On Robin’s criterion for the Riemann Hypothesis

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Abstract

Robin’s criterion states that the Riemann Hypothesis (RH) is true if and only if Robin’s inequality

$$\sigma(n) := \sum_{d \mid n} d < e^\gamma n \log \log n$$

is satisfied for $n \geq 5041$, where $\gamma$ denotes the Euler(-Mascheroni) constant. We show by elementary methods that if $n \geq 37$ does not satisfy Robin’s criterion it must be even and is neither squarefree nor squarefull. Using a bound of Rosser and Schoenfeld we show, moreover, that $n$ must be divisible by a fifth power $>1$. As consequence we obtain that RH holds true iff every natural number divisible by a fifth power $>1$ satisfies Robin’s inequality.

1 Introduction

Let $\mathcal{R}$ be the set of integers $n \geq 1$ satisfying $\sigma(n) < e^\gamma n \log \log n$. This inequality we will call Robin’s inequality. Note that it can be rewritten as

$$\sum_{d \mid n} \frac{1}{d} < e^\gamma \log \log n.$$ 

Ramanujan [8] (in his original version of his paper on highly composite integers, only part of which, due to paper shortage, was published, for the shortened version see [7, pp. 78-128]) proved that if RH holds then every sufficiently large integer is in $\mathcal{R}$. Robin [9] proved that if RH holds, then actually every integer $n \geq 5041$ is in $\mathcal{R}$. He also showed that if RH is false, then there are infinitely many integers that are not in $\mathcal{R}$. Put $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040\}$. The set $\mathcal{A}$ consists of the integers $n \leq 5040$ that do not satisfy Robin’s inequality. Note that none of the integers in $\mathcal{A}$ is divisible by a 5th power of a prime.

In this paper we are interested in establishing the inclusion of various infinite subsets of the natural numbers in $\mathcal{R}$. We will prove in this direction:

**Theorem 1** Put $\mathcal{B} = \{2, 3, 5, 6, 10, 30\}$. Every squarefree integer that is not in $\mathcal{B}$ is an element of $\mathcal{R}$.

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A similar result for the odd integers will be established:

**Theorem 2** Any odd positive integer $n$ distinct from $1, 3, 5$ and $9$ is in $\mathcal{R}$.

On combining Robin’s result with the above theorems one finds:

**Theorem 3** The RH is true if and only for all even non-squarefree integers $\geq 5044$ Robin’s inequality is satisfied.

It is an easy exercise to show that the even non-squarefree integers have density $\frac{1}{2} - \frac{2}{\pi^2} = 0.2973\ldots$ (cf. Tenenbaum [11, p. 46]). Thus, to wit, this paper gives at least half a proof of RH!

Somewhat remarkably perhaps these two results will be proved using only very elementary methods. The deepest input will be Lemma 1 below which only requires pre-Prime Number Theorem elementary methods for its proof (in Tenenbaum’s [11] introductory book on analytic number theory it is already derived within the first 18 pages).

Using a bound of Rosser and Schoenfeld (Lemma 4 below), which ultimately relies on some explicit knowledge regarding the first so many zeros of the Riemann zeta-function, one can prove some further results:

**Theorem 4** The only squarefull integers not in $\mathcal{R}$ are $1, 4, 8, 9, 16$ and $36$.

We recall that an integer $n$ is said to be squarefull if for every prime divisor $p$ of $n$ we have $p^2|n$. An integer $n$ is called $t$-free if $p^t \nmid m$ for every prime number $p$. (Thus saying a number is squarefree is the same as saying that it is $2$-free.)

**Theorem 5** All 5-free integers satisfy Robin’s inequality.

Together with the observation that all exceptions $\leq 5040$ to Robin’s inequality are 5-free and Robin’s criterion, this result implies the following alternative variant of Robin’s criterion.

**Theorem 6** The RH holds iff for all integers $n$ divisible by the fifth power of some prime we have $\sigma(n) < e^{\gamma} n \log \log n$.

The latter result has the charm of not involving a finite range of integers that has to be excluded (the range $n \leq 5040$ in Robin’s criterion). We note that a result in this spirit has been earlier established by Lagarias [5] who, using Robin’s work, showed that the RH is equivalent with the inequality

$$\sigma(n) \leq h(n) + e^{h(n)} \log(h(n)),$$

where $h(n) = \sum_{k=1}^{n} 1/k$ is the harmonic sum.
2 Proof of Theorem 1 and Theorem 2

Our proof of Theorem 1 requires the following lemmata.

Lemma 1
1) For $x \geq 2$ we have

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right),$$

where the implicit constant in Landau’s symbol does not exceed $2(1 + \log 4) < 5$ and

$$B = \gamma + \sum_p \left(\log(1 - \frac{1}{p}) + \frac{1}{p}\right) = 0.2614972128 \cdots$$

denotes the (Meissel-)Mertens constant.

2) For $x \geq 5$ we have

$$\sum_{p \leq x} \frac{1}{p} \leq \log \log x + \gamma.$$

Proof. 1) This result can be proved with very elementary methods. It is derived from scratch in the book of Tenenbaum [11], p. 16. At p. 18 the constant $B$ is determined.

2) One checks that the inequality holds true for all primes $p$ satisfying $5 \leq p \leq 3673337$. On noting that

$$B + \frac{2(1 + \log 4)}{\log 3673337} < \gamma,$$

the result then follows from part 1. 

Remark 1. More information on the (Meissel-)Mertens constant can be found e.g. in the book of Finch [4, §2.2].

Remark 2. Using deeper methods from (computational) prime number theory Lemma 1 can be considerably sharpened, see e.g. [10], but the point we want to make here is that the estimate given in part 2, which is the estimate we need in the sequel, is a rather elementary estimate.

We point out that 15 is in $\mathcal{R}$.

Lemma 2 If $r$ is in $\mathcal{A}$ and $q \geq 7$ is a prime, then $rq$ is in $\mathcal{R}$, except when $q = 7$ and $r = 12, 120$ or 360.

Corollary 1 If $r$ is in $\mathcal{B}$ and $q \geq 7$ is a prime, then $rq$ is in $\mathcal{R}$.

Proof of Lemma 2 One verifies the result in case $q = 7$. Suppose that $r$ is in $\mathcal{A}$. Direct computation shows that $11r$ is in $\mathcal{R}$. From this we obtain for $q \geq 11$ that

$$\sigma(rq) \leq (1 + \frac{1}{q})\sigma(r) = 12\sigma(r) \leq e^\gamma \log \log(11r) \leq e^\gamma \log \log(qr).$$

Proof of Theorem 1 By induction with respect to $\omega(n)$, that is the number of distinct prime factors of $n$. Put $\omega(n) = m$. The assertion is easily provable for those integers
with \( m = 1 \) (the primes that is). Suppose it is true for \( m - 1 \), with \( m \geq 2 \) and let us consider the assertion for those squarefree \( n \) with \( \omega(n) = m \). So let \( n = q_1 \cdots q_m \) be a squarefree number that is not in \( B \) and assume w.l.o.g. that \( q_1 < \cdots < q_m \). We consider two cases:

**Case 1:** \( q_m \geq \log(q_1 \cdots q_m) = \log n \).

If \( q_1 \cdots q_{m-1} \) is in \( B \), then if \( q_m \) is not in \( B \), \( n = q_1 \cdots q_m \) is in \( R \) (by the corollary to Lemma 2) and we are done, and if \( q_m \) is in \( B \), the only possibility is \( n = 15 \) which is in \( R \) and we are also done.

If \( q_1 \cdots q_{m-1} \) is not in \( B \), by the induction hypothesis we have

\[
(q_1 + 1) \cdots (q_{m-1} + 1) < e^\gamma q_1 \cdots q_{m-1} \log \log(q_1 \cdots q_{m-1}),
\]

and hence

\[
(q_1 + 1) \cdots (q_{m-1} + 1)(q_m + 1) < e^\gamma q_1 \cdots q_{m-1}(q_m + 1) \log \log(q_1 \cdots q_{m-1}). \tag{1}
\]

We want to show that

\[
e^\gamma q_1 \cdots q_{m-1}(q_m + 1) \log \log(q_1 \cdots q_{m-1})
\]

\[
\leq e^\gamma q_1 \cdots q_{m-1}q_m \log \log(q_1 \cdots q_{m-1}q_m) = e^\gamma n \log \log n. \tag{2}
\]

Indeed (2) is equivalent with \( q_m \log \log(q_1 \cdots q_{m-1}q_m) \geq (q_m + 1) \log \log(q_1 \cdots q_{m-1}) \), or alternatively

\[
\frac{q_m(\log \log(q_1 \cdots q_{m-1}q_m) - \log \log(q_1 \cdots q_{m-1}))}{\log q_m} \geq \frac{\log \log(q_1 \cdots q_{m-1})}{\log q_m}. \tag{3}
\]

Suppose that \( 0 < a < b \). Note that we have

\[
\frac{\log b - \log a}{b - a} = \frac{1}{b - a} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{4}
\]

Using this inequality we infer that (3) (and thus 2) is certainly satisfied if the next inequality is satisfied:

\[
\frac{q_m}{\log(q_1 \cdots q_m)} \geq \frac{\log \log(q_1 \cdots q_{m-1})}{\log q_m}.
\]

Note that our assumption that \( q_m \geq \log(q_1 \cdots q_m) \) implies that the latter inequality is indeed satisfied.

**Case 2:** \( q_m < \log(q_1 \cdots q_m) = \log n \).

It is easy to see that \( \sigma(n) < e^\gamma n \log \log n \) is equivalent with

\[
\log(q_1 + 1) - \log q_1 + \cdots + \log(q_m + 1) - \log q_m < \gamma + \log \log \log(q_1 \cdots q_m). \tag{5}
\]

Note that

\[
\log(q_1 + 1) - \log q_1 = \int_{q_1}^{q_1 + 1} \frac{dt}{t} < \frac{1}{q_1}.
\]

In order to prove (5) it is thus enough to prove that

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \log \log \log(q_1 \cdots q_m). \tag{6}
\]
Since \( q_m \geq 7 \) we have by part 2 of Lemma 1 and the assumption \( q_m < \log(q_1 \cdots q_m) \) that
\[
\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \log \log q_m < \gamma + \log \log(q_1 \cdots q_m),
\]
and hence (6) is indeed satisfied.

Theorem 2 will be derived from the following stronger result.

**Theorem 7** For all odd integers except 1, 3, 5, 9 and 15 we have
\[
\frac{n}{\varphi(n)} < e^\gamma \log \log n,
\]
where \( \varphi(n) \) denotes Euler’s totient function.

To see that this is a stronger result, let
\[
n = \prod_{i=1}^k p_i^{e_i}
\]
be the prime factorisation of \( n \) and note that for \( n \geq 2 \) we have
\[
\frac{\sigma(n)}{n} = \prod_{i=1}^k \frac{1 - p_i^{-e_i-1}}{1 - p_i^{-1}} < \prod_{i=1}^k \frac{1}{1 - p_i^{-1}} = \frac{n}{\varphi(n)}.
\]

We let \( \mathcal{N} \) (\( \mathcal{N} \) in acknowledgement of the contributions of J.-L. Nicolas to this subject) denote the set of integers \( n \geq 1 \) satisfying (7). Our proofs of Theorems 2 and 7 use the next lemma.

**Lemma 3** Put \( S = \{3^a \cdot 5^b \cdot q^c : q \geq 7 \text{ is prime, } a, b, c \geq 0\} \). All elements from \( S \) except 1, 3, 5 and 9 are in \( \mathcal{R} \). All elements from \( S \) except 1, 3, 5, 9 and 15 are in \( \mathcal{N} \).

**Proof.** If \( n \) is in \( S \) and \( n \geq 31 \) we have
\[
\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)} \leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{q}{q-1} \leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} < e^\gamma \log \log n.
\]
Using this observation the proof is easily completed. \( \square \)

**Remark.** Let \( y \) be any integer. Suppose that we have an infinite set of integers all having no prime factors \( > y \). Then \( \sigma(n)/n \) and \( n/\varphi(n) \) are bounded above on this set, whereas \( \log \log n \) tends to infinity. Thus only finitely many of those integers will not be in \( \mathcal{R} \), respectively \( \mathcal{N} \). It is a finite computation to find them all (cf. the proof of Lemma 3).

**Proof of Theorem 7** As before we let \( m = \omega(n) \). If \( m \leq 1 \) then, by Lemma 3, \( n \) is in \( \mathcal{N} \), except when \( n = 1, 3, 5 \) or 9. So we may assume \( m \geq 2 \). Let \( \kappa(n) = \prod_{p|n} p \) denote the squarefree kernel of \( n \). Since \( n/\varphi(n) = \kappa(n)/\varphi(\kappa(n)) \) it follows that if \( r \) is a squarefree number satisfying (7), then all integers \( n \) with \( \kappa(n) = r \) satisfy (7) as well. Thus we consider first the case where \( n = q_1 \cdots q_m \) is an odd squarefree integer with \( q_1 < \cdots < q_m \). In this case \( n \) is in \( \mathcal{N} \) iff
\[
\frac{n}{\varphi(n)} = \prod_{i=1}^m \frac{q_i}{q_i - 1} < e^\gamma \log \log n.
\]
Note that
\[ \frac{q_i}{q_i - 1} \leq \frac{3}{2} \quad \text{and} \quad \frac{q_i}{q_i - 1} < \frac{q_{i-1} + 1}{q_{i-1}}, \]
and hence
\[ \frac{n}{\varphi(n)} = \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{3}{2} \prod_{i=1}^{m-1} \frac{q_i + 1}{q_i} = \frac{\sigma(n_1)}{n_1}, \]
where \( n_1 = 2n/q_m < n \). Thus, \( n/\varphi(n) < \sigma(n_1)/n_1 \). If \( n_1 \) is in \( \mathcal{R} \), then invoking Theorem 1 we find
\[ \frac{n}{\varphi(n)} < \frac{\sigma(n_1)}{n_1} < e^\gamma \log \log n_1 < e^\gamma \log \log n, \]
and we are done.

If \( n_1 \) is not in \( \mathcal{R} \), then by Theorem 1 it follows that \( n \) must be in \( \mathcal{S} \). The proof is now completed on invoking Lemma 3.

Proof of Theorem 2: One checks that 1, 3, 5 and 9 are not in \( \mathcal{R} \), but 15 is in \( \mathcal{R} \). The result now follows by Theorem 7 and inequality (8).

2.1 Theorem 7 put into perspective

Since the proof of Theorem 7 can be carried out with such simple means, one might expect it can be extended to quite a large class of even integers. However, even a superficial inspection of the literature on \( n/\varphi(n) \) shows this expectation to be wrong.

Rosser and Schoenfeld [10] showed in 1962 that
\[ \frac{n}{\varphi(n)} \leq e^\gamma \log \log n + \frac{5}{2 \log \log n}, \]
with one exception: \( n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \). They raised the question of whether there are infinitely many \( n \) for which
\[ \frac{n}{\varphi(n)} > e^\gamma \log \log n, \quad (9) \]
which was answered in the affirmative by J.-L. Nicolas [9]. More precisely, let \( N_k = 2 \cdot 3 \cdot \ldots \cdot p_k \) be the product of the first \( k \) primes, then if the RH holds true [9] is satisfied with \( n = N_k \) for every \( k \geq 1 \). On the other hand, if RH is false, then there are infinitely many \( k \) for which [9] is satisfied with \( n = N_k \) and there are infinitely many \( k \) for which [9] is not satisfied with \( n = N_k \). Thus the approach we have taken to prove Theorem 2 namely to derive it from the stronger result Theorem 7 is not going to work for even integers.

3 Proof of Theorem 4

The proof of Theorem 4 is an immediate consequence of the following stronger result.

Theorem 8 The only squarefull integers \( n \geq 2 \) not in \( \mathcal{N} \) are 4, 8, 9, 16, 36, 72, 108, 144, 216, 900, 1800, 2700, 3600, 44100 and 88200.
Its proof requires the following two lemmas.

**Lemma 4** [10]. For \( x > 1 \) we have

\[
\prod_{p \leq x} \frac{p}{p - 1} \leq e^\gamma (\log x + \frac{1}{\log x}).
\]

**Lemma 5** Let \( p_1 = 2, \ p_2 = 3, \ldots \) denote the consecutive primes. If

\[
\prod_{i=1}^m \frac{p_i}{p_i - 1} \geq e^\gamma \log(2 \log(p_1 \cdots p_m)),
\]

then \( m \leq 4 \).

**Proof.** Suppose that \( m \geq 26 \) (i.e. \( p_m \geq 101 \)). It then follows by Theorem 10 of [10], which states that \( \theta(x) := \sum_{p \leq x} \log p > 0.84x \) for \( x \geq 101 \), that \( \log(p_1 \cdots p_m) = \theta(p_m) > 0.84p_m \). We find that

\[
\log(2 \log(p_1 \cdots p_m)) > \log p_m + \log 1.64 \geq \log p_m + \frac{1}{\log p_m},
\]

and so, by Lemma [10] that

\[
\prod_{i=1}^m \frac{p_i}{p_i - 1} \leq e^\gamma \left( \log p_m + \frac{1}{\log p_m} \right) < e^\gamma \log(2 \log(p_1 \cdots p_m)).
\]

The proof is then completed on checking the inequality directly for the remaining values of \( m \). \( \square \)

**Proof of Theorem** Suppose that

\[
\frac{n}{\varphi(n)} \geq e^\gamma \log \log n.
\]

Put \( \omega(n) = m \). Then

\[
\prod_{i=1}^m \frac{p_i}{p_i - 1} \geq \frac{n}{\varphi(n)} \geq e^\gamma \log \log n \geq e^\gamma \log(2 \log(p_1 \cdots p_n)).
\]

By Lemma [5] it follows that \( m \leq 4 \). In particular we must have

\[
2 \cdot 3 \cdot 5 \cdot 7 \cdot \frac{35}{8} = \frac{35}{8} \geq e^\gamma \log \log n,
\]

whence \( n \leq \exp(\exp(e^{-\gamma}35/8)) \leq 116144 \). On numerically checking the inequality for the squarefull integers \( \leq 116144 \), the proof is then completed. \( \square \)

**Remark.** The squarefull integers \( \leq 116144 \) are easily produced on noting that they can be uniquely written as \( a^2 b^3 \), with \( a \) a positive integer and \( b \) squarefree.
4 On the ratio $\sigma(n)/(n \log \log n)$ as $n$ ranges over various sets of integers

We have proved that Robin’s inequality holds for large enough odd numbers, square-free and squarefull numbers. A natural question to ask is how large the ratio $f_1(n) := \sigma(n)/(n \log \log n)$ can be when we restrict $n$ to these sets of integers. We will consider the same question for the ratio $f_2(n) := n/(\varphi(n) \log \log n)$. Our results in this direction are summarized in the following result:

**Theorem 9** We have

\[\begin{align*}
(1) \quad & \limsup_{n \to \infty} f_1(n) = e^\gamma, \\
(2) \quad & \limsup_{n \to \infty} f_1(n) = \frac{6e^\gamma}{\pi^2}, \\
(3) \quad & \limsup_{n \to \infty} f_1(n) = \frac{e^\gamma}{2},
\end{align*}\]

and, moreover,

\[\begin{align*}
(4) \quad & \limsup_{n \to \infty} f_2(n) = e^\gamma, \\
(5) \quad & \limsup_{n \to \infty} f_2(n) = e^\gamma, \\
(6) \quad & \limsup_{n \to \infty} f_2(n) = e^\gamma.
\end{align*}\]

Furthermore,

\[\begin{align*}
(7) \quad & \limsup_{n \to \infty} f_1(n) = e^\gamma, \\
(8) \quad & \limsup_{n \to \infty} f_2(n) = e^\gamma.
\end{align*}\]

(The fact that the corresponding lim infs are all zero is immediate on letting $n$ run over the primes.)

Part 4 of Theorem 9 was proved by Landau in 1909, see e.g. [1, Theorem 13.14], and the remaining parts can be proved in a similar way. Gronwall in 1913 established part 1. Our proof makes use of a lemma involving $t$-free integers (Lemma 6), which is easily proved on invoking a celebrated result due to Mertens (1874) asserting that

\[
\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \sim e^\gamma \log x, \quad x \to \infty. \tag{10}
\]

**Lemma 6** Let $t \geq 2$ be a fixed integer. We have

\[\begin{align*}
(1) \quad & \limsup_{t \text{-free integers}} f_1(n) = \frac{e^\gamma}{\zeta(t)}, \\
(2) \quad & \limsup_{\text{odd } t \text{-free integers}} f_1(n) = \frac{e^\gamma}{2\zeta(t)(1 - 2^{-t})}.
\end{align*}\]

**Proof.** 1) Let us consider separately the prime divisors of $n$ that are larger than $\log n$. Let us say there are $r$ of them. Then $(\log n)^r < n$ and thus $r < \log n / \log \log n$. Moreover, for $p > \log n$ we have

\[
\frac{1 - p^{-t}}{1 - p^{-1}} < \frac{1 - (\log n)^{-t}}{1 - (\log n)^{-1}}.
\]

Thus,

\[
\prod_{p \mid n \atop p > \log n} \frac{1 - p^{-t}}{1 - p^{-1}} < \left(\frac{1 - (\log n)^{-t}}{1 - (\log n)^{-1}}\right)^{\log n / \log \log n}.
\]
Let \( p_k \) denote the largest prime factor of \( n \). We obtain

\[
\frac{\sigma(n)}{n} = \prod_{i=1}^{k} \frac{1 - p_i^{-e_i - 1}}{1 - p_i^{-1}} \leq \prod_{i=1}^{k} \frac{1 - p_i^{-t}}{1 - p_i^{-1}} < \left( \frac{1 - (\log n)^{-t}}{1 - (\log n)^{-1}} \right) \prod_{p \leq \log n} \frac{1 - p^{-t}}{1 - p^{-1}},
\]

where in the derivation of the first inequality we used that \( e_i < t \) by assumption. Note that the factor before the final product satisfies \( 1 + O((\log \log n) - 1) \) and thus tends to 1 as \( n \) tends to infinity. On invoking (10) and noting that \( \prod_{p \leq \log n} (1 - p^{-t}) \sim \zeta(t)^{-1} \), it follows that the \( \limsup \) satisfies

\[
\frac{\sigma(n)}{n} \log \log n \leq e^\gamma \zeta(t) \cdot (1 + o(1)).
\]

Thus, in particular, for a given \( \epsilon > 0 \) there are infinitely many \( n \) such that

\[
\frac{\sigma(n)}{n \log \log n} > \frac{e^\gamma}{\zeta(t)}(1 - \epsilon).
\]

2) Can be proved very similarly to part 1. Namely, the third product in (11) will extend over the primes \( 2 < p \leq \log n \) and for the \( \geq \) part we consider the integers \( n \) of the form \( n = \prod_{p \leq x} p^{t-1} \).

Remark. Robin [9] has shown that if RH is false, then there are infinitely many integers \( n \) not in \( \mathcal{R} \). As \( n \) ranges over these numbers, then by part 1 of Lemma 6 we must have \( \max\{e_i\} \to \infty \), where \( n = \prod_{i=1}^{k} p_i^{e_i} \).

Proof of Theorem 9

1) Follows from part 1 of Lemma 6 on letting \( t \) tend to infinity. A direct proof (similar to that of Lemma 6) can also be given, see e.g. [3]. This result was proved first by Gronwall in 1913.

2) Follows from part 1 of Lemma 6 with \( t = 2 \).

3) Follows on letting \( t \) tend to infinity in part 2 of Lemma 6.

4) Landau (1909).

5) Since \( f_2(n) \leq f_2(\kappa(n)) \), part 5 is a consequence of part 4.

6) A consequence of part 4 and the fact that for odd integers \( n \) and \( a \geq 1 \) we have \( f_2(2^a n) = 2f_2(n)(1 + O((\log n \log \log n)^{-1})) \).
7) Consider numbers of the form $n = \prod_{p \leq x} p^{t-1}$ and let $t$ tend to infinity. These are squarefull for $t \geq 3$ and using them the $\geq$ part of the assertion follows. The $\leq$ part follows of course from part 3.

8) It is enough here to consider the squarefull numbers of the form $n = \prod_{p \leq x} p^2$. □

5 Reduction to Hardy-Ramanujan integers

Recall that $p_1, p_2, \ldots$ denote the consecutive primes. An integer of the form $\prod_{i=1}^{s} p_i^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_s \geq 0$ we will call an Hardy-Ramanujan integer. We name them after Hardy and Ramanujan who in a paper entitled ‘A problem in the analytic theory of numbers’ (Proc. London Math. Soc. 16 (1917), 112-132) investigated them. See also [7, pp. 241-261], where this paper is retitled ‘Asymptotic formulae for the distribution of integers of various types’.

**Proposition 1** If Robin’s inequality holds for all Hardy-Ramanujan integers $5041 \leq n \leq x$, then it holds for all integers $5041 \leq n \leq x$. Asymptotically there are

$$\exp((1 + o(1))2\pi \sqrt{\log x/3 \log \log x})$$

Hardy-Ramanujan numbers $\leq x$.

Hardy and Ramanujan proved the asymptotic assertion above. The proof of the first part requires a few lemmas.

**Lemma 7** For $e > f > 0$, the function

$$g_{e,f} : x \to \frac{1 - x^{-e}}{1 - x^{-f}}$$

is strictly decreasing on $(1, +\infty]$.

*Proof.* For $x > 1$, we have

$$g_{e,f}'(x) = \frac{ex^f - fx^e + f - e}{x^{e+f+1}(1 - x^{-f})^2}. $$

Let us consider the function $h_{e,f} : x \to ex^f - fx^e + f - e$. For $x > 1$, we have $h_{e,f}'(x) = efx^f(1 - x^{e-f}) < 0$. Consequently $h_{e,f}$ is decreasing on $(1, +\infty]$ and since $h_{e,f}(1) = 0$, we deduce that $h_{e,f}(x) < 0$ for $x > 1$ and so $g_{e,f}(x)$ is strictly decreasing on $(1, +\infty]$. □

**Remark.** In case $f$ divides $e$, then

$$\frac{1 - x^{-e}}{1 - x^{-f}} = 1 + \frac{1}{x^f} + \frac{1}{x^{2f}} + \cdots + \frac{1}{x^e},$$

and the result is obvious.

**Lemma 8** If $q > p$ are primes and $f > e$, then

$$\frac{\sigma(p^f q^e)}{p^f q^e} > \frac{\sigma(p^e q^f)}{p^e q^f}. \quad (12)$$
Thus we may assume that the integer $n > 5040$ that does not satisfy Robin’s inequality. Let $n$ be the smallest such integer. Then $P(n) < \log n$, where $P(n)$ denotes the largest prime factor of $n$. 

Proof. Write $n = r \cdot q$ with $P(n) = q$ and note that $r$ is $t$-free. The minimality assumption on $n$ implies that either $r \leq 5040$ and does not satisfy Robin’s inequality or that $r$ is in $\mathcal{R}$. First assume we are in the former case. Since 720 is the largest integer $a$ in $\mathcal{A}$ with $P(a) \leq 5$ and $5 \cdot 720 \leq 5040$, it follows that $q \geq 7$. By Lemma 8 we then infer, using the assumption that $n > 5040$, that $n = qr$ is in $\mathcal{R}$; a contradiction. Thus we may assume that $r$ is in $\mathcal{R}$ and therefore $r \geq 7$. We will now show that this together with the assumption $q \geq \log n$ leads to a contradiction, whence the result follows.

6 The proof of Theorem 5

Our proof of Theorem 5 makes use of lemmas 11, 12 and 13.

Lemma 11 Let $t \geq 2$ be fixed. Suppose that there exists a $t$-free integer exceeding 5040 that does not satisfy Robin’s inequality. Let $n$ be the smallest such integer. Then $P(n) < \log n$, where $P(n)$ denotes the largest prime factor of $n$. 

Proof. Write $n = r \cdot q$ with $P(n) = q$ and note that $r$ is $t$-free. The minimality assumption on $n$ implies that either $r \leq 5040$ and does not satisfy Robin’s inequality or that $r$ is in $\mathcal{R}$. First assume we are in the former case. Since 720 is the largest integer $a$ in $\mathcal{A}$ with $P(a) \leq 5$ and $5 \cdot 720 \leq 5040$, it follows that $q \geq 7$. By Lemma 8 we then infer, using the assumption that $n > 5040$, that $n = qr$ is in $\mathcal{R}$; a contradiction. Thus we may assume that $r$ is in $\mathcal{R}$ and therefore $r \geq 7$. We will now show that this together with the assumption $q \geq \log n$ leads to a contradiction, whence the result follows.
So assume that \( q \geq \log n \). This implies that
\[
\frac{q \log q}{\log n} \geq \frac{\log \log r}{\log q},
\]
and hence
\[
\frac{q(\log \log n - \log \log r)}{\log q} > \frac{\log \log r}{\log q},
\]
where we used that
\[
\frac{\log \log n - \log \log r}{\log q} = \frac{1}{\log n} - \frac{\log \log r}{\log q}.
\]
Inequality (13) is equivalent with
\[
(1 + \frac{1}{q}) \log \log r < \log \log n
\]
where we used that \( \sigma \) is submultiplicative (that is \( \sigma(qr) \leq \sigma(q)\sigma(r) \)). The inequality (14) contradicts our assumption that \( n \not\in R \).

Lemma 12 All 5-free Hardy-Ramanujan integers \( n > 5040 \) with \( P(n) \leq 73 \) satisfy Robin’s inequality.

Proof. There are 12649 5-free Hardy-Ramanujan integers \( n \) with \( P(n) \leq 73 \) that are easily produced using MAPLE. A further MAPLE computation learns that all integers exceeding 5040 amongst these (12614 in total) are in \( R \).

Remark. On noting that \( \prod_{p \leq x} (1 - p^{-t}) = \frac{\sigma(qr)}{qr} \leq \left( 1 + \frac{1}{q} \right) \frac{\sigma(r)}{r} < \left( 1 + \frac{1}{q} \right) e^\gamma \log r < e^\gamma \log \log n \), the inequality (12) contradicts our assumption that \( n \not\in R \).

Lemma 13 For \( x \geq 3 \) and \( t \geq 2 \) we have that
\[
\sum_{p \leq x} \log \left( \frac{1 - p^{-t}}{1 - p^{-1}} \right) \leq -\log \zeta(t) + \frac{t}{t - 1} x^{1-t} + \gamma + \log \log x + \log \left( 1 + \frac{1}{\log^2 x} \right).
\]
The proof of this lemma on its turn rests on the lemma below.

Lemma 14 Put \( R_t(x) = \prod_{p > x} (1 - p^{-t})^{-1} \). For \( x \geq 3 \) and \( t \geq 2 \) we have that
\[
\log(R_t(x)) \leq tx^{1-t}/(t - 1).
\]

Proof. We have
\[
R_t(x) = -\sum_{p > x} \log \left( 1 - \frac{1}{p^t} \right) = \sum_{p > x} \sum_{m=1}^\infty \frac{1}{mp^m} \leq \sum_{p > x} \sum_{m=1}^\infty \frac{1}{(p^m)^t} \\
\leq \sum_{n > x} \frac{1}{n^t} \leq \frac{1}{x^t} + \sum_{n > x + 1} \frac{1}{n^t} \leq \frac{1}{x^{1-t}} + \int_x^{\infty} \frac{du}{u^t} = \frac{t}{t - 1} x^{1-t}.
\]

Proof of Lemma 14 On noting that \( \prod_{p \leq x} (1 - p^{-t}) = R_t(x)/\zeta(t) \) and invoking Lemma 14 we obtain
\[
\sum_{p \leq x} \log \left( 1 - \frac{1}{p^t} \right) = -\log \zeta(t) + \log(R_t(x)) \leq -\log \zeta(t) + \frac{t}{t - 1} x^{1-t}.
\]

On combining this estimate with Lemma 4 the estimate then follows.
Lemma 15 Let $m$ be a 5-free integer such that $P(m) < \log m$ and $m$ does not satisfy Robin’s inequality. Then $P(m) \leq 73$.

Proof. Put $t = 5$. Write $P_t(x) = \prod_{p \leq x} (1 - p^{-1})/(1 - p^{-1})$. Put $z = \log m$. The assumptions on $m$ imply that $\sigma(m)/m \leq P_t(z)$. This inequality in combination with Lemma 13 yields

$$\log \left( \frac{\sigma(m)}{m} \right) \leq -\log \zeta(t) + \frac{t}{(t-1)z^{-t-1}} + \gamma + \log \log z + \log \left( 1 + \frac{1}{\log^2 z} \right).$$

Once

$$-\log \zeta(t) + \frac{t}{t-1} z^{1-t} + \gamma + \log \log z + \log \left( 1 + \frac{1}{\log^2 z} \right) < \gamma + \log \log z,$$

Robin’s inequality is satisfied. We infer that once we have found a $z_0 \geq 3$ such that

$$\frac{t}{t-1} z_0^{1-t} + \log \left( 1 + \frac{1}{\log^2 z_0} \right) - \log \zeta(t) < 0,$$

then Robin’s inequality will be satisfied in case $z \geq z_0$. One finds that $z_0 = 196$ will do. It follows that $z < 196$ and hence $\sigma(m)/m < P_5(193) = 9.18883221\ldots$. Note that if $e^\gamma \log \log m \geq P_5(193)$, then Robin’s inequality is satisfied. We thus conclude that $\log m \leq \exp(P_5(193)e^{-\gamma}) = 174.017694\ldots$. Since 173 is the largest prime $< 175$ we know that $m$ must satisfy $\sigma(m)/m < P_5(173) = 8.992602079\ldots$. We now proceed as before, but with $P_5(193)$ replaced by $P_5(173)$. Indeed, this ‘cascading down’ can be repeated several times before we cannot reduce further. This is at the point where we have reached the conclusion that $z = \log m \leq 73$. Then we cannot reduce further since $\exp(P_5(73)e^{-\gamma}) > 73$. \hfill \Box

Proof of Theorem. By contradiction. So suppose a 5-free integer exceeding 5040 exists that does not satisfy Robin’s inequality. We let $n$ be the smallest such integer. By Lemma 10 it follows that $P(n) < \log n$, whence by Lemma 15 we infer that $P(n) \leq 73$. We will now show that $n$ is a Hardy-Ramanujan number. On invoking Lemma 12 the proof is then completed.

It thus remains to establish that $n$ is a Hardy-Ramanujan number. Let $\bar{e}$ denote the factorisation pattern of $n$. Note that $m(\bar{e})$ is 5-free and that $m(\bar{e}) \leq n$. By the minimality of $n$ and part 1 of Lemma 10 it follows that we cannot have that $5041 \leq m(\bar{e}) < n$ and so either $m(\bar{e}) = n$, in which case we are done as $m(\bar{e})$ is a Hardy-Ramanujan number, or $m(\bar{e}) \leq 5040$. In the latter case we must have $n = p_1^e p_2^{e_2} p_3^{e_3} p_4^{e_4} p_5^{e_5}$ (since $\max\{\omega(r) : r \leq 5040\} = 5$) and so

$$\frac{\sigma(n)}{n} \leq \prod_{p \leq 11} \frac{1 - p^{-5}}{1 - p^{-1}} = 4.6411\ldots$$

and

$$\prod_{p \leq 11} \frac{1 - p^{-5}}{1 - p^{-1}} \geq e^\gamma \log \log n,$$

whence $\log n \leq 13.55$. A MAPLE computation now shows that $n \notin \mathcal{R}$, contradicting our assumption that $n \notin \mathcal{R}$. \hfill \Box
By the method above we have not been able to replace 5-free by 6-free in Theorem 5 (this turns out to require a substantial computational effort). Recently J.-L. Nicolas kindly informed the authors of an approach (rather different from the one followed here and being less self-contained) that might lead to a serious improvement of the 5-free. It would certainly be interesting to pursue Nicolas’s idea further and this might be part of a follow-up paper.

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References

[1] T.M. Apostol, Introduction to analytic number theory, Undergraduate Texts in Mathematics. (Springer-Verlag, New York-Heidelberg, 1976).

[2] K. Briggs, Abundant numbers and the Riemann hypothesis, to appear in Experimental Mathematics.

[3] J.H. Bruinier, Primzahlen, Teilersummen und die Riemannsche Vermutung, Math. Semesterber. 48 (2001) 79–92.

[4] S.R. Finch, Mathematical constants, Encyclopedia of Mathematics and its Applications 94, (Cambridge University Press, Cambridge, 2003).

[5] J.C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, Amer. Math. Monthly 109 (2002) 534–543.

[6] J.-L. Nicolas, Petites valeurs de la fonction d’Euler, J. Number Theory 17 (1983) 375–388.

[7] S. Ramanujan, Collected Papers, (Chelsea, New York, 1962).

[8] S. Ramanujan, Highly composite numbers. Annotated and with a foreword by J.-L. Nicolas and G. Robin, Ramanujan J. 1 (1997) 119–153.

[9] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. Pures Appl. (9) 63 (1984) 187–213.

[10] J.B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962) 64–94.

[11] G. Tenenbaum, Introduction to analytic and probabilistic number theory, Cambridge Studies in Advanced Mathematics 46, (Cambridge University Press, Cambridge, 1995).
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