Associative Yang-Baxter equation for quantum (semi-)dynamical R-matrices

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Abstract

In this paper we propose versions of the associative Yang-Baxter equation and higher order R-matrix identities which can be applied to quantum dynamical R-matrices. As is known quantum non-dynamical R-matrices of Baxter-Belavin type satisfy this equation. Together with unitarity condition and skew-symmetry it provides the quantum Yang-Baxter equation and a set of identities useful for different applications in integrable systems. The dynamical R-matrices satisfy the Gervais-Neveu-Felder (or dynamical Yang-Baxter) equation. Relation between the dynamical and non-dynamical cases is described by the IRF-Vertex transformation. An alternative approach to quantum (semi-)dynamical R-matrices and related quantum algebras was suggested by Arutyunov, Chekhov and Frolov (ACF) in their study of the quantum Ruijsenaars-Schneider model. The purpose of this paper is twofold. First, we prove that the ACF elliptic R-matrix satisfies the associative Yang-Baxter equation with shifted spectral parameters. Second, we directly prove a simple relation of the IRF-Vertex type between the Baxter-Belavin and the ACF elliptic R-matrices predicted previously by Avan and Rollet. It provides the higher order R-matrix identities and an explanation of the obtained equations through those for non-dynamical R-matrices. As a by-product we also get an interpretation of the intertwining transformation as matrix extension of scalar theta function likewise R-matrix is interpreted as matrix extension of the Kronecker function. Relations to the Gervais-Neveu-Felder equation and identities for the Felder’s elliptic R-matrix are also discussed.
1 Introduction and summary

We start with a brief review of the Yang-Baxter structures under consideration, and then give a summary of the paper.

1.1 Brief review

Quantum non-dynamical $R$-matrices of $\text{GL}(N, \mathbb{C})$ type in fundamental representation are elements of $\text{Mat}(N, \mathbb{C})^\otimes 2$ satisfying the quantum Yang-Baxter equation \cite{27, 6, 25}

$$R^h_{12}(z_1, z_2) R^h_{13}(z_1, z_3) R^h_{23}(z_2, z_3) = R^h_{23}(z_2, z_3) R^h_{13}(z_1, z_3) R^h_{12}(z_1, z_2)$$

(1.1)

together with the unitarity condition which we write in some special $R$-matrix normalization\footnote{The normalization (1.2) implies in fact that we deal with a special class of $R$-matrices which includes Baxter-Belavin’s elliptic one and some of its trigonometric and rational degenerations.}:

$$R^h_{12}(z_1, z_2) R^h_{21}(z_2, z_1) = \phi(h, z_1 - z_2) \phi(h, z_2 - z_1) \ 1 \otimes 1 \ = \ (\varphi(h) - \varphi(z_1 - z_2)) \ 1 \otimes 1,
\quad (1.2)$$

where $\phi(h, z)$ is the Kronecker function (A.7) and $\varphi(z)$ is the Weierstrass $\wp$-function (A.8), (A.11). The parameter $h$ is called the Planck constant, and $z_1, z_2$ – spectral parameters. The Baxter-Belavin (6, 8) elliptic solution of (1.1), (1.2) is of the form:

$$R^h_{12}(h, z_1, z_2) = R^h_{12}(h, z_1 - z_2) = \frac{1}{N} \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi^h_a(z_1 - z_2) T_a \otimes T_{-a},
\quad (1.3)$$

where $\{T_a\}$ is a special basis (A.4) in $\text{Mat}(N, \mathbb{C})$, and $\{\varphi^h_a(z)\}$ is a set of related functions (A.12).
Quantum dynamical $R$-matrices: Gervais-Neveu-Felder equation. The dynamical $R$-matrices depend on additional parameters $u_1,...,u_N$. In the classical Hamiltonian mechanics of integrable many-body systems they are the coordinates of particles, while in quantum case these are the parameters entering the Boltzmann weights in IRF statistical models [14, 15].

The quantum dynamical $R$-matrices are described by the Gervais-Neveu-Felder equation [12]:

$$R^h_{12}(z_1, z_2 \mid u) R^h_{13}(z_1, z_3 \mid u + h^{(2)}) R^h_{23}(z_2, z_3 \mid u) =$$

$$= R^h_{23}(z_2, z_3 \mid u + h^{(1)}) R^h_{13}(z_1, z_3 \mid u) R^h_{12}(z_1, z_2 \mid u + h^{(3)}),$$

where the shifts of dynamical arguments $u$ are performed as follows:

$$R^h_{12}(z_1, z_2 \mid u + h^{(3)}) = P^h_3 R^h_{12}(z_1, z_2 \mid u) P^{-h}_3, \quad P^h_3 = \sum_{k=1}^N 1 \otimes 1 \otimes E_{kk} \exp(h \frac{\partial}{\partial u_k})$$

with notation $\{E_{ij}\}$ for the standard basis in Mat($N, \mathbb{C}$): $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$. The weight zero condition implies that $[R^h_{12}(z_1, z_2 \mid u), P^h_{12}] = 0$.

The elliptic solution of (1.4), (1.2) is given by the Felder’s $R$-matrix [11]:

$$R^E_{12}(h, z_1, z_2 \mid u) = R^E_{12}(h, z_1 - z_2 \mid u) =$$

$$= \sum_{i \neq j} E_{ii} \otimes E_{jj} \phi(h, u_{ij}) + \sum_{i \neq j} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, -u_{ij}) + \phi(h, z_1 - z_2) \sum_i E_{ii} \otimes E_{ii},$$

where $u_{ij} = u_i - u_j$ and $\phi(\eta, z)$ is the Kronecker function $\langle \eta, i \rangle$.

IRF-Vertex (or Vertex-Face) correspondence provides explicit relation between dynamical and non-dynamical $R$-matrices [7, 14, 13]. Its applications to classical integrable systems, 1+1 models and monodromy preserving equations can be found in [16, 17]. Consider the following matrix $g \in$ Mat($N, \mathbb{C}$):

$$g_{ij}(z, u) = \vartheta \left[ \frac{1}{2} - i \frac{N}{2} \right] \left( z + Nu_j - \sum_{m=1}^N u_m | N \tau \right) \prod_{k \neq j} \frac{1}{\vartheta(u_k - u_j)},$$

where theta-functions with characteristics are defined in (A.2). The IRF-Vertex relation between the quantum $R$-matrices (1.3) and (1.6) has the form:

$$g_2(z_2, u) g_1(z_1, u - h^{(2)}) R^E_{12}(h, z_1 - z_2 \mid u) = R^h_{12}(h, z_1 - z_2) g_1(z_1, u) g_2(z_2, u - h^{(1)}).$$

The trigonometric and rational analogues of (1.3) and (1.7) can be found in [2] and [17].

Quantum dynamical $R$-matrices: Arutyunov-Chechkov-Frolov (ACF) approach. An alternative to (1.4) quantization of dynamical $r$-matrix structure was suggested in [3] and then studied in [4, 5], where it was called semi-dynamical Yang-Baxter equation:

$$R^h_{12}(z_1, z_2 \mid u) R^h_{13}(z_1 - h, z_3 - h \mid u) R^h_{23}(z_2, z_3 \mid u) =$$

$$= R^h_{23}(z_2 - h, z_3 - h \mid u) R^h_{13}(z_1, z_3 \mid u) R^h_{12}(z_1 - h, z_2 - h \mid u).$$
Notice that in (1.9) there are no shifts (1.5) of the dynamical parameters instead. The elliptic $R$-matrix satisfying the unitarity condition (1.2) and the Yang-Baxter equation (1.9) was found in [3]. It is of the form:

$$R_{12}^{\text{ACF}}(\hbar, z_1, z_2 | u) = \sum_{i \neq j} E_{ii} \otimes E_{jj} \phi(\hbar, -u_{ij}) + \sum_{i \neq j} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, -u_{ij}) -$$

$$- \sum_{i \neq j} E_{ij} \otimes E_{jj} \phi(z_1 + h, -u_{ij}) + \sum_{i \neq j} E_{jj} \otimes E_{ij} \phi(z_2, -u_{ij}) +$$

$$(1.10)$$

$$+ (E_1(h) + E_1(z_1 - z_2) + E_1(z_2) - E_1(z_1 + h)) \sum_i E_{ii} \otimes E_{ii},$$

where $u_{ij} = u_i - u_j$ and $E_1(z) = \vartheta'(z)/\vartheta(z)$ is the first Eisenstein function (A.8).

The relation between $R_{12}^{\text{ACF}}$ and $R^p$ was obtained

$$R_{12}^{\text{ACF}}(\hbar, z_1, z_2 | u) = \tilde{R}_{12}(\hbar, z_1 | u - h^{(2)}) R_{12}^{p}(h, z_1 - z_2 | u) \tilde{R}_{12}^{-1}(h, z_2 | u - h^{(1)}) (1.11)$$

or

$$R_{12}^{\text{ACF}}(h, z_1, z_2 | u) = \tilde{R}_{21}(h, z_2 | u) R_{12}^{p}(h, z_1 - z_2 | u) \tilde{R}_{12}^{-1}(h, z_1 | u) (1.12)$$

in terms of explicitly given twist matrix\(^2\)

$$\frac{\vartheta'(0)}{\vartheta(\hbar)} \tilde{R}_{12}(\hbar, z | u) =$$

$$= \sum_{i \neq j} E_{ii} \otimes E_{jj} \phi(\hbar, -u_{ij}) - \sum_{i \neq j} E_{ij} \otimes E_{jj} \phi(z + h, -u_{ij}) - \phi(z + h, -h) \sum_i E_{ii} \otimes E_{ii} (1.13)$$

and its inverse

$$\frac{\vartheta'(0)}{\vartheta(\hbar)} \tilde{R}_{12}^{-1}(h, z | u) = \sum_{i, j} E_{ii} \otimes E_{jj} \phi(h, u_{ij} - h) - \sum_{i, j} E_{ij} \otimes E_{jj} \phi(z, h - u_{ij}). (1.14)$$

The origin of the ACF type Yang-Baxter equation (1.9), i.e. its relation to the Yang-Baxter equation (1.1) or the Gervais-Neveu-Felder equation (1.4) was described in [22, 23]. We will give a direct proof of relation between $R$-matrices $R_{12}^{\text{ACF}}$ and $R^a$ (and therefore $R^p$) and corresponding Yang-Baxter equations as well as higher order $R$-matrix identities (see Theorem 2 below).

**Associative Yang-Baxter equation** was originally introduced in [1] for constant $R$-matrices and then generalized by Polishchuk [22] to the form:

$$R_{12}^h R_{23}^h = R_{13}^h R_{12}^{\eta_h} + R_{23}^{\eta_h} R_{13}^h, \quad R_{ab}^h = R_{ab}^h (z_a - z_b). (1.15)$$

\(^2\)We use different sign for the dynamical parameters $u$, and $R$-matrix normalization is chosen as in (1.2).

\(^3\)The shifts $h^{(1)}$, $h^{(2)}$ in (1.11) are defined as in (1.5).

\(^4\)A certain relation between (1.9) and (1.1) was observed indirectly in the original paper [3]: it was shown that any representation of quantum algebra underlying $R_{12}^{\text{ACF}}$: $R_{12}^{\text{ACF}}(z, w)L_1(z)\tilde{R}_{21}(w)L_2(w) = L_2(w)\tilde{R}_{12}(z)L_1(z)\tilde{R}_{21}(w)$ turns into a representation of the exchange relations $R_{12}^h(z - w)L_1(z)\tilde{R}_{21}(w) = \tilde{L}_2(w)L_1(z)\tilde{R}_{12}(z - w)$ via the IRF-Vertex transformation (1.8).
It was shown in [22] that the Baxter-Belavin $R$-matrix written in Richey-Tracy [23] form satisfies (1.15). The unitarity condition (1.2) was not required. The $R$-matrix (1.3) was rather considered as deformation of the classical one. In fact, the quantum $R$-matrix (1.3) satisfies also the skew-symmetry property

$$R^h_{12}(z_1 - z_2) = -R^{-h}_{21}(z_2 - z_1)$$

which can be viewed as the classical analogue of the unitarity condition (1.2). The relation of (1.15) to the quantum Yang-Baxter equation (1.1) was studied separately [24].

The most natural and simple dynamical solution of (1.15) was proposed by Burban and Henrich [9] (see also [21]):

$$R_{12}^{BH}(h, z_1, z_2 | u) = R_{12}^{BH}(h, z_1 - z_2 | u) = \sum_{i,j} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, h - u_{ij}).$$

(1.17)

It is skew-symmetric (1.16) but the unitarity condition is not valid:

$$R_{12}^{BH}(h, z_1 - z_2 | u)R_{21}^{BH}(h, z_2 - z_1 | u) = \sum_{i,j} E_{ii} \otimes E_{jj} \left(\varphi(h - u_{ij}) - \varphi(z_1 - z_2)\right)$$

(1.18)

as well as the quantum Yang-Baxter equation (1.1). Therefore, it behaves more like a classical $r$-matrix.

Later [18, 19] the equation (1.15) found applications in integrable systems, the KZB and Painlevé equations. In particular, it was mentioned in [18, 20] that a unitary (1.2) and skew-symmetric (1.16) solution of the associative Yang-Baxter equation (1.15) (in particular, the Baxter-Belavin (1.3) one) satisfies also the following cubic identity:

$$R_{12}^{\eta} R_{13}^{h} R_{23}^{\eta} - R_{23}^{h} R_{13}^{\eta} R_{12}^{h} = R_{13}^{h+\eta} \left(\varphi(\eta) - \varphi(h)\right),$$

(1.19)

where $R_{ab}^h = R_{ab}^{\eta}(h, z_a, z_b)$. For $\eta = h$ the latter equation provides the Yang-Baxter one (1.1) while for $\eta = -h$ it leads to

$$R_{12}^{h} R_{23}^{h} R_{31}^{h} + R_{13}^{h} R_{32}^{h} R_{21}^{h} = -\varphi'(h) 1 \otimes 1 \otimes 1.$$

(1.20)

Higher order analogues of (1.20) can be found in [28]. They are discussed below.

### 1.2 Summary

The purpose of paper is to find solutions of the associative Yang-Baxter equation (1.15) and to prove identities of type (1.19), (1.20) for quantum (semi-)dynamical $R$-matrices. Before we proceed further let us mention that (1.15) can be considered as matrix generalization of the Fay identity (A.9), which for $z = z_1 - z_2$ and $w = z_2 - z_3$ takes the form

$$\phi(h, z_{12})\phi(\eta, z_{23}) = \phi(\eta, z_{13})\phi(h - \eta, z_{12}) + \phi(\eta - h, z_{23})\phi(h, z_{13}), \quad z_{ab} = z_a - z_b.$$
Indeed, the Baxter-Belavin $R$-matrix in scalar case (for $N = 1$) is exactly the Kronecker function $\phi(h, z_1 - z_2)$. This analogy, in fact, underlies the results of papers [22] and [18, 19, 28]. In this sense the unitarity condition (1.2) is analogue of (A.11).

Similarly to the Baxter-Belavin case the Felder’s $R$-matrix (1.6) is also unitary and skew-symmetric. Moreover, for $N = 1$ it equals to the Kronecker function $\phi(h, z_1 - z_2)$. However, we have not found a quadratic equation of type (1.15) for $R^\phi$. Equations for $R^\phi$ follows from those for $R^h$ via the IRF-Vertex transformation (1.8) but the twist matrix (1.7) is not cancelled out from the final answers. We discuss it in Section 4. At the same time it appears that the quadratic equation of type (1.15) holds true for $R^{ACF}$.

**Theorem 1** The ACF $R$-matrix (1.10) satisfies the following modification of the associative Yang-Baxter equation (1.13):

$$R^h_{12}(z_1 + \eta, z_2 + \eta)R^h_{23}(z_2 + h, z_3 + h) = R^h_{13}(z_1 + h, z_3 + h)R^h_{12}(z_1 + \eta, z_2 + \eta) + R^h_{23}(z_2 + h, z_3 + h)R^h_{13}(z_1 + \eta, z_3 + \eta),$$

where $R^h_{ab}(z, w) = R^{ACF}_b(h, z, w|u)$, and the cubic identity

$$R^n_{12}(z_1, z_2) R^h_{13}(z_1 - h, z_3 - h) R^n_{23}(z_2, z_3) -$$

$$- R^n_{23}(z_2 - h, z_3 - h) R^n_{13}(z_1, z_3) R^h_{12}(z_1 - h, z_2 - h) = R^{h+\eta}_{13}(z_1 - h, z_3 - h) \left(\wp(\eta) - \wp(h)\right).$$

See Section 2 for the proof. As a conclusion of this theorem we also obtain the Yang-Baxter equation (1.9) ($\eta = h$ in (1.23)) and the unchanged identity (1.19) ($\eta = -h$ in (1.23)).

The results of Theorem 1 are valid in trigonometric and rational cases as well. The functions (A.7), (A.8) entering the ACF $R$-matrix are given for these cases.

The equations (1.22) and (1.23) can be derived from their non-dynamical analogues (1.15) and (1.19) using the IRF-Vertex like relation between $R^{ACF}$ (1.10) and $R^h$ (1.3) predicted in [4]:

**Theorem 2** The ACF $R$-matrix (1.10) and the twist matrix (1.13) are expressed in terms of IRF-Vertex transformation matrix (1.7) and the Baxter-Belavin $R$-matrix (1.5) as follows:

$$R^n_{12}(h, z_1 - z_2) = g_1(z_1 + h, u) g_2(z_2, u) R^{ACF}_{12}(h, z_1, z_2|u) g_2^{-1}(z_2 + h, u) g_1^{-1}(z_1, u) \quad (1.24)$$

and

$$R_{12}(h, z|u) = g_1^{-1}(z + h, u + h^{(2)}) g_1(z, u). \quad (1.25)$$

See the proof in Section 3. As we will see it follows from (1.24) that the ACF $R$-matrix satisfies also $n$-th order identities proved for $R^n$ in [28]:

$$\sum_{1 \leq i_1 \ldots i_n \leq n} R^h_{a_1 i_1} R^h_{i_1 i_2} \ldots R^h_{i_{n-2} i_{n-1}} R^h_{i_{n-1} a_n} = 1 \otimes \ldots \otimes 1 (-1)^n \frac{d^{(n-2)}}{d\eta^{(n-2)}} \wp(\eta) \bigg|_{\eta = h}. \quad (1.26)$$

where $R^h_{ij} = R^{ACF}_{ij}(h, z_i, z_j|u)$, $a$ is a fixed index $1 \leq a \leq n$ and $n \geq 3$. For $n = 3$ it is (1.20).
As already mentioned the associative Yang-Baxter equation (1.13) in non-dynamical case is a matrix analogue of the Fay identity (1.21) for the Kronecker function. At the same time the scalar \((N = 1)\) ACF \(R\)-matrix is not the Kronecker function. It is equal to

\[
\phi^h(z_1, z_2) = E_1(h) + E_1(z_1 - z_2) + E_1(z_2) - E_1(z_1 + h) \quad \text{(4.10)}
\]

Neverthless this function satisfies (1.22) due to the Fay identity (1.21) because the products of additional multiples \((\phi(h, z_j)/\phi(h, z_j))\) are equal for each term of (1.22). See (A.13).

It is also notable that (1.24) leads to interpretation of the intertwining matrix (1.7) as a matrix analogue of the Kronecker function. See (3.14). Finally, in Section 4 we discuss identities for the Felder’s \(R\)-matrix arising from the IRF-Vertex relations.

## 2 Associative YB equation for ACF \(R\)-matrix

The proof of (1.22) is given in the Appendix. Let us derive the cubic identity (1.23) likewise it was made in [20] for derivation of (1.19) using (1.15), (1.2) and (1.16). For this purpose we also need an analogue of the skew-symmetry property (1.16) for \(R^{ACF}\). It is given in [3]:

\[
R^h_{12}(z_1, z_2) = -R^{-h}_{21}(z_2 + h, z_1 + h),
\]

where \(R^h_{12}(z_1, z_2) = R^{ACF}(h, z_1, z_2| u)\).

**Proof of cubic identity (1.23):**

Multiplying equation (1.22) by \(R^{h-\eta}_{23}(z_2 + \eta, z_3 + \eta)\) form the left and using (1.2), (2.1) we obtain

\[
R^{h-\eta}_{23}(z_2 + \eta, z_3 + \eta)R^h_{12}(z_1 + \eta, z_2 + \eta)R^\eta_{23}(z_2 + h, z_3 + h) =\]

\[
= R^{h-\eta}_{23}(z_2 + \eta, z_3 + \eta)R^\eta_{13}(z_1 + h, z_3 + h)R^{h-\eta}_{12}(z_1 + \eta, z_2 + \eta) -
\]

\[-(\phi(h - \eta) - \phi(z_2 - z_3)) R^h_{13}(z_1 + \eta, z_3 + \eta).\]

Consider (1.22) with interchanged indices 2 and 3:

\[
R^h_{13}(z_1 + \eta, z_3 + \eta)R^\eta_{32}(z_3 + h, z_2 + h) =
\]

\[
R^\eta_{12}(z_1 + h, z_2 + h)R^{h-\eta}_{13}(z_1 + \eta, z_3 + \eta) + R^{h-h}_{32}(z_3 + h, z_2 + h)R^{h}_{12}(z_1 + \eta, z_2 + \eta).
\]

Multiplying it by \(R^\eta_{23}(z_2 + h, z_3 + h)\) from the right and using (1.2), (2.1) we get:

\[
R^h_{13}(z_1 + \eta, z_3 + \eta) (\phi(\eta) - \phi(z_2 - z_3)) =
\]

\[
= R^\eta_{12}(z_1 + h, z_2 + h)R^{h-\eta}_{13}(z_1 + \eta, z_3 + \eta)R^\eta_{23}(z_2 + h, z_3 + h) +
\]

\[
+ R^{\eta-h}_{32}(z_3 + h, z_2 + h)R^h_{12}(z_1 + \eta, z_2 + \eta)R^\eta_{23}(z_2 + h, z_3 + h).
\]
Subtracting (2.4) from (2.2) yields:

\[ R^h_{12}(z_1 + h, z_2 + h) R^{h-\eta}_{13}(z_1 + \eta, z_3 + \eta) R^\eta_{23}(z_2 + h, z_3 + h) - \]

\[ - R^{h-\eta}_{23}(z_2 + \eta, z_3 + \eta) R^\eta_{13}(z_1 + h, z_3 + \eta) R^{h-\eta}_{12}(z_1 + \eta, z_2 + \eta) = \]

\[ (\varphi(\eta) - \varphi(h - \eta)) R^h_{13}(z_1 + \eta, z_3 + \eta) . \]

Redefinition \( h := h + \eta \) and \( z_{1,2,3} := z_{1,2,3} - h - \eta \) gives (1.23). \( \blacksquare \)

The special case \( \eta = h \) for (1.23) obviously reproduces the Yang-Baxter equation (1.9). At the same time the case \( \eta = -h \) yields (1.20)

\[ R^h_{12}(z_1, z_2) R^h_{23}(z_2, z_3) R^h_{31}(z_3, z_1) + R^h_{13}(z_1, z_3) R^h_{32}(z_3, z_2) R^h_{21}(z_2, z_1) = -\varphi'(h) \ 1 \otimes 1 \otimes 1 \] (2.6)

via the usage of skew-symmetry (2.1) and due to

\[ \lim_{h \to 0} [h \ R^{ACF}_{12}(h, z_1, z_2 | u)] = 1 \otimes 1 . \] (2.7)

## 3 IRF-Vertex for ACF \( R \)-matrix and higher identities

We start with

**Proof of Theorem 2**

Suppose (1.25) holds true. Then (1.24) follows from (1.8) and (1.11). Indeed, plugging (1.25) into (1.11) we get for \( R^p \)

\[ R^p_{12}(h, z_1 - z_2 | u) = \]

\[ = g_1^{-1}(z_1, u - \hbar^{(2)}) g_1(z_1 + h, u) R^{ACF}_{12}(h, z_1, z_2 | u) g_2^{-1}(z_2 + h, u) g_2(z_2, u - \hbar^{(1)}) . \] (3.1)

On the other hand, it follows from the IRF-Vertex relation (1.8) that \( R^p_{12}(h, z_1 - z_2 | u) \) equals

\[ g_1^{-1}(z_1, u - \hbar^{(2)}) g_2^{-1}(z_2, u) R^{p}_{12}(h, z_1 - z_2 | u) g_1(z_1, u) g_2(z_2, u - \hbar^{(1)}) . \] (3.2)

Compared together (3.1) and (3.2) yield (1.24).

Let us prove now (1.25), which can be rewritten in the form

\[ g_1(z, u) \tilde{R}^{-1}_{12}(h, z | u) = g_1(z + h, u + \hbar^{(2)}) . \] (3.3)

Substituting \( g(z, u) \) (1.7) and \( \tilde{R}^{-1}_{12} \) (1.14) into (3.3). Then we obtain for its l.h.s.:

\[ \frac{\partial(h)}{\partial(0)} \sum_{i,j,k} E_{ij} \otimes E_{kk} g_{ij}(z, u) \phi(h, u_{jk} - h) + E_{ij} \otimes E_{jj} g_{ik}(z, u) \phi(z, h - u_{kj}) . \] (3.4)

Using the definition (1.5), the r.h.s. of (3.3) equals

\[ \sum_{i,j,k} E_{ij} \otimes E_{kk} \exp \left( h \frac{\partial}{\partial u_k} \right) g_{ij}(z + h, u) \exp \left( -h \frac{\partial}{\partial u_k} \right) . \] (3.5)
Comparing (3.4) and (3.5) taking into account that \(\exp(h \partial_u) (\sum_m u_m) \exp(-h \partial_u) = h + \sum_m u_m\) for any \(k\). For \(k \neq j\) type terms equality of (3.4) and (3.5) is equivalent to

\[
\frac{\partial(h)}{\partial^2(0)} g_{ij}(z, u) \phi(h, u_{jk} - h) = g_{ij}(z, u) \frac{\partial(u_{kj})}{\partial(u_{kj} + h)}.
\]

(3.6)

It is the definition of the Kronecker function. For \(k = j\) the first term in (3.4) equals zero \((\phi(h, -h) = 0)\). The equality of (3.4) and (3.5) takes the form:

\[
\frac{\partial(h)}{\partial^2(0)} \sum_k g_{ik}(z, u) \phi(z, -u_{kj} + h) = g_{ij}(z + N h, u) \prod_{m \neq j} \frac{\partial(u_{mj})}{\partial(u_{mj} - h)}.
\]

(3.7)

The latter is the statement of [13] about factorization of the Lax operator for the Ruijse naars-Schneider model.$$

\textbf{Identities for ACF } R\text{-matrix.} \text{ The statement of Theorem } 2 \text{ allows to derive results of the previous Section from those for non-dynamical } R\text{-matrices. Let us show that (1.22) follows from the associative Yang-Baxter equation (1.15) for non-dynamical } R\text{-matrices. To see it we will use that the l.h.s. (i.e. the Baxter-Belavin } R\text{-matrix) of (1.24) depends on difference of spectral parameters, that is its r.h.s. is independent of the shift } z_1 \to z_1 + c \text{ and } z_2 \to z_2 + c.

\text{Substitute (1.24) into } R_{12}^g(h_1, z_1 - z_2)R_{23}^g(\eta, z_2 - z_3) \text{ (the l.h.s. of (1.15)) with the constant } c = \eta \text{ for } R_{12}^g \text{ and with } c = h \text{ for } R_{23}^g:

R_{12}^g(h_1, z_1 - z_2)R_{23}^g(\eta, z_2 - z_3) = g_1(z_1 + h + \eta)g_2(z_2 + \eta)g_3(z_3 + h) \times

\times R_{12}^{\text{ACF}}(h, z_1 + \eta, z_2 + \eta)R_{23}^{\text{ACF}}(\eta, z_2 + h, z_3 + h)g_1^{-1}(z_1 + \eta)g_2^{-1}(z_2 + h)g_3^{-1}(z_3 + h + \eta).

(3.8)

In the same way substitute (1.24) into \(R_{13}^g(\eta, z_1 - z_3)R_{12}^g(h - \eta, z_1 - z_2)\) (the first term in the r.h.s. of (1.15)) with the constant \(c = h\) for \(R_{13}^g\) and with \(c = \eta\) for \(R_{12}^g:

R_{13}^g(\eta, z_1 - z_3)R_{12}^g(h - \eta, z_1 - z_2) = g_1(z_1 + h + \eta)g_2(z_2 + \eta)g_3(z_3 + h) \times

\times R_{13}^{\text{ACF}}(\eta, z_1 + h, z_3 + h)R_{12}^{\text{ACF}}(h - \eta, z_1 + \eta, z_2 + \eta)g_1^{-1}(z_1 + \eta)g_2^{-1}(z_2 + h)g_3^{-1}(z_3 + h + \eta).

(3.9)

At last substitute (1.24) into \(R_{23}^g(\eta - h, z_2 - z_3)R_{13}^g(h, z_1 - z_3)\) (the second term in the r.h.s. of (1.15)) with the constant \(c = h\) for \(R_{23}^g\) and with \(c = \eta\) for \(R_{13}^g:

R_{23}^g(\eta - h, z_2 - z_3)R_{13}^g(h, z_1 - z_3) = g_1(z_1 + h + \eta)g_2(z_2 + \eta)g_3(z_3 + h) \times

\times R_{23}^{\text{ACF}}(\eta - h, z_2 + h, z_3 + h)R_{13}^{\text{ACF}}(h, z_1 + \eta, z_3 + \eta)g_1^{-1}(z_1 + \eta)g_2^{-1}(z_2 + h)g_3^{-1}(z_3 + h + \eta).

(3.10)

The products of \(g\) matrices in (3.8), (3.9), (3.10) are the same. Therefore, from (3.8), (3.9), (3.10) and (1.15) we get (1.22).

Similarly, one can get the Yang-Baxter equation (1.9) from (1.1) and more general cubic relation (1.23) from (1.19). In the latter case we get

\[\text{eq. (1.19)}\]

= \(g_1(z_1 + \eta)g_2(z_2)g_3(z_3 - h) \left[\text{eq. (1.23)}\right] g_1^{-1}(z_1 - h)g_2^{-1}(z_2)g_3^{-1}(z_3 + \eta)\)

(3.11)

Finally, in the same way one can verify that the higher order (in \(R\)) identities (1.26) are also valid for ACF \(R\)-matrix. It happens because each term of the sum in (1.26) contains all distinct...
indices. Therefore, each matrix $g$ is either cancelled out or can be removed to the right or to the left. This reasoning shows that the substitution of (1.24) into the identity (written for the Baxter-Belavin $R$-matrix) yields (1.26) for the ACF $R$-matrix conjugated by

$$g_a(z_a + \hbar) \prod_{c \neq a} g_c(z_c).$$

**Modification of bundles as matrix theta function.** It was shown in [16, 17] that the intertwining matrix (1.7) is of the same form in classical mechanics, where it plays the role of special gauge transformation relating Lax pairs of Calogero-Moser (Ruijsenaars-Schneider) model and integrable (relativistic) elliptic top. This approach treats the Lax operator of an integrable system as section of some bundle over complex curve (with local coordinate $z$). The gauge transformation (1.7) changes its characteristic class (e.g. degree of underlying vector bundle) because it is degenerated at $z = 0$:

$$\det g(z, u) = c(\tau) \vartheta(z) \prod_{j > k} \vartheta(u_j - u_k).$$

(3.12)

Such gauge transformations are called modifications of bundles, and the gauge equivalence of a set of integrable systems related to different characteristic classes is called the symplectic Hecke correspondence. Here we argue that $g(z, u)$ matrix can be considered as a matrix analogue of theta function (A.2) in the same way as $R$-matrix (1.3) is a matrix analogue of the Kronecker function (A.7). See [22] and [18, 19] for details.

First, notice that by definition (1.7) $g(z)$ is indeed $\vartheta(z)$ in scalar ($N = 1$) case. Next, consider (1.24) written as follows

$$g^{-1}_2(z_2, u) R_{12}^\alpha(h, z_1 - z_2) = g_1(z_1 + h, u) R_{12}^{ACF}(h, z_1, z_2|u) g_2^{-1}(z_2 + h, u) g_1^{-1}(z_1, u).$$

(3.13)

As functions of $z_2$ both parts of (3.13) have simple poles at $z_2 = 0$. Taking residues at $z_2 = 0$ we get

$$\tilde{g}_2(0, u) R_{12}^\alpha(h, z) = g_1(z + h, u) \mathcal{O}_{12} g_2^{-1}(h, u) g_1^{-1}(z, u),$$

(3.14)

where

$$\tilde{g}(0, u) = \text{Res}_{z=0} g^{-1}(z)$$

(3.15)

is a matrix analogue of theta constant $1/\vartheta'(0)$ while $\mathcal{O}_{12}$ is the following (degenerated) matrix:

$$\mathcal{O}_{12} = \text{Res}_{z_2=0} R_{12}^{ACF}(h, z_1, z_2|u) = \sum_{i,j} E_{ii} \otimes E_{ji}.$$ (3.16)

When $N = 1$ all the elements in (3.14) become scalar, $\mathcal{O}_{12} |_{N=1} = 1$, and we reproduce the definition of the Kronecker function (A.7).

### 4 Equations for Felder’s $R$-matrix

Let us derive equations for the Felder’s $R$-matrix (1.6) which follow from those for the Baxter-Belavin case via the IRF-Vertex transformation (1.8). Rewrite (1.8) as follows:

$$R_{12}^\alpha(h, z_1 - z_2) = g_2(z_2, u) g_1(z_1, u - \hbar^{(2)}) R_{12}^{e}(h, z_1 - z_2|u) g_2^{-1}(z_2, u - \hbar^{(1)}) g_1^{-1}(z_1, u),$$ (4.1)
$R_{12}^a(h, z_1 - z_2) = g_1(z_1, u) g_2(z_2, u + \hbar^{(1)}) R_{12}^e(h, z_1 - z_2| u) g_1^{-1}(z_1, u + \hbar^{(2)}) g_2^{-1}(z_2, u)$.  \hfill (4.2)

The latter follows from (1.8) and the skew-symmetry (1.16) valid for $R^a$ and $R^e$. Let us mention that (4.2) together with (1.12) reproduces the relation (1.24) between the Baxter-Belavin and ACF $R$-matrices. Consider for example transformation of $R^a_{12}(h, z_1 - z_2)$ and $R^a_{23}(\eta, z_2 - z_3)$:

$$R_{12}^a(h, z_1 - z_2) R_{23}^a(\eta, z_2 - z_3) = (P^{-\eta}_3 R_{12}^a(h, z_1 - z_2) P^\eta_3) R_{23}^a(\eta, z_2 - z_3) =$$

$$= g_3(z_3, u) g_1(z_1, u - \eta^{(3)}) g_2(z_2, u + \hbar^{(1)} - \eta^{(3)}) R_{12}^e(h, z_1 - z_2| u - \eta^{(3)}) \times$$

$$g_1^{-1}(z_1, u + \hbar^{(2)} - \eta^{(3)}) R_{23}^e(h, z_2 - z_3| u) g_3^{-1}(z_3, u - \eta^{(2)}) g_2^{-1}(z_2, u).$$  \hfill (4.3)

Here we used (4.2) for $R_{12}^a(h, z_1 - z_2)$ and (4.1) for $R_{23}^a(\eta, z_2 - z_3)$. Another possibility to keep a single $g$ multiple between $R^e$ is

$$R_{12}^a(h, z_1 - z_2) R_{23}^a(\eta, z_2 - z_3) = R_{12}^a(h, z_1 - z_2) (P^{-h}_1 R_{23}^a(\eta, z_2 - z_3) P^h_1) =$$

$$= g_2(z_2, u) g_1(z_1, u - h^{(2)}) R_{12}^e(h, z_1 - z_2| u) g_3(z_3, u - h^{(1)} + \eta^{(2)}) \times$$

$$R_{23}^e(h, z_2 - z_3| u - h^{(1)}) g_2^{-1}(z_2, u + \eta^{(3)} - h^{(1)}) g_3^{-1}(z_3, u - h^{(1)}).$$  \hfill (4.4)

Here we used (4.1) for $R_{12}^a(h, z_1 - z_2)$ and (4.2) for $R_{23}^a(\eta, z_2 - z_3)$.

Combining application of (4.1) and (4.2) to $R_{ab}^a(h, z_a - z_b)$ we can generate identities for the Felder’s $R$-matrix starting from those for $R_{ab}^a(h, z_a - z_b)$, but the $g$ multiples are not cancelled out from the obtained expressions. Let us however write down the transformed cubic identity (1.19). Again we use that $R_{12}^a(h) = P^\eta_3 R_{12}^a(h) P^{-\eta}_3$ since it is non-dynamical:

$$R_{12}^a(\eta, z_1 - z_2) (P^\eta_3 R_{13}^a(h, z_1 - z_3) P^{-\eta}_3) R_{23}^a(\eta, z_2 - z_3) =$$

$$= (\varphi(\eta) - \varphi(h)) R_{13}^a(h + \eta, z_1 - z_3).$$  \hfill (4.5)

By applying (4.2) we obtain:

$$R_{12}^e(\eta, z_1 - z_2| u) g_3(z_3, u + h^{(1)} + \eta^{(2)}) R_{13}^e(h, z_1 - z_3| u + \eta^{(2)}) \times$$

$$g_1^{-1}(z_1, u + h^{(3)} + \eta^{(2)}) R_{23}^e(\eta, z_2 - z_3| u) -$$

$$- g_3(z_3, u + h^{(2)} + \eta^{(1)}) R_{23}^e(h, z_2 - z_3| u + \eta^{(1)}) g_2^{-1}(z_2, u + h^{(3)} + \eta^{(1)}) \times$$

$$R_{13}^e(\eta, z_1 - z_3| u) g_2(z_2, u + h^{(1)} + \eta^{(3)}) R_{12}^e(h, z_1 - z_2| u + \eta^{(3)}) g_1^{-1}(z_1, u + h^{(2)} + \eta^{(3)}) =$$

$$= (\varphi(\eta) - \varphi(h)) g_2^{-1}(z_2, u + \eta^{(1)}) g_3(z_3, u + h^{(1)} + \eta^{(1)}) R_{13}^a(h + \eta, z_1 - z_3) \times$$

$$g_1^{-1}(z_1, u + h^{(3)} + \eta^{(3)}) g_2(z_2, u + \eta^{(3)}).$$
It can be useful by the following reason. In the case $\hbar = \eta$ its r.h.s. equals zero, and the l.h.s. provides the Gervais-Neveu-Felder equation (1.4). It happens because in this case one can apply the weight zero condition $P_1^\hbar P_2^\hbar R_{12}^\hbar = R_{12}^\eta P_1^\hbar P_2^\hbar$, and the $g$ multiples cancel out (this is the proof of the IRF-Vertex transformation). On the other hand, the case $\hbar = -\eta$ applied to (1.19) provides (1.20) which leads to commutativity of the KZB connections $[\nabla_a, \nabla_\tau] = 0$. Therefore, it would appear reasonable that (4.6) leads to commutativity of the dynamical (Felder’s) KZB connections. We will discuss it in our next paper.

Remark: Let us mention that while the associative Yang-Baxter equation (1.15) (without any shifts of arguments) is not valid for the Felder’s $R$-matrix, the disappearance error has quite simple form in the rational case. Consider $R_{12}^{\hbar} = R_{12}^\eta (\hbar, z_a - z_b | u)$ (1.6) with $\phi(\eta, z) = 1/\eta + 1/z$, then

$$R_{12}^{\hbar} R_{23}^{\eta} - R_{13}^{\eta} R_{12}^{\hbar} - R_{23}^{\eta-h} R_{13}^{h} =$$

$$= \sum_{i \neq j} \frac{1}{(u_i - u_j)^2} \left( E_{ij} \otimes E_{jj} \otimes E_{ji} + E_{ii} \otimes E_{ij} \otimes E_{ji} + E_{ij} \otimes E_{ji} \otimes E_{ii} - E_{ii} \otimes E_{ii} \otimes E_{jj} - E_{ii} \otimes E_{jj} \otimes E_{ii} \right)$$

(4.7)

i.e. the r.h.s. is independent of spectral parameters and the Planck constants.

5 Appendix

5.1 Elliptic functions

The Riemann theta-functions [10] with characteristics on an elliptic curve $\Sigma_\tau = \mathbb{C}^2/(\mathbb{Z} \oplus \tau \mathbb{Z})$ (with moduli $\tau$, $\text{Im} \tau > 0$) are defined for some integer $N \geq 2$:

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z | \tau) = \sum_{j \in \mathbb{Z}} \exp \left( 2\pi i (j + a)^2 \tau / 2 + 2\pi i (j + a)(z + b) \right), \quad a, b \in \frac{1}{N} \mathbb{Z}. \quad (A.1)$$

It has the following quasi-periodic properties (i.e. behavior on the lattice $\mathbb{Z} \oplus \tau \mathbb{Z}$):

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + 1 | \tau) = \exp(2\pi i a) \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z | \tau),$$

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + a' \tau | \tau) = \exp \left( -2\pi i a'^2 \tau / 2 - 2\pi i a'(z + b) \right) \theta \left[ \begin{array}{c} a + a' \\ b \end{array} \right] (z | \tau).$$

A shorthand notation for the odd theta-function is used

$$\vartheta(\zeta | \tau) \equiv \vartheta(\zeta) \equiv \theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z | \tau). \quad (A.2)$$

The space of functions (A.1) is a natural module for the action of the Heisenberg group [23]. Its finite dimensional representation is generated by a pair of matrices $Q, \Lambda \in \text{Mat}(N, \mathbb{C})$:

$$Q_{kl} = \delta_{kl} \exp \left( \frac{2\pi i k}{N} \right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \text{mod} N}, \quad Q^N = \Lambda^N = 1_{N \times N}. \quad (A.3)$$
Then for
\[ T_a = T_{a_1 a_2} = \exp \left( \frac{\pi i}{N} a_1 a_2 \right) Q^{a_1} \Lambda^{a_2}, \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N \] (A.4)
the following relations hold
\[ T_\alpha T_\beta = \kappa_{\alpha, \beta} T_{\alpha + \beta}, \quad \kappa_{\alpha, \beta} = \exp \left( \frac{\pi i}{N} (\beta_1 \alpha_2 - \beta_2 \alpha_1) \right), \] (A.5)
where \( \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \). The permutation operator takes the form
\[ P_{12} = \frac{1}{N} \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} T_\alpha \otimes T_{-\alpha} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji}. \] (A.6)

The Kronecker function [26] can be defined in terms of (A.2):
\[ \phi(\eta, z) = \begin{cases} 
\frac{1}{\eta} + \frac{1}{z} & \text{rational case,} \\
\coth(\eta) + \coth(z) & \text{trigonometric case,} \\
\frac{\vartheta'(0) \vartheta(\eta + z)}{\vartheta(\eta) \vartheta(z)} & \text{elliptic case.}
\end{cases} \] (A.7)

We also need the first Eisenstein (odd) function and the Weierstrass (even) \( \wp \)-function. In rational, trigonometric and elliptic cases they are given by
\[ E_1(z) = \begin{cases} 
\frac{1}{z}, \\
\coth(z), \\
\vartheta'(z)/\vartheta(z),
\end{cases} \quad \varphi(z) = \begin{cases} 
\frac{1}{z^2}, \\
\frac{1}{\sinh^2(z)}, \\
-\vartheta_2 E_1(z) + \frac{1}{3} \vartheta''(0).
\end{cases} \] (A.8)

The properties and identities for (A.7)-(A.8) can be found in [10]. See also [19] and the Appendix in [20], where the same notations are used. Here we give only the most important. It is the Fay trisecant identity
\[ \phi(\hbar, z)\phi(\eta, w) = \phi(\hbar - \eta, z)\phi(\eta, z + w) + \phi(\eta - \hbar, w)\phi(\hbar, z + w) \] (A.9)
and its degenerations
\[ \phi(\eta, z)\phi(\eta, w) = \phi(\eta, z + w)(E_1(\eta) + E_1(z) + E_1(w) - E_1(z + w + \eta)), \] (A.10)
\[ \phi(\hbar, z)\phi(\hbar, -z) = \varphi(\hbar) - \varphi(z). \] (A.11)

The quantum Baxter-Belavin \( R \)-matrix (1.3) also uses the set of (sections of bundle over \( \Sigma_r \)) functions
\[ \varphi^h_a(z) = \exp(2\pi i \frac{a_2}{N} z) \phi(z, \frac{\hbar + a_1 + a_2 \tau}{N}), \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N. \] (A.12)

\footnote{The definition of the Baxter-Belavin \( R \)-matrix in [17]-[20], [28] slightly differs from [13], (A.12). The relation is as follows: \( R^h(z) := NR^{N\hbar}(z) \).}
5.2 Direct proof of associative Yang-Baxter equation
for Arutyunov-Chekhov-Frolov $R$-matrix

Let us compare the left and the right hand sides of (1.22) rewriting them with the definition of $R^{\mathrm{CF}}_{13}$ (1.10). It meant here that in all summands $i \neq j, i \neq k, j \neq k$.

The cancellation in some components directly follows from definition (A.7):

\[
E_{ij} \otimes E_{ik} \otimes E_{ji} : \quad 0 = -\phi(z_{13}, -u_{ij})\phi(\eta - h, -u_{ij}) + \phi(\eta - h, -u_{ij})\phi(z_{13}, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{ik} \otimes E_{kj} : \quad 0 = \phi(z_{1} + h + \eta, -u_{ij})\phi(\eta - h, -u_{ij}) - \phi(\eta - h, -u_{ij})\phi(z_{1} + h + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{ij} \otimes E_{ii} : \quad \phi(h, -u_{ij})\phi(z_{3} + h, -u_{ij}) = \phi(z_{3} + h, -u_{ij})\phi(h, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{jk} \otimes E_{ij} : \quad -\phi(z_{12}, -u_{ij})\phi(-z_{3} - h, -u_{ij}) = -\phi(-z_{3} - h, -u_{ij})\phi(z_{12}, -u_{ij}) ,
\]
\[
E_{ii} \otimes E_{ij} \otimes E_{ij} : \quad -\phi(h, -u_{ij})\phi(z_{32}, -u_{ij}) = -\phi(h, -u_{ij})\phi(z_{32}, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{ij} \otimes E_{ij} : \quad -\phi(\eta, -u_{ij})\phi(-z_{2} - \eta, -u_{ij}) = -\phi(\eta, -u_{ij})\phi(-z_{2} - \eta, -u_{ij}) ,
\]
\[
E_{jj} \otimes E_{jj} \otimes E_{jj} : \quad 0 = \phi(z_{13}, -u_{ij})\phi(z_{2} + \eta, -u_{ij}) - \phi(z_{2} + \eta, -u_{ij})\phi(z_{13}, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{ij} \otimes E_{ij} : \quad 0 = -\phi(z_{1} + h + \eta, -u_{ij})\phi(z_{2} + \eta, -u_{ij}) + \phi(z_{2} + \eta, -u_{ij})\phi(z_{1} + h + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{kk} \otimes E_{ij} : \quad \phi(z_{ij}, -u_{ij})\phi(\eta, -u_{ij}) = \phi(\eta, -u_{ij})\phi(z_{12}, -u_{ij}) ,
\]
\[
E_{jj} \otimes E_{jk} \otimes E_{kk} : \quad \phi(z_{2} + \eta, -u_{ij})\phi(\eta, -u_{ik}) = \phi(\eta, -u_{ik})\phi(z_{2} + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{jk} \otimes E_{ki} : \quad -\phi(z_{1} + h + \eta, -u_{ij})\phi(z_{23}, -u_{jk}) = -\phi(z_{23}, -u_{jk})\phi(z_{1} + h + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{jk} \otimes E_{kj} : \quad \phi(h, -u_{ij})\phi(z_{23}, -u_{jk}) = \phi(z_{23}, -u_{jk})\phi(h, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{kk} \otimes E_{kj} : \quad \phi(h, -u_{ij})\phi(z_{3} + h, -u_{kk}) = \phi(z_{3} + h, -u_{kk})\phi(h, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{ij} \otimes E_{kk} : \quad \phi(h, -u_{ij})\phi(z_{3} + h, -u_{kk}) = \phi(z_{3} + h, -u_{kk})\phi(h, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{ij} \otimes E_{kj} : \quad \phi(z_{2} + \eta, -u_{ij})\phi(z_{3} + h, -u_{kk}) = \phi(z_{3} + h, -u_{kk})\phi(z_{2} + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{kk} \otimes E_{jj} : \quad 0 = -\phi(z_{1} + h + \eta, -u_{ij})\phi(h - \eta, -u_{jk}) - \phi(h - \eta, -u_{jk})\phi(z_{1} + h + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{kk} \otimes E_{kk} : \quad 0 = \phi(z_{13}, -u_{ij})\phi(h - \eta, -u_{kk}) + \phi(\eta - h, -u_{kk})\phi(z_{13}, -u_{ij}) ,
\]
\[
E_{jj} \otimes E_{kk} \otimes E_{jj} : \quad 0 = -\phi(z_{1} + h + \eta, -u_{ij})\phi(z_{2} + \eta, -u_{jk}) + \phi(z_{2} + \eta, -u_{jk})\phi(z_{1} + h + \eta, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{kk} \otimes E_{jj} : \quad 0 = \phi(z_{13}, -u_{ij})\phi(z_{2} + \eta, -u_{kk}) - \phi(z_{2} + \eta, -u_{kk})\phi(z_{13}, -u_{ij}) .
\]

Other identities can be proven by direct use of (A.9):

\[
E_{ii} \otimes E_{jj} \otimes E_{kk} : \quad \phi(h, -u_{ij})\phi(\eta, -u_{jk}) = \phi(\eta - h, -u_{ij})\phi(\eta, -u_{jk}) = \phi(\eta - h, -u_{ij})\phi(h, -u_{jk}) = \phi(h, -u_{ij})\phi(\eta - h, -u_{jk}) = \phi(\eta, -u_{jk})\phi(h, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{jj} \otimes E_{kk} : \quad -\phi(z_{1} + h + \eta, -u_{ij})\phi(\eta, -u_{jk}) = -\phi(\eta, -u_{jk})\phi(z_{1} + h + \eta, -u_{ij}) ,
\]
\[
E_{ik} \otimes E_{kk} \otimes E_{ji} : \quad 0 = \phi(z_{13}, -u_{ij})\phi(z_{1} + h, -u_{ik}) - \phi(z_{1} + h, -u_{ik})\phi(z_{13}, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{jk} \otimes E_{ki} : \quad \phi(z_{12}, -u_{ij})\phi(\eta, -u_{ik}) = \phi(\eta, -u_{ik})\phi(z_{12}, -u_{ij}) ,
\]
\[
E_{jj} \otimes E_{ik} \otimes E_{kj} : \quad \phi(z_{2} + \eta, -u_{ij})\phi(\eta, -u_{ik}) = \phi(\eta, -u_{ik})\phi(z_{2} + \eta, -u_{ij}) ,
\]
\[
E_{ii} \otimes E_{jk} \otimes E_{kk} : \quad -\phi(h, -u_{ij})\phi(\eta, -u_{jk}) = -\phi(\eta, -u_{jk})\phi(h, -u_{ij}) = -\phi(\eta, -u_{jk})\phi(h, -u_{ij}) = -\phi(h, -u_{ij})\phi(\eta, -u_{jk}) ,
\]
\[
E_{kk} \otimes E_{ii} \otimes E_{jk} : \quad 0 = \phi(\eta - h, -u_{ij})\phi(z_{3} + h, -u_{jk}) + \phi(z_{3} + h, -u_{jk})\phi(\eta - h, -u_{ij}) ,
\]
\[
E_{ij} \otimes E_{jk} \otimes E_{kk} : \quad -\phi(z_{12}, -u_{ij})\phi(z_{2} + h + \eta, -u_{ik}) + \phi(z_{2} + h + \eta, -u_{ik})\phi(z_{12}, -u_{ij}) = -\phi(z_{12}, -u_{ij})\phi(z_{2} + h + \eta, -u_{ik}) + \phi(z_{2} + h + \eta, -u_{ik})\phi(z_{12}, -u_{ij}) .
\]

One can rewrite the rest of identities using (A.10) and prove them by comparing obtained
summands with $E_1$-functions:

\[
E_{ii} \otimes E_{ij} \otimes E_{ji} : \\
\frac{\phi(h, z_2 + \eta)}{\phi(h, z_1 + h)} \phi(h, z_{12}) \phi(z_{23}, -u_{ij}) = \frac{\phi(h, z_3 + \eta)}{\phi(h, z_1 + \eta)} \phi(h, z_{13}) \phi(z_{23}, -u_{ij}) - \\
\frac{\phi(h, z_3 + \eta)}{\phi(h, z_1 + \eta)} \phi(h, z_{12}) \phi(z_{23}, -u_{ij}) + \\
- \phi(h, z_{12}) \phi(h, z_{23}, -u_{ij}) = - \phi(h, z_3 + h, -u_{ij}) \phi(-z_3 - \eta, -u_{ij}) + \\
\phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) - \phi(h, z_3 + \eta) \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}).
\]

\[
E_{ii} \otimes E_{jj} \otimes E_{ii} : \\
- \phi(h, -u_{ij}) \phi(h, -u_{ij}) = - \phi(h, z_3 + h, -u_{ij}) \phi(-z_3 - \eta, -u_{ij}) + \\
\phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) - \phi(h, z_3 + \eta) \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{ij} : \\
\phi(z_{23}, -u_{ij}) \phi(z_{23}, -u_{ij}) = \frac{\phi(h, z_3 + \eta)}{\phi(h, z_2 + h)} \phi(h, z_{13}) \phi(z_{23}, -u_{ij}) + \\
\phi(h, z_{13}) \phi(z_{23}, -u_{ij}) = \phi(h, z_{23}, -u_{ij}) \phi(z_{23}, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{jj} : \\
- \phi(z_{23}, -u_{ij}) \phi(z_{23}, -u_{ij}) = - \phi(z_3 + h, -u_{ij}) \phi(-z_3 - \eta, -u_{ij}) + \\
\phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) - \phi(h, z_3 + \eta) \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}).
\]

\[
E_{ij} \otimes E_{ji} \otimes E_{ij} : \\
- \frac{\phi(h, z_3 + \eta)}{\phi(h, z_2 + h)} \phi(h, z_{13}) \phi(z_{23}, -u_{ij}) + \\
\phi(h, z_{13}) \phi(z_{23}, -u_{ij}) = \phi(h, z_{23}, -u_{ij}) \phi(z_{23}, -u_{ij}).
\]

\[
E_{ji} \otimes E_{jj} \otimes E_{ij} : \\
- \phi(h, -u_{ij}) \phi(h, -u_{ij}) = - \phi(h, z_3 + h, -u_{ij}) \phi(-z_3 - \eta, -u_{ij}) + \\
\phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) - \phi(h, z_3 + \eta) \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}).
\]

\[
E_{ji} \otimes E_{ji} \otimes E_{ii} : \\
- \phi(z_3 + h, -u_{ij}) \phi(-z_3 - \eta, -u_{ij}) + \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}).
\]

\[
E_{jj} \otimes E_{ii} \otimes E_{ii} : \\
- \phi(h, z_3 + \eta) \phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) + \\
\phi(h, z_{13}) \phi(h, z_{23}, -u_{ij}) = \phi(h, z_{23}, -u_{ij}) \phi(h, z_{23}, -u_{ij}).
\]

\[
E_{ii} \otimes E_{ii} \otimes E_{jj} : \\
\phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \phi(h, z_3 + \eta) \phi(h, -u_{ij}) + \\
\phi(h, z_{12}) \phi(h, -u_{ij}) - \phi(h, z_3 + \eta) \phi(h, -u_{ij}).
\]

\[
E_{ii} \otimes E_{ii} \otimes E_{jj} : \\
\phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \phi(h, z_3 + \eta) \phi(h, -u_{ij}) + \\
\phi(h, z_{12}) \phi(h, -u_{ij}) - \phi(h, z_3 + \eta) \phi(h, -u_{ij}).
\]

\[
E_{ii} \otimes E_{ii} \otimes E_{jj} : \\
- \phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, -u_{ij}) - \phi(h, z_{12}) \phi(h, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{ij} : \\
- \phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, -u_{ij}) - \phi(h, z_{12}) \phi(h, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{ij} : \\
- \phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, -u_{ij}) - \phi(h, z_{12}) \phi(h, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{ij} : \\
- \phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, -u_{ij}) - \phi(h, z_{12}) \phi(h, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{ij} : \\
- \phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, -u_{ij}) - \phi(h, z_{12}) \phi(h, -u_{ij}).
\]

\[
E_{ij} \otimes E_{jj} \otimes E_{ij} : \\
- \phi(h, z_2 + \eta) \phi(h, z_{12}) \phi(h, -u_{ij}) = \\
\phi(h, z_3 + \eta) \phi(h, -u_{ij}) - \phi(h, z_{12}) \phi(h, -u_{ij}).
\]
The last identity

\[ E_{ii} \otimes E_{ii} \otimes E_{ii} : \phi(h, z_2 + \eta) \phi(\eta, z_3 + h) \phi(h, z_{12}) \phi(\eta, z_{23}) = \]

\[ = \frac{\phi(h - \eta, z_2 + \eta) \phi(\eta, z_3 + h)}{\phi(h - \eta, z_1 + \eta) \phi(\eta, z_2 + h)} \phi(\eta, z_{13}) \phi(h - \eta, z_{12}) + \frac{\phi(h - \eta, z_3 + \eta) \phi(h, z_2 + \eta)}{\phi(h - \eta, z_1 + \eta) \phi(h, z_1 + \eta)} \phi(h - \eta, z_{23}) \phi(h, z_{13}) \]

follows from (A.9) and the statement

\[ \frac{\phi(h, z_2 + \eta) \phi(\eta, z_3 + h)}{\phi(h, z_1 + \eta) \phi(\eta, z_2 + h)} = \frac{\phi(h - \eta, z_2 + \eta) \phi(\eta, z_3 + h)}{\phi(h - \eta, z_1 + \eta) \phi(\eta, z_1 + h)} = \]

\[ = \frac{\phi(\eta - h, z_3 + h)}{\phi(\eta - h, z_2 + h)} \frac{\phi(h, z_3 + h)}{\phi(h, z_1 + \eta)} \frac{\phi(h, z_2 + \eta)}{\phi(h, z_1 + \eta)} = \frac{\partial(z_1 + \eta) \partial(z_2 + h) \partial(z_3 + \eta + h)}{\partial(z_1 + h + \eta) \partial(z_2 + \eta) \partial(z_3 + h)}. \]  

(A.13)

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