Origin of maximal symmetry breaking in even $\mathcal{PT}$-symmetric lattices

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By investigating a parity and time-reversal ($\mathcal{PT}$) symmetric, $N$-site lattice with impurities $\pm i\gamma$ and hopping amplitudes $t_0(t_b)$ for regions outside (between) the impurity locations, we probe the origin of maximal $\mathcal{PT}$-symmetry breaking that occurs when the impurities are nearest neighbors. Through a simple and exact derivation, we prove that the critical impurity strength is equal to the hopping amplitude between the impurities, $\gamma_c = t_b$, and the simultaneous emergence of $N$ complex eigenvalues is a robust feature of any $\mathcal{PT}$-symmetric hopping profile. Our results show that the threshold strength $\gamma_c$ can be widely tuned by a small change in the global profile of the lattice, and thus have experimental implications.

Introduction: The discovery of “complex extension of quantum mechanics” by Bender and coworkers [1, 2] set in motion extensive mathematical [3-5] and theoretical investigations [6] of non-Hermitian Hamiltonians $H_{\mathcal{PT}} = \hat{K} + \hat{V}$ that are symmetric with respect to combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) operations. Such continuum or lattice Hamiltonians [7, 10] usually consist of a Hermitian kinetic energy part, $\hat{K} = \hat{K}^\dagger$, and a non-Hermitian, $\mathcal{PT}$-symmetric potential part, $\hat{V} = \mathcal{PT}\hat{V}\mathcal{PT} \neq \hat{V}^\dagger$. Although it is not Hermitian, $H_{\mathcal{PT}}$ has purely real eigenvalues $E = E^\ast$ over a range of parameters, and its eigenfunctions are simultaneous eigenfunctions of the combined $\mathcal{PT}$-operation; this range is defined as the $\mathcal{PT}$-symmetric region. The breaking of $\mathcal{PT}$-symmetry, along with the attendant non-reciprocal behavior, was recently observed in two coupled optical waveguides [11, 12] and has ignited further interest in $\mathcal{PT}$-symmetric lattice models. These evanescently coupled waveguides provide an excellent realization [13] of an ideal, one-dimensional lattice with tunable hopping [14], disorder [15], and non-Hermitian, on-site, impurity potentials [16, 17].

Recently nonuniform lattices with site-dependent hopping $t_n(k) = t_0[k(N - k)]^{n/2}$ and a pair of imaginary impurities $\pm i\gamma$ at positions $(m, \bar{m})$ have been extensively explored [17-20], where $\bar{m} = N + 1 - m$ and $N \gg 1$ is the number of lattice sites. The $\mathcal{PT}$-symmetric phase in such a lattice is robust when $a \geq 0$, the loss and gain impurities $\pm i\gamma$ are closest to each other, and $\gamma \leq \gamma_c$ where the critical impurity strength is proportional to the bandwidth of the clean lattice, $\gamma_c \propto 4t_0(N/2)^a$. For a generic impurity position $m$, when the impurity strength $\gamma > \gamma_c(m)$ increases the number of complex eigenvalues increases sequentially from four to $N - 1$ when $N$ is odd and to $N$ when it is even. In an exceptional contrast, when $m = N/2$ - nearest neighbor impurities on an even lattice - all eigenvalues simultaneously become complex at the onset of $\mathcal{PT}$-symmetry breaking. This maximal symmetry breaking is accompanied by unique signatures in the time-evolution of a wavepacket [20].

These results raise the following questions: Is this exceptional behavior limited to lattices with $a$-dependent hopping or is it generic? Which factors truly determine the critical impurity strength $\gamma_c(N/2)$ in the exceptional case? How does the critical impurity strength $\gamma_c(m)$ depend upon lattice parameters and impurity positions?

In this Brief Report, we investigate an $N$-site lattice with impurities $\pm i\gamma$ at positions $(m, \bar{m})$ and a constant hopping amplitude $t_0(t_b)$ for sites outside (between) the parity-symmetric impurity locations. Our two salient results are as follows: i) When $m = N/2$, we analytically prove that all eigenvalues simultaneously become complex when $\gamma > \gamma_c(N/2) = t_b$. This robust result is true for any symmetric distribution of real hopping amplitudes. ii) When $t_b \gg t_0$, the critical impurity strength $\gamma_c(m) \rightarrow t_b$ irrespective of the impurity position $m$. When $t_b < t_0$, the critical impurity strength $\gamma_c(m) \sim t_b^\eta$ where the exponent $\eta(d) \sim d$ increases monotonically with the distance $d = N + 1 - 2m$ between the impurities. Thus, the $\mathcal{PT}$-symmetry breaking threshold can be substantially tuned without significant changes in the global hopping-amplitude profile of the lattice, and the exceptional nature of the $m = N/2$ case is due to the ability to partition the system into two, and exactly two, pieces.

Tight-binding Model: We start with the Hamiltonian for a one-dimensional, tight-binding, non-uniform lattice

$$H_{\mathcal{PT}} = -\sum_{i=1}^{N-1} t(i) \left( a_{i+1}^\dagger a_i + a_i^\dagger a_{i+1} \right) + i\gamma \left( a_m^\dagger a_m - a_m^\dagger a_m \right),$$

where $a_n^\dagger(a_n)$ is the creation (annihilation) operator for a state localized at site $n$, and the hopping function is given by $t(i) = t_b > 0$ for $m \leq i \leq \bar{m} - 1$, and $t(i) = t_0 > 0$ otherwise. This Hamiltonian continuously extrapolates from that for a lattice of length $d = N + 1 - 2m$ with impurities at its end when $t_b \gg t_0$, to that of a pair of disconnected lattices, one with the gain impurity and the other with the loss impurity, when $t_b \ll t_0$. Note that the critical impurity strengths in these two limits are known [17, 21]. Due to the constant hopping amplitude outside or between the impurity locations, an arbitrary eigenfunction $|\psi\rangle = \sum_{n=1}^N \psi(n) a_n^\dagger |0\rangle$ with energy $E$ can
be expressed using the Bethe ansatz as
\[
\psi(n) = \begin{cases} 
A \sin(kn), & 1 \leq n \leq m, \\
B \sin(kn), & m < n < \bar{m}, \\
P \sin(k' n) + Q \cos(k' n), & m < n < \bar{m}, 
\end{cases}
\] (2)

Here \( E(k, k') = -2t_0 \cos(k) = -2t_0 \cos(k') \) defines the relation between the quasimomenta \( k, k' \). In the \( PT \)-symmetric phase, the energy spectrum of Eq. (1) is particle-hole symmetric \([22]\), and the eigenenergies satisfy \(|E| \lesssim 2 \max(t_0, tb)\). Note that the relative phases of \( \psi(n) \) are the same at different points within each of the three regions, although there may be a phase difference between wavefunctions in different regions. Therefore, without loss of generality, we may choose \( \psi(n) \) to be real for \( 1 \leq n \leq m \). By considering the eigenvalue equation \( H_{PT} |\psi\rangle = E |\psi\rangle \) at points \( m, m + 1 \) and their reflection counterparts, it follows that the quasimomenta \( (k, k') \) obey the equation \([21]\)
\[
M(k, k') \equiv \left[ \sin^2(k(m + 1)) + \sin^2(k(m)) \right] \\
+ \sin[k'(N + 1 - 2m)] + T_b^2 \sin^2(k(m)) \\
\times \sin[k'(N - 1 - 2m)] - 2T_b \sin(k(m)) \\
\times \sin(k(m + 1)) \sin[k'(N - 2m)] = 0,
\] (3)

where \( \Gamma = \gamma/t_0 \) and \( T_b = tb/t_0 \) denote the dimensionless impurity strength and hopping amplitude respectively. Note that when \( 2 \min(t_0, tb) < |E| \leq 2 \max(t_0, tb) \), \( k \) is real and \( k' \) is purely imaginary (or vice versa), whereas for \( |E| \leq 2 \min(t_0, tb) \), both \( k, k' \) are real. Thus, Eq. (3) represents two distinct equations in these two cases.

The right-hand panel in Fig. 1 shows the dimensionless critical impurity strength \( \Gammac(d) = \gamma_c(m)/t_0 \) as a function of \( Tb = tb/t_0 \geq 1 \) for various inter-impurity-distances \( d = N + 1 - 2m \) in an \( N = 20 \) even lattice; we obtain similar results for an odd lattice. We find that \( \gamma_c \to tb \) quickly for \( tb/t_0 > 1 \); when \( tb/t_0 \gg 1 \), the lattice reduces to one with \( d + 1 \) sites, impurities at its end points, and the result \( \gamma_c = tb \) is expected \([21]\). The left-hand panel shows \( \Gamma_c(d) \) vs. \( T_b \) on a logarithmic scale in \( N = 20 \) and \( N = 21 \) lattices for \( T_b < 1 \). As the distance \( d \) between the impurities increases, corresponding critical impurity strength decreases as a power-law, \( \Gamma_c(d) \propto T_b^{-\nu(d)} \) where the exponent \( \eta(d) \sim d \). This behavior can be qualitatively understood as follows: the system is in the \( PT \)-symmetric region if the frequency \( \sim \gamma/t_0 \) at which particles are created at the gain-impurity site \( m \) is lower than rate at which these excess particles can hop over to the loss-impurity site, where they are absorbed at frequency \( \sim \gamma/t_0 \). Since \( tb \) is the hopping amplitude at sites between the impurities, it follows that the effective frequency of hopping from the gain- to the loss-site decreases with \( d = T_b^d \). Indeed, when \( tb/t_0 \ll 1 \), the system is divided into two, non \( PT \)-symmetric, uniform lattices, one with the loss impurity and the other with the gain. It follows, then, that \( \gamma_c \to 0 \) as \( tb/t_0 \to 0 \).

**Origin of Maximal Symmetry Breaking:** Now let us consider the \( m = N/2 \) case, where Eq. (3) reduces to
\[
t_0^2 \sin^2 \left[ k \left( \frac{N}{2} + 1 \right) \right] = (b^2 - \gamma^2)^2 \sin^2 \left( k \frac{N}{2} \right).
\] (4)

It follows from Eq. (4) that the \( PT \)-symmetry breaks maximally when \( \gamma > \gamma_c(N/2) = tb \) and is accompanied by the simultaneous emergence of \( N \) complex (not purely imaginary) quasimomenta and eigenenergies. Since the bandwidth of the clean lattice is determined by both hoppings \((t_0, tb)\), it follows that the critical impurity strength is independent of the lattice bandwidth.

To generalize this result, we consider the system with an arbitrary, \( PT \)-symmetric, position-dependent hopping profile \( t_k = t_{N-k} \) and real energy eigenvalues. Since the hopping and eigenvalues are real, the eigenvalue difference equations imply that for any eigenfunction \( |\psi\rangle \), we can choose the coefficients \( \psi(k) \) to be real for \( 1 \leq k \leq m \). A real eigenvalue \( \epsilon \) and the (real) coefficients \( \alpha = \phi(N/2) \) and \( \beta = \phi(N/2 - 1) \) of its corresponding eigenfunction \( |\phi\rangle \equiv \sum_{i=1}^{N} \phi(i)|i\rangle \) satisfy
\[
\det \begin{bmatrix}
t_{N/2-1} \beta + (\epsilon - i\gamma) \alpha \\
t_{N/2} \alpha \\
t_{N/2-1} \beta + (\epsilon + i\gamma) \alpha 
\end{bmatrix} = 0,
\] (5)

where we have used the \( PT \)-symmetric nature of eigenfunctions to deduce that \( \phi(N/2 + 1) = e^{i\lambda} \alpha \), \( \phi(N/2 + 2) = e^{i\lambda} \beta \). Thus, when \( \gamma > \gamma_c = t_{N/2} = tb \), the eigenvalue \( \epsilon \) must become complex. Since this result is true for all eigenfunctions, it follows that the \( PT \)-symmetry breaks maximally and the critical impurity strength is solely determined by the hopping amplitude between the two impurities. This robust result also explains the fragile nature of \( PT \)-symmetric phase in lattices with hopping.
function \( t_\alpha(k) \) for \( \alpha < 0 \) [20]: in this case, the lattice bandwidth \( \Delta_\alpha \sim N^{-|\alpha|/2} \) whereas the hopping amplitude between the two nearest-neighbor impurities scales as \( t_b \sim N^{-|\alpha|} \). Therefore the critical impurity strength \( \gamma_c/\Delta_\alpha \sim N^{-|\alpha|/2} \rightarrow 0 \) as \( N \rightarrow \infty \). A similar analysis for closest impurities in an odd-\( N \) lattice shows that, due to the presence of a lattice site between the two impurity positions \( m = (N - 1)/2 \) and \( \bar{m} = (N + 3)/2 \), the corresponding critical impurity strength \( \gamma_c \) depends on the details of the eigenfunction.

Thus, the maximal symmetry breaking only occurs in an even, \( \mathcal{PT} \)-symmetric lattice with nearest-neighbor impurities, and its origin is the ability to naturally partition such a lattice into exactly two components.

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