RIEMANN HYPOTHESIS AND THE ARC LENGTH OF THE Riemann \( Z(t) \)-CURVE

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ABSTRACT. On Riemann hypothesis it is proved in this paper that the arc length of the Riemann \( Z \)-curve is asymptotically equal to the double sum of local maxima of the function \( Z(t) \) on corresponding segment. This paper is English remake of our paper [9], with short appendix concerning new integral generated by Jacob’s ladders added.

1. INTRODUCTION AND RESULT

1.1. Main object of this paper is the study of the integral

\[
\int_{T}^{T+H} \sqrt{1 + \{Z'(t)\}^2} \, dt,
\]

i.e. the study of the arc length of the Riemann curve

\( y = Z(t), \ t \in [T, T + H], \ T \to \infty, \)

where (see [13], pp. 79, 329)

\[
Z(t) = e^{i\vartheta(t)} \left( \frac{1}{2} + it \right),
\]

(1.2)

\[
\vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + it \right) = \frac{t}{2\pi} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O \left( \frac{1}{t} \right).
\]

Remark 1. Let us remind that the formula

\[
\{Z(t) = \} \ e^{i\vartheta(t)} \left( \frac{1}{2} + it \right) =
\]

(1.3)

\[
= 2 \sum_{n \leq \sqrt{T}} \frac{1}{\sqrt{n}} \cos \{ \vartheta(t) - t \ln n \} + O(t^{-1/4}), \ \bar{t} = \sqrt{T} / 2\pi
\]

was known to Riemann (see [11], p. 60, comp. [12], p. 98).

Next, we will denote the roots of the equations

\( Z(t) = 0, \ Z'(t) = 0, \ t_0 \neq \gamma \)

by the symbols

\( \{\gamma\}, \ \{t_0\}, \)

correspondingly.

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Remark 2. On the Riemann hypothesis, the points of the sequences \( \{\gamma\} \) and \( \{t_0\} \) are separated each from other (see [3], Corollary 3), i.e., in this case we have
\[ \gamma' < t_0 < \gamma'' , \]
where \( \gamma', \gamma'' \) are neighboring points of the sequence \( \{\gamma\} \). Of course, \( Z(t_0) \) is local extremum of the function \( Z(t) \) located at \( t = t_0 \).

1.2. In this paper we use the Riemann hypothesis together with some synthesis of properties of the sequences \( \{t_0\}, \{h_\nu(\tau)\} \), where the numbers \( h_\nu(\tau) \) are defined by the equation (comp. (1.2))
\[ \vartheta_1[h_\nu(\tau)] = \pi \nu + \tau + \frac{\pi}{2}, \nu = 1, 2, \ldots, \tau \in [-\pi, \pi] , \]
\[ \vartheta_1(t) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} , \]
\[ \vartheta(t) = \vartheta_1(t) + O \left( \frac{1}{t} \right) , \]
in order to obtain the following theorem.

Theorem. On the Riemann hypothesis we have the asymptotic formula
\[ \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} dt = 2 \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \Theta H + O \left( \frac{\sqrt{H}}{\ln H} \right) , \]
\[ \Theta = \Theta(T, H) \in (0, 1), H = T^\epsilon, T \to \infty \]
for every fixed \( \epsilon > 0 \).

Remark 3. Geometric meaning of our asymptotic formula (1.5) is as follows: the arc length of the Riemann curve
\[ y = Z(t), t \in [T, T+H] \]
is asymptotically equal to the double of the sum of local maxima of the function
\[ |Z(t)|, t \in [T, T+H] . \]

2. DISCRETE FORMULAE – LEMMA 1

2.1. In this part of the paper we use the following formula
\[ Z'(t) = -2 \sum_{n<P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sin \{ \vartheta - t \ln n \} + \]
\[ + O(T^{-1/4} \ln T), P = \sqrt{\frac{2\pi}{T}} \]
that we have obtained in our work [6], (see (2.1)). Next, we obtain from (2.1) in the case
\[ \vartheta \to \vartheta_1 \]
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(see (1.4)) that

\[ Z'(t) = -2 \sum_{n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sin \{ \vartheta_1 - t \ln n \} + \]

\[ + \mathcal{O}(T^{-1/4} \ln T), \quad H \in (0, \sqrt{T}). \]

Let \( S(a, b) \) denotes elementary trigonometric sum

\[ S(a, b) = \sum_{a \leq n \leq b} n^i, \quad 1 \leq a < b \leq \ldots \]

Then we obtain from (2.2) in the case of the sequence \( h_\nu(\tau) \) (see (1.4)) the following

Lemma 1. If

\[ |S(a, b)| \leq A(\Delta) \sqrt{\pi}, \quad \Delta \in (0, 1/6] \]

then \((h_\nu = h_\nu(0))\)

\[ \sum_{T \leq h_\nu \leq T + H} Z'[h_\nu(\tau)] = -\frac{1}{\pi} H \ln^2 P \cos \tau + \mathcal{O}(T^\Delta \ln^2 T), \]

\[ \sum_{T \leq h_\nu + 1 \leq T + H} Z'[h_\nu(\tau)] = \frac{1}{\pi} H \ln^2 P \cos \tau + \mathcal{O}(T^\Delta \ln^2 T), \]

where \( \mathcal{O} \)-estimates are uniform for \( \tau \in [-\pi, \pi] \).

Proof. We obtain from (2.2) by (1.4)

\[ Z'[h_\nu(\tau)] = 2(-1)^{\nu+1} \ln P \cos \tau - \]

\[ - \frac{1}{2} \sum_{2 \leq n \leq P} \frac{1}{\sqrt{n}} \ln P \frac{\cos \{ \pi \nu - h_\nu(\tau) \ln n + \tau \} +}{n} \]

\[ + \mathcal{O}(T^{-1/4} \ln T), \quad h_\nu(\tau) \in [T, T + H]. \]

\[ \square \]

2.2. Since (see [5], (23))

\[ \sum_{T \leq h_\nu \leq T + H} 1 = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + \mathcal{O}(1) = \frac{1}{\pi} H \ln P + \mathcal{O}(1), \]

then we obtain from (2.5) (comp. [1], (59)-(61), [5], (51)-(53)) that

\[ \sum_{T \leq h_\nu \leq T + H} Z'[h_\nu(\tau)] = -2 \tilde{\omega}(T, H; \tau) + \mathcal{O}(\ln^2 T), \]

where

\[ \tilde{\omega} = \frac{1}{2} (-1)^\nu \sum_{n} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos \varphi + \]

\[ + \frac{1}{2} (-1)^{N+\nu} \sum_{n} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \cos(\omega N + \varphi) + \]

\[ + \frac{1}{2} (-1)^{\nu} \sum_{n} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \tan \frac{\omega}{2} \sin \varphi + \]

\[ + \frac{1}{2} (-1)^{N+\nu+1} \sum_{n} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \tan \frac{\omega}{2} \sin(\omega N + \varphi), \]

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where
\[ \omega = \pi \frac{\ln n}{\ln P}, \quad \varphi = h_\nu(\tau) \ln n - \tau, \quad n \in [2, P), \]
and
\[ \bar{\nu} = \min\{\nu : h_\nu \in [T, T + H]\}, \quad \bar{\nu} + N = \max\{\nu : h_\nu \in [T, T + H]\}. \]

Of course, we have
\[ \sum_{T \leq h_\nu(T) \leq T + H} 1 = \sum_{T \leq h_\nu \leq T + H} 1 + O(1) \]
for any fixed \( \tau \in [-\pi, \pi] \). Now, it is clear that the method \([6]\), (54)-(64) implies by (2.3) that
\[ \bar{w} = O(T^\Delta \ln^2 T) \]
uniformly for \( \tau \in [-\pi, \pi] \), and consequently we obtain (see (2.7)) the estimate
\[ (2.8) \quad \sum_{T \leq h_\nu \leq T + H} Z'[h_\nu(\tau)] = O(T^\Delta \ln^2 T) \]
uniformly for \( \tau \in [-\pi, \pi] \).

2.3. Next, we have (see (2.5), (2.6))
\[ \sum_{T \leq h_\nu \leq T + H} (-1)^\nu Z'[h_\nu(\tau)] = -\frac{2}{\pi} H \ln^2 P \cos \tau - 2R + O(\ln^2 P), \]
\[ R = \sum_{2 \leq n < P} \frac{1}{\sqrt{n}} \ln \frac{P}{n} \sum_{T \leq h_\nu \leq T + H} \cos\{h_\nu(\tau) \ln n - \tau\}. \]

Since by (2.3) and \([6]\), (65)-(79) we have the estimate
\[ R = O(T^\Delta \ln^2 T) \]
then we obtain the formula
\[ (2.9) \quad \sum_{T \leq h_\nu \leq T + H} (-1)^\nu Z'[h_\nu(\tau)] = -\frac{2}{\pi} H \ln^2 P \cos \tau + O(T^\Delta \ln^2 T) \]
uniformly for \( \tau \in [-\pi, \pi] \).

Finally, from (2.8), (2.9) formulae (2.4) follow.

3. Integrals over disconnected sets – Lemma 2

Let (comp. \([7]\), (3))
\[ G_{2\nu}(x) = \{t : h_{2\nu}(-x) < t < h_{2\nu}(x), \quad t \in [T, T + H]\}, \quad x \in (0, \pi/2], \]
\[ G_{2\nu+1}(y) = \{t : h_{2\nu+1}(-y) < t < h_{2\nu+1}(y), \quad t \in [T, T + H]\}, \quad y \in (0, \pi/2], \]
\[ G_1(x) = \bigcup_{T \leq h_{2\nu} \leq T + H} G_{2\nu}(x), \]
\[ G_2(y) = \bigcup_{T \leq h_{2\nu+1} \leq T + H} G_{2\nu+1}(y). \]

The following lemma holds true.
Lemma 2. (2.3) implies
\[
\int_{G_1(x)} Z'(t) dt = -\frac{2}{\pi} H \ln P \sin x + O(x T^\Delta \ln T),
\]
(3.2)
\[
\int_{G_2(y)} Z'(t) dt = \frac{2}{\pi} H \ln P \sin y + O(y T^\Delta \ln T).
\]

Proof. First of all we have (see (1.4), comp. [7], (51))
\[
\left(\frac{dh_{2\nu}(\tau)}{d\tau}\right)^{-1} = \vartheta_1'[h_{2\nu}(\tau)] = \ln P + O\left(\frac{H}{T}\right).
\]

Next, from (2.2) by (2.3) we obtain the estimate
\[
Z'(t) = O\left(T^\Delta \ln^2 T\right), \quad t \in [T, T + H]
\]
(ABEL transformation). Then we have (comp. [7], (52)) that
\[
\int_{-x}^{x} Z'[h_{2\nu}(\tau)] d\tau = \int_{-x}^{x} Z'[h_{2\nu}(\tau)] \left(\frac{dh_{2\nu}(\tau)}{d\tau}\right)^{-1} \frac{d}{d\tau} h_{2\nu}(\tau) d\tau =
\]
(3.3)
\[
= \ln P \int_{G_0(x)} Z'(t) dt + O\left(x H T^\Delta \ln T\right) = \ln P \int_{G_0(x)} Z'(t) dt + O\left(x H T^\Delta \ln T\right).
\]

Consequently, we obtain from the first formula in (2.4) by (2.6), (3.1), (3.3) the following asymptotic equality
\[
\int_{G_1(x)} Z'(t) dt = -\frac{2}{\pi} H \ln P \sin x + O(x T^\Delta \ln T) + O\left(x H^2 T^{-5/6} \ln T\right),
\]
i.e. the first integral in (3.2). The second integral can be derived by a similar way. \(\square\)

4. An estimate from below – Lemma 3

The following lemma holds true.

Lemma 3. From (2.3) the estimate
\[
\int_{T}^{T+H} |Z'(t)| dt > \frac{4}{\pi} (1 - \epsilon) H \ln P, \quad P = \sqrt{\frac{T}{2\pi}}, \quad H \in [T^{\Delta+\epsilon}, \sqrt{T}]
\]
follows, where \(\epsilon > 0\) is an arbitrarily small number.

Proof. Let (comp. [8], (10))
\[
G_1^+(x) = \{t : Z'(t) > 0, \ t \in G_1(x)\}, \\
G_1^- (x) = \{t : Z'(t) < 0, \ t \in G_1(x)\}, \\
G_1^0 (x) = \{t : Z'(t) = 0, \ t \in G_1(x)\},
\]
and the symbols
\[
G_2^+(y), G_2^-(y), G_2^0 (y)
\]
have similar meaning. Of course
\[
m\{G_1^+(x)\} = m\{G_2^+(y)\} = 0.
\]
Since the expressions (3.2) in the case
\[ H \in [T^{\Delta + \epsilon}, \sqrt{T}], \quad x, y \in (0, \pi/2) \]
are asymptotic formulae then from them we obtain the following inequalities
\[ \frac{2}{\pi} (1 - \epsilon) H \ln P < - \int_{G_1(\pi/2)} Z'(t) dt \leq \]
\[ \leq - \int_{G_1^-(\pi/2)} Z'(t) dt = \int_{G_1^{-} (\pi/2)} |Z'(t)| dt, \]
\[ \frac{2}{\pi} (1 - \epsilon) H \ln P < \int_{G_2(\pi/2)} Z'(t) dt \leq \int_{G_2^- (\pi/2)} |Z'(t)| dt. \]

Since \( G_1^{-} (\pi/2) \cup G_2^{-} (\pi/2) \subset [T, T + H], \ G_1^{-} (\pi/2) \cap G_2^{-} (\pi/2) = \emptyset \)
then by (4.2) needful estimate
\[ \int_T^{T+H} |Z'(t)| dt \geq \int_{G_1^- (\pi/2)} |Z'(t)| dt + \int_{G_2^- (\pi/2)} |Z'(t)| dt > \]
\[ > \frac{4}{\pi} (1 - \epsilon) H \ln P. \]
follows. \( \square \)

5. Quadrature formula – Lemma 4

The following lemma holds true.

**Lemma 4.** On Riemann hypothesis we have the following asymptotic formula
\[ \int_T^{T+H} |Z'(t)| dt = 2 \sum_{T \leq t_0 \leq T+H} |Z(t_0)| + \]
\[ + O \left( T^{\frac{\Delta}{1+\mu}} \right), \quad H \in [T^\mu, \sqrt{T}], \]
where \( 0 < \mu \) is an arbitrary small number.

**Proof.** First of all, we have on Riemann hypothesis the following two Littlewood’s estimates
\[ \gamma'' - \gamma' < \frac{A}{\ln \ln \gamma}, \quad \gamma' \to \infty \]
(see [2], p. 237), and
\[ Z(t) = O \left( t^{\frac{\Delta}{1+\mu}} \right), \quad t \to \infty \]
(see [13], p. 300). Next, on Riemann hypothesis we have the following basic configuration (see Remark 2)
\[ \gamma' < t_0 < \gamma'' \ ; \ t_0 \in [T, T + H]. \]

Now, there are following possibilities (see (5.4)): either
\[ Z(t) > 0, \ t \in (\gamma', \gamma'') \Rightarrow \]
\[ Z'(t) > 0, \ t \in (\gamma', t_0), \ Z'(t) < 0, \ t \in (t_0, \gamma''). \]
or
\[ Z(t) < 0, \ t \in (\gamma', \gamma'') \Rightarrow \]
\[ Z'(t) < 0, \ t \in (\gamma', t_0), \ Z'(t) > 0, \ t \in (t_0, \gamma''). \]
Consequently, (5.5) and (5.6) imply that
\[ \int_{\gamma'}^{\gamma''} |Z'(t)| \, dt = 2|Z(t_0)|, \ \forall t_0 \in [T, T + H]. \]
Similarly, we obtain (see (5.2), (5.3)) the estimates
\[ \int_{\beta'}^{\beta''} |Z'(t)| \, dt, \int_{\bar{\gamma}'}^{\bar{\gamma}''} |Z'(t)| \, dt = O \left( \frac{T^{\frac{\mu}{2} + \epsilon}}{\ln \ln T} \right) \]
in the following cases
\[ \bar{\gamma}' < T \leq t_0 < \bar{\gamma}'', \ \bar{\beta}' < t_0 \leq T + H < \bar{\beta}''. \]
Now, our formula (5.1) follows from (5.7), (5.8).

6. Proof of Theorem

We use the following formula
\[ \int_T^{T+H} \sqrt{1 + \{Z'(t)\}^2} \, dt = \]
\[ = \int_T^{T+H} |Z'(t)| \, dt + \int_T^{T+H} \frac{1}{\sqrt{1 + \{Z'(t)\}^2 + |Z'(t)|}} \, dt. \]
Since
\[ 0 < \frac{1}{\sqrt{1 + \{Z'(t)\}^2 + |Z'(t)|}} \leq 1 \]
and
\[ \left| \frac{1}{\sqrt{1 + \{Z'(t)\}^2 + |Z'(t)|}} \right|_{t=t_0} = 1, \ t_0 \in [T, T + H], \]
i.e. the inequality (6.2) holds true for the finite set of values, then the mean-value theorem gives
\[ \int_T^{T+H} \frac{1}{\sqrt{1 + \{Z'(t)\}^2 + |Z'(t)|}} \, dt = \Theta H, \ \Theta = \Theta(T, H) \in (0, 1). \]
Next, we obtain by (4.1), (5.1), (\( \mu \leq \epsilon \)), the inequality
\[ \frac{4}{\pi} (1 - \epsilon) H \ln P < \int_T^{T+H} |Z'(t)| \, dt = \]
\[ = 2 \sum_{T \leq t_0 \leq T+H} |Z'(t_0)| + O \left( T^{\frac{\mu}{2} + \epsilon} \right). \]
Hence, by (6.1)-(6.4) the formula (1.5) follows for
\[ H \in [T^{\Delta + \epsilon}, T]. \]
Since the Riemann hypothesis implies Lindelöf hypothesis a it implies that \( \Delta = \epsilon \) (comp. [1], p. 89), then we obtain from (6.5) that
\[ H = T^{2\epsilon}, \ 2\epsilon \rightarrow \epsilon, \]
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APPENDIX A. INFLUENCE OF JACOB’S LADDERS

If
\[ \varphi_1([\hat{T}, \hat{T} + H]) = [T, T + H], \]
then from (1.5) we obtain (see [10], (9.7)) the formula
\[
\int_{T}^{T + H} \sqrt{1 + \{Z'_{\varphi_1}(t)\}^2} \left| \frac{1}{2} + it \right|^2 dt \sim \\
\sim \left\{ 2 \sum_{T \leq t_0 \leq T + H} \left| Z(t_0) \right| + \Theta H + \mathcal{O} \left( T^{\frac{n}{\ln T}} \right) \right\} \ln T, \ T \to \infty.
\]

From (A.1) we obtain by mean-value theorem that
\[
\int_{T}^{T + H} \sqrt{1 + \{Z'_{\varphi_1}(t)\}^2} dt \sim \\
\sim \frac{\ln T}{\left| \frac{1}{2} + i\alpha \right|^2} \left\{ 2 \sum_{T \leq t_0 \leq T + H} \left| \zeta \left( \frac{1}{2} + it_0 \right) \right| + \Theta H + \mathcal{O} \left( T^{\frac{4}{\ln T}} \right) \right\},
\]
\[ \alpha \in (\hat{T}, \hat{T} + H). \]

Remark 4. Since we have (see [10], (8.5))
\[
\rho([T, T + H]; [\hat{T}, \hat{T} + H]) \sim (1 - c)\pi(T) > (1 - c)(1 - c) \frac{T}{\ln T}, \ T \to \infty,
\]
where \( \rho \) denotes the distance of corresponding segments and \( \pi(T) \) is the prime-counting function and \( c \) is the Euler constant, then the formula (A.2) gives strongly non-local expression for the integral on the left-hand side of (A.2).

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