THE SCHWARZ GENUS OF THE STIEFEL MANIFOLD AND COUNTING GEOMETRIC CONFIGURATIONS

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Abstract. In this paper we compute: the Schwarz genus of the Stiefel manifold $V_k(\mathbb{R}^n)$ with respect to the action of the Weyl group $W_k := (\mathbb{Z}/2)^k \rtimes S_k$, and the Lusternik–Schnirelmann category of the quotient space $V_k(\mathbb{R}^n)/W_k$. Furthermore, these results are used in estimating the number of: critically outscribed parallelotopes around the strictly convex body, and Birkhoff–James orthogonal bases of the normed finite dimensional vector space.

Dedicated to Professor Ljubiša Kočinac on the occasion of his 65th birthday

1. Introduction

The classical problem of estimating the number of the periodic billiard trajectories in a strictly convex body, introduced by G. Birkhoff in early 1990s, was recently studied in series of papers by Farber [6], Farber & Tabachnikov [7], and Karasev [10] via topological methods. The problem of counting the number of periodic billiard trajectories was connected with the problem of determining the critical points of the appropriately defined length function on a cyclic configuration space. Then the number of critical points was estimated from below by the Lusternik–Schnirelmann category of the quotient of the cyclic configuration space.

In this paper we study the Schwarz genus of the Stiefel manifold $V_k(\mathbb{R}^n)$ with respect to the action of the Weyl group $W_k := (\mathbb{Z}/2)^k \rtimes S_k$ and consequently the Lusternik–Schnirelmann category of the quotient manifold $V_k(\mathbb{R}^n)/W_k$, Section 2. In particular, in Corollary 2.6 we prove that:

$$g_{W_k}(V_k(\mathbb{R}^n)) = \text{cat}_{V_k(\mathbb{R}^n)/W_k} = nk - \frac{k(k+1)}{2} + 1.$$

Following the spirit of arguing as in the billiard problem we use the result on the Lusternik–Schnirelmann category of the quotient manifold $V_k(\mathbb{R}^n)/W_k$ to estimate the number of: critically outscribed parallelotopes and Birkhoff–James orthonormal bases, Section 3. We prove that:

- Every strictly convex body $K \subset \mathbb{R}^n$ has at least $\frac{n(n-1)}{2} + 1$ distinct critically outscribed parallelotopes;
- Every smooth norm on $\mathbb{R}^n$ has at least $\frac{n(n-1)}{2} + 1$ Birkhoff–James orthonormal bases.

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2. The Schwarz genus of the Stiefel manifold

In this section we evaluate the Schwarz genus of the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal $k$-frames in $\mathbb{R}^n$ with respect to the action of the group $W_k := (\mathbb{Z}/2)^k \rtimes S_k$ and prove that

$$g_{W_k}(V_k(\mathbb{R}^n)) = \dim V_k(\mathbb{R}^n) + 1 = nk - \frac{k(k+1)}{2} + 1.$$

Moreover, as a consequence we obtain the following estimate for the Lusternik–Schnirelmann category of the unordered configuration space of the projective space

$$\text{cat}_{B_k(\mathbb{R}^n)} \geq nk - \frac{k(k-1)}{2} + 1.$$
2.1. The Schwarz genus. Following the original paper by Schwarz and adapting the notion of the genus of a regular fibration [8 Chapter V] we introduce the Schwarz genus for a free $G$-space. For more details and applications consult [11] and [15].

**Definition 2.1.** The Schwarz genus $g_G(X)$ of a free $G$-space $X$ is the smallest number $n$ such that $X$ can be covered by $n$ open $G$-invariant subsets $X_1, \ldots, X_n$ with the property that for every $1 \leq i \leq n$ there exists a $G$-equivariant map $X_i \rightarrow G$. The Schwarz genus of a free $G$-space $X$ will be denoted by $g_G(X)$.

For a paracompact $G$-space $X$ the Schwarz genus coincides with the smallest number $n$ such that there exists a $G$-equivariant map $X \rightarrow G^n$. Here $G^n$ denotes the $n$-fold join of the group $G$ with diagonal $G$-action. The group $G$ is considered as a 0-dimensional simplicial complex.

Let $E_rG$ denote a compact free $r$-dimensional $G$-space that is also $(r-1)$-connected. The following natural generalization of the Borsuk–Ulam theorem

*There is no $G$-equivariant map $E_rG \rightarrow E_{r-1}G,*

implies the $g_G(E_rG) = r + 1$.

The classical notion of Lusternik–Schnirelmann category of a topological space can be defined as follows.

**Definition 2.2.** The **Lusternik–Schnirelmann category** $\text{cat}(X)$ of a space $X$ is the smallest integer $n$ for which $X$ can be covered by $n$ open subsets $X_1, X_2, \ldots, X_n$ such that the inclusions $X_i \rightarrow X$ are nullhomotopic.

The Schwarz genus of a free $G$-space and the Lusternik–Schnirelmann category of its quotient $X/G$ are connected via the following lemma.

**Lemma 2.3.** For a free $G$-space $X$ the following inequality holds

$$\text{cat}(X/G) \geq g_G(X).$$

**Proof.** Let $m = \text{cat}(X/G)$ and $Y_1 \cup \cdots \cup Y_m$ be the corresponding open covering of $X/G$ with nullhomotopic inclusions $Y_i \subseteq X/G$. Then the maps $X_i = \pi^{-1}(Y_i) \rightarrow Y_i$ defined as restrictions of the natural projection $\pi : X \rightarrow X/G$ are trivial coverings. This means that there exist $G$-equivariant homeomorphisms $X_i \rightarrow Y_i \times G$ where the actions on $Y_i \times G$ are given by $g \cdot (y, h) := (y, gh)$. Composing these homeomorphisms with the projection on the second factor we get $G$-equivariant maps $X_i \rightarrow G$. Thus, by the definition of the Schwarz genus $g_G(X) \leq m$.

Use of the Schwarz genus is sometimes more convenient, compared to the Lusternik–Schnirelmann category, because the Schwarz genus is monotone with respect to inclusions of $G$-invariant spaces. This property will be essentially used in the proofs of Corollaries 2.7, 8.2 and 8.3.

One of the main tools to estimate the Schwarz genus and consequently the Lusternik–Schnirelmann category of the quotient is the following cohomological criterion that is a particular case of [8 Theorem 12, page 91].

**Lemma 2.4.** Let $G$ be a finite group, $R$ be a commutative ring with unit, $M$ be an $R[G]$-module and $n > 0$ be an integer. If the map in the equivariant cohomology $\pi^* : H^n_G(pt; M) \rightarrow H^n_G(X; M)$ is nonzero, then

$$g_G(X) \geq n + 1.$$ 

The map $\pi^*$ is induced by the $G$-equivariant projection $\pi : X \rightarrow pt$.

2.2. The Schwarz genus of the Stiefel manifold of orthonormal $k$-frames in $\mathbb{R}^d$. Let $V_k(\mathbb{R}^n)$ be the Stiefel manifold of all orthonormal $k$-frames in $\mathbb{R}^n$, and $G \subseteq O(k)$. Then $G$ acts naturally on the Stiefel manifold $k$-frames by

$$(v_1, \ldots, v_k) \cdot g = \left(\sum_{j=1}^{k} v_j g_{j1}, \ldots, \sum_{j=1}^{k} v_j g_{jk}\right).$$

Here $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ and $g = (g_{ij})_{i,j=1}^{k} \in O(k)$. The action is from the right, but it transforms into a left action by $g \cdot (v_1, \ldots, v_k) := (v_1, \ldots, v_k) \cdot g^{-1}$.

We consider the action of the group $W_k := (\mathbb{Z}/2)^k \rtimes S_k \subset O(k)$ on the Stiefel manifold $V_k(\mathbb{R}^n)$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be generators of the component $(\mathbb{Z}/2)^n$ and $\pi \in S_k$. Then for $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ we set $\varepsilon_i \cdot (v_1, \ldots, v_k) = (v_1', \ldots, v_k')$. 

The map $\pi^*$ is induced by the $G$-equivariant projection $\pi : X \rightarrow pt$. 

**Theorem 2.5.** Let $G$ be a finite group, $R$ be a commutative ring with unit, $M$ be an $R[G]$-module and $n > 0$ be an integer. If the map in the equivariant cohomology $\pi^* : H^n_G(pt; M) \rightarrow H^n_G(X; M)$ is nonzero, then

$$g_G(X) \geq n + 1.$$
where \( v'_i = -v_i \) and \( v'_j = v_j \) for all \( j \neq i \), and
\[
\pi \cdot (v_1, \ldots, v_k) = (v_{\pi(1)}, \ldots, v_{\pi(k)}).
\]

The space \( M_k \) of all real \( k \times k \)-matrices can be considered as a real \( O(k) \)-representation with respect to the matrix conjugation action. Consequently, \( M_k \) is also a real \( W_k \)-representation. We consider the following real \( W_k \)-subrepresentations of \( M_k \):

- \( R_k \) the space of all symmetric \( k \times k \) matrices with zeroes on the diagonal, and
- \( P_k \) the space of all \( k \times k \) matrices with zeroes outside the diagonal.

Notice that \( \dim R_k = \frac{k(k-1)}{2} \) and \( \dim P_k = k \).

Let \( X \) be a free \( G \)-space, and \( V \) be a \( d \)-dimensional real \( G \)-representation. Then \( \xi_{X,V} \) denotes the following flat vector bundle
\[
V \to X \times_G V \to X/G,
\]
and \( w_{X,V} := w_d(\xi_{X,V}) \in H^d(X/G; \mathbb{F}_2) \cong H^d_G(X; \mathbb{F}_2) \) its top Stiefel–Whitney class.

**Theorem 2.5.** Let \( n \geq 3, 1 \leq k \leq n, \ d := \dim V_k(\mathbb{R}^n) = nk - \frac{k(k+1)}{2} \) and \( U := R_k \oplus P_k^{(n-k)} \). The map in equivariant cohomology induced by the natural projection \( \pi : X \to \text{pt} \):
\[
\pi^* : H^d_{W_k}(\text{pt}; \mathbb{F}_2) \to H^d_{W_k}(V_k(\mathbb{R}^n); \mathbb{F}_2)
\]
is non-zero. In particular,
\[
w_{V_k(\mathbb{R}^n),U} \in H^d(V_k(\mathbb{R}^n)/W_k; \mathbb{F}_2) \cong H^d_{W_k}(V_k(\mathbb{R}^n); \mathbb{F}_2)
\]
does not belong to the kernel of \( \pi^* \), i.e., is the generator of the group \( H^d(V_k(\mathbb{R}^n)/W_k; \mathbb{F}_2) \).

**Proof.** We prove the theorem in the following steps:

- we first equip the Stiefel manifold \( V_k(\mathbb{R}^n) \) with the free \( W_k \)-CW-structure, then
- identify the Stiefel–Whitney class \( w_{V_k(\mathbb{R}^n),U} \) with the equivariant (mod 2)-primary obstruction
\[
\gamma_{W_k}(V_k(\mathbb{R}^n), S(U)) \in H^d_{W_k}(V_k(\mathbb{R}^n); \mathbb{F}_2),
\]
and finally

- prove that the equivariant primary obstruction which decides the existence of \( W_k \)-equivariant map \( V_k(\mathbb{R}^n) \to S(U) \):
\[
\gamma_{W_k}(V_k(\mathbb{R}^n), S(U)) \in H^d_{W_k}(V_k(\mathbb{R}^n); Z),
\]

being also the preimage of \( \gamma_{W_k}(V_k(\mathbb{R}^n), S(U)) \) under the coefficient reduction \( W_k \)-morphism \( Z \to \mathbb{F}_2 \), is the generator of the group \( H^d_{W_k}(V_k(\mathbb{R}^n); Z) \cong \mathbb{Z}/2 \). Moreover,

- our proof will yield that coefficient reduction \( W_k \)-morphism \( Z \to \mathbb{F}_2 \) induces an isomorphism of the groups
\[
H^d_{W_k}(V_k(\mathbb{R}^n); \mathbb{F}_2) \cong H^d_{W_k}(V_k(\mathbb{R}^n); Z).
\]

Here \( Z \) denotes the homotopy group \( \pi_{d-1}(S(U)) \) of the sphere \( S(U) \) considered as a \( W_k \)-module. In particular, it is important to deduce that for any generator \( \varepsilon_i \) of the subgroup \( \langle \varepsilon \rangle \) of the subgroup \( (\mathbb{Z}/2)^k \) of \( W_k \) and \( z \in Z \) the \( W_k \)-action is described by \( \varepsilon_i \cdot z = (-1)^{n-1}z \).

(1) The Stiefel manifold \( V_k(\mathbb{R}^n) \) is a \( d \)-dimensional smooth closed \( W_k \)-manifold. The smooth structure on \( V_k(\mathbb{R}^n) \) can be assumed to be invariant with respect to the action of \( W_k \). Thus, the quotient manifold \( V_k(\mathbb{R}^n)/W_k \) is a \( d \)-dimensional smooth closed manifold. Therefore, \( V_k(\mathbb{R}^n)/W_k \) can be triangulated [4 Theorem, page 389]. Moreover, for a given point \( x \in V_k(\mathbb{R}^n)/W_k \) the triangulation can be chosen in such a way that \( x \) belongs to an interior of some maximal cell. Since the action of the group \( W_k \) on \( V_k(\mathbb{R}^n) \) is free, the triangulation of the quotient \( V_k(\mathbb{R}^n)/W_k \) transforms the Stiefel manifold \( V_k(\mathbb{R}^n) \) into a free \( W_k \)-CW-complex. The \( W_k \)-CW-structure on \( V_k(\mathbb{R}^n) \) has an additional property that the preimages of \( x \), with respect to the quotient map \( V_k(\mathbb{R}^n) \to V_k(\mathbb{R}^n)/W_k \), belong to interiors of different maximal cells.

(2) Since \( V_k(\mathbb{R}^n) \) is a \( d \)-dimensional connected free \( W_k \)-CW-complex and \( \dim U = d \), then

- using [3 Lemma 5.3] the Stiefel–Whitney class \( w_{V_k(\mathbb{R}^n),U} \) can be identified with the equivariant (mod 2)-primary obstruction
\[
\gamma_{W_k}(V_k(\mathbb{R}^n), S(U)) \in H^d_{W_k}(V_k(\mathbb{R}^n); \mathbb{F}_2) \cong H^d(V_k(\mathbb{R}^n)/W_k; \mathbb{F}_2),
\]

- the quotient space \( V_k(\mathbb{R}^n)/W_k \) is a compact manifold, and consequently
\[
H^d(V_k(\mathbb{R}^n)/W_k; \mathbb{F}_2) \cong \mathbb{F}_2.
\]
The equivariant (mod 2)-primary obstruction $\gamma_{W_k}(V_k(\mathbb{R}^n), S(U))$ is the image of the equivariant primary obstruction $\gamma^W(V_k(\mathbb{R}^n), S(U)) \in H^d_{W_k}(V_k(\mathbb{R}^n); Z)$ under the coefficient reduction $W_k$-morphism $Z \rightarrow \mathbb{F}_2$, [3 Section 5.1]. The ambient group of the obstruction can be identified via the Equivariant Poincaré duality, as explained in [2] Theorem 1.4, page 2638,

$$H^d_{W_k}(V_k(\mathbb{R}^n); Z) \cong H^0(\mathbb{V}; V \otimes \mathcal{V})$$

where $\mathcal{V} \cong \mathbb{A}_k Z$ is the orientation character of the Stiefel manifold $V_k(\mathbb{R}^n)$ with respect to the group $W_k$. The action of $W_k$ on $\mathcal{V}$ is given by

$$\varepsilon_i \cdot v = (-1)^i v, \quad \pi \cdot v = \text{sign}(\pi)^n v$$

for $v \in \mathcal{V}$, $\varepsilon_i$ a generator of the subgroup $\langle Z/2 \rangle_k$ and $\pi \in \mathfrak{S}_k \subset W_k$. Here sign($\pi$) $\in \{1, -1\}$ denotes the sign of permutation $\pi$.

In the case when the Stiefel manifold $V_k(\mathbb{R}^n)$ is simply connected we have a sequence of isomorphisms, as in [2] Section 1.5, page 2639,

$$H^0(\mathbb{V}; V \otimes \mathcal{V}) \cong H_0(\mathbb{W}; \mathbb{Z} \otimes \mathcal{V}) \cong \mathbb{Z} \otimes \mathcal{V}_{W_k},$$

where the last group is the group of $W_k$-coinvariants of the $W_k$-module $\mathbb{Z} \otimes \mathcal{V}$. Since every element $\varepsilon_i$ of the group $W_k$ acts on the $W_k$-module $\mathbb{Z} \otimes \mathcal{V}$ by multiplication by “$(-1)^i$” we get that the group of coinvariants is isomorphic with $\mathbb{Z}/2$. When the Stiefel manifold $V_k(\mathbb{R}^n)$ is not simply connected, for $k \in \{n - 1, n\}$, direct calculation yields the same conclusion.

Thus, we have proved that

$$H^d_{W_k}(V_k(\mathbb{R}^n); Z) \cong \mathbb{Z}/2.$$

(3) Finally, we evaluate the equivariant primary obstruction $\gamma^W(V_k(\mathbb{R}^n), S(U))$ using the general position map scheme [2] Definition 1.5, page 2639 and already described $W_k$-CW-structure on $V_k(\mathbb{R}^n)$ where the distinguished point $x \in V_k(\mathbb{R}^n)/W_k$ will be chosen later.

Let $\mathbb{R}^k \subset \mathbb{R}^n$ be the standard inclusion and $x_1, \ldots, x_n : \mathbb{R}^n \rightarrow \mathbb{R}$ denote coordinate functionals. Consider a symmetric quadratic form $\phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ that has generic restriction $\phi|_{\mathbb{R}^k \times \mathbb{R}^k}$, i.e., the restriction form $\phi|_{\mathbb{R}^k \times \mathbb{R}^k}$ has pairwise distinct eigenvalues. The equivariant map in general position $\tau = (\tau_0, \tau_1, \ldots, \tau_{n-k}) : V_k(\mathbb{R}^n) \rightarrow U = R_k \oplus P_k^{\mathbb{R}^{(n-k)}}$ is defined by

$$\tau_0(v_1, \ldots, v_k) = [\phi(v_1, v_j)]_{1 \leq j \leq k},$$

and for $1 \leq j \leq n - k$ by

$$\tau_j(v_1, \ldots, v_k) = (x_{k+j}(v_1), \ldots, x_{k+j}(v_k)).$$

Now we are interested in the solution of the system of equations $\tau(v_1, \ldots, v_k) = 0 \in U$ modulo the group action $W_k$. The subsystem of equations

$$\tau_1(v_1, \ldots, v_k) = (x_{k+1}(v_1), \ldots, x_{k+1}(v_k)) = 0$$
$$\tau_2(v_1, \ldots, v_k) = (x_{k+2}(v_1), \ldots, x_{k+2}(v_k)) = 0$$
$$\ldots$$
$$\tau_{n-k}(v_1, \ldots, v_k) = (x_n(v_1), \ldots, x_n(v_k)) = 0$$

implies that the $k$-frame $v_1, \ldots, v_k$ belongs to $\mathbb{R}^k$. The condition of generic restriction on $\phi$ with the remaining set of equations $\tau_0(v_1, \ldots, v_k) = 0 \in R_k$ implies that the restricted symmetric quadratic form $\phi|_{\mathbb{R}^k \times \mathbb{R}^k}$ on the solution frame $v_1, \ldots, v_k$ has to be diagonal. Thus, a solution of the system of equations $\tau(v_1, \ldots, v_k) = 0 \in U$ exists and solution frame $v_1, \ldots, v_k$ is unique up to the action of the group $W_k$. Moreover, it can be verified that the solution is non-degenerated.

If the $W_k$-CW-structure on $V_k(\mathbb{R}^n)$ is chosen in such a way that the distinguished point is the solution frame $v_1, \ldots, v_n$, then the equivariant obstruction cocycle induced by the map $\tau$, that lives in $C^d_{W_k}(V_k(\mathbb{R}^n); Z)$, evaluates non-trivially only on the cell that contains distinguished point. More precisely, after suitable choice of orientations it evaluates to $1 \in Z$. Thus, the equivariant primary obstruction $\gamma^W(V_k(\mathbb{R}^n), S(U))$ is the generator of the group $H^d_{W_k}(V_k(\mathbb{R}^n); Z)$ and therefore does not vanish. This concludes the proof of the theorem.

A direct consequence of the previous theorem is the following lower bound estimate for the Schwarz genus of the Stiefel manifold.

**Corollary 2.6.**

$$g_{W_k}(V_k(\mathbb{R}^n)) = \text{cat } V_k(\mathbb{R}^n)/W_k = \dim V_k(\mathbb{R}^n) + 1 = nk - \frac{k(k+1)}{2} + 1.$$
Proof. From Lemmas 2.4 and 2.5 we get:
\[
\text{cat } V_k(\mathbb{R}^n)/W_k \geq g_{W_k}(V_k(\mathbb{R}^n)) \geq nk - \frac{k(k+1)}{2} + 1.
\]
On the other hand the dimension bounds the Lusternik–Schnirelmann category from above [5] Theorem 1.7, page 4] and thus
\[
\text{cat } V_k(\mathbb{R}^n)/W_k \leq \dim V_k(\mathbb{R}^n) + 1 = nk - \frac{k(k+1)}{2} + 1.
\]
Therefore, the equality holds in both cases. \(\square\)

2.3. The Lusternik–Schnirelmann category of the unordered configuration space of the projective space. Let \(G\) be a finite group that acts on the topological space \(X\), and \(k > 1\) be an integer. The \(G\)-ordered configuration space of \(k\) pairwise distinct points in \(X\) is defined to be:
\[
F_G(X, k) := \{(x_1, \ldots, x_k) \in X^k : x_i \neq g \cdot x_j \text{ for } i \neq j \text{ and } g \in G\}.
\]
The symmetric group \(\Sigma_k\) acts naturally on the \(G\)-ordered configuration space by permuting the factors. The quotient space \(B_G(X, k) := F_G(X, k)/\Sigma_k\) is called the \(G\)-unordered configuration space of \(k\) pairwise distinct points in \(X\). When \(G\) is the trivial group, the \(G\)-ordered configuration space \(F_G(X, k)\) coincides with the classical ordered configuration space \(F(X, k)\); and similarly for unordered configuration space \(B(X, k)\).

Using the results of the previous section we give a lower bound on the Lusternik–Schnirelmann category of the unordered configuration space of the projective space \(B(\mathbb{R}P^d, k)\) when \(d \geq 3\).

Corollary 2.7. For \(d \geq 3\):
\[
\text{cat } B(\mathbb{R}P^d, k) \geq dk - \frac{k(k-1)}{2} + 1.
\]

Proof. Let \(\mathbb{Z}/2\) act antipodally on the sphere \(S^d\). The group \(W_k\) acts naturally on the \(\mathbb{Z}/2\)-configuration space \(F_{\mathbb{Z}/2}(S^d, k)\). Then there is a homeomorphism:
\[
B(\mathbb{R}P^d, k) \equiv B_{\mathbb{Z}/2}(S^d, k)/W_k.
\]
Since \(V_k(\mathbb{R}^{d+1}) \subset B_{\mathbb{Z}/2}(S^d, k)\), then the monotonicity of the Schwarz genus under equivariant inclusion (see [5] or [11]) implies:
\[
g_{W_k}(B_{\mathbb{Z}/2}(S^d, k)) \geq g_{W_k}(V_k(\mathbb{R}^{d+1})) = (d+1)k - \frac{k(k+1)}{2} + 1 = dk - \frac{k(k-1)}{2} + 1.
\]
Application of Lemma 2.5 concludes the proof. \(\square\)

Remark 2.8. For \(k = 2\) the space \(B_{\mathbb{Z}/2}(S^d, k)\) is \(W_k\)-equivariantly homotopy equivalent to \(V_k(\mathbb{R}^{d+1})\) and therefore we have the equality:
\[
\text{cat } B_{\mathbb{Z}/2}(\mathbb{R}P^d) = 2d.
\]

Remark 2.9. Results of [9], with more detailed proofs in [3], give a lower bound for the Lusternik–Schnirelmann category of \(B(M, k)\) in the form \((d-1)(k - D_p(k)) + 1\) for a \(d\)-dimensional manifold \(M\) and \(D_p(k)\) the sum of digits in the \(p\)-adic expansion of \(k\) for the prime \(p\). Thus, for small \(k\) the bound from Corollary 2.7 improves this general bound.

3. Counting geometric configurations

In this section we use the results of Section 2 in order to count different geometric configurations: critically outscribed parallelotopes and Birkhoff–James orthonormal bases.

3.1. Critically outscribed parallelotopes. Now we are going to consider parallelotopes outscribed around a given convex body and estimate the possible number of critical ones.

A parallelotope is a convex polytope \(P\) in \(\mathbb{R}^n\), bounded by \(2n\) pairwise distinct hyperplanes
\[
H_0^1, H_1^1, H_0^2, H_2^2, \ldots, H_0^n, H_n^1
\]
such that \(H_0^i\) and \(H_1^i\) are parallel for every \(i\), and \(H_0^0, H_1^0, \ldots, H_n^n\) are in general position. The class of parallelotopes naturally extends the class of parallelograms from the plane to higher dimensions.

Definition 3.1. A parallelotope \(P \subset \mathbb{R}^n\) is critically outscribed around the convex body \(K\) if:
(1) \(K \subseteq P\), and
(2) for every pair of its parallel defining hyperplanes \(H_0^i\) and \(H_1^i\) there exist points \(x_0 \in H_0^i \cap K\) and \(x_1 \in H_1^i \cap K\) such that the line determined by \(x_0\) and \(x_1\) is parallel with all the remaining hyperplanes \(H_0^0, H_1^0\) where \(j \neq i\).
The notion of a critically outscribed parallelotope arises naturally in the context of finding parallelotopes of minimal volume containing convex body \( K \). Every parallelotope containing \( K \) that has the minimal volume must be critically outscribed. Thus, it is important to estimate the possible number of critically outscribed parallelotope around a convex body.

**Corollary 3.2.** Every strictly convex body \( K \subset \mathbb{R}^n \) has at least \( \frac{n(n-1)}{2} + 1 \) distinct critically outscribed parallelotopes.

**Proof.** Consider the manifold \( M \) of all \( n \)-frames \((e_1, \ldots, e_n)\), with unit vectors \( e_i \) not necessarily orthogonal to each other. The manifold \( M \) contains \( V_n(\mathbb{R}^n) \) as a \( W_n \)-equivariant subspace. Therefore by Corollary 3.1 and the monotonicity of the Schwarz genus we have that
\[
\text{cat } M/W_n \geq g_{W_n}(M) \geq g_{W_n}(V_n(\mathbb{R}^n)) = \frac{n(n-1)}{2} + 1.
\]

Consider a function \( f : M \to \mathbb{R} \) defined as follows. For any \( n \)-frame \((e_1, \ldots, e_n) \in M \), find the support hyperplanes \( H_i^0, H_i^1 \) to \( K \) orthogonal to \( e_i \) such that \( \langle e_i, H_i^1 \rangle > \langle e_i, H_i^0 \rangle \). Here \( \langle \cdot, \cdot \rangle \) denotes the standard inner product of \( \mathbb{R}^n \). These hyperplanes bound a parallelotope \( P(e_1, \ldots, e_n) \). Put
\[
f(e_1, \ldots, e_n) := \text{vol}(P(e_1, \ldots, e_n)).
\]
The function \( f: M \to \mathbb{R} \) is a smooth \( W_n \)-invariant proper function on \( M \). Here proper function means that the preimage of every compact set is also compact. Thus, \( f \) has at least \( \text{cat } M/W_n \) critical \( W_n \)-orbits by the standard Lusternik–Schnirelmann theory, [3, Theorem 1.15, page 7].

Finally, we note that distinct critical orbits of the function \( f : M \to \mathbb{R} \) correspond to the distinct critically outscribed parallelotopes around \( K \). Indeed, consider the dependence of \( \text{vol}(P(e_1, \ldots, e_n)) \) on some \( e_i \). Without loss of generality let it be \( e_1 \). The support hyperplanes to \( K \) with fixed normals \( \pm e_2, \ldots, \pm e_n \) form a cylinder \( Z \). Now varying supporting hyperplanes \( H_i^0, H_i^1 \) with normals \( -e_1 \) and \( e_1 \) cut the parallelotope \( P(e_1, \ldots, e_n) \) out of the cylinder \( Z \). Let \( \ell \) be the unique line orthogonal to \( e_2, \ldots, e_n \). If we only vary \( e_1 \) then the volume \( \text{vol}(P(e_1, \ldots, e_n)) \) is proportional to the length \( L(e_1) \) of the segment cut on \( \ell \) by the hyperplanes \( H_i^0 \) and \( H_i^1 \). Parameterizing \( \ell \) by \( t \cdot v \) for some unit vector \( v \) and \( t \in \mathbb{R} \) and introducing the support function
\[
s(K, y) = \sup_{x \in K} \langle y, x \rangle
\]
we obtain the expression
\[
L(e_1) = \frac{s(K, e_1)}{\langle v, e_1 \rangle} - \frac{s(K, -e_1)}{\langle v, -e_1 \rangle}.
\]
This expression does not depend on the length of \( e_1 \). Thus we can normalize \( e_1 \) by \( \langle e_1, v \rangle = 1 \) instead of \( \| e_1 \| = 1 \). Under this constraint the expression simplifies to \( L(e_1) = s(K, e_1) + s(K, -e_1) \). Recall that the directional differential of the support function can be identified in the case of strict convexity with the support point. For \( s(K, e_1) \) this is \( p_+ = H_1^1 \cap K \) and for \( s(K, -e_1) \) this is \( p_- = H_1^0 \cap K \). Hence
\[
dL(e_1) = p_+ - p_-.
\]
Recalling the constraint \( \langle v, e_1 \rangle = 1 \) we see that the condition for a critical point is that \( p_+ - p_- \parallel \ell \). Applied for each base vector \( e_i \) this gives the definition of a critically outscribed parallelotope. \( \square \)

**Remark 3.3.** Using the general Stiefel manifold \( V_k(\mathbb{R}^n) \) it is possible to generalize Corollary 3.1 to \((n-k)\)-cylinders over \( k \)-dimensional parallelotopes. An \((n-k)\)-cylinder is a convex region in \( \mathbb{R}^n \) bounded by \( 2k \) pairwise distinct hyperplanes \( H_0^0, H_1^0, \ldots, H_n^0, H_0^1, \ldots, H_{n-k}^1 \) that are parallel in pairs, \( H_0^0 \parallel H_1^0 \), and \( H_0^1, \ldots, H_{n-k}^1 \) are in general position. Each such convex region has a \( k \)-dimensional cross-section volume. We see that among the \((n-k)\)-cylinders over \( k \)-dimensional parallelotopes outscribed around the convex body \( K \subset \mathbb{R}^n \) there are at least \( nk - \frac{k(k+1)}{2} + 1 \) critical ones with respect to the cross-section volume. This “criticality” may be described in terms of segments of opposite support points, like in Definition 3.1.

### 3.2. Birkhoff–James orthogonal bases.

Similarly, as in Corollary 3.2, we estimate the number of Birkhoff–James orthonormal bases in a finite dimensioned normed space. Assume that the norm \( \| \cdot \| \) in \( \mathbb{R}^n \) is smooth.

**Definition 3.4.** The vectors \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) are **Birkhoff–James orthogonal** with respect to \( \| \cdot \| \), and denoted by \( x \perp_{BJ} y \), if \( \| x + ty \| = 0 \) for all \( t \).

This relation needs not to be symmetric, that is \( x \perp_{BJ} y \) is not equivalent to \( y \perp_{BJ} x \). In fact, for \( n \geq 3 \), the Birkhoff–James orthogonality is symmetric if and only if the norm is Euclidean. When considering a Birkhoff–James orthogonal basis \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \) it is important to be aware that \( e_i \perp_{BJ} e_j \) and \( e_j \perp_{BJ} e_i \) are two different conditions.
Corollary 3.5. Every smooth norm on $\mathbb{R}^n$ has at least $\frac{n(n-1)}{2} + 1$ Birkhoff–James orthonormal bases.

Proof. Consider again the manifold $M$ of all bases $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ such that $\|e_1\| = \cdots = \|e_n\| = 1$. The function $f: M \to \mathbb{R}$ defined by

$$f(e_1, \ldots, e_n) := \frac{1}{\det(e_1, \ldots, e_n)}$$

is proper, smooth, and $W_n$-invariant on $M$. Hence it has at least $\text{cat} M/W = \frac{n(n-1)}{2} + 1$ critical points.

Now, for a pair of distinct indexes $(i, j)$ and the fixed basis $(e_1, \ldots, e_n)$ we consider the function $\phi_{(i,j)}: \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_{(i,j)}(t) := \frac{1}{\det(e_1, \ldots, e_i+te_j, \ldots, e_n)}.$$

The first derivative of this function evaluated at $t = 0$ is

$$\phi'_{(i,j)}(0) = \left( \frac{\partial}{\partial t} \det(e_1, \ldots, e_i+te_j, \ldots, e_n) \right)_{t=0} = \left( \frac{\partial}{\partial t} \det(e_1, \ldots, e_i, \ldots, e_n) \right)_{t=0} = \frac{\|e_i+te_j\|_{t=0}}{\det(e_1, \ldots, e_i+te_j, \ldots, e_n)}.$$

Let $E_{ij}(t) \in GL(n)$ be the elementary matrix with $t$ at position $(i, j)$, units on the diagonal, and zeroes elsewhere. Then $\phi'_{(i,j)}(0) = \frac{d}{dt}f((e_1, \ldots, e_n) \cdot E_{ij}(t))$, where “$\cdot$” denotes the right action defined in Section 2.2. For a critical point of $f$, all such derivatives must vanish and thus a critical point of the function $f$ is also a Birkhoff–James orthonormal basis. This concludes the proof. □

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