \times \mathbb{P}(V_{n-1})\) is equal to \(2n + 4\), which by lemma 4.5 is equal to the dimension of \(Z\), so that we easily deduce that the codimension of \(Z\) in \(\mathbb{P}(V_n) \times \mathbb{P}^2\) is equal to 3. We then have \([Z] = \beta_1(h_n)t^2 + \beta_2(h_n)t + \beta_3(h_n)\).

**Lemma 4.7.** We have \(\text{im}(pr_{1 \ast}) = (\beta_1(h_n), \beta_2(h_n), \beta_3(h_n))\).

The lemma above reduces the computation of the generators of \(\text{im}(pr_{1 \ast})\) to the computation of the class \([Z]\) inside \(A^*_GL_3(\mathbb{P}(V_n) \times \mathbb{P}^2)\). Before proving lemma 4.7, we need some preliminary results.
Lemma 4.8. The closed subscheme \( Z \) is a projective subbundle of \( \mathbb{P}(V_n) \times \mathbb{P}^2 \) over the universal quadric \( Q \subset \mathbb{P}^2 \times S \).

Proof. First recall that \( Q = \{(q,p) \text{ such that } q(p) = 0\} \). We can work Zariski-locally on \( Q \), so that \( Z \) is described by the equations of lemma 4.1 which are linear in the coefficients of \( f \), thus proving the lemma. □

Lemma 4.9. The image of \( i_*: A^*_{GL_3}(Z) \to A^*_{GL_3}(\mathbb{P}(V_n) \times \mathbb{P}^2) \) is generated as an ideal by \( i_*1 = [Z] \).

Proof. We claim that the equivariant Chow ring of \( Z \) is generated as a \( pr_2^*(A^*_{GL_3}(\mathbb{P}^2)) \)-algebra by \( i^*h_n \), so that every element is a sum of monomials of the form \( pr_2^*\xi \cdot i^*h_n^r \). The cartesian square

\[
\begin{array}{ccc}
\mathbb{P}(V_n) \times \mathbb{P}^2 & \xrightarrow{pr_2} & \mathbb{P}^2 \\
\downarrow{pr_1} & & \downarrow{pr_1} \\
\mathbb{P}(V_n) & \rightarrow & \text{Spec}(k)
\end{array}
\]

plus the compatibility formula and the projection formula imply that

\[
i_* (pr_2^*\xi \cdot i^*h_n^r) = i_* pr_2^*\xi \cdot h_n^r
\]

which is equal to zero unless \( \xi = \eta \cdot t^2 \), where \( \eta \) come from \( A^*_{GL_3} \), in which case we have \( i_* pr_2^*\xi \cdot h_n^r = \eta \cdot h_n^r \). From this the lemma follows.

So we are only left to prove the initial claim. From lemma 4.8 we know that the equivariant Chow ring of \( Z \) is generated by \( i^*h_n \) as \( A^*_{GL_3}(Q) \)-algebra, where \( Q \) is the universal quadric.

Consider the trivial vector bundle \( \mathbb{P}^2 \times \mathbb{A}(2,2) \) over \( \mathbb{P}^2 \), which contains the vector subbundle \( Q' \) defined by the equation \( q(p) = 0 \), which is linear in the coefficients of \( q \). Clearly, the equivariant Chow ring of \( Q' \) is isomorphic to the one of \( \mathbb{P}^2 \) via the pullback along the projection map. But \( Q \) is an open subscheme of \( Q' \), thus its Chow ring has the same generators. This conclude the proof of the lemma. □

Proof of lemma 4.7. From lemma 4.8 we deduce that \( \text{im}(pr_1^*) \) is generated, as an ideal, by elements of the form \( pr_1^*([Z] \cdot pr_1^*\xi \cdot pr_2^*\eta) \). Applying the projection formula and the usual arguments, we obtain that the image of \( pr_1^* \) is actually generated by the cycles \( pr_1^*([Z] \cdot t^i) \) for \( i = 0, 1, 2 \), which are equal to \( \beta_3, \beta_2 \) and \( \beta_1 \) respectively. □

Now, proving proposition 4.2 is then equivalent to computing the \( GL_3 \)-equivariant class \([Z]\).

Consider

\[
\phi: \mathbb{P}(\mathbb{P}(2,n-2)) \to \mathbb{P}(\mathbb{P}(2,n)) \quad (q,f) \longrightarrow (q,qf)
\]

Then it is easy to see that \( \mathbb{P}(\mathbb{P}(2,n)) \setminus \text{im}(\phi) \) is a vector bundle over \( \mathbb{P}(V_n) \).

We can then equivalently compute the cycle class of the pullback of \( Z \) of \( (\mathbb{P}(2,n) \times S \setminus \text{im}(\phi)) \times \mathbb{P}^2 \). Let \( Z' \) be the \( GL_3 \)-invariant, closed subscheme of \( \mathbb{P}(2,n) \times \mathbb{P}(S) \times \mathbb{P}^2 \) whose points are triples \((q,f,p)\) such that

\[
q(p) = f(p) = 0, \quad \det J(q,f)(p) = 0, \quad i = 0, 1, 2
\]

Then we can pull back the cycle class \([Z']\) along the projection

\[
\mathbb{P}(2,n) \times S \times \mathbb{P}^2 \longrightarrow \mathbb{P}(2,n) \times \mathbb{P}(S) \times \mathbb{P}^2
\]

and then restrict it to \((\mathbb{P}(2,n) \times S \setminus \text{im}(\phi)) \times \mathbb{P}^2\): what we obtain in the end is exactly \([Z]\).
Let \( s \) be the hyperplane class of \( \mathbb{P}(\mathcal{S}) \): then \( \mathcal{S} \to \mathbb{P}(\mathcal{S}) \) coincides with the \( \mathbb{G}_m \)-torsor \( \mathcal{O}(-1) \otimes \mathcal{D} \), where \( \mathcal{D} \) is the determinant representation of \( \text{GL}_3 \): by the usual argument of [Vis98, pg.638] we have that pulling back cycles along

\[
P(2, n) \times \mathcal{S} \to \mathbb{P}(2, n) \times \mathbb{P}(\mathcal{S}) \times \mathbb{P}^2
\]

is equivalent to substituting \( s \) with \( c_1 \), which is zero in the equivariant Chow ring of \( \mathcal{S} \) (see [I]). Moreover, from [Z] we see that restricting a class to \( (\mathbb{P}(2, n) \times \mathcal{S}) \text{im}(\beta)) \times \mathbb{P}^2 \) is equivalent to set \( 2c_3 = 0 \).

Therefore, all we have to do for determining \( [\mathcal{Z}] \) is to compute \( [\mathcal{Z}'] \) and impose \( c_1 = 0, 2c_3 = 0 \). Observe that \( \mathcal{Z}' \) is not a complete intersection, but it becomes so if we restrict to the open subscheme of \( \mathbb{P}^2 \) consisting of points where one of the homogeneous coordinates does not vanish. Let us consider the auxiliary cycle class \( [\mathcal{Z}']_2 \), where \( \mathcal{Z}'_2 \) is the closed subscheme defined by the equations

\[
q(p) = f(p) = 0, \quad \det J(q, f)(p) = 0
\]

It is easy to check that this locus has two irreducible components, \( \mathcal{Z}'_2 \) and \( \mathcal{W}_2 \), where \( \mathcal{W}_2 \) is defined by the equations \( q(p) = 0, f(p) = 0 \) and \( p_2 = 0 \) (here \( p_2 \) stands for the third homogeneous coordinate of \( \mathbb{P}^2 \)).

Thus, we would like to write \( [\mathcal{Z}'] = [\mathcal{Z}']_2 - [\mathcal{W}_2] \). Unfortunately, the subschemes \( \mathcal{Z}'_2 \) and \( \mathcal{W}_2 \) are not \( \text{GL}_3 \)-invariant. Then we use the following trick: we first pass to the action of the maximal subtorus \( T \subset \text{GL}_3 \), and we observe that \( \mathcal{Z}'_2 \) and \( \mathcal{W}_2 \) are equivariant with respect to the \( T \)-action. Then we compute their \( T \)-equivariant classes in the \( T \)-equivariant Chow ring of \( \mathbb{P}(2, n) \times \mathbb{P}(2, 2) \times \mathbb{P}^2 \): by standard results, their difference will coincide with the \( \text{GL}_3 \)-equivariant cycle \( [\mathcal{Z}] \). With some simple computations we obtain:

\[
[\mathcal{Z}'] = [\mathcal{Z}']_2 - [\mathcal{W}_2] = (s + 2t)(h_n + nt)(s + h_n + nt - \lambda_1 - \lambda_2) - (s + 2t)(h_n + nt)(t + \lambda_3) = (s + 2t)(h_n + nt)(s + h_n + (n - 1)t - c_1)
\]

Observe that the two classes were not symmetric with respect to \( \lambda_i \) but they become so when combined together, precisely how we expected. In order to find the coefficients \( \beta_1, \beta_2 \) and \( \beta_3 \) we have to put this expression in its canonical form, substituting \( t^3 \) with \( -c_3 - c_2 t - c_1 t^2 \). In the end we obtain

\[
[\mathcal{Z}'] = s^2h_n + s^2h_n - sh_n c_1 - 2n(n - 1)c_1 + (n - 1)s^2 + (2n + 1)sh_n - ns c_1 + 2h_n^2 - 2h_n c_1 - 2n(n - 1)c_2 t
\]

\[
((n - 1)^2 + 3n - 1)s + (4n - 2)h_n - 2nc_1 - 2n(n - 1)c_1 t^2
\]

Substituting \( s = c_1 = 0 \) and \( 2c_3 = 0 \), we obtain \( \beta_1 = (4n - 2)h_n, \beta_2 = 2h_n^2 - 2n(n - 1)c_2 \) and \( \beta_3 = 0 \). This proves proposition [12].

Remark 4.10. The content of proposition [4.2] could also have been deduced by [FV11] proposition 5.2, exploiting the usual commutative diagrams of equivariant Chow rings. Nevertheless, we preferred to give an independent proof of this fact.

Corollary 4.11. The Chow ring of \( \mathcal{H}_g \) is a quotient of the ring

\[
\mathbb{Z}[\tau, c_2, c_3]/(4(2g + 1)\tau, 8\tau^2 - 2g(g + 1)c_2, 2c_3)
\]

Proof. The only thing we need to prove is that \( p_n(-2\tau) \) is contained in the ideal above. This works exactly as in [FV11] proposition 6.4].

The next section will be devoted to check if there are other relations in the Chow ring of \( \mathcal{H}_g \), or if they all come from the pullback to \( A_{\text{GL}_3 \times \mathbb{G}_m}^0(V_n) \) of \( \text{im}(\pi_1^g) \).
5. Other generators of $\text{im}(i_*)$

This section is divided in two parts, just as is done in [FV11]. In the first one, we complete the computation of $\text{im}(i_*)$ with $\mathbb{Z}[\frac{1}{2}]$-coefficients, which means that we do the computations in the equivariant Chow ring tensored over $\mathbb{Z}$ with $\mathbb{Z}[\frac{1}{2}]$. The main result is the following proposition:

**Proposition 5.1.** We have $\text{im}(\pi_1^*) \otimes \mathbb{Z}[\frac{1}{2}] \subset \text{im}(\pi_1^*) \otimes \mathbb{Z}[\frac{1}{2}]$.

In other terms, using $\mathbb{Z}[\frac{1}{2}]$-coefficients, the ideal $\text{im}(i_*)$ coincides with the ideal $\text{im}(\pi_1^*)$, whose generators we computed in the previous section (proposition 4.2).

In the second part, we work with $\mathbb{Z}(2)$-coefficients. What we deduce at the end is the following result:

**Proposition 5.2.** We have $\text{im}(i_*) \otimes \mathbb{Z}(2) = (2h_1^3 - 2n(n-1)c_2, 4(n-2)h_n, \pi_2^*(h_1^2 \times [P(V_{n-2})])$ and the inclusion $\text{im}(\pi_1^*) \otimes \mathbb{Z}(2) \subset \text{im}(i_*) \otimes \mathbb{Z}(2)$ is strict.

The last two propositions together imply:

**Corollary 5.3.** We have $\text{im}(i_*) = (2h_1^3 - 2n(n-1)c_2, 4(n-2)h_n, \pi_2^*(h_1^2 \times [P(V_{n-2})])$.

Using proposition 4.3 we see that the corollary above, interpreted in the PGL$_2$-equivariant setting, says that the image of

$$i_* : A_{\text{PGL}_2}(\mathbb{P}(1, 2n)_{\text{sing}}) \longrightarrow A_{\text{PGL}_2}(\mathbb{P}(1, 2n))$$

is equal to the image of

$$\pi_1^* : A_{\text{PGL}_2}(\mathbb{P}(1, 1) \times \mathbb{P}(1, 2n - 2)) \longrightarrow A_{\text{PGL}_2}(\mathbb{P}(1, 2n))$$

plus the cycle $\pi_2^*(H^2 \times 1)$, where $H$ is the hyperplane section of $\mathbb{P}(1, 2)$ and the morphism $\pi_2$ is

$$\pi_2 : \mathbb{P}(1, 2) \times \mathbb{P}(1, 2n - 4) \longrightarrow \mathbb{P}(1, 2n), \quad (f, g) \longmapsto f^2 g$$

and the inclusion $\text{im}(\pi_1^*) \subset \text{im}(i_*)$ is strict. Instead, in [FV11] proposition 5.3 was erroneously stated that $\text{im}(\pi_1^*) \subset \text{im}(\pi_1^*)$.

In this first part, we follows closely the ideas of [FV11] subsection 5.2, adapting their language to our different setting.

In the second part we initially work with GL$_3$-equivariant Chow rings, but then we start using the $T$-equivariant ones, and we complete the computation of $\text{im}(i_*) \otimes \mathbb{Z}(2)$ in this different setting. Then, using what we have found exploiting the $T$-equivariant Chow rings, we go back to the GL$_3$-context and we finish the computation of the generators of $\text{im}(i_*) \otimes \mathbb{Z}(2)$.

In this second part we initially follow the path of [FV11] but at a certain point we diverge. Indeed, as said before, the computation is completed in the $T$-equivariant setting, mainly because we start working with cycle classes of subvarieties that are only $T$-invariant and not GL$_3$-invariant. In particular, these classes do not have an analogue in the PGL$_2$-equivariant setting that is adopted in [FV11]. This is where we really need the new presentation given by theorem 2.8.

5.1. Computations with $\mathbb{Z}[\frac{1}{2}]$-coefficients. Let us recall the content of [FV11] lemma 5.4:

**Lemma 5.4.** Let $X$ be a smooth scheme on which PGL$_n$ acts, and consider the induced action of SL$_n$ via the quotient map SL$_n \rightarrow$ PGL$_n$. Then the kernel of the natural, surjective pullback map $A_{\text{PGL}_n}(X) \rightarrow A_{\text{SL}_n}(X)$ is of $n$-torsion.
From now until the end of the current subsection, every Chow ring is assumed to be tensored over \( \mathbb{Z} \) with \( \mathbb{Z}[\frac{1}{2}] \). The strategy adopted here is substantially the same as the one used in [FV11]. Consider first the commutative square

\[
\begin{array}{ccc}
A^*_\text{GL}_3(\mathbb{P}(V_s) \times \mathbb{P}(V_{n-2s})) & \longrightarrow & A^*_\text{GL}_3(\mathbb{P}(V_n)) \\
\cong & & \cong \\
A^*_\text{PGL}_2(\mathbb{P}(1, 2s) \times \mathbb{P}(1, 2n-4s)) & \longrightarrow & A^*_\text{PGL}_2(\mathbb{P}(1, 2n)) \\
& & \\
A^*_\text{SL}_2(\mathbb{P}(1, 2s) \times \mathbb{P}(1, 2n-4s)) & \longrightarrow & A^*_\text{SL}_2(\mathbb{P}(1, 2n))
\end{array}
\]

where the three horizontal arrows are the pushforward along the maps \( \pi'_{2s} \), \( \pi_{2s} \) and \( \pi'_{2s} \). Recall that \( \pi'_{2s}(f, g) := f^2g \), and \( \pi_{2s} \) is defined in the same way. Then from lemma 5.3 we deduce that the two last vertical arrows, when using \( \mathbb{Z}[\frac{1}{2}] \)-coefficients, are injective, so that it is enough to prove proposition 5.1, for the \( \text{SL}_2 \)-equivariant Chow rings. This is can be done following [EF09, section 4], by simply adding the relation \( c_1 = 0 \).

The case \( r = 2s + 1 \) is handled similarly, using the commutative square

\[
\begin{array}{ccc}
A^*_\text{GL}_3(\mathbb{P}(V_{2s+1} \times \mathbb{P}(V_{n-2s-1})) & \longrightarrow & A^*_\text{GL}_3(\mathbb{P}(V_n)) \\
\cong & & \cong \\
A^*_\text{PGL}_2(\mathbb{P}(1, 2s + 1) \times \mathbb{P}(1, 2n-4s-2)) & \longrightarrow & A^*_\text{PGL}_2(\mathbb{P}(1, 2n)) \\
& & \\
A^*_\text{SL}_2(\mathbb{P}(1, 2s + 1) \times \mathbb{P}(1, 2n-4s-2)) & \longrightarrow & A^*_\text{SL}_2(\mathbb{P}(1, 2n))
\end{array}
\]

This concludes the proof of proposition 5.1 stated at the beginning of the section.

5.2. Computations with \( \mathbb{Z}(2) \)-coefficients, first part. Throughout this subsection, we will assume that every Chow ring and every ideal appearing is tensored over \( \mathbb{Z} \) with \( \mathbb{Z}(2) \), also when is not explicitly written. The main result is the next proposition.

**Proposition 5.5.** The ideal \( \text{im}(i_*) \otimes \mathbb{Z}(2) \) is equal to the sum of the ideal \( \text{im}(\pi'_{1s}) \otimes \mathbb{Z}(2) \) and the ideal generated by the elements \( \pi'_{2s+1}(h^2r \cdot 1) \) for \( s = 1, ..., n/2 \).

Recall from lemma 5.1 that the ideal \( \text{im}(i_*) \) is the sum of the ideals \( \text{im}(\pi'_{r+1}) \). The proof of the proposition above then splits into two parts: in the first one we will show, following [FV11], that there are no generators of \( \text{im}(i_*) \) of the form \( \pi'_{2s+1} \xi \) other than the ones that we have already found. In other terms we have:

\[
\text{im}(\pi'_{2s+1}) \otimes \mathbb{Z}(2) \subset \text{im}(\pi'_{1s}) \otimes \mathbb{Z}(2)
\]

In the second part, we will show that the elements of proposition 5.5 are actually enough to generate \( \text{im}(i_*) \).

The following proof can also be deduced directly from [FV11 lemma 5.5]. For the sake of completeness, we decided to give here an independent proof in the \( \text{GL}_3 \)-equivariant setting, though the idea is exactly the same of [FV11 lemma 5.5].

**Lemma 5.6.** We have \( \text{im}(\pi'_{2s+1}) \otimes \mathbb{Z}(2) \subset \text{im}(\pi'_{1s}) \otimes \mathbb{Z}(2) \).
Proof. Consider the commutative square

\[
\begin{array}{c}
\mathcal{Y}_1 \times S \mathbb{P}(V_s) \times S \mathbb{P}(V_{n-2s-1}) \\
\phi \times \text{id} \\
\mathcal{Y}_{2s+1} \times S \mathbb{P}(V_{n-2s-1}) \\
\pi'_{2s+1} \\
\end{array}
\xrightarrow{\psi} \begin{array}{c}
\mathcal{Y}_1 \times S \mathbb{P}(V_{n-1}) \\
\pi'_1 \\
\mathbb{P}(V_n) \\
\end{array}
\]

where the top horizontal arrow \( \psi \) sends a tuple \((q, l, f, g)\) to \((q, l, f^2g)\). Observe that the vertical arrow on the left is finite of degree \(2s+1\), thus the pushforward induces an isomorphism at the level of Chow rings (we are using \(\mathbb{Z}(2)\)-coefficients). The commutativity of the square implies that

\[
(\pi'_{2s+1})_* = (\pi'_1)_* \circ (\psi)_* \circ (\phi \times \text{id})_*^{-1}
\]

and this implies the lemma.

Now we want to study the image of \(\pi'_{2s+1}\). Observe that this ideal is generated by the pushforward of the classes \(h^i_{2s} \cdot h_{n-2s}^j\), for \(i = 0, \ldots, 2s\) and \(j = 0, \ldots, 2n - 4s\), where we use the notational shorthand \(h^i_{2s} \cdot h_{n-2s}^j\) to indicate what should be more correctly denoted as \(\text{pr}_1^i h^i_{2s} \cdot \text{pr}_2^j h_{n-2s}^j\). An intermediate result is the following lemma:

**Lemma 5.7.** We have that \(\pi'_{2s+1}(h^i_{2s} \cdot h_{n-2s}^j)\) is in \(\text{im}(\pi'_{1s})\) for \(i = 0, \ldots, 2s - 1\) and \(j = 0, \ldots, 4n - 2s\).

In order to prove the lemma above we need a technical result, which can be found also in [FV11], with the exception that there the authors claim the result also for \(i = 2s\). The proof works exactly in the same way.

**Lemma 5.8.** We have that \(\pi'_{2s+1}(h^i_{2s} \cdot h_{n-2s}^j)\) is 2-divisible for \(i = 0, \ldots, 2s - 1\) and \(j = 0, \ldots, 4n - 2s\).

**Proof.** We start with the case \(s = n/2\). Observe that \(\pi'_{n*}(h^i_{n/2})\), for \(i = 0, \ldots, n - 1\), is 2-divisible if and only if \(\pi'_{n*}(h^i_{n/2}) \cdot h_n\) is 2-divisible. This follows from the uniqueness of the representation of cycles in \(A^*_n(\mathbb{P}(V_n))\) as polynomials in \(h_n\) of degree less or equal to \(2n\). We also have that \(\pi'_{n*}h_n = 2h_{n/2}\), and from this we deduce that

\[
\pi'_{n*}(h^i_{n/2}) \cdot h_n = \pi'_{n*}((h^i_{n/2}) \cdot \pi'_{n*}h_n) = 2\pi'_{n*}h_{n/2}^{i+1}
\]

Now consider the general case, and observe that we have a factorization of \(\pi'_{2s}\) as follows:

\[
\begin{array}{c}
\mathbb{P}(V_s) \times S \mathbb{P}(V_{n-2s}) \\
\pi''_{2s} \times \text{id} \\
\mathbb{P}(V_{2s}) \times S \mathbb{P}(V_{n-2s}) \\
\pi''_{2s,n-2s} \rightarrow \mathbb{P}(V_n)
\end{array}
\]

where \(\pi''_{2s}((q, f)) = (q, f^2)\). At the level of Chow rings, the first morphism coincides with

\[
A^*_{\text{GL}_s}(\mathbb{P}(V_s)) \otimes A^*_{\text{GL}_s}(\mathbb{P}(V_{n-2s})) \xrightarrow{\pi''_{2s} \otimes \text{id}} A^*_{\text{GL}_s}(\mathbb{P}(V_{2s})) \otimes A^*_{\text{GL}_s}(\mathbb{P}(V_{n-2s}))
\]

From the previous case we deduce than that

\[
\pi'_{2s+1}(h^i_{2s} \cdot h_{n-2s}^j) = 2\pi_{2s,n-2s}(\pi''_{2s}h_{2s}^{i+1} \cdot h_{n-2s}^j)
\]

which concludes the proof of the lemma.

**Remark 5.9.** We cannot extend the previous Lemma to the classes in which \(h^{2s}_{2s}\) appears. Indeed, for \(s = n/2\), write \(\pi'_{n*}h_{n/2}^n\) as a polynomial \(\alpha_0 h_{n/2}^n + \alpha_1 h_{n/2}^{n-1} + \ldots + \alpha_{2n}\). Then we have

\[
\pi'_{n*}h_{n/2}^n \cdot h_n = (\alpha_0 h_{n/2}^n + \alpha_1 h_{n/2}^{n-1} + \ldots + \alpha_{2n}) \cdot h_n = \alpha_0(h_n^{n+1} - p_n(h_n)) + \ldots + \alpha_{2n}h_n
\]

and the first coefficient will always be 2-divisible, no matter if \(\alpha_0\) is even or not.
Recall that we defined in the third section the apex. Moreover, we will keep using the morphisms between $T$ to every $s$.

Proof of lemma 5.7. Consider again the commutative diagram

\[
\begin{array}{c}
A_{\text{GL}_2}(\mathbb{P}(V_s) \times \mathbb{P}(V_{n-2s})) \\
\cong \\
A_{\text{PSL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(2,2n-4s)) \\
\cong \\
A_{\text{PSL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(2,2n-4s)) \rightarrow A_{\text{PSL}_2}(\mathbb{P}(2,2n))
\end{array}
\]

where the three horizontal arrows are respectively the pushforward along the morphisms $\pi_2^*, \pi_3^*$, and $\pi_3^*$.

Recall that the kernel of the two last vertical maps is $(c_3)$ and that, from [EP09], we already know that $\text{im}(\pi_{2s}^{\text{PSL}_2})$ is contained in $\text{im}(\pi_{2s}^{\text{PSL}_2})$. This implies that there exists a cycle $\xi$ such that $\pi_{2s}^*(h_s^i \cdot h_{n-2s}^j) + c_3 \cdot \xi$ is contained in $\text{im}(\pi_{1s}^*)$.

Observe that this last ideal is contained in $(2)$ and, by lemma 5.8, so is $\pi_{2s}^*(h_s^i \cdot h_{n-2s}^j)$. From this we deduce that $c_3, \xi$ is contained in $(c_3)$ and $(2)$, but $(c_3) \cap (2) = (0)$, thus $c_3 \cdot \xi = 0$ and consequently $\pi_{2s}^*(h_s^i \cdot h_{n-2s}^j)$ is contained in $\text{im}(\pi_{1s}^*)$.

So far we have proved that the ideal $\text{im}(i_s)$ is equal to the sum of the ideal $\text{im}(\pi_{1s}^*)$ and the ideal generated by the elements $\pi_{2s}^*(h_s^i \cdot h_{n-2s}^j)$ for $s = 1, \ldots, n/2$ and $j = 0, \ldots, 2n-4s$. We are in position to prove the main result of this subsection.

Proof of proposition 5.2. The key observation is that $\pi_{2s}^* h_n = 2h_s + h_{n-2s}$. This implies the following chain of equalities:

\[
\begin{align*}
\pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j) &= \pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j \cdot (h_{n-2s} + 2h_s - 2h_s)) \\
&= \pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j \cdot \pi_2^*(h_{n}) - 2\pi_{2s}^*(h_s^{2+1} \cdot h_{n-2s}^j)) \\
&= \pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j \cdot h_n - 2\pi_{2s}^*(\alpha_0 h_s^2 + \ldots + \alpha_{2s}) \cdot h_{n-2s}^j)
\end{align*}
\]

By lemma 5.7 we see that the cycle $\pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j)$ is in the ideal $\text{im}(\pi_{1s}^*) + (\pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j))$. Iterating this argument, we see that for every $j > 0$ the cycle $\pi_{2s}^*(h_s^2 \cdot h_{n-2s}^j)$ is contained in the ideal $\text{im}(\pi_{1s}^*) + (\pi_{2s}^*(h_s^2 \cdot 1))$. Applying this to every $s$ we conclude the proof of the lemma.

5.3. Interlude: computations in the $T$-equivariant setting. Let again $T \subset \text{GL}_2$ be the maximal subtorus of diagonal matrices. In this subsection the fact that we work with the $T$-equivariant Chow ring will be essential. For the sake of clarity, the morphisms between $T$-equivariant Chow rings will be denoted with a $T$ in the apex. Moreover, we will keep using $\mathbb{Z}_{(2)}$-coefficients, so that every ring and ideal is $\mathbb{Z}_{(2)}$-invariant subvarieties $W_{n,r,t}$ of $\mathbb{P}(V_n)$ whose points are the pairs $(g, X^t X^t_t)$ (see definition 5.3). Then what we are going to prove now is the following result:

**Proposition 5.10.** We have $\text{im}(i_s^T) = (2h_n^2 - 2n(n-1)c_2, 4(n-2)h_n, [W_{n,2,0}])$ inside $A_{\text{PSL}_2}(\mathbb{P}(V_n))$.

All what we need in order to prove the proposition above is the following lemma:

**Lemma 5.11.** We have $\text{im}(i_s^T) = \text{im}(\pi_{1s}^T) + ([W_{n,2,2,s-2}])_{s=1,\ldots,n/2}$. 

Proof. From lemma 3.8 we know that the cycle class \([W_{1,s} - 1]\) contained in the \(T\)-equivariant Chow ring of \(\mathbb{P}(V_5)\) is a monic polynomial in \(h^s\) of degree \(2s\). We already know from lemma 5.7 that the cycles \(\pi^T(h^s, 1)\), for \(i < 2s\), are in \(\text{im}(\pi^T)\).

Combining this with our initial observation, we get
\[
\text{im}(\pi^T) + \pi^T(h^s, 1) \subset \text{im}(\pi^T) + (\pi^T([W_{1,s} - 1] \times 1))
\]

because we have
\[
\pi^T(h^s, 1) = \pi^T([W_{1,s} - 1] \times 1) - \sum \xi_i \pi^T(h_i^s)^{1}
\]
for \(i < 2s\). The other inclusion is obvious because the ideal on the right is by construction contained in \(\text{im}(\pi^T)\), that coincides with the ideal on the left. Thus we actually proved that we have an equality. To finish the proof of the lemma, is enough to observe that, by lemma 3.8 we have \(\pi^T([W_{1,s} - 1] \times 1) = [W_{n,2s} - 2]\).

Proof of proposition 5.10. From lemma 3.7 we see that the ideal \([W_{n,2s} - 2]\), where \(s = 1, \ldots, n/2\), is actually generated by \([W_{n,2,0}].

This plus lemma 5.7 implies that \(\text{im}(i^2) = \text{im}(\pi^T) + ([W_{n,2,0}])\). The fact that the inclusion \(\text{im}(\pi^T) \subset \text{im}(i^2)\) is strict follows from the fact that \(\text{im}(\pi^T) \subset (2)\) whereas \([W_{n,2,0}]\) is not 2-divisible, as lemma 3.8 shows.

5.4. Computations with \(\mathbb{Z}/2\)-coefficients, part two. We want to deduce from proposition 5.10 what are the generators of \(\text{im}(i^2) \otimes \mathbb{Z}/2\). Again, all the ideals and the Chow rings will be assumed to be tensorized over \(\mathbb{Z}/2\), where not explicitly stated. Observe that also the ideal \(\text{im}(i^2)\) is equal to the sum of ideals \(\text{im}(\pi^T)\), where \(r\) ranges from 1 to \(n\). In particular, proposition 5.10 remains true also in the \(T\)-equivariant setting, so that we actually know that
\[
\text{im}(i^2) = \text{im}(\pi^T) + (\pi^T(h^s)^{1})_{s = 1, \ldots, n/2}
\]
and from lemma 5.7 which also stays true in the \(T\)-equivariant setting, we know that the cycles of the form \(\pi^T(h^s)^{1}\) are in \(\text{im}(\pi^T)\) for \(i = 0, \ldots, 2s - 1\) and \(j = 0, \ldots, 4n - 2s\).

We proved in proposition 5.10 that \(\text{im}(i^2)\) is equal to \(\text{im}(\pi^T)\) plus the ideal generated by the cycle class \([W_{n,2,0}] = \pi^T([W_{1,1,0}] \times 1)\). The equality
\[
[W_{1,1,0}] = (h_1 - \lambda_2)(h_1 - \lambda_3) = h_1^2 - (\lambda_2 + \lambda_3)h_1 + h_2 \lambda_3
\]
is immediate to check, using the fact that \(\mathbb{P}(V_1) = \mathbb{P}(2,1) \times S\). Putting all together, we readily deduce that
\[
\text{im}(i^2) = \text{im}(\pi^T) + ([W_{n,2,0}]) \subset \text{im}(\pi^T) + (\pi^T(h^s)^{1}) \subset \text{im}(i^2)
\]
which implies the following result:

Corollary 5.12. We have \(\text{im}(i^2) = \text{im}(\pi^T) + (\pi^T(h^s)^{1})\)

Let us recall the following result (\cite[Lemma 2.1]{FV}), which enables us to pass from the \(T\)-equivariant setting to the \(G\)-equivariant one:

Proposition 5.13. Let \(G\) be a special algebraic group and let \(T \subset G\) be a maximal torus.

(i) Let \(X\) be a smooth \(G\)-space and let \(I \subset A_G(X)\) be an ideal, then
\[
IA_T(X) \cap A_G(X) = I
\]

(ii) Let \(\{x_1, \ldots, x_r\}\) be a set of variables and let \(I \subset A_G[x_1, \ldots, x_r]\) be an ideal, then
\[
IA_T[x_1, \ldots, x_r] \cap A_G[x_1, \ldots, x_r] = I
\]
Then we see that the ideal $\text{im}(i_*)$ inside $A^*_\text{GL}_4(\mathbb{P}(V_n))$ is equal to the symmetric elements of the image of $i^*_T$ inside the $T$-equivariant Chow ring of $\mathbb{P}(V_n)$. The corollary above gives explicit generators for the ideal $\text{im}(i^*_T)$ that are also symmetric in the $A_i$. From this we immediately deduce proposition \textbf{5.2} stated at the beginning of the section.

6. The Chow ring of $\mathcal{H}_g$: end of the computation

In this section we finish the computation of $A^*(\mathcal{H}_g)$. Recall that in corollary \textbf{4.11}, we proved that

$$\text{im}(i_*) = (2h_n^2 - 2n(n-1)c_2, 4(n-2)h_n, n_2, h^2[\mathbb{P}(V_n-2)])$$

In order to obtain the relations inside the Chow ring of $\mathcal{H}_g$, we need to pull back the generators of the ideal above along the $\mathbb{G}_m$-torsor $p : V_n \setminus \sigma_0 \to \mathbb{P}(V_n)$, where $\sigma_0$ denotes the image of the zero section $S \to V_n$.

We have already computed some of these relations in the third section (see corollary \textbf{4.11}). Let us call $I$ the ideal appearing in that corollary, that is

$$I = (4(2g+1)\tau, 8c_2 - 2g(g+1)c_2, 2c_3)$$

By construction, it coincides with the pullback of the ideal $\text{im}(\pi_1^*)$. Unfortunately, the ideal $\text{im}(i_*)$ is not equal to $\text{im}(\pi_1^*)$ inside $A^*_\text{GL}_4(\mathbb{P}(V_n))$, so we can’t conclude that $I$ is the whole ideal of relations, though this claim may still be true, because when pulling back the only generator of $\text{im}(i_*)$ not in $\text{im}(\pi_1^*)$ we may obtain a cycle contained in $I$.

Moreover, the fact that we do not know an explicit expression for the last generator, namely $\pi_2^*(h_1^2[\mathbb{P}(V_n-2)])$, prevent us from finishing the computation in a direct way.

Recall that in the previous section we also deduced that

$$\text{im}(i^*_T) = \text{im}(\pi_1^*) + ([W_{n,2,0}])$$

inside the $T$-equivariant Chow ring of $\mathbb{P}(V_n)$, and observe that the relations inside the Chow ring of $\mathcal{H}_g$ are exactly the pullback of these symmetric elements along the $\mathbb{G}_m$-torsor $p : V_n \setminus \sigma_0 \to \mathbb{P}(V_n)$.

If $\xi$ is in $\text{im}(i_*)$, seeing it as an element of the $T$-equivariant Chow ring through the embedding $A^*_\text{GL}_4(\mathbb{P}(V_n)) \hookrightarrow A^*_T(\mathbb{P}(V_n))$, then we have $\xi = a \cdot \pi_1^* \xi + \beta : [W_{n,2,0}]$. To proceed with our discussion, we need the following technical result:

**Lemma 6.1.** Let $\xi = \alpha_0 h_n^{2n} + \alpha_1 h_n^{2n-1} + \ldots + \alpha_{2n}$ be a cycle in $\text{im}(i_*)$, considered as an ideal in the $\text{GL}_4$-equivariant Chow ring of $\mathbb{P}(V_n)$, and suppose that $\alpha_{2n}$ is 2-divisible. Then $\pi^*\xi$ is in $I$.

The proof of the lemma is postponed to the end of the section. Write $\xi$ as a polynomial in $h_n$ of degree less or equal to $2n$. If we prove that $\xi$, written in this form and evaluated in $h_n = 0$, is 2-divisible, then by lemma \textbf{6.1} we can conclude that $\pi^*\xi$ must be in $I$, thus $I$ is the whole ideal of relations.

We already know that every element in the image of $\pi_1^*$ is 2-divisible, so that we only need to check that $\beta : [W_{n,2,0}]$, seen as a polynomial in $h_n$ of degree less or equal to $2n$ and evaluated in $h_n = 0$, is 2-divisible.

Clearly, it is enough to prove this claim when $\beta = h_n^d$, where $d = 0, \ldots, 2n$. For matters of clarity, let us work with $\mathbb{Z}/2\mathbb{Z}$-coefficients, so that what we need to prove is that $\beta : [W_{n,2,0}]$, seen as a polynomial in $h_n$ of degree less or equal to $2n$ and evaluated in $h_n = 0$, is equal to 0.

Write $[W_{n,2,0}]$ as $h_n^d + \omega_1 \cdot h_n^{2d} + \ldots + \omega_d$. Observe that, with these coefficients, we have $h_n^{2n+1} = 0$. This implies that the product $h_n^d \cdot [W_{n,2,0}]$ is equal to

$$\chi_4 h_n^{d+4} + \chi_3 \omega_1 h_n^{d+3} + \ldots + \chi_0 \omega_d h_n^d$$
where \( \chi_j = 0 \) for \( j + d > 2n \), and equal to 1 otherwise. In particular, for \( d > 0 \) the evaluation of this polynomial in \( h_n = 0 \) is zero. In other terms, we have showed that \( h_n^d \cdot [W_{n,2,0}] \), written as a polynomial in \( h_n \), of degree less or equal to \( 2n \) and evaluated in \( h_n = 0 \), is 2-divisible for \( d > 0 \).

Now we only need to prove that \([W_{n,2,0}]\) itself has this property. Recall from lemma 3.6 that

\[
[W_{n,2,0}] = \prod (h_n - k \cdot \lambda) + 2\lambda_3 \xi \text{ for } k \text{ s.t. } |k| = n, k_2 < 2, k_0 < 2
\]

The 2-divisibility of \([W_{n,2,0}]\) when evaluated in \( h_n = 0 \) is then equivalent to studying the 2-divisibility of the product \( \prod (-\frac{k}{\lambda}) \), where \( |k| = n, k_2 < 2, k_0 < 2 \).

Observe that there are only four triples \( k \) that verify the conditions above, namely \((0,n,0), (0,n-1,1), (1,n-1,0) \) and \((1,n-1)\). This implies that the product above is a multiple of \( n\lambda_3 \), thus it is 2-divisible because \( n = g + 1 \) is even. We now give a proof of the technical lemma.

**Proof of lemma 6.1.** We can assume that \( \xi \) is not in \( \text{im}(\pi_1^{'}) \), otherwise the conclusion is obvious. Moreover we can also assume that \( \xi \) is in \( \text{im}(\pi_2^{'}) \), because we have proved in the last section that \( \text{im}(\pi_2^{'}) \) is contained in \( \text{im}(\pi_1^{'}) \). Consider again the commutative diagram

\[
\begin{array}{ccc}
A^\ast_{\text{GL}_3}(\mathbb{P}(V_4) \times s \mathbb{P}(V_{n-2s})) & \longrightarrow & A^\ast_{\text{GL}_3}(\mathbb{P}(V_n)) \\
\cong & & \cong \\
A^\ast_{\text{PGL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(2,2n-4s)) & \longrightarrow & A^\ast_{\text{PGL}_2}(\mathbb{P}(2,2n)) \\
A^\ast_{\text{SL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(2,2n-4s)) & \longrightarrow & A^\ast_{\text{SL}_2}(\mathbb{P}(2,2n))
\end{array}
\]

Then it must be true that \( \xi = \pi_1^{'\prime} \eta + c_3 \cdot \eta' \); indeed we know from [EF09] that the image of the last horizontal map is contained in the image of \( \pi_1^{'\prime} \), and that the kernel of the last two vertical maps (which are surjective) is generated as an ideal by \( c_3 \).

Observe that we can assume that \( \eta \), seen as a polynomial in \( h_n \), has only odd coefficients: indeed, if we write \( \eta = \eta' + 2\eta'' \) then

\[
c_3 \cdot \eta = c_3 \cdot \eta' + 2c_3 \cdot \eta'' = c_3 \cdot \eta'
\]

We deduce then that \( \eta \) must be equal to \( h_n \cdot \gamma \), because by hypothesis when we evaluate \( \xi \) in \( h_n = 0 \) we must obtain something even, and \( \eta \) has only odd coefficients. In the end, we have that \( \xi = \pi_1^{'\prime} \eta + h_n \cdot c_3 \cdot \gamma \). We now pull back \( \xi \) to \( V_n \setminus \sigma_0 \), which we saw to be equivalent to substituting \( h_n \) with \(-2\tau\), so we get:

\[
p^\ast \xi = p^\ast \pi_1^{'\prime} \eta - 2\tau \cdot c_3 \cdot p^\ast \gamma = p^\ast \pi_1^{'\prime} \eta
\]

where in the last equality we used the relation \( 2c_3 = 0 \). This concludes the proof of the lemma. \( \square \)

Putting all together, we have finally proved the

**Theorem 6.2.** We have \( A^\ast(\mathcal{H}_g) = \mathbb{Z}[\tau, c_2, c_3]/(4(2g+1)\tau, 8\tau^2 - 2g(2g + 1)c_2, 2c_3) \).

We want to give a geometrical interpretation of the generators of \( A^\ast(\mathcal{H}_g) \). Recall that in order to do the computations of the last three sections we used the isomorphism \([U'/\text{GL}_3 \times \mathbb{G}_m] \) obtained in theorem 2.8, where \( U' \) is the open subscheme of \( V_{g+1} \) whose points are pairs \((q, f)\) such that \( Q \) and \( F \) intersect transversely.
We also showed, at the end of section 2, that the rank 4 vector bundle over $\mathcal{H}_g$ associated to the $GL_3 \times \mathbb{G}_m$-torsor $U'$ is the vector bundle $\mathcal{E} \oplus \mathcal{L}$, where $\mathcal{L}$ is the line bundle over $\mathcal{H}_g$ functorially defined as

$$\mathcal{L}((\pi : C \to S, i)) = \pi_* \omega_{C/S}^{\otimes \frac{g+1}{2}} \left( 1 - \frac{g}{2} W \right)$$

and $\mathcal{E}$ is the rank 3 vector bundle over $\mathcal{H}_g$ functorially defined as

$$\mathcal{E}((\pi : C \to S, i)) = \pi_* \omega_{C/S}^{\vee} (W)$$

Then by construction the generator $\tau$ coincides with $c_1(\mathcal{L})$ and $c_2$ and $c_3$ coincide respectively with $c_2(\mathcal{E})$ and $c_3(\mathcal{E})$, whereas $c_1(\mathcal{E}) = 0$. This analysis agrees with the one made in the last section of [FV11].

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