THE INCOMPLETE GENERALIZED $\tau$-HYPERGEOMETRIC AND SECOND $\tau$-APPELL FUNCTIONS

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Abstract. Motivated mainly by certain interesting recent extensions of the generalized hypergeometric function [Integral Transforms Spec. Funct. 23 (2012), 659–683] and the second Appell function [Appl. Math. Comput. 219 (2013), 8332–8337] by means of the incomplete Pochhammer symbols $(\lambda;\kappa)_n$ and $[\lambda;\kappa]_n$, we introduce here the family of the incomplete generalized $\tau$-hypergeometric functions $2\gamma^2_1(z)$ and $2\Gamma^2_1(z)$. The main object of this paper is to study these extensions and investigate their several properties including, for example, their integral representations, derivative formulas, Euler-Beta transform and associated with certain fractional calculus operators. Further, we introduce and investigate the family of incomplete second $\tau$-Appell hypergeometric functions $\Gamma_{2,\tau}^{1,\tau}$ and $\gamma_{2,\tau}^{1,\tau}$ of two variables. Relevant connections of certain special cases of the main results presented here with some known identities are also pointed out.

1. Introduction, definitions and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{C}$ denote the sets of positive integers, negative integers and complex numbers, respectively,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$ 

The familiar incomplete Gamma functions $\gamma(s, \kappa)$ and $\Gamma(s, \kappa)$ defined by

$$\gamma(s, \kappa) := \int_0^\kappa t^{s-1} e^{-t} \, dt \quad (\Re(s) > 0; \ \kappa \geq 0) \quad \text{and}$$

$$\Gamma(s, \kappa) := \int_\kappa^\infty t^{s-1} e^{-t} \, dt \quad (\kappa \geq 0; \ \Re(s) > 0 \ \text{when} \ \kappa = 0),$$

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respectively, satisfy the following decomposition formula:

\[(1.3)\quad \gamma(s, \kappa) + \Gamma(s, \kappa) := \Gamma(s) \quad (\Re(s) > 0).\]

Each of these functions plays an important role in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, e.g., [1, 3, 7, 11, 14, 15, 17, 20, 21, 22, 29, 30, 31, 40, 43]).

Recently, Srivastava et al. [28] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

\[(1.4)\quad p\gamma_q \left[ \left( \alpha_1, \kappa \right), \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1; \kappa)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},\]

and

\[(1.5)\quad p\Gamma_q \left[ \left( \alpha_1, \kappa \right), \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z \right] = \sum_{n=0}^{\infty} \frac{[\alpha_1; \kappa]_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},\]

where, in terms of the incomplete Gamma functions \(\gamma(s, \kappa)\) and \(\Gamma(s, \kappa)\) defined by (1.1) and (1.2), respectively, the incomplete Pochhammer symbols \((\lambda; \kappa)_\nu\) and \([\lambda; \kappa]_\nu\) \((\lambda, \nu \in \mathbb{C}; \kappa \geq 0)\) are defined as follows:

\[(1.6)\quad (\lambda; \kappa)_\nu := \frac{\gamma(\lambda + \nu, \kappa)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0),\]

and

\[(1.7)\quad [\lambda; \kappa]_\nu := \frac{\Gamma(\lambda + \nu, \kappa)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0),\]

so that, obviously, these incomplete Pochhammer symbols \((\lambda; \kappa)_\nu\) and \([\lambda; \kappa]_\nu\) satisfy the following decomposition relation:

\[(1.8)\quad (\lambda; \kappa)_\nu + [\lambda; \kappa]_\nu := (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C}; \kappa \geq 0).\]

Here, and in what follows, \((\lambda)_\nu\) \((\lambda, \nu \in \mathbb{C})\) denotes the Pochhammer symbol (or the shifted factorial) which is defined (in general) by

\[(1.9)\quad (\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases},\]

it being understood conventionally that \((0)_0 := 1\) and assumed tacitly that the \(\Gamma\)-quotient exists (see, for details, [31, p. 21 et seq.]).

As already observed by Srivastava et al. [28], the definitions (1.4) and (1.5) readily yield the following decomposition formula:

\[(1.10)\quad p F_q \left[ \left( \alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z \right) \right] = p F_q \left[ \left( \alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z \right) \right].\]
for the familiar generalized hypergeometric function \( pF_q \) [26].

More recently, Çetinkaya [6] introduced and studied various properties of the following two families of the incomplete second Appell hypergeometric functions \( \gamma_2 \) and \( \Gamma_2 \):

\[
\gamma_2[(\alpha, \kappa), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] = \sum_{m,p=0}^{\infty} \frac{(\alpha; \kappa)m+p(\beta_1)m(\beta_2)p}{(\gamma_1)m(\gamma_2)p} \frac{x_1^m x_2^p}{m! p!}
\]

(1.11)

and

\[
\Gamma_2[(\alpha, \kappa), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] = \sum_{m,p=0}^{\infty} \frac{(\alpha; \kappa)m+p(\beta_1)m(\beta_2)p}{(\gamma_1)m(\gamma_2)p} \frac{x_1^m x_2^p}{m! p!}
\]

(1.12)

In 2001, Virchenko et al. [42, p. 90, Eq. (5)] have studied and investigated (see also [12]) the following generalized \( \tau \)-hypergeometric function:

\[
2R_1^\tau (a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!}
\]

(\( \tau > 0; \ |z| < 1; \ \Re(c) > \Re(b) > 0 \))

They gave the Euler type integral representation as follows [42, p. 91, Eq. (6)]:

\[
2R_1 (a, b; c; \tau; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1} (1 - zt^{\tau})^{-a} dt.
\]

(\( \tau > 0; \ \arg(1 - z) < \pi; \ \Re(c) > \Re(b) > 0 \))

The special case when \( \tau = 1 \) in (1.13) and (1.14) yields the familiar representations of Gauss’s hypergeometric function [23].

Moreover, Al-Shammery and Kalla [2] introduced and studied various properties of second \( \tau \)-Appell’s hypergeometric functions as:

\[
F^{\tau_1, \tau_2}_2[\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] := \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \sum_{m_1, m_2=0}^{\infty} \frac{(\alpha)_{m_1+m_2} \Gamma(\beta_1+\tau_1 m_1)\Gamma(\beta_2+\tau_2 m_2)}{\Gamma(\gamma_1+\tau_1 m_1)\Gamma(\gamma_2+\tau_2 m_2)} \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!}
\]

(\( \tau_1, \tau_2 > 0; \ |x_1| + |x_2| < 1 \)).

Motivated essentially by the demonstrated potential for applications of these incomplete hypergeometric functions \( p\gamma_q \) and \( p\Gamma_q \), and the incomplete second Appell hypergeometric functions \( \gamma_2 \) and \( \Gamma_2 \) in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [6, 28] and the references cited therein), we aim here at systematically investigating the family of the incomplete generalized \( \tau \)-hypergeometric function \( 2\gamma_1^\tau(z) \) and \( 2\Gamma_1^\tau(z) \). For each of these incomplete generalized \( \tau \)-hypergeometric function, we obtain integral representations, derivative formula, Euler-Beta transform and associated with the fractional calculus operators. Further, we introduce and investigate the family of incomplete second \( \tau \)-Appell hypergeometric functions
\[ \Gamma_2^{\tau_1,\tau_2} \] and \( \gamma_2^{\tau_1,\tau_2} \) of two variables. Some interesting special cases of our main results are also pointed out. For various other investigations involving generalizations of the hypergeometric function \( pF_q \) of \( p \) numerator and \( q \) denominator parameters, which were motivated essentially by the pioneering work of Srivastava et al. [28], the interested reader may be referred to several recent papers on the subject (see, e.g., [6, 8, 9, 16, 27, 33, 35, 36, 37, 38, 39] and the references cited in each of these papers).

2. The incomplete generalized \( \tau \)-hypergeometric function

In terms of the incomplete Pochhammer symbol \((\lambda;\kappa)_n\) and \([\lambda;\kappa]_n\) defined by (1.6) and (1.7), we introduce the families of the incomplete generalized \( \tau \)-hypergeometric function \( 2\gamma_1^\tau(z) \) and \( 2\Gamma_1^\tau(z) \) as follows: For \( a, b, c \in \mathbb{C} \) and \( c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \), we have

\[
2\gamma_1^\tau(z) = 2\gamma_1^\tau((a,\kappa), b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;\kappa)_n \Gamma(b + \tau n) z^n}{\Gamma(c + \tau n)} \frac{z^n}{n!}
\]

\[ (\kappa \geq 0; \ \tau > 0; \ \n|z| < 1; \ \Re(c) > \Re(b) > 0 \ \text{when} \ \kappa = 0) \]

and

\[
2\Gamma_1^\tau(z) = 2\Gamma_1^\tau((a,\kappa), b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{[a;\kappa]_n \Gamma(b + \tau n) z^n}{\Gamma(c + \tau n)} \frac{z^n}{n!}
\]

\[ (\kappa \geq 0; \ \tau > 0; \ \n|z| < 1; \ \Re(c) > \Re(b) > 0 \ \text{when} \ \kappa = 0). \]

In view of (1.8), these families of incomplete generalized \( \tau \)-hypergeometric function satisfy the following decomposition formula:

\[
2\gamma_1^\tau((a,\kappa), b; c; z) + 2\Gamma_1^\tau((a,\kappa), b; c; z) = 2R_1^\tau(a, b; c; z),
\]

where \( 2R_1^\tau(z) \) is the generalized \( \tau \)-hypergeometric function [12, 42].

It is noted in passing that, in view of the decomposition formula (2.3), it is sufficient to discuss the properties and characteristics of the incomplete generalized \( \tau \)-hypergeometric function \( 2\Gamma_1^\tau(z) \).

Remark 1. The special cases of (2.1) and (2.2) when \( \tau = 1 \) are easily seen to reduce to the known families of the incomplete Gauss hypergeometric functions [28, p. 664, Eq. (3.1)] and [28, p. 664, Eq. (3.2)]:

\[
2\gamma_1((a,\kappa), b; c; z) = \sum_{n=0}^{\infty} \frac{(a;\kappa)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad \text{and}
\]

\[
2\Gamma_1((a,\kappa), b; c; z) = \sum_{n=0}^{\infty} \frac{[a;\kappa]_n (b)_n}{(c)_n} \frac{z^n}{n!}
\]
respectively. Also, the special cases of (2.1) and (2.2) when \( \tau = 1 \) and \( \kappa = 0 \) is seen to yield the classical Gauss’s hypergeometric function (see, e.g., [15, 23]):

\[
\binom{2}{1}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad (|z| < 1; \Re(c) > \Re(b) > 0).
\]

### 2.1. Integral representations

In this section, we present certain integral representations of the incomplete generalized \( \tau \)-hypergeometric function \( 2\Gamma_{\tau}^1(z) \) by applying (1.2) and (1.7).

**Theorem 1.** The following integral representation for \( 2\Gamma_{\tau}^1(z) \) in (2.2) holds true:

\[
2\Gamma_{\tau}^1[(a, \kappa), b; c; z] = \frac{1}{\Gamma(a)} \int_{\kappa}^{\infty} e^{-t} t^{a-1} \Phi_{\tau}^1(b; c; zt) dt,
\]

\((\kappa \geq 0; \Re(z) < 1, \Re(a) > 0 \text{ when } \kappa = 0)\)

where \( \Phi_{\tau}^1(b; c; z) \) is the \( \tau \)-confluent hypergeometric function introduced by Virchenko [41]:

\[
\Phi_{\tau}^1(z) = \sum_{n=0}^{\infty} \frac{\Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!} \quad (\tau > 0, \Re(c) > \Re(b) > 0).
\]

**Proof.** Using the definition of the incomplete Pochhammer symbol \([a; \kappa]_n\) in (2.2) by considering the integral representation resulting from (1.2) and (1.7) and using (2.5), we are led to the desired result (2.4) asserted by Theorem 1. \(\square\)

**Theorem 2.** The following integral representation for \( 2\Gamma_{\tau}^1(z) \) in (2.2) holds true:

\[
2\Gamma_{\tau}^1[(a, \kappa), b; c; z] = \frac{1}{B(b, c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1} 1\Gamma_{0}[(a, \kappa); zt^\tau] dt
\]

\((\tau > 0, \Re(c) > \Re(b) > 0; \kappa \geq 0)\).

**Proof.** Considering the following elementary identity involving the Beta function:

\[
\frac{\Gamma(c) \Gamma(b + \tau n)}{\Gamma(b) \Gamma(c + \tau n)} = \frac{(b)_{\tau n}}{(c)_{\tau n}} = \frac{B(b + \tau n, c-b)}{B(b, c-b)}
\]

\[
= \frac{1}{B(b, c-b)} \int_{0}^{1} t^{b+\tau n-1}(1-t)^{c-b-1} dt
\]

\((\tau > 0, \Re(c) > \Re(b) > 0)\)
in (2.2) and interchanging summation and integration under the stated conditions, we have
\[ 2\Gamma_\tau^\alpha[(a, \kappa), b; c; z] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{n=0}^{\infty} (a; \kappa)_n \frac{(zt)^n}{n!}. \]

Finally, using the definition (1.5), we get the desired integral representation (2.6) asserted by Theorem 2. □

**Theorem 3.** The following relationship with the incomplete gamma function \( \Gamma(s, \kappa) \) defined by (1.2) holds true:
\[ 2\Gamma_\tau^\alpha[(a, \kappa), b; b; z] = \left(1 - z\right)^{-a} \frac{1}{\Gamma(a)} \Gamma(a, \kappa(1-z)) \quad (|z| < 1; \kappa \geq 0). \]

**Proof.** Putting \( c = b \) in (2.4), we immediately obtain the following simplified form:
\[ 2\Gamma_\tau^\alpha[(a, \kappa), b; b; z] = \frac{1}{\Gamma(a)} \int_{1-z}^\infty e^{-t(1-z)} t^{a-1} dt. \]

Now by setting \( t = \frac{u}{1-z} \) and \( dt = \frac{du}{1-z} \) in (2.8), we have
\[ 2\Gamma_\tau^\alpha[(a, \kappa), b; b; z] = \left(1 - z\right)^{-a} \frac{1}{\Gamma(a)} \int_{\kappa(1-z)}^\infty e^{-u} u^{a-1} du \quad (|z| < 1), \]
which is precisely the assertion (2.7) asserted by Theorem 3. □

**Theorem 4.** The following relationship with the complementary error function \( \text{erfc}(z) \) holds true:
\[ 2\Gamma_1^\alpha\left(\left(\frac{1}{2}, \kappa\right), b; 1-z\right) = \frac{1}{\sqrt{z}} \text{erfc}(\sqrt{\kappa z}) \quad (\kappa \geq 0). \]

**Proof.** Upon replacing \( z \) by \( 1-z \) and putting \( a = \frac{1}{2} \) in (2.7), we find that the incomplete generalized \( \tau \)-hypergeometric function \( 2\Gamma_1^\alpha(z) \) reduces to the complementary error function \( \text{erfc}(z) \) (see, e.g., [22, p. 726]) as follows:
\[ 2\Gamma_1^\alpha\left(\left(\frac{1}{2}, \kappa\right), b; 1-z\right) = \frac{z^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \Gamma\left(\frac{1}{2}, \kappa z\right) = \frac{1}{\sqrt{z}} \text{erfc}(\sqrt{\kappa z}) \]
□

**Remark 2.** The special cases of (2.6) and (2.4) when \( \tau = 1 \) are easily seen to reduce to the known integral representations of the incomplete Gauss hypergeometric functions [28, p. 665, Eq. (3.6)] and [28, p. 672, Eq. (3.53)]:
\[ 2\Gamma_1[(a, \kappa), b; c; z] = \frac{1}{\Gamma(a)} \int_{\kappa}^{\infty} e^{-t} t^{a-1} \binom{1}{1}(b; c; z) dt \]
and
\[ 2\Gamma_1[(a, \kappa), b; c; z] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \binom{1}{1}(a; \kappa; ; z) dt \]
respectively. Also, the special cases of (2.6) and (2.4) when $\tau = 1$ and $\kappa = 0$ are seen to yield the classical integral representations of Gauss’s hypergeometric function (see, e.g., [23]):

\[
2F_1[a, b; c; z] = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} 1F_1(b; c; zt) dt
\]

and

\[
2F_1[a, b; c; z] = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt.
\]

respectively.

2.2. Derivative formula

Theorem 5. Each of the following derivative formula for $2\Gamma_1^\tau(z)$ holds true:

(2.15) \[
\frac{d^n}{dz^n} [2\Gamma_1^\tau((a, \kappa), b; c; z)] = \frac{(a)_n \Gamma(c) \Gamma(b + \tau n)}{\Gamma(b) \Gamma(c + \tau n)} 2\Gamma_1^\tau((a + n, \kappa), b + \tau n; c + \tau n; z]
\]

and

(2.16) \[
\left( \frac{d}{dz} \right)^m [z^{c-1} 2\Gamma_1^\tau((a, \kappa), b; c; \omega z^\tau)] = \frac{\Gamma(c) \sum_{n=0}^{\infty} [a; \kappa]_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \omega^n \left( \frac{d}{dz} \right)^m [z^{c+\tau n-1}]
\]

where $a, b, c, \omega \in \mathbb{C}; \Re(\tau) > 0, \Re(a) > 0, \Re(b) > 0, \Re(c) > 0; \Re(\kappa) \geq 0$.

Proof. Differentiating $n$ times both sides of (2.2) with respect to $z$, we can easily obtain a derivative formula for the incomplete generalized $\tau$-hypergeometric function $2\Gamma_1^\tau(z)$ asserted by (2.15).

Next, according to the uniform convergence of the series (2.2), differentiating term by term under the sign of summation, we have

\[
\left( \frac{d}{dz} \right)^m [z^{c-1} 2\Gamma_1^\tau((a, \kappa), b; c; \omega z^\tau)]
\]

\[
= \frac{\Gamma(c) \sum_{n=0}^{\infty} [a; \kappa]_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \omega^n \left( \frac{d}{dz} \right)^m [z^{c+\tau n-1}]
\]

\[
= \frac{\Gamma(c) \sum_{n=0}^{\infty} [a; \kappa]_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \omega^n z^{c+\tau n-m-1}
\]

which, in view of the definition (2.2), yields the desired representation (2.16). \qed
2.3. Euler-Beta transform

The Euler-Beta transform of the function \( f(z) \) is defined, as usual, by

\[
B\{f(z); \mu, \nu\} = \int_0^1 z^{\mu-1}(1-z)^{\nu-1} f(z) \, dz.
\]

**Theorem 6.** The following Euler-Beta transform representation for the \( 2\Gamma_1^\tau(z) \) in (2.2) holds true:

\[
B\{2\Gamma_1^\tau[(a, \kappa), b; \mu c, \nu]; \omega z \tau\} := B(c, \nu)\Gamma_1^\tau[(a, \kappa), b; c+\nu; \omega z \tau\}.
\]

\( \tau > 0; \Re(a) > 0, \Re(\nu) > 0, \Re(b) > 0, \Re(c) > 0 \).

**Proof.** Using the definition (2.17) of the Euler-Beta transform, we find from (2.2)

\[
B\{2\Gamma_1^\tau[(a, \kappa), b; \omega z \tau\} : c, \nu\}
\]

\[
= \int_0^1 z^{c-1}(1-z)^{\nu-1} 2\Gamma_1^\tau[(a, \kappa), b; \omega z \tau\} \, dz.
\]

Upon interchanging the order of integration and summation in (2.19), which can easily be justified by uniform convergence under the constraints stated with (2.17), we get

\[
B\{2\Gamma_1^\tau[(a, \kappa), b; \omega z \tau\} : c, \nu\}
\]

\[
= \Gamma(c)\sum_{n=0}^{\infty} \frac{[a; \kappa]_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{\omega^n}{n!}\Gamma(c + \nu + \tau n)\Gamma(\nu)
\]

\[
\Gamma(b)\Gamma(c + \nu + \tau n)\Gamma(c + \nu + \tau n)\Gamma(\nu)
\]

Using the definition (2.2), we get the desired representation (2.18). \( \square \)

2.4. Fractional calculus approach

In this section, we derive certain interesting properties of the incomplete generalized \( \tau \)-hypergeometric function \( 2\Gamma_1^\tau(z) \) in (2.2) associated with right-sided Riemann-Liouville fractional integral operator \( I_{\alpha^+}^\mu \) and the right-sided Riemann-Liouville fractional derivative operator \( D_{\alpha^+}^\mu \), which are defined as follows (see, e.g., [14, 19, 25]):

\[
(I_{\alpha^+}^\mu \varphi)(x) = \frac{1}{\Gamma(\mu)} \int_{\alpha}^{x} \frac{\varphi(t)}{(x-t)^{1-\mu}} \, dt \quad (\mu \in \mathbb{C}, \Re(\mu) > 0)
\]
and
\begin{align}
(2.21) \quad (D_{\alpha+}^{\mu} \varphi) (x) &= \left( \frac{d}{dx} \right)^n \left( I_{\alpha+}^{n-\mu} \varphi \right) (x) \quad (\mu \in \mathbb{C}, \mathbb{R}(\mu) > 0; \ n = [\mathbb{R}(\mu)] + 1),
\end{align}

where \([x]\) means the greatest integer not exceeding real \(x\).

Another generalization of Riemann-Liouville fractional derivative operator \(D_{\alpha+}^\mu\) in (2.21) by introducing a right-sided Riemann-Liouville fractional derivative operator \(D_{\alpha+}^{\mu, \nu}\) of order \(0 < \mu < 1\) and type \(0 \leq \nu \leq 1\) with respect to \(x\) by Hilfer (see, e.g., [13]) is given as follows:
\begin{align}
(2.22) \quad (D_{\alpha+}^{\mu, \nu} \varphi) (x) &= \left( \int_{0}^{x} \left( x - \tau \right)^{\mu - 1} \frac{d}{d\tau} \left( I_{\alpha+}^{(1-\nu)(1-\mu)} \varphi \right) \right) (x) \\
&= \left( \frac{d}{dx} \right)^n \left( I_{\alpha+}^{n-\mu} \varphi \right) (x) \quad (\mu \in \mathbb{C}, \mathbb{R}(\mu) > 0; \ n = [\mathbb{R}(\mu)] + 1).
\end{align}

The generalization (2.22) yields the classical Riemann-Liouville fractional derivative operator \(D_{\alpha+}^\mu\) when \(\nu = 0\).

**Theorem 7.** Let \(\alpha \in \mathbb{R}_+ = [0, \infty), \ a, b, c, \mu, \omega \in \mathbb{C} \) and \(\mathbb{R}(a) > 0, \mathbb{R}(b) > 0, \mathbb{R}(c) > 0, \mathbb{R}(\mu) > 0, \tau > 0\). Then, for \(x > \alpha\), the following relations hold true:
\begin{align}
(2.23) \quad (I_{\alpha+}^{\mu, \nu} \left( (t-\alpha)^{\nu-1} 2 \Gamma^n \{(a, \kappa), b; \omega(t-\alpha)^\tau\} \right)) (x) &= \frac{(x-\alpha)^{c+\mu-1} \Gamma(c)}{\Gamma(c+\mu)} 2 \Gamma^n \{(a, \kappa), b; \omega(x-\alpha)^\tau\}, \\
(2.24) \quad (D_{\alpha+}^{\mu, \nu} \left( (t-\alpha)^{\nu-1} 2 \Gamma^n \{(a, \kappa), b; \omega(t-\alpha)^\tau\} \right)) (x) &= \frac{(x-\alpha)^{c-\mu-1} \Gamma(c)}{\Gamma(c-\mu)} 2 \Gamma^n \{(a, \kappa), b; \omega(x-\alpha)^\tau\}
\end{align}

and
\begin{align}
(2.25) \quad (D_{\alpha+}^{\mu, \nu} \left( (t-\alpha)^{\nu-1} 2 \Gamma^n \{(a, \kappa), b; \omega(t-\alpha)^\tau\} \right)) (x) &= \frac{(x-\alpha)^{c-\mu-1} \Gamma(c)}{\Gamma(c-\mu)} 2 \Gamma^n \{(a, \kappa), b; \omega(x-\alpha)^\tau\}.
\end{align}

**Proof.** By virtue of the formulas (2.20) and (2.2), the term-by-term fractional integration and the application of the relation [25]:
\begin{align}
(2.26) \quad (I_{\alpha+}^{\alpha, \beta} \left( (t-\alpha)^{\beta-1} \right)) (x) &= \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} (x-a)^{\alpha + \beta - 1} \quad (\alpha, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0)
\end{align}
yields, for \(x > \alpha\),
\begin{align}
(2.27) \quad (I_{\alpha+}^{\mu, \nu} \left( (t-\alpha)^{\nu-1} 2 \Gamma^n \{(a, \kappa), b; \omega(t-\alpha)^\tau\} \right)) (x) &= \left( I_{\alpha+}^{\mu} \left( \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{[a; \kappa]_n \Gamma(b + \tau n)}{(c + \tau n)!} \omega^n (t-\alpha)^{c+\tau n-1} \right) \right)
\end{align}
Next, by (2.21) and (2.2), we find that
\[ (D_{a+}^{\mu} [(t - \alpha)^{c-1}]^{2}\Gamma_1 \left((a, \kappa), b; \omega(t - \alpha)^\tau\right)) (x) \]
\[ = \left( \frac{d}{dx} \right)^n \left[ \frac{(x - \alpha)^{c+n-\mu} \Gamma(c)}{\Gamma(c - \mu + n)} \right]^{2}\Gamma_1 \left((a, \kappa), b; \omega(x - \alpha)^\tau\right) \].

Applying (2.16), we are led to the desired result (2.24).

Finally, by (2.22) and (2.2), we have
\[ (D_{a+}^{\mu, \nu} [(t - \alpha)^{c-1}]^{2}\Gamma_1 \left((a, \kappa), b; \omega(t - \alpha)^\tau\right)) (x) \]
\[ = \left( \frac{d}{dx} \right)^n \left[ \frac{(x - \alpha)^{c+n-\mu} \Gamma(c)}{\Gamma(c - \mu + n)} \right]^{2}\Gamma_1 \left((a, \kappa), b; \omega(x - \alpha)^\tau\right) \].

Using the known relation of Srivastava and Tomovski [34, p. 203, Eq. (2.18)]
\[ (D_{a+}^{\mu, \nu} [(t - \alpha)^{c-1}]^{2}\Gamma_1 \left((a, \kappa), b; \omega(t - \alpha)^\tau\right)) (x) \]
\[ = \left( \frac{d}{dx} \right)^n \left[ \frac{(x - \alpha)^{c+n-\mu} \Gamma(c)}{\Gamma(c - \mu + n)} \right]^{2}\Gamma_1 \left((a, \kappa), b; \omega(x - \alpha)^\tau\right) \].

in (2.29), we are led to the desired result (2.25).

Remark 3. The special cases of (2.23)-(2.25) when \( \tau = 1 \) yield the corresponding known relations for the generalized \( \tau \)-hypergeometric function [24].

3. The incomplete second \( \tau \)-Appell functions

Further, we introduce the incomplete second Appell \( \tau \)-hypergeometric functions \( \gamma_{1,2}^{\tau} \) and \( \Gamma_{2}^{\tau} \) of two variables as follows: For \( a, b_1, b_2 \in \mathbb{C} \) and \( c_1, c_2 \in \mathbb{C} \setminus \mathbb{Z}_0 \), we have
\[ \gamma_{1,2}^{\tau}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] \]
\[ = \frac{\Gamma(c_1) \Gamma(c_2)}{\Gamma(b_1) \Gamma(b_2)} \sum_{m_1, m_2=0}^{\infty} \frac{(a, \kappa)_{m_1+m_2} \Gamma(b_1 + \tau_1 m_1) \Gamma(b_2 + \tau_2 m_2) \, x_1^{m_1} x_2^{m_2}}{m_1! m_2!} \]
\[ (\kappa \geq 0; \, \tau_1, \tau_2 > 0; \, |x_1| + |x_2| < 1 \, \text{when} \, \kappa = 0) \]
and
\[ \Gamma_{2}^{\tau}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] \]
3.1. Integral representations

\[
\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(b_2)} \sum_{m_1, m_2 = 0}^{\infty} \frac{[\alpha; \kappa]_{m_1 + m_2}}{\Gamma(c_1 + \tau_1 m_1)\Gamma(c_2 + \tau_2 m_2)} \frac{x_1^{m_1} x_2^{m_2}}{m_1! m_2!}
\]

\((\kappa \geq 0; \tau_1, \tau_2 > 0; |x_1| + |x_2| < 1 \text{ when } \kappa = 0).\)

In view of (1.8), these families of incomplete second \(\tau\)-Appell function satisfy the following decomposition formula:

\[
\Gamma_{2}^{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] + \Gamma_{2}^{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2]
\]

\[
= F_{2}^{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2],
\]

where \(F_{2}^{\tau_1, \tau_2}\) is the second \(\tau\)-Appell function [2].

Remark 4. The special cases of (3.1) and (3.2) when \(\tau_1 = 1 = \tau_2\) are easily seen to reduce to the known families of the incomplete second Appell functions (1.11) and (1.12), respectively.

Also the special cases of (3.1) and (3.2) when \(\tau_1 = 1 = \tau_2\) and \(\kappa = 0\) are easily seen to reduce to the classical second Appell functions [4, 5].

In view of the decomposition formula (3.3), it is sufficient to discuss the properties and characteristics of the incomplete second \(\tau\)-Appell function \(\Gamma_{2}^{\tau_1, \tau_2}\).

3.1. Integral representations

Theorem 8. The following integral representation for \(\Gamma_{2}^{\tau_1, \tau_2}\) in (3.2) holds true:

\[
\Gamma_{2}^{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] = \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-t} t^{a-1} \Phi_{1}^{\tau_1} \left[\begin{array}{c}
\frac{b_1}{c_1} \frac{t}{x_1} \\
\frac{b_2}{c_2} \frac{t}{x_2}
\end{array}\right] dt
\]

\((\kappa \geq 0; \tau_1, \tau_2 > 0; \Re(x_1 + x_2) < 1, \Re(a) > 0 \text{ when } \kappa = 0).\)

Proof. Using the integral representation of the Pochhammer symbol \((a)_{m_1 + m_2}\) and the definition of \(\tau\)-confluent hypergeometric function (2.5) in (3.2), we are led to the desired result (3.4) asserted by Theorem 8. \(\square\)

Theorem 9. The following integral representation for \(\Gamma_{2}^{\tau_1, \tau_2}\) in (3.2) holds true:

\[
\Gamma_{2}^{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] = \frac{1}{B(b_1, c_1 - b_1)B(b_2, c_2 - b_2)}
\]

\[
\times \int_{0}^{1} \int_{0}^{1} t^{b_1 - 1} s^{b_2 - 1} (1 - t)^{c_1 - b_1 - 1} (1 - s)^{c_2 - b_2 - 1}
\]

\[
\times (1 - x_1 t^{\tau_1} - x_2 s^{\tau_2})^{-a} \frac{\Gamma(a, \kappa(t^{\tau_1} + s^{\tau_2}))}{\Gamma(a)} dt ds
\]

\((\kappa \geq 0; \tau_1, \tau_2 > 0; \Re(c_j) > \Re(b_j) > 0 \text{ when } \kappa = 0).\)
Proof. Considering the following elementary identity involving the Beta function $B(\beta, \gamma)$:

\[
\frac{(\beta)_\nu}{(\gamma)_\nu} = \frac{B(\beta + \nu, \gamma - \beta)}{B(\beta, \gamma - \beta)} = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta+\nu-1}(1-t)^{\gamma-\beta-1} \, dt
\]

in (3.2), we have

\[
(\Re(\gamma) > \Re(\beta) > \max\{0, -\Re(\nu)\})
\]

in (3.2), we have

\[
\Gamma_1^\tau_1,\tau_2[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2]
\]

\[
= \frac{1}{B(b_1, c_1 - b_1)B(b_2, c_2 - b_2)} \int_0^1 \int_0^1 t^{b_1-1}s^{b_2-1}(1-t)^{c_1-b_1-1}(1-s)^{c_2-b_2-1}
\times \Gamma_1[(a, \kappa), b_1, b_2; b_1, b_2; x_1t^{\tau_1}, x_2s^{\tau_2}] \, dt \, ds
\]

and using the relation [6, p. 8334, Eq. (22)]:

\[
\Gamma_2[(a, \kappa), b_1, b_2; x_1, x_2] = (1 - x_1t - x_2s)^{-\alpha} \frac{\Gamma(a, \kappa(1 - x_1t - x_2s))}{\Gamma(a)}
\]

we get the desired multiple integral representation (3.5) asserted by Theorem 9.

□

Remark 5. The special cases of (3.4) and (3.5) when $\tau_1 = 1 = \tau_2$ are easily seen to reduce to the known integral representations of the incomplete second Appell functions [6, p. 8333, Eq. (10)] and [6, p. 8335, Eq. (24)]:

\[
(3.8) \quad \Gamma_2[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] = \frac{1}{\Gamma(a)} \int_\kappa^\infty e^{-t} t^{a-1} \, _1F_1 \left[ b_1; c_1; x_1t \right] \, _1F_1 \left[ b_2; c_2; x_2t \right] \, dt
\]

\[
(\kappa \geq 0; \Re(x_1 + x_2) < 1, \Re(a) > 0 \text{ when } \kappa = 0)
\]

and

\[
(3.9) \quad \Gamma_2[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] = \frac{1}{B(b_1, c_1 - b_1)B(b_2, c_2 - b_2)} \int_0^1 \int_0^1 t^{b_1-1}s^{b_2-1}(1-t)^{c_1-b_1-1}(1-s)^{c_2-b_2-1}
\times (1 - x_1t - x_2s)^{-\alpha} \frac{\Gamma(a, \kappa(1 - x_1t - x_2s))}{\Gamma(a)} \, dt \, ds
\]

\[
(\kappa \geq 0; \Re(c_j) > \Re(b_j) > 0 \text{ when } \kappa = 0) \quad (j = 1, 2)
\]

respectively.
Also, the special cases of (2.6) and (2.4) when \( \tau_1 = 1 = \tau_2 \) and \( \kappa = 0 \) are seen to yield the classical integral representations of second Appell functions (see, e.g., [4, 31]).

\[
F_2[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \, F_1 \left[ \begin{array}{c} b_1; \\ c_1; \end{array} \right] x_1 t \, F_1 \left[ \begin{array}{c} b_2; \\ c_2; \end{array} \right] x_2 t \, dt
\]

\((\Re(x_1 + x_2) < 1, \Re(a) > 0)\)

and

\[
F_2[(a, \kappa), b_1, b_2; c_1, c_2; x_1, x_2] = \frac{1}{B(b_1, c_1 - b_1)B(b_2, c_2 - b_2)} \int_0^1 \int_0^1 (1 - t)^{c_1 - b_1 - 1}(1 - s)^{c_2 - b_2 - 1}(1 - x_1 t - x_2 s)^{-a} \, dt \, ds
\]

\((\Re(c_j) > \Re(b_j) > 0 \ (j = 1, 2))\),

respectively.

### 3.2. Connections with certain known functions

**Theorem 10.** The following relationship with the incomplete gamma function \( \Gamma(s, \kappa) \) defined by (1.2) holds true:

\[
\Gamma_{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; b_1, b_2; x_1, x_2] = \frac{(1 - x_1 - x_2)^{-a}}{\Gamma(a)} \Gamma(a, \kappa(1 - x_1 - x_2))
\]

\((|x_1 + x_2| < 1; \kappa \geq 0)\).

**Proof.** Putting \( c_1 = b_1 \) and \( c_2 = b_2 \) in (3.4), we obtain the following simplified form:

\[
\Gamma_{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; b_1, b_2; x_1, x_2] = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t(1-x_1-x_2)} \, t^{a-1} dt.
\]

Now by setting \( t = \frac{u}{1-x_1-x_2} \) and \( dt = \frac{du}{1-x_1-x_2} \) in (3.13), we have

\[
\Gamma_{\tau_1, \tau_2}[(a, \kappa), b_1, b_2; b_1, b_2; x_1, x_2] = \frac{(1 - x_1 - x_2)^{-a}}{\Gamma(a)} \int_{\kappa(1-x_1-x_2)}^\infty e^{-u} \, u^{a-1} du \ (|z| < 1),
\]

which is precisely the assertion (3.12) asserted by Theorem 10. \( \square \)
Theorem 11. The following relationship with the complementary error function $\text{erfc}(z)$ holds true:

$$\Gamma_{2}^{\tau_{1}, \tau_{2}} \left[ \left( \frac{1}{2}, \kappa \right), b_{1}, b_{2}; b_{1}, b_{2}; 0, 1 - z \right] = \Gamma_{2}^{\tau_{1}, \tau_{2}} \left[ \left( \frac{1}{2}, \kappa \right), b_{1}, b_{2}; b_{1}, b_{2}; 1 - z, 0 \right]$$

(3.15)

$$= \frac{1}{\sqrt{2}} \text{erfc}(\sqrt{\kappa z}) \quad (\kappa \geq 0).$$

Proof. Upon replacing $x_{1} = 0$ and $x_{2} = 1 - z$ or $x_{2} = 0$ and $x_{1} = 1 - z$ and putting $a = \frac{1}{2}$ in (3.12), we can easily find that the incomplete second $\tau$-Appell function $\Gamma_{2}^{\tau_{1}, \tau_{2}}$ reduces to the complementary error function $\text{erfc}(z)$ as follows:

$$\Gamma_{2}^{\tau_{1}, \tau_{2}} \left[ \left( \frac{1}{2}, \kappa \right), b_{1}, b_{2}; b_{1}, b_{2}; 0, 1 - z \right] = \Gamma_{2}^{\tau_{1}, \tau_{2}} \left[ \left( \frac{1}{2}, \kappa \right), b_{1}, b_{2}; b_{1}, b_{2}; 1 - z, 0 \right]$$

$$= \frac{1}{\sqrt{2}} \text{erfc}(\sqrt{\kappa z}).$$

Theorem 12. Each of the following relationship with the incomplete generalized $\tau$-hypergeometric function $2\Gamma_{1}(z)$ in (2.2) holds true for $\kappa \geq 0$:

(3.16) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, c_{2}; x_{1}, 0] = 2\Gamma_{1}^{\tau_{1}} [(a, \kappa), b_{1}, x_{1}];$

(3.17) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, c_{2}; 0, x_{2}] = 2\Gamma_{1}^{\tau_{2}} [(a, \kappa), b_{2}, x_{2}];$

(3.18) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, c_{2}; x_{1}, x_{2}] = (1 - x_{1})^{-a} 2\Gamma_{1}^{\tau_{2}} [(a, \kappa(1 - x_{1})), b_{2}, c_{2}; \frac{x_{1}}{1 - x_{1}}];$

(3.19) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, b_{2}; x_{1}, x_{2}] = (1 - x_{2})^{-a} 2\Gamma_{1}^{\tau_{1}} [(a, \kappa(1 - x_{2})), b_{1}, c_{1}; \frac{x_{2}}{1 - x_{2}}].$

Proof. The proofs of (3.16) to (3.19) are direct consequences of the definitions (3.2) and (3.4).

Corollary 1. Each of the following relationship with the incomplete Gauss hypergeometric function $2\Gamma_{1}(z)$ in (1.5) holds true for $\kappa \geq 0$:

(3.20) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, c_{2}; x_{1}, 0] = 2\Gamma_{1} [(a, \kappa), b_{1}; c_{1}; x_{1}];$

(3.21) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, c_{2}; 0, x_{2}] = 2\Gamma_{1} [(a, \kappa), b_{2}; c_{2}; x_{2}];$

(3.22) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, c_{2}; x_{1}, x_{2}] = (1 - x_{1})^{-a} 2\Gamma_{1} [(a, \kappa(1 - x_{1})), b_{2}, c_{2}; \frac{x_{1}}{1 - x_{1}}];$

(3.23) $\Gamma_{2}^{\tau_{1}, \tau_{2}} [(a, \kappa), b_{1}, b_{2}; c_{1}, b_{2}; x_{1}, x_{2}] = (1 - x_{2})^{-a} 2\Gamma_{1} [(a, \kappa(1 - x_{2})), b_{1}, c_{1}; \frac{x_{2}}{1 - x_{2}}].$
4. Concluding remarks and observations

In our present investigation, with the help of the incomplete Pochhammer symbols \( (\lambda; \kappa)_\nu \) and \( [\lambda; \kappa]_\nu \), we have introduced the incomplete generalized \( \tau \)-hypergeometric function \( 2 \gamma_2^\tau (z) \) and \( 2 \Gamma_2^\tau (z) \) and investigated their diverse properties such mainly as integral representations, derivative formula, Euler-beta transform and obtain Riemann-Liouville fractional integration and differentiation formulæ. Further, we have introduced and investigated the family of incomplete second \( \tau \)-Appell hypergeometric functions \( \Gamma^{\tau_1, \tau_2}_2 \) and \( \gamma^{\tau_1, \tau_2}_2 \) of two variables. The special cases of the results presented here when \( \kappa = 0 \) would reduce to the corresponding well-known results for the generalized \( \tau \)-hypergeometric function (see, for details, \([10, 12, 18, 41, 42]\)) and second \( \tau \)-Appell function (see, for details, \([2]\)).

The expressions of the integrals, which we have evaluated in this paper, are (presumably) new and generalize the results in the existing literature (see, for details, \([4, 23, 31, 32]\)).

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References

[1] M. Abramowitz and I. A. Stegun (Editors), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, 55 For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964.

[2] A. H. Al-Shammery and S. L. Kalla, *An extension of some hypergeometric functions of two variables*, Rev. Acad. Canaria Cienc. 12 (2000), no. 1-2, 189–196.

[3] L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Company, New York, 1984.

[4] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert-Hafner, New York, 1964.

[5] Yu. A. Brychkov and N. Saad, *On some formulas for the Appell function \( F_2[a, b, b'; c, c'; w, z] \)*, Integral Transforms Spec. Funct. 25 (2014), no. 2, 111–123.

[6] A. Ç etinkaya, *The incomplete second Appell hypergeometric functions*, Appl. Math. Comput. 219 (2013), no. 15, 8332–8337.

[7] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall (CRC Press Company), Boca Raton, London, New York and Washington, D.C., 2001.

[8] J. Choi, R. K. Parmar, and P. Chopra, *The incomplete Lauricella and first Appell functions and associated properties*, Honam Math. J. 36 (2014), no. 3, 531–542.

[9] M. Dotsenko, *On some applications of Wright’s hypergeometric function*, C. R. Acad. Bulgare Sci. 44 (1991), no. 6, 13–16.

[10] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions, Vol. I*, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[11] L. Galué, A. Al-Zamel, and S. L. Kalla, *Further results on generalized hypergeometric functions*, Appl. Math. Comput. 136 (2003), no. 1, 17–25.
[13] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000.
[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
[15] Y. L. Luke, Mathematical Functions and Their Approximations, Academic Press, New York, San Francisco and London, 1975.
[16] M.-J. Luo and R. K. Raina, Extended generalized hypergeometric functions and their applications, Bull. Math. Anal. Appl. 5 (2013), no. 4, 65–77.
[17] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Third Enlarged edition, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtung der Anwendungsgebiete, Bd. 5, Springer-Verlag, Berlin, Heidelberg and New York, 1966.
[18] V. Malovichko, On a generalized hypergeometric function and some integral operators, Math. Phys. 19 (1976), 99–103.
[19] A. M. Mathai, R. K. Saxena, and H. J. Haubold, The H-Functions: Theory and Applications, Springer, New York, 2010.
[20] K. B. Oldham, J. Myland, and J. Spanier, An Atlas of Functions with Equator, The Atlas Function Calculator, Second edition [With 1 CD-ROM (Windows)], Springer-Verlag, Berlin, Heidelberg and New York, 2009.
[21] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Editors), NIST Handbook of Mathematical Functions, [With 1 CD-ROM (Windows, Macintosh and UNIX)], US Department of Commerce, National Institute of Standards and Technology, Washington, D.C., 2010; Cambridge University Press, Cambridge, London and New York, 2010.
[22] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series, Vol. II, Gordon and Breach Science Publishers, New York, 1990.
[23] E. D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
[24] S. B. Rao, J. C. Prajapati, A. K. D. Patel, and A. K. Shukla, Some properties of Wright-type generalized hypergeometric function via fractional calculus, Adv. Difference Equ. 2014 (2014), 119.
[25] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, 1993.
[26] L. J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, London, and New York, 1966.
[27] H. M. Srivastava, A. Cetinkaya, and İ. O. Kıymaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions, Appl. Math. Comput. 226 (2014), 484–491.
[28] H. M. Srivastava, M. A. Chaudhry, and R. P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms Spec. Funct. 23 (2012), no. 9, 659–683.
[29] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[30] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
[31] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
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[33] H. M. Srivastava, R. K. Parmar, and P. Chopra, A Class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms 1 (2012), 238–258.

[34] H. M. Srivastava and Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl. Math. Comput. 211 (2009), no. 1, 198–210.

[35] R. Srivastava, Some properties of a family of incomplete hypergeometric functions, Russian J. Math. Phys. 20 (2013), no. 1, 121–128.

[36] R. Srivastava, Some generalizations of Pochhammer’s symbol and their associated families of hypergeometric functions and hypergeometric polynomials, Appl. Math. Inform. Sci. 7 (2013), no. 6, 2195–2206.

[37] R. Srivastava, Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions, Appl. Math. Comput. 243 (2014), 132–137.

[38] R. Srivastava and N. E. Cho, Generating functions for a certain class of incomplete hypergeometric polynomials, Appl. Math. Comput. 219 (2012), no. 6, 3219–3225.

[39] R. Srivastava, Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials, Appl. Math. Comput. 234 (2014), 277–285.

[40] N. Temme, Special Functions: An Introduction to Classical Functions of Mathematical Physics, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1996.

[41] N. Virchenko, On some generalizations of the functions of hypergeometric type, Fract. Calc. Appl. Anal. 2 (1999), no. 3, 233–244.

[42] N. Virchenko, S. L. Kalla, and A. Al-Zamel, Some results on a generalized hypergeometric function, Integral Transf. Spec. Funct. 12 (2001), no. 1, 89–100.

[43] G. N. Watson, A Treatise on the Theory of Bessel Functions, Second edition, Cambridge University Press, Cambridge, London and New York, 1944.

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