Non-Abelian $U$-duality for membranes

Yuho Sakatani* and Shozo Uehara

Department of Physics, Kyoto Prefectural University of Medicine, Kyoto 606-0823, Japan

*E-mail: yuho@koto.kpu-m.ac.jp

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The $T$-duality of string theory can be extended to the Poisson–Lie (PL) $T$-duality when the target space has a generalized isometry group given by a Drinfel’d double. In M-theory, $T$-duality is understood as a subgroup of $U$-duality, but the non-Abelian extension of $U$-duality is still a mystery. In this paper we study membrane theory on a curved background with a generalized isometry group given by the $E_n$ algebra. This provides a natural setup to study non-Abelian $U$-duality because the $E_n$ algebra has been proposed as a $U$-duality extension of the Drinfel’d double. We show that the standard treatment of Abelian $U$-duality can be extended to the non-Abelian setup. However, a famous issue in Abelian $U$-duality still exists in the non-Abelian extension.

Subject Index B20, B23, B25

1. Introduction

Abelian $T$-duality is a symmetry of string theory when the target space has $D$ commuting Killing vector fields. This $T$-duality can be extended to the Poisson–Lie (PL) $T$-duality [1,2] when the target geometry has a certain symmetry generated by the Lie algebra of the Drinfel’d double. For the PL $T$-duality, the usual Killing vector fields are not necessary and we can consider the extended $T$-duality in a more general class of target spaces. Similar to Abelian $T$-duality, the PL $T$-duality is a symmetry of the supergravity equations of motion (see, e.g, Ref. [3]), and it can generate various supergravity solutions (see, e.g., Refs. [2,4–16]). To be more precise, some dual geometries do not solve the standard supergravity equations but rather the modified ones, known as the generalized supergravity equations [17,18]. However, as shown in Refs. [21,22], the generalized supergravity equations can be derived from double field theory (DFT) [23–27], which is a $T$-duality-manifest formulation of supergravity. Thus, now the PL $T$-duality has been understood as a symmetry of DFT [14,28–31]. Recently, various aspects of the PL $T$-duality, in particular its relation to the Yang–Baxter deformation [7,32–35] (i.e. a class of integrable deformations of string theory) have been clarified and are still actively studied.

Type IIA string theory compactified on a flat $D$-torus has the $O(D,D)$ Abelian $T$-duality symmetry. From the perspective of M-theory compactified on a flat $n$-torus ($n \equiv D + 1$), this $O(D,D)$ Abelian $T$-duality group is understood as a subgroup of the $E_n$ $U$-duality group. By using the $U$-duality-manifest formulation of supergravity, known as the exceptional field theory (EFT) [36–40], the $U$-duality symmetry in supergravity has been clearly understood. EFT also exhibits the duality between M-theory and type IIB supergravity, and it also provides a useful framework to study various non-geometric backgrounds or non-trivial compactifications. Moreover, by using EFT, a $U$-duality extension of the PL $T$-duality has been discussed recently in Refs. [41,42] for $n \leq 4$, where

1 Note that the modified equations for the NS–NS sector have been discussed earlier in Refs. [19,20].
the Drinfeld double is realized as a subalgebra of the proposed $E_n$ algebra. There still remain many things to be clarified, but it has been expected that this $E_n$ algebra is the symmetry underlying the non-Abelian extension of $U$-duality.

In contrast to the success in supergravity, $U$-duality symmetry in membrane theory remains mysterious. In the case of string theory, the equations of motion in a flat space have been successfully expressed in a $T$-duality-covariant form [43]. By closely following this approach, the equations of motion for a membrane have been expressed in a $U$-duality-covariant form [44] (see also Refs. [45,46]). However, as pointed out in Ref. [47], certain integrability is broken under general $U$-duality transformations, and it has been concluded that only a subgroup of Abelian $U$-duality is the symmetry of the membrane equations of motion. Only when the dimension of the target space is $n = 3$ (where the membrane is space-filling and called the topological membrane) is the full $SL(2) \times SL(3)$ $U$-duality consistently realized [47] (see also Refs. [48–50]).

In this paper, focusing on the successful case $n = 3$, we investigate non-Abelian $U$-duality in membrane theory. Our main results are as follows. In the $T$-duality-covariant formulation of string theory, the displacement $dx^m(\sigma) (m = 1, \ldots, D)$ is extended to the generalized displacement $\mathcal{P}_M(\sigma) = dx^M(\sigma) (M = 1, \ldots, 2D)$. In the setup of the PL $T$-duality, this $\mathcal{P}_M$ is further extended to the Maurer–Cartan (MC) form satisfying $d\mathcal{P}^A = \frac{1}{2} \mathcal{F}_{BC}^A \mathcal{P}^B \wedge \mathcal{P}^C (A = 1, \ldots, 2D)$, where $\mathcal{F}_{BC}^A$ denote the structure constants of the Drinfeld double. Locally, it can be parameterized as $\mathcal{P}^A T_A = dl l^{-1}$ by using a group element of a Drinfel’d double $l(\sigma)$, and this reduces to $\mathcal{P}_M = dx^M$ when the Drinfeld double is Abelian. In the $U$-duality-covariant formulation of the topological membrane in flat space ($n = 3$), the generalized displacement satisfies $d\mathcal{P}^l = 0 (l = 1, \ldots, \frac{n(n+1)}{2})$, and it can be locally express as $\mathcal{P}^l = dx^l$. This paper studies its extension to the case where the target space is curved. By requiring the target space to have a symmetry of the $E_n$ algebra, we show that the generalized displacement satisfies the MC equation $d\mathcal{P}^A = \frac{1}{2} \mathcal{F}_{BC}^A \mathcal{P}^B \wedge \mathcal{P}^C (A = 1, \ldots, \frac{n(n+1)}{2})$, where $\mathcal{F}_{BC}^A$ are the structure constants of the $E_n$ algebra. The MC equation does not depend on the choice of the generators $T_A$ in the $E_n$ algebra, and it is manifestly covariant under the change of generators $T'_A = C_A^B T_B$, where the constant matrix $C_A^B$ is an element of the $U$-duality group $E_3 \equiv SL(2) \times SL(3)$. This arbitrariness in the choice of generators is what we call non-Abelian $U$-duality. It naturally unifies the PL $T$-duality and Abelian $U$-duality.

For clarity, we note here several subtleties. In string theory, the number of equations of motion for the scalar fields $x^m$ is $D$, and the number of non-trivial MC equations is also $D$. Accordingly, we can show that the equations of motion are equivalent to the MC equations. In contrast, in membrane theory, the number of equations of motion for the scalar fields $x^i$ is $n$ while the number of non-trivial MC equations is $\frac{n(n-1)}{2}$ (see Sect. 2.2). For $n \geq 4$, the number of non-trivial MC equations is greater than the number of equations of motion, and it is impossible to realize the full MC equations even under the equations of motion. Therefore, we can show the MC equations only when $n = 3$. In $n = 3$, both the equations of motion and the MC equations are identically satisfied and it is not so clear whether we can claim that the non-Abelian $U$-duality is the symmetry of the membrane equations of motion. However, the situation is completely the same as Abelian $U$-duality. Our result is a natural non-Abelian extension of the standard Abelian $U$-duality, and the fact that the generalized displacement $\mathcal{P}^A$ satisfies the MC equations is non-trivial.

The structure of this paper is as follows. In Sect. 2 we review Abelian $T$-duality and $U$-duality. The famous issue in $U$-duality is also reviewed. In Sect. 3, we discuss the non-Abelian extensions. The PL $T$-duality in string theory is reviewed in Sect. 3.1. In Sect. 3.2 we discuss non-Abelian $U$-duality.
in membrane theory; Sect. 4 is devoted to the summary and discussion. Technical details are given in the Appendices.

2. Abelian T-duality and U-duality

In this section we review Abelian T-duality in string theory. We also review Abelian U-duality in membrane theory and explain a notorious issue specific to U-duality.

2.1. Abelian T-duality in string theory

In order to perform Abelian T-duality, the target space needs to have Abelian Killing vectors. Thus, we consider here string theory in a D-dimensional flat space, where the supergravity fields are constant. The string equations of motion can be expressed as

\[ dJ_m = 0, \quad J_m = g_{mn} * dx^n + B_{mn} dx^n. \]  

(2.1)

We also have a trivially conserved current,

\[ dJ^m = 0, \quad J^m = dx^m, \]  

(2.2)

and Abelian T-duality can be understood as a permutation of the equations of motion and the Bianchi identities [43],

\[ J_M \rightarrow J'_M = C^N_M J_N, \quad (J_M) = \left( \frac{J_m}{J^m} \right) = \left( \frac{g_{mn} * dx^n + B_{mn} dx^n}{dx^m} \right). \]  

(2.3)

If we introduce the generalized metric

\[ (\mathcal{H}^N_M) = \begin{pmatrix} (Bg^{-1})^m_n & (g - Bg^{-1}B)_{mn} \\ g_{mn} & -(g^{-1}B)^m_n \end{pmatrix}, \]  

(2.4)

which is an O(D,D) matrix preserving the O(D,D) metric invariant

\[ \mathcal{H}^P_M \mathcal{H}^Q_N \eta_{PQ} = \eta_{MN} \quad (\eta_{MN}) = \begin{pmatrix} 0 & \delta^m_n \\ \delta_m^n & 0 \end{pmatrix}, \]  

(2.5)

we find that the 1-form fields \( J_M(\sigma) \) satisfy the self-duality relation,

\[ J_M = \mathcal{H}^N_M J_N. \]  

(2.6)

In order to keep this relation under the rotation in Eq. (2.3), \( \mathcal{H}^N_M \) should also be transformed as

\[ \mathcal{H}'^N_M = C^P_M \mathcal{H}^Q_P (C^{-1})_Q^N. \]  

(2.7)

By requiring that the transformed metric \( \mathcal{H}'^N_M \) is still an O(D,D) matrix, the matrix \( C^N_M \) is required to be an O(D,D) element. This O(D,D) symmetry is the standard Abelian T-duality.

Now, let us introduce 1-form fields \( P^M(\sigma) \) through

\[ J_M = \mathcal{H}^N_M P^N, \quad (P^M) = \begin{pmatrix} dx^m \\ g_{mn} * dx^n + B_{mn} dx^n \end{pmatrix}. \]  

(2.8)
Here and hereafter, we raise or lower the indices \(M, N\) by using the matrix \(\eta\); e.g., \(\mathcal{H}_{MN} = \mathcal{H}_M^P \eta_{PN}\). Under the equations of motion \(d\mathcal{J}_M = 0\), the 1-form fields \(\mathcal{P}^M\) satisfy

\[
d \ast \mathcal{P}^M = 0. \tag{2.9}
\]

From the self-duality relation in Eq. (2.6), the \(\mathcal{P}^M\) also satisfy

\[
\mathcal{P}^M = \mathcal{H}^M_{\ast N} \ast \mathcal{P}^N. \tag{2.10}
\]

Then, since \(\mathcal{H}^{MN}_{\ast N}\) is constant and invertible, the equations of motion in Eq. (2.9) are equivalent to

\[
d \mathcal{P}^M = 0. \tag{2.11}
\]

This shows that, under the equations of motion, we can locally express the 1-form fields as

\[
\mathcal{P}^M(\sigma) = dx^M(\sigma), \quad (x^M) = (x^\tilde{m}). \tag{2.12}
\]

The scalar fields \(x^M(\sigma)\) are interpreted as the embedding functions into a 2D-dimensional doubled space, and \(\mathcal{P}^M = dx^M\) is interpreted as the generalized displacement.

### 2.2. Abelian U-duality in membrane theory

In Ref. [44], the same idea has been applied to membrane theory in a flat space. By following Refs. [44,45], we consider the dynamics of a membrane in an \(n\)-dimensional Lorentzian spacetime (\(n \leq 4\)). Similar to the string case, the equations of motion are expressed as

\[
d \mathcal{J}_i = 0, \quad \mathcal{J}_i \equiv g_{ij} \ast dx^j \quad \quad - \frac{1}{2} \quad C_{ijk} \quad dx^j \wedge dx^k, \tag{2.13}
\]

where \(i, j = 1, \ldots, n\) and \(\ast\) is the Hodge star operator associated with the induced metric \(h_{\alpha\beta} \equiv g_{ij} \partial_\alpha x^i \partial_\beta x^j\). The trivially conserved current, known as the topological current, is defined as \(\mathcal{J}^i_{\ast j} \equiv dx^i \wedge dx^j\), and we consider the combination

\[
(\mathcal{J}_i) \equiv \left( \begin{array}{c} \mathcal{J}_{i_{12}} \\ \frac{\mathcal{J}^i_{\ast j}}{\sqrt{2!}} \end{array} \right) = \left( g_{ij} \ast dx^j \quad - \frac{1}{2} \quad C_{ijk} \quad dx^j \wedge dx^k \right). \tag{2.14}
\]

Similar to the Abelian T-duality of Eq. (2.3), Abelian U-duality can be understood as a permutation of the equations of motion \((d\mathcal{J}_i = 0)\) and the Bianchi identities \((d\mathcal{J}^i_{\ast j} = 0)\),

\[
\mathcal{J}_i \rightarrow \mathcal{J}^i_j = C_{ij} \mathcal{J}_j. \tag{2.15}
\]

### 2.2.1. Definitions

In order to see that the matrix \(C_{ij}^j\) is restricted to the \(E_n\) U-duality group, let us make several definitions. The generalized metric in EFT,\(^2\) which is a \(U\)-duality-covariant combination of supergravity fields, is defined as

\[
(\mathcal{H}_{ij}) \equiv \left( \begin{array}{c} g_{ij} + \frac{1}{2} \quad C_{ij}^{k_1 k_2} \quad C_{k_1 k_2 j} \quad - \frac{C_{i_{12}}^{j}}{\sqrt{2!}} \quad g_{i_{12}, j_{12}} \quad \sqrt{2!} \end{array} \right), \tag{2.16}
\]

\(^2\)The generalized metric \(\mathcal{H}_{ij}\) has “effective weight” 0 while \(\mathcal{M}_{ij} \equiv |g|^{-1/2} \mathcal{H}_{ij} \in E_n\) has weight 0 [38].
where \( g_{ij}^{i_1 j_1 j_2} \equiv g_{i_1}^{i_1} g_{j_1 j_2} \), and the inverse is denoted as

\[
(\mathcal{H}^{IJ}) = \begin{pmatrix}
g^{ij} & \frac{C_{i_1 j_1}}{\sqrt{2!}} \\
\frac{C_{i_2 j_2}}{\sqrt{2!}} & g_{i_1 i_2 j_1 j_2} + \frac{1}{2} \, C_{i_1 i_2} \, C_{j_1 j_2}
\end{pmatrix}.
\]  

(2.17)

We also introduce the \( U \)-duality-invariant tensor \( \eta_{IJ;K} \), where \( K \) denotes the index for the so-called \( R_2 \)-representation of the \( E_n \) group that can be decomposed as \( (\eta_{IJ;K}) = \left( \eta_{IJ;K}, \frac{\eta_{I(i_1 j_1) j_2 K}}{\sqrt{4!}} \right) \). For \( n \leq 4 \), they are explicitly defined as [51]

\[
(\eta_{IJ}) = \begin{pmatrix}
0 & \frac{2 \delta_{i_1 j_1}}{\sqrt{2!}} \\
\frac{2 \delta_{i_2 j_2}}{\sqrt{2!}} & 0
\end{pmatrix}, \quad (\eta_{IJ;k_1 \cdots k_4}) = \begin{pmatrix}
0 & 0 \\
\frac{4 \delta_{i_1 j_1} \delta_{i_2 j_2}}{\sqrt{2!}} & 0
\end{pmatrix}.
\]  

(2.18)

where \( \delta_{i_1 \cdots i_p} \equiv \delta_{i_1} \cdots \delta_{i_p} \). As discussed in Ref. [52], in order to formulate membrane theory in a \( U \)-duality-covariant manner, it is important to introduce the charge vector for a membrane,

\[
(q^T) = \left( q^I, \frac{q_{i_1 \cdots i_4}}{\sqrt{4!}} \right).
\]  

(2.19)

We then define a 1-form, which we call the \( \eta \)-form,

\[
(\eta_{IJ}) = \begin{pmatrix}
0 & \delta_{i_1 j_1} \\
\delta_{i_2 j_2} & 0
\end{pmatrix}.
\]  

(2.20)

The membrane charge vector \( q^T \) transforms covariantly under the \( U \)-duality transformation in Eq. (2.15), and accordingly, the \( \eta \)-form also transforms covariantly as

\[
\eta_{IJ} \to \eta'_{IJ} = C_{I}^{K} \, C_{J}^{L} \, \eta_{KL}.
\]  

(2.21)

When we consider the M2-brane (without any M5-charge induced), the charge vector should be chosen as [52]

\[
(q^T) = \frac{1}{2} \left( dx^i, 0 \right),
\]  

(2.22)

and then the \( \eta \)-form becomes

\[
(\eta_{IJ}) = \begin{pmatrix}
0 & \delta_{i_1 j_1} \\
\delta_{i_1 j_1} & 0
\end{pmatrix}.
\]  

(2.23)

We also define the matrix

\[
(\mathcal{H}_T) = (\eta_{IK} \, \mathcal{H}^{KJ}) = \begin{pmatrix}
\frac{1}{2} \, C_{ik} \, dx^k & \frac{g_{i_1 j_1} \, dx^k + \frac{1}{2} \, C_{i_1 j_1} \, C_{j_1 j_2} \, dx^k}{\sqrt{2!}} \\
\frac{g_{i_2 j_2} \, dx^k + \frac{1}{2} \, C_{i_2 j_2} \, C_{j_1 j_2} \, dx^k}{\sqrt{2!}} & \frac{1}{2} \, C_{i_1 i_2} \, dx^k
\end{pmatrix},
\]  

(2.24)

which corresponds to the matrix in Eq. (2.4) defined in string theory, although here it is a 1-form.

2.2.2. \( E_n \) \( U \)-duality

By using the above definitions, we find the self-duality relation

\[
J_I = \mathcal{H}_T^I \wedge * J_I,
\]  

(2.25)
which corresponds to Eq. (2.6). Since the matrix $\mathcal{H} \eta^I$ transforms covariantly only under the $E_n$ $U$-duality transformations, the transformation matrix $C_I^J$ given in Eq. (2.15) should be an element of the $E_n$ group.

Similar to the string case, we introduce 1-form fields $\mathcal{P}^I(\sigma)$ as

$$\mathcal{J}_I \equiv \mathcal{H}_{IJ} \ast \mathcal{P}^J, \quad (\mathcal{P}^I) = \left( \frac{dx_i}{\sqrt{2 \Lambda_1}} \right). \tag{2.26}$$

Then, from the relation in Eq. (2.25), we obtain the self-duality relation for $\mathcal{P}^I$,

$$\mathcal{P}^I = *(\mathcal{H}^I \wedge \mathcal{P}^J) \quad [\mathcal{H}^I = \mathcal{H}_{IJ} \eta_K = (\mathcal{H}^I)^J]. \tag{2.27}$$

This corresponds to the relation in Eq. (2.10) in string theory. Since $\mathcal{J}_I$ and $\mathcal{H}_{IJ}$ transform as

$$\mathcal{J}'_I = C_I^J \mathcal{J}_J, \quad \mathcal{H}'_{IJ} = C_I^K C_J^L \mathcal{H}_{KL} \tag{2.28}$$

under $U$-duality transformations, the 1-form $\mathcal{P}^I(\sigma)$ should be transformed as

$$\mathcal{P}^I(\sigma) = (C^{-1})_J^I \mathcal{P}^J(\sigma). \tag{2.29}$$

The $U$-duality-covariant equations of motion $d\mathcal{J}_I = 0$ can also be expressed as

$$d(\mathcal{H}_{IJ} \ast \mathcal{P}^I) = 0 \iff d \ast \mathcal{P}^I = 0. \tag{2.30}$$

2.2.3. An issue specific to $U$-duality

So far, everything is parallel to the string case. However, as pointed out in Ref. [47], the transformation in Eq. (2.29) generally causes an issue. Here, we explain the issue by following the presentation given in Ref. [52]. By using the self-duality relation of Eq. (2.27), the equations of motion in Eq. (2.30) are equivalent to

$$d(\mathcal{H}^I \wedge \mathcal{P}^J) = -\mathcal{H}^I \wedge d\mathcal{P}^J = 0. \tag{2.31}$$

Unlike the string case, $\mathcal{H}^I$ is not invertible and they are not equivalent to $d\mathcal{P}^I = 0$. To be more precise, the equations of motion are weaker than the (Abelian) MC equation $d\mathcal{P}^I = 0$. Indeed, Ref. [47] gives an explicit solution of membrane theory where the equations of motion in Eq. (2.31) are satisfied but $d\mathcal{P}^I \neq 0$. By the definition of $\mathcal{P}^I$ given in Eq. (2.26), the first component $\mathcal{P}^i$ trivially satisfies $d\mathcal{P}^i = 0$, but for the second component $\mathcal{P}^{i_1 i_2}$, $d\mathcal{P}^{i_1 i_2} = 0$ is not ensured.

Let us suppose that we have a solution $x^i(\sigma)$ satisfying $d\mathcal{P}^{i_1 i_2}(\sigma) \neq 0$. Under a particular $U$-duality transformation,$^3$

$$(C_I^J) = \left( \begin{array}{cc} \Lambda^k_i & 0 \\ 0 & (\Lambda^{-1})_{k_1}^{i_1} (\Lambda^{-1})_{k_2}^{i_2} \end{array} \right) \left( \begin{array}{cc} \delta^j_k & -c_i^{k_1 k_2} \\ 0 & \delta_{j_1 j_2}^{k_1 k_2} \end{array} \right) \tag{2.32}$$

with $\Lambda^k_i$ and $c_i^{k_1 k_2}$ constants, the (constant) supergravity fields are transformed as

$$g_{i_1 i_2} = \Lambda_i^k \Lambda^j_l g_{kl}, \quad C_{i_1 i_2 i_3} = \Lambda_i^{j_1} \Lambda_i^{j_2} \Lambda_i^{j_3} (C_{j_1 j_2 j_3} + c_{j_1 j_2 j_3}). \tag{2.33}$$

$^3$ The GL(n) matrix contained in the $E_n$ group has the form $|\det(\Lambda)| \frac{1}{\sqrt{n}} \left( \begin{array}{cc} \Lambda^j_i & 0 \\ 0 & (\Lambda^{-1})_{j_1}^{i_1} (\Lambda^{-1})_{j_2}^{i_2} \end{array} \right)$, but since we are using the generalized tensors with the effective weight 0, the determinant factor is dropped out.
At the same time, the 1-form fields are transformed as

$$
\mathcal{P}^I \rightarrow \mathcal{P}'^I = \left( C_{i12j} (\Lambda^{-1})^j_i \frac{d\mathcal{P}^j}{\sqrt{2!}} - g_{i12,j12} (\Lambda^{-1})^j_i (\Lambda^{-1})^j_2 \frac{\ast(dx^1 \wedge dx^2)}{\sqrt{2!}} \right). 
$$

(2.34)

This shows that

$$
x'^i(\sigma) = (\Lambda^{-1})^j_i x^j(\sigma) 
$$

is a solution of membrane theory in the dual geometry, and the (geometric) $U$-duality in Eq. (2.32) always maps a solution to the dual solution. On the other hand, a serious problem happens if we consider the (non-geometric) $\Omega$-transformation,

$$
(C_I^J) = \begin{pmatrix} \delta_I^J & 0 \\ \frac{\omega_{i12}}{\sqrt{2!}} & \delta_{i12} \end{pmatrix}. 
$$

(2.36)

After the $\Omega$-transformation, we obtain

$$
\mathcal{P}'^I \equiv \frac{\partial \mathcal{P}^I}{\partial x^I} = \left( \frac{dx^I}{\sqrt{2!}} + \frac{1}{2} \frac{\omega_{i12}}{\sqrt{2!}} \mathcal{P}_{i12} \right). 
$$

(2.37)

By assumption, we have $d\mathcal{P}_{i12} \neq 0$, and thus $d\mathcal{P}'^I(\sigma) \neq 0$. This shows that we cannot parameterize the dualized 1-form field $\mathcal{P}'^I(\sigma)$ as

$$
\mathcal{P}'^I = \left( C_{i12j} \frac{dx^j}{\sqrt{2!}} - g_{i12,j12} \frac{\ast(dx^1 \wedge dx^2)}{\sqrt{2!}} \right). 
$$

(2.38)

because the integrability $d\mathcal{P}'^I = d^2x'^I(\sigma) = 0$ is now violated.

In general, $E_n$ $U$-duality transformations (for $n \leq 4$) are generated by the geometric transformations of Eq. (2.32) and the $\Omega$-transformation of Eq. (2.36), but only the former preserve the integrability. Thus, it was concluded in Ref. [47] that only the geometric subgroup of $U$-duality is the (classical) symmetry of membrane theory. A resolution was discussed in Ref. [52], but even in their approach it is impossible to realize the full MC equation $d\mathcal{P}'^I(\sigma) = 0$. Therefore, unlike the string case, we cannot express the 1-form as

$$
\mathcal{P}^I(\sigma) = dx^I(\sigma), \quad \left( x^I \equiv \frac{x_i}{\sqrt{2!}} \right). 
$$

(2.39)

In summary, the point is that the equations of motion $\mathcal{H}^I_J \wedge d\mathcal{P}^J = 0$ are weaker than the MC equation $d\mathcal{P}^I = 0$ and we cannot realize $\mathcal{P}^I(\sigma) = dx^I(\sigma)$ even under the equations of motion. Accordingly, unlike the string case, we cannot interpret that the membrane is fluctuating in an extended spacetime with coordinates $x^I$.

2.2.4. An exceptional case where $n = 3$

As discussed in Ref. [47], the case $n = 3$ is exceptional. There, the membrane is called the topological membrane because it is non-dynamical. Indeed, by using the identity

$$
\ast(dx^i \wedge dx^j) = \varepsilon^i_{jk} dx^k \left( \varepsilon^{012} = \frac{1}{\sqrt{3!}} \right),
$$

(2.40)
the equations of motion in Eq. (2.13) are identically satisfied. Moreover, as is clear from

\[ (\mathcal{P}^I) = \left( \frac{dx^I}{\sqrt{2}} \right), \]

the (Abelian) MC equation

\[ d\mathcal{P}^I(\sigma) = 0 \]

is also identically satisfied, and, at least locally, we can express the 1-form as

\[ \mathcal{P}^I(\sigma) = dx^I(\sigma). \]

Here, \( x^I \) describes the embedding of the membrane into the six-dimensional extended space. We can freely rotate a given solution \( x^I(\sigma) \) as

\[ x^I(\sigma) \rightarrow x'^I(\sigma) = (C^{-1})^I_J x^J(\sigma) \]

under the full SL(2) \( \times \) SL(3) U-duality transformation, and here the U-duality group is not restricted to the geometric subgroup.

For \( n \geq 4 \), only part of \( d\mathcal{P}^I(\sigma) = 0 \) can be derived from the equations of motion. As discussed in Sect. 1, naively we have only \( n \) equations of motion, but the number of non-trivial components of the MC equation \( d\mathcal{P}_{ij} = 0 \) is \( \frac{n(n-1)}{2} \). Therefore, these coincide only when \( n = 3 \) (see Ref. [49] for a similar discussion). For \( n \geq 4 \), we cannot expect to obtain the full components of \( d\mathcal{P}_{ij} = 0 \). If any component of the MC equation is not satisfied we obtain \( d\mathcal{P}'_{i} \neq 0 \) after a certain \( \Omega \)-transformation, and the integrability is broken.

Of course, as is well discussed in EFT, at the level of supergravity, the Lagrangian or the equations of motion have the \( E_n \) U-duality symmetry for an arbitrary \( n \leq 8 \) (or perhaps \( n \leq 11 \)). The issue arises only when we try to realize the symmetry in membrane theory. A membrane is only a member of the supersymmetric branes, which form a U-duality multiplet. In order to realize the full U-duality symmetry, we will need to formulate a brane theory which describes all of the supersymmetric branes in a unified manner (see Sect. 4 for more discussion on such a formulation). At present, such a formulation has not been found, and we can realize the U-duality symmetry only for the topological membrane. Accordingly, as we discuss below, we can realize non-Abelian U-duality only for the topological membrane.

3. Non-Abelian T-/U-duality

In this section we study the non-Abelian extension of U-duality.

3.1. PL T-duality in string theory

Before studying non-Abelian U-duality, we review the PL T-duality in string theory [1,2].

3.1.1. PL T-dualizability

In order to perform the PL T-duality, the target geometry is required to satisfy the differential equations [1,2]

\[ \mathcal{L}_{\nu_\beta} E_{mn} = -\hat{\mathcal{G}}^{bc}_{d} E_{mp} v^{P}_{\nu_\beta} v^{Q}_{\nu_\gamma} E_{qn}. \]
where $E_{mn}(x) \equiv g_{mn}(x) + B_{mn}(x)$ and $v^m_a (a = 1, \ldots, D)$ is a set of vector fields satisfying the algebra $[v_a, v_b] = f_{ab}^c v_c$. Under this setup, the string equations of motion are expressed as the MC equation,

$$dJ_a - \frac{1}{2} f_{a}^{bc} J_b \wedge J_c = 0, \quad J_a = v^m_a (g_{mn} \ast dx^n + B_{mn} dx^n).$$

(3.2)

As discussed in Refs. [1,2], Eq. (3.1) suggests that $f_{abc}$ and $\tilde{f}_{a b c}$ can be identified with the structure constants of the Lie algebra of the Drinfel’d double,

$$[T_a, T_b] = f_{abc} T_c, \quad [\tilde{T}_a, \tilde{T}_b] = \tilde{f}_{a b c} \tilde{T}_c, \quad [T_a, \tilde{T}_b] = \tilde{f}_{b a c} T_c - f_{a c b} \tilde{T}_c.$$  

(3.3)

This is sometimes expressed as $[T_A, T_B] = \mathcal{F}_{AB}^C T_C$ by denoting the set of generators as $\{T_A\} \equiv \{T_a, \tilde{T}^a\}$. In addition, an ad-invariant bilinear form is defined for the generators,

$$\langle T_A, T_B \rangle \equiv \eta_{AB}, \quad (\eta_{AB}) \equiv \begin{pmatrix} 0 & \delta_b^a \\ \delta_b^a & 0 \end{pmatrix}.$$

(3.4)

We denote a subgroup $G$ generated by $\{T_a\}$ as the physical subgroup while a subgroup $\tilde{G}$ generated by $\{\tilde{T}^a\}$ as the dual group. If we assume that the target space is a group manifold of $G$ and identify the vector fields $v^m_a$ with the left-invariant vector fields, we can solve the differential equation in Eq. (3.1) as follows [1]:

$$E_{mn}(x) = [\hat{E} (1 - \Pi(x) \hat{E})^{-1}]_{ab} r^a_m(x) r^b_m(x).$$

(3.5)

Here, $\hat{E} \equiv (\hat{E}_{ab})$ is a constant matrix and several quantities are defined as follows. For a group element $g(x) \in G$, we define a matrix $M_A^B(x)$ as

$$g^{-1} T_A g \equiv M_A^B T_B.$$  

(3.6)

From the structure of the algebra in Eq. (3.3), the matrix $M_A^B(x)$ can be generally parameterized as follows by using two matrices $a^b_a(x)$ and $\Pi^{ab}(x) = -\Pi^{ba}(x)$:

$$(M_A^B) \equiv \begin{pmatrix} \delta^c_a & 0 \\ -\Pi^{ac} & \delta^c_b \end{pmatrix} \begin{pmatrix} a^b_a & 0 \\ 0 & (a^{-1})_b^c \end{pmatrix}.$$  

(3.7)

The left- and right-invariant 1-forms are denoted as

$$\ell \equiv g^{-1} dg, \quad r \equiv dg g^{-1} \quad (r^a_m = a^b_a r^b_m),$$

(3.8)

and they are dual to the left- and right-invariant vectors $v^m_a$ and $e^m_a$,

$$\ell^a_m v^m_b = \delta^a_b, \quad r^a_m e^m_b = \delta^a_b.$$  

(3.9)

Now, the solution in Eq. (3.5) can be understood from these definitions. When a target geometry takes the form of Eq. (3.5), we can perform the PL $T$-duality.

---

4 Ad-invariance means $\langle [T_C, T_A], T_B \rangle + \langle T_A, [T_C, T_B] \rangle = 0$. 
3.1.2. Manifest T-duality

The solution in Eq. (3.5) can be rewritten in a nicer form by using the generalized metric $\mathcal{H}_{MN}$. Indeed, the solution in Eq. (3.5) can be expressed as

$$\mathcal{H}_{MN}(x) = E_M^A(x) E_N^B(x) \hat{\mathcal{H}}_{AB}.$$ (3.10)

Here, $\hat{\mathcal{H}}_{AB}$ is a constant matrix associated with $\hat{E}_{ab} \equiv (\hat{g} + \hat{B})_{ab}$ ($\hat{g}_{ab} \equiv \hat{E}_{(ab)}$, $\hat{B}_{ab} \equiv \hat{E}_{(ab)}$), and the coordinate dependence is contained only in the twist matrix $E_M^A(x)$,

$$\hat{E}_{ab} \equiv (\hat{g} - \hat{B} \hat{g}^{-1})_{ab}, \quad (E_M^A) \equiv \begin{pmatrix} r_a^m & 0 \\ -e^m_{ca} & e_a^m \end{pmatrix}. \quad (3.11)$$

We note that the inverse of the twist matrix, denoted as $E_{AM}$, is known as the generalized frame fields and, in fact, they satisfy the relation $[28]$

$$\hat{\xi}_{E_A} E^M_B = -\mathcal{F}^C_{AB} E_C^M,$$ (3.12)

where $\hat{\xi}$ denotes the generalized Lie derivative in DFT. Then, we can show that the generalized metric satisfies the equation

$$\hat{\xi}_{E_A} \mathcal{H}_{MN} = \mathcal{F}^P_{AM} \mathcal{H}_{PN} + \mathcal{F}^P_{AN} \mathcal{H}_{MP} \quad (\mathcal{F}^N_{AM} \equiv \mathcal{F}^C_{AB} E^B_M E^C_N),$$ (3.13)

which shows that the target space has the symmetry of the Drinfel’d double.

Now, we rewrite the equations of motion in Eq. (3.2) into a T-duality-manifest form. Similar to the Abelian case, we define 1-form fields

$$(P^A) = \begin{pmatrix} \ell^a_m \\mathcal{J}_a \end{pmatrix} = \begin{pmatrix} \ell^a_m dx^m \\ \nu^m_{\alpha} (g_{mn} * dx^n + B_{mn} dx^n) \end{pmatrix},$$ (3.14)

which reduce to Eq. (2.8) in the Abelian case (where $\ell^a_m = \delta^a_m$ and $\nu^m_{\alpha} = \delta^m_{\alpha}$). For convenience, we also define

$$P(\sigma) \equiv P^A(\sigma) T_A = \ell + \mathcal{J} \quad (\ell \equiv \ell^a T_a, \quad \mathcal{J} \equiv \mathcal{J}_a \tilde{T}^a).$$ (3.15)

By further acting the adjoint action, we define

$$P(\sigma) \equiv P^A(\sigma) T_A \equiv g P(\sigma) g^{-1} = g (\ell + \mathcal{J}) g^{-1} \quad \left[ \Leftrightarrow P^A = (M^{-1})_B^A P^B = E_M^A P^M, \quad (P^M) \equiv \begin{pmatrix} dx^m \\ g_{mn} * dx^n + B_{mn} dx^n \end{pmatrix} \right].$$ (3.16)

Equation (3.2) suggests that, under the equations of motion, $\mathcal{J}$ can be identified with the right-invariant 1-form $\tilde{r} \equiv d\tilde{g} \tilde{g}^{-1}$ associated with a dual group element $\tilde{g}(\tilde{x})$, and we obtain

$$P(\sigma) = g (g^{-1} dg + d\tilde{g} \tilde{g}^{-1}) g^{-1} = dl^{-1} \quad (l \equiv g \tilde{g}).$$ (3.17)

This shows that the 1-form field $P(\sigma)$ is the right-invariant 1-form on the Drinfel’d double, which satisfies

$$dP - P \wedge P = 0 \quad \text{or} \quad dP^A - \frac{1}{2} \mathcal{F}^{BC}_A P^B \wedge P^C = 0.$$ (3.18)
Similar to the Abelian case, the 1-form fields are subjected to the self-duality relation,

\[ P^A = \mathcal{H}^A_B \ast P^B \quad (\mathcal{H}^A_B \equiv \hat{\mathcal{H}}^{AC} \eta_{CB}), \]  

(3.19)

and only \( D \) components are independent. Thus, the 2D MC equations in Eq. (3.18) are equivalent to the \( D \) equations of motion given in Eq. (3.2).

### 3.1.3. PL \( T \)-duality

The PL \( T \)-duality (or the PL \( T \)-plurality \[6\]) is a symmetry under redefinitions of the generators

\[ T'_A = C^B_A T_B. \]  

(3.20)

Under the redefinition, the structure constants are transformed as

\[ \mathcal{F}'_{AB}^\ C = C^D_A C^F_B (C^{-1})_F^\ C \mathcal{F}_{DE}^F. \]  

(3.21)

By requiring that the redefined algebra is also a Lie algebra of the Drinfel’d double, the metric \( \eta_{AB} \) [i.e. the bilinear form of Eq. (3.4)] must be preserved,

\[ C^A_C C^B_D \eta_{CD} = \eta_{AB}. \]  

(3.22)

Namely, the constant matrix \( C^A_B \) should be an element of the \( O(D, D) \) group. After the redefinition, we introduce new group elements \( g'(\sigma) \) and \( \tilde{g}'(\tilde{\sigma}) \) such that \( g \tilde{g} = l = l' = g' \tilde{g}' \) is satisfied. Then, we obtain \( P(\sigma) = dl l^{-1} = dl' l'^{-1} \equiv P'(\sigma) \), or, equivalently,

\[ P'^A = (C^{-1})_B^\ A P^B. \]  

(3.23)

This shows that the equations of motion in Eqs. (3.18) and (3.19) are covariantly transformed if \( \mathcal{H}_{AB} \) is also transformed as

\[ \mathcal{H}'_{AB} = C^A_C C^B_D \mathcal{H}_{CD}. \]  

(3.24)

In this sense, the PL \( T \)-duality is an \( O(D, D) \) transformation that covariantly transforms the equations of motion of string theory.

In summary, the essential point of the PL \( T \)-duality is that the string equations of motion are expressed as the MC equations in Eq. (3.18) for a 1-form \( P(\sigma) \) that satisfy the self-duality relation in Eq. (3.19). These equations are manifestly covariant under the \( O(D, D) \) PL \( T \)-duality transformations given in Eqs. (3.20), (3.23), and (3.24).

### 3.1.4. Dual solution

Although the equations of motion are manifestly covariant, the procedure to obtain the dual string solution may be rather complicated. For the explicit computation, we need to fix the parameterizations of the group elements (e.g. \( g(x) = e^{x^\alpha T_\alpha} \) and \( \tilde{g}(\tilde{x}) = e^{\tilde{x}^\alpha \tilde{T}_\alpha} \)). Given these, we can compute the original target geometry by using Eq. (3.5). After an \( O(D, D) \) rotation, we again provide parameterizations of group elements, such as \( g'(x') = e^{x'^\alpha T_\alpha} \) and \( \tilde{g}'(\tilde{x}') = e^{\tilde{x}'^\alpha \tilde{T}_\alpha} \), and then obtain the dual geometry

\[ E'''_{mn}(x) = \left[ \hat{E}' \left( 1 - \Pi' \hat{E}' \right)^{-1} \right]_{ab} \alpha^a_m \beta^b_n. \]  

(3.25)

In order to relate the two geometries, we require

\[ g(x) \tilde{g}(x) = l = g'(x') \tilde{g}'(x'). \]  

(3.26)
Then, in principle, we can find the relation between the two coordinates,

\[ x'^a = x^a(x^a, \tilde{x}_a), \quad \tilde{x}'_a = \tilde{x}_a'(x^a, \tilde{x}_a). \quad (3.27) \]

Using this relation, we can map a string solution in the original geometry to the dual solution. From a given solution \( x^a(\sigma) \), we can compute the 1-form \( \mathcal{J}_a \) defined in Eq. (3.2). Then, solving the differential equations \( \mathcal{J} = d\tilde{g}^{-1} \), we find \( \tilde{x}_a(\sigma) \). Finally, substituting the solutions \( x^a(\sigma) \) and \( \tilde{x}_a(\sigma) \) into Eq. (3.27), we obtain the dual solution \( x'^a(\sigma) \).

Another, easier, method is as follows. From a given solution \( x^a(\sigma) \), we can map a string solution in the original geometry to the dual solution. From a given solution \( x^a(\sigma) \), we can easily compute the 1-form field \( \mathcal{P}(\sigma) \). Expanding \( \mathcal{P} \) by means of the redefined generators \( T'_a \), we obtain the dual solution \( x'^a(\sigma) \).

Either way, we can map a solution \( x^m(\sigma) \) to a new solution \( x'^m(\sigma) \) of the dual sigma model.

### 3.2. Non-Abelian U-duality

Here, we study the \( U \)-duality extension of the PL \( T \)-duality in membrane theory. We note that our analysis is restricted to \( n \leq 4 \).

#### 3.2.1. The \( \mathcal{E}_n \) algebra

The PL \( T \)-duality is based on the Lie algebra of the Drinfel’d double, and, similarly, non-Abelian \( U \)-duality will be based on a new algebra that extends the Lie algebra of the Drinfel’d double. Such an algebra was recently proposed in Refs. [41,42], and we call it the \( \mathcal{E}_n \) algebra, following Ref. [41]. For \( n \leq 4 \), the algebra is given by

\[
T_a \circ T_b = f_{ab}^\ c T_c, \\
T_a \circ T_b \circ T_c = f_{ab}^{\ [b_1 b_2} T_c^{\ b_2]} + 2f_{ab}^{[b_1} T^{b_2]}_{\ ]}, \\
T^{a_1 a_2} \circ T_b = -f_b^{a_1 a_2 c} T_c + 3f_{c [a_2} [a_1 b_2]} T^{c 1 c}, \\
T^{a_1 a_2} \circ T_b = -2f_d^{a_1 a_2 [b_1} T^{b_2]}_{\ ]}. \\
(3.28)
\]

where \( a, b = 1, \ldots, n \). The indices of the generators are antisymmetric, \( T_{ab} = -T_{ba} \), and the structure constants have symmetries \( f_{ab}^\ c = f_{[ab]}^\ c \) and \( f_a^{b_1 b_2 b_3} = f_a^{[b_1 b_2 b_3]} \). For simplicity, we denote the algebra as

\[
T_A \circ T_B = \mathcal{F}_{AB}^\ C T_C, \quad (T_A) \equiv (T_A, \frac{T^{a_1 a_2}}{\sqrt{2}}). \\
(3.29)
\]

Since the first two indices of the structure constants are not antisymmetric (i.e. \( \mathcal{F}_{AB}^\ C \neq -\mathcal{F}_{BA}^\ C \)), this is a Leibniz algebra rather than a Lie algebra. The Leibniz identity,

\[
T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C + T_B \circ (T_A \circ T_C), \\
(3.30)
\]

requires that the structure constants should satisfy [41]

\[
0 = f_{[ab}^\ e f_e^{\ e d]} , \\
0 = f_{bc}^\ e f_e^{a_1 a_2 d} + 6f_{eb}^\ [d f_e^{a_1 a_2]} e, \\
0 = f_{d_1 d_2}^{a_1 a_2} f_e^{d_1 d_2} e, \\
(3.31)
(3.32)
(3.33)
\]
The right-invariant 1-form \(3.2.2\). Target space with the \(E\)
can construct a target geometry as \(\frac{\eta_{\text{IAB}}}{\Pi_{1}}\) where \(E\)\(\eta_{\text{IAB}}\) of \(E\)\(\eta_{\text{IAB}}\) is a constant matrix and \(E\) of \(E\) has the form

\[
\mathcal{H}_{IJ}(x) = E_{I}^{A}(x) E_{J}^{B}(x) \hat{\mathcal{H}}_{AB}
\]

by using the \(E\) algebra. Here, \(\hat{\mathcal{H}}_{AB}\) is a constant matrix and \(E_{I}^{A}\) has the form

\[
(E_{I}^{A}) = \begin{pmatrix}
\frac{\eta_{\text{IAB}}}{\Pi_{1}} & 0 \\
\eta_{[a_{1}a_{2}]} & \eta_{[a_{1}a_{2}]}
\end{pmatrix} \frac{\eta_{\text{IAB}}}{\Pi_{1}}
\]

The right-invariant 1-form \(r_{a}^{b}\) and its dual \(e_{a}^{b}\) has been defined by using a physical group element \(g(x)\), which we parameterize as \(g(x) = e^{x^{T}T_{A}}\). In addition, similar to Eqs. (3.6) and (3.7), the tri-vector \(\Pi_{ab}^{c}\) is defined as

\[
\mathcal{H}_{IJ} = F_{AIJ}^{\Gamma} \mathcal{H}_{ijkl} + F_{AIJ}^{C} \mathcal{H}_{ijk} \quad \left(F_{AIJ}^{\Gamma} = F_{AB}^{C} E_{I}^{B} E_{C}^{J}\right).
\]

If we introduce the dual metric \(\hat{g}_{ij}\) and \(\Omega_{ijk}\) through the non-geometric parameterization of the generalized metric (see, for example, Refs. [53–55])

\[
(\mathcal{H}_{IJ}) = \begin{pmatrix}
\delta_{i}^{k} & 0 \\
\delta_{i}^{l} & 0 \\
\delta_{j}^{k} & \delta_{j}^{l} & 0
\end{pmatrix}
\]

we find

\[
\hat{g}_{ij} = e_{i}^{a} e_{j}^{b} \hat{g}_{ab} = \eta_{i}^{a} \eta_{j}^{b} \hat{g}_{ab}, \quad \Omega_{ijk} = \Omega^{abc} e_{i}^{a} e_{j}^{b} e_{c}^{k} \left(\Omega^{abc} = \Pi^{abc} + \hat{\Omega}^{abc}\right),
\]

Similar to the Drinfel’d double, a \(U\)-duality-invariant metric has been defined as

\[
\langle T_{A}, T_{B} \rangle_{C} = \eta_{\text{IAB}} C,
\]

where \((\eta_{AB};C) = (\eta_{AB};C)\) has the same matrix elements as Eq. (2.18).
where we have parameterized the constant matrix \( \hat{H}_{AB} \) as
\[
(\hat{H}_{AB}) = \begin{pmatrix}
\delta^*_a & 0 \\
\frac{\delta^*_a}{\sqrt{2}} & \delta_{c_1 c_2}^a d_2 \\
0 & \hat{g}_{c_1 c_2} d_2 \\
0 & \hat{g}_{c_1 c_2} d_2 \\
\end{pmatrix},
\]
where \( \hat{g}^{abc} \) and \( \hat{g}_{ab} \) are constants that are assumed to satisfy
\[
f_{a(b'} \hat{g}_{c)d} = 0, \quad f_{de}^{[a} \hat{g}^{bc]e} = 0.
\]
Then, we find that the dual fields satisfy [41]
\[
\mathcal{L}_{v_a} \hat{g}_{ij} = 0, \quad \mathcal{L}_{v_a} \Omega^{ijk} = f_a^{bcd} v_b^i v_c^j v_d^k.
\]
The target space constructed in this way is the setup to discuss non-Abelian \( U \)-duality.

In order to identify the standard supergravity fields, we make the following identification between the standard fields \( (g_{ij}, C_{ijk}) \) and the dual fields \( (\tilde{g}_{ij}, \tilde{\Omega}^{ijk}) \) [53,54] (see also Ref. [44]):
\[
|\tilde{g}|^{1/2} \begin{pmatrix}
\delta^*_l \\
\frac{\delta^*_l}{2} \\
0 \\
0 \\
\end{pmatrix} \begin{pmatrix}
\tilde{g}_{kl} \\
\frac{\tilde{g}_{kl}}{2} \\
0 \\
0 \\
\end{pmatrix} = |g|^{1/2} \begin{pmatrix}
\delta^*_i \\
\frac{\delta^*_i}{2} \\
0 \\
0 \\
\end{pmatrix} \begin{pmatrix}
g_{kl} \\
\frac{g_{kl}}{2} \\
0 \\
0 \\
\end{pmatrix}.
\]

where the density factors are needed in order to remove the weight of the generalized metric. From this relation, the standard supergravity fields (for \( n \leq 4 \)) are obtained as follows:
\[
g_{ij} = K^{1/2} \left( K^{-1} \tilde{g}_{ij} - \frac{1}{2} \Omega_{ijkl} \tilde{\Omega}^{klj} \right), \quad C^{ijk} = -K \Omega^{ijk},
\]
where \( K^{-1} = 1 + \frac{1}{2} \Omega^{ijk} \Omega_{ijk} \) and the indices of \( C_{ijk} \) and \( \Omega^{ijk} \) are raised or lowered by the metric \( g_{ij} \) and the dual metric \( \tilde{g}_{ij} \), respectively.

In terms of the standard fields, the relations in Eq. (3.46) read\(^5\)
\[
\mathcal{L}_{v_a} g_{ij} = -\frac{2}{3} \frac{\ell_a}{3!} f_{a}^{bcd} C_{bcd} g_{ij} + f_{a}^{bcd} C_{bc(i} gJD_{j)d},
\]
\[
\mathcal{L}_{v_a} C_3 = -\frac{1}{3!} f_{a}^{bcd} \ell_b \wedge \ell_c \wedge \ell_d + \frac{1}{3!} f_{a}^{bcd} C_{bcd} C_3,
\]

where \( \ell_a \equiv g_{ab} \ell^b \), and the curved indices \( i, j \) of \( g_{ij} \) and \( C_{ijk} \) have been converted to the indices \( a, b \) by using \( \ell^i \) (e.g. \( C_{abc} \equiv \ell^a v^b v^c C_{ijk} \)).

We also note that the metric \( G_{ab} \equiv e_a^i e_b^j g_{ij} \) also satisfies
\[
f_{a(b'} \hat{G}_{c)d} = 0.
\]
Indeed, in \( n = 3 \), \( g_{ij} \propto \tilde{g}_{ij} \) and Eq. (3.50) is trivial. In \( n = 4 \), we can parameterize \( \Omega^{ijk} \) as \( \tilde{\Omega}^{ijk} = \varepsilon^{ijkl} \Omega^l (\tilde{\epsilon}^{0123} = \frac{1}{\sqrt{|g|}}) \) (see Appendix A), and then we obtain
\[
g_{ij} = K^{1/2} \left( \tilde{g}_{ij} - \Omega_i \Omega_j \right), \quad K = \frac{1}{1 - \tilde{g}_{ij} \Omega_i \Omega_j}.
\]

\(^5\) They can be checked by using relations specific to \( n = 3, 4 \) given in Appendices A and B.
By assuming \( f_{ab}^d = 0 \), Eq. (3.45) leads to \( f_{abc}^c \Omega_c = 0 \), where \( \Omega_c = \hat{\varepsilon}^{abcd} \hat{\Omega_d} \). Moreover, from the identity in Eq. (C.12), we obtain \( f_{abc}^c (\Omega^{abd} - \hat{\Omega}^{abd}) = 0 \), which is equivalent to \( f_{ab}^d (\Omega_c - \hat{\Omega}_c) = 0 \). Then, we obtain

\[
f_{abc}^c \Omega_c = f_{abc}^c (\Omega_c - \hat{\Omega}_c) = -2 f_{c[a}^c (\Omega_{b]} - \hat{\Omega}_{b]}), \tag{3.52}
\]

This shows the desired relation in \( n = 4 \),

\[
f_{ab}^d G_{c[d} = K^2 (f_{ab}^d \hat{g}_{c]} - f_{ab}^d \Omega_c) \Omega_d) = 0. \tag{3.53}
\]

Thus, both in \( n = 3 \) and \( n = 4 \) (with \( f_{ab}^d = 0 \)), \( G_{ab} \) is an invariant metric, and we have

\[
G_{ab} = (a^{-1}) a^c (a^{-1}) b^d G_{cd} = \nu^i_a \nu^j_b g_{ij} = g_{ab}. \tag{3.54}
\]

### 3.2.3. Membrane theory

Now, let us consider membrane theory. In a general curved spacetime, the equations of motion for the scalar fields \( x^i(\sigma) \) become

\[
\partial_\alpha \left( \sqrt{-h} g_{ij} h^{\alpha\beta} \partial_\beta x^j + \frac{1}{2} C_{ijk} \varepsilon^{\alpha\beta\gamma} \partial_\beta x^j \partial_\gamma x^k \right) = \frac{1}{2} \partial_i g_{jk} h^{\alpha\beta} \partial_\alpha x^k \partial_\beta x^j + \frac{1}{3!} \partial_i C_{k_1k_2k_3} \varepsilon^{\alpha\beta\gamma} \partial_\alpha x^{k_1} \partial_\beta x^{k_2} \partial_\gamma x^{k_3}, \tag{3.55}
\]

where \( \varepsilon^{012} = 1 \). By contracting the free index \( i \) with a set of vector fields \( v^i_a \), we obtain

\[
d J_a = \frac{1}{2} \nu_v g_{ij} dx^i \wedge * dx^j - \nu_v C_3, \tag{3.56}
\]

where we have defined

\[
J_a = * \ell_a - \frac{1}{2} C_{abc} \ell^b \wedge \ell^c. \tag{3.57}
\]

In the target geometry given by Eqs. (3.37) and (3.48), by choosing the vector fields \( \nu^i_a \) as the left-invariant vector fields, the equations of motion become

\[
d J_a = \frac{1}{3!} f_{a}^{bcd} \left[ \ell_b \wedge \ell_c \wedge \ell_d + (3 C_{bce} H_{d}^e - C_{bcd}) * 1 - C_{bcd} C_3 \right], \tag{3.58}
\]

where \( H_b^a \equiv h^{\alpha\beta} \ell_i^a \ell_j^b g_{i\beta} \partial_\alpha x^j \partial_\beta x^i \). Note that \( H_b^a \) is a projector satisfying \( H_b^a H_b^c = H_b^a \) in \( n = 4 \), while \( H_b^a = \delta_b^a \) in \( n = 3 \). Note also that Eq. (3.58) reduces to the equations of motion in Eq. (2.13) in the Abelian case, where \( f_{a}^{bcd} = 0 \) and \( \ell^a = \delta_a^i dx^i \).

For the manifest \( U \)-duality, we define the combination

\[
(J_A) = \left( J_a \epsilon^1 \wedge \epsilon^2 \right) / (2^1), \tag{3.59}
\]

similar to the Abelian case of Eq. (2.14). Similar to Eq. (2.26), we also define the Hodge dual of the 2-form field \( J_A \) through

\[
J_A = \mathcal{H}_{AB} \ast P^B, \tag{3.60}
\]
where

\[
(H_{AB}) \equiv \begin{pmatrix}
\delta^c_d & -C_{abc}^1C_{abc}^2 \\
\delta^d_c & 0
\end{pmatrix}
\begin{pmatrix}
g_{cd} & 0 \\
g^{c1c2d1d2} & g^{d1d2}
\end{pmatrix}
\begin{pmatrix}
\delta^d_b \\
\delta^b_c
\end{pmatrix}
= \begin{pmatrix}
\frac{C_{d1d2}}{\sqrt{2!}} & 0 \\
\frac{C_{d1d2}}{\sqrt{2!}} & \frac{g^{d1d2}}{\delta^b_c}
\end{pmatrix}.
\]

(3.61)

Then, we find that the 1-form fields have the form

\[
(P^A) \equiv \begin{pmatrix}
P^a \\
P_{a1a2}
\end{pmatrix}
\begin{pmatrix}
\ell^a \\
\frac{1}{2^!} P_{a1a2}^{\ell a}
\end{pmatrix}.
\]

(3.62)

Similar to the case of the PL T-duality [see Eq. (3.16)], we redefine the 1-form fields as

\[
\mathcal{P} \equiv \mathcal{P}^A T_A = g \circ \left( P^a T_a + \frac{1}{2} P_{a1a2} T^{a1a2} \right) \Leftrightarrow \mathcal{P}^A = (M^{-1})_B^A \mathcal{P}^B.
\]

Then, we obtain

\[
(\mathcal{P}^A) \equiv \begin{pmatrix}
\mathcal{P}^a \\
\mathcal{P}_{a1a2}
\end{pmatrix}
\begin{pmatrix}
\ell^a \\
\frac{1}{2^!} P_{a1a2}^{\ell a}
\end{pmatrix}.
\]

(3.64)

Similar to Eq. (3.19), this satisfies the self-duality relation

\[
\mathcal{P}^A = \ast (\ast \mathcal{H}^A_B \wedge \mathcal{P}^B) \quad (\mathcal{H}^A_B \equiv \ast \mathcal{H}^{AC} \eta_{CB}),
\]

(3.65)

where we have defined the \( \eta \)-form as

\[
(\eta_{AB}) \equiv \begin{pmatrix}
0 & \frac{\delta^{01} \ell^{p1}}{\sqrt{2!}} \\
\frac{\delta^{p1} \ell^{01}}{\sqrt{2!}} & 0
\end{pmatrix}.
\]

(3.66)

3.2.4. Equations of motion

By using the identity

\[
\ell^b \wedge P_{ba} = \ell^b \wedge \left[ C_{bac} \ell^c - \ast (\ell_b \wedge \ell_a) \right] = 2 \mathcal{J}_d,
\]

(3.67)

the equations of motion in Eq. (3.58) can be expressed as

\[
\ell^b \wedge dP_{ba} = -\frac{1}{2} f_{cd}^b \ell^c \wedge \ell^d \wedge P_{ba}
\]

\[
- \frac{1}{3} f_{a}^{bcd} \left[ \ell_b \wedge \ell_c \wedge \ell_d + (3 C_{bce} H_d^e - C_{bcd}) \ast 1 - C_{bce} C_3 \right],
\]

(3.68)

where we have used \( d\ell^b = -\frac{1}{2} f_{cd}^b \ell^c \wedge \ell^d \). We then consider a projection,

\[
H_a^c \ell^b \wedge dP_{bc} = H_a^c \left[ -\frac{1}{2} f_{cd}^b \ell^c \wedge \ell^d \wedge P_{bc}
\right.
\]

\[
- \frac{1}{3} f_{e}^{bcd} \left[ \ell_b \wedge \ell_c \wedge \ell_d + (3 C_{bce} H_d^e - C_{bcd}) \ast 1 - C_{bce} C_3 \right].
\]

(3.69)

When \( n = 3 \), \( H_a^c = \delta_a^c \) and they are equivalent to the equations of motion in Eq. (3.68), while when \( n = 4 \), one equation has been projected out. In fact, as we show in Appendix D, Eq. (3.69) is equivalent to

\[
H_a^c \ell^b \wedge \left( dP_{bc} - \frac{1}{2} f_{lo}^{def} P_{cd} \wedge P_{ef} + \frac{1}{2} f_{bc}^d P^e \wedge P_{de} \right) = 0.
\]

(3.70)
To be more precise, when $n = 3$ we can show the equivalence without any assumption, but when $n = 4$ we need to assume $f_{ab}^c = 0$. Then, the equations of motion in Eq. (3.70) imply
\begin{equation}
\frac{dP_{ab}}{d\sigma} = \frac{1}{2} f_{a}^{cde} P_{b|c} \wedge P_{d|e} - \frac{1}{2} f_{ab}^{c} P^{c} \wedge P_{c|d}.
\end{equation}

In fact, we can directly show that Eq. (3.71) is identically satisfied in $n = 3$ (see Appendix B). Thus, the equations of motion in $n = 3$ are automatically satisfied and the membrane is non-dynamical even in the curved background given in Eq. (3.37). In $n = 4$, the projected equations in Eq. (3.70) are satisfied under the equations of motion, but they do not lead to Eq. (3.71).

In fact, as we show in Appendix C, the relation in Eq. (3.71), which is suggested by the equations of motion, is equivalent to the MC equation of the $E_n$ algebra,
\begin{equation}
dP^A = \frac{1}{2} F_{BC}^{A} P^B \wedge P^C,
\end{equation}
where the $F_{BC}^{A}$ are the structure constants of the $E_n$ algebra. Thus, in $n = 3$, the generalized displacement $P^A$ satisfies the MC equation, which generalizes the Abelian one given in Eq. (2.42). In $n = 4$, $P^A$ does not satisfy the MC equation, similar to the Abelian case, and we cannot perform the full $U$-duality transformation.

### 3.2.5. Non-Abelian $U$-duality

In $n = 3$, non-Abelian $U$-duality is realized as a redefinition of the $E_n$ generators,
\begin{equation}
T'_{A} = C_{A}^{B} T_{B},
\end{equation}
where $C_{A}^{B}$ is an element of the $U$-duality group $SL(2) \times SL(3)$. Under the redefinition, the structure constants are transformed as
\begin{equation}
F'_{AB}^{C} = C_{A}^{D} C_{B}^{E} (C^{-1})_{F}^{C} F_{DE}^{F}.
\end{equation}

In order to keep the MC 1-form $P$ invariant, the components should be transformed as
\begin{equation}
P'^{A} = (C^{-1})_{B}^{A} P^{B}.
\end{equation}

Then, the MC equation in Eq. (3.72) is manifestly covariant under the non-Abelian $U$-duality in Eq. (3.73). The $\eta$-form is also transformed covariantly,
\begin{equation}
\eta'_{AB} = C_{A}^{C} C_{B}^{D} \eta_{CD},
\end{equation}
and by further transforming the constant matrix as
\begin{equation}
\hat{H}'_{AB} = C_{A}^{C} C_{B}^{D} \hat{H}_{CD},
\end{equation}
the self-duality relation of Eq. (3.65) is also manifestly covariant under Eq. (3.73).

If a solution $x^{i}(\sigma)$ of membrane theory is given, we can explicitly compute the 1-form fields $P^{A}(\sigma)$. After the change of generators in Eq. (3.73), the 1-form fields are transformed as $P'^{A} = (C^{-1})_{B}^{A} P^{B}$. We can also introduce a new group element $g'(x') = e^{\epsilon_{0}^{A} T_{a}'}$ and, through the relation in Eq. (3.63), we can compute the 1-form $P'^{A} = M_{B}^{A} P^{B}$ in the dual theory. Since the first component $P^{a}$ has been identified as the left-invariant 1-form $\ell^{a}$, by solving
\begin{equation}
P'^{a} T_{a}' = g'^{-1}(x') \, dg'(x')
\end{equation}
we can in principle determine the dual solution $x'^{i}(\sigma)$. 


4. Discussion

We have studied membrane theory in a curved background, Eq. (3.37), which has the symmetry of the $E_n$ algebra. Similar to the case of Abelian $U$-duality, we can show that the generalized displacement $\mathcal{P}^A$ satisfies the MC equation of the $E_n$ algebra only when $n = 3$. Both the MC equation and the self-duality relation for $\mathcal{P}^A$ are manifestly covariant under the non-Abelian $U$-duality of Eq. (3.73) (which is a redefinition of the $E_n$ generators) and we have naturally extended the standard story of Abelian $U$-duality to the non-Abelian setup. In $n = 4$, we face the difficulty already known in the Abelian case, and $\mathcal{P}^A$ do not satisfy the MC equation even under the equations of motion.

In addition to the membrane, M-theory contains the M5-brane as well (see Refs. [52,56,57] for M5-brane theory in $U$-duality-covariant approaches). Again in the M5-brane theory, the equations of motion will not generally provide the MC equation. The only exceptional case will be $n = 6$, where the M5-brane becomes space-filling. There, the generalized displacement is extended as $\mathcal{P}^A = (p^a, p^{a_1 a_2} \sqrt{\frac{2}{3}}, p^{a_1 ... a_5} \sqrt{\frac{5}{3}})$ (see, for example, Ref. [52]), and the number of non-trivial components of the MC equations is $\frac{n!}{2(n-2)!} + \frac{n!}{5!(n-5)!}$, corresponding to $\mathcal{P}_{a_1 a_2}$ and $\mathcal{P}_{a_1 ... a_5}$. The dynamical fields on the M5-brane are $x^i$ and the 2-form gauge field $A_{\alpha \beta}$ ($\alpha, \beta = 0, \ldots, 5$) and, naively, the number of equations of motion coincides with the number of non-trivial MC equations when $n = 6$. Thus, we expect that $U$-duality symmetry in the M5-brane theory can be realized for $n = 6$. For $n > 6$, the number of equations of motion is smaller and the full MC equation will not be reproduced. In order to examine this possibility, it is important to construct the $E_n$ algebra for $n = 6$ or higher.

As is well known, when the (self-dual) field strength on the M5-brane is non-vanishing, the M2-brane is induced on the M5-brane. Then, the dynamics of the induced M2-brane will be described by the 2-form gauge fields $A_{\alpha \beta}$ on the M5-brane. For example, in Sect. 6 of Ref. [58], the gauge fields are dualized to the embedding functions $x^i$ of the M2-brane, and the membrane action is reproduced from the M5-brane action. Then, it is interesting to consider the following possibility. As we have discussed, in $n = 6$, membrane theory does not have the $E_6$ $U$-duality symmetry. However, if the $E_6$ $U$-duality symmetry is realized in the topological (or space-filling) M5-brane theory, it is interesting to interpret the topological M5-brane theory as the $E_6$-covariant membrane theory. Since the M5-brane is space-filling $x^i$ will be non-dynamical, and only the gauge fields $A_{\alpha \beta}$ are dynamical, which describe the fluctuation of the membrane. If $\mathcal{P}^A$ satisfies the MC equation, we can perform non-Abelian $U$-duality. This approach may resolve the issue of $U$-duality in membrane theory. Moreover, in the approach of Refs. [52,57], gauge fields on the worldvolume are introduced as the diffeomorphism parameters along the dual direction in the extended spacetime. In other words, the gauge fields are interpreted as the fluctuation along the dual directions in the extended spacetime. Since the number of diffeomorphism parameters along the dual direction is always the same as the number of non-trivial components of the MC equations, naively we can expect that the MC equation is realized under the equations of motion even for higher $n$. For example, in $n = 8$, it will be impossible to realize the $E_8$ duality symmetry in M5-brane theory. However, there, the Kaluza–Klein monopole (KKM) is space-filling, and its worldvolume theory may have the $E_8$ $U$-duality symmetry. If so, it may be possible to regard the topological KKM theory as the $E_8$ M5-brane theory. We hope to work on this in the future.

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6 Here we have not taken into account the self-duality relation for the gauge field.
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Note added

To clarify the connection with the $\mathcal{E}$-model [59], we show here the classical current algebra. One of the defining properties of the $\mathcal{E}$-model is the current algebra,

$$\{ j_A(\sigma), j_B(\sigma') \} = \mathcal{F}_{AB}^{\mathcal{C}} j_C(\sigma) \delta(\sigma - \sigma') + \eta_{AB} \partial_x \delta(\sigma - \sigma). \quad (4.1)$$

In the context of the PL $T$-duality, the spatial component of $\mathcal{J}_A \equiv \hat{H}_{AB} \ast \mathcal{P}^B$ plays the role of the current:

$$j_A(\sigma) = \mathcal{J}_{A}\sigma = E_A^M(\chi(\sigma)) Z_M(\sigma), \quad Z_M(\sigma) \equiv (\mathcal{H}_{MN} \ast P^N)_{\sigma} = \left( \frac{p_m}{\partial_{x^m}} \right). \quad (4.2)$$

Here, $p_m \equiv -g_{mn} \sqrt{-h} h^{0a} \partial_{x^a} x^n + B_{mn} \epsilon^{0a} \partial_{x^a} x^n$ is the canonical conjugate momenta of $x^m(\sigma)$ in the Hamiltonian formulation. As shown in Refs. [24,60], this current satisfies the algebra in Eq. (4.1) by means of the equal-time Poisson bracket $\{ x^m(\sigma), p_n(\sigma') \} = \delta^m_n \delta(\sigma - \sigma')$.

We provide here a brief sketch of the extension of the current algebra to the case of membrane theory. For this purpose, we employ the analysis of Ref. [46], where a canonical analysis of membrane theory in a flat space was worked in the SL(5)-covariant manner. There, the generalized momenta $Z_M(\sigma)$ defined in Eq. (4.2) are extended to $Z_I(\sigma)$, which are defined as the spatial components of $\mathcal{J}_I$ defined in Eq. (2.14). Namely, $Z_I(\sigma)$ is given by

$$Z_I(\sigma) \equiv \frac{1}{2} \epsilon^{0ab} \mathcal{J}_{Iab} = \left( \frac{\epsilon^{0ab} \partial_{x^a} \partial_{x^b} x^1 x^2}{\sqrt{2\pi}} \right), \quad (4.3)$$

$$p_i = -g_{ij} \sqrt{-h} h^{0a} \partial_{x^a} x^j - \frac{1}{2} \epsilon^{0ab} C_{ijk} \partial_{x^i} \partial_{x^j} x^k. \quad (4.4)$$

Then, the equal-time Poisson bracket $\{ x^i(\sigma), p_j(\sigma') \} = \delta^i_j \delta^2(\sigma - \sigma')$ leads to

$$\{ Z_I(\sigma), Z_J(\sigma') \} = \rho^I_{J\sigma}(\sigma) \partial_\sigma \delta^2(\sigma - \sigma') - \rho^I_{J\sigma}(\sigma) \eta_{IJ,k} \partial_{x^k}(\sigma). \quad (4.5)$$

In our case of interest ($n \leq 4$), defining the current $j_A(\sigma)$ as

$$j_A(\sigma) \equiv E_A^I(\chi(\sigma)) Z_I(\sigma), \quad (4.6)$$

we obtain a natural extension of the current algebra in Eq. (4.1),

$$\{ j_A(\sigma), j_B(\sigma') \} = \mathcal{F}_{AB}^{\mathcal{C}} j_C(\sigma) \delta(\sigma - \sigma') - \left( \rho_{J\sigma}^I E_A^I E_B^I(\sigma') \partial_\sigma \delta^2(\sigma - \sigma') \right), \quad (4.7)$$

---

7 We would like to thank an anonymous referee for the suggestion.

8 Here we have used the following identity [46] for $n \leq 4$ and arbitrary vectors $\Lambda^I_1$ and $\Lambda^I_2$ (the arbitrary parameter $K$ in Ref. [46] is chosen as $K = 1$):

$$\{ \Lambda^I_1(\chi(\sigma)) Z_I(\sigma), \Lambda^I_2(\chi(\sigma')) Z_I(\sigma') \} = -(\hat{\epsilon}_{\Lambda^I_1, \Lambda^I_2}^I(\chi(\sigma)) Z_I(\sigma) \delta^2(\sigma - \sigma')$$

$$- \left( \rho_{J\sigma}^I \Lambda^I_1(\sigma') \partial_\sigma \delta^2(\sigma - \sigma') \right).$$
where the coefficient in the second term is no longer constant. It would be interesting to study the membrane extension of the $E$-model that is defined by the current algebra in Eq. (4.7) and the Hamiltonian $H \equiv \frac{1}{2} \int d^2 \sigma \tilde{t}^{AB} j_A(\sigma) j_B(\sigma)$.

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Appendix A. Formulas in $n = 4$
In $n = 4$, the relations between the non-geometric fields and the standard fields are given as

$$\tilde{g}_{ij} = K^{\frac{1}{2}} E_{ij}, \quad E_{ij} \equiv g_{ij} + \frac{1}{2} C_i^{kl} C_{klij}, \quad \Omega^{ijk} = -E_{il}^{\phantom{i}m} g_{kn} C_{lmn} = -K C^{ijk},$$

$$\Omega_{ijk} = -K^{-1} C_{ijk}, \quad \text{det}(E_{ij}) = K^{-3} \text{det}(g_{ij}), \quad \sqrt{-\tilde{g}} = K^{-\frac{3}{2}} \sqrt{-g}, \quad (A.1)$$

where the indices of $C_{ijk}$ and $\Omega^{ijk}$ are raised or lowered with the metric $g_{ij}$ and $\tilde{g}_{ij}$, respectively. In $n = 4$, it is useful to parameterize $C_{ijk}$ and $\Omega^{ijk}$ as

$$C_{ijk} = \epsilon_{ijkl} C^l, \quad \Omega^{ijk} = \epsilon^{ijkl} \Omega_l, \quad (A.2)$$

and then we find the following relations:

$$\Omega^i = -K^{-\frac{1}{2}} C^i, \quad \Omega_i = -K^{\frac{1}{2}} C_i, \quad K = \frac{1}{1 - g_{ij} C^i C^j} = \frac{1}{1 - \tilde{g}^{ij} \Omega_i \Omega_j},$$

$$g_{ij} = K^{\frac{3}{2}} (g_{ij} - \Omega_i \Omega_j), \quad \tilde{g}_{ij} = K^{\frac{3}{2}} (g_{ij} - C_i C_j). \quad (A.3)$$

Appendix B. Results specific to $n = 3$
In this appendix, by considering $n = 3$, we show Eq. (3.71), namely,

$$dP_{ab} = \frac{1}{2} f^{a c d e} P_{b}^{c} \wedge P_{de} = \frac{1}{2} f_{ab}^{\phantom{ab}c} P^{c} \wedge P_{cd}. \quad (B.1)$$

Using the dual metric $\tilde{g}_{ij}$ defined in Eq. (3.43), we define the anti-symmetric tensor

$$\tilde{\epsilon}_{ijk} \equiv \sqrt{g} | \epsilon_{ijk}, \quad \tilde{\epsilon}^{ijk} \equiv \frac{1}{\sqrt{|g|}} \epsilon^{ijk}, \quad \epsilon^{012} \equiv 1 \equiv -\epsilon_{012}. \quad (B.2)$$

In $n = 3$, the tri-vector $\Omega^{ijk}$ should be proportional to $\tilde{\epsilon}^{ijk}$, and we define

$$\Omega^{ijk} \equiv \omega \tilde{\epsilon}^{ijk}. \quad (B.3)$$

From Eq. (3.48) we obtain the standard supergravity fields as

$$g_{ij} = \frac{1}{(1 - \omega^2)^{2/3}} \tilde{g}_{ij}, \quad C_{ijk} = -\frac{\omega}{1 - \omega^2} \tilde{\epsilon}_{ijk} = -\omega \epsilon_{ijk}, \quad (B.4)$$

where we have defined

$$\epsilon_{ijk} \equiv \sqrt{|g|} \epsilon_{ijk} = \frac{1}{1 - \omega^2} \tilde{\epsilon}_{ijk}, \quad \epsilon^{ijk} \equiv \frac{1}{\sqrt{|g|}} \epsilon^{ijk} = (1 - \omega^2) \tilde{\epsilon}^{ijk}. \quad (B.5)$$
Using $\varepsilon_{ij} = 0$ (which follows from $\varepsilon_{ij}g_{ij} = 0$), Eq. (3.46) reduces to

$$
\varepsilon_{ij} \varepsilon^{kl} = 0 \quad \Rightarrow \quad d\omega = -\frac{1}{3!} f^{bcd}_{a} \bar{g}_{bcd} \ell^{a}.
$$

(B.6)

Another important relation specific to $n = 3$ is

$$
\varepsilon^{abc} \ell^{c} = * (\ell^{a} \wedge \ell^{b}).
$$

(B.7)

This allows us to simplify $P_{ab}$ as

$$
P_{ab} \equiv C_{abc} \ell^{c} - * (\ell^{a} \wedge \ell^{b}) = -\frac{\omega}{1 - \omega} \bar{g}_{abc} \ell^{c} - \frac{1}{1 - \omega} \bar{g}_{abc} \ell^{c} = -\frac{1}{1 - \omega} \bar{g}_{abc} \ell^{c},
$$

and we obtain

$$
dP_{ab} = -\frac{d\omega}{(1 - \omega)^{2}} \bar{g}_{abc} \wedge \ell^{c} - \frac{d \ln |\bar{g}|}{1 - \omega} \bar{g}_{abc} \wedge \ell^{c} - \frac{1}{1 - \omega} \bar{g}_{abc} d\ell^{c}.
$$

(B.9)

Now, let us rewrite each term on the right-hand side. The first term is

$$
-\frac{d\omega}{(1 - \omega)^{2}} \bar{g}_{abc} \wedge \ell^{c} = -\frac{\bar{g}_{abc}}{3! (1 - \omega)^{2}} f_{e}^{d_{1}d_{2}d_{3}} \bar{g}_{d_{1}d_{2}d_{3}} \ell^{e} \wedge \ell^{c}
$$

$$
= -\frac{\bar{g}_{abc}}{3! (1 - \omega)^{2}} f_{e}^{d_{1}d_{2}d_{3}} \bar{g}_{d_{1}d_{2}d_{3}} \ell^{e} \wedge P_{ab}
$$

$$
= -\frac{1}{12} f_{[a}^{d_{1}} d_{d_{2}d_{3}} \bar{g}_{d_{1}d_{2}d_{3}} \ell^{e_{[e} e_{e_{e}^{2} e_{3}^{3}]} P_{e_{e_{2} e_{3}}} \wedge P_{a b]}
$$

$$
= \frac{1}{2} f_{[a}^{c_{1} c_{2} c_{3}]} P_{b] c_{1}} \wedge P_{c_{2} c_{3}},
$$

(B.10)

where we have used the Schouten identity, $f_{[e_{1}^{d_{1} d_{2} d_{3}}} P_{e_{2} c_{3} \wedge a b]} = 0$. The second term is

$$
-\frac{d \ln |\bar{g}|}{1 - \omega} \bar{g}_{abc} \wedge \ell^{c} = \frac{f_{d e}^{d}}{1 - \omega} \bar{g}_{abc} \ell^{e} \wedge \ell^{c},
$$

(B.11)

and the third term is

$$
-\frac{1}{1 - \omega} \bar{g}_{abc} d\ell^{c} = \frac{1}{2} \frac{1}{1 - \omega} \bar{g}_{abc} f_{d e}^{c} \ell^{d} \wedge \ell^{e} = \frac{3}{2} \frac{1}{1 - \omega} \bar{g}_{d [a b] e}^{c} \ell^{d} \wedge \ell^{e}
$$

$$
= \frac{3}{2} \frac{1}{1 - \omega} \bar{g}_{d [a b] e}^{c} \ell^{d} \wedge \ell^{e}
$$

$$
= -\frac{f_{d e}^{d}}{1 - \omega} \bar{g}_{abc} \ell^{e} \wedge \ell^{c} - \frac{1}{2} f_{a b}^{c} \ell^{d} \wedge P_{c d},
$$

(B.12)

Thus, we obtain

$$
dP_{ab} = \frac{1}{2} f_{[a}^{c_{1} c_{2} c_{3}]} P_{b] c_{1}} \wedge P_{c_{2} c_{3}} - \frac{1}{2} f_{a b}^{c} P_{d}^{a} \wedge P_{c d},
$$

(B.13)

which is an identity in $n = 3$. 

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Appendix C. Maurer–Cartan equation for the $E_n$ algebra

In this appendix we show that, for $n \leq 4$,

$$dP_{ab} = \frac{1}{2} f_{abc}^\ell d P_c \wedge P_{de} - \frac{1}{2} f_{abc}^d P^d \wedge P_{cd}$$

(C.1)

given in Eq. (3.71) is equivalent to the $U$-duality-manifest MC equation

$$dP^A - \frac{1}{2} F_{BC}^A P^B \wedge P^C = 0,$$

(C.2)

where

$$(P^A) \equiv \left( \frac{P^a}{\sqrt{2!}} \right) = \left( t^a + \frac{1}{3} \delta_{ab} b_1 b_2 P_{b_1 b_2} \right).$$

(C.3)

By using the explicit form of $F_{BC}^A$, The MC equation in Eq. (C.2) is equivalent to

$$dP^a = \frac{1}{2} f_{bc}^a P^b \wedge P^c + \frac{1}{2} f_{b c}^{(1,2)} a \ P^b \wedge P_{c_2},$$

(C.4)

$$dP_{a_1 a_2} = 2 f_{a_1 a_2} b \ P^b \wedge P_{a_2 a_3} + \frac{1}{2} f_{a_1 a_2}^b P^c \wedge P_{c b} + \frac{1}{2} f_{a_1 a_2}^{b_1 b_2 c} P_{a_2 a_3} \wedge P_{b_1 b_2}.$$  (C.5)

As we show later, the former follows from the latter. Thus, in the following, we show that the latter is equivalent to Eq. (C.1). To this end, we employ the following identities [41]:

$$d a_a^b = a_d^b f_{de}^b e^c,$$  (C.6)

$$d \Pi^{a_1 a_2 a_3} = f_{b}^{a_1 a_2 a_3} b^b + 3 f_{c a}^{a_1} \Pi^{a_2 a_3} c^d,$$  (C.7)

$$(a^{-1})^e_a \left( a^{-1} ight)_b^f a g^c f_{e f}^g = f_{a b}^c,$$  (C.8)

$$a_a^b (a^{-1})_b^f b_1 (a^{-1})_f^a b_2 (a^{-1})_b^c a e f d f_2 e = 3 f_a^{b_1 b_2 b_3} + 3 f_{a \ c} b_1 b_2 c,$$  (C.9)

$$3 \left( f_{e c}^{a_1} d_{a_2}^{b_2} \Pi^{b_1 b_2} + f_{e c}^{a_2} d_{a_1}^{b_2} \Pi^{b_1 b_2} \right) + f_{c d}^{a_1} a_2 b_1 b_2 = 0,$$  (C.10)

$$f_d^{b_1 b_2 c} \Pi^{a_1 a_2 d} - 3 f_d^{a_1 a_2 b_1} \Pi^{b_2 c} = 0,$$  (C.11)

$$f_{a b}^c \Pi^{a b d} = 0.$$  (C.12)

By using Eqs. (C.6), (C.8), and (C.9), Eq. (C.5) becomes

$$dP_{a_1 a_2} = -2 f_{a_1 a_2} b \ P_{a_2 a_3} + 2 f_{a_1 a_2} \ P_{a_2 a_3} + \frac{1}{2} f_{a_1 a_2} b \tilde{P}^c \wedge P_{c b}$$

$$+ \frac{1}{2} f_{a_1 a_2} c_2 d P_{a_2 a_3} + \frac{3}{2} f_{e c}^{a_1} \tilde{P}_{a_2 a_3} \wedge P_{c_1 c_2}$$

$$= \frac{1}{2} f_{a_1 a_2} c d e P_{a_2 a_3} \wedge P_{c d} - \frac{1}{2} f_{a_1 a_2} b \tilde{P}^c \wedge P_{b c}$$

$$+ \frac{1}{4} \left( 6 f_{e c}^{a_1} d_{a_2}^{b c} \tilde{P}_{a_2 a_3} + f_{a_1 a_2} d_{a_2}^{b c} \tilde{P}_{a_2 a_3} \right) P_{d b} \wedge P_{c_1 c_2},$$

(C.13)

where we have defined

$$\tilde{P}^a \equiv a_b^a P^b, \quad \tilde{P}^{a b c} \equiv a_d^a a_e^b a_f^c \Pi^{d e f}.$$  (C.14)
By further using Eq. (C.10), we can show that the last line of Eq. (C.13) vanishes. Then, we have shown that Eq. (C.5) is equivalent to Eq. (C.1).

In the remainder of this appendix we show that Eq. (C.4) is trivially satisfied under Eq. (C.5). By using the explicit form of $P^a$, the left-hand side of Eq. (C.4) is

$$dP^a = \frac{1}{2} f^a_{bc} r^b \wedge r^c + \frac{1}{2} f^a_{bc} b^a r^c \wedge P_{b_1 b_2} + \frac{3}{2} r^c f^a_{cd} [b_1 \Pi^{b_2 a l} P_{b_1 b_2} + \frac{1}{2} \Pi^{b_1 b_2 a} dP_{b_1 b_2}. \tag{C.15}$$

Then, by using Eqs. (C.5) and (C.12), Eq. (C.4) is equivalent to

$$\frac{1}{2} \Pi^{a_1 a_2 a} f_{b a_1} c \Pi^{b d_1 d_2} P_{d_1 d_2} \wedge P_{a_2 c} + \frac{1}{4} f^b_{c_1 c_2} a \Pi^{b e_1 e_2} P_{c_1 c_2} \wedge P_{e_1 e_2}$$

$$- \frac{1}{4} f^a_{b_1 b_2 c_1} a \Pi^{a_2 a_1} P_{c a_2} \wedge P_{b_1 b_2} - \frac{1}{8} f^a_{b c} a \Pi^{b e_1 e_2} P_{e_1 e_2} \wedge P_{f f_2} = 0. \tag{C.16}$$

By further using the identity

$$\frac{1}{4} f^b_{c_1 c_2} a \Pi^{b e_1 e_2} P_{c_1 c_2} \wedge P_{e_1 e_2} - \frac{1}{4} f^a_{b_1 b_2 c_1} a \Pi^{a_2 a_1} P_{c a_2} \wedge P_{b_1 b_2} - \frac{1}{8} f^a_{b c} a \Pi^{b e_1 e_2} P_{e_1 e_2} \wedge P_{f f_2}$$

$$+ \frac{1}{4} f^b_{c d} b \Pi^{b d_1 d_2} P_{b_1 b_2} \wedge P_{d_1 d_2} + \frac{1}{2} f^a_{d e} a \Pi^{a_2 d_1} P_{b_1 b_2} \wedge P_{a_2 d_1} + \frac{1}{2} f^a_{d e} a \Pi^{a_2 d_1} P_{b_1 b_2} \wedge P_{a_1 a_2} = 0. \tag{C.17}$$

which follows from Eq. (C.11), we can show that Eq. (C.16) is equivalent to

$$\frac{1}{4} f^b_{c b} b \Pi^{b d_1 c} a \Pi^{a_2 d_1} P_{b_1 b_2} \wedge P_{d_1 d_2} + \frac{1}{2} f^a_{d e} a \Pi^{a_2 d_1} P_{b_1 b_2} \wedge P_{a_1 a_2} = 0. \tag{C.18}$$

We can easily see that this equality follows from Eq. (C.10). Thus, Eq. (C.4) is always satisfied when Eq. (C.5) is satisfied.

**Appendix D. Rewriting the equations of motion**

Here, we rewrite the equations of motion in Eq. (3.68),

$$e^b \wedge dP_{ba} = -\frac{1}{2} f^b_{cd} b \varepsilon^c \wedge \varepsilon^d \wedge P_{ba}$$

$$- \frac{1}{3} f^b_{a b c d} [\varepsilon^a \wedge \varepsilon^c \wedge \varepsilon^d + (3 C_{b c e} H^e_d - C_{b c d} \varepsilon^e) \wedge C_{b c d} C^c] \tag{D.1}$$

into a more convenient form. Only when $n = 4$ do we assume that $f_{ab} a = 0$.

We begin by rewriting the first term on the right-hand side as

$$-\frac{1}{2} f^b_{c d} b \varepsilon^c \wedge \varepsilon^d \wedge P_{ba} = -\frac{1}{2} f^b_{c d} b \varepsilon^c \wedge \varepsilon^d \wedge [C_{b c a e} \varepsilon^e + \varepsilon^a \wedge \varepsilon^b]. \tag{D.2}$$

Here, by using Eq. (3.50), the second term vanishes:

$$f^b_{c d} b \varepsilon^c \wedge \varepsilon^d \wedge \varepsilon^a \wedge \varepsilon^b = 2 H^e_a f^e_{c d} b g_{e b} H^d \wedge \varepsilon^e. \tag{D.3}$$

Then, in $n = 3$, by using the Schouten identity $A_{[c d a e]} = 0$, we obtain

$$-\frac{1}{2} f^b_{c d} b \varepsilon^c \wedge \varepsilon^d \wedge P_{ba} = \frac{1}{2} f^b_{a b c d} \varepsilon^c \wedge \varepsilon^d \wedge \varepsilon^e. \tag{D.4}$$

In $n = 4$, by using the Schouten identity $A_{[c d b a e]} = 0$, we obtain the same relation,

$$-\frac{1}{2} f^b_{c d} b \varepsilon^c \wedge \varepsilon^d \wedge P_{ba} = \frac{1}{2} f^b_{a b c d} \varepsilon^c \wedge \varepsilon^d \wedge \varepsilon^e. \tag{D.5}$$
although the assumption $f_{ab}^a = 0$ has been used. Then, in both $n = 3$ and $n = 4$, we obtain

$$-\frac{1}{2} f_{cd}^b \epsilon^c \land \ell^d \land P_{ba} = \frac{1}{2} f_{ac}^b C_{bde} \epsilon^c \land \ell^d \land \ell^e = \frac{1}{2} f_{ac}^b \epsilon^c \land \ell^d \land P_{bd}, \quad \text{(D.6)}$$

where we have used $f_{ac}^b g_{bd} \epsilon^c \land \epsilon^d = 0$. The equations of motion then become

$$\ell^b \land dP_{ba} = \ell^c \land \left( -\frac{1}{2} f_{ca}^b \epsilon^d \land P_{bd} \right) - \frac{1}{3} f_{bcd}^a \left[ \ell_b \land \epsilon_c \land \ell_d + (3 C_{bce} H_d^e - C_{bcd}) * 1 - C_{bcd} C_3 \right]. \quad \text{(D.7)}$$

Now, we rewrite the second line of Eq. (D.7). In $n = 3$, we can easily rewrite it as

$$-\frac{1}{3} f_{abcd}^a \left[ \ell_b \land \epsilon_c \land \ell_d + 2 C_{bcd} * 1 - C_{bcd} C_3 \right] = \frac{1}{3} f_{abcd}^a (1 + 2 \omega + \omega^2) \epsilon_{bcd} * 1. \quad \text{(D.8)}$$

Using the identity $P_{ab} = -(1 + \omega) \epsilon_{abc} \ell^c$ in $n = 3$, we also find that

$$\ell^b \land \frac{1}{2} f_{bdef}^d \epsilon_d \land P_{ef} = (1 + \omega)^2 f_{bdef}^d \epsilon_{d} \land P_{ef} * 1 = \frac{1}{3} (1 + \omega)^2 f_{abcd}^a \epsilon_{bcd} * 1, \quad \text{(D.9)}$$

and these show the following relation:

$$-\frac{1}{3} f_{abcd}^a \left[ \ell_b \land \epsilon_c \land \ell_d + 2 C_{bcd} * 1 - C_{bcd} C_3 \right] = \ell^b \land \left( \frac{1}{2} f_{bdef}^d \epsilon_{def} \land P_{ef} \right). \quad \text{(D.10)}$$

Then, the equations of motion of Eq. (D.7) become

$$\ell^b \land \left( dP_{ba} - \frac{1}{2} f_{bdef}^d \epsilon_{def} \land P_{ef} + \frac{1}{2} f_{bdef}^d \epsilon_{def} \land P_{de} \right) = 0. \quad \text{(D.11)}$$

In $n = 4$, as we show below, we obtain the projected relation

$$H^c_a \left( -\frac{1}{3} f_{bcd}^b \left[ \ell_b \land \epsilon_c \land \ell_d + (3 C_{bce} H_d^e - C_{bcd}) * 1 - C_{bcd} C_3 \right] \right)$$

$$= H^c_a \ell^b \land \left( \frac{1}{2} f_{bdef}^d \epsilon_{def} \land P_{ef} \right). \quad \text{(D.12)}$$

Then, by combining Eqs. (D.7) and (D.12), we obtain

$$H^c_a \ell^b \land \left( dP_{bc} - \frac{1}{2} f_{bdef}^d \epsilon_{def} \land P_{ef} + \frac{1}{2} f_{bdef}^d \epsilon_{def} \land P_{de} \right) = 0. \quad \text{(D.13)}$$

In $n = 3$ this is equivalent to the equations of motion in Eq. (3.68), while in $n = 4$ it is equivalent to the projected components of the equations of motion.

**Appendix D.1. Derivation of Eq. (D.12)**

In the following we use several relations specific to $n = 4$ (see Appendix A) and also use the Schouten identity, $A_{[a_1 \cdots a_3]} = 0$.

Let us begin by considering the following expansion:

$$\frac{1}{2} f_{aefg}^b \ell^a \land P_{b|e} \land P_{fg}$$

$$= \frac{1}{2} f_{aefg}^b \ell^a \land *(\ell_b \land \ell_e) \land *(\ell_f \land \ell_g) + \frac{1}{2} f_{aefg}^b C_{b|e} P_{fg} \ell^a \land \ell^p \land \ell^q \quad \text{(A)}$$

$$= \frac{1}{2} f_{aefg}^b \ell^a \land \ell^p \land \ell^q \quad \text{(B)}$$

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Then, by acting the projection, which removes terms proportional to \( n_a \), we have defined \( N_b \) as a membrane. The second term is complicated and it may not be useful to show all of the computation. By using the Schouten identity, we find

\[
\frac{1}{2} f_{a[efg} \left[ C_{b]e} \epsilon^a \land \epsilon^p \land * \left( \ell_f \land \ell_g \right) + C_{fg} \epsilon^a \land * \left( \ell_{b]} \land \ell_e \right) \land \epsilon^q \right].
\]  

We rewrite each term as follows. The first term is

\[
(A) = \frac{1}{2} \left( f_{a[e} \epsilon^g H_b^{\gamma} \epsilon^\gamma_{\ell_f} \epsilon^\beta \epsilon^\gamma_{\ell_g} + f_{b} \epsilon_{\ell_f} \epsilon_{\ell_g} \epsilon^a \epsilon^\beta \epsilon^\gamma \right) \ast 1
\]

\[
= \left( \frac{1}{3!} f_{a[e} \epsilon^g H_b^{\gamma} \epsilon^\gamma_{\ell_f} \epsilon^\beta \epsilon^\gamma_{\ell_g} + \frac{1}{2} f_{b} \epsilon_{\ell_f} \epsilon_{\ell_g} \epsilon^a \epsilon^\beta \epsilon^\gamma \right) \ast 1
\]

\[
= \frac{1}{3} f_{b} \epsilon_{\ell_f} \epsilon_{\ell_g} + \frac{1}{3} f_{a[e} \epsilon_{\ell_f} \epsilon_{\ell_g} \epsilon^a \epsilon^\beta \epsilon^\gamma \ast 1,
\]

where we have defined \( N_b^a \equiv \delta^b_a - H_a \), which is a projector along the orthogonal directions of the membrane. The second term is

\[
(B) = \frac{1}{2} f_{a[e} \epsilon^g C_{b]e} \epsilon^p q r \epsilon^a \land \epsilon^p \land \epsilon^q = \frac{1}{3!} f_{a[e} \epsilon^g C_{b]e} \epsilon^a \land \epsilon^p \land \epsilon^q
\]

\[
= \frac{1}{2 \cdot 3!} f_{a[e} \epsilon_{bpq} \epsilon^a \land \epsilon^p \land \epsilon^q + \frac{1}{2 \cdot 3!} f_{b} \epsilon_{pq} \epsilon^a \land \epsilon^p \land \epsilon^q
\]

\[
= \frac{1}{3!} C^r f_{r[e} \epsilon^g C_{efg} \epsilon_{bpq} \epsilon^a \land \epsilon^p \land \epsilon^q + \frac{1}{3} f_{b} \epsilon_{pq} \epsilon^a \land \epsilon^p \land \epsilon^q
\]

\[
= \frac{1}{3} f_{a[e} \epsilon^g C_{efg} C_{3 \cdot} \frac{1}{3!} \epsilon_{b} C^e f_{r[e} \epsilon^g C_{efg},
\]

where we have defined \( n_a = \frac{1}{3!} \epsilon_{abcd} \epsilon^b \land \epsilon^c \land \epsilon^d \), which satisfies \( H_a \) \( n_b = 0 \). The third term is rather complicated and it may not be useful to show all of the computation. By using the Schouten identity, we find

\[
(C) = \left( \frac{1}{3} f_{b} \epsilon_{efg} C_{efg} - f_{b} \epsilon_{efg} C_{fg} H_e^p \right) \ast 1 + N_b^a \left( \frac{1}{2} f_{a[e} \epsilon^g C_{efg} - \frac{1}{3} f_{a[e} \epsilon^g C_{efg} \right) \ast 1.
\]

Then, by acting the projection, which removes terms proportional to \( n_a \) and \( N_b^a \), we obtain

\[
H_a \epsilon^b \land \left( \frac{1}{2} f_{b[def} P_{c]d} \land \epsilon_{ef} \right)
\]

\[
= -\frac{1}{3} H_a \epsilon_{efg} \left[ \epsilon^b \land \epsilon_{ef} + \left( 3 C_{fgp} H_e^p - C_{efg} \right) \ast 1 - C_{efg} C_3 \right].
\]

This is precisely the desired equation, Eq. (D.12).

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