Real and Complex Supersymmetric $d = 1$ Sigma Models With Torsions

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ABSTRACT

We derive and discuss, at both the classical and the quantum levels, generalized $\mathcal{N} = 2$ supersymmetric quantum mechanical sigma models describing the motion over an arbitrary real or an arbitrary complex manifold with extra torsions. We analyze the relevant vacuum states to make explicit the fact that their number is not affected by adding the torsion terms.
1 Introduction

Sigma model is a theory where the configuration space on which the dynamic variables are defined is not the flat space $\mathbb{R}^D$, but represents a nontrivial target manifold of dimension $D$. The number of physical space-time coordinates can vary. In the simplest case of mechanical system, all variables depend only on time. In the absence of external fields, the bosonic Lagrangian of such $d = 1$ sigma model is then given by

$$L^{bos} = \frac{1}{2} g_{MN}(x) \dot{x}^M \dot{x}^N.$$  \hfill (1.1)

It describes the free motion over the manifold with coordinates $x^M$, $M = 1, \ldots, D$, and the metric $g_{MN}(x)$.

The Lagrangian (1.1) can be supersymmetrized in different ways, yielding, after quantization, various versions of supersymmetric quantum mechanics (SQM). One can, e.g., introduce $D$ real $\mathcal{N} = 1$ superfields $X^M = x^M + i\theta \psi^M$, and write the action \[1\]

$$S = \frac{i}{2} \int d\theta d\bar{\theta} d\bar{\psi} d\psi g_{MN}(X) \mathcal{D}X^M \dot{X}^N$$

$$= \frac{1}{2} \int dt g_{MN}(x)(\dot{x}^M \dot{x}^N + i\psi^M \nabla \psi^N),$$  \hfill (1.2)

with $\mathcal{D}^2 = \partial_\theta - i\theta \partial_\bar{\psi}$, $g_{MN} = g_{NM}$ and $\nabla \psi^N = \dot{\psi}^N + \Gamma^N_{PQ} \dot{x}^P \psi^Q$. The quantum version of the corresponding supercharge can be associated with the Dirac operator $\slashed{D}$.

Another possibility is to introduce the real $\mathcal{N} = 2$ superfields with twice as many fermion degrees of freedom,\[2\]

$$X^M = x^M + \theta \psi^M + \bar{\psi}^M \bar{\theta} + F^M \theta \bar{\theta},$$  \hfill (1.4)

The action can be then written as \[4\]

$$S = -\frac{1}{2} \int dt d\bar{\theta} d\psi \dot{x}^M \dot{X}^N$$

$$= \frac{1}{2} \int dt \left[ g_{MN}(x)(\dot{x}^M \dot{x}^N + i[\psi^M \nabla \psi^N - \nabla \psi^M \dot{\psi}^N]) + R_{MNPQ} \psi^M \psi^N \psi^P \psi^Q \right]$$  \hfill (1.5)

$^1\mathcal{N}$ counts the number of real supersymmetries.

$^2$There exists also another quantum supercharge $\hat{Q} = \slashed{D} \Gamma^{D+1}$ where $\gamma^{D+1} = \prod_A \gamma^A \equiv \prod_A \psi^A (\gamma^A = \psi^M e^A_M)$, i.e. the quantum Hamiltonian enjoys here the $\mathcal{N} = 2$ supersymmetry required to make the spectrum double degenerate. Note that the quantum supersymmetry algebra

$$Q^2 = \hat{Q}^2 = H, \quad \{ Q, \hat{Q} \} = 0$$  \hfill (1.3)

cannot be preserved at the classical level, because the Poisson bracket $\{ Q, \hat{Q} \}_{PB}$ vanishes \[2\]. Thus, we are facing here an interesting phenomenon of the classical anomaly of supersymmetry \[3\] (quantum anomalies when one cannot keep a classical Lagrangian symmetry at the quantum level are, of course, much better known).

$^3$This $\mathcal{N} = 2, d = 1$ multiplet can be conveniently denoted as $(1, 2, 1)$, where the numerals count the numbers of the physical bosonic, physical fermionic and auxiliary bosonic fields \[5\]. In this notation, the previous $\mathcal{N} = 1$ multiplet is $(1, 1, 0)$, and the $\mathcal{N} = 2$ multiplet $(1, 2, 1)$ is split into the direct sum of $\mathcal{N} = 1$ multiplets as $(1, 2, 1) = (1, 1, 0) \oplus (0, 1, 1)$. 

\hspace{1cm}
with

\[ D = \frac{\partial}{\partial \theta} - i \bar{\theta} \frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial \bar{\theta}} + i \theta \frac{\partial}{\partial t}. \quad (1.6) \]

It involves a four-fermion term with the Riemann tensor. While passing in (1.5) to the component action, we have eliminated the auxiliary fields \( F^M \) via their algebraic equations of motion.

It is well known that the system (1.5) has a nice geometric interpretation [6]: the quantum supercharges can be interpreted as the exterior derivative operator \( d \) and its conjugate \( d^\dagger \) of the de Rahm complex.

The Lagrangians (1.2) and (1.5) can be written for an arbitrary manifold. When the manifold is of some special type, supersymmetric sigma models with further extended supersymmetries can be defined. For instance, for a 3-dimensional manifold with conformally flat metric and for the manifolds of dimension \( 3n \) with metrics satisfying certain special conditions, the so-called symplectic \( \mathcal{N} = 4 \) supersymmetric sigma model can be defined [7]. For 5n- dimensional manifolds with the metric satisfying similar conditions supplemented by the harmonicity conditions, one can write an interesting \( \mathcal{N} = 8 \) model [8]. Actually, by now the whole “zoo” of \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \) models is known, in both the manifestly supersymmetric superfield off-shell formulations and the on-shell component ones (see, e.g., [9] and [10], and references therein). The general constraints on the target geometry required for one or another type of extended supersymmetry were given, e.g., in [11], [12] and [13]. The characteristic feature of such geometries is that, in general, they involve torsion, though torsionless geometries are admissible as well. In particular, it is well known that, when the manifold is Kähler, the Lagrangian (1.5) admits a second pair of supercharges [14, 15] and, when it is hyper-Kähler, three extra such pairs exist [16] (extending the supersymmetry up to \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \), respectively). Also, the Lagrangian (1.2) for these two types of the bosonic geometry admits an extension to the \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) supersymmetric ones.

In a recent paper [17], a certain special \( \mathcal{N} = 2 \) SQM sigma model for a generic complex manifold of real dimension \( D = 2n \) was constructed and studied. One can introduce chiral superfields

\[ Z^j = z^j + \sqrt{2} \theta \psi^j - i \theta \bar{\psi} z^j, \quad \bar{Z}^j = \bar{z}^j - \sqrt{2} \bar{\theta} \bar{\psi} \bar{z}^j + i \theta \theta \bar{z}^j, \quad (1.7) \]

\( j, \bar{j} = 1, \ldots, n; \quad DZ = \bar{D} \bar{Z} = 0 \), which describe \( \mathcal{N} = 2, d = 1 \) multiplets \( (2, 2, 0) \). The action was chosen in the form [13]

\[ S = \int dt d^2 \theta \left( \mathcal{L}_\sigma + \mathcal{L}_{\text{gauge}} \right), \]

\[ \mathcal{L}_\sigma = -\frac{1}{4} h_{j\bar{k}}(Z, \bar{Z}) DZ^j \bar{D} \bar{Z}^\bar{k}, \quad \mathcal{L}_{\text{gauge}} = W(Z, \bar{Z}) \quad (1.8) \]

with arbitrary superfunctions \( h_{j\bar{k}} \) (the metric, \( ds^2 = 2h_{j\bar{k}} dz^j \bar{dz}^\bar{k} \)) and \( W \) (the prepotential from which the coupling to the background gauge potential \( A_M = (-i \partial_j W, i \partial_{\bar{j}} W) \) is derived). The component action corresponding to (1.8) involves the torsion terms which
disappear only for Kähler manifolds, when the metric satisfies the constraint $\partial_j h_{ij} = 0$ (and its c.c.). The relevant quantum $\mathcal{N} = 2$ supercharges can be interpreted as the holomorphic exterior derivative $\partial$ and its conjugate $\partial^\dagger$, forming twisted or untwisted Dolbeault complexes.

Each of the actions (1.2), (1.5), and (1.8) can be deformed to include extra torsions. Consider first the action (1.2). It can be deformed by adding a term 

$$S = -\frac{1}{12} \int d\theta dt C_{KLM} D\chi^K D\chi^L D\chi^M .$$  \hspace{1cm} (1.9)

This gives the following component Lagrangian

$$L = \frac{1}{2} g_{MN}(x) \left( \dot{x}^M \dot{x}^N + i \psi^M \hat{\nabla} \psi^N \right) - \frac{1}{12} \partial_P C_{KLM} \psi^K \psi^L \psi^M ,$$  \hspace{1cm} (1.10)

where the covariant derivative $\hat{\nabla}$ involves now the torsionful affine connection

$$\hat{\Gamma}_{K,LM} = \Gamma_{K,LM} + \frac{1}{2} C_{KLM} ,$$  \hspace{1cm} (1.11)

$\Gamma_{K,LM}$ being the standard Christoffel symbol.

The quantum supercharge derived from the action (1.10) with generic $g_{MN}, C_{KLM}$ has the form [18]

$$Q = \psi^M \left[ \Pi_M - \frac{i}{2} \Omega_{M,BC} \psi^B \psi^C \right] + \frac{i}{12} C_{KLM} \psi^K \psi^L \psi^M ,$$  \hspace{1cm} (1.12)

where $\Omega_{M,BC}$ are the standard spin connections satisfying the Cartan-Maurer equation

$$de_A + \Omega_{AB} \wedge e_B = 0 , \text{ whence } \Omega_{M,AB} = e_{AK} (\partial_M e^K_B + \Gamma_{KLM} e^K_L) .$$  \hspace{1cm} (1.13)

The supercharge (1.12) can be interpreted as a torsionful Dirac operator, where the torsions enter with an extra factor $1/3$ [18, 19]. Indeed, the last term in (1.12) can be absorbed into the following redefinition of the spin connection (cf. (1.11))

$$\Omega_{M,BC} \rightarrow \tilde{\Omega}_{M,BC} = e_{BK} (\partial_M e^K_B + \Gamma_{KLM} e^K_L), \quad \tilde{\Gamma}_{K,LM} = \Gamma_{K,LM} + \frac{1}{6} C_{KLM} .$$

The action (1.5) for the $(1, 2, 1)$ multiplet that corresponds to de Rham complex can also be deformed. In the geometrical language, the simplest such deformation [6] is described as

$$d_W \mathcal{O} = d\mathcal{O} - dW \wedge \mathcal{O} ,$$
$$d_W^\dagger \mathcal{O} = d^\dagger \mathcal{O} + \langle dW, \mathcal{O} \rangle ,$$  \hspace{1cm} (1.14)

where $W$ is an arbitrary regular function and $\langle dW, \mathcal{O} \rangle$ stands for the interior product. For a $p$-form $\mathcal{O}$,

$$\langle dW, \mathcal{O} \rangle = p (\partial_M W) \mathcal{O}^M_{M_2 \ldots M_p} dx^{M_2} \wedge \cdots \wedge dx^{M_p} .$$
The deformation (1.14) corresponds to adding the potential term

\[ \int d^2 \theta dt W(X^M) \]  

(1.15)
to the action.

One can consider also a deformation of a different type \[20, 21\]

\[ d_\mathcal{B} \mathcal{O} = d\mathcal{O} - d\mathcal{B} \wedge \mathcal{O}, \]

\[ d^\dagger_\mathcal{B} \mathcal{O} = d^\dagger \mathcal{O} - \langle d\bar{\mathcal{B}}, \mathcal{O} \rangle, \]  

(1.16)

where \( \mathcal{B} \) is a regular 2-form (generically, complex) and \( \langle d\bar{\mathcal{B}}, \mathcal{O} \rangle \) is the interior product involving the contraction of all the indices in \( d\bar{\mathcal{B}} \). When \( \mathcal{O} \) is a p-form with \( p < 3 \), it vanishes. The precise definition of \( \langle d\bar{\mathcal{B}}, \mathcal{O} \rangle \) will be given in (2.22) below. The exterior derivative of \( \mathcal{B} \) can be associated with the torsion \( \mathcal{C} \).

One should note here that, in contrast to the systems with \((1, 1, 0)\) or \((2, 2, 0)\) multiplets, the analogy between \( d\mathcal{B} \) and the torsion in the case of the \((1, 2, 1)\) multiplet is not quite direct. First of all, the torsions are usually assumed real, while \( \mathcal{B} \) and \( d\mathcal{B} \) can be complex. Second, as we will see in Sect. 3, in this case, the torsions enter covariant derivatives not in the same way as standard Christoffel symbols: \( \sim \mathcal{C}\psi \bar{\psi} \) vs. \( \sim \Gamma \bar{\psi} \psi \). All this notwithstanding, we will call \( \mathcal{C} \) torsion also in this case.

One can easily observe that the operators \( d_\mathcal{B}, d^\dagger_\mathcal{B} \) (as well as the operators \( d_W, d^\dagger_W \)) are nilpotent. They form thus a minimal supersymmetry algebra by the same token as the operators \( d, d^\dagger \) do. This deformed supersymmetry system can be realized in the superfield language. To this end, one should add the term

\[ S_2 = \frac{1}{2} \int d^2 \theta dt \mathcal{B}_{MN}(X^P) DX^M DX^N + \text{c.c.} \]  

(1.17)

to the action (1.5) \[12\]. The expressions for the quantum supercharges and the Hamiltonian for this model (in the case of real \( \mathcal{B}_{MN} \)) can be found in \[22\].

One can deform the system (1.5) even further by adding the exterior derivative \( d\mathcal{B}_4 \) of an arbitrary 4-form \( \mathcal{B}_4 \) to \( d_\mathcal{B} \). The deformed operator \( d_W, d^\dagger_W \mathcal{B}_4 \) is still nilpotent. In superfield language, that corresponds to adding the structure

\[ S_4 = \frac{1}{2} \int d^2 \theta dt \mathcal{B}_{MNPQ}(X^S) DX^M DX^N DX^P DX^Q + \text{c.c.} \]  

(1.18)

to the action. The component Lagrangian of a model with an extra 4-form will be written in Sect. 2 below. One can further add the exterior derivative of a 6-form, etc. These higher even-dimensional forms can be dubbed \textit{generalized} torsions. It should be pointed out that these additional superfield terms do not bring in the component Lagrangians any terms of higher order in time derivatives of the involved fields.

The complex sigma model Lagrangian (1.8) can be generalized along similar lines. The generic Lagrangian is obtained by adding the terms

\[ \sim \mathcal{B}_{jk}(Z, \bar{Z}) DZ^j DZ^k + \text{c.c.} \]  

(1.19)
to $L_\sigma$ \[13\]. By the same token as in the sigma model involving real $(1, 2, 1)$ multiplets, one can also add the terms $\propto B_{jklp}$, etc. The terms, associated with extra torsions (torsions coming from (1.19) should be added to the torsions which are already present in (1.8) in non-Kähler case), as well as with the generalized torsions, were not considered in \[17\]. The corresponding supercharges define some $B$-deformation of the Dolbeault complex.

The present paper is devoted to filling some gaps existing in the literature on these subjects. In particular, we give the explicit form of the $\mathcal{N}=2$ supercharges for the torsionful sigma models based on the multiplets $(1, 2, 1)$ and $(2, 2, 0)$, with taking into account, in the first case, both interactions (1.17) and (1.18), as well as the potential term (1.15). We also find the vacuum states in these sigma models and demonstrate that the inclusion of the torsion terms does not influence their number.

The plan of the paper is the following. In Sect. 2, we discuss the $(1, 2, 1)$ multiplet. We write the Lagrangian and present both the classical and the quantum supercharges in the system that includes the torsions and generalized torsions.

In Sect. 3, we discuss the complex sigma model. We write a generic component Lagrangian, the supercharges and the Hamiltonian, both at the classical and the quantum levels for the model involving the terms (1.19).

In Sect. 4, we are addressing the supersymmetric vacua of our models. There is a simple mathematical argument saying that cohomology classes of the deformed de Rham complex (1.14), (1.16) are the same as for the undeformed one. A similar reasoning applies to the $B$-deformed Dolbeault complex too. To illustrate and confirm these statements, we present some explicit calculations for the wave functions of deformed vacuum states on the spheres $S^n$ for the real sigma model and on the $\mathbb{CP}^n$ manifolds for the complex one.

## 2 Torsions and generalized torsions for $(1, 2, 1)$ sigma model

Consider the supermultiplet (1.4). $\mathcal{N} = 2$ supersymmetry acts there as

$$\delta X^M = -(\epsilon Q + \bar{\epsilon} \bar{Q})X^M, \quad Q = \frac{\partial}{\partial \theta} + i\bar{\theta} \partial_t, \quad \bar{Q} = \frac{\partial}{\partial \bar{\theta}} + i\theta \partial_t,$$

whence we obtain the transformations of the component fields

$$\delta x^M = -(\epsilon \psi^M - \bar{\epsilon} \bar{\psi}^M), \quad \delta \psi^M = \epsilon (i \dot{x}^M - F^M), \quad \delta \bar{\psi}^M = -\epsilon (i \dot{\bar{x}}^M + F^M), \quad \delta F^M = i(\epsilon \dot{\psi}^M + \bar{\epsilon} \dot{\bar{\psi}}^M).$$  (2.2)

Write the action as the sum $S = S_g + S_2 + S_4$, where $S_g$ is the standard action given by the sum of (1.5) and (1.15), while $S_2$ and $S_4$ are the terms (1.17) and (1.18) describing the torsions and generalized torsions.

The corresponding component actions have the form

$$S_g = \int dt \left[ \frac{1}{2} g_{MN}(\dot{x}^M \dot{x}^N + F^M F^N) + \frac{i}{2} g_{MN} (\bar{\psi}^M \nabla \psi^N - \nabla \bar{\psi}^M \psi^N) \right].$$
\[ + \Gamma_{MNP}\psi^N\bar{\psi}^P F^M - \frac{1}{2} \partial_M \Gamma_{NPQ} \psi^M \bar{\psi}^Q \psi^P \bar{\psi}^N + F^M \partial_M W + \partial_M \partial_N W \psi^M \bar{\psi}^N, \]

\[ S_2 = \frac{1}{2} \int dt \left[ (\partial_M C_{NPQ} \psi^N \psi^P \psi^Q \bar{\psi}^M + \partial_M \bar{C}_{NPQ} \psi^N \bar{\psi}^P \psi^Q \psi^M) + 3 C_{MNP}(F^M - i\bar{x}^M)\psi^P \bar{\psi}^N - 3 \bar{C}_{MNP}(F^M + i\bar{x}^M)\psi^N \bar{\psi}^P, \right. \]

\[ + 5 C_{MNPQS}(F^M - i\bar{x}^M)\psi^N \psi^P \psi^Q \psi^S + c.c. \] (2.3)

Here

\[ \nabla \psi^M = \dot{\psi}^M + \Gamma^M_{NP} \dot{\bar{x}}^N \bar{\psi}^P, \quad \Gamma_{MNP} = \frac{1}{2} \left[ \partial_N g_{MP} + \partial_P g_{MN} - \partial_M g_{NP} \right], \]

\[ C_{MNP} = \frac{1}{3} \left\{ \partial_M \mathcal{B}_{NP} + \text{cycle}(M, N, P) \right\}, \]

\[ C_{MNPQS} = \frac{1}{5} \left\{ \partial_M \mathcal{B}_{NPQS} + \text{cycle}(M, N, P, Q, S) \right\}. \] (2.4)

The \( F^M \) equation of motion (with \( W = 0 \) for simplicity) yields:

\[ F^M = -\Gamma^M_{NP} \psi^N \bar{\psi}^P - \frac{1}{2} C^M, \] (2.5)

where

\[ C^M = 3 \left[ C^{(2)M}(\psi)^2 - \bar{C}^{(2)M}(\bar{\psi})^2 \right] + 5 \left[ C^{(4)M}(\psi)^4 + \bar{C}^{(4)M}(\bar{\psi})^4 \right], \] (2.6)

and we used the condensed notation

\[ C^{(2)M}(\psi)^2 := C_{NP}^M \psi^N \psi^P, \quad C^{(4)M}(\psi)^4 := C_{NPQS}^M \psi^N \psi^P \psi^Q \psi^S, \quad \text{etc.} \] (2.7)

After substitution of this expression back into the sum of actions (2.3) - (2.5) (with \( W = 0 \)) we obtain the on-shell form of the total action

\[ S_{\text{on–sh}} = \frac{1}{2} \int dt \left[ g_{MN} \dot{x}^M \dot{x}^N + i g_{MN}(\bar{\psi}^M \nabla \psi^N - \nabla \bar{\psi}^M \psi^N) + R_{MPNO} \bar{\psi}^M \psi^N \bar{\psi}^P \psi^Q \right. \]

\[ + \left( \nabla_M C_{NPQ} \psi^N \psi^P \psi^Q \bar{\psi}^M + \nabla_M \bar{C}_{NPQ} \psi^N \bar{\psi}^P \psi^Q \psi^M \right) - 3i \dot{x}^M(C_{MNP} \psi^N \psi^P + \bar{C}_{MNP} \bar{\psi}^N \bar{\psi}^P) \]

\[ + \left( \nabla_M C_{NPQST} \psi^N \psi^P \psi^Q \psi^S \psi^T \psi^M - \nabla_M \bar{C}_{NPQST} \bar{\psi}^N \bar{\psi}^P \bar{\psi}^Q \bar{\psi}^S \bar{\psi}^T \bar{\psi}^M \right) \]

\[ - 5i \dot{x}^M(C_{MNPSQ} \psi^N \psi^P \psi^Q \psi^S - \bar{C}_{MNPSQ} \bar{\psi}^N \bar{\psi}^P \bar{\psi}^Q \bar{\psi}^S) - \frac{1}{4} C^M C_M \right], \] (2.8)

where

\[ R_{MPNO} = g_{MT} \left( \partial_P \Gamma_{QN}^T - \partial_Q \Gamma_{PN}^T + \Gamma_{PS}^T \Gamma_{QN}^S - \Gamma_{QS}^T \Gamma_{PN}^S \right) \] (2.9)
is the Riemann tensor.

When deriving the supercharges, it is more convenient not to eliminate the auxiliary field $F^M$ until the final step. The classical conserved Noether supercharge calculated from the infinitesimal transformations (2.2) that leave invariant the off-shell action $S = S_g + S_2 + S_4$ has the following form,

$$Q = \psi^M (\tilde{\Pi}_M + i \partial_M W) - \frac{i}{2} \partial_M g_{NP} \psi^M \psi^N \bar{\psi}^P + i C_{MNP} \psi^M \psi^N \bar{\psi}^P \psi Q \psi^S,$$

(2.12)

where $\tilde{\Pi}_M$ is the canonical momentum,

$$\tilde{\Pi}_M = \frac{\partial L}{\partial \dot{x}^M} = g_{MN} \dot{x}^N - \frac{i}{2} \Gamma_{NMP} (\psi^P \dot{\bar{\psi}}^N - \psi^N \dot{\psi}^P) - \frac{i}{2} S_M,$$

(2.13)

with

$$S_M := 3 \left[ C^{(2)}(\psi)^2 + \bar{C}^{(2)}(\bar{\psi})^2 \right] + 5 \left[ C^{(4)}(\psi)^4 - \bar{C}^{(4)}(\bar{\psi})^4 \right].$$

Correspondingly,

$$\overline{Q} = \bar{\psi}^M (\tilde{\Pi}_M - i \partial_M W) - \frac{i}{2} \partial_M h_{NP} \bar{\psi}^M \bar{\psi}^N \psi^P + i \bar{C}_{MNP} \bar{\psi}^M \bar{\psi}^N \psi^P - i \bar{C}_{MNPQS} \bar{\psi}^M \bar{\psi}^N \psi^P \bar{\psi}^Q \bar{\psi}^S.$$

(2.14)

Even though the auxiliary fields $F^M$ were kept in the Lagrangian, they do not explicitly appear in the supercharges.

The partial derivative (2.13) was calculated assuming fixed $\psi^M$, $\bar{\psi}^M$. The latter variables are not, however, canonically conjugated, their Poisson bracket being

$$\{ \bar{\psi}^M, \psi^N \} = g^{MN}.$$

As a preliminary step to quantization, one should define the tangent space canonically conjugated fermion variables $\psi^A = e^A_M \psi^M$, $\bar{\psi}^A = e^A_M \bar{\psi}^M$ and express the classical supercharges through these variables and the new bosonic canonical momentum

$$\Pi_M = \frac{\partial L}{\partial \dot{x}^M} \bigg|_{\text{fixed } \psi^A} = \tilde{\Pi}_M - \frac{\partial \psi^A}{\partial \dot{x}^M} \frac{\partial L}{\partial \dot{\psi}^A} - \frac{\partial \bar{\psi}^A}{\partial \dot{x}^M} \frac{\partial L}{\partial \dot{\bar{\psi}}^A}.$$

(2.15)

Finally, we derive the following classical expression for $Q$ (and analogously for $\overline{Q}$):

$$Q = \psi^A e^{AM} (\Pi_M + i \partial_M W) - i \Omega^{CAB} \psi^C \psi^A \bar{\psi}^B + i C^{ABC} \psi^A \psi^B \psi^C + C^{ABCDE} \psi^A \psi^B \psi^C \psi^D \psi^E.$$

(2.16)

Here,

$$\Omega^{C,AB} = e^{MC} \Omega^{AB}_M, \quad \Omega^{AB}_M = e^A_M (\partial_M e^{NB} + \Gamma^N_{MT} e^{TB}),$$

(2.17)

is the standard spin connection.

To derive the quantum supercharges, one has to replace $\Pi_M$ and $\bar{\psi}^A$ by differential operators $\Pi_M \rightarrow -i \partial / \partial x^M$, $\bar{\psi}^A \rightarrow \partial / \partial \psi^A$ and to resolve the ordering ambiguities problem.
To make a selection between many different quantum theories corresponding to a given classical one, we require that the supersymmetry algebra remains intact at the quantum level and that $Q_{qu}$ and $\bar{Q}_{qu}$ are Hermitian conjugate to each other. This fixes the quantum supercharges and Hamiltonian.

As was shown in [23], the general recipe of such a symmetry-preserving quantization is as follows.

- Take the expressions for the classical supercharges and order them according to the symmetric Weyl prescription.
- The supercharges thus obtained are nilpotent. Their anticommutator gives the quantum Hamiltonian. Generically, it does not coincide with the operator obtained from the classical Hamiltonian by Weyl ordering.
- This procedure gives the quantum supercharges and the Hamiltonian acting in the “flat” Hilbert space with the measure

$$\sim \left( \prod_M dx^M \right) d(\text{fermions})$$

in the inner product. If we want to obtain the expressions for the covariant operators acting in the Hilbert space with the measure involving the factor $\sqrt{\det g}$ (such operators have a nicer geometric interpretation), an appropriate similarity transformation

$$(Q^{\text{cov}}, \bar{Q}^{\text{cov}}) = (\det g)^{-1/4} \left( Q^{\text{flat}}, \bar{Q}^{\text{flat}} \right)(\det g)^{1/4}$$  \hspace{1cm} (2.18)

should be performed.

We finally obtain the quantum supercharges as

$$Q^{\text{cov}} = -i\psi^A e^{MA} (\partial_M - \partial_M W) - i\Omega^{C,AB} \psi^C \bar{\psi}^A \psi^B + i C^{ABC} \psi^A \psi^B \psi^C,$$

$$\bar{Q}^{\text{cov}} = -i\bar{\psi}^{\bar{A}} e^{\bar{M}A} (\partial_M + \partial_M W) - i\bar{\Omega}^{C,AB} \bar{\psi}^C \psi^A \bar{\psi}^B + i \bar{C}^{ABC} \bar{\psi}^{\bar{A}} \bar{\psi}^B \bar{\psi}^C$$ \hspace{1cm} (2.19)

The torsion-free part of these expressions is well known [15, 24]. The terms involving the torsions $C_{ABC}$ were written in Ref. [22] (for real $C_{ABC}$). The supercharges thus obtained constitute the $\mathcal{N}=2, d=1$ Poincaré superalgebra

(a) $(Q^{\text{cov}})^2 = (\bar{Q}^{\text{cov}})^2 = 0$, (b) $\{Q^{\text{cov}}, \bar{Q}^{\text{cov}}\} = 2H$, (c) $[Q^{\text{cov}}, H] = [\bar{Q}^{\text{cov}}, H] = 0$ \hspace{1cm} (2.20)

and are isomorphic to the twisted de Rham operators (1.16) with $\mathcal{B} = \mathcal{B}_2 + \mathcal{B}_4$. For our further purposes, we will not need the explicit form of the quantum Hamiltonian $H$. For real $C^{ABC}$ and vanishing $C^{ABCDE}$, it was given in [22]. The isomorphism between the
supercharges (2.19) and the operators appearing in the geometric setting (1.16) implies, in particular, the correspondence

$$\langle d\mathcal{B}, \mathcal{O} \rangle \Leftrightarrow C^{ABC} \frac{\partial}{\partial \psi^A} \frac{\partial}{\partial \bar{\psi}^B} \frac{\partial}{\partial \bar{\psi}^C} \psi^{D_1} \cdots \psi^{D_n} \mathcal{O}^{D_1 \cdots D_n},$$

(2.21)

which gives the following explicit definition for the inner product in the second line of (1.16):

$$\langle d\mathcal{B}, \mathcal{O} \rangle = -\frac{p!}{(p-3)!} C^{MNP} \mathcal{O}_{MNP\cdots R_a} dx^{R_4} \wedge \cdots \wedge dx^{R_p}$$

(2.22)

for a $p$-form $\mathcal{O}$ ($p \geq 3$). We will use it in what follows.

The supercharges (2.19) can be represented in the form

$$Q^{cov} = e^{W + \psi^L \bar{\psi}^B \mathcal{B}_{KL} + \psi^L \bar{\psi}^M \psi^N \mathcal{B}_{KLMN}} Q_0^{cov} e^{-W - \psi^L \bar{\psi}^B \mathcal{B}_{KL} - \psi^L \bar{\psi}^M \psi^N \mathcal{B}_{KLMN}},$$

$$\bar{Q}^{cov} = e^{-W + \psi^L \bar{\psi}^B \mathcal{B}_{KL} - \psi^L \bar{\psi}^M \psi^N \mathcal{B}_{KLMN}} \bar{Q}_0^{cov} e^{W - \psi^L \bar{\psi}^B \mathcal{B}_{KL} + \psi^L \bar{\psi}^M \psi^N \mathcal{B}_{KLMN}},$$

(2.23)

where $Q_0^{cov}$, $\bar{Q}_0^{cov}$ are the supercharges with the torsion and potential terms being suppressed. Note that the derivatives in $Q_0^{cov}$, $\bar{Q}_0^{cov}$ act here not only on $W$ and $\mathcal{B}$, but also on $\psi^K = e^{KA} \bar{\psi}^A$, etc. These terms are exactly canceled by the terms coming from the commutators of the structures $\sim \Omega \psi \bar{\psi} \psi$ and $\sim \Omega \bar{\psi} \bar{\psi} \bar{\psi}$ in $Q_0^{cov}$ and $\bar{Q}_0^{cov}$ with the torsion structures.

In the differential form language, this notable representation of the supercharges has a rather transparent meaning. The first line in (2.23) means that

$$d_{W,B} = e^{W+B} de^{- W-B},$$

(2.24)

which is a direct corollary of the definitions (1.14), (1.16). The second line is Hermitian conjugate of the first one. The representation (2.23), (2.24) will be used while finding the explicit form of the ground state wave functions in Section 4.

### 3 Complex model with torsions

We start from the action (1.8) and add to it the term

$$S_{\text{extra torsion}} = \frac{1}{4} \int dt d^3 \theta \left( \mathcal{B}_{jk}(Z, \bar{Z}) DZ^j DZ^k - \bar{\mathcal{B}}_{jk}(Z, \bar{Z}) D\bar{Z}^j D\bar{Z}^k \right)$$

(3.1)

with arbitrary antisymmetric complex superfunction $\mathcal{B}_{jk}$ and its conjugate $\bar{\mathcal{B}}_{jk}$.

The component form of the full action is

$$S = \int dt \left\{ h_{jk} \left[ \dot{\psi}^j \dot{\bar{\psi}}^k + \frac{i}{2} \left( \psi^j \bar{\psi}^k - \psi^k \bar{\psi}^j \right) \right] + \left( \partial_t h_{jk} \right) \psi^i \psi^j \bar{\psi}^i \bar{\psi}^j \right. \right.$$

$$- \frac{i}{2} \left[ (2 \partial_j h_{ik} - \partial_i h_{jk}) \dot{\psi}^j - (2 \partial_i h_{jk} - \partial_j h_{ik}) \dot{\psi}^i \right] \dot{\bar{\psi}}^k + 2 \partial_j \partial_k \bar{W} \psi^j \bar{\psi}^k - i \left( \partial_j W \bar{\psi}^j - \partial_k W \bar{\psi}^k \right)$$

$$- 3i \partial (m \mathcal{B}_{ik}) \bar{\psi}^m \bar{\psi}^i \psi^k \bar{\psi}^k - 3i \partial (m \bar{\mathcal{B}}_{ik}) \bar{\psi}^m \bar{\psi}^i \psi^k \bar{\psi}^k$$

$$\left. - \partial_i \partial_m \bar{\mathcal{B}}_{ik} \bar{\psi}^n \bar{\psi}^m \psi^i \psi^k - \partial_m \partial_i \mathcal{B}_{ik} \psi^m \bar{\psi}^i \bar{\psi}^k \right\},$$

(3.2)
This action can be cast in the $\mathcal{N} = 1$ superfield notations as the sum of the terms \((1.2)\), \((1.9)\) and the term

\[
S_{\text{gauge}} = -i \int dt d\theta A_M(\mathcal{X}^P)D\mathcal{X}^M, \tag{3.3}
\]

with $M = \{j, \bar{j}\}$, $A_M = \{-i\partial j W, i\partial j W\}$. This gives in components

\[
L = L_\sigma + L_{\text{extra torsion}} + L_{\text{gauge}}
= \frac{1}{2} \left[ g_{MN} \dot{z}^M \dot{z}^N + ig_{MN} \psi^M \hat{\nabla} \psi^N - \frac{1}{6} \partial_P C_{KLM} \psi^K \psi^L \psi^M \right]
+ A_M \dot{z}^M - \frac{i}{2} F_{MN} \bar{\psi}^M \psi^N, \tag{3.4}
\]

where $F_{MN} = \partial_M A_N - \partial_N A_M$ and $z^M \equiv x^M = (z^j, \bar{z}^\bar{j}), \psi^M = (\psi^j, \psi^{\bar{j}})$, and $\hat{\nabla}$ is the exterior holomorphic derivative.

The non-vanishing components of the totally antisymmetric torsion tensor $C_{KLM}$ are

\[
C_{klm} = - (\partial_k h_{lm} - \partial_l h_{km}), \quad C_{\bar{k}l\bar{m}} = (C_{klm})^\ast = - (\partial_k h_{\bar{m}l} - \partial_l h_{\bar{m}k}),
C_{klm} = 12 \partial_{[k} \mathcal{B}_{lm]}, \quad C_{\bar{k}l\bar{m}} = 12 \partial_{[k} \bar{\mathcal{B}}_{lm}] . \tag{3.5}
\]

We see that the terms $\propto \mathcal{B}_{jk}$, $\bar{\mathcal{B}}_{\bar{j}k}$ in the Lagrangian bring about the holomorphic components of the torsion $C_{klm}$, $C_{\bar{k}l\bar{m}}$. The (3,0)-form $C_{klm} dz^k \wedge dz^j \wedge dz^m$ is obtained from the action of the exterior holomorphic derivative $\partial$. Besides, there are mixed components of the torsion tensor $C_{klm}$, $C_{\bar{k}l\bar{m}}$ which are not arbitrary, but are strictly related to the metric $h_{jk}$. When the manifold is Kähler, i.e., $h_{jk} = \partial_j \partial_k K$, these components vanish. In this case (and when $\mathcal{B}_{jk} = 0$), the Lagrangian coincides with \((1.2)\).

Generically, the Lagrangian in Eqs. \((3.2)\), \((3.3)\) involves a 4-fermion term. Note that, if the form $C_{MNK} dz^M \wedge dz^N \wedge dz^K$ is closed, the 4-fermion term is absent (this case was addressed in \([19]\)) Note also that the “new” terms brought about by the extra torsion terms $\propto \mathcal{B}, \bar{\mathcal{B}}$ show up only starting from the complex target dimension $n = 3$. For $n = 2$ (and, of course, for $n = 1$) they vanish identically.

The classical supercharges can be calculated by the Nöther theorem in a standard way. We obtain

\[
Q = \sqrt{2} e^k \bar{\psi}^c \left[ \Pi_k - i \Omega_{k,\dot{a}b} \bar{\psi}^a \psi^b + i \psi^a \bar{\psi}^{\dot{b}} e_{a}^{\dot{b}} e_{\dot{b}}^{l} \partial_{l} \mathcal{B}_{jl} \right] ,
Q = \sqrt{2} e^k \bar{\psi}^c \left[ \bar{\Pi}_k - i \bar{\Omega}_{k,\dot{a}b} \psi^a \bar{\psi}^{\dot{b}} + i \bar{\psi}^a \psi^{\dot{b}} e_{a}^{\dot{b}} e_{\dot{b}}^{l} \partial_{l} \bar{\mathcal{B}}_{jl} \right] , \tag{3.6}
\]

where

\[
\Pi_k = P_k + i \partial_k W , \quad \bar{\Pi}_k = P_{\bar{k}} - i \partial_{\bar{k}} W , \tag{3.7}
\]

and $P_k, P_{\bar{k}}$ are the canonical momenta (obtained by varying the Lagrangian with respect to $\dot{z}^k, \dot{z}^{\bar{k}}$ at fixed $\psi^a, \bar{\psi}^{\dot{b}}$). The spin connections $\Omega_{k,\dot{a}b}$ and $\bar{\Omega}_{k,\dot{a}b}$ are the corresponding components of the standard real spin connections $\Omega_{M,AB}$ satisfying \((1.13)\). The terms
$\propto \partial \mathcal{B}$ in $Q, \bar{Q}$ can be interpreted as the holomorphic components $\hat{\Omega}_{k,ab}$ and $\hat{\Omega}_{k,ab}$ of the connection

$$\hat{\Omega}_{M,AB} = e_{AN}(\partial M e^N_B + \hat{\Gamma}_{MK}^N e^K_B) = \Omega_{M,AB} + \frac{1}{2} e^K_A e^K_B C_{MLK},$$

(3.8)

in which the torsion is taken into account and which satisfies the following generalization of (1.13)

$$\bar{d}e_A + \hat{\Omega}_{AB} \wedge e_B = C_A = C_{ABC} dz^B \wedge dz^C.$$  

(3.9)

One can be convinced that, for the particular torsion whose components are displayed in Eq. (3.5), the sum $Q + \bar{Q}$ of the supercharges (3.6) coincides with the $(1, 2, 1)$ supercharge (1.12). Note that for a generic complex manifold, the spin connections involve the components $\Omega_{k,ab}$ and $\Omega_{k,ab}$ which vanish in the Kähler case. One can observe that their contribution to the first term of Eq. (1.12) exactly cancels out the contributions due to $C_{jkl}, C_{jkl}$ in the second term.

The canonical classical Hamiltonian $H_{cl}$ can be represented in the following compact form:

$$H_{cl} = h^{kj} \bar{P}_j \bar{P}_k - e^i_a e^j_b e^k_c (\partial_i \partial_j \hat{h}_{k}) \psi^a \psi^b \bar{\psi}^c \bar{\psi}^d + e^m_a e^j_b e^k_c (\partial_m \partial_j B_k) \bar{\psi}^b \psi^a \psi^c + e^m_a e^j_b e^k_c (\partial_m \partial_j B_k) \psi^d \bar{\psi}^a \bar{\psi}^b \bar{\psi}^c - 2 e^i_a e^j_b (\partial_i \partial_j W) \psi^a \bar{\psi}^b,$$

(3.10)

where

$$\bar{P}_M = \Pi_M - \frac{i}{2} \hat{\Omega}_{M,AB} \bar{\psi}^A \psi^B.$$  

(3.11)

In contrast to the supercharge (1.12), the Hamiltonian (3.10) involves the conventional “hatted” spin connections (3.8).

Let us now turn to quantum theory. As in the previous section, we resolve the ordering ambiguities as prescribed in [23], i.e. use the symmetric Weyl ordering for the supercharges supplemented by a similarity transformation (2.18). We obtain the following expressions for the covariant quantum supercharges:

$$Q_{\text{cov}} = \sqrt{2} e^k_c \bar{\psi}^c \left[ \Pi_k - \frac{\partial}{\partial \ln det \bar{e}} \left( \frac{i}{2} \Omega_{k,ab} \psi^b \bar{\psi}^a + i \psi^a \psi^b e^j_b \partial_j B_{kj} \right) \right],$$

$$\bar{Q}_{\text{cov}} = \sqrt{2} e^k_c \bar{\psi}^c \left[ \Pi_k - \frac{\partial}{\partial \ln det e} \left( \frac{i}{2} \bar{\Omega}_{k,ab} \bar{\psi}^b \psi^a + i \bar{\psi}^a \bar{\psi}^b e^j_b \partial_j B_{kj} \right) \right].$$

(3.12)

These expressions almost coincide by form with (3.6) (note, however, the presence of important terms $\propto \partial \ln det e, \partial \ln det \bar{e}$), but $\bar{\psi}^a$ are now operators, $\bar{\psi}^a = \partial/\partial \psi^a$, and, similarly, $\Pi_M = -i \partial M - A_M$, with $A_M = \{-i \partial_j W, i \partial_j W\}$.

The quantum Hamiltonian is

$$H_{\text{qu}}^{\text{cov}} = -\frac{1}{2} \Delta_{\text{cov}} + \frac{1}{8} \left( R - \frac{1}{2} h^{ii} h^{ij} h^{kk} C_{ij,k} C_{i,j,k} - \frac{1}{6} h^{ii} h^{ij} h^{kk} C_{ij,k} C_{i,j,k} \right)$$

$$- 2 \langle \psi^a \bar{\psi}^c \rangle e^i_a e^j_b (\partial_i \partial_j W) - \langle \psi^a \psi^b \bar{\psi}^c \bar{\psi}^d \rangle e^i_a e^j_b e^k_c (\partial_i \partial_j h_{kl}) + \langle \bar{\psi}^d \psi^a \psi^b \psi^c \rangle e^i_a e^j_b e^k_c (\partial_m \partial_i B_{jk}) + e^m_a e^j_b e^k_c \langle \psi^d \bar{\psi}^a \bar{\psi}^b \bar{\psi}^c \rangle (\partial_m \partial_i B_{jk}).$$

(3.13)
Here, \( \langle \ldots \rangle \) denotes the Weyl-ordered products of fermions, \( R \) is the standard scalar curvature of the metric \( h_{jk} \), and \( \triangle^{\text{cov}} \) is the covariant Laplacian calculated with the hatted affine and spin connections,

\[
- \triangle^{\text{cov}} = h^{kj} \left( P_j \tilde{P}_k + i \tilde{\Gamma}^q_{jk} P_q + \tilde{P}_k P_j + i \tilde{\Gamma}^s_{kj} P_s \right),
\]

(3.14)

where \( P_M \) are still given by Eq. (3.11) with \( P_M \rightarrow -i \partial M \).

The structure \( \sim R - \text{CC} \) entering the expression (3.13) can be written in the real notation as \( \tilde{R} = R - \frac{1}{12} C_{MNP} C^{MNP} \), that is to be compared with the "hatted" counterpart of \( R \) calculated with non-symmetric affine connections \( \hat{\Gamma} \),

\[
\hat{R} = R - \frac{1}{4} C_{MNP} C^{MNP}.
\]

Let us say a few words about geometric interpretation of our system.

By the same token as in the Kähler torsionless case [25, 17] (the extra term \( \sim C \psi^3 \) in the classical supercharge (1.12) does not create ordering problems), the sum \( Q^{\text{cov}} \) of the supercharges (3.12) can be interpreted as the Dirac operator on the manifold equipped with the torsion \( \frac{1}{3} C_{MNP} \),

\[
Q^{\text{cov}} = Q^{\text{cov}} + \bar{Q}^{\text{cov}} \equiv i \gamma^M \hat{\nabla}_M,
\]

(3.15)

where \( \hat{\nabla}_M = \Pi_M - \frac{i}{2} \hat{\Omega}_{M,BC} \gamma^B \gamma^C \) and

\[
\hat{\Omega}_{M,BC} = \Omega_{M,BC} + \frac{1}{6} e_B^L e_C^K C_{LKM}.
\]

(3.16)

The difference \( Q^{\text{cov}} - \bar{Q}^{\text{cov}} \) is isomorphic to the operator \( \gamma^M I_M^N \hat{\nabla}_N \), where \( \hat{\nabla}_M = \partial_M + \frac{1}{2} \hat{\Omega}_{M,AB} \gamma^A \gamma^B \) and \( I_M^N \) is the covariantly constant complex structure matrix (\( I^2 = -1 \), \( \nabla_T I_M^N = 0 \)).

It was noticed in [17] that, with the vanishing extra torsion terms \( \propto B, \bar{B} \) and for a particular choice \( W = \frac{1}{4} \ln \det h \), the supercharges (3.12) realize the Dolbeault complex. It involves the operator of the exterior holomorphic derivative \( \partial \) and its conjugate \( \partial^\dagger \).

When \( W = -\frac{1}{4} \ln \det h \), the supercharges are isomorphic to the operators \( \bar{\partial} \) and \( \bar{\partial}^\dagger \) of the anti-Dolbeault complex. For other choices of \( W \), they realize a twisted Dolbeault complex with \( \partial_A = \partial - A \), where \( A = \partial W \) is an exact \( (1,0) \)-form that can be interpreted as a gauge field. The latter might be nontrivial. The fact that \( \partial A = 0 \) does not mean that the real gauge field \( A_M = (-i \partial_j W, i \partial_j W) \) has a zero curl.

In a more general case we have considered in this section, we are dealing with the torsion-deformed twisted Dolbeault complex [26] involving the operators [cf. Eq. (1.16) ]

\[
\partial_{W,B} \mathcal{O} = \partial \mathcal{O} - \partial W \wedge \mathcal{O} - \partial \mathcal{B} \wedge \mathcal{O},
\]

\[
\partial^\dagger_{W,B} \mathcal{O} = \partial^\dagger \mathcal{O} + \langle \bar{\partial} W, \mathcal{O} \rangle - \langle \bar{\partial} \mathcal{B}, \mathcal{O} \rangle.
\]

(3.17)

The notation \( \langle X, Y \rangle \) stands now for the complex interior product. For example, if \( X \) is a \((0,1)\)-form and \( Y \) is a \((1,0)\)-form, \( \langle X, Y \rangle = h^{jk} X_j Y_k \).

Note that the quantum supercharges (3.12) and the Hamiltonian (3.13) depend on \( \mathcal{B} \) only via its exterior derivative \( \partial \mathcal{B} \). This means that \( \mathcal{B} \) is defined only up to a gauge transformation \( \mathcal{B} \rightarrow \mathcal{B} + \partial A \).
4 Vacuum states

In this section we will find the zero-energy vacuum states wave functions $\Phi_B$ in the $(1, 2, 1)$ and $(2, 2, 0)$ SQM sigma-models with nonzero torsions produced by the terms (1.17)-(1.19) with tensors $B$. These wave functions are solutions of the generic equations

$$Q^{\text{cov}} \Phi_B = 0, \quad \bar{Q}^{\text{cov}} \Phi_B = 0. \quad (4.1)$$

As a result, the vacuum wave functions $\Phi_B$ in the presence of torsions will prove to be certain deformations of the torsionless wave functions $\Phi_{B=0}$, thus encompassing the same number of states as $\Phi_{B=0}$. An important tool for obtaining the general solution for the vacuum wave functions $\Phi_B$ will be the representation (2.23) for the quantum supercharges, where the terms with torsions (as well as the potential terms) are absorbed into a similarity transformation of the “undeformed” supercharges. We will essentially exploit the geometric correspondence with the de Rham complex in the $(1, 2, 1)$ case [such that (2.23) acquires the form (2.24)] and the Dolbeault complexes in the $(2, 2, 0)$ case. We will limit our analysis to the spheres $S^n$ and $\mathbb{CP}^n$ manifolds, in the first and the second cases, respectively.

4.1 de Rham complex with torsions

For the de Rham complex (i.e. for the $(1, 2, 1)$ SQM model of Sect. 2), the Witten index $\text{Tr} \left\{ (-1)^F \right\}$ coincides with the Euler characteristic $\chi$ of the manifold. Consider $S^n$ as the simplest example.

When $n$ is even, $\chi = 2$, which suggests the presence of two bosonic zero modes. When the torsions are absent, these zero modes are seen explicitly - it is the constant 0-form and the volume $n$-form[^1]. Witten index cannot change under a smooth deformation. This assures the presence of two bosonic zero modes in the spectrum also for a deformed complex.

When $n$ is odd, the Euler characteristic vanishes. If the manifold has an isometry, one can consider another index, the so called Lefshetz number $\text{Tr} \left\{ (-1)^F K \right\}$, where $K$ is an isometry commuting with the Hamiltonian, for example - a reflection of one of the coordinates. For a “round” odd-dimensional sphere, this Lefshetz number is equal to 2, which means, again, the presence of two zero modes in the deformed complex if the deformation respects this isometry[^3].

For a “crumbled” sphere without any isometry (or when the isometry is not respected by the deformation), this argument does not work. Still, one can prove that the number of supersymmetric vacua is left unchanged.

The situation is especially simple for the deformation (1.14) where the deformed vacua can be found explicitly. Indeed, the operator $d_W = e^W d e^{-W}$ annihilates the 0-form $e^W$ (being 0-form, it is automatically annihilated by $d_W^\dagger = e^{-W} d e^W$). Likewise, the operator $d_W^\dagger = e^{-W} d^\dagger e^W$ annihilates the form $e^{-W} \mathcal{V}_n = e^{-W} \sqrt{g} dx^1 \wedge \cdots \wedge dx^n$, which is also automatically annihilated by $d_W$.\[^4\]

[^1]: Such zero modes exist for any target Riemann geometry, not only for spheres. In the generic case, other zero modes can be present.
To prove the same for the deformation (1.16), a little more elaborate reasoning is required. Let us first prove the theorem of existence - show that the equations (4.1) or, in the differential form language,
\[ d_B \Phi_B = d_B^\dagger \Phi_B = 0 , \tag{4.2} \]
have two nontrivial solutions. In fact, the proof goes in the same way, irrespective of whether \( n \) is odd or even.

Consider the form \( \Phi = e^B \). From (2.24), we immediately deduce that it is closed in the deformed sense, \( d_B \Phi = 0 \). However, it cannot be exact, \( \Phi \neq d_B \Phi \). Indeed, the identity \( e^B = d_B \Psi = e^B d e^{-B} \Psi \) would mean that \( 1 = d \left( e^{-B} \Psi \right) \). But a 0-form cannot be \( d \)-exact as the operator \( d \) increases the order of the form by one.

Note now that any form of even order and, in particular, \( \Phi \) can be represented as
\[ \Phi = d_B \mathcal{X} + d_B^\dagger \mathcal{Y} + \Phi_B , \tag{4.3} \]
where \( \mathcal{X} \) and \( \mathcal{Y} \) are some odd-order forms, while the form \( \Phi_B \) satisfies (4.2) and is thus \( d_B \)-harmonic. A mathematician will recognize in (4.3) a variant of the Hodge decomposition theorem [28]. Its physical meaning is rather transparent. It simply says that the Hilbert space of any SQM system is spanned by (i) zero modes of the Hamiltonian (i.e. \( \Phi_B \)), (ii) the states annihilated by the supercharge \( Q \) but not by the supercharge \( \bar{Q} \) (i.e. \( d_B \mathcal{X} \)) and (iii) the states annihilated by \( \bar{Q} \) but not by \( Q \) (i.e. \( d_B^\dagger \mathcal{Y} \)).

For our form \( \Phi \), annihilated by the action of \( Q \equiv d_B \), the second term in the expansion (4.3) must be absent. As the form is not exact, there should be some nontrivial nonzero \( \Phi_B = e^B - d_B \mathcal{X} \) (belonging to the same cohomology class as \( e^B \)). This is a first solution of (4.2).

To find the second solution, consider the volume form \( \mathcal{V}_n \). It is \( d \)-closed and also \( d_B \)-closed. It is the zero mode of the untwisted complex and hence cannot be \( d \)-exact. It follows that neither it is \( d_B \)-exact. Indeed, \( \mathcal{V}_n = e^B d e^{-B} \mathcal{X} \) would mean that \( e^B d e^{-B} \Psi = \mathcal{V}_n = d \left( e^{-B} \mathcal{X} \right) \). Using the same reasoning as above, we derive that the form \( \mathcal{V}_n \) can be presented as
\[ \mathcal{V}_n = d_B Z + \tilde{\Phi}_B , \tag{4.4} \]
where \( \tilde{\Phi}_B \) is a nontrivial \( d_B \)-harmonic form of the same order as \( \mathcal{V}_n \). In the physical language, this is the second bosonic zero mode for even-dimensional spheres and a fermionic zero mode for the odd-dimensional ones. This fermionic vacuum state is needed to compensate the bosonic zero mode and so to ensure the vanishing Witten index in the case of odd-dimensional spheres.

The theorem is proven.

It is interesting, however, to find the solutions of the equation (4.2) explicitly. It is possible to do this perturbatively to any order of perturbation theory in \( \mathcal{B} \). Let us see how it works.

---

5This argument is specific for \( S^n \) (only which we study here), but the isomorphism of the cohomologies of the twisted de Rham complex and the untwisted one is a quite general fact [27]. Indeed, using the property (2.24), it is easy to observe that a form \( \alpha \) is \( d_W \)-closed if and only if the form \( e^{-W-B} \alpha \) is \( d \)-closed and the form \( \alpha \) is \( d_W \)-exact if and only if the form \( e^{-W-B} \alpha \) is \( d \)-exact.
The simplest nontrivial case is $S^3$ (for 2-manifolds, the deformation vanishes). Let us look for the solution to the equations in the form
$$\Phi = 1 + Y_2,$$
where $Y_2$ is a 2-form (this Ansatz asserts that the undeformed function is just $\Phi_0 = 1$). For $S^3$, Eq. (1.2) implies
$$S^3: \quad dY_2 = dB, \quad d^\dagger Y_2 = 0. \quad (4.6)$$
The solution of the first equation in (4.6) is
$$Y_2 = B + dA_1, \quad (4.7)$$
with an arbitrary 1-form $A_1$. The latter can be written as $A_1 = d\omega_0 + d^\dagger \omega_2$, where $\omega_0$ and $\omega_2$ are, respectively, some 0- and 2-forms (the 0-form term is just a gauge freedom). Such representation is guaranteed by the Hodge decomposition theorem with respect to the usual de Rham complex $d, d^\dagger$, bearing in mind that there are no zero-mode 1-form — the Betty number $\beta_1$ for $S^3$ vanishes. The term $d\omega_0$, being a gauge freedom, does not affect the solution (4.7) and we are safe to disregard it and choose the gauge $A_1 = d^\dagger \omega_2$ such that $d^\dagger A_1 = 0$. Then the second equation in (4.7) yields
$$\triangle A_1 = -d^\dagger B,$$
where $\triangle \equiv dd^\dagger + d^\dagger d$ is the covariant Laplacian. The latter can be inverted (again, we are exploiting the fact that it does not have zero modes in the Hilbert space of 1-forms), which gives the solution
$$A_1 = -\triangle^{-1} d^\dagger B \quad (4.8)$$
for any $B$.
For $S^4$, we may seek for the solution in the same form (4.5) as for $S^3$. We obtain the same equations (4.6) and the same solution (4.7), (4.8). Note that in this case we could also add some 4-form $Y_4$ in the Ansatz (4.5) but the equations (4.2) would imply that $Y_4 = 0$.
For $S^5$ and for $S^6$, the Ansatz (4.5) is not compatible with the equations (4.2) and we are forced to extend it by adding a 4-form $Y_4$
$$\Phi = 1 + Y_2 + Y_4. \quad (4.9)$$
Putting (4.9) in (4.2), we derive the following equations for the forms $Y_{2,4}$:
$$S^{5,6}: \quad
dY_2 = dB, \quad d^\dagger Y_2 - \langle dB, Y_4 \rangle = 0, \quad dY_4 = dB \wedge Y_2, \quad d^\dagger Y_4 = 0. \quad (4.10)$$
A generic solution of the equations with the operator $d$ (left column of (4.10)) is
$$Y_2 = B + dA_1, \quad Y_4 = \frac{1}{2} B \wedge B + B \wedge dA_1 + dA_3 \quad (4.11)$$
where \( A_3 \) is an arbitrary 3-form defined up to gauge transformations \( A_3 \rightarrow A_3 + d\omega_2 \). By the same token as above, we use this gauge freedom to choose the gauge \( A_3 = d^\dagger \omega_4 \), in which case \( d^\dagger A_3 = 0 \). We also assume as before that \( d^\dagger A_1 = 0 \) (by choosing the proper gauge).

Then the equations in right column of (4.10) amount to the following system

\[
\begin{align*}
\triangle A_1 &= -d^\dagger B + \langle d^\dagger B, \frac{1}{2} \mathcal{B} \wedge \mathcal{B} \rangle + \langle d^\dagger \mathcal{B}, \mathcal{B} \wedge dA_1 + \langle d^\dagger \mathcal{B}, dA_3 \rangle, \\
\triangle A_3 &= -\frac{1}{2}d^\dagger (\mathcal{B} \wedge \mathcal{B}) - d^\dagger (\mathcal{B} \wedge dA_1),
\end{align*}
\]

which allows one to define \( A_1 \) and \( A_3 \).

The solution to this set of equations can be found as a perturbation series with respect to the torsion field \( \mathcal{B} \). It is nothing but a standard quantum mechanical perturbation series, which is, however, essentially simplified due to supersymmetry. Indeed, the vacuum energy remains zero, and so we have to solve not the second order Schrödinger equation, but rather the first order equations.

For \( S^7,8 \), we are obliged to include a 6-form in the Ansatz, \( \Phi = 1 + Y_2 + Y_4 + Y_6 \). We obtain the following equations

\[
\begin{align*}
dY_2 &= dB, & d^\dagger Y_2 - \langle d^\dagger B, Y_2 \rangle &= 0, \\
dY_4 &= dB \wedge Y_2, & d^\dagger Y_4 - \langle d^\dagger B, Y_6 \rangle &= 0, \\
dY_6 &= dB \wedge Y_4, & d^\dagger Y_6 &= 0.
\end{align*}
\]

A general solution of the equations in the left column is

\[
\begin{align*}
Y_2 &= \mathcal{B} + dA_1, \\
Y_4 &= \frac{1}{2} \mathcal{B} \wedge \mathcal{B} + \mathcal{B} \wedge dA_1 + dA_3, \\
Y_6 &= \frac{1}{6} \mathcal{B} \wedge \mathcal{B} \wedge \mathcal{B} + \frac{1}{2} \mathcal{B} \wedge \mathcal{B} \wedge dA_1 + \mathcal{B} \wedge dA_3 + dA_5
\end{align*}
\]

with arbitrary \( A_{1,3,5} \). Choosing the gauge \( d^\dagger A_1 = d^\dagger A_3 = d^\dagger A_5 = 0 \), using the expansions (4.14), (4.15) and

\[ A_5 = A_5^{(3)} + A_5^{(5)} + A_5^{(7)} + A_5^{(9)} + \cdots \]
we find all the components in the decompositions (4.14), (4.15), (4.18) step by step from the equations in the right column in (4.16), similarly to the $S^{5,6}$ case.

The solutions can be represented in the following nice form, 

\[ \Phi_B = e^B (1 + dA_1 + dA_3 + \ldots + dA_{n-2}) \] (4.19) 

for odd-dimensional spheres and 

\[ \Phi_B = \left[ e^B - \frac{1}{(n/2)!} B^{n/2} \right] (1 + dA_1 + dA_3 + \ldots + dA_{n-3}) \] (4.20) 

for even-dimensional ones.

Up to now we only constructed the solution obtained by a perturbation of the constant 0-form due to nonzero $B$. But the same analysis can be done for the volume form $V_n$ by duality. For example, for $S^7$, we can start from the Ansatz  

\[ \Phi = V_7 - \langle \bar{Y}_2, V_7 \rangle + \langle \bar{Y}_4, V_7 \rangle - \langle \bar{Y}_6, V_7 \rangle, \] (4.21) 

with arbitrary $\bar{Y}_{2,4,6}$. The latter satisfy exactly the same equations as before, with the same solutions.

The solution (4.19) can be recast as $\Phi_B = e^B + d_B [e^B (A_1 + \ldots + A_{n-2})]$, i.e. it belongs to the cohomology class 

\[ \Phi_B = e^B + d_B X. \] (4.22) 

The solution (4.20) can be presented as $e^B - \frac{1}{(n/2)!} B^{n/2} + d_B [e^B (A_1 + \ldots + A_{n-2})]$. It belongs to a mixture of the cohomology class (4.22) and the class  

\[ \hat{\Phi}_B = V_n + d_B \hat{X}. \] (4.23) 

The results (4.19) and (4.20) can be easily translated into the “physical” notation. For example, (4.19) describes the wave functions of the form 

\[ \Phi_B = e^{\psi M \psi N B_{MN}} (1 + \psi M_1 \psi N \partial_M A_N + \ldots + \psi M_n \ldots \psi M_n \partial_{M_1} A_{M_2} \ldots A_{M_n}). \] (4.24) 

This expression for the ground state wave function matches well with the representation (2.23) for the supercharges.

### 4.2 Dolbeault complex with torsions

Consider first the Lagrangian (1.8) without the extra torsion terms. The number of vacuum states is given by the Atiyah-Singer theorem. The latter is widely known when the manifold is Kähler and the index of the Dolbeault operator coincides with the standard Dirac index.\(^6\) For example, in the $\mathbb{C}P^n$ case with the additional condition  

\[ W = \frac{q}{2(n+1)} \ln \det h \newline = -\frac{q}{2} \ln (1 + \bar{z}z) \] (4.25) 

\(^6\)In the non-Kähler case, there are certain complications, but the Atiyah-Singer theorem still can be formulated and proven. The physical proof was given in a recent paper [29].
(this choice of $W$ defines what is called the **canonical** or **determinant** bundle), the index is equal to

$$I_{\mathbb{C}P^n} = \left( q + \frac{(n-1)}{2} \right),$$

(4.26)

where $q$ must be integer for odd $n$ and half-integer for even $n$. For other values of $q$, one cannot consistently define the Hilbert space, where the spectrum of the Hamiltonian is supersymmetric [30]. This means that, in contrast to the real case, introduction of the term $\sim W$ in the action cannot be considered as a smooth deformation, and this is the reason why the index (4.26) depends on $q$.

When $|q| < \frac{n+1}{2}$, the zero-energy vacuum states are absent, indicating the spontaneous breaking of supersymmetry in this case

When $q = \pm \frac{n+1}{2}$, we are facing the Dolbeault (respectively, anti-Dolbeault) complex and there is only one vacuum state in the sector $(p, q) = (0, 0)$ (respectively, in the sector $(p, q) = (n, n)$). For larger values of $|q|$, we are dealing with the twisted Dolbeault complex (with an additional gauge field) and there are several such states. Let us first discuss the pure Dolbeault complex with $q = \frac{n+1}{2}$. The supercharges $Q, \bar{Q}$ can be interpreted in this case as the operators of exterior holomorphic derivative and its Hermitian conjugate.

The torsion term (3.1) can be introduced as a smooth deformation, and the index cannot change. Thus, one can expect the system (3.12) to have exactly the same number of vacuum states (4.26) as in the case of $B = 0$. These states can be constructed along the same lines as for the de Rahm complex.

In simplest nontrivial cases, $\mathbb{C}P^3, \quad q = 2$, and $\mathbb{C}P^4, \quad q = \frac{5}{2}$, the deformed vacuum wave functions can be found *exactly*. Indeed, we can adopt the Ansatz $\Phi = 1 + Y_{(2,0)}$, where $Y_{(2,0)}$ is a $(2,0)$-form. By analogy with (4.7), (4.8), the solution to the equations $\partial_B \Phi = \partial_B^\dagger \Phi = 0$ is $Y_{(2,0)} = B - \partial \triangle^{-1} \partial^\dagger B$, where $\triangle$ is the Laplacian (for Kähler manifolds, there is only one covariant Laplacian, $\triangle = \partial \partial^\dagger + \partial^\dagger \partial = \bar{\partial} \bar{\partial}^\dagger + \bar{\partial}^\dagger \bar{\partial}$).

The analysis for $\mathbb{C}P^n$ with $n > 4$ repeats without changes the analysis given above for $S^n, n > 4$. One should replace $\langle dB, \cdot \rangle$ in all formulas by $\langle \partial B, \cdot \rangle$ and also substitute everywhere $\partial, \partial^\dagger$ for $d, d^\dagger$. The solutions are

$$\Phi_B = e^B \left( 1 + \partial A_{(1,0)} + \partial A_{(3,0)} + \ldots + \partial A_{(n-2,0)} \right)$$

(4.27)

for odd $n$ and

$$\Phi_B = e^B - \left( B^{n/2} \right) \left( 1 + \partial A_{(1,0)} + \partial A_{(3,0)} + \ldots + \partial A_{(n-3,0)} \right)$$

(4.28)

for even $n$. All $(n,0)$-form $A_{(n,0)}$ can be defined as series in $B$ after fixing the gauges $\partial^\dagger_{(n,0)} A_{(n,0)} = 0$.

Like for the de Rham complex, the presence of the multiplier $e^B$ in the solution (4.27) is rather natural, bearing in mind the representation $\partial_B = e^B \partial e^{-B}$. In the physical notations, it reads (cf. (2.23)):

$$Q^{\text{cov}} = e^{\psi^k B_{ik}} \|_{B=0} e^{-\psi^k B_{ik}}$$

(4.29)

This in turn is related to spontaneous breaking of supersymmetry in $d = 3$ supersymmetric Yang-Mills-Chern-Simons theory [31, 32].
where $Q_{B=0}^{\text{cov}}$ is the supercharge (3.12) without torsion terms. For convenience of the reader who prefers the more traditional language to the language of differential forms (which is most convenient and adequate, in our opinion), in Appendix we present solutions of the vacuum equations (4.11) for some particular $\mathbb{C}P^n$ cases with $q = \frac{n+1}{2}$, using an equivalent tensorial notation.

Note that both (4.27) and (4.28) belong to the cohomology class $e^B + \partial_B X$. For (4.27), it is clear, and (4.28) differs from (4.27) by the $(n,0)$ - form $B_{n/2}$. In contrast to an n-form for $S^n$, it is exact, $B_{n/2} = \partial Y$. It is guaranteed by the Hodge decomposition theorem for the untwisted complex, where the term $\partial^\dagger Z$ is absent because it is the highest holomorphic form and the zero modes are absent because $\beta_{(n,0)} = 0$ for $\mathbb{C}P^n$. We finally note that $Y$ is also $\partial$-exact as, for a $(n-1,0)$ - form $Y$, $\partial Y = \partial_B Y$.

When $q > \frac{n+1}{2}$, we are facing the twisted Dolbeault complex. The equations are then somewhat more complicated. When $B = 0$, the vacuum states $\Phi_0$ should be defined by the equation

$$\partial_{W'} \Phi_0 = e^{W'} \partial \left( e^{-W'} \Phi_0 \right) = 0, \quad (4.30)$$

where $W'$ is the superpotential renormalized by the shift $q \to 2s := q - \frac{n+1}{2}$ (see [17] for details),

$$W' = -s \ln(1 + \bar{z}z). \quad (4.31)$$

The solution is then

$$\Phi_0 = e^{W'} R(\bar{z}), \quad (4.32)$$

where $R(\bar{z})$ is a polynomial of $\bar{z}^j$ of the degree not higher than $2s$ (to keep normalizability of (4.32)) [33]. The index (4.26) is none other than a number of coefficients in this polynomial.

Consider the simplest nontrivial $\mathbb{C}P^3$ case. Seek for the deformed vacuum wave function in the form

$$\Phi_B = (1 + Y_{(2,0)}) \Phi_0.$$

The equations for $Y_{(2,0)}$ are

$$\partial_{W'}(Y_{(2,0)} \Phi_0) = \partial_{W'}(B \Phi_0),$$

$$\partial_{W'}^\dagger (Y_{(2,0)} \Phi_0) = 0 \quad (4.33)$$

with $\partial_{W'} = e^{W'} \partial_e^{e^{-W'}}$. A generic solution of the first equation in (4.33) is

$$Y_{(2,0)} = B + \Phi_0^{-1} \partial_{W'} A_{(1,0)}. \quad (4.34)$$

By the Hodge decomposition theorem with respect to the operators $\partial_{W'}$, $\partial_{W'}^\dagger$ (it is valid as the operators $\partial_{W'}$ and $\partial_{W'}^\dagger$ satisfy the standard $\mathcal{N} = 2$ supersymmetry algebra) and from the fact that no zero modes of the Hamiltonian $H_{W'} = \partial_{W'} \partial_e^{e^{-W'}} + \partial_{W'} \partial_{W'}^\dagger$ exist in the $(1,0)$ sector, the form $A_{(1,0)}$ can be represented as $A_{(1,0)} = \partial_{W'} \omega^{(0,0)} + \partial_{W'}^\dagger \omega^{(2,0)}$. 

19
Let us choose a gauge, in which the first term is absent, so that $\partial \tilde{W} A_{(1,0)} = 0$. Then the second equation in (4.33) acquires the form

$$H_{W'} A_{(1,0)} = -\partial W B.$$ 

The Hamiltonian $H_{W'}$ is positive-definite in the sector of $(1,0)$-forms and so can be inverted. This gives us the form $A_{(1,0)}$ and the solution (4.34).

The solutions for $\mathbb{C}P^n$ manifolds with higher $n$ have the form (4.27), (4.28), where one should make the substitution $\partial O \rightarrow \Phi^{-1} \partial W O = \partial (\Phi^{-1} O)$.

5 Summary and outlook

In this paper we have studied the models of torsionful $\mathcal{N}=2$ supersymmetric quantum mechanics based on the supermultiplets $(1,2,1)$ and $(2,2,0)$. These models are more general than those which were studied in the literature up to now. For instance, the general $\mathcal{N}=2$ model based on a sum of the superfield Lagrangians (1.8) and (3.1) was considered before basically at the classical level, while its quantum version, including the explicit form of the relevant $\mathcal{N}=2$ supercharges, was known only for a few particular cases [19, 17]. Also, the quantum models associated with the multiplet $(1,2,1)$ were known either for the basic action (1.5) (plus the potential term (1.15)), or for its modification obtained by adding the action (1.17) with the real torsion potential [22] and without including any higher-order terms like (1.18). It should be pointed out that torsions of a certain special form always appear for non-Kähler complex sigma model [17], but in the present paper we were interested in the models involving some extra torsion terms in the Lagrangians and supercharges that are not related to the bosonic target space metric.

In all considered cases we constructed the corresponding quantum $\mathcal{N}=2$ superalgebra. The general prescription is the use of the Weyl-ordered supercharges with subsequent passing to the covariant supercharges which act in the Hilbert space with the geometrically motivated inner product. Knowing these quantum supercharges and interpreting them in terms of de Rahm (in the $(1,2,1)$ case) and Dolbeault (in the $(2,2,0)$ case) complexes allowed us to explicitly find the vacuum states in the considered models and check that their number does not change after switching on the torsions. Such invariance is ensured by the index theorem, enforced by a simple mathematical argument that the cohomologies for the twisted de Rham complex and for the untwisted complex are the same [27]. The explicit construction of these states (we did it in the framework of the perturbative expansion over the torsion $B$) is a new result.

In this paper we have considered $\mathcal{N}=2, d=1$ supersymmetric models which are in one-to-one correspondence with the Rham complex and the Dolbeault complexes (untwisted and twisted). It is interesting to explore, along the same geometric lines, the sigma models associated with various $\mathcal{N}=4$ supermultiplets. First, the number of different off-shell $\mathcal{N}=4$ supermultiplets is considerably larger than that of $\mathcal{N}=2$ supermultiplets, which could lead to more opportunities for the geometrical treatment of the corresponding models in terms of various complexes. In particular, we expect to recover the so called quaternionic Dolbeault complex (see, e.g., [34] and refs. therein) within such a context.
Second, these systems are much richer, and so they could require some new means for the construction of the corresponding quantum theories.

Even in the $\mathcal{N}=2$ case, there exists a class of models which until now were not well studied and geometric interpretation of which is unknown. They are based on the multiplets $(1,2,1)$ described by the superfield action in (1.5) in which the metric $g_{MN}(X^P)$ contains both symmetric and antisymmetric parts. Though the existence of such $\mathcal{N}=2$ models was mentioned in [11, 12], no special attention was paid to them afterwards.

Finally, in this paper we only studied sigma models with $\mathcal{N}=2$ supermultiplets of the same type. Of interest are also the models in which different types of supermultiplets enter simultaneously. In particular, it is worthwhile to consider the models with isometries, a part of which is gauged (see [35] for the $d=1$ gauging procedure).

We will try to address these issues in the future.

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Appendix

Sometimes it is perhaps useful to have a more explicit form of the equations for the vacuum wave functions (4.1) and their solutions. Here we present a few examples related to the $B$-deformed $\mathbb{CP}^n$ models with the condition (4.25) and $q = \frac{n+1}{2}$ (i.e. those associated with the untwisted Dolbeault complex).

In the deformed $\mathbb{CP}^3$ case with $q = 2$ Eqs. (4.1) for the wave function $\Phi = 1 + Y_{ik}^{(2)}\psi^i\psi^k$ amount to the following set of explicit equations

$$
\partial_{[i}(Y_{kl}^{(2)} - B)_{kl]} = 0, \quad Y_{kl}^{(2)} = B_{kl} + \partial_{[k}A_{l]},
$$

(A.1)

$$
h^{ik}\partial_k Y_{il}^{(2)} = 0.
$$

(A.2)

Using the gauge freedom $A_t \rightarrow A_t + \partial_t \omega$, one can choose the gauge $h^{ik}\partial_k A_t = 0$ and reduce (A.2) to

$$
\Delta A_t = -2h^{ik}\partial_k B_{il},
$$

(A.3)

where $\Delta = h^{ik}\nabla_i \partial_k$ is the covariant Laplace-Beltrami operator. Then Eq. (A.3) can be solved for $A_t$ in terms of $\partial_k B_{il}$,

$$
A_t = -2\Delta^{-1}h^{ik}\partial_k B_{il}.
$$

(A.4)
For the $\mathbb{C}P^4$ case, with $q = \frac{5}{2}$ and $\Phi = 1 + Y^{(2)}_{ik} \psi^i \psi^k + Y^{(4)}_{iklm} \psi^i \psi^j \psi^l \psi^m$, we have the following system

\[
\partial_{[i} (Y^{(2)} - B)_{kl]} = 0 ,
\]
\[
h^{ik} \partial_k Y^{(2)}_{il} + 12 h^{ik} h^{m} \partial_{[k} \bar{B}_{ij]} Y^{(4)}_{ilm} = 0 ,
\]
\[
h^{ik} \partial_k Y^{(4)}_{ilmn} = 0 .
\]

The last equation implies $\partial_k Y^{(4)}_{ilmn} = 0$, whence

\[
Y^{(4)}_{ilmn} = 0
\]

and we are left with the same solution as in the $\mathbb{C}P^3$ case.

In the $\mathbb{C}P^5$ case, with $q = 3$ and $\Phi = 1 + Y^{(2)}_{ik} \psi^i \psi^k + Y^{(4)}_{iklm} \psi^i \psi^j \psi^l \psi^m$, Eqs. (A.5) - (A.7) are supplemented by the following new equation

\[
\partial_{[i} Y^{(4)}_{kjlm]} - \partial_{[i} B_{kj} Y^{(2)}_{lm]} = 0 ,
\]

which can be solved as

\[
Y^{(4)}_{iklm} = \frac{1}{2} B_{[ik} B_{lm]} + B_{[ik} \partial_l A_m] + \partial_{[i} \Omega_{klm]} ,
\]

where $\Omega_{klm}$ is a new totally antisymmetric function with its own gauge freedom $\Omega_{klm} \rightarrow \Omega_{klm} + \partial_{[k} \omega_{lm]}$. Eqs. (A.6) and (A.7) can be used to solve for $A_m$ and $\Omega_{klm}$ in terms of $B_{ik}$, $\bar{B}_{ik}$ as perturbation series with respect to these fields. For instance, in the lowest order $A_l$ is still given by the expression (A.4), while $\Omega_{klm}$ subjected to the gauge condition $h^{ik} \partial_k \Omega_{ilm} = 0$ is determined from Eq. (A.7) as

\[
\Omega_{ilm} = -2 \Delta^{-1} h^{ik} \partial_k \left[ B_{[il} B_{mn]} - 2 B_{[il} \partial_m \Delta^{-1} h^{ip} \partial_p B_{tm]} \right] .
\]

The solutions (A.1), (A.8) and (A.10) nicely match with the general formulas (4.27) and (4.28).

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