Tensor Multiplets

in Six-Dimensional \((2,0)\) Supergravity

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Abstract

We construct the complete coupling of \((2,0)\) supergravity in six dimensions to \(n\) tensor multiplets, extending previous results to all orders in the fermi fields. The truncation to \((1,0)\) supergravity coupled to tensor multiplets exactly reproduces the complete couplings recently obtained.

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1 Introduction

The massless representations of $(2,0)$ supersymmetry in six dimensions, labeled by their $SU(2) \times SU(2)$ representations, are the gravity multiplet $((1,1) + 4(1, \frac{1}{2}) + 5(1,0))$ and the tensor multiplet $((0,1) + 4(0, \frac{1}{2}) + 5(0,0))$. In particular $(2,0)$ supergravity coupled to 21 tensor multiplets is free of gravitational anomalies [1], and the corresponding six-dimensional vacua naturally arise as compactifications of Type-IIB superstrings on $K3$.

In this paper we construct the coupling of $(2,0)$ supergravity to an arbitrary number of tensor multiplets, thus completing [2] to all orders in the fermi fields. In [3] (see also [4] for a brief review) the complete $(1,0)$ supergravity coupled to tensor and vector multiplets was obtained, generalizing the results of [5] to an arbitrary number of tensors and completing the results of [6, 7, 8]. This theory contains reducible gauge and supersymmetry anomalies induced by tensor couplings, that are completely determined solving Wess-Zumino consistency conditions [9]. Here we follow the construction of [3], but the derivation is simpler since in the $(2,0)$ case there are no vector multiplets and the low-energy theory does not present a supersymmetry anomaly.

In [2] $(1,0)$ supergravity coupled to $n$ tensor multiplets was obtained as a truncation of the corresponding $(2,0)$ model to lowest order in the fermi fields. Here we show how the higher order fermion terms of the complete $(1,0)$ model [3] can also be recovered from a truncation of the complete $(2,0)$ supergravity. The flat-space limit of $(1,0)$ supergravity coupled to vector and tensor multiplets was recently studied [10], and two different global limits were found, but in the case of $(2,0)$ supergravity coupled to tensor multiplets the flat-space limit does not display similar subtleties.

In Section 2 we derive the complete model, and in Section 3 we truncate it to $(1,0)$ supergravity. The Appendix explains the notations and collects some useful identities. The conventions follow those used in [3].
2 Complete (2, 0) Supergravity in Six Dimensions Coupled to $n$ Tensor Multiplets

Chiral extended supersymmetry in six dimensions is generated by four left spinorial charges $Q^a$ ($a = 1, 2, 3, 4$), obeying the symplectic Majorana condition

$$Q^a = \Omega^{ab}C\bar{Q}^T_b,$$  \hspace{1cm} (2.1)

where $\Omega^{ab}$ is the invariant tensor of $Sp(4)$. Using the isomorphism between $Sp(4)$ and $SO(5)$ the index $a$ can be considered as an index in the spinorial representation of $SO(5)$. All fermi fields appear in this representation, and we will denote with $\Gamma^i$ the gamma matrices of $SO(5)$.

$(2, 0)$ supergravity coupled to $n$ tensor multiplets is described by the vielbein $e_\mu^a$ (now $a$ is a Lorentz index), a left-handed gravitino $\psi_\mu$, $(n+5)$ antisymmetric tensors $B^r_{\mu\nu}$ ($r = 0, ..., n+4$) obeying (anti)self-duality conditions, $n$ right-handed “tensorini” $\chi^m$ ($m = 1, ..., n$) and $5n$ scalars. The scalars parametrize the coset space $SO(5, n)/SO(5) \times SO(n)$, and are associated to the $SO(5, n)$ matrix

$$V = \begin{pmatrix} v^i_r \\ x^m_r \end{pmatrix},$$  \hspace{1cm} (2.2)

where $i = 1, ..., 5$ is an index in the vector representation of $SO(5)$. The matrix elements satisfy the constraints

$$v^{ir}v^j_r = \delta^{ij},$$

$$x^{mr}x^r_n = -\delta^{mn},$$

$$v^{ir}x^m_r = 0,$$

$$v^i_r v^i_s - x^m_r x^m_s = \eta_{rs}.$$  \hspace{1cm} (2.3)

The $SO(5)$ connection is

$$Q^{ij}_\mu = v^i_r(\partial_\mu v^{jr})$$  \hspace{1cm} (2.4)

while the $SO(n)$ connection is

$$S^{mn}_\mu = (\partial_\mu x^m_r)x^{nr}.$$  \hspace{1cm} (2.5)
We start considering the theory to lowest order in the fermi fields \( \Phi \). The (anti)self-duality conditions for the tensor fields are

\[
G_{rs} H_{\mu\nu\rho}^s = \frac{1}{6e} \epsilon_{\mu\nu\rho\alpha\beta\gamma} H_r^{\alpha\beta\gamma},
\]

(2.6)

where

\[
G_{rs} = v_i^r v_s^i + x_m^r x_s^m .
\]

(2.7)

These conditions imply that \( v_i^r H_{\mu\nu\rho}^r \) are self dual, while \( x_m^r H_{\mu\nu\rho}^r \) are antiself dual. The divergence of eq. (2.6) yields the second-order tensor equation

\[
D_\mu (G_{rs} H_{\mu\nu\rho}^s) = 0 .
\]

(2.8)

In [11] it was shown how to obtain a lagrangian formulation of (anti)self-dual two-forms. It involves an additional scalar field and additional gauge-invariance, and the (anti)self-duality condition, as well as the elimination of the scalar from the spectrum, results in a gauge-fixed version of the complete model. Thus, the lagrangian

\[
e^{-1} \mathcal{L}_H = \frac{1}{12} G_{rs} H_{\mu\nu\rho}^r H_{\mu\mu\rho}^s ,
\]

(2.9)

that would not make sense for (anti)self-dual forms, is the gauge-fixed version of the complete tensor lagrangian, and it gives the second-order tensor equation for (anti)self-dual two-forms, i.e. the divergence of the self-duality condition. With this in mind, we write the lagrangian (see the Appendix for the notations)

\[
e^{-1} \mathcal{L} = -\frac{1}{4} R + \frac{1}{12} G_{rs} H_{\mu\nu\rho}^r H_{\mu\mu\rho}^s + \frac{1}{4} x_m^r x_m^s (\partial_\mu v^i_r) (\partial^\nu v^i_s) - \frac{i}{2} (\bar{\psi}_\mu^r \gamma^{\mu\nu\rho} D_\nu \psi)^i
\]

\[
- \frac{i}{2} v_i^r H_{\mu\nu\rho}^r (\bar{\psi}_\mu^r \gamma^{\nu\rho} \psi)^i + \frac{i}{2} (\bar{\chi}_m^r \gamma^{\mu\nu\rho} D_\nu \chi^m)^i - \frac{i}{24} v_i^r H_{\mu\mu\rho}^r (\bar{\chi}_m^r \gamma^{\mu\nu\rho} \chi^m)^i
\]

\[
+ \frac{1}{2} x_m^r H_{\mu\nu\rho}^r (\bar{\psi}_\mu^r \gamma^{\nu\rho} \chi^m)^i - \frac{1}{2} x_m^r (\partial_\nu v^r)^i (\bar{\psi}_\mu^r \gamma^{\nu\rho} \chi^m)^i ,
\]

(2.10)

invariant under the supersymmetry transformations

\[
\delta e_\mu^a = -i (\bar{\epsilon} \gamma^a \psi_\mu) ,
\]

\[
\delta B_{\mu\nu}^r = i v^r (\bar{\psi}_\mu^r \gamma_{\nu})^i + \frac{1}{2} x_m^r (\bar{\chi}_m^r \gamma^{\mu\nu} \epsilon)^i ,
\]

\[
\delta v_i^r = x_m^r (\bar{\epsilon} \chi^m)^i ,
\]

invariant under the supersymmetry transformations

\[
\delta e_\mu^a = -i (\bar{\epsilon} \gamma^a \psi_\mu) ,
\]

\[
\delta B_{\mu\nu}^r = i v^r (\bar{\psi}_\mu^r \gamma_{\nu})^i + \frac{1}{2} x_m^r (\bar{\chi}_m^r \gamma^{\mu\nu} \epsilon)^i ,
\]

\[
\delta v_i^r = x_m^r (\bar{\epsilon} \chi^m)^i ,
\]

invariant under the supersymmetry transformations
\[ \delta x^m_r = v^i_r (\bar{\epsilon} \chi^m)^i , \]
\[ \delta \psi_\mu = D_\mu \epsilon + \frac{1}{4} v^i_r H^r_{\mu \nu \rho} \Gamma^i \gamma_\mu \epsilon \]
\[ \delta \chi^m = \frac{i}{2} x^m_r \partial_\mu v^i_r \Gamma^i \gamma_\mu \epsilon + \frac{i}{12} x^m_r H^r_{\mu \nu \rho} \gamma^{\mu \nu \rho} \epsilon \]  
(2.11)

to lowest order in the fermi fields and using the self-duality condition of eq. (2.6). The commutator of two supersymmetry transformations on the bosonic fields closes on the local symmetries:

\[ [\delta_1, \delta_2] = \delta_{\text{gct}}(\xi^\mu) + \delta_{\text{tens}}(A^r_\mu = -\frac{1}{2} v^i_r \xi^i_\mu - \xi^\mu B^r_{\mu \nu}) + \delta_{\text{SO}(n)}(A^{mn} = -\xi^\mu S^{mn}) \]
\[ + \delta_{\text{SO}(5)}(A^{ij} = -\xi^\mu Q^{ij}_\mu) + \delta_{\text{Lorentz}}(\Omega^{ab} = -\xi^\mu \omega^{\mu ab} + \xi^i_\mu v^i_r H^{r \mu ab}) \]  
(2.12)

where

\[ \xi_\mu = -i (\bar{\epsilon}_1 \gamma_\mu \epsilon_2) , \quad \xi^i_\mu = -i (\bar{\epsilon}_1 \gamma_\mu \epsilon_2)^i . \]  
(2.13)

To this order, one can not see the local supersymmetry transformation in the gauge algebra, since the expected parameter, \( \xi^\mu \psi_\mu \), is generated by bosonic variations. As usual, the spin connection satisfies its equation of motion, that to lowest order in the fermi fields is

\[ D_\mu e^a_\nu - D_\nu e^a_\mu = 0 \]  
(2.14)

and implies the absence of torsion.

Varying the lagrangian of eq. (2.10) with respect to the fermi fields one obtains

\[ \gamma^{\mu \nu \rho} D_\nu \psi_\rho + v^i_r H^{r \mu \nu \rho} \Gamma^i \gamma_\nu \psi_\rho - \frac{i}{2} x^m_r H^{r \mu \nu \rho} \gamma_\nu \psi_\rho + \frac{i}{2} x^m_r \partial_\nu v^i_r \Gamma^i \gamma_\mu \psi_\rho = 0 \]  
(2.15)

and

\[ \gamma^\mu D_\mu \chi^m - \frac{1}{12} v^i_r H^{r \mu \nu \rho} \Gamma^i \gamma_\mu \rho \chi^m - \frac{i}{2} x^m_r H^{r \mu \nu \rho} \gamma_\mu \psi_\rho - \frac{i}{2} x^m_r \partial_\nu v^i_r \Gamma^i \gamma_\nu \chi^m = 0 \]  
(2.16)

while varying it with respect to the scalar fields and to the metric, and considering only terms without fermions, one obtains

\[ D_\mu (x^m_r \partial^\mu v^i_r) + \frac{2}{3} x^m_r v^i_r H^r_{\alpha \beta \gamma} H^{s \alpha \beta \gamma} = 0 \]  
(2.17)

and

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - x^{mr} x^{ms} \partial_r v^i_s \partial_\nu v^i_s + \frac{1}{2} g_{\mu \nu} x^{mr} x^{ms} \partial_\alpha v^i_r \partial^\alpha v^i_s - G_{rs} H^r_{\mu \alpha \beta} H^s_{\nu \alpha \beta} = 0 \]  
(2.18)
Completing these equations will require terms cubic in the fermi fields in the fermionic equations, and terms quadratic in the fermi fields in their supersymmetry transformations. Supersymmetry will then determine corresponding modifications of the bosonic equations, and the (anti)self-duality conditions (2.10) will also be modified by terms quadratic in the fermi fields. Supercovariance actually fixes all terms containing the gravitino in the first-order equations and in the supersymmetry variations of fermi fields.

The supercovariant forms

\[
\hat{\omega}_{\mu\nu\rho} = \omega_{\mu\nu\rho}^0 - \frac{i}{2} \left\{ (\bar{\psi}_\mu \gamma_\nu \psi_\rho) + (\bar{\psi}_\nu \gamma_\rho \psi_\mu) + (\bar{\psi}_\rho \gamma_\mu \psi_\nu) \right\},
\]

(2.19)

\[
\hat{H}^r_{\mu\nu\rho} = H^r_{\mu\nu\rho} - \frac{1}{2} x^{mr} \left\{ (\bar{\chi}^m \gamma_\mu \psi_\rho) + (\bar{\chi}^m \gamma_\nu \psi_\mu) + (\bar{\chi}^m \gamma_\rho \psi_\nu) \right\}
- \frac{i}{2} v^r \left\{ (\bar{\psi}_\mu \gamma_\nu \psi_\rho)^i + (\bar{\psi}_\nu \gamma_\rho \psi_\mu)^i + (\bar{\psi}_\rho \gamma_\mu \psi_\nu)^i \right\} ,
\]

(2.20)

\[
\partial_\mu v^r_i = \partial_\mu v^i_r - x^m_r (\bar{\chi}^m \gamma_\mu \psi_\rho)^i,
\]

(2.21)

where

\[
\omega_{\mu\nu\rho}^0 = \frac{1}{2} e_{\rho a} (\partial_\mu e_{\nu}^a - \partial_\nu e_{\mu}^a) - \frac{1}{2} e_{\nu a} (\partial_\nu e_{\rho}^a - \partial_\rho e_{\nu}^a) + \frac{1}{2} e_{\rho a} (\partial_\rho e_{\mu}^a - \partial_\mu e_{\rho}^a) \quad (2.22)
\]

is the standard spin connection in the absence of torsion, do not generate derivatives of the parameter under supersymmetry. So one can consider the supercovariant transformations

\[
\delta \psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4} v^i_r \hat{H}^r_{\mu\nu\rho} \Gamma^{i\nu\rho} \epsilon ,
\]

\[
\delta \chi^m = \frac{i}{2} x^m_r (\partial_\mu \hat{\omega}^i_{\nu\rho}) \Gamma^{i\nu\rho} \epsilon + \frac{i}{12} x^m_r \hat{H}^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon .
\]

(2.23)

The tensorino transformation is complete, while the gravitino transformation could include additional terms quadratic in the tensorini. On the other hand, one does not expect modifications of the bosonic transformations in the complete theory.

The algebra (2.12) has been obtained varying only the fermi fields in the bosonic supersymmetry transformations. The next step is to compute the commutator completely, varying the bosonic fields as well. On \( v^i_r, x^m_r \) and on the vielbein \( e_\mu^a \) this only modifies the local Lorentz parameter,

\[
\Omega^{ab} = -\xi^\mu \hat{\omega}^{ab}_\mu + \xi^{i\mu} v^i_r \hat{H}^r_{\mu}^{ab} ,
\]

(2.24)
the $SO(n)$ parameter,

$$A^{mn} = -\xi^\mu S^{mn}_\mu + (\bar{\chi}^m \epsilon_2)^i (\bar{\chi}^n \epsilon_1)^i - (\bar{\chi}^m \epsilon_1)^i (\bar{\chi}^n \epsilon_2)^i ,$$  \hspace{1cm} (2.25)

and the $SO(5)$ parameter,

$$A^{ij} = -\xi^\mu Q^{ij}_\mu + (\bar{\chi}^m \epsilon_2)^i (\bar{\chi}^m \epsilon_1)^j - (\bar{\chi}^m \epsilon_1)^i (\bar{\chi}^m \epsilon_2)^j$$  \hspace{1cm} (2.26)

together with the inclusion in the algebra of a supersymmetry transformation of parameter

$$\zeta = \xi^\mu \psi_\mu .$$  \hspace{1cm} (2.27)

These results are obtained using the torsion equation for $\hat{\omega}$,

$$\hat{D}_\mu e^a_\nu - \hat{D}_\nu e^a_\mu = 2 S^a_{\mu \nu} = -i (\bar{\psi}_\mu \gamma^a \psi_\nu) .$$  \hspace{1cm} (2.28)

New results come from the complete commutator on $B^r_{\mu \nu}$, where one needs to use the (anti)self-duality conditions. First of all, since these conditions are first-order equations, they must be supercovariant. In general one can require that the tensor

$$\hat{\mathcal{H}}^r_{\mu \nu \rho} = \hat{H}^r_{\mu \nu \rho} + i \alpha v^r (\bar{\chi}^m \gamma_{\mu \nu \rho} \chi^m)$$  \hspace{1cm} (2.29)

with $\alpha$ real coefficient, satisfy the (anti)self-duality conditions

$$G^s_{rs} \hat{\mathcal{H}}^s_{\mu \nu \rho} = \frac{1}{6} \epsilon_{\mu \nu \rho \beta \gamma} \hat{\mathcal{H}}^{\alpha \beta \gamma} .$$  \hspace{1cm} (2.30)

Using eqs. (2.3), one can see that the new $\chi^2$ terms contribute only to the self-duality condition, while the tensors $x^m_r \hat{H}^r_{\mu \nu \rho}$ remain antself dual without extra $\chi^2$ terms. Consequently, the commutator on the tensor fields generates all local symmetries in the proper form, aside from the extra terms

$$[\delta_1, \delta_2]_{\text{extra}} B^r_{\mu \nu} = \frac{\alpha}{4} v^{ir} (\bar{\epsilon}_2 \gamma_{\mu \rho \sigma} \epsilon_1)^i j (\bar{\chi}^m \gamma^\rho \chi^m)^j - \frac{\alpha}{4} v^{ir} (\bar{\epsilon}_2 \gamma_{\nu \rho \sigma} \epsilon_1)^i j (\bar{\chi}^m \gamma^\rho \chi^m)^j$$

$$+ \frac{1}{2} v^{ir} (\bar{\epsilon}_1 \chi^m)^i (\bar{\chi}^m \gamma_{\mu \nu} \chi^2) - \frac{1}{2} v^{ir} (\bar{\epsilon}_2 \chi^m)^i (\bar{\chi}^m \gamma_{\mu \nu} \chi^2)$$  \hspace{1cm} (2.31)

that may be canceled adding $\chi^2$ terms to the transformation of the gravitino. The most general expression one can add is

$$\delta' \psi_\mu = ia (\bar{\chi}^m \gamma_{\mu \nu} \chi^m)^i \gamma^\nu \epsilon + ib (\bar{\chi}^m \gamma_{\mu \nu} \chi^m)^i \Gamma^i \gamma^\nu \epsilon$$

$$+ ic (\bar{\chi}^m \gamma_{\mu \nu} \chi^m)^i j \Gamma^i j \epsilon + id (\bar{\chi}^m \gamma_{\mu \nu} \chi^m)^i j \Gamma^i j \gamma_{\mu \nu} \epsilon$$  \hspace{1cm} (2.32)
with $a$, $b$, $c$ and $d$ real coefficients, and the total commutator on $B_{\mu\nu}^\tau$ then determines all these parameters together with the parameter $\alpha$ to be

$$a = -\frac{1}{64} \quad , \quad b = -\frac{1}{64} \quad , \quad c = \frac{3}{64} \quad , \quad d = -\frac{1}{64} \quad , \quad \alpha = -\frac{1}{8} \quad . \quad (2.33)$$

The commutator on $e_\mu^a$ now closes with a local Lorentz parameter modified by the addition of

$$\Delta \Omega^{ab} = -\frac{i}{16} (\bar{\chi}^m \gamma^{ab} \chi^m) \xi_\rho - \frac{i}{16} (\bar{\chi}^m \gamma^{ab} \chi^m) \xi_i + \frac{i}{32} [\bar{\chi}^m \gamma_\rho \chi^m] ij \xi_\rho \quad , \quad (2.34)$$

where

$$\xi_{ij} = -i [\bar{\epsilon}_1 \gamma_{\mu\rho} \xi_2]_{ij} \quad , \quad (2.35)$$

while the commutators on the scalar fields are not modified.

One can now start to compute the commutators on fermi fields, that as usual close only on shell. Following [12], we will actually use this result to derive the complete fermionic equations. Let us begin with the commutator on the tensorini, using eq. (2.23). Supercovariance determines the field equation of the tensorini up to a term proportional to $\chi^3$. Closure of the algebra fixes this additional term, and the end result is

$$\gamma^\mu \hat{D}_\mu \chi^m - \frac{1}{12} \psi_i \hat{H}^{i\mu\nu\rho} i\gamma^\mu\nu\rho \chi^m - \frac{i}{2} x^m \hat{H}^{\mu\nu\rho} \chi^m \chi^\mu \chi^\nu \chi^\rho $$

$$- \frac{i}{2} x^m (\partial_\nu \hat{v}^{\mu\nu}) i\gamma^\mu \gamma^\nu \psi_{\mu} + \frac{i}{16} [\bar{\chi}^m \gamma_\mu \chi ] ij i\gamma^\mu \chi^m = 0 \quad . \quad (2.36)$$

The complete commutator of two supersymmetry transformations on the tensorini is then

$$[\delta_1, \delta_2] \chi^m = \delta_{\text{get}} \chi^m + \delta_{\text{Lorentz}} \chi^m + \delta_{\text{SO}(n)} \chi^m + \delta_{\text{SO}(5)} \chi^m + \delta_{\text{susy}} \chi^m$$

$$+ \frac{3}{8} \xi_\alpha \gamma^\alpha [\text{eq. } \chi^m] - \frac{1}{8} \xi_\alpha \gamma^\alpha [\text{eq. } \chi^m] \quad . \quad (2.37)$$

A similar result can be obtained for the gravitino. In this case the complete equation,

$$\gamma^\mu \hat{D}_\nu \psi_{\rho} + \frac{1}{4} \psi_i \hat{H}^{\nu}_{\alpha\beta} \Gamma^i \gamma^{\mu\rho} \chi^m \gamma^\alpha \psi_{\rho} - \frac{i}{2} x^m \hat{H}^{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho \chi^m$$

$$+ \frac{i}{8} (\bar{\chi}^m \gamma^{\mu\rho} \chi^m) \gamma^\rho \psi_{\mu} + \frac{i}{32} (\bar{\chi}^m \gamma_{\mu\rho\sigma} \chi^m) \gamma^{\mu\rho} \gamma^\sigma \psi_{\mu} - \frac{i}{32} (\bar{\chi}^m \gamma^{\mu\rho} \chi^m) \gamma_{\sigma\rho} \psi_{\sigma}$$

$$+ \frac{i}{8} (\bar{\chi}^m \gamma^{\mu\rho} \chi^m) \Gamma^i \gamma_{\mu} \psi_{\rho} + \frac{i}{32} (\bar{\chi}^m \gamma_{\mu\rho\sigma} \chi^m) \Gamma^i \gamma^{\mu\rho} \psi_{\sigma} - \frac{i}{32} (\bar{\chi}^m \gamma^{\mu\rho} \chi^m) \Gamma^i \gamma_{\sigma\rho} \psi_{\sigma}$$

$$+ \frac{i}{16} [\bar{\chi}^m \gamma_\nu \chi ] ij i\gamma^\nu \psi_{\nu} - \frac{i}{16} [\bar{\chi}^m \gamma_\nu \chi ] ij i\gamma^\nu \psi_{\nu} = 0 \quad , \quad (2.38)$$
is fixed by supercovariance, and the commutator closes up to terms proportional to a particular combination of eq. (2.38) and its $\gamma$-trace. Precisely, one obtains

$$\delta_1, \delta_2 \psi_\mu = \delta_{\text{gct}} \psi_\mu + \delta_{\text{Lorentz}} \psi_\mu + \delta_{\text{susy}} \psi_\mu + \delta_{\text{SO}(5)} \psi_\mu$$

$$+ \frac{5}{16} \xi_\sigma \gamma^\sigma \{ \text{eq. } \psi_\mu \} - \frac{7}{64} \xi_\sigma \gamma^\sigma \gamma_\mu [\gamma - \text{trace}] + \frac{1}{4} \xi_\sigma \gamma_\mu \{ \text{eq. } \psi^\sigma \} - \frac{1}{4} \gamma^\sigma [\gamma - \text{trace}]$$

$$+ \frac{1}{16} \xi_\sigma \Gamma^i \gamma^\sigma \{ \text{eq. } \psi_\mu \} + \frac{1}{64} \xi_\sigma \Gamma^i \gamma_\mu \{ \text{eq. } \psi^\sigma \} - \frac{1}{4} \gamma^\sigma [\gamma - \text{trace}]$$

$$- \frac{1}{384} \xi_{\sigma \delta \tau} \Gamma^{ij} \gamma^{\sigma \delta \tau} \{ [\text{eq. } \psi_\mu ] - \frac{1}{4} \gamma_\mu [\gamma - \text{trace}] \} .$$

(2.39)

Summarizing, from the algebra we have obtained the complete fermionic equations of (2,0) six-dimensional supergravity coupled to $n$ tensor multiplets. In addition, the modified 3-form

$$\hat{H}^r_{\mu \nu \rho} = \hat{H}^r_{\mu \nu \rho} - \frac{i}{8} \nu^{ir} (\bar{\chi}^m \gamma_{\mu \nu \rho} \chi^m)^i$$

satisfies the (anti)self-duality conditions

$$G_{rs} \hat{H}^s_{\mu \nu \rho} = \frac{1}{6} \epsilon_{\mu \nu \rho \alpha \beta \gamma} \hat{H}_{\alpha \beta \gamma} .$$

(2.41)

We have also identified the complete supersymmetry transformations, that we collect here for convenience:

$$\delta e_\mu^a = -i (\bar{\epsilon} \gamma^a \psi_\mu) ,$$

$$\delta B^r_{\mu \nu} = i v^{ir} (\bar{\psi}_\mu \gamma_\nu \epsilon)^i + \frac{1}{2} x^{mr} (\bar{\chi}^m \gamma_{\mu \nu} \epsilon) ,$$

$$\delta v_r^i = \bar{v}_r^m (\bar{\chi}^m)^i ,$$

$$\delta x_m^r = v_r^i (\bar{\chi}^m)^i ,$$

$$\delta \psi_\mu = \hat{D}_{\mu} \epsilon + \frac{1}{4} v_r^i \hat{H}^r_{\mu \nu \rho} \Gamma^i \gamma^{\nu \rho} \epsilon - \frac{i}{64} (\bar{\chi}^m \gamma_{\mu \nu \rho} \chi^m) \gamma^{\nu \rho} \epsilon - \frac{i}{64} (\bar{\chi}^m \gamma_{\mu \rho} \chi^m) \Gamma^i \gamma^{\nu \rho} \epsilon$$

$$+ \frac{3 i}{64} [\bar{\chi}^m \gamma_\mu \chi^m]^{ij} \Gamma^{ij} \epsilon - \frac{i}{64} [\bar{\chi}^m \gamma_\nu \chi^m]^{ij} \Gamma^{ij} \gamma_{\mu \rho} \epsilon ,$$

$$\delta \chi^m = \frac{i}{2} x_r^m (\bar{\partial}_r \nu^{ir}) \Gamma^{i} \gamma^\alpha \epsilon + \frac{i}{12} x_r^m \hat{H}^r_{\alpha \beta \gamma} \gamma^{\alpha \beta \gamma} \epsilon .$$

(2.42)

From the equations of the fermi fields one obtains the complete Lagrangian

$$e^{-1} \mathcal{L} = - \frac{1}{4} R + \frac{1}{12} G_{rs} H^r_{\mu \nu \rho} H^{s \mu \nu \rho} + \frac{1}{4} x^{mr} x^{ms} (\partial_r v_r^i) (\partial^m v_s^i)$$

$$- \frac{i}{2} \bar{\psi}_\mu \gamma^{\mu \rho} D_\nu \left( \frac{1}{2} (\omega + \hat{\omega}) \right) \psi_\rho - \frac{i}{8} v_r^i \left[ H + \hat{H} \right]^{\mu \rho} (\bar{\psi}_\mu \gamma_\nu \psi_\rho)^i$$
\[
\begin{align*}
+ \frac{i}{48} v^r_i [H + \hat{H}]^r_{\alpha \beta \gamma} (\bar{\psi}_\mu \gamma^{\mu \nu \alpha \beta \gamma} \psi_\nu)^i + \frac{i}{2} (\bar{\chi}^m \gamma^\mu D_\mu (\hat{\omega}) \chi^m) \\
- \frac{i}{24} v^r_i \hat{H}^r_{\mu \nu \rho} (\bar{\chi}^m \gamma^{\mu \nu \rho} \chi^m)^i + \frac{1}{4} x^m_r [\partial_\nu v^i r + \partial_\nu \hat{v}^i r] (\bar{\psi}_\mu \gamma^\nu \gamma^\mu \chi^m)^i \\
- \frac{1}{8} x^m_r [H + \hat{H}]^{\mu \nu \rho} (\bar{\psi}_\mu \gamma_{\nu \rho} \chi^m) + \frac{1}{24} x^m_r [H + \hat{H}]^{\mu \nu \rho} (\bar{\psi}_\mu \gamma_{\nu \rho} \chi^m) \\
+ \frac{1}{8} (\bar{\chi}^m \gamma^{\mu \nu \rho} \chi^m) (\bar{\psi}_\mu \gamma_\nu \psi_\rho) - \frac{1}{64} [\bar{\chi}^m \gamma_\mu \chi^m]^{ij} [\bar{\chi}^n \gamma^\mu \chi^n]^{ij},
\end{align*}
\]

where, in the 1.5 order formalism, the spin connection

\[
\omega^{\mu \nu \rho} = \omega_0^{\mu \nu \rho} - \frac{i}{2} \{ (\bar{\psi}_\mu \gamma_\nu \psi_\rho) + (\bar{\psi}_\nu \gamma_\rho \psi_\mu) + (\bar{\psi}_\nu \gamma_\mu \psi_\rho) \}
- \frac{i}{4} (\bar{\psi}_\alpha \gamma_{\mu \nu \rho} \psi^\beta) - \frac{i}{4} (\bar{\chi}^m \gamma_{\mu \nu \rho} \chi^m)
\]

satisfies its equation of motion, and is thus kept fixed in all variations.

As we anticipated, varying \( \mathcal{L} \) with respect to the antisymmetric tensor \( B^r_{\mu \nu} \) yields the second-order tensor equation, the divergence of eq. (2.41).

Supersymmetry is finally proved showing that

\[
\begin{align*}
\delta F \frac{\delta \mathcal{L}}{\delta F} + \delta B \frac{\delta \mathcal{L}}{\delta B} = 0,
\end{align*}
\]

where \( F \) and \( B \) denote collectively the fermi and bose fields aside from the antisymmetric tensors, that are constrained to satisfy eq. (2.41). We would like to stress that the equations for the fermi fields defined from the gauge algebra differ from the lagrangian equations by overall factors that may be simply identified.

In order to study the flat-space limit, one has to rescale the fields following [10] and then let the gravitational constant \( \kappa \) tend to zero. The end result is a free field theory of \( n \) antiseft-dual tensors, \( 5n \) scalars and \( n \) right-handed spinors. In this limit the supersymmetry transformations become

\[
\begin{align*}
\delta B^m_{\mu \nu} &= \frac{1}{2} (\bar{\epsilon} \gamma_\mu \epsilon^m) , \\
\delta \phi^m_{i} &= (\bar{\epsilon} \chi^m)^i \\
\delta \chi^m &= -\frac{i}{2} (\partial_\mu \phi^m) \Gamma^i \gamma^{\mu} \epsilon + \frac{i}{12} H^m_{\mu \rho \nu} \gamma^{\mu \nu \rho} \epsilon
\end{align*}
\]

and the algebra closes on the equations of \( \chi^m \) to give a translation and a gauge transformation.
3 Truncation to $(1, 0)$ Supergravity

In [2] $(1, 0)$ supergravity coupled to $n$ tensor multiplets was obtained to lowest order in the fermi fields as a truncation of $(2, 0)$ supergravity. Here we generalize the construction, showing how the truncation acts on the $Sp(4)$ bilinears to give the $Sp(2)$ bilinears of ref. [3].

The scalars $v^i_r$, as well as the bilinears $(\bar{\psi}\chi)^i$, where $\psi$ and $\chi$ are generic spinors of different chirality, lose their $Sp(4)$ index and give $v_r$ and $(\bar{\psi}\chi)$ in the notations of [3]. In this way, for example, one can obtain the supersymmetry transformations of $\psi_\mu$ and $\chi^m$ to lowest order in the fermi fields [2]. On the other hand, the product of two bilinears with anomalous behavior under Majorana-flip,

$$[\bar{\psi}\chi]^{ij}[\bar{\lambda}\epsilon]^{ij}$$

becomes

$$-4[\bar{\psi}\chi]^i[\bar{\lambda}\epsilon]^i$$
in $(1, 0)$ notations. Following these rules, the complete supersymmetry transformation of the gravitino becomes in the $(1, 0)$ model

$$\delta\psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4}v_r \hat{H}^r_{\nu\rho} \gamma^{\nu\rho} \epsilon - \frac{i}{32}(\bar{\chi}^m \gamma_{\mu\nu\rho} \chi^m)\gamma^{\nu\rho} \epsilon$$

$$-\frac{3i}{16}[\bar{\chi}^m \gamma_\mu \chi^m]^i\sigma^i \epsilon + \frac{i}{16}[\bar{\chi}^m \gamma^\nu \chi^m]^i\sigma^i \gamma_{\mu\nu} \epsilon \ ,$$

(3.1)

that coincides with the fourth of eqs. (2.38) of [3] after some Fierz rearrangement. In the same way one can show that all other truncations correspond to terms already found in [3].

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Spinors satisfy the symplectic Majorana condition

\[
\psi^a = \Omega^{ab} C \bar{\psi}_b^T ,
\]  
(4.1)

where

\[
\bar{\psi}_a = (\psi^a)^\dagger \gamma_0 .
\]  
(4.2)

Any bilinear \( \bar{\psi}_a \chi^b \) carries a pair of \( Sp(4) \) indices, and can be decomposed in terms of the identity, the five \( \Gamma^i \) matrices and the ten \( \Gamma^{ij} \) matrices. Indeed, one can form the bilinears

\[
(\bar{\psi} \chi) = \bar{\psi}_a \chi^a , \quad (\bar{\psi} \chi)^i = \Gamma^{ia}_b \bar{\psi}_a \chi^b , \quad [\bar{\psi} \chi]^{ij} = \Gamma^{ij a}_b \bar{\psi}_a \chi^b ,
\]  
(4.3)

and using the \( SO(5) \) Clifford algebra one has

\[
\bar{\psi}_b \chi^a = \frac{1}{4} \delta^a_b (\bar{\psi} \chi) + \frac{1}{4} \Gamma^{ia}_b [\bar{\psi} \chi]^{i} - \frac{1}{8} \Gamma^{ij a}_b [\bar{\psi} \chi]^{ij} .
\]  
(4.4)

Using eq. (4.1), one can then see that the fermi bilinears \( (\bar{\psi} \chi) \) and \( (\bar{\psi} \chi)^i \) have standard behavior under Majorana flip, namely

\[
(\bar{\psi} \chi) = (\bar{\chi} \psi) , \quad (\bar{\psi} \chi)^i = (\bar{\chi} \psi)^i ,
\]  
(4.5)

while all ten bilinears \( [\bar{\psi} \chi]^{ij} \) have the anomalous behavior

\[
[\bar{\psi} \chi]^{ij} = -[\bar{\chi} \psi]^{ij} .
\]  
(4.6)

One can study Fierz relations between spinor bilinears using eq. (4.4). In particular, if \( \psi \) and \( \chi \) have the same chirality, one has

\[
\psi^a \bar{\chi}_b - \chi^a \bar{\psi}_b = -\frac{1}{8} (\bar{\chi} \gamma^\alpha \psi) \delta^a_b \gamma_\alpha - \frac{1}{8} (\bar{\chi} \gamma^\alpha \psi)^i \Gamma^{ia}_b \gamma_\alpha - \frac{1}{192} [\bar{\chi} \gamma^{\alpha \beta \gamma} \psi]^{ij} \Gamma^{ij a}_b \gamma_\alpha \beta \gamma
\]  
(4.7)

and

\[
\psi^a \bar{\chi}_b + \chi^a \bar{\psi}_b = \frac{1}{16} [\bar{\chi} \gamma^\alpha \psi]^{ij} \Gamma^{ij a}_b \gamma_\alpha + \frac{1}{96} (\bar{\chi} \gamma^\alpha \beta \gamma \chi) \delta^a_b \gamma_\alpha \beta \gamma + \frac{1}{96} (\bar{\chi} \gamma^\alpha \beta \gamma \chi)^i \Gamma^{ia}_b \gamma_\alpha \beta \gamma .
\]  
(4.8)
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