Solvability of Backward Stochastic Differential Equations with Quadratic Growth

Revaz Tevzadze

Georgian–American University, Business School, 3, Alleyway II, Chavchavadze Ave. 17a, Georgian Technical University, 77 Kostava str., 0175, Institute of Cybernetics, 5 Euli str., 0186, Tbilisi, Georgia (e-mail: reztev@yahoo.com)

We prove the existence of the unique solution of a general Backward Stochastic Differential Equation with quadratic growth driven by martingales. Some kind of comparison theorem is also proved.

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1 Introduction

In this paper we show a general result of existence and uniqueness of Backward Stochastic Differential Equation (BSDE) with quadratic growth driven by continuous martingale. Backward stochastic differential equations have been introduced by Bismut [1] for the linear case as equations of the adjoint process in the stochastic maximum principle. A nonlinear BSDE (with Bellman generator) was first considered by Chitashvili [4]. He derived the semimartingale BSDE (or SBE), which can be considered as a stochastic version of the Bellman equation for a stochastic control problem, and proved the existence and uniqueness of a solution. The theory of BSDEs driven by the Brownian motion was developed by Pardoux and Peng [22] for more general generators. The results of Pardoux and Peng were generalized by Kobylansky [11], Lepeltier and San Martin [12] for generators with quadratic growth. In the work of Hu at all [8] BMO-martingales were used for BSDE with quadratic generators in Brownian setting and in [15], [16], [17], [18], [19], [21]
for BSDEs driven by martingales. By Chitashvili [4], Buckdahn [3], and El Karoui and Huang [7] the well posedness of BSDE with generators satisfying Lipschitz type conditions was established. Here we suggest new approach including an existence and uniqueness of the solution of general BSDE with quadratic growth. In the earlier papers [15], [16], [17], [18], [19] we studied, as well as Bobrovnytska and Schweizer [2], the particular cases of BSDE with quadratic nonlinearities related to the primal and dual problems of Mathematical Finance. In these works the solutions were represented as a value function of the corresponding optimization problems.

The paper is organized as follows. In Section 2 we give some basic definitions and facts used in what follows. In Section 3 we show the solvability of the system of BSDEs for sufficiently small initial condition and further prove the solvability of one dimensional BSDE for arbitrary bounded initial data. At the end of Section 4 we prove the comparison theorem, which generalizes the results of Mania and Schweizer [14], and apply this results to the uniqueness of the solution.

2 Some basic definitions and assumptions

Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)\) be filtered probability space satisfying the usual conditions. We assume that all local martingales with respect to \(\mathbb{F}\) are continuous. Here the time horizon \(T < \infty\) is a stopping time and \(\mathcal{F}_T = \mathcal{F}_T\). Let us consider Backward Stochastic Differential Equation (BSDE) of the form

\[
dY_t = -f(t, Y_t, \sigma_t^*Z_t)dK_t - d\langle N \rangle_t g_t + Z_t^*dM_t + dN_t, \tag{2.1}
\]

\[
Y_T = \xi \tag{2.2}
\]

We suppose that

- \((M_t, t \geq 0)\) is an \(\mathbb{R}^n\)-valued continuous martingale with cross-variations matrix \(\langle M \rangle_t = (\langle M^i, M^j \rangle_t)_{1 \leq i,j \leq n}\),

- \((K_t, t \geq 0)\) is a continuous, adapted, increasing process, such that \(\langle M \rangle_t = \int_0^t \sigma_s \sigma_s^*dK_s\) for some predictable, non-degenerate \(n \times n\) matrix \(\sigma\),

- \(\xi\) is \(\mathcal{F}\)-measurable an \(\mathbb{R}^d\)-valued random variable,

- \(f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{n \times d} \to \mathbb{R}^d\) is a stochastic process, such that for any \((y, z) \in \mathbb{R}^d \times \mathbb{R}^{n \times d}\) the process \(f(\cdot, \cdot, y, z)\) is predictable,

- \(g : \Omega \times \mathbb{R}^+ \to \mathbb{R}^{d \times d}\) is a predictable process.

The notation \(\mathbb{R}^{n \times d}\) here denotes the space of \(n \times d\)-matrix \(C\) with Euclidian norm \(|C| = \sqrt{\text{tr}(CC^*)}\). For some stochastic process \(X_t\) and stopping times \(\tau, \nu\), such that \(\tau \geq \nu\) we denote \(X_{\tau,\nu} = X_\tau - X_\nu\). For all unexplained notations concerning the martingale theory used below we refer [9], [5] and [13]. About BMO-martingales see [6] or [10].

A solution of the BSDE is a triple \((Y, Z, N)\) of stochastic processes, such that (2.1), (2.2) is satisfied and
• $Y$ is an adapted $R^d$-valued continuous process,
• $Z$ is an $R^{n \times d}$-valued predictable process,
• $N$ is an $R^d$-valued continuous martingale, orthogonal to the basic martingale $M$.

One says that $(f, g, \xi)$ is a generator of BSDE (2.1),(2.2).

We introduce the following spaces

• $L^\infty(R^d) = \{X : \Omega \to R^d, \mathcal{F}_t \text{ measurable}, ||X||_\infty = \text{ess sup}_\omega |X(\omega)| < \infty\}$,
• $S^\infty(R^d) = \{\varphi : \Omega \times R^+ \to R^d, \text{ continuous, adapted}, ||\varphi||_\infty = \text{ess sup}_{[0,T]} |\varphi(t, \omega)| < \infty\}$,

\[ H^2(R^{n \times d}, \sigma) = \{\varphi : \Omega \times R^+ \to R^{n \times d}, \text{ predictable}, ||\varphi||^2_H = \text{ess sup}_{[0,T]} E(\int_t^T |\sigma_s^* \varphi_s|^2 dK_s|\mathcal{F}_t) \leq \text{ess sup}_{[0,T]} E(\text{tr}(\varphi \cdot M)_{tt}|\mathcal{F}_t) < \infty\}, \] (2.3)

• $\text{BMO}(Q) = \{N, R^d \text{ valued } Q \text{ martingale } ||N||^2_Q = \text{ess sup}_{[0,T]} E^Q(\text{tr}(N)_{tt}|\mathcal{F}_t) < \infty\}$

We also use the notation $|r|_{2,\infty}$ for the norm $||\int_0^T r_s^2 dK_s||_\infty$.

The norm of the triple is defined as

\[ ||(Y, Z, N)||^2 = ||Y||^2 + ||Z||^2_H + ||N||^2_Q. \]

Throughout the paper we use the condition

A) There exist a constant $\theta$ and predictable processes

\[ \alpha : \Omega \times R^+ \to R^d, \Gamma : \Omega \times R^+ \to \text{Lin}(R^{n \times d}, R^d), r : \Omega \times R^+ \to R, \]

such that the following conditions $\int_0^T r_s^2 dK_s, \int_0^T r_s^2 dK_s \in L^\infty, \Gamma(\sigma^{-1}) \in H^2_T, |\alpha_t| \leq r_t, |g_t| \leq \theta^2$ and

\[ |f(t, y_1, z_1) - f(t, y_2, z_2) - \alpha_t(y_1 - y_2) - \Gamma_t(z_1 - z_2)| \]

\[ \leq (r_t |y_1 - y_2| + \theta |z_1 - z_2|)(r_t(|y_1| + |y_2|) + \theta(|z_1| + |z_2|)). \] (2.4)

are satisfied.

Sometimes we use the more restrictive conditions

B1) $\int_0^T |f(t, 0, 0)| dK_t + |g_t| \leq \theta^2$ for all $t \in [0, T],$

B2) $|f_y(t, y, z)| \leq r_t, |f_z(t, y, z)| \leq r_t + \theta |z|$ for all $(t, y, z),$

B3) $|f_{yy}(t, y, z)| \leq r^2_t, |f_{yz}(t, y, z)| \leq \theta r_t, |f_{zz}(t, y, z)| \leq \theta^2$ for all $(t, y, z).$
Remark 1. Condition A) follow from conditions B1)-B3), since using notations \( \delta y = y_1 - y_2, \ \delta z = z_1 - z_2 \) for \( \alpha_t = f_y(t, 0, 0), \ \Gamma_t = f_z(t, 0, 0) \) by the mean value theorem we have

\[
|f(t, y_1, z_1) - f(t, y_2, z_2) - \alpha_t \delta y - \Gamma_t(\delta z)|
\]

\[
= |f_y(t, \nu y_1 + (1 - \nu)y_2, \nu z_1 + (1 - \nu)z_2)\delta y - f_y(t, 0, 0)\delta y| 
+ f_z(t, \nu y_1 + (1 - \nu)y_2, \nu z_1 + (1 - \nu)z_2)(\delta z) - f_z(t, 0, 0)(\delta z)|,
\]

for some \( \nu \in [0, 1] \). Using again mean value theorem we obtain that

\[
|f(t, y_1, z_1) - f(t, y_2, z_2) - \alpha_t \delta y - \Gamma_t(\delta z)|
\]

\[
\leq (|\nu y_1 + (1 - \nu)y_2| \max_{y,z} |f_{y_{y}}(t, y, z)| + |\nu z_1 + (1 - \nu)z_2| \max_{y,z} |f_{y_{z}}(t, y, z)|)\delta y|
+ (|\nu y_1 + (1 - \nu)y_2| \max_{y,z} |f_{y_{z}}(t, y, z)| + |\nu z_1 + (1 - \nu)z_2| \max_{y,z} |f_{z_{z}}(t, y, z)|)\delta z|
\]

\[
\leq \left[ r^2 t(|y_1| + |y_2|) + r\theta(|z_1| + |z_2|)\right]|\delta y| + \left[ r\theta(|y_1| + |y_2|) + \theta^2(|z_1| + |z_2|)\right]|\delta z|
= (r\theta t|\delta y| + \theta|\delta z|)(r\theta(|y_1| + |y_2|) + \theta(|z_1| + |z_2|)).
\]

Remark 2. If \( d = 1 \) the operator \( \Gamma_t \) is given by an \( n \)-dimensional vector \( \gamma_t \) such that \( \Gamma_t(z) = \gamma_t^* z \). Thus inequality in A) can be rewritten as

\[
|f(t, y_1, z_1) - f(t, y_2, z_2) - \alpha_t \delta y - \gamma_t^* \delta z|
\]

\[
\leq (r\theta t|\delta y| + \theta|\delta z|)(r\theta(|y_1| + |y_2|) + \theta(|z_1| + |z_2|)).
\]

The main statement of the paper is the following

Theorem 1. Let \( \xi \in L^\infty, \ d = 1 \) and conditions B1)-B3) are satisfied. Then there exists a unique triple \( (Y, Z, N) \), where \( Y \in S^\infty, Z \in H^2, N \in BMO \), that satisfies equation (2.1), (2.2).

3 Existence of the solution

First we prove the existence and uniqueness of the solution for a sufficiently small initial data.

Proposition 1. Let \( f \) and \( g \) satisfy condition A) with \( \alpha = 0 \) and \( \gamma_t = 0 \). Then for \( \xi \) with the norm \( ||\xi||_\infty < \frac{1}{32\beta} \), \( \beta = 8 \max(|r|^2_{2,\infty}, \theta^2) \) there exists a unique solution \( (Y, Z, N) \) of BSDE

\[
dY_t = (f(t, 0, 0) - f(t, Y_t, \sigma_t^* Z_t))dK_t + d\langle N \rangle_t g_t + Z_t^* dM_t + dN_t,
\]

\[
Y_T = \xi,
\]

with the norm \( ||(Y, Z, N)|| \leq R \), where \( R \) is a constant satisfying the inequality \( 4||\xi||_\infty^2 + \beta^2 R^4 \leq R^2 \), namely \( R = 2\sqrt{2}||\xi||_\infty \).

Moreover if \( ||\xi||_\infty + ||\int_0^\infty |f(s, 0, 0)|dK_s||_\infty \) is small enough then BSDE (2.1) admits a unique solution.
Proof. We define the mapping \((Y, Z, N) = F(y, z, n)\), \(n\) is orthogonal to \(M, (y, z \cdot M + n) \in S^\infty_T \times \text{BMO}(P)\) by the relation
\[
dY_t = (f(t, 0, 0) - f(t, y_t, \sigma^*_t z_t))dK_t + d\langle n \rangle_t g_t + Z_t^*dM_t + dN_t, \\
y_T = \xi. \tag{3.2}
\]
Using the Ito formula for \(|Y_t|^2\) we obtain that
\[
|Y_t|^2 = |\xi|^2 + 2 \int_t^T Y^*_s(f(s, y_s, \sigma^*_s z_s) - f(s, 0, 0))dK_t \\
+ 2 \int_t^T Y^*_s d\langle n \rangle_s g_s - \int_t^T \text{tr}Z^*_s d\langle M \rangle_s Z_s - \text{tr}(N)_{tT} - \int_t^T Y^*_s Z^*_s dM_s - \int_t^T Y^*_s dN_s.
\]
If we take the conditional expectation and use (2.3) and the elementary inequality \(2ab \leq \frac{1}{4}a^2 + 4b^2\) we get
\[
|Y_t|^2 + E(\int_t^T |\sigma^*_s Z_s|^2dK_s + \text{tr}(N)_{tT}|\mathcal{F}_t) \leq ||\xi||^2 + \frac{1}{4}||Y||^2_\infty \\
+ 4E^2(\int_t^T |f(s, y_s, \sigma^*_s z_s) - f(s, 0, 0)|dK_s + \int_t^T |g_s|d\text{tr}(n)_s|\mathcal{F}_t). \tag{3.3}
\]
Thus using condition A), identities
\[
\text{tr}(z \cdot M)_t = \text{tr} \int_0^t z^*_s d\langle M \rangle_s z_s = \int_0^t \text{tr}(z^*_s \sigma_s \sigma^*_s z_s)dK_s = \int_0^t |\sigma^*_s z_s|^2dK_s \tag{3.4}
\]
and explicit inequalities
\[
\frac{1}{2}(|Y|^2_\infty + ||Z \cdot M + N||^2_{\text{BMO}}) \leq \max(||Y||^2_\infty, ||Z \cdot M + N||^2_{\text{BMO}})
\]
\[
\leq \text{ess sup}||Y||^2 + E(\int_t^T |\sigma^*_s Z_s|^2dK_s + \text{tr}(N)_{tT}|\mathcal{F}_t)
\]
we obtain from (3.3)
\[
\frac{1}{4}||Y||^2_\infty + \frac{1}{2}||Z \cdot M + N||^2_{\text{BMO}} \leq ||\xi||^2 \\
+ 4\text{ess sup}E^2(\int_t^T |f(s, y_s, \sigma^*_s z_s) - f(s, 0, 0)|dK_s + \theta^2 d\text{tr}(n)_{tT}|\mathcal{F}_t)
\]
\[
\leq ||\xi||^2 + 16\text{ess sup}E^2(\int_t^T r^2 s^2 |dK_s + \theta^2 \text{tr}(z \cdot M + n)_{tT}|\mathcal{F}_t)
\]
\[
\leq ||\xi||^2 + 16|r|^4_\infty ||y|^4_\infty + 16\theta^4||z \cdot M + n||^4_{\text{BMO}}.
\]
Therefore
\[
||Y||^2_\infty + ||Z \cdot M + N||^2_{\text{BMO}} \leq 4||\xi||^2 \\
+ 64|r|^4_\infty ||y|^4_\infty + 64\theta^4||z \cdot M + n||^4_{\text{BMO}}
\]
\[
\leq 4||\xi||^2 + \beta^2(||y||^2_\infty + ||z \cdot M + n||^2_{\text{BMO}})^2,
\]
where $\beta = 8 \max(|r|_{2,\infty}^2, \theta^2)$. We can pick $R$ such that

$$4||\xi||^2 + \beta^2 R^4 \leq R^2$$

if and only if $||\xi||_{\infty} \leq \frac{1}{4\beta}$. For instance $R = 2\sqrt{2}||\xi||_{\infty}$ satisfies this quadratic inequality. Therefore the ball

$$\mathcal{B}_R = \{ (Y, Z \cdot M + N) \in S^{\infty} \times \text{BMO}, N \perp M, ||Y||_{\infty}^2 + ||Z \cdot M + N||_{\text{BMO}}^2 \leq R^2 \}$$

is such that $F(\mathcal{B}_R) \subset \mathcal{B}_R$.

Similarly for $(y^j, z^j \cdot M + n^j) \in \mathcal{B}_R$, $j = 1, 2$ using the notations $\delta y = y^1 - y^2$, $\delta z = z^1 - z^2$, $\delta n = n^1 - n^2$ we can show that

$$||\delta Y||_{\infty}^2 + ||\delta Z \cdot M + \delta N||_{\text{BMO}}^2 \leq 4 \text{ess sup} E^2 \left( \int_{[0,T]} |f(s, y^1_s, \sigma_s z^1_s) - f(s, y^2_s, \sigma_s z^2_s)| dK_s + \int_{[0,T]} |g_s| d\text{var}(\langle \delta n, n^1 + n^2 \rangle)_s |\mathcal{F}_t) \right)$$

$$\leq 8 \text{ess sup} E \left( \int_{[0,T]} (r_s^2 |\delta y_s|^2 + \theta^2 |\sigma_s \delta z_s|^2) dK_s |\mathcal{F}_t) \right) \times E \left( \int_{[0,T]} (r_s (|y^1_s|^2 + |y^2_s|^2)) + \theta (|\sigma_s z^1_s|^2 + |\sigma_s z^2_s|^2)) dK_s |\mathcal{F}_t) \right) + \theta^2 E (\text{tr} \langle \delta n \rangle_{\text{tr}} |\mathcal{F}_s) E (\text{tr} n^1 + n^2)_{\text{tr}} |\mathcal{F}_t)$$

Again using the equalities (3.4) we can pass to the norm. Thus

$$||\delta Y||_{\infty}^2 + ||\delta Z \cdot M + \delta N||_{\text{BMO}}^2 \leq 8 (|r|_{2,\infty}^2 ||\delta y||_{\infty}^2 + \theta^2 ||\delta z \cdot M||_{\text{BMO}}^2) \times (|r|_{2,\infty}^2 (||y^1||_{\infty}^2 + ||y^2||_{\infty}^2) + \theta^2 (||z^1 \cdot M||_{\text{tr}}^2 + ||z^2 \cdot M||_{\text{tr}}^2) \right)^2 \times (||\delta z||_{\text{BMO}}^2 (||n^1||_{\text{BMO}}^2 + ||n^2||_{\text{BMO}}^2)^2).$$

Since $||z^1 \cdot M||, ||z^2 \cdot M|| \leq R, ||n^1||, ||n^2|| \leq R$ we get

$$||\delta Y||_{\infty}^2 + ||\delta Z \cdot M + \delta N||_{\text{BMO}}^2 \leq 128 \beta^2 R^2 (||\delta y||_{\infty}^2 + ||\delta z \cdot M||_{\text{BMO}}^2) + 4 \beta^2 R^2 ||\delta n||_{\text{BMO}}^2 \leq 128 \beta^2 R^2 (||\delta y||_{\infty}^2 + ||\delta z \cdot M + \delta n||_{\text{BMO}}^2).$$

Now we can take $R = 2\sqrt{2}||\xi||_{\infty} < \frac{1}{8\sqrt{2}\beta}$. This means that $||\xi||_{\infty} < \frac{1}{32\beta}$ and $F$ is contraction on $\mathcal{B}_R$. By contraction principle the mapping $F$ admits a unique fixed point, which is the solution of (3.1). \Box

From now we suppose that $d = 1$.

**Lemma 1.** Let condition A) is satisfied. Then the generator $(\hat{f}, \hat{g}, \hat{\xi})$, where

$$\hat{f}(t, \bar{y}, \bar{z}) = e^{\int_0^t \alpha_s dK_s} (f(t, e^{-\int_0^t \alpha_s dK_s} \bar{y}, e^{-\int_0^t \alpha_s dK_s} \bar{z}) - f(t, 0, 0)) - \alpha_t \bar{y} - \gamma_t \bar{z},$$


\[g_t = e^{-\int_0^t \alpha_s dK_s} g_t \quad \text{and} \quad \xi = e^{\int_0^T \alpha_s dK_s} \xi,\]
satisfies condition A) with \(\alpha = 0, \gamma = 0, \bar{r}_t = r_t e^{\|\int_0^\infty r_s dK_s\|_\infty}, \) and \(\bar{\theta} = \theta e^{\|\int_0^T r_s dK_s\|_\infty}.\)

Moreover, \((Y, Z, N)\) is a solution of BSDE (3.1) if and only if

\[
(Y_t, Z_t, N_t) = (e^{\int_0^t \alpha_s dK_s} Y_t, e^{\int_0^t \alpha_s dK_s} Z_t, \int_0^t e^{\int_0^s \alpha_u dK_u} dN_u)
\]
is a solution w.r.t. measure \(d\bar{P} = \mathcal{E}_T((\gamma \sigma^{-1}) \cdot M) dP\) of BSDE

\[
d\bar{Y}_t = -\bar{f}(t, Y_t, \sigma_t^* \bar{Z}_t) dK_t - d\langle \bar{N} \rangle_t \bar{g}_t + \bar{Z}_t^* d\bar{M}_t + d\bar{N}_t,
\]
where \(\bar{M}_t = M_t - \langle (\gamma \sigma^{-1}) \cdot M, M \rangle_t.\)

**Proof.** Condition A) for \((\bar{f}, \bar{g}, \bar{\xi})\) is satisfied since by (2.3)

\[
|\bar{f}(t, \bar{g}_1, \bar{z}_1) - \bar{f}(t, \bar{g}_2, \bar{z}_2)|
\]
\[
\leq e^{\int_0^t \alpha_s dK_s} (r_t |\delta \bar{g}| + \theta |\delta \bar{z}|) (r_t (|\bar{g}_1| + |\bar{g}_2|) + \theta (|\bar{z}_1| + |\bar{z}_2|))
\]
\[
\leq (\bar{r}_t |\delta \bar{g}| + \bar{\theta} |\delta \bar{z}|) (\bar{r}_t (|\bar{g}_1| + |\bar{g}_2|) + \bar{\theta} (|\bar{z}_1| + |\bar{z}_2|)).
\]

On the other hand using the Itô formula we have

\[
d\bar{Y}_t = e^{\int_0^t \alpha_s dK_s} dY_t + \alpha_t e^{\int_0^t \alpha_s dK_s} Y_t dK_t
\]
\[
e^{\int_0^t \alpha_s dK_s} (f(t, 0, 0) - f(t, Y_t, \sigma_t^* Z_t)) dK_t + e^{\int_0^t \alpha_s dK_s} d\langle N \rangle_t \bar{g}_t
\]
\[
+ e^{\int_0^t \alpha_s dK_s} Z_t^* d\bar{M}_t + e^{\int_0^t \alpha_s dK_s} dN_t + \alpha_t \bar{Y}_t dK_t
\]
Taking into account that

\[
e^{\int_0^t \alpha_s dK_s} (f(t, 0, 0) - f(t, Y_t, \sigma_t^* Z_t)) + \alpha_t \bar{Y}_t
\]
\[
= -\bar{f}(t, \bar{Y}_t, \sigma_t^* \bar{Z}_t) - \gamma_t \sigma_t^* \bar{Z}_t,
\]
\[
e^{\int_0^t \alpha_s dK_s} d\langle N \rangle_t \bar{g}_t = d\langle N \rangle_t e^{-\int_0^t \alpha_s dK_s} g_t = d\langle N \rangle_t \bar{g}_t
\]
and

\[
Z \cdot M - \int_0^t \gamma_t \sigma_t^* Z_t dK_t = Z \cdot M - \int_0^t \gamma_t \sigma_t^{-1} \sigma_t^* Z_t dK_t
\]
\[
= \bar{Z} \cdot M - \int_0^t \gamma_t \sigma_t^{-1} d\langle M \rangle_t \bar{Z}_t = \bar{Z} \cdot M - \langle (\gamma \cdot \sigma^{-1}) \cdot M, \bar{Z} \cdot M \rangle = \bar{Z} \cdot \bar{M}
\]
we obtain

\[
d\bar{Y}_t = -\bar{f}(t, \bar{Y}_t, \sigma_t^* \bar{Z}_t) dK_t - d\langle \bar{N} \rangle_t \bar{g}_t + \bar{Z}_t d\bar{M}_t + d\bar{N}_t.
\]

Here \(\bar{M}\) is a local martingale w.r.t. \(\bar{P}\) by Girsanov theorem.

**Corollary 1.** Let \(f\) and \(g\) satisfy condition A) and \(\|\xi\|_\infty \leq \frac{1}{128^2} \exp(-2\|\int_0^T r_s dK_s\|_\infty)\).

Then there exist the solution of (3.1) with the norm \(\|Y\|_\infty^2 + \|Z \cdot \bar{M} + N\|_{\BMO(\bar{P})}^2 \leq \frac{1}{128^2}.\)
Proof. Obviously that

\[ ||Y||^2 + ||Z \cdot \tilde{M} + N||^2_{BMO(P)} \leq \left( ||\tilde{Y}||^2 + ||\tilde{Z} \cdot \tilde{M} + \tilde{N}||^2_{BMO(P)} \right) \exp(2|| \int_0^T r_s dK_s ||_{\infty}) \]

\[ \leq 8||\hat{\xi}||^2 \exp(2|| \int_0^T r_s dK_s ||_{\infty}) \leq 8||\xi||^2 \exp(4|| \int_0^T r_s dK_s ||_{\infty}). \]

From \( ||\xi||_{\infty} \leq \frac{1}{128\beta} \exp(-2|| \int_0^T r_s dK_s ||_{\infty}) \) follows that \( 8||\xi||_{\infty} \exp(4|| \int_0^T r_s dK_s ||_{\infty}) \leq \frac{1}{128\beta}. \) Hence we get \( ||Y||^2_{\infty} + ||Z \cdot \tilde{M} + N||^2_{BMO(P)} \leq \frac{1}{128\beta}. \)

**Corollary 2.** Let generator \((f, g, \xi)\) satisfies conditions B1)-B3) and \((\tilde{Y}_t, \tilde{Z}_t, \tilde{N}_t)\) be a solution of (3.1). Then BSDE

\[ d\tilde{Y}_t = (f(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t) - f(t, \tilde{Y}_t + \tilde{Z}_t, \sigma^*_t \tilde{Z}_t + \sigma^*_t \tilde{Z}_t))dK_t \]

\[ -d(\langle \tilde{N} \rangle_t + 2(\tilde{N}, \tilde{N}) g_t + \tilde{Z}_t dM_t) + d\tilde{N}_t, \]

\[ \tilde{Y}_T = \tilde{\xi} \]

satisfy condition A) with \( \hat{f}(t, y, z) = f(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t) - f(t, y + \tilde{Y}_t, z + \sigma^*_t \tilde{Z}_t), \alpha_t = f_y(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t), \gamma_t = f_z(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t) \) and the new probability measure \( \mathcal{E}_T(2g \cdot \tilde{N}) dP. \) Moreover (3.7) admits a unique solution \((\tilde{Y}_t, \tilde{Z}_t, \tilde{N}_t)\) if \( ||\xi||_{\infty} \leq \frac{1}{128\beta} \exp(-2|| \int_0^T r_s dK_s ||_{\infty}). \)

**Proof.** Using a change of measure the equation (3.7) reduces to equation of type (3.1).

By previous corollary we obtain the existence and uniqueness of the BSDE.

**Lemma 2.** Let conditions B1)-B3) be satisfied and random variables \( \tilde{\xi} \) and \( \hat{\xi} \) be such that \( \max(||\tilde{\xi}||_{\infty}, ||\hat{\xi}||_{\infty}) \leq \frac{1}{128\beta} e^{-\frac{1}{2}2|| \int_0^T r_s dK_s ||_{\infty}}. \) Then there exist solutions of BSDEs (3.7) and

\[ dY_t = (f(t, 0, 0) - f(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t))dK_t - d(\langle \tilde{N} \rangle_t g_t + \tilde{Z}_t^* dM_t) + d\tilde{N}_t, \]

\[ Y_T = \tilde{\xi} \]

and the triple \((Y, Z, N) = (\tilde{Y} + \dot{Y}, \tilde{Z} + \dot{Z}, \tilde{N} + \dot{N})\) satisfies BSDE

\[ dY_t = (f(t, 0, 0) - f(t, Y_t, \sigma^*_t Z_t))dK_t - d(\langle N \rangle_t g_t + Z^*_t dM_t) + dN_t, \]

\[ Y_T = \tilde{\xi} + \hat{\xi}. \]

**Proof.** Similarly to the Remark from Section 1 we can show that for \( \hat{f}(t, y, z) = f(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t) - f(t, y + \tilde{Y}_t, \sigma^*_t z + \sigma^*_t \tilde{Z}_t), \alpha_t = f_y(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t), \gamma_t = f_z(t, \tilde{Y}_t, \sigma^*_t \tilde{Z}_t) \) the estimate

\[ ||\hat{f}(t, y, z, z) - \hat{f}(t, y, z, z) - \alpha_t \delta y - \gamma_t \delta z|| \]

\[ \leq (r_t |\delta y| + \theta |\delta z|)(r_t (|y_1| + |y_2|) + \theta (|z_1| + |z_2|)). \]

holds.

Now by Lemma 1 and Corollary 2 of Lemma 1 we obtain the solvability of both equations (3.8), (3.7).

**Proposition 2.** Let \( f \) and \( g \) satisfy condition B1)-B3) and \( \xi \in L^\infty. \) Then BSDE (2.1) admits a solution \((Y, Z \cdot M + N) \in S^\infty \times BMO). \)
Proof. An arbitrary $\xi \in L^\infty(R)$ can be represented as sum $\xi = \sum_{i=1}^{m} \xi_i$ with $||\xi_i||_\infty \leq \frac{1}{32\beta} \exp(-2\|r_s dK_s\|_\infty)$. Denote by $(Y^j, Z^j, N^j)$, $j = 1, ..., m$ the solution of

$$
\begin{align*}
\text{d}Y^j_t &= (f(t, Y^j_0 + ... + Y^j_{t-1})_t, \sigma^*_t(Z^j_0 + ... + Z^j_{t-1}))
- f(t, Y^j_0 + ... + Y^j_t, \sigma^*_t(Z^j_0 + ... + Z^j_t))dK_t \\
- d\langle N^j \rangle_t + 2\langle N^j, N^0 + ... + N^{j-1} \rangle_t d_t + Z^j_t dM_t + dN^j_t,
\end{align*}
$$

(3.9)

By Corollary 1 we get

$$
||Y^j||^2_\infty + ||Z^j \cdot M^j + N^j||^2_{\text{BMO}(P)} \leq \frac{1}{128\beta^2},
$$

where $dP^j = \mathcal{E}_T (f_0 f(s, Y^j_0 + ... Y^j_{t-1}, \sigma^*_s(Z^j_0 + ... + Z^j_{t-1}))\sigma^*_s dM_s dP$, and $M^j = M - \langle f(\cdot, Y^j_0 + ... + Y^j_{t-1}, \sigma^*(Z^j_0 + ... + Z^j_{t-1}))\sigma^* dM, M = M \rangle$.

Using Lemma 2 we get the existence of a solution for BSDE

$$
\begin{align*}
\text{d}Y_t &= (f(t, 0, 0) - f(t, Y_t, \sigma^*_t Z_t))dK_t - d\langle N \rangle_t g_t + Z^*_t dM_t + dN_t, \\
Y_T &= \xi.
\end{align*}
$$

Since $\int_0^T f(t, 0, 0) dK_t$ is bounded we can apply the above argument with $f$ replaced by $\bar{f}(t, y, z) = f(t, y - \int_0^t f(s, 0, 0) dK_s, z)$ to get the existence of solution

$$
\begin{align*}
\text{d}\bar{Y}_t &= (f(t, 0, 0) - f(t, \bar{Y}_t - \int_0^t f(s, 0, 0) dK_s, \sigma^*_t Z_t))dK_t - d\langle N \rangle_t g_t + Z^*_t dM_t + dN_t, \\
\bar{Y}_T &= \xi + \int_0^T f(s, 0, 0) dK_s.
\end{align*}
$$

Obviously $Y_t = \bar{Y}_t - \int_0^t f(s, 0, 0) dK_s$ is a solution of BSDE $\text{(2.1)}, (2.2)$.

### 4 A comparison theorem for BSDEs

Let us consider BSDE $\text{(2.1)}, (2.2)$ in the case $d = 1$.

**Lemma 3.** Let $\xi \in L^\infty$ and assume that there are positive constants $C(f), C(g)$, increasing function $\lambda : R^+ \rightarrow R^+$, bounded on all bounded subsets and a predictable process $k \in H^2(R, 1)$ such that

$$
|f(t, y, z)| \leq k_t^2 \lambda(|y|) + C(f) z^2,
$$

(4.1)

$$
|g(t)| \leq C(g).
$$

(4.2)
Then the martingale part of any bounded solution of (2.1), (2.2) belongs to the space $\text{BMO}(P)$.

Proof. Let $Y$ be a solution of (2.1), (2.2) and there is a constant $C > 0$ such that
$$|Y_t| \leq C \quad \text{a.s for all } t.$$

Applying the Itô formula for $\exp\{\beta Y_T\} - \exp\{\beta Y_\tau\}$ and using the boundary condition $Y_T = \xi$ we have
$$\frac{\beta^2}{2} \int_\tau^T e^{\beta Y_s} Z_s^* d\langle M \rangle_s Z_s + \frac{\beta^2}{2} \int_\tau^T e^{\beta Y_s} d\langle N \rangle_s
- \beta \int_\tau^T e^{\beta Y_s} f(s, Y_s, Z_s) dK_s - \beta \int_\tau^T e^{\beta Y_s} g(s) d\langle N \rangle_s
+ \beta \int_\tau^T e^{\beta Y_s} Z_s^* dM_s + \beta \int_\tau^T e^{\beta Y_s} dN_s = e^{\beta \xi} - e^{\beta Y_\tau} \leq e^{\beta C},$$

where $\beta$ is a constant yet to be determined.

If $Z \cdot M$ and $N$ are square integrable martingales taking conditional expectations in (1.3) we obtain
$$\frac{\beta^2}{2} E\left( \int_\tau^T e^{\beta Y_s} Z_s^* d\langle M \rangle_s Z_s | F_\tau \right) + \frac{\beta^2}{2} E\left( \int_\tau^T e^{\beta Y_s} d\langle N \rangle_s | F_\tau \right)
\leq e^{\beta C} + \beta E\left( \int_\tau^T e^{\beta Y_s} f(s, Y_s, Z_s) | dK_s | F_\tau \right) + \beta E\left( \int_\tau^T e^{\beta Y_s} g(s) | d\langle N \rangle_s | F_\tau \right)$$

Now if we use the estimates (4.1), (4.2) we get
$$\frac{\beta^2}{2} E\left( \int_\tau^T e^{\beta Y_s} Z_s^* d\langle M \rangle_s Z_s | F_\tau \right) + \frac{\beta^2}{2} E\left( \int_\tau^T e^{\beta Y_s} d\langle N \rangle_s | F_\tau \right)
\leq e^{\beta C} + \beta \lambda(C) E\left( \int_\tau^T e^{\beta Y_s} k_s^2 dK_s | F_\tau \right)
+ \beta C(f) E\left( \int_\tau^T e^{\beta Y_s} \sigma_s^2 Z_s^2 dK_s | F_\tau \right) + \beta E\left( \int_\tau^T e^{\beta Y_s} g(s) | d\langle N \rangle_s | F_\tau \right)
\leq e^{\beta C} + \beta \lambda(C) E\left( \int_\tau^T e^{\beta Y_s} k_s^2 dK_s | F_\tau \right)
+ \beta C(f) E\left( \int_\tau^T e^{\beta Y_s} Z_s^2 d\langle M \rangle_s Z_s | F_\tau \right) + C(g) \beta E\left( \int_\tau^T e^{\beta Y_s} d\langle N \rangle_s | F_\tau \right).$$

Conditions (4.1) and (4.2) imply that
$$\frac{\beta^2}{2} - \beta C(f)) E\left( \int_\tau^T e^{\beta Y_s} Z_s^* d\langle M \rangle_s Z_s | F_\tau \right) +$$
$$+ \frac{\beta^2}{2} - \beta C(g)) E\left( \int_\tau^T e^{\beta Y_s} d\langle N \rangle_s | F_\tau \right) \leq$$
$$\leq e^{\beta C} + \beta \lambda(C) E\left( \int_\tau^T e^{\beta Y_s} k_s^2 dK_s | F_\tau \right).$$
Taking $\beta = 4\overline{C}$, where $\overline{C} = \max(C(f), C(g))$, from (4.4) we have

$$4\overline{C}[E\left(\int_{\tau}^{T} e^{\beta s} Z_s^* d\langle M \rangle_s Z_s^* | F_{\tau}\right) + E\left(\int_{\tau}^{T} e^{\beta s} d\langle N \rangle_s | F_{\tau}\right)] \leq$$

$$\leq e^{4\overline{C}}(4\overline{C} \lambda(C) ||k||_H + 1).$$

Since $Y \geq -C$, from the latter inequality we finally obtain the estimate

$$E\left(\langle Z \cdot M \rangle_{\tau T} | F_{\tau}\right) + E\left(\langle N \rangle_{\tau T} | F_{\tau}\right) \leq$$

$$\leq \frac{e^{8\overline{C}}[4\overline{C} \lambda(C) ||k||_H + 1]}{4\overline{C}^2}$$

for any stopping time $\tau$, hence $Z \cdot M, N \in BMO$.

For general $Z \cdot M$ and $N$ we stop at $\tau_n$ and derive (4.5) with $T$ replaced $\tau_n$. Letting $n \to \infty$ then completes the proof. \hfill \qed

Further we use some notations. Let $(Y, Z), (\tilde{Y}, \tilde{Z})$ be two pairs of processes and $(f, g, \xi), (\tilde{f}, \tilde{g}, \tilde{\xi})$ two triples of generators. Then we denote:

$$\delta f = f - \tilde{f}, \ \delta g = g - \tilde{g}, \ \delta \xi = \xi - \tilde{\xi},$$

$$\partial_y f(t, Y_t, \tilde{Y}_t, Z_t) = \partial f_y(t) = \frac{f(t, Y_t, Z_t) - f(t, \tilde{Y}_t, Z_t)}{Y_t - \tilde{Y}_t}$$

for all $j = 1, \ldots, n, \ \partial_j f(t, \tilde{Y}_t, Z_t, \tilde{Z}_t) \equiv \partial_j f(t)

$$= \frac{f(t, \tilde{Y}_t, Z_t^1, \ldots, Z_t^{j-1}, Z_t^j, \tilde{Z}_t^{j+1}, \ldots, \tilde{Z}_t^n) - f(t, \tilde{Y}_t, Z_t^1, \ldots, Z_t^{j-1}, \tilde{Z}_t^j, \tilde{Z}_t^{j+1}, \ldots, \tilde{Z}_t^n)}{Z_t^j - \tilde{Z}_t^j},$$

$$\nabla f(t) = (\partial_t f(t), \ldots, \partial_n f(t))^*$$

Thus we have

$$f(t, Y_t, Z_t) - f(t, \tilde{Y}_t, \tilde{Z}_t) = \partial_y f(t) \delta Y_t + \nabla f(t)^* \delta Z_t.$$  \hfill (4.6)

**Theorem 2.** Let $Y$ and $\tilde{Y}$ be the bounded solutions of SBE (2.1) with generators $(f, g, \xi)$ and $(\tilde{f}, \tilde{g}, \tilde{\xi})$ respectively, satisfying the conditions of Lemma 3.

If $\xi \geq \tilde{\xi}$ (a.s), $f(t, y, z) \geq \tilde{f}(t, y, z) \ (\mu^K-\text{a.e.}), \ g(t) \geq \tilde{g}(t) \ (\mu^N-\text{a.e.})$ and $f$ (or $\tilde{f}$) satisfies the following Lipschitz condition:

L1) for any $Y, \tilde{Y}, Z$

$$\frac{f(t, Y_t, Z_t) - f(t, \tilde{Y}_t, Z_t)}{Y_t - \tilde{Y}_t} \in S^\infty,$$

L2) for any $Z, \tilde{Z} \in H^2$ and any bounded process $Y$

$$(\sigma_t \sigma_t^*)^{-1} \nabla f(t, Y_t, Z_t, \tilde{Z}_t) \in H^2(R^n, \sigma),$$

then $Y_t \geq \tilde{Y}_t$ a.s. for all $t \in [0, T]$. \hfill 11
Proof. Taking the difference of the equations (2.1), (2.2) with generators \((f, g, \xi)\) and \((\tilde{f}, \tilde{g}, \tilde{\xi})\) respectively, we have

\[
Y_t - \tilde{Y}_t = Y_0 - \tilde{Y}_0 \\
- \int_0^t [f(s, Y_s, Z_s) - f(s, \tilde{Y}_s, \tilde{Z}_s)]dK_s \\
- \int_0^t [f(s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s)]dK_s - \int_0^t [g(s) - \tilde{g}(s)]d\langle N \rangle_s \\
- \int_0^t \tilde{g}(s)d(\langle N \rangle_s - \langle \tilde{N} \rangle_s) + \int_0^t (Z_s - \tilde{Z}_s)dM_s + N_t - \tilde{N}_t. \tag{4.7}
\]

Let us define the measure \(Q\) by

\[
dQ = E_T(\Lambda) dP,
\]

where

\[
\Lambda_t = \int_0^t \nabla f(s)^*(\sigma_s\sigma_s^*)^{-1}dM_s + \int_0^t \tilde{g}(s)d(N_s + \tilde{N}_s).
\]

By Lemma 3 \(Z, \tilde{Z} \in H^2\) and \(N, \tilde{N}\) are BMO- martingales. Therefore Condition \(L1, L2\) and (4.2) imply that \(\Lambda \in BMO\) and hence \(Q\) is a probability measure equivalent to \(P\).

Denote by \(\bar{\Lambda}\) the martingale part of \(\delta Y = Y - \tilde{Y}\), i.e.,

\[
\bar{\Lambda} = (Z - \tilde{Z}) \cdot M + N - \tilde{N}.
\]

Therefore, by Girsanov’s Theorem and by (1.6) the process

\[
\delta Y_t + \int_0^t (\partial_y f(s)\delta Y_s + \nabla f(s)^*\delta Z_s)dK_s \\
+ \int_0^t \delta f(s, \tilde{Y}_s, \tilde{Z}_s)dK_s + \int_0^t \delta g(s)d(\langle N \rangle_s) \\
= \delta Y_t + \int_0^t (\partial_y f(s)\delta Y_s + \delta f(s, \tilde{Y}_s, \tilde{Z}_s))dK_s \\
+ \int_0^t \nabla f(s)^*(\sigma_s\sigma_s^*)^{-1}d\langle M \rangle_s dZ_s + \int_0^t \delta g(s)d\langle N \rangle_s \\
= - \int_0^t \tilde{g}(s)d(\langle N \rangle_s - \langle \tilde{N} \rangle_s) + \int_0^t (Z_s - \tilde{Z}_s)dM_s + N_t - \tilde{N}_t \\
= \bar{\Lambda}_t - \langle \Lambda, \bar{\Lambda} \rangle_t,
\]

is a local martingale under \(Q\). Moreover, since by Lemma 3 \(\tilde{N} \in BMO\), Proposition 11 of [6] implies that

\[
\bar{\Lambda}_t - \langle \Lambda, \bar{\Lambda} \rangle_t \in BMO(Q).
\]

Thus, using the martingale property and the boundary conditions \(Y_T = \xi, \tilde{Y}_T = \tilde{\xi}\) we have

\[
Y_t - \tilde{Y}_t =
\]
\[ E^Q \left( e_t^T \partial_y f_s dK_s (\xi - \tilde{\xi}) \right) \]
\[ + \int_t^T e_s^T \partial_y f_s dK_s (f(s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s)) dK_s |F_t \]
\[ + E^Q \left( \int_t^T e_s^T \partial_y f_s dK_s (g(s) - \tilde{g}(s)) d\langle N \rangle_s |F_t \right), \]

which implies that \( Y_t \geq \tilde{Y}_t \) a.s. for all \( t \in [0, T] \).

**Corollary.** Let condition A) be satisfied. Then if the solution of (2.1), (2.2) exists it is unique.

The proof of **Theorem 1** follows now from the last corollary and Proposition 2.

**Remark.** Condition L1), L2) is satisfied if there is constant \( C > 0 \) such that

\[ |f(t, y, z) - f(t, \tilde{y}, \tilde{z})| \leq C|y - \tilde{y}| + C|z - \tilde{z}||z| + |\tilde{z}| \]

and \( tr(\sigma_t \sigma_t^*)^{-1} \leq C \) for all \( y, \tilde{y} \in R, z, \tilde{z} \in R^n \) \( t \in [0, T] \). Conditions L1), L2) are also fulfilled if \( f(t, y, z) \) satisfies the global Lipschitz condition and \( M \in BMO \).

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