A generalization for the infinite integral over three spherical Bessel functions

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Abstract

A new formula is derived that generalizes an earlier result for the infinite integral over three spherical Bessel functions. The analytical result involves a finite sum over associated Legendre functions, \( P_m^l(x) \) of degree \( l \) and order \( m \). The sum allows for the values of \(|m|\) that are greater than \( l \). A generalization for the associated Legendre functions to allow for any rational \( m \) for a specific \( l \) is also shown.

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1. Introduction

The calculations of infinite integrals that involve a product of spherical Bessel functions have been the focus of many papers, amongst which are [1–22]. In particular, [12] showed a derivation for an analytical evaluation of an infinite integral over three spherical Bessel functions, given by the form

\[
I(\lambda_1, \lambda_2, \lambda_3; k_1 k_2 k_3) = \int_0^\infty r^2 j_{\lambda_1}(k_1 r) j_{\lambda_2}(k_2 r) j_{\lambda_3}(k_3 r) dr.
\]

(1.1)

This type of integral has received considerable attention due to its use in nuclear scattering theory [9–14]. In this paper, we extend this result to analytically evaluate infinite integrals of the form

\[
I(\lambda; \lambda_1, \lambda_2, \lambda_3; k_1 k_2 k_3) = \int_0^\infty r^{2-\lambda} j_{\lambda_1}(k_1 r) j_{\lambda_2}(k_2 r) j_{\lambda_3}(k_3 r) dr.
\]

(1.2)

This type of integral has in the past been attempted both analytically [8] and numerically [17]. The authors believe that the evaluation carried out here is more compact, easier to work with and extendible. The new result, like the earlier one when \( \lambda = 0 \), only works when \(|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2 \), i.e. when the integer indices \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) satisfy the triangular condition, and \( \lambda_1 + \lambda_2 + \lambda_3 \) is even. However, unlike the earlier result where the triangular condition is satisfied, the new result does not impose any such condition.
condition forces $k_1, k_2$ and $k_3$ to form the sides of a triangle, this generalized integral can have non-zero values when the $k$'s do not form the sides of a triangle. Our new result is only derived for the case when the $k$'s do form the sides of a triangle. We also show the general result, i.e. for any values of $k_1$, $k_2$ and $k_3$, when $\lambda = 1$. This new result involves finite sums over the associated Legendre function, $P^m_l(x)$, of degree $l$ and order $m$, whereas the earlier result involved sums over the Legendre polynomial, $P_l(x)$. The sums can involve values for $|m|$ that are larger than $l$. An extension for the associated Legendre functions which produces a formula that allows for such values is shown in appendix A. An application is shown to evaluate an integral involving four spherical Bessel functions and is used in the angular integration of the homogeneous and velocity isotropic Boltzmann equation. This integral, which is highly oscillatory and needs special treatment when evaluated numerically, is reduced to an integral over an associated Legendre function and a Legendre polynomial combined with algebraic factors which can easily be evaluated numerically. A comparison is shown in appendix B between our analytical result and the analytical evaluation carried out in [8].

2. Generalizing the integral over three spherical Bessel functions

An earlier result [12] showed that an infinite integral over three spherical Bessel functions can be written as

$$I(\lambda_1, \lambda_2, \lambda_3; k_1 k_2 k_3) = \frac{\pi \beta(\Delta)}{4k_1 k_2 k_3} \left[ \frac{\lambda_1}{k_1} \right]^{\lambda_1} \left[ \frac{\lambda_2}{k_2} \right]^{\lambda_2} \left[ \frac{\lambda_3}{k_3} \right]^{\lambda_3} \sum_{l=0}^{\lambda_1} \left( \frac{2\lambda_3}{2k_1} \right)^{1/2} \left( \frac{k_2}{k_1} \right)^{\ell} \sum_{l'=0}^{\lambda_3} (2l+1) \left( \frac{\lambda_1 - \lambda_3 - \ell - l'}{0} \right) \right] P_l(\Delta),$$

(2.1)

where $\Delta = (k_1^2 + k_2^2 - k_3^2)/2k_1k_2$ and $\beta(\Delta) = \theta(1 - \Delta)\theta(1 + \Delta)$ with $\theta$ the Heaviside function in half-maximum convention. $P_l(x)$ is a Legendre polynomial of degree $l$, $\left( \frac{\lambda_1}{k_1} \right)$ is a $3j$ symbol and $\left( \frac{\lambda_1}{k_1} \right) \left( \frac{\lambda_2}{k_2} \right) \left( \frac{\lambda_3}{k_3} \right)$ is a $6j$ symbol which can be found in any standard angular momentum text [23, 24]. Note that the summand in $l$ vanishes unless $|\lambda_1 - (\lambda_3 - \ell)| \leq l \leq \lambda_1 + \lambda_3 - \ell$. Now, multiply both sides by $k_3^{\lambda_3+2}$ and integrate over $k_3$ from 0 to $K$, where $K$ can have any positive value. The left-hand side involves the integral (see [25], equation (5.52-1), p 661)

$$\int_0^K k_3^{\lambda_3+2} j_{\lambda_3}(k_3r)dk_3 = \frac{1}{r} K^{\lambda_3+2} j_{\lambda_3+1}(Kr).$$

(2.2)

The right-hand side involves the integral

$$J = \int_0^K \beta(\Delta) k_3 P_l(\Delta)dk_3.$$

(2.3)

Now, $k_3dk_3 = -k_1 k_2 d\Delta$. So, when $k_3 = 0$ then $k_1 = k_2$ and $\Delta = 1$. Also, when $k_3 = K$, $\Delta = \Delta'$, where $\Delta' = (k_1^2 + k_2^2 - K^2)/2k_1k_2$. Hence,

$$J = k_1 k_2 \int_{\Delta'}^1 \beta(\Delta) P_l(\Delta)d\Delta.$$

(2.4)

If $\Delta' > 1$, i.e. $K < |k_2 - k_1|$, then $J = 0$. If $-1 \leq \Delta' \leq 1$, then

$$J = k_1 k_2 \beta(\Delta')(1 - \Delta')^{1/2} P^{-1}_l(\Delta'),$$

(2.5)
using [15]

\[ P_l^m(x) = (1 - x^2)^{-m/2} \int_{-1}^{1} \cdots \int_{-1}^{1} P_l(x)(dx)^m, \tag{2.6} \]

where \( P_l^m(x) \) is the associated Legendre function of the first kind of degree \( l \) and order \( m \). If \( \Delta' < -1 \), then

\[ J = \int_{-1}^{1} P_l(\Delta)d\Delta = 2k_1k_2\theta[K - (k_1 + k_2)] \delta_{l,0}. \tag{2.7} \]

Hence, the result is

\[
\begin{aligned}
&\left( \lambda_1 \lambda_2 \lambda_3 \right) I(1; \lambda_1, \lambda_2, \lambda_3; k_1k_2K) = \frac{\pi \beta(\Delta')}{4K^2} l^{\lambda_1+\lambda_2-\lambda_3}(2\lambda_3 + 1)^{1/2} \left( \frac{k_1}{K} \right)^{\lambda_1} (1 - \Delta')^{1/2} \\
&\quad \times \sum_{L=0}^{\lambda_3} \left( \frac{2\lambda_3}{2L} \right)^{1/2} \left( \frac{k_2}{k_1} \right)^L \sum_{L=0}^{\lambda_1} \left( \frac{2L + 1}{2L} \right) \left( \frac{\lambda_1 - L - l}{0} \right) \left( \frac{\lambda_2 - L - l}{0} \right) \\
&\quad \times \left\{ \frac{\lambda_1 \lambda_2 \lambda_3}{L \lambda_3 - L - l} \right\} P_l^{\lambda_3-\lambda_3}(\Delta') + (-1)^{\lambda_3} \pi \frac{k_1^2 k_2^2}{2K^{\lambda_3+2}} \frac{\sqrt{\lambda_3 + 1}}{(2\lambda_1 + 1)(2\lambda_2 + 1)} \\
&\quad \times \left( \frac{2\lambda_1 \lambda_2 \lambda_3}{2\lambda_2} \right)^{1/2} \theta[K - (k_1 + k_2)] \delta_{\lambda_1,\lambda_2,\lambda_3}. \tag{2.8} \end{aligned}
\]

Result (2.8) applies for any values of \( k_1, k_2 \) and \( K \). The second term on the right-hand side of (2.8) is also consistent with equation (6.578-4) of [25], when \( \lambda_3 = \lambda_1 + \lambda_2 \). This equation applies for \( K > k_1 + k_2 \) and can be written in the form

\[
\int_0^\infty \rho^{\lambda_1+\lambda_2-\lambda_3+1} j_{\lambda_1}(k_1r) j_{\lambda_2}(k_2r) j_{\lambda_3+1}(Kr) dr = 2^{\lambda_1+\lambda_2-\lambda_3-2} \pi^{3/2} \frac{k_1^2 k_2^2}{K^{\lambda_3+2}} \frac{\Gamma(\lambda_3 + 3/2)}{\Gamma(\lambda_2 + 3/2)\Gamma(\lambda_1 + 3/2)}. \tag{2.9} \]

If we assume that \( K \) satisfies the triangular condition, then the second term in (2.8) vanishes, and by repeated integration one arrives at (renaming \( K \) back to \( k_3 \))

\[
I(\lambda; \lambda_1, \lambda_2, \lambda_3; k_1k_2k_3) = \frac{\pi \beta(\Delta)}{4k_1k_2k_3} l^{\lambda_1+\lambda_2-\lambda_3}(2\lambda_3 + 1)^{1/2} \left( \frac{k_1}{k_3} \right)^{\lambda_3} \left( \frac{k_2}{k_3} \right)^{\lambda_1} \left( \frac{k_3}{k_3} \right)^{\lambda_2} (1 - \Delta')^{1/2} \tag{2.10} \]

\[
\times \left\{ \frac{\lambda_1 \lambda_2 \lambda_3}{L \lambda_3 - L - l} \right\} P_l^{\lambda_3-\lambda_3}(\Delta),
\]

where \( \lambda, \lambda_1, \lambda_2 \) and \( \lambda_3 \) are all greater than or equal to 0 to ensure convergence of the integral. Equation (2.10) is a new result that generalizes the infinite integral over three spherical Bessel functions. In general \( \lambda \) can exceed the value for \( l \). In this case the definition of \( P_l^m(x) \) needs to be extended for \( |m| > l \) in a way compatible with equation (2.6). Appendix A discusses such a generalization for the associated Legendre functions. Several explicit results for equation (2.10) are presented in appendix B. For some small values of the indices, these have been compared with a different existing result for the integral as well be discussed there.
3. Special cases and identities

Equation (2.10) can be reduced to a known integral [3] for the case \( \lambda_3 = 0 \), i.e. \( \lambda_1 = \lambda_2 \equiv \lambda' \) with the result

\[
I(\lambda; \lambda', \lambda', 0; k_1k_2k_3) = \frac{\pi \beta(\Delta)}{4k_1k_2k_3} \left( \frac{k_1k_2}{k_3} \right)^\lambda (1 - \Delta^2)^{\lambda/2} P_\lambda^\lambda(\Delta),
\]

(3.1)

when \( k_1, k_2 \) and \( k_3 \) satisfy the triangular condition

\[
|k_1 - k_2| \leq k_3 \leq k_1 + k_2.
\]

If we set \( \lambda = \lambda_2 = 0 \) and \( \lambda_1 = \lambda_3 = \lambda' \) in equation (2.10) and equate it to equation (3.1) after setting \( \lambda = 0 \) and interchanging \( k_2 \) and \( k_3 \), the following sum rule for the Legendre polynomial is obtained:

\[
\beta(\eta) \sum_{\ell=0}^{\lambda} \binom{\lambda}{\ell} \left( \frac{-k_2}{k_1} \right)^\ell P_\ell(\eta) = \beta(\eta') \binom{\lambda}{k_1} P_\lambda(\eta'),
\]

(3.3)

where \( \eta = (k_1^2 + k_2^2 - k_3^2)/2k_1k_2 \) and \( \eta' = (k_1^2 + k_3^2 - k_2^2)/2k_1k_3 \). Equation (3.3) is consistent with the result of [26]. In a triangle of sides \( k_1, k_2 \) and \( k_3 \), \( \eta \) is the cosine of the angle facing side \( k_3 \) and \( \eta' \) is the cosine of the angle facing side \( k_2 \).

4. Applications

One application is an integral over four spherical Bessel functions which arises in the angular integration of the homogeneous and velocity isotropic Boltzmann equation [19, 20]

\[
I(L; N + M - L, 0, N, M; k_1k_2k_3k_4) \equiv \int_0^\infty r^{2L - N} j_{N+M-L}(k_1r) j_0(k_2r) j_N(k_3r) j_M(k_4r) dr,
\]

(4.1)

where \( L, N, M \) are non-negative integers with \( N + M \geq L \) and \( k_1, k_2, k_3, k_4 \) the sides of a quadrilateral. Using the closure relation for the Bessel functions [2, 19], this integral can be written as

\[
\frac{2}{\pi} \int_0^\infty K^2 dK I(L; N + M - L, 0, N + M - L; k_1k_2K) I(0; N, M, N + M; k_3k_4K).
\]

(4.2)

Now, using equation (2.10)

\[
I(L; N + M - L, 0, N + M - L; k_1k_2K) = \frac{\pi \beta(\xi)}{4k_1k_2K} \left( \frac{k_1}{K} \right)^{N+M-L} \times \left( \frac{k_1k_2}{K} \right)^L (1 - \xi^2)^{L/2} \sum_{\ell=0}^{N+M-L} (-k_2/k_1)^\ell \binom{N + M - L}{\ell} P_\ell^{-L}(\xi),
\]

(4.3)

where \( \xi = (k_1^2 + k_2^2 - K^2)/2k_1k_2 \) and using

\[
\left( \frac{2l}{2l'} \right)^{1/2} \left( \begin{array}{ccc} l & l - l' & l' \\ 0 & 0 & 0 \end{array} \right) = \left( -1 \right)^l \frac{l}{\sqrt{2l+1}} \left( \begin{array}{c} l \\ l' \end{array} \right).
\]

(4.4)
the resulting integral is

\[ I(L; N + M - L, 0, N, M; k_1 k_2 k_3 k_4) = \frac{\pi^2}{16 k_1 k_2 k_3 k_4} (k_1 k_3)^{N + M} k_2^L \sqrt{2(N + M) + 1} \]

\[ \times \left( \begin{array}{ccc} N & M & N + M \\ 0 & 0 & 0 \end{array} \right) \sum_{\ell = 0}^{N + M - L} (-k_2/k_1)^{\ell/2} \ell (k_4/k_3)^{\ell} \left( \frac{2N + 2M}{2\ell} \right)^{1/2} \]

\[ \times \left( \begin{array}{ccc} N + M - L \\ \ell \end{array} \right) \sum_{l} (2l + 1) \left( \begin{array}{ccc} N & N + M - \ell' & l \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} M & \ell' & l \end{array} \right) S(k_1 k_2 k_3 k_4; L M N \ell'), \quad (4.5) \]

where, using \( \xi' = (k_1^2 + k_2^2 - K^2)/2k_3k_4 \),

\[ S(k_1 k_2 k_3 k_4; L M N \ell') = \int_0^\infty \frac{\beta(\xi)\beta(\xi')}{K^{2(N+M)}} (1 - \xi')^{L/2} P^{L-L}_\ell(\xi) P_{j}(\xi') dK, \quad (4.6) \]

is an integral that is easily done numerically as it converges rapidly.

5. Conclusions

A new generalized analytical formula for the infinite integral over three spherical Bessel functions was shown. It involves finite sums over the associated Legendre function combined with angular momentum coupling coefficients 3-j and 6-j symbols. The sums involved values of the order which exceeded the degree, which is in conflict with the usual definition of the associated Legendre functions. An extension for the associated Legendre functions is shown in appendix A.

Appendix A. A generalized solution to the associated Legendre equation

The associated Legendre function of the first kind, \( P_l^m(x) \), is a solution of the differential equation [25]

\[ (1 - x^2) \frac{d}{dx} P_l^m(x) - 2x \frac{d}{dx} P_l^m(x) - \left[ l(l + 1) - \frac{m^2}{1 - x^2} \right] P_l^m(x) = 0, \quad (A.1) \]

given by

\[ P_l^m(x) = (-1)^m \frac{(1 - x^2)^{m/2}}{2^l l!} \frac{d^{lm}}{dx^{lm}} (x^2 - 1)^l, \quad (A.2) \]

defined for integer degree \( l \geq 0 \) and integer order \( m \) (with \( |m| \leq l \)). In this appendix, it will be shown that another solution exists for this differential equation that extends the associated Legendre function to any rational \( |m| \) \((|m| < \infty)\) for integer \( l \). This solution is irregular at \( x = 1 \) when \( m > l \) and at \( x = -1 \) for \( m < l \). The formula can be used to derive closed form expressions for \( P_l^m \) for a specific \( l \) and any rational \( m \).

If we set \( l = 0 \) in equation (A.1), it is easy to show that the following function, \( P_0^m(x) \), is a solution:

\[ P_0^m(x) = a_m \left( \frac{1 - x}{1 + x} \right)^{-m/2}, \quad (A.3) \]

where \( a_m \) is a coefficient that depends on \( m \) (for general \( m \) this restricts \( x \) to \(-1 < x < 1\)). Now, if we assume that the dependence on \( l \) can be separated, then in general \( P_l^m \) can be written as

\[ P_l^m(x) = A_{l,m}(x) P_0^m(x), \quad (A.4) \]
where $A_{l,m}(x)$ is a coefficient that depends on $l$, $m$ and $x$. To determine the coefficient (up to a constant), substitute equation (A.4) back into equation (A.1). The resulting differential equation for $A_{l,m}$ is

$$(1 - x^2) \frac{d^2 A_{l,m}(x)}{dx^2} + (2m - 2x) \frac{dA_{l,m}(x)}{dx} + l(l + 1)A_{l,m}(x) = 0.$$  \hspace{1cm} (A.5)

This identifies $A_{l,m}$ as the Jacobi polynomial, \cite{25} $P_{l}^{-m,m}(x)$, defined by

$$P_{l}^{m,b}(x) = \frac{(-1)^l}{2^{l}l!} (1 - x)^{-m}(1 + x)^{-b} \frac{d^l}{dx^l}[(1 - x)^{l+m}(1 + x)^{l+b}].$$  \hspace{1cm} (A.6)

Hence, equation (A.4) becomes

$$P_{l}^{m}(x) = b_{l,m} \left( \frac{1 - x}{1 + x} \right)^{-m/2} P_{l}^{-m,m}(x),$$  \hspace{1cm} (A.7)

where $b_{l,m}$ is a constant that depends on $l$ and $m$. Using the normalization for the associated Legendre functions

$$\int_{-1}^{1} \left[ P_{l}^{m}(x) \right]^2 dx = \frac{2}{2l + 1} \frac{(l + m)!}{(l - m)!},$$  \hspace{1cm} (A.8)

and the normalization for the Jacobi polynomials

$$\int_{-1}^{1} (1 - x)^a(1 + x)^b \left[ P_{l}^{m,b}(x) \right]^2 dx = 2^{a+b+1} \frac{(l + a)!(l + b)!}{l!(a + b + 2l + 1)(l + a + b)!},$$  \hspace{1cm} (A.9)

one can square equation (A.7) and integrate it over $x$ from $-1$ to $1$, then compare it with equation (A.8) to obtain an expression for $b_{l,m}$:

$$b_{l,m} = \frac{l!}{|l - m|!},$$  \hspace{1cm} (A.10)

where the absolute value in $|l - m|!$ is introduced to allow $m$ to go from $-\infty$ to $\infty$, and the expression reduces to $(l - m)!$ for $-l \leq m \leq l$. We are interested here in integer $m$. $|l - m|!$ is to be replaced with $\Gamma(|l - m| + 1)$ here and below, when $m$ is not an integer. Equation (A.7) then becomes

$$P_{l}^{m}(x) = \frac{(-1)^l}{2^{l}l!} \left( \frac{1 - x}{1 + x} \right)^{m/2} \frac{d^l}{dx^l} \left[(1 - x^2)^l \left( \frac{1 + x}{1 - x} \right)^m \right].$$  \hspace{1cm} (A.11)

which agrees with equation (2.4), page 12 of \cite{27}, and is valid for integer degree $l \geq 0$, rational order $-\infty < m < \infty$ and $-1 < x < 1$.

When $-l \leq m \leq l$, equations (A.11) and (A.2) are equal, leading to the identity

$$\frac{d^{l+m}}{dx^{l+m}}(1 - x^2)^l = (-1)^m \frac{l!}{(l - m)!} (1 + x)^{-m} \frac{d^l}{dx^l} \left[(1 - x^2)^l \left( \frac{1 + x}{1 - x} \right)^m \right],$$  \hspace{1cm} (A.12)

for $-l \leq m \leq l$.

Another property that can easily be found is

$$P_{l}^{m}(x) = \frac{(-1)^l}{|l - m|!} \frac{l!}{|l - m|!} P_{l}^{-m}(x),$$  \hspace{1cm} (A.13)

for positive integer $l$ and any rational $m$.

Using equations (A.7) and (A.10), one can express the associated Legendre function in terms of the Jacobi polynomial as

$$P_{l}^{m}(x) = \frac{l!}{|l - m|!} \left( \frac{1 - x}{1 + x} \right)^{-m/2} P_{l}^{-m,m}(x).$$  \hspace{1cm} (A.14)
Using the recurrence relation for the Jacobi polynomials [25]

\[ 2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)P_n^{(\alpha, \beta)}(x) \]
\[ = (2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)x + \alpha^2 \beta^2 P_n^{(\alpha, \beta)}(x) \]
\[ - 2(n + \alpha)(n + \beta)(2n + \alpha + \beta + 2)P_{n-1}^{(\alpha, \beta)}(x), \]  

(A.15)

we find the following recurrence relation for the generalized associated Legendre functions:

\[ |l - m + 1|P_{m+1}^l(x) = (2l + 1)x P_m^l(x) - \frac{(l^2 - m^2)}{|l - m|} P_{m+1}^l(x). \]  

(A.16)

Using this recurrence relation and equation (A.11) one finds the following closed-form expressions for the associated Legendre functions at specific \( l \)'s and any rational \( m \):

\[ P_0^m(x) = \frac{1}{|m|!} \left( \frac{1 + x}{1 - x} \right)^{m/2}, \]  

(A.17)

\[ P_1^m(x) = \frac{1}{|1 - m|!} (x - m) \left( \frac{1 + x}{1 - x} \right)^{m/2}, \]  

(A.18)

\[ P_2^m(x) = \frac{1}{|2 - m|!} (3x^2 - 3xm - 1 + m^2) \left( \frac{1 + x}{1 - x} \right)^{m/2}, \]  

(A.19)

\[ P_3^m(x) = \frac{1}{|3 - m|!} (15x^3 - 15x^2m - 9x + 6xm^2 + 4m - m^3) \left( \frac{1 + x}{1 - x} \right)^{m/2}, \]  

(A.20)

\[ P_4^m(x) = \frac{1}{|4 - m|!} (105x^4 - 105x^3m - 90x^2 + 45x^2m^2 + 55xm - 10xmn^3 + 9 - 10m^2 + m^4) \left( \frac{1 + x}{1 - x} \right)^{m/2}. \]  

(A.21)

Appendix B. Explicit expressions

Here we provide some explicit expressions for integral (1.2) and compare to those obtained from a result given in [8], which reads in our notation

\[ I(\lambda; \lambda_1, \lambda_2, \lambda_3; k_1k_2k_3) = -\pi^{\lambda_1+\lambda_2+\lambda_3} (\text{sgn}(k_1 + k_2 + k_3)) F_{\lambda_1\lambda_2\lambda_3}(k_1, k_2, k_3) \]
\[ + (-1)^{\lambda_1} \text{sgn}(-k_1 - k_2 + k_3) F_{\lambda_1\lambda_1\lambda_3}(k_1, k_2, k_3) \]
\[ + (-1)^{\lambda_2} \text{sgn}(k_1 - k_2 + k_3) F_{\lambda_2\lambda_1\lambda_3}(k_1, k_2, k_3) \]
\[ + (-1)^{\lambda_3} \text{sgn}(-k_1 + k_2 + k_3) F_{\lambda_3\lambda_1\lambda_3}(k_1, k_2, k_3), \]  

(B.1)

with

\[ F_{\lambda_1\lambda_2\lambda_3}(k_1, k_2, k_3) = \sum_{m_1=0}^{\lambda_1} \frac{(\lambda_1 + m_1)!(-1)^{m_1}}{(\lambda_1 - m_1)!m_1!(2k_1)^{m_1}} \sum_{m_2=0}^{\lambda_2} \frac{(\lambda_2 + m_2)!(-1)^{m_2}}{(\lambda_2 - m_2)!m_2!(2k_2)^{m_2+1}} \]
\[ \times \sum_{m_3=0}^{\lambda_3+\lambda_2} \frac{(\lambda_3 + \lambda + m_3)!(-1)^{m_3}}{(\lambda_3 + \lambda - m_3)!m_3!(2k_3)^{m_3+1}} \]
\[ \frac{(k_1 + k_2 + k_3)^{\lambda_1+\lambda_2+\lambda_3+m_1+m_2+m_3+\lambda}}{(m_1 + m_2 + m_3 + \lambda)!}. \]  

(B.2)
Restricting ourselves to values of $k_1$, $k_2$ and $k_3$ which satisfy the triangular condition and using symmetries of $F_{\lambda_1,\lambda_2,\lambda_3}$, (B.1) can be simplified to

$$I(\lambda; \lambda_1, \lambda_2, \lambda_3; k_1, k_2, k_3) = -\pi \beta(\Delta) \frac{\pi}{k_1 k_2 k_3} \lambda \sum_{m_1=0}^{\lambda_1} \frac{(-1)^{m_1} (\lambda_1 + m_1)!}{(\lambda_1 - m_1)! m_1! (2k_1)^{m_1+1}}$$

$$\times \sum_{m_2=0}^{\lambda_2} \frac{(-1)^{m_2} (\lambda_2 + m_2)!}{(\lambda_2 - m_2)! m_2! (2k_2)^{m_2+1}} \sum_{m_3=0}^{\lambda_3} \frac{(-1)^{m_3} (\lambda_3 + \lambda + m_3)!}{(\lambda_3 + \lambda - m_3)! m_3! (2k_3)^{m_3+1}}$$

$$\times \frac{1}{(m_1 + m_2 + m_3 + \lambda)!} \left( (k_1 + k_2 + k_3)^{m_1+m_2+m_3+\lambda} - (-1)^{\lambda_1+m_1} c_1^{m_1+m_2+m_3+\lambda} - (-1)^{\lambda_2+m_2} c_2^{m_1+m_2+m_3+\lambda} - (-1)^{\lambda_3+m_3} c_3^{m_1+m_2+m_3+\lambda} \right), \quad (B.3)$$

with $c_1 = (-k_1+k_2+k_3)$, $c_2 = (k_1-k_2+k_3)$ and $c_3 = (k_1+k_2-k_3)$. For numerical computations this expression is more suitable than (B.1) in which almost cancellations between the different terms including sgn functions can lead to large relative errors. For some small values of $\lambda$, $\lambda_1$, $\lambda_2$ and $\lambda_3$ this yields the following explicit results:

$$I(0; 0, 0; 0; k_1 k_2 k_3) = \frac{1}{4} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3}$$

$$I(0; 1, 0; 1; k_1 k_2 k_3) = -\frac{1}{16} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (-2k_1 k_3 + k_1^2 + 2k_1 c_1 - 2k_1 c_2 - 2k_1 c_3 + k_3^2 - 2k_3 c_1 - 2k_3 c_2 - 2k_3 c_3 - k_2^2 + c_2^2 - c_3^2)$$

$$I(0; 1, 1; 0; k_1 k_2 k_3) = -\frac{1}{16} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (-2k_1 k_2 + k_1^2 + 2k_1 c_1 - 2k_1 c_2 - 2k_1 c_3 + k_2^2 - 2k_2 c_1 - 2k_2 c_2 - 2k_2 c_3 - k_3^2 + c_3^2$$

$$I(1; 0, 0; 0; k_1 k_2 k_3) = \frac{1}{16} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (k_1^3 - 2k_1 c_1 - 2k_1 c_2 + k_1 c_3 - k_2^2 + c_1^2$$

$$I(1; 1, 0; 1; k_1 k_2 k_3) = \frac{1}{64} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (2k_1^2 k_3^2 - 4k_2^2 c_1^2 - 12k_1 c_1^2 k_3 + c_2^4 + 8k_1 c_3^2)$$

$$- 8k_1 k_1 c_3^2 + 8k_1 k_2 c_2$$

$$I(1; 1, 1; 0; k_1 k_2 k_3) = \frac{1}{192} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (-k_2 c_1^3 - 6k_2^2 k_1^2 c_3 + 2k_1 k_2 k_3 c_2 - 12k_1 c_1^2 k_2$$

$$+ c_2^4 + 12k_1 k_3 c_2^2 - 12k_1 c_1 k_3 - 12k_2 k_3 c_2^2$$

$$I(1; 1, 1; 0; k_1 k_2 k_3) = \frac{1}{192} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (-k_2 c_1^3 - 6k_2^2 k_1^2 c_3 + 2k_1 k_2 k_3 c_2 - 12k_1 c_1^2 k_2$$

$$+ c_2^4 + 12k_1 k_3 c_2^2 - 12k_1 c_1 k_3 - 12k_2 k_3 c_2^2$$

$$I(1; 1, 1; 0; k_1 k_2 k_3) = \frac{1}{192} \frac{\pi \beta(\Delta)}{k_1 k_2 k_3} (-k_2 c_1^3 - 6k_2^2 k_1^2 c_3 + 2k_1 k_2 k_3 c_2 - 12k_1 c_1^2 k_2$$

$$+ c_2^4 + 12k_1 k_3 c_2^2 - 12k_1 c_1 k_3 - 12k_2 k_3 c_2^2$$

Higher order expressions are not shown here for the sake of brevity. On the other hand we obtain from equation (2.10)

$$I(\lambda; \lambda_1, \lambda_2, \lambda_3; k_1 k_2 k_3) = \frac{\pi \beta(\Delta)}{2^{\lambda+2} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda)!} \times k_1^{-(\lambda+1)} k_2^{-(\lambda+1)} k_3^{-(\lambda+1)} A(\lambda; \lambda_1, \lambda_2, \lambda_3), \quad (B.5)$$
with
\[
A(0; 0, 0, 0) = 1
\]
\[
A(0; 1, 0, 1) = -2k_1(-k_1 + k_2\Delta),
\]
\[
A(0; 1, 1, 0) = 2k_1k_2\Delta,
\]
\[
A(1; 0, 0, 0) = 2k_1k_2(1 - \Delta),
\]
\[
A(1; 1, 0, 1) = -6k_1^2k_2(-2k_1 + k_2\Delta + k_2)(1 - \Delta),
\]
\[
A(1; 1, 1, 0) = 6k_1^2k_2^2(\Delta + 1)(1 - \Delta),
\]
\[
A(2; 0, 0, 0) = 4(\Delta - 1)^2k_2^2k_1^2,
\]
\[
A(2; 1, 0, 1) = -16(\Delta - 1)^2(-3k_1 + 2k_2 + k_2\Delta)k_2^2k_1^3,
\]
\[
A(2; 1, 1, 0) = 16(\Delta - 1)^2(2 + \Delta)k_2^3k_1^3.
\]

To test our new result we have compared several explicit analytic expressions obtained from (2.10) and (B.3), respectively (including the ones given above), and found full agreement. The advantage of equations (B.5) over (B.4) is their compact form in terms of generalized Legendre functions. This property will be exploited in [28] where we study the integral
\[
\int_0^\infty e^{-r/\Delta_1} r^\alpha j_{\lambda_1}(k_1 r) j_{\lambda_2}(k_2 r) \, dr.
\]

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