Multiple $SU(3)$ algebras in shell model and interacting boson model

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Abstract

Rotational $SU(3)$ algebraic symmetry continues to generate new results in the shell model (SM). Interestingly, it is possible to have multiple $SU(3)$ algebras for nucleons occupying an oscillator shell $\eta$. Several different aspects of the multiple $SU(3)$ algebras are investigated using shell model and also deformed shell model based on Hartree-Fock single particle states with nucleons in $sdg$ orbits giving four $SU(3)$ algebras. Results show that one of the $SU(3)$ algebra generates prolate shapes, one oblate shape and the other two also generate prolate shape but one of them gives quiet small quadrupole moments for low-lying levels. These are inferred by using the standard form for the electric quadrupole transition operator and using quadrupole moments and $B(E2)$ values in the ground $K = 0^+$ band in three different examples. Multiple $SU(3)$ algebras extend to interacting boson model and using $sdg$IBM, the structure of the four $SU(3)$ algebras in this model are studied by coherent state analysis and asymptotic formulas for $E2$ matrix elements. The results from $sdg$IBM further support the conclusions from the $sdg$ shell model examples.

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Elliott has recognized way back in 1958 that shell model (SM) admits $SU(3) \supset SO(3)$ algebra and this will generate rotational spectra in nuclei starting with the interacting particle picture [1, 2]. Following this, $SU(3)$ algebra was developed in considerable detail by various groups and this includes methods to obtain $SU(3)$ irreducible representations (irreps) and $SU(3)$ Wigner-Racah algebra with codes for calculating $SU(3) \supset SO(3)$ and $SU(3) \supset SU(2) \times U(1)$ reduced Wigner coefficients, $SU(3)$ Racah coefficients, $SU(3)$ coefficients of fractional parentage and so on [3–10]. By mid 60’s it was recognized that the $SU(3)$ symmetry is good for $1p$ and $2s1d$ shell nuclei but due to the strong spin-orbit force it will be a badly broken symmetry for $1p2f$ shell nuclei and beyond. Hecht, Draayer and others later recognized [11–15] that for heavy deformed nuclei, pseudo-$SU(3)$ based on pseudo spin and pseudo Nilsson orbits will be a useful symmetry and it gave rise to many new results. Very recently, a proxy-$SU(3)$ scheme by Bonatsos, Casten and others [16–18] has appeared within SM with definite prediction for prolate dominance over oblate shape in heavy deformed nuclei. This $SU(3)$ model is currently being investigated in more detail. In addition, in the multishell situation again $SU(3)$ appears within the $Sp(6, R)$ model of Rowe and Rosensteel [19–21] and this has given rise to the no-core-sympletic shell model [22, 23]. Going beyond SM, a major basis for the interacting boson model (IBM) of atomic nuclei is that with $s$ and $d$ bosons the spectrum generating algebra (SGA) is $U(6)$ and it has $SU(3)$ as a subalgebra generating rotational spectrum [24, 25]. Similarly, $sdg$IBM [26, 27], $sdpf$IBM [28, 29] and also IBM-3 with isospin ($T$) and IBM-4 with spin-isospin ($ST$) degrees of freedom [25, 30] all contain $SU(3)$ symmetry generating rotational spectra. In addition, in IBM-3 and IBM-4 models, $SU(3)$ also appears for isospin ($T$) and spin-isospin ($ST$) degrees of freedom respectively. Similarly, for odd-A nuclei we have $SU^{BF}(3) \times SU^{F}(2)$ symmetry in IBFM model with Nilsson correspondence [31]. This extends to $SU(3)$ in IBFFM for odd-odd nuclei [32, 33] and $SU(3)$ in IBF$^2$M for two quasi-particle excitations [34]. With $SU(3)$ generating rotational spectra within both SM and IBM, it is natural to look for new perspectives for $SU(3)$ symmetry in nuclei.

One curious aspect of $SU(3)$ in nuclei is that in a given oscillator shell $\eta$, there will be multiple $SU(3)$ algebras. Very early it is recognized that in SM with $s$ and $d$ orbits there will be two $SU(3)$ algebras [35] but its consequences are not explored in any detail. Similarly,
in sdIBM there are two $SU(3)$ algebras [25] and they are applied in phase transition studies [36]. Finally, it was also recognized that there will be four $SU(3)$ algebras in $sdgIBM$ [27]. Except for the $sdIBM$, properties of multiple $SU(3)$ algebras are not investigated in any detail in the past. As we will show, for a given oscillator shell with major shell number $\eta$, there will be $2^{[\frac{\eta}{2}]}$ number of $SU(3)$ algebras where $[\frac{\eta}{2}]$ is the integer part of $\eta/2$. In the present paper, following the recent investigation of multiple pairing algebras in SM and IBM [37], several different aspects of multiple $SU(3)$’s in SM and IBM are investigated. Now, we will give a preview.

In Section 2, multiple $SU(3)$ algebras in SM generated by angular momentum operator $L^1_q$ and quadrupole moment operator $Q^2_q$ with different signs for the $\ell \to \ell \pm 2$ matrix elements are identified and the matrix elements for the corresponding $Q \cdot Q$ operators are given. Using these, correlations between different $Q \cdot Q$ operators are studied. In Section 3, Spectra and electric quadrupole ($E2$) properties of these algebras are studied using shell model codes and also deformed shell model based on Hartree-Fock single particle states (called DSM [30]). Used here are examples with 6 protons, 6 protons plus 2 neutrons and 6 protons plus 6 neutrons systems. In Section 4, results for multiple $SU(3)$ algebras in IBM’s (with no internal degrees of freedom for the the bosons) are presented. Finally, Section 5 gives conclusions.

II. PHASE CHOICE AND MULTIPLE $SU(3)$ ALGEBRAS IN SHELL MODEL

Let us consider the situation where valence nucleons in a nucleus occupying an oscillator shell with major shell number $\eta$. With the spin-isospin degrees of freedom for the nucleons, the spectrum generating algebra (SGA) is $U(4N)$ and decomposing the space into orbital and spin-isospin ($ST$) parts, we have $U(4N) \supset U(N) \times SU(4)$. Here, $N = (\eta + 1)(\eta + 2)/2$ and $SU(4)$ is the Wigner’s spin-isospin $SU(4)$ algebra; see for example [30, 38–41]. Also, for a given $\eta$, the the single particle (sp) orbital angular momentum $\ell$ takes values $\ell = \eta$, $\eta - 2$, . . . , 0 or 1. Note that, for nuclei with only valence protons or neutrons $SU(4)$ changes to $SU(2)$ generating spin $S$. As Elliott has established, the orbital $U(N)$ algebra admits $SU(3)$ subalgebra with $U(N) \supset SU(3) \supset SO(3)$ where $SO(3)$ generates orbital angular momentum. The eight generators of $SU(3)$ are the orbital angular momentum operators $L^1_q$ and quadrupole moment operators $Q^2_q$. In $LST$ coupling and using fermion creation ($a^\dagger$)
and annihilation (\(a\)) operators,

\[
L^1_q = 2 \sum_{\ell} \sqrt{\ell(\ell+1)(2\ell+1)/3} \left( \alpha_{\ell\frac{1}{2}} \tilde{a}_{\ell\frac{1}{2}} \right)^{1,0,0}.
\]

(1)

Note that \(\tilde{a}_{\ell-m,\frac{1}{2}-m_s,\frac{1}{2}-m_t} = (-1)^{\ell-m+\frac{1}{2}-m_s+\frac{1}{2}-m_t} a_{\ell m,\frac{1}{2} m_s,\frac{1}{2} m_t}\) where \(m_s\) and \(m_t\) are the \(S_z\) and \(T_z\) quantum numbers for a single nucleon. Similarly, the quadrupole operator is

\[
Q^2_q = 2 \sum_{\ell_f,\ell_i} \langle \eta, \ell_f || Q^2 || \eta, \ell_i \rangle \left( \alpha_{\ell\frac{1}{2}} \tilde{a}_{\ell\frac{1}{2}} \right)^{2,0,0}.
\]

(2)

Closure examination of the reduced matrix element \(\langle \eta, \ell_f || Q^2 || \eta, \ell_i \rangle\) of the quadrupole operator in the orbital space allows us to recognize that there will be multiple \(SU(3)\) subalgebras in \(U(N)\). We will turn to this now.

As Elliott considered [1], the quadrupole operator is

\[
Q^2_q = \sqrt{\frac{4\pi}{5}} \left[ r^2 Y^2_2(\theta, \phi) + p^2 Y^2_2(\theta_p, \phi_p) \right] \quad \text{with oscillator length parameter } b = 1.
\]

For a single shell, this is equivalent to using

\[
Q^2_q = \sqrt{\frac{16\pi}{5}} r^2 Y^2_2(\theta, \phi).
\]

Therefore, the reduced matrix elements of \(Q^2\) decompose into the radial part and angular part,

\[
\langle \eta, \ell_f || Q^2 || \eta, \ell_i \rangle = \left( \alpha_{\ell,\ell+2} \right) \sqrt{\frac{6(\ell+1)(\ell+2)}{(2\ell+3)(2\ell-1)}}.
\]

(3)

with the angular part given by [42],

\[
\langle \eta, \ell || \sqrt{\frac{16\pi}{5}} Y^2_2(\theta, \phi) || \eta, \ell \rangle = -2 \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{(2\ell+3)(2\ell-1)}},
\]

\[
\langle \eta, \ell || \sqrt{\frac{16\pi}{5}} Y^2_2(\theta, \phi) || \eta, \ell + 2 \rangle = \left( \alpha_{\ell,\ell+2} \right) \sqrt{\frac{6(\ell+1)(\ell+2)}{(2\ell+3)}}.
\]

(4)

Similarly, the radial matrix elements are

\[
\langle \eta, \ell || r^2 || \eta, \ell \rangle = \frac{2\eta+3}{2},
\]

\[
\langle \eta, \ell || r^2 || \eta, \ell + 2 \rangle = \left( \alpha_{\ell,\ell+2} \right) \sqrt{(\eta-\ell)(\eta+\ell+3)};
\]

\[
\alpha_{\ell,\ell+2} = \alpha_{\ell+2,\ell} = \pm 1.
\]

(5)

The phase factor \(\alpha_{\ell,\ell+2}\) arises as there is freedom in choosing the phases of the radial wavefunctions of a 3D oscillator. In SM studies, the standard convention is to use \(\alpha_{\ell,\ell+2} = -1\) for all \(\ell\) [41–43]. However, Elliott in his \(SU(3)\) introductory paper [1] and in \(sd\) as well as
we will first consider the quadrupole-quadrupole interaction generated by \((pfhj)\) in literature before is \((sdg)\).

Angular momentum algebra is that the eight operators \((L)\) now, the most important result that can be proved by using the tedious but straightforward example of \(\eta\).

\[\alpha = (\alpha_{0,2}, \alpha_{2,4}, \ldots, \alpha_{\eta-2,\eta})\] for \(\eta\) even,

\[\alpha = (\alpha_{1,3}, \alpha_{3,5}, \ldots, \alpha_{\eta-2,\eta})\] for \(\eta\) odd,

\[\alpha = (\pm 1, \pm 1, \ldots)\].

Now, the most important result that can be proved by using the tedious but straightforward angular momentum algebra is that the eight operators \((L^1_q, Q^2_q(\alpha))\) generate \(SU(3)\) algebra independent of the choice of the \(\alpha\)'s and they satisfy the commutation relations [1, 41],

\[\begin{align*}
[L^1_q, L^1_{q'}] &= -\sqrt{2} \langle 1q 1q' | 1q + q' \rangle L^1_{q+q'}, \\
[L^1_q, Q^2_q(\alpha)] &= -\sqrt{6} \langle 1q 2q' | 2q + q' \rangle Q^2_{q+q'}(\alpha), \\
[Q^2_q(\alpha), Q^2_{q'}(\alpha)] &= 3\sqrt{10} \langle 2q 2q' | 1q + q' \rangle L^1_{q+q'}.
\end{align*}\]

Thus, we have multiple \(SU(3)\) algebras \(SU^\alpha(3)\) in SM spaces generated by the operators in Eq. (6). Clearly for a given \(\eta\), there will be \(2^{[\eta/2]}\) number of \(SU(3)\) algebras; \(\frac{\eta}{2}\) is the integer part of \(\eta/2\). Then, we have two \(SU(3)\) algebras in \(sd\) (\(\eta = 2\)) and \(pf\) (\(\eta = 3\)) shells, four \(SU(3)\) algebras in \(sdg\) (\(\eta = 4\)) and \(pfh\) (\(\eta = 5\)) shells, eight \(SU(3)\) algebras in \((sdgi)\) (\(\eta = 6\)) and \((pfhj)\) (\(\eta = 7\)) shells and so on. Thus, the first non-trivial situation that is not discussed in literature before is \(sdg\) or \(\eta = 4\) shell with four \(SU(3)\) algebras \(SU^{(-,-)}(3), SU^{(+,-)}(3), SU^{(-,+)}(3)\) and \(SU^{(+,+)}(3)\). Here, \(\alpha = (\alpha_{sd}, \alpha_{dg})\) and \((-,-)\) means \((\alpha_{sd}, \alpha_{dg}) = (-1, -1)\) and similarly for other choices of \((\alpha_{sd}, \alpha_{dg})\). In the reminder of this paper, we will use the example of \(\eta = 4\) shell to present some results from multiple \(SU(3)\) algebras. Before this, we will first consider the quadrupole-quadrupole interaction generated by \(Q^2_q(\alpha)\).
A. Matrix elements of Quadrupole-quadrupole interaction from multiple $SU(3)$ algebras

Investigation of multiple $SU(3)$ algebras in shell model spaces needs firstly the single particle energies (spe) and two-body matrix elements (TBME) of the quadrupole-quadrupole interaction operator $Q^2(\alpha) \cdot Q^2(\alpha)$ for all phase choices $\alpha$ (also the spe and TBME for the simpler $L \cdot L$ operator). The methods for obtaining these are well known [42] and we will give only the final formulas. In order to derive formulas for the spe and TBME generated by $Q^2(\alpha) \cdot Q^2(\alpha)$ operators, firstly notice that the $Q^2_q$ operator can be written as,

$$Q^2_q(\alpha) = 2 \sum_{\ell_j, \ell_i} C_{\ell_j, \ell_i}^{\alpha} \left(a_{\ell_j}^{\dagger} a_{\ell_i}^{\dagger} a_{\ell_j} a_{\ell_i}\right)^{2,0,0}_{q}.$$  \hspace{1cm} (8)

The $C_{\ell_j, \ell_i}^{\alpha}$ follow easily from Eq. (6). From now on we will drop '2' and $\alpha$ in $Q^2_q(\alpha)$ when there is no confusion. For a many particle system,

$$Q \cdot Q = \sum_{i=1}^{m} Q(i) \cdot Q(i) + 2 \sum_{i<k=1}^{m} Q(i) \cdot Q(k)$$  \hspace{1cm} (9)

where $i$ and $k$ are particle indices and $m$ is number of particles. The first sum generates spe and the second term TBME. Given the shell model single particle ($n\ell j$)-orbits (note that the oscillator shell number $\eta = 2n + \ell$), matrix elements of $Q(1) \cdot Q(2)$ in the two-particle antisymmetric states (called a.s.m.) can be written in terms of the matrix elements in the two-particle non-antisymmetric states (called n.a.s.m.) as,

$$\langle j_a j_b JT | Q(1) \cdot Q(2) | j_c j_d JT \rangle_{\text{a.s.m.}} = \langle j_a j_b JT | Q(1) \cdot Q(2) | j_c j_d JT \rangle_{\text{n.a.s.m.}} + (-1)^{J+T-j_c-j_d} \langle j_a j_b JT | Q(1) \cdot Q(2) | j_d j_c JT \rangle_{\text{n.a.s.m.}}.$$  \hspace{1.5cm} (10)

Using angular momentum algebra it is easy to recognize that,

$$\langle j_a j_b JT | Q(1) \cdot Q(2) | j_c j_d JT \rangle_{\text{n.a.s.m.}} = (-1)^{j_a+j_b+j_c+J} \begin{pmatrix} \ell_a & j_b & J \\ j_d & j_c & 2 \end{pmatrix} \times \langle j_a || Q || j_c \rangle \langle j_b || Q || j_d \rangle.$$  \hspace{1cm} (11)

The reduced matrix elements $\langle || Q || \rangle$ are given by,

$$\langle \eta, \ell_j, j_f || Q^2(\alpha) || \eta, \ell_i, j_i \rangle = (-1)^{\ell_f + \frac{1}{2} + j_i + 2} \times \sqrt{5(2j_i + 1)(2j_f + 1)} \begin{pmatrix} \ell_f & j_f & \frac{1}{2} \\ j_i & \ell_i & 2 \end{pmatrix} C_{\ell_f, \ell_i}^{\alpha}.$$  \hspace{1cm} (12)
Combining Eqs. (11) and (12) with Eq. (10) and Eq. (9) will give the TBME of the $Q^2(\alpha) \cdot Q^2(\alpha)$ operator. The spe $\epsilon_{ij}^\alpha$ of the $Q^2(\alpha) \cdot Q^2(\alpha)$ are simply given by

$$\epsilon_{ij}^\alpha = \frac{5}{2\ell + 1} \sum_{\ell^\prime} |C_{\ell\ell'}^\alpha|^2. \quad (13)$$

An important property of the $Q^2(\alpha) \cdot Q^2(\alpha)$ operator is that it is related to the quadratic Casimir invariant ($C_2$) of $SU^{\alpha}(3)$ in a simple manner,

$$-Q^2(\alpha) \cdot Q^2(\alpha) = -C_2(SU^{\alpha}(3)) + \frac{3}{4} L \cdot L. \quad (14)$$

The procedure described above will also give the spe and TBME of $L \cdot L$ operator. Let us mention that the eigenvalue of $C_2(SU^{\alpha}(3))$ over a $SU^{\alpha}(3)$ irrep $(\lambda\mu)$ is $\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)$. Also, note that the dot product in Eqs. (14) and (9) is with respect to the orbital space.

**B. Correlation between different $Q \cdot Q$ operators**

In order to gain some insight into the differences between different $SU^{\alpha}(3)$ algebras, we will consider the correlation in $m$ nucleon spaces between different $Q(\alpha) \cdot Q(\alpha)$ operators. For this, we will use the example of $\eta = 4$ shell giving $(n\ell j)$ to be $(2,0,1/2), (1,2,3/2), (1,2,5/2), (0,4,7/2)$ and $(0,4,9/2)$. In this space, spe and TBME are obtained for $Q^2(\alpha) \cdot Q^2(\alpha)$ operators with $\alpha = (\alpha_{sd}, \alpha_{dg}) = (+, +), (+, -), (-, +)$ and $(-, -)$ using the results in Section IIA.

Given an operator $\mathcal{O}$ acting in $m$ particle spaces ($\mathcal{O}$ is assumed to be real), its trace over the $m$ particle space is $\langle\langle \mathcal{O} \rangle\rangle^m = \sum_{\gamma} \langle m, \gamma | \mathcal{O} | m, \gamma \rangle$. Note that $|m, \gamma\rangle$ are $m$-particle states. Similarly, the $m$-particle average is $\langle \mathcal{O} \rangle^m = [d(m)]^{-1} \langle\langle \mathcal{O} \rangle\rangle^m$ where $d(m)$ is $m$-particle space dimension. Using the spectral distribution method of French [45, 46], a geometry can be defined [46] with norm (or size or length) of an operator $\mathcal{O}$ given by $|| \mathcal{O} ||_m = \sqrt{\langle\langle \tilde{\mathcal{O}} \tilde{\mathcal{O}} \rangle\rangle^m}$; $\tilde{\mathcal{O}}$ is the traceless part of $\mathcal{O}$. Following this, given any two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, the correlation coefficient

$$\zeta(\mathcal{O}_1, \mathcal{O}_2) = \frac{\langle\langle \tilde{\mathcal{O}}_1 \tilde{\mathcal{O}}_2 \rangle\rangle^m}{|| \mathcal{O}_1 ||_m || \mathcal{O}_2 ||_m}, \quad (15)$$

gives the cosine of the angle between the two operators. Thus, $\mathcal{O}_1$ and $\mathcal{O}_2$ are same within a normalization constant if $\zeta = 1$ and they are orthogonal to each other if $\zeta = 0$ [45]. Most
TABLE I. Correlation coefficient $\zeta$ between $Q \cdot Q$ operators with different values for the phases $(\alpha_{sd}, \alpha_{dg})$ in sdg shell model $m$-particle spaces ($m$ is number of nucleons). Note that the total number of single particle states (with spin and isospin) is 60. The $\zeta$ values in column 3 are for $m = 4, 8, 12, 20, 30, 40, 50$ and 56. See text for other details.

| $(\alpha_{sd}, \alpha_{dg})$ | $(\alpha'_{sd}, \alpha'_{dg})$ | $\zeta$ |
|-----------------------------|-------------------------------|--------|
| $(-, -)$                    | $(+, -)$                      | 0.39, 0.36, 0.35, 0.35, 0.35, 0.36, 0.39 |
| $(-, +)$                    | $(+, +)$                      | 0.14, 0.1, 0.08, 0.08, 0.08, 0.09, 0.14 |
| $(+, +)$                    | $(+, +)$                      | 0.07, 0.02, 0.01, 0.0, 0.0, 0.0, 0.01, 0.07 |
| $(+, -)$                    | $(-, +)$                      | 0.07, 0.02, 0.01, 0.0, 0.0, 0.0, 0.01, 0.07 |
| $(-, +)$                    | $(+, +)$                      | 0.14, 0.1, 0.08, 0.08, 0.08, 0.08, 0.09, 0.14 |
| $(-, +)$                    | $(+, +)$                      | 0.39, 0.36, 0.35, 0.35, 0.35, 0.36, 0.39 |

recent application of norms and correlation coefficients is in understanding the structure of multiple pairing algebras in shell model [37].

Applying Eq. (15), we have calculated $\zeta$ between the operators $Q^2(\alpha_{sd}, \alpha_{dg}) \cdot Q^2(\alpha_{sd}, \alpha_{dg})$ and $Q^2(\alpha'_{sd}, \alpha'_{dg}) \cdot Q^2(\alpha'_{sd}, \alpha'_{dg})$ for all possible combinations of $\alpha$’s and $(\alpha')$’s. Some results for $\zeta$ are given in Table I. It is seen from the table that $Q^2(-, -) \cdot Q^2(-, -)$ is strongly correlated with $Q^2(+, -) \cdot Q^2(+, -)$. Similarly, the $Q \cdot Q$’s with $(\alpha_{sd}, \alpha_{dg}) = (+, +)$ and $(-, +)$ are strongly correlated. However, the correlations between other pairs of $Q^2 \cdot Q^2$ are quite small. Thus, $SU^{(-,-)}(3)$ and $SU^{(+,-)}(3)$ are expected to give similar results but quite different from $SU^{(+,+)}(3)$ and $SU^{(-,+)}(3)$. This is seen in the results of detailed calculations presented in the next section. It is important to stress that all the four $SU^\alpha(3)$ algebras generate the same spectrum for $H(\alpha) = Q^2(\alpha_{sd}, \alpha_{dg}) \cdot Q^2(\alpha_{sd}, \alpha_{dg})$ independent of $(\alpha_{sd}, \alpha_{dg})$. We will consider these in more detail in the following.
III. RESULTS FOR SPECTRA, QUADRUPOLE MOMENTS AND $E2$ TRANSITION STRENGTHS FROM SM AND DSM

With the $sdg$ example, we have four $Q \cdot Q$ Hamiltonians,

$$
H_Q^{(-,-)} = -Q^2(-,-) \cdot Q^2(-,-), \\
H_Q^{(+,-)} = -Q^2(+,-) \cdot Q^2(+,-), \\
H_Q^{(-,+)} = -Q^2(-,+) \cdot Q^2(-,+), \\
H_Q^{(+,+)} = -Q^2(+,+) \cdot Q^2(+,+).
$$

(16)

In this section we will present the results generated by these four $H$’s for the yrast levels, quadrupole moments $Q_2(J)$ of these levels and the $B(E2)$’s along the yrast line for $J$ up to 10. Used for this purpose are the Antoine shell model code [47] and also the deformed shell model (DSM) based on Hartree-Fock states [30]. DSM is particularly important for bringing out shape information in a transparent manner and also it is useful for larger particle numbers where SM calculations are impractical. We will test the SM results with analytical results derived using $SU(3)$ algebra and also test DSM using SM results. We will first present some analytical results from $SU(3)$ algebra.

A. Analytical results from $SU(3)$ algebra

With $SU(3)$ symmetry of the $H_Q$ Hamiltonians, the shell model space for a $m$ nucleon system decomposes into $SU(3)$ irreducible representations (irreps) due to the equivalence between $H_Q$ and $C_2(SU(3))$ as given by Eq. (14). If we have identical nucleons (protons or neutrons), the ground band belongs to the leading $SU(3)$ irrep $(\lambda_H, \mu_H)$ with spin $S = 0$ and $J = L$ for even $m$ (similarly with $S = 1/2$ for odd $m$). It is easy to write a formula for obtaining $(\lambda_H, \mu_H)$ as given in [48]. The irreps for $m$ identical nucleons in $\eta = 4$ shell are given in Table II. Similarly, for $m$ nucleons with isospin $T$, we need to consider the lowest spin-isospin $SU(4)$ irrep allowed for this system [38, 40] and this will then give $(\lambda_H, \mu_H)$ [48]. The irreps $(\lambda_H, \mu_H)$ for $m$ nucleons with $T = |T_z|$ are given in Table II. The eigenstates of
TABLE II. Ground state or leading $SU(3)$ irrep $(\lambda_H, \mu_H)$ for a given number $m$ of identical nucleons and also for a given number $m$ of nucleons with isospin $T = |T_Z|$. Results are given for the oscillator shell $\eta = 4$. The $(\lambda_H, \mu_H)$ are given in the table as $(\lambda_H, \mu_H)^m$ for identical nucleons with $m \geq 2$ and $(\lambda_H, \mu_H)^{m,T}$ for nucleons with $T = |T_z|$ and $3 \leq m \leq 15$; for odd $m$ values, $2T$ value given instead of $T$ value. More complete results are available in [48].

$\eta = 4$: identical nucleons

$(8, 0)^2, (10, 1)^3, (12, 2)^4, (15, 1)^5, (18, 0)^6, (18, 2)^7, (18, 4)^8, (19, 4)^9, (20, 4)^{10}, (22, 2)^{11}, (24, 0)^{12}$,
$(22, 3)^{13}, (20, 6)^{14}, (19, 7)^{15}, (18, 8)^{16}, (18, 7)^{17}, (18, 6)^{18}, (19, 3)^{19}, (20, 0)^{20}, (16, 4)^{21}, (12, 8)^{22},$
$(9, 10)^{23}, (6, 12)^{24}, (4, 12)^{25}, (2, 12)^{26}, (1, 10)^{27}, (0, 8)^{28}, (0, 4)^{29}, (0, 0)^{30}$

$\eta = 4$: even number of nucleons

$(16, 0)^{4,0}, (14, 1)^{4,1}, (12, 2)^{4,2}, (20, 2)^{6,0}, (20, 2)^{6,1}, (19, 1)^{6,2}, (18, 0)^{6,3}, (24, 4)^{8,0}, (25, 2)^{8,1}$,
$(26, 0)^{8,2}, (22, 2)^{8,3}, (18, 4)^{8,4}, (30, 2)^{10,0}, (30, 0)^{10,1}, (28, 3)^{10,2}, (26, 4)^{10,3}, (23, 4)^{10,4}, (20, 4)^{10,5},$
$(36, 0)^{12,0}, (33, 3)^{12,1}, (30, 6)^{12,2}, (29, 5)^{12,3}, (28, 4)^{12,4}, (26, 2)^{12,5}, (24, 0)^{12,6}, (36, 4)^{14,0}, (36, 4)^{14,1},$
$(34, 5)^{14,2}, (32, 6)^{14,3}, (32, 3)^{14,4}, (32, 0)^{14,5}, (26, 3)^{14,6}, (20, 6)^{14,7}$

$\eta = 4$: odd number of nucleons

$(12, 0)^{3,1}, (10, 1)^{3,3}, (18, 1)^{5,1}, (16, 2)^{5,3}, (15, 1)^{5,5}, (22, 3)^{7,1}, (23, 1)^{7,3}, (22, 0)^{7,5}$,
$(18, 2)^{7,7}, (27, 3)^{9,1}, (28, 1)^{9,3}, (26, 2)^{9,5}, (22, 4)^{9,7}, (19, 4)^{9,9}, (33, 1)^{11,1}, (30, 4)^{11,3},$
$(28, 5)^{11,5}, (27, 4)^{11,7}, (24, 4)^{11,9}, (22, 2)^{11,11}, (36, 2)^{13,1}, (33, 5)^{13,3}, (31, 6)^{13,5}, (30, 5)^{13,7},$
$(30, 2)^{13,9}, (28, 0)^{13,11}, (22, 3)^{13,13}, (36, 6)^{15,1}, (37, 4)^{15,3}, (35, 5)^{15,5}, (34, 4)^{15,7}, (34, 1)^{15,9},$
$(30, 3)^{15,11}, (24, 6)^{15,13}, (19, 7)^{15,15}$

$H_Q$ are $|m; (\lambda_H \mu_H)KL; S : JT\rangle$ and the $(\lambda_H \mu_H) \rightarrow L$ reduction is well known giving,

$$(\lambda \mu) \rightarrow L : K = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \cdots, 0 \text{ or } 1,$$
$L = K, K + 1, K + 2, \cdots, K + \max(\lambda, \mu) \text{ for } K \neq 0$,
$L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \cdots, 0 \text{ or } 1 \text{ for } K = 0$,

$$(\lambda, \mu) \rightarrow L \iff (\mu, \lambda) \rightarrow L.$$  \hspace{1cm} (17)

It is easy to see that the energies of the yrast levels in a even $m$ system (assuming spin $S = 0$) are given by,

$$E(J = L) = -(\lambda_H^2 + \mu_H^2 + \lambda_H \mu_H + 3(\lambda_H + \mu_H)) + \frac{3}{4}L(L + 1).$$  \hspace{1cm} (18)
In the examples presented ahead in the present paper we will only consider even \( m \) systems with \((\lambda_H \mu_H) = (\lambda 0)\) and then \( \lambda \) is even. A \((\lambda, 0)\) irrep with \( \lambda \) even, as seen from Eq. (17), generates the ground band with \( J = 0, 2, 4, \ldots, \lambda \). The ground state energy \( E_{gs} = (\lambda^2 + 3\lambda) \) and the energies of the \( J \) levels with respect to \( E_{gs} \) are just \( 3J(J + 1)/4 \). In addition, if we choose the \( E2 \) transition operator to be the \( Q \) of one of the \( H_Q \), then formulas for \( Q_2(J) \) and \( B(E2) \) will be simple for the \((\lambda, 0)\) irrep of the corresponding \( SU(3) \) algebra. Just as it is considered in SM and DSM codes, we will take the \( E2 \) operator \( T^{E2} \) for identical nucleon systems to be

\[
T^{E2} = Q^2(-, -) \, e_{eff} b^2 \tag{19}
\]

where \( b \) is the oscillator length parameter and \( e_{eff} \) is effective charge. Then, analytical formulas for the quadrupole moments \( (Q(J)) \) of the yrast levels and \( B(E2) \)'s among them follow from the simple \( SU(3) \) algebra for the eigenstates obtained for \( H_Q^{(-,-)} \) as they belong to \( SU^{(-,-)}(3) \). Using the results in [6, 14, 49, 50], we have, \( H_Q^{(-,-)} \) in Eq. (16) with \( T^{E2} \) in Eq. (19),

\[
Q((\lambda, 0) : J = L) = -\frac{L}{2L + 3} (2\lambda + 3) \, e_{eff} b^2 ,
\]

\[
B(E2; (\lambda, 0) \rightarrow J-2 = L-2) = \frac{5}{16\pi} \left\{ \frac{6J(J-1)(\lambda - J + 2)(\lambda + J + 1)}{(2J-1)(2J+1)} \right\} (e_{eff})^2 b^4 . \tag{20}
\]

However, for systems with valence protons and neutrons, the \( E2 \) transition operator is taken to be

\[
T^{E2} = [e^p_{eff} \, Q^2_{q}(-, -; p) + e^n_{eff} \, Q^2_{q}(-, -; n)] \, b^2 \tag{21}
\]

where \( e^p_{eff} \) and \( e^n_{eff} \) are proton and neutron effective charges. Again, using eigenstates obtained for \( H_Q^{(-,-)} \) as they belong to \( SU^{(-,-)}(3) \) and the \( T^{E2} \) in Eq. (21), a simple formula is obtained for \( Q(J) \) and \( B(E2) \)'s in the situation where the ground band is given by \( |(\lambda_\pi, 0)(\lambda_\nu, 0)(\lambda_\pi + \lambda_\nu, 0)K = 0, L, S = 0, J = L) \) for a system with protons \( (\pi) \) and neutrons \( (\nu) \). Now, carrying out the \( SU(3) \) algebra using the mathematical formulation and analytical results given in [6, 14, 49, 50] we have,

\[
Q((\lambda, 0) : J = L) = -\frac{L}{2L + 3} (2\lambda + 3) \, X_{eff} b^2 ,
\]

\[
B(E2; (\lambda, 0) \rightarrow J-2 = L-2) = \frac{5}{16\pi} \left\{ \frac{6J(J-1)(\lambda - J + 2)(\lambda + J + 1)}{(2J-1)(2J+1)} \right\} (X_{eff})^2 b^4 ;
\]

\[
X_{eff} = \frac{e^p_{eff} \, (\lambda^2_\pi + 3\lambda_\pi + \lambda_\pi \lambda_\nu) + e^n_{eff} \, (\lambda^2_\nu + 3\lambda_\nu + \lambda_\nu \lambda_\pi)}{(\lambda^2 + 3\lambda)} , \quad \lambda = \lambda_\pi + \lambda_\nu . \tag{22}
\]
Tests of Eqs. (18), (20) and (22) are carried out using SM and DSM in the next three subsections.

It is important to stress that in the event we use the eigenstates of other $H_Q^{\alpha}$, the ground band generated by them will belong to the $(\lambda 0)$ irrep of the corresponding $SU^{\alpha}(3)$. However, then the $Q$’s in $T_{E2}^{\alpha}$ in Eqs. (19) and (21) are no longer generators of these $SU^{\alpha}(3)$’s and hence the formulas in Eqs. (20) and (22) will not apply. In this situation, we have to use $Q_q^{2}(\alpha) + \Delta Q$ and $\Delta Q$ follows easily from Eq. (6). Then, one has to carry out the $SU(3)$ tensorial decomposition of $\Delta Q$ with respect to $SU^{\alpha}(3)$ and use the $SU(3)$ Wigner-Racah algebra as described for example in [6, 14, 49, 50] for obtaining the matrix elements of $\Delta Q$ in the $|(\lambda 0)K = 0, L\rangle$ states. This exercise is postponed to a future publication and instead we will present results of full (without any truncation) SM results along with some DSM results in the next two subsections and only DSM results in the third subsection. In addition, to gain more insight into the other $SU^{\alpha}(3)$ algebras, we will use the asymptotic formulas for quadrupole moments and $B(E2)$’s in $sdg$IBM in Section IV.

B. SM and DSM results for multiple $SU(3)$ algebras: $(sdg)^{6_p}$ example

In our first example, we have analyzed a system of 6 protons in $\eta = 4$ shell, i.e. $(sdg)^{6_p}$ system by carrying out SM calculations using the four $H_Q$ Hamiltonians in the full SM space (matrix dimension in the $m$-scheme is $\sim 10^5$) using the Antoine code. For this system, the leading $SU(3)$ irrep (see Table II) is $(18, 0)$ with $S = 0$. Then, Eq. (18) gives $E_{gs} = 378$ and SM calculations for all four $H_Q$’s are in agreement with this $SU(3)$ result. Also, in the SM results the excitation energies of the yrast $J$ states or ground band members ($J = 0, 2, 4, 6, \ldots$) are seen to follow for all the four $H_Q$’s the $3J(J+1)/4$ law as given by $SU(3)$. Thus, it is verified by explicit SM calculations that all the four $H_Q$’s give $SU(3)$ symmetry. Though the energy spectra are same, the wavefunctions of the yrast $J$ states are different. This is established by calculating $Q(J)$ and $B(E2)$’s for the ground band members using $T_{E2}^{\alpha}$ given by Eq. (19). In all the calculations, $e_{eff} = 1e$ and $b^2 = A^{1/3} \, fm^2$ with $A = 86$ are used. Results from SM for the four $H_Q$’s are given in Tables III. It is easy to see that the results for $H_Q^{(-,-)}$ are in complete agreement with the results the $SU(3)$ formulas given by Eq. (20). This is expected as $T_{E2}^{\alpha}$ in Eq. (19) is a generator of $SU^{(-,-)}(3)$ generated by $H_Q^{(-,-)}$. However, the results from the other three $H_Q$’s are quite different.
TABLE III. Shell model results for quadrupole moments $Q(J)$ and $B(E2; \Delta J = 2)$ values for the ground $K = 0^+$ band members for a system of 6 protons in $\eta = 4$ shell. Results are given for the four Hamiltonians in Eq. (16). In the table $(-, -)$ means we are using the wavefunctions obtained using $H_Q^{-,-}$ and similarly others. For other details see text.

| $J$   | $Q(J)$ $e fm^2$ | $B(E2; \Delta J = 2)$ $e^2 fm^4$ |
|-------|-----------------|----------------------------------|
|       | $(-, -)$ (+, -) $(-, +)$ $(+, +)$ | $(-, -)$ (+, -) $(-, +)$ $(+, +)$ |
| $2^+_1$ | -49.18 -33.90 -1.85 13.44 | 585.97 291.34 0.42 42.20 |
| $4^+_1$ | -62.59 -40.16 -4.71 17.72 | 815.31 388.79 1.38 58.58 |
| $6^+_1$ | -68.85 -39.78 -9.12 19.97 | 853.68 377.33 3.90 61.14 |
| $8^+_1$ | -72.48 -37.06 -14.96 20.46 | 827.24 325.68 9.24 58.93 |
| $10^+_1$ | -74.84 -34.57 -21.99 18.28 | 760.52 254.56 18.38 53.81 |

and do not follow the $SU(3)$ results in Eq. (20) as the $T^{E2}$ chosen is not a generator of the $SU(3)$'s generated by the three $H_Q$'s. It is seen from Tables III that the results for $Q(J)$ and $B(E2)$'s from $H_Q^{(+,-)}$ are closer to those from $H_Q^{-,-}$ and this is consistent with the correlation coefficients shown in Table II. The $B(E2)$'s from $H_Q^{(-,+)}$ are much smaller in magnitude. Moreover, $H_Q^{-,-}$ generates prolate shape and $H_Q^{(+,+)}$ oblate as seen clearly from Table III. Quadrupole moments show that $H_Q^{(+,-)}$ and $H_Q^{(-,+)}$ also generate prolate shapes but the deformation from $H_Q^{(-,+)}$ is quite small for the low-lying levels. To gain more insight into these results, we have performed DSM calculations using the four $H_Q$'s with results as follows.

Starting with the same model space, sp energies and two-body interaction, in DSM one solves Hartree-Fock (HF) sp equations self-consistently assuming axial symmetry. The
FIG. 1. Hartree-Fock sp spectrum (it is same for both protons and neutrons) and the lowest intrinsic state for the \((sdg)^{6p,6n}\) system generated by the four \(H_Q\) operators in Eq. (16). In the figure, the symbol \(\times\) denotes neutrons and 0 denotes protons. Shown in the figure are the \(k\) values of the sp orbits and each orbit is doubly degenerate with \(|k\rangle\) and \(|-k\rangle\) states. The spectrum is same for all the four Hamiltonians although the sp wavefunctions are different. The HF energy \(E_{HF}\) for the lowest intrinsic state is -1351.73 (note that \(E\) is unit less and the unit MeV has to be put back after multiplying with an appropriate scale factor if the results are used for a real nucleus) for all the four Hamiltonians. The intrinsic quadrupole moments (in units of \(b^2\)), calculated using 
\[ T^{E2} = Q_q^2(-,-) b^2 \] as the quadrupole operator, for \(H_Q^{(-,-)}, H_Q^{(+,-)}, H_Q^{(-,+)}\) and \(H_Q^{(+,+)}\) are 71.95, 47.43, 5.06 and -19.45 respectively. See text for other details.
The lowest-energy prolate or oblate intrinsic state for the nucleus in question is then obtained. The various excited intrinsic states then are obtained by making particle-hole (p-h) excitations over the lowest-energy intrinsic state (lowest configuration). Carrying out angular momentum projection from each intrinsic state and performing band mixing, orthonormalized $|JK⟩$ states are obtained. See [30] for full details and many applications of DSM. Latest application of DSM is to dark matter studies [51]. In the present DSM calculations, only the lowest intrinsic state is considered. It is found that the four $H_Q$’s generate the same HF spectrum and it is same as shown in Fig. 1 ahead except for some scale factors. The lowest intrinsic state is obtained by putting two protons each in the $\frac{1}{2}^1$, $\frac{1}{2}^2$ and $\frac{3}{2}^1$ states. The intrinsic quadrupole moments (in units of $b^2$) for $H_Q(−,−)$, $H_Q(+,−)$, $H_Q(−,+)$ and $H_Q(+,+)$ are $+35.85$, $24.4$, $1.83$ and $−9.63$ respectively. Thus, $H_Q(−,−)$ generates prolate shape and $H_Q(+,+)$ generates oblate shape in agreement with SM. It is important to emphasize that the intrinsic quadrupole moments are calculated using $T^2_E = Q^2_q(−,−) b^2$ as the quadrupole operator. The ground state energy for the 6 proton system is found to be, for all the four $H_Q$’s same as the exact $SU(3)$ values within less than 1% deviation. The energies of the yrast $J$ states from the ground state are also same for four $H_Q$’s and they follow the $3J(J+1)/4$ law. Similarly, the results for $Q(J)$’s and $B(E2)$’s are essentially same as the SM values. For example for $H_Q(−,−)$, the $Q(J)$ values (in efm$^2$ unit) are $−49.04$, $−62.38$, $−68.56$, $−72.07$ and $−74.29$ for $J = 2$, $4$, $6$, $8$ and $10$ respectively. The corresponding $B(E2; J \rightarrow J−2)$ values (in e$^2fm^4$ unit) are $582.78$, $810.44$, $848.78$, $822.26$ and $755.63$ respectively. Thus, for larger particle systems where SM calculations are not possible, one can use with confidence DSM for further insight into the results from the four $H_Q$’s, i.e. from multiple $SU(3)$ algebras and this is used in Section III-D.

C. SM results for multiple $SU(3)$ algebras: $(sdg)^{6p,2n}_{η=4}$ example

In our second example, we have considered a system of 6 protons and 2 neutrons in $\eta = 4$ shell, i.e. $(sdg)^{6p,2n}$ system and carried out SM calculations using the four $H_Q$ Hamiltonians in the full SM space (dimension in the $m$-scheme is $\sim 2 \times 10^7$) using Antoine code. For this system, the leading $SU(3)$ irrep (see Table II) is $(26,0)$ with $S = 0$ and $T = 2$. Then, Eq. (18) gives $E_{gs} = 754$ and SM calculations for all four $H_Q$’s is in agreement with this $SU(3)$ result. Also, in the SM results the excitation energies of the yrast $J$ states or ground band
TABLE IV. Shell model results for quadrupole moments $Q(J)$ and $B(E2; J \rightarrow J - 2)$ values for the ground $K = 0^+$ band members for a system of 6 protons and 2 neutrons in $\eta = 4$ shell. Results are given for the four Hamiltonians in Eq. (16). In the table $(-, -)$ means we are using the wavefunctions obtained using $H_Q^{(-,-)}$ and similarly others. For other details see text.

| $J$   | $Q(J)\,\text{efm}^2$ | $B(E2; J \rightarrow J - 2)\,\text{e}^2\text{fm}^4$ |
|-------|---------------------|---------------------------------------------|
|       | $(-, -)$ $(-, +)$ $(-, +)$ $(+, +)$ | $(+, -)$ $(+, +)$ $(-, +)$ $(+, +)$ |
| $2_1^+$ | $-83.34$ $-50.54$ $-8.84$ $23.96$ | $1687.70$ $629.26$ $17.14$ $140.57$ |
| $4_1^+$ | $-106.07$ $-62.96$ $-12.96$ $30.15$ | $2379.04$ $876.18$ $25.78$ $198.76$ |
| $6_1^+$ | $-116.68$ $-67.10$ $-17.17$ $32.42$ | $2556.85$ $920.91$ $30.98$ $214.70$ |
| $8_1^+$ | $-122.82$ $-67.95$ $-22.17$ $32.70$ | $2580.68$ $899.62$ $36.37$ $218.34$ |
| $10_1^+$ | $-126.82$ $-67.38$ $-28.13$ $32.32$ | $2521.87$ $841.70$ $42.61$ $215.48$ |

members ($J = 0, 2, 4, 6, \ldots$) are seen to follow for all the four $H_Q$’s the $3J(J + 1)/4$ law as given by $SU(3)$. Thus, it is again verified by explicit SM calculations that all the four $H_Q$’s give $SU(3)$ symmetry. The wavefunctions of the yrast $J$ states are investigated by calculating $Q(J)$ and $B(E2)$’s for the ground band members using $T^{E2}$ in Eq. (21). In all the calculations, $e_{eff}^p = 1.5\text{e}$, $e_{eff}^n = 0.5\text{e}$ and $b^2 = A^{1/3}\text{ fm}^2$ with $A = 88$ are used. Note that the ground $(26, 0)$ irrep arises from the strong coupling of the $(18, 0)$ irrep for the 6 protons (see the previous Section) and the $(8, 0)$ irrep for the two neutrons. Therefore, formulas in Eq. (22) will apply for the states from $H_Q^{(-,-)}$. Results from SM for the four $H_Q$’s are given in Tables IV. It is easy to see that the results for $H_Q^{(-,-)}$ are in complete agreement with the formulas in Eq. (22). This is expected as the proton and neutron parts of $T^{E2}$ in Eq. (21) are generators of $SU^{(-,-)}(3)$ for protons and neutrons respectively. However,
TABLE V. Deformed shell model results for quadrupole moments $Q(J)$ and $B(E2; J \to J - 2)$ values for the ground $K = 0^+$ band members for a system of 6 protons and 6 neutrons (with $T = 0$) in $\eta = 4$ shell. Results are given for the four Hamiltonians in Eq. (16). In the table $(-, -)$ means we are using the wavefunctions obtained using $H_Q^{(-,-)}$ and similarly others. Numbers in the brackets in the second column are exact $SU(3)$ results for $H_Q^{(-,-)}$. For other details see text.

| $J$   | $Q(J) \text{ e.fm}^2$ |
|-------|------------------------|
|       | $(-,-)$    | $(+,-)$ | $(-,+)$ | $(+,+)$ |
| $2^+_1$ | $-96.67(-95.31)$ | $-65.95$ | $-4.63$ | $26.1$ |
| $4^+_1$ | $-123.04(-123.12)$ | $-82.61$ | $-7.02$ | $33.41$ |
| $6^+_1$ | $-135.34(-135.43)$ | $-88.65$ | $-9.66$ | $37.03$ |
| $8^+_1$ | $-142.45(-142.58)$ | $-90.30$ | $-12.91$ | $39.24$ |
| $10^+_1$ | $-147.09(-147.21)$ | $-89.60$ | $-16.86$ | $40.63$ |

| $J$   | $B(E2; J \to J - 2) \text{ e.fm}^4$ |
|-------|----------------------------------|
|       | $(-,-)$    | $(+,-)$ | $(-,+)$ | $(+,+)$ |
| $2^+_1$ | $2273.96(2276.93)$ | $1069.13$ | $4.61$ | $164.89$ |
| $4^+_1$ | $3225.34(3229.59)$ | $1503.04$ | $7.43$ | $233.99$ |
| $6^+_1$ | $3506.56(3511.13)$ | $1608$ | $9.98$ | $254.61$ |
| $8^+_1$ | $3601.29(3605.99)$ | $1613.12$ | $13.44$ | $261.78$ |
| $10^+_1$ | $3605.81(3610.51)$ | $1565.64$ | $18.32$ | $262.44$ |

the results from the other three $H_Q$’s are quite different as in the previous $(sdg)^6p$ example. Again, it is seen from Tables IV that the results for $Q(J)$ and $B(E2)$’s from $H_Q^{(+,-)}$ are closer to those from $H_Q^{(-,-)}$. The $B(E2)$’s from $H_Q^{(-,+)}$ and $H_Q^{(+,+)}$ are much smaller in magnitude. Moreover, $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ oblate as in the previous example. Finally, let us mention that we have also carried out DSM calculations for this example and they are all in agreement with SM results.
D. DSM results for multiple $SU(3)$ algebras: $(sdg)^{(6_p,6_n)T=0}$ example

In our final example we have considered a system of 12 nucleons with $T = 0$ in $\eta = 4$ shell, i.e. $(sdg)^{(6_p,6_n)T=0}$ system. Here the dimension in the $m$-scheme in SM is $\sim 10^{10}$ and therefore SM calculations are not possible with our computational facilities. Thus, in this example DSM gives the predictions for four $H_Q$’s and only for $H_Q^{(-,-)}$ we have exact $SU(3)$ results (they will be same as SM results if performed) from Section III.A. Carrying out DSM calculations for this system, it is found that the four $H_Q$’s generate the same HF sp spectrum as shown in Fig. 1. Using the lowest intrinsic shown in Fig. 1, it is seen from the intrinsic quadrupole moments for the four $H$’s that $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ generates oblate shape in agreement with SM. The ground state energy for the system is found to be $-1402.4$ for all four $H_Q$’s against the exact $SU(3)$ value $-1404$ giving less than 1% deviation. Note that the $SU(3)$ irrep for the ground band is $(36,0)$ and this generated by the irrep $(18,0)$ for the 6 protons and $(18,0)$ for the 6 neutrons. The energies of the yrast $J$ states are also same for four $H_Q$’s and they are also within 1% deviation from the $3J(J+1)/4$ law. Turning to $Q(J)$ and $B(E2)$’s, in the calculations used are $e_{eff}^p = 1.5e$, $e_{eff}^n = 0.5e$ and $b^2 = A^{1/3} fm^2$ with $A = 92$. Note that the ground $(36,0)$ irrep arises from the strong coupling of the $(18,0)$ irreps of the 6 protons and the 6 neutrons. Therefore, formulas in Eq. (22) will apply for the states from $H_Q^{(-,-)}$. DSM results for $H_Q^{(-,-)}$, as shown in Table V are in complete agreement with the formulas in Eq. (22) as expected. However, the results from the other three $H_Q$’s are quite different as in the previous $(sdg)^{6_p}$ and $(sdg)^{6_p,2n}$ examples. Again, it is seen from Tables V that the results for $Q(J)$ and $B(E2)$’s from $H_Q^{(+,-)}$ are closer to those from $H_Q^{(-,-)}$. The $B(E2)$’s from $H_Q^{(-,+)}$ and $H_Q^{(+,+)}$ are much smaller in magnitude. Moreover, $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ oblate as in the previous examples. Thus, the results in Tables III-V are generic results for the four $H_Q$’s.
IV. MULTIPLE SU(3) ALGEBRAS IN INTERACTING BOSON MODEL

In the interacting boson models with $sd$ ($\ell = 0, 2$) or $sdg$ ($\ell = 0, 2, 4$) bosons (and their appropriate generalizations to $pf$, $sdgi$ etc.), the eight operators $(L_q^1, Q_q^2(\alpha))$ are

$$L_q^1 = \sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{3}} (b_\ell^\dagger \tilde{b}_\ell)_q^1,$$

$$Q_q^2(\alpha) = -(2\eta + 3) \sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{5(2\ell+3)(2\ell-1)}} (b_\ell^\dagger \tilde{b}_\ell)_q^2,$$

$$+ \sum_{\ell<\eta} \alpha_{\ell,\ell+2} \sqrt{\frac{6(\ell+1)(\ell+2)(\eta-\ell)(\eta+\ell+3)}{5(2\ell+3)}} \left[ (b_{\ell+2}^\dagger \tilde{b}_{\ell+2})_q^2 + (b_{\ell+2}^\dagger \tilde{b}_\ell)_q^2 \right] ; \alpha_{\ell,\ell+2} = \pm 1.$$

(23)

Note that $b^\dagger$ and $b$ are boson creation and annihilation operators and $\tilde{b}_{\ell m} = (-1)^{\ell-m} b_{\ell-m}$.

Again, after some tedious angular momentum algebra, it is easy to prove that for all choices of $\alpha_{\ell,\ell+2} = \pm 1$, Eq. (7) is valid and therefore giving a SU(3) algebra for each choice of the $\alpha$'s. With $\alpha_{\ell,\ell+2}$ taking $+1$ or $-1$ value, for a given $\eta$ there will be $2^{[\eta/2]}$ number of SU(3) algebras in IBM's just as in SM. It is important to stress that $\alpha_{\ell,\ell+1} = +1$ for all $\ell$ values is the standard choice in $sd$IBM and $sdg$IBM. As an example, in $sd$IBM with $\eta = 2$, the $(L_q^1, Q_q^2)$ operators generating multiple SU(3) algebra are,

$$L_q^1 = \sqrt{10} \left( d^\dagger d \right)_q^1,$$

$$Q_q^2(\alpha_{sd}) = \sqrt{2} \left[ -\frac{\sqrt{7}}{2} \left( d^\dagger d \right)_q^2 + \alpha_{sd} \left( s^\dagger d + d^\dagger s \right)_q^2 \right] ; \alpha_{sd} = \pm 1.$$

(24)

giving two $SU^\alpha(3)$ algebras. In $sd$IBM they are discussed in the context of quantum phase transitions (QPT) [36]. The $\alpha_{sd} = +1$ and $-1$ generate prolate and oblate shapes respectively as discussed ahead. In $sdg$IBM with $\eta = 4$ there will be four $SU^\alpha(3)$ algebras generated by,

$$L_\mu^1 = \sqrt{10} \left( d^\dagger d \right)_\mu^1 + 2\sqrt{15} \left( (g^\dagger g) \right)_\mu^1,$$

$$Q_\mu^2(\alpha_{sd}, \alpha_{dg}) = \sqrt{\frac{3}{4}} \left\{ -11 \sqrt{\frac{21}{2}} \left( d^\dagger d \right)_\mu^2 - 2 \frac{33}{7} \left( g^\dagger g \right)_\mu^2 \right. + \alpha_{sd} 4 \sqrt{\frac{7}{15}} \left( s^\dagger d + d^\dagger s \right)_\mu^2 \left. + \alpha_{dg} \frac{36}{\sqrt{105}} \left( d^\dagger g + g^\dagger d \right)_\mu^2 \right\},$$

(25)

with $\alpha_{sd} = \pm 1$ and $\alpha_{dg} = \pm 1$. 

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\[\gamma = 0, (\alpha_{sd}, \alpha_{dg}) = (+, +)\]
\[\gamma = \pi/3, (\alpha_{sd}, \alpha_{dg}) = (-, -)\]
\[\gamma = 0, (\alpha_{sd}, \alpha_{dg}) = (+, -)\]
\[\gamma = \pi/3, (\alpha_{sd}, \alpha_{dg}) = (-, +)\]

\[\gamma = 0\], \(\gamma = \pi/3\)

\[\alpha_{sd}, \alpha_{dg}\] = \((-,+), (-,-)\)

\[\alpha_{sd}, \alpha_{dg}\] = \((-,-), (+,+)\)

**FIG. 2.** Energy functional \(E = E_{SU_{sdg}(3)}/N^2\) as a function of \(\beta_2\) and \(\beta_4\). (a) plot for \((\alpha_{sd} = 1, \alpha_{dg} = 1)\) with \(\gamma = 0^\circ\) and \((\alpha_{sd} = -1, \alpha_{dg} = -1)\) with \(\gamma = 60^\circ\). Note that the energy functional is same for both of these choices as can be seen from Eq.(30). (b) Same as (a) but for \((\alpha_{sd} = 1, \alpha_{dg} = -1)\) with \(\gamma = 0^\circ\) and \((\alpha_{sd} = -1, \alpha_{dg} = +1)\) with \(\gamma = 60^\circ\).

**A. Geometry of multiple SU(3) algebras in sdIBM and sdlgIBM**

In order to have some insight into the multiple SU(3) algebras in IBM, let us examine the geometric shapes generated by them using coherent states. Starting with sdIBM, the coherent state is

\[|N; \beta_2; \gamma\rangle = \left[ N! \left(1 + \beta_2^2\right)^N \right]^{-1/2} \left\{ s_0^\dagger + \beta_2 \left[ \cos \gamma \, d_0^\dagger + \sqrt{\frac{1}{2}} \sin \gamma \left( d_2^\dagger + d_{-2}^\dagger \right) \right] \right\}^N, \quad (26)\]

where \(\beta_2 \geq 0\) and \(0^\circ \leq \gamma \leq 60^\circ\). Now, let us consider the SU(3) Hamiltonian

\[H_{SU_{sd}(3)}^{\alpha_{sd}} = -Q^2(\alpha_{sd}) \cdot Q^2(\alpha_{sd}) \quad (27)\]

and \(-Q^2(\alpha_{sd}) \cdot Q^2(\alpha_{sd}) = -C_2(SU^{\alpha_{sd}(3)}) + \frac{3}{4} L \cdot L\). It is important to note that \(H_{SU_{sd}(3)}\) generates the same spectrum for the two choices of \(\alpha_{sd}\). In the \(N \to \infty\) limit, the coherent
state expectation value of $H_{SU_{sd}(3)}$ is given by

$$E_{SU_{sd}(3)}(N; \beta_2, \gamma) = \langle N; \beta_2, \gamma | -Q^2(\alpha_{sd}) \cdot Q^2(\alpha_{sd}) | N; \beta_2, \gamma \rangle$$

$$= -\frac{2N^2}{(1 + \beta_2^2)^2} \left[ 4\beta_2^2 + \frac{\beta_4^2}{2} + 2\sqrt{2}\alpha_{sd}\beta_2^3 \cos 3\gamma \right]. \quad (28)$$

Minimizing the $SU(3)$ energy functional $E_{SU_{sd}(3)}(N; \beta_2, \gamma)$ gives the equilibrium solutions $(\beta_2^0, \gamma^0)$ to be $\beta_2^0 = \sqrt{2}$ and $\gamma^0 = 0^\circ$ for $\alpha_{sd} = +1$ and $\gamma^0 = 60^\circ$ for $\alpha_{sd} = -1$. Also, for both situations the equilibrium energy is $-4N^2$ and this is same as the large $N$ eigenvalue of $-C_2(SU(3))$ in the h.w. $(2N, 0)$ irrep [also for the lowest weight $(0, 2N)$ irrep]. Note that the eigenvalue of $C_2(SU(3))$ in a $SU(3)$ irrep $(\lambda\mu)$ is simply $\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)$. Also, the formula in Eq. (28) is good in the limit $N \to \infty$ and in this limit $L \cdot L$ will not contribute as only terms of the order of $N^2$ will survive. Thus, $\alpha_{sd} = \pm 1$ will give prolate and oblate solutions and these results for $sd$IBM are well known [25, 36].

First non-trivial situation happens with $sdg$IBM and for this we will consider the three parameter coherent state used in [44, 52] in terms of $(\beta_2, \beta_4, \gamma)$ parameters for a $N$ boson system,

$$| N; \beta_2; \beta_4, \gamma \rangle = \left[ N! (1 + \beta_2^2 + \beta_4^2)^N \right]^{-1/2} \left\{ s_0^+ + \beta_2 \left[ \cos \gamma \, d_0^+ + \sqrt{\frac{1}{2}} \sin \gamma \left( d_2^+ + d_{-2}^+ \right) \right] + \frac{1}{6} \beta_4 \left[ (5\cos^2 \gamma + 1) \, g_0^+ + \sqrt{\frac{15}{2}} \sin 2\gamma \left( g_2^+ + g_{-2}^+ \right) + \sqrt{\frac{45}{2}} \sin^2 \gamma \left( g_4^+ + g_{-4}^+ \right) \right] \right\}^N | 0 \rangle. \quad (29)$$

Note that $\beta_2 \geq 0$, $-\infty \leq \beta_4 \leq +\infty$ and $0^\circ \leq \gamma \leq 60^\circ$ respectively. Using the results given [26, 52], the $SU(3)$ energy functional is given by

$$E_{SU_{sdg}(3)}(N; \beta_2, \beta_4, \gamma) = \langle N; \beta_2, \beta_4, \gamma | -Q^2(\alpha) \cdot Q^2(\alpha) | N; \beta_2, \beta_4, \gamma \rangle$$

$$= -\frac{3N^2}{4 (1 + \beta_2^2 + \beta_4^2)^2} \left[ \frac{448}{15} \alpha_{sd}^2 \beta_2^2 + \frac{384\sqrt{14}}{35} \alpha_{sd}\alpha_{dg} \beta_2^3 \beta_4 \right.$$

$$+ \frac{352\sqrt{35}}{105} \alpha_{sd}\beta_2^3 \cos 3\gamma + \frac{64\sqrt{35}}{21} \alpha_{sd}\beta_2 \beta_4^3 \cos 3\gamma + \frac{3456}{245} \alpha_{dg}^2 \beta_2^2 \beta_4^2$$

$$+ \frac{156\sqrt{10}}{245} \alpha_{dg} \beta_2^3 \beta_4 \cos 3\gamma + \frac{484}{147} \beta_2^4 + \frac{192\sqrt{10}}{49} \alpha_{dg} \beta_2 \beta_4^3 \cos 3\gamma$$

$$+ \frac{880}{441} \left( 4 - \cos^2 3\gamma \right) \beta_2^2 \beta_4^2 + \frac{400}{1323} (16 - 7\cos^2 3\gamma) \beta_4^4 \right\} \quad (30).$$

Note that $\alpha = (\alpha_{sd}, \alpha_{dg})$. Minimizing $E_{SU_{sdg}(3)}(N; \beta_2, \beta_4, \gamma)$ with respect to $\beta_2, \beta_4$ and $\gamma$ will give the equilibrium (ground state) shape parameters $(\beta_2^0, \beta_4^0, \gamma^0)$ and the corresponding equilibrium energy $E_{SU_{sdg}(3)}^0$. Results are given in Table VI. As seen from the Table VI, the four values of $(\alpha_{sd}, \alpha_{dg})$ generate four combinations of $(\beta_2^0, \beta_4^0, \gamma^0)$. These can be easily
TABLE VI. Equilibrium shapes for the four $SU(3)$ algebras in $sdg$ IBM. For $(\alpha_{sd}, \alpha_{dg}) = (-1, +1)$ and $(-1, -1)$, shown are the $\beta_0^0$ and $\beta_4^0$ values for both $\gamma^0 = 0^\circ$ and $60^\circ$ and they are equivalent.

| $\alpha_{sd}$ | $\alpha_{dg}$ | $\beta_2^0$ | $\beta_4^0$ | $\gamma^0$ | $E_{SU_{sdg}(3)}^0$ |
|---------------|---------------|-------------|-------------|-----------|------------------|
| +1            | +1            | $\sqrt{20/7}$ | $\sqrt{8/7}$ | 0$^\circ$ | $-16N^2$         |
| +1            | -1            | $\sqrt{20/7}$ | $-\sqrt{8/7}$ | 0$^\circ$ | $-16N^2$         |
| -1            | +1            | $\sqrt{20/7}$ | $-\sqrt{8/7}$ | 60$^\circ$ | $-16N^2$         |
| -1            | -1            | $\sqrt{20/7}$ | $\sqrt{8/7}$  | 0$^\circ$ | $-16N^2$         |

understood from the symmetries under $\beta_2 \rightarrow -\beta_2$, $\beta_4 \rightarrow -\beta_4$ and $\gamma = 0^\circ \rightarrow 60^\circ$. We have for example $E(\beta_2, \beta_4, \gamma; \alpha_{sd} = 1, \alpha_{dg} = 1) = E(\beta_2, -\beta_4, \gamma; \alpha_{sd} = 1, \alpha_{dg} = -1)$, $E(\beta_2, \beta_4, \gamma; \alpha_{sd} = 1, \alpha_{dg} = 1) = E(\beta_2, \beta_4, \gamma + 60^\circ; \alpha_{sd} = -1, \alpha_{dg} = -1) = E(-\beta_2, -\beta_4, \gamma; \alpha_{sd} = -1, \alpha_{dg} = 1) = E(-\beta_2, -\beta_4, \gamma; \alpha_{sd} = 1, \alpha_{dg} = 1)$. These also show that the solutions with $\gamma = 60^\circ$ can be changed to $\gamma = 0^\circ$ with $\beta_2 \rightarrow -\beta_2$ as given in Table VI. More importantly, for all the four solutions, the $E_{SU_{sdg}(3)}^0 = -16N^2$. This energy value is same as the large $N$ eigenvalue of $-C_2(SU(3))$ in $(4N, 0)$ irrep. This then implies that the internal structure of the $(4N, 0)$ irrep is different for the four solutions as discussed ahead. The energy functional is shown in Fig. 2 as a function of $\beta_2$ and $\beta_4$ for $\gamma = 0^\circ$ and $60^\circ$ for the four choices of $(\alpha_{sd}, \alpha_{gd})$.

B. Large $N$ results for quadrupole moments and $B(E2)$’s

For further understanding of the four solutions for $SU_{sdg}(3)$, we have examined quadrupole moments and $B(E2)$ values in the ground $K = 0$ band generated by the four solutions in Table VI. Note that the intrinsic state structure for the $K = 0$ ground band is

$$ |N; K = 0\rangle = (N!)^{-1/2} \left(x_0 s_0^\dagger + x_2 d_0^\dagger + x_4 g_0^\dagger\right)^N |0\rangle $$

(31)

where $x_0 = \sqrt{1/5}$, $x_2 = \beta_0^0/\sqrt{5}$ and $x_4 = \beta_4^0/\sqrt{5}$ with $\gamma^0 = 0^\circ$. It is easy to construct the angular momentum projected states $|N; K = 0, L, M\rangle$ and calculate quadrupole moments $Q(L)$ and $B(E2; L \rightarrow L - 2)$ for the ground band. The formulation for these is given in
detail in [53] and valid to order $1/N^2$ where $N$ is the boson number. Then we have,

$$Q(L) = \langle LL \mid Q^2_0 \mid LL \rangle = \frac{\langle LL \mid 20 \mid LL \rangle}{\sqrt{2L+1}} \langle L \parallel Q^2 \parallel L \rangle,$$

$$B(E2; L \rightarrow L - 2) = \frac{5}{16\pi} \frac{|\langle L - 2 \parallel Q^2 \parallel L \rangle|^2}{(2L+1)};$$

$$\langle N; K = 0, L_f \parallel Q^2 \parallel N; K = 0, L_i \rangle = \left[ N\sqrt{(2L_i + 1)} \right] \langle L_i 0 \ 20 \mid L_f, 0 \rangle \times$$

$$\left[ B_{00} + \frac{1}{N} \left( B_{00} - \frac{B_{10} - 3B_{00}}{a} \right) - \frac{L_i(L_f + 1)}{aN^2} \right] \left\{ B_{00} + \frac{F_1}{4a} \delta_{L_f,L_i}$$

$$- \frac{F_2}{12a} \delta_{L_f,L_i+2} \right\}; \quad L_f = L_i \text{ or } L_f = L_i + 2$$

$$B_{mn} = \sum_{\ell', \ell} \left[ \ell' (\ell' + 1) \right]^m \left[ \ell (\ell + 1) \right]^n \langle \ell' 0 \ 0 \mid 20 \rangle \ t_{\ell', \ell} \ x_{\ell' \ell_0} \ x_{\ell_0},$$

$$F_1 = B_{20} - B_{11} - 10B_{10} + 12B_{00}, \quad F_2 = B_{20} - B_{11} + 6B_{10} - 12B_{00},$$

$$a = \sum_{\ell} \ell (\ell + 1) (x_{\ell})^2, \quad \ell = 0, 2, 4.$$

In Eq. (32), the $t_{\ell', \ell}$ are the coefficients in the $E2$ transition operator and they are chosen as,

$$T^{E2} = \sum_{\ell', \ell} t_{\ell', \ell} \left( b_{\ell'}^\dagger \ d_{\ell} \right)^2 = Q^2_q (\alpha_{sd} = +1, \alpha_{sg} = +1).$$

See Eq. (25) for $Q^2_q (\alpha_{sd} = +1, \alpha_{sg} = +1)$. Using the $T^{E2}$, the solutions in Table VI and Eq. (32), results are obtained for $Q(2^+_1)$, $Q(4^+_1)$, $B(E2; 2^+_1 \rightarrow 0^+_1)$ and $B(E2; 4^+_1 \rightarrow 2^+_1)$ for a 10 boson system and the results are given in Table VII. It is seen that the $SU^{(+,+)}(3)$ and $SU^{(+,-)}(3)$ are closer generating prolate shape and $SU^{(-,-)}(3)$ generating oblate shape. The $SU^{(-,+)}(3)$ though generates prolate shape, the quadrupole moments are very small. Thus, $sdg$IBM substantiates the general structures observed in $sdg$ shell model examples presented in Section III.

V. CONCLUSIONS

Multiple $SU(3)$ algebras appear in both shell model and interacting boson model spaces and they open a new paradigm in the applications of $SU(3)$ symmetry in nuclei. In the first detailed attempt made in this paper, using three ($sdg$) space examples in SM, we showed that the four $SU(3)$ algebras in this space exhibit quite different properties with regard to quadrupole collectivity as brought out by the quadrupole moments $Q(J)$ and $B(E2)$’s in the
TABLE VII. Quadrupole moments and $B(E2)$ values for low-lying states in the ground band for a 10 boson system generated by the four $SU(3)$ algebras in $sdg$IBM. Note that $T^{E2}$ Eq. (33) is unit-less and therefore $Q(L)$ and $B(E2)$’s in the table are unit-less.

| $\alpha_{sd}$ | $\alpha_{dg}$ | $Q(2^+_1)$ | $Q(4^+_1)$ | $B(E2; 2^+_1 \rightarrow 0^+_1)$ | $B(E2; 4^+_1 \rightarrow 2^+_1)$ |
|---------------|---------------|-------------|-------------|------------------------------|------------------------------|
| +1            | +1            | -13.69      | -17.43      | 45.68                        | 64.87                        |
| +1            | -1            | -6.15       | -7.89       | 9.15                         | 12.61                        |
| -1            | +1            | -2.16       | -2.98       | 1.05                         | 1.59                         |
| -1            | -1            | 5.38        | 6.55        | 7.33                         | 10.51                        |

Going beyond the present investigations, in future the structure of the low-lying $\gamma$ (also $\beta$) band generated by the multiple $SU(3)$ algebras will be investigated using SM and DSM. Here, we need to deal with the $SU(3)$ integrity basis operators that are 3 and 4-body, as the leading $SU(3)$ irrep in general will be of the type $(\lambda\mu)$ with $\mu \neq 0$ [14]. For example, as seen from Table II, for 8 nucleons with $T = 0$ the leading $SU(3)$ irrep is $(24, 4)$. Let us add that the method for dealing with 3-body operators in DSM was described in [30]. In addition, applications of the $H_Q$’s in Eq. (16) to quantum phase transitions (QPT) may give new insights. For example, using $H = \sum_{\alpha} c_{\alpha} Q^2(\alpha) \cdot Q^2(\alpha)$ and varying the parameters $c_{\alpha}$, it is possible to study QPT; for a similar study using multiple pairing algebras in SM and IBM see [37]. Also studies using $Q^2(\alpha, p) \cdot Q^2(\alpha', n)$ with $\alpha \neq \alpha'$ and $p$ ($n$) denoting protons (neutrons) will be of interest; results of such a study in $sd$IBM are known [54]. In $sdg$IBM a more general CS in terms of $(\beta_2, \beta_4, \gamma, \gamma_4, \delta_4)$ given in [55–57] may prove to be important in understanding further the four $SU(3)$ algebras in this model. Also, it is possible to examine
the properties of $\beta$ and $\gamma$ bands in this model using the results in [53, 58]. All these will be addressed and the results will be reported in a future publication.

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