EXACT CONFIDENCE NETS BASED ON FINITE REFLECTION GROUPS

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Abstract. Confidence nets — that is, collections of confidence intervals that fill out parameter space and whose exact coverage can be computed — are familiar in nonparametric statistics. Here the distributional assumptions are based on invariance under the action of a finite reflection group. Exact confidence nets are exhibited for a single parameter, based on the root system of the group. The main result is a formula for the generating function of the interval probabilities. The proof makes use of the theory of “buildings” and the Chevalley factorization theorem for the length distribution on Cayley graphs.

1. Introduction

It is well known, and usually attributed to Wilks [31], that the order statistics from a random sample provide non-parametric confidence intervals for percentiles from a distribution: every interval formed by the order statistics covers the given percentile with a computable probability. For the median the probabilities are of binomial form. We shall refer to the situation in which the set of coverage intervals cover the real line and the coverage probability of each interval is computable as a confidence net.

An interesting example is given by Hartigan [15, 16] for the median, given an independent sample from a distribution symmetric about the median. There, the net is based on all sub-sample means: for a sample \( \{ y_i | i \in \mathbb{n} = \{1, \ldots, n\} \} \) and \( S \subset \mathbb{n} \), a subsample mean is \( \frac{1}{|S|} \sum_{i \in S} y_i \), in which each of the \( 2^n \) intervals has coverage probability \( \frac{1}{2^n} \). The work has had impact in the theory of the bootstrap and resampling (see Efron [7], Efron and Tibshirani [8]). It was a long-standing interest in the Hartigan result which gave birth to the current paper.

A third example is pairwise means, sometimes called Walsh averages. These are the basis for one version of the Hodge-Lehmann estimator for a mean [17], which is the empirical median of all pairwise means (including the single observation). There are strong connections to the
Wilcoxon signed rank (sum) test where the same generating function as here (see below) applies to obtain critical values \([24, 29]\). Indeed, the current paper could be represented as a group theoretic generalisation of the generating function approach of these papers, or, given the duality between testing and confidence intervals, as a way to invert certain permutation tests (see for example Trichler \([28]\)). As with the above examples we shall also study a one dimensional parameter case.

We first give an account of a general construction of non-parametric confidence interval nets and then specialise to the case of finite reflection groups in the following section, showing the relation to the root systems of the groups. Finite reflection groups (defined geometrically) have been classified completely up to isomorphism, and via this classification are also known as finite Coxeter groups (which have a purely algebraic definition based on their presentations). We will not elaborate on this classification here but refer the reader to \([19]\) or \([1]\). Sections 4 and 5 cover in some detail the case of the finite Coxeter groups of type \(B_n\) (the hyperoctahedral groups) and type \(D_n\). In both cases, simply by manipulating inequalities, we can describe the chamber boundaries and find the probability generating function for the distributions. It turns out that in the case of \(B_n\) the interval boundaries are the pairwise means, mentioned above, including the single observations. In the case of \(D_n\) they are the pairwise means, but excluding the single observations. The generating functions turn out to be familiar from the theory of partitions.

In Section 6 the main result of the paper is given, namely a generating function for the interval probabilities for a general finite Coxeter group (with one exception). The proof is given in Section 8 and relies on showing that the probabilities have the same “distribution” as the Coxeter length function for the quotient of the Coxeter group by the symmetric (permutation) group. To translate the geometry of the confidence net into the group theory requires the theory of buildings (given in Section 7) and specifically the mapping of intervals into “chambers” and the full problem into a discussion of “galleries”. Because of the strong link into group theory, we also put this paper forward as a contribution to the rapidly developing area of “algebraic statistics”, in which there has been renewed interest in permutation tests — see for example Morton et al \([25]\).

There is a long tradition of the study of “statistics” (also called indices) such as the length function, on groups. For example Reiner \([27]\) studied the extension of such statistics from the symmetric groups to \(B_n\). Such statistics have many and varied applications. For example, Adin and Roichman \([2]\) defined a new index called the flag major index whose length was equidistributed with length in the type \(B\) case. They used this to study group actions on polynomial rings. Geometric distance problems in genomic rearrangements can be reduced to Coxeter
length problems [10, 20]. In statistics, Diaconis [6, Chapter 4], makes
the connection between length distributions and non-parametric tests
and this paper could be seen as being in the same tradition.

Our formula is verified by corroborating the answer for $B_n$ and $D_n$
already derived in Sections 4 and 5. The generating functions for the
exceptional groups $E_6$, $E_7$, $E_8$ and for the groups of type $A_n$ are given
as examples after the main proof. The net in the case of $E_8$ has a
remarkable 93 cells.

The paper concludes with short sections on an example not in the
group class, some simple asymptotics, a link to majorization and some
concluding remarks.

2. Confidence nets

Let $Y$ be a random $n$-vector with distribution of density $f(y, \theta)$,
where $\theta$ is an unknown $p$-dimensional parameter. We assume that $Y$
can be transformed by a measurable transformation $T(y, \theta)$, typically
$\theta$-dependent, to a random variable $Z$:

$$Z = T(Y, \theta),$$

which is also $n$-dimensional and has a distribution some of whose prop-
ties are known, independently of $\theta$.

Our construction takes the following form.

There is a finite collection of sets $C_i$, for $i = 1, \ldots, m$, such that

1. $\bigcup C_i = \mathbb{R}^n$,
2. The measure with respect to $Z$ of any intersection $C_i \cap C_j$, $i \neq j$,
is zero,
3. $\text{prob}\{Z \in C_i\} = \alpha_i$, $i = 1, \ldots, m$.
4. The $\alpha_i$ are positive, do not depend on $\theta$, and $\sum_{i=1}^m \alpha_i = 1$.

With regard to point (2) above, it will simpler for every low dimen-
sional set to have measure zero. By taking the pre-image of any $C_i$ we
obtain a coverage statement for $\theta$

$$\text{prob}\{S_i(y) \ni \theta\} = \alpha_i,$$

where $S_i(y) := T^{-1}(y)(\cdot)$ is the inverse of the function $T(y, \theta)$ in its
second argument, for each $y$.

We should note that typically $p$ (the dimension of $\theta$) is very much
smaller than $n$. The construction is based on a coarsening in the de-
scription of the coverage sets $S_i$ using geometric considerations. We
shall refer to the following situation as exact.

Suppose that there are $N$ random sets $U_j(Y)$, $j = 0, \ldots, N - 1$ in $\theta$-
space whose intersections cover $\theta$ with zero probability, $\bigcup_{j=0}^{N-1} U_j = \mathbb{R}^p$,
and such that given any $j = 1, \ldots, m$

$$S_i(y) = U_{u(i)}(y)$$
for some computable injective mapping $u : i \mapsto u(i)$ that is independent of $y$. (The reason for indexing from 0 to $N - 1$ will become apparent in the connection with the group length function). This implies that the $U_j(y)$ are themselves coverage sets with coverage probabilities

$$p_i = \text{prob}\{U_j \ni \theta\} = \sum_{i \in u^{-1}(j)} \alpha_i$$

for $j = 0, \ldots, N - 1$. Note also that since $\sum_{j=0}^{N-1} \alpha_i = 1$, we have $\sum_{i=0}^{N-1} p_i = 1$. We refer to the set $\{U_j\}$ as a confidence net. The indexing by $j$ is critical in the interpretation of the sets as coverage sets.

Here we take a classical statistical approach to coverage nets. Thus, the notion is that the user declares the sets $U_j$. We are aware that coverage set based theories of inference can be thought of as part of a well developed theory of belief function based on upper and lower probabilities and the theory of random sets based on Choquet capacities (see [5, 30]). In terms of the former, a coverage set is a theory of random sets in which the upper and lower probabilities coincide, and in which the Choquet capacity functional is additive over the $\sigma$-algebra of unions of sets. That is to say, for the sets $\{U_j\}$ we have for $i \neq j$,

$$\text{prob}\{U_i \cup U_j \ni \theta\} = \text{prob}\{U_i \ni \theta\} + \text{prob}\{U_j \ni \theta\},$$

and so on.

3. Reflection group construction

In this construction the special sets $C_i$ will be the cones of a reflection group. We state the construction again from the beginning. Let $Y$ be an $n$-dimensional random vector and $\theta \in \mathbb{R}$ an unknown 1-dimensional parameter. Define

$$Z = (Y_1 - \theta, Y_2 - \theta, \ldots, Y_n - \theta)^T.$$

Let $G$ be a finite reflection group of order $m$ acting on $\mathbb{R}^n$ and let $C_i, (i = 1, \ldots, m)$ be the set of cones in $\mathbb{R}^n$ that are transformations under $G$ (including the identity) of the fundamental cone $C_1$. Assume that every such cone has the same probability content with respect to the distribution of $Z$:

$$\text{prob}\{Z \in C_i\} = \frac{1}{m}, \; i = 1, \ldots, m.$$

In vector notation we define

$$(3.1) \quad Z(y, \theta) = y - \theta j,$$

where $j = (1,1,\ldots,1)^T$. For fixed $y$ we shall refer to the one dimensional affine subspace defined by (3.1), as $\theta$ varies, as the ray from $y$, 

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Denoted $E_y$:

$$E_y = \{ z : z = y - \theta j, \ -\infty < \theta < \infty \}.$$

Each statement $\{ y \in C_i \}$ can be inverted to obtain an interval for $S_i(y)$, which depends on $y$. To find $\{ U_j \}$ we need to provide the mapping $i \mapsto u(i)$, not dependent on $y$. The $U_j$ are intervals and it is enough to give their endpoints.

Every finite reflection group is defined by its roots. These are vectors $a_j$ that define the perpendiculars to the defining hyperplanes $H_j = \{ x : a_j^T x = 0 \}$ forming the walls of the cones $C_i$. Roots come in pairs: $\pm a_j$, which are important in the classification of these groups (for more details on root systems and the classification of finite reflection groups, see for instance [19]). However, we shall need just one representative when we refer to a root in what follows.

Lemma 3.1. Let $\{ C_i \}$ be the collection of all cones generated in the standard way by a finite reflection group, and let $y$ be a non-zero vector.

1. If $y$ is in general position (not lying in any defining hyperplane $H_i$) then the ray $E_y$ intersects the faces of a fixed number $N$ of the cones $C_i$ at values $\theta_1(y) < \theta_2(y) < \ldots < \theta_{N-1}(y)$.

2. Any $\theta_j(y)$ is given by

$$\theta_j(y) = \frac{a_j^T y}{a_i^T a},$$

for some root $a$ which is not orthogonal to $j$.

3. $N - 1$ is the number of roots not orthogonal to $j$.

Proof. By elementary geometry, when $y$ is in general position the ray $E_y$ intersects every defining hyperplane $H_i$ exactly once except when $j$ lies in an $H_i$, in which case it does not intersect that $H_i$. Because for any hyperplane $H_i$, its root $a_i$ spans the orthogonal subspace to $H_i$, the latter condition is equivalent to being orthogonal to $a_i$. Each cone has two intersection points except for the end cones when the intersection are at $\theta_1(y)$ and $\theta_{N-1}(y)$. Part (2) follow since the intersection points satisfy: $a_j^T (y - \theta j) = 0$.

We refer to hyperplanes $H_i$ as being live if their roots are not orthogonal to $j$. The intervals we require are

$$U_0 = (-\infty, \theta_1(y)], \quad U_1 = [\theta_1(y), \theta_2(y)], \quad \ldots,$$

$$U_{N-2} = [\theta_{N-2}(y), \theta_{N-1}(y)], \quad U_{N-1} = [\theta_{N-1}(y), \infty).$$

Define the mapping $u(i) = j$ whenever $E_y \cap C_i = U_j(y)$. That is, effectively, $u$ maps cone $C_i$ to its position $j$ in the list of cones intersecting the ray $E_y$. Then $u^{-1}(j)$ indexes the set of cones giving the $j$-th chamber. Thus, for each $j$ define $n_j = |u^{-1}(j)|$, namely the number of
cones associated with \( U_j \). Then under our assumptions

\[
p_j = \text{prob} \{ U_j \ni \theta \} = \frac{n_j}{m}, \quad j = 0, \ldots, N - 1.
\]

Fortunately, although the orders of the groups can be very large, the number of roots is orders of magnitude smaller. The non-orthogonality condition cuts down the number of live intervals, \( N \), even further.

A main aim of this paper is to find the generating function for the \( n_j \):

\[
G(q) = \sum_{j=0}^{N-1} n_j q^j.
\]

4. The hyperoctahedral groups: type \( B_n \)

The group of type \( B_n \) (we will refer to the group simply as “\( B_n \)” ) operates on points \( z \in \mathbb{R}^n \) by permutation and sign change of the coordinates. It has order \( 2^n n! \). Its fundamental cone \( C_1 \) is given by

\[
z_1 \geq z_2 \geq \cdots \geq z_n \geq 0,
\]

and it has fundamental roots given by each of the inequalities above. In standard notation the roots are

\[
\{ e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n \},
\]

where the \( e_i \) are unit vectors. The roots are thus

\[
(1, -1, 0, \ldots, 0)^T, (0, 1, -1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1)^T.
\]

It is important to note that all other roots come from transformation of these roots under the group. Each cone \( C_i \) is obtained by transformation of \( C_1 \) under a suitable group element. We can describe the cones compactly by inequalities:

\[
\pm z_{\pi_1} \geq \pm z_{\pi_2} \geq \cdots \pm z_{\pi_n} \geq 0,
\]

where \( \pi = (\pi_1, \ldots, \pi_n) \in S_n \) ranges over all \( n! \) permutations of \( \{1, \ldots, n\} \).

Substituting \( z_i = y_i - \theta \) for \( i = 1, \ldots, n \), we have

\[
\pm (y_{\pi_1} - \theta) \geq \pm (y_{\pi_2} - \theta) \geq \cdots \geq \pm (y_{\pi_n} - \theta) \geq 0.
\]

Thus, as expected, every cone \( C_i \) induces a set of inequalities for \( \theta \), each of which yields an interval for \( \theta \).

For clarity we look at the case \( n = 4 \). Fix \( \pi \) as the identity so that we begin with the fundamental cone and then change signs. Now we count the number of intervals possible created from these inequalities. Using Lemma 3.1 we obtain \( N = 11 \) intervals which we label generically \( U_0, \ldots, U_{10} \). But since there are 16 inequalities (four choices of sign) the pigeonhole principle demands at least one interval to be represented by at least two sets of inequalities. There are 10 live roots (taking only one from each pair): \( (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1) \).
After some work we see that, for example, the inequality
\[ y_1 - \theta \geq -(y_2 - \theta) \geq y_3 - \theta \geq y_4 - \theta \geq 0 \]
is equivalent to
\[ y_2 \leq \frac{y_2 + y_3}{2} \leq \theta \leq \frac{y_2 + y_4}{2} \leq y_4 \leq \frac{y_3 + y_4}{2} \leq y_3 \leq \frac{y_1 + y_4}{2} \leq y_1. \]
This is the only set of inequalities giving \( \theta \) covered by the third interval. On the other hand the inequality
\[ -(y_1 - \theta) \geq (y_2 - \theta) \geq y_3 - \theta \geq -(y_4 - \theta) \geq 0 \]
yields
\[ y_1 \leq \frac{y_2 + y_3}{2} \leq y_4 \leq \frac{y_1 + y_3}{2} \leq \theta \leq \frac{y_3 + y_4}{2} \leq y_3 \leq \frac{y_2 + y_4}{2} \leq y_2, \]
giving the central interval. But the inequality
\[ y_1 - \theta \geq -(y_2 - \theta) \geq -(y_3 - \theta) \geq y_4 - \theta \geq 0 \]
also yields the central interval.

From this example we see how to evaluate the vector \((n_0, \ldots, n_{N-1})\) for \(B_n\). Thus, for any \(C_i\) the \(k\)-th interval covers \(\theta\) if and only if there are exactly \(k\) elements from the set of possible boundaries \(\{y_i, \frac{y_i + y_j}{2}, i < j\}\) less than or equal to \(\theta\). Thus, for our initial permutation
\[ n_k = |\{i : y_i \leq \theta\}| + |\{(i, j) : i \leq j; \frac{y_i + y_j}{2} \leq \theta\}|. \]
Now consider which sign combinations lead to a contribution to \(n_k\). The possibilities are:

1. \(y_i \leq \theta\): a single \(-\) at position \(i\).
2. \(y_i + y_j \leq \theta\): a pair \(-, +\) in positions \(i < j\), respectively.
3. \(\frac{y_i + y_j}{2} \leq \theta\): a pair \(-, -\) in positions \(i < j\), respectively, (inferred from (1)).

Define an indicator function given by \(x_i = 1, 0\) for \(-, +\) in the \(i\)-th position respectively. We can then set up a full indicator function for the \(i\)-th position:
\[ \phi(x) = \sum_{i=1}^{n} x_i + \sum_{i,j,i<j}^{n} x_i(1-x_j) + \sum_{i<j}^{n} x_ix_j \]
\[ = \sum_{i=1}^{n} (n - i + 1)x_i \]
\[ = \sum_{i=1}^{n} ix_i. \]
Note: had we started with the more standard order in statistics \(0 \leq x_1 \leq x_2 \leq \ldots \leq x_n\), we would not have to make this latter change.
Using a simple convolution argument considering the $x_i$ as independent Bernoulli random variables with probability $\frac{1}{2}$ we see that

$$G_n(q) = (1 + q)(1 + q^2) \cdots (1 + q^n).$$

For $n = 3$ we obtain

$$G_3(q) = 1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6,$$

while for $n = 4$

$$G_4(q) = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + q^9 + q^{10}.$$

We confirm $N = 11$ intervals with frequencies

$$(1, 1, 1, 2, 2, 2, 2, 1, 1, 1).$$

5. The groups of type $D_n$

The group of type $D_n$ (which we again will call simply the group $D_n$) is the group of permutations with an even number of sign changes, and has order $2^{n-1}n!$. It has fundamental cone

$$z_1 \geq z_2 \geq \cdots \geq z_n, \quad z_{n-1} + z_n \geq 0.$$  

The roots are

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n - e_{n-1} + e_n\},$$

giving

$$(1, -1, 0, \ldots, 0)^T, (0, 1, -1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1, -1)^T, (0, \ldots, 0, 1, 1)^T.$$

In the case $n = 4$ the inequalities are

$$z_1 \geq z_2 \geq z_3 \geq z_4; \quad z_3 \geq -z_4,$$

giving

$$y_1 - \theta \geq y_2 - \theta \geq y_3 - \theta \geq y_4 - \theta; \quad y_3 - \theta \geq -(y_4 - \theta).$$

Now $D_4$ allows permutations of the coordinates as well as even numbers of sign changes. Therefore the possible signs are as follows (the vertical and horizontal lines will be explained shortly):

|   |   |   |   |
|---|---|---|---|
| + | + | + | + |
| - | - | + | + |
| - | + | - | + |
| + | - | - | + |
| - | + | + | - |
| + | - | + | - |
| + | + | - | - |
| - | - | - | - |
The second line, for example, gives
\[-(y_1 - \theta) \geq -(y_2 - \theta) \geq y_3 - \theta \geq y_4 - \theta; \quad y_3 - \theta \geq -(y_4 - \theta),\]
from which we deduce
\[\theta \geq \frac{y_1 + y_3}{2}, \frac{y_1 + y_4}{2}, \frac{y_2 + y_3}{2}, \frac{y_2 + y_4}{2}, \frac{y_1 + y_2}{2};\]
\[\theta \geq \frac{y_3 + y_4}{2}.
\]
The inequality \(\theta \geq \frac{y_1 + y_2}{2}\) is found by first noting that \(\theta \geq y_2 \geq y_1\). It is tempting to include the singletons \(y_i\) in the set of boundary points, but not all \(y_j\) can be determined in this way which means that intervals using the \(y_i\) are not fully computable.

Following the last remark, we determine coverage of \(\theta\) given by all pair means \(\frac{y_i + y_j}{2}, i < j\). We shall need to account for the following possibilities using slightly more complicated rules than for \(B_n\):

1. \(y_i \leq \theta\): a single \(-\) at position \(i\), for \(i \in \{1, \ldots, n-1\}\). This is used to help place the pair-means.
2. \(\frac{y_i + y_n}{2} \leq \theta\): \(-\) in position \(i\) and \(-\) in position \(n\) for \(i \in \{1, \ldots, n-1\}\).
3. \(\frac{y_i + y_j}{2} \leq \theta\) for \(1 \leq i < j \leq n\): a pair \(-, +\) in positions \(i, j\) respectively.
4. \(\frac{y_i + y_j}{2} \leq \theta\) for \(1 \leq i < j \leq n-1\): a pair \(-, -\) in positions \(i, j\) respectively. This follows by noting that \(y_i, y_j \leq \theta\), as in part 1.

We can split rule 3 into two cases: when the \(+\) is in position \(1, \ldots, n-1\); and when the \(+\) is in position \(n\) (hence the vertical line in the preceding figure). The latter can be combined with rule 2 to give rule 1. Again we take the indicator with \(x_i = 1\) for \(-\) and \(x_i = 0\) for \(+\). Then
\[\phi(x) = \sum_{i=1}^{n-1} x_i + \sum_{i<j}^{n-1} x_i(1-x_j) + \sum_{i<j}^{n-1} x_i x_j \]
\[= \sum_{i=1}^{n-1} (n-i)x^i.
\]
Similarly to \(B_n\) we prefer, without loss of generality:
\[\phi(x) = \sum_{i=1}^{n-1} ix^i.
\]
The generating function for the \(n_i\), again using a convolution argument, is
\[G_{n-1}(q) = (1 + q^2)(1 + q^3) \cdots (1 + q^{n-1}).\]
6. THE GENERAL CASE

As mentioned in the introduction we will use some theory developed around the concept of indices (sometimes call “statistics”) attached to an element $g$ of a group $G$. MacMahon [22] discussed, for the symmetric group, descent, excedance, length and the major index. Authors are often interested in the frequency of the distinct values of an index as $g$ ranges over the whole group, and there are strong combinatorial results, going back to MacMahon, showing that one index has the same distribution as another, even though the actual indices (as mappings) are different. This work is relevant for us because (i) we have a special index which is the value $j$ of our interval $U_j$ of the net construction; (ii) generating functions play an important role; and (iii) the study of such indices is being extended to finite reflection groups such as $B_n$ and $D_n$.

A starting point for the construction of these indices is the Cayley graph of a group. If $S$ is the set of generators of our group, then the Cayley graph is a graph $(E, V)_G$ where each vertex $v_g \in V$ is labelled by a group element $g \in G$ and each edge $e_{g,h}$ by a single right multiplication by a generator $s \in S : h = gs$; only generators may be used. The length, $l(g)$ of a group element $g \in G$ is the length of the minimal path on the graph from the identity $e$ to $g$, when each edge counts unity:

$$l(g) = \min\{k \geq 0 : g = s_{i_1}s_{i_2}\cdots s_{i_k}, \text{ for } s_{i_1} \in S\}.$$

The Cayley graph for the group $B_2$ has two generators which we may take as (i) $s_1$ reflection in the line $z_1 = z_2$ and (ii) $s_2$ reflection in the $z_1 = 0$ axis. The Cayley graph and corresponding lengths are given in Figure 1.

The length frequency distribution is $\{f_0, f_1, \ldots, f_m\}$ where $f_j = \#\{g : l(g) = j, g \in G\}$ and $m$ is the diameter of the group. The
Table 1. Degrees for the crystallographic Coxeter groups

| Type | \(d_1, d_2, \ldots\) |
|------|------------------|
| \(A_n\) | \(2, 3, \ldots, n + 1\) |
| \(B_n, C_n\) | \(2, 4, \ldots, 2n - 2, 2n\) |
| \(D_n\) | \(2, 4, 6, \ldots, 2n - 2, n\) |
| \(E_6\) | \(2, 5, 6, 8, 9, 12\) |
| \(E_7\) | \(2, 6, 8, 10, 12, 14, 18\) |
| \(E_8\) | \(2, 8, 12, 14, 18, 20, 24, 30\) |
| \(F_4\) | \(2, 6, 8, 12\) |
| \(G_2\) | \(2, 6\) |

The length distribution for the symmetric group is

\[ G_S(q) = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q}. \]

In the main theorem, which follows, the formula is obtained by dividing the generating function for the length distribution of our group given in Theorem 6.1, by that for the symmetric group.
Theorem 6.2. The generating function for the frequency distribution for the intervals of the confidence net based on a finite irreducible Coxeter group $G$ of any type except $F_4$ is given by

$$G(q) = \frac{\prod_{j=1}^{m}(1 - q^{d_j})}{\prod_{i=1}^{n}(1 - q^{j})},$$

where $d_1, \ldots, d_m$ are the basic invariant degrees of the group.

The exclusion of $F_4$ in the theorem statement is necessary because the result depends on the symmetric group being a maximal parabolic subgroup of $G$, which is it in all cases except $F_4$ (see for instance [11, Appendix A]). We leave the calculation of the generating function for $F_4$ as an exercise along the lines of the examples in Sections 4 and 5.

Before we prove this theorem, we demonstrate with two examples that its results agree with those calculated in Sections 4 and 5.

For $B_n$ the formula in Theorem 6.2 gives

$$\frac{G_W(q)}{G_n(q)} = \frac{\prod_{i=1}^{n}(1 - q^{2i})}{\prod_{i=1}^{n}(1 - q^{i})} = \prod_{i=1}^{n}(1 + q^i),$$

as expected. Note that we have two ways of counting the number of intervals: the number of live roots, following Lemma 3.1, and the degree of $G(s)$:

$$n + \left(\begin{array}{c} n \\ 2 \end{array}\right) = \sum_{j=1}^{n} j.$$

For $D_n$ the formula is

$$\frac{G_W(q)}{G_n(q)} = \frac{\prod_{j=1}^{n-1}(1 - q^{2j})(1 - q^{n})}{\prod_{i=1}^{n}(1 - q^{j})} = \prod_{j=1}^{n-1}(1 + q^i),$$

again, as expected.

7. Chamber graphs and Cayley graphs

To prove Theorem 6.2, we need to introduce some of the group-theoretic geometry behind it. An excellent reference for further reading on this topic is [1, Chapter 1].

The chamber graph of a finite reflection group has cones (called chambers in this context) as vertices, with two cones having an edge if they share a common face. A path in the chamber graph is called a gallery: imagine a walk through chambers with doors in the common wall (face). With each edge given length unity, distance between chambers is defined (as for a Cayley graph) by the shortest distance between the chambers, and we call the corresponding gallery minimal. A gallery is minimal if it does not cross any wall more than once ([1] Proposition 1.56). Since a straight line in general position (in an obvious sense) cannot cut any wall of a chamber more than once, it defines a minimal
gallery. For both the Cayley graph and the chamber graph, $C_e$ is the cone corresponding to the identity element $e$ of the group, and we call this the fundamental cone.

Following Lemma 3.1, the ray $E_y$ defines a gallery that we denote $G_y$. This gallery starts in the identity chamber $C_e$ and has length $N$ (there are $N$ chambers along it). The index $j$ of a chamber $C_g$ yielding the interval $[\theta_j(y), \theta_{j+1}(y)]$ is the distance in the gallery $G_y$ from $C_e$ to $C_g$. Consequently, the number of cones $u^{-1}(j)$ that map into a given index $j$ is the number of group elements of distance $j$ from $C_e$ along the gallery $G_y$.

Now consider reflections in the walls of a chamber. Suppose this chamber is a translation by $w$ of the fundamental chamber $C_e$, so that its faces are translations of the fundamental hyperplanes that are the faces of $C_e$. If $H_s$ is a face of $C_e$ (for a generator $s$ of $G$), then it is translated by $w$ to $wH_s$, and reflection in this hyperplane corresponds to action by the reflection $ws w^{-1}$ (in general this is not a fundamental reflection). Thus, reflection in the face $wH_s$ of $wC_e$ gives the chamber given by the left multiplication of $w$ by $ws w^{-1}$, namely $ws w^{-1} w = ws$, or $ws C_e$. In other words we move from the chamber $wC_e$ to the chamber $ws C_e$. Thus, movement along a gallery corresponds to right multiplication by a generator.

As an aside, it is worth noting that the movement along the gallery by right multiplication provides a correspondence between the chamber graph and the Cayley graph, in which the movement along edges is given by left multiplication $w \rightarrow sw$. The chamber graph, however, is the natural place for our results because it has a very direct link with the geometry.

8. Proof of Theorem 6.2

Theorem 6.2 states, in effect, that the distribution of distances of group elements along galleries defined by the rays $E_y$ is the same as the distribution of lengths of minimal coset representatives when the quotient of $G$ is taken by the symmetric group $S_n$. This is because the numerator of this generating function is the Poincaré polynomial of the group, and the denominator is that of the symmetric group (see [19, Section 1.11] for more details). For background reading on the theory of reflection groups and buildings there are many good sources, but we recommend in particular Humphreys [19], Abramenko and Brown [1] and Kane [21].

So to prove Theorem 6.2, it suffices to show that the set of group elements along the galleries defined by the rays $E_y$ is precisely the set of minimal length coset representatives of $S_n$ in $G$, for $G$ of the types given in the Theorem. We prove this in Proposition 8.3, below, but first a short lemma.
Lemma 8.1. The roots from $S_n$ are all orthogonal to $j = (1, 1, \ldots, 1)^T$.

Proof. Action by any element of $S_n$ fixes $j$; that is, reflection in any root from $S_n$ fixes $j$, which means that the root must be orthogonal to $j$. □

There are some well-known facts about Coxeter groups that we will refer to in what follows, listed in the following Lemma.

Lemma 8.2. Let $W$ be a finite Coxeter group, and $W'$ a parabolic subgroup of $W$.

1. The minimal length elements of the cosets of $W'$ in $W$ are unique.
2. The minimal length elements of the cosets of $W'$ in $W$ add in length when multiplied by any element of $W'$.
3. If $W'$ is a parabolic subgroup of $W$, then the length distribution of distinguished coset representatives of $W'$ in $W$ is given by $G_W(t)/G_{W'}(t)$, where $G_W(t)$ and $G_{W'}(t)$ are the respective Poincaré polynomials.
4. The length of the longest word $w_0$ in $S_n$ is the number of positive roots in the root system of $S_n$.

Proof. These are all given in various texts, but in particular all are in [19]: for 1 and 2 see [19, Section 1.10]; for 3 see [19, Section 1.11]; and for 4 see [19, Section 1.8]. □

Proposition 8.3. For chambers defined by the action of a finite Coxeter group on $\mathbb{R}^n$, let the gallery $G_y$ be the series of adjacent chambers beginning with $C_e$, defined by the ray $E_y$ where $y$ is some point in $C_e$.

(i) The group elements labelling chambers in $G_y$ are all minimal length $S_n$-coset representatives.
(ii) Every $S_n$-coset has its minimal length element appearing on a gallery $G_y$ for some $y$.

Proof. First note that $S_n$ is a parabolic subgroup of every finite Coxeter group $G$ except $F_4$, and we consider $G$ acting on $\mathbb{R}^n$ [11, Appendix A]. Explicitly: $S_n$ is a parabolic subgroup of the groups of types $A_n$, $B_n$ and $D_n$ that we consider acting on $\mathbb{R}^n$, and of the groups of types $E_n$ for $n = 6, 7, 8$ (acting on $\mathbb{R}^n$); $S_3$ and $S_4$ are parabolic subgroups of the groups $H_3$ and $H_4$ acting on $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively; and $S_2$ is (rather trivially) a parabolic subgroup of each of the dihedral groups $I_2(m)$ acting on $\mathbb{R}^2$.

Part (i).

We begin by showing that the element corresponding to the last chamber in the gallery is in the same coset as the longest word in the group. The length of the longest word $w_0$ in $S_n$ is the number of positive roots in the root system of $S_n$ (Lemma 8.2(4)), which is the number of roots orthogonal to $j$ (Lemma 8.1). Let $C$ be the $S_n$-coset
containing \(w_0\). Because minimal coset elements add in length with any element of \(S_n\) (Lemma 8.2(2)), the minimal coset representative in \(C\) must have length the number of positive roots not orthogonal to \(j\).

An element of \(C\) appears in some gallery \(G_y\), for some \(y\), by Lemma 8.3. The number of chambers in each gallery is the number of roots not orthogonal to \(j\). Since crossing each hyperplane from one chamber to the next along the gallery adds at most 1 in length, the longest element on any gallery is at most length the number of roots not orthogonal to \(j\). Therefore the longest word of the group must be in a coset whose minimal length element is precisely the number of roots not orthogonal to \(j\). This can only be the last chamber in the gallery.

We now show that all other group elements on the gallery are minimal right coset representatives.

Take a minimal right cost representative \(s_{i_1} \ldots s_{i_m}\) on the gallery \(G_y\). We claim that the preceding element on the gallery, \(s_{i_2} \ldots s_{i_m}\), is also a minimal length coset representative. If not, then there is a \(w \in S_n\) satisfying \(\ell(s_{i_2} \ldots s_{i_m}w) < \ell(w) + (m - 1)\) (length is additive for minimal coset representatives in Coxeter groups; see Lemma 8.2(2)). But then \(\ell(s_{i_1}s_{i_2} \ldots s_{i_m}w) < \ell(w) + (m - 1) + 1\) since multiplying by a generator can add at most 1 to the minimal length. That is, \(\ell(s_{i_1}s_{i_2} \ldots s_{i_m}w) < \ell(w) + m\), contradicting the minimality of \(s_{i_1} \ldots s_{i_m}\). It follows that all elements corresponding to cones along a gallery \(G_y\) are minimal coset representatives.

Part (ii).

It suffices to show that each \(S_n\) coset contains an element in the gallery \(G_y\). Consider points \(Z = (Z_1, \ldots, Z_n)\) in the fundamental cone that are also on the ray \(E_y = \{y - \theta j \mid \theta \in \mathbb{R}\}\). The inequality they must satisfy is

\[
Z_n \geq \cdots \geq Z_1 \geq 0 \tag{8.1}
\]

for \(Z = (Z_1, \ldots, Z_n)\). Because the entries in \(j\) are all equal, this means \(y_n \geq \cdots \geq y_1 \geq 0\) for \(y = (y_1, \ldots, y_n)\).

Moving along the ray \(E_y\) in the positive direction (decreasing \(\theta\)) does not change the inequalities in Equation (8.1) (all components stay positive) and hence the ray stays in the fundamental cone. Increasing \(\theta\) moves the ray through the gallery into different cones as first \(Z_n \geq \cdots \geq Z_3 \geq 0 \geq Z_1\), then \(Z_n \geq \cdots \geq Z_3 \geq 0 \geq Z_2 \geq Z_1\) and so on.

On the other hand, acting by \(S_n\) on a point on the ray permutes the entries, but this fixes the entries of \(j\) and simply permutes the entries of \(y\). So a chamber on the gallery corresponds to an ordering of form \(Z_n \geq \cdots \geq Z_{i+1} \geq 0 \geq Z_i \geq \cdots \geq Z_1\), and the other chambers in its \(S_n\) coset are obtained by permuting these entries.

Now consider an arbitrary point \(v\) in \(\mathbb{R}^n\) and denote the cone it is contained within by \(C_w(v)\). The action of \(S_n\) permutes the entries of \(v\), and there is a permutation that puts the entries in increasing order.
Every point that is in increasing order is in a cone that is on a ray-
gallery, so we are done. □

We are now in a position to prove our main result. Recall that this
gives a generating function for the frequency distribution of inter-
vals in the confidence net based on an irreducible finite Coxeter group.

\textbf{Proof of Theorem 6.2.} The set of all rays \( E_y \) from the identity chamber
to the last chamber (labelled by \( g \)) gives the set of all possible galleries
from 1 to \( g \). Each group element in each gallery is a minimum length
coset representative of \( S_n \) in \( G \), and that all \( S_n \)-cosets have their mini-
mal length representative occurring in such a gallery (Proposition 8.3).

Recall that \( u^{-1}(j) \) is the set of chambers that are in the \( j \)’th position
along a ray \( E_y \). When we start with the identity chamber, this is simply
the set of group elements of length \( j \) that appear in galleries. From
Proposition 8.3, this is the set of minimal coset representatives of length
\( j \). So the number \( n_j \) is the number of minimal coset representatives of
length \( j \), and this is given by our formula, by Lemma 8.2(3). □

\section{9. Further examples}

In Sections 4 and 5 we gave examples of confidence nets from our
theory for types \( B \) and \( D \) respectively, and recalculated them using
Theorem 6.2 after its statement. Here we add types \( E_6 \), \( E_7 \), \( E_8 \) and
\( A_n \). The remaining types of finite Coxeter group (omitted) are types
\( H_3 \) and \( H_4 \), the dihedral groups \( I_2(m) \), and the group \( F_4 \) (to which
Theorem 6.2 doesn’t apply).

\subsection{9.1. Type \( E \).}

Inserting the \( d_j \) values for \( E_6 \), \( E_7 \) and \( E_8 \) from Table 1
and obtaining help in factorization from Maple we have the following
formulae:

\begin{align*}
\text{\emph{E}}_6 & : \quad (q + 1)(q^2 + 1)(q^2 - q + 1)(q^4 - q^2 + 1)(q^2 + q + 1)(q^6 + q^3 + 1)(q^4 + 1) \\
\text{\emph{E}}_7 & : \quad (q + 1)^2(q^2 - q + 1)^2(q^6 + q^3 + 1)(q^6 - q^3 + 1) \\
& \quad (q^6 - q^5 + q^4 - q^3 + q^2 - q + 1) \\
& \quad (q^2 + 1)(q^2 + q + 1)(q^4 - q^2 + 1)(q^4 - q^3 + q^2 - q + 1)(q^4 + 1) \\
\text{\emph{E}}_8 & : \quad (q^4 + q^3 + q^2 + q + 1)(q^6 + q^3 + 1)(q^6 - q^3 + 1)(q^4 + 1) \\
& \quad (q^6 - q^5 + q^4 - q^3 + q^2 - q + 1)(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1) \\
& \quad (q^8 + q^7 - q^5 - q^4 - q^3 + q + 1)(q^8 - q^6 + q^4 - q^2 + 1)(q^8 - q^4 + 1) \\
& \quad (q^2 + q + 1)^2(q^4 - q^3 + q^2 - q + 1)^2(q^2 + 1)^2 \\
& \quad (q^4 - q^2 + 1)^2(q^2 - q + 1)^3(q + 1)^4.
\end{align*}
Let us consider $E_8$ in a little more detail. It has order $2^{143}3^52^7 = 696729600$. This means that $\mathbb{R}^8$ is split into this many cones. The root system is described in the standard way as:

$$\{\pm e_i \pm e_j : 1 \leq i < j\}, \quad \left\{ \frac{1}{2} \sum_{i=1}^{8} \lambda_i E_i : \lambda_i = \pm 1, \prod_{i=1}^{8} \lambda_i = 1 \right\}.$$ 

Again we have two ways of counting. The number of live roots are those not orthogonal to $j = (1, 1, 1, 1, 1, 1, 1, 1)^T$. From the first set above we have those of the form $(1, 1, 0, \ldots)$, namely $\binom{8}{2}$. From the second set we have all those for which the number of ones and zeros is different and even, being careful not to double count. This gives

$$\binom{8}{2} + 1 + \binom{8}{2} + \binom{8}{6} = 92.$$ 

On the other hand $G(s) = a_0 + a_1 q + \ldots$ is a polynomial of degree 92 whose $N=93$ coefficients are laid out below to show the symmetry.

|   | 1  | 1  | 1  | 2  | 3  | 6  | 6  | 8  | 10 | 13 | 17 |
|---|----|----|----|----|----|----|----|----|----|----|----|
| 1 | 21 | 26 | 32 | 38 | 46 | 55 | 64 | 74 | 86 | 98 | 112|
| 2 | 127| 142| 157| 175| 193| 211| 230| 249| 267| 287| 307|
| 3 | 325| 343| 361| 377| 393| 409| 421| 432| 443| 452| 458|
| 4 | 464| 466| 466| 466| 466| 464| 458| 452| 443| 325| 307|
| 5 | 307| 287| 267| 249| 230| 211| 193| 175| 157| 142| 127|
| 6 | 112| 98 | 86 | 74 | 64 | 55 | 46 | 38 | 32 | 26 | 21 |
| 7 | 17 | 13 | 10 | 8  | 6  | 4  | 3  | 2  | 1  | 1  | 1  |

9.2. **Type $A_n$.** The usual interpretation of the action of type $A_n$ is as the restriction of the symmetric group $S_{n+1}$ to the hyperplane: $H : \sum_{i=1}^{n} x_i = 0$. When $n = 2$ this yields a figure in 2-dimensions with cones with apex angle $\frac{1}{3}\pi$. The role of $S_n$ in the above examples is now played by $A_{n-1}$. Referring to the first entry in Table 1 this gives the generating function

$$G(q) = \frac{\prod_{i=1}^{n} (1 - q^i)}{\prod_{i=1}^{n} (1 - q^i)} = \sum_{i=0}^{n} q^n,$$

giving a discrete uniform distribution on the net chambers.

The statistical interpretation takes a little care. There are different choices one can make for the representation of $A_{n-1}$ as a subgroup of $A_n$. A simple choice is for $A_{n-1}$ to be the restriction to the hyperplane $H$ of the group that permutes the first $n$ coordinates. Let us require that the “data” $Y$ also lies in $H$ and that the model is given by

$$Y = \theta k + Z$$

where $\text{prob}\{Z \in C_i\} = \frac{1}{(n+1)!}$ and the $C_i$ are the cones of $A_n$ in $H$ (with similar assumptions as is in the introduction). The key is to make the
vector \( \mathbf{k} \), which is the analogue of the previous \( \mathbf{j} \), to be invariant under \( A_{n-1} \). Thus, we can take

\[
\mathbf{k} = (1, 1, \ldots, 1, -n)^T.
\]

Following Lemma 3.1, we find the boundary of the net chamber by taking the intersection of the ray \( Y - \theta \mathbf{k} \) with the live root of \( A_n \) that is all those not orthogonal to \( \mathbf{k} \). These are

\[
(1, 0 \ldots, 0 - 1)^T, (1, 0 \ldots, 0 - 1)^T, \ldots, (0, 0 \ldots, 0, 1, -1)^T.
\]

Taking the \( j \)-th member of this list first we see that the boundary is given by

\[
y_j - \theta - (y_{n+1} + n\theta) = y_j - y_{n+1} - (n + 1)\theta.
\]

But since \( Y \in H \) we have \( y_{n+1} = -\sum_{i=1}^{n} y_i \). This means that the boundaries are the \( n \) sample quantities

\[
\frac{1}{n + 1} \left( 2y_j + \sum_{i \neq j} x_i \right), \ j = 1, \ldots, n,
\]

giving \( n + 1 \) chambers, as expected.

10. A non-group cone example

Exact coverage nets also arise for the situation in which \( \mathbb{R}^n \) is divided into cones that are congruent, but not generate as the fundamental cones of a reflections group. Consider the partition of the positive orthant into \( n \) cones generate by a “long diagonal” and \( n - 1 \) principal axes. Leaving out the first principal axes we obtain generators:

\[
(1, 1, \ldots, 1)^T, (0, 1, 0, \ldots, 0)^T, (0, 0, 1, \ldots, 0)^T, \ldots, (0, 0, \ldots, 1)^T.
\]

The other cones generated by successively omitting principal axes. Now take all sign changes to reach all other quadrants. This divides \( \mathbb{R}^n \) into \( n^2 \) congruent cones.

Assume the \( Z \)-probability content of each cone is equal and apply the method used for the other examples. We first check how many, and which, are walls are cut by a typical ray, and group together the cones which lead to the same “index”, as above. The number of planes is \( 2n + 1 \). After a little work it turns the intervals formed by the order statistics \( y_{(1)} < y_{(2)} < \ldots < y_{(n)} \) and all neighbour pair \( \frac{y_{(i)} + y_{(i+1)}}{2} \) form a net of \( 2n \) intervals.

The successive net vectors (ignoring commas) are the rows below for \( n = 2, \ldots, 6 \).

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 \\
1 & 2 & 3 & 3 & 3 & 1 & 1 \\
1 & 1 & 4 & 4 & 6 & 6 & 4 & 4 & 1 & 1 \\
1 & 1 & 5 & 5 & 10 & 10 & 10 & 5 & 5 & 1 & 1 \\
\end{array}
\]
Note how each row is constructed by repeating the integer of the previous row of the Pascal triangle e.g. the row $1, 6, 15, 20, 15, 6, 1$ is split $6 \rightarrow (1, 5)$, $15 \rightarrow (5, 10)$, $20 \rightarrow (10, 10)$, giving the last row of the tableau above. The generating function is

$$(1 + q)(1 + q^2)^{n-1}.$$ 

11. Some asymptotics

The generating function for $B_n$ is well-known in the theory of partitions. It is the generating function for the number partition of an integer into at most $n$ distinct parts. The infinite version $G(q) = \prod (1 + q^i)_{i=1}^\infty$ gives the number of partitions into $j$ distinct parts, with no other restrictions, and the two generating functions are identical up to $q^n$. The general $G(q)$ has a long history. Following their celebrated work on partitions [14], Hardy and Ramanujan also studied this case, giving an asymptotic formula, see [18] [13]. For an extensive review see [3].

Noting the convergence of the Binomial distribution to the Normal (and following computer experimentation), it is natural to conjecture that for $G_n(s)$ the $\{a_n\}$ follow an asymptotic distribution, and indeed this is the case. The associated probability distribution is that of the random variable

$$U = \sum_{j=1}^n j V_j,$$

where the $V_i$ are iid Bernoulli random variables with probability $\frac{1}{2}$. Then, a theorem of Hájek and Sidák [12] for sums of independent random variable with unequal means and variance gives $U \sim N(\mu, \sigma)$ where $\mu = \frac{1}{2} \sum_{j=1}^n j = \frac{1}{4} n(n+1)$ and $\sigma^2 = \frac{1}{4} \sum_{j=1}^n j^2 = \frac{1}{24} n(n+1)(2n+1)$.

An Edgeworth-type expansion shows that the standardized random variable $\frac{U - \mu}{\sigma}$ can be approximated by

$$\phi(u) \left(1 + \frac{\kappa_4}{24} H_4(u)\right),$$

where $\phi$ is the standard Normal density, $\kappa_4$ is the fourth cumulant of the standardized variable, and $H_4 = u^4 - 6u^2 + 3$ is the order 4 standard Hermite polynomial.

After a little work we derive, for $B_n$,

$$\kappa_4 = -\frac{12}{5} \frac{3n^2 + 3n - 1}{n(n+1)(2n+1)}.$$ 

Keeping the $O\left(\frac{1}{n}\right)$ terms we have the approximation

$$\phi(u) \left(1 - \frac{3}{20} H_4(u) \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right).$$

There are similar result for $D_n$. 
For $E_8$ the distribution mean and variance are $(\mu, \sigma^2) = (46, \frac{6811}{3})$ and probabilities roughly follow a normal distribution with this mean and variance. The approximation using $H_4$ is surprisingly good. For the standardized distribution

\[ \frac{\kappa_4}{24} = -\frac{365311}{28896080} = -0.01264 \ldots. \]

Converting the approximation back to the original cell probabilities the maximum absolute deviation and the root mean squared error are approximately $1.5 \times 10^{-4}$ and $7.3 \times 10^{-5}$ respectively. For the Edgeworth-type approximation, the integral (or in this case the sum) of the approximate probability will typically not be unity. In this case the sum of the approximands is 1.0001534\ldots so that the error is of the same order as the maximum deviation.

12. A MAJORIZATION RESULT

It is possible to obtain geometric results concerning the actual intervals $U_i$ depending on the nature of the vector $Z$ and related to the finite reflection group $G$. Following [23], for two vectors $Z, Z'$ in $\mathbb{R}^n$ we say that $Z' \prec_G Z$ if and only if $Z$ lies in the convex hull of the orbit of $Z'$ under $G$:

\[ Z \in \mathcal{C}(Z') = \text{conv}\{g(Z') : g \in G\}. \]

Consider, within the theory described above, a live hyperplane defined by root $a_i$ : $H_i = \{ z : a_i^T z = 0 \}$. Its intersection \( a_i \neq a_j \), with the ray $Y = Z - \theta_j$ occurs at

\[ \theta_{i,Z} = \frac{a_i^T Z}{a_i^T j}. \]

The difference between two such values is

\[ \theta_{i,Z} - \theta_{j,Z} = \left( \frac{a_i^T}{a_i^T j} - \frac{a_j^T}{a_j^T j} \right) Z. \]

Now assume $Z \prec_G Z'$, then there are $\alpha_g \geq 0$, $g \in G$, $\sum_{g \in G} \alpha_g = 1$ such that

\[ Z = \sum_{g \in G} \alpha_g g(Z'). \]

Then,

\[ \theta_{i,Z} - \theta_{j,Z} = \left( \frac{a_i^T}{a_i^T j} - \frac{a_j^T}{a_j^T j} \right) \sum_{g \in G} \alpha_g g(Z') \]

\[ = \sum_{g \in G} \alpha_g \left( \frac{a_i^T}{a_i^T j} - \frac{a_j^T}{a_j^T j} \right) g(Z'). \]
Thus,
\[ |\theta_i, Z - \theta_j, Z| \leq \sum_{g \in G} \alpha_g \left| \left( \frac{a_i^T}{a_j^T} \right) g(Z') \right|, \]
or
\[ |\theta_i, Z - \theta_j, Z| \leq \sum_{g \in G} \alpha_g \left| \left( \frac{a_i^T}{a_j^T} \right) g(Z') \right|, \]
giving
\[ |\theta_i, Z - \theta_j, Z| \leq \sum_{g \in G} \alpha_g |\theta_{i,g}(Z') - \theta_{j,g}(Z')|. \]

These moduli can be interpreted as the length of the interval between coverage intervals, but with not necessarily neighbouring end points.

We now use a standard procedure to create an invariant:
\[
\sum_{h \in G} |\theta_{i,h}(Z) - \theta_{j,h}(Z)| \leq \sum_{h \in G} \sum_{g \in G} \alpha_g |\theta_{i,gh}(Z') - \theta_{j,gh}(Z')| 
= \sum_{g \in G} \alpha_g \sum_{h \in G} |\theta_{i,gh}(Z') - \theta_{j,gh}(Z')|. 
\]

Fixing \( g \) but taking all \( h \in G \) implies that \( gh \) also ranges over all of \( G \) so that the internal sum on the right-hand side does not depend on \( g \). Noting that \( \sum \alpha_g = 1 \) and relabelling the group elements with \( g \) we have proved the following lemma.

**Lemma 12.1.** For any finite reflection group \( G \), let \( Y = Z + \theta_j \) and \( Y' = Z' + \theta_j \) be two samples in which \( Z \prec_G Z' \). Then under the invariance described in Section 1, for any \( i \neq j \) (corresponding to live roots) the length of the coverage intervals satisfy
\[
\sum_{g \in G} |\theta_{i,g}(Z) - \theta_{j,g}(Z)| \leq \sum_{g \in G} |\theta_{i,g}(Z') - \theta_{j,g}(Z')|. 
\]

This says that the measure of concentration \( \sum_{g \in G} |\theta_{i,g}(Z) - \theta_{j,g}(Z)| \), is a \( \prec_G \) order preserving function uniformly over all roots \( a_i \neq a_j \).

**13. Conclusions and further work**

This paper is a contribution to coverage problems in which, essentially, there are only symmetry conditions on the underlying distribution. Clearly, then, this can be seen as a species of classical non-parametric statistics. The most obvious limitation of the present paper is that it only covers a single parameter. But the authors consider that it contains the ingredients of a more ambitious programme to develop coverage nets based on chambers in more than one dimension. Roughly, the requirements are (i) a large group that houses the distributional assumptions, (ii) a smaller sub-group under which the (linear) model is invariant, (iii) a valid quotient or coset operation, (iv) the use...
of the Chevalley factorization formula to perform the counting, and (v) the use of the theory of buildings.

Another challenge is to apply the methods to infinite groups and Weyl groups are an example. It may be easier to carry out the mathematical treatment than to find statistical application, but there is hope in that such groups already have applications in physics, material science and genomics [4], [26], [9].

Finally, note that throughout we have derived exact nets. In cases where this is hard one may be satisfied with upper and lower probabilities and it seems an open problem to apply the above methods.

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