Some relations between the topological and geometric filtration for smooth projective varieties

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Abstract

In the first part of this paper, we show that the assertion “$T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$” (which is called the Friedlander-Mazur conjecture) is a birationally invariant statement for smooth projective varieties $X$ when $p = \dim(X) - 2$ and when $p = 1$. We also establish the Friedlander-Mazur conjecture in certain dimensions. More precisely, for a smooth projective variety $X$, we show that the topological filtration $T_p H_{2p+1}(X, \mathbb{Q})$ coincides with the geometric filtration $G_p H_{2p+1}(X, \mathbb{Q})$ for all $p$. (Friedlander and Mazur had previously shown that $T_p H_{2p}(X, \mathbb{Q}) = G_p H_{2p}(X, \mathbb{Q})$). As a corollary, we conclude that for a smooth projective threefold $X$, $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ for all $k \geq 2p \geq 0$ except for the case $p = 1, k = 4$. Finally, we show that the topological and geometric filtrations always coincide if Suslin’s conjecture holds.

1 Introduction

In this paper, all varieties are defined over $\mathbb{C}$. Let $X$ be a projective variety with dimension $n$. Let $Z_p(X)$ be the space of algebraic $p$-cycles.

The Lawson homology $L_p H_k(X)$ of $p$-cycles is defined by

\[ L_p H_k(X) = \pi_{k-2p}(Z_p(X)) \quad \text{for} \quad k \geq 2p \geq 0, \]
where $Z_p(X)$ is provided with a natural topology (cf. [F1], [L1]). For general background, the reader is referred to Lawson’ survey paper [L2].

In [FM], Friedlander and Mazur showed that there are natural maps, called cycle class maps

$$\Phi_{p,k} : L_p H_k(X) \to H_k(X).$$

**Definition 1.1**

$$L_p H_k(X)_{\text{hom}} := \ker \{\Phi_{p,k} : L_p H_k(X) \to H_k(X)\};$$

$$T_p H_k(X) := \text{Image} \{\Phi_{p,k} : L_p H_k(X) \to H_k(X)\};$$

$$T_p H_k(X, \mathbb{Q}) := T_p H_k(X) \otimes \mathbb{Q}.$$ 

It was shown in [FM], §7 that the subspaces $T_p H_k(X, \mathbb{Q})$ form a decreasing filtration:

$$\cdots \subseteq T_p H_k(X, \mathbb{Q}) \subseteq T_{p-1} H_k(X, \mathbb{Q}) \subseteq \cdots \subseteq T_0 H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q})$$

and $T_p H_k(X, \mathbb{Q})$ vanishes if $2p > k$.

**Definition 1.2** (FM) Denote by $G_p H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$ the $\mathbb{Q}$-vector subspace of $H_k(X, \mathbb{Q})$ generated by the images of mappings $H_k(Y, \mathbb{Q}) \to H_k(X, \mathbb{Q})$, induced from all morphisms $Y \to X$ of varieties of dimension $\leq k - p$.

The subspaces $G_p H_k(X, \mathbb{Q})$ also form a decreasing filtration (called geometric filtration):

$$\cdots \subseteq G_p H_k(X, \mathbb{Q}) \subseteq G_{p-1} H_k(X, \mathbb{Q}) \subseteq \cdots \subseteq G_0 H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$$

If $X$ is smooth, the Weak Lefschetz Theorem implies that $G_0 H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q})$. Since $H_k(Y, \mathbb{Q})$ vanishes for $k$ greater than twice the dimension of $Y$, $G_p H_k(X, \mathbb{Q})$ vanishes if $2p > k$.

The following results have been proved by Friedlander and Mazur in [FM]:

**Theorem 1.1** (FM) Let $X$ be any projective variety.

1. For non-negative integers $p$ and $k$,

$$T_p H_k(X, \mathbb{Q}) \subseteq G_p H_k(X, \mathbb{Q}).$$

2. When $k = 2p$,

$$T_p H_{2p}(X, \mathbb{Q}) = G_p H_{2p}(X, \mathbb{Q}).$$

**Question** (FM, [L2]): Does one have equality in Theorem 1.1 when $X$ is a smooth projective variety?

Friedlander [F2] has the following result:
Theorem 1.2 ([F2]) Let $X$ be a smooth projective variety of dimension $n$. Assume that Grothendieck’s Standard Conjecture B ([Gro]) is valid for a resolution of singularities of each irreducible subvariety of $Y \subset X$ of dimension $k - p$, then

$$T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q}).$$

Remark 1.1 ([Lew], §15.32) The Grothendieck’s Standard Conjecture B is known to hold for a smooth projective variety $X$ in the following cases:

1. dim$X \leq 2$.
2. Flag manifolds $X$.
3. Smooth complete intersections $X$.
4. Abelian varieties (due to D. Lieberman [Lieb]).

In this paper, we will use the tools in Lawson homology and the methods given in [H] to show the following main results:

Theorem 1.3 Let $X$ be a smooth projective variety of dimension $n$. If the conclusion in Theorem 1.2 holds (without the assumption of Grothendieck’s Standard Conjecture B) for $X$ with $p = 1, (\text{resp.} p = n - 2)$ (k arbitrary), then it also holds for any smooth projective variety $X'$ which is birationally equivalent to $X$ with $p = 1, (\text{resp.} p = n - 2)$.

Theorem 1.4 For any smooth projective variety $X$,

$$T_p H_{2p+1}(X, \mathbb{Q}) = G_p H_{2p+1}(X, \mathbb{Q}).$$

As corollaries, we have

Corollary 1.1 Let $X$ be a smooth projective 3-fold. We have $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ for all $k \geq 2p \geq 0$ except for the case $p = 1, k = 4$.

Corollary 1.2 Let $X$ be a smooth projective 3-fold with $H^{2,0}(X) = 0$. Then $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ for any $k \geq 2p \geq 0$. In particular, it holds for $X$ a smooth hypersurface and a complete intersection of dimension 3.

By using the Künneth formula in homology with rational coefficient, we have

Corollary 1.3 Let $X$ be the product of a smooth projective curve and a smooth simply connected projective surface. Then $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ for any $k \geq 2p \geq 0$. 

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Corollary 1.4 For 4-folds $X$, the assertion that $T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q})$ holds for all $k \geq 2p \geq 0$ is a birational invariant statement. In particular, if $X$ is a rational manifold with $\text{dim}(X) \leq 4$, then the conclusion in Theorem 1.2 holds for any $k \geq 2p \geq 0$ without assumption of Grothendieck’s Standard Conjecture B.

Remark 1.2 A Conjecture given by Suslin (see [PHW], §7) implies that $L_p H_{n+p}(X^n) \cong H_{n+p}(X^n)$.

As an application of Theorem 1.4 and Proposition 3.1, we have the following result:

Corollary 1.5 If the Suslin’s Conjecture is true, then the topological filtration is the same as the geometric filtration for a smooth projective variety.

The main tools to prove this result are: the long exact localization sequence given by Lima-Filho in [L], the explicit formula for Lawson homology of codimension-one cycles on a smooth projective manifold given by Friedlander in [F1], (and its generalization to general irreducible varieties, see below), and the weak factorization theorem proved by Wlodarczyk in [W] and in [AKMW].

2 The Proof of the Theorem 1.3

Let $X$ be a smooth projective manifold of dimension $n$ and $i_0 : Y \hookrightarrow X$ be a smooth subvariety of codimension $r \geq 2$. Let $\sigma : X_Y \to X$ be the blowup of $X$ along $Y$, $\pi : D = \sigma^{-1}(Y) \to Y$ the nature map, and $i : D = \sigma^{-1}(Y) \hookrightarrow X_Y$ the exceptional divisor of the blowup. Set $U := X - Y \cong X_Y - D$. Denote by $j_0$ the inclusion $U \subset X$ and $j$ the inclusion $U \subset X_Y$.

Now I list the Lemmas and Corollaries given in [H].

Lemma 2.1 For each $p \geq 0$, we have the following commutative diagram

\[ \cdots \to L_p H_k(D) \xrightarrow{i_*} L_p H_k(X_Y) \xrightarrow{j_*} L_p H_k(U) \xrightarrow{\delta_*} L_p H_{k-1}(D) \to \cdots \]

\[ \downarrow \pi_* \quad \downarrow \sigma_* \quad \downarrow \delta \quad \downarrow \pi_* \]

\[ \cdots \to L_p H_k(Y) \xrightarrow{(i_0)_*} L_p H_k(X) \xrightarrow{j_0^*} L_p H_k(U) \xrightarrow{(\delta_0)_*} L_p H_{k-1}(Y) \to \cdots \]

Remark 2.1 Since $\pi_*$ is surjective (there is an explicitly formula for the Lawson homology of $D$, i.e., the Projective Bundle Theorem proved by Friedlander and Gabber, see [EG]), it is easy to see that $\sigma_*$ is surjective.

Corollary 2.1 If $p = 0$, then we have the following commutative diagram

\[ \cdots \to H_k(D) \xrightarrow{i_*} H_k(X_Y) \xrightarrow{j_*} H_k^BM(U) \xrightarrow{\delta_*} H_{k-1}(D) \to \cdots \]

\[ \downarrow \pi_* \quad \downarrow \sigma_* \quad \downarrow \delta \quad \downarrow \pi_* \]

\[ \cdots \to H_k(Y) \xrightarrow{(i_0)_*} H_k(X) \xrightarrow{j_0^*} H_k^BM(U) \xrightarrow{(\delta_0)_*} H_{k-1}(Y) \to \cdots \]

Moreover, if $x \in H_k(D)$ maps to zero under $\pi_*$ and $i_*$, then $x = 0 \in H_k(D)$. 
Corollary 2.2 If \( p = n - 2 \), then we have the following commutative diagram

\[
\cdots \to L_{n-2}H_k(D) \overset{i_*}{\to} L_{n-2}H_k(\tilde{X}Y) \overset{j^*}{\to} L_{n-2}H_k(U) \overset{\delta}{\to} L_{n-2}H_{k-1}(D) \to \cdots \\
\downarrow \pi_* \hspace{1cm} \downarrow \sigma_* \hspace{1cm} \downarrow \cong \hspace{1cm} \downarrow \pi_* \\
\cdots \to L_{n-2}H_k(Y) \overset{(i_0)_*}{\to} L_{n-2}H_k(X) \overset{j^*_0}{\to} L_{n-2}H_k(U) \overset{(\delta_0)_*}{\to} L_{n-2}H_{k-1}(Y) \to \cdots 
\]

Lemma 2.2 For each \( p \geq 0 \), we have the following commutative diagram

\[
\cdots \to L_pH_k(D) \overset{i_*}{\to} L_pH_k(\tilde{X}Y) \overset{j^*}{\to} L_pH_k(U) \overset{\delta}{\to} L_pH_{k-1}(D) \to \cdots \\
\downarrow \Phi_{p,k} \hspace{1cm} \downarrow \Phi_{p,k} \hspace{1cm} \downarrow \Phi_{p,k} \hspace{1cm} \downarrow \Phi_{p,k-1} \\
\cdots \to H_k(D) \overset{i_*}{\to} H_k(\tilde{X}Y) \overset{j^*}{\to} H_k^BM(U) \overset{\delta}{\to} H_{k-1}(D) \to \cdots 
\]

In particular, it is true for \( p = 1, n - 2 \).

Proof. See \( [Li] \) and also \( [FM] \).

Lemma 2.3 For each \( p \geq 0 \), we have the following commutative diagram

\[
\cdots \to L_pH_k(Y) \overset{(i_0)_*}{\to} L_pH_k(X) \overset{j^*_0}{\to} L_pH_k(U) \overset{(\delta_0)_*}{\to} L_pH_{k-1}(Y) \to \cdots \\
\downarrow \Phi_{p,k} \hspace{1cm} \downarrow \Phi_{p,k} \hspace{1cm} \downarrow \Phi_{p,k} \hspace{1cm} \downarrow \Phi_{p,k-1} \\
\cdots \to H_k(Y) \overset{(i_0)_*}{\to} H_k(X) \overset{j^*_0}{\to} H_k^BM(U) \overset{(\delta_0)_*}{\to} H_{k-1}(Y) \to \cdots 
\]

In particular, it is true for \( p = 1, n - 2 \).

Proof. See \( [Li] \) and also \( [FM] \).

Remark 2.2 The smoothness of \( X \) and \( Y \) is not necessary in the Lemma 2.3.

Remark 2.3 All the commutative diagrams of long exact sequences above remain commutative and exact when tensored with \( \mathbb{Q} \). We will use these Lemmas and Corollaries with rational coefficients.

The following result will be used several times in the proof of our main theorem:

Theorem 2.1 (Friedlander \( [F1] \)) Let \( W \) be any smooth projective variety of dimension \( n \). Then we have the following isomorphisms

\[
\begin{cases}
L_{n-1}H_{2n}(W) \cong \mathbb{Z}, \\
L_{n-1}H_{2n-1}(W) \cong H_{2n-1}(X, \mathbb{Z}), \\
L_{n-1}H_{2n-2}(W) \cong H_{n-1,n-1}(X, \mathbb{Z}) = NS(W), \\
L_{n-1}H_k(X) = 0 \quad \text{for} \quad k > 2n.
\end{cases}
\]
The proof of Theorem 1.3 (\( p = n - 2 \)):

There are two cases:

Case 1. If \( T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q}) \), then \( T_p H_k(\tilde{X}_Y, \mathbb{Q}) = G_p H_k(\tilde{X}_Y, \mathbb{Q}) \).

The injectivity of \( T_p H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow G_p H_k(\tilde{X}_Y, \mathbb{Q}) \) has been proved by Friedlander and Mazur in [FM]. We only need to show the surjectivity. Note that the case for \( k = 2p + 1 \) holds for any smooth projective variety (Theorem 1.4). We only need to consider the cases where \( k \geq 2p + 2 \). In these cases, \( k - p \geq p + 2 = n \), from the definition of the geometric filtrations, we have \( G_p H_k(\tilde{X}_Y, \mathbb{Q}) \) and \( G_p H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q}) \).

Let \( b \in G_p H_k(\tilde{X}_Y, \mathbb{Q}) \), and \( a \) be the image of \( b \) under the the map \( \sigma_* : H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q}) \), i.e., \( \sigma_*(b) = a \). By assumption, there exists an element \( \tilde{a} \in L_{n-2} H_k(X) \otimes \mathbb{Q} \) such that \( \Phi_{n-2,k}(\tilde{a}) = a \). Since \( \sigma_* : L_{n-2} H_k(\tilde{X}_Y) \otimes \mathbb{Q} \rightarrow L_{n-2} H_k(X) \otimes \mathbb{Q} \) is surjective ([H]), there exists an element \( \tilde{b} \in L_{n-2} H_k(X) \otimes \mathbb{Q} \) such that \( \sigma_*(\tilde{b}) = \tilde{a} \). By the following commutative diagram

\[
\begin{array}{ccc}
L_{n-2} H_k(\tilde{X}_Y) \otimes \mathbb{Q} & \xrightarrow{\sigma_*} & L_{n-2} H_k(X) \otimes \mathbb{Q} \\
\downarrow \Phi_{n-2,k} & & \downarrow \Phi_{n-2,k} \\
H_k(\tilde{X}_Y, \mathbb{Q}) & \xrightarrow{=} & H_k(X, \mathbb{Q})
\end{array}
\]

we have \( \Phi_{n-2,k}(\tilde{b}) - b \) maps to zero in \( H_k(X, \mathbb{Q}) \). By the commutative diagram in Corollary 2.1, \( j^*(\Phi_{n-2,k}(\tilde{b}) - b) = 0 \in H_k^{BM}(U, \mathbb{Q}) \). From the exactness of the upper long exact sequence in Corollary 2.1, there exists an element \( c \in H_k(D, \mathbb{Q}) \) such that \( i_*(c) = \Phi_{n-2,k}(\tilde{b}) - b \). From Theorem 2.1, we find that \( \Phi_{n-2,k} : L_{n-2} H_k(D) \otimes \mathbb{Q} \rightarrow H_k(D) \otimes \mathbb{Q} \) is an isomorphism for \( k \geq 2n - 2 \). Hence there exists an element \( \tilde{c} \in L_{n-2} H_k(D) \otimes \mathbb{Q} \) such that \( i_*(\Phi_{n-2,k}(\tilde{c})) = \Phi_{n-2,k}(\tilde{b}) - b \). Therefore \( \Phi_{n-2,k}(\tilde{b} - i_*(\tilde{c})) = b \), i.e., the surjectivity of \( T_p H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow G_p H_k(\tilde{X}_Y, \mathbb{Q}) \).

On the other hand, we need to show

Case 2. If \( T_p H_k(\tilde{X}_Y, \mathbb{Q}) = G_p H_k(\tilde{X}_Y, \mathbb{Q}) \), then \( T_p H_k(X, \mathbb{Q}) = G_p H_k(X, \mathbb{Q}) \).

This part is relatively easy. By Theorem 1.4, we only need to consider the cases that \( k \geq 2p + 2 = 2n - 2 \). Let \( a \in G_p H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q}) \). From the blow up formula for singular homology (cf. [GH]), we know \( \sigma_* : H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q}) \) is surjective. Then there exists an element \( b \in H_k(\tilde{X}_Y, \mathbb{Q}) \) such that \( \sigma_*(b) = a \). By assumption, we can find an element \( \tilde{b} \in L_{n-2} H_k(\tilde{X}_Y, \mathbb{Q}) \) such that \( \Phi_{n-2,k}(\tilde{b}) = b \). Set \( \tilde{a} = \sigma_*(\tilde{b}) \). Then \( \Phi_{n-2,k}(\tilde{a}) = a \) under the natural map \( \Phi_{n-2,k} \). This is exactly the surjectivity we want.

This completes the proof for a blow-up along a smooth codimension at least two subvariety \( Y \) in \( X \).

\( \square \)

The proof of Theorem 1.3 (\( p = 1 \)):

The injectivity of the map \( T_1 H_k(W, \mathbb{Q}) \rightarrow G_1 H_k(W, \mathbb{Q}) \) has been proved for any smooth projective variety \( W \) by Friedlander and Mazur in [FM]. We only need to show the surjectivity under certain assumption.
Similar to the case $p = n - 2$, we also have two cases:

**Case A.** If $T_1 H_k(X, \mathbb{Q}) = G_1 H_k(X, \mathbb{Q})$, then $T_1 H_k(\tilde{X}_Y, \mathbb{Q}) = G_1 H_k(\tilde{X}_Y, \mathbb{Q})$.

From Theorem 1.4, the case where $k = 3$ holds for any smooth projective variety. We only need to consider the cases where $k \geq 4$.

Let $b \in G_1 H_k(\tilde{X}_Y, \mathbb{Q})$. Denote by $a$ the image of $b$ under the the map $\sigma_* : H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow H_0(X, \mathbb{Q})$, i.e., $\sigma_*(b) = a$. From the blow up formula for singular homology and the definition of the geometric filtration, we have $\sigma_*(G_1 H_k(\tilde{X}_Y, \mathbb{Q})) = G_1 H_k(X, \mathbb{Q})$.

Case B. $L_1 H_k(\tilde{X}_Y) \otimes \mathbb{Q} \rightarrow L_1 H_k(X) \otimes \mathbb{Q}$ is surjective ([H]), there exists an element $\tilde{a} \in L_1 H_k(\tilde{X}_Y) \otimes \mathbb{Q}$ such that $\sigma_*(\tilde{a}) = \tilde{a}$. By the following commutative diagram

$$
\begin{array}{ccc}
L_1 H_k(\tilde{X}_Y) \otimes \mathbb{Q} & \rightarrow & L_1 H_k(X) \otimes \mathbb{Q} \\
\downarrow \Phi_{1,k} & & \downarrow \Phi_{1,k} \\
H_k(\tilde{X}_Y, \mathbb{Q}) & \rightarrow & H_k(X, \mathbb{Q})
\end{array}
$$

we have $\Phi_{1,k}(\tilde{b}) - b$ maps to zero in $H_k(X, \mathbb{Q})$. By the commutative diagram in Corollary 2.1, $j^*(\Phi_{1,k}(\tilde{b}) - b) = 0 \in H_{k+1}^{BM}(U, \mathbb{Q})$. From the exactness of the upper long exact sequence in Corollary 2.1, there exists an element $c \in H_k(D, \mathbb{Q})$ such that $i_*(c) = \Phi_{1,k}(\tilde{b}) - b$. Set $\tilde{d} = \pi_*(c) \in H_k(Y, \mathbb{Q})$. By the commutative diagram in Corollary 2.1, $d$ maps to zero under $(i_0)_*: H_k(Y, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q})$. Hence there exists an element $e \in H_{k+1}(U, \mathbb{Q})$ such that whose image is $d$ under the boundary map $(\delta_0)_*$. Let $\tilde{d} \in H_k(D, \mathbb{Q})$ be the image of $e$ under this boundary map $\delta_* : H_{k+1}(U, \mathbb{Q}) \rightarrow H_k(D, \mathbb{Q})$. Therefore, the image of $c - \tilde{d}$ is zero under $\pi_* : H_k(Y, \mathbb{Q})$ and is also zero under $i_* : H_k(\tilde{X}_Y, \mathbb{Q})$. Note that $D$ is a bundle over $Y$ with projective spaces as fibers. From the “projective bundle theorem” for the singular homology (cf.[GH]), we have $H_k(D, \mathbb{Q}) \cong H_k(Y, \mathbb{Q}) \oplus H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y, \mathbb{Q})$. From this, we have $c - \tilde{d} \in H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y, \mathbb{Q})$. By the revised Projective Bundle Theorem ([FG] and [H]) the revised case essentially due to Complex Suspension Theorem ([L]) and Dold-Thom Theorem ([D]), we have $L_1 H_k(D, \mathbb{Q}) \cong L_1 H_k(Y, \mathbb{Q}) \oplus L_0 H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus L_{2-r} H_{k-2r+2}(Y, \mathbb{Q}) \cong L_1 H_k(Y, \mathbb{Q}) \oplus H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y, \mathbb{Q})$, where $r$ is the codimension of $Y$. Since $c - \tilde{d} \in H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y, \mathbb{Q})$ and $L_0 H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus L_{2-r} H_{k-2r+2}(Y, \mathbb{Q}) \cong H_{k-2}(Y, \mathbb{Q}) \oplus \cdots \oplus H_{k-2r+2}(Y, \mathbb{Q})$, there exists an element $f \in L_1 H_k(D, \mathbb{Q})$ such that $\Phi_{1,k}(f) = c - \tilde{d}$. Therefore we obtain $\Phi_{1,k}(\tilde{b} - i_*(f)) = b$. This is the surjectivity we need.

**Case B.** If $T_1 H_k(\tilde{X}_Y, \mathbb{Q}) = G_1 H_k(\tilde{X}_Y, \mathbb{Q})$, then $T_1 H_k(X, \mathbb{Q}) = G_1 H_k(X, \mathbb{Q})$.

This part is also relatively easy. Note that $k \geq 4$. Let $a \in G_1 H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$, then there exists an element $b \in G_1 H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\sigma_*(b) = a$. By assumption, we can find an element $\tilde{b} \in L_1 H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\Phi_{1,k}(\tilde{b}) = b$. Set $\tilde{a} = \sigma_*(\tilde{b})$. Then $\Phi_{1,k}(\tilde{a}) = a$ under the natural transformation $\Phi_{1,k}$. This is exactly the surjectivity in these cases.
This completes the proof for one blow-up along a smooth codimension at least two subvariety $Y$ in $X$. \hfill \Box

Now recall the weak factorization Theorem proved in [AKMW] (and also [W]) as follows:

**Theorem 2.2** ([AKMW] Theorem 0.1.1, [W]) Let $\varphi: X \to X'$ be a birational map of smooth complete varieties over an algebraically closed field of characteristic zero, which is an isomorphism over an open set $U$. Then $f$ can be factored as a sequence of birational maps

$$X = X_0 \varphi_1 \to X_1 \varphi_2 \to \cdots \varphi_{n+1} X_n = X'$$

where each $X_i$ is a smooth complete variety, and $\varphi_{i+1} : X_i \to X_{i+1}$ is either a blowing-up or a blowing-down of a smooth subvariety disjoint from $U$. \hfill \Box

**Remark 2.4** From the proof of the Theorem 1.3, we can draw the following conclusions:

1. If

$$T_r H_k(Y, Q) = G_r H_k(Y, Q)$$

for all $k$ is true for algebraic $r$-cycles with $r \geq p$ for $\dim(Y) = n$, then

"$T_{p-1} H_k(X, Q) = G_{p-1} H_k(X, Q), \ \forall k$"

is a birationally invariant statement for smooth projective varieties $X$ with $\dim(X) \leq n+2$.

2. If

$$T_r H_k(Y, Q) = G_r H_k(Y, Q)$$

for all $k$ is true for $r$-algebraic cycles with $r \leq p$ for $\dim(Y) = n$, then

"$T_{p+1} H_k(X, Q) = G_{p+1} H_k(X, Q), \ \forall k$"

is a birationally invariant statement for smooth projective varieties $X$ with $\dim(X) \leq n+2$.

3 **The Proof of the Theorem 1.4**

**Proposition 3.1** For any irreducible projective variety $Y$ of dimension $n$, we have

\[
\begin{cases}
L_{n-1} H_{2n}(X) \cong \mathbb{Z}, \\
L_{n-1} H_{2n-1}(X) \cong H_{2n-1}(X, \mathbb{Z}), \\
L_{n-1} H_{2n-2}(X) \to H_{2n-2}(X, \mathbb{Z}) \text{ is injective}, \\
L_{n-1} H_k(X) = 0 \text{ for } k > 2n.
\end{cases}
\]
Remark 3.1 When $Y$ is smooth projective, Friedlander have drawn a stronger conclusion, i.e., besides those in the proposition, $L_{n-1}H_{2n-2}(Y) \cong H_{n-1,n-1}(X, \mathbf{Z}) = NS(X)$.

Proof. Set $S = \text{Sing}(Y)$, the set of singular points. Then $S$ is the union of proper irreducible subvarieties. Set $S = (\cup_i S_i) \cup S'$, where $\dim(S_i) = n - 1$ and $S'$ is the union of subvarieties with dimension $\leq n - 2$. Let $V = Y - S$ be the smooth open part of $Y$. According to Hironaka [Hi], we can find $\tilde{V}$. Let $\tilde{V}$ be the smooth open part of $Y$. Denote by $i_0 : S \hookrightarrow Y$ and $i : D \hookrightarrow \tilde{Y}$ the inclusions of closed sets. Denote by $j_0 : V \hookrightarrow Y$ and $j : V \hookrightarrow \tilde{Y}$ the inclusions of open sets.

There are a few cases:

Case 1: $k \geq 2n$.

By the localization long exact sequence in Lawson homology

\[ \cdots \to L_{n-1}H_k(S) \to L_{n-1}H_k(Y) \to L_{n-1}H_k(V) \to L_{n-1}H_k(S) \to \cdots, \]

we have

\[ L_{n-1}H_k(Y) \cong L_{n-1}H_k(V) \quad \text{for} \quad k \geq 2n \]

since $L_{n-1}H_k(S) = 0$ for $k \geq 2n - 1$.

By the localization exact sequence in homology

\[ \cdots \to H_k(S) \to H_k(Y) \to H_k^{BM}(V) \to H_{k-1}(S) \to \cdots, \]

we have

\[ H_k(Y) \cong H_k^{BM}(V) \quad \text{for} \quad k \geq 2n \]

since $H_k(S) = 0$ for $k \geq 2n - 1$. Here $H_k^{BM}(V)$ is the Borel-Moore homology.

Similarly,

\[ L_{n-1}H_k(\tilde{Y}) \cong L_{n-1}H_k(V) \quad \text{for} \quad k \geq 2n \]

and

\[ H_k(\tilde{Y}) \cong H_k^{BM}(V) \quad \text{for} \quad k \geq 2n. \]

Since $\tilde{Y}$ is smooth, we have $L_{n-1}H_k(\tilde{Y}) \cong H_k(\tilde{Y})$ for $k \geq 2n$(cf. [F]). This completes the proof for the case $k \geq 2n$.

Case 2: $k = 2n - 1$.

Applying Lemma 2.3 to the pair $(Y, S)$ for $p = n - 1$, we have the commutative diagram of the long exact sequence

\[
\begin{array}{ccccccccc}
0 & \to & L_{n-1}H_{2n-1}(Y) & \overset{j_0^*}{\to} & L_{n-1}H_{2n-1}(V) & \overset{(\delta_0)^*}{\to} & L_{n-1}H_{2n-2}(S) & \overset{(i_0)^*}{\to} & L_{n-1}H_{2n-2}(Y) & \to & \cdots \\
\downarrow \Phi_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-1} & & \downarrow \Phi_{n-1,2n-2} & & \downarrow \Phi_{n-1,2n-2} & & \downarrow \Phi_{n-1,2n-2} & \\
0 & \to & H_{2n-1}(Y) & \overset{j_0^*}{\to} & H_{2n-1}^{BM}(V) & \overset{(\delta_0)^*}{\to} & H_{2n-2}(S) & \overset{(i_0)^*}{\to} & H_{2n-2}(Y) & \to & \cdots 
\end{array}
\]
Similarly, applying Lemma 2.3 to the pair $(\tilde{Y}, D)$ for $p = n - 1$, we have the commutative diagram of the long exact sequence

$$0 \to L_{n-1}H_{2n-1}(\tilde{Y}) \xrightarrow{j^*} L_{n-1}H_{2n-1}(V) \xrightarrow{\delta} L_{n-1}H_{2n-2}(D) \xrightarrow{i^*} L_{n-1}H_{2n-2}(\tilde{Y}) \to \cdots$$

\[
\downarrow \Phi_{n-1,2n-1} \quad \downarrow \Phi_{n-1,2n-1} \quad \downarrow \Phi_{n-1,2n-2} \quad \downarrow \Phi_{n-1,2n-2} \\
0 \to H_{2n-1}(\tilde{Y}) \xrightarrow{j^*} H_{2n-1}^B(V) \xrightarrow{\delta} H_{2n-2}(D) \xrightarrow{i^*} H_{2n-2}(\tilde{Y}) \to \cdots
\] (2)

Note that $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(\tilde{Y}) \to H_{2n-2}(\tilde{Y})$ is injective, $\Phi_{n-1,2n-1} : L_{n-1}H_{2n-1}(\tilde{Y}) \cong H_{2n-1}(\tilde{Y})$ and $\Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(D) \cong H_{2n-2}(D) \cong \mathbb{Z}^m$, where $m$ is the number of irreducible varieties of $D$. From (2) and the Five Lemma, we have the isomorphism

$$\Phi_{n-1,2n-1} : L_{n-1}H_{2n-1}(V) \cong H_{2n-1}^B(V).$$

(3)

From (1), (3) and the Five Lemma, we have the following isomorphism

$$\Phi_{n-1,2n-1} : L_{n-1}H_{2n-2}(Y) \cong H_{2n-2}(Y).$$

**Case 3:** $k = 2n - 2$.

Now the commutative diagram (1) is rewritten in the following way:

$$\cdots \to L_{n-1}H_{2n-1}(V) \xrightarrow{\delta^0} L_{n-1}H_{2n-2}(S) \xrightarrow{(i_0)^*} L_{n-1}H_{2n-2}(Y) \xrightarrow{j^*} L_{n-1}H_{2n-2}(V) \to 0$$

\[
\downarrow \Phi_{n-1,2n-1} \quad \downarrow \Phi_{n-1,2n-2} \quad \downarrow \Phi_{n-1,2n-2} \quad \downarrow \Phi_{n-1,2n-2} \\
\cdots \to H_{2n-1}^B(V) \xrightarrow{\delta^0} H_{2n-2}(S) \xrightarrow{(i_0)^*} H_{2n-2}(Y) \xrightarrow{j^*} H_{2n-2}^B(V) \to 0
\] (4)

In the commutative diagram (2), we can show that the injective maps

$$j^* : H_{2n-1}(\tilde{Y}) \to H_{2n-1}^B(V)$$

(5)

and

$$j^* : L_{n-1}H_{2n-1}(\tilde{Y}) \to L_{n-1}H_{2n-1}(V)$$

(6)

are actually isomorphisms. Hence the commutative diagram (2) reduces to the following diagram:

$$0 \to L_{n-1}H_{2n-2}(D) \to L_{n-1}H_{2n-2}(\tilde{Y}) \to L_{n-1}H_{2n-2}(V) \to 0$$

\[
\downarrow \Phi_{n-1,2n-2} \quad \downarrow \Phi_{n-1,2n-2} \quad \downarrow \Phi_{n-1,2n-2} \\
0 \to H_{2n-2}(D) \to H_{2n-2}(\tilde{Y}) \to H_{2n-2}^B(V) \to 0
\] (7)
To see (5) are surjective, by the exactness of the rows in (2) we only need to show that the maps \( i_* : H_{2n-2}(D) \to H_{2n-2}(Y) \) are injective. Note that \( Y \) is a compact Kähler manifold, and the homology class of an algebraic subvariety is nontrivial in the homology of the Kähler manifold. From these, we get the injectivity of \( i_* \). The surjectivity of (6) follows from the same reason.

We need the following lemma.

**Lemma 3.1** The natural transformation \( \Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(V) \to H_{2n-2}^{BM}(V) \) is injective.

**Proof.** Let \( a \in L_{n-1}H_{2n-2}(V) \) such that \( \Phi_{n-1,2n-2}(a) = 0 \in H_{2n-2}^{BM}(V) \). Since the map \( j^* : L_{n-1}H_{2n-2}(Y) \to L_{n-1}H_{2n-2}(V) \) is surjective, there exists an element \( b \in L_{n-1}H_{2n-2}(Y) \) such that \( j^*(b) = a \). Set \( \tilde{b} = \Phi_{n-1,2n-2}(b) \in H_{2n-2}(Y) \). By the commutativity of the diagram, we have \( j^*(\tilde{b}) = 0 \) under the map \( j^* : H_{2n-2}(Y) \to H_{2n-2}^{BM}(V) \). By the exactness of the bottom row in the commutative diagram (7), there exists an element \( \tilde{c} \in H_{2n-2}(D) \) such that the image of \( \tilde{c} \) under the map \( i_* : H_{2n-2}(D) \to H_{2n-2}(Y) \) is \( \tilde{b} \). Now note that \( \Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(D) \to H_{2n-2}(D) \) is an isomorphism, there exists an element \( c \in L_{n-1}H_{2n-2}(D) \) such that \( \Phi_{n-1,2n-2}(c) = \tilde{c} \). Hence \( \Phi_{n-1,2n-2}(i_*(c) - b) = 0 \). Note that \( \Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(Y) \to H_{2n-2}(Y) \) is injective since \( Y \) is smooth and of dimension \( n \) (cf. [F1]). Hence we get \( i_*(c) = b \), i.e., \( b \) is in the image of the map \( i_* : L_{n-1}H_{2n-2}(D) \to L_{n-1}H_{2n-2}(Y) \). Therefore \( a = 0 \) by the exactness of the top row of the commutative diagram (7).

\( \square \)

We need to show that \( \Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(Y) \to H_{2n-2}(Y) \) is injective. For \( a \in L_{n-1}H_{2n-2}(Y) \) such that \( \Phi_{n-1,2n-2}(a) = 0 \in H_{2n-2}(Y) \). By the commutative diagram (4) and the Lemma 3.1, the image of \( a \) under \( j_0^* : L_{n-1}H_{2n-2}(Y) \to L_{n-1}H_{2n-2}(V) \) is zero. Hence there exists an element \( b \in L_{n-1}H_{2n-2}(S) \) such that the image of \( (i_0)_* : L_{n-1}H_{2n-2}(S) \to L_{n-1}H_{2n-2}(Y) \) is \( a \), i.e., \( (i_0)_*(b) = a \). Set \( \tilde{b} = \Phi_{n-1,2n-2}(b) \). Then the image of \( \tilde{b} \) under the map \( (i_0)_*: H_{2n-2}(S) \to H_{2n-2}(Y) \) is zero. By exactness of the bottom row in the commutative diagram (4), there exists an element \( \tilde{c} \) such that its image under the map \( H_{2n-2}^{BM}(V) \to H_{2n-2}(S) \) is \( \tilde{b} \). By the result in Case 2, \( \Phi_{n-1,2n-1} : L_{n-1}H_{2n-1}(V) \to H_{2n-1}^{BM}(V) \) is an isomorphism. Hence there exists an element \( c \in L_{n-1}H_{2n-1}(V) \) such that \( \Phi_{n-1,2n-1}(c) = \tilde{c} \). Now since \( \Phi_{n-1,2n-2} : L_{n-1}H_{2n-2}(S) \to H_{2n-2}(S) \) is an isomorphism, the image of \( c \) under the map \( L_{n-1}H_{2n-1}(V) \to L_{n-1}H_{2n-2}(S) \) is exactly \( b \). Now the exactness of the top row of the commutative diagram (4) implies the vanishing of \( a \).

The proof of the proposition is done.

\( \square \)

By using this proposition, we will give a proof of Theorem 1.4.

**Proof of Theorem 1.4:**
For any smooth projective variety $X$, the injectivity of $T_pH_{2p+1}(X, \mathbb{Q}) \to G_pH_{2p+1}(X, \mathbb{Q})$ has been proved in \cite{FM}, §7. We only need to show the surjectivity of $T_pH_{2p+1}(X, \mathbb{Q}) \to G_pH_{2p+1}(X, \mathbb{Q})$. For any subvariety $i : Y \subset X$, we denote by $V =: X - Y$ the complementary of $Y$ in $X$. We have the following commutative diagram of the long exact sequences (Lemma 2.3, or \cite{Li}):

$$
\cdots \to L_pH_{2p+1}(Y) \to L_pH_{2p+1}(X) \to L_pH_{2p+1}(V) \to L_pH_{2p}(Y) \to \cdots
$$

$$
\downarrow \Phi_{p,2p+1} \quad \downarrow \Phi_{p,2p+1} \quad \downarrow \Phi_{p,2p+1} \quad \downarrow \Phi_{p,2p}
$$

$$
\cdots \to H_{2p+1}(Y) \to H_{2p+1}(X) \to H_{2p+1}^B(V) \to H_{2p}(Y) \to \cdots
$$

Obviously, the above commutative diagram holds when tensored with $\mathbb{Q}$. In the following, we only consider the commutative diagrams with $\mathbb{Q}$-coefficient.

Now let $a \in G_pH_{2p+1}(X, \mathbb{Q})$, by definition, we can assume that $a$ lies in the image of the map $i_* : H_{2p+1}(Y, \mathbb{Q}) \to H_{2p+1}(X, \mathbb{Q})$ for some subvariety $Y \subset X$ with dimension $\dim Y = (2p + 1) - p = p + 1$. Hence there exists an element $b \in H_{2p+1}(Y, \mathbb{Q})$ such that $i_*(b) = a$. By the Proposition 3.1, we know that $\Phi_{p,2p+1} : L_pH_{2p+1}(Y) \otimes \mathbb{Q} \to H_{2p+1}(Y, \mathbb{Q})$ is an isomorphism. Therefore there exists an element $\tilde{b} \in L_pH_{2p+1}(Y) \otimes \mathbb{Q}$ such that $\Phi_{p,2p+1}(\tilde{b}) = b$. Set $\tilde{a} = i_*(\tilde{b})$. Then $\tilde{a}$ maps to $a$ under the map $L_pH_{2p+1}(X) \otimes \mathbb{Q} \to H_{2p+1}(X, \mathbb{Q})$. By the definition of the topological filtration, $a \in T_pH_{2p+1}(X, \mathbb{Q})$. This completes the proof of surjectivity of $T_pH_{2p+1}(X, \mathbb{Q}) \to G_pH_{2p+1}(X, \mathbb{Q})$.

\[\square\]

**Remark 3.2** In the proof of the surjectivity of Theorem 1.4, the assumption of smoothness is not necessary, more precisely, for any irreducible projective variety $X$, the image of the natural transformation $\Phi_{p,2p+1} : L_pH_{2p+1}(X, \mathbb{Q}) \to H_{2p+1}(X, \mathbb{Q})$ contains $G_pH_{2p+1}(X, \mathbb{Q})$.

**Remark 3.3** Independently, M. Warker has recently also obtained this result (\cite{Wa}, Prop. 2.5).

Now we prove the corollaries 1.2-1.5.

**The proof of Corollary 1.1:** By Theorem 1.1 and 1.4, Dold-Thom Theorem and Proposition 3.1, we only need to show the cases that $p = 1, k \geq 5$. Now the following commutative diagram (\cite{FM}, Prop.6.3)

$$
L_2H_k(X) \otimes \mathbb{Q} \to L_1H_k(X) \otimes \mathbb{Q}
$$

$$
\downarrow \Phi_{2,k} \quad \downarrow \Phi_{1,k}
$$

$$
H_k(X, \mathbb{Q}) \cong H_k(X, \mathbb{Q}).
$$

shows that if $L_2H_k(X) \otimes \mathbb{Q} \to H_k(X, \mathbb{Q})$ is a surjective, then $L_1H_k(X) \otimes \mathbb{Q} \to H_k(X, \mathbb{Q})$ must be surjective. Proposition 3.1 gives the needed surjectivity for $k \geq 5$ even if $X$ is singular variety of dimension 3.

\[\square\]
The proof of Corollary 1.2: By Corollary 1.1, we only need to show that $T_1H_4(X, \mathbb{Q}) = G_1H_4(X, \mathbb{Q})$. By the assumption and Poincaré duality, $H_4(X, \mathbb{Q}) \cong H_2(X, \mathbb{Q}) \cong \mathbb{Q}$. Therefore, $G_1H_4(X, \mathbb{Q}) = H_4(X, \mathbb{Q}) \cong \mathbb{Q}$ and again by the commutative diagram

$$
\begin{array}{ccc}
L_2H_k(X) \otimes \mathbb{Q} & \rightarrow & L_1H_k(X) \otimes \mathbb{Q} \\
\downarrow \Phi_{2,k} & & \downarrow \Phi_{1,k} \\
H_k(X, \mathbb{Q}) & \cong & H_k(X, \mathbb{Q}),
\end{array}
$$

we have the surjectivity of $L_1H_4(X) \otimes \mathbb{Q} \rightarrow H_4(X, \mathbb{Q})$.

The proof of Corollary 1.3: Suppose $X = S \times C$, where $S$ is a smooth projective surface and $C$ is a smooth projective curve. We only need to consider the surjectivity of $L_1H_4(X) \otimes \mathbb{Q} \rightarrow H_4(X, \mathbb{Q})$ because of Corollary 1.1. Now the Künneth formula for the rational homology of $H_4(S \times C, \mathbb{Q})$ and Theorem 2.1 for $S$ and $C$ gives the surjectivity in this case.

The proof of Corollary 1.4: This follows directly from Theorem 1.3.

The proof of Corollary 1.5: By Theorem 1.4, we only need to show that $T_pH_k(X, \mathbb{Q}) = G_pH_k(X, \mathbb{Q})$ for $k \geq 2p + 2$. By the definition of geometric definition, an element $a \in G_pH_k(X, \mathbb{Q})$ comes from the linear combination of elements $b_j \in H_k(Y_j, \mathbb{Q})$ for subvarieties $Y_j$ of $\dim(Y_j) \leq k - p$. From the following commutative diagram

$$
\begin{array}{ccc}
i_* : L_pH_k(Y) \otimes \mathbb{Q} & \rightarrow & L_pH_k(X) \otimes \mathbb{Q} \\
\downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} \\
i_* : H_k(Y, \mathbb{Q}) & \rightarrow & H_k(X, \mathbb{Q}),
\end{array}
$$

it is enough to show that $L_pH_k(Y) \rightarrow H_k(Y)$ is surjective for any irreducible subvariety $Y \subset X$ with $\dim(Y) = k - p$. By Suslin’s conjecture, this is true for any smooth variety $Y$ since $\dim(Y) = k - p$. Now we need to show that it is also true for singular irreducible varieties if the Suslin Conjecture is true.

Using induction, we will show the following lemma:

**Lemma 3.2** If the Suslin Conjecture is true for every smooth projective variety, then it is also true for every quasi-projective variety.

**Proof.** Suppose that $Y$ is an irreducible quasi-projective variety with $\dim(Y) = m$, $S$ is an irreducible quasi-projective variety with $\dim(S) = n < m$ and

$$
\begin{align*}
\left\{ \begin{array}{l}
L_pH_{n+p-1}(S) \rightarrow H_{n+p-1}(S) \text{ is injective}, \\
L_pH_{n+q}(S) \cong H_{n+q}(S) \text{ for } q \geq p.
\end{array} \right.
\end{align*}
$$

Denote by $\overline{Y}$ a projective closure of $Y$ and $S = \text{sing}(\overline{Y})$ the singular point set of $\overline{Y}$. Let $U = \overline{Y} - S$ Let $\sigma : \overline{Y} \rightarrow \overline{Y}$ be a desingularization of $\overline{Y}$ and denote by $D := \overline{Y} - U$.  

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The existence of a smooth $\tilde{Y}$ is guaranteed by Hironaka [Hi]. Then $D$ is the union of irreducible varieties with dimension $\leq m - 1$.

By Lemma 2.3, we have the following commutative diagram

$$
\cdots \rightarrow L_pH_k(Z) \rightarrow L_pH_k(V) \rightarrow L_pH_k(U) \rightarrow L_pH_{k-1}(Z) \rightarrow \cdots \\
\downarrow \Phi_{p,k} \downarrow \Phi_{p,k} \downarrow \Phi_{p,k} \downarrow \Phi_{p,k-1} \\
\cdots \rightarrow H_k(Z) \rightarrow H_k(V) \rightarrow H_{BM}^k(U) \rightarrow L_pH_{k-1}(Z) \rightarrow \cdots,
$$

where $U \subset V$ are quasi-projective varieties of $\dim(V) = \dim(U) = m$ and $Z = V - U$ is a closed subvariety of $V$.

Claim: By inductive assumption, the above commutative diagram and the Five Lemma, we have the equivalence between

$$
\left\{ \begin{array}{l}
L_pH_{m+p-1}(U) \rightarrow H_{m+p-1}(U) \text{ is injective,} \\
L_pH_{m+q}(U) \cong H_{m+q}(U) \text{ for } q \geq p.
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{l}
L_pH_{m+p-1}(V) \rightarrow H_{m+p-1}(V) \text{ is injective,} \\
L_pH_{m+q}(V) \cong H_{m+q}(V) \text{ for } q \geq p.
\end{array} \right.
$$

The proof of the claim is obvious.

By using the claim for finite times beginning from $V = \tilde{Y}$, we have the result for any quasi-projective variety $U$. The proof of Lemma 3.2 is done.

By Lemma 3.2, we know that the Suslin’s Conjecture is also true for singular varieties. This completes the proof of Corollary 1.4.

\[\Box\]

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