Regularization of ill-posed mixed variational inequalities with non-monotone perturbations

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Abstract
In this paper, we study a regularization method for ill-posed mixed variational inequalities with non-monotone perturbations in Banach spaces. The convergence and convergence rates of regularized solutions are established by using a priori and a posteriori regularization parameter choice that is based upon the generalized discrepancy principle.

Keywords: monotone mixed variational inequality, non-monotone perturbations, regularization, convergence rate

1 Introduction
Variational inequality problems in finite-dimensional and infinite-dimensional spaces appear in many fields of applied mathematics such as convex programming, nonlinear equations, equilibrium models in economics, and engineering (see [1-3]). Therefore, methods for solving variational inequalities and related problems have wide applicability. In this paper, we consider the mixed variational inequality: for a given \( f \in X^* \), find an element \( x_0 \in X \) such that

\[
\langle Ax_0 - f, x - x_0 \rangle + \varphi(x) - \varphi(x_0) \geq 0, \quad \forall x \in X,
\]

where \( A : X \to X^* \) is a monotone-bounded hemicontinuous operator with domain \( D(A) = X \), \( \phi : X \to \mathbb{R} \) is a proper convex lower semicontinuous functional and \( X \) is a real reflexive Banach space with its dual space \( X^* \). For the sake of simplicity, the norms of \( X \) and \( X^* \) are denoted by the same symbol \( || \cdot || \). We write \( \langle x^*, x \rangle \) instead of \( x^*(x) \) for \( x^* \in X^* \) and \( x \in X \).

By \( S_0 \) we denote the solution set of the problem (1). It is easy to see that \( S_0 \) is closed and convex whenever it is not empty. For the existence of a solution to (1), we have the following well-known result (see [4]):

**Theorem 1.1.** If there exists \( u \in \text{dom} \phi \) satisfying the coercive condition

\[
\lim_{||x|| \to \infty} \frac{\langle Ax, x - u \rangle + \varphi(x)}{||x||} = \infty,
\]

then (1) has at least one solution.
Many standard extremal problems can be considered as special cases of (1). Denote \( \phi \) by the indicator function of a closed convex set \( K \) in \( X \),

\[
\psi(x) = \begin{cases} 
0 & \text{if } x \in K, \\
\infty & \text{otherwise}. 
\end{cases}
\]

Then, the problem (1) is equivalent to that of finding \( x_0 \in K \) such that

\[
\langle Ax_0 - f, x - x_0 \rangle \geq 0, \quad \forall x \in K. 
\]

In the case \( K \) is the whole space \( X \), the later variational inequality is of the form of the following operator equation:

\[
Ax = f. 
\]

When \( A \) is the Gâteaux derivative of a finite-valued convex function \( F \) defined on \( X \), the problem (1) becomes the nondifferentiable convex optimization problem (see [4]):

\[
\min_{x \in X} [F(x) + \psi(x)]. 
\]

Some methods have been proposed for solving problem (1), for example, the proximal point method (see [5]), and the auxiliary subproblem principle (see [6]). However, the problem (1) is in general ill-posed, as its solutions do not depend continuously on the data \( (A, f, \phi) \), we used stable methods for solving it. A widely used and efficient method is the regularization method introduced by Liskovets [7] using the perturbative mixed variational inequality:

\[
\langle A_h x^*_h + \alpha U(x^*_h - x_\alpha) - f_\alpha, x - x^*_h \rangle + \varphi_\epsilon(x) - \varphi_\epsilon(x^*_h) \geq 0, \quad \forall x \in X, 
\]

where \( A_h \) is a monotone operator, \( \alpha \) is a regularization parameter, \( U \) is the duality mapping of \( X \), \( x_\alpha \in X \) and \( (A_h, f_\alpha, \phi_\epsilon) \) are approximations of \( (A, f, \phi) \), \( \tau = (h, \delta, \epsilon) \). The convergence rates of the regularized solutions defined by (6) are considered by Buong and Thuy [8].

In this paper, we do not require \( A_h : x_\alpha \in X \) to be monotone. In this case, the regularized variational inequality (6) may be unsolvable. In order to avoid this fact, we introduce the regularized problem of finding \( x^*_h \in X \) such that

\[
\langle A_h x^*_h + \alpha U(x^*_h - x_\alpha) - f_\alpha, x - x^*_h \rangle + \varphi_\epsilon(x) - \varphi_\epsilon(x^*_h) \geq 0, \quad \forall x \in X, \quad \mu \geq h, 
\]

where \( \mu \) is positive small enough, \( U^\epsilon \) is the generalized duality mapping of \( X \) (see Definition 1.3) and \( x_\alpha \) is in \( X \) which plays the role of a criterion of selection, \( g \) is defined below.

Assume that the solution set \( S_0 \) of the inequality (1) is non-empty, and its data \( A, f, \phi \) are given by \( A_h, f_\alpha, \phi_\epsilon \) satisfying the conditions:

1. \( ||f - f_\alpha|| \leq \delta, \delta \to 0; \)
2. \( A_h : X \to X^* \) is not necessarily monotone, \( D(A_h) = D(A) = X \), and

\[
||A_h x - Ax|| \leq h\delta(||x||), \quad \forall x \in X, \quad h \to 0, 
\]

with a non-negative function \( g(t) \) satisfying the condition

\[
g(t) \leq g_0 + g_1 t^v, \quad v = s - 1, \quad g_0, \quad g_1 \geq 0; 
\]
(3) \( \phi_{\varepsilon} : X \to \mathbb{R} \) is a proper convex lower semicontinuous functional for which there exist positive numbers \( c_{\varepsilon} \) and \( r_{\varepsilon} \) such that
\[
\phi_{\varepsilon}(x) \geq -c_{\varepsilon}||x|| \quad \text{as } ||x|| > r_{\varepsilon}
\]
and
\[
|\phi_{\varepsilon}(x) - \phi(x)| \leq \varepsilon d(||x||), \quad \forall x \in X, \varepsilon \to 0,
\]
\[
|\phi_{\varepsilon}(x) - \phi_{\varepsilon}(y)| \leq C_{0}||x - y||, \quad \forall x, y \in X,
\]
where \( C_{0} \) is some positive constant, \( d(t) \) has the same properties as \( g(t) \).

In the next section we consider the existence and uniqueness of solutions \( x_{n}^{\varepsilon} \) of (7), for every \( \alpha > 0 \). Then, we show that the regularized solutions \( x_{n}^{\varepsilon} \) converge to \( x_{0} \in S_{0} \) in the \( x_{\varepsilon} \)-minimal norm solution defined by
\[
||x_{0} - x_{\varepsilon}|| = \arg\min_{x \in S_{0}} ||x - x_{\varepsilon}||.
\]

The convergence rate of the regularized solutions \( x_{n}^{\varepsilon} \) to \( x_{0} \) will be established under the condition of inverse-strongly monotonicity for \( A \) and the regularization parameter choice based on the generalized discrepancy principle.

We now recall some known definitions (see [9-11]).

**Definition 1.1.** An operator \( A : D(A) = X \to X^{*} \) is said to be
(a) hemi-continuous if \( A(x + t_{n}y) - Ax \) as \( t_{n} \to 0^{+}, x, y \in X \), and demicontinuous if \( x_{n} \to x \) implies \( Ax_{n} \to Ax \);
(b) monotone if \( \langle Ax - Ay, x - y \rangle \geq 0 \), \( \forall x, y \in X \);
(c) inverse-strongly monotone if
\[
\langle Ax - Ay, x - y \rangle \geq m_{A}||Ax - Ay||^{2}, \quad \forall x, y \in X,
\]
where \( m_{A} \) is a positive constant.

It is well-known that a monotone and hemi-continuous operator is demicontinuous and a convex and lower semicontinuous functional is weakly lower semicontinuous (see [9]). And an inverse-strongly monotone operator is not strongly monotone (see [10]).

**Definition 1.2.** It is said that an operator \( A : X \to X^{*} \) has \( S \)-property if the weak convergence \( x_{n} \rightharpoonup x \) and \( \langle Ax_{n} - Ax, x_{n} - x \rangle \to 0 \) imply the strong convergence \( x_{n} \to x \) as \( n \to \infty \).

**Definition 1.3.** The operator \( U^{\varepsilon} : X \to X^{*} \) is called the generalized duality mapping of \( X \) if
\[
U^{\varepsilon}(x) = \{x^{*} \in X^{*} : \langle x^{*}, x \rangle = ||x^{*}|| \cdot ||x|| : ||x^{*}|| = ||x||^{s} \}, \quad s \geq 2.
\]
When \( s = 2 \), we have the duality mapping \( U \). If \( X \) and \( X^{*} \) are strictly convex spaces, \( U^{\varepsilon} \) is single-valued, strictly monotone, coercive, and demicontinuous (see [9]).

Let \( X = L^{p}(\Omega) \) with \( p \in (1, \infty) \) and \( \Omega \subset \mathbb{R}^{m} \) measurable, we have
\[
U(\psi) = ||\psi||^{2-p}_{L^{p}(\Omega)}|\psi(t)|^{p-2}\psi(t), \quad t \in \Omega.
\]
Assume that the generalized duality mapping \( U^{\varepsilon} \) satisfies the following condition:
\[
\langle U^{\varepsilon}(x) - U^{\varepsilon}(y), x - y \rangle \geq m_{A}||x - y||^{s}, \quad \forall x, y \in X,
\]
where $m_s$ is a positive constant. It is well-known that when $X$ is a Hilbert space, then $U^s = I$, $s = 2$ and $m_s = 1$, where $I$ denotes the identity operator in the setting space (see [12]).

### 2 Main result

**Lemma 2.1.** Let $X^*$ be a strictly convex Banach space. Assume that $A$ is a monotone-bounded hemicontinuous operator with $D(A) = X$ and conditions (2) and (3) are satisfied. Then, the inequality (7) has a non-empty solution set $S$, for each $\alpha > 0$ and $f_0 \in X^*$.

**Proof.** Let $x_\alpha \in \text{dom } \phi_\alpha$. The monotonicity of $A$ and assumption (3) imply the following inequality:

$$\frac{\langle Ax + \alpha U^f(x - x_\alpha), x - x_\alpha \rangle + \psi_\epsilon(x)}{|x|} \geq \alpha \frac{||x - x_\alpha||^{-1} (||x - x_\alpha|| - ||x - x_\epsilon||)}{|x|} - ||Ax_\epsilon|| \left(1 + \frac{||x_\epsilon||}{|x|}\right) - c_\epsilon, \quad s \geq 2,$$

for $|x| > r_\alpha$. Consequently, (2) is fulfilled for the pair $(A + \alpha U^f, \phi_\epsilon)$. Thus, for each $\alpha > 0$ and $f_0 \in X^*$, there exists a solution of the following inequality:

$$\langle Ax + \alpha U^f(x - x_\alpha) - f_0, z - x \rangle + \psi_\epsilon(z) - \psi_\epsilon(x) \geq 0, \quad \forall z \in X, x \in X. \quad (14)$$

Observe that the unique solvability of this inequality follows from the monotonicity of $A$ and the strict monotonicity of $U^f$. Indeed, let $x_1$ and $x_2$ be two different solutions of (14). Then,

$$\langle Ax_1 + \alpha U^f(x_1 - x_\alpha) - f_0, z - x_1 \rangle + \psi_\epsilon(z) - \psi_\epsilon(x_1) \geq 0, \quad \forall z \in X \quad (15)$$

and

$$\langle Ax_2 + \alpha U^f(x_2 - x_\alpha) - f_0, z - x_2 \rangle + \psi_\epsilon(z) - \psi_\epsilon(x_2) \geq 0, \quad \forall z \in X. \quad (16)$$

Putting $z = x_2$ in (15) and $z = x_1$ in (16) and add the obtained inequalities, we obtain

$$\langle Ax_1 - Ax_2, x_2 - x_1 \rangle + \alpha \langle U^f(x_1 - x_\alpha) - U^f(x_2 - x_\alpha), x_2 - x_1 \rangle \geq 0.$$

Due to the monotonicity of $A$ and the strict monotonicity of $U^f$, the last inequality occurs only if $x_1 = x_2$.

Let $x^\delta_{\alpha}$ be a solution of (14), that is,

$$\langle Ax^\delta_{\alpha} + \alpha U^f(x^\delta_{\alpha} - x_\alpha) - f_0, z - x^\delta_{\alpha} \rangle + \psi_\epsilon(z) - \psi_\epsilon(x^\delta_{\alpha}) \geq 0, \quad \forall z \in X. \quad (17)$$

For all $h > 0$, making use of (8), from (17) one gets

$$\langle Ax^\delta_{\alpha} + \alpha U^f(x^\delta_{\alpha} - x_\alpha) - f_0, z - x^\delta_{\alpha} \rangle + \psi_\epsilon(z) - \psi_\epsilon(x^\delta_{\alpha}) \geq -h g(||x^\delta_{\alpha}||)||z - x^\delta_{\alpha}||, \quad \forall z \in X. \quad (18)$$

Since $\mu \geq h$, we can conclude that each $x^\delta_{\alpha}$ is a solution of (7).

Let $x^\mu_{\alpha}$ be a solution of (7). We have the following result.

**Theorem 2.1.** Let $X$ and $X^*$ be strictly convex Banach spaces and $A$ be a monotone-bounded hemicontinuous operator with $D(A) = X$. Assume that conditions (1)-(3) are
satisfied, the operator \(U^t\) satisfies condition (13) and, in addition, the operator \(A\) has the \(S\)-property. Let

\[
\lim_{\alpha \to 0} \frac{\mu + \delta + \varepsilon}{\alpha} = 0. \tag{19}
\]

Then \(\{x^t_n\}\) converges strongly to the \(x_s\)-minimal norm solution \(x_0 \in S_0\).

**Proof.** By (1) and (7), we obtain

\[
\begin{align*}
\langle A_h x^t_n + \alpha U^t(x^t_n - x_s) - f_s, x_0 - x^t_n \rangle + \varphi_t(x_0) - \varphi_t(x^t_n) \\
+ \langle Ax_0 - f_s, x^t_n - x_0 \rangle + \varphi(x^t_n) - \varphi(x_0) \geq -\mu g(||x^t_n||)||x_0 - x^t_n||.
\end{align*}
\]

This inequality is equivalent to the following

\[
\begin{align*}
\alpha \langle U^t(x^t_n - x_s) - U^t(x_0 - x_s), x^t_n - x_0 \rangle &\leq \alpha \langle U^t(x_0 - x_s), x_0 - x^t_n \rangle \\
+ \langle A_h x^t_n - Ax^t_n, x_0 - x^t_n \rangle \\
+ \langle Ax_0 - Ax^t_n, x^t_n - x_0 \rangle + (f - f_s, x_0 - x^t_n) \\
+ \varphi_t(x_0) - \varphi_t(x_0) + \varphi(x^t_n) - \varphi_t(x^t_n) \\
+ \mu g(||x^t_n||)||x_0 - x^t_n||.
\end{align*}
\]

The monotonicity of \(A\), assumption (1), and the inequalities (8), (9), (13) and (20) yield the relation

\[
m_s||x^t_n - x_0||^t \leq \left[ \frac{h + \mu}{\alpha} g(||x^t_n||) + \frac{\delta}{\alpha} \right] ||x_0 - x^t_n||
\]

\[
+ \frac{\varepsilon}{\alpha} [d(||x_0||) + d(||x^t_n||)] + \langle U^t(x_0 - x_s), x_0 - x^t_n \rangle. \tag{21}
\]

Since \(\mu/\alpha \to 0\) as \(\alpha \to 0\) (and consequently, \(h/\alpha \to 0\)), it follows from (19) and the last inequality that the set \(x^t_n\) are bounded. Therefore, there exists a subsequence of \(x^t_n\) weakly converging to \(\bar{x} \in X\).

We now prove the strong convergence of \(\{x^t_n\}\) to \(\bar{x}\). The monotonicity of \(A\) and \(U^t\) implies that

\[
0 \leq \langle Ax^t_n - A\bar{x}, x^t_n - \bar{x} \rangle
\]

\[
\leq \langle Ax^t_n + \alpha U^t(x^t_n - x_s) - A\bar{x} - \alpha U^t(\bar{x} - x_s), x^t_n - \bar{x} \rangle
\]

\[
= \langle Ax^t_n + \alpha U^t(x^t_n - x_0), x^t_n - \bar{x} \rangle - \langle A\bar{x} + \alpha U^t(\bar{x} - x_s), x^t_n - \bar{x} \rangle. \tag{22}
\]

In view of the weak convergence of \(\{x^t_n\}\) to \(\bar{x}\), we have

\[
\lim_{\alpha \to 0} \langle A\bar{x} + \alpha U^t(\bar{x} - x_s), x^t_n - \bar{x} \rangle = 0. \tag{23}
\]

By virtue of (8),

\[
\langle Ax^t_n + \alpha U^t(x^t_n - x_s), x^t_n - \bar{x} \rangle
\]

\[
= \langle Ax^t_n - A_h x^t_n + \alpha U^t(x^t_n - x_s), x^t_n - \bar{x} \rangle
\]

\[
\leq \langle A_h x^t_n + \alpha U^t(x^t_n - x_s), x^t_n - \bar{x} \rangle + h g(||x^t_n||)||x^t_n - \bar{x}||. \tag{24}
\]

Using further (7), we deduce

\[
\langle A_h x^t_n + \alpha U^t(x^t_n - x_s), x^t_n - \bar{x} \rangle
\]

\[
= \langle A_h x^t_n + \alpha U^t(x^t_n - x_s) - f_s, x^t_n - \bar{x} \rangle + (f_s, x^t_n - \bar{x})
\]

\[
\leq (f_s, x^t_n - \bar{x}) + \varphi_t(\bar{x}) - \varphi_t(x^t_n) + \mu g(||x^t_n||)||\bar{x} - x^t_n||. \tag{25}
\]
Since $x^*_a \to \tilde{x}$ and $\phi$ is proper convex weakly lower semicontinuous, we have from (25) that
\[
\lim_{\alpha \to 0} \langle A\alpha x^*_a + \alpha U'(x^*_a - x), x^*_a - \tilde{x} \rangle \leq 0. \tag{26}
\]
By (22)-(24) and (26), it results that
\[
\lim_{\alpha \to 0} \langle Ax^*_a - Ax^*_\alpha, x^*_a - \tilde{x} \rangle = 0.
\]
Finally, the S property of $A$ implies the strong convergence of $\{x^*_a\}$ to $\tilde{x} \in X$. We show that $\tilde{x} \in S_0$. By (8) and take into account (7) we obtain
\[
\langle Ax^*_a + \alpha U'(x^*_a - x) - f, x - x^*_a \rangle + \phi(x) - \phi(x^*_a) 
\geq -(h + \mu)g(||x^*_a||)||x - x^*_a||, \quad \forall x \in X. \tag{27}
\]
Since the functional $\phi$ is weakly lower semicontinuous,
\[
\phi(\tilde{x}) \leq \lim_{\alpha \to 0} \inf \phi(x^*_a). \tag{28}
\]
Since $\{x^*_a\}$ is bounded, by (9), there exists a positive constant $c_2$ such that
\[
\phi(x^*_a) \leq \phi(\tilde{x}) + c_2 \varepsilon. \tag{29}
\]
By letting $\alpha \to 0$ in the inequality (7), provided that $A$ is demicontinuous, from (8), (9), (28), (29) and condition (1) imply that
\[
\langle Ax^*_\alpha - f, x - \tilde{x} \rangle + \phi(x) - \phi(\tilde{x}) \geq 0, \quad \forall x \in X.
\]
This means that $\tilde{x} \in S_0$.
We show that $\tilde{x} = x_0$. Applying the monotonicity of $U'$ and the inequalities (8), (9) and (13), we can rewrite (17) as
\[
\langle U'(x - x_\alpha), x^*_\alpha - x \rangle \leq \left[ \frac{h + \mu}{\alpha} g(||x^*_\alpha||) + \frac{\delta}{\alpha} \right] ||x - x^*_\alpha|| + \frac{\varepsilon}{\alpha} [d(||x||) + d(||x^*_\alpha||)], \quad \forall x \in S_0.
\]
Since $\alpha \to 0$, $\omega/\alpha$, $\delta/\alpha$, $\mu/\alpha \to 0$ (and $h/\alpha \to 0$), the last inequality becomes
\[
\langle U'(x - x_\alpha), \tilde{x} - x \rangle \leq 0, \quad \forall x \in S_0.
\]
Replacing $x$ by $t\tilde{x} + (1 - t)x$, $t \in (0, 1)$ in the last inequality, dividing by $(1 - t)$ and then letting $t$ to 1, we get
\[
\langle U'(\tilde{x} - x_\alpha), \tilde{x} - x \rangle \leq 0, \quad \forall x \in S_0
\]
or
\[
\langle U'(\tilde{x} - x_\alpha), \tilde{x} - x \rangle \leq \langle U'(\tilde{x} - x_\alpha), x - x_\alpha \rangle, \quad \forall x \in S_0.
\]
Using the property of $U'$, we have that $||\tilde{x} - x_\alpha|| \leq ||x - x_\alpha||$, $\forall x \in S_0$. Because of the convexity and the closedness of $S_0$, and the strictly convexity of $X$, we can conclude that $\tilde{x} = x_0$. The proof is complete.
\]
Now, we consider the problem of choosing posteriori regularization parameter $\tilde{\alpha} = \alpha(\mu, \delta, \varepsilon)$ such that
\[
\lim_{\mu, \delta, \varepsilon \to 0} \alpha(\mu, \delta, \varepsilon) = 0 \quad \text{and} \quad \lim_{\mu, \delta, \varepsilon \to 0} \frac{\mu + \delta + \varepsilon}{\alpha(\mu, \delta, \varepsilon)} = 0.
\]

To solve this problem, we use the function for selecting \( \tilde{\alpha} = \alpha(\mu, \delta, \varepsilon) \) by generalized discrepancy principle, i.e. the relation \( \tilde{\alpha} = \alpha(\mu, \delta, \varepsilon) \) is constructed on the basis of the following equation:

\[
\rho(\tilde{\alpha}) = (\mu + \delta + \varepsilon)^p \tilde{\alpha}^{-q}, \quad p, q > 0,
\]

with \( \rho(\tilde{\alpha}) = \tilde{\alpha} (c + ||x^*_u - x_\alpha||^{-1}) \), where \( \tilde{x}^*_u \) is the solution of (7) with \( \alpha = \tilde{\alpha} \), \( c \) is some positive constant.

**Lemma 2.2.** Let \( X \) and \( X^* \) be strictly convex Banach spaces and \( A : X \to X^* \) be a monotone-bounded hemicontinuous operator with \( D(A) = X \). Assume that conditions (1), (2) are satisfied, the operator \( U^* \) satisfies condition (13). Then, the function \( \rho(\alpha) = \alpha (c + ||x^*_u - x_\alpha||^{-1}) \) is single-valued and continuous for \( \alpha \geq \alpha_0 > 0 \), where \( x^*_u \) is the solution of (7).

**Proof.** Single-valued solvability of the inequality (7) implies the continuity property of the function \( \rho(\alpha) \). Let \( \alpha_1, \alpha_2 \geq \alpha_0 \) be arbitrary \( (\alpha_0 > 0) \). It follows from (7) that

\[
\alpha_1 \langle U^*(x^*_{u_1} - x_u), x^*_{u_1} - x^*_{u_2} \rangle + \alpha_2 \langle U^*(x^*_{u_2} - x_u), x^*_{u_1} - x^*_{u_2} \rangle + \langle A_h x^*_{u_1} - A_h x^*_{u_2}, x^*_{u_1} - x^*_{u_2} \rangle \geq -\mu (g(||x^*_{u_1}||) + g(||x^*_{u_2}||)) ||x^*_{u_1} - x^*_{u_2}||,
\]

where \( x^*_{u_1} \) and \( x^*_{u_2} \) are solutions of (7) with \( \alpha = \alpha_1 \) and \( \alpha = \alpha_2 \). Using the condition (2) and the monotonicity of \( A \), we have

\[
\alpha_1 \langle U^*(x^*_{u_1} - x_u) - U^*(x^*_{u_2} - x_u), x^*_{u_1} - x^*_{u_2} \rangle \leq (\alpha_2 - \alpha_1) \langle U^*(x^*_{u_2} - x_u), x^*_{u_1} - x^*_{u_2} \rangle + (h + \mu) (g(||x^*_{u_1}||) + g(||x^*_{u_2}||)) ||x^*_{u_1} - x^*_{u_2}||.
\]

It follows from (13) and the last inequality that

\[
m_\gamma ||x^*_{u_1} - x^*_{u_2}|| \leq \frac{|\alpha_1 - \alpha_2|}{\alpha_0} ||x^*_{u_2} - x_u||^{-1} + (h + \mu) (g(||x^*_{u_1}||) + g(||x^*_{u_2}||)).
\]

Obviously, \( x^*_{u_1} \to x^*_u \) as \( \mu \to 0 \) and \( \alpha_1 \to \alpha_2 \). It means that the function \( ||x^*_u - x_\alpha||^{-1} \) is continuous on \( [\alpha_0, +\infty) \). Therefore, \( \rho(\alpha) \) is also continuous on \( [\alpha_0, +\infty) \).

**Theorem 2.2.** Let \( X \) and \( X^* \) be strictly convex Banach spaces and \( A : X \to X^* \) be a monotone-bounded hemicontinuous operator with \( D(A) = X \). Assume that conditions (1)- (3) are satisfied, the operator \( U^* \) satisfies condition (13). Then

(i) there exists at least a solution \( \tilde{\alpha} \) of the equation (30),

(ii) let \( \mu, \delta, \varepsilon \to 0 \). Then

\[
(1) \quad \tilde{\alpha} \to \alpha
\]

(2) if \( 0 < p < q \) then \( \frac{\mu + \delta + \varepsilon}{\tilde{\alpha}} \to 0 \), \( x^*_{u_0} \to x_0 \in S_0 \) with \( x_0 \)-minimal norm and there exist constants \( C_1, C_2 > 0 \) such that for sufficiently small \( \mu, \delta, \varepsilon > 0 \) the relation

\[
C_1 \leq (\mu + \delta + \varepsilon)^p \tilde{\alpha}^{-1-q} \leq C_2
\]

holds.
Proof.
(i) For $0 < \alpha < 1$, it follows from (7) that
\[
\langle A\alpha x_\alpha^\tau + \alpha U'(x_\alpha^\tau - x_\ast) - f_\ast, x_\ast - x_\alpha^\tau \rangle + \varphi_\varepsilon(x_\ast) - \varphi_\varepsilon(x_\alpha^\tau) \\
\geq -\mu g(||x_\alpha^\tau||)||x_\ast - x_\alpha^\tau||.
\]
Hence,
\[
\alpha \langle U'(x_\alpha^\tau - x_\ast), x_\alpha^\tau - x_\ast \rangle \leq \mu g(||x_\alpha^\tau||)||x_\ast - x_\alpha^\tau|| + \varphi_\varepsilon(x_\ast) - \varphi_\varepsilon(x_\alpha^\tau) \\
+ \langle A\alpha x_\alpha^\tau - Ax_\alpha^\tau, Ax_\alpha^\tau - Ax_\ast + f + f - f_\ast, x_\ast - x_\alpha^\tau \rangle.
\]
We invoke the condition (1), the monotonicity of $A$, (8), (10), (12), and the last inequality to deduce that
\[
\alpha||x_\alpha^\tau - x_\ast||^{t-1} \leq (h + \mu)g(||x_\alpha^\tau||) + C_0 + ||Ax_\ast - f|| + \delta.
\] (33)
It follows from (33) and the form of $\rho(\alpha)$ that
\[
\alpha^q \rho(\alpha) = \alpha^{1+q}(\varepsilon + ||x_\alpha^\tau - x_\ast||^{t-1}) \\
= c\alpha^{1+q} + \alpha^q \times \alpha||x_\alpha^\tau - x_\ast||^{t-1} \\
\leq c\alpha^{1+q} + \alpha^q[(h + \mu)g(||x_\alpha^\tau||) + C_0 + ||Ax_\ast - f|| + \delta].
\]
Therefore, $\lim_{\alpha \to +0} \alpha^q \rho(\alpha) = 0$.
On the other hand,
\[
\lim_{\alpha \to +\infty} \alpha^q \rho(\alpha) \geq c \lim_{\alpha \to +\infty} \alpha^{1+q} = +\infty.
\]
Since $\rho(\alpha)$ is continuous, there exists at least one $\bar{\alpha}$ which satisfies (30).
(ii) It follows from (30) and the form of $\rho(\bar{\alpha})$ that
\[
\bar{\alpha} \leq c^{-1/(1+q)} \mu + \delta + \varepsilon)^{p/(1+q)}.
\]
Therefore, $\bar{\alpha} \to 0$ as $\mu, \delta, \varepsilon \to 0$.
If $0 < p < q$, it follows from (30) and (32) that
\[
\left[ \frac{\mu + \delta + \varepsilon}{\bar{\alpha}} \right]^p \\
= \left( [\mu + \delta + \varepsilon]^{1-q} \bar{\alpha}^{1-q} \right)^p \\
= \left[ c\bar{\alpha} + \bar{\alpha}||x_\alpha^\tau - x_\ast||^{t-1} \right]^{p-1} \\
\leq c\alpha^{1+q-p} + \alpha^{q-p} [2\mu g(||x_\alpha^\tau||) + C_0 + ||Ax_\ast - f|| + \delta].
\]
So,
\[
\lim_{\mu, \delta, \varepsilon \to 0} \left[ \frac{\mu + \delta + \varepsilon}{\bar{\alpha}} \right]^p = 0.
\]
By Theorem 2.1 the sequence $x_\alpha^\tau$ converges to $x_\ast \in S_0$ with $x_\ast$-minimal norm as $\mu, \delta, \varepsilon \to 0$.
Clearly,
\[
(\mu + \delta + \varepsilon)^q \bar{\alpha}^{-1-q} = \bar{\alpha}^{-1} \rho(\bar{\alpha}) = (c + ||x_\alpha^\tau - x_\ast||^{t-1}),
\]
therefore, there exists a positive constant $C_2$ such that (32). On the other hand, because $c > 0$ so there exists a positive constant $C_1$ satisfied (32). This finishes the proof.
\[\square\]
Theorem 2.3. Let $X$ be a strictly convex Banach space and $A$ be a monotone-bounded hemicontinuous operator with $D(A) = X$. Suppose that

(i) for each $h$, $\delta$, $\varepsilon > 0$ conditions (1)-(3) are satisfied;
(ii) $U^*$ satisfies condition (13);
(iii) $A$ is an inverse-strongly monotone operator from $X$ into $X^*$, Fréchet differentiable at some neighborhood of $x_0 \in S_0$ and satisfies
$$||A(x) - A(x_0) - A'(x_0)(x - x_0)|| \leq \tilde{\tau}||A(x) - A(x_0)||;$$

(iv) there exists $z \in X$ such that
$$A'(x_0)^*z = U^*(x_0 - x_*);$$
then, if the parameter $\alpha = \alpha(\mu, \delta, \varepsilon)$ is chosen by (30) with $0 < p < q$, we have
$$||A^{\alpha}(x_0) - A(x_0)|| = O\left(\sqrt{\delta + \mu + \varepsilon + \alpha}\right).$$

Proof. By an argument analogous to that used for the proof of the first part of Theorem 2.1, we have (21). The boundedness of the sequence $\{x^{\alpha}_n\}$ follows from (21) and the properties of $g(t), d(t)$ and $\alpha$. On the other hand, based on (20), the property of $U^*$ and the inverse-strongly monotone property of $A$ we get that
$$\|A(x^{\alpha}_n) - A(x_0)\|^2 \leq m_1^{-1}\left\{\left[(h + \mu)g(\|x^{\alpha}_n\|) + \delta + \alpha\|x^{\alpha}_n - x_*\|^{r-1}\right]\|x_0 - x^{\alpha}_n\| + \varepsilon [d(\|x_0\|) + d(\|x^{\alpha}_n\|)]\}.$$

Hence,
$$||A(x^{\alpha}_n) - A(x_0)|| = O\left(\sqrt{\delta + \mu + \varepsilon + \alpha}\right).$$

Further, by virtue of conditions (iii), (iv) and the last estimate, we obtain
$$\langle U^*(x_0 - x_*), x_0 - x^{\alpha}_n \rangle = \langle z, A'(x_0)(x_0 - x^{\alpha}_n) \rangle \leq \|z\|(1 + \tilde{\tau})\|A(x^{\alpha}_n) - A(x_0)\| \leq \|z\|(1 + \tilde{\tau})O\left(\sqrt{\delta + \mu + \varepsilon + \alpha}\right).$$

Consequently, (21) has the form
$$m_n\|x^{\alpha}_n - x_0\|^r \leq \frac{2\mu g(\|x^{\alpha}_n\|) + \delta}{\alpha}\|x_0 - x^{\alpha}_n\| + \|z\|(1 + \tilde{\tau})O\left(\sqrt{\delta + \mu + \varepsilon + \alpha}\right) + \\frac{\varepsilon}{\alpha}[d(\|x_0\|) + d(\|x^{\alpha}_n\|)].$$

When $\alpha$ is chosen by (30), it follows from Theorem 2.1 that
$$\alpha(\mu, \delta, \varepsilon) \leq C_1^{-1/(1+q)}(\mu + \delta + \varepsilon)^{\eta/(1+q)}.$$
and
\[
\frac{\mu + \delta + \varepsilon}{\alpha(\mu, \delta, \varepsilon)} \leq C_2(\mu + \delta + \varepsilon)^{1-\eta} a(\mu, \delta, \varepsilon)
\]
\[
\leq C_2 C_1^{-\eta/(1+\eta)}(\mu + \delta + \varepsilon)^{1-\eta/(1+\eta)}.
\]

Therefore, it follows from (35) that
\[
m_s ||x^\tau_{(\mu, \delta, \varepsilon)} - x_0||^s \leq \tilde{C}_1(\mu + \delta + \varepsilon)^{1-\eta} ||x^\tau_{(\mu, \delta, \varepsilon)} - x_0||
\]
\[
+ \tilde{C}_2(\mu + \delta + \varepsilon)^{1-\eta} + \tilde{C}_3(\mu + \delta + \varepsilon)^{\eta/(2+\eta)},
\]
where \(\tilde{C}_i, i = 1, 2, 3,\) are the positive constants. Using the implication
\[
a, b, c \geq 0, \quad s > t, \quad d' \leq ba^t + c \Rightarrow d' = O(b^{(t-1)} + c),
\]
we obtain
\[
||x^\tau_{(\mu, \delta, \varepsilon)} - x_0|| = O((\mu + \delta + \varepsilon)^{\mu_2}).
\]

**Remark 2.1** If \(\alpha\) is chosen a priori such that \(\alpha \sim (\mu + \delta + \varepsilon)^\eta,\) \(0 < \eta < 1,\) it follows from (35) that
\[
m_s ||x^\tau_{(\mu, \delta, \varepsilon)} - x_0||^s \leq \tilde{C}_4(\mu + \delta + \varepsilon)^{1-\eta} ||x_0 - x^\tau_{(\mu, \delta, \varepsilon)}||
\]
\[
+ \tilde{C}_5(\mu + \delta + \varepsilon)^{\eta/2} + \tilde{C}_6(\mu + \delta + \varepsilon)^{1-\eta}.
\]

Therefore,
\[
||x^\tau_{(\mu, \delta, \varepsilon)} - x_0|| = O((\mu + \delta + \varepsilon)^{\mu_2}), \quad \mu_2 = \min \left\{ \frac{1 - \eta}{s}, \frac{\eta}{2s} \right\}.
\]

**Remark 2.2** Condition (34) was proposed in [13] for studying convergence analysis of the Landweber iteration method for a class of nonlinear operators. This condition is used to estimate convergence rates of regularized solutions of ill-posed variational inequalities in [14].

**Remark 2.3** The generalized discrepancy principle for regularization parameter choice is presented in [15] for the ill-posed operator equation (4) when \(A\) is a linear and bounded operator in Hilbert space. It is considered and applied to estimating convergence rates of the regularized solution for equation (4) involving an accretive operator in [16].

**Competing interests**

The author declares that they have no competing interests.

Received: 10 February 2011 Accepted: 21 July 2011 Published: 21 July 2011

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Cite this article as: Thuy: Regularization of ill-posed mixed variational inequalities with non-monotone perturbations. Journal of Inequalities and Applications 2011 2011:25.

doi:10.1186/1029-242X-2011-25

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