Differential Form Valued Forms and Distributional Electromagnetic Sources

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Abstract

Properties of a fundamental double-form of bi-degree \((p, p)\) for \(p \geq 0\) are reviewed in order to establish a distributional framework for analysing equations of the form

\[
\Delta \Phi^{(p)} + \lambda^2 \Phi^{(p)} = S^{(p)}
\]

where \(\Delta\) is the Hodge-de Rham operator on \(p\)-forms \(\Phi^{(p)}\) on \(\mathbb{R}^3\). Particular attention is devoted to singular distributional solutions that arise when the source \(S^{(p)}\) is a singular \(p\)-form distribution. A constructive approach to Dirac distributions on (moving) submanifolds embedded in \(\mathbb{R}^3\) is developed in terms of (Leray) forms generated by the geometry of the embedding. This framework offers a useful tool in electromagnetic modeling where the possibly time dependent sources of certain physical attributes, such as electric charge, electric current and polarization or magnetization, are concentrated on localized regions in space.\(^1\)

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\(^1\) 02.40.Hw Classical differential geometry, 03.50.De Classical electromagnetism 41.20.-q Applied classical electromagnetism
1 Introduction

Mathematical distributions [1] play an all pervading role in the physical sciences. They underpin the theoretical formulation of quantum mechanics and many linear systems in classical field theory and are used to extend the solution space for linear partial differential equations to include non-smooth fields with singularities. They also offer a useful tool in electromagnetic modeling where the sources of certain physical attributes, such as electric charge, electric current, polarization or magnetization, are concentrated on localized regions in space [2], [3], [4], [5], [6] or spacetime. These aspects of distributions often occur together in problems in electromagnetic theory and this article explores this symbiosis in the unifying language of exterior double-forms. Such a formulation has direct relevance to many contemporary design problems in accelerator science where one is confronted with a number of difficult issues in electromagnetic computation. The effective resolution of such problems is often contingent on reliable computationally expensive numerical analysis which can greatly benefit from reliable analytic treatments. The formulation below will be illustrated by a number of applications in applied electro-magnetics.

2 Notation

The formulation exploits the geometric language of exterior differential forms [7] since this is ideally suited to accommodate local changes of coordinates that can be used to simplify the description of boundary and initial-value problems and naturally encapsulates intrinsic global properties of domains. In the following the distinction between smooth \( \mathcal{C}^\infty \) forms on some regular domain and those with possible singularities or discontinuities will be important. Furthermore smooth forms with compact support will play a pivotal role. These will be referred to as test forms [8], [9], [10] and distinguished below by a superposed hat. It will also be useful sometimes to notationally distinguish \( p \)-forms on different manifolds. Thus \( \alpha_M \) will denote a \( p \)-form \( ^{(p)} \)

on some manifold \( M \) while \( \gamma_{(p,q)} (X,Y) \) will denote a double-form of bi-degree \( (p,q) \) (see section [3]) on the product manifold \( X \times Y \). Each \( n \)-dimensional

\[ ^{(p)} \text{ i.e. } \alpha_M \in \mathcal{S}\Lambda^p(M), \text{ the space of sections of the exterior bundle of } p \text{-forms over } M \]
manifold \( M \) will be assumed orientable and endowed with a preferred \( n \)-form \( \star 1 \) induced from a metric tensor \( g_M \). One then has \([7]\) the linear Hodge operator \( \star \) that maps \( p \)-forms to \((n - p)\)-forms on \( M \). If \( g_M \) has signature \( t_g \) one may write

\[
g_M = \sum_{i=1}^{n} e^i_M \otimes e^i_M \eta_{ij} \tag{1}
\]

where \( \eta_{ij} = \text{diag}(\pm 1, \pm 1, \ldots \pm 1) \) and

\[
\star (M) 1 = e^1_M \wedge e^2_M \wedge \ldots \wedge e^n_M \tag{2}
\]

with \( t_g = \text{det}(\eta_{ij}) \) and \( \{e^i_M\} \) is a set of basis \( 1 \)-forms in \( \Gamma T^*(M) \).

If \( \alpha \) and \( \beta \) are integrable \( p \)-forms on \( U \subset \mathbb{R}^n \) denote

\[
\int_U \alpha \wedge (\star (\mathbb{R}^n)) \beta \tag{3}
\]

by

\[
(\alpha, \beta)_U = (\beta, \alpha)_U \tag{4}
\]

Furthermore if one of the forms has compact support on \( U \) (and belongs to the space of Schwartz test functions) an alternative notation is

\[
\alpha^{D}_{(p)} [\hat{\phi}] = (\alpha, \hat{\phi})_U \tag{5}
\]

When \( U = \mathbb{R}^n \) the bracket subscript will be omitted.

If \( (\alpha, \hat{\phi}) \) is well defined (i.e. \( \alpha \) is integrable with respect to the bracket \((..,..)\) ) then one says that \( \alpha^D \) is a regular Schwartz \( p \)-form distribution associated with \( \alpha \). Not all distributions are regular and associated with smooth forms.

Suppose \( Z \) is a smooth vector field on \( \mathbb{R}^n \) then \( \tilde{Z} \equiv g(Z, -) \) is a smooth \( 1 \)-form. Using the property

\[
\star \star = t_g \eta^{n-1} \tag{6}
\]

where the involution \( \eta \) on forms satisfies \( \eta \alpha_{(p)} \equiv (-1)^p \alpha_{(p)} \) then it is straightforward to verify the algebraic identities.

\[
\star \alpha^D_{(q)} [\hat{\phi}]_{(n-q)} = t_g \eta^{n-1} \alpha^D_{(q)} [\star^{-1}_{(n-q)} \hat{\phi}]_{(n-q)} \tag{7}
\]
The operator \( \delta \equiv (-1) d \ast \eta \) is a formal adjoint of the exterior derivative \( d \) with respect to the bracket \( (.,.) \) in the sense that

\[
d\alpha \hat{\phi} = \alpha [\delta \hat{\phi}]
\]

Similarly

\[
\delta \alpha \hat{\psi} = \alpha [d\hat{\psi}]
\]

These relations are used to define \( d\alpha^D \) and \( \delta \alpha^D \). If \( f \) is a smooth 0-form then we denote by \( f\alpha^D \) the distribution defined by \( f\alpha^D[\hat{\chi}] = \alpha^D[f\hat{\chi}] \).

If one defines the Lie derivative of \( \alpha^D \) with respect to a smooth vector field \( Z \) by

\[
\mathcal{L}_Z \alpha^D = (i_Z d + d i_Z) \alpha^D
\]

then

\[
\mathcal{L}_Z \alpha^D \hat{\chi} = \alpha^D[\delta \tilde{Z} \wedge \hat{\chi}] + (\tilde{Z}\eta + \tilde{Z}) \delta \hat{\chi} - \nabla_Z \hat{\chi}
\]

where \( \nabla \) denotes covariant differentiation with respect to the Levi-Civita connection for which

\[
\delta \equiv -g^{ab} i_X a \nabla_X .
\]

In terms of \( d \) and \( \delta \) one defines\(^3\) the Hodge de Rham operator \( \Delta = d\delta + \delta d \). A number of important identities follow by integrating the Leibnitz relation involving smooth forms

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (\eta \alpha) \wedge d\beta
\]

over regular domains \( \mathcal{U} \). These include

\[
(d\phi, \psi)_{\mathcal{U}} = (\phi, \delta\psi)_{\mathcal{U}} + \int_{\partial \mathcal{U}} \phi \wedge \ast \psi
\]

\[
(\delta \phi, \psi)_{\mathcal{U}} = (\phi, d\psi)_{\mathcal{U}} - \int_{\partial \mathcal{U}} \psi \wedge \ast \phi
\]

\(^3\)The traditional Laplacian operator on forms is \(-\Delta\)
\[(\Delta \phi, \psi)_U - (d\phi, d\psi)_U - (\delta \phi, \delta \psi)_U = \int_{\partial U} (\delta \phi \wedge \ast \psi - \psi \wedge \ast d\phi) \] (15)

\[(\Delta \phi, \psi)_U - (\phi, \Delta \psi)_U = \int_{\partial U} (\delta \phi \wedge \ast \psi - \delta \psi \wedge \ast \phi + \phi \wedge \ast d\psi - \psi \wedge \ast d\phi)\]

(16)

From these elementary identities further useful identities involving singular double-forms can be obtained.

### 3 Double-Forms on \( \mathbb{R}^n \)

The notion of a double-form was introduced in order to analyse problems in potential theory in \( \mathbb{R}^3 \) [11]. Double-forms have since been used by Duff [12] and others as a powerful tool to study exterior equations on more general manifolds. A double-form of bi-degree \((p, q)\) over the product manifold \(X \times Y\) may be regarded for each point in \(X\) as a \(p\)-form valued \(q\)-form on \(Y\) or for each point in \(Y\) as a \(q\)-form valued \(p\)-form on \(X\). To illustrate these notions suppose \(X = Y = \mathbb{R}^n\). In the natural coordinate system \((x^i)\) for \(X\) and \((y^j)\) for \(Y\) one has:

\[\gamma(X, Y) = \sum_{(p,q)} \sum_{i_1<...<i_p \ j_1<...<j_q} \gamma_{i_1...i_p\ j_1...j_q}(X, Y) (dx^{i_1} \wedge ... \wedge dx^{i_p}) \odot (dy^{j_1} \wedge ... \wedge dy^{j_q})\] (17)

The symmetric product \(\odot\) is defined so that:

\[(dx^i \odot dy^j) \wedge (dx^k \odot dy^s) = (dx^i \odot dx^k) \wedge (dy^j \odot dy^s)\] (18)

and this implies

\[\gamma \wedge \gamma' = (-1)^{pp'+qq'} \gamma' \wedge \gamma\] (19)

for all \(\gamma, \gamma'\). The Euclidean metric tensors in natural coordinates and their associated Hodge maps will be designated:

\[g_X = \sum_{i=1}^{n} dx^i \otimes dx^i\] (20)

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4If \(f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) we denote its image by \(f(X, Y)\) with a similar notation for double-forms.
\[ g_Y = \sum_{i=1}^{n} d y^i \otimes d y^i \]  \hspace{1cm} (21)

and
\[ \star \mathbf{1} = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \]  \hspace{1cm} (22)
\[ \star \mathbf{1} = dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n \]  \hspace{1cm} (23)

The operators \( d_X, d_Y, \delta_X, \delta_Y, \star \) then act naturally on forms in the appropriate space.

The fundamental double-form of bi-degree \((p, p)\) is defined as
\[ \gamma \left( (X,Y) \right) = \sum_{i_1 < i_2 < \ldots < i_p} f(X,Y)(dx^{i_1} \wedge \ldots \wedge dx^{i_p})_X \odot (dy^{i_1} \wedge \ldots \wedge dy^{i_p})_Y \]  \hspace{1cm} (24)

where
\[ f(X,Y) = \begin{cases} \frac{1}{(n-2)V_{n-2}} \frac{1}{|x-y|^{n-2}} & n > 2 \\ \frac{1}{2\pi} \log \frac{1}{|x-y|} & n = 2 \end{cases} \]  \hspace{1cm} (25)

and
\[ V_{n-1} = V_{n-2} \int_0^\pi \sin^{n-2} \theta d\theta \]  \hspace{1cm} (26)

with \( V_1 = 2\pi, \ V_0 = 1 \) and \(|x - y|^2 = \sum_{i=1}^{n} (x^i - y^i)^2\). This double-form is singular on \( X \times Y \) when \( x = y \) and satisfies
\[ \gamma \left( (X,Y) \right) = \gamma \left( (Y,X) \right) \]  \hspace{1cm} (27)
\[ \Delta_X \gamma \left( (X,Y) \right) = 0 \hspace{1cm} X \neq Y \]  \hspace{1cm} (28)
\[ d_X \gamma \left( (X,Y) \right) = \delta_Y \gamma \left( (X,Y) \right) \]  \hspace{1cm} (29)
\[ \delta_X \gamma \left( (X,Y) \right) = d_Y \gamma \left( (X,Y) \right) \]  \hspace{1cm} (30)

\[ ^5V_n \] is the volume of the unit \( n \)-sphere in \( \mathbb{R}^n \).
$$\Delta_X \gamma_{(p,p)}(X,Y) = \Delta_Y \gamma_{(p,p)}(X,Y) \quad (31)$$

The singularity at $x = y$ in $\gamma_{(p,p)}(X,Y)$ permeates many of the integrands of integrals that feature below. Such integrals, when they exist, are then defined implicitly as limits. In general integrals $\int_{\mathcal{U}} \alpha$ of $n$-forms $\alpha$, over a domain $\mathcal{U} \subset \mathbb{R}^n$, that contain such a singularity are understood as

$$\lim_{\epsilon \to 0} \int_{\mathcal{U} - B_\epsilon} \alpha$$

where $B_\epsilon$ is an $n$-ball of radius $\epsilon$ centred on the location of the singularity.

For any smooth $p$–form $\alpha$ on $Y$ the double-form $\gamma_{(p,p)}(X,Y)$ may be used to define the type-preserving integral operator $\Gamma$: $\alpha_X \rightarrow (\Gamma \alpha)_Y$ mapping $p$–forms on $X$ to $p$–forms on $Y$: where

$$(\Gamma \alpha)_Y = \int_X \alpha_X \wedge \gamma_{(p,p)}(X,Y) = \int_X \gamma_{(p,p)}(X,Y) \wedge \alpha_X$$

From (13) and (14) it follows that for all test forms $\hat{\phi}_X$, the differential operators $d_X$ and $\delta_X$ commute with the integral operator $\Gamma$:

$$d_X(\Gamma \hat{\phi}_X) = \Gamma(d_X \hat{\phi}_X) \quad (33)$$

$$\delta_X(\Gamma \hat{\phi}_X) = \Gamma(\delta_X \hat{\phi}_X) \quad (34)$$

If one writes

$$(\Gamma \hat{\phi}_X)_Y = \left( \gamma_{(p,p)}(X,Y), \hat{\phi}_X \right)$$

and notes that $\Gamma$ is a symmetric operator:

$$(\Gamma \alpha, \beta) = (\alpha, \Gamma \beta) = (\Gamma \beta, \alpha)$$

then it follows from Green’s identity applied to forms with compact support that:

$$\hat{\psi} = \Gamma \Delta \hat{\psi}$$

(36)
Thus

\[(\Gamma \phi, \Delta \hat{\chi}) = (\phi, \Gamma \delta \hat{\chi}) = (\phi, \hat{\chi})\] (37)

But \((\Gamma \phi, \Delta \hat{\chi}) = (\Delta \Gamma \phi, \hat{\chi})\) hence

\[(\Delta \Gamma \phi, \hat{\chi}) = (\phi, \hat{\chi})\] (38)

or

\[\Delta \Gamma \phi[\hat{\chi}] = \phi[\hat{\chi}]\] (39)

and \(\Gamma\) is seen to be an inverse of \(\Delta\).

For boundary value problems it is useful to introduce singular double-forms on \(U\) that differ from the fundamental double-forms \(\gamma_{(p,p)}(X,Y)\) by regular solutions to Laplace’s equation \(\Delta \phi = 0\). Such double-forms inherit the singularity structure of \(\gamma_{(p,p)}(X,Y)\) but can be tailored to satisfy certain conditions on the boundary \(\partial U\).

As an example suppose one requires \(p\)-form solutions to the generalized Poisson equation

\[\Delta_{(p)} \phi_{(p)} = S_{(p)}\] (40)

that vanish on \(\partial U\). Denote by

\[G_{XY}^p = \gamma_{(p,p)}(X,Y) - \gamma_{(p,p)}(X,Y)\] (41)

with \(\gamma_{(p,p)}(X,Y)\) a symmetric non-singular solution of \(\Delta \gamma_{(p,p)}(X,Y) = 0\) that coincides with the singular solution \(\gamma_{(p,p)}(X,Y)\) on \(\partial U\). Thus

\[\Delta G_{XY}^p = 0 \quad X \neq Y\] (42)

\[G_{XY}^p = G_{XY}^p\] (43)

and

\[G_{XY}^p|_{\partial U} = 0\] (44)
Suppose more generally now that $G^p_{XY}$ is a double-form satisfying chosen conditions on $\partial U$. For any smooth $p$–form on $\mathbb{R}^n$ define the integral operator $G$ by

$$ (G\phi)_Y = (G^p_{XY}, \phi_X) $$ \hfill (45)

If $y \in U$ the integral is singular at $x = y$ and

$$ (G^p_{XY}, \phi_X) = \lim_{\epsilon \to 0} (G^p_{XY}, \phi)_{U-B_\epsilon}. $$ \hfill (46)

One may use (16) together with the limits (derived by explicit computation in spherical polar coordinates with origin at the center of $B_\epsilon$):

$$ \lim_{\epsilon \to 0} \int_{S_\epsilon} \left( \delta_X G^p_{XY} \wedge \star \phi_X - \phi_X \wedge \star d_X G^p_{XY} \right) = \nu \phi_Y $$

$$ \lim_{\epsilon \to 0} \int_{S_\epsilon} \left( \delta_X \phi_X \wedge \star G^p_{XY} - G^p_{XY} \wedge \star d_X \phi_X \right) = 0 $$

where $S_\epsilon$ is the $n-1$ dimensional surface of the ball $B_\epsilon$, to verify

$$ \nu \phi_Y = (G \Delta \phi)_Y $$

$$ - \int_{\partial U} \left( \delta \phi \wedge \star G^p_{XY} - G^p_{XY} \wedge \star (d\phi) X - \delta_X G^p_{XY} \wedge \star \phi_X + \phi_X \wedge \star G^p_{XY} \right) $$ \hfill (47)

In this expression, $\nu = 0$ if $y$ is outside $U$, $\nu = \frac{1}{2}$ if $y \in \partial U$ and $\nu = 1$ if $y \in U$. If the first term on the right of this equation is written in terms of the source $S$ in the equation (40) then (47) offers a representation of the solution of this equation in terms of this source and values of $\delta \phi$, $d\phi$, $\star \phi$ and $\phi$ on $\partial U$. It should be stressed however that this representation is not an explicit solution in general since $\delta \phi$, $d\phi$, $\star \phi$ and $\phi$ cannot in general be assigned values arbitrarily on $\partial U$. Nevertheless this representation is the cornerstone of many developments in potential theory [2]. The above identities and their subsequent uses are often attributed to Green.

4 Electromagnetic Fields in Spacetime

Maxwell’s equations for an electromagnetic field in an arbitrary medium can be written

$$ d F = 0 \quad \text{and} \quad d \star G = j, $$ \hfill (48)
where $F \in \Gamma \Lambda^2 M$ is the Maxwell 2-form, $G \in \Gamma \Lambda^2 M$ is the excitation 2-form and $j \in \Gamma \Lambda^3 M$ is the 3-form electric current source. To close this system, “electromagnetic constitutive relations” relating $G$ and $j$ to $F$ are necessary.

The electric 4-current $j$ describes both (mobile) electric charge and effective (Ohmic) currents in a conducting medium. The electric field $e \in \Gamma \Lambda^1 M$ and magnetic induction field $b \in \Gamma \Lambda^1 M$ associated with $F$ are defined with respect to an arbitrary unit future-pointing timelike 4-velocity vector field $U \in \Gamma TM$ by

$$e = i_U F \quad \text{and} \quad c_0 b = i_U F. \quad (49)$$

Thus $i_U e = 0$ and $i_U b = 0$ and since $g(U, U) = -1$

$$F = e \wedge \tilde{U} - \star (c_0 b \wedge \tilde{U}). \quad (50)$$

The field $U$ may be used to describe an observer frame on spacetime and its integral curves model idealized observers.

Likewise the displacement field $d \in \Gamma \Lambda^1 M$ and the magnetic field $h \in \Gamma \Lambda^1 M$ associated with $G$ are defined with respect to $U$ by

$$d = i_U G, \quad \text{and} \quad h/c_0 = i_U \star G. \quad (51)$$

Thus

$$G = d \wedge \tilde{U} - \star ((h/c_0) \wedge \tilde{U}), \quad (52)$$

and $i_U d = 0$ and $i_U h = 0$.

## 5 Time dependent Maxwell Systems in Space

In this article we restrict to fields on Minkowski spacetime which can be globally foliated by 3-dimensional spacelike hyperplanes. The Minkowski metric induces a metric with Euclidean signature on each hyperplane. Furthermore each hyperplane contains events that are deemed simultaneous with respect to a clock attached to any integral curve of a future-pointing unit time-like

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6 All tensors in this article have dimensions constructed from the SI dimensions $[M], [L], [T], [Q]$ where $[Q]$ has the unit of the Coulomb in the MKS system. We adopt $[g] = [L^2], [G] = [j] = [Q], [F] = [Q]/[\epsilon_0]$ where the permittivity of free space $\epsilon_0$ has the dimensions $[Q^2 T^2 M^{-1} L^{-3}]$ and $c_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ denotes the speed of light in vacuo. Note that, with $[g] = [L^2]$, for $p$–forms $\alpha$ in $n$ dimensions one has $[\star \alpha] = [\alpha][L^{n-2p}]$. 

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vector field $U = \frac{1}{c} \frac{\partial}{\partial t}$ on spacetime and the Hodge map $\star$ induces a Hodge map $\#$ on each hyperplane by the relation
\[ \star 1 = c dt \wedge \# 1 \]

The spacetime Maxwell system can now be reduced to a family of exterior systems on $\mathbb{R}^3$. Each member is an exterior system involving forms on $\mathbb{R}^3$ depending parametrically on time $t$. Let the $3+1$ split of the 4-current 3-form with respect to the foliation be
\[ j = - J \wedge dt + \rho \# 1, \quad (53) \]

with $\dot{\rho} J = 0$. Then, from (48)
\[ d j = 0, \quad (54) \]
yields
\[ d J + \dot{\rho} \# 1 = 0. \quad (55) \]

Here and below an over-dot denotes (Lie) differentiation with respect to the parameter $t$ and $\dot{\alpha} \equiv \mathcal{L}_{\dot{\alpha}}$. It is convenient to introduce on each hyperplane the (Hodge) dual forms:
\[ E := \# e, \quad D := \# d, \quad B := \# b, \quad H := \# h, \quad j := \# J \quad (2) \]
so that the $3+1$ split of the spacetime covariant Maxwell equations (48) with respect to $\tilde{U} = - c dt$ becomes
\[ d e = - \dot{B}, \quad (56) \]
\[ d B = 0, \quad (57) \]
\[ d h = J + \dot{D}, \quad (58) \]
\[ d D = \rho \# 1. \quad (59) \]
All \( p \)-forms \((p \geq 0)\) in these equations are independent of \( dt \) but have components that may depend parametrically on \( t \).

In the following it is assumed that \( b = \mu h \) and \( d = \epsilon e \) where \( \epsilon = \epsilon_1 \epsilon_0 \), \( \mu = \mu_1 \mu_0 \). Thus in terms of \( e, h, E, H \):

\[
\begin{align*}
\text{d} e^{(1)} &= -\mu \dot{H}^{(2)}, \\
\text{d} H^{(2)} &= 0, \\
\text{d} h^{(1)} &= \epsilon \dot{E}^{(2)} + J^{(2)}, \\
\epsilon \text{d} E^{(2)} &= \rho \#1.
\end{align*}
\]

It is straightforward to show from this exterior system that the fields \( e^{(1)} \) and \( h^{(1)} \) satisfy

\[
\begin{align*}
\Delta h^{(1)} + \epsilon \mu \ddot{h}^{(1)} &= \delta J + \# \left( \text{d} e^{(1)} \land \#^{-1} \left( J - \text{d} h^{(1)} \right) \right), \\
\Delta e^{(1)} + \epsilon \mu \dot{e}^{(1)} &= -d \left( \frac{\rho}{\epsilon} \right) - \dot{j} + \# \left( d \mu \land \# \left( \mu^{-1} \text{d} e^{(1)} \right) \right).
\end{align*}
\]

Thus for homogeneous media with zero conductivity these equations each reduce to driven wave equations. Henceforth \( \epsilon, \mu \) are assumed to be constant scalars.

Locally on spacetime one has an equivalence class of \( 1 \)-forms whose elements \( A \) differ by the addition of any exact \( 1 \)-form. Then since \( F \) is closed one may write locally \( F = \text{d} A \). Decomposing \( A \) (with respect to a particular \( \tilde{U} \)) into 0–form \( \phi \) and 1-form \( \mathbf{A} \) potentials yields

\[
A = -\phi \tilde{U} - c \mathbf{A}
\]

with \( i_U A = 0 \). With \( \tilde{U} = -cdt \) and \( \ast 1 = \#1 \land \tilde{U} \) one has in terms of \( t \)-parameterised forms on \( \mathbb{R}^3 \)

\[
e^{(1)} = -d \phi - \dot{\mathbf{A}}
\]
The Maxwell system then reduces to

$$\delta d \phi + \delta \dot{A} = \frac{\rho}{\epsilon}$$  \hspace{1cm} (67)$$

$$\delta d A + \epsilon \mu \ddot{A} = \mu j - \epsilon \mu d \dot{\phi}$$  \hspace{1cm} (68)$$

In a gauge with

$$\delta A - \epsilon \mu \dot{\phi} = 0$$

the Maxwell system above requires that the potentials must satisfy

$$\Delta A + \epsilon \mu \ddot{A} = \mu j$$  \hspace{1cm} (69)$$

$$\Delta \phi + \epsilon \mu \ddot{\phi} = \frac{\rho}{\epsilon}$$  \hspace{1cm} (70)$$

for sources satisfying (55). Physical solutions correspond to those satisfying physically motivated boundary conditions (in both space and time). Alternatively in a gauge with

$$\delta A = 0$$  \hspace{1cm} (71)$$

the Maxwell system becomes

$$\Delta \phi = \frac{\rho}{\epsilon}$$  \hspace{1cm} (72)$$

$$\Delta A + \epsilon \mu \ddot{A} = \mu j - \epsilon \mu d \dot{\phi}$$  \hspace{1cm} (73)$$

In these equations on $\mathbb{R}^3$ with the Euclidean metric, the Hodge map on all forms satisfies $## = 1$ so $\delta = \# d \# \eta$ and $\Delta = \# d \# \eta d + d \# d \# \eta$. Henceforth, attention is devoted mainly to exterior systems on $\mathbb{R}^3$ so it is unnecessary to maintain the notational distinction between $d$ and $d$, $\delta$ and $\delta$, $\Delta$ and $\Delta$. 

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6 Electrostatics in $\mathbb{R}^3$ with Sources in Domains with Boundaries

Integral operators analogous to $\Gamma$ may be used to solve electrostatic boundary value problems in $\mathbb{R}^3$ involving the Hodge-de Rham operator $\Delta$ above and electric fields independent of $t$. As a basic example consider the Dirichlet problem of finding the electrostatic $0$–form potential $\phi$ on $\mathcal{U} \subset \mathbb{R}^3$ that must satisfy

$$
\begin{cases}
\Delta \phi = \bar{\rho}^{(0)} \\
\phi|_{\partial \mathcal{U}} = \bar{\phi}
\end{cases}
$$

(74)

for some data $\bar{\rho} \equiv \bar{\rho} \in \mathcal{U}, \bar{\phi} \in \partial \mathcal{U}$.

The solution may be represented in terms of the integral operator $\mathcal{G}$ defined by

$$(\mathcal{G} \bar{\rho})_Y = \int_{\mathcal{U}(X)} \tilde{\rho}_X \# \mathcal{G}_{XY}(X)$$

(75)

and is given by

$$\phi_Y = (\mathcal{G} \bar{\rho})_Y - \int_{\partial \mathcal{U}} \bar{\phi}_X \# d_x \mathcal{G}_{XY}(X)$$

(76)

where

$$\mathcal{G}_{XY} = \gamma_{(0,0)}(X,Y) - \gamma_{(0,0)}(\tilde{X},\tilde{Y})$$

(77)

with $\gamma_{(0,0)}(X,Y)$ a symmetric non-singular solution of $\Delta \gamma_{(0,0)}(X,Y) = 0$ that coincides with the singular solution $\gamma_{(0,0)}(X,Y)$ on $\partial \mathcal{U}$. Thus

$$\Delta \mathcal{G}_{XY} = 0 \quad X \neq Y$$

(78)

$$\mathcal{G}_{XY} = \mathcal{G}_{YX}$$

(79)

and

$$\mathcal{G}_{XY}|_{\partial \mathcal{U}} = 0$$

(80)
If the data is smooth it defines regular distributions \( \bar{\rho}^D, \bar{\phi}^D \) with supports on \( U \) and \( \partial U \) respectively by

\[
\bar{\rho}^D[\hat{\chi}] = (\bar{\rho}, \hat{\chi}) \quad \bar{\phi}^D[\hat{\chi}] = -\int_{\partial U(X)} \hat{\phi}_X \, d\gamma'(\hat{\chi})_X
\]  \hspace{1cm} (81)

Now

\[
(G \bar{\rho})[\hat{\chi}] = (G \bar{\rho}, \hat{\chi}) = (\bar{\rho}, G \hat{\chi}) = \bar{\rho}[G \hat{\chi}]
\]  \hspace{1cm} (82)

Thus for singular distributional sources \( \bar{\rho}^D \) one then has distributional Dirichlet solutions:

\[
\bar{\phi}^D[\hat{\chi}] = \bar{\rho}^D[\hat{\chi}] + \bar{\phi}^D[\hat{\chi}]
\]  \hspace{1cm} (83)

If the eigenvalues \( \lambda_M \) and Dirichlet eigenfunctions \( \Phi_N \) of \( \Delta \) on \( U \) can be found by solving

\[
\begin{cases}
\Delta \Phi_M = \lambda_M \Phi_M \\
\Phi_M|_{\partial U} = 0
\end{cases}
\]  \hspace{1cm} (84)

then a traditional way to satisfy the conditions required of \( G_{XY} \) is to express it as a Fourier expansion in the eigenmodes \( \Phi_M(Y) \) for each point of \( Y \in U \).

If the real modes are labeled by a discrete index set \( M \) the expansion takes the form

\[
G_{XY} = \sum_M G_M(X) \Phi_M(Y)
\]  \hspace{1cm} (85)

If such modes (with support on \( U \)) are ortho-normalised so that \( (\Phi_M, \Phi_N) = \delta_{MN} \) then

\[
G_N(X) = (G_{XY}, \Phi_N(Y))
\]  \hspace{1cm} (86)

Since the solutions \( \Phi_N \) are zero on \( \partial U \) it follows that

\[
\Phi_N(Y) = \lambda_N \int_U \Phi_N(X) \, d\gamma'(X) = \lambda_N(\Phi_N(X), G_{XY}) = \lambda_N G_N(Y)
\]  \hspace{1cm} (87)

or

\[
G_N(X) = \frac{1}{\lambda_N} \Phi_N(X) \quad \lambda_N \neq 0
\]

Thus

\[
G_{XY} = \sum_N \frac{1}{\lambda_N} \Phi_N(X) \Phi_N(Y)
\]  \hspace{1cm} (88)
This series must be regarded as weakly convergent and the summations become integrations when the eigenvalues are continuous.

For example consider $\mathcal{U}$ to be the space between two perfectly conducting parallel plates at $z = 0$ and $z = L$, separated by a distance $L$ in vacuo. Taking cartesian coordinates $\{x, y, z\}$ for $X$ and $\{x', y', z'\}$ for $Y$ the eigenvalues $N = \{k_x, k_y, n\}$ with $-\infty < k_x < \infty$, $-\infty < k_y < \infty$, $n = 1, 2, \ldots$ and (in complexified form):

$$\Phi_N(x, y, z) = \frac{1}{2\pi} \sqrt{\frac{2}{L}} \exp(ik_x + ik_y) \sin \frac{n\pi z}{L}$$  \hspace{1cm} (89)

This gives

$$\mathcal{G}(x, y, z; x', y', z') = \frac{2}{4\pi^2 L} \sum_n \sin \frac{n\pi z}{L} \sin \frac{n\pi z'}{L} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_x dk_y \exp(ik_x(x-x') + ik_y(y-y'))}{k_x^2 + k_y^2 + \frac{n^2\pi^2}{L^2}}$$  \hspace{1cm} (90)

If the source $\bar{\rho}$ is not smoothly distributed in $\mathcal{U}$ one needs further technology to evaluate the solution and this will be developed below.

7 Magnetostatics in $\mathbb{R}^3$ with Smooth Sources

Since magnetic charge is absent in Nature a typical problem in magnetostatics is the determination of a static magnetic field from a stationary electric current. The magnetostatic equations are a subset of Maxwell’s equations and may be written:

$$\delta h^{(1)} = 0$$ \hspace{1cm} (91)

$$d h^{(1)} = J^{(2)}$$ \hspace{1cm} (92)

in terms of forms on $\mathbb{R}^3$. If the current is localized in space (vanishing at $\infty$) the boundary conditions can be accommodated by using $\gamma(X, Y)$ for the solution. If one sets $\nabla \cdot h = \#dA/\mu_0$ then (91) is immediately satisfied.

$^7$Since $d B^{(2)} = 0$ one has $B^{(2)} = d A^{(1)}$ in regular domains and in vacuo $\#B = \mu_0 h$ where $\mu_0$ is the permeability of free space.
Writing $\bar{J} = -\mu_0 \# J$ one must have from (92)

$$d\# \bar{J} = 0 \quad (93)$$

and in the gauge with $\delta A = 0$ equation (92) becomes

$$\Delta A = \bar{J} \quad (94)$$

Thus in free space with a smooth source $\bar{J}$

$$A = \Gamma \bar{J} \quad (95)$$

since

$$\Delta A = \Delta \Gamma \bar{J} = \bar{J} \quad (96)$$

i.e.

$$A_Y = \int_X \gamma (X,Y) \wedge \# \bar{J} \quad (X) \quad (97)$$

with

$$\gamma (X,Y) = \frac{1}{4\pi |x-y|} \sum_{j=1}^{3} dx^j \odot dx'^j \quad (98)$$

Note if $\bar{J}$ is bounded

$$\delta_Y A_Y = \int_X \delta_Y \gamma (X,Y) \wedge \# \bar{J} = \int_X d_X \gamma (X,Y) \wedge \# \bar{J} \quad (X)$$

$$= - \int_X \gamma (X,Y) d_X \# \bar{J} = 0 \quad (99)$$

Since $\Delta h = \delta J$ one also has directly

$$h(Y) = \int_X \gamma (X,Y) \wedge \# (\delta J)_X \quad (X) \quad (100)$$

That this also furnishes a solution to (91) and (92) will be verified in section [10].
Now for a smooth (regular) source $\mathcal{J}$

$$A[\hat{\psi}] = \Gamma \mathcal{J}[\hat{\psi}] = \int_Y (\Gamma \mathcal{J})_Y \wedge \# \hat{\psi}_Y$$

$$= \int_Y \left( \int_X (\gamma (X, Y) \wedge \# \mathcal{J}_X) \right) \wedge \# \hat{\psi}_Y$$

$$= \int_X \int_Y \left( \gamma (X, Y) \wedge \# \hat{\psi}_Y \right) \wedge \# \mathcal{J}_X$$

$$= \int_X (\Gamma \hat{\psi})_X \wedge \# \mathcal{J}_X$$

$$= \int_X \mathcal{J}_X \wedge \# (\Gamma \hat{\psi})_X$$

$$\equiv \mathcal{J}^D[\Gamma \hat{\psi}] \quad (100)$$

If $\mathcal{J}$ is a smooth 1–form in $\mathbb{R}^3$ the integral $\mathcal{J}^D[\Gamma \hat{\psi}]$ furnishes a solution for $A$. However if the current has support on a curve in $\mathbb{R}^3$ it must be regarded as a distributional 1–form source not associated with a smooth 1–form. Similarly the charge density in the electrostatic problem may be restricted to a curve or surface in $\mathbb{R}^3$ in which case a distributional source $\rho^D$ not associated with a smooth 0–form must be specified.

8 Dirac Distributions on Submanifolds

To define singular distributions with support on (possibly disjoint) submanifolds of a manifold $M$ it is convenient to describe each submanifold parametrically as an embedding in $M$. Suppose that a distribution has support on a collection of submanifolds $S_0^{(r)}$, i.e. on the chain $\sum_r S_0^{(r)}$. Recall that an $n - k$ dimensional submanifold $S_0^{(r)}$ (possibly with boundary) in an $n$–dimensional manifold $M$ can also be prescribed in terms of $k$ 0–forms $f_1^{(r)}, f_2^{(r)}, \ldots, f_k^{(r)}$ on $M$, such that $df_1^{(r)} \wedge df_2^{(r)} \ldots \wedge df_k^{(r)} \neq 0$. Such forms generate a local foliation $S^{(r)}$ in the neighborhood of each submanifold $S_0^{(r)}$, where each leaf $S_t^{(r)}$ of the foliation is an $n - k$ dimensional embedding given by $f_1^{(r)} = c_1, f_2^{(r)} = c_2, \ldots, f_k^{(r)} = c_k$ for some constants $c_1, c_2, \ldots c_k$ and $c = (c_1, c_2, \ldots c_k)$. We may choose the leaf $S_0^{(r)}$ with all these constants zero and denote it $\Sigma_n^{n-k}$ with $\mathbf{f}^{(r)} \equiv (f_1^{(r)}, f_2^{(r)}, \ldots, f_k^{(r)})$. It is sufficient to
establish the distributional framework for a single component of a general chain so henceforth the label \((r)\) will be omitted.

If \(M\) is endowed with a metric \(g\) (and associated Hodge map \(\star\)), the forms \(\{df_j\}\) give rise to a class of forms \(\Omega_f\) on \(M\) defined with respect to \(\star 1\) by

\[
\star 1 = df_1 \land df_2 \land \ldots \land df_k \land \Omega_f \quad (n-k)
\]

This class restricts to a natural class of measures on \(\Sigma^{n-k}_f\). Thus each representative induces the measure

\[
\omega_f = \Omega_f \mid_{\Sigma^{n-k}_f} \quad (n-k)
\]

on the submanifold \(\Sigma^{n-k}_f\). Members of the class on \(M\) are equivalent if they differ by any combination of the \(df_j\). This gauge freedom is of no significance on \(\Sigma^{n-k}_f\) since the \(df_j\) vanish there under pull-back. Thus it is sufficient to represent the class by an element satisfying

\[
i_{df_j} \Omega_f = 0 \quad j = 1 \ldots k.\]

In those situations where the leaves form orthogonal families, i.e. \(g^{-1}(df_i, df_j) = 0\) for \(i \neq j\) then

\[
\Omega_f = \star \frac{(df_1 \land df_2 \land \ldots \land df_k)}{\prod_{j=1}^k |df_j|^2} \quad (n-k)
\]

where \(|df_j|^2 = \tilde{df}_j(df_j) \neq 0\) and the sign of \(|df_j|\) is chosen so that the direction of the vector field \(df_j \equiv g^{-1}(df_j, -)\) is in the direction of increasing values of \(f_j\). Since \(df_j \neq 0\) one has on \(\Sigma^{n-k}_f\) two fields of "unit normals": \(n_j = \pm \frac{df_j}{|df_j|}\). Then with \(n_j = \frac{df_j}{|df_j|}\)

\[
\Omega_f = \star \prod_{j=1}^k \left( \frac{n_j}{|df_j|} \right) \quad (n-k)
\]

For non-orthogonal leaves an \(\Omega_f\) can be chosen directly from (101). This condition is equivalent to the equation

\[
\pm 1 = \star (df_1 \land df_2 \land \ldots \land df_k \land \Omega_f) \quad (n-k)
\]
which yields freedom to choose $\omega^{(n-k)}_f$ with non-singular coordinate components when restricted to $\Sigma^{n-k}_f$. The singular Dirac 0--form with support on $\Sigma^{n-k}_f$ is denoted $\delta^{(0)}_{\Sigma^{n-k}_f}$ and defined by

$$\delta^{(0)}_{\Sigma^{n-k}_f}[\hat{\phi}] = \int_{\Sigma^{n-k}_f} \hat{\phi} \omega^{(n-k)}_f$$

(107)

Furthermore if $Y$ is a smooth vector field on $M$ one may define the directional derivative $\delta^Y_{\Sigma^{n-k}_f}$ of $\delta^{(0)}_{\Sigma^{n-k}_f}$ by:

$$\delta^Y_{\Sigma^{n-k}_f}[\hat{\phi}] = \int_{\Sigma^{n-k}_f} \omega^Y_f (\hat{\phi})$$

(109)

where for orthogonal leaves:

$$\omega^Y_f (\hat{\phi}) = \frac{1}{\Pi_{j=1}^k |df_j|^2} i_{\tilde{d}f_k} \ldots i_{\tilde{d}f_1} d(\star \tilde{Y} \hat{\phi}) |_{\Sigma^{n-k}_f}$$

(110)

Since

$$\delta (f (\tilde{Y} \psi)) = f \delta (\tilde{Y} \psi) - \psi (Y f)$$

$$\omega^Y_f (\hat{\phi}) = - (\hat{\phi} \delta \tilde{Y} - Y \hat{\phi}) * \left( \frac{df_1}{|df_1|^2} \wedge \ldots \wedge \frac{df_k}{|df_k|^2} \right)$$

(111)

Additional directional derivatives yield the distributions

$$\delta^{Y_1 Y_2 \ldots Y_q}_{\Sigma^{n-k}_f} \equiv Y^1 Y^2 \ldots Y^q \delta^{(0)}_{\Sigma^{n-k}_f}$$

For example

$$\left( Y^1 Y^2 \delta^{(0)}_{\Sigma^{n-k}_f} \right)[\hat{\phi}] = \delta^{(0)}_{\Sigma^{n-k}_f} [\delta (\tilde{Y}_1 \wedge \delta (\tilde{Y}_2 \hat{\phi}))]$$

(112)
If one regards any point \( p_0 \in M \) as a 0–dimensional space then the original singular Dirac distribution \( \delta^D_{p_0} \) with support on \( p_0 \) is defined by:

\[
\delta^{D}_{p_0}(\hat{\phi}) = \hat{\phi}(p_0) \quad (113)
\]

It follows that

\[
\delta^{\Sigma}_0 = \omega_{t} \delta^{D}_{p_0} \quad (114)
\]

The above singular 0–form Dirac distributions can be used to construct singular \( p \– form Dirac distributions with support on submanifolds in various ways. Thus in a spherical polar chart for \( \mathbb{R}^3 \) a 1–form 0–layer Dirac distribution on the unit 2–sphere centred at the origin may be represented as

\[
\delta_{\Sigma_t}^2 (\mu_\theta(\theta, \phi) \, d\theta + \mu_\phi(\theta, \phi) \, d\phi) \quad (115)
\]

where \( \mu_\theta \) and \( \mu_\phi \) are smooth functions on the sphere. A 1–form 1–layer Dirac distribution on the sphere (associated with some vector field \( Y \) on \( \mathbb{R}^3 \)) may take the form

\[
\delta_{\Sigma_t}^1 (p_\theta(\theta, \phi) \, d\theta + p_\phi(\theta, \phi) \, d\phi) \quad (116)
\]

for smooth \( p_\theta \) and \( p_\phi \). Similarly a 2–form 0–layer Dirac distribution on the sphere takes the form

\[
\delta_{\Sigma_t}^2 q(\theta, \phi) \, d\theta \wedge d\phi \quad (117)
\]

and a 2–form 1–layer Dirac distribution takes the form

\[
\delta_{\Sigma_t}^1 Y^s(\theta, \phi) \, d\theta \wedge d\phi \quad (118)
\]

for smooth \( q \) and \( s \) respectively. More generally in coordinates

\[
(\xi^1, \ldots, \xi^k, \sigma_1, \ldots, \sigma_p)
\]

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adapted to a \( p \)-dimensional submanifold in \( \mathbb{R}^n \), one has for \( j = 0, 1, \ldots p \), the \( j \)-form, “\( r \)-layer” singular Dirac distribution with support on \( \Sigma_{r-k}^n \):

\[
\mathcal{J}^{(j)} = \lambda^{(0)}(\sigma^1 \ldots \sigma^p) \delta^{Y_1 \ldots Y_r}_{\Sigma_{r-k}^n(0)} \left( \sum_{i_1 < i_2 < \ldots < i_j} \sigma_{i_1} \ldots \sigma_{i_j} \right) d\sigma_{i_1} \wedge \sigma_{i_2} \wedge \ldots \wedge d\sigma_{i_j}
\]

(119)

For a point electrostatic dipole with dipole moment \( Z_X = p^i \frac{\partial}{\partial x^i} \) located at \( x_0 \in X \) the electrostatic potential distribution \( \phi_Y^D \) is given by the singular distributional source \( \rho_Y^D \) where

\[
\phi_Y^D[\hat{x}] = \rho_Y^D[\hat{x}] = \left( L_{Z_X}(\gamma_{(0,0)}(X,Y)) \right) \delta^{D}_{x_0}[\hat{x}] = p \cdot \nabla \left( \frac{1}{|y - x_0|} \right) \hat{x}(x_0)
\]

(120)

9 Line and Surface Currents in \( \mathbb{R}^3 \)

The source of a smooth magnetostatic field is the smooth current 2-form \( J \) in \( \mathbb{R}^3 \) and the total electric current passing through a surface \( S \subset \mathbb{R}^3 \) is \( I[S] = \int_S J \), measured in amps in MKS units. In some circumstances this current density may be concentrated in the vicinity of 1-dimensional submanifolds \( \Sigma_1 \subset \mathbb{R}^3 \) or 2-dimensional submanifolds \( \Sigma_2 \subset \mathbb{R}^3 \). Electric current filaments on material curves in space (wires) and current sheets in material surfaces in space are idealizations of such localized sources and may be modeled by singular distribution-valued 1-forms with support on curves and surfaces respectively. In these situations new physical current densities are introduced so that the total current in a segment of wire or a region of a surface is finite and measurable. These singular distributional sources can be mathematically modeled in terms of the distributions \( \delta_{\Sigma_r^f} \) for \( r = 1, 2 \) and a vector field \( W \) in \( \mathbb{R}^3 \) with support on \( \Sigma_f^r \).

For a wire described by the curve with image \( \Sigma_1^f \) introduce the distributional 1-form

\[
\mathcal{I}^{D}_{\Sigma_1^f(1)} = I_0 \mathcal{W} \delta_{\Sigma_1^f(0)}
\]

(121)
where \( I_0 \) is a smooth function on the wire. Hence

\[ T^{D}_{\Sigma^1_f} [\hat{\psi}] = \int_{\mathbb{R}^3} \hat{\psi} \wedge J \]

and with \( J = \# J \):

\[ T^{D}_{\Sigma^1_f} [\hat{\psi}] = \delta_{\Sigma^1_f} [I_0 i_W \hat{\psi}] = \int_{\Sigma^1_f} I_0 (i_W \hat{\psi}) \omega_f = \int_{\mathbb{R}^3} \hat{\psi} \wedge J = \int_{\mathbb{R}^3} \hat{\psi} \wedge \# J = J^D[\hat{\psi}] \]

\[ = \int_{\Sigma^1_f} \hat{\psi} I_0 (i_W \omega_f) \equiv \int_{\Sigma^1_f} \hat{\psi} \tilde{I} \]

since \( \hat{\psi} \wedge \omega_f \) is zero on \( \Sigma^1_f \). It does not make sense to ascribe a physical dimension to \( T^{D}_{\Sigma^1_f} \) since it is a functional not a value. However if \( \{ J^D[\hat{\psi}] \} \) denotes the physical dimension of \( J^D[\hat{\psi}] \) one has in MKS units

\[ \{ J^D[-]\} = \{ J \}{-} = \text{amp} \{-\} \]

and \( \{ J \} = \{ \# J \} = \text{amp} \text{ m}^{-1} \) with \( \{ I_0 \}\{ \tilde{W} \}\{ \omega_f \} = \{ J \} = \text{amp} \). The (smooth) 0-form density

\[ \tilde{I}(s) \equiv I_0(i_W \omega_f) \]

belonging to \( \Gamma \Lambda^0 \Sigma^1_f \) is the total current (amps) flowing along the wire at any point \( s \in \Sigma^1_f \)

In a similar way, for the 1-form distribution on \( \mathbb{R}^3 \):

\[ K^{D}_{\Sigma^2_f} = \kappa_0 \tilde{W} \delta_{\Sigma^2_f} \]

one has

\[ K^{D}_{\Sigma^2_f} [\hat{\psi}] = \int_{\mathbb{R}^3} \hat{\psi} \wedge J = \delta_{\Sigma^2_f} [\kappa_0 i_W \hat{\psi}] \]

\[ = \int_{\Sigma^2_f} \kappa_0 (i_W \hat{\psi}) \omega_f \equiv \int_{\Sigma^2_f} \hat{\psi} \wedge \omega_f = \int_{\Sigma^2_f} \hat{\psi} \wedge \tilde{k} \]

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since \( \kappa_0(i_W \hat{\psi}) \omega_f = \kappa_0 \hat{\psi} \wedge (i_W \omega_f) \) on \( \Sigma^2_f \). Here

\[
\tilde{\kappa} = \kappa_0 i_W(\omega_f) \in \Gamma \Lambda^1 \Sigma^2_f
\]

is a smooth 1-form (measured in amps) on the surface \( \Sigma^2_f \). If \( C_1 \in \mathbb{R}^3 \) is any space-curve lying on this surface then \( \int_{C_1} \tilde{\kappa} \) is the total current in amps crossing this curve in the direction \( W \), on \( \Sigma^2_f \). This may be compared with the definition above of the total current (in amps), \( \int_S J \), crossing the surface \( S \in \mathbb{R}^3 \) in the direction \( \hat{\#} \tilde{J} \). If \( C_1^* \frac{\partial}{\partial s} \) is a unit tangent vector \( \hat{\#} \) to \( C_1 \) then \( i_{W^*_{(1)}} \tilde{\kappa} \) is the surface current density in amp \( m^{-1} \).

It is worth noting that \( I_0 \in \Gamma \Lambda^0 \mathbb{R}^3 \), \( \kappa_0 \in \Gamma \Lambda^0 \mathbb{R}^3 \), \( W \in \Gamma \mathbb{T} \mathbb{R}^3 \) and \( \Sigma^1_f \), \( \Sigma^2_f \) are the primary objects used to define the distributions \( \mathcal{I}^D_{\Sigma^1_f} \) and \( \mathcal{K}^D_{\Sigma^2_f} \) in \( \mathbb{R}^3 \) in terms of which are defined the smooth 0-forms \( \tilde{I} \in \Gamma \Lambda^0 \Sigma^1_f \) and 1-forms \( \tilde{\kappa} \in \Gamma \Lambda^1 \Sigma^2_f \).

10 Magnetostatic Fields from a Singular Distributional Stationary Current Source

Suppose a uniform electric current confined to a wire in \( \mathbb{R}^3 \) is modeled in terms of a 1-form 0-layer Dirac distribution \( \delta_{\Sigma^1_f} \) with support on the wire. A distributional source \( \mathcal{J}^D \) will generate a distributional 1-form \( \mathcal{A}^D \). Thus define \( \mathcal{A}^D \) by

\[
\mathcal{A}^D[\hat{\psi}] = \mathcal{J}^D[\Gamma \hat{\psi}]
\]

Suppose \( \mathcal{J}^D \) has a support on the locus described by \( \mathbf{f} = (f_1, f_2) \) with \( f_1 = r - a \), and \( f_2 = z \) in cylindrical coordinates \((r, \phi, z)\). This will be used to model a uniform stationary current source in a circular loop of radius \( a \) if

\[
\mathcal{J}^D = \lambda \delta_{\Sigma^1_f} d\phi
\]

8i.e. in terms of the Euclidean metric tensor \( g \) in \( \mathbb{R}^3 \), \( g(C_1^* \frac{\partial}{\partial s}, C_1^* \frac{\partial}{\partial s}) = 1 \)
with some constant \( \lambda \). Thus

\[
\tilde{J}^D[\Gamma \hat{\psi}] = \lambda \delta_{\Sigma^1(1)}[i_{\bar{\phi}}(\Gamma \hat{\psi})]
\]

\[= \lambda \int_{\Sigma^1(1)} \left(i_{\bar{\phi}}(\Gamma \hat{\psi})_Y \right)_Y \omega^Y_X \tag{128}\]

Since

\[(\Gamma \hat{\psi})_Y = \int_X (X, Y) \wedge \# \hat{\psi}_X \tag{129}\]

and

\[i_{\bar{\phi}}(\gamma)(\Gamma \hat{\psi})_Y = \int_X \left(i_{\bar{\phi}}(\gamma)(X, Y) \right)_Y \wedge \# \hat{\psi}_X \tag{130}\]

then

\[A^D[\hat{\psi}] = \int_X A_X \wedge \# \hat{\psi}_X \tag{131}\]

where

\[A_X = \lambda \int_{\Sigma^1(1)} \left(i_{\bar{\phi}}(\gamma)(X, Y) \right)_Y \omega^Y_X \tag{132}\]

In cylindrical polar coordinates, \( \mathbf{x} = (r \cos \phi, r \sin \phi, z) \), \( \mathbf{y} = (r' \cos \phi', r' \sin \phi', z') \)

\[| \mathbf{x} - \mathbf{y} |^2 \equiv R^2(r, \phi, z; r', \phi', z') = r^2 + r'^2 + z^2 + z'^2 - 2zz' - 2rr' \cos(\phi - \phi') \]

and the Euclidean metric tensor is

\[g = dr \otimes dr + r^2 d\phi \otimes d\phi + dz \otimes dz \tag{133}\]

with

\[g^{-1} = \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + r^{-2} \frac{\partial}{\partial \phi} \otimes \frac{\partial}{\partial \phi} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \tag{134}\]

Hence \( \tilde{\phi}' = \frac{1}{r^2} \frac{\partial}{\partial \phi} \). With \( dx' = drr' \cos \phi' - r' \sin \phi' d\phi', dy' = drr' \sin \phi' + r' \cos \phi' d\phi', i_{\frac{\partial}{\partial \phi}} dx' = -r' \sin \phi', i_{\frac{\partial}{\partial \phi}} dy' = r' \cos \phi' \), \( i_{\frac{\partial}{\partial \phi}} d z' = 0 \)

\[i_{\bar{\phi}}(\gamma)(X, Y)_{(1,1)}|_{r' = a, z' = 0} = \frac{1}{aR} \left( dr \sin(\phi - \phi') + r d\phi \cos(\phi - \phi') \right) \tag{135}\]
Further with
\[ \# 1 = r' \, d \, r' \wedge d \, z \wedge d \, \phi' = d \, f_1 \wedge d \, f_2 \wedge \omega_{r}^Y \]
and one has
\[ \omega_{r}^Y = a \, d \, \phi' \]
and one has\(^9\)
\[ A_X(r, \phi, z) = r \, d \phi \frac{1}{4\pi} \int_{0}^{2\pi} \frac{\lambda \cos(\phi - \phi')}{(r^2 + a^2 + z^2 - 2ar \cos(\phi - \phi'))^{1/2}} d \phi' \quad (136) \]
or with \( \Psi = \phi' - \phi \) and noting that the integrand is an even periodic function of \( \Psi \):
\[ A_X(r, \phi, z) = -r \, d \phi \int_{0}^{2\pi} \frac{\lambda \cos \Psi \, d \Psi}{(r^2 + a^2 + z^2 - 2ar \cos \Psi)^{1/2}} \quad (137) \]
with magnitude independent of \( \phi \). For a constant current \( \bar{I} \) in the circular loop the constant \( \lambda = 4\pi \mu_0 \bar{I} a \). In MKS units the physical unit for \( h \) is the ampere and that for \( \bar{J} \) is \( \mu_0 \) ampere/m. The methodology here illustrated for a circular planar coil is directly applicable to any open or closed current carrying conductor of arbitrary shape in space and possibly composed of piecewise smooth connected segments.

To verify that when \( A^D[\hat{\psi}] = \int_X A \wedge \# \hat{\psi} \) the distribution \( A^D \) lies in the gauge satisfying \( \delta A^D = 0 \) one must compute
\[ \delta A^D[\hat{\xi}] = A^D[d \hat{\xi}] \]
\[ = \int_X A \wedge \# d \hat{\xi} = \int_X d \hat{\xi} \wedge \# A \]
\[ = -\int_X \hat{\xi} d \# A = -\int_X \hat{\xi} \# \delta A \]
\[ = -\lambda \int_X \hat{\xi} \# \int_{\Sigma_f(Y)} \frac{i}{d\phi(Y)} \delta_X \gamma (X, Y) \omega_{r}^Y \quad (138) \]
\[ ^9\text{The term in the integrand proportional to } \sin(\phi - \phi') \text{ integrates to zero.} \]
But

\[ \delta_X \gamma (X,Y) = d_Y \gamma (X,Y) \]

\[ i \frac{d}{d\phi(Y)} d_Y \gamma (0,0) (X,Y) = -\frac{\partial}{\partial \phi} \gamma (0,0) (X,Y) \]

so

\[ \delta A^D[\hat{\xi}] = 0 \]

since \( \omega^{Y_1}_{(1)} = a \, d \phi' \) and \( \Sigma^1_f(Y) \) is a closed curve. This is physically equivalent to the statement that the current in the coil is “conserved”.

One can just as easily work in a gauge invariant manner by solving the equation

\[ \Delta h = \delta J \tag{139} \]

All solutions to (91) and (92) must satisfy (139). As noted above (99), for a smooth source \( \delta J \), the solution in free space is

\[ h^{(1)}(Y) = \int_{X^{(1,1)}} \gamma (X,Y) \wedge \# (\delta J)_X. \tag{140} \]

For a distributional source the solution

\[ \mu_0 h^{D[\hat{\psi}]} = \mathcal{J}^D[\Gamma' \hat{\psi}] \tag{141} \]

(cf (100)) is modeled on the representation following directly from (140):

\[ \mu_0 h^{D[\hat{\psi}]} = \int_X \int_{Y^{(1,1)}} \mathcal{J}_{Y} \wedge \# \gamma' (X,Y) \wedge \# \hat{\psi}_X \tag{142} \]

In these equations the operator \( \Gamma' \) is defined as \( \Gamma \) but with \( \gamma (X,Y) \) replaced by \( \gamma' (X,Y) \) where

\[ \gamma' (X,Y) = \delta_Y \# \gamma (X,Y). \tag{143} \]

Thus for the distributional source

\[ \mathcal{J}^{D[\hat{\psi}]} = \lambda \mathcal{W} \delta \Sigma^1_f \]

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for some vector field $W$ on $\mathbb{R}^3$ one has

$$\mu_0 h \frac{D}{\partial t} \hat{\psi} + \mathcal{H}_X = \int_X \frac{\mathcal{H}_X}{(1)} \wedge \#X_{(1)}$$

(144)

where

$$\mathcal{H}_X = \int_{\Sigma^1_t} \lambda i W_v \gamma'(X,Y) \omega^Y_{(1)}$$

(145)

Following (123) the choice of $W$ and $\Sigma^1_t$ determines the physical dimensions of $\lambda$. \footnote{For a loop with a non-constant but steady current, $\tilde{W} = d\phi$ and $\Sigma^1_t = a d\phi$ one could take $\lambda = 4\pi \mu_0 I_0 a f(\phi)$ for some physically dimensionless function $f$ of $\phi$ to describe a coil with varying resistivity.}

Although (140) solves (139), in order to qualify as a magnetostatic field one must verify that it satisfies $\delta h = 0$ and $d h = J$. Now for some $U \in \mathbb{R}^3$ containing the smooth source $J$ with

$$h_{Y} = \left( \gamma'(X,Y), j_X \right)_{U}$$

one has

$$(\delta h)_{Y} = \left( \gamma'(X,Y), j_X \right)_{U} = \left( d_X \gamma'(X,Y), j_X \right)_{U}$$

$$= \left( \gamma'(X,Y), \delta_X j_X \right)_{U} + \int_{\partial U} \gamma'(X,Y) \wedge \#X_{(0)}$$

$$= \left( \gamma'(X,Y), \#X d j_X \right)_{U} + \int_{\partial U} \gamma'(X,Y) \wedge J_X$$

Hence with $dJ = 0$, and $J|_{S_\infty} = 0$ and (by explicit calculation)

$$\lim_{\epsilon \to 0} \int_{S_\epsilon} \gamma'(X,Y) \wedge J_X = 0$$

one has

$$\delta h = 0$$

(146)
Writing $\phi = h$ and $\psi = \hat{\psi}$ in (15)

$$
\left( \delta J, \hat{\psi} \right)_U - \left( dh, d\hat{\psi} \right)_U - \left( \delta h, \delta \hat{\psi} \right)_U = \int_{\partial U} (\delta h \wedge \# \hat{\psi} - \hat{\psi} \wedge \# dh)
$$

But $\delta h = 0$ and $\left( \delta J, \hat{\psi} \right)_U = \left( J, d\hat{\psi} \right)_U$. It remains to calculate $\int_{\partial U} \# dh \wedge \hat{\psi}$ with $h = \left( \gamma (X,Y), \delta J \right)_U$ and so $(dh)_Y = \left( \delta X \gamma (X,Y), \delta X J_X \right)_U$. Thus

$$
\int_{(\partial B_v)Y} (\#dh)_Y \wedge \hat{\psi}_Y = \int_{(\partial B_v)Y} \left( \#_Y \delta X \gamma (X,Y), \delta X J_X \right)_U \wedge \hat{\psi}_Y
$$

$$
= (\delta X \Lambda X, \delta X J_X)_U
$$

where $\Lambda X = \int_{(\partial B_v)Y} \#_Y \gamma (X,Y) \wedge \hat{\psi}_Y$. In the limit as $\epsilon \to 0$ one has by explicit computation that this integral is zero, hence

$$
\left( dh, d\hat{\psi} \right)_U = \left( J, d\hat{\psi} \right)_U
$$

But since $d\hat{\psi}$ is an arbitrary test form it follows that

$$
dh = J \quad \text{on } U \tag{147}
$$

These arguments generalize immediately to the singular distributional case.

11 Magnetostatic Fields from a Steady Helical Line Current

The derivation above of the potential for the magnetic field due to a circular line current was somewhat labored. However no further steps are required to find the potentials due to more complex geometries once one parametrises the source in terms of the geometry of its support in space.

Suppose a helical wire with pitch $p > 0$ and radius $a$ is the space curve $(r = a, \phi = \sqrt{1-p^2} \sigma, z = p \sigma)$ in cylindrical polar coordinates. The parameter $\sigma$ is the arc-length parameter so $0 < \sigma < L$ for a helix of length $L$. It is convenient to introduce $P > 0$ with $a^2 P^2 + p^2 = 1$. Then the unit tangent
to the helix is the direction $W = p \frac{\partial}{\partial z} + a P \frac{\partial}{\partial \phi}$. If one chooses to describe the helix in terms of the foliating set $\mathbf{f} = \{f_1 = r - a, f_2 = z - \frac{p}{P} \phi\}$ this yields a Leray form $\Omega_\mathbf{f} = -rd\phi$. Then, in this case, the source $\mathcal{J}_D = \lambda \tilde{W} \delta_{\Sigma_\mathbf{f}}^{(0)}$ with support on the helical wire yields the magnetostatic potential 1–form

$$\mathcal{A}(r, \phi, z) = \mathcal{A}_r(r, \phi, z) \, dr + \mathcal{A}_\phi(r, \phi, z) \, r \, d\phi + \mathcal{A}_z(r, \phi, z) \, dz$$  \hspace{1cm} (148)$$
where

$$\mathcal{A}_r = \frac{\lambda}{4\pi} \int_0^L \frac{d\sigma}{R} \sin (P\sigma - \phi)$$  \hspace{1cm} (149)$$

$$\mathcal{A}_\phi = -\frac{\lambda a^2 P^2}{4\pi} \int_0^L \frac{d\sigma}{R} \cos (P\sigma - \phi)$$  \hspace{1cm} (150)$$

$$\mathcal{A}_z = -\frac{\lambda apP}{4\pi} \int_0^L \frac{d\sigma}{R}$$  \hspace{1cm} (151)$$
with

$$R^2 = r^2 + a^2 + z^2 + \sigma^2 p^2 - 2z p \sigma - 2a \, r \cos (P\sigma - \phi)$$  \hspace{1cm} (152)$$

For a constant current $\tilde{I}_0$ in the helix the constant $\lambda = 4\pi \mu_0 \tilde{I}_0 a$. The potential $\mathcal{A}$ yields a global description of the field in terms of elliptic integrals that depend on the geometry of the helix specified by the parameters $L, a$ and $p$ or $P$. Such parameters offer natural scales that are useful for defining dimensionless variables that in turn can be used to generate multipole or asymptotic expansions of the above integrals. The practical generation of high intensity uniform magnetic fields by carefully designing coils with complex helical windings is of paramount importance in the construction of undulators and free electron lasers.

### 12 Magnetostatic Fields from a Steady Helical Surface Current

It is natural to model a solenoid composed of closely wound current-carrying turns by a surface current source. Such a current, regarded as a vector field on the solenoid surface $\Sigma_\mathbf{f}^2$, can have an arbitrary direction $W|_{\Sigma_\mathbf{f}^2}$ and the distributions $\mathcal{J} = \kappa_0 \tilde{W} \delta_{\Sigma_\mathbf{f}^2}$ are well suited to model such a source. Suppose $\Sigma_\mathbf{f}^2$ is a right circular cylinder of radius $a$ and length $L_0$ and the surface
current is “painted” on it with helical strokes of pitch \( p = \sqrt{1 - a^2 P^2} \). Thus the integral curves of \( W = p \frac{\partial}{\partial z} + a P \frac{\partial}{\partial \phi} \) are each similar to the helix above. To construct \( \delta_{\Sigma^2} \) one parametrises the solenoid surface as \( \{ r = a, z = \rho, \phi = \sigma \} \) in cylindrical polars with \( 0 < \sigma < 2\pi \) and \( 0 < \rho < L_0 \) and takes \( f = r - a \) to generate \( \Omega_f = r d\phi \wedge dz \). It follows that, in this case, the source \( \bar{\mathcal{J}}^D = \kappa_0 W \delta_{\Sigma^2} \) with support on the cylindrical surface yields the magnetostatic potential

\[
\mathcal{A}(r, \phi, z) = \mathcal{A}_r(r, \phi, z) \, dr + \mathcal{A}_\phi(r, \phi, z) \, r \, d\phi + \mathcal{A}_z(r, \phi, z) \, dz \tag{153}
\]

where

\[
\mathcal{A}_r = a^2 P \frac{\kappa_0}{4\pi} \int_0^{2\pi} \left( \int_0^{L_0} \frac{d\rho}{R} \sin(\sigma - \phi) \right) \, d\sigma \tag{154}
\]

\[
\mathcal{A}_z = -aP \frac{\kappa_0}{4\pi} \int_0^{2\pi} \left( \int_0^{L_0} \frac{d\rho}{R} \right) \, d\sigma \tag{155}
\]

\[
\mathcal{A}_\phi = -a^2 P \frac{\kappa_0}{4\pi} \int_0^{2\pi} \left( \int_0^{L_0} \frac{d\rho}{R} \cos(\sigma - \phi) \right) \, d\sigma \tag{156}
\]

with

\[
R^2 = r^2 + a^2 + z^2 + \rho^2 - 2z\rho - 2ar \cos(\sigma - \phi)
\]

The double integrals above can be reduced to quadratures involving Elliptic integrals.

13 Electrostatics with a Singular Distributional Charge Source in a Domain with a Boundary

Returning to the electrostatic example discussed in section [6] in Cartesian coordinates, suppose a uniform charged straight wire is inserted in the direction of the \( y \) axis, between the grounded (\( \tilde{\phi} = 0 \)) planes at \( z = 0 \) and \( z = L \), at a position with \( z = z_0 \) (\( 0 < z_0 < L \)) and \( x = 0 \). Then with \( f = \{ x, z - z_0 \} \) one has a 0-form \( \theta \)-layer distributional source

\[
\bar{\rho}^D = \lambda \delta_{\Sigma^2}^f \tag{157}
\]
and
\[ \phi^D[\hat{\chi}] = \rho^D[\mathcal{G}\hat{\chi}] = \int_U \mathcal{P}_X \# \hat{\chi}_X \] \tag{158}
with
\[ \mathcal{P}_X = \lambda \int_{\Sigma_t^1} \mathcal{G}_{XY} \omega_t \] \tag{159}

Since \( \omega^Y_t = dy' \) one finds from (90)

\[ \mathcal{P}_X(x, y, z) = \frac{\lambda}{\pi L} \sum_n \sin \frac{n \pi z_0}{L} \sin \frac{n \pi z}{L} \int_{-\infty}^{\infty} \frac{dk_x}{k_x^2 + n^2 \pi^2 \frac{L}{L}} e^{ik_x x} \] \tag{160}

The \( k_x \) integration can be done by contour integration in the complex \( k_x \) plane. One has

\[ \begin{cases} \int_{-\infty}^{\infty} \frac{e^{isx} ds}{s^2 + b^2} = \frac{\pi}{b} e^{-bx} & x > 0 \\ \int_{-\infty}^{\infty} \frac{e^{isx} ds}{s^2 + b^2} = \frac{\pi}{b} e^{bx} & x < 0 \end{cases} \] \tag{161}

yielding

\[ \mathcal{P}_X(x, y, z) = \begin{cases} \frac{\lambda}{\pi} \sum_n \frac{1}{n} \sin \frac{n \pi z_0}{L} \sin \frac{n \pi z}{L} e^{-\frac{n \pi x}{L}} & x > 0 \\ \frac{\lambda}{\pi} \sum_n \frac{1}{n} \sin \frac{n \pi z_0}{L} \sin \frac{n \pi z}{L} e^{\frac{n \pi x}{L}} & x < 0 \end{cases} \] \tag{162}

This agrees with the computation in [13]

### 14 Time Dependent Electromagnetic fields and Smooth Sources

When the smooth electromagnetic sources \( \rho \) and \( \mathbf{J} \) depend on \( t \) they may generate time dependent electromagnetic fields that must satisfy (60), (61), (62), (63). These equations offer a well-posed initial-boundary value problem that is traditionally approached by finding time-dependent potentials \( \phi \) and \( \mathbf{A} \) that satisfy the source-driven wave-equations (69), (70) (subject to initial and boundary conditions) in the Lorentz gauge. When the sources are distributional one seeks distributional potentials subject to similar conditions.
The distributional formulation given above for real static field configurations generalizes without difficulty to the time dependent situation. The essential modification is to define functionals on the space of $t$ dependent complex test forms on $\mathbb{R}^3$ and exploit the Fourier transform $|13|$ of such forms. Thus if $\hat{\psi}$ is a $t$-dependent test $p$-form on $X = \mathbb{R}^3$ its Fourier transform is the $\omega$-dependent test $p$-form $\mathcal{F}\hat{\psi}$ on $X = \mathbb{R}^3$ defined by

$$ (\mathcal{F}\hat{\psi})(X, \omega) = \frac{1}{\sqrt{2\pi}} \int_R \hat{\psi}(X, t) \exp(i \omega t) \, dt $$

(163)

with inverse

$$ \hat{\psi}(X, t) = \frac{1}{\sqrt{2\pi}} \int_R (\mathcal{F}\hat{\psi})(X, \omega) \exp(-i \omega t) \, d\omega $$

(164)

The fundamental $t$-dependent $\langle p, p \rangle$ type double-form is the real part of the Fourier transform of $\bar{K}$

$$ K_{\omega}(X, Y) = \exp\left(-\frac{i}{c} |x - y| \right) \gamma_{\langle p, p \rangle}(X, Y). $$

(165)

It is useful, then, to define the singular complex-linear integral operator $\Gamma_{\omega}$ on all $p$-forms $\psi$ by:

$$ (\Gamma_{\omega}\psi)(Y, \omega) = \int_X K_{\omega}(X, Y) \wedge \# \psi(X, \omega) $$

(166)

Suppose that $\bar{J}$ is a smooth $t$-dependent $p$-form source (regular at spatial infinity) that enters into the equation

$$ \Delta_X C(X, t) + \frac{1}{c^2} \ddot{C}(X, t) = \bar{J}(X, t) $$

(167)

for the time-dependent $p$-form $C$ on $X = \mathbb{R}^3$. Then

$$ (\mathcal{F}C)(X, \omega) = (\Delta_X - \frac{\omega^2}{c^2})^{-1} (\mathcal{F}\bar{J})(X, \omega) $$

(168)

11Note that the phases in $|163|$ and $|165|$ are chosen so that for $0 < r < \infty$, $t > 0$

$$ \exp(\pm i \dot{\phi} r) \exp(\mp i \omega t) = \exp(\pm i \dot{\phi}(r - ct)) $$

describes a radially outgoing wave.
Since the singularity structure of $K_{\omega}(X,Y)$ is the same as that for $\gamma_{(p,p)}(X,Y)$ it is straightforward to show that a particular solution, regular at spatial infinity, is given by

$$(\mathcal{F}C)(X,\omega) = \int_{Y} K_{\omega}(X,Y) \wedge (\gamma_{(p,p)}(X,Y)) \quad (169)$$

For each $\omega$ the function $\mathcal{F}C$ may be associated with a functional by applying the operation $\int_{X} \# \hat{\alpha}(X,\omega)$ to define:

$$\mathcal{F}C[\hat{\alpha}](\omega) = \int_{X} (\mathcal{F}C)(X,\omega) \wedge \# \hat{\alpha}(X,\omega) \quad (170)$$

But from (169) this may be expressed

$$(\mathcal{F}C)[\hat{\alpha}](\omega) = \int_{Y} (\mathcal{F}C)(Y,\omega) \wedge \# (\Gamma_{\omega} \hat{\alpha})(Y,\omega) \equiv (\mathcal{F}C)^D[\Gamma_{\omega} \hat{\alpha}](\omega) \quad (171)$$

Thus from (165) and Fourier inversion

$$C(X,t) = \int_{Y} \gamma_{(p,p)}(X,Y) \wedge \# \tilde{\mathcal{J}}(Y, t - \frac{|x - y|}{c}) \quad (172)$$

and

$$C[\hat{\alpha}](t) = \frac{1}{\sqrt{2\pi}} \int_{R} dt' \int_{Y} \mathcal{J}(Y,t') \wedge \# \int_{X} \gamma_{(p,p)}(X,Y) \wedge \# (\mathcal{F} \hat{\alpha})(X, t' - \frac{|x - y|}{c}) \quad (173)$$

or

$$C[\hat{\alpha}](t) = \frac{1}{\sqrt{2\pi}} \int_{R} dt' \int_{Y} \tilde{\mathcal{J}}(Y, t - t' - \frac{|x - y|}{c}) \wedge \# \int_{X} \gamma_{(p,p)}(X,Y) \wedge \# (\mathcal{F} \hat{\alpha})(X, t') \quad (174)$$

Motivated by (172) the distribution $C^D$ associated with the smooth $p$–form $C(X,t)$ is defined by

$$C^D[\hat{\beta}] = \int_{R} dt \int_{X} C(X,t) \wedge \# \hat{\beta}(X,t) \quad (175)$$
Hence

$$C^D[\hat{\beta}] = \int_R dt \int_X \int_{Y (p,p)} \gamma (X, Y) \wedge \# \check{J} \left( Y, t - \frac{|x - y|}{c} \right) \wedge \# \check{\beta}(X, y)$$  \hspace{1cm} (176)

or with \( t \mapsto t' - t - \frac{|x - y|}{c} \) and interchange of \( X \) and \( Y \)

$$C^D[\hat{\beta}] = \int_R dt' \int_X (\Gamma_{\omega R} \hat{\beta})(X, t') \wedge \# \check{J}(X, t')$$  \hspace{1cm} (177)

where

$$(\Gamma_{R} \hat{\beta})(X, t') \equiv \int_Y \gamma (X, Y) \wedge \# \hat{\beta}(Y, t' + \frac{|x - y|}{c})$$

Thus finally

$$C^D[\hat{\beta}] = \int_R dt' \check{J}(X, t') \wedge \# (\Gamma_{R} \hat{\beta})(X, t') \equiv \check{J}^D[\Gamma_{R} \hat{\beta}]$$  \hspace{1cm} (178)

where \( \check{J}^D \) is the distribution associated with the smooth time-dependent \( p \)-form \( \check{J} \). If one now contemplates a distributional source \( \check{J}^D \) that is not associated with a smooth time-dependent \( p \)-form on \( \mathbb{R}^3 \) then (178) offers a particular distributional solution \( C^D \) to (167) based on a retarded extension of the fundamental solution \( \gamma (X, Y) \).

The above solution is immediately applicable to the problem of finding the retarded electromagnetic potentials \( A \) and \( \phi \) in free space, where \( c^2 \epsilon \mu = 1 \), for any smooth time dependent source. Thus from section (5) one sees that with a smooth source \( \check{J} = -\frac{\rho}{c} \) in equation (167) the smooth 0-form solution \( \phi \) associated with \( C^D \) describes a free-space scalar potential solution to (70) and that with a smooth source \( \check{J} = \mu \# J \) in equation (167) the smooth 1-form solution \( A \) associated with \( C^D \) describes a free-space vector potential solution to (69).

In the absence of losses due to conduction the convective current space-time 3-form is given as \( j = \rho_0 \star \tilde{V} \) where the unit future-pointing time-like vector field \( V \) convects proper-charge density \( \rho_0 \). In a local spacetime co-basis \( \{ cdt, e^1, e^2, e^3 \} \) adapted to \( U \) the source velocity \( \tilde{V} = \gamma ( -cdt + \frac{\vec{v}}{c} ) \) where \( \gamma^{-2} = 1 - \frac{|\vec{v}|^2}{c^2} \) and \( \vec{v} = \sum_{k=1}^3 v_k e^k \) is the instantaneous time-dependent (Newtonian) 3-velocity 1-form in \( \mathbb{R}^3 \). Since

$$j = -J \wedge dt + \rho \# 1 = \rho_0 \star \tilde{V}$$  \hspace{1cm} (179)
one finds immediately

\[ \#J = -\rho \tilde{V} \]

The analysis above also offers distributional solutions to (69) and (70) in terms of scalar and 1-form distributional sources. The electromagnetic field associated with an arbitrarily moving charge point source is now be modeled in terms of a moving Dirac distribution in space.

## 15 Time Dependent Electromagnetic fields and Distributional Sources: Moving Dirac Distributions

The above section formulates a distributional solution to (167) associated with a \( t \)-dependent \( p \)-form distributional source \( \mathcal{J}^D \) on \( \mathbb{R}^3 \). 0-layer singular distributions associated with electromagnetic sources constrained to moving curves or surfaces in space can be constructed in terms of a family of distributions \( \delta_{\Sigma^3-k \times R}(0) \) defined for \( k = 1, 2 \) by

\[
\delta_{\Sigma^3-k \times R}(0) \left[ \hat{\beta} \right] = \int_R \delta_{\Sigma^3-k}(0) \left[ \hat{\beta} \right] \, dt
\]  

Then

\[
\tilde{W} \delta_{\Sigma^3-k \times R}(0) \left[ \hat{\beta} \right] = \int_R \tilde{W}(t) \delta_{\Sigma^3-k}(0) \left[ \hat{\beta} \right] \, dt
\]  

for some \( t \)-dependent 1-form \( \tilde{W} \) in \( \mathbb{R}^3 \) with support on \( \Sigma^3-k \).

For the case \( k = 3 \) the support of \( \delta_{\Sigma^3-k}(0) \) is a moving point \( p_0(t) \) in \( X = \mathbb{R}^3 \).

Then the worldline history of the moving point is encoded into the support of a distribution \( \delta_{p_0 \times R}(0) \) where

\[
\delta_{p_0 \times R}(0) \left[ \hat{\beta} \right] = \int_R \hat{\beta}(p_0(t), t) \, dt
\]  

If the history of \( p_0 \in X \) is parameterized by \( t \) it is the spacetime curve

\[
t' \mapsto (x, t) = (x_0(t'), t')
\]  

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Then
\[
\begin{aligned}
\delta_{p_0 \times R}^{(0)}[\hat{\beta}] &= \int_R \hat{\beta}(x_0(t'), t') \, dt' \\
&= \int_R \int_R \gamma(x_0(t'), Y) \# \hat{\beta}(Y, t + \frac{|x_0(t') - y|}{c}) \\
&= \int_R dt' \int_Y \gamma(x_0(t'), Y) \# \hat{\beta}(Y, t + \frac{|x_0(t') - y|}{c})
\end{aligned}
\] (184)

and
\[
\begin{aligned}
\tilde{W} \delta_{p_0 \times R}^{(0)}[\hat{\beta}] &= \int_R (i_{W(t')}(t'))(x_0(t'), t') \, dt' \\
&= \int_R dt' \int_Y \gamma(x_0(t'), Y) \# \hat{\beta}(Y, t + \frac{|x_0(t') - y|}{c})
\end{aligned}
\] (185)

One may now easily calculate
\[
\begin{aligned}
\delta_{p_0 \times R}^{(0)}[\Gamma_R \hat{\beta}] &= \int_R (\Gamma_R \hat{\beta})(x_0(t'), t') \, dt' \\
&= \int_R dt' \int_Y \gamma(x_0(t'), Y) \# \hat{\beta}(Y, t + \frac{|x_0(t') - y|}{c})
\end{aligned}
\] (186)

Changing variable \( t' \mapsto t = t' + \frac{|x_0(t') - y|}{c} \) at fixed \( Y \) with Jacobian \( Q \) defined by
\[
dt = Q^{-1}(Y, t') \, dt'
\]
yields
\[
\begin{aligned}
\delta_{p_0 \times R}^{(0)}[\Gamma_R \hat{\beta}] &= \int_R dt \int_Y \left( Q(Y, t') \gamma(x_0(t'), Y) \right) \# \hat{\beta}(Y, t) \\
&= \int_R dt \int_Y \gamma(x_0(t'), Y) \# \hat{\beta}(Y, t + \frac{|x_0(t') - y|}{c})
\end{aligned}
\] (187)

where \( t' \equiv \hat{t}(Y, t) \) solves the equation \( t' = t - \frac{|x_0(t') - y|}{c} \). Thus
\[
Z(Y, t) = Q(Y, t') \gamma(x_0(t'), Y)
\] (188)

is the 0–form associated with the distributional source \( \delta_{p_0 \times R}^{(0)}[\Gamma_R \hat{\beta}] \) at the field point \( Y \) at the instant \( t \). It is straightforward to calculate the Jacobian in terms of \( x_0(t) \) and its derivative \( \dot{x}_0(t) \):
\[
Q^{-1}(Y, t) = \left( 1 + \frac{1}{c} \frac{(x_0(t') - y) \cdot (\dot{x}_0(t'))}{|x_0(t') - y|} \right)
\] (189)

The instantaneous Newtonian 3–velocity of the point support is \( \dot{v}(t) \equiv \dot{x}_0(t) \) and if one introduces the Euclidean unit vector
\[
n(Y, t) = \frac{y - x_0(t')}{|y - x_0(t')|}
\]
connecting the field point \( y \) at time \( t \) to the source point \( x_0(t') \) at the earlier time \( t' = \hat{t}'(Y, t) \), then the Jacobian inverse takes the form

\[
Q^{-1}(Y, t) = \left( 1 - g \left( \frac{v(t')}{c}, n(Y, t) \right) \right)
\]  
(190)

where \( t' \equiv \hat{t}'(Y, t) \) solves the equation \( t' = t - \frac{|x_0(t') - y|}{c} \) and \( g \) denotes the Euclidean metric tensor on \( \mathbb{R}^3 \).

By contrast to the smooth electromagnetic sources \( \tilde{J} \) discussed in section (14) one models a moving point source (with electric charge \( q \) and 3–velocity \( v(t) \)) for the scalar potential \( \phi^D \) by

\[
\tilde{J}^D(0) = -\frac{q}{\epsilon} \delta_{p_0 \times R}^{(0)}
\]  
(191)

and for the potential 1–form \( A^D \) by

\[
\tilde{J}^D(1) = -q \mu \tilde{v}(t) \delta_{p_0 \times R}^{(0)}
\]  
(192)

It then follows immediately from (178) that the distribution \( \phi^D \) may be associated with

\[
P(Y, t) = -\frac{q}{\epsilon} Q(Y, t) \gamma_{(0,0)}(x(t'), Y)
\]  
(193)

while the distribution \( A^D \) may be associated with

\[
A(Y, t) = -q \mu Q(Y, t) i_{v(t')} \gamma_{(1,1)}(x(t'), Y)
\]  
(194)

where \( t' = \hat{t}'(Y, t) \) as above. These are the classic Lienard-Weichert potentials for a moving point charge [2]. It is of interest to verify from \#J^D = -\tilde{v}p^D that the distributions \( A^D \) and \( \phi^D \) satisfy the gauge condition \( \delta (A^D + \epsilon \mu \phi^D) = 0 \). It may also be noted that by using the appropriate fundamental double-forms in place of \( \gamma_{(0,0)} \) and \( \gamma_{(1,1)} \) these potentials maintain their structure for solutions in \( D \subset Y \) that satisfy appropriate boundary conditions on \( \partial D \).
16 Conclusions

With the aid of properties of the fundamental double-form of bi-degree \((p, p)\) associated with the Hodge-de Rham operator \(\Delta\) on differential forms, a distributional framework for analysing equations of the form

\[
\Delta \Phi^{(p)} + \lambda^2 \Phi^{(p)} = S^{(p)}
\]

on \(\mathbb{R}^3\) has been established. A set of \(r\)-layer Dirac singular distributions with supports on (moving) embeddings in \(\mathbb{R}^3\) has been constructed and finds application in electromagnetic source modeling. The framework has been illustrated by explicitly calculating the fields associated with a current carrying circular and helical coil, a finite length solenoid with a helical surface current density, a uniformly charged wire between two conducting plates and an arbitrarily moving point charge in free space. With the aid of \(r\)-layer Dirac singular distributions many fields with more complex distributional sources can be readily reduced to quadratures once one parametrises such sources in terms of the geometry of their support in space.

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