Chaos is associated to extreme complexity and “unpredictability”. Very few exact results are therefore known, however several exponents have been introduced to provide measures of the complexity and further, to classify chaotic systems \[ \lambda \] . The most popular are the Lyapunov exponents, which have a clear and intuitive interpretation, but require the choice of a metric and of an invariant measure. They can vary considerably under a very tiny change of the parameters of the dynamical system (see [2] p237): probing fine details, they are not universal. The same remarks also apply to the metric entropy \[ h \]. By contrast there also exist exponents which do not involve any assumption on the phase space. These exponents, obviously, give a less detailed description of the system, but are more universal. They provide a mean for a classification of dynamical systems. The topological entropy \[ \log h \] and the Arnold complexity \[ 1 \] are two examples. The topological entropy probes the growth of the number of stable cycles as a function of their length, and the Arnold complexity probes the growth of the number of intersections of a line with its successive iterate. These two notions give general informations not sensitive to specific details. From an intuitive point of view one can understand the importance of the topological entropy, since the asymptotic behavior of a dynamical system heavily depends on its fixed points and stable cycles: an initial point is often very close to many basins of attraction, which results in a chaotic motion. Precisely we introduce the fixed points generating function \[ H(t) = \sum_{n} h_n \cdot t^n \] where \[ h_n \] is the number of real, or complex, fixed points of the \( n \)-th power \[ k^n \] of the mapping. The same information is also coded in the so-called dynamical zeta function \[ \zeta(t) \] introduced by Artin and Mazur \[ 4 \] and related to the generating function \[ H(t) \] by \[ H(t) = t \frac{\partial}{\partial t} \ln(\zeta(t)) \). Both functions only depend on the number of fixed points, and not on their particular properties or localization: functions \[ H(t) \] and \[ \zeta(t) \] are invariant under topological conjugacy (see Smale \[ 10 \] for this notion). They do not depend on a specific choice of variable. \( h \), the exponential of the topological entropy characterizes how the coefficients \( h_n \) grow with \( n \): \( h_n \sim h^n \), so that \( h \) is the modulus of the inverse of the smallest modulus pole of \( H(t) \), if rational. For a linear dynamic on a torus, the cat map, the exponential of the topological entropy has been calculated and found algebraic, i.e. solution of polynomial with integer coefficients. We are not aware of other non trivial dynamical systems where an algebraic value for the exponential of the topological entropy has been calculated. In this letter we will provide such an example of a discrete dynamical system with a rational dynamical zeta function, and, consequently, algebraic value for the exponential of...
the topological entropy. This result should not be considered like a mathematical curiosity: it is similar to the rationality of critical exponents in conformal theory. This algebraicity is the signing of deeper “rigid” structures (like Feigenbaum cascades \[ \alpha = 0 \] are).

In the context of rational mappings it is easy to see that the Arnold complexity can be replaced by the complexity growth of the successive iterations. To define the complexity growth \( \lambda \) we introduce the complexity generating function \( G(t) = \sum \alpha_n \cdot t^n \) where \( \alpha_n \) is the degree of any of the numerator, or denominator, of the components of the successive iterates of the rational mapping under consideration. When common polynomials factorize in the numerators and denominators, the coefficients \( \alpha_n \) grow slower than expected. We stress that this definition only apply to rational mappings. The complexity growth \( \lambda \) characterizes how coefficients \( \alpha_n \) grow with \( n \): \( \alpha_n \sim \lambda^n \). Like \( h \), the exponential of the topological entropy, complexity \( \lambda \) is the modulus of the inverse of the smallest modulus pole of \( G(t) \). In this letter we claim that the complexity growth \( \lambda \) and \( h \), the exponential of the topological entropy, are equal

\[
h = \lambda \quad (1)
\]

This will be tested successfully for a particular class of mappings, for which both generating functions are conjectured to be rational and, consequently, the complexity growth and the exponential of the topological entropy are algebraic. We will also give an effective semi-numerical method to compute these two characteristic numbers.

Let us introduce the discrete rational mapping \( k_{\alpha,\epsilon} \) which associates \( (u_{n+1}, v_{n+1}) \) to \( (u_n, v_n) \)

\[
\begin{align*}
u_{n+1} &= 1 - u_n + u_n/v_n \\
v_{n+1} &= \epsilon + v_n - v_n/u_n + \alpha \cdot (1 - u_n + u_n/v_n)
\end{align*} \quad (2)
\]

This mapping originates from the study of the symmetries of models of lattice statistical mechanics [13]. Depending on the actual values of the parameters \( \alpha \) and \( \epsilon \), the mapping can have completely different behaviors. For example, for \( \epsilon = 0 \) and whatever \( \alpha \), as well as for \( \alpha = 0 \) and \( \epsilon = -1 \), \( 0 \), \( 1/2 \), \( 1/3 \) or \( 1 \), the mapping is integrable, whereas for all other values it is not [13]. A simple calculation shows that \( k_{\alpha,\epsilon} \) is invertible and that its inverse is also rational. This property of birationality is of importance in our study. We have formally calculated the successive powers of \( k_{\alpha,\epsilon} \) for arbitrary \( \alpha \) and \( \epsilon \), from which we propose

\[
G_{\alpha,\epsilon}(t) = \frac{(1 + t)^2}{1 - t - 2t^2 - t^3} \quad (3)
\]

The expansion of the conjectured expression Eq. (3) coincides with our results up to the largest power \( n = 7 \) we were able to compute. Another rational expression with the same denominator is also obtained if one uses a matrixial representation of the mapping Eq. (3) [13]. The expression Eq. (3) of \( G_{\alpha,\epsilon} \) yields \( \lambda \simeq 2.14789 \). To support this conjecture, we have devised a semi-numerical method to estimate complexity \( \lambda \). It consists in iterating \( k_{\alpha,\epsilon} \) over the field of rationals. During the first steps, some “accidental” cancellations between numerators and denominators can arise, but after this transient regime, the numerators and denominators get extremely large, and cancellations are only due to formal simplifications. We then determine how the magnitude of the four numerators, or denominators, grows with \( n \). With this method it is possible to raise \( k_{\alpha,\epsilon} \) to the 15-th power, and moreover it is easy to scan a large number of values of the parameters \( \alpha \) and \( \epsilon \). The calculations are performed with an infinite precision C-library [17]. Obviously, this method works only for rational mappings. On Fig. 1 one clearly sees that, for most of the values of \( \epsilon \), the complexity \( \lambda \) is extremely close to the expected value. We call “specific” the values of the parameters for which the complexity \( \lambda \) is different from 2.14789, they will be discussed later. We also have formally computed for arbitrary \( \alpha \) and \( \epsilon \) the fixed points of the powers of \( k_{\alpha,\epsilon} \) using, once again, the rationality of the mapping. This gives

\[
H_{\alpha,\epsilon}(t) = 2t + 2t^2 + 11t^3 + 18t^4 + 47t^5 + 95t^6 + 212t^7 + \cdots \quad (4)
\]

From this expression we propose the rational expression for the generating function of the number of fixed points \( k_{\alpha,\epsilon} \)

\[
H_{\alpha,\epsilon}(t) = \frac{(2 + 3t^2 + t^3) \cdot t}{(1 - t^2) \cdot (1 - t - 2t^2 - t^3)} \quad (5)
\]

or, equivalently, the dynamical zeta function reads

\[
\zeta_{\alpha,\epsilon}(t) = \frac{(1 + t)(1 - t^2)}{(1 - t - 2t^2 - t^3)} \quad (6)
\]

Note that the total number of fixed points of \( k_{\alpha,\epsilon} \) does not depend on the actual generic values of \( \alpha \) and \( \epsilon \), however the number of real fixed points is extremely dependent on these two parameters. Let us also mention a local area preserving property: the determinant of the Jacobian of the \( n \)-th power of \( k_{\alpha,\epsilon} \), evaluated at each fixed points of \( k_{\alpha,\epsilon} \), is equal to one. The “visual” complexity of the phase diagram takes its origin in the real fixed points, and therefore varies considerably with \( \alpha \) and \( \epsilon \) [18]. One sees that the two polynomials giving exponents \( \alpha \) and \( h \) are the same, and consequently we have the equality \( h = \lambda \). This equality holds for generic values of the parameters, however, as shown on Fig. 1 there exist specific values. These specific values include \( \epsilon = 1/3, 1/2, 3/5 \). It is then natural to investigate if the equality of the complexity growth and the exponential of the topological entropy also holds for the specific values. We have performed calculations for these values and found that
equality \( \lambda \) is always true. The polynomials giving the value of \( \lambda \) and \( h \) are presented in table Tab. I. Probably, other specific values exist, but they lead to simplifications occurring at very high orders, and the corresponding \( \lambda \) is too close to the generic value to be distinguished from it with our method.

Besides the specific values mentioned above, extra simplifications also happen for \( \alpha = 0 \), and the complexity is further reduced. We hence study this special case \( \alpha = 0 \). In that case, a change of variables \([16]\), turns \( k_{0,\epsilon} \) into a simpler mapping, \( k_{\epsilon} \)

\[
y_{n+1} = z_{n} + 1 - \epsilon \\
z_{n+1} = y_{n} \cdot \frac{z_{n} - \epsilon}{z_{n} + 1}
\]  

(7)

From now on, the degrees, and the fixed points, are those of \( k_{\epsilon} \). Since the complexity is lower, the semi-numerical method presented above is more efficient and it is possible to perform calculations beyond the 20-th power. The results are displayed on Fig. 3, where the existence of integrable values, and non generic values, is clearly seen. It is simple to see that, if there is no simplification, \( d_{n+1} = d_{n} + d_{n-1} \) where \( d_{n} \) was introduced in the definition of generating function \( G(t) \). In that case \( G_{\epsilon}(t) - 2t - 1 = t \cdot G_{\epsilon}(t) - 1 + t^2 \cdot G_{\epsilon}(t) \). Up to the 20-th power there is no simplification and consequently we conjecture that, except for the specific values, the generating function of the complexity growth for \( k_{\epsilon} \) is the following rational expression

\[
G_{\epsilon}(t) = \frac{1 + t}{1 - t - t^2}
\]  

(8)

The corresponding complexity growth is \( \lambda \approx 1.61803 \), in excellent agreement with Fig. 3. We have studied the possible equality between \( h \) and \( \lambda \) for the example \( \epsilon = 13/25 = 0.52 \). We have chosen this value, for which we present a detailed analysis, because it is generic. We give in table Tab. I the number of fixed points, as well as their properties. The corresponding phase portrait is very complicated and dominated by the real fixed points \([13]\) which are all saddle or elliptic. We note that the same properties also holds for the complex fixed points. The expansion of \( H_{\epsilon} \) can be deduced, up to order eleven, from Tab. I. This expansion is compatible with the very simple rational form for the generating function of the number of fixed points for \( k_{\epsilon} \)

\[
H_{\epsilon}(t) = \frac{(1 + t^2) \cdot t}{(1 - t^2) \cdot (1 - t - t^2)}
\]  

(9)

or, equivalently, the dynamical zeta function is

\[
\zeta_{\epsilon}(t) = \frac{1 - t^2}{1 - t - t^2}
\]  

(10)

As expected, the two polynomials determining the exponential of the topological entropy and the complexity growth are equal, and so are \( \lambda \) and \( h \). Both are algebraic numbers.

Coming back to Fig. 3 we now analyze, for \( \alpha = 0 \), the specific values of \( \epsilon \). Let us recall that \( \epsilon = -1, 0, 1/3, 1/2, 1 \) lead to integrable mappings \([13]\). This corresponds to a polynomial growth of complexity and of the number of fixed points, that is, \( \lambda = h = 1 \). This is seen on Fig. 3 except for \( \epsilon = -1 \), which is out of scale but for which this is also true. The other specific values are non integrable and can be partitioned in two sets: \( \{1/m; \ m > 3\} \) and \( \{(m-1)/(m+1); \ m > 3\} \). In all cases the polynomials giving the complexity growth and the exponential of the topological entropy are the same. These polynomials are listed in Tab. II.

In conclusion, we have given an example of two-parameter family of two-dimensional discrete dynamical system with rational dynamical zeta function and rational degree generating function \( G(t) \). On this example the exponential of the topological entropy and the Arnold complexity have the same algebraic value. A semi-numerical method, applying to rational transformations only, has been given, which enables to calculate the complexity growth, and to localize possible integrable points. In fact, and this will be detailed in forthcoming publications, this mapping belongs to a “huge” family of transformations, for which similar results are also obtained. This family of transformations is so large that (if one believes in “some” universality of dynamical systems) most of the dynamical systems should be very closely “approximated” by transformations having algebraic complexity values.

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FIG. 1. The complexity growth $\lambda$ as a function of $\epsilon$ for $\alpha = 27/20$. $\epsilon$ is taken of the form $j/720$. The arrow indicates the conjectured generic value.

![Complexity growth](image1)

FIG. 2. The complexity growth $\lambda$ as a function of $\epsilon$ for $\alpha = 0$. $\epsilon$ is taken of the form $j/720$, the values $\alpha = 1/7, 1/11, 1/13, 5/7$ have also been added. The arrow indicates the conjectured generic value.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|----|----|
| # n-cycles | 1 | 0 | 1 | 1 | 2 | 2 | 4 | 5 | 8 | 11 | 18 |
| # elliptic | 1.0 | 0.0 | 0.0 | 0.0 | 1.0 | 0.0 | 1.0 | 0.0 | 1.0 | 1.3 | 1.0 |
| # saddle real | 0.0 | 0.0 | 1.0 | 1.0 | 1.0 | 0.2 | 1.2 | 1.4 | 3.4 | 1.6 | 5.12 |
| # total fixed points | 1 | 1 | 4 | 5 | 11 | 16 | 29 | 44 | 76 | 121 | 199 |

**TABLE I.** Number of real (first number) and complex (second number) fixed points of $k_{13/25}^n$. n-cycle means cycle of minimum length n.

| $\alpha$ generic | $\epsilon = 1/3$ | $\epsilon = 1/2$ | $\epsilon = \frac{1}{m} \; m > 3$ | $\epsilon = \frac{m-1}{m+1} \; m > 3$ |
|------------------|-----------------|-----------------|-----------------|-----------------|
| $\alpha = 0$ | $t$ is n-th root of unity | $t$ is n-th root of unity | $t$ is n-th root of unity | $t$ is n-th root of unity |
| $\epsilon = 1/3$ | $1 - t - t^2 - 2t^3 - t^5 - t^7$ | $1 - t - t^2 - 2t^3 - t^5 - 2t^7 - t^9 - t^{11}$ | generic see [1] | (*) |
| $\epsilon = 1/2$ | $t$ is n-th root of unity | $t$ is n-th root of unity | $1 - t - t^2 + t^{m+1}$ | $1 - t - t^2 - t^{m+1}$ |

**TABLE II.** The polynomials giving $\lambda$ and $h$ in various specific cases. The symbol(*) means that $\alpha \neq 0$ and $\epsilon = (m-1)/(m+1)$ are not generic, however the exponents are extremely close to the generic value, preventing us to compute them reliably. The case $\alpha \neq 0$ and $\epsilon = 1/m$ is generic for $m > 3$. 

\[ 1 - t - t^2 - 2t^3 - t^5 - t^7 \]