Strong necessary conditions and Cauchy problem

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Abstract

Some exact solutions of boundary or initial conditions formulated for Bogomolny equations (derived by using the strong necessary conditions and associated with some ordinary equation and some partial differential equations), have been found. Besides, a degeneracy of the hamiltonian for the restricted baby Skyrme model has been established.

keywords: action integral, Bogomolny equations, Bogomol’nyi equations, boundary and initial conditions, Cauchy problem, degeneracy of hamiltonian, degeneracy of energy, energy degeneracy, strong necessary conditions

1 Introduction

There are several approaches to solving of nonlinear partial differential equations (see for e.g.: [3], [6], [7], [10], [12], [15], [16], [17], [19], [20], [21], [22], [23], [24], [25], [28], [29], [30], [31], [27], [32], [33], [34], [35], [36], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [50], [51], [52], [53], [54]. In 2001 Professor Bolesław Szafirski has pointed out that it is unknown, how to implement boundary and initial conditions as well as how to set the Cauchy problem in SNCM.

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This paper shows a way for satisfying His requirement. In the section 2 we present briefly the concept of strong necessary conditions. The next section is devoted to solving Cauchy problem associated with Bogomolny equation (which is an ordinary differential equation) of 1-dimensional harmonic oscillator. In the section 4 we present solutions of Cauchy problem associated with Bogomolny equations for continous Heisenberg model and restricted baby Skyrme model. The section 5 includes some conclusions.

2 A short presentation of the method

The main idea of the concept of strong necessary conditions \([39, 40, 41, 43, 44, 45, 46]\), is such that instead of considering the Euler-Lagrange equations,

$$
F_{,u} - \frac{d}{dx}F_{,u,x} - \frac{d}{dy}F_{,u,y} = 0,
$$

which follow from the varying of the functional

$$
\Phi[u] = \int_{E^2} F(u, u_x, u_y) \, dx \, dy,
$$

one considers strong necessary conditions, \([39, 40, 41, 43, 44, 45, 46]\):

$$
F_{,u} = 0,
F_{,u,x} = 0,
F_{,u,y} = 0,
$$

where \(F_{,u} = \frac{\partial F}{\partial u}\), etc.

Obviously, the set of the solutions of the system of the equations \([3, 4, 5]\), is a subset of the set of the solutions satisfying the Euler-Lagrange equation \([1]\). On the other hand, even if this subset is non-empty, its elements (solutions of the system \([3, 4, 5]\)), are very often trivial solutions. So, in order to extend this subset, we consider the gauged functional \([2]\)

$$
\tilde{\Phi} = \Phi + I,
$$

where \(I\) is such functional that its local variation vanishes, with respect to \(u(x, y)\): \(\delta I = 0\).

Owing to this feature, the Euler-Lagrange equations \([1]\) possess the same form as the Euler-Lagrange equations resulting from requiring of the extremum of \([0]\). On the other hand, the equations following from applying of the strong necessary conditions \([3, 4, 5]\), to \([4]\), do not possess the same form as the equations following from applying the strong necessary conditions \([3, 4, 5]\), to \([2]\). Hence, there is an opportunity to obtain non-trivial solutions. Let us note that the order of the system of the partial differential equations, constituted by strong necessary conditions \([3, 4, 5]\), is less than the order of Euler-Lagrange equations.
The method of derivation Bogomolny equations (Bogomolny decomposition), by using the strong necessary conditions, was included and applied in [39], [42], [47], and developed in [2]. As we see, this approach differs from the approach of deriving of Bogomolny equations, presented in [8], [9], [11], [14], [26]. In [50], the Bogomolny equations for baby Skyrme models, were derived, by using the concept of strong necessary conditions. The idea that the lagrangian after adding to it a total derivative of a function dependent only on field variable, generates the same Euler-Lagrange equations as the original lagrangian, had been known very well in the literature (c.f. for e.g. [5], [13]). However, gauging the lagrangian on a complete set of invariants and applying this to derivation of Bogomolny equations, by using the concept of strong necessary conditions, was firstly presented just in [39].

3 Ordinary Differential Equations

In this section we present application of SNCM in an initial conditions problem. As an introductory example we consider linear equation resulting from the SNCM applied to Lagrangian of the one dimensional harmonic oscillator:

$$L = \frac{m}{2} \left( \left( \frac{dx}{dt} \right)^2 - \omega^2 x^2 \right). \quad (7)$$

In order to set the strong necessary conditions we perform the gauge transformation of (7) using the following topological invariant density $G(x) \frac{dx}{dt}$, where $G(x)$ is an arbitrary function and $G(x) \in C^1$:

$$\tilde{L} = \frac{m}{2} \left( \left( \frac{dx}{dt} \right)^2 - \omega^2 x^2 \right) + G(x) \frac{dx}{dt}. \quad (8)$$

Note that $L$ depends on the two functions $L = L(x, \frac{dx}{dt})$. According to the strong necessary conditions we have to optimize the action functional, by regarding both, $x$ and $\frac{dx}{dt}$:

$$\frac{\partial \tilde{L}}{\partial x} = 0, \quad \frac{\partial \tilde{L}}{\partial (\frac{dx}{dt})} = 0. \quad (9)$$

Equations (9) read:

$$-m\omega^2 x + G_x \frac{dx}{dt} = 0, \quad (10)$$

$$G + m \frac{dx}{dt} = 0. \quad (11)$$

We eliminate $\frac{dx}{dt}$, from this system, and we get the equation, which has to be satisfied by the function $G$:

$$GG_x + m^2 \omega^2 x = 0. \quad (12)$$
Hence
\[
\frac{1}{2} (G^2),_x + m^2 \omega^2 x = 0. \tag{13}
\]
The solution of this equation has the form
\[
G = \pm \sqrt{c_1 - m^2 \omega^2 x^2}. \tag{14}
\]
Then, we formulate Cauchy problem
\[
-m\omega^2 x + G_x \frac{dx}{dt} = 0, \tag{15}
\]
\[
G + m \frac{dx}{dt} = 0, \tag{16}
\]
\[
x(0) = c_3. \tag{17}
\]
Solving (15) - (16), provided that (14), where we take into account "plus" sign, we get
\[
x(t) = \sqrt{c_1} \tan (\omega (c_2 - t)) \sqrt{\frac{m \omega c_3}{\sqrt{c_1 - c_2^2 \omega^2 m^2}}}, \tag{18}
\]
where \(c_2\) is the integration constant. Now we take into account (17), hence
\[
c_2 = \frac{1}{\omega} \arctan \frac{m \omega c_3}{\sqrt{c_1 - c_2^2 \omega^2 m^2}}. \tag{19}
\]
If we take into account the Euler-Lagrange equations for this problem
\[
m \frac{d^2 x(t)}{dt^2} + m \omega^2 x(t) = 0, \tag{20}
\]
them its solution is
\[
x(t) = A \sin (\omega t) + B \cos (\omega t), \tag{21}
\]
where \(A = \text{const}, B = \text{const}\), and this does not satisfy the Bogomolny equations (15) - (16), where \(G\) is given by (14). Obviously, the solution of Bogomolny equations, given by (18), with and without providing that (19), satisfies (20).
4 Partial Differential Equations

4.1 Field Equations and Cauchy problem associated with \( \pi_2(S^2) \) homotopy group

As an example we consider the continuous Heisenberg model represented by the following Hamiltonian, [8]:

\[
H = \int_{E^2} \left( \frac{\nabla w \cdot \nabla w^*}{(1 + w \cdot w^*)^2} + I_1 \right) dx dy,
\]

(22)

where the field variable \( w \) consists of classical spin components:

\[
w = \frac{(S^x + iS^y)}{(1 + S^z)}
\]

(23)

where \( S^\alpha \) are components of the classical spin. \( I_1 \) is density of the topological invariant:

\[
I_1 = G_1(w, w^*)(w, w^* - w, w^*) - G_2(w, w^*) - G_3(w, w^*) + G_4(w, w^*) = 0.
\]

(24)

We apply the strong necessary conditions to (22) and we obtain the system of dual equations, which can be also obtained as a two-dimensional version of the system of the dual equations derived in [42]:

\[
- \frac{2w^* \nabla w \nabla w^*}{(1 + w w^*)^3} + G_{1,w}(w, x w_y^* - w, y w_x^*) + D_x G_{1,w}(w, w^*) + D_y G_{2,w}(w, w^*) = 0,
\]

(25)

\[
\text{c.c.,}
\]

(26)

\[
\frac{w_x^*}{(1 + w w^*)^2} + G_{1,w}^* + G_{2,w} = 0,
\]

(27)

\[
\frac{w_y^*}{(1 + w w^*)^2} - G_{1,w}^* + G_{3,w} = 0,
\]

(28)

\[
\text{c.c.}
\]

(29)

We make this system self-consistent by choosing \( G_n = \text{const} \ (n = 2, 3) \) and \( G_1 = \frac{i}{(1 + w w^*)^2} \). Next, expressing the complex fields \( w \) and \( w^* \) by real fields:

\[
w = U(x, y) + iV(x, y), \quad w^* = U(x, y) - iV(x, y),
\]

(30)

we derive from (27) - (29), the pair of equations, governing real fields \( V(x, y) \) and \( U(x, y) \):

\[
\frac{\partial}{\partial y} V(x, y) - \frac{\partial}{\partial x} U(x, y) = 0.
\]

(31)

\[
\frac{\partial}{\partial x} V(x, y) + \frac{\partial}{\partial y} U(x, y) = 0
\]

(32)
Solving (31) and (32) we get:

\[ U(x, y) = F_1(y - ix) + F_2(y + ix), \]  
\[ V(x, y) = -iF_1(y - ix) + iF_2(y + ix) + C_1, \]

where \( F_1(\cdot) \) and \( F_2(\cdot) \) are some functions. After taking into account the formula (30), we obtain that \( F_1, F_2 \) are connected with \( w, w^* \), by the formulas

\[ F_1 = \frac{1}{2}(w - iC_1), \]
\[ F_2 = \frac{1}{2}(w^* + iC_1) \]
and \( C_1 \) is an arbitrary real constant.

Basing on the general solutions (33), (34) of (31), (32) we present the Cauchy problem for partial differential equations of the first order created by the strong necessary conditions. The considered example consists of two independent variables \( x \) and \( y \) and two functions. Therefore it is possible to formulate the following constrains for the general solutions:

\[ U(x, 0) = f_1(x), \quad V(x, 0) = f_2(x), \]

where \( f_1(x) \) and \( f_2(x) \) are given functions.

It is possible for the considered Heisenberg model to derive analogous relations to \( U(0, y) \) and \( V(0, y) \), which relate integration constants to initial or boundary conditions. Constraining (33) and (34) to (37) and substituting \( y = 0 \) we obtain:

\[ f_1(x) = F_1(-ix) + F_2(ix), \]
\[ f_2(x) = -iF_1(-ix) + iF_2(ix) + C_1, \]

Since \( f_1(x) \) and \( f_2(x) \) are given therefore \( F_1 \) and \( F_2 \) can’t be arbitrary:

\[ F_1(-ix) = if_2(x) + f_1(x)/2 + f_2(x)/2 - i/2 C_1 \]
\[ F_2(-ix) = \frac{if_1(x) + f_2(x) - C_1}{2i}. \]

Therefore, the only freedom for \( F_1 \) and \( F_2 \) is gauge transformation regarding \( C_1 \) constant. This full solution can be extended by applying semi-strong necessary conditions concept (this concept was presented in [43]).

5 Field Equations and Cauchy problem for the restricted baby Skyrme model

The restricted baby Skyrme model has the following Hamiltonian
\[ H = -4\beta \frac{(\omega_x \omega^*_y - \omega_y \omega^*_x)^2}{(1 + \omega \omega^*)^4} + V(\omega, \omega^*), \]  

(42)

In \[50\], the Bogomolny decomposition for this model, was derived by using the concept of strong necessary conditions (the Bogomolny equations for this model, but for some special forms of the potential, and by another way, and some solutions of these equations, were derived in \[1\]). We apply this concept to the hamiltonian gauged on the invariants is as follows, \[50\]

\[ \tilde{H} = -4\beta \frac{(\omega_x \omega^*_y - \omega_y \omega^*_x)^2}{(1 + \omega \omega^*)^4} + V(\omega, \omega^*) + G_1(\omega_x \omega^*_y - \omega_y \omega^*_x) + D_x G_2 + D_y G_3, \]  

(43)

where \( G_i, (i = 1, 2, 3) \) are some unspecified functions of \( \omega, \omega^* \) (of course, \( G_i \in \mathcal{C} \)). If \( G_1 = \frac{4i\sqrt{\beta}}{(1 + \omega \omega^*)^2} \sqrt{V(\omega, \omega^*)}, \ G_k = const, (k = 2, 3), \)  

we can derive the Bogomolny decomposition, which in this case, has the following form, \[50\]

\[ \omega_x \omega^*_y - \omega_y \omega^*_x = \frac{i}{2\sqrt{\beta}} \sqrt{V(\omega, \omega^*)}(1 + \omega \omega^*)^2. \]  

(45)

We find now an exact localized solution (with localised density of energy), of the Bogomolny decomposition (45), for the case of the so-called, "Mexican hat" potential: \( V = \lambda^3 (\omega \omega^* - \gamma^2)^2 \). We use "hedgehog ansatz":

\[ \omega = \frac{\sin (f(r)) \cos (N\theta) + i \sin (f(r)) \sin (N\theta)}{1 + \cos (f(r))}, \text{c.c.,} \]  

(46)

where \((r, \theta)\) are polar coordinates in the cartesian \( x - y \) plane.

After inserting this ansatz into (45), we formulate the Cauchy problem

\[ \frac{(\cos (f(r)) + 1)Nf'(r) \sin (f(r))}{r} = \sqrt{\frac{\lambda^3}{\beta} \left[ \cos (f(r)) (\gamma^2 + 1) + \gamma^2 - 1 \right]}, \]  

(47)

\[ f(0) = c_0 = const, \]  

(48)

where in this case we put \( c_0 = 2 \).

We are interested in obtaining some localized solution, then we impose also the conditions:

\[ \lim_{r \to \pm \infty} f(r) = const, \]  

(49)

\[ \lim_{r \to \pm \infty} \tilde{H} = const, \]  

(50)

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We solve this problem and we have

$$f(r) = \pi - \arccos \left( X_1 \right), \quad (51)$$

where

$$X_1 = \frac{1}{\gamma^2 + 1} \left( \gamma^2 - \exp \left( \frac{1}{\sqrt{\beta} N} \left( - 4 \text{Lambert} \left( \frac{1}{2} \exp \left( \frac{1}{N \sqrt{\beta}} X_2 \right) \right) \times N \sqrt{\beta} + X_2 \right) \right) - 1 \right), \quad (52)$$

$$\exp \left( \frac{1}{\sqrt{\beta} N} \left( - 4 \text{Lambert} \left( \frac{1}{2} \exp \left( \frac{1}{N \sqrt{\beta}} X_2 \right) \right) \times N \sqrt{\beta} + X_2 \right) \right) - 1 \right), \quad (53)$$

$$X_2 = \frac{N \ln \left( (\gamma^2 + 1) \cos (2) + \gamma^2 - 1 \right)}{\sqrt{\beta} + 2 \left( \frac{N (\gamma^2 + 1)(\cos (2) + \gamma^2 - 1)\sqrt{\beta} + \sqrt{\lambda^2 \gamma^2 (\gamma^2 + 1)^2}}{4} \right)}, \quad (54)$$

Lambert(Y) is the so-called Lambert function, which satisfies the equation

$$\text{Lambert}(Y) \exp(\text{Lambert}(Y)) = Y.$$  

For $\gamma = 2, N = 1, \lambda_3 = 1, \beta = 1$:

$$f(r) = \arccos \left\{ \frac{2 \text{Lambert} \left( \frac{1}{2} \exp \left( \frac{\lambda^2 \gamma^2 (\gamma^2 + 1)^2}{4} \right) (5 \cos (2) + 3) \right) - 3}{5} \right\}. \quad (55)$$

We present a figure of this above solution on FIG 1.

Now, it has turned out that if we insert the found solution of Cauchy problem into the ungauged and gauged hamiltonian densities (42), (43), correspondingly, then ungauged hamiltonian density is nonzero and its figure is on FIG 2.

The gauged hamiltonian density is zero (of course, the condition (44) and Bogomolny equations (45) hold):

$$\tilde{H} = 0. \quad (56)$$

Thus, we can tell here about a degenerate hamiltonian, (the problem of degenerate hamiltonian in the case of theory of gravity, was investigated in [37]; there in [18] was proven the existence of an infinite number of Lagrangians for a given second-order ODE). It corresponds to the fact that if we consider two versions of a field-theoretical lagrangian: ungauged and gauged on total derivatives of any function of field variables, then energy-momentum tensors corresponding to each of these lagrangians, will be different. [4].
6 Conclusions

The first conclusion concerns just possibility to solve the ordinary differential equations subjected to the strong necessary conditions. In the case of linear ODE, the solution of Cauchy problem is given by (18) - (19).

The formulas (31) and (32) establish Cauchy-Riemann system, which is a start point for the theory of analytic functions. Hence, because of Riemann theorem, this may be a step to the investigations of conformal maps.

Moreover, as far as the Cauchy problems for Heisenberg model and for restricted baby Skyrme model, are concerned, after using of strong necessary conditions and deriving Bogomolny equation for this problem, one can formulate Cauchy problem and solve it.
We have also obtained some localized solution of Cauchy problem of Bogomolny equation for restricted baby Skyrme model. An example of such solution is given by (57). Besides, we have also showed on the example of restricted baby Skyrme model, that there exists degeneracy of hamiltonian, i.e. the values of hamiltonians: ungauged and gauged one, are different for the solution of Cauchy problem for Bogomolny equations, and these both hamiltonians generate the same Euler-Lagrange equations.

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