DENSITY CONVERGENCE OF A FULLY DISCRETE FINITE DIFFERENCE METHOD FOR STOCHASTIC CAHN–HILLIARD EQUATION

JIALIN HONG, DIANCONG JIN, AND DERUI SHENG

Abstract. This paper focuses on investigating the density convergence of a fully discrete finite difference method when applied to numerically solve the stochastic Cahn–Hilliard equation driven by multiplicative space-time white noises. The main difficulty lies in the control of the drift coefficient that is neither globally Lipschitz nor one-sided Lipschitz. To handle this difficulty, we propose a novel localization argument and derive the strong convergence rate of the numerical solution to estimate the total variation distance between the exact and numerical solutions. This along with the existence of the density of the numerical solution finally yields the convergence of density in $L^1(\mathbb{R})$ of the numerical solution. Our results partially answer positively to the open problem emerged in [J. Cui and J. Hong, J. Differential Equations (2020)] on computing the density of the exact solution numerically.

1. Introduction

The density of the exact solution of a stochastic system characterizes all relevant probabilistic information and has wide applications in the probability potential theory. When a numerical method is applied to the original system, a natural question is whether the numerical solution provides an effective approximation of the density of the exact solution, which has received much attention recently. For instance, for stochastic differential equations (SDEs) whose coefficients are smooth vector fields with bounded derivatives, [3, 25, 28] obtained the convergence of density of the numerical solution based on Itô–Taylor type dicretizations under Hörmander’s condition. For stochastic Langevin equations with non-globally monotone coefficients, [17] used the splitting method to derive an approximation for the density of the exact solution. Relatively, the research of approximations for densities of exact solutions of stochastic partial differential equations (SPDEs) is still at its infancy. And we are only aware of [12], where the authors investigated the existence and convergence of densities of numerical dicretizations for stochastic heat equations with additive noise. Following this line of investigation, the present work makes further contributions on numerical approximations of densities of exact solutions of SPDEs with polynomial nonlinearity and multiplicative noises.

This paper is concerned with the following stochastic Cahn–Hilliard equation

$$\partial_t u + \Delta^2 u = \Delta f(u) + \sigma(u)\dot{W}, \quad \text{in } [0, T] \times \mathcal{O} \quad (1.1)$$

with the initial value $u(0, \cdot) = u_0$ and Dirichlet boundary conditions (DBC$s$) $u = \Delta u = 0$ on $\partial\mathcal{O}$. Here, $\mathcal{O} := (0, \pi)$, $T > 0$, and $\dot{W}$ is the formal derivative of a Brownian sheet $W = $
The stochastic Cahn–Hilliard equation is a model arising in non-equilibrium dynamics of metastable states \([4, 5, 7, 13, 30]\). For example, \((1.1)\) can describe the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably \([6, 19]\). The unknown quantity \(u\) in \((1.1)\) represents the concentration, and \(f(u) = u^3 - u\) is the derivative of the double well potential. For the existence of a unique solution to \((1.1)\) under suitable assumptions, we refer to \([1, 8, 14]\) and references therein. Concerning the density of the exact solution, \([8]\) and \([9, 15]\) respectively proved the existence and strict positivity of the density \(\{p_{t,x}\}_{(t,x) \in (0,T) \times \mathcal{O}}\) of \(\{u(t, x)\}_{(t,x) \in (0,T) \times \mathcal{O}}\) under the non-degeneracy condition \(|\sigma(\cdot)| > 0\). From the practical point of view, numerically approximating the density \(p_{t,x}\) is of prime importance in understanding intrinsic properties, beyond the existence, of the density, and we will resort to numerical methods to handle this problem.

Numerical methods have been successfully applied to solve the stochastic Cahn–Hilliard equation; see \([11, 21]\) for that of the linearized Cahn–Hilliard equation, \([18, 22, 29, 36]\) for the multiplicative noise case, and \([14, 15, 38, 21]\) for the additive noise case. Among them, the stochastic Cahn–Hilliard equation is interpreted as an SDEs in Hilbert spaces, which is discretized by the finite element method or the spectral Galerkin method in space. In order to numerically approximating the density of the exact solution, we understand the exact solution \(u: [0,T] \times \mathcal{O} \rightarrow L^2(\Omega)\) as a random field and apply the spatial finite difference method (FDM) to discretize \((1.1)\). By introducing a uniform spatial stepsize \(h = \pi/n, n \geq 2\), the spatial FDM of \((1.1)\) can be formulated into an \((n-1)\)-dimensional SDE

\[
dU(t) + A_n^2 U(t) dt = A_nF_n(U(t)) dt + \sqrt{n/\pi} \Sigma_n(U(t)) d\beta_t. \tag{1.2}
\]

Here \(A_n\) is the matrix form of the discrete Dirichlet Laplacian, \(F_n\) and \(\Sigma_n\) are respectively determined by \(f\) and \(\sigma\) (see \([2,9]\)), and \(\{\beta_t\}_{t \in [0,T]}\) is some \((n-1)\)-dimensional Brownian motion related to \(W\) (see subsection 2.2 for more details). Further, by denoting \(\tau := T/m\) \((m \in \mathbb{N}_+)\) the uniform time stepsize, we discretize \((1.2)\) by the backward Euler scheme in time and obtain a fully discrete FDM method

\[
U^{i+1} - U^i + \tau A_n^2 U^{i+1} = \tau A_n F_n(U^{i+1}) + \sqrt{n/\pi} \Sigma_n(U^i)(\beta_{t_{i+1}} - \beta_{t_i}), \tag{1.3}
\]

where \(t_i := i\tau\) for \(i \in \{0,1,\ldots,m\}\). The \(k\)th component \(U_k(t_i)\) (resp. \(U_k^i\)) of \(U(t_i)\) (resp. \(U^i\)) approximates formally to \(u(t_i, kh)\) for every \(k \in \mathbb{Z}_{n-1} := \{1, \ldots, n-1\}\). Taking advantage of the local weak monotonicity of \(A_nF_n\), we prove the well-posedness of \((1.2)\) and \((1.3)\), and that for every \(k \in \mathbb{Z}_{n-1}\), both \(\{U_k(t)\}_{t \in (0,T]}\) and \(\{U_k^i\}_{i=1,\ldots,m}\) admit densities under the non-degeneracy condition.

Our first main result is the strong convergence rates of the spatial FDM and fully discrete FDM for \((1.1)\), which will be used to derive the convergence of densities of numerical methods, and is of independent interest. Inspired by regularity estimates of original systems in \([1, 8]\), we first use the interpolation approach to establish in Proposition 4.4 a uniform moment bound for the numerical solution \(U(t)\) in the discrete Sobolev norm \(\|(-A_n)^{\frac{1}{2}} \cdot \|_{l_2^n}\). To overcome the difficulty that the drift coefficient is neither globally Lipschitz nor one-sided Lipschitz, we introduce an auxiliary process \(\bar{U}(t)\) (see \((1.1)\)), and focus mainly on estimating the error \(E(t) := \bar{U}(t) - U(t)\). With the aid of the one-sided Lipschitz property of \(A_n F_n\) in the discrete negative Sobolev norm \(\|(-A_n)^{-\frac{1}{2}} \cdot \|_{l_2^n}\), we are able to estimate \((-A_n)^{-\frac{1}{2}} E(t)\) in Proposition 4.3 and meanwhile the linear part \(-A_n^2 E(t)\) leads to an upper bound for the \(L^4(\Omega; L^2(0,T; \|(-A_n)^{\frac{1}{2}} \cdot \|_{l_2^n})\text{-}norm of \(E(t)\). Further, building on the local Lipschitz continuity
of $F_n$ in the norm $\|(-A_n)^{1/2} \cdot \|_{L^2}$, we attain the strong convergence order 1 of the spatial FDM. By essentially exploiting the discrete analogue of previous arguments, we also show that the fully discrete FDM converges strongly to the spatial FDM with order nearly $\frac{3}{8}$ in time. The above convergence orders are optimal in the sense that they coincide with the spatial and temporal Hölder continuity exponents of the exact solution, respectively.

Our second main result is the convergence of density in $L^1(\mathbb{R})$ of the numerical solutions for (1.1), which is realized by a localization argument to deal with the non-globally monotone coefficient $\Delta f$. Let us illustrate our idea by taking the spatial semi-discrete numerical solution for example. First, we establish a criterion for reducing the total variation distance of random variables to that of their localizations in Proposition 6.1. Based on this criterion, the estimate of the total variation distance between $u$ and $u^n$ boils down to estimating that between $u_R$ and $u^n_R$. Here, $u_R$ is the localization of $u$ and solves the localized stochastic Cahn–Hilliard equation

$$\partial_t u_R + \Delta^2 u_R = \Delta f_R(u_R) + \sigma(u_R) \dot{W}, \quad R \geq 1,$$

where $f_R = f K_R$ with $K_R$ being a smooth cut-off function supported on $[-R - 1, R + 1]$. In addition, for any fixed $R \geq 1$, define the localization $u^n_R$ of $u^n$ as the spatial FDM numerical solution of $u_R$. Second, in order to control the total variation distance between $u^n_R$ and $u_R$, we apply a criterion for the convergence in total variation distance provided by [33], whose prerequisites contain the negative moment estimate of the Malliavin derivative $Du_R(t, x)$ and the convergence of $u^n_R(t, x)$ in the Malliavin–Sobolev space $D^{1,2}$. These are accomplished by making full use of the globally Lipschitz condition of $f_R$ and the strong regularizing effect of the linear part. Finally, together with the existence of density of the spatial FDM, we obtain that the density of the spatial FDM converges in $L^1(\mathbb{R})$ to that of the exact solution. In a similar manner, we also show that the density of the fully discrete FDM converges in $L^1(\mathbb{R})$ to that of the exact solution.

We summarize main contributions of this work as follows.

- We give the optimal strong convergence rate of a fully discrete FDM for stochastic Cahn–Hilliard equations with polynomial nonlinearity and multiplicative noise.
- We are the first to give the convergence of density for numerical approximations of SPDEs with polynomial nonlinearity. The results on the existence and convergence of density of the numerical solutions for stochastic Cahn–Hilliard equations partially respond positively to an open problem on computing the density of the exact solution numerically proposed in [15, Section 5].
- We propose a criterion for reducing the total variation distance of random variables to that of their localizations. And it is successfully applied to derive the density convergence of a fully discrete numerical method for (1.1). We believe that this localization argument is also available for other SPDEs with non-globally Lipschitz coefficients such as stochastic Allen–Cahn equations.

The rest of this paper is organized as follows. Section 2 is devoted to introducing the mild solution, the spatial and fully discrete FDMs for (1.1). In subsection 2.2 we also prove the existence of densities of the numerical solutions. The regularity estimates of the numerical solutions are presented in Section 3. The strong convergence rate of the spatial FDM and the fully discrete FDM are proved in Sections 4 and 5, respectively. Finally, Section 6 is reserved for the convergence of densities in $L^1(\mathbb{R})$ of the numerical solutions.
2. Preliminaries

Let $C^{α}(O)$ be the space of $α$-Hölder continuous functions on $O$ for $α \in (0, 1)$, and the space of $α$ times continuously differentiable functions on $O$ for $α \in \mathbb{N}$. For $d \in \mathbb{N}_{+}$, we denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the Euclidean norm and inner product of $\mathbb{R}^{d}$, respectively. Given a measurable space $\langle M, \mathcal{M}, m \rangle$ and a Banach space $\langle H, \| \cdot \|_{H} \rangle$, let $L^{p}(M; H)$ be the space of measurable functions $g : M \rightarrow H$ endowed with the usual norm $\|g\|_{L^{p}(M; H)} := (\int_{M} \|g\|_{H}^{p} \, dm)^{\frac{1}{p}}$. Especially, we write $L^{p}(M) := L^{p}(M; \mathbb{R})$ for short. For $N \in \mathbb{N}_{+}$, denote $Z_{N} := \{1, \ldots, N\}$ and $Z_{N}^{0} := \{0, 1, \ldots, N\}$. We use $C$ to denote a generic positive constant that may change from one place to another and depend on several parameters but never on the stepsize $h$.

Given a random field $v = \{v(t, x), (t, x) \in [0, T] \times O\}$ and a kernel $S : [0, T] \times O \times O \rightarrow \mathbb{R}$, we denote $S \ast f(v)$ and $S \circ \sigma(v)$ the deterministic and stochastic convolutions, respectively, namely for any $(t, x) \in [0, T] \times O$,

\[ S \ast f(v)(t, x) := \int_{0}^{t} \int_{O} S_{t-s}(x, y)f(v(s, y))dyds, \]
\[ S \circ \sigma(v)(t, x) := \int_{0}^{t} \int_{O} S_{t-s}(x, y)\sigma(v(s, y))W(ds, dy). \]

If for any $p \geq 1$, there exists some constant $C_{p}$ such that $\|v(t, x)\|_{L^{p}(O)} \leq C_{p}$ for all $(t, x) \in [0, T] \times O$, then it can be verified that for any $0 \leq s \leq t \leq T$ and $x, y \in O$,

\[ \|S \ast f(v)(t, x) - S \ast f(v)(s, y)\|_{L^{p}(\Omega)} \leq C \int_{0}^{s} \int_{O} |S_{t-r}(x, z) - S_{s-r}(x, z)|dzdr \]
\[ + C \int_{s}^{t} \int_{O} |S_{t-r}(x, z)|dzdr + C \int_{0}^{s} \int_{O} |S_{s-r}(x, z) - S_{r}(y, z)|dzdr, \]

and

\[ \|S \circ \sigma(v)(t, x) - S \circ \sigma(v)(t, y)\|_{L^{p}(\Omega)}^{2} \leq C \int_{0}^{s} \int_{O} |S_{t-r}(x, z) - S_{s-r}(y, z)|^{2}dzdr \]
\[ + C \int_{0}^{s} \int_{O} |S_{s-r}(x, z) - S_{s-r}(x, z)|^{2}dzdr + C \int_{s}^{t} \int_{O} |S_{t-r}(x, z)|^{2}dzdr. \]

2.1. Mild solution. The physical importance of the Dirichlet problem lies in that it governs the propagation of a solidification front into an ambient medium which is at rest relative to the front [19]; see for instance [15, 10, 20] for the study of Cahn–Hilliard equations with DBCs. In this case, the Green function associated to $\partial_{t} + \Delta^{2}$ is given by $G_{t}(x, y) = \sum_{j=1}^{\infty} e^{-\lambda_{j}t} \phi_{j}(x)\phi_{j}(y), t \in [0, T], x, y \in O$, where $\lambda_{j} = -j^{2}$, $\phi_{j}(x) = \sqrt{2/\pi} \sin(jx)$, $j \geq 1$. It is known that $\{\phi_{j}\}_{j \geq 1}$ forms an orthonormal basis of $L^{2}(O)$. As pointed out in [10] p.19, there exist $C, c > 0$ such that

\[ |G_{t}(x, y)| \leq \frac{C}{t^{1/4}} \exp\left(-c \frac{|x - y|^{4/3}}{|t|^{1/3}} \right), \]
\[ |\Delta G_{t}(x, y)| \leq \frac{C}{t^{3/4}} \exp\left(-c \frac{|x - y|^{4/3}}{|t|^{1/3}} \right), \]

which corresponds to [10] formula (1.2) with $a = b = 0$ and $a = 2, b = 0$, respectively.

Without further explanations, we always assume in the text that $u_{0} : O \rightarrow \mathbb{R}$ is nonrandom and continuous, $f(x) = x^{3} - x$, and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is bounded and satisfies the globally Lipschitz condition. These assumptions ensure that [L1] admits a unique mild solution.
u = \{u(t, x), (t, x) \in [0, T] \times \mathcal{O}\} given by (cf. \cite{15,16})

\[
\begin{align*}
  u(t, x) &= \mathcal{G}_tu_0(x) + \int_0^t \int_\mathcal{O} \Delta G_{t-s}(x, y) f(u(s, y)) \, dy \, ds \\
  &\quad + \int_0^t \int_\mathcal{O} G_{t-s}(x, y) \sigma(u(s, y)) W(ds, dy), \quad (t, x) \in [0, T] \times \mathcal{O}.
\end{align*}
\]

Hereafter, \( \mathcal{G}_tv(x) := \int_\mathcal{O} G_t(x, y)v(y) \, dy \) for \( v \in C(\mathcal{O}) \). Moreover, as shown in \cite{16} Proposition 5.2, the exact solution to (1.1) satisfies

\[ \sup_{t \in [0,T]} \mathbb{E} \left[ \sup_{x \in \mathcal{O}} |u(t, x)|^p \right] \leq C(u_0, T, p). \tag{2.7} \]

Similar to \cite{8} Lemma 1.8, we have the following regularity estimate of \( G \).

Lemma 2.1. For \( \alpha \in (0, 1) \), there exists \( C = C_\alpha \) such that for \( x, y \in \mathcal{O} \) and \( t > s \),

\[
\begin{align*}
  &\int_0^t \int_\mathcal{O} |G_{t-r}(x, z) - G_{t-r}(y, z)|^2 \, dz \, dr \leq C|x - y|^2, \\
  &\int_0^s \int_\mathcal{O} |G_{t-r}(x, z) - G_{s-r}(x, z)|^2 \, dz \, dr + \int_s^t \int_\mathcal{O} |G_{t-r}(x, z)|^2 \, dz \, dr \leq C|t - s|^{\frac{3}{2}}, \\
  &\int_0^t \int_\mathcal{O} |\Delta G_{t-r}(x, z) - \Delta G_{t-r}(y, z)| \, dz \, dr \leq C|x - y|, \\
  &\int_0^s \int_\mathcal{O} |\Delta G_{t-r}(x, z) - \Delta G_{s-r}(x, z)| \, dz \, dr + \int_s^t \int_\mathcal{O} |\Delta G_{t-r}(x, z)| \, dz \, dr \leq C|t - s|^{\frac{5}{3}}.
\end{align*}
\]

Then we are able to investigate the Hölder continuity of \( u \).

Lemma 2.2. Let \( u_0 \in C^2(\mathcal{O}) \) and \( \alpha \in (0, 1) \). Then for \( p \geq 1 \), there exists some constant \( C = C(\alpha, p, T) \) such that

\[ ||u(t, x) - u(s, y)||_{L^p(\Omega)} \leq C(|t - s|^{\frac{3}{2}} + |x - y|) \quad \forall \ (t, x), (s, y) \in [0, T] \times \mathcal{O}. \]

Proof. Without loss of generality, let \( p \geq 2 \) and \( t > s \). Using \cite{8} Lemma 2.3 and the assumption \( u_0 \in C^2(\mathcal{O}) \), we get

\[ |\mathcal{G}_tu_0(x) - \mathcal{G}_tu_0(y)| + |\mathcal{G}_tu(x) - \mathcal{G}_su_0(x)| \leq C(|t - s|^{\frac{1}{2}} + |x - y|). \]

Then by (2.3) with \( S = \Delta G \) and (2.4) with \( S = G \), as well as Lemma 2.1, we finish the proof. \( \square \)

2.2. Spatial FDM. In this part, we introduce the spatial FDM for (1.1) and present the regularity estimates of the spatial discrete Green function. Given a function \( w \) defined on the mesh \( \{0, h, 2h, \ldots, \pi\} \) with \( h = \frac{2}{\pi} \), define the difference operator

\[ \delta_h w_i := \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}, \quad i \in \mathbb{Z}_{n-1}, \]

where \( w_i := w(ih) \). Notice that

\[ \delta^2_h w_i = \frac{w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}}{h^4}, \quad i \in \mathbb{Z}_{n-1}. \]

The compatibility conditions \( u_0(0) = u_0(\pi) = 0 \) and \( u_0'(0) = u_0'(\pi) = 0 \) are direct results of DBCs and the initial condition. One can approximate \( u(t, kh) \) via \( \{u^n(t, kh)\}_{n \geq 2} \), where \( u^n(0, kh) = u_0(kh) \) and

\[ du^n(t, kh) + \delta^2_h u^n(t, kh)dt \tag{2.8} \]
By virtue of \cite[Theorem 3.1.1]{35}, (2.12), (2.13) and the Lipschitz continuity of \(\kappa\), that (1.2) admits a unique solution for every \(t, x\) to (1.1) admits a density.  

Based on (2.10) and (2.11), it can be verified that there exist \(K_n(R), K_n > 0\) such that for all \(R \in (0, \infty)\), \(x, y \in \mathbb{R}^{n-1}\) with \(\|x\|, \|y\| \leq R\),

\[
\langle x - y, A_n F_n(x) - A_n F_n(y) \rangle \leq K_n(R)\|x - y\|^2 \quad \text{(local weak monotonicity)},
\]

\[
\langle x, A_n F_n(x) \rangle \leq K_n(1 + \|x\|^2) \quad \text{(weak coercivity)}.
\]

By virtue of \cite[Theorem 3.1.1]{35}, (2.12), (2.13) and the Lipschitz continuity of \(\sigma\), we obtain that (1.2) admits a unique solution \(\{U(t), t \in [0, T]\}\), which is a.s. continuous and \(\{\mathcal{F}_t\}\)-adapted.

By virtue of \cite[Theorem 5]{8} and \cite[Remark 5.3(ii)]{10}, we know that if \(\sigma(x) \neq 0\) for any \(x \in \mathbb{R}\), then for any \((t, x) \in (0, T) \times \mathcal{O}\), the exact solution \(u(t, x)\) to (1.1) admits a density. As a numerical counterpart, Theorem 2.4 implies the existence of density of \(u^n(t, kh) = \hat{U}_k(t)\) for every \(k \in \mathbb{Z}_{n-1}\) and \(t \in (0, T]\).
Theorem 2.4. Let $\sigma$ be continuously differentiable and $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$. Then for any $t \in (0, T]$, the law of $U(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$.

Proof. Introduce $Z(t) := (-A_n)^{-\frac{1}{2}}U(t)$ for $t \in [0, T]$. By (1.2),

$$dZ(t) + A_n^{-\frac{1}{2}}Z(t)dt = -(-A_n)^{-\frac{1}{2}}F_n((-A_n)^{-\frac{1}{2}}Z(t))dt + \sqrt{\frac{n}{\pi}}(-A_n)^{-\frac{1}{2}}\Sigma_n((-A_n)^{-\frac{1}{2}}Z(t))d\beta_t.$$  

For $t \in (0, T]$, to prove the absolute continuity of the law of $U(t)$, it suffices to show that the law of $Z(t)$ is absolutely continuous. Next, we apply [26] Theorem 5.2 to show the absolute continuity of the law of $Z(t)$, where the following conditions (i) and (ii) are required.

(i) Assumption 3.1 of [26], which mainly contains properties (1)-(3) below.

1. $\mathbb{R}^{n-1} \ni x \mapsto \tilde{b}(x) := -(-A_n)^{-\frac{1}{2}}F_n((-A_n)^{-\frac{1}{2}}x)$ satisfies the one-sided Lipschitz condition;
2. $\mathbb{R}^{n-1} \ni x \mapsto \tilde{\sigma}(x) := \sqrt{n/\pi}(-A_n)^{-\frac{1}{2}}\Sigma_n((-A_n)^{-\frac{1}{2}}x)$ satisfies the global Lipschitz condition;
3. both $\tilde{b}$ and $\tilde{\sigma}$ are continuously differentiable.

(ii) For any $\mathbb{R}^{n-1} \ni z \neq 0$ and $s \leq t$,

$$z^\top \Sigma_n(U(s))\Sigma_n(U(s))^\top z > \lambda(s, t)\|z\|^2 \geq 0, \text{ a.s.,}$$

for some function $\lambda : \mathbb{R}_+^n \to \mathbb{R}$ with $\int_0^t \lambda(s, t)ds > 0$.

We first prove property (1). In view of (2.10), we have that for any $x, y \in \mathbb{R}^{n-1}$,

$$-\langle x - y, F_n(x) - F_n(y) \rangle = \sum_{k=1}^{n-1} (x_k - y_k)(f(y_k) - f(x_k)) \leq \|x - y\|^2.$$

Hence, it follows from the symmetry of $(-A_n)^{-\frac{1}{2}}$ that

$$\langle x - y, -(-A_n)^{-\frac{1}{2}}F_n((-A_n)^{-\frac{1}{2}}x) + (-A_n)^{-\frac{1}{2}}F_n((-A_n)^{-\frac{1}{2}}y) \rangle = -\langle (-A_n)^{-\frac{1}{2}}x - (-A_n)^{-\frac{1}{2}}y, F_n((-A_n)^{-\frac{1}{2}}x) - F_n((-A_n)^{-\frac{1}{2}}y) \rangle \leq \|(-A_n)^{-\frac{1}{2}}(x - y)\|^2 \leq (n - 1)^2\|x - y\|^2,$$

which yields the desired property (1). Similarly, using the Lipschitz continuity of $\sigma$ and the continuous differentiability of $f$ and $\sigma$, one could see that properties (2) and (3) are fulfilled. Besides, since $\sigma(\cdot) \neq 0$, the square matrix $\Sigma_n(U(s))\Sigma_n(U(s))^\top$ has positive minimum eigenvalue, which is denoted by $\lambda_{\min}(s, \omega)$. Then the property (ii) follows immediately by choosing $\lambda(s, t) = \frac{1}{2}\lambda_{\min}(s, \omega) > 0$ for $s \leq t$.

Making use of the variation of constants formula, we obtain from (1.2) that

$$U(t) = \exp(-A_n^2t)U(0) + \int_0^t A_n \exp(-A_n^2(t - s))F_n(U(s))ds + \sqrt{\frac{n}{\pi}} \int_0^t \exp(-A_n^2(t - s))\Sigma_n(U(s))d\beta_s, \text{ } t \in [0, T].$$

For $j \in \mathbb{Z}_{n-1}$, $e_j = (e_j(1), \ldots, e_j(n - 1))^\top$ given by

$$e_j(k) = \sqrt{\sigma/n}\phi_j(kh) = \sqrt{2/n}\sin(jkh), \text{ } k \in \mathbb{Z}_{n-1},$$

(2.15)
is an eigenvector of $A_n$ associated with the eigenvalue $\lambda_{j,n} = -j^2c_{j,n}$, where $c_{j,n} := \sin^2\left(\frac{j\pi}{2n}\right)/\left(\frac{j^2\pi^2}{2n}\right)^2$ satisfies $\frac{4}{\pi^2} \leq c_{j,n} \leq 1$. The vectors $\{e_i\}_{i=1}^{n-1}$ form an orthonormal basis of $\mathbb{R}^{n-1}$ (see e.g., [24]). In particular,

$$\langle e_i, e_j \rangle = \frac{n}{\pi} \sum_{k=1}^{n-1} \phi_i(kh)\phi_j(kh) = \int_0^\pi \phi_i(\kappa_n(y))\phi_j(\kappa_n(y))dy = \delta_{ij}.$$ 

It is verified that $1 - \frac{\sin^2\alpha}{\alpha} \leq \frac{1}{6}a^2$ for all $\alpha \in [0, \frac{\pi}{2})$, which indicates that for $j \in \mathbb{Z}_{n-1}$,

$$0 \leq 1 - c_{j,n} = \left(1 + \sin\left(\frac{j\pi}{2n}\right)/\left(\frac{j^2\pi^2}{2n}\right)^2\right)\left(1 - \sin\left(\frac{j\pi}{2n}\right)/\left(\frac{j^2\pi^2}{2n}\right)\right) \leq \frac{\pi^2j^2}{12n^2}. \quad (2.16)$$

Introduce the discrete kernel

$$G_t^n(x, y) = \sum_{j=1}^{n-1} \exp(-\lambda_{j,n}^2t)\phi_{j,n}(x)\phi_j(\kappa_n(y)),$$

where $\phi_{j,n} := \Pi_n(\phi_j)$. Define the discrete Dirichlet Laplacian $\Delta_n$ by $\Delta_nw(y) = 0$ for $y \in [0, h)$, and

$$\Delta_nw(y) = \frac{n^2}{\pi^2}\left(2w(\kappa_n(y)) - 2w(\kappa_n(y)) + w(\kappa_n(y) - \frac{\pi}{n})\right), \quad (2.17)$$

for $y \in [h, \pi)$, where $w : \mathcal{O} \to \mathbb{R}$ with $w(0) = w(\pi) = 0$. Since $\Delta_n\phi_j(\kappa_n(y)) = \lambda_{j,n}\phi_j(\kappa_n(y))$, it follows that

$$\Delta_nG_t^n(x, y) = \sum_{j=1}^{n-1} \lambda_{j,n} \exp(-\lambda_{j,n}^2t)\phi_{j,n}(x)\phi_j(\kappa_n(y)).$$

Similar to [24] Section 2, based on (2.14), the diagonalization of the matrix $A_n$, (2.15) and $u^n(t, kh) = \sum_{j=1}^{n-1} (U(t), e_j)e_j(k)$, one has

$$u^n(t, x) = \int_0^t \int_\mathcal{O} G_t^n(x, y)u_0(\kappa_n(y))dy + \int_0^t \int_\mathcal{O} \Delta_nG_{t-s}^n(x, y)f(u^n(s, \kappa_n(y)))dyds \quad \text{for } y \in \mathcal{O}.$$

We have the following regularity estimate of $G^n$, whose proof is analogous to that of Lemma 2.1 and thus is omitted.

Lemma 2.5. Let $\alpha \in (0, 1)$. Then for any $x, y \in \mathcal{O}$ and $0 \leq s < t \leq T$,

$$\int_0^s \int_\mathcal{O} |G^n_{t-r}(x, z) - G^n_{s-r}(y, z)|^2dzdr \leq C_\alpha|\alpha - y|^2 + |t - s|^\frac{3\alpha}{4},$$

and

$$\int_s^t \int_\mathcal{O} |G^n_{t-r}(x, z)|^2dzdr \leq C|t - s|^\frac{3\alpha}{4},$$

and

$$\int_0^s \int_\mathcal{O} \left|\Delta_nG^n_{t-r}(x, z) - \Delta_nG^n_{s-r}(y, z)\right|dzdr \leq C_\alpha|\alpha - y| + |t - s|^\frac{3\alpha}{4},$$

and

$$\int_s^t \int_\mathcal{O} |\Delta_nG^n_{t-r}(x, z)|dzdr \leq C_\alpha|t - s|^\frac{3\alpha}{4}.$$
2.3. Fully discrete FDM. We utilize the implicit Euler method with the time stepsize \( \tau = T/m \) to further discretize (1.2) and then obtain the fully discrete numerical method (1.3) with \( U^0 = U(0) \). In order to illustrate that (1.3) is uniquely solvable, by introducing \( Z^i := (-A_n)^{-1}U^i \), one can see that (1.3) is equivalent to

\[
Z^{i+1} - Z^i = \tau A_n^2 Z^{i+1} + \tau \tilde{b}(Z^{i+1}) + \tilde{\sigma}(Z^i)(\beta_{t_{i+1}} - \beta_{t_i}),
\]

(2.19)

where \( \tilde{b} \) is one-sided Lipschitz continuous and \( \tilde{\sigma} \) is globally Lipschitz continuous (see the proof of Theorem 2.4 for the expressions of \( \tilde{b} \) and \( \tilde{\sigma} \)). By means of [32, Lemma 3.1], we obtain the unique solvability of (2.19), which implies that (1.3) is uniquely solvable. In virtue of (1.3), we have

\[
U^i = (I + \tau A_n^2)^{-i}U^0 + \tau \sum_{l=0}^{i-1} (I + \tau A_n^2)^{-(i-l)}A_n F_n(U^{l+1})
\]

\[
+ \sum_{l=0}^{i-1} \sqrt{n^2/\pi} (I + \tau A_n^2)^{-(i-l)} \sum_{l=0}^{i-1} (U^{l+1}) \beta_{t_{i+1}}(U^i - \beta_{t_i}), \quad i \in \mathbb{Z}_m^0,
\]

where \( \sum_{l=0}^{i-1} \) is viewed as 0 by convention. Denote \( \eta_r(t) := \tau \lfloor \frac{t}{\tau} \rfloor \), i.e., \( \eta_r(t) = t_i \) for \( t \in (t_i, t_{i+1}) \).

Notice that for any \( i \in \mathbb{Z}_m^0, U^i = (U^i_1, \ldots, U^i_{n-1})^T \) is the temporal numerical approximation of the spatial semi-discrete numerical solution \( U(t_i) = (u^n(t_i, h), \ldots, u^n((n-1)h))^T \). Hence we denote \( u^{n,\tau}(t_i, kh) := U^i_k \) for \( k \in \mathbb{Z}_{n-1} \) and \( u^{n,\tau}(t_i, kh) := 0 \) for \( k \in \{0, n\} \), in view of the DBCs. In addition, we define \( u^{n,\tau}(t_i, x) := \Pi_n(u^{n,\tau}(t_i, \cdot))(x) \) as the fully discrete numerical solution of \( u(t_i, x) \) for every \( i \in \mathbb{Z}_m^0 \) and \( x \in \mathcal{O} \). Introduce the fully discrete Green function

\[
G^{n,\tau}_t(x, y) := \sum_{j=1}^{n-1} (1 + \tau \lambda_j^2)^{-\lfloor \frac{t}{\mu_i} \rfloor} \phi_j(x) \phi_j(y),
\]

and then by (2.17),

\[
\Delta_n G^{n,\tau}_t(x, y) = \sum_{j=1}^{n-1} \lambda_j \phi_j(x)(\kappa_n(y)).
\]

Analogously to (2.18), one has that for \( i \in \mathbb{Z}_m^0 \) and \( x \in \mathcal{O} \),

\[
u^{n,\tau}_n(t_i, x) = \int_{\mathcal{O}} G^{n,\tau}_t(x, y) u_0(\kappa_n(y)) dy
\]

\[
+ \int_0^{t_i} \int_{\mathcal{O}} \Delta_n G^{n,\tau}_{t_i-s + \tau}(x, y) f(u^{n,\tau}(\eta_r(s) + \tau, \kappa_n(y))) ds dy ds
\]

\[
+ \int_0^{t_i} \int_{\mathcal{O}} G^{n,\tau}_{t_i-s + \tau}(x, y) \sigma(u^{n,\tau}(\eta_r(s), \kappa_n(y))) W(ds, dy).
\]

By the polygonal interpolation in time, we define \( \{u^{n,\tau}(t, x)\}_{(t, x) \in [0, T] \times \mathcal{O}} \) by

\[
u^{n,\tau}(t, x) := u^{n,\tau}(t_i, x) + \frac{t - t_i}{\tau} (u^{n,\tau}(t_{i+1}, x) - u^{n,\tau}(t_i, x)), \quad t \in (t_i, t_{i+1}], \quad i \in \mathbb{Z}_{m-1}^0.
\]

For \( R \geq 1 \), let \( K_R : \mathbb{R} \to \mathbb{R} \) be an even smooth cut-off function satisfying

\[
K_R(x) = 1, \quad \text{if } |x| < R; \quad K_R(x) = 0, \quad \text{if } |x| \geq R + 1,
\]

(2.21)

and \( |K_R| \leq 1, |K_R'| \leq 2 \). We are now ready to present the existence of the density of the fully discrete numerical solution.
Theorem 2.6. Let $\sigma$ be continuously differentiable and $\sigma(x) \neq 0$ for any $x \in \mathbb{R}$. Then for sufficiently small $\tau > 0$, the law of $\{U^i\}_{i \in \mathbb{Z}_m}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{n-1}$.

Proof. As in the proof of Theorem 2.4, we only need to show that for any $i \in \mathbb{Z}_m$, the law of $Z^i = (-A_n)^{-\frac{1}{2}}U^i$ is absolutely continuous. For every $R \geq 1$, define $\{Z^i_R\}_{i \in \mathbb{Z}_m}$ recursively by

$$Z^i_R = \tau A_n^2 Z^i_R + \tau \tilde{b}_R(Z^i_R) + \tilde{\sigma}(Z^i_R^{-1})(\beta_{t_i} - \beta_{t_{i-1}}), \quad i \in \mathbb{Z}_m,$$

and $Z^0_R = Z^0$. Here $\tilde{b}_R(x) = \tilde{b}(x)K_R(||x||)$ is globally Lipschitz continuous.

Fix $i \in \mathbb{Z}_m$. Then $Z^i_R = Z^i$ on the set $\{\omega \in \Omega : \sup_{t \in \mathbb{Z}_m} ||Z^i|| \leq R\}$ whose probability converges to 1 as $R \to \infty$. By means of the globally Lipschitz continuity of $\tilde{b}_R$ and $\tilde{\sigma}$, one can prove that each component of $Z^i_R$ belongs to $\mathbb{D}^{1,2}$. Together with [33] Theorem 2.1.2, once we prove that the Malliavin covariance matrix $\gamma_i := \int_0^T D_r Z^i(D_r Z^i)^\top dr$ of $Z^i$ is invertible a.s., it will follow that the law of $Z^i$ is absolutely continuous (see Appendix A for the more details about the Malliavin derivative $D$ and the Malliavin–Sobolev space $\mathbb{D}^{1,2}$). Taking the Malliavin derivative on both sides of (2.19), it holds that for a.s. $r \in [t_{i-1}, t_i)$,

$$D_r Z^i = \tau(A_n^2 + \nabla \tilde{b}(Z^i)) D_r Z^i + \tilde{\sigma}(Z^i)^{-1},$$

By the one-sided Lipschitz continuity of $\tilde{b} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, there exists some constant $K_n$ depending on $n$ such that $y^\top (A_n^2 + \nabla \tilde{b}(Z^i)) y \leq K_n ||y||^2$. Hence for any $\tau \in (0, 1/K_n)$,

$$||y||^2 - \tau y^\top (A_n^2 + \nabla \tilde{b}(Z^i)) y \geq (1 - \tau K_n)||y||^2 \quad \forall y \in \mathbb{R}^{n-1}.$$

This implies that the matrix $I - \tau(A_n^2 + \nabla \tilde{b}(Z^i))$ is invertible for sufficiently small $\tau > 0$. Hence it follows from the invertibility of $\tilde{\sigma}$ that for any $y \in \mathbb{R}^{n-1}$ with $||y|| \neq 0$,

$$y^\top \gamma_i y \geq \int_{t_{i-1}}^{t_i} y^\top D_r Z^i(D_r Z^i)^\top y dr > 0,$$

which means that the Malliavin covariance matrix $\gamma_i$ of $Z^i$ is invertible a.s. \hfill \Box

3. Discrete $H^1$-regularity

We introduce the discrete $L^2$-inner product and the discrete $L^p$-norm ($1 \leq p \leq \infty$), respectively, as

$$\langle a, b \rangle_{l_n^2} = \pi \sum_{i=1}^{n-1} a_i b_i, \quad ||a||_{l_n^p} = \begin{cases} \left( \frac{1}{n} \sum_{i=1}^{n-1} |a_i|^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_{1 \leq i \leq n-1} |a_i|, & p = \infty, \end{cases}$$

for vectors $a = (a_1, \ldots, a_{n-1})^\top$ and $b = (b_1, \ldots, b_{n-1})^\top$.

This section presents the discrete $H^1$-regularity of the numerical solutions, which is crucial for the strong convergence analysis of the numerical solutions in Sections 3.4. We begin with the discrete versions of embedding and interpolation theorems.

Lemma 3.1. Let $2 \leq p \leq \infty$, $n \geq 2$, and $t > 0$. Then for any $a \in l_n^\infty$,

$$||a||_{l_n^\infty} \leq \sqrt{\pi} ||(-A_n)^{\frac{1}{2}} a||_{l_n^2}, \quad (3.1)$$

$$||a||_{l_n^p} \leq C ||A_n a||_{l_n^2} ||a||_{l_n^2}^{\frac{1}{2}}, \quad (3.2)$$

$$||e^{-A_n^2 t} a||_{l_n^p} \leq C t^{-\frac{1}{2}(\frac{1}{2} - \frac{1}{p})} ||a||_{l_n^2}, \quad (3.3)$$
where \( C > 0 \) is a constant independent of \( a, n \) and \( t > 0 \).

**Proof.** Let \( a = (a_1, \ldots, a_{n-1})^\top \) and \( a_0 = a_n = 0 \).

(i) It follows from the definition of \( A_n \) that

\[
\|(-A_n)^{\frac{1}{2}} a\|^2_{l^n_2} = \langle -A_n a, a \rangle_{l^n_2} = \frac{n}{\pi} \sum_{j=1}^{n} |a_j - a_{j-1}|^2.
\]

Hence, by the triangle and Cauchy–Schwarz inequalities, for \( k \in \mathbb{Z}_{n-1} \),

\[
|a_k| \leq \sum_{j=1}^{k} |a_j - a_{j-1}| \leq \left( \sum_{j=1}^{k} |a_j - a_{j-1}|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{k} \frac{n}{\pi} |a_j - a_{j-1}|^2 \right)^{\frac{1}{2}} \leq \sqrt{\frac{n}{\pi}} \|(-A_n)^{\frac{1}{2}} a\|_{l^n_2}.
\]

(ii) By the Cauchy–Schwarz inequality and (3.4),

\[
|a_k^2| \leq \sum_{j=1}^{k} \left| a_j^2 - a_{j-1}^2 \right| \leq \left( \frac{n}{\pi} \sum_{j=1}^{k} |a_j - a_{j-1}|^2 \right)^{\frac{1}{2}} \left( \frac{n}{\pi} \sum_{j=1}^{k} |a_j + a_{j-1}|^2 \right)^{\frac{1}{2}} \leq \|(-A_n)^{\frac{1}{2}} a\|_{l^n_2} \left( \frac{4n}{\pi} \sum_{j=1}^{n-1} |a_j|^2 \right)^{\frac{1}{2}} = 2 \|(-A_n)^{\frac{1}{2}} a\|_{l^n_2} \|a\|_{l^n_2} \quad \forall \ k \in \mathbb{Z}_{n-1},
\]

from which we deduce \( \|a\|_{l^n_2} \leq C \|(-A_n)^{\frac{1}{2}} a\|_{l^n_2} \|a\|_{l^n_2} \). This gives

\[
\|a\|_{l^n_2}^6 \leq \frac{n}{\pi} \sum_{j=1}^{n} |a_j|^2 \|a\|_{l^n_2}^4 \leq C \|(-A_n)^{\frac{1}{2}} a\|_{l^n_2}^2 \|a\|_{l^n_2}^4,
\]

which together with \( \|(-A_n)^{\frac{1}{2}} a\|_{l^n_2}^2 = \langle -A_n a, a \rangle_{l^n_2} \leq \|A_n a\|_{l^n_2} \|a\|_{l^n_2} \) yields (3.2).

(iii) Since \( \lambda_{j,n} \leq -\frac{4}{\pi} e^{2}, \sum_{j=1}^{n} e^{-2\lambda_{j,n}^2 t} < 1 \) \( \leq t^{-\frac{1}{4}} \int_{0}^{\infty} e^{-\frac{32}{\pi} z^4} dz \) \( =: C_0 t^{-\frac{1}{4}} \), where \( C_0 := \int_{0}^{\infty} \exp(-\frac{32}{\pi} z^4) dz < \infty \). By the Cauchy–Schwarz inequality and (2.15),

\[
\|e^{-A_n^2 t} a\|_{l^n_2} = \sup_{1 \leq k \leq n-1} \left| \sum_{j=1}^{n-1} e^{-\lambda_{j,n}^2 t} \langle a, e_j \rangle e_j(k) \right| \leq \left( \frac{n}{\pi} \sum_{j=1}^{n-1} |\langle a, e_j \rangle|^2 \right)^{\frac{1}{2}} \left( \frac{n}{\pi} \sum_{j=1}^{n-1} e^{-2\lambda_{j,n}^2 t} \|2/n\|^2 \right)^{\frac{1}{2}} \leq \|a\|_{l^n_2} C_0 t^{-\frac{1}{4}},
\]

where in the last step, we have used the fact that \( \{e_j\}_{j=1}^{n-1} \) forms an orthonormal basis of \( \mathbb{R}^{n-1} \). This proves (3.3) for \( p = \infty \). Besides, the Parseval identity leads to

\[
\|e^{-A_n^2 t} a\|_{l^n_2}^2 = \frac{n}{\pi} \|e^{-A_n^2 t} a\|^2_2 = \frac{n}{\pi} \sum_{j=1}^{n-1} e^{-2\lambda_{j,n}^2 t} |\langle a, e_j \rangle|^2 \leq \frac{n}{\pi} \sum_{j=1}^{n-1} |\langle a, e_j \rangle|^2 = \|a\|_{l^n_2}^2,
\]

which implies (3.3) for \( p = 2 \). By the Riesz–Thorin interpolation theorem (see e.g., [23, Theorem 1.3.4]), we obtain (3.3) for \( p \in (2, \infty) \). \( \square \)
3.1. Spatial FDM. Introduce \( O(t) := \sqrt{\frac{2}{\pi}} \int_0^t \exp(-A_n^2(t-s)) \Sigma_n(U(s))d\beta_s \) and \( V(t) := U(t) - O(t) \) for \( t \in [0, T] \). Then \( V \) solves
\[
\dot{V}(t) = -A_n^2 V(t) + A_n F_n(U(t)), \quad V(0) = U(0), \tag{3.5}
\]
where \( \cdot \) denotes the derivative with respect to time. In this part, we give the \( H^1 \)-regularity estimate of the spatial semi-discrete numerical solution \( U(t) \) by dealing with \( O(t) \) and \( V(t) \) separately.

By the elementary identity
\[
\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha} dr = \frac{\pi}{\sin(\pi \alpha)} \quad \forall \ 0 \leq r \leq t, \quad \text{with} \ \alpha \in (0, 1),
\tag{3.6}
\]
we shall use the factorization method to write
\[
(-A_n)^{1/2} O(t) = \frac{\sin(\pi \alpha)}{\pi} \int_0^t \exp(-A_n^2(t-r))(t-r)^{\alpha-1} Y(r) dr,
\]
where \( Y(r) := \sqrt{n/\pi} \int_0^t \exp(-A_n^2(r-s))(-A_n)^{1/2}(r-s)^{-\alpha} \Sigma_n(U(s))d\beta_s \). Then by \( \| \exp(-A_n^2(t-r)) \|_2 \leq 1 \) and the Hölder inequality, for any \( p > \frac{1}{\alpha} \),
\[
\|(-A_n)^{1/2} O(t)\|^p \leq C(\alpha, T, p) \int_0^t \| Y(r) \|^p dr \quad \forall \ t \in [0, T].
\]

Hereafter, \( \| \cdot \|_2 \) and \( \| \cdot \|_F \) denote the Euclidean and Frobenius norms of matrices, respectively. Notice that \( \| \Sigma_n(U(s))e_l \|^2 = \sum_{k=1}^{n-1} |\sigma(U_k(s)) e_l(k)\|^2 \leq C \), thanks to the boundedness of \( \sigma \) and \( |e_l(k)| \leq \sqrt{2/n} \) for all \( l, k \in \mathbb{Z}_{n-1} \). For any \( \alpha_1 > 0 \),
\[
e^{-x} \leq C_{\alpha_1, x} x^{-\alpha_1} \quad \forall \ x > 0.
\tag{3.7}
\]

Hence, by the symmetry of \( \Sigma_n(U(s)) \) and \( \|3.7| \) with \( \alpha_1 = \frac{3}{4} + \epsilon, \ 0 < \epsilon \ll 1 \),
\[
\|(-A_n)^{1/2} \exp(-A_n^2(r-s)) \Sigma_n(U(s))\|^2_F
\tag{3.8}
\]
\[
= \sum_{k,l=1}^{n-1} \langle (-A_n)^{1/2} \exp(-A_n^2(r-s)) \Sigma_n(U(s)) e_k, e_l \rangle^2
\]
\[
= \sum_{l=1}^{n-1} (-\lambda_{l,n}) \exp(-2\lambda_{l,n}^2(r-s)) \| \Sigma_n(U(s)) e_l \|^2
\]
\[
\leq C \sum_{l=1}^{n-1} (-\lambda_{l,n})^{-1/2} - 2\epsilon (r-s)^{-\frac{3}{4} - \epsilon} \leq C_\epsilon (r-s)^{-\frac{3}{4} - \epsilon}
\]
for any \( 0 \leq s < r \leq T \) since \( -\lambda_{l,n} \geq \frac{4}{\pi l^2} \). As a consequence, by the Burkholder inequality and choosing \( \alpha \in (0, \frac{1}{8} - \frac{3}{4}) \), we derive that for any \( p > \frac{1}{\alpha} \),
\[
\mathbb{E} \left[ \| Y(r) \|^p \right] = \mathbb{E} \left[ \| \int_0^r (-A_n)^{1/2} \exp(-A_n^2(r-s))(r-s)^{-\alpha} \Sigma_n(U(s))d\beta_s \|^p \right]
\]
\[
\leq C(p) \mathbb{E} \left[ \left| \int_0^r (r-s)^{-2\alpha} \|(-A_n)^{1/2} \exp(-A_n^2(r-s)) \Sigma_n(U(s))\|^2_F ds \right|^\frac{p}{2} \right]
\leq C(p, T).
\]
Therefore, we have that for any $p \geq 1$,
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|(-A_n)^{\frac{1}{2}} O(t)\|_{l_n^2}^p \right] \leq C_{p, T} \int_0^T \mathbb{E} \left[ \|Y(r)\|_{l_n^2}^p \right] dr \leq C(T, p). \tag{3.9}
\]

Taking (3.4) into account, it further yields that for any $p \geq 1$,
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|O(t)\|_{l_n^p}^p \right] \leq C(p, T). \tag{3.10}
\]

Taking advantage of the special form of $F_n$ and (3.10), we are able to show that $V$ is bounded in $L^\infty(0, T; L^p(\Omega; l_n^2))$ for $p \geq 2$.

**Lemma 3.2.** Let $u_0 \in C^1(\Omega)$ and $q \geq 1$. Then for any $t \in [0, T]$,
\[
\mathbb{E} \left[ \|V(t)\|_{l_n^2}^{2q} \right] + \mathbb{E} \left[ \int_0^t \|A_n V(s)\|_{l_n^2}^q ds \right] \leq C(q, T).
\]

**Proof.** Due to (3.4) and $u_0 \in C^1(\Omega)$,
\[
\|(-A_n)^{\frac{1}{2}} U(0)\|_{l_n^2}^2 = \frac{n}{\pi} \sum_{j=1}^n |u_0(jh) - u_0((j-1)h)|^2 \leq n \sum_{j=1}^n \frac{C}{n^2} \leq C,
\]
which together with the symmetry of $A_n$ and $\frac{4}{\pi^2} \leq \lambda_{\min}(-A_n) \leq \lambda_{\max}(-A_n) \leq (n-1)^2$ implies
\[
\|(-A_n)^{-\frac{1}{2}} V(0)\|_{l_n^2} \leq C \|V(0)\|_{l_n^2} \leq C \|(-A_n)^{\frac{1}{2}} V(0)\|_{l_n^2} \leq C. \tag{3.11}
\]
Here, $\lambda_{\min}(-A_n)$ and $\lambda_{\max}(-A_n)$ denote the minimal and maximal eigenvalues of $-A_n$, respectively. Notice that for $i \in \{1, 2, 3\}$ and $\delta \in (0, 1)$, $a^4 - b^4 \leq \delta a^4 + C \delta b^4$ for any $a, b \in \mathbb{R}$, from which it follows that for any $\epsilon \in (0, 1)$,
\[
\langle A_n F_n(U(t)), (-A_n)^{-1} V(t) \rangle_{l_n^2} = -\langle F_n(U(t)), V(t) \rangle_{l_n^2} \tag{3.12}
\]
\[
= -\frac{\pi}{n} \sum_{j=1}^{n-1} \left[ (V_j(t) + O_j(t))^2 - (V_j(t) + O_j(t)) \right] V_j(t)
\]
\[
\leq - (1 - \epsilon) \|V(t)\|_{l_n^4}^4 + C(\epsilon) (\|O(t)\|_{l_n^4}^4 + 1).
\]

Taking the inner product $\langle \cdot, (-A_n)^{-1} V(t) \rangle_{l_n^2}$ on both sides of (3.11) gives
\[
\frac{1}{2} \frac{d}{dt} \langle V(t), (-A_n)^{-1} V(t) \rangle_{l_n^2} = \langle A_n V(t), V(t) \rangle_{l_n^2} + \langle A_n F_n(U(t)), (-A_n)^{-1} V(t) \rangle_{l_n^2}
\]
\[
\leq -\|(-A_n)^{\frac{1}{2}} V(t)\|_{l_n^2}^2 - (1 - \epsilon) \|V(t)\|_{l_n^4}^4 + C(\epsilon) (\|O(t)\|_{l_n^4}^4 + 1).
\]

Then integrating with respect to time and using (3.11), we get that for $t \in (0, T]$,
\[
\|(-A_n)^{-\frac{1}{2}} V(t)\|_{l_n^2}^2 + \int_0^t 2\|(-A_n)^{\frac{1}{2}} V(s)\|_{l_n^2}^2 ds + \int_0^t (2 - 2\epsilon) \|V(s)\|_{l_n^4}^4 ds \tag{3.13}
\]
\[
\leq \|(-A_n)^{-\frac{1}{2}} V(0)\|_{l_n^2}^2 + C\epsilon \int_0^t 1 + \|O(s)\|_{l_n^4}^4 ds \leq C + C\epsilon \int_0^t \|O(s)\|_{l_n^4}^4 ds.
\]

By (3.13) with $\epsilon = \frac{1}{2}$, (3.10) and the Hölder inequality, we arrive at
\[
\mathbb{E} \left[ \int_0^t \|(-A_n)^{\frac{1}{2}} V(s)\|_{l_n^2}^2 ds \right] + \mathbb{E} \left[ \int_0^t \|V(s)\|_{l_n^4}^4 ds \right] \tag{3.14}
\]
\begin{align*}
& \leq C(p) + C(p, T) \mathbb{E} \int_0^t \|O(s)\|_{L^q} \, ds \leq C(p, T) \quad \forall t \in (0, T].
\end{align*}

Based on (3.11), we proceed to estimate \(\|V(t)\|_{L^2}^2\). Taking the inner product \(\langle \cdot, V(t) \rangle_{L^2_{\tau}}\) on both sides of (3.3), it follows that

\[ \frac{1}{2} \frac{d}{dt} \|V(t)\|_{L^2_{\tau}}^2 + \|A_n V(t)\|_{L^2_{\tau}}^2 = \langle F_n(U(t)) - F_n(V(t)), A_n V(t) \rangle_{L^2_{\tau}} + \langle A_n F_n(V(t)), V(t) \rangle_{L^2_{\tau}}. \]

The inequality \(ab \leq \frac{a^2}{4} + \frac{b^2}{4}\) for \(a, b \in \mathbb{R}\), and (2.11) give that for any \(\epsilon \in (0, 1)\),

\[ \langle F_n(U(t)) - F_n(V(t)), A_n V(t) \rangle_{L^2_{\tau}} \leq \frac{\epsilon}{4} \|A_n V(t)\|_{L^2_{\tau}}^2 + \frac{1}{\epsilon} \|F_n(U(t)) - F_n(V(t))\|_{L^2_{\tau}}^2 \leq \frac{\epsilon}{4} \|A_n V(t)\|_{L^2_{\tau}}^2 + C(\epsilon) \left( 1 + \|V(t)\|_{L^4_{\tau}}^4 + \|O(t)\|_{L^4_{\tau}}^4 \right) \|O(t)\|_{L^\infty_{\tau}}^2. \]

By \(a^3b \leq \frac{3}{4}a^4 + \frac{1}{4}b^4\) for \(a, b \in \mathbb{R}\), one can verify \(\langle x, A_n(x^3, \ldots, x^3_{n-1})^T \rangle \leq 0\). Hence,

\[ \langle A_n F_n(V(t)), V(t) \rangle_{L^2_{\tau}} \leq -\langle A_n F_n(x), V(t) \rangle_{L^2_{\tau}} \leq \frac{\epsilon}{4} \|A_n V(t)\|_{L^2_{\tau}}^2 + C(\epsilon)(\|V(t)\|_{L^4_{\tau}}^4 + 1). \]

Combining the above estimates with (3.11) produces

\begin{align*}
\|V(t)\|_{L^2_{\tau}}^2 & + (2 - \epsilon) \int_0^t \|A_n V(s)\|_{L^2_{\tau}}^2 \, ds \\
& \leq \|V(0)\|_{L^2_{\tau}}^2 + C(\epsilon) \int_0^t \left( 1 + \|V(s)\|_{L^4_{\tau}}^4 + \|O(s)\|_{L^4_{\tau}}^4 \right) \|O(s)\|_{L^\infty_{\tau}}^2 + 1 \, ds \\
& \leq C + C(\epsilon) \left( \sup_{t \in [0, T]} \|O(t)\|_{L^4_{\tau}}^4 + 1 \right) \int_0^t \left( 1 + \|V(s)\|_{L^4_{\tau}}^4 + \|O(s)\|_{L^4_{\tau}}^4 \right) \, ds.
\end{align*}

Taking \(\epsilon = \frac{1}{2}\) in the above inequality, we obtain from the Young inequality, (3.10) with \(p = 8q\), and (3.14) with \(p = 2q\) that

\[ \mathbb{E} \left[ \left( \int_0^t \|A_n V(s)\|_{L^2_{\tau}}^2 \, ds \right)^q \right] \leq C \sup_{t \in [0, T]} \|O(t)\|_{L^4_{\tau}}^4 + 1 + \mathbb{E} \left[ \left( \int_0^t 1 + \|V(s)\|_{L^4_{\tau}}^4 + \|O(s)\|_{L^4_{\tau}}^4 \, ds \right)^2q \right] \leq C, \]

which completes the proof. \(\square\)

Now we proceed to derive the discrete \(H^1\)-regularity estimate of the spatial semi-discrete numerical solution \(U(t)\), which guarantees \(U \in L^\infty(0, T; L^p(\Omega; L^2_{\tau})).\)

**Proposition 3.3.** Let \(u_0 \in C^1(\Omega)\) and \(p \geq 1\). Then for any \(t \in [0, T]\),

\[ \mathbb{E} \left[ \|(-A_n)^{\frac{1}{2}} U(t)\|_{L^2_{\tau}}^2 \right] \leq C(T, p). \]
Proof. It follows from (3.1), (3.3) with $p = 2$ and (3.11) that

$$
\|e^{-A^2_n t}V(0)\|_{l^p_n} \leq C\|e^{-A^2_n t}V(0)\|_{l^\infty_n} \leq C\|(-A_n)^{\frac{1}{2}} e^{-A^2_n t}V(0)\|_{l^p_n} \leq C.
$$

From the spectrum mapping theorem and the symmetry of $A_n$, we get

$$
\|e^{-\frac{1}{2}A^2_n (t-s)}(-A_n)\gamma\|^2_2 = \max_{1 \leq j \leq n} e^{-\frac{1}{2}\lambda_j^2 (t-s)}(-\lambda_j n)\gamma \leq C(t-s)^{-\gamma} \tag{3.18}
$$

for any $\gamma > 0$, since $x \mapsto x^\gamma e^{-\frac{1}{2}x^2}$ is uniformly bounded on $[0, \infty)$. Applying the variation of constants formula to (3.5), we infer from (3.3) with $\gamma > 0$, we obtain (3.20). Furthermore, (3.20)

$$
\|V(t)\|_{l^p_n} \leq \|e^{-A^2_n t}V(0)\|_{l^p_n} + \int_0^t \|e^{-\frac{1}{2}A^2_n (t-s)}e^{-\frac{1}{2}A^2_n (t-s)}A_n F_n(U(s))\|_{l^p_n} ds \tag{3.19}
$$

We claim that for any $\theta \in (0, \frac{3}{2})$ and sufficiently large $p \geq 1$,

$$
E \left[ \left\| \int_0^t (t-s)^{-\theta}\|F_n(U(s))\|_{l^2_n}^2 ds \right\|^p \right] \leq C(p, \theta, T). \tag{3.20}
$$

Indeed, by the definition of $F_n$, (3.2), and the Hölder inequality,

$$
\int_0^t (t-s)^{-\theta}\|F_n(U(s))\|_{l^2_n}^2 ds \leq C \int_0^t (t-s)^{-\theta} \left( 1 + \|V(s)\|_{l^p_n}^2 + \|O(s)\|_{l^p_n}^2 \right) ds
$$

$$
\leq C \int_0^t (t-s)^{-\theta} \left( 1 + \|V(s)\|_{l^p_n}^2 \right) ds + \sup_{t \in [0, T]} \|O(t)\|_{l^p_n}^3 + \left( \int_0^t (t-s)^{-\theta} \|V(s)\|_{l^p_n}^2 ds \right)^{\frac{3}{4}} \left( \int_0^T \|A_n V(s)\|_{l^p_n}^3 ds \right)^{\frac{1}{4}}
$$

$$
\leq C \left( 1 + \sup_{t \in [0, T]} \|O(t)\|_{l^p_n}^3 \right) + C \left( \int_0^T \|V(s)\|_{l^p_n}^{\frac{10p_2}{3p_1}} ds \right)^{\frac{3p_1}{3p_2}} \left( \int_0^T \|A_n V(s)\|_{l^p_n}^3 ds \right)^{\frac{1}{4}},
$$

where $p_2 = \frac{p_1}{p_1 - 1}$ with $p_1 = \frac{5}{2} + \frac{3}{\theta}$. Taking $p$th moment on both sides of the above inequality, then using the Hölder inequality, (3.10) and Lemma 3.2 we obtain (3.20). Furthermore, (3.20) (with $\theta = \frac{7}{12}$) and (3.19) give that for any $p \geq 1$,

$$
E \left[ \left\| V(t) \right\|_{l^p_n}^p \right] \leq C(p, T) \quad \forall \ t \in [0, T]. \tag{3.21}
$$

Due to (3.5), (3.11), and (3.18) with $\gamma = \frac{3}{2}$, for any $t \in [0, T],

$$
\|(-A_n)^{\frac{1}{2}} V(t)\|_{l^p_n} = \|(-A_n)^{\frac{1}{2}} e^{-A^2_n t}V(0)\|_{l^p_n} + \int_0^t \|(-A_n)^{\frac{1}{2}} e^{-A^2_n (t-s)}A_n F_n(U(s))\|_{l^p_n} ds
$$

$$
\leq C + C \int_0^t (t-s)^{-\frac{3}{4}}\|F_n(U(s))\|_{l^p_n} ds
$$

$$
\leq C + C \int_0^t (t-s)^{-\frac{3}{4}} \left( 1 + \|V(s)\|_{l^p_n}^3 + \|O(s)\|_{l^p_n}^3 \right) ds. \tag{3.22}
$$

Taking $p$th moment on both sides of (3.22), and using (3.10), (3.21), we have

$$
E \left[ \left\| (-A_n)^{\frac{1}{2}} V(t) \right\|_{l^p_n}^p \right] \leq C(p, T) \quad \forall \ t \in [0, T]. \tag{3.23}
$$

Due to $U = V + O$, (3.17) follows from (3.23) and (3.19).
Since $u^n(t, x) = \Pi_n(u^n(t, \cdot))(x)$ and $u^n(t, 0) = u^n(t, \pi) = 0$, it holds that
\[
\sup_{x \in \Omega} |u^n(t, x)| = \sup_{1 \leq j \leq n-1} |U_j(t)| = \|U(t)\|_{l_\infty} \leq \sqrt{n} \|(-A_n)^{1/2} U(t)\|_{l_\infty},
\]
thanks to (3.1). As a consequence, Proposition 3.3 implies the moment boundedness of the spatial semi-discrete numerical solution $u^n(t, x)$, i.e., for any $p \geq 1$,
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \sup_{x \in \Omega} |u^n(t, x)|^p \right] \leq C \sup_{t \in [0, T]} \mathbb{E} \left[ \|(-A_n)^{1/2} U(t)\|_{l_\infty}^p \right] \leq C(T, p). \tag{3.24}
\]

3.2. Fully discrete FDM. Introduce $O^i := \sqrt{n} \int_0^{t_i} (I + \tau A_n^2)^{-((i-\frac{1}{2}))} \sum_n(U^{(2)} \Sigma_n(U^{(1)}))d\beta_s$ and $V^i := U^i - O^i$ for $i \in \mathbb{Z}_m$. Then for $i \in \mathbb{Z}_m - 1$,
\[
V^{i+1} - V^i + \tau A_n V^{i+1} = \tau A_n F_n(U^{i+1}), \tag{3.25}
\]
with the initial value $V^0 = U^0 = U(0)$. Similarly to the spatial semi-discrete case, this part gives the $H^1$-regularity estimate of the fully discrete numerical solution $U^i$ by estimating $O^i$ and $V^i$ separately.

We first use the factorization method to estimate $O^i$. By (3.6) with $t = t_i$, \[
(-A_n)^{1/2} O^i = \frac{\sin(\pi \alpha)}{\pi} \int_0^{t_i} (I + \tau A_n^2)^{-(i-\frac{1}{2})-1}(t_i - r)^{\alpha-1} Y^\tau(r)dr,
\]
where $Y^\tau(r) := \sqrt{n} \int_0^r (I + \tau A_n^2)^{-(t-\frac{1}{2})}(A_n)^{1/2}(r - s)^{-\alpha} \Sigma_n(U^{(1)}) d\beta_s$. We have the smooth effect of $(I + \tau A_n^2)^{-t}$, namely for any $\gamma \in [0, 2]$ and $i \geq 1$,
\[
\|(-A_n)^{\gamma}(I + \tau A_n^2)^{-t}\|_2 = \max_{1 \leq k \leq n-1} (-\lambda_{k,n})^\gamma (1 + \tau \lambda_{k,n}^2)^{-t} \tag{3.26}
\]
and
\[
\|(-A_n)^{1/2}(I + \tau A_n^2)^{-t}\|_2 \leq C \times (\tau)^{-\frac{3}{4}-\gamma}. \tag{3.27}
\]
Here we used Bernoulli’s inequality $(1 + z)^{-\alpha} \leq (1 + \alpha z)^{-1}$ for $z > -1$ and $\alpha \geq 1$. Then proceeding as in (3.8), it holds that for any $0 < \epsilon < 1$ and $j \in \mathbb{Z}_m$,
\[
\|(-A_n)^{1/2}(I + \tau A_n^2)^{-t}\|_2 \leq C \epsilon (\tau)^{-\frac{3}{4}-\gamma} \quad \forall \ t \geq 1.
\]
Furthermore, the Burkholder inequality gives that for any $\alpha \in (0, \frac{1}{\alpha} - \frac{3}{4})$ and $p \geq 1$,
\[
\mathbb{E} \left[ \|Y^\tau(r)\|^p_{l_\infty} \right] \leq C(p) \left[ \int_0^r (r - s)^{-2\alpha} (\eta_r(r) - \eta_r(s)) + (\tau)^{-\frac{3}{4}-\gamma} ds \right]^\frac{p}{2} \leq C(p, T),
\]
since $\tau + \eta_r(r) - \eta_r(s) \geq r - s$. Hence similarly to (3.9), for any $p > \frac{1}{\alpha}$,
\[
\mathbb{E} \left[ \left| \left| \left| (-A_n)^{1/2} O^i \right| \right|_{l_\infty}^p \right| \right] \leq C(p, T) \int_0^T \mathbb{E} \left[ \|Y^\tau(r)\|^p_{l_\infty} \right] dr \leq C, \tag{3.27}
\]
which together with (3.1) and the Hölder inequality further ensures
\[
\mathbb{E} \left[ \sup_{i \in \mathbb{Z}_m} \|O^i\|_{l_\infty}^p \right] \leq C(p, T) \quad \forall \ p \geq 1. \tag{3.28}
\]

**Lemma 3.4.** Let $u_0 \in C^1(\Omega)$ and $q \geq 1$. Then for any $i \in \mathbb{Z}_m$,
\[
\mathbb{E} \left[ \|V^i\|_{l_\infty}^q \right] + \mathbb{E} \left[ \left| \sum_{j=0}^{i-1} \tau \|A_n(\tau^{j+1})\|_{l_\infty}^q \right| \right] \leq C(q, T).
\]
Proof. In view of the relation $V^0 = V(0) = U(0)$ and (3.11),

$$\|(-A_n)^{-\frac{1}{2}} V^0\|_{l^n_2} \leq C \|V^0\|_{l^n_2} \leq C \|(-A_n)^{-\frac{1}{2}} V^0\|_{l^n_2} \leq C.$$  \(3.29\)

Let us recall a fundamental identity

$$\langle x - y, x \rangle = \frac{1}{2} (\|x\|^2 - \|y\|^2 + \|x - y\|^2) \quad \forall \ x, y \in \mathbb{R}^d, d \geq 1.$$  \(3.30\)

Applying $\langle \cdot, (-A_n)^{-1} V^{i+1}\rangle_{l^n_2}$ on both sides of (3.25), it follows from (3.30) and a similar estimate of (3.12) that

$$\frac{1}{2}\|(-A_n)^{-\frac{1}{2}} V^{i+1}\|_{l^n_2}^2 - \frac{1}{2}\|(-A_n)^{-\frac{1}{2}} V^i\|_{l^n_2}^2 + \frac{1}{2}\|(-A_n)^{-\frac{1}{2}} (V^{i+1} - V^i)\|_{l^n_2}^2 = -\tau \|(-A_n)^{-\frac{1}{2}} V^{i+1}\|_{l^n_2}^2 + \tau \langle A_n F_n(U^{i+1}), (-A_n)^{-1} V^{i+1}\rangle_{l^n_2}$$

$$\leq -\tau \|(-A_n)^{-\frac{1}{2}} V^{i+1}\|_{l^n_2}^2 - \frac{1}{2}\|V^{i+1}\|_{l^n_4}^4 + C \tau \|O^{i+1}\|_{l^n_4}^4 + 1.$$  \(3.29\)

Hence by (3.28),

$$\|(-A_n)^{-\frac{1}{2}} V^i\|_{l^n_2}^2 + \sum_{j=0}^{i-1} \tau \|(-A_n)^{-\frac{1}{2}} V^{j+1}\|_{l^n_2}^2 + \sum_{j=0}^{i-1} \tau \|V^{j+1}\|_{l^n_4}^4 \leq C + C \tau \sum_{j=0}^{i-1} \|O^{j+1}\|_{l^n_4}^4.$$  \(3.30\)

Furthermore, according to (3.28) and the Hölder inequality, we deduce that

$$E \left[ \left( \sum_{j=0}^{i-1} \tau \|V^{j+1}\|_{l^n_4}^4 \right)^p \right] \leq C \quad \forall \ p \geq 1.$$  \(3.31\)

By (3.30), taking the inner product $\langle \cdot, V^{i+1}\rangle_{l^n_2}$ on both sides of (3.25) gives

$$\frac{1}{2}\|V^{i+1}\|_{l^n_2}^2 - \frac{1}{2}\|V^i\|_{l^n_2}^2 + \frac{1}{2}\|V^{i+1} - V^i\|_{l^n_2}^2 = -\tau \|A_n V^{i+1}\|_{l^n_2}^2 + \tau \langle F_n(U^{i+1}), A_n V^{i+1}\rangle_{l^n_2}.$$  \(3.28\)

Similarly to (3.15) and (3.16) with $\epsilon = \frac{1}{7}$,

$$\langle F_n(U^{i+1}), A_n V^{i+1}\rangle_{l^n_2} = \langle F_n(U^{i+1}) - F_n(V^{i+1}), A_n V^{i+1}\rangle_{l^n_2} + \langle F_n(V^{i+1}), A_n V^{i+1}\rangle_{l^n_2}$$

$$\leq \frac{1}{4}\|A_n V^{i+1}\|_{l^n_2}^2 + C \left( 1 + \|V^{i+1}\|_{l^n_4}^4 + \|O^{i+1}\|_{l^n_4}^4 \right) \left( \|O^{i+1}\|_{l^n_4}^2 + 1 \right).$$

Collecting the above estimates yields

$$\|V^i\|_{l^n_2}^2 + \tau \sum_{j=0}^{i-1} \|A_n V^{j+1}\|_{l^n_2}^2$$

$$\leq \|V^0\|_{l^n_2}^2 + C \left( \sup_{i \in \mathbb{Z}_m} \|O^i\|_{l^n_4}^2 + 1 \right) \sum_{j=0}^{i-1} \tau (1 + \|V^{j+1}\|_{l^n_4}^4 + \|O^{j+1}\|_{l^n_4}^4).$$

Taking $q$th moments on both sides of the above inequality and using (3.29), (3.28) and (3.31) finally yield the desired result. \(\square\)

**Proposition 3.5.** Let $u_0 \in C^1(O)$ and $p \geq 1$. Then for any $i \in \mathbb{Z}_m$,

$$E \left[ \|(-A_n)^{-\frac{1}{2}} U^i\|_{l^n_2}^p \right] \leq C(T, p).$$
Proof. According to (3.25), for any \( i \in \mathbb{Z}_m^0 \),
\[
V^i = (I + \tau A_n^2)^{-i}V^0 + \tau \sum_{j=0}^{i-1} (I + \tau A_n^2)^{-(i-j)}A_n(U^{j+1}).
\]

By virtue of the fact that \( \sum_{j=1}^{n-1}(1 + \tau \lambda_j^2)^{-2}\lambda_j \leq \sum_{j=1}^{n-1}(1 + 2\tau \lambda_j^2)^{-1} \leq C(\tau)^{-\frac{1}{3}} \), similarly to (3.3), it can be verified that for any \( 2 \leq p \leq \infty \),
\[
\|(I + \tau A_n^2)^{-i}a\|_{L_n^p} \leq C(p)(\tau)^{-\frac{1}{3}}(\frac{1}{p} - \frac{1}{p})\|a\|_{L_n^p} \quad \forall \ a \in L_n^\infty.
\] (3.32)

Applying (3.26), (3.32) and \( \|(I + \tau A_n^2)^{-i}V^0\|_{L_n^p} \leq \|V^0\|_{L_n^p} \leq C \), we arrive at
\[
\|V^i\|_{L_n^p} \leq C + \tau \sum_{j=0}^{i-1} \|(I + \tau A_n^2)^{-\frac{1}{3}(i-j)}A_n(I + \tau A_n^2)^{-\frac{1}{3}(i-j)}F_n(U^{j+1})\|_{L_n^p}
\leq C + C\tau \sum_{j=0}^{i-1} t_{i-j}^{\frac{7}{12}}\|A_n(I + \tau A_n^2)^{-\frac{1}{3}(i-j)}\|_2\|F_n(U^{j+1})\|_{L_n^p}
\leq C + C\tau \sum_{j=0}^{i-1} t_{i-j}^{\frac{7}{12}}\|F_n(U^{j+1})\|_{L_n^p}.
\]

Proceeding as in (3.20), one can show that for \( \theta \in [0, \frac{3}{4}) \) and sufficiently large \( p \geq 1 \),
\[
\mathbb{E}\left[\left(\tau \sum_{j=0}^{i-1} t_{i-j}^{-\theta}\|F_n(U^{j+1})\|_{L_n^p}\right)^p\right] \leq C(p, \theta, T),
\]
based on Lemma 3.4. In particular, the case \( \theta = \frac{7}{12} \) yields that for any \( p \geq 1 \),
\[
\mathbb{E}\left[\|V^t\|_{L_n^p}^p\right] \leq C(p, T) \quad \forall \ t \in [0, T].
\] (3.33)

Then analogously to (3.22), by (3.29) and (3.26),
\[
\|(A_n)^{\frac{3}{4}}V^i\|_{L_n^p} \leq \|(A_n)^{\frac{3}{4}}(I + \tau A_n^2)^{-i}V^0\|_{L_n^p}
\leq C + C\tau \sum_{j=0}^{i-1} (t_i - t_j)^{-\frac{7}{12}}(1 + \|V^{j+1}\|_{L_n^p}^3 + \|O^{j+1}\|_{L_n^p}^3).
\] (3.34)

Then one can use (3.33) and (3.28) to obtain that
\[
\mathbb{E}[\|(A_n)^{\frac{3}{4}}V^i\|_{L_n^p}^p] \leq C(T, p) \quad \forall \ i \in \mathbb{Z}_m^0,
\]
which along with (3.27) and \( U^i = V^i + O^i \) completes the proof. \( \square \)

In a similar manner to (3.24), Proposition 3.35 guarantees that for any \( p \geq 1 \),
\[
\sup_{i \in \mathbb{Z}_m^0} \mathbb{E}\left[\sup_{x \in \mathcal{G}} \left|u^{n, \tau}(t_i, x)\right|^p\right] \leq C(T, p).
\] (3.35)
4. Strong convergence analysis (I)

In this section, we study the strong convergence rate of the spatial FDM. For $n \geq 2$, denote by $\bar{U}(t) := (u(t, h), \ldots, u(t, (n-1)h))$ the exact solution to (1.1) on grid points, where the explicit dependence of $U(t)$ on $n$ is omitted. We introduce the following auxiliary process $\{\tilde{U}(t), t \in [0, T]\}$ by

$$d\tilde{U}(t) + A_n^T \tilde{U}(t) dt = A_n F_n(\tilde{U}(t)) dt + \sqrt{n/\pi} \Sigma_n(\tilde{U}(t)) d\beta_t, \quad t \in (0, T) \tag{4.1}$$

with initial value $\tilde{U}(0) = U(0)$. Let $\tilde{u}^n = \{\tilde{u}^n(t, x), (t, x) \in [0, T] \times \mathcal{O}\}$ satisfy

$$\tilde{u}^n(t, x) = \int_0^t \int_\mathcal{O} G^n_t(x, y) u_0(\kappa_n(y)) dy + \int_0^t \int_\mathcal{O} \Delta_n G^n_{t-s}(x, y) f(u(s, \kappa_n(y))) dy ds$$

$$+ \int_0^t \int_\mathcal{O} G^n_{t-s}(x, y) \sigma(u(s, \kappa_n(y))) W(ds, dy).$$

Then $\tilde{U}_k(t) = \tilde{u}^n(t, kh)$ for $k \in \mathbb{Z}_{n-1}$ and $t \in [0, T]$.

4.1. Error estimate between $\tilde{u}^n$ and $u$. This part deals with the error between the exact solution $u$ and the auxiliary process $\tilde{u}$, which will rely on the following estimates of the discrete Green function.

Lemma 4.1. There exists $C = C(T)$ such that for any $(t, x) \in (0, T) \times \mathcal{O}$,

$$\int_0^t \int_\mathcal{O} |\Delta_n G^n_s(x, y) - \Delta_s G_s(x, y)| dy ds \leq C n^{-1}, \quad (4.2)$$

$$\int_0^t \int_\mathcal{O} |G^n_s(x, y) - G_s(x, y)|^2 dy ds \leq C n^{-2}. \quad (4.3)$$

Proof. Due to the H"older inequality, it suffices to prove that for $\mu \in \{0, 1\}$,

$$\int_0^T \left( \int_\mathcal{O} (-\Delta_n)^\mu G^n_s(x, y) - (-\Delta)^\mu G_s(x, y) \right)^2 dy ds \leq C n^{-2(2-\mu)}.$$

In the remainder of the proof, we always assume $\mu \in \{0, 1\}$ and $0 < \epsilon \ll 1$. Denote

$$M_{1, \mu}^{s,x,y} := \sum_{j=1}^{n-1} (-\lambda_{j,n})^\mu e^{-\lambda_{j,n}^2 s} \phi_{j,n}(x) (\phi_{j}(\kappa_n(y)) - \phi_{j}(y)),$$

$$M_{2, \mu}^{s,x,y} := \sum_{j=1}^{n-1} \left((-\lambda_{j,n})^\mu e^{-\lambda_{j,n}^2 s} \phi_{j,n}(x) - (-\lambda_{j})^\mu e^{-\lambda_{j}^2 s} \phi_{j}(x) \right) \phi_{j}(y),$$

$$M_{3, \mu}^{s,x,y} := \sum_{j=n}^{\infty} (-\lambda_{j})^\mu e^{-\lambda_{j}^2 s} \phi_{j}(x) \phi_{j}(y), \quad s \in [0, T], x, y \in \mathcal{O}.$$

Since $\{\phi_{j}\}_{j \geq 1}$ is an orthonormal basis of $L^2(\mathcal{O})$, it holds that

$$\int_\mathcal{O} |(-\Delta_n)^\mu G^n_s(x, y) - (-\Delta)^\mu G_s(x, y)|^2 dy \leq 3 \sum_{i=1}^{3} \int_\mathcal{O} |M_{i, \mu}^{s,x,y}|^2 dy. \quad (4.4)$$

By virtue of $|\phi_j(x)| \leq 1$ and (4.7) with $\alpha_1 = \frac{2}{2-\mu}(1-\epsilon) > \mu + \frac{1}{4}$,

$$\int_0^T \left( \int_\mathcal{O} |M_{i, \mu}^{s,x,y}|^2 dy \right)^{\frac{1}{2}} ds \leq \int_0^T \left( \sum_{j=n}^{\infty} j^{4\mu} e^{-2j^4 s} \right)^{\frac{1}{2}} ds$$
\[ \leq C \int_{0}^{T} \left( \sum_{j=n}^{\infty} j^{4 \mu - 4 \alpha_1} s^{-\alpha_1} \right)^{1-\frac{\mu}{2}} ds \leq C n^{-4 \left[ 1 - (\mu + \frac{1}{2}) \left( 1 - \frac{n}{4} \right) \right] - \epsilon} \leq C n^{-(2-\mu)}, \]

thanks to \(4 \left[ 1 - (\mu + \frac{1}{2}) \left( 1 - \frac{n}{4} \right) \right] > 2 - \mu\) for all \(\mu \in [0, 1]\).

We recall the following inequality in the proof of [24, Lemma 3.2]:

\[ \int_{\mathcal{O}} |w(y) - w(\kappa_n(y))|^2 dy \leq Cn^{-2} \int_{\mathcal{O}} \left| \frac{d}{dy} w(y) \right|^2 dy, \quad \text{for } w \in C^1(\mathcal{O}). \quad (4.5) \]

By (4.5) and (3.1), for any \(\rho > 0\),

\[ \int_{\mathcal{O}} |M_{i,\mu}^{s,x,y}|^2 dy \leq C n^{-2} \int_{\mathcal{O}} \left| \sum_{j=1}^{n-1} (-\lambda_{j,n})^\mu e^{-\lambda_{j,n} s} \phi_{j,n}(x) \cos(jy) \right|^2 dy \]
\[ \leq C n^{-2} \sum_{j=1}^{n-1} j^{4 \mu + 2 - 2 \lambda_{j,n}^2 s} \phi_{j,n}(x)^2 \leq C n^{-2} \sum_{j=1}^{n-1} j^{4 \mu + 2 - 4 \rho s - \rho}. \]

Observe that for any \(\alpha \in (0, 1]\),

\[ 1 - e^{-x} \leq C_n x^\alpha, \quad x \geq 0. \quad (4.6) \]

In view of (2.16), \(|\lambda_j - \lambda_{j,n}| \leq C j^2/n^2\). Thus, \(|(-\lambda_j)^\mu - (-\lambda_{j,n})^\mu| \leq \mu (-\lambda_{j,n})^{\mu-1} |\lambda_j - \lambda_{j,n}| \leq C j^{2 \mu + 2}/n^2\) and \(\lambda_j^2 - \lambda_{j,n}^2 = |\lambda_j - \lambda_{j,n}| |\lambda_j + \lambda_{j,n}| \leq C j^6/n^2\), which along with (4.6) yields that for \(\rho_1 > 0\),

\[ |e^{-\lambda_j^2 n s} - e^{-\lambda_{j,n}^2 n s}| \leq e^{-\lambda_j^2 n s} j^6 n^{-2} s \leq C n^{-2} j^6 j^{2 \rho_1} s^{1 - \frac{\mu}{2}}. \]

Besides, it can be verified that \(|\phi_{j,n}(x) - \phi_{j}(x)| \leq C j/n\). Therefore, for \(\rho, \rho_1 > 0\),

\[ \int_{\mathcal{O}} |M_{i,\mu}^{s,x,y}|^2 dy = \sum_{j=1}^{n-1} \left| (-\lambda_{j,n})^\mu e^{-\lambda_{j,n} s} \phi_{j,n}(x) - (-\lambda_j)^\mu e^{-\lambda_j s} \phi_j(x) \right|^2 \]
\[ \leq 3 \sum_{j=1}^{n-1} \left| (-\lambda_{j,n})^\mu - (-\lambda_j)^\mu \right|^2 e^{-2 \lambda_{j,n} s} \phi_{j,n}(x) \phi_{j}(x) \]
\[ + 3 \sum_{j=1}^{n-1} \lambda_j^2 e^{-2 \lambda_j s} \phi_{j,n}(x) \phi_{j}(x) \]
\[ \leq C n^{-4} \sum_{j=1}^{n-1} j^{4 \mu + 1 - 4 \rho s - \rho} + C n^{-4} \sum_{j=1}^{n-1} j^{12 + 4 \mu - 4 \rho s} - \rho + C n^{-2} \sum_{j=1}^{n-1} j^{4 \mu + 2 - 4 \rho s - \rho} \]
\[ \leq C n^{-2} \sum_{j=1}^{n-1} j^{4 \mu + 2 - 4 \rho s - \rho}, \]

where in the last step we set \(\rho_1 = \rho + 2\). By choosing \(\rho \in (\mu + \frac{3}{4}, \frac{2}{2-\mu})\), we have

\[ \int_{0}^{T} \left( \int_{\mathcal{O}} |M_{i,\mu}^{s,x,y}|^2 dy \right)^{1-\frac{\mu}{2}} ds \leq C \int_{0}^{T} \left( n^{-2 s - \rho} \right)^{1-\frac{\mu}{2}} ds \leq C n^{-(2-\mu)}, \quad i = 1, 2. \]

Finally inserting the estimates on \(\{M_{i,\mu}^{s,x,y}\}_{i=1,2,3}\) into (1.4) finishes the proof. \qed

By virtue of Lemmas 2.2 and 4.1 we now estimate the error between \(u\) and \(\bar{u}\).
Proposition 4.2. Let \( u_0 \in C^3(\Omega) \). Then for any \( p \geq 1 \), there exists some constant \( C = C(p, T) \) such that for any \( (t, x) \in [0, T] \times \Omega \),
\[
\|\bar{u}^n(t, x) - u(t, x)\|_{L^p(\Omega)} \leq Cn^{-1}.
\]

**Proof.** For fixed \( (t, x) \in [0, T] \times \Omega \), \( u^n(t, x) - u(t, x) = \sum_{j=1}^5 I_j \), where
\[
I_1 := \int_\Omega G_t^n(x, y)u_0(\kappa_n(y))dy - G_tu_0(x), \\
I_2 := \int_0^t \int_\Omega [G^n_{t-s}(x, y) - G^n_{t-s}(x, y)]\sigma(u(s, \kappa_n(y)))W(ds, dy), \\
I_3 := \int_0^t \int_\Omega G_{t-s}(x, y)[\sigma(u(s, \kappa_n(y))) - \sigma(u(s, y))]W(ds, dy), \\
I_4 := \int_0^t \int_\Omega [\Delta_n G^n_{t-s}(x, y) - \Delta G^n_{t-s}(x, y)]f(u(s, \kappa_n(y)))dyds, \\
I_5 := \int_0^t \int_\Omega \Delta G_{t-s}(x, y) [f(u(s, \kappa_n(y))) - f(u(s, y))]dyds.
\]

Following the proof of [8] Lemma 2.3], we use the PDE satisfied by \( G \) to write \( G_tu_0(x) = u_0(x) - \int_0^t \int_\Omega G_r(x, z)u_0^n(z)dzdr \). As a numerical counterpart,
\[
\int_\Omega G_t^n(x, y)u_0(\kappa_n(y))dy - \bar{u}^n(0, x) = \int_\Omega \int_0^t \frac{\partial}{\partial r}G^n_r(x, z)u_0(\kappa_n(z))dzdr \\
- \int_0^t \int_\Omega \Delta^2_n G^n_r(x, z)u_0(\kappa_n(z))dzdr = - \int_0^t \int_\Omega \Delta_n G^n_r(x, z)\Delta_n u_0(z)dzdr,
\]
where \( \bar{u}^n(0, x) = \Pi_n(u_0)(x) \) and in the last step we have used the fact that
\[
\int_\Omega \Delta_n v(z)w(\kappa_n(z))dz = \int_\Omega v(\kappa_n(z))\Delta_n w(z)dz,
\]
for \( v, w : \Omega \to \mathbb{R} \) with \( v = w = 0 \) on \( \partial \Omega \). In particular, when \( u_0 \in C^1(\Omega) \),
\[
|\bar{u}^n(0, x) - \bar{u}^n(0, y)| \leq C|x - y|, \quad x, y \in \Omega.
\]

By \( u_0 \in C^3(\Omega) \), (2.17) and the Taylor expansion, there exist \( \theta_1, \theta_2 \in (0, 1) \) such that for \( z \in [h, \pi] \),
\[
|u^n_0(z) - \Delta_n u_0(z)| = |u^n_0(z) - \frac{1}{2}u''_0(\kappa_n(z) + \theta_1 \pi_n) - \frac{1}{2}u''_0(\kappa_n(z) - \theta_2 \pi_n)| \leq Cn^{-1},
\]
and for \( z \in [0, h] \), \( |u^n_0(z) - \Delta_n u_0(z)| = |u''_0(z)| = |u''_0(z) - u''_0(0)| \leq Cn^{-1} \). Therefore, using (2.3) and (4.2), a direct calculation gives
\[
|I_1| \leq Cn^{-1} + \int_0^t \int_\Omega |\Delta_n G^n_r(x, z) - \Delta G_r(x, z)||u''_0(z)|dzdr \\
+ \int_0^t \int_\Omega |\Delta G_r(x, z)||u''_0(z) - \Delta_n u_0(z)|dzdr \leq Cn^{-1}.
\]

Then we apply the Burkholder inequality, the boundedness and Lipschitz continuity of \( \sigma \), (4.3), (2.5), and Lemma 2.2 to obtain
\[
\|I_2 + I_3\|_{L^p(\Omega)} ^2 \leq C \int_0^t \int_\Omega |G^n_{t-s}(x, y) - G_{t-s}(x, y)|^2dyds
\]
\[ + C \int_0^t \int_\mathcal{O} G_{t-s}^2(x,y)\|u(s,\kappa_n(y)) - u(s,y)\|_{L^p(\Omega)}^2 dyds \leq Cn^{-2}. \]

It follows from \(|f(x)| \leq C(1 + |x|^3)\) and (2.27) that \(\|u(t,x)\|_{L^p(\Omega)}\) is uniformly bounded for all \((t,x) \in [0,T] \times \mathcal{O}\). This together with (4.2) indicates

\[ \|\mathcal{I}_4\|_{L^p(\Omega)} \leq C \int_0^t \int_\mathcal{O} |\Delta_n G^m_{t-s}(x,y) - \Delta G_{t-s}(x,y)| \|u(s,\kappa_n(y))\|_{L^p(\Omega)} dyds \leq Cn^{-1}. \]

In addition, making use of (2.11), the Hölder inequality, Lemma 2.2 and (2.7) yields

\[ \|f(u(s,\kappa_n(y))) - f(u(s,y))\|_{L^p(\Omega)} \leq C\|u(s,\kappa_n(y)) - u(s,y)\|_{L^p(\Omega)} (1 + \|u(s,\kappa_n(y))\|_{L^p(\Omega)} + \|u(s,y)\|_{L^p(\Omega)}) \leq Cn^{-1} \]

for all \((s,y) \in [0,T] \times \mathcal{O}\). Hence taking advantage of (2.3), we have \(\|\mathcal{I}_5\|_{L^p(\Omega)} \leq Cn^{-1}\). Gathering the above estimates finally completes the proof.

\[ \square \]

The next result reveals that \(u^n\) has the same Hölder continuity exponent as \(u\).

**Lemma 4.3.** Let \(u_0 \in C^2(\mathcal{O})\). Then for any \(\alpha \in (0,1)\) and \(p \geq 1\), there exists some constant \(C = C(p,T,\alpha)\) such that for any \(0 \leq s < t \leq T\), \(x, y \in \mathcal{O}\) and \(n \geq 2\),

\[ \|u^n(t,x) - u^n(s,y)\|_{L^p(\Omega)} \leq C(|t - s|^{\alpha/p} + |x - y|). \]

**Proof.** Since \(u_0 \in C^2(\mathcal{O})\), \(|\Delta_n u_0(z)| \leq C\) for \(z \in \mathcal{O}\). Hence it follows from (4.7), (4.8) and Lemma 2.5 that

\[ \left| \int_\mathcal{O} G^n_t(x,z)u_0(\kappa_n(z))dz - \int_\mathcal{O} G^n_s(y,z)u_0(\kappa_n(z))dz \right| \]

\[ \leq C|x-y| + \int_0^s \int_\mathcal{O} |\Delta_n G^n_s(x,z) - \Delta_n G^n_s(y,z)||\Delta_n u_0(z)|dzdr \]

\[ + \int_s^t \int_\mathcal{O} |\Delta_n G^n_s(x,z)||\Delta_n u_0(z)|dzdr \leq C(|x-y| + |t-s|^{\alpha/p}). \]

In virtue of Lemma 2.5, by further applying (2.3) with \(S = \Delta_n G^n\) and (2.4) with \(S = G^n\), we obtain the desired result. \[ \square \]

### 4.2. Error estimate between \(\tilde{u}^n\) and \(u^n\).

This part carries out the error estimate between the auxiliary process \(\tilde{u}^n\) and the numerical solution \(u^n\). This will be accomplished by studying the moment estimates of \(E(t) := \tilde{U}(t) - U(t)\), since \(\sup_{x \in \mathcal{O}} |\tilde{u}^n(t,x) - u^n(t,x)| = \|E(t)\|_{L^\infty(\mathcal{O})}\).

**Proposition 4.4.** Let \(u_0 \in C^3(\mathcal{O})\). Then there exists some constant \(C = C(T)\) such that for any \(t \in [0,T]\),

\[ \mathbb{E}\left[\|(-A_n)^{-\frac{3}{2}} E(t)\|^4_{l^8}\right] + \mathbb{E}\left[\int_0^t \|(-A_n)^{-\frac{3}{2}} E(s)\|^2_{l^8} ds\right]^2 \leq Cn^{-4}. \]

**Proof.** The proof is divided into two steps.

**Step 1:** We show that for any \(p \geq 2\), there exists \(C = C(p,T)\) such that

\[ \int_0^t \mathbb{E}\left[\|(-A_n)^{-\frac{3}{2}} E(s)\|_{l^8}^{p-2}\] \(\|(-A_n)^{-\frac{3}{2}} E(s)\|_{l^8}^2\) ds \leq Cn^{-p} \forall t \in [0,T]. \]

Subtracting (1.2) from (1.1) leads to

\[ dE(t) + A_n^2 E(t)dt \]

(4.11)
= A_n \{ F_n(U(t)) - F_n(U(t)) \} dt + \sqrt{n/\pi} \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \} d\beta_t.

Applying Itô's formula to $$\|(-A_n)^{-\frac{1}{2}} E(t)\|^p$$ ($$p \geq 2$$) reveals

\[
d\|(-A_n)^{-\frac{1}{2}} E(t)\|^p = -p\|(-A_n)^{-\frac{1}{2}} E(t)\|^{p-2}\|(-A_n)^{\frac{1}{2}} E(t)\|^2 dt
\]

\[
+ \frac{n}{2\pi} p\|(-A_n)^{-\frac{1}{2}} E(t)\|^{p-2}\|(-A_n)^{-\frac{1}{2}} \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \}\|^2 dt
\]

\[
+ \frac{n}{2\pi} p(p-2)\|(-A_n)^{-\frac{1}{2}} E(t)\|^{p-4}\| E(t)^\top (-A_n)^{-\frac{1}{2}} \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \}\|^2
\]

\[
+ \frac{p}{\sqrt{\pi}} \|(-A_n)^{-\frac{1}{2}} E(t)\|^{p-2} \langle (-A_n)^{-1} E(t), \sqrt{n} \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \} d\beta_t \rangle.
\]

For $$t \in [0, T]$$, we introduce

\[
J_1(t) := \langle E(t), F_n(U(t)) - F_n(U(t)) \rangle,
\]

\[
J_2(t) := n \|(-A_n)^{-\frac{1}{2}} \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \}\|^2_F,
\]

\[
J_3(t) := n \| E(t)^\top (-A_n)^{-1} \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \}\|^2.
\]

Since $$\|z^\top B_1\| \leq \|B_1\|_{F} \|z\|$$ for any $$z \in \mathbb{R}^{n-1}$$ and $$B_1 \in \mathbb{R}^{(n-1) \times (n-1)}$$, we have

\[
J_3(t) \leq J_2(t) \|(-A_n)^{-\frac{1}{2}} E(t)\|^2.
\]

Due to (4.12), (4.13), and $$E(0) = 0$$, it holds that for any $$p \geq 2$$,

\[
E \left[ \|(-A_n)^{-\frac{1}{2}} E(t)\|^p \right] + p \int_0^t E \left[ \|(-A_n)^{-\frac{1}{2}} E(s)\|^{p-2}\|(-A_n)^{\frac{1}{2}} E(s)\|^2 \right] ds
\]

\[
\leq \frac{1}{2} p^2 \int_0^t E \left[ \|(-A_n)^{-\frac{1}{2}} E(s)\|^{p-2} (J_1(s) + J_2(s)) \right] ds.
\]

The Lipschitz continuity of $$\sigma$$ and $$e_l(k) \leq \sqrt{2/n}$$ for $$l, k \in \mathbb{Z}_{n-1}$$ imply

\[
\| \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \} e_l \|^2 = \sum_{k=1}^{n-1} \| (\sigma(U_k(t)) - \sigma(U_k(t))) e_l(k) \|^2
\]

\[
\leq C n \sum_{k=1}^{n-1} \| U_k(t) - U_k(t) \|^2 = C n \| U(t) - U(t) \|^2 \quad \forall l \in \mathbb{Z}_{n-1}.
\]

Analogous to (3.8), by (4.15) and the symmetry of $$A_n$$ and $$\Sigma_n(U(t)) - \Sigma_n(U(t))$$,

\[
J_2(t) = n \sum_{l=1}^{n-1} (-\lambda_{l,n})^{-1} \| \{ \Sigma_n(U(t)) - \Sigma_n(U(t)) \} e_l \|^2
\]

\[
\leq C \| U(t) - U(t) \|^2 \leq C \| U(t) - U(t) \|^2 + C \| E(t) \|^2,
\]

due to $$\sum_{l=1}^{n-1} (-\lambda_{l,n})^{-1} \leq C$$. By (2.11) and the Hölder inequality, for $$1 \leq q < \infty$$,

\[
\| F_n(U(t)) - F_n(U(t)) \|^2 \leq C \sum_{k=1}^{n-1} (1 + U_k(t)^2 + \tilde{U}_k(t)^2)^2 \| U_k(t) - \tilde{U}_k(t) \|^2
\]

\[
\leq C n \| U(t) \|_{L^q}^4 + \| U(t) \|_{L^q}^4 \| U(t) - \tilde{U}(t) \|_{L^q}^2,
\]
where \( q' = q/(q - 1) \). Denote \( K(t) := \left( 1 + \| U(t) \|_{L^p}^4 + \| \tilde{U}(t) \|_{L^p}^4 \right) \| U(t) - \tilde{U}(t) \|^2_{L^q} \) for \( t \in [0, T] \).

Utilizing (2.10) and the Young inequality, we get
\[
J_1(t) = \langle E(t), F_n(U(t)) - F_n(\tilde{U}(t)) \rangle + \langle E(t), F_n(\tilde{U}(t)) - F_n(U(t)) \rangle \\
\leq \frac{3}{2} \| E(t) \|^2 + \frac{1}{2} \| F_n(U(t)) - F_n(\tilde{U}(t)) \|^2,
\]
which, along with (4.17) (with \( q = 2 \)), (4.16) and \( \| \cdot \|_{L^q} \leq C \| \cdot \|_{L^4} \), indicates
\[
J_1(t) + J_2(t) \leq C \| E(t) \|^2 + CnK(t)
\]
for \( \epsilon_1 > 0 \). By Proposition 4.2 and the Hölder inequality, for any \( q \geq \theta \geq 1 \),
\[
\mathbb{E} \left[ \| U(s) - \tilde{U}(s) \|_{L^q}^q \right] = \mathbb{E} \left[ \left( \frac{\pi}{n} \sum_{l=1}^{n-1} |u(s, lh) - \tilde{u}^n(s, lh)|^q \right)^{\frac{1}{q}} \right] \leq Cn^{-q}
\]
for any \( s \in [0, T] \). Thanks to (2.7), we have that for any \( q \geq \theta \geq 1 \) and \( t \in [0, T] \),
\[
\mathbb{E} \left[ \| U(t) \|_{L^q}^q \right] \leq C \mathbb{E} \left[ \| U(t) \|_{L^\theta}^\theta \right] \leq C \| \sup_{x \in \Omega} |u(t, x)|^q \| \leq C(q, T).
\]
A combination of (4.19) and (4.20) yields that for any \( q \geq \theta \geq 1 \),
\[
\mathbb{E} \left[ \| \tilde{U}(s) \|_{L^q}^q \right] \leq C(\theta, q, T) \quad \forall s \in [0, T].
\]
The previous three estimates and the Hölder inequality ensure that for any \( p \geq 1 \),
\[
\| K(s) \|_{L^p(\Omega)} \leq C (1 + \| U(s) \|_{L^p(\Omega; L^q)}^q + \| \tilde{U}(s) \|_{L^p(\Omega; L^q)}^q) \| U(s) - \tilde{U}(s) \|^2_{L^q(\Omega; L^p)} \\
\leq Cn^{-2} \quad \forall s \in [0, T].
\]
Since \( \sqrt{\pi} \| \cdot \| = \sqrt{\pi} \| \cdot \|_{L^q} \), it follows from (4.14) and (4.18) with \( \epsilon_1 = 1/p^2 \) that
\[
\mathbb{E} \left[ \| (-A_n)^{-\frac{1}{2}} E(t) \|_{L^p}^p \right] + (p - \frac{1}{2}) \int_0^t \mathbb{E} \left[ \| (-A_n)^{-\frac{1}{2}} E(s) \|_{L^p}^{p-2} \| (-A_n)^{\frac{1}{2}} E(s) \|_{L^q}^2 \right] ds \\
\leq C \int_0^t \mathbb{E} \left[ \| (-A_n)^{-\frac{1}{2}} E(s) \|_{L^p}^p \right] ds + C \int_0^t \mathbb{E} \left[ \| (-A_n)^{-\frac{1}{2}} E(s) \|_{L^p}^{p-2} \| K(s) \|_{L^q}^2 \right] ds \\
\leq C \int_0^t \mathbb{E} \left[ \| (-A_n)^{-\frac{1}{2}} E(s) \|_{L^p}^p \right] ds + C \int_0^t \mathbb{E} \left[ \| K(s) \|_{L^q}^2 \right] ds.
\]
This together with the Gronwall lemma and (4.21) leads to
\[
\mathbb{E} \left[ \| (-A_n)^{-\frac{1}{2}} E(t) \|_{L^p}^p \right] \leq Cn^{-p}
\]
for any \( t \in [0, T] \), and consequently, we obtain (4.10).

Step 2: We prove (4.9). Taking \( p = 2 \) in (4.12) and using (4.18), we deduce
\[
\| (-A_n)^{-\frac{1}{2}} E(t) \|^2 + \int_0^t \| (-A_n)^{\frac{1}{2}} E(s) \|^2 ds \leq \int_0^t 2J_1(s) + J_2(s) ds + nM(t)
\]
\[
\leq 2\epsilon_1 \int_0^t \| (-A_n)^{\frac{1}{2}} E(s) \|^2 ds + \int_0^t C\epsilon_1 \| (-A_n)^{-\frac{1}{2}} E(s) \|^2 + nK(s) ds + nM(t)
\]
for $\epsilon_1 \in (0, 1)$. Here $M(t) := \frac{2}{\sqrt{n}} \int_0^t \langle (-A_n)^{-1} E(s), \{\Sigma_n(U(s)) - \Sigma_n(U(s)) \} \rangle \, d\beta_n$ is a martingale since (4.10) and the boundedness of $\sigma$ imply $\mathbb{E}[|M(t)|^2] < \infty$. Moreover, by the Itô isometry, (4.13), (4.16), and the Young inequality,

$$\mathbb{E}[|M(t)|^2] = \frac{4}{\pi n} \int_0^t \mathbb{E}[|\{\Sigma_n(U(s)) - \Sigma_n(U(s))\}(-A_n)^{-1} E(s)|^2] \, ds$$

$$\leq C \int_0^t \mathbb{E}[J_3(s)] \, ds \leq C \int_0^t \mathbb{E}[|(-A_n)^{-\frac{3}{2}} E(s)|^2] \, ds$$

$$\leq C \int_0^t \mathbb{E}[|U(s) - \bar{U}(s)|^4] \, ds + C n^{-2} \int_0^t \mathbb{E}[|(-A_n)^{-\frac{3}{2}} E(s)|^4] \, ds$$

$$+ C \int_0^t \mathbb{E}[|(-A_n)^{-\frac{3}{2}} E(s)|^2] \, ds.$$

Then (4.22) with $\epsilon_1 = \frac{1}{2}$ and the Young inequality give

$$\mathbb{E} \left[ \left| (-A_n)^{-\frac{3}{2}} E(t) \right|^4 \right] + \mathbb{E} \left[ \int_0^t \left| (-A_n)^{-\frac{3}{2}} E(s) \right|^2 \, ds \right]^2$$

$$\leq C \mathbb{E} \left[ \int_0^t \left| (-A_n)^{-\frac{3}{2}} E(s) \right|^2 \, ds \right]^2 + \mathbb{E}[|M(t)|^2] + C \mathbb{E} \left[ \int_0^t K(s) \, ds \right]^2$$

$$\leq C \int_0^t \mathbb{E}[|(-A_n)^{-\frac{3}{2}} E(s)|^4] \, ds + C \int_0^t \mathbb{E}[|U(s) - \bar{U}(s)|^4] \, ds$$

$$+ C \int_0^t \mathbb{E}[|(-A_n)^{-\frac{3}{2}} E(s)|^2] \, ds + C \int_0^t \mathbb{E}[K(s)^2] \, ds$$

$$\leq C \int_0^t \mathbb{E}[|(-A_n)^{-\frac{3}{2}} E(s)|^4] \, ds + C n^{-4},$$

where we have used (4.10) with $p = 4$, (4.19), and (4.21) in the last step. Thus, (4.9) follows immediately from the Gronwall lemma. \qed

Similar to Proposition 3.3, we have the following regularity estimate of $\bar{U}(t)$.

**Lemma 4.5.** Let $u_0 \in C^1(\mathcal{O})$ and $p \geq 1$. Then

$$\mathbb{E}[\left| (-A_n)^{\frac{3}{2}} \bar{U}(t) \right|_p^p] \leq C(T, p) \quad \forall \, t \in [0, T].$$

**Proof.** For $t \in [0, T]$, denote $\bar{O}(t) := \sqrt{n/\pi} \int_0^t \exp(-A_n^2(t - s)) \Sigma_n(U(s)) \, d\beta_s$ and $\bar{V}(t) := \bar{U}(t) - \bar{O}(t)$. Then $\bar{V}$ satisfies

$$\frac{d}{dt} \bar{V}(t) = -A_n^2 \bar{V}(t) + A_n F_n(U(t)), \quad t \in (0, T],$$

and $\bar{V}(0) = U(0)$. Analogously to (3.22), we have that for $t \in [0, T]$,

$$\left| (-A_n)^{\frac{3}{2}} \bar{V}(t) \right|_p^p \leq C + C \int_0^t (t - s)^{-\frac{3}{2}} \left| F_n(U(s)) \right|_p^p \, ds$$

$$\leq C + C \int_0^t (t - s)^{-\frac{3}{2}} \left| 1 + \left| U(s) \right|_p^3 \right|_p^p \, ds,$$

which along with the Minkowski inequality and (4.20) yields that for $p \geq 2$,

$$\mathbb{E}[\left| (-A_n)^{\frac{3}{2}} \bar{V}(t) \right|_p^p] \leq C \quad \forall \, t \in [0, T].$$
Since $\sigma$ is bounded, a similar argument of (3.4) yields $\mathbb{E}[\|(-A_n)^{\frac{1}{2}}\hat{O}(t)\|^2_{\ell^2_n}] \leq C(T,p)$ for all $t \in [0,T]$. Combining the above estimates finally finishes the proof. \hfill \Box

Further, we present the local Lipschitz continuity of $F_n$ under the $\|(-A_n)^{\frac{1}{2}} \cdot \|_2^2$-norm, which is crucial for the strong convergence analysis in Theorem 4.7.

**Lemma 4.6.** For any $a, b \in \mathbb{R}^{n-1}$,

$$\|(-A_n)^{\frac{1}{2}} (F_n(a) - F_n(b))\|^2_{\ell^2_n} \leq C (1 + \|(-A_n)^{\frac{1}{2}} a\|^2_{\ell^2_n} + \|(-A_n)^{\frac{1}{2}} b\|^2_{\ell^2_n}) \|(-A_n)^{\frac{1}{2}} (a - b)\|^2_{\ell^2_n}.$$ 

**Proof.** Let $a = (a_1, \ldots, a_{n-1})$, $b = (b_1, \ldots, b_{n-1})$, and $a_0 = b_0 = a_n = b_n = 0$. Then by (3.4) and $f(x) = x^3 - x$,

$$\|(-A_n)^{\frac{1}{2}} (F_n(a) - F_n(b))\|^2_{\ell^2_n} = \frac{n}{\pi} \sum_{j=1}^{n} |f(a_j) - f(b_j) - f(a_{j-1}) + f(b_{j-1})|^2 \leq \frac{2n}{\pi} \sum_{j=1}^{n} |a_j^3 - b_j^3 - a_{j-1}^3 + b_{j-1}^3|^2 + 2\|(-A_n)^{\frac{1}{2}} (a - b)\|^2_{\ell^2_n}.$$

By noticing that

$$\|a_j^3 + b_j^3 + \sigma_j b_j - a_{j-1}^3 - b_{j-1}^3 + a_{j-1}b_{j-1}\|^2 \leq 3|a_j^3 - a_{j-1}^3|^2 + |b_j^3 - b_{j-1}^3|^2 + |\sigma_j b_j - a_{j-1}b_{j-1}|^2 \leq 12(|a_j - a_{j-1}|^2 + |b_j - b_{j-1}|^2 + \|b\|_{T_n}^2 |a_j - a_{j-1}|^2) + 6(\|b\|_{T_n}^2 + \|a\|_{T_n}^2 |b_j - b_{j-1}|^2) \leq C (\|a_j - a_{j-1}|^2 + |b_j - b_{j-1}|^2)(\|a\|_{T_n}^2 + \|b\|_{T_n}^2),$$

we obtain from (3.4) that

$$\frac{n}{\pi} \sum_{j=1}^{n} |a_j^3 - b_j^3 - a_{j-1}^3 + b_{j-1}^3|^2 \leq \frac{2n}{\pi} \sum_{j=1}^{n} |a_j^3 + b_j^3 + \sigma_j b_j - a_{j-1}^3 - b_{j-1}^3 + a_{j-1}b_{j-1}|^2 (a_j - b_j)^2 + \frac{2n}{\pi} \sum_{j=1}^{n} |a_{j-1}^3 + b_{j-1}^3 + \sigma_{j-1} b_{j-1} - a_j b_{j-1}|^2 (a_j - a_{j-1} + b_{j-1})^2 \leq C\|a - b\|_{T_n}^2 \sum_{j=1}^{n} (\|a_j - a_{j-1}|^2 + |b_j - b_{j-1}|^2)(\|a\|_{T_n}^2 + \|b\|_{T_n}^2) + C \max_{1 \leq j \leq n} (a_j^3 + b_j^3 - 1)\|(-A_n)^{\frac{1}{2}} (a - b)\|^2_{\ell^2_n} \leq C\left(\|(-A_n)^{\frac{1}{2}} a\|^2_{\ell^2_n} + \|(-A_n)^{\frac{1}{2}} b\|^2_{\ell^2_n}\right) (\|a\|_{T_n}^2 + \|b\|_{T_n}^2)\|a - b\|_{T_n}^2 + C(\|a\|_{T_n}^2 + \|b\|_{T_n}^2)\|(-A_n)^{\frac{1}{2}} (a - b)\|^2_{\ell^2_n}.$$ 

Finally, a combination of (3.1) and (4.23) completes the proof. \hfill \Box

Now we are ready to present the main result of this section on the strong convergence rate of the numerical solution associated with the spatial FDM.
Theorem 4.7. Let \( u_0 \in C^3(O) \) and \( \zeta \in [1, 2) \). Then there exists some constant \( C = C(\zeta, T) \) such that for any \( (t, x) \in [0, T] \times O \),
\[
\mathbb{E} \left[ |u(t, x) - u^n(t, x)|^\zeta \right] \leq Cn^{-\zeta}.
\]

Proof. Due to Proposition 4.22, it remains to show that for any \( (t, x) \in [0, T] \times O \),
\[
\mathbb{E} \left[ |\bar{u}^n(t, x) - u^n(t, x)|^\zeta \right] \leq Cn^{-\zeta}.
\] (4.24)

Notice that \( \sup_{x \in O} \|\bar{u}^n(t, x) - u^n(t, x)\|_{L^\zeta(O)} \leq \|\bar{U}(t) - U(t)\|_{L^\zeta(\Omega, L^\infty)} = \|E(t)\|_{L^\zeta(\Omega, L^\infty)}. \) In view of (3.1), a sufficient condition for (4.24) is
\[
\|(-A_n)^{1/2}E(t)\|_{L^\zeta(\Omega, L^2_{\mathbb{C}})} \leq Cn^{-1} \quad \forall \ t \in [0, T].
\] (4.25)

Fix \( t \in (0, T] \). For \( \mu \in [0, \frac{1}{2}] \) and \( s \in [0, t] \), we denote
\[
K_\mu(s) := e^{-A_n^2(t-s)}(-A_n)^{1+\mu} \{F_n(U(s)) - F_n(U(s))\},
\]
\[
L_\mu(s) := \sqrt{n}(-A_n)^{\mu}e^{-A_n^2(t-s)} \{\Sigma_n(U(s)) - \Sigma_n(U(s))\}.
\]

Then the variation of constants formula applied to (4.11) and \( E(0) = 0 \) produce
\[
(-A_n)^{\mu}E(t) = -\int_0^t K_\mu(s)ds + \frac{1}{\sqrt{\pi}} \int_0^t L_\mu(s)d\beta_s.
\]

By the Burkholder inequality, it holds that for any \( p \geq 1/2 \),
\[
\mathbb{E} \left[ \left\| (-A_n)^{\mu}E(t) \right\|^{2p} \right] \leq CNE \left[ \left( \int_0^t \|K_\mu(s)\|ds \right)^{2p} \right] + CNE \left[ \left( \int_0^t \|L_\mu(s)\|^2ds \right)^p \right].
\] (4.26)

Similarly to (3.8), we have
\[
\|L_\mu(s)\|^2_F = n \sum_{l=1}^{n-1} (-\lambda_{l,n})^{2\mu}e^{-2\lambda_t^l(t-s)}\|\Sigma_n(U(s)) - \Sigma_n(U(s))\|e_i^2.
\]

Further, taking (4.15) and (3.7) with \( \alpha_1 = \frac{1}{2} + \mu + \epsilon \) into account, we arrive at
\[
\|L_\mu(s)\|^2_F \leq C \sum_{l=1}^{n-1} (-\lambda_{l,n})^{2\mu}e^{-2\lambda_t^l(t-s)}\|U(s) - U(s)\|^2
\] (4.27)
\[
\leq C(t-s)^{-\frac{1}{2} - \mu - \epsilon}\|U(s) - U(s)\|^2
\]
\[
\leq C(t-s)^{-\frac{1}{2} - \mu - \epsilon}\|U(s) - \bar{U}(s)\|^2 + C(t-s)^{-\frac{1}{2} - \mu - \epsilon}\|E(s)\|^2,
\]
where \( 0 < \epsilon \ll 1 \). The remainder of the proof of (4.25) is separated into two steps.

Step 1: In this step, we take \( \mu = 0 \) and estimate \( \|E(t)\|_{l_2^n} \).

By (3.18), (4.17) with \( q = 1 \), and Lemma 4.6,
\[
\|K_0(s)\|_{l_2^n} \leq \|e^{-A_n^2(t-s)}(-A_n)\{F_n(\bar{U}(s)) - F_n(U(s))\}\|_{l_2^n}
\]
\[
+ \|e^{-A_n^2(t-s)}(-A_n)\{F_n(U(s)) - F_n(\bar{U}(s))\}\|_{l_2^n}
\]
\[
\leq C(t-s)^{-\frac{3}{2}}(1 + \|(-A_n)^{1/2}\|_{l_2^n} + \|(-A_n)^{1/2}U(s)\|_{l_2^n})\|(-A_n)^{1/2}E(s)\|_{l_2^n}
\]
\[
+ C(t-s)^{-\frac{3}{2}}(1 + \|U(s)\|_{l_2^n} + \|U(s)\|_{l_2^n})\|U(s) - \bar{U}(s)\|_{l_2^n}.
\]
Plugging this inequality and (4.27) (with $\mu = 0$) into (4.26) (with $p = 1$), we obtain
\[
\mathbb{E} \left[ \| E(t) \|_{l_2^n}^2 \right] \leq C \mathbb{E} \left[ \left( \int_0^t \| K_0(s) \|_{l_2^n}^2 \, ds \right)^2 \right] + \frac{C}{n} \int_0^t \mathbb{E} \left[ \| L_0(s) \|_{l_2^n}^2 \right] \, ds
\]
\[
\leq C \mathbb{E} \left[ \left( \int_0^t (t-s)^{-\frac{1}{4}} \left( 1 + \| (A_n)^{\frac{1}{2}} U(s) \|_{l_2^n}^2 + \| (A_n)^{\frac{1}{2}} U(s) \|_{l_2^n}^2 \right) \right)^2 ds \right]
\]
\[
+ C \mathbb{E} \left[ \left( \int_0^t (t-s)^{-\frac{1}{4}} \left( 1 + \| U(s) \|_{l_2^n}^2 + \| U(s) \|_{l_2^n}^2 \right) \right)^2 ds \right]
\]
\[
+ C \varepsilon \int_0^t (t-s)^{-\frac{1}{4}-\varepsilon} \mathbb{E} \left[ \| \mathbb{U}(s) - \mathbb{U}(s) \|_{l_2^n}^2 \right] ds + C \varepsilon \int_0^t (t-s)^{-\frac{1}{4}-\varepsilon} \mathbb{E} \left[ \| E(s) \|_{l_2^n}^2 \right] ds
\]
\[=: C \text{Err}_1 + C \text{Err}_2 + C \varepsilon \text{Err}_3 + C \varepsilon \int_0^t (t-s)^{-\frac{1}{4}-\varepsilon} \mathbb{E} \left[ \| E(s) \|_{l_2^n}^2 \right] ds.
\]
We proceed to estimate $\text{Err}_1$, $\text{Err}_2$ and $\text{Err}_3$. By the Hölder inequality, Proposition 4.4, Proposition 5.5 and Lemma 4.5, we obtain
\[
\text{Err}_1 \leq Cn^{-2} \int_0^t (t-s)^{-\frac{1}{4}} \left( 1 + \| (A_n)^{\frac{1}{2}} U(s) \|_{L^4(\Omega; l_2^n)}^4 + \| (A_n)^{\frac{1}{2}} U(s) \|_{L^4(\Omega; l_2^n)}^4 \right) ds
\]
\[
\leq Cn^{-2}.
\]
Besides, the Minkowski inequality and (3.1) imply that $\sqrt{\text{Err}_2}$ is bounded by
\[
C \int_0^t (t-s)^{-\frac{1}{4}} \left( 1 + \| U(s) \|_{L^4(\Omega; l_2^n)}^2 + \| (A_n)^{\frac{1}{2}} U(s) \|_{L^4(\Omega; l_2^n)}^2 \right) \| U(s) - U(s) \|_{L^4(\Omega; l_2^n)} ds,
\]
which together with (4.20), Lemma 4.5 and (4.19) leads to
\[
\sqrt{\text{Err}_2} \leq C \int_0^t (t-s)^{-\frac{1}{4}-n^{-1}} ds \leq Cn^{-1}.
\]
Similarly, (4.19) also gives $\sqrt{\text{Err}_3} \leq Cn^{-1}$. Gathering the above estimates yields
\[
\mathbb{E} \left[ \| E(t) \|_{l_2^n}^2 \right] \leq Cn^{-2} + C \varepsilon \int_0^t (t-s)^{-\frac{1}{4}-\varepsilon} \mathbb{E} \left[ \| E(s) \|_{l_2^n}^2 \right] ds.
\]
Then the Gronwall lemma with weak singularities (see e.g., [24, Lemma 3.4]) implies $\mathbb{E}[\| E(t) \|_{l_2^n}^2] \leq Cn^{-2}$, which along with (4.19) with $\theta = q = 2$ gives
\[
\mathbb{E} \left[ \| \mathbb{U}(t) - U(t) \|_{l_2^n}^2 \right] \leq Cn^{-2}. \tag{4.28}
\]

Step 2: In this step, we take $\mu = \frac{1}{2}$ and estimate $\| (A_n)^{\frac{1}{2}} E(t) \|_{l_2^n}$.

By (3.18) and a similar argument of (4.17) with $q = 1$,
\[
\| K_{\frac{1}{2}}(s) \|_{l_2^n} \leq C(t-s)^{-\frac{1}{4}} \| F_n(\mathbb{U}(s)) - F_n(\mathbb{U}(s)) \|_{l_2^n}
\]
\[
\leq C(t-s)^{-\frac{1}{4}} \left( 1 + \| \mathbb{U}(s) \|_{l_2^n}^2 + \| U(s) \|_{l_2^n}^2 \right) \| \mathbb{U}(s) - U(s) \|_{l_2^n}.
\]

Hence, the Minkowski and Hölder inequalities, (2.7), (4.28), (3.1) and Proposition 3.3 give that for $q \in [1, 2)$ and $r = \frac{2q}{2-q}$,
\[
\left\| \int_0^t \| K_{\frac{1}{2}}(s) \|_{l_2^n} ds \right\|_{L^q(\Omega)}
\]
Combining the above two inequalities and (4.26) with \( p \), which proves (4.25). The proof is completed.

Lemma 5.1. By denoting \( \tilde{\eta} \) similar to (3.34), we also have

\[ \tilde{E}(\eta) \leq \tilde{E}(\eta) \]

satisfies \( \tilde{E}(\eta) \) which along with Proposition 3.3 yields

\[ \tilde{E}(\eta) \]

for \( i \) so that \( \tilde{\eta} \in E \). As in (2.20), let

\[ \tilde{\eta} = \tilde{\eta}(z) = \int_{\Omega} G_{\eta, \tau}(x, y) u_0(\eta(y))dy \]

so that \( \tilde{\eta}(z, t) = \tilde{u}(t) \) is the \( k \)th component of \( \tilde{u}(t) \) and \( \tilde{\eta}(t, x) = \Pi_n(\tilde{u}(t, \cdot))(x) \).

Lemma 5.1. Let \( u_0 \in C^1(\Omega) \) and \( p \geq 1 \). Then for any \( i \in \mathbb{Z}_m^0 \),

\[ \mathbb{E}\left[ \left\| (A_n)^{1/2} \tilde{\eta} \right\|_{l_2}^p \right] \leq C(\tau, p). \]

Proof. By denoting \( \tilde{\eta} := \sqrt{T_n^i} \int_{\Omega} (I + \tau A_n^2)^{-1/2}(\tau^{i+1}) \Sigma_n(\tilde{U}(\eta))d\beta \), we have that \( \tilde{\eta} \) satisfies \( \tilde{\eta} = \tilde{\eta} - \tilde{\eta} \)

\( \tilde{\eta} = \tilde{\eta} + \tau A_n^2 \tilde{\eta} \)

for \( i \in \mathbb{Z}_m^0 \) and \( \tilde{\eta} = \tilde{\eta} \). Then by repeating the derivation of (3.27), one can prove \( \mathbb{E}[\sup_{i \in \mathbb{Z}_m^0} \| (A_n)^{1/2} \tilde{\eta} \|_{l_2}^p] \leq C \). Moreover, similar to (3.34), we also have

\[ \left\| (A_n)^{1/2} \tilde{\eta} \right\|_{l_2}^p \leq C \tau \sum_{j=0}^{i-1} (t_i - t_j)^{-1/2}(1 + \|U(t_j+1)\|_{l_2}^3), \]

which along with Proposition 5.1 yields \( \mathbb{E}[\| (A_n)^{1/2} \tilde{\eta} \|_{l_2}^p] \leq C \) for all \( i \in \mathbb{Z}_m \). \( \square \)
5.1. Error estimate between $\tilde{u}^{n,\tau}$ and $u^n$. This part estimates the error between the auxiliary process $\tilde{u}^{n,\tau}$ and the spatial semi-discrete numerical solution $u^n$. We begin with the following error analysis between the fully discrete Green function $G_{s+\tau}^{n}(x,y)$ and the semi-discrete Green function $G_s^n(x,y)$.

**Lemma 5.2.** For any $0 < \epsilon \ll 1$, there exists $C = C(T, \epsilon)$ such that for any $x \in \mathcal{O}$,

$$
\int_0^{t_i} \int_{\mathcal{O}} \left| G_{s+\tau}^{n}(x,y) - G^{n}(x,y) \right|^2 \, dy \, ds \leq C \tau^{\frac{2}{3} - \epsilon}, \quad i \in \mathbb{Z}_m,
$$

$$
\int_0^{t_i} \int_{\mathcal{O}} \left| \Delta_n G_{s+\tau}^{n}(x,y) - \Delta_n G^{n}(x,y) \right| \, dy \, ds \leq C \tau^{\frac{2}{3} - \epsilon}, \quad i \in \mathbb{Z}_m.
$$

**Proof.** By the Hölder inequality, it suffices to show that for $\mu \in \{0, 1\}$,

$$
\int_0^T \left( \int_{\mathcal{O}} |(-\Delta_n)^\mu G_{s+\tau}^{n}(x,y) - (-\Delta_n)^\mu G^{n}(x,y)|^2 \, dy \right)^{1-\frac{\mu}{2}} \, ds \leq C \tau^{1-(\frac{\mu+1}{2})(1-\frac{1}{2})-\epsilon}.
$$

Using the orthogonality of $\{\phi_j \circ \kappa_n\}_{j=1}^{n-1}$ and $|\phi_j,n(x)| \leq 1$,

$$
\int_{\mathcal{O}} |(-\Delta_n)^\mu G_{s+\tau}^{n}(x,y) - (-\Delta_n)^\mu G^{n}(x,y)|^2 \, dy
\leq \sum_{j=1}^{n-1} (-\lambda_j,n)^{2\mu} \left| \left(1 + \tau \lambda_j^2,n \right)^{\frac{\mu+1}{2}} - e^{-\lambda_j^2,n s} \right|^2 \leq J_1^n(s) + 2J_2^n(s) + 2J_3^n(s),
$$

where

$$
J_1^n(s) := \sum_{j=1}^{\lceil \frac{\tau}{\frac{1}{2}} \rceil - 1} (-\lambda_j,n)^{2\mu} \left| \left(1 + \tau \lambda_j^2,n \right)^{\frac{\mu+1}{2}} - e^{-\lambda_j^2,n s} \right|^2
$$

$$
J_2^n(s) := \sum_{j=\lceil \frac{\tau}{\frac{1}{2}} \rceil}^{n-1} (-\lambda_j,n)^{2\mu} e^{-2\lambda_j^2,n s}, \quad J_3^n(s) := \sum_{j=\lceil \frac{\tau}{\frac{1}{2}} \rceil}^{n-1} (-\lambda_j,n)^{2\mu} \left(1 + \tau \lambda_j^2,n \right)^{-2\left(1 + \frac{\mu}{4}\right)}.
$$

Since $\frac{4}{\tau^2} j^2 \leq \lambda_j,n \leq j^2$ and $j \geq \frac{j+1}{2}$ for all $j \geq 1$,

$$
J_2^n(s) \leq \sum_{j=\lceil \frac{\tau}{\frac{1}{2}} \rceil}^{n-1} j^{4\mu} e^{-\frac{32}{\tau^4} j^4} \leq \sum_{j=\lceil \frac{1}{4} \rceil}^{n-1} \int_{\frac{1}{4}}^{j+1} z^{4\mu} e^{-\frac{32}{\tau^4} z^4} \, dz \leq \int_{\frac{1}{4}}^{\infty} z^{4\mu} e^{-\frac{32}{\tau^4} z^4} \, dz \leq \int_{\frac{1}{2}}^{\frac{1}{4}} z^{4\mu} e^{-\frac{32}{\tau^4} z^4} \, dz,
$$

by supposing without loss of generality that $\frac{1}{2} \tau^{-\frac{1}{4}} \leq \tau^{-\frac{1}{4}} - 1 \leq \lceil \frac{1}{\frac{1}{4}} \rceil$. Likewise,

$$
J_3^n(s) \leq \int_{\frac{1}{2}}^{\frac{1}{4}} z^{4\mu} \left(1 + \frac{\tau}{\frac{1}{4}}\right)^{-2\left(1 + \frac{\mu}{4}\right)} \, dz. \quad (5.3)
$$

By taking the change of variables $\bar{z} = z\tau^{-\frac{1}{2}}$ and $\bar{s} = s/\tau$ in turn, we obtain

$$
\int_0^T \left| J_2^n(s) \right|^{1-\frac{\mu}{2}} \, ds \leq C \int_0^{T/\pi^4} \left| \int_{\frac{1}{4} \tau^{-\frac{1}{2}}}^{\frac{1}{2}} z^{4\mu} e^{-2\bar{z}^4} \, dz \right|^{1-\frac{\mu}{2}} \, ds
\leq C \tau^{1-(\frac{\mu+1}{2})(1-\frac{1}{2})} \int_0^{\infty} \left| \int_{\frac{1}{4} \tau^{-\frac{1}{2}}}^{\frac{1}{2}} \bar{z}^{4\mu} e^{-2\bar{z}^4} \, d\bar{z} \right|^{1-\frac{\mu}{2}} \, d\bar{s} \leq C \tau^{1-(\frac{\mu+1}{2})(1-\frac{1}{2})}.
$$
Here the last integral is finite since by (3.7), for any $\alpha_1 \in \left( \mu + \frac{1}{4}, \frac{2}{2-\mu} \right)$ and $\alpha_2 > \frac{2}{2-\mu}$,
\[
\int_0^\infty \left| \int_{\frac{1}{2}}^\infty z^{4\mu} e^{-2z^4 t} dz \right|^{1-\frac{\mu}{2}} dt 
\leq C \int_0^1 \left| \int_{\frac{1}{2}}^\infty z^{4\mu-4\alpha_1 t-\alpha_1} dz \right|^{1-\frac{\mu}{2}} dt + C \int_1^\infty \left| \int_{\frac{1}{2}}^\infty z^{4\mu-4\alpha_2 t-\alpha_2} dz \right|^{1-\frac{\mu}{2}} dt < \infty.
\]
In a similar manner, applying the change of variables $\tilde{z} = z^{\frac{1}{4}}$ to (5.3) leads to
\[
J_{\mu}^\epsilon (s) \leq C \tau^{-(\mu+\frac{1}{4})} \int_{\frac{1}{2}}^\infty \left( 1 + \pi^{-4} z^4 \right)^{-2} |\tilde{z}^2 - 2(\tilde{z}^2) - 2 \tilde{z}^{4\mu} d\tilde{z} \leq C \tau^{-(\mu+\frac{1}{4})} (1 + \frac{1}{16\pi^4})^{-2(\frac{1}{4})},
\]
which implies that
\[
\int_0^T |J_{\mu}^\epsilon (s)|^{1-\frac{\mu}{2}} ds \leq C \tau^{-(\mu+\frac{1}{4}) (1-\frac{\mu}{2})} \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} (1 + \frac{1}{16\pi^4})^{-k(2-\mu)} ds \leq C \tau^{-(\mu+\frac{1}{4}) (1-\frac{\mu}{2})}.
\]
In order to estimate $J_{\mu}^\epsilon (s)$, we notice that for any $1 \leq j \leq \lfloor \tau^{-\frac{1}{4}} \rfloor - 1$,
\[
\left| (1 + \tau^{2\lambda^2_{j,n}})^{-\frac{1-\epsilon}{2 \mu}} - e^{-\lambda^2_{j,n} s} \right|
\leq e^{-\frac{1-\epsilon}{2 \mu} \ln (1+\tau^{2\lambda^2_{j,n}})} \left| 1 - e^{-\frac{1-\epsilon}{2 \mu} (\frac{\lambda^2_{j,n}}{\ln (1+\tau^{2\lambda^2_{j,n}})} - \frac{1}{2}) \ln (1+\tau^{2\lambda^2_{j,n}})} \right| + e^{-\lambda^2_{j,n} s} \left| e^{-\frac{1-\epsilon}{2 \mu} (\frac{\lambda^2_{j,n}}{\ln (1+\tau^{2\lambda^2_{j,n}})} - \frac{1}{2}) \ln (1+\tau^{2\lambda^2_{j,n}})} - 1 \right|
\leq e^{-\frac{k}{\mu} \ln (1+\tau^{2\lambda^2_{j,n}})} \left| 1 - e^{-\frac{1-\epsilon}{2 \mu} (\frac{\lambda^2_{j,n}}{\ln (1+\tau^{2\lambda^2_{j,n}})} - \frac{1}{2}) \ln (1+\tau^{2\lambda^2_{j,n}})} \right| + e^{-\lambda^2_{j,n} s} \left| e^{-\frac{1-\epsilon}{2 \mu} (\frac{\lambda^2_{j,n}}{\ln (1+\tau^{2\lambda^2_{j,n}})} - \frac{1}{2}) \ln (1+\tau^{2\lambda^2_{j,n}})} - 1 \right|
\leq e^{-\frac{k}{\mu} \ln (1+\tau^{2\lambda^2_{j,n}})} \left| 1 - e^{-\frac{1-\epsilon}{2 \mu} (\frac{\lambda^2_{j,n}}{\ln (1+\tau^{2\lambda^2_{j,n}})} - \frac{1}{2}) \ln (1+\tau^{2\lambda^2_{j,n}})} \right| + e^{-\lambda^2_{j,n} s} \left| e^{-\frac{1-\epsilon}{2 \mu} (\frac{\lambda^2_{j,n}}{\ln (1+\tau^{2\lambda^2_{j,n}})} - \frac{1}{2}) \ln (1+\tau^{2\lambda^2_{j,n}})} - 1 \right|
\leq C e^{-k\lambda^2_{j,n}} (s + \tau) \lambda^4_{j,n} + C e^{-\lambda^2_{j,n} s} \tau \lambda^2_{j,n} \leq C e^{-k\lambda^2_{j,n}} (s + \tau) \lambda^4_{j,n} + C e^{-k\lambda^2_{j,n} s} \tau \lambda^2_{j,n},
\]
where we used (1.10) and the fact that there exist some constants $c_0 \in (0, 1)$ and $c_1 > 0$ such that $\ln (1 + z) \geq c_0 z$ and $0 \leq z - \ln (1 + z) \leq c_1 z^2$ for all $z \in [0, 1]$. Hence by virtue of (3.7), for $\alpha_3 := \frac{2}{2-\mu}(1-\epsilon)$ with $0 < \epsilon \ll 1$,
\[
(\lambda^2_{j,n})^{2\mu} e^{-2c_0 \lambda^2_{j,n}} (s^2 \lambda^4_{j,n} + 1) \tau^2 \lambda^2_{j,n}
\leq C (\lambda^2_{j,n})^{2\mu} (s^2 \lambda^4_{j,n} + 1) \tau^2 \lambda^2_{j,n} \leq C (\lambda^2_{j,n})^{2\mu} \left( s^2 \lambda^4_{j,n} + 1 + c_0 \right) \tau^2 \lambda^2_{j,n} \leq C \tau^{4\mu + 8 - \alpha_3 \tau^2 \lambda^2_{j,n}},
\]
which indicates that for any $0 < \epsilon \ll 1$,
\[
\int_0^T |J_{\mu}^\epsilon (s)|^{1-\frac{\mu}{2}} ds \leq C \left| \int_0^T \int_0^{\tau^{-\frac{1}{4}}} j^{4\mu + 8 - \alpha_3 \tau^2 \lambda^2_{j,n}} \right|^{1-\frac{\mu}{2}} \leq C \tau^{1-(\mu+\frac{1}{4})(1-\frac{\mu}{2})}.\]
Finally, collecting the above estimates finishes the proof.

**Proposition 5.3.** Let $u_0 \in C^2(\Omega)$ and $0 < \epsilon \ll 1$. Then for any $p \geq 1$, there exists some constant $C = C(p, T, \epsilon)$ such that for any $i \in \mathbb{Z}_m$ and $x \in \Omega$,
\[
\|u^n(t_i, x) - \bar{u}^{i, \tau}(t_i, x)\|_{L^p(\Omega)} \leq C \tau^{\frac{3}{p} - \epsilon}.
\]

**Proof.** By (2.18) and (5.2), $u^n(t_i, x) - \bar{u}^{i, \tau}(t_i, x) = \sum_{j=1}^6 Y_j^\tau$, where
\[
Y_1^\tau := \int_\Omega G^u_{ti} (x, y) u_0(\kappa_n(y)) dy - \int_\Omega G^u_{ti} (x, y) u_0(\kappa_n(y)) dy,
\]
\[
Y_2^\tau := \int_0^{t_i} \int_\Omega \Delta_n G^u_{ti-s} (x, y) [f(u^n(s, \kappa_n(y))) - f(u^n(\eta_r(s) + \tau, \kappa_n(y)))] dy ds,
\]
\[ Y_3^\tau := \int_0^{t_i} \int_\mathcal{O} [\Delta_n G_{t_i-s}^n(x, y) - \Delta_n G_{t_i-s+\tau}^n(x, y)] f(u^n(\eta_\tau(s) + \tau, \kappa_n(y))) ds dy, \]
\[ Y_4^\tau := \int_0^{t_i} \int_\mathcal{O} G_{t_i-s}^n(x, y) [\sigma(u^n(s, \kappa_n(y))) - \sigma(u^n(\eta_\tau(s), \kappa_n(y)))] W(ds, dy), \]
\[ Y_5^\tau := \int_0^{t_i} \int_\mathcal{O} [G_{t_i-s}^n(x, y) - G_{t_i-s+\tau}^n(x, y)] \sigma(u^n(\eta_\tau(s), \kappa_n(y))) W(ds, dy), \]
\[ Y_6^\tau := \int_0^{t_i} \int_\mathcal{O} G_{t_i-s+\tau}^n(x, y) [\sigma(u^n(\eta_\tau(s), \kappa_n(y))) - \sigma(\tilde{u}^{n, \tau}(\eta_\tau(s), \kappa_n(y)))] W(ds, dy). \]

Taking advantage of the following identity
\[(I + \tau A_n^2)^{-i} U_0 - U_0 = -\tau \sum_{k=1}^i A_n (I + \tau A_n^2)^{-k} A_n U_0 \quad \forall \ i \in \mathbb{N}, \]
and the fact \(\tilde{u}^{n, \tau}(0, x) = \tilde{u}^n(0, x)\), it can be verified that for \(i \in \mathbb{Z}_m\),
\[ \int_\mathcal{O} G_{t_i}^{n, \tau}(x, y) u_0(\kappa_n(y)) dy - \tilde{u}^n(0, x) \]
\[ = -\int_0^{t_i} \int_\mathcal{O} \Delta_n G_{t_i+s}^{n, \tau}(x, z) \Delta_n u_0(\kappa_n(z)) dz dr. \]

This together with (4.7) and \(\Delta_n u_0(\kappa_n(y)) = \Delta_n u_0(y)\) yields
\[ Y_1^\tau = \int_0^{t_i} \int_\mathcal{O} [\Delta_n G_{t_i+s}^{n, \tau}(x, z) - \Delta_n G_{t_i+s+\tau}^{n, \tau}(x, z)] \Delta_n u_0(\kappa_n(z)) dz dr. \]

According to the assumption \(u_0 \in C^2(\mathcal{O})\), Lemma 5.2 and 3.21,
\[ \|Y_1^\tau\|_{L^p(\Omega)} + \|Y_2^\tau\|_{L^p(\Omega)} + \|Y_5^\tau\|_{L^p(\Omega)} \leq C \tau^{\frac{3}{2} - \epsilon}. \]

By the expression of \(G_t^n\) (resp. \(G_{t_i+s}^{n, \tau}\)) and (3.17) (resp. (3.20)), for any \(0 < \epsilon \ll 1, \epsilon\)
\[ |G_t^n(x, y)| \leq C_t e^{-\frac{4}{3} \epsilon}, \quad |\Delta_n G_t^n(x, y)| \leq C t^{-\frac{4}{3} \epsilon} \quad \forall \ t \in (0, T], \ x, y \in \mathcal{O}, \]
\[ |G_{t_i+s}^{n, \tau}(x, y)| \leq C_{t_i} e^{-\frac{4}{3} \epsilon}, \quad |\Delta_n G_{t_i+s}^{n, \tau}(x, y)| \leq C_{t_i} t_i^{-\frac{4}{3} \epsilon} \quad \forall \ i \in \mathbb{Z}_m, \ x, y \in \mathcal{O}. \]

Hence using Lemma 4.3 and 3.21 produces \(\|Y_2^\tau\|^2_{L^p(\Omega)} + \|Y_5^\tau\|_{L^p(\Omega)} \leq C \tau^{\frac{3}{2} - \epsilon}\) and
\[ \|Y_6^\tau\|_{L^p(\Omega)} \leq \int_0^{t_i} (t_i - s + \tau)^{-\frac{4}{3} - 2\epsilon} |u^n(\eta_\tau(s), \kappa_n(y)) - \tilde{u}^{n, \tau}(\eta_\tau(s), \kappa_n(y))| \leq \int_0^{t_i} (t_i - s + \tau)^{-\frac{4}{3} - 2\epsilon} |u^n(\eta_\tau(s), \kappa_n(y)) - \tilde{u}^{n, \tau}(\eta_\tau(s), \kappa_n(y))| ds. \]

Gathering the above estimates on \(\{Y_i^\tau\}_{i=1}^6\) and using the singular Gronwall inequality (see e.g., [24] Lemma 3.4) finally complete the proof. \(\square\)

We close this part by giving the Hölder regularity of the fully discrete FDM.

Lemma 5.4. Let \(u_0 \in C^2(\mathcal{O})\). Then for any \(\alpha \in (0, 1)\), there exists some constant \(C = C(\alpha, T, p)\) such that for any \(0 \leq s < t \leq T\) and \(x, y \in \mathcal{O}, \)
\[ \|u^{n, \tau}(t, x) - u^{n, \tau}(s, y)\|_{L^p(\Omega)} \leq C(\tau - s)^\frac{\alpha}{2} + |x - y|^{\alpha}. \]

Proof. We set \(\mu \in \{0, 1\}, \alpha \in (0, 1 - 2\epsilon)\) and \(0 < \epsilon \ll 1\) throughout this proof. Since \(|\phi_{k,n}(x) - \phi_{k,n}(y)| \leq C \min\{k|x - y|, 1\} \leq C(-\lambda_{k,n})\frac{\alpha}{2} |x - y|^{\alpha}\), we have
\[ |(-\Delta_n)\mu G_{t_i-s+\tau}(x, z) - (-\Delta_n)\mu G_{t_i-s+\tau}(y, z)| \]
\[
\leq C \sum_{k=1}^{n-1} (-\lambda_{k,n})^\mu (1 + \tau \lambda_{k,n}^2)^{-\frac{t_i-r+\tau}{\tau}} (-\lambda_{k,n})^\alpha |x-y|^\alpha
\]
\[
\leq C \left( \frac{t_i-r+\tau}{\tau} \right)^{-\frac{1}{2}(\mu+\frac{1}{2}+\epsilon)} |x-y|^\alpha \leq C(t_i-r)^{-\frac{1}{2}(\mu+\frac{1}{2}+\epsilon)} |x-y|^\alpha,
\]
thanks to (3.26) with \(\gamma = \mu + \frac{\alpha}{2} + \frac{1}{2} + \epsilon\). Hence
\[
\int_{t_i}^{t_i} \int_{\mathcal{O}} |(-\Delta_n)^\mu G_{t_i-r+\tau}(x,z) - (-\Delta_n)^\mu G_{t_i-r+\tau}(y,z)|^{2-\mu} dz dt \leq C |x-y|^{(2-\mu)}. 
\]
By (1.6) and \(\ln(1+x) \leq x\) for \(x > 0\), one obtains that for any \(\alpha \in (0,1]\),
\[
1 - (1 + \tau \lambda_{k,n}^2)^{-(i-i)} = 1 - e^{-(i-i)\ln(1+\tau \lambda_{k,n}^2)} \leq (t_j-t_i)^{\alpha_1} \lambda_{k,n}^{\alpha_1},
\]
which together with (3.26) indicates that for any \(\alpha \in (0,1-2\epsilon)\) and \(0 < \epsilon \ll 1\),
\[
|(-\Delta_n)^\mu G_{t_i-r+\tau}(x,z) - (-\Delta_n)^\mu G_{t_i-r+\tau}(x,z)|
\]
\[
\leq C \sum_{k=1}^{n-1} (-\lambda_{k,n})^\mu (1 + \tau \lambda_{k,n}^2)^{-\frac{t_i-r+\tau}{\tau}} (t_j-t_i)^{\alpha_1} \lambda_{k,n}^{\alpha_1}
\]
\[
\leq C(t_j-r)^{-\frac{1}{2}(\mu+\frac{1}{2}+\epsilon)} (t_j-t_i)^{\alpha_1}
\]
for \(0 \leq i < j \leq m\). On this basis, we arrive at
\[
\int_{t_i}^{t_j} \int_{\mathcal{O}} |(-\Delta_n)^\mu G_{t_j-r+\tau}(x,z) - (-\Delta_n)^\mu G_{t_j-r+\tau}(x,z)|^{2-\mu} dz dt \leq C |t_j-t_i|^{\frac{1}{2}(2-\mu)}.
\]
Again by using (3.26) with \(\gamma = \mu + \frac{1}{2} + \epsilon\),
\[
|(-\Delta_n)^\mu G_{t_j-r+\tau}(x,z)| \leq C \sum_{k=1}^{n-1} (-\lambda_{k,n})^\mu (1 + \tau \lambda_{k,n}^2)^{-\frac{t_j-r+\tau}{\tau}} \leq C(t_j-r)^{-\frac{1}{2}(\mu+\frac{1}{2}+\epsilon)}.
\]
Consequently, it holds that for any \(0 \leq i < j \leq m\),
\[
\int_{t_i}^{t_j} \int_{\mathcal{O}} |(-\Delta_n)^\mu G_{t_j-r+\tau}(x,z)|^{2-\mu} dz dt \leq C |t_j-t_i|^{\frac{1}{2}(2-\mu)},
\]
\[1 - \frac{\gamma}{2}(2-\mu) \geq \frac{\alpha}{2}(2-\mu) \text{ for } \alpha \in (0,1-2\epsilon) \text{ and } \mu \in \{0,1\}.\]
By means of (5.4) and \(u_0 \in C^2(\mathcal{O})\), for any \(0 \leq i < j \leq m\),
\[
\left| \int_{\mathcal{O}} G_{t_i}^{\eta,\tau}(x,z) u_0(\kappa(z)) dz - \int_{\mathcal{O}} G_{t_j}^{\eta,\tau}(y,z) u_0(\kappa(z)) dz \right|
\]
\[
\leq |\tilde{u}^n(0,x) - \tilde{u}^n(0,y)| + \int_{t_i}^{t_j} \int_{\mathcal{O}} |\Delta_n G_{t_i}^{\eta,\tau}(x,z) - \Delta_n G_{t_j}^{\eta,\tau}(y,z)||\Delta_n u_0(\kappa(z))| dz dt 
\]
\[
+ \int_{t_i}^{t_j} \int_{\mathcal{O}} |\Delta_n G_{t_j}^{\eta,\tau}(x,z)||\Delta_n u_0(\kappa(z))| dz dt \leq C(|t_j-t_i|^\frac{\alpha}{2} + |x-y|^\alpha).
\]
In addition, by virtue of (2.3) and (2.4), it follows from (2.20) and (3.35) that (5.7) holds for all \(s = t_i\) and \(t = t_j\) with \(0 \leq i < j \leq m\). Thanks to the triangle inequality, it suffices to prove (5.7) for the following two cases.

**Case 1:** \(t = s\) and \(x \neq y\). By the definition of \(u^{n,\tau}(t,x)\),
\[
||u^{n,\tau}(t,x) - u^{n,\tau}(t,y)||_{L^p(\Omega)} \leq \frac{\eta(t) + \tau - t}{\tau} ||u^{n,\tau}(\eta(t),x) - u^{n,\tau}(\eta(t),y)||_{L^p(\Omega)}
\]
Case 2: $x = y$ and $t \geq s$. If $\eta_r(t) = \eta_r(s)$, then
\[
\|u^{n,\tau}(t, x) - u^{n,\tau}(s, x)\|_{L^p(\Omega)} 
\leq C\tau^{-1}(t-s)\|u^{n,\tau}(\eta_r(t) + \tau, x) - u^{n,\tau}(\eta_r(t), x)\|_{L^p(\Omega)} \leq C(t-s)^{\frac{2}{p}}.
\]
If $\eta_r(t) \geq \eta_r(s) + \tau$, then based on \eqref{5.8},
\[
\|u^{n,\tau}(t, x) - u^{n,\tau}(s, x)\|_{L^p(\Omega)} 
\leq \|u^{n,\tau}(t, x) - u^{n,\tau}(\eta_r(t), x)\|_{L^p(\Omega)} + \|u^{n,\tau}(\eta_r(t), x) - u^{n,\tau}(\eta_r(s) + \tau, x)\|_{L^p(\Omega)} 
+ \|u^{n,\tau}(\eta_r(s) + \tau, x) - u^{n,\tau}(s, x)\|_{L^p(\Omega)} 
\leq C(t - \eta_r(t))^{\frac{2}{p}} + C(\eta_r(t) - \eta_r(s) - \tau)^{\frac{2}{p}} + C(\eta_r(s) + \tau - s)^{\frac{2}{p}} \leq C(t-s)^{\frac{2}{p}}.
\]
The proof is completed. \hfill \Box

Remark 5.5. Based on the standard Picard argument, it can be verified that when the co-efficients $f$ and $\sigma$ satisfy the global Lipschitz condition, the stochastic Cahn–Hilliard equation \eqref{1.1} admits a unique mild solution $u = \{u(t, x), (t, x) \in [0, T] \times \Omega\}$ satisfying $\sup_{(t,x)\in[0,T]\times \Omega} E[|u(t, x)|^p] \leq C(p, T)$. Since the discussions in subsections 4.4 and 5.1 are mainly based on the properties of the Green function $G$ and the discrete Green functions $G^n$ and $G^{n,\tau}$, we remark that Lemmas 2.2 and 4.3 as well as Propositions 4.2 and 5.3 are valid as well when the coefficients $f$ and $\sigma$ satisfy the global Lipschitz condition.

5.2. Error estimate between $\tilde{u}^{n,\tau}$ and $u^{n,\tau}$. This part presents the error estimate between the fully discrete numerical solution $u^{n,\tau}$ and the auxiliary process $\tilde{u}^{n,\tau}$. As in subsection 4.2, this will be accomplished by estimating $E^i := \tilde{U}^i - U^i$.

Proposition 5.6. Let $u_0 \in C^3(\Omega)$ and $0 < \epsilon \ll 1$. Then there exists some constant $C = C(T, \epsilon)$ such that for any $i \in \mathbb{Z}_m$,
\[
E\left[\|(-A_n)^{-\frac{1}{2}} E^i \|_{L^2_{\Omega}}^2 \right] + E\left[\tau \sum_{j=0}^{i-1} \|(-A_n)^{-\frac{1}{2}} E^{i+1} \|_{L^2_{\Omega}}^2 \right] \leq C\tau^{\frac{p}{2} - 4\epsilon}.
\]

Proof. By subtracting \eqref{1.3} from \eqref{5.1},
\[
E^{i+1} - E^i + \tau A_n^2 E^{i+1} = \tau A_n\{F_n(U(t_{i+1})) - F_n(U^{i+1})\} + \sqrt{n/\pi}\{\Sigma_n(\tilde{U}^i) - \Sigma_n(U^i)\}\{\beta_{t_{i+1}} - \beta_t\}.
\]
Then applying $\langle \cdot, (-A_n)^{-1} E^{i+1} \rangle$ on both sides of \eqref{5.9}, one has
\[
\langle E^{i+1} - E^i, (-A_n)^{-1} E^{i+1} \rangle + \tau \|(-A_n)^{-\frac{1}{2}} E^{i+1} \|^2 
= \tau \langle F_n(U^{i+1}) - F_n(\tilde{U}^{i+1}), E^{i+1} \rangle + \tau \langle F_n(U(t_{i+1})) - F_n(U^{i+1}) \rangle, E^{i+1} \rangle 
+ \sqrt{n/\pi}\{\Sigma_n(\tilde{U}^i) - \Sigma_n(U^i)\}\{\beta_{t_{i+1}} - \beta_t\} (-A_n)^{-1} (E^{i+1} - E^i) 
+ \sqrt{n/\pi}\{\Sigma_n(U^i) - \Sigma_n(U^i)\}\{\beta_{t_{i+1}} - \beta_t\} (-A_n)^{-1} E^i , \quad i \in \mathbb{Z}_{m-1}.
\]
Utilizing the identity \eqref{3.30}, the Young inequality and \eqref{2.10}, it holds that
\[
\frac{1}{2}\|(-A_n)^{-\frac{1}{2}} E^{i+1} \|^2 - \frac{1}{2}\|(-A_n)^{-\frac{1}{2}} E^i \|^2 + \tau\|(-A_n)^{-\frac{1}{2}} E^{i+1} \|^2 
\leq \frac{3}{4}\|E^{i+1} \|^2 + \frac{1}{2}\|F_n(\tilde{U}^{i+1}) - F_n(U(t_{i+1}))\|^2
\]
Consequently, one has
\[
\frac{1}{4} \| (A_n) - \frac{1}{2} E^{i+1} \|^4 + \frac{1}{4} \| (A_n) - \frac{1}{2} E^{i+1} \|^4 + \frac{3}{4} \| (A_n) - \frac{1}{2} E^{i+1} \|^2 \left\| (A_n) - \frac{1}{2} E^{i+1} \right\|^2
\]
\[
\leq C(\epsilon) \| (A_n) - \frac{1}{2} E^{i+1} \|^4 + C(\epsilon) \| (A_n) - \frac{1}{2} E^{i+1} \|^4 + C \| F_n(U^{i+1}) - F_n(U(t_{i+1})) \|^4
\]
\[
+ \| (A_n) - \frac{1}{2} E^{i+1} \|^2 \left\| (A_n) - \frac{1}{2} E^{i+1} \right\|^2
\]
\[
+ \| \epsilon n^2 \| (A_n) - \frac{1}{2} E^{i+1} \|^2 \| (A_n) - \frac{1}{2} E^{i+1} \|^2
\]
\[
= C(\epsilon) \| (A_n) - \frac{1}{2} E^{i+1} \|^4 + C \| F_n(U^{i+1}) - F_n(U(t_{i+1})) \|^4
\]
\[
+ \| (A_n) - \frac{1}{2} E^{i+1} \|^2 \left\| (A_n) - \frac{1}{2} E^{i+1} \right\|^2
\]
where 0 < \epsilon \ll 1 is to be determined. Since \( E^0 = 0 \), for any \( i \in Z_m \),
\[
\frac{1}{4} \| (A_n) - \frac{1}{2} E^{i} \|^4 + \frac{3}{4} \sum_{j=0}^{i-1} \| (A_n) - \frac{1}{2} E^{j+1} \|_2^2 \| (A_n) - \frac{1}{2} E^{j+1} \|_2^2
\]
\[
\leq C\| E \^ \sum_{j=0}^{i-1} \| (A_n) - \frac{1}{2} E^{j+1} \|_2^4 + C \| F_n(U^{j+1}) - F_n(U(t_{j+1})) \|^4
\]
\[
+ \| (A_n) - \frac{1}{2} E^{j+1} \|^2 \left\| (A_n) - \frac{1}{2} E^{j+1} \right\|^2
\]
\[
+ \| E \| \sum_{j=0}^{i-1} \| (A_n) - \frac{1}{2} E^{j+1} \|_2^4 \]
\[
\leq C\| E \| \sum_{j=0}^{i-1} \| (A_n) - \frac{1}{2} E^{j+1} \|_2^4 + C \| F_n(U^{j+1}) - F_n(U(t_{j+1})) \|^4
\]
\[
+ \| (A_n) - \frac{1}{2} E^{j+1} \|^2 \left\| (A_n) - \frac{1}{2} E^{j+1} \right\|^2
\]
\[
+ \| E \| \sum_{j=0}^{i-1} \| (A_n) - \frac{1}{2} E^{j+1} \|_2^4 \]
Using the similar arguments for proving (3.8) and (4.15), it can be verified that
\[
\|(-A_n)^{-\frac{1}{2}}\{\Sigma_n(\tilde{U}^j) - \Sigma_n(U^j)\}\|_{L^2_t L^2_x}^2 \leq C(\sigma)n^{-1}\|E^j\|_{L^2_t L^2_x}^2 \leq C_1(\sigma)\|E^j\|_{L^2_t L^2_x}^2. \tag{5.12}
\]
Further, thanks to the Burkholder–Davis–Gundy inequality and Hölder inequality,
\[
E\left[\|(-A_n)^{-\frac{1}{2}}\{\Sigma_n(\tilde{U}^j) - \Sigma_n(U^j)\}(\beta_{t_j+1} - \beta_t)\|^4\right] \leq C\tau^2 E\left[\|(-A_n)^{-\frac{1}{2}}\{\Sigma_n(\tilde{U}^j) - \Sigma_n(U^j)\}\|_{L^2_t L^2_x}^4\right] \leq C_2(\sigma)\tau^2 E\left[\|E^j\|_{L^2_t L^2_x}^4\right]. \tag{5.13}
\]
Similarly to (4.17) and (4.21), it follows from Proposition 5.3 that for any \( p \geq 1 \),
\[
\|F_n(\tilde{U}^{j+1}) - F_n(U(t_{j+1}))\|_{L^{2p}(\Omega; L^p_x)}^2 \leq C \left( 1 + \|\tilde{U}^{j+1}\|_{L^{2p}(\Omega; L^p_x)}^4 + \|U(t_{j+1})\|_{L^{2p}(\Omega; L^p_x)}^4 \right) \|\tilde{U}^{j+1} - U(t_{j+1})\|_{L^{2p}(\Omega; L^p_x)}^2 \leq C\tau^{-\frac{3}{2} - 2\epsilon}. \tag{5.14}
\]
As a consequence, we derive that
\[
\frac{1}{4} E\left[\|(-A_n)^{-\frac{1}{2}}E^i\|_{L^2_t L^2_x}^4\right] + \frac{3}{4} \sum_{j=0}^{i-1} E\left[\|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^2\right] \leq C_\varepsilon \tau \sum_{j=0}^{i-1} E\left[\|E^j\|_{L^2_t L^2_x}^4\right] + \varepsilon C_2(\sigma)\tau \sum_{j=0}^{i-1} E\left[\|E^j\|_{L^2_t L^2_x}^4\right] + C\tau^{\frac{3}{2} - 4\epsilon} \leq C_\varepsilon \tau \sum_{j=0}^{i-1} E\left[\|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^4\right] + \varepsilon C_2(\sigma)\tau \sum_{j=0}^{i-1} E\left[\|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^2\right] \leq C_\varepsilon \tau \sum_{j=0}^{i-1} E\left[\|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^2\right] + C\tau^{\frac{3}{2} - 4\epsilon}.
\]
Hence by choosing \( \varepsilon \) small enough so that \( C_2(\sigma)\varepsilon \leq \frac{1}{2} \), one could use the discrete Gronwall inequality to obtain
\[
E\left[\|(-A_n)^{-\frac{1}{2}}E^i\|_{L^2_t L^2_x}^4\right] + \tau \sum_{j=0}^{i-1} E\left[\|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^2\right] \leq C\tau^{\frac{3}{2} - 4\epsilon}. \tag{5.15}
\]

**Step 2:** Based on (5.10), we utilize \( E^0 = 0 \) and (5.11) with \( \beta = \frac{1}{2} \) to deduce
\[
\frac{1}{2} \|(-A_n)^{-\frac{1}{2}}E^i\|_{L^2_t L^2_x}^2 + \frac{1}{2} \tau \sum_{j=0}^{i-1} \|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^2 \leq C\tau \sum_{j=0}^{i-1} \|(-A_n)^{-\frac{1}{2}}E^{j+1}\|_{L^2_t L^2_x}^2 + \tau \sum_{j=0}^{i-1} \|F_n(\tilde{U}^{j+1}) - F_n(U(t_{j+1}))\|_{L^2_t L^2_x}^2.
\]
\[
+ \frac{n}{\pi} \sum_{j=0}^{i-1} \langle\langle \Sigma_n(\tilde{U}^j) - \Sigma_n(U^j)\rangle\rangle_{(\beta_{t_j+1} - \beta_t)}(\beta_{t_j+1} - \beta_t), (-A_n)^{-1}E^j\rangle.
\]
In order to estimate the last term, let us introduce
\[ M^i := \sqrt{\frac{\pi}{n}} \int_0^{\tau_i} \langle (-A_n)^{-1}E^{\frac{1}{2}} | \{\Sigma_n(U)^{\frac{1}{2}} - \Sigma_n(U^{\frac{1}{2}})\} \rangle d\beta_s, \quad i \in \mathbb{Z}_m. \]

Note that \( \{M^i\}_{i \in \mathbb{Z}_m} \) is a discrete martingale and satisfies
\[
\mathbb{E} \left[ |M^i|^2 \right] \leq C n^{-1} \int_0^{\tau_i} \mathbb{E} \left[ \left\| (E^{\frac{1}{2}})^\top ( -A_n )^{-1} \{ \Sigma_n(U)^{\frac{1}{2}} - \Sigma_n(U^{\frac{1}{2}}) \} \right\|^2 \right] ds = C n^{-2} \int_0^{\tau_i} \mathbb{E} \left[ \left\| ( -A_n )^{-\frac{1}{2}} E^{\frac{1}{2}} \right\|^2 \left\| E^{\frac{1}{2}} \right\|^2 \right] ds \\
\leq \frac{3}{4} \sum_{j=0}^{i-1} \mathbb{E} \left[ \left\| ( -A_n )^{\frac{1}{2}} E^j \right\|_{L_2}^2 \left\| ( -A_n )^{-\frac{1}{2}} E^j \right\|_{L_2}^2 \right] + C \tau \sum_{j=0}^{i-1} \mathbb{E} \left[ \left\| ( -A_n )^{\frac{1}{2}} E^j \right\|_{L_2}^4 \left\| ( -A_n )^{-\frac{1}{2}} E^j \right\|_{L_2}^4 \right],
\]
due to (5.12) and (5.11). Besides, taking (5.13) into account,
\[
\mathbb{E} \left[ \sum_{j=0}^{i-1} \frac{n}{\pi} \left\| ( -A_n )^{-\frac{1}{2}} \{ \Sigma_n(U^j) - \Sigma_n(U) \} ( \beta_{t_{j+1}} - \beta_{t_j} ) \right\|_{L_2}^2 \right] \\
\leq C \tau \sum_{j=0}^{i-1} \mathbb{E} \left[ \left\| E^j \right\|_{L_2}^4 \right] \leq C \tau \sum_{j=0}^{i-1} \mathbb{E} \left[ \left\| ( -A_n )^{\frac{1}{2}} E^j \right\|_{L_2}^2 \left\| ( -A_n )^{-\frac{1}{2}} E^j \right\|_{L_2}^2 \right].
\]

Taking second order moments on both sides of (5.16), then (5.14)–(5.13) yield
\[
\frac{1}{4} \mathbb{E} \left[ \left\| ( -A_n )^{-\frac{1}{2}} E^j \right\|_{L_2}^4 \right] + \frac{1}{4} \mathbb{E} \left[ \tau \sum_{j=0}^{i-1} \left\| ( -A_n )^{\frac{1}{2}} E^{j+1} \right\|_{L_2}^2 \right] \\
\leq C \tau \sum_{j=0}^{i-1} \mathbb{E} \left[ \left\| ( -A_n )^{-\frac{1}{2}} E^{j+1} \right\|_{L_2}^4 \right] + C \tau^{\frac{3}{2} - 4\epsilon}.
\]

Finally, applying the discrete Gronwall inequality leads to the desired result. \( \square \)

**Theorem 5.7.** Let \( u_0 \in C^3(O), \) \( 0 < \epsilon \ll 1, \) and \( \zeta \in [1, 2). \) Then there exists some constant \( C = C(\zeta, T, \epsilon) \) such that for any \( i \in \mathbb{Z}_m \) and \( x \in O, \)
\[
\mathbb{E} \left[ |u^{n, \tau}(t_i, x) - u^n(t_i, x)|^{\zeta} \right] \leq C \tau^{\frac{3}{2} - \epsilon + \zeta}.
\]

**Proof.** For \( i \in \mathbb{Z}_m \) and \( j = 0, 1, \ldots, i - 1, \) we denote
\[
K_{\mu}^{i,j} := (-A_n)^{1+\mu}(I + \tau A_n^2)^{(i-j)} \{ F_n(U(t_{j+1})) - F_n(U^{j+1}) \}, \\
L_{\mu}^{i,j} := \frac{\sqrt{n/\pi}}{\mu}(-A_n)^{\mu}(I + \tau A_n^2)^{(i-j)} \{ \Sigma_n(U_j) - \Sigma_n(U) \}.
\]

It then follows from (5.9) and \( E^0 = 0 \) that for any \( \mu \in [0, 1], \)
\[ (-A_n)^{\mu} E^i = -\sum_{j=0}^{i-1} \tau K_{\mu}^{i,j} + \sum_{j=0}^{i-1} L_{\mu}^{i,j} (\beta_{t_{j+1}} - \beta_{t_j}). \]

Repeating the proof of (4.27), it can be shown that for any \( \mu \in [0, \frac{1}{2}] \) and \( 0 < \epsilon \ll 1, \)
\[ \| L_{\mu}^{i,j} \|_{F}^2 \leq n \sum_{l=1}^{n-1} (-\lambda_{l,n})^{2\mu}(1 + \tau \lambda_{l,n}^2)^{-2(i-j)} \left\| \{ \Sigma_n(U^{j}) - \Sigma_n(U) \} e_l \right\|^2. \]
where we used \( (3.26) \) with \( \gamma = \frac{1}{2} + 2\mu + 2\varepsilon \). Using the Burkholder inequality and gathering the above estimates give that for any \( p \geq 1/2, \)

\[
\mathbb{E}[\|(-A_n)\nu E^i\|_{l_n^2}^p] \leq C\mathbb{E}[\pi \sum_{j=0}^{i-1} \tau |K_{i,j}^{i,j}|_{l_n^2}^{2p}] + C\mathbb{E}[\pi \sum_{j=0}^{i-1} \tau t_{i-2}^{\frac{3}{2} - \mu - \varepsilon} \|E_j\|_{l_n^2}^p]. \tag{5.17}
\]

**Step 1:** In this step, we take \( \mu = 0 \) and estimate \( \|E^i\|_{l_n^2} \).

By Lemma 4.6 and (3.26), for any \( j = \ldots, i - 1, \)

\[
\|K_{0,j}^{i,j}\|_{l_n^2} \leq \|(-A_n)^\frac{3}{4} (I + \tau A_n^2)^{-i-j} (-A_n)^{-\frac{1}{2}} \{F_n(U^{j+1}) - F_n(U^{j+1})\}\|_{l_n^2} + \|(-A_n)(I - \tau A_n^2)^{-i-j} \{F_n(U(t_{j+1})) - F_n(U^{j+1})\}\|_{l_n^2} \leq C t_{i-2}^{\frac{3}{2}} (1 + \|(-A_n)^\frac{3}{4} U^{j+1}\|_{l_n^2}^2 + \|(-A_n)^\frac{3}{4} U^{j+1}\|_{l_n^2}^2) \|(-A_n)\tau E^{j+1}\|_{l_n^2}^2 + C t_{i-2}^{\frac{3}{2}} (1 + \|U(t_{j+1})\|_{l_n^2}^{2} + \|U^{j+1}\|_{l_n^2}^{2}) \|U(t_{j+1}) - \tilde{U}^{j+1}\|_{l_n^2},
\]

where the second term on the right hand side is handled in the same way as in (4.17) with \( q = 1 \). Then one can apply Proposition 5.6 to derive that

\[
\mathbb{E}[\sum_{j=0}^{i-1} \tau |K_{0,j}^{i,j}|_{l_n^2}^2] \leq C \tau \sum_{j=0}^{i-1} \tau t_{i-2}^{\frac{3}{2}} (1 + \|(-A_n)^\frac{3}{4} U^{j+1}\|_{L^q(\Omega;l_n^2)}^4 + \|(-A_n)^\frac{3}{4} U^{j+1}\|_{L^q(\Omega;l_n^2)}^4) \|U(t_{j+1}) - \tilde{U}^{j+1}\|_{L^q(\Omega;l_n^2)}^2, \]

which can be further bounded by \( C \tau \tau^{-2\varepsilon} \), due to Proposition 5.3, (3.24), (3.35) and Lemma 5.1. Plugging this estimate into (5.17) with \( \mu = 0 \) and \( p = 1 \) yields

\[
\mathbb{E}[\|E^i\|_{l_n^2}^2] \leq C \tau \tau^{-2\varepsilon} + C \sum_{j=0}^{i-1} \tau t_{i-2}^{\frac{3}{2}} \mathbb{E}[\|E^j\|_{l_n^2}^2],
\]

which along with the discrete Gronwall inequality gives \( \mathbb{E}[\|E^i\|_{l_n^2}^2] \leq C \tau \tau^{-2\varepsilon} \). Hence taking Proposition 5.6 into account yields

\[
\mathbb{E}[\|U(t_i) - U^i\|_{l_n^2}^2] \leq C \tau \tau^{-2\varepsilon} \quad \forall i \in \mathbb{Z}_m^0. \tag{5.18}
\]

**Step 2:** In this step, we take \( \mu = \frac{1}{2} \) and estimate \( \|(-A_n)^\frac{3}{4} E^i\|_{l_n^2} \).

In view of (3.26) and a similar argument of (4.17) (with \( q = 1 \)),

\[
\|K_{0,j}^{i,j}\|_{l_n^2} \leq C t_{i-2}^{\frac{3}{2}} (1 + \|U(t_{j+1})\|_{l_n^2}^2 + \|U^{j+1}\|_{l_n^2}^2) \|U(t_{j+1}) - U^{j+1}\|_{l_n^2}.
\]

By the Hölder inequality, (3.24) and (3.35), for any \( q \in [1, 2] \),

\[
\|K_{0,j}^{i,j}\|_{L^q(\Omega;l_n^2)} \leq C \|U(t_{j+1}) - U^{j+1}\|_{L^2(\Omega;l_n^2)} \leq C \tau \tau^{-\varepsilon},
\]
Then it holds that by virtue of (C1) and (C2), for any $(C1)–(C4)$ following conditions

\[ \sigma \text{ is the Borel } \]

that of their localizations, which will be applied to prove the density convergence in $p$

Proposition 6.1.

where $\Phi$ is the set of continuous functions $X$ the total variation distance between

in view of (5.18). This in combination with (5.17) indicates that for any $(C2)$ and $\Phi$-

Proposition 5.3.

(C2) Given a discretization parameter $(C3)$ there exists

Assume that $\{d\}$ approximate of $X$

such that for any $\upsilon$

$\upsilon$

and $\upsilon$

in $\upsilon$

\[ E \left[ \left| \left( \sum_{t=0}^{\infty} \tau \left| K_{t+1}^j \right| \right) \right| ^2 \right] \leq C_T \left( \right) \]

In this way, we obtain the required result, thanks to (3.1) and Proposition 5.3. □

6. CONVERGENCE OF DENSITY

Given $d \in N_+$, for two $\mathbb{R}^d$-valued random variables $X, Y$, we write $d_{TV}(X, Y)$ to indicate the total variation distance between $X$ and $Y$, i.e.,

\[ d_{TV}(X, Y) = 2 \sup_{A \in \mathcal{B} \left( \mathbb{R}^d \right)} \left\{ \left| \mathbb{P}(X \in A) - \mathbb{P}(Y \in A) \right| \right\} = \sup_{\phi \in \Phi} \left[ E[\phi(X)] - E[\phi(Y)] \right], \]

where $\Phi$ is the set of continuous functions $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ which are bounded by 1, and $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-algebra of $\mathbb{R}^d$. Furthermore, if $\{X_n\}_{n \geq 1}$ and $X_\infty$ have the densities $\{p_{X_n}\}_{n \geq 1}$ and $p_{X_\infty}$ respectively, then

\[ d_{TV}(X_n, X_\infty) = \|p_{X_n} - p_{X_\infty}\|_{L^1(\mathbb{R}^d)}. \tag{6.1} \]

We now present a criterion for reducing the total variation distance of random variables to that of their localizations, which will be applied to prove the density convergence in $L^1(\mathbb{R})$ for the numerical discretizations.

**Proposition 6.1.** Let $\mathcal{T} := \bigcup_{i=1}^{d} [a_i, b_i]$ (a$_i$ < b$_i$) be an interval or a rectangle in $\mathbb{R}^d$ and $\mathcal{X} = \{X(t), t \in \mathcal{T}\}$ an $\mathbb{R}^d$-valued random field defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous trajectories a.s. For every $R \geq 1$, denote $\Omega_R := \{ \omega \in \Omega : \sup_{t \in \mathcal{T}} \|X(t, \omega)\| \leq R \}$. Assume that the following conditions (C1)–(C4) hold.

(C1) Assume that $\{\{(\Omega_R, X_R)\}_{R \geq 1}\}$ is a localization of $\mathcal{X}$, i.e., $X_R = X$ on $\Omega_R \subset \Omega$ for every $R \geq 1$ and $\lim_{R \to \infty} \mathbb{P}(\Omega_R) = 1$.

(C2) Given a discretization parameter $N \in N_+$, denote by $X^N$ (resp. $X^N_R$) the a numerical approximation of $X$ (resp. $X_R$). Assume that for sufficiently large $N$ and $R$, $X^N = X^N_R$ on $\Omega_{R,N} := \{ \omega \in \Omega : \sup_{t \in \mathcal{T}} \|X^N(t, \omega)\| \leq R \}$.

(C3) There exists $v_1 > 0$ and $q > 0$ such that

\[ E \left[ \left| X^N(t) - X(t) \right| ^q \right] \leq CN^{-qv_1} \quad \forall \ t \in \mathcal{T}. \]

(C4) There exists $v_2 > 0$ such that for any $p \geq 1$, there exists $C = C(p)$ independent of $N$ such that for any $t_1, t_2 \in \mathcal{T}$,

\[ E \left[ \left| X^N(t_1) - X^N(t_2) \right| ^p \right] + E \left[ \left| X(t_1) - X(t_2) \right| ^p \right] \leq C \|t_1 - t_2\| ^{p+2}. \]

Then it holds that

\[ \limsup_{N \to \infty} d_{TV}(X^N(t), X(t)) \leq \limsup_{R \to \infty} \limsup_{N \to \infty} d_{TV}(X^N_R(t), X_R(t)). \tag{6.2} \]

**Proof.** Step 1. By virtue of (C1) and (C2), for any $\phi \in \Phi$ and $R \geq 1$, 

\[ \left| E[\phi(X^N_R(t))] - E[\phi(X(t))] \right| \leq \left| E[\phi(X^N(t))1_{N>0}] - E[\phi(X(t))1_{N>0}] \right| + \left| E[\phi(X^N_R(t))] - E[\phi(X_R(t))1_{N>0}] \right| + \left| E[\phi(X^N_R(t))] - E[\phi(X_R(t))1_{N>0}] \right| \]
\begin{align}
&\leq 2\mathbb{P}(\Omega^N_R) + \left| \mathbb{E}[\phi(X^N_R(t))] - \mathbb{E}[\phi(X(t))] \right| + 2\mathbb{P}(\Omega^N_{R,N}),
\end{align}

where \(I_A\) denotes the indicator function on the set \(A\). Since \(\sup_{t \in \mathbb{T}} \|X^N(t)\| \leq \sup_{t \in \mathbb{T}} \|X^N(t) - X(t)\| + \sup_{t \in \mathbb{T}} \|X(t)\|\), we have
\[
\mathbb{P}(\Omega^N_{R,N}) \leq \mathbb{P}(\Omega^N_{R-1}) + \mathbb{P}(\sup_{t \in \mathbb{T}} \|X^N(t) - X(t)\| \geq 1).
\]

Taking supremum over \(\phi \in \Phi\), we obtain
\[
d_{TV}(X^N(t), X(t)) \leq 4\mathbb{P}(\Omega^N_{R-1}) + 2\mathbb{P}(\sup_{t \in \mathbb{T}} \|X^N(t) - X(t)\| \geq 1) \tag{6.3}
\]
\[
+ d_{TV}(X^N_R(t), X_R(t))
\]

**Step 2.** Let \(\delta \in (0, qv_1/d_1)\) be arbitrarily fixed. For \(k \in \mathbb{N}\) and \(i \in \{1, \ldots, d_1\}\), denote \(s^k_i := a_i + k(b_i - a_i)N^{-\delta}\). Then for any \(t = (t_1, \ldots, t_{d_1}) \in \mathbb{T}\) and \(i \in \{1, \ldots, d_1\}\), there is a unique integer \(0 \leq k_i(t) \leq \lceil N^\delta \rceil\) such that \(t_i \in [s^k_i(t), s^{k_i+1}(t) \land b_i)\). Hence
\[
\|X^N(t) - X(t)\|^q \leq 3q\|X^N(t) - X^N(s^k_i(t), \ldots, s^{d_1}(t))\|^q
\]
\[
+ 3q\|X^N(s^k_i(t), \ldots, s^{d_1}(t)) - X(s^k_i(t), \ldots, s^{d_1}(t))\|^q
\]
\[
+ 3q\|X(t) - X(s^k_i(t), \ldots, s^{d_1}(t))\|^q
\]
for \(q > 0\). As a result,
\[
\mathbb{P}\left(\sup_{t \in \mathbb{T}} \|X^N(t) - X(t)\| \geq 1\right) \leq \sum_{i=1}^3 \mathbb{P}(R^N_i \geq 3^{-(q+1)}), \tag{6.4}
\]
where \(R^N_1 := \sum_{s \in \mathbb{T}_\delta} \|X^N(s) - X(s)\|^q\), and
\[
R^N_2 := \sup_{s \in \mathbb{T}_\delta} \sup_{t:|s-t| \leq c(T)N^{-\delta}} \|X(s) - X(t)\|^q,
\]
\[
R^N_3 := \sup_{s \in \mathbb{T}_\delta} \sup_{t:|s-t| \leq c(T)N^{-\delta}} \|X^N(s) - X^N(t)\|^q
\]
with \(c(T) := (\sum_{i=1}^{d_1} |a_i - b_i|^2)^{1/2}\) and
\[
\mathbb{T}_\delta := \{t \in \mathbb{T} \mid \text{for any } i = 1, \ldots, d_1, t_i = s^k_i \text{ for some } 0 \leq k_i \leq \lceil N^\delta \rceil\}\).

The Markov inequality, (C3) and \(qv_1 > d_1 \delta\) reveal
\[
\mathbb{P}(R^N_1 \geq 3^{-(q+1)}) \leq 3^{q+1} \sum_{s \in \mathbb{T}_\delta} \mathbb{E}\left[\|X^N(s) - X(s)\|^q\right] \leq CN^{-qv_1 + d_1 \delta} \to 0, \quad \text{as } N \to \infty.
\]

By (C4) and [17, Theorem C.6], one has that for sufficiently large \(p \geq 1\),
\[
\mathbb{E}\left[\sup_{t_1, t_2 \in \mathbb{T}, t_1 \neq t_2} \frac{\|X(t_1) - X(t_2)\|^p}{\|t_1 - t_2\|^{\nu_2(1 - \frac{d_1}{p\nu_2})}}\right] \leq C_1.
\]

Thus the Markov inequality gives that for sufficiently large \(p \geq \max\{q, 2d/v_2\}\),
\[
\mathbb{P}(R^N_2 \geq 3^{-(q+1)}) \leq 3^{q+1} \mathbb{E}\left[\sup_{s \in \mathbb{T}_\delta} \sup_{t:|s-t| \leq c(T)N^{-\delta}} \|X(s) - X(t)\|^p\right]
\]
\[
\leq CN^{-\delta v_2(1 - \frac{d_1}{p\nu_2})} \to 0, \quad \text{as } N \to \infty.
\]
Analogously, the Hölder regularity assumption of $X^N$ also yields
\[ \mathbb{P}(R_3^N \geq 3^{-(q+1)}) \to 0, \quad \text{as } N \to \infty. \]
Plugging the above estimates on $\{R_j^N\}_{j=1,2,3}$ into (6.4), we obtain
\[ \lim_{N \to \infty} \mathbb{P}\left( \sup_{t \in T} \|X^N(t) - X(t)\| \geq 1 \right) = 0. \tag{6.5} \]

**Step 3.** Letting $N \to \infty$ on both sides of (6.3), it follows from (6.5) that
\[ \limsup_{N \to \infty} d_{TV}(X^N(t), X(t)) \leq 4\mathbb{P}(\Omega^R_{R-1}) + \limsup_{N \to \infty} d_{TV}(X^N_R(t), X_R(t)), \]
which together with $\lim_{R \to \infty} \mathbb{P}(\Omega_R) = 1$ in (C1) leads to (6.2).

In order to apply Proposition 6.1 with $X = u$ and $T = [0,T] \times \mathcal{O}$, we first construct the localization of $u$. Denote $\Omega_R := \{ \omega \in \Omega : \sup_{t,x} |u(t,x,\omega)| \leq R \}$. Set $f_R(x) = K_R(x)f(x)$ for $x \in \mathbb{R}$, where the cut-off function $K_R$ is defined in (2.21). Then we consider the following localized Cahn–Hilliard equation
\[ \partial_t u_R + \Delta^2 u_R = \Delta f_R(u_R) + \sigma(u_R) \dot{W}, \quad R \geq 1 \tag{6.6} \]
with $u_R(0,\cdot) = u_0$ and DBCs. By the fact that $u$ has a.s. continuous trajectories and the local property of stochastic integrals, one has
\[ u = u_R \text{ on } \Omega_R \text{ a.s., and } \lim_{R \to \infty} \mathbb{P}(\Omega_R) = 1. \tag{6.7} \]

For $n \geq 2$, consider the spatial FDM numerical solution $u^n_R$ and the fully discrete FDM numerical solution $u^{n,\tau}_R$ of (6.6), i.e., $u^n_R$ and $u^{n,\tau}_R$ respectively solve (2.18) and (2.20) with $f$ replaced by $f_R$. Similarly to (6.7), by setting $\Omega_{R,n} := \{ \omega \in \Omega : \sup_{t,x} |u^n(t,x,\omega)| \leq R \}$ and $\Omega_{R,n,m} := \{ \omega \in \Omega : \sup_{t,x} |u^{n,\tau}(t,x,\omega)| \leq R \}$, we have
\[ u^n = u^n_R \text{ on } \Omega_{R,n} \text{ a.s., and } \lim_{R \to \infty} \mathbb{P}(\Omega_{R,n}) = 1, \tag{6.8} \]
\[ u^{n,\tau} = u^{n,\tau}_R \text{ on } \Omega_{R,n,m} \text{ a.s., and } \lim_{R \to \infty} \mathbb{P}(\Omega_{R,n,m}) = 1. \tag{6.9} \]

The following uniform non-degeneracy condition is instrumental for the convergence of density of the numerical solution.

**Assumption 1.** There exists some $\sigma_0 > 0$ such that $|\sigma(x)| > \sigma_0$ for any $x \in \mathbb{R}$.

Since for a fixed $R \geq 1$, $f_R$ in (6.6) is infinitely differentiable with bounded derivatives of any order, Proposition 6.2 is a direct consequence of Proposition A.6.

**Proposition 6.2.** Let $u_0 \in \mathcal{C}^3(\mathcal{O})$ and $\sigma$ be twice differentiable with bounded first and second order derivatives. Then for any $x \in \mathcal{O}$,
\[ \lim_{n \to \infty} d_{TV}(u_R(T,x), u^n_R(T,x)) = 0, \quad \text{for any fixed } R \geq 1, \]
\[ \lim_{\tau \to 0} d_{TV}(u^n_R(T,x), u^{n,\tau}_R(T,x)) = 0, \quad \text{for any fixed } R \geq 1 \text{ and } n \geq 2. \]

**Proposition 6.3.** Let $u_0 \in \mathcal{C}^3(\mathcal{O})$, $\sigma$ be twice differentiable with bounded first and second order derivatives, and Assumption 1 hold. Then
\[ \lim_{n \to \infty} d_{TV}(u^n(T,x), u(T,x)) = 0, \quad x \in \mathcal{O}, \tag{6.10} \]
\[ \lim_{\tau \to 0} d_{TV}(u^{n,\tau}(T,x), u(T,x)) = 0, \quad x \in \mathcal{O}. \tag{6.11} \]
Proof. (i) Let $X = u$, $T = [0, T] \times \mathcal{O}$, $N = n$, $X_R = u_R$, $X^N = u^n_R$, $X^N_R = u^{n,\tau}_R$ in Proposition 6.1. Note that (C1) and (C2) follow from (6.7) and (6.8) respectively. In addition, (C3) can be ensured by Theorem 4.7, and (C4) is a consequence of Lemmas 2.2 and 4.3. Hence an application of Proposition 6.1, together with Proposition 6.2, proves the first assertion (6.10).

(ii) For fixed $n$, let $X = u^n$, $T = [0, T] \times \mathcal{O}$, $N = m = T/\tau$, $X_R = u^n_R$, $X^N = u^n_{\tau,\tau}$, $X^N_R = u^{n,\tau}_R$ in Proposition 6.1. Then (C1) and (C2) come from (6.8) and (6.9) respectively. Theorem 5.7 implies (C3), and (C4) is a consequence of Lemmas 4.3 and 5.4. Therefore, using Propositions 6.1, 6.2 yields that for fixed $n$,

$$\lim_{\tau \to 0} d_{TV}(u^{n,\tau}(T, x), u^n(T, x)) = 0, \quad x \in \mathcal{O}.$$ 

This together with the triangle inequality and (6.10) yields (6.11). \hfill \square

Recall that for any $t \in (0, T]$ and $x \in \mathcal{O}$, $p_{t,x}$ is the density of the exact solution $u(t, x)$ to (1.1). In Theorems 2.4 and 2.6 we have also shown that for $k \in \mathbb{Z}_{n-1}$, both the spatial FDM numerical solution $\{u^n(t, kh)\}_{t \in [0, T]}$ and the fully discrete FDM numerical solution $\{u^{n,\tau}(t_i, kh)\}_{t_i \in \mathbb{Z}_m}$ admit densities, which are denoted by $\{p^n_{t, kh}\}_{t \in [0, T]}$ and $\{p^{n,\tau}_{t_i, kh}\}_{t_i \in \mathbb{Z}_m}$, respectively. In view of (6.1) and Proposition 6.3 we have the following convergence of density in $L^1(\mathbb{R})$ of the numerical solutions.

**Theorem 6.4.** Under the assumptions of Proposition 6.3, for any $k \in \mathbb{Z}_{n-1}$,

$$\lim_{n \to \infty} \int_{\mathbb{R}} |p^n_{T, kh}(\xi) - p_{T, kh}(\xi)| d\xi = 0,$$

$$\lim_{n \to \infty} \lim_{\tau \to 0} \int_{\mathbb{R}} |p^{n,\tau}_{T, kh}(\xi) - p_{T, kh}(\xi)| d\xi = 0.$$

**APPENDIX A. CONVERGENCE OF DENSITY: LIPSCHITZ CASE**

This section is devoted to studying the density convergence of the spatial and fully discrete FDMs for the stochastic Cahn–Hilliard equation with Lipschitz nonlinearities. This is motivated by proving Proposition 6.2. We adopt a slight abuse of notation in the appendix. The coefficient $f$ in (1.1) is no longer $f(x) = x^3 - x$, but a general Lipschitz continuous function. More precisely, we will work under the following assumption.

**Assumption 2.** $f$ and $\sigma$ are twice differentiable with bounded derivatives of first and second order.

The main approach is based on the Malliavin calculus. Let us introduce some notations in the context of the Malliavin calculus with respect to the space-time white noise (see e.g., [34]). The isonormal Gaussian family $\{W(h), h \in \mathbb{H}\}$ corresponding to $\mathbb{H} := L^2([0, T] \times \mathcal{O})$ is given by the Wiener integral $W(h) = \int_0^T \int_{\mathcal{O}} h(s, y) W(ds, dy)$. Denote by $\mathcal{S}$ the class of smooth real-valued random variables of the form

$$X = \varphi(W(h_1), \ldots, W(h_n)), \quad (A.1)$$

where $\varphi \in C^\infty_p(\mathbb{R}^n)$, $h_i \in \mathbb{H}$, $i = 1, \ldots, n$, $n \geq 1$. Here $C^\infty_p(\mathbb{R}^n)$ is the space of all $\mathbb{R}$-valued smooth functions on $\mathbb{R}^n$ whose partial derivatives have at most polynomial growth. The Malliavin derivative of $X \in \mathcal{S}$ of the form (A.1) is an $\mathbb{H}$-valued random variable given by

$$DX = \sum_{i=1}^n \partial_i \varphi(W(h_1), \ldots, W(h_n)) h_i,$$

which is also a random field $DX = \{D\theta, \xi X, (\theta, \xi) \in [0, T] \times \mathcal{O}\}$ with $D\theta, \xi X = \sum_{i=1}^n \partial_i \varphi(W(h_1), \ldots, W(h_n)) h_i(\theta, \xi)$ for almost everywhere $(\theta, \xi, \omega) \in [0, T] \times \mathcal{O}$.
[0, T] \times \mathcal{O} \times \Omega. For any \( p \geq 1 \), we denote the domain of \( D \) in \( L^p(\Omega; \mathbb{R}) \) by \( \mathbb{D}^{1,p} \), meaning that \( \mathbb{D}^{1,p} \) is the closure of \( \mathcal{S} \) with respect to the norm
\[
\|X\|_{1,p} = \left( \mathbb{E} \left[ |X|^p + \|DX\|_{\mathbb{H}^k}^p \right] \right)^{\frac{1}{p}}.
\]

We define the iteration of the operator \( D \) in such a way that for \( X \in \mathcal{S} \), the iterated derivative \( D^k X \) is an \( \mathbb{H}^\otimes k \)-valued random variable. More precisely, for \( k \in \mathbb{N}_+ \), \( D^k X = \{D_{r_1, \theta_1} \cdots D_{r_k, \theta_k} X, (r_i, \theta_i) \in [0, T] \times \mathcal{O}\} \) is a measurable function on the product space \(([0, T] \times \mathcal{O})^k \times \Omega\). Then for \( p \geq 1, k \in \mathbb{N} \), denote by \( \mathbb{D}^{k,p} \) the completion of \( \mathcal{S} \) with respect to the norm
\[
\|X\|_{k,p} = \left( \mathbb{E} \left[ |X|^p + \sum_{j=1}^k \|D^j X\|_{\mathbb{H}^\otimes j}^p \right] \right)^{\frac{1}{p}}.
\]

Define \( \mathbb{D}^{k,\infty} := \bigcap_{p \geq 1} \mathbb{D}^{k,p} \) and \( \mathbb{D}^{\infty} := \bigcap_{k \geq 1} \mathbb{D}^{k,\infty} \) to be topological projective limits. The following proposition allows one to obtain the convergence of density of a sequence of random variables from the convergence in \( \mathbb{D}^{1,2} \).

**Proposition A.1.** [33 Theorem 4.2] Let \( \{X_N\}_{N \geq 1} \) be a sequence in \( \mathbb{D}^{1,2} \) such that each \( X_N \) admits a density. Let \( X_\infty \in \mathbb{D}^{2,4} \) and let \( 0 < \alpha \leq 2 \) be such that \( \mathbb{E}[\|DX_\infty\|_{\mathbb{H}^k}^\alpha] < \infty \). If \( X_N \to X_\infty \) in \( \mathbb{D}^{1,2} \), then there exists a constant \( c > 0 \) depending only on \( X_\infty \) such that for any \( N \geq 1 \),
\[
d_{TV}(X_N, X_\infty) \leq c \|X_N - X_\infty\|_{1,2}^{\frac{\alpha}{2}}.
\]

From [43 Proposition 3.1] or [8 Lemma 3.2], one can see that if \( f \) and \( \sigma \) in [11] is continuously differentiable with bounded derivatives, then for any \( (t, x) \in [0, T] \times \mathcal{O} \), \( u(t, x) \in \mathbb{D}^{1,2} \) satisfies
\[
D_{r,z} u(t, x) = G_{t-r}(x, z) \sigma(u(r, z)) + \int_r^t \int_{\mathcal{O}} \Delta G_{t-s}(x, y) f'(u(s, y)) D_{r,z} u(s, y) dy ds,
\]
\[
+ \int_r^t \int_{\mathcal{O}} G_{t-s}(x, y) \sigma'(u(s, y)) D_{r,z} u(s, y) W(ds, dy),
\]
if \( r \leq t \), and \( D_{r,z} u(t, x) = 0 \), if \( r > t \). Further, we study the regularity of the exact solution \( u(t, x) \) to [11] in the Malliavin–Sobolev space \( \mathbb{D}^{k,p} \).

**Lemma A.2.** Given \( k \in \mathbb{N}_+ \), let \( f \) and \( \sigma \) be \( k \)-th differentiable with bounded derivatives up to order \( k \). Then \( u(t, x) \in \mathbb{D}^{k,\infty} \) for any \( (t, x) \in [0, T] \times \mathcal{O} \). Moreover, for any \( p \geq 1 \), there exists \( C = C(k, p, T) \) such that for any \( (t, x) \in [0, T] \times \mathcal{O} \),
\[
\|u(t, x)\|_{k,p} \leq C.
\]

**Proof.** Define the Picard approximation by \( w^0(t, x) = u_0(x) \), and for \( i \in \mathbb{N} \),
\[
w^{i+1}(t, x) = G_{t,u_0(x)} + (\Delta G) * f(w^i)(t, x) + G \circ \sigma(w^i)(t, x), \quad (t, x) \in [0, T] \times \mathcal{O};
\]
see [2.1] and [2.2] for more details. Fix \( (t, x) \in [0, T] \times \mathcal{O} \). In view of [41 Lemma 1.5.3], the proof of \( u(t, x) \in \mathbb{D}^{k,\infty} \) boils down to proving that
\[(i) \ \{w^p(t, x)\}_{p \geq 1} \text{ converges to } u(t, x) \text{ in } L^p(\Omega) \text{ for every } p \geq 1;
\[(ii) \ \text{for any } p \geq 1, \sup_{i \geq 0} \|w^i(t, x)\|_{k,p} < \infty.
\]

Property \( (i) \) and property \( (ii) \) with \( k = 1 \) and \( p = 2 \) can be obtained in the same way as in [8 Lemma 3.2] (the sequence \( \{w^p(t, x)\}_{p \geq 1} \) corresponds to \( \{u_{n,k}(t, x)\}_{k \geq 1} \) in [8]). The proof of property \( (ii) \) with general \( k, p \geq 1 \) is omitted since it is standard and similar to those for other kinds of SPDEs with Lipschitz continuous coefficients; see [2 Proposition 4.3] for the case of stochastic heat equations and [37 Theorem 1] for the case of stochastic wave equations. \( \square \)
Similar to properties (i) and (ii), the standard Picard approximation also shows that under Assumption 2, \( u^n(t, x) \in \mathbb{D}^{1,2} \) for any \((t, x) \in [0, T] \times \mathcal{O} \) and \( u^{n,\tau}(t_i, x) \in \mathbb{D}^{1,2} \) for any \( i \in \mathbb{Z}_m \) and \( x \in \mathcal{O} \).

**Lemma A.3.** Let \( u_0 \in C^3(\mathcal{O}) \) and Assumption 2 hold. Then for every \( p \geq 1 \) and \( 0 < \varepsilon \ll 1 \), there exist some constants \( C_1 = C(p, T) \) and \( C_2 = C(p, T, \varepsilon) \) such that for all \( \varepsilon \in (0, 1), \) if \( \varepsilon \ll 1 \), then

\[
\|u^n(t, x) - u(t, x)\|_{L^p(\Omega)} \leq C_1 \varepsilon^{-1},
\]

\[
\|u^{n,\tau}(t_i, x) - u^n(t_i, x)\|_{L^p(\Omega)} \leq C_2 \varepsilon^{-\frac{3}{2}}.
\]

**Proof.** Notice that \( u^n - u = (u^n - \tilde{u}^n) + (\tilde{u}^n - u) \) and \( u^{n,\tau} - u^n = (u^{n,\tau} - \tilde{u}^{n,\tau}) + (\tilde{u}^{n,\tau} - u^n) \). The errors \( \|\tilde{u}^n(t, x) - u(t, x)\|_{L^p(\Omega)} \) and \( \|\tilde{u}^{n,\tau}(t_i, x) - u^n(t_i, x)\|_{L^p(\Omega)} \) have been tackled in Propositions 4.2 and 5.3 (see Remark 5.5), respectively. Since \( f \) and \( \sigma \) are globally Lipschitz, the estimates of \( \|u^n(t, x) - \tilde{u}^n(t, x)\|_{L^p(\Omega)} \) and \( \|u^{n,\tau}(t_i, x) - \tilde{u}^{n,\tau}(t_i, x)\|_{L^p(\Omega)} \) are standard by using (5.3), (5.6) and the singular Gronwall inequality. \( \square \)

**Proposition A.4.** Let \( u_0 \in C^3(\mathcal{O}) \) and Assumption 2 hold. Then there exists some constant \( C \) such that for any \( (t, x) \in [0, T] \times \mathcal{O} \),

\[
\|u^n(t, x) - u(t, x)\|_{1,2} \leq C n^{-1}.
\]

The proof of Proposition A.4 is standard and thus is omitted. In order to apply Proposition A.3 with \( X_\infty = u(T, x) \), we further investigate the inverse moment estimate of \( \|Du(t, x)\|_{\mathbb{H}} \).

**Lemma A.5.** Under Assumptions 4.2, for any \( x \in \mathcal{O} \), there is \( \rho \in (0, 1] \) such that

\[
\mathbb{E} \left[ \|Du(T, x)\|_{\mathbb{H}}^{-2\rho} \right] \leq C(\rho, T).
\]

**Proof.** We need to use [15], Proposition 3.2, which is summarized as follows: under Assumption 4 if \( x_i \in \mathcal{O}, i = 1, \ldots, d \), are distinct points, then for some \( p_0 > 0 \), there exists \( \varepsilon_0 = \varepsilon_0(p_0) \) such that for all \( \varepsilon \in (0, \varepsilon_0) \),

\[
\sup_{\xi \in \mathbb{R}^d, \|\xi\| = 1} \mathbb{P} \left( \xi^\top C(t) \xi \leq \varepsilon \right) \leq \varepsilon^{p_0}, \tag{A.2}
\]

where \( C(t) := \langle (Du(t, x_i), Du(t, x_j))_{\mathbb{H}} \rangle_{1 \leq i, j \leq d} \) denotes the Malliavin covariance matrix of the random vector \((u(t, x_1), \ldots, u(t, x_d))\) (the notation \( u \) corresponds to \( X_R \) in [15]). As a consequence of (A.2) with \( d = 1 \) and \( t = T \), we have that for all \( 0 < \varepsilon \leq \varepsilon_0 \), \( \mathbb{P}(\|Du(T, x)\|_{\mathbb{H}}^2 \leq \varepsilon) \leq \varepsilon^{p_0} \), which implies that for any \( \rho < p_0 \),

\[
\sum_{n=1}^{\infty} n^{\rho - 1} \mathbb{P}(\|Du(T, x)\|_{\mathbb{H}}^{-2} \geq n) \leq \sum_{n=1}^{\lfloor \varepsilon_0^{-1} \rfloor} n^{\rho - 1} + \sum_{n=\lfloor \varepsilon_0^{-1} \rfloor + 1}^{\infty} n^{\rho - 1} n^{-p_0} \leq C(\rho, \varepsilon_0).
\]

Then we have that for \( 0 < \rho < \min\{p_0, 1\} \) and \( Z := \|Du(T, x)\|_{\mathbb{H}}^{-2} \),

\[
\mathbb{E}[Z^\rho] \leq 1 + \sum_{n=1}^{\infty} (n+1)^\rho \mathbb{P}(n \leq Z < n+1) \leq 2 + \sum_{n=1}^{\infty} ((n+1)^\rho - n^\rho) \mathbb{P}(Z \geq n)
\]

\[
\leq 2 + \rho \sum_{n=1}^{\infty} n^{\rho - 1} \mathbb{P}(Z \geq n) \leq C(\rho, \varepsilon_0).
\]

The proof is completed. \( \square \)
In view of the Boule and Hirsch’s criterion (see e.g., [34, Theorem 2.1.2]), Lemmas A.2 and A.5 imply that under Assumptions 1 and 2, for any \((t,x) \in [0,T] \times \mathcal{O}\), the exact solution \(u(t,x)\) to (1.1) admits a density. We are ready to give the main result of the appendix.

**Proposition A.6.** Under Assumptions 1–2, if \(u_0 \in C^3(\mathcal{O})\), then for any \(x \in \mathcal{O}\),

\[
\lim_{n \to \infty} d_{TV}(u(T,x), u^n(T,x)) = 0, \tag{A.3}
\]

and for any fixed \(n \geq 2\),

\[
\lim_{\tau \to 0} d_{TV}(u^n(T,x), u^{n,\tau}(T,x)) = 0. \tag{A.4}
\]

**Proof.** (i) Lemma A.2, Proposition A.4, and Lemma A.5 indicate that the conditions of Proposition A.1 are fulfilled with \(\alpha = 2\rho\), \(X_n = u^n(T,x)\) and \(X_\infty = u(T,x)\). Thus (A.3) can be obtained from Proposition A.1. (ii) In a similar manner, (A.4) can be proved applying Proposition A.1 with \(X_\infty = u^n(T,x)\) and \(X_m = u^{n,\tau}(T,x)\), and we omit the details. \(\square\)

In Sections 4–6, we give the strong convergence orders and density convergence of the spatial and fully discrete FDMs applied to (1.1) for Case 1: \(f(x) = x^3 - x\). In the appendix, we also present that the above results also hold for Case 2: \(f\) is twice differentiable with bounded derivatives of first and second order. Although the results are the same in both cases, the main techniques are essentially different. We take the spatial FDM as instance to point out some differences between these two cases below.

(1) In both Case 1 and Case 2, the strong convergence analysis of the spatial FDM relies on the introduction of the auxiliary process \(\tilde{u}\) and the error estimate between \(u\) and \(\tilde{u}\) in Proposition A.2 (see Theorem 4.7 and Lemma A.3 for Case 1 and Case 2, respectively). However, we emphasize that the introduction of the auxiliary process \(\tilde{u}\) is mainly used to deal with Case 1, and is not necessary for Case 2. Alternatively, Lemma A.3 can be proved based on the following decomposition

\[
u^n(t,x) - u(t,x) = \int_0^t \int_\mathcal{O} [\Delta_n G^n_{t-s}(x,y) - \Delta G_{t-s}(x,y)] f(u^n(s,\kappa_n(y)))dyds
+ \int_0^t \int_\mathcal{O} \Delta G_{t-s}(x,y)[f(u^n(s,\kappa_n(y))) - f(u^n(s,y))]dyds
+ \int_0^t \int_\mathcal{O} \Delta G_{t-s}(x,y)[f(u^n(s,y)) - f(u(s,y))]dyds
+ \int_0^t \int_\mathcal{O} [G^n_{t-s}(x,y) - G_{t-s}(x,y)] \sigma(u^n(s,\kappa_n(y)))W(ds,dy)
+ \int_0^t \int_\mathcal{O} G_{t-s}(x,y)[\sigma(u^n(s,\kappa_n(y))) - \sigma(u^n(s,y))]W(ds,dy)
+ \int_0^t \int_\mathcal{O} G_{t-s}(x,y)[\sigma(u^n(s,y)) - \sigma(u(s,y))]W(ds,dy).
\]

However, the above way of decomposition does not work for the proof of Theorem 4.7 due to the absence of the Lipschitz property of \(f\) in Case 1.

(2) In Case 2, the key to deriving the density convergence of the numerical solution is the application of Proposition A.1 whose prerequisite involves the Malliavin differentiability of the exact solution. However, in Case 1, we are only aware that the exact solution \(u(t,x)\) is locally Malliavin differentiable, and it is still unclear to us whether \(u(t,x)\) belongs to \(D^{1,2}\) or not. This brings difficulty in applying Proposition A.1 to prove the density convergence for Case 1. Instead, we propose a novel localization
argument, which enables us to convert the proof of the density convergence in Case 1 into the strong convergence analysis of the numerical method in Case 1 and the density convergence of the numerical solution in Case 2.

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LSEC, ICMSEC, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, AND SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, CHINA

Email address: hjl@lsec.cc.ac.cn

School of Mathematics and Statistics, Huazhong University of Science and Technology, and Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, CHINA

Email address: jindc@hust.edu.cn

LSEC, ICMSEC, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, AND SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100049, CHINA

Email address: sdr@lsec.cc.ac.cn