The onset of layer undulations in smectic A liquid crystals due to a strong magnetic field

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Abstract

We investigate the effect of a strong magnetic field on a three dimensional smectic A liquid crystal. We identify a critical field above which the uniform layered state loses stability; this is associated to the onset of layer undulations. In a previous work García-Cervera and Joo (2012 Arch. Ration. Mech. Anal. 203 1–43), García-Cervera and Joo considered the two dimensional case and analyzed the transition to the undulated state via a simple bifurcation. In dimension \( n = 3 \) the situation is more delicate because the first eigenvalue of the corresponding linearized problem is not simple. We overcome the difficulties inherent to this higher dimensional setting by identifying the irreducible representations for natural actions on the functional that take into account the invariances of the problem thus allowing for reducing the bifurcation analysis to a subspace with symmetries. We are able to describe at least two bifurcation branches, highlighting the richer landscape of energy critical states in the three dimensional setting. Finally, we analyze a reduced two dimensional problem, assuming the magnetic field is very strong, and are able to relate this to a model in micromagnetics studied in Alouges \textit{et al}.
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(Some figures may appear in colour only in the online journal)

1. Introduction

The main thrust of this work is to understand the undulation phenomena in smectic A liquid crystals caused by a strong magnetic field. The mathematical framework for our study is the model introduced by de Gennes [11]. The Landau de Gennes energy describes the state of the liquid crystal in terms of a director \( n \) and a complex order parameter

\[ \Psi = \rho(x)e^{iq\omega(x)}, \]

for the layered structure. The molecular mass density is defined by

\[ \delta(x) = \rho_0(x) + \frac{1}{2}(\Psi(x) + \Psi^*(x)) = \rho_0(x) + \rho(x) \cos(q\omega(x)), \]

where \( \rho_0 \) is a locally uniform mass density, \( \rho(x) \) is the mass density of the smectic layers, and \( \omega \) parametrizes the layers so that \( \nabla \omega \) is the direction of the layer normal. Also, \( q \) is the wave number and \( 2\pi/q \) is the layer thickness. The state \( \Psi = 0 \) corresponds to no layered structure (the sample is in a nematic phase). If \( |\Psi| = \rho \) is a nonzero constant, then the sample is in a smectic state throughout. The contribution to the energy due to the external magnetic field is accounted for by the magnetic free energy density

\[ -\chi_aH^2(n \cdot h)^2, \]

where \( \chi_a \) is the magnetic anisotropy, \( H \) is the magnitude of the magnetic field and \( h \) is a unit vector representing the direction of the field. We assume that \( \chi_a < 0 \). As a consequence, the director has a preferred orientation perpendicular to the direction of the applied magnetic field.\footnote{In [21], \( \chi_a \) is assumed to be positive; in that case the preferred orientation for the director is parallel to the field which fully specifies the orientation in that 2d setting.}

In the one constant approximation case for the Oseen-Frank nematic energy, the total free energy for smectic A is given by,

\[
F_0(\Psi, n) = \int_{\Omega} \left[ C |\nabla \Psi - iqn\Psi|^2 + K|\nabla n|^2 + \frac{g_0}{2} \left( |\Psi|^2 - \frac{T}{g_0} \right)^2 - \chi_aH^2(n \cdot h)^2 \right],
\]

where the material parameters \( C, K, g_0 \) and temperature dependent parameter \( r = T_{NA} - T \) are fixed positive constants and

\[
\Omega = (-L_1, L_1) \times (-L_2, L_2) \times (-d_0, d_0)
\]

is a rectangular box. This models a smectic A liquid crystal with smectic layers parallel to the top and bottom boundaries.

In our study we assume that the magnetic field is applied in the \( z \)-direction, perpendicular to the layers, that is \( h = e_3 \). With the director constraint \(|n| = 1\), we rewrite the magnetic energy density
by omitting the constant term.

1.1. Nondimensionalization

We define the dimensionless order parameter, \( \psi = \frac{\sqrt{\pi}}{r} \Psi \), and do the change of variables \( \tilde{x} = x / d \) with \( d = 2d_0 / \pi \), and obtain the following nondimensionalized energy

\[
F(\psi, n) := \frac{\varepsilon}{d\lambda} F_0(\psi, n) = \int_{\Omega} \left( \varepsilon \varepsilon \nabla \psi - \varepsilon \varepsilon \nabla \nabla \psi + \varepsilon \varepsilon \nabla n \right) + \frac{\varepsilon}{2(1 - \varepsilon \varepsilon)} (1 - \varepsilon \varepsilon)^2 - \tau (n_1^2 + n_2^2),
\]

where

\[
\varepsilon = \frac{\lambda}{d \sqrt{g_0}}, \quad \lambda = \frac{K}{\sqrt{C q^2}}, \quad \tau = \frac{|\chi_d|d^2 d \varepsilon}{K}, \quad c = \frac{Cr}{Kg_0}, \quad g = \frac{r}{C q^2},
\]

and \( \Omega = ( -\pi/a, \pi/a ) \times ( -\pi/b, \pi/b ) \times ( -\pi/2, \pi/2 ) \) with \( a = 2d_0 / l_1 \) and \( b = 2d_0 / l_2 \). The values \( d = 1 \text{ mm} \) and \( \lambda = 20 \text{ A} \) are employed in [12]. The dimensionless parameter \( \varepsilon \) is in fact the ratio of the layer thickness to the sample thickness. The case \( \varepsilon \ll 1 \) corresponds to a lower dimensional approximation and this is the point of view in the works [4, 21, 23, 33] where the emergence of interesting structures is captured in minimizers via a \( \Gamma \)-convergence approach. We point out that this kind of variational dimension reduction has already been considered in the analysis of the related Ginzburg–Landau model [6, 8, 9]. One strength of our approach is that we do not require \( \varepsilon \) to be small but just that it avoids values at which resonances occur. The other advantage is that the branches we find do not only start near the lowest critical field but at higher bifurcation values as well, so that metastable bifurcations are also obtained.

1.2. The onset of layer undulations

We are interested in studying the stability of the uniformly layered state:

\( (\psi_0, n_0) \equiv (e^{i\psi_0}, e_3) \).

To that end, we impose periodic boundary conditions for \( \psi \) and \( n \) in the \( x \) and \( y \) directions, while we assume Dirichlet boundary conditions for \( \psi \) and \( n \) on the boundary plates:

\[
\psi(x, y, \pm \pi/2) = e^{i\pm\pi/2 \varepsilon} \quad \text{and} \quad n(x, y, \pm \pi/2) = e_3,
\]

for all \((x, y) \in [-\pi/a, \pi/a] \times [-\pi/b, \pi/b] \). With this boundary condition, we assume that \( \Omega = T^2 \times I \), where \( T^2 \) is the two torus identified with \([-\pi/a, \pi/a] \times [-\pi/b, \pi/b] \) and \( I = [-\pi/2, \pi/2] \).

It is known that the application of an electric or magnetic field of large enough strength, or bare mechanical tension [13, 18, 19, 27, 35–38], can trigger the emergence of undulations in the smectic layers: this is called the Helfrich–Hurlaut effect [25, 26]. In [21], the effect of a magnetic field on a 2d smectic A liquid crystal was considered; a threshold value of the intensity, above which periodic layer undulations are observed, was estimated and a nontrivial solution curve was found. The proof relied on Crandall–Rabinowitz bifurcation theory for simple eigenvalues and thus the solution curve branches off of the lowest eigenvalue. In our situation, the spectral analysis of the linearized operator about the uniformly layered state gives a lowest eigenvalue which is not simple and so we cannot follow the same route to study the instability

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of the undeformed state at the critical magnetic field. Instead, we work in a space of maximal symmetries where the resonances are avoided; This is akin to the Palais principle of symmetric criticality [32]. The previous analysis yields the existence of branches stemming from the critical magnetic field with the symmetries of these one dimensional fixed point subspaces of irreducible representations (see section 4 for more details). A similar approach has been used to obtain multi-pole periodic solutions to the Gross–Pitaevskii equation in the plane [7]; these solutions arise as global bifurcations of a nonlinear Schrödinger equation. We also mention the work of Chillingworth [5] on a free energy function for biaxial nematics where the author describes bifurcations from isotropy using the symmetries of the problem and a normal form approach.

Our main result is the following.

**Theorem 1.** There exists a critical magnetic field $\tau_c$ such that the uniformly layered state has two bifurcating solutions starting from $\tau_c$, except for a finite union of hypersurfaces in the set of parameters $(\alpha, \beta, \epsilon)$.

In section 4 we give more precise information about these branches, in particular we find their symmetries and we provide their local asymptotic profile at the end of the section. We are also able to retrieve other bifurcation branches not covered by theorem 1, namely for the resonant case $\alpha = \beta$ (see proposition 12 for more details). We remark that while it is known that the uniformly layered state must lose its stability to one branch stemming at the critical value $\tau_c$, we do not know if the ones we find are stable or not; our method of construction allows, at best, to study stability within the restricted subspaces where the branches are found. Nevertheless, numerical simulations at the end of this article (see figure 2) provide supporting evidence that the branch we find with finite isotropy group, is the ground state.

Next, we investigate what happens at each cross section of the domain under the assumption that the magnetic field is large enough to render a planar configuration. We find this reduced model bears strong resemblance with the one for micromagnetics considered in [1] and from this we derive important qualitative information about minimizers in the limit $\epsilon \to 0$, including a general compactness statement and a sharp lower bound that allows us to compare between natural candidates for the optimal pattern. Finally, we provide some numerical evidence in support of our findings.

We now give an outline of the paper. In the next section we introduce notation and the functional setting where our bifurcation problem is posed. We then proceed to study the spectral problem of the linearized operator at the undeformed state and derive the critical field in section 3. In section 4 we study the irreducible representations associated to $O(2) \times O(2) \times \mathbb{Z}_2$ and the corresponding isotropy groups; from these we obtain two fixed one dimensional subspaces were we find the aforementioned bifurcation branches. We move to the 2d dimensional problem in section 5 and deduce the periodicity of minimizers. In the final section we present the numerical simulations depicting the 2d planar configurations and layers at the onset of undulations, confirming that the estimate of the critical field from our theory is consistent with the numerical results.

2. Setting up the problem

We start by introducing some notation which is going to be essential for the bifurcation analysis.

2.1. Some notions of actions on functionals

Let $G$ be a group. We recall that a Hilbert space $\mathcal{H}$ is a $G$-representation if there is a morphism of groups given by a map $\rho : G \to GL(\mathcal{H})$. Correspondingly, a set $\Omega$ is said to be $G$-invariant.
if $\rho(\gamma)x \in \Omega$ for all $\gamma \in G$, $x \in \Omega$. An irreducible representation is an invariant subspace of $\mathcal{H}$ without a proper invariant subspace.

For a subgroup $H \subset G$, we consider the fixed point space

$$\text{Fix}(H) = \{x \in \mathcal{H} : \rho(\gamma)x = x \text{ for all } \gamma \in H\}.$$  

The orbit and the isotropy group of a point $x \in \mathcal{H}$ are defined as

$$Gx = \{\rho(\gamma)x : \gamma \in G\} \text{ and } G_x = \{\gamma : \rho(\gamma)x = x\} \text{ respectively.}$$

It is said that a map $f : \mathcal{H} \to \mathcal{H}$ is $G$-equivariant if

$$f(\rho(\gamma)x) = \rho(\gamma)f(x), \text{ for all } \gamma \in G.$$ 

Finally, a map $F : \mathcal{H} \to \mathbb{R}$ is $G$-invariant if $F(x) = F(\rho(\gamma)x)$ for all $\gamma \in G$.

The following properties are used in this article. For a proof, see also [2, 3, 15, 17, 24] for equivariant bifurcation theory.

**Lemma 2 ([28]).**

1. If $f : \mathcal{H} \to \mathcal{H}$ is $G$-equivariant and $x \in \text{Fix}(H)$, then the linear map $f'(x)$ is $H$-equivariant.

2. If $\rho(\gamma)$ is an isometry in $\mathcal{H}$ for all $\gamma \in G$, i.e. $\rho : G \to \mathcal{O}(\mathcal{H})$, then the gradient of a $G$-invariant map $F$ is $G$-equivariant, i.e.

$$f(x) = \nabla F : \mathcal{H} \to \mathcal{H}$$

is $G$-equivariant.

3. The restriction of an $G$-equivariant map $f : \mathcal{H} \to \mathcal{H}$ to $\text{Fix}(H)$ is well defined, i.e. $f(x) \in \text{Fix}(H)$ for all $x \in \text{Fix}(H)$.

The previous definitions and statement are adapted for a manifold $M \subset \mathcal{H}$, if $M$ is invariant by the linear transformation $\rho(\gamma)$. In this case the action is

$$\rho(\gamma)u = \rho(\gamma, u) : G \times M \to M.$$ 

2.2. Natural actions on the Landau–de Gennes energy

The Landau–de Gennes energy $F$ introduced in (2) is well defined in the manifold

$$\mathcal{M} = \{(\psi, n) \in H^2(T^2 \times I; \mathbb{C} \times S^2) : (\psi, n)(x, y, \pm \pi/2) = (e^{\pi i/2}e, e_3)\}.$$ 

Since $\mathcal{M}$ is contained in the Hilbert space $L^2(T^2 \times I; \mathbb{C} \times \mathbb{R}^2)$, we define the action of the group

$$G := \mathcal{O}(2) \times \mathcal{O}(2) \times \mathbb{Z}_2$$

in this Hilbert space as follows.

1. The rotations.  
   For $(\varphi, \theta) \in [0, 2\pi)^2 = \mathcal{O}(2) \times \mathcal{O}(2)$,

$$\rho(\varphi, \theta)(\psi, n) = (\psi, n)(x + \varphi la, y + \theta lb, z),$$

2. The reflections.  
   The action of the planar reflections $\kappa_x, \kappa_y \in \mathcal{O}(2) \times \mathcal{O}(2)$ is given by

$$\rho(\kappa_x)(\psi, n) = (\psi, R_{\varphi}n)(-x, y, z),$$

$$\rho(\kappa_y)(\psi, n) = (\psi, R_{\varphi}n)(x, -y, z),$$

where $R_\varphi = \text{diag}(-1, 1, 1)$ and $R_\gamma = \text{diag}(1, -1, 1)$.  

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The above defines the action of the direct product of the two copies of the orthogonal group.

3. The $\mathbb{Z}_2$ component.
We define the action of the reflection $\kappa \in \mathbb{Z}_2$ by

$$\rho(\kappa)(\psi, n) = (\psi, R_x R_y n)(x, y, z).$$

We have:

**Lemma 3.** The map $F : \mathcal{M} \to \mathbb{R}$ is $G$-invariant.

**Proof.** Since $F$ is invariant by translations and reflections in $x$ and $y$, due to the boundary conditions, then it is invariant by the action of $O(2) \times O(2)$. Moreover, it is clear that all the terms in $F$ are invariant by the action of $\kappa$ except maybe for $|c \varepsilon \nabla \psi - i n \psi|^2$. To see that this term is invariant too, let us denote $(\psi, \tilde{n}) = \rho(\kappa)(\psi, n)$, then $\nabla \tilde{\psi} = R_x R_y \nabla \psi(x, y, z)$ and using that $R_x = -R_y R_z$, we have

$$|c \varepsilon \nabla \tilde{\psi} - i n \tilde{\psi}|^2 = |c \varepsilon R_x R_y \nabla \psi(x, y, z) - i R_y n \psi|^2 = |c \varepsilon \nabla \psi - i n \psi|^2.$$

Therefore, we conclude $F$ is invariant under the action of the group $G$. ■

2.3. The linearization at the uniformly layered state

The uniformly layered state $x_0 = (\psi_0, n_0) \equiv (e^{i c x}, e_x)$. is a trivial critical point of $F$ in $\mathcal{M}$. Since all the elements of $G$ leave this critical point invariant, the orbit $G x_0 = \{ x_0 \}$ is trivial, and $G_{x_0} = G$, that is the isotropy group of $x_0$ is all of $G$.

We may parameterize a submanifold of $\mathcal{M}$ by writing

$$\psi = \psi_0(1 + iv/c \varepsilon) \text{ and } n(x, y, z) = (w, 1)/[w, 1],$$

where $v \in H^2_0(T^2 \times I; \mathbb{R})$ and $w \in H^2_0(T^2 \times I; \mathbb{R}^2)$, where hereafter the subindex 0 means a zero boundary conditions on $v(x, y, \pm \pi/2) = 0$ and $w(x, y, \pm \pi/2) = 0$.

The map $(w, v) \to (\psi, n)$ defines a diffeomorphism from the space

$$\mathcal{H} = H^2_0(T^2 \times I; \mathbb{R}^3)$$

into the submanifold

$$\mathcal{N} = \{ (\psi, n) \in \mathcal{M} : \psi - \psi_0 \in i \mathbb{R}, n(x, y, z) \in S_+^3 \}.$$

This is the set of functions in $\mathcal{M}$ such that $n$ takes values in the upper half of the unit sphere and $\psi - \psi_0$ is purely imaginary.

Let us define $u = (w, v) \in \mathcal{H}$, writing the energy functional $F$ using this parametrization of $\mathcal{N}$, we are led to the map $\tilde{F} : \mathcal{H} \to \mathbb{R}$ defined by

$$\tilde{F}(u) := F(\psi(u), n(u)),$$

thus $u = 0$ corresponds to the uniformly layered state $(\psi_0, n_0)$, see (5).

The action of $G$ in these coordinates $u = (w, v)$ is given by

$$\rho(\varphi, \theta) u = u(x + \varphi/a, y + \theta b, z).$$
and
\[ \rho(\kappa_x)u = R_xu(-x, y, z), \rho(\kappa_y)u = R_yu(x, -y, z) \text{ and } \rho(\kappa_z)u = -u(x, y, -z). \]

The first variation \( \delta F(u) \) is characterized by the relation
\[ \delta F(u)w = \langle \nabla F(u), w \rangle_{L^2}. \]

We define the map \( f : \mathcal{H} \to L^2 \) by
\[ f(u) = \nabla F(u), \]
which is \( G \)-equivariant, because the action of \( G \) is isometric, by property 2 of lemma 2.

In order to obtain the second variation of the energy (2) around the flat layer state \( \psi_n, 0 \), we follow the definition and derivation of the critical magnetic field in [30]. In these coordinates, consider a perturbation \( (\psi_n, n_t) = (\psi(nu), n(nu)) \), i.e. we have
\[ \psi_n = \psi_0 + \frac{\psi_0}{c \varepsilon}, \quad n_t = \frac{e_3 + (nw, 0)}{|e_3 + (nw, 0)|} = n_0 + t_2 + \mathcal{O}(r^3), \]
where \( n_0 = e_3, n_1 = (w, 0) \) and \( n_2 = -\frac{1}{2}w^T e_3 \), since \( e_3 \) and \( w \) are orthogonal. Then, the second variation around the flat layer state \( u = 0 \) is
\[ \delta^2 F(0)u = \frac{1}{2} \int_\Omega \frac{1}{\varepsilon} |\nabla v - (w, 0)|^2 + \varepsilon |\nabla w|^2 - \tau |w|^2. \]  
(6)

Now, since we have that \( \delta F(u)w = \langle \nabla F(u), w \rangle \), then
\[ \delta^2 F(0)u = \langle f'(0)u, u \rangle. \]

From the associated Euler–Lagrange equations, we have that the second variation at the flat layered state applied to \( u = (w, v) \) is
\[ \delta^2 F(0)u = \frac{1}{2} \int_\Omega \frac{1}{\varepsilon} |\nabla v - (w, 0)|^2 + \varepsilon |\nabla w|^2 - \tau |w|^2 = \langle Lu, u \rangle_{L^2}, \]
then the operator \( L = f'(0) : \mathcal{H} \to L^2 \) is given by
\[ L(w, v) = \left( -\varepsilon \Delta w - \frac{1}{\varepsilon} \nabla \cdot w + \frac{1}{\varepsilon} w - \tau w, -\frac{1}{\varepsilon} \Delta v + \frac{1}{\varepsilon} \nabla \cdot w \right). \]  
(7)

where \( \nabla \cdot = (\partial_x, \partial_y) \).

Of course, the linearized operator \( L \) depends on the parameter \( \tau \) and in the next section we make this dependence explicit and write \( L = L(\tau) \) when we present the bifurcation theorem needed for our main result.

2.4. An abstract bifurcation theorem

The solutions described in theorem 1 correspond to critical points of \( \tilde{F} : \mathcal{H} \to \mathbb{R} \). We obtain these solutions as zeros of the map
\[ f(u) = f(u; \tau) = L(\tau)u + O(\|u\|_{H^1}) : \mathcal{H} \to L^2. \]
Moreover, since $f(u)$ is $G$-equivariant, then the map $f^H : \text{Fix}(H) \cap \mathcal{H} \times \mathbb{R} \to \text{Fix}(H)$ is well defined by property 3 of lemma 2, where

$$\text{Fix}(H) = \{u \in L^2 : \rho(\gamma)u = u \text{ for } \gamma \in H\}.$$ 

Therefore, we may consider

$$f^H(u, \tau) = L^H(\tau)u + O(|u|_H^2),$$

where $L^H$ is the restriction of $L$ to $\text{Fix}(H)$ and $|\cdot|_H$ is the restriction of $|\cdot|_{\mathcal{H}}$ to $\text{Fix}(H)$. To obtain our bifurcation branches we apply theorem 5 below to $f^H$ for suitable choices of $H$.

**Definition 4.** Assume that for some linear operator $L$ and some $H < G$, $L^H(\tau)$ is a self-adjoint Fredholm operator, $\ker L^H(\tau_0)$ is non trivial and $L^H(\tau)$ is invertible for $\tau$ close but different from $\tau_0$. Then we can define $n^H(\tau)$ as the Morse index of $L(\tau)$ restricted to $\ker L^H(\tau_0)$ for $\tau$ close to $\tau_0$ and $\eta^H(\tau_0)$ as the net crossing number of eigenvalues of $L^H$ at $\tau_0$, that is

$$\eta^H(\tau_0) := \lim_{\varepsilon \to 0} (n^H(\tau_0 - \varepsilon) - n^H(\tau_0 + \varepsilon)).$$

**Theorem 5.** Let $g : \mathcal{H} \to L^2$ be a family of $G$-equivariant maps (indexed by $\tau$) satisfying

$$g(u, \tau) = L(\tau)u + O(|u|_{\mathcal{H}}^2)$$

If $L^H(\tau)$ is a self-adjoint Fredholm operator and $\eta^H(\tau_0)$ is odd, then $g^H$ has a local bifurcation of zeros from $(\tau_0, 0)$ in $\text{Fix}(H) \cap \mathcal{H} \times \mathbb{R}$.

**Proof.** We may adapt the theorem 3.5.1 in [31] that uses degree theory to prove that the bifurcation branch form a local continuum. In the case $\dim \ker L^H(\tau_0) = 1$ and $\eta^H(\tau_0) = \pm 1$, the theorem can be obtained with an application of the implicit function theorem. This is the condition of having a simple bifurcation which allows us to give estimates in the local branches parametrized by the amplitude, using the bifurcation equation as in [10].

**Remark 6.** Actually, applying a result [29] based on index for gradient maps, we can prove that if $\eta^H(\tau_0)$ is different from zero, there is local bifurcation from $(0, \tau_0)$. However, this local branch does not need to be a continuum.

**3. The spectrum: identifying the critical field**

For the bifurcation analysis we need to characterize the spectrum of the linear map

$$Lu = f'(0)u.$$

As a starting point we already know that the space $u \in \mathcal{H}$ has a spectral representation in Fourier series given by

$$u = \sum_{m,n \in \mathbb{Z}} u_{m,n}(z)e^{i(m \alpha x + n \beta y)},$$

where $u_{m,n}(z) \in H^3(I; \mathbb{C})$ with $u_{m,-n} = \bar{u}_{m,n}$. Next we proceed to further decompose the $z$-dependent Fourier coefficients $u_{m,n}$.

**Proposition 7.** The space $u_{m,n} \in H^3(I; \mathbb{C})$ has a representation as

$$u_{m,n}(z) = \sum_{l \in \mathbb{Z}^+} u_{m,n,l}(z),$$

where $u_{m,n,l}(z) \in H^3(I; \mathbb{C})$ for each $l$. The coefficients $u_{m,n,l}$ represent the local bifurcation branches at the critical field.
where \( u_{m,n,l} \in \mathbb{C}^3 \) and \( e_{m,n,l}(z) \) is the eigenfunction of \( L \) with eigenvalue \( \lambda_{m,n,l} \). Moreover, the eigenfunction \( \lambda_{m,n,l}(\tau) \) crosses zero only at the critical field

\[
\tau_{m,n,l} = \frac{p^2}{\xi} + \frac{1}{\xi}(\xi p)^2,
\]

where

\[
p^2 = (am)^2 + (bn)^2 + l^2.
\]

In addition, the eigenfunctions satisfy the relations

\[
e_{m,n,l}(z) = R_\alpha e_{m,n,l}(z) = R_\beta e_{m,-n,-l}(z) = (-1)^{l+1}e_{m,n,l}(-z).
\]

**Proof.** Recalling the definition of \( L \) in (7) we see that the eigenfunctions \( e_{m,n,l}(z) \) and eigenvalues \( \lambda_{m,n,l} \) are such that

\[
L(e_{m,n,l}(z)e^{i(\alpha m + \beta n)\tau}) = \lambda_{m,n,l}e_{m,n,l}(z)e^{i(\alpha m + \beta n)\tau}.
\]

From here after we omit the index \( m, n, l \) to simplify notation. We set \( e_{m,n}(z) = (w_1, w_2, v) \).

\[
\alpha = am \text{ and } \beta = bn.
\]

We have that the condition (8) is equivalent to

\[
-\varepsilon w''_1 + \varepsilon(\alpha^2 + \beta^2)w_1 - i\alpha v/\varepsilon + (\varepsilon^{-1} - \tau)w_1 = \lambda w_1,
\]

\[
-\varepsilon w''_2 + \varepsilon(\alpha^2 + \beta^2)w_2 - i\beta v/\varepsilon + (\varepsilon^{-1} - \tau)w_2 = \lambda w_2,
\]

\[
-v''/\varepsilon + (\alpha^2 + \beta^2)v/\varepsilon + i(\alpha w_1 + \beta w_2)/\varepsilon = \lambda v.
\]

Writing

\[
f = \alpha w_1 + \beta w_2, \quad g = \alpha w_1 - \beta w_2, \quad \text{and } h = iv/\varepsilon,
\]

the system (9) becomes

\[
-\varepsilon f'' + [\varepsilon(\alpha^2 + \beta^2) + (\varepsilon^{-1} - \tau)]f - (\alpha^2 + \beta^2)h = \lambda f,
\]

\[
-\varepsilon g'' + [\varepsilon(\alpha^2 + \beta^2) + (\varepsilon^{-1} - \tau)]g - (\alpha^2 - \beta^2)h = \lambda g,
\]

\[
-\varepsilon h'' + [\varepsilon(\alpha^2 + \beta^2)]h - f = \lambda\varepsilon^2 h.
\]

with boundary conditions \((f, g, h)(\pm\pi/2) = 0\).

We consider the linear operator

\[
Lh = -\varepsilon h'' + [\varepsilon(\alpha^2 + \beta^2) - \varepsilon^2 \lambda]h,
\]

defined in \( \mathcal{D}(L) = \{ h \in H^2(I; \mathbb{R}) | h(\pm\pi/2) = 0 \} \). Let the inverse be \( T = L^{-1} \), it is well known that \( T \) is compact. Then, the system (11) and (13) can be written as \( h = T f \) and

\[
f = (\lambda(1 - \varepsilon^2) + \tau - \varepsilon^{-1})Tf + (\alpha^2 + \beta^2)T^2f = p(T)f,
\]

where we have defined the polynomial

\[
p(z) = (\lambda(1 - \varepsilon^2) + \tau - \varepsilon^{-1})z + (\alpha^2 + \beta^2)z^2.
\]
Let $\kappa$ be an eigenvalue of $T$, i.e. $Tf = \kappa f$, then $f$ satisfies the equation $Lf = \kappa^{-1}f$. From the Dirichlet boundary conditions we see that the solutions are

$$f_j(x) = \begin{cases} \sin(l\pi x) & \text{for } l \in 2\mathbb{Z}^+ \\ \cos(l\pi x) & \text{for } l \in 2\mathbb{Z}^+ - 1 \end{cases}.$$ 

Therefore, the eigenvalues $\kappa$ need to satisfy

$$\kappa^{-1} = \varepsilon(\alpha^2 + \beta^2 + l^2) - \varepsilon^2 \lambda. \tag{15}$$

Since $T$ is compact, from (14) and the fact that $\sigma(p(T)) = p(\sigma(T))$ where $\sigma(T)$ is the spectrum of $T$ and $p$ is a polynomial, we have $1 \in p(\sigma(T))$. This can be written as

$$\kappa^{-2} = (\lambda(1 - \varepsilon^2) + \tau - \varepsilon^{-1})\kappa^{-1} + (\alpha^2 + \beta^2). \tag{16}$$

The eigenvalues are determined by the relations (15) and (16). Using

$$p^2 = \alpha^2 + \beta^2 + l^2$$

we can combine these to obtain

$$[\varepsilon p^2 - \varepsilon^2 \lambda]^2 = (\lambda(1 - \varepsilon^2) + \tau - \varepsilon^{-1})[\varepsilon p^2 - \varepsilon^2 \lambda] + (\alpha^2 + \beta^2)$$

The previous relation is equivalent to

$$(p^2 - \lambda\varepsilon)(\varepsilon^2 p^2 - \lambda\varepsilon - \tau\varepsilon) + l^2 - \lambda\varepsilon = 0.$$ 

Now, $\lambda = 0$ when

$$\tau_0 = \varepsilon p^2 + \frac{1}{\varepsilon} \frac{l^2}{p^2}.$$ 

Taking derivative with respect to $\tau$, we have that $\lambda(\tau)$ satisfies

$$(-\lambda\varepsilon)(\varepsilon^2 p^2 - \lambda\varepsilon - \tau\varepsilon) + (p^2 - \lambda\varepsilon)(-\lambda\varepsilon - \varepsilon) - \lambda\varepsilon = 0.$$ 

Evaluating at the critical value $\tau_0$ we have $\lambda = 0$, from where we deduce

$$\lambda(\tau_0) = \frac{p^2}{(lp)^2 - p^2 - 1} \neq 0$$

as $p \neq 0$. Therefore, the eigenvalue $\lambda$ crosses zero at $\tau_0$.

The solution to (12) is given by

$$g = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}f,$$ \tag{17}

which leads to the eigenfunction

$$(w_1, w_2, v) = \left( \frac{am}{\alpha^2 + \beta^2}, \frac{bm}{\alpha^2 + \beta^2}, \frac{-i}{p^2 - \lambda\varepsilon} \right)f_l(z).$$
Moreover, we would like to choose the eigenfunctions such that the condition \( u_{-m,-n} = \tilde{u}_{m,n} \) becomes \( u_{-m,-n,l} = \tilde{u}_{m,n,l} \). Then, the eigenfunction \( e_{m,n,l}(z) \) such that \( e_{m,n,l}(z) = \tilde{e}_{m,n,l}(z) \) are

\[
e_{m,n,l}(z) = \left\{ \frac{ami}{\alpha^2 + \beta^2}, \frac{bni}{\alpha^2 + \beta^2}, \frac{1}{p^2 - \lambda c} \right\} f(z).
\] (18)

Thus the eigenvalue problem (9) has eigenvalues \( \lambda \) which are zero at the critical value \( \tau \) and corresponding eigenfunctions \( e_{m,n,l}(z) \).

We have considered the domain with sides \( \Omega \) in order to simplify the spectrum, as it appears in terms of \((am)^2, (bn)^2 \) and \( l^2 \). We conclude:

**Proposition 8.** For all \( u \in \mathcal{H} \) we have

\[
u = \sum_{m,n \in \mathbb{Z}, l \in \mathbb{Z}^*} u_{m,n,l} e_{m,n,l}(z) e^{(amx + bny)},
\]

and

\[
Lu = \sum_{m,n \in \mathbb{Z}, l \in \mathbb{Z}^*} \lambda_{m,n,l} u_{m,n,l} e_{m,n,l}(z) e^{(amx + bny)}.
\]

**Remark 9.** In order to find the critical value of \( \tau \), we minimize (6) over functions satisfying \( \int_{\Omega} (n_1^2 + n_2^2) = 1 \). Therefore, one can see that the first critical field is the minimum of the \( \tau_{m,n,l} \)

\[
\tau_c = \min_{m,n,l} \tau_{m,n,l}.
\] (19)

Following the proof of proposition 7, one can see that \( \tau_{m,n,l} \) can be achieved from the system (11)–(13) with \( \lambda = 0 \). Noticing that (11) and (13) are decoupled from the system, we rewrite them, with \( \delta = \varepsilon (\alpha^2 + \beta^2) \) and \( \psi = \delta h \),

\[
\begin{align*}
-\varepsilon f'' + \left( \delta + \frac{1}{\varepsilon} \right) f - \frac{1}{\varepsilon} \psi &= \tau f, \\
-\psi'' + \frac{\delta}{\varepsilon} \psi &= \frac{\delta}{\varepsilon} f.
\end{align*}
\]

Thus this system determines \( \tau_c \) in (19) and it has been studied for layer undulations in two dimensions [21] with the magnetic field applied in the \( x \) direction. Thus the first eigenvalue obtained in [21] for two dimensions also provides the estimate of the critical field for our three dimensional case. In particular, theorem 4 of [21] proves that the first eigenvalue is estimated as \( \pi \) when the interval for \( z \) is \((-1, 1)\). Since the scaling used in this paper gives the interval \((-\pi/2, \pi/2)\) for \( z \), it is easy to see that the first eigenvalue is estimated

\[
\tau_c \sim 2.
\] (20)

### 4. Bifurcation

As anticipated, the first step is to identify the irreducible representations associated to \( G \). Due to Schur’s lemma, all eigenvalues \( \lambda \) of \( L \) corresponding to this representation are the same, and then, if one does not consider the symmetries the resonances make the proof of the bifurcation difficult or impossible.
In the second step, the strategy is to identify the isotropy groups that have a fixed point space of dimension one in the irreducible representation, and then, using the implicit function theorem in the restricted point space of these isotropy groups, obtain simple bifurcation, that is without resonances. However, we do not claim to find them all, although the two we find are enough to characterize the loss of stability of the uniform state to a preferred branch while at the same time shows the emergence of other interesting solutions satisfying different symmetries, as soon as one crosses the critical field $\tau_c$.

The symmetries of the bifurcation solutions are given by the isotropy groups with these properties. The symmetries of the solutions are presented in the third part of this section, and the bifurcation theorem in the fourth section.

4.1. Irreducible representations

Now, we proceed to find the irreducible representations. It suffices to characterize the action on the elements of the Fourier basis.

**Proposition 10.** For fixed $m, n, l > 0$, the action on $(u_1, u_2) = (u_{m,n,l}, u_{m,-n,l})$ is given by

\[
\rho(\varphi, \theta)(u_1, u_2) = e^{im\varphi}(e^{in\theta}u_1, e^{-in\theta}u_2),
\]

\[
\rho(\kappa_x)(u_1, u_2) = (\hat{u}_2, \hat{u}_1),
\]

\[
\rho(\kappa_y)(u_1, u_2) = (u_2, u_1),
\]

\[
\rho(\kappa_z)(u_1, u_2) = (-1)^l(u_1, u_2).
\]

**Proof.** The action of $G$ in each Fourier component is given by

\[
\rho(\varphi, \theta)u(x, y) = u(x + \varphi/a, y + \theta/b) = \sum_{m,n \in \mathbb{Z}} (e^{im\varphi}e^{in\theta}u_{m,n})e^{i(amx + by)}.
\]

From this we obtain

\[
\rho(\varphi, \theta)u_{m,n}(z) = e^{im\varphi}e^{in\theta}u_{m,n}(z).
\]

In a similar way, we have that

\[
\rho(\kappa_x)u_{m,n} = R_xu_{-m,n},
\]

\[
\rho(\kappa_y)u_{m,n} = R_yu_{m,-n},
\]

\[
\rho(\kappa_z)u_{m,n}(z) = -u_{m,n}(-z).
\]

Moreover, since $R_xe_{m,n,l}(z) = e_{m,n,l}(z)$, then

\[
R_xu_{-m,n} = \sum_{l \in \mathbb{Z}} u_{-m-n,l}R_xe_{m,n,l}(z) = \sum_{l \in \mathbb{Z}} u_{-m,n,l}e_{m,n,l}(z),
\]

and then the reflection $\kappa_x$ acts according to

\[
\rho(\kappa_x)u_{m,n,l} = u_{-m,n,l} = b_{m,-n,l}.
\]

In a similar way, using the fact that $R_ye_{m,n,l}(z) = e_{m,n,l}(z)$, one can prove that

\[
\rho(\kappa_y)u_{m,n,l} = u_{m,-n,l}.
\]

Furthermore, since $-e_{m,n,l}(-z) = (-1)^le_{m,n,l}(z)$, then

\[
\rho(\kappa_z)u_{m,n}(z) = -u_{m,n}(-z) = \sum_{l \in \mathbb{Z}} (-1)^lu_{m,n,-l}e_{m,n,l}(z).
\]
We conclude that the irreducible representations correspond to \((u_1, u_2) \in \mathbb{C}^2\) with the previous action when \(m, n \neq 0\). In the case that \(m\) or \(n\) are zero, the irreducible representations are \(\mathbb{C}\), however we do not analyze this case because this produces a bifurcation of stationary solutions in \(x\) or \(y\), which may be analyzed directly [21].

### 4.2. Isotropy groups

For each representation there are at least two kinds of isotropy groups that have fixed point spaces of dimension one. Bear in mind that, as mentioned earlier, in this section we do not claim to find all of them, but at least two. Hereafter, we denote

\[(u_1, u_2) = (u_{m,n,l}, u_{-m,-n,l}) \in \mathbb{C}^2\]

the irreducible representation for a fixed \(m, n, l\).

The irreducible representation for \(l\) even is fixed by the element \(\kappa_z\). Moreover, the point \((u_1, u_2) = (r, 0)\) for \(r \in \mathbb{R}\) has an isotropy group given by

\[O(2) \times \mathbb{Z}_2 = \{(\varphi/m, -\varphi/n), \kappa_x, \kappa_y, \kappa_z\},\]

where \((g_1, g_2, \ldots, g_0)\) denotes, as usual, the subgroup generated by the elements \(g_1, \ldots, g_0\) of \(G\).

The other isotropy groups corresponds to \((u_1, u_2) = (r, r)\) and is given by

\[D \times \mathbb{Z}_2 = \{(\pi/m, -\pi/n), \kappa_x, \kappa_y, \kappa_z\}.

For \(l\) odd one isotropy group corresponds to \((u_1, u_2) = (ir, 0)\) and is given by

\[\tilde{D}(2) = \{(\varphi/m, -\varphi/n), \kappa_x, \kappa_y\}\]

The second isotropy group corresponds to \((u_1, u_2) = (r, -r)\) is given by

\[\tilde{D} = \{(\pi/m, -\pi/n), \kappa_x, \kappa_y\}\]

It is not hard to prove that the fixed point spaces of these groups have real dimension equal to one, where the fixed point spaces are generated by \((1, 0); (1, 1); (i, 0)\) and \((1, -1)\) respectively.

### 4.3. Symmetries

The previous isotropy groups are relevant because they give the symmetries of the bifurcating solutions. Before proving the existence of these branches, we want to present the symmetries of the bifurcating solutions, i.e. the fixed point spaces of the previous groups.

Since the elements \((\varphi/m_0, -\varphi/n_0), \kappa_x, \kappa_y\) and \(\kappa_z\) generate \(O(2) \times \mathbb{Z}_2\), solutions with this isotropy group must satisfy

\[
u(x, y, z) = R_x R_y u(-x, -y, z) = -u(x, y, -z) = u(x + \varphi/\alpha, y - \varphi/\beta, z),
\]

where \(\alpha = am_0\) and \(\beta = bn_0\).

From the first symmetry, we have that

\[
u = \sum_{m,n \in \mathbb{Z}} e^{i(m/n_0) - (m/n_0)\varphi} u_{m,n} e^{(m\alpha + n\beta)y}.
\]

Then \(u_{m,n} = 0\) unless \((m/n_0) - (n/m_0) = 0\) or \(m = m_0/n_0\), then \(m = jm_0\) and \(n = jn_0\) for \(j \in \mathbb{Z}\) and
\[
    u = \sum_{j \in \mathbb{Z}} u_j(z) e^{i(jamx + bmz)}
\]

with \( u_j(z) = u_{jm_0,jn_0}(z) \) with

\[
    u_j(z) = \bar{u}_j(z) = R_r R_l \bar{u}_j(z) = -u_j(-z).
\]

Since the elements \((\varphi/m_0, -\varphi/n_0), \kappa, \kappa_0, \kappa_z\) generate \(\hat{O}(2)\), they must satisfy

\[
    u(x, y, z) = u(x + \varphi/\alpha, y - \varphi/\beta, z) = R_r u(-x, -y, -z),
\]

where \(R_r = -R_r\). Therefore, \(u\) is as before but with \( u_j(z) = u_{jm_0,jn_0}(z) \) such that

\[
    u_j(z) = \bar{u}_j(z) = R_r \bar{u}_j(-z).
\]

The isotropy group \(D \times \mathbb{Z}_2\) has the generators \(\kappa_x, \kappa_y, \kappa_z\) and \((\pi/m_0, -\pi/n_0)\), then solutions with this isotropy group must satisfy

\[
    u(x, y, z) = R_l u(-x, y, -z) = R_l u(x, -y, -z) = -u(x, y - z)
    = u(x + \pi/\alpha, y - \pi/\beta, z),
\]

where \(\alpha = am_0, \beta = bn_0\). In this case

\[
    u = \sum_{m,n \in \mathbb{Z}} e^{i(m/m_0 - (n/n_0))\pi} u_{m,n}(z) e^{i(amx + bny)},
\]

then \(u_{m,n} = 0\) unless \((m/m_0) - (n/n_0) \in 2\mathbb{Z}\). Moreover, we have that

\[
    u_{m,n}(z) = \bar{u}_{-m,-n}(z) = R_l u_{-m,-n}(z) = -u_{m,n}(z).
\]

The group \(\hat{D}\) has generators \(\kappa_z, \kappa_x, \kappa_y, \kappa_z\) and \((\pi/m_0, -\pi/n_0)\). Then, solutions with this isotropy group \(\hat{D}\) must satisfy

\[
    u(x, y, z) = -R_r u(-x, y, -z) = -R_r u(x, -y, -z) = -u(x, y - z)
    = u(x + \pi/\alpha, y - \pi/\beta, z).
\]

As before, we have that \(u_{m,n} = 0\) unless \((m/m_0) - (n/n_0) \in 2\mathbb{Z}\) and

\[
    u_{m,n}(z) = \bar{u}_{-m,-n}(z) = -R_l u_{-m,-n}(z) = -R_l u_{m,n}(-z).
\]

### 4.4. Bifurcation theorem

**Theorem 11.** For each fixed \(m_0, n_0, \ell_0 \geq 1\), the uniformly layered state \(u = 0\) has two bifurcations of critical points of (2) starting from the critical field \(\tau_{m_0,n_0,\ell_0}\) in \(\mathcal{H}\), except for a finite union of hypersurfaces in the set of parameters \((a, b, \varepsilon)\). One of the bifurcations has the symmetries of \(O(2) \times \mathbb{Z}_2\) and the other one of \(D \times \mathbb{Z}_2\) for \(\ell_0\) even, and \(\hat{O}(2)\) and \(D\) for \(\ell_0\) odd.

**Proof.** We appeal to theorem 5. Since the eigenvalue \(\lambda_{m,n,l}\) is zero only at \(\tau_{m,n,l}\), and \(\tau_{m,n,l} \to \infty\) as \(m, n, l \to \infty\), then \(\lambda_{m,n,l}(\tau_{m_0,n_0,\ell_0})\) is different from zero except for a finite number of \((m, n, l) \in 2\mathbb{Z}^3\). Moreover, the condition \(\tau_{m,n,l} - \tau_{m_0,n_0,\ell_0} = 0\) defines a hypersurface in the space of parameters \((a, b, \varepsilon)\). On the other hand \(\tau_{m,n,l} - \tau_{m_0,n_0,\ell_0}\) is identically zero only if \(l^2 = \ell_0^2\) and \(m_0^2 = m^2\) and \(n_0^2 = n^2\).
Therefore, there are no resonances with other eigenvalues \( \lambda_{m_0,n_0,l_0} \) except for the set of parameters \((a, b, \varepsilon)\) in this finite number of hypersurfaces. Thus, we conclude that the only eigenvalues \( \lambda_{m,n,l} \) close to zero at \( \tau_{m_0,n_0,l_0} \) are \( \lambda_{m_0,\pm n_0,l_0} \). Moreover, the kernel of \( L \) is given by the space corresponding to the irreducible representation \((u_{m_0,n_0,l_0}, u_{m_0,-n_0,l_0}) \in \mathbb{C}^2\), i.e. the eigenvalue \( \lambda_{m_0,n_0,l_0} \) has real dimension equal to four. We then take advantage of the actions of the isotropy groups that have a fixed point space of real dimension equal to one in \( \mathbb{C}^2 \), in either one of the fixed point spaces \((O(2) \times \mathbb{Z}_2 \text{ and } D \times \mathbb{Z}_2\) for \( l_0 \) even, and \( \tilde{O}(2) \text{ and } \tilde{D}\) for \( l_0 \) odd\) of these groups the kernel has real dimension equal to one. Therefore, the restriction of the map \( f \) to the fixed point space of the isotropy group has only one eigenvalue that crosses zero, which then yields the desired result as an application of theorem 5.

Unfortunately, there are resonances in the hypersurface of parameters \( a = b \), i.e. we have that \( \lambda_{m_0,n_0,l_0} = \lambda_{m_0,m_0,l_0} \). In this case, the previous theorem does not apply, but we can improve the results of the previous theorem for this particular case.

**Proposition 12.** In the case that \( a = b \), there is always a bifurcation of solutions from \( \tau_{m,n,l} \) for the groups \( O(2) \times \mathbb{Z}_2 \text{ and } O(2) \), as in the previous theorem, and for the groups \( D \times \mathbb{Z}_2 \text{ and } \tilde{D} \) if in addition the only solutions of

\[
m^2 + n^2 = m_0^2 + n_0^2
\]

are \((m, n)\) is equal to \((m_0, n_0)\) or \((n_0, m_0)\).

**Proof.** In this case, we can use the same arguments as in the previous theorem, we are only left to verify that there are no resonances, that is, we only need to consider the cases when \( \tau_{m,n,l} - \tau_{m_0,n_0,l_0} \) is identically zero (viewing both expressions as functions of \( \varepsilon \)). In the fixed point space \( O(2) \times \mathbb{Z}_2 \text{ and } \tilde{O}(2) \) other resonances cannot appear, since the fixed point space has functions \( u \) with components \( u_{m,n} \) equal to zero unless \( (m, n) = f(m_0, n_0) \) for \( j \in \mathbb{Z} \), and then, we have that \( \lambda_{m,n,l} = \lambda_{m_0,n_0,l_0} \) for \( n > 0 \) unless \( (m, n) = (m_0, n_0) \).

In the case of the groups \( D \times \mathbb{Z}_2 \text{ and } \tilde{D} \), we have from the assumptions that \( m^2 + n^2 = m_0^2 + n_0^2 \) only for \((m, n)\) equal to \((m_0, n_0)\) or \((n_0, m_0)\), then we have the double eigenvalue \( \lambda_{m_0,n_0,l_0} = \lambda_{n_0,m_0,l_0} \). Actually, this double eigenvalue is a consequence of the invariance of \( F \) by another action \( \kappa \in \mathbb{Z}_2 \) given by \( \rho(\kappa)u(x, y) = ut(y, x) \), due to \( a = b \). Using this extra symmetry, it is not hard to see that in the fixed point space of \( \kappa \) and the group \( D \times \mathbb{Z}_2 \text{ or } \tilde{D} \), no additional resonances exist by assumptions. Therefore, the desired conclusion follows again in this case thanks to theorem 5.

**Remark 13.** The assumptions in the case \( a = b \) are true for many \((m_0, n_0)\), including the case \( m_0 = n_0 = 1 \).

**Remark 14.** As all the eigenvalues \( \lambda_{m_0,n_0,l_0} \) cross in the same direction, then \( \tau_{m_0,n_0,l_0} \) is always a bifurcation point regardless of the multiplicity. But the previous theorems allows us to obtain specific information about different branches, the symmetries of the branches, the local continuity of the branches, and local estimates given in the following section.

### 4.5. Local estimates

We complete the bifurcation analysis by providing the asymptotic portrait of the branches found in theorem 1. As mentioned in the introduction, these properties follow from standard bifurcation arguments [10].
Since the bifurcation is simple, we can estimate the local bifurcating branches in the cases $a \neq b$.

The bifurcation with group $O(2) \times \mathbb{Z}_2$ has leading term $(u_{m_0,n_0,0}, u_{m_0,-n_0,0}) = (r/2, 0)$, from where we gather

$$u = \frac{r}{2} (e_{m_0,n_0,0}(z) e^{i(a m_0 + b n_0 y)} + e_{-m_0,-n_0,0}(z) e^{-i(a m_0 + b n_0 y)}) + O(r^2)$$

where $O(r^2)$ is a function in the fixed point space of $O(2) \times \mathbb{Z}_2$ of order $r^2$. Moreover, using (18) we conclude that

$$u = r \left( \frac{-a m_0 \sin(a m_0 x + b n_0 y)}{\alpha^2 + \beta^2}, \frac{-b n_0 \sin(a m_0 x + b n_0 y)}{\alpha^2 + \beta^2}, \frac{\cos(am_0 x + bn_0 y)}{p^2 - \lambda \varepsilon} \right) \times \sin(l_0 z) + O(r^2).$$

The previous solutions satisfy the corresponding symmetry (21). Also, notice that $u$ is the parameterization of $(\psi(u), n(u))$ given in (5), then one can approximately characterize these functions in the local branches

$$\psi = \psi_0 + \frac{i \psi_0}{\varepsilon} \left( \frac{\cos(am_0 x + bn_0 y)}{p^2 - \lambda \varepsilon} \right) r \sin(l_0 z) + O(r^2),$$

$$n = \left( \frac{-a m_0 \sin(a m_0 x + b n_0 y)}{\alpha^2 + \beta^2}, \frac{-b n_0 \sin(a m_0 x + b n_0 y)}{\alpha^2 + \beta^2}, \frac{\cos(am_0 x + bn_0 y)}{p^2 - \lambda \varepsilon} \right) r \sin(l_0 z), 1 + O(r^2).$$

For the group $D \times \mathbb{Z}_2$ we have that $(u_{m_0,n_0,0}, u_{m_0,-n_0,0}) = (r/2, r/2)$, thus

$$u = r \Re(e_{m_0,n_0,0}(z) e^{i(a m_0 + b n_0 y)} + e_{m_0,-n_0,0}(z) e^{i(a m_0 - b n_0 y)}) + O(r^2).$$

and as before we have the local solutions that satisfy the symmetry (23) given by

$$u = \sum_{n = -a m_0} r \left( \frac{-a m_0 \sin(a m_0 x + b n y)}{\alpha^2 + \beta^2}, \frac{b n \sin(a m_0 x + b n y)}{\alpha^2 + \beta^2}, \frac{\cos(am_0 x + b n y)}{p^2 - \lambda \varepsilon} \right) \times \sin(l_0 z) + O(r^2).$$

For the group $\tilde{O}(2)$ we have that $(u_{m_0,n_0,0}, u_{m_0,-n_0,0}) = (-ir/2, 0)$, and so

$$u = r \Im(e_{m_0,n_0,0}(z) e^{i(a m_0 + b n_0 y)} + O(r^2).$$

Using (18) we have that the solution that satisfy (22) locally is

$$u = r \left( \frac{a m_0 \cos(a m_0 x + b n_0 y)}{\alpha^2 + \beta^2}, \frac{b n_0 \cos(a m_0 x + b n_0 y)}{\alpha^2 + \beta^2}, \frac{\sin(am_0 x + b n_0 y)}{p^2 - \lambda \varepsilon} \right) \times \cos(l_0 z) + O(r^2).$$

For the group $\tilde{D}$ we have that $(u_{m_0,n_0,0}, u_{m_0,-n_0,0}) = (r/2, -r/2)$,

$$u = r \Re(e_{m_0,n_0,0}(z) e^{-i(a m_0 x + b n_0 y)} - e_{m_0,-n_0,0}(z) e^{i(a m_0 x + b n_0 y)}) + O(r^2),$$

where a similar expression with the symmetry (24) may be obtained.
5. 2D planar configuration

In this section, we study the de Gennes free energy in the cross section of the rectangular box while assuming the constant smectic order parameter and find the characteristics of the minimizers well above the critical field. We also assume a sufficiently strong magnetic field which forces a planar configuration. Thus by taking $\psi = e^{i\varphi/\epsilon}$ and $\tau = 1/\epsilon^2$ with $1 < \delta < 2$, the energy (2) becomes

$$F_\epsilon(\varphi, n) = \int_{\mathbb{T}^2} \left( \epsilon |\nabla n|^2 + \frac{1}{\epsilon} |\nabla \varphi - n||^2 + \frac{1}{\epsilon^2} n_3^2 \right)$$

(25)

where $\varphi = \varphi(x, y), n = (n^1, n_3) = (n_1, n_2, n_3)$. We assume that $\mathbb{T}^2$ is a flat torus identified with $[0, 1)^2$. If $(\varphi, n)$ is a minimizer of $F_\epsilon$, then $\varphi = \varphi_n$ satisfies

$$\Delta \varphi_n = \nabla \cdot n_\parallel \text{ in } \mathbb{T}^2 \text{ and } \int_{\mathbb{T}^2} \varphi_n = 0.$$  

(26)

Letting $\varphi_n$ be a solution to (26) and defining

$$G_\epsilon(n) = \int_{\mathbb{T}^2} \left( \epsilon |\nabla n|^2 + \frac{1}{\epsilon} |\nabla \varphi_n - n||^2 + \frac{1}{\epsilon^2} n_3^2 \right),$$

(27)

we have

$$\inf_{(\varphi, n) \in H^1(\mathbb{T}^2, \mathbb{R}) \times H^1(\mathbb{T}^2, S^1)} F_\epsilon(\varphi, n) = \inf_{n \in H^1(\mathbb{T}^2, S^1)} G_\epsilon(n).$$

We therefore study $G_\epsilon$ in (27) with $\varphi_n$ satisfying (26) to obtain the configuration of minimizer of $F_\epsilon$. This formulation was also used in [30] where the weak critical field was achieved.

The functional $G_\epsilon$ in two dimensions is analogous to a free energy for micromagnetics studied in [1]. To see this, we let

$$m = (m^1, n_3), \quad m^1 = n^1 = (n_2, n_1), \quad \text{and} \quad \nabla \cdot m = (\varphi_n, \varphi_n).$$

Then by setting $H = m^1 - \nabla \cdot \varphi, G_\epsilon$ becomes

$$\int_{\mathbb{T}^2} \epsilon |\nabla m|^2 + \frac{1}{\epsilon} |H|^2 + \frac{1}{\epsilon^2} m_3^2,$$

(28)

where the demagnetizing field $H : \mathbb{T}^2 \to \mathbb{R}^2$ is a solution to

$$\nabla \times H = 0 \quad \text{and} \quad \nabla \cdot (H - m^1) = 0 \text{ in } \mathbb{T}^2.$$  

(29)

This energy is not the same as a micromagnetics energy (28) studied in [1] due to the presence of a nonlocal term in (28). The following results, however, will follow directly from [1].

While the limiting $m$ in [1] is a divergence-free vector field tangent to $\partial \Omega$, the limit $n$ in our case is a curl-free $S^1$-valued vector field in $\mathbb{T}^2$ whose components satisfy zero mass constraint. We restate the following theorem from [1] in terms of $n$ with the addition of mass constraint.

Let $X \in [0, \frac{\pi}{2}]$ be a geometric half-angle between $m^1$ and $m^1$, where $m^1$ are traces on each side of the one-dimensional jump set of $m^1$. In terms of $n$, $X$ is simply the half the angle between $n_\perp$ and $n_\perp$.

**Proposition 15.** Let the sequences $\{\epsilon_j\}_{j=1}^\infty \subset (0, \infty), \text{ and } \{n_j\}_{j=1}^\infty \subset H^1(\mathbb{T}^2, S^1)$ be such that $\epsilon_j \to 0$ and $\{G_\epsilon(n_j)\}_{j=1}^\infty$ is bounded.
Then there exist a subsequence \( \{n_j\} \), \( n = (n_1, n_2, 0) \in \cap_{p<\infty} L^p(\mathbb{T}^2, \mathbb{S}^1) \), and \( \varphi \in \cap_{p<\infty} W^{1,p}(\mathbb{T}^2) \) such that

\[
n_j \to n \quad \text{in} \quad \cap_{p<\infty} L^p(\mathbb{T}^2), \\
n = \nabla \varphi \quad \text{in} \quad \mathbb{T}^2, \quad |\nabla \varphi| = 1, \\
\int_{\mathbb{T}^2} n_1 = \int_{\mathbb{T}^2} n_2 = 0.
\]  

(30)

If, in addition, \( n \in BV(\mathbb{T}^2, \mathbb{S}^1) \), then

\[
\liminf_{j \to \infty} G_\varepsilon(n_j) \geq \int_{\Sigma_0} A(X) \, d\mathcal{H}^d
\]

(31)

where

\[
A(X) = 4|\sin X - X \cos X| \quad \text{for} \quad X \in \left[0, \frac{\pi}{4}\right],
\]

(32)

\[
A(X) = 4\left|X - \frac{\pi}{2}\right| \cos X - \sin X + \sqrt{2} \quad \text{for} \quad X \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]
\]

(33)

**Proof.** The compactness result may follow directly from lemma 2.1 in [1] and the mass constraints of \( n_1 \) and \( n_2 \) in (30) result from \( n = \nabla \varphi \) and periodicity of \( \varphi \). The lower bound inequality (31) follows from theorem 1 of [1].

Note that constant configurations are not allowed due to the zero mass constraint in (30) and periodicity of \( n \). Therefore, the line singularity appears, which is also observed in [18]. Here, we consider horizontal or vertical lines for the one-dimensional jump set rather than diagonal lines, in order to minimize the arc length of the jump set. One can see that \( n \) has at least two internal jumps due to the periodic boundary condition on \( n \). Therefore we should consider two situations, squares with two jumps on both horizontal and vertical 1d tori (figure 1(a)) and only vertical strips (or horizontal strips) with two parallel 1d-tori (figure 1(b)). For the latter case, we must have 180° wall to ensure the mass constraint (30).

By applying proposition 15, we would like to compare these two configurations, 2d square pattern (figure 1(a)) and 1d stripe configuration (figure 1(b)). For the 1d stripes, the formula (33) with \( X = \pi/2 \) gives the lower bound

\[
G_\varepsilon^{1d} \geq 8(\sqrt{2} - 1) \approx 3.3.
\]

(34)

On the other hand, for the 2d square pattern, the formula (32) with \( X = \pi/4 \) gives the better lower bound

\[
G_\varepsilon^{2d} \geq 2\sqrt{2}(4 - \pi) \approx 2.4.
\]

(35)

It was proved in [34] that the lower bound in (32) can be achieved by one dimensional structure in the wall for jumps less than or equal to \( \pi/2 \). Thus the lower bound in (35) is optimal and hence we can conclude that 2d square pattern gives the lower energy than 1d stripe structure.

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6. Numerical simulations

6.1. Complex de Gennes energy

We consider the gradient flow (in $L^2$) of the energy (2) and study the behavior of the solutions. The gradient flow equations are

$$\begin{align*}
\frac{\partial n}{\partial t} &= \Pi_n (\varepsilon \Delta n - c J \psi (\nabla \psi)^T + \tau [(n \cdot e_1) e_1 + (n \cdot e_2) e_2]), \\
\frac{\partial \psi}{\partial t} &= c \Delta \psi - 2 \varepsilon n \cdot \nabla \psi - \varepsilon (\nabla \cdot n) \psi - \frac{1}{\varepsilon} \psi + \frac{g}{\varepsilon} (1 - |\psi|^2) \psi, \\
\end{align*}$$

where we have defined, for a given vector $f \in \mathbb{R}^3$, the orthogonal projection onto the plane orthogonal to the vector $n$ as

$$\Pi_n (f) = f - (n \cdot f) n.$$ 

This projection appears as a result of the constraint $n \in S^2$. This method has been used for smectic A liquid crystals [20–22]. As initial condition, we consider a small perturbation from the undeformed state. More precisely, for all $(x, y, z) \in \Omega$,

$$\begin{align*}
n(x, y, z, 0) &= \frac{(\eta u_1, \eta u_2, 1 + \eta u_3)}{[\eta u_1, \eta u_2, 1 + \eta u_3]}, \\
\psi(x, y, z, 0) &= e^{i \varepsilon x} + \eta \psi_0,
\end{align*}$$

where a small number $\eta = 0.1$ and $u_1, u_2, u_3$ and $\psi_0$ are arbitrarily chosen. We impose strong anchoring condition for the director field and Dirichlet boundary condition on $\psi$ at the top and the bottom plates;

$$\begin{align*}
n(x, y, \pm \frac{\pi}{2}, t) &= e_y, \quad \text{and} \quad \psi(x, y, \pm \frac{\pi}{2}, t) = e^{i \varepsilon x}.
\end{align*}$$

Periodic boundary conditions are imposed for both $n$ and $\psi$ in the $x$ and $y$ directions.

We use a Fourier spectral discretization in the $x$ and $y$ directions, and second order finite differences in the $z$ direction. The fast Fourier transform is computed using the FFTW libraries [16]. For the temporal discretization, we combine a projection method for the variable $n$ [14], with a semi-implicit scheme for $\psi$: Given $(\psi^k, n^k)$, we solve

$$\frac{n^k - n^{k+1}}{\Delta t} = \varepsilon \Delta n^k - c J \psi k (\nabla \psi^k)^T + \tau [(n^k \cdot e_1) e_1 + (n^k \cdot e_2) e_2],$$

Figure 1. 2d square pattern versus 1d stripe pattern.
The second step (39) ensures that \( |n^{k+1}| = 1 \) at each grid point. Note that \( |n^0| \neq 0 \) in (39) since we consider the case where there are no point defects in the liquid crystal. The consistency and convergence of the projection method are given in [14]. The method is first order accurate in time and second order accurate in space due to the first order accuracy of the projection method (38)–(39). To solve the implicit system, we perform a discrete Fourier transform in the \( x \) and \( y \) direction. The resulting tridiagonal systems in the \( z \) direction are solved using Gauss elimination.

We take the domain size \( L_1 = L_2 = 3d_0 \), i.e. in the case of \( a = b \), and \( \varepsilon = 0.3 \). The number of grid points in the \( x \), \( y \) and \( z \) directions are all 128. Here we use parameters

\[
K = 0.001, \quad C = 0.01, \quad g_0 = 0.5, \quad r = 0.25, \quad \text{and} \quad q = 10,
\]

and then we obtain from (3)

\[
c = \sqrt{5}, \quad \text{and} \quad g = 0.25.
\]

We have solved this system in [20] to understand the minimizer at a various field strength in a two dimensional domain, \( \Omega = (0, L)^2 \) and \( n \in S^2 \). Here we consider a three dimensional domain with \( n \in S^2 \). In figure 3 we show the layer structures in the middle of the sample \( z = 0 \) in response to various magnetic field strengths \( \tau \). In figures, we show the contour maps of \( \psi \) since the level sets of \( \psi \) represent the smectic layers. One can see that the undeformed state is stable for values of \( \tau \) below the critical field \( \tau_c \sim 2 \) as seen in the left column of figure 2. If \( \tau \) increases and reaches \( \tau_c \), layer undulations of 2d square patterns occur (the second column of figure 2). The estimates of the critical field, \( \tau_c \sim 2 \), is consistent with our analytical result (20). The third column of figure 2 shows that the amplitude of undulations grows in the increase of the field strengths. Figure 3 depicts the contour map of the phase of \( \psi \) in two vertical sections, \( y = 0 \) and \( x = 0 \). As in the classic Helfrich–Hurault theory and result from two dimension [21], the figures show that the maximum undulations occur in the middle of the domain and layer perturbations decreases as approaching the bounding plates.
Figure 3. The layer structure on the vertical cross sections, $y = 0$ on the left column and $x = 0$ on the right column, near the onset of the undulations.

Figure 4. 2D planar configuration: director in the plane, vector field plot of $\mathbf{n}$ and contour map of $\phi$ in the first row, contour maps of $n_1$ and $n_2$ in the second row.
6.2. Planar description

In this section, we present numerical simulations of liquid crystal equilibrium states which minimize (27) to support analytical results given in section 5. We consider the gradient flow (in $L^2$) of the energy (27) in $T^2 = (0, 1)^2$ with $\delta = 1.5$. The gradient flow equations are

\[
\frac{\partial \varphi}{\partial t} = \frac{1}{\varepsilon}(\Delta \varphi - \nabla \cdot n),
\]

\[
\frac{\partial n}{\partial t} = \Pi_n \left( \varepsilon \Delta n + \frac{1}{\varepsilon} (\nabla \varphi - n) - \frac{1}{\varepsilon^2} (n \cdot e_3) e_3 \right),
\]

(41)

where we have used the same notation as in the section 6.1.

As initial condition, we consider, for all $(x, y) \in T^2$,

\[
n(x, y, 0) = \frac{(\eta u_1, \eta u_2, 1 + \eta u_3)}{|(\eta u_1, \eta u_2, 1 + \eta u_3)|},
\]

\[
\varphi(x, y, 0) = \eta \varphi_0,
\]

where a small number $\eta = 0.1, u_1, u_2, u_3$ and $\varphi_0$ are arbitrarily chosen.

We take $\varepsilon = 0.005$ and 1024 grid points in the $x$ and $y$ direction, which ensures that the transition layers are accurately resolved.

In figures 4(a) and (b) we illustrate the director and layer configuration and confirm the analytical results in section 5. Numerical simulations illustrate 2d square patterns as an equilibrium state. (See figures 1(a) and 4(a).) Furthermore, the contour maps of $n_1$ and $n_2$ in figures 4(c) and (d) clearly demonstrate the zero mass constraint given in (30).

The numerical values of energy (27) at the equilibrium state with various values of $\varepsilon$ are calculated and written in the table 1. These values agree with the lower bound given in (35).

| $\varepsilon$ | 0.01 | 0.005 | 0.003 | 0.001 |
|----------------|--------|--------|--------|--------|
| Energy (27)    | 3.50   | 2.82   | 2.56   | 2.48   |

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