DUAL SPACES OF MIXED-NORM MARTINGALE HARDY SPACES

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(Communicated by William O. Bray)

Abstract. In this paper, we generalize the Doob's maximal inequality for mixed-norm \( L_{\vec{p}} \) spaces. We consider martingale Hardy spaces defined with the help of mixed \( L_{\vec{r}} \)-norm. A new atomic decomposition is given for these spaces via simple atoms. The dual spaces of the mixed-norm martingale Hardy spaces is given as the mixed-norm \( BMO_{\vec{r}}(\vec{\alpha}) \) spaces. This implies the John-Nirenberg inequality \( BMO_1(\vec{\alpha}) \sim BMO_{\vec{r}}(\vec{\alpha}) \) for \( 1 < \vec{r} < \infty \). These results generalize the well known classical results for constant \( p \) and \( r \).

1. Introduction. Fefferman [5] proved that the dual space of the classical Hardy space is equivalent to the \( BMO \) space of functions of bounded mean oscillation introduced by John and Nirenberg [23]. John and Nirenberg [23] obtained their famous inequality, i.e., that the \( BMO_p \) spaces are equivalent. One year later, Fefferman and Stein [6] characterized the dual space of \( H_p \) (\( 0 < p < 1 \)) as a Lipschitz space. Since that time, the theory of Hardy spaces has been developed very quickly. Recently several papers were published about the generalization of Hardy spaces. For example, (anisotropic) Hardy spaces with variable exponents were considered in Nakai and Sawano [27], Yan, et al. [34], Jiao et al. [22], Liu et al. [24] and [25]. Moreover Musielak-Orlicz-Hardy spaces were studied in Yang et al. [35]. The mixed norm classical Hardy spaces have been developed in Cleanthous et al. [4] and intensively studied by Huang et al. in [12, 13, 14, 15, 16].

Parallel, a similar theory was evolved for martingale Hardy spaces (see e.g. Garsia [7], Long [26] and Weisz [30]). Amongst others, a martingale analogue of \( H_1-BMO \) duality can be found in these books. For dyadic martingales, Herz [8] obtained the dual space of \( H_p \) (\( 0 < p < 1 \)). In 1990, Weisz [29] characterized the dual space of \( H_p \) (\( 0 < p < 1 \)) for general martingales via atomic decomposition. For a regular stochastic basis, the \( BMO_p \) spaces are equivalent in the martingale case, too. Recently, these results were extended to more general cases. Jiao et al. investigated martingale Hardy-Lorentz spaces in [19, 20] and variable martingale Hardy spaces in [21, 18, 17]. Martingale Musielak–Orlicz Hardy spaces were investigated in Xie et al. [31, 33, 32].

2020 Mathematics Subject Classification. Primary: 42B35; Secondary: 42B30, 60G42, 42B25, 46E30.

Key words and phrases. Mixed Lebesgue spaces, mixed martingale Hardy spaces, atomic decompositions, Doob’s inequality, \( BMO \) spaces, John-Nirenberg inequality.

This research was supported by the Hungarian National Research, Development and Innovation Office-NKFIH, KH130426.
The mixed Lebesgue spaces were introduced in 1961 by Benedek and Panzone [1] (see also Hörmander [11]). They considered the Descartes product $(\Omega, \mathcal{F}, \mathbb{P})$ of the probability spaces $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$, where $\Omega = \prod_{i=1}^d \Omega^i$, $\mathcal{F}$ is generated by $\bigcup_{i=1}^d \mathcal{F}^i$ and $\mathbb{P}$ is generated by $\bigcup_{i=1}^d \mathbb{P}^i$. The mixed $L_{\vec{p}}$-norm of the measurable function $f$ is defined as a number obtained after taking successively the $L_{p_i}$-norm of $f$ in the variable $x_1$, the $L_{p_2}$-norm in the variable $x_2$, ..., the $L_{p_d}$-norm in the variable $x_d$. Some basic properties of the spaces $L_{\vec{p}}$ were proved in [1], such as the well known Hölder’s inequality and the duality theorem. Mixed-norm Lebesgue and Hardy spaces were investigated in a great number of papers (e.g. in [2, 3, 4, 9, 10, 12, 13, 14, 15, 16]).

Huang et al. [12, 16, 15] verified the atomic decomposition of mixed-norm (anisotropic) Hardy spaces defined on $\mathbb{R}^n$ and containing tempered distributions. They [13, 15] generalized the well known $H_1$-BMO duality result mentioned at the beginning and proved that the dual of $H_{\vec{p}}(\mathbb{R}^n)$ is the space $BMO_{\vec{r}}(\vec{\alpha})$, where $0 < \vec{p} \leq 1$ (this inequality is understood in the coordinate-wise sense), $\vec{\alpha} = 1/\vec{p} - 1$ and $1 \leq r < \infty$ (their notation is a little bit different from ours). This implies the generalization of the John and Nirenberg theorem: if $0 \leq \vec{\alpha} < \infty$ and $1 < r < \infty$, then $BMO_{\vec{r}}(\vec{\alpha})$ is equivalent to $BMO_{\vec{r}}(\vec{\alpha})$.

In this paper, we will prove similar results for mixed-norm martingale Hardy spaces. In [28], we have developed the theory of these spaces. There we proved the atomic decomposition of different types of Hardy spaces and some martingale inequalities. However, that atomic decomposition cannot be used to prove duality results. In this paper, a finer atomic decomposition is needed. We decompose the martingale Hardy space $H_{\vec{p}}^M$ defined by the maximal operator $M$ into the sum of the so called simple $(\vec{p}, \vec{r})$-atoms. One of the basic ideas is that the second coordinate $\vec{r}$ is also a vector, i.e., in the definition of the atom, we suppose that $\|s(a)\|_r \leq \|\chi_I\|_{\vec{p}}$, where $I$ is the support of the atom $a$ and $\chi_H$ denotes the characteristic function of the set $H$. Usually, $\vec{r}$ is a constant, see e.g. [12, 13, 16, 15, 28]. To this atomic decomposition, we need a new version of Doob’s maximal inequality, which is also proved here. Next, we prove that if the stochastic basis is regular, then the dual of $H_{\vec{p}}^M$ is the space $BMO_{\vec{r}}(\vec{\alpha})$, where $0 < \vec{p} \leq 1$, $\vec{\alpha} = 1/\vec{p} - 1$ and $1 < \vec{r} < \infty$. Consequently, we obtain the generalization of the John and Nirenberg theorem for martingale spaces: if $0 \leq \vec{\alpha} < \infty$ and $1 < \vec{r} < \infty$, then $BMO_{\vec{r}}(\vec{\alpha}) = BMO_{\vec{r}}(\vec{\alpha})$ with equivalent norms. Note that as we mentioned above, in the results in [12, 13, 15] $\vec{r}$ was a constant. Finally, we prove also some duality results for non-regular stochastic bases. For constant $p$ and $r$, we get back the well known classical results for martingale spaces proved in [30].

We denote by $C$ a positive constant, which can vary from line to line, and denote by $C_p$ a constant depending only on $p$. The symbol $A \sim B$ means that there exist constants $\alpha, \beta > 0$ such that $\alpha A \leq B \leq \beta A$ and $A \lesssim B$ means that there exist $C > 0$ such that $A \leq CB$.

I would like to thank the referees for reading the paper carefully and for their useful comments and suggestions.

2. **Mixed Lebesgue spaces.** Let $1 \leq d \in \mathbb{N}$ and $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$ be probability spaces for $i = 1, \ldots, d$, and $\vec{p} := (p_1, \ldots, p_d)$ with $0 < p_i \leq \infty$. Consider the product space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \prod_{i=1}^d \Omega^i$, $\mathcal{F}$ is generated by $\bigcup_{i=1}^d \mathcal{F}^i$ and $\mathbb{P}$ is generated by $\bigcup_{i=1}^d \mathbb{P}^i$. A measurable function $f : \Omega \to \mathbb{R}$ belongs to the mixed $L_{\vec{p}}$ space if

$$\|f\|_{\vec{p}} := \|f\|_{(p_1, \ldots, p_d)}$$
3. Doob’s inequality. Suppose that the \( \sigma \)-algebra \( \mathcal{F}_n^i \subset \mathcal{F}_i \) \( (n \in \mathbb{N}, i = 1, \ldots, d) \), \( (\mathcal{F}_n^i)_{n \in \mathbb{N}} \) is increasing and \( \mathcal{F} = \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n^i) \). Let \( \mathcal{F}_n = \sigma(\prod_{i=1}^d \mathcal{F}_n^i) \). The expectation and conditional expectation operators relative to \( \Omega, \Omega^i, \mathcal{F}_n, \) and \( \mathcal{F}_n^i \) are denoted by \( E, E^i, E_n, \) and \( E_n^i \) \( (i = 1, \ldots, d, n \in \mathbb{N}) \), respectively. Obviously, \( E_n f = E_n^1 \circ \ldots \circ E_n^d f \).

An integrable sequence \( f = (f_n)_{n \in \mathbb{N}} \) is said to be a martingale if

(a) \( (f_n)_{n \in \mathbb{N}} \) is adapted, that is for all \( n \in \mathbb{N}, f_n \) is \( \mathcal{F}_n \)-measurable;

(b) \( E_n f_m = f_n \) in case \( n \leq m \).

We define the Doob’s maximal function by

\[
M(f) := \sup_{n \in \mathbb{N}} |f_n|.
\]

Of course,

\[
M(f) \leq M_d \circ M_{d-1} \circ \ldots \circ M_1(f),
\]

where, for any \( f \in L_1 \) and \( i = 1, \ldots, d \),

\[
M_i(f) := \sup_{n \in \mathbb{N}} |E_n^i f|.
\]

In [28], we generalized the well-known Doob’s inequality, we proved the boundedness of the maximal operators \( M_d \) and \( M \) on \( L_{\vec{p}} \).
Theorem 3.1. Suppose that $1 < \tilde{p} < \infty$ or
\[
\tilde{p} = (\infty, \infty, \ldots, \infty, p_{k+1}, \ldots, p_d), \quad 1 < p_{k+1}, \ldots, p_d < \infty
\] (3.1)
for some $k \in \{1, \ldots, d\}$. Then, for all $f \in L_{\tilde{p}}$,
\[
\|M_d(f)\|_{\tilde{p}} \leq C\|f\|_{\tilde{p}}.
\]
This implies

Theorem 3.2. Under the same conditions as in Theorem 3.1, for all $f \in L_{\tilde{p}}$,
\[
\|M(f)\|_{\tilde{p}} \leq C\|f\|_{\tilde{p}}.
\]
Note that this theorem is not true for all $1 < \tilde{p} \leq \infty$ (see [28]). A weighted version of Doob’s inequality can be found in Chen et al. [2].

In this paper, we need a new maximal function defined by
\[
M_{\tilde{q}}(f) := \sup_{n \in \mathbb{N}} \left( E_n^d \left( E_n^{d-1} \cdots \left( E_n^1 \left( \mathbb{E} \left( |f|^{q_1} \right)^{\frac{q_2}{q_2}} \cdots \right)^{\frac{q_d}{q_d}} \right)^{\frac{1}{q_d}} \right)^{\frac{1}{q_d}} \right),
\]
where $0 < \tilde{q} < \infty$. Now, we can show that under some conditions, this operator is bounded on $L_{\tilde{p}}$, too.

Theorem 3.3. Let $0 < \tilde{q} < \infty$ and $0 < \tilde{p} < \infty$ or
\[
\tilde{p} = (\infty, \infty, \ldots, \infty, p_{k+1}, \ldots, p_d), \quad 0 < p_{k+1}, \ldots, p_d < \infty
\]
for some $k \in \{1, \ldots, d\}$. Suppose that
\[
\begin{aligned}
& p_1 > q_1, q_2, \ldots, q_d, \\
& p_2 > q_2, \ldots, q_d, \\
& \quad \cdots \\
& p_d > q_d.
\end{aligned}
\]
Then, for all $f \in L_{\tilde{p}}$,
\[
\|M_{\tilde{q}}(f)\|_{\tilde{p}} \leq C\|f\|_{\tilde{p}}.
\]

Proof. Observe that
\[
M_{\tilde{q}}(f) \leq \left( M_{d} \left( M_{d-1} \cdots \left( M_{2} \left( M_{1} \left| f \right|^{q_1} \right)^{\frac{q_2}{q_2}} \cdots \right)^{\frac{q_d}{q_d}} \right)^{\frac{1}{q_d}} \right)^{\frac{1}{q_d}}.
\]
By Theorem 3.1,
\[
\|M_{\tilde{q}}(f)\|_{\tilde{p}} \leq \left\| \left( M_{d} \left( M_{d-1} \cdots \left( M_{2} \left( M_{1} \left| f \right|^{q_1} \right)^{\frac{q_2}{q_2}} \cdots \right)^{\frac{q_d}{q_d}} \right)^{\frac{1}{q_d}} \right)^{\frac{1}{q_d}} \right\|_{\tilde{p}}
\]
\[
= \left\| M_{d} \left( M_{d-1} \cdots \left( M_{2} \left( M_{1} \left| f \right|^{q_1} \right)^{\frac{q_2}{q_2}} \cdots \right)^{\frac{q_d}{q_d}} \right)^{\frac{1}{q_d}} \right\|_{\tilde{p}}
\]
\[
\leq \left( M_{d-1} \cdots \left( M_{2} \left( M_{1} \left| f \right|^{q_1} \right)^{\frac{q_2}{q_2}} \cdots \right)^{\frac{q_d}{q_d}} \right)^{\frac{1}{q_d}}.
\]
\[
\|M_{d-1} \left( M_{d-2} \cdots \left( M_2 \left( M_1 |f|^{q_1} \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_3}{q_2}} \cdots \right) \right\|_{\tilde{p}} \leq \left\| M_{d-1} \left( M_{d-2} \cdots \left( M_2 \left( M_1 |f|^{q_1} \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_3}{q_2}} \cdots \right) \right\|_{\tilde{p}}^{\frac{1}{q_1}} \frac{1}{q_1},
\]
whenever \( \tilde{p} > q_d \). Applying again Theorem 3.1 in the coordinates \((p_1, \ldots, p_{d-1})\), we get that
\[
\left\| M_{d-1} \left( M_{d-2} \cdots \left( M_2 \left( M_1 |f|^{q_1} \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_3}{q_2}} \cdots \right) \right\|_{\tilde{p}} \leq \left\| \left( M_{d-2} \cdots \left( M_2 \left( M_1 |f|^{q_1} \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_3}{q_2}} \cdots \right) \right\|_{\tilde{p}}^{\frac{1}{q_1}} \frac{1}{q_1}
\]
whenever \((p_1, \ldots, p_{d-1}) > q_{d-1}\). Hence
\[
\|M_\tilde{p}(f)\|_{\tilde{p}} \leq \|\left( M_{d-2} \cdots \left( M_2 \left( M_1 |f|^{q_1} \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_3}{q_2}} \cdots \right)\|_{\tilde{p}}^{\frac{1}{q_1}} \frac{1}{q_1}
\]
Continuing this process, we obtain
\[
\|M_\tilde{p}(f)\|_{\tilde{p}} \leq \|\left( M_{d-2} \cdots \left( M_2 \left( M_1 |f|^{q_1} \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_3}{q_2}} \cdots \right)\|_{\tilde{p}}^{\frac{1}{q_1}} \frac{1}{q_1}
\]
and
\[
\|M_1 |f|^{q_1}\|_{\tilde{p}}^{\frac{1}{q_1}} \leq \|f\|_{p_1}
\]
which implies the inequality in the theorem.

\[\Box\]

4. Mixed martingale Hardy spaces. For \( n \in \mathbb{N} \), the martingale difference is defined by \( d_n f := f_n - f_{n-1} \), where \( f = (f_n)_{n \in \mathbb{N}} \) is a martingale and \( f_0 := f_{-1} := 0 \). The map \( \nu : \Omega \to \mathbb{N} \cup \{\infty\} \) is called a stopping time relative to \((\mathcal{F}_n)\) if for all \( n \in \mathbb{N} \), \( \{\nu = n\} \in \mathcal{F}_n \).

For a martingale \( f = (f_n) \) and a stopping time \( \nu \), the stopped martingale is defined by
\[
f_\nu = \sum_{m=0}^{\nu} d_m f \chi_{\nu \geq m}.
\]

Let us define the quadratic variation and the conditional quadratic variation of the martingale \( f \) relative to \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n \in \mathbb{N}})\) by
\[
S_m(f) := \left( \sum_{n=0}^{m} |d_n f|^2 \right)^{1/2}, \quad S(f) := \left( \sum_{n=0}^{\infty} |d_n f|^2 \right)^{1/2}
\]
\[
s_m(f) := \left( \sum_{n=0}^{m} \mathbb{E}_{n-1} |d_n f|^2 \right)^{1/2}, \quad s(f) := \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n f|^2 \right)^{1/2}
\]
The set of the sequences \((\lambda_n)_{n \in \mathbb{N}}\) of non-decreasing, non-negative and adapted functions with \( \lambda_{\infty} := \lim_{n \to \infty} \lambda_n \) is denoted by \( \Lambda \). With the help of the previous
operators, the mixed martingale Hardy spaces can be defined as follows:

\[ H^M_{\vec{p}} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \| f \|_{H^M_{\vec{p}}} := \| M(f) \|_{\vec{p}} < \infty \right\}; \]

\[ H^S_{\vec{p}} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \| f \|_{H^S_{\vec{p}}} := \| S(f) \|_{\vec{p}} < \infty \right\}; \]

\[ H^S_{\vec{p}} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \| f \|_{H^S_{\vec{p}}} := \| s(f) \|_{\vec{p}} < \infty \right\}; \]

\[ Q_{\vec{p}} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \exists (\lambda_n)_{n \in \mathbb{N}} \in \Lambda, \text{ such that } S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in L_{\vec{p}} \right\}, \]

\[ \| f \|_{Q_{\vec{p}}} := \inf_{(\lambda_n) \in \Lambda} \| \lambda_\infty \|_{\vec{p}}; \]

\[ P_{\vec{p}} := \left\{ f = (f_n)_{n \in \mathbb{N}} : \exists (\lambda_n)_{n \in \mathbb{N}} \in \Lambda, \text{ such that } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in L_{\vec{p}} \right\}, \]

\[ \| f \|_{P_{\vec{p}}} := \inf_{(\lambda_n) \in \Lambda} \| \lambda_\infty \|_{\vec{p}}. \]

The following corollary comes from Theorem 3.2. It is well-known for classical Hardy spaces with \( \vec{p} = (p, \ldots, p) \) (see e.g. [30]).

**Corollary 1.** If \( 1 < \vec{p} < \infty \) or \( \vec{p} \) satisfies (3.1), then \( H^M_{\vec{p}} \) is equivalent to \( L_{\vec{p}} \).

**Definition 4.1.** The stochastic basis \( (\mathcal{F}_n) \) is said to be regular, if there exists \( R > 0 \) such that for all nonnegative martingales \( (f_n) \),

\[ f_n \leq R f_{n-1}. \]

The following result was proved in [28].

**Theorem 4.2.** If the stochastic basis \( (\mathcal{F}_n) \) is regular, then the Hardy spaces are equivalent, that is

\[ H^S_{\vec{p}} = Q_{\vec{p}} = P_{\vec{p}} = H^M_{\vec{p}} = H^S_{\vec{p}} \quad (0 < \vec{p} < \infty) \]

with equivalent quasi-norms.

5. **Atomic decomposition.** In this section, we consider a new atomic characterization of mixed Hardy spaces. In [28], we proved an atomic decomposition of \( H^S_{\vec{p}} \) using \( (s, \vec{p}, \infty) \)-atoms defined by stopping times. In what follows, we assume that every \( \sigma \)-algebra \( (\mathcal{F}_n) \) is generated by countably many atoms. We denote by \( A(\mathcal{F}_n) \) the set of all atoms in \( \mathcal{F}_n \). Here we show that \( H^S_{\vec{p}} \) has atomic decomposition via simple \( (s, \vec{p}, \vec{r}) \)-atoms. We begin this section with the definition of the simple atoms.

**Definition 5.1.** Let \( 1 < \vec{r} \leq \infty \). A measurable function \( a \) is called a simple \( (s, \vec{p}, \vec{r}) \)-atom if there exist \( j \in \mathbb{N}, I \in A(\mathcal{F}_j) \) such that

(a) the support of \( a \) is contained in \( I \),

(b) \( \| s(a) \|_{\vec{r}} \leq \frac{\| s \|_{\vec{p}}}{\| s \|_{\vec{r}}}, \)

(c) \( \mathbb{E}^j_s(a) = 0. \)

If \( s(a) \) in (ii) is replaced by \( S(a) \) (or \( M(a) \)), then the function \( a \) is called simple \( (S, \vec{p}, \vec{r}) \)-atom (or simple \( (M, \vec{p}, \vec{r}) \)-atom).

We state the main result of this section. The atomic decomposition via simple \( (s, \vec{p}, \vec{r}) \)-atoms are much more complicated than the atomic decomposition via \( (s, \vec{p}, \infty) \)-atoms proved in [28]. To this, we need the condition that every \( \sigma \)-algebra is generated by countably many atoms.
Theorem 5.2. Let \( 1 < \vec{r} \leq \infty \) and
\[
\begin{align*}
p_1 < r_1, r_2, \ldots, r_d, \\
p_2 < r_2, \ldots, r_d, \\
\vdots \\
p_d < r_d.
\end{align*}
\]
(5.1)
A martingale \( f = (f_n)_{n \in \mathbb{N}} \in \mathcal{H}^p_{\vec{r}} \) if and only if there exist a sequence \((a^{k,j,i})_{k,j,i}\) of simple \((s, \vec{p}, \vec{r})\)-atoms associated with \((I_{k,j,i})_{k,j,i} \subset A(F_j)\), which are disjoint for fixed \(k\), and a sequence \((\mu_{k,j,i})_{k,j,i} \in \mathbb{N}^i\) of positive real numbers such that
\[
f_n = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} E_n a^{k,j,i} \quad \text{a.e. } \quad (n \in \mathbb{N})
\]
(5.2)
and
\[
\|f\|_{\mathcal{H}^p_{\vec{r}}} \sim \inf \left\| \left( \sum_{k \in \mathbb{Z}} \left( \sum_{j=0}^{\infty} \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\vec{p}}} \right)^t \right)^{1/t} \right\|_{\mathcal{F}_T}^{t/p},
\]
where \(0 < t < \min\{p_-, 1\}\) and the infimum is taken over all decompositions of the form (5.2).

Note that \(p_-\) was defined in (2.1).

Proof. For \(f \in \mathcal{H}^p_{\vec{r}}\), let us define the stopping times
\[
\tau_k := \inf \{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \}.
\]
For fixed \(k, j\), there exist disjoint atoms \((I_{k,j,i})_{i} \subset F_j\) such that
\[
\bigcup_i I_{k,j,i} = \{ \tau_k = j \} \subset F_j.
\]
It is easy to see that
\[
f_n = \sum_{k \in \mathbb{Z}} (f_{n}^{\tau_k} - f_{n}^{\tau_k}) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \chi_{I_{k,j,i}} (f_{n}^{\tau_k} - f_{n}^{\tau_k}) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} a^{k,j,i}_n,
\]
where
\[
\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{\vec{p}} \quad \text{and} \quad a^{k,j,i}_n = \chi_{I_{k,j,i}} \frac{f_{n}^{\tau_k} - f_{n}^{\tau_k}}{\mu_{k,j,i}}.
\]
Since \(I_{k,j,i} \subset \{ \tau_k = j \}\) and
\[
\chi_{I_{k,j,i}} (f_{n}^{\tau_k} - f_{n}^{\tau_k}) = \chi_{I_{k,j,i}} \sum_{m=j+1}^{n} d_m f \chi_{\{\tau_{m+1} \geq m > \tau_k\}},
\]
we conclude that \(E_j (a^{k,j,i}_n) = 0\). Moreover,
\[
s((a^{k,j,i}_n)_n) \leq \frac{1}{\|\chi_{I_{k,j,i}}\|_{\vec{p}}},
\]
thus \((a^{k,j,i}_n)_n\) is an \(L_2\)-bounded martingale and so there exists \(a^{k,j,i} \in L_2\) such that
\[
E_n (a^{k,j,i}) = a^{k,j,i}_n \quad \text{and} \quad s(a^{k,j,i}) \leq \frac{1}{\|\chi_{I_{k,j,i}}\|_{\vec{p}}}.
\]
This means that $a^{k,j,i}$ is a simple $(s, \vec{p}, \infty)$-atom. Furthermore,
\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left( \sum_{j=0}^{\infty} \frac{\mu_{k,i,j} \chi_{I_k,j,i}}{\| \chi_{I_k,j,i} \|_{p'}} \right)^t \right)^{1/t} \right\|_{p'} = \left\| \left( \sum_{k \in \mathbb{Z}} (3 \cdot 2^k \chi_{\{t_k \leq \infty\}})^t \right)^{1/t} \right\|_{p'} \leq \|f\|_{H^s_p},
\]
where the second inequality was proved in [28]. Since every simple $(s, \vec{p}, \infty)$-atom is a simple $(s, \vec{p}, \vec{r})$-atom, the first half of the theorem is proved.

Conversely, if $f$ has a decomposition of the form (5.2), then
\[
s(f) \leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} s(a^{k,j,i}).
\]
Since $0 < t < \min\{p_-, 1\} \leq 1$, we obtain
\[
\|f\|_{H^s_p} \leq \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \left| s(a^{k,j,i}) \right|^t \right)^{1/t} \right\|_{p'} = \left\| \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \left| s(a^{k,j,i}) \right|^t \right\|_{p'}.\]

By Lemma 2.2, there exists a function $g \in L_{q}((\vec{p}, t')')$ with $\|g\|_{(\vec{p}, t')'} \leq 1$, such that
\[
\|f\|_{H^s_p} \leq \int_{\Omega} \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \left| s(a^{k,j,i}) \right|^t g \, d\mathbb{P}.
\]
The fact $a^{k,j,i} = a^{k,j,i} \chi_{I_k,j,i}$ implies that the support of $s(a^{k,j,i})$ is also $I_k,j,i$. Since $t < \vec{r}$, the previous expression can be estimated by Hölder’s inequality
\[
\|f\|_{H^s_p} \leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \int_{\Omega} \chi_{I_k,j,i} \left| s(a^{k,j,i}) \right|^t g \, d\mathbb{P}
\leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \left\| s(a^{k,j,i}) \right\|_{(\vec{r}, t')'} \left\| \chi_{I_k,j,i} \right\|_{(\vec{r}, t')'}
\leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \left\| \chi_{I_k,j,i} \right\|_{p'} \left\| \chi_{I_k,j,i} \right\|_{(\vec{r}, t')'}.
\]

Since $I_{k,j,i} = I_{k,j,i}^1 \times \cdots I_{k,j,i}^d$, it is easy to see that
\[
\left\| \chi_{I_{k,j,i}} \right\|_r = \mathbb{P}(I_{k,j,i}^1)^{1/r_1} \cdots \mathbb{P}(I_{k,j,i}^d)^{1/r_d}
\]
and
\[
\left\| \chi_{I_{k,j,i}} \right\|_{p'} \left\| \chi_{I_{k,j,i}} \right\|_{(\vec{r}, t')'} = \mathbb{P}(I_{k,j,i})^{t/r_1} \cdots \mathbb{P}(I_{k,j,i})^{t/r_d}
\]
\[
= \mathbb{P}(I_{k,j,i})^{t/r_1} \cdots \mathbb{P}(I_{k,j,i})^{t/r_d}
\]
\[
\left( \int_{I_{k,j,i}^1} \cdots \left( \int_{I_{k,j,i}^d} \right)^{(r_1/t)'} d x_1 \right)^{(r_1/t)'} \cdots d x_d
\]
\[
= \int_{I_{k,j,i}} \left( \frac{1}{\mathbb{P}(I_{k,j,i})} \right) \int_{I_{k,j,i}^1} \cdots
\]
Consequently, of Theorem 5.2, which proves the theorem.

\[
\left( \frac{1}{\mathbb{P}(I_{k,j,i})} \int_{I_{k,j,i}} |g(x_1, \ldots, x_d)|^{(r_1/t)'} dx_1 \right)^{(r_2/t)'} \left( \int_{I_{k,j,i}} \cdots dx_d \right)^{(r_d/t)'} \frac{1}{\mathbb{P}(I_{k,j,i})} \cdot d\mathbb{P} \\
= \int_{I_{k,j,i}} \left( \mathbb{E}_n^d \left( \mathbb{E}_n^{d-1} \cdots \left( \mathbb{E}_n^1 \left| g \right|^{(r_1/t)'} \right)^{(r_2/t)'} \right) \cdots \right)^{(r_d/t)'} \frac{1}{\mathbb{P}(I_{k,j,i})} \cdot d\mathbb{P} \\
\leq \int_{I_{k,j,i}} M(\mathcal{F}/t')'(g) d\mathbb{P}.
\]

Here we have used that

\[
\mathbb{E}_n^i(g) = \sum_{A \in A(A(F_n^i))} \left( \frac{1}{\mathbb{P}(A)} \int_A g d\mathbb{P} \right) \chi_A \quad (n \in \mathbb{N}, i = 1, \ldots, d).
\]

Then

\[
\|f\|_{H_\mathcal{F}^t} \leq \int_{\Omega} \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\mathbb{P}}^t} M(\mathcal{F}/t')'(g) d\mathbb{P} \\
\leq \left\| \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\mathbb{P}}^t} \right\|_{\mathbb{P}/t} \|M(\mathcal{F}/t')'(g)\|_{\mathbb{P}/t'}.
\]

Since (5.1) is equivalent to

\[
\begin{align*}
(p_1/t')' > (r_1/t)', (r_2/t)', \ldots, (r_d/t)', \\
(p_2/t')' > (r_2/t)', \ldots, (r_1/d)', \\
\cdots \\
(p_d/t')' > (r_d/t)',
\end{align*}
\]

the boundedness of \(M\) (see Theorem 3.3) implies

\[
\|f\|_{H_\mathcal{F}^t} \leq \left\| \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\mathbb{P}}^t} \right\|_{\mathbb{P}/t} \|g\|_{(\mathcal{F}/t')'} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \sum_{j=0}^\infty \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\mathbb{P}}^t} \right)^{1/t} \right\|_{\mathbb{P}/t}.
\]

Consequently,

\[
\|f\|_{H_\mathcal{F}^t} \leq \left\| \sum_{k \in \mathbb{Z}} \left( \left( \sum_{j=0}^\infty \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\mathbb{P}}^t} \right)^{1/t} \right) \right\|_{\mathbb{P}/t} = \left\| \sum_{k \in \mathbb{Z}} \left( \sum_{j=0}^\infty \sum_{i} \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\mathbb{P}}^t} \right)^{1/t} \right\|_{\mathbb{P}/t},
\]

which proves the theorem. \(\square\)

For the classical martingale Hardy space \(H_\mathcal{F}^t\), this result is due to the author (see [30]). We present the following result without proof because it is similar to the one of Theorem 5.2.

**Theorem 5.3.** Let \(1 < \mathfrak{r} \leq \infty\) satisfy (5.1). A martingale \(f = (f_n)_{n \in \mathbb{N}} \in P_{\mathcal{F}}\) (resp. \(Q_{\mathcal{F}}\)) if and only if there exist a sequence \((a^{k,j,i})_{k,j,i}\) of simple \((\mathcal{F}, \mathfrak{r}, \mathfrak{r}')\)-atoms (resp. simple \((S, \mathfrak{r}, \mathfrak{r}')\)-atoms) associated with \((I_{k,j,i})_{k,j,i} \subset A(A(F_j))\), which are disjoint for
fixed $k$, and a sequence $(\mu_{k,j,i})_{k,j,i \in \mathbb{N}}$ of positive real numbers such that (5.2) holds and

$$\|f\|_{P_p} (\text{resp. } \|f\|_{Q_p}) \sim \inf \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \chi_{I_{k,j,i}} \right)^t \right\|_p,$$

where $0 < t < \min \{p-, 1\}$ and the infimum is taken over all decompositions of the form (5.2).

This implies that if the stochastic basis is regular, then $H^M_{\vec{p}}$ and $H^S_{\vec{p}}$ have also atomic decompositions.

6. The dual spaces of mixed Hardy spaces. In this section, we study the dual spaces of mixed martingale Hardy spaces.

**Definition 6.1.** Let $0 \leq \vec{\alpha} < \infty$ and $1 \leq \vec{q} < \infty$. Define $BMO_{\vec{q}}(\vec{\alpha})$ as the space of functions $f \in L^\infty_{\vec{q}}$ for which

$$\|f\|_{BMO_{\vec{q}}(\vec{\alpha})} = \sup_{n \geq 0} \sup_{I \in \mathcal{A}(F_n)} \|\chi_I\|^{-1}_{\vec{q}} \|\chi_I\|_{(\vec{q})'} \|f - f_n\|_{\vec{q}} < \infty.$$

If $q$ is a constant and $\vec{\alpha} = 0$, then this definition goes back to the classical martingale $BMO_q$ space. If both $q$ and $\vec{\alpha}$ are non-zero constants, then this definition becomes the classical martingale Lipschitz space investigated in Weisz [30].

**Proposition 1.** If $0 < \vec{p} \leq 1$ and $1 < \vec{r} \leq \infty$, then

$$\sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \lesssim \|f\|_{H^s_{\vec{p}}} \quad (f \in H^s_{\vec{p}}),$$

where $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{\vec{p}}$. An analogous result holds also for $P_{\vec{p}}$.

**Proof.** By Lemma 2.3, we have

$$\sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \lesssim \sum_{k \in \mathbb{Z}} 2^k \sum_{j=0}^{\infty} \sum_{i} \|\chi_{I_{k,j,i}}\|_{\vec{p}} \leq \sum_{k \in \mathbb{Z}} 2^k \sum_{j=0}^{\infty} \sum_{i} \chi_{I_{k,j,i}} \|f\|_{H^s_{\vec{p}}},$$

which finishes the proof.

Now we are ready to characterize the dual of $H^s_{\vec{p}}$.

**Theorem 6.2.** If $0 < \vec{p} \leq 1$ and $\vec{\alpha} = 1/\vec{p} - 1$, then

$$(H^s_{\vec{p}})^* = BMO_2(\vec{\alpha})$$

with equivalent norms.

**Proof.** For $\varphi \in BMO_2(\vec{\alpha}) \subset L_2$, define

$$l_\varphi(f) = \mathbb{E}(f \varphi) \quad (f \in L_2).$$

By Hölder’s inequality,

$$\|f\|_{H^s_{\vec{p}}} = \|s(f)\|_2 \leq \|s(f)\|_2 = \|f\|_2 \quad (f \in L_2).$$

Thus $L_2$ can be embedded continuously in $H^s_{\vec{p}}$. Theorem 5.2 implies that for each $f \in L_2$,

$$f = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} a^{k,j,i} \quad (6.1)$$
and the convergence holds also in the $L_2$-norm, where $a^{k,j,i}$ is a simple $(s,\vec{p},2)$-atom and 
$\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{\vec{p}}$. Hence

$$l_\varphi(f) = \mathbb{E}(f \varphi) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_i \mu_{k,j,i} \mathbb{E}(a^{k,j,i} \varphi).$$

Since

$$\mathbb{E}(a^{k,j,i} \varphi) = \mathbb{E}(a^{k,j,i}(\varphi - \varphi_j)),$$

we have

$$|l_\varphi(f)| \leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_i \mu_{k,j,i} \left| \int_\Omega a^{k,j,i}(\varphi - \varphi_j) d\mathbb{P} \right|$$

$$\leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_i \mu_{k,j,i} \|a^{k,j,i}\|_2 (\varphi - \varphi_j) \chi_{I_{k,j,i}} \|_2$$

$$\leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^\infty \sum_i \mu_{k,j,i} \|a^{k,j,i}\|_{BMO_2(\vec{a})} \|\varphi\|_{BMO_2(\vec{a})} \| \varphi \|_{BMO_2(\vec{a})} \| \varphi \|_{BMO_2(\vec{a})}$$

because of Proposition 1. Since by Theorem 5.2, $L_2$ is dense in $H_{\vec{p}}^s$, thus $l_\varphi$ can be uniquely extended to a linear functional on $H_{\vec{p}}^s$.

Conversely, let $l$ be an arbitrary bounded linear functional on $H_{\vec{p}}^s$. Since $L_2$ can be embedded continuously to $H_{\vec{p}}^s$, there exists $\varphi \in L_2$ such that

$$l(f) = l_\varphi(f) = \mathbb{E}(f \varphi) \quad (f \in L_2).$$

For $I \in A(\mathcal{F}_j)$, set

$$a = \frac{(\varphi - \varphi_j) \chi_I}{\|((\varphi - \varphi_j) \chi_I\|_2 \| \chi_I \|_{\frac{1}{\vec{p}}} \| \chi_I \|_2^{-1}}.$$ Then the function $a$ is a simple $(s,\vec{p},2)$-atom and so $a \in H_{\vec{p}}^s$ with $\|a\|_{H_{\vec{p}}^s} \lesssim 1$.

Finally,

$$\|l\| \geq l(a) = \mathbb{E}(a(\varphi - \varphi_j)) = \| \chi_I \|_{\frac{1}{s+1}} \| \chi_I \|_2 \|((\varphi - \varphi_j) \chi_I\|_2.$$ This means that

$$\|\varphi\|_{BMO_2(\vec{a})} \lesssim \|l\|$$

and the theorem is shown.

Let us denote by $(P_{\vec{p}})_1^*$ those elements $l$ from $(P_{\vec{p}})^*$ for which there exists $\varphi \in L_1$ such that $l(f) = \mathbb{E}(f \varphi)$ ($f \in L_\infty$). We can verify the following result similarly to Theorem 6.2.

**Theorem 6.3.** If $0 < \vec{p} \leq 1$ and $\vec{a} = 1/\vec{p} - 1$, then

$$(P_{\vec{p}})_1^* = BMO_1(\vec{a})$$

with equivalent norms.
Proof. Let \( \varphi \in BMO_1(\tilde{\alpha}) \subset L_1 \) and define

\[
l_{\varphi}(f) = \mathbb{E}(f \varphi) \quad (f \in L_\infty).
\]

By Theorem 4.2,

\[
f = \sum_{k \in \mathbb{Z}} (f^{\tau_k+1} - f^{\tau_k}) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{n-1} \sum_i \mu_{k,j,i} a^{k,j,i}
\]

and the partial sums can be majorized by \( 2M(f) \in L_\infty \) if \( f \in L_\infty \). Hence, by the dominated convergence theorem,

\[
l_{\varphi}(f) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \mathbb{E}(a^{k,j,i} \varphi)
\]

where \( a^{k,j,i} \) is a simple \((M, \tilde{p}, \infty)\)-atom and \( \mu_{k,j,i} = 3 \cdot 2^k \| \chi_{I_{k,j,i}} \|_{\tilde{p}} \). Similarly to Theorem 6.2,

\[
|l_{g}(f)| \lesssim \|f\|_{\tilde{p}} \|g\|_{BMO_1(\tilde{\alpha})}.
\]

Since \( L_\infty \) is dense in \( P_{\tilde{p}} \) (see Theorem 4.2), \( l_{\varphi} \) can be extended to a continuous linear functional on \( P_{\tilde{p}} \).

Conversely, if \( l \in (P_{\tilde{p}})^* \), then there exists \( \varphi \in L_1 \) such that

\[
l(f) = l_{\varphi}(f) = \mathbb{E}(f \varphi) \quad (f \in L_\infty).
\]

For \( I \in A(\mathcal{F}_n) \), set

\[
h = \text{sign}(\varphi - \varphi_j), \quad a = \frac{1}{2} \| \chi_I \|^{-\frac{1}{\tilde{p}} - 1} (h - h_j) \chi_I.
\]

Then \( a \) is a simple \((M, \tilde{p}, \infty)\)-atom. From this it follows that

\[
\|l\| \gtrless l(a) = \mathbb{E}(a(\varphi - \varphi_j)) = \frac{1}{2} \| \chi_I \|^{-\frac{1}{\tilde{p}} + 1} \mathbb{E}(h(\varphi - \varphi_j) \chi_I) = \frac{1}{2} \| \chi_I \|^{-\frac{1}{\tilde{p}} + 1} \| (\varphi - \varphi_j) \chi_I \|_1,
\]

that is to say \( \|l\|_{MO_1(\tilde{\alpha})} \lesssim \|l\| \).

Corollary 2. If \( 0 < \tilde{p} \leq 1, \tilde{\alpha} = 1/\tilde{p} - 1 \) and \((\mathcal{F}_n)\) is regular, then

\[
(P_{\tilde{p}})^* = BMO_1(\tilde{\alpha})
\]

with equivalent norms.

Proof. If \((\mathcal{F}_n)\) is regular, then, by Theorem 4.2, \( P_{\tilde{p}} \) is equivalent to \( H_{\tilde{p}}^* \). We know that \( L_2 \) can be embedded continuously to \( H_{\tilde{p}}^* \). Thus, for \( l \in P_{\tilde{p}} \), there exists \( \varphi \in L_2 \subset L_1 \) such that \( l(f) = \mathbb{E}(f \varphi) \) for any \( f \in L_2 \supset L_\infty \). Hence \( (P_{\tilde{p}})^1 = (P_{\tilde{p}})^* \) and the corollary follows from Theorem 6.3.

For a regular stochastic basis \( (\mathcal{F}_n) \), we can prove sharper results.

Theorem 6.4. If \( 0 < \tilde{p} \leq 1, \tilde{\alpha} = 1/\tilde{p} - 1, 1 < (\tilde{r})' < \infty \) and \((\mathcal{F}_n)\) is regular, then

\[
(H_{\tilde{p}})^* = BMO_{(\tilde{r})'}(\tilde{\alpha})
\]

with equivalent norms.

Proof. Let \( \varphi \in BMO_{(\tilde{r})'}(\tilde{\alpha}) \subset L_{(\tilde{r})'} \) and define

\[
l_{\varphi}(f) = \mathbb{E}(f \varphi) \quad (f \in L_{\tilde{r}}).
\]
By Theorem 4.2, $H^M_p$ is equivalent to $P^\rho$, so we can apply Theorem 5.3. Thus the atomic decomposition (6.1) converge in the $L^\rho$-norm if $f \in L^\rho$. Here $a^{k,j,i}$ is a simple $(M, p, \rho)$-atom and $\mu_{k,j,i} = 3 \cdot 2^k |\chi_{l_{k,j,i}}|$. Hence

$$l_{\varphi}(f) = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} \mu_{k,j,i} \mathbb{E}(a^{k,j,i} \varphi).$$

The inequality

$$|l_{\varphi}(f)| \lesssim \|f\|_{H^M_p} \|\varphi\|_{BMO(\rho, \rho)}$$

is obtained as in Theorem 6.2. By Theorem 5.2, $L^\rho$ is dense in $H^M_p$, thus $l_{\varphi}$ can be extended to a continuous linear functional on $H^M_p$.

Conversely, let $l$ be an arbitrary bounded linear functional on $H^M_p$. Since $L^\rho$ can be embedded continuously to $H^M_p$, there exists $\varphi \in L^\rho$ such that

$$l(f) = l_{\varphi}(f) = \mathbb{E}(f \varphi) \quad (f \in L^\rho).$$

For $I \in A(F_j)$ and $1 < (\rho)' < \infty$, set

$$h = \text{sign}(\varphi - \varphi_j)((\varphi - \varphi_j)\chi_I)|r_1^{i-1}| \times \frac{\|\varphi - \varphi_j\| \chi_I(r_1^i - r_1) \|\varphi - \varphi_j\| \chi_I(r_1^{i-1}) \cdots \|\varphi - \varphi_j\| \chi_I(r_1^{i-1}) \|\varphi - \varphi_j\| \chi_I(r_1^{i-1})}{C \|\varphi - \varphi_j\| \chi_I \|\chi_I\|^2_{\rho} \|\chi_I\|_{\rho}^{(r_1 - 1)}}$$

and

$$a := h - h_j.$$

Since $r_1(r_1^j - 1) = r_1$, we conclude

$$\|h\|_{(r_1)} = \left(\int_{\Omega} |h(x_1, \ldots, x_d)|^{r_1} \, d\mathcal{P}(x_1)\right)^{1/r_1} = \left(\int_{\Omega} |(\varphi - \varphi_j)\chi_I(r_1^i - r_1) \varphi_j \chi_I(r_1^{i-1}) \cdots \|\varphi - \varphi_j\| \chi_I(r_1^{i-1}) \|\varphi - \varphi_j\| \chi_I(r_1^{i-1})}{C \|\varphi - \varphi_j\| \chi_I \|\chi_I\|^2_{\rho} \|\chi_I\|_{\rho}^{(r_1 - 1)}}

Taking the $r_2, \ldots, r_d$-norm in the variables $x_2, \ldots, x_d$, we obtain

$$\|h\|_{\rho} = \frac{\|\chi_I\|_{\rho}}{C \|\chi_I\|_{\rho}}.$$

Theorem 3.2 means that, with a suitable constant $C$,

$$\|M(a)\|_{\rho} \leq 2\|M(h)\|_{\rho} \leq C\|h\|_{\rho} \leq \frac{\|\chi_I\|_{\rho}}{C \|\chi_I\|_{\rho}}.$$

Thus $a$ is a simple $(M, p, \rho)$-atom and $\|a\|_{H^M_p} \lesssim 1$. Consequently,

$$\|l\| \geq l(a) = \mathbb{E}(a(\varphi - \varphi_j)) = \mathbb{E}^d \cdots \mathbb{E}^1(h(\varphi - \varphi_j))$$
If Corollary 3.

inequality.

which implies that

<ref>From Theorem 6.4, we get a generalization of the well known John-Nirenberg inequality.</ref>

Corollary 3. If $0 \leq \vec{\alpha} < \infty$, $1 < (\vec{r})' < \infty$ and $(\mathcal{F}_n)$ is regular, then

\[ BMO_{1}(\vec{\alpha}) = BMO_{(\vec{r})'}(\vec{\alpha}) \]

with equivalent norms.

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Received May 2020; revised October 2020.

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