Quantum Computers and Dissipation

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Abstract

We analyse dissipation in quantum computation and its destructive impact on efficiency of quantum algorithms. Using a general model of decoherence we study the time evolution of a quantum register of arbitrary length coupled with an environment of arbitrary coherence length. We discuss relations between decoherence and computational complexity and show that the quantum factorisation algorithm must be modified in order to be regarded as efficient and realistic.

1 Introduction

Quantum computers can accept input states which represent a coherent superposition of many different possible inputs and subsequently evolve them into a corresponding superposition of outputs. Computation, i.e. a sequence of unitary transformations, affects simultaneously each element of the superposition generating a massive parallel data processing albeit within one piece of quantum hardware. As the result quantum computers can efficiently solve some problems which are believed to be intractable on any classical computer (Deutsch 1985, Deutsch and Jozsa 1992, Bernstein and Vazirani 1993, Simon 1994, Shor 1994). The most striking example is the factoring problem: to factor a number $N$ of $L$ digits on any classical computer requires an execution time that grows exponentially with $L$ (approximately $\exp L^{1/3}$ for the best known algorithms such as the Number Field Sieve (Lenstra et al. 1990)). In contrast Shor (1994) has shown that quantum
computers require an execution time that grows only as a polynomial function of $L$ ($\approx L^2$). A noteworthy consequence for cryptology is the possibility of breaking public key cryptosystems such as RSA (Rivest et al. 1979). In this context a practical implementation of quantum computation is a most important issue.

This paper shows that Shor’s algorithm, in order to be efficient from the computational complexity perspective, must be modified (i.e. supplemented with an exponentially efficient quantum error correction). This is because the error rate, which is due to the interaction with the environment (a thermal reservoir), grows as an exponential function of $L$ regardless the type and the strength of the computer - environment coupling. Consequently, in order to obtain factors of $N$ with a prescribed probability of success one has to repeat the computation $k$ times, where $k$ grows exponentially with $L$. Let us stress that this observation is not related to any technological limitations but is a consequence of the computational complexity requirements imposed on any realistic model of computation. Any technological progress in suppressing dissipation will, of course, help to factor bigger and bigger numbers but will not change an exponential increase of the error rate.

Our analysis is quite general and covers cases where qubits interact with a single reservoir and the distance between the qubits is comparable with the correlation length of the reservoir. In the limit of a short correlation length (i.e. coupling to the independent reservoirs) our results coincide with those by Unruh (1995). Also we provide a plausibility argument that any model of dissipation will lead to errors that grow exponentially with $L$.

The structure of the paper is as follows: Section 2 links the computational complexity of randomized algorithms with dissipation, Section 3 outlines basic features of the dissipation model we adopted for this paper (the model is known under the label “decoherence”), Section 4 explains details of decoherence of a single qubit using both semi-classical and fully quantised approaches, Section 5 extends the fully quantum analysis to the two qubit case first and then to the general case of an $L$-qubit register. The results are summarised and discussed in the last section.

2 Errors in Randomized Algorithms

An algorithm is said to be efficient if its execution time increases no faster than a polynomial function of the size of the input. For example, any efficient factoring algorithm will factor a number $N$ of size $L = \log N$ (the base of the logarithm is 10 when the size is measured in decimal digits and 2 when it is measured in binary digits) in time proportional to a polynomial function of $L$. Algorithms do not have to be deterministic; random decisions can be taken at certain junctures in the execution of the algorithm. The so-
called randomized algorithms may sometimes produce an incorrect solution, however; they have the property that the probability of error can be made arbitrarily small.

Consider, for example, an efficient factoring algorithm which runs successfully only with probability $1 - \epsilon$ ($0 < \epsilon < 1$). Clearly we know when the algorithm is successful because we can check it efficiently by a trial division. Assuming that $\epsilon$ is independent of the input the success probability of the algorithm can be amplified (boosted) arbitrarily close to 1 by repeating the computation several times. After $k$ runs the probability of having at least one success is

$$P(k, \epsilon) = 1 - \epsilon^k. \quad (1)$$

If we fix $P(k, \epsilon)$ (i.e. when we agree upon the desired probability of successful factorisation) then the number of runs $k$ depends on the error rate $\epsilon$. In order to retain efficiency of the algorithm we may let $\epsilon$ increase with $L$ but not faster than a polynomial. Indeed, Shor’s quantum factoring algorithm is of this type; in a single run it provides factors of $N$ with probability $1 - \epsilon(L)$ and the probability of error $\epsilon(L)$ grows only as a polynomial function of $L$. However, errors in Shor’s algorithm are of purely “mathematical” origin and correspond to the random decisions which rely on the prime number theorem (i.e. on the distribution of prime numbers). In a realistic scenario the probability of error $\epsilon(L)$ has also contributions which are due to the computer - environment coupling. From the computational complexity point of view it is irrelevant how weak this coupling is for a fixed $L$, it is its dependence on $L$ which matters. If the interaction with the environment causes an exponential increase of the error rate i.e. $\epsilon(L) = 1 - A \exp(-\alpha L)$, where $A$ and $\alpha$ are positive constants, then $k(L) \approx \exp(\alpha L)$ and the randomized algorithm cannot be regarded as efficient any more!

Indeed, our results show that the computer-environment interaction leads to this unwelcome exponential increase of the error rate. Shor’s algorithm, as it stands at the moment, is not robust to the environmental noise and requires modifications. This statement, however, should not be taken as a fundamental argument against possibilities of experimental implementation of quantum factorisation. It simply shows that further efforts, both theoretical (e.g. designing an exponentially efficient quantum error correction) and experimental (e.g. customizing the environmental noise), are necessary to make quantum computation practical.

## 3 Decoherence

Let us consider a quantum register composed of $L$ qubits; each qubit being a two-state quantum system described by a two-dimensional Hilbert space.
$\mathcal{H}_2$ with computational basis states labeled as $|0\rangle$ and $|1\rangle$. The $2^L$ dimensional Hilbert space of the whole register is an $L$ term tensor product of $\mathcal{H}_2$ pertaining to each qubit. Any quantum state of the register can be described by a density operator of the form

$$\rho(t) = \sum_{i,j=0}^{2^L-1} \rho_{ij}(t) |i\rangle \langle j|,$$  \hspace{1cm} (2)

where the computational basis $|i\rangle$ is defined as a tensor product of the qubit basis states:

$$|i\rangle = |i_{L-1}\rangle \otimes |i_{L-2}\rangle \otimes \ldots \otimes |i_0\rangle.$$  \hspace{1cm} (3)

The r.h.s. is the binary decomposition of number $i = \sum_{i=0}^{L-1} 2^l i_l$, e.g. for a two qubit register $|2\rangle = |1\rangle \otimes |0\rangle$.

Quantum computation takes its power from quantum interference and entanglement. The degree of the interference and entanglement in an $L$ qubit register is quantified by the coherences i.e. the off-diagonal elements of the density operator in the computational basis $\rho_{ij} (i \neq j)$. When a quantum computer is in contact with an environment, here modeled as a thermal reservoir, the resulting dissipation destroys the coherences and changes the populations (the diagonal elements). When a reservoir is in a thermal state the coherences change on a much shorter time scale than populations and it is decoherence which is responsible for destroying quantum parallelism. However, it should be mentioned that in some cases populations can be more fragile than coherences. The time scales depend on the type of the system-environment coupling and on the quantum state of the reservoir (which does not have to be thermal), for example, in some optically driven systems coherences change more slowly than populations (Kimble & Mabuchi 1995).

The notion of decoherence can be intuitively understood as follows. Consider a computer and a reservoir, initially both in pure states:

$$|\Psi(0)\rangle = \left(\sum_i c_i(0) |i\rangle\right) \otimes |\alpha_0\rangle.$$  \hspace{1cm} (4)

A unitary evolution of the composed system results in an entangled computer-reservoir state which can be written as

$$|\Psi(t)\rangle = \sum_i c_i(t) |i\rangle \otimes |\alpha_i(t)\rangle,$$  \hspace{1cm} (5)
where, in general, $\langle \alpha_i | \alpha_j \rangle \neq 0$ for $i \neq j$. The elements of the register density matrix evolve as

$$\rho_{ij}(0) = c_i(0)c_j^*(0) \rightarrow \rho_{ij}(t) = c_i(t)c_j^*(t)\langle \alpha_i(t) | \alpha_j(t) \rangle.$$  \hfill (6)

In a popular model of decoherence (Zurek 1991), where the environment effectively acts as a measuring apparatus, the evolution affects the reservoir states $\{ | \alpha_i \rangle \}$ which become more and more orthogonal to each other whilst the coefficients $\{ c_i \}$ remain unchanged. Consequently the off-diagonal elements $\rho_{ij}$ disappear due to the $\langle \alpha_i(t) | \alpha_j(t) \rangle$ factors. We will analyse this type of decoherence in detail in the following sections.

The characteristic time for the off-diagonal elements to disappear is known as the decoherence time. Its value depends on the type of qubits and their interaction with the environment and can vary from $10^4$ s for nuclear spins in a paramagnetic atom to $10^{-12}$ s for electron-hole excitations in a bulk of a semiconductor (for a table of decoherence times for various physical systems see for example DiVincenzo 1995).

## 4 Single Qubit Dephasing Mechanism

In this section we want to describe in detail the basic features of the decoherence model we adopted for this paper. For this purpose it suffices to analyse an interaction between a single qubit and the reservoir. The case involving two and more qubits will be analysed in the subsequent sections.

The density operator for a single qubit has the form

$$\rho(t) = \sum_{i,j=0}^1 \rho_{ij}(t) | i \rangle \langle j |,$$  \hfill (7)

and we are interested in the time evolution of the coherence $\rho_{01}(t) = \rho_{10}^*(t)$.

### 4.1 Semiclassical Picture

In a semiclassical analysis of the single qubit decoherence it is assumed that the reservoir is a fluctuating classical magnetic field. This approach excludes many interesting quantum features, most notably the qubit-environment entanglement; however, we have decided to include it here because it can simplify some calculations.

We describe our physical qubit as a fictitious spin 1/2 system following the standard correspondence between the density operator $\rho$ and the spin
vector \( \vec{s} \), also referred to as the Bloch vector (Allen & Eberly 1975, Cohen-Tannoudji et al 1977)

\[
\rho(t) = \frac{1}{2} (1 + \vec{s}(t) \cdot \vec{\sigma}) = \frac{1}{2} (1 + \sum_{a=x,y,z} s_a(t) \sigma_a),
\]

where \( \sigma_a, a = x, y, z \) are the Pauli spin matrices. Components \( s_x \) and \( s_y \) correspond respectively to the real and imaginary parts of \( \rho_{01} \), and \( s_z \), the so called population inversion, is the difference between the two diagonal terms \( \rho_{11} - \rho_{00} \). We are mainly interested in \( s_x \) and \( s_y \) because they describe the internal coherence of the qubit.

Consider the time evolution of \( \vec{s} \) in the presence of an external magnetic field \( B_0 \), which points in the \( z \) direction. We label the two eigenstates of the \( \sigma_z \) spin operator with \( |0\rangle \) (spin down) and \( |1\rangle \) (spin up). These eigenstates are separated by an energy gap \( \omega_0 = g B_0 \), where \( g \) is the gyromagnetic ratio. Hence the spin vector rotates along the \( z \) axis with an angular frequency \( \omega_0 \) (see Fig. 1).

Let us suppose that the external noise can be modeled as a stochastic time dependent field \( B_z(t) \). Then the \( xy \) component of the spin vector, i.e. \( s_x \vec{e}_x + s_y \vec{e}_y \), rotates around the \( z \) axis with an instantaneous angular frequency \( \omega_0 + \Omega(t) \), where \( \Omega(t) = g B_z(t) \). After time \( t \) the average phase factor acquired by \( s_x \) and \( s_y \) due to the presence of external noise becomes \( \langle \exp \{ i \int_0^t \Omega(t')dt' \} \rangle \), where the average is to be taken over all the possible field configurations.

In order to evaluate the average over the field configurations it is convenient to decompose the stochastic field into a Fourier series as follows:

\[
B_z(t) = \sum_k b_k^* e^{i \omega_k t} + b_k e^{-i \omega_k t} \tag{9}
\]

where \( b_k, b_k^* \) are Gaussian time independent random variables. This allows us to write

\[
\langle \exp \{ i \int_0^t \Omega(t')dt' \} \rangle = \prod_k \langle \exp \{ -i g \int_0^\infty b_k e^{i \omega_k t'} dt' + c.c. \} \rangle \tag{10}
\]

which shows how the overall phase factor can be expressed as a product of independent contributions from each of the Fourier component of the fluctuating field. These contributions add incoherently and destroy the spin coherences.

The process of decoherence described above is similar to what is known as \( T_2 \) decay in nuclear magnetic resonance (Allen & Eberly 1975). This is to be distinguished from \( T_1 \) decay, which implies a damping of the spin energy into the field. Our model does not describe \( T_1 \) processes.
To illustrate the destruction of coherences in the semiclassical picture we simulate the energy fluctuations by explicitly introducing the random energy fluctuations $B_z(t)$ while calculating the time evolution of $\vec{s}$. We do this numerically in discrete time steps $\Delta t$. For each time step we select a random number $\kappa$ between 0 and 1. If $\kappa < 0.1$ then $B_z(t + \Delta t) = B_z(t) + B'$, where $B'$ is some constant field. Correspondingly, if $\kappa > 0.9$ then $B_z(t + \Delta t) = B_z(t) - B'$. Otherwise $\vec{B}$ remains unchanged. We scale our variables dimensionless, set $B' = 0.1B_0$ and set $g = 1$. In figure 2 we show examples of such simulations. In a more realistic model the size of fluctuations should be randomized as well, but this simple model with fixed-size fluctuations is good enough for our illustration purposes.

Each stochastic calculation of time evolution will produce a practically unique history for $\vec{s}$. These histories form an ensemble, and if we average over its all possible members, we obtain a statistical average which then corresponds to the one shown in (10). In figure 2(a) and 2(b) we show two examples of single ensemble members, and in figure 2(c) we show an ensemble average of 100 such random ensemble members. We see that as the fluctuations average out, the changes introduced to the coherences $s_x$ and $s_y$ remain. And as seen in figure 2(d), with 500 members in the ensemble, they eventually disappear altogether as $t$ becomes large, i.e., $t \gg \omega_0^{-1}$.

Our semiclassical simulations for a single quantum bit demonstrate how the full statistical result is obtained by adding the individual members of the ensemble. It should be noted that in the time evolution of a single member of the ensemble the amplitude for the oscillations in $s_x$ and $s_y$ remains constant. Hence there is no real decay of the coherences present in a single ensemble member, but merely a shift in phase.

In the semiclassical picture it is possible to regard a single member of the ensemble as a fully physical entity. It could describe e.g. collisionally induced energy fluctuations in qubits formed by ions in a trap; these arise when the atoms of the background gas collide with the ions and perturb the ionic energy level structure. However, when we consider the effect of the thermal and vacuum fluctuations, such a semiclassical view on the interaction of the qubit with the reservoir is not adequate, and one has to apply a fully quantum approach, which we shall present next.

### 4.2 Fully Quantised Model

The above description of our decoherence mechanism can be rigorously formulated in quantum terms. The quantised environment is modeled as a continuum of field modes. The state of the combined system i.e. qubit + environment is described by a density operator $\rho(t)$ which at time $t = 0$ is
assumed to be in the form

$$\rho(0) = \rho(0) \otimes \prod_k R_{kT},$$  \hspace{1cm} (11)$$

where $R_{kT}$ is the thermal density matrix of the $k$ mode of the field and we have taken advantage of the fact that the density operator of the field in thermal equilibrium factorizes into the tensor product of the density operator of each of its modes.

We assume that the dynamics of a single qubit + environment is described by the following Hamiltonian

$$H = \frac{1}{2} \sigma_z \omega_0 + \sum_k b_k^\dagger b_k \omega_k + \sum_k \sigma_z (g_k b_k^\dagger + g_k^* b_k)$$  \hspace{1cm} (12)$$

where the first and the second term on the r.h.s. describe respectively the free evolution of the qubit and the environment, and the third term describes the interaction between the two ($b_k, b_k^\dagger$ are now creation and annihilation field operators, here and in the following we put $\hbar = 1$). Hamiltonian (12) is equivalent to the one introduced in connection with the tunnelling problem (Leggett et al. 1987, Gardiner 1991) and used to model decoherence in quantum computers (Unruh 1995).

Since $[\sigma_z, H] = 0$ the populations of the qubit density matrix, $\rho(t) = Tr_\rho(t)$ are not affected by the environment and the coupling with the environment in our model simply erodes quantum coherence. This means that also in our fully quantised model there is no exchange of energy between qubit and environment and consequently no $T_1$ type of decay takes place. This is not a limitation of the model since, as will became clear in the following, it describes adequately the decoherence mechanism which usually takes place on a shorter timescale than the energy dissipation. Furthermore this model is exactly soluble and allows a clear analysis of the mechanism of entanglement between qubit and environment which is believed to be at the core of most decoherence processes.

We will postpone a detailed discussion on the form of the coupling $g_k$ between qubit and field modes which will depend on the specific characteristics of the physical system.

In order to study the time evolution of our system it will be convenient to move to the interaction picture, where the time evolution operator takes the form

$$U(t) = \exp \left\{ -i \int_0^t \sum_k \sigma_z \left( g_k b_k^\dagger e^{i\omega_k t'} + g_k^* b_k e^{-i\omega_k t'} \right) dt' \right\}$$  \hspace{1cm} (13)$$
\[
\sigma_2 \frac{1}{2} \sum_k \left( b_k^\dagger \xi_k(t) - b_k \xi_k^*(t) \right)
\]

with

\[
\xi_k(t) = 2g_k \frac{1 - e^{i\omega_k t}}{\omega_k}.
\] (14)

Here \( U(t) \) can be described as a conditional displacement operator for the field, the sign of the displacement being dependent on the logical value of the qubit. In particular for any pure state \( |\Psi\rangle \) of the field

\[
U(t) |0\rangle \otimes |\Psi\rangle = |0\rangle \otimes \prod_k D(- \frac{1}{2} \xi_k(t)) |\Psi\rangle
\]

\[
U(t) |1\rangle \otimes |\Psi\rangle = |1\rangle \otimes \prod_k D(+ \frac{1}{2} \xi_k(t)) |\Psi\rangle
\] (15)

where the displacement operator \( D(\xi_k) \) is defined as

\[
D(\xi_k) = \exp \left\{ b_k^\dagger \xi_k - b_k \xi_k^* \right\}.
\] (16)

The above discussion makes it clear that \( U(t) \) induces entanglement between qubit states and field states. For example, if at time \( t = 0 \) the state of the system composed by the qubit and the \( k \) mode of the field is the tensor product of a general qubit state times the vacuum state for the field mode, then the interaction Hamiltonian will, at time \( t \), generate an entangled state

\[
(c_0|0\rangle + c_1|1\rangle) \otimes |0_k\rangle \xrightarrow{U(t)} c_0|0\rangle - \frac{1}{2} \xi_k(t) + c_1|1\rangle + \frac{1}{2} \xi_k(t))
\] (17)

where \( |\frac{1}{2} \xi_k(t)\rangle \) is a coherent state of amplitude \( \frac{1}{2} \xi_k(t) \). Such correlations between qubit and environment cannot be accounted for in the semiclassical approach outlined at the beginning of this section. It is precisely this entanglement which, in the fully quantised description we have adopted, is responsible for the decoherence process. Indeed the off-diagonal elements of the reduced density matrix of the qubit decay due to the fact that the overlap between the different field states with which the qubit becomes entangled diminishes in time. We can formulate in rigorous terms the analysis we have outlined above by taking into account all the field modes in thermal equilibrium. The matrix elements of the reduced density operator of the qubit are defined as

\[
\rho_{ij}(t) = \langle i | \text{Tr}_R U(t) \rho(0) U^{-1}(t) | j \rangle.
\] (18)
Using Eq. (15) we check that, as anticipated, \( \rho_{00}(t) = \rho_{00}(0) \), \( \rho_{11}(t) = \rho_{11}(0) \) and for the coherence \( \rho_{10} \) we obtain

\[
\rho_{10}(t) = \prod_k \text{Tr}_k \{ R_k T D(\xi_k(t)) \} \rho_{10}(0) = e^{-\Gamma(t)} \rho_{10}(0).
\]  

Equation (19) is exact, i.e. no approximation has been made to obtain it. To evaluate \( \exp\{-\Gamma(t)\} \) we have to calculate the average value of the displacement operator for each mode in a thermal state. This is also known as the symmetric order generating function for a harmonic oscillator in thermal equilibrium (Hillery et al. 1984, Gardiner 1991). It can be shown that

\[
\text{Tr}_k \{ R_k T D(\xi_k) \} = \exp\left\{ -\frac{|\xi_k|^2}{2} \coth \left( \frac{\omega_k}{2T} \right) \right\},
\]

(20)

(we put the Boltzmann constant \( k_B = 1 \)). Thus, in the continuum limit

\[
\Gamma(t) \propto \int d\omega dk |g(\omega)|^2 \coth \left( \frac{\omega_k}{2T} \right) \frac{1 - \cos \omega_k t}{\omega_k^2}
\]

\[
\propto \int d\omega dk \frac{d\omega}{d\omega} G(\omega)|g(\omega)|^2 (1 + 2\langle n(\omega) \rangle_T) \frac{1 - \cos \omega t}{\omega^2}
\]

(21)

where \( G(\omega) \) is the density of modes at frequency \( \omega \) (we have dropped the superfluous index \( k \)), \( \langle n(\omega) \rangle_T = \exp(-\omega/2T) \text{cosech}(\omega/2T) \) is the average number of field excitations at temperature \( T \) and \( (dk/d\omega) \) is the dispersion relation. In Eq. (21) we can separate the effects due to thermal noise from the one due to purely quantum fluctuations. This formal separation allows us to identify the existence of various timescales in the decoherence process (see also Unruh 1995). Let us first note that thermal fluctuations can affect the qubit dynamics only for times longer than the characteristic thermal frequency \( T \). For \( t < T^{-1} \) only vacuum quantum fluctuations contribute to the dephasing process.

Furthermore the quantity \( G(\omega)|g(\omega)|^2 \) is in general characterized by a cutoff frequency whose specific value depends on the particular nature of the physical qubit under investigation. For example if the noise field is a phonon field the natural cutoff can be identified with the Debye frequency. More generally we can think of the cutoff as due to some characteristic length scale in our system below which the qubit-environment coupling decreases
rapidly. Therefore we assume \((dk/d\omega)G(\omega)|g(\omega)|^2 \propto \omega^n e^{-\omega/\omega_c}\). The exponent \(n\) will depend on the number of dimensions of the field. In our analysis we will concentrate our attention to the case of one-dimensional field, for which \(n = 1\) and of three-dimensional field, for which \(n = 3\). It is evident that the so-called quantum vacuum fluctuations will contribute to the dephasing process only for times \(t > \omega_c^{-1}\).

We can identify three time regimes of decoherence:

- a "quiet" regime, for \(t < \omega_c^{-1}\), where the fluctuations are ineffective in the decoherence process
- a quantum regime, for \(\omega_c^{-1} < t < T^{-1}\), where the main cause of coherence loss are the quantum vacuum fluctuations
- a thermal regime, for \(t > T^{-1}\), where thermal fluctuations play the major role in eroding the qubit coherence.

In order to have some semiquantitative picture of the time dependence of the decoherence we need to specify the frequency dependence of the density of states and of the coupling. The case of one-dimensional field \((n = 1)\) has already received some attention in literature, in this case

\[
\Gamma(t) \propto \int d\omega e^{-\omega/\omega_c} \coth \left( \frac{\omega}{2T} \right) \frac{1 - \cos(\omega t)}{\omega}.
\] (22)

An analytic solution of (22) can be obtained in one dimension in the low temperature limit \((\omega_c \gg T)\):

\[
\Gamma(t) \propto \ln(1 + \omega_c^2 t^2) + 2 \ln \left[ \frac{1}{\pi T t} \sinh(\pi T t) \right].
\] (23)

The first term arises from the quantum vacuum fluctuations while the second is due to the thermal ones. Expression (23) reduces to \(\Gamma(t) \sim \omega_c^2 t^2\) for \(t < \omega_c^{-1}\), \(\Gamma(t) \sim 2 \ln \omega_c t\) for \(\omega_c^{-1} < t < T^{-1}\), and \(\Gamma(t) \sim T t\) for \(t > T^{-1}\).

These three regimes can be easily identified in Fig.(4), which shows the decoherence of a single qubit induced by a one-dimensional field for the particular choice \(\omega_c/T = 100\).

It is also interesting to consider the three dimensional field case \((n = 3)\), where the integral can be evaluated exactly:

\[
\Gamma(t) \propto \int d\omega \omega e^{-\omega/\omega_c} \coth \left( \frac{\omega}{2T} \right) [1 - \cos(\omega t)]
\]
\[2 \pi T^2 \left\{ 2 \zeta \left[ 2, \frac{T}{\omega_c} \right] - \zeta \left[ 2, \frac{T}{\omega_c} (1 + i \omega_c t) \right] - \zeta \left[ 2, \frac{T}{\omega_c} (1 - i \omega_c t) \right] \right\} \]
\[+ \omega_c^2 \left[ \frac{1}{(1 + i \omega_c t)^2} + \frac{1}{(1 - i \omega_c t)^2} - 2 \right]. \tag{24}\]

where \(\zeta(x, y)\) is the generalized Riemann zeta function. In Fig. (5) is shown the decay of a single qubit as a function of time and as a function of the ratio \(\eta = \omega_c/T\) in the case of a one-dimensional (a) and three-dimensional (b) fields. In (a) the coherence always decays to zero. For \(\eta \sim 1\) this decay is dominated by the thermal fluctuations of the reservoir and it is therefore exponential. However, for large \(\eta\) there is an intermediate region where the vacuum fluctuations dominate and the decay is roughly linear before the thermal regime takes over. In all cases there is a short \(t\) region where almost no decay is present. As \(\eta\) decreases, the decay onset moves towards larger \(t\). This is because the cut-off frequency \(\omega_c\) determines the extent of the ”quiet” regime. In (b) the system shows a very different behaviour: the decoherence saturates to a value determined by \(\eta\). This difference in behaviour is reminiscent of the dimensional differences observed in various models of phase transitions (Ma 1976) and is due to the suppressed influence of low frequency fluctuations in three dimensions. It should however be pointed out that at longer timescales, when \(T_1\) processes involving exchange of energy between qubit and environment take place, this residual coherence will disappear.

### 5 Decoherence of Quantum Registers

In the present literature on decoherence processes in quantum computation it is usually assumed that in a register of length \(L\) each qubit interacts individually with a different reservoir. In this case all the analysis of the relevance of dissipation from the complexity viewpoint is done assuming that \(\Gamma_L(t) \sim L \Gamma(t)\) (see next section). In this section we discuss in which circumstances this assumption is justified and we analyse the consequences of collective interaction on the complexity analysis, when such effects need to be taken into account.

We will start by considering a system of two qubits at positions \(r_a, r_b\) respectively. Their dynamics is described by the following Hamiltonian

\[H = \frac{1}{2} \sigma_z^a \omega_0^a + \frac{1}{2} \sigma_z^b \omega_0^b + \sum_k b_k^\dagger b_k \omega_k \]
\[+ \sum_k \left( \sigma_z^a (g_k b_k^\dagger + g_k^* b_k) + \sigma_z^b (g_k^* b_k^\dagger + g_k b_k) \right). \tag{25}\]
Hamiltonian (25) is a straightforward generalization of (12) where \( g_a^k, g_b^k \) are position dependent couplings. In (25) we are not considering any direct interaction between the two qubits which would be necessary for the conditional dynamics (Barenco et al 1995). Here we concentrate our attention to the simplest case of a system of two qubits coupled only to the environment.

In the interaction picture the time evolution operator takes the form

\[
U(t) = \exp \left\{ \frac{1}{2} \sum_k b_k^\dagger \left[ \sigma_z^a \xi_k^a(t) + \sigma_z^b \xi_k^b(t) \right] - b_k \left[ \sigma_z^a \xi_k^a(t) + \sigma_z^b \xi_k^b(t) \right] \right\} \tag{26}
\]

Here again the unitary evolution produces entanglement between register states and environment states. It should be noted however that \( U(t) \) acts as a displacement operator on the field with a displacement amplitude depending on the logic value of both qubits of the register. The following simple example will help to clarify this point. Let us consider the following two initial states

\[
| \Phi^-(t) \rangle = (c_{10}|1_a, 0_b\rangle + c_{01}|0_a, 1_b\rangle) \otimes |0_k\rangle, \tag{27}
\]

\[
| \Phi^+(t) \rangle = (c_{00}|0_a, 0_b\rangle + c_{11}|1_a, 1_b\rangle) \otimes |0_k\rangle, \tag{28}
\]

where we consider for simplicity only the vacuum state of the \( k \) mode and where the \( c_{ij} \) are arbitrary complex amplitudes. Due to the qubit-field interaction the system will evolve into the states

\[
| \Phi^-(t) \rangle = c_{10}|1_a, 0_b\rangle + \frac{1}{2}(\xi_k^a - \xi_k^b) + c_{01}|0_a, 1_b\rangle + \frac{1}{2}(\xi_k^a - \xi_k^b), \tag{29}
\]

\[
| \Phi^+(t) \rangle = c_{00}|0_a, 0_b\rangle - \frac{1}{2}(\xi_k^a + \xi_k^b) + c_{11}|1_a, 1_b\rangle + \frac{1}{2}(\xi_k^a + \xi_k^b). \tag{30}
\]

Since the amount of decoherence is measured by the overlap between the two different field states with which the qubit states become entangled, the states \( | \Phi^+(t) \rangle, | \Phi^-(t) \rangle \) will be characterized by different decoherence times. In particular, in the limit \( r_a \approx r_b \) we have \( \xi_k^a \approx \xi_k^b \) and therefore in the ”sub-decoherent” state \( | \Phi^-(t) \rangle \) the qubits are not entangled with the field, while in the ”superdecoherent” state \( | \Phi^+(t) \rangle \) the qubits are collectively entangled with a coherent state with an amplitude which is twice that of the single qubit case.

We can formulate in more rigorous terms the analysis we have just sketched above by taking into account all the modes of the field in thermal equilibrium. We will again concentrate our attention on the reduced density
matrix of the two qubit system whose matrix elements can be conveniently expressed as

$$\rho_{ia,ib}^{ia,ib}(t) = \langle ia, ib|Tr_R\{g(t)\}|ja, jb\rangle.$$  \hspace{1cm} (31)

It is straightforward to verify that the analogous of Eq. (19) is

$$\rho_{ia,ib}^{ia,ib}(t) = \rho_{ia,ib}^{ia,ib}(0) \prod_k Tr_k \left\{ R_{kT}D[(ia - ja)\xi^a_k]D[(ib - jb)\xi^b_k] \right\}. \hspace{1cm} (32)$$

Matrix elements of the form $\rho_{ia,ia,ib,ib}$ show no collective decay, in fact

$$\rho_{ia,ia,ib,ib}(t) = \rho_{ia,ia,ib,ib}(0) \prod_k Tr_k \left\{ R_{kT}D[(ib - ja)\xi^b_k] \right\}. \hspace{1cm} (33)$$

which for $ib = ja$ reduces to $\rho_{ia,ia,ia,ia}(t) = \rho_{ia,ia,ia,ia}(0)$ while for $ib \neq ja$ we have $\rho_{ia,ia,ib,ib}(t) = \rho_{ia,ia,ia,ia}(0)e^{-\Gamma(t)}$. Analogous results hold of course for matrix elements of the form $\rho_{ia,ia,ia,ia}$.

Collective decay is instead apparent in the matrix elements of the form $\rho_{ia,ja,ib,jb}$ with $ia \neq ja, ib \neq jb$:

$$\rho_{10,10}(t) = \rho_{10,10}(0) \prod_k Tr_k \left\{ R_{kT}D(\xi^a_k + \xi^b_k) \right\}, \hspace{1cm} \rho_{10,01}(t) = \rho_{10,01}(0) \prod_k Tr_k \left\{ R_{kT}D(\xi^a_k - \xi^b_k) \right\}, \hspace{1cm} (34)$$

where $R = r_a - r_b$.

The expression for $\Gamma^{\pm}$ can obtained from (20) with the obvious substitution $\xi_k \rightarrow \xi^a_k \pm \xi^b_k$. If we suppose $g^a_k = g_k e^{ik \cdot r_a}$ and $g^b_k = g_k e^{ik \cdot r_b}$, then, in the continuum limit, the collective decay rates are

$$\Gamma^{\pm}(R, t) \propto \int dk |g_k|^2 \coth \left( \frac{\omega_k}{2T} \right) \frac{1 - \cos(\omega_k t)}{\omega_k^2} \left[ 1 \pm \cos(\mathbf{k} \cdot \mathbf{R}) \right], \hspace{1cm} (36)$$

where $R = r_a - r_b$.

The origin of the position dependence of the collective decay constants $\Gamma^{\pm}(R, t)$ can be easily explained both in semiclassical and in quantum terms. We have already mentioned that in a system of two qubits the entanglement
with the field will in general depend on the qubit separation. In a semiclassical picture if the distance between the two qubits is smaller than the correlation length of the bath they “feel” the same fluctuation. More generally the random phase-shifts due to the bath fluctuations of wavelength shorter than the inverse of the qubit separation are the same for both qubits. This produces a reduction - or an enhancement - of the decay rates depending on whether the phase shifts on the two qubits add or subtract.

In order to avoid possible confusions it must be stressed that the phenomenon of superdecoherence vs. subdecoherence that we have just described is not the same as the process of superradiance vs. subradiance more commonly encountered in literature (see e.g Allen & Eberly 1975). While superdecoherence is due to collective entanglement between qubits and environment with no exchange of energy, superradiance is a process of collective radiation by a group of closely spaced atoms. Indeed the conditions for the two processes to occur are quite different: superdecoherence requires the qubit to be in a region of space smaller than the coherence length of the environment, which is a much more stringent condition that the one needed for superradiance to take place, i.e. that the atoms are at a distance smaller than the wavelength of the resonant field modes. Furthermore, it must be noted that some superradiant states are subdecoherent. In general the set of states which are robust against the dephasing action of the environment will depend on the specific form of qubit-environment coupling.

In the one-dimensional and three-dimensional field cases, the collective decay constants are, respectively, given by

$$\Gamma_{1D}^\pm (R, t) \propto 2 \int_0^\infty d\omega e^{-\omega/\omega_c} \coth \left( \frac{\omega}{2T} \right) \left( 1 - \frac{\cos \omega t}{\omega} \right) [1 \pm \cos(\omega t_s)]$$ (37)

$$\Gamma_{3D}^\pm (R, t) \propto 2 \int_0^\infty d\omega e^{-\omega/\omega_c} \coth \left( \frac{\omega}{2T} \right) \left( 1 - \frac{\cos \omega t}{\omega} \right) \left( 1 \pm \frac{\sin(\omega t_s)}{\omega t_s} \right)$$ (38)

where we have introduced the transit time $t_s (\omega t_s = k \cdot R)$.

The analysis carried out for a two-qubit register can be easily extended to the case of a register of $L$ qubits, whose density matrix elements can be written as

$$\rho_{\{i_n, j_n\}}(t) = \langle i_{L-1}, i_{L-2}, ..., i_0 | Tr_R \{ g(t) \} | j_{L-1}, j_{L-2}, ..., j_0 \rangle$$

$$= \rho_{\{i_n, j_n\}}(0) \times \prod_k Tr_k \left\{ R_{kT} \left[ D((i_{L-1} - j_{L-1})\xi_k^{(L-1)}) \right] D((i_{L-2} - j_{L-2})\xi_k^{(L-2)}) \right. \left. \left[ \cdots D(i_0 - j_0)\xi_k^{(0)} \right] \right\}. $$

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From the viewpoint of the following complexity analysis it is interesting to consider the limiting case in which all the qubits of the register are in the same position. In this case all the $\xi_k^{(i)}$ are equal and the matrix elements with the fastest decay are $\rho_{\{1_n,0_n\}}$ and $\rho_{\{0_n,1_n\}}$ for which we have

$$\rho_{\{1_n,0_n\}}(t) = \rho_{\{1_n,0_n\}}(0)e^{-L^2\Gamma(t)}.$$  \hspace{1cm} (40)

In general,

$$\rho_{\{i_n,j_n\}}(t) = \rho_{\{i_n,j_n\}}(0)\prod_k Tr_k \left\{ R_{kT} D \left[ \sum_{n=0}^{L-1} (i_n - j_n)\xi_k \right] \right\}$$

$$= \rho_{\{i_n,j_n\}}(0)\exp \left\{ -\sum_{n=0}^{L-1} (i_n - j_n)^2 \Gamma(t) \right\}$$  \hspace{1cm} (41)

which should be compared with the expression for the decoherence in the case of independent reservoirs

$$\rho_{\{i_n,j_n\}}(t) = \rho_{\{i_n,j_n\}}(0)\exp \left\{ -\sum_{n=0}^{L-1} |(i_n - j_n)| \Gamma(t) \right\}$$ \hspace{1cm} (42)

where $\sum_{n=0}^{L-1} |(i_n - j_n)|$ is the Hamming distance between the two qubit states. In the independent reservoir case therefore in the worst case the density matrix elements decay as $\exp\{-L\Gamma(t)\}$.

6 Discussion and Conclusions

Our analysis shows that decoherence destroys quantum interference and entanglement in quantum computers, thus decreasing the probability of successful computation exponentially with the input size $L$. Reservoirs with large coherence lengths introduce only asymmetry in the decoherence rates of various off-diagonal elements of the density matrix with the same Hamming distance between its indices (compare Eq. (41) with Eq. (42)). In this case while some coherences are not affected by the decoherence some others are destroyed more rapidly. Assuming that all coherences are equally important and that for the complexity analysis we choose the worst case (i.e. the off-diagonal element which decays at the fastest rate) then we are forced to conclude that reservoirs with a large coherence length can be
more of a hindrance than a help. In the independent reservoirs limit the fastest decoherence is proportional to \( \exp(-L \Gamma(t)) \) whereas for the reservoirs with a large coherence length there are off-diagonal elements which decay as \( \exp(-\text{poly}(L) \Gamma(t)) \). However, if one can support quantum computation with some selected coherences then a large coherence length can be utilised.

Let us suppose that we can manufacture in our laboratory a quantum register of \( 2L \) qubits composed of pairs of qubits close enough to each other so that each pair is effectively interacting with the same reservoir. Different pairs can interact with different reservoirs although the results we are going to illustrate are not modified if all the qubits interact with the same reservoir. The idea is that if we use a pair of qubits to encode a bit we might effectively decouple the register from its environment. To give a simplified example of how this might work let us consider a register of 4 qubits in which the two pairs of qubits interact with two different reservoirs. The interaction will evolve the system as

\[
|i_a, j_a\rangle |i_b, j_b\rangle |0_k^{a}\rangle |0_k^{b}\rangle \rightarrow |i_a, j_a\rangle |i_b, j_b\rangle \left((-1^{i_a} - 1^{j_a}) \xi_k^{a}\right) \left((-1^{i_b} - 1^{j_b}) \xi_k^{b}\right)
\]  

where for simplicity we have considered only the vacuum state of one mode for each reservoir, labelled \( a, b \), respectively. It is obvious that states for which \( i_a \neq j_a \) and \( i_b \neq j_b \) do not get entangled with the reservoir and therefore maintain their coherence. We might therefore use the following encoding

\[
|\tilde{0}\rangle = |0, 1\rangle, \\
|\tilde{1}\rangle = |1, 0\rangle.
\]  

This “noiseless” encoding does not pose complexity problems since to encode \( L \) qubits we need simply \( 2L \) qubits. Of course, the experimental realization of a register with the above characteristics encounters practical problems; it might, for instance, be difficult to excite and manipulate only the stable coherences. The main point we want to make, however, is that appropriate encoding could be a way to overcome some of the problems posed by couplings with external noise.

Is the computational complexity analysis dependent on the character of decoherence i.e. on the type of coupling with the environment? We believe that as long as the reservoir is in a thermal state, any model of decoherence will lead to the same conclusions. In the independent reservoirs limit the fastest decoherence is proportional to \( \exp(-L \Gamma(t)) \) regardless the functional dependence \( \Gamma(t) \). For reservoirs with a large coherent length the fastest
decoherence will be proportional to $\exp(-\text{poly}(L)\Gamma'(t))$ leading to the same unwelcome exponential increase of the error rate. However, let us emphasise again that this should not be taken as a fundamental argument against possibilities of experimental implementation of quantum factorisation. Quantum error correction, modification of the existing quantum algorithms, customizing the environmental noise, and probably many other techniques can be employed to improve the efficiency of quantum algorithms.

Even if quantum algorithms for factorisation are not efficient from the complexity theory point of view, they may still permit factoring of numbers which are much bigger than those that can be factored using classical algorithms. Technological progress in isolating quantum computers from the environment and suppressing decoherence will increase the size of the biggest number that can be factored by such computers. For if the elementary computational step takes time $\tau$ and $t$ is a decoherence time of a single qubit then the requirement for the coherent computation to be completed within the decoherence time of the computer can be written as

$$\tau L^2 < t/L.$$  

(45)

The l.h.s. represents the time needed to factor number of size $L$ with the Shor algorithm and the r.h.s. represents the decoherence time of $L$ qubits (in the independent reservoirs approximation). From the equation above follows that the size $L$ is bounded by

$$L < \left(\frac{t}{\tau}\right)^{1/3}.$$  

(46)

Thus the ratio $t/\tau$, which depends on the technology employed, determines the limits of the algorithm and it is unrealistic to assume that this ratio can be made infinite. For a review of possible values of $t$ and $\tau$ see DiVincenzo 1995.

Clearly the decoherence problem calls for a satisfactory solution if we want to make quantum factoring realistic and eventually practical. The authors believe that an efficient quantum error correction is needed in order to achieve this goal.

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Figure 1: The Bloch vector $\vec{s}$ and its projections in the Cartesian coordinate system. The field vector $B_0 \vec{e}_z$ lies in the $z$-direction. When $B_0$ is constant the Bloch vector simply rotates around the field vector and its tip follows a closed, circular trajectory, which is parallel to the $xy$-plane. If the qubit is in an equal superposition of states 0 and 1, then $s_z = 0$ and the Bloch vector itself lies in the $xy$-plane.

Figure 2: Results from a semiclassical stochastic simulation. In (a) and (b) the time evolution of $s_x$ and $s_y$ (solid lines) generated by the fluctuating field $B_z$ is shown for two different ensemble members. The dotted lines represent the unperturbed evolution. Time $t$ is in units of $\omega_0^{-1}$. We have chosen $c_0 = c_1 = 2^{-1/2}$ (i.e. $s_z = 0$). In (c) and (d) we show averages over 100 and 500 ensemble members, respectively. $B_z$ averages rapidly to zero as the ensemble size increases.

Figure 3: The interaction between qubit and reservoir causes a conditional displacement of the field mode. The sign of the displacement depends on the qubit state. In the displaced modes the field state oscillates with amplitude $\pm \frac{1}{2} \xi_k$. This conditional displacement causes a qubit-field entanglement.
Figure 4: Decoherence of a single qubit for a one-dimensional field density of states for $\omega_c/T = 100$. Time is in units of $T^{-1}$ and the proportionality factor is equal to 0.1. The three decay regimes can be easily identified.

Figure 5: Decoherence of a single qubit as a function of time and of the ratio $\eta = \omega_c/T$. Time is in units of $T^{-1}$ and the proportionality factor has been set equal to 0.1. Here (a) shows the result of a numerical integration of Eq. (22) for the one-dimensional density of states while (b) shows of the exact solution for the three-dimensional density of states (Eq. 24).

Figure 6: The decay of two qubit coherence in the case of the shared reservoir with one-dimensional density of states (Eq. 37). We have set $\eta = 1$ and the proportionality factor is chosen such that at the limit of large $t_s$ we get the results of Fig. 5(a). In (a) we see how the decay is cancelled out, and in (b) we see how it is amplified, when $t_s$ is small. The onset of decay does not change with $t_s$.

Figure 7: The decay of two qubit coherence in the case of the shared reservoir with three-dimensional density of states (Eq. 38). We have set $\eta = 1$ and the proportionality factor is chosen such that at the limit of large $t_s$ we get the results of Fig. 5(b). In (a) we see how the decay is cancelled out, and in (b) we see how it is amplified, when $t_s$ is small. The saturation of the decay is present in both cases, though.
The figure shows the function $e^{-\Gamma(t)}$ as a function of $t$. The graph is divided into three regimes:

1. **"quiet" regime**
   - This regime is characterized by a steep decrease in the function values as $t$ increases.

2. **Quantum regime**
   - The function values decrease more gradually compared to the "quiet" regime.

3. **Thermal regime**
   - The function values continue to decrease slowly, indicating a further transition towards equilibrium.

The x-axis represents time $t$, while the y-axis represents the function values of $e^{-\Gamma(t)}$. The graph visually represents the transition from the "quiet" to the quantum and then to the thermal regime as time progresses.
$e^{-\Gamma(t)}$
$e^{-\Gamma_{1D}^+ (t)}$
$e^{-\Gamma^+_{3D}(t)}$