Monte Carlo computation of the effective potential for the three dimensional Ising system

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Using a novel finite size scaling Monte Carlo technique, we calculate the four, six and eight point renormalized coupling constants defined at zero momentum for the three dimensional Ising system. Our values of the six and eight point coupling constants are significantly different from those obtained from other methods.

1. INTRODUCTION

Effective action is a useful formalism to describe long distance, low momentum behavior of a system. In this work we are interested in the Monte Carlo calculation of the the effective potential for the three dimensional (3D) Ising model. The Ising model is equivalent to the single component \( \lambda \phi^4 \) theory in the limit \( \lambda \to \infty \), so that the use of the standard weak coupling perturbation method is plagued because of the strong coupling nature of the model. Hence a variety of new analytical methods have recently been developed. Due to the non-perturbative nature of the problem, Monte Carlo methods have also been used. However, standard Monte Carlo measurements of relevant physical quantities suffer from high statistical noise. In this work we employ a novel finite size scaling (FSS) technique combined with a cluster flipping Monte Carlo algorithm in order to calculate the coefficients of the Taylor expansion of the effective potential, which are closely related to the dimensionless N-point renormalized coupling constants (RCC).

In terms of the renormalized field \( \varphi_R \) the effective potential can be written as

\[
V_{\text{eff}}(\varphi_R) = \frac{1}{2} m_R^2 \varphi_R^2 + \frac{1}{4!} m_R^4 g_R^{(4)}(\varphi_R)^4 + \frac{1}{6!} g_R^{(6)}(\varphi_R)^6 + \frac{1}{8!} m_R^8 g_R^{(8)}(\varphi_R)^8 + \ldots ,
\]

where \( m_R \) and \( g_R^{(N)} \) represent respectively the renormalized mass (inverse correlation length (\( \xi \))) and the N-point RCC defined at zero momentum. It is easy to show that

\[
g_R^{(4)} = -(Z_{\phi}^2/m_R) W_2^{-4} W_4
\]

\[
g_R^{(6)} = -Z_{\phi}^3 W_2^{-6} [W_6 - 10 W_2^2 W_4^{-1}]
\]

\[
g_R^{(8)} = -m_R Z_{\phi}^4 W_2^{-8} \times [W_8 - 56 W_6 W_4 W_2^{-1} + 280 W_4^3 W_2^{-2}].
\]

Here \( Z_{\phi} \) represents the field strength renormalized factor (\( Z_{\phi} = \chi m_R^2 \), with \( \chi \) denoting magnetic susceptibility), and \( W_N \) is the Fourier transformed N-point connected Green function of zero momentum.

2. THE MONTE CARLO METHOD

Since we are concerned with the low momentum properties of the model when it is in the scaling regime (but not at criticality), our goal is to calculate the thermodynamic values (the infinite volume limit value) of the RCC measured at zero momentum in the limit \( m_R \to 0 \). \( g_R^{(N)} \) is expressed in terms of the combinations of the various connected Green functions, so that the statistical noise in its numerical estimation becomes increasingly large as \( N \) increases. For a fixed \( N \), we empirically observe that the statistical error increases drastically for approximately \( L/\xi \geq 4 \). To illustrate we measured various physical quantities at an arbitrary (inverse) temperature with increasing linear size of the lattice \( L \). Table(1) clearly shows that the statistical error in the measured values of \( g_R^{(N)} \) increases with \( N \) for any given value of \( L \). For each quantity, the corresponding value increases monotonically with \( L \).
Size dependence of the various physical quantities at $\beta = 0.217$.

| $L$ | $\xi_L$ | $g_R^{(4)}$ | $g_R^{(6)}$ | $g_R^{(8)} \times 10^{-4}$ |
|-----|---------|-------------|-------------|---------------------------|
| 8   | 3.93(0) | 9.86(2)     | 492.5(1.9)  | 5.45(3)                   |
| 12  | 4.85(1) | 13.59(3)    | 852(4)      | 10.8(1)                   |
| 16  | 5.30(1) | 17.74(6)    | 1285(10)    | 16.0(3)                   |
| 20  | 5.50(1) | 20.95(11)   | 1591(15)    | 16.4(6)                   |
| 24  | 5.59(1) | 23.3(1)     | 1768(36)    | 15.4(1.8)                 |
| 28  | 5.60(1) | 24.6(1)     | 1958(69)    | 18.7(4.7)                 |
| 32  | 5.62(1) | 25.1(3)     | 1990(187)   | 18.3(26.4)                |
| 36  | 5.62(1) | 26.0(4)     | 2089(278)   |                           |

(see also Table(2)) until it becomes $L$-independent, and can be regarded as the corresponding thermodynamic value. The standard theory of FSS shows that the thermodynamic limit is reached beyond a certain universal value of $L/\xi$, regardless of the value of the temperature.

The correlation length in Table(1) measured under the condition $L/\xi_L \geq 4$ seems to be almost $L$-independent. However, this is certainly not the case for other renormalized quantities which continue to increase significantly until $L/\xi_L \approx 6$. Note also that for the six and eight point renormalized couplings the relative statistical errors increase drastically for larger values of $L$, whereas those for the correlation length remains negligibly small. Evidently, the requirement of larger $L$ for the traditional Monte Carlo measurements of the thermodynamic values of the RCC makes it virtually impossible to measure these quantities precisely in a deep scaling regime due to the large statistical fluctuations.

We would like to stress that the number of our Monte Carlo measurements is very large. For each $L$ we typically generated tens of millions of the single cluster configurations. For $L=28$, for example, Monte Carlo measurements were taken every 20th configuration out of a total of $2 \times 10^8$ configurations that were generated; yet, the relative statistical errors for the $g_R^{(6)}$ and $g_R^{(8)}$ amount to approximately 4% and 25% respectively.

In order to overcome the difficulty closer to $T_c$, we make use of a new FSS function $A_L(t) = A(t)Q_A(x(L,t))$, defined by the expression

$$A_L(t) = A(t)Q_A(x(L,t)), \quad x(L,t) \equiv \xi_L(t)/L. \quad (1)$$

Here $A_L(t)$ represents the quantity $A$ measured on a finite lattice of linear size $L$ at a reduced temperature $t$, with its corresponding thermodynamic value $A(t)$ scaling as $A(t) \sim t^{-\rho}$.

The technique is especially useful for our purpose, because it enables us to extract accurate thermodynamic values based on the Monte Carlo measurements on much smaller lattices. We just outline the FSS extrapolation technique used in this work. For a detailed explanation, we refer the readers to Ref.[3].

1. For a certain $t_0$, measure $A_L(t_0)$ and $x(L,t_0) = \xi_L(t_0)/L$ for increasing $L$.
2. Determine $A(t_0)$ by measuring $A_L(t_0)$ such that it is $L$ independent.
3. Fit $(x(L,t_0), A_L(t_0)/A(t_0))$ data to a suitable functional form. In this work we used the ansatz, $Q(x) = 1 + c_1 e^{-1/x} + c_2 e^{-2/x} + \ldots$.
4. For any other $\rho$, choose a suitable $L$, measure the value of $x(L,t) \equiv \xi_L/L$ and $A_L(t)$, and interpolate $Q(x(L,t))$.
5. Extract $A(t)$ by plugging $A_L(t)$ and $Q(x(L,t))$ into Eq.(1).

Our $t_0$ is $\beta = 0.217$ and $0.220$, and our largest value of $L$ for the extraction of the thermodynamic values up to $\beta = 0.2213$ is 72.

### 3. Result and Discussion

At $\beta = 0.217$ we have been unable to determine the precise value of the thermodynamic value of
Table 3
Thermodynamic values of the four and six point RCC extracted by the FSS technique for some temperatures over $0.217 \leq \beta \leq 0.2213$.

| $\beta$   | 0.217   | 0.219   | 0.220   | 0.2206  | 0.2210  | 0.2212  | 0.2213  |
|-----------|---------|---------|---------|---------|---------|---------|---------|
| $g_R(4)$  | 25.6(4) | 25.3(1.1)| 24.7(5) | 24.6(7) | 24.5(3) | 24.5(3) | 24.4(3) |
| $g_R(6)$  | 2100(100)| 2074(144)| 1982(157)| 2006(157)| 1949(146)| 1966(135)| 1952(127)|

$g_R(6)$ due to the large statistical noise in the measurements for $L=32$ and 36, but we estimate that $g_R(6)(\beta = 0.217) \simeq 2100(100)$. With this value, we have been able to determine the thermodynamic values of the $g_R(6)$ up to $\beta = 0.2213$ (where $\xi \simeq 29.2(1)$). Of course, all other quantities including correlation length have been determined without any ambiguity since their thermodynamic values can be accurately measured. We summarize our results for the RCC in Table(3), from $\beta = 0.217$ to $\beta = 0.2213$.

We observe that $g_R(4)$ decreases quite slowly as $\xi$ increases, but eventually the value becomes stabilized around 24.5(2). $g_R(6)$ has a tendency to decrease as well, but within the statistical fluctuations it remains a constant for $\beta \geq 0.2210$, i.e., $g_R(6) \simeq 1956(145)$. The statistics of our $g_R(8)$ data at $\beta = 0.217$ to $\beta = 0.2213$ are not good enough for the application of our FSS technique. At $\beta = 0.2210$, however, we have been able to measure $g_R(8)$ quite precisely (Table(2)) by varying $L/\xi_L$ up to 3.73, where we find $g_R(8) = 1.63(33) \times 10^5$. Again we observe a systematic increase of $g_R(8)$ with increasing $L$, and estimate $g_R(8) \simeq 1.75(25) \times 10^5$. Our results indicate that all the RCC “scale” with the corresponding values of the critical exponents zero. Since the susceptibity and the correlation length already “scale” over this regime of the temperature, such pathological behavior as a sudden change of them closer to criticality is very unlikely.

Our value of $g_R(4)$ is in reasonable agreement with other estimates, but our estimate of $g_R(6)$ is more than 40% larger than those obtained from the measurement of the probability density of the order parameter, and from average action method and linked cluster expansion. The disagreement in $g_R(8)$ is more conspicuous; our value is approximately three times larger than those estimated from the other methods. The Monte Carlo measurements for the probability distribution in Ref.[8], however, were made under the condition $L/\xi_L \simeq 4$, which is probably too small. Our result contradicts the claims in the recent literature that the effective potential can be very well approximated by the terms up to $\phi^{10}$. On the other hand, our estimates of the renormalized couplings are significantly smaller than those estimated by the dimensional expansion. It has been noticed lately, however, that the higher order terms in the expansion become negative.

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