Bounding generalized relative entropies: non-asymptotic quantum speed limits

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Information theory has become an ubiquitous and increasingly important research field to better understand quantum mechanics. Noteworthy, it covers both foundational and applied perspectives, also offering a common technical language to study a variety of research fields. Remarkably, one of the key information-theoretic quantities is given by the relative entropy, or divergence, which quantifies how difficult is to tell apart two probability distributions, or even two quantum states. Such a quantity rests at the core of fields like metrology, quantum thermodynamics, quantum communication and quantum information. Given this broadness of applications, it is desirable to understand how this quantity changes under a quantum process. By considering a general unitary channel, we establish a bound on the generalised relative entropies (Rényi and Tsallis) between the output and the input of the channel. As an application of our bounds, we derive a family of quantum speed limits based on relative entropies. Possible connections between this family with thermodynamics, quantum coherence, asymmetry and single-shot information theory are briefly discussed.

Introduction — Since its basic formulation decades ago by Shannon [1], information theory has played a major role in both applied and fundamental science, ranging from neuroscience [2] to quantum gravity [3, 4], and along the way has impacted thermodynamics [5], finance [6], and evolutionary biology [7]. Shannon entropy also plays a role on the speed of evolution of classical stochastic processes [8]. A central element in this theory is the Shannon entropy, which is measure on how much information is contained in a probability distribution. However, when the basic assumptions of the theory do not hold, e.g., extensivity or very large data set (non-asymptotic regime), another information measures appear as generalizations of the Shannon entropy. Indeed, such family of information-theoretic measures include the paradigmatic cases of Tsallis [9] and Rényi [10] entropies.

Each of these developments is based on the idea that a physical process could be understood as an information processing protocol. In such tasks, distinguishing quantum states or even classical probability distributions plays a fundamental role. One of the most important information-theoretic distinguishability metrics, which in turn exhibits several operational meanings in distinct fields, is given by the relative entropy, also called divergence [11]. For instance, the relative entropy (quantum or classical) [12] quantifies the dissipated work in a driven evolution [13], the amount of entanglement [14] and of quantum coherence in a given state [15, 16]. Moreover, it unveils the role of entropy production in thermal relaxation processes [17–19], and also the asymmetry of a state or process [20]. Similarly, the Rényi relative entropy determine an entire family of second laws of thermodynamics in the quantum regime [21], which also applies to black hole physics [22], and a cut-off rate in the theory of hypothesis testing [23], just to name a few results. Furthermore, Rényi relative entropy is linked to an entropic energy-time uncertainty relation for time-independent systems [24], also being related to the concept of multiple quantum coherences [25].

In turn, Tsallis entropy is mainly considered in the field of non-extensive statistical mechanics [26]. However, important applications of this theory also appear in several other areas [27]. Interestingly, it has been shown that Tsallis relative entropies define a bona fide quantum coherence quantifier [28]. Furthermore, Tsallis relative entropy satisfies a class of bounds derived from Pinsker and Fannes type inequalities [29].

Here we consider the fundamental problem of bounding the change in the generalized relative entropies under an arbitrary unitary process. Specifically, we derive an upper bound on both the asymmetric and the symmetrized versions of both Rényi and Tsallis relative entropies between the initial state and the transformed one. As one of the several of applications of this result, we show that this upper bound implies an entirely new family of quantum speed limits.

The importance of the results presented in this article is twofold. First, from a general perspective, it establishes a bound on entropic quantifiers that are employed as key quantities in several fields, from quantum communication to biology [30, 31]. Therefore, since the practical computation of relative entropies are difficult in general, our main result can directly be applied in all of these fields by providing bounds on central quantities. Secondly, our family of quantum speed limits provides, from one side, non-asymptotic bounds on the time evolution of quantum systems in the sense of the so-called single-shot information theory [32]. Furthermore, due

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to the broadness application of Rényi relative entropy, it provides a bridge among the speed of quantum evolution, thermodynamics [33] and quantum resources, like entanglement, coherence and asymptotics [34]. Importantly, since our results apply for Tsallis relative entropy, it also provides a non-extensive version of the quantum speed limit, which can found a plethora of applications, both on fundamental and practical aspects [35].

In the following, we begin by stating the problem and the main result of the paper in the next two sections. After this, we derive the new family of quantum speed limits. We close the paper with a discussion on these results and also on possible new applications.

**Generalized relative entropies** — Let us start by defining our physical system, which is described by a finite-dimensional Hilbert space $\mathcal{H}$. In general, the state of the system will be given by a density matrix $\rho \in \Omega$, where $\Omega = \{\rho \in \mathcal{H} | \rho^\dagger = \rho, \rho \geq 0, \text{Tr}(\rho) = 1\}$, defines the convex space of density operators. In this setting, given two states $\rho, \omega \in \Omega$, the Rényi (RRE) and Tsallis (TRE) relative entropies are defined, respectively, as

$$R_\alpha(\rho||\omega) = \frac{1}{\alpha - 1} \ln [g_\alpha(\rho, \omega)], \quad \text{for } 0 < \alpha < 1,$$

$$H_\alpha(\rho||\omega) = \frac{1}{1 - \alpha} [1 - g_\alpha(\rho, \omega)], \quad \text{for } 0 < \alpha < 1.$$

where $g_\alpha(\rho, \omega) = \text{Tr} (\rho^\alpha\omega^{1-\alpha})$ is the $\alpha$-relative purity, with the parameter $\alpha \in (0, 1) \cup (1, +\infty)$ labelling the family of quantum relative entropies [37]. Importantly, relative purity satisfies the property $g_\alpha(\omega, \rho) = g_{1-\alpha}(\rho, \omega)$, i.e., it is skew symmetric with respect to $\alpha$. In particular, when $\rho = \omega$ we have $g_\alpha(\rho, \rho) = 1$ for all $\alpha$, and thus both relative entropies vanish, i.e., $R_\alpha(\rho||\rho) = H_\alpha(\rho||\rho) = 0$. Noteworthy, for $\alpha = 1/2$ one recovers the so-called quantum affinity, $g_{1/2}(\rho, \omega) = \text{Tr} (\sqrt{\rho} \sqrt{\omega})$, which is related to Hellinger angle [38]. In turn, Hellinger angle is associated to Wigner-Yanase skew information metric, and characterizes the length of the geodesic path connecting states $\rho$ and $\omega$ in the space of quantum states [39, 40].

In the following we summarize some properties of these entropies that will be useful for our discussion. A more complete presentation can be found in Ref. [41]. On the one hand, by starting with Rényi relative entropy, the limit $\alpha \rightarrow 1$ recovers the well-known quantum relative entropy $R_1(\rho||\omega) = S(\rho||\omega) := \text{Tr}(\rho \ln \rho - \rho \ln \omega)$. On the other hand, for $\alpha = 0$, it reduces to the min-relative entropy $R_0(\rho||\omega) = -\ln \text{Tr}(\Pi_\omega)$, with $\Pi_\omega$ being the projector onto the support of the state $\rho$ [42]. Noteworthy, for $0 \leq \alpha \leq 2$, RRE satisfies the data-processing inequality, i.e., $R_\alpha(\Lambda(\rho)||\Lambda(\omega)) \leq R_\alpha(\rho||\omega)$, thus being monotonic under any completely positive and trace preserving map $\Lambda(\bullet)$ [43]. This is a fundamental inequality not only within information theory, but also for physics (see, for instance, Ref. [44] where the second law of thermodynamics is obtained from such inequality). Moving to Tsallis relative entropy, it has been shown that, for $0 \leq \alpha < 1$, TRE is (i) nonnegative, i.e., $H_\alpha(\rho||\omega) \geq 0$ for all $\rho, \omega \in \Omega$, with the equality holding if and only if $\rho = \omega$; (ii) jointly convex; (iii) nonadditive; and (iv) contractive under completely positive and trace preserving maps [45–47]. It is worthwhile to note that TRE also recovers the standard quantum relative entropy in the limit $\alpha \rightarrow 1$, i.e., $H_1(\rho||\omega) = S(\rho||\omega)$.

We shall stress that the aforementioned Rényi and Tsallis relative entropies are asymmetric with respect to the input states, also being skew-symmetric for most values of $\alpha$. However, often in information geometry it is desirable to work with a symmetrized quantity. Quite recently, the so-called quantum Jensen-Shannon divergence, i.e., the square-root of the symmetrized version of the quantum relative entropy, was proved to be a metric on the cone of positive matrices [48]. For the case at hand, and also motivated by the refereed information-theoretic quantity, the above entropies can be symmetrized as

$$O_\alpha(\rho : \omega) := O_\alpha(\rho||\omega) + O_\alpha(\omega||\rho), \quad \text{for } \alpha > 0,$$

where index $O \equiv \{R, H\}$ labels Rényi and Tsallis relative entropies, respectively. We are now ready to present our main result.

**Bounds on generalized relative entropies** — The dynamics of our system is governed by a general, possibly time-dependent Hamiltonian $H_t \in B(\mathcal{H})$, with $B(\mathcal{H})$ being the set of bounded operators acting on $\mathcal{H}$. In general, the Hamiltonian $H_t$ is not self-commuting at different times, i.e., $[H_s, H_t] \neq 0$ for $s \neq t$. The initial state $\rho_0 \in \Omega$ undergoes the unitary evolution $\rho_t = U_t \rho_0 U_t^\dagger$, for $t \in [0, T]$, where $U_t = T e^{-i t} ds H_s$ is the time-ordered unitary evolution operator satisfying the Schrödinger equation $-i (dU_t/dt) = H_s U_t$. From now on, we will work in natural units and set $\hbar = k_B = 1$.

Based on the aforementioned physical setting, our main goal is to provide a class of nontrivial upper bounds for the symmetric relative entropies. Specifically, for $\alpha \in (0, 1)$, we prove in Appendix A that RRE and TRE satisfy the following inequalities

$$O_\alpha(\rho_t \parallel \rho_0) \leq \frac{1}{|1 - \alpha|} \int_0^T dt \, G_\alpha^\mathcal{O}(\rho_t, \rho_0),$$

with

$$G_\alpha^\mathcal{O}(A, B) := \Phi_\alpha^\mathcal{O}(A, B) \parallel B^{1-\alpha} \parallel_2 \parallel [U_t^\dagger H_s U_t, B^\alpha] \parallel_2.$$

Here, $\Phi_\alpha^\mathcal{O}$ is an auxiliary functional which reads $\Phi_\alpha^\mathcal{O}(A, B) = [g_\alpha(A, B)]^{-1}$ and $\Phi_1^\mathcal{O}(A, B) = 1$, while $\parallel A \parallel = \sqrt{\text{Tr}(A^\dagger A)}$ stands for the Schatten 2-norm. Importantly, Eq. (4) is the first main result of this article.

Naturally, the above bound can also be obtained for the case where the arguments on the left-hand side of inequality in Eq. (4) are swapped. Indeed, the corresponding inequality for both symmetrized forms of RRE and
TRE [see Eq. (3)] is then obtained, roughly speaking, by combining the two non-symmetric upper bounds (see details in Appendix A). Before discussing the physical significance of this result, next, we make use of it to obtain our second main result, a family of quantum speed limits.

**Quantum speed limit** — The family of quantum speed limits is obtained by time averaging the right-hand side of Eq. (4), thus followed by a rearrangement of the resulting inequality. The time \( \tau \) required for an arbitrary unitary evolution driving a closed quantum system from \( \rho_0 \) to \( \rho_\tau \) is lower bounded as \( \tau \geq \tau_0^\alpha := \max\{\tau_0^C(\rho_\tau || \rho_0), \tau_0^O(\rho_\tau || \rho_\tau), \tau_0^\alpha(\rho_\tau : \rho_\tau)\} \), where

\[
\tau_0^C(A || B) := \frac{|1 - \alpha| O_\alpha(A || B)}{\langle G_\alpha^O(A, B) \rangle_\tau},
\]

and

\[
\tau_0^\alpha(\rho_0 : \rho_\tau) := \frac{|1 - \alpha| O_\alpha(\rho_0 : \rho_\tau)}{\langle G_\alpha^C(\rho_1, \rho_0) + G_\alpha^O(\rho_1, \rho_0) \rangle_\tau},
\]

with \( \langle \bullet \rangle_\tau := \tau^{-1} \int_0^\tau \bullet \ dt \) denoting time average. Equation (7) presents the QSL time due to symmetrized relative entropies [for completeness, see Eq. (3)].

Recently, a related family of QSLs, based on the relative entropy, were derived in Ref. [49] bounding the time it takes to generate or consume a given quantum resource such as entanglement, asymmetry, and athermality. These bounds, dubbed as resource speed limits (RSL) were shown to be tighter than QSLs in several instances. However, as RSLs are constructed using relative entropy, they are only meaningful in the asymptotic limit. RSLs for single shot scenarios requires working \( \alpha \)-Rényi entropies. Here, we have taken the first step in this direction. This is our second main result of this paper, thus establishing a novel family of entropic quantum speed limits (QSL), i.e., RRE and TRE provide lower bounds on the time of evolution between the initial and the final states of the quantum system. In the last section we present a discussion regarding the role played by the order parameter \( \alpha \) into our family of quantum speed limits.

We shall stress that the previous discussion is only valid when \( \alpha \in (0, 1) \). In the following, we will discuss the limiting cases \( \alpha \to 1 \) and \( \alpha \to 0 \), which crucially reduce to the standard relative entropy and the so-called min-relative entropy, respectively. Importantly, these results cannot be simply obtained from the main results above. While the case \( \alpha \to 1 \) is clearly delicate from the definition of RRE and TRE given in Eqs. (1) and (2), the case \( \alpha \to 0 \) must also be carefully considered since simply taking \( \alpha = 0 \) in Eq. (4) would provide us a trivial bound, independently of the initial state and the dynamics.

**Limiting case of \( \alpha \to 0 \)** — Let us start by considering the limit \( \alpha \to 1 \), in which both RRE and TRE recover the Umegaki relative entropy. In this case, it can be proved that the following upper bound applies

\[
O_1(\rho_\tau || \rho_0) \leq \ln \rho_0 \parallel_2 \int_0^\tau dt \| [H_t, \rho_\tau] \|_2, \tag{8}
\]

where \( O_1(\rho_\tau || \rho_0) = S(\rho_\tau || \rho_0) \) define the standard quantum relative entropy. The details of this calculation are fully presented in Appendix B. The corresponding family of QSL is \( \tau \geq \tau_1^{RE} := \max\{\tau_1^{RE}(\rho_\tau || \rho_0), \tau_1^{RE}(\rho_\tau || \rho_\tau), \tau_1^{RE}(\rho_\tau : \rho_\tau)\} \) where

\[
\tau_1^{RE}(A || B) := \frac{S(A || B)}{\ln \rho_0 \parallel_2 \| [H_t, A] \parallel_2}_\tau, \tag{9}
\]

and

\[
\tau_1^{RE}(\rho_0 : \rho_\tau) := \frac{O_1(\rho_0 : \rho_\tau)}{\ln \rho_0 \parallel_2 \| [H_t, \rho_0] \parallel_2 + \| [H_t, \rho_\tau] \parallel_2}_\tau. \tag{10}
\]

When the Hamiltonian is time-independent, i.e., \( H_t \equiv H \), thus one obtains the identity

\[
\langle \| [H, \rho_0] \parallel_2 \rangle_\tau = 4\sqrt{I_L(\rho_0, H)}, \text{ where we have used the fact that } \| [H, \rho_0] \|_2^2 = 4I_L(\rho_0, H),
\]

with \( I_L(\rho_0, H) = -(1/4) \text{Tr}(\rho_0 H^2) \) being a quantum coherence quantifier which sets a lower bound on Wigner-Yanase skew information [50, 51]. Now, since \( I_L(\rho_0, H) \leq (\Delta H)^2 \), where \( (\Delta H)^2 = \text{Tr}(\rho_0 H^2) - [\text{Tr}(\rho_0) H]^2 \) is the squared deviation associated with the Hamiltonian, thus Eq. (9) implies the lower bound \( \tau_1^{RE}(\rho_0 || \rho_\tau) \geq S(\rho_0 || \rho_\tau)/(4 \Delta H \ln \rho_0 \parallel_2) \).

**Limiting case of \( \alpha \to 0 \)** — Considering now the case \( \alpha \to 0 \), we show in Appendix C, that the Rényi min-relative entropy is upper bounded as

\[
R_0(\rho_\tau || \rho_0) \leq \int_0^\tau dt Q_0(\rho_\tau, \Pi_{\rho_0}) , \tag{11}
\]

with

\[
Q_0(A, B) := \frac{\| A \parallel_2 \| [U_t^\dagger H_t U_t, B] \parallel_2}{|\text{Tr}(U_t A U_t^\dagger)|}. \tag{12}
\]

Here \( \Pi_{\rho_0} \) is the projector onto the support of the initial state \( \rho_0 \). From Eq. (11), we can derive the QSL time as

\[
\tau_0^R(\rho_\tau || \rho_0) := \frac{R_0(\rho_\tau || \rho_0)}{\langle Q_0(\Pi_{\rho_0}, \rho_0) \rangle}_\tau , \tag{13}
\]

and

\[
\tau_0^R(\rho_0 : \rho_\tau) := \frac{R_0(\rho_0 || \rho_\tau)}{\langle Q_0(\Pi_{\rho_0}, \rho_0) \rangle}_\tau. \tag{14}
\]

For completeness, the QSL respective to the symmetric min-entropy is given as follows

\[
\tau_0^R(\rho_0 : \rho_\tau) := \frac{R_0(\rho_\tau : \rho_0)}{\langle Q_0(\rho_0, \Pi_{\rho_0}) + Q_0(\Pi_{\rho_0}, \rho_0) \rangle}_\tau. \tag{15}
\]
TABLE I. Theoretical-information quantifiers related to the single qubit state $\rho_0 = (1/2)(I + \vec{r} \cdot \vec{\sigma})$, evolving under the general time-dependent Hamiltonian $H_I = \omega I + \tilde{n}_I \cdot \vec{\sigma}$. As defined in the main text, $\dot{\tilde{n}}_t = \tilde{u}_t/|\tilde{u}_t|$, with $\tilde{u}_t = \int_0^t ds \tilde{n}_s$. Moreover, note that $U_t^I H_I U_t = \omega I + \tilde{\mu}_t \cdot \vec{\sigma}$, with the unit vector $\tilde{\mu}_t := \tilde{n}_t - \sin(2|\tilde{u}_t|)(\tilde{u}_t \times \tilde{n}_t) + 2\sin^2(|\tilde{u}_t|)(|\tilde{u}_t| - \tilde{n}_t)\tilde{u}_t - \tilde{n}_t$ (see Appendix D1). If the Hamiltonian is time-independent, $\tilde{n}_t = \tilde{n}$, one must apply the changes $\tilde{u}_t \to \tilde{n}$, $|\tilde{u}_t| \to t$, and $\tilde{\mu}_t \to \tilde{n}$ into the listed quantities.

| Quantifier | Analytical value |
|------------|------------------|
| $||[U_t^I H_I U_t, \rho_0]||_2$ | $\xi_\alpha \sqrt{2(1 - (\mu_t \cdot \vec{r})^2)}$ |
| $||[H_I, \rho_0]||_2$ | $r \sqrt{2(1 - (\mu_t \cdot \vec{r})^2)}$ |
| $||[H_I, \rho_0]||_2$ | $r \sqrt{2(1 - (\tilde{n}_t \cdot \vec{r})^2)}$ |
| $S(\rho_0||\rho_0)$ | $r \ln \left(1 + \frac{1}{\xi_\alpha^2}ight) (1 - (\tilde{n}_t \cdot \vec{r})^2) \sin^2(|\tilde{u}_t|)$ |
| $||\ln \rho_0||_2$ | $\sqrt{\ln^2 \left(\frac{1 + \xi_\alpha^2}{2}\right) + \ln^2 \left(\frac{1 + \xi_\alpha^2}{2}\right)}$ |
| $||\rho_0||_2$ | $\sqrt{2} \xi_2 \alpha$ |

Noteworthy, the ‘speed’ contribution $Q^I_0$ is closely related to the QSL derived by means of Euclidean distance in the generalized Bloch sphere [52, 53]. Importantly, when the density matrix $\rho_0$ has full-rank, i.e., $\dim(\rho_0) = \text{supp}(\rho_0)$, thus Rényi min-relative entropy vanishes as $R_0(\rho_t||\rho_0) = -\ln \text{Tr}(\Pi_\rho^t \rho_0) = 0$, and also $R_0(\rho_0||\rho^r) = -\ln \text{Tr}(\Pi_\rho^r \rho_0^r) = 0$. Indeed, in this case one may verify the right-hand side of Eq. (11) is also identically zero since $Q^I_0(\rho_0, \Pi_\rho^r) = 0$. In contrast, the quantity $Q^I_0(\Pi^r_\rho^t, \rho_0) = \sqrt{d} ||U_t^{-1} H_I U_t, \rho_0||_2$ will remains finite for $t \in [0, \tau]$, where $d = \dim(\mathcal{H})$ stands for the dimension of the Hilbert space $\mathcal{H}$.

We now provide an example that can be analytically computed in order to make the physical implications of our results more clear.

Example — Let us consider a single-qubit system, whose Bloch sphere representation can be written as $\rho_0 = (1/2)(I + \vec{r} \cdot \vec{\sigma})$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, $\vec{r} = r \hat{r}$ is the Bloch vector, with $\hat{r} = \{\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\}$, $0 < r < 1$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, while $I$ is the $2 \times 2$ identity matrix. The dynamics of the system is governed by the time-dependent Hamiltonian $H_I = \omega I + \tilde{n}_I \cdot \vec{\sigma}$, where $\tilde{n}_I = (n_x^I, n_y^I, n_z^I)$ is a time-dependent unit vector, $|\tilde{n}_I| = 1$, and $\omega \in \mathbb{R}$. In this case, the time ordered evolution operator becomes $U_t = e^{-i\omega t I} \cos(|\tilde{u}_t|) I - i \sin(|\tilde{u}_t|)(\tilde{u}_t \cdot \vec{\sigma})$, where $\tilde{u}_t = \tilde{u}_t/|\tilde{u}_t|$ is a unit vector, with $\tilde{u}_t := \int_0^t ds \tilde{n}_s$. In particular, if $H_I \equiv H$ is time-independent, i.e., $\tilde{n}_t = \tilde{n}$ is a constant unit vector, we directly obtain $\tilde{u}_t = \tilde{n}$.

Considering the range $0 < \alpha < 1$, the evolved density operator $\rho_t^\alpha = U_t \rho_0^\alpha U_t^\dagger$ can be written as

$$\rho_t^\alpha = \frac{1}{2} \left[ \xi_\alpha^+ \mathbb{1} + \xi_\alpha^- (\tilde{u}_t \cdot \vec{\sigma}) \right],$$

with

$$\xi_\alpha^\pm = 2^{-\alpha} \left[ (1 + r)^\alpha \pm (1 - r)^\alpha \right],$$

and

$$\tilde{u}_t := \tilde{r} + \sin(2|\tilde{u}_t|)(\tilde{u}_t \times \tilde{r}) + 2 \sin^2(|\tilde{u}_t|)(|\tilde{u}_t| - \tilde{n}_t)\tilde{u}_t - \tilde{n}_t.$$ 

Particularly, for $t = 0$, Eq. (18) implies that $\tilde{u}_0 = \tilde{r}$, and thus Eq. (16) reduces to $\rho_0^\alpha = (1/2) \left[ \xi_\alpha^+ \mathbb{1} + \xi_\alpha^- (\tilde{r} \cdot \vec{\sigma}) \right]$. Moving forward, based on Eq. (16), the $\alpha$-relative purity is written as

$$g_\alpha(\rho_t, \rho_0) = 1 - \xi_\alpha^- \xi_\alpha^- (1 - (\tilde{u}_t \cdot \vec{r})^2) \sin^2(|\tilde{u}_t|).$$

Interestingly, since relative purity is skew symmetric over the index $\alpha$, Eq. (19) thus implies that $g_\alpha(\rho_t, \rho_0) = g_{1-\alpha}(\rho_t, \rho_0) = g_\alpha(\rho_0, \rho_t)$. In turn, both Rénnyi and Tsallis relative entropies satisfy the constraint $O_\alpha(\rho_t||\rho_0) = O_\alpha(\rho_0||\rho_t)$. Similarly, although relative entropy is not a symmetric function over its entries, we point out that $S(\rho_t||\rho_0) = S(\rho_0||\rho_t)$ for a single-qubit state $\rho_0$ undergoing the referred unitary evolution (see Appendix D1).

From Eq. (19), note that $\alpha$-relative purity is equal to 1 for $|\tilde{u}_t| = n\pi$, with $n \in \mathbb{Z}$ and $t \in [0, \tau]$. Furthermore, $g_\alpha(\rho_t, \rho_0) = 1$ if vectors $\tilde{u}_t$ and $\tilde{r}$ are parallel. Conversely, $\alpha$-relative purity becomes $g_\alpha(\rho_t, \rho_0) = 1 - \xi_\alpha^- \xi_\alpha^- \sin^2(|\tilde{u}_t|)$ if vectors $\tilde{u}_t$ and $\tilde{r}$ are orthogonal. For completeness, in Table I we summarize the quantities required to evaluate the QSL bounds $\tau^I_\alpha$ in Eqs. (6) and (7), $\tau^{\text{RE}}_\alpha$ in Eqs. (9) and (10), and $\tau^K_\alpha$ in Eqs. (13), (14), and (15), for the case of a single-qubit state. For more details, see Appendix D1.

In order to illustrate our findings, here we will consider the time-dependent Hamiltonian $H_I = \tilde{n}_t \cdot \vec{\sigma}$, with $\tilde{n}_t = \gamma^{-1}\{\Delta, 0, v t\}$ and $\gamma := \sqrt{\Delta^2 + (vt)^2}$, where $v$ stands as a ‘level velocity’ of the energies of the system, and $\Delta$ is the level splitting [54]. Figure 1 shows the QSL $\tau^K_\alpha$ as function of the evolution time $\tau$ and the entropy label $\alpha$, for the initial single-qubit state with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$. For this figure we chosen the ratio $\Delta/v = 0.5$. In Appendix D2-D5 we provide a detailed numerical study in order to show that the qualitative behaviour depicted in Fig. 1 is also present while considering Tsallis entropy, and when we change the physical parameter of the system.

Discussion — The main contribution of this paper is to provide an upper bound on the change in the generalized divergences when the considered system undergoes a unitary transformation.

As the first application of our bound we derived a family of quantum speed limits, presented in Eqs. (6) and (7). From this result, we see that the minimum time
required for the state transformation is inversely proportional to a quantity that involves the average energy of the system, which in turn determines the speed of the transformation while being directly proportional to \( O_\alpha \), thus implying that these entropies play the role of distances. Moreover, our derivations of QSLs, based on \( \alpha \)-Rényi entropies, is first step toward resource speed limits quantifying consumption of resource in single shot scenarios [49].

Another interesting connection can be built based on the results presented in Ref. [20], where the so-called asymmetry monotones were introduced. These quantities characterize conservation laws for general quantum systems, in the sense of Noether’s theorem. In short, an asymmetry monotone is a function \( f : B(\mathcal{H}) \rightarrow \mathbb{R} \) that quantifies how much the state of the system breaks the symmetry in question. Mathematically, the action of a symmetry group \( G \) on a quantum system can be represented by the operation \( U_g(\rho) = U_g \rho U_g^\dagger \), with \( g \) being a variable labelling the group elements. The key idea behind the asymmetry monotones is to recognize the orbit of each quantum state as an encoding process, while the transformation \( \rho \rightarrow \sigma \), which implies the map \( U_g(\rho_0) \rightarrow U_g(\rho) \), is viewed as data processing. This implies that we can employ any contractive information measure to characterize the orbit of each state, which leads to a measure of asymmetry \( f \), the asymmetry monotone, such that \( f(U_g(\rho)) \leq f(\rho) \) [20].

Considering the range of the parameter \( \alpha \) where both Rényi and Tsallis relative entropies are contractive under the action of a completely-positive and trace-preserving map, we immediately see that the quantities appearing in Eq. (3) (as well as its asymmetric versions) define an entire family of asymmetry monotones. Indeed, the relative entropy \( (\alpha \rightarrow 1) \) was previously considered as an asymmetry monotone [55].

Now, since the result presented in Eq. (4) is valid for any unitary transformation encoding an unknown parameter into the state of the system, i.e., it goes beyond the paradigmatic case of time evolutions, we can replace \( t \) in such equation by the group variable \( g \). This can be seen from the fact that the unitary representation of the symmetry group leads to the evolution equation \( d\rho_g / dg = -i[H, \rho_g] \), with \( K \) being the generator of the transformation. Therefore, our result provides an upper bound on how much the state under consideration breaks the symmetry generated by \( K \). This, in turn, sets an upper bound of how much the conservation of the associated physical quantity can be broken. Remarkably, in the specific case of the quantum speed limit, Eqs. (6) and (7), this implies that the minimum evolution time is determined by the asymmetry monotone \( O_\alpha(\rho_t || \rho_0) \), which in turn stands as a measure of how much the initial state breaks the time-translation symmetry, related with the energy conservation.

Finally, we would like to discuss the relation between our results and the concept of non-equilibrium entropy production. We shall begin by setting the initial state of the system as a thermal one, \( \rho_0 = \rho_\beta = \exp(-\beta H_0) / Z \), where \( Z \) is the thermodynamic partition function, and \( H_0 \) is the ‘bare’ Hamiltonian of the system, i.e., \([H_0, H_i] \neq 0 \) for all \( t \neq 0 \). From Eq. (9), one may verify the lower bound on the time of evolution is proportional to the relative entropy \( S(\rho_t || \rho_\beta) \), which is turn stands as the entropy production associated with the process under consideration. Therefore, a natural question arising here is about the extension of this connection to more general entropies and systems. Indeed, such a general picture could be possible by exploiting the entire family of second laws of thermodynamics based on Rényi relative entropies, which has been derived in Ref. [21] (also see Ref. [33]). This may open an avenue for the comprehension of quantum speed limits [56–62], asymmetry monotones [55] and quantum thermodynamics [19, 63–65], based on a strictly geometric framework.

Beyond the open questions mentioned above, many more raise from the results presented here. First, we could consider the extension of our results to open system dynamics, describing general, non-unitary evolutions. Moreover, given the recently claimed link involving quantum coherence and the Rényi and Tsallis relative entropies [66], it would be interesting to investigate the trade-off among entropy production, QSL and quantum coherence in this general scenario. Furthermore, since our results also apply for the min-entropy, i.e., the limit \( \alpha \rightarrow 0 \) regarding to Rényi relative entropy, they can be employed in the single-shot information theory, where the relations among asymmetry, quantum speed limit and thermodynamics can be further developed into the non-asymptotic regime.
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Appendix A: Bounds on relative entropies

Here we provide the details involved in the derivation of Eq. (4) presented in the main text. We shall begin noticing that, for $0 < \alpha < 1$, Rényi (RRE) and Tsallis (TRE) relative entropies are skew symmetric with respect to the parameter $\alpha$, i.e., $(1 - \alpha) \mathcal{O}_\alpha(\rho || \sigma) = \alpha \mathcal{O}_{1-\alpha}(\sigma || \rho)$, for all $\rho, \sigma \in \Omega$. By using this fact, the symmetric relative entropy is written as

$$\mathcal{O}_\alpha(\rho : \sigma) := \mathcal{O}_\alpha(\rho || \sigma) + \mathcal{O}_\alpha(\sigma || \rho) = \mathcal{O}_\alpha(\rho || \sigma) + \frac{\alpha}{1 - \alpha} \mathcal{O}_{1-\alpha}(\sigma || \rho).$$

(A1)

From now on, we will focus on the symmetric relative entropy $\mathcal{O}_\alpha(\rho_t : \rho_0)$, where $\rho_0$ is the initial state of the system, and $\rho_t = U_t \rho_0 U_t^\dagger$ its respective evolved state, with $U_t = T e^{-i \int_0^t ds H_s}$ being the unitary time-ordered evolution

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operator. Let $|d\mathcal{O}_\alpha(\rho_t : \rho_0)/dt|$ be the absolute value of the time-derivative of symmetric relative entropy between states $\rho_0$ and $\rho_t$. By using Eq. (A1), and also applying the triangle inequality $|a_1 + a_2| \leq |a_1| + |a_2|$, one may conclude that
\begin{equation}
\left| \frac{d}{dt}\mathcal{O}_\alpha(\rho_t : \rho_0) \right| \leq \frac{\alpha}{|1 - \alpha|} \left| \frac{d}{dt}\mathcal{O}_{1-\alpha}(\rho_t : \rho_0) \right|.
\end{equation}
But now notice that
\begin{equation}
\left| \frac{d}{dt}\mathcal{O}_s(\rho_t : \rho_0) \right| = \frac{\Phi^O_s(\rho_t, \rho_0)}{|s - 1|} \left| \frac{d}{dt}g_s(\rho_t : \rho_0) \right|,
\end{equation}
where the auxiliary functional $\Phi^O_s(x, y)$ read as
\begin{equation}
\Phi^O_s(x, y) = \begin{cases} [g_s(x, y)]^{-1}, & \text{for } \mathcal{O} \equiv \mathcal{R} \\ 1, & \text{for } \mathcal{O} \equiv \mathcal{H}. \end{cases}
\end{equation}
The time-derivative of relative purity $g_s(\rho_t, \rho_0) = \text{Tr}(\rho_t^{1-s} \rho_0^s)$ is evaluated as follows. Because $\rho_t$ evolves unitarily, it is possible to verify the operator $\rho_t^s = U_t \rho_0^s U_t^\dagger$ satisfies the von Neumann equation
\begin{equation}
\frac{d}{dt}\rho_t^s = -i[H_t, \rho_t^s],
\end{equation}
where we used the identity $U_t(dU_t^\dagger / dt) = -(dU_t / dt)U_t^\dagger = -iH_t$. Hence, we thus have that
\begin{equation}
\frac{d}{dt}g_s(\rho_t, \rho_0) = -i \text{Tr}(\rho_t^{1-s} [H_t, \rho_t^s])
\end{equation}
\begin{equation}
= -i \text{Tr}(U_t^\dagger \rho_0^{1-s} U_t U_t^\dagger H_t U_t \rho_0^s),
\end{equation}
where the first equality follows from Eq. (A5), and the last one is obtained by using the cyclic property of trace. By taking the absolute value of Eq. (A6), and applying the Cauchy-Schwarz inequality, $|\text{Tr}(A_1 A_2)| \leq \|A_1\|_2 \|A_2\|_2$, with $\|A\|_2 = \sqrt{\text{Tr}(A^\dagger A)}$, one gets
\begin{equation}
\left| \frac{d}{dt}g_s(\rho_t, \rho_0) \right| \leq \|\rho_0^{1-s}\|_2 \|[U_t^\dagger H_t U_t, \rho_0^s]\|_2.
\end{equation}
By substituting Eq. (A7) into Eq. (A3), we thus obtain
\begin{equation}
\left| \frac{d}{dt}\mathcal{O}_s(\rho_t : \rho_0) \right| \leq |s - 1|^{-1} G^O_s(\rho_t, \rho_0),
\end{equation}
where we define the functional
\begin{equation}
G^O_s(A, B) := \Phi^O_s(A, B) \|B^{1-s}\|_2 \|[U_t^\dagger H_t U_t, B^s]\|_2.
\end{equation}
Finally, by plugging Eq. (A8) into Eq. (A2), and thus integrating it over the interval $0 \leq t \leq \tau$, one finds the upper bound
\begin{equation}
|\mathcal{O}_\alpha(\rho_\tau : \rho_0)| \leq |\alpha - 1|^{-1} \int_0^\tau dt \left[ G^O_\alpha(\rho_t, \rho_0) + G^O_{1-\alpha}(\rho_t, \rho_0) \right],
\end{equation}
where we have applied the inequality $\int dx f(x) \leq \int dx |f(x)|$.

Appendix B: Recovering relative entropy

In this Appendix we present the details in the calculation of the limit $\alpha \to 1$. The idea here is going back few steps and pinpoint the main features needed to properly address such nontrivial limit. We shall begin evaluating the quantity $\lim_{\alpha \to 1} d\mathcal{O}_\alpha(\rho_t : \rho_0)/dt$, i.e., the limiting case $\alpha \to 1$ of the time-derivative of symmetric relative entropy
between states $\rho_0$ and $\rho_t = U_t \rho_0 U_t^\dagger$. By taking its absolute value, i.e., $|\lim_{\alpha \to 1} dO_\alpha(\rho_t : \rho_0)|/dt|$, and thus applying Eq. (A1), one gets

$$\left| \lim_{\alpha \to 1} \frac{d}{dt} O_\alpha(\rho_t : \rho_0) \right| \leq \left| \lim_{\alpha \to 1} \frac{d}{dt} O_\alpha(\rho_t || \rho_0) \right| + \left| \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} O_{1-\alpha}(\rho_t || \rho_0) \right|, \quad (B1)$$

where we have used the triangle inequality $|a_1 + a_2| \leq |a_1| + |a_2|$. Now, by integrating Eq. (B1) over interval $t \in [0, \tau]$, we thus have

$$\int_0^\tau \left| \lim_{\alpha \to 1} \frac{d}{dt} O_\alpha(\rho_t : \rho_0) \right| \leq \int_0^\tau \left| \lim_{\alpha \to 1} \frac{d}{dt} O_\alpha(\rho_t || \rho_0) \right| + \int_0^\tau \left| \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} O_{1-\alpha}(\rho_t || \rho_0) \right|. \quad (B2)$$

Quite interestingly, given the relative entropy $S(A||B) = \text{Tr}[A(\ln A - \ln B)]$, note that one may write down

$$|S(\rho_t || \rho_0) + S(\rho_t || \rho_0)| = \left| \lim_{\alpha \to 1} O_\alpha(\rho_t : \rho_0) \right| = \left| \int_0^\tau d\alpha \left( \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} O_\alpha(\rho_t : \rho_0) \right) \right| \leq \int_0^\tau \left| \lim_{\alpha \to 1} \frac{d}{dt} O_\alpha(\rho_t : \rho_0) \right|. \quad (B3)$$

Just to clarify, here we assume that RRE and TRE are continuous real-valued functions over the set $\alpha \in (0, 1) \cup [1, +\infty)$ and $t \in [0, \tau]$. In this sense, we are formally able to switch the limit on parameter $\alpha$ with the integration sign over variable $t$. Thus, by combining Eqs. (B2) and (B3), one readily obtains

$$\left| \lim_{\alpha \to 1} \frac{d}{dt} O_\alpha(\rho_t : \rho_0) \right| = \left| \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} O_\alpha(\rho_t : \rho_0) \right|, \quad (B5)$$

and

$$\left| \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} O_{1-\alpha}(\rho_t : \rho_0) \right| = \left| \lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} O_{1-\alpha}(\rho_t : \rho_0) \right|, \quad (B6)$$

with the auxiliary functional $\Phi^\alpha_\beta(A, B)$ defined in Eq. (A4). Interestingly, one may verify the right-hand side of Eq. (B4) exhibits an indeterminacy in the limit $\alpha \to 1$. Indeed, since $(d\rho_t^\alpha/dt) = -i[H, \rho_t^\alpha]$, it follows that $\lim_{\alpha \to 1} [d g_\alpha(\rho_t, \rho_0)/dt] = (-i) \lim_{\alpha \to 1} \text{Tr} (\rho_t^{1-\alpha}[H, \rho_t^\alpha]) = 0$, and also $\lim_{\alpha \to 1} [d g_{1-\alpha}(\rho_t, \rho_0)/dt] = (-i) \lim_{\alpha \to 1} \text{Tr} (\rho_t^{1-\alpha}[H, \rho_t^{1-\alpha}]) = 0$. Similarly, one readily verifies that $\lim_{\alpha \to 1} (1-\alpha)[\Phi^\alpha_\beta(\rho_t, \rho_0)]^{-1} = 0$, where we used the fact that $\lim_{\alpha \to 1} [\Phi^\alpha_\beta(\rho_t, \rho_0)]^{-1} = 1$ and $\lim_{\alpha \to 1} [\Phi^\alpha_\beta(\rho_t, \rho_0)]^{-1} = 1$ for $\beta \equiv [R, H]$. In this case, we thus have that

$$\lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} g_\alpha(\rho_t, \rho_0) \rightarrow 0, \quad (B7)$$

and

$$\lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} g_{1-\alpha}(\rho_t, \rho_0) \rightarrow 0, \quad (B8)$$

which in turn implies the right-hand side of Eq. (B4) is not well-behaved.

However, such issue is readily circumvented by applying the L’Hospital rule, leading us to

$$\lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} g_\alpha(\rho_t, \rho_0) = \lim_{\alpha \to 1} \frac{d}{d(1-\alpha)} [\Phi^\alpha_\beta(\rho_t, \rho_0)]^{-1} / d\alpha, \quad (B9)$$

and

$$\lim_{\alpha \to 1} \frac{\alpha}{(1-\alpha)} \frac{d}{dt} g_{1-\alpha}(\rho_t, \rho_0) = \lim_{\alpha \to 1} \frac{d}{d(1-\alpha)} [\Phi^\alpha_{1-\alpha}(\rho_t, \rho_0)]^{-1} / d\alpha. \quad (B10)$$
In the following we will discuss in details each contribution in the right-hand side of Eqs. (B9) and (B10). Let us start by evaluating the following derivatives

\[
\lim_{\alpha \to 1} \frac{d}{d\alpha} \left( \frac{d}{dt} g_\alpha(\rho_t, \rho_0) \right) = -i \lim_{\alpha \to 1} \frac{d}{d\alpha} (\text{Tr} (\rho_0^{1-\alpha} [H_t, \rho_0^{\alpha}])),
\]

\[
= -i \lim_{\alpha \to 1} \sum_{j, \ell} \frac{d}{d\alpha} \left( p_j^{1-\alpha} p_\ell^\alpha \right) \langle \psi_j | [H_t, U_t] | \psi_\ell \rangle \langle \psi_\ell | U_t^\dag | \psi_j \rangle
\]

\[
= -i \sum_{j, \ell} p_\ell (\ln p_\ell - \ln p_j) \langle \psi_j | [H_t, U_t] | \psi_\ell \rangle \langle \psi_\ell | U_t^\dag | \psi_j \rangle
\]

\[
= i \text{Tr} \left( \ln \rho_0 [H_t, \rho_0] \right),
\]

(B11)

and

\[
\lim_{\alpha \to 1} \frac{d}{d\alpha} \left( \frac{d}{dt} g_\alpha(\rho_t, \rho_0) \right) = -i \lim_{\alpha \to 1} \frac{d}{d\alpha} (\text{Tr} (\rho_0^\alpha [H_t, \rho_t^{1-\alpha}])),
\]

\[
= -i \lim_{\alpha \to 1} \sum_{j, \ell} \frac{d}{d\alpha} \left( p_j^\alpha p_\ell^{1-\alpha} \right) \langle \psi_j | [H_t, U_t] | \psi_\ell \rangle \langle \psi_\ell | U_t^\dag | \psi_j \rangle
\]

\[
= i \sum_{j, \ell} p_\ell (\ln p_\ell - \ln p_j) \langle \psi_j | [H_t, U_t] | \psi_\ell \rangle \langle \psi_\ell | U_t^\dag | \psi_j \rangle
\]

\[
= -i \text{Tr} \left( U_t \ln \rho_0 U_t^\dag [H_t, \rho_0] \right),
\]

(B12)

where we have used that \( d(p_j^{1-\alpha} p_\ell^\alpha)/d\alpha = (\ln p_\ell - \ln p_j) p_j^{1-\alpha} p_\ell^\alpha \). Moving forward, note that

\[
\frac{d}{d\alpha} \left( (1 - \alpha) \Phi_\alpha(\rho_t, \rho_0) \right)^{-1} = -\Phi_\alpha(\rho_t, \rho_0)^{-1} + (1 - \alpha) \frac{d}{d\alpha} \Phi_\alpha(\rho_t, \rho_0)^{-1},
\]

(B13)

and

\[
\frac{d}{d\alpha} \left( (1 - \alpha) \Phi_{1-\alpha}(\rho_t, \rho_0) \right)^{-1} = -\Phi_{1-\alpha}(\rho_t, \rho_0)^{-1} + (1 - \alpha) \frac{d}{d\alpha} \Phi_{1-\alpha}(\rho_t, \rho_0)^{-1}.
\]

(B14)

From Eqs. (B13) and (B14), we point out that \( \lim_{\alpha \to 1} \Phi_\alpha(\rho_t, \rho_0)^{-1} = 1 \) and \( \lim_{\alpha \to 1} \Phi_{1-\alpha}(\rho_t, \rho_0)^{-1} = 1 \), for \( \mathcal{O} = \{ \mathcal{R}, \mathcal{H} \} \). Hence, from now on it suffices to proceed by showing the derivatives \( d(\Phi_\alpha(\rho_t, \rho_0))^{-1}/d\alpha \) and \( d(\Phi_{1-\alpha}(\rho_t, \rho_0))^{-1}/d\alpha \) are indeed well-behaved for \( \alpha \to 1 \). On the one hand, for Tsallis relative entropy the auxiliary functional is given by \( \Phi_\alpha^\mathcal{H}(\rho_t, \rho_0) = \Phi_{1-\alpha}^\mathcal{H}(\rho_t, \rho_0) = 1 \), for all \( \alpha \), and the aforementioned derivatives are identically zero. In this case, from Eqs. (B13) and (B14), one obtains

\[
\frac{d}{d\alpha} \left( (1 - \alpha) \Phi_\alpha^\mathcal{H}(\rho_t, \rho_0) \right)^{-1} = \frac{d}{d\alpha} \left( (1 - \alpha) \Phi_{1-\alpha}^\mathcal{H}(\rho_t, \rho_0) \right)^{-1} = -1.
\]

(B15)

On the other hand, for Rényi relative entropy the auxiliary functional behave as \( \Phi_\alpha^\mathcal{R}(\rho_t, \rho_0)^{-1} = g_\alpha(\rho_t, \rho_0) \) and \( \Phi_{1-\alpha}^\mathcal{R}(\rho_t, \rho_0)^{-1} = g_{1-\alpha}(\rho_t, \rho_0) \), and the calculation is far from trivial. To see this, let \( \rho_t = \sum_{\ell} \rho_\ell |\psi_\ell \rangle \langle \psi_\ell|U_t^\dag \rangle \) be the spectral decomposition of the initial state, with \( 0 \leq \rho_\ell \leq 1 \) and \( \sum_\ell \rho_\ell = 1 \). In this case, given the evolved state \( \rho_t = U_t \rho_0 U_t^\dag \), we thus have that \( \rho_t^\alpha = \sum_\ell \rho_\ell^\alpha U_t \rho_\ell |\psi_\ell \rangle \langle \psi_\ell|U_t^\dag \rangle \), and the relative purity becomes \( g_\alpha(\rho_t, \rho_0) = \sum_{j, \ell} p_j^{1-\alpha} p_\ell^\alpha |\langle \psi_j|U_t|\psi_\ell \rangle|^2 \). Hence, the derivative with respect to the parameter \( \alpha \) is simply given by

\[
\lim_{\alpha \to 1} \frac{d}{d\alpha} \Phi_\alpha^\mathcal{R}(\rho_t, \rho_0)^{-1} = \lim_{\alpha \to 1} \frac{d}{d\alpha} \text{Tr}(\rho_\ell^\alpha \rho_0^{1-\alpha})
\]

\[
= \lim_{\alpha \to 1} \sum_{j, \ell} \rho_\ell^\alpha (\ln p_\ell - \ln p_j) |\langle \psi_j|U_t|\psi_\ell \rangle|^2
\]

\[
= S(\rho_t || \rho_0),
\]

(B16)

and

\[
\lim_{\alpha \to 1} \frac{d}{d\alpha} \Phi_{1-\alpha}^\mathcal{R}(\rho_t, \rho_0)^{-1} = \lim_{\alpha \to 1} \frac{d}{d\alpha} \text{Tr}(\rho_t^{1-\alpha} \rho_0^\alpha)
\]

\[
= -\lim_{\alpha \to 1} \sum_{j, \ell} \rho_\ell^{1-\alpha} (\ln p_\ell - \ln p_j) |\langle \psi_j|U_t|\psi_\ell \rangle|^2
\]

\[
= S(\rho_0 || \rho_t).
\]

(B17)
Hence, by combining Eqs. (B13), (B14), (B15), (B16), and (B17), we get the result
\[
\lim_{\alpha \to 1} \frac{d}{d\alpha} \left( (1 - \alpha) \left[ \Phi^\alpha_0 (\rho_\ell, \rho_\text{p}) \right]^{-1} \right) = \lim_{\alpha \to 1} \frac{d}{d\alpha} \left( (1 - \alpha) \left[ \Phi^{1-\alpha}_0 (\rho_t, \rho_0) \right]^{-1} \right) = -1 .
\] (B18)

Finally, by inserting Eqs. (B11), (B12), (B15) and (B18), into Eqs. (B9) and (B10), we conclude
\[
\lim_{\alpha \to 1} \Phi^\alpha_0 (\rho_t, \rho_0) \frac{d}{dt} g_{1-\alpha} (\rho_t, \rho_0) = -i \text{Tr} \left( \rho_\text{p} [H_t, \rho_t] \right) ,
\] (B19)
and
\[
\lim_{\alpha \to 1} \Phi^{1-\alpha}_0 (\rho_t, \rho_0) \frac{d}{dt} g_{1-\alpha} (\rho_t, \rho_0) = i \text{Tr} \left( U_t \rho_0 U_t^\dagger [H_t, \rho_0] \right) .
\] (B20)

Finally, by substituting Eqs. (B19) and (B20) into Eq. (B4), and then applying the Cauchy-Schwarz inequality, we get
\[
|\text{Tr}(A_1 A_2)| \leq \|A_1\|_2 \|A_2\|_2 ,
\]
with \( \|A\|_2 = \sqrt{\text{Tr}(A^\dagger A)} \), it yields the result
\[
|S(\rho_t \| \rho_0) + S(\rho_0 \| \rho_t)| \leq \| \rho_0 \|_2 \int_0^\tau dt \left( \|H_t, \rho_t\|_2 + \|H_t, \rho_0\|_2 \right) .
\] (B21)

**Appendix C: Recovering min-relative entropy**

In this Appendix we will discuss the case \( \alpha = 0 \) for symmetric Rényi relative entropy, which is related to the min-relative entropy
\[
R_0(\rho : \omega) := R_0(\rho \| \omega) + R_0(\omega \| \rho) ,
\] (C1)
where \( R_0(\rho \| \omega) = -\ln \text{Tr}(\Pi_\rho \omega) \), with \( \Pi_\rho \) being the projector onto the support of the state \( \rho \). Here we will focus on the time-independent initial state \( \rho_0 \), and its evolved state \( \rho_t = U_t \rho_0 U_t^\dagger \). By using the triangle inequality \( |a_1 + a_2| \leq |a_1| + |a_2| \), the absolute value of the time-derivative of Eq. (C1) is written as
\[
\left| \frac{d}{dt} R_0(\rho_t : \rho_0) \right| \leq \left| \frac{d}{dt} R_0(\rho_t \| \rho_0) \right| + \left| \frac{d}{dt} R_0(\rho_0 \| \rho_t) \right| .
\] (C2)

From now on we will discuss the evaluation of each contribution in right-hand side of Eq. (C2). In order to do so, let \( \rho_0 = \sum_j p_\ell |\psi_\ell \rangle \langle \psi_\ell| \) be the spectral decomposition of the initial state \( \rho_0 \) into the basis \( \{ |\psi_\ell \rangle \}_{\ell = 1, \ldots, d} \), with \( 0 \leq p_\ell \leq 1 \) and \( \sum_\ell p_\ell = 1 \). By hypothesis, the support of \( \rho_0 \) has dimension \( d_{\rho_0} := \dim \text{supp}(\rho_0) \), and is given by \( \text{supp}(\rho_0) = \text{span}\{ |\psi_\ell \rangle : p_\ell \neq 0 \} \). Thus, the projector onto the support of state \( \rho_0 \) is defined as \( \Pi_{\rho_0} := \sum_{\ell : p_\ell \neq 0} |\psi_\ell \rangle \langle \psi_\ell| \).

The evolved state is given by \( \rho_t = U_t \rho_0 U_t^\dagger = \sum_j p_\ell |\psi_\ell \rangle \langle \psi_\ell| \), with \( |\psi_\ell \rangle := U_t |\psi_\ell \rangle \), and its support is defined as \( \text{supp}(\rho_t) = \text{span}\{ |\psi_\ell \rangle : p_\ell \neq 0 \} \). Noteworthy, since the unitary evolution does not change the purity of the initial state, i.e., both states \( \rho_0 \) and \( \rho_t \) share the same set of eigenvalues, we thus have \( \dim \text{supp}(\rho_t) = \dim \text{supp}(\rho_0) \). The projector \( \Pi_{\rho_t} \) onto the support of \( \rho_t \) read as
\[
\Pi_{\rho_t} = \sum_{\ell : p_\ell \neq 0} |\psi_\ell \rangle \langle \psi_\ell| = \sum_{\ell : p_\ell \neq 0} U_t |\psi_\ell \rangle \langle \psi_\ell| U_t^\dagger = U_t \rho_0 U_t^\dagger .
\] (C3)

Interestingly, starting from Eq. (C3), the projector \( \Pi_{\rho_t} \) fulfills the von Neumann-like equation \( (d \Pi_{\rho_t} / dt) = -i [H_t, \Pi_{\rho_t}] \), where we applied the identity \( U_t (dU_t^\dagger / dt) = -(dU_t / dt) U_t^\dagger = -i H_t \). Thus, the time-derivative of min-relative entropy \( R_0(\rho_t \| \rho_0) = -\ln \text{Tr}(\Pi_{\rho_t} \rho_0) \) read as
\[
\frac{d}{dt} R_0(\rho_t \| \rho_0) = \frac{i \text{Tr}(\rho_\text{p} [H_t, \Pi_{\rho_t}])}{\text{Tr}(\Pi_{\rho_t} \rho_0)} = \frac{i \text{Tr}(U_t^\dagger \rho_0 U_t [U_t^\dagger H_t U_t, \Pi_{\rho_0}])}{\text{Tr}(\Pi_{\rho_0} U_t^\dagger \rho_0 U_t)} ,
\] (C4)
where we have explicitly used the property obtained in Eq. (C3). By taking the absolute value of Eq. (C4), and thus applying the Cauchy-Schwarz inequality, |Tr(A_1 A_2)| ≤ ∥A_1∥_2 ∥A_2∥_2, with ∥A∥_2 = √Tr(A^† A), one obtains
\[
\left| \frac{d}{dt} R_0(\rho_t || \rho_0) \right| \leq \frac{\|\rho_0\|_2 \|[U_t^\dagger H_t U_t, \Pi_{\rho_0}]\|_2}{\|\text{Tr}(\Pi_{\rho_0} U_t^\dagger \rho_0 U_t)\|} . \tag{C5}
\]

Let us now move to the the second term in the right-hand side of Eq. (C2), which is related to the time-derivative of R_0(\rho_0 || \rho_t) = -ln Tr(\Pi_{\rho_0} \rho_t). In this case, one readily obtains
\[
\left| \frac{d}{dt} R_0(\rho_0 || \rho_t) \right| = \frac{i \text{Tr}(\Pi_{\rho_0} [H_t, \rho_t])}{\text{Tr}(\Pi_{\rho_0} \rho_t)} \quad = \frac{i \text{Tr}(U_t^\dagger \Pi_{\rho_0} U_t [U_t^\dagger H_t U_t, \rho_0])}{\text{Tr}(\Pi_{\rho_0} U_t^\dagger \rho_0 U_t^\dagger)} , \tag{C6}
\]
where we used the fact that \rho_t fulfills the von Neumann equation \(d\rho_t/dt = -i [H_t, \rho_t]\). By taking the absolute value of Eq. (C6), and thus applying the aforementioned Cauchy-Schwarz inequality, one obtains
\[
\left| \frac{d}{dt} R_0(\rho_0 || \rho_t) \right| \leq \frac{\|\Pi_{\rho_0}\|_2 \|[U_t^\dagger H_t U_t, \rho_0]\|_2}{\|\text{Tr}(\Pi_{\rho_0} U_t^\dagger \rho_0 U_t^\dagger)\|} . \tag{C7}
\]

Hence, by substituting Eqs. (C5) and (C7) into Eq. (C2), we thus have that
\[
\left| \frac{d}{dt} R_0(\rho_t : \rho_0) \right| \leq Q_0^r(\rho_0, \Pi_{\rho_0}) + Q_0^l(\Pi_{\rho_0}, \rho_0) , \tag{C8}
\]
where the functional Q_0(\rho_t, \rho) is defined as follows
\[
Q_0(A,B) := \frac{\|[U_t^\dagger H_t U_t, B]\|_2}{\|\text{Tr}(AU_t^\dagger BU_t^\dagger)\|} . \tag{C9}
\]

Finally, by integrating Eq. (C8) over the interval \(t \in [0, \tau]\), and thus applying the inequality \|\int dx f(x)\| \leq \int |dx| |f(x)|, one gets the inequality
\[
|R_0(\rho_\tau : \rho_0)| \leq \int_0^\tau dt \left[ Q_0^r(\rho_0, \Pi_{\rho_0}) + Q_0^l(\Pi_{\rho_0}, \rho_0) \right] . \tag{C10}
\]

Appendix D: Example: single-qubit state

1. Basic properties

In this Appendix we will present details for QSL bounds respective to Rényi and Tsallis relative entropies for the single-qubit state. Let \rho_0 = (1/2)(I + \hat{r} \cdot \sigma) be the Bloch sphere representation of the initial state of the system, where \sigma = (\sigma_x, \sigma_y, \sigma_z) is the vector of Pauli matrices, \hat{r} = r \hat{\sigma} is the Bloch vector, with \hat{r} = \{sin \theta \cos \phi, sin \theta \sin \phi, \cos \theta\}, 0 < r < 1, \theta \in [0, \pi] and \phi \in [0, 2\pi], while I is the 2 × 2 identity matrix. Particularly, for \(0 < \alpha < 1\), the operator \rho_0^\alpha respective to the initial single-qubit state is written as
\[
\rho_0^\alpha = \frac{1}{2} \left[ \xi_\alpha^+ I + \xi_\alpha^- (\hat{r} \cdot \hat{\sigma}) \right] , \tag{D1}
\]
where
\[
\xi_\alpha^\pm = 2^{-\alpha} [(1 + r)^\alpha \pm (1 - r)^\alpha] . \tag{D2}
\]
Therefore, starting from Eq. (D1) it is straightforward to verify that
\[
\|\rho_0^\alpha\|_2 = \sqrt{\xi_\alpha^+} . \tag{D3}
\]
Moreover, it can be proved that
\[ \| \ln \rho_0 \|_2 = \sqrt{\ln^2 \left( \frac{1-r}{2} \right) + \ln^2 \left( \frac{1+r}{2} \right)} . \] (D4)

For simplicity, we assume the dynamics of the system is governed by the time-dependent Hamiltonian \( H_t = \varpi \mathbb{1} + \hat{n}_t \cdot \vec{\sigma} \), where \( \hat{n}_t = \{ n^x_t, n^y_t, n^z_t \} \) is a unit vector with \( |\hat{n}_t| = 1 \), and \( \varpi \in \mathbb{R} \). In this case, the evolution operator thus reads as
\[ U_t = e^{-it\varpi} \cos(|\vec{u}_t|) \mathbb{1} - i \sin(|\vec{u}_t|) (\hat{u}_t \cdot \vec{\sigma}) , \] (D5)

where \( \vec{u}_t := \int_0^t ds \hat{n}_s \) is a time-dependent vector, while \( \hat{u}_t = \vec{u}_t / |\vec{u}_t| \) stands as its respective unit vector. Since \( H_t \) is not a self-commuting Hamiltonian at different times, i.e., \( [H_t, H_{t'}] \neq 0 \) for \( t \neq t' \), Eq. (D5) implies that
\[ U_t^\dagger H_t U_t = \varpi \mathbb{1} + \hat{\mu}_t \cdot \vec{\sigma} , \] (D6)

where the unit vector \( \hat{\mu}_t \) is defined by
\[ \hat{\mu}_t := \hat{n}_t - \sin(2|\vec{u}_t|) (\hat{u}_t \times \hat{n}_t) + 2 \sin^2(|\vec{u}_t|) [(\hat{u}_t \cdot \hat{n}_t)\hat{u}_t - \hat{n}_t] . \] (D7)

From Eq. (D6) one shall prove the commutator \([U_t^\dagger H_t U_t, \rho_0] = i r (\hat{\mu}_t \times \hat{r}) \cdot \vec{\sigma}) \) which in turn gives rise to the Schatten 2-norm as follows
\[ \| [H_t, \rho_0] \|_2 = \| [U_t^\dagger H_t U_t, \rho_0] \|_2 = r \sqrt{2 (1 - (\hat{\mu}_t \cdot \hat{r})^2)} . \] (D8)

Furthermore, by performing some straightforward calculations, one obtains
\[ \| [H_t, \rho_0] \|_2 = r \sqrt{2 (1 - (\hat{n}_t \cdot \hat{r})^2)} . \] (D9)

Now, starting from Eqs. (D1) and (D5), by performing a lengthy but straightforward calculation, one may verify the evolved density matrix \( \rho^n_t = U_t \rho^n_0 U_t^\dagger \) becomes
\[ \rho^n_t = \frac{1}{2} \left[ \xi^n \mathbb{1} + \xi^{{n}^*} (\hat{\mu}_t \cdot \vec{\sigma}) \right] , \] (D10)

with
\[ \hat{\mu}_t := \hat{r} + \sin(2|\vec{u}_t|) (\hat{u}_t \times \hat{r}) + 2 \sin^2(|\vec{u}_t|) [(\hat{u}_t \cdot \hat{r})\hat{u}_t - \hat{r}] . \] (D11)

Moreover, starting from Eqs. (D1) and (D6) we verify the commutator \([U_t^\dagger H_t U_t, \rho^n_0] = i \xi^n (\hat{\mu}_t \times \hat{r}) \cdot \vec{\sigma}) \) which immediately implies the Schatten 2-norm as follows
\[ \| [U_t^\dagger H_t U_t, \rho^n_0] \|_2 = \| [H_t, \rho^n_t] \|_2 = \xi^n \sqrt{2 (1 - (\hat{\mu}_t \cdot \hat{r})^2)} . \] (D12)

Let us now move to the \( \alpha \) relative purity. Based on Eqs. (D1) and (D10), the \( \alpha \) relative purity is written as
\[ g_{\alpha}(\rho_t, \rho_0) = 1 - \xi^{-\alpha} \xi^{1-\alpha} (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2(|\vec{u}_t|) . \] (D13)

Interestingly, we point out that relative purity in Eq. (D13) satisfies the symmetry property
\[ g_{\alpha}(\rho_t, \rho_0) = g_{1-\alpha}(\rho_t, \rho_0) = g_{\alpha}(\rho_0, \rho_t) . \] (D14)

Note that, even though relative entropy is asymmetric a priori, for the single-qubit state \( \rho_0 = (1/2)(\mathbb{1} + \hat{r} \cdot \vec{\sigma}) \) and \( \rho_t = U_t \rho_0 U_t^\dagger \), with \( U_t \) given in Eq. (D5), one may verify that \( S(\rho_t \| \rho_0) = S(\rho_0 \| \rho_t) \), i.e.,
\[ S(\rho_t \| \rho_0) = S(\rho_0 \| \rho_t) = r \ln \left( \frac{1+r}{1-r} \right) (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2(|\vec{u}_t|) . \] (D15)
2. Tsallis relative entropy (TRE)

From Appendix A [see Eq. (A10)], it is straightforward to prove that Tsallis relative entropy (TRE) implies the quantum speed limit time as follows

$$\tau_H^\alpha (\rho_0 : \rho_\tau) = \frac{|\alpha - 1| |H_\alpha (\rho_\tau \| \rho_0) + H_\alpha (\rho_0 \| \rho_\tau)|}{\langle G^H_\alpha (\rho_\tau, \rho_0) + G^H_{1-\alpha} (\rho_\tau, \rho_0) \rangle_\tau}. \quad (D16)$$

Thus, for an initial single-qubit state evolving unitarily according the physical setting presented in Sec. D1, the symmetric Tsallis relative entropy is given by

$$H_\alpha (\rho_\tau \| \rho_0) + H_\alpha (\rho_0 \| \rho_\tau) = \frac{2 - g_\alpha (\rho_\tau, \rho_0) - g_\alpha (\rho_0, \rho_\tau)}{1 - \alpha}$$

$$= \frac{2 [1 - g_\alpha (\rho_\tau, \rho_0)]}{1 - \alpha}$$

$$= \frac{2 \xi_\alpha \xi_{1-\alpha} |1 - (\hat{u}_\tau \cdot \hat{r})^2| \sin^2(|\hat{u}_\tau|)}{1 - \alpha}, \quad (D17)$$

where we have applied the property $g_\alpha (\rho_0, \rho_\tau) = g_{1-\alpha} (\rho_\tau, \rho_0) = g_\alpha (\rho_\tau, \rho_0)$, and also used Eq. (D13). Hence, by using Eqs. (D3) and (D12), the time average $\sum_{s=\{1, 0\}} \langle G^s_H (\rho_t, \rho_0) \rangle_\tau$ for the single-qubit state read as

$$\langle G^H_\alpha (\rho_t, \rho_0) + G^H_{1-\alpha} (\rho_t, \rho_0) \rangle_\tau = \| \rho_0^{1-\alpha} \|_2 \frac{1}{\tau} \int_0^\tau dt \| [U_t^\dagger H_t U_t, \rho_0^\alpha] \|_2 + \| \rho_0^\alpha \|_2 \frac{1}{\tau} \int_0^\tau dt \| [U_t^\dagger H_t U_t, \rho_0^{1-\alpha}] \|_2$$

$$= \sqrt{2} \left( \sqrt{\xi_{2-2\alpha} \xi_\alpha} + \sqrt{\xi_{2\alpha} \xi_{1-\alpha}} \right) \frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{\mu}_t \cdot \hat{r})^2}. \quad (D18)$$

**FIG. 2.** (Color online) Density plot of QSL time $\tau_H^\alpha$, as a function of time $\tau$ and $\alpha$, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \hat{\sigma}$, with $\hat{n}_t = N^{-1} (\Delta, 0, vt)$ and $N := \sqrt{\Delta^2 + (vt)^2}$. Here we choose the initial single qubit state $\rho_0 = (1/2) (I + \hat{r} \cdot \hat{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also setting the ratio (a) $\Delta/v = 0.5$, (b) $\Delta/v = 1$, (c) $\Delta/v = 5$, and (d) $\Delta/v = 10$. 
Finally, by substituting Eqs. (D17) and (D18) into Eq. (D16), QSL time related to Tsallis relative entropy for the single-qubit state, thus becomes

$$
\tau_{\alpha}^H (\rho_0 : \rho_r) = \frac{\sqrt{2} \xi^- \xi_{1-\alpha}^- |1 - (\hat{u}_\tau \cdot \hat{r})^2| \sin^2 (|\tilde{u}_\tau|)}{\left( \sqrt{\xi_{2-2\alpha}^- \xi^- + \sqrt{\xi_{2\alpha}^+ \xi_{1-\alpha}^-}} \right)} \left( \frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{u}_t \cdot \hat{r})^2} \right)^{-1}.
$$

(D19)

In Fig. 2 we plot the QSL time $\tau_{\alpha}^H$, as a function of time $\tau$ and $\alpha$, for the initial single qubit state $\rho_0 = (1/2)(I + \vec{r} \cdot \vec{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also varying the ratio (a) $\Delta/v = 0.5$, (b) $\Delta/v = 1$, (c) $\Delta/v = 5$, and (d) $\Delta/v = 10$.

3. Rényi relative entropy (RRE)

From Appendix A [see Eq. (A10)], it is straightforward to verify that the symmetrized Rényi relative entropy implies the quantum speed limit time as follows

$$
\tau_{\alpha}^R (\rho_0 : \rho_r) = \frac{|\alpha - 1| |R_\alpha (\rho_r \|= \rho_0) + R_\alpha (\rho_0 \|= \rho_r)|}{\langle G^R_\alpha (\rho_t, \rho_0) + G^R_{1-\alpha} (\rho_t, \rho_0) \rangle}. \quad (D20)
$$

By considering the initial single-qubit state evolving unitarily according the physical setting presented in Sec. D1, the symmetric Rényi relative entropy is given by

$$
R_\alpha (\rho_r \|= \rho_0) + R_\alpha (\rho_0 \|= \rho_r) = \frac{\ln [g_\alpha (\rho_r || \rho_0)] + \ln [g_\alpha (\rho_0 || \rho_r)]}{\alpha - 1} = \frac{\ln (1 - \xi^- \xi_{1-\alpha}^- (1 - (\hat{u}_\tau \cdot \hat{r})^2) \sin^2 (|\tilde{u}_\tau|))}{\alpha - 1}, \quad (D21)
$$

where we have applied the property $g_\alpha (\rho_0 || \rho_r) = g_{1-\alpha} (\rho_r || \rho_0) = g_\alpha (\rho_r || \rho_0)$, and also used Eq. (D13). Hence, by using Eqs. (D3) and (D12), the time average $\sum_{s=\{1,\alpha\}} \langle G_s^R (\rho_t, \rho_0) \rangle_{\tau}$ for the single-qubit state read as

$$
\langle G^R_\alpha (\rho_t, \rho_0) + G^R_{1-\alpha} (\rho_t, \rho_0) \rangle = \int_0^\tau \left( \sqrt{\xi_{2-2\alpha}^- \xi^- + \sqrt{\xi_{2\alpha}^+ \xi_{1-\alpha}^-}} \right) \frac{1}{\tau} dt \int_0^\tau \frac{\sqrt{1 - (\hat{u}_t \cdot \hat{r})^2}}{1 - \xi^- \xi_{1-\alpha}^- (1 - (\hat{u}_\tau \cdot \hat{r})^2) \sin^2 (|\tilde{u}_\tau|)}. \quad (D22)
$$

Finally, by substituting Eqs. (D21) and (D22) into Eq. (D20), QSL time related to Tsallis relative entropy for the single-qubit state, thus becomes

$$
\tau_{\alpha}^R (\rho_0 : \rho_r) = \frac{\sqrt{2} |\ln (1 - \xi^- \xi_{1-\alpha}^- (1 - (\hat{u}_\tau \cdot \hat{r})^2) \sin^2 (|\tilde{u}_\tau|))|}{\left( \sqrt{\xi_{2-2\alpha}^- \xi^- + \sqrt{\xi_{2\alpha}^+ \xi_{1-\alpha}^-}} \right)} \left( \frac{1}{\tau} \int_0^\tau dt \sqrt{1 - (\hat{u}_t \cdot \hat{r})^2} \right)^{-1}. \quad (D23)
$$

In Fig. 3 we plot the QSL time $\tau_{\alpha}^R$, as a function of time $\tau$ and $\alpha$, for the initial single qubit state $\rho_0 = (1/2)(I + \vec{r} \cdot \vec{\sigma})$ with $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, and also varying the ratio (a) $\Delta/v = 0.5$, (b) $\Delta/v = 1$, (c) $\Delta/v = 5$, and (d) $\Delta/v = 10$.

4. Relative entropy (RE)

From Appendix B [see Eq. (B21)], one may verify that relative entropy (RE) implies the following quantum speed limit time

$$
\tau_{\alpha}^{RE} (\rho_0 : \rho_r) := \frac{|S(\rho_r || \rho_0) + S(\rho_0 || \rho_r)|}{\| \rho_0 \|^2 \langle \| H_t, \rho_t \| \rangle + \| H_t, \rho_0 \|^2 \rangle}). \quad (D24)
$$
FIG. 3. (Color online) Density plot of QSL time $\tau_{RE}^R$, as a function of time $\tau$ and $\alpha$, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = N^{-1}\{\Delta, 0, vt\}$ and $N := \sqrt{\Delta^2 + (vt)^2}$. Here we choose the initial single qubit state $\rho_0 = (1/2)(I + \vec{r} \cdot \vec{\sigma})$ with $(r, \theta, \phi) = \{1/4, \pi/4, \pi/4\}$, and also setting the ratio $(a) \Delta/v = 0.5$, $(b) \Delta/v = 1$, $(c) \Delta/v = 5$, and $(d) \Delta/v = 10$.

By considering the initial single-qubit state evolving unitarily according the physical setting presented in Sec. D 1, the symmetric relative entropy is given by [see Eq. (D15)]

$$S(\rho_\tau \| \rho_0) + S(\rho_0 \| \rho_\tau) = 2r \ln \left( \frac{1 + r}{1 - r} \right) \left( 1 - (\vec{u}_\tau \cdot \vec{r})^2 \right) \sin^2(|\vec{u}_\tau|)$$ \hspace{1cm} (D25)

where we have used Eq. (D15). Now, by using Eqs. (D9) and (D8) we thus obtain the following time average

$$\langle \| [H_t, \rho_t] \|_2 + \| [H_t, \rho_0] \|_2 \rangle_\tau = \frac{1}{\tau} \int_0^\tau dt \left( \| [H_t, \rho_t] \|_2 + \| [H_t, \rho_0] \|_2 \right),$$

$$= \sqrt{2} r \frac{1}{\tau} \int_0^\tau dt \left( \sqrt{1 - (\vec{u}_t \cdot \vec{r})^2} + \sqrt{1 - (\vec{n}_t \cdot \vec{r})^2} \right).$$ \hspace{1cm} (D26)

Finally, by substituting Eqs. (D4), (D25), and (D26) into Eq. (D24), QSL time related to the Umegaki’s relative entropy for the single-qubit state read as

$$\tau_{RE}^{\text{QSL}}(\rho_0 : \rho_\tau) = \frac{\sqrt{2} \ln \left( \frac{1 + r}{1 - r} \right) \left( 1 - (\vec{u}_\tau \cdot \vec{r})^2 \right) \sin^2(|\vec{u}_\tau|)}{\sqrt{\ln^2 \left( \frac{1 + r}{1 - r} \right) + \ln^2 \left( \frac{1 + r}{1 - r} \right)}} \left( \int_0^\tau dt \left( \sqrt{1 - (\vec{u}_t \cdot \vec{r})^2} + \sqrt{1 - (\vec{n}_t \cdot \vec{r})^2} \right) \right)^{-1}.$$ \hspace{1cm} (D27)

In Fig. 4 we plot the QSL time $\tau_{RE}^{\text{QSL}}$, as a function of time $\tau$ and $\Delta/v$, for the initial single qubit state $\rho_0 = (1/2)(I + \vec{r} \cdot \vec{\sigma})$ with (a) $(r, \theta, \phi) = \{1/4, \pi/4, \pi/4\}$, (b) $(r, \theta, \phi) = \{1/4, \pi/3, \pi/4\}$, (c) $(r, \theta, \phi) = \{1/2, \pi/4, \pi/4\}$, and (d) $(r, \theta, \phi) = \{1/2, \pi/3, \pi/4\}$. 
We shall begin our analysis with the following statement: if the state initial single qubit state $\rho$ generated by the time-dependent Hamiltonian $H$ defined as follows
$$
|H\rangle = \frac{(1/2)(1 + r \cdot \hat{\sigma})}{\sqrt{\Delta^2 + (\Delta t)^2}}.
$$
Here we choose the initial single qubit state $\rho_0 = (1/2)(1 + r \cdot \hat{\sigma})$ with (a) $\{r, \theta, \phi\} = \{1/4, \pi/4, \pi/4\}$, (b) $\{r, \theta, \phi\} = \{1/4, \pi/3, \pi/4\}$, (c) $\{r, \theta, \phi\} = \{1/2, \pi/4, \pi/4\}$, and (d) $\{r, \theta, \phi\} = \{1/2, \pi/3, \pi/4\}$.

5. Min-relative entropy

From Appendix C [see Eq. (C10)], it can be readily proved that for $\alpha = 0$ the min-relative entropy implies the quantum speed limit time

$$
\tau_R^{(\rho_0 \to \rho_T)} := |R_0(\rho_T \| \rho_0) + R_0(\rho_0 \| \rho_T) - \langle Q_0(\rho_0, \Pi_{\rho_0}) + Q_0(\Pi_{\rho_0}, \rho_0) \rangle| \tau,
$$

(D28)

where $R_0(\rho \| \omega) = -\ln \text{Tr}(\Pi_{\rho} \omega)$, with $\Pi_{\rho}$ being the projector onto the support of the state $\rho$, and the functional $Q_0$ defined as follows

$$
Q_0(A, B) := \frac{\|A\|_2 \|U_H^\dagger H U_{\ell}, B\|_2}{|\text{Tr}(AU_H^\dagger U_{\ell}^\dagger)|}.
$$

We shall begin our analysis with the following statement: if the state $\rho_0 = \sum p_\ell |\psi_\ell\rangle \langle \psi_\ell|$ is a full-rank density matrix, then $\dim[\text{supp}(\rho_0)] = \dim[\text{rank}(\rho_0)]$, Eq. (D28) thus gives rise to the trivial bound $\tau_R^{(0)} = 0$. In order to see this, note the projector $\Pi_{\rho_0}$ onto the support of the full-rank state $\rho_0$ is equal to the identity, $\Pi_{\rho_0} = \sum_{\ell, p_\ell \neq 0} |\psi_\ell\rangle \langle \psi_\ell| = I$. Hence, it is straightforward to verify the symmetric min-entropy is identically zero because $R_0(\rho_T \| \rho_0) = -\ln[\text{Tr}(U_H^\dagger \rho_0 U_{\ell}^\dagger)] = 0$ and $R_0(\rho_0 \| \rho_T) = -\ln[\text{Tr}(U_{\ell}^\dagger \rho_0 U_H^\dagger)] = 0$, while the functional $Q_0$ reads $Q_0(\rho_0, \Pi_{\rho_0}) = 0$ and $Q_0(\Pi_{\rho_0}, \rho_0) = \sqrt{d} \|U_H^\dagger H U_{\ell}, \rho_0\|_2$, where $d = \dim(\mathcal{H})$ stands for the dimension of the Hilbert space $\mathcal{H}$. Therefore, we prove the aforementioned statement.

From now on, we will choose the initial state $\rho_0$ being a pure one, i.e., a non full-rank density matrix. In particular, for a single-qubit state such a condition is equivalent to imposing the purity value $r = 1$, i.e., $\rho_0 = (1/2)(I + r \cdot \hat{\sigma})$, which in turn implies the spectral decomposition $\rho_0 = \sum_{\ell, p_\ell \neq 0} p_\ell |\psi_\ell\rangle \langle \psi_\ell|$, with eigenvalues $p_+ = 1$ and $p_- = 0$, and eigenstates $|\psi_+\rangle = |\theta, \phi\rangle$ and $|\psi_-\rangle = |\theta - \pi, \phi\rangle$, with $|\theta, \phi\rangle := \cos(\theta/2)|0\rangle + e^{-i\phi}\sin(\theta/2)|1\rangle$. Just to clarify, here $|0\rangle = (1 \quad 0)^T$ and $|1\rangle = (0 \quad 1)^T$ define the standard states of the computational basis. In this case, the projector onto
the initial pure single qubit state $\rho$ the support of $\tau$

FIG. 5. (Color online) Density plot of QSL time $\tau_0^\Delta$, as a function of time $\tau$ and the ratio $\Delta/v$, respective to the unitary evolution generated by the time-dependent Hamiltonian $H_t = \hat{n}_t \cdot \vec{\sigma}$, with $\hat{n}_t = N^{-1} \{ \Delta, 0, \nu t \}$ and $N := \sqrt{\Delta^2 + (\nu t)^2}$. Here we choose the initial pure single qubit state $\rho_0 = (1/2)(1 + \hat{r} \cdot \vec{\sigma})$ with $r = 1$ and (a) $\{ \theta, \phi \} = \{ \pi/4, \pi/4 \}$; (b) $\{ \theta, \phi \} = \{ \pi/3, \pi/4 \}$; (c) $\{ \theta, \phi \} = \{ \pi/4, \pi/3 \}$; and (d) $\{ \theta, \phi \} = \{ \pi/3, \pi/3 \}$.

the support of $\rho_0$ read as $\Pi_{\rho_0} = |\psi_+\rangle \langle \psi_+|$. Moving forward, one may proceed the calculation as follows

$$\begin{align*}
\text{Tr}(\Pi_{\rho_0} U_\tau \rho_0 U_\tau^\dagger) &= \text{Tr}(\rho_0 U_\tau \Pi_{\rho_0} U_\tau^\dagger) \\
&= |\langle \psi_+ | U_\tau | \psi_+ \rangle|^2 \\
&= 1 - (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2(|\hat{u}_t|),
\end{align*}$$

where we have applied the expression of $U_\tau$ presented in Eq. (D9). Now, by using the result of Eq. (D30), one may evaluate the symmetric min-entropy as

$$\begin{align*}
R_0(\rho_0 \| \rho_0) + R_0(\rho_0 \| \rho_\tau) &= -\ln[\text{Tr}(\Pi_{\rho_0} U_\tau \rho_0 U_\tau^\dagger)] - \ln[\text{Tr}(\rho_0 U_\tau \Pi_{\rho_0} U_\tau^\dagger)] \\
&= -2 \ln (1 - (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2(|\hat{u}_t|)).
\end{align*}$$

Furthermore, given that $\langle \psi_+ | (U_\tau^\dagger H_t U_\tau)^2 | \psi_+ \rangle = 1 + \varpi^2 + 2 \varpi (\hat{\mu}_t \cdot \hat{r})$ and also $\langle \psi_+ | U_\tau^\dagger H_t U_\tau | \psi_+ \rangle = \varpi + \hat{\mu}_t \cdot \hat{r}$, one obtains

$$\begin{align*}
|||U_\tau^\dagger H_t U_\tau | \rho_0||_2^2 &= |||U_\tau^\dagger H_t U_\tau, \Pi_{\rho_0}||_2^2 \\
&= 2 \left( \langle \psi_+ | (U_\tau^\dagger H_t U_\tau)^2 | \psi_+ \rangle - \langle \psi_+ | U_\tau^\dagger H_t U_\tau | \psi_+ \rangle^2 \right) \\
&= 2 \left( 1 - (\hat{\mu}_t \cdot \hat{r})^2 \right).
\end{align*}$$

From Eqs. (D30), and (D32), and also using the Schatten 2-norms $|||\rho_0||_2 = |||\Pi_{\rho_0}||_2 = 1$, the time average exhibited in $\tau_0^{(\Delta)}$ read as

$$\begin{align*}
\text{Tr} \left[ Q_0^\Delta(\rho_0, \Pi_{\rho_0}) + Q_0^\Delta(\Pi_{\rho_0}, \rho_0) \right]_{\tau} = \frac{2\sqrt{2}}{\tau} \int_0^\tau dt \frac{\sqrt{1 - (\hat{\mu}_t \cdot \hat{r})^2}}{1 - (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2(|\hat{u}_t|)}. 
\end{align*}$$
Finally, by substituting Eqs. (D31) and (D33) into Eq. (D28), we thus obtain

\[
\tau_0^R (\rho_0 : \rho_r) = \frac{1}{\sqrt{2}} \left| \ln \left( 1 - (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2 (|\vec{u}_t|) \right) \right| \left[ \frac{1}{\tau} \int_0^\tau dt \sqrt{\frac{1 - (\hat{u}_t \cdot \hat{r})^2}{1 - (1 - (\hat{u}_t \cdot \hat{r})^2) \sin^2 (|\vec{u}_t|)}} \right]^{-1}.
\]

(D34)

In Fig. 5 we plot the QSL time \( \tau_0^R \) as a function of time \( \tau \) and \( \Delta / \nu \), for the initial pure single qubit state \( \rho_0 = (1/2)(\mathbb{I} + \hat{r} \cdot \vec{\sigma}) \) with \( r = 1 \) and (a) \( \{\theta, \phi\} = \{\pi/4, \pi/4\} \); (b) \( \{\theta, \phi\} = \{\pi/3, \pi/4\} \); (c) \( \{\theta, \phi\} = \{\pi/4, \pi/3\} \); and (d) \( \{\theta, \phi\} = \{\pi/3, \pi/3\} \).