ADDITIVE STRUCTURE OF TOTALLY POSITIVE QUADRATIC INTEGERS

TOMÁŠ HEJDA AND VÍTĚZSLAV KALA

Abstract. Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field. We obtain a presentation of the additive semigroup $\mathcal{O}_K^+(\mathbb{Z})$ of totally positive integers in $K$; its generators (indecomposable integers) and relations can be nicely described in terms of the periodic continued fraction for $\sqrt{D}$. We also characterize all uniquely decomposable integers in $K$ and estimate their norms. Using these results, we prove that the semigroup $\mathcal{O}_K^+(\mathbb{Z})$ completely determines the real quadratic field $K$.

1. Introduction

The additive semigroup of totally positive integers $\mathcal{O}_K^+$ in a totally real number field $K$ has long played a fundamental role in algebraic number theory, even though more attention has perhaps been paid to the multiplicative structure of the ring $\mathcal{O}_K$, for example, to its units and unique factorization into primes. Most prominent purely additive objects are the indecomposable elements, i.e., totally positive integers $\alpha \in \mathcal{O}_K^+$ that cannot be decomposed into a sum $\alpha = \beta + \gamma$ of totally positive integers $\beta, \gamma \in \mathcal{O}_K^+$. For example, in 1945, Siegel used them (under the name “extremal elements”) to prove that if $K$ is a number field different from $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$, then there is a totally positive integer in $K$ that cannot be written as sum of any number of squares [Sie45].

In the real quadratic case $K = \mathbb{Q}(\sqrt{D})$, indecomposables can be nicely characterized in terms of continued fraction (semi-)convergents to $\sqrt{D}$ ([Per13, DSS2], cf. Section 2); in the second cited work, Dress and Scharlau proved an upper bound on the norm of each indecomposable $N(\alpha) \leq D$, which was recently refined by Jang, Kim, and the second author [JK16, Kal16a]. This stands in contrast to the situation of a general totally real field $K$, where it is much harder to describe indecomposables: Brunotte [Bru83] proved an upper bound on their norm in terms of the regulator, but otherwise their structure remains quite mysterious.

The goal of this short article is to study the structure of the whole additive semigroup $\mathcal{O}_K^+(\mathbb{Z})$. This is an interesting problem in itself, but it seems also necessary for certain applications (such as the recent progress in the study of universal quadratic forms and lattices over $K$ by Kim, Blomer, and the second author [Kim00, BK15, Kal16b, BK17]).
In particular, as indecomposable elements are precisely the generators of $\mathcal{O}_K^+(+)$, we need to determine the relations between them. While the description of indecomposables in terms of the continued fraction is fairly straightforward, it is a priori not clear at all if the same will be the case for relations, as there could be some "random" or "accidental" ones. Perhaps surprisingly, it turns out that this is not the case and that the presentation of the semigroup $\mathcal{O}_K^+(+)$ (given in Theorem 1) is quite elegant. A key tool in the proof of the presentation is the fact that each totally positive integer can be uniquely written as a $\mathbb{Z}^+$-linear combination of two consecutive indecomposables (Proposition 3).

One of course cannot hope to have an analogue of unique factorization in the additive setting, but nevertheless, some elements can be uniquely decomposed as a sum of indecomposables. In Theorem 4 we characterize all such uniquely decomposable elements and obtain again a very explicit result depending only on the continued fraction. This then yields a direct proof of Theorem 6 that $\mathcal{O}_K^+(+)$ (viewed as an abstract semigroup) completely determines $D$ and the number field $K$. Let us briefly remark that this result can be viewed alongside a number of beautiful results concerning the (im)possibility of reconstructing a number field from some of its invariants, such as the absolute Galois group, Dedekind zeta-function, or Dirichlet $L$-series, see, e.g., [Gaß26, Kub57, Neu69, Uch76, CdSL+17].

A natural question to ask is of course whether an analogue of our Theorem 6 holds also for totally real number fields of higher degree, although this may be quite hard, as we lack a good understanding of indecomposable elements.

Finally, we use our results to estimate the norms of totally positive integers, and in particular, analogously to the results that norms of convergents and indecomposables are at most $2D^{1/2}$ and $D$, respectively, we show that the norm of a uniquely decomposable elements is at most of the order $D^{3/2}$ (Theorem 10).

2. Preliminaries

Throughout the work, we will use the following notation. We fix a squarefree integer $D \geq 2$ and consider the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ and its ring of integers $\mathcal{O}_K$; we know that $\{1, \omega_D\}$ forms an integral basis of $\mathcal{O}_K$, where

$$\omega_D := \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod{4}, \\ \frac{1 + \sqrt{D}}{2} & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

By $\Delta$ we denote the discriminant of $K$, i.e., $\Delta = 4D$ if $D \equiv 2, 3 \pmod{4}$ and $\Delta = D$ otherwise. The norm and trace from $K$ to $\mathbb{Q}$ are denoted by $N$ and $\text{Tr}$, respectively.

An algebraic integer $\alpha \in \mathcal{O}_K$ is totally positive iff $\alpha > 0$ and $\alpha' > 0$, where $\alpha'$ is the Galois conjugate of $\alpha$, we write this fact as $\alpha > 0$; for $\alpha, \beta \in \mathcal{O}_K$ we denote by $\alpha \succ \beta$ the fact that $\alpha - \beta > 0$, and by $\mathcal{O}_K^+$ the set of all totally positive integers. We say that $\alpha \in \mathcal{O}_K^+$ is indecomposable iff it can not be written as a sum of two totally positive integers or equivalently iff there is no algebraic integer $\beta \in \mathcal{O}_K^+$ such that $\alpha \succ \beta$. We say that $\alpha \in \mathcal{O}_K^+$ is uniquely decomposable iff there is a unique way how to express it as a sum of indecomposable elements.

It will be slightly more convenient for us to work with a purely periodic continued fraction, and so let $\sigma_D = [\overline{u_0, u_1, \ldots, u_{s-1}}]$ be the periodic continued fraction
expansion of
\[ \sigma_D := \omega_D + [-\omega_D'] = \begin{cases} \sqrt{D} + \lfloor \sqrt{D} \rfloor & \text{if } D \equiv 2, 3 \pmod{4}, \\ \frac{1 + \sqrt{D}}{2} + \lfloor -\frac{1 + \sqrt{D}}{2} \rfloor & \text{if } D \equiv 1 \pmod{4} \end{cases} \]
(with positive integers \( u_i \)). We then have that \( \omega_D = \lfloor u_0/2 \rfloor, u_1, \ldots, u_\infty \). It is well known that \( u_1, u_2, \ldots, u_{s-1} \) is a palindrome and that \( u_0 = u_s \) is even if and only if \( D \equiv 2, 3 \pmod{4} \), hence \( \lfloor u_0/2 \rfloor = (u_s + \text{Tr}(\omega_D))/2 \).

Denote the convergents to \( \omega_D \) by \( p_i/q_i := \lfloor u_0/2 \rfloor, u_1, \ldots, u_i \) and recall that the sequences \((p_i), (q_i)\) satisfy the recurrence
\[ X_{i+2} = u_{i+2}X_{i+1} + X_i \quad \text{for} \quad i \geq -1 \]
with the initial condition \( q_{-1} = 0, p_{-1} = q_0 = 1 \), and \( p_0 = \lfloor u_0/2 \rfloor \). Denote \( \alpha_i := p_i - q_i\omega_D \) and \( \alpha_i, r = \alpha_i + r\alpha_{i+1} \). Then we have the following classical facts (see, e.g., [Per13, DS82]):

- The sequence \((\alpha_i)\) satisfies the recurrence (1).
- \( \alpha_i > 0 \) if and only if \( i \geq -1 \) is odd.
- The indecomposable elements in \( \mathcal{O}_K^+ \) are \( \alpha_i, r \) with odd \( i \geq -1 \) and \( 0 \leq r \leq u_i+2-1 \), together with their conjugates.
- We have that \( \alpha_{i, u_i+2} = \alpha_{i+2, 0} \).
- The indecomposables \( \alpha_i, r \) are increasing with increasing \((i, r)\) (in the lexicographic sense).
- The indecomposables \( \alpha_i', r \) are decreasing with increasing \((i, r)\).

We also denote \( \varepsilon > 1 \) the fundamental unit of \( \mathcal{O}_K \); we have that \( \varepsilon = \alpha_{s-1} \).

Furthermore, we denote \( \varepsilon^+ > 1 \) the smallest totally positive unit \( > 1 \); we have that \( \varepsilon^+ = \varepsilon \) if \( s \) is even and \( \varepsilon^+ = \varepsilon^2 = \alpha_{2s-1} \) if \( s \) is odd. Furthermore, we denote \( \gamma_0 = \omega_D \) and \( \gamma_i = [u_i, u_{i+1}, u_{i+2}, \ldots] \) for \( i \geq 1 \); we have that \( u_i < \gamma_i = u_i + \frac{1}{\gamma_{i+1}} < u_i + 1 \) for \( i \geq 1 \).

In the final Section 5 we will estimate norms of totally positive integers and particularly uniquely decomposable integers. To this end, we will use the following additional notation: For a convergent \( \alpha_i \), we set
\[ N_i := |N(\alpha_i)| = (-1)^{i+1}N(\alpha_i) = \begin{cases} |p_i^2 - Dq_i^2| & \text{if } D \equiv 2, 3 \pmod{4}, \\ |p_i^2 - p_iq_i - q_i^2D - 1| & \text{if } D \equiv 1 \pmod{4}. \end{cases} \]
Recall that we have \( p_{i+1}q_i - p_iq_{i+1} = (-1)^i \) and let us define \( T_i \) so that \( \alpha_i\alpha'_{i+1} = T_{i+1} + (-1)^i\omega_D \). This means that we have \( T_i = p_i(p_i - q_i - q_i - \frac{D-1}{4}) \) or \( T_i = p_ip_{i-1} - Dq_iq_{i-1} \) when \( D \equiv 1 \pmod{4} \) or \( 2, 3 \pmod{4} \), respectively.

3. Presentation of the semigroup \( \mathcal{O}_K^+(+) \)

In this section we will prove the following theorem that gives a presentation of the semigroup \( \mathcal{O}_K^+(+) \). We recall that \( \langle S \mid \mathcal{R} \rangle \) is a presentation of a semigroup \( G(+) \) if \( G \) is generated by \( S \) and all (additive) relations between elements of \( S \) are generated by the relations in \( \mathcal{R} \).

**Theorem 1.** The additive semigroup \( \mathcal{O}_K^+(+) \) is presented by
\[ \mathcal{O}_K^+ = \langle \mathcal{A} \cup \mathcal{A}' \cup \{1\} \mid \mathcal{R}, \mathcal{R}', \mathcal{R}_0 \rangle, \]
where \( \mathcal{A} := \{ \alpha_{i,r} : i \geq -1 \text{ odd and } 0 \leq r \leq u_{i+2} - 1 \} \setminus \{1\} \) are the indecomposable elements \( > 1 \), \( \mathcal{A}' := \{ y' : y \in \mathcal{A} \} \), and the relations are the following:

\[
\begin{align*}
(2) \quad &\mathcal{R} : \alpha_{i,r-1} - 2\alpha_{i,r} + \alpha_{i,r+1} = 0 \quad \text{for odd } i \geq -1 \text{ and } 1 \leq r \leq u_{i+2} - 1, \\
&\alpha_{i-2,u_{i-1}} - (u_{i+1} + 2)\alpha_{i,0} + \alpha_{i,1} = 0 \quad \text{for odd } i \geq 1;
\end{align*}
\]

\( \mathcal{R}' : \) some relations as in \( \mathcal{R} \) after applying the isomorphism \( (') \);

\( \mathcal{R}_0 : \alpha_{i,1} - (u_0 + 2) \cdot 1 + \alpha_{-1,1} = 0. \)

For convenience, we introduce an alternative notation of the indecomposables. We define \( \beta_j, j \in \mathbb{Z} \) by the condition that \( \cdots < \beta_{-3} < \beta_{-2} < \beta_{-1} < \beta_0 = 1 < \beta_1 < \beta_2 < \beta_3 < \cdots \) is the increasing sequence of the indecomposables. Note that we have \( \beta_j' = \beta_{-j} \) for all \( j \in \mathbb{Z} \).

**Lemma 2.** For each \( j \in \mathbb{Z} \) we have that

\[
v_j\beta_j = \beta_{j-1} + \beta_{j+1},
\]

where

\[
v_j := \begin{cases} 
2 & \text{if } \beta_{|j|} = \alpha_{i,r} \text{ with odd } i \geq -1 \text{ and } 1 \leq r \leq u_{i+2} - 1, \\
u_{i+1} + 2 & \text{if } \beta_{|j|} = \alpha_{i,0} \text{ with odd } i \geq -1.
\end{cases}
\]

**Proof.** As \( \beta_{-j} = \beta_j' \), we can assume \( j \geq 0 \). We have \( \beta_j = \alpha_{i,r} \) for some odd \( i \geq -1 \) and \( 0 \leq r \leq u_{i+2} - 1 \).

If \( 1 \leq r \leq u_{i+2} - 1 \), then \( \beta_{j-1} = \alpha_{i,r-1} = \alpha_i + (r-1)\alpha_{i+1}, \beta_j = \alpha_{i,r} = \alpha_i + r\alpha_{i+1} \) and \( \beta_{j+1} = \alpha_{i,r+1} = \alpha_i + (r+1)\alpha_{i+1} \) form an arithmetic sequence, whence \( \beta_j - \beta_{j-1} = \beta_{j+1} - \beta_j \), which is the statement.

If \( r = 0 \) and \( i \geq 1 \), we have a multiple of \( \alpha_{i,0} = \alpha_i \) as the left-hand side and \( \alpha_{i-2,u_{i-1}} + \alpha_{i,1} \) as the right-hand side. We use the definition of \( \alpha_{i,r} \) and the recurrence (1) for \( \alpha_j \) to see that

\[
\alpha_{i-2,u_{i-1}} + \alpha_{i,1} = \alpha_{i-2} + (u_i - 1)\alpha_{i-1} + \alpha_i + \alpha_{i+1} \\
= \alpha_i - \alpha_{i-1} + \alpha_i + u_{i+1}\alpha_i + \alpha_{i-1} = (u_{i+1} + 2)\alpha_i = (u_{i+1} + 2)\alpha_{i,0}.
\]

Finally, consider \( r = 0 \) and \( i = -1 \), i.e., the case \( j = 0 \). We have \( \beta_1 = \alpha_{-1,1} = \alpha_1 + 1 = [u_0/2] - \omega' + 1, \) whence

\[
\beta_1 + \beta_{-1} = \Tr(\beta_1) = \Tr([u_0/2] - \omega'_D + 1) = 2[u_0/2] - \Tr(\omega_D) + 2 = u_0 + 2. \quad \Box
\]

Note that with the notation from Lemma \( 2 \) we can rewrite the relations (2) in a unified way in terms of \( \beta_j \) and \( v_j \) as follows:

\[
(3) \quad \mathcal{R}, \mathcal{R}', \mathcal{R}_0 : \beta_{j-1} - v_j\beta_j + \beta_{j+1} = 0 \quad \text{for } j \in \mathbb{Z}.
\]

We can now show that each totally positive integer is a linear combination of two consecutive indecomposables with non-negative integral coefficients. A variant of this statement concerning sums of powers of units was used by Kim, Blomer, and the second author [Kim00, BK17] in the construction of universal quadratic forms.

**Proposition 3.** Let \( x \in \mathcal{O}_K^+ \) be given as a finite sum \( x = \sum k_j\beta_j \) with \( k_j \in \mathbb{Z} \). Then there exist unique \( j_0, e, f \in \mathbb{Z} \) with \( e \geq 1 \) and \( f \geq 0 \) such that \( x = e\beta_{j_0} + f \beta_{j_0+1} \). Moreover, the relation \( e\beta_{j_0} + f \beta_{j_0+1} - \sum k_j\beta_j = 0 \) is a \( \mathbb{Z} \)-linear combination of the relations (3).
Proof. As the sequence \((\beta_j)_{j \in \mathbb{Z}}\) is strictly increasing from 0 to \(+\infty\) and the sequence \((\beta'_j)_{j \in \mathbb{Z}}\) is strictly decreasing from \(+\infty\) to 0, we know that the sequence \((\beta_j/\beta'_j)_{j \in \mathbb{Z}}\) is strictly increasing from 0 to \(+\infty\). As \(x > 0\), we know that \(x/x' > 0\). Hence we can fix the unique index \(j_0\) such that

\[
\frac{\beta_{j_0}}{\beta'_{j_0}} \leq \frac{x}{x'} < \frac{\beta_{j_0+1}}{\beta'_{j_0+1}}.
\]

We will first show that \(x = \sum_{j=\min}^{\max} k_j \beta_j\) can be rewritten to

\[
x = e \beta_{j_0} + f \beta_{j_0+1}
\]

with \(e, f \in \mathbb{Z}\) using only relations (3). We assume that \(j_{\min} \leq j_0\) and \(j_{\max} \geq j_0 + 1\) (if not, we pad the sum by zeros to achieve this). We use induction on the length of the sum \(L = j_{\max} - j_{\min} + 1 \geq 2\):

- Suppose \(L = 2\). Then \(j_{\min} = j_0\) and \(j_{\max} = j_0 + 1\). Putting \(e = k_{j_0}\) and \(f = k_{j_0+1}\) gives the statement.
- Suppose \(L \geq 3\) and \(j_{\min} \leq j_0 - 1\). Then

\[
x = (k_{j_{\min}+1} + v_{j_{\min}+1} k_{j_{\min}}) \beta_{j_{\min}+1} + (k_{j_{\min}+2} - k_{j_{\min}}) \beta_{j_{\min}+2} + \sum_{j=j_{\min}+3}^{j_{\max}} k_j \beta_j
\]

(here we used (3) for \(j = j_{\min} + 1\)) is a sum of the same form with length \(L - 1\).
- Suppose \(L \geq 3\) and \(j_{\min} = j_0\). Then \(j_{\max} = j_0 + L - 1 \geq j_0 + 2\) and

\[
x = \sum_{j=j_{\min}}^{j_{\max}-3} k_j \beta_j + (k_{j_{\max}+2} - k_{j_{\max}}) \beta_{j_{\max}+2} + (k_{j_{max}+1} + v_{j_{max}+1} k_{j_{\max}}) \beta_{j_{\max}+1}
\]

(here we used (3) for \(j = j_{\max} - 1\)) is a sum of the same form with length \(L - 1\).

This finishes the proof of (5) (note that so far we have not used the property of \(j_0\)).

We now plug (5) into (4) to derive that

\[
0 < f(\beta_{j_0+1} \beta'_{j_0} - \beta_{j_0} \beta'_{j_0+1}) \quad \text{and} \quad 0 < e(\beta_{j_0+1} \beta'_{j_0} - \beta_{j_0} \beta'_{j_0+1}).
\]

As \(\beta'_{j_0} > 0\) and \(\beta_{j_0+1} > \beta_{j_0} > 0\), we have that \(\beta_{j_0+1} \beta'_{j_0} - \beta_{j_0} \beta'_{j_0+1} > 0\), hence \(e > 0\) and \(f \geq 0\).

To prove the uniqueness of \(j_0\) from the statement of the proposition, suppose to the contrary that \(j_1 \neq j_0\) and \(x = e_1 \beta_{j_1} + f_1 \beta_{j_1+1}\) with \(e_1 > 0\) and \(f_1 \geq 0\). We first consider the case when \(j_1 \leq j_0 - 1\), i.e.,

\[
\frac{\beta_{j_1}}{\beta'_{j_1}} < \frac{\beta_{j_1+1}}{\beta'_{j_1+1}} \leq \frac{x}{x'} = \frac{e_1 \beta_{j_1} + f_1 \beta_{j_1+1}}{e_1' \beta_{j_1} + f_1' \beta_{j_1+1}}.
\]

Easy manipulation leads to \(e_1(\beta_{j_1+1} \beta'_{j_1} - \beta_{j_1} \beta'_{j_1+1}) \leq 0\), hence \(e_1 \leq 0\). Likewise, in the case \(j_1 \geq j_0 + 1\) we get \(f_1 < 0\).

Once \(j_0\) is fixed, the uniqueness of \(e, f\) follows from the linear independence of \(\beta_{j_0}, \beta_{j_0+1}\) over \(\mathbb{Q}\).

We now use this fact to prove the presentation of \(\mathcal{O}_K^+(+)\) as our first main result. \(\square\)
Proof of Theorem 4. The indecomposable elements are generators of $\mathcal{O}_K^+$ by definition (by considering the trace $\text{Tr} : K \to \mathbb{Q}$ it is easy to see that there are no infinite descending chains in $\mathcal{O}_K^+$ with respect to $\succ$), and the relations $\mathcal{R}, \mathcal{R}', \mathcal{R}_0$ hold in $\mathcal{O}_K^+$ by Lemma 2. So it remains to show that if any other (additive) relation holds between indecomposables, it can be derived from (3).

To this end, consider any relation between indecomposables $\sum k_j \beta_j = 0$ (where of course only finitely many $k_j$’s are non-zero). Let us write this as two sums,

$$
\sum_{k_j > 0} k_j \beta_j - \sum_{k_j < 0} (k_j) \beta_j = 0, \quad \text{i.e.,} \quad x := \sum_{k_j > 0} k_j \beta_j = \sum_{k_j < 0} (-k_j) \beta_j.
$$

Clearly $x \in \mathcal{O}_K^+$. By Proposition 3 it is uniquely written as $x = e \beta_j + f \beta_{j+1}$ for some $j \in \mathbb{Z}$, $e \geq 1$, $f \geq 0$. Let us start with the trivial identity $e \beta_j + f \beta_{j+1} = e \beta_j + f \beta_{j+1}$. From Proposition 3 we know that applying the relations (3) we can derive (6), as desired.

Note that it is easy to see from (3) that the given set of relations is minimal in the sense that none of them can be removed.

4. Uniquely decomposable elements

We now characterize all the uniquely decomposable elements in $\mathcal{O}_K^+$ using the results of the previous section.

Theorem 4. All uniquely decomposable elements $x \in \mathcal{O}_K^+$ are the following:

(a) $\alpha_{i,r}$ with odd $i \geq -1$ and $0 \leq r \leq u_{i+2} - 1$;
(b) $e \alpha_{i,0}$ with odd $i \geq -1$ and with $2 \leq e \leq u_{i+1} + 1$
(c) $\alpha_{i,u_{i+2}-1} + f \alpha_{i+2,0}$ with odd $i \geq -1$ odd such that $u_{i+2} \geq 2$ and with $1 \leq f \leq u_{i+3}$;
(d) $e \alpha_{i,0} + \alpha_{i,1}$ with odd $i \geq -1$ such that $u_{i+2} \geq 2$ and with $1 \leq e \leq u_{i+1}$;
(e) $e \alpha_{i,0} + f \alpha_{i+2,0}$ with odd $i \geq -1$ such that $u_{i+2} = 1$ and with $1 \leq e \leq u_{i+1} + 1$, $1 \leq f \leq u_{i+3} + 1$, $(e, f) \neq (u_{i+1} + 1, u_{i+3} + 1)$;
(f) Galois conjugates of all of the above.

Note that the first item lists all the indecomposables (which are clearly uniquely decomposable) and the second one comprises small multiples of all convergents $\alpha_i > 0$, including positive integers $1, 2, \ldots, u_0 + 1$.

Proof. The statement of the theorem, rewritten using $\beta_j$ and $v_j$ (defined in Lemma 2), says that $x$ is uniquely decomposable if and only if $x = e \beta_j + f \beta_{j+1}$ with

$$
1 \leq e \leq v_j - 1, \quad 0 \leq f \leq v_{j+1} - 1 \quad \text{and} \quad (e, f) \neq (v_j - 1, v_{j+1} - 1).
$$

Let $x$ be uniquely decomposable. By Proposition 3 it can be written as $x = e \beta_j + f \beta_{j+1}$ for some $e, f, j \in \mathbb{Z}$ with $e \geq 1, f \geq 0$. Suppose now that $e, f$ do not satisfy (7). This can happen only in one of the following ways:

- $e \geq v_j$. Then $x = e \beta_j + f \beta_{j+1} = \beta_{j-1} + (e - v_j) \beta_j + (f + 1) \beta_{j+1}$ are two decompositions of $x$ (see Lemma 2).
- $f \geq v_{j+1}$. Then $x = e \beta_j + f \beta_{j+1} = (e + 1) \beta_j + (f - v_{j+1}) \beta_{j+1} + \beta_{j+2}$ are two decompositions of $x$.
- $(e, f) = (v_j - 1, v_{j+1} - 1)$. Then $x = (v_j - 1) \beta_j + (v_{j+1} - 1) \beta_{j+1} = \beta_{j-1} + \beta_{j+2}$ are two decompositions of $x$. 

Conversely, let us now show that the condition (7) is sufficient. First note that (7) implies
\[ x + \beta_j \leq v_j \beta_j + v_{j+1} \beta_{j+1} \quad \text{or} \quad x + \beta_{j+1} \leq v_j \beta_j + v_{j+1} \beta_{j+1}. \]

Suppose now that \( x = e \beta_j + f \beta_{j+1} \) with \( e, f \) satisfying (7) is not uniquely decomposable. This can happen only in one of the following ways:

- \( x \) has a decomposition that contains some \( \beta_i \) with \( i \geq j + 2 \), i.e., \( x \geq \beta_i \geq \beta_{j+2} \). Simultaneously, we have from (8) that \( x + \beta_{j-1} \leq v_j \beta_j + v_{j+1} \beta_{j+1} = \beta_{j-1} + \beta_{j+2} \), whence \( x < \beta_{j+2} \). This is a contradiction.
- \( x \) has a decomposition that contains some \( \beta_i \) with \( i \leq j - 1 \), i.e., \( x \geq \beta_i \). Then \( x' \geq \beta'_i \geq \beta'_{j-1} \). We also have from (8) that \( x' + \beta'_{j+2} \leq v_j \beta'_j + v_{j+1} \beta'_{j+1} = \beta'_{j-1} + \beta'_{j+2} \), and so \( x' < \beta'_{j-1} \), a contradiction.
- \( x = e \beta_j + f \beta_{j+1} = e_1 \beta_j + f_1 \beta_{j+1} \) for \( (e,f) \neq (e_1,f_1) \). This is impossible as \( \beta_j, \beta_{j+1} \) are linearly independent over \( \mathbb{Q} \).

Note that similarly to the classical formula for the number of indecomposables as a sum of some coefficients \( u_i \) (e.g., [BK15]), we can now express the number of uniquely decomposable elements.

**Corollary 5.** The number of uniquely decomposable elements of \( \mathcal{O}_K^+ \) modulo totally positive units (i.e., powers of \( \varepsilon^+ \)) is equal to
\[
\sum_{i=1}^{s} u_i + 2 \sum_{i=2, \text{even}}^{s} u_i + \sum_{i=1, \text{odd}}^{s-1} u_{i-1} u_{i+1} \quad \text{in case} \ s \text{ even,}
\]
\[
4 \sum_{i=1}^{s} u_i + \sum_{i=1, \text{odd}}^{s} u_{i-1} u_{i+1} \quad \text{in case} \ s \text{ odd.}
\]

**Proof.** Denote \( s^+ = s \) for \( s \) even and \( s^+ = 2s \) for \( s \) odd. Then by direct computation from Theorem 4 we obtain that the number is, for each item in the theorem statement:

(a) \( \sum_{u_i=1} u_{i+2} \); \quad (b) \( \sum_{u_i=1} u_{i+1} \); \quad (c) \( \sum_{u_i=2} u_{i+3} \); \quad (d) \( \sum_{u_i=1} u_{i+1} \);

(e) \( \sum_{u_i=1} ((u_{i-1} + 1)(u_{i+3} + 1) - 1) = \sum_{u_i=1}^s (u_{i-1} + 1) + \sum_{u_i=1}^s u_{i-1} u_{i+1} \),

where all the sums are over odd \( i \) between 1 and \( s^+ \), because \( \alpha_{i+s+} = \varepsilon^+ \alpha_i \) for all odd \( i \geq -1 \) so this restriction on \( i \) picks one representative from the uniquely decomposable integers for each class modulo powers of \( \varepsilon^+ \). Rearranging the sums and using that \( u_{i+s} = u_i \) we get the result \( \sum_{i=1}^{s^+} u_i + 2 \sum_{i=1, \text{even}}^{s^+} u_i + \sum_{i=1, \text{odd}}^{s^+-1} u_{i-1} u_{i+1} \); this finishes the proof for \( s \) even.

For \( s \) odd, note that all the summands are periodic with period \( s \), hence a sum over all \( 1 \leq i \leq s^+ \) is twice the sum over all \( 1 \leq i \leq s \) and a sum over odd \( 1 \leq i \leq s^+ \) is equal to the sum over all \( 1 \leq i \leq s \).

As another application, we use Theorem 4 to prove that the additive semigroups of totally positive integers of different real quadratic fields are non-isomorphic.
This is in stark contrast to the situation of the groups $\mathcal{O}_K(\pm)$, which are all isomorphic to $\mathbb{Z}^2(\pm)$.

**Theorem 6.** The additive semigroups $\mathcal{O}_K^\pm$, for real quadratic fields $K$, are pairwise not isomorphic.

**Proof.** Assume that we are given $\mathcal{O}_K^\pm(\pm)$ as an abstract semigroup $S(\pm)$. To prove the uniqueness of $K = \mathbb{Q}(\sqrt{D})$, we shall reconstruct the continued fraction for $\sigma_D$ using the additive structure of $S$. Note that the indecomposable and uniquely decomposable elements are well-defined in $S$ as they are defined intrinsically just from the semigroup structure. Since $S$ is a cancellative semigroup, we can define the Grothendieck group of differences $S - S$ and have $S \subset S - S$.

Consider all the indecomposables such that their double is uniquely decomposable:

$$A := \{ \alpha \in S : \alpha \text{ indecomposable and } 2\alpha \text{ uniquely decomposable} \}.$$ 

From Theorem 4 we know that $A$ is exactly the set of convergents and their conjugates, $A = \{ \alpha_i, \alpha_i' : i \text{ odd} \}$. For $\alpha \in A$, denote $k_\alpha$ the maximum integer such that $k_\alpha \alpha$ is uniquely decomposable. From Theorem 4 we know that for each $\alpha \in A$ there are exactly two $\beta \in S$ such that

$$k_\alpha \alpha + \beta \text{ is uniquely decomposable},$$

namely if $\alpha = \alpha_i$ then $\beta = \alpha_{i,1} = \alpha + \alpha_{i+1}$ or $\beta = \alpha_{i-2,u_i-1} = \alpha_{i-2,u_i} - \alpha_{i-1} = \alpha - \alpha_{i-1}$.

Consider now an infinite (bipartite) graph $G$ with vertices $A \cup B$, where

$$B = \{ \{\alpha - \beta, \beta - \alpha\} : \alpha \in A, \alpha, \beta \text{ satisfy } (9) \},$$

where $\alpha - \beta \in S - S$. Note that $B$ actually contains pairs $\{\alpha_i+1, -\alpha_{i+1}\}$ and $\{\alpha_{i+1}', -\alpha_{i+1}'\}$ with odd $i$ (but we have no intrinsic way of distinguishing $\alpha_{i+1}$ from $-\alpha_{i+1}$). The edges in $G$ are defined as follows: There is an edge between $\alpha \in A$ and $\{\gamma, -\gamma\} \in B$ iff $\beta = \alpha - \gamma$ or $\beta = \alpha + \gamma$ satisfy (9). Then $G$ is actually an infinite chain corresponding to

$$\ldots, \alpha_3, \{\alpha_2', -\alpha_2\}, \alpha_1', \{\alpha_0', -\alpha_0\}, \alpha_{-1} = \alpha_{-1}' = 1, \{\alpha_0, -\alpha_0\}, \alpha_1, \{\alpha_2, -\alpha_2\}, \alpha_3, \ldots$$

We add labels on each vertex of $G$ in the following way: We label $\alpha \in A$ by $k_\alpha - 1$. For $\{\gamma, -\gamma\} \in B$ and its neighbors $\alpha, \tilde{\alpha}$, we have that $\alpha - \tilde{\alpha} = l_{\gamma} \gamma$ for some $l_\gamma \in \mathbb{Z}$ and we label $\{\gamma, -\gamma\}$ by $|l_\gamma|$. To see the motivation behind this, assume that $\gamma = \alpha_{i+1}$ for some odd $i$; then $\alpha - \tilde{\alpha} = \pm(\alpha_{i+2} - \alpha_i) = \pm(\alpha_i + u_i + 2\alpha_{i+1} - \alpha_i) = \pm u_i + 2\alpha_{i+1}$.

Whence we know that for $i$ odd, $\alpha_i$ is labelled by $u_{i+1}$ and $\{\alpha_{i+1}, -\alpha_{i+1}\}$ is labelled by $u_{i+2}$ (with the same being true for $\alpha'_i$). This means that the infinite chain of the labels is equal to

$$\ldots, u_{s-1}, u_s = u_0, u_1, u_2, \ldots, u_{s-2}, u_{s-1}, u_s = u_0, u_1, \ldots$$

(where it does not matter in which direction we read the chain as it is a palindrome). We know that $u_0$ is the maximal value in the chain and that $s$ is the (shortest) period of the chain. This means that we have reconstructed $\sigma_D = [u_0; u_1, \ldots, u_{s-1}]$ just from the intrinsic properties of $S(\pm)$. 

\[\Box\]
5. Estimating norms

We have seen in Proposition 3 that (up to conjugacy) every element of $O_K^+$ can be uniquely written in the form $e\alpha_i + r\alpha_{i+1}$, where $i \geq -1$ is odd, $0 \leq r \leq u_{i+2} - 1$, $e \geq 1$, and $f \geq 0$. We will now estimate the norm of such an element, generalizing the results concerning indecomposables by Dress, Scharlau, Jang, Kim, the second author (and others) [DS82, JK16, Kal16a]; here we will use the notation introduced at the end of Section 2.

Let us start by recalling the following classical fact (for the proof see, e.g., [Kal16a] Proposition 5) and [BK17] Lemma 3).

**Lemma 7.** For all $i \geq -1$, we have

$$ N_{i+1} = \frac{\sqrt{\Delta}}{\gamma_{i+2}} - \frac{N_i}{\gamma_{i+2}}, \quad T_{i+1} = (-1)^{i+1} \left( \omega_D - \frac{N_i}{\gamma_{i+2}} \right). $$

We are interested in

$$ e\alpha_{i,r} + f\alpha_{i,r+1} = (e + f)\alpha_i + (re + (r+1)f)\alpha_{i+1}, $$

and so let us first prove a general formula for the norm of the element on the right-hand side.

**Lemma 8.** Let $m, n \in \mathbb{Z}$ and $i \geq -1$ odd. Then

$$ N(m\alpha_i + n\alpha_{i+1}) = \left( m - \frac{n}{\gamma_{i+2}} \right) \left( n\sqrt{\Delta} + mN_i - n\frac{N_i}{\gamma_{i+2}} \right). $$

**Proof.** Let’s prove the result only when $D \equiv 2, 3 \pmod{4}$, as the other case is very similar. Using the definitions and the previous lemma, we compute

$$ N(m\alpha_i + n\alpha_{i+1}) = m^2N(\alpha_i) + n^2N(\alpha_{i+1}) + mn(\alpha_i\alpha_{i+1} + \alpha_i\alpha_{i+1}) $$

$$ = m^2N_i - n^2N_{i+1} + 2mnT_{i+1} $$

$$ = m^2N_i - n^2 \left( \frac{2\sqrt{D}}{\gamma_{i+2}} - \frac{N_i}{\gamma_{i+2}} \right) + 2mn \left( \frac{\sqrt{D}}{\gamma_{i+2}} - \frac{N_i}{\gamma_{i+2}} \right) $$

$$ = \left( m - \frac{n}{\gamma_{i+2}} \right) \left( 2n\sqrt{D} + mN_i - n\frac{N_i}{\gamma_{i+2}} \right). \square $$

Now we return to the original situation when $m = e + f$ and $n = re + (r+1)f$ and estimate the norm of totally positive integers.

**Proposition 9.** Consider $\alpha = e\alpha_{i,r} + f\alpha_{i,r+1} \in O_K^+$ with $i \geq -1$ odd, $0 \leq r < u_{i+2}$, $e \geq 1$, and $f \geq 0$. Then we have the following upper bounds on $N(\alpha)$:

$$ N(\alpha) < \sqrt{\Delta}((r+1)e + (r+2)f)(e + f) \quad \text{and} \quad N(\alpha) < (e + f)^2 \frac{\Delta}{4N_{i+1}}. $$

For a lower bound we distinguish four cases:

(a) If $f > 0$ and $r = 0$, then $N(\alpha) > ef\sqrt{\Delta}$.

(b) Let $c \in (0, 1)$. If $f > 0$ and $1 \leq r \leq cu_{i+2} - 1$, then $N(\alpha) > (1 - c)(e + f)^2\sqrt{\Delta}$.

(c) If $f > 0$ and $\frac{u_{i+2} + 1}{2} < r \leq u_{i+2} - 1$, then $N(\alpha) > \frac{\sqrt{\Delta}}{2} e(e + f)$.

(d) If $f = 0$ and $r > 0$, then $N(\alpha) > e^2(1 - \frac{1}{u_{i+2}})\sqrt{\Delta}$. 
Note that when \( f = r = 0 \), then \( \alpha = e\alpha_i \) is a multiple of a convergent and we have a good control on the size of its norm by Lemma \(^7\). Hence we have not included the lower bound for this case in the proposition.

**Proof.** Let \( m = e + f \) and \( n = re + (r+1)f \). By the previous proposition, we want to estimate

\[
N(m\alpha_i + n\alpha_{i+1}) = \left( m - \frac{n}{\gamma_{i+2}} \right) \left( n\sqrt{\Delta} + mN_i - n\frac{N_i}{\gamma_{i+2}} \right).
\]

We have \( A = e(1 - \frac{r}{\gamma_{i+2}}) + f(1 - \frac{r+1}{\gamma_{i+2}}) \) and \( B = n\sqrt{\Delta} + AN_i \).

We start with the upper bounds. We have \( A < e + f \) and \( B < n\sqrt{\Delta} + (e + f)\sqrt{\Delta} = \sqrt{\Delta}(r+1)e + (r+2)f \), where we used the easy consequence of Lemma \(^7\) that \( N_i < \sqrt{\Delta} \).

For the second estimate we use a slightly different argument (which we again give only in the case \( D \equiv 2, 3 \pmod{4} \)):

\[
N(\alpha) = N\left( \frac{(m\alpha_i + n\alpha_{i+1})\alpha'_{i+1}}{\alpha'_i} \right) = \frac{N(m\alpha_i\alpha'_{i+1} + nN(\alpha_{i+1}))}{N(\alpha_{i+1})} = \frac{N(mT_{i+1} - nN_{i+1} - m\sqrt{D})}{-N_{i+1}} = \frac{(mT_{i+1} - nN_{i+1})^2 - m^2D}{N_{i+1}} < \frac{m^2D}{N_{i+1}}.
\]

We finish with the lower bounds. Let us assume first that \( f > 0 \) and distinguish three cases according to the size of \( r \).

- **Case \( r = 0 \).** Then \( A > e \) and \( B > \sqrt{\Delta}f \).
- **Case \( 1 \leq r \leq cu_{i+2} - 1 \).** Then \( r + 1 \leq cu_{i+2} < c\gamma_{i+1} \), and so \( 1 - \frac{r}{\gamma_{i+2}} > 1 - \frac{r+1}{\gamma_{i+2}} > 1 - c \). Thus \( A = e(1 - \frac{r}{\gamma_{i+2}}) + f(1 - \frac{r+1}{\gamma_{i+2}}) > (1 - c)(e + f) \). Moreover, \( B > \sqrt{\Delta}(e + f)r \).
- **Case \( \frac{u_{i+2} + 1}{2} < r < u_{i+2} - 1 \).** Again, \( B > \sqrt{\Delta}(e + f)r \). Note that the function \( r \mapsto (1 - \frac{r}{\gamma_{i+2}})r \) is increasing for \( r > \frac{u_{i+2} + 1}{2} \). If we apply this observation, we obtain \( Ar > e(1 - \frac{r}{\gamma_{i+2}})r > e\left( 1 - \frac{u_{i+2} - 1}{\gamma_{i+2}} \right)(u_{i+2} - 1) = e\frac{1 + (\gamma_{i+2} - u_{i+2})}{u_{i+2} + (\gamma_{i+2} - u_{i+2})}(u_{i+2} - 1) > e\frac{u_{i+2} - 1}{u_{i+2}} > \frac{r}{2} \), where we use the fact that \( u_{i+2} > r \geq 1 \).

Finally, if \( f = 0 \), then \( N(\alpha) = e^2N(\alpha_{i,r}) \), so proving the case \( e = 1 \) is sufficient. If \( r = 0 \), then estimates on the norm of the convergent \( \alpha_i \) are well-known, see, e.g., [BK17 Lemma 5]. Assume hence \( 1 \leq r \leq u_{i+2} - 1 \). We have

\[
N(\alpha_{i,r}) = \left( 1 - \frac{r}{\gamma_{i+2}} \right) \left( r\sqrt{\Delta} + N_i - r\frac{N_i}{\gamma_{i+2}} \right) > \left( 1 - \frac{r}{\gamma_{i+2}} \right) r\sqrt{\Delta}.
\]

Since the minimum of the function \( r \mapsto (1 - \frac{r}{\gamma_{i+2}})r \) is at one of the endpoints of the considered interval \( 1 \leq r \leq u_{i+2} - 1 \), we conclude that \( N(\alpha_{i,r}) > (1 - \frac{1}{u_{i+2}})\sqrt{\Delta} \). \( \square \)
Let us conclude the paper with some applications of the previous proposition. From the theory of continued fractions it quite easily follows that all elements with absolute value of norm less than $\sqrt{\Delta}/4$ are convergents \cite{BK17} Proposition 7. However, often it is useful to know all elements of norm less than $\sqrt{\Delta}$ (or other small multiples of $\sqrt{\Delta}$): the characterization of all such totally positive integers follows easily from our proposition.

As another application we prove an upper bound on the norm of uniquely decomposable elements, similar to the bounds on norms of convergents follows easily from our proposition.

If $\alpha \in \mathcal{O}_K^+$ is uniquely decomposable, then

$$N(\alpha) < \sqrt{\Delta}(2\sqrt{\Delta} + 1)(3\sqrt{\Delta} + 2).$$

Proof. All uniquely decomposable $\alpha$ are described in Theorem \ref{thm:10} and so we just need to check the estimate on the norm in each of these cases. This is very easy when $\alpha$ is an indecomposable or a multiple of a convergent. In the rest of the proof we will often use the easy observation that $u_j \leq u_s < \sqrt{\Delta}$ for all $j$.

Case $\alpha = e\alpha_i + f\alpha_{i+2}$ with odd $i \geq -1$ such that $u_{i+2} = 1$ and with $1 \leq e \leq u_{i+1} + 1$, $1 \leq f \leq u_{i+3} + 1$, $(e, f) \neq (u_{i+1} + 1, u_{i+3} + 1)$. In the notation of Proposition \ref{prop:10} we have $r = 0$, and so the first upper bound gives $N(\alpha) < \sqrt{\Delta}(e+2f)(e+f)$. Thus the estimate follows from the range of $e, f$ using $u_j < \sqrt{\Delta}$.

Case $e\alpha_{i,0} + a_{i,1}$ with odd $i \geq -1$ such that $u_{i+2} \geq 2$ and with $1 \leq e \leq u_{i+1}$. We have $f = 1$ and $r = 0$, and so we similarly as in the previous case get $N(\alpha) < \sqrt{\Delta}(\sqrt{\Delta}+1)(\sqrt{\Delta}+2)$.

Case $\alpha_{i,ui+2-1} + f\alpha_{i+2,0}$ for $i \geq -1$ odd and $1 \leq f \leq u_{i+3}$. Now $e = 1$ and $r = u_{i+2} - 1$. In the notation (10) we have $A = \left(1 - \frac{u_{i+2}-1}{\gamma_{i+2}}\right) + \frac{1}{\gamma_{i+2}} < \frac{2}{u_{i+2}} + \frac{f}{u_{i+2}} < \frac{\sqrt{\Delta}+2}{u_{i+2}}$. Because $n < (f+1)(r+1)$, we also have $B < (f+1)u_{i+2}\sqrt{\Delta} + AN_\alpha < (\sqrt{\Delta} + 1)u_{i+2}\sqrt{\Delta} + \frac{\sqrt{\Delta}+2}{u_{i+2}}\sqrt{\Delta}$. Finally,

$$N(\alpha) = AB < \frac{\sqrt{\Delta}+2}{u_{i+2}}(\sqrt{\Delta} + 1)u_{i+2}\sqrt{\Delta} + (\sqrt{\Delta}+2)^2\sqrt{\Delta} < (\sqrt{\Delta}+2)\sqrt{\Delta}(2\sqrt{\Delta} + 3).$$

The bound is essentially sharp in the first case discussed in the proof. However, one could refine it in terms of sizes of the coefficients $u_j$, similarly as was done in \cite{JK16} for the norms of indecomposables.

References

\cite{BK15} Valentin Blomer and Vítězslav Kala, Number fields without $n$-ary universal quadratic forms, Math. Proc. Cambridge Philos. Soc. \textbf{159} (2015), no. 2, 239–252.

\cite{BK17} Valentin Blomer and Vítězslav Kala, Arity of universal quadratic forms over real quadratic fields, 2017, preprint, 17 pp., arXiv:1705.03871

\cite{Bru83} Horst Brunotte, Zur Zerlegung totalpositiver Zahlen in Ordnungen totalrealer algebraischer Zahlkörper [On the decomposition of totally positive numbers in orders of totally real algebraic number fields], Arch. Math. (Basel) \textbf{41} (1983), no. 6, 502–503, (in German).
12

[ CdSL +17 ] Gunter Cornelissen, Bart de Smit, Xin Li, Matilde Marcolli, and Harry Smit, *Reconstructing global fields from Dirichlet L-series*, 2017, preprint, 14 pp., arXiv:1706.04515.

[ DS82 ] Andreas Dress and Rudolf Scharlau, *Indecomposable totally positive numbers in real quadratic orders*, J. Number Theory 14 (1982), no. 3, 292–306.

[ Gaß26 ] Fritz Gaßmann, *Bemerkungen zur vorstehenden arbeit von Hurwitz: Über beziehungen zwischen den primideal en eines algebraischen körpers und den substitutionen seiner gruppen* [Note on the hereinabove work of Hurwitz: Relations between the primitives of an algebraic body and the substitutions of its groups], Math. Z. 25 (1926), 665–675.

[ JK16 ] Se Wook Jang and Byeong Moon Kim, *A refinement of the Dress-Scharlau theorem*, J. Number Theory 158 (2016), 234–243.

[ Kal16a ] Vítězslav Kala, *Norms of indecomposable integers in real quadratic fields*, J. Number Theory 166 (2016), 193–207.

[ Kal16b ] Vítězslav Kala, *Universal quadratic forms and elements of small norm in real quadratic fields*, Bull. Aust. Math. Soc. 94 (2016), no. 1, 7–14.

[ Kim00 ] Byeong Moon Kim, *Universal octonary diagonal forms over some real quadratic fields*, Comment. Math. Helv. 75 (2000), no. 3, 410–414.

[ Kub57 ] Tomio Kubota, *Galois group of the maximal abelian extension over an algebraic number field*, Nagoya Math. J. 12 (1957), 177–189.

[ Neu69 ] Jürgen Neukirch, *Kennzeichnung der p-adischen und der endlichen algebraischen Zahlkörper* [Characteristic of p-adic and finite algebraic number field extensions], Invent. Math. 6 (1969), 296–314.

[ Per13 ] Oskar Perron, *Die lehre von den Kettenbrüchen* [The theory of continued fractions], First ed., B. G. Teubner Verlag, 1913, (in German).

[ Sie45 ] Carl Ludwig Siegel, *Sums of mth powers of algebraic integers*, Ann. of Math. (2) 46 (1945), 313–339.

[ Uch76 ] Kôji Uchida, *Isomorphisms of Galois groups*, J. Math. Soc. Japan 28 (1976), no. 4, 617–620.

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 18600 Praha 8, Czech Republic
University of Chemistry and Technology, Prague, Department of Mathematics, Studentská 6, 16000 Praha 6, Czech Republic
E-mail address: tohecz@gmail.com

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 18600 Praha 8, Czech Republic
University of Götingen, Mathematisches Institut, Bunsenstr. 3–5, D-37073 Göttin gen, Germany
E-mail address: vita.kala@gmail.com