Properties of the frequency operator do not imply the quantum probability postulate

Carlton M. Caves\textsuperscript{1,*} and Rüdiger Schack\textsuperscript{2,†}

\textsuperscript{1}Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131-1156, USA
\textsuperscript{2}Department of Mathematics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK
(Dated: January 20, 2022)

We review the properties of the frequency operator for an infinite number of systems and disprove claims in the literature that the quantum probability postulate can be derived from these properties.

I. INTRODUCTION

Although quantum theory is thought by many physicists to provide a complete description of the physical world, the account it gives is strange and counter-intuitive. Nobody can claim fully to understand the quantum description, yet we find it appealing because it provides exquisitely precise predictions for the results of experiments.

The counter-intuitive description provided by quantum mechanics would be more palatable were it the consequence of a set of compelling, physically motivated assumptions about the way the world works. In contrast to this desire, however, the postulates usually given for quantum theory are notable for their abstract, mathematical character. As an illustration, consider how the theory’s foundations are introduced in one standard graduate-level textbook \cite{ref}. A chapter of 54 pages is devoted to stating and explaining five foundational postulates, paraphrased as follows: (i) the state of a physical system is a normalized vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$; (ii) every measurable quantity is described by a Hermitian operator (observable) $A$ acting in $\mathcal{H}$; (iii) the only possible result of measuring a physical quantity is one of the eigenvalues of the corresponding observable $A$; (iv) the probability for obtaining eigenvalue $\lambda$ in a measurement of $A$ is $\Pr(\lambda) = \langle \psi | P_\lambda | \psi \rangle$, where $P_\lambda$ is the projector onto the eigensubspace of $A$ having eigenvalue $\lambda$; and (v) the post-measurement state in such a measurement is $P_\lambda |\psi\rangle / \sqrt{\Pr(\lambda)}$. Compared to the crisp postulates of special relativity—the laws of physics and the speed of light are the same in all inertial frames—which are physically motivated and stated directly in terms of physical concepts and quantities, the quantum postulates make up a baggy set that can only be described as more mathematical than physical (see, however, Refs. \cite{2,3,4} for a derivation of quantum mechanics from axioms for states and measurements that do not presuppose a Hilbert-space structure and Refs. \cite{5,6,7} for a derivation that places quantum theory on a foundation of information-theoretic postulates).

A skeptic, on being exposed to the quantum postulates, would balk after just the first few of the 54 pages and question the entire mathematical construction: Who ordered the complex vector space, which seems to have nothing to do with the arena of ordinary experience? How

\begin{itemize}
  \item [*] Electronic address: caves@info.phys.unm.edu
  \item [†] Electronic address: r.schack@rhul.ac.uk
\end{itemize}
could anyone think that states of physical systems are vectors in this abstract space? How could anyone think that observables are operators in this space?

One avenue to enlightenment might be to reduce the number of postulates. A signal result of this sort, of which we are particularly fond, is provided by Gleason’s theorem [8]. The theorem assumes postulates (ii) and (iii) and that the task of the theory is to provide probabilities for measurement outcomes, which in accordance with (iii), are associated with complete, orthogonal sets of projection operators. The key assumption is that these probabilities are noncontextual [9], which means that the probability associated with a projection operator is independent of which other projectors complete the set of outcomes. Put differently, noncontextuality is the assertion that if two observables share an outcome, i.e., have a shared eigensubspace, then the probability associated with this outcome is the same for both observables. The content of Gleason’s theorem is that in Hilbert-space dimensions ≥ 3, probabilities that are noncontextual must be derived from a density operator using the mixed-state generalization of the quantum probability rule in (iv). This is particularly pleasing, since it gets the state-space structure of (i), generalized to density operators, and the quantum probability rule of (iv), both from the one assumption of noncontextuality for probabilities of the allowed outcomes in (iii). Gleason’s theorem doesn’t answer the skeptic’s fundamental question—who ordered the complex vector space—but it does suggest that we can focus our attention on measurements, trying to figure out why they are described in a Hilbert space, and we can let states come along for the ride.

Another approach to reducing the number of postulates has been to try to derive the quantum probability rule (iv) from the frequency properties of repeated measurements of an observable on a finite or infinite number of copies of a system, where all copies are in the same state. A critical analysis of programs of this sort is the subject of this paper. Such programs are the quantum analogue of the classical attempt to define probabilities as the frequencies of outcomes in a finite or infinite number of trials on identical systems. There are many problems with this frequentist approach to defining probabilities, and these problems are succinctly summarized in Ref. [10]. Ultimately, the program comes down to an attempt to define probabilities using the weak or strong law of large numbers. The program founders because the laws of large numbers are statements within probability theory, which cannot even be formulated without reference to probabilities. For this reason, attempts to define probabilities using the laws of large numbers are inherently circular. The laws of large numbers are indeed important mathematical results that connect frequencies to probabilities, but their form illustrates the crucial point: Inferences always run not from frequencies to probabilities, but from probabilities to statistical properties of frequencies.

There is, however, some reason to hope that the frequentist program can be salvaged within a quantum-mechanical framework, because of the Hilbert-space setting of quantum theory. The hope is that the Hilbert-space inner product provides additional structure, not available in the classical setting, that allows one to make statements about repeated measurements on an infinite number of copies, statements that are independent of the quantum probability rule relating inner products to probabilities. The mathematical object that embodies this hope is the frequency operator associated with repeated measurements of an observable, and the hope has motivated a number of researchers [11, 12, 13, 14] to investigate properties of the frequency operator.

The program followed by these investigators is to consider an infinite number of copies of a quantum system, all in the same state \(|\psi\rangle\) and all subjected to a measurement of the same observable. The chief technical object of the program is to demonstrate that the resulting
infinite repetition state, $|\Psi_\infty\rangle = |\psi\rangle^{\otimes \infty} \equiv |\psi\rangle \otimes |\psi\rangle \otimes \cdots$, is an eigenstate of the frequency operator associated with a particular measurement outcome, with the eigenvalue given by the absolute square of the inner product for that outcome. This object accomplished, the program invokes the rule that when an observable is measured on a system that is in an eigenstate of the observable, the result is guaranteed to be the corresponding eigenvalue. The conclusion is that an infinite number of measurements of an observable on an infinite-repetition state yields each outcome with a frequency given by the corresponding inner product. By identifying long-run frequencies with probabilities, this is then interpreted to mean that the probability for each outcome is given by the appropriate inner product.

The proponents of this program portray it as replacing the quantum probability rule (iv) with a weaker hypothesis, which makes no reference to probabilities, that hypothesis being that a measurement of an observable on a system in an eigenstate of the observable yields the eigenvalue with certainty. It is useful to formalize these two postulates for later reference in the paper. The standard quantum probability postulate has the following form.

**Quantum Probability Postulate (QPP):** Let $B = \sum \lambda P_\lambda$ be an observable, where $\lambda$ denotes the different eigenvalues of $B$ and the operators $P_\lambda$ are orthogonal projectors onto the eigenspaces of $B$. If $B$ is measured on a system in state $|\psi\rangle$, the probability of outcome $\lambda$ is $||P_\lambda |\psi\rangle||^2 = \langle \psi | P_\lambda |\psi\rangle$.

Here $|| \cdot ||$ denotes the Euclidean norm, i.e., $|||\psi\rangle|| = \sqrt{\langle \psi | \psi \rangle}$. Throughout this paper, it is sufficient for us to deal with repeated measurements of a nondegenerate observable $B$. In this case, we denote the eigenvalues of $B$ by $\lambda_j$ and the corresponding eigenvectors by $|B, j\rangle$, i.e., $B|B, j\rangle = \lambda_j |B, j\rangle$; the probability for obtaining outcome $\lambda_j$ in a measurement of $B$ becomes $|\langle B, j |\psi\rangle|^2$. The aim of the frequentist program is to replace QPP with the weaker postulate of definite outcomes.

**Postulate of Definite Outcomes (PDO):** If an observable $O$ is measured on a system in an eigenstate $|\psi\rangle$ of $O$, i.e., $O|\psi\rangle = \lambda |\psi\rangle$, the outcome is $\lambda$ with certainty.

The purpose of this paper is to subject the quantum frequentist program to critical analysis. As noted above, the proponents of the program see their main task as establishing a technical mathematical property of the frequency operator for an infinite number of copies of a system, to wit, that an infinite repetition state is an eigenstate of the infinite-copy frequency operator associated with eigenvalue $\lambda_j$ of observable $B$, with the eigenvalue given by $|\langle B, j |\psi\rangle|^2$. This technical demonstration is indeed at the heart of the program, and its analysis occupies the main part of our paper; we show that the desired property of the frequency operator follows only if one assumes QPP from the outset. At the end of the paper, we go beyond this technical result to analyze other aspects of the program, and we find the program to be flawed at every step. Our conclusion is that the inner-product structure of quantum mechanics does not buy additional power for defining probabilities in terms of infinite frequencies. Even in quantum mechanics, inferences run from probabilities to frequencies, not the other way around.

The paper is organized as follows. Following Finkelstein [11] and Hartle [12], Sec. IIA defines the finite-copy frequency operator $F^N$ and reviews the Finkelstein-Hartle theorem, a probability-independent statement about the frequency operator that is mathematically akin to the classical weak law of large numbers. Section IIB argues that the Finkelstein-Hartle
Theorem cannot be used to establish properties of the infinite-copy frequency operator $F^\infty$, as was claimed by Finkelstein and Hartle, and thus does not show that infinite-repetition states are eigenstates of $F^\infty$.

Section III takes up the definition of the infinite-copy frequency operator $F^\infty$. Following Farhi, Goldstone, and Gutmann [13], Sec. III A reviews the construction of the nonseparable infinite-copy Hilbert space, and then Sec. III B uses a method due to Gutmann [14] to define $F^\infty$. Gutmann’s definition assumes QPP and thus arrives at an $F^\infty$ that has the property that infinite-repetition states are eigenstates of $F^\infty$, with the eigenvalues given by the quantum probabilities. Gutmann’s construction makes clear, however, that the definition of $F^\infty$ depends on a choice of probability measure for the space of outcome sequences, and we show that probability measures other than that dictated by QPP lead to other eigenvalues for the frequency operator. This shows that the definition of $F^\infty$ is not unique and thus that properties of $F^\infty$ cannot be used to pick out the quantum probability rule. In Sec. III C we examine a derivation of $F^\infty$ due to Farhi, Goldstone, and Gutmann [13], which purports to derive the unique $F^\infty$ that is consistent with QPP solely from the inner-product structure. Cassinello and Sánchez-Gómez [15] criticized this derivation, but their criticism turns out not to be justified. We point out in Sec. III C that though the derivation does pick out a unique measure, other measures can be used to define $F^\infty$. Section III D shows that an assumption of noncontextual infinite frequencies is both necessary and sufficient for picking out the unique $F^\infty$ that is consistent with QPP, thus making the entire quantum frequentist program merely an elaborate device for placing the Gleason derivation of QPP within the unnecessary context of infinite frequencies.

In Sec. IV we grant the proposition that there is a unique frequency operator $F^\infty$ such that $|\psi\rangle^{\otimes\infty}$ is an eigenstate of $F^\infty$ with eigenvalue $|\langle B, j| \psi \rangle|^2$, but argue that this cannot be used to conclude anything about single-copy probabilities. In Sec. IV A we point out that the eigenvalue equation for $F^\infty$ is really a probability-1 statement and that probability 1 does not imply certainty in uncountable sample spaces—thus PDO does not hold in nonseparable Hilbert spaces—and in Sec. IV B we discuss how tail properties such as the limiting frequency of an outcome sequence are irrelevant to probabilities for a finite number of copies.

Section V summarizes our findings and their implications for the frequentist program to derive the quantum probability rule.

Throughout the paper we use “copies” to refer to multiple versions of the same quantum system. Thus we talk about having a finite or infinite number of copies of a system. For example, in the former case we refer to the finite-copy Hilbert space $\mathcal{H}^{\otimes N}$ and the finite-copy frequency operator $F^N$ of $N$ copies; likewise, in the latter case we have the infinite-copy Hilbert space $\mathcal{H}^{\otimes\infty}$ and the infinite-copy frequency operator $F^\infty$. We reserve the words “repeated” and “repetition” to refer to situations where the same state or the same measurement applies to all of the copies, finite or infinite. Thus we talk about the repetition state $|\Psi_N\rangle = |\psi\rangle^{\otimes N} \equiv |\psi\rangle \otimes \cdots \otimes |\psi\rangle$ of $N$ copies, where $|\psi\rangle$ is a single-copy state, the infinite-repetition state $|\Psi_\infty\rangle = |\psi\rangle^{\otimes\infty} \equiv |\psi\rangle \otimes |\psi\rangle \otimes \cdots$ of an infinite number of copies, and repeated measurements of an observable $B$ on a finite or infinite number of copies.
II. FREQUENCY OPERATOR: FINITE NUMBER OF COPIES

A. Construction of the finite-copy frequency operator

Consider $N$ copies of a quantum system, each copy described in the same $D$-dimensional Hilbert space $\mathcal{H}$. Suppose we measure the same observable, $B$, on each of the $N$ systems. We assume for simplicity that the eigenvalues of $B$ are nondegenerate, although none of our conclusions depends on this assumption. We denote by $\lambda_j$ the eigenvalues of $B$ and by $|B,j\rangle$ the corresponding eigenvectors, i.e.,

$$B|B,j\rangle = \lambda_j|B,j\rangle, \quad j = 0, \ldots, D - 1. \quad (1)$$

We now single out the outcome $j = 0$ (i.e., the eigenvalue $\lambda_0$) as the outcome of interest. We are not interested in which of the other outcomes occurs nor in the order of the outcomes, only in the number of times, $n$, that outcome $j = 0$ occurs. The frequency of outcome $j = 0$ is the fraction $n/N \equiv f$.

To describe a repeated measurement of $B$ as a single measurement, we introduce the tensor-product Hilbert space $\mathcal{H}^\otimes N \equiv \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ for $N$ copies of the system. The projector corresponding to the sequence $j_1, \ldots, j_N$ of measurement outcomes is given by the tensor product $|B,j_1\rangle\langle B,j_1| \otimes \cdots \otimes |B,j_N\rangle\langle B,j_N|$. The corresponding frequency of outcome $j = 0$ is

$$f = \frac{1}{N} \sum_{r=1}^{N} \delta_{0j_r}. \quad (2)$$

Of course, many different outcome sequences give rise to the same frequency $f$ or occurrence number $n = Nf$. Adding all the projectors that lead to this frequency gives the multi-dimensional projector onto the subspace corresponding to this frequency (or occurrence number):

$$\Pi^N_n = \sum_{j_1,\ldots,j_N} |B,j_1\rangle\langle B,j_1| \otimes \cdots \otimes |B,j_N\rangle\langle B,j_N| \delta\left(n, \sum_{r=1}^{N} \delta_{0j_r}\right)$$

$$= \sum_{k_1,\ldots,k_N \in \{0,1\}} P^1_{k_1} \otimes \cdots \otimes P^N_{k_N} \delta\left(n, \sum_{r=1}^{N} \delta_{0k_r}\right). \quad (3)$$

Here $P^r_0 = |B,0\rangle\langle B,0|$, $P^r_1 = 1 - P^r_0$, and the superscripts, $r = 1, \ldots, N$, label which copy the projector applies to. The projectors $P^r_0$ and $P^r_1$ make up a binary POVM that describes retaining only the information about whether the $r$th measurement yields outcome $j = 0$, i.e., $k_r = 0$, or some other outcome, i.e., $k_r = 1$.

The projectors $\Pi^N_n$ clearly add up to the unit operator, $1^\otimes N$, on $\mathcal{H}^\otimes N$ and thus form a POVM made up of orthogonal multi-dimensional projectors. This POVM describes a measurement that can be realized by making $N$ successive measurements of $B$, after which the outcomes $j \neq 0$ are placed in a single bin and information about the ordering of the outcomes is discarded, leaving only the outcome frequencies for $j = 0$ as measurement results.

By associating the appropriate outcome frequency with each projector $\Pi^N_n$, the same measurement can be described as a measurement of a frequency observable, which is the finite-copy frequency operator for outcome $j = 0$,

$$F^N = \sum_{n=0}^{N} \frac{n}{N} \Pi^N_n. \quad (4)$$
Measuring the frequency operator is equivalent to measuring the POVM consisting of the projection operators \( (3) \).

The frequency operator \( (4) \) can be put in other useful forms through the following manipulations:

\[
F^N = \frac{1}{N} \sum_{n=0}^{N} n \sum_{k_1, \ldots, k_N \in \{0,1\}} \delta \left( n, \sum_{r=1}^{N} \delta_{0kr} \right) P_{k_1}^1 \otimes \cdots \otimes P_{k_N}^N \\
= \frac{1}{N} \sum_{r=1}^{N} \sum_{k_1, \ldots, k_N \in \{0,1\}} \delta_{0kr} \ P_{k_1}^1 \otimes \cdots \otimes P_{k_N}^N \\
= \frac{1}{N} \sum_{r=1}^{N} 1^{\otimes(r-1)} \otimes P_0^r \otimes 1^{\otimes(N-r)} \\
= \frac{1}{N} (P_0^1 + \cdots + P_0^N). \tag{5}
\]

The second form on the right is the form in which the frequency operator was introduced by Hartle \[12\]. In the final expression on the right, the frequency operator is written as the average over the \( N \) systems of the projectors onto the chosen outcome. This final form says that to measure the frequency operator for a particular outcome, one should measure on each copy the observable that is the projection operator onto the chosen outcome—the POVM for the \( r \)th copy consists of the projection operators \( P_r^1 \) and \( P_r^0 \)—and then average the results. The final form is the form of the frequency operator that comes from the work of Finkelstein \[11\], although Finkelstein actually dealt with the average of a general observable, rather than specifically with the average of a projector onto a particular outcome.

Regardless of which form one prefers for the frequency operator, one can see that measuring the frequency operator can be accomplished through repeated measurements on the separate copies, this despite the joint operators that appear in Eqs. \( (3) \) and \( (5) \). Failure to appreciate this point has led to some confusing discussions in the literature \[16\].

An important property of the finite-copy frequency operator is provided by the Finkelstein-Hartle theorem \[11, 12\]: Let \(|\psi\rangle \in \mathcal{H}\) be a single-copy state, and let

\[
|\Psi_N\rangle = |\psi\rangle^{\otimes N} \equiv |\psi\rangle \otimes \cdots \otimes |\psi\rangle \tag{6}
\]

be the corresponding \( N \)-copy repetition state; then there is a unique number \( q \) such that

\[
\lim_{N \to \infty} \langle F^N|\Psi_N\rangle - q|\Psi_N\rangle \| = \lim_{N \to \infty} \Delta_N = 0, \tag{7}
\]

namely

\[
q = \langle \psi | P_0 | \psi \rangle = |\langle B, 0 | \psi \rangle|^2. \tag{8}
\]

If we start from the final form of \( F^N \) in Eq. \( (5) \), the proof of the theorem is straightforward. Substituting Eq. \( (5) \) into Eq. \( (7) \), we obtain

\[
\Delta_N^2 = \left( q - \langle \psi | P_0 | \psi \rangle \right)^2 + \frac{\langle \psi | P_0 | \psi \rangle(1 - \langle \psi | P_0 | \psi \rangle)}{N}, \tag{9}
\]

and the Finkelstein-Hartle theorem follows immediately. We can also start from the form \( (4) \) of \( F^N \). Assuming \( q \) is given by Eq. \( (5) \), we see that \( \Delta_N^2 \) is the variance of a random variable \( f^N \), which has possible values \( n/N, \ n = 0, \ldots, N \), and distribution

\[
\Pr(f^N = n/N) = \|\Pi_n^N|\Psi_N\rangle\|^2 = \binom{N}{n} q^n (1-q)^{N-n}, \ n = 0, \ldots, N, \tag{10}
\]
which is a binomial distribution. This variance is

\[ ||F^N|\Psi_N\rangle - q|\Psi_N\rangle||^2 = E\left((f^N - q)^2\right) \]

\[ = \sum_{n=0}^{N} \left(\frac{n}{N} - q\right)^2 ||P_n^N|\Psi_N\rangle||^2 \]

\[ = \frac{q(1-q)}{N} \to 0 \text{ as } N \to \infty. \quad (11) \]

Here \(E\) denotes the expectation value with respect to the distribution (10). Notice that the Finkelstein-Hartle theorem is a purely mathematical statement, which uses the results of probability theory without assuming any probabilistic interpretation of the above expressions. The interpretation of the finite-copy frequency operator is the subject of Sec. III B below.

To end the current subsection, we draw attention to another simple consequence of the binomial distribution (10), which in the context of the frequency operator was first pointed out by Squires [17]. Squires’s observation was that the repetition state \(|\Psi_N\rangle\) becomes orthogonal to all frequency subspaces as \(N\) becomes large,

\[ \max_{0 \leq w \leq N} ||\Pi_n^N|\Psi_N\rangle||^2 \to 0 \text{ as } N \to \infty, \quad (12) \]

except in the trivial cases \(q = 0\) and \(q = 1\). This is, of course, nothing but the fact that the maximum of the binomial distribution approaches zero as the number of trials increases.

B. Status of the finite-copy frequency operator

The Finkelstein-Hartle theorem suggests that the quantum probability postulate, QPP, follows directly from the Hilbert-space structure. Indeed, Finkelstein interpreted Eq. (7) to mean that, for sufficiently large \(N\), the \(N\)-copy repetition state \(|\Psi_N\rangle\), representing a finite ensemble of systems in the same quantum state, is “nearly an eigenstate” of the average operator \(F^N\), with the deviation from being an eigenstate measured by the error \(\Delta_N\) [11]. Vague though this assertion is, one way to see that it is not warranted is to refer to Squires’s observation that in the limit \(N \to \infty\), the repetition state \(|\Psi_N\rangle\) becomes orthogonal to any eigenstate of the frequency operator.

Independently of Finkelstein, Hartle [12] considered an infinite number of copies and the infinite-repetition state,

\[ |\Psi_{\infty}\rangle = |\psi\rangle^{\otimes\infty} \equiv |\psi\rangle \otimes |\psi\rangle \otimes \cdots, \quad (13) \]

and defined the action of an infinite-copy frequency operator, \(F^\infty\), through the limit

\[ F^\infty|\Psi_{\infty}\rangle = \lim_{N \to \infty} (F^N|\Psi_{\infty}\rangle). \quad (14) \]

He then used Eq. (17) to show that the infinite-repetition state \(|\Psi_{\infty}\rangle\) is an eigenstate of \(F^\infty\) with eigenvalue \(q\), i.e.,

\[ F^\infty|\Psi_{\infty}\rangle = q|\Psi_{\infty}\rangle. \quad (15) \]

We must defer a detailed discussion of the flaw in Hartle’s approach till Sec. III B, but for the present we note that the finite-copy frequency operator does not have a unique extension to the infinite-copy Hilbert space, so the operator \(F^\infty\) in Eq. (14) is not well defined. More
generally, the $N \to \infty$ limit of $N$-copy expressions is not sufficient to establish the properties of infinite-copy expressions.

These points were first stated clearly by Farhi, Goldstone, and Gutmann (p. 370 of Ref. [13]). The Finkelstein-Hartle theorem is about limits of finite-copy quantities, but the finite-copy repetition state $|\psi\rangle^\otimes N$ is not an eigenstate of the frequency operator. The result is that finite-copy considerations do not determine the infinite-copy frequency operator; one must have a procedure to extend the definition of the frequency operator to the nonseparable infinite-copy Hilbert space. In Sec. III we give a thorough discussion of the infinite-copy Hilbert space and show in detail why the finite-copy analysis does not determine the action of the infinite-copy frequency operator on infinite-repetition states.

This situation is reminiscent of the distinction in probability theory between the weak and strong laws of large numbers. The weak law of large numbers is about the $N \to \infty$ limit of probabilities for $N$ trials, where $N$ is finite, just as the Finkelstein-Hartle theorem is about the $N \to \infty$ limit of a Hilbert-space norm for $N$ copies. In contrast, the strong law of large numbers is about probabilities for infinite sequences of trials, just as the putative eigenvalue equation (15) is a direct statement about infinite-repetition states. The strong law does not follow from the weak law, nor does the eigenvalue equation (15) follow from the Finkelstein-Hartle theorem. Section III describes a quantum version of the strong law of large numbers.

If the postulate QPP is assumed, as in the standard approach to quantum theory, only finite-copy considerations are needed to establish a tight connection between single-trial probabilities and frequencies in repeated measurements. It follows directly from QPP that the probability of measuring a frequency $f = n/N$ is given by the binomial distribution (10), which for large $N$ becomes strongly peaked near the single-trial probability $q$. For large $N$, the measured frequency is close to $q$ with probability close to 1. This is the familiar connection between measured frequency and probability expressed by the weak law of large numbers. It is important that in the formulation of the weak law, frequency and probability are two separate concepts, with probability being the primary concept: a single-trial probability distribution is assumed to be given from the start (in quantum mechanics, provided by QPP for individual copies); it is then shown that the derived probability distribution for the measured frequency (a random variable) obeys certain bounds that are called the weak laws of large numbers.

To summarize, the Finkelstein-Hartle theorem establishes a connection between the squared inner product and the measured frequency only if the quantum probability postulate QPP is assumed from the outset. To overcome this problem, Farhi, Goldstone, and Gutmann [13] proposed to derive, directly from the properties of Hilbert space, a unique frequency operator $F^\infty$ defined for an infinite number of copies of the system, i.e., for an infinite number of measurements. An analysis of this derivation and closely related work by Gutmann [14] is the topic of the next section.

III. FREQUENCY OPERATOR: INFINITE NUMBER OF COPIES

A. Construction of the infinite-copy Hilbert space

Consider $N$ copies of a system with a $D$-dimensional Hilbert space $\mathcal{H}$. The $N$ copies are described by the $D^N$-dimensional tensor-product Hilbert space $\mathcal{H}^\otimes N = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. As before, we denote the orthonormal basis of eigenstates of the measured observable $B$ in
\( \mathcal{H} \) by \( |B, j\rangle \), where \( j = 0, \ldots, D - 1 \). This basis gives rise to a product orthonormal basis \( |B, j_1\rangle \otimes \cdots \otimes |B, j_N\rangle \) for \( \mathcal{H}^\otimes N \).

We want to construct the infinite tensor-product space \( \mathcal{H}^\otimes \infty = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \). This is a nonseparable Hilbert space, meaning that it has an uncountable orthonormal basis. Even this means more than one might think initially. The orthonormal products of eigenstates of \( B \),

\[
|B; \{ j \}\rangle \equiv |B, j_1\rangle \otimes |B, j_2\rangle \otimes \cdots ,
\]

are in one-to-one correspondence with the real numbers in the interval \([0, 1)\) and thus make up an uncountable set of orthonormal vectors. Though the finite-copy case might lead one to expect these vectors to span \( \mathcal{H}^\otimes \infty \), they must be augmented by an uncountable number of other orthonormal vectors to produce a basis for \( \mathcal{H}^\otimes \infty \).

We follow the discussion contained in Secs. V and IX of Ref. [13], which constructs \( \mathcal{H}^\otimes \infty \) as a direct sum of an uncountable number of separable Hilbert spaces called components. As noted in Ref. [13], this construction is ideally suited to an analysis of infinite frequencies, because questions about infinite frequencies can be handled within the separate components. To begin, let

\[
\{ \psi \} \equiv |\psi_1\rangle, |\psi_2\rangle, \ldots
\]

denote an infinite sequence of normalized vectors in \( \mathcal{H} \), and let

\[
|\{ \psi \}\rangle \equiv |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots
\]

be the corresponding infinite product vector. The magnitude of the inner product of two product vectors,

\[
|\langle \{ \phi \}|\{ \psi \}\rangle| = \prod_{r=1}^{\infty} |\langle \phi_r|\psi_r\rangle|,
\]

lies in the interval \([0, 1]\). Indeed, the inner product goes to zero unless the product vectors have tail sequences that are essentially identical.

One defines two sequences to be equivalent, written \( \{ \phi \} \sim \{ \psi \} \), if there exists \( N \geq 1 \) such that

\[
\prod_{r=N}^{\infty} |\langle \phi_r|\psi_r\rangle| > 0 ,
\]

which is equivalent to saying the series

\[
\sum_{r=N}^{\infty} (-\log |\langle \phi_r|\psi_r\rangle|)
\]

converges absolutely. From the properties of absolutely convergent series, it is easy to see that two sequences are equivalent if and only if for any \( \epsilon > 0 \), there exists \( N \geq 1 \) such that

\[
\prod_{r=N}^{\infty} |\langle \phi_r|\psi_r\rangle| > 1 - \epsilon .
\]

Thus two sequences are equivalent if and only if they have tails that are essentially identical. It remains to be shown that \( \sim \) defines an equivalence relation, but this is not hard to do, the only part that requires a little work being transitivity. Notice that vectors corresponding to inequivalent sequences are orthogonal.
The component associated with an equivalence class is defined to be the subspace spanned by the infinite product vectors in the class. To show that each component is a separable Hilbert space, one constructs a countable orthonormal basis for the component in the following way. Select a sequence \( \{ \psi \} = |\psi_1>, |\psi_2>, \ldots \) from the equivalence class that defines the component, and call the component \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \). For each vector \( |\psi_r\rangle \) in the representative sequence, choose an orthonormal basis \( |\psi_r, 0>, \ldots, |\psi_r, D - 1\rangle \) such that \( |\psi_r, 0\rangle = |\psi_r\rangle \). Now define the sequences \( \{ i \} = i_1, i_2, \ldots \), where \( i_k = 0, \ldots, D - 1 \), to be those with a finite number of nonzero elements. These sequences are countable. The corresponding sequences of vectors, \( |\psi_1, i_1>, |\psi_2, i_2>, \ldots \), are clearly in the equivalence class of \( \{ \psi \} \). What we want to show is that the corresponding product vectors,

\[
|\psi; \{ i \}\rangle \equiv |\psi_1, i_1\rangle \otimes |\psi_2, i_2\rangle \otimes \cdots ,
\]  

(23)

span \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \). To do so, one needs to show that the vector corresponding to any sequence \( \{ \phi \} \sim \{ \psi \} \) can be expanded in terms of the vectors \( |\psi; \{ i \}\rangle \). The expansion should look like

\[
|\{ \phi \}\rangle = \sum_{\{ i \}} |\psi; \{ i \}\rangle \langle \psi; \{ i \}| \{ \phi \}\rangle .
\]  

(24)

The vectors \( |\psi; \{ i \}\rangle \) are complete in \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \) if and only if the inner-product expansion coefficients satisfy the completeness condition,

\[
\sum_{\{ i \}} \langle \psi; \{ i \}| \{ \phi \}\rangle^2 = 1 \quad \text{for all } \{ \phi \} \sim \{ \psi \} .
\]  

(25)

The inner-product expansion coefficients have the explicit form

\[
\langle \psi; \{ i \}| \{ \phi \}\rangle = \prod_{r=1}^\infty \langle \psi_r, i_r| \phi_r\rangle .
\]  

(26)

Once past the finite number of nonzero entries in \( \{ i \} \), the terms in Eq. (26) are identical to those in \( \langle \{ \psi \}| \{ \phi \}\rangle \). By rephasing the vectors \( |\phi_r\rangle \) (at the expense of introducing an overall phase into \( |\{ \phi \}\rangle \)), one can make all the inner products \( \langle \psi_r| \phi_r\rangle \) real and nonnegative. Then the terms in the tail of the inner product \( \langle \psi; \{ i \}| \{ \phi \}\rangle \) are just \( \langle \psi_r| \phi_r\rangle \).

To demonstrate the completeness condition (25), one begins by noting that for any \( \epsilon > 0 \), there exists \( N \) such that Eq. (20) is satisfied, which implies that

\[
\prod_{r=N+1}^\infty \langle \psi_r| \phi_r\rangle^2 > (1 - \epsilon)^2 > 1 - 2\epsilon .
\]  

(27)

Now consider the sequences \( \{ i \} \) such that \( i_r = 0 \) for \( r > N \). These sequences run over all possibilities in the first \( N \) slots and are completed by 0’s in the tail slots \( r > N \). For these sequences, we can write

\[
\langle \psi; \{ i \}| \{ \phi \}\rangle = \prod_{r=1}^N \langle \psi_r, i_r| \phi_r\rangle \prod_{r=N+1}^\infty \langle \psi_r| \phi_r\rangle .
\]  

(28)

Now summing only over the sequences such that \( i_r = 0 \) for \( r > N \) (sum denoted by a prime), we have

\[
\sum_{\{ i \}} \langle \psi; \{ i \}| \{ \phi \}\rangle^2 = \sum_{i_1=0}^{D-1} \langle \psi_1, i_1| \phi_1\rangle^2 \cdots \sum_{i_N=0}^{D-1} \langle \psi_N, i_N| \phi_N\rangle^2 \prod_{r=N+1}^\infty \langle \psi_r| \phi_r\rangle^2 > 1 - 2\epsilon ,
\]  

(29)
since the $N$ sums are all equal to 1. This result implies that the unrestricted sum converges to 1, completing the demonstration that the sequences $|\psi;\{i\}\rangle$ span $\mathcal{H}^{\otimes\infty}_{\{\psi\}}$.

To summarize, there are an uncountable number of components, each corresponding to an equivalence class of product states that are essentially identical in the tail. Each component is a separable Hilbert space, spanned by a countable orthonormal basis. Different components are orthogonal, and the entire Hilbert space $\mathcal{H}^{\otimes\infty}$ is the direct sum of the components. We draw attention to the fact that every component contains the entire Hilbert space $\mathcal{H}^{\otimes N}$ for the first $N$ copies for any finite value of $N$; i.e., in mathematical language, $\mathcal{H}^{\otimes\infty}_{\{\psi\}} = \mathcal{H}^{\otimes N}_{\{\psi\}} \otimes \mathcal{H}^{\otimes\infty}_{\{\psi'\}}$, where $\{\psi'\}$ denotes the defining sequence $\{\psi\}$ with the first $N$ vectors omitted. This means that the difference between components lies entirely in the tails of the equivalence classes defining the components. The subspace spanned by the product vectors $|B;\{j\}\rangle$ is already a direct sum of an uncountable number of components, yet there are an uncountable number of other components.

B. Infinite-copy frequency operator

The task now is to define a frequency operator $F^\infty$ on the infinite-copy Hilbert space $\mathcal{H}^{\otimes\infty}$. In doing so, the finite-copy frequency operator $F^N$ is of little help. All that $F^N$ tells us is how to define the action of $F^\infty$ on the products of eigenstates, $|B;\{j\}\rangle = |B,j_1\rangle \otimes |B,j_2\rangle \otimes \cdots$, but this doesn’t go very far, because there is an uncountable number of other components of $\mathcal{H}^{\otimes\infty}$ on which we still have to define $F^\infty$. To define $F^\infty$, we must extend its action to these other components, containing states like the infinite-repetition states $|\Psi^\infty\rangle$, which are our main interest. The treatment of the extension in this subsection follows closely the account given by Gutmann [14].

We begin by defining the frequency of a sequence, $\{j\}$, to be

$$ f(\{j\}) = \frac{1}{2} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \delta_{0j_r} + \liminf_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \delta_{0j_r} \right), \quad (30) $$

where, for a sequence $a_N$, $\liminf_{N \to \infty} a_N = \lim_{N \to \infty} (\inf_{k \geq N} a_k)$ and $\limsup_{N \to \infty} a_N = \lim_{N \to \infty} (\sup_{k \geq N} a_k)$. If there is a limiting frequency, we can dispense with the lim sup’s and lim inf’s, writing

$$ f(\{j\}) = \lim_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \delta_{0j_r}. \quad (31) $$

We need the following generalization of the strong law of large numbers, which is a simple consequence of Theorem 3 in Sec. VII.8 of Ref. [18]. Let $\{q\} = q_1, q_2, \ldots$ be an arbitrary sequence of probabilities, i.e., $0 \leq q_r \leq 1$ for all $r$, and let $X_1, X_2, \ldots$ be a sequence of independent binary random variables such that $X_r \in \{0, 1\}$ and $\Pr(X_r = 0) = q_r$ for all $r$. Define the average probability

$$ f_{\{q\}} = \frac{1}{2} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} q_r + \liminf_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} q_r \right), \quad (32) $$

and the frequency random variable

$$ f^\infty = \frac{1}{2} \left( \limsup_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \delta_{0X_r} + \liminf_{N \to \infty} \frac{1}{N} \sum_{r=1}^{N} \delta_{0X_r} \right), \quad (33) $$
where \( \delta_{0X_r} = 1 - X_r \). Then \( f^\infty = f_q \) with probability 1.

Consider now a component represented by a sequence \( \{ \psi \} \). As discussed in the preceding subsection, the component \( \mathcal{H}^{\otimes \infty}_{\{ \psi \}} \) is spanned by a countable orthonormal basis \( |\psi;\{ i \}\rangle = |\psi, i_1\rangle \otimes |\psi, i_2\rangle \otimes \cdots \), where the sequences \( \{ i \} \) have only a finite number of nonzero entries. Following Gutmann [14], we now associate with a given state \( |\psi;\{ i \}\rangle \) a probability measure, \( d\nu_{|\psi;\{ i \}\rangle}(\{ j \}) \), on the space of infinite sequences \( \{ j \} \) of outcomes for repeated measurements of \( B \). Gutmann chooses the measure associated with all sequences beginning with \( j_1, \ldots, j_N \) to be that given by the quantum probability rule,

\[
\nu_{|\psi;\{ i \}\rangle}(j_1, \ldots, j_N) = \int d\nu_{|\psi;\{ i \}\rangle}(\{ j' \}) \prod_{r=1}^N \delta_{j_r,j'_r} = \prod_{r=1}^N q_{|\psi, i_r\rangle}(j_r),
\]

where

\[
q_{|\psi, i_r\rangle}(j_r) = |\langle \psi, i_r | B, j_r \rangle|^2.
\]

These \( N \)-copy conditions (for all \( N \)) determine the measure \( d\nu_{|\psi;\{ i \}\rangle} \).

Two points should be emphasized here. First, if \( \{ \psi \} \) is not equivalent to any of the products of eigenstates of \( B \) (i.e., the states \( |B;\{ j \}\rangle \)), then the product of inner products in Eq. (34) goes to zero as \( N \) goes to infinity, because vectors from different components are orthogonal. This is essentially Squires’s observation, and it means only that the probability for any infinite sequence is zero. Second, the measure \( d\nu_{|\psi;\{ i \}\rangle}(\{ j \}) \) depends on the particular state \( |\psi;\{ i \}\rangle \). This must be the case in order to get the “right measure” for the initial part of a sequence \( \{ j \} \). Nevertheless, since all the sequences \( \{ i \} \) have the same tails, consisting entirely of zeroes, the tail terms in the above product of inner products are independent of \( \{ i \} \), given by \( |\langle \psi, i_r | B, j_r \rangle|^2 \). A function of outcome sequences whose value is determined by the tail of \( \{ j \} \), i.e., is independent of \( j_1, \ldots, j_N \) for any finite value of \( N \), is called a tail property [15]. When integrating a tail property over \( d\nu_{|\psi;\{ i \}\rangle}(\{ j \}) \), all the measures give the same result, independent of \( \{ i \} \).

An example of a tail property is the frequency (30) of outcome \( j = 0 \). The average frequency is given by the integral

\[
\int d\nu_{|\psi;\{ i \}\rangle}(\{ j \}) f(\{ j \}) = f_q,
\]

where \( f_q \) is the average probability of Eq. (32), with the sequence of probabilities, \( \{ q \} \), given by Eq. (35) with \( j_r = 0 \), i.e.,

\[
q_r = \int d\nu_{|\psi;\{ i \}\rangle}(\{ j \}) \delta_{0j_r} = |\langle \psi, i_r | B, 0 \rangle|^2 = q_{|\psi, i_r\rangle}(0).
\]

Once past the finite number of nonzero entries in \( \{ i \} \), the tail terms in \( \{ q \} \) are independent of \( \{ i \} \), given by \( q_r = |\langle \psi, i_r | B, 0 \rangle|^2 \). The frequency of a sequence is determined by the tail, so the limit for \( f_q \) is independent of \( \{ i \} \). Thus we write

\[
f_q = f_{\psi},
\]

emphasizing that this is a unique frequency associated with the component \( \mathcal{H}^{\otimes \infty}_{\{ \psi \}} \).

We now define the projector \( \Pi_f^\infty \) that projects onto frequency \( f \). Following Gutmann, we define the action of \( \Pi_f^\infty \) on the component \( \mathcal{H}^{\otimes \infty}_{\{ \psi \}} \) by requiring

\[
||\Pi_f^\infty|\psi;\{ i \}\rangle||^2 = \int d\nu_{|\psi;\{ i \}\rangle}(\{ j \}) \Pi_f(\{ j \}),
\]
where

$$\Pi_f(\{j\}) = \begin{cases} 1, & \text{if } f(\{j\}) = f, \\ 0, & \text{if } f(\{j\}) \neq f. \end{cases} \quad (40)$$

is the indicator function for frequency \(f\), meant to characterize the desired properties of the projection operator. The generalization of the strong law of large numbers quoted above gives us immediately that

$$||\Pi_f^\infty |\psi;\{i\}||^2 = \int d\nu_{|\psi\{i\}\rangle(\{j\})} \Pi_f(\{j\}) = \begin{cases} 1, & \text{if } f = f_{\{i\}}, \\ 0, & \text{if } f \neq f_{\{i\}}. \end{cases} \quad (41)$$

We can thus proceed to define

$$\Pi_f^\infty |\psi;\{i\} = \begin{cases} |\psi;\{i\}>, & \text{if } f = f_{\{i\}}, \\ 0, & \text{if } f \neq f_{\{i\}}. \end{cases} \quad (42)$$

Since the vectors \(|\psi;\{i\}\rangle\) span \(\mathcal{H}_{\{\psi\}}^\infty\), this can be extended to all vectors \(|\Psi\rangle \in \mathcal{H}_{\{\psi\}}^\infty\),

$$\Pi_f^\infty |\Psi\rangle = \begin{cases} |\Psi>, & \text{if } f = f_{\{i\}}, \\ 0, & \text{if } f \neq f_{\{i\}}. \end{cases} \quad (43)$$

This result is a quantum version of the strong law of large numbers, following directly from the classical strong law in the form expressed above. It is clear now that we can define the infinite-copy frequency operator by

$$F^\infty |\Psi\rangle = f_{\{i\}} |\Psi\rangle. \quad (44)$$

Each component is an eigensubspace of the infinite-copy frequency operator; i.e., all vectors in a component are eigenvectors of \(F^\infty\), all having the same frequency eigenvalue.

All of this simplifies in the situation of most interest, where the component under consideration has a representative sequence that consists of identical vectors, i.e., \(\{\psi\} = |\psi\rangle, |\psi\rangle, \ldots\). Then the tail of \(\{q\}\) is independent of \(r\), i.e., \(q_r = |\langle\psi|B,0\rangle|^2\), and the frequency associated with \(\mathcal{H}_{\{\psi\}}^\infty\) is \(f_{\{i\}} = |\langle\psi|B,0\rangle|^2\).

It is instructive to pause here and ponder what all this means. Equation (33) equates two ways of writing the probability for frequency \(f\) in an infinite number of measurements of \(B\) on the product state \(|\psi;\{i\}\rangle\). The indicator function (40) restricts the integral on the right of Eq. (39) to sequences of measurement outcomes that have a particular frequency \(f\), so the integral reports the probability for finding that frequency, just what the matrix element on the left of Eq. (39) is supposed to be. The strong law of large numbers is invoked to evaluate the integral as being either 0 or 1, thus defining the frequency projection operators. The definition is determined by the measure \(d\nu_{|\psi\{i\}\rangle(\{j\})}\) of Eq. (31), which is the unique choice if one has already identified the absolute square of inner products with probabilities. This is all Gutmann [11] has in mind, since he is interested not in deriving the quantum probability rule, but rather in defining projection operators on \(\mathcal{H}_{\{\psi\}}^\infty\) and deriving properties of these operators using the strong law of large numbers for the quantum probabilities.

On the other hand, if one is trying to derive the quantum probability rule from the result that \(|\psi;\{i\}\rangle\) is an eigenstate of \(F^\infty\) with the “right” eigenvalue, then one needs to think harder about the procedure used to get to this result. The starting point is Eq. (39), which says that the overlap of \(|\psi;\{i\}\rangle\) with the subspace of sequences with frequency \(f\) is to be identified with the integral of the indicator function \(\Pi_f(\{j\})\) over the quantum probability
measure \( d\nu_{|\psi;i\rangle}\{\{j\}\}\). The measure is clearly the whole story, and one has to justify the choice of the quantum probability measure in the absence of any \textit{a priori} connection between inner products and probabilities.

One might argue that \( d\mu_{|\psi;i\rangle}\{\{j\}\}\) is the only possible measure, given the inner-product structure on Hilbert space, but it is easy to see that there are other choices. Suppose, for example, that one adopts a measure \( d\mu_{|\psi;i\rangle}\{\{j\}\}\) specified by

\[
\mu_{|\psi;i\rangle}\{j_1, \ldots, j_N\} = \int d\mu_{|\psi;i\rangle}\{\{j'\}\} \prod_{r=1}^{N} \delta_{j_r j'_r} = \prod_{r=1}^{N} q_{\psi,r,i_r} (j_r) , \tag{45}
\]

where the terms in the product are now given by

\[
q_{\psi,r,i_r} (j_r) = N_r^{-1} g(|\langle \psi_r , i_r | B , j_r \rangle|) . \tag{46}
\]

Here \( g \) is a function that maps the interval \([0,1]\) to itself, satisfying \( g(0) = 0 \) and \( g(1) = 1 \), and the normalization factor \( N_r \) is given by

\[
N_r = \sum_{j_r = 0}^{D-1} g(|\langle \psi_r , i_r | B , j_r \rangle|) . \tag{47}
\]

The standard quantum measure corresponds to \( g(x) = x^2 \), but we can equally well use any other function, such as \( g(x) = x^4 \), if we don’t already have the quantum probability rule in mind. With the measure specified by Eq. (45), Eq. (37) is replaced by

\[
q_r = \int d\mu_{|\psi;i\rangle}\{\{j\}\} \delta_{0j_r} = N_r^{-1} g(|\langle \psi_r , i_r | B , 0 \rangle|) = q_{\psi,r,i_r} (0) , \tag{48}
\]

which determines the frequency \( f_{\{\psi\}} \) associated with \( \mathcal{H}_\{\psi\}^{\otimes \infty} \). In the case of a sequence of identical vectors, the frequency becomes

\[
f_{\{\psi\}} = \frac{g(|\langle \psi | B , 0 \rangle|)}{\sum_{j} g(|\langle \psi | B , j \rangle|)} . \tag{49}
\]

The upshot of this discussion is that there is no unique extension of the finite-copy frequency operator to the infinite-copy Hilbert space. Indeed, there are even more general choices of measure in which one allows \( q_r \) to depend on the phase of the inner product \( \langle \psi_r , i_r | B , 0 \rangle \), but there is no need for us to consider these more general measures here.

Another way of illustrating the lack of uniqueness is to consider infinite tensor products of eigenstates of the measured observable, i.e., the states \( |B;\{j\}\rangle \). These states being eigenstates of \( F^N \), the finite-copy frequency operator tells us—this is the only thing it tells us the infinite limit—that these states should be eigenstates of \( F^\infty \), with the eigenvalue given by the frequency (30) of the binary sequence \( \delta_{0j_1} , \delta_{0j_2} , \ldots \). To see how this frequency arises from the measures identified in Eqs. (15) and (16), suppose the component’s representative sequence corresponds to the product state \( \{|\psi\rangle\} = |B;\{j\}\rangle \). Then, as a consequence of the requirements \( g(0) = 0 \) and \( g(1) = 1 \) and nothing else, no matter what product vector \( |\psi;\{i\}\rangle \) is chosen to define the measure, the sequence of probabilities, \( \{q\} \), of Eq. (18) has a tail that is precisely the required binary sequence, i.e., \( q_r = \delta_{0j_r} \) in the tail. The component’s frequency \( f_{\{\psi\}} = f_{\{q\}} \) is thus the frequency of this binary sequence, as required. To summarize, the only thing that the finite-copy frequency operator tells us in the infinite limit
is the endpoints of the function \( g \), in which case we are dealing entirely with measurement results that are certainties. The nonuniqueness of the measure comes from the fact that the behavior of \( g \) away from the endpoints is determined by what we assume for the single-copy measurement probabilities.

The lack of a unique extension is what dooms Hartle’s approach \[\text{[12]}\]. In a Mathematical Appendix to his classic 1968 paper, Hartle defines a particular extension \( F^\infty \) and proceeds to show that the infinite repetition states \( |\Psi^\infty\rangle \) are eigenstates of \( F^\infty \) with the “right” eigenvalue. The work in this subsection shows, however, that the extension cannot be unique, so Hartle’s extension contains implicitly an assumption of the quantum probability rule for infinite repetition states. We note in addition that the particular operator \( F^\infty \) that Hartle defines is not a reasonable extension of \( F^N \), because it is defined to give the “right” frequencies only for states in the symmetric subspace, i.e., states in the subspace spanned by the infinite-repetition states. This means, in particular, that Hartle’s extension does not give the right frequencies for infinite tensor products of eigenstates of the measured observable, frequencies that are determined uniquely in the limit.

C. Derivation of the infinite-copy frequency operator in Farhi et al.

The importance of Ref. [13] is that the authors claim to derive the quantum form of the measure, given in Eqs. (34) and (35), without assuming the quantum probability rule, QPP. This derivation thus deserves close attention, because it purports to determine the quantum form of the measure solely from the inner-product structure of \( \mathcal{H}^{\otimes \infty} \), as expressed in the properties of unitary transformations between bases. This subsection is restricted to the case of a two-dimensional system, \( D = 2 \), in order to match the analysis in Sec. V of Ref. [13].

In Sec. V of Ref. [13], the authors set out to construct simultaneous eigenstates \( |b;\{j\}\rangle \) of the measured observables, \( B_1, B_2, \ldots \), within a component \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \) whose representative sequence, \( \{\psi\} = |\psi\rangle, |\psi\rangle, \ldots \), consists of identical vectors. (It is easy to generalize to other components, but there is no need to do so.) Here as previously, the superscript on the measured observable \( B \) specifies which copy the operator acts on.

The states \( |b;\{j\}\rangle \) are in one-to-one correspondence with the eigenstate products \( |B;\{j\}\rangle \), but they are not the eigenstate products except in the uninteresting case where \( |\psi\rangle \) is one of the eigenstates of the measured observable. Indeed, the states \( |b;\{j\}\rangle \) are unnormalizable, as is pointed out in Ref. [13] and as must be true since they are an uncountable basis for the separable component \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \). The states \( |b;\{j\}\rangle \) are determined by defining the inner products \( \langle \psi; \{i\} | b;\{j\} \rangle \), which allows one to write the states \( |b;\{j\}\rangle \) in terms of the basis states \( |\psi; \{i\}\rangle \) for the component in question. Not surprisingly, this transformation hinges on a measure \( d\mu(\{j\}) \) for integrating over the uncountable infinity of unnormalizable states \( |b;\{j\}\rangle \). As shown by Farhi, Goldstone, and Gutmann [13], this measure is uniquely determined by the inner-product structure of the component, and hence it is the measure \( d\mu(\{j\}) \) that is associated with the standard inner-product quantum probability rule for the representative sequence \( \{i\} = \{0\} \).

It is tempting to view the states \( |b;\{j\}\rangle \) as playing the role of the eigenstate products \( |B;\{j\}\rangle \) and thus as dictating the definition of \( F^\infty \) in the component \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \), and this is what is done in Sec. VII of Ref. [13]. This procedure is not justified, however, because the states \( |b;\{j\}\rangle \) do not lie in the component \( \mathcal{H}_{\{\psi\}}^{\otimes \infty} \)—indeed, as unnormalizable states, they do not lie
in the nonseparable infinite-copy Hilbert space $\mathcal{H}^{\otimes\infty}$—although as bras they are legitimate dual vectors for the component. The procedure used in Sec. VII of Ref. [13] is formally attractive, but it has no physically motivated justification. In particular, that the standard measure emerges from the transformation to the states $|b;\{j\}\rangle$ does not mean that it is the only measure that can be used to define the infinite-copy frequency operator. This is clear from the other measures exhibited in Sec. IIIB, all of which can be used to define $F^{\infty}$.

To think that the measure involved in the transformation between the states $|\psi;\{i\}\rangle$ and the unnormalizable states $|b;\{j\}\rangle$ dictates QPP is equivalent to regarding the unitary transformation between two orthonormal bases as determining QPP. Unitary transformations between bases are an expression of the inner-product structure of Hilbert space, since they are the unique transformations that preserve inner products. This means that probabilities derived from QPP transform in a particularly simple way, when compared to other possible rules, but this is not sufficient to reject other probability rules without further assumptions.

D. Status of the infinite-copy frequency operator

The work in this section is devoted entirely to the definition and mathematical properties of the infinite-copy frequency operator $F^{\infty}$. This work shows convincingly, we think, that absent the quantum probability rule, there is no justification for choosing the measure that makes infinite repetition states eigenstates of $F^{\infty}$ with the frequency eigenvalue given by the quantum probability rule. Indeed, you can get any frequency eigenvalue you want, unless you have already assumed the quantum probability rule to fix the choice of measure.

To derive particular eigenvalues for the frequency operator requires additional assumptions. Specifically, if one wishes the limiting frequencies to be those given by the quantum probability rule, then one of these assumptions must be that of noncontextual limiting frequencies, i.e., that the limiting frequency of the selected outcome (chosen to be $j = 0$ in our previous discussion) in repeated measurements does not depend on the other possible outcomes of the measurement. That one must assume noncontextuality or its equivalent is illustrated by the alternative measures identified in Eqs. (45) and (46). For all these measures, except the quantum measure $g(x) = x^2$, the normalization factor $N_r$ in the quantities $q_r$ of Eq. (48) means that for dimensions $D \geq 3$, these quantities and, hence, the limiting frequencies are contextual, depending on eigenstates of the measured observable other than the eigenstate for the selected outcome.

This argument can be put in the formal context of Gleason’s theorem [8]. Suppose one has an infinite number of independent copies of a quantum system on which one makes repeated measurements of the observable $B$. Without saying anything about the state of the copies, one can say that the measure and, hence, the limiting frequency for a selected outcome $j'$ is determined by a sequence $\{q\}$, where the entry for the $r$th copy has the form

$$q_r = q_r(j';\{|B,j\}\rangle) ,$$

signifying that it generally depends both on the set of eigenstates of the measured observable and on the selected outcome $j'$. Since some outcome occurs for each copy, we must have

$$\sum_{j'=0}^{D-1} q_r(j';\{|B,j\}\rangle) = 1 .$$
The assumption of noncontextual limiting frequencies asserts that for repeated measurements of any other observable $C$ that shares an eigenstate with $B$, i.e., $|C, k'\rangle = |B, j'\rangle$ for some $k'$, we must have
\[ q_r(j'; \{ |B, j\rangle \}) = q_r(k'; \{ |C, k\rangle \}). \] (52)

This means that $q_r$ depends only on the eigenstate of the selected outcome and not on the other eigenstates of the measured observable. Technically, since limiting frequencies are determined by the tail of the sequence $\{q\}$, Eq. (52) only needs to hold in the tail, i.e., for copies beyond any finite number. There being no difference between the tail copies and the leading copies, however, we extend Eq. (52) to all copies. In the jargon of Gleason’s theorem, Eqs. (51) and (52) mean that $q_r$ is a frame function, i.e., a function on pure states that sums to a constant (here equal to 1) on orthonormal bases. Gleason’s theorem [8] implies that in dimensions $D \geq 3$, any such function has the form $q_r = \langle B, j | \rho_r | B, j \rangle$ for some normalized density operator $\rho_r$. The infinite product state $\rho_1 \otimes \rho_2 \otimes \cdots$ thus becomes the state of the infinite-copy system, and the elements of the sequence $\{q\}$ and, hence, the eigenvalues of $F^\infty$ are computed using the quantum probability rule.

Although this argument does show that the assumption of noncontextual limiting frequencies picks out the quantum probability rule, what it really shows is the bankruptcy of the program of deriving quantum probabilities from limiting frequencies. The first step in the argument, that the limiting frequency is derived from a sequence $\{q\}$, is only justified if one interprets the $r$th element of the sequence, the quantity $q_r$ of Eq. (50), as the probability for obtaining outcome $j$ in a measurement of $B$ on the $r$th copy. With this realization, the elaborate superstructure of repeated measurements on an infinite number of copies collapses, revealed as irrelevant to an argument that really deals directly with single-copy probabilities. With the superstructure swept away, the argument stands forth in its original form as the pristine Gleason derivation of the state-space structure of quantum mechanics and the quantum probability rule from the assumption of noncontextual probabilities for quantum measurements [9].

IV. DERIVING QPP FROM PDO IS FLAWED FROM THE OUTSET

We could stop at the end of the preceding section, having shown that the eigenvalues of $F^\infty$ are not uniquely determined without reference to single-copy probabilities, but a critical analysis of the program to derive QPP from the properties of $F^\infty$ would not be complete without a discussion of the points in this section. In this section, we grant the proposition that there is a unique infinite-copy frequency operator whose eigenvalues are given by the quantum probability rule, but we argue that one still cannot derive QPP by applying
PDO to this result. The two arguments made in this section are really arguments within classical probability theory. They can be made against the classical frequentist program to derive probabilities from the strong law of large numbers, but here we put these arguments specifically within the quantum context.

A. Certainty versus probability 1

For finite or countably infinite sample spaces, if a subset has probability 1, then in selecting an alternative from the sample space, one of the alternatives in the subset is certain to occur; likewise, probability 0 means impossibility. These statements are no longer true for uncountable sample spaces, where probability 1 does not imply certainty. Any alternative or any subset of measure zero in an uncountable sample space has zero probability. Any alternative or any subset of measure zero can be moved from a set of probability 1 to the complementary subset of probability 0 without changing these probabilities. If one believed that probability 0 implied impossibility, one would conclude that all alternatives were impossible. An example of this is provided by the uncountable sample space of outcome sequences for an infinite sequence of trials, where the strong law of large numbers establishes that with probability 1 the frequency of occurrence of an outcome is equal to its probability, but does not imply that the frequency is certain to be equal to the probability. The strong law of large numbers is a statement within probability theory; its only interpretation is as a precise mathematical statement about the probability measure on the uncountable set of outcome sequences.

In the quantum context, we have already seen in Sec. III B that the strong law of large numbers, applied to QPP, implies that infinite-repetition states are eigenstates of the frequency operator, with eigenvalues given by the absolute square of the inner product. In particular, when we invoke the strong law of large numbers to evaluate the integral on the right side of Eq. (39), we are explicitly using a probability-1 statement on the uncountable sample space of outcome sequences to define the frequency operator. Thus, within standard quantum mechanics, starting from QPP, a repeated measurement on an infinite repetition state yields the frequency eigenvalue with probability 1, not with certainty. The point is that PDO is not a part of standard quantum mechanics for observables on a nonseparable Hilbert space; probability-1 predictions for measurements of such observables do not mean that the eigenvalue occurs with certainty, but rather can only be interpreted as statements within probability theory. In quantum theory, just as for classical probabilities, we cannot interpret probability-1 statements about infinite frequencies without reference to an underlying notion of probabilities, and thus these statements cannot be used to define probabilities.

An alternative to this point of view would be to assert PDO for the frequency operator even though it is not a consequence of the probability rule one is trying to derive. Doing so, however, would replace QPP not with an underlying weaker postulate, but with a strictly stronger postulate, which would make quantum measurements different from classical random processes, such as coin tossing, in an incomprehensible and ultimately untestable way.

B. Tail properties

There is no problem with probability theorists’ deriving purely mathematical properties of infinite sequences of measurement outcomes and of the infinite-copy frequency operator,
for these are well defined mathematical objects within probability theory. The problem arises when one tries to derive properties of finite objects from the purely mathematical properties of infinite sequences, because there is no way to give an operational definition of a measurement of an infinite sequence, no way to interpret the infinite objects outside the mathematical formalism in which they reside. Thus, to understand finite objects, it should not be necessary to refer to the properties of infinite objects such as $F^\infty$.

In this subsection we pursue this line of reasoning in the quantum setting by granting (i) that $|\psi\rangle^{\otimes\infty}$ is an eigenvector of $F^\infty$ with eigenvalue $|\langle\psi|B,0\rangle|^2$ and (ii) the PDO conclusion that a measurement of $F^\infty$ on $|\psi\rangle^{\otimes\infty}$ gives $|\langle\psi|B,0\rangle|^2$ with certainty. Even granting all this, we argue that one cannot reach any conclusions about finite-copy probabilities. The reason is that the frequency (30) of an outcome sequence is a tail property, which means that the limiting frequency is independent of any finite number of initial outcomes. It follows that any initial finite sequence is independent of the limiting frequency. In other words, the fact that the limiting frequency is equal to $|\langle\psi|B,0\rangle|^2$ is of no consequence whatsoever for the probability of an initial finite sequence of measurement outcomes.

It is useful to emphasize precisely how this argument appears in the quantum setting developed in Sec. III. As noted there, every component contains the entire Hilbert space $H^{\otimes N}$ for the first $N$ copies for any finite value of $N$; i.e., $H^{\otimes\infty} = H^{\otimes N} \otimes H^{\otimes\infty}_{\{\psi'\}}$, where $\{\psi'\}$ denotes the component’s defining sequence in the first $N$ vectors omitted. All components are identical for any finite number of copies. Every component can thus accommodate any state for a finite number of copies and the analysis of any measurement on those copies. The difference between components is wholly due to the tails of the equivalence classes defining the components and is thus entirely irrelevant to finite-copy considerations.

Gutmann [14] notes that by the classical Kolmogorov zero-one law [18], the integral over outcome sequences of the indicator function for any tail property is equal to 0 or 1, in the same way that the integral (41) of the frequency indicator function is equal to 0 or 1. This means that we can use the analogue of Eq. (41) to define a quantum operator for any tail property and that the operator has the property that infinite-copy Hilbert-space components are eigensubspaces of the operator. In particular, Gutmann reports (see also Ref. [19]) that Coleman and Lesniewski (unpublished) have defined a randomness observable, $R^\infty$, such that eigenvalue 1 means that the outcome sequence has the “Kolmogorov-Martin-Löf randomness property.” In accordance with the general properties of tail operators, Coleman and Lesniewski showed that an infinite-repetition state is an eigenstate of $R^\infty$ with eigenvalue 1, i.e., $R^\infty|\psi\rangle^{\otimes\infty} = |\psi\rangle^{\otimes\infty}$, provided $|\psi\rangle$ is not one of the eigenstates of the measured observable $B$. From our discussion above, it follows that this eigenvalue equation—or, indeed, the analogous eigenvalue equation for the observable associated with any tail property—is irrelevant to the probabilities for outcomes of any finite number of measurements.

V. CONCLUSION

Probabilities play an important role in nearly every human endeavor. They are central to all the sciences and especially to quantum physics, where they appear in the very foundations of the theory. We have our own favorite way to interpret probabilities, the Bayesian interpretation [20], which posits that probabilities represent an agent’s subjective beliefs about a set of alternatives. This philosophical inclination has led us to propose that even quantum probabilities are Bayesian probabilities [9, 21]. In the Bayesian approach, it is
clear that single-trial probabilities are a primary concept and that one derives properties of frequencies in repeated trials from single-trial probabilities.

One does not have to adopt the Bayesian view—though we recommend it—to realize that the program for defining probabilities in terms of frequencies is bankrupt. This paper charges an additional debt to this bankruptcy, by showing that the inner-product structure of quantum mechanics does not provide any additional leverage in the attempt to derive probabilities from frequencies. As we have seen, the quantum frequentist program turns out to be an elaborate apparatus for rephrasing Gleason’s theorem in terms of long-run frequencies instead of directly in terms of the quantities of interest, single-copy probabilities. Even this elaborate apparatus must be supported by further unjustified assumptions that connect the desired single-copy probabilities to infinite-copy frequencies. Jettisoning the entire infinite-copy apparatus allows us to deal directly with single-trial probabilities, thus avoiding both the technical mathematics needed to understand the nonseparable infinite-copy Hilbert space and the need for assumptions to relate probabilities to long-run frequencies.

The final lesson of our story is clear, and it is the same for classical and quantum probabilities: Probabilities, not frequencies, are the primary concept; inferences always run not from frequencies to probabilities, but from probabilities to statistical properties of frequencies.

Acknowledgments

We profited from discussions with J. B. Hartle. This work was supported in part by the U.S. Office of Naval Research Grant No. N00014-03-1-0426.

[1] C. Cohen-Tannoudji, B. Diu, and F. Laloe, *Quantum Mechanics*, Vol. I, (Wiley, New York, 1977), Chap. III.
[2] L. Hardy, in *Quantum Theory: Reconsideration of Foundations*, edited by A. Khrennikov (Växjö University Press, Växjö, Sweden 2002), pp. 117–130, e-print quant-ph/0101012.
[3] L. Hardy, Stud. Hist. Phil. Mod. Phys. 34, 381 (2003).
[4] R. Schack, Found. Phys. 33, 1461 (2003).
[5] R. Clifton, J. Bub, and H. Halvorson, Found. Phys. 33, 1561 (2003).
[6] J. Bub, “Why the quantum?” unpublished, e-print quant-ph/0402149.
[7] J. Bub, “Quantum mechanics is about quantum information,” unpublished, e-print quant-ph/0408020.
[8] A. M. Gleason, J. Math. Mech. 6, 885 (1957).
[9] C. M. Caves, C. A. Fuchs, and R. Schack, Phys. Rev. A 65, 022305 (2002).
[10] D. M. Appleby, “Probabilities are single-case, or nothing,” unpublished, e-print quant-ph/0408058.
[11] D. Finkelstein, Transactions of the New York Academy of Sciences 25, 621 (1963).
[12] J. B. Hartle, Am. J. Phys. 36, 704 (1968).
[13] E. Farhi, J. Goldstone, and S. Gutmann, Ann. Phys. 192, 368 (1989).
[14] S. Gutmann, Phys. Rev. A 52, 3560 (1995).
[15] A. Cassinello and J. L. Sánchez-Gómez, Found. Phys. 26, 1357 (1996).
[16] Y. Aharonov and B. Reznik, Phys. Rev. A 65, 052116 (2002).
[17] E. J. Squires, Phys. Lett. A 145, 67 (1990).
[18] W. Feller, *An Introduction to Probability Theory and Its Applications. Volume II*, 2nd Ed. (Wiley, New York, 1971).

[19] J. Preskill, Lecture Notes for Physics 229: Quantum Information and Computation, 1998, available at http://www.iqi.caltech.edu

[20] J. M. Bernardo and A. F. M. Smith, *Bayesian Theory* (Wiley, Chichester, England, 1994).

[21] C. M. Caves, C. A. Fuchs, and R. Schack, J. Math. Phys. 43, 4537 (2002).