The interplay between flattening and damping of single particle spectra in strongly correlated Fermi systems

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Abstract

The self-consistent theory of the fermion condensation, a specific phase transition which results in a rearrangement of the single particle degrees of freedom in strongly correlated Fermi systems is developed. Beyond the phase transition point, the single particle spectra are shown to be flat. The interplay between the flattening and the damping of the single particle spectra at $T \to 0$ is investigated. The width $\gamma(\varepsilon)$ of the single particle states is found to grow up linearly with $\varepsilon$ over a wide range of energy as in a marginal Fermi liquid. Our results gain insight into the success of the phenomenological theory of the normal states of high-temperature superconductors by Varma et al.

1. Introduction

In the article [1] within the Landau-Migdal (LM) quasiparticle pattern, a new phase transition in strongly correlated Fermi systems, the so called fermion condensation, has been uncovered. Its salient feature is in the appearance of the fermion condensate (FC), a group of degenerate states whose energy at $T = 0$ coincides with the chemical potential $\mu$. Owing to the degeneracy, the quasiparticle occupation numbers $n(p, T = 0)$ are no longer 1 or 0. They are determined by a variational condition [1, 2, 3]

$$\delta E_0(n)/\delta n(p) = \mu, \quad p \in \Omega, \quad (1)$$

where $E_0(n(p))$ stands for the ground state energy, while $\Omega$ denotes the FC region whose boundaries are determined by eq. (1) itself. According to Landau, the l.h.s. of (1) is nothing but the quasiparticle energy $\varepsilon(p)$ and therefore this equation implies smearing the Fermi surface — its metamorphosis it into a volume in the three-dimensional systems or into a surface in the two-dimensional ones. In anisotropic systems, say, in crystals, the FC may occupy only part of the old Fermi surface, e.g. a region, adjacent to the van Hove singularities [4, 5].

At $T > 0$ the quasiparticle distribution $n(p, T)$ is found with the help of the usual Fermi-liquid-theory variational equation $\delta F/\delta n(p) = 0$ where $F = E_0 - \mu N - TS$ is the free energy. The solution is written in the standard Fermi-Dirac form

$$n(p, T) = \left[ 1 + \exp \left( \frac{\xi(p,n)}{T} \right) \right]^{-1}, \quad (2)$$

but the energy $\xi(p) = \varepsilon(p) - \mu$ depends on the distribution $n$ itself. For both equations (1), (2) to be consistent, the spectrum $\xi(p)$ should exhibit a strong $T$-dependence in the FC region

$$\xi_{FC}(p, T) \simeq T \ln \frac{1 - n_0(p)}{n_0(p)}, \quad p \in \Omega, \quad (3)$$
where \(n_0(p)\) is the solution of eq. (1). We see that the plateau in the spectrum \(\xi(p)\) existing at \(T = 0\) is converted into an inclined plane with the slope proportional to \(T\). Such a significant liability of single particle characteristics with respect to \(T\) is a calling card of fermion condensation.

The flattening of the single particle spectra \(\xi(p)\) has been experimentally observed in the ARPES study of electron systems of some high-\(T_c\) superconductors [6, 7, 8] but observable single particle peaks have much larger widths \(\gamma\) than those calculated employing Fermi liquid theory. As a result, the applicability of the quasiparticle formalism to such systems has been put in question and alternative explanations have been proposed [9, 10].

As known in Landau theory, the single particle spectrum \(\xi(p)\) is described by the only parameter — the effective mass \(M^*\), characterizing the slope \((d\xi(p)/dp)_F \sim (M^*)^{-1}\) of the spectrum at the Fermi surface. Usually the effective mass is treated as a \(T\)-independent quantity. But, as seen from (3), beyond the FC phase transition point, the slope \(d\xi_{FC}/dp\) is just proportional to \(T\), i.e. for the FC states \(M^* \sim T^{-1}\). This peculiarity of the fermion condensation alters the Landau theory predictions even if the quasiparticle picture survives in the systems with the FC. However, the ratio \(\gamma_{FC}(T)/\xi_{FC}(p, T)\) calculated within the Landau approach [11, 12, 13] does not vanish at \(T \to 0\) in contrast to fundamentals of the Fermi liquid theory. In other words, at finite \(T\), the theory of the fermion condensation based on the LM quasiparticle concept suffers from inconsistency.

On the other hand, at \(T = 0\), the quasiparticle picture is known to hold as long as the critical index \(\nu\) characterizing behavior of the imaginary part of the mass operator \(\text{Im } \Sigma(p, \varepsilon) \sim \varepsilon^\nu\), exceeds 1 (here \(\varepsilon\) is measured from the chemical potential \(\mu\)). This conclusion stems from the formula for the renormalization constant \(z\) given by

\[
z = \left[1 - \text{Re} \left(\frac{\partial \Sigma(p_F, \varepsilon)}{\partial \varepsilon} \right)_{\varepsilon=0}\right]^{-1}.
\]

Straightforward calculations with taking into account the well known dispersion relation [14, 15]

\[
\text{Re} \left(\Sigma(p, \varepsilon) - \Sigma(p, 0)\right) = \frac{1}{\pi} P \int \frac{\text{Im } \Sigma(p, \varepsilon_1) \text{sgn } \varepsilon_1}{\varepsilon_1 (\varepsilon_1 - \varepsilon)} d\varepsilon_1,
\]

demonstrate that the factor \(z\) vanishes provided \(\nu \leq 1\). Thus if \(\nu \leq 1\) the conventional quasiparticle picture is destroyed. We shall see that this just occurs beyond the point of the fermion condensation.

The interplay between damping and flattening beyond this point is quite subtle. Indeed, let the solution with the FC, i.e. with the plateau in \(\xi(p)\) at \(T = 0\), exist neglecting damping effects. When damping effects get involved the plateau in \(\xi(p)\) should somehow change. If the flat portion of \(\xi(p)\) were completely destroyed then the spectrum \(\xi(p)\) could be characterized by the single constant \(M^*\) that results in reviving Fermi liquid theory and the FC solution, as well. The way out is that the flat portion in the single particle spectrum survives but flattening and damping should somehow counterbalance each other.

In this article, we develop the self-consistent theory of the fermion condensation proceeding from the standard many-body-theory approach and bypassing the variational condition (1) which is not adapted to take damping effects into account. The first step is to derive
a general relation between the spectrum $\xi(p)$ and the distribution $n(p)$ assuming the LM quasiparticle pattern to be valid. The second step is to verify that this relation considered as an equation has a class of peculiar solutions for which $\xi(n(p)) = 0$ in a whole region $\Omega$ of the momentum space rather than at several isolated points. These are just the FC solutions. However, as we shall see, the LM quasiparticle picture is nothing more than a reasonable approximation for the treatment of the systems with the FC. These systems do possess features inherent in marginal Fermi liquids for which the quasiparticle weight in the single particle state adjacent to the Fermi surface approaches zero. The next step is to extend the Landau formalism to marginal Fermi liquids including the derivation of the relation between the single particle spectrum and the distribution function. Then one needs to calculate irregular contributions to the single particle mass operator beyond the FC phase transition point and evaluate how strong the plateau in the spectrum of the single particle excitations is inclined due to the damping.

2. The fermion condensation in the quasiparticle approximation

We already mentioned above that the systems with the FC possess the properties inherent in marginal Fermi liquids. In particular, the single particle mass operator $\Sigma(p, \varepsilon)$ contains irregular terms precluding expansion in a Taylor series in the energy $\varepsilon$. However, these terms rapidly die out if $\varepsilon$ exceeds a characteristic FC energy $\varepsilon_{FC}$. Therefore the quasiparticle picture seems to be a reasonable first approximation for studying the energy dependence of quantities of our interest.

In the system with the FC, the quasiparticle Green function $G^q$ is found by retaining leading terms in the expansion of the inverse Green function $G^{-1}$ in the limit $\varepsilon \to 0$. This expansion reads as

$$G^{-1}(p, \varepsilon \to 0) = \varepsilon - \xi^0_p - \Sigma(p, \varepsilon=0) - \varepsilon \left( \frac{\partial \Sigma(p_c, \varepsilon)}{\partial \varepsilon} \right)_0,$$

(6)

where $\xi^0_p = p^2/2M - \mu$. Here we neglected the variation of the derivative $(\partial \Sigma(p_c, \varepsilon)/\partial \varepsilon)_0$ in the FC region and calculated this derivative at a fixed point $p_c \in \Omega$.

The pole term $G^q$ has the accustomed form

$$G^q(p, \varepsilon) = \left( \varepsilon - \xi(p) \right)^{-1}.$$

However, the quasiparticle spectrum $\xi(p)$ near the Fermi surface is given by a different formula

$$\xi(p) = z (\xi^0_p + \Sigma(p, \varepsilon=0)),$$

(7)

than in the Landau theory of normal Fermi liquid since the momentum expansion with retaining terms of the type $(p^2 - p^2_F)(\partial \Sigma/\partial p^2)_F$ makes no sense in the system with the FC.

The quasiparticle momentum distribution is evaluated as usual

$$n(\xi(p)) = \text{Im} \int G^q(p, \varepsilon) \frac{d\varepsilon}{2\pi i},$$

(8)
yielding the step function $\theta(\xi(p))$.

Another relation between $\xi$ and $n$ is found proceeding from the Landau-Pitaevskii identity

$$\frac{\partial G^{-1}(p, \varepsilon)}{\partial p_\alpha} = \frac{p_\alpha}{M} + \int U(p, \varepsilon, p_1, \varepsilon_1) G(p_1, \varepsilon_1) \frac{\partial G^{-1}(p_1, \varepsilon_1)}{\partial p_\alpha_1} G(p_1, \varepsilon_1) \frac{d^4 p_1}{(2\pi)^4 i}.$$ (9)

Here $U$ stands for the block of diagrams irreducible in the p-h channel.

In doing the renormalization of eq. (9) a little modification of the standard decomposition [14, 15] of the product $GG$ into a regular part $B$ and a singular one $A$ is necessary. It refers to the singular term $A$. Now it is defined as follows

$$A(p, q, \varepsilon, \omega) = z^2 \delta(\varepsilon) \int G^q(p, \varepsilon) G^q(p+q, \varepsilon+\omega) \frac{d\varepsilon}{2\pi i}.$$ (10)

Then

$$\int \frac{\partial G^{-1}(p, \varepsilon)}{\partial p_\alpha} A(p, \varepsilon) \frac{d\varepsilon}{2\pi i} = z^2 \int G^q(p, \varepsilon) \frac{\partial G^{-1}(p, \varepsilon=0)}{\partial p_\alpha} G^q(p, \varepsilon) \frac{d\varepsilon}{2\pi i} = z \frac{\partial n(p)}{\partial p_\alpha}.$$ 

In obtaining this result, we have used relation

$$\frac{\partial G^{-1}(p, \varepsilon=0)}{\partial p_\alpha} = z^{-1} \frac{\partial (G^q(p, \varepsilon))^{-1}}{\partial p_\alpha}.$$

As a result, after conventional transformations, we arrive at the well known relation [14, 15]

$$\frac{\partial \xi(p, n)}{\partial p_\alpha} = \frac{p_\alpha}{M} + \int \mathcal{F}(p, p_1; n) \frac{\partial n(p_1)}{\partial p_\alpha_1} d\tau_1.$$ (11)

The phenomenological quantity $\mathcal{F} \sim (1 - BU)^{-1} U$, being $\omega$-independent, plays the role of an effective interaction potential between particles in medium. Assuming this function to be known including possible dependence of $\mathcal{F}$ on the distribution $n(\xi)$ itself, eq. (11) can be integrated and conveniently rewritten as

$$\xi(p) = \frac{p^2}{2M} + \int H(p, p_1; n) n(\xi(p_1)) d\tau_1,$$ (12)

the effective Hartree-like term $H(p, p_1)$ being related to the effective interaction $\mathcal{F}$ by the formula

$$\mathcal{F} = H + n \frac{\delta H}{\delta n}.$$ 

As mentioned above, the momentum distribution function $n(\xi)$ is simply $\theta(\xi)$ so that eq. (12) can be treated as an equation for finding the spectrum $\xi(p)$. Usually $\mathcal{F}$ and, hence, $H$ are nearly momentum independent functions. Then eq. (12) has the single solution $\xi(p)$ for any $p$, with $d\xi/dp > 0$, i.e. there is one-to-one correspondence between $\xi$ and $p$.

The situation drastically changes in strongly correlated systems where momentum dependent components of the effective interaction can be strong. The case of homogeneous
matter where particle interact by means of limited long range effective forces, i.e. \( H(p, p_1) = V \delta(p - p_1) \), analyzed in \([3]\) is especially impressive. In this case, relation between \( \xi \) and the distribution \( n \) becomes algebraic:

\[
\xi = \xi_0^p + V n(\xi). \tag{13}
\]

This formula considered as an equation for finding \( \xi(p) \) is rewritten in the form

\[
L(\xi, p) \equiv \frac{\xi - \xi_0^p}{V} = n(\xi), \tag{14}
\]

where \( n(\xi) = \theta(\xi) \). This equation has a simple graphical solution (see Fig. 1) that furnishes a transparent explanation of how the FC solution of eq. (12) arises in more complicated cases. Indeed, the l.h.s. \( L(\xi, p) \) of eq. (14) depending explicitly on the momentum \( p \) as a parameter is depicted by a set of straight lines with the slope \( \sim 1/V \). On the other hand, the r.h.s. \( n(\xi) \), made up of two horizontal lines with the kink at \( \xi = 0 \), is independent of \( p \). Crossing points of the set \( L(\xi, p) \) with the horizontal pieces of \( n(\xi) \) are of no interest while in the entire interval \( p_i < p < p_f \) satisfying the inequality

\[
-V < \xi_0^p < 0, \tag{15}
\]

the straight lines cross the vertical line \( \xi = 0 \). Thus all particles with the momenta \( p \) varying in this interval turn out to have the same energy \( \xi(p) = 0 \), i.e. they belong to the FC region. It is worth noting that the FC solution is stable provided \( V > 0 \).

The graphical procedure is easily generalized to the case of nonzero temperatures where the distribution function \( n(p) \) is given by eq. (2). This model is also applied to anisotropic systems \([4]\) where the energy \( \xi_0^p \) becomes angle-dependent that results in the anisotropy of the FC occupation. So, in a two-dimensional tight-binding model with the single particle spectrum given by

\[
\xi_0^p = -\varepsilon_0 \cos px + \cos py + t \cos px \cos py - \mu, \tag{16}
\]

and \( t > 0 \), the condition (15) of the model for particles becoming trapped into the FC reads as

\[
0 < \cos px + \cos py + t \cos px \cos py + \frac{\mu}{\varepsilon_0} < \frac{V}{\varepsilon_0}. \tag{17}
\]

It is worth noting that often components of the effective interaction between electrons in solids have quite narrow peaks. So, the spin-spin component considered in the first quadrant of the quadratic two-dimensional Brillouin zone has the peak close to the point \( Q = (\pi, \pi) \) associated with antiferromagnetism \([16, 17]\). The resonance part of this interaction can be crudely written in the form

\[
S(p_1, p_2, q) = -\lambda \tilde{\sigma}_1 \tilde{\sigma}_2 \phi(p_1) \phi(p_2) \delta(q - Q). \tag{18}
\]

This is an exchange spin fluctuation term. Inserting it into the RPA formula for the mass operator \( \Sigma \) an essentially momentum dependent contribution

\[
\Sigma(p) = \frac{\lambda}{2} \phi(p) \phi(p - Q) n(p - Q) \tag{19}
\]
arises (see Fig. 2). Assuming other contributions to be reduced to a renormalization of the chemical potential a system of two equations

\[
\begin{align*}
\xi_1 & = \xi^0(p) + \frac{\lambda}{2} \phi(p) \phi(p-Q) n(\xi_2), \\
\xi_2 & = \xi^0(p-Q) + \frac{\lambda}{2} \phi(p) \phi(p-Q) n(\xi_1)
\end{align*}
\]

is obtained. We notice that the graphical procedure of solution requires operating in the three-dimensional space. Analyzing the landscape \(\xi(p)\) we see that the interaction term does not affect the region close to the center \((0,0)\). At the same time, its involving gives rise to an elevation of the area adjacent to the van Hove points that eventually results in the appearance of the FC in this region since otherwise energy of the occupied states would exceed that of unoccupied ones. Results of numerical calculations are given in Fig. 3, where the shadded areas correspond to the regions of the FC.

The models with limited long range forces are not exceptional. Similar results are obtained, e.g., in a model with finite range forces given by the Yukawa formula in the momentum space: \(H(p_1 - p_2) = g \exp(-\beta|p_1 - p_2|)\) \([2, 3]\). Now the l.h.s. of eq. (14) is recast as

\[
L(\xi, p) \equiv (-\Delta + \beta^2) \frac{\xi - \xi^0}{g} = n(\xi).
\]

In this case, the graphical representation of solution is complicated by the presence of the term \(\Delta \xi\). Fortunately, in the FC region, this term vanishes and we come back to eq. (14). Some complications emerge because of the existence of discontinuities in the term \(\Delta \xi\) at the boundaries of the FC region \([3]\) giving rise to jumps in the distribution \(n(p)\) at these points. The range of the jumps, which drops with increasing the coupling constant \(g\), at relatively small \(g\) may attain 1 and then the FC solution ceases to exist. As a result, requirements for the appearance of the FC solution become more severe: the constant \(g\) should exceed a nonzero positive critical value \(g_c\).

To make the next step, let us recast eq. (12) as follows

\[
\int H^{-1}(p, p_1) \xi(p_1) d\tau_1 - X_0(p) = n(\xi),
\]

where \(X_0 = (H^{-1}\xi^0_p)\). The model (21) is recovered provided \(H^{-1}(p, p_1) = g^{-1}(\beta^2 - \Delta) \delta(p - p_1)\). For a certain class of the momentum dependent operators \(H^{-1}\) this expansion can be extended to

\[
H^{-1}(p, p_1) \xi(p_1) = (A_0 - A_2\Delta + A_4\Delta^2 \ldots) \delta(p - p_1).
\]

Upon substituting this equation into eq. (22) one finds

\[
(A_0 - A_2\Delta + A_4\Delta^2 \ldots)(\xi - \xi^0_p) = n(\xi).
\]

Again all the terms containing the Laplace operators applied to \(\xi\) vanish in the FC region where \(\xi = 0\), and the FC solutions ceases to exist provided the jumps of \(n\) at the boundaries
of the FC region attain the unity. Thus we infer that for a quite wide class of the effective single particle Hamiltonian $H$, the FC solutions $\xi = 0$ exists. The extension of this result to the general form of the operator $H(p, p_1)$ is straightforward. We seek the FC solutions in an equation

$$0 = \xi_0^p + \int H(p, p_1; n) n(\xi(p_1)) \, d\tau_1, \quad p \in \Omega,$$

obtained from eq. (12) setting there $\xi = 0$.

It is the usual integral Fredholm equation of the first kind. As known, numerical solving such equations encounters difficulties. Specialized methods have been developed to overcome these difficulties which are often called inverse problems (see e.g. [20]). We are going to analyze them in a separate paper.

Thus we see that the FC solutions can be obtained proceeding from the Landau-Pitaevskii relation between the single particle spectrum $\xi(p)$ and the quasiparticle momentum distribution $n(p)$, without addressing to the variational condition (1). What is remarkable is that the Migdal jump in the distribution function $n$ at the point $\xi = 0$, being a building block of Fermi liquid theory in systems with velocity independent forces, becomes a principal factor for the appearance of the FC solutions in systems where such a dependence turns out to be sufficiently strong.

To conclude, having at hand the FC solution $\xi(p \in \Omega) = 0$, one can recover the variational condition (1) suggesting $\varepsilon(p; n) = \delta E_0(n)/\delta n(p)$ and remembering that $\xi(p) = \varepsilon(p) - \mu$. In principle, the functional $E_0(n)$ can be found from this variational equation provided the dependence of $\varepsilon(p, n)$ on the distribution $n$ is supposed to be known.

3. The extension of the Landau formalism to marginal Fermi liquid

If the systems with the FC were subject to the Landau theory then the development of the self-consistent theory of the fermion condensation would be completed. However, this is not the case. As we shall see, the mass operator $\Sigma(p, \varepsilon)$ in these systems contains the irregular logarithmic term $\sim \varepsilon \ln |\varepsilon|$ inherent in marginal Fermi liquid. This term makes it impossible expand $\Sigma(\varepsilon)$ in a Taylor series in energy. Although the pole in the single particle Green function $G$ survives but the residue turns out to be logarithmically small vanishing at the Fermi surface (see Fig. 4).

In spite of this fact, there is good reason to believe that underlying ideas of the Landau theory remain robust including that of the decomposition of the single particle Green function $G$ into a sum $G = G^r + ZG^s$ with a singular part $G^s$ (we shall call it the pseudoparticle propagator) containing the logarithmic components of the mass operator. In homogeneous matter, we write $G^s$ in a simple form

$$G^s(p, \varepsilon) = \frac{Z^{-1}}{\varepsilon - \sigma(\varepsilon) - s(p) - i\kappa(\varepsilon)}.$$  

The connection between the quantities $\sigma$ and $\kappa$ stems from the dispersion relation (3) which implies [21]

$$\kappa(\varepsilon \to 0) = -\kappa \varepsilon, \quad \sigma(\varepsilon \to 0) = \frac{2}{\pi} \kappa \varepsilon \ln \frac{|\varepsilon|}{\varepsilon_L},$$  

(27)
where $\kappa$ and $\varepsilon_L \sim \varepsilon_F^0 = p_F^2/2M$ are numerical constants.

In the Landau theory, the quasiparticle distribution $n(\xi)$ is the universal function with the kink at $\xi = 0$. In marginal liquids, the pseudoparticle distribution function $n(\xi)$ ceases to be universal. Analogously to eq. (8), it is evaluated from relation

$$n(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im} G^s(p, \varepsilon) d\varepsilon$$

$$\equiv \frac{1}{2} + \frac{1}{2\pi} \left[ \int_{0}^{\infty} \frac{\gamma(\varepsilon) d\varepsilon}{[\varepsilon - \sigma_L(\varepsilon) - \xi]^2 + \gamma^2(\varepsilon)} - \int_{0}^{\infty} \frac{\gamma(\varepsilon) d\varepsilon}{[\varepsilon - \sigma_L(\varepsilon) + \xi]^2 + \gamma^2(\varepsilon)} \right], \quad (28)$$

where we introduced notations $\xi = Z_s, \gamma = Z_\kappa$ and $\sigma_L(\varepsilon) = 2\gamma \varepsilon \ln |\varepsilon|/\varepsilon_L$. The constant $\varepsilon_L$ in (26) can be chosen in such a way as to ensure the equality between the particle and the "pseudoparticle" numbers, i.e.

$$\rho = \text{Tr} \int G^s(p, \varepsilon) d\tau \frac{d\varepsilon}{2\pi i}.$$ \quad (29)

We notice that if the ratio $\gamma(\xi)/\xi$ is relatively small then the integral (28) is close to 1, provided $\xi < 0$, or to 0 in case $\xi > 0$. On the other hand, if this ratio is of order of unity (marginal regime) then there $n(\xi)$ essentially deviates from $\theta(\xi)$. As a result, the strength of the FC states turns out to be distributed over all this interval and it is its upper boundary $\varepsilon_{FC}$ that should be compared to the characteristic FC energy.

The renormalization of eq. (9) in the marginal region is implemented along the same lines as before with the natural redefinition of the singular part $A$ which now contains the product $G^s G^s$. Furthermore in contrast to the quasiparticle approximation, $\xi(p)$ does not coincide with the spectrum of the single particle excitations determined by the location of the pole of $G^s(p, \varepsilon)$ although both these quantities are related to each other unambiguously. After some algebra one finds

$$\frac{\partial \xi(p)}{\partial p_\alpha} = \frac{p_\alpha}{M} + \int \mathcal{F}(p, p_1) \frac{\partial n(p_1)}{\partial p_{1\alpha}} d\tau_1. \quad (30)$$

In obtaining this result, we ignored the possible alteration of the regular part $B$ due to corrections related to singular terms in the mass operator.

Again the integration of this equation yields

$$\xi(p) = \frac{p^2}{2M} + \int H(p, p_1) n(p_1) d\tau_1.$$ \quad (31)

However, in marginal Fermi liquid, the dependence of the distribution $n(\xi)$ on $\xi$ is intricate and for finding the flat portion of the spectrum, one needs to solve eqs. (31) and (28) self-consistently. The result crucially depends on the magnitude of the marginal term in the mass operator $\Sigma$. 

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4. The evaluation of the irregular part of $\Sigma$ beyond the phase transition point

The purpose of this chapter is to calculate the imaginary part of the mass operator $\Sigma(p, \varepsilon)$ in the presence of a flat portion in the single particle spectrum. In doing so, the temperature $T$ is supposed to exceed a temperature $T_c$ of a phase transition, say, Cooper pairing or crystallization, that results in lifting the degeneracy of $\xi(p)$ implied by the variational condition (1). In this article, we investigate behavior of the mass operator $\Sigma(p, \varepsilon)$ in the limit $T \ll \varepsilon$ and therefore set $T = 0$. Without loss of generality, we restrict ourselves to the decay of a particle implying $\varepsilon > 0$ in all the formulas. Different contributions to $\text{Im} \Sigma(p, \varepsilon > 0)$ fall into two categories: i) the particle decay into a particle and a particle-hole pair and ii) the decay into a particle and a collective state. The evaluation of the last contribution requires special calculations of the collective spectra in the systems with the FC. This problem is beyond the scope of this article. It will be studied later. With these restrictions, $\text{Im} \Sigma(p, \varepsilon)$ is given by [14, 18]

$$\text{Im} \Sigma(p, \varepsilon > 0) = \int \int d\tau d\tau' \int_0^\varepsilon \frac{d\omega}{2\pi} \int_0^\omega \frac{d\varepsilon_3}{2\pi} W(p, \varepsilon, p_1, \varepsilon_1, p_2, \varepsilon_2, p_3, \varepsilon_3) \times \text{Im} G(p_1, \varepsilon_1) \text{Im} G(p_2, \varepsilon_2) \text{Im} G(p_3, \varepsilon_3).$$

(32)

The 4-momenta $p, p_2$ refer to incoming particles while $p_1, p_3$, outgoing ones: $p_3 = p + p_2 - p_1$; $\varepsilon_3 = \varepsilon + \varepsilon_2 - \varepsilon_1 = \varepsilon_2 + \omega$. The limits of integration over $\varepsilon_3$ are imposed by the restrictions: i) $\varepsilon_3 > 0$ since the 4-vector $p_3$ corresponds to the particle and ii) $\varepsilon_3 < \omega$ since $\varepsilon_2 < 0$ because the 4-vector $p_2$ corresponds to the hole. In the three-dimensional system $d\tau = d^3q/(2\pi)^3$. But most of experimental data on properties of strongly correlated Fermi systems are obtained in studies of anisotropic quasi-two-dimensional crystals where $d\tau = d^2q/(2\pi)^2$. The transition probability $W$ is related to the square of the effective interaction amplitude between particles averaged over spin variables $[18]$

$$W = \frac{1}{4} |\Gamma_{\uparrow\uparrow}|^2 + \frac{1}{2} |\Gamma_{\uparrow\downarrow}|^2. $$

(33)

Our prime goal is to establish the energy dependence of the mass operator $\Sigma(p, \varepsilon)$ beyond the FC phase transition point and further numerical factors will be omitted. In this article, we also neglect terms containing $G^r$ since their contributions are reduced to a renormalization of input parameters.

Usually, the interaction $\Gamma$ varies slowly versus its arguments. As a result, the integration is essentially facilitated [15]. But it is not always true. We shall see that in the systems with the FC, behavior of $\Gamma$ as a function of $\omega$ drastically changes that has a dramatic impact on the energy dependence of the mass operator $\Sigma$. At the same time, dealing with the damping effects the momentum dependence of $\Gamma$ is of no crucial importance. Therefore to avoid unjustified complications we restrict ourselves to the momentum-independent zero Landau harmonic of the interaction amplitude. Then $\Gamma_{\uparrow\uparrow} = 0$ while $\Gamma_{\uparrow\downarrow}$ is expressed in terms of the scalar component $\Gamma$ of the interaction amplitude $\Gamma_{\uparrow\downarrow} = 2\Gamma$. This suggestion being consistent
with our choice (26) of the form of $G^s$ furnishes the opportunity to integrate over $p_3$ and rewrite (32) in the form

$$\gamma(p, \varepsilon) \sim \int d\tau \int_0^\varepsilon d\omega \frac{\gamma(p-q, \varepsilon-\omega) |\Gamma(q, \omega)|^2 \text{Im} A(q, \omega)}{2\pi |\varepsilon - \omega - \sigma_L(p-q, \varepsilon-\omega) - \xi(p-q)|^2 + \gamma^2(p-q, \varepsilon-\omega)},$$

(34)

where $q = p - p_1$.

The interaction $\Gamma$ is related to the effective interaction potential $F$ by the well known Landau equation

$$\Gamma(q, \omega) = F(q) + F(q) A(q, \omega) \Gamma(q, \omega) = \frac{F(q)}{1 - F(q) A(q, \omega)},$$

(35)

developed in the particle-hole (p-h) channel. As usual, the factor $Z^2$ is already included into the definition of the effective interaction $\Gamma$. Then the propagator $A(q, \omega)$ given by

$$A(q, \omega) = 2 \int G^s(p, \varepsilon) G^s(p-q, \varepsilon-\omega) d\tau \frac{d\varepsilon}{2\pi i},$$

(36)

is conveniently rewritten [14] as

$$A(q, \omega) \equiv N_n(0) \left( a(q, \omega) + ib(q, \omega) \right),$$

with $N_n(0) = p_F M^*/\pi^2$ being the density of states of the normal Fermi liquid and $M^*$, the effective mass of the normal quasiparticle, and

$$b(q, \omega) = \frac{1}{N_n(0)} \int_0^\omega \text{Im} G^s(p, \varepsilon) \text{Im} G^s(p-q, \varepsilon-\omega) d\tau \frac{d\varepsilon}{\pi},$$

$$a(q, \omega) = \frac{1}{N_n(0)} \int_{-\infty}^\infty \text{Im} G^s(p, \varepsilon) \text{Re} \left[ G^s(p-q, \varepsilon-\omega) + G^s(p-q, \varepsilon+\omega) \right] d\tau \frac{d\varepsilon}{\pi}.$$ 

(37)

In ordinary Fermi systems, the magnitude of $|A|/N_n(0)$ is of order of 1.

In the strongly correlated Fermi systems, such as liquid He-3, dense neutron matter or electrons in high-temperature superconductors the amplitude $F$ is repulsive ($F > 0$) and strong so that, as a rule, the term $|FA|$ in the denominator of (35) significantly exceeds the unity. Omitting 1 in eq. (35), we are left with [19]

$$\Gamma(q, \omega) = -\frac{1}{A(q, \omega)}.$$ 

(38)

In this limit, the interaction amplitude $\Gamma$ is completely independent of characteristics of the effective interaction potential $F$. It holds over a wide range of frequencies being violated only close to lines $\omega_s(q)$ corresponding to the collective excitations of the system.

Irregular contributions to $\Sigma$ may also emerge in the particle-particle (p-p) channel, that is important, e.g. in superfluid systems. Here we study the systems for which the critical
temperature $T_c$ of the superfluid phase transition is very low (e.g. Sr$_2$RuO$_4$ with $T_c \simeq 1$ K) and consider the case $T > T_c$, at which the p-p correlations make no difference and therefore below we omit these contributions.

In what follows we consider a crystal with a cubic or square lattice assuming the FC to be positioned in the vicinity of the van Hove points $\mathbb{4}$, $\mathbb{3}$. We also set that the FC density $\rho_c = \eta \rho$ is rather small: $\eta < 1$. The total damping $\gamma$ from (34) is decomposed into a sum

$$\gamma(p, \varepsilon) = \gamma_0(\varepsilon) + \gamma_1(p, \varepsilon) + \gamma_2(p, \varepsilon).$$

(39)

The term $\gamma_0$ does not contain the final FC Green functions. Contributions containing the single final FC Green function are included into the term $\gamma_1$, those with two FC ones, into the last term $\gamma_2$. It should be indicated that for the square or cubic lattice the decay of the FC particle into the final state with no less than two FC states unambiguously results in the appearance all three FC final states.

Dealing with the momentum dependence of $\gamma(p)$, two regions can be distinguished: i) the FC region $\Omega$ where the single particle spectrum $\varepsilon(p)$ is flat and ii) the ordinary region where the gradient $d\xi / dp_n$ has the usual order of value $\simeq p_F / M_0^*$ where $M_0^*$ being independent of $T$ is the effective mass of the ordinary quasiparticle. Further we shall neglect a possible variations of the relevant quantities inside these regions and use the subscript $c$ for the FC region and the subscript $n$, for the ”normal” one, e.g. $\gamma_0(p \in \Omega) \equiv \gamma_c$ or $\gamma_0(p \notin \Omega) \equiv \gamma_n$. We also use the notation $\Gamma_{00}$ for the set of diagrams of the interaction amplitude without the FC states and the notation $A_n(q, \omega)$, for the particle-hole propagator of the normal Fermi liquid.

In many respects, the evaluation of the quantities $\Gamma_{00}(q, \omega)$ and $A_n(q, \omega)$ at $q \sim p_F$ is equivalent to that in normal homogeneous Fermi liquid. The momentum integration in Feynman diagrams is carried out with the help of the standard formula

$$dp_x dp_y dp_z = dS dp_\nu = \frac{dS d\xi}{|d\xi / dp_\nu|},$$

(40)

where $\nu$ indicates the normal to the Fermi surface direction.

Overwhelming contributions to $\gamma_0$ come from the momentum region quite far from the van Hove points. Here the components $b_0$ and $a_0$ of the propagator $A_n(q, \omega)$ and the function $|\Gamma_{00}(q, \omega)|^2$ are known to vary slowly enough that allows us to replace them by averaged values:

$$b_n \sim \omega \frac{M^*}{p_F^2}, \quad |\Gamma_{00}|^2 \sim \frac{1}{N_n^2(0)} \sim \frac{1}{(M^*)^2}.$$  

(41)

Upon substituting (41) into eq. (34) the integrals cease to depend on $p$ at all, and we arrive at the common result

$$\gamma_0(\varepsilon) = -\frac{\gamma_0 M^* \varepsilon^2}{p_F^2},$$

(42)

where $\gamma_0$ is a dimensionless positive constant.

When evaluating $\gamma_1(p, \varepsilon)$, two ordinary Green functions entering eq. (32) are combined into the particle-hole propagator if the FC particle is generated, or into particle-particle
one when the FC particle is annihilated. As before, we concentrate on the analysis of the contribution $\gamma_{1}^{ph}$. Then the contribution $\gamma_{1n}$ is given by (see Fig. 5)

$$\gamma_{1n}(\varepsilon) \sim N_n(0) \int_{\Omega} d\tau d\varepsilon_1 \text{Im} G_c(p_1, \varepsilon_1) |\Gamma_{01}(p, p_1, \varepsilon-\varepsilon_1)|^2 b_n(\varepsilon-\varepsilon_1),$$

where

$$\text{Im} G_c(p_1, \varepsilon_1) = \frac{\gamma_c(p_1, \varepsilon_1)}{[\varepsilon_1 - \sigma_c(p_1, \varepsilon_1) - \xi_c(p_1)]^2 + \gamma_c^2(p_1, \varepsilon_1)}.$$

The region $\Omega$ of the momentum integration in this formula consists of "patches" of the FC adjacent to the van Hove points.

It can be verified that the interaction amplitude $\Gamma_{01}$ with the single (final) FC state has the same order of value as $\Gamma_{00} \sim 1/N_n(0)$. Then eq. (43) becomes

$$\gamma_{1n}(\varepsilon) \sim N_n(0) p_F^2 \int_{\Omega} d\tau d\varepsilon_1 \text{Im} G_c(p_1, \varepsilon_1) (\varepsilon_1 - \varepsilon).$$

In obtaining this result the relation (41) for the averaged value $b_0$ was used.

The integral (45) considered versus energy $\varepsilon$, the upper limit of integration, is, apparently, a monotonic function. If the upper limit were extended to infinity, the integral value would be equal to $(1 - n_0(p))$ provided $\xi(p) > 0$. In the opposite case $\xi(p) < 0$, it is close to zero (recall, we consider the particle decay). Integrating over momenta, one finds

$$\gamma_{1n}(\varepsilon) \sim \frac{M^*}{N_n(0) p_F^2} \int_{\Omega} d\tau d\varepsilon_1 \text{Im} G_c(p_1, \varepsilon_1) (\varepsilon_1 - \varepsilon).$$

The evaluation of this integral is facilitated if the energy $\varepsilon$ exceeds the characteristic FC energy $\varepsilon_{FC}$. In this case, in contrast to the density of the states $N_0(\varepsilon)$, which varies slowly with $\varepsilon$, the density of the FC states $N_c(\varepsilon)$ does extinct at $\varepsilon > \varepsilon_{FC}$. Therefore it can be approximated by the $\delta$-function: $N_c(\varepsilon) = \rho_c \delta(\varepsilon)$. As a result, at $\varepsilon > \varepsilon_{FC}$ the FC contribution to $\gamma_n$ turns out to be linear function of energy rather than the quadratic one as the ordinary term $\gamma_0$:

$$\gamma_{1n}(\varepsilon > \varepsilon_{FC}) \simeq -\gamma_{1n} \eta \varepsilon, \quad \sigma_{1n}(\varepsilon > \varepsilon_{FC}) \simeq \frac{2\gamma_{1n} \eta}{\pi} \varepsilon \ln \frac{|\varepsilon|}{\varepsilon_L},$$

where $\gamma_{1n}$ is a constant of order unity.

As for the corresponding contribution $\gamma_{1c}(\varepsilon)$ to the FC damping (see Fig. 6a), the equation for its evaluation has the form

$$\gamma_{1c}(\varepsilon) \sim N_n(0) \int_{\Omega} d\tau d\varepsilon_1 \text{Im} G_n(p_1, \varepsilon_1) |\Gamma_{20}(p, p_1, \varepsilon-\varepsilon_1)|^2 b_n(\varepsilon-\varepsilon_1),$$

where $\Gamma_{20}$ stands for the amplitude of the transition of the normal p-h pair into the FC pair.

The energy dependence of the quantities in this integral is more important than the momentum one and therefore, in the first approximation, the amplitude $\Gamma_{20}(p, p_1, \varepsilon-\varepsilon_1)$ can
be replaced by the averaged over the FC region value $\Gamma_{20}(\varepsilon-\varepsilon_1)$. Then after implementation of the momentum integration one finds

$$
\gamma_{1c}(\varepsilon) \sim N_n(0) \int_0^\varepsilon N_c(\varepsilon_1) |\Gamma_{20}(\varepsilon-\varepsilon_1)|^2 b_n(\varepsilon-\varepsilon_1) d\varepsilon_1. \quad (49)
$$

Two final FC states also come into play when the contribution $\gamma_{2n}(\varepsilon)$ to the damping $\gamma(\varepsilon)$ is considered. Again we concentrate on the p-h contribution (see Fig. 6b) given by

$$
\gamma_{2n}(\varepsilon) \sim N_n(0) \int_0^\varepsilon d\tau_1 d\varepsilon_1 \text{Im} G_n(p_1, \varepsilon_1) |\Gamma_{20}(p, p_1, \varepsilon-\varepsilon_1)|^2 b_c(p-p_1, \varepsilon-\varepsilon_1). \quad (50)
$$

We introduced a special notation $\Omega'$ for the integration region in (50), determined by the requirement for the difference $p-p_1$ to have the value appropriate for the pair of the FC states. Neglecting the momentum dependence of the amplitude $\Gamma_{20}$ and the propagator $b_2$ this equation is recast to

$$
\gamma_{2n}(\varepsilon) \sim N_n^2(0) \int_0^\varepsilon |\Gamma_{20}(\varepsilon-\varepsilon_1)|^2 b_n(\varepsilon-\varepsilon_1) d\varepsilon_1. \quad (51)
$$

At last, dealing with the contribution $\gamma_{2c}(\varepsilon)$, all four states belong to the FC (see Fig. 6c), so that

$$
\gamma_{2c}(\varepsilon) \sim N_n(0) \int_0^\varepsilon d\tau_1 d\varepsilon_1 \text{Im} G_c(p_1, \varepsilon_1) |\Gamma_{22}(p, p_1, \varepsilon-\varepsilon_1)|^2 b_c(p-p_1, \varepsilon-\varepsilon_1). \quad (52)
$$

This integral is reduced to

$$
\gamma_{2c}(\varepsilon) \sim N_n(0) \int_0^\varepsilon N_c(\varepsilon_1) |\Gamma_{22}(\varepsilon-\varepsilon_1)|^2 b_c(\varepsilon-\varepsilon_1)d\varepsilon_1. \quad (53)
$$

The averaged over the FC region propagator $b_c(\omega)$ entering this formula is defined in terms of

$$
b_c(q, \omega) \sim \frac{1}{N_n(0)} \int_0^\omega \text{Im} G^s(p_1, \varepsilon) \text{Im} G^s(p_1-q, \varepsilon-\omega) d\tau d\varepsilon \sim \frac{1}{N_n(0)} \int_0^\omega d\tau \frac{\gamma_c(\varepsilon) \gamma_c(\varepsilon-\omega) d\varepsilon}{([\varepsilon-\sigma_c(\varepsilon)-\xi_c(p_1)]^2 + \gamma^2_c(\varepsilon)) ([\varepsilon-\omega-\sigma_c(\varepsilon-\omega)-\xi_c(p_1-q)]^2 + \gamma^2_c(\varepsilon-\omega))}. \quad (54)
$$

Primarily, we study the case $\varepsilon \geq \varepsilon_{FC}$. In this situation, the integrand in eq. (54) is made up of two separate peaks. The first is positioned at $\varepsilon \sim \varepsilon_{FC}$ and contributes only if $\xi_c(p_1) > 0$ (i.e. if $n_0(p_1) < 1/2$) independently of the sign of the other term $\xi_c(p_1)$. The second is located
at \((\omega - \varepsilon) \sim \varepsilon_{FC}\) and contributes provided \(\xi_c(p - q) < 0\) (i.e. \(n_0(p - q) > 1/2\)) independently of \(\xi_c(p_1)\). Both these terms supplement each other and we arrive at

\[
\begin{align*}
\left. b_c(\omega) \right|_{\omega \geq \varepsilon_{FC}} & \approx \frac{\eta \rho}{N_n(0)} \frac{\gamma_c(\omega)}{[\omega - \sigma_c(\omega)]^2 + \gamma_c^2(\omega)} \left[ \int_{\omega}^{\infty} \frac{\gamma_c(\varepsilon - \omega) d\varepsilon}{(\varepsilon - \omega - \sigma_c(\varepsilon - \omega))^2 + \gamma_c^2(\varepsilon - \omega)} \right. \\
& \left. + \int_{0}^{\omega} \frac{\gamma_c(\varepsilon) d\varepsilon}{(\varepsilon - \sigma_c(\varepsilon))^2 + \gamma_c^2(\varepsilon)} \right].
\end{align*}
\]

(55)

Again we can use the fact that the integrands on the r.h.s. of this equation are \(\delta\)-like functions. Then after simple integration we are left with

\[
\begin{align*}
\left. b_c(\omega) \right|_{\omega \geq \varepsilon_{FC}} & \sim \frac{\eta}{N_n(0)} \frac{\gamma_c(\omega)}{M^* f^2(\omega)},
\end{align*}
\]

(56)

where

\[
f^2(\omega) = [\omega - \sigma_c(\omega)]^2 + \gamma_c^2(\omega).
\]

(57)

To correctly estimate the damping one also needs to know the behavior of the real part \(a_c\) of the propagator \(A_c\) averaged over the FC region. It can be traced from the formula

\[
\begin{align*}
a_c(q, \omega) & \sim \frac{1}{N_n(0)} \int_{-\infty}^{\infty} \text{Im} G^s(p_1, \varepsilon) \text{Re} \left[ G^s(p_1 - q, \varepsilon - \omega) + G^s(p_1 + q, \varepsilon + \omega) \right] d\tau d\varepsilon \\
& \approx \frac{1}{N_n(0)} \int_{-\infty}^{\infty} d\tau \left[ \frac{\gamma_c(\varepsilon) d\varepsilon}{\varepsilon - \sigma_c(\varepsilon) - \xi_c(p_1)} \right]^2 + \gamma_c^2(\varepsilon) \\
& \times \left[ \frac{\varepsilon - \omega - \sigma_c(\varepsilon) - \xi_c(p_1 - q)}{\varepsilon - \omega - \sigma_c(\varepsilon) - \xi_c(p_1 - q)^2 + \gamma_c^2(\varepsilon) - \omega} + \frac{\varepsilon + \omega - \sigma_c(\varepsilon + \omega) - \xi_c(p_1 + q)}{\varepsilon + \omega - \sigma_c(\varepsilon + \omega) - \xi_c(p_1 + q)^2 + \gamma_c^2(\varepsilon + \omega)} \right].
\end{align*}
\]

(58)

After simple algebra, the following estimate for the averaged value \(a_c(\omega)\) is obtained

\[
|a_c(\omega)| \sim \frac{\eta p_F^2 \varepsilon_{FC}}{M^* \left( [\omega - \sigma_c(\omega)]^2 + \gamma_c^2(\omega) \right)}.
\]

(59)

Comparing \(|a_c|\) to \(|b_c|\) and bearing in mind that the value of \(\varepsilon_{FC}\) depends by itself on \(\gamma_c\) we infer that, in the first approximation, the \(a_c\) contribution can be neglected.

It is worth noting that the integrands of (19) and (34) are peaked at \(\varepsilon_1 \sim \varepsilon_{FC}\) due to the \(\delta\)-like behavior of \(\text{Im} G_c(p_1, \varepsilon_1)\) and rather smooth behavior of the products \(b_n|\Gamma_{20}|^2\) and \(b_c|\Gamma_{22}|^2\). To verify this assertion consider the set of equations for the amplitudes \(\Gamma_{22}, \Gamma_{02}\). In the symbolic form, it reads as

\[
\begin{align*}
\Gamma_{22} \quad & = \mathcal{F}_{22} + \mathcal{F}_{22} A_c \Gamma_{22} + \mathcal{F}_{20} A_n \Gamma_{02} \\
\Gamma_{02} \quad & = \mathcal{F}_{02} + \mathcal{F}_{02} A_c \Gamma_{22} + \mathcal{F}_{00} A_n \Gamma_{02}.
\end{align*}
\]

(60)
This system is easily solved

\[
\Gamma_{20} = \frac{\mathcal{F}_{20}}{(1-\mathcal{F}_{00}A_n)(1-\mathcal{F}_{22}A_c) - \mathcal{F}_{20}^2 A_c A_n},
\]
\[
\Gamma_{22} = \frac{\mathcal{F}_{22}(1-\mathcal{F}_{00}A_n) + \mathcal{F}_{20}^2 A_n}{1 - \mathcal{F}_{00}A_n - A_c[\mathcal{F}_{22}(1-\mathcal{F}_{00}A_n) + \mathcal{F}_{20}^2 A_n]}.
\] (61)

From the last formula we can infer that in the strong coupling limit the value \(|A_c\Gamma_{22}|\) is of order of 1. In what follows we employ an interpolation formula

\[
N_n^2(0) |\Gamma_{22}(\epsilon)|^2 \sim \frac{1}{1 + b_c^2(\epsilon)}.
\] (62)

which is correct for a moderate interaction and in the strong coupling limit, as well. As for the ratio \(|\Gamma_{20}/\Gamma_{22}| \sim \mathcal{F}_{20}/\mathcal{F}_{22}\), its value is, probably, considerably less than 1. Upon substituting these results into eqs. (49) and (53) we are led to the following equations

\[
\gamma_{1c}(\epsilon) \sim -\eta \frac{\epsilon}{1+b_c^2(\epsilon)}, \quad \gamma_{2c}(\epsilon) \sim \eta p_F^2 \frac{b_c(\epsilon)}{M^* (1+b_c^2(\epsilon))}.
\] (63)

As a result, the equation for finding \(\gamma_c(\epsilon)\) takes the form

\[
\gamma_c(\epsilon) = -\gamma_0 \frac{M^* \epsilon^2}{p_F^2} - C \eta \frac{\epsilon}{1+b_c^2(\epsilon)} + A \eta p_F^2 \frac{b_c(\epsilon)}{M^* (1+b_c^2(\epsilon))}.
\] (64)

Upon substituting here the formula (56) for the propagator \(b_c\) and simple transformations one obtains

\[
\gamma_c(\epsilon) \left(1 - A \frac{\delta^2 f^2(\epsilon)}{f^4(\epsilon) + B^2 \delta^2 \gamma^2(\epsilon)}\right) = -\gamma_0 \frac{\epsilon^2}{\epsilon_F^2} - C \eta \frac{\epsilon f^4(\epsilon)}{f^4(\epsilon) + B^2 \delta^2 \gamma^2(\epsilon)},
\] (65)

where \(\delta_F = \eta p_F^2/M^*\) while the constants \(A, B, C\) are of order of 1. At \(\epsilon > \delta_F\), the magnitude of the propagator \(b_c\) is less than 1 and therefore the second term in the brackets of eq. (65) is relatively small. This implies

\[
\gamma_c(\epsilon > \delta_F) \sim -C \eta \epsilon,
\] (66)

where the constant \(C\) is of order of 1. We infer that at these energies the magnitude of the marginal terms in the mass operator is quite small. It should be indicated that, according to (28), the presence of the marginal term in \(\Sigma(\epsilon \to 0)\), no matter how small it is, alters the Landau distribution \(n(\xi)\) giving rise to the an inclination of the plateau in the single particle spectrum proportional to \(\eta\) if this behavior persists in the limit \(\epsilon \to 0\).

However, this smallness cannot always hold as \(\epsilon\) together with \(f(\epsilon)\) go down to zero, otherwise the sum of the terms in the brackets on the l.h.s. of eq. (65) would change its sign. This circumstance makes the function \(\gamma(\epsilon)\) rapidly enhance so as to both the terms in the
brackets almost cancel each other approaching $\varepsilon \simeq \delta_F$. Indeed, let us seek the solution of a simplified equation

$$1 - A \frac{\delta_F^2 f^2(\varepsilon)}{f^4(\varepsilon) + B^2 \delta_F^2 \gamma_c^2(\varepsilon)} = 0.$$ 

It can be easily verified that no solution of this equation exists at $\varepsilon$ exceeding a critical value $\varepsilon_{cr} = \delta_F \sqrt{A}$ and $\gamma_c(\varepsilon_{cr}) = 0$. As $\varepsilon$ goes down, $\gamma_c(\varepsilon)$ grows up so that the ratio $r_c = \gamma_c(\varepsilon)/\varepsilon$ attains in the region $\varepsilon \sim \delta_F$ values of order of the unity in contrast to perturbation theory estimate $r_c \simeq \eta \ll 1$. The alteration of behavior of the damping from the perturbative regime (66) to a "strong coupling limit"

$$\gamma_c(\varepsilon) \simeq \varepsilon, \quad \varepsilon < \delta_F,$$  

occurs in a quite narrow region of energy that is confirmed in numerical calculations.

The result $r_c(\delta_F) \sim 1$ is crucial for finding out behavior of $\gamma_c(\varepsilon \to 0)$ where a scaling approach can be employed. We assume that $\gamma_c(\varepsilon \to 0) \sim \varepsilon^\nu$. For the evaluation of the critical index $\nu$, we calculate the propagator $b_c$ proceeding from an assumption that the damping term exceeds all the rest ones in $\text{Im} \ G_c(p, \varepsilon)$. Then

$$b_c(\omega) \sim \eta \frac{\omega}{\gamma_c^2(\omega)}.$$ 

This function diverges at $\omega \to 0$ provided $\nu > 1/2$ while the product

$$|\Gamma_{22}(\varepsilon)|^2 b_c(\varepsilon) \sim \frac{\gamma_c^2(\omega)}{\eta \omega}.$$ 

Then the folding with $\text{Im} \ G_c(p, \varepsilon) \sim 1/\gamma_c$ leads to the strong-coupling-limit equation for $\gamma_c$ which reads as

$$\gamma_c(\varepsilon \to 0) \simeq \int_0^\varepsilon \gamma_c(\omega) \frac{d\omega}{\omega}.$$  

(68)

Significantly that $\eta$ does not enter this equation. This allows us to find the correct solution $\gamma(\varepsilon) \simeq \varepsilon$ which simultaneously meets eq. (68) and the matching condition

$$r_c(\delta_F) \simeq 1.$$ 

Thus the marginal behavior of the mass operator $\Sigma(\varepsilon)$ in the system with the FC holds over a quite wide region of energy up to $\varepsilon \sim \delta_F$. This means that beyond the FC phase transition point there is no room for the Landau theory in the vicinity of the origin $\varepsilon = 0$. Otherwise the ratio $r_c(\varepsilon \to 0)$ should vanish and then some completely flat portion $\xi = 0$ in the spectrum of the single particle excitations should survive. But in this case calculations along the same lines as done above results in the nonzero value of $r_c(\varepsilon \to 0)$. This contradiction proves the inapplicability of the Landau theory.

Now we return to the damping $\gamma_n$. From eq. (51) one has

$$\gamma_{2n}(\varepsilon) = C_{2n} \int_0^\varepsilon \frac{b_c(\varepsilon_1)}{1 + b_c^2(\varepsilon_1)} d\varepsilon_1.$$  

(69)
with the constant $C_{2n}$ whose value is less than 1. In principle, this integral is a linear in $\varepsilon$ function in the region $\varepsilon \simeq \delta_F$ so that the total damping $\gamma_n(\varepsilon)$ is

$$\gamma_n(\varepsilon) = -\gamma_n \varepsilon, \quad \varepsilon \leq \delta_F,$$

(70)

the value of the constant $\gamma_n$, proportional to $\mathcal{F}_{20}/\mathcal{F}_{22}$ being, probably, considerably less than 1.

To establish the structure of the spectrum of the single particle excitations beyond the FC phase transition point one needs to solve self-consistently the system (28) and (31) accounting for the energy dependence of the damping $\gamma(\varepsilon)$ given by eqs. (66), (67) and (70).

One of the ways to obtain the solution is the iterative procedure. We start with the FC distribution $n(p)$ found from eq. (25) assuming the explicit form of this equation, where the damping is completely ignored, to be known. With this distribution at hand, we evaluate the parameters determining the damping at different energies. Having found these constants we substitute them into eq. (28), evaluate the new distribution function $n(p)$ and insert it into eq. (31) to find the new single particle spectrum and then repeat all the procedure till its convergence.

It should be kept in mind that the results depend on $\eta$ as an input parameter. What actually happens is that the FC density $\rho_c$ is not the input parameter. It is given by an integral

$$\rho_c = \int_\Omega n(p) d\tau.$$

(71)

The integration in (71) is performed over regions where the slope of the single particle spectrum is small compared to the usual one. In the quasiparticle approximation, the boundaries of this region are determined unambiguously, since this slope is identically 0, and the density $\rho_c$ is also evaluated unambiguously being proportional to the excess $\mathcal{F} - \mathcal{F}_{cr}$. However, the damping implicitly influences on the size of the FC region so that the integral (71) turns out to be an equation for obtaining $\rho_c$ and with it the set of equations describing the basic properties of the system beyond the FC phase transition point becomes complete.

The numerical solution of this system is beyond of the scope of this article. We restrict ourselves to several comments. Since the ratio $\gamma_c(\varepsilon)/\varepsilon$ exceeds 1 from 0 to $\delta_F$ the kink at $\xi = 0$ in the distribution $n(\xi)$ evaluated in the quasiparticle approximation is split: its upper part moves to the left while the low one, to the right and the magnitude of this shift is of order of $\delta_F$. Steep pieces of the curve $n(\xi)$ are connected by a bridge the length of which and the slope depend on input parameters. If it is almost horizontal, then the insertion of such a distribution $n(\xi)$ into eq. (28) results in splitting the plateau in the single particle spectrum, one part of which is shifted upward while the other, downward of the Fermi level resembling a zigzag. The lower part occupied by electrons can be investigated in ARPES studies while the upper one, not. In this case, the spectrum of the former FC states remains practically flat but it is shifted from the Fermi surface. Its value $\simeq \delta_F$ can be treated as the characteristic FC energy $\varepsilon_{FC}$.

On the other hand, if the slope of the bridge is not so small then the zigzag gets smeared.
Then the average slope of the spectrum is estimated from an approximate formula

\[ \varepsilon_{FC} \simeq \int_{\Omega} \frac{d\varepsilon}{dp} dp \simeq \frac{d\varepsilon}{dp} L. \]

In the two-dimensional problem, the characteristic size \( L \) of the FC region is proportional to \( \eta^{\frac{2}{3}} \), in the three-dimensional system it is proportional to \( \eta^{\frac{2}{3}} \). Substitution here \( \varepsilon_{FC} \simeq \frac{\delta}{F} \) yields in the two-dimensional case the result \( d\varepsilon/dp \sim \eta^{\frac{2}{3}} \) while in the three-dimensional system \( d\varepsilon/dp \sim \eta^{\frac{2}{3}} \). Thus we infer that the plateau in the single particle spectrum is inclined but the slope turns out to be small: it vanishes together with the condensate density \( \rho_c \).

### 5. Comparison with the phenomenological theory by Varma et al. [21]

Now we are able to compare the results obtained with postulates of the phenomenological theory by Varma et al. [21] of normal states of high-\( T_c \) superconductors. The first postulate is concerned with the polarizability \( P(q, \omega) \) and requires it to be an \( \omega \)-independent quantity in contrast to the ordinary linear \( \omega \)-dependence. The second is related to the mass operator \( \Sigma(p, \varepsilon) \). It is assumed to have the form (27) inherent in marginal Fermi liquid. As we have seen, such a behavior of the mass operator \( \Sigma \) beyond the FC phase transition point is due to flattening the single particle spectra.

As for the polarizability \( P(q, \omega) \) which determines a relation between an external field \( V_0 \) and a real one \( V \) acting inside of the system: \( V(q, \omega) = P(q, \omega) V_0(q, \omega) \), we evaluate it neglecting connections between the normal and the FC regions, i.e. suggesting \( \Gamma_{20} \equiv 0 \). Then

\[ P^{-1}(q, \omega) = 1 - \mathcal{F}(q) A_n(q, \omega) \]

and

\[ \text{Im } P(q, \omega) = \text{Im } A_n(q, \omega) |P(q, \omega)|^2 \simeq N_n(0) \frac{b_n(q, \omega)}{|1 - \mathcal{F}(q) A_n(q, \omega)|^2}. \quad (72) \]

In the optical conductivity problem, the value of the wave vector \( q = \omega/c \) is so small that the damping effects exhibit themselves in full force. In the normal region,

\[ b_n(q=0, \omega) \sim \frac{1}{N_n(0)} \int_0^\omega \frac{\gamma_n(\varepsilon)\gamma_n(\varepsilon-\omega)}{\left[ (\varepsilon-\sigma_n(\varepsilon)-\xi_n(p))^2 + \gamma_n^2(\varepsilon) \right] \left[ (\varepsilon-\omega-\sigma_n(\varepsilon-\omega)-\xi_n(p))^2 + \gamma_n^2(\varepsilon-\omega) \right]} \, d\varepsilon \, dp \, dS. \quad (73) \]

This integral is evaluated analytically and the result

\[ b_n(q \to 0, \omega) \simeq -\int_0^\omega \frac{d\varepsilon}{|\gamma_n(\varepsilon)| + |\gamma_n(\varepsilon-\omega)|}. \quad (74) \]

is drastically enhanced compared to the conventional one (41), proportional to \( \omega \). Upon substituting into (73) the result \( \gamma_n(\varepsilon) = -\gamma_n \varepsilon \) and simple integration one finds

\[ b_n(q \to 0, \omega) \simeq -\frac{1}{\gamma_n} \quad (75) \]
where $\gamma_n \sim 1$. With this result, simple algebra leads to the following estimate

$$\text{Im } P(q < q_{cr}, \omega) \sim -N_0(0).$$

(76)

Thus the first postulate of the theory [21] consisting of the independence of the polarizability $P(\omega)$ of the frequency $\omega$ is reproduced in our analysis, too.

With increasing $\omega$ the damping grows up and the magnitude of $\text{Im } P$ drops. As $\omega$ increases from $\varepsilon_{FC}$ the damping $\gamma(\varepsilon)$ rapidly drops. As a result, the value of $b_n(q \to 0, \omega)$ grows up. However, simultaneously the denominator of (72) increases, even faster than the numerator, and $\text{Im } P(q \to 0, \omega)$ falls rapidly to values of order of $\eta N_0(0)$. As a result, the optical conductivity of the system with the FC has the maximum at frequencies $\omega \simeq \eta \varepsilon_F^0$. Thus in the systems with the FC there exists a new mechanism, different from the polaron one, for the appearance of irregular behavior of the optical conductivity $\sigma(\omega)$.

The behavior of the real part $a_n(q, \omega)$ in the static limit also differs from the Fermi liquid one. Indeed

$$a_n(q=0, \omega=0) \sim \frac{1}{N_n(0)} \int_{-\infty}^{\infty} \frac{\gamma_n(\varepsilon) \left[ \varepsilon - \sigma_n(\varepsilon) - \xi_n(p) \right] d\varepsilon \, d^3p}{\left[ \left( \varepsilon - \sigma_n(\varepsilon) - \xi_n(p) \right)^2 + \gamma_n^2(\varepsilon) \right]^2}.$$

(77)

In contrast to the quasiparticle-approximation formula, this integrand has no singularities and therefore replacing integration over $p$ by that over $t = \varepsilon - \sigma_n(\varepsilon) - \xi_n(p)$ and assuming the derivative $dt/dp = -d\xi_n(p)/dp$ to be constant one finds

$$\text{Re } A(q=0, \omega=0) = 0.$$

(78)

The Landau result is restored only if $q > q_{cr}$. Of course, this result holds only if the momentum dependence of the group velocity $\xi_n(p)$ is negligible but usually this dependence is sufficiently weak that results in a significant suppression of the real part of the particle-hole propagator and, in its turn, in the suppression of the well known results for the spin and charge susceptibility. From these results one obtains that the well known singular behavior of the propagator $A(q, \omega)$ in the static limit $q \to 0, \omega \to 0$ is absent in the systems with the FC and the reason for that is the linear in $\varepsilon$ damping of the single particle degrees of freedom.

6. Conclusion

In this article, proceeding from the standard many-body-theory formalism we analyzed the interplay between the flattening and the damping in the single particle spectra of the strongly correlated Fermi systems beyond the FC phase transition point assuming the FC density $\rho_c$ to be relatively small: $\eta = \rho_c/\rho < 1$. We concentrated on crude features of this phenomenon not caring about numerical factors in quantities of our interest. We have found that in the quasiparticle approximation there exists a class of solutions of the many body theory equation for the spectrum $\xi(p)$ with the zero value of the quasiparticle group velocity at the Fermi surface. The necessary condition for the appearance of these solutions is a sufficiently strong momentum dependence of the mass operator $\Sigma(p, \varepsilon)$. Such a strong
momentum dependence may stem from the fact that in the vicinity of many phase transition points, e.g. antiferromagnetism, the effective interaction $\Gamma(q)$ has quite narrow peaks. Employing the RPA for the evaluation of the contribution of these resonance components into the mass operator $\Sigma(p, \varepsilon)$ a well pronounced momentum dependence does emerge in this function. Thus we can infer that in strongly correlated electron systems of solids, the fermion condensation could be entailed by the spin fluctuations and, hence, the fermion condensation is the precursor of antiferromagnetism.

We demonstrated that in the vicinity of the Fermi surface there is no room for the Landau theory: the system with the FC does behave itself as marginal Fermi liquid, i.e. the damping $\gamma(\varepsilon)$ of the states adjacent to the Fermi surface is linear in energy. The enhancement of $\gamma$ should entail a significant spread of the FC states and influences on the intensity of the peaks observed in ARPES studies in the region occupied by the FC. This intensity is proportional to the factor $\left(1 - \frac{\partial \Sigma(p, \varepsilon)}{\partial \varepsilon}|_{\varepsilon=\varepsilon(p)}\right)^{-1}$ and therefore it is suppressed in the FC region adjacent to the van Hove points. Of course, the magnitude of this suppression greatly depends on input parameters and varies from one compound to another.

Two different marginal regimes of the energy dependence of $\gamma$ have been established. The perturbative regime $\gamma(\varepsilon) \sim \eta \varepsilon$ is adequate at energies $\varepsilon > \delta_F \simeq \eta \varepsilon_0^F$. The strong coupling regime with $\gamma(\varepsilon) \simeq \varepsilon$ is involved in the vicinity of $\varepsilon = 0$. The transition from the first to the second one is, probably, quite smooth, but dealing with the states near the van Hove points this transition could be well pronounced providing an unaccustomed shape of the energy dependence of $\gamma(\varepsilon)$.

We established that behavior of the imaginary part of the polarization operator $P(q = 0, \omega)$ of the system with the FC drastically differs from that assigned by the Fermi liquid theory: its magnitude remains constant over the low-frequency region. This effect can lead to anomalies in the optical conductivity $\sigma(\omega)$ conventionally associated with the polaron mechanism. We also uncovered an alteration of the real part of the particle-hole propagator that entails a suppression of the spin and charge susceptibility of the strongly correlated systems.

It is worth noting that the presence of the marginal term in the mass operator should change many formulas of the BCS theory of superfluid systems with the FC including the BCS relation between $\Delta(T = 0)$ and the critical temperature $T_c$. This problem will be investigated in the next article in more detail.

In conclusion, that the analysis of many experimental data on the base of Luttinger liquid where the renormalization constant $z = 0$ is an appropriate guide to understanding of behavior of strongly correlated electron systems became clear, probably, ten years ago [21, 23]. At the same time, many properties of these systems are known to be successfully treated proceeding from Fermi liquid theory. What we demonstrated in our article is that the fermion condensation model serves as a ”bridge” connecting both these approaches.

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Figures captions

Fig. 1. Graphical solution of eq. (14). The distribution $n(\xi)$ (bold line) is crossed by the set of straight lines $(\xi - \xi^0_p)/V$. The crossing points of these lines with the vertical line $\xi = 0$ are included into the FC region $\Omega$.

Fig. 2. The graphical representation of the momentum dependent contribution to the mass operator (19) resulting from exchange spin fluctuation term (18) of the effective interaction.

Fig. 3. FC regions in the square lattice within the model with the spin-spin effective interaction (18) peaked at the vector $Q = (\pi, \pi)$. The formfactors $\phi(p)$ are assumed to vanish far from the van Hove points. The parameters used are: $t=0.7, \mu=1.1\varepsilon_0 t, \lambda=0.4\varepsilon_0$.

Fig. 4. The residue of the single-particle Green function in marginal Fermi liquid with $\kappa(\varepsilon) = -\varepsilon$ as a function of energy $\varepsilon$ measured in $\varepsilon_F$.

Fig. 5. The graphical representation for the contribution $\gamma_{1n}$ to the total damping.

Fig. 6. The same for $\gamma_{1c}$ (a), $\gamma_{2n}$ (b) and $\gamma_{2c}$ (c).
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
Fig. 6

a) $\gamma_{1c}$

b) $\gamma_{2n}$

c) $\gamma_{2c}$