Further Development of the Tetron Model

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Abstract

After a prologue which clarifies some issues left open in my last paper, the main features of the tetron model of elementary particles are discussed in the light of recent developments, in particular the formation of strong and electroweak vector bosons and a microscopic understanding of how the observed tetrahedral symmetry of the fermion spectrum may arise.
1 Prologue

In the left-right symmetric standard model with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ there are 24 left-handed and 24 right-handed fermion fields which including antiparticles amounts to 96 degrees of freedom, i.e. this model has right handed neutrinos as well as righthanded weak interactions.

In a recent paper a new ordering scheme for the observed spectrum of quarks and leptons was presented, which relies on the structure of the group of permutations $S_4$ of four objects, and a mechanism was proposed, how 'germs' of the Standard Model interactions might be buried in this symmetry. In the following I want to extend this analysis in several directions. First, I will show that it is possible to embed the discrete $S_4$-symmetry in a larger continuous symmetry group. Afterwards, we shall see how the appearance of gauge bosons can be understood as well as obtain some hints about how the underlying microscopic structure may look like.

The permutation group $S_4$ consists of 5 classes with altogether 24 elements $\sigma = \overline{abcd}$ where $a, b, c, d \in \{1, 2, 3, 4\}$. It has 5 representations $A_1$, $A_2$, $E$, $T_1$ and $T_2$ of dimensions 1, 1, 2, 3 and 3 and is isomorphic to the symmetry group $T_d$ of a regular tetrahedron (and also to the subgroup $O$ of proper rotations of the symmetry group $O_h$ of a cube), cf table 1. The observed fermion symmetry will therefore be synonymously called $T_d$ or $S_4$ in the following, depending on whether a geometrical or an algebraic viewpoint is taken.

An important subgroup of $S_4$ is $A_4$, the group of even permutations, which is sometimes called the 'symmetric group' and will be relevant in the discussion of gauge bosons in section 5. $A_4$ has 3 representations $A$, $E$ and $T$ of dimensions 1, 2 and 3 and is isomorphic to the symmetry group of proper
|       | $S_4$                          | $T_d$                         | $O$                          |
|-------|-------------------------------|-------------------------------|-------------------------------|
| I     | 1234 (id)                     | identity-rotation             | identity-rotation             |
| 3$C_2$| 2143 3412 4321                | rotations by $\pi$ about     | rotation by $\pi$ about      |
|       |                               | the coordinate axes           | the coordinate axes           |
| 8$C_3$| 2314 3124 3241 1342           | rotations by $\frac{2}{3}\pi$| rotations by $\frac{2}{3}\pi$|
|       | 1423 2431 4132 4213           | about diagonals of the cube   | about diagonals of the cube   |
| 6$C_4$| 6 transpositions (i ↔ j) like | 6 reflections on planes       | rotations by $\pm\frac{1}{2}\pi$|
|       | (1 ↔ 2) = 2134                | through the center and two    | about the coordinate axes     |
|       |                               | edges i and j                 |                               |
| 6$C'_2$| 2341 3142                     | 6 rotoreflections by $\frac{1}{2}\pi$ | rotations by $\pi$ about     |
|       | 2413 3421                      | axes parallel to the 6        | the coordinate axes           |
|       | 4123 4312                      | face diagonals                |                               |

Table 1: Classes I, $C_2$, $C_3$, $C_4$ and $C'_2$ of the groups $S_4$, $T_d$ and $O$ making their isomorphy explicit. Classes I, $C_2$ and $C_3$ form the 12-element subgroup $A_4$ of even permutations, which will be important in our analysis of vector bosons in section 5. The notation $C_4$ and $C'_2$ is normally used only for rotations in $O$, whereas the classes of reflections in $T_d$ are usually called $\sigma$ and $S_4$ in the literature.

rotations of a regular tetrahedron.

The starting point of ref. [2] was the observation that there is a natural one-to-one correspondence between the fermion states and the elements of $S_4$. This feature is made explicit in table 2 where the elements of $S_4$ are associated to the fermions.

I use the term 'natural' because the color, isospin and family structure of fermions corresponds to $K$, $Z_2$ and $Z_3$ subgroups of $S_4$, where $Z_n$ is the
Table 2: List of elements of $S_4$ ordered in 3 families. $k_i$ denote the elements of $K$ and $(a \leftrightarrow b)$ a simple permutation where $a$ and $b$ are interchanged. Permutations with a 4 at the last position form a $S_3$ subgroup of $S_4$ and may be thought of giving the set of lepton states. It should be noted that this is only a heuristic assignment. Actually one has to consider linear combinations of permutation states as discussed in section 2.
abelian) symmetric group of \( n \) elements and \( K \) is the so-called Kleinsche Vierergruppe which consists of the 3 even permutations \( 2143, \ 3412, \ 4321 \), where 2 pairs of numbers are interchanged (class \( C_2 \)), plus the identity. In fact, \( S_4 \) is a semi-direct product \( S_4 = K \circ Z_3 \circ Z_2 \) where the \( Z_3 \) factor is the family symmetry and \( Z_2 \) and \( K \) can be considered to be the 'germs' of weak isospin and color (cf \[2\] and section 5). At low energies this product cannot be distinguished from the direct product \( K \times Z_3 \times Z_2 \) but has the advantage of being a simple group and having a rich geometric and group theoretical interpretation and will also lead to a new ordering scheme for the Standard Model vector bosons in section 5.

If one wants to include antiparticles and the spin of the fermions in this analysis, one can do the following: relativistically the situation seems very simple. Spin and antiparticles each double the degrees of freedom, so that one has the structure of table 2 for \( f_L, \ f_R, \bar{f}_L \) and \( \bar{f}_R \) separately. This is enough, as long as one continues to consider quarks and leptons as pointlike objects, and asks questions like how under the assumption of the \( S_4 \) symmetry vector boson formation can be interpreted (section 5), and as long as one keeps the (discrete) inner and spatial symmetries completely separate - but it would not suffice any more, as soon as one would consider the possibility of compositeness and a spatial extension of the observed fermions, in particular in the form of a micro-geometric tetrahedral substructure \[1,2\].

In that case the situation becomes much more difficult. The point is that a tetrahedron is not relativistically invariant and one does not have a relativistic description of such an extended object. As an alternative one may try \[2\] to use a nonrelativistic approach to spin and antiparticles by going from \( T_d \) to \( \tilde{O}_h \), which is the covering group of the octahedral group \( O_h \). \( O_h \) is in fact just the direct product \( T_d \times P \), where \( P \) is the space inversion symmetry. Going from \( T_d \) to \( \tilde{O}_h \) amounts more or less to adding 2 factors of \( Z_2 \) to \( T_d \),
one corresponding to spin and one for antiparticles (complex conjugation). In addition to the ordinary representations one then has to include the representation $G_1$ of the covering group [6]. As can be shown, this amounts to introducing two functions $f_{\sigma}^+$ and $f_{\sigma}^-$ where the spin averaged wave function is given by the sum

$$f_{\sigma} = f_{\sigma}^+ + f_{\sigma}^- \quad (1)$$

whereas the spin content is contained in the difference $f_{\sigma}^+ - f_{\sigma}^-$, and means that including the spin degrees of freedom one has now 48 wave functions instead of the 24 given in table 1.

One may visualize this approach by a geometrical picture, where one has a cube which contains two tetrahedra (one for particles and the other one for antiparticles) which transform into each other by a CP-transformation so that for example in the process of vector boson formation $\bar{F} \gamma_\mu f$ the fermion $f$, which spreads over the first tetrahedron, and antifermion $\bar{F}$, which spreads over the other, join together to form a cube.

It should be noted that even if one rejects the constituent and spatial extension picture it is possible to give a meaning to the tetrahedra describing $f_L$ and $f_R$ and being connected by parity. For example, in the SU(4) model which will be introduced in section 4, they do not live in physical space but exist as weight diagrams of the fundamental SU(4) representation. If one follows such an approach (which will be done for the most part of the paper) a correct relativistic treatment can be maintained without any difficulty.
The Use of Symmetry adapted Wave Functions and the Origin of strong and electroweak Charges

In [2] a sort of seesaw mechanism was derived which is able to accommodate all observed hierarchies in the quark and lepton masses. This mechanism relies on the introduction of $S_4$ symmetry functions to describe fermion fields, where the given Dirac fields of quarks and leptons are written as symmetry adapted linear combinations of more fundamental fields $\psi_\sigma$, $\sigma \in S_4$.

The linear coefficients are essentially given by the $A_1$, $A_2$, $E$, $T_1$ and $T_2$ representation matrices of $S_4$. This is due to the group theoretic theorem that from an arbitrary function $f(x)$ orthonormal sets of symmetry functions of a discrete group $G$ can be obtained as

$$f_{ij} = \frac{\text{dim}(D)}{|G|} \sum_{g \in G} D_{ij}(g)f(g^{-1}x)$$

where $D$ is any representation of $G$. (In general this will yield $\text{dim}(D)$ sets of $\text{dim}(D)$ orthonormal symmetry functions corresponding to the representation $D$.) Therefore to obtain the symmetry adapted functions one just has to take as linear coefficients the appropriate representation matrix entries $D_{ij}$ which are well known in the realm of finite symmetry groups and for convenience given in tables 3 and 4 [5]. The resulting functions were already given in ref. [2].

In order to explain the observed parity violation of the weak and the $V - A$ structure of the strong interaction it was suggested that the two tetrahedra describing fermions and antifermions are intertwined in the following sense: field components $\psi_g$ corresponding to even permutations $g \in S_4$ live on one
|      | x | y | z | xyz | xy\bar{z} | \bar{x}yz | xy\bar{z} | \bar{x}yz | xy\bar{z} | \bar{x}yz | xy\bar{z} | \bar{x}yz |
|-----|---|---|---|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| \(A_1\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(A_2\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \((E)_{11}\) | 1 | 1 | 1 | c | c | c | c | c | c | c | c | c |
| \((E)_{21}\) | 0 | 0 | 0 | 0 | s | s | s | s | -s | -s | -s | -s |
| \((E)_{12}\) | 0 | 0 | 0 | -s | -s | -s | -s | s | s | s | s | s |
| \((E)_{22}\) | 1 | 1 | 1 | c | c | c | c | c | c | c | c | c |
| \((T_1)_{11}\) | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \((T_1)_{21}\) | 0 | 0 | 0 | 0 | 1 | -1 | 1 | -1 | 0 | 0 | 0 | 0 |
| \((T_1)_{31}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 1 |
| \((T_1)_{12}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 1 |
| \((T_1)_{22}\) | 1 | -1 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \((T_1)_{32}\) | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 0 | 0 | 0 | 0 |
| \((T_1)_{13}\) | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 |
| \((T_1)_{23}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 1 | -1 | 1 |
| \((T_1)_{33}\) | 1 | -1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \((T_2)_{11}\) | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \((T_2)_{21}\) | 0 | 0 | 0 | 0 | 1 | -1 | 1 | -1 | 0 | 0 | 0 | 0 |
| \((T_2)_{31}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 1 |
| \((T_2)_{12}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 1 |
| \((T_2)_{22}\) | 1 | -1 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \((T_2)_{32}\) | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 0 | 0 | 0 | 0 |
| \((T_2)_{13}\) | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 |
| \((T_2)_{23}\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 1 | -1 | 1 |
| \((T_2)_{33}\) | 1 | -1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 3: Matrices for the irreducible representations of \(S_4 = T_d\) fixing the coefficients of the symmetry adapted functions as given in [5]. I have used the abbreviation \(c = \cos(\frac{2}{3}\pi) = -\frac{1}{2}\) and \(s = \sin(\frac{2}{3}\pi) = \frac{\sqrt{3}}{2}\).
|   | \( \bar{x}y \) | xy | \( \bar{x}z \) | xz | \( \bar{y}z \) | yz | z | z | S_4 | S_4 | S_4 | S_4 | S_4 | S_4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( A_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( A_2 \) | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| \( (E)_{11} \) | 1 | c | 1 | 1 | 1 | c | c | c | 1 | 1 | c | c | c | c |
| \( (E)_{21} \) | 0 | 0 | s | s | -s | -s | 0 | 0 | s | s | -s | -s | -s | -s |
| \( (E)_{12} \) | 0 | 0 | s | s | -s | -s | 0 | 0 | s | s | -s | -s | -s | -s |
| \( (E)_{22} \) | -1 | -1 | -c | -c | -c | -c | -1 | -1 | -c | -c | -c | -c | -c | -c |
| \( (T_1)_{11} \) | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| \( (T_1)_{21} \) | -1 | 1 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( (T_1)_{31} \) | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 |
| \( (T_1)_{12} \) | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( (T_1)_{22} \) | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| \( (T_1)_{32} \) | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | -1 | -1 | 1 | 1 | 1 | 1 |
| \( (T_1)_{13} \) | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 |
| \( (T_1)_{23} \) | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 1 | -1 | 1 | 1 |
| \( (T_1)_{33} \) | -1 | -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( (T_2)_{11} \) | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | -1 |
| \( (T_2)_{21} \) | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( (T_2)_{31} \) | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 |
| \( (T_2)_{12} \) | 1 | -1 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( (T_2)_{22} \) | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 |
| \( (T_2)_{32} \) | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | -1 |
| \( (T_2)_{13} \) | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 |
| \( (T_2)_{23} \) | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -1 | 1 | 1 | 1 |
| \( (T_2)_{33} \) | 1 | 1 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4: Continuation of table 3: representation matrices for the reflection operations in \( T_d \). The symbols above the symmetry operations indicate their orientation relative to the axes.
tetrahedron, whereas components $\psi_u$ corresponding to odd permutations $u \in S_4$ live on the other. In other words, the symmetry adapted functions for left handed fermions have the generic form $f_L = \psi_g + P\psi_u$ and those for the right handed $f_R = P\psi_g + \psi_u$. The point is that fermions of opposite isospin differ by an odd permutation (as is explicit from table 2), so that parity violation/conservation for weak bosons/gluons is obtained.[2]

Having made extensive use of symmetry adapted functions in various directions, it is time to discuss the legitimacy and drawbacks of such an approach, which have to do with the fact that one is combining fields with different Standard Model charges into linear combinations. As a consequence no definite strong and electroweak charges can be associated to single state components $\psi_\sigma, \sigma = abcd \in S_4$, but only to the symmetry adapted linear combinations giving the quarks and leptons. In other words, such an approach can only be valid, if the Standard Model charges arise as derived entities from secondary dynamical causes and are not really fundamental. Fundamental are only the interactions behind the $S_4$-symmetry (resp. SU(4)-symmetry in section 4) or the superstrong forces between the possible constituents, whereas the Standard Model interactions of the fermions do not exist a priori but are just a consequence of the relative position of a, b, c and d in the permutations. In order to understand this more clearly it was suggested in [2] to introduce nondiagonal charge operators so that not the permutation fields $\psi_\sigma$ but their symmetry combinations are eigenfunctions of the Standard Model charge operators - in much the same way as they are not eigenfunctions of the mass operator.

If one does not like this approach and wants to stick to the viewpoint that charge operators must be diagonal and have to be associated not to linear combinations of fields but to the fields $\psi_\sigma$ themselves, one has to give up the symmetry adapted linear combinations. The only linear combinations
which may then be used are $Z_3$-adapted functions, because they are not associated to any charges but to the family symmetry. In other words, since for example the 3 neutrinos, for which permutations of the first 3 indices are relevant (cf table 2), have identical Standard Model charges, one may use linear combinations of the form

\begin{align*}
\nu_e &= \psi_{1234} + \psi_{2314} + \psi_{3124} \tag{3} \\
\nu_\mu &= \psi_{1234} + \epsilon \psi_{2314} + \epsilon^* \psi_{3124} \tag{4} \\
\nu_\tau &= \psi_{1234} + \epsilon^* \psi_{2314} + \epsilon \psi_{3124} \tag{5}
\end{align*}

and similarly for electron, muon and tau-lepton

\begin{align*}
e &= \psi_{3214} + \psi_{1324} + \psi_{2134} \tag{6} \\
\mu &= \psi_{3214} + \epsilon \psi_{1324} + \epsilon^* \psi_{2134} \tag{7} \\
\tau &= \psi_{3214} + \epsilon^* \psi_{1324} + \epsilon \psi_{2134} \tag{8}
\end{align*}

These equations are easily understood because $Z_3$-symmetry combinations always have the generic form $f_0 + f_1 + f_2$, $f_0 + \epsilon f_1 + \epsilon^* f_2$ and $f_0 + \epsilon^* f_1 + \epsilon f_2$, where $\epsilon = \text{exp}(2\pi i/3)$.

Gauge bosons may be re-expressed using these combinations. For example one obtains for the leptonic part of the neutral weak W-boson

\begin{align*}
W_{3\mu} &= \bar{e} \gamma_\mu e - \bar{\nu}_e \gamma_\mu \nu_e + \bar{\mu} \gamma_\mu \mu + \bar{\nu}_\mu \gamma_\mu \nu_\mu + \bar{\tau} \gamma_\mu \tau - \bar{\nu}_\tau \gamma_\mu \nu_\tau \\
&= 3(\bar{\psi}_{1234} \gamma_\mu \psi_{1234} + \bar{\psi}_{2314} \gamma_\mu \psi_{2314} + \bar{\psi}_{3124} \gamma_\mu \psi_{3124} \\
&- \bar{\psi}_{3214} \gamma_\mu \psi_{3214} - \bar{\psi}_{1324} \gamma_\mu \psi_{1324} - \bar{\psi}_{2134} \gamma_\mu \psi_{2134}) \tag{9}
\end{align*}

Note that eqs. (3)-(10) hold separately for left and right handed lepton and W fields.
3 The two main Problems

In the remainder of this work I will deal with the two fundamental problems, which have to be solved, if the tetron approach is to make sense:

• First to understand in a natural way the appearance of vector bosons as linear combinations of products of fermion fields. In particular the question why among the many fermion-antifermion products which can in principle be formed, precisely and only those corresponding to the Standard Model gauge groups arise. The idea which reduces the number of possible combinations and produces the Standard Model gauge bosons will be that when product states are formed from two fermions each with $T_d$- resp. $O_h$-symmetry a final state object appears, which again has a symmetry of (a subgroup of) $T_d$.

• Secondly what the underlying origin of the tetrahedral symmetry may be. It is plausible although not compelling that the observed $S_4$-symmetry points to a substructure of quarks and leptons with four constituents. In this scenario the main question is how the spin-$\frac{1}{2}$ nature of the observed fermions can be obtained. One possibility, which will be followed in a separate publication [4], is to give up continuous spatial rotation symmetry on the microscopic level and replace it by a discrete (tetrahedral or octahedral) symmetry and then to consider $Z_4$-extensions of the tetrahedral group instead of the $Z_2$-extension defined by the covering group. There is then the possibility that for this $Z_4$-extension quaternion instead of complex quantum mechanics may play a role.

I consider the first problem more important, in particular in view of the highly speculative nature of the second one.
4 Discrete versus continuous inner Symmetry

I have repeatedly mentioned the argument of ref.[2] that $S_4$-symmetry transformations may serve as 'germs' for the gauge symmetries which in modern times are used to describe the strong and electroweak interactions.

Discrete symmetry as an ordering scheme for quarks and leptons and a possible source for their interactions? At this point particle physicists may feel a bit uneasy, because it can hardly be imagined that the rich and rather involved structure of the Standard Model gauge theories can be derived in a strict sense from a discrete symmetry structure.

Therefore, one may look for alternative ideas, and one possibility is that the appearant $S_4$-symmetry of quarks and leptons is part of a larger (continuous) symmetry group like SU(4) or Sp(4). In these groups the $S_4$-symmetry adapted functions naturally appear as part of the product states in $4 \otimes 4 \otimes 4 \otimes 4$, where 4 is the fundamental representation of SU(4), the representation space being spanned by 'tetron' states a, b, c and d, just like in the $SU(3)_{flavor}$ quark model the fundamental representation 3 is spanned by fields u, d and s. The point is that if one considers fourfold tensor products $4 \otimes 4 \otimes 4 \otimes 4$, among the corresponding 256 possible states one will automatically encounter the 24 linear combinations of product states $\psi_{abcd} = a \times b \times c \times d$ and their permutations, or more precisely the symmetry adapted linear combinations thereof - just like in the $SU(3)_{flavor}$ quark model among the 27 baryonic states in $3 \otimes 3 \otimes 3$ there are 6 linear combinations like for example $\Lambda^0 = \frac{1}{\sqrt{12}}[sdu - sud + usd - dsu + 2(uds - dus)]$ which can be interpreted as symmetry adapted functions of the permutation group $S_3$. This is not astonishing but has to do with the fact that $S_4(S_3)$ is a distinct
particle symmetry of the product states in $4 \otimes 4 \otimes 4 \otimes 4$ ($3 \otimes 3 \otimes 3$).

Since the fundamental representation of SU(4) can be geometrically visualized as a tetrahedron which lives in a 3-dimensional weight diagram spanned by the SU(4) generators $\lambda_{3,8,15}$, we arrive at more or less the same geometrical picture as described in section 1 for the discrete $S_4$-symmetry. Even the formation of vector bosons as compounds $F\gamma_\mu f$ from two tetrahedral configurations, which can be transformed into another by CP and where a cube is formed in the combined weight diagram of particles and antiparticles, can be understood in this model.

There are 3 questions left open:

- how the Standard Model charges and interactions can arise from an SU(4) 'hyperflavor' interaction just by a permutation of constituents. This question will be tackled in section 5.

- how products of 4 constituents can make up for fermions with their spin-$\frac{1}{2}$ transformation properties under spatial rotations. This will be discussed in a forthcoming publication [4].

- and finally why only 'distinct'-tetron states arise, whereas all the rest of the 256 product states (those where one of the tetrons appears at least twice) are not observed (or have a much higher mass).

As for the last problem I formulate the following exclusion principle for tetrons: quarks and leptons consist of 4 tetron states a,b,c,d. Only states where all tetrons are different are allowed. In order to include vector bosons and their treatment in section 5 one may extend this principle as follows: for an arbitrary state to be physical the exclusion principle demands that it is part of a $S_4$ permutation multiplet.
Note that this is a weaker condition (i.e. allows more states) than for example the color singlet principle of $SU(3)_{\text{color}}$-QCD, which demands that among all $3 \otimes 3 \otimes 3$ only the $A_2$ singlet function $\epsilon(i, j, k)q_iq_jq_k$ is allowed.

In conclusion one may say that one has two options which match the phenomenological fermion spectrum equally well: either one uses a continuous inner symmetry group like $SU(4)$ together with an exclusion principle or one sticks to the discrete tetrahedral=permutation symmetry.

One can make the connection between these two approaches explicit by writing down the $T_d$-content of the relevant $SU(4)$ representations. Namely, within the discrete approach the 24 fermion states can be classified according to the $T_d$ representations $A_1$, $A_2$, $E$, $T_1$ and $T_2$, i.e. the 18 $T_1$- and $T_2$-functions are used to describe up- and down-type quarks degrees of freedom respectively, whereas the 6 $A_1$-, $A_2$- and $E$-functions are responsible for leptons. (This is just the use of the symmetry adapted functions discussed before and in [2].) On the other hand, in the continuous symmetry approach the 256 $SU(4)$-states of $4 \otimes 4 \otimes 4 \otimes 4$ may be decomposed according to

$$4 \otimes 4 \otimes 4 \otimes 4 = 3 \times 45(T_1) + 3 \times 15(T_2) + 2 \times 20(E) + 35(A_1) + 1(A_2) \quad (11)$$

Here one finds in brackets, which kind of $T_d$ symmetry functions are contained in the corresponding $SU(4)$ representations. For example, there are three $SU(4)$ representations of dimension 45 each containing a set of 3 $T_1$-functions, i.e. all in all the 9 functions used to describe the up-type quarks. More precisely, the 3 functions of the n-th $T_1$ in (11) are to describe the family triplet $u_n$, $c_n$ and $t_n$, where n=1,2,3 is the color index. Similarly there are 3 sets of 3 $T_2$-functions in the 3 15-dimensional representations to describe the down-type quarks. Furthermore, $A_1$ and $A_2$ describes the electron and its neutrino, whereas one E-representation in (11) contains $\mu$ and $\tau$ and the other $\nu_\mu$ and $\nu_\tau$. 

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It is an interesting observation that this way only particles of the same Standard Model charges (but belonging to different families) are put together in a SU(4) multiplet. The alternative would be to put quarks of different color into one SU(4) multiplet (like $u_1$, $u_2$ and $u_3$ into one 45) and similarly for leptons of different isospin (e.g. $\mu$ and $\nu_\mu$ into one 20).

It should further be noted that the fermion mass relations derived in [2] on the basis of the discrete $T_d$-symmetry can be rederived as SU(4) mass relations that are analogous to the mass relations for hadrons derived in the $SU(3)_{\text{flavor}}$ quark model.

5 Vector Boson Formation

In this section I will not make any assumptions about possible substructures of quarks and leptons, but will only use the apparent $S_4$-symmetry of their spectrum table 2. On the basis of this symmetry I want to show that the vector bosons of the left-right symmetric Standard Model can be ordered in a similar manner and according to the same principle as the fermions. The idea is that the tetrahedral (resp octahedral) symmetry of the quarks and leptons is more or less retained when the vector bosons are formed. More precisely, I shall assume that the vector boson states can be ordered according to the subgroup $A_4$ of $S_4$ (the so called symmetric group of even permutations). This reduces the a priori large number of possible fermion-antifermion interactions, because it means that whatever internal dynamical reordering takes place in the process of vector boson formation $\bar{F}\gamma_\mu f$ from two fermions $F$ and $f$, the resulting state has to have $A_4$ symmetry. For example, the long discussion in ref.[2] of how to avoid leptoquarks is completely superfluous in this approach simply because within the $A_4$-symmetry with its 12 degrees of freedom there
Table 5: List of Standard Model vector bosons ordered heuristically according to their proposed $A_4$ symmetry. $A_4$ is composed of 3 classes $I$, $3C_2$, $8C_3$ (cf. table 1) and the proposed ordering follows this line. Note that just as table 2 for fermions these are only preliminary assignments. Later we shall see, how to construct the correct vector bosons states in terms of symmetry adapted functions.

| $B_\mu$       | $G_{3\mu}$   | $G_{8\mu}$   |
|---------------|-------------|-------------|
| $1234$        | $2314$      | $3124$      |
| $W_{3\mu}$   | $G_{1\mu}$  | $G_{2\mu}$  |
| $2143$        | $3241$      | $1342$      |
| $W_{1\mu}$   | $G_{4\mu}$  | $G_{5\mu}$  |
| $3412$        | $1423$      | $2431$      |
| $W_{2\mu}$   | $G_{6\mu}$  | $G_{7\mu}$  |
| $4321$        | $4132$      | $4213$      |

Table 5: List of Standard Model vector bosons ordered heuristically according to their proposed $A_4$ symmetry. $A_4$ is composed of 3 classes $I$, $3C_2$, $8C_3$ (cf. table 1) and the proposed ordering follows this line. Note that just as table 2 for fermions these are only preliminary assignments. Later we shall see, how to construct the correct vector bosons states in terms of symmetry adapted functions.

There is no space for additional gauge bosons.

The two possible types of vector bosons $V_{\mu L} = \bar{F}_L\gamma_\mu f_L$ and $V_{\mu R} = \bar{F}_R\gamma_\mu f_R$ can be accounted for by including parity $P : V_{\mu L} \leftrightarrow V_{\mu R}$ so that one arrives at the so called pyritohedral symmetry $A_4 \times P$, a subgroup of the octahedral group $O_h$. Note that since the gauge bosons have spin 1, no covering group has to be considered. Note further that since I work in the relativistic limit (which I can do since $S_4$ and $A_4$ are just inner symmetries of pointlike particles) no vector boson spin-0 component appears.

In table 5 I present a heuristic ordering of the observed vector bosons according to the proposed $A_4$-symmetry. Phenomenologically, there are 8 gluons $G_\mu$, one $(B - L)$-photon $B_\mu$ and 3 weak bosons $W_{1,2,3\mu}$. The argument of why only the weak bosons appear in a right- and a left-handed version $W_R$ and $W_L$, whereas for gluons and photon one has $G_{\mu L} = G_{\mu R}$ and $B_{\mu L} = B_{\mu R}$ can be taken over from ref [2].

This table, which may look miraculous at first sight, is not difficult to un-
derstand. For example, in [2] it was argued that the weak bosons $W_{1,2,3}$ arise naturally from the Kleinsche Vierergruppe $K$ (the subgroup of $A_4$ formed by the classes $I$ and $3C_2$) because it is isomorphic to $Z_2 \times Z_2$ where the two $Z_2$ factors stand for the germs of weak isospin of the fermion resp antifermion.

To go beyond such a heuristic understanding one should use symmetry adapted linear combinations of functions $\Psi_\sigma, \sigma \in A_4$ instead of the simple assignments of table 5. The linear coefficients could in principle be taken from table 3 (dropping the contributions from improper rotations). However we shall instantly see how to construct them explicitly from fermion-antifermion bilinears in order to obtain the combinations relevant in particle physics.

Using $S_4$-Clebsch-Gordon coefficients for the fermion-antifermion tensor products [7], I want to show, that and how from the $24 \times 24 = 512$ possible fermion-antifermion-product states 12 are selected in order to describe the final states (the vector bosons). From the point of principle this is in fact no question: if the final states are to have $A_4$-symmetry then their number must boil down to 12. In practice these states can be explicitly constructed by evaluating fermion-antifermion products using the $S_4$-symmetry adapted functions for the fermions whose benefits and deficiencies have been discussed in section 2, also in connection with their appearance in the continuous SU(4) model in section 4, cf. eq. (11), projecting them to $A_4 \subset S_4$ and comparing the result with the observed vector boson spectrum.

Before I start I want to remind the reader that the 24 $S_4$-functions for fermions divide into 9 symmetry functions from $T_1$ used for the up-type-quarks, 9 functions from $T_2$ for the down-type-quarks and 6 functions from $A_1, A_2$ and $E$ for the lepton degrees of freedom and that they all can be obtained from table 3. Clebsch-Gordon(CG) coefficients appear when one
calculates tensor products of two representations $D_1$ and $D_2$ as direct sums

$$D_1 \otimes D_2 = D_3 \oplus ...$$  \hfill (12)

and wants to determine a set of symmetry functions for $D_3$ from symmetry functions $f^i_1$ and $f^j_2$ of $D_1$ and $D_2$. Namely they are given

$$f^k_3 = \sqrt{\text{dim}(D_3)} \sum_{i,j} V(D_1, D_2, D_3, i, j, k) f^i_1 f^j_2$$  \hfill (13)

where the sum runs over sets of symmetry functions that span the representation spaces, $i = 1, ..., \text{dim}(D_1)$ and $j = 1, ..., \text{dim}(D_2)$. Eq. (13) will be used as the defining equation for the normalization of the CG-coefficients. (In fact we are using so-called V-coefficients which have the advantage of being invariant under simultaneous permutations of representations and indices in their argument.)

Consider for example the product $T_1 \otimes T_1$. Since $T_1$ corresponds to the up-type quarks, the product $T_1 \otimes T_1$ will yield 9 up-quark bilinears. Within $S_4$ these can be decomposed according to

$$T_1 \otimes T_1 = A_1 \oplus E \oplus T_1 \oplus T_2$$  \hfill (14)

Taking the 3 up-quark color components $u_1$, $u_2$ and $u_3$ as $T_1$-functions on the LHS and evaluating the corresponding Clebsch-Gordon coefficients leads to

- a representation of the ($B - L$)-photon as

$$B_\mu = \bar{u}_1 \gamma_\mu u_1 + \bar{u}_2 \gamma_\mu u_2 + \bar{u}_3 \gamma_\mu u_3$$  \hfill (15)

This stems from the representation $A_1$ on the right hand side of eq. (14) and from the corresponding Clebsch-Gordon coefficient [7]

$$V(T_1, T_1, A_1; i, j, 1) = \frac{1}{\sqrt{3}} \delta_{ij}$$  \hfill (16)
• a representation of the gluon octet stemming from the remaining part $E \oplus T_1 \oplus T_2$ of the decomposition eq. (14). Namely, the CG-coefficients can be written in terms of the Gell-Man $\lambda$-matrices as

$$V(T_1, T_1, T_1; i, j, k) = \frac{1}{\sqrt{6}} \epsilon_{ijk}$$  \hspace{1cm}  (17)

$$= \frac{i}{\sqrt{6}} \lambda_{7,5,2ij} \ \text{for} \ k = 1, 2, 3 \ \hspace{1cm}  (18)$$

$$V(T_1, T_1, T_2; i, j, k) = \frac{1}{\sqrt{6}} |\epsilon_{ijk}|$$  \hspace{1cm}  (19)

$$= \frac{1}{\sqrt{6}} \lambda_{6,4,1ij} \ \text{for} \ k = 1, 2, 3 \ \hspace{1cm}  (20)$$

$$V(T_1, T_1, E; i, j, 1) = \frac{1}{2} \lambda_{8ij}$$  \hspace{1cm}  (21)

$$V(T_1, T_1, E; i, j, 2) = \frac{1}{2} \lambda_{3ij}$$  \hspace{1cm}  (22)

Note that the difference in the coefficients $\frac{1}{2}$ of $V(T_1, T_1, E)$ and $\frac{1}{\sqrt{6}}$ of $V(T_1, T_1, T_1, 2)$ is an artefact of the normalization factor $\sqrt{\text{dim}(D_3)}$ in eq. (13). All in all we obtain

$$G_{3\mu} = \bar{u}_1 \gamma_\mu u_1 - \bar{u}_2 \gamma_\mu u_2$$  \hspace{1cm}  (24)

$$G_{8\mu} = \frac{1}{\sqrt{3}} (\bar{u}_1 \gamma_\mu u_1 + \bar{u}_2 \gamma_\mu u_2 - 2 \bar{u}_3 \gamma_\mu u_3)$$  \hspace{1cm}  (25)

and similarly for the other $\lambda$-matrices.

The fact that formally the same bilinear combinations are created as needed in $SU(3)_{\text{color}}$-QCD is no accident but has to do with the fact that $S_4 = T_d$ is a subgroup $T_d \subset SO(3) \subset SU(3)$. The result is therefore an elaboration on the claim formulated in [2] that the apparent tetrahedral symmetry of quarks and leptons is able to provide 'germs’ of the Standard Model gauge interactions.

It should further be noted that there is no problem of antifermions being
involved here, because on the $S_4$ level there is no difference in the treatment of fermion-fermion and fermion-antifermion bilinears, because the group tensor product states do not care whether they are formed with particles or antiparticles.

Nevertheless, one could have the suspicion of being cheated here in that one obtains complex fields from real representations of a discrete group. To be on the safe side, one may embed these considerations in the framework of the $SU(4)$ model presented in section 4. In that model the physical vector bosons will be states in the representation $\left(\bar{4} \otimes \bar{4} \otimes \bar{4} \otimes \bar{4}\right) \otimes \left(4 \otimes 4 \otimes 4 \otimes 4\right)$. What is done in this section is to select the 12 physical vector bosons among the $4^8$ states in that representation by applying the exclusion principle (‘any physical particle must be a permutation state’) proposed in section 4.

As a next step the results eqs. (14)-(25) have to be projected from $S_4$ to $A_4$ of the vector bosons. This can be done by symmetrization in the family (u,c,t) and the isospin (up,down) degrees of freedom. Doing that the gluons turn out all right, but the $(B - L)$-photon is still missing its lepton contributions.

The point is that $A_4$ has a 3-dimensional representation $T$ (for which 9 symmetry functions are needed), a 2-dimensional representation $E$ (with only 2 functions because it is separably degenerate) and the totally symmetric representation $A$. Interpreted on this basis we obtain from the RHS of eq. (14):

- i) the symmetry function for the totally symmetric representation $A$
- ii) the two symmetry functions for the representation $E$
- iii) 6 of the 9 $T$-functions (3 from $T_1$ and 3 from $T_2$).

The 3 missing $T$-functions, which will be used to describe the weak bosons, can be obtained, for example, from the product

$$E \otimes E = A_1 \oplus A_2 \oplus E$$

(26)
Namely, taking $\mu$ and $\nu_\mu$ as basis functions for $E$ on the LHS and evaluating the corresponding Clebsch-Gordon coefficients leads to

- a representation of the $(B - L)$-photon as $B_\mu = \bar{\nu}_\mu \gamma_\mu \nu_\mu + \bar{\nu}_\mu \gamma_\mu \nu_\mu$ which is due to the $A_1$-term in eq. (26) and, after symmetrization over the family index, gives in fact the missing lepton part of the quark-lepton symmetrized representation of $B_\mu$.

- a representation of the weak boson triplet stemming from the remaining part $A_2 \oplus E$ of the decomposition eq. (26). Namely, the CG-coefficients $V(E, E, A_2)$ and $V(E, E, E)$ are given by

$$V(E, E, A_2; 1, 1, 1) = 0 \quad V(E, E, A_2; 1, 2, 1) = \frac{1}{\sqrt{2}} \quad (27)$$

$$V(E, E, A_2; 2, 1, 1) = -\frac{1}{\sqrt{2}} \quad V(E, E, A_2; 2, 2, 1) = 0 \quad (28)$$

and

$$V(E, E, E; 1, 1, 1) = -\frac{1}{2} \quad V(E, E, E; 1, 2, 1) = 0 \quad (29)$$

$$V(E, E, E; 2, 1, 1) = 0 \quad V(E, E, E; 2, 2, 1) = \frac{1}{2} \quad (30)$$

$$V(E, E, E; 1, 1, 2) = 0 \quad V(E, E, E; 1, 2, 2) = \frac{1}{2} \quad (31)$$

$$V(E, E, E; 2, 1, 2) = \frac{1}{2} \quad V(E, E, E; 2, 2, 2) = 0 \quad (32)$$

leading to the combinations

$$W_1 = \bar{\nu}_\mu \gamma_\mu \nu_\mu + \bar{\nu}_\mu \gamma_\mu \nu_\mu \quad (33)$$

$$iW_2 = \bar{\nu}_\mu \gamma_\mu \nu_\mu - \bar{\nu}_\mu \gamma_\mu \nu_\mu \quad (34)$$

$$W_3 = \bar{\nu}_\mu \gamma_\mu \nu_\mu - \bar{\nu}_\mu \gamma_\mu \nu_\mu \quad (35)$$
Writing the CG-coefficients eqs. (27)-(32) in terms of Pauli matrices $\sigma$

\[
V(E, E, A_2; i, j, 1) = \frac{i}{\sqrt{2}} \sigma_{2ij} \quad (36)
\]

\[
\frac{1}{\sqrt{2}} V(E, E, i, j, 2) = \frac{1}{\sqrt{2}} \sigma_{1ij} \quad (37)
\]

\[
\frac{i}{\sqrt{2}} V(E, E, i, j, 1) = \frac{1}{\sqrt{2}} \sigma_{3ij} \quad (38)
\]

it becomes apparent that they are formally a $SU(2)_{\text{weak}}$ triplet. Since the $T$-representation of $A_4$ is the restriction of the triplet representation to $A_4$ considered as a subgroup of $SU(2)_{\text{weak}}$ they can be used as the set of missing symmetry functions for $T$.

As before the result eq. (33)-(35) has to be symmetrized in the family and the quark and lepton degrees of freedom.

## 6 Summary

It is a remarkable observation, that quarks, leptons and gauge bosons can be ordered with the help of essentially the same symmetry group, the permutation group $S_4$.

Starting from that paradigm we have seen, that and how from the $24 \times 24 = 512$ possible fermion-antifermion product states 12 are selected to describe the gauge bosons, and - though lacking an understanding of the underlying dynamics responsible for this selection - by inspection of Clebsch-Gordon coefficients we have tried to follow the path of how this dynamics works on the level of gauge bosons.

Realizing that there is a connection of the $S_4$-states to representations of $SU(4)$ we have found two options which match the phenomenological fermion
and gauge boson spectrum equally well: either one uses a continuous inner symmetry group like SU(4) or Sp(4) together with an exclusion principle or one sticks to the discrete permutation symmetry.

The discussion of SU(4) suggests the existence of a fundamental quartet of 'tetron' constituents. Up to this point the new symmetry can be kept completely independent from spacetime symmetries. However, since it is difficult to generate the spin-$\frac{1}{2}$ behavior of quarks and leptons from 4 such constituents by conventional means, in ref. [4] a somewhat different viewpoint will be taken, in which $S_4$ is not really an inner but a micro-geometric symmetry, where in physical space one has clouds of 4 tetronic constituents which surround a tetrahedral skeleton and tries to generate a (discrete) spin-$\frac{1}{2}$ behavior from that picture.

This scenario is complicated by the fact that the spatial tetrahedral symmetry should in principle be relativistically generalized to a subgroup of the Lorentz group. In this connection one may even speculate whether there is a relation of the tetrons to the graviton, i.e. whether the underlying unknown interaction of tetrons may also be used to describe gravity.

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