CONSTRUCTION OF A $C^*$-ALGEBRAIC QUANTUM GROUPOID FROM A WEAK MULTIPLIER HOPF ALGEBRA

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Abstract. Van Daele and Wang developed a purely algebraic notion of weak multiplier Hopf algebras, which extends the notions of Hopf algebras, multiplier Hopf algebras, and weak Hopf algebras. With an additional requirement of an existence of left or right integrals, this framework provides a self-dual class of algebraic quantum groupoids. The aim of this paper is to show that from this purely algebraic data, with only some relatively minimal additional requirements (“quasi-invariance”), one can construct a $C^*$-algebraic quantum groupoid of separable type, recently defined by the author, with Van Daele. The $C^*$-algebraic quantum groupoid is represented as an operator algebra on the Hilbert space constructed from the left integral, and the comultiplication is determined by means of a certain multiplicative partial isometry $W$, which is no longer unitary. In the last section (Appendix), we obtain some results in the purely algebraic setting, which have not appeared elsewhere.

0. INTRODUCTION

In a series of papers, Van Daele and Wang introduced a purely algebraic notion of weak multiplier Hopf algebras, which extends the notions of Hopf algebras, multiplier Hopf algebras, and weak Hopf algebras. In short, a weak multiplier Hopf algebra is a pair $(A, \Delta)$, where $A$ is a non-degenerate idempotent algebra and $\Delta$ is a comultiplication, satisfying some number of conditions. They are natural generalizations of Hopf algebras (when $A$ is unital and $\Delta$ is non-degenerate), multiplier Hopf algebras (when $A$ is non-unital and $\Delta$ is non-degenerate), and weak Hopf algebras (when $A$ is unital, but $\Delta(1) \neq 1 \otimes 1$). For a weak multiplier Hopf algebra, the algebra is not assumed to be unital and the comultiplication is no longer assumed to be non-degenerate.

Going further, in [36], they considered the situation in which a weak multiplier Hopf algebra possesses (a faithful family of) left/right invariant functionals (or “integrals”). While in a purely algebraic setting, this framework is known to include all compact and discrete quantum groups, as well as all weak Hopf algebras and finite quantum groupoids. It is true that not all quantum groups/groupoids are contained in this framework, and some classical groups are left out. Nevertheless, it is shown that a dual object can be constructed within this framework, thereby giving rise to a nice self-dual class of algebraic quantum groupoids.

This purely algebraic framework provided a strong motivational basis for a $C^*$-algebraic framework of locally compact quantum groupoids of separable type, by Van Daele and the author [10], [11]. There is a strong resemblance between the purely algebraic framework (= weak multiplier Hopf algebras) and the $C^*$-framework (= locally compact quantum groupoids of

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separable type). Having said this, it has never been made explicit whether there is indeed a direct pathway from the purely algebraic setting to the $C^*$-setting without too many additional requirements. Naively speaking, this is about constructing a $C^*$-completion of the algebra. But there are some subtle issues to consider.

The main purpose of this paper is to clarify that we can indeed carry out the construction of a $C^*$-algebraic locally compact quantum groupoid (in the sense of [10], [11]) out of a weak multiplier Hopf $^*$-algebra equipped with a faithful left (or right) integral. Just as the work of Kustermans and Van Daele of similar nature ([16]), constructing a $C^*$-algebraic quantum group from a purely algebraic object of a multiplier Hopf algebra, made fundamental contributions to the development of the theory of locally compact quantum groups, we hope that the current work can help us understand better the theory of quantum groupoids.

The paper is organized as follows: In Section 1, we give an overview of the purely algebraic framework of weak multiplier Hopf algebras. This is needed, not only for motivational purposes but for clarifying the ingredients necessary for the construction of the $C^*$-algebraic quantum groupoid in what follows. The definition and the properties of weak multiplier Hopf algebras are given in this section, though most of the detailed proofs are skipped (Instead, the relevant theorems elsewhere are referred to.) We do not, however, describe the original definition (as in [34]). Instead, we take a more recent but equivalent approach, as in [8]: In that paper, it is shown that a regular weak multiplier bialgebra (in the sense of [2]) becomes a weak multiplier Hopf algebra, if it has sufficient number of left and right integrals. Being more recent is one thing, but actually, for our purposes of working with the integrals and studying the duality, this characterization turns out to be more convenient.

One aspect of note is that we get to consider the case of a weak multiplier Hopf $^*$-algebra here. While the discussion about the case with an involution has appeared in the original literatures on weak multiplier Hopf algebras and separability idempotents, typically they have appeared in a scattered way.

In Section 2, from the purely algebraic data given above, we construct the “base” $C^*$-algebras $B$ and $C$, as well as their multiplier algebras $M(B)$ and $M(C)$. They are essentially the source algebra and the target algebra. They are equipped with certain KMS-weights $\nu$ and $\mu$.

The construction of the $C^*$-algebra $A$ and the comultiplication $\Delta$ is carried out in Section 3, whose representation is given in terms of a certain partial isometry $W$. The construction of the canonical idempotent $E \in M(A \otimes A)$ is also given in this section.

In Section 4, we discuss the polar decomposition of the antipode map, which allows it to be properly defined at the operator algebra level. We also collect some relevant technical results.

The aim of Section 5 is to carry out a construction of a left invariant weight and a right invariant weight. For technical reasons, the weights are considered at the von Neumann algebra level first, before the $C^*$-algebraic weights are constructed. To make things work, we require a certain quasi-invariance condition at the $^*$-algebra level. This allows an existence of a Radon–Nikodym relationship between our two weights in terms of a certain modular operator, which plays a central role. Eventually, we are able to construct two KMS weights, $\varphi$ and $\psi$, on $A$, satisfying the left invariance condition and the right right invariant condition, respectively.
In this way, in Section 6, we can verify that we have successfully constructed a $C^*$-algebraic quantum groupoid, that fits well in the $C^*$-algebraic framework developed in [10], [11], as expected.

We chose not to pursue the construction of the dual object here, which should be more or less similar. The reason for not doing this is because the $C^*$-algebraic framework for the duality of the locally compact quantum groupoids of separable type is still in the works [12]. In that paper, we plan to give a clarification of the duality picture in the $C^*$-algebraic framework (for a related work, refer also to [7]). We will postpone to a future occasion to verify the expected result that the $C^*$-algebraic counterpart of the algebraic dual $(\hat{A}, \hat{\Delta})$ is indeed isomorphic to the $C^*$-algebraic dual of $(A, \Delta)$ obtained here.

In Appendix (Section 7), we gathered some results in the purely algebraic framework regarding the modular element $\delta$. Even though this is done in the purely algebraic setting, the author could not find a suitable reference, and some of the results here may be new. As such, all proofs are given for the results in the Appendix section.

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1. The algebraic framework: Weak multiplier Hopf algebra with integrals

1.1. Preliminaries. As indicated in Introduction, we will primarily follow the description in [8]. We will only consider associative algebras over $\mathbb{C}$. We require that the algebras are non-degenerate, which means that the product on an algebra is non-degenerate as a bilinear form. So $\mathcal{A}$ is a non-degenerate algebra, if $[ab = 0, \forall b \in \mathcal{A}] \Rightarrow [a = 0]$; also $[ba = 0, \forall b \in \mathcal{A}] \Rightarrow [a = 0]$. We also require that the algebras are idempotent, written $\mathcal{A}^2 = \mathcal{A}$, meaning that every element in the algebra can be written as a sum of products of two elements. It is evident that if an algebra $\mathcal{A}$ is unital or has local units, it is automatically non-degenerate and idempotent. Typically, however, we do not expect our algebras to be unital. We will denote by $\mathcal{A}^*$ the dual vector space of $\mathcal{A}$, consisting of the linear functionals on $\mathcal{A}$.

We will mostly consider *-algebras, equipped with an involution. For a non-degenerate *-algebra $\mathcal{A}$, we can define its multiplier algebra $M(\mathcal{A})$, which is a unital *-algebra containing $\mathcal{A}$ as an essential self-adjoint ideal. It is the largest such, and is unique. It can be characterized in terms of double centralizers. If $\mathcal{A}$ is unital, then we have $M(\mathcal{A}) = \mathcal{A}$. We can also consider $\mathcal{A} \odot \mathcal{A}$, the algebraic tensor product, and its multiplier algebra $M(\mathcal{A} \odot \mathcal{A})$.

1.2. Comultiplication. Let $\mathcal{A}$ be a non-degenerate idempotent *-algebra. By a comultiplication on $\mathcal{A}$, we mean a *-homomorphism $\Delta$ from $\mathcal{A}$ into $M(\mathcal{A} \odot \mathcal{A})$ such that

\[(\Delta a)(1 \otimes b) \in \mathcal{A} \odot \mathcal{A}, \quad \text{and} \quad (c \otimes 1)(\Delta a) \in \mathcal{A} \odot \mathcal{A}, \quad \text{for all } a, b, c \in \mathcal{A}, \quad (1.1)\]
which is also required to satisfy the following “weak coassociativity” condition:
\[(c \otimes 1 \otimes 1)((\Delta \otimes \text{id})(1 \otimes b)) = (\text{id} \otimes \Delta)(c \otimes 1)(\Delta a)(1 \otimes b), \quad \text{for } a, b, c \in A. \tag{1.2}\]
Note that condition (1.1) is needed to formulate the weak coassociativity (1.2).

As \(A\) is a \(*\)-algebra and \(\Delta\) is a \(*\)-homomorphism, it automatically follows from condition (1.1) that we also have:
\[(\Delta a)(c \otimes 1) \in A \odot A, \quad \text{and} \quad (1 \otimes b)(\Delta a) \in A \odot A, \quad \text{for all } a, b, c \in A. \tag{1.3}\]

There is also another version of the weak coassociativity, as follows:
\[(\Delta \otimes \text{id})((1 \otimes b)(\Delta a))(c \otimes 1 \otimes 1) = (1 \otimes 1 \otimes b)(\text{id} \otimes \Delta)((\Delta a)(c \otimes 1)), \quad \text{for } a, b, c \in A. \tag{1.4}\]

Remark. Having \((A, \Delta)\) further satisfy (1.3) and (1.4) means that our comultiplication is “regular”, in the sense of [34] (see Definition 1.1 of that paper).

The comultiplication is also assumed to be “full”. This means that the left and the right legs of \(\Delta(A)\) are all of \(A\) (see [33, 34], also see [2]). A way to characterize the fullness of \(\Delta\) is as follows:
\[
\begin{align*}
\text{span}\{\text{id} \otimes \omega)((\Delta a)(1 \otimes b)) : a, b \in A, \omega \in A^*\} &= A, \\
\text{span}\{(\omega \otimes \text{id})(b \otimes 1)(\Delta a) : a, b \in A, \omega \in A^*\} &= A.
\end{align*}
\]

For \((A, \Delta)\), we have the following result:

**Lemma 1.1.** Suppose there exists a self-adjoint idempotent element \(E \in M(A \odot A)\) such that \(\Delta(A)(A \odot A) = E(A \odot A), \quad \text{and} \quad (A \odot A)\Delta(A) = (A \odot A)E\).
\(\)Then this idempotent is unique. It is the smallest idempotent \(E \in M(A \odot A)\) satisfying \(E(\Delta a) = \Delta a, \quad (\Delta a)E = \Delta a, \quad \forall a \in A\).

**Proof.** See Lemmas 3.3 and 3.5 of [33]. \(\square\)

**Remark.** The requirement to have \(E\) self-adjoint is natural, because we are working with \(*\)-algebras. The existence of the idempotent \(E \in M(A \odot A)\) as above is referred to as \(\Delta\) being weakly non-degenerate. Note that when \(E = 1 \otimes 1\), we would indeed have the non-degeneracy of the comultiplication. Moreover, if such an idempotent \(E\) exists (called the canonical idempotent), then it can be shown that \(\Delta\) has a unique extension to a \(*\)-homomorphism \(\tilde{\Delta} : M(A) \to M(A \odot A)\), such that \(\tilde{\Delta}(m) = E \tilde{\Delta}(m) = \tilde{\Delta}(m)E, \quad \forall m \in M(A)\). For proof of this result and more details, see Appendix of [33]. For convenience, we will just denote the extension map also by \(\Delta\).

It is also the case that we can make sense of the maps \(\Delta \otimes \text{id}\) and \(\text{id} \otimes \Delta\) as \(*\)-homomorphisms naturally extended to \(M(A \odot A)\), such that
\[(\Delta \otimes \text{id})(m) = (E \otimes 1)((\Delta \otimes \text{id})(m)) = ((\Delta \otimes \text{id})(m))(E \otimes 1), \quad \forall m \in M(A \odot A),\]
and similarly for \(\text{id} \otimes \Delta\). These results mean that when extended to the multiplier algebra level, we have \(\Delta(1) = E\), and that \((\Delta \otimes \text{id})(1 \otimes 1) = E \otimes 1\) and \((\text{id} \otimes \Delta)(1 \otimes 1) = 1 \otimes E\).
As a consequence of the weak coassociativity, namely Equations (1.2) and (1.4), now knowing that the maps $\Delta \otimes id$ and $id \otimes \Delta$ are extended, we obtain the following coassociativity property:

\[(\Delta \otimes id)(\Delta a) = (id \otimes \Delta)(\Delta a), \quad \forall a \in A.\]  

(1.5)

See again Appendix of [33].

We will require one more condition on $\Delta$, which is also a part of the axioms for a weak multiplier Hopf algebra (see Definition 1.14 of [34]). The following condition is referred to as the weak comultiplicativity of the unit:

\[(id \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1).\]  

(1.6)

This condition already appeared in the theory of weak Hopf algebras (see Definition 2.1 in [3]).

1.3. **Separability idempotent.** The canonical idempotent is further required to be a separability idempotent, in the sense of [30]. See also Appendix B of [8]. In the below is a short summary.

Suppose $B$ and $C$ are non-degenerate $\ast$-algebras. Consider a self-adjoint idempotent $E \in M(B \circ C)$ such that

\[E(1 \otimes c) \in B \circ C, \quad (b \otimes 1)E \in B \circ C, \quad \forall b \in B, \forall c \in C.\]

As $E$ is self-adjoint, we also have $(1 \otimes c)E \in B \circ C, \quad E(b \otimes 1) \in B \circ C,$ for $b \in B, c \in C$. Assume also that $E$ is “full”, which means that its left and the right legs are all of $B$ and $C$, respectively. Or equivalently, we have

\[
\text{span}\{ (id \otimes \omega)(E(1 \otimes c)) : c \in C, \omega \in C^* \} = B,
\]

\[
\text{span}\{ (\omega \otimes id)((b \otimes 1)E) : b \in B, \omega \in B^* \} = C.
\]

Then, it can be shown that $E \in M(B \circ C)$ automatically becomes a regular separability idempotent, which means that we have:

\[E(B \circ 1) = E(1 \circ C) \quad \text{and} \quad (B \circ 1)E = (1 \circ C)E.\]  

(1.7)

See Proposition 3.7 of [30]. The regularity condition implies $B$ and $C$ have local units (see [30], v1). The other aspect of the regularity of $E$ is that there exist two anti-isomorphisms $S_B : B \rightarrow C$ and $S_C : C \rightarrow B$, characterized by

\[E(b \otimes 1) = E(1 \otimes S_B(b)), \quad b \in B,\]  

(1.8)

\[(1 \otimes c)E = (S_C(c) \otimes 1)E, \quad c \in C.\]  

(1.9)

It can be also shown that $(S_B \circ S_C)(E) = \varsigma E$, where $\varsigma$ is the flip map between $B \circ C$ and $C \circ B$.

The maps $S_B$ and $S_C$ are in general not involutive, but they satisfy the following relations:

\[S_C(S_B(b)^*)^* = b \quad \text{and} \quad S_B(S_C(c)^*)^* = c, \quad \forall b \in B, \forall c \in C.\]  

(1.10)

The existence of such a self-adjoint separability idempotent element $E \in M(B \circ C)$ restricts the possible structure of the algebras $B$ and $C$, which are typically direct sums of finite-dimensional matrix algebras. We will not go too deep into this discussion here. See section 4 of [30].
By the general theory of regular separability idempotents [30], we have the existence of the following distinguished linear functionals, namely \( \nu \) on \( B \) and \( \mu \) on \( C \), such that
\[
(\nu \otimes \text{id})(E) = 1, \quad \text{and} \quad (\text{id} \otimes \mu)(E) = 1.
\]
[The notations used in [30] for the distinguished linear functionals are actually \( \varphi_B \) and \( \varphi_C \), but we are denoting them as \( \nu \) and \( \mu \) here, mainly to avoid a future confusion with the total linear functional \( \varphi \) on \( A \).]

The distinguished functionals \( \nu \) and \( \mu \) are uniquely determined and are faithful. Being a faithful functional means that we have \( \nu(bk) = 0, \forall k \in B \) \( \Rightarrow \) \( b = 0 \), and similarly for \( \mu \). Meanwhile, with the algebras \( B \) and \( C \) being \( *\)-algebras, and \( E \) being self-adjoint, it also turns out that \( \nu \) and \( \mu \) become positive linear functionals (see section 4 of [30]).

The general theory also tells us that the functional \( \nu \) is equipped with a KMS-type automorphism \( \sigma_\nu \) on \( B \), such that
\[
\nu \circ \sigma_\nu = \nu \quad \text{and} \quad \nu(bb') = \nu(b' \sigma_\nu(b)), \quad \forall b, b' \in B,
\]
given by \( \sigma_\nu = S_B^{-1} \circ S_C^{-1} \). As for \( \mu \) also, there is a KMS-type automorphism \( \sigma_\mu \) on \( C \), such that \( \mu \circ \sigma_\mu = \mu \) and
\[
\mu(cc') = \mu(c' \sigma_\mu(c)), \quad \forall c, c' \in C,
\]
given by \( \sigma_\mu = S_B \circ S_C \). The existence of such automorphisms is referred to as the weak KMS-property. See Proposition 2.8 of [30] for more details.

In addition, we can show that
\[
\mu = \nu \circ S_C \quad \text{and} \quad \nu = \mu \circ S_B,
\]
by using the result \((S_B \otimes S_C)E = \varsigma E\) and the uniqueness of the functionals.

Here is a Radon–Nikodym type theorem on the functionals on the algebras \( B \) and \( C \):

**Proposition 1.2.** (1) Consider the distinguished linear functional \( \nu \) on \( B \). For any other linear functional \( g \) on \( B \), there is a unique element \( y \in M(B) \) such that \( g(b) = \nu(by) \) for all \( b \in B \). Moreover, the functional \( g \) is faithful if and only if \( y \) is invertible in \( M(B) \).

(2) Similar results hold true for linear functionals on \( C \).

**Proof.** This is a consequence of the regularity property of \( E \). See Proposition 1.2 of [31], which is actually more general than the result given here. \(\Box\)

As a consequence of Proposition 1.2 above, we can show that any faithful linear functional on \( B \) (or on \( C \)) is equipped with a KMS-type automorphism:

**Proposition 1.3.** (1) Any faithful linear functional on \( B \) has the weak KMS property.

(2) Any faithful linear functional on \( C \) has the weak KMS property.

**Proof.** See Proposition 1.3 of [31]. The case for \( B \) is below. The case for \( C \) is similar.

If \( f \) is a faithful linear functional on \( B \), then by Proposition 1.2, we can write it as \( f(\ ) : b \mapsto \nu(by) \), where \( y \) is invertible in \( M(B) \). Then for any \( b, b' \in B \), we have:
\[
f(bb') = \nu(bb'y) = \nu(b'y\sigma_\nu(b)) = f(b'y\sigma_\nu(b)y^{-1}).
\]
This shows that $b \mapsto y\sigma^\nu(b)y^{-1}$ determines a KMS-type automorphism for $f$. □

1.4. Algebraic quantum groupoid. Let $\mathcal{A}$ be a non-degenerate idempotent $^*$-algebra, and $\Delta$ a regular, full comultiplication that is weakly non-degenerate (See §1.2). This means the existence of a unique canonical idempotent element $E \in M(\mathcal{A} \otimes \mathcal{A})$, which is also assumed to satisfy the weak comultiplicativity of the unit, or Equation (1.6).

We will further assume that $E$ is a regular, full separability idempotent (See §1.3), such that there exist two $^*$-subalgebras $\mathcal{B}$ and $\mathcal{C}$ of $M(\mathcal{A})$ sitting in a non-degenerate way, and we have $E \in M(\mathcal{B} \odot \mathcal{C})$. When we say the subalgebras $\mathcal{B}$ and $\mathcal{C}$ sit in a non-degenerate way inside $M(\mathcal{A})$, it means that $BA = AB = A$, and $CA = AC = A$. Then it is easy to see that $\mathcal{B}$ and $\mathcal{C}$ are non-generate algebras, and that $M(\mathcal{B})$ and $M(\mathcal{C})$ can be regarded as subalgebras in $M(\mathcal{A})$ as well.

Remark. It turns out that the subalgebras $\mathcal{B}$ and $\mathcal{C}$ are completely determined by the conditions given above. See Proposition 3.1 of [8]. As such, it would be all right to just say that “$E$ is a regular separability idempotent”, without having to specify $\mathcal{B}$ and $\mathcal{C}$ explicitly.

On $M(\mathcal{B})$ and $M(\mathcal{C})$, regarded as subalgebras in $M(\mathcal{A})$, it turns out that the comultiplication acts the following way (which turns out to characterize the subalgebras):

**Proposition 1.4.** If $x \in M(\mathcal{B})$, then we have $\Delta x = E(1 \otimes x) = (1 \otimes x)E$.

If $y \in M(\mathcal{C})$, then we have $\Delta y = (y \otimes 1)E = E(y \otimes 1)$.

**Proof.** See Proposition 3.4 in [8]. □

Let us next give the definition for left and right integrals. See Definition 3.5 of [8].

**Definition 1.5.** A linear functional $\varphi$ on $\mathcal{A}$ is said to be left invariant, if

$$(\text{id} \otimes \varphi)(\Delta a) \in M(\mathcal{C}), \quad \forall a \in \mathcal{A}.$$ 

Similarly, a linear functional $\psi$ on $\mathcal{A}$ is said to be right invariant, if

$$(\psi \otimes \text{id})(\Delta a) \in M(\mathcal{B}), \quad \forall a \in \mathcal{A}.$$ 

Any non-zero left-invariant linear functional on $\mathcal{A}$ is called a left integral, and any non-zero right-invariant linear functional on $\mathcal{A}$ is called a right integral.

**Remark.** While $\Delta a \in M(\mathcal{A} \otimes \mathcal{A})$, we can still make sense of $(\text{id} \otimes \varphi)(\Delta a)$ as an element in $M(\mathcal{A})$. To see this, remember that $(b \otimes 1)(\Delta a) \in \mathcal{A} \otimes \mathcal{A}$, for all $a, b \in \mathcal{A}$. For any $b \in \mathcal{A}$, we may regard $b(\text{id} \otimes \varphi)(\Delta a) = (\text{id} \otimes \varphi)((b \otimes 1)(\Delta a)) \in \mathcal{A}$. What the left invariance of $\varphi$ means is that $(\text{id} \otimes \varphi)(\Delta a)$ is actually contained in $M(\mathcal{C}) \subset M(\mathcal{A})$. Similarly, we can also make sense of $(\psi \otimes \text{id})(\Delta a) \in M(\mathcal{B})$.

By the result of Proposition 1.4, we can actually sharpen the statements in Definition 1.5:

**Proposition 1.6.** Let $\varphi$ and $\psi$ are, respectively, left and right integrals on $\mathcal{A}$. Then we have:

$$(\text{id} \otimes \varphi)(\Delta x) \in \mathcal{C}, \quad \forall x \in \mathcal{A},$$

$$(\psi \otimes \text{id})(\Delta x) \in \mathcal{B}, \quad \forall x \in \mathcal{A}.$$
Proof. Let $a \in A$ and let $c \in C$. By a reasoning similar to the one given in Remark above, it is easy to see that we can make sense of $\text{id} \otimes \varphi((\Delta a)(c \otimes 1))$. By the left invariance, we know $(\text{id} \otimes \varphi)(\Delta a) \in M(C)$. Therefore, we have: 

$$(\text{id} \otimes \varphi)((\Delta a)(c \otimes 1)) = (\text{id} \otimes \varphi)(\Delta a)c \in C.$$ 

But by Proposition 1.4, we know $\Delta c = E(c \otimes 1)$. So we have $(\Delta a)(c \otimes 1) = (\Delta a)E(c \otimes 1) = (\Delta a)(\Delta c) = \Delta(ac)$. Considering the result from the above, we conclude that $(\text{id} \otimes \varphi)(\Delta(ac)) = (\text{id} \otimes \varphi)((\Delta a)(c \otimes 1)) \in C$. This is true for any $a \in A$ and any $c \in C$. But then, as $C$ sits in a non-degenerate way inside $M(A)$, or $AC = A$, we know that any $x \in A$ can be expressed as a finite sum of the elements of the form $ac$, where $a \in A$, $c \in C$. It follows that $(\text{id} \otimes \varphi)(\Delta x) \in C$, for any $x \in A$.

The proof for $(\psi \otimes \text{id})(\Delta x) \in B$, $x \in A$, can be done in exactly the same way for a right invariant functional $\psi$. \hfill \square

Here are some additional consequences of the left/right invariance of $\varphi$ and $\psi$:

Proposition 1.7. Denote

$$
F_1 = (\text{id} \otimes S)(E), \quad F_2 = (S \otimes \text{id})(E), \quad F_3 = (\text{id} \otimes S^{-1})(E), \quad F_4 = (S^{-1} \otimes \text{id})(E),
$$

which are elements in $M(A \odot A)$. If $\varphi$ is a left integral and if $\psi$ is a right integral, we have:

$$(\text{id} \otimes \varphi)(\Delta a) = (\text{id} \otimes \varphi)(F_2(1 \otimes a)) = (\text{id} \otimes \varphi)((1 \otimes a)F_4),$$

$$(\psi \otimes \text{id})(\Delta a) = (\psi \otimes \text{id})((a \otimes 1)F_1) = (\psi \otimes \text{id})(F_3(a \otimes 1)),$$

for all $a \in A$.

Proof. See Proposition 1.4 of [36] and Proposition 3.7 of [8]. \hfill \square

Proposition 1.8 below gives a relationship between the integrals $\varphi$, $\psi$ and the expressions $(\psi \otimes \text{id})(\Delta x)$ and $(\text{id} \otimes \varphi)(\Delta x)$, for $x \in A$. These result hold true for any left integral and any right integral. The proof is fundamentally no different than the one given in Proposition 4.9 of [10], but now done at the $\ast$-algebra level.

Proposition 1.8. For any right integral $\psi$ and any left integral $\varphi$, we have:

- $\nu((\psi \otimes \text{id})(\Delta x)) = \psi(x)$, for $x \in A$.
- $\mu((\text{id} \otimes \varphi)(\Delta x)) = \varphi(x)$, for $x \in A$.

Proof. By Definition 1.3 and Proposition 1.6, we know $(\psi \otimes \text{id})(\Delta x) \in B$ and $(\text{id} \otimes \varphi)(\Delta x) \in C$, for any $x \in A$. As such, both expressions on the left sides above make sense.

Let $a \in A$. Consider $(\psi \otimes \text{id})(\Delta a)$, then apply $\Delta$. On the one hand, we have:

$$
\Delta((\psi \otimes \text{id})(\Delta a)) = (\psi \otimes \text{id} \otimes \text{id})((\text{id} \otimes \Delta)(\Delta a)) = (\psi \otimes \text{id} \otimes \text{id})((\Delta \otimes \text{id})(\Delta a)),
$$

where we used the coassociativity of $\Delta$. Meanwhile, by Proposition 1.4 we have:

$$
\Delta((\psi \otimes \text{id})(\Delta a)) = E(1 \otimes (\psi \otimes \text{id})(\Delta a)) = (\psi \otimes \text{id} \otimes \text{id})((1 \otimes E)\Delta_{13}(a)).
$$

It thus follows that

$$(\psi \otimes \text{id} \otimes \text{id})((\Delta \otimes \text{id})(\Delta a)) = (\psi \otimes \text{id} \otimes \text{id})((1 \otimes E)\Delta_{13}(a)). \quad (1.13)$$
Let \( y = \tilde{yc} \), where \( \tilde{y} \in A \) and \( c \in C \). Recall that \( AC = A \), so such elements span \( A \). Multiply \( 1 \otimes y = 1 \otimes \tilde{yc} \) to both sides of Equation (1.13), from left. Then the equation becomes:
\[
(\psi \otimes \text{id} \otimes \text{id})( (\Delta \otimes \text{id})((1 \otimes \tilde{yc})(\Delta a)) ) = (\psi \otimes \text{id} \otimes \text{id})( (1 \otimes 1 \otimes \tilde{yc})(1 \otimes E)\Delta_{13}(a) ) .
\]
Let \( \omega \in A^* \), and apply \( \text{id} \otimes \omega \) to the equation above. Then it becomes:
\[
(\psi \otimes \text{id})(\Delta((\text{id} \otimes \omega)(1 \otimes \tilde{yc})(\Delta a)))
= (\psi \otimes \text{id})((\text{id} \otimes \text{id} \otimes \omega)((1 \otimes 1 \otimes \tilde{y})(1 \otimes 1 \otimes c)(1 \otimes E)\Delta_{13}(a))) .
\]
Apply \( \nu \) to both sides. By the property of \( \nu \), we know \( (\nu \otimes \text{id})(E) = 1 \). So we have:
\[
\nu((\psi \otimes \text{id})(\Delta((\text{id} \otimes \omega)(1 \otimes \tilde{yc})(\Delta a))) )
= \psi((\text{id} \otimes \omega)((1 \otimes \tilde{y})(1 \otimes (\nu \otimes \text{id})(1 \otimes c)E))(\Delta a))
= \psi((\text{id} \otimes \omega)(1 \otimes \tilde{yc})(\Delta a)) .
\]
(1.14)

For convenience, write: \( x = (\text{id} \otimes \omega)(1 \otimes \tilde{yc})(\Delta a) \). By the “fullness” of the comultiplication, such elements span all of \( A \). Then Equation (1.14) can be expressed as \( \nu((\psi \otimes \text{id})(\Delta x)) = \psi(x) \), which would be true for all \( x \in A \).

Similarly, we can show that for any left integral \( \varphi \), we have \( \mu((\text{id} \otimes \varphi)(\Delta x)) = \varphi(x) \), for \( x \in A \). \( \square \)

In general, while we can have a faithful left integral and a faithful right integral on \( A \), it may have neither. Instead, it may be possible to have a faithful set of left integrals \( \{\varphi_a\} \), in the sense that if \( a \in A \) is such that \( \varphi_a(ax) = 0 \) for all \( x \in A \) and for all left integrals \( \phi_a \), then we must have \( a = 0 \). Similarly, if \( \varphi_a(xa) = 0 \) for all \( x \in A \) and for all \( \phi_a \), then we must have \( a = 0 \). We can make sense of a faithful set of right integrals in a similar way.

The following proposition is a consequence of having a faithful set of left integrals and a faithful set of right integrals. Again, proof is skipped.

**Proposition 1.9.**

1. If there is a faithful set of left integrals, then
\[
A = \text{span}\{ (\text{id} \otimes \varphi)((\Delta a)(1 \otimes b)) : \varphi \text{ is a left integral, } a, b \in A \} .
\]

2. If there is a faithful set of right integrals, then
\[
A = \text{span}\{ (\psi \otimes \text{id})(b \otimes 1)((\Delta a)) : \psi \text{ is a left integral, } a, b \in A \} .
\]

3. If there is a faithful set of left integrals, then
\[
\Delta(A)(1 \otimes A) = E(A \otimes A) \quad \text{and} \quad (1 \otimes A)\Delta(A) = (A \otimes A)E .
\]

4. If there is a faithful set of right integrals, then
\[
(A \otimes 1)\Delta(A) = (A \otimes A)E \quad \text{and} \quad \Delta(A)(A \otimes 1) = E(A \otimes A) .
\]

The main result of [8] is that given data \( (A, \Delta, E) \) as above, if there is a faithful set of left integrals and a faithful set of right integrals, then we have a regular weak multiplier Hopf algebra, in the sense of [34]. In particular, the existence of the antipode, \( S \), can be proved, using the result of Proposition 1.9. One can also construct the counit, \( \varepsilon \). It then becomes equivalent
to the situation of a regular weak multiplier Hopf algebra equipped with a faithful set of left integrals, whose right integrals can be obtained using the antipode map.

In [30], [31], regular weak Hopf algebras with a faithful set of integrals is referred to as algebraic quantum groupoids. It is shown there that they form a self-dual category. Note, however, that this notion is different from Timmermann’s notion of an algebraic quantum groupoid (see [25], [24]), which is based on the framework of multiplier Hopf algebroids. Some discussion on the relationship between these two frameworks can be found in [26].

1.5. Weak multiplier Hopf ∗-algebra with a single faithful integral. According to the general theory on weak multiplier Hopf algebras, the existence of a faithful family of (left) integrals is required for the duality picture to be complete. See [30]. Unlike in the case of multiplier Hopf algebras (see [28], [29]), having a left or right integral does not necessarily mean that it is also faithful. There are known examples of weak (multiplier) Hopf algebras where enough integrals exist to form a faithful family, but not a single faithful one [6].

Having said this, in many examples there exists a single faithful integral. In particular, it has been observed that for finite-dimensional weak Hopf algebras, a single faithful integral exists if and only if the underlying algebra is a Frobenius algebra, for instance a finite-dimensional C∗-algebra (See Theorem 3.16 in [3].). Infinite-dimensional case is not fully understood. Nonetheless, considering that our aim is to eventually construct a C∗-algebraic version, this observation seems to suggest that it may not be too restrictive to require the existence of a single faithful integral. We will do so here. Also, as we are working in the ∗-algebra setting, we will further require that the integrals are positive linear functionals.

**Definition 1.10.** Let (A, ∆, E) be as above, and assume that there exists a single positive faithful left integral ϕ and a single positive faithful right integral ψ. We will call (A, ∆) a weak multiplier Hopf ∗-algebra with a faithful integral.

**Remark.** The positivity of the functionals ϕ and ψ is not automatic, even under the ∗-structure and the self-adjointness of E. There are examples even in the case of ∗-algebraic quantum groups of compact type, for which the left invariant functional is not positive. As such, we explicitly require here the positivity of the invariant functionals.

As noted earlier, having the ∗-structure means this is a regular weak multiplier Hopf algebra. Meanwhile, having a single faithful left integral and a single faithful right integral just means that we have one-element families of left/right integrals, so the main results outlined in the previous subsection still hold, including the existence of the counit and the antipode:

**Theorem 1.11.** Let A be a non-degenerate idempotent ∗-algebra, with a full comultiplication ∆ : A → M(A ⊙ A) as defined in §1.2. Assume also that there exists the canonical idempotent E ∈ M(A ⊙ A), satisfying the properties given in §1.2 and §1.3. In addition, we assume that there exists a single positive faithful left integral ϕ and a single positive faithful right integral ψ.

Then (A, ∆) becomes a regular weak multiplier Hopf ∗-algebra, in the sense of Van Daele and Wang [34]. In particular, the counit ε and the antipode S exists.

**Proof.** See Theorem 3.15 of [8]. See also the last two paragraphs of Section §1.4 above. □
We will gather some additional results before we end this subsection, which we will use down the road. Next proposition summarizes a characterization of the antipode map, with its properties:

**Proposition 1.12.** (1) The antipode map, $S$, is anti-multiplicative and bijective on $A$, and can be characterized as follows:

$$ S : (\text{id} \otimes \varphi)((\Delta a)(1 \otimes b)) \mapsto (\text{id} \otimes \varphi)((1 \otimes a)(\Delta b)), \quad \forall a, b \in A. $$

(2) Similarly, the antipode $S$ can be also characterized as follows:

$$ S : (\psi \otimes \text{id})((a \otimes 1)(\Delta b)) \mapsto (\psi \otimes \text{id})((\Delta a)(b \otimes 1)), \quad \forall a, b \in A. $$

(3) The antipode $S$ of $(A, \Delta)$ can be extended to the multiplier algebra $M(A)$, and when restricted to $B$ and $C$, it coincides with the maps $S_B$ on $B$ and $S_C$ on $C$.

(4) We have: $(S \otimes S)(\Delta x) = \Delta^{\text{cop}}(S(x)) = \varsigma \Delta(S(x))$, for $x \in A$, where $\varsigma$ is the flip map on $M(A \odot A)$.

(5) With respect to the $\ast$-structure, we have: $S(S(x)^\ast) = x$, for $x \in A$.

**Proof.** For the characterizations (1), (2), see Proposition 1.5 of [36], which are true not just for faithful integrals but for any left integral and any right integral. See also Proposition 3.16 of [8]. In view of the earlier Proposition 1.9, we see that the antipode is bijective.

For (3), see Proposition 3.18 of [8]. For (4) and (5), see Propositions 4.4 and 4.11 of [34]. For more results on the antipode on a regular multiplier Hopf algebra, see Section 4 of [34]. □

Even though a characterization for the antipode, $S$, is given above in terms of the left integral $\varphi$ (and alternatively in terms of the right integral $\psi$), it is actually intrinsic. Indeed, the antipode $S$ is independent of any particular choice of the integrals $\varphi$ and $\psi$. This seemingly ambiguous aspect is because the invariant functionals are not uniquely determined. Nonetheless, the results (1) and (2) of Proposition 1.12 mean that there exists a weak sense of uniqueness for the left/right integrals. More discussions are given below.

Here are some additional consequences of having a single faithful integral $\varphi$. The following are general results (see section 1 of [30]), which do not require the positivity of the functionals. As these are purely algebraic results that can be found in [30], we will skip the details and the proofs.

**Proposition 1.13.** There exists an automorphism $\sigma$ of $A$ such that

$$ \varphi(ab) = \varphi(b \sigma(a)), \quad \forall a, b \in A. $$

We also have $\varphi \circ \sigma = \varphi$.

**Proof.** See Proposition 1.7 of [36] for the existence of $\sigma$. □

The automorphism $\sigma$ above will be referred to as the **modular automorphism for $\varphi$**. The terminology is motivated by the theory of weights on $C^*$-algebras. Note that while any faithful linear functional on a finite-dimensional algebra admits such an automorphism, that is not always the case in the infinite-dimensional case. So the above result indicates that there is more going on with $\varphi$ than it being just a faithful linear functional.
The next result gives a relation between two left integrals:

**Proposition 1.14.** Let \( \varphi \) be a faithful left integral, and let \( \varphi_1 \) be a left integral (not necessarily faithful). Then there is an element \( y \in M(B) \) such that \( \varphi_1(x) = \varphi(xy) \), for all \( x \in A \).

**Proof.** See Proposition 1.8 of [36]. \( \square \)

Going the other way, it is not difficult to show that for any left integral \( \varphi \) and \( y \in M(B) \), the functionals \( \varphi(\cdot y) \) and \( \varphi(y \cdot) \) again become left invariant. As a consequence, we can see that \( \sigma \) leaves \( M(B) \) invariant.

By a similar argument used to prove the above results, we obtain the following proposition, relating any right integral \( \psi_1 \) with the faithful left integral \( \varphi_1 \):

**Proposition 1.15.** Let \( \varphi \) be a faithful left integral, and let \( \psi_1 \) be a right integral (not necessarily faithful). Then there exists an element \( \delta_1 \in M(A) \) such that \( \psi_1(x) = \varphi(x\delta_1) \) for all \( x \in A \).

In particular, if \( \psi_1 \) is also faithful, then \( \delta_1 \) is invertible in \( M(A) \).

**Proof.** See Proposition 1.9 of [36]. \( \square \)

Of particular interest is the modular element \( \delta \), which relates the left integral \( \varphi \) with the functional \( \varphi \circ S \), which turns out to be right invariant. Namely, we have \( (\varphi \circ S)(x) = \varphi(x\delta) \), for \( x \in A \). By the faithfulness of \( \varphi \), it is not difficult to show that the modular element is invertible and is uniquely determined. See Section 7 (Appendix), where some more results on \( \delta \) are gathered.

### 1.6. The dual algebra

Let \( (A, \Delta) \) be a (regular) weak multiplier Hopf *-algebra, with its counit \( \varepsilon \) and the antipode \( S \), as in [34]. Suppose there exists a faithful left integral \( \varphi \). Then by the general theory, we can construct its dual object. For details, see Section 2 of [36]. First, we consider \( \hat{A} \), the space of linear functionals on \( A \) spanned by the elements of the form \( \varphi(\cdot a) \), for \( a \in A \). It can be given a weak multiplier Hopf *-algebra structure, as follows.

For \( \omega, \omega' \in \hat{A} \) and \( x \in A \), define the multiplication \( \omega \omega' \in \hat{A} \) by

\[
(\omega \omega')(x) := (\omega \otimes \omega')(\Delta x).
\]

For \( \omega \in \hat{A} \) and \( x \in A \), define the involution on \( \hat{A} \) by \( \omega^*(x) := \omega(S(x)^*) \). One can show that \( \hat{A} \) becomes a non-degenerate idempotent *-algebra.

As for the comultiplication, we define \( \hat{\Delta} \) on \( \hat{A} \) in such a way that

\[
\hat{\Delta}(\omega)(x \otimes y) = \omega(xy), \quad \text{for } \omega \in \hat{A}, \ x, y \in A.
\]

It becomes a full coassociative comultiplication. However, making sense of this needs some care, as we need to consider \( M(\hat{A} \odot \hat{A}) \) inside the dual space \( (A \odot A)^* \) in a proper way. See Propositions 2.7, 2.8, 2.9 of [36].

The antipode map, \( \hat{S} : \hat{A} \to \hat{A} \), is given by

\[
\hat{S}(\omega)(x) = \omega(S(x)), \quad \text{for } \omega \in \hat{A}, \ x \in A.
\]
Meanwhile, the canonical idempotent \( \hat{E} \in M(A \odot A) \) should be such that \( \hat{E} = \hat{\Delta}(1_{M(\hat{A})}) \), so it is defined by

\[
\hat{E}(x \otimes y) = \varepsilon(xy), \quad \text{for } x, y \in A,
\]

where \( \varepsilon \) is the counit. It can be shown that these structure maps make \((\hat{A}, \hat{\Delta})\) a regular weak multiplier Hopf \( \ast \)-algebra. See Theorem 2.15 of [36].

Finally, as we are assuming that \((A, \Delta)\) is equipped with a single faithful integral, it can be shown that \((\hat{A}, \hat{\Delta})\) is also equipped with a single faithful integral. (See Theorem 2.21 of [36].) We will skip the details, but if \( \varphi \) is a faithful left integral for \((A, \Delta)\) and \( \omega = \varphi(\cdot c) \in \hat{A}, c \in A, \) then one can consider the functional \( \hat{\psi} \) on \( \hat{A} \), such that \( \hat{\psi}(\omega) := \varepsilon(c) \). It can be shown that such a functional \( \hat{\psi} \) becomes a faithful right integral on \((\hat{A}, \hat{\Delta})\).

To summarize, from a regular multiplier Hopf algebra \((A, \Delta)\) equipped with a faithful left integral \( \varphi \), one can construct its dual \((\hat{A}, \hat{\Delta})\), which is also a regular multiplier Hopf algebra, with a faithful right integral \( \hat{\psi} \). A main theorem is that if we consider the dual of \((\hat{A}, \hat{\Delta})\), then the resulting object \((\hat{A}, \hat{\Delta})\) is canonically isomorphic to the original \((A, \Delta)\), which is a generalized Pontryagin duality. See Section 2 of [36].

2. The base \( C^\ast \)-algebras

From this point on, we are going to systematically construct a \textit{C*-algebraic quantum groupoid of separable type} in the sense of [10], [11], out of our purely algebraic object of a weak multiplier Hopf \( \ast \)-algebra \((A, \Delta)\) equipped with a faithful left integral \( \varphi \) (and also a right integral \( \psi \)). We will use the notations and properties summarized in Section 1 including the subalgebras \( B \) and \( C \), the distinguished linear functionals \( \nu \) and \( \mu \) on them, the canonical idempotent \( E \), and the antipode \( S \).

2.1. Construction of the \( C^\ast \)-algebras \( B \) and \( C \). We will begin with the construction of the base \( C^\ast \)-algebras \( B \) and \( C \), which will be essentially the “source algebra” and the “target algebra”. This will be done by completing the algebras \( B \) and \( C \) in an appropriate sense.

First, recall (from Section 1.3) that there exists a distinguished linear functional \( \nu \) on the \( \ast \)-algebra \( B \), which is positive and faithful. Using \( \nu \), we can provide \( B \) with an inner product, as follows:

\[
\langle x_1, x_2 \rangle := \nu(x_2^*x_1), \quad \text{for } x_1, x_2 \in B.
\]

Form the completion of \( B \) with respect to the induced norm, and obtain a Hilbert space \( \mathcal{H}_B \) with the natural inclusion \( \Lambda_B : B \to \mathcal{H}_B \).

In a similar way, using the distinguished linear functional \( \mu \) on the \( \ast \)-algebra \( C \), we can equip \( C \) with an inner product, by \( \langle y_1, y_2 \rangle := \mu(y_2^*y_1), \) for \( y_1, y_2 \in C \). As above, we obtain a Hilbert space \( \mathcal{H}_C \) with the natural inclusion \( \Lambda_C : C \to \mathcal{H}_C \).
Between the Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_C$, there exists a unitary map given by the anti-homomorphism $S_B$. More precisely, note that for any $b_1, b_2 \in \mathcal{B}$, we have:

$$
\mu(S_B(b_1)^* S_B(b_2)) = \mu(S_C^{-1}(b_1^*) S_B(b_2)) = \mu(S_B(b_2) S_B(b_1^*))
$$

= $(\mu \circ S_B)(b_1^* b_2) = \nu(b_1^* b_2),$

because $S_C(S_B(b_1)^*)^* = b_1$ and $\sigma_C = S_B \circ S_C$, while $\mu \circ S_B = \nu$. This shows that $S_B : \mathcal{B} \to \mathcal{C}$ provides an isometry with respect to the inner products on $\mathcal{H}_B$ and $\mathcal{H}_C$. So it lifts to a unitary $\tilde{J}_B : \mathcal{H}_B \to \mathcal{H}_C$, by $\tilde{J}_B \Lambda_B(x) = \Lambda_C(S_B(x)), \forall x \in \mathcal{B}$.

We can formulate the Hilbert space tensor product $\mathcal{H}_B \otimes \mathcal{H}_C$. On this Hilbert space, note that our canonical idempotent $E \in M(\mathcal{B} \otimes \mathcal{C})$ naturally defines an operator, $\Pi(E)$, by

$$
\Pi(E)(\Lambda_B(x) \otimes \Lambda_C(y)) = (\Lambda_B \otimes \Lambda_C)(E(x \otimes y)), \quad \text{for } x \in \mathcal{B}, y \in \mathcal{C}.
$$

As $\mathcal{B}$ and $\mathcal{C}$ are dense in $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively, it is clear that $\Pi(E)$ is a densely-defined operator acting on $\mathcal{H}_B \otimes \mathcal{H}_C$. By the property of $E$, it is also clear that $\Pi(E)$ is self-adjoint and idempotent. This means $\Pi(E)$ extends to an orthogonal projection, so it is a bounded self-adjoint operator in $\mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_C)$.

Consider an element of the form $x = (\text{id} \otimes \omega)(E) \in \mathcal{B}$, where $\omega \in \mathcal{C}^*$ is defined by $\omega = \mu(c_1^* \cdot c_2)$, for $c_1, c_2 \in \mathcal{C}$. We saw in §1.3 that such elements span the algebra $\mathcal{B}$. In terms of the bounded operator $\Pi(E)$ above, it is clear that for this $x$, we can consider the operator $(\text{id} \otimes \omega_{\Lambda_C(c_2),\Lambda_C(c_1)})(\Pi(E)) \in \mathcal{B}(\mathcal{H}_B)$, such that

$$(\text{id} \otimes \omega_{\Lambda_C(c_2),\Lambda_C(c_1)})(\Pi(E)) \Lambda_B(b) = \Lambda_B(xb), \quad \text{for all } b \in \mathcal{B}.$$ 

Here, we are using the standard notation that $\omega_{\xi,\eta}(T) = \langle T \xi, \eta \rangle$, for $T \in \mathcal{B}(\mathcal{H}_C)$ and $\xi, \eta \in \mathcal{H}_C$. In other words, any element of the form $x = (\text{id} \otimes \omega)(E) \in \mathcal{B}$ can be regarded as a bounded operator on $\mathcal{H}_B$ by left multiplication. Since such elements span all of $\mathcal{B}$, the same can be said true for any element in $\mathcal{B}$. This allows us to define the GNS-representation $\pi_B$ of $\nu$:

**Definition 2.1.** Define $\pi_B$ from $\mathcal{B}$ into $\mathcal{B}(\mathcal{H}_B)$, by

$$
\pi_B(x) \Lambda_B(b) = \Lambda(xb), \quad \text{for all } x, b \in \mathcal{B}.
$$

Then $\pi_B$ is an injective $*$-homomorphism, which is the GNS-representation of $\mathcal{B}$, such that $\pi_B(\mathcal{B})\mathcal{H}_B$ is dense in $\mathcal{H}_B$.

**Remark.** In general, even if we have a positive linear functional on a $*$-algebra $\mathcal{B}$, resulting in an inner product and a Hilbert space $\mathcal{H}_B$, it is not always possible to represent the algebra as an algebra of left multiplication operators. Some elements may become unbounded operators. Observe that in our case, the existence of our self-adjoint idempotent $E$ allowed the construction of the GNS-representation. Meanwhile, note that the density statement in the last sentence of the definition is a quick consequence of the fact that $\mathcal{B}$ is a non-degenerate idempotent algebra. We can also extend $\pi_B$ to the level of $M(\mathcal{B})$.

By a similar argument, using $E$ and considering its other leg, we can also define the GNS-representation $\pi_C$ of $\mu$:
**Definition 2.2.** Define $\pi_C$ from $\mathcal{C}$ into $\mathcal{B}(\mathcal{H}_C)$, by
\[
\pi_C(y)\Lambda_C(c) = \Lambda(yc), \quad \text{for all } y, c \in \mathcal{C}.
\]
Then $\pi_C$ is an injective $^*$-homomorphism, which is the GNS-representation of $\mathcal{C}$, such that $\pi_C(\mathcal{C})\mathcal{H}_C$ is dense in $\mathcal{H}_C$. We can also extend $\pi_C$ to the level of $\mathcal{M}(\mathcal{C})$.

The GNS-representations allow us to properly define our $C^*$-algebras $\mathcal{B}$ and $\mathcal{C}$:

**Definition 2.3.**
1. Define $\mathcal{B} := \overline{\pi_B(\mathcal{B})}^\|$, as a non-degenerate $C^*$-subalgebra of $\mathcal{B}(\mathcal{H}_B)$. Or, equivalently,
\[
\mathcal{B} = \overline{\{ (\mathrm{id} \otimes \omega)(E) : \omega \in \mathcal{B}(\mathcal{H}_C)^* \}}^\| \subseteq \mathcal{B}(\mathcal{H}_B),
\]
where we wrote $E = \Pi(E) \in \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_C)$, for convenience.
2. Define $\mathcal{C} := \overline{\pi_C(\mathcal{C})}^\|$, as a non-degenerate $C^*$-subalgebra of $\mathcal{B}(\mathcal{H}_C)$. Or, equivalently,
\[
\mathcal{C} = \overline{\{ (\omega \otimes \mathrm{id})(E) : \omega \in \mathcal{B}(\mathcal{H}_B)^* \}}^\| \subseteq \mathcal{B}(\mathcal{H}_C).
\]

We may also get to work with their enveloping von Neumann algebras, namely,
\[
\mathcal{N} := \pi_B(\mathcal{B})'' \subseteq \mathcal{B}(\mathcal{H}_B) \quad \text{and} \quad \mathcal{L} := \pi_C(\mathcal{C})'' \subseteq \mathcal{B}(\mathcal{H}_C).
\]

**2.2. The left Hilbert algebras.** Eventually, we wish to lift the functionals $\nu$ and $\mu$ to the $C^*$-algebra level. Observe first that $\Lambda_B(\mathcal{B}) \subseteq \mathcal{H}_B$ and $\Lambda_C(\mathcal{C}) \subseteq \mathcal{H}_C$ are left Hilbert algebras, as in the Tomita–Takesaki modular theory (see [23]):

**Proposition 2.4.** The subspaces $\Lambda_B(\mathcal{B}) \subseteq \mathcal{H}_B$ and $\Lambda_C(\mathcal{C}) \subseteq \mathcal{H}_C$ are left Hilbert algebras, with respect to the multiplications and the $^*$-structures inherited from $\mathcal{B}$ and $\mathcal{C}$, respectively.

**Proof.** For any $x \in \mathcal{B}$, we have already shown that $\pi_B(x) : \Lambda_B(b) \mapsto \Lambda_B(xb)$, $b \in \mathcal{B}$, given by the multiplication, determines a bounded operator. The involution, $x \mapsto x^*$, $x \in \mathcal{B}$, is such that
\[
\langle \Lambda_B(xb), \Lambda_B(b') \rangle = \nu(b^*xb) = \nu(b'\sigma^{-1}(x^*)b) = \nu(x^*b^*b) = \langle \Lambda_B(b), \Lambda(x^*b') \rangle,
\]
for $b, b' \in \mathcal{B}$. This shows that for any $\xi, \eta \in \Lambda_B(\mathcal{B})$, we have $\langle \Lambda_B(x)\xi, \eta \rangle = \langle \xi, \Lambda_B(x^*)\eta \rangle$.

To see that the involution is pre-closed, note that for any fixed $b \in \mathcal{B}$ and any $x_n \in \mathcal{B}$, we have:
\[
\langle \Lambda_B(b), \Lambda_B(x_n^*) \rangle = \nu(x_n b) = \nu(\sigma^{-1}(\sigma^*)^{-1}(b)x_n) = \langle \Lambda_B(x_n), \Lambda_B(\sigma^{-1}(\sigma^*)^{-1}(b)^*) \rangle.
\]
Since $\Lambda_B(\mathcal{B})$ is dense in $\mathcal{H}_B$, we can quickly see that if $x_n \to 0$ and $x_n^* \to z$ in $\mathcal{B}$, then $z = 0$. So the involution is pre-closed. We already know that $\mathcal{B}^2 = \mathcal{B}$, and $\Lambda_B(\mathcal{B})^2$ is dense in $\mathcal{H}_B$.

In this way, we showed that $\Lambda_B(\mathcal{B})$ becomes a left Hilbert algebra (see Definition 1.1 in [23]). Similarly, we can show that $\Lambda_C(\mathcal{C})$ is also a left Hilbert algebra.

The modular theory associates to the left Hilbert algebra $\Lambda_B(\mathcal{B})$ a von Neumann algebra, which should be none other than $\mathcal{N} = \pi_B(\mathcal{B})''$. Also by the general theory on left Hilbert algebras, we obtain a normal semi-finite faithful (n.s.f.) weight $\tilde{\nu}$ on $\mathcal{N}$. Consider the associated spaces $\mathfrak{M}_B = \{ x \in \mathcal{N} : \tilde{\nu}(x^*x) < \infty \}$ and $\mathfrak{M}_B^\# = \mathfrak{M}_B^*\mathfrak{M}_B$. The general theory provides us with the following properties:
We have a closed linear map $\Lambda_B : \mathfrak{M}_\nu \to \mathcal{H}_B$ (the same Hilbert space), which is the GNS-map for the weight $\tilde{\nu}$. The map $\Lambda_B$ extends $\Lambda_B$, such that $\pi_B(\mathcal{B}) \subseteq \mathfrak{M}_\nu$ and $\Lambda_B \circ \pi_B = \Lambda_B$. Note that the weight $\tilde{\nu}$ extends the functional $\nu$. In particular, we have $\tilde{\nu}(\pi_B(x)^* \pi_B(x)) = \nu(x^* x)$, for all $x \in \mathcal{B}$. For any $b \in \mathfrak{M}_\nu$, there exists a sequence $(x_n)_n$ in $\mathcal{B}$ such that $\Lambda_B(x_n) \xrightarrow{\text{in} \mathcal{H}_B} \Lambda_B(b)$ and $\pi_B(x_n) \xrightarrow{(\sigma\text{-strong})} b$.

Denote by $T_\nu$ the closure of the involution $\Lambda_B(x) \mapsto \Lambda_B(x^*)$ on $\Lambda_B(\mathcal{B})$. There exists a polar decomposition, $T_\nu = J_\nu \nabla_\nu^{\frac{1}{2}}$, where $\nabla_\nu$ is the modular operator, given by $\nabla_\nu = T_\nu^* T_\nu$, and $J_\nu$ is the modular conjugation, which is anti-unitary.

According to the modular theory in the von Neumann algebra setting, the modular operator defines a strongly continuous one-parameter group of automorphisms $\sigma_\nu^t$, by $\sigma_\nu^t(b) = \nabla_\nu^{it} b \nabla_\nu^{-it}$, for $b \in \mathcal{B}$, $t \in \mathbb{R}$, leaving the von Neumann algebra $N$ invariant. We have $\tilde{\nu} \circ \sigma_\nu^t = \tilde{\nu}$ for $t \in \mathbb{R}$, and $\tilde{\nu}$ satisfies a certain KMS boundary condition. In particular, the weak KMS property at the $^*$-algebra level, $\nu(bb^*) = \nu(b'\sigma(\nu(b)))$, $b, b' \in \mathcal{B}$, extends to the von Neumann algebra as $\tilde{\nu}(xx^*) = \tilde{\nu}(x'\sigma_{\nu}(x))$, $x \in \mathfrak{M}_\nu$, $x' \in \mathcal{D}(\sigma_{\nu})$. Meanwhile, the modular conjugation $J_\nu$ can be characterized by $J_\nu \Lambda_\nu(x) = \Lambda_\nu(\sigma_{\nu}^t(x^*))$, for $x \in \mathfrak{M}_\nu$.

Observe that the elements in $\mathcal{B}$ are actually analytic elements for $\nu$.

**Proposition 2.5.** Consider the KMS weights $\nu$ on $\mathcal{B}$. Then any element $b \in \mathcal{B}(\subseteq \mathcal{B}$) is an analytic element for $\nu$.

**Proof.** Recall the weak KMS property of $\nu$, or Equation (1.11), such that there exists an automorphism $\sigma$ on $\mathcal{B}$ satisfying $\nu(bb^*) = \nu(b'\sigma_{\nu}(b))$, for all $b, b' \in \mathcal{B}$. As such, we can see that for any $b \in \mathcal{B}$, we have $b \in \mathcal{D}(\sigma_{\nu}^t)$ and that $\sigma_{\nu}^t(b) = \nabla_{\nu}^{-1} b \nabla_{\nu} = \sigma_{\nu}(b)$. Then we can see quickly that we also have $b \in \mathcal{D}(\sigma_{\nu}^m)$ and that $\sigma_{\nu}^m(b) = \nabla_{\nu}^{-m} b \nabla_{\nu}^m$ for all $m \in \mathbb{Z}$. Continuing, it is not difficult to see that $b \in \mathcal{D}(\sigma_{\nu}^z)$ and that $\sigma_{\nu}^z(b) = \nabla_{\nu}^{iz} b \nabla_{\nu}^{-iz}$, for any $z \in \mathbb{C}$. \hfill $\square$

Meanwhile, we can also do the same with the left Hilbert algebra $\Lambda_C(C)$, obtaining another n.s.f. weight $\tilde{\mu}$ on $L = \pi_C(C)$, as well as the modular operator $\nabla_{\tilde{\mu}}$, the modular conjugation $J_{\tilde{\mu}}$, the modular automorphism group $\sigma_{\tilde{\mu}}$. The elements in $C$ will be analytic elements for $\mu$.

However, having gathered all these results from the left Hilbert algebra theory and the modular theory, we have to point out that they are not quite sufficient for our purposes. Since we wish to develop a $C^*$-algebraic framework, two main issues arise: (i) The modular automorphism group $\sigma_{\nu}$ leaves the von Neumann algebra $N$ invariant, but we also want it to leave the $C^*$-algebra $B$ invariant; (ii) While $(\sigma_{\nu}^t)_{t \in \mathbb{R}}$ is strongly continuous, we want it to be norm continuous as well. As (i) and (ii) are not automatic consequences of the modular theory, some more work is needed. To remedy this situation, let us gather below some additional results on the canonical idempotent $E$.

### 2.3. The idempotent $E$.

We saw earlier that we may regard our canonical idempotent $E \in M(\mathcal{B} \otimes C)$ as a bounded operator $\Pi(E) \in B(\mathcal{H}_B \otimes \mathcal{H}_C)$. In fact, we can now see that $\Pi = \pi_B \otimes \pi_C$, and that $(\pi_B \otimes \pi_C)(E)$ is an element of the tensor product von Neumann algebra $N \otimes L$. It is
also apparent that \((\pi_B \otimes \pi_C)(E) \in M(B \otimes C)\), where \(\otimes\) is now a (spatial) \(C^*\)-tensor product. For convenience, we will regard \(E = (\pi_B \otimes \pi_C)(E)\) in what follows.

Also for convenience, we may regard \(x \in B\) as \(x = \pi_B(x) \in B \subseteq N \subseteq B(H_B)\), and regard \(y \in C\) as \(y = \pi_C(y) \in C \subseteq L \subseteq B(H_C)\).

Recall from Equation (1.8) that for \(b \in \mathcal{B}\), we have:

\[
(\nu \otimes \text{id})(E(b \otimes 1)) = (\nu \otimes \text{id})(E(1 \otimes S_B(b))) = (\nu \otimes \text{id})(E)S_B(b) = S_B(b).
\]

As the weight \(\nu\) extends the functional \(\nu\), we can use the above observation to define the map \(\gamma_N : N \to \tilde{L}\), by

\[
\gamma_N(b) := (\nu \otimes \text{id})(E(b \otimes 1)), \quad b \in \mathcal{B}.
\]  

(2.1)

This map may be unbounded, but as \(\mathcal{B} = \pi_B(\mathcal{B})\) is dense in \(N\) and \(S_B(\mathcal{B}) = \mathcal{C}\) is dense in \(L\), we see that \(\gamma_N : N \to \tilde{L}\) is a densely-defined map having a dense range, which is an injective anti-homomorphism because \(S_B\) is. However it is not a \(*\)-map, as \(S_B\) is not.

Consider instead \(\tilde{R} := \gamma_N \circ \sigma^\nu_{-\frac{i}{2}}\), where \(\sigma^\nu_{-\frac{i}{2}}\) is the analytic generator for \((\sigma^\nu_t)_{t \in \mathbb{R}}\), at \(z = -\frac{i}{2}\).

Since \(\sigma^\nu_{-\frac{i}{2}}\) is an automorphism and since \(\gamma_N\) is an anti-homomorphism, we can see quickly that \(\tilde{R}\) is anti-multiplicative. As all elements of \(\mathcal{B}\) are analytic, we see that \(\sigma^\nu_{-\frac{i}{2}}\) is densely-defined, and so is \(\tilde{R}\). In addition, the following observation shows that \(\tilde{R}\) is involutive. Note that for \(b \in \mathcal{T}_L\), we have:

\[
\tilde{R}(b^*) = (\nu \otimes \text{id})(E(\sigma^\nu_{-\frac{i}{2}}(b^*)) \otimes 1) = (\nu \otimes \text{id})(\sigma^\nu_{-\frac{i}{2}}(\sigma^\nu_{-\frac{i}{2}}(b^*)) \otimes 1)E = (\nu \otimes \text{id})(\sigma^\nu_{\frac{i}{2}}(b^*) \otimes 1)E,
\]

(2.2)

because of the KMS property of \(\nu\). At the same time,

\[
\tilde{R}(b^*) = [(\nu \otimes \text{id})(E(\sigma^\nu_{-\frac{i}{2}}(b \otimes 1)))^* = (\nu \otimes \text{id})(\sigma^\nu_{\frac{i}{2}}(b^*) \otimes 1)E,
\]

because \(E\) is self-adjoint. Comparing, we see that \(\tilde{R}(b^*) = \tilde{R}(b^*)^*\). This shows that \(\tilde{R}\) is a \(*\)-map, which means that it is actually a \(*\)-anti-homomorphism, so bounded. Therefore, we can extend \(\tilde{R}\) to all of \(N\). In fact, as \(\tilde{R}\) is a bounded map from \(N\) to \(\tilde{L}\), injective, densely-defined, having a dense range, it extends to a \(*\)-anti-isomorphism \(\tilde{R} : N \to \tilde{L}\). Meanwhile, from the definition of \(\tilde{R}\), it is immediate that we have \(\gamma_N = \tilde{R} \circ \sigma^\nu_{-\frac{i}{2}}\), which is essentially like a polar decomposition.

Next, consider the n.s.f. weight \(\tilde{\mu}\) on \(\tilde{L}\), extending the functional \(\mu\) on \(\mathcal{C}\). In an analogous way as above, we can consider the extension of the map \(S_C : \mathcal{C} \to \mathcal{B}\) to the von Neumann algebra level, namely the densely-defined anti-homomorphism \(\gamma_L : \tilde{L} \to \tilde{N}\). Analogous to Equation (2.1) for \(\gamma_N\), we can characterize it by

\[
\gamma_L(c) = (\text{id} \otimes \tilde{\mu})(1 \otimes c)E, \quad c \in \mathcal{C}.
\]

(2.3)

In the following proposition, we gather some useful relationships between the weights \(\nu, \tilde{\mu}\), and the maps \(\gamma_N\) and \(\gamma_L\).

Proposition 2.6. Let the weights \(\nu\) on \(N\) and \(\tilde{\mu}\) on \(\tilde{L}\) be the extensions of the functionals \(\nu\) and \(\mu\), and let \(\gamma_N : N \to \tilde{L}\) and \(\gamma_L : \tilde{L} \to \tilde{N}\) be the densely-defined anti-homomorphisms as
in Equations (2.1) and (2.3), extending the maps $S_B$ and $S_C$. Also let $\tilde{R} = \gamma_N \circ \sigma_{\tilde{\nu}}$ be the *-anti-isomorphism from $N$ to $L$ obtained above. Then

1. \( \tilde{\nu} = \mu \circ \gamma_N \) and \( \tilde{\nu} = \tilde{\mu} \circ \tilde{R} \).
2. \( \gamma_N = \tilde{R} \circ \sigma_{\tilde{\nu}} = \sigma_{\tilde{\nu} \tilde{\nu}} \circ \tilde{R} \).
3. \( \gamma_L = \sigma_{\tilde{\nu}}^- \circ \tilde{R}^{-1} = \tilde{R}^{-1} \circ \sigma_{\tilde{\nu}}^- \).
4. For any \( t \in \mathbb{R} \), we have \( (\sigma_t \tilde{\nu} \circ \sigma_{\tilde{\nu}} \tilde{\nu})(E) = E \).
5. \( (\gamma_N \circ \gamma_L)(E) = \varsigma E \) and \( (\gamma_L \circ \gamma_N)(\varsigma E) = E \).
6. \( (\tilde{R} \circ \tilde{R}^{-1})(E) = \varsigma E \) and \( (\tilde{R}^{-1} \circ \tilde{R})(\varsigma E) = E \).

**Proof.** (1). Recall that at the *-algebra level, we have \( \nu = \mu \circ S_B : B \to C \). Extending this to the von Neumann algebra level, we have \( \tilde{\nu} = \tilde{\mu} \circ \gamma_N \), or equivalently, \( \tilde{\nu} \circ \gamma_N^{-1} = \tilde{\mu} \).

From \( \tilde{R} = \gamma_N \circ \sigma_{\tilde{\nu}}^{-1} \), we can write \( \gamma_N^{-1} = (\tilde{R} \circ \sigma_{\tilde{\nu}})^{-1} = \sigma_{\tilde{\nu} \tilde{\nu}}^{-1} \circ \tilde{R}^{-1} \). So from \( \tilde{\nu} \circ \gamma_N^{-1} = \tilde{\mu} \), we have \( \tilde{\nu} \circ \sigma_{\tilde{\nu} \tilde{\nu}}^{-1} \circ \tilde{R}^{-1} = \tilde{\mu} \). Since we know \( \tilde{\nu} \circ \sigma_{\tilde{\nu}}^{-1} = \tilde{\nu} \), \( \forall t \), it follows that \( \tilde{\nu} \circ \tilde{R}^{-1} = \tilde{\mu} \). Or equivalently, \( \tilde{\nu} = \tilde{\mu} \circ \tilde{R} \).

(2). Since \( \tilde{\nu} = \tilde{\mu} \circ \tilde{R} \), it is easy to see that the modular automorphism groups have the following relation:

\[
\sigma_t^{\tilde{\nu}} = \tilde{R}^{-1} \circ \sigma_{\tilde{\nu} \tilde{\nu}} \circ \tilde{R}, \quad \forall t \in \mathbb{R}.
\]

(3). At the *-algebra level, from Equation (1.11), we know about the KMS-type automorphism \( \sigma^{\nu} = S_B^{-1} \circ S_C^{-1} \). At the von Neumann algebra level, this extends to \( \sigma_{\tilde{\nu} \tilde{\nu} \tilde{\nu}} = \gamma_N^{-1} \circ \gamma_L^{-1} \). We thus have

\[
\gamma_L = (\gamma_N \circ \sigma_{\tilde{\nu} \tilde{\nu}})^{-1} = \sigma_t^{\tilde{\nu} \tilde{\nu}} \circ \gamma_N^{-1} = \sigma_t^{\tilde{\nu} \tilde{\nu}} \circ (\tilde{R} \circ \sigma_{\tilde{\nu}})^{-1} = \sigma_t^{\tilde{\nu}} \circ \sigma_{\tilde{\nu} \tilde{\nu}}^{-1} \circ \tilde{R}^{-1} = \sigma_t^{\tilde{\nu}} \circ \tilde{R}^{-1}.
\]

Alternatively, by Equation (2.4), we have \( \gamma_L = (\tilde{R}^{-1} \circ \sigma_{\tilde{\nu} \tilde{\nu}} \circ \tilde{R}) \circ \tilde{R}^{-1} = \tilde{R}^{-1} \circ \sigma_{\tilde{\nu} \tilde{\nu}} \).

(4). For arbitrary \( b \in D(\sigma_{\tilde{\nu}} \tilde{\nu}) \) and \( t \in \mathbb{R} \), observe that

\[
(\tilde{\nu} \otimes \text{id})(\sigma_t^{\tilde{\nu}} \otimes \sigma_{\tilde{\nu} \tilde{\nu}})(E)(b \otimes 1)) = (\tilde{\nu} \otimes \text{id})(\sigma_t^{\tilde{\nu}} \otimes \sigma_{\tilde{\nu}})(E(\sigma_t^{\tilde{\nu}}(b) \otimes 1)) = (\tilde{\nu} \otimes \text{id})(\text{id} \otimes \sigma_{\tilde{\nu}})(E(\sigma_t^{\tilde{\nu}}(b) \otimes 1)) = \sigma_{\tilde{\nu}}(\gamma_N(\sigma_{\tilde{\nu} \tilde{\nu}})(b)) = (\sigma_{\tilde{\nu} \tilde{\nu}} \circ \tilde{R} \circ \sigma_{\tilde{\nu} \tilde{\nu}})(b) = (\sigma_{\tilde{\nu} \tilde{nu}} \circ \tilde{R} \circ \sigma_{\tilde{\nu} \tilde{nu}})(b) = (\tilde{R} \circ \sigma_{\tilde{\nu}} \circ \tilde{R}^{-1})(b) = \gamma_N(b) = (\tilde{\nu} \otimes \text{id})(E(b \otimes 1)) = \gamma_N(b).
\]
We used the fact that $\sigma_t^\flat$ is an automorphism for the first equality; for the second, we used $\tilde{\nu} \circ \sigma_t^\flat = \tilde{\nu}$; and in the rest, we used the definition of $\gamma_N$ and Equation (2.4). This is true for any $b \in D(\sigma_t^\flat)$ and $\tilde{\nu}$ is faithful, so we see that $(\sigma_t^\flat \otimes \sigma_t^\flat)(E) = E$, for any $t \in \mathbb{R}$.

(5). At the $*$-algebra level, it is known that $(S_B \otimes S_C)(E) = \zeta E$, where $\zeta$ is the flip map between $B \otimes C$ and $C \otimes B$. It thus follows that at the von Neumann algebra level, we have $(\gamma_N \otimes \gamma_L)(E) = \zeta E$. Also $(\gamma_L \otimes \gamma_N)(\zeta E) = E$.

(6). Combine the results of (4) and (5). Since $\tilde{R} = \gamma_N \circ \sigma_t^\flat$ and $\tilde{R}^{-1} = \gamma_L \circ \sigma_t^\flat$, we have:

$$(\tilde{R} \otimes \tilde{R}^{-1})(E) = (\gamma_N \otimes \gamma_L)((\sigma_t^\flat \otimes \sigma_t^\flat)(E)) = (\gamma_N \otimes \gamma_L)(E) = \zeta E.$$  

We also have $(\tilde{R}^{-1} \otimes \tilde{R})(\zeta E) = E$. $\square$

2.4. The KMS weights on the $C^*$-algebras $B$ and $C$. Note that by restricting the weight $\tilde{\nu}$ on $N = \pi_B(B)^\prime$ to the $C^*$-algebra $B = \pi_B(B)^\prime \| \|$, represented on the same Hilbert space $\mathcal{H}_B$, we obtain a faithful lower semi-continuous weight. For convenience, we will denote this weight by $\nu$, as it is also an extension of the functional $\nu$ at the $*$-algebra level. We can consider the associated spaces $\mathcal{M}_\nu = \{ x \in B : \nu(x^*x) < \infty \}$ and $\mathcal{M}_\nu = \mathcal{M}_\nu \mathcal{M}_\nu$.

We can also consider the operator $T_\nu$, the closure of $\Lambda_B(x) \mapsto \Lambda_B(x^*), x \in B$. It is apparent that it will exactly coincide with $T_\nu$ earlier, and the polar decomposition will also remain exactly same, $T_\nu = J_\nu \nabla_\nu^\frac{1}{2}$, with $\nabla_\nu = \nabla_\nu$ and $J_\nu = J_\nu$. However, the stumbling issue in 2.2 was the question whether the associated modular automorphism group $\sigma_t^\nu : x \mapsto \nabla_\nu^{it} x \nabla_\nu^{-it}$ leaves the $C^*$-algebra $B$ invariant, and whether $(\sigma_t^\nu)_{t \in \mathbb{R}}$ forms a norm-continuous one-parameter group. With the results gathered in 2.3, we are now in a position to resolve this issue in the affirmative.

Proposition 2.7. Consider the weight $\nu$ on $B$, restricted from $\tilde{\nu}$ on $N$. Then

1. The automorphism group $(\sigma_t^\nu)_{t \in \mathbb{R}}$ leaves $B$ invariant. So we can consider $\sigma_t^\nu := \sigma_t^\nu | B$, for $t \in \mathbb{R}$.
2. $\nu$ becomes a KMS weight on $B$, equipped with the automorphism group $(\sigma_t^\nu)_{t \in \mathbb{R}}$, which is norm-continuous.

Proof. (1). Consider $\omega \in B(\mathcal{H}_C)$, and consider $(id \otimes \omega)(E) \in B$. Such elements are dense in $B$. For any $t \in \mathbb{R}$ we know from Proposition 2.6(4) that $(\sigma_t^\nu \otimes \sigma_t^\nu)(E) = E$. we thus have

$$\sigma_t^\nu ((id \otimes \omega)(E)) = \sigma_t^\nu ((id \otimes \omega)(\sigma_t^\nu \otimes \sigma_t^\nu)(E))) = (id \otimes (\omega \circ \sigma_t^\nu))(E) \in B. \tag{2.5}$$

This shows that $\sigma_t^\nu (B) = B$, for all $t \in \mathbb{R}$. We will just write $\sigma_t^\nu := \sigma_t^\nu | B$.

(2). As we noted above, it is clear that $\nu$ is a faithful lower semi-continuous weight because $\tilde{\nu}$ is an n.s.f. weight. In addition, we know that $\nu$ is semi-finite because it extends the distinguished functional $\nu$, which is defined on a dense subalgebra $B \subseteq B$.

Meanwhile, since $\sigma_t^\nu (B) = B$, we can consider the one-parameter group of automorphisms $(\sigma_t^\nu)_{t \in \mathbb{R}}$. At present, we only know that it is strongly continuous. But, the strong continuity together with Equation (2.5) show us that $t \mapsto \sigma_t^\nu (b)$ is indeed norm-continuous.
Proposition 2.8. The restriction $E := \hat{E}|_{\mathcal{B}}$. For any $b \in \mathcal{D}(\sigma^\nu_1)$, we have

$$\nu(x^*x) = \nu(\sigma^\nu_1 \sigma^\nu_1(x)^*).$$

In this way, we show that $\nu$ is a KMS weight (see [23], [14], [13]). \hfill $\square$

Note, by the way, that the *-anti-isomorphism $R : N \rightarrow L$ can be restricted to the $C^*$-algebra level. So consider $R := \hat{R}|_{\mathcal{B}}$. Then for $(\text{id} \otimes \omega)(E) \in \mathcal{B}$, we have:

$$R((\text{id} \otimes \omega)(E)) = \hat{R}((\omega \otimes \text{id})(\xi(E))) = \hat{R}((\omega \otimes \text{id})(\hat{R} \otimes \hat{R}^{-1})(E)) = ((\omega \circ \hat{R}) \otimes \text{id})(E) \in \mathcal{C},$$

where we used the result of Proposition 2.6 (6). This shows that $\hat{R} : N \rightarrow L$ restricts to $R : \mathcal{B} \rightarrow \mathcal{C}$. It becomes a $C^*$-anti-isomorphism.

This means that together with the *-anti-isomorphism $R : \mathcal{B} \rightarrow \mathcal{C}$ and the KMS weight $\nu$ on $\mathcal{B}$, it turns out that $(E, B, \nu)$ forms a separability triple, in the sense of [9]:

**Proposition 2.9.** The restriction $R = \hat{R}|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{C}$ is a $C^*$-anti-isomorphism. The self-adjoint idempotent $E \in M(\mathcal{B} \otimes \mathcal{C})$ is a separability idempotent, in the sense that

1. $(\nu \otimes \text{id})(E) = 1$
2. For $b \in \mathcal{D}(\sigma^\nu_1)$ we have: $(\nu \otimes \text{id})(E(b \otimes 1)) = (R \circ \sigma^\nu_1)(b)$.

Proof. We showed that $R$ and $\sigma^\nu_1$ are now valid at the level of the $C^*$-algebra $\mathcal{B}$. Then (1) is just recognizing the fact that $\nu$ extends the distinguished functional $\nu$ on $\mathcal{B}$, and (2) is just noting that $R \circ \sigma^\nu_1 = \gamma_B = \gamma_{\mathcal{B}}$, extending $S_B$. \hfill $\square$

Remark. While we gave results only regarding the weight $\nu$ on $\mathcal{B}$, a very much similar argument can be given for the weight $\mu$ on the $C^*$-algebra $\mathcal{C}$, as a restriction of $\tilde{\mu}$ on $L$. It would extend the distinguished functional $\mu$, and become a KMS weight on the $C^*$-algebra $\mathcal{C}$, equipped with the norm-continuous one-parameter group $(\sigma^\mu_1)_{t \in \mathbb{R}}$ given by the modular operator.

In Proposition 2.5 we saw that the elements in $\mathcal{B}$ are analytic elements for $\nu$. Similarly, the elements of $\mathcal{C}$ are analytic elements for $\mu$. These results suggest that the elements in $\Lambda_B(\mathcal{B})$ and $\Lambda_C(\mathcal{C})$ are right bounded vectors in $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively. (See Definition 1.7 of [23] for the notion of right bounded vectors.) To make this point clearer, see the proposition below.

**Proposition 2.9.**

- For any $b \in \mathcal{B}$, the vector $\Lambda_B(b) \in \mathcal{H}_B$ is right bounded. This means that the map $\pi^B_R(b) : \Lambda_B(x) \mapsto \Lambda_B(xb)$ is a bounded operator on $\mathcal{H}_B$.

- For any $c \in \mathcal{C}$, the vector $\Lambda_C(c) \in \mathcal{H}_C$ is right bounded. Or the map $\pi^C_R(c) : \Lambda_C(y) \mapsto \Lambda_C yc$ is a bounded operator on $\mathcal{H}_C$.

Proof. (1). For any $x \in \mathcal{B}$, we know $\pi^B_R(x) \Lambda_B(b) = \Lambda_B(xb)$. Recall next the unitary operator $\hat{J}_B : \mathcal{H}_B \rightarrow \mathcal{H}_C$ defined earlier (see [2.1], given by $\hat{J}_B \Lambda_B(x) = \Lambda_C(\hat{S}_B(x))$. Note that we can
write:

\[ [\hat{J}_B]^* \pi_C(S_B(b)) \hat{J}_B \Lambda_B(x) = [\hat{J}_B]^* \pi_C(S_B(b)) \Lambda_C(S_B(x)) = [\hat{J}_B]^* \Lambda_C(S_B(b)S_B(x)) \]

\[ = [\hat{J}_B]^* \Lambda_C(S_B(xb)) = \Lambda_B((S_B^{-1} \circ S_B)(xb)) = \Lambda_B(xb). \]

Combining, we observe that \( \pi_B(x) \Lambda_B(b) = \Lambda_B(xb) = \pi_B^R(b) \Lambda_B(x) \), where \( \pi_B^R(b) \) is the bounded operator \( [\hat{J}_B]^* \pi_C(S_B(b)) \hat{J}_B \). This proves that \( \Lambda_B(b), b \in \mathcal{B}, \) is right bounded in \( \mathcal{H}_B \).

(2). Similarly, we can show that \( \pi_C(y) \Lambda_C(c) = \Lambda_C(yc) = \pi_C^R(c) \Lambda_C(y) \), where \( \pi_C^R(c) = \hat{J}_B \pi_B(S_B^{-1}(c)) [\hat{J}_B]^* \), a bounded operator. So \( \Lambda_C(c), c \in \mathcal{C}, \) is right bounded in \( \mathcal{H}_C \). \( \square \)

3. The \( \mathcal{C}^* \)-bialgebra \( (\mathcal{A}, \Delta) \)

Recall that our weak multiplier Hopf \( \mathcal{C}^* \)-algebra \( (\mathcal{A}, \Delta) \) is equipped with a faithful positive left integral \( \varphi \). As \( \varphi \) is a positive linear functional, we can equip \( \mathcal{A} \) with an inner product:

\[ \langle x, y \rangle := \varphi(y^* x), \quad \text{for } x, y \in \mathcal{A}. \]

As usual, complete \( \mathcal{A} \) with respect to the induced norm, and obtain a Hilbert space \( \mathcal{H} \) with the natural inclusion \( \Lambda : \mathcal{A} \to \mathcal{H} \). (Note that \( \Lambda \) is injective because \( \varphi \) is faithful.) We are planning to represent our \( \mathcal{C}^* \)-algebra as an operator algebra in \( \mathcal{B}(\mathcal{H}) \), but at present it is not clear if the left multiplication of the elements of \( \mathcal{A} \) are bounded. Some work is needed.

3.1. Representations of \( B \) and \( C \) on \( \mathcal{H} \). Note that \( \mathcal{C} \mathcal{A} = \mathcal{A} \mathcal{C} = \mathcal{A} \). This suggests us to define the map \( \rho_C : \mathcal{A} \to \mathcal{L}(\mathcal{H}_C, \mathcal{H}) \), by

\[ \rho_C(a) \Lambda_C(y) = \Lambda(ya), \quad \text{for } a \in \mathcal{A}, y \in \mathcal{C}. \]

The next proposition shows that \( \rho_C(a), a \in \mathcal{A}, \) is bounded.

Proposition 3.1. Consider \( \rho_C : \mathcal{A} \to \mathcal{L}(\mathcal{H}_C, \mathcal{H}) \) above. Then \( \rho_C(a) \) is a bounded element in \( \mathcal{L}(\mathcal{H}_C, \mathcal{H}) \), for any \( a \in \mathcal{A} \).

Proof. Let \( a \in \mathcal{A} \) and \( y \in \mathcal{C} \). Then

\[ \left\| \rho_C(a) \Lambda_C(y) \right\|^2 = \langle \Lambda(ya), \Lambda(ya) \rangle = \varphi(a^* y^* ya) = \varphi(y^* ya \sigma(a^*)), \]

where \( \sigma \) is the modular automorphism for \( \varphi \), as noted in Proposition 1.13. Apply here the result of Proposition 1.18 knowing that the weight \( \mu \) extends the functional \( \mu \) on \( \mathcal{C} \). Then we have:

\[ \left\| \rho_C(a) \Lambda_C(y) \right\|^2 = \varphi(y^* ya \sigma(a^*)) = \mu((id \otimes \varphi)(\Delta(y^* ya \sigma(a^*))), \]

Note that by Proposition 1.14 since \( y^* y \in \mathcal{C} \), we have \( \Delta(y^* y) = (y^* y \otimes 1)E \). So we have

\[ \Delta(y^* ya \sigma(a^*)) = \Delta(y^* y) \Delta(a \sigma(a^*)) = (y^* y \otimes 1)E \Delta(a \sigma(a^*)) = (y^* y \otimes 1) \Delta(a \sigma(a^*)). \]

Putting this in the previous equation, we see that

\[ \left\| \rho_C(a) \Lambda_C(y) \right\|^2 = \mu(y^* y(id \otimes \varphi)(\Delta(a \sigma(a^*)))) = \mu(y^* yc) = \langle \Lambda_C(yc), \Lambda_C(y) \rangle_{\mathcal{H}_C}, \]

where \( c = (id \otimes \varphi)(\Delta(a \sigma(a^*))) \in M(\mathcal{C}) \), by the left invariance property of \( \varphi \).
By Proposition 2.9, we can write $\Lambda_C(yC) = \pi^R_C(c)\Lambda_C(y)$, where $\pi^R_C(c)$ is a bounded operator. So the previous equation becomes:

$$\|\rho_C(a)\Lambda_C(y)\|^2_H = \langle \pi^R_C(c)\Lambda_C(y), \Lambda_C(y) \rangle_{\mathcal{H}_C} \leq \|\pi^R_C(c)\|\|\Lambda_C(y)\|^2_{\mathcal{H}_C},$$

showing that $\|\rho_C(a)\| \leq \|\pi^R_C(c)\|^\frac{1}{2}$. □

**Corollary.** For any $a, b \in A$, we have $\rho_C(b)^*\rho_C(a) \in B(\mathcal{H}_C)$, a bounded operator on $\mathcal{H}_C$.

**Proof.** By the previous proposition, we know $\rho_C(a), \rho_C(b) \in \mathcal{L}(\mathcal{H}_C, \mathcal{H})$ are bounded, which also means $\rho_C(b)^*$ is a bounded element in $\mathcal{L}(\mathcal{H}, \mathcal{H}_C)$. It follows that $\rho_C(b)^*\rho_C(a)$ is also bounded, such that $\rho_C(b)^*\rho_C(a) \in B(\mathcal{H}_C)$.

See below that any operator of the form $\rho_C(b)^*\rho_C(a) \in B(\mathcal{H}_C)$, $a, b \in A$, commutes with the elements of the $C^*$-algebra $C = \pi(\mathcal{C})$. Where we used Propositions 1.8 and 1.4, we can write

$$\langle \rho_C(b)^*\rho_C(a)\Lambda_C(y_1), \Lambda_C(y_2) \rangle_{\mathcal{H}_C} = \langle \rho_C(a)\Lambda_C(y_1), \rho_C(b)^*\Lambda_C(y_2) \rangle_{\mathcal{H}} = \langle \Lambda(y_1a), \Lambda(y_2b) \rangle_{\mathcal{H}} = \varphi(b^*y_2^*y_1a) = \varphi(y_2^*y_1a\sigma(b^*)),$$

By the same argument as in the proof of Proposition 3.1, where we used Propositions 1.8 and 1.4, we can write

$$\varphi(y_2^*y_1a\sigma(b^*)) = \cdots = \mu(y_2^*y_1\tilde{c}),$$

where $\tilde{c} = (id \otimes \varphi)(\Delta(aa\sigma(b^*))) \in M(\mathcal{C})$. So we can write $\rho_C(b)^*\rho_C(a)\Lambda_C(y_1) = \Lambda_C(y_1\tilde{c})$, which means that $\rho_C(b)^*\rho_C(a) \in B(\mathcal{H}_C)$ is none other than $\pi^R_C(\tilde{c})$.

As such, for $\pi_C(c) \in C$ and any $y \in \mathcal{C}$, we have:

$$\rho_C(b)^*\rho_C(a)\pi_C(c)\Lambda_C(y) = \rho_C(b)^*\rho_C(a)\Lambda_C(cy) = \pi^R_C(\tilde{c})\Lambda_C(cy	ilde{c}) = \Lambda_C(cy\tilde{c}),$$

$$\pi_C(c)\rho_C(b)^*\rho_C(a)\Lambda_C(y) = \pi_C(c)\pi^R_C(\tilde{c})\Lambda_C(cy) = \pi_C(c)\Lambda_C(cy\tilde{c}) = \Lambda_C(cy\tilde{c}),$$

showing that $\rho_C(b)^*\rho_C(a)$ commutes with any $\pi_C(c) \in C$. □

We are now ready to construct a *-representation of the $C^*$-algebra $C$ into $B(\mathcal{H})$. See below:

**Proposition 3.3.** Consider any $c \in \mathcal{C}$, which we regard as $c = \pi_C(c)$, an element of the $C^*$-algebra $C$. Define $\alpha(\pi_C(c)) \in \mathcal{L}(\mathcal{H})$, by

$$\alpha(\pi_C(c))\Lambda(a) = \Lambda(ca), \quad a \in A.$$

Then

1. $\alpha(\pi_C(c)), c \in \mathcal{C}$, is a bounded operator on $\mathcal{H}$.
2. $\alpha$ extends to a (bounded) $C^*$-representation $\alpha : C \to B(\mathcal{H})$.
3. $\alpha : C \to B(\mathcal{H})$ becomes a non-degenerate *-representation. It also extends to the *-representation at the level of the multiplier algebra $M(\mathcal{C})$. 
Proof. (1). Without loss of generality, we may consider the vectors of the type \( \Lambda(ya) \in \mathcal{H} \), where \( y \in \mathcal{C}, \ a \in \mathcal{A} \), because \( \mathcal{C} \mathcal{A} = \mathcal{A} \). Note that we can write

\[
\alpha(\pi_C(c))\Lambda(ya) = \Lambda(cya) = \rho_C(a)\Lambda_C(cy) = \rho_C(a)\pi_C(c)\Lambda_C(y).
\]

We know that \( \rho_C(a) \) is bounded. We thus have:

\[
\|\alpha(\pi_C(c))\Lambda(ya)\|^2 = \langle \rho_C(a)\pi_C(c)\Lambda_C(y), \rho_C(a)\pi_C(c)\Lambda_C(y) \rangle_{\mathcal{H}} = \langle \rho_C(a)^*\rho_C(a)\pi_C(c)\Lambda_C(y), \pi_C(c)\Lambda_C(y) \rangle_{\mathcal{H}} \\
= \langle \pi_C(c)^*\rho_C(a)\Lambda_C(y), \pi_C(c)\Lambda_C(y) \rangle_{\mathcal{H}} \\
\leq \|\pi_C(c)\|^2\langle \rho_C(a)^*\rho_C(a)\Lambda_C(y), \Lambda_C(y) \rangle_{\mathcal{H}} = \|\pi_C(c)\|^2\langle \rho_C(a)\Lambda_C(y), \rho_C(a)\Lambda_C(y) \rangle_{\mathcal{H}} \\
= \|\pi_C(c)\|^2\langle \Lambda(ya), \Lambda(ya) \rangle_{\mathcal{H}} = \|\pi_C(c)\|^2\|\Lambda(ya)\|^2.
\]

Note that the third equality is because \( \rho_C(a)^*\rho_C(a) \) commutes with \( \pi_C(c) \in \mathcal{C} \) (see Proposition 3.2). This observation shows that \( \alpha(\pi_C(c)) \) is bounded, with \( \|\alpha(\pi_C(c))\| \leq \|\pi_C(c)\| \).

(2). It is not difficult to see that \( \alpha \) preserves multiplication. Note that for any \( c_1, c_2 \in \mathcal{C} \) and for any \( a \in \mathcal{A} \), we have

\[
\alpha(\pi_C(c_1))\alpha(\pi_C(c_2))\Lambda(a) = \alpha(\pi_C(c_1))\Lambda(c_2a) = \Lambda(c_1c_2a) = \alpha(\pi_C(c_1c_2))\Lambda(a).
\]

As \( \pi_C(c_1c_2) = \pi_C(c_1)\pi_C(c_2) \), and since the vectors \( \Lambda(a), \ a \in \mathcal{A} \), are dense in \( \mathcal{H} \), it is evident that \( \alpha(\pi_C(c_1))\alpha(\pi_C(c_2)) = \alpha(\pi_C(c_1)\pi_C(c_2)) \).

To see if \( \alpha \) preserves the involution, note that for any \( c \in \mathcal{C} \) and any \( a_1, a_2 \in \mathcal{A} \), we have

\[
\langle \alpha(\pi_C(c))\Lambda(a_1), \Lambda(a_2) \rangle = \langle \Lambda(c_1), \Lambda(a_2) \rangle = \varphi(a_2^\ast c_1) = \varphi((c^\ast a_2)^\ast a_1) \\
= \langle \Lambda(a_1), \alpha(\pi_C(c^\ast))\Lambda(a_2) \rangle,
\]

since \( \pi_C(c^\ast) = \pi_C(c)^\ast \). Since \( a_1, a_2 \in \mathcal{A} \) are arbitrary, we see that \( \alpha(\pi_C(c))^\ast = \alpha(\pi_C(c)^\ast) \).

This means \( \alpha : \pi_C(\mathcal{C}) \to \mathcal{B}(\mathcal{H}) \) is a *-representation, which is automatically bounded. It follows that \( \alpha \) extends to \( \mathcal{C} = \pi_C(\mathcal{C})\| \), giving us the *-representation of the \( C^\ast \)-algebra \( \mathcal{C} \).

(3). The non-degeneracy of \( \alpha \) is easy to see, using the fact that \( \mathcal{C} \mathcal{A} = \mathcal{A} \). As a result, it is clear that \( \alpha \) naturally extends to the level of the multiplier algebra \( M(\mathcal{C}) \) \( \Box \).

An analogous procedure can be carried out for \( \mathcal{B} \). First, we can consider \( \rho_B : \mathcal{A} \to \mathcal{L}(\mathcal{H}_B, \mathcal{H}) \), as follows:

**Proposition 3.4.** Let \( \rho_B : \mathcal{A} \to \mathcal{L}(\mathcal{H}_B, \mathcal{H}) \), by

\[
\rho_B(a)\Lambda_B(x) = \Lambda(xa), \quad \text{for } a \in \mathcal{A}, \ x \in \mathcal{B}.
\]

Then \( \rho_B(a) \) is a bounded element in \( \mathcal{L}(\mathcal{H}_B, \mathcal{H}) \), for any \( a \in \mathcal{A} \).

**Proof.** The proof is essentially no different from that of Proposition 3.1. Use again Propositions 1.3 and 1.4, but this time use the right invariance of \( \psi \). \( \Box \)

We can also define a *-representation of the \( C^\ast \)-algebra \( \mathcal{B} \) into \( \mathcal{B}(\mathcal{H}) \):
Proposition 3.5. For any \( b \in \mathcal{B} \), regarded as an element of the \( C^* \)-algebra \( B \) by \( b = \pi_B(b) \), define \( \beta(\pi_B(b)) \in \mathcal{L}(\mathcal{H}) \), by
\[
\beta(\pi_B(b))\Lambda(a) = \Lambda(ba), \quad a \in \mathcal{A}.
\]
Then
\begin{enumerate}
\item \( \beta(\pi_B(b)) \in \mathcal{B}(\mathcal{H}) \), for any \( b \in \mathcal{B} \).
\item \( \beta \) extends to a (bounded) \( C^* \)-representation \( \beta : B \to \mathcal{B}(\mathcal{H}) \).
\item \( \beta : B \to \mathcal{B}(\mathcal{H}) \) becomes a non-degenerate \( * \)-representation. It also extends to the \( * \)-representation at the level of the multiplier algebra \( \mathcal{M}(B) \).
\end{enumerate}

Proof. Proof can be carried out using essentially the same argument as in Propositions 3.2 and 3.3 above, in the construction of the \( C^* \)-representation \( \alpha : C \to \mathcal{B}(\mathcal{H}) \). □

3.2. The \( C^* \)-algebra \( \mathcal{A} \). Unlike the subalgebras \( \mathcal{B} \) and \( \mathcal{C} \), it is not clear at this stage whether the elements of \( \mathcal{A} \) can be similarly all regarded as bounded operators on \( \mathcal{H} \). For this purpose, let us define the following operator \( W \), using the left invariance property of \( \varphi \). This will help us construct the left regular representation of \( \mathcal{A} \):

Proposition 3.6. There exists a bounded operator \( W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \) satisfying the following:
\begin{enumerate}
\item For any \( a, b \in \mathcal{A} \), we have
\[
W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((\Delta b)(a \otimes 1)).
\]
\item For any \( a, b \in \mathcal{A} \), we have
\[
W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1)),
\]
where \( S \) denotes the antipode on \( (\mathcal{A}, \Delta) \).
\item For any \( a, b \in \mathcal{A} \), we have
\[
W^*W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(E(a \otimes b)).
\]
\end{enumerate}

Proof. Let \( a, b \in \mathcal{A} \) be arbitrary. Define \( W^* : \Lambda(a) \otimes \Lambda(b) \mapsto W^*(\Lambda(a) \otimes \Lambda(b)) \), by
\[
W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((\Delta b)(a \otimes 1)). \tag{3.1}
\]
It is well-defined on \( (\Lambda \otimes \Lambda)(\mathcal{A} \otimes \mathcal{A}) \), but at present we do not know it is bounded.

From the definition of \( W^* \) given in Equation (3.1), we can find an expression for its adjoint operator, \( W \). To compute this, let \( c, d \in \mathcal{A} \) be arbitrary. Then
\[
\langle W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle
= \langle \Lambda(a) \otimes \Lambda(b), W^*(\Lambda(c) \otimes \Lambda(d)) \rangle
= \langle \Lambda(a) \otimes \Lambda(b), (\Lambda \otimes \Lambda)((\Delta d)(c \otimes 1)) \rangle
= (\varphi \otimes \varphi)((c^* \otimes 1)(\Delta(d^*))(a \otimes b)) = \varphi(c^*(\text{id} \otimes \varphi)(\Delta(d^*)(1 \otimes b))a).
\]
Here, we may use the characterization of the antipode map \( S \), given in Proposition 1.12(1), so that we have \( (\text{id} \otimes \varphi)((\Delta(d^*))(1 \otimes b)) = S^{-1}((\text{id} \otimes \varphi)((1 \otimes d^*)(\Delta(b))) \). Then the above becomes
\[
= \varphi(c^*S^{-1}((\text{id} \otimes \varphi)((1 \otimes d^*)(\Delta(b)))a) = (\varphi \otimes \varphi)((c^* \otimes d^*)(S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1))
= \langle (\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1)), \Lambda(c) \otimes \Lambda(d) \rangle.
\]
As $c,d \in \mathcal{A}$ are arbitrary, this shows that for any $a,b \in \mathcal{A}$, we have
\[
W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1)).
\] (3.2)

Next, let us combine Equations (3.1) and (3.2), to obtain an expression for $W^* W$. Observe that for $a,b \in \mathcal{A}$, we have
\[
W^* W(\Lambda(a) \otimes \Lambda(b)) = W^* ((\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1))).
\]

We may use the Sweedler notation to write $(S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1) = \sum_b [S^{-1}(b_{(1)})a \otimes b_{(2)}]$. Then applying $W^*$, it becomes:
\[
W^* W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda) \left( \sum_b [b_{(2)}S^{-1}(b_{(1)})a \otimes b_{(3)}] \right).
\]

Use here a result from the algebraic framework, namely, Proposition 4.3 of [34], which says that
\[
\sum_b [b_{(2)}S^{-1}(b_{(1)})a \otimes b_{(3)}] = E(a \otimes b).
\]

It follows that
\[
W^* W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(E(a \otimes b)).
\] (3.3)

Recall that $E \in M(B \otimes C)$. Since we know that the elements of $M(B)$ and $M(C)$ are considered as bounded operators by the representations $\alpha$ and $\beta$, respectively, it follows that
\[
W^* W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(E(a \otimes b)) = (\alpha \otimes \beta)(E)(\Lambda \otimes \Lambda)(a \otimes b).
\]

This shows that $W^* W = E$, where we regard $E$ as the operator $(\alpha \otimes \beta)(E) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. In this way, we show that $W^* W$ is a bounded operator, which in turn means that $W$ and $W^*$ are also bounded. Equations (3.1) and (3.2) characterize the operators $W^*$ and $W$, respectively. □

**Remark.** In general, the operator $W$ is not unitary, unless $E = 1 \otimes 1$. In fact, the observation $W^* W = E$ in the above proof indicates that $W$ is actually a partial isometry. We will further discuss this aspect in the next subsection.

Here is one more result regarding $W$, which will be useful in defining the GNS representation of $A$ in $\mathcal{H}$:

**Proposition 3.7.** For any $a,p,q \in \mathcal{A}$, we have:
\[
(id \otimes \omega_{\Lambda(p),\Lambda(q)})(W)\Lambda(a) = \Lambda((id \otimes \varphi)(\Delta(q^*)(1 \otimes p))a).
\]

**Proof.** For any $d \in \mathcal{A}$, we have:
\[
\langle (id \otimes \omega_{\Lambda(p),\Lambda(q)})(W)\Lambda(a), \Lambda(d) \rangle = \langle W(\Lambda(a) \otimes \Lambda(p)), \Lambda(d) \otimes \Lambda(q) \rangle = \langle \Lambda(a) \otimes \Lambda(p), W^*(\Lambda(d) \otimes \Lambda(q)) \rangle = \langle \Lambda(a) \otimes \Lambda(p), (\Lambda \otimes \Lambda)(\Delta(q)(d \otimes 1)) \rangle = \langle \varphi \otimes \varphi)((d^* \otimes 1)\Delta(q^*)(a \otimes p)) \rangle = \varphi(d^*(id \otimes \varphi)(\Delta(q^*)(1 \otimes p))a) = \langle \Lambda((id \otimes \varphi)(\Delta(q^*)(1 \otimes p))a), \Lambda(d) \rangle.
\]

As $d \in \mathcal{A}$ was arbitrary, this proves the result. □
Recall that by the fullness assumption of $\Delta$, we know that the elements of the form $x = (\text{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))$, where $p, q \in \mathcal{A}$, span all of $\mathcal{A}$. Therefore what Proposition 3.7 is saying is that for any $x \in \mathcal{A}$, we can find a bounded operator $X \in \mathcal{B}(\mathcal{H})$ such that $X\Lambda(a) = \Lambda(xa)$, for all $a \in \mathcal{A}$. In this way, we can define the GNS-representation $\pi$ of $\varphi$:

**Definition 3.8.** Define $\pi$ from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ by

$$\pi(x)\Lambda(a) = \Lambda(xa), \text{ for all } x, a \in \mathcal{A}.$$  

Then $\pi$ is an injective $\ast$-homomorphism, which is the GNS representation of $\mathcal{A}$, such that $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$.

The last statement on the non-degeneracy of $\pi$ is a consequence of $\mathcal{A}^2 = \mathcal{A}$. The GNS-representation allows us to properly define our $C^\ast$-algebra $\mathcal{A}$:

**Definition 3.9.** Define $\mathcal{A} := \pi(\mathcal{A})\|\|$, as a non-degenerate $C^\ast$-subalgebra of $\mathcal{B}(\mathcal{H})$. It can be also characterized as

$$\mathcal{A} = \{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_\ast\}\|\|.$$

The alternative characterization of $\mathcal{A}$ is obtained by noting from Proposition 3.7 that for $x = (\text{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))$, for any $p, q \in \mathcal{A}$, we have

$$\pi(x) = \pi((\text{id} \otimes \varphi)(\Delta(q^*)(1 \otimes p))) = (\text{id} \otimes \omega_{\Lambda(p),\Lambda(q)})(W).$$

As $\pi$ is a non-generate $\ast$-representation, it can be naturally extended to the level of the multiplier algebra $M(\mathcal{A})$. We will often regard $\mathcal{A} = \pi(\mathcal{A})$ and $M(\mathcal{A}) = \pi(M(\mathcal{A}))$.

At the $\ast$-algebra level, we saw that $\mathcal{B}$ and $\mathcal{C}$ are subalgebras in $M(\mathcal{A})$. As these algebras are now all represented on $\mathcal{H}$ by left multiplications, and in turn completed to the $C^\ast$-algebras $\beta(B)$, $\alpha(C)$, $\pi(M(\mathcal{A}))$, respectively, it is apparent that $\alpha = \pi|_C$, $\beta = \pi|_B$. It is thus natural to regard $B = \beta(B) = \pi(B) \subset M(\mathcal{A})$ and $C = \alpha(C) = \pi(C) \subset M(\mathcal{A})$, as operator algebras in $\mathcal{B}(\mathcal{H})$. We also have $M(B) \subset M(\mathcal{A})$ and $M(C) \subset M(\mathcal{A})$. As for our idempotent $E$, we may regard $E = (\alpha \otimes \beta)(E) = (\pi \otimes \pi)(E) \in M(\mathcal{A} \otimes \mathcal{A})$. While it is true that in Section 2 we considered the $C^\ast$-algebras $\mathcal{B}$ and $\mathcal{C}$ as represented on $\mathcal{H}_B$ and $\mathcal{H}_C$, respectively, and such aspect may still be needed down the road, we will be able to tell from the context on which space they are represented.

### 3.3. The comultiplication on $\mathcal{A}$

We next wish to define the comultiplication at the $C^\ast$-algebra level, extending the comultiplication on $\mathcal{A}$. We have our candidate below using the operator $W$, analogous to the quantum group case. We still need some work to clarify that this is indeed a correct definition.

**Definition 3.10.** Define the map $\tilde{\Delta}$ from the $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, by $\tilde{\Delta}(x) = W^\ast(1 \otimes x)W$, for all $x \in \mathcal{B}(\mathcal{H})$.

The next proposition shows that $\tilde{\Delta}$ is an extension of the comultiplication on $\mathcal{A}$.

**Proposition 3.11.** For any $a \in \mathcal{A}$, we have $\tilde{\Delta}(\pi(a)) = (\pi \otimes \pi)(\Delta a)$. 
Proof. Let $c,d \in \mathcal{A}$ be arbitrary. Using the characterization of $W$ given in Proposition 3.6 (2), we have

$$W^*(1 \otimes \pi(a))W(\Lambda(c) \otimes \Lambda(d)) = W^*(1 \otimes \pi(a))((\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta d)(c \otimes 1)))$$

$$= W^*((\Lambda \otimes \Lambda)((1 \otimes a)(S^{-1} \otimes \text{id})(\Delta d)(c \otimes 1)))$$

$$= W^*\left((\Lambda \otimes \Lambda)\left(\sum_{(d)}[S^{-1}(d(1))c \otimes ad(2)]\right)\right),$$

where we again used the Sweedler notation. Thus, by applying the characterization of $W^*$ given in Proposition 3.6 (1), the above becomes:

$$W^*(1 \otimes \pi(a))W(\Lambda(c) \otimes \Lambda(d)) = (\Lambda \otimes \Lambda)\left(\Delta a\sum_{(d)}[d(2)S^{-1}(d(1))c \otimes d(3)]\right).$$

As before use the algebraic result Proposition 4.3 of [34]. Then we have

$$W^*(1 \otimes \pi(a))W(\Lambda(c) \otimes \Lambda(d)) = (\Lambda \otimes \Lambda)((\Delta a)E(c \otimes d)) = (\Lambda \otimes \Lambda)((\Delta a)(c \otimes d))$$

$$= (\pi \otimes \pi)(\Delta a)(\Lambda(c) \otimes \Lambda(d)).$$

As $c,d \in \mathcal{A}$ can be arbitrary, this shows $\tilde{\Delta}(\pi(a)) = W^*(1 \otimes \pi(a))W = (\pi \otimes \pi)(\Delta a)$. \hfill \Box

In particular, as for our canonical idempotent $E$, which we consider as an operator $E = (\alpha \otimes \beta)(E) = (\pi \otimes \pi)(E) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ by the GNS-representation $\pi$, the following result is evident.

**Proposition 3.12.** We have:

1. $E = \tilde{\Delta}(1_{M(\mathcal{A})})$
2. $E = W^*W$

Proof. As an element of $M(\mathcal{A} \otimes \mathcal{A})$, we know that $E = \Delta(1_{M(\mathcal{A})})$. So, by Proposition 3.11 we have:

$$E = (\pi \otimes \pi)(E) = \tilde{\Delta}(1_{M(\mathcal{A})})) = W^*(1 \otimes \pi(1_{M(\mathcal{A})}))W = W^*W.$$

Indeed, the observation $W^*W = E \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ has been already made in the proof of Proposition 3.6 \hfill \Box

The operator $W$ is essentially like the multiplicative unitary operator (in the sense of [1, 37]) in the framework of locally compact quantum groups [15]. In our setting, however, as we have $E \neq 1 \otimes 1$, the operator $W$ is not unitary. But then, it turns out that it is a partial isometry. See below:

**Proposition 3.13.** We have

1. $W^*(1 \otimes x) = (\tilde{\Delta}x)W^*$, for any $x \in \mathcal{A}$.
2. $EW^* = W^*$.
3. $W$ is a partial isometry, satisfying $W^*WW^* = W^*$ and $WW^*W = W$. 
Proof. (1). For \( a \in \mathcal{A} \), and for any \( c,d \in \mathcal{A} \), note that
\[
W^* (1 \otimes \pi(a))(\Lambda(c) \otimes \Lambda(d)) = W^* (\Lambda(c) \otimes \Lambda (ad)) = (\Lambda \otimes \Lambda)(\Delta(ad)(c \otimes 1))
\]
\[= (\pi \otimes \pi)(\Delta a)(\Lambda \otimes \Lambda)(\Delta(d)(c \otimes 1))
\]
\[= \tilde{\Delta}(\pi(a))W^*(\Lambda(c) \otimes \Lambda(d)).
\]
As \( c,d \in \mathcal{A} \) are arbitrary, this shows that \( W^* (1 \otimes \pi(a)) = \tilde{\Delta}(\pi(a))W^* \). Since the \( \pi(a), a \in \mathcal{A} \), are dense in \( \mathcal{A} \), it follows that we have \( W^*(1 \otimes x) = (\tilde{\Delta}x)W^* \), for any \( x \in \mathcal{A} \).

(2). It is evident that the result of (1) will hold true also for all \( m \in M(\mathcal{A}) \). So, in particular, if \( m = 1_{M(\mathcal{A})} = \pi(1_{M(\mathcal{A})}) \), we have:
\[W^* = W^*(1 \otimes 1_{M(\mathcal{A})}) = \tilde{\Delta}(1_{M(\mathcal{A})})W^* = EW^*.
\]

(3). We know from Proposition 3.12 that \( E = W^*W \). Combining this with \( EW^* = W^* \), we see that \( W^*WW^* = W^* \). Also \( WW^*W = W \). So \( W \) is a partial isometry. □

For convenience, write \( \Delta = \tilde{\Delta}|_\mathcal{A} \), which is reasonable since \( \tilde{\Delta} \) extends \( \Delta \) at the \( * \)-algebra level on \( \mathcal{A} \) and \( \mathcal{A} \) is dense in \( \mathcal{A} \). The theorem below shows that \( \Delta : x \mapsto W^*(1 \otimes x)W \) determines a valid comultiplication on the \( * \)-algebra \( \mathcal{A} \).

**Theorem 3.14.** The map \( \Delta = \tilde{\Delta}|_\mathcal{A} \) is a \( * \)-representation of \( \mathcal{A} \) into \( M(\mathcal{A} \otimes \mathcal{A}) \). It extends to a \( * \)-representation from \( M(\mathcal{A}) \) into \( M(\mathcal{A} \otimes \mathcal{A}) \), which we may still denote by \( \Delta \).

We also have:

1. \( (\Delta x) (1 \otimes y) \in A \otimes A \) and \( (x \otimes 1)(\Delta y) \in A \otimes A \), for all \( x,y \in A \).
2. The following spaces are norm-dense in \( A \):
   - \( \text{span}\{(\text{id} \otimes \omega)((\Delta x)(1 \otimes y)) : \omega \in A^*, x,y \in A\} \),
   - \( \text{span}\{((\omega \otimes \text{id}))(x \otimes 1)(\Delta y) : \omega \in A^*, x,y \in A\} \).
3. The coassociativity condition holds:
   \[(\Delta \otimes \text{id})(\Delta x) = (\text{id} \otimes \Delta)(\Delta x), \quad \forall x \in A.
   \]

**Proof.** It is easy to see that \( \Delta(x^*) = W^*(1 \otimes x^*)W = (W^*(1 \otimes x)W)^* = \Delta(x)^* \). In addition, for \( x,y \in A \), by Proposition 3.13 we have,
\[\Delta(x)\Delta(y) = (\Delta x)W^*(1 \otimes y)W = W^*(1 \otimes x)(1 \otimes y)W = W^*(1 \otimes xy)W = \Delta(xy).
\]
This shows that \( \Delta \) is a \( * \)-representation.

Next, let \( a,b \in \mathcal{A} \) be arbitrary. Then
\[\pi(a) \Delta(\pi(b)) = \pi(a) \Delta \pi(b) = (\pi \otimes \pi)((a \otimes 1)(\Delta b)) \in (\pi \otimes \pi)(\mathcal{A} \otimes \mathcal{A}) \subset A \otimes A,
\]
because we know from Equation (111) that \( (a \otimes 1)(\Delta b) \in \mathcal{A} \otimes \mathcal{A} \) at the \( * \)-algebra level. As \( \pi(\mathcal{A}) \) is dense in \( \mathcal{A} \), this shows that \( (x \otimes 1)(\Delta y) \in A \otimes A \), for all \( x,y \in A \). Similarly, we can also show that \( (\Delta x)(1 \otimes y) \in A \otimes A \) for all \( x,y \in A \). These observations prove (1).
As an immediate consequence, we can see that for any \( x, y, z \in A \), we have \( (\Delta x)(y \otimes z) \in A \otimes A \) and \( (y \otimes z)(\Delta x) \in A \otimes A \), showing that \( \Delta(A) \subseteq M(A \otimes A) \). In other words, we see that \( \Delta \) is a *-representation from \( A \) into \( M(A \otimes A) \).

Let \( m, n \in M(A) \), and consider the expression \( (p \otimes q)\tilde{\Delta}(m)\tilde{\Delta}(n) = (p \otimes q)E\tilde{\Delta}(m)\tilde{\Delta}(n) \), for arbitrary \( p, q \in A \). Here, we used the fact that \( EW^* = W^* \) (Proposition \ref{prop}), from which we have \( E\tilde{\Delta}(m) = EW^*(1 \otimes m)W = W^*(1 \otimes m)W = \tilde{\Delta}(m) \). Since \( (A \otimes A)\Delta(A) = (A \otimes A)E \) (see Lemma \ref{lem}), we know \( (p \otimes q)E \) can be written as a sum of the expressions \( (a \otimes b)(\Delta c) \). But then, we know that for any \( c \in A \) and any \( m \in M(A) \), we have

\[
\Delta(c)\tilde{\Delta}(m) = (\Delta c)W^*(1 \otimes m)W = W^*(1 \otimes c)(1 \otimes m)W = W^*(1 \otimes cm)W = \Delta(cm),
\]

by Proposition \ref{prop}. Using this result twice, it follows that

\[
(a \otimes b)(\Delta c)\tilde{\Delta}(m)\tilde{\Delta}(n) = (a \otimes b)\Delta(cm)\tilde{\Delta}(n) = (a \otimes b)\Delta(cmn) = (a \otimes b)(\Delta c)\tilde{\Delta}(mn).
\]

As noted above, any \( (p \otimes q)E \) can be written as a sum of the expressions \( (a \otimes b)(\Delta c) \). This means that we have:

\[
(p \otimes q)\tilde{\Delta}(m)\tilde{\Delta}(n) = (p \otimes q)E\tilde{\Delta}(m)\tilde{\Delta}(n) = (p \otimes q)E\tilde{\Delta}(mn) = (p \otimes q)\tilde{\Delta}(mn).
\]

This result is true for arbitrary \( p, q \in A \). Therefore, we can conclude that \( \tilde{\Delta}(m)\tilde{\Delta}(n) = \tilde{\Delta}(mn) \), showing that \( \tilde{\Delta}|_{M(A)} \) preserves the multiplicativity. The *-property is immediate, and it is evident that \( \tilde{\Delta}(M(A)) \subseteq M(A \otimes A) \). Therefore, we see that \( \tilde{\Delta}|_{M(A)} \) is a *-representation from \( M(A) \) to \( M(A \otimes A) \), which naturally extends the comultiplication on \( A \) and on \( A \). As such, we may from this point on write \( \Delta = \tilde{\Delta}|_{M(A)} \).

Turning to the rest of the results in the proposition, the “fullness” property, given in (2), is a consequence of the fullness of \( \Delta \) at the *-algebra level, as the spanned space is exactly \( A \), which is norm-dense in \( A \).

Finally, the coassociativity property: We have already seen that \( \Delta \) is well established at the level of both \( A \) and \( M(A) \), and that it extends the comultiplication at the dense *-algebra level. As one can expect, it indeed satisfies the coassociativity at the \( C^* \)-algebra level. To confirm this, consider \( a, b, c \in A \). We have:

\[
(\Delta \otimes \text{id})(\Delta(\pi(a)))(\pi(b) \otimes 1 \otimes \pi(c)) = (\Delta \otimes \text{id})(\Delta(\pi(a))(1 \otimes \pi(c)))(\pi(b) \otimes 1 \otimes 1)
\]

\[
= (\pi \otimes \pi \otimes \pi)((\Delta \otimes \text{id})(\Delta(\pi(a))(1 \otimes c))(b \otimes 1 \otimes 1))
\]

\[
= (\pi \otimes \pi \otimes \pi)((\Delta \otimes \text{id})(\Delta(a)(b \otimes 1))(1 \otimes 1 \otimes c))
\]

\[
= (\text{id} \otimes \Delta)(\Delta(\pi(a))(\pi(b) \otimes 1))(1 \otimes 1 \otimes \pi(c))
\]

\[
= (\text{id} \otimes \Delta)(\Delta(\pi(a)))(\pi(b) \otimes 1 \otimes \pi(c)).
\]

As \( A \) acts non-degenerately, this shows that \( (\Delta \otimes \text{id})(\Delta(\pi(a))) = (\text{id} \otimes \Delta)(\Delta(\pi(a))) \), \( \forall a \in A \). Since \( \pi(\mathcal{A}) \) is dense in \( A \), we see that \( (\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \) on \( A \). \( \square \)

In this way, we have shown that \( (A, \Delta) \) is a \( C^* \)-bialgebra, with the comultiplication \( \Delta \) satisfying all the conditions prescribed in Definition 3.1 of \ref{ref10}.
3.4. **The multiplicative partial isometry** $W$. We observed above that the behavior of our partial isometry $W$ quite resembles that of a multiplicative unitary operator (in the sense of [1], [37]) in the quantum group setting. While it is not unitary, it actually satisfies relations such that it can be referred to as a *multiplicative partial isometry*. Such a notion has been noted in the finite-dimensional setting of weak $C^*$-Hopf algebras [4]. See also [7], where an axiomatic discussion on multiplicative partial isometries is given for a class of $C^*$-algebraic quantum groupoids.

Here indeed, we can show that our partial isometry $W$ satisfies the defining axioms given in [4] for being a multiplicative partial isometry (The conditions in [7] are slightly different, but equivalent.):

**Proposition 3.15.** The operator $W$ satisfies the following conditions:

$$
W_{12}W_{13}W_{23} = W_{23}W_{12} \tag{3.4}
$$

$$
W_{12}^*W_{13} = W_{13}W_{23}W_{23}^* \tag{3.5}
$$

$$
W_{23}^*W_{23}W_{12} = W_{12}W_{23}^*W_{23} \tag{3.6}
$$

$$
W_{12}W_{12}^*W_{23} = W_{23}W_{12}W_{12}^* \tag{3.7}
$$

**Proof.** Let $a, b, c \in A$. By the characterization of $W^*$ given in Proposition 3.6 we have

$$
W_{12}^*W_{23}^*(\Lambda(a) \otimes \Lambda(b) \otimes \Lambda(c)) = W_{12}^*(\Lambda \otimes \Lambda \otimes \Lambda)((a \otimes (\Delta c))(1 \otimes b \otimes 1))
$$

$$
= (\Lambda \otimes \Lambda \otimes \Lambda)((\Delta \otimes \text{id})(\Delta c)((\Delta b) \otimes 1)(a \otimes 1 \otimes 1)), \tag{3.8}
$$

while we also have

$$
W_{23}^*W_{13}W_{12}^*(\Lambda(a) \otimes \Lambda(b) \otimes \Lambda(c)) = W_{23}^*W_{13}^*(\Lambda \otimes \Lambda \otimes \Lambda)((\Delta b) \otimes c)(a \otimes 1 \otimes 1))
$$

$$
= W_{23}^*(\Lambda \otimes \Lambda \otimes \Lambda)((\Delta c)_{13}(\Delta b) \otimes 1)(a \otimes 1 \otimes 1))
$$

$$
= (\Lambda \otimes \Lambda \otimes \Lambda)((\text{id} \otimes \Delta)(\Delta c)((\Delta b) \otimes 1)(a \otimes 1 \otimes 1)). \tag{3.9}
$$

Comparing Equations (3.8) and (3.9), with the knowledge that $\Delta$ is coassociative, namely $(\Delta \otimes \text{id})(\Delta c) = (\text{id} \otimes \Delta)(\Delta c)$, we conclude that $W_{12}^*W_{23} = W_{23}^*W_{13}W_{12}^*$. Or equivalently, we have $W_{12}W_{13}W_{23} = W_{23}W_{12}$. This is in fact the famous “pentagon equation”.

In the case of a unitary, the other three equations of the proposition, namely Equations (3.5), (3.6), (3.7), would not need a proof. That is not so in our case, and we need separate proofs for these three cases. We will just prove Equation (3.5) here.

For this, we will quote a result from the algebraic framework, namely Equation (4.4) of [34] that appear in the proof of Proposition 4.3 of that paper: It says

$$
(c \otimes d)E = \sum_{(c)} c_{(1)} \otimes dS^{-1}(c_{(3)})c_{(2)}, \quad \text{for } c, d \in A.
$$

Modifying it a little, we can obtain the following:

$$
(c \otimes 1)E(1 \otimes b) = \sum_{(c)} c_{(1)} \otimes S^{-1}(c_{(3)})c_{(2)}b, \quad \text{for } c, b \in A. \tag{3.10}
$$
To prove Equation (3.5), let \( a, b, c \in A \) be arbitrary, and compute. Note first that

\[
W_{13}^* W_{12} W_{12} (\Lambda(a) \otimes \Lambda(b) \otimes \Lambda(c)) = W_{13}^* (\Lambda \otimes \Lambda \otimes \Lambda) (E(a \otimes b) \otimes c)
\]

\[
= (\Lambda \otimes \Lambda \otimes \Lambda) ( (\Delta c)_{13} (E(a \otimes b) \otimes 1) ),
\]

(3.11)

because \( W^* W = (\pi \otimes \pi)(E) \). Meanwhile, we have

\[
W_{23} W_{23}^* W_{13}^* (\Lambda(a) \otimes \Lambda(b) \otimes \Lambda(c)) = W_{23} W_{23}^* (\Lambda \otimes \Lambda \otimes \Lambda) ((\Delta c)_{13} (a \otimes b \otimes 1))
\]

\[
= W_{23} (\Lambda \otimes \Lambda \otimes \Lambda) \left( \sum_c c(1) a \otimes c(2) b \otimes c(3) \right)
\]

\[
= (\Lambda \otimes \Lambda \otimes \Lambda) \left( \sum_c c(1) a \otimes S^{-1}(c(3)) c(2) b \otimes c(4) \right),
\]

where we are using the characterization of \( W \) given in Proposition 3.6 (2), expressed in the Sweedler notation. Apply here the algebraic result Equation (3.10). Then it becomes

\[
W_{23} W_{23}^* W_{13}^* (\Lambda(a) \otimes \Lambda(b) \otimes \Lambda(c)) = (\Lambda \otimes \Lambda \otimes \Lambda) ( (\Delta c)_{13} (E(a \otimes b) \otimes 1) ).
\]

(3.12)

Comparing Equations (3.11) and (3.12), we conclude that \( W_{13}^* W_{12} W_{12} = W_{23} W_{23}^* W_{13}^* \). Or equivalently, we have \( W_{12} W_{12} W_{13} = W_{13} W_{23} W_{23}^* \).

The other two can be proved similarly, using analogous algebraic results. But as they are not directly needed below, and since they can be proved using an alternative method (see [11]) once the full construction of the quantum groupoid is carried out, we will skip the details here. \( \square \)

**Proposition 3.16.** The operator \( W \) satisfies the following conditions:

\[
W_{23} W_{12} W_{23} = W_{12} W_{13}
\]

(3.13)

\[
W_{12}^* W_{23} W_{12} = W_{13} W_{23}
\]

(3.14)

\[
W_{12} W_{23} = W_{23} W_{12}
\]

(3.15)

\[
W_{12}^* W_{23} = W_{13} W_{23} W_{12}
\]

(3.16)

\[
W_{13} W_{13} W_{23} = W_{23} W_{12} W_{12}
\]

(3.17)

\[
W_{12} W_{13} W_{13} = W_{23} W_{23}^* W_{12}
\]

(3.18)

**Proof.** These can be all proved as a quick consequence of the four conditions that appear in Proposition 3.15. Since we will be needing only the second one later, we will just give the proof for Equation (3.14). For more discussion, refer to [7].

From Proposition 3.15 we saw that \( W_{23} W_{12} = W_{12} W_{13} W_{23} \). Multiply here \( W_{12}^* \), from the left, to obtain:

\[
W_{12} W_{23} W_{12} = W_{12}^* W_{12} W_{12} W_{13} W_{23} = W_{13} W_{23} W_{23}^* W_{23} = W_{13} W_{23},
\]

using Equation (3.5) and the fact that \( W W^* W = W^* \), being a partial isometry. \( \square \)
3.5. The idempotent \( E \). We already noted that the canonical idempotent element \( E \) at the \(*\)-algebra level can be considered as the operator \( E = (\pi \otimes \pi)(E) \in M(A \otimes A) \subset B(\mathcal{H} \otimes \mathcal{H}) \), by the GNS-representation \( \pi \). As such, its properties will be inherited from those at the \(*\)-algebra level, which we gather below.

**Proposition 3.17.** Consider the canonical idempotent \( E \), regarded as \( E = (\pi \otimes \pi)(E) \in M(A \otimes A) \subset B(\mathcal{H} \otimes \mathcal{H}) \), such that \( E^* = E \) and \( E^2 = E \). We have:

1. \( \Delta(A)(A \otimes A) \parallel E(A \otimes A) \) and \( (A \otimes A)\Delta(A) \parallel (A \otimes A)E \)
2. \( E(\Delta x) = \Delta x = (\Delta x)E \), for all \( x \in A \)
3. \( E \otimes 1 \) and \( 1 \otimes E \) commute, and we also have
   \( (\text{id} \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1) = (\Delta \otimes \text{id})(E) \).
4. There exists an \(*\)-anti-isomorphism \( R = R_{BC} : B \to C \), and together with the KMS weight \( \nu \) on \( B \) and \( E \in M(B \otimes C) \), we obtain a separability triple \( (E, B, \nu) \), in the sense of [9].

**Proof.** (1), (2) are the consequences of Lemma 1.1 at the \(*\)-algebra level. Since \( A \) is norm dense in \( A \), the results follow immediately. As shown in Proposition 3.3 in [10], this uniquely determines \( E \).

(3) is the weak comultiplicativity of the unit, noted already in Equation (1.6).

(4). As \( E \in M(B \otimes C) \), it can be also considered as an element in \( M(B \otimes C) \), where \( B \) and \( C \) are the \( C^*\)-subalgebras of \( M(A) \) we saw earlier. We saw in Proposition 2.8 that \( E \) satisfies the properties of being a separability idempotent, in the \( C^*\)-algebraic sense (see [9]).

\[ \square \]

In this way, we have shown that \( E \in M(B \otimes C) \subseteq M(A \otimes A) \) satisfies all the conditions for being the canonical idempotent for \( (A, \Delta) \), as prescribed in Definition 3.7 of [10].

3.6. The weight \( \bar{\varphi} \) and the modular automorphism group. So far, our \( \varphi \) has been only a linear functional on \( A \). Going forward, we will need to consider its extension to the operator algebra level. The full construction of a KMS weight on the \( C^*\)-algebra \( A \), extending the functional \( \varphi \) and satisfying a proper left invariant condition, needs some additional preparation so will be postponed to a later section. However, we can do some initial work in this subsection.

Observe first that \( \Lambda(A) \subseteq \mathcal{H} \) obtained from the functional \( \varphi \) at the \(*\)-algebra level, as given in the beginning part of Section 3, is a left Hilbert algebra, with respect to the multiplication and the \(*\)-structure inherited from \( A \). We will skip the proof, which is essentially no different in nature from that of Proposition 2.4 in Section 2.

By the general theory on left Hilbert algebras (see [23]), we can associate to \( \Lambda(A) \) a von Neumann algebra \( M \). In our case, in terms of the GNS-representation \( \pi \), it would be exactly \( M = \pi(A)^\prime \). Also by the general theory on left Hilbert algebras, we obtain a normal semi-finite faithful (n.s.f.) weight \( \bar{\varphi} \) on \( M \).

We can consider the associated spaces \( \mathfrak{N}_{\bar{\varphi}} = \{ x \in M : \bar{\varphi}(x^*x) < \infty \} \) and \( \mathfrak{M}_{\bar{\varphi}} = \mathfrak{N}_{\bar{\varphi}}^* \mathfrak{N}_{\bar{\varphi}} \). The associated GNS map \( \Lambda_{\bar{\varphi}} \) is an injective map from \( \mathfrak{N}_{\bar{\varphi}} \) to \( \mathcal{H} \) (same Hilbert space), which
extends Λ. By the standard left Hilbert algebra theory, we know that the weight \( \tilde{\varphi} \) extends the functional \( \varphi \). In particular, we have \( \tilde{\varphi}(\pi(a)^*\pi(a)) = \varphi(a^*a) \), for all \( a \in A \). For any \( x \in \mathcal{M}_{\tilde{\varphi}} \), there exists a sequence \( (a_n) \) in \( A \) such that \( \Lambda(a_n) \xrightarrow{\text{(in } H)} \Lambda_{\tilde{\varphi}}(x) \) and \( \pi(a_n) \xrightarrow{\text{(σ-strong-)}} x \).

Denote by \( T \) the closure of the involution \( \Lambda(x) \mapsto \Lambda(x^*) \) on \( \Lambda(A) \). As before, there exists a polar decomposition, \( T = J\nabla^{\frac{1}{2}} \), where \( \nabla = T^*T \) is the modular operator, and \( J \) is the modular conjugation, which is anti-unitary. Note also that \( J\nabla J = \nabla^{-1} \), so we can also write \( T = \nabla^{-\frac{1}{2}} J \).

According to the modular theory in the von Neumann algebra setting (\cite{23, 22}), the modular operator defines a strongly continuous one-parameter group of automorphisms \( \tilde{\sigma} = (\tilde{\sigma}_t)_{t \in \mathbb{R}} \), by \( \tilde{\sigma}_t(a) = \nabla^{it}a\nabla^{-it} \), for \( a \in M \), \( t \in \mathbb{R} \). We have \( \tilde{\varphi} \circ \tilde{\sigma}_t = \tilde{\varphi}, \ t \in \mathbb{R} \), and \( (\tilde{\sigma}_t) \) satisfies a certain KMS boundary condition. In particular, the weak KMS property at the *-algebra level, \( \varphi(ab) = \varphi(b\sigma(a)) \), \( a, b \in A \), extends to the von Neumann algebra as \( \tilde{\varphi}(xy) = \tilde{\varphi}(y\tilde{\sigma}_i(x)) \), \( x \in \mathcal{M}_{\tilde{\varphi}}, \ y \in \mathcal{D}(\tilde{\sigma}_i) \). Meanwhile, the modular conjugation \( J \) can be characterized by \( J\Lambda_{\tilde{\varphi}}(x) = \Lambda_{\tilde{\varphi}}(\tilde{\sigma}_i(x)^*) \), for \( x \in \mathcal{M}_{\tilde{\varphi}} \).

From the n.s.f. weight \( \tilde{\varphi} \) on the von Neumann \( M = \pi(A)^\sigma \), we can restrict it to the level of the C*-algebra \( A = \overline{\pi(A)}^\sigma \), to obtain a faithful lower semi-continuous weight \( \varphi \) on \( A \). However, as was the case for the weights \( \tilde{\nu} \) and \( \tilde{\mu} \) earlier (Section 2), the main issue is whether the restriction of the modular automorphism group \( (\tilde{\sigma}_t) \) to the C*-algebra level would leave \( A \) invariant, and whether the restriction is norm-continuous. These are not automatic consequences of the modular theory, so this needs more work. We will return to this matter in Section 5.

There is also the matter of properly establishing the left invariance property of the weight \( \varphi \), and we also wish to construct a right-invariant weight \( \psi \). That is also postponed to Section 5.

4. Polar decomposition of the antipode

At present, the antipode map, \( S \), is considered only at the *-algebra level. It is an anti-isomorphism of \( A \) onto itself, which can be extended to the multiplier algebra \( M(A) \), such that \( S(B) = C \) and \( S(C) = B \).

To regard it as a map at the operator algebra level, a convenient way is to express it as a polar decomposition. This is much like the case of a locally compact quantum group \cite{16, 18, 15, 32}, which has been also adopted to the cases of quantum groupoids \cite{17, 5, 11}. As such, some portion of the construction process below would look familiar.

4.1. The involutive operator \( K \) and its polar decomposition. Let us first define an anti-linear involutive operator \( K \) below, which incorporates the antipode at the operator level:

**Proposition 4.1.** For \( a \in A \), define:

\[
K_0 \Lambda(a) := \Lambda(S(a)^*)..
\]

Then:

- (1) \( K_0 \) is a well-defined map, from \( \Lambda(A) \) into \( H \).
(2) \( K_0 \) is closable, so we may consider its closure \( K \). It becomes a closed, densely-defined operator from \( \mathcal{H} \) into \( \mathcal{H} \), such that \( \Lambda(A) \) forms its core and it has a dense range.

(3) \( K \) is anti-linear and involutive.

**Proof.** As \( S \) is well-defined from \( A \) onto itself, and since \( \Lambda(A) \) is dense in \( \mathcal{H} \), it is clear that \( K_0 \) is well-defined, densely-defined, and has a dense range. Meanwhile, for \( a, b \in A \), note that

\[
\langle \Lambda(S(a)^*), \Lambda(b) \rangle = \varphi(b^*S(a)^*) = \varphi(S(a)b) = (\varphi \circ S)(S^{-1}(b)a) = \varphi(S^{-1}(b)a\delta) = \varphi(a\delta\sigma(S^{-1}(b))),
\]

where we used the fact that \( \varphi \) is positive, and \( \delta \in M(A) \) is the modular element (see the last paragraph of [15]). From this, it is not difficult to show that \( K_0 \) is closable. We can denote by \( K \) its closure. Thus \( K \) becomes a closed, densely-defined operator having a dense range, with \( \Lambda(A) \) forming a core.

Finally, note that \( K \) is anti-linear because \( S \) is, and note also that \( K \) is involutive by the property of the antipode:

\[
K^2\Lambda(a) = K\Lambda(S(a)^*) = K\Lambda(S^{-1}(a^*)) = \Lambda(S(S^{-1}(a^*))^*) = \Lambda(a),
\]

for any \( a \in A \). \( \square \)

We can consider the polar decomposition of this linear involutive operator \( K \), given by \( K = IL^{\frac{1}{2}} \), where \( L = K^*K \) is a positive operator, and \( I \) is a conjugate linear isomorphism. We will have \( I^* = I \), \( I^2 = 1 \), and \( ILI = L^{-1} \). Also \( K = L^{-\frac{1}{2}}I \). In addition, we can consider the “scaling group” \( (\tau_t)_{t \in \mathbb{R}} \), a norm-continuous automorphism group on \( B(\mathcal{H}) \) given by \( \tau_t(\cdot) = L^{it} \cdot L^{-it} \).

Here is a useful result that relates the operator \( W \) with \( L \) and \( \nabla \), where \( \nabla \) is the operator arising from the modular theory for the n.s.f. weight \( \tilde{\varphi} \), such that \( T = J\nabla^\frac{1}{2} \).

**Proposition 4.2.** For any \( a, b \in A \), we have:

\[
W(L \otimes \nabla)(\Lambda(a) \otimes \Lambda(b)) = (L \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)).
\]

**Proof.** For \( a, b \in A \), we have:

\[
W^*(K \otimes T)(\Lambda(a) \otimes \Lambda(b)) = W^*(\Lambda(S(a)^*) \otimes \Lambda(b^*))
\]

\[
= (\Lambda \otimes \Lambda)((\Delta(b^*)(S(a)^* \otimes 1)) = (\Lambda \otimes \Lambda)([(S(a) \otimes 1)(\Delta b)^*]^{1})
\]

\[
= (\Lambda \otimes \Lambda)([(S \otimes \text{id})(S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1)]^{1})
\]

\[
= (K \otimes T)(\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1))
\]

\[
= (K \otimes T)W(\Lambda(a) \otimes \Lambda(b)), \quad (4.1)
\]

using Proposition [x,6]. On the other hand, we also have

\[
W(K \otimes T)(\Lambda(a) \otimes \Lambda(b)) = W(\Lambda(S(a)^*) \otimes \Lambda(b^*)) = (\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta(b^*)(S(a)^* \otimes 1))
\]

\[
= (\Lambda \otimes \Lambda)([(S(a) \otimes 1)(S \otimes \text{id})(\Delta b)]^{*})
\]

\[
= (\Lambda \otimes \Lambda)([(S \otimes \text{id})(\Delta b)(a \otimes 1)]^{*}) = (K \otimes T)(\Lambda \otimes \Lambda)((\Delta b)(a \otimes 1))
\]

\[
= (K \otimes T)W^*(\Lambda(a) \otimes \Lambda(b)), \quad (4.2)
\]
from which we have
\[ W(K^* \otimes T^*)(\Lambda(a) \otimes \Lambda(b)) = (K^* \otimes T^*)W^*(\Lambda(a) \otimes \Lambda(b)), \quad \text{for } a, b \in \mathcal{A}, \quad (4.3) \]
by taking the adjoint.

Combining Equations (4.1) and (4.3), we have:
\[ W(L \otimes \nabla)(\Lambda(a) \otimes \Lambda(b)) = W(K^*K \otimes T^*T)(\Lambda(a) \otimes \Lambda(b)) = (K^* \otimes T^*)W^*(K \otimes T)(\Lambda(a) \otimes \Lambda(b)) \]
\[ = (K^*K \otimes T^*T)W(\Lambda(a) \otimes \Lambda(b)) = (L \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)). \]
\[ \square \]

Note, by the way, that since we are working with unbounded operators \( L \) and \( \nabla \), which are only densely-defined, the above result does not necessarily mean \( W(L \otimes \nabla) = (L \otimes \nabla)W \).
As it is possible that some of the elements contained in \( \text{Ker}(W) \) may not be contained in \( \mathcal{D}(L \otimes \nabla) \), we would have \( \mathcal{D}(W(L \otimes \nabla)) \subset \mathcal{D}((L \otimes \nabla)W) \). So, to be precise, we can only write \( W(L \otimes \nabla) \subseteq (L \otimes \nabla)W \), or \( W^*(L \otimes \nabla) \subseteq (L \otimes \nabla)W^* \).

If \( W \) was a unitary operator, then there exists a clever method one can use to quickly establish \( W(L \otimes \nabla) = (L \otimes \nabla)W \) (see, for instance, Lemma 5.9 in [15]), which helps us proceed along. That, however, is not the case here. We need a more roundabout approach. We can almost verbatim follow the steps carried out in Propositions 4.13 – 4.17 in [11], and we obtain the following. We will skip the details of the proof, but see the remark following the proposition:

**Proposition 4.3.** (1) For any \( z \in \mathbb{C} \), we have:
\[ W(L^z \otimes \nabla^z) \subseteq (L^z \otimes \nabla^z)W \quad \text{and} \quad W^*(L^z \otimes \nabla^z) \subseteq (L^z \otimes \nabla^z)W^*. \]

(2) Let \( t \in \mathbb{R} \). Then the following is true on the whole space \( \mathcal{H} \otimes \mathcal{H} \):
\[ W(L^{it} \otimes \nabla^{it}) = (L^{it} \otimes \nabla^{it})W \quad \text{and} \quad W^*(L^{it} \otimes \nabla^{it}) = (L^{it} \otimes \nabla^{it})W^*. \]

**Remark.** Our Proposition 4.2 gives rise to the same conclusion as in Proposition 4.12 in [11]. The contexts are different, so of course the proofs are not the same. Nonetheless, the steps taken in the paragraphs following that proposition can be taken almost word for word, up to Propositions 4.17 in [11]. That is because this is more about dealing with the unbounded operators in general than any specifics about quantum groupoids.

The results (1), (2) in Proposition 4.3 above are basically Propositions 4.17 and 4.18 of [11]. The key point is that for any \( z \in \mathbb{C} \), it turns out that \( (L^z \otimes \nabla^z)|_{\text{Ran}(E)} \) acts as an operator on \( \text{Ran}(E) \), where \( E = W^*W \), and also that \( (L^z \otimes \nabla^z)|_{\text{Ran}(E)^\perp} \) acts as an operator on \( \text{Ran}(E)^\perp = \text{Ker}(W) \). Since \( \mathcal{H} \otimes \mathcal{H} = \text{Ran}(W^*W) \otimes \text{Ker}(W) \), the partial isometry \( W \) behaves as a unitary map from \( \text{Ran}(W^*W) \) onto \( \text{Ker}(W^*)^\perp = \text{Ran}(WW^*) \), and similarly for the partial isometry \( W^* \), allowing us the result that resembles the case when \( W \) is unitary.

Since \( W \) is not unitary, we only have “\( \subseteq \)” in general. On the other hand, the situation improves if \( z \in \mathbb{C} \) is purely imaginary, or \( z = it, t \in \mathbb{R} \). If so, the operators \( L^{it} \) and \( \nabla^{it} \) are bounded, so the domain of \( L^{it} \otimes \nabla^{it} \) becomes the whole space \( \mathcal{H} \otimes \mathcal{H} \).

Next is another useful result, relating the operator \( W \) with the anti-unitary operators \( I \) (coming from \( K = IL^{\frac{1}{2}} \)) and \( J \) (coming from \( T = J\nabla^{\frac{1}{2}} \)).
Proposition 4.4. We have:

\[(I \otimes J)W(I \otimes J) = W^*.\]

Proof. Let \(a, b \in A\) be arbitrary. By Proposition 4.3 (1), when \(z = \frac{1}{2}\), we have:

\[
\begin{align*}
(I \otimes J)W(L_2^\frac{1}{2} \otimes \nabla_2^\frac{1}{2})(\Lambda(a) \otimes \Lambda(b)) &= (I \otimes J)(L_2^\frac{1}{2} \otimes \nabla_2^\frac{1}{2})W(\Lambda(a) \otimes \Lambda(b)) \\
&= (K \otimes T)W(\Lambda(a) \otimes \Lambda(b)) = W^*(K \otimes T)(\Lambda(a) \otimes \Lambda(b)) \\
&= W^*(I \otimes J)((L_2^\frac{1}{2} \otimes \nabla_2^\frac{1}{2})(\Lambda(a) \otimes \Lambda(b))),
\end{align*}
\]

where we used Equation (4.1). Even though these expressions involve unbounded operators, note that \((L_2^\frac{1}{2} \otimes \nabla_2^\frac{1}{2})\) has a dense range in \(\mathcal{H} \otimes \mathcal{H}\), while \((I \otimes J)W\) and \(W^*(I \otimes J)\) are bounded operators, so no domain issues. Therefore, we can conclude that

\[ (I \otimes J)W = W^*(I \otimes J), \]

on the whole space \(\mathcal{H} \otimes \mathcal{H}\). This is equivalent to saying that \((I \otimes J)W(I \otimes J) = W^*\). \(\square\)

4.2. Antipode map in terms of its polar decomposition. Now that we have gathered some results regarding the operators \(K\) and \(T\), including their polar decompositions, we can carry on with the construction of the polar decomposition for the antipode.

Note first that the scaling group on \(\mathcal{B}(\mathcal{H})\) can be defined at the level of the \(C^*-\text{algebra}\) \(A\) and also at the level of the von Neumann algebra \(M\):

Proposition 4.5. For \(x \in A\), define \(\tau_t(x) := L^txL^{-it}\), for \(t \in \mathbb{R}\). Then \(\tau_t(A) = A\), and \((\tau_t)_{t \in \mathbb{R}}\) is a norm-continuous one-parameter group of automorphisms on \(A\), referred to as the scaling group on \(A\).

Similarly, \(M \in z \mapsto \tau(z) = L^tzL^{-it}, t \in \mathbb{R}\), defines a strongly-continuous one-parameter group of automorphisms on \(M\).

Proof. From Proposition 4.3 (2), we know \((L^t \otimes \nabla^t)W = W(L^t \otimes \nabla^t)\), for all \(t \in \mathbb{R}\). Multiply \((\text{id} \otimes \nabla^{-it})\) from the left and multiply \((L^{-it} \otimes \text{id})\) from the right, to obtain:

\[
(L^t \otimes \text{id})W(L^{-it} \otimes \text{id}) = (\text{id} \otimes \nabla^{-it})W(\text{id} \otimes \nabla^t).
\]

Then we can see that for any \(\omega \in \mathcal{B}(\mathcal{H})_*,\) we have:

\[
L^t(\text{id} \otimes \omega)(W)L^{-it} = (\text{id} \otimes \omega(\nabla^{-it} \cdot \nabla^t))(W). \tag{4.4}
\]

Recall that the elements of the form \(x = (\text{id} \otimes \omega)(W)\) generate the \(C^*-\text{algebra}\) \(A = \overline{\pi(A)}^\|\) (see Definition 3.9) as well as the von Neumann algebra \(M = \pi(A)'\). Therefore, we can see quickly from Equation (4.4) that \(\tau_t(A) = A\) and \(\tau_t(M) = M\), for all \(t \in \mathbb{R}\). The norm-continuity of \((\tau_t|_A)_{t \in \mathbb{R}}\) and the strong continuity of \((\tau_t|_M)_{t \in \mathbb{R}}\) also follow immediately. \(\square\)

The result of Proposition 4.5 indicates that the elements of the form \((\text{id} \otimes \omega)(W), \omega \in \mathcal{B}(\mathcal{H})_*,\) span a core for \(\tau\). We can also say that the elements \(\pi(a), a \in \mathcal{A}\), are analytic elements of \(\tau\).

Proposition 4.6. Let \(a \in \mathcal{A}\) Then \(\pi(a)\) is an analytic element of \(\tau\), and that

\[
\tau_{ni}(\pi(a)) = \pi(S^{-2n}(a)), \quad \text{for any } n \in \mathbb{Z}.
\]
Proof. Let $a \in A$. For any $b \in A$, note that
\[
\pi(a)K\Lambda(b) = \pi(a)\Lambda(S(b)\ast) = \Lambda(aS(b)\ast) = \Lambda([S(b)a\ast] \ast) = \Lambda([S(S^{-1}(a)\ast)b] \ast)
\]
\[
= \Lambda([S(S(a)\ast)b]) \ast = K\Lambda(S(a)\ast)b = K\pi(S(a)\ast)\Lambda(b).
\]
From this, we can conclude that $\pi(a)K \subseteq K\pi(S(a)\ast)$. Similarly, we have $\pi(a)K^* \subseteq K^*\pi(S(a)\ast)$.

Recall that $L = K^*K$. The above observations mean that we have the following:
\[
\pi(a)L = \pi(a)K^*K \subseteq K^*\pi(S(a)\ast)K
\]
\[
\subseteq K^*K\pi(S(S(a)\ast)) = K^*K\pi(S^{-1}(S(a)\ast)) = K^*K\pi(S^{-1}(S^{-1}(a))) = L\pi(S^{-2}(a)).
\]
It follows that for any $n \in \mathbb{Z}$, we have:
\[
\pi(a)L^n \subseteq L^n\pi(S^{-2n}(a)),
\]
which is true for all $a \in A$.

In this way, we can see that for any $a \in A$, we have $\pi(a) \in \mathcal{D}(\tau_n)$, and that $\tau_n(\pi(a)) = \pi(S^{-2n}(a))$, for all $n \in \mathbb{Z}$. Continuing, we can also see that $\pi(a) \in \mathcal{D}(\tau_z)$, for any $z \in \mathbb{C}$. \qed

The other main ingredient for the polar decomposition of the antipode is the following *-anti-isomorphism $R_A : A \to A$, called the unitary antipode.

**Proposition 4.7.** For $x \in A$, define $R_A(x)$ by $R_A(x) =Ix^\ast I$. It is a *-anti-isomorphism from $A$ onto $A$, and is involutive. In particular, for any $\omega \in \mathcal{B}(\mathcal{H})_\ast$, we have:
\[
R : (\text{id} \otimes \omega)(W) \mapsto (\text{id} \otimes \theta)(W),
\]
where $\theta \in \mathcal{B}(\mathcal{H})_\ast$ is such that $\theta = \bar{\omega}(J \cdot J)$.

It has a natural extension $\bar{R}_M : M \to M$, which is a *-anti-isomorphism from $M$ onto $M$.

**Proof.** Since $I$ is an anti-unitary, it is evident that $R_A : x \mapsto Ix^\ast I$ is anti-multiplicative and that $R_A(x^\ast) = R_A(x)^\ast$, for all $x \in A$.

To see if $R_A$ is indeed a map from $A$ onto itself, recall first Proposition 4.4, saying $(I \otimes J)W = W^\ast(I \otimes J)$. It may be re-written as
\[
(I \otimes 1)W^\ast(I \otimes 1) = (1 \otimes J)W(1 \otimes J).
\]
Consider $x = (\text{id} \otimes \omega)(W)$, for $\omega \in \mathcal{B}(\mathcal{H})_\ast$. Then by Equation (4.6), we have:
\[
Ix^\ast I = I(\text{id} \otimes \omega)(W^\ast)I = (\text{id} \otimes \bar{\omega})(I \otimes 1)W^\ast(I \otimes 1)
\]
\[
= (\text{id} \otimes \bar{\omega})(I \otimes 1)W(1 \otimes J) = (\text{id} \otimes \theta)(W),
\]
where $\theta \in \mathcal{B}(\mathcal{H})_\ast$ is such that $\theta = \bar{\omega}(J \cdot J)$. Since $A$ (and $M$) are generated by the elements of the form $(\text{id} \otimes \omega)(W)$, for $\omega \in \mathcal{B}(\mathcal{H})_\ast$, we can see that the map $x \mapsto Ix^\ast I$ sends a dense subspace of $A$ into a dense subspace of $A$. But then, since $R_A$ is a *-anti-homomorphism, it must be bounded. We can thus conclude that $R_A$ extends to all of $A$, becoming a *-anti-automorphism from $A$ onto $A$. We can say the same for $\bar{R}_M : M \to M$, which is a *-anti-automorphism from $M$ onto $M$. \qed
Remark. Since $R_A$ is a $^*$-anti-isomorphism, it can be naturally extended to the level of the multiplier algebra $M(A)$. Or as a restriction of $R_M : M \to M$ to the level of $M(A)$. For this map $R_A : M(A) \to M(A)$, it can be shown (see [11]) that $R_A(B) = C$ and that the restriction of $R_A$ to $B$ coincides with the map $R_{BC} : B \to C$ that appears in Proposition 3.17. For this reason, we have been rather lazy with the notations in Sections 2 and 3, and we will continue to do so rest of the way: The $^*$-anti-isomorphism will be all written as $R$, whether it is considered as the unitary antipode $R_A$ on $A$, or as $R_{BC} : B \to C$, or its inverse $R_{CB} : C \to B$.

**Proposition 4.8.** $R$ and $\tau_t$ commute, for all $t \in \mathbb{R}$. That is, $R \circ \tau_t = \tau_t \circ R$.

**Proof.** We know that $ILL = L^{-1}$. Since $I$ is conjugate linear, we thus have $IL^tI = L^{it}$. Therefore, we have

$$R(\tau_t(x)) = I\tau_t(x)^*I = IL^tIx^*L^{-it}I = L^{it}Ix^*L^{-it} = \tau_t(R(x)).$$

□

Consider $\tau_{-\frac{t}{2}}$, the analytic generator of $\tau = (\tau_t)$ at $t = -\frac{1}{2}$. From Proposition 4.6, we can deduce that it is a densely-defined map. Using this map, we are now ready to describe the polar decomposition of the antipode.

**Theorem 4.9.** Consider the densely-defined map $S$ on $A$, defined by

$$S := R \circ \tau_{-\frac{1}{2}}.$$ 

This is the antipode, extending the antipode map at the level of $A$. In fact, for any $a \in A$, we have:

$$\pi(S(a)) = R(\tau_{-\frac{1}{2}}(\pi(a))).$$

At the von Neumann algebra level, on $M = \pi(A)''$, we can consider the extended antipode map, $\tilde{S} = \tilde{R} \circ \tau_{-\frac{1}{2}}$.

**Proof.** We saw in the proof of Proposition 4.6 that for any $a \in A$, we have $\pi(a)K \subseteq K\pi(S(a)^*)$. Since $K = IL\frac{1}{2} = L^{-\frac{1}{2}}I$ (shown earlier, which can be proved using $ILL = L^{-1}$), it becomes

$$\pi(a)L^{-\frac{1}{2}}I \subseteq L^{-\frac{1}{2}}I\pi(S(a)^*).$$

Then we have

$$\pi(a)L^{-\frac{1}{2}} \subseteq L^{-\frac{1}{2}}I\pi(S(a)^*)I = L^{-\frac{1}{2}}R(\pi(S(a))).$$

It follows that $\tau_{-\frac{1}{2}}(\pi(a)) = R(\pi(S(a)))$, or equivalently, $R(\tau_{-\frac{1}{2}}(\pi(a))) = \pi(S(a))$, because $R$ is involutive. □

Theorem 4.9 agrees with Definition 4.25 of [11]. In this way, the antipode map can be properly considered at the operator algebra level. The definition also resembles the quantum group case, such as Definition 5.21 of [15].

We can refer to sections 4 and 5 of [11] for more properties of the antipode map at the $C^*$-algebra level, once our full construction is done. Nonetheless, some of those results are actually needed for our purposes in a later section. Among those, here are some results involving $R$, $\tau$,
The following four propositions (Propositions 4.10, 4.11, 4.12, 4.13) are none other than Propositions 5.1, 5.2, 5.4, and 5.5 of [11].

**Proposition 4.10.** For all \( x \in A \) and \( t \in \mathbb{R} \), we have: \( \Delta(\tilde{\sigma}_t(x)) = (\tau_t \otimes \tilde{\sigma}_t)(\Delta x) \).

**Proof.** Let \( x \in A \). Then
\[
\Delta(\tilde{\sigma}_t(x)) = W^*(1 \otimes \tilde{\sigma}_t(x))W = W^*(1 \otimes \nabla_{it}x\nabla_{-it})W
= W^*(L_{it} \otimes \nabla_{it})(1 \otimes x)(L_{-it} \otimes \nabla_{-it})W
= (L_{it} \otimes \nabla_{it})W^*(1 \otimes x)W(L_{-it} \otimes \nabla_{-it}) = (\tau_t \otimes \tilde{\sigma}_t)(\Delta x),
\]
by Proposition 4.13(2).

**Proposition 4.11.** For all \( x \in A \) and \( t \in \mathbb{R} \), we have: \( \Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)(\Delta x) \).

**Proof.** For any \( a \in A \), by the coassociativity of \( \Delta \) and by Proposition 4.10, we have
\[
(\Delta \otimes \text{id})\Delta(\tilde{\sigma}_t(a)) = (\text{id} \otimes \Delta)\Delta(\tilde{\sigma}_t(a)) = (\tau_t \otimes \tau_t \otimes \tilde{\sigma}_t)((\text{id} \otimes \Delta)(\Delta a)).
\]
Applying again the proposition (in the left) and the coassociativity (in the right), this becomes:
\[
((\Delta \circ \tau_t) \otimes \tilde{\sigma}_t)(\Delta a) = (\tau_t \otimes \tau_t \otimes \tilde{\sigma}_t)((\text{id} \otimes \Delta)(\Delta a)).
\]
By applying the automorphism \( \tilde{\sigma}_{-t} \) to the third leg, this becomes:
\[
((\Delta \circ \tau_t) \otimes \text{id})(\Delta a) = ((\tau_t \otimes \tau_t) \circ \Delta \otimes \text{id})(\Delta a).
\]
Apply now \( \text{id} \otimes \text{id} \otimes \omega \), for \( \omega \in A^* \). Then we have:
\[
\Delta(\tau_t((\text{id} \otimes \omega)(\Delta a))) = (\tau_t \otimes \tau_t)(\Delta((\text{id} \otimes \omega)(\Delta a))).
\]
Note that the elements of the form \( (\text{id} \otimes \omega)(\Delta a) \), for \( a \in A \), \( \omega \in A^* \), generate all of \( A \). It follows that \( \Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)(\Delta x) \), for all \( x \in A \).

**Proposition 4.12.** For all \( x \in A \), we have:
\[
(R \otimes R)(\Delta x) = \Delta^{\text{cop}}(R(x)),
\]
where \( \Delta^{\text{cop}} \) denotes the coopposite comultiplication, given by \( \Delta^{\text{cop}}(x) = \varsigma(\Delta(x)) \).

At the von Neumann algebra level, we have: \((\tilde{R} \otimes \tilde{R})(\Delta x) = \Delta^{\text{cop}}(\tilde{R}(x))\), for \( x \in M \).

**Proof.** We know that the elements of the form \((\text{id} \otimes \omega)(W)\), for \( \omega \in \mathcal{B}(\mathcal{H})_+ \), generate all of \( A \) (and \( M \)). Therefore, to prove the above result, it is sufficient to prove the following:
\[
(R \otimes R)(\Delta((\text{id} \otimes \omega)(W))) = \Delta^{\text{cop}}(R((\text{id} \otimes \omega)(W))).
\]
Remembering the definition of \( R \) (see Proposition 4.7), we have:
\[
(R \otimes R)(\Delta((\text{id} \otimes \omega)(W))) = (I \otimes I)\Delta((\text{id} \otimes \omega)(W)^*)(I \otimes I)
= (I \otimes I)W^*((1 \otimes (\text{id} \otimes \omega)(W^*))W(I \otimes I)
= (I \otimes I)(\text{id} \otimes \text{id} \otimes \omega)(W_{12}^*W_{23}^*W_{13}^*)(I \otimes I)
= (I \otimes I)(\text{id} \otimes \text{id} \otimes \omega)(W_{23}^*W_{13}^*)(I \otimes I)
= (\text{id} \otimes \text{id} \otimes \omega)(I \otimes I \otimes 1)W_{23}^*W_{13}^*(I \otimes I \otimes 1)),
\]
where we used Equation (3.14). Since \((I \otimes J)W^*(I \otimes J) = W\) from Proposition 4.4, we see that 
\((I \otimes 1)W^*(I \otimes 1) = (1 \otimes J)W(1 \otimes J)\). Using this, the above expression becomes:
\[
(R \otimes R)(\Delta((\text{id} \otimes \omega)(W))) = (\text{id} \otimes \text{id} \otimes \omega)((1 \otimes 1 \otimes J)W_{23}W_{13}(1 \otimes 1 \otimes J)).
\] (4.7)

Meanwhile, we also have
\[
\Delta(R((\text{id} \otimes \omega)(W))) = \Delta(I((\text{id} \otimes \omega)(W^*)I) = W^*(1 \otimes I((\text{id} \otimes \omega)(W^*)I)W
= (\text{id} \otimes \text{id} \otimes \omega))(W_{12}^*(1 \otimes I \otimes 1)W_{23}^*(1 \otimes I \otimes 1)W_{12})
= (\text{id} \otimes \text{id} \otimes \omega))(W_{12}^*(1 \otimes I \otimes J)W_{23}(1 \otimes I \otimes J)W_{12})
= (\text{id} \otimes \text{id} \otimes \omega)((1 \otimes 1 \otimes J)W_{12}W_{23}W_{12}(1 \otimes 1 \otimes J))
= (\text{id} \otimes \text{id} \otimes \omega)((1 \otimes 1 \otimes J)W_{13}W_{23}(1 \otimes 1 \otimes J)),
\]
using again \((I \otimes 1)W^*(I \otimes 1) = (1 \otimes J)W(1 \otimes J)\) and also using Equation (3.14). Applying the flip map, we then have:
\[
\Delta^{\cop}(R((\text{id} \otimes \omega)(W))) = (\text{id} \otimes \text{id} \otimes \omega)((1 \otimes 1 \otimes J)W_{23}W_{13}(1 \otimes 1 \otimes J)).
\] (4.8)

Comparing Equations (4.7) and (4.8), we prove the result: 
\[
(R \otimes R)(\Delta((\text{id} \otimes \omega)(W))) = \Delta^{\cop}(R((\text{id} \otimes \omega)(W))).
\]

\[\square\]

**Proposition 4.13.** As an immediate consequence of Propositions 4.10, 4.11, 4.12, we have, for all \(t \in \mathbb{R}\), the following results:

1. \((\tau_t \otimes \sigma_t)(E) = E\)
2. \((\tau_t \otimes \tau_t)(E) = E\)
3. \((R \otimes R)(E) = \varsigma E\)

\[\square\]

**Proof.** The results follow from the earlier propositions, with \(E = \Delta(1)\).

5. **Left and right invariant weights**

We have so far established our \(C^*\)-algebra \(A\); the comultiplication \(\Delta\); the base \(C^*\)-algebra \(B\) (and \(C\)) as a \(C^*\)-subalgebra of \(M(A)\); a KMS weight \(\nu\) on \(B\) (and a KMS weight \(\mu\) on \(C\)); as well as the canonical idempotent \(E\). We have formulated a polar decomposition for the antipode \(S\), making it properly defined as an unbounded map at the \(C^*\)-algebra level. In view of the definition of a \(C^*\)-algebraic quantum groupoid of separable type (Definition 4.8 of [10] and Definition 1.2 of [11]), what remains is showing the existence of a suitable left invariant weight and a right invariant weight.

Some initial work was carried out in §3.6 where we constructed an n.s.f. weight \(\tilde{\psi}\) at the von Neumann algebra level, extending the functional \(\varphi\) at the \(*\)-algebra level. We looked at some of its modular theoretic data, such as the operators \(\nabla, J\), and the modular automorphism group \((\tilde{\sigma}_t)_{t \in \mathbb{R}}\). However, there are still a good amount of work remaining: We need to obtain a KMS weight \(\varphi\) at the \(C^*\)-algebra level; We also need to establish the left invariance of the weight \(\varphi\); Finally, there is also the task of constructing a right invariant weight \(\psi\). These are the tasks that we will carry out in the subsections below.
5.1. The left invariance property of $\tilde{\varphi}$. Consider the weight $\tilde{\varphi}$, which we constructed in subsection §3.6. It is an n.s.f. weight at the von Neumann algebra level, which we noted to extend the functional $\varphi$ on $A$. Note also that by the general modular theory (see Theorem VI.1.26 of [23]), for any $x \in \mathcal{M}_{\tilde{\varphi}}$, there exists a sequence $(a_n)_n$ in $A$ such that $\Lambda(a_n) \rightarrow \Lambda_{\tilde{\varphi}}(x)$ and $\pi(a_n) \xrightarrow{\text{strong-*}} x$. As such, we expect that the left invariance property of $\varphi$ (Definition 1.5) would carry over to $\tilde{\varphi}$. Let us discuss this matter here.

**Proposition 5.1.**  
(1) Let $x \in \mathcal{M}_{\tilde{\varphi}}$ and let $\omega \in A^+_+$. Then $(\omega \otimes \text{id})(\Delta(x^*x)) \in \mathcal{M}^+_{\tilde{\varphi}}$.

(2) Let $x \in \mathcal{M}_{\tilde{\varphi}}$ and let $\omega \in A^*$. Then $(\omega \otimes \text{id})(\Delta x) \in \mathcal{M}_{\tilde{\varphi}}$.

**Proof.** (1). Without loss of generality, we may write $\omega \in A^+_+$ as $\omega = \theta(y^* \cdot y)$, for some $\theta \in A^*_+$ and $y \in A$. Find sequences $(a_n)$ and $(b_n)$ in $A$, such that $(b_n)_{n=1}^\infty \rightarrow y$, $(a_n)_{n=1}^\infty \rightarrow x$, and $(\Lambda(a_n))_{n=1}^\infty \rightarrow \Lambda_{\tilde{\varphi}}(x)$. For convenience, we may regard $a_n = \pi(a_n)$ and $b_n = \pi(b_n)$, so that they can be also viewed as elements in $\pi(A)$. Then we will have $((\theta(b_n^* \cdot b_n) \otimes \text{id})\Delta(a_n^*a_n))_{n=1}^\infty \rightarrow (\omega \otimes \text{id})\Delta(x^*x)$.

Apply $\tilde{\varphi}$. Since $\tilde{\varphi}$ extends the functional $\varphi$ on $A$, we have

$$\tilde{\varphi}((\theta(b_n^* \cdot b_n) \otimes \text{id})\Delta(a_n^*a_n)) = (\varphi \otimes \varphi)((b_n^* \otimes 1)\Delta(a_n^*a_n)(b_n \otimes 1)) = \theta(b_n^*c_nb_n),$$

where we wrote $c_n = (\text{id} \otimes \varphi)(\Delta(a_n^*a_n)) \in \mathcal{M}(C)$, by the left invariance property of $\varphi$. As $\Delta$ is a *-representation, we have $\|(\pi \otimes \pi)(\Delta(a^*a))\| \leq \|\pi(a^*a)\|$, for $a \in A$. So for each $n$, we can see that $\|\pi(c_n)\| \leq |\varphi(a_n^*a_n)|$.

Since $\tilde{\varphi}$ is an n.s.f. weight, we have

$$\tilde{\varphi}((\omega \otimes \text{id})\Delta(x^*x)) \leq \liminf_{n \to \infty} (\varphi((\theta(b_n^* \cdot b_n) \otimes \text{id})\Delta(a_n^*a_n))) \leq \liminf_{n \to \infty} |\varphi(a_n^*a_n)|\theta(b_n^*b_n) = \varphi(x^*x)\|\omega\|.$$

This shows that $(\omega \otimes \text{id})(\Delta(x^*x)) \in \mathcal{M}^+_{\tilde{\varphi}}$.

(2). By standard theory on linear functionals (see, for instance Lemma 4.4 of [10] or Proposition 3.6.7 of [20]), there exists a unique positive linear functional $|\omega| \in A^*$ such that $\|\omega\| = \|\omega\|$ and $\omega(a^*a) = |\omega(a)|^2 \leq \|\omega\||\omega(a^*a)|$, for $a \in A$. Then we have

$$(\omega \otimes \text{id})(\Delta x)^*(\omega \otimes \text{id})(\Delta x) \leq \|\omega\||(\omega \otimes \text{id})(\Delta(x^*x)),$$

for all $x \in A$. By using (1), we see that

$$\tilde{\varphi}((\omega \otimes \text{id})(\Delta x)^*(\omega \otimes \text{id})(\Delta x)) \leq \|\omega\|^2\tilde{\varphi}(\Delta(x^*x)) \leq \|\omega\|^2\tilde{\varphi}(x^*x).$$

This shows that $(\omega \otimes \text{id})(\Delta x) \in \mathcal{M}_{\tilde{\varphi}}$. 

The results of the above proposition are true for any functional $\omega$. It suggests that for any $x \in \mathcal{M}_{\tilde{\varphi}}$, we have $\Delta x \in \mathcal{M}_{\text{id} \otimes \tilde{\varphi}}$. Moreover, we can actually show the left invariance of $\tilde{\varphi}$ at the von Neumann algebra level. See below:

**Proposition 5.2.** Let $x \in \mathcal{M}_{\tilde{\varphi}}$. Then $\Delta x \in \mathcal{M}_{\text{id} \otimes \tilde{\varphi}}$. We also have:

$$\text{id} \otimes \tilde{\varphi}(\Delta x) \in \pi(C)^{''},$$

where $\pi(C)^{''}$ is the enveloping von Neumann algebra of $C$, containing $\mathcal{M}(C)$. 

Proof. Let \( y \in \mathfrak{M}_\varphi \), and let \( z \in A \) be arbitrary. Find sequences \((a_n)\) and \((b_n)\) in \( A \), such that
\( (\pi(a_n))_{n=1}^\infty \rightarrow y \), \( (\pi(b_n))_{n=1}^\infty \rightarrow z \), and \( (\Lambda(a_n))_{n=1}^\infty \rightarrow \varphi(y) \). Note that
\[
\left\| (\mathrm{id} \otimes \varphi)(\Delta(a_n)(\pi(b_n)(\pi(a_n)(\pi(b_n)(1)))\right\|^2 = \left\| \pi((\mathrm{id} \otimes \varphi)((b_n \otimes 1)\Delta(a_n)a_n(b_n \otimes 1))) \right\| = \left\| \pi(b_n^*c_nb_n) \right\|
\]
where we wrote \( c_n = (\mathrm{id} \otimes \varphi)(\Delta(a_n^*a_n)) \in M(\mathcal{C}) \), by the left invariance of \( \varphi \).

As observed in the proof of Proposition 5.1, we know that \( \left\| \pi(c_n) \right\| \leq \left| \varphi(a_n^*a_n) \right| \), for each \( n \). As \((\Lambda(a_n))_{n=1}^\infty \rightarrow \varphi(y) \), which also means \((\varphi(a_n^*a_n))_{n=1}^\infty \rightarrow \varphi(y)^*y \), this implies that the sequence \((\pi(c_n))_{n=1}^\infty \) must be Cauchy, hence convergent. Combining these results, we can see that \( \Delta(y)(z \otimes 1) \in \mathfrak{M}_\varphi \), or equivalently that \( (z^* \otimes 1)\Delta(y^*y)(z \otimes 1) \in \mathfrak{M}_\varphi \), such that
\[
(\mathrm{id} \otimes \varphi)(\Delta(y^*y)) = \lim_{n=1} \pi(c_n) \in \pi(\mathcal{C})''
\]
By polarization, it follows that for any \( x \in \mathfrak{M}_\varphi \), we have \( \Delta x \in \mathfrak{M}_\varphi \), and that
\[
(\mathrm{id} \otimes \varphi)(\Delta x) \in \pi(\mathcal{C})''.
\]

\( \square \)

5.2. The weight \( \tilde{\psi} \). Again at the von Neumann algebra level, let us define another weight \( \tilde{\psi} \), as follows:

**Definition 5.3.** Define the weight \( \tilde{\psi} = \varphi \circ \tilde{R} \). Then \( \tilde{\psi} \) is an n.s.f. weight on \( M = \pi(A)'' \).

Since \( \tilde{R} : M \rightarrow M \) is an involutive *-anti-isomorphism, it is not too difficult to see that it is an n.s.f. weight, with its associated modular automorphism group \( \tilde{\sigma}^t := \sigma^\psi \) satisfying \( \tilde{\sigma}_t^t = \tilde{R} \circ \tilde{\sigma}_{-t} \circ \tilde{R} \), for all \( t \).

Here is a result that is analogous to Proposition 4.10, this time in terms of the modular automorphism group \( (\tilde{\sigma}_t^t) \).

**Proposition 5.4.** For all \( x \in A \) and \( t \in \mathbb{R} \), we have: \( \Delta(\tilde{\sigma}_t^t(x)) = (\tilde{\sigma}_t^t \otimes \tau_{-t})(\Delta x) \).

**Proof.** We can use the fact \( \tilde{\sigma}_t^t = \tilde{R} \circ \tilde{\sigma}_{-t} \circ \tilde{R} \), together with the result of Proposition 4.12 saying \( (\tilde{R} \otimes \tilde{R})(\Delta x) = \Delta^{\text{cop}}(\tilde{R}(x)) \). See below:
\[
\Delta(\tilde{\sigma}_t^t(x)) = \Delta((\tilde{R} \circ \tilde{\sigma}_{-t} \circ \tilde{R})(x)) = \varsigma(\tilde{R} \otimes \tilde{R})(\Delta(\tilde{\sigma}_{-t}(\tilde{R}(x)))) = \varsigma(\tilde{R} \otimes \tilde{R})(\tau_{-t} \circ \tilde{\sigma}_{-t})(\Delta(\tilde{R}(x)))
\]
\[
= (\tilde{R} \otimes \tilde{R})(\tilde{\sigma}_{-t} \otimes \tau_{-t})(\Delta^{\text{cop}}(\tilde{R}(x))) = ((\tilde{R} \otimes \tilde{R}) \circ (\tilde{\sigma}_{-t} \otimes \tau_{-t}) \circ (\tilde{R} \otimes \tilde{R}))(\Delta x)
\]
\[
= (\tilde{\sigma}_t^t \otimes \tau_{-t})(\Delta x),
\]
because \( \tau_t \) commutes with \( \tilde{R} \). \( \square \)
As a consequence of Propositions 4.10 and 5.4, we obtain the following result, which is essentially a converse result of Proposition 1.4, providing us with useful characterizations of $M(B)$ and $M(C)$.

**Proposition 5.5.** Let $x, y \in M(A)$. We have:

1. $x \in M(C)$ if and only if $\Delta x = (x \otimes 1)E = E(x \otimes 1)$,
2. $y \in M(B)$ if and only if $\Delta y = E(1 \otimes y) = (1 \otimes y)E$.

**Proof.** We can use mostly the same proof as in Proposition 5.22 in [11], with only some minor modifications. Even though our weights $\tilde{\varphi}$ and $\tilde{\psi}$ are at the von Neumann algebra level at present, the results still stand because the proof characterizes $x \in M(C)$ as a double centralizer for $C$, and similarly for $y \in M(B)$. The first result uses Proposition 4.10 and the second one uses Proposition 5.4. □

Here are a couple more useful results, regarding the base algebras $M(B)$ and $M(C)$:

**Proposition 5.6.** The scaling group $(\tau_t)$ leaves both $M(B)$ and $M(C)$ invariant.

**Proof.** See Proposition 5.23 (1) of [11] for the proof. Since it uses only the characterizations of $M(B)$ and $M(C)$ noted in Proposition 5.5 as well as the results of Propositions 4.11 and 4.13, this is all right. □

**Proposition 5.7.** We have: $R(M(B)) = M(C)$ and $R(M(C)) = M(B)$.

At the von Neumann algebra level, we have: $\hat{R}(\pi(B)'') = \pi(C)''$ and $\hat{R}(\pi(C)'') = \pi(B)''$.

**Proof.** See Proposition 5.23 (2) of [11] for the proof of the first result. Since it uses only the characterizations of $M(B)$ and $M(C)$ noted in Proposition 5.5 as well as the results of Propositions 4.12 and 4.13, this is all right. The second result is a direct consequence of the first result. □

**Remark.** In section 5 of [11], there are more results obtained regarding $S$, $R$, $\tau$. In particular, not only $R(M(B)) = M(C)$ and $R(M(C)) = M(B)$, but in fact, it turns out that $R|_B = R_{BC} : B \to C$, the $^*$-anti-isomorphism that appear in Proposition 3.17. Not all the results in [11] can be used (or only allowed with an alternative proof), because the full construction is not yet done. However, eventually all these results would be obtained as consequences, once we successfully carry out the construction of our $C^*$-algebraic quantum groupoid.

With these results regarding the isomorphism $\hat{R}$ gathered, we can now show that the weight $\hat{\psi}$ satisfies the right invariance property at the von Neumann algebra level:

**Proposition 5.8.** Let $x \in \mathfrak{M}_{\hat{\psi}}$. Then $\Delta x \in \mathfrak{M}_{(\hat{\psi} \otimes \text{id})}$. We also have:

$$(\hat{\psi} \otimes \text{id})(\Delta x) \in \pi(B)'',$$

where $\pi(B)''$ is the enveloping von Neumann algebra of $B$, containing $M(B)$. 


Proof. Let \( x \in \mathcal{M}_R \). Since \( \tilde{\psi} = \tilde{\varphi} \circ \tilde{R} \), this immediately means that \( \tilde{R}(x) \in \mathcal{M}_R \), which in turn means \( \Delta(\tilde{R}(x)) \in \mathcal{M}_{id \otimes \tilde{\varphi}} \), by Proposition 5.2. It follows that

\[
(\tilde{\psi} \otimes id)(\Delta x) = ((\tilde{\varphi} \circ \tilde{R}) \otimes id)(\Delta x) = (\tilde{\varphi} \circ \tilde{R})(\Delta x) = (\tilde{\varphi} \circ \tilde{R})(\Delta(x)) = \tilde{R}((id \otimes \tilde{\varphi})(\Delta(\tilde{R}(x)))),
\]

which is a valid element in \( \tilde{R}(\pi(C)''') \), by Proposition 5.2. Since \( \tilde{R}(\pi(C)''') = \pi(B)''' \), we prove that \( (\tilde{\psi} \otimes id)(\Delta x) \in \pi(B)''' \). \( \square \)

With this result, we see that the weight \( \tilde{\psi} \) is a right invariant weight. One word of caution, however: Unlike \( \tilde{\varphi} \), which extends the functional \( \varphi \) on \( \mathcal{A} \), it may not necessarily be the case that \( \tilde{\psi} = \tilde{\varphi} \circ \tilde{R} \) is an extension of the functional \( \psi \) we began with. There is no complete uniqueness property for the right invariant functionals and weights, so while they are related they may not be exactly same.

Meanwhile, for the operator algebraic theory to work properly, we not only need the weights \( \tilde{\varphi} \) and \( \tilde{\psi} \), but we actually need a certain condition such that we are able to have a Radon–Nikodym derivative between them. This means that we need an additional requirement, already at the \(*\)-algebra level. See the discussion in the next subsection.

5.3. Quasi-invariance assumption. So far, at least at the von Neumann algebra level, we have established that the weights \( \tilde{\varphi} \) and \( \tilde{\psi} \) are left and right invariant, respectively.

Unlike the purely algebraic theory, however, for things to work properly in the operator algebraic setting, we need the two weights to allow a Radon–Nikodym derivative between them. Even in the classical setting of locally compact groupoids, certain quasi-invariance condition was assumed as a part of the definition (see [21, 19]).

Since we are constructing an operator algebraic object from purely algebraic data, some form of an additional condition needs to be required at the algebra level to allow this development. With this in mind, we introduce the following quasi-invariance assumption at the purely algebraic level. This is the one small (but necessary) additional condition we are going to require:

**Assumption (Quasi-invariance).** We will assume that \( \sigma|_B \), the restriction of \( \sigma \) to the base algebra \( B \), leaves \( B \) invariant, and that \( \nu \circ \sigma|_B = \nu \).

**Remark.** This is a purely algebraic condition, where \( \nu \) is the distinguished functional on \( B \), and \( \sigma \) is the automorphism on \( \mathcal{A} \), as given in Proposition 1.13. It can be shown that the restriction \( \sigma|_C \) leaves \( C \) invariant with \( \mu \circ \sigma|_C = \mu \), because it turns out that \( \sigma|_C = \sigma^B \). Similarly, it turns out that \( \sigma^{\phi_0 S}|_B = \sigma^\nu \), thus it leaves \( B \) invariant with \( \nu \circ \sigma^{\phi_0 S}|_B = \nu \). See Proposition 7.4 in Appendix (Section §7). On the other hand, we do not know whether similar properties hold for the restrictions \( \sigma|_B \) and \( \sigma^{\phi_0 S}|_C \). There is no reason why they should. As such, our quasi-invariance requirement is an extra assumption.

**Remark.** It is possible to actually prove the invariance of \( B \) under \( \sigma|_B \), but not the \( \nu \circ \sigma|_B = \nu \) result. Symmetrically, if the assumption holds, then we can quickly show that \( C \) is invariant under \( \sigma^{\phi_0 S}|_C \) and that \( \mu \circ \sigma^{\phi_0 S}|_C = \mu \). We termed this assumption as “quasi-invariance”,
because this assumption indeed gives rise to the quasi-invariance property at the \( C^* \)-algebraic framework. See the remarks given later in the subsection.

In this subsection, we will collect some consequences of the quasi-invariance assumption. Many of these results refer to \( \varphi \) and \( \sigma \), as well as \( \varphi \circ S \) and \( \sigma^{\varphi S} \). See \S 7.1, 7.2 of Appendix (Section 7) for discussion on some algebraic results on these objects.

**Lemma 5.9.** Under the above quasi-invariance assumption, we have:

1. \( (\varphi \circ S)((\sigma \circ S^{-2})(a)) = (\varphi \circ S)(a), \) for all \( a \in \mathcal{A} \).
2. \( (\sigma^{\varphi S} \circ \sigma \circ S^{-2})(a) = (\sigma \circ S^{-2} \circ \sigma^{\varphi S})(a), \) for all \( a \in \mathcal{A} \).

**Proof.** (1). Let \( a \in \mathcal{A} \). Since \( \varphi \circ S \) is a right integral, by Proposition 7.3 (1) (or Proposition 1.8), we have:

\[
(\varphi \circ S)((\sigma \circ S^{-2})(a)) = \nu(((\varphi \circ S) \otimes \text{id})(\Delta((\sigma \circ S^{-2})(a)))) = \nu((\sigma \circ S^{-2})(((\varphi \circ S) \otimes \text{id})(\Delta a))).
\]

Note here that we also used Corollary of Proposition 7.5.

In the right side of the above equation, since we know \(((\varphi \circ S) \otimes \text{id})(\Delta a) \in M(\mathcal{B})\) by the right invariance of \( \varphi \circ S \), we may regard the maps \( \sigma \) and \( S^{-2} \) above as \( \sigma|_\mathcal{B} \) and \( S^{-2}|_\mathcal{B} \), respectively. As \( \nu \circ \sigma|_\mathcal{B} = \nu \) (by the quasi-invariance assumption) and \( \nu \circ S^{-2}|_\mathcal{B} = \nu \circ \sigma\nu = \nu \) (by Proposition 7.4), the equation becomes:

\[
(\varphi \circ S)((\sigma \circ S^{-2})(a)) = \nu(((\varphi \circ S) \otimes \text{id})(\Delta a)) = (\varphi \circ S)(a).
\]

(2). Let \( a, x \in \mathcal{A} \) be arbitrary. We have:

\[
(\varphi \circ S)(ax) = (\varphi \circ S)(x\sigma^{\varphi S}(a)) = (\varphi \circ S)((\sigma \circ S^{-2})(x\sigma^{\varphi S}(a)))
\]

\[
= (\varphi \circ S)((\sigma \circ S^{-2})(x)(\sigma \circ S^{-2} \circ \sigma^{\varphi S})(a)), \quad (5.1)
\]

by (1) above and the fact that \( \sigma \) and \( S^{-2} \) are multiplicative (being isomorphisms on \( \mathcal{A} \)).

On the other hand, we also have:

\[
(\varphi \circ S)(ax) = (\varphi \circ S)((\sigma \circ S^{-2})(ax)) = (\varphi \circ S)((\sigma \circ S^{-2})(a)(\sigma \circ S^{-2})(x))
\]

\[
= (\varphi \circ S)((\sigma \circ S^{-2})(x)(\sigma^{\varphi S} \circ \sigma \circ S^{-2})(a)). \quad (5.2)
\]

Comparing Equations (5.1) and (5.2), since \( x \in \mathcal{A} \) is arbitrary and since \( \varphi \circ S \) is faithful, it follows that \( (\sigma \circ S^{-2} \circ \sigma^{\varphi S})(a) = (\sigma^{\varphi S} \circ \sigma \circ S^{-2})(a) \), for all \( a \in \mathcal{A} \). \( \Box \)

This lemma helps us prove the following commutativity results:

**Proposition 5.10.** Under the quasi-invariance assumption above, we have:

1. \( \sigma \circ S^2 = S^2 \circ \sigma \)
2. \( \sigma^{\varphi S} \circ S^2 = S^2 \circ \sigma^{\varphi S} \)
3. \( \sigma \circ \sigma^{\varphi S} = \sigma^{\varphi S} \circ \sigma \)
Proof. (1). Let \( a \in A \) be arbitrary. By Corollary following Proposition 7.5 (in Appendix), and by using the fact that \( \sigma \circ S^{-2} = [\sigma \circ S^{-1}] \circ (\sigma \circ S^{-2}) \circ \sigma \circ S^{-1} \) (from Lemma 5.9), we have:

\[
(id \otimes (\sigma \circ S^{-2}))(\Delta a) = \Delta((\sigma \circ S^{-2})(a)) = \Delta(((\sigma \circ S^{-1}) \circ (\sigma \circ S^{-2}) \circ (\sigma \circ S^{-1}))(a)).
\] (5.3)

Meanwhile, by Proposition 7.5(2), where we have \( \Delta((\sigma \circ S^{-1}) \circ (\sigma \circ S^{-2}) \circ (\sigma \circ S^{-1})) = \sigma \circ S^{-2}(x) \) for all \( x \in A \), we know that \((\sigma \circ S^{-1}) \circ (\sigma \circ S^{-2}) \circ (\sigma \circ S^{-1})(\Delta a) = \Delta a \). Letting \( x = [\sigma \circ S^{-1}]^{-1}(a) \), for \( a \in A \), we have:

\[
\Delta((\sigma \circ S^{-1})^{-1}(a)) = \sigma \circ S^{-2}(\Delta a).
\] (5.4)

Combining Equation (5.3) and (5.4), and using again Corollary of Proposition 7.5, we have:

\[
(id \otimes (\sigma \circ S^{-2}))(\Delta a) = \Delta((\sigma \circ S^{-2})(a)) = \Delta((\sigma \circ S^{-2})(\sigma \circ S^{-1})(\sigma \circ S^{-1}))(\Delta a).
\]

Note that \( \Delta((\sigma \circ S^{-1})(a)) = (\sigma \circ S^{-1} \circ S^{-2})(\Delta a) \), by Proposition 7.5. As a result, we thus obtain the following:

\[
(id \otimes (\sigma \circ S^{-2}))(\Delta a) = (id \otimes (S^2 \circ \sigma \circ S^{-2} \circ S^{-2}))(\Delta a).
\]

Since \( \Delta \) is full, it follows that \( \sigma \circ S^{-2} = S^2 \circ \sigma \circ S^{-2} \circ S^{-2} \). Since \( S^2 \) and \( S^{-2} \) are automorphisms on \( A \), we thus have:

\[
\sigma \circ S^{-2} = S^2 \circ \sigma = \sigma \circ S^2.
\]

(2). Note (see proof of Proposition 7.1) that \( \sigma \circ S^{-1} = S^{-1} \circ \sigma^{-1} \circ S \). By (1), we know \( S^2 \circ \sigma = \sigma \circ S^2 \), or equivalently \( \sigma^{-1} = S^2 \circ \sigma^{-1} \). So we have:

\[
\sigma \circ S^{-2} = S^2 \circ \sigma \circ S^{-2} = S \circ \sigma \circ S = S^2 \circ \sigma \circ S^{-1} \circ S = S^2 \circ \sigma \circ S^{-1} \circ S.
\]

(3). Let \( a \in A \) be arbitrary. We have:

\[
\Delta((\sigma \circ S^{-1})(a)) = (S^2 \otimes \sigma)(\Delta((\sigma \circ S^{-1})(a)))
\]

\[
= ((S^2 \otimes (\sigma \circ S^{-2})) \otimes (\sigma \circ S^{-2}))(\Delta a) = ((\sigma \circ S^{-1} \circ S^2 \circ \sigma))(\Delta a)
\]

\[
= (\sigma \circ S^{-1} \otimes S^{-2})(\Delta(\sigma(a)) = \Delta((\sigma \circ S^{-1})(\sigma(a))).
\]

by Proposition 7.8 and by (1), (2) above (the commutativity of \( \sigma \) and \( S^2 \), and of \( \sigma \circ S^{-1} \) and \( S^2 \)). As \( \Delta \) is injective, this shows that \( \sigma \circ \sigma \circ S^{-1} = \sigma \circ S^{-1} \circ \sigma \). \( \square \)

So far, the results we obtained in this subsection have been all at the *-algebra level, under the quasi-invariance assumption (formulated at the *-algebra level), and using only the algebraic properties summarized in Appendix (7). However, the results of Proposition 5.10 now allows us to obtain some results at the operator algebra level:

**Proposition 5.11.** Under the quasi-invariance assumption, we have the following commutativity results involving the automorphism groups \( (\tau_t) \), \( (\tilde{\sigma}_t) \), and \( (\tilde{\sigma}_t') \).

1. The automorphism groups \( (\tau_t) \) and \( (\tilde{\sigma}_t) \) commute.
2. The automorphism groups \( (\tilde{\sigma}_t) \) for the weight \( \tilde{\varphi} \) and \( (\tilde{\sigma}_t') \) for the weight \( \tilde{\psi} \) commute.
Proof. (1). From Proposition 5.10(1), we saw that $\sigma \circ S^2 = S^2 \circ \sigma$, on $\mathcal{A}$. Since $\tilde{\sigma}_{-i} \left( \pi(a) \right) = \sigma(a)$ and $\tau_{-i} \left( \pi(a) \right) = S^2(a)$, while $\mathcal{A}$ forms a core for both, this means that $\tilde{\sigma}_{-i} \circ \tau_{-i} = \tau_{-i} \circ \tilde{\sigma}_{-i}$.

Note that $\tilde{\sigma}_{-i}$ is an analytic generator for the automorphism group $(\tilde{\sigma}_t)$ at $-i$, while $\tau_{-i}$ is an analytic generator for $(\tau_t)$. Since the analytic generators of the two automorphism groups commute, we can show without difficulty that the automorphism groups $(\tau_t)$ and $(\tilde{\sigma}_t)$ commute.

For instance, we have $\tilde{\sigma}_{-i} \tau_{-2i} = \tilde{\sigma}_{-i} \tau_{-i} \tau_{-i} = \tau_{-i} \tilde{\sigma}_{-i} \tau_{-i} = \tau_{-2i} \tilde{\sigma}_{-i}$. Also, we have $(\tilde{\sigma}_{-i} \tau_{-2i}) \circ (\tilde{\sigma}_{-i} \tau_{-i}) = \tilde{\sigma}_{-i} \tau_{-i} \tilde{\sigma}_{-i} = \tau_{-i}$, so $\tilde{\sigma}_{-i} \tau_{-i} \tilde{\sigma}_{-i} = \tilde{\sigma}_{-i} \circ \tau_{-i} = \tilde{\sigma}_{-i} \circ \tau_{-i} \circ \tilde{\sigma}_{-i}$. That also means $\tilde{\sigma}_{-i} \tau_{-2i} = \tau_{-2i} \tilde{\sigma}_{-i}$. Continuing, we can show that $\tilde{\sigma}_{-i} \tau_s = \tau_s \tilde{\sigma}_{-i}$, for all $s$. We can do the same for $\tilde{\sigma}_t$, so we would have

$$\tau_s \circ \tilde{\sigma}_t = \tilde{\sigma}_t \circ \tau_s,$$

for all $s, t \in \mathbb{C}$. (5.5)

(2). In Proposition 5.10(3), we saw that $\sigma \circ \sigma^{\circ \sigma} = \sigma^{\circ \sigma} \circ \sigma$, on $\mathcal{A}$. This means that on $\mathcal{A}$, we have: $\sigma \circ \left( S^{-1} \circ \sigma^{-1} \circ S \right) = \left( S^{-1} \circ \sigma^{-1} \circ S \right) \circ \sigma$.

Going up to the operator algebra level, recall that $\tilde{\psi} = \varphi \circ R$, with $\tilde{\sigma}'_t = R \circ \tilde{\sigma}_{-t} \circ R$. In particular, at $t = -i$, we have

$$\tilde{\sigma}'_{-i} = R \circ \tilde{\sigma}_i \circ R = R \circ \tau_{\frac{i}{2}} \circ \tilde{\sigma}_i \circ \tau_{-\frac{i}{2}} \circ \tilde{\sigma}_i \circ S = S^{-1} \circ \tilde{\sigma}_i \circ S,$$

because $\tau$ commutes with $R$ (Proposition 4.8) and also with $\tilde{\sigma}$ (see (1) above). We also used the polar decomposition of the antipode.

Note that $\tilde{\sigma}_i$ coincides with $\sigma^{-1}$ on $\mathcal{A}$. So on $\mathcal{A}$, the map $S^{-1} \circ \tilde{\sigma}_i \circ S$ coincides with $S^{-1} \circ \sigma^{-1} \circ S$, which has been observed to be commuting with $\sigma$. What all this means is that $\tilde{\sigma}_{-i}$ commutes with $\tilde{\sigma}_{-i}$.

As above, since $\tilde{\sigma}'_{-i}$ is an analytic generator for $(\tilde{\sigma}'_t)$ and since $\tilde{\sigma}_{-i}$ is an analytic generator for $(\tilde{\sigma}_t)$, the commutativity of the analytic generators mean the commutativity of the automorphism groups $(\tilde{\sigma}'_t)$ and $(\tilde{\sigma}_t)$. In particular, we have $\tilde{\sigma}'_s \circ \tilde{\sigma}_t = \tilde{\sigma}_t \circ \tilde{\sigma}'_s$, for all $s, t$.

The commutativity of the modular automorphism groups of the weights $\varphi$ and $\tilde{\psi}$ is significant, because we have the following general result by Vaes:

**Proposition 5.12. (Vaes [27]):** Let $\varphi$ and $\tilde{\psi}$ be two n.s.f. weights on a von Neumann algebra $M$. Then the following are equivalent:

(i). The modular automorphism groups $\sigma^{\tilde{\psi}}$ and $\sigma^\varphi$ commute.

(ii). There exists a strictly positive operator $\delta$ affiliated with $M$ and a strictly positive operator $\lambda$ affiliated with the center of $M$ such that $\sigma^\varphi \left( \tilde{\delta}^{it} \right) = \lambda^{ist} \tilde{\delta}^{it}$ for all $s, t \in \mathbb{R}$ and such that $\psi = \varphi_\delta = \varphi \left( \tilde{\delta}^{\frac{i}{2}} \cdot \tilde{\delta}^{\frac{i}{2}} \right)$.

(iii). There are a strictly positive operator $\delta$ affiliated with $M$ and a strictly positive operator $\lambda$ affiliated with the center of $M$ such that $\left[ D\psi : D\varphi \right]_t = \lambda^{\frac{1}{2}it^2} \tilde{\delta}^{it}$ for all $t \in \mathbb{R}$.

**Proof.** See Proposition 5.2 in [27].

In our case, we should consider the weights $\tilde{\varphi}$, $\tilde{\psi}$, and the commuting modular automorphism groups $(\tilde{\sigma}_t)$, $(\tilde{\delta}'_t)$. The commutativity is a consequence of our (purely algebraic) quasi-invariance
assumption. As Vaes’s result (Proposition 5.12 above) indicates, this assures us the existence of a suitable Radon–Nikodym derivative between our left and right invariant weights, justifying the term “quasi-invariance”. Recall that in the theory of (classical) locally compact groupoids (see [21], [19]), the existence of a Radon–Nikodym derivative between the left integral and the right integral is typically assumed as part of the definition, which is referred to as the quasi-invariance condition.

Note also the existence of a positive operator $\tilde{\delta}$ affiliated with $M$, and also an existence of another positive operator $\lambda$. We re-named Vaes’s $\delta$ as $\tilde{\delta}$ here, to avoid confusion with our modular element $\delta$ at the $\ast$-algebra level. But then, it seems reasonable to expect that the operator $\tilde{\delta}$ is closely related with the modular element $\delta$. See discussion below. As for $\lambda$, it seems to be a generalization of the “scaling constant” in the quantum group theory, but now no longer a scalar. In the rest of the subsection, we will attempt to make a connection between the operator $\tilde{\delta}$ and the modular element $\delta \in \mathcal{A}$.

At the $\ast$-algebra level, we have been working with the functionals $\varphi$ and $\varphi \circ S$, which are faithful left and right integrals, respectively. On the other hand, at the operator algebra level, we have been considering the weights $\tilde{\varphi}$ (which extends $\varphi$) and $\tilde{\psi} = \tilde{\varphi} \circ \tilde{R}$. As such, any discussion regarding the modular element $\delta$ and the modular operator $\tilde{\delta}$ should begin with clarifying the relationship between the functional $\varphi \circ S$ and the weight $\tilde{\psi}$. For this, we will make use of the polar decomposition of the antipode: $S = R \circ \tau_{-\frac{1}{2}}$.

Let us consider the weight $\tilde{\varphi} \circ \tau_{\frac{1}{2}}$. It is not difficult to show that it satisfies the left invariance property:

**Proposition 5.13.** The weight $\tilde{\varphi} \circ \tau_{\frac{1}{2}}$ is left invariant:

$$(\text{id} \otimes (\tilde{\varphi} \circ \tau_{\frac{1}{2}}))(\Delta x) \in M(C)'', \quad \text{for all } x \in M_{\tilde{\varphi}}.$$ 

*Proof.* Let $x \in M_{\tilde{\varphi}}$. By Proposition 4.11, we have

$$(\text{id} \otimes (\tilde{\varphi} \circ \tau_{\frac{1}{2}}))(\Delta x) = (\tau_{-\frac{1}{2}} \otimes \tilde{\varphi})((\tau_{\frac{1}{2}} \otimes \tau_{\frac{1}{2}})(\Delta x)) = (\tau_{-\frac{1}{2}} \otimes \tilde{\varphi})((\Delta(\tau_{\frac{1}{2}}(x))))
= \tau_{-\frac{1}{2}}((\text{id} \otimes \tilde{\varphi})(\Delta(\tau_{\frac{1}{2}}(x)))) \in \tau_{-\frac{1}{2}}(M(C)'') = M(C)''.
$$

We used also the left invariance property of $\tilde{\varphi}$ (Proposition 5.2) and also Proposition 5.6. \Box

While $\tilde{\varphi} \circ \tau_{\frac{1}{2}}$ is a weight at the von Neumann algebra level, it can be considered as a functional on $\mathcal{A}$, by $(\tilde{\varphi} \circ \tau_{\frac{1}{2}})|_{\mathcal{A}}(a) := \tilde{\varphi}(\tau_{\frac{1}{2}}(\pi(a)))$. Note that all the elements $\pi(a), \ a \in \mathcal{A}$, are analytic elements of $\tau$ (Proposition 4.6), and the commutativity between $\tau$ and $\tilde{\sigma}$ as noted in Equation 4.12 implies that all $\tau_{\frac{1}{2}}(\pi(a)), \ a \in \mathcal{A}$, are analytic for $\tilde{\sigma}$. As such, we see that $\tau_{\frac{1}{2}}(\pi(a)) \in M_{\tilde{\varphi}}$, making $(\tilde{\varphi} \circ \tau_{\frac{1}{2}})|_{\mathcal{A}}$ a valid functional defined on all of $\mathcal{A}$. This functional turns out to be a left integral. See below:

**Proposition 5.14.** The functional $(\tilde{\varphi} \circ \tau_{\frac{1}{2}})|_{\mathcal{A}}$ is a left integral on $\mathcal{A}$.

*Proof.* Let $a \in \mathcal{A}$. Note that $x = (\text{id} \otimes ((\tilde{\varphi} \circ \tau_{\frac{1}{2}})|_{\mathcal{A}})(\Delta a) \in M(\mathcal{A})$, and by Proposition 5.13 we have $\pi(x) \in M(C)'$. Then it must be the case that $\Delta x = E(x \otimes 1)$. Therefore, by a result at
the algebra level (see Proposition 2.9 in [8] and Proposition 2.16 in [35]), this means \( x \in M(C) \). Or \((\text{id} \otimes (\tilde{\phi} \circ \tau_{\frac{i}{2}}))|_A)(\Delta a) \in M(C)\), for all \( a \in A \).

**Corollary.** There exists an element \( y \in M(B) \) such that \((\hat{\phi} \circ \tau_{\frac{i}{2}})|_A(x) = \varphi(xy), \) for all \( x \in A \).

**Proof.** This is a consequence of \((\hat{\phi} \circ \tau_{\frac{i}{2}})|_A\) being a left integral on \( A \). See Proposition 1.14.

**Corollary.** There exists an element \( z \in M(B) \) such that \((\hat{\phi} \circ \tau_{\frac{i}{2}})(\cdot) = \hat{\varphi}(\cdot z)\).

**Proof.** Take \( z = \pi(y) \), where \( y \in M(B) \) is as in the previous Corollary.

Let us now consider the weight \( \tilde{\psi} = \hat{\varphi} \circ R \). First, from Proposition 5.12 we saw that we can write

\[
\tilde{\psi} = \hat{\varphi}(\delta^\frac{1}{2} \cdot \delta^\frac{1}{2}),
\]

where \( \delta \) is the modular operator. By the polar decomposition \( S = R \circ \tau_{\frac{1}{2}} \), we know \( \tilde{R} = \tau_{\frac{1}{2}} \circ S \).

So we can write the above as

\[
(\hat{\varphi} \circ \tau_{\frac{i}{2}})(S(x)) = \hat{\varphi}(\delta^\frac{1}{2} x \delta^\frac{1}{2}), \quad x \in M_{\tilde{\psi}}.
\]

At the same time, by Corollary above, there exists \( z = \pi(y), \ y \in M(B), \) such that

\[
(\hat{\varphi} \circ \tau_{\frac{i}{2}})(S(x)) = \hat{\varphi}(S(x)z), \quad x \in M_{\tilde{\psi}}.
\]

In particular, if \( a \in A \), then it becomes:

\[
\varphi(S(a)y) = \hat{\varphi}(S(\pi(a))z) = (\hat{\varphi} \circ \tau_{\frac{i}{2}})(S(\pi(a))) = \hat{\varphi}(\delta^\frac{1}{2} \pi(a) \delta^\frac{1}{2})
\]

\[
= \hat{\varphi}(\pi(a) \delta^\frac{1}{2} \sigma_{-i}(\delta^\frac{1}{2})) = \hat{\varphi}(\pi(a) \delta^\frac{1}{2} \lambda^{-\frac{1}{4}} \delta^\frac{1}{2}) = \hat{\varphi}(\lambda^{-\frac{1}{4}} \pi(a) \delta)
\]

(5.6)

by Proposition 5.12, where it is noted that \( \sigma_{-i}(\delta^it) = \lambda^{ist} \delta^it \) and that \( \lambda \) is central. We took \( s = -i, \ t = -\frac{i}{2} \) in Equation (5.6). Note also that

\[
\varphi(S(a)y) = (\varphi \circ S)(S^{-1}(y)a) = \varphi(S^{-1}(y)a\delta).
\]

(5.7)

Compare Equations (5.6) and (5.7). Since \( \varphi \) is faithful on \( A \) (a core for \( \hat{\varphi} \)), which in turn means the uniqueness of \( \delta \) and \( y \), this has to mean that \( \delta \equiv \delta \) and \( S^{-1}(y) \equiv \lambda^{-\frac{1}{2}} \), modulo possibly multiplication by positive real numbers. In particular, it follows that \( \delta \) is positive (self-adjoint) and that \( y \) is central. This is not saying that \( y \) has to be a scalar, but that \( y \in M(B) \cap M(C) \). Also, this is not saying that \( \tilde{\delta} \) extends \( \delta \), but that \( \tilde{\delta} \) is possibly an extension of \( p\delta \), where \( p \) is a positive real number. See below:

**Proposition 5.15.** Let \( \delta \) be the modular element, as defined in Proposition 1.15 and Proposition 7.3(3), such that \((\varphi \circ S)(x) = \varphi(x\delta), \) for all \( x \in A \). Then under the quasi-invariance assumption, it turns out that \( \delta \) is a positive self-adjoint element in \( A \).

**Proof.** See the discussion given in the preceding paragraphs.
This observation means that under the quasi-invariance assumption, the modular element $\delta$ is self-adjoint, so the results of Proposition 7.13 (in Appendix) can be used. Since the assumption was made at the $\ast$-algebra level and the results are at the $\ast$-algebra level, this indicates that there is possibly a way to give a direct, purely algebraic proof of the self-adjointness of the modular element $\delta$ from the quasi-invariance assumption. We will not pursue that question here.

Meanwhile, the operator $\lambda$ would be a generalization of the “scaling constant” in the quantum group theory (see Proposition 6.8 in [15]). For our current purposes, its role will be downplayed. In a future paper (such as [12], when we study the duality theory for $C\ast$-algebraic quantum groupoids), we will have more occasions to discuss further implications of having $\tilde{\delta}$ and $\lambda$.

5.4. Some additional consequences of the quasi-invariance condition. We will gather here a few additional technical results that are consequences of our quasi-invariance assumption. They will be useful in the next subsection.

In Proposition 7.3 (in Appendix §7), we gathered some results on the modular element $\delta$ under the modular automorphisms $\sigma$ and $\sigma^{\varphi \circ S}$. With the quasi-invariance assumption, we can prove another result:

**Proposition 5.16.** Consider the modular automorphisms $\sigma$ and $\sigma^{\varphi \circ S}$, which can be naturally extended to the multiplier algebra level. Under the quasi-invariance assumption, we have:

$$\sigma^{-1}(a) = \delta[\sigma^{\varphi \circ S}]^{-1}(a)\delta^{-1} \quad \text{and} \quad [\sigma^{\varphi \circ S}]^{-1}(a) = \delta^{-1}\sigma^{-1}(a)\delta,$$

for any $a \in A$.

**Proof.** As a consequence of the quasi-invariance assumption, we can use Proposition 5.10 (3), the commutativity of $\sigma$ and $\sigma^{\varphi \circ S}$. Since $\sigma^{\varphi \circ S} \circ \sigma = \sigma \circ \sigma^{\varphi \circ S}$, we can see quickly that $[\sigma^{\varphi \circ S}]^{-1} \circ \sigma = \sigma \circ [\sigma^{\varphi \circ S}]^{-1}$.

Applying this commutativity result to Proposition 7.3 (4), we obtain:

$$[\sigma^{\varphi \circ S}]^{-1}(\sigma(x)) = \sigma([\sigma^{\varphi \circ S}]^{-1}(x)) = \delta^{-1}x\delta.$$  

Here let $x = \sigma^{-1}(a)$, for $a \in A$. Then it becomes:

$$[\sigma^{\varphi \circ S}]^{-1}(a) = \delta^{-1}\sigma^{-1}(a)\delta,$$

true for any $a \in A$. Equivalently, we have: $\sigma^{-1}(a) = \delta[\sigma^{\varphi \circ S}]^{-1}(a)\delta^{-1}$, $\forall a \in A$.  

In the below is one more consequence of the quasi-invariance assumption and the self-adjointness of $\delta$ (itself a consequence of the quasi-invariance):

**Proposition 5.17.** Given the quasi-invariance assumption, we have:

$$(\sigma^{-1} \otimes \sigma^{\varphi \circ S})(\Delta x) = \Delta(S^{-2}(x)).$$

for any $x \in A$. 
Proof. Note that for any \( a \in \mathcal{A} \), we have \( \sigma^{\circ S}(a) = \delta \sigma(a) \delta^{-1} \) by Proposition \( \ref{Proposition 7.8}(3) \), and due to the quasi-invariance, we have \( \sigma^{-1}(a) = \delta [\sigma^{\circ S}]^{-1}(a) \delta^{-1} \), by Proposition \( \ref{Proposition 5.16} \). We thus have, for any \( x \in \mathcal{A} \),

\[
(\sigma^{-1} \otimes \sigma^{\circ S})(\Delta x) = (\delta \otimes \delta)(([\sigma^{\circ S}]^{-1} \otimes \sigma)(\Delta x)(\delta^{-1} \otimes \delta^{-1}).
\]

Apply here the result \( \Delta([\sigma^{\circ S}]^{-1}(x)) = ([\sigma^{\circ S}]^{-1} \otimes S^2)(\Delta x) \), which is essentially Proposition \( \ref{Proposition 7.8}(2) \). Then the above expression becomes:

\[
(\sigma^{-1} \otimes \sigma^{\circ S})(\Delta x) = \cdots = (\delta \otimes \delta)[(\text{id} \otimes (\sigma \circ S^{-2}))(\Delta([\sigma^{\circ S}]^{-1}(x)))](\delta^{-1} \otimes \delta^{-1})
\]

\[
= (\delta \otimes \delta)[(\text{id} \otimes (S^{-2} \circ \sigma))(\Delta([\sigma^{\circ S}]^{-1}(x)))](\delta^{-1} \otimes \delta^{-1}),
\]

as \( \sigma \) and \( S^{-2} \) commute, again by the quasi-invariance (see Proposition \( \ref{Proposition 5.10} \)).

Use here the result \( \Delta(\sigma(a)) = (S^2 \otimes \sigma)(\Delta a), \ a \in \mathcal{A} \), from Proposition \( \ref{Proposition 7.5} \) which can be also written as \( (S^{-2} \otimes \text{id})\Delta(\sigma(a)) = (\text{id} \otimes \sigma)(\Delta a) \). Then the above expression becomes:

\[
= (\delta \otimes \delta)[(S^{-2} \otimes S^{-2})(\Delta(\sigma([\sigma^{\circ S}]^{-1}(x))))](\delta^{-1} \otimes \delta^{-1})
\]

\[
= (\delta \otimes \delta)[(S^{-2} \otimes S^{-2})(\Delta(\delta^{-1} x \delta))](\delta^{-1} \otimes \delta^{-1}),
\]

where we used the result \( \sigma([\sigma^{\circ S}]^{-1}(x)) = \delta^{-1} x \delta \), from Proposition \( \ref{Proposition 7.8}(4) \).

Note that by Proposition \( \ref{Proposition 7.13} \) (because \( \delta \) is self-adjoint), we have:

\[
\Delta(\delta^{-1} x \delta) = \Delta(\delta^{-1})(\Delta x) \Delta(\delta) = (\delta^{-1} \otimes \delta^{-1})E(\Delta x)E(\delta \otimes \delta) = (\delta^{-1} \otimes \delta^{-1})(\Delta x)(\delta \otimes \delta).
\]

Combining all these observations, we thus have:

\[
(\sigma^{-1} \otimes \sigma^{\circ S})(\Delta x) = \cdots = (\delta \otimes \delta)[(S^{-2} \otimes S^{-2})(\Delta(\delta^{-1} x \delta))](\delta^{-1} \otimes \delta^{-1})
\]

\[
= (\delta \otimes \delta)[(S^{-2} \otimes S^{-2})(\Delta(\delta^{-1} x \delta))](\delta^{-1} \otimes \delta^{-1})
\]

\[
= (\delta \otimes \delta)[(\delta^{-1} \otimes \delta^{-1})[(S^{-2} \otimes S^{-2})(\Delta x)](\delta \otimes \delta)](\delta^{-1} \otimes \delta^{-1})
\]

\[
= \Delta(S^{-2}(x)).
\]

Here, we used the result that \( S(\delta) = \delta^{-1} \) and \( S^2(\delta) = \delta \), from Proposition \( \ref{Proposition 7.13} \) and the property of the antipode that \( (S \otimes S)(\Delta a) = \Delta^{\text{op}}(S(\delta)), \ \forall a \in \mathcal{A} \), from Proposition \( \ref{Proposition 1.12} \) applied twice. \( \square \)

5.5. The KMS weight \( \varphi \). We have been working with an n.s.f. weight \( \tilde{\varphi} \) at the von Neumann algebra level, but the time has come to consider its restriction to the \( C^* \)-algebra level. By restricting the weight \( \tilde{\varphi} \) on the von Neumann algebra \( M = \pi(A)^{\prime\prime} \) to the level of the \( C^* \)-algebra \( A = \overline{\pi(A)}^{\prime\prime} \), we obtain a faithful lower semi-continuous weight \( \varphi \) on \( A \).

As noted earlier, the weight \( \varphi \) extends the linear functional \( \varphi \) on \( A \), in the sense that \( \varphi(\pi(a)) = \varphi(a) \), for \( a \in \mathcal{A} \). For convenience, let us use the same notation for our \( C^* \)-algebra weight as the linear functional at the \( * \)-algebra level. Denote the associated spaces by \( \mathfrak{M}_\varphi = \{ x \in A : \varphi(x^* x) < \infty \} \) and \( \mathfrak{N}_\varphi = \mathfrak{N}^*_\varphi \mathfrak{M}_\varphi \). Write \( \Lambda_\varphi \) to denote the GNS map from \( \mathfrak{M}_\varphi \) into \( \mathcal{H} \), where we can take our Hilbert space to be the same as before.

We can consider the operators \( T, \nabla, J \) as before, because the Hilbert space remains the same. However, as noted in \( \S \ref{Section 3.6} \) we do not know whether the restriction of the modular
automorphism group \( (\tilde{\sigma}_t) \) to the \( C^* \)-algebra level would leave \( A \) invariant, and whether the restriction is norm-continuous. These are not automatic consequences of the modular theory. Earlier, for the weights \( \nu \) and \( \mu \) at the base \( C^* \)-algebra level, we were benefitted by the existence of the canonical idempotent \( E \). However, that is not possible this time. We need a different approach.

Let us return back down to the \( \ast \)-algebra level, and consider the modular element \( \delta \in M(A) \). For its properties, see Appendix (Section 7). Note that due to our quasi-invariance assumption (see discussion given in §5.3, in particular Proposition 5.15), we can use the fact that \( \delta \) is positive self-adjoint. Also, the results of §5.4 can be all used.

Using \( \delta \), define a new Hilbert space \( \mathcal{H}_\delta \), as follows:

**Proposition 5.18.** Let \( a, b \in A \). Then as \( \delta \) is a positive element, we can define a positive sesquilinear form:

\[
(a, b) \mapsto \varphi(b^\ast \delta a).
\]

In this way, we can define a Hilbert space \( \mathcal{H}_\delta \), together with an injective linear map \( \Lambda_\delta : A \to \mathcal{H}_\delta \), having a dense range in \( \mathcal{H}_\delta \), such that

\[
\langle \Lambda_\delta(a), \Lambda_\delta(b) \rangle = \varphi(b^\ast \delta a), \quad \text{for all } a, b \in A.
\]

**Proof.** As \( \delta \) is positive, it is clear that \( \varphi(a^\ast \delta a) \) is positive, for any \( a \in A \). Next, assume that \( \varphi(a^\ast \delta a) = 0 \). Then by the Schwarz inequality, we have, for any \( b \in A \),

\[
|\varphi(b^\ast \delta a)|^2 \leq \varphi(b^\ast \delta b)\varphi(a^\ast \delta a),
\]

so \( \varphi(b^\ast \delta a) = 0 \). Since \( \varphi \) is faithful and since this is true for any \( b \in A \), this means that \( \delta a = 0 \). As \( \delta \) is invertible, we must have \( a = 0 \). We see that \( \varphi(a^\ast \delta a) = 0 \) if and only if \( a = 0 \).

We thus have the positive definiteness, and so we obtain an inner product on \( A \). By completing \( A \) with respect to the induced norm, we thereby obtain the Hilbert space \( \mathcal{H}_\delta \), with the natural inclusion \( \Lambda_\delta : A \to \mathcal{H}_\delta \). □

We define an anti-linear, closed (unbounded) operator \( Z \) from \( \mathcal{H} \) to \( \mathcal{H}_\delta \), in the following proposition:

**Proposition 5.19.** For \( a \in A \), define:

\[
Z_0\Lambda(a) := \Lambda_\delta(S(a^\ast)).
\]

Then:

1. \( Z_0 \) is a well-defined map from \( \Lambda(A) \) into \( \mathcal{H}_\delta \).
2. \( Z_0 \) is a closable, so we can consider its closure \( Z \). Then \( Z \) becomes a closed, densely-defined, injective operator from \( \mathcal{H} \) into \( \mathcal{H}_\delta \), such that \( \Lambda(A) \) forms a core and \( Z \) has a dense range.
3. \( Z \) is anti-linear.
4. \( \Lambda_\delta(A) \) forms a core for \( Z^* \), which is also a densely-defined, injective, and has a dense range, and given by

\[
Z^*\Lambda_\delta(a) = \Lambda(\delta^{-1}S(a^\ast)\delta), \quad \text{for } a \in A.
\]
Proof. As \( S \) is well-defined from \( \mathcal{A} \) onto itself, and since \( \Lambda(\mathcal{A}) \) is dense in \( \mathcal{H} \) while \( \Lambda_\delta(\mathcal{A}) \) is dense in \( \mathcal{H}_\delta \), with respect to the relevant norms, it is clear that \( Z_0 \) is well-defined, densely-defined, and has a dense range. Meanwhile, for \( a, b \in \mathcal{A} \), note that
\[
\langle \Lambda_\delta(S(a^*)), \Lambda_\delta(b) \rangle = \varphi(b^* \delta S(a^*)) = \varphi(S(a^* \delta^{-1} S^{-1}(b^*))) = \varphi(S(a^* \delta^{-1} S(b^*)));
\]
because \( S(\delta) = \delta^{-1} \) (see Proposition 7.13 in Appendix) and \( S(b^*)^* = b \). Since \( \varphi \circ S = \varphi(\cdot \delta) \), this becomes:
\[
\langle \Lambda_\delta(S(a^*)), \Lambda_\delta(b) \rangle = \varphi(a^* \delta^{-1} S(b^*) \delta) = \langle \Lambda(\delta^{-1} S(b^*) \delta), \Lambda(a) \rangle.
\]
From this, it is not difficult to show that \( Z_0 \) is closable. Let us denote by \( Z \) its closure. Then \( Z \) becomes closed, densely-defined, and has a dense range, with \( \Lambda(\mathcal{A}) \) forming a core. It is anti-linear since \( S \).

With \( Z^* \Lambda_\delta(b) = \Lambda(\delta^{-1} S(b^*) \delta) \), for \( b \in \mathcal{A} \), this result can be expressed as
\[
\langle Z \Lambda(a), \Lambda_\delta(b) \rangle = \langle Z^* \Lambda_\delta(b), \Lambda(a) \rangle = \langle \Lambda(a), Z^* \Lambda_\delta(b) \rangle.
\]
From this, it is easy to notice that \( Z \) is injective. It is also apparent that \( Z^* \) is itself a closed, injective, anti-linear operator, that is densely-defined with a dense range. \( \square \)

Define \( P := Z^* Z \). As a consequence of Proposition 5.19 we see that \( P \) is a closed, positive, injective operator on \( \mathcal{H} \), which is densely-defined and has a dense range. It is clear that \( \Lambda(\mathcal{A}) \) forms a core for \( P \). Moreover, we have:
\[
P \Lambda(a) = Z^* Z \Lambda(a) = Z^* \Lambda(\delta^{-1} S(a^*) \delta) = \Lambda(\delta^{-1} S(\delta^{-1} S^{-1}(a^* \delta))) = \Lambda(\delta^{-1} S^{-2}(a \delta)), \quad (5.8)
\]
because \( S(x^*) = S^{-1}(x^*) \), which is applied twice. Next proposition gives a useful relationship between the operators \( W, \nabla (= T^* T), \) and \( P (= Z^* Z) \):

**Proposition 5.20.** For any \( a, b \in \mathcal{A} \), we have:
\[
W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)) = (\nabla \otimes \nabla) W(\Lambda(a) \otimes \Lambda(b)).
\]

**Proof.** Let \( a, b, c, d \in \mathcal{A} \) be arbitrary. Then
\[
\begin{align*}
\langle W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle &= \langle T^* T \Lambda(a) \otimes Z^* Z \Lambda(b), W^* (\Lambda(c) \otimes \Lambda(d)) \rangle \\
&= \langle T \Lambda(a) \otimes Z \Lambda(b), (T \otimes Z)(\Lambda \otimes \Lambda)((\Delta d)(c \otimes 1)) \rangle \\
&= \langle (\Lambda \otimes \Lambda_\delta)((c^* \otimes 1)(\text{id} \otimes S)(\Delta(d^*))), \Lambda(a^*) \otimes \Lambda_\delta(S(b^*))) \rangle,
\end{align*}
\]
using the characterization of \( W^* \) as in Proposition 3.6, the definitions of \( T \) and \( Z \), and the fact that \( S(b^*)^* = S^{-1}(b) \). Note that the inner product is in \( \mathcal{H} \otimes \mathcal{H}_\delta \). Continuing, this becomes:
\[
\text{RHS} = (\varphi \otimes \varphi)((a \otimes S^{-1}(b))(1 \otimes \delta)(c^* \otimes 1)(\text{id} \otimes S)(\Delta(d^*))) \\
= (\varphi \otimes \varphi)((ac^* \otimes S^{-1}(b) \delta)(\text{id} \otimes S)(\Delta(d^*))).
\]
We thus have, so far:
\[
\langle W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle = (\varphi \otimes \varphi)((ac^* \otimes S^{-1}(b) \delta)(\text{id} \otimes S)(\Delta(d^*))). \quad (5.9)
\]
Meanwhile, we have:
\[
\langle (\nabla \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle = \langle (T^* T \otimes T^* T)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle 
\]
\[
= \langle (T \otimes T)(\Lambda \otimes \Lambda)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1)), T\Lambda(c) \otimes T\Lambda(d) \rangle 
\]
\[
= \langle \Lambda(c^*) \otimes \Lambda(d^*), (\Lambda \otimes \Lambda)((a^* \otimes 1)(S \otimes \text{id})(\Delta(b^*))) \rangle 
\]
\[
= (\varphi \otimes \varphi)((S^{-1} \otimes \text{id})(\Delta b)(a \otimes 1)(c^* \otimes d^*)) = \varphi(S^{-1}[(\text{id} \otimes \varphi)((\Delta b)(1 \otimes d^*))ac^*]) 
\]
by the characterization of \( W \) given in Proposition 3.6(2), and the fact that \( S^{-1}(x^*) = S(x^*) \).

By using a characterization of \( S \) given in Proposition 1.12(1), we can go further:
\[
(RHS) = \varphi(S^{-2}[(\text{id} \otimes \varphi)((1 \otimes b)\Delta(d^*))ac^*]) 
\]
\[
= (\varphi \otimes \varphi)((1 \otimes b)(S^{-2} \otimes \text{id})(\Delta(d^*)))(ac^* \otimes 1) 
\]
\[
= (\varphi \otimes \varphi)((ac^* \otimes b(\sigma \otimes \text{id})[(S^{-2} \otimes \text{id})(\Delta(d^*))])]. \quad (5.10)
\]

Note here that we used the modular automorphism \( \sigma \).

In Proposition 5.17, we saw that \( (\sigma^{-1} \otimes \sigma^{-\infty}S)(\Delta x) = \Delta(S^{-2}(x)) = (S^{-2} \otimes S^{-2})(\Delta x) \), for \( x \in A \). As a consequence, it follows that for all \( x \in A \),
\[
(\sigma \otimes \text{id})[(S^{-2} \otimes \text{id})(\Delta x)] = (\text{id} \otimes (S^2 \circ \sigma^{-\infty}S))(\Delta x) = (\text{id} \otimes (S \circ \sigma^{-1}))(\text{id} \otimes S)(\Delta x),
\]
because \( \sigma^{\infty}S = S^{-1} \circ \sigma^{-1} \circ S \) (see proof of Proposition 7.1). Then Equation (5.10) becomes:
\[
\langle (\nabla \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle = \ldots 
\]
\[
= (\varphi \otimes \varphi)((ac^* \otimes b)(\text{id} \otimes (S \circ \sigma^{-1}))[(\text{id} \otimes S)(\Delta(d^*))]) 
\]
\[
= (\varphi \otimes (\varphi \circ S))((ac^* \otimes 1)(\text{id} \otimes \sigma^{-1})[(\text{id} \otimes S)(\Delta(d^*))](1 \otimes S^{-1}(b))) 
\]
\[
= (\varphi \otimes \varphi)((ac^* \otimes 1)(\text{id} \otimes \sigma^{-1})[(\text{id} \otimes S)(\Delta(d^*))](1 \otimes S^{-1}(b)\delta)) 
\]
\[
= (\varphi \otimes \varphi)((ac^* \otimes S^{-1}(b)\delta)(\text{id} \otimes S)(\Delta(d^*))).
\]

We thus have:
\[
\langle (\nabla \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle = (\varphi \otimes \varphi)((ac^* \otimes S^{-1}(b)\delta)(\text{id} \otimes S)(\Delta(d^*))). \quad (5.11)
\]

Compare Equations (5.9) and (5.11). We conclude that
\[
\langle W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle = \langle (\nabla \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)), \Lambda(c) \otimes \Lambda(d) \rangle, 
\]
true for any \( c, d \in A \). So we have:
\[
W(\nabla \otimes P)(\Lambda(a) \otimes \Lambda(b)) = (\nabla \otimes \nabla)W(\Lambda(a) \otimes \Lambda(b)),
\]
for any \( a, b \in A \).

Note, by the way that since we are working with unbounded operators \( P \) and \( \nabla \), which are only densely-defined, the above result does not necessarily mean \( W(\nabla \otimes P) = (\nabla \otimes \nabla)W \). To be precise, considering the domains, this should be written as \( W(\nabla \otimes P) \subseteq (\nabla \otimes \nabla)W \).

The situation becomes similar to what we had earlier for \( W(L \otimes \nabla) \subseteq (L \otimes \nabla)W \) (in (5.11)). As was in that case, the non-unitarity of \( W \) (being only a partial isometry) means the need
for a more roundabout approach. By following more or less the same procedure (similar to Propositions 4.13 – 4.17 in [11]), we obtain the following:

**Proposition 5.21.** We have:

1. The restrictions \((\nabla \otimes P)|_{\text{Ran}(E)}, (\nabla \otimes P)|_{\text{Ker}(W)}, (\nabla \otimes \nabla)|_{\text{Ran}(G)}, (\nabla \otimes \nabla)|_{\text{Ker}(W^*)}\) become valid operators on the subspaces \(\text{Ran}(E), \text{Ker}(W), \text{Ran}(G), \text{Ker}(W^*)\), respectively.

2. For any \(z \in \mathbb{C}\), we have:
   \[W(\nabla^2 \otimes P^2) \subseteq (\nabla^2 \otimes \nabla^2)W \quad \text{and} \quad W^*(\nabla^2 \otimes \nabla^2) \subseteq (\nabla^2 \otimes P^2)W^*.\]

3. Let \(t \in \mathbb{R}\). The following results hold on the whole space \(\mathcal{H} \otimes \mathcal{H}\):
   \[
   (\nabla^{it} \otimes \nabla^{it})W(\nabla^{-it} \otimes P^{-it}) = W,
   \]
   \[
   (\nabla^{it} \otimes 1)W(\nabla^{-it} \otimes 1) = (1 \otimes \nabla^{-it})W(1 \otimes P^{it}).
   \]

**Remark.** We will skip the detailed proof. Modify the procedure taken in Propositions 4.13 – 4.17 in [11], and see also the comments made in the remark following Proposition 4.3. While not exactly the same, the overall idea is similar. This is fundamentally about dealing with the unbounded operators in relation to a partial isometry, not really using any specific results on quantum groupoids. Note that when \(z = it\), the operators \(\nabla^{it}\) and \(P^{it}\) become bounded, so the domain issue becomes simpler.

As a consequence of Proposition 5.21, we are now ready to resolve our question on our modular automorphism group. For this, consider \(\omega \in \mathcal{B}(\mathcal{H})_\ast\) and let \(t \in \mathbb{R}\). Apply \(\text{id} \otimes \omega\) to the result (3) of Proposition 5.21. Then we have

\[
\nabla^{it}(\text{id} \otimes \omega)(W)\nabla^{-it} = (\text{id} \otimes \omega)(W),
\]

where \(\theta \in \mathcal{B}(\mathcal{H})_\ast\) is such that \(\theta(X) = \omega(\nabla^{-it}XP^{it})\), for \(X \in \mathcal{B}(\mathcal{H})\).

As elements of the form \((\text{id} \otimes \omega)(W), \omega \in \mathcal{B}(\mathcal{H})_\ast\), generate the \(C^\ast\)-algebra \(A\), the observation above shows that for any \(t \in \mathbb{R}\), we have \(\nabla^{it}a\nabla^{-it} \in A\), for any \(a \in A\). We can also observe that \(t \mapsto \nabla^{it}a\nabla^{-it}\) is norm-continuous.

We can thus justify the following:

**Definition 5.22.** Define the norm-continuous one-parameter group \(\sigma = (\sigma_t)\) on the \(C^\ast\)-algebra \(A\), by

\[
\sigma_t(x) = \nabla^{it}a\nabla^{-it},
\]

for all \(t \in \mathbb{R}, a \in A\).

The one-parameter group \((\sigma_t)\) is a restriction of the modular automorphism group \((\tilde{\sigma}_t)\) for the n.s.f. weight \(\tilde{\varphi}\). With \((\sigma_t)\), the faithful lower semi-continuous weight \(\varphi\) becomes a KMS weight on \(A\). Its KMS properties are inherited from the properties of \(\tilde{\varphi}\). In particular, we have \(\varphi \circ \sigma_t = \varphi\); and for any \(x \in \mathcal{D}(\sigma_{\tilde{\varphi}})\), we have \(\varphi(x^*x) = \varphi(\sigma_{\tilde{\varphi}}(x)\sigma_{\tilde{\varphi}}(x)^*)\). Also note that \(\Lambda_{\varphi}(\sigma_t(x)) = \nabla^{it}\Lambda_{\varphi}(x)\), for any \(t \in \mathbb{R}\) and any \(x \in \mathcal{N}_{\varphi}\).

**Proposition 5.23.** Let \(\varphi\) denote the weight on the \(C^\ast\)-algebra \(A\), given by \(\varphi = \tilde{\varphi}|_A\). Similarly, let \(\psi\) denote the weight on the \(C^\ast\)-algebra \(A\), given by \(\psi = \tilde{\psi}|_A\).
Then $\varphi$ and $\psi$ are KMS weights on $A$. Their modular automorphism group $(\sigma_t)_{t \in \mathbb{R}}$ for $\varphi$ and $(\sigma'_t)_{t \in \mathbb{R}}$ for $\psi$ are restrictions of $(\tilde{\sigma}_t)_{t \in \mathbb{R}}$ and $(\tilde{\sigma}'_t)_{t \in \mathbb{R}}$, respectively, and they leave $A$ invariant and are norm-continuous.

We also have $\psi = \varphi \circ R$, where $R$ is the unitary antipode.

Proof. With Definition 5.22 and the paragraph following it, we showed that $\varphi$ is a KMS weight on $A$, with the modular automorphism group $(\sigma_t)_{t \in \mathbb{R}}$. Its properties are inherited from those of the n.s.f. weight $\tilde{\varphi}$.

Since we know $\tilde{\psi} = \tilde{\varphi} \circ \tilde{R}$ and $\tilde{\sigma}'_t = \tilde{R} \circ \tilde{\sigma}_{-t} \circ \tilde{R}$, for all $t \in \mathbb{R}$, while $\tilde{R}$ restricts to the anti-isomorphism $R$ on $A$, it is evident that $\psi = \varphi \circ R$ and $\sigma'_t = R \circ \sigma_{-t} \circ R, \forall t \in \mathbb{R}$. It is also clear that $(\sigma'_t)_{t \in \mathbb{R}}$ leaves $A$ invariant and is norm-continuous. $\square$

6. The $C^*$-algebraic locally compact quantum groupoid

In [10], [11], Van Daele and the author developed a $C^*$-algebraic framework of a class of $C^*$-algebraic locally compact quantum groupoids (quantum groupoids of separable type). The definition is given below (see Definition 4.8 of [10] and Definition 1.2 of [11]):

Definition 6.1. The data $(A, \Delta, E, B, \nu, \varphi, \psi)$ defines a locally compact quantum groupoid of separable type, if

- $A$ is a $C^*$-algebra.
- $\Delta : A \to M(A \otimes A)$ is a comultiplication on $A$.
- $B$ is a non-degenerate $C^*$-subalgebra of $M(A)$.
- $\nu$ is a KMS weight on $B$.
- $E$ is the canonical idempotent of $(A, \Delta)$. That is,
  - (1) $\Delta(A)(A \otimes A)$ is dense in $E(A \otimes A)$ and $(A \otimes A)\Delta(A)$ is dense in $(A \otimes A)E$;
  - (2) there exists a $C^*$-subalgebra $C \cong B^{\text{op}}$ contained in $M(A)$, with a *-anti-isomorphism $R = R_{BC} : B \to C$, so that $E \in M(B \otimes C)$ and the triple $(E, B, \nu)$ forms a separability triple;
  - (3) $E \otimes 1$ and $1 \otimes E$ commute, and we have:
    \[(\text{id} \otimes \Delta)(E) = (E \otimes 1)(1 \otimes E) = (1 \otimes E)(E \otimes 1) = (\Delta \otimes \text{id})(E).\]
- $\varphi$ is a KMS weight, and is left invariant.
- $\psi$ is a KMS weight, and is right invariant.
- There exists a (unique) one-parameter group of automorphisms $(\theta_t)_{t \in \mathbb{R}}$ of $B$ such that $\nu \circ \theta_t = \nu$ and that $\sigma^\nu_t|_B = \theta_t, \forall t \in \mathbb{R}$.

Remark. We will refer the details to the main papers. For instance, the notion of the canonical idempotent is summarized in Definition 3.7 of [10].

This definition is similar, but different from that of measured quantum groupoids, in the von Neumann algebra setting [17], [5]. The von Neumann algebra setting may be a bit more general, which is rather related to the algebraic framework of multiplier Hopf algebroids [25]. There are some subtle differences between the two locally compact frameworks.
Let us re-cap the constructions we carried out so far: Starting from a purely algebraic, weak multiplier Hopf $^\ast$-algebra with a faithful integral (see Definition 1.10), without any additional conditions other than the quasi-invariance assumption (see Subsection §5.3), we wish to verify that the data indeed gives us a $C^\ast$-algebraic quantum groupoid of separable type, as in Definition 6.1 above.

Our $C^\ast$-algebra was defined in Definition 3.9, extending the $^\ast$-algebra $A$. The comultiplication $\Delta : A \to M(A \otimes A)$ was given in Definition 3.10 and Theorem 3.14. The base $C^\ast$-algebras $B$ and $C$ were defined in Definition 2.3. They are equipped with KMS weights $\nu$ and $\mu$, respectively, which actually extends the distinguished linear functionals at the $^\ast$-algebra level (see Proposition 2.7). There exists a $C^\ast$-anti-isomorphism $R : B \to C$, while the canonical idempotent $E$ at the $^\ast$-algebra level extends to the separability idempotent $E \in M(B \otimes C)$. See Proposition 2.8.

The idempotent $E$ was further shown to satisfy additional properties, making it a valid canonical idempotent at the $C^\ast$-level. See Proposition 3.17.

Finding a suitable left-invariant weight $\varphi$ and a right-invariant weight $\psi$ was rather tricky. We first extended the left integral $\varphi$ at the $^\ast$-algebra level to an n.s.f. weight $\tilde{\varphi}$, but it took a while to establish that its restriction to the $C^\ast$-algebra level is a KMS weight because it relied on results that use the quasi-invariance assumption. As for $\psi$, we could not just attempt to extend the right integral at the $^\ast$-algebra level. Instead, we used results at the purely algebraic level saying the existence of both a left integral and a right integral imply the existence of the antipode (Theorem 3.15 of [8]), then carried out a polar decomposition of the antipode to have it established at the operator algebra level, which allowed us obtain various technical results. In the end, the right weight was chosen to be $\varphi \circ R$, where $R$ is the unitary antipode that come from the polar decomposition of the antipode. This weight does not necessarily have to be an extension of the original right integral.

Along the way, an important role was played by the modular element $\delta$ at the $^\ast$-algebra level, as well as its operator algebraic counter part $\tilde{\delta}$, which provided a relationship between the extended weights $\tilde{\varphi}$ and $\tilde{\psi}$. Having the quasi-invariance assumption was necessary for this to work, which was to be expected because some form of a quasi-invariance had to be assumed already in the framework of classical locally compact groupoids.

What remains to be shown is the result verifying that the KMS weights $\varphi$ and $\psi$ thus obtained are indeed left invariant and right invariant, respectively. This is not too difficult, because we already have corresponding results at the von Neumann algebra level (Proposition 5.2 and Proposition 5.8). See below:

**Proposition 6.2.** Let $\varphi$ and $\psi$ be the KMS weights established in Proposition 5.23 Then

1. For any $a \in \mathcal{M}_\varphi$, we have $\Delta a \in \mathcal{M}_{\text{id} \otimes \varphi}$ and $(\text{id} \otimes \varphi)(\Delta a) \in M(C)$.
2. For any $a \in \mathcal{M}_\psi$, we have $\Delta a \in \mathcal{M}_{\psi \otimes \text{id}}$ and $(\psi \otimes \text{id})(\Delta a) \in M(B)$.

**Proof.** (1). We showed in Proposition 5.2 a corresponding result for the weight $\tilde{\varphi}$. We can just use the same proof, replacing $\tilde{\varphi}$ with $\varphi$ and replacing the strong convergence with the norm convergence. Since $\overline{M(C)}^\| = M(C)$, we can see that for any $a \in \mathcal{M}_\varphi$, we have $\Delta a \in \mathcal{M}_{\text{id} \otimes \varphi}$.
and that 
\[(\text{id} \otimes \varphi)(\Delta a) \in M(C).\]

(2) Since \(\psi = \varphi \circ R\), and since we know \((R \otimes R)(\Delta x) = \Delta^{\text{cop}}(R(x))\) from Proposition 4.12, the right invariance of \(\psi\) follows from the left invariance of \(\varphi\).

More specifically, if \(a \in M_{\psi}\), which means \(R(a) \in M_{\varphi}\), we would have \(\Delta(R(a)) \in M_{\text{id} \otimes \varphi}\), and that \((\text{id} \otimes \varphi)(\Delta(R(a))) \in M(C)\). But then, we have
\[(\psi \otimes \text{id})(\Delta a) = ((\varphi \circ R) \otimes \text{id})(\Delta a) = R((\varphi \otimes \text{id})(R \otimes R)(\Delta a))) = R((\text{id} \otimes \varphi)(\Delta(R(a))) \in R(M(C)) = M(B).\]

Finally, observe the last requirement in Definition 6.1 about the restriction of \(\sigma\) to the base \(C^*\)-algebra \(B\). But this is an immediate consequence of none other than the quasi-invariance assumption we required earlier, as \(\sigma|_B\) would play the role of an analytic generator for \((\theta_t)\).

We noted earlier that as we are developing a locally compact theory, we need some form of a quasi-invariance condition, just as in the classical locally compact groupoid case [21], [19]. The last condition in Definition 6.1 is needed for that purpose. What we are noticing is that for a purely algebraic object of a weak multiplier Hopf \(*\)-algebra (with a faithful integral) to allow a construction of a \(C^*\)-algebraic quantum groupoid, some form of the quasi-invariance property is required even at the algebra level, which turns out to be the quasi-invariance assumption we required in Section 5.3.

Summarizing, we have the following conclusion:

**Theorem 6.3.** Let \((A, \Delta, E)\) be a weak multiplier Hopf \(*\)-algebra with a single faithful integral, as in Definition 1.10. With the quasi-invariance assumption (as in 5.3), we can construct from it a \(C^*\)-algebraic quantum groupoid of separable type, in the sense of [10], [11].

Now that the construction is done, we can take full advantage of the already-developed theory in [11]. There are alternative representations for the \(C^*\)-algebra and there are multiple equivalent characterizations for the antipode, among other results. Some more relations between the base algebras \(B, C\) and the total algebra \(A\) can be found.

On the other hand, we did not pursue the duality aspect in this paper (see §1.6). Ideally, it would make the picture complete if one can confirm that the \(C^*\)-extension of the dual weak multiplier Hopf \(*\)-algebra \((\hat{A}, \hat{\Delta})\) is the dual in the \(C^*\)-context of the \(C^*\)-algebraic quantum groupoid. We will postpone that project to a future occasion, after the paper on the duality theory for the \(C^*\)-algebraic quantum groupoids ([12], in preparation) is finished.

7. **Appendix: The modular element at the \(*\)-algebra level**

In this Appendix, we gather some purely algebraic results for a weak multiplier Hopf \(*\)-algebra. As in §1.5 we assume the existence of a single faithful positive left integral \(\varphi\).

In the purely algebraic setting, we noted in Proposition 1.13 the existence of the modular automorphism \(\sigma\) for \(\varphi\). We also noted the existence of an invertible element \(\delta \in M(A)\), called
the modular element, relating the functionals \( \varphi \) and \( \varphi \circ S \). The modular element behaves like a modular function in the classical setting.

In this Appendix, we gather some results regarding the functional \( \varphi \circ S \), the modular automorphisms for \( \varphi \) and \( \varphi \circ S \), and the modular element \( \delta \). While what appear below are all purely algebraic results, and some results are likely already known, the author could not find a good reference for some of these results (especially regarding \( \delta \)), and some results here may be new. As such, unlike in [11], all the proofs are given here.

### 7.1. The functional \( \varphi \circ S \)

Consider the functional \( \varphi \circ S \), where \( \varphi \) is our faithful left integral \( \varphi \), and \( S \) is the antipode map. Since \( S \) is a bijection on \( A \), we see that \( \varphi \circ S \) is also a faithful linear functional on \( A \). However, there is no reason to expect that it is positive.

Recall the existence of the modular automorphism, \( \sigma \), for the functional \( \varphi \). This means that there exists a similar object for the functional \( \varphi \circ S \), which would be the modular automorphism \( \sigma^{\varphi \circ S} \) for \( \varphi \circ S \).

**Proposition 7.1.** There exists an automorphism \( \sigma^{\varphi \circ S} \) of \( A \), such that

\[
(\varphi \circ S)(ab) = (\varphi \circ S)(b\sigma^{\varphi \circ S}(a)), \quad \forall a, b \in A,
\]

and also \( (\varphi \circ S)(\sigma^{\varphi \circ S}(a)) = (\varphi \circ S)(a) \), for all \( a \in A \).

**Proof.** For all \( a, b \in A \), using Proposition [1.13] and knowing \( S \) is an anti-isomorphism, we have

\[
(\varphi \circ S)(ab) = \varphi(S(b)S(a)) = \varphi(\sigma^{-1}(S(a))S(b)) = (\varphi \circ S)(bS^{-1}(\sigma^{-1}(S(a)))).
\]

This shows that with \( \sigma^{\varphi \circ S}(a) := (S^{-1} \circ \sigma^{-1} \circ S)(a) \), \( a \in A \), we have the desired automorphism. We can also see that

\[
(\varphi \circ S)(\sigma^{\varphi \circ S}(a)) = (\varphi \circ S)((S^{-1} \circ \sigma^{-1} \circ S)(a)) = \varphi(\sigma^{-1}(S(a))) = (\varphi \circ S)(a).
\]

\( \square \)

It turns out that \( \varphi \circ S \) is a right integral, which is sort of expected:

**Proposition 7.2.** The functional \( \varphi \circ S \) is right invariant, in the sense of Proposition [1.12]:

\[
((\varphi \circ S) \otimes \text{id})(\Delta a) \in M(B), \quad \text{for all } a \in A.
\]

**Proof.** Let \( a \in A \). By Proposition [1.12](4), we have:

\[
((\varphi \circ S) \otimes \text{id})(\Delta a) = S^{-1}((\varphi \otimes \text{id})((S \otimes S)(\Delta a))) = S^{-1}((\text{id} \otimes \varphi)(\Delta(S(a))))
\]

Since \( (\text{id} \otimes \varphi)(\Delta(S(a))) \in M(B) \) by the left invariance of \( \varphi \), and since \( S^{-1}(M(B)) = M(C) \) by Proposition [1.12](3), this means that \( ((\varphi \circ S) \otimes \text{id})(\Delta a) \in M(C) \).

\( \square \)

As \( \varphi \circ S \) is a right integral on \( A \), the results for right integrals apply:

**Proposition 7.3.** The function \( \varphi \circ S \) is a right integral. Therefore the following results hold:

1. \( \nu(((\varphi \circ S) \otimes \text{id})(\Delta x)) = (\varphi \circ S)(x) \), for all \( x \in A \).
(2) For all \( a \in \mathcal{A} \), we have
\[
((\varphi \circ S) \otimes \text{id})(\Delta a) = ((\varphi \circ S) \otimes \text{id})((a \otimes 1)F_1) = ((\varphi \circ S) \otimes \text{id})(F_3(a \otimes 1)),
\]
where \( F_1 = (\text{id} \otimes S)(E) \in M(\mathcal{A} \otimes \mathcal{A}) \) and \( F_3 = (\text{id} \otimes S^{-1})(E) \in M(\mathcal{A} \otimes \mathcal{A}) \).

(3) There exists a unique invertible element \( \delta \in M(\mathcal{A}) \) such that \( (\varphi \circ S)(x) = \varphi(x\delta) \), for all \( x \in \mathcal{A} \). We refer to \( \delta \) as the modular element.

(4) There is an alternative characterization of the antipode map \( S \), in terms of the functional \( \varphi \circ S \):
\[
S : ((\varphi \circ S) \otimes \text{id})((a \otimes 1)(\Delta b)) \mapsto ((\varphi \circ S) \otimes \text{id})((\Delta a)(b \otimes 1)), \quad \forall a, b \in \mathcal{A}.
\]

**Proof.** See Proposition 1.8 for (1), and Proposition 1.7 for (2), which are consequences of \( \varphi \circ S \) being a right integral. For (3), see Proposition 1.15 for (2). The uniqueness of \( \delta \) is due to \( \varphi \) and \( \varphi \circ S \) being faithful. Finally, (4) is basically Proposition 1.12(2), which is really a result that is true for any right integral (see Proposition 1.5 of [36]). \( \square \)

### 7.2. Relationships between \( \sigma \), \( \sigma^{\varphi S} \) and the antipode \( S \)

Let \( \sigma \), \( \sigma^{\varphi S} \) be the modular automorphisms for \( \varphi \) and \( \varphi \circ S \), respectively. Recall also that \( \sigma^{\varphi S} = S^{-1} \circ \sigma^{-1} \circ S \), which can be seen in the proof of Proposition 7.1

Here are some results regarding their restrictions to the level of the base algebras:

**Proposition 7.4.**

1. The restriction of \( \sigma \) to \( \mathcal{C} \) leaves \( \mathcal{C} \) invariant, and we have:
   \[
   \sigma|_{\mathcal{C}} = S^2|_{\mathcal{C}} = S_{\mathcal{B}} \circ S_{\mathcal{C}} = \sigma^\mu.
   \]

2. The restriction of \( \sigma^{\varphi S} \) to \( \mathcal{B} \) leaves \( \mathcal{B} \) invariant, and we have:
   \[
   \sigma^{\varphi S}|_{\mathcal{B}} = S^{-2}|_{\mathcal{B}} = S_{\mathcal{B}}^{-1} \circ S_{\mathcal{C}}^{-1} = \sigma^\nu.
   \]

3. We have: \( \mu \circ \sigma|_{\mathcal{C}} = \mu \) and \( \nu \circ \sigma^{\varphi S}|_{\mathcal{B}} = \nu \).

**Proof.** (1). Let \( y \in \mathcal{C} \) and let \( a \in \mathcal{A} \) be arbitrary. Note that
\[
\varphi(ya) = \mu((\text{id} \otimes \varphi)(\Delta(ya))) = \mu((\text{id} \otimes \varphi)((y \otimes 1)(\Delta a))) = \mu(y(\text{id} \otimes \varphi)(\Delta a))
\]
\[
= \mu((\text{id} \otimes \varphi)(\Delta a)\sigma^\mu(y)) = \mu((\text{id} \otimes \varphi)((\Delta a)(\sigma^\mu(y) \otimes 1)))
\]
\[
= \mu((\text{id} \otimes \varphi)(\Delta(a\sigma^\mu(y)))) = \varphi(a\sigma^\mu(y)).
\]
We used here the result of Proposition 1.8, and the fact that \( \Delta y = (y \otimes 1)E = E(y \otimes 1) \) and \( \Delta(\sigma^\mu(y)) = (\sigma^\mu(y) \otimes 1)E = E(\sigma^\mu(y) \otimes 1) \), because \( y, \sigma^\mu(y) \in \mathcal{C} \) (Proposition 1.4).

Meanwhile, note that \( \varphi(ya) = \varphi(a\sigma(y)) \). Combining the two observations, we see that
\[
\varphi(a\sigma(y)) = \varphi(a\sigma^\mu(y)).
\]
Since \( \varphi \) is faithful and since the result is true for any \( a \in \mathcal{A} \), this shows that \( \sigma(y) = \sigma^\mu(y) \), for all \( y \in \mathcal{C} \), that is, \( \sigma|_{\mathcal{C}} = \sigma^\mu \). We already know from 1.3 that \( \sigma^\mu = S_{\mathcal{B}} \circ S_{\mathcal{C}} \). We also know that \( S|_{\mathcal{B}} = S_{\mathcal{B}} \) and \( S|_{\mathcal{C}} = S_{\mathcal{C}} \) (Proposition 1.12), so we have: \( \sigma|_{\mathcal{C}} = \sigma^\mu = S^2|_{\mathcal{C}} \).

(2). The proof for the restriction \( \sigma^{\varphi S}|_{\mathcal{B}} = \sigma^\nu = S^{-2}|_{\mathcal{B}} \) is similar, with Proposition 7.3(1).

(3). As it is known that \( \mu \circ \sigma^\mu = \mu \) and \( \nu \circ \sigma^\nu = \nu \), the results follow immediately from (1) and (2). \( \square \)
The following results show how $\sigma$ and $\sigma^{\varphi S}$ behave when the comultiplication map is applied:

**Proposition 7.5.** We have:

1. $\Delta(\sigma(a)) = (S^2 \otimes \sigma)(\Delta a)$, for all $a \in A$,
2. $\Delta(\sigma^{\varphi S}(a)) = (\sigma^{\varphi S} \otimes S^{-2})(\Delta a)$, for all $a \in A$.

**Proof.** (1). Let $a,x \in A$ be arbitrary. By a characterization of the antipode $S$ given in Proposition 1.12(1), we have:

\[
(id \otimes \varphi)((1 \otimes x)\Delta((\sigma(a)))) = (id \otimes \varphi)((\Delta x)(1 \otimes \sigma(a))) = S((id \otimes \varphi)((\Delta x)(1 \otimes \sigma(a))))
\]

\[
= S((id \otimes \varphi)((1 \otimes a)(\Delta x))) = S^2((id \otimes \varphi)((\Delta a)(1 \otimes x)))
\]

\[
= (id \otimes \varphi)((1 \otimes x)(S^2 \otimes \sigma)(\Delta a)).
\]

Since $\varphi$ is faithful and since $x \in A$ is arbitrary, this shows that $\Delta(\sigma(a)) = (S^2 \otimes \sigma)(\Delta a)$, for all $a \in A$.

(2). Proof for $\Delta(\sigma^{\varphi S}(a)) = (\sigma^{\varphi S} \otimes S^{-2})(\Delta a)$, $\forall a \in A$, is similar, using an alternative characterization of $S$, namely, $S(((\varphi \circ S) \otimes \text{id})(a \otimes 1)(\Delta x)) = ((\varphi \circ S) \otimes \text{id})(\Delta a)(x \otimes 1)$, noted in Proposition 7.3(4). □

**Remark.** For $a \in A$, we know that $\sigma(a) = \hat{\sigma}_{-i}(\pi(a))$ and that $\tau_{-i}(\pi(a)) = S^2(\pi(a)) = S^2(a)$ (by the polar decomposition of $S$). As such, the result (1) above is essentially Proposition 4.10 when $t = -i$. The second result is analogous to Proposition 5.4.

**Corollary.** We have:

\[
\Delta(\sigma(S^{-2}(a))) = (id \otimes (\sigma \circ S^{-2}))(\Delta a), \quad \forall a \in A.
\]

**Proof.** By Proposition 1.12(4), applied twice, we know $\Delta(S^{-2}(a)) = (S^{-2} \otimes S^{-2})(\Delta a)$. Combine this result with (1) of Proposition 7.5. □

### 7.3. Some results on the modular element

In Proposition 7.3(3), we noted the existence of a unique invertible element $\delta \in M(A)$ such that $(\varphi \circ S)(a) = \varphi(a \delta)$, for all $a \in A$. In what follows, we will gather some additional results about $\delta$.

Let us begin with a lemma, which gives a similar result for $\varphi \circ S^{-1}$, also a right invariant functional:

**Lemma 7.6.** We have: $(\varphi \circ S^{-1})(a) = \varphi(\delta^* a)$, for all $a \in A$.

**Proof.** Let $a \in A$ be arbitrary. Recall from Proposition 1.12(5) that $S(S(a)^*)^* = a$. It follows that $S^{-1}(a)^* = S(a^*)$. As $\varphi$ is a positive functional, we thus have:

\[
(\varphi \circ S^{-1})(a) = \varphi(S^{-1}(a)) = \varphi(S^{-1}(a)^*) = \varphi(S(a^*)) = \varphi(a^* \delta) = \varphi((\delta^* a)^*) = \varphi(\delta^* a).
\]

□
In the below is a result showing what happens when the antipode map $S$, when extended to the multiplier algebra level, is applied to $\delta$:

**Proposition 7.7.** We have: $S(\delta) = (\delta^*)^{-1}$.

*Proof.* Let $x \in \mathcal{A}$ be arbitrary. As $S$ is anti-multiplicative, we have:

$$\varphi(x) = (\varphi \circ S)(S^{-1}(x)) = \varphi(S^{-1}(x)\delta) = (\varphi \circ S^{-1})(S(\delta)x) = \varphi(\delta^* S(\delta)x),$$

where we used the result of Lemma 7.6.

As $\varphi$ is faithful and since $x \in \mathcal{A}$ is arbitrary, it follows that $\delta^* S(\delta) = 1$. So we have: $S(\delta)^{-1} = \delta^*$ and $S(\delta) = (\delta^*)^{-1}$. \hfill $\square$

Next, we gather some results regarding the modular automorphisms $\sigma$ and $\sigma^{\varphi S}$:

**Proposition 7.8.** Under the modular automorphisms $\sigma$ and $\sigma^{\varphi S}$, which can be naturally extended to the multiplier algebra level, we have:

1. $\sigma^{-1}(\delta) = [\sigma^{\varphi S}]^{-1}(\delta)$;
2. $\sigma([\sigma^{\varphi S}]^{-1}(\delta)) = \delta$ and $\sigma^{\varphi S}(\sigma^{-1}(\delta)) = \delta$;
3. $\sigma^{\varphi S}(a) = \delta \sigma(a) \delta^{-1}$ and $\sigma(a) = \delta^{-1} \sigma^{\varphi S}(a) \delta$, for any $a \in \mathcal{A}$;
4. $\sigma([\sigma^{\varphi S}]^{-1}(a)) = \delta^{-1} a \delta$, for any $a \in \mathcal{A}$;

*Proof.* (1). Let $a \in \mathcal{A}$. We have:

$$(\varphi \circ S)(a) = \varphi(a \delta) = \varphi(\sigma^{-1}(\delta)a).$$

Meanwhile, we have:

$$(\varphi \circ S)(a) = (\varphi \circ S)(a \delta^{-1} \delta) = (\varphi \circ S)(([\sigma^{\varphi S}]^{-1}(\delta)a \delta^{-1}) = \varphi([\sigma^{\varphi S}]^{-1}(\delta)a).$$

Compare the two equations. Since $\varphi$ is faithful and since $a \in \mathcal{A}$ is arbitrary, this shows that $\sigma^{-1}(\delta) = [\sigma^{\varphi S}]^{-1}(\delta)$.

(2). As a consequence of (1), we have: $\sigma([\sigma^{\varphi S}]^{-1}(\delta)) = \delta$ and $\sigma^{\varphi S}(\sigma^{-1}(\delta)) = \delta$.

(3). Let $a, x \in \mathcal{A}$. We have:

$$(\varphi \circ S)(ax) = \varphi(ax \delta) = \varphi(x \delta \sigma(a)).$$

Meanwhile,

$$(\varphi \circ S)(ax) = (\varphi \circ S)(x \sigma^{\varphi S}(a)) = \varphi(x \sigma^{\varphi S}(a) \delta).$$

Compare the two expressions. Since $\varphi$ is faithful and since $x \in \mathcal{A}$ is arbitrary, this shows that $\delta \sigma(a) = \sigma^{\varphi S}(a) \delta$. Or, equivalently, $\sigma^{\varphi S}(a) = \delta \sigma(a) \delta^{-1}$ and $\sigma(a) = \delta^{-1} \sigma^{\varphi S}(a) \delta$, true for any $a \in \mathcal{A}$.

(4). We may consider $x = [\sigma^{\varphi S}]^{-1}(a)$, for $a \in \mathcal{A}$, and apply (3). Then we have:

$$\sigma([\sigma^{\varphi S}]^{-1}(a)) = \sigma(x) = \delta^{-1} \sigma^{\varphi S}(x) \delta = \delta^{-1} a \delta.$$  \hfill $\square$

The following result is a consequence of $\varphi$ and $\varphi S$ being left and right invariant, respectively. See Proposition 7.7 for the definitions of $F_1, F_2, F_3, F_4$, which are elements in $M(\mathcal{A} \odot \mathcal{A})$. 

Proposition 7.9. Let \( a \in \mathcal{A} \). We have:
\[
(\varphi \otimes \text{id})(\Delta a) = (\text{id} \otimes \varphi)(F_1(1 \otimes a))\delta = \delta^*((\text{id} \otimes \varphi)((1 \otimes a)F_3)).
\]
Here, \( F_1 = (\text{id} \otimes S)(E) \) and \( F_3 = (\text{id} \otimes S^{-1})(E) \in M(\mathcal{A} \otimes \mathcal{A}) \).

Proof. Let \( a, x \in \mathcal{A} \). Observe:
\[
\varphi(x(\varphi \otimes \text{id})(\Delta a)) = (\varphi \otimes \varphi)((1 \otimes x)(\Delta a)) = \varphi((\text{id} \otimes \varphi)((1 \otimes x)(\Delta a)))
\]
\[
= \varphi(S((\text{id} \otimes \varphi)((\Delta x)(1 \otimes a)))) = \varphi(((\varphi \otimes S) \otimes \text{id})(\Delta x)a)
\]
\[
= \varphi(((\varphi \circ S) \otimes \text{id})((x \otimes 1)F_1(1 \otimes a))) = (\varphi \circ S)(x(\text{id} \otimes \varphi)(F_1(1 \otimes a)))
\]
\[
= \varphi(x(\text{id} \otimes \varphi)(F_1(1 \otimes a))\delta).
\]

Third equality is using the characterization of the antipode given in Proposition 1.12 (2). The fifth is an application of Proposition 7.3 (2), as \( \varphi \circ S \) is a right integral. The last equality is remembering \( (\varphi \circ S)(\cdot) = \varphi(\cdot \delta) \).

Since \( x \in \mathcal{A} \) is arbitrary and since \( \varphi \) is faithful, this shows that
\[
(\varphi \otimes \text{id})(\Delta a) = (\text{id} \otimes \varphi)(F_1(1 \otimes a))\delta, \quad \text{for all } a \in \mathcal{A}.
\]

Taking the adjoint, we have: \( (\varphi \otimes \text{id})(\Delta(a^*)) = \delta^*((\text{id} \otimes \varphi)((1 \otimes a^*)F_1^*)) \), \( \forall a \in \mathcal{A} \), since \( \varphi \) is a positive functional. Or, equivalently, \( (\varphi \otimes \text{id})(\Delta a) = \delta^*((\text{id} \otimes \varphi)((1 \otimes a)F_1^*)) \). Note here that since \( F_1 = (\text{id} \otimes S)(E) \) and since \( S(a^*) = S^{-1}(a^*) \) for any \( a \in \mathcal{A} \), we have \( F_1^* = (\text{id} \otimes S^{-1})(E^*) = (\text{id} \otimes S^{-1})(E) = F_3 \). In other words, we have:
\[
(\varphi \otimes \text{id})(\Delta a) = \delta^*((\text{id} \otimes \varphi)((1 \otimes a)F_3)), \quad \text{for all } a \in \mathcal{A}.
\]

Eventually, we wish to find what \( \Delta(\delta) \) is. At present it is not clear, but the following proposition will help us in that direction.

Proposition 7.10. Let \( p, q \in \mathcal{A} \), and \( \delta \in M(\mathcal{A}) \) be the modular element. We have:
\[
(\varphi \otimes \varphi)((p \otimes q)\Delta(\delta)) = ((\varphi \circ S) \otimes (\varphi \circ S))(p \otimes q)E),
\]
\[
(\varphi \otimes \varphi)(\Delta(\delta^*)(p \otimes q)) = ((\varphi \circ S) \otimes (\varphi \circ S))(E(p \otimes q)).
\]

Proof. The second result can be obtained immediately from the first one by taking the adjoint. So let us just prove the first result.

Let \( a, b \in \mathcal{A} \) be arbitrary. By Proposition 7.9, we have:
\[
(\varphi \otimes \varphi)((1 \otimes a)(\Delta b)\Delta(\delta)) = \varphi(a(\varphi \otimes \text{id})(\Delta(b\delta))) = \varphi(a(\text{id} \otimes \varphi)(F_1(1 \otimes b\delta))\delta).
\]

Since \( \varphi(\cdot \delta) = \varphi \circ S \), this becomes:
\[
(\varphi \otimes \varphi)((1 \otimes a)(\Delta b)\Delta(\delta)) = ((\varphi \circ S) \otimes (\varphi \circ S))((a \otimes 1)F_1(1 \otimes b)) = ((\varphi \circ S) \otimes (\varphi \circ S))((1 \otimes a)\varsigma F_1(b \otimes 1)),
\]
where \( \varsigma \) denotes taking the flip on \( M(\mathcal{A} \otimes \mathcal{A}) \).
Note that $\varsigma F_1 = \varsigma G_1 = \varsigma (\text{id} \otimes S)E = (S \otimes \text{id})(\varsigma E)$. Since $\varsigma E = (S^{-1} \otimes S^{-1})(E)$, we thus have $\varsigma F_1 = (\text{id} \otimes S^{-1})(E) = F_3$. Apply this result to the above, then we have:

$$(\varphi \otimes \varphi)(1 \otimes a)(\Delta b)\Delta(\delta) = ((\varphi \circ S) \otimes (\varphi \circ S))((1 \otimes a)F_3(b \otimes 1)) = ((\varphi \circ S) \otimes (\varphi \circ S))((1 \otimes a)(\Delta b)),
$$

(7.1)

where we used here the right invariance result $((\varphi \circ S) \otimes \text{id})(F_3(b \otimes 1)) = ((\varphi \circ S) \otimes \text{id})(\Delta b)$, as in Proposition 7.3 (2).

Equation (7.1) holds true for any $a, b \in A$. But note that the elements of the form $(1 \otimes a)(\Delta b)$ span $(A \otimes A)E$. Therefore, it is equivalent to saying

$$((\varphi \circ S) \otimes (\varphi \circ S))(p \otimes q)E = qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p))$$

for all $p, q \in A$.

Before we find $\Delta(\delta)$, we first prove the following lemma, which is yet another consequence of the right invariance of the functional $\varphi \circ S$.

**Lemma 7.11.** Let $p, q \in A$. Then we have:

$$(\varphi \circ S \otimes \text{id})(p \otimes q)E = qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p)).$$

**Proof.** Let $\omega \in A^*$ be arbitrary. Then:

$$\omega(((\varphi \circ S) \otimes \text{id})(p \otimes q)E) = (\varphi \circ S)(p \otimes q)E$$

where $x = (\text{id} \otimes \omega)(1 \otimes q)E$. As $E$ is full, note that such elements span the base algebra $B$. By Proposition 7.3 (1), we have:

$$(\varphi \circ S)(px) = \nu(((\varphi \circ S) \otimes \text{id})(\Delta(p)Ex)) = \nu(((\varphi \circ S) \otimes \text{id})(\Delta(p)E(1 \otimes x)) = \nu(((\varphi \circ S) \otimes \text{id})(\Delta p)(1 \otimes x)).$$

Here, we used the fact that since $x \in B$, we have $\Delta x = E(1 \otimes x)$ by Proposition 1.4, from which it follows that $\Delta(p)E(1 \otimes x) = (\Delta(p)E)(1 \otimes x) = (\Delta p)(1 \otimes x)$.

By Proposition 7.2, we know $((\varphi \circ S) \otimes \text{id})(\Delta p)E \in M(B)$. Therefore, combining the results:

$$\omega(((\varphi \circ S) \otimes \text{id})(p \otimes q)E) = \nu(((\varphi \circ S) \otimes \text{id})(\Delta p)x) = \omega(q \otimes \nu \otimes \text{id})(((\varphi \circ S) \otimes \text{id})(\Delta p)(1 \otimes 1))E).$$

As $\omega \in A^*$ is arbitrary, this means that

$$((\varphi \circ S) \otimes \text{id})(p \otimes q)E = qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p)E)$$

Since $((\varphi \circ S) \otimes \text{id})(\Delta p)E \in M(B)$, using the map $S_C$, as characterized in Equation (1.9), we have: $[((\varphi \circ S) \otimes \text{id})(\Delta p)E]E = [1 \otimes S_C^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p))]E$. Note also that $(\nu \otimes \text{id})(E) = 1$. Putting all these together, we thus obtain:

$$((\varphi \circ S) \otimes \text{id})(p \otimes q)E = qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p)) = qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p)),$$

as $S^{-1}$ extends $S_C^{-1}$.

The next proposition provides us with some characterizations of $\Delta(\delta)$ and $\Delta(\delta^*)$.

**Proposition 7.12.** Let $\delta \in M(A)$ be the modular element. We have:

- $\Delta(\delta) = (\delta \otimes S^{-1}(\delta^{-1}))E = (\delta \otimes S^{-2}(\delta^*))E = E(\delta \otimes S^{-2}(\delta^*))E$.
- $\Delta(\delta^*) = E(\delta^* \otimes S^2(\delta)) = E(\delta^* \otimes S^2(\delta))E$.
\( \Delta(\delta^*) = (\delta \otimes \delta^*)E = E(\delta \otimes \delta^*)E \)
\( \Delta(\delta) = E(\delta^* \otimes \delta) = E(\delta^* \otimes \delta)E \)
\( \Delta(\delta) = E(\delta \otimes \delta)E \)
\( \Delta(\delta^*) = E(\delta^* \otimes \delta^*)E \)

Proof. (1). Recall from Proposition 7.10 that \((\varphi \otimes \varphi)((p \otimes q)\Delta(\delta)) = ((\varphi \circ S)(\varphi \circ S))((p \otimes q)E)\), for any \(p, q \in A\). By Lemma 7.11, we have:
\[
(\varphi \otimes \varphi)((p \otimes q)\Delta(\delta)) = (\varphi \circ S)(((\varphi \circ S) \otimes \text{id})(\varphi \circ S)E)) = (\varphi \circ S)(qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p))). \tag{7.2}
\]
Meanwhile, note that
\[
((\varphi \circ S) \otimes \text{id})(\Delta p) = ((\varphi \circ S) \otimes \text{id})(F_3(p \otimes 1)) = (\varphi \otimes \text{id})(F_3(p \delta \otimes 1))
= (\id \otimes \varphi)(F_3(1 \otimes p\delta)) = (\id \otimes \varphi)(F_1(1 \otimes p\delta))
= (\varphi \otimes \text{id})(\Delta(p\delta))\delta^{-1} = \delta^*\id(\delta \otimes \varphi)((1 \otimes p\delta)F_3)\delta^{-1}. \tag{7.3}
\]
The first equality is Proposition 7.3 (2); The second used \(\varphi \circ S = \varphi \cdot \delta\); In the third and fourth, the flip map is applied, together with the observation earlier (see proof of Proposition 7.10) that \(\varepsilon F_1 = F_3\); The fifth and sixth used the result of Proposition 7.3.

Insert into Equation (7.2) the result of Equation (7.3). Then we have:
\[
(\varphi \otimes \varphi)((p \otimes q)\Delta(\delta)) = (\varphi \circ S)(qS^{-1}(((\varphi \circ S) \otimes \text{id})(\Delta p)))
= (\varphi \circ S)(qS^{-1}(\delta^*\id(\delta \otimes \varphi)|(1 \otimes p\delta)F_3)\delta^{-1})) = \varphi(qS^{-1}((\id \otimes \varphi)|(1 \otimes p\delta)F_3)\delta^{-1}))
= \varphi(qS^{-1}(\delta^{-1})S^{-1}((\id \otimes \varphi)|(1 \otimes p\delta)F_3))). \tag{7.4}
\]
The fourth and sixth equalities used the anti-multiplicativity \(S^{-1}\); In the fifth, we used the result of Proposition 7.7, namely \(S(\delta) = (\delta^*)^{-1}\).

Note that by applying the flip map and using again \(\varepsilon F_1 = F_3\), we have:
\[
S^{-1}((\id \otimes \varphi)|(1 \otimes p\delta)F_3)) = S^{-1}((\varphi \circ \varphi)|(\varphi \circ \delta \otimes 1)F_3)) = (\varphi \circ \varphi)|(\varphi \circ \delta \otimes 1)E, \tag{7.5}
\]
because \(F_1 = (\id \otimes S)(E)\). Insert the result of Equation (7.5) into Equation (7.4), to obtain:
\[
(\varphi \otimes \varphi)((p \otimes q)\Delta(\delta)) = \varphi(qS^{-1}(\delta^{-1}))(\varphi \circ \varphi)|(\varphi \circ \delta \otimes 1)E)
= (\varphi \otimes \varphi)((p \delta \otimes qS^{-1}(\delta^{-1}))E)
= (\varphi \otimes \varphi)((p \otimes q)(\delta \otimes S^{-1}(\delta^{-1}))E).
\]

Here, note that \(\varphi\) is faithful and that \(p, q \in A\) are arbitrary, which means that we have
\(\Delta(\delta) = (\delta \otimes S^{-1}(\delta^{-1}))E\).
Or \(\Delta(\delta) = (\delta \otimes S^{-2}(\delta^*))E\), by using the fact \(\delta^* = S(\delta^{-1})\). Meanwhile, noting that \(\Delta(\delta) = E \Delta(\delta)\), we can also write this as \(\Delta(\delta) = E(\delta \otimes S^{-2}(\delta^*))E\).

(2). From (1), we know \(\Delta(\delta) = (\delta \otimes S^{-2}(\delta^*))E\). Take the adjoint, to obtain \(\Delta(\delta^*) = E(\delta^* \otimes S^2(\delta))\), using the property of \(S\). Also, as we should have \(\Delta(\delta^*) = \Delta(\delta^*)E\), we can also write this as \(\Delta(\delta^*) = E(\delta^* \otimes S^2(\delta))E\).
(3). From (1), we saw $\Delta(\delta) = (\delta \otimes S^{-1}(\delta^{-1}))E$. As the comultiplication preserves multiplication and since $\Delta(1) = E$, we can see from this quickly that
$$\Delta(\delta^{-1}) = E(\delta^{-1} \otimes [S^{-1}(\delta^{-1})]^{-1}) = E(\delta^{-1} \otimes S^{-1}(\delta)).$$
As $S(\delta^{-1}) = \delta^*$, and by using the result we just obtained on $\Delta(\delta^{-1})$, we have:

$$\Delta(\delta^*) = \Delta(S(\delta^{-1})) = (S \otimes S)\Delta^{\text{cop}}(\delta^{-1}) = (S \otimes S)(\varsigma E(S^{-1}(\delta) \otimes \delta^{-1}))$$

$$= (\delta \otimes S(\delta^{-1}))(S \otimes S)(\varsigma E) = (\delta \otimes \delta^*)E.$$

Again, as we should have $\Delta(\delta^*) = E\Delta(\delta^*)$, we can also write this as $\Delta(\delta^*) = E(\delta \otimes \delta^*)E$.

(4). From (3), we saw $\Delta(\delta^*) = (\delta \otimes \delta^*)E$. Take the adjoint, to obtain: $\Delta(\delta) = E(\delta^* \otimes \delta)$. As before, this can be also written as $\Delta(\delta) = E(\delta^* \otimes \delta^*)E$.

(5). Consider $x = (\text{id} \otimes \phi)(F_1(1 \otimes a)) = (\text{id} \otimes \phi)(\phi S)(S^{-1}(a) \cdot))(E)$, for $a \in A$. It is an element in $B$, and it is evident that such elements span $B$. Note that since $x \in B$, we have $\Delta x = E(1 \otimes x) = (1 \otimes x)E$. We thus have: $(1 \otimes x)\Delta(\delta) = (1 \otimes x)E\Delta(\delta) = \Delta(x\delta)$. But note that by Proposition 4.9, we have:

$$x\delta = (\text{id} \otimes \phi)(F_1(1 \otimes a))\delta = \delta^*(\text{id} \otimes \phi)((1 \otimes a)F_3) = \delta^* x \hat{x},$$

where $\hat{x} = (\text{id} \otimes \phi)((1 \otimes a)F_3)$, also an element of $B$, so we have $\Delta(\hat{x}) = E(1 \otimes \hat{x}) = (1 \otimes \hat{x})E$.

Combining these observations, we have:

$$(1 \otimes x)\Delta(\delta) = \Delta(x\delta) = \Delta(\delta^* \hat{x}) = \Delta(\delta^*)(1 \otimes \hat{x})$$

$$= E(\delta \otimes \delta^*)E(1 \otimes \hat{x}) = E(\delta \otimes \delta^*)(1 \otimes \hat{x})E$$

$$= E(\delta \otimes \delta^* \hat{x})E = E(\delta \otimes x\delta)E = E(1 \otimes x)(\delta \otimes \delta)E$$

$$= (1 \otimes x)E(\delta \otimes \delta)E,$$

where we used $\Delta(\delta^*) = E(\delta \otimes \delta^*)E$, observed in (3) above.

As noted above, the elements $x = (\text{id} \otimes \phi)(F_1(1 \otimes a))$, $a \in A$, span $B$. Note also that $B$ is non-degenerate. So this result means that

$$\Delta(\delta) = E(\delta \otimes \delta)E.$$

(6). From $\Delta(\delta) = E(\delta \otimes \delta)E$, take the adjoint, to obtain $\Delta(\delta^*) = E(\delta^* \otimes \delta^*)E$. 

We do not know whether $\delta$ is self-adjoint, so $\Delta(\delta)$ and $\Delta(\delta^*)$ can be different and we made a distinction so far.

However, under the quasi-invariance assumption (see Section 5.3), it seems that the modular element $\delta$ becomes self-adjoint. See discussion given in Proposition 4.9 and the paragraphs following it. Considering this, we revisit in the below some of the results we obtained so far for the modular element $\delta$, when we restrict ourselves to the situation when $\delta$ is self-adjoint. The self-adjointness of $\delta$ makes things simpler.

**Proposition 7.13.** Assume that $\delta$ is self-adjoint. Then we have the following simpler results:

1. $(\phi \circ S)(a) = \phi(\delta a)$ and $(\phi \circ S^{-1})(a) = \phi(\delta a)$, for all $a \in A$.
2. $S(\delta) = \delta^{-1}$ and $S^2(\delta) = \delta$. 


(3) \((\varphi \otimes \text{id})(\Delta a) = (\text{id} \otimes \varphi)(F_1(1 \otimes a))\delta = \delta(\text{id} \otimes \varphi)((1 \otimes a)F_3)\), for all \(a \in A\).

(4) \(\Delta(\delta) = (\delta \otimes \delta)E = E(\delta \otimes \delta) = E(\delta \otimes \delta)E\).

Proof. (1). See Lemma 7.6 and let \(\delta^* = \delta\).

(2). See Proposition 7.7 and let \(\delta^* = \delta\).

(3). See Proposition 7.9 and let \(\delta^* = \delta\).

(4). See Proposition 7.12 and let \(\delta^* = \delta\). \(\square\)

Remark. As discussed in §5.3 and as indicated above, it seems to be the case that the quasi-invariance Assumption leads to the self-adjointness of \(\delta\). The reasoning requires going up to the von Neumann algebra level and back down, so not purely algebraic. It may be possible to find a more direct proof, but as the focus of the current paper is on the construction of a \(C^*\)-algebraic object, we did not make an attempt to develop such a proof. Moreover, in the purely algebraic setting, as there is no reason to have to require the quasi-invariance condition, the need for such a proof is probably not too significant.

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