ON GYROSCOPIC PRECESSION

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PACS : 04.20.Cv, 04.20.−q, 04.80.Cc, 04.90.+e.

Abstract

The vorticity of a congruence is often considered to be the rate of rotation for the precession of a gyroscope moving along a world-line belonging to that congruence. Our aim here was to determine the evolution equation for the angular momentum of a gyroscope with respect to an arbitrary time-like congruence: i.e., a reference congruence not containing the curve described by the gyroscope. In particular, we show the specific conditions needed to support the introductory statement about the vorticity. We thus establish a well-founded theoretical description for the analysis of the precession of gyroscopes, providing suitable conclusions for possible experiments.

1. INTRODUCTION

Analysis of the evolution undergone by the angular momentum of a gyroscope in the presence of a gravitational field is a problem in General Relativity that has reignited the interest aroused by the expected (although delayed) launching of the artificial satellite Gravity Probe B (GP-B) [1]. Thus, in recent years a series of articles [2]-[8] studying different aspects of this issue has been published, showing explicitly that despite the time elapsed since the early pioneering works on the topic [9]-[12], there are still some obscure aspects that require further elucidation.

Among such articles, one by Rindler and Perlick [2] is outstanding. That paper establishes the foundations for studying the evolution of a point-like gyroscope (i.e., of negligible size) with respect to a reference system considered as a congruence of time-like world-lines. This point of view is essential in order to gain both a model of the structure supporting the gyroscope and any other reference that can be used to evaluate the physical magnitudes involved in the problem. In this paper we propose a specific procedure that allows us to recover the classical post-newtonian results of Fokker–de Sitter and Schiff in a fairly simple way.

In particular, we review article [2] because it explicitly calculates the orientation of the gyroscope (changes in a certain angle) after its revolution along the orbit and -as explained below- this is the result that we are specifically interested in. To be fair, however, other works should also be mentioned. In this sense, Massa and Zordan [13] conducted a rigorous study of the motion of a gyroscopically stabilized point compass in a given frame, applying the theory of space

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tensors. To a certain extent, those authors managed to modernise the Cattaneo-Zel’tmanov [14-17] approach to the splitting of space–times. Later on, exhaustive work by Jantzen, Carini and Bini [18] updated these techniques within the framework of the spatial gravitational forces used to treat gyroscope precession. In two papers [6],[19], one of those authors (D. Bini) devoted extensive space to studying gyroscope evolution and spinning test particles in general relativity, including the effects of gravitational waves effects on the behaviour of gyroscopes.

The aim of the present work is to offer a complementary analysis with respect to the above-mentioned study by Rindler and Perlick [2] as regards two main aspects. First, we wished to analyse the evolution of a gyroscope’s angular momentum with respect to an arbitrary congruence; i.e., a congruence which does not contain the curve along which the gyroscope is moving. Thus, our aim is to show how the intrinsic quantities defined by the congruence on the quotient manifold (acceleration or Newtonian field, rotation or Coriolis field and, deformation rate) affect the evolution of the gyroscope, as done in [13]. In particular, we obtain the key equation of Rindler and Perlick when the gyroscope is assumed to be at rest with respect to the congruence, which, additionally, has no deformation rate (Born’s congruence). As stated by the authors, this result implies that the precession rate of the gyroscope coincides with the rotation tensor of the congruence.

Second, we wished to determine the change in the orientation of a gyroscope after one revolution in its orbit as appreciated by some arbitrary observer. To do so, we make use of a triad of connectors transported by parallelism in the sense of the spatial metric and we take into account the holonomy group of such connectors transported along closed curves in a Riemannian space.

In [2], the authors are mainly interested in the final rotated angle made by the gyroscope after a given period in a closed orbit and do not take into account the influence of gravitational fields associated with an arbitrary congruence. Accordingly, the problem about the change from the “proper” congruence to an arbitrary one can be solved intuitively by taking the standard coordinates of the Schwarzschild metric (and the Boyer–Lindquist coordinates of the Kerr metric) at the level of the spherical coordinates of Minkowski flat space. We thus assume that the tangent vectors to the radial coordinates rotate an angle of \(2\pi\) when a displacement of one revolution is made along a circular orbit on the equatorial plane. We propose the prior definition of a new angular momentum by means of a suitable boost. This approach has already been used by Bel and one of us, (Martín) [20], to transform a spin vector from one frame of reference to another within the context of the predictive relativistic mechanics of systems with spin, and it has also been used more recently by Bini [6] to study the evolution of gyroscopes. We then evaluate the rotation of this spin (new angular momentum) with respect to a triad that is transported by parallelism in the sense of the quotient
metric associated with the congruence. Finally, we see that it is possible to recover
the results of Rindler and Perlick if the proper rotation of the triad is considered
for a complete revolution of the circular orbit.

The work is presented as follows. Section 2 offers a brief review of the time-
like congruences of space-time, placing emphasis on certain aspects to be deployed
later on, such as the concept of natural and orthogonal connector triads.

Section 3 is divided into three subsections. In the first, we explore the evol-
elution of a gyroscope with respect to a congruence and the construction of a new
spin by means of a local Lorentzian boost. In the second, we obtain the evolution
equation of the new spin in the three-dimensional formalism associated with
the reference congruence. Finally, in the third subsection we introduce a field of
non-natural connector triads in order to generalise the Cartesian connectors of
Minkowski space-time.

Finally, in order to illustrate this issue, in section 4 we evaluate the precession
for Schwarzschild and Kerr metrics. Our aim in doing this is not only to show
that this methodology provides the correct (and well known) results but also to
highlight the difficulties that would be encountered on handling this problem in
more general cases.

2. TIME-LIKE CONGRUENCES

In this section we explicitly introduce the definitions to be used later on
regarding a time-like congruence. Let \( \mathcal{C} \) be a congruence of time-like world-lines
on some domain \( \mathcal{D} \) of the space-time manifold \( (\mathcal{V}_4; g_{\alpha\beta}) \), and let
\[
x^\alpha = f^\alpha(p, z^i)
\]
\((i, j, \ldots = 1, 2, 3; \alpha, \beta, \ldots = 0, 1, 2, 3)\) be the parametric equations of this congru-
ence. The unit time-like tangent vector field is:
\[
u^\alpha(x) = \xi^{-1} \xi^\alpha[p(x), z^i(x)], \quad g_{\alpha\beta} u^\alpha u^\beta = -1,
\]
with
\[
\xi^\alpha \equiv \frac{\partial f^\alpha}{\partial p}, \quad g_{\alpha\beta} \xi^\alpha \xi^\beta \equiv -\xi^2 < 0,
\]
where
\[
\begin{aligned}
p &= p(x^\alpha) \\
z^i &= z^i(x^\alpha)
\end{aligned}
\]
are the inverse functions of (1). The time-like parameter \( p \) of the congruence can
be choosen, up to the gauge transformation \( \tau_R \to \tau_R + A(z^i) \), to be the proper
time of the congruence, such that:
\[
\tau_R : u^\alpha = \frac{\partial x^\alpha}{\partial \tau_R}.
\]
The three tensor fields associated intrinsically with the time-like congruence - the Deformation rate $\Sigma_{\alpha\beta}$, the Rotation $\Omega_{\alpha\beta}$ (sometimes called the Coriolis force or Gravitomagnetic field) and the Acceleration $b_{\alpha}$ (Newtonian or Gravitoelectric field up to sign) - are given by:

$$\nabla_{\alpha} u_{\beta} = \Sigma_{\alpha\beta} + \Omega_{\alpha\beta} - u_{\alpha} b_{\beta}, \quad (6)$$

$$\begin{cases}
\Sigma_{\alpha\beta} = \frac{1}{2} \hat{g}_\alpha^\lambda \hat{g}_\beta^\mu (\nabla_\lambda u_\mu + \nabla_\mu u_\lambda) \\
\Omega_{\alpha\beta} = \frac{1}{2} \hat{g}_\alpha^\lambda \hat{g}_\beta^\mu (\nabla_\lambda u_\mu - \nabla_\mu u_\lambda) \\
b_{\alpha} = u^\rho \nabla_\rho u_{\alpha}, \quad (7)
\end{cases}$$

where

$$\hat{g}_\alpha^\lambda \equiv \delta_\alpha^\lambda + u^\lambda u_{\alpha} \quad (8)$$

is the projector tensor orthogonal to $u^\alpha$ (usually considered to be the metric on the 3-space quotient manifold).

Since we are interested in the evolution of a vector orthogonal to $u^\alpha$, to refer to it we first define a 3-frame orthogonal to $u^\alpha$. The natural way to do this is as follows: since each line of the congruence is characterized by a parameter $z^i$, a vector connecting two lines of (1) and orthogonal to $u^\alpha$ is given by a linear combination (with functions independent of the parameter $p$) of the spatial projection of the three derivatives:

$$Q_i^\alpha \equiv \frac{\partial f^\alpha}{\partial z^i}. \quad (9)$$

i.e.

$$q_i^\alpha \equiv \hat{g}_\alpha^\lambda Q_i^\lambda = \hat{g}_\alpha^\lambda \frac{\partial f^\lambda}{\partial z^i}, \quad (10)$$

which we shall henceforth refer to as the natural orthogonal connectors triad $\{q_i^\alpha\}$. These connectors are the components of the following vector-field:

$$\hat{\partial}_i \equiv q_i^\alpha \frac{\partial}{\partial x^\alpha} = \varphi_i \frac{\partial}{\partial p} + \frac{\partial}{\partial z^i}, \quad \varphi_i \equiv \xi^{-1}(u Q_i). \quad (11)$$

Thus, a tetrad of space-time is given by $\{e_a^\alpha\} \equiv \{u^\alpha, q_i^\alpha\}$, ($a, b, ... = 0, 1, 2, 3$), which satisfies the orthogonality conditions:

$$g_{\alpha\beta} u^\alpha q_i^\beta = 0, \quad (12)$$

where

$$g_{\alpha\beta} q_i^\alpha q_j^\beta \equiv \hat{g}_{ij} \quad (13)$$
are the components of the quotient metric with respect to the triad \( \{ q^\alpha_i \} \). The related co-base of one-forms can be constructed to give \( \{ \theta^a_\alpha \equiv \{ -u_\alpha, p^i_\alpha \} \) such that:

\[
\begin{align*}
    p^i_\alpha dx^\alpha &= \frac{\partial z^i}{\partial x^\alpha} dx^\alpha = dz^i \\
    q^i_\alpha p^j_\alpha &= \delta^i_j.
\end{align*}
\] (14)

It is worthwhile noting the behaviour of orthogonal connectors triads under a change in the parameters of the congruence (1). If a general change is performed, such as:

\[
\begin{align*}
    p &\rightarrow p' = p'(p, z^i) \\
    z^i &\rightarrow z^{k'} = z^{k'}(z^i),
\end{align*}
\] (15)

one then has:

\[
Q_i^\alpha = \xi' u_\alpha \frac{\partial p'}{\partial z^i} + Q_{k'}^\alpha \frac{\partial z^{k'}}{\partial z^i},
\] (16)

with

\[
\xi' = \left( \frac{\partial p'}{\partial p} \right)^{-1} \xi.
\] (17)

Therefore, the natural orthogonal connector changes like a tensor on the quotient manifold, whatever the time parameter \( p' \) considered:

\[
q^\lambda_i = q^\lambda_{k'} \frac{\partial z^{k'}}{\partial z^i}.
\] (18)

For further calculations it will be useful to obtain the coefficients \( \gamma^c_{ab} \) of the connection with respect to the tetrad \( \{ e^\alpha_a \equiv \{ u_\alpha, q^i_\alpha \} \) in terms of the geometrical objects defining the congruence. As is well known, we have:

\[
e^\lambda_b \nabla^\lambda e^\mu_a = \gamma^c_{ab} e^\mu_c.
\] (19)

By contrast, we also have:

\[
d\theta^c = -\frac{1}{2} C^c_{ae} \theta^a \wedge \theta^e,
\] (20)

\( C^c_{ae} \) being the coefficients appearing in the Lie brackets of the tetrad vectors; i.e.,

\[
[\vec{e}_a, \vec{e}_b] = C^c_{ab} \vec{e}_c.
\] (21)

Now using the definition of connectors, we obtain:

\[
\begin{align*}
    [\vec{u}, \vec{q}_j] &= (b q_j) \vec{u} \\
    [\vec{q}_i, \vec{q}_j] &= 2 \tilde{\Omega}_{ij} \vec{u},
\end{align*}
\] (22)

and it is therefore easy to derive the following expressions for the coefficients of the connection:
where the following notation has been used (the latin indices are raised and lowered with the quotient metric $\hat{g}_{ij}$):

\[
\begin{align*}
\hat{b}_i &\equiv q_i^\alpha b_\alpha \\
\hat{\Omega}_{ij} &\equiv q_i^\alpha q_j^\beta \Omega_{\alpha\beta} = \frac{1}{2} \xi (\hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i) \\
\hat{\Sigma}_{ij} &\equiv q_i^\alpha q_j^\beta \Sigma_{\alpha\beta} = \frac{1}{2} \xi^{-1} \frac{\partial \hat{g}_{ij}}{\partial \varphi}.
\end{align*}
\]

where $\hat{\Gamma}^i_{jk}$ is the Zel’manov-Cattaneo connection [16],[21]-[22]:

\[
\hat{\Gamma}^i_{jk} = \frac{1}{2} \hat{g}^{ih} (\hat{\partial}_j \hat{g}_{kh} + \hat{\partial}_k \hat{g}_{jh} - \hat{\partial}_h \hat{g}_{jk}).
\]

3. THE EVOLUTION OF A GYROSCOPE WITH RESPECT TO AN ARBITRARY CONGRUENCE

Let us now consider a point-like gyroscope (of negligible size) moving along an arbitrary time-like curve. As usual, we can dispense with the Papapetrou equations [23] and assume that the spin of the gyroscope is Fermi-Walker-transported along the curve. The aim of this section is to provide an exact description of the evolution of this spin with respect to a certain reference time-like congruence, and hence in terms of the geometric objects associated with it: acceleration, rotation, deformation rate and Cattaneo’s connection. This generalizes, for example, the conclusions concerning the postnewtonian experiment of a rocket gyroscope in orbit around the Earth [24]. In addition, as we will see, the question of recovering the well known Thomas or Fokker–de Sitter and Schiff postnewtonian precession terms is not a trivial matter, because evaluation of the angle rotated by the spin of the gyroscope is not an issue of superficial geometric considerations.

A) Fermi-Walker transport and local covariant Lorentzian boost

Let $\hat{S}^\alpha$ be a Fermi-Walker-transported (FWT) vector along a time-like curve with a unitary tangent vector $w^\alpha$ and an acceleration of $a^\alpha$:

\[
\begin{align*}
x^\alpha &= \varphi^\alpha(\tau) : \quad \begin{cases} w^\alpha &\equiv \frac{d\varphi^\alpha}{d\tau}, \\
a^\alpha &\equiv \frac{\nabla w^\alpha}{d\tau} \quad , \quad g_{\alpha\beta} w^\alpha w^\beta = -1, \end{cases}
\end{align*}
\]

(26a)
\[
\frac{\nabla S^\alpha}{d\tau} + (a^\alpha w_\lambda - a_\lambda w^\alpha) S^\lambda = 0.
\]  
\text{(26b)}

The first two integrals of this differential equation are given by \( S_\mu w^\mu \) and the length of \( S^\alpha \), i.e. \( g_{\alpha\beta} S^\alpha S^\beta \). Therefore, we can take \( S_\mu w^\mu = 0 \), and so from here onwards \( S^\alpha \) will represent the intrinsic spin vector of a gyroscope moving along such a curve.

Let \( C \) now be a congruence of time-like world-lines defined as usual:

\[
C : x^\alpha = f^\alpha(\tau_R, z^i) , \quad \begin{cases} 
\tau_R = \tau_R(x^\alpha) \\
\tau_i = \tau_i(x^\alpha)
\end{cases}
\]  
\text{(27)}

\( \tau_R \) being the proper time and \( u^\alpha \) the unit tangent vector

\[
u^\alpha(x) = \frac{\partial f^\alpha}{\partial \tau_R}[\tau_R(x), z^i(x)] , \quad g_{\alpha\beta} u^\alpha u^\beta = -1.
\]  
\text{(28)}

An observer evolving with the reference congruence sees the gyroscope in motion because in general \( u^\alpha \) is not co-linear with \( w^\alpha \). One therefore needs to know what the components of the spin vector \( S^\alpha \) are as seen by this observer. To do so, we define a covariant local Lorentzian boost \( B : \{u^\alpha\} \rightarrow \{w^\alpha\} \), thereby transforming this observer at rest for the congruence into an observer tied to the gyroscope. By applying the boost to the spin-vector \( S^\alpha \), we have:

\[
S^\alpha \rightarrow N^\alpha = S^\alpha + \frac{(Su) + (Sw)}{1 - (uw)} (u^\alpha + w^\alpha) - 2(Sw)u^\alpha .
\]  
\text{(29)}

This transformation fulfills the following properties:

\[
\begin{aligned}
(Nu) &= (Sw) \\
N^2 &= S^2 \\
(Nw) &= -(Su) - 2(Sw)(uw) .
\end{aligned}
\]  
\text{(30)}

In particular, and as shown, this preserves the length of the spin-vector. Moreover, because the spin-vector \( S^\alpha \) is orthogonal to \( w^\alpha \), we have:

\[
\begin{aligned}
(Nu) &= 0 \\
(Nw) &= -(Su) ,
\end{aligned}
\]  
\text{(31)}

and hence the new spin vector \( N^\alpha \) is orthogonal to the reference congruence.

Let us now write the evolution equation of the transformed spin-vector \( N^\alpha \) along the curve \( \varphi^\alpha(\tau) \). Since \( S^\alpha \) satisfies the FWT equation (26b), from (29) it is straightforward to obtain:

\[
\frac{\nabla N^\alpha}{d\tau} = \frac{(Na)}{1 + \gamma} (w^\alpha - \gamma u^\alpha) - \frac{(Nw)}{1 + \gamma} [a^\alpha + (ua)u^\alpha]
\]  
\[+ \frac{1}{1 + \gamma} \left[ N_\rho(u^\alpha + w^\alpha) - (Nw)\delta^\alpha_\rho \right] \frac{\nabla u^\rho}{d\tau} ,
\]  
\text{(32)}
with
\[
\frac{\nabla u^\rho}{d\tau} = w^\lambda (\Sigma_\lambda^\alpha + \Omega_\lambda^\alpha - u_\lambda b^\alpha),
\]
where \( \gamma \equiv -(u \cdot w) \) is analogous to the factor appearing in the Lorentz transformation for Special Relativity.

**B) Evolution equation of the spin \( N^\alpha \) in a three-dimensional formalism**

We shall now refer \( N^\alpha \), as well as the tangent unit vector \( w^\alpha \) and the acceleration \( a^\alpha \) of the curve, to the tetrad of space–time \( \{u^\alpha, q_i^\alpha\} \):

\[
\begin{align*}
N^\alpha &= \hat{N}^i q_i^\alpha \\
w^\alpha &= \gamma u^\alpha + \hat{w}^i q_i^\alpha \\
a^\alpha &= -(ua) u^\alpha + \hat{a}^i q_i^\alpha.
\end{align*}
\]

(34)

As mentioned in the introduction, techniques for splitting space-time have already been used in previous works by other authors (Bini, Carini, Massa,...,[6],[13],[18-19], and by Bel, Llosa, Martín and Molina [25-26]) following the pioneer work by Cattaneo. From the first expression in (34) we can write the covariant derivative of \( N^\alpha \) thus

\[
\frac{\nabla N^\alpha}{d\tau} = \frac{d\hat{N}^i}{d\tau} q_i^\alpha + \hat{N}^i \frac{\nabla q_i^\alpha}{d\tau},
\]

(35)

where, using the second expression of the decomposition (34), \( \frac{\nabla q_i^\alpha}{d\tau} \) is:

\[
\frac{\nabla q_i^\alpha}{d\tau} = w^\rho \nabla_\rho q_i^\alpha = \gamma u^\rho \nabla_\rho q_i^\alpha + \hat{w}^k q_k^\rho \nabla_\rho q_i^\alpha.
\]

(36)

Now using the coefficients (23) of the connection (Ricci’s rotation coefficients) with respect to the above tetrad, we have:

\[
\frac{\nabla q_i^\alpha}{d\tau} = -(uw) \left[(bq_i) u^\alpha + q_i^\lambda (\Sigma_\lambda^\alpha + \Omega_\lambda^\alpha)\right] + \hat{w}^k \left[\hat{\Sigma}_{ki} + \hat{\Omega}_{ki}\right] u^\alpha + \hat{\Gamma}_{kj}^i q_j^\alpha.
\]

(37)

With this result and from (32), we can extract from (35) the evolution equation of the components of \( N^\alpha \) in the connectors frame \( \{q_i^\alpha\} \):

\[
\frac{d\hat{N}^i}{d\tau} + \hat{\Gamma}_{kj}^i \hat{w}^j \hat{N}^k = B_k^i \hat{N}^k,
\]

(38a)

with,

\[
B_k^i \equiv \gamma (\hat{\Omega}_{ki} - \hat{\Sigma}_{ki})
\]

\[
+ \frac{1}{1 + \gamma} \left[\hat{w}^i (\gamma \hat{b}_k + \hat{a}_k) - \hat{w}_k (\gamma \hat{b}^i + \hat{a}^i)\right]
\]

\[
+ \frac{1}{1 + \gamma} \left[\hat{w}^i \hat{w}^j (\hat{\Sigma}_{jk} + \hat{\Omega}_{jk}) - \hat{w}_k \hat{w}^j (\hat{\Sigma}_{j}^i + \hat{\Omega}_{j}^i)\right].
\]

(38b)
Equation (38) describes the evolution of a gyroscope moving along a curve with 3-velocity $\hat{w}^i$ with respect to an arbitrary congruence. As can be seen, the expression involves not only the geometric objects of the congruence (such as rotation, deformation, acceleration, or the Zel’manov–Cattaneo connection) but also the velocity and acceleration of the curve described by the gyroscope. On the other hand, in general equation (38) is not a “precession” equation, but something more complicated; this is because of the presence of $\hat{\Sigma}_{ij}$ and the symmetric part of the covariant component of $\hat{\Gamma}_{ij}^k \hat{w}^j$. In this sense, for further applications it is suitable to write equation (38) in its separate symmetric and antisymmetric parts; i.e.:

$$\frac{d\hat{N}^i}{d\tau} = (A^i_k + S^i_k)\hat{N}^k,$$

with

$$A_{jk} \equiv \hat{g}_{ji} A^i_k \equiv \gamma \hat{\Omega}_{jk} + \frac{1}{1 + \gamma} \left[ \hat{w}_j (\gamma \hat{b}_k + \hat{a}_k) - \hat{w}_k (\gamma \hat{b}_j + \hat{a}_j) \right]$$

$$+ \frac{1}{1 + \gamma} \left[ \hat{w}_j (\hat{\Sigma}_{lk} + \hat{\Omega}_{lk}) - \hat{w}_k (\hat{\Sigma}_{lj} + \hat{\Omega}_{lj}) \right] \hat{w}^l \quad (39b)$$

$$+ \frac{1}{2} (\hat{\partial}_j \hat{g}_{kl} - \hat{\partial}_k \hat{g}_{jl})w^l,$$

$$S_{jk} \equiv \hat{g}_{ji} S^i_k \equiv -\gamma \hat{\Sigma}_{jk} - \frac{1}{2} \hat{w}^l \hat{\partial}_l \hat{g}_{jk} \quad (39c).$$

Expression (39b) shows a precession rate that generalizes the classical rate of precession appearing in postnewtonian calculations (see, for instance, Weinberg [27]), since the geodetic-precession terms (Thomas’s gravitational precession) and those referring to the spin-spin interaction (gravitomagnetic effects) are obviously more complicated.

It should be stressed that equation (38) is a tensorial expression on the quotient manifold since the left hand side is the covariant derivative of $\hat{N}^i$ in the sense of the Zel’manov–Cattaneo connection. However, the evaluation of the angle rotated by the vector $\hat{N}^i$ (see Appendix), after one revolution on a closed orbit, depends on the parametrization of the congruence because this angle is defined up to an integer multiple of $2\pi$. As an example, we mention the simpler case of Minkowski space-time. If we consider a gyroscope moving on a circular orbit at the “equatorial” plane and we refer its evolution to the congruence defined only by the variation of the time coordinate, which is irrotational and has no deformation rate, equation (38) is simplified. By solving this equation in Cartesian coordinates ($\hat{\Gamma}_{jk}^i = 0$), in one orbital period the gyroscope precesses by an angle $2\pi(\gamma - 1)$, which is the correct Thomas precession, whereas by solving the same equation in spherical coordinates the angle is $2\pi\gamma$. It is clear, for Minkowski space–time, why the difference between both calculations of the angle rotated is $2\pi$: the Cartesian
connectors evolve by parallelism, whereas the connectors in spherical coordinates rotate exactly $2\pi$ after one period of the orbit.

The problem becomes slightly more complicated if we consider, for example, Schwarzschild or Kerr space–time (also with an equatorial circular orbit and the congruence defined by $t$ varying alone) because the quotient manifold is a non-flat $3$–Riemannian metric, and we therefore have no “Cartesian connectors” that we can use as a reference system.

C) Triads of non-natural orthogonal connectors

In order to avoid ambiguity in the determination of the rotation of the gyroscope in the general case, it is necessary to introduce a “Cartesian-like connectors” system on the quotient space. We first define a general triad of connectors as a combination of the orthogonal connectors:

$$h^\alpha_i = \hat{h}^k_i q^\alpha_k, \quad \det(\hat{h}^k_i) > 0,$$

where the coefficients $\hat{h}^k_i$ only depend on the space parameters of the congruence -i.e., $\hat{h}^k_i = \hat{h}^k_i(z^j)$- in order to preserve the meaning given to them. If we now look for changes in the evolution equation of $N^\alpha$ (38), we end up with the following:

$$\frac{d\tilde{N}^i}{d\tau} + \tilde{q}^i_j \hat{\nabla} \hat{h}^j_k = \tilde{B}^i_k \tilde{N}^k,$$

where $\hat{\nabla}$ stands for the covariant derivative with respect to $\hat{\Gamma}^i_{jk}$ and $\tilde{B}^i_k$ denotes the analogous expression to (38b) with respect to the new connectors, i.e, the “tilde” is a notation for a decomposition similar to (34) but relative to the new connectors $h^\alpha_i$. The matrix $\tilde{q}^i_j$ denotes the inverse of (40), i.e. $q^\alpha_k = \tilde{q}^i_k h^\alpha_i$.

Obviously, equation (41) can be simplified by assuming that the new connectors are transported by parallelism, in the sense of $\hat{\Gamma}$, along the curve described by the gyroscope which, as is known, corresponds to spatial Fermi-Walker transport. This assumption implies an unambiguous criterion for the evaluation of the gyroscope–orientation change although, as we shall see, for each case it is necessary to evaluate the holonomy group on the quotient manifold. At the same time it represents a way to intrinsically generalize the connectors associated with Cartesian coordinates in Minkowski space–time.

With this choice of the new connectors, and focusing on Born’s congruences ($\tilde{\Sigma}_{ij} = 0$), equation (41) proves to be:

$$\frac{d\tilde{N}^i}{d\tau} = \tilde{A}^i_k \tilde{N}^k$$

$$\tilde{A}^i_k = \frac{1}{1 + \gamma} \left[ \tilde{w}^i \tilde{w}^j \tilde{\Omega}^k_{jk} - \tilde{w}^j \tilde{w}^k \tilde{\Omega}^i_{jk} \right] (42a)$$

$$+ \frac{1}{1 + \gamma} \left[ \tilde{w}^i (\gamma \tilde{b}^k + \tilde{a}_k) - \tilde{w}^k (\gamma \tilde{b}^i + \tilde{a}_i) \right], \quad (42b)$$
which is an authentic precession equation for the evolution of the gyroscope. We shall illustrate this situation in the next Section with Schwarzschild and the Kerr metrics.

4. GYROSCOPE ORBITING IN SCHWARZSCHILD AND KERR SPACE–TIME

A) Schwarzschild space-time

Let now use Schwarzschild space-time as an example to calculate the precession of a gyroscope moving in an equatorial circular orbit. In standard coordinates, this metric is written:

\[
\begin{align*}
ds^2 &= - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \\
&= (43)
\end{align*}
\]

On one hand, we have the standard reference time-like congruence:

\[
\mathcal{C} : \begin{cases} 
t = p \\
r = z^1 , \quad \theta = z^2 , \quad \phi = z^3,
\end{cases}
\]

which has the following quantities associated with it:

\[
\begin{align*}
\xi^\lambda &= \frac{\partial x^\alpha}{\partial p} = (1, 0, 0, 0) \\
Q^\lambda_i &= \frac{\partial x^\alpha}{\partial z^i} = \delta^\alpha_i , \\
\xi^2 &= 1 - \frac{2m}{r} , \quad \varphi_i = 0 ,
\end{align*}
\]

and

\[
\begin{align*}
\hat{g}_{ij} &= \text{diag} \left(1 - \frac{2m}{r}, r^2, r^2 \sin^2 \theta \right) \\
\hat{\Omega}_{ij} &= 0 , \quad \hat{\Sigma}_{ij} = 0 , \quad b_i = \frac{m/r^2}{1 - \frac{2m}{r}} (1, 0, 0) .
\end{align*}
\]

On the other hand, we consider a circular orbit in the equatorial plane with constant angular velocity:

\[
\begin{cases} 
t = p(\tau) \\
r = R , \quad \theta = \frac{\pi}{2} \\
\varphi = \omega_{\pm} t , \quad (\omega_{\pm} = \pm \omega , \omega = \text{Cte} > 0),
\end{cases}
\]

(48)
where ± denotes the sense of rotation (direct or retrograde) of the orbit and \( p(\tau) \) must be chosen such that \( w^\alpha w_\alpha = -1 \); i.e.,

\[
w^\alpha = \frac{dx^\alpha}{d\tau} = \frac{1}{X}(1, 0, 0, \omega), \quad \text{(49)}
\]

where

\[
X \equiv \left( \frac{dp}{d\tau} \right)^{-1} = \left( 1 - \frac{2m}{R} - \omega^2 R^2 \right)^{1/2}. \quad \text{(50)}
\]

Thus, the acceleration of the gyroscope along the orbit is:

\[
a^\alpha = \nabla w^\alpha = \frac{\xi^2}{X^2} \frac{X}{R^2 - \omega^2 R} (0, 1, 0, 0). \quad \text{(51)}
\]

Since (42b) is a tensorial expression with respect to changes of connectors triads of the type shown in (40), and since we are only interested in evaluating the precession angle of the gyroscope, we can carry out the calculation by using the connectors associated with the standard parametrization (44). Thus, we first have:

\[
A_{ik} = \frac{1}{1 + \gamma} \left[ \dot{w}_i (\gamma \dot{a}_k + \dot{a}_k) - \ddot{w}_k (\gamma \dot{b}_i + \ddot{a}_i) \right], \quad \text{(52)}
\]

where

\[
\gamma = -g_{\alpha\beta} w^\alpha w^\beta = \frac{\xi}{X}, \quad \text{(53)}
\]

Therefore, from (47), (49) and (51) we have:

\[
\begin{align*}
A_{12} &= A_{23} = 0 \\
A_{31} &= \pm \frac{\omega R}{\xi X^2} \left[ 1 - \frac{3m}{R} - \frac{\xi X}{R^2} \right].
\end{align*} \quad \text{(54)}
\]

The precession angle and the sense of rotation can be obtained from the following dual vector (see Appendix):

\[
\Omega^i = -\frac{1}{2 \sqrt{g}} e^{ijk} A_{jk} = -\frac{1}{\sqrt{g}} A_{31} \delta^i_2, \quad \text{(55)}
\]

which turns out to be:

\[
\Omega^i = \pm \frac{\omega}{RX^2} \left[ 1 - \frac{3m}{R} - \frac{\xi X}{R^2} \right] (0, 1, 0). \quad \text{(56)}
\]

• By taking in (56) the orbit to be geodesic \( (\omega^2 = m/R^3) \), we have:

\[
\Omega^\theta = \pm \frac{\sqrt{m/R}}{R^2 X_g} \left[ \sqrt{1 - \frac{2m}{R}} - \sqrt{1 - \frac{3m}{R}} \right] \begin{cases} < 0, \quad \text{if} \quad \text{the orbit is geodesic} \\ > 0, \quad \text{otherwise} \end{cases}, \quad \text{(57)}
\]
where $X_g$ stands for the value of $X$ when the orbit is geodesic ($X_g \equiv \sqrt{1 - 3m/R}$). Hence, the precession is direct or retrograde for direct or retrograde orbits respectively, and the rotated angle after one period of proper time is:

$$\Delta \alpha = \pm \Omega \frac{2\pi X_g}{\omega} = \pm 2\pi \left[ \sqrt{1 - \frac{2m}{R}} - \sqrt{1 - \frac{3m}{R}} \right] , \quad (58)$$

where the corresponding sign is considered for the respective sense of the orbit.

- On other hand, by taking $m = 0$ in expression (56) we can recover the result for Minkowski space-time. As is known, for the case of a direct orbit the precession is retrograde and vice-versa, which is clear from the component:

$$\Omega^\theta = \frac{\pm \omega}{R(1 - \omega^2 R^2)} \left[ 1 - \sqrt{1 - \omega^2 R^2} \right] \begin{cases} > 0 \quad \{ \geq 0 \\ < 0 \end{cases} , \quad (59)$$

The rotated angle is:

$$\Delta \alpha \equiv \mp \Omega \frac{2\pi X_m}{\omega} = \mp 2\pi \left[ (1 - \omega^2 R^2)^{-1/2} - 1 \right] , \quad (60)$$

$X_m$ being the value of $X$ (50) for $m = 0$, and the signs $\mp$ standing for direct or retrograde orbits, respectively.

As can be checked, results (58) and (60) do not correspond to the expressions obtained by Rindler and Perlick [2]. The expressions obtained by these authors come simply from adding an angle of $2\pi$ to the result obtained if the congruence defined by all the circular orbits centered in the symmetry axes and “orthogonal” to it is considered (in this way the matrix $B^i_k$ of (38) is reduced to $\hat{\Omega}^i_k$ and $\hat{\Gamma}^i_{kj} = 0$).

This procedure is quite reasonable for simple examples, but it does not seem to be supported in the general case and, moreover, it only gives the rotated angle but allows no conclusions to be drawn about the influence of fields associated with some suitable reference congruence. In any case, the question is as follows: what criterion is used to say that the precession angle is any given angle? A possible answer for this question comes from the use of a triad of connectors transported by parallelism in the sense of $\hat{g}_{ij}$, as we have shown, although obviously there is no unique answer and some other similarly reasonable criterion may be used.

With respect to the difference between results (58), (60) and those obtained by Rindler and Perlick, it should be noted that this is due to the fact that the connectors transported by parallelism along a closed curve in a Riemannian space rotate at a certain angle (holonomy), which is zero in a flat Minkowsky space-time. Therefore, we rely on a procedure suitable for calculating the “total” angle that the gyroscope rotates by adding the angle rotated by the parallel transported connectors to expressions (58) and (60). To evaluate the angle rotated, after one revolution, by the triad transported by parallelism, we avoid possible ambiguity by forcing the angle to be zero at the Minkowskian limit.
To this end, let us consider the parallel transport for the connectors in the quotient space of Schwarzschild space-time. The final equation is:

\[
\frac{dq^i}{d\tau} = P^i_j q^j ,
\]

with

\[
P^i_j : P^i_j \equiv -\frac{1}{2} \sqrt{g} \epsilon^{ijk} \hat{g}_{jl} P^l_k = \pm \frac{\omega}{R} \sqrt{\frac{1 - 2m}{R}} (0, 1, 0) .
\]

By using the techniques shown in the Appendix, it is trivial to conclude that the connector rotates, after a period \( T_p \), by an angle:

\[
|\Delta \beta| \equiv \frac{2\pi}{T_p} (P^i P_i)^{1/2} = \left| 2\pi \sqrt{1 - \frac{2m}{R}} + 2k\pi \right|.
\]

Since this angle must be zero at the Minkowskian limit, we have that the corresponding angles for the respective direct and retrograde orbit cases are:

\[
\Delta \beta = \pm 2\pi \left[ 1 - \sqrt{1 - \frac{2m}{R}} \right] .
\]

Henceforth, the precession for the geodesic case can be corrected as follows:

\[
\Delta \alpha + \Delta \beta = \pm 2\pi \left( 1 - \sqrt{1 - \frac{3m}{R}} \right) ,
\]

which is consistent with the result obtained by Rindler and Perlick.

**B) Kerr space-time**

Finally, we should like to complete this analysis by showing the results obtained for Kerr space-time.

We first calculate the precession angle of the gyroscope by using the connectors associated with the standard parametrization (44) of the reference time-like congruence, as well as the circular orbit on the equatorial plane (48), \( \{t, r, \theta, \varphi\} \) being Boyer-Lindquist coordinates. In this case, if the orbit is a geodesic, the constant angular velocity turns out to be:

\[
\omega_{\pm} = \frac{\omega_s}{a\omega_s \pm 1} ,
\]

with \( \omega_s \equiv +\sqrt{m/R^3} \) and where the respective sign shows that the orbit is direct or retrograde.

It is straightforward to calculate that (55) becomes:

\[
\Omega^\theta = \mp \frac{\omega_s^2}{R\xi^3 X_k} C ,
\]
where
\begin{align}
C &\equiv -a(\omega_s \pm \xi^2) \pm \frac{\omega_s \xi^2 R^2 (\xi^2 + a^2 / R^2)}{a \omega_s \pm \xi^2 \pm \xi X_k} > 0 \\
\xi &\equiv \sqrt{1 - 2m / R} \\
X_k &\equiv \sqrt{1 - 3m / R \pm 2a \omega_s} .
\end{align}

The angle rotated after one revolution (with its corresponding sign for the respective direct or retrograde orbit) turns out to be:
\[ \Delta \alpha = \pm 2\pi \frac{\omega_s}{\xi^3} \left[ -a(\omega_s \pm \xi^2) \pm \frac{R^2 \omega_s \xi^2 (\xi^2 + a^2 / R^2)}{a \omega_s \pm \xi^2 \pm \xi X_k} \right] . \]

It can be readily checked that the reduction \( a = 0 \) provides the results obtained for the Schwarzschild case. As we did for the Schwarzschild case, the evaluation of the parallel transport for the connectors in the quotient metric of Kerr space-time leads to an equation like (61), with:
\[ P^i = \pm \frac{\omega_s (\xi^4 - \omega_s^2 a^2)}{X_k \xi^3 R} (0, 1, 0) . \]

And the angle rotated by the connectors (with the good Minkowskian limit) is:
\[ \Delta \beta = \pm 2\pi \left[ 1 - \frac{(\xi^4 - \omega_s^2 a^2)}{\xi^3} \right] . \]

Since we have that \( \omega_s C - (\xi^4 - \omega_s^2 a^2) = -X_k \xi^3 \), we thus have that the total precession angle is:
\[ \Delta \alpha + \Delta \beta = \mp 2\pi \left[ \sqrt{1 - 3m / R \pm 2a \omega_s} - 1 \right] . \]

APPENDIX

In this Appendix we wish to show the fundamental aspects of a differential precession equation on a three-dimensional Riemannian manifold \( (\mathcal{V}_3, g_{ij}) \); that is, a differential equation of the following type:
\[ \frac{dN^i}{d\tau} = A_k^i(\tau) N^k , \quad A_{jk} \equiv g_{ij} A_k^i = -A_{kj} . \]

The general solution of this kind of equation is:
\[ N^i(\tau) = \Phi_k^i(\tau) N_{(0)}^k , \]
where

\[
\begin{cases}
\dot{\Phi}_j^i = A_{jk}^i(\tau) \Phi_j^k \\
\Phi_j^i(0) = \delta_j^i.
\end{cases}
\]  \hspace{1cm} (A3)

Although it is already quite well known, we show that \((A1)\) is an authentic equation of precession. Indeed, by introducing the dual vector (up to a sign) of \(A_{jk}\) in the sense of \(g_{ij}\)

\[
\Omega^i \equiv -\frac{1}{2} \eta^{ijk} A_{jk}, \quad \eta^{ijk} = \frac{1}{\sqrt{g}} \epsilon^{ijk},
\]  \hspace{1cm} (A4)

we see that:

\[
A_k^i N^k = \eta^{ijk} \Omega_j N_k \equiv + (\vec{\Omega} \wedge \vec{N})^i,
\]  \hspace{1cm} (A5)

where the vectorial product is also understood in the sense of \(g_{ij}\).

Furthermore, the matrix \(A_k^i\) has the following and well-known interesting property (because of the antisymmetric character of \(A_{jk}\)):

\[
A_j^i A_k^j A_l^k = -\Omega^2 A_l^i, \quad (A^3 = -\Omega^2 A),
\]  \hspace{1cm} (A6)

where

\[
\Omega^2 \equiv \Omega^i \Omega_i = -\frac{1}{2} A_j^i A_l^j \equiv -\frac{1}{2} \text{tr} A^2.
\]  \hspace{1cm} (A7)

Let us now assume that all components of \(A_{ij}\) are \textit{constants}. Then, the general solution for \((A3)\) is:

\[
\Phi(\tau) = e^{A \tau}.
\]  \hspace{1cm} (A8)

By virtue of \((A6)\), we have:

\[
\Phi(\tau) = I + \frac{\sin \Omega \tau}{\Omega} A + \frac{1 - \cos \Omega \tau}{\Omega^2} A^2,
\]  \hspace{1cm} (A9)

and the solution of \((A1)\) can therefore be written as:

\[
\vec{N}(\tau) = \vec{N}_0 + \frac{\sin \Omega \tau}{\Omega} \vec{\Omega} \wedge \vec{N}_0
\]

\[
+ \frac{1 - \cos \Omega \tau}{\Omega^2} \vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{N}_0),
\]  \hspace{1cm} (A10)

which can be simplified to give:

\[
\vec{N}(\tau) = (\vec{n} \cdot \vec{N}_0) \vec{n} + \sin \Omega \tau (\vec{n} \wedge \vec{N}_0)
\]

\[
- \cos \Omega \tau \left[ (\vec{n} \cdot \vec{N}_0) \vec{n} - \vec{N}_0 \right],
\]  \hspace{1cm} (A11)

with

\[
\vec{n} \equiv \frac{\vec{\Omega}}{\Omega}, \quad \vec{n} \cdot \vec{N}_0 \equiv g_{ik} n^i N_0^k.
\]  \hspace{1cm} (A12)
This expression shows that the angle between $\vec{\Omega}$ and $\vec{N}$ remains unchanged:

$$\vec{n} \cdot \vec{N}(\tau) = \vec{n} \cdot \vec{N}_0 , \quad \forall \tau$$  \hspace{1cm} (A13)

So, if we take for instance $\vec{n} \cdot \vec{N}_0 = 0$, this leads to the final expression:

$$\vec{N}(\tau) = N_0 (\cos \Omega \tau \vec{n}_1 + \sin \Omega \tau \vec{n}_2) , \quad \forall \tau$$  \hspace{1cm} (A14)

with $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ a triad of orthonormal vectors with the following notation:

$$\begin{cases}
\vec{n}_1 \equiv \frac{\vec{N}_0}{N_0} \\
\vec{n}_2 \equiv \vec{n}_3 \wedge \vec{n}_1 \\
\vec{n}_3 \equiv \vec{n}.
\end{cases} \hspace{1cm} (A15)$$

Finally, it is clear from above, (A14), that the precession angle after a value of the parameter $\tau$ is equal to:

$$\Delta \alpha = \Omega \tau , \hspace{1cm} (A16)$$

which is a positive defined quantity because $\Omega$ is the norm of the dual vector $\Omega^i$. Nevertheless, in practice it is useful to assign a sign (positive or negative) to this angle, as done in this article, according to whether the rotation defined by the vector $\vec{\Omega}$ (from $\vec{n}_1$ to $\vec{n}_2$) is direct or retrograde for each particular context.

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