On variation of zeros of classical discrete orthogonal polynomials

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Abstract
The purpose of this note is to establish, from the hypergeometric-type difference equation introduced by Nikiforov and Uvarov, new tractable sufficient conditions for the monotonicity with respect to a real parameter of zeros of classical discrete orthogonal polynomials. This result allows one to carry out a systematic study of the monotonicity of zeros of classical orthogonal polynomials on linear, quadratic, q-linear, and q-quadratic grids. In particular, we analyze in a simple and unified way the monotonicity of the zeros of Hahn, Charlier, Krawtchouk, Meixner, Racah, dual Hahn, q-Meixner, quantum q-Krawtchouk, q-Krawtchouk, affine q-Krawtchouk, q-Charlier, Al-Salam-Carlitz, q-Hahn, little q-Jacobi, little q-Laguerre/Wal, q-Bessel, q-Racah and dual q-Hahn polynomials.

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1. Introduction
The properties of Jacobi polynomials, \( P_n^{(\alpha,\beta)}(x) \) \((n = 1, 2, \ldots; \alpha > -1, \beta > -1)\), may well consult in [23, Chapter IV]; they are orthogonal on \([-1, 1]\) with the

\[ \sum_{n=0}^{\infty} \frac{(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n + 1)} \frac{1}{n!^2} z^n = \frac{\sqrt{1-z}}{(1-\alpha z - \beta z)}. \]
weight function \((1 - X)^\alpha(1 + X)^\beta\), and

\[
y = P_n^{(\alpha, \beta)}(X) = \frac{1}{2F_1 \left( \begin{array}{c} n, n + \alpha + \beta + 1 \\ 1 + \alpha \end{array} \right)} \frac{1 - X}{2},
\]

satisfy the homogeneous differential equation of second order

\[
a y'' + b y' + c y = 0, \tag{1.1}
\]

where

\[
a = a(X) = -X^2 + 1 \quad \text{and} \quad b = b(X; \alpha, \beta) = -(\alpha + \beta + 2)X - \alpha + \beta.
\]

(Having the explicit expression of \(P_n^{(\alpha, \beta)}\), the parameter \(c\) does not play any interesting role and it can be easily calculated.) Recall that the hypergeometric function \(iF_j\) is formally defined by the series (see \([1, (2.1.2)]\))

\[
iF_j \left( \begin{array}{c} \alpha_1, \ldots, \alpha_i \\ \beta_1, \ldots, \beta_j \end{array} \right) X = \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_i)_k}{(\beta_1, \ldots, \beta_j)_k} X^k
\]

where

\[
(\alpha_1, \ldots, \alpha_i)_k = (\alpha_1)_k \cdots (\alpha_i)_k, \quad (\alpha_1)_k = \prod_{j=1}^{k} (\alpha_1 + j - 1);
\]

by convention, the empty product is 1. From \((1.1)\), Stieltjes proves an important statement concerning the dependence of the zeros of Jacobi polynomials on the parameters \(\alpha\) and \(\beta\) (see \([21]\)). Soon after his work has been accepted for publication, Stieltjes implicitly acknowledges, in a note added at the end of the manuscript itself and also in a letter of February 3, 1887 to Hermite (see \([22, \text{Lettre 106}]\)), that the monotonicity of the zeros of Jacobi polynomials was previously proved by A. Markov in \([3]\). However, in Stieltjes’ work, \textit{ut in multis alius rebus}, the “How” is more important than the “What” and, as he wrote to Hermite, “\textit{la démonstration que j’ai développée pour les Acta Mathematica est différente de celle de M. Markoff}.” Historically, Stieltjes proves that given a positive definite real symmetric matrix with non-positive off-diagonal elements\(^1\), its inverse is also positive definite. As a consequence, putting aside a clever manipulation of the differential equation \((1.1)\), he shows that, since

\[
\frac{b}{a} = \frac{-(\alpha + \beta + 2)X - \alpha + \beta}{-X^2 + 1}
\]

is a strictly decreasing function of \(X \in (-1, 1)\), and

\[
\frac{\partial}{\partial \alpha} \left( \frac{b}{a} \right) = \frac{1}{X - 1} < 0, \quad \frac{\partial}{\partial \beta} \left( \frac{b}{a} \right) = \frac{1}{X + 1} > 0,
\]

for each \(X \in (-1, 1)\), the zeros of Jacobi polynomials are strictly decreasing functions of \(\alpha\) on \((-1, \infty)\) and strictly increasing functions of \(\beta\) on \((-1, \infty)\) (see \([24, \text{Definition 3.4}]\)).

\(^1\)These matrices are now known as Stieltjes’ matrices (see \([24, \text{Definition 3.4}]\)).
The proof of Markov is entirely different from that of Stieltjes and is based on the weight function. While it is true that Stieltjes worked directly with Jacobi polynomials—and Markov proves his result through a general theorem—his argument furnishes similar results for a more general differential equation (see [23], Section 6.22). Stieltjes himself considered the ultraspherical case $\alpha = \beta$ and Szegő noted that “the same method applies to Laguerre polynomials” (see [23], p. 123). In particular, by definition, the old classical orthogonal polynomials on the real line (Hermite, Jacobi, and Laguerre) are solutions of a differential equation of the same type of (1.1) (see [20], Section 4.2). However, only Jacobi and Laguerre polynomials depend on a real parameter and, therefore, the Stieltjes result is no longer applicable in this framework.

From a truly practical point of view—for example in Physics, which was traditionally the birthplace of some of the most beautiful families (see [25])—, it rarely will require more than the classical orthogonal polynomials. For classical (discrete) orthogonal polynomials on a uniform grid (Charlier, Krawtchouk, Hahn, and Meixner polynomials), Markov’s theorem can be used (see [4], Chapter 7) and, of course, for other discrete families. Here we propose an alternative approach, establishing a bridge between two works separated in time by almost one century, on one hand the work of Stieltjes and, on the other hand, the work [14] by Nikiforov and Uvarov. This allows one to carry out a systematic study of the monotonicity of zeros of classical orthogonal polynomials on linear, quadratic, q-linear, and q-quadratic grids. Indeed, the purpose of this note is to prove that under suitable regularity conditions the Stieltjes result for Jacobi polynomials remains valid if we replace the differential equation (1.1), with $a$ and $b$ arbitrary polynomials of degree at most 2 and 1, respectively, and $c$ an arbitrary constant (from now on, whenever we refer to (1.1), we are assuming these conditions), by the following difference equation introduced in [14], (5) (see also [15], p. 127 and [11], p. 71):

$$a(X) \frac{\Delta}{\Delta X} (s-1/2) \left( \frac{\nabla y(X)}{\nabla X} \right) + b(X) \frac{\nabla y(X)}{2 \Delta X} + c y(X) = 0,$$

(1.4)

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2 We write “classical orthogonal polynomials on the real line” rather than simply “classical orthogonal polynomials” because, for instance, from the algebraic point of view of Maroni (see [9]), the Jacobi polynomials exist and are “classical” even when $-\alpha, -\beta, -\alpha - \beta + 1 \notin \mathbb{N}$ (see [15], Chapters 8 and 9 for a recent survey on the subject). Moreover, the Bessel polynomials are classical in the same sense as the other three systems. As Maroni says, “comme dans le roman d’Alexandre Dumas, les trois mousquetaires étaient quatre en réalité”.

3 A non-empty (totally) ordered set of equidistant (respectively, non-equidistant) points is called uniform (respectively, non-uniform) grid, which in turn is an elementary example of lattice.

4 For a “continuous” case as the Askey-Wilson polynomials, Askey and Wilson used a consequence of Markov’s theorem, which goes back to Szegő (see [23], Theorem 6.12.2), to study the monotonicity of zeros of these polynomials (see [2], Section 7).
or, equivalently,

$$a(s) \frac{\Delta}{\Delta x(s - 1/2)} \left( \nabla y(X) \right) + b(X) \frac{\Delta y(X)}{\Delta X} + cy(X) = 0, \quad (1.5)$$

where

$$a(s) = a(X) - \frac{1}{2} b(X) \Delta x \left( s - \frac{1}{2} \right).$$

$X = x(s)$ defines a class of grids with, generally nonuniform, step-size $\Delta X = \Delta x(s) = x(s + 1) - x(s)$ and $\nabla X = \nabla x(s) = \Delta x(s - 1)$. (By abuse of notation, we use the same letter $a$ for the function $a(s)$ and the polynomial $a(X)$.) In what follows, we assume that $x$ is a real-valued function defined on an interval of the real line. Any solution of (1.4) can be brought in correspondence with the solution of (1.1) by replacing $s$ by $s/h$ and then taking limit $h \to 0$, whenever it exists. It is important to highlight that (1.4) has polynomial solutions, in $X$, whose difference-derivatives satisfy equations of the same kind if and only if, for $q \neq 1$ fixed, $x$ is a linear, quadratic, q-linear, or q-quadratic grid of the form

$$x(s) = \begin{cases} 
C_1 s^2 + C_2 s, \\
C_3 q^{-s} + C_4 q^s,
\end{cases}$$

where $(C_1, C_2) \neq (0, 0)$ and $(C_3, C_4) \neq (0, 0)$ (see [3, (1.68)]). The grids that depend on “q” are called q-linear if $C_1$ or $C_4$ is zero; otherwise it is q-quadratic. By using linear transformations (see [12, (3.4.1)]) we can reduce the expressions for the grids to simpler forms. In what follows, we assume that the grid $x$ takes on the following canonical forms (see [12, p. 74]):

$$x(s) = \begin{cases} 
s, \\
s(s + 1), \\
q^s \quad (q > 1), \\
\frac{1}{2}(q^s - q^{-s}) \quad (q > 1), \\
\frac{1}{2}(q^s + q^{-s}) \quad (q > 1), \\
\frac{1}{2}(q^s + q^{-s}) \quad (q = e^{2i\theta}, \ 0 < \theta < \pi/2).
\end{cases} \quad (1.6)$$

Under the above notation, we adopt the following definition of classical discrete orthogonal polynomials on the real line (COPRL), which is enough for our purpose:

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5In first edition of “Special Functions of Mathematical Physics” (see [13]) the author only consider the case $x(s) = s$. The second edition (see [15]) was significantly enriched with the equation (1.4); although A. A. Samarskii’s preface is the same in both editions.
Definition 1.1. Fix \( a \in \mathbb{R} \cup \{-\infty\} \) and \( N \in \mathbb{N} \cup \{\infty\} \) and define \( b = a + N \). Fix \( q \) and let \( X = x(s) \) be a real-valued function given by (1.6), where the variable \( s \) ranges over the finite interval \([a, b]\) or the infinity interval \([a, \infty)\). A sequence of polynomials, \( (P_n(X))_{n=0}^N \), is said to be sequence of classical discrete orthogonal polynomials on the set \( \{x(a), x(a+1), \ldots, x(b-1)\} \) or, simply, COPRL if:

\[ (i) \quad P_n \text{ satisfy (1.4), } x \text{ being a strictly monotone function on } [a, b] \text{ or } [a, \infty) \text{ given, up to a linear transformation, by (1.6);} \]

\[ (ii) \quad \text{a positive weight function } \omega \text{ satisfying the boundary conditions (1.7)} \]

\[ \omega(s)a(s)x^k \left(s - \frac{1}{2}\right) \bigg|_{a,b} = 0 \quad (k = 0, 1, \ldots) \]

\[ (iii) \quad \text{the difference equation (1.8)} \]

\[ \Delta x \left(s - \frac{1}{2}\right) \left(\omega(s)a(s)\right) = \omega(s)b(X) \]

Subsequently, when we say that a certain polynomial is a COPRL, we are assuming the definition and notation given in Definition 1.1. From (1.7), (1.8), and (1.9), we conclude that the COPRL satisfy the orthogonality condition (see (12, (3.3.4)))

\[ \sum_{s=a}^{b-1} P_m(X)P_n(X)\omega(s)\Delta x \left(s - \frac{1}{2}\right) = 0 \quad (m \neq n). \] (1.9)

The polynomials \( (P_n(X))_{n=0}^N \) given in Definition 1.1 are called simply discrete orthogonal polynomials, in \( X \), on the set \( \{x(a), x(a+1), \ldots, x(b-1)\} \) with respect to a positive weight function \( \omega \) if they satisfy the relation (1.3) instead of the requirements (i) – (iii). Of course, COPRL are a special case of discrete orthogonal polynomials. From (1.9), we can see that the zeros of discrete orthogonal polynomials on \( \{x(a), x(a+1), \ldots, x(b-1)\} \) are real and distinct and are located in \( (\min\{x(a), x(b-1)\}, \max\{x(a), x(b-1)\}) \) (see [23, Theorem 3.3.1]). In concluding this section we remark that it is possible to obtain a series repre-
sentation of COPRL (see [4, (4.19)] and [16, Section 3]):

\[
P_n(X) = (-1)^n \gamma_n \sum_{j=0}^{n} \binom{n}{j} q^{-n/2} \frac{\Delta x(s - (n-1)/2 + j)}{\prod_{k=0}^{n} \Delta x(s + (j - k + 1)/2)}
\]

\[
\times \prod_{k=0}^{n} a(s - n + j + k) \prod_{l=0}^{j} \frac{a(x(s + l - 1)) + 1/2 b(x(s + l - 1)) \Delta x(s + l - 1/2)}{a(x(s + l)) - 1/2 b(x(s + l)) \Delta x(s + l + 1/2)}
\]

where \(\gamma_n\) is a constant and

\[
\binom{n}{j} = \frac{(q;q)_n}{(q;q)_j(q;q)_{n-j}}.
\]

Furthermore (see [16]), COPRL represent special cases of hypergeometric series or q-hypergeometric series, the latter formally defined by (see [1, (10.9.4)]

\[
i \phi_j \left( \begin{array}{c} \alpha_1, \ldots, \alpha_i \\ \beta_1, \ldots, \beta_j \end{array} \mid q, X \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_i; q)_k}{(\beta_1, \ldots, \beta_j; q)_k} (-1)^{1-i+j} k \left( \begin{array}{c} k \\ 2 \end{array} \right) X^k (q; q)_k.
\]

The outline of this note is as follows. In Section 2 we present an extension of the Stieltjes work in the framework of COPRL. In Section 3 we study the variation of zeros of some families of COPRL according to the type of underlying grid.

2. Main results

Unless otherwise stated we assume that \(a, b,\) and \(c\) appearing in (1.4) depend on a parameter \(t\) varying in a non-degenerate open interval of the real line. Rewrite (1.4) in the more suggestive form

\[
Ay(s - 1) + By(s + 1) + Cy(s) = 0,
\]

where \(y(s) = y(X; t)\) and

\[
A = A(s; t) = \frac{a(s; t)}{\nabla X \Delta x(s - 1/2)}, \quad B = B(s; t) = \frac{a(s; t) + b(X; t) \Delta x(s - 1/2)}{\Delta X \Delta x(s - 1/2)},
\]

\[
C = C(s; t) = c(t) - A(s; t) - B(s; t).
\]

The following example will help motivate our main result.

**Example 2.1.** In 1960, Karlin and McGregor proved (see [4, (1.3)] and [2, (5.1)]) that the COPRL known as Hahn polynomials (see [4, Section 9.5])

\[
y(s) = H_n^{(\alpha, \beta)}(X) = {}_3F_2 \left( \begin{array}{c} -n, -X, \alpha + \beta + n + 1 \\ \beta + 1, 1 - N \end{array} \mid 1 \right)
\]
(n = 1, . . . , N − 1; a = 0, b = N; α > −1, β > −1), which constitute the finite discrete analogue of Jacobi polynomials considered by Stieltjes in [21]—satisfy

\[ A = A(X; \alpha) = X(−X + \alpha + N), \quad B = B(X; \beta) = (X + \beta + 1)(−X + N − 1). \]

The function

\[ \frac{B}{A} = \frac{(X + \beta + 1)(−X + N − 1)}{X(−X + \alpha + N)} \] (2.13)

is a positive and strictly decreasing function of \( X \in (a, b − 1) \), and

\[ \frac{\partial}{\partial \alpha} \left( \frac{B}{A} \right) = \frac{(X + \beta + 1)(X − N + 1)}{X(−X + \alpha + N)^2} < 0, \] (2.14)

\[ \frac{\partial}{\partial \beta} \left( \frac{B}{A} \right) = \frac{−X + N − 1}{X(−X + \alpha + N)} > 0. \] (2.15)

for each \( X \in (a, b − 1) \). Since this example corresponds to the grid (I), from Markov’s theorem, Ismail proves that the zeros of \( H_n^{(\alpha, \beta)} \) are decreasing functions of \( \alpha \) on \((-1, \infty)\) and increasing functions of \( \beta \) on \((-1, \infty)\) (see [7, Theorem 7.1.2]). Comparison of (1.12) and (2.14), and (1.13) and (2.15) suggests that, as for Jacobi polynomials, the information on the monotonicity of the zeros of Hahn polynomials is stored in the rational function \( B/A \). Therefore, it is not unreasonable to conjecture that the same happens with any COPRL.

The following two mathematical objects play a central role in our exposition.

**Definition 2.1.** Let \( A \) and \( B \) be given by (2.12). The function \( f \), from now on called monotonicity function, is defined by

\[ f(s; t) = \frac{B(s; t)}{A(s; t)} = \frac{a(s; t) + b(X; t)\Delta x(s − 1/2)\Delta X}{a(s; t)} \] (2.16)

**Definition 2.2.** Let \( X = x(s) \) be given by (1.1) and let \( x(y_j(t)) \) \( (j = 1, . . . , n) \) be the zeros of a COPRL of degree \( n \), say \( P_n(X; t) \), depending on a parameter \( t \) taking values on \( I \subseteq \mathbb{R} \). For any set \( I \subseteq \mathbb{R} \), the (nonempty) subset \( S^{(t)}_I(P) \) of \((a, b − 1)\) is defined by

\[ S^{(t)}_I(P_n) = \left\{ y \in (a, b − 1) \mid t \in I \cap j \wedge (\forall j \in \{1, . . . , n\}) \{ y = y_j(t) \} \right\}. \]

We will write it simply \( S_I(P_n) \), \( S^{(t)}_I \) or \( S_I \) when no confusion can arise.\(^7\)

To prove our main result we need two lemmas. The first one was proved for the Hahn polynomials by Levit (see [7, Theorem 3]). Here we reproduce in a more general framework, mutatis mutandis, his arguments.

\(^7\)Note that \( S_J = S_\mathbb{R} \) and \( S_I \subseteq S_J \) whenever \( I \subseteq J \).
Lemma 2.1. Let $X = x(s)$ be given by (1.6). Let $x(y(t))$ and $x(z(t))$ be consecutive zeros of a COPRL depending on a real parameter $t$. Let $f$ be the monotonicity function given by (2.10). Then $|z(t) - y(t)| > 1$ for those values of $t$ such that $f(\cdot; t) > 0$ on $S_R$.

Proof. We give the proof only for the case in which $x$ is a strictly increasing function. The same arguments apply to the case in which $x$ is a strictly decreasing function. There is no loss of generality in assuming that $x(y(t)) < x(z(t))$ are the greatest pair of consecutive zeros of a COPRL, say $P(X; t)$. (Recall that the zeros of $P$ are distinct and are located in $(a, b - 1)$.) Moreover, under our assumptions, $y(t) < z(t)$. Suppose that there exists $t_0$ such that $z(t_0) = y(t_0) + 1$. Replacing $s$ by $z(t_0)$ and $t$ by $t_0$ in (2.11) we have $P(x(z(t_0) + 1); t_0) = 0$, because $f(z(t_0); t_0) < 0$, which contradicts the assumption that $x(z(t_0))$ is the greatest zero of $P(\cdot; t_0)$. (Here we have also used the fact that $x$ is, in particular, a strictly increasing function on $[a, b]$).

Suppose now that $z(t_0) < y(t_0) + 1$. Replacing again $s$ by $z(t_0)$ and $t$ by $t_0$ in (2.11) we have

$$-P(x(z(t_0) - 1); t_0) P(x(z(t_0) + 1); t_0) = f(z(t_0); t_0) > 0.$$ 

Since $z(t_0) < y(t_0) + 1$, there is an odd number of zeros of $P(\cdot; t_0)$ in $[x(z(t_0) - 1), x(z(t_0) + 1)]$. We next claim that there exists at least one integer $m$ ($1 \leq m \leq N - 1$) such that $y(t_0) < a + m \leq z(t_0)$ or, what is the same, $x(y(t_0)) < x(a + m) \leq x(z(t_0))$. Suppose the assertion were false. Hence

$$\sum_{s=a}^{b-1} \frac{P^2(X; t_0)}{(X - x(y(t_0)))(X - x(z(t_0)))} \omega(s; t_0) \Delta x \left( s - \frac{1}{2} \right) > 0,$$

a contradiction with (1.9). Since $z(t_0) < y(t_0) + 1$, $m$ is unique. Thus the only zeros of $P(\cdot; t_0)$ in $[x(z(t_0) - 1), x(z(t_0) + 1)]$ are $x(y(t_0))$ and $x(z(t_0))$. This contradicts the fact that there is an odd number of zeros of $P(\cdot; t_0)$ in $[x(z(t_0) - 1), x(z(t_0) + 1)]$, and the lemma follows.

Lemma 2.2. Let $X = x(s)$ be given by (1.6). Let $P$ be a non-constant COPRL, in $X$, satisfying (2.11) and depending on a parameter $t$ varying in a non-degenerate open interval of the real line containing $t_0$. Suppose that $A(\cdot; t)$ and $B(\cdot; t)$ given by (2.12) admit partial derivatives with respect to $t$ on a neighbourhood of $t_0$. Assume that $P(Y_0; t_0) = 0$. Then there exist $\epsilon > 0$ and $\delta > 0$ such that $(Y_0 - \delta, Y_0 + \delta) \times (t_0 - \epsilon, t_0 + \epsilon)$ is on the neighbourhood where $P$ is defined, and there exists $Y : (t_0 - \epsilon, t_0 + \epsilon) \to (Y_0 - \delta, Y_0 + \delta)$, such that

$$P(Y(t); t) = 0$$

and, for each $t \in (t_0 - \epsilon, t_0 + \epsilon)$, $Y$ is the unique solution of (2.17) with $Y(t) \in (Y_0 - \delta, Y_0 + \delta)$. Moreover, $Y$ possess a continuous derivative on $(t_0 - \epsilon, t_0 + \epsilon).$
Proof. From (1.10) we see that the coefficients of \( P(\cdot; t) \) are differentiable functions of \( t \). Moreover, \( P(Y_0; t_0) = 0 \); from this it follows that
\[
\left. \frac{\partial P}{\partial X}(X; t) \right|_{X = Y_0, t = t_0} \neq 0,
\]
because the zeros of \( P(\cdot; t_0) \) are distinct. Thus, the result is a direct consequence of the implicit function theorem (see [19, Theorem 3.4.2]). \( \square \)

We shall refer to Theorem 2.1 below as discrete Stieltjes theorem.

**Theorem 2.1.** Assume the hypotheses and notation of Lemma 2.2. Let \( f \) be the monotonicity function given by (2.10). Denote \( f_1(s; \cdot) = (\partial f/\partial s)(s; \cdot) \) and \( f_2(s; t) = (\partial f/\partial t)(s; t) \). Suppose that \( f(s; t) > 0 \) and \( f_1(s; t) < 0 \) on \( S_{(t_0 - \epsilon, t_0 + \epsilon)} \). In the case of the grid (IV), assume also that \( n f(s; \cdot) + f_1(s; \cdot) \leq 0 \). Suppose furthermore that \( f_2(s; t) > 0 \) (respectively, \( f_2(s; t) < 0 \)) on \( S_{(t_0 - \epsilon, t_0 + \epsilon)} \). Then \( Y \) is a strictly increasing (respectively, decreasing) function on \( (t_0 - \epsilon, t_0 + \epsilon) \) if \( x \) is a strictly increasing function, or else \( Y \) is a strictly decreasing (respectively, increasing) function on \( (t_0 - \epsilon, t_0 + \epsilon) \).

**Proof.** We give the proof only for the case in which \( x \) is a strictly increasing function. The same arguments apply to the case in which \( x \) is a strictly decreasing function. Assume that \( P \) is monic and has fixed degree \( n \). Let \( Y_j \) \( (j = 1, \ldots, n) \) denote the zeros of \( P \). Since the indeterminate of \( P \) takes values on the real (open) interval \( x((a, b - 1)) \), there exist functions \( y_j \) defined on a neighbourhood of \( t_0 \) and taking values on \( (a, b - 1) \) such that \( Y_j(t) = x(y_j(t)) \). Moreover, since \( x \) is strictly increasing on \( (a, b - 1) \), \( x^{-1} \) is differentiable on \( x((a, b - 1)) \). Hence, by Lemma 2.2 there exists a neighbourhood of \( t_0 \) where \( y_j \) is differentiable.

Let
\[
P(X; t) = \prod_{j=1}^{n} (X - Y_j(t))
\]
defined in a neighbourhood of \( t_0 \). Replacing \( s \) by \( y_j(t) \) in (2.11) we have
\[
f(y_j(t); t) = -\frac{P(x(y_j(t) - 1); t)}{P(x(y_j(t) + 1); t)}.
\]
(2.18)

Taking the partial derivative of (2.18) with respect to \( t \) on the neighbourhood of \( t_0 \) leads to
\[
y_j'(t)f_1(y_j(t); t) + f_2(y_j(t); t)
\]
\[
= \frac{P(x(y_j(t) - 1); t) \frac{\partial P}{\partial t}(x(y_j(t) + 1); t) - P(x(y_j(t) + 1); t) \frac{\partial P}{\partial t}(x(y_j(t) - 1); t)}{P^2(x(y_j(t) + 1); t)}.
\]
where

\[
\frac{\partial P}{\partial t}(x(y(t) \pm 1); t) = P(x(y(t) \pm 1); t) \sum_{k=1}^{n} \frac{\frac{dx}{ds}(s \pm 1)}{x(y(t) \pm 1) - Y_k(t)} \frac{\frac{dy_j(t)}{ds}(s) - \frac{dy_j(t)}{ds}(s)}{x(y(t) \pm 1) - Y_k(t)}
\]

Hence

\[
f_2(y(t); t) = \sum_{k=1}^{n} a_{jk}(t) y_k'(t),
\]

where

\[
a_{jj}(t) = -f_1(y(t); t) + f(y(t); t) \sum_{k \neq j \neq k} b_{jk}(t) - \sum_{k=1}^{n} a_{jk},
\]

with

\[
b_{jk}(t) = \frac{\frac{dx}{ds}(s - 1)}{x(y(t) - 1) - Y_k(t)} - \frac{\frac{dx}{ds}(s) - \frac{dx}{ds}(s)}{x(y(t) + 1) - Y_k(t)},
\]

and

\[
a_{jk}(t) = f(y(t); t) c_{jk}(t) (j \neq k),
\]

where

\[
c_{jk}(t) = \begin{pmatrix} 1 \\ x(y(t) + 1) - Y_k(t) - x(y(t) - 1) - Y_k(t) \end{pmatrix} \frac{\frac{dx}{ds}(s)}{s=y_k(t)}.
\]

We claim that \(a_{jk}(t) < 0\) \((j \neq k)\). From now on, without loss of generality, we assume \(Y_j(t) < Y_k(t)\). Then, by Lemma 2.3, \(x(y(t) + 1) < Y_k(t)\). Since \(\frac{dx}{ds}(s)_{s=y_k(t)} > 0\), \(a_{jk}(t) < 0\) as required. Consider now the canonical forms \(1.3\) to conclude that \(a_{jj}(t) > 0\). We next claim that, for the grids (I) – (III) and (V) – (VI), \(b_{jk}(t) > 0\). Indeed, for the grids (I) and (III), the proof is straightforward. For the grid (II), we have

\[
b_{jk}(t) = \frac{4}{(y_j(t) + y_k(t))(y_j(t) + y_k(t) + 2)}
\]

and \(y_j(t) \in S_{(t_0 - \epsilon, t_0 + \epsilon)} \subset (-1/2, \infty)\), the latter because \(x\) is strictly increasing on \((-1/2, \infty)\). Hence, by Lemma 2.4, \(b_{jk}(t) > 0\) as claimed. Since \(f(y(t); t) > 0\), \(f_1(y(t); t) < 0\), \(a_{jk}(t) < 0\), and \(b_{jk}(t) > 0\), \(2.20\) implies \(a_{jj}(t) > 0\). The corresponding result for the grid (V) follows in the same way after noting that

\[
b_{jk}(t) = 2\theta \sinh(2\theta) \csc (2\theta) \csc ((y_k(t) + y_j(t) - 1)(\theta) \csc ((y_k(t) + y_j(t) + 1)(\theta)),
\]
for \( q = e^{2\theta} (\theta > 0) \), and \( y_j(t) \in S_{(t_0 - \epsilon, t_0 + \epsilon)} \subset (0, \infty) \). And the same goes for the grid (VI) using hyperbolic identities and noting that \( y_j(t) \in S_{(t_0 - \epsilon, t_0 + \epsilon)} \subset (-\pi/(2\theta), 0) \). Define

\[
f(t) = (f_2(y_1(t); t), \ldots, f_2(y_n(t); t))^T,
\]

\[
A(t) = (a_{jk}(t)), \quad y(t) = \left( y'_1(t), \ldots, y'_n(t) \right)^T,
\]

and rewrite (2.19) as \( f(t) = A(t)y(t) \). Observe that \( A(t) \) has positive diagonal entries and negative of off-diagonal entries. Moreover, from (2.20) we get

\[
|a_{jj}(t)| > \sum_{k=1, k\neq j}^n |a_{jk}(t)|.
\]

Hence \( A(t) \) is a real irreducibly diagonally dominant matrix, and so, by Corollary 1, p. 85 all the entries of \( A^{-1}(t) \) are positive. Thus all the entries of \( y(t) = A^{-1}(t)f(t) \) are positive, and the theorem is proved for the grids (I) – (III) and (V) – (VI). The above argument does not work for the grid (IV), because, for \( q = e^{2\theta} (\theta > 0) \),

\[
b_{jk}(t) = -2\theta \sinh(2\theta) \operatorname{sech} \left( (y_j(t) + y_k(t) + 1)\theta \right) \operatorname{sech} \left( (y_j(t) + y_k(t) - 1)\theta \right) < 0.
\]

Indeed, \(-1 < b_{jk}(t) < 0\) and, therefore, under the additional hypothesis for this grid, the theorem follows from (2.20).

Remark 2.1. In some cases, the hypotheses of the discrete Stieltjes theorem are fulfilled in \((a, b - 1) \supset S_R \). However, there are several, and important, examples where this is only true on a subset of \((a, b - 1)\) containing all the elements of \( S_R \); see, for instance, the Racah polynomials in Section 3.4 below. Of course, since the precise location of the zeros of \( P \) is not known, these cases require a more careful analysis.

In Example 2.1 we can write \( H_n^{(\alpha, \beta, N)} \) instead of \( H_n^{(\alpha, \beta)} \). The interlacing property between the zeros of \( H_n^{(\alpha, \beta, N)} \) and \( H_n^{(\alpha, \beta, N+1)} \) was observed by Levit (see [3], Theorem 6). It was proved independently, and almost simultaneously, by Mesztenyi that [7, Theorem 6] is always true for discrete orthogonal polynomials on the grid (I) when the variable \( s \) ranges over a finite interval (see [11, Lemma 3]). In this way, the following general result might be of interest to the reader.

Theorem 2.2. Fix \( a \in \mathbb{R} \) and \( N \in \mathbb{N} \) and define \( b = a + N \). Let \( X = x(s) \) be a real-valued function, where the variable \( s \) ranges over the finite interval \([a, b+1]\). Let \( (P_n(X; N))_{n=1}^{N-1} \) be the sequence of discrete orthogonal polynomials, in \( X \), on the set \( \{x(a), x(a+1), \ldots, x(b-1)\} \). Let \( (P_n(X; N+1))_{n=1}^N \) be

---

\[\text{Indeed, } A(t) \text{ is a Stieltjes matrix.}\]
the sequence of discrete orthogonal polynomials, in \( X \), on the set \( \{ x(a), x(a + 1), \ldots, x(b) \} \). Assume that both sequence are orthogonal with respect to the same positive weight function. Let \( Y_1 < Y_2 < \cdots < Y_n \) be the zeros of \( P_n(\cdot; N) \). Then one of the following situations holds:

i) The zeros of \( P_n(\cdot; N + 1) \) are those of \( P_n(\cdot; N) \), if \( P_n(x(b); N) = 0 \);

ii) \( P_n(\cdot; N + 1) \) has a zero on \((Y_k, Y_{k+1})\), if \( P_n(x(b); N) \neq 0 \) and \( x(b) \notin (Y_k, Y_{k+1}) \) for fixed \( k \in \{1, \ldots, n - 1\} \);

iii) \( P_n(\cdot; N + 1) \) has a zero on each of the intervals \((Y_k, x(b))\) and \((x(b), Y_{k+1})\), if \( P_n(x(b); N) \neq 0 \) and \( x(b) \in (Y_k, Y_{k+1}) \) for fixed \( k \in \{1, \ldots, n - 1\} \).

**Proof.** To shorten notation, write \( P_n^{(N)} \) and \( P_n^{(N+1)} \) instead of \( P_n(\cdot; N) \) and \( P_n(\cdot; N + 1) \), respectively. The reader may check for himself that by expressing \( P_n^{(N+1)} \) as a linear combination of the elements of the set \( \{1, P_1^{(N)}, \ldots, P_n^{(N)}\} \) and using the orthogonality property, we obtain

\[
P_n^{(N+1)}(X) = P_n^{(N)}(X)
\]

(2.21)

where \( \eta_n = \omega(b) \Delta x(b - 1/2) \| P_{n-1}^{(N)} \|^{-2} \) and

\[
\zeta_n = 1 + \eta_n \left( P_{n-1}^{(N)}(x(b)) \frac{dP_n^{(N)}}{dX}(X) \bigg|_{X=x(b)} - \frac{dP_{n-1}^{(N)}}{dX}(X) \bigg|_{X=x(b)} P_n^{(N)}(x(b)) \right).
\]

Following a standard procedure, the rest of the proof follows as [10, Lemma 3].

We end this section with the following consequence of Theorem 2.2, which is valid, in particular, for COPRL.

**Corollary 2.1.** Assume the hypotheses and notation of Theorem 2.2. Suppose furthermore that \( x \) is a strictly monotone function on \([a, b + 1]\). Set \( Y_0 = Y_{n+1} = x(b) \). Then \( P_n(\cdot; N + 1) \) has a zero on each interval \((Y_k, Y_{k+1})\) for all \( k \in \{1, \ldots, n\} \) if \( x \) is a strictly increasing function, or else \( P_n(\cdot; N + 1) \) has a zero on each interval \((Y_k, Y_{k+1})\) for all \( k \in \{0, \ldots, n - 1\} \).

**Proof.** We give the proof only for the case in which \( x \) is a strictly increasing function. The same arguments apply to the case in which \( x \) is a strictly decreasing function. From (2.21), it follows that

\[
\text{sgn} \left( P_n(Y_n; N + 1) \right) = -\text{sgn} \left( P_n(Y_{n+1}; N) \right) = -1.
\]

Hence \( P_n(\cdot; N + 1) \) has a zero on \((Y_n, Y_{n+1})\). The rest of the proof is a direct consequence of Theorem 2.2 ii).
3. Applications

In this section we apply the discrete Stieltjes theorem to specific families of COPRL. (Of course, Corollary 2.1 is applicable to any family of COPRL on a finite grid, see for instance the Hahn, Krawtchouk, Racah, dual Hahn, q-Hahn, q-Krawtchouk, quantum q-Krawtchouk, q-Racah, and dual q-Hahn polynomials below.) We prove, or sketch the proof in similar cases, only of some illustrative examples; the other cases are stated and the proofs are left as exercises for the reader. The reader also should satisfy himself that the hypotheses of Lemma 2.2 are fulfilled. As far as we know, only the monotonicity of zeros of COPRL on the grid \((I)\) has been studied (see [4, Chapter 7]). We include this case for the sake of completeness.

3.1. The grid \((I)\)

Examples of COPRL on \(X = x(s) = s\) (Hahn, Charlier, Krawtchouk, and Meixner polynomials).

The Hahn polynomials, \(H_n^{(\alpha, \beta)}\), are defined in Example 2.1.

Proposition 3.1. The zeros of \(H_n^{(\alpha, \beta)}\) are strictly decreasing functions of \(\alpha\) on \((-1, \infty)\) and strictly increasing functions of \(\beta\) on \((-1, \infty)\).

Proof. It suffices to use (2.13) and (2.14)-(2.15) together with the discrete Stieltjes theorem.

The Charlier polynomials (see [6, Section 9.14]),

\[ y(s) = C_n^{(\alpha)}(X) = \binom{-n}{-X} - \frac{1}{\alpha} \]

\((n = 1, 2, \ldots; a = 0, b = \infty; \alpha > 0)\), satisfy (2.11) with \(A\) and \(B\) given by

\[ A = A(X) = X, \quad B = B(\alpha) = \alpha. \]

Proposition 3.2. The zeros of \(C_n^{(\alpha)}\) are strictly increasing functions of \(\alpha\) on \((0, \infty)\).

The Krawtchouk polynomials (see [6, Section 9.11]),

\[ y(s) = K_n^{(\alpha)}(X) = \binom{-n}{-X} \frac{1}{1 - \alpha} \]

\((n = 1, \ldots, N - 1; a = 0, b = N; 0 < \alpha < 1)\), satisfy (2.11) with \(A\) and \(B\) given by

\[ A = A(X; \alpha) = (1 - \alpha)X, \quad B = B(X; \alpha) = \alpha(-X + N - 1). \]

\(^9\)Recall that for a continuous case on the grid \((VI)\) as the Askey-Wilson polynomials, the monotonicity of their zeros was studied in [2, Section 7].
Proposition 3.3. The zeros of \( K_n^{(\alpha)} \) are strictly increasing functions of \( \alpha \) on \((0, 1)\).

The Meixner polynomials (see [6, Section 9.10]),

\[
y(s) = M_n^{(\alpha, \beta)}(X) = 2F_1 \left( \begin{array}{c} -n, -X \\ \beta \end{array} \right| 1 - \frac{1}{\alpha} \right) \\
(n = 1, 2, \ldots; a = 0, b = \infty; 0 < \alpha < 1, \beta > 0),
\]

satisfy (2.11) with \( A \) and \( B \) given by

\[
A = A(X) = X, \quad B = B(X; \alpha, \beta) = \alpha(X + \beta).
\]

Proposition 3.4. The zeros of \( M_n^{(\alpha, \beta)} \) are strictly increasing functions of \( \alpha \) on \((0, 1)\) and strictly increasing functions of \( \beta \) on \((0, \infty)\).

3.2. The grid (II)

Examples of COPRL on \( X = x(s) = s(s + 1) \) (Racah and dual Hahn polynomials).

The Racah polynomials (see [16, p. 236]),

\[
y(s) = R_n^{(\alpha, \beta)}(X) = 4F_3 \left( \begin{array}{c} -n, \alpha + \beta + n + 1, a-s, s+a+1 \\ 2a + \alpha + N + 1, \beta + 1, 1-N \end{array} \right| 1 \right) \\
(n = 1, \ldots, N - 1; a > -1/2, b = a + N; \alpha > -1, -1 < \beta < 2a + 1),
\]

satisfy (2.11) with \( A \) and \( B \) given by

\[
A = A(s; \alpha, \beta) = \frac{(s-a)(s+a+N)(s-a-\alpha-N)(s+a-\beta)}{2s(2s+1)},
\]

\[
B = B(s; \alpha, \beta) = \frac{(s+a+1)(s-a-N+1)(s+a+\alpha+N+1)(s-a+\beta+1)}{2(s+1)(2s+1)}.
\]

Proposition 3.5. The zeros of \( R_n^{(\alpha, \beta)} \) are strictly decreasing functions of \( \alpha \) on \((-1, \infty)\) and strictly increasing function of \( \beta \) on \((-1, 2a + 1)\) if \( a \geq 0 \), or else the zeros of \( R_n^{(\alpha, \beta)} \) are strictly decreasing functions of \( \alpha \) on \((-1, \infty)\) for each \( \beta \in (a, 2a + 1) \) and strictly increasing function of \( \beta \) on \((a, 2a + 1)\).

Proof. We give the proof only for the case in which \( a \geq 0 \). The proof for \(-1/2 < a < 0\) is similar. Define the interval \( K = \{ \max \{ a, \beta - a \}, a + N - 1 \} \).

The monotonicity function

\[
f = \frac{B}{A} = \frac{s(s+a+1)(s-a-N+1)(s+a+\alpha+N+1)(s-a+\beta+1)}{(s+1)(s-a)(s+a+N)(s-a-\alpha-N)(s+a-\beta)}
\]
is a positive and strictly decreasing function of \( s \in K \), and

\[
\frac{\partial f}{\partial \alpha} = \frac{s(2s + 1)(s + a + 1)(s - a - N + 1)(s - a + \beta + 1)}{(s + 1)(s - a)(s + a + N)(s - a - \alpha - N)^2(s + a - \beta)} < 0, \tag{3.24}
\]

\[
\frac{\partial f}{\partial \beta} = \frac{s(2s + 1)(s + a + 1)(s - a - N + 1)(s + a + \alpha + N + 1)}{(s + 1)(s - a)(s + a + N)(s - a - \alpha - N)^2(s + a - \beta)^2} > 0, \tag{3.25}
\]

for each \( s \in K \). It is immediate that \( S^{(\beta)}_{(-1,2a]} \subseteq K \). We next claim that \( S^{(\beta)}_{(2a,2a+1]} \subseteq K \). Indeed, this is equivalent to prove that \( R^{(\alpha,\beta)}_n \) has no zeros on \( (a(a + 1), (\beta - a)(\beta - a + 1)) \) for \( \beta \in (2a, 2a + 1) \). For \( \beta \in (2a, 2a + 1) \),

\[
R^{(\alpha,\beta)}_n((\beta - a)(\beta - a + 1)) = \frac{3F_2}{(2a + \alpha + N + 1)_n(\alpha + \beta + N + 1)_n} > 0,
\]

the last equality being a consequence of Sheppard’s identity (see [1, Corollary 3.34]). Hence \( R^{(\alpha,\beta)}_n \) has no zeros on \( (a(a + 1), (\beta - a)(\beta - a + 1)) \) or at least two zeros there. Suppose the second of these possibilities is true. From the proof of Lemma 2.4 we have \( x(a + 1) < x(\beta - a) \), which is impossible. Thus \( S^{(\beta)}_{(2a,2a+1]} \subseteq K \) as claimed. The same proof actually shows that \( S^{(\alpha)}_{(-1,\infty)} \subseteq K \). The result follows from the discrete Stieltjes theorem.

**Remark 3.1.** In [6, Section 9.2], the Racah polynomials (interchanging \( \alpha \) and \( \beta \)) are defined by

\[
\overline{R}^{(\alpha,\beta,\delta)}_n(\overline{x}(s)) = 4F_3\left(\begin{array}{c}
\begin{array}{c}
-n, \alpha + \beta + n + 1, -s, s + \delta - N + 1
\end{array}
\alpha + \delta + 1, \beta + 1, 1 - N
\end{array}\right| 1
\right)
\]

\((n = 1, \ldots, N - 1; a = 0, b = N; \alpha > -1, -1 < \beta < \delta - N + 1, \delta > N - 1)\), where \( \overline{x}(s) = s(s + \delta - N + 1) \). In (3.22), we can write \( R^{(\alpha,\beta,a)}_n \) instead of \( R^{(\alpha,\beta)}_n \); although, implicitly, we have the agreement to omit those parameters that are assumed fixed. Clearly, \( \overline{x} \) is not one of the canonical forms (1.6). However, \( \overline{x} \) and \( x \) are related by a linear transformation. Therefore, for \( a = (\delta - N)/2 \) fixed,

\[
R^{(\alpha,\beta,(\delta - N)/2)}_n(x(s)) = \overline{R}^{(\alpha,\beta,\delta)}_n(\overline{x}\left(s - \frac{\delta - N}{2}\right)),
\]

where

\[
\overline{x}\left(s - \frac{\delta - N}{2}\right) = x(s) - \frac{(\delta - N + 1)^2 - 1}{4}.
\]

Consequently, Proposition 3.5 remains valid if we replace \( R^{(\alpha,\beta)}_n \) by \( \overline{R}^{(\alpha,\beta,\delta)}_n \).
The dual Hahn polynomials (see [16, p. 236]),
\[ y(s) = W_n^{(\alpha)}(X) = 3F_2 \left( \begin{array}{c} -n, a-s, s+a+1 \\ \alpha+1, 1-N \end{array} \middle| 1 \right) \]  \hspace{1cm} (3.26)
\((n = 1, \ldots, N-1; a > -1/2, b = a+N; -1 < \alpha < 2a+1),\) satisfy (2.11) with 
\(A\) and \(B\) given by
\[ A = A(s; \alpha) = \frac{(s-a)(s+a+N)(s+a-\alpha)}{2s(2s+1)}, \]
\[ B = B(s; \alpha) = \frac{(s+a+1)(-s+a+N-1)(s-a+\alpha+1)}{2(s+1)(2s+1)}. \]

**Proposition 3.6.** The zeros of \(W_n^{(\alpha)}\) are strictly increasing functions of \(\alpha\) on \((-1, 2a+1)\) if \(a \geq 0\), or else the zeros of \(W_n^{(\alpha)}\) are strictly increasing functions of \(\alpha\) on \((a, 2a+1)\).

**Proof.** We sketch the proof only for the case in which \(a \geq 0\). The proof for \(-1/2 < a < 0\) is similar. Define the interval \(K = (\max\{a, \alpha-a\}, a+N-1)\). Thus we only need to prove that \(S((-1, 2a+1)) \subset K\). Note that \(W_n^{(\alpha)}(a(a+1)) = 1\) and \(W_n^{(\alpha)}((\alpha-a)(\alpha-a+1)) = 2F_1 \left( \begin{array}{c} -n, 2a-\alpha \\ 1-N \end{array} \middle| 1 \right) = \frac{(1-N+\alpha-2a)_n}{(1-N)_n} > 0,\)
the last equality being a consequence of Chu-Vandermonde’s identity ([1, Corollary 2.2.3]). The rest of the proof runs as in Proposition 3.5. \(\square\)

**Remark 3.2.** In [6, Section 9.6], the dual Hahn polynomials are defined by
\[ \tilde{W}_n^{(\alpha, \beta)}(\tilde{x}(s)) = 3F_2 \left( \begin{array}{c} -n, -s, s+\alpha+\beta+1 \\ \alpha+1, 1-N \end{array} \middle| 1 \right) \]
\((n = 1, \ldots, N-1; a = 0, b = N; \alpha > -1, \beta > -1\) or \(\alpha < 1-N, \beta < 1-N),\)
where \(\tilde{x}(s) = s(s+\alpha+\beta+1).\) In (3.26), we can write \(W_n^{(\alpha,a)}\) instead of \(W_n^{(\alpha)}\).

Hence, for \(a = (\alpha+\beta)/2\) fixed,
\[ W_n^{(\alpha,(\alpha+\beta)/2)}(x(s)) = \tilde{W}_n^{(\alpha,\beta)} \left( \tilde{x} \left( s - \frac{\alpha+\beta}{2} \right) \right), \]
where
\[ \tilde{x} \left( s - \frac{\alpha+\beta}{2} \right) = x(s) - \frac{(\alpha+\beta+1)^2-1}{4}.\]
Consequently, Proposition 3.6 remains valid if we replace \(W_n^{(\alpha)}\) by \(\tilde{W}_n^{(\alpha,\beta)}\) and assume that \(\alpha + \beta\) is constant.

\[^{10}\text{It is immediate that } S_{(-1,2a)} \subset K.\]
3.3. The grid (III)

3.3.1. Examples of COPRL on $X = x(s) = q^{-s}$ ($0 < q < 1$) ($q$-Meixner, Al-Salam-Carlitz, $q$-Charlier, $q$-Hahn, $q$-Krawtchouk, affine $q$-Krawtchouk, and quantum $q$-Krawtchouk) and some related cases ($q$-Charlier and big $q$-Laguerre).

The $q$-Meixner polynomials (see [6, Section 14.13]),

$$y(s) = M_n^{(\alpha,\beta)}(X; q) = 2\phi_1\left( q^{-n}, X \left| \frac{q^n+1}{\alpha} \right. \right)$$

(n = 1, 2, ..., $a = 0, b = \infty; \alpha > 0, 0 \leq \beta < q^{-1}$), satisfy (2.11) with $A$ and $B$ given by

$$A = A(s; \alpha, \beta) = (1 - q^{-s})(1 + \alpha \beta q^{-s}), \quad B = B(s; \alpha, \beta) = \alpha q^s(1 - \beta q^{s+1}).$$

**Proposition 3.7.** The zeros of $M_n^{(\alpha,\beta)}(\cdot; q)$ are strictly increasing functions of $\alpha$ on $(0, \infty)$ and strictly decreasing functions of $\beta$ on $[0, q^{-1})$.

**Proof.** The monotonicity function

$$f = \frac{B}{A} = \frac{\alpha q^s(1 - \beta q^{s+1})}{(1 - q^s)(1 + \alpha \beta q^s)}$$

is a positive and strictly decreasing function of $s \in (0, \infty)$, and

$$\frac{\partial f}{\partial \alpha} = \frac{q^s(1 - \beta q^{s+1})}{(1 - q^s)(1 + \alpha \beta q^s)^2} > 0,$$

$$\frac{\partial f}{\partial \beta} = -\frac{\alpha q^{2s}(\alpha + q)}{(1 - q^s)(1 + \alpha \beta q^s)^2} < 0,$$

for each $s \in (0, \infty)$. The result follows from the discrete Stieltjes theorem. □

**Remark 3.3.** The $q$-Charlier polynomials (see [6, Section 14.23]) are given by $C_n^{(\alpha)}(\cdot; q) = M_n^{(\alpha,0)}(\cdot; q)$.

The second family of Al-Salam-Carlitz polynomials (see [6, Section 14.25]),

$$y(s) = V_n^{(\alpha)}(X; q) = (-\alpha)^n q^{-\binom{n}{2}} \phi_0\left( q^{-n}, X \left| q, \frac{q^n}{\alpha} \right. \right)$$

(n = 1, 2, ..., $a = 0, b = \infty; 0 < \alpha < q^{-1}$), satisfy (2.11) with $A$ and $B$ given by

$$A = A(s; \alpha) = (1 - q^{-s})(\alpha - q^{-s}), \quad B = B(s; \alpha) = \alpha q^s.$$

**Proposition 3.8.** The zeros of $V_n^{(\alpha)}(\cdot; q)$ are strictly increasing functions of $\alpha$ on $(0, q^{-1})$.

**Proof.** Define the interval $K = (\max\{-\log q \alpha, 0\}, \infty)$. Note that the hypotheses of the discrete Stieltjes theorem are fulfilled in $K$. Thus we only need to prove that $S_{(1, q^{-1})} \subset K^{11}$. The rest of the proof runs as in Proposition 3.5.
though given the simplicity of this case, we do not need to use q-hypergeometric identities.

**Remark 3.4.** The first family of Al-Salam-Carlitz polynomials is given by $U_n^{(\alpha)}(; q^{-1}) = V_n^{(\alpha)}(; q)$.

The $q$-Hahn polynomials (see [6, Section 14.6]),

$$y(s) = H_n^{(\alpha, \beta)}(X; q) = 3\phi_2 \left( \begin{array}{c} q^{-n}, \alpha \beta q^{n+1}, X \\ \alpha q, q^{1-N} \end{array} \middle| q, q \right)$$

$(n = 1, \ldots, N - 1; a = 0, b = N; 0 < \alpha < q^{-1}, 0 < \beta < q^{-1})$, satisfy (2.11) with $A$ and $B$ given by

$$A = A(s; \alpha, \beta) = \alpha q(1 - q^s)(\beta - q^{s-N})$$

$$B = B(s; \alpha) = (1 - q^{s-N+1})(1 - \alpha q^{s+1})$$

**Proposition 3.9.** The zeros of $H_n^{(\alpha, \beta)}(; q)$ are strictly decreasing functions of $\alpha$ on $(0, q^{-1})$ and strictly increasing functions of $\beta$ on $(0, q^{-1})$.

The $q$-Krawtchouk polynomials (see [6, Section 14.15]),

$$y(s) = K_n^{(\alpha)}(X; q) = 3\phi_2 \left( \begin{array}{c} q^{-n}, -\alpha q^n, X \\ 0, q^{1-N} \end{array} \middle| q, q \right)$$

$(n = 1, \ldots, N - 1; a = 0, b = N; \alpha > 0)$, satisfy (2.11) with $A$ and $B$ given by

$$A = A(s; \alpha) = \alpha(q^s - 1), \quad B = B(s; \alpha) = 1 - q^{s-N+1}$$

**Proposition 3.10.** The zeros of $K_n^{(\alpha)}(; q)$ are strictly decreasing functions of $\alpha$ on $(0, \infty)$.

The affine $q$-Krawtchouk polynomials (see [6, Section 14.16]),

$$y(s) = \hat{K}_n^{(\alpha)}(X; q) = 3\phi_2 \left( \begin{array}{c} q^{-n}, 0, X \\ \alpha q, q^{1-N} \end{array} \middle| q, q \right)$$

$(n = 1, \ldots, N - 1; a = 0, b = N; 0 < \alpha < q^{-1})$, satisfy (2.11) with $A$ and $B$ given by

$$A = A(s; \alpha) = \alpha q^{s-N+1}(q^s - 1), \quad B = B(s; \alpha) = (1 - q^{s-N+1})(1 - \alpha q^{s+1})$$

**Proposition 3.11.** The zeros of $\hat{K}_n^{(\alpha)}(; q)$ are strictly decreasing functions of $\alpha$ on $(0, q^{-1})$.

**Remark 3.5.** The big $q$-Laguerre polynomials (see [6, Section 14.16]) are given by

$$L_n^{(\alpha, \beta)}(X; q) = 3\phi_2 \left( \begin{array}{c} q^{-n}, 0, X \\ \alpha q, \beta q \end{array} \middle| q, q \right)$$

$(n = 1, 2, \ldots; 0 < \alpha < q^{-1}, \beta < 0)$. In particular, $\hat{K}_n^{(\alpha)}(; q) = L_n^{(\alpha, q^{-N})}(X; q)$ (see also Remark 3.6).
The quantum q-Krawtchouk polynomials (see [6, Section 14.14]),
\[ y(s) = \tilde{K}_n^{(\alpha)}(X; q) = \frac{(q^{-N}, q)_n}{\alpha^n q^{n^2}} \Phi_1 \left( \begin{array}{c} q^{-n}, X \\ q^{1-N} \end{array} \bigg| q, \alpha q^{n+1} \right) \]
for \( n = 1, \ldots, N - 1; a = 0, b = N; \alpha > q^{1-N} \), satisfy (2.11) with \( A \) and \( B \) given by
\[ A = A(s; \alpha) = (1 - q^s)(\alpha - q s - N), \quad B = B(s) = -q^s(1 - q s - N + 1). \]

**Proposition 3.12.** The zeros of \( \tilde{K}_n^{(\alpha)}(\cdot; q) \) are strictly decreasing functions of \( \alpha \) on \((q^{-N}, \infty)\).

**Proof.** Define the interval \( K = \left[ \max\{0, \log_q \alpha + N\}, N - 1 \right) \). Note that the hypotheses of the discrete Stieltjes theorem are fulfilled in \( K \). Thus we only need to prove that \( S_{(q^{-N}, q^{-N})} \subset K \). The rest of the proof runs as in Proposition 3.5 although given the simplicity of this case, we do not need to use q-hypergeometric identities.

### 3.3.2. Examples of COPRL on \( X = x(s) = q^s \ (0 < q < 1) \) (q-Bessel, little q-Jacobi, and little q-Laguerre/Wall) and some related cases (big q-Jacobi, big q-Laguerre, and q-Laguerre).

The q-Bessel (see [6, Section 14.22]),
\[ y(s) = B_n^{(\alpha)}(X; q) = \Phi_1 \left( \begin{array}{c} q^{-n}, -\alpha q^n \\ 0 \end{array} \bigg| q, qX \right) \]
for \( n = 1, 2, \ldots; a = 0, b = \infty; \alpha > 0 \), satisfy (2.11) with \( A \) and \( B \) given by
\[ A = A(s) = q^s - 1, \quad B = B(s; \alpha) = \alpha. \]

**Proposition 3.13.** The zeros of \( B_n^{(\alpha)}(\cdot; q) \) are strictly decreasing functions of \( \alpha \) on \((0, \infty)\).

The little q-Jacobi polynomials (see [6, Section 14.12]),
\[ y(s) = P_n^{(\alpha, \beta)}(X; q) = \Phi_1 \left( \begin{array}{c} q^{-n}, \alpha \beta q^{n+1} \\ \alpha q \end{array} \bigg| q, qX \right) \]
for \( n = 1, 2, \ldots; a = 0, b = \infty; 0 < \alpha < q^{-1}, \beta < q^{-1} \), satisfy (2.11) with \( A \) and \( B \) given by
\[ A = A(s) = q^{-s}(q^s - 1), \quad B = B(s; \alpha, \beta) = \alpha q^{-s}(\beta q^{s+1} - 1). \]

**Proposition 3.14.** The zeros of \( P_n^{(\alpha, \beta)}(\cdot; q) \) are strictly decreasing functions of \( \alpha \) on \((0, q^{-1})\) and strictly increasing functions of \( \beta \) on \((-\infty, q^{-1})\).

\[ ^{12} \text{It is immediate that } S_{[q^{-N}, \infty)} \subset K. \]
Examples of COPRL on $X = (q^s + q^{-s})/2$ $(0 < q < 1)$ (q-Racah and dual q-Hahn polynomials).

The q-Racah polynomials (see [16, p. 239]),

$$y(s) = R_n^{(\alpha, \beta)}(X; q) = 4\phi_3\left(\begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{s+a} \\ q^{2s+\alpha+N}, q^{\beta+1}, q^{1-N} \end{array} \middle| q, q \right)$$

(3.27)

$(n = 1, \ldots, N - 1; a > 0, b = a + N; \alpha > -1, -1 < \beta < 2a)$, satisfy (2.11) with $A$ and $B$ given by

\begin{align*}
A &= A(s; \alpha, \beta) \\
&= -\frac{4q^{\alpha+\beta+5/2}(q^{s-a} - 1)(q^{s+a+N-1} - 1)(q^{s-a-N} - 1)(q^{s+a-\beta} - 1)}{(q - 1)^2(q^{2s} - 1)(q^{2s-1} - 1)}, \\
B &= B(s; \alpha, \beta) \\
&= -\frac{4q^{3/2}(q^{s+a} - 1)(q^{s-a} - 1)(q^{s+a+N-1} - 1)(q^{s+a+\alpha+N} - 1)(q^{s-a+\beta+1} - 1)}{(q - 1)^2(q^{2s} - 1)(q^{2s+1} - 1)}. \\
\end{align*}
is equivalent to prove that $R_n^{(\alpha,\beta)}(\cdot;q)$ are strictly decreasing functions of $\alpha$ on $(-1,\infty)$ and strictly increasing function of $\beta$ on $(-1,2a)$ if $a \geq 1/2$, or else the zeros of $R_n^{(\alpha,\beta)}(\cdot;q)$ are strictly decreasing functions of $\alpha$ on $(-1,\infty)$ for each $\beta \in (a-1/2,2a)$ and strictly increasing function of $\beta$ on $(a-1/2,2a)$.

Proof. We give the proof only for the case in which $a \geq 1/2$. The proof for $0 < a < 1/2$ is similar. Define the interval $K = \{ \max \{a,\beta-a+1\}, a+N-1 \}$. The monotonicity function

$$f = \frac{B}{A} = \frac{(q^{s+a} - 1)(q^{s-a-N+1} - 1)(q^{s+a+b-N} - 1)(q^{s-a+b+1} - 1)}{q^{s-a+b+1}(q^{2s+1} - 1)(q^{s-a} - 1)(q^{s+a+N} - 1)(q^{s-a-N} - 1)q^{2s+1} - 1},$$

is a positive and strictly decreasing function of $s \in K$, and

$$0 > \frac{\partial f}{\partial \alpha} = \log q \frac{(q^{2s} - 1)(q^{2s-1} - 1)(q^{s+a} - 1)(q^{s-a-N+1} - 1)(q^{s-a+b+1})}{q^{s+a+b+1}(q^{2s+1} - 1)(q^{s-a} - 1)(q^{s+a+N} - 1)(q^{s-a-N} - 1)q^{2s+1} - 1},$$

$$0 < \frac{\partial f}{\partial \beta} = \log q \frac{(q^{2s} - 1)(q^{2s-1} - 1)(q^{s+a} - 1)(q^{s-a-N+1} - 1)(q^{s+a+b+N})}{q^{s+a+b+1}(q^{2s+1} - 1)(q^{s-a} - 1)(q^{s+a+N} - 1)(q^{s-a-N} - 1)q^{2s+1} - 1},$$

for each $s \in K$. Thus we only need to prove that $S_{(2a-1,2a)}^{(\beta)} \subset K$. Indeed, this is equivalent to prove that $R_n^{(\alpha,\beta)}(\cdot;q)$ has no zeros on $\{(q^{a} + q^{-a})/2, (q^{\beta-a+1} + q^{a-N})/2\}$ for $\beta \in (2a-1,2a)$. For $\beta \in (2a-1,2a)$, $R_n^{(\alpha,\beta)}((q^{a} + q^{-a})/2; q) = 1$ and

$$R_n^{(\alpha,\beta)}((q^{\beta-a+1} + q^{a-N})/2; q) = \phi_2 \left( \begin{array}{c} q^{a-N}, q^{a+b+n+1}, q^{2a-b-1} \\ q^{2a+a+N}, q^{1-N} \end{array} \right| q, q \right) = \frac{(q^{2a-b+N-n-1}; q)_n(q^{a+b+N+1}; q)_n}{(q^{2a+a+N}; q)_n(q^{N-n}; q)_n} > 0,$$

the last equality being a consequence of q-Pfaff-Saalschütz’s identity (see [1, (10.10.3)]). We thus get, as in Proposition 3.3, $S^{(\beta)}_{(-1,2a-1)} \subset K$. The same proof actually shows that $S^{(\alpha)}_{(-1,\infty)} \subset K$. The result follows from the discrete Stieltjes theorem.

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13It is immediate that $S^{(\beta)}_{(-1,2a-1)} \subset K$. 

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Remark 3.7. In [16, Section 14.2], the q-Racah polynomials (replacing $\alpha$ by $q^\delta$ and $\beta$ by $q^\gamma$) are defined by

$$R_n^{(\alpha, \beta, \gamma)}(x; q) = \phi_3 \left( \begin{array}{c} q^{-n}, q^\gamma, q^\delta + q^n + 1, q^{\gamma + \delta + n} + 1 \\ q^{\alpha + \delta + n}, q^{\beta + \delta + n}, q^{\gamma + \delta + n + 1} \end{array} \right)$$

$n = 1, \ldots, N-1; a = 0, b = N; \alpha > -1, -1 < \beta < 1 - N + \log_2 \delta, 0 < \delta < q^{-N}$,

where $x(s) = \delta q^{-N+1} + q^{-s}$. In [3.27], we can write $R_n^{(\alpha, \beta, \gamma)}(x; q)$ instead of $R_n^{(\alpha, \beta)}(x; q)$.

Consequently, Proposition 3.17 remains valid if we replace $R_n^{(\alpha, \beta)}(x; q)$ by $R_n^{(\alpha, \beta)}(x; q)$.

The dual q-Hahn polynomials (see [16, p. 239]),

$$y(s) = W_{n^{(\alpha)}}(X; q) = \phi_2 \left( \begin{array}{c} q^{-n}, q^\gamma, q^\delta + q^n + 1, q^{\gamma + \delta + n} + 1 \\ q^\alpha, q^{\beta + \gamma}, q^{\gamma + \delta + n + 1} \end{array} \right)$$

$n = 1, \ldots, N-1; a > 0, b = a + N; -1 < \alpha < 2a$), satisfy (2.11) with $A$ and $B$ given by

$$A = A(s; \alpha) = -\frac{4q^{-a+\alpha-N+5/2}(q^{-a}-1)(q^{\alpha+\gamma+N-1}-1)(q^{\alpha-a-\gamma-1}-1)}{(q-1)^2(q^{2a}-1)(q^{2a-1}-1)},$$

$$B = B(s; \alpha) = -\frac{4q^{a+3/2}(q^{\alpha-a}-1)(q^{\alpha-a-N+1}-1)(q^{\alpha-a+\gamma+1}-1)}{(q-1)^2(q^{2a}-1)(q^{2a+1}-1)}.$$

Proposition 3.17. The zeros of $W_{n^{(\alpha)}}(x; q)$ are strictly increasing functions of $\alpha$ on $(-1, 2a)$ if $a \geq 1/2$, or else the zeros of $W_{n^{(\alpha)}}(x; q)$ are strictly increasing functions of $\alpha$ on $(a-1/2, 2a)$.

Proof. We sketch the proof only for the case in which $a \geq 1/2$. The proof for $0 < a < 1/2$ is similar. Define the interval $K = \max \{a, a+1\}, a+N-1$. Note that the hypotheses of the discrete Stieltjes theorem are fulfilled in $K$.

We only need to prove that $S_{(2a-1, 2a)} \subset K$ [16]. Note that $W_{n^{(\alpha)}}((q^\alpha + q^{-\alpha})/2; q) = 1$ and

$$W_{n^{(\alpha)}}((q^\alpha + q^{-\alpha})/2; q) = \phi_4 \left( \begin{array}{c} q^{-n}, q^{2a-\alpha-1} \\ q^{\alpha-N} \end{array} \right) \frac{q^n(2a-\alpha-1)(q^{2a+\alpha-N-2}; q)_n}{(q^{1-N}; q)_n} > 0,$$

14 It is immediate that $S_{(-1, 2a-1]} \subset K$. 

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the last equality being a consequence of q-Chu-Vandermonde’s identity (6, (1.11.5)). The rest of the proof runs as in Proposition 3.16.

Remark 3.8. In [6, Section 14.7], the dual q-Hahn polynomials are defined by

$$\tilde{W}_n^{(\alpha,\beta)}(\tilde{x}(s); q) = \sum_{n=0}^{N-1} \frac{q^{-n} q^{a-s} q^{s+a}}{q^{a+1} q^{1-N} q^n}$$

$$(n = 1, \ldots, N-1; a = 0, b = N; \alpha > -1, \beta > -1 \text{ or } \alpha < -N, \beta < -N),$$

where $\tilde{x}(s) = q^{s+\alpha+\beta+1} + q^{-s}$. In (3.28), we can write $W_n^{(\alpha,\lambda)}(\cdot; q)$ instead of $W_n^{(\alpha)}(\cdot; q)$. Hence, for $a = (\alpha + \beta + 1)/2$ fixed,

$$W_n^{(\alpha,\beta+1+/2)}(x(s); q) = \tilde{W}_n^{(\alpha,\beta)}(\tilde{x}(s-(\alpha+\beta+1)/2); q),$$

where

$$\tilde{x}(s-(\alpha+\beta+1)/2) = 2q^{(\alpha+\beta+1)/2} x(s).$$

Consequently, Proposition 3.17 remains valid if we replace $W_n^{(\alpha)}(\cdot; q)$ by $\tilde{W}_n^{(\alpha,\beta)}(\cdot; q)$ and assume that $\alpha + \beta$ is constant.

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References

[1] G. E. Andrews, R. Askey, and R. Roy. Special Functions, volume 71 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999.

[2] R. Askey and J. Wilson. Some basic hypergeometric orthogonal polynomials that generalize jacobi polynomials. Mem. Amer. Math. Soc., 54(319), 1985.

[3] N. M. Atakishiyev, M. Rahman, and S. K. Suslov. On classical orthogonal polynomials. Constr. Approx., 11:181–226, 1995.
[4] M. E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 2005.

[5] S. Karlin and J. L. McGregor. The Hahn polynomials, formulas and an application. Technical Report 2, Applied Mathematics and Statistics Laboratories, Stanford University, Stanford, CA, April 1960.

[6] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*. Springer Monographs in Mathematics, 2010.

[7] R. J. Levit. The zeros of the Hahn polynomials. *SIAM Rev.*, 9:191–203, 1967.

[8] A. Markoff. Sur les racines de certaines équations (second note). *Math. Ann.*, 27:177–182, 1886.

[9] P. Maroni. Une théorie algébrique des polynômes orthogonaux. application aux polynômes orthogonaux semi-classiques. In C. Brezinski, L. Gori, and A. Ronveaux, editors, *Orthogonal Polynomials and Their Applications*, volume 9 of *IMACS Annals Comput Appl Math.*, pages 95–130, 1991.

[10] Ch. K. Mesztenyi. Orthogonal polynomials on a finite set. Technical Report TR-66-30, University of Maryland, 1966.

[11] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. Классические ортогональные полиномы дискретной переменной (Russian) [Classical orthogonal polynomials of a discrete variable]. “Nauka”, Moscow, 1985.

[12] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. *Classical orthogonal polynomials of a discrete variable*. Translated from the Russian. Springer Series in Computational Physics. Springer-Verlag, 1991.

[13] A. F. Nikiforov and V. B. Uvarov. Специальные функции математической физики. (Russian) [Special Functions of Mathematical Physics] With a preface by A. A. Samarskii. “Nauka”, Moscow, 1978.

[14] A. F. Nikiforov and V. B. Uvarov. Классические ортогональные полиномы дискретной переменной на неравномерных сетках (russian) [classical orthogonal polynomials of a discrete variable]. Preprint 17, Keldysh Inst. Appl. Math., 1983.

[15] A. F. Nikiforov and V. B. Uvarov. Специальные функции математической физики. (Russian) [Special Functions of Mathematical Physics] With a preface by A. A. Samarskii. “Nauka”, Moscow, second edition, 1984.
[16] A. F. Nikiforov and V. B. Uvarov. Polynomial solutions of hypergeometric type difference equations and their classification. *Integral Transforms Spec. Funct.*, 1:223–249, 1993.

[17] J. Petronilho. Orthogonal polynomials and special functions. Class notes for a course given in the UC|UP Joint PhD Program in Mathematics, University of Coimbra, 2018.

[18] G.-C. Rota and D. Sharp. Mathematics, Philosophy, and Artificial Intelligence... a dialogue with Gian-Carlo Rota and David Sharp. *Los Alamos Science*, Spring/Summer(12):92–104, 1985.

[19] B. Simon. *Basic Complex Analysis. A Comprehensive Course in Analysis, Part 2A*. American Mathematical Society, Providence, RI, 2015.

[20] B. Simon. *Operator Theory. A Comprehensive Course in Analysis, Part 4*. American Mathematical Society, Providence, RI, 2015.

[21] T. J. Stieltjes. Sur les racines de l’équation $X_n = 0$. *Acta Math.*, 9:385–400, 1887.

[22] T. J. Stieltjes and Ch. Hermite. *Correspondance d’Hermite et de Stieltjes. Vol. I*. Gauthier-Villars, Paris, 1905.

[23] G. Szegö. *Orthogonal polynomials*, volume 23. Amer. Math. Soc. Coll. Publ., Amer. Math. Soc., Providence, R. I., revised edition, 1959.

[24] R. S. Varga. *Matrix iterative analysis*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.

[25] N. Ja. Vilenkin and U. A. Klimyk. *Representation of Lie groups and special functions. Vol. 1. Simplest Lie groups, special functions and integral transforms. Translated from the Russian by V. A. Groza and A. A. Groza.*, volume 72 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.