Temporal aspects of one-dimensional completed scattering: An alternative view

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(Dated: November 10, 2018)

A completed scattering of a particle on a static one-dimensional (1D) potential barrier is a combined quantum process to consist from two elementary sub-processes (transmission and reflection) evolved coherently at all stages of scattering and macroscopically distinct at the final stage. The existing model of the process is clearly inadequate to its nature: all one-particle "observables" and "tunneling times", introduced as quantities to be common for the sub-processes, cannot be experimentally measured and, consequently, have no physical meaning; on the contrary, quantities introduced for either sub-process have no basis, for the time evolution of either sub-process is unknown in this model. We show that the wave function to describe a completed scattering can be uniquely presented as the sum of two solutions to the Schrödinger equation, which describe separately the sub-processes at all stages of scattering. For symmetric potential barriers such solutions are found explicitly. For either sub-process we define the time spent, on the average, by a particle in the barrier region. We define it as the Larmor time. As it turned out, this time is just Buttiker’s dwell time averaged over the corresponding localized state. Thus, firstly, we justify the known definition of the local dwell time introduced by Hauge and co-workers as well by Leavens and Aers, for now this time can be measured; secondly, we confirm that namely Buttiker’s dwell time gives the energy-distribution for the tunneling time; thirdly, we state that all the definitions are valid only if they are based on the wave functions for transmission and reflection found in our paper. Besides, we define the exact and asymptotic group times to be auxiliary in timing the scattering process.

I. INTRODUCTION

For a long time scattering a particle on one-dimensional (1D) static potential barriers have been considered in quantum mechanics as a representative of well-understood phenomena. However, solving the so-called tunneling time problem (TTP) (see reviews \[1, 2, 3, 4, 5, 6, 7\] and references therein) showed that this is not the case.

At present there is a variety of approaches to introduce characteristic times for a 1D scattering. They are the group (Wigner) tunneling times (more known as the "phase" tunneling times) \[8, 9, 10, 11\], different variants of the dwell time \[10, 11, 12, 13, 14, 15, 16, 17, 18\], the Larmor time \[17, 21, 22, 23, 24, 25, 26\], and the concept of the time of arrival which is based on introducing either a suitable time operator (see, e.g., \[27, 28, 29, 30, 31\]) or the positive operator valued measure (see review \[8\]). A particular class of approaches to study the temporal aspects of a 1D scattering includes the Bohmian \[32, 33, 34, 35, 36\], Feynman and Wigner ones (see \[37, 38, 39, 40, 41\] as well as \[2, 5\] and references therein). One has also point out the papers \[42, 43, 44\] to study the characteristic times of "the forerunner preceding the main tunneling signal of the wave created by a source with a sharp onset".

As is known (see \[1\]), the main question of the TTP is that of the time spent, on the average, by a particle in the barrier region in the case of a completed scattering.

Setting this problem implies that the particle’s source and detectors are located at a considerable distance from the potential barrier. The answer to this question, for a given potential and initial state of a particle, is evident now this time can be measured; secondly, we confirm that namely Buttiker’s dwell time gives the energy-distribution for the tunneling time; thirdly, we state that all the definitions are valid only if they are based on the wave functions for transmission and reflection found in our paper. Besides, we define the exact and asymptotic group times to be auxiliary in timing the scattering process.

One has to recognize that the answer has not yet been found, and this elastic scattering process looks at present like an unexplained phenomenon surrounded by paradoxes. We bear in mind, in particular, 1) the lack of a causal relationship between the transmitted and incident wave packets \[45\]; 2) a superluminal propagation of a particle through opaque potential barriers (the Hartman effect) \[46, 47, 48, 49\]; 3) accelerating (on the average) a transmitted particle, in the asymptotic region, as compared with an incident one \[45\]; 4) aligning the average particle’s spin with the magnetic field \[50\]; 5) the Larmor precession of the reflected particles under the non-zero magnetic field localized beyond the barrier on the side of transmission \[23\].

At the first glance the Bohmian mechanics provides an adequate description of the temporal aspects of a completed scattering (see, e.g., \[32, 33, 34, 35, 36\]). For its "causal" one-particle trajectories exclude, a priory, the appearance of the above paradoxes. For example, the Hartman effect does not appear in this approach: in the case of opaque rectangular barriers, the Bohmian dwell time, unlike Smith’s and Buttiker’s dwell times, increases exponentially together with the barrier’s width (see also Section \[IVD\]).

It should be stressed however that the Bohmian model of a 1D completed scattering is not free of paradoxes. As is well known, the region of location of the particle’s
source consists in this model from two parts separated by some critical point. This point is such that all particles starting from the sub-region, adjacent to the barrier region, are transmitted by the barrier; otherwise they are reflected by it. That is, the subensembles of transmitted and reflected particles are macroscopically distinct in this model at all stages of scattering, what clearly contradicts the main principles of quantum mechanics.

Note, the position of the critical point depends on the barrier’s shape. For a particle impinging the barrier from the left, this point approaches the left boundary of the barrier when the latter becomes less transparent. Otherwise, the critical point approaches minus infinity on the OX-axis. This property means, in fact, that particles feel the barrier’s shape, being however far from the barrier region. Of course, this fact evidences, too, that the existing "causal" trajectories of the Bohmian mechanics give an improper description of the scattering process.

From our viewpoint, all difficulties and paradoxes to arise in studying the temporal aspects of a completed scattering result from the fact that setting this problem in the existing framework of quantum mechanics is contradictory. On the one hand, in the case of a completed scattering an observer deals only either with transmitted or reflected particles, and, consequently, all one-particle observables must be introduced individually for each sub-process. On the other hand, quantum mechanics, as it stands, does not imply the introduction of observables for the sub-processes, for its formalism does not provide the wave functions for transmission and reflection, needed for computing the expectation values of observables.

So, in the case of a completed scattering, a conflicting situation arises already at the stage of setting the problem: the nature of this process requires a separate description of transmission and reflection; while quantum mechanics, as it stands, does not allow such a description. This conflict underlies all controversy and paradoxes to arise in solving the TTP: in fact, in the existing framework of quantum theory, there are no observables which can be consistently introduced for this process.

Note, this concerns not only characteristic times but also all observables to have Hermitian operators. For example, averaging the particle’s position and momentum over the whole ensemble of particles does not give the expectation (i.e., most probable) values of these quantities. As regards characteristic times, we have to stress once more that among the existing time concepts neither separate nor common times for transmission and reflection give the time spent by a particle in the barrier region. In the first case, there is no basis to distinguish (theoretically and experimentally) transmitted and reflected particles in the barrier region. In the second case, characteristic times introduced cannot be properly interpreted (see, e.g., discussion of the dwell and Larmor times in [10]); these times describe neither transmitted nor reflected particles (ideal transmission and reflection are exceptional cases).

At the same time there is a viewpoint that all the time scales introduced for a completed scattering are valid: one has only to choose a suitable clock (operational procedure) for each of them. This viewpoint is based on the assumption that timing a quantum particle, without influencing the scattering process, is impossible in principle. By this viewpoint the time measured should always depend on the clock used for this purpose.

However, quantum phenomena, such as a completed scattering, have their own, intrinsic spatial and temporal scales, and our main task is to learn to measure these scales without influencing their values. In this paper we show that for the problem under consideration this is possible. The above conflict can be resolved in the framework of conventional quantum mechanics, and characteristic times for transmission and reflection can be introduced. For measuring these time scales without affecting the scattering process, one can exploit the Larmor precession of the particle’s spin under the infinitesimal magnetic field.

The plan of this paper is as follows. In (Section II) we introduce the concept of combined and elementary quantum processes and states. By this concept, the state of the whole quantum ensemble of particles, at the problem at hand, is a combined one to represent a coherent superposition of two (elementary) states of the (to-be-)transmitted and (to-be-)reflected subensembles of particles. In Section III we present two solutions to the Schrödinger equation to describe transmission and reflection at all stages of scattering. On their basis we define the group, dwell and Larmor times for transmission and reflection (Section IV).

II. THE SCHÖDINGER’S CAT PARADOX AND 1D COMPLETED SCATTERING: THE CONCEPT OF COMBINED AND ELEMENTARY STATES.

For our purposes it is relevant to address the well-known Schrödinger’s cat paradox which displays explicitly a principal difference between macroscopically distinct quantum states and their superpositions.

As is known, macroscopically distinct quantum states are symbolized in this paradox by the ‘dead-cat’ and ‘alive-cat’ ones. Either may be associated with a single, really existing cat which can be described in terms of one-cat observables. As regards a superposition of these two states, it cannot be associated with a cat to exist really (a cat cannot be dead and alive simultaneously). To calculate the expectation values of one-cat observables for this state is evident to have no physical sense.

As is known, quantum mechanics as it stands does not distinguish between the ‘dead-cat’ and ‘alive-cat’ states and their superposition. It postulates that all its rules should be equally applied to macroscopically distinct states and their superpositions. From our point of view, the main lesson of the Schrödinger’s cat paradox is just that this postulate is erroneous. Quantum me-
III. WAVE FUNCTIONS FOR TRANSMISSION AND REFLECTION

A. Setting the problem for a 1D completed scattering

Let us consider a particle incident from the left on the static potential barrier \( V(x) \) confined to the finite spatial interval \([a, b]\) (\( a > 0 \); \( d = b - a \) is the barrier width). Let its in-state, \( \psi_{\text{in}}(x) \), at \( t = 0 \) be a normalized function to belong to the set \( S_\infty \) consisting from infinitely differentiable functions vanishing exponentially in the limit \( |x| \to \infty \). The Fourier-transform of such functions are known to belong to the set \( S_\infty \), too. In this case the position, \( \hat{x} \), and momentum, \( \hat{p} \), operators both are well-defined. Without loss of generality we will suppose that

\[
<\psi_{\text{in}}|\hat{x} |\psi_{\text{in}}>=0, \quad <\psi_{\text{in}}|\hat{p} |\psi_{\text{in}}>=\hbar k_0 >0,
\]

\[
<\psi_{\text{in}}|\hat{x}^2 |\psi_{\text{in}}>=l_0^2; \quad (1)
\]

here \( l_0 \) is the wave-packet’s half-width at \( t = 0 \) (\( l_0 << a \)).

We consider a completed scattering. This means that the average velocity, \( \hbar k_0/m \), is large enough, so that the transmitted and reflected wave packets do not overlap each other at late times. As for the rest, the relation of the average energy of a particle to the barrier’s height may be any by value.

We begin our analysis with the derivation of expressions for the incident, transmitted and reflected wave packets to describe, in the problem at hand, the whole ensemble of particles. For this purpose we will use the variant (see [51]) of the well-known transfer matrix method [52]. Let the wave function \( \psi_{\text{full}}(x; k) \) to describe the stationary state of a particle in the out-of-barrier regions be written in the form

\[
\psi_{\text{full}}(x; k) = e^{ikx} + b_{\text{out}}(k)e^{ik(2a-x)}, \quad \text{for} \quad x \leq a; \quad (2)
\]

\[
\psi_{\text{full}}(x; k) = a_{\text{out}}(k)e^{ik(x-a)}, \quad \text{for} \quad x > b; \quad (3)
\]

here \( k = \sqrt{2mE}/\hbar \); \( E \) is the energy of a particle; \( m \) is its mass.

The coefficients entering this solution are connected by the transfer matrix \( Y \):

\[
\begin{pmatrix}
1 \\
b_{\text{out}}e^{2ika}
\end{pmatrix} = Y
\begin{pmatrix}
a_{\text{out}}e^{-ikd} \\
0
\end{pmatrix}, \quad Y = \begin{pmatrix} q & p \\ p^* & q^* \end{pmatrix}; \quad (4)
\]

\[
q = \frac{1}{\sqrt{T(k)}} \exp \left[ i\left( kd - J(k) \right) \right],
\]

\[
p = \sqrt{\frac{R(k)}{T(k)}} \exp \left[ i \left( \frac{\pi}{2} + F(k) - ks \right) \right] \quad (5)
\]

where \( T(k) \) (the transmission coefficient) and \( J(k) \) (phase) are even and odd functions of \( k \), respectively; \( F(-k) = \pi -
$F(k); R(k) = 1 - T(k); s = a + b$. We will suppose that the tunneling parameters have already been calculated.

In the case of many-barrier structures, for this purpose one may use the recurrence relations obtained in [51] just for these real parameters. For the rectangular barrier of height $V_0$,

$$T = \left[1 + \vartheta_{(+)}^2 \sinh^2(\kappa d)\right]^{-1},$$
$$J = \arctan(\vartheta_{(-)} \tanh(\kappa d)),$$
$$F = 0, \quad \kappa = \sqrt{2m(V_0 - E)/\hbar},$$

if $E < V_0$; and

$$T = \left[1 + \vartheta_{(-)}^2 \sin^2(\kappa d)\right]^{-1},$$
$$J = \arctan(\vartheta_{(+)} \tan(\kappa d)),$$
$$F = \begin{cases} 0, & \text{if } \vartheta_{(-)} \sin(\kappa d) \geq 0 \\
\pi, & \text{otherwise}, \end{cases} \quad \kappa = \sqrt{2m(E - V_0)/\hbar},$$

if $E \geq V_0$; in both cases $\vartheta_{(+)} = \sqrt{\frac{\kappa}{2} \left(\frac{\kappa}{2} + \frac{\kappa}{2}\right)}$ (see [51]).

Now, taking into account Eqs. (4) and (5), we can write in-asymptote, $\psi \text{in}(x, t)$, and out-asymptote, $\psi \text{out}(x, t)$, for the time-dependent scattering problem (see [54]):

$$\psi \text{in}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\text{in}(k, t) e^{ikx} dk;$$
$$f_\text{in}(k, t) = A_\text{in}(k) \exp[-iE(k)t/\hbar];$$

(8)

$$\psi \text{out}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_\text{out}(k, t) e^{ikx} dk;$$
$$f_\text{out}(k, t) = f_\text{tr}(k, t) + f_\text{ref}(k, t);$$

(9)

$$f_\text{tr}(k, t) = \sqrt{T(k)} A_\text{in}(k) \exp[i(J(k) - kd - E(k)t/\hbar)];$$

(10)

$$f_\text{ref}(k, t) = \sqrt{R(k)} A_\text{in}(k) \exp[-i(J(k) - F(k) - \pi/2 + 2ka + E(k)t/\hbar)];$$

(11)

where Eqs. (8), (10) and (11) describe, respectively, the incident, transmitted and reflected wave packets. Here $A_\text{in}(k)$ is the Fourier-transform of $\psi \text{in}(x)$. For example, for the Gaussian wave packet to obey condition $\text{U}$, $A_\text{in}(k) = c \cdot \exp(-l_0^2(k - k_0)^2)$; $c$ is a normalization constant.

B. Incoming waves for transmission and reflection

Let us now show that by the final states (9)-(11) one can uniquely reconstruct the prehistory of the subensembles of transmitted and reflected particles at all stages of scattering. Let $\psi \text{tr}$ and $\psi \text{ref}$ be searched-for wave functions for transmission (TWF) and reflection (RWF), respectively. By our approach their sum should give the (full) wave function $\psi \text{full}(x, t)$ to describe the whole combined scattering process. From the mathematical point of view our task is to find, for a particle impinging the barrier from the left, such two solutions $\psi \text{tr}$ and $\psi \text{ref}$ to the Schrödinger equation that, for any $t$,

$$\psi \text{full}(x, t) = \psi \text{tr}(x, t) + \psi \text{ref}(x, t);$$

(12)

in the limit $t \to \infty$,

$$\psi \text{tr}(x, t) = \psi \text{out}(x, t), \quad \psi \text{ref}(x, t) = \psi \text{ref}(x, t);$$

(13)

where $\psi \text{out}(x, t)$ and $\psi \text{ref}(x, t)$ are the transmitted and reflected wave packets whose Fourier-transforms presented in (10) and (11).

We begin with searching for the stationary wave functions for reflection, $\psi \text{ref}(x, k)$, and transmission, $\psi \text{tr}(x, k)$. Let for $x \leq a$

$$\psi \text{ref}(x, k) = A_\text{in} \text{ref} e^{i\lambda x} + b_\text{out} e^{i(k(2a - x))},$$
$$\psi \text{tr}(x, k) = A_\text{in} \text{tr} e^{i\lambda x};$$

(14)

where $A_\text{in} \text{ref} + A_\text{in} \text{tr} = 1$.

Since the RWF describes only reflected particles, which are expected to be absent behind the barrier, the probability flux for $\psi \text{ref}(x, k)$ should be equal to zero -

$$|A_\text{in} \text{ref}|^2 - |b_\text{out}|^2 = 0.$$

(15)

In its turn, the probability flux for $\psi \text{full}(x, k)$ and $\psi \text{tr}(x, k)$ should be the same -

$$|A_\text{in} \text{tr}|^2 = T(k).$$

(16)

Then, taking into account that $\psi \text{tr} = \psi \text{full} - \psi \text{ref}$, we can exclude $\psi \text{tr}$ from Eq. (16). As a result, we obtain

$$\Re \left(A_\text{in} \text{ref}\right) - |b_\text{out}|^2 = 0.$$

(17)

Since $|b_\text{out}|^2 = R$, from Eqs. (15) and (17) it follows that $A_\text{in} \text{ref} = \sqrt{R} (\sqrt{R} \pm i\sqrt{T}) = \sqrt{R} \exp(i\lambda); \lambda = \pm \arctan(\sqrt{T/R}).$

So, a coherent superposition of the incoming waves to describe transmission and reflection, for a given $E$, yields the incoming wave of unite amplitude, that describes the whole ensemble of incident particles. In this case, not only $A_\text{in} \text{in} + A_\text{in} \text{ref} = 1$, but also $|A_\text{in} \text{in}|^2 + |A_\text{in} \text{ref}|^2 = 1!$ Besides, the phase difference for the incoming waves to describe reflection and transmission equals $\pi/2$ irrespective of the value of $E$.

Our next step is to show that only one root of $\lambda$ gives a searched-for $\psi \text{ref}(x, k)$. For this purpose the above solution should be extended into the region $x > a$. To do this, we will restrict ourselves by symmetric potential barriers, though the above derivation is valid for all barriers.
C. Wave functions for transmission and reflection in the case of symmetric potential barriers

Let \( V(x) \) be such that \( V(x-x_c) = V(x_c-x); x_c = (a+b)/2 \). As is known, for the region of a symmetric potential barrier, one can always find odd, \( u(x-x_c) \), and even, \( v(x-x_c) \), solutions to the Schrödinger equation. We will suppose here that these functions are known. For example, for the rectangular potential barrier (see Exps. (6) and (7)),

\[
u(x) = \sinh(x), \quad v(x) = \cosh(x), \quad \text{if} \quad E \leq V_0;
\]

\[
u(x) = \sin(x), \quad v(x) = \cos(x), \quad \text{if} \quad E \geq V_0.
\]

Note, \( \frac{du}{dx} - \frac{dv}{dx} \) is a constant, which equals \( \kappa \) in the case of the rectangular barrier. Without loss of generality we will keep this notation for any symmetric potential barrier.

Before finding \( \psi_{\text{ref}}(x; k) \) and \( \psi_{\text{tr}}(x; k) \) in the barrier region, we have firstly to derive expressions for the tunneling parameters of symmetric barriers. Let in the barrier region \( \psi_{\text{full}}(x; k) = a_{\text{full}} u(x-x_c, k) + b_{\text{full}} v(x-x_c, k) \). “Sewing” this expression together with Exps. (6) and (7) at the points \( x = a \) and \( x = b \), we obtain

\[
a_{\text{full}} = \frac{1}{\kappa} (P + P^* b_{\text{out}}) e^{ika} = \frac{1}{\kappa} P^* a_{\text{out}} e^{ika},
b_{\text{full}} = \frac{1}{\kappa} (Q + Q^* b_{\text{out}}) e^{ika} = \frac{1}{\kappa} Q^* a_{\text{out}} e^{ika},
\]

\[
Q = \left. \left( \frac{du(x-x_c)}{dx} + iku(x-x_c) \right) \right|_{x=b},
P = \left. \left( \frac{dv(x-x_c)}{dx} + ikv(x-x_c) \right) \right|_{x=b}.
\]

As a result,

\[
a_{\text{out}} = \frac{1}{2} \left( \frac{Q}{Q^*} - \frac{P}{P^*} \right); \quad b_{\text{out}} = -\frac{1}{2} \left( \frac{Q}{Q^*} + \frac{P}{P^*} \right).
\]

As it follows from (11), \( a_{\text{out}} = \sqrt{T} \exp(iJ) \), \( b_{\text{out}} = \sqrt{R} \exp(i (J - F - \frac{\pi}{2})) \). Hence \( T = |a_{\text{out}}|^2, R = |b_{\text{out}}|^2 \), \( J = \arg(a_{\text{out}}) \). Besides, for symmetric potential barriers \( F = 0 \) when \( \Re(QP^*) > 0 \); otherwise, \( F = \pi \).

Then, one can show that “sewing” the general solution \( \psi_{\text{ref}}(x; k) \) in the barrier region together with Exp. (14) at \( x = a \), for both the roots of \( \lambda \), gives odd and even functions in this region. For the problem considered, only the former has a physical meaning. The corresponding roots for \( A_{\text{in}}^{\text{ref}} \) and \( A_{\text{in}}^{\text{tr}} \) read as

\[
A_{\text{in}}^{\text{ref}} = b_{\text{out}} (b_{\text{out}}^* - a_{\text{out}}^*); \quad A_{\text{in}}^{\text{tr}} = a_{\text{out}}^* (a_{\text{out}} + b_{\text{out}})
\]

One can easily show that in this case

\[
Q^* = \frac{A_{\text{in}}^{\text{ref}}}{b_{\text{out}}} = \frac{A_{\text{in}}^{\text{tr}}}{a_{\text{out}}},
\]

for \( a \leq x \leq b \)

\[
\psi_{\text{ref}} = \frac{1}{\kappa} \left( PA_{\text{in}}^{\text{ref}} + P^* b_{\text{out}} \right) e^{ika} u(x-x_c).
\]

The extension of this solution onto the region \( x \geq b \) gives

\[
\psi_{\text{ref}} = -b_{\text{out}} e^{ik(x-d)} - A_{\text{in}}^{\text{ref}} e^{-ik(x-s)}.
\]

Let us now show that the searched for RWF is, in reality, zero to the right of the barrier’s midpoint. Indeed, as is seen from Exp. (21), \( \psi_{\text{ref}}(x; k) = 0 \) for all values of \( k \). In this case the probability flux, for any time-dependent wave function formed from \( \psi_{\text{ref}}(x; k) \), is equal to zero at the barrier’s midpoint for any value of time. This means that reflected particles impinging the symmetric barrier from the left do not enter the region \( x \geq x_c \). Thus, \( \psi_{\text{ref}}(x; k) \equiv 0 \) for \( x \geq x_c \). In the region \( x \leq x_c \) it is described by Exps. (14) and (21). For this solution, the probability density is everywhere continuous and the probability flux is everywhere equal to zero.

As regards the searched-for TWF, one can easily show that

\[
\psi_{\text{tr}} = a_{\text{tr}}^l u(x-x_c) + b_{\text{tr}} v(x-x_c) \text{ for } a \leq x \leq x_c; (22)
\]

\[
\psi_{\text{tr}} = a_{\text{tr}}^b u(x-x_c) + b_{\text{tr}} v(x-x_c) \text{ for } x_c \leq x \leq b; (23)
\]

\[
\psi_{\text{tr}} = a_{\text{tr}} e^{ik(x-d)} \text{ for } x \geq b,
\]

where

\[
a_{\text{tr}}^l = \frac{1}{\kappa} PA_{\text{in}}^{\text{tr}} e^{ika}, \quad b_{\text{tr}} = b_{\text{full}} = \frac{1}{\kappa} Q^* a_{\text{out}} e^{ika},
\]

\[
a_{\text{tr}}^r = a_{\text{full}} = \frac{1}{\kappa} P^* a_{\text{out}} e^{ika}
\]

Like \( \psi_{\text{ref}}(x; k) \), the TWF is everywhere continuous and the corresponding probability flux is everywhere constant (we have to stress once more that this flux has no discontinuity at the point \( x = x_c \), though the first derivative of \( \psi_{\text{tr}}(x; k) \) on \( x \) is discontinuous at this point). As in the case of the RWF, wave packets formed from \( \psi_{\text{tr}}(x; k) \) should evolve in time with a constant norm.

So, for any value of \( t \)

\[
T = < \psi_{\text{tr}}(x, t)|\psi_{\text{tr}}(x, t) > = \text{const};
R = < \psi_{\text{ref}}(x, t)|\psi_{\text{ref}}(x, t) > = \text{const};
\]

\( T \) and \( R \) are the average transmission and reflection coefficients, respectively. Besides,

\[
< \psi_{\text{full}}(x, t)|\psi_{\text{full}}(x, t) > = T + R = 1.
\]

From this it follows, in particular, that the scalar product of the wave functions for transmission and reflection, \( < \psi_{\text{tr}}(x, t)|\psi_{\text{ref}}(x, t) > \), is a purely imaginary quantity to approach zero when \( t \to \infty \).
IV. CHARACTERISTIC TIMES FOR TRANSMISSION AND REFLECTION

Now we are ready to proceed to the study of temporal aspects of a 1D completed scattering. The wave functions for transmission and reflection presented in the previous section permit us to introduce characteristic times for either sub-process. Our main aim is to find, for each sub-process, the time spent, on the average, by a particle in the barrier region. In doing so, we have to bear in mind that there may be different approximations of this quantity. However, we have to remind that its true value must not depend, for a completed scattering, on the choice of ”clocks”.

Measuring the tunneling time, under such conditions, implies that a particle has its own, internal ”clock” to remember the time spent by the particle in the spatial region investigated. This means that the only way to measure the tunneling time for a completed scattering is to exploit the internal degrees of freedom of quantum particles. As is known, namely this idea underlies the Larmor-time concept based on the Larmor precession of the particle’s spin under the infinitesimal magnetic field.

In the above context, the concepts of the group and dwell times are rather auxiliary ones, since they cannot be verified. Nevertheless, they may be useful for a better understanding of the scattering process.

A. Group times for transmission and reflection

We begin our analysis from the group time concept to give the time spent by the wave-packet’s CM in the spatial regions. In other words, both for transmitted and reflected particles, we begin with timing ”mean-statistical particles” of these subensembles (their motion is described by the Ehrenfest equations). In doing so, we will distinguish exact and asymptotic group times.

1. Exact group times

Let \( t_{1}^{tr} \) and \( t_{2}^{tr} \) be such moments of time that

\[
\frac{1}{T} \langle \psi_{tr}(x, t_{1}^{tr}) | \hat{x} | \psi_{tr}(x, t_{1}^{tr}) \rangle = a; \tag{26}
\]

\[
\frac{1}{T} \langle \psi_{tr}(x, t_{2}^{tr}) | \hat{x} | \psi_{tr}(x, t_{2}^{tr}) \rangle = b. \tag{27}
\]

Then, one can define the transmission time \( \Delta t_{tr}(a, b) \) as the difference \( t_{2}^{tr} - t_{1}^{tr} \) where \( t_{1}^{tr} \) is the smallest root of Eq. (26), and \( t_{2}^{tr} \) is the largest root of Eq. (27).

Similarly, for reflection, let \( t_{(+)} \) and \( t_{(-)} \) be such values of \( t \) that

\[
\frac{1}{R} \langle \psi_{ref}(x, t_{+}) | \hat{x} | \psi_{ref}(x, t_{+}) \rangle = a, \tag{28}
\]

Then the exact group time for reflection, \( \Delta t_{ref}(a, b) \), is

\[
\Delta t_{ref}(a, b) = t_{(+)} - t_{(-)}. \]

Of course, a serious shortcoming of the exact characteristic times is that they fit only for sufficiently narrow (in \( x \)-space) wave packets. For wide packets these times give a very rough estimation of the time spent by a particle in the barrier region. For example, one may a priori say that the exact group time for reflection, for a sufficiently narrow potential barrier and/or wide wave packet, should be equal to zero. In this case, the wave-packet’s CM does not enter the barrier region.

2. Asymptotic group times for transmission and reflection

Note, the potential barrier influences a particle not only when its most probable position is in the barrier region. For a completed scattering it is useful also to introduce asymptotic group times to describe the passage of the particle in the sufficiently large spatial interval \([a - L_{1}, b + L_{2}]; \) where \( L_{1}, L_{2} \gg t_{0}\).

It is evident that in this case, instead of the exact wave functions for transmission and reflection, we may use the corresponding in- and out-asymptotes derived in \( k \)-representation. The ”full” in-asymptote, like the corresponding out-asymptote, represents the sum of two wave packets:

\[
f_{in}(k, t) = f_{tr}^{in}(k, t) + f_{ref}^{in}(k, t); \tag{29}
\]

\[
f_{in}^{tr}(k, t) = \sqrt{T} A_{in} \exp[i(\lambda - \frac{\pi}{2} - E(k)t/\hbar)]; \tag{30}
\]

\[
\lambda = \arg(A_{in}^{ref}) \ (\text{see} \ [19]). \quad \text{One can easily show that} \quad |\lambda'(k)| = \frac{e^{-\gamma k}}{2\pi i}; \quad \text{hereinafter, the prime denotes the derivative with respect to} \ k.
\]

For the average wave numbers in the asymptotic spatial regions we have

\[
<k>_{in}^{tr} = <k>_{out}^{tr}, \quad <k>_{in}^{ref} = - <k>_{out}^{ref}.
\]

Besides, at early and late times

\[
<k>_{in}^{tr} = \frac{\hbar}{m} <k>_{in}^{tr} - <\lambda'>_{in}^{tr}; \tag{31}
\]

\[
<k>_{out}^{tr} = \frac{\hbar}{m} <k>_{out}^{tr} - <\lambda'>_{out}^{tr} + d;
\]

\[
<k>_{in}^{ref} = \frac{\hbar}{m} <k>_{in}^{ref} - <\lambda'>_{in}^{ref}; \tag{32}
\]

\[
<k>_{out}^{ref} = \frac{\hbar}{m} <k>_{out}^{ref} + <\lambda' - F'>_{out}^{ref} + 2a
\]

(henceforth, angle brackets denote averaging over the corresponding in- or out-asymptotes).
As it follows from Exps. (31) and (32), the average starting points \(x_{\text{start}}^{tr}\) and \(x_{\text{start}}^{ref}\), for the subensembles of transmitted and reflected particles, respectively, read as

\[
x_{\text{start}}^{tr} = -<\lambda'>_{in}^{tr}, \quad x_{\text{start}}^{ref} = -<\lambda'>_{in}^{ref}.
\]  
(33)

The implicit assumption made in the standard wave-packet analysis is that transmitted and reflected particles start, on the average, from the origin (in the above setting the problem). However, by our approach, just \(x_{\text{start}}^{tr}\) and \(x_{\text{start}}^{ref}\) are the average starting points of transmitted and reflected particles, respectively. They are the initial values of \(<\dot{x} >^{tr}_{in}\) and \(<\dot{x} >^{ref}_{in}\), which have the status of the expectation values of the particle’s position. They behave causally in time. As regards the average starting point of the whole ensemble of particles, its coordinate is the initial value of \(<\dot{x} >_{in}\), which behaves non-causally in the course of scattering. This quantity has no status of the expectation value of the particle’s position.

Let us take into account Exps. (31), (32) and analyze the motion of a particle in the spatial interval \([a-L_1, b+L_2]\). In particular, let us define the transmission time for this region, making use the asymptotes of the TWF. We will denote this time as \(\Delta t_{tr}^{as}(a-L_1, b+L_2)\). The equations for the arrival times \(t_{1}^{tr}\) and \(t_{2}^{tr}\) for the extreme points \(x = a - L_1\) and \(x = b + L_2\), respectively, read as

\[
<\dot{x} >^{tr}_{in}(t_{1}^{tr}) = a - L_1; \quad <\dot{x} >^{tr}_{out}(t_{2}^{tr}) = b + L_2.
\]

Considering (34), we obtain from here that the transmission time for this interval is

\[
\Delta t_{tr}^{as} = \frac{m}{\hbar} \frac{<J' >^{tr}_{out} - <\lambda' >^{tr}_{in} + L_1 + L_2}{<\dot{x} >^{tr}_{in} - h < k >^{tr}_{in}}.
\]

Similarly, for the reflection time \(\Delta t_{ref}^{as}(a-L_1, b+L_2)\), where \(\Delta t_{ref}(a-L_1, b+L_2) = t_{2}^{ref} - t_{1}^{ref}\), we have

\[
<\dot{x} >^{ref}_{in}(t_{1}^{ref}) = a - L_1; \quad <\dot{x} >^{ref}_{out}(t_{2}^{ref}) = a - L_1.
\]

Considering (32), one can easily show that

\[
\Delta t_{ref}^{as} = \frac{m}{\hbar} \frac{<J' >^{ref}_{out} - <\lambda' >^{ref}_{in} + 2L_1}{<\dot{x} >^{ref}_{in} - h < k >^{ref}_{in}}.
\]

The times \(\tau_{tr}^{as} = \Delta t_{tr}^{as}(a,b)\) and \(\tau_{ref}^{as} = \Delta t_{ref}^{as}(a,b)\) are, respectively, the searched-for asymptotic group times for transmission and reflection, for the barrier region:

\[
\tau_{tr}^{as} = \frac{m}{\hbar} \frac{<J' >^{tr}_{out} - <\lambda' >^{tr}_{in}}{h < k >^{tr}_{in}},
\]

\[
\tau_{ref}^{as} = \frac{m}{\hbar} \frac{<J' >^{ref}_{out} - <\lambda' >^{ref}_{in}}{h < k >^{ref}_{in}}.
\]

Note, unlike the exact group times, the asymptotic ones may be negative by value: they do not give the time spent by a particle in the barrier region (see also Fig.1).

The lengths \(d_{eff}^{tr}\) and \(d_{eff}^{ref}\), where

\[
d_{eff}^{tr} = <J' >^{tr}_{out} - <\lambda' >^{tr}_{in},
\]

\[
d_{eff}^{ref} = <J' >^{ref}_{out} - <\lambda' >^{ref}_{in},
\]

may be treated as the effective barrier’s widths for transmission and reflection, respectively.

3. Average starting points and asymptotic group times for rectangular potential barriers

Let us consider the case of the rectangular barrier and obtain explicit expressions for \(d_{eff}(k)\) (now, both for transmission and reflection, \(d_{eff}(k) = J'(k) - \lambda'(k)\) since \(F'(k) \equiv 0\) which can be treated as the effective width of the barrier for a particle with a given \(k\). Besides, we will obtain the corresponding expressions for the expectation value, \(x_{start}(k)\), of the starting point for this particle: \(x_{start}(k) = -\lambda'(k)\). It is evident that in terms of \(d_{eff}\) the above asymptotic times for a particle with the well-defined momentum \(\hbar k_0\) read as

\[
\tau_{tr}^{as} = \tau_{ref}^{as} = \frac{md_{eff}(k_0)}{\hbar k_0}.
\]

Using Exps. (40) and (77), one can show that, for the below-barrier case \((E \leq V_0)\):

\[
d_{eff}(k) = \frac{4}{\kappa} \frac{[ k^2 + \kappa_0^2 \sinh^2(\kappa d/2)] - [\kappa_0^2 \sinh(\kappa d) - k^2 d]}{4k^2 \kappa^2 + \kappa_0^4 \sin^2(\kappa d)}
\]

\[
x_{start}(k) = -\frac{\beta \kappa_0^2}{\kappa} (k^2 - k^2 \sinh(\kappa d) + k^2 \kappa d \cosh(\kappa d))
\]

for the above-barrier case \((E \geq V_0)\)

\[
d_{eff}(k) = \frac{4}{\kappa} \frac{[ k^2 - \beta \kappa_0^2 \sin(\kappa d)]}{4k^2 \kappa^2 + \kappa_0^4 \sin^2(\kappa d)}
\]

\[
x_{start}(k) = -\frac{\beta \kappa_0^2}{\kappa} (k^2 + k^2 \sinh(\kappa d) - k^2 \kappa d \cosh(\kappa d))
\]

where \(\kappa_0 = \sqrt{2m|V_0|/\hbar^2}\); \(\beta = 1\), if \(V_0 > 0\); otherwise, \(\beta = -1\).

Note, \(d_{eff} \rightarrow d\) and \(x_{start}(k) \rightarrow 0\), in the limit \(k \rightarrow \infty\). For infinitely narrow in \(x\)-space wave packets, this property ensures the coincidence of the average starting points for both subensembles with that for all particles. For wide barriers, when \(kd \gg 1\) and \(E \leq V_0\), we have \(d_{eff} \approx 2/\kappa\) and \(x_{start}(k) \approx 0\). That is, the asymptotic group transmission time saturates with increasing the width of an opaque potential barrier.

It is important to stress that for the \(\delta\)-potential, \(V(x) = W(\delta(x-a)), d_{eff} \equiv 0\). The subensembles of transmitted and reflected particles start, on the average, from the point \(x_{start}(k): x_{start}(k) = -2m\hbar^2 W/(\hbar^4 k^2 + m^2 W^2)\).
B. Dwell times

Let us now consider the stationary scattering problem. It describes the limiting case of a scattering of wide wave packets, when the group-time concept leads to a large error in timing a particle.

1. Dwell time for transmission

Note, in the case of transmission the density of the probability flux, \( I_{tr} \), for \( \psi_{tr}(x; k) \) is everywhere constant and equal to \( T \cdot \hbar k / m \). The velocity, \( v_{tr}(x, k) \), of an infinitesimal element of the flux, at the point \( x \), equals \( v_{tr}(x) = I_{tr} / |\psi_{tr}(x; k)|^2 \). Outside the barrier region the velocity is everywhere constant: \( v_{tr} = \hbar k / m \). In the barrier region it depends on \( x \). In the case of an opaque rectangular potential barrier, \( v_{tr}(x) \) decreases exponentially when the infinitesimal element approaches the midpoint \( x_c \). One can easily show that \( |\psi_{tr}(a; k)| = |\psi_{tr}(b; k)| = \sqrt{T} \), but \( \psi_{tr}(x_c; k) \sim \sqrt{T} \exp(ikd/2) \).

Thus, any selected infinitesimal element of the flux passes the barrier region for the time \( \tau_{dwell}^{tr} \), where

\[
\tau_{dwell}^{tr}(k) = \frac{1}{I_{tr}} \int_{a}^{b} |\psi_{tr}(x; k)|^2 dx. \tag{36}
\]

By analogy with \[15\] we will call this time scale the dwell time for transmission. For the rectangular barrier this time reads (for \( E < V_0 \) and \( E \geq V_0 \), respectively) as

\[
\tau_{dwell}^{tr} = \frac{m}{2\hbar k n^3} \left[ (\kappa^2 - k^2) \kappa d + \kappa_0^2 \sinh(\kappa d) \right], \tag{37}
\]

\[
\tau_{dwell}^{tr} = \frac{m}{2\hbar k n^3} \left[ (\kappa^2 + k^2) \kappa d - \beta \kappa_0^2 \sin(\kappa d) \right]. \tag{38}
\]

2. Dwell time for reflection

In the case of reflection the situation is less simple. The above arguments are not applicable here, for the probability flux for \( \psi_{ref}(x; k) \) is zero. However, as is seen, the dwell time for transmission coincides, in fact, with Buttiker’s dwell time introduced however on the basis of the wave function for transmission. Therefore, making use of the arguments by Buttiker, let us define the dwell time for reflection, \( \tau_{dwell}^{ref} \), as

\[
\tau_{dwell}^{ref}(k) = \frac{1}{I_{ref}} \int_{a}^{b} |\psi_{ref}(x, k)|^2 dx; \tag{39}
\]

where \( I_{ref} = R \cdot \hbar k / m \) is the incident probability flux for reflection. Again, for the rectangular barrier

\[
\tau_{dwell}^{ref} = \frac{mk}{\hbar k} \frac{\sinh(\kappa d) - \kappa d}{\kappa^2 + \kappa_0^2 \sinh^2(\kappa d/2)} \quad \text{for} \quad E \leq V_0; \tag{40}
\]

As is seen, for rectangular barriers the dwell times for transmission and reflection do not coincide with each other, unlike the asymptotic group times.

We have to stress once more that Exps. \[30\] and \[39\], unlike Smith’s, Buttiker’s and Bohmian dwell times, are defined in terms of the TWF and RWF. As will be seen from the following, the dwell times introduced can be justified in the framework of the Larmor-time concept.

C. Larmor times for transmission and reflection

As was said above, both the group and dwell time concepts do not give the way of measuring the time spent by a particle in the barrier region. This task can be solved in the framework of the Larmor time concept. As is known, the idea to use the Larmor precession as clocks was proposed by Baz’ \[21\] and developed later by Rybachenko \[22\] and Büttiker \[15\] (see also \[23, 25\]). However, the known concept of Larmor time has a serious shortcoming. It was introduced in terms of asymptotic values (see \[15, 23, 25\]). In this connection, our next step is to define the Larmor times for transmission and reflection, taking into account the expressions for the corresponding wave functions in the barrier region.

1. Preliminaries

Let us consider the quantum ensemble of electrons moving along the \( x \)-axis and interacting with the symmetrical time-independent potential barrier \( V(x) \) and small magnetic field (parallel to the \( z \)-axis) confined to the finite spatial interval \([a, b]\). Let this ensemble be a mixture of two parts. One of them consists from electrons with spin parallel to the magnetic field. Another is formed from particles with antiparallel spin.

Let at \( t = 0 \) the in state of this mixture be described by the spinor

\[
\Psi_{in}(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \psi_{in}(x), \tag{42}
\]

where \( \psi_{in}(x) \) is a normalized function to satisfy conditions \[11\]. So that we will consider the case, when the spin coherent in state \[12\] is the eigenvector of \( \sigma_x \) with the eigenvalue 1 (the average spin of the ensemble of incident particles is oriented along the \( x \)-direction); hereinafter, \( \sigma_x, \sigma_y \) and \( \sigma_z \) are the Pauli spin matrices.

For electrons with spin up (down), the potential barrier effectively decreases (increases), in height, by the value \( \hbar \omega_L / 2 \); here \( \omega_L \) is the frequency of the Larmor precession: \( \omega_L = 2 \mu B / \hbar \), \( \mu \) denotes the magnetic moment. The
corresponding Hamiltonian has the following form,
\[ \hat{H} = \frac{\hat{p}^2}{2m} + V(x) = \frac{\hbar \omega L}{2} \sigma_z, \text{ if } x \in [a, b]; \]
\[ \hat{H} = \frac{\hat{p}^2}{2m}, \text{ otherwise.} \] (43)
For \( t > 0 \), due to the influence of the magnetic field, the states of particles with spin up and down become different. The probability to pass the barrier is different for them. Let for any value of \( t \) the spinor to describe the state of particles read as
\[ \Psi_{full}(x, t) = \frac{1}{\sqrt{2}} \left( \psi_{tr}^{(\uparrow)}(x, t) + \psi_{ref}^{(\downarrow)}(x, t) \right). \] (44)
In accordance with (12), either spinor component can be uniquely presented as a coherent superposition of two probability fields to describe transmission and reflection:
\[ \psi_{full}^{(\uparrow)}(x, t) = \psi_{tr}(x, t) + \psi_{ref}(x, t); \] (45)
note that \( \psi_{ref}(x, t) \equiv 0 \) for \( x \geq x_c \). As a consequence, the same decomposition takes place for spinor \( \Psi_{full}(x, t) = \Psi_{tr}(x, t) + \Psi_{ref}(x, t) \).
We will suppose that all the wave functions for transmission and reflection are known. It is important to stress here (see (28)) that
\[ \begin{align*}
\langle \psi_{full}^{(\uparrow)}(x, t) | \psi_{full}^{(\uparrow)}(x, t) \rangle &= T^{(\uparrow)} + R^{(\uparrow)} = 1; \\
T^{(\uparrow)} &= \langle \psi_{tr}^{(\uparrow)}(x, t) | \psi_{tr}^{(\uparrow)}(x, t) \rangle = \text{const}; \\
R^{(\uparrow)} &= \langle \psi_{ref}^{(\downarrow)}(x, t) | \psi_{ref}^{(\downarrow)}(x, t) \rangle = \text{const};
\end{align*} \] (46)
\( T^{(\uparrow)} \) and \( R^{(\uparrow)} \) are the (real) transmission and reflection coefficients, respectively, for particles with spin up \( \uparrow \) and down \( \downarrow \). Let further \( T = (T^{(\uparrow)} + T^{(\downarrow)})/2 \) and \( R = (R^{(\uparrow)} + R^{(\downarrow)})/2 \) be quantities to describe all particles.

2. Time evolution of the spin polarization of particles
To study the time evolution of the average particle’s spin, we have to find the expectation values of the spin projections \( \hat{S}_x, \hat{S}_y \) and \( \hat{S}_z \). Note, for any \( t \)
\[ \begin{align*}
\langle \hat{S}_x \rangle_{full} &= \frac{\hbar}{2} \sin(\theta_{full}) \cos(\phi_{full}) \\
&= \hbar \Re(\langle \psi_{full}^{(\uparrow)} \rangle^{(\downarrow)}); \\
\langle \hat{S}_y \rangle_{full} &= \frac{\hbar}{2} \sin(\theta_{full}) \sin(\phi_{full}) \\
&= \hbar \Im(\langle \psi_{full}^{(\uparrow)} \rangle^{(\downarrow)}); \\
\langle \hat{S}_z \rangle_{full} &= \frac{\hbar}{2} \cos(\theta_{full}) \\
&= \frac{\hbar}{2} \left( \langle \psi_{full}^{(\uparrow)} \rangle^{(\uparrow)} - \langle \psi_{full}^{(\downarrow)} \rangle^{(\downarrow)} \right). \end{align*} \] (47)

Similar expressions are valid for transmission and reflection:
\[ \begin{align*}
\langle \hat{S}_x \rangle_{tr} &= \frac{\hbar}{T} \Re(\langle \psi_{tr}^{(\uparrow)} \rangle^{(\downarrow)}), \\
\langle \hat{S}_y \rangle_{tr} &= \frac{\hbar}{T} \Im(\langle \psi_{tr}^{(\uparrow)} \rangle^{(\downarrow)}), \\
\langle \hat{S}_z \rangle_{tr} &= \frac{\hbar}{2T} \left( \langle \psi_{tr}^{(\uparrow)} \rangle^{(\uparrow)} - \langle \psi_{tr}^{(\downarrow)} \rangle^{(\downarrow)} \right), \\
\langle \hat{S}_x \rangle_{ref} &= \frac{\hbar}{R} \Re(\langle \psi_{ref}^{(\uparrow)} \rangle^{(\downarrow)}), \\
\langle \hat{S}_y \rangle_{ref} &= \frac{\hbar}{R} \Im(\langle \psi_{ref}^{(\uparrow)} \rangle^{(\downarrow)}), \\
\langle \hat{S}_z \rangle_{ref} &= \frac{\hbar}{2R} \left( \langle \psi_{ref}^{(\uparrow)} \rangle^{(\uparrow)} - \langle \psi_{ref}^{(\downarrow)} \rangle^{(\downarrow)} \right). \end{align*} \]

Note, \( \theta_{full} = \pi/2 \), \( \phi_{full} = 0 \) at \( t = 0 \). However, this is not the case for transmission and reflection. Namely, for \( t = 0 \) we have
\[ \begin{align*}
\phi_{tr, ref}^{(0)} &= \arctan \left( \frac{\Im(\langle \psi_{tr, ref}^{((0)} \rangle^{(0)})}{\Re(\langle \psi_{tr, ref}^{((0)} \rangle^{(0)})} \right); \\
\theta_{tr, ref}^{(0)} &= \arccos \left( \langle \psi_{tr, ref}^{(0)} \rangle^{(0)} \right). \end{align*} \]

Since the norms of \( \psi_{tr}^{(\uparrow)}(x, t) \) and \( \psi_{tr}^{(\uparrow)}(x, t) \) are constant, \( \theta_{tr}(t) = \theta_{tr}^{(0)} \) and \( \theta_{ref}(t) = \theta_{ref}^{(0)} \) for any value of \( t \). For the z-components of spin we have
\[ \begin{align*}
\langle \hat{S}_z \rangle_{tr} &= \frac{\hbar}{T} \left( \langle T^{(\uparrow)} - T^{(\downarrow)} \rangle, \\
\langle \hat{S}_z \rangle_{ref} &= \frac{\hbar}{R} \left( \langle R^{(\uparrow)} - R^{(\downarrow)} \rangle. \end{align*} \] (48)

So, since the operator \( \hat{S}_z \) commutes with Hamiltonian (48), this projection of the particle’s spin should be constant, on the average, both for transmission and reflection. From the most beginning the subensembles of transmitted and reflected particles possess a nonzero average z-component of spin (though it equals zero for the whole ensemble of particles, for the case considered) to be conserved in the course of scattering. By our approach it is meaningless to use the angles \( \theta_{tr}^{(0)} \) and \( \theta_{ref}^{(0)} \) as a measure of the time spent by a particle in the barrier region.

3. Larmor precession caused by the infinitesimal magnetic field confined to the barrier region

As in (15), (27), we will suppose further that the applied magnetic field is infinitesimal. In order to introduce
characteristic times let us find the derivations \(d\phi_{tr}/dt\) and \(d\phi_{ref}/dt\). For this purpose we will use the Ehrenfest equations for the average spin of particles:

\[
\begin{align*}
\frac{d <\hat{S}_x>_{tr}}{dt} &= -\frac{\hbar \omega_L}{T} \int_a^b \Im[(\psi^{(1)}_{tr}(x,t))^* \psi^{(1)}_{tr}(x,t)]dx, \\
\frac{d <\hat{S}_y>_{tr}}{dt} &= -\frac{\hbar \omega_L}{T} \int_a^b \Re[(\psi^{(1)}_{tr}(x,t))^* \psi^{(1)}_{tr}(x,t)]dx, \\
\frac{d <\hat{S}_x>_{ref}}{dt} &= -\frac{\hbar \omega_L}{R} \int_a^b \Im[(\psi^{(1)}_{ref}(x,t))^* \psi^{(1)}_{ref}(x,t)]dx, \\
\frac{d <\hat{S}_y>_{ref}}{dt} &= -\frac{\hbar \omega_L}{R} \int_a^b \Re[(\psi^{(1)}_{ref}(x,t))^* \psi^{(1)}_{ref}(x,t)]dx.
\end{align*}
\]

Note, \(\phi_{tr} = \arctan (<\hat{S}_y>_{tr}/<\hat{S}_x>_{tr})\), \(\phi_{ref} = \arctan (<\hat{S}_y>_{ref}/<\hat{S}_x>_{ref})\). Hence, in the case of infinitesimal magnetic field and chosen initial conditions, when \(|<\hat{S}_y>_{tr,ref}| >> |<\hat{S}_x>_{tr,ref}|\), we have

\[
\begin{align*}
\frac{d\phi_{tr}}{dt} &= \frac{1}{<\hat{S}_x>_{tr}} \frac{d <\hat{S}_y>_{tr}}{dt}, \\
\frac{d\phi_{ref}}{dt} &= \frac{1}{<\hat{S}_x>_{ref}} \frac{d <\hat{S}_y>_{ref}}{dt}.
\end{align*}
\]

Then, considering the above expressions for the spin projections and their derivatives on \(t\), we obtain

\[
\begin{align*}
\frac{d\phi_{tr}}{dt} &= \omega_L \int_a^b \Re[(\psi^{(1)}_{tr}(x,t))^* \psi^{(1)}_{tr}(x,t)]dx; \\
\frac{d\phi_{ref}}{dt} &= \omega_L \int_a^b \Re[(\psi^{(1)}_{ref}(x,t))^* \psi^{(1)}_{ref}(x,t)]dx.
\end{align*}
\]

Or, taking into account that in the first order approximation on \(\omega_L\), when \(\psi^{(1)}_{tr}(x,t) = \psi^{(1)}_{tr}(x,t)\) and \(\psi^{(1)}_{ref}(x,t) = \psi^{(1)}_{ref}(x,t)\), we have

\[
\begin{align*}
\frac{d\phi_{tr}}{dt} &\approx \frac{\omega_L}{T} \int_a^b |\psi_{tr}(x,t)|^2 dx; \\
\frac{d\phi_{ref}}{dt} &\approx \frac{\omega_L}{R} \int_a^b |\psi_{ref}(x,t)|^2 dx;
\end{align*}
\]

note, in this limit, \(T \rightarrow T\) and \(R \rightarrow R\).

As is supposed in our setting the problem, both at the initial and final instants of time, a particle does not interact with the potential barrier and magnetic field. In this case, without loss of exactness, the angles of rotation (\(\Delta\phi_{tr}\) and \(\Delta\phi_{ref}\)) of spin under the magnetic field, in the course of a completed scattering, can be written in the form,

\[
\begin{align*}
\Delta\phi_{tr} &= \frac{\omega_L}{T} \int_{-\infty}^{\infty} dt \int_a^b dx |\psi_{tr}(x,t)|^2, \\
\Delta\phi_{ref} &= \frac{\omega_L}{R} \int_{-\infty}^{\infty} dt \int_a^b dx |\psi_{ref}(x,t)|^2. \quad (49)
\end{align*}
\]

On the other hand, both the quantities can be written in the form: \(\Delta\phi_{tr} = \omega_L \tau_{tr}^{L} \) and \(\Delta\phi_{ref} = \omega_L \tau_{ref}^{L} \), where \(\tau_{tr}^{L}\) and \(\tau_{ref}^{L}\) are the Larmor times for transmission and reflection. Comparing these expressions with (44), we eventually obtain

\[
\begin{align*}
\tau_{tr}^{L} &= \frac{1}{T} \int_{-\infty}^{\infty} dt \int_a^b dx |\psi_{tr}(x,t)|^2, \\
\tau_{ref}^{L} &= \frac{1}{R} \int_{-\infty}^{\infty} dt \int_a^b dx |\psi_{ref}(x,t)|^2. \quad (50)
\end{align*}
\]

These are just the searched-for definitions of the Larmor times for transmission and reflection.

As is seen, if the state of a particle is described by a normalized wave function \(\psi\), then the time \(\tau(\psi; a, b)\) spent by the particle in the barrier region is

\[
\tau(\psi; a, b) = \int_{-\infty}^{\infty} dt \int_a^b dx |\psi(x,t)|^2. \quad (51)
\]

This definition is just that introduced in [11, 12, 13, 14] on the basis of classical mechanics (see also [6, 7, 19]): note that in both cases the integrals are calculated over the whole completed scattering. Thus, on the other hand, our approach justifies the definition (B2), since this expression is obtained now as the Larmor time. As a consequence, it can be verified experimentally. On the other hand, we correct the domain of the validity of this expression. By our approach, it is meaningful only in the framework of the separate description of transmission and reflection, based on the solutions \(\psi_{tr}(x,t)\) and \(\psi_{ref}(x,t)\) found first in the present paper.

Our next step is to transform Exps. (50). Note, for transmission, \(\psi_{tr}(x,t)\) reads as

\[
\psi_{tr}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_{in}(k) \psi_{tr}(x,k)e^{-iE(k)t/h} dk;
\]

where \(\psi_{tr}(x,k)\) is the stationary wave function for transmission (see Section III). Then the integral \(I = \int_{-\infty}^{\infty} dt \int_a^b dx |\psi_{tr}(x,t)|^2\) in (50) can be reduced, by integrating on \(t\), to the form

\[
I = \frac{h}{\pi} \int_{-\infty}^{\infty} dk dk' A_{in}^{*}(k')A_{in}(k) \int_a^b dx \psi_{tr}^{*}(x,k')\psi_{tr}(x,k) \times \lim_{\Delta t \rightarrow \infty} \frac{\sin[(E(k') - E(k))\Delta t/h]}{E(k') - E(k)}
\]

However,

\[
\lim_{\Delta t \rightarrow \infty} \frac{\sin[(E(k') - E(k))\Delta t/h]}{E(k') - E(k)} = \frac{\pi}{h} \delta(E(k') - E(k))/h = \frac{\pi m}{h^2 k} \delta(k' - k) - \delta(k' + k).
\]

Making use of a symmetrized expression for the real integral \(I\), one can show that the second term to contain \(\delta(k' + k)\)
leads to zero input into I. As a result, for the Larmor transmission time, we obtain
\[ \tau_{tr}^L = \frac{m}{\mathcal{T} h} \int_{-\infty}^{\infty} dk |A_{in}(k)|^2 k^{-1} \int_{a}^{b} dx |\psi_{tr}(x, k)|^2. \]

Or, taking into account Exp. 639 as well as the relationship \( \psi_{tr}(x, -k) = \psi^{*}_{tr}(x, k) \), we eventually obtain that
\[ \tau_{tr}^L = \frac{1}{\mathcal{T}} \int_{0}^{\infty} \varpi(k) T(k) \tau_{dwell}^L(k) dk, \] (52)

where \( \varpi(k) = |A_{in}(k)|^2 - |A_{in}(-k)|^2 \).

A similar expression takes place for \( \tau_{ref}^L \) -
\[ \tau_{ref}^L = \frac{1}{\mathcal{T}} \int_{0}^{\infty} \varpi(k) R(k) \tau_{dwell}^L(k) dk. \] (53)

The integrands in both these expressions are evident to be non-singular at \( k = 0 \).

Thus, the Larmor times for transmission and reflection are, like the local dwell time (see 1, 11, 11), the average values of the corresponding dwell times.

In the end of this section it is useful again to address rectangular barriers. For the stationary case, in addition to Larmor times (37), (38), (40) and (41), we present explicit expressions for the initial angles \( \theta_{tr}^{(0)} \) and \( \phi_{tr}^{(0)} \).

To the first order in \( \omega_L \), we have \( \theta_{tr}^{(0)} = \frac{\pi}{2} - \omega_L \tau_z \), \( \phi_{tr}^{(0)} = \omega_L \tau_0 \), \( \theta_{ref}^{(0)} = \frac{\pi}{2} + \omega_L \tau_z \), \( \phi_{ref}^{(0)} = -\omega_L \tau_0 \), where
\[ \tau_z = \frac{mk_0^2}{\hbar^2} \left( \kappa^2 - k^2 \right) \sinh(\kappa d) + \kappa^2 \kappa^2 \sinh(\kappa d) \]
\[ \tau_z = \frac{mk_0^2}{\hbar^2} \frac{\kappa^2 \kappa^2 \cos(\kappa d) - \beta(\kappa^2 + k^2) \sin(\kappa d)}{4k^2 \kappa^2 + \kappa^4 \sinh^2(\kappa d)} \sin(\kappa d), \] (54)

for \( E < V_0 \) and \( E \geq V_0 \), respectively:
\[ \tau_0 = \frac{2mk}{\hbar} \left( \kappa^2 - k^2 \right) \sin(\kappa d) + \kappa^2 \kappa^2 \cos(\kappa d) \]
\[ \tau_0 = \frac{2mk}{\hbar} \frac{\beta \kappa^2 \kappa^2 \cos(\kappa d) - (\kappa^2 + k^2) \sin(\kappa d)}{4k^2 \kappa^2 + \kappa^4 \sinh^2(\kappa d)}, \]

for \( E < V_0 \) and \( E \geq V_0 \), respectively.

Note that \( \tau_z \) is just the characteristic time introduced in 13 (see Exp. (2.20a)). However we have to stress once more that this quantity does not describe the duration of the scattering process (see the end of Section IV.C.2). As regards \( \tau_0 \), this quantity is directly associated with timing a particle in the barrier region. It describes the initial position of the "clock-pointers", which they have before entering this region.

D. Tunneling a particle through an opaque rectangular barrier

Let us now show that the case of tunneling a particle, with a well defined energy, through an opaque rectangular potential barrier is the most suitable one to verify our approach. Let us denote the measured azimuthal angle as \( \phi_{tr}^{(\infty)} \). By our approach \( \phi_{tr}^{(\infty)} = \phi_{tr}^{(0)} + \Delta \phi_{tr} \). That is, the final time to be registered by the particle’s ”clocks” should be equal to \( \tau_0 + \tau_{tr}^L \).

As is seen, in the general case there is a problem to distinguish the inputs \( \tau_0 \) and \( \tau_{tr}^L \). However, for a particle tunneling through an opaque rectangular barrier this problem disappears. The point is that for \( \kappa d \gg 1 \), \( |\tau_0| \ll \tau_{tr}^L \) (see Exps. (57) and (58)).

Note, in the case considered, Smith’s dwell time (\( \tau_{dwell}^{Smith} \)), which coincides with the ”phase” time, and Buttiker’s dwell time (see Exps. (3.2) and (2.20b) in 15) saturate with increasing the barrier’s width. Just this property of the tunneling times is interpreted as the Hartman effect. At the same time, our approach denies the existence of the Hartman effect: transmission time \( \tau_{tr}^L \) increases exponentially when \( d \rightarrow \infty \).

Note that the Bohmian approach formally denies this effect, too. It predicts that the time, \( \tau_{dwell}^{Bohm} \), spent by a transmitted particle in the opaque rectangular barrier is
\[ \tau_{dwell}^{Bohm} = \frac{1}{\tau_{dwell}^L} \tau_{dwell}^{Smith} = \frac{m}{2\hbar k^2} \left( \frac{(\kappa^2 - k^2) k^2 \kappa d}{2} + \kappa^4 \sinh(2kd)/2 \right). \]

Thus, for \( \kappa d \gg 1 \) we have \( \tau_{dwell}^{Bohm} / \tau_{dwell}^L \sim \cosh(kd) \), i.e.,
\[ \tau_{dwell}^{Bohm} \gg \tau_{dwell}^L \gg \tau_{dwell}^{Smith} \sim \tau_{dwell}^{Butt}. \]

As is seen, in comparison with our definition, \( \tau_{dwell}^{Bohm} \) overestimates the duration of dwelling transmitted particles in the barrier region. In the final analysis, this sharp difference between \( \tau_{dwell}^L \) and \( \tau_{dwell}^{Bohm} \) is explained by the fact that \( \tau_{dwell}^{Bohm} \) to describe transmission was obtained in terms of \( \psi_{full} \). One can show that the input of the to-be-reflected subensemble of particles into \( \int_{0}^{\infty} |\psi_{full}(x, k)|^2 dx \) dominates inside the region of an opaque potential barrier. Therefore treating this time scale as a characteristic time for transmission has no basis.

As was said (see Sections II and III), the trajectories of transmitted and reflected particles are ill-defined in the Bohmian mechanics. However, we have to stress that our approach does not at all deny the Bohmian one. It suggests only that causal trajectories for these particles should be redefined. An incident particle should have two possibility (to be transmitted or to reflected by the barrier) irrespective of the location of its starting point. This means that just two causal trajectories should evolve from each staring point: on the \( OX \)-axis one should lead to plus infinity, but another should approach minus infinity. Both sets of causal trajectories must be defined on the basis of \( \psi_{tr}(x, t) \) and \( \psi_{ref}(x, t) \). As to the rest, all mathematical tools developed in the Bohmian mechanics (see, e.g., 32, 50) remain in force.

Tunneling is useful also to display explicitly the role of the exact and asymptotic group times. Fig.1 shows the time dependence of the mean value of the particle’s position for transmission, where \( a = 200nm, b = 215nm, \).
$V_0 = 0.2eV$. At $t = 0$ the (full) state of the particle is described by the Gaussian wave packet peaked around $x = 0$; its half-width $10nm$; the average energy of the particle $0.05eV$.

As is seen, the exact group time gives the time spent by the CM of the transmitted wave packet in the barrier region. But the asymptotic time displays its lag, long after the scattering event, with respect to the CM of a packet, to start from the point $x_{start}^tr$ and move freely with the velocity $\hbar < k >_{out}/m$.

In this case the exact group transmission time is equal approximately to $0.155ps$, the asymptotic one is of $0.01ps$, and $\tau_{free} \approx 0.025ps$. As is seen, the dwell and exact group times for transmission, both evidence that, though the asymptotic group time for transmission is small for this case, transmitted particles spend much time in the barrier region. Note, also that the times spent by transmitted and reflected particles in the barrier region do not coincide even for symmetric barriers.

V. CONCLUSION

It is shown that a 1D completed scattering is a combination of two sub-processes, transmission and reflection, evolved coherently. In the case of symmetric potential barrier we find explicitly two solutions to the Schrödinger equation, which describe these sub-processes at all stages of scattering. Their sum gives the wave function to describe the whole combined process.

On the basis of these solutions, for either sub-process, we define the time spent, on the average, by a particle in the barrier region. For this purpose we reconsider the well-known group, dwell and Larmor-time concepts. The group time concept is suitable for timing a particle in a well-localized state, when the width of a wave packet is smaller than the barrier’s width. The dwell time concept is introduced for timing a particle in the stationary state. The Larmor ”clock” is the most universal instrument for timing the motion of transmitted and reflected particles, without influence on the scattering event. It is applicable for any wave packets. We found that the Larmor times for transmission and reflection are the average values of the corresponding dwell times. The results of our theory can be verified experimentally.

[1] E.H. Hauge and J.A. Støvneng, Rev. Mod. Phys. 61, 917 (1989).
[2] R. Landauer and Th. Martin, Rev. Mod. Phys. 66, 217 (1994).
[3] V.S. Olkhovsky and E. Recami, Phys. Repts. 214, 339 (1992).
[4] A.M. Steinberg, Phys. Rev. 118, 1201 (1961).
[5] J.G. Muga, C.R. Leavens, Phys. Repts. 338, 353 (2000).
[6] C.A.A. Carvalho, H.M. Nussenzveig, Phys. Repts. 364, 83 (2002).
[7] V.S. Olkhovsky, E. Recami, J. Jakied, Phys. Repts. 398, 133 (2004).
[8] E.P. Wigner, Phys. Rev. 98, 145 (1955).
[9] T.E. Hartman, J. Appl. Phys. 33, 3427 (1962).
[10] E.H. Hauge, J.P. Falck and T.A. Fjeldly, Phys. Rev. B 36, 4203 (1987).
[11] N. Teranishi, A.M. Kriman and D.K. Ferry, Superlatt. and Microstrs. 3, 509 (1987).
[12] F.T. Smith, Phys. Rev. 118, 349 (1960).
[13] W. Jaworski and D.M. Wardlaw, Phys. Rev. A 37, 2843 (1988).
[14] W. Jaworski and D.M. Wardlaw, Phys. Rev. A 38, 5404 (1988).
[15] M. Buttiker, Phys. Rev. B 27, 6178 (1983).
[16] C.R. Leavens and G.C. Aers, Phys. Rev. B 39, 1202 (1989).
[17] H.M. Nussenzveig, Phys. Rev. A 62, 042107 (2000).
[18] Mario Goto, Hiromi Iwamoto, Verissimo M. Aquino, Valdir C. Aguilera-Navarro and Donald H. Kobe, J. Phys. A 37, 3599 (2004).
[19] J.G. Muga, S. Brouard and R. Sala, Phys. Lett. A 167, 24 (1992).

[20] Christian Bracher, Manfred Kleber, and Mustafa Riza, Phys. Rev. A 60, 1864 (1999).
[21] A.I. Baz, Yad. Fiz. 4, 252 (1966).
[22] V.F. Rybichenko, Yad. Fiz. 5, 895 (1966).
[23] C.R. Leavens and G.C. Aers, Phys. Rev. B 40, 5387 (1989).
[24] M. Buttiker, in Time in Quantum Mechanics, (Lecture Notes in Physics vol M72) ed J.G. Muga, R.S. Mayato and I.L. Egusquiza (Berlin: Springer), 256 (2002).
[25] Z.J. Li, J.Q. Liang, and D.H. Kobe, Phys. Rev. A 64, 043112 (2001).
[26] Z.J.Q. Liang, Y.H. Nie, J.J. Liang and J.Q. Liang, J. Phys. A 36, 6563 (2003).
[27] Y. Aharonov, D. Bohm, Phys. Rev. 122, 1649 (1961).
[28] S. Brouard, R. Sala, and J.G. Muga, Phys. Rev. A 49, 4312 (1994).
[29] G.E. Hahne, J. Phys. A 36, 7149 (2003).
[30] J.W. Noh, A. Fougeres, and L. Mandel, Phys. Rev. Let. 67, 1426 (1991).
[31] C.R. Leavens, Phys. Rev. A 58, 840 (1998).
[32] G. Grubl and K. Rheinberger, J. Phys. A 35, 2907 (2002).
[33] S. Kreidl, G. Grubl and H.G. Embacher, J. Phys. A 36, 8851 (2003).
[34] D. Sokolovski and L.M. Baskin, Phys. Rev. A 6563 (1961).
[35] J.W. Noh, A. Fougeres, and L. Mandel, Phys. Rev. Let. 67, 1426 (1991).
[36] C.R. Leavens, Phys. Rev. A 58, 840 (1998).
[37] J.G. Muga, S. Brouard and R. Sala, Phys. Lett. A 167, 24 (1992).
Phys. Rev. A 67, 052106 (2003).
[40] Norifumi Yamada, Phys. Rev. Let. 93, 170401 (2004).
[41] P. Krekora, Q. Su, and R. Grobe, Phys. Rev. A 64, 022105 (2001).
[42] G. Garcia-Calderon and J. Villavicencio, Phys. Rev. A 64, 012107 (2001).
[43] G. Garcia-Calderon, J. Villavicencio, F. Delgado, and J.G. Muga, Phys. Rev. A 66, 042119 (2002).
[44] F. Delgado, J.G. Muga, A. Ruschhaupt, G. Garcia-Calderon, and J. Villavicencio, J. Phys. A 68, 032101 (2003).
[45] M. Buttiker and R. Landauer, Phys. Rev. Let. 49, 1739 (1982).
[46] J.G. Muga, I.L. Egusquiza, J.A. Damborenea, F. Delgado, Phys. Rev. A 66, 042115 (2002).
[47] H.G. Winful, Phys. Rev. Let. 91, 260401 (2003).
[48] V.S. Olkhovsky, V. Petrillo, and A.K. Zaichenko, Phys. Rev. A 70, 034103 (2004).
[49] D. Sokolovski, A.Z. Msezane, V.R. Shaginyan, Phys. Rev. A 71, 064103 (2005).
[50] N.L. Chuprikov, e-prints quant-ph/0405028.
[51] N.L. Chuprikov, Sov. Semicond. 26, 2040 (1992).
[52] E. Merzbacher, Quantum mechanics (John Wiley & Sons, INC. New York), (1970).
[53] J.R. Taylor, Scattering theory: the quantum theory on nonrelativistic collisions (John Wiley & Sons, INC. New York - London - Sydney), (1972).
[54] N.L. Chuprikov, Semicond. 31, 427 (1997).

Figure captions

Fig.1 The $t$-dependence of the average position of transmitted particles (solid line); the initial (full) state vector represents the Gaussian wave packet peaked around the point $x = 0$, its half-width equals to 10nm, the average kinetic particle's energy is $0.05eV$; $a = 200nm$, $b = 215nm$. 
