On the spectral properties of the Bloch-Torrey equation in infinite periodically perforated domains

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To Ari Laptev on the occasion of his 70th birthday.

Abstract

We investigate spectral and asymptotic properties of the particular Schrödinger operator (also known as the Bloch-Torrey operator), \(-\Delta + igx\), in infinite periodically perforated domains of \(\mathbb{R}^d\). We consider Dirichlet realizations of this operator and formalize a numerical approach proposed in [18] for studying such operators. In particular, we discuss the existence of the spectrum of this operator and its asymptotic behavior as \(g \to \infty\).

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1 Introduction

The aim of this paper is to formalize on the mathematical side a numerical approach proposed in [18] for analyzing the Bloch-Torrey equation in infinite periodically perforated domains. More precisely, we consider the Dirichlet realization of the Bloch-Torrey operator

\[ B := -\Delta + igx, \]

(1.1)
denoted respectively by \(B^D\) where \(\Delta\) is the Laplace operator, \(x\) is one of the Cartesian coordinates, and \(g\) is a real nonzero parameter. The Neumann and Robin realizations, which are important in physical applications [7, 8, 9, 18], could also be treated by the...
same techniques but the details of the proof are omitted in the present paper. The major novelty of this work (as compared to former studies in [10, 11, 2, 3]) is that we consider the Bloch-Torrey operator in a periodically perforated domain \((d \geq 2)\),
\[
\Omega = \mathbb{R}^d \setminus \{ \cup_{y \in \mathbb{Z}^d} H_y \},
\]
where
\[
H_y = \{ x \in \mathbb{R}^d : x - y \in H_0 \}
\]
and \(H_0 \subset (-1/2, 1/2)^d\) is a domain with a smooth boundary.

A typical example is the case when \(H_y = B_2(y, r)\) where \(B_2(y, r)\) is the disk of radius \(r < \frac{1}{4}\) centered at \(y\). One of the major difficulties in the definition and study of such non-self-adjoint operators is that the potential \(igx\) is not periodic, unbounded and changing sign.

The Bloch-Torrey operator describes the diffusion-precession of spin-bearing particles in nuclear magnetic resonance experiments and helps in understanding the intricate relation between the geometric structure of a studied sample (domain) and the measured signal [7, 8, 9]. The spectral properties of this operator and its asymptotic behavior play thus a crucial role in this analysis.

The paper is organized as follows. In Sec. 2, we provide a rigorous definition of the considered realization of the Bloch-Torrey operator and describe its basic properties. In Sec. 3, we use the periodicity in \(y\)-direction via the Floquet theory to reduce the operator on the planar perforated domain to a family (indexed by a Floquet parameter) of operators on the infinite perforated cylinder. In particular, we formulate here two conjectures about their spectral properties. Section 4 analyzes the role of the pseudo-periodicity in \(x\)-direction, which is specific to the Bloch-Torrey operator. In Sec. 5, we formulate in an asymptotic regime the main results concerning the non-emptiness of the spectrum and the asymptotic properties. Finally, Sec. 6 concludes the paper and discusses the extensions in the proofs to more general settings.

## 2 The Bloch-Torrey operator in the perforated whole plane

We start from the Dirichlet realization of the Bloch-Torrey operator
\[
B_g := -\Delta_{x,y} + igx,
\]
in \(\Omega = \mathbb{R}^2 \setminus \cup_{y \in \mathbb{Z}^2} H_y\) as defined in (1.2)-(1.3) and where \(g\) is a non zero real parameter. We introduce
\[
\mathcal{H} := L^2(\Omega) \quad \text{and} \quad \mathcal{V} := \{ u \in H_0^1(\Omega) : |x|^{\frac{1}{2}} u \in L^2(\Omega) \}
\]
and we can now extend the operator initially defined on \(C_0^\infty(\Omega)\) to get a closed operator on \(\mathcal{H}\) by the following variant of the Friedrichs extension.
**Proposition 2.1.** The Dirichlet realization $\mathcal{B}_g^D$ of $B_g$ with domain

$$D(\mathcal{B}_g^D) = \{ u \in \mathcal{V} \, , \, \mathcal{B}_g^D u \in \mathcal{H} \}$$

is a closed accretive operator which generates a continuous semi-group on $L^2(\Omega)$.

**Proof.** We take

$$\mathcal{H} := L^2(\Omega) \text{ and } \mathcal{V} := \{ u \in H^1_0(\Omega) \, , \, |x|^{\frac{1}{2}} u \in L^2(\Omega) \}$$

and apply Theorem A.1 to the quadratic form

$$a(u, u) = \int_{\Omega} |\nabla u|^2 \, dx \, dy + i \int_{\Omega} x |u|^2 \, dx \, dy + C_0 \|u\|^2,$$

where $\|u\|$ denotes the $L^2(\Omega)$ norm and $C_0 > 0$ is a large positive constant to be determined.

For $\Phi_1 = \Phi_2$ we take the multiplication operator by $x/\sqrt{1 + x^2}$.

We simply then observe that

$$\Re a(u, u) = \int_{\Omega} |\nabla u|^2 \, dx \, dy + C_0 \|u\|^2,$$

and then consider $\Im a(u, \Phi_1(u))$.

By an easy computation, we can show the existence of $C > 0$ such that, for all $u \in \mathcal{V}$,

$$\Im a(u, \Phi_1(u)) \geq \int_{\Omega} x^2 (1 + x^2)^{-1/2} |u|^2 \, dx \, dy - C \|u\|^2_{H^1(\Omega)}$$

Hence the assumptions of the theorems are satisfied if we choose $C_0$ sufficiently large.

What we call $\mathcal{B}_g^D$ is the operator $S - C_0$ where $S$ is given by Theorem A.1. To get the semi-group property, we then apply the Hille-Yosida Theorem. \( \square \)

### 3 Floquet approach for $\gamma$-periodic problems

The goal is now to analyze the spectrum of this operator and to possibly consider asymptotic problems in function of $g$. Starting with a qualitative analysis, we omit the reference to $g > 0$.

#### 3.1 Floquet decomposition

Let $\tau_2$ be the translation by $(0, 1)$, i.e.

$$\tau_2 u(x, y) = u(x, y - 1) \text{ for } u \in L^2(\Omega).$$

Observing that

$$\tau_2 \circ \mathcal{B}^D = \mathcal{B}^D \circ \tau_2,$$

(3.1)
we can, at least formally, apply a Floquet theorem in the $y$ variable and get the family of operators ($q \in \mathbb{R}$)

$$B_{g,q} := -\left(\frac{d}{dy} - iq\right)^2 - \frac{d^2}{dx^2} + igx,$$

in

$$\Omega_0 = \left(\mathbb{R} \times (-1/2, 1/2)\right) \setminus \{\cup_{n \in \mathbb{Z}} H_{(n,0)}\},$$

where we put the Dirichlet condition at the boundary of each $H_{(n,0)}$ and the periodicity condition on the remaining boundary

$$u(x,-1/2) = u(x,1/2), \quad \partial_y u(x,-1/2) = \partial_y u(x,1/2), \quad \forall x \in \mathbb{R}. \quad (3.3)$$

Nevertheless the best way is to consider it as an operator on the infinite perforated cylinder

$$\hat{\Omega}_0 := \left(\mathbb{R} \times \mathbb{T}^1\right) \setminus \{\cup_{n \in \mathbb{Z}} H_{(n,0)}\},$$

where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ with Dirichlet condition on $\partial\hat{\Omega}_0$.

### 3.2 Spectral analysis

We now study the spectral properties of the Dirichlet realization $B_d^D$ of $B_{g,q}$. We omit from now on the reference to $g > 0$ which is unimportant for the qualitative analysis.

**Proposition 3.1.** For any $q \in \mathbb{R}$, we can extend $B_q$ as a closed operator $B_d^D$ with the following properties:

1. $B_d^D$ is a closed accretive operator on $L^2(\Omega_0)$ and is the generator of a continuous semi-group.

2. $B_d^D$ has compact resolvent.

**Proof.** The proof is the same as for Proposition 2.1 but this time we take

$$\mathcal{H}_0 := L^2(\Omega_0) \text{ and } \mathcal{V}_0 := \{u \in H^1(\Omega_0), |x|^{1/2}u \in L^2(\Omega_0) \text{ and } (BC) \text{ holds}\},$$

where by $(BC)$ we mean that $u$ satisfies the Dirichlet condition at the boundary of the $H_{(0,n)}$ and that we have the periodicity condition $u(x, -\frac{1}{2}) = u(x, \frac{1}{2})$ for $x \in \mathbb{R}$.

Actually, it is better to deal with $\hat{\Omega}_0$ as mentioned above and to consider

$$\hat{\mathcal{H}}_0 := L^2(\hat{\Omega}_0) \text{ and } \hat{\mathcal{V}}_0 := \{u \in H^1_0(\hat{\Omega}_0), |x|^{1/2}u \in L^2(\hat{\Omega}_0)\}.$$

The compact resolvent property is a consequence of the compact injection of $\mathcal{V}_0$ in $\mathcal{H}_0$. 

Noting the inequality
\[ \Re \langle B_q^D u, u \rangle \geq 0, \quad \forall u \in \mathcal{V}_0, \]
we can construct by the Hille-Yosida theorem the associated continuous semi-group and consider (as in [18])
\[ \mathbb{R}_+ \ni t \mapsto \exp(-tB_q^D). \]

\[ \Box \]

### 3.3 Application of Floquet decomposition to spectral theory

The interest of the Floquet theory is to relate the spectrum of the initial operator to the union of the spectra of a family of simpler operators. It is based on the so-called Floquet decomposition whose definition is independent of the operator but is only related to the symmetry group property satisfied by the operator. The application to spectral theory is standard in the self-adjoint case (see for example in Reed-Simon [19]) but we are not aware of a reference for the application in the non-self-adjoint case (see nevertheless [16, 17]). Therefore it is not completely clear that the proof easily goes on in the non-self-adjoint case and this is why we present in this subsection an approach covering this situation. Our main result is:

**Proposition 3.2.**

\[ \bigcup_{q \in \mathbb{R}} \sigma(B_q^D) \subset \sigma(B^D), \quad (3.4) \]

where, for \( A \subset \mathbb{C} \), \( \overline{A} \) denotes the closure of \( A \).

**Proof.**

**Step 1** We first observe that \( B_q^D \) is unitarily equivalent to \( B_{q+2\pi}^D \). Hence it is enough to consider the problem for \( q \in [0, 2\pi] \).

**Step 2** This is a simple version of the Schnoll theorem relating the spectrum and the existence of generalized polynomially bounded eigenfunctions. Let \( q \in \mathbb{R} \). We know from Proposition 3.1 that \( \sigma(B_q^D) \) is either empty or consists of eigenvalues with finite multiplicity. There is nothing to prove if \( \sigma(B_q^D) \) is empty. Let us assume now that it is not empty and let \( \mu \) be an eigenvalue of \( B_q^D \) associated to a normalized eigenfunction \( u_\mu(x,y) \). We extend \( u_\mu \) as a periodic function in the whole domain \( \Omega \) and consider a cutoff function \( \chi \) which equals 1 on \((-\frac{1}{2}, \frac{1}{2})\) with support in \((-1, 1)\).

We then consider
\[ u_{\mu,n}(x,y) := e^{i\gamma q \chi(y/n)}u_\mu(x,y). \]

We have, for \( n \in \mathbb{N}^* \),
\[ ||u_{\mu,n}||^2_{L^2(\Omega)} \geq \int_{\Omega \cap \{|y| < \frac{1}{2}\}} |u_\mu|^2 dxdy = n \int_{\Omega \cap \{|y| < \frac{1}{2}\}} |u_\mu|^2 dxdy = n, \quad (3.5) \]
and
\[ f_n := (B^D - \mu) u_{\mu,n} = -\frac{2}{n} \chi'(y/n) \partial_y u_\mu(x,y) - \frac{1}{n^2} \chi''(y/n) u_\mu(x,y). \]

It is immediate to see that there exists a constant \( C \) such that
\[ \|f_n\|_{L^2(\Omega)} \leq C/\sqrt{n}. \]

If \( \mu \) was not in the spectrum of \( B^D \), we would have
\[ u_{\mu,n} = (B^D - \mu)^{-1} f_n = O(1/\sqrt{n}), \]
in contradiction with (3.5).

We know that in the self-adjoint case (i.e. the case of the Schrödinger operator with real periodic potential) the converse inclusion holds true. We notice in our situation two conjectures.

**Conjecture 3.3.**
\[ \bigcup_{q \in \mathbb{R}} \sigma(B^D_q) = \sigma(B^D). \] (3.6)

We will see below that this conjecture is a consequence of the second one:

**Conjecture 3.4.** \( B^D \) has the following property:
\[ \rho(B^D) = \mathbb{C}, \] (3.7)

where \( \rho(B^D) \) denotes the resolvent set of \( B^D \).

**Remarks.**

1. Property (3.7) would have been automatically satisfied if \( B^D \) was self-adjoint (because the spectrum would have been real).

2. The proof of the second conjecture (which would imply the first one as can be seen from the proof below) does not seem at the moment easier as the first one.

3. The validity of Conjecture 3.4 would probably be quite specific of the Bloch-Torrey operator and of the case \( d = 2 \). For instance, we suspect that the resolvent set of the operator \(-\Delta + \cos x + i \cos y\) in \( \mathbb{R}^2 \) has not this property.

We indeed suspect that the spectrum can be described by a Floquet theory as \( \bigcup_{p,q,\lambda_1(p) + \lambda_2(q)} \), where \( \lambda_1(p) \) is the Floquet eigenvalue of \(-\frac{d^2}{dx^2} + \cos x\) and \( \lambda_2(q) \) is the Floquet eigenvalue of \(-\frac{d^2}{dy^2} + i \cos y\) which is not real valued.
4. It could be easier to show that the set
\[ \Omega^R := \mathbb{C} \setminus (\cup_{q \in [0, 2\pi]} \sigma(\mathcal{B}_q^D)) , \]
satisfies
\[ \overline{\Omega^R} = \mathbb{C} . \]
The set \( \cup_{q \in \mathbb{R}} \sigma(\mathcal{B}_q^D) \) seems indeed a union of piecewise analytic curves in \( \mathbb{C} \). We do not know if this property will lead to the proof of Conjecture 3.3. Again this question would be specific of the case \( d = 2 \).

5. One could imagine that (for \( g \) large) some of these curves \( [0, 2\pi] \ni q \mapsto \lambda_k(q) \) are simple closed curves in \( \mathbb{C} \) implying the non-connectedness of \( \Omega^R \). This situation is of course excluded in the self-adjoint case.

We now prove a weaker version of Conjecture 3.3 for which we need more definitions. Observing that all our operators are accretive, we immediately see that
\[ \mathbb{C}_- := \{ z \in \mathbb{C} , \Re z < 0 \} \subset \rho(\mathcal{B}^D) \cap \Omega^R . \] (3.8)

We now denote by \( \omega_D \) the connected component in \( \Omega_D := \rho(\mathcal{B}^D) \) containing \( \mathbb{C}_- \) and by \( \omega_R \) the connected component in \( \Omega^R \) containing \( \mathbb{C}_- \).

From Proposition 3.2 we know that
\[ \omega_D \subset \omega_R . \]

We then prove:

**Proposition 3.5.** \( \omega_D = \omega_R \).

**Proof.** By Floquet theory, there is a unitary transform \( \mathcal{U} \) such that
\[ \mathcal{U}^{-1}(\mathcal{B}^D - \lambda)^{-1}\mathcal{U} = \int_0^{2\pi} (\mathcal{B}_q^D - \lambda)^{-1} dq . \] (3.9)

It is clear that this formula holds if \( \lambda \notin \sigma(\mathcal{B}^D) \) but we will show that, under assumption (3.7) this can be extended if the right hand side is well defined.

Suppose indeed that \( \lambda \notin \cup_{q \in [0, 2\pi]} \sigma(\mathcal{B}_q^D) \). Hence \( (\mathcal{B}_q^D - \lambda)^{-1} \) exists for any \( q \in [0, 2\pi] \) and we have to verify (to give a meaning of the right hand side in (3.9)) that we have a uniform control with respect to \( q \). For this we observe that
\[ (\mathcal{B}_q^D - \lambda) \circ (\mathcal{B}_q^D - \lambda)^{-1} = I + 2i(q - q_0)\partial_y(\mathcal{B}_q^D - \lambda)^{-1} + (q^2 - q_0)^2(\mathcal{B}_q^D - \lambda)^{-1} . \]

For \( |q - q_0| \) small enough this implies
\[ (\mathcal{B}_q^D - \lambda)^{-1} = (\mathcal{B}_q^D - \lambda)^{-1}(I + 2i(q - q_0)\partial_y(\mathcal{B}_q^D - \lambda)^{-1} + (q^2 - q_0)^2(\mathcal{B}_q^D - \lambda)^{-1})^{-1} . \]

By compactness, we get the uniformity and define an holomorphic \( L^2 \)-valued function \( \tilde{R}(\lambda) \) extending the resolvent \( R(\lambda) \) to the open set \( \Omega^R \). Let us assume by contradiction
that $\omega_D$ is strictly included in $\omega_R$. Let us consider $\lambda_0 \in \omega_R \setminus \omega_D$. Using Lemma B.1 and the remark there, we can construct a sequence $u_n$ of normalized functions such that $(B^D - \lambda_0)u_n$ tends to zero. Now we consider the holomorphic functions $\omega_R \ni \lambda \mapsto v_n(\lambda) = \tilde{R}(\lambda)(B^D - \lambda)u_n$. On $\omega_D$, one has $v_n(\lambda) = u_n$, therefore $v_n(\lambda)$ is constant and equal to $u_n$ over $\omega_R$. In particular, $||v_n(\lambda)|| = 1$ for all integer $n$ and all $\lambda \in \omega_R$. However, one has $||v_n(\lambda_0)|| = ||\tilde{R}(\lambda_0)(B^D - \lambda_0)u_n|| \to 0$, hence we get a contradiction. □

**Remark.** Note that the proposition implies that $\partial \omega_D = \partial \omega_R$. One could hope that $\partial \omega_R = \partial \Omega^R$ which would correspond to the guess that all the other components of $\Omega^R$ are bounded by spectral curves (as mentioned above).

4 The Bloch-Torrey operator in a perforated cylinder continued.

Here the analysis is quite specific of the Bloch-Torrey operator.

4.1 The pseudo-invariance in the $x$-variable

Let $\tau_1$ be the translation by 1 in the $x$ variable, which is defined for $u \in L^2_{loc}(\overline{\Omega_0})$ by

$$\tau_1 u(x, y) = u(x - 1, y).$$

Using this translation, one obtains that:

If $\lambda \in \sigma(B^D_q)$ then $\lambda + ig \in \sigma(B^D_q)$.

(4.1)

This indeed results from the commutation relation

$$\tau_1 \circ B^D_q = (B^D_q - ig) \circ \tau_1.$$  

(4.2)

Here it is important to note in the argument that we have used that

$$\tau_1 \Omega_0 = \Omega_0.$$

**Remark.** This commutation relation does not permit to use the standard Floquet theory (see [19, 16, 17, 22]). We can try to recover this invariance by considering on the Hilbert space $\oplus_{n \in \mathbb{Z}} H_n$ (with $H_n$ isomorphic to $L^2(\Omega_0)$) the operator $\oplus_{n \in \mathbb{Z}} (\mathcal{B}^D_q + i gn)$. If we denote by $\tau_0$ the translation on $\ell^2(\mathbb{Z})$, we obtain the commutation of this operator with respect to $\tau_0^{-1} \tau_1$. One can then perform a Floquet decomposition in this context. This will give another way to obtain our next results but not more.

Using Proposition 3.1 (second assertion), we get
Proposition 4.1. For any \( q \in \mathbb{R} \), there is a discrete sequence (finite, infinite or possibly empty) of eigenvalues \( \lambda_k(q) \in \mathbb{R}_+ + i g (-\frac{1}{2}, \frac{1}{2}) \) such that the spectrum is given by

\[
\sigma(\mathcal{B}_q^D) = \cup_{k,n} (\lambda_k(q) + i g n).
\]

Moreover, if \( u \) is an eigenfunction of \( \mathcal{B}_q^D \) for \( \lambda_k(q) \), then \( \tau_1^n u \) is an eigenfunction for \( \lambda_{k,n}(q) := \lambda_k(q) + i g n \).

Nevertheless it remains to analyze these sequences \( \lambda_k(q) \) and to determine in particular if the spectrum is not empty (a question which is specific of the non-self-adjoint situation).

We now consider the associate semi-group. We deduce from (4.2)

\[
\tau_1 \circ \exp(-t \mathcal{B}_q^D) = \exp(i t g) \exp(-t \mathcal{B}_q^D) \circ \tau_1.
\]

Choosing

\[ t_g = \frac{2\pi}{g}, \]

we get

\[
\tau_1 \circ \exp(-t_g \mathcal{B}_q^D) = \exp(-t_g \mathcal{B}_q^D) \circ \tau_1.
\]

The idea is now to consider the spectrum of the operator

\[
K_{g,q} := \exp(-t_g \mathcal{B}_q^D),
\]

and to perform an adapted Floquet decomposition for this operator since we have recovered the commutation relation.

A natural question is then to ask if \( K_{g,q} \) is a compact operator on \( L^2(\hat{\Omega}_0) \). This would be clear if \( A := \mathcal{B}_q^D \) was self-adjoint or more generally sectorial. In this case, one can indeed prove that for any \( T > 0 \), \( t A \exp(-t A) \) is bounded uniformly for \( t \in (0, T) \) (see for example [6], Chapter 2 and references therein). Hence the operator \( \exp(-t A) = (A + \lambda_0)^{-1} \left((A + \lambda_0) \exp(-t A)\right) \) is compact as the composition of a compact operator and a bounded operator. But \( \mathcal{B}_q^D \) is not sectorial (at least if the spectrum is not empty) due to (4.3).

On the other hand, the guess is that its spectrum is given by

Conjecture 4.2.

\[
\sigma(K_{g,q}) \setminus \{0\} = \cup_k \exp(-t_g \lambda_k(q)).
\]

The first result, which may be non intuitive, is:

Proposition 4.3. If \( K_{g,q} \) is a compact operator, then the spectrum of \( \mathcal{B}_q^D \) is empty.

Proof.
Step 1 If \( \lambda_k \) is an eigenvalue of \( B^D_q \) and \( u_k \) is the corresponding normalized eigenfunction, then \( u_k \) is an eigenfunction of \( \exp(-t g B^D_q) \) associated with \( \lambda = \exp(-t g \lambda_k) \). This proves at least that:

\[
\bigcup_k \exp(-t g \lambda_k(q)) \subset \sigma(K_{g,q}) \setminus \{0\}.
\]

Step 2 By (4.5) \( \tau_1 u_k \) is also an eigenfunction. More generally \( \tau_1^n u_k \) is an eigenfunction for any \( n \in \mathbb{Z} \).

It remains to show that the vector space \( \mathcal{U} \) formed by the \( \tau_1^n u_k \) is not of finite dimension which will give a contradiction to the compactness.

Suppose by contradiction that \( \mathcal{U} \) is of finite dimension \( N_0 \), then there exists \( N_1 \) such that this space is generated by the \( \tau_1^n u_k \) for \( |n| \leq N_1 \). Hence (uniform localization of a normalized element belonging to a finite subspace in \( L^2(\Omega_0) \)) for any \( \epsilon > 0 \), there would exist \( R > 0 \) such that

\[
||\tau_1^n u||_{L^2((-R,R) \times (-\frac{1}{2},\frac{1}{2}) \cap \Omega_0)} \geq 1 - \epsilon, \quad \forall n \in \mathbb{Z}.
\]

But this cannot be true as \( |n| \to +\infty \), because, for fixed \( R \), \( ||\tau_1^n u||_{L^2((-R,R) \times (-\frac{1}{2},\frac{1}{2}) \cap \Omega_0)} \) tends to 0.

Remark. In the self-adjoint case, we could have directly used that the eigenfunctions corresponding to different eigenvalues are orthogonal. This is not the case here. In the case \( q = 0 \), we can observe that if \( u \) is an eigenfunction of \( B^D_q \) associated with \( \lambda \), then \( \tilde{u} \) is an eigenfunction of the realization of \( -\Delta - ix \) for the eigenvalue \( \tilde{\lambda} \). Hence, we get

\[
\langle u_{k,n}, \tilde{u}_{k,n'} \rangle = 0, \quad \text{for} \ n \neq n'.
\]

Remark. In the case without holes, the emptiness of the spectrum for the operators

\[
-\frac{d^2}{dx^2} + ig x - \frac{d^2}{dy^2} \text{ in } \mathbb{R}^2 \text{ or } \mathbb{R} \times (-\frac{1}{2},\frac{1}{2})
\]

is known for a long time (see [12] and references therein). Nevertheless our guess is that this is not always the case for our perforated situation (see for example [3] for the exterior problem).

4.2 Floquet decomposition in the \( x \)-direction.

The authors of [18] analyze directly the spectrum of \( K_{q,g} \). Observing as in [18] that \( K_{q,g} \) commutes with the translation \( \tau_1 \), we now consider a Floquet decomposition (relative to the translation in \( x \)). We recall from this decomposition that, if \( u \) is in \( L^2(\Omega_0) \), then

\[
u_{p} := \sum_{n} e^{-inp} \tau_1^n u \quad (4.10)
\]

satisfies the \( p \)-Floquet condition

\[
\tau_1 u_p = e^{ip} u_p,
\]
is well defined in $L^2_{loc}$ and satisfies
\[
\int_{\Omega_0} |u(x, y)|^2 \, dx \, dy = \int_{(0, 2\pi)} ||u_p(x, y)||^2_{L^2(\Omega_0 \cap (-\frac{1}{2}, \frac{1}{2})^2)} \, dp \, .
\] (4.11)

Conversely, if we have a measurable family (with respect to $\tilde{\mathcal{D}}$) of functions $u_p$ in $L^2(\Omega_0)$ satisfying the $p$-Floquet condition and such that
\[
\int_{(0, 2\pi)} ||u_p(x, y)||^2_{L^2(\Omega_0 \cap (-\frac{1}{2}, \frac{1}{2})^2)} \, dp ,
\]
then
\[
u = \frac{1}{2\pi} \int_{(0, 2\pi)} u_p(x, y) \, dp ,
\]
is well defined in $L^2(\Omega_0)$ and satisfies (4.11).

Now, if $u$ is an eigenfunction associated with an eigenvalue $\lambda$ of $\mathcal{B}^D_q$, then $u_p$ satisfies
\[
(\exp(-t_\gamma \mathcal{B}_q^D)) u_p = \exp(-t_\gamma \lambda) u_p .
\]

More generally, what one hopes from Floquet theory is that
\[
\sigma(K_{q,g}) \setminus \{0\} = \bigcup_{p \in \mathbb{R}} (\sigma(K_{q,g,p}) \setminus \{0\})
\] (4.12)
where $K_{q,g,p}$ is the restriction of $K_{q,g}$ to the functions satisfying the $p$-Floquet condition with respect to the translation $\tau_1$.

The authors formally observe the following rather surprising property:

**Proposition 4.4.** Suppose that $u_p$ is a Floquet eigenfunction of $K_{q,q,p}$ then, for any $t > 0$, the function
\[
u_{p,t} := \exp(-t\mathcal{B}_{q,p}^D) u_p
\]
satisfies the Floquet condition with parameter $p - t_\gamma$ and
\[
\exp(-t_\gamma \mathcal{B}_{q,p}^D) u_{p,t} = \exp(-t_\gamma \lambda) u_{p,t} .
\]

**Proof.** By (4.4), we have
\[
\tau_1 (\exp(-t\mathcal{B}_{q}^D)) u_p = \exp(it_\gamma) \exp(-t\mathcal{B}_{q}^D) \tau_1 u_p = \exp(-i(p - t_\gamma)) u_{p,t} .
\]

As an application, this justifies at least formally to look numerically at the periodic problem associated with $\exp(-t_\gamma \mathcal{B}_{q}^D)$ and to recover the spectrum of $\mathcal{B}_q$ by considering $-\log \mu/t_\gamma + ig\mathbb{Z}$ (for $\mu \in \sigma(K_{q,g,0})$). Note that at this stage we have only proved (see (4.8)) that
\[
\exp(-t_\gamma \sigma(\mathcal{B}_{q}^D)) \subset \sigma(K_{q,g}) .
\] (4.13)
From Floquet eigenfunctions of $K_{q,g}$ to eigenfunctions of $B^D_q$.

We get from formula (4.10) that for an eigenfunction $u$ of $B^D_q$

$$u = \frac{1}{2\pi} \int_0^{2\pi} u_p \, dp.$$  \hspace{1cm} (4.14)

This formula is standard due to the particular choice of the $u_p$ through formula (4.10). Note for example that

$$\tau_1 u = \frac{1}{2\pi} \int_0^{2\pi} \tau_1 u_p \, dp = \frac{1}{2\pi} \int_0^{2\pi} e^{i p} u_p \, dp.$$  

Hence the infinite dimensional space $\text{Span}(\tau^n_1 u)$ is recovered by $\frac{1}{2\pi} \int_0^{2\pi} \beta_p u_p \, dp$ for some function $\beta_p$.

The next proposition is a kind of converse statement.

**Proposition 4.5.** If $\mu$ is an eigenvalue of $K_{q,p=0}$ with corresponding eigenfunction $u_0$, then $\log \mu + ig\mathbb{Z}$ belongs to the spectrum of $B^D_q$ and, for each $k$, we can construct starting from $u_0$ an eigenfunction $u_{\lambda_k}$ associated with $\lambda_k := \log \mu + igk$.

**Proof.**

**Step 1: The heuristics behind the proof.** Heuristically, we will proceed in the following way. We start from an eigenfunction $u_0$ and associate with it

$$u_p = \exp \left( -\frac{p}{g} (B^D_q - \lambda_0) \right) u_0.$$  \hspace{1cm} (4.15)

Defining $u$ by (4.14), we obtain formally:

$$(B^D_q - \lambda_0) u = \left( \frac{1}{2\pi} \int_0^{2\pi} (B^D_q - \lambda_0) \exp(-\frac{p}{g} (B^D_q - \lambda_0)) \, dp \right) u_0$$

$$= \left( \frac{g}{2\pi} \int_0^{2\pi} \left( -\frac{d}{dp} \exp(-\frac{p}{g} (B^D_q - \lambda_0)) \right) \, dp \right) u_0$$

$$= \frac{g}{2\pi} \left( I - \exp(-\frac{2\pi}{g} (B^D_q - \lambda_0)) \right) u_0$$  \hspace{1cm} (4.16)

We finally observe that $u$ is not 0, we have indeed formally

$$\sum_n \tau^n_1 u = u_0.$$

**Step 2: Mathematical proof**

In Step 1, we have been very formal, omitting in particular the questions relative to the domain for $u$. We now give a rigorous proof. Let us first state the following proposition
**Proposition 4.6.** For any $s \in \mathbb{R}$,

$$\exp(-t B^D_q) L^{2,s} (\Omega_0) \subset L^{2,s} (\Omega_0), \quad \forall t \geq 0,$$

and there exists $\omega_s$ such that

$$|| \exp(-t B^D_q) v ||_{L^{2,s} (\Omega_0)} \leq \exp(\omega_s t) ||v||_{L^{2,s} (\Omega_0)}, \quad \forall v \in L^{2,s} (\Omega_0),$$

where $L^{2,s}$ is the weighted space

$$L^{2,s} (\Omega_0) = \{ v \in L^2_{loc} (\Omega_0), \langle x \rangle^{s/2} v \in L^2 (\Omega_0) \},$$

with $\langle x \rangle = \sqrt{1 + x^2}$.

We can now give a sense to (4.14)-(4.15). We observe that $u_0$ belongs to $L^{2,-s} (\Omega_0)$ for $s > \frac{1}{2}$. This implies by the proposition that $u$ is well defined in particular in $L^{2,-s} (\Omega_0)$.

We now prove that $u$ is actually in $L^2$. For this we use the Floquet theory. We observe that

$$u_p := \exp \left( -\frac{p}{g} (B^D_q - \lambda_0) \right) u_0 \in L^{2,-s},$$

and that $u_p$ satisfies the $p$-Floquet condition and the uniform bound:

$$||u_p||_{L^2 ((0,1)^2 \cap \Omega_0)} \leq \tilde{C} ||u_0||_{L^2 ((0,1)^2 \cap \Omega_0)}, \quad \forall p \in [0, 2\pi],$$

for some constant $\tilde{C} > 0$. This implies, by the reverse Floquet decomposition formula (see around (4.11)), that $u$ is indeed in $L^2 (\Omega_0)$.

We now compute $\exp(-t (B^D_q - \lambda_0)) u$. It is clear using the periodicity of $\hat{s} \mapsto \exp(-\hat{s} (B^D_q - \lambda_0)) u_0$, that

$$\exp(-t (B^D_q - \lambda_0)) u = u \quad \text{in} \ L^{2,-s}, \quad \forall t \in [0, 2\pi / g].$$

By semi-group theory, we immediately get that $u \in D(B^D_q)$ and that $(B^D_q - \lambda_0) u = 0$ as claimed in (4.16). This achieves the proof of Proposition 4.5 modulo the proof of Proposition 4.6.

\[ \square \]

**Proof of Proposition 4.6.** To prove this proposition, we have to analyze the semi-group

$$t \mapsto \langle x \rangle^{s/2} \exp(-t B^D_0) \langle x \rangle^{-s/2},$$

on $L^2 (\Omega_0)$.

Its associated infinitesimal generator is the unbounded closed operator

$$C^D_s := \langle x \rangle^{s/2} B^D_0 \langle x \rangle^{-s/2}.$$
on $L^2(\Omega_0)$. The associated differential operator is, with $\alpha_s := \langle x \rangle^{s/2}$,

$$
-\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + i gx - 2\alpha_s'\alpha_s^{-1} \frac{d}{dx} - \alpha_s(\alpha_s^{-1})'',
$$

which can be rewritten in the form

$$
-\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + igx + a_s(x) \frac{d}{dx} + b_s(x),
$$

where $a_s$ et $b_s$ are bounded.

The introduced perturbation does not change the form domain and using Hille-Yosida theorem and the results of Appendix A, we get the existence of $\omega_s$ such that

$$
||\langle x \rangle^{s/2} \exp(-t\mathcal{B}_0^D) \langle x \rangle^{-s/2}||_{L(L^2(\Omega_0))} \leq \exp(\omega_s t), \quad \forall t > 0.
$$

This proves our proposition.

**Remark.** We guess but have not proved that $K_{q,p=0}$ has only point spectrum, i.e. that its spectrum only consists of eigenvalues. Note that we could think of replacing in the above proof $u_0$ by a sequence of approximate eigenfunctions $u_0^{(n)}$.

Formally (4.16) gives

$$
(B_0^D - \lambda_0)u^{(n)} = \frac{g}{2\pi} \left( I - \exp\left( -\frac{2\pi}{g} (B_0^D - \lambda_0) \right) \right) u_0^{(n)}.
$$

But on the right hand side we have only a sequence of periodic functions which only tends to 0 on a fundamental domain.

Hence, we do not have a Weyl sequence for $B_0^D$ relative to $\lambda_0$. It remains some hope to proceed like in the proof of the Schnoll theorem by introducing cut-off functions.

## 5 Quasi-modes and non-emptiness of the spectrum

We would like to analyze the behavior of the operator as $g \to +\infty$ using the techniques of [14, 2, 3]. We take the semi-classical representation and look in the 2-dimensional case at the asymptotic limit as $h \to 0$ of the spectrum of the semi-classical realization $\mathcal{A}_h^D$ of $-h^2\Delta + ix$.

### 5.1 Main results

The main result is:
Theorem 5.1. Under Assumptions (1.2)-(1.3) and assuming that $H_0$ is a strictly convex set in $\mathbb{R}^2$ with boundary of positive curvature, we have

$$
\lim_{h \to 0} \frac{1}{h^{2/3}} \inf \{ \Re \sigma(\mathcal{A}_h^D) \} = \frac{|a_1|}{2},
$$

(5.1)

where $a_1 < 0$ is the rightmost zero of the Airy function $\text{Ai}$.

Moreover, for every $\varepsilon > 0$, there exist $h_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$
\forall h \in (0, h_\varepsilon), \quad \sup_{\gamma \leq |a_1|, \nu \in \mathbb{R}} \|(\mathcal{A}_h^D - (\gamma - \varepsilon)h^{2/3} - i\nu)^{-1}\| \leq \frac{C_\varepsilon}{h^{2/3}}.
$$

(5.2)

In particular the spectrum of $\mathcal{A}_h^D$ is not empty.

If one now looks at the reduced problem on the cylinder $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$, the main result would be the same for $\mathcal{A}_{h,g}$ with the difference that we know that the spectrum is discrete.

Theorem 5.2. Under the same assumptions as in Theorem 5.1, there exists $\lambda(h, q)$ such that

$$
\lim_{h \to 0} (\lambda(h, q) - ir)h^{-\frac{2}{3}} = |a_1|e^{-i\frac{\pi}{3}}
$$

and such that

$$
\lambda(h, q) + ik \in \sigma(\mathcal{A}_h^{D,g}), \quad \forall k \in \mathbb{Z}.
$$

This result is essentially a reformulation of the results stated by Almog in [1] and in the case of the exterior problem in [3].

Remark. According to [14, 2, 3], similar results can be formulated for Neumann and Robin boundary conditions, by adapting the proofs from [3].

5.2 Proofs.

5.2.1 Lower bound

The proof is identical to the exterior case considered in [3] (Subsection 2.2). The fact that there is an infinite number of holes instead one hole does not change the proof. The assumptions that $V(x, y) = x$ and the strict convexity of $H_0$ permits to verify all the assumptions appearing in this subsection. We recall the main lines with the simplifications that our potential $V$ is simply $V(x, y) = x$.

By lower bound, we mean

$$
\lim_{h \to 0} \frac{1}{h^{2/3}} \inf \{ \Re \sigma(\mathcal{A}_h^D) \} \geq \frac{|a_1|}{2}.
$$

(5.3)
We keep the notation of [2, Section 6] and [3]. For some $1/3 < q < 2/3$ and for every $h \in (0, h_0]$, we choose two sets of indices $\mathcal{J}_i(h)$, $\mathcal{J}_0(h)$, and a set of points such that the closed balls $\bar{\Omega}$ such that, for every $\mathcal{J}_{i}(h)$\footnote{Supp $\chi_{j,h} \subset B(a_j(h), h^\varrho)$ for $j \in \mathcal{J}_i(h)$,}

\begin{equation}
\{ a_j(h) \in \Omega : j \in \mathcal{J}_i(h) \} \cup \{ b_k(h) \in \partial \Omega : k \in \mathcal{J}_0(h) \},
\end{equation}

such that $B(a_j(h), h^\varrho) \subset \Omega$, \begin{equation}
\hat{\Omega} \subset \bigcup_{j \in \mathcal{J}_i(h)} B(a_j(h), h^\varrho) \cup \bigcup_{k \in \mathcal{J}_0(h)} B(b_k(h), h^\varrho),
\end{equation}

and such that the closed balls $\bar{B}(a_j(h), h^\varrho/2)$, $\bar{B}(b_k(h), h^\varrho/2)$ are all disjoint.

Now we construct in $\mathbb{R}^2$ two families of functions

\begin{equation}
(\chi_{j,h})_{j \in \mathcal{J}_i(h)} \quad \text{and} \quad (\zeta_{j,h})_{j \in \mathcal{J}_0(h)},
\end{equation}

such that, for every $x \in \hat{\Omega}$,

\begin{equation}
\sum_{j \in \mathcal{J}_i(h)} \chi_{j,h}(x)^2 + \sum_{k \in \mathcal{J}_0(h)} \zeta_{k,h}(x)^2 = 1,
\end{equation}

and such that

- $\text{Supp } \chi_{j,h} \subset B(a_j(h), h^\varrho)$ for $j \in \mathcal{J}_i(h)$,
- $\text{Supp } \zeta_{j,h} \subset B(b_j(h), h^\varrho)$ for $j \in \mathcal{J}_0(h)$,
- $\chi_{j,h} \equiv 1$ (respectively $\zeta_{j,h} \equiv 1$) on $\bar{B}(a_j(h), h^\varrho/2)$ (respectively $\bar{B}(b_j(h), h^\varrho/2)$).

We now define the approximate resolvent as in [3]

\begin{equation}
\mathcal{R}_h(z) = \sum_{j \in \mathcal{J}_i(h)} \chi_{j,h}(\mathcal{A}_h - z)^{-1} \chi_{j,h} + \sum_{j \in \mathcal{J}_0(h)} q_{j,h} R_{j,h}(z) q_{j,h},
\end{equation}

where $R_{j,h}(z)$ is given by [3, Eq. (6.14)], and $q_{j,h} = 1_{\Omega} \zeta_{j,h}$.

We write

\begin{equation}
\mathcal{R}_h(z) \circ (\mathcal{A}_h^D - z) = I + \mathcal{E}(h, z),
\end{equation}

where

\begin{equation}
\mathcal{E}(h, z) = -h^2 [\Delta, \chi_{j,h}] (\mathcal{A}_h - z)^{-1} \chi_{j,h} + \sum_{j \in \mathcal{J}_0(h)} (\mathcal{A}_h - z) q_{j,h} R_{j,h}(z) q_{j,h}.
\end{equation}

The control can be achieved as in [3]. We may thus conclude that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for sufficiently small $h$

\begin{equation}
\sup_{z \in h^2/3(\{ |z| \} - \varepsilon)} \| \mathcal{E}(h, z) \| \leq C \left( h^{2-2\rho-\frac{2}{3}} + h^{2\rho-\frac{2}{3}} \right).
\end{equation}

Since for sufficiently small $h$, $(I + \mathcal{E}(h, z))$ becomes invertible, we can now use (5.6) to conclude that for any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ and $h_\varepsilon > 0$ such that for any $h \in (0, h_\varepsilon]$

\begin{equation}
\sup_{z \in h^2/3(\{ |z| \} - \varepsilon)} \| (\mathcal{A}_h^D - z)^{-1} \| \leq \frac{C_\varepsilon}{h^{2/3}}.
\end{equation}

This completes the proof of (5.3).
5.2.2 The proof of upper bounds

According to Proposition 3.2, it is enough to prove Theorem 5.2. Without loss of generality it is enough to present the proof for \( q = 0 \). Up to multiplication by \( e^{iyq} \), the quasimode is the same and we can take it localized near the point \( (r, 0) \) (see Formula (7.2) in [2]). To prove that

\[
\lim_{h \to 0} \frac{1}{h^{2/3}} \inf \{ \Re \sigma(A_{h}^{D,q}) \} \leq \frac{|a_1|}{2},
\]

we use the same procedure presented in [2, Section 7].

6 Conclusion

The goal of the paper was to analyze mathematically the approach of [18] for the spectral analysis of the Bloch-Torrey equation in periodic perforated domains in \( \mathbb{R}^2 \).

We have proved that, at least in an asymptotic regime \( (g \to +\infty) \), the presence of holes creates indeed spectrum and that it was natural to detect these eigenvalues by analyzing an associate bounded operator on \( L^2(\mathbb{T}^2) \) through an extension of the Floquet theory.

We have focused on the proofs for the Dirichlet realization \( B^D \) in planar domains. The precise definition of the realizations for the other boundary conditions is treated in detail in the (1D)-case in [10] (see also [11] in (2D)) referring to the same variant of the Lax-Milgram theorem recalled in the appendix. The semi-classical analysis of Section 5 relies on Refs. [14, 4, 2, 3, 11]. Note that Ref. [14] is written for problems without the restriction to the dimension. Nevertheless, as mentioned in the remarks of Subsection 3.3 some of the results and conjectures are specific of the dimension 2.

A Generalized Lax-Milgram Theorem

In this appendix we recall some results established in Almog-Helffer [4]. For other applications of these results we also mention [15] where magnetic Laplacians are considered.

We consider two Hilbert spaces \( \mathcal{V} \) and \( \mathcal{H} \) such that \( \mathcal{V} \subset \mathcal{H} \), and that for some \( C > 0 \) and any \( u \in \mathcal{V} \), we have

\[
\|u\|_{\mathcal{H}} \leq C \|u\|_{\mathcal{V}}.
\]

Suppose further that

\[ \mathcal{V} \text{ is dense in } \mathcal{H}. \]

Consider a continuous sesquilinear form \( a \) defined on \( \mathcal{V} \times \mathcal{V} \):

\[
(u, v) \mapsto a(u, v).
\]
Let

\[ D(S) = \{ u \in \mathcal{V} \mid v \mapsto a(u, v) \text{ is continuous on } \mathcal{V} \text{ in the norm of } \mathcal{H} \} , \quad (A.3) \]

and define the operator \( S : D(S) \to \mathcal{H} \) by

\[ a(u, v) = \langle Su, v \rangle_{\mathcal{H}}, \quad \forall u \in D(S) \text{ and } \forall v \in \mathcal{V} . \quad (A.4) \]

We have the following theorem:

**Theorem A.1.** Let \( a \) be a continuous sesquilinear form satisfying for some \( \Phi_1, \Phi_2 \in \mathcal{L}(\mathcal{V}) \)

\[ |a(u, u)| + |a(u, \Phi_1(u))| \geq \alpha \| u \|_{\mathcal{V}}^2, \quad \forall u \in \mathcal{V} . \quad (A.5) \]

\[ |a(u, u)| + |a(\Phi_2(u), u)| \geq \alpha \| u \|_{\mathcal{V}}^2, \quad \forall u \in \mathcal{V} . \quad (A.6) \]

Assume further that \( \Phi_1 \) and \( \Phi_2 \) extend into continuous linear maps in \( \mathcal{L}(\mathcal{H}) \) and let \( S \) be defined by (A.3)-(A.4). Then

1. \( S \) is bijective from \( D(S) \) onto \( \mathcal{H} \) and \( S^{-1} \in \mathcal{L}(\mathcal{H}) \);
2. \( D(S) \) is dense in both \( \mathcal{V} \) and \( \mathcal{H} \);
3. \( S \) is closed;
4. Let \( b \) denote the conjugate sesquilinear form of \( a \), i.e.

\[ (u, v) \mapsto b(u, v) := \overline{a(v, u)} . \quad (A.7) \]

Let \( S_1 \) denote the closed linear operator associated with \( b \) by the same construction. Then

\[ S^* = S_1 \text{ and } S_1^* = S . \quad (A.8) \]

**Remark.** We recall that the Hille-Yosida theorem (see Theorem 13.22 in [12]) can be applied to the above defined operator \( S \), if we have

\[ \Re \langle Su, u \rangle \geq -C \| u \|^2_{\mathcal{H}}, \quad \forall u \in \mathcal{V} . \]

**B  Spectrum and Weyl’s sequences**

In this section, we adapt to the non-self-adjoint situation a characterization of the spectrum for a self-adjoint operator through Weyl’s sequences.

**Lemma B.1.** Let \( A \) be a closed operator in a Hilbert space \( \mathcal{H} \) which has the property that, for any \( \lambda \in \mathbb{C} \)

\[ (A - \lambda) \text{ injective implies } (A^* - \overline{\lambda}) \text{ injective}. \quad (B.1) \]

Under this assumption, the two assertions are equivalent:
Bloch-Torrey equation

1. \( \lambda \in \sigma(A) \).

2. There exists a sequence \( u_n \) in \( D(A) \) such that \( ||u_n|| = 1 \) and \( (A - \lambda)u_n \to 0 \).

Proof. The only difficulty is to prove that (1) implies (2). If there is no sequence \( (u_n) \in D(A) \) such that (2) is satisfied, then there exists \( c > 0 \) such that

\[ ||(A - \lambda)u|| \geq c||u||, \quad \forall u \in D(A). \]

From this we deduce that \( (A - \lambda) \) is injective with closed range. Now we have

\[ \text{Range}(A - \lambda) = \left( \text{Ker}(A^* - \bar{\lambda}) \right)^\perp. \]

By (B.1) \( (A^* - \bar{\lambda}) \) is injective. Hence we get the surjectivity. \( \square \)

Remarks.

- The property is evidently satisfied in the self-adjoint case because the spectrum is real.

- The property (B.1) is satisfied if \( H \) is a complex Hilbert space and if there is an antilinear involution \( \Gamma \) such that \( \Gamma D(A) \subset D(A) \) and

\[ \Gamma A = A^* \Gamma. \quad \text{(B.2)} \]

In particular it holds for our Bloch-Torrey operators by taking as \( \Gamma \) the complex conjugation.

- The lemma is not true without property (B.1). As suggested by the referee, one can indeed consider \( H = L^2(0, 1), A = -i\frac{d}{dx}, D(A) = H^1_0(0, 1) \). We have \( \sigma(A) = \mathbb{C} \) and there is no a Weyl sequence for any \( \lambda \in \mathbb{C} \). On the other hand \( (A - \lambda) \) is injective for any \( \lambda \) and \( (A^* - \bar{\lambda}) \) is non-injective for any \( \lambda \).

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