EXISTENCE OF SOLUTIONS TO CHEMOTAXIS DYNAMICS WITH LOGISTIC SOURCE

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ABSTRACT. This paper is concerned with a chemotaxis system with nonlinear diffusion and logistic growth term \( f(b) = \kappa b - \mu |b|^{\alpha-1}b \) with \( \kappa > 0, \mu > 0 \) and \( \alpha > 1 \) under the no-flux boundary condition. It is shown that there exists a local solution to this system for any \( L^2 \)-initial data and that under a stronger assumption on the chemotactic sensitivity there exists a global solution for any \( L^2 \)-initial data. The proof is based on the method built by Marinoschi [8].

1. Introduction and main results.

1.1. Introduction. This paper is a continuation of our previous work in [11] for a chemotaxis model with Lipschitz growth \( f(b, c) \) or superlinear growth \( |b|^{\alpha-1}b \) with \( \alpha \leq 4 \) \((n = 1)\), \( \alpha < 1 + \frac{4}{n} \) \((n = 2, 3)\). We shall study the case of logistic growth; more precisely, we consider existence of solutions to the chemotaxis model

\[
\begin{aligned}
\frac{\partial b}{\partial t} - \Delta D(b) + \nabla \cdot (K(b, c)b\nabla c) &= f(b, c) \quad \text{in } (0, \infty) \times \Omega, \\
-\Delta c &= b - c \quad \text{in } (0, \infty) \times \Omega, \\
(-\nabla D(b) + K(b, c)b\nabla c) \cdot \nu &= 0, \quad \frac{\partial c}{\partial \nu} = 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
b(0, x) &= b_0(x), \quad x \in \Omega,
\end{aligned}
\]

(KS)

with logistic source

\[
f(b, c) = \kappa b - \mu |b|^{\alpha-1}b, \quad \kappa > 0, \quad \mu > 0, \quad \alpha > 1,
\]

where \( b_0 \in L^2(\Omega) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \leq 3 \), \( \partial \Omega \) is \( C^2 \)-class, \( D \) and \( K \) satisfy some conditions which will be given in Section 1.2. We denote by \( \nu \) and \( \frac{\partial c}{\partial \nu} \) the unit outward normal vector and the normal derivative of \( c \) to \( \partial \Omega \), respectively. We emphasize that there is no restriction on an upper bound of \( \alpha \). It is also possible to consider the Dirichlet boundary condition for \( b \) instead of the no-flux boundary condition. The system (KS) was introduced by Keller and Segel [6] in 1970. The system describes a part of the life cycle of cellular slime molds with chemotaxis. In more detail, slime molds move towards higher concentrations of a chemical substance when they plunge into hunger. Here \( b(t, x) \) represents the density of the cell population and \( c(t, x) \) shows the concentration of the signal substance at time \( t \) and place \( x \). A number of variations of the original Keller-Segel system with logistic source have been studied. Specially in the case that logistic source is added to the original model, i.e., \( D(b) = b \) and \( K(b, c) \) is constant, global existence has been proved (see, e.g., [1, 9, 10]). However, to our knowledge there are few results via operator theory when \( D \) and \( K \) are general forms in (KS) with logistic source. Employing the theory for nonlinear \( m \)-accretive

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operators (see Barbu [3]). Marinoschi [8] succeeded in showing local existence of solutions to (KS) for sufficiently small initial data in the case that \( f \) is Lipschitz continuous. The result in [8] was improved in our previous paper [11]. More precisely, we proved in [11] that there exists a local solution for any \( L^2 \)-initial data when \( f \) is Lipschitz continuous or superlinear. Ardeleanu and Marinoschi also dealt with the superlinear case in [2]. The purpose of this paper is to establish existence of solutions to (KS) with logistic source by the theory for nonlinear \( m \)-accretive operators.

1.2. Main results. In this paper we make the following assumption on \( D \) and \( K \):

\[
D \in C^1(\mathbb{R}), \quad K \in C(\mathbb{R}^2), \quad D(0) \geq 0, \quad 0 < D_0 \leq D'(r) \leq D_\infty < \infty \quad \text{for all} \quad r \in \mathbb{R},
\]

\[
(r_1, r_2) \mapsto K(r_1, r_2) r_1 \quad \text{is Lipschitz continuous on} \quad \mathbb{R}^2.
\]

To state our results we define the Hilbert spaces \( H, V_1 \) and the Banach space \( V_2 \) as

\[
H := L^2(\Omega), \quad V_1 := H^1(\Omega), \quad V_2 := L^{\alpha+1}(\Omega).
\]

Moreover, we set the Banach space \( V \) as

\[
V := V_1 \cap V_2, \quad \|v\|_V := \max\{\|v\|_{V_1}, \|v\|_{V_2}\}.
\]

Note that the dual space of \( V \) is given by

\[
V' = V_1' + V_2', \quad \|v\|_{V'} = \inf\{\|v_1\|_{V_1'} + \|v_2\|_{V_2'} \mid v = v_1 + v_2, \quad v_i \in V_i\}
\]

and we have the continuous injections

\[
V \hookrightarrow H = H' \hookrightarrow V',
\]

where the first one is compact. Now we define weak solutions to (KS).

**Definition 1.1.** A pair \((b, c)\) is called a weak solution to (KS) on \([0, T]\) \((T > 0)\) if

(a) \( b \in C([0, T]; H) \cap L^2(0, T; V_1) \cap L^{\alpha+1}(0, T; V_2) \cap H^1(0, T; V') \),

(b) \( c \in C([0, T]; H^2(\Omega)) \cap H^1(0, T; V_1) \),

(c) \((b, c)\) satisfies (KS) in the following sense: for any \((\psi_1, \psi_2) \in V \times V_1\),

\[
\left\langle \frac{db}{dt}(t), \psi_1 \right\rangle_{V', V} + \int_\Omega \nabla D(b) \cdot \nabla \psi_1 + \int_\Omega \nabla c \cdot \nabla \psi_2 + \int_\Omega c \psi_2
\]

\[
= \int_\Omega K(b, c) b \nabla c \cdot \nabla \psi_1 + \int_\Omega f(b, c) \psi_1 + \int_\Omega b \psi_2 \quad \text{a.a.} \quad t \in (0, T).
\]

In particular, if \( T > 0 \) can be taken arbitrarily, then \((b, c)\) is called a global weak solution to (KS).

**Remark 1.** The definition of weak solutions is different from the one in our previous paper [11]. However, if there is some restriction on the size of \( \alpha \), the definition coincides with the previous one.

The first main theorem asserts local existence in (KS) and reads as follows.

**Theorem 1.2.** Suppose that \( n \leq 3 \) and \( D, K \) and \( f \) satisfy (2)–(4) and (1). Then for any \( b_0 \in H = L^2(\Omega) \) there exists \( T_1 > 0 \) such that (KS) possesses a weak solution \((b, c)\) on
Moreover, the following estimates hold:
\[ \|b(t)\|_H \leq M_1, \quad t \in [0, T_1], \]  
\[ \|b\|_{L^2(0,T_1;V_1)} \leq M_2, \]  
\[ \|b\|_{L^{n+1}(0,T;V_2)} \leq M_3, \]  
\[ \|\frac{db}{dt}\|_{L^{(n+1)'}(0,T_1;V')} \leq M_4, \]
where \( M_1, M_2, M_3, M_4 \) are positive constants which depend on \( \|b_0\|_H, n, \Omega \).

The second theorem is concerned with global existence in (KS).

**Theorem 1.3.** Assume the hypotheses of Theorem 1.2, and suppose that
\[ |K(r_1, r_2)| \leq M, \quad (r_1, r_2) \in \mathbb{R}^2 \]  
for some \( M > 0 \). Then for any \( b_0 \in H = L^2(\Omega) \), (KS) possesses a global weak solution \((b, c)\). Moreover, for every \( T > 0 \) the following estimates hold:
\[ \|b(t)\|_H \leq M_5, \quad t \in [0, T], \]  
\[ \|b\|_{L^2(0,T;V_1)} \leq M_6, \]  
\[ \|b\|_{L^{n+1}(0,T;V_2)} \leq M_7, \]  
\[ \|\frac{db}{dt}\|_{L^{(n+1)'}(0,T;V')} \leq M_8, \]
where \( M_5, M_6, M_7, M_8 \) are positive constants which depend on \( \|b_0\|_H, n, \Omega, T \).

**Remark 2.** In [8] and [11] there is the assumption on \( D \) as \( D(0) = 0 \). More precisely, we used the following inequality derived by the above assumption:
\[ \int_{\Omega} D(u)u \geq D_0 \|u\|_{L^2(\Omega)}^2. \]
To prove the theorems it suffices to use the inequality
\[ \int_{\Omega} D(u)u \geq D_0 \|u\|_{L^2(\Omega)}^2 - \frac{1}{4} \left( \frac{D(0)}{D_0} \right)^2, \]
which is derived by (3) and \( D(0) \geq 0 \) instead of the above inequality.

This paper is organized as follows. Section 2 gives an abstract formulation for (KS), introduces approximate problems, and provides some basic inequalities. We will prove key lemmas and our main results in Section 3.

2. Approach and preliminaries.

2.1. Problem approach. As introduced in [11] we rewrite (KS) as an abstract Cauchy problem in \( V_1' \). We define the inner product on \( V_1' \) as
\[ (\theta, \overline{\theta})_{V_1'} := \langle \theta, (I + A_\Delta)^{-1}\overline{\theta} \rangle_{V_1', V_1} \quad \text{for} \ \theta, \overline{\theta} \in V_1', \]
where the maximal monotone operator \( A_\Delta : D(A_\Delta) \subset H \to H \) is defined as
\[ A_\Delta := -\Delta \quad \text{with} \quad D(A_\Delta) := \left\{ u \in H^2(\Omega) \mid \frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega \right\}. \]
It is known that by a simple argument
\[ \|\theta\|_{V_1'} = \|(I + A_\Delta)^{-1}\theta\|_{V_1} \quad \text{for} \ \theta \in V_1'. \]
Then we rewrite (KS) as
\[
\begin{cases}
\frac{db}{dt}(t) + Ab(t) = 0 & \text{a.a. } t \in (0, T), \\
b(0) = b_0,
\end{cases}
\] (15)
where \( A : D(A) := \{ b \in V_1 \mid D(b) \in V_1 \} = V_1 \subset V_1' \rightarrow V_1' \) is the nonlinear operator defined as
\[
\langle Ab, \psi \rangle_{V_1', V_1} := \int_{\Omega} \nabla D(b) \cdot \nabla \psi - \int_{\Omega} K(b, c_b)b \nabla c_b \cdot \nabla \psi - \int_{\Omega} f(b, c_b)\psi
\]
for any \( \psi \in V_1 \), where we have denoted
\[
c_b := (I + A_\Delta)^{-1}b.
\]
To show that (15) has a solution we proceed to construct an approximate problem as in [11]. For each \( N > 0 \) we define \( F_N : H \rightarrow H \) as
\[
F_N(b) := \begin{cases}
b & \text{if } \|b\|_H \leq N, \\
N \frac{b}{\|b\|_H} & \text{if } \|b\|_H > N,
\end{cases}
\] (16)
and consider the Yosida approximation \( g_\lambda \) of the nonlinear part: \( g(b) := |b|^{\alpha-1}b \) in the logistic term. More precisely, by defining \( g : D(g) := \{ g \in H \mid g(b) \in H \} \subset H \rightarrow H \) as \( (gb)(x) := g(b(x)) \), \( J_\lambda \) and \( g_\lambda \) are given by
\[
J_\lambda := (I + \lambda g)^{-1}, \quad \lambda > 0, \\
g_\lambda := \frac{1}{\lambda}(I - J_\lambda) = g(J_\lambda), \quad \lambda > 0.
\]
Then we consider the following approximate problem with \( N > 0, \varepsilon > 0, \lambda > 0 \):
\[
\begin{cases}
\frac{db}{dt}(t) + A_{N,\varepsilon,\lambda}b(t) = 0 & \text{a.a. } t \in (0, T), \\
b(0) = b_0,
\end{cases}
\] (17)
where \( A_{N,\varepsilon,\lambda} : D(A_{N,\varepsilon,\lambda}) := D(A) \subset V_1' \rightarrow V_1' \) is defined as
\[
\langle A_{N,\varepsilon,\lambda}b, \psi \rangle_{V_1', V_1} := \int_{\Omega} \nabla D(b) \cdot \nabla \psi - \int_{\Omega} K(b, c_b)b^{N} \nabla (J_\lambda c_b) \cdot \nabla \psi \\
+ \kappa \int_{\Omega} b \, dx - \mu \int_{\Omega} g_\lambda(b)\psi
\] (18)
for any \( \psi \in V_1 \), where \( b^N, c_b, J_\lambda \) are defined as
\[
b^N := F_N(b), \quad c_b := (I + A_\Delta)^{-1}b, \quad J_\lambda := (I + \varepsilon A_\Delta)^{-1}, \quad \varepsilon > 0.
\] (19)
Since the mapping \( b \mapsto b - g_\lambda(b) \) is Lipschitz continuous on \( H \) due to the property of the Yosida approximation, we obtain the following existence lemma as in [11].

Lemma 2.1 (Existence of approximate solutions). Let \( N > 0, \varepsilon > 0, \lambda > 0 \) and \( T > 0 \). Assume the hypothesis of Theorem 1.2. Then (17) has a unique solution \( b_{N,\varepsilon,\lambda} \) such that
\[
b_{N,\varepsilon,\lambda} \in W^{1,\infty}(0, T; V_1') \cap L^{\infty}(0, T; D(A)) \cap C([0, T]; V_1').
\]
2.2. Basic inequalities. We next state some known lemmas to prove the theorems. The following two lemmas are stated in [11].

Lemma 2.2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) (\( n \in \mathbb{N} \)) with \( C^2 \)-boundary. Let \( b \in L^2(\Omega) \) and \( c_0 = (I + A_\Delta)^{-1}b \). Denote by \( J_\Delta^\varepsilon \) the resolvent of \( A_\Delta \) as in (19), i.e., \( J_\Delta^\varepsilon := (I + \varepsilon A_\Delta)^{-1} \) for each \( \varepsilon > 0 \). Then \( c_0 \) and \( J_\Delta^\varepsilon c_0 \) belong to \( H^2(\Omega) \) with the following estimates:

\[
\|c_0\|_{L^2(\Omega)} \leq \|b\|_{L^2(\Omega)},
\|c_0\|_{H^1(\Omega)} \leq \|b\|_{L^2(\Omega)},
\|J_\Delta^\varepsilon c_0\|_{H^2(\Omega)} \leq C_\varepsilon \|b\|_{L^2(\Omega)},
\]

where \( C_\varepsilon \) is a positive constant independent of \( \varepsilon \).

The following inequality will be used in the estimate for approximate solutions.

Lemma 2.3. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) (\( n \leq 3 \)) with \( C^2 \)-boundary. Then there exist \( a \in (0,1) \) and \( C'_\text{GN} > 0 \) such that for all \( u, v \in H^1(\Omega) \),

\[
\|uv\|_{L^2(\Omega)} \leq C'_\text{GN} \|u\|_{L^2(\Omega)}^{a} \|u\|_{H^1(\Omega)}^{1-a} \|v\|_{H^1(\Omega)}.
\]

3. Proof of the main results. We will prove Theorems 1.2 and 1.3. Since the starting points of the proofs are different, we divide this section into two subsections.

3.1. Proof of Theorem 1.2. The following lemma is concerned with time-in-local estimates for approximate solutions and is the main part of this paper.

Lemma 3.1. Let \( T > 0 \) and \( b_0 \in L^2(\Omega) \). Assume the same hypothesis as in Theorem 1.2. Let \( b_{N,\varepsilon,\lambda} \) be a unique solution to (17) on \([0,T]\). Then there exists \( T_1 = T_1(\|b_0\|_{L^2(\Omega)}) \in (0,T) \) such that the following estimates hold:

\[
\|b_{N,\varepsilon,\lambda}(t)\|^2_{H^1} \leq C_1, \quad t \in [0,T_1], \tag{20}
\|b_{N,\varepsilon,\lambda}\|_{L^2(0,T_1;V_1)}^2 \leq C_2, \tag{21}
\|J_\Delta^\varepsilon b_{N,\varepsilon,\lambda}\|_{L^{(\alpha+1)}(0,T_1;L^{\alpha+1}(\Omega))}^{\alpha+1} \leq C_2, \tag{22}
\left| \frac{db_{N,\varepsilon,\lambda}}{dt} \right|_{L^{(\alpha+1)'}(0,T_1;V')} \leq C_3 \tag{23}
\]

where \( C_1, C_2, \) and \( C_3 \) are positive constants which do not depend on \( N, \varepsilon, \lambda \) but depend on \( \|b_0\|_{L^2(\Omega)} \).

Proof. We use the notation \( b \) instead of \( b_{N,\varepsilon,\lambda} \) for simplicity. We first show (20). Testing the first equation in (17) by \( b \) and using the assumption \( D'(r) \geq D_0 \) give

\[
\frac{1}{2} \frac{d}{dt} \|b(t)\|^2_{L^2(\Omega)} + D_0 \|\nabla b(t)\|^2_{L^2(\Omega)} \leq \int_\Omega |K(b^N, c_0^N) b^N| \|\nabla (J_\Delta^\varepsilon c_0^N)\| \|\nabla b\| + \int_\Omega (\mu b^2 - \mu g(b)b)
\]

\[
=: I_1 + I_2. \tag{24}
\]

Denoting by \( K_L \) the Lipschitz constant of \((r_1, r_2) \rightarrow K(r_1, r_2)r_1 \) (see (4)), we observe that

\[
|K(r_1, r_2)r_1| \leq K_L(r_1^2 + r_2^2)^{1/2} \leq K_L(|r_1| + |r_2|), \quad (r_1, r_2) \in \mathbb{R}^2.
\]
We estimate $I_1$ in the same way as in the proof of [11, Proposition 3.2]. For convenience we will present the detail of the proof. It follows from Lemma 2.3 that for some $a \in (0, 1)$,

$$I_1 \leq K_L \int_\Omega \left( |b^N| + |c_b^N| \right) |\nabla (J_{\Delta N} c_b^N)| |\nabla b|
$$

$$\leq K_L \left( \|b^N \nabla (J_{\Delta N} c_b^N)\|_{L^2(\Omega)} + \|c_b^N \nabla (J_{\Delta N} c_b^N)\|_{L^2(\Omega)} \right) \|\nabla b\|_{L^2(\Omega)}
$$

$$\leq K_L C'_{GN} \left( \|b^N\|_{H^1(\Omega)}^a \|b^N\|_{L^2(\Omega)}^{1-a} \|J_{\Delta N} c_b^N\|_{H^2(\Omega)} + \|c_b^N\|_{H^1(\Omega)}^a \|c_b^N\|_{L^2(\Omega)}^{1-a} \|J_{\Delta N} c_b^N\|_{H^2(\Omega)} \right) \|b\|_{H^1(\Omega)}.
$$

Noting by (16) and Lemma 2.2 that

$$\|b^N\|_{H^1(\Omega)} \leq \|b\|_{H^1(\Omega)} \cdot \|b^N\|_{L^2(\Omega)} \leq \|b\|_{L^2(\Omega)}, \quad \|c_b^N\|_{H^1(\Omega)} \leq \|b\|_{H^1(\Omega)}, \quad \|c_b^N\|_{L^2(\Omega)} \leq \|b\|_{L^2(\Omega)}, \quad \|J_{\Delta N} c_b^N\|_{H^2(\Omega)} \leq C_R \|b\|_{L^2(\Omega)},$$

we have

$$I_1 \leq 2K_L C'_{GN} C_R \|b\|_{H^1(\Omega)}^a \|b\|_{L^2(\Omega)}^{1-a} \|b\|_{L^2(\Omega)} \|b\|_{H^1(\Omega)}$$

$$= \left( \frac{D_0}{2} \|b\|_{H^1(\Omega)}^2 \right)^{\frac{a}{1-a}} \left( \frac{2}{D_0} \right)^{\frac{1-a}{2}} 2K_L C'_{GN} C_R \|b\|_{L^2(\Omega)}^{2-a}.$$

Now employing Young’s inequality, we consequently obtain

$$I_1 \leq \frac{D_0}{2} \|b\|_{H^1(\Omega)}^2 + c_1 \|b\|_{L^2(\Omega)}^\beta$$

with $\beta := \frac{2(2-a)}{1-a} > 2$, $c_1 := \left( \frac{2}{D_0} \right)^{\frac{1-a}{2}} \left( 2K_L C'_{GN} C_R \right)^{\frac{1-a}{2}}$. Next we consider the estimate for $I_2$. The definition of the Yosida approximation implies

$$g_\lambda(b)b = g_\lambda(b)(b - J_\lambda^1(b)) + g_\lambda(b)J_\lambda^1(b)$$

$$= \lambda |g_\lambda(b)|^2 + |J_\lambda^1(b)|^{a-1} J_\lambda^1(b) \cdot J_\lambda^1(b)$$

$$\geq |J_\lambda^1(b)|^{a+1},$$

and thus

$$I_2 \leq \kappa \|b\|_{L^2(\Omega)}^2 - \mu \|J_\lambda^1(b)\|_{L^{a+1}(\Omega)}^{a+1}.$$

Combining the estimates for $I_1$ and $I_2$ with (24), we infer that

$$\frac{1}{2} \frac{d}{dt} \|b(t)\|_{L^2(\Omega)}^2 + \frac{D_0}{2} \|b(t)\|_{H^1(\Omega)}^2 + \mu \|J_\lambda^1(b)\|_{L^{a+1}(\Omega)}^{a+1} \leq c_2 \|b\|_{L^2(\Omega)}^\beta + c_2 \|b\|_{L^2(\Omega)}^2,$$

where $c_2 := \max\{c_1, D_0 + \kappa\}$. Applying the Gronwall type inequality (see e.g., [5, Theorem 21]) gives

$$\|b(t)\|_{L^2(\Omega)}^2 \leq \left( -1 + e^{-\frac{2}{\pi^2}c_2 t} \left( 1 + \|b_0\|_{L^2(\Omega)}^{-(\beta-2)} \right) \right)^{-\frac{2}{\pi^2}}, \quad t \in [0, T_0),$$

where $T_0 := \frac{3}{(D_0 + \kappa)(\beta-2)} \log \left( 1 + \|b_0\|_{L^2(\Omega)}^{-(\beta-2)} \right) > 0$. Now fix $T_1 \in (0, T_0)$. Then (20) holds with

$$C_1 := \left( -1 + e^{-\frac{2}{\pi^2}c_2 T_1} \left( 1 + \|b_0\|_{L^2(\Omega)}^{-(\beta-2)} \right) \right)^{-\frac{2}{\pi^2}}.$$
We next prove (21) and (22). Integrating (25) and using (20), we infer that for $t \in [0, T_1]$,
\[
\frac{1}{2} \|b(t)\|_{L^2(\Omega)}^2 + \frac{D_0}{2} \int_0^t \|b(s)\|_{H^1(\Omega)}^2 \, ds + \mu \int_0^t \|J_{\epsilon}^\alpha(b(s))\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} \, ds \\
geq c_2 C_1^2 t + c_2 C_1^2 t + \frac{1}{2} \|b_0\|_{L^2(\Omega)}^2.
\]
Thus we arrive at (21) and (22) with
\[
C_2 := \left( \frac{2}{D_0} + \frac{1}{\mu} \right) \left( c_2 C_1^2 T_1 + c_2 C_1^2 T_1 + \frac{1}{2} \|b_0\|_{L^2(\Omega)}^2 \right).
\]
Finally we estimate $\left\| \frac{db}{dt} \right\|_{L^{(\alpha+1)'}([0, T_1]; V^{'})}$. Now we rewrite $\frac{db}{dt}$ as
\[
\frac{db}{dt} = \Delta D(b) - \nabla \cdot \left( K(b^N, c_{b^N}) b^N \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right) + \kappa b - \mu g_{\lambda}(b).
\]
From (20)–(22), the terms $\Delta D(b)$ and $\kappa b - \mu g_{\lambda}(b)$ are bounded in $V'$ and $V''$, respectively. Moreover, noting that
\[
\left\| K(b^N, c_{b^N}) b^N \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right\|_{L^2(\Omega)} \leq K_L \left\| b^N \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right\|_{L^2(\Omega)} \leq K_L \left\| b^N \right\|_{L^4(\Omega)} \left\| \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right\|_{L^4(\Omega)} \leq K_L C_S^2 \left\| b^N \right\|_{H^1(\Omega)} \left\| \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right\|_{H^1(\Omega)} \leq C_1 K_L C_R C_S^2 \left\| b^N \right\|_{H^1(\Omega)}
\]
due to the Sobolev inequality, Lemma 2.2 and (20), we see by the general theory for Sobolev spaces (see e.g., [4, p. 291]) that
\[
\left\| \nabla \cdot \left( K(b^N, c_{b^N}) b^N \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right) \right\|_{H^1(\Omega)} \leq \left\| K(b^N, c_{b^N}) b^N \nabla \left( J_{\epsilon}^\alpha c_{b^N} \right) \right\|_{L^2(\Omega)} \leq C_1 K_L C_R C_S^2 \left\| b^N \right\|_{H^1(\Omega)},
\]
where $C_S$ is a Sobolev constant. Thus in view of (5) we infer that $\frac{db}{dt}$ is bounded in $L^{(\alpha+1)'}([0, T_1]; V^{'})$ and hence (23) holds with a positive constant $C_3$.

We are now in a position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Estimates (21) and (23) enable us to use the Lions-Aubin theorem (see Lions [7, p. 57]). As a consequence, there exist a subset of $(b_{N,\varepsilon,\lambda})_{\varepsilon,\lambda>0}$ (still denoted by $(b_{N,\varepsilon,\lambda})_{\varepsilon,\lambda>0}$) and a function $b_N \in L^2(0, T_1; H)$ such that
\[
b_{N,\varepsilon,\lambda} \to b_N \quad \text{in} \quad L^2(0, T_1; H) \quad \text{as} \quad \varepsilon, \lambda \to 0.
\]
The $m$-accretivity of $g$ in $L^{(\alpha+1)'}(0, T; V')$ entails that
\[
g_{\Lambda}(b_{N,\varepsilon,\lambda}) \to g(b_N) \quad \text{weakly in} \quad L^{(\alpha+1)'}(0, T_1; V'_{\varepsilon,\lambda}) \quad \text{as} \quad \varepsilon, \lambda \to 0,
\]
and thus by using a similar way to that in [11] we conclude that $b_N$ satisfies the equation
\[
\left\langle \frac{db_N}{dt}, \psi \right\rangle_{V'_{\varepsilon,\lambda}, V_1} = \int_{\Omega} \nabla D(b_N) \cdot \nabla \psi - \int_{\Omega} K(b_N, b_N) \nabla c_{b_N} \cdot \nabla \psi \\
+ \kappa \int_{\Omega} b_N \psi - \mu \int_{\Omega} |b_N|^{\alpha-1} b_N \psi, \quad \psi \in V_1.
\]
Recalling the definition (16) and choosing $N$ large enough, we see from (20) that $F_N(b_N)$ is equal to $b_N$ and hence $b_N$ is the desired solution to (KS) on $[0, T_1]$. Estimates (6)–(9) follow from (20)–(23).
3.2. Proof of Theorem 1.3. First we establish the time-in-global estimates for approximate solutions under the hypothesis of Theorem 1.3.

Lemma 3.2. Let $T > 0$ and $b_0 \in L^2(\Omega)$. Assume the same hypothesis as in Theorem 1.3. Let $b_{N,\varepsilon,\lambda}$ be a unique solution to (17) on $[0, T]$. Then the following estimates hold:

\[
\begin{align*}
\|b_{N,\varepsilon,\lambda}(t)\|_{H^1} &\leq C_4, \quad t \in [0, T], \\
\|b_{N,\varepsilon,\lambda}\|_{L^2(0,T; V_1)} &\leq C_5, \\
\|J_g b_{N,\varepsilon,\lambda}\|_{L^{\alpha+1}(0,T; L^{\alpha+1}(\Omega))} &\leq C_5, \\
\left\|\frac{db_{N,\varepsilon,\lambda}}{dt}\right\|_{L^{(\alpha+1)'(0,T; V')}} &\leq C_6,
\end{align*}
\]

where $C_4$, $C_5$ and $C_6$ are positive constants which do not depend on $N, \varepsilon, \lambda$ but depend on $\|b_0\|_{L^2(\Omega)}$.

Proof. We first show (26). Testing the first equation in (17) by $b$ and using the assumption $D'(r) \geq D_0$, we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|b(t)\|_{L^2(\Omega)}^2 + D_0 \|\nabla b(t)\|_{L^2(\Omega)}^2 &\leq \int_\Omega |K(b^N, c_0^N)| b^N |\nabla (J_\Delta c_0^N)| \|\nabla b\| + \kappa \int_\Omega |b|^2 - \mu \int_\Omega g\lambda(b)b \\
&=: I_3 + I_4.
\end{align*}
\]

It follows from the assumption (10) that

\[
I_3 \leq K_M \int_\Omega |\nabla (J_\Delta c_0^N)| \|\nabla b\| \leq K_M \|\nabla (J_\Delta c_0^N)\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)} \leq K_M \|(J_\Delta c_0^N)\|_{H^1(\Omega)} \|\nabla b\|_{L^2(\Omega)}.
\]

From Lemma 2.2 we have

\[
I_3 \leq C \|b\|_{L^2(\Omega)} \|b\|_{H^1(\Omega)} \leq \frac{1}{2} \frac{C^2}{D_0} \|b\|_{L^2(\Omega)}^2 + \frac{D_0}{2} \|b\|_{H^1(\Omega)}^2.
\]

On the other hand, as in the proof of Theorem 1.2, we have

\[
I_4 \leq \kappa \|b\|_{L^2(\Omega)}^2 - \mu \|J_g^\lambda(b)\|_{L^{\alpha+1}(\Omega)}^{\alpha+1}.
\]

The estimates for $I_3$ and $I_4$ imply

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|b(t)\|_{L^2(\Omega)}^2 + D_0 \|b\|_{H^1(\Omega)}^2 + \mu \|J_g^\lambda(b)\|_{L^{\alpha+1}(\Omega)}^{\alpha+1} \leq c_4 \|b\|_{L^2(\Omega)}^2,
\end{align*}
\]

where $c_4 := \frac{1}{4} \frac{C^2}{D_0} + \kappa$. Hence we infer from Gronwall’s inequality that

\[
\|b(t)\|_{L^2(\Omega)} \leq 2e^{2c_4t}, \quad t \in (0, T).
\]

Thus (26) holds with $C_4 := 2e^{2c_4T}$. Estimates (27) and (29) can be shown by a similar way as in the proof of Lemma 3.2. We note that the time for which estimates (27) and (29) are valid can be taken one in (26).

We are now in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. In the same way as in the proof of Theorem 1.2, we have the desired weak solution to (KS) on some interval. In view of the proof of Theorem 1.2, since the estimates for the approximate solutions are valid on $(0, T)$, the approximate solutions converge on $(0, T)$. Thus we conclude that the weak solution exists globally.
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