Local convertibility of the ground state of the perturbed Toric code

Siddhartha Santra  
*Department of Physics and Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, California 90089, USA*

Alioscia Hamma  
*Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, P.R. China*

Lukasz Cincio  
*Perimeter Institute for Theoretical Physics, 31 Caroline St. N, N2L 2Y5, Waterloo ON, Canada*

Yigit Subasi  
*Joint Quantum Institute and Maryland Center for Fundamental Physics, University of Maryland, College Park, Maryland 20742*

Paolo Zanardi  
*Department of Physics and Center for Quantum Information Science and Technology, University of Southern California, Los Angeles, California 90089, USA*

Luigi Amico  
*CNR-MATIS-IMM & Dipartimento di Fisica e Astronomia Università di Catania, Via S. Sofia 64, 95127 Catania, Italy and Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543*

We elaborate on the idea that topologically ordered ground states can be distinguished through a specific property related to the notion of LOCC, known as differential local convertibility. Such a property can be expressed in terms of the response of the Renyi entropies to an external perturbation. Here, we consider the toric code Hamiltonian with several perturbations and using analytical and numerical methods we show that while a subset of these entropies increase with the correlation length, others decrease in the topologically ordered phase; compared to the response in the trivial phase where all entropies increase with increasing correlation length. We conjecture that this characteristic perturbative response should hold for a wider class of states such as quantum double models, cluster states and other quantum spin liquids whereas symmetry broken states should always show monotonic behaviour of their entropies. The findings here hold experimental promise because the subsystem size can be independent of the correlation length, in addition we discuss their implications for quantum information processing tasks using topologically ordered states.

I. INTRODUCTION

In recent years a central thrust of research in quantum many-body theory and quantum information science has been the identification and characterization of novel phases of matter which cannot be adequately described by the Landau symmetry breaking mechanism [1]. These phases are generically exhibited by ground states of strongly interacting systems in two spatial dimensions. Quantum spin liquids [2], topological insulators [3], and anyonic systems [4], are examples that are of immediate interest to the condensed matter community and important for quantum information processing tasks as well [4–8]. Because the low energy states of these gapped systems do not break any symmetry of the Hamiltonian there exists no local observable whose expectation values may be taken as an order parameter denoting the phase [1]; however despite sharing the same symmetries there may exist phases that exhibit different physical properties [9]. The non-symmetry-breaking quantum order [4] in such systems thus needs careful definition and characterization. To this end, methods of varying reliability and feasibility have been proposed [4, 10–15].

In this work we focus on a certain class of spin liquids providing topologically ordered phases of matter. According to the most common definition, topological phases of matter are those that have a ground state degeneracy (w.r.t. all local observables) protected by the topology of the lattice on which the spin Hamiltonian is defined [4, 16, 17]. Using insights from quantum information theory an alternative definition of a topologically non-trivial state is one which cannot be adiabatically connected to a product state (in the lattice degrees of freedom) using unitary circuits (with finite range) of constant depth (not scaling with lattice size). This means that local unitary quantum circuits cannot completely remove the entanglement and the residue can be thought of as long range entanglement [18]. The order in such states can then be identified through the values (zero for topologically trivial states) of carefully constructed quantities, such as the topological entanglement entropy [4, 10–14, 75], which characterize the correlation between different subregions of the many-body system, or as the Wilson loop [19]. We note that such figures of merit for Topological Order (TO) are efficient, provided that length scales of the system much larger than the correlation length ξ are inspected. This makes the detection of the topological order experimentally challenging, because it involves a state tomography of a macroscopic portion of the system.

In this paper, we elaborate on ideas proposed in [20] and
studied further in [21] which show that the detection of topological quantum phases is possible through the study of its local convertibility as implied by the notion of pure state manipulation using Local Operations and Classical Communications (LOCC)[22–24]. More precisely, the question we address is whether the response of a state in the ground state manifold to an external perturbation can be rendered by using LOCC, restricted to the two parts of the bipartition. The picture behind, is that the global character of the correlations encoded in topologically ordered phases, is expected to pose specific constraints on the locality of such response. The figure of merit we employ, is the differential local convertibility of the many-body state introduced by Cui et al. in [25]. Quantitatively, this amounts to studying the response of the Rényi entropies w.r.t. $\lambda$. These entropies are functions of the eigenvalues of the reduced density matrix and the monotonicity of the entire set of entropies depends on their relative majorization, which is a partial order on the set of probability vectors (the vector of eigenvalues) [40]. Finally we check if all Rényi entropies show monotonic behaviour within a phase or does a subset of them show opposing behaviour from the rest. In order to achieve this, we need to solve for the ground state $|\psi(\lambda)\rangle$ and then obtain the reduced density matrix as a function of the parameters $\lambda$.

We employ both analytical and numerical methods to find the ground state and compute the Rényi entropies of the model. Analytically, we resort to two models. One, the Castelnovo-Chamon model, possesses an exact form for the ground state. We are able to compute exactly all the Rényi entropies by using group theoretic methods [41]. We also study the toric code in external magnetic fields, where the field is only acting on a subset of spins. This model maps into free fermions [4, 42, 43], and is thus exactly solvable. In [42, 44], an expression was derived for the Rényi entropy for a particular subsystem in terms of correlation functions. Here, we achieve a general expression for the Rényi entropy of a generic subsystem of this model. These results are more general and can be applied to any lattice gauge theory. Finally, we study the toric code in presence of Ising couplings in both the $x$ and $z$ direction. This model is non exactly solvable. We attack the problem numerically using a version of infinite DMRG in two dimensions [45–47], based on a Matrix Product State (MPS) representation of the ground state manifold for a cylinder of infinite length and finite width. This method has proven very useful to study topological phases [48].

### B. Rényi Entropies

Consider a multipartite pure quantum state $|\psi\rangle \in \otimes_i^N \mathcal{H}_i$. The entanglement spectrum $\tilde{\nu} = \{\nu_1, \nu_2, \ldots, \nu_d\}$ of the state, is defined as the set of eigenvalues of the reduced density matrix $\rho_A = \text{Tr}_A(|\psi\rangle \langle \psi|)$, where $A$ is a subset of local Hilbert space indices, $A \subset [N]$, with the associated Hilbert space given by $\mathcal{H}_A = \otimes_{j \notin A} \mathcal{H}_j$. We call $A$ the subsystem. The complement of the subsystem $A$ then is $\bar{A} = [N] \setminus A$ with its associated Hilbert space $\mathcal{H}_{\bar{A}} = \otimes_{j \in \bar{A}} \mathcal{H}_j$.

The entanglement spectrum of a state is the crucial ingredient in the definition of Rényi entropies for the reduced density matrix $\rho_A$ defined as:

$$S_\alpha(\rho_A) := \frac{1}{1-\alpha} \log \text{Tr}(\rho_A^\alpha) = \frac{1}{1-\alpha} \log \left(\sum_j \nu_j^\alpha\right) \quad \forall \alpha \geq 0$$

(1)
Knowledge about the entire set of Rényi entropies $S_\alpha(\rho_A) \forall \alpha \in [0, \infty)$ is equivalent to complete knowledge about the spectrum of the state itself. At specific values of the continuous parameter $\alpha$, the Rényi entropies provide operationally important information about the state: $S_{\alpha=0} = \log R - R$ being the Schmidt rank is a measure of bipartite entanglement for the state that serves as a criteria for efficient classical representation of the state [49] while $\lim_{\alpha \to 1} S_{\alpha} = S_{\psi_N}$ is the entanglement entropy of the pure state $|\psi\rangle$, that is a measure of its distillable entanglement, entanglement cost and that of its distillable entanglement, entanglement cost and that of formation, relative entropy of entanglement and squashed entanglement [22]. Also a linear combination of 2-Rényi entropies $S_2$ calculated for suitably chosen bipartitions, can be used as a probe of topological order [42, 44]. For product states $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_A$, the entanglement spectrum collapses to unity for one eigenvalue and zero for all others: $\rho_A^2 = \rho_A$, which means that all Rényi entropies are zero as well.

C. Manifold of topologically ordered ground states

We define the ground state manifold, $\mathcal{M}$, of a Hamiltonian $H(\lambda)$ as the continuous set of ground states $|\psi(\lambda)\rangle$ (in a particular topological sector) for all possible values of the control parameters $\lambda$. So $\mathcal{M} = \{|\psi(\lambda)\rangle \ s.t. \ |\psi(\lambda)\rangle \text{ is the ground state of } H(\lambda) \forall \lambda = (\lambda^1, ..., \lambda^n) \in \mathbb{R}^n\}$. As the Hilbert space is endowed with a definite tensor product structure $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, which defines a bipartition of the system, we can consider the set of reduced density matrices $\rho_A(\lambda)$ to the subsystem $A$ as a function of $\lambda$, and study the behaviour of the set of Rényi entropies $S_\alpha(\lambda)$ with $\lambda$ and $\alpha$:

$$S_\alpha(\lambda) = S_\alpha(\rho_A(\lambda)) = S_\alpha(\text{Tr}_A(|\psi(\lambda)\rangle\langle\psi(\lambda)|)) \forall \alpha \geq 0, \quad (2)$$

In the next section, we show, on the back of specific examples, that the monotonicity of the entire set $S_\alpha(\lambda) \forall \alpha$ is a characteristic of the phase unless the perturbation and/or the choice of bipartition is fine tuned. The collective behaviour can be captured succinctly by the sign of the derivative $\text{Sign}[\partial_\lambda S_\alpha(\lambda)] \forall \alpha$, which remains constant in the topologically disordered phase - negative as the perturbation is tuned away from the critical point; whereas in the ordered phase $\partial_\lambda S_\alpha(\lambda) < 0$ for $\alpha < \alpha_c$, while it is positive for $\alpha \geq \alpha_c$, as we move away from the quantum critical point.

D. Differential local convertibility on the ground state manifold

Differential local convertibility (dLOCC) is a property of a submanifold $\mathcal{M}_i \subset \mathcal{M}$ of the ground state manifold $\mathcal{M}$ that determines whether LOCC operations may be used to transform from $|\psi(\lambda)\rangle \in \mathcal{M}_i$ to another $|\psi(\lambda + \delta\lambda)\rangle \in \mathcal{M}_i$. The LOCC operations we refer to are general quantum operations restricted to two parts of some bipartition of the system augmented with unlimited both-ways classical communication.

Mathematically we say that, $\mathcal{M}_i$ is dLOCC iff,

$$\text{Sign}[\partial_\lambda S_\alpha(\lambda)] = \text{constant} \forall \alpha \geq 0 \forall |\psi(\lambda)\rangle \in \mathcal{M}_i$$

(3)

A negative sign in the R.H.S. of the condition above implies dLOCC property of $\mathcal{M}_i$ in the direction of increasing $\lambda$. In this work we focus on submanifolds $\mathcal{M}_i$ that are regions of the ground state manifold pertaining to the different phases, labelled by $i$, for the different Hamiltonian models we consider. Thus we frequently refer to a phase being dLOCC as well.

The quantity: $\text{Sign}[\partial_\lambda S_\alpha(\lambda)] \forall \alpha$, has operational significance w.r.t. traversing $\mathcal{M}_i$ using LOCC. The results of [23, 27, 29–33, 40], imply that one can use LOCC operations to transform a ground state $|\psi(\lambda)\rangle \in \mathcal{M}_i$ to another $|\psi(\lambda + \delta\lambda)\rangle \in \mathcal{M}_i$, which may require access to a shared entangled state $|\phi\rangle$ (entanglement catalyst) between $A, \bar{A}$ (bipartition), with probability 1, at proximal values of $\lambda, \lambda + \delta\lambda$ within a phase, iff the vector of Schmidt coefficients of the product state $|\psi(\lambda + \delta\lambda)\rangle |\phi\rangle$ at the target parameter value $\lambda + \delta\lambda$, majorizes the vector of Schmidt coefficients of the state $|\psi(\lambda)\rangle |\phi\rangle$ at the initial point.

Majorization is a partial order on the set of positive vectors $\vec{\nu}_\lambda, \vec{\nu}_{\lambda+\delta\lambda}$ which, for our purposes here, are the vectors of Schmidt coefficients of the states $|\psi(\lambda)\rangle |\phi\rangle$ and $|\psi(\lambda + \delta\lambda)\rangle |\phi\rangle$ respectively w.r.t. the $A, \bar{A}$ bipartition. It compares the disorder in one vector w.r.t. another. Arranging the entries of the vectors $\vec{\nu}_{\lambda+\delta\lambda}, \vec{\nu}_{\lambda}$ in a non-increasing manner: $(\nu_{\lambda+\delta\lambda})_1 \geq (\nu_{\lambda+\delta\lambda})_2 \geq (\nu_{\lambda+\delta\lambda})_3 \geq \geq (\nu_{\lambda+\delta\lambda})_d$ and $(\nu_{\lambda})_1 \geq (\nu_{\lambda})_2 \geq \geq (\nu_{\lambda})_d$, we say $\vec{\nu}_{\lambda+\delta\lambda}$ majorises $\vec{\nu}_{\lambda}$, i.e. $\vec{\nu}_{\lambda} < \vec{\nu}_{\lambda+\delta\lambda}$ iff:

$$\sum_{j=1}^{k} (\nu_{\lambda+\delta\lambda})_j \leq \sum_{j=1}^{k} (\nu_{\lambda})_j \forall k = 1, 2, ..., d_A \quad (4)$$

Which may be called the catalytic majorization relation since the vectors represent the Schmidt coefficients of states that are a tensor product with the catalyst state $|\phi\rangle$.

It should be clear that not all pairs of states $|\psi(\lambda + \delta\lambda)\rangle$ and $|\psi(\lambda)\rangle$ will require a catalyst for dLOCC conversion. For such states their respective vectors of Schmidt Coefficients $\gamma_{\lambda+\delta\lambda}, \gamma_{\lambda}$ follow a majorization relation $\gamma_{\lambda} < \gamma_{\lambda+\delta\lambda}$ without the need for the ancillary entanglement catalyst $|\phi\rangle$. The necessary and sufficient condition for dLOCC conversion, with or without the need for a catalyst is succintly captured by the condition [27]:

$$S_\alpha(\bar{\gamma}_{\lambda}) \geq S_\alpha(\bar{\gamma}_{\lambda+\delta\lambda}) \forall \alpha$$  \quad (5)

which implies Eq. (3). In words, one can use LOCC transformations, possibly assisted by entanglement catalysis, to transform from $|\psi(\lambda)\rangle$ to $|\psi(\lambda + \delta\lambda)\rangle$ provided all Rényi entropies show monotonically decreasing behavior in going from the initial parameter value to the final one.

Thus catalytic majorization and monotonic behaviour in $\alpha$ of the whole set of Rényi entropies are mutual implications. For $\alpha = 1$ for e.g. Ineq. (5) implies that a necessary condition
for LOCC operations to be used to transform to the new state $|\psi(\lambda + \delta\lambda)\rangle$ is for it to have a lower value of the entanglement entropy w.r.t. the underlying bipartition $[22]$.

E. The models

Here, we present the models we will be dealing with in the rest of the paper. We consider three different perturbations $V(\lambda)$, to Kitaev’s Toric code (TC) model $H_{TC}[16]$.

The TC Hamiltonian $H_{TC}$ is defined on a 2-D system of spin-1/2 particles living on the edges of a square lattice with periodic boundary conditions in both directions, Fig. (1). The Hilbert space size of the system defined on a square lattice of size $L \times L$ is $N = 2^{2L^2}$. There are two different kinds of mutually commuting operators that appear in the Hamiltonian: stars $A_s = \prod_{i} \hat{\sigma}^z_i$ defined at the vertices of the lattice that are the products of Pauli matrices $\hat{\sigma}^z$ acting on the 4 edges shared by a vertex and plaquettes $B_p = \prod_{j} \hat{\sigma}^z_j$ that are products of $\hat{\sigma}^z$ on the 4 edges of a unit cell. The operators $A_s, B_p$ have eigenvalues $\pm 1$. All our Hamiltonians then have the form

$$H = H_{TC} + V(\lambda) := -\sum_s A_s - \sum_p B_p + V(\lambda) \quad (6)$$

Note that, because $\prod_i A_s = \prod_p B_p = 1$, there are only $L^2 - 1$ independent operators of each kind. They constitute a complete set of commuting operators with $H_{TC}$, and therefore all excitations of the unperturbed Hamiltonian $H_{TC}$ may be labelled by the $\pm 1$ eigenvalues of the $2 \times (L^2 - 1)$ operators. This means that there are $g^{2L^2-2}$ excited states corresponding to each of the $2^{2L^2}/(2(2L^2-2)) = 4$ degenerate ground states which is consistent with the fact that the ground state degeneracy for a topologically ordered Hamiltonian of spin-1/2 defined on a torus is $4^g$ with $g = 1$ being the genus of the surface. For our purposes though, one can work in a gauge fixed sector with all $B_p = +1$, that corresponds to an effective low energy theory with $\mathbb{Z}_2$-gauge symmetry since $[A_s, \prod_i \hat{\sigma}^z_i] = 0$ $\forall s$, and the only excitations are those of stars, so that in this sector the Hilbert space dimension is $2^{2L^2-1}$ again with 4 degenerate ground states. In this gauge fixed sector all eigenstates of $H_{TC}$ are superpositions of loop operators $g = \prod_{i} \hat{\sigma}^z_i$ that are products of spin-flips on spins that are crossed by contractible closed loops in the dual lattice. The loop operators are elements of the group $G$ that is generated by the stars. The four degenerate ground states, $|\psi\rangle, W_1|\psi\rangle, W_2|\psi\rangle, W_1W_2|\psi\rangle$, each define a particular topological sector within the gauge fixed sector and are related to each other by spin flips on non-contractible loops $W_1, W_2$, along the two non-contractible directions of the Torus.

In our work we focus on the simplest ground state $|\psi\rangle$ i.e. a fixed topological sector within the gauge. Restricting our attention to this sector, which we call $TS_1$, essentially captures all the phenomenology we want to highlight as well as simplifies the calculations. Thus our analytical results pertain to this sector where in subsections $III\;A, III\;B$ we consider gauge invariant perturbations to $H_{TC}$ that take drive the system across a quantum critical point between a topologically ordered and disordered phase. For a discussion of the critical point see $[50–52]$. The more general perturbation $III\;C$ is studied numerically. The tool used here is a two dimensional density matrix renormalization group extended to infinite cylinders $[53]$. The ability to study a Hamiltonian on an infinite cylinder allows us to obtain the entire set of quasi-degenerated ground states. From that set we choose a ground state in a given topological sector and make sure that the same choice was made for every value of $\lambda_g$ and $\lambda_z$ in Eq. (9). This can be done by looking at the expectation value of certain loop operators around the cylinder. For small perturbations studied here, they are close to $\pm 1$, which allows one to identify the topological sector. All DMRG results presented here are converged in bond dimension, which is a refinement parameter in this calculation.

Here we list the perturbations studied in the current paper:

a. The Castelnovo-Chamon model

This perturbation has an exponential form,

$$V_1(\lambda) = \sum_s e^{-\lambda \Sigma_{i} \hat{\sigma}^z_i}, \quad (7)$$

that commutes with all the plaquette operators $[B_p, V_2(\lambda)] = 0 \forall p$ i.e. it is a gauge invariant perturbation. This system shows a phase transition from a topologically ordered phase to a paramagnetic phase at the critical value of $\lambda \approx 0.44$.

b. Toric code Hamiltonian with magnetic field along spins on rows. The perturbation here is a $\hat{\sigma}^z$ magnetic field applied only to the spins along the rows of the square lattice (we call this direction the horizontal direction),

$$V_2(\lambda) = \lambda \sum_{i \in \text{horiz}} \hat{\sigma}^z_i. \quad (8)$$
Since \( [B_p, V_3(\lambda)] = 0 \) \( \forall p \) this is a gauge invariant perturbation as well that drives the TC model from a topologically ordered phase across the critical point at \( \lambda = 1 \) to a paramagnetic one.

c. The Toric-Ising Model

Here the perturbation,
\[
V_3(\lambda_x, \lambda_z) = - \sum_{i, \mu = x, y} \left( \lambda_x \hat{\sigma}^x_i \hat{\sigma}^x_{i+\mu} + \lambda_z \hat{\sigma}^z_i \hat{\sigma}^z_{i+\mu} \right),
\]
(9)
describes the interplay between topological and antiferromagnetic orders. For generic \( \lambda_x \) and \( \lambda_z \), the perturbation breaks the \( \mathbb{Z}_2 \) gauge symmetry. The latter is preserved for either \( \lambda_x = 0 \) or \( \lambda_z = 0 \). When \( \lambda_x(\lambda_z) = 0 \), the topological and antiferromagnetic orders are separated by a continuous quantum phase transition occurring at the critical value of \( \lambda_x(\lambda_z) = \lambda_c \approx 1/6 \) [54].

III. RESULTS

In this section we present analytical and numerical results that exhibit the relationship between differential local convertibility and correlation length for Hamiltonians \( H = H_{TC} + V(\lambda) \), where \( V(\lambda) = V_1, V_2, V_3 \) described in the previous section.

A. The Castelnovo-Chamon model, \( V = V_1 \)

We start by observing here that the perturbation \( V_1 \) is such that the spin-spin correlation function \( \langle \hat{\sigma}^x_i \hat{\sigma}^x_j \rangle_\lambda \) in a ground state within the topological sector \( TS_1 \) of the Hamiltonian \( H = H_{TC} + \sum_x e^{-\lambda \sum \sigma^x_i} \) is zero for all values of \( \lambda \). In the sector \( TS_1 \), we pick a ground state \( |\xi\rangle \) given by [41]:
\[
|\xi\rangle = \frac{1}{\sqrt{Z}} \sum_{g \in G} e^{\lambda/2 \sum g \sigma^x(g)} |g\rangle
\]
(10)
where \( g |0\rangle \) is the state obtained by acting with \( g = \prod_i A_i, i \in G \), that is the product of star operators, on the totally polarized all spins-up (in the \( z \)-basis) reference state \( |0\rangle \) and the term \( \sigma^x(g) = \langle g | \hat{\sigma}^x | g \rangle \) in the exponent takes the value of \(-1 \) if the spin at edge \( i \) has been flipped and \(+1 \) otherwise. \( Z = Z(\lambda) = \sum_{g \in G} e^{\lambda \sum \sigma^x(g)} \) is a normalization constant. Note that with \( G \) denoting the set of all spins, \( \sum_{g \in G} \sigma^x(g) = N - L(\lambda) \), i.e. the sum counts the total number of spins in a state less the number that have been flipped by the operator \( g \in G \) which are closed loops or products of closed loops in the dual lattice.

In order to analyze the DLOCC properties of this model we need the reduced density matrix for a subset of spins \( A \), on the whole lattice \( \Lambda = A \cup B \), when the whole system is in state (10):
\[
\rho_A(\lambda) = \frac{1}{Z} \sum_{g \in G} e^{\frac{\lambda}{2} (N-L(g))} e^{\frac{\lambda}{2} (N-L(g'))} x_A |0\rangle_A (g) \langle g' | x'_A
\]
(11)
where the group \( G_A = \{ g \in G | g = g_A \otimes I_B \} \) is the subgroup of \( G \) generated by stars operators acting non-trivially only on the spins in \( A \) and \( x_A^g \) is the restriction of the operators \( g \in G \) to just the subsystem \( A \) (for details see [10, 11]). We will also need the subgroup \( G_B = \{ g \in G | g = I_A \otimes g_B \} \) which includes all products of star operators that act non-trivially only on the spins in \( B \). Then the \( \alpha \)-Rényi entropy by given by:
\[
S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \sum_{g \in G} e^{-\lambda L(g)} \langle g \rangle^{1-\alpha} = \sum_{g \in G} e^{-\lambda L(g)} \langle g \rangle^{1-\alpha} = \sum_{g \in G} e^{-\lambda L(g)} \langle g \rangle^{1-\alpha} \lambda, g \rangle
\]
(12)
where \( E_g = L(g) - N \) and \( w(\lambda, g) := \sum_{h \in \Lambda} \sum_{h \in \Lambda} e^{-\lambda L(h)} \) with all the \( \lambda \) dependence made explicit.

After a straightforward but tedious calculation one can obtain the derivative of (12) w.r.t. the parameter \( \lambda \) and it is given by the expression:
\[
\partial_\lambda S_\alpha(\lambda) = \left\langle \{ E_g \} w(\lambda, g) \right\rangle \tilde{Z}(\lambda, \alpha) + \frac{\alpha}{1-\alpha} \sum_{g \in G} \langle g \rangle \right\rangle \tilde{Z}(\lambda, \alpha) - \frac{1}{1-\alpha} \sum_{g \in G} \langle g \rangle \tilde{Z}(\lambda, \alpha)
\]
(13)
Here \( \tilde{Z}(\lambda, \alpha) := \sum_{g \in G} e^{-\lambda L(g)} | g \rangle \langle g | \) and we use averages w.r.t. the functions \( f(g) = w(\lambda, g) \), \( \tilde{Z}(\lambda, \alpha), Z(\lambda) \) defined as usual: \( \langle E(g) f(g) \rangle = \sum_{g \in G} \langle g \rangle \langle g | E(g) \langle g | f(g) \rangle \rangle \). One can now evaluate the R.H.S. of Eq. (13) in the limit \( \lambda \rightarrow 0 \) which corresponds to small perturbations of the TC model and find that \( \partial_\lambda S_\alpha(\lambda) \leq 0 \forall \alpha \). This implies that all \( \alpha \) Rényi entropies decrease as we move away from the point in the phase diagram with a flat entanglement spectrum. Under the assumption that the slopes of Rényi entropies for fixed \( \alpha \) do not change within a phase we find that this model has DLOCC within the topologically ordered phase. Similarly if one considers the \( \lambda \rightarrow \infty \) limit one finds that all the slopes are negative as well implying that the particular form of the perturbation \( V_1 \) leads to DLOCC in both, the TO and the paramagnetic, phases of the model.

B. Toric code with magnetic field along spins on rows, \( V = V_2 \)

The gauge invariant perturbation \( V_2(\lambda) \) lets us analyse a model with a non-constant correlation length \( \xi(\lambda) \). The Gauge fixed \( (B_p = 1 \forall p) \) Hamiltonian (6) up to a constant offset is thus:
\[
H = - \sum \lambda \sum_{h \in \text{horiz}} \hat{\sigma}^z_h
\]
(14)
where by \( h \in \text{horiz} \), we mean that the external field is applied only to spins on edges along the rows that we take to be the horizontal direction, Fig. (2).

To solve Eq. (14) we map it to an exactly solvable model that preserves the local algebra of the terms. We first observe that the star operators have eigenvalues \( \pm 1 \). Then we note
that each $\sigma^z_h$ operator on a horizontal link has two neighboring star operators acting on the vertices connected by the edge. Because the action of $\sigma^z_h$ is to flip the sign of both the star operators that share the spin ‘h’, $\{A_s, \sigma^z_h\} = 0$ for these neighboring stars and we can move to an alternate picture where the star operators at a vertex are replaced by pseudo-spin operators, $\hat{\sigma}^z_i$, at the same vertex with eigenvalues ±1. The action of $\sigma^z_h$ then corresponds to the action of $\hat{\sigma}^z_x \hat{\sigma}^z_{i+1}$ when the vertices $s, s + 1$ share the edge labelled ‘h’ i.e. it flips both neighboring pseudo-spins. We will call $A_s, \hat{\sigma}^z_h$ operators in the ‘σ-picture‘ in contrast to the ‘τ-picture‘ for operators in terms of the pseudo-spin operators $\hat{\tau}$. The map is thus given by:

$$A_s \rightarrow \hat{\sigma}^z_i$$

$$\hat{\sigma}^z_h \rightarrow \hat{\sigma}^z_x \hat{\sigma}^z_{i+1}$$

(15)

which maps the Hamiltonian (14) to:

$$\tilde{H} = -\sum_{s \text{ all vertices}} \hat{\sigma}^z_s - \lambda \sum_{s \text{ all rows } s \text{ even}} \sum_{s \text{ s even}} \hat{\sigma}^x_s \hat{\sigma}^x_{s+1}$$

$$= -\sum_{s \text{ all rows } s \text{ even}} \hat{\sigma}^z_s - \lambda \sum_{s \text{ all rows } s \text{ even}} \sum_{s \text{ s even}} \hat{\sigma}^x_s \hat{\sigma}^x_{s+1}$$

$$= \sum_{s \text{ all rows } s \text{ even}} (- \sum_{s \text{ s even}} \hat{\sigma}^z_s - \lambda \sum_{s \text{ s even}} \hat{\sigma}^x_s \hat{\sigma}^x_{s+1})$$

$$= \Theta_{\text{all rows}} H_{\text{row}},$$

$$H_{\text{row}} = -\sum_{s = 1}^{L} \hat{\sigma}^z_s - \lambda \sum_{s \text{ s even}} \hat{\sigma}^x_s \hat{\sigma}^x_{s+1}$$

(16)

Eq.(16) implies that the new Hamiltonian is a direct sum of 1-D quantum Ising Hamiltonians on the $L$ rows. The ground state of $\tilde{H}$ is thus given by the tensor product of the ground states of each individual row i.e. $|\psi\rangle = \otimes_{j \in \text{all rows}} |\psi_j\rangle$. Each row Hamiltonian $H_{\text{row}}$ in the expression above is solved by mapping the Pauli spins via the Jordan-Wigner transformation to Fermions and then a Bogoliubov transformation diagonalizes the Hamiltonian to a free Fermionic form [55]. In the present paper, we consider the symmetric ground state enjoying the global spin flip symmetry of the Hamiltonian and thus $< \hat{\tau}_i^x > = 0$ in the ground state.

This model exhibits two phases as well: a topologically ordered one for weak magnetic field and a disordered one beyond the critical value $\lambda = 1$ [42, 43]. The results of this model, which follow in the next subsections, demonstrate that for fine-tuned perturbations one might indeed obtain differential local convertibility for specially chosen bipartitions. We remark that although we considered the symmetric ground state of the system this does not result in a loss of generality and at the same time eases the analytical presentation.

For special choices of subsystems (we call these ‘thin’ subsystems for reasons that become clear in the following) we can determine the exact eigenvalues of the reduced density matrix for all values of the perturbing field $\lambda$ and hence all the Rényi entropies $S_\alpha$ which show monotonic perturbative behaviour for all $\alpha$. On the other hand, for systems with a ‘bulk’ some Rényi entropies have a different behaviour with increasing $\lambda$ than others.

1. ‘Thin’ subsystems

A drastic simplification in the exact calculation of the Rényi entropies for the ground state of gauge theories (of which the toric code is the simplest example, the $Z_2$ gauge theory) can be obtained by choosing some particular partitions[42, 44]. A ‘thin’ subsystem $A$, in the lattice for the Toric code model is one where there are no star operators that can act on spins which exclusively belong to $A$. For example, the bipartition of spins on the lattice where subsystem $A$ is comprised only of rows (columns) with the columns (rows) forming the complement $B$. Mathematically this means that the group $G_A$ only contains the identity, $I$, which in turn implies that the reduced density matrix, $\rho_A$, is diagonal in the $\sigma$-basis of the $\sigma$-spins [11]. All loops on the real lattice are other examples, the shortest such loop being a plaquette, Fig. (3). Intuitively, ‘thin’ subsystems are those wherein all the degrees of freedom are maximally entangled, even in the unperturbed toric code model, while respecting the gauge constraints. Thus increasing correlation length cannot lead to newer non-zero values appearing in the entanglement spectrum.

Such is the subsystem $A$ that we now investigate. The reduced density matrix for a plaquette with 4 spins is a matrix of size $2^4 \times 2^4$. However because of the gauge constraint, $B_p = 1$, only three spins are independent which means that the maximal rank of the reduced density matrix is $2^3 = 8$. The diagonal entries of this matrix, $(\rho_A)_{ss}$, correspond to expectation values of the projector onto the different spin configurations, $s = (s_1, s_2, s_3) \in \{-1, 1\}^3$, in the ground state of the Hamilt-
nian (14) of the three independent spins i.e.:

\[
\langle \rho_A \rangle_{x,z} = \frac{1}{2^3} \left( \langle 1 + s_1 \hat{\sigma}^+_1 \rangle(1 + s_2 \hat{\sigma}^+_2)(1 + s_3 \hat{\sigma}^+_3) \right) \langle \psi \rangle
\]

\[
= \frac{1}{2^3} \left( 1 + s_1 \langle \hat{\sigma}^+_1 \rangle + s_2 \langle \hat{\sigma}^+_2 \rangle + s_3 \langle \hat{\sigma}^+_3 \rangle 
+ s_1 s_2 \langle \hat{\sigma}^+_1 \hat{\sigma}^+_2 \rangle + s_2 s_3 \langle \hat{\sigma}^+_2 \hat{\sigma}^+_3 \rangle + s_3 s_1 \langle \hat{\sigma}^+_3 \hat{\sigma}^+_1 \rangle 
+ s_1 s_2 s_3 \langle \hat{\sigma}^+_1 \hat{\sigma}^+_2 \hat{\sigma}^+_3 \rangle \right)
\]

\[
= \frac{1}{2^3} \left( 1 + s_1 \langle \hat{\tau}^+_1 \rangle + s_2 \langle \hat{\tau}^+_2 \rangle + s_3 \langle \hat{\tau}^+_3 \rangle 
+ s_1 s_2 \langle \hat{\tau}^+_1 \hat{\tau}^+_2 \rangle + s_2 s_3 \langle \hat{\tau}^+_2 \hat{\tau}^+_3 \rangle + s_3 s_1 \langle \hat{\tau}^+_3 \hat{\tau}^+_1 \rangle 
+ s_1 s_2 s_3 \langle \hat{\tau}^+_1 \hat{\tau}^+_2 \hat{\tau}^+_3 \rangle \right)
\]

where in the last line above we have used the mapping (15) to express the diagonal entries in terms of the \( \tau \)-spins (see Appendix B 1). Notice that the only non-trivial expectation values of the \( \tau \)-spins are those of two point functions since \( \langle \hat{\tau}^+_i \rangle = 0 \) in the symmetric ground state. The thermodynamic limit expressions \([53, 56]\) in the entire domain of \( \lambda \) is:

\[
\langle \hat{\tau}^+_i \hat{\tau}^+_i \rangle = \frac{1}{\pi} \int_0^\pi \cos(\phi) \left[ \cos(\phi) - 1/\lambda \right] + \sin^2(\phi) \left[ \left( 1/\lambda - \cos(\phi) \right)^2 + \sin^2(\phi) \right]^{1/2} \, d\phi \quad 0 < \lambda
\]

\[
(18)
\]

Thus we can calculate the trace of arbitrary powers of the reduced density matrix, \( \text{Tr}(\rho_A^\gamma) = \sum_{x_1, x_2, x_3 = -1, 1} \langle \rho_A \rangle_{x_1}^\gamma \), using which the Rényi entropies are given by (with \( T(\lambda) = \langle \hat{\tau}^+_i \hat{\tau}^+_i \rangle \)):

\[
S_\alpha(\lambda) = \frac{1}{1 - \alpha} \log \left[ \frac{1}{2^{3\alpha}} \left( 2(1 + T(\lambda))^{2\alpha} + 2(1 - T(\lambda))^{2\alpha} + 4(1 - T(\lambda)^2)^{\alpha} \right) \right]
\]

\[
(19)
\]

From the plot of Eq. (19) in Fig. (4) we observe that for all values of \( \alpha = .01, .1, .5, 1, .01, 2 \), the entropies show monotonic behaviour with \( \lambda \) in both the phases. While in the topologically ordered phase, \( \lambda < 1 \), the entropies decrease as we approach the quantum critical point, for the disordered region it decreases as we move away from it.

\[\text{FIG. 3. (color online) Subsystem A, shown in the shaded region, of one plaquette with the spins 1, 2, 3, 4, on the edges. The eigenvalues of the reduced density matrix} \rho_A, \text{involves calculating expectation values of operators on the 4 pseudo-spins } i, i+1, j, j+1, \text{at the shown vertices (see Appendix (B 1)).}\]

\[\text{FIG. 4. (color online) Rényi entropies for a subsystem A of one plaquette (shown in Fig. 3), at different values of } \alpha. \text{ All entropies show monotonic behavior in both the phases: they decrease monotonically with increasing correlation length } \xi(\lambda) \text{ for } \lambda < \lambda_c = 1 \text{ while they increase with } \xi(\lambda) \text{ for } \lambda > 1.\]

\[\text{FIG. 5. (color online) Subsystem A, shown as the shaded region, comprised of a total of seven spins which form two overlapping stars.}\]

2. **General treatment**

On the lattice, we call systems with a ‘bulk’ those that have at least one or more star operators that act on spins exclusively belonging to \( A \). This means that the group \( G_A \) is non-trivial and the reduced density matrix for the subsystem is not diagonal anymore \([11]\). Consequently, the analysis of this case is considerably more involved. We refer to \([66]\) for an introduction to the technique used to treat a gauge theory. Since the perturbation we consider is gauge invariant, indeed, we can represent the state as the sum over element of a group, and this makes the calculation possible in the formalism. We can compute exactly the reduced density matrix (See Appendix (B 2 b) for details). Moreover, we can find an exact expression for the purity:

\[
P(\lambda) = \frac{|G_B|}{|G|} \sum_{g \in G_A, \tilde{z} \in Z_A} |\langle \psi(\lambda) | g z | \psi(\lambda) \rangle|^2, \quad (20)
\]
where, $|\psi(\lambda)\rangle$ is the ground state of the Hamiltonian (14) and $|G_B\rangle$ is the cardinality of the group of star operators acting *exclusively* in the complement of $A$ i.e. $G_B = \{g \in G | g = \mathbb{1}_A \otimes g_B\}$. As before, $G_A$ is the group of star operators exclusively in $A$ while $Z_A$ is the group generated by products of $\hat{\sigma}^z$’s acting on spins in $A$. This expression can be generalized to general gauge theories and quantum double models, and to a general Rényi entropy of index $\alpha$, and constitutes one of the main results of this paper.

Although in principle we can calculate the entropies $S_\alpha(\lambda)$ for each integer $\alpha$, we focus on the 2–Rényi entropy only. In particular, we demonstrate that it has a monotonic behaviour in both the phases. The monotonicity of $S_2(\lambda)$ is sufficient to show that all higher entropies obey the same monotonicity because of the continuity of the entropies in $\alpha$ and because of their ordering relation: $S_{\alpha'} \leq S_\alpha \forall \alpha' \geq \alpha$. On the other hand in the Toric code limit at $\lambda = 0$, the eigenspectrum is flat with there being $2^5$ equal eigenvalues summing to 1 with the remaining $2^7 - 2^5 = 96$ eigenvalues, all zero. Turning on the perturbation has the effect of making some of these zero eigenvalues non-zero which shows up as an increase of $\lim_{\alpha \to 0} S_\alpha$ and other Rényi entropies with $\alpha$ close to zero. Alternatively put: the Schmidt rank of the state $|\psi(\lambda)\rangle$ increases with $\lambda$ w.r.t. bipartitions with a bulk.

To analyze this case while keeping the presentation simple, we choose a subsystem $A$ which includes the 7 spins of two neighboring stars, Fig. (5). For the calculations, here we use the symmetric ground state in the $TS_1$ sector.

The evaluation of the R.H.S of Eq. (20) again relies on the $\sigma - \tau$ correspondence (15) and we get for the purity:

$$P = \frac{1}{27} \left\{ (1 + 3 < \hat{\tau}_1^x \hat{\tau}_2^x >^2 + 2 < \hat{\tau}_1^x \hat{\tau}_3^x >^2 + < \hat{\tau}_1^x \hat{\tau}_4^x >^2 + < \hat{\tau}_2^x \hat{\tau}_3^x >^2 + < \hat{\tau}_2^x \hat{\tau}_4^x >^2 + < \hat{\tau}_3^x \hat{\tau}_4^x >^2 + < \hat{\tau}_1^x \hat{\tau}_2^x \hat{\tau}_3^x >^2 + < \hat{\tau}_1^x \hat{\tau}_2^x \hat{\tau}_4^x >^2 + < \hat{\tau}_1^x \hat{\tau}_3^x \hat{\tau}_4^x >^2 + < \hat{\tau}_2^x \hat{\tau}_3^x \hat{\tau}_4^x >^2 + < \hat{\tau}_1^x \hat{\tau}_2^x \hat{\tau}_3^x \hat{\tau}_4^x >^2 \right\}$$

(21)

![Fig. 6](image-url) (color online) The 2-Rényi entropy of a subsystem comprised of two stars $A$ (shown in Fig. 5) across the phase transition at $\lambda = 1$ for $H = H_{TC} + V_2(\lambda)$. The monotonic behavior in both the phases for $S_2$ implies similar behavior for $S_\alpha \forall \alpha \geq 2$; whereas the general arguments presented in the text imply that for $\alpha \to 0$ they should increase till the quantum critical point. The dotted line is the inverse of the energy gap between the ground and first excited states for the transverse field Ising model to which the perturbed gauge-fixed Toric code Hamiltonian is mapped.

The 2-Rényi entropy $S_2(\lambda) = -\log(P(\lambda))$ is shown in Fig. (6). Just as for the *thin* subsystem case, we find similar monotonicity in the approach and departure from the quantum critical point.

![Fig. 7](image-url) (color online) Subsystem $A$, shown as the shaded region, comprised of a total of six spins which make up the spins on a plaquette and two neighboring spins to its northeast corner.

### C. The Toric-Ising model, $V = V_3$

Here we consider the subsystem $A$ consisting of a plaquette with two adjoining spins pictured in Fig. 7 and numerically show that for the perturbation $V = V_3(\lambda_x, \lambda_z) = -\sum_{i,\mu=x,z}(\lambda_x \hat{\sigma}_i^x \hat{\sigma}_i^\mu + \lambda_z \hat{\sigma}_i^z \hat{\sigma}_i^\mu)$, which takes the Toric code Hamiltonian from a TO phase to a ferromagnetic phase, the set of Rényi entropies in the TO phase show the splitting behavior. Note that neither the perturbation here nor the choice of the subsystem is fine-tuned. In other words, the lack of differential local convertibility is a robust property of the topologically ordered phase and is universal. However, the value of $\alpha$ for the Rényi index such that the sign of the derivative $\partial_\lambda S_\alpha(\lambda)$ changes, is non universal and is numerically found here to be $\alpha \approx 1.3$, see Fig. 8. The space of the parameters spanned is deep in the topological phase, with $|\lambda_x, \lambda_z| \leq 0.05$. For high $\lambda$ values i.e. in the ferromagnetic phase, the sign is
found to be the same (not shown in the plot) for every value of the Rényi index $\alpha$.

Thus even in this model where a phase transition occurs from a TO phase to a ferromagnetic one the latter exhibits differential local convertibility whereas the former does not.

**D. Summary of results**

Here we collect the main results of this section that will help formulate, in the conclusions, the conjecture about the splitting phenomenon of the Rényi’s entropies.

- For perturbations (III A) with constant correlation length and any bipartition the behaviour of the Rényi entropies is monotonic and there is no splitting phenomenon.

- For perturbations (III B) with non-constant correlation length and thin bipartitioning the behaviour of the Rényi entropies is monotonic and there is no splitting phenomenon.

- For perturbations (III B) with non-constant correlation length and bulk bipartitioning the Rényi’s entropies split.

- For general perturbations (III C) the splitting behaviour of the entropies is robust and happens without reference to the size of the subsystem as long as the subsystem has some bulk.

**IV. DISCUSSION AND CONCLUSION**

In this paper, we have considered a paradigmatic class of topological phases, as those ones arising from the toric code with a perturbation driven by a set of control parameters $\lambda = (\lambda_1, \ldots, \lambda^n)$. We focused on the case where the energy gap can vanish, giving rise to a quantum phase transition to a topologically trivial phase (paramagnet). The perturbation studied affects the correlation length of the system $\xi$ (that is vanishing exactly for the toric code $\lambda = 0$).

We have shown that the two phases can be distinguished through a specific notion related to LOCC. This notion is known as differential local convertibility: Bipartitioning the system in $A$ and $B$, the result is that two adjacent states in the topological quantum phase, generically cannot be connected by LOCC in $A$ and $B$ (even in the presence of a catalyst); in the paramagnetic phases, in contrast, the states are locally convertible. This is consistent with the fact that in the topologically trivial phases it is always possible to transform the ground state to a totally factorized state in the physical degrees of freedom by using a local unitary quantum circuit of fixed depth. The locally convertible character of a phase is intimately connected with the adiabatic computational power since a phase that allows LOCC traversal between two points cannot offer a computational advantage upon use in any quantum information protocol [35, 57].

The figure of merit for the locality of the response to the perturbation is expressed in terms of the Rényi entropies associated to a subsystem $A$ of contiguous spins: The non-local convertible phase features a splitting behaviour of the entropies, with their partial derivative along the control parameter $\lambda^i$ changing sign for a particular value of the Rényi index $\alpha$. The splitting phenomenon is observed within the whole topological phase irrespective of the particular form of the perturbation or of the subsystem $A$, unless it is very fine tuned - such as the ones without any bulk. The value of $\alpha$ at which the splitting occurs is instead dependent on the details of the model.

The understanding of this phenomenon relies on the structure of the entanglement spectrum around a special point in the phase. Indeed, in the TO phase of this model there exists an extremal point with a flat entanglement spectrum and zero correlation length, $\xi = 0$. As we perturb away from this point, if the correlation length $\xi$ also increases then newer degrees of freedom get involved in the entanglement spectrum as a result of which the lower ($\alpha \to 0$) entropies increase, on the other hand the higher $\alpha$ entropies decrease because of the algebraic suppression of the contributions from the new small but non-zero values in the spectrum and loss of contributions from the previously non-zero larger eigenvalues. We comment that since similar phenomenology in the entanglement spectrum is known to be displayed in cluster states [58, 59], or more generally in all graph states [60], similar findings in the Rényi entropies response should apply to those as well. Our work here should be seen as supporting a growing body of evidence [21, 35] that this characteristic perturbative response should hold for a wider class of states such as quantum double models, cluster states and other quantum spin liquids. In the toric code case knowledge about the ground state degeneracy can additionally distinguish its TO ground states from the latter. Compared to this, ground states of all symmetry broken phases exhibit monotonic behaviour of their Rényi entropies with an increase in correlation length, and are thus always locally convertible [35].

In order to compute the Rényi entropies for the perturbed toric code, we have resorted to two methods. For general per-
turbations that break gauge invariance, and also make the system non integrable, we resort to a 2D DMRG method, which can treat infinite cylinders [48]. On the other hand, for the gauge invariant perturbation, we find a general expression for the Rényi entropies, that can be generalized to every gauge theory [69]. Moreover, for a particular form of the perturbation, the system is integrable, and we can find an exact analytical formula for the Rényi entropy. This result is technically relevant, and would allow to study several problems, including stability issues at zero [81–83] and finite temperature [70, 72, 73], the confinement problem [78], and the identification of relevant correlations [79, 85]. A very important arena including stability issues at zero [81–83] and finite temperature is the system non integrable, we resort to a 2D DMRG method, which can treat infinite cylinders [48]. On the other hand, for the perturbations that break gauge invariance, and also make the system non integrable, we resort to a 2D DMRG method, which can treat infinite cylinders [48].

In perspective, it would also be interesting to see if the local convertibility properties -or failure of thereof- hold for more general TO states without flat entanglement spectra such as fractional quantum Hall states [61–63] and chiral spin liquids [64].

\[ \partial_\lambda S_\alpha(\rho_A) = \partial_\lambda \left( \frac{1}{1-\alpha} \log \text{Tr}[\rho_A^\alpha] \right) \]

\[ = \frac{1}{1-\alpha} \text{Tr}[\rho_A^\alpha] \partial_\lambda \left( \text{Tr}[\rho_A^\alpha] \right) \]

\[ = \frac{1}{1-\alpha} \text{Tr}[\rho_A^\alpha] \partial_\lambda \left( \frac{1}{Z(\lambda)} \sum_{g \in G} e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} \right) \]

\[ = \frac{1}{1-\alpha} \text{Tr}[\rho_A^\alpha] \frac{1}{Z(\lambda)} \left\{ \sum_{g \in G} \left[ -E_g e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} + (\alpha-1) w'(\lambda, g) w^{\alpha-2}(\lambda, g) e^{-\lambda E_g} \right] \right. \]

\[ - \frac{\partial Z'(\lambda)}{Z(\lambda)} \sum_{g \in G} e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} \}

\[ = \frac{1}{1-\alpha} \text{Tr}[\rho_A^\alpha] \frac{1}{Z(\lambda)} \left\{ \sum_{g \in G} \left[ \frac{\partial}{\partial \lambda} \log Z(\lambda) \right] \sum_{g \in G} \left[ -E_g e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} + (\alpha-1) w'(\lambda, g) w^{\alpha-2}(\lambda, g) e^{-\lambda E_g} \right] \right. \]

\[ + \frac{\partial Z'(\lambda)}{Z(\lambda)} \sum_{g \in G} e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} \}

\[ + \frac{1}{1-\alpha} \text{Tr}[\rho_A^\alpha] \frac{1}{Z(\lambda)} \left\{ \sum_{g \in G} \left[ \frac{\partial}{\partial \lambda} \log Z(\lambda) \right] \sum_{g \in G} \left[ -E_g e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} + (\alpha-1) w'(\lambda, g) w^{\alpha-2}(\lambda, g) e^{-\lambda E_g} \right] \right. \]

\[ + \frac{\partial Z'(\lambda)}{Z(\lambda)} \sum_{g \in G} e^{-\lambda E_g w^{\alpha-1}(\lambda, g)} \}

In the second last line above we have used the fact that \( \frac{Z'(\lambda)}{Z(\lambda)} = -\sum_{g \in G} E_g e^{-\lambda E_g} Z(\lambda) \). Next we define certain averages that appear in eq.(A1). For any function \( f(h, g, k), g \in G, h \in G_A, k \in G_B \), we have:

\[ \langle f(g) \rangle_{Z(\lambda)} := \sum_{g \in G} f(g) e^{-\lambda E_g} Z(\lambda), \quad Z(\lambda) := \sum_{g \in G} e^{-\lambda E_g} \]

\[ \langle f(h, g, k) \rangle_{w(\lambda, g)} := \sum_{h \in G_A, k \in G_B} f(h, g, k) e^{-\lambda E_{h, k}} w(\lambda, g) \]

\[ w(\lambda, g) := \sum_{h \in G_A, k \in G_B} e^{-\lambda E_{h, k}} \]

\[ \begin{align*}
G, h \in G_A, k \in G_B, f(g) = f(h = 1_A, g, k = 1_B) \quad &\text{we have:} \\
\langle f(g) \rangle_{Z(\lambda)} := \sum_{g \in G} f(g) e^{-\lambda E_g} Z(\lambda), \quad Z(\lambda) := \sum_{g \in G} e^{-\lambda E_g} \\
\langle f(h, g, k) \rangle_{w(\lambda, g)} := \sum_{h \in G_A, k \in G_B} f(h, g, k) e^{-\lambda E_{h, k}} w(\lambda, g) \\
w(\lambda, g) := \sum_{h \in G_A, k \in G_B} e^{-\lambda E_{h, k}}
\end{align*} \]
Observe now that the term outside the sum in eq.(A1) has in the denominator the product \(\sum_{\alpha \in G} e^{-\lambda E_g} w^{\alpha-1}(\lambda, g) = \hat{Z}(\lambda, \alpha)\). This implies that the R.H.S. of eq.(A1) is really an average w.r.t. the new partition function \(\hat{Z}(\lambda, \alpha)\) i.e.:

\[
\partial_\lambda S_\alpha(\lambda) = \frac{1}{(1-\alpha)} \sum_{\alpha \in G} T(\alpha, \lambda, g) e^{-\lambda E_g} w^{\alpha-1}(\lambda, g) \frac{Z(\lambda, \alpha)}{\hat{Z}(\lambda, \alpha)}
\]  

(A3)

where \(T(\alpha, \lambda, g) := \left[ (\alpha-1)w'(\lambda, g) \right] \) w.r.t. the new partition function \(\hat{Z}(\lambda, \alpha)\).

Further note that \(w'(\lambda, g) \) is a function of \(g \in G\) whereas \(Z'(\lambda) / Z(\lambda) = -\langle E_g \rangle_{w(\lambda, g)}\) is independent of \(g \in G\). Equation (A3) thus takes the form of a sum of averages:

\[
\partial_\lambda S_\alpha(\lambda) = \left( \langle E_g \rangle_{w(\lambda, g)} \right) \hat{Z}(\lambda, \alpha) + \frac{\alpha}{1-\alpha} \langle E_g \rangle_{Z(\lambda)} - \frac{1}{1-\alpha} \langle E_g \rangle_{\hat{Z}(\lambda, \alpha)}
\]  

(A4)

2. Perturbations around the toric code limit

One can perform a small \(\lambda\) expansion of eq.(A4) to see that the model permits DLOCC for any bipartition for small perturbations to the Toric Code limit of \(\lambda = 0\). To see this let us note the following:

\[
Z(\lambda) \approx \sum_{\alpha \in G} (1 - \lambda E_g) = \left| G \right| - \lambda \sum_g E_g
\]

\[
w(\lambda, g) \approx \sum_{h \in G_A, k \in G_B} (1 - \lambda E_{h\lambda k}) = \left| G_A \right| \left| G_B \right| - \lambda \sum_{h \in G_A, k \in G_B} E_{h\lambda k}
\]

\[
\hat{Z}(\lambda, \alpha) \approx \sum_{\alpha \in G} (1 - \lambda E_g) \left| G_A \right| \left| G_B \right| - \lambda \sum_{h \in G_A, k \in G_B} E_{h\lambda k})^{\alpha-1}
\]

\[
\approx \sum_{\alpha \in G} (\left| G_A \right| \left| G_B \right|)^{\alpha-1} (1 - \lambda E_g) (1 - \frac{\lambda(\alpha - 1)}{\left| G_A \right| \left| G_B \right|}) \sum_{h \in G_A, k \in G_B} E_{h\lambda k})
\]

\[
\approx (\left| G_A \right| \left| G_B \right|)^{\alpha-1} (1 - \lambda E_g) \sum \frac{\lambda(\alpha - 1)}{\left| G_A \right| \left| G_B \right|} \sum_{h \in G_A, k \in G_B} E_{h\lambda k})
\]

\[
= (\left| G_A \right| \left| G_B \right|)^{\alpha-1} (1 - \lambda \sum_g E_g)
\]  

(A5)

Using the weights \((1 - \lambda E_g), (1 - \lambda E_{h\lambda k}), (1 - \lambda \alpha E_g)\) for the evaluation of the averages w.r.t. \(Z(\lambda), w(\lambda, g), \hat{Z}(\lambda, \alpha)\) respectively we find that :

\[
\partial_\lambda S_\alpha(\lambda) = 0 + \lambda (C_1 \alpha + C_2) + \text{higher order terms in } \lambda
\]

with

\[
C_1 = \frac{\sum_g E_g^2}{\left| G \right|^2} - \frac{\sum_g \sum_{h \in G_A, k \in G_B} E_{g} E_{h\lambda k}}{\left| G \right| \left| G_A \right| \left| G_B \right|}
\]

\[
C_2 = -\frac{\sum_g E_g}{\left| G \right|} + \frac{\sum_g \sum_{h \in G_A, k \in G_B} \sum_{\alpha \in G} \lambda E_{h\lambda k} E_{g\lambda k}}{\left| G \right| \left| G_A \right| \left| G_B \right|}
\]  

(A6)

To prove that \(C_1, C_2 \leq 0\) we note that cosets w.r.t. the subgroup \(G_A \times G_B\) of the group \(G\) divide the group into disjoint subsets. If \(q\) labels these unique subsets then one can write:

\[
\sum_g \sum_{h \in G_A, k \in G_B} E_{g} E_{h\lambda k} \equiv \sum_{q \in Q \setminus (G_A \times G_B)} \sum_{h \in G_A, k \in G_B} E_{h\lambda k} E_{h\lambda k}
\]

\[
= \sum_{q \in Q \setminus (G_A \times G_B)} \sum_{h \in G_A, k \in G_B} E_{h\lambda k} E_{h\lambda k}
\]

\[
= \sum_{q \in Q \setminus (G_A \times G_B)} \left( \sum_{h \in G_A, k \in G_B} E_{h\lambda k} \right)^2
\]  

(A7)

Let us now note that each \(E_g \geq 0\) \(\forall g \in G\). Thus to prove that \(C_1 \leq 0\) one needs to prove that for a collection of \(\left| G \right|\) positive numbers \(E_1, E_2, \ldots, E_{\left| G \right|}\) any grouping of \(\left| G_A \right| \times \left| G_B \right|\) numbers such that \(mod(\left| G \right|, \left| G_A \right| \times \left| G_B \right|) = 0\) yields (with \(k = \left| G \right| / \left| G_A \times G_B \right|\))
\[
\frac{\left( (E_1 + E_2 + \ldots + E_{|G_A||G_B|})^2 \right) + \left( (E_{|G_A||G_B|+1} + \ldots + E_{2|G_A||G_B|})^2 \right) + \left( (E_{(k-1)|G_A||G_B|} + \ldots + E_{k|G_A||G_B|})^2 \right)}{|G||G_A||G_B|} \geq \frac{(E_1 + E_2 + \ldots + E_{|G|})^2}{|G|^2} .
\] (A8)

with equality holding iff \( E_1 = E_2 = \ldots = E_{|G|} \). If one represents the sum of the energies in each coset by \( S_i, i = 1, 2, \ldots, k \) then condition (A8) is equivalent to proving:

\[
S_1^2 + \ldots + S_k^2 \geq \frac{(S_1 + S_2 + \ldots + S_k)^2}{k} \implies (k-1)[S_1^2 + \ldots + S_k^2] \geq 2 \sum_{i<j} S_i S_j
\]

which is the sum of several inequalities all of which are of the form \((S_i^2 + S_j^2) \geq 2S_i S_j\). The same inequalities are used to prove \( C_2 \leq 0 \) by noticing:

\[
\sum_{g} \sum_{\text{h} \in G_A} \sum_{\text{k} \in G_B} E_{\text{h}g} E_{\text{k}g} = \sum_{q} \sum_{\text{h} \in G_A} \sum_{\text{k} \in G_B} E_{\text{h}q} E_{\text{k}q} = |G_A||G_B| \sum_{q} \sum_{\text{h} \in G_A} \sum_{\text{k} \in G_B} E_{\text{h}q} E_{\text{k}q} = |G_A||G_B| \left( \sum_{q} E_{\text{h}q} \right)^2
\] (A9)

### 3. Large-\( \lambda \) : Spin Polarized Phase

For the large-\( \lambda \) case note that successive contributions to the partition functions get suppressed by factors of \( e^{-2\lambda} \). This is because the possible lengths of loops increase in steps of two after the shortest non-trivial length of 4 i.e. \( E_g = 0, 4, 6, 8, 10, \ldots \). Although the number of loops of each length increases algebraically in the number of sites in the lattice, the exponential suppression means that we can consider only the maximally contributing term in a proper limit of \( \lambda \). Thus,

\[
Z(\lambda) = \sum_{g \in G} e^{-\lambda E_g} = 1 + L^2 e^{-4\lambda} + O(e^{-6\lambda}) \approx 1 + L^2 e^{-4\lambda}
\]

The partition function \( w(\lambda, g) \) depends on the particular value of the element \( g \in G \) and hence admits three possibilities:

\[
\begin{align*}
\text{case(1) When } g &= g_A \times g_B \in G_A \times G_B, \\
\text{we have: } w(\lambda, g) &= \sum_{\text{h} \in G_A} e^{-\lambda E_{(\text{h}g_A)(\text{k}g_B)}} \\
&= \sum_{\text{h} \in G_A} e^{-\lambda E_{(\text{h}g)}} \\
&= \sum_{\text{h} \in G_A} e^{-\lambda E_{\text{h}}} \\
&= ( \sum_{\text{h} \in G_A} e^{-\lambda E_{\text{h}}}) \left( \sum_{\text{k} \in G_B} e^{-\lambda E_{\text{k}}} \right) \\
&= (1 + n_A e^{-4\lambda} + O(e^{-6\lambda}))(1 + n_B e^{-4\lambda} + O(e^{-6\lambda})) \\
&\approx 1 + (n_A + n_B) e^{-4\lambda}
\end{align*}
\] (A10)

where \( n_A, n_B \) are respectively the number of independent star operators in \( A \) and \( B \) - the two parts of the bipartition.

\text{case(2) When } g \notin G_A \times G_B \text{ there are two subcategories of such operators.}

\text{case(2a) For } g = A_{\partial A} (g_A \times g_B) \text{ i.e. a product of a single boundary star operator and an element from the subgroup } \text{G_A \times G_B} \text{ the only non-vanishing contribution to } w(\lambda, g) \text{ comes from a loop of length 4 and thus } w(\lambda, g) = e^{-4\lambda}

\text{case(2b) For all other loop operators } g \in G, w(\lambda, g) = 0 \text{ in the limit that we are working in.}

Thus a complete list of \( w(\lambda, g) \) for any \( g \in G \) is as follows:

\[
w(\lambda, g) = \begin{cases} 
1 + n_A e^{-4\lambda} & \forall g \in G_A \times G_B \\
e^{-4\lambda} & \forall g = A_{\partial A} (g_A \times g_B) \\
0 & \text{otherwise}
\end{cases}
\] (A11)

with \( n_{AB} = n_A + n_B \).

At this point let us also evaluate the partition function \( \tilde{Z}(\lambda, \alpha) = \sum_{g \in G} e^{-\lambda E_g} w^{-\alpha^{-1}}(\lambda, g) \). Note that because of the dependence on \( \alpha \) in the different terms of the partition function we get different forms for \( \tilde{Z}(\lambda, \alpha) \) for \( \alpha > 1 \) and \( \alpha < 1 \).

\[
\tilde{Z}(\lambda, \alpha) = \begin{cases} 
1 + n_A e^{-4\lambda} & \text{for } \alpha > 1 \\
1 + n_A e^{-4\lambda} + L_{\partial A} e^{-4\alpha} & \text{for } \alpha < 1
\end{cases}
\] (A12)

where \( L_{\partial A} \) is the length of the boundary of the bipartition.

Now we evaluate the 3 different expectation values of the loop
lengths and find that:

$$\langle E_g \rangle_{Z(\lambda)} = \frac{4L^2 e^{-4\lambda}}{1 + L^2 e^{-4\lambda}}$$

$$\langle E_g \rangle_{\tilde{Z}(\lambda,\alpha)} = \frac{4\alpha L^2 e^{-4\lambda}}{1 + \alpha L^2 e^{-4\lambda}} \text{ for } \alpha > 1$$

$$\langle E_g \rangle_{\tilde{Z}(\lambda,\alpha)} = \frac{4\alpha L^2 e^{-4\lambda}}{1 + \alpha L^2 e^{-4\lambda}} \text{ for } \alpha = 1$$

Using the expressions (A13) in eq.(A4) we get the derivative of the Renyi entropy in the two domains of $\alpha$ to be:

$$\partial_\lambda S_\alpha(\lambda) = \frac{4\alpha e^{-4\lambda}}{1 - \alpha} \left[ \frac{(L^2 - n_{AB}) + (\alpha - 1)n_{AB}L^2 e^{-4\lambda}}{(1 + L^2 e^{-4\lambda})(1 + \alpha n_{AB} e^{-4\lambda})} \right], \quad \alpha > 1$$

$$\partial_\lambda S_\alpha(\lambda) = \frac{4\alpha e^{-4\lambda}}{1 - \alpha} \left[ \frac{(L^2 - n_{AB}) - L_{\theta A} e^{-4\lambda}(\alpha - 1)}{(1 + L^2 e^{-4\lambda})(1 + \alpha n_{AB} e^{-4\lambda} + L_{\theta A} e^{-4\lambda})} \right], \quad \alpha < 1$$

Note that in the above equation for $\alpha > 1$ the numerator is clearly positive for the term in the square bracket whereas the factor $\frac{1}{1 - \alpha}$ provides the overall negative sign. For the $\alpha < 1$ region that the numerator in the square brackets yields a negative sign can be seen as follows:

$$(L^2 - n_{AB}) - L_{\theta A} e^{-4\lambda}(\alpha - 1) < 0$$

$$\implies \frac{1}{1 - \alpha} \log\left( \frac{L^2 - n_{AB}}{L_{\theta A}} \right) < \lambda \quad \implies 0 < \lambda$$

which is always true and where we use the fact that $L^2 - n_{AB} = L_{\theta A}$.

**Appendix B: Calculations for the Toric Code with external field along Horizontal rows**

1. Thin systems

To show how the correspondence of the $\sigma$ and $\tau$ operators, (15), is used we give two examples (the pseudospin operators are the blue stars at the vertices in Fig. (9)):

$$\langle \hat{\sigma}_1^x \rangle = \langle \hat{\tau}_i^x \hat{\tau}_{i+1}^x \rangle$$

$$\langle \hat{\sigma}_2^z \hat{\sigma}_3^z \rangle = \langle \hat{\tau}_i^x \hat{\tau}_j^z \hat{\tau}_j^z \hat{\tau}_{j+1}^x \rangle = \langle \hat{\tau}_i^x \hat{\tau}_{j+1}^x \rangle \langle \hat{\tau}_j^x \rangle \langle \hat{\tau}_{j+1}^x \rangle$$

Using these relations and the fact that $\langle \hat{\tau}_i^x \rangle = 0$ for $i = j$, $\langle \hat{\tau}_i^x \rangle = 0$ for $i = j+1$, $\langle \hat{\tau}_i^x \rangle = \frac{1}{2}$ for $i = j$, and $\langle \hat{\tau}_i^x \rangle = -\frac{1}{2}$ for $i = j+1$, we can further simplify:

$$\langle \hat{\tau}_i^x \hat{\tau}_j^x \rangle = \langle \hat{\tau}_i^x \rangle \langle \hat{\tau}_j^x \rangle = \frac{1}{4}$$

$$\langle \hat{\tau}_i^z \hat{\tau}_j^z \rangle = \langle \hat{\tau}_i^z \rangle \langle \hat{\tau}_j^z \rangle = \frac{1}{2}$$

$$\langle \hat{\tau}_i^0 \hat{\tau}_j^0 \rangle = \langle \hat{\tau}_i^0 \rangle \langle \hat{\tau}_j^0 \rangle = 0$$

Thus we arrive at the operator identity:

$$\langle \hat{\tau}_i^x \hat{\tau}_j^x \rangle \langle \hat{\tau}_i^z \hat{\tau}_j^z \rangle \langle \hat{\tau}_i^0 \hat{\tau}_j^0 \rangle = \frac{1}{4} \cdot \frac{1}{2} \cdot 0 = 0$$

2. General treatment: systems with a bulk

a. The reduced density matrix

A state within the gauge theory of TC model with $B_p = 1 \forall p$ can be written in two different ways:

$$|\psi\rangle = \sum_{g \in G} a(g) g |\psi\rangle$$

$$|\psi\rangle = \sum_{z \in Z} b(z) z |\psi\rangle$$

$$\hat{\sigma}_i^z |\psi\rangle = |\psi\rangle \quad \forall i$$

$$|\psi\rangle = |G|^{-1/2} \sum_{g \in G} g |\psi\rangle$$

where $\langle \psi |$ is the state with all the spins pointing up in the z-basis and $G$ is the group generated by the $N^2 - 1$ independent star operators $A_i$ (or equivalently closed loops of $\hat{\sigma}_i^z$ operators). The ground state of the ground state of the TCM is indicated by $|\psi\rangle$ and $Z$ is the group generated by all the open string operators of the form $\hat{\sigma}_i^z \hat{\sigma}_j^z \cdots \hat{\sigma}_k^z$.

Combining Eqs. (B3) and (B5) we get:

$$|\psi\rangle = |G|^{-1/2} \sum_{z \in Z} b(z) \sum_{g \in G} g |\psi\rangle$$

Note that $g$ is a product of closed loops of $\sigma^z$ operators and $z$ is a string of $\sigma^z$ operators. If we try to commute these two operators we get a negative sign for every spin that is common to both of these strings. Let us introduce the following notation: given $g \in G$ and $z \in Z$ we denote by $g \cap z$ the number of spins that gets acted upon non-trivially by each of these operators. Thus we arrive at the operator identity:

$$gz = zg(-1)^{g \cap z}$$

Using this formula and the fact that $z |\psi\rangle = |\psi\rangle$ Eq. (B6) can be further simplified:

$$|\psi\rangle = |G|^{-1/2} \sum_{z \in Z} b(z) \sum_{g \in G} (-1)^{g \cap z} g |\psi\rangle$$
The density matrix associated with this pure state is given by:

\[ \rho = |G|^{-1} \sum_{z,z' \in \mathbb{Z}} \tilde{b}(z') \tilde{b}(z) \sum_{g,g' \in G} (-1)^{g \otimes z + g' \otimes z'} |x_A x_B \rangle \langle x_A' x_B'| \]

where we have adopted the notation: \( g | \rangle = |x_A x_B \rangle \)

The reduced density matrix of subsystem \( A \) can be obtained by tracing over the spins in \( B \):

\[ \rho_A = |G|^{-1} \sum_{z,z' \in \mathbb{Z}} \tilde{b}(z') \tilde{b}(z) \sum_{g,g' \in G} (-1)^{g \otimes z + g' \otimes z'} \langle x_B'|x_B \rangle_{x_A} \langle x_A'|x_A \rangle_{x_A'} \]

(B9)

Note that \( \langle x_B'|x_B \rangle \) imposes the condition \( g' = g \tilde{g}, \) where \( \tilde{g} \in G \setminus G_B = G_A \) and \( G_A \) is the group of star operations that leave the spin configuration in subsystem \( B \) unchanged.

We have the following formula for the reduced density operator of subsystem \( A \):

\[ \rho_A = |G|^{-1} \sum_{z,z' \in \mathbb{Z}} \tilde{b}(z') \tilde{b}(z) \sum_{g \in G_A} (-1)^{g \otimes z + g \tilde{g} \otimes z'} |g \rangle_{x_A} \langle g \rangle_{x_A} \]

(B10)

An expression for the purity of the subsystem \( A \) follows directly from (B11):

\[ P = |G|^{-2} \sum_{z,z' \in \mathbb{Z}} \sum_{g_1,g_2 \in G} \tilde{b}(z_1) \tilde{b}(z_2) \tilde{b}(z_1') \tilde{b}(z_2') \times (-1)^{g_1 \otimes z_1 + g_2 \otimes z_2 + g_1 \otimes z_1 + g_2 \otimes z_2} |g \rangle_{x_A} \langle g \rangle_{x_A} \]

(B12)

\[ \times \langle g_2 x_2 A | x_1 A \rangle = \langle g_2 x_2 A | x_1 A \rangle = \langle g \tilde{g} x_2 A | x_1 A \rangle = \langle g_2 \tilde{g} x_2 A | x_1 A \rangle \]

(B13)

First we focus on the last term. If a product of \( g \) and \( z \) operators are commuted, the result can be expressed in two different ways. First, one can apply Eq. (B7) to the products themselves, since any product of \( g \)'s and \( z \)'s is another member of the group \( G \) or \( \mathbb{Z} \) respectively.

\[ g_1 \ldots g_k z_1 \ldots z_l = z_1 \ldots z_l g_1 \ldots g_k (-1)^{g_1 \otimes z_1 \ldots z_l} \]

(B15)

However, one can also choose to commute each \( g_i \) and \( z_j \) one at a time, picking a sign \((-1)^{g_i \otimes z_j}\), for each pair. This procedure results in:

\[ g_1 \ldots g_k z_1 \ldots z_l = (-1)^{\Sigma_{i=1}^{k} g_i \otimes z_i} \]

(B16)

This shows that we can manipulate the terms involving powers of \(-1\) by separating them and regrouping back together in different ways. Thus we can rewrite the last summations in Eq. (B14) as:

\[ \sum_{g \in G_B} (-1)^{g \otimes z_1 + g \tilde{g} \otimes z_2} \sum_{g \in G} (-1)^{g \otimes z_1 z_2} \sum_{g \in G} (-1)^{g \otimes z_2} \]

(B17)

First, we work on the term appearing in the first sum above. From Eq. (B3) we have:

\[ \tilde{b}(z_2) = (0 \mid z_2 \mid \psi), \quad \tilde{b}(z_1') = (\psi \mid z_1' \mid 0) \]

(B18)

Using the above formulae, Eq. (B7) and the fact that that \( \tilde{g}_1 \rho \tilde{g}_1 \) gives:

\[ (-1)^{g \otimes z_1 + g \tilde{g} \otimes z_2} \tilde{b}(z_2) = b(z_2 \tilde{g}_1) \tilde{b}(z_1') \]

(B19)

Next, we work on the last two sums in Eq. (B17). Let us consider the following expression:

\[ \sum_{g \in G_R} (-1)^{g \otimes z} \]

(B20)

which is a function of \( z \). Here \( G_R \) is a subgroup of \( G \). Since \( g \otimes z \) appears as the power of \((-1)^{\ast}\) its exact value is not relevant but only its parity:

\[ (-1)^{g \otimes z} = (-1)^{g \otimes z \mod 2} \]

(B21)

Upon inspection one can see that a star \( A_s \) and a string operator \( z \) have overlap on an odd number of spins if the string \( z \), when acting on the ground state, leads to an excitation at the star at \( s \), i.e. \( A_s \otimes |0\rangle = -|z\rangle \otimes |0\rangle \). Then we can think of \( z \) as a set of excitations of stars rather than a string of \( z \) operators as far as \( g \setminus z \mod 2 \) is concerned. In this picture \( g \) and \( z \) live in the same space, i.e. the vertices of the lattice. From now on we will refer to \( g \otimes z \) as the overlap of \( g \) and \( z \), by which we mean the number of stars that are common to \( g = A_1, A_2 \ldots \) and \( z \) identified with the stars at the ends of open strings.

Let \( z_R \) be the restriction of \( z \) to the vertices of the lattice spanned by the stars in \( G_R \). Then \( g \otimes z = g \otimes z_R \). Once phrased in terms of the overlap, the summation in Eq. (B20) becomes a problem of combinatorics:

\[ \sum_{g \otimes z_R \otimes \text{even}} (-1)^{g \otimes z_R \otimes \text{even}} = \sum_{g \otimes z_R \otimes \text{even}} 1 - \sum_{g \otimes z_R \otimes \text{odd}} 1 \]

(B22)

Given \( z, g \otimes z_R \otimes \text{mod 2} \) does only depend whether \( g \) has stars on the vertices where \( z_R \) has excitations. This is because any \( g' \) that differs on other sites from \( g \) will still give the same overlap with \( z_R \). Let assume that \( z_R \) has \( k \neq 0 \) excitations. The sum over \( g \) involves all the combinations of star operators on these \( k \) vertices. There are \( \binom{k}{m} \) elements \( g \in G_R \) that
Finally, from Eq.(B18) and (B3) we have
\[ \sum_{z \in Z} \langle A_z | b(z) B \rangle = \sum_{z \in Z} \langle \psi | A_z | 0 \rangle \langle 0 | z B | \psi \rangle = \langle \psi | A B | \psi \rangle \]
\[ \text{(B26)} \]
where in the last line we used the fact that \( \sum_{z \in Z} z | 0 \rangle \langle 0 | z = 1 \) within the gauge sector we are working in.

Using Eq.(B26) in Eq.(B25) we arrive at our final result:
\[ P = \left| \frac{G_B}{|G|} \sum_{z \in Z_A} \langle \psi | g z | \psi \rangle \langle \psi | z g | \psi \rangle \right|^2 = \left| \frac{G_B}{|G|} \sum_{z \in Z_A} \langle \psi | g z | \psi \rangle \right|^2 \]
\[ \text{(B27)} \]

Using the technique developed here we also obtained the following, more general result:
\[ \text{Tr}_{A}[\rho_A^n] = \left| \frac{G_B}{|G|} \right|^{n-1} \sum_{g_1, \ldots, g_{n-1} \in Z_A} \langle \psi | g_1 z_1 \ldots z_{n-1} | \psi \rangle \langle \psi | g_2 z_1 g_1 | \psi \rangle \ldots \langle \psi | g_{n-1} z_{n-2} g_{n-2} | \psi \rangle \langle \psi | z_{n-1} g_{n-1} | \psi \rangle \]
\[ \text{(B28)} \]

\[ b. \text{ Evaluation of the purity for a system with 2 adjoining stars} \]

Working with the symmetric state considerably eases the analytical calculations as all operators that anticommute with the global spin flip (or parity in the fermionic picture), \( \prod_i \hat{\tau}_i \), have a zero expectation value in the ground state. This implies that many operators in the product: \( g z \) that have an odd number of \( \hat{\tau}_i \) operators in any row, have zero expectation. The expression for purity (B27) involves expectation values of operators in the \( \sigma \)-picture. Our strategy is to calculate the product of operators appearing in Eq. (B27) by separating the different contributions based on the number of star operators

| subtype | \( E \) | \( F \) |
|---------|------|------|
| a | 0 | 0 | \( \mathbb{I} \times \{ \mathbb{I}, \sigma_{1-2}, \sigma_{2-3}, \sigma_{1-2}^{\dagger} \sigma_{2-3}^{\dagger}, \sigma_{1-2} \sigma_{2-3} \} \) |
| b | 0 | 0 | \( \mathbb{I} \times \{ \mathbb{I}, \tau_1, \tau_2, \tau_3 \} \) |
| c | 0 | 0 | \( \mathbb{I} \times \{ \mathbb{I}, \tau_1, \tau_2, \tau_3 \} \) |
| d | 0 | 0 | \( \mathbb{I} \times \{ \mathbb{I}, \tau_1, \tau_2, \tau_3, \tau_4 \} \) |

TABLE 1. All possible priori non-zero operators of type \( i \) arranged into 4 subtypes. For each subtype the first row gives the operators in the \( \sigma \)-picture and the corresponding operator in the \( \tau \)-picture appears in the second row. An entry of 1 against \( E \) means that both \( \hat{\sigma}_{1-2} \) and \( \hat{\sigma}_{2-3} \) appear as factors in the operator product \( z \in \mathcal{Z}_A \).
FIG. 10. (color online) Subsystem $A$ with a bulk has spins comprised of those that form two adjoining stars (green ovals). The pseudospins (blue stars) that appear in the calculation are labelled by the numbers $1, 2, 3, 4, 5, 6, 7, 8$. in the product. Schematically we represent this as:

$$
\text{type i} \quad \quad \quad \quad \quad \quad \quad \text{type ii} \quad \quad \quad \quad \quad \quad \quad \text{type iii}
\begin{align*}
&\text{All operators of the form } g_z = \text{operator products with no stars} \\
&+ \text{operator products with only 1 star} \\
&+ \text{operator products with both stars}
\end{align*}
$$

(B29)

Now we collect all terms of type $i$ as follows. From Fig. (10) we find that only those operators which have either both or none of the $\hat{\sigma}_z$ on edges between vertices labelled $(5-2), (6-3)$ in the product contribute. Similarly only those operators that have the product of both or none of $\hat{\sigma}_z$ on edges between vertices labelled $(2-7), (3-8)$ contribute. However all possible products of $\hat{\sigma}_z$ on the row of spins labelled $D$ in the same figure are apriori non zero. This means that out of a total of $2^7$ operators of the type $i$ - we need to consider only those that have products of both the $\hat{\sigma}_z$‘s in the oval marked $E$ or both the $\hat{\sigma}_z$‘s in the oval marked $F$ as factors as shown in Fig. (10). However all possible products of $\sigma_z$‘s along the row marked $A$ in the same figure are apriori non-zero. This means that we need to consider a total number of $2^2 \times 2^3$ operators of $\sigma_z$ where the factor $2^2$ comes from the fact that $E, F$ can be turned on/none present in 4 different ways (subtypes) for each of the $2^3$ operator products of $\sigma_z$‘s along row $D$. We can then write down a table corresponding to the possible operators we need to calculate expectation values for, in the $\tau$-picture by using the map (15). For eg. $\hat{\sigma}_{z-2}$ which is an operator on the spin on the edge connecting vertices 1, 2 is mapped to the product $\hat{\tau}_1^{z} \hat{\tau}_2^{z}$. The table(I) tabulates the operators in both the $\sigma$ and $\tau$ pictures. Note that operators of each of the 4 subtypes are products of elements from the group of operators labelled $Q$ which are products of $\hat{\tau}_z$‘s only along row $A$ and depending on whether $E$ or $F$ is turned on - product of $\hat{\tau}_5 \hat{\tau}_6$ or/and $\hat{\tau}_3 \hat{\tau}_4$ on rows $B$ and $C$. Because operators of each subtype factorize into operators from the group $Q$, which belong to one particular row, and other operators on adjacent rows, we need to evaluate only 8 correlation functions to determine all expectation values of operators of type $i$.

One can similarly tabulate all operators of type $ii$ and type $iii$ in the $\sigma$ and $\tau$ representations, which we omit here for the sake of brevity, and evaluate the sum of expectation values in Eq. (B27) leading to Eq. (21).
[1] N. Goldenfeld, *Lectures on phase transitions and the renormalization group* (Addison Wesley, New York, 1992).

[2] S. Yan, D. Huse, and S. White, *Science*, 332, 1173, (2011).

[3] M. Hasan and C. Kane, *Rev. Mod. Phys.* 82, 3045-3067 (2010).

[4] X. G. Wen, *Quantum field theory of many body systems* (Oxford university press, 2004).

[5] S. Flammia, A. Hamma, T. Hughes, and X.-G. Wen, *Phys. Rev. Lett.* 103, 261601 (2009).

[6] H. Briegel and R. Raussendorf, *Phys. Rev. Lett.* 86, 910-913 (2001).

[7] M. H. Freedman, A. Kitaev, and Z. Wang, *Commun. Math. Phys.* 227, 587-603 (2002).

[8] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, *Rev. Mod. Phys.* 80, 10831159 (2008).

[9] H. Stormer, *Rev. Mod. Phys.* 71, 875-889 (1999).

[10] A. Hamma, R. Ionicioiu, and P. Zanardi, *Phys. Lett. A* 337, 22-28 (2005).

[11] A. Hamma, R. Ionicioiu, and P. Zanardi, *Phys. Rev. A* 71, 022315 (2005).

[12] A. Y. Kitaev and J. Preskill, *Phys. Rev. Lett.* 96, 110404, 2006.

[13] M. Levin and X.-G. Wen, *Phys. Rev. Lett.* 96, 110405 (2006).

[14] Z.-G. Gu and X. Wen, *Phys. Rev. B* 80, 155131 (2009).

[15] X.-G. Wen, *Advances in Physics, Vol. 44, 5* (1995).

[16] A. Y. Kitaev, *Ann. Phys. (N.Y.)* 303, 1, 2-30 (2003).

[17] X.-G. Wen and Q. Niu, *Phys. Rev. B* 41, 9377-9396 (1990).

[18] X. Chen, Z. Gu, and X. Wen, *Phys. Rev. B* 82, 155138 (2010).

[19] E. Fradkin, *Field theories of condensed matter physics* (Cambridge, 2013).

[20] A. Hamma, L. Cincio, S. Santra, P. Zanardi, and L. Amico, *Phys. Rev. Lett.* 110, 210602 (2013).

[21] J. Cui, L. Amico, H. Fan, M. Gu, A. Hamma, and V. Vedral, *Phys. Rev. B* 88, 125117 (2013).

[22] S. Dusuel, M. Kamfor, R. Orús, K. P. Schmidt, and J. Vidal, *Phys. Rev. Lett.* 103, 261601 (2009).

[23] Y. R. Sanders and G. Gour, *Phys. Rev. A* 79, 054302 (2009).

[24] A. Hamma, L. Amico, H. Fan, M. Gu, A. Hamma, and V. Vedral, *Nature commun.* 3, 812 (2012).

[25] J. Cui, M. Gu, L. Kwek, M. Santos, H. Fan, and V. Vedral, *Quantum Hall effects* (Wiley, New York, 1997 Eds.).

[26] T. Chakraborty and P. Pietiläinen, *The Quantum Hall effects* (Springer Series in Solid-State Sciences Volume 85, 1995, pp 32-38).

[27] S. Papanikolaou, K. Raman, and E. Fradkin, *Phys. Rev. B* 76, 224421 (2007).

[28] M. Kamfor, S. Dusuel, J. Vidal, K. P. Schmidt, arXiv:1308.6150.

[29] A. Hamma, Radu Ionicioiu, Paolo Zanardi, *Quantum Information Processing*, 169.

[30] Yirun Arthur Lee, Guifre Vidal, arXiv:1306.5711.

[31] Claudio Castelnovo, arXiv:1306.4990.

[32] A. Hamma et al. *in preparation*.

[33] D.I. Tsomokos, A. Hamma, W. Zhang, S. Haas, R. Fazio, *Phys. Rev. A* 80, 060302(R) (2009).

[34] M.B. Hastings, *Phys. Rev. Lett.* 107, 210501 (2011).

[35] D. Mazac and A. Hamma, *Ann. Phys.* 327, 2096 (2012).

[36] M. D. Schulz, S. Dusuel, R. Orús, J. Vidal, K. P. Schmidt, *New J. Phys.* 14, 025005 (2012).

[37] Tarkan Grover, Ari M. Turner, Ashvin Vishwanath, *Phys. Rev. B* 84, 195120 (2011).

[38] J. Ignacio Cirac, Didier Poilblanc, Norbert Schuch, Frank Ver...
[77] S.V. Isakov, P. Fendley, A.W.W. Ludwig, S. Trebst, M. Troyer, Phys. Rev. B 83, 125114 (2011)
[78] K. Gregor, David A. Huse, R. Moessner, S. L. Sondhi, New J. Phys. 13:025009 (2011)
[79] Hao Wang, B. Bauer, M. Troyer, V. W. Scarola, Phys. Rev. B 83, 115119 (2011)
[80] Armin Rahmani, Claudio Chamon, Phys. Rev. B 82, 134303 (2010)
[81] I. Klich, Annals of Physics, Volume 325, Issue 10, p. 2120-2131 (2010)
[82] S. Bravyi, M. B. Hastings, Commun. Math. Phys. 307, 609 (2011)
[83] Sergey Bravyi, Matthew Hastings, Spyridon Michalakis, J. Math. Phys. 51 093512 (2010)
[84] Zohar Nussinov and Gerardo Ortiz, Annals of Physics, Volume 324, Issue 5, May 2009, Pages 977-1057
[85] Zohar Nussinov and Gerardo Ortiz, Phys. Rev. B 77, 064302 (2008)