Projectability of stable, partially free $\mathcal{H}$-surfaces in the non-perpendicular case

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Abstract

A projectability result is proved for surfaces of prescribed mean curvature (shortly called $H$-surfaces) spanned in a partially free boundary configuration. Hereby, the $H$-surface is allowed to meet the support surface along its free trace non-perpendicularly. The main result generalizes known theorems due to Hildebrandt-Sauvigny and the author himself and is in the spirit of the well known projectability theorems due to Radó and Kneser. A uniqueness and an existence result are included as corollaries.

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1 Introduction

Let us write $B^+ := \{ w = (u,v) = u + iv : |w| < 1, \ v > 0 \}$ for the upper unit half disc in the plane. Its boundary is divided into
\[ \partial B^+ = I \cup J, \quad I := (-1,1), \quad J := \partial B^+ \setminus I = \{ w \in \overline{B^+} : |w| = 1 \}. \]
In the present paper, a surface of prescribed mean curvature $H = H(p) \in C^0(\mathbb{R}^3, \mathbb{R})$ or, shortly, an $H$-surface is a mapping $x = x(w) : B^+ \to \mathbb{R}^3 \in C^2(B^+, \mathbb{R}^3)$, which solves the system
\[
\Delta x = 2H(x)x_u \wedge x_v \quad \text{in } B^+, \\
|x_u| = |x_v|, \quad x_u \cdot x_v = 0 \quad \text{in } B^+.
\] (1.1)
Here, $y \wedge z$ and $y \cdot z$ denote the cross product and the standard scalar product in $\mathbb{R}^3$, respectively.

Observe that an $H$-surface is not supposed to be a regular surface, that means, it may possess branch points $w_0 \in B^+$ with $x_u \wedge x_v(w_0) = 0$.

We consider $H$-surfaces spanned in a projectable, partially free boundary configuration, which means the following:

Definition 1. (Projectable boundary configuration)

Let $S = \Sigma \times \mathbb{R} \subset \mathbb{R}^3$ be an embedded cylinder surface over the planar closed Jordan arc $\Sigma = \pi(S)$ of class $C^3$; here $\pi$ denotes the orthogonal projection onto
the $x^1, x^2$-plane. Furthermore, let $\Gamma \subset \mathbb{R}^3$ be a closed Jordan arc which can be represented as a $C^3$-graph over the planar closed $C^3$-Jordan arc $\Gamma = \pi(\Gamma)$. Finally, assume $\Gamma \cap \Sigma = \{\pi_1, \pi_2\}$, where $\pi_1, \pi_2$ are the distinct end points of $\Gamma$ as well as $\Sigma$, and $\Gamma$ and $\Sigma$ meet with a positive angle at the respective points $p_1, p_2 \in \Gamma \cap \Sigma$ correlated by $\pi_j = \pi(p_j)$, $j = 1, 2$. Then we call $\{\Gamma, \Sigma\}$ a projectable (partially free) boundary configuration.

To be precise, in Definition 1, the phrase "$\Gamma$ and $\Sigma$ meet with a positive angle at the respective points $p_1, p_2 \in \Gamma \cap \Sigma$" means that the tangentional vector of $\Gamma$ is not an element of the tangential plane of $\Sigma$ at these points. A partially free $\mathcal{H}$-surface is a solution $x \in C^2(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3)$ of (1.1), which satisfies the boundary conditions

$$
x(w) \in S \quad \text{for all } w \in I,
$$
$$
x|_{J}: J \to \Gamma \text{ strictly monotonic},
$$
$$
x(-1) = p_1, \ x(+1) = p_2
$$

for a given projectable boundary configuration $\{\Gamma, \Sigma\}$. Roughly speaking, we aim to show that any such partially free $\mathcal{H}$-surface is itself projectable. This is in the spirit of the famous projectability result for minimal surfaces by Radó and Kneser and will be proved under additional assumptions on the $\mathcal{H}$-surface and the configuration $\{\Gamma, \Sigma\}$, namely: The boundary configuration shall be R-admissible in the sense of Definition 2 below and the $\mathcal{H}$-surface shall be Hölder-continuous on $\overline{B^+}$, stationary w.r.t. some energy functional $E_Q$ and stable w.r.t. the corresponding generalized area functional $A_Q$. Here $Q$ is a given vector field which satisfies a natural smallness condition and which possesses a suitable normal component w.r.t. $\Sigma$ as well as the divergence $\text{div} \ Q = 2H$; see Section 2 for details.

The first results of this type were given by Hildebrandt-Sauvigny [HS1]-[HS3]. They considered the special case of minimal surfaces; a generalization to $F$-minimal surfaces can be found in [MW]. Concerning partially free $\mathcal{H}$-surfaces the only projectability result known to the author was proved in [M3]. There, the above mentioned vector field $Q$ was supposed to be tangential along the support surface $\Sigma$, which forces the corresponding stationary $\mathcal{H}$-surface to meet $\Sigma$ perpendicularly along its free trace $x|_{J}$. This condition was essential at many points of the proof in [M3], in particular, while deriving the second variation formula for $A_Q$ and establishing a boundary condition for the third component of the surface normal of our $\mathcal{H}$-surface. One motivation for writing the present paper was to drop this restriction and to study $\mathcal{H}$-surfaces which meet $\Sigma$ non-perpendicularly.

Methodically, we orientate on [M3] which in turn is based on the work of Hildebrandt and Sauvigny in [HS3] and on Sauvigny’s paper [S1], where a corresponding projectability result for stable $\mathcal{H}$-surfaces subject to Plateau type boundary conditions has been proven.

The paper is organized as follows: In Section 2 we fix notations, specify our assumptions and state the main projectability result, Theorem 1, as well as some preliminary results on the $\mathcal{H}$-surface and its normal. The consequential unique solvability of the studied partially free problem is captured in Corollary 1. In Section 3 we derive the second variation formula for the functional $A_Q$ allowing boundary perturbations on the free trace $x|_{J}$. Then, Section 4 contains the
crucial boundary condition for the third component of the surface normal and the proof of Theorem 1. We close with an exemplary application of Theorem 1 to the existence question for a mixed boundary value problem for the non-parametric $H$-surface equation, Corollary 2.

2 Notations and main result

We start by specifying our additional assumptions on the boundary configuration: Let $\{\Gamma, S\}$ be a projectable boundary configuration in the sense of Definition 1. Let $\sigma = \sigma(s), s \in [0, s_0]$, parametrize $\Sigma = \pi(S)$ by arc length, that is,

$$\sigma \in C^3([0, s_0], \mathbb{R}^2), \quad |\sigma'| \equiv 1 \text{ on } [0, s_0], \quad \text{and } s_0 = \text{length}(\Sigma) > 0.$$ 

Setting $e_3 := (0, 0, 1)$ we define $C^2$-unit tangent and normal vector fields $t, n$ on $S$ as follows:

$$t(p) := (\sigma'(s), 0), \quad n(p) := t(p) \wedge e_3 \quad \text{for } p \in \{\sigma(s)\} \cap \mathbb{R}, \quad s \in [0, s_0]. \quad (2.1)$$

Furthermore, we can write $\Gamma = \{(x^1, x^2, \gamma(x^1, x^2)) \in \mathbb{R}^3 : (x^1, x^2) \in \Gamma\}$, where $\Gamma = \pi(\Gamma)$ is a closed $C^3$-Jordan arc and $\gamma \in C^3(\Gamma)$ is the height function. For the end points $p_1, p_2$ of $\Gamma$ we assume to have representations

$$p_1 = (\sigma(0), \gamma(\sigma(0))), \quad p_2 = (\sigma(s_0), \gamma(\sigma(s_0))).$$

The set $\Gamma \cup \Sigma$ bounds a simply connected domain $G \subset \mathbb{R}^2$, that is, $\partial G = \Gamma \cup \Sigma$, and we have $\Gamma \cap \Sigma = \{\pi_1, \pi_2\}$ with $\pi_j = \pi(p_j), j = 1, 2$. With $\alpha_j \in (0, \pi)$ we denote the interior angle between $\Gamma$ and $\Sigma$ at $\pi_j$ w.r.t. $G$ ($j = 1, 2$). Finally, we assume that $\Sigma$ is parametrized such that $\nu := \pi(n)$ points to the exterior of $G$ along $\Sigma$.

**Definition 2.** A projectable boundary configuration $\{\Gamma, S\}$ is called $R$-admissible, if the following hold:

(i) $\Gamma \cup S \subset Z := \{(p^1, p^2, p^3) \in \mathbb{R}^3 : |(p^1, p^2)| < R\}$ for some $R > 0$.

(ii) $G$ is $\frac{1}{R}$-convex, i.e., for any point $\xi \in \partial G$ there is an open disc $D_\xi \subset \mathbb{R}^2$ of radius $R$ such that $G \subset D_\xi$ and $\xi \in \partial D_\xi$.

For a given $R$-admissible boundary configuration $\{\Gamma, S\}$, we define the class $C(\Gamma, S; Z)$ of mappings $x \in H^2_2(B^+, Z)$, which satisfy the boundary conditions (1.2) weakly, i.e.,

$$x(w) \in S \quad \text{for a.a. } w \in I,$$

$$x|_J : J \to \Gamma \text{ continuously and weakly monotonic}, \quad (2.2)$$

$$x(-1) = p_1, \quad x(+1) = p_2.$$ 

For arbitrary $\mu \in [0, 1)$, we additionally define its subsets

$$C_\mu(\Gamma, S; Z) := \left\{ x \in C(\Gamma, S; Z) : \begin{array}{l} x \in C^\mu(B^+, Z), \\ x|_J : J \to \Gamma \text{ strictly monotonic} \end{array} \right\}. \quad (2.3)$$
Now let $Q = Q(p) \in C^1(\mathbb{Z}, \mathbb{R}^3)$ be a vector field satisfying
\begin{equation}
\sup_{p \in \mathbb{Z}} |Q(p)| < 1, \quad \text{div} \ Q(p) = 2H(p) \quad \text{for all} \ p \in \mathbb{Z}.
\end{equation}
Here the function $H = H(p)$ belongs to $C^{1,\alpha}(\mathbb{Z})$ for some $\alpha \in (0,1)$ and fulfills
\begin{equation}
\sup_{p \in \mathbb{Z}} |H(p)| \leq \frac{1}{2R}.
\end{equation}
We introduce the functional
\[ E_Q(x) := \int_{B^+} \left\{ \frac{1}{2} |\nabla x(w)|^2 + Q(x) \cdot x_u \wedge x_v\right\} du \ dv, \quad x \in H^1(B^+; \mathbb{Z}), \tag{2.6} \]
and consider the variational problem
\[ E_Q(x) \to \min, \quad x \in C(\Gamma; \mathbb{Z}). \tag{2.7} \]

The following lemma collects some well known results concerning the existence and regularity of solutions of (2.7) as well as stationary points of $E_Q$.

**Lemma 1.** (Heinz, Hildebrandt, Tomi)

Let $\{\Gamma, S\}$ be an $R$-admissible boundary configuration $\{\Gamma, S\}$ and assume $Q \in C^1(\mathbb{Z}, \mathbb{R}^3)$, $H \in C^{1,\alpha}(\mathbb{Z})$ to satisfy (2.4) and (2.5). Then there exists a solution $x = x(w)$ of (2.7), $x$ belongs to the class $C_0(\Gamma, S; \mathbb{Z}) \cap C^{3,\alpha}(\mathbb{R}^+; \mathbb{Z})$ for some $\alpha \in (0,1)$ and satisfies the system (1.1), i.e., $x$ is a partially free $H$-surface.

More generally, any stationary point $x \in C_0(\Gamma, S; \mathbb{Z})$ of $E_Q$ solves (1.1) and belongs to the class $C^{3,\alpha}(\mathbb{R}^+; \mathbb{Z})$. Here, stationarity means
\[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \{E_Q(x_{\varepsilon}) - E_Q(x)\} \geq 0 \]
for all inner and outer variations $x_{\varepsilon} \in C_0(\Gamma, S; \mathbb{Z})$, $\varepsilon \in [0,\varepsilon_0)$ with sufficiently small $\varepsilon_0 > 0$; see Definition 2 in [DHT] Section 5.4 for the definition of inner and outer variations.

We also associate the *generalized area functional* to $Q$:
\[ A_Q(x) := \int_{B^+} \left\{ |x_u \wedge x_v| + Q(x) \cdot x_u \wedge x_v\right\} du \ dv, \quad x \in H^1(B^+; \mathbb{Z}). \tag{2.8} \]
A stationary, partially free $H$-surface $x \in C_0(\Gamma, S; \mathbb{Z})$ is called *stable*, if it is stable w.r.t. $A_Q$, that means, the second variation $\frac{d^2}{d\varepsilon^2} A_Q(\bar{x}(\cdot,\varepsilon)) \big|_{\varepsilon=0}$ of $A_Q$ is nonnegative for all outer variations $\bar{x}(\cdot,\varepsilon) \in C_0(\Gamma, S; \mathbb{Z})$, $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, for which this quantity exists; note that $x$ has its image $x(B^+)$ in $\mathbb{Z}$, according to Lemma 1. Since the first variation of $A_Q$ w.r.t. such variations $\bar{x}$ vanishes for stationary $x$, any relative minimizer of $A_Q$ in $C_0(\Gamma, S; \mathbb{Z})$ is stable. In Definition 4 below, we give an exact definition of stability, which is used in the present paper and which is somewhat less stringent than the above mentioned requirement.

We are now in a position to state our main result:
Theorem 1. Let \( \{ \Gamma, S \} \) be an admissible boundary configuration and let \( Q \in C^{1,\alpha}(\mathbb{Z}, \mathbb{R}^3) \) be chosen such that (2.4) is fulfilled with some \( H \in C^{1,\alpha}(\mathbb{Z}) \), \( \alpha \in (0,1) \), satisfying (2.5). In addition, we assume
\[
\frac{\partial}{\partial p^3} H(p) \geq 0 \quad \text{for all } p \in \mathbb{Z}
\] (2.9)
as well as
\[
(Q \cdot n)(p) = (Q \cdot n)(p^1, p^2, 0) \quad \text{for all } p = (p^1, p^2, p^3) \in S,
\]
\[
\|Q \cdot n(p_j)\| < \cos \alpha_j, \quad j = 1, 2. \] (2.10)

Then any stable \( H \)-surface \( x \in C_\mu(\Gamma, S; \mathbb{Z}) \), \( \mu \in (0,1) \), possesses a graph representation over \( \mathcal{G} \). More precisely, \( x \) is immersed and can be represented as the graph of some function \( \zeta : \mathcal{G} \to \mathbb{R} \in C^{3,\alpha}(\mathcal{G}) \cap C^{2,\alpha}(\mathcal{G} \setminus \{\pi_1, \pi_2\}) \cap C^0(\mathcal{G}) \), which satisfies the mixed boundary value problem
\[
\text{div} \left( \frac{\nabla \zeta}{\sqrt{1 + |\nabla \zeta|^2}} \right) = 2H(\cdot, \zeta) \quad \text{in } G, \quad (2.11)
\]
\[
\frac{\nabla \zeta \cdot \nu}{\sqrt{1 + |\nabla \zeta|^2}} = \psi \quad \text{on } \Sigma \setminus \{\pi_1, \pi_2\}, \quad \zeta = \gamma \quad \text{on } \Gamma. \quad (2.12)
\]
Here \( \nu = \pi(n) \) denotes the exterior unit normal on \( \Sigma \) w.r.t. \( G \) and we defined \( \psi := Q \cdot n|_{\Sigma} \in C^1(\Sigma) \).

As a consequence of Theorem 1 we obtain the following

Corollary 1. Let the assumptions of Theorem 1 be satisfied. Then, apart from reparametrization, there exists exactly one stable \( H \)-surface \( x \in C_\mu(\Gamma, S; \mathbb{Z}) \) with some \( \mu \in (0,1) \).

Proof. The existence of a stable \( H \)-surface \( x \in C_\mu(\Gamma, S; \mathbb{Z}) \) for some \( \mu \in (0,1) \) is assured by Lemma 1. According to Theorem 1, we can represent \( x \) as a graph over \( G \), and the height function \( \zeta \) solves the boundary value problem (2.11), (2.12).

If there would exist another stable \( H \)-surface \( \tilde{x} \in C_\mu(\Gamma, S; \mathbb{Z}) \) with some \( \tilde{\mu} \in (0,1) \) and if \( \tilde{\zeta} \) denotes the height function of its graph representation, which also solves (2.11), (2.12) by Theorem 1, we consider the difference function
\[
f := \zeta_1 - \zeta_2. \]
As is well known, \( f \) solves an elliptic differential equation in \( G \), which is subject to the maximum principle according to assumption (2.9); cf. [S2] Chap. VI, §2. Consequently, \( f \) assumes its maximum and minimum on \( \partial G = \Sigma \cup \Gamma \).

Assume that \( f \) has a positive maximum at \( p_0 \in \Sigma \setminus \{\pi_1, \pi_2\} \). Then Hopf’s boundary point lemma implies
\[
\nabla f(p_0) = (\nabla f(p_0) \cdot \nu(p_0)) \nu(p_0) \quad \text{with } \nabla f(p_0) \cdot \nu(p_0) > 0.
\]
On the other hand, the first boundary condition in (2.12) yields (\( M(p_0)\nabla f(p_0) \cdot \nu(p_0) = 0 \), where we have abbreviated
\[
M(p) := \int_0^1 Dh(t\nabla \zeta_1(p) + (1-t)\nabla \zeta_2(p)) \, dt, \quad p \in \Sigma,
\]
Let the assumptions of Theorem 1 be satisfied and let \( \mathbf{x} \). Due to Lemma 1, \( \nabla \mathbf{x} \) gives towards the projectability of our analytical and geometrical regularity results and first important informations we deduce that \( f \geq 0 \) on \( \mathcal{G} \). Hence, we conclude \( f \geq 0 \) on \( \mathcal{G} \). This gives \( \mathbf{x} \equiv \xi, \zeta \) on \( \mathcal{G} \), which yields \( \mathbf{x} = \mathbf{x} \circ \omega \) with some positively oriented parameter transformation \( \omega : \mathcal{B}^+ \to \mathcal{B}^+ \). This proves the corollary.

We complete this section with a preparatory lemma, which collects some analytical and geometrical regularity results and first important informations towards the projectability of our \( \mathcal{H} \)-surfaces:

**Lemma 2.** Let the assumptions of Theorem 1 be satisfied and let \( \mathbf{x} = \mathbf{x}(w) \in C_\mu(\Gamma, S; \mathcal{Z}) \) be an \( \mathcal{H} \)-surface which is stationary w.r.t. \( E_\mathcal{Q} \). Then there follow:

(i) \( \mathbf{x} \in C^{3,\alpha}(B^+, \mathcal{Z}) \cap C^{2,\alpha}(\mathcal{B}^+ \setminus [-1, +1], \mathcal{Z}) \), and there holds

\[
(x_0 + Q(x) \land x_w)(w) \perp T_{x(w)}S \quad \text{for all } w \in I, \tag{2.13}
\]

where \( T_pS \) denotes the tangential plane of \( S \) at the point \( p \in S \).

(ii) \( f(\mathcal{B}^+) \subset \mathcal{G} \) for the projection mapping \( f := \pi(x) \).

(iii) \( \nabla \mathbf{x}(w) \neq 0 \) for all \( w \in \partial B^+ \setminus [-1, +1] \), and \( \nabla \mathbf{x} = 0 \) for at most finitely many points in \( B^+ \).

(iv) Set \( W := \{x_0 \land x_w\}, B' := \{w \in B^+ : W(w) > 0\} \), and define the surface normal \( N(w) := W^{-1}x_0 \land x_w \) as well the Gaussian curvature \( K = K(w) \) of \( \mathbf{x} \) for points \( w \in B' \). Then \( N \) and \( KW \) can be extended to mappings

\[
N \in C^{2,\alpha}(B^+, \mathbb{R}^3) \cap C^{1,\alpha}(\mathcal{B}^+ \setminus [-1, +1], \mathbb{R}^3) \cap C^0(\mathcal{B}^+, \mathbb{R}^3),
\]

\[
KW \in C^{1,\alpha}(B^+),
\]

and \( N \) satisfies the differential equation

\[
\Delta N + 2(2H(x)^2 - K - \langle \nabla H(x) \cdot N \rangle)W N = -2W \nabla H(x) \quad \text{in } B^+. \tag{2.14}
\]

**Proof.** (ii) Due to Lemma 1, \( \mathbf{x} \) is a stationary, partially free \( \mathcal{H} \)-surface of class \( C^{3,\alpha}(B^+, \mathcal{Z}) \). In addition, we have \( f(\partial B^+) = \partial G \) due to the geometry of our boundary configuration. An inspection of the proof of Hilfssatz 4 of [S1] shows, that this boundary condition, the smallness condition (2.5) and the \( \frac{1}{2} \)-convexity of \( G \) imply \( f(\mathcal{B}^+) \subset \mathcal{G} \).

(i), (iii) A well known regularity result according to E. Heinz [He] implies \( \mathbf{x} \in C^{2,\alpha}(B^+ \cup J, \mathcal{Z}) \). And from Theorem 1 in [M6] we obtain \( \mathbf{x} \in C^{1,\frac{1}{2}}(B^+ \cup I, \mathcal{Z}) \). Setting

\[
I' := \{w \in I : f(w) = (\pi \circ \mathbf{x})(w) \notin \{\pi_1, \pi_2\}\},
\]

with \( h(z) := \frac{z}{\sqrt{1 + |z|^2}}, z \in \mathbb{R}^2 \). If we finally note

\[
(Dh(z)) \cdot \xi = \frac{\xi^3}{(1 + |z|^2)^{\frac{3}{2}}} > 0, \quad \xi \in \mathbb{R}^2 \setminus \{0\}, \quad z \in \mathbb{R}^2,
\]

we deduce that \( M \) is positive definite on \( \Sigma \) and arrive at the contradiction
the stationarity yields the natural boundary condition (2.13) on \( I' \).

Due to (ii), the arguments from Satz 2 in [S1] yield \( \nabla x(w) \neq 0 \) for all \( w \in J \). Assume that \( w_0 \in I \) is a branch point of \( x \) and set \( B^*_+ (w_0) := \{ w \in B^+ : |w-w_0| < \delta \} \). Then the asymptotic expansion from Theorem 2 in [M6] imply that \( x|_{B^*_+ (w_0)} \), \( 0 < \delta \ll 1 \), looks like a whole perturbed disc.

Consequently, the projection \( f|_{B^*_+ (w_0)} \) would meet the complement of \( \overline{G} \), in contrast to \( f(B) \subset \overline{G} \). Indeed, for \( w_0 \in I' \) this effects from the natural boundary condition (2.13), which can be rewritten as \( (Q \cdot n)(x) = -N \cdot n(x) \) on \( I' \); see Remark 1 below. And for \( w_0 \in I \setminus I' \), i.e. \( f(w_0) \notin \{ \pi_1, \pi_2 \} \), this is trivial by geometry. Consequently, we have a contradiction and \( \nabla x(w) \neq 0 \) for \( w \in I \) follows; this completes the proof of the first part of (iii).

Next we show \( I' = I \), i.e. \( f(I) = \Sigma \setminus \{ \pi_1, \pi_2 \} \). From [HJ] or [M5] we then obtain \( x \in C^{2,\alpha}(B^+ \cup I, Z) \) and (2.13) holds on \( I \); this will complete the proof of (i).

Assume there exists \( w^* \in I \) with \( f(w^*) = \pi_1 \). Then there would be a maximal point \( w_0 \in I \) with \( f(w_0) = \pi_1 \) and \( f(w) \in \Sigma \setminus \{ \pi_1, \pi_2 \} \) for \( w \in (w_0, w_0+\varepsilon) \subset I \), \( 0 < \varepsilon \ll 1 \). Consequently, the boundary condition (2.13) holds on \( (w_0, w_0+\varepsilon) \) and, in particular, we get

\[ (x_0 + Q(x) \wedge x_a) \cdot t(x) = 0 \quad \text{on} \quad (w_0, w_0+\varepsilon). \]  

(2.15)

By continuity, (2.15) remains valid for \( w = w_0 \). In addition, the geometry of \( S \) yields \( x_a = \pm |x_a|e_3 \). This and the relation \( n = t \wedge e_3 \) on \( S \) imply

\[ x_a \cdot t(x) = \pm |x_a| (Q(x) \cdot n(x)) \quad \text{in} \quad w_0. \]  

(2.16)

According to the conformality relations and \( \nabla x \neq 0 \) on \( I \), we have \( |x_a| = |x_a| \neq 0 \) in \( w_0 \). Denote the angle between \( x_a(w_0) \) and \( t(x(w_0)) \) by \( \beta_1 \). Then (2.16) and condition (2.10) imply

\[ |\cos \beta_1| = |(Q(x(w_0)) \cdot n(x(w_0)))| < \cos \alpha_1 \quad \text{or} \quad \beta_1 \in (\alpha_1, \pi - \alpha_1), \]

where \( \alpha_1 \in (0, \frac{\pi}{2}) \) denotes the interior angle between \( \Gamma \) and \( \Sigma \) at \( \pi_1 \) w.r.t. \( G \). A simple application of the mean-value theorem then yields a contradiction to the inclusion \( f(B^+) \subset \overline{G} \). Analogously, one shows that there cannot exist \( w^{**} \in I \) with \( f(w^{**}) = \pi_2 \). In conclusion, we have \( I' = I \) and (i) is proved.

We finally show the finiteness of branch points in \( B^+ \), completing the proof of (iii): Hildebrandt’s asymptotic expansions at interior branch points [Hi] imply the isolated character of these points. By \( \nabla x \neq 0 \) on \( I \cup J \), the only points where branch points could accumulate are the corner points \( w = \pm 1 \). But this is impossible, too, according to the asymptotic expansions near these points proven in [M4] Theorem 2.2; see Corollary 7.1 there. We emphasize that the cited result is applicable, since \( \Gamma \) and \( S \) meet with positive angles \( \gamma_j \in (0, \alpha_j) \) at \( p_j \) by Definition 1, and since we assume

\[ |Q(p_j) \cdot n(p_j)| < \cos \alpha_j \leq \cos \gamma_j, \quad j = 1, 2. \]

(Note that a simple reflection of \( S \) can be used to assure \( \{ \Gamma, S \} \) and \( x \) to fulfill the assumptions of [M4] Corollary 7.1.)
(iv) The interior regularity \( N \in C^{2,\alpha}(B^+,\mathbb{R}^3) \), \( KW \in C^{1,\alpha}(B^+) \) as well as equation (2.14) were proven by F. Sauvigny in [S1] Satz 1. The global regularity \( N \in C^{1,\alpha}(\overline{B^+} \setminus \{-1,1\},\mathbb{R}^3) \) follows from (i) and (iii). Finally, the continuity of \( N \) up to the corner points \( w = \pm 1 \) was proven in [M4] Theorem 5.4; see the remarks above concerning the applicability of this result.

\[ \square \]

**Remark 1.** By taking the cross product with \( x_u \in T_x S \), the natural boundary condition (2.13) can be written in the form

\[ Q(x) \cdot n(x) = -N \cdot n(x) \quad \text{on } I. \tag{2.17} \]

This relation describes the well-known fact that the normal component of \( Q \) w.r.t. to \( S \) prescribes the contact angle between a stationary \( \mathcal{H} \)-surface and the support surface \( S \).

### 3 The second variation of \( A_Q \), stable \( \mathcal{H} \)-surfaces

Let us choose an \( \mathcal{H} \)-surface \( x \in C_\mu(\Gamma,S;Z) \), \( \mu \in (0,1) \), which is stationary w.r.t. \( E_Q \) (and thus belongs to \( C^{3,\alpha}(B^+,Z) \cap C^2(\overline{B^+} \setminus \{-1,1\},Z) \) according to Lemma 2 (i)). Consider a one-parameter family \( \tilde{x} = \tilde{x}(w,\varepsilon) \), which belongs to the class \( C^\mu(\Gamma,S;Z) \cap C^2(\overline{B^+} \setminus \{-1,1\},\mathbb{R}^3) \) for any fixed \( \varepsilon \in (-\varepsilon_0,\varepsilon_0) \) and which depends smoothly on \( \varepsilon \) together with its first and second derivatives w.r.t. \( u,v \). We call \( \tilde{x} \) an **admissible perturbation** of \( x \), if we have:

(i) \( \tilde{x}(w,0) = x(w) \) for all \( w \in \overline{B^+} \),

(ii) \( \text{supp}(\tilde{x}(\cdot,\varepsilon) - x) \subset B^+ \cup I \) for all \( \varepsilon \in (-\varepsilon_0,\varepsilon_0) \),

(iii) \( y := \frac{\partial}{\partial \varepsilon} \tilde{x}(\cdot,\varepsilon) \big|_{\varepsilon=0} \in C^2(B^+ \cup I,\mathbb{R}^3) \), \( z := \frac{\partial^2}{\partial \varepsilon^2} \tilde{x}(\cdot,\varepsilon) \big|_{\varepsilon=0} \in C^1(B^+ \cup I,\mathbb{R}^3) \).

The direction \( y = \frac{\partial}{\partial \varepsilon} \tilde{x}(\cdot,\varepsilon) \big|_{\varepsilon=0} \) of an admissible perturbation \( \tilde{x} \) satisfies

\[ y(w) \in T_{x(w)} S \quad \text{for all } w \in I. \quad \quad \tag{3.1} \]

On the other hand, choosing an arbitrary vector-field \( y \in C^2_c(B^+ \cup I,\mathbb{R}^3) \) with the property (3.1), one may construct an admissible perturbation \( \tilde{x} \) as described above by using a flow argument (compare, e.g., [DHT] pp. 32–33).

In the present section, we compute the second variation \( \frac{\partial^2}{\partial \varepsilon^2} A_Q(\tilde{x}(\cdot,\varepsilon)) \big|_{\varepsilon=0} \) for admissible perturbations. To this end, we have to examine the quantity

\[ \frac{\partial^2}{\partial \varepsilon^2} \left( |\tilde{x}_u \wedge \tilde{x}_v| + Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right) \bigg|_{\varepsilon=0} = \frac{\partial^2}{\partial \varepsilon^2} \left( |\tilde{x}_u \wedge \tilde{x}_v| \right) \bigg|_{\varepsilon=0} + \frac{\partial^2}{\partial \varepsilon^2} \left( Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right) \bigg|_{\varepsilon=0} \tag{3.2} \]

We first compute (3.2) on \( B^+ \cup I \) with

\[ B^+ = \{ w \in B^+ : W(w) > 0 \}, \quad W = |x_u \wedge x_v| = |x_u|^2 = |x_v|^2, \]

and then observe that the resulting formula can be extended continuously to \( B^+ \cup I \). We start with the first addend on the right-hand side of (3.2):
Proposition 1. Let $\tilde{x}$ be an admissible perturbation of a stationary $H$-surface $x \in C_0(\Gamma, S, \mathbb{Z})$ as described above. Define $\varphi := y \cdot N \in C^2(B \cup I, \mathbb{R}^3)$. Then there holds

$$\frac{\partial^2}{\partial \varepsilon^2} (|\tilde{x}_u \wedge \tilde{x}_v|) = |\nabla \varphi|^2 + 2KW\varphi^2 - 2H(x)y \cdot (y_u \wedge x_v + x_u \wedge y_v)$$

$$+ 2H(x)[\varphi(x_u \cdot y_u) + \varphi(x_v \cdot y_v) + (x_u \cdot y)\varphi_u + (x_v \cdot y)\varphi_v]$$

$$- [\varphi(N_u + 2H(x)x_u) \cdot y]_u - [\varphi(N_v + 2H(x)x_v) \cdot y]_v$$

$$+ [N \cdot (y \wedge y_v)]_u + [N \cdot (y_u \wedge y)]_v$$

$$- 2H(x)z \cdot (x_u \wedge x_v) + (z \cdot x_u)_u + (z \cdot x_v)_v \quad \text{on } B',$$

where $K$ denotes the Gaussian curvature of $x$.

Proof. 1. We start by noting the relation

$$\frac{\partial^2}{\partial \varepsilon^2} (|\tilde{x}_u \wedge \tilde{x}_v|)_{|\varepsilon=0} = \frac{1}{2W} \frac{\partial^2}{\partial \varepsilon^2} (|\tilde{x}_u \wedge \tilde{x}_v|^2)_{|\varepsilon=0} - \frac{1}{4W^2} \left( \frac{\partial}{\partial \varepsilon} (|\tilde{x}_u \wedge \tilde{x}_v|^2) \right)^2_{|\varepsilon=0}$$

on $B'$. Expanding $\tilde{x}$ w.r.t. $\varepsilon$, we infer

$$\tilde{x} (\cdot, \varepsilon) = x + \varepsilon y + \frac{\varepsilon^2}{2} z + o(\varepsilon^2) \quad \text{on } B^+$$

and, consequently,

$$\tilde{x}_u \wedge \tilde{x}_v = W N + \varepsilon (x_u \wedge y_v + y_u \wedge x_v) + \varepsilon^2 y_u \wedge y_v + \frac{\varepsilon^2}{2} (x_u \wedge z_v + z_u \wedge x_v) + o(\varepsilon^2) \quad \text{on } B'$$

as well as

$$|\tilde{x}_u \wedge \tilde{x}_v|^2 = W^2 + 2\varepsilon W N \cdot (x_u \wedge y_v + y_u \wedge x_v)$$

$$+ \varepsilon^2 |x_u \wedge y_v + y_u \wedge x_v|^2 + 2\varepsilon^2 W N \cdot (y_u \wedge y_v)$$

$$+ \varepsilon^2 W N \cdot (x_u \wedge z_v + z_u \wedge x_v) + o(\varepsilon^2).$$

Combining (3.3) with (3.6) gives

$$\frac{\partial^2}{\partial \varepsilon^2} (|\tilde{x}_u \wedge \tilde{x}_v|)_{|\varepsilon=0} = W^{-1} |x_u \wedge y_v + y_u \wedge x_v|^2 + 2N \cdot (y_u \wedge y_v)$$

$$+ N \cdot (x_u \wedge z_v + z_u \wedge x_v)$$

$$- W^{-1} \left[ N \cdot (x_u \wedge y_v + y_u \wedge x_v) \right]^2$$

$$= (y_u \cdot N)^2 + (y_v \cdot N)^2 + 2N \cdot (y_u \wedge y_v)$$

$$+ N \cdot (x_u \wedge z_v + z_u \wedge x_v).$$

And since $x$ is a conformally parametrized $H$-surface, we have

$$N \cdot (x_u \wedge z_v + z_u \wedge x_v) = z_v \cdot x_u + z_u \cdot x_v$$

$$= (z \cdot x_u)_u + (z \cdot x_v)_v - 2H(x)Wz \cdot N \quad \text{on } B'.$$
arriving at
\[
\frac{\partial^2}{\partial \varepsilon^2} ((\mathbf{x}_u \wedge \mathbf{x}_v)) \bigg|_{\varepsilon=0} = (\mathbf{y}_u \cdot \mathbf{N})^2 + (\mathbf{y}_v \cdot \mathbf{N})^2 + 2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) + (z \cdot x_u)_u + (z \cdot x_u)_v - 2\mathcal{H}(x)Wz \cdot \mathbf{N} \quad \text{on } B'.
\] (3.7)

2. In the following, we sometimes write \( u^1 := u, u^2 := v \) and use Einstein's convention summing up tacitly over sub- and superscript latin indices from 1 to 2. Furthermore, we set \( \lambda^j := W^{-1}x_{u^j} \cdot y \) for \( j = 1, 2 \) obtaining \( y = \lambda^j x_{u^j} + \varphi \mathbf{N} \quad \text{on } B' \).

Writing \( g_{jk} := x_{u^j} \cdot x_{u^k}, g_{jk}, \Gamma^l_{jk} \), and \( h_{jk} := x_{u^j} \cdot \mathbf{N} = -x_{u^j} \cdot \mathbf{N} \) for the coefficients of the first fundamental form, its inverse and Christoffel symbols, and the coefficients of the second fundamental form, respectively, we then infer
\[
y_{u^k} = (\lambda^j_{u^k} + \lambda^l \Gamma^j_{lk} - \varphi h_{kl}g^{lj})x_{u^j} + (\lambda^1 h_{jk} + \varphi_{u^k})\mathbf{N} \quad \text{on } B'.
\] (3.8)

Due to the conformal parametrization of the \( \mathcal{H} \)-surface \( x \), we have
\[
g_{jk} = W\delta_{jk}, \quad g_{jk} = \frac{\delta_{jk}}{W},
\]
\[
\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \frac{W_u}{2W},
\]
\[
\Gamma^2_{22} = -\Gamma^1_{11} = \Gamma^1_{21} = \Gamma^1_{12} = \frac{W_v}{2W},
\]
\[
h_{11} + h_{22} = 2W\mathcal{H}(x), \quad h_{11}h_{22} - (h_{12})^2 = W^2K \quad \text{on } B',
\]
where \( \delta_{jk} = \delta^{jk} \) denotes the Kronecker delta.

3. We now evaluate the first line of the right-hand side in (3.7): Using (3.8) and (3.9), the first two terms can be written as
\[
(\mathbf{y}_u \cdot \mathbf{N})^2 + (\mathbf{y}_v \cdot \mathbf{N})^2 = (\lambda^1 h_{11} + \lambda^2 h_{12} + \varphi_u)^2 + (\lambda^1 h_{12} + \lambda^2 h_{22} + \varphi_v)^2
\]
\[
= |\nabla \varphi|^2 + [(\lambda^1)^2 + (\lambda^2)^2](h_{12})^2 + (\lambda^1)^2(h_{11})^2 + (\lambda^2)^2(h_{22})^2
\]
\[
+4\lambda^1\lambda^2 h_{12} W\mathcal{H}(x) + 4(\lambda^1 \varphi_u + \lambda^2 \varphi_v) W\mathcal{H}(x)
\]
\[
+2(\lambda^2 h_{12} - \lambda^1 h_{22}) \varphi_u + 2(\lambda^1 h_{12} - \lambda^2 h_{11}) \varphi_v \quad \text{on } B'.
\] (3.10)

We next write the third term on the right-hand side of (3.7) as
\[
2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) = [\mathbf{N} \cdot (\mathbf{y} \wedge \mathbf{y}_v)]_u + [\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y})]_v
\]
\[
- \mathbf{N}_u \cdot (\mathbf{y} \wedge \mathbf{y}_v) - \mathbf{N}_v \cdot (\mathbf{y}_u \wedge \mathbf{y}) \quad \text{on } B'.
\] (3.11)

Using the relations \( \mathbf{N} \wedge \mathbf{x}_u = \mathbf{x}_v, \mathbf{N} \wedge \mathbf{x}_v = -\mathbf{x}_u \), we get from (3.8):
\[
\mathbf{y} \wedge \mathbf{y}_{u^k} = -\varphi(\lambda^1_{u^k} + \lambda^1 \Gamma^1_{1k} + \lambda^2 \Gamma^2_{2k} - \varphi h_{k2}W^{-1})\mathbf{x}_u
\]
\[
+ \varphi(\lambda^1_{u^k} + \lambda^1 \Gamma^1_{1k} + \lambda^2 \Gamma^2_{2k} - \varphi h_{k1}W^{-1})\mathbf{x}_v
\]
\[
+ \lambda^2(\lambda^1 h_{1k} + \lambda^2 h_{2k} + \varphi_{u^k})\mathbf{x}_u
\]
\[
- \lambda^1(\lambda^1 h_{1k} + \lambda^2 h_{2k} + \varphi_{u^k})\mathbf{x}_v + (\ldots)\mathbf{N} \quad \text{on } B',
\]
where \((\ldots)\mathbf{N}\) denotes the normal part of \(\mathbf{y} \wedge \mathbf{y}_\nu\). This identity, formula (3.9), and the Weingarten equations \(\mathbf{N}_\nu = -h_{jk}g^{kl}\mathbf{x}_\nu\) on \(B'\) yield
\[-\mathbf{N}_u \cdot (\mathbf{y} \wedge \mathbf{y}_u) - \mathbf{N}_v \cdot (\mathbf{y}_u \wedge \mathbf{y}) = W^{-1}\left[(h_{11}\mathbf{x}_u + h_{12}\mathbf{x}_u) \cdot (\mathbf{y} \wedge \mathbf{y}_v) - (h_{21}\mathbf{x}_u + h_{22}\mathbf{x}_u) \cdot (\mathbf{y} \wedge \mathbf{y}_u)\right]
\]
\[= 2(\varphi)^2WK + (\lambda_1^1h_{22} - \lambda^2h_{12})\varphi_u - (\lambda^1h_{11} - \lambda_2h_{11})\varphi_v,
\]
\[+ \varphi[\lambda_1^1h_{12} - \lambda^1h_{22} - \lambda^1W_u\mathbf{H}(\mathbf{x})] - \varphi[\lambda_2^1h_{11} - \lambda_2^2h_{12} + \lambda^2W_v\mathbf{H}(\mathbf{x})]
\]
\[+ [(\lambda_1)^2 + (\lambda_2)^2][h_{11}h_{22} - (h_{12})^2]\quad\text{on } B'.
\]
According to the Codazzi-Mainardi equations
\[h_{21,u} - h_{22,u} + W_u\mathbf{H} = 0, \quad h_{11,u} - h_{12,u} - W_v\mathbf{H} = 0,
\]
we infer
\[
\begin{align*}
\lambda_1^1h_{12} - \lambda^1h_{22} &= \lambda^1W_u\mathbf{H}(\mathbf{x}) = (\lambda^1h_{12})_v - (\lambda^1h_{22})_u, \\
\lambda_2^2h_{11} - \lambda_2^2h_{12} &= \lambda^2W_v\mathbf{H}(\mathbf{x}) = (\lambda^2h_{11})_v - (\lambda^2h_{12})_u \quad\text{on } B'.
\end{align*}
\]
Inserting these identities into (3.12) and the resulting relation into (3.11), we arrive at
\[2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v) = [\mathbf{N} \cdot (\mathbf{y} \wedge \mathbf{y}_v)]_u + [\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y})]_v
\]
\[+ 2(\varphi)^2WK + (\lambda_1^1h_{22} - \lambda^2h_{12})\varphi_u - (\lambda^1h_{11} - \lambda_2h_{11})\varphi_v
\]
\[- \varphi(\lambda_1^1h_{22} - \lambda^2h_{12})_u + \varphi(\lambda^1h_{11} - \lambda^2h_{11})_v
\]
\[+ [(\lambda_1)^2 + (\lambda_2)^2][h_{11}h_{22} - (h_{12})^2]\quad\text{on } B'.
\]
Adding (3.10) and (3.13) we now find
\[\mathbf{y}_u \cdot \mathbf{N}^2 + (\mathbf{y}_u \cdot \mathbf{N})^2 + 2\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y}_v)
\]
\[= [\nabla\varphi]^2 + 2(\varphi)^2KW + [\mathbf{N} \cdot (\mathbf{y} \wedge \mathbf{y}_v)]_u + [\mathbf{N} \cdot (\mathbf{y}_u \wedge \mathbf{y})]_v
\]
\[- [\varphi(\lambda^1h_{22} - \lambda^2h_{12})]_u + [\varphi(\lambda^1h_{11} - \lambda^2h_{11})]_v
\]
\[+ 2W\mathbf{H}(\mathbf{x})[(\lambda_1)^2h_{11} + (\lambda_2)^2h_{22} + 2\lambda^1\lambda^2h_{12} + 2(\lambda^1\varphi_u + \lambda^2h_{22})]
\]
(3.14)
on \(B'\). Finally, we calculate via the Weingarten equations and (3.9)
\[
\begin{align*}
\lambda^1h_{22} - \lambda^2h_{12} &= W^{-1}(h_{22}\mathbf{x}_u - h_{12}\mathbf{x}_u) \cdot \mathbf{y} = (\mathbf{N}_u + 2\mathbf{H}(\mathbf{x})\mathbf{x}_u) \cdot \mathbf{y}, \\
\lambda^1h_{12} - \lambda^2h_{11} &= W^{-1}(h_{12}\mathbf{x}_u - h_{11}\mathbf{x}_u) \cdot \mathbf{y} = -(\mathbf{N}_v + 2\mathbf{H}(\mathbf{x})\mathbf{x}_u) \cdot \mathbf{y}
\end{align*}
\]
as well as
\[
(\lambda^1)^2 h_{11} + (\lambda^2)^2 h_{22} + 2\lambda^1 \lambda^2 h_{12} + 2(\lambda^1 \varphi_u + \lambda^2 \varphi_v)
\]
\[
= \lambda^1(\lambda^1 h_{11} + \lambda^2 h_{12}) + \lambda^2(\lambda^1 h_{12} + \lambda^2 h_{22}) + 2(\lambda^1 \varphi_u + \lambda^2 \varphi_v)
\]
\[
= -\lambda^1(N_u \cdot y - \lambda^2(N_v \cdot y) + 2(\lambda^1 \varphi_u + \lambda^2 \varphi_v)
\]
\[
= W^{-1}[(x_u \cdot y)(N \cdot y_u) + (x_v \cdot y)(N \cdot y_v)] + (\lambda^1 \varphi_u + \lambda^2 \varphi_v)
\]
\[
= W^{-1}[(\varphi(x_u \cdot y_u) + \varphi(x_v \cdot x_v) + (x_u \cdot y)\varphi_u + (x_v \cdot y)\varphi_v]
\]
\[
- W^{-1}[y \cdot (y_u \wedge x_u) + y \cdot (x_u \wedge y_u)].
\]
(3.16)

Inserting (3.15) and (3.16) into (3.14), the asserted identity follows from the resulting relation and formula (3.7).

Proof. Using (2.4) and the general relation
\[
[Ma] \cdot (b \wedge c) + a \cdot ([Mb] \wedge c) + a \cdot (b \wedge [Mc]) = (\text{tr} M)[a \cdot (b \wedge c)]
\]
(3.17)
for arbitrary vectors \(a, b, c \in \mathbb{R}^3\) and matrices \(M \in \mathbb{R}^{3 \times 3}\) with trace \(\text{tr} M\), we first compute
\[
\partial^2 \|Q(\tilde{x}) \cdot (\tilde{x}_u \wedge \tilde{x}_v)\|_{\varepsilon=0}
\]
\[
= 2W\varphi^2[\nabla H(x) \cdot N - 2H(x)^2] + 2H(x)y \cdot (x_u \wedge y_u + y_u \wedge x_u)
\]
\[
- 2H(x)[\varphi(x_u \cdot y_u) + \varphi(x_v \cdot y_v) + (x_u \cdot y)\varphi_u + (x_v \cdot y)\varphi_v]
\]
\[
+ 2[\varphi H(x)(x_u \cdot y)_v] + 2[\varphi H(x)(x_v \cdot y)]_v + 2H(x)z \cdot (x_u \wedge x_v)
\]
\[
+ [(DQ(x)y) \cdot (y \wedge x_v)]_v + [Q(x) \cdot (x_u \wedge y)]_v + [Q(x) \cdot (y \wedge y)]_v
\]
on \(B'\).

Proposition 2. Under the assumptions of Proposition 1, there holds
\[
\frac{\partial^2}{\partial \varepsilon^2} [Q(\tilde{x}) \cdot (\tilde{x}_u \wedge \tilde{x}_v)]_{\varepsilon=0}
\]
\[
= 2W\varphi^2[\nabla H(x) \cdot N - 2H(x)^2] + 2H(x)y \cdot (x_u \wedge y_u + y_u \wedge x_u)
\]
\[
+ 2H(x)y \cdot (x_u \wedge y_v + y_v \wedge x_u)
\]
\[
- 2H(x)[\varphi(x_u \cdot y_u) + \varphi(x_v \cdot y_v) + (x_u \cdot y)\varphi_u + (x_v \cdot y)\varphi_v]
\]
\[
+ 2[\varphi H(x)(x_u \cdot y)_v] + 2[\varphi H(x)(x_v \cdot y)]_v + 2H(x)z \cdot (x_u \wedge x_v)
\]
\[
+ [(DQ(x)y) \cdot (y \wedge x_v)]_v + [Q(x) \cdot (x_u \wedge y)]_v + [Q(x) \cdot (y \wedge y)]_v
\]
on \(B'\).
Writing again $y = \lambda' x_{ij} + \varphi N$ on $B'$ with $\lambda' = W^{-1} x_{ij} \cdot y$ and employing (1.1), the assertion follows from (3.18) and the identity

$$2[\nabla H(x) \cdot y] y \cdot (x_u \wedge x_v)$$

$$= 2W \varphi^2 \nabla H(x) \cdot N + 2\varphi \lambda' W H(x)_{uv}$$

$$= 2W \varphi^2 \nabla H(x) \cdot N + 2[\varphi H(x)(x_u \cdot y)]_u + 2[\varphi H(x)(x_v \cdot y)]_v$$

$$- 2H(x)[\varphi(x_u \cdot y_u) + \varphi(x_v \cdot y_v) + (x_u \cdot y)\varphi_u + (x_v \cdot y)\varphi_v]$$

$$- 4W \varphi^2 H(x)^2.$$  

As already announced, the right-hand sides in the results of Propositions 1 and 2 can be extended continuously onto $B^+ \cup I$, according to Lemma 2. Hence we can compute the second variation via the divergence theorem for any admissible one-parameter family $\tilde{x}(\cdot, \varepsilon)$ with direction $y \in C^2_0(B^+ \cup I, \mathbb{R}^3)$ satisfying (3.1). Nevertheless, we concentrate on directions of the form

$$y(w) := \frac{\varphi(w)}{1 + Q(x(w)) \cdot N(w)} [Q(x(w)) + N(w)], \quad (3.19)$$

with some function $\varphi \in C^2_0(B^+ \cup I)$. Note that $y$ is well-defined according to assumption (2.4), belongs to $C^2_0(B^+ \cup I, \mathbb{R}^3)$, and satisfies $y \cdot N \equiv \varphi$ as well as (3.1); for the latter, see Remark 1.

**Definition 3.** For given $\varphi \in C^2_0(B^+ \cup I)$ we define $y \in C^2_0(B^+ \cup I, \mathbb{R}^3)$ by (3.19) and consider the admissible perturbation $\tilde{x}(\cdot, \varepsilon)$ with direction $y$. Then we set

$$\delta^2 A_Q(x, \varphi) := \left. \frac{d^2}{d\varepsilon^2} A_Q(\tilde{x}(\cdot, \varepsilon)) \right|_{\varepsilon = 0}$$

for the second variation of $A_Q(x)$ with dilation $\varphi$.

In order to compute $\delta^2 A_Q(x, \varphi)$, we introduce the curvature of the cylindrical support surface $S$ defined by

$$\kappa(p) := -(a''(s), 0) \cdot n(p) \quad \text{for } p \in \{a(s)\} \times \mathbb{R}, \ s \in [0, s_0], \quad (3.20)$$

compare Section 2. Note that, due to the cylindrical structure of $S$, we have the relation

$$[Dn(p)\zeta_1] \cdot \zeta_2 = \kappa(p) [\zeta_1 \cdot t(p)] [\zeta_2 \cdot t(p)] \text{ for all } \zeta_1, \zeta_2 \in T_p S, \ p \in S, \quad (3.21)$$

interpreting $Dn$ as the Weingarten map of $S$.

**Lemma 3.** Let $x \in C_\mu(\Gamma, S; \mathbb{Z})$, $\mu \in (0, 1)$, be a stationary $\mathcal{H}$-surface w.r.t. $E_Q$ and let $\varphi \in C^2_0(B^+ \cup I)$ be chosen. Setting

$$q(w) := [2H(x(w))^2 - K(w) - \nabla H(x(w)) \cdot N(w)] W(w), \quad w \in B^+ \cup I, \quad (3.22)$$
we then have
\[
\delta^2 A_Q(x, \varphi) = \int_{B^+} \left\{ \frac{\partial}{\partial \varepsilon^2} \left[ (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right] \right\}_{\varepsilon=0} \, dv \, du
+ \int_{B^+} \left\{ \frac{\partial}{\partial \varepsilon} \left[ (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right] \right\}_{\varepsilon=0} \, dv \, du
\]
\[
+ \int_I \left\{ \frac{\partial}{\partial \varepsilon} \left[ (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right] \right\}_{\varepsilon=0} \, dv \, du
\]
\[
+ \int_I \left\{ \left[ \frac{\partial^2}{\partial x^2} (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot (\tilde{x}_u \wedge \tilde{x}_v) \right] \right\}_{\varepsilon=0} \, dv \, du.
\]
(3.23)

Proof. We add the results of Propositions 1 and 2 obtaining
\[
\delta^2 A_Q(x, \varphi) = \int_{B^+} \left\{ \frac{\partial}{\partial \varepsilon} \left[ (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right] \right\}_{\varepsilon=0} \, dv \, du
+ \int_I \left\{ \frac{\partial}{\partial \varepsilon} \left[ (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot \tilde{x}_u \wedge \tilde{x}_v \right] \right\}_{\varepsilon=0} \, dv \, du
\]
\[
+ \int_I \left\{ \left[ \frac{\partial^2}{\partial x^2} (\tilde{x}_u \wedge \tilde{x}_v) + Q(\tilde{x}) \cdot (\tilde{x}_u \wedge \tilde{x}_v) \right] \right\}_{\varepsilon=0} \, dv \, du.
\]
(3.24)

Due to the special choice (3.19) of \( y \), the first three terms on the right-hand side of (3.24) are identical with those in the announced relation (3.23). In order to identify the fourth terms of (3.23) and (3.24), we recall Lemma 2 (i) and deduce
\[
z \cdot (x_v + Q(x) \wedge x_u) = (z \cdot n(x)) \left[ (x_v + Q(x) \wedge x_u) \cdot n(x) \right] \quad \text{on } I.
\]
(3.25)

Similar to [HS3] p. 431, we compute \( z \cdot n(x) \) on \( I \): Since \( \tilde{x}(w, \varepsilon) \in S \) holds for all \( w \in I \) and \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), we have \( \frac{\partial}{\partial \varepsilon} \tilde{x}(w, \varepsilon) \cdot n(\tilde{x}(w, \varepsilon)) = 0 \) and, consequently,
\[
\frac{\partial^2}{\partial x^2} \tilde{x}(w, \varepsilon) \cdot n(\tilde{x}(w, \varepsilon)) + \frac{\partial}{\partial \varepsilon} \tilde{x}(w, \varepsilon) \cdot [Dn(\tilde{x}(w, \varepsilon)) \frac{\partial}{\partial \varepsilon} \tilde{x}(w, \varepsilon)] = 0
\]
for \( w \in I \) and \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \). For \( \varepsilon = 0 \) we employ (3.21) and infer
\[
z \cdot n(x) = -\kappa(x) [y \cdot t(x)]^2 \quad \text{on } I.
\]
Together with (3.25), we arrive at
\[
z \cdot (x_v + Q(x) \wedge x_u) = -\kappa(x) [(x_v + Q(x) \wedge x_u) \cdot n(x)] [y \cdot t(x)]^2 \quad \text{on } I.
\]
Putting this relation into (3.24), proves the assertion. □
By a standard approximation argument, dilations $\varphi \in H^1_2(B^+) \cap C_0^\infty(B^+ \cup I)$ are admissible in the second variation $\delta^2 A_Q(x, \varphi)$ due to formula (3.23).

Definition 4. A partially free $H$-surface $x \in C_0(\Gamma, S; \mathbb{Z})$ with $\delta^2 A_Q(x, \varphi) \geq 0$ for any dilation $\varphi \in H^1_2(B^+) \cap C_0^\infty(B^+)$ is called stable.

4 Boundary condition for the surface normal and proof of the theorem

In order to deduce the crucial relation $N^3 > 0$ on $\overline{B^+}$ for the third component of the surface normal of our stable $H$-surface, we will combine formula (3.23) with the following boundary condition:

Lemma 4. Let the assumptions of Theorem 1 be satisfied and let a stationary $H$-surface $x \in C_0(\Gamma, S; \mathbb{Z})$, $\mu \in (0, 1)$, be given. Then, the third component $N^3$ of the surface normal of $x$ fulfills the boundary condition

$$N^3_u = \left\{ \frac{N_u \cdot Q(x)}{1 + Q(x) \cdot N} + \frac{[DQ(x)(Q(x) + N)] \cdot [x_u + Q(x) \wedge x_u]}{(1 + Q(x) \cdot N)^2} + \kappa(x) \frac{[(x_u + Q(x) \wedge x_u) \cdot n(x)] \cdot [(Q(x) + N) \cdot t(x)]^2}{(1 + Q(x) \cdot N)^2} \right\} N^3 \quad \text{on } I,$$

(4.1)

where $t$, $n$, and $\kappa$ were defined in (2.1), (3.20).

Proof.

1. From (1.1) and Lemma 2 (iv) we get the well known relations

$$N_u = N \wedge N_u - 2H(x)x_u, \quad N_v = -N \wedge N_u - 2H(x)x_v \quad \text{on } B^+ \cup I.$$  

(4.2)

Writing $H = H(x)$, $Q = Q(x)$, $\kappa = \kappa(x)$ etc. and employing (4.2) as well as (2.17), we compute

$$\langle N_u \cdot Q \rangle N^3 = \left\{ [(Q + N) \cdot N]N \right\} \cdot e_3$$

$$= -\left\{(N \wedge N_v) \wedge (Q + N) - [N \cdot (Q + N)]N_v \right\} \cdot e_3$$

$$= -\left\{N_u \wedge (Q + N) + 2Hx_u \wedge (Q + N) - [1 + (Q \cdot N)]N_v \right\} \cdot e_3$$

$$= (N \wedge e_3)_u \cdot (Q + N) + [1 + (Q \cdot N)]N^3_v \quad \text{on } I.$$

Consequently, the asserted relation (4.1) is equivalent to the identity

$$\langle N \wedge e_3 \rangle_u \cdot (Q + N) = -\left\{ [DQ(Q + N)] \cdot (x_v + Q \wedge x_u)$$

$$+ \kappa[(x_u + Q \wedge x_u) \cdot n][Q + N] \cdot t]^2 \right\} \frac{N^3}{1 + Q \cdot N} \quad \text{on } I.$$  

(4.3)
2. Next, we manipulate the left-hand side of (4.3): Having (2.17) in mind, we find

\[(Q + N) \wedge e_3 = (Q + N) \wedge (n \wedge t) = [(Q + N) \cdot t]n\] on \(I\).

Together with (3.21), we infer

\[\begin{align*}
[(Q + N) \wedge e_3]u \cdot (Q + N) &= [(Q + N) \cdot t] \{[(Dn)x_u] \cdot (Q + N)\} \\
&= \kappa [(Q + N) \cdot t]^2 (x_u \cdot t)\] on \(I\). 
\end{align*}\]

(4.4)

On the other hand, we calculate

\[\begin{align*}
(x_u \cdot t)(1 + Q \cdot N) &= (x_u \cdot t)[N \cdot (Q + N)] \\
&= [x_u \wedge (Q + N)] \cdot (t \wedge N) - (x_u \cdot N) [t \cdot (Q + N)] \\
&= \{[x_u \wedge (Q + N)] \cdot n\} [n \cdot (t \wedge N)] \\
&= -[(x_u + Q \wedge x_u) \cdot n] N^3 \] on \(I\).
\end{align*}\]

or, equivalently,

\[x_u \cdot t = -\frac{N^3}{1 + Q \cdot N} [(x_u + Q \wedge x_u) \cdot n]\] on \(I\). (4.5)

From (4.4) and (4.5) we now deduce

\[\begin{align*}
(N \wedge e_3)u \cdot (Q + N) &= [(Q + N) \wedge e_3]u \cdot (Q + N) - (Q \wedge e_3)u \cdot (Q + N) \\
&= -\kappa [(x_u + Q \wedge x_u) \cdot n] [(Q + N) \cdot t]^2 \frac{N^3}{1 + Q \cdot N} \\
&= -(Q \wedge e_3)u \cdot (Q + N)\] on \(I\). (4.6)

By inserting (4.6) into (4.3), the claimed relation (4.1) becomes equivalent to

\[(Q \wedge e_3)u \cdot (Q + N) = [(DQ)(Q + N)] \cdot (x_u + Q \wedge x_u) \frac{N^3}{1 + Q \cdot N}\] on \(I\). (4.7)

3. In the next step, we observe that (4.7) is equivalent to the identity

\[\begin{align*}
[(DQ)x_u] \cdot [e_3 \wedge (Q + N)] + x_u \cdot \{e_3 \wedge [(DQ)(Q + N)]\} &= 0\] on \(I\). (4.8)
\end{align*}\n
Indeed, the left hand side of (4.7) can be written as

\[\begin{align*}
(Q \wedge e_3)u \cdot (Q + N) &= \{[(DQ)x_u] \wedge e_3\} \cdot (Q + N) = [(DQ)x_u] \cdot [e_3 \wedge (Q + N)],
\end{align*}\n
whereas we compute in the right hand side

\[\begin{align*}
[(DQ)(Q + N)] \cdot (x_u + Q \wedge x_u) N^3 \\
&= [(x_u + Q \wedge x_u) \wedge N] \cdot \{[(DQ)(Q + N)] \wedge e_3\} \\
&= (1 + Q \cdot N) x_u \cdot \{[(DQ)(Q + N)] \wedge e_3\}\] on \(I\).
\end{align*}\n
This proves the claimed equivalence.
4. It remains to prove (4.8). Applying the relation (3.17) with $a = x_u$, $b = e_3$, $c = Q + N$, and $M = DQ$, we obtain

$$[(DQ)x_u] \cdot [e_3 \wedge (Q + N)] + x_u \cdot [e_3 \wedge [(DQ)(Q + N)]]$$

$$= -x_u \cdot \{[(DQ)e_3] \wedge (Q + N)\} + (\text{tr} DQ) \{x_u \cdot [e_3 \wedge (Q + N)]\}$$

$$= [(DQ)e_3] \cdot [x_u \wedge (Q + N)] \text{ on } I,$$

where we also used $Q + N \parallel T_x S$. For the same reason, $x_u \wedge (Q + N)$ is normal to $S$ along $I$ and, as a consequence, the right hand side of the above identity vanishes. Indeed, we have

$$[DQ(p)e_3] \cdot n(p) = \left[\frac{\partial}{\partial p^3} Q(p)\right] \cdot n(p) = \frac{\partial}{\partial p^3} [Q(p) \cdot n(p)] = 0 \text{ on } S,$$

by assumption. This completes the proof of (4.8), and (4.1) is confirmed.

q.e.d.

We are now able to give the

**Proof of Theorem 1.** 1. According to Lemma 2 (iv), the surface normal $N = (N^1, N^2, N^3)$ of $x$ belongs to $C^{2,\alpha}(B^+) \cap C^{1,\alpha}(\overline{B^+} \setminus \{-1, +1\}) \cap C^0(\overline{B^+})$. In addition, the inclusion $f(B) \subset \overline{C}$ and the $\frac{1}{2}$-convexity of $G$ imply $N^3 > 0$ on $J \setminus \{-1, +1\}$ as was shown in [S1] Satz 2. The behaviour of the surface normal near the corner points $\pm 1$ was studied in [M4] Theorem 5.4; the applicability of the cited result follows – after reflecting $S$ and rotating appropriately in $\mathbb{R}^3$ – from the assumption $|(Q \cdot n)(p)| < \cos \alpha_j \leq \cos \gamma_j$ for $j = 1, 2$, where $\gamma_j$ denote the angles between $\Gamma$ and $S$ at $p_j$ ($j = 1, 2$). In particular, $N^3(\pm 1)$ cannot vanish and, by continuity, we infer $N^3(\pm 1) > 0$. Consequently, the dilation $\omega := (N^3)^{-} = \max\{0, -N^3\} \in C^0(B^+ \cup I) \cap H^1(B^+)$ is admissible in the second variation of $A_Q(x)$. Writing $\omega^2 = -\omega N^3$ and $|\nabla \omega|^2 = -\nabla \omega \cdot \nabla N^3$, we obtain from Lemmas 3 and 4:

$$\delta^2 A_Q(x, \omega) = \iint_{B^+} (|\nabla \omega|^2 - 2q\omega^3) \, du \, dv - \int_I \omega N^3 \, du$$

$$= -\iint_{B^+} \{\text{div}(\omega(\nabla N^3) + \omega(\Delta N^3 + 2q N^3))\} \, du \, dv - \int_I \omega N^3 \, du$$

$$= \iint_{B^+} (\omega(\Delta N^3 + 2q N^3)) \, du \, dv = -2 \iiint_{B^+} \omega \mathcal{H}(x) W \, du \, dv \leq 0,$$

where we have applied Gauss’ theorem, equation (2.14), and assumption (2.9) in the last line. The stability of $x$ thus yields $\delta^2 A_Q(x, \omega) = 0$.

2. Now we choose $\xi \in C^\infty_c(B^+)$ arbitrarily. Then also $\omega + \varepsilon \xi$ is admissible in $\delta^2 A_Q(x, \cdot)$ for any $\varepsilon \in \mathbb{R}$. The function $\Xi(\varepsilon) := \delta^2 A_Q(x, \omega + \varepsilon \xi)$ depends
smoothly on $\varepsilon \in \mathbb{R}$ and satisfies $\Xi \geq 0$ as well as $\Xi(0) = 0$. Consequently, we have $\Xi'(0) = 0$, which means

$$\int_B \{\nabla \omega \cdot \nabla \xi - 2q\omega \xi\} \, du \, dv = 0 \quad \text{for any } \xi \in C^\infty_c(B^+) ,$$

due to formula (3.23). From $\omega = 0$ near $J$, we conclude $\omega \equiv 0$ by means of the weak Harnack inequality. Hence, we have $N^3 \geq 0$ in $\overline{B}^+$. Due to assumption (2.9) and equation (2.14), we further have $\Delta N^3 + 2qN^3 \leq 0$ in $B^+$. Therefore, Harnack’s inequality, in conjunction with $N^3 > 0$ near $J$, yields $N^3 > 0$ in $B^+ \cup J$. Finally, we have $N^3 > 0$ on $I$ and hence everywhere on the closed half disc $\overline{B}^+$. Indeed, if $N^3(w_0) = 0$ would be true for some point $w_0 \in I$, relation (4.1) would imply $N^3(0) = 0$. But this is impossible due to Hopf’s boundary point lemma.

3. Since we have no branch points on $\partial B^+ \setminus \{-1, +1\}$ according to Lemma 2 (iii), the relation $N^3 > 0$ on $\partial B^+$ implies $x_4^2 x_6^2 - x_5^2 x_7^2 > 0$ on $\partial B^+ \setminus \{-1, +1\}$. Consequently, the projection $f = \pi(x) = (x^1, x^2) : \overline{B}^+ \to \mathbb{R}^2$ maps $\partial B^+$ topologically and positively oriented onto $\partial G$. As in [S1] Hilfsatz 7, an index argument now shows that $f : \overline{B}^+ \to \overline{G}$ is a homeomorphism, $x$ has no branch points in $\overline{B}^+$, and $J_f > 0$ is satisfied in $\overline{B}^+ \setminus \{-1, +1\}$. By the inverse mapping theorem and the regularity of $x$, the mapping $f : \overline{G} \to \overline{B}^+$ belongs to $C^2(\overline{G} \setminus \{p_1, p_2\}) \cap C^0(\overline{G})$, where we abbreviated $p_j = \pi(p_j^0)$, $j = 1, 2$.

Now we consider $\zeta := x^3 \circ f^{-1} \in C^2(\overline{G} \setminus \{p_1, p_2\}) \cap C^0(\overline{G})$. Since we have $(x^1, x^2, \zeta(x^1, x^2)) = x \circ f^{-1}(x^1, x^2)$, $\zeta$ is the desired graph representation over $\overline{G}$ satisfying the differential equation (2.11) and the second boundary condition in (2.12). In addition, we compute

$$\psi(x) = Q(x) \cdot \mathbf{n}(x) \quad \overset{(2.17)}{=} \quad -\mathbf{N} \cdot \mathbf{n}(x)$$

$$= \frac{1}{\sqrt{1 + |\nabla \zeta|^2}} (\zeta_1, \zeta_2, -1) \cdot (\nu(x), 0)$$

$$= \frac{\nabla \zeta \cdot \nu(x)}{\sqrt{1 + |\nabla \zeta|^2}} , \quad x = (x^1, x^2, \zeta(x^1, x^2)), \quad (x^1, x^2) \in \Sigma .$$

Hence, $\zeta$ is a solution of the boundary value problem (2.11), (2.12), and standard elliptic theory yields $\zeta \in C^{3, \alpha}(G) \cap C^{2, \alpha}(\overline{G} \setminus \{p_1, p_2\})$ according to the regularity assumptions on $Q$, $\mathcal{H}$, $S$, and $\Gamma$. This completes the proof.

We finally give an example of how to apply Theorem 1 to the existence question for the mixed boundary value problem (2.11), (2.12).

**Corollary 2.** Let $G \subset B_R := \{(x^1, x^2) \in \mathbb{R}^2 : |(x^1, x^2)| < R\}$ be a \(\frac{1}{2}\)-convex domain with boundary $\partial G = \Gamma \cup \Sigma$, where $\Gamma, \Sigma \in C^2$ are closed Jordan arcs, which satisfy $\Gamma \cap \Sigma = \{\pi_1, \pi_2\}$ and which meet with interior angles $\alpha_j \in (0, \frac{\pi}{2}]$ w.r.t. $G$ at the distinct points $\pi_j$ ($j = 1, 2$). In addition, assume that $\Sigma$ can be written as a graph

$$\Sigma = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 = g(x^1), \quad a \leq x^1 \leq b\} , \quad -R < a < b < R,$$
with some function $g \in C^3([-R, R])$. Moreover, let $\mathcal{H} \in C^{1,\alpha}(\overline{B_R})$, $\psi \in C^{1,\alpha}(\Sigma)$ and $\gamma \in C^4(\Gamma)$ be given functions and abbreviate $h_0 := \sup_{B_R} |H|$, $\psi_0 := \sup_\Sigma |\psi|$, $g_0 := \sup_{[-R, R]} |g'|$. Finally, suppose the conditions

$$4Rh_0 + \psi_0 \sqrt{1 + g_0^2} < 1, \quad |\psi(\pi_j)| < \cos \alpha_j, \quad j = 1, 2,$$

(4.9)

to be satisfied. Then, the boundary value problem (2.11), (2.12) has a unique solution $\zeta \in C^3(\Gamma, S, \overline{Z}) \cap C^2(\Gamma, S, \overline{Z} \setminus \{\pi_1, \pi_2\}) \cap C^0(\overline{Z})$.

**Remark 3.** Note that the prescribed mean curvature function $\mathcal{H}$ in Corollary 2 does not depend on the height $p^3$. If one wants to allow such a dependence, one has to use estimates for the length of the free trace as given in [M2]; see [M3] sec. 6 for a description of the required arguments.

**Proof of Corollary 2.** We assume w.l.o.g. that the exterior normal $\nu$ w.r.t. $G$ is given by $\nu(x) = (1 + (g'(x))^2)^{-\frac{1}{2}}(g'(x), -1)$ along $\Sigma$ and set

$$Q_2(p^1, p^2) := 2 \int_{g(p^1)}^{p^2} H(p^1, \eta) \eta - \psi(p^1, g(p^1)) \sqrt{1 + g''(p^1)}, \quad (p^1, p^2) \in \overline{B_R}.$$  

We use the notations $Z = B_R \times \mathbb{R}$, $\Gamma = \text{graph} \, \varphi$, $S = \Sigma \times \mathbb{R}$, $n = (\nu, 0, \ldots)$ from above and set $Q(p) := (0, Q_2(p^1, p^2), 0)$ for $p = (p^1, p^2, p^3) \in \overline{Z}$. Then, $Q$ belongs to $C^{1,\alpha}(\overline{Z}, \mathbb{R}^3)$ and satisfies

$$\text{div} \, Q = Q_2, \quad \text{in} \, \overline{Z}, \quad Q \cdot n = \psi \quad \text{on} \, \Sigma.$$  

In addition, $Q$ fulfills relations (2.10) and $\sup_{\overline{Z}} |Q| < 1$, according to our assumptions (4.9). Consequently, the preconditions of Theorem 1 and Corollary 1 are satisfied. The graph representation of the existing (and unique) stable $\mathcal{H}$-surface $x \in C^3(\Gamma, S, \overline{Z})$ yields the desired solution of (2.11), (2.12). 

\[\square\]

**References**

[DHT] U. Dierkes, S. Hildebrandt, A. J. Tromba: *Regularity of minimal surfaces*. Grundlehren math. Wiss. 340. Springer, Heidelberg, 2010.

[He] E. Heinz: *Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern*. Math. Z. 113, 99–105 (1970).

[Hi] S. Hildebrandt: *Einige Bemerkungen über Flächen beschränkter mittlerer Krümmung*. Math. Z. 115, 169–178 (1970).

[HJ] S. Hildebrandt, W. Jäger: *On the regularity of surfaces with prescribed mean curvature at a free boundary*. Math. Z. 118, 289–308 (1970).

[HS1] S. Hildebrandt, F. Sauvigny: *Embeddedness and uniqueness of minimal surfaces solving a partially free boundary value problem*. J. Reine Angew. Math. 422, 69–89 (1991).

[HS2] S. Hildebrandt, F. Sauvigny: *On one-to-one harmonic maps and minimal surfaces*. Nachr. Akad. Wiss. Göttingen, II. Math.-Phys. Kl., 73–93 (1992).
S. Hildebrandt, F. Sauvigny: Uniqueness of stable minimal surfaces with partially free boundaries. J. Math. Soc. Japan 47, 423–440 (1995).

F. Müller: On the analytic continuation of $H$-surfaces across the free boundary. Analysis 22, 201–218 (2002).

F. Müller: A priori bounds for surfaces with prescribed mean curvature and partially free boundaries. Analysis 26, 471–489 (2006).

F. Müller: On stable surfaces of prescribed mean curvature with partially free boundaries. Calc. Var. 24, 289–308 (2005).

F. Müller: The asymptotic behaviour of surfaces with prescribed mean curvature near meeting points of fixed and free boundaries. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 6, 529–559 (2007).

F. Müller: On the regularity of $H$-surfaces with free boundaries on a smooth support manifold. Analysis 28, 401–419 (2008).

F. Müller: On $C^{1,1/2}$-regularity of $H$-surfaces with a free boundary. Preprint, Universität Duisburg-Essen (2014).

F. Müller, S. Winklmann: Projectability and uniqueness of $F$-stable immersions with partially free boundaries. Pac. J. Math. 230, 409–426 (2007).

F. Sauvigny: Flächen vorgeschriebener mittlerer Krümmung mit eineindeutiger Projektion auf eine Ebene. Math. Z. 180, 41–67 (1982).

F. Sauvigny: Partial Differential Equations 1 - Foundations and Integral Representations. Springer, Berlin Heidelberg, 2006.
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