A review of progress in the physics of open quantum systems: theory and experiment

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Abstract

This report on progress explores recent advances in our theoretical and experimental understanding of the physics of open quantum systems (OQSs). The study of such systems represents a core problem in modern physics that has evolved to assume an unprecedented interdisciplinary character. OQSs consist of some localized, microscopic, region that is coupled to an external environment by means of an appropriate interaction. Examples of such systems may be found in numerous areas of physics, including atomic and nuclear physics, photonics, biophysics, and mesoscopic physics. It is the latter area that provides the main focus of this review, an emphasis that is driven by the capacity that exists to subject mesoscopic devices to unprecedented control. We thus provide a detailed discussion of the behavior of mesoscopic devices (and other OQSs) in terms of the projection-operator formalism, according to which the system under study is considered to be comprised of a localized region ($Q$), embedded into a well-defined environment ($P$) of scattering wavefunctions (with $Q + P = 1$). The $Q$ subspace must be treated using the concepts of non-Hermitian physics, and of particular interest here is: the capacity of the environment to mediate a coupling between the different states of $Q$; the role played by the presence of exceptional points (EPs) in the spectra of OQSs; the influence of EPs on the rigidity of the wavefunction phases, and; the ability of EPs to initiate a dynamical phase transition (DPT). EPs are singular points in the continuum, at which two resonance states coalesce, that is where they exhibit a non-avoided crossing. DPTs occur when the quantum dynamics of the open system causes transitions between non-analytically connected states, as a function of some external control parameter. Much like conventional phase transitions, the behavior of the system on one side of the DPT does not serve as a reliable indicator of that on the other. In addition to discussing experiments on mesoscopic quantum point contacts that provide evidence of the environmentally-mediated coupling of quantum states, we also review manifestations of DPTs in mesoscopic devices and other systems. These experiments include observations of resonance-trapping behavior in microwave cavities and open quantum dots, phase lapses in tunneling through single-electron transistors, and spin swapping in atomic ensembles. Other possible manifestations of this phenomenon are presented, including various superradiant phenomena in low-dimensional semiconductors. From these discussions a generic picture of OQSs emerges in which the environmentally-mediated coupling between different quantum states plays a critical role in governing the system behavior. The ability to control or manipulate this interaction may even lead to new applications in photonics and electronics.
List of principal symbols and acronyms

| Symbol | Definition |
|--------|------------|
| A      | Number of nucleons in a nucleus |
| β      | Transmission phase |
| b_kl   | Mixing coefficient for states k and l due to their environmental coupling |
| c, c'  | Open decay channel |
| D, D, Δ, δ | Quantum level spacing |
| DPT    | Dynamical phase transition |
| E      | Energy of the system |
| E_k    | Energy of the discrete eigenstate k of H^0 |
| ε_k    | Complex eigenvalue of H |
| E_k    | Energy of the eigenstate k of H |
| ε = Re(ε_k) | Real part of the diagonal matrix elements of the non-Hermitian Hamiltonian matrix |
| Ep     | Exceptional point |
| Φ_k    | Eigenfunction of H^0 (orthogonal) |
| Φ_k    | Eigenfunction of H_0 (biorthogonal) |
| G, G_x, G_d | Conductance |
| G_P^{(k)} | Green function in the environment P |
| Γ, Γ'  | Resonance line width |
| Γ/2    | Width of the eigenstate k of H |
| γ/2    | Imaginary part (line width) of the diagonal matrix element of the non-Hermitian Hamiltonian |
| γ_kc   | Partial width amplitude of the isolated state k relative to channel c |
| Ω      | Wavefunction of the resonance state k |
| ξ      | Scattering wavefunctions of the P subspace in channel c at energy E |
| χ      | Incoming wave in channel c at energy E |

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(Some figures may appear in colour only in the online journal)
1. Introduction

A core problem in modern physics, and one which has evolved to assume an unprecedented interdisciplinary character, is related to the study of open quantum systems (OQSs). In the broadest sense, these systems may be defined as consisting of some localized, microscopic, region that is coupled to an external environment by means of an appropriate interaction. In discussions of the crossover from quantum mechanics to classical physics, the role of such an environment is often invoked to describe the influence of a classical measuring apparatus. Even if one is to remove such an apparatus, however, the description of most open systems may nonetheless be reduced to one in which the properties of some microscopic region are influenced by its coupling to its own, ‘natural’, environment. In contrast to the aforementioned measurement problem, the influence of this environment cannot be deleted but exists at all times, independent of any observer. In essence, this environment functions as an ‘intrinsic’ (natural) measuring apparatus, introducing a coupling between different states of the open system. Two very different cases in which this situation applies are provided by the decay of unstable states in nuclei, and the transport of electrons through mesoscopic quantum dots. In the former case, the environment consists of a continuum of scattering wavefunctions outside of the nucleus, which may mediate the escape of either a neutron or a proton from that structure [1]. Similarly, in discussions of quantum-dot transport, the discrete quantized states of an isolated cavity develop a broadening when the cavity is coupled to macroscopic reservoirs, through the addition of appropriate leads. In this latter case, the environment is provided by the states of these reservoirs, and the broadening of the discrete cavity levels is strongly dependent upon the mutual overlap with these states, as mediated through the leads [2].

There are numerous examples of OQSs, from a broad spectrum of disciplines within physics. While obvious examples include nuclei, atoms, and molecules, a much broader range of systems is provided by lasers and optically-active media, biomolecules and molecular networks, nanophotonic structures, and mesoscopic electronic devices. It is the latter systems (namely mesoscopic devices) that we focus on in this review, motivated by two of their important features. The first of these is their capacity to exhibit a variety of rich phenomena arising from their environmental interaction, while the second is the ability to subject them to sophisticated external control.

Quite generally, mesoscopic devices are small metallic or semiconducting structures, in which carrier motion is constrained on a spatial scale that is comparable to, or even smaller than, the fundamental length scales associated with transport. These length scales include the elastic mean free path, the phase-coherence length, and the inelastic scattering length, all of which may exceed the size of sub-micron scale devices at sufficiently low temperatures [3]. In terms of their general structure, these devices consist of some central scattering region, in the form of a quasi-one dimensional wire or a quasi-zero-dimensional quantum dot (QD), into which carriers may be injected, and subsequently extracted, to allow for transport. Such transport is achieved by connecting the central scattering region of the device to macroscopic charge reservoirs, by means of appropriate lead structures. When compared with systems such as nuclei, the great advantage that mesoscopic devices offer is the capacity to externally control the key parameters of the system. These include the system size, and thus the nature of the quantum states involved in transport, the strength of the coupling between the system and its environment, and the energy of the conduction electrons involved in transport. This control makes these structures ideally suited to the study of forefront issues associated with OQSs. We demonstrate this here for few-electron QDs [3, 4], which are the solid-state analog of scattering billiards, and for quasi-one-dimensional quantum point contacts (QPCs), in which the presence of a strong environmental coupling may modify the very nature of the quantum states responsible for transport [5].

In this report, we focus on providing a detailed discussion of the behavior of OQSs in terms of the projection operator formalism of quantum mechanics. In this approach, as indicated in figure 1, we consider the system under study to be comprised of a localized region (Q) that is embedded in a well-defined environment (P) of scattering wavefunctions. The complete function space satisfies the requirement $Q + P = 1$, and is described in terms of a Hermitian Hamiltonian. The Hamiltonian describing the subspace of interest (Q) is non-Hermitian, however, and has complex eigenvalues that define both the energy and broadening of its states [1]. These eigenstates are coherently coupled to each other, via an interaction that is mediated by the environment (P). This projection operator formalism differs from that in which complex scaling is used to analyze non-Hermitian systems, an approach that has been summarized in [6] and that we will not discuss. We also note that the role of the environment considered here is very different to that which is often invoked in discussions of decoherence, arising from the uncontrolled interaction of a quantum system with its environment (such as an electron or phonon bath). This latter problem has been treated in other works (see, for example, the reviews of [7, 8]), where the feedback from the environment onto the quantum system is not explicitly considered. This is in contrast with the situation here, where we are concerned, above all, with understanding the influence of the natural environment (P) on the system (Q). This environmental influence remains even when sources of decoherence have been suppressed (by lowering the temperature, for example). As such, the approach that we outline is expected to be useful in describing the behavior of small systems, in which the number of particles active in transport is small, and in which the number of quantum states they occupy is similarly restricted in number. Under such conditions we will see that, far from acting as a source of decoherence, the coupling to the environment can in fact mediate an extremely robust and coherent interaction.

The objective of this review is to focus on critical implications of non-Hermiticity, namely the presence of exceptional points (EPs) in the spectrum of OQSs, their influence
on the rigidity of the wavefunction phases, and their ability to initiate a dynamical phase transition (DPT). As we discuss in further detail below, these features can be observed because of our ability to vary the strength of the environmental interaction in OQSs, by means of some suitable control parameter. EPs are singular points in the continuum, at which two resonance states coalesce, that is where they exhibit a non-avoided crossing. This behavior should be contrasted with the well-known avoided crossings, exhibited by Hermitian systems at so-called diabolic points. Due to the coalescence of the two resonance states at an EP, the time evolution of the system becomes undefined in its neighborhood. At the same time, the EPs are branch points in the complex plane and are responsible for the generation of double poles of the $S$ matrix, with associated implications for transmission.

An important quantity that indicates the range of influence of an EP is the phase rigidity of its two eigenfunctions. In isolated (Hermitian) systems the phases of the eigenfunctions are said to be rigid, a statement that expresses the well-known orthogonality of these functions. In open (non-Hermitian) systems, however, the presence of the environmentally-mediated coupling between the eigenstates means that they are no longer orthogonal, but rather exhibit the property of biorthogonality. This allows the rigidity of the eigenfunction phases to be reduced, particularly near an EP where the rigidity actually vanishes. This surprising behavior expresses the very strong influence of the environment on the localized system at an EP.

While the physical significance of EPs for the dynamics of OQSs has only recently begun to be studied (see, for example, the review [1]), they are nonetheless understood to be intimately connected to the observation of DPTs in different open systems. The term DPT has been coined to refer to a phenomenon in which the quantum dynamics of the open system undergoes a phase transition between non-analytically connected states as a function of some suitable control parameter. Here, time-dependent approaches to the description of the problem fail [9]. Instead, much like a conventional phase transition in, for example, magnetism or superconductivity, the physical behavior of the system on one side of the transition does not serve as a reliable indicator of that on the other. The connection of DPTs to the EPs is provided by the phenomenon of width bifurcation, which results in a spectroscopic redistribution in the system into long-lived states with narrow linewidths and a much smaller number of short-lived, strongly-broadened, ones. An important objective of this review is to connect the theoretical concepts of EPs and DPTs to the physical behavior manifested in experimental investigations of OQSs.

One of the earliest works to explicitly invoke the notion of a DPT involved studies of spin swapping in atomic systems [10]. In these experiments, the authors studied Rabi oscillations due to spin flips in the $^{13}$C–$^1$H system, and showed a transition to strongly damped motion by increasing the coupling to an environment formed by a spin bath. Although not explicitly discussed as such, evidence of DPTs is apparent, also, in earlier work on resonance trapping in microwave cavities [11], and on einselection [12–14] in mesoscopic quantum dots. In both of these systems, the focus was on understanding the manner in which the states of a quantized cavity are affected by coupling them to an external environment. The common phenomenon revealed in both cases was of width bifurcation, with certain eigenstates actually becoming narrower when the environmental coupling was increased over a specific range. The accompanying long-lived eigenstates appear as a background with which the long-lived ones interfere, a behavior that has been observed experimentally in microwave cavities [11]. Ultimately, the long-lived states may even become discrete, forming so-called bound states in the continuum [15]. With the renewed interest in such problems, it is interesting to note that the appearance of long-lived ‘merkwürdigen’ (remarkable) states is actually a problem that dates back to the earliest days of quantum mechanics [16]!

Elsewhere, evidence of a DPT has also been provided in work where a multi-level quantum dot was embedded into one of the arms of an Aharonov–Bohm interferometer, allowing the evolution of the transmission phase to be monitored across a sequence of resonance states [17–19]. These
experiments revealed the presence of unexpected regularity in the measured scattering phases (so-called ‘phase lapeses’), when the number of states occupied by electrons in the dot was sufficiently large. While this behavior could not be fully explained within approaches based upon Hermitian quantum theory, see for example [20, 21], it has recently been established that the phase lapses can be attributed to the non-Hermitian character of this mesoscopic system, and to a DPT that occurs as the number of electrons in the dot is varied [22]. The observed regularity arises from the overlap of the many long-lived states with a short-lived one, all of which are formed due to the DPT. More recently, studies of transport in ballistic quantum wires may have also revealed a DPT, involving the formation of a protected channel for conduction under strongly nonequilibrium conditions. In these experiments the environment is essentially provided by the phonon system, whose influence is controlled by means of the applied source bias and by the strong quantum confinement of the carriers induced within the wire [23]. A well-known phenomenon from optics that may also involve a DPT is that of Dicke superradiance [24]. This refers to the effect in which an ensemble of excited atoms within a cavity does not emit radiation randomly, but rather does so in a correlated manner when the atoms experience the same radiation field. Recently, superradiance has been demonstrated for solid-state systems, namely ensembles of self-assembled quantum dots [25] and a dense semiconductor electron–hole plasma [26].

In this review, we will use the various experiments described above to connect the behavior exhibited by OQSs to the key concepts of non-Hermiticity, EPs, and DPTs. We will also introduce the important concept of wavefunction phase rigidity, and the significance of time in open systems, in which we are unable to describe a DPT by means of time-dependent approaches. We emphasize again that the focus of our review will be on discussions of the properties of open systems consisting of a small number of particles. As such, we explore the behavior in a very different limit to that relevant, for example, in heavy nuclei. These contain a much larger number of particles, as well as many closely-neighboring states, and are adequately described by the concepts of random-matrix theory [27]. In fact, with the exception of a few specific examples, we will not address phenomena arising in nuclear systems, preferring to note instead that this topic was recently excellently served by an associated review in this journal [28]. Due to the continued importance of mesoscopic systems for the investigation of fundamental quantum phenomena, we believe that the focus of our review will prove to be a particularly useful one.

The remainder of this is organized as follows. In the next section, we introduce the concept of small quantum systems that are coupled to an environment with which they interact. We begin by reviewing some of the well known properties of isolated (Hermitian) quantum systems, emphasizing concepts such as the real nature of their eigenvalues and the rigidity (orthogonality) of their eigenfunctions. Following this, we next introduce the projection-operator formalism for OQSs, in which the total system is described in terms of two coupled subspaces ($Q$ & $P$). The complex character of the eigenvalues and eigenfunctions of the non-Hermitian operator of the $Q$ subspace is described, and its physical implications are discussed. We introduce also the concept of gain and loss in OQSs. In the last part of this section, some physical examples of the environmentally-mediated coupling of quantum states are presented, focusing on their manifestations in mesoscopic structures. In section 3, we discuss the importance of EPs to the behavior of OQSs. We formulate this discussion by first of all focusing on the interaction of two resonance states near an EP, following which we study the influence of an EP on the eigenvalues and eigenfunctions of the non-Hermitian Hamiltonian. In section 3.4 we discuss the implications of EPs for the $S$-matrix of the system, a problem with important implications for the analysis of transmission. In section 4, we consider the role of EPs in giving rise to DPTs in open systems subjected to some form of external control. In the presence of this control, DPTs are found to occur when the range of influence of several different EPs develops a sufficiently strong overlap. The connection of DPTs to the notion of width bifurcation is emphasized, and several experimental demonstrations of such phase transitions are presented. These include demonstrations of resonance trapping in microwave cavities, studies of phase lapses in tunneling through quantum dots, and measurements of spin swapping in atomic systems. Emphasizing our belief that DPTs are, in fact, an inherent feature of OQSs in general, in section 5 we discuss further examples of physical problems in which DPTs may occur. In section 6, we conclude this review by summarizing its main points and by identifying important issues for further study. In an appendix to this report we make a few additional remarks on the non-Hermitian Hamiltonian (with and without using perturbation theory), and give expressions for the coupling matrix elements (the partial width amplitudes) between the system and the environment. A list of principal symbols and acronyms used has been provided immediately after the abstract.

2. The natural environment of an open quantum system

In this section we introduce some of the essential features of non-Hermitian systems, most specifically focusing on the concept of the natural environment of such systems. We begin, however, by firstly reviewing the essential aspects of Hermitian quantum mechanics, as applied to the treatment of closed quantum systems. Following this, we present the non-Hermitian projection-operator formalism for the treatment of OQSs. In this discussion we emphasize the physical implications of non-Hermiticity for OQSs, rather than the formal mathematical details of this problem. The latter have been treated in a separate review [1] and we refer the interested reader to this publication. In this section we instead discuss how the function space of the entire system may be broken into two distinct subspaces; one of which contains the wavefunctions of some localized region ($Q$), while the other contains scattering
wavefunctions that represent its natural environment \((P)\). An important feature of such systems is shown to be a coupling between their different states, which is mediated through their coupling to the common environment. We conclude this section by presenting the results of recent experiments on mesoscopic devices, which provide direct evidence of the role of such coupling.

2.1. Hermitian quantum physics and its limits

In introductory discussions of quantum mechanics, the Schrödinger equation is solved for various closed systems using a Hermitian Hamiltonian whose real eigenvalues \(E_k\) provide the energy of the various states of the system. The eigenfunctions associated with these states are normalized according to \(\left\langle \Phi_k^Q | \Phi_l^Q \right\rangle = \delta_{kl}\), where the superscripts in this expression indicate that these are the eigenfunctions of a closed system. This expression also indicates the orthogonal character of the different \((k \neq l)\) eigenfunctions, whose phases are said to be ‘rigid’ with respect to one another. The validity of these basic statements has been confirmed through nearly a century of careful experimental study.

In spite of the successes of Hermitian quantum mechanics there are, nonetheless, a number of problems that remain unresolved with its implementation. One of these concerns the fact that the lifetime of the states of a decaying quantum system cannot be derived from first principles, but instead requires the introduction of additional assumptions. In practice, this is achieved by assuming that particles may tunnel from the system to an environment of scattering states. If the original states of the closed system are isolated from one another, and feedback from the environment to this system can be neglected, one is able to obtain results that agree well with experiment. Problems arise, however, when the states are close to one another and when the coupling between the system and its environment is strong. Here, the environment may mediate an interaction between the initially-orthogonal eigenstates (see the following section), making it essentially impossible to distinguish the relation of the open system to its closed counterpart. This problem is highlighted in figure 2, which shows the results of numerical calculations of the inelastic-scattering cross section of protons from \(^{15}\text{N}\) nuclei. In these calculations, the influence of the environment on the states of the nucleus was treated in an essentially ad-hoc manner, by varying the separation of its shell-model states from one another. When the overlap between the various resonance states is small, the scattering cross-section exhibits a series of sharp resonances in accordance with the structure of the shell-model states (bottom panel). As the overlap is increased, however, this correspondence is lost and one ultimately obtains features in the cross section that resemble Ericsson oscillations (top panel). Under such conditions, it becomes impossible to extract spectroscopic information from the cross section. Unfortunately, a direct experimental demonstration of this behavior has proven impossible to date, since in nuclear systems one has very little control of the system properties [29]. Nonetheless, this example serves to demonstrate how the coupling to the natural environment may alter the behavior of an initially isolated system.

The fact that a coupling to external states must be introduced to compute transition probabilities for realistic quantum systems, as well as the lifetime of their various states, suggests it would be better from the very outset to treat them as open structures that interact with their natural environment. In performing such an analysis, we should require that the results obtained pass smoothly into those of Hermitian quantum physics at low level density, and for small coupling strength between the system and its environment. In the next section we describe how this may be achieved, by making use of key concepts from non-Hermitian quantum mechanics.

2.2. Open quantum systems and non-Hermitian quantum physics

Our discussion of the non-Hermitian nature of OQSs is based on an extension of the projection-operator formalism, elaborated many years ago in nuclear physics by Feshbach [31]. In this approach, the system under study is broken into two subspaces (as illustrated in figure 1), with the \(Q\) subspace corresponding to the localized system, while the \(P\) subspace is related to the extended environment (with \(Q + P = 1\)). The advantage of this approach is that the quantum-mechanical problem may be separately solved in each subspace, using well-established methods. The solutions for the system as a whole may then be obtained by combining those for the two subspaces. Before outlining the features of this scheme, let us first take a few moments to provide some important clarifications regarding terminology. In the discussion that follows, we shall refer to the ‘whole’ or ‘complete’ system to mean the full Hermitian space comprised of both the \(Q\) and \(P\) subspaces (with \(Q + P = 1\)). The term ‘closed’ or ‘localized’ system will then be used to identify the subspace \(Q\), before it is coupled to any environment, under which conditions \(Q\) is fully Hermitian. Finally, the term ‘OQS’ will be used to refer to the (non-Hermitian) system \(Q\) under the influence of the environment \(P\). To provide some context for these statements, we refer again to the open quantum dot depicted in figure 1. Here, the central cavity represents the OQS and is non-Hermitian when its two leads are open to provide coupling to the reservoirs. If these lead structures are closed completely, however, the influence of the environment on \(Q\) is suppressed and this isolated space thus becomes Hermitian.

While the complete system, comprised of the \(P\) and \(Q\) subspaces, is considered to be closed and described by a Hermitian Hamiltonian, the subspace \(Q\) is non-Hermitian in the presence of its coupling to \(P\) and its Hamiltonian reads (for details see [1])

\[
\mathcal{H} = H^Q + V_{QP} + V_{PQ}. \tag{1}
\]

According to perturbation theory due to Feshbach [31], the first term on the right-hand side of this equation \((H^Q)\) is the Hermitian Hamiltonian of the closed system \((Q)\) and has real eigenvalues, while the second term, and therefore \(\mathcal{H}\), is
non-Hermitian. As we describe in appendix, a non-perturbative approach to this problem is also possible, by replacing the Hermitian operator $H_Q$ with the non-Hermitian one $H_0$. In numerical calculations, the differences between these approaches are typically negligible, however, since the non-Hermiticity of $H$ is determined, above all, by the coupling of the states via the environment (the second term in (1)). This coupling is only large in the neighborhood of EPs, as will be discussed in the next section.

Reflecting the fact that the system and its states are embedded into a continuum of scattering wavefunctions (the $P$ subspace), the eigenstates of the Hamiltonian $H$ consist of a set of resonance states with complex eigenvalues. The real components of these eigenvalues define the energies ($E_k$) of the different states ($k$), while the imaginary parts ($\Gamma_k/2$) are inversely proportional to their lifetimes. The interaction of the states via the environment is described by the second term on the right-hand side of equation (1), in which $V_{QP}$ and $V_{PQ}$ denote the interaction between the system and the environment, while $G_P^{(+)}$ is the Green function in the environment. The corresponding matrix elements describing the coupling that develops between the different states of $Q$ are typically complex, consisting of real and imaginary parts. Writing $\langle \Phi_k^Q | V_{QP}G_P^{(+)}V_{PQ} | \Phi_l^Q \rangle \equiv W_{kl} = \text{Re}(W_{kl}) + i \text{Im}(W_{kl})$, these two parts may be expressed as [1]

$$\text{Re}(W_{kl}) = \text{Re} \left( \langle \Phi_k^Q | H | \Phi_l^Q \rangle - E_k^Q \delta_{kl} \right) = \frac{1}{2\pi} \sum_c \mathcal{P} \int dE' \frac{\gamma_k^Q \gamma_l^Q}{E - E'}$$

(2)

and

$$\text{Im}(W_{kl}) = \text{Im} \left( \langle \Phi_k^Q | H | \Phi_l^Q \rangle \right) = \frac{1}{2} \sum_c \xi_k^c \xi_l^c$$

(3)

Equation (2) contains the principal-value integral $\mathcal{P}$, while equation (3) corresponds to the residuum. $c$ denotes an open decay channel of the environment, and $E$ is the energy of the system and $\Phi_k^Q$ and $E_k^Q$ are, respectively, the eigenfunctions and eigenvalues of the Hermitian operator $H^Q$. Equations (2) and (3) express an important difference between Hermitian and non-Hermitian systems, indicating that the environment

Figure 2. Calculated cross section ($\sigma$, left) and width ($\Gamma$, right) for the formation of the $^{16}$O nucleus by the inelastic scattering of protons from $^{15}$N nuclei. The four different plots shown in each of the two panels correspond to results obtained by varying the degree of overlap of the nuclear levels (expressed here as the ratio of the ‘average width’ ($\langle \Gamma \rangle$) of the different levels of the $^{16}$O nucleus to their average separation ($D$)). Figure reproduced with permission from [30]. Copyright 1985 American Physical Society.
of an open system may mediate an additional interaction between its different states. This interaction is governed by the (energy-dependent) matrix elements \( \gamma_k^Q = \sqrt{2\pi} \langle \Phi^Q\mid V\xi_k^E \rangle \) and \( \gamma_k^l = \sqrt{2\pi} \langle \Phi_l\mid V\xi_k^E \rangle \), each of which describe the coupling of a specific state (\( k \) or \( l \)) of subspace \( Q \) to the environment, i.e., to the scattering wavefunctions \( \xi_k^E \). These matrix elements are commonly referred to as the partial width amplitudes of the isolated state \( k \) and \( l \), respectively, relative to channel \( c \). The combined effect of these matrix elements is to introduce an interaction between states \( k \) and \( l \), mediated via the environment, and experimental evidence for this form of environmentally-induced coupling has recently been provided in experiments performed on mesoscopic devices [32] (see the discussion in section 2.3 below).

The relative magnitude of the real and imaginary components of the coupling matrix element \( W_{kl} \) depends strongly on the energy separation of the two states \( k \) and \( l \), as can be understood by appealing to some relatively straightforward arguments. Namely, in the principal-value integral of equation (2), the contribution from states \( k \) and \( l \) is integrated over all energies. With the energy \( E \) lying in the interval \( |E_k^Q - E_l^Q| < k \hbar \), this contribution is large as long as the two resonances are distant from one another. The residue of equation (3), however, is only nonvanishing at the energy \( E \) of the system and therefore large when \( E_k^Q = E_l^Q \). Consequently, \( W_{kl} \) is (almost) real when the two levels \( k \) and \( l \) are well separated in energy, but (almost) imaginary when they are near to one another.

Another important concept that should be clarified concerns the use of the term ‘channel’, which has related yet distinct meaning in nuclear and mesoscopic physics (refer to the schematics shown in figure 3). When describing the particle decay of nuclei, this term is used to refer to a transition from a specific state of an initial nucleus (with \( A \) nucleons) to a specific one of a final nucleus (with \( A-1 \) nucleons) by emitting a single neutron or proton to the continuum. In single-channel decay, only one decay channel is allowed energetically while in a multi-channel process several such channels are involved. In problems involving transport through some mesoscopic structure, one is concerned with a discussion where carriers are injected from one lead, into some microscopic region, before eventually leaving via a second lead. In two-channel transport, the carriers enter and exit the structure via a single mode in each lead, while in multi-channel transport the conductance is determined by summing the transmission between many different modes in both leads.

Thus far we have considered the description of OQSs that may emit particles into the continuum of scattering states in which they are embedded. Under certain conditions, however, it is possible that the system may even absorb particles from this continuum. The projection-operator formalism outlined above may also be applied to the description of such systems. The formal difference between the loss of particles from the \( Q \) subspace and the gain of particles by the same space is simply given in terms of the sign of the width \( \Gamma_k \) of the relevant states \( k \) involved; this sign is positive in the case of gain and negative in the case of loss. The direct relation (inverse proportionality) of the width \( \Gamma_k \) to the lifetime of the state \( k \) is meaningful however only in the case of loss. Nevertheless, there is considerable interest in the properties of systems with gain, most notably in those in which loss and gain may be
introduced in different physical layers. Examples of this type will be considered later in this review.

2.3. Experiment: coupling quantum states through a continuum

It has already been noted in the introduction that mesoscopic devices are ideally suited to investigating the predictions of the non-Hermitian formalism, most notably to studying the nature of the interaction between the $P$ and $Q$ subspaces. Quite generally, the usefulness of these devices arises from the capacity that they offer to ‘engineer’ the nature of this interaction, by using advanced fabrication techniques to define nanostructured devices with arbitrary geometries. In our discussions here, we will focus explicitly on the behavior exhibited by two distinct types of mesoscopic device, namely quantum dots (QDs) and quantum point contacts (QPCs). The fabrication of these devices typically starts from a GaAs/AlGaAs heterostructure, in which a crystalline layer of AlGaAs is formed directly on top of a similarly crystalline GaAs substrate. Through the deliberate addition of donors to the AlGaAs, electrons may be transferred into the undoped GaAs where, under conditions of thermal equilibrium, they accumulate in an essentially two-dimensional layer that is confined near the interface of the two materials. For almost forty years now the study of this two-dimensional electron gas has driven fundamental discovery in condensed-matter physics, perhaps most spectacularly in the form of the integer and fractional quantum Hall effects [33].

For investigations of mesoscopic phenomena, it is necessary to confine the two-dimensional electron system in GaAs to some quasi-one-, or quasi-zero-, dimensional structure. In the experiments that we shall discuss, this is most-commonly achieved by forming nanostructured metal gates on the top surface of the heterostructure. By biasing these gates with a suitably-large negative voltage the regions of electron gas immediately underneath them can be depleted, leaving mobile electrons confined to the gaps between the gates. The great advantage of this technique is that the specific size and geometry of the remaining electron puddle is largely determined by the lithographic shape of the gates. A simple pair of ‘split-gates’, separated by a sub-micron scale gap, for example, can be used to define a QPC, a narrow channel through which electrons are transmitted ballistically at low temperatures. In essence, these structures may be viewed as the electron analog of microwave waveguides, since the strong lateral confinement that electrons experience as they pass through the QPC quantizes their spectrum to form a series of transverse ‘channels’, ‘modes’, or ‘subbands’. Each such subband is characterized by a uniquely-quantized value of its transverse momentum, corresponding to accommodating an integer number of half electron wavelengths into the confinement potential induced by the gates. The existence of these quantum subbands gives rise to one of the most important phenomena in mesoscopic physics, namely that of one-dimensional conductance quantization (see, for example, chapter 5 of [3]). According to this phenomenon, which provides the starting point for the discussion of many more-complex mesoscopic effects, the conductance of each occupied subband is quantized at a value of $2e^2/h$ when transport is ballistic.

In addition to electrostatically-gated QDs, the other mesoscopic system that we will focus on in this review is realized by utilizing QPCs that are configured very close to pinch-off [34]. This condition can be realized by increasing the gate biasing to generate strongly-repulsive electric fields in the gap between the split gates, thereby driving almost all mobile electrons out of this region. Under such conditions, the occupancy of even the lowest-energy subband drops to zero and the QPC essentially functions as tunable barrier to the flow of current. In a simple picture, one neglects the role of electron interactions in the vicinity of the QPC, and the shape of this barrier is typically expected to correspond to a simple two-dimensional saddle [3]. Recent theoretical work [5, 35] suggests a more interesting scenario, however, in which the build up of electrons on either side of the barrier near pinch-off may actually alter the self-consistent electrostatics of this problem. Indeed, on the basis of numerical solution of the full many-body problem in this regime, using approaches such as spin-density functional theory, it has been suggested that this charge build-up may cause the QPC to spontaneously develop a local minimum at the center of its barrier. This well is expected to give rise to the formation of a quasi-bound state at the QPC center, allowing the QPC to essentially function as an ‘on-demand’ quantum state. With this well occupied by a single electron, the large Coulomb energy cost (or ‘on-site $U$’) associated with adding a second one to such a small region should be prohibitively large, in which case the QPC is essentially expected to function as an ‘artificial’ magnetic impurity (formed by a single-electron spin). Evidence for the existence of this novel many-body state was first provided in experiments revealing apparent hints of Kondo physics in the QPC conductance in this regime [36]. From the perspective of the problem of interest here, the key point to note is that the QPC directly manifests the interplay between the $P$ and $Q$ subspaces, and as such should allow for rich investigations of the influence of the environment on the QOQS. Most notable here is the fact that the environment is a many-body one, which is actually able to strongly modify the states of the original quantum system. This is an issue that we shall return to again, when we discuss the possible observations of DPTs in atomic ensembles [10] and in mesoscopic wires [23].

A key prediction of non-Hermitian theory involves the notion that the environment may be used to mediate an effective interaction between the different states of the open system (recall the discussion of equations (2) and (3)). The presence of such coupling has been nicely demonstrated in experiments performed on pairs of QPCs, which interact with each other as a result of their mutual overlap with a common continuum. In the simplest realization of these experiments, one QPC is used as a ‘detector’ that monitors the state of another QPC (the ‘swept-QPC’) nearby [37–42]. The concept of this experiment in indicated in figure 4, in which we show one example of a multi-gate device that has been used to realize the coupled-QPC system [41]. By biasing suitable combinations of gates, this one device may be used to implement coupled QPCs in various configurations, differing in terms of the spatial...
separation between the two QPCs and of their exact geometry. Regardless of this geometry, the common observation in such experiments is of an isolated Fano resonance in the conductance of the detector, which occurs as the swept-QPC is driven to pinch-off [37, 41, 42]. Since this resonance occurs precisely in the regime where a bound-state is expected to form in the swept QPC, we have suggested that it arises from an interference process in which the conductance of the detector QPC is modified by a resonant scattering process. This occurs as the variation of the swept-QPC gate voltage drives its bound state up through the Fermi level in the two-dimensional continuum. When this occurs, quantum interference of electron partial waves that travel from the detector to the bound state, and then scatter back to the detector, modify its conductance.

In [39], a simple model reproducing the features of experiment was formulated, the most important element of which was a mutual wavefunction overlap between the bound state and the quantum states of the detector, mediated through the intervening continuum. In experiment, this continuum is provided by the two-dimensional electron gas that separates the two QPCs. At the low temperatures where the experiments are performed, it is the states at the Fermi level of this continuum that are most important for the discussion of the detector resonance. In early work on this system, effort was largely focused on utilizing the characteristics of the observed resonance as a means to probe the microscopic properties of the QPC bound state. This included studies of the temperature and magnetic-field dependence of the resonance, yielding information on the confinement strength of the bound state and on its spin structure [41]. In [42] it was shown that the lineshape of the resonance is strongly dependent upon the spatial separation of the two QPCs, indicating that the strength of their environmentally-mediated coupling may be manipulated in this manner.

From the discussion above it will be apparent that coupled QPCs provide a useful system to investigate the nature of the environmental interaction in OQSs. The experimental approach that we have described is essentially ‘modular’, and can be extended to allow the realization of even more sophisticated systems. One example is provided in [32], where the same gate geometry as that shown in figure 4 was used to configure two bound states, coupled to each other through a mutual continuum. The manner in which this was achieved experimentally is indicated in figure 5, where we see how the activation of an additional ‘control’ gate could be used to realize two QPC bound states with a separation of just a few hundred nanometers. These bound states are coupled through a region of two-dimensional electron gas (referred to in the figure as ‘Region (1)’), while a second similar region (‘Region (2)’) provides the coupling between...
one of the bound states and the detector. By variation of the gate voltages that define the swept- and control-QPCs, two distinct resonances could be observed in the detector, with one of each of these arising from the coupling of the detector to a particular bound state.

Moving beyond the simple QPC, more complex gate geometries may be used to form a quantum dot (QD), essentially the solid-state realization of a scattering billiard [12, 13, 43–46]. In such devices, electrons are confined within a small cavity (recall figure 1) and undergo multiple scattering from its internal boundaries before finally escaping to the reservoirs via appropriate leads. These leads therefore regulate the interaction of the QOVS with its environment, much like the situation that is found in microwave billiards. Indeed, the strong overlap between the properties of microwave cavities and mesoscopic QDs was demonstrated nicely in a comparative study by Kim and co-workers [47]. They showed clear similarities between the properties of the transmission fluctuations exhibited by microwave cavities, and the regular oscillations exhibited in the low-temperature magneto-conductance of QDs. In section 4.3.1, we return to consider these similarities in greater detail.

Later on in this review we will be concerned with a discussion of the role of so-called exceptional points (EPs) in QOVSs, which are points of zero measure at which two eigenvalues of the non-Hermitian Hamiltonian coalesce. Experimentally, specific signatures are expected in the vicinity of these points, most notably the phenomena of width bifurcation and non-avoided level crossings. While such phenomena were not considered in the study of figure 5, the system shown here is one that should be suitable for such investigations. By appropriate tuning of the gate voltages, the two resonances arising from the bound states formed in the swept- and control-QPCs could be brought towards coincidence in experiment. As can be seen in figure 5, this resulted in the observation of an avoided crossing, indicating the presence of an interaction between the two bound states. While this interaction is indirect in nature, being mediated through an intervening continuum, the experiment suggests that the non-local coupling actually supports a robust coupling of the two bound states. To illustrate this

![Figure 5](attachment:coupling-two-bound-states-through-a-continuum)
point, it is helpful to compare the findings of this experiment to the more-widely studied problem in which two QDs interact with each other by means of their coupling through a common tunnel barrier [48]. In experiments of this type, the two-state interaction arises from the mutual overlap of the electronic states of the QPCs, bringing the bound state in the swept-QPC in close proximity to the continuum formed by its one-dimensional subbands. Under these conditions, the detector resonance now arises from an interference scheme in which electron partial waves leaving the detector may tunnel onto either the bound state or this second continuum before returning back to the detector. Formally, this problem corresponds to a three-path Fano resonance in which the usual two-path scheme, involving the coupling of a discrete level (the bound state in the swept QPC) to a continuum (provided by the states of the detector) is modified through its coupling to an additional continuum (in this case comprising the one-dimensional subbands of the swept-QPC). In [50] this latter continuum was referred to as an ‘intruder’, in analogy with discussions of the role of intruder states in modifying the Fano resonance of Rydberg atoms [51]. The capacity to manipulate the influence of this intruder in experiments performed on mesoscopic systems may provide new avenues for quantum-state control.

3. Exceptional points in open quantum systems

A classic problem in Hermitian quantum mechanics concerns the coupling of quantum states between two potential wells, separated from each other by a common tunnel barrier. This problem provides a model system for understanding the concepts of chemical bonding, and can be realized also in semiconductor multilayers configured to form a superlattice potential. A well-known consequence of the non-zero wavefunction

Figure 6. Influence of an intruder continuum on the detector Fano resonance exhibited by coupled QPCs. The experimental geometry is the same as that shown in the micrograph on the left of figure 4, except that now a magnetic field of varying strength is applied perpendicular to the plane of the system. (a) The blue curves show the detector conductance and its variation with swept-QPC gate voltage ($V_g$), while the red curves indicate the corresponding variation of the swept-QPC conductance ($G_d$). In these panels, which were obtained at the different magnetic fields indicated, the resonant variation of the detector conductance ($G_d$), obtained after subtracting the linear background apparent in figure 4) is actually plotted. The red dotted lines indicate the pinch-off of the swept-QPC and the measurement temperature is 4.2 K. (b) A comparison of the measured variation of the detector resonance with magnetic field (values indicated) with the results of a theoretical model for an intruder-modified Fano resonance. The intruder continuum is coupled to the original Fano system in these calculations by means of a matrix element $t$. Variation of $t$ is seen to closely mimic the influence of the magnetic field found in experiment. Reprinted with permission from [50]. Copyright 2014 American Chemical Society.
overlap between such wells is the appearance of level repulsion. It is this interaction that is the source of the bonding and anti-bonding states in molecular structures, and of the avoided level crossing exhibited by tunnel-coupled QDs [3].

In our discussion of the properties of the resonance states of QQSs, we have seen that the coupling mediated by the environment leads to an effective interaction between the different states of the open system. In this section we discuss how these interacting states may exhibit a behavior that is very different to that of the discrete states of Hermitian systems. Under the influence of the environmental coupling, so-called exceptional points (EPs) appear in the spectrum of the open system. At these singular points, two complex eigenvalues of the non-Hermitian operator coalesce while their eigenfunctions lose their orthogonality and differ by only a phase factor from one another [1]. That is, they exhibit a non-avoided crossing where their real components are identical, as are their imaginary ones. At the EP, and in its vicinity, the imaginary part (equation (3)) of the environmentally-induced coupling matrix element ($W_{ij}$) can become much larger than its real component (equation (2)). As a result, width bifurcation ultimately occurs, causing the system to be characterized by two very-different time scales. The bifurcation involves the formation of a short-lived (collective) state that is strongly coupled to the environment, and a long-lived one that may be almost completely decoupled from it. In this section we review these, and other, critical properties of the EPs.

### 3.1. The interaction of two resonance states near an exceptional point

To describe the properties of EPs, we start from a relatively simple model of two resonance states that interact with one another via their common environment of scattering wavefunctions. Since the two states coalesce at the EP, the influence of all other states on the interaction of these two can be neglected. Following arguments from section 2.2 and appendix, the non-Hermitian Hamiltonian of the two states may be expressed as a symmetric $2 \times 2$ matrix

$$
\mathcal{H}^{(2)} = \begin{pmatrix}
\epsilon_1 & \omega_{12} \\
\omega_{21} & \epsilon_2
\end{pmatrix},
$$

(4)

where $\gamma_1 \leq 0$ for decaying states. The diagonal elements of equation (4) contain the energies ($\epsilon_{1,2}$) and widths ($\gamma_{1,2}$) of the two states when their coupling matrix elements vanish, $\omega_{12} = \omega_{21} = 0$. In the discussion that follows we take $\omega_{12} = \omega_{21} = \omega$ (which, according to equations (2) and (3), is generally complex), and the self-energy of the states is assumed to be included into $\epsilon_1$ and $\epsilon_2$. The eigenvalues of $\mathcal{H}^{(2)}$ consist of real and imaginary parts and may be expressed as

$$
\mathcal{E}_{1,2} \equiv \mathcal{E}_{1,2} + \frac{i}{2} \Gamma_{1,2} \equiv \frac{\epsilon_{1,2} \pm \sqrt{\gamma_1 \gamma_2}}{2} \pm Z;
$$

$$
Z \equiv \frac{1}{2} \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\omega^2}.
$$

(5)

In contrast to the definition that is typically used in, for example, nuclear physics, we define the complex energies before and after diagonalization of $\mathcal{H}$ by $\mathcal{E}_1 = \epsilon_1 + \frac{i}{2} \gamma_1$ and $\mathcal{E}_2 = \epsilon_2 + \frac{i}{2} \gamma_2$, respectively, with $\gamma_1 \leq 0$ and $\Gamma \leq 0$ for decaying states. This definition will be useful when discussing systems with gain (positive widths) and loss (negative widths), as will be described briefly in sections 3.3 and 4.3.4.

Now consider the behavior that arises when the energy detuning of the two states is varied, bringing them towards coalescence. Dependent upon the value of the coupling strength ($\omega$), the two states may repel each other in accordance with Re(Z), or they may alternatively undergo width bifurcation in accordance with Im(Z). When $Z = 0$ the two states cross each other, at a point that has been referred to by Kato [52] as an EP. The EP is a branch point in the complex plane where the $S$-matrix has a double pole [53–55]. The parametric dependence of the eigenvalues $\mathcal{E}_{1,2}$ is non-analytical in the vicinity of this point, with the widths, in particular, showing variations that are inconsistent with the predictions of Fermi’s golden rule (for an example, see figure 4 of [56]).

As an illustration of these concepts, in figure 7 we show calculated eigenvalue trajectories ($\mathcal{E}_{1,2}(a)$ and $\Gamma_{1,2}(a)Z$), obtained in a scenario in which the control parameter $a$ is used to vary the energy ($\epsilon_{1,2}$) of the two states. The calculations assume a real coupling ($\omega$) to the continuum, and correspond to different values of the resonance widths ($\gamma_{1,2}$), with fixed ratio $\gamma_1 \gamma_2$. As a result, we observe the characteristic behaviors of both level repulsion (at small $\gamma_{1,2}$) and width bifurcation (at large $\gamma_{1,2}$). The two eigenvalues cross when $\gamma_1 = 1.0$, the critical value corresponding to the position of the EP. It is also interesting to note from this figure how the level repulsion of narrow resonances at small $\gamma_1$ (panel(b)) passes smoothly into the well-known level repulsion of discrete states at $\gamma_1 = 0$ (panel(c)).

The requirement for the presence of an EP is $Z = 0$ and may be expressed by reformulating equation (5) as

$$
(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 + 4\text{Re}(\omega^2) = 0
$$

(6)

$$
(e_1 - e_2)(\gamma_1 - \gamma_2) + 4\text{Im}(\omega^2) = 0.
$$

(7)

These conditions cannot be satisfied for a Hermitian operator (when $e_1 = e_2$), due to its real eigenvalues ($\gamma_1 = \gamma_2 = 0$) and real coupling strengths ($\omega$), expressing the well-known result from standard quantum theory that two discrete states always avoid crossing [58]. This is true also for a non-Hermitian system with narrow resonance states, for which the $\gamma_i$ are small and $\omega$ is almost real. For broader resonance states, however, their non-zero widths may allow equations (6) and (7) to be satisfied. In this case, $\mathcal{H}^{(2)}$ actually gives rise to two EPs as can easily be seen from equation (5) by assuming imaginary coupling $\omega = i\omega_0$. The expression

$$
Z = \frac{1}{2} \sqrt{(e_1 - e_2)^2 - \frac{1}{4}(\gamma_1 - \gamma_2)^2 + i(e_1 - e_2)(\gamma_1 - \gamma_2) - 4\omega_0^2}
$$

(8)
will then be complex, and the condition $Z = 0$ can be fulfilled when $$(e_1 - e_2)^2 - \frac{1}{2}(\gamma_1 - \gamma_2)^2 = 4\omega^2_0 \quad \text{and} \quad (e_1 - e_2)(\gamma_1 - \gamma_2) = 0,$$ i.e. when $\gamma_1 = \gamma_2$ (while $e_1 = e_2$). In this case, two EPs appear. It furthermore holds that


textual content...
one point in the continuum, with a corresponding measure of zero, the width bifurcation that it gives rise to in its neighborhood determines the dynamics of OQSs (as will be discussed in section 4).

Finally in this section, we comment that the topology of an open system is different to that of a closed system. Importantly, the geometrical phase related to the EP of a two-level system differs from the usual Berry phase by a factor of two (see, e.g. [1]).

### 3.2. Non-Hermitian eigenfunctions in the vicinity of an exceptional point

Having discussed the behavior of the eigenvalues of the non-Hermitian Hamiltonian $H$ in the vicinity of an EP, in this section we discuss how the eigenfunctions of non-Hermitian operators also exhibit unusual properties near such a point. This novel behavior is a consequence of the very different nature of non-Hermitian eigenfunctions, when compared to their Hermitian counterparts. This is apparent already for the two-level system described by equation (4), whose eigenfunctions (and eigenvalues, $\epsilon_{1,2}$) exhibit a global character arising from many-body forces induced by the coupling $\omega_{12}$. As discussed already, this environmentally-mediated interaction not only occurs between pairs of different states, but may also involve self-energy terms.

To discuss the properties of non-Hermitian eigenfunctions we begin from the most fundamental viewpoint, noting that the eigenfunctions of every Hamilton operator $H$ must fulfill the conditions $H|\Phi_k\rangle = \epsilon_k|\Phi_k\rangle$ and $\langle \Psi_l|H = \epsilon_l|\Psi_l\rangle [61, 62]$. Here, $\epsilon_k$ is an eigenvalue of $H$ and the vectors $|\Phi_k\rangle$ and $|\Psi_l\rangle$ denote its right and left eigenfunctions, respectively, which may, in principle, differ from one another. When $H$ is a Hermitian operator, the $\epsilon_k$ are real and we arrive at the well-known relation $|\langle \Psi_l|H = \langle \Phi_k\rangle|$. This then allows the eigenfunctions to be normalized using the functional $\langle \Phi_k|\Psi_l\rangle$, whose value is real. The normalization is made possible by the orthogonal character of the Hermitian eigenfunctions, which describe stationary states whose overlap integrals vanish perfectly if $k \neq l$. In such systems, it is therefore only by introducing an external perturbation that a coupling may arise between the different eigenfunctions, allowing for transitions between their associated states.

Moving now to the non-Hermitian case, specifically that of the eigenfunctions of the symmetric Hamiltonian $H^{(2)}$, the situation is very different. The environmentally-induced coupling that is responsible for the non-Hermitian nature of the eigenstates results in them developing an intrinsic overlap, implying that their eigenfunctions are no longer orthogonal. Rather, they exhibit the characteristic of biorthogonality, and should be normalized by computing the functional $\langle \Phi_k|\Psi_l\rangle$. Formally, this condition follows from the fact that, for the symmetric non-Hermitian Hamiltonian equation (4), we have $|\langle \Psi_l|H = \langle \Phi_k\rangle|$. In marked contrast to the Hermitian case, the value of this functional is, generally, complex (for further details see sections 2.2 and 2.3 of [1]). To smoothly describe the transition from a closed system with discrete states, to a weakly-open one with narrow resonance states, we normalize the $\Phi_k$ according to

$$\langle \Phi_k^\dagger|\Phi_l\rangle = \delta_{kl}. \quad (12)$$

Related to the normalization condition equation (12) is the notion of the phase rigidity of the non-Hermitian

\[ \text{Figure 8. Landscape of the } S \text{ matrix elements } |S_{kl}(E)| \text{ (elastic scattering channel) over the complex energy plane in the case of four resonances at low (top left) and high (top right) level density. Quite generally, the } S \text{ matrix relates the initial and final states of a physical system whose particles undergo scattering. The different matrix elements } S_{kl} \text{ define the scattering from initial channel } k \text{ to the final channel } l, \text{ and the poles of the } S \text{ matrix in the complex plane are identified with the presence of resonance states. The top-left panel corresponds to the situation of low level density, in which all four resonances can be distinctly identified from their corresponding poles. The top-right and bottom panels, on the other hand, are for high level density where only a single, broad, resonance remains. The bottom panel is an enlarged view of the area enclosed approximately by the red line in the upper-right panel for high level density. The enlarged view reveals that the three very narrow resonances are “quenched” by the broad one. Re}(E) \text{ and } \text{Im}(E) \text{ are given in MeV, and the contour line distance is } 0.5 \text{ MeV (top) and } 0.1 \text{ Mev (bottom). Figure reproduced from [60]. With kind permission from Springer Science and Business Media.} \]
eigenfunctions, which is expressed mathematically according to

$$ r_k \equiv \frac{\langle \Phi_k^0 | \Phi_k \rangle}{\langle \Phi_k | \Phi_k \rangle}. \quad (13) $$

For Hermitian systems whose eigenfunctions are real ($\Phi_k^0 = \Phi_k$), the phase rigidity is equal to unity. As noted already, this expresses the fact that these eigenfunctions are orthogonal to one another, and do not experience any coupling. Almost the same is also true for weakly decaying systems, for which one has well-separated resonance states. Under such conditions Hermitian physics still represents a reasonable approximation to the description of the open system, and different eigenfunctions remain almost orthogonal.

As a variation of the system parameters is used to approach an EP, the phase rigidity of the eigenfunctions undergoes dramatic changes that reflect their strongly-coupled character near this point. Most significantly, it is now understood from analytical studies, as well as from numerics and experiment [1, 9], that, in the vicinity of the EP, the eigenfunctions only differ from one another by a phase factor

$$ \Phi_1 \to i \Phi_2^*; \quad \Phi_2 \to -i \Phi_1^*. \quad (14) $$

As the EP is approached, it follows from the normalization condition of equation (12) that $\langle \Phi_k | \Phi_k \rangle \to \infty$ so that the phase rigidity $r_k \to 0$. In other words, the relative phase of the two eigenfunctions changes dramatically when the crossing point is approached. This non-rigidity of the phases follows directly from the fact that $\langle \Phi_k^0 | \Phi_k \rangle$ is a complex number (in difference to the norm $\langle \Phi_k | \Phi_k \rangle$, which is real) so that the normalization condition (12) can be fulfilled only by the additional requirement $\text{Im}(\Phi_k^0 | \Phi_k) = 0$ (corresponding to a rotation away from the complex plane). As noted already, this causes the different states of the system to develop a coupling through the continuum, the influence of which is large when $r_k < 1$. Thus we see that the biorthogonality of the eigenfunctions $\Phi_k$ causes perceptible physical effects in the neighborhood of an EP.

Generally speaking, the phase rigidity takes values between zero and one, with a value of $r_k = 1$ obtained for Hermitian systems with real eigenfunctions. Near an EP in a non-Hermitian system, however, the two eigenfunctions are no longer orthogonal but are rather linearly dependent (according to equation (14)) so that $r_k \ll 1$. This non-rigidity of the eigenfunction phases is the most important difference between Hermitian and non-Hermitian eigenfunctions. From a mathematical perspective, the lack of phase rigidity near the EP leads very naturally to the appearance of nonlinear effects in OQSs [1]. Physically, it allows one of the states of the Q system to align with the common environment, by which we mean that the matrix element governing the overlap of this eigenfunction with the wavefunctions of the $P$ subspace becomes very large [1]. (This is not to be confused with achieving a specific alignment of the energy of the state.)

Because of its strong coupling to the environment, the aligned state is extremely short lived and therefore corresponds to the strongly broadened state that emerges from the aforementioned width bifurcation (compare figure 8). In the presence of this broadening, the remaining states decouple from the environment, essentially becoming trapped within the localized region. This process is reflected also in the eigenfunctions $\Phi_k$ of $\mathcal{H}^{(2)}$. In order to show this, we represent the eigenfunctions $\Phi_k$ by using the set of basis wavefunctions $\Phi^0_k$ of $\mathcal{H}_0$ (corresponding to vanishing $\omega$),

$$ \Phi_k = \sum_{i=1}^{N} b_{kl} \Phi^0_i; \quad b_{kl} = |b_{kl}| e^{i \theta_{kl}}, \quad (15) $$

where the mixing coefficients $b_{kl}$ are normalized according to the biorthogonality relations for the wavefunctions $\{ \Phi_k \}$. The phase angle $\theta_{kl}$ that appears in this expression is related to the coefficients $b_{kl}$ according to $\tan(\theta_{kl}) = \text{Im}(b_{kl})/\text{Re}(b_{kl})$. From the normalization condition (12), it may then be inferred that $|b_{kl}| \to \infty$ on approaching an EP [1]. In other words, we see that the entanglement of the wavefunctions is large in the neighborhood of an EP, e.g. [63, 64].

As an illustration of the ideas above, in figure 9 we plot the two eigenvalues of $\mathcal{H}^{(2)}$ in the vicinity of their expected crossing (see panel (a)). The eigenvalues are plotted as a function of the parameter $d$, the distance between the two unperturbed energies $e_1$ and $e_2$. Panel (a): Eigenergies, $E_1$, Panel (b): Level widths, $\Gamma/2$. Panel (c): Phase rigidity, $r_r = r_2$. Panel (d): Mixing coefficients $|b_{kl}|$. The calculations assume the following parameters: $e_1 = 2/3$ and $e_2 = 2/3 + d$ (the dotted lines in panel (a)); $\gamma_2/2 = -0.5; \gamma_2/2 = -0.5; \omega = 0.05i$. Figure reproduced from [65].
below one in the vicinity of the two EPs, reflecting the associated mixing of the eigenfunctions. Confirming this idea, the mixing coefficients (panel (d)) show a strong increase at the EPs. Particularly interesting is the behavior near \( d = 0 \), where the two eigenfunctions remain strongly mixed (1 : 1) but the phase rigidity nonetheless approaches unity. That is, the two eigenfunctions are (almost) orthogonal near \( d = 0 \), an extremely surprising result but one that is straightforward to understand. Essentially, it reflects nothing more than the fact that the system has separated into short- and long-lived parts, which exist independent of one another at different time scales. This point is discussed in more detail in [59, 64].

The wavefunctions associated with the long- and short-lived states that form in-between the two EPs cannot be analytically connected to those of the original eigenstates. Rather, as \( d = 0 \) is approached from the direction of either increasing or decreasing \( d \), the system breaks into two components that bear no resemblance to those at \( d \neq 0 \). Consequently, it is impossible to continuously trace the evolution of the system from \( d < 0 \) to \( d > 0 \) (or vice versa), since each such a path would cross a region that must be described by a Schrödinger equation with non-linear terms and analytically unrelated eigenfunctions. It is interesting to note that the two eigenfunctions of \( H^{(2)} \) are never exactly orthogonal. Instead, what we observe is that they become almost orthogonal when the two levels are either distant in energy or distant in lifetime [59, 64, 65], compare figure 9, panels (c) and (d).

The relationship between the real and imaginary parts of the eigenfunctions near an EP according to equation (14), as well as the associated reduction in phase rigidity, have been seen in experimental investigations of microwave billiards [66] (see, also, the related discussion in [1, 9]). Elsewhere, in a recent theoretical study [67], a basic model for wave transport in open ballistic cavities was developed. The model makes use of imaginary potentials which mimic, for example, the contacts to the source and drain of a quantum dot. By investigating the properties of the eigenfunctions for different strengths of the complex potential the predictions of equation (14) could be confirmed. The phase rotation expected from the reduction of phase rigidity near the EP can be clearly seen in panels (b) and (c) of figure 10, in which the real and imaginary components of two nearby eigenstates are ‘swapped’ relative to one another, in accordance with equation (14).

Before concluding this section, we note that the notion of phase rigidity has also been exploited in discussions of the wavefunction statistics of open chaotic billiards [68]. The connection of the rigidity to the analysis of EPs was not considered in that work, however.

3.3. Gain in non-Hermitian systems

Thus far we have discussed the coupling between the \( P \) and \( Q \) subspaces as a means of escape (or loss), as is typical of most quantum systems found in nature. Recently, however, there have been theoretical [69] and experimental [70] studies of non-Hermitian systems that are capable of absorbing particles from the environment. Experimental work in this area is still in its early stages and has focused largely on the use of photonics [70], where the analogy of the optical wave equation to the Schrödinger equation [71] can be exploited. These studies reveal the manifestations of EPs in non-Hermitian systems, as shown in [72, 73] by making use of the projection operator formalism described above. The important role played by EPs in non-Hermitian systems with gain can also be found in many other papers and we simply refer the reader here to [74]. We emphasize, therefore, that the non-Hermitian formalism can not only be applied to the description of decaying systems (which exist in nature), but also to systems that are able to absorb particles from the environment. Having said that, it should also be made clear that the projection operator formalism cannot directly be applied in optics, where the equivalence of the optical wave equation to the Schrödinger equation must instead be utilized.

3.4. The scattering matrix in the vicinity of an exceptional point

In situations where one is interested how an EP influences transmission (or transitions) within an open system, it is natural to consider the structure of the \( S \)-matrix. In the simplest problem of a single level that is coupled to just one channel, the \( S \)-matrix describes a Breit–Wigner resonance,

\[
S = 1 + \frac{i \Gamma_{1,2}}{E - E_{1,2} - \frac{1}{2}i \Gamma_{1,2}}. 
\]  

(16)

where \( E_{1,2} \) and \( \Gamma_{1,2} \) are defined in (5) with \( \Gamma_{1,2} \leq 0 \) for decaying states. This expression can be rewritten as [75]

\[
S = \frac{E - E_{1} + \frac{1}{2}i \Gamma_{1}}{E - E_{1} - \frac{1}{2}i \Gamma_{1}}. 
\]  

(17)

which is explicitly unitary when the energy dependence of \( E_{1} \) and \( \Gamma_{1} \) is properly taken into account [1] (see section 2.2 and appendix). If we now extend this problem to describe the behavior of two resonance states (with energies \( E_{1} \) and \( E_{2} \) that are nearby in energy, and coupled to a common continuum, the unitary representation of the \( S \)-matrix reads (up to a background term)

\[
S = \left( \frac{E - E_{1} + \frac{1}{2}i \Gamma_{1}}{E - E_{1} - \frac{1}{2}i \Gamma_{1}} \right) \left( \frac{E - E_{2} + \frac{1}{2}i \Gamma_{2}}{E - E_{2} - \frac{1}{2}i \Gamma_{2}} \right). 
\]  

(18)

It was mentioned already that the \( S \)-matrix has a double pole at the EP. In order to explicitly illustrate this point, equation (18) may be rewritten [75] to read

\[
S = 1 + 2i \frac{\Gamma_{d}}{E - E_{d} - \frac{1}{2}i \Gamma_{d}} - \frac{\Gamma_{d}^{2}}{(E - E_{d} - \frac{1}{2}i \Gamma_{d})^{2}}. 
\]  

(19)

where \( E_{d} = E_{2} \equiv E_{d} \) and \( \Gamma_{1} = \Gamma_{2} \equiv \Gamma_{d} \). The second term on the right-hand side of this expression is the usual linear term obtained for a single state (see equation (16)), albeit multiplied by a factor of two since the EP involves the coincidence of two states. The third term is quadratic in the energy and once again reflects the presence of the double pole at the EP.
This term is manifested through the appearance of an interference minimum in the $S$-matrix (more precisely, in the quantity $1 - S^2$), at the energy of the EP and, in the presence of which, the $S$-matrix appears to consist of two well-separated resonance states (with asymmetric lineshapes).

This latter point is illustrated in figure 11, in the insets of which we plot the energy-dependent variation of $1 - S^2$ (which determines the total cross section with two eigenstates of $H$) for three different values of the coupling strength (indicated). For weak environmental coupling ($\alpha = 0.8$, lower-right inset), such that there is little mutual interaction between the two levels that ultimately give rise to the EP, the $S$-matrix contains two isolated resonances with Breit–Wigner form. As the coupling to the environment is increased ($\alpha = 1$, upper-right inset), however, the $S$ matrix approaches the form of equation (19) and the two resonances develop an asymmetric line shape while avoiding overlap with one another. Ultimately, in the limit of strong coupling ($\alpha = 4$, upper-left inset), the $S$-matrix exhibits a broad resonance as a function of energy, with a narrow dip located at its center. This behavior is reminiscent of the features produced by width bifurcation, which gives rise to a long-lived resonance state together with a much broader (short-lived) structure (figure 11, main panel).

The discussion of figure 11 reveals several noteworthy aspects of the $S$-matrix and its behavior near an EP. On the one hand, we see that the influence of the EP is felt over a wide range of parameter space that encompasses this point. This very property, however, makes it difficult to unambiguously identify the specific parameter value at which the EP occurs. In order to establish the presence of an EP it is therefore necessary instead to study the spectroscopic properties of the system (such as the energy and width of the resonance states), and to trace the biorthogonality ($\langle \Phi_2 | \Phi_2 \rangle \to \infty$) or phase rigidity ($\tau_2 \to 0$) while approaching the EP. Since this information is typically not directly accessible in many experimental studies, such as those performed on microwave billiards and open QDs, the importance of EPs for the dynamical properties of OQSSs does not appear to have been widely appreciated to date.
Extending the discussion above to the case of two open decay channels, as required for transmission between an entrance and exit channel, the resonance-reaction part of the S-matrix (with N resonance states) reads [1]

\[-S^{(2)}_{c'} = -i \sqrt{2\pi} \sum_{k=1}^{N} \langle \chi'^{kj}_{c'} | V_{pq} | \Omega_k \rangle \cdot \frac{\gamma_{kc}}{E - \xi_k} \].

(20)

Here \( \Omega_k = (1 + G_p | V_{pq} | G_C) \Phi_k \) is the wavefunction of resonance state \( k \) and \( \chi'^{kj}_{c'} \) is that of the incoming wave in channel \( c' \). After some manipulation (see section 3.2 of [1]), equation (20) may be expressed as

\[-S^{(2)}_{c'} = -i \sum_{k=1}^{N} \frac{\gamma_{kc} \gamma_{kc'}}{E - \xi_k} \],

(21)

where the partial width amplitude \( \gamma_{kc} = \sqrt{2\pi} \langle \chi'^{kj}_{c'} | V_{pq} | \Omega_k \rangle \) is related to the incoming wave in channel \( c' \) while \( \gamma_{kc} = \sqrt{2\pi} \langle \Phi_k | V_{pq} | \chi'^{kj}_{c'} \rangle \) is due to the outgoing one in channel \( c \). Both the numerator and denominator in this summation are energy dependent (see section 2.2 and appendix) and \( \gamma_{kc} \) shows a resonance-like variation at the EP [75]. While equation (21) contains only the product \( \gamma_{kc} \gamma_{kc'} \), the structure of equation (20) indicates that one of these factors represents the excitation of the (extended) resonance state \( \Omega_k \) and the other one the decay of the (localized) eigenstate \( \Phi_k \). In the case where \( c = c' \), equation (21) describes the Breit–Wigner lineshape (equation (16)) of a single resonance with \( \gamma_{kc}^2 = \Gamma_k \). (The relation \( \gamma_{kc}^2 = \Gamma_k \) should not be confused with the relation \( \gamma_{kc'}^2 = \Gamma_k \) considered in random matrix theory).

By making use of a discretized (tight-binding) scheme, equation (21) for the resonance part of the S-matrix can be applied to describe the transmission through a quantum dot [76, 77]

\[ I_{c'} = -i \sum_{k=1}^{N} \frac{\gamma_{kc} \gamma_{kc'}}{E - \xi_k} \]

(22)

with two channels \( c \) and \( c' \) that differ dependent upon the geometry of the two attached leads. This expression is identical (for details see [78]) to that obtained by Datta [79], who partitioned the overall Green’s function into two submatrices \( (G_p \& G_C) \) with symmetric coupling \( G_{pc} \) and \( G_{cp} \)

\[ \left[ \begin{array}{cc} G_p & G_{pc} \\ G_{cp} & G_C \end{array} \right] = \left[ (E + i\eta)I - H_p - \tau_p \right]^{-1} \left[ \begin{array}{cc} \tau^+_p & -H_C \end{array} \right]. \]

(23)

Here, the matrix \( (E + i\eta)I - H_p \) represents the isolated lead, while the matrix \( E - H_C \) represents the isolated dot (with the unit matrix \( I \) in both cases). In the tight-binding approach, the coupling matrix between lead and cavity is non-zero only for adjacent points \( i \) and \( p_j \), Correspondingly, the coupling is denoted by \( \tau_p \) and \( \tau^+_p \) in (23) and \( \tau_p(p_j, i) = t \). The advantage of using equation (22) is the direct relation of this expression to the spectroscopic values (eigenvalues and eigenfunctions of \( \mathcal{H} \)) which gives us the possibility of studying the influence of EPs. For the purpose of illustration, in figure 12 we present results obtained for a tight-binding model of a one-dimensional box (of \( N \) sites), connected at either end to a pair of leads

\[ \mathcal{H} = - \sum j |j | (j + 1) + \text{c.c.} \]

(24)

The index \( j \) in this expression runs over all sites of the system, including the leads at the ends of the box, and the calculations in figure 12 are performed for different values of the coupling strength \( \langle v_{pq} \rangle \) between the box and these leads. In the left plot of panel (b), the transmission shows that this coupling gives rise to a shift in the energy of the various resonances of the box. The right figure of panel (c) shows clear evidence of width bifurcation, which occurs beyond a critical value of the coupling strength. In the vicinity of this value, two broad resonances `disappear` from the transmission window, leaving just three resonances as shown in the right plot of panel (b).

Very similar results to those of figure 12 have been obtained in studies of the relation between steady-state currents and resonance states in molecular junctions [80]. In that work it was shown that, in the weak-coupling regime, the current through...
such junctions is proportional to the sum of the resonance widths, and initially increases with increasing coupling. With further increase of the coupling strength, however, the current was eventually found to saturate before decreasing for even stronger coupling. The decrease was associated with a bifurcation of the widths of the different molecular states, forming long-lived resonances that dominate the transmission, and broad ones associated with localized states at the electrodes. As in the case of figure 12, the nature of the transport could be shown to be determined by a parameter that measures the degree of ‘overlap’ among the various resonances of the system. Developed for a molecule consisting of five sites, the results of this study revealed many features in common with those of figure 12, which were similarly obtained for a QD consisting of five sites.

4. Dynamical phase transitions

As we have seen in section 3, interesting and unexpected behavior arises in OQSs when their states undergo a mutual interaction that is mediated by a common environment. At the heart of this behavior is the existence of EPs, which give rise to width bifurcation over a non-zero parameter range. While such behavior is counterintuitive from the standpoint of Hermitian physics, even richer behavior yet may be obtained under conditions where more than two resonance states undergo a similar, environmentally-mediated, interaction. As we discuss here, the environment may establish a global interaction among such states, endowing them with a highly correlated character. An important phenomenon that arises under such circumstances is that of a dynamical phase transition.

\[
\begin{align*}
\text{(a)} & \quad j = -2 \quad -1 \quad 0 \quad 1 \quad 2 \ldots \quad N - 1 \quad N \quad N + 1 \quad N + 2 \quad N + 3 \\
\text{(b)} & \quad \text{TRANSMISSION} \\
\text{(c)} & \quad \text{Im}(z_k) \\
\end{align*}
\]

Figure 12. (a) One-dimensional model for quantum transport according to equation (24), \( j \) runs over all sites of the system, including left and right leads and the one-dimensional box consisting of \( N \) sites. The hopping matrix elements between leads and the box are \( t_j = v_L \) if \( j = 1, t_j = v_R \) if \( j = N \) and \( t_j = 1 \) otherwise. (b) Transmission probability versus wave number of the incident particle. The left panel is for \( N = 5 \) and \( v_L = v_R = 0.5 \), while the right panel is for \( N = 5 \) and \( v_L = v_R = 2 \). The position of the eigenvalues of the closed system \( (v_L = v_R = 0) \) are indicated by the red filled symbols. The shifts of the transmission peaks relative to the red symbols are caused by the coupling of the box to the leads. (c) Real and imaginary parts of the five eigenvalues \( z_k \) of \( \hat{H} \), as a function of \( v_L (=v_R) \) at \( E = 1 \). The resulting shift of the energies (left panel) and width bifurcation (right panel) are due to the coupling of the box to its leads. At a critical value of \( v_L \), a DPT occurs as two of the states separate in energy and width from the other three. Below the DPT transmission is mediated by five resonances but beyond it it is governed by just three (see panel (b)). ©IOP Publishing. Reproduced from [76] by permission of IOP Publishing. All rights reserved.
(DPT), according to which the spectroscopic features of the system change radically as a function of some suitable control parameter. In this section, we introduce the concept of the DPT by exploring the properties of systems consisting of three or more resonance states, from which we will see that the picture of a DPT quickly emerges. Our discussion is formulated initially for a one-channel picture, appropriate for problems such as nuclear decay. We also discuss, however, the two-channel problem appropriate to mesoscopic transport, as well as the multi-channel one for which approximation methods are often required. We conclude this section by discussing the manifestation of DPTs in microwave cavities and open QDs, in single-electron transistors, atomic ensembles, and quantum systems with gain and loss.

4.1. Environmentally-mediated interaction of several resonance levels

Thus far we have discussed the influence of a single EP on the dynamics of an OQS, by considering the environmentally-mediated interaction between just two quantum states. In this section, however, we extend this discussion to treat the behavior of systems containing larger numbers of states. The simplest case is that in which just one extra state is close to an EP, a problem that was explored recently in [64]. There it was shown that the system exhibits all of the characteristic features that arise from the influence of an EP. Consistent with our discussions in section 3, the phase rigidity of the eigenfunctions is found to be reduced in a finite parameter range, where significant nonlinearities also appear. Under such conditions, the three states exhibit strong mixing, or entanglement, losing their individual character once the distinct regions of influence of the various EPs overlap with one another. As a result of the loss of their individual spectroscopic features, it is not possible for more than two states to coalesce at a single point. Instead, a unique spectroscopic redistribution occurs in order to achieve a dynamical stabilization of the system. According to this process, the system evolves so that its total coupling strength to the environment is mediated (in the one-channel case) mostly via a single state, while the remaining states become largely decoupled from the environment. These effects persist when the number of interacting states is increased beyond three and it is this phenomenon that is responsible for the width bifurcation observed in figure 8. It should be underlined here that this process of stabilization occurs globally, with the participation of all states of the system, and that, in the two-level case, it corresponds to nothing more than the width bifurcation discussed in sections 3.1 and 3.2. The behavior exhibited by many interacting states has come to be referred to as a DPT.

In the discussion that follows, we show that the common feature of a DPT is the presence of EPs. This can be seen already, of course, in the two-level problem involving just a single EP. In the multi-level case it is harder to establish, however, since the role of the EPs is concealed by the overlap of their regions of influence, and the impossibility of analytically determining the position of the various EPs. However, numerical calculations clearly show that, with many closely-neighboring states present, EPs are responsible for the spectroscopic reordering processes that ultimately cause the DPT (when $\text{Im}(\omega) \gg \text{Re}(\omega)$) [64]. According to these results, in the region of the DPT, the non-Hermitian Hamiltonian of equation (1) can be quite well approximated as

$$\mathcal{H} = H^0 - i\alpha V V^+, \quad (25)$$

where $V V^+$ denotes the residuum of equation (3) and the principal value integral of equation (2) can be neglected due to its small value near the DPT. There are many reports (see, for example [11, 22, 60, 61]) that make use of the Hamiltonian of equation (25), with $\alpha$ treated as a control parameter. Starting from an initial distribution of resonance levels, equation (25) can then be used to describe the main features of the spectroscopic reordering over the critical range of $\alpha$. An important result is the suppression of avoided level crossings, which are replaced instead by non-avoided crossings and accompanying width bifurcation. At the same time, the quantum dynamics on either side of this transition becomes non-analytically connected.

To explicitly highlight the connection of the DPT to more conventional ones, we may consider the example of the ferromagnetic phase transition known to appear in an infinite system. Here, the control parameter is temperature, the critical parameter is the Curie temperature, and the material undergoes a magnetic-ordering transition on passing through this point. Obviously, the non-ordered paramagnetic state that is observed above the Curie temperature provides no indication of the magnetic order that awaits on passing through this critical point, and it is this sense that these two different ground states are non-analytically connected.

Although the non-analytical character of the eigenvalues of a non-Hermitian operator is apparent on passing through a DPT, the question of whether this is a true phase transition, in the sense of thermodynamics, is nontrivial. In [81], this question was addressed by making use of the Hamiltonian of equation (25) and varying the parameter $\alpha$. Both on the basis of analytical theory, and of numerical simulations, it was found that the spectral redistribution associated with the DPT may indeed be viewed as a second-order phase transition. More specifically, it was shown that the reorganization of the spectrum occurs globally, over its entire range, at a critical parameter value. Such behavior may be understood as arising from the fact that all states act cooperatively, causing the degree of non-Hermiticity of the Hamiltonian to diverge.

Thus far we have discussed the manifestation of DPTs in one-channel systems, in which the appearance of the phase transition can be clearly traced. DPTs may also influence the transmission through small systems with, at least, two channels. Such systems have been investigated in several different works [4, 15, 76, 77, 82, 83], by combining tight-binding approaches together with the formalism described in the foregoing sections. A characteristic finding of these studies is an enhancement of the transmission due to the influence of EPs, as well of a correlation between reduced phase rigidity and enhanced transmission [4, 83]. For illustration, in figure 13 we show numerical results obtained for the conductance and exit channels at its opposite ends.
The conductance and phase rigidity exhibit a clear anticorrelation with regards to one another (see figure 13(c)), indicating that high transmission is associated with low phase rigidity. Further examples of this anticorrelation can be found in [4].

Apparent in figure 13(b) is a broad plateau in energy where the conductance is strongly enhanced, corresponding to a transition from stationary to traveling modes. This DPT occurs as the coupling strength between the chain and the two wires (v) is increased from zero to around one, but for larger values than this the conductance once again decreases [82].

Similar plateau-like features have also been found in numerical studies of the influence of whispering-gallery modes on the conductance of open quantum billiards [55]. Over the range of parameter space where the conductance reaches the plateau, the picture of transmission is very different from that which holds as v → 0 (the weak-coupling regime) or for v ≫ 1 (the strong-coupling regime, where the transmission arises from narrow resonances with a superimposed direct component). Near v = 1, the conductance depends only weakly on the spectral details of the closed system. It also cannot be described by a master-type equation, due to the existence of EPs at which two states cross and the time evolution of the system becomes undefined.

While the details of the DPT can often be well described by equation (25), simply by variation of α and without explicit consideration of EPs, the enhanced transmission that occurs in the critical parameter range can only be understood by addressing the influence of these special points on the DPT.

4.2. Coupling of a multi-level system to many channels

While we have thus far focused on the emergence of EPs and DPTs in systems that contain a small number of channels, in this section we briefly discuss the corresponding behavior exhibited by a multi-level system that is coupled to many open channels. We have learned already from the study of several
different open systems how, at some critical value of the coupling strength, short-lived collective modes arise together with long-lived trapped ones. The critical coupling strength at which this occurs is determined by \( \Gamma / (\mathcal{K} \mathcal{D}) \approx 1 \), where \( \Gamma \) denotes the average width of the states, \( \mathcal{D} \) is their average energy separation, and \( \mathcal{K} \) is the number of open channels. This result holds true for decaying systems, such as nuclei \([1]\), and for microwave billiards with one lead whose width can be varied \([54]\). The collective states contain contributions in their wavefunctions from many states of the isolated system, including those lying at energies much more distant than the range covered by their widths. Due to the formation of these collective short-lived states, all relation to the original spectroscopic details of the system is lost. The larger the number of open channels, the larger the effect. Consequently, the properties of the long-lived states are typically described, in this case, by random matrix theory.

According to random matrix theory, each state of the system is characterized by just two numbers, the energy and width of the respective resonance state. As a consequence, an \( N \) level system may be described by different parameter sets. For example, the widths of all \( N \) states may be assumed to be similar to one another, or the width of one of the states may be assumed to be much larger than those of the \( N - 1 \) remaining states (corresponding to the so-called doorway picture of nuclear physics). In both cases, the system is described by means of 2 \( N \) parameters, and the results obtained by means of these two different parameter sets cannot be distinguished from one another, as was shown many years ago \([85]\). In order to infer the nature of the dynamics, it is necessary to obtain additional information by parametrically controlling the system properties. For this purpose, the imaginary part of the coupling \( \omega \) between the system and environment is usually varied over a large range of \( \alpha \) in equation \((25)\), while leaving the real component unaltered \([28]\). This approximation provides good results when the number of channels is large since the formation of collective short-lived states is a very robust effect.

The situation is very different when the system is described as being open from the very beginning, as in the formalism described in sections 2 and 3. In such a case, the energies and widths of the states are not numbers but are rather energy-dependent functions (since, as described in sections 2 and 3 and in appendix, they are eigenvalues of the energy-dependent Hamiltonian \( \mathcal{H} \)). While the results of \([85]\) are not applicable in this case, control of the system dynamics by some appropriate parameter is nonetheless necessary in order to draw reliable conclusions on the dynamics. For this purpose, both \( \text{Re}(\omega) \) and \( \text{Im}(\omega) \) can be varied individually, allowing the transition from level repulsion of well-separated levels (caused by \( \text{Re}(\omega) \), according to equation \((2)\)) to width bifurcation of closely-neighboring states (caused by \( \text{Im}(\omega) \), according to equation \((3)\)) to be traced. An example of such a parametric variation is shown in figure 14, for a system with only one channel.

Before concluding this section, we remark on the significance of the averaged quantity \( \bar{\Gamma} / \bar{\mathcal{D}} = \frac{1}{N} \sum_k (\Gamma_k / \mathcal{D}_k) \), which is often taken to quantify the ‘degree of resonance overlap’ in an OQS. In the presence of a DPT, this quantity is only physically meaningful when the sum over states used to determine the average broadening includes contributions only from the long-lived states, with the influence of the short-lived states produced by width bifurcation excluded. Under such constraints, \( \frac{1}{N} \sum_k (\Gamma_k / \mathcal{D}_k) < 1 \) is consistent with the presence of well separated resonances, not only below the DPT but also above it. This point was appreciated more than a decade ago, in studies of the relationship between the decay rates and widths of multi-level systems with closely-neighboring states \([86, 87]\). The essential point here is that, beyond the DPT, the decay rates and widths of the long-lived states both saturate, and the fundamental quantum-mechanical connection between these quantities therefore continues to hold.

### 4.3. Experimental observations

In spite of the important role that DPTs can play in governing the dynamical behavior of open systems, it is only recently that their significance has begun to be appreciated. With the benefit of such hindsight, the role of this transition in governing the dynamics of long-studied open systems becomes apparent. A good example is provided by discussions from nuclear physics, where, for many years, it was thought that the complex behavior of the heavy, so-called compound, nuclei could not be described by the existence of individual (shell-model) states, in contrast to the well established applicability of such concepts to atoms. This issue was highlighted by Bohr \([88]\) in an address to the Copenhagen Academy on January 27th, 1936, where he stated: ‘In the atom and in the nucleus...’
we have indeed to do with two extreme cases of mechanical many-body problems, for which a procedure of approximation resting on a combination of one-body problems, so effective in the former case, loses any validity in the latter". What Bohr was unaware of at that time, however, is that nuclear spectra exhibit a very characteristic transition as a function of level density and excitation energy; the low-lying states in light nuclei are in fact well described by a shell model [89], with most states being sharply-resolved, while the states in the heavy compound nuclei, lying far above the ground state, have more universal features. In this latter regime: ‘we, from the very beginning, have to do with essential collective aspects of the interplay between the constituent particles’ [88]. In fact, one finds a small number of collective states, the so-called ‘neutron resonances’, which coexist with many extremely long-lived compound-nucleus states. In other words, even in this earliest problem from quantum mechanics we see the characteristic features of a DPT! The main reason that the existence of this transition has largely gone unappreciated is due, as we have already noted, to the lack of external control in nuclear systems. In the discussion that follows, we present examples of DPTs from a variety of different OQSs. The common feature of these studies will be the capacity to subtly control the dynamics of the open system, thereby revealing the full character of the DPT.

4.3.1. Wave transmission in microwave cavities and open quantum dots. The first experimental verification of width bifurcation was provided in experiments performed on open microwave cavities [11]. In these studies, the equivalence between the electromagnetic spectrum of the cavities, and the eigenspectrum of their quantum-mechanical counterparts [90], was exploited, as a means to investigate the influence of environmentally-mediated level coupling. The cavity used in the experiments was of the Sinai type, and was attached to a single waveguide lead by means of an adjustable slit. By varying the width \( d \) of this slit, the coupling strength of the cavity to the decay channels in its lead could then be varied.

In the experiments of [11], the energies \( E_R \) and widths \( \Gamma_R \) of the various cavity resonances were obtained at a series of fixed coupling strengths to the lead (i.e. fixed \( d \)). The motion of the resonance poles was then traced as a function of the slit opening, starting from an initial condition close to the real axis (where the \( \Gamma_R \) are small). By increasing the slit opening, the energy and width of the resonances could then be followed continuously, into the regime of width bifurcation. The results of these studies are plotted in figure 15 and show clear evidence of resonance trapping. In this phenomenon, the width of each of the resonances initially increases as the slit is opened. Eventually, however, many of the resonances show a reversal of this behavior, with their widths decreasing as \( d \) is further increased (the reader should note that the linewidths are plotted on a logarithmic scale in figure 15). Figure 15(b) shows the variation of the linewidth of four resonances as a function of their eigenenergies, and the resonance trapping is indicated by a non-monotonic dependence of the \( \Gamma_R \) on \( E_R \). The shift of each eigenenergy to lower values with increasing \( d \) in this figure is induced by the weakening confinement as the slit is opened. In figure 15(c), the linewidth is plotted as a function of \( d \) for the four resonances. In each case, we see that the linewidth initially increases, before reaching a local maximum and then decreasing for further increase of \( d \).

As discussed in sections 3.1 and 4.1, resonance trapping (i.e. decoupling of some resonance states from the environment at strong coupling strength and closely neighboring states) is a characteristic feature of a DPT. A key detail of this phenomenon is that it is not expected to depend on the specific details of the microwave cavity [54]. Instead, the only requirement is that the level density, and the strength of the...
coupling to the environment, should be sufficiently strong to give rise to width bifurcation. Consistent with these ideas, investigations of quantum transport in open mesoscopic cavities [12, 13] have been found to reveal many features in common with those of the microwave studies. This is illustrated in figure 16, where we reproduce the results of numerical calculations from [13]. In this numerical study, the authors computed the eigenspectrum of closed cavities with geometries of the form shown in the top panel of figure 16(a), and compared the energy-dependent structure of this spectrum to that of the conductance obtained with the cavities opened to macroscopic reservoirs. The coupling between the isolated cavity and these reservoirs was implemented by making use of waveguide leads, a feature that provides a direct connection to experimental implementations of open dots [12] and to the microwave studies mentioned above.

In figure 16(b), the energy dependent variation of the QD conductance, computed via a transmission-matrix approach, is compared to a decomposition of the open-cavity probability density. This decomposition expresses the contribution of individual eigenstates of the closed cavity to the probability density, and shows a close correlation to the computed conductance (indicated by the dashed line). The latter shows three Fano resonances over the indicated energy range, which occur at energies that closely correspond to specific eigenenergies of the closed system. At these energies, the probability density of the open dot corresponds closely to that of individual eigenstates, as we indicate in the lower two panels of figure 16(a). These calculations were performed for two different values of the waveguide opening (w) and both show the signature of the same bouncing-ball eigenstate. In this situation, we therefore see that the probability density exhibits the signature of a single eigenstate of the closed dot, which also is manifested as the appearance of a resonance in the conductance.

To achieve the close correspondence between the properties of the open and closed systems, revealed in figure 16, proper choice of the closed counterpart of the open system was found to be essential. It will be noted in the upper panel of figure 16(a) that the closed system was taken to consist of a cavity with perturbing lead structures, rather than just simply being formed by an isolated stadium billiard. The eigenstates of this modified cavity differ from those of its simpler counterpart, and it is only when the leads are included into the cavity structure that the connection between closed and open systems becomes apparent.

The strong similarity between the behaviors exhibited by open quantum dots and microwave cavities is indicated in figure 16(c). Here we plot, as a grayscale contour, the variation of the QD conductance as a function of both energy and lead opening. Dark regions in the figure correspond to high conductance, from which identification it becomes clear that the conductance clearly shows the signature of narrow resonances, at least while the lead width is held small. As the leads are opened, however, several of the resonances broaden in linewidth and shift to lower energy, consistent with a weakening of the confinement as the cavity is opened. Other resonances, on the other hand, remain clearly resolved, neither shifting significantly in energy nor becoming broadened.

Figure 16. Manifestation of resonance trapping in open QDs. (a) The upper panel shows the hard-walled geometry of the QD, whose geometry is modified by the addition of the two “lead” structures. The bottom two panels show the calculated probability density of the open dot, at the same energy but for two different values of the lead opening (w). (b) Comparison of the energy-dependent variation of the QD conductance (dashed line, right axis) with the decomposition of the probability density of the open system (solid line, left axis). Energy is plotted in units of the average level spacing of the cavity (Δ). (c) Grayscale contour showing conductance of the open QD as a function of energy and cavity opening. Darker coloring indicates higher conductance. Reprinted with permission from [13]. Copyright 2002, AIP Publishing LLC.

These features appear reminiscent of the resonance trapping discussed above for microwave cavities, with the key feature being that the linewidth of eigenstates of the closed dot evolves in a strongly state-specific manner as the system is opened. In this effect, the coupling of the cavity eigenstates to the environment is not the same for all states, but rather depends on the specific structure of the individual eigenfunctions. In [13] it was demonstrated that the states least affected by the environmental coupling are, quite naturally, those whose eigenfunctions have the smallest probability density in the regions where the leads are added. In [12], by varying the relative orientation of the two leads, it was shown experimentally that
this characteristic could be exploited to ‘select’ which states participate in transport. In spite of the fact that the experimental work on microwave cavities and quantum-dot transport described above took place over a similar time period, the interpretation of their results was framed in terms of very different vocabulary. While the resonance-trapping phenomenon was emphasized in the microwave studies, a different formulation was emphasized for the QDs. This involved connecting the observed behavior to the notion of \textit{einsel} 

\[ \beta = \int \langle T|e^{i\beta} \rangle \] 

formulated in \cite{19}. In this interpretation, the interaction between a closed system and its environment is viewed as giving rise to a decoherence process in which those states of the closed system that couple strongly to the environment are quickly washed out. This in turn leaves other, orthogonal, \textit{pointer states} that are characterized by a strong robustness in the presence of this coupling (for recent reviews of these ideas, see \cite{2}). In subsequent work, Brunner et al. extended these ideas to investigate the nature of transport in QD \textit{arrays}, in which the environment of any single dot is determined by its coupling to its neighbors \cite{92, 93}. In a certain sense, these studies may be viewed as exploring how to ‘engineer’ the environment of the system, by forming it from a small number of discrete states. In \cite{93}, it was shown that the introduction of such an environment may actually give rise to \textit{new} states, not associated with either the \textit{P} or \textit{Q} subspaces alone. Arising instead from the coupling between the subspaces, these so-called \textit{bipartite} pointer states are associated with an environmental interaction that appears to go beyond the projection-operator formalism that we have considered here.

Most recently, evidence of resonance trapping was found in a experimental study of quantum-chaotic resonators, realized from two-dimensional photonic crystals \cite{94}. These experiments were performed on microstructured cavities, whose design was tailored to achieve strong transparency without absorption losses. Light entering the resonator could then be used to excite cavity resonances that decay due to the coupling of the system to the environment. For sufficiently small losses, the resonances are weakly coupled to the environment and are therefore found to decay independent of one another. Above a certain loss threshold, however, a DPT occurs as the resonances interact with one another to form a collective set of states. Beyond this transition, the resonance widths and (frequency) eigenvalues extracted from the transmitted power-density spectrum clearly show a background that arises from a large number of long-lived modes, together with a smaller number of states with very short lifetimes. Consistent with the examples described above, this phenomenon was found to be very robust and could be described without explicit consideration of EPs. This last point may actually give rise to the projection-operator formalism that we have considered here.

\textit{4.3.2. Phase lapses in multi-level quantum dots.} The next example of a DPT is provided by measurements of the phase change during tunneling in few-electron QDs \cite{17–19}. In experiments of this type, the focus is on understanding the evolution of the phase \((\beta)\) governing electron transmission \((T = |T|e^{i\beta})\) as the energy is swept through an individual resonance. To determine the phase, the QD is embedded in one of the arms of an Aharonov–Bohm interferometer \cite{95}, as indicated in figure 17(a). In the absence of a QD, such ring structures exhibit periodic magneto-conductance oscillations, arising from the coherent interference between electron partial waves that propagate along the two separate arms of the ring \cite{96}. Insertion of the QD into one of these arms now adds an additional phase shift to partial waves propagating through that arm, an effect that can be detected directly as a shift in the phase of the magneto-conductance oscillations. In a series of beautiful experiments, Heiblum and co-workers have used this approach to investigate the evolution of the phase \(\beta\) associated with individual resonances. Early experiments \cite{17, 18} were performed on relatively large QDs, and revealed a surprising universality in the phase evolution, independent of the size, shape and occupancy of the dot. According to this behavior, the transmission phase was found to increase monotonically by \(\pi\) on passing through each resonance, before returning suddenly to its initial value (figure 17(c)).

In contrast to the ‘universal’ phase behavior described above, more recent measurements of the phase evolution in smaller QDs reveal very different behavior \cite{19}. When the dot contains more than ten or so electrons, the universal phase evolution is observed, as in earlier experiments. For smaller electron occupancy, however, a crossover to a so-called ‘mesoscopic’ regime occurs, in which the evolution of the phase becomes strongly specific to the particular state involved (see figure 17(b)). In spite of much effort, the crossover from the mesoscopic to the universal regime with increasing electron number could not be explained within the framework of standard, Hermitian quantum mechanics \cite{20}. In \cite{21}, for example, Oreg determined the QD transmission probability and tunneling phase by means of a noninteracting toy model, whose key assumption was that the QD spectrum consists of a ladder of narrow states and a single broad resonance. Using this ad hoc model, he found that the interference of electron waves transmitted through the QD exhibits universal phase behavior (see figure 18). This universal behavior was then attributed to a Fano-type process, taking place between a narrow resonance and the broad one. At the same time, through suitable tuning of the model parameters, it was also shown that the Fano process could be modified, so that the universal behavior was suppressed and the mesoscopic evolution was instead recovered. While these features capture the essential characteristics of the experiment, the justification for the toy model itself remained an open issue within the framework of Hermitian quantum physics. Later, however, it was demonstrated \cite{22} that the features of the toy model may viewed as arising quite naturally from width bifurcation and a DPT. According to this interpretation, the universal regime corresponds to the system after the DPT has occurred, under which conditions the long-lived states generated by width bifurcation each exhibit a universal phase evolution, since they are superposed with the short-lived state that is created together with them \cite{22}. In this sense, we see that the individual spectroscopic properties of the states are lost at high level density due to the DPT. In contrast to this, the mesoscopic regime corresponds to the situation for which the DPT has not yet occurred. The evolution of
the transmission phase $\beta$ under such conditions then reflects the individual features of the resonance states, and is consequently different for the different conductance peaks.

4.3.3. Spin swapping operation in a two-spin system. In this section we consider an example of an environmentally induced DPT that is very different in character to those described in the previous two subsections. This problem involves a two-spin system embedded in an environment of neighboring spins [10], and is realized by performing cross-polarization NMR studies of polycrystalline samples of ferrocene (Fe(C$_2$H$_5$)$_2$). Through appropriate control of the microwave excitation frequency, it is possible to select a specific pair of interacting spins within such molecules, namely those on a $^{13}$C atom and its directly bonded $^1$H. With these two spins representing the closed quantum systems ($Q$), a swapping gate may be implemented in which exchange of the two degenerate spins is achieved by means of the spin–spin interaction ($b$). The interaction induces a splitting of the two spin levels, causing the system to undergo Rabi oscillations with a natural frequency $\omega$ (figure 19). These oscillations are damped, however, by the interaction of the spins with an environment that is formed by the neighboring spins. This problem may therefore be interpreted as one in which the swapping dynamics of two coupled spins (qubits) is damped through their interaction with a spin bath. From this perspective, the environment is represented as a stroboscopic process that may instantaneously interrupt the system evolution through measurements and/or injections.

In describing the damped dynamics of the two-spin system, its interaction with its environment may be taken to be inversely proportional to some characteristic time ($\tau_{SE}$), which degrades the spin-swapping oscillations on some 'decoherence time' ($\tau_{\phi}$). Working on quite general grounds, we might expect to find that $\omega \propto b$ and $\tau_{\phi} \propto \tau_{SE}^{-1}$. Very different behavior was in fact found in experiment [10], however, where two distinct dynamical regimes were observed, figure 19. In the first of these, the expected proportionalities were indeed demonstrated as the interaction strength between the spins was increased. On exceeding a critical environmental interaction, however, the swapping was instead found to freeze while the decoherence rate dropped according to $\tau_{\phi}^{-1} \propto b^2 \tau_{SE}$. The transition between these two dynamical regimes was not smooth, but rather had the characteristics of a critical phenomenon, occurring once $\omega$ becomes imaginary. For such conditions,
damping of the spin motion decreases with increasing coupling to the environment, in marked contrast to the behavior obtained when $\omega$ is real. In discussing the results of their experiment in terms of a DPT, the authors of [10] noted that it is only below the phase transition that the dephasing rate ($\tau_{\phi}$) scales in proportion to ($\tau_{SE}$), as would be expected from Fermi’s golden rule. When the critical interaction strength is exceeded, however, the dephasing rate no longer conforms to the behavior expected from the golden rule. Further support for these ideas was provided in [97], in which the authors related the DPT to the presence of an EP by means of an exactly solvable model.

We conclude this section by noting that the results discussed here for spin swapping in the presence of an environmental interaction show the characteristic features of DPTs discussed for open systems in section 4.1. As such, they indicate the universality of the fundamental laws that govern the behavior of OQSs, independent of the specific system details.

4.3.4. Quantum systems with gain and loss. It was mentioned already in section 3.3 that significant theoretical and experimental activity is presently focused on the study of systems with gain and loss. The original motivation for these studies came from the mathematical finding that a wide class of non-Hermitian Hamiltonians may have completely real spectra [69], as long as certain (so-called PT-) symmetry conditions are fulfilled. Breaking such symmetries causes the eigenvalues to become complex, however, behavior that was first demonstrated experimentally in [70]. As a result of these and subsequent experimental works, it is generally understood today that the eigenvalues of a non-Hermitian Hamilton operator may be real over a finite parameter range, when the physical system is not isolated and loss and gain are balanced [98].

Experimental realizations of systems with gain and loss have typically been implemented in optical lattices, in which the formal equivalence of the optical-wave equation to the Schrödinger equation can be exploited [71]. These experiments [70] have demonstrated the existence of a DPT, at which the real spectra become complex as some appropriate system parameter is varied. In related theoretical work, Hamiltonians with Hermitian tunneling- and non-Hermitian perturbation-terms have been investigated [99–102], mostly by means of a tight-binding approach. An important finding of these studies is that DPTs may be realized in optics by creating media with alternating regions of gain and loss, so that the creation and absorption of photons occurs in a balanced manner. The net loss or gain of such systems is therefore zero and the eigenvalues of their non-Hermitian Hamiltonian are real [100, 102].

Figure 18. Computed phase lapses within a toy model for tunneling through a QD. (a) In the mesoscopic regime the phase lapses are not universal. Plotted in the figure are the conductance ($G$, blue line) and the evolution of the phase ($\text{Arg}(t)$, green line), which may vary by more than $\pi$ between successive tunneling events. (b) In the universal regime the phase lapses are universal. Plotted in the figure are the conductance ($G$, blue line) and the evolution of the phase ($\text{Arg}(t)$, green line), which varies by no more than $\pi$ between successive tunneling events. In both panels (a) and (b) the conductance and phase variations are plotted as a function of the plunger-gate voltage ($V_g$), expressed in units of the QD level spacing, $\delta$. Reprinted by permission from [21]. ©IOP Publishing and Deutsche Physikalische Gesellschaft. CC BY-NC-SA.
It has furthermore been shown that the boundary between the phase with real eigenvalues, and the broken one with complex eigenvalues, exhibits the characteristics of an EP [101]. Similar results have been obtained in many other works, and some important examples are listed in [103]. The general conclusions of these studies is that it should be possible to realize OQSs with both gain and loss. This capacity not only makes open systems of interest from a fundamental perspective, but it also endows them with significant technological potential.

The experimental results obtained on optical lattices have been confirmed by theoretical studies that start from the description of OQSs by making use of the equivalence of the optical wave equation to the Schrödinger equation and by using the non-Hermitian Hamiltonian along the lines sketched in sections 2 and 3. In [59, 64] it was shown that real eigenvalues may appear over a large parameter range between two EPs, allowing the connection of the DPT to its EPs to be revealed [72]. In other cases, width bifurcation results in the formation of eigenstates with non-vanishing widths of opposite sign (corresponding to gain and loss) over the range of parameter space between the two EPs, and a balance of these values causes the eigenvalues to be real (also see [59, 64]). A DPT may also occur, however, without full balance of gain and loss, in circumstances where the eigenvalues cease to be real over the whole parameter range between the EPs [59, 64, 73]. Although most calculations have been performed on the basis of equation (4), with simple assumptions for the individual matrix elements, the results serve to demonstrate that systems with gain and loss may be realized under very different conditions. Nonetheless, the equivalence of the optical wave equation to the Schrödinger equation needs to be studied in further detail, not only from the viewpoint of gain and loss and their balance, but also for an understanding of Dicke superradiance (see the next section).

Recently, experimental studies of the role of gain and loss have been performed on systems other than optical lattices. Work on the mutually-coupled modes of a pair of active RLC circuits, one of which exhibited amplification while the other showed equivalent attenuation, was found to show how gain and loss mechanisms may break Hermiticity [104]. At a critical value of the gain (and loss), the eigenfrequencies were found to undergo a spontaneous phase transition from real to complex values, while the eigenmodes were similarly found to coalesce. The parametric evolution of the eigenfrequencies, as a function of the normalized gain and loss parameters, clearly showed the existence of an EP at a critical value. These results are fully consistent with those discussed above and can also be expected in systems without any special symmetry (see the discussion in section 3.3). DPTs have also been observed in other systems with balanced loss and gain, with one example being provided by work on whispering gallery microwave-cavities [105].

5. Other possible manifestations of dynamical phase transitions

As discussed in section 4, the characteristic feature of a DPT in an OQS is the appearance of collective modes. According to theory, these modes do not arise on their own but are always accompanied by the emergence of long-lived (so-called, “trapped”) resonance states. In most experiments, these states
are not detected directly but are rather inferred by their influence on other observables, or by an additional spectroscopic analysis. Good examples of DPTs in mesoscopic systems are provided by magnetoconductance studies of QQSs [12, 13], and by measurements of the transmission phase in tunneling through single-electron transistors [17–19] (see sections 4.3.1 and 4.3.2). However, no systematic effort has thus far been undertaken to identify both the collective short-lived modes, and the trapped long-lived ones, in QQSs with closely neighboring states (and/or strong coupling between the isolated system and its environment). In this section, we review some experimental results that may hint at further possible manifestations of DPTs in open systems.

Before proceeding with our discussion we first briefly note that, in the research literature, the dynamics of QQSs in the presence of strong coupling between system and environment is often referred to as ’superradiant dynamics’, suggesting a similarity to the phenomenon of Dicke superradiance in optics. Some sixty years on from the interpretation of this phenomenon as a collective one [24], its details are still not fully understood and continue to be debated in many papers (see the following section 5.1). While there is no doubt that the existence of collective superradiant modes in optics has been clearly demonstrated experimentally, their correlation with the existence of long-lived, so-called subradiant, modes has not thus far been proven experimentally [106]. That is not to say that such subradiant modes do not exist, but rather that these weakly-coupled modes are much more difficult to identify than strongly-coupled ones, especially if both types of state are correlated and appear simultaneously. Because of these problems, we prefer to avoid using the term ’superradiant dynamics’ or ’superradiance transition’ when characterizing DPTs in QQSs. Rather, we will use the term ’superradiance’ to describe the enhanced emission of particles or radiation from different experimental systems.

5.1. Dicke superradiance

Some 60 years ago, Dicke introduced the concept of superradiance [24] to describe the cooperative spontaneous emission of photons from an ensemble of two-level atoms. This phenomenon corresponds to an enhancement of the emission rate compared to the natural decay rate, and does not satisfy the exponential law characteristic of the decay of a group of independent atoms. Following on from the pioneering work of Dicke, superradiance has now been investigated in numerous reports (see, e.g. [106–113]), in which its appearance under various conditions has been discussed. Theoretically, this effect is typically discussed within a time-dependent approach, in which it is analyzed in terms of an acceleration (i.e. superradiance), and of a slowing (i.e. subradiance), of emission. An issue of interest in this section concerns the possible relation of Dicke superradiance to the appearance of a DPT in open many-body systems.

The resemblance of Dicke superradiance to a DPT has been demonstrated in several numerical studies. One example is provided in [108], in which the problem of collective spontaneous emission was investigated without making recourse to the rotating-wave approximation. In this approach, it was only by making the assumption of small system volume (i.e. low level density) that the states were found to approximate the exponential decay characteristic of a group of independent atoms. Furthermore, resembling our discussions of DPTs, it has been shown elsewhere that Fermi’s golden rule cannot adequately describe the features of Dicke superradiance, and that so-called ’timed’ Dicke states exist in addition to the standard ones [107]. Such results represent a clear signature of a DPT, in which short-lived (superradiant) modes are formed together with long-lived (subradiant) ones. Another interesting result that might also be related to a DPT, is provided by the experimental observation of the collective Lamb shift [109, 112].

Using the formalism outlined in section 2 (equations (18) and (19)), the influence of EPs on the cross section of two states was studied in [63]. Under conditions for which the two states are well separated from one another, the cross section consisted of two well-separated resonances. By varying the coupling coefficient between system and environment, from a real to an imaginary value, the cross-section could be made to evolve into a single broad peak with an extremely sharp dip near its center. This behavior agrees qualitatively with that obtained in theoretical [110] and experimental [111] studies of the transition from Autler–Townes-splitting to electromagnetically-induced transparency. In [110], two different models were developed to describe the behavior observed at small and large Rabi frequency. The transition between these two different mechanisms was found to occur over a critical range, while controlling the Rabi frequency. In [63], the transition from Autler–Townes-splitting to electromagnetically-induced transparency was discussed as the result of a spectroscopic redistribution taking place at a DPT. Below this transition, the environmentally-mediated interaction of the two distant states is characterized by an (almost) real coupling and the system can be described quite well by Hermitian quantum physics. Beyond the DPT, however, the two states are close to one another, and the coupling coefficient is (almost) imaginary (see figure 14). The spectroscopic features of the non-Hermitian system therefore differ very strongly above and below the DPT, and it is this result that eventually gives rise to the two different mechanisms considered in [110]. The sharp dip observed in the electromagnetically-induced transparency at small Rabi frequency is caused by the long-lived state formed in the DPT, which is superimposed upon the short-lived state responsible for the broad bump.

As discussed in section 3.4, the existence of an EP cannot be inferred directly from the an analysis of the scattering cross section (or the transmission), since the S matrix varies smoothly in the neighborhood of such a point. For this reason it is difficult to convincingly prove the existence of a DPT without any further information, especially that involving the spectroscopic properties of the states. Consequently, and in spite of the similarities noted above, the issue of whether Dicke superradiance may be considered as a DPT (according to the description given in section 4) remains, at present, an open problem.
5.2. Superradiance in quantum dots

Experimentally, there have recently been some interesting demonstrations of Dicke superradiance in semiconductor systems, which highlight the issues discussed above. In the work described by Scheibner et al [25], the authors studied the photoluminescence spectra of arrays of self-assembled (CdSe/ZnSe) QDs (see figure 20), in which the individual dot size was in the range of 6–10 nm. Strong quantum confinement of electrons within these structures gives rise to a quantization of the energy spectrum, allowing these structures to essentially function as ‘artificial atoms’. In [25], the authors exploited this characteristic to investigate the recombination of excitons associated with these atomic states, and to investigate how this photoluminescence is modified by the mutual interaction between the QDs (see the right panel of figure 20). As with usual discussions of Dicke superradiance, the source of this interaction was presumed to be the common radiation field arising from the luminescence from different dots. The novel feature of their approach involved monitoring the emission from ensembles comprised of varying numbers of QDs. When sampling just a small number of such dots, the emission spectra consisted of a small number of discrete resonances, arising from the independent emission of photons from the uncorrelated dots. By increasing the mesa size, however, such that a larger ensemble of dots was explored, the spectral lines merged to form a single, wide, peak, as expected from discussions of width bifurcation associated with a DPT (e.g. figure 8). While the luminescence decay time was found to be strongly influenced, no evidence of a sharp dip was found to accompany this peak. More specifically, for a sufficiently-large ensemble of dots, the excitonic decay was found to occur more rapidly than for isolated dots, and to be strongly frequency dependent, suggestive of an interpretation consistent with Dicke superradiance. No evidence of the subradiant mode was found to accompany this behavior, however, making it difficult to conclusively relate these observations to a DPT. Nonetheless, by varying the number of QDs in the system, the authors were able to use the observation of Dicke superradiance to infer

![Figure 20](https://example.com/figure20.png)

(a) Photoluminescence spectra, measured for QD ensembles with varying numbers of dots. QD number was varied by changing the area of the sample included in the photoluminescence measurements. From top to bottom, respectively, the sample size was 25 μm, 1 μm, 350 nm, and 175 nm. These values should be compared with the average size and separation of the QDs, of 6–10 nm and >35 nm, respectively. Consequently, the largest sample area contains a very large number (tens of thousands) of QDs, while the smallest contains just a few dots. (b) In addition to single mesas (ensembles), mesas arranged in grid-like patterns were also etched into the sample, to study the radiation of an ensemble of QDs (a gas of artificial atoms) versus the radiation of an ensemble of single QDs (a diluted gas of artificial atoms). The grid-like patterns serve the purpose of statistical averaging. Reprinted by permission from Macmillan Publishers Ltd: Nature Physics [25], copyright (2007).

![Figure 21](https://example.com/figure21.png)

Figure 21. Observation of delayed bursts of radiation from a photo-excited semiconductor plasma. The color contour shown in this figure is a streak-camera image of the emission intensity from a stack of 15 InGaAs/GaAs quantum wells, as a function of the resulting photon energy and delay time. The delay time in this case refers to a pump–probe measurement, in which the pump probe is used to create an extremely dense electron–hole plasma, following which, after a certain delay, an ultrashort burst of coherent radiation emerges. This radiation reflects the presence of a self-organizing process, in which many-body interactions lead to the emergence of a cooperative state of phased electron–hole ‘dipoles’ at high magnetic fields and low temperatures. Reprinted by permission from Macmillan Publishers Ltd: Nature Physics [26], copyright (2012).
the effective range of interaction between the dots. From their investigations they determined a range of 150 nm, significantly larger than the size of the individual dots and comparable to the wavelength of the radiation field.

In a separate study to [25], superradiance was investigated in a magneto-plasma induced in a photo-excited semiconductor [26]. In these experiments, the carriers were not confined within QDs but were rather created by an intense femtosecond laser pulse that generated an extremely dense electron–hole plasma. This system was found to undergo a novel process of self-organization, in which excitons formed by electron–hole pairs spontaneously developed a macroscopic coherence, or giant dipole. Referred to as a ‘self-phasing’ stage by the authors, this process is driven by quantum fluctuations associated with the individual excitonic dipole moments. Following this stage, the authors observed the emission of ‘superfluorescent’ bursts of photons, corresponding to superradiance (figure 21). A magnetic field was used in these experiments

Figure 22. (a) QPC transient-measurement set-up. The pulse generator supplies voltage pulses with rise times of a nanosecond or less, and of duration tens of nanoseconds. The transient current through the QPC is detected at the 50-Ω input of a sampling oscilloscope. (b) Transient response of a QPC for various gate voltages. The dotted line shows the input pulse (V_p in (a)), and the pulses were obtained for different gate voltages (numbers of subbands). The multimode 1D regime corresponds to the situation where many 1D subbands are occupied. The QPC shows clear signatures of heating here, as indicated by the slowly-increasing voltage (current) at the top of the pulse. In the few-mode 1D regime, however, the current is carried by just one or two subbands and the time-dependent heating is completely suppressed. Instead, the transient current rises quickly to its steady-state value, suggesting the QPC has become immune to locally-generated heat. (c) The calculated subband bottom for the ten lowest sub-bands as a function of the source-drain bias (V_{sd}). The black line denotes the bottom of the lowest subband and the subband index increases systematically from 1 to 10 on moving up from the lowest curve. The vertical dotted line indicates the optical-phonon energy of 36 meV for GaAs and the red dashed line identifies the position of the electrochemical potential (measured from the edge of the lowest subband). Error bars represent level broadening due to dephasing introduced by inter-subband transitions. Reprinted by permission from Macmillan Publishers Ltd: Nature Nanotechnology [23], copyright (2014).
to induce Landau-level quantization, and to thereby modify the emission energy of the resulting magneto-excitons. This allowed a transition from ordinary spontaneous emission to superfluorescence to be observed, although, once again, the connection of this phenomenon to a DPT was not studied explicitly and therefore remains unclear.

5.3. Protected-subband formation

Earlier, we described a DPT that was driven by the interaction of localized spins with a spin bath, with the latter functioning as a source of decoherence. Related behavior has recently been reported in studies of the nonlinear transport through QPCs, subjected to extremely strong biasing [23] (see figure 22). In these experiments, electrons in the source are injected into the quantum subbands of a QPC, and are accelerated through to the drain by the strong electric fields induced by the applied source-drain voltage. For sufficiently strong biasing, the accelerated electrons surpass the threshold for the emission of optical phonons, and so rapidly lose energy via this process as they pass to the drain. Under such conditions, quantum transport through the QPC therefore takes place in the presence of an “environment” that is provided by a non-equilibrium distribution of phonons. As the bias voltage was increased in the experiment, a sudden change in the heating characteristics of electrons was observed (see figure 22(b)), which was explained by a process in which induced electron-phonon scattering leads to a strong renormalization of the QPC spectrum. According to this interpretation, the lowest subband within the QPC was found to split-off strongly from the higher ones (figure 22(c)), thereby forming a ‘protected’ ground subband that is immune to the influence of locally-generated heating [23]. In these studies, the source voltage therefore serves as the control parameter, that determines the strength of the interaction between the quantum system (the QPC) and its environment (locally-generated phonons). The divergence of the lowest subband away from the higher ones, in a manner determined by the control parameter, is clearly suggestive of our discussions of width bifurcation in DPTs.

5.4. Photosynthetic light-harvesting systems

An interesting problem, of very different character to that which we have considered thus far, concerns the efficiency of energy transport in photosynthetic light-harvesting complexes. Recently, this problem has been analyzed within the non-Hermitian formalism, by coupling the light-harvesting complex to a reaction center that serves as a sink with a quasi-continuous energy spectrum [114]. The presence of this quasi-continuum is found to strongly influence the energy transfer. Calculations were performed for the Hamiltonian of equation (25), in a tight-binding implementation consisting of a ring of several sites that were taken to be coupled to the sink at the ring center. By taking a realistic electric-dipole interaction into account, and varying the strength of the coupling to the reaction center, it was shown that maximal efficiency in energy transport is achieved in the vicinity of a DPT (referred to as a ‘superradiance transition’ by the authors). For coupling strengths close to the critical one for which the DPT occurs, calculated energy-transfer times were found to be in good agreement with those (of order a few picoseconds) observed in experiment. These predictions regarding the efficiency of transport are furthermore fully consistent with those obtained theoretically for the transmission through QDs (see sections 3.4 and 4.1). From such systems we know that the efficiency of transport is directly related to the reduced phase rigidity, resulting from the role of the EPs at the DPT. In [115], the robustness of the energy transport in photosynthetic light-harvesting systems with regards to dephasing was investigated. Experimental studies are required, however, to prove the connection of the efficiency of energy transport in these materials to a DPT.

6. Conclusions

In this Report on Progress, we have reviewed the key physical attributes of OQSs in the regime of closely-neighboring states and have described a number of different experimental systems in which they are manifested. At the most basic level, these attributes may be understood as arising from the fact that the Hamiltonians needed to describe the dynamics of open systems are non-Hermitian, in marked contrast to their Hermitian counterparts that so accurately describe the behavior of isolated quantum systems. In the non-Hermitian formalism, the open system is viewed as being comprised of a localized region \( Q \), embedded in a well-defined environment \( P \) of scattering wavefunctions, and the total system \( (Q + P = 1) \) is Hermitian. The coupling between the two subspaces is found to mediate an effective interaction among the original states of the localized system. This natural (intrinsic) environment does not serve as a source of decoherence (the role of which is not considered here) and its influence can never be deleted from the system. Instead, it gives rise to complex eigenvalues, whose real component defines the energy of the state and whose imaginary part describes an effective level broadening. The eigenfunctions also exhibit an important difference with those of Hermitian systems, in that they are no longer orthogonal to one another but are instead biorthogonal. Physically, the biorthogonality expresses the fact that there is an environmentally-induced coupling between the different states of the system. This should be contrasted with the case of Hermitian quantum mechanics, which yields orthogonal stationary states that require a perturbation of the system in order to be coupled. The description of open systems in terms of this approach is found to be well suited to the solution of a broad range of problems, providing proper care is taken to accurately identify the subspaces \( Q \) and \( P \). Under such conditions, this approach is expected to provide a good description of small systems, in which the number of relevant quantum states, as well as the number of channels, is restricted. In larger systems that do not conform to these constraints, other approaches, such as random-matrix theory, should be more effective. Such approaches have been extensively treated in the literature and have therefore not been considered here.
We have seen in this review how the eigenstates of the non-Hermitian Hamiltonian exhibit fundamentally new behavior not encountered in Hermitian physics. The first example is provided by the appearance of singular points, referred to as exceptional points (EPs). Arising when an appropriate system parameter (such as the coupling strength between the $Q$ and $P$ subsystems) is varied, the EP involves the coalescence of two eigenvalues of the non-Hermitian operator and (up to a phase) their corresponding eigenfunctions. Although a point of measure zero, the EP exerts a strong influence on the spectral properties of the system, extending over a wider region than the singular point at which the two eigenvalues coalesce.

An important property that indicates the range of influence of the EP is the phase rigidity of the eigenfunctions. Essentially, the phase rigidity is a measure of the biorthogonality of different eigenfunctions, and, therefore, of the strength of the environmentally-induced interaction between the different states. In the region of the EP the phase rigidity is significantly reduced, expressing the fact that this interaction via the environment is very strong at the EP. As a result, the wavefunctions of these states are no longer orthogonal with one another but are equal, up to a phase. Over the range where the phase rigidity is reduced, the two eigenstates undergo the phenomenon of width bifurcation, with one state becoming strongly broadened while the other becomes more stable as it decouples from the environment. Under this condition, transport through the system occurs with high efficiency. Simultaneous with this, the eigenvalues may cross in energy. This is in complete contrast to the behavior of Hermitian systems, which may only ever exhibit an avoided level crossing when two of the original states develop the strong broadening, leaving all but these two states stabilized. Much of this phenomenon occurs when all states of the system are orthogonal with one another. The DPT involves a change in the dynamics of the system, which arises when width bifurcation occurs hierarchically for many states. The simplest example of this phenomenon occurs when all states of the system are coupled to a single channel, resulting in the formation of just one strongly-broadened state. The remaining states are then essentially stabilized, through a process in which they strongly decouple from the environment. In the two-channel case relevant for transport, essentially the same picture holds although now two of the original states develop the strong broadening, leaving all but these two states stabilized. Much like conventional phase transitions in thermodynamics, the details of the dynamics on either side of the DPT are not analytically connected to each other. That is, the behavior on one side of the transition does not serve as a reliable indicator of that on the other. Both the width bifurcation caused by the EPs, and the DPT that these give rise to, result from the fact that the coupling strength to the environment has an imaginary component.

Experimental evidence in support of these various concepts has been provided throughout the course of this review, with a particular emphasis having been placed on their manifestations in mesoscopic systems. At the heart of the non-Hermitian scheme is the idea of environmentally-mediated coupling, which yields an effective interaction among the states of the quantum system. We have seen how this coupling is manifested directly in experiments involving non-locally coupled quantum point contacts (QPCs), in which an intervening continuum serves as the $P$ subsystem and mediates a coupling between remote pairs of QPCs. We have described how this coupling gives rise to novel Fano-resonance phenomenology, and have also observed that the environmentally-mediated interaction may provide a stronger coupling between quantum states than a simple tunneling overlap. This may be attributed to the fact that the environmental interaction is supported by a large number of states that comprise a continuum, resulting in a much stronger effect than would be expected to arise from the direct interaction between the different states of $Q$.

Several experimental demonstrations of DPTs in physical systems were also presented. These examples included: observations of resonance-trapping and its related phenomena, in open microwave cavities and QDs; a crossover from mesoscopic to universal behavior in the phase lapses found in single-electron tunneling through Coulomb-blockaded QDs; observations of spin-swapping dynamics in ensembles of interacting spins, and; energy-transfer processes in photosynthesis. Similar phenomena are observed also in optics, most notably in the form of the long-studied problem of Dicke superradiance, as well as more recent investigations of systems with gain and loss. Here, however, a better understanding of the equivalence between the optical-wave and Schrödinger equations is needed before convincing conclusions can be drawn. In addition, we have also highlighted several other examples of complex physical phenomena that may well be related to the occurrence of a DPT. These include superradiance in QDs, and the observation of protected-subband formation in nonequilibrium transport of hot electrons through QPCs. These observations by no means represent an exhaustive list, but rather suggest instead that DPTs are a common property of QQSs in general.

Having summarized the general status of this field, we can now identify a number of important issues that should be resolved in future studies. While the different experiments described in this review provide several hints that DPTs do indeed occur in QQSs, the presence of the distinct time scales that are expected to accompany this phenomenon has yet to be directly demonstrated. According to the description of width bifurcation given above, the environmentally-induced interaction in the vicinity of an EP is expected to result in the formation of eigenstates with very different lifetimes; the first of these is very short and associated with the strongly-broadened state, while the other is much longer and related to the stabilized state. Experiments that can probe the existence of these different time scales are therefore desired. A problem in attempting to design such experiments, however, comes from...
the fact that this problem must first be clearly formulated from a theoretical perspective. A simpler task may be to provide a demonstration of the width bifurcation that should serve as the signature of the different time scales, but, to the best of our knowledge, this has not yet been done experimentally, in either mesoscopic or any other physical systems.

Time-resolved investigations of OQSs may also be useful in exploring issues associated with irreversibility in the vicinity of the DPT. The issues here can be discussed by referring to the features of figure 9, in which we show the system properties and their variation in the vicinity of two EPs. As either EP is approached from the left or the right of the figure, the width bifurcation begins at the EP and becomes maximal once \( d = 0 \). At this point the phase rigidity returns to unity, indicating that the two eigenfunctions are orthogonal. Physically, this can be understood to result from the fact that the two states are now associated with very different time scales, and so no longer interact with one another. While the results of figure 9 suggest that, by passing continuously through \( d = 0 \) and the point of maximum bifurcation, it should be possible to return to two distinct states, in practice this is not possible. In essence, one of the original states has now been ‘lost’ to the environment, and the remaining stable state is therefore unable to interact directly with it. This is completely consistent with the notion of a DPT; the spectroscopic properties of the system on one side of the transition are not analytically connected to those on the other, and nonlinear terms in the Schrödinger equation play an important role in this transition.

Another important property that is of interest in characterizing the properties of OQSs is the phase rigidity of their eigenfunctions. The value of this parameter becomes vanishingly small near an EP, and it is useful also for establishing the range of influence of such points. While the rigidity itself can probably not be measured directly in a physical system, we have seen here that it is related to a high efficiency of transport (recall figure 13). While this connection may not have been widely appreciated to date, it implies that measurements of the conductance of mesoscopic systems can provide a flexible probe of this important system property. Moreover, the enhancement of the conductance around a DPT is, by itself, of interest for possible applications.

Throughout this review, we have provided a number of different examples of DPTs, occurring in a variety of physical systems. From these examples we suggest that the DPT should be a very general characteristic of open systems, with broader consequences than have been appreciated to date. Indeed, the stability properties of classical systems are also governed by non-Hermitian degeneracies (EPs), as has been nicely shown in [116]. Our hope therefore is that this review can stimulate interest in the study of this still relatively unexplored problem, and ultimately lead to a deeper understanding of its implications.

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Note added in proof. Subsequent to the acceptance of our report we became aware of recently published work by Ruderman et al [118], demonstrating the possibility of a quantum dynamical phase transition in molecular chemical bond formation and dissociation. The occurrence of this transition was connected to the non-Hermitian nature of the Hamiltonian in the molecular system, consistent with the discussions of dynamical phase transitions given here.

Appendix. The non-Hermitian operator and the coupling matrix elements

In a consistent description of any open system by a non-Hermitian Hamiltonian without using perturbation theory, the Hermitian operator \( H^0 \) in equation (1) should strictly be replaced by a non-Hermitian one, \( H^0 \rightarrow H_0 \) where \( H_0 \equiv \mathcal{H} - V_0 \ell / V_0 q_0 \). At the same time, the orthogonal eigenfunctions \( \Phi^0_k \) of \( H_0 \) that appear in the partial width amplitudes in equations (2) and (3) should also be replaced by the biorthogonal eigenfunctions \( \Phi_k^0 \) of \( H_0 \). This replacement results in negligible changes [59, 65], however; the differences between the Hermitian and non-Hermitian descriptions of the \( Q \) subspace arise, above all, from the (complex) matrix elements \( W_k \) that describe the coupling of states \( k \) and \( l \) via the environment, and this is involved in \( \mathcal{H} \) and not in \( H_0 \).

Numerical results demonstrating this have been provided in [59], and are also apparent in panels (c) and (d) of figure 9. The coupling coefficients (or partial width amplitudes) of specific isolated states (\( k \)) of the \( Q \) subspace to channel \( \ell \) of the environment (that is, to the scattering states \( \xi_\ell \)), are then \( \gamma^0_k \equiv \gamma_k^0 \equiv \sqrt{2\pi} \langle \Phi^0_k \mid \xi_\ell \rangle \). It is only in the partial width amplitudes \( \gamma_k \equiv \sqrt{2\pi} \langle \Phi_k \mid \xi_\ell \rangle \), where \( \Phi_k \) is an eigenfunction of \( \mathcal{H} \), that the coupling of states of the \( Q \) subspace via the environment (the continuum of scattering wavefunctions) is involved. These partial width amplitudes are important part of the \( S \) matrix, and have dimensions of square-root of energy. It follows that \( \gamma_k^2 = \Gamma_k \) in the neighborhood of, and beyond, an EP, but that \( |\gamma_k^0|^2 > \Gamma_k \) [1]. This last relation once again expresses the influence of an EP on its neighborhood.

The energy dependence of the partial width amplitudes is one of the sources of the energy dependence of the eigenvalues of the non-Hermitian Hamiltonian (\( \mathcal{H} \)), as was demonstrated previously in nuclear physics for an isolated resonance state coupled to two decay channels [117]. In this case, the width (\( \Gamma_k \)) of state \( k \) is equal to the sum of the partial widths, and the strong energy dependence of \( \Gamma_k \) (when the second channel opens) may cause the lineshape of the isolated resonance to change from the standard Breit–Wigner form to that of a cusp (consistent with the behavior found in experiment). Another source of the energy dependence of \( \Gamma_k \) is the environmentally-mediated interaction among the nuclear levels. This smoothly varying interaction influences the symmetry of the resonance lineshape, giving rise to a ‘tail’ in the resonance at high (low) energy when \( \Gamma_k \) increases (decreases) with energy. Correctly allowing for the energy dependence of the eigenvalues of \( \mathcal{H} \) guarantees the unitarity of the \( S \) matrix [1, 75].
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