In Limbo: Three Triangle Centers

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Abstract. Yet more candidates are proposed for inclusion in the Encyclopedia of Triangle Centers. Our focus is entirely on simple calculations.

The best-known triangle centers:

- incenter (intersection of three angle bisectors)
- centroid (intersection of three medians)
- circumcenter (intersection of three perpendicular bisectors)
- orthocenter (intersection of three altitudes)

are the first four listed in Kimberling’s famous database [1]. Thousands more appear. A recent addition is the electrostatic center [2].

Rigorous definition of a triangle center involves a real function $f$ defined on the set of all possible triples $(a, b, c)$ of triangle sides and satisfying certain properties. We forego such requirements (hence the phrase “in limbo”) and informally propose three more triangle centers:

- equiareal disk center (associated with Fraenkel asymmetry [3, 4])
- illuminating center (also called a Shibata streetlight [5, 6])
- thermodynamic center (also called a “hot spot” [7, 8])

in the hope that someone else can pick up where we leave off. The latter notion, like the electrostatic center, has its origins in physics.

Consider three triangles $T_1, T_2, T_3$ with vertices

- $\{0, 0\}, \{1, 0\}, \{0, 1\}$, isosceles right triangle;
- $\{0, 0\}, \{1, 0\}, \{0, \sqrt{3}\}$, $30^\circ$-$60^\circ$-$90^\circ$ triangle;
- $\{0, 0\}, \{6, 0\}, \left\{-\frac{13}{3}, \frac{4\sqrt{35}}{3}\right\}$, 6-9-13 triangle.

Our humble contribution is the calculation of triangle centers for these cases. We make no claim of originality or special insight. If our paper starts a conversation, leading perhaps to future inclusion of the three centers in [1], then our efforts will be justified.

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1. **Equiareal Disk Center**

Given a triangle $T$, let $|T|$ denote its area. The Fraenkel asymmetry of $T$ is defined to be

$$\alpha(T) = \inf \left\{ \frac{|(T \setminus D) \cup (D \setminus T)|}{|T|} : D \text{ a disk with } |D| = |T| \right\}.$$

The numerator contains the symmetric difference of $T$ and $D$. One could imagine a similar definition involving disks having the same perimeter as $T$, rather than area, but we leave this variation for other people to explore. The infimum $\alpha(T)$ is achieved for some disk centered at a unique interior point of $T$. Locating this point is a challenging exercise in calculus. We illustrate the necessary partitioning of $T_1, T_2, T_3$ in Figures 1, 2, 3 respectively. The color red is used for vertical rectangles of width $dx$; green is used for horizontal rectangles of width $dy$; black is used to further subdivide certain cells of the partition. Details of other feasible configurations of the triangle and disk are omitted for brevity’s sake.

1.1. **Isosceles Right Triangle.** For $T_1$, the center must be on the diagonal line $y = x$ by symmetry. Since $|T_1| = 1/2$, the optimal circle $C$ has the form

$$(x - t)^2 + (y - t)^2 = \frac{1}{2\pi}.$$ 

The intersection of the line $x + y = 1$ and $C$ yields two points $(p_1, q_1), (p_2, q_2)$ with $p_1 < p_2, q_1 > q_2$. The intersection of the line $x = 0$ and $C$ yields two points $(p_3, q_3), (p_4, q_4)$ with $p_3 = p_4 = 0, q_3 > q_4$. The intersection of the line $y = 0$ and $C$ yields two points $(p_5, q_5), (p_6, q_6)$ with $p_5 < p_6, q_5 = q_6 = 0$. The intersection of the line $y = x$ and $C$ yields two points; we select $(p_7, q_7)$ to be the point with the larger $x$-coordinate. Everything can easily be made explicit, for example,

$$p_1 = \frac{1}{2} \left( 1 - \sqrt{\frac{1}{\pi} - (1 - 2t)^2} \right), \quad p_5 = t - \sqrt{\frac{1}{2\pi} - t^2}, \quad p_7 = \frac{1}{2\sqrt{\pi}} + t.$$ 

The northwestern triangle has area

$$\int_0^{p_1} \left[ (1 - x) - \left\{ t + \sqrt{\frac{1}{2\pi} - (t - x)^2} \right\} \right] dx$$ 

identical to the southeastern triangle. The southern circular cap has area

$$2\int_{p_5}^{t} \left[ 0 - \left\{ t - \sqrt{\frac{1}{2\pi} - (t - x)^2} \right\} \right] dx.$$
Figure 1: Isosceles right triangle $T_1$
Figure 2: $30^\circ$-$60^\circ$-$90^\circ$ triangle $T_2$
Figure 3: 6-9-13 triangle $T_3$
identical to the western circular cap. The southwestern triangle has area
\[
\int_0^{p_5} \left[ \left\{ t - \sqrt{\frac{1}{2\pi} - (t - x)^2} \right\} - 0 \right] dx
\]
and the northeastern circular cap has area
\[
2 \int_{p_1}^{1/2} \left[ \left\{ t + \sqrt{\frac{1}{2\pi} - (t - x)^2} \right\} - (1 - x) \right] dx + 2 \int_{1/2}^{p_7} \left[ \left\{ t + \sqrt{\frac{1}{2\pi} - (t - x)^2} \right\} - x \right] dx.
\]
Adding these areas and differentiating with respect to \( t \), we find that the best \( t \) is
\[
t = \frac{1}{4} \left( 1 + \frac{1}{\pi} \right) = 0.32957747154594766768844418...
\]
corresponding to an asymmetry \( \alpha(T_1) \approx 0.450 \).

1.2. 30°-60°-90° Triangle. Since \( |T_2| = \sqrt{3}/2 \), the optimal circle \( C \) has the form
\[
(x - s)^2 + (y - t)^2 = \frac{\sqrt{3}}{2\pi}.
\]
The intersection of the line \( \sqrt{3}x + y = \sqrt{3} \) and \( C \) yields two points \((p_1, q_1), (p_2, q_2)\) with \( p_1 < p_2, q_1 > q_2 \). The intersection of the line \( x = 0 \) and \( C \) yields two points \((p_3, q_3), (p_4, q_4)\) with \( p_3 = p_4 = 0, q_3 > q_4 \). The intersection of the line \( y = 0 \) and \( C \) yields two points \((p_5, q_5), (p_6, q_6)\) with \( p_5 < p_6, q_5 = q_6 = 0 \). Everything can be made explicit, for example,
\[
p_1, p_2 = \frac{1}{4} \left( 3 + s - \sqrt{3}t \mp \sqrt{-3 + \frac{2\sqrt{3}}{\pi} + 6s - 3s^2 + 2\sqrt{3t - 2\sqrt{3}st - t^2}} \right),
\]
\[
q_1, q_2 = \frac{1}{4} \left( \sqrt{3} - \sqrt{3}s + 3t \pm \sqrt{-9 + \frac{6\sqrt{3}}{\pi} + 18s - 9s^2 + 6\sqrt{3t - 6\sqrt{3}st - 3t^2}} \right),
\]
\[
p_5 = s - \sqrt{\frac{3}{2\pi} - t^2}, \quad q_4 = t - \sqrt{\frac{3}{2\pi} - s^2}.
\]
The northwestern triangle has area
\[
\int_0^{p_1} \left[ \sqrt{3}(1 - x) - \left\{ t + \sqrt{\frac{\sqrt{3}}{2\pi} - (s - x)^2} \right\} \right] dx
\]
and the southeastern triangle has area

$$\int_0^{q_2} \left[ \frac{1}{\sqrt{3}} (\sqrt{3} - y) - \left\{ s + \sqrt{\frac{3}{2\pi}} - (t - y)^2 \right\} \right] \, dy.$$ 

The southern circular cap has area

$$2 \int_{p_5}^{s} \left[ 0 - \left\{ t - \sqrt{\frac{3}{2\pi}} - (s - x)^2 \right\} \right] \, dx$$

and the western circular cap has area

$$2 \int_{q_4}^{t} \left[ 0 - \left\{ s - \sqrt{\frac{3}{2\pi}} - (t - y)^2 \right\} \right] \, dy.$$ 

The southwestern triangle has area

$$\int_{p_5}^{p_5} \left[ \left\{ t - \sqrt{\frac{3}{2\pi}} - (s - x)^2 \right\} - 0 \right] \, dx$$

and the northeastern circular cap has area

$$\int_{q_2}^{q_2} \left[ \left\{ s + \sqrt{\frac{3}{2\pi}} - (t - y)^2 \right\} - \frac{1}{\sqrt{3}} \left( \sqrt{3} - y \right) \right] \, dy.$$ 

Adding these areas and differentiating with respect to both $s$ and $t$, we find that the best $(s, t)$ is

$$s = 0.3719164279770188862100673...,$$

$$t = 0.4794617554511785131491672...$$

corresponding to an asymmetry $\alpha(T_2) \approx 0.517$.

**1.3. 6-9-13 Triangle.** Since $|T_3| = 4\sqrt{35}$, the optimal circle $C$ has the form

$$(x - s)^2 + (y - t)^2 = \frac{4\sqrt{35}}{\pi}.$$ 

The intersection of the line $4\sqrt{35}x + 31y = 24\sqrt{35}$ and $C$ yields two points $(p_1, q_1)$, $(p_2, q_2)$ with $p_1 < p_2$, $q_1 > q_2$. The intersection of the line $4\sqrt{35}x + 13y = 0$ and $C$
yields two points; we select \((p_3, q_3)\) to be the point with the larger \(y\)-coordinate. The intersection of the line \(y = 0\) and \(C\) yields two points \((p_4, q_4), (p_5, q_5)\) with \(p_4 < p_5, q_4 = q_5 = 0\). Everything can be made explicit, for example,

\[
p_1, p_2 = \frac{\kappa \mp 31\sqrt{\frac{\pi \xi}{1521\pi}}}{1521}, \quad q_1, q_2 = \frac{4\left(\lambda \pm \sqrt{\frac{35\xi}{s}}\right)}{1521}
\]

where

\[
\xi = 6084\sqrt{35} - 20160\pi + 6720\pi s - 560\pi s^2 + 1488\sqrt{35\pi} t - 248\sqrt{35\pi} st - 961\pi t^2,
\]

\[
\kappa = 3360\pi + 961\pi s - 124\sqrt{35\pi} t, \quad \lambda = 186\sqrt{35} - 31\sqrt{35}s + 140t;
\]

\[
p_3 = \frac{13}{729\pi}\left(\mu - \sqrt{\frac{\pi \eta}{t}}\right), \quad q_3 = \frac{4}{729}\left(\nu + \sqrt{\frac{35\pi}{\eta}}\right)
\]

where

\[
\eta = 2916\sqrt{35} - 560\pi s^2 - 104\sqrt{35\pi} st - 169\pi t^2,
\]

\[
\mu = 13\pi s - 4\sqrt{35\pi} t, \quad \nu = -13\sqrt{35}s + 140t;
\]

\[
p_4, p_5 = s + \sqrt{\frac{4\sqrt{35}}{\pi} - t^2}.
\]

The northwestern triangle has area

\[
\int_{\frac{-13}{3}}^{p_3} \left[ \left\{ -\frac{4\sqrt{35}}{31}(x - 6) \right\} - \left\{ -\frac{4\sqrt{35}}{13}x \right\} \right] dx + \int_{p_3}^{p_1} \left[ \left\{ -\frac{4\sqrt{35}}{31}(x - 6) \right\} - \left\{ t + \sqrt{\frac{4\sqrt{35}}{\pi} - (s - x)^2} \right\} \right] dx
\]

and the southeastern triangle has area

\[
\int_{p_5}^{p_2} \left[ \left\{ t - \sqrt{\frac{4\sqrt{35}}{\pi} - (s - x)^2} \right\} - 0 \right] dx + \int_{p_2}^{6} \left[ \left\{ -\frac{4\sqrt{35}}{31}(x - 6) \right\} - 0 \right] dx.
\]

The southwestern circular cap has area

\[
2\int_{p_4}^{s} \left[ 0 - \left\{ t - \sqrt{\frac{4\sqrt{35}}{\pi} - (s - x)^2} \right\} \right] dx + \int_{0}^{q_3} \left[ \left\{ -\frac{13}{4\sqrt{35}}y \right\} - \left\{ s - \sqrt{\frac{4\sqrt{35}}{\pi} - (t - y)^2} \right\} \right] dy
\]

and the northeastern circular cap has area

\[
2\int_{p_1}^{s} \left[ \left\{ t + \sqrt{\frac{4\sqrt{35}}{\pi} - (s - x)^2} \right\} - q_1 \right] dx + \int_{q_2}^{q_1} \left[ \left\{ s + \sqrt{\frac{4\sqrt{35}}{\pi} - (t - y)^2} \right\} - \left\{ 6 - \frac{31}{4\sqrt{35}}y \right\} \right] dy.
\]
Adding these areas and differentiating with respect to both $s$ and $t$, we find that the best $(s, t)$ is 
\[ s = 0.9999634051829363409671652..., \]
\[ t = 2.4097948974186280609774486... \]
corresponding to an asymmetry $\alpha(T_3) \approx 0.694$.

The fairly arbitrary triangle $T_3$, in particular, serves as a benchmark in [1] to distinguish various centers. Our preceding value $t$ is the perpendicular distance from the equiareal disk center to the shortest triangle side. Since the numerical value 2.409... does not appear in the database, we infer that this center is new.

2. Illuminating Center

Let $\Lambda$ denote a light source in three-dimensional space of luminosity $L$. The amount of light an observer $\Theta$ receives from $\Lambda$ is called its brightness, measured in lumens per unit area. Brightness is inversely proportional to the square of distance between $\Lambda$ and $\Theta$. (Reason: brightness is constant on the sphere of radius $R$, center $\Lambda$ and thus is equal to $L/(4\pi R^2)$). Although we will focus solely on light sources in the plane (streetlights in a triangular park), the preceding spatial definition of brightness is the basis of our model.

Given a triangle $T$ (more precisely, its interior), the total brightness

\[
\iint_T \frac{1}{(x-s)^2 + (y-t)^2} \, dx \, dy
\]

is the quantity we would wish to maximize with respect to $(s, t) \in T$. It turns out that this integral is divergent and

\[
\iint_{T \setminus D_\varepsilon} \frac{1}{(x-s)^2 + (y-t)^2} \, dx \, dy
\]

must be examined instead, where $D_\varepsilon$ is the disk of radius $\varepsilon > 0$, center $(s, t)$. Further, $(s, t)$ cannot be close to $\partial T$ (the boundary of $T$), that is, we must restrict $(s, t) \in T \setminus (T \cap N_\varepsilon)$ where $N_\varepsilon$ is the $\varepsilon$-tubular neighborhood of $\partial T$. Under such conditions, in the limit as $\varepsilon \to 0^+$, a geometric characterization of the maximum point $P = (s, t)$ becomes available.

Let $T$ possess vertices $A, B, C$. Select any two distinct vertices $U, V$ and note that the semiperimeter of subtriangle $UPV$ is

\[
\sigma = \frac{\sqrt{(U-P) \cdot (U-P)} + \sqrt{(V-P) \cdot (V-P)} + \sqrt{(U-V) \cdot (U-V)}}{2}.
\]
Using Heron’s formula, it follows that the ratio of inner angle to area:

\[ \rho(UPV) = \frac{\angle UPV}{|UPV|} \]

\[ = \frac{\arccos \left( \frac{(U-P) \cdot (V-P)}{\sqrt{(U-P) \cdot (U-P)} \sqrt{(V-P) \cdot (V-P)}} \right)}{\sqrt{\left( \sigma - \sqrt{(U-P) \cdot (U-P)} \right) \left( \sigma - \sqrt{(V-P) \cdot (V-P)} \right) \left( \sigma - \sqrt{(U-V) \cdot (U-V)} \right)}} \]

satisfies \( \rho(APB) = \rho(BPC) = \rho(CPA) \), a remarkable fact \( [5] \)!

For \( T_1 \), we solve the resulting equations, obtaining

\[ s = t = 0.3082756986146550422567206...; \]

for \( T_2 \),

\[ s = 0.35168768876632055410277..., \]
\[ t = 0.4491286165669552235961426...; \]

and, for \( T_3 \),

\[ s = 0.8345011650594754190821304..., \]
\[ t = 2.0031487728161056257679347.... \]

Our preceding value \( t \) is the perpendicular distance from the illuminating center to the shortest triangle side. Since the numerical value 2.003... does not appear in the ETC database, we infer that this center is new.

Let us revisit the definition of brightness. Had a planar definition been adopted – measured in lumens per unit length – then brightness would be inversely proportional to the distance itself between \( \Lambda \) and \( \Theta \). (Reason: brightness would be constant on the circle of radius \( R \), center \( \Lambda \) and thus would be equal to \( L / (2\pi R) \)). This scenario yields exactly the same formulation as that underlying the electrostatic center \( [2] \).

3. Thermodynamic Center

Here, the triangle \( T \) (more precisely, its interior) is assumed to be a heat conductor with initial temperature 1 while its boundary \( \partial T \) is held at temperature 0 always. Heat \( u \) will dissipate as time \( t \) increases according to the following initial-boundary value problem:

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{for} \ (x, y, t) \in T \times (0, \infty), \]
\[ u(x, y, t) = 1 \quad \text{for} \ (x, y, t) \in T \times \{0\}, \]
\[ u(x, y, t) = 0 \quad \text{for} \ (x, y, t) \in \partial T \times (0, \infty) \]
but does so non-uniformly: its density gathers around a unique maximum point $(x_\infty, y_\infty)$ as $t \to \infty$ [7, 8]. The point $(x_\infty, y_\infty)$ is, in fact, the unique extreme point of the first Laplacian eigenfunction for $T$. The eigenfunction for $T_1$ is [9, 10, 11, 12, 13, 14]

$$\sin(\pi x) \sin(2\pi y) + \sin(2\pi x) \sin(\pi y)$$

with

$$x_\infty = y_\infty = \frac{1}{\pi} \arccsc \left( \sqrt{3} \right) = 0.3040867239846963649145722...$$

and the eigenfunction for $T_2$ is

$$\sin \left( \frac{\pi x}{3} \right) \sin \left( \sqrt{3} \pi y \right) + \sin \left( \frac{4\pi x}{3} \right) \sin \left( \frac{2\pi y}{\sqrt{3}} \right) + \sin \left( \frac{5\pi x}{3} \right) \sin \left( \frac{\pi y}{\sqrt{3}} \right)$$

with

$$x_\infty = 0.3558473606263811208579681..., \quad y_\infty = 0.4255359610370576630888604...$$

It remains to compute the eigenfunction for $T_3$, but no closed-form expression for this exists. A numerical computation using the Matlab pdeig tool yields $(x_\infty, y_\infty) \approx (0.88, 1.91)$. See Figure 4. Significantly higher precision will be needed to ascertain whether $y_\infty$ appears in the ETC database; we are hopeful that techniques in [15] might save the day.
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