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on Some Hyperbolic Graphs

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GROWTH SERIES AND RANDOM WALKS ON SOME HYPERBOLIC GRAPHS

LAURENT BARTHOLDI AND TULLIO G. CECCHERINI-SILBERSTEIN

Abstract. Consider the tessellation of the hyperbolic plane by \( m \)-gons, \( \ell \) per vertex. In its 1-skeleton, we compute the growth series of vertices, geodesics, tuples of geodesics with common extremities. We also introduce and enumerate holly trees, a family of reduced loops in these graphs.

We then apply Grigorchuk’s result relating cogrowth and random walks to obtain lower estimates on the spectral radius of the Markov operator associated with a symmetric random walk on these graphs.

1. Introduction

We consider the graphs \( \mathcal{X}_{\ell,m} \) introduced by Floyd and Plotnick in [FP87]. These graphs are \( \ell \)-regular and are the 1-skeleton of a tessellation of the sphere (if \( (\ell - 2)(m - 2) < 4 \)), the Euclidean plane (if \( (\ell - 2)(m - 2) = 4 \)) or the hyperbolic plane (if \( (\ell - 2)(m - 2) > 4 \)) by regular \( m \)-gons. These tessellations were studied by Coxeter [Cox54]. When \( m = \ell = 4g \), then \( \mathcal{X}_{\ell,m} \) is the Cayley graph of the fundamental group \( J_g = \pi_1(\Sigma_g) \) of an orientable compact surface \( \Sigma_g \) of genus \( g \), with respect to the usual set of generators \( S_g = \{a_1, b_1, \ldots, a_g, b_g\} \):

\[
J_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] \rangle.
\]

The growth series for \( J_g \) with respect to \( S_g \), namely

\[
F_{J_g}(X) = \sum_{s \in J_g} X^{|s|} = \sum_{n=0}^{\infty} a_n X^n,
\]

where \( |s| = \min \{t : s = s_1 \ldots s_t, s_i \in S_g \cup S_g^{-1}\} \) denotes the word length of \( s \) with respect to \( S_g \) and \( a_n = |\{s \in J_g : |s| = n\}| \), was computed by James Cannon and Wagreich in [Can83] and [CW92] and shown to be rational, indeed

\[
F_{J_g}(X) = \frac{1 + 2X + \cdots + 2X^{2g-1} + X^{2g}}{1 - (4g - 2)X - \cdots - (4g - 2)X^{2g-1} + X^{2g}};
\]

moreover they showed that the denominator is a Salem polynomial.

In [FP87] and [FP94], Floyd and Plotnick, among other things, extended the calculations of Cannon and Wagreich to the family \( \mathcal{X}_{\ell,m} \). Fixing arbitrarily a base point \( * \in V(\mathcal{X}_{\ell,m}) \) and denoting by \( |x| \) the graph distance between the vertices \( x \) and \( * \), they obtained the following formula for the growth series \( F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{2g-1} + X^{2g}}{1 - (4g - 2)X - \cdots - (4g - 2)X^{2g-1} + X^{2g}}. \)
\[ \sum_{x \in V(\mathcal{X}_{\ell,m})} x^{[x]} \text{ for } m \text{ even}: \]
\[ F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{\frac{m}{2}-1} + X^{\frac{m}{2}}}{1 - (\ell - 2)X - \cdots - (\ell - 2)X^{\frac{m}{2}-1} + X^{\frac{m}{2}}} \]

(they also gave a formula for \( m \) odd).

In [BC01] it is shown that even in this more general setting the denominator is, after simplification by \( 1 + X \) in case \( m \equiv 2 \mod 4 \), a Salem polynomial. As a consequence of this the growth rates of these graphs are Salem numbers. One also obtains more precise information about the growth coefficients: if \( F_{\ell,m}(X) = \sum_{n \geq 0} a_n X^n \), then there exist constants \( K, \lambda \) and \( R \) such that
\[ K\lambda^n - R < a_n < K\lambda^n + R \]
holds for all \( n \). This improves on a result by Coornaert [Coo93].

The calculations in [Can83], [CW92] and [FP87] are based on linear relations among generating functions. We recover these as derivations in linear grammars, and describe a context-free grammar enumerating more complicated objects (see Subsection 3.6).

We then compute the growth series of finite geodesics in \( \mathcal{X}_{\ell,m} \) starting from \(*\):
\[ G_{\ell,m}(X) = \sum_{(\gamma, s) \text{ geodesic starting at } *} X^{[\gamma]} = \sum_{s \in V(\mathcal{X}_{\ell,m})} \lambda(s) X^{[s]}, \]
where \( \lambda(s) \) is the number of geodesics from \(*\) to \( s \in V(\mathcal{X}_{\ell,m}) \), obtaining for instance for \( m \) even
\[ G_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{\frac{m}{2}-1} + X^{\frac{m}{2}} + (X^{\frac{m}{2}} - X^{\frac{m}{2}-1})}{1 - (\ell - 2)X - \cdots - (\ell - 2)X^{\frac{m}{2}-1} + X^{\frac{m}{2}} + (X^{\frac{m}{2}} - X^{\frac{m}{2}-1})} \]
(see the formula in subsection 3.4 for \( m \) odd); and the growth series of pairs of geodesics starting at \(*\) and with same endpoint:
\[ H_{\ell,m}(X) = \sum_{(\gamma, \delta) \text{ geodesics starting at } *} X^{[\gamma]} \frac{1}{2} \sum_{s \in V(\mathcal{X}_{\ell,m})} \lambda(s)^2 X^{[s]}, \]
obtaining again for \( m \) even
\[ H_{\ell,m}(X) = 1 + \frac{\ell}{1 - (\ell - 2)X - \cdots - (\ell - 2)X^{\frac{m}{2}-1} + X^{\frac{m}{2}} + X^{\frac{m}{2}-1}(X - 1)(2X^{\frac{m}{2}-1} - 3)} \]
(see subsection 3.5 for \( m \) odd). By computing the radius of convergence of this last series we obtain estimates on the asymptotic number of closed paths starting at \(*\).

Better estimates in this direction are obtained by counting a larger category of closed paths we introduced and called holly trees; all these calculations are based on generating functionological methods [Wil90].

Upper estimates for the spectral radius of the Markov operators associated to a simple random walk on fundamental groups of surfaces were calculated with various methods in [BCCH97]. Here, applying Grigorchuk’s theorem which relates cogrowth and spectral radius, we obtain lower estimates for the graphs \( \mathcal{X}_{\ell,m} \). We remark that the estimates we obtain in the case of surface groups are sharper than any other we know of, including Paschke’s [Pas93]. Explicitly, for the surface group of genus 2 we obtain the estimate of the spectral radius \( \mu_2 \) of \( \mathcal{X}_{8,8} \)
\[ \mu_2 \geq 0.6623. \]
The paper is organized as follows. In Section 2 we recall notions about graphs, groups, random walks, growth and cogrowth as well as some facts about the $X_{t,m}$ graphs. In Section 3 we describe grammars and compute using them, growth series of vertices, geodesics, pairs of geodesics and holly trees (a family of reduced closed curves) in $X_{t,m}$. Finally in Section 4 we use the results of Section 3 to obtain estimates on the spectral radius of the $X_{t,m}$.

2. Preliminaries

2.1. Cayley graphs. Our results are best expressed in terms of graph theory, but the subject was first approached through group theory. We stress the connection between these views.

2.1.1. Groups. Let $G$ be a finitely generated group, and denote by $S$ a finite symmetric ($S = S^{-1}$) generating subset of $G$. Writing $S = S_+ \cup S_+^{-1}$ we view $G$ as a quotient of $F_{S_+}$, the free group on $S_+$, by a normal subgroup $N$, namely $G = F_{S_+}/N$. In other words, if $G$ has presentation $(S_+ \mid \mathcal{R})$, then $N$ is the normal closure in $F_{S_+}$ of the set of relations $\mathcal{R}$.

As $G = \bigcup_{n \geq 0} S^n$, we define the word length of an element $g \in G$ by

$$|g| = \min\{n : g \in S^n\},$$

and the distance between two elements $g, h \in G$ by

$$d(g, h) = |g^{-1}h|.$$ 

This way $(G, d)$ becomes a metric space. (If $S$ is replaced by another finite generating set $S'$, the new distance $d'$ will be equivalent to $d$.)

2.1.2. Graphs. Let now $\mathcal{G}$ be a graph, with vertex set $V(\mathcal{G})$ and edge set $E(\mathcal{G}) \subset V(\mathcal{G}) \times V(\mathcal{G})$. We suppose $E(\mathcal{G})$ is symmetrical, i.e. is invariant under the flip $F$ on $V(\mathcal{G}) \times V(\mathcal{G})$ defined by $F([x, y]) = (y, x)$, and that $E(\mathcal{G})$ contains no self-loop $(x, x)$.

An orientation of $\mathcal{G}$ is a (choice of) a subset $\overrightarrow{E} \subset E(\mathcal{G})$ such that $E(\mathcal{G}) = \overrightarrow{E} \sqcup F(\overrightarrow{E})$.

If $(x, y) \in E(\mathcal{G})$ we shall refer to the vertices $x$ and $y$ as neighbours and write $x \sim y$. The degree $d(x)$ of a vertex $x$ is the number of its neighbours: $d(x) = |\{y : y \sim x\}|$; if this number is independent of $x$, namely $d(x) = d(y) = d$ for all $x, y \in V(\mathcal{G})$, we say that $\mathcal{G}$ is regular of degree $d$.

A path in $\mathcal{G}$ is a sequence $\gamma = (\gamma_0, \ldots, \gamma_n)$ of vertices of $\mathcal{G}$ with $\gamma_i \sim \gamma_{i+1}$. We call $\gamma_0$ and $\gamma_n$ respectively the start and end of $\gamma$. The integer $n = |\gamma|$ is the length of $\gamma$. The inverse of the path $\gamma = (\gamma_0, \ldots, \gamma_n)$ is $\gamma^{-1} = (\gamma_n, \ldots, \gamma_0)$; its start is the end of $\gamma$, and its end is the start of $\gamma$.

From now on we will suppose an arbitrary base point $* \in V(\mathcal{G})$ has been fixed and will refer to $(\mathcal{G}, *)$ as a pointed graph. We will also suppose $\mathcal{G}$ is connected, i.e. any two points are connected by a path, and locally finite, i.e. $d(x) < \infty$ for all $x \in V(\mathcal{G})$. (It then follows that for any $n$ there is a finite number of paths of length $n$ starting at $*$.)

The vertex set $V(\mathcal{G})$ becomes a metric space when equipped with the distance

$$d(x, y) = \min\{n : \text{there exists a path } \gamma \text{ with } \gamma_0 = x, \gamma_n = y\};$$

this metric is called the graph distance. For $x \in V(\mathcal{G})$ we write $|x| = d(*, x)$.

A path $\gamma$ of length $n$ is said to be proper if $\gamma_{i-1} \neq \gamma_{i+1}$ for all $i = 1, \ldots, n - 1$; closed if $\gamma_n = \gamma_0$; geodesic if $d(\gamma_0, \gamma_n) = n$ (and thus $d(\gamma_i, \gamma_j) = |i - j|$ for all $i, j$); and a loop if it is both proper and closed. We write $\mathfrak{P}$, $\mathfrak{C}$, $\mathfrak{S}$, $\mathfrak{L}$ for the set
It is obvious that all trees are two-colourable. Any graph \( G \) such that \( G \) is regular, \( \sim \) is a tree. There is a canonical map \( \pi : \tilde{G} \rightarrow G \) given by \( \pi(\gamma) = \gamma[\gamma] \) that is a graph homomorphism. If \( G \) is regular, \( \tilde{G} \) is also regular, of the same degree.

If \( G \) is a tree (or more generally a two-colourable graph) we may orient \( \tilde{G} \) by choosing for \( E \) the edges \( (x,y) \) with \( x \sim y \) and \( |x| < |y| \). We call this orientation the \textit{radial orientation}.

\subsection{Cayley graphs.} One associates with a finitely generated group \( G \) and finite symmetric generating system \( S \) (i.e., an \( S \) with \( |S| < \infty \), \( S = S^{-1} \), \( 1 \not\in S \) and \( S \mapsto G \)) its \textit{Cayley graph} \( \mathcal{G} = \text{Cay}(G, S) \), where \( V(\mathcal{G}) = G \) and \( E(\mathcal{G}) = \{(g, gs) : g \in G, s \in S\} \). The edge \( e = (g, gs) \) is labelled by \( \lambda(e) = s \). The base point \( * \) of \( \mathcal{G} \) is the vertex corresponding to the neutral element \( 1 \) in \( G \). Cayley graphs are connected and regular of degree \( |S| \). More generally, if \( G = \mathbb{F}_{S_+}/N \), the universal cover \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \) is the Cayley graph of \( \mathbb{F}_{S_+} \) w.r.t. \( S_+ \), and \( \pi : \tilde{\mathcal{G}} \rightarrow G \) is induced by the canonical quotient map \( \pi : \mathbb{F}_{S_+} \rightarrow G \).

There is a one-to-one correspondence between paths in \( \mathcal{G} \) starting at \( 1 \) and words \( w \) on the alphabet \( S \); it is given explicitly by

\[ \gamma = (\gamma_0, \ldots, \gamma_n) \mapsto \lambda(\gamma_0, \gamma_1) \cdots \lambda(\gamma_{n-1}, \gamma_n) \]

and

\[ w = w_1 \cdots w_n \mapsto (1, w_1, w_1w_2, \ldots, w_1 \cdots w_n). \]

Under this correspondence a path \( \gamma \) is proper if and only if \( w_i \neq w_{i+1}^{-1} \) for all \( i \); the word corresponding to a proper path from \( 1 \) to \( w \) is the normal form of the element \( w \in \mathbb{F}_{S_+} \), and in this case the correspondence is even an isometry: \( |\gamma| = |w| \). Also, \( \gamma \) is geodesic if and only if \( |w_1 \cdots w_i| = i \) for all \( i \), and \( \gamma \) is closed if and only if \( \pi(w) = 1 \). Loops starting at \( 1 \) correspond to elements in \( N = \ker(\pi : \mathbb{F}_{S_+} \rightarrow G) \).

We omit the proof of the following easy lemma:

\textbf{Lemma 2.1.} For a Cayley graph \( \mathcal{G} = \text{Cay}(G, S) \) the following conditions are equivalent:

\begin{itemize}
    \item[(1)] \( \mathcal{G} \) is two-colourable;
    \item[(2)] all loops in \( \mathcal{G} \) have even length;
    \item[(3)] all closed paths in \( \mathcal{G} \) have even length;
    \item[(4)] all relators in \( N \) have even length;
    \item[(5)] there is a morphism \( \phi : G \rightarrow \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \) with \( \phi(S) = \{1\} \).
\end{itemize}

\subsection{Growth and cogrowth.} Given a pointed graph \((\mathcal{G}, *)\), its \textit{growth function} is

\[ \gamma_{\mathcal{G}}(n) = |\{x \in V(\mathcal{G}) : |x| = n\}|. \]
Its generating series is
\[ F_G(X) = \sum_{n \geq 0} \gamma_G(n)X^n. \]

The growth rate of \( G \) is
\[ \rho_G = \limsup_{n \to \infty} \sqrt[n]{\gamma_G(n)} = R^{-1}, \]
where \( R \) is the radius of convergence of \( F_G \). Note that \( \rho_G \) is not dependent on the choice of \( * \), although \( F_G \) is. We say \( G \) has exponential growth if \( \rho_G > 1 \).

If \( G \) is a group generated by a finite set \( S \) and \( G \) is the corresponding Cayley graph, the growth function of \( G \) (relative to \( S \)) is that of the pointed graph \((G, 1)\), where \( 1 \) is the unit element in \( G \).

If \( Y \subseteq V(G) \), the relative growth of \( Y \) in \( G \) is
\[ \gamma_Y^G(n) = \{|y \in Y : |y| = n\}|. \]

We define similarly its generating series and relative growth rate. In particular, the relative growth rate of the set of loops \( L_+ \) viewed as a subset of \( V(G) \) is called the cogrowth of \( G \), which we shall denote by \( \alpha_G \); again this number \( \alpha_G \) is not dependent on the choice of \( * \). In case \( G = \text{Cay}(G, S) \), this is the relative growth rate of \( N = \ker(\pi) \) in \( F_S \), and is called the cogrowth of \((G, S)\).

The following estimates for a regular graph of degree \( d \geq 2 \), due to Grigorchuk [Gri80], hold:
\[ \sqrt{d-1} \leq \alpha_G \leq d - 1, \]
unless \( G \) is a tree; in that case \( \alpha_G \) is defined to be 1, even though the associated growth series has infinite radius of convergence.

2.3. Random walks. Let \( G \) be a connected graph, regular of degree \( d \) and denote by \( \ell^2(V(G)) = \{ f : V(G) \to \mathbb{C} \text{ s.t. } \|f\| = \sum_{x \in V(G)} |f(x)|^2 < \infty \} \) the Hilbert space of square integrable complex functions defined on the vertex set of \( G \). The simple random walk [Woe00] on \( G \) is given by the stochastic transition matrix
\[ P = (p(x, y))_{x, y \in V(G)}, \]
with
\[ p(x, y) = \begin{cases} 1/d & \text{if } x \sim y \\ 0 & \text{otherwise}. \end{cases} \]

The associated Markov operator is the bounded linear operator \( M : \ell^2(V(G)) \to \ell^2(V(G)) \) given by
\[ M(\phi)(x) = \frac{1}{d} \sum_{y \sim x} \phi(y) \quad \text{for } \phi \in \ell^2(V(G)), x \in V(G). \]

Let us denote by \( \{\delta_x : x \in V(G)\} \) the canonical orthonormal basis of \( \ell^2(V(G)) \); then \( p(x, y) = (M\delta_x | \delta_y) \) expresses the probability of going to \( y \) in one step, starting at \( x \). More generally \( p^{(n)}(x, y) = (M^n\delta_x | \delta_y) \) is the probability of going to \( y \) in \( n \) steps, starting at \( x \).

The spectral radius of the simple random walk on \( G \) is then
\[ \mu = \limsup_{n \to \infty} \sqrt[n]{p^{(n)}(\ast, \ast)} = \|M\|. \]

One can link \( \mu \) to the asymptotic growth of closed paths in \( G \); there are approximately \((\mu d)^n\) such paths of length \( n \), at least when \( n \) is even.
Kesten obtained in [Kes59] the following estimates:
$$\frac{2\sqrt{d-1}}{d} \leq \mu \leq 1,$$
with equality on the left if and only if \(G\) is the regular tree of degree \(d\), and equality on the right if and only if \(G\) is amenable: see for instance [DK88] and [CGH99].

2.4. The Grigorchuk formula. In his thesis, Grigorchuk [Gri80] found the following relation between the cogrowth \(\alpha\) of \(G\) and the spectral radius \(\mu\) of the simple random walk on a regular graph \(G\) of degree \(d\), namely

\[
\mu = \begin{cases} 
\frac{2\sqrt{d-1}}{d} & \text{if } \alpha \leq \sqrt{d-1}, \\
\frac{\sqrt{d-1}\left(\frac{\sqrt{d-1}}{\alpha} + \frac{\phi}{\sqrt{d-1}}\right)}{d} & \text{if } \alpha > \sqrt{2d-1}.
\end{cases}
\]

A generalization of (1), along with a simple proof can be found in [Bar99].

Since the function \(\mu(t) = (\sqrt{d-1}/d)\left(\sqrt{d-1}/t + t/\sqrt{2d-1}\right)\) is an increasing function of \(t\) for \(t \in [\sqrt{d-1}, d-1]\), any lower bound on \(\alpha\) yields a lower bound on \(\mu\).

2.5. Surface groups. Let \(J_g = \pi_1(\Sigma_g)\) denote the fundamental group of an orientable surface of genus \(g\). This group is finitely generated and admits the following presentation:

\[J_g = \left\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] \right\rangle.\]

Similarly denoting by \(J'_g = \pi(\Sigma'_g)\) the fundamental group of a non-orientable surface of genus \(g \geq 1\), we have

\[J'_g = \left\langle a_1, \ldots, a_g \mid \prod_{i=1}^g [a_i^2] \right\rangle.\]

A simple justification of these presentations is given in the first chapters of [Mas67].

The Cayley graphs corresponding to these groups with generators specified as above constitute the 1-skeleton of planar tessellations, the underlying space being the sphere \(S^2\) for \(J_g\) and \(J'_1\); the Euclidean plane \(\mathbb{R}^2\) for \(J_1\) and \(J'_2\); and the hyperbolic plane \(\mathbb{H}^2\) for \(J_g\) such that \(g \geq 2\) and \(J'_g\), \(g \geq 3\). The 2-cells of the tessellation are \(4g\)-gons (respectively \(2g\)-gons) for \(J_g\) (respectively \(J'_g\)). These facts are used and developed in [Can83] and [LS70, chapter III, §5].

2.6. The graphs \(X_{\ell,m}\). We shall be concerned with the more general graphs \(X_{\ell,m}\) that are the 1-skeleton of regular tessellations of a constant-curvature surface consisting of \(m\)-gons, with vertex angles \(2\pi/\ell\) [FP87]. We assume that \(\ell \geq 3\) and \(m \geq 3\). It is proved for instance in [Cox54] that such a tiling by polygons exists and is unique up to isometry. We fix once and for all an arbitrary base point \(* \in V(\mathcal{X}_{\ell,m})\).

When \((\ell-2)(m-2) > 4\), these graphs can be embedded quasi-isometrically in \(\mathbb{H}^2\), the Poincaré-Lobachevskii plane. When \((\ell-2)(m-2) = 4\), they embed in Euclidean \(\mathbb{R}^2\) space. When \((\ell-2)(m-2) < 4\), they embed in \(S^2\), the sphere. For instance, if \((\ell, m) = (3, 3)\) we have the tetrahedral tesselation of the sphere; if \((\ell, m) = (4, 3)\) the octahedral tesselation; if \((\ell, m) = (5, 3)\) the icosahedral tesselation; if \((\ell, m) = (3, 4)\) the cubical tesselation; and if \((\ell, m) = (3, 5)\) the dodecahedral tesselation of the sphere. For \((\ell, m) = (3, 6), (4, 4), (6, 3)\) we have respectively the hexagonal, square
and triangular Euclidean tessellations; and in all other cases a hyperbolic tessellation. When \( \ell = m = 4g \) they are the Cayley graphs of \( J_g \) and when \( \ell = m = 2g \) the Cayley graphs of \( J'_g \), with respect to their canonical generating sets.

By Lemma 2.1, these graphs are two-colourable if and only if \( m \) is even.

Thanks to the underlying space and the presentation of these graphs as tessellations, there is a natural notion of cell or face. These are closures of connected components of the complement of \( X_{\ell,m} \) in the embedding space. Cell boundaries are precisely the simple loops in \( X_{\ell,m} \) of length \( m \).

Recall the graph-theoretical notion of dual graph. Given a graph \( \mathcal{G} = (V, E) \) embedded in a surface with set of cells \( C \), its dual is the graph \( \mathcal{G}' = (C, F) \), with an edge \((C, D) \in F \) whenever \( C \) and \( D \), viewed as two-cells in \( \mathcal{G} \), share a common edge in \( E \). In our setting the dual of \( X_{\ell,m} \) is \( X_{m,\ell} \). In particular, the Cayley graphs of surface groups are self-dual.

**Definition 2.1.** Let \( \mathcal{G} \) be a pointed graph. The cone of a vertex \( x \in V(\mathcal{G}) \) is the subgraph spanned by the set of vertices

\[ \{ y \in V(\mathcal{G}) : \exists \gamma \in \mathcal{G}_* \text{ with } \gamma_i = x, \gamma_j = y, \text{ for some } i \leq j \}. \]

This is the set of vertices which may be joined starting from the base point \( * \) by a geodesic passing through \( x \), namely \( \gamma = (*, x, \ldots, y) \).

Two cones are said to be equivalent if there exists a graph isomorphism mapping one cone onto the other. The cone type of a vertex \( x \) is the class \( t(x) \) of its cone modulo this equivalence.

Call a vertex \( x \) a successor of \( y \) if \( x \sim y \) and \(|x| > |y| \) (hence \(|x| = |y| + 1\)). Symmetrically \( y \) is a predecessor of \( x \). If \( x \sim y \) and \(|x| = |y| \) call \( x \) and \( y \) peers.

Graphically, if \( y \) is a successor of \( x \), we orient the edge \((x, y)\) in the direction \( x \to y \) and we say that this arrow exits from \( x \) and enters in \( y \). It follows easily from the definition that the cone type \( t(x) \) determines the number \( \ell(t) \) and the cone types of its successors.

We label the edges of \( X_{\ell,m} \) using the alphabet \( \{ e_1, \ldots, e_\ell, e_p \} \), as follows: arbitrarily label clockwise \( e_1, \ldots, e_\ell \) the edges exiting from \( * \). At each vertex \( x \in \text{Vert}(X_{\ell,m}) \), label from left to right \( e_1, \ldots, e_{\ell(t)} \) the edges exiting from \( x \). If \( x \) has a peer \( y \), label \( e_p \) the edge between \( x \) and \( y \).

Note that if \( x \) and \( y \) have the same cone type, then there is a edge-label-preserving isomorphism between their cones.

The graphs \( X_{\ell,m} \) have a finite number of cone types. This was first proven by Cannon for the Cayley graphs of surface groups [Can83] and later extended to the hyperbolic \( X_{\ell,m} \) by Floyd and Plotnick [FP87]. The cone types can be described in several equivalent ways among which we propose the following. First suppose \( m \) is even.

**Definition 2.2.** For \( x \in \text{Vert}(X_{\ell,m}) \), with \( m \) even, set

\[ t(x) = \max_{(F,y)}: F \ni x \subseteq F, y \in F (|x| - |y|) = |x| - \min_{(F,y)}: F \ni x, y \in F |y|. \]

In the expression above \( F \) denotes a cell of the tessellation, i.e. an \( m \)-gon.

One has \( t(x) \in \{0, \ldots, m/2\} \). Also, \( t(x) = 0 \) holds only for \( x = * \).

Furthermore, cone types appear in two mirror versions, depending on which side (left or right) contains the 2-cell closest to origin. We extend the cone types of vertices in \( X_{\ell,m} \) as follows: if \( v \) has type 0 or 1, then its extended type is the same. If \( v \) has type \( t \geq 2 \), and \( v \) is on the left side of its lowest adjacent 2-cell, then its
The cone type data for even $m$

| $t$ | $1$ | $2$ | $t+1$ | predecessors |
|-----|-----|-----|-----|--------------|
| 0   | $\ell$ | 0  | 0   | 0            |
| $0 < t < \frac{m}{2}$ | $\ell - 3$ | 1  | 1   | 1            |
| $\frac{m}{2}$ | $\ell - 4$ | 2  | 0   | 2            |

Table 1. The cone type data for even $m$

Extended type is $(L, t)$. If $v$ is on the right side of that 2-cell, its extended type is $(R, t)$.

Extended types will be used in the computations of Section 3. For now, we describe diagrammatically the (left) cone types, letting $t$ be any type in $\{1, \ldots, m/2\}$:

The data we will use are summarized in Table 1. For instance, a vertex of type $m/2$ has two successors of type 2, $\ell - 4$ of type 1 and two predecessors of type $m/2 - 1$.

We now consider the case when $m$ is odd. As said earlier $\mathcal{X}_{t,m}$ is not two-colourable: in fact there exist peers in $\mathcal{X}_{t,m}$. If we suppress all edges $(x, y)$ connecting peers we obtain a tesselation by $(2m - 2)$-gons, which we call temporarily $\overline{\mathcal{X}}_{t,m}$ (note that the corresponding graph is no more regular).

Definition 2.3. For $x \in \text{Vert}(\mathcal{X}_{t,m})$, with $m$ odd, set

$$t(x) = \max_{(F, y) : F \ni x, y \in F} (|x| - |y|) = |x| - \min_{(F, y) : F \ni x, y \in F} |y|.$$ 

In the expression above $F$ denotes a cell of $\overline{\mathcal{X}}_{t,m}$, i.e. a $(2m - 2)$-gon.

Obviously $t(x) \in \{0, \ldots, m - 1\}$. Also, $t(x) = 0$ holds only for $x = *$. 
As before, the (left) cone types can be represented by

```
1 1 1 1
1 1 1 1
1 1 1 1
```

The data we will use are summarized in Table 2. The two predecessors of a vertex of type \( m - 1 \) are of type \( m - 2 \); the peer of a vertex of type \( (m - 1)/2 \) also has type \( (m - 1)/2 \).

2.7. Hyperbolicity of the graphs \( X_{t,m} \). Let \((X,d)\) be a geodesic metric space and \( \delta > 0 \). We say \( X \) is Gromov-\( \delta \)-hyperbolic if geodesic triangles are \( \delta \)-thin in the following sense: for all \( x, y, z \in X \) one has

\[
\forall t \in [t, t+1], \quad d(t, \gamma) < \delta
\]

where \( \gamma \) is a geodesic segment between \( y \) and \( z \).

As a special case, two geodesics \( \gamma \) and \( \gamma' \) from \( x \) and \( y \) in \( X \) are at distance at most \( \delta \) apart:

\[
\forall t \in \gamma, \quad \exists t' \in \gamma' \text{ such that } d(t, t') < \delta
\]

(take \( z = y \) in (2)).

A finitely generated group \( G \) is hyperbolic if its Cayley graph \( G = \text{Cay}(G,S) \) is hyperbolic as a graph (with a constant of hyperbolicity \( \delta_S \) which depends on the
generating system $S$): this is well defined since for any other generating system $S'$ the graph $G' = \text{Cay}(G, S')$ will still be hyperbolic (with a possibly different constant of hyperbolicity $\delta_{G'}$).

Finite graphs and finite groups are clearly hyperbolic.

The groups $J_g$ are hyperbolic [GH90, Gro93] as soon as $g \neq 1$, and $J'_g$ are hyperbolic as soon as $g \neq 1, 2$. The other cases are well-known: $J_0$ is trivial, $J_1$ is $\mathbb{Z}^2$, $J'_2$ is the two-group and $J'_3$ is a twofold extension of $\mathbb{Z}^2$.

Our graphs $X_{r,m}$ are hyperbolic whenever $(l - 2)(m - 2) \neq 4$: indeed one may take $\delta = w$ for even $m = 2w$ and odd $m = 2w + 1$.

The following lemma strengthens for the graphs $X_{r,m}$ the situation described in (3):

**Lemma 2.2.** Let $G$ be a hyperbolic $X_{r,m}$; let $\gamma$ and $\gamma'$ be two geodesics in $G$ of length $n$, with same extremities $\gamma_0 = \gamma'_0$ and $\gamma_n = \gamma'_n$. Then for all $i \in \{0, \ldots, n\}$, there exists a cell $F_i$ in $G$ such that both $\gamma_i$ and $\gamma'_i$ belong to $F_i$.

**Proof.** The proof is not difficult and relies on Euler characteristic considerations; see [BCCH97] and [Zuk97].

3. **Recursive Computations**

We present in this section the methods used to compute the growth series of various objects related to $X_{r,m}$: vertices, geodesics, etc. The unifying notion is that of *production grammar*. The procedure we will follow for each family of objects is: construct a grammar and compute the growth of its associated language.

3.1. **Grammars.** A *grammar* is a tuple $(N, T, S, R)$, with $N$ and $T$ disjoint finite sets, called respectively the non-terminal and terminal alphabets; $S \in N$ called the *axiom*; and $R$ a finite subset of $N \times (N \cup T)^*$ called the *set of rules*, where for an alphabet $A$, we denote by $A^*$ the set of finite-length words over $A$. A rule $(X, w) \in R$ is conveniently written $X \to w$.

Let $\Gamma = (N, T, S, R)$ be a grammar, and $u, v \in (N \cup T)^*$ two words. We say $v$ is *derived from* $u$, and write $u \Rightarrow v$, if there is a rule $(X, w) \in R$ and factorizations $u = aXb$, $v = awb$ for some $a \in (N \cup T)^*$ and $b \in T^*$. Let $\Rightarrow$ be the transitive closure of $\Rightarrow$: one has $u \Rightarrow v$ precisely when there is a sequence $u = u_0 \Rightarrow u_1 \Rightarrow \cdots \Rightarrow u_n = v$. The *language* of $\Gamma$ is

$$L(\Gamma) = \{w \in T^* : S \Rightarrow w\}.$$

The *intermediate derivations* of $\Gamma$ are the $w \in (N \cup T)^*$ such that $S \Rightarrow w$.

In the standard literature [Har78] our grammars are called *context-free grammars*. We restrict to this class of grammars. If all rules $(X, w) \in R$ satisfy $w \in T^*N$, the grammar is called *right-linear*. If they all satisfy $w \in NT^*$, the grammar is called *left-linear*. If for any word $w \in L(\Gamma)$ there is a unique sequence of derivations that yield $w$ from $S$, the grammar is called *unambiguous*. The *growth series* of $\Gamma$ is the formal power series

$$f_{\Gamma}(X) = \sum_{w \in L(\Gamma)} X^{|w|}.$$

As examples, consider $\Gamma = (\{S\}, \{a\}, S, \{S \to a, S \to aaS\})$. Then $L(\Gamma)$ consists precisely of those words $a^n$ for which $n$ is odd. Indeed the only derivation path is $S \to aaS \cdots \Rightarrow a^{2n} S \Rightarrow a^{2n+1}$. This grammar is right-linear and unambiguous. Consider also $\Gamma' = (\{S\}, \{(, )\}, S, \{S \to (S), S \to SS, S \to \varepsilon\})$, where $\varepsilon$ denotes the
empty sequence. \( L(\Gamma') = \{ \varepsilon, (, )(, ) \ldots \} \) consists precisely of those words over \( \{ (, ) \} \) that correspond to legal bracket nestings. This grammar is not regular.

We quote without proof the following results:

**Theorem 3.1** (Schützenberger [CS63]). Let \( \Gamma \) be an unambiguous grammar. Then its growth series is an algebraic function, i.e. there is a polynomial \( 0 \neq P(X,Y) \in \mathbb{Z}[X,Y] \) such that \( P(X, f_\Gamma(X)) = 0 \).

**Theorem 3.2** ([SS78, Har78]). Let \( \Gamma \) a (left- or right-)linear grammar. Then its growth series is rational, i.e. there exist polynomials \( P, Q \in \mathbb{Z}[X] \) with \( f_\Gamma(X) = P(X)/Q(X) \).

### 3.2. Growth series

By Theorem 3.2, the growth series of a (say, right-)linear grammar \( \Gamma \) is a rational function. We explicit here the computational method we use. Let \( V \) be the free \( \mathbb{R}[X] \)-module spanned by the \( f_A \) for all \( A \in \mathcal{N} \), the non-terminal alphabet. Let \( u \in V \) be the sum, over all rules \( A \rightarrow w \in T^* \), of \( X^{|w|} f_A \).

Writing \( f_A \otimes f_B^* \) for the matrix unit with a single \( 1 \) at position \( (A,B) \), let \( M \) be the sum, over all rules \( A \rightarrow wB \), of \( X^{|w|} f_A \otimes f_B^* \).

**Proposition 3.3.** Let \( \Gamma \) be an unambiguous grammar. Then the growth series of \( \Gamma \) is \( f_\Gamma = f_\Delta(1 - M)^{-1}u \).

**Proof.** By power series expansion, \( f_\Delta(1 - M)^{-1}u = \sum_{n \geq 0} f_\Delta^* M^nu. \) Any word produced by \( \Gamma \) has a unique derivation path \( S \Rightarrow w_1A_1 \Rightarrow w_1w_2A_2 \Rightarrow \ldots \Rightarrow w_1 \ldots w_n \). It corresponds to the term \( f_\Delta(X^{|w|} f_S \otimes f_{A_1}) \cdots (X^{|w_{n-1}|} f_{A_{n-1}} \otimes f_{A_n}) (X^{|w_n|} f_{A_n}) = X^{|w_1 \ldots w_n|} \).

### 3.3. Growth of vertices in \( X_{\ell,m} \)

We identify the set of vertices in \( X_{\ell,m} \) with a language over \( \{ \varepsilon, \ldots, \varepsilon_\ell, \varepsilon_p \} \), by selecting for each vertex \( x \in V(X_{\ell,m}) \) the label of the unique left-most geodesic from \( \ast \) to \( x \). This language can be described by a right-linear grammar, which we first present for even \( m = 2w \).

- its set of non-terminals is \( N = \{ X_0, X_1, X_{L,2}, \ldots, X_{L,w}, X_{R,2}, \ldots, X_{R,w-1} \} \);
- its set of terminals is \( T = \{ \varepsilon_1, \ldots, \varepsilon_\ell \} \);
- its axiom is \( X_0 \);
- its rules are

\[
\begin{align*}
X_0 & \rightarrow \varepsilon e_1 X_1 \ldots e_\ell X_1 \\
X_1 & \rightarrow \varepsilon e_1 X_{R,2} e_2 X_1 \ldots e_\ell-2 X_1 e_\ell-1 X_{L,2} \\
X_{L,t} & \rightarrow \varepsilon e_1 X_{R,2} e_2 X_1 \ldots e_\ell-2 X_1 e_\ell-1 X_{L,t+1} \text{ for } 2 \leq t \leq w - 1 \\
X_{R,t} & \rightarrow \varepsilon e_1 X_{R,t+1} e_2 X_1 \ldots e_\ell-2 X_1 e_\ell-1 X_{L,2} \text{ for } 2 \leq t \leq w - 2 \\
X_{R,w-1} & \rightarrow \varepsilon e_2 X_1 \ldots e_\ell-2 X_1 e_\ell-1 X_{L,2} \\
X_{L,w} & \rightarrow \varepsilon e_1 X_{R,2} e_2 X_1 \ldots e_\ell-3 X_1 e_\ell-2 X_{L,2}.
\end{align*}
\]

**Proposition 3.4.** The above grammar’s language is the set of vertices of \( X_{\ell,m} \), represented by their leftmost geodesic from \( \ast \).

**Proof.** The non-terminal \( X_0 \) expresses the cone of \( \ast \), that is, the whole graph; \( X_1 \) expresses the cone of a vertex of type 1. The non-terminal \( X_{L,t} \) expresses the cone of a type-1 vertex on the left of a 2-cell, and similarly for \( X_{R,t} \).

The set of rules for a non-terminal \( X_t \), \( X_{L,t} \) or \( X_{R,t} \) corresponds to the decomposition of the cone of a vertex of extended cone type \( t, (L,t) \) or \( (R,t) \) into subcones at its successors.
The vertices in the cone at a vertex \( x = e_{i_1} \ldots e_{i_t} \) are derived from an intermediate derivation of the form \( e_{i_1} \ldots e_{i_s} X_s \), for some \( s \in \{0, 1, (L, 2), \ldots, (L, w - 1), (R, 2), \ldots, (L, w)\} \). For instance, \( x \) itself is derived from \( xX_s \) by the rule \( X_s \rightarrow \epsilon \).

Note that the only difference between left and right is the extra rule \( e_{\ell - 1}X_{L,w} \) which has no right counterpart. In that way the vertex of type \( w \) derived from \( X_{L,w} \), which has two predecessors, is produced only once from its left predecessor.

We now present the grammar for \( \text{Vert}(X_{\ell,m}) \) for odd \( m = 2w + 1 \), omitting its analogous proof:

- its set of non-terminals is \( N = \{X_0, X_1, X_{L,2}, \ldots, X_{L,m}, X_{R,2}, \ldots, X_{R,m-1}\} \);
- its set of terminals is \( T = \{e_1, \ldots, e_{\ell}\} \);
- its axiom is \( X_0 \);
- its rules are

\[
\begin{align*}
X_0 & \rightarrow e_1X_1 \ldots e_{\ell}X_{\ell} \\
X_1 & \rightarrow e_1X_{R,2} \ldots e_{\ell-2}X_1 e_{\ell-1}X_{L,2} \\
X_{L,t} & \rightarrow e_1X_{R,2} \ldots e_{\ell-2}X_1 e_{\ell-1}X_{L,t+1} \quad \text{for } 2 \leq t \leq m - 1, t \neq w \\
X_{R,t} & \rightarrow e_1X_{R,t+1} e_2X_1 \ldots e_{\ell-2}X_1 e_{\ell-1}X_{L,2} \quad \text{for } 2 \leq t \leq m - 2, t \neq w \\
X_{L,w} & \rightarrow e_1X_{R,2} e_2X_1 \ldots e_{\ell-3}X_1 e_{\ell-2}X_{L,t+1} \\
X_{R,w} & \rightarrow e_1X_{R,t+1} e_2X_1 \ldots e_{\ell-3}X_1 e_{\ell-2}X_{L,2} \\
X_{R,m-1} & \rightarrow e_2X_1 \ldots e_{\ell-2}X_1 e_{\ell-1}X_{L,2} \\
X_{L,m} & \rightarrow e_1X_{R,2} e_2X_1 \ldots e_{\ell-3}X_1 e_{\ell-2}X_{L,2}.
\end{align*}
\]

A simple application of Proposition 3.3 yields:

**Corollary 3.5.** The growth series \( F_{\ell,m} \) of the vertices in \( X_{\ell,m} \) with even \( m = 2w \) is

\[
F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + X^w}{1 - (\ell - 2)X - \cdots - (\ell - 2)X^{w-1} + X^w};
\]

for odd \( m = 2w + 1 \), it is

\[
F_{\ell,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + 4X^w + 2X^{w+1} + \cdots + 2X^{m-2} + X^m}{1 - (\ell - 2)X - \cdots - (\ell - 4)X^w - \cdots - (\ell - 2)X^{m-2} + X^m}.
\]

### 3.4. Growth of geodesics in \( X_{\ell,m} \)

We now compute the growth series associated to finite geodesics starting at \( \ast \); that is,

\[
G_{\ell,m}(X) = \sum_{\gamma \in \Phi_\ast} X^{[\gamma]}.
\]

The grammar for the set of geodesics can be obtained from the previous ones just by adding the extra rules

\[
X_{R,w-1} \rightarrow x_1X_{L,w}
\]

for even \( m \), and

\[
X_{R,m-1} \rightarrow x_1X_{L,m}
\]

for odd \( m \). Indeed one can reach a vertex of type \( w \) (resp. of type \( m \)) both from the right and the left. While in the grammar for vertices such a vertex is derived only from the left-side (otherwise we would count it twice!) here the two geodesic ending at this point are distinct.
A simple application of Proposition 3.3 yields:

**Theorem 3.6.** The growth series $G_{t,m}$ of the geodesics in $X_{t,m}$ with even $m = 2w$ is

$$G_{t,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + X^w + (X^w - X^{w-1})}{1 - (\ell - 2)X - \cdots - (\ell - 4)X^{w-2}}.$$ 

for odd $m = 2w + 1$ it is

$$G_{t,m}(X) = \frac{1 + 2X + \cdots + 2X^{w-1} + 4X^w + 2X^{w+1} + \cdots + 2X^{m-2} + X^{m-1} + (X^{m-1} - X^{m-2})}{1 - (\ell - 2)X - \cdots - (\ell - 4)X^{w-2} - (\ell - 2)X^{w+1} - \cdots - (\ell - 2)X^{m-2} + X^{m-1} + (X^{m-1} - X^{m-2})}.$$ 

### 3.5. Growth of ordered pairs of geodesics in $X_{t,m}$

We compute the growth series of ordered pairs of geodesics, both starting at the base point $* \in V(G)$ and having the same endpoint; that is,

$$H_{t,m}(X) = \sum_{\gamma, \delta \in \Gamma_{t,m}, |\gamma| = |\delta|} \chi^{|\gamma|}.$$ 

We express this set of pairs of geodesics as a language in $T^* \times T^*$, that is, a direct product of free monoids. This means that, for instance, $(e_{i_1}, e_{i_2})(e_{i_3}, e_{i_4}) = (e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})$, where $e_{ij} \in T$.

We start with even $m = 2w$:

- its set of non-terminals is $N = \{X_0, X_1, X_{L,2}, \ldots, X_{L,w}, X_{R,2}, \ldots, X_{R,w}, X_{L,\Lambda}, X_{R,\Lambda}\};$
- its set of terminals is $T \times T$, with $T = \{e_1, \ldots, e_{\ell}\};$
- its axiom is $X_0;$
- its rules, with $2 \leq t \leq w - 1,$ are

$X_0 \rightarrow \varepsilon|\varepsilon|e_{t-1}^t|e_{t-1}^t|X_1|\cdots|\varepsilon|e_{t-1}^t|X_1|$

$(e_1 e_{t-1}^t, e_2 e_{t-1}^t) X_{L,\Lambda} \cdot \cdots \cdot (e_{t-1}^t e_{t-1}^t, e_{t-1}^t e_{t-1}^t) X_{L,\Lambda}$

$(e_2 e_{t-1}^t, e_1 e_{t-1}^t) X_{R,\Lambda} \cdot \cdots \cdot (e_{t-1}^t e_{t-1}^t, e_{t-1}^t e_{t-1}^t) X_{R,\Lambda}$

$X_1 \rightarrow \varepsilon|\varepsilon|e_{t-1}^t|e_{t-1}^t|X_{R,2}|(e_2 e_{t-1}^t) X_1|\cdots|\varepsilon|e_{t-1}^t|X_{R,2}|(e_2 e_{t-1}^t) X_1|\varepsilon|e_{t-1}^t|X_{L,2}|$

$(e_1 e_{t-1}^t, e_2 e_{t-1}^t) X_{L,\Lambda} \cdot \cdots \cdot (e_{t-1}^t e_{t-1}^t, e_{t-1}^t e_{t-1}^t) X_{L,\Lambda}$

$(e_2 e_{t-1}^t, e_1 e_{t-1}^t) X_{R,\Lambda} \cdot \cdots \cdot (e_{t-1}^t e_{t-1}^t, e_{t-1}^t e_{t-1}^t) X_{R,\Lambda}$

$\rightarrow (e_2 e_{t-1}^t, e_1 e_{t-1}^t) X_{R,\Lambda} \cdot \cdots \cdot (e_{t-1}^t e_{t-1}^t, e_{t-1}^t e_{t-1}^t) X_{R,\Lambda}$
Proposition 3.7. The above grammar’s language is the set of ordered pairs of geodesics with common extremities.

Proof. Given a geodesic \( \gamma = (\gamma_0, \ldots, \gamma_n) \) the truncation of \( \gamma \) is \( \gamma' = (\gamma_0, \ldots, \gamma_{n-1}) \); more generally, for \( i \leq n \), define its \( i \)-step truncation \( \gamma^{(i)} = (\gamma^{(i-1)})' = (\gamma_0, \ldots, \gamma_{n-i}) \), the geodesic obtained by deleting its last \( i \) edges.

We first explain the meaning of the non-terminals. If \( (\gamma, \delta) \) ends in a vertex of extended type \( t \), then \( X_0 \Rightarrow (\gamma, \delta) X_t \) is an intermediate derivation.

Therefore \( X_{L,t} \) expresses the possible continuations of a pair of geodesics that pass through a common point of type \( t \) on the left of a 2-cell — and analogously for \( X_{R,t} \).

If \( (\gamma, \delta) \) is a pair of geodesics of length \( n \) with \( \gamma_{n-1} \neq \delta_{n-1} \), then (by Lemma 2.2) there is a unique 2-cell \( F \) below \( \gamma_n \) and between \( \gamma \) and \( \delta \). If \( \gamma \) is on the left and \( \delta \) is on the right of \( F \), then \( X_0 \Rightarrow (\gamma', \delta') X_{L,\wedge} \) is an intermediate derivation. If \( \gamma \) is on the right and \( \delta \) is on the left of \( F \), then \( X_0 \Rightarrow (\gamma', \delta') X_{R,\wedge} \) is an intermediate derivation.

Therefore \( X_{L,\wedge} \) and \( X_{R,\wedge} \) express the possible continuations of the 1-step truncation of a pair of geodesics surrounding a 2-cell.

Let \( (\gamma, \delta) \) be a pair of geodesics of length \( n \). We show by induction on \( n \) that \( \gamma \wedge \delta \) is uniquely derived by the above grammar. Write \( t \) the extended type of \( \gamma_n \).

We distinguish several cases:

- if \( \gamma_{n-1} = \delta_{n-1} \); then there is a unique derivation
  \[ X_0 \Rightarrow (\gamma', \delta') X_t \Rightarrow (\gamma, \delta), \]

- where \( t' \) is the extended type of \( \gamma_{n-1} \);

- if \( \gamma_{n-1} \neq \delta_{n-1} \); let \( i \) be minimal such that \( \gamma_{n-1-i}(w-1) = \delta_{n-1-i}(w-1) \). Then there is a unique derivation
  \[ X_0 \Rightarrow (\gamma^{(1+i)(w-1)} \delta^{(1+i)(w-1)}) X_t \Rightarrow (\gamma^{(1+i)(w-1)} \delta^{(1+i)(w-1)}) X_{S,\wedge} \Rightarrow \cdots \Rightarrow (\gamma', \delta') X_{S,\wedge} \Rightarrow (\gamma, \delta) X_t \Rightarrow (\gamma, \delta), \]
where $t'$ is the extended type of $\gamma_{n-1-i(w-1)}$ and $S \in \{L, R\}$ determines which of $\gamma$ and $\delta$ is on the left between $\gamma_{n-1-i(w-1)}$ and $\gamma_n$.

We now present the grammar for odd $m = 2w + 1$; we omit the proof — completely analogous to that of Proposition 3.7 — that it describes pairs of geodesics, since

- its set of non-terminals is
  \[ N = \{ X_0, X_1, X_{L,2}, \ldots, X_{L,m}, X_{R,2}, \ldots, X_{R,m}, X_{L,\wedge}, X_{R,\wedge} \}; \]
- its set of terminals is $T = \{ e_1, \ldots, e_\ell \}$;
- its axiom is $X_0$;
- its rules are
  \[
  X_0 \rightarrow \epsilon([e_1, e_1)X_1] \cdots [(e_\ell, e_\ell)X_1] \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_\ell-2e_{\ell-1}^{-m-2}, e_\ell-1e_1^{-m-2})X_{L,\wedge} \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{R,\wedge} \cdots [(e_\ell-1e_\ell^{-m-2}, e_\ell-2e_{\ell-1}^{-m-2})X_{R,\wedge} \]
  \[
  X_1 \rightarrow \epsilon([e_1, e_1)X_{R,2}[(e_2, e_2)X_1] \cdots [(e_\ell-2, e_\ell-2)X_1[(e_\ell-1, e_\ell-1)X_{L,2} \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_\ell-2e_{\ell-1}^{-m-2}, e_\ell-1e_1^{-m-2})X_{L,\wedge} \]
  \[
  \rightarrow (e_{\ell-1}e_{\ell-1}^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{R,\wedge} \cdots [(e_{\ell-1}e_1^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{R,\wedge} \]
  \[
  X_{L,t} \rightarrow \epsilon([e_1, e_1)X_{R,2}[(e_2, e_2)X_1] \cdots [(e_\ell-2, e_\ell-2)X_1[(e_\ell-1, e_\ell-1)X_{L,t+1} \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_\ell-2e_{\ell-1}^{-m-2}, e_\ell-1e_1^{-m-2})X_{L,\wedge} \]
  \[
  \rightarrow (e_1e_{\ell-2}^{-m-2}, e_2e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_\ell-1e_\ell^{-m-2}, e_\ell-2e_{\ell-1}^{-m-2})X_{L,\wedge} \]
  \[
  X_{R,t} \rightarrow \epsilon([e_1, e_1)X_{R,t+1}[(e_2, e_2)X_1] \cdots [(e_\ell-2, e_\ell-2)X_1[(e_\ell-1, e_\ell-1)X_{L,2} \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_\ell-2e_{\ell-1}^{-m-2}, e_\ell-1e_1^{-m-2})X_{L,\wedge} \]
  \[
  \rightarrow (e_1e_{\ell}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_{\ell-1}e_1^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{L,\wedge} \]
  \[
  X_{R,w} \rightarrow \epsilon([e_1, e_1)X_{R,w+1}[(e_2, e_2)X_1] \cdots [(e_\ell-3, e_\ell-3)X_1[(e_\ell-2, e_\ell-2)X_{L,2} \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{L,\wedge} \cdots [(e_\ell-3e_{\ell-1}^{-m-2}, e_\ell-2e_{\ell-1}^{-m-2})X_{L,\wedge} \]
  \[
  \rightarrow (e_1e_{\ell-2}^{-m-2}, e_1e_{\ell-2}^{-m-2})X_{L,\wedge} \cdots [(e_{\ell-2}e_{\ell-1}^{-m-2}, e_{\ell-3}e_{\ell-1}^{-m-2})X_{L,\wedge} \]
  \[
  X_{L,w} \rightarrow \epsilon([e_1, e_1)X_{R,2}[(e_2, e_2)X_1] \cdots [(e_\ell-3, e_\ell-3)X_1[(e_\ell-2, e_\ell-2)X_{L,w+1} \]
  \[
  (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-2}^{-m-2})X_{L,\wedge} \cdots [(e_{\ell-3}e_{\ell}^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{L,\wedge} \]
  \[
  \rightarrow (e_1e_{\ell-1}^{-m-2}, e_1e_{\ell-1}^{-m-2})X_{R,\wedge} \cdots [(e_{\ell-1}e_1^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{R,\wedge} \]
  \[
  X_{R,m} \rightarrow \epsilon([e_1, e_1)X_{R,2}[(e_2, e_2)X_1] \cdots [(e_\ell-3, e_\ell-3)X_1[(e_\ell-2, e_\ell-2)X_{L,2} \]
  \[
  (e_1e_{\ell-2}^{-m-2}, e_1e_{\ell-2}^{-m-2})X_{L,\wedge} \cdots [(e_{\ell-3}e_{\ell-1}^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{L,\wedge} \]
  \[
  \rightarrow (e_1e_{\ell-2}^{-m-2}, e_1e_{\ell-2}^{-m-2})X_{R,\wedge} \cdots [(e_{\ell-2}e_{\ell-1}^{-m-2}, e_{\ell-3}e_{\ell-1}^{-m-2})X_{R,\wedge} \]
  \[
  X_{L,m} \rightarrow X_{R,m} \]
  \[
  X_{L,\wedge} \rightarrow (e_{\ell-1}, e_1)X_{L,m}[(e_{\ell-1}, e_1)X_{L,\wedge}[(e_{\ell-2}e_{\ell-1}^{-m-2}, e_{\ell-1}e_1^{-m-2})X_{L,\wedge} \]
  \[
  X_{R,\wedge} \rightarrow (e_1, e_\ell)X_{R,m}[(e_1e_\ell^{-m-2}, e_\ell-1e_1^{-m-2})X_{R,\wedge}[(e_{\ell-2}e_{\ell-1}^{-m-2}, e_{\ell-2}e_{\ell-1}^{-m-2})X_{R,\wedge} \]
Theorem 3.8. The growth series $H_{t,m}$ of the pair of geodesics in $X_{t,m}$ with even $m = 2w$ is
\[
H_{t,m}(X) = 1 + \ell \frac{X + X^2 + \cdots + X^{w-1}}{1 - (\ell - 2)X - \cdots - (\ell - 2)X^{w-1} + X^w + X^{w-1}(X - 1)(2X^{w-1} - 3)};
\]
for odd $m = 2w + 1$, it is
\[
H_{t,m}(X) = 1 + \ell \frac{X + X^2 + \cdots + X^{m-2}}{1 - (\ell - 2)X - \cdots - (\ell - 4)X^{w-1} - \cdots - (\ell - 2)X^{m-2} + X^{m-1} - X^{m-1}(1 - X)(3 - 4X^w + 2X^{m-2})}.
\]

3.6. Holly trees. Regular grammars were used in the previous subsection to describe fellow-travelling pairs of geodesics, giving in the next section a lower estimation on the cogrowth (and hence the spectral radius) of $X_{t,m}$, using loops of the form $\gamma\rho\delta^{-1}$, where $(\gamma, \delta)$ is a pair of geodesics with common extremities $*$ and $x$, and $\rho$ is the loop around a 2-cell at $x$, not tangent to $\gamma$ nor $\delta$.

We give here a context-free grammar producing a larger class of reduced loops, which we call holly trees. To define them, call $H_x$ the set of holly trees at the vertex $x \in \text{Vert}(X_{t,m})$; then $H_x$ is a set of loops based at $x$, and lying entirely in the cone of $x$. Then

1. if $\gamma, \delta \in H_x$ and $\gamma \delta$ is a reduced path, then $\gamma \delta \in H_x$;
2. if $y$ is a successor of $x$ and $\gamma \in H_y$, then $\gamma y \in H_x$, where $v$ is the label of the edge from $x$ to $y$;
3. if $P$ is a 2-cell in the cone of $x$ and touching $x$, whose perimeter starting at $x$ is labelled $\gamma$, then $\gamma \in H_x$;
4. no other loop belongs to $H_x$.

We note that the growth of $H_x$ depends only on the cone type of $x$, since all holly trees at $x$ lie within $x$’s cone. We may thus consider $L_t(X)$, the growth series of holly trees at any fixed vertex of type $t$. That this function is algebraic follows from the fact that holly trees can be described by the unambiguous context-free grammar given below (and compare with Theorem 3.1).

Note also that all holly-trees are non-trivial reduced paths. Write $e$ for the empty path.

For simplicity, assume $m = 2w$ is even. Consider the nonterminal $L_{t,e}$ for all cone types $t$, and all labels $e$ of the successors of a vertex of type $t$. We then have $t \in \{0, \ldots, w\}$ and $e \in \{e_1, \ldots, e_t\}$. The variable $L_{t,e}$ expresses the set of all holly trees in the cone (at a vertex of) type $t$, whose first edge is labelled $e_t$. For commodity, define also variables $L_t$, counting all holly trees in a cone of type $t$, and $L_{t,\hat{e}}$, expressing all holly trees in a cone of type $t$ whose first edge is not labelled $e$.

The terminal alphabet is $\{e_t^\pm, \ldots, e_1^\pm\}$, describing paths in $X_{t,m}$. The axiom is $L_0$. The derivations are
\[
L_t \rightarrow L_{t,e_1} \cdots L_{t,e_t} \text{ for } t \in \{1, \ldots, w - 1\}
\]
$L_{t,\hat{e}} \Rightarrow$ the same, but excluding $L_{t,e}$
$L_{t,e} \Rightarrow eL_u e^{-1}|eL_u e^{-1}L_{t,\hat{e}}|e_1 \cdots e_m|e_1 \cdots e_m L_{t,\hat{e}}$, where $e$ leads to a vertex of type $u$ and $(e = e_1, \ldots, e_m)$ is the boundary of a 2-cell.

The computations of growth series are more complicated, and the results, for $m = t = 8$, appear in the next section.
We end this section by remarking that also Cayley graphs of hyperbolic groups, or, more generally strongly-transitive hyperbolic graphs have a finite number of cone types. As a consequence, mutatis mutandis, all our above computations (enumeration of vertices; of geodesics; of ordered pairs of geodesics or, more generally, of ordered $N$-tuples of geodesics; of holly trees) can be performed there as well (always leading to rational growth series, except for the holly trees for which the growth series is again algebraic).

4. Estimates for Simple Random Walks

4.1. Upper estimates. Upper estimates for the spectral radius of the Markov operator associated with a simple random walk on the fundamental group of an orientable surface have been obtained in [BCCH97]. In that paper various methods were described, yielding as best estimate for 
\[ \mu_2 \leq \frac{\sqrt{4g-2}}{2g} + \frac{1}{4g}, \]
and in particular
\[ \mu_2 \leq 0.7374. \]

These estimates (again for \( \ell = m = 4g \)) have been improved by Žuk [Žuk97] and by Nagnibeda [Nag97], who obtained the best known estimate from above:
\[ \mu_2 \leq 0.6629. \]

Both methods easily extend to all graphs \( X_{\ell,m} \).

4.2. Lower estimates. In this section, using the Grigorchuk formula (1) and the combinatorial results of section 3, we obtain tighter and tighter lower estimates for \( \mu_{\ell,m} \), the spectral radius of the simple random walk on \( X_{\ell,m} \), and present numerical results for \( \mu_{8,8} = \mu_2 \).

We start by re-obtaining Kesten’s estimate. Consider the loops consisting of a proper path \( \gamma = (*, \ldots, x) \) starting at *, followed by the boundary of a cell containing \( x \) and inside its cone, followed by \( \gamma^{-1} \). There are \( \ell(\ell-1)^{n-1} \) paths \( \gamma \) of length \( n \), and \( 2(\ell-2) \) choices for the cell’s boundary; thus there are at least
\[ \beta_n = 2(\ell-2)\ell(\ell-1)^{n-1} \]
paths of length \( 2n + m \), whence we reobtain Grigorchuk’s estimate
\[ \alpha \geq \limsup_{n \to \infty} 2^{n+\sqrt{\beta_n}} = \sqrt{1 - \frac{1}{\ell}}, \]
and Kesten’s estimate
\[ \mu_{\ell,m} \geq \frac{2\sqrt{1 - \frac{1}{\ell}}}{\ell}; \]
in particular
\[ \mu_2 \geq 0.66143. \]

Taking into account the boundaries of two cells neighbouring * as other constituents of the loops we count, it is possible to obtain slightly tighter results. As the gain is negligible, we shall not describe the counting in detail, but refer to the paper by Kesten [Kes59, Theorem 4.15].

The first non-trivial estimate is obtained by considering the growth of unimodular loops. A unimodular loop of weight \( n \) is a loop \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{2n}) \) of length \( 2n \) such that \( |\gamma_n| = n \) and such that \( \gamma_{n-1} \neq \gamma_{n+1} \). In other words
can be viewed as a couple of distinct geodesics of length $n$, say $\delta = (\delta_0, \ldots, \delta_n)$ and $\beta = (\beta_0, \ldots, \beta_n)$ with common starting and end-points, namely $\delta_0 = \beta_0$ and $\delta_n = \beta_n$ but with no common last edge, namely $\delta_{n-1} \neq \beta_{n-1}$. We then regard $\gamma$ as $(\delta_0, \ldots, \delta_n, \beta_{n-1}, \ldots, \beta_0)$.

The growth series $U$ for unimodular loops is closely related to the function $H_{\ell,m}$ computed in subsection 3.5: both functions have the same denominator, and hence the same radius of convergence. This is because $U(X) < H_{\ell,m}(X)$ and $H_{\ell,m}(X) < X^m U(X)$ coefficient-wise: any unimodular loop comes from a pair of geodesics, and any pair of geodesics can be completed in at worst $m$ steps in an unimodular loop. For $\ell = m = 8$, for instance, the radius of convergence of $U$ is

$$\rho_U \approx 1/7.0248,$$

so that $\alpha_2 \geq \sqrt{1/\rho_U} \approx 2.65$, and thus, by the Grigorchuk formula,

$$\mu_2 \geq 0.66144.$$

This last estimate is weaker than that obtained by Paschke [Pas93, Theorem 3.2], whose result applied to $X_8;8$ gives

$$\mu_2 \geq 0.6616.$$

A sharper estimate is given by the growth of holly trees as computed in subsection 3.6. Again in the case $\ell = m = 8$, the algebraic growth function was computed using the computer algebra program MAPLE, as a solution $L(X)$ of $P(X, L(X)) \equiv 0$. The algebraic function $L$ has a singularity at all vanishing points $\rho$ of the discriminant of $P(X, Y)$, so the radius of convergence of $L$ is at most the absolute value of a minimal such $\rho$. The discriminant turns out to be a degree-179 polynomial vanishing at

$$\rho \approx 0.12887$$

from which we deduce the asymptotic growth of loops is at least

$$\alpha \geq \frac{1}{\sqrt{\rho}} \approx 2.7856,$$

so by Grigorchuk’s formula we get the following:

**Theorem 4.1.** The spectral radius $\mu_2$ associated with a simple random walk on the fundamental group $J_2$ of a surface of genus 2 (with respect to the canonical set of generators) is bounded below by

$$\mu_2 \geq 0.6623.$$

In [Bar01], the first author introduced a class of loops, called *cactus trees*, which contains the class of holly trees and obtained a slightly better estimate:

$$\mu_2 \geq 0.6624.$$

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References

[Bar99] Laurent Bartholdi, Counting paths in graphs, Enseignement Math. 45 (1999), 83–131.

[Bar01] Laurent Bartholdi, Lower estimates on the spectral radius of surface groups, and cactus trees, unpublished, 2001.

[BC01] Laurent Bartholdi and Tullio G. Ceccherini-Silberstein, Salem numbers and growth series of some hyperbolic graphs, to appear in Geom. Dedicata, 2001.

[BCCH97] Laurent Bartholdi, Serge Cantat, Tullio G. Ceccherini-Silberstein, and Pierre de la Harpe, Estimates for simple random walks on fundamental groups of surfaces, Colloq. Math. 72 (1997), no. 1, 173–193.

[Can83] James W. Cannon, The growth of the closed surface groups and the compact hyperbolic Coxeter groups, unpublished, March 1983.

[CGH99] Tullio G. Ceccherini-Silberstein, Rostislav I. Grigorchuk, and Pierre de la Harpe, Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces, Trudy Mat. Inst. Steklov. 224 (1999), no. Algebra. Topol. Differ. Uravn. i ikh Prilozh., 68–111, Dedicated to Academician Lev Semenovich Pontryagin on the occasion of his 90th birthday (Russian).

[Coo93] Michel Coornaert, Mesures de Patterson-Sullivan sur le bord d’un espace hyperbolique au sens de Gromov, Pacific J. Math. 159 (1993), no. 2, 241–270.

[CS63] Noam Chomsky and Marcel-Paul Schützenberger, The algebraic theory of context-free languages, Computer programming and formal systems, North-Holland, Amsterdam, 1963, pp. 118–161.

[CW92] James W. Cannon and Philip Wagreich, Growth functions of surface groups, Math. Ann. 293 (1992), no. 2, 239–257.

[DK88] Joseph Dodziuk and Leon Karp, Spectral and function theory for combinatorial Laplacians, Contemp. Math 73 (1988), 25–40.

[FP87] William J. Floyd and Steven P. Plotnick, Growth functions on Fuchsian groups and the Euler characteristic, Invent. Math. 88 (1987), no. 1, 1–29.

[FP94] William J. Floyd and Steven P. Plotnick, Growth functions for semi-regular tilings of the hyperbolic plane, Geom. Dedicata 53 (1994), 1–23.

[GH90] Étienne Ghys and Pierre de la Harpe, Sur les groupes hyperboliques d’après Mikhael Gromov, Progress in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[Gri80] Rostislav I. Grigorchuk, Symmetrical random walks on discrete groups, Multicomponent random systems, Dekker, New York, 1980, pp. 285–325.

[Gro93] Mikhael Gromov, Asymptotic invariants of infinite groups, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[Har78] Michael A. Harrison, Introduction to formal language theory, Addison-Wesley Publishing Co., Reading, Mass., 1978.

[Kes59] Harry Kesten, Symmetric random walks on groups, Trans. Amer. Math. Soc. 92 (1959), 336–354.

[LS76] Roger C. Lyndon and Paul E. Schupp, Combinatorial group theory, Springer-Verlag, 1970.

[Mas67] William S. Massey, Algebraic topology: an introduction, Harcourt, Brace and World (New York), 1967.

[Nag97] Tatiana Nagnibeda, An upper bound for the spectral radius of a random walk on surface groups, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 240 (1997), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 2, 154–165, 293–294.
[Pas93] William L. Paschke, Lower bound for the norm of a vertex-transitive graph, Math. Z. 213 (1993), no. 2, 225–239.

[SS78] Arto Salomaa and Matti Soittola, Automata-theoretic aspects of formal power series, Springer-Verlag, 1978.

[Wil90] Herbert S. Wilf, Generatingfunctionology, Academic Press Inc., Boston, MA, 1990.

[Woe00] Wolfgang Woess, Random walks on infinite graphs and groups, Cambridge University Press, Cambridge, 2000.

[Zuk97] Andrzej Żuk, A remark on the norm of a random walk on surface groups, Colloq. Math. 72 (1997), no. 1, 195–206.

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