Kubo Formulae for Second-Order Hydrodynamic Coefficients

Guy D. Moore and Kiyoumars A. Sohrabi
Department of Physics, McGill University, 3600 rue University, Montréal QC H3A 2T8, Canada
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At second order in gradients, conformal relativistic hydrodynamics depends on the viscosity \(\eta\) and on five additional “second-order” hydrodynamical coefficients \(\tau_1, \kappa, \lambda_1, \lambda_2,\) and \(\lambda_3\). We derive Kubo relations for these coefficients, relating them to equilibrium, fully retarded 3-point correlation functions of the stress tensor. We show that the coefficient \(\lambda_3\) can be evaluated directly by Euclidean means and does not in general vanish.

Results from the RHIC experiments, particularly the measurement of a large transverse flow \([1]\), appear to show that the Quark-Gluon plasma can be well described by hydrodynamics with a surprisingly small viscosity \([2]\). A major future goal for heavy ion experiment and theory is to quantify how small the viscosity of the plasma is. This requires the numerical treatment of relativistic viscous hydrodynamics \([3]\). It has long been known \([4–6]\) that the relativistic Navier-Stokes equations are acausal and unstable. But the Navier-Stokes equations are just the result of a first-order (Chapman-Enskog \([7]\)) expansion in gradients. Extending the expansion to second order yields numerically stable equations after a certain reorganization is applied \([5]\). The drawback is that it adds unknown coefficients. In the conformal case (which we will consider for simplicity), besides the equation of state \(P(\epsilon)\) at zero order and the shear viscosity \(\eta\) at first order, there are five new transport coefficients: \(\tau_1, \kappa, \lambda_1, \lambda_2, \lambda_3\) in the notation of \([8]\). These have been evaluated in strongly coupled \(N=4\) super-Yang-Mills theory in the limit of many colors \([8, 9]\) and at leading order in weakly coupled QCD \([10]\). In each case \(\lambda_3 = 0\) at lowest order in the respective (strong or weak coupling) expansion.

Baier et al have also presented Kubo formulae for two of these coefficients, \(\tau_1\) and \(\kappa\), which relate them to well defined, equilibrium correlation functions of the stress tensor. Presumably, the remaining three coefficients \(\lambda_{1,2,3}\) can also be expressed in terms of stress tensor correlation functions. Doing so would put the definition of these coefficients on a solid footing and might aid in their physical interpretation and their theoretical calculation. In the remainder of this paper we will derive such Kubo relations for the three remaining second-order coefficients. We do this first by showing how the first and second order hydrodynamic coefficients can be related to the stress tensor in a background spacetime with perturbatively small geometrical curvature. Then we expand in the metric as an external background field \(a la\) Kubo \([11]\) and derive a relation between \(\lambda_{1,2,3}\) and certain fully retarded 3-point stress-tensor correlation functions. This allows us to determine the previously unknown perturbative behavior of the coefficient \(\lambda_3\) (which is not zero) and to say something about its physical interpretation.

We restrict attention to conformal fluids mostly to simplify the presentation; in the nonconformal case there are more coefficients \([12]\) but there are no conceptual or technical obstacles to treating this case with the same methodology developed here.

**Constitutive Relations for Second Order Coefficients**

We begin by defining the second order coefficients. The expectation value of the stress-energy tensor operator for a fluid can be decomposed in terms of a local equilibrium piece and an extra piece,

\[
\langle T^\mu_\nu \rangle = T^\mu_\nu_{eq}(\mu^\mu, \epsilon) + \Pi^\mu_\nu,
\]

\[
T^\mu_\nu_{eq} \equiv (\epsilon + P) u^\mu u^\nu + Pg^\mu_\nu. \quad (1)
\]

Here \(g^{}_{\mu_\nu}, \epsilon, P, u^\mu\) are the spacetime metric (in the mostly-plus convention), energy density, pressure as given by the equation of state, and flow 4-velocity. We work in the Landau-Lifshitz frame, \(u_0^{\mu} \Pi^{\mu_\nu} = 0\), which makes the division between \(T_{eq}^{\mu_\nu}\) and \(\Pi^{\mu_\nu}\) unique; we normalize \(u^\mu\) so that \(u^\mu u_\mu = -1\). While \(u^\mu, \epsilon,\) and \(g^{\mu_\nu}\) are ordinary functions of \(x\), \(T^{\mu_\nu}\) is a Heisenberg-picture operator; \(\langle T^{\mu_\nu} \rangle\) represents its trace in the density matrix describing the fluid.

The key idea of hydrodynamics is that, for a system which varies slowly in space and time, \(\Pi^{\mu_\nu}\) arises only due to the nonuniformity of the system and should therefore be expressible in terms of a gradient expansion in that nonuniformity. To write out \(\Pi^{\mu_\nu}\) to second order, we introduce some notation. We define \(\Delta^{\mu_\nu} \equiv g^{\mu_\nu} + u^\mu u^\nu\), which is the projector to spatial directions in the local rest frame. Angular brackets around a pair of Lorentz indices, \(\langle \mu_\nu \rangle\), mean that the indices are to be symmetrized, space-projected, and trace-subtracted; that is,

\[
A^{\langle \mu_\nu \rangle} \equiv \frac{1}{2} \Delta^{\alpha_\beta} \Delta^{\gamma_\delta} (A_{\alpha_\beta} + A_{\gamma_\delta}) - \frac{1}{3} \Delta^{\alpha_\beta} \Delta^{\gamma_\delta} A_{\alpha_\beta}. \quad (2)
\]

The shear and vorticity tensors are defined as

\[
\sigma^{\mu_\nu} \equiv 2\nabla^{\langle \mu_\nu \rangle}, \quad (3)
\]

\[
\Omega^{\mu_\nu} \equiv \frac{1}{2} \Delta^{\alpha_\beta} \Delta^{\gamma_\delta} (\nabla^\alpha u_\beta - \nabla^\beta u_\alpha). \quad (4)
\]

\(R^{\mu_\nu}\) and \(R^{\mu_\nu_\alpha_\beta}\) are the Ricci tensor and curvature tensor respectively. In terms of these quantities, the most general form for \(\Pi^{\mu_\nu}\) compatible with conformal symmetry...
We derive Kubo relations for $\lambda_1$ etc. by considering a system where some nonuniformity, either in the initial conditions or in the spacetime geometry, forces $\sigma^{\mu\nu}$ etc. to be nonzero. It is particularly convenient to consider an initially uniform, equilibrium system in flat space but to introduce perturbatively weak and slowly varying spacetime nonuniformity which causes the fluid to experience shear and vorticity. Writing the metric as $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ ($\eta_{\mu\nu}$ the flat-space metric), one expands perturbatively in $h_{\mu\nu}$. Since $h_{\mu\nu}$ couples to the stress tensor $T^{\mu\nu}$, this generates an expansion in correlation functions of multiple stress tensors, whose coefficients are the response of the stress tensor to fluid nonuniformities.

Consider the expectation value $\langle T^{\mu\nu}(0) \rangle$ for a system initially (time $t_0 \ll 0$) in equilibrium at temperature $T$, subject to a spacetime dependent metric perturbation $h_{\alpha\beta}(x)$, with $h_{\mu\nu}(t \leq t_0) = 0$. The stress tensor is determined by

$$\langle T^{\mu\nu}(0) \rangle = \text{Tr} e^{-\beta H} \text{Exp} \left( \int_{t_0}^{0} dt' H[h(t')] \right) T^{\mu\nu} \times \text{Exp} \left( \int_{t_0}^{0} dt' (-i) H[h(t')] \right)$$

(with $\text{Exp}$ and $\text{Anti-time ordered} \times \text{Exp}$ the anti-time ordered and time-ordered exponentials respectively, $H[h(t)]$ the Hamiltonian, showing explicitly its dependence on the metric, and $\beta = T^{-1}$ the inverse temperature). This is best treated using the Schwinger-Keldysh (closed time path) formalism (see [13, 14]; we follow the conventions in [14]). We introduce independent metric perturbations for the $T$-ordered and $\tilde{T}$-ordered evolution operators in the above expression and define the generating functional

$$W[h_1, h_2] \equiv \ln \text{Tr} e^{-\beta H} \text{Exp} \left( i \int_{t_0}^{\infty} dt' H[h_2(t')] \right) \times \text{Exp} \left( -i \int_{t_0}^{\infty} dt' H[h_1(t')] \right)$$

$$= \ln \int [\Phi_1, \Phi_2, \Phi_3] e \int \sqrt{-g} d^4x \mathcal{L}[\Phi_1(x), h_1] \times \mathcal{L}[\Phi_2(x), h_2] \times - \int d^4x \mathcal{L}[\Phi_3(x)] e^{-i \int \sqrt{-g} d^4x \mathcal{L}[\Phi_2(y), h_2]}.$$  

One then defines the average metric perturbation $h_r \equiv \frac{h_1 + h_2}{2}$ and stress tensor $T_s \equiv \frac{T_1 + T_2}{2}$, and the difference variables $h_a \equiv h_1 - h_2$, $T_a \equiv T_1 - T_2$. Variation with respect to $h_a$ gives $T_s$, explicitly

$$\frac{-2i}{\sqrt{-g}} \frac{\partial W}{\partial h_{\alpha\beta}(x)} = \langle T^{\mu\nu}(x) \rangle.$$  

We use such a variation to pull down the $T^{\mu\nu}(0)$ factor we want to evaluate. After taking this $h_a$ derivative, we set $\dot{h}_a = 0$ and $\dot{h}_r = 0$, since we are interested in the case of a classical background value $h_1 = h_2 = h_r = h$. (The difference $h_a$ represents possible quantum fluctuations in the metric which we do not want to consider.) We then expand order by order in $h_{\alpha\beta\mu\nu}$ to obtain a series expansion of $\langle T^{\mu\nu} \rangle$ in powers of $h$. Explicitly, we find

$$\langle T^{\mu\nu}_r \rangle_h = G^{\mu\nu}_r(0) - \frac{1}{2} \int d^4x G^{\mu\nu\alpha\beta}_{rad}(0, x) h_{\alpha\beta}(x)$$

$$+ \frac{1}{8} \int d^4x d^4y G^{\mu\nu\alpha\beta\gamma\delta}_{rad}(0, x, y) h_{\alpha\beta}(x) h_{\gamma\delta}(y)$$

plus terms of order $h^3$. Here $G^{\mu\nu\alpha\beta\gamma\delta}_{rad}(0, x, y)$ is the correlation function of one $T_r$ and 0 or more $T_a$’s,

$$G^{\mu\nu\alpha\beta\gamma\delta}_{rad}(0, x, y) = (-i)^{n-1} (-2i)^n \frac{\partial^n W}{\partial g_{\alpha\beta}(0) \partial g_{\gamma\delta}(x) \cdots g_{\mu\nu}(y)}.$$  

The expectation value is with respect to the flat-space, equilibrium density matrix. $G_{rad}$ is a fully retarded correlation function [14], which is a nested commutator, from earliest to latest time, with $T_r$ at the last time and innermost in the commutator, $\eta g$ when $x^0 < y^0 < \cdots < 0$ the correlator is $\langle [T(x), [T(y), \ldots, [T(0)]\ldots]\rangle$. Here (c.t.) refers to the contact terms which are built into our definition of the n-point stress tensor correlation functions. This is discussed in [13]; the contact terms turn out not to be important for evaluating $\eta$ but will contribute to the evaluation of $\lambda_{1,2,3}$.

**KUBO FORMULAE**

First we review the derivation of Kubo formulae for the “linear” transport coefficients $\eta, \gamma_1, \kappa$ [8]. Consider $\langle T^{xy} \rangle$ in the presence of $h_{xy}(z, t)$. According to Eq. (9), at first order

$$\langle T^{xy}_r \rangle_h = - \int d^4x h_{xy}(x) G^{xy}_{ra}(0, x) + O(h^2).$$  

Using Eq. (11) and $\nabla_\mu T^{\mu\nu} = 0$ (energy-momentum conservation), we derive that $u^i = 0$ at $\mathcal{O}(h)$. We then evaluate $\sigma^{\mu\nu}$, $u \cdot \nabla (\sigma^{\mu\nu})$ etc. explicitly for this $u^i$ and $h_{\mu\nu}$, finding for instance that $\sigma^{xy} = \partial_y h_{xy}$. Substituting into Eq. (5), we find

$$\langle T^{xy}_r \rangle_h = - P h_{xy} - \eta \partial_t h_{xy} + \eta \nabla_\mu \partial_\mu h_{xy}$$

$$- \frac{\kappa}{2} \left( \partial_1^2 h_{xy} + \partial_2^2 h_{xy} \right) + O(\partial^3, h^2).$$  


defining $G^{xy}_{ra}(\omega, k) = \int d^4 x e^{i(\omega t-kz)}G^{xy}_{ra}(0,-x)$ and equating Eqs. (11)-(12) order by order in derivatives, we find

$$\eta = i\partial_\omega G^{xy}_{ra}(\omega, k)\bigg|_{\omega=0}=0,$$  \hspace{1cm} (13)

$$\kappa = -\partial^2_{kz}G^{xy}_{ra}(\omega, k)\bigg|_{\omega=0}=0,$$  \hspace{1cm} (14)

$$\eta_\tau = -\frac{1}{2} \left( \partial^2_{\omega} G^{xy}_{ra}(\omega, k) - \partial^2_{kz} G^{xy}_{ra}(\omega, k) \right)\bigg|_{\omega=0}=0.$$  \hspace{1cm} (15)

These reproduce the Kubo relations obtained by \cite{3}.

To obtain higher order Kubo formulae for the nonlinear coefficients, we continue this procedure to $O(h^2)$, for a background choice which allows nonzero shear flow and vorticity. To do so, we will consider $\Pi^{xy}$ term. Since

$$\text{find equilibrium value of } \eta \text{ in this background to second order, we find}$$

$$\text{These reproduce the Kubo relations obtained by } \eta.$$  

In this background to second order, we find

$$(T^{xy}) = P(h_{xz} h_{yz} - h_{xt} h_{yt}) + \eta (h_{xz} h_{yz,t} + h_{xt} h_{yt}) + \frac{\eta_\tau}{2} (h_{xz} h_{yt,z} + h_{xt} h_{yz,z} - 2h_{xz} h_{yt,t} - 2h_{xt} h_{yz,t}) + \lambda_1 (h_{xz} h_{yt,t} - \frac{\lambda_3}{4} (h_{xt} h_{yt} + h_{xz} h_{yt})) + \lambda_3 (h_{xt} h_{yt}).$$  

Equating with the $h^2$ part of Eq. (9), and defining

$$G^{\mu\nu,\alpha\beta,\sigma\lambda}_{ra}(p, q) \equiv \int d^4x d^4y e^{i(p x + q y)} G^{\mu\nu,\alpha\beta,\sigma\lambda}_{ra}(0, x, y) \text{ (16)}$$

we find the following Kubo relations:

$$\lambda_1 = \eta_\tau - \lim_{p^0, q^0 \to 0} \frac{p^0}{p^0} \lim_{\eta \to 0} G^{xy,zz,qt}_{ra}(p, q).$$  \hspace{1cm} (17)

$$\lambda_2 = 2\eta_\tau - 4 \lim_{p^0, q^0 \to 0} \frac{p^0}{p^0} \lim_{\eta \to 0} G^{xy,zz,mt}_{ra}(p, q).$$  \hspace{1cm} (18)

$$\lambda_3 = -4 \lim_{p^0, q^0 \to 0} \frac{p^0}{p^0} \lim_{\eta \to 0} G^{xy,zt,qt}_{ra}(p, q).$$  \hspace{1cm} (19)

These Kubo relations are our main result.

We also find extra Kubo relations for $\eta$, $\kappa$, and $\tau$:

$$i\eta = \lim_{p^0 \to 0} \frac{\partial}{(dp^0)^2} G^{xy,zz,qt}_{ra}(p, q).$$  \hspace{1cm} (21)

$$\kappa = 2 \lim_{p^0 \to 0} \frac{\partial^2}{(dp^0)^2} G^{xy,zz,mt}_{ra}(p, q).$$  \hspace{1cm} (22)

$$2\eta_\tau - \kappa = 2 \lim_{p^0 \to 0} \frac{\partial^2}{(dp^0)^2} G^{xy,zz,qt}_{ra}(p, q).$$  \hspace{1cm} (23)

where besides the differentiated variable all other $p, q$ components are taken to zero first. These extra relations require inter-relations between $G^{\mu\nu,\alpha\beta}$ and $G^{\mu\nu,\alpha\beta,\gamma\delta}$.

Each extra Kubo formula involves one stress tensor at zero external 4-momentum, arising from an undifferentiated $h_{\mu\nu}$ in Eq. (16). We can always force $h_{\mu\nu} = 0$ at $x = 0$ where $T^{xy}$ is evaluated by a coordinate “gauge” choice. The invariance of the theory to such gauge choice enforces (Ward) relations between two point functions and three point functions with a $T^a_{\mu\nu}$ at zero 4-momentum. Consider a stress tensor two-point function in a spacetime-independent, background $h_{\mu\nu}$:

$$\langle T^\mu_{T} T_{a}^{\alpha\beta}(x) \rangle = i G^{\mu\nu,\alpha\beta}(0, x)$$  \hspace{1cm} (24)

The gauge change which eliminates $h_{\mu\nu}$ is $x^\mu \rightarrow x^\mu + \xi^\mu$ with $\xi_{\mu,\nu} + \xi_{\nu,\mu} = h_{\mu\nu}$. Applying the gauge change to the lefthand side of Eq. (24), we re-express it in terms of $h_{\mu\nu}$ and the flat-space correlation functions; 

$$h_{\gamma\delta} (\left[ \eta^{\alpha\beta} G^{\mu\nu,\alpha\beta}(p) + (\mu \leftrightarrow \nu) \right]$$

$$+ \left[ \eta^{\gamma\delta} G^{\mu\nu,\gamma\delta}(p) + (\alpha \leftrightarrow \beta) \right] + (\gamma \leftrightarrow \delta)$$

$$= 2 \eta_{\gamma\delta} G^{\mu\nu,\alpha\beta,\gamma\delta}(p, 0).$$  \hspace{1cm} (25)

Choosing $\mu = xy$, $\alpha = xz$, $\gamma = yz$, 

$$G^{xy}_{ra}(p) + G^{xz}_{ra}(p) = 2G^{xy,zz}_{ra}(p, 0).$$  \hspace{1cm} (26)

Now $\partial_{\alpha} G^{xy}_{ra} = \eta_y$ by Eq. (13) and $\partial_{\beta} G^{xz}_{ra} = \eta_y$ by rotational invariance, so Eq. (21) follows. The same procedure applies for the other linear coefficients.

**DISCUSSION**

Our derivation had two goals. First, we wanted relations, shown in Eqs. (18)-(19)-(20), for the second-order nonlinear transport coefficients in terms of equilibrium energy-momentum tensors. Second, we hoped that these relations would shed some light on the nature or properties of these transport coefficients. The most mysterious of these transport coefficients is $\lambda_3$, which is found to vanish in $N=4$ SYM theory in the limit of many colors and large coupling $g$ and which is zero at order $g^{-8}$ in the weak coupling expansion, the order where $\lambda_1, 2$ are nonzero. Is it identically zero? Romatschke \cite{12} studied this problem (among others) using a generalized entropy current and showed that $\lambda_3$ is related to a certain modification of the entropy density in the presence of vorticity. Our Kubo relation allows for a direct evaluation of $\lambda_3$ in weakly coupled field theory.
\( G_E \) the Euclidean correlation function. In particular, 
\[ G^{\mu_\alpha, \nu_\beta, \sigma_\tau}_{\gamma_\rho} (-i \omega_1 - i \omega_2) = i^{n_0} G^{\mu_\alpha, \nu_\beta, \sigma_\tau}_{\gamma_\rho} (\omega_1, \omega_2) \] 
for Matsubara frequencies \( \omega_{1,2} = 2 \pi T n_{1,2} \). Here \( n_0 \) is the number of indices \( \mu, \nu, \alpha, \beta, \sigma, \tau \) which are 0, since there is a factor of \( i \) arising from the Euclidean continuation of \( a \) index. This relation shows that the zero-frequency raa and Euclidean correlation functions are equal up to factors of \( i \). Hence

\[
\lambda_3 = 4 \lim_{\vec{p}, \vec{q} \to 0} \frac{\partial^2}{\partial p_x \partial q_x} G_E^{xy,0,0}(p, q). \tag{27}
\]

One usually considers such Euclidean correlation functions to carry only thermodynamical information; \( \lambda_3 \) should not be thought of as a dynamical coefficient but as a thermodynamic response to vorticity \[21\].

At weak coupling we can directly evaluate Eq. \[27\] diagrammatically in the Matsubara formalism. This contrasts with the case of \( \eta, \tau_{11}, \lambda_1 \), and \( \lambda_2 \), where time derivatives mean that \( G_{\text{raa}} \) must be evaluated at small nonzero frequency where the continuation cannot be so simply applied. Therefore the weak coupling expansion of \( \lambda_3 \) (and \( \kappa [13] \)) will start at \( g^0 \), while the expansions for \( \lambda_{1,2} \) can involve inverse powers of \( g [10, 17] \).

We have evaluated the correlation function in Eq. \[27\] for a one-component scalar field theory at leading order in weak coupling. Two diagrams contribute; a triangle diagram, \( \partial_{p_x} \partial_{q_y} \langle T^{xy}(-p - q) T^{00}(p) T^{00}(q) \rangle = \frac{T^2}{144} \) and a contact term involving \( X^{xy0} = 20 T x \partial y \), \( \partial_{p_x} \partial_{q_y} \langle T^{xy}(-p - q) X^{0y0}(p + q) \rangle = \frac{T^2}{72} \). Hence

\[
\lambda_3 = \frac{T^2}{12}, \quad 1 \text{ weak-coupled real scalar field.} \tag{28}
\]

We get the same answer using the scalar field stress tensor from \[13\]. The important observations are that \( \lambda_3 \) can be quite easily evaluated at weak coupling via Euclidean techniques, and the result is not in general zero.

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[17] S. Jeon, Phys. Rev. D 52, 3591 (1995).

[18] C. G. Callan, S. R. Coleman and R. Jackiw, Annals Phys. 59, 42 (1970).

[19] Technically \( h_{x0}, h_{y0} \) must depend on \( z, t \) since it must vanish in the initial conditions. It is essential to “turn on” this perturbation very slowly, on a timescale \( t > (\epsilon + P)/(k^2 \eta) \) \( k \) the wave number for \( h_{x0} \), and to include the viscous term \( -\eta p^{\mu\nu} \) in Eq. \[5\], to correctly derive that \( u^i = 0 \) after fully turning on the \( h_{x0}, h_{y0} \) perturbations.

[20] We are indebted to Peter Arnold, Diana Vaman, Chaolun Wu, and Wei Xiao for pointing out an error in the original version of this paper (see \[18\]). At this point we proposed investigating \( \Pi^{xx} \) using nontrivial \( h_{y0}(z, t), h_{x0}(y) \). However, in this case \( g^{\mu\nu} \) arises at \( O(h^0) \), and so \( O(h^2) \) corrections to the pressure \( P \) must be evaluated. We failed to do so, and therefore the Kubo relations in the original version of this paper were in error.

[21] For instance, there are curved but time-independent geometries and density matrix choices where \( \kappa \) and \( \lambda_3 \) contribute to \( T^{\mu\nu} \) but the fluid is in equilibrium and no entropy production is occurring.