Abstract

We study the path realization of Demazure crystals related to solvable lattice models in statistical mechanics. Various characters are represented in a unified way as the sums over one dimensional configurations which we call unrestricted, classically restricted and restricted paths. As an application characters of Demazure modules are obtained in terms of $q$-multinomial coefficients for several level 1 modules of classical affine algebras.
0 Introduction

Let $U_q(g)$ be a quantum affine algebra and $V(\lambda)$ be the integrable $U_q(g)$-module with highest weight $\lambda \in P^+$. Given a Weyl group element $w$, the associated Demazure module $V^w(\lambda)$ is the finite dimensional subspace of $V(\lambda)$ generated from the external weight space $V(\lambda)_w$ by the $e_i$ generators. In [1], Kashiwara introduced its crystal $B^w(\lambda)$, which is a finite subset of the crystal $B(\lambda)$ for $V(\lambda)$. With this aid the character of the Demazure module $V^w(\lambda)$ is expressed as

$$chV^w(\lambda) = \sum_{p \in B^w(\lambda)} e^{wt_p}. \quad (0.1)$$

The subject of this paper is to calculate (0.1) systematically by using the path realization of the Demazure crystal $B^w(\lambda)$ studied in [2], [3]. The realization is based on the earlier one for $B(\lambda)$ [4], [5] and has an origin in the analyses of solvable lattice models [6], [7], [8]. In these works the object called one dimensional configuration sums (1dsums) played an essential role and we studied extensively either in their "infinite lattice limits $j \to \infty$" or "finite truncations $j < \infty$". In this paper we consider three kinds of 1dsums $g_j, X_j$ and $\overline{X}_j$, which we call unrestricted, classically restricted and restricted 1dsums, respectively. The unrestricted $g_j$ is relevant to vertex models and so is $X_j$ to the restricted solid-on-solid (RSOS) type models. In section 2 we apply the main theorem in [2] to relate the Demazure character with the 1dsum $g_j$ for finite $j$. We shall also clarify the relations among the three kinds of 1dsums and thereby give a unified picture to understand the Demazure characters and various branching functions. (See Table 1.) This enables us to evaluate these quantities explicitly from several known results on the 1dsums. As an application, in section 3 we shall give $q$-multinomial formulae for $chV^w(\lambda)$ for many level 1 modules $V(\lambda)$ over $U_q(g)$ for $g$ of classical types $g = A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. We hope to report on higher level cases in a future publication.

1 Path realization of Demazure crystals

1.1 Perfect crystals

Let us recall relevant facts and notations from [2], [3], [4], [5]. $\alpha_i, h_i, \Lambda_i (i \in I)$ are the simple roots, coroots and fundamental weights, respectively. We put $\rho = \sum_{i \in I} \Lambda_i$ and let $\delta$ denote the null root. $P = \oplus_i \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta$ and $P_{cl} = \oplus_i \mathbb{Z} \Lambda_i \subset P$ are the weight and the classical weight lattices, respectively. Set further $P^+(P^+_{cl}) = \{ \lambda \in P(P_{cl}) \mid \langle \lambda, h_i \rangle \geq 0 \text{ for any } i \}$ and $(P^+_{cl})^l = \{ \lambda \in P^+_{cl} \mid \langle \lambda, c \rangle = l \}$, where $c$ is the canonical central element. For $\lambda \in P^+$ we let $(L(\lambda), B(\lambda))$ denote the crystal base of the irreducible $U_q(\mathfrak{g})$-module $V(\lambda)$ with highest weight $\lambda$. For a crystal base of a finite dimensional $U_q(\mathfrak{g})$-module we use the symbol $(L, B)$. Let $B$ be a perfect crystal of level $l$. See Definition 4.6.1
in [4] for its definition. Then for any \( \lambda \in (P_+^+)_I \), there uniquely exists \( b(\lambda) \in B \) such that \( \varphi(b(\lambda)) = \lambda \). Here we recall that \( \varepsilon_i(b) = \max \{ k \mid \tilde{e}_i^k b \neq 0 \} \), \( \varphi_i(b) = \max \{ k \mid f_i^k b \neq 0 \} \), \( \varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i \) and \( \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i \). Let \( \sigma \) be the automorphism of \((P^+_I)_I\) given by \( \sigma \lambda = \varepsilon(b(\lambda)) \). We put \( b_k = b(\sigma^{k-1} \lambda) \) and \( \lambda_k = \sigma^k \lambda \). Then perfectness assures that we have the isomorphism of crystals

\[
\mathcal{B}(\lambda_{k-1}) \simeq \mathcal{B}(\lambda_k) \otimes \mathcal{B}.
\]  

(1.1)

Define the set of paths \( \mathcal{P}(\lambda, B) \) by

\[
\mathcal{P}(\lambda, B) = \{ p = \cdots \otimes p(2) \otimes p(1) \mid p(j) \in B, p(k) = \overline{b}_k \text{ for } k \gg 0 \},
\]

By iterating (1.1) we have an isomorphism of crystals

\[
\mathcal{B}(\lambda) \simeq \mathcal{P}(\lambda, B).
\]

(1.2)

In particular, the image of the highest weight vector \( u_\lambda \in \mathcal{B}(\lambda) \) is given by \( \overline{p} = \cdots \otimes \overline{b}_k \otimes \cdots \otimes \overline{b}_2 \otimes \overline{b}_1 \). We call \( \overline{p} \) the ground-state path. The actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( \mathcal{P}(\lambda, B) \) are determined explicitly by the signature rule. See section 1.3 of [3].

To describe the weights on \( \mathcal{P}(\lambda, B) \) it is necessary to introduce the energy function \( H : B \otimes B \to \mathbb{Z} \). Up to an additive constant it is determined by requiring the following for any \( b, b' \in B \) and \( i \in I \) such that \( \tilde{e}_i(b \otimes b') \neq 0 \).

\[
H(\tilde{e}_i(b \otimes b')) = \begin{cases} 
H(b \otimes b') & \text{if } i \neq 0 \\
H(b \otimes b') + 1 & \text{if } i = 0 \text{ and } \varphi_0(b) \geq \varepsilon_0(b') \\
H(b \otimes b') - 1 & \text{if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b').
\end{cases}
\]

(1.3)

Under the isomorphism (1.2), the weight of a path \( p = \cdots \otimes p(2) \otimes p(1) \) is given by (Proposition 4.5.4 in [4])

\[
wt p = \lambda + \sum_{i=1}^\infty (wt p(i) - wt \overline{b}_i) \\
- \left( \sum_{i=1}^\infty i(H(p(i+1) \otimes p(i)) - H(\overline{b}_{i+1} \otimes \overline{b}_i)) \right) \delta.
\]

(1.4)

We remark the weight relation

\[
\lambda_j = \lambda - \sum_{i=1}^j wt \overline{b}_i,
\]

(1.5)

which is valid for any \( j \geq 0 \).
1.2 Demazure modules

Let \( \{ r_i \}_{i \in I} \) be the set of simple reflections, and let \( W \) be the Weyl group. For \( \lambda \in (P^+_\mathbb{Z})_I \) we consider the Demazure module \( V_w(\lambda) \) generated from the extremal weight space \( V(\lambda) \). By definition its character is given by \( \text{ch} V_w(\lambda) = \sum_{\mu} \dim(V_w(\lambda))_{\mu} e^{\mu} \). For \( \mu \in P, i \in I \) define the operator \( D_i : Z[P] \to Z[P] \) by

\[
D_i(e^{\mu}) = \frac{e^{\mu+\rho} - e^{-r_i(\mu+\rho)}}{1 - e^{-\alpha_i}} e^{-\rho}.
\]

Let \( w = r_{i_k} \cdots r_{i_1} \in W \) be a reduced expression. Then the following character formula is well known \[9\], \[10\], \[11\].

\[
\text{ch} V_w(\lambda) = D_{i_k} \cdots D_{i_2} D_{i_1}(e^{\mu}).
\]

(1.6)

From this one has a recursion relation

\[
\text{ch} V_{r_i w}(\lambda) = D_i(\text{ch} V_w(\lambda)) \quad \text{if} \quad r_i w \succ w.
\]

(1.7)

Let \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) be the crystal base of \( V(\lambda) \). In \[1\] Kashiwara showed that for each \( w \in W \), there exists a subset \( \mathcal{B}_w(\lambda) \) of \( \mathcal{B}(\lambda) \) such that

\[
\frac{V_w(\lambda) \cap \mathcal{L}(\lambda)}{V_w(\lambda) \cap q\mathcal{L}(\lambda)} = \bigoplus_{b \in \mathcal{B}_w(\lambda)} Qu.
\]

(1.8)

Furthermore, \( \mathcal{B}_w(\lambda) \) has the following recursive property.

If \( r_i w \succ w \), then

\[
\mathcal{B}_{r_i w}(\lambda) = \bigcup_{n \geq 0} \mathcal{f}_{i}^{n} \mathcal{B}_w(\lambda) \setminus \{0\},
\]

(1.9)

which is analogous to \[1.7\]. We call \( \mathcal{B}_w(\lambda) \) a Demazure crystal. One can now express \( \text{ch} V_w(\lambda) \) as in \[1.3\]). It affords an efficient way to calculate the Demazure character through the path realization of \( \mathcal{B}_w(\lambda) \) given in the next subsection and \[1.4\]. Relations with the formula \[1.6\] and the 1dsums will also be explained in section 2.

1.3 Path realization

In \[2\] the image of \( \mathcal{B}_w(\lambda) \) under the isomorphism \[1.2\] is determined for a suitably chosen Weyl group element \( w \). Let us recall the main theorem therein, which gives a path realization of the Demazure crystal. There appears the mixing index \( \kappa \) specified from \( \lambda \) and \( B \). (See section 2.3 of \[2\].) In this paper we shall only consider the case \( \kappa = 1 \). Let \( \lambda \) be an element of \( (P^+_\mathbb{Z})_I \), and let \( B \) be a classical crystal. For the theorem, we need to assume four conditions (I-IV).
(I) $B$ is perfect of level $l$.

Thus, we can assume an isomorphism between $B(\lambda)$ and the set of paths $\mathcal{P}(\lambda, B)$. Let $\overline{p} = \cdots \otimes \overline{b}_2 \otimes \overline{b}_1$ denote the ground state path. Fix a positive integer $d$. For a set of elements $i_a^{(j)}$ ($j \geq 1, 1 \leq a \leq d$) in $I$, we define $B_a^{(j)}$ ($j \geq 1, 0 \leq a \leq d$) by

$B_0^{(j)} = \{\overline{b}_j\}$, \quad $B_a^{(j)} = \bigcup_{n \geq 0} \tilde{f}_i^{n} B_{a-1}^{(j)} \setminus \{0\}$ \quad ($a = 1, \cdots, d$).

(II) For any $j \geq 1$, $B_d^{(j)} = B$.

(III) For any $j \geq 1$ and $1 \leq a \leq d$, $\langle \lambda, h_{i_a^{(j)}} \rangle \leq \varepsilon_{i_a^{(j)}}(b)$ for all $b \in B_a^{(j)}$.

We now define an element $w^{(k)}$ of the Weyl group $W$ by

$w^{(0)} = 1$, \quad $w^{(k)} = r_{i_a^{(j)}} w^{(k-1)}$ for $k > 0$,

where $j$ and $a$ are fixed from $k$ by $k = (j - 1)d + a$, $j \geq 1, 1 \leq a \leq d$.

(IV) $w^{(0)} \prec w^{(1)} \prec \cdots \prec w^{(k)} \prec \cdots$.

See [2], [3] on how to check the last condition.

Finally we define a subset $\mathcal{P}^{(k)}(\lambda, B)$ of $\mathcal{P}(\lambda, B)$ as follows. We set $\mathcal{P}^{(0)}(\lambda, B) = \{\overline{p}\}$. For $k > 0$,

\[ \mathcal{P}^{(k)}(\lambda, B) = \cdots \otimes B_0^{(j+2)} \otimes B_0^{(j+1)} \otimes B_a^{(j)} \otimes B^{(j-1)}, \]

where $j \geq 1$ and $1 \leq a \leq d$ are uniquely specified by $k = (j - 1)d + a$.

Now we have

**Theorem 1.1 ([2])** Under the assumptions (I-IV), we have

$B_{w^{(k)}}(\lambda) \simeq \mathcal{P}^{(k)}(\lambda, B)$.

The proof is done by showing the recursion relation (1.9) in the path realization.

### 2 One dimensional sums

Here we first introduce the unrestricted 1 dimensional sum (1dsum) $g_j$ and express the Demazure characters in terms of it. After establishing its fundamental properties we then introduce classically restricted 1dsums $X_j$ and restricted 1dsums $\overline{X}_j$ and study their relations. In the working below we shall use the variable

$q = e^{-\delta}$. 

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2.1 Unrestricted 1dsum

For \( j \in \mathbb{Z}_{\geq 0}, b \in B \) and \( \mu \in P \), put

\[
P_j(b, \mu) = \{ b \otimes b_j \otimes \cdots \otimes b_1 \in B^\otimes (j+1) | wt(b_j \otimes \cdots \otimes b_1) = cl(\mu) \}.
\] (2.1)

In the sequel we always assume the relation \( k = (j - 1)d + a \). Comparing (2.1) with (1.10) we have

\[
P(k)(\lambda, B) = \sqcup_{\mu \in P} \sqcup_{b \in B_j} P_{j-1}(b, \mu).
\]

For \( j \in \mathbb{Z}_{\geq 0}, b \in B \) and \( \mu \in P \) we define the (unrestricted) 1dsum as follows.

\[
g_j(b, \mu) = q^{(\Lambda_0, \mu)} \sum_{b_j+1 \otimes \cdots \otimes b_1 \in P_j(b, \mu)} q^{\sum_{i=1}^j iH(b_{i+1} \otimes b_i)}. \tag{2.2}
\]

Defining a map \( E : B^\otimes (j+1) \to \mathbb{Z} \) by

\[
E(b_j+1 \otimes \cdots \otimes b_1) = \sum_{i=1}^j iH(b_{i+1} \otimes b_i), \tag{2.3}
\]

one can write (2.2) as

\[
g_j(b, \mu) = q^{(\Lambda_0, \mu)} \sum_{p \in P_j(b, \mu)} q^{E(p)}. \tag{2.4}
\]

Note that

\[
g_j(b, \mu) = 0 \text{ unless } \langle \mu, c \rangle = 0 \quad \text{and} \quad g_j(b, \mu + m\delta) = q^m g_j(b, \mu) \text{ for any } \mu \in P \text{ and } m \in \mathbb{Z}\. \tag{2.5}
\]

For the Weyl group element \( w^{(k)} \) in (III), the Demazure character (0.1) is expressed in terms of the 1dsum.

**Proposition 2.1**

\[
chV_{w^{(k)}}(\lambda) = q^{-c_j} \sum_{\mu \in P_{ct}} e^{\lambda_j + \mu} \sum_{b \in B_j} q^{iH(\overline{b}_{j+1} \otimes b)} g_{j-1}(b, \mu - wt(b)),
\]

\[
c_j = \sum_{i=1}^j iH(\overline{b}_{i+1} \otimes b_i). \tag{2.6}
\]

**Proof.** Substitute (1.5) and (2.2) into the rhs. Setting \( p = \cdots \otimes b_{j+1} \otimes b_j \otimes \cdots \otimes b_1 \) and noting (1.4) one finds that the result is equal to \( \sum_{p \in P^{(k)}(\lambda, B)} e^{wt(p)} \). Thus the assertion follows from Theorem 1.1. \( \square \)

The 1dsums are uniquely characterized also from the recursion relation and the initial condition as follows.
Proposition 2.2

\[ g_j(b, \mu) = \sum_{b' \in B} q^{iH(b \otimes b')} g_{j-1}(b', \mu - wt(b')) \]

\[ g_0(b, \mu) = \delta_{b0} \mu. \] (2.7)

Proof. In the definition (2.2) put \( b' = b_j \) and notice \( b_{j+1} = b \) due to (2.1). Thus Proposition 2.2 simplifies the sum in (2.3) into

\[ \text{chV}_{w(i,j)}(\lambda) = q^{-e_j} \sum_{\mu \in P_i} e^{\lambda_j + \mu} g_j(b_{j+1}, \mu). \] (2.8)

The following relation is of primary importance in later discussion. See Remark 2.3 and the proof of Proposition 2.7.

Proposition 2.3 For \( b \in B \), let \( m = \varphi_i(b) \). Then we have

\[ \sum_{i=0}^{m} g_j(\tilde{f}_i^1 b, \mu + t\alpha_i) q^{t\delta_{\alpha_i}} = \sum_{i=0}^{m} g_j(\tilde{f}_i^1 b, r_i(\mu + (m-t)\alpha_i)) q^{t\delta_{\alpha_i}}. \] (2.9)

For the proof we need a few Lemmas. The following is an immediate consequence of the signature rule.

Lemma 2.1 For any \( b_1, b_2 \in B \) and \( i \in I \) we have

\[ \varphi_i(b_1 \otimes b_2) \geq \varphi_i(b_1) + \langle h_i, wt b_2 \rangle \]

\[ \varepsilon_i(b_1 \otimes b_2) \geq -\langle h_i, wt (b_1 \otimes b_2) \rangle. \] (2.10)

Lemma 2.2 Let \( n = \langle h_i, \mu \rangle + m \) and assume \( n \geq 0 \). Then the map

\[ \tilde{f}_i^n : \sqcup_{t=0}^m \mathcal{P}_j(\tilde{f}_i^1 b, \mu + t\alpha_i) \to \sqcup_{t=0}^m \mathcal{P}_j \left( \tilde{f}_i^1 b, r_i(\mu + (m-t)\alpha_i) \right) \] (2.11)

is a bijection.

Proof. The image is certainly within the rhs by the weight reason unless it is zero. Since \( \tilde{f}_i^n p = p' \) is equivalent to \( e_i^n p' = p \), it suffices to show for \( 0 \leq t \leq m \) that \( \varphi_i(p) \geq n \) for any \( p \in \mathcal{P}_j(\tilde{f}_i^1 b, \mu + t\alpha_i) \) and \( \varepsilon_i(p') \geq n \) for any \( p' \in \mathcal{P}_j \left( \tilde{f}_i^1 b, r_i(\mu + (m-t)\alpha_i) \right) \). Applying Lemma 2.1 one has

\[ \varphi_i(p) \geq m - t + \langle h_i, cl(\mu + t\alpha_i) \rangle = n + t, \]

\[ \varepsilon_i(p') \geq \varepsilon_i(\tilde{f}_i^1 b) - \varphi_i(\tilde{f}_i^1 b - \langle h_i, cl(\mu + (m-t)\alpha_i) \rangle) \geq 2t - m + \langle h_i, \mu + (m-t)\alpha_i \rangle = n, \]

which completes the proof. \( \square \)
Lemma 2.3 Let \( b \in B, m = \varphi_t(b), \xi \in P \). For \( 0 \leq t \leq s \leq m \), assume that \( p \in P_j(f_t b, \xi) \) and \( \tilde{e}_i^p \in P_j(f_t b, \xi + (n + t - s)\alpha_i) \). Then we have

\[
E(\tilde{e}_i^p) = E(p) + ((j + 1)(s-t) - n) \delta_{0i}.
\]

Proof. Use (1.3) in (2.3). \( \blacksquare \)

Proof of Proposition 2.3. Put \( n = \langle \mathbf{h}_i, \mu \rangle + m \). Since (2.9) is invariant under the change \( \mu \rightarrow r_i(\mu + m\alpha_i) \), we may assume \( n \geq 0 \) with no loss of generality. The lhs of (2.9) is written as

\[
\sum_{0 \leq t \leq s \leq m} q^{(\Lambda_0, \mu + t\alpha_i) + tj\delta_{0i}} q' E(p'),
\]

where \( \sum_{p'} \) extends over those \( p' \in P_j(f_t b, \mu + t\alpha_i) \) such that \( \tilde{f}_i^p p' \in P_j(f_t b, r_i(\mu + (m-s)\alpha_i)) \). The sum \( \sum_{0 \leq t \leq s \leq m} \sum_{p'} \) in total ranges over the set appearing in the lhs of (2.11). Thus Lemma 2.2 allows a change of the summation variable into \( p = \tilde{f}_i^p p' \), and thereby transforms (2.12) into

\[
\sum_{0 \leq t \leq s \leq m} q^{(\Lambda_0, \mu + t\alpha_i) + t\delta_{0i}} \sum_{p} q' E(p).
\]

Here \( \sum_{p''} \) extends over those \( p \in P_j(f_t b, r_i(\mu + (m-s)\alpha_i)) \) such that \( \tilde{e}_i^p p' \in P_j(f_t b, \mu + t\alpha_i) \). Setting \( \xi = r_i(\mu + (m-s)\alpha_i) \), one can apply Lemma 2.3 to rewrite (2.13) as

\[
\sum_{0 \leq t \leq s \leq m} q^{(\Lambda_0, r_i(\mu + (m-s)\alpha_i)) + t\delta_{0i}} \sum_{p} q' E(p)
= \sum_{0 \leq t \leq s \leq m} q^{(\Lambda_0, r_i(\mu + (m-s)\alpha_i)) + t\delta_{0i}} \sum_{p \in P_j(f_t b, r_i(\mu + (m-s)\alpha_i))} q E(p).
\]

By (2.4), the last expression is the rhs of (2.9). \( \blacksquare \)

The relation (1.7) for the Demazure character (2.6) implies yet another recursion relation for the 1dsums than Proposition 2.2.

Proposition 2.4 For any \( 0 < a < d - 1 \) and \( \mu \in P_t \) one has

\[
\sum_{\mathbf{b} \in B_{a+1}^{(j)} \setminus B_a^{(j)}} q^{iH(\overline{\mathbf{b}_{a+1}} \otimes \mathbf{b})} g_{j-1} (b, \mu - wt(b))
= \sum_{\mathbf{b} \in B_{a+1}^{(j)}} q^{iH(\overline{\mathbf{b}_{a+1}} \otimes \mathbf{b})} g_{j-1} (b, \mu + \alpha_i) - wt(b)
- \sum_{\mathbf{b} \in B_{a+1}^{(j)}} q^{iH(\overline{\mathbf{b}_{a+1}} \otimes \mathbf{b})} g_{j-1} (b, r_i(\mu + \lambda_j) - \lambda_j - \rho - wt(b)).
\]
Proof. Suppose $1 \leq a \leq d - 1$. Substitute (2.6) into (1.7) with $w = w^{(k)}$ and $i = i_{a+1}^{(j)}$. Note that $D_i(e^{\mu + z \delta}) = e^{z \delta}D_i(e^{\mu})$ holds for any $z \in \mathbb{Z}$ and $\mu \in P_d$ because of $r_i(\delta) = \delta$. After multiplying both sides by $1 - e^{-\alpha_i}$, comparison of the coefficients of $e^{\lambda_j + \mu}$ leads to the above relation. The case $a = 0$ is similar.

Remark 2.1 It is possible to prove Proposition 2.4 without using (1.7) and (2.6) but only from (2.9). In this sense, (2.9) is the most fundamental relation of the 1dsums implied from the Demazure recursion relation (4.3).

2.2 Classically restricted and restricted 1dsum

The 1dsums discussed so far is related to vertex models. Now we proceed to the two variants of them related to restricted solid-on-solid (RSOS) type models (cf. [6], [7]) and Kostka-type polynomials (cf. [12], [13]). Here they shall be called the restricted 1dsums and the classically restricted 1dsums, respectively. Let

\[ \Lambda_i = \Lambda_i - \langle c, \Lambda_i \rangle \Lambda_0 \]

and put

\[ P_+^{cl} = \bigoplus_{i \in I \setminus \{0\}} \mathbb{Z}_{\geq 0} \Lambda_i. \]

Fix a non-negative integer $l'$. Given $\xi, \eta \in (P_+^{cl})_{l+l'}$ (resp. $\xi \in P_+^{cl}$) and $b \in B$ we define

\[ (\xi, b) \text{ is admissible } \iff e_i^{(h_i, \xi_i)} b = 0 \forall i \in I, \]

\[ (\xi, b) \text{ is classically admissible } \iff e_i^{(h_i, \xi_i)} b = 0 \forall i \in I \setminus \{0\}. \]

Recall that $l$ is the level of the perfect crystal $B$. It is easy to see that if $(\xi, b)$ is admissible (resp. $(\xi, b)$ is classically admissible) then $\xi + wt(b) \in (P_+^{cl})_{l+l'}$ (resp. $\xi + wt(b) \in P_+^{cl}$). The admissibility condition has been introduced by [14] in the study of $q$-vertex operators by the crystal base theory. For $j \in \mathbb{Z}_{\geq 0}, b \in B$ and $\xi, \eta \in (P_+^{cl})_{l+l'}$ such that $(\xi - wt(b), b)$ is admissible, (resp. $(\xi - wt(b), b)$ is classically admissible), we define $q$-polynomials $X_j(b, \xi, \eta)$ and $\overline{X}_j(b, \xi, \eta)$ to be the sum

\[ \sum_{b, \ldots, b_1 \in B, b_{j+1} = b} q \sum_{i=1}^{j} iH(b_{i+1} \otimes b_i). \]

Here the outer sum $\sum$ is taken over $b_j, \ldots, b_1 \in B$ under the following conditions for each case.

- $X_j(b, \xi, \eta)$ case: $\xi_i + wt(b_i) = \xi_{i-1}$ for $1 \leq i \leq j$, $\xi_j = \xi$, $\xi_0 = \eta$.
  $(\xi_i, b_i)$ is admissible for $1 \leq i \leq j$.

- $\overline{X}_j(b, \xi, \eta)$ case: $\overline{\xi}_i + wt(b_i) = \overline{\xi}_{i-1}$ for $1 \leq i \leq j$, $\overline{\xi}_j = \overline{\xi}$, $\overline{\xi}_0 = \overline{\eta}$.
  $(\overline{\xi}_i, b_i)$ is classically admissible for $1 \leq i \leq j$. 

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We define $X_j(b, \xi, \eta)$ (resp. $\overline{X}_j(b, \bar{\xi}, \bar{\eta})$) to be zero if $(\xi - wt(b), b)$ is not admissible (resp. if $(\bar{\xi} - wt(b), b)$ is not classically admissible). This implies that $\varphi_i(b) \leq \langle h_i, \bar{\xi} \rangle$ for all $i \in I$ (resp. $\varphi_i(b) \leq \langle h_i, \xi \rangle$ for all $i \in I \setminus \{0\}$). We shall call $X_j(b, \xi, \eta)$ and $\overline{X}_j(b, \bar{\xi}, \bar{\eta})$ the (level $l + l'$) restricted 1dsums and the classically restricted 1dsums, respectively.

For any $\bar{\xi} \in \mathcal{P}_l^+, \xi + (l + l')\Lambda_0$ belongs to $(\mathcal{P}_{l+1}^+)_l$ for sufficiently large $l'$. Moreover the pair $(\bar{\xi}, b)$ is classically admissible if and only if $(\xi + (l + l')\Lambda_0, b)$ is admissible in the limit $l' \to \infty$. Therefore we have

**Proposition 2.5** For any $\bar{\xi}, \bar{\eta} \in \mathcal{P}_l^+$, 

$$\overline{X}_j(b, \bar{\xi}, \bar{\eta}) = \lim_{l' \to \infty} X_j(b, \xi + (l + l')\Lambda_0, \xi + (l + l')\Lambda_0).$$

As Proposition 2.2, the 1dsums $X_j(b, \xi, \eta)$ and $\overline{X}_j(b, \bar{\xi}, \bar{\eta})$ are characterized by the recursion relation and the initial condition as follows.

**Proposition 2.6**

$$X_j(b, \xi, \eta) = \sum_{b' \in B, (\xi, b') \text{admissible}} q^{jH(b \otimes b')} X_{j-1}(b', \xi + wt(b'), \eta),$$
$$X_0(b, \xi, \eta) = \delta_{\xi \eta},$$
$$\overline{X}_j(b, \xi, \eta) = \sum_{b' \in B, (\xi, b') \text{classically admissible}} q^{jH(b \otimes b')} \overline{X}_{j-1}(b', \bar{\xi} + wt(b'), \eta),$$
$$\overline{X}_0(b, \xi, \eta) = \delta_{\xi \eta}.$$

In order to express $X_j$ and $\overline{X}_j$ in terms of $g_j$, we need to assume

**Conjecture 2.1** For any $\xi \in (\mathcal{P}_{l+1}^+)_l$, there exists a disjoint union decomposition (not necessarily unique) as

$$\{ b \in B \mid (\xi, b) \text{ is not admissible} \} = \bigsqcup_{b' \in B, e_i(b') = (h_i, \bar{\xi}) + 1 \text{ for some } i \in I} \{ \tilde{\varphi}_i b' \mid 0 \leq \varphi_i(b') \}. \quad (2.14)$$

We have proved this for $U_q(A_n^{(1)})$, $B = l$-fold symmetric tensor case and have checked several other cases. This property seems reflecting an intrinsic combinatorial nature of perfect crystals. If the conjecture holds, it follows by the same argument as for Proposition 2.5 that for any $\bar{\xi} \in \mathcal{P}_l^+$, there exists a disjoint union decomposition (not necessarily unique) as

$$\{ b \in B \mid (\bar{\xi}, b) \text{ is not classically admissible} \} = \bigsqcup_{b' \in B, e_i(b') = (h_i, \xi) + 1 \text{ for some } i \in I \setminus \{0\}} \{ \tilde{\varphi}_i b' \mid 0 \leq \varphi_i(b') \}. \quad (2.15)$$
Proposition 2.7 If Conjecture 2.1 holds, the restricted and classically restricted 1dsums are expressed as linear superpositions of the 1dsum $g_j$ over the affine Weyl group $W$ and the classical Weyl group $W$, respectively as

$$X_j(b, \xi, \eta) = \sum_{w \in W} \det w g_j(b, w(\eta + \rho) - \xi - \rho),$$

(2.16)

$$X_j(b, \bar{\xi}, \eta) = \sum_{w \in W} \det w g_j(b, w(\eta + \rho) - \bar{\xi} - \rho).$$

(2.17)

Proof. We show (2.16) from (2.14). (2.17) can be verified from (2.15) analogously. Let $F_{j}(b, \xi, \eta)$ denote the rhs of (2.16). We are to show that $F_{j}(b, \xi, \eta)$ fulfills the properties in Proposition 2.6. To check the initial condition is easy. By using (2.7) one has

$$F_{j}(b, \xi, \eta) = \sum_{b' \in B} q^{j H(b \otimes b')} F_{j-1}(b', \xi + \text{wt}(b'), \eta).$$

The sum here is similar to the one in Proposition 2.6 but without the constraint $(\xi, b')$: admissible. Thus it is enough to show the cancellation of those unwanted contributions from non-admissible $b'$, namely,

$$0 = \sum_{b' \in B, (\xi, b') \text{non-admissible}} q^{j H(b \otimes b')} F_{j-1}(b', \xi + \text{wt}(b'), \eta).$$

Under the assumption (2.14) it suffices to show the further decomposed form of this as

$$0 = \sum_{t=0}^{m} q^{j H(b \otimes \tilde{f}^t b')} F_{j-1}(\tilde{f}^t b', \xi + \text{wt}(\tilde{f}^t b'), \eta)$$

for each $b' \in B$ such that $\epsilon_i(b') = \langle h_i, \xi \rangle + 1$. Here $m = \varphi_i(b') = \langle h_i, \xi + \rho + \text{wt}(b') \rangle$. From $\varphi_i(b) \leq \langle h_i, \xi \rangle$ one has $H(b \otimes \tilde{f}^t b') = H(b \otimes b') + t \delta_0$. By using $\text{wt}(\tilde{f}^t b') = \text{wt}(b') - t \alpha_i + t \delta_0 \delta$ and (2.5) further, the rhs of the above is expressed as

$$q^{j H(b \otimes b')} \sum_{w \in W} \det w \sum_{t=0}^{m} g_{j-1}(\tilde{f}^t b', w(\eta + \rho) - \xi - \rho - \text{wt}(b') + t \alpha_i) q^{j(j-1) \delta_0}.$$

Upon applying (2.9), one finds that this quantity is precisely equal to itself with $w(\eta + \rho)$ replaced by $r, w(\eta + \rho)$. Thus it vanishes because of $\det r w = - \det w$. $lacksquare$

2.3 Relation with affine Lie algebra and coset characters

Given a $U_q(\mathfrak{g})$ module $M$ and $\mu \in P$, let $[M : \mu]$ (resp. $[M : \mu]_{cl}$) denote the dimension of the linear space $\{ v \in M \mid \text{wt}(v) = \mu, e_i v = 0 \text{ for all } i \in I \}$ (resp. $i \in I$)
\[ \lambda, \lambda_j \in (P^+_cl)_l, V(\lambda), V_\nu(\lambda), b_\lambda, b(\lambda), d \] and \( \sigma \) be as in section 1. Put \( \overline{X}_j = \lambda_j - t\Lambda_0 \). The \( j \to \infty \) limits of \( g_j, X_j \) and \( \overline{X}_j \) give rise to various branching functions. We summarize them in

**Proposition 2.8** For \( \mu \in P, \xi \in (P^+_cl)_\nu, \eta \in (P^+_cl)_{l+l'} \) and \( \overline{\eta} \in P^+_{cl} \) we have

\[
\begin{align*}
\lim_{j \to \infty} q^{-c_j} g_j(\overline{\eta}_{j+1}, \mu) &= \sum_i (\text{dim } V(\lambda)_{\mu-i\delta}) q^i, & (2.18) \\
\lim_{j \to \infty} q^{-c_j} X_j(\overline{b}_{j+1}, \xi + \lambda_j, \eta) &= \sum_i [V(\xi) \otimes V(\lambda) : \eta - i\delta] q^i, & (2.19) \\
\lim_{j \to \infty} q^{-c_j} \overline{X}_j(\overline{b}_{j+1}, \overline{\lambda}_j, \overline{\eta}) &= \sum_i [V(\lambda) : \overline{\eta} + t\Lambda_0 - i\delta]_{cl} q^i. & (2.20)
\end{align*}
\]

*Proof.* \( 2.18 \) is due to [1]. \( 2.19 \) is due to [14]. To show \( 2.20 \) recall that any path can be written in the form \( p = u_{\lambda_j} \otimes b_j \otimes \cdots \otimes b_1 \) for sufficiently large \( j \), where one may identify \( u_{\lambda_j} = \cdots \otimes \overline{b}_{j+2} \otimes \overline{b}_{j+1} \). For a path \( p = u_{\lambda_j} \otimes b_j \otimes \cdots \otimes b_1 \), the condition \( \overline{e}_i p = 0 \) \( \forall i \in I \setminus \{0\} \) is equivalent to the requirement that \( (\overline{X}_j + \text{wt}(b_j) + \cdots + \text{wt}(b_{l+1}), b_i) \) is classically admissible for \( 1 \leq i \leq j \). Since the weight of the path \( p \) is given by \( \overline{\eta} + t\Lambda_0 - (E(\overline{b}_{j+1} \otimes b_j \otimes \cdots \otimes b_1) - c_j) \delta \), this completes the proof of \( 2.20 \). \( \blacksquare \)

Up to an overall power of \( q \), \( 2.18 \) is a string function \([13], 2.13\) is a branching coefficient of the module \( V(\eta) \) in the tensor product \( V(\xi) \otimes V(\lambda) \), \( 2.20 \) is the branching coefficient of the irreducible \( U_q(\mathfrak{g}) \) module with highest weight \( \overline{\eta} \) within the integrable highest weight module \( V(\lambda) \), where \( U_q(\mathfrak{g}) \) stands for the subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, t_i (i \in I \setminus \{0\}) \).

**Remark 2.2** Multiply \( q^{-c_j} \) on both sides of \([2.14]\) and take \( j \to \infty \) limit. From \([2.14] \) and \([2.13] \), the result turns out to be equivalent with Theorem 3.1 in [14] when \( \mu \) there is dominant integral. In this sense \([2.14] \) is a finite \( j \) analogue of it.

One can interpret the Kostka-Foulkes polynomial \( K_{\xi\mu}(q) \) [17] as a classically restricted 1dsum for \( A_n^{(1)} \). Consider the level \( l \) perfect crystal \( B \) corresponding to the \( l \)-fold symmetric tensor representation [3]. It is parametrized by semistandard tableaux of shape \( (l) \) and entries from \( \{0, 1, \ldots, n\} \). In particular \( b(l\Lambda_0) \) is the one with all entries being \( n \). Let \( 0 \leq x_1 \leq \cdots \leq x_l \leq n \) and \( 0 \leq y_1 \leq \cdots \leq y_l \leq n \) stand for the semistandard tableaux for \( b \) and \( b' \in B \), respectively. Then the \( H \)-function \([13] \) is given by \( H(b \otimes b') = \min \theta(\sum_{i=1}^l \theta(x_i \geq y_{r(i)})). \) Here \( \theta(\text{true}) = 1, \theta(\text{false}) = 0 \) and the minimum extends over the degree \( l \) symmetric group. Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_{n+1}) \) be any partition of \( lj \) (depth \( l(\xi) \leq n + 1 \)) and identify it with \( \sum_{i=1}^n (\xi_i - \xi_{i+1}) \overline{X}_i \in P^+_{cl} \). Then we have

\[
K_{\xi(\nu)}(q) = q^{-lj} \overline{X}_j(b(l\Lambda_0), 0, \xi). \tag{2.21}
\]
This is just an interpretation of a special case of a theorem in [13] via the classically restricted 1dsums. See also [18] for another extension. Our picture can be summarized roughly in the following table.

| 1dsum   | $g_j$ | $X_j$ | $X_j$ |
|---------|------|------|------|
| path    | unrestricted | classically restricted | restricted |
| $j < \infty$ | string function of \(\text{Demazure module}\) $g$-Kostka | restricted $g$-Kostka |
| $j \to \infty$ | string function | $g_l/\overline{g}$ | $(g_{l'} \oplus g_l)/g_{l+l'}$ |

Here $g_l$ denotes the affine Lie algebra $g$ at level $l$. By $g$-Kostka we generally mean the branching coefficients of the irreducible $U_q(g)$-modules in the Demazure module $V_{\omega(\mu)}(\sigma^{-J}(\Lambda_l))$. See [3], [18] for the $U_q(g)$ invariance of the Demazure modules.

**Remark 2.3** Combining (2.21) and (2.17) one can express $K_{\xi(l)}(q)$ as an alternating sum over $W$. However the resulting formula is different from the one on p244 in [17].

### 3 $q$-multinomial formula for $g_j(b, \mu)$

In this section we present explicit formulae for the 1dsums $g_j(b, \mu)$ in terms of $q$-multinomial coefficients. We shall also attach the data $B, d, i^{(j)}_b, B_0^{(j)}$, etc from [3], which satisfy (I) - (IV) in section 1. Combined with Proposition 2.1 or (2.8) they yield a character formula for the Demazure module $V_{\omega(k)}(\lambda)$. We shall only consider level 1 cases of $U_q(g)$ with $g$ being classical types: $A^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$, $A_2^{(2)}$, $A_n^{(2)}$ and $D_n^{(2)}$. Other cases, especially higher level cases will be treated elsewhere. Except the $D_n^{(2)}$ case, the $q$-multinomial formulae for $g_j(b, \mu)$ have been effectively known in earlier works [19], [7], [20] on solvable lattice models. They can be proved by establishing the recursion relation (2.7).

Given a crystal $B$ and an integer vector with \#B-components $\gamma = (\gamma_b)_{b \in B}$ we shall employ the notations

$$\begin{bmatrix} j \\ \gamma \end{bmatrix}_q = \begin{cases} \prod_{b \in B} (q)^{\gamma_b} & \text{if } j = \sum_{b \in B} \gamma_b \text{ and } \gamma_b \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}$$
\[(q)_m = \prod_{i=1}^{m} (1 - q^i) \text{ for } m \in \mathbb{Z}_{\geq 0}.\]

We shall also use
\[\varepsilon_s = \begin{cases} 0 & \text{if } s \text{ is even} \\ 1 & \text{if } s \text{ is odd} \end{cases}.\]

In view of (2.5) we shall assume \(\mu\) is a level 0 integral weight, i.e., \(\mu \in P_{cl}, \langle \mu, c \rangle = 0\) in the rest of the paper.

3.1 \((A_n^{(1)}, B(\Lambda_1))\) case

The level 1 perfect crystal \(B = B(\Lambda_1) = \{0,1,\ldots,n\}\) can be depicted in the crystal graph

![Crystal Graph]

Elements of \(B\) have the weights
\[wt(b) = \Lambda_{\overline{b+1}} - \Lambda_b \text{ for } b \in B,\]
where \(\overline{x}\) is uniquely specified from \(x\) by \(\overline{x} \equiv x \mod n + 1\) and \(0 \leq \overline{x} \leq n\). The energy function is given by
\[H(b \otimes b') = \begin{cases} 0 & \text{if } b < b' \\ 1 & \text{if } b \geq b' \end{cases}.

Due to the Dynkin diagram symmetry it suffices to consider the case \(\lambda = \Lambda_0\). Then we have the result \(\mathfrak{B}\):

\[d = n, \quad \lambda_j = \Lambda_{-j}, \quad \overline{b}_j = -j, \quad P_n^{(j)} = \{-j, -j + 1, \ldots, -j + a\} \quad 1 \leq a \leq n, \quad i_a^{(j)} = -j + a.\]

**Proposition 3.1** (cf. [19], [16]) For \(j \in \mathbb{Z}_{\geq 0}, b \in B\) and \(\mu = (\mu_i)_{i \in B} = (\mu_n - \mu_0)\Lambda_0 + (\mu_0 - \mu_1)\Lambda_1 + \cdots + (\mu_{n-1} - \mu_n)\Lambda_n \in P_{cl}\), we have
\[g_j(b, \mu) = q^{\sum_{i \in B} \mu_i (\mu_i - 1) + \sum_{i \in B} H(b \otimes i) \mu_i \left[ \begin{array}{c} j \\ [\mu]_q \end{array} \right]}.

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3.2 \((B_n^{(1)}, B(Λ_1))\) Case

The level 1 perfect crystal \(B = B(Λ_1) = \{1, 2, \ldots, n, 0, \overline{n}, \ldots, \overline{1}\}\) is depicted by the crystal graph

```
1 1 2 2 \ldots \rightarrow n-2 n-1 n \rightarrow n 0

\begin{array}{c}
T 1 2 2 \ldots n-2 n-1 \rightarrow n-1 n
\end{array}
```

Elements of \(B\) have the weights

\[
wt(b) = -wt(\overline{b}) = \begin{cases}
\Lambda_b - \Lambda_{b-1} & b = 1 \text{ or } 3 \leq b \leq n - 1 \\
\Lambda_2 - \Lambda_1 - \Lambda_0 & b = 2 \\
2\Lambda_n - \Lambda_{n-1} & b = n
\end{cases}
\]

\[
wt(0) = 0.
\]

We introduce an order \(\prec\) on \(B\) by

\[
1 \prec \cdots \prec n \prec 0 \prec \overline{n} \prec \cdots \prec \overline{1}.
\]

This and similar \(\prec\) will be used in the subsequent subsections just for convenience and should not be confused with the Bruhat order. The energy function is given by

\[
H(b \otimes b') = \begin{cases}
0 & \text{if } b \prec b' \\
1 & \text{if } b \succ b'.
\end{cases}
\]

with the exceptions:

\[
H(0 \otimes 0) = 0, \quad H(1 \otimes \overline{1}) = -1.
\]

There are 3 level 1 dominant integral weights \((P^+_d)^1 = \{0, \Lambda_1, \Lambda_n\}\). Due to the Dynkin diagram symmetry it suffices to consider \(\lambda = \Lambda_0\) and \(\Lambda_n\). In both cases we have \(d = 2n - 1\). The other data given in \[3\] reads

\[\lambda = \Lambda_0\] case:

\[
\begin{align*}
\lambda_j &= \Lambda_{\epsilon_j}, \quad \overline{b}_j = \begin{cases}
1 & j \text{ : even} \\
\overline{1} & j \text{ : odd}
\end{cases}, \\
B_a^{(j)} &= \begin{cases}
\{b_j, 2, \ldots, a + 1\} & 1 \leq a \leq n - 1 \\
\{\overline{b}_j, 2, \ldots, n, 0, \overline{n}, \ldots, 2n - a\} & n \leq a \leq 2n - 2
\end{cases}, \\
\varepsilon_a^{(j)} &= \begin{cases}
\epsilon_{1-j} & a = 1 \text{ or } a = 2n - 1 \\
\min(a, 2n - a) & 2 \leq a \leq 2n - 2
\end{cases},
\end{align*}
\]

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\[ \lambda = \Lambda_n \text{ case:} \]
\[ \lambda_j = \Lambda_n, \quad b_j = 0, \]
\[ B_a^{(j)} = \begin{cases} 
\{0, \pi, n-1, \ldots, n+1-a\} & 1 \leq a \leq n-1 \\
\{0, \pi, \ldots, \pi \} & a = n \\
\{0, \pi, n-1, \ldots, 1, 1, 2, \ldots, a-n+1\} & n+1 \leq a \leq 2n-1 
\end{cases}, \]
\[ i_a^{(j)} = \begin{cases} 
 n+1-a & 1 \leq a \leq n-1 \\
 a & n \leq a \leq 2n-1. 
\end{cases} \]

For \( \lambda = \Lambda_n \) one can also make another choice of \( B_a^{(j)} \) and \( i_a^{(j)} \) as
\[ B_a^{(j)} = \begin{cases} 
\{0, \pi, n-1, \ldots, n+1-a\} & 0 \leq a \leq n \\
\{0, \pi, n-1, \ldots, \pi, 1, 2, \ldots, a-n+1\} & n+1 \leq a \leq 2n-1 
\end{cases}, \]
\[ i_a^{(j)} = \begin{cases} 
 n+1-a & 1 \leq a \leq n+1 \\
 a & n+2 \leq a \leq 2n-1. 
\end{cases} \]

In any case, \( B_0^{(j)} = \{b_j\} \) and \( B_d^{(j)} = B \) hold. Let us parametrize the level 0 elements \( \mu \in \mathcal{P}_{cl} \) by \( (\mu_i)_{i=1}^n \in \mathbb{Z}^n \) as
\[ \mu = (-\mu_1 - \mu_2)\Lambda_0 + (\mu_1 - \mu_2)\Lambda_1 + \cdots + (\mu_{n-1} - \mu_n)\Lambda_{n-1} + 2\mu_n\Lambda_n. \]

**Proposition 3.2** (cf. [7]) For \( j \in \mathbb{Z}_{\geq 0}, b \in B \) and the above \( \mu \in \mathcal{P}_{cl} \), we have
\[ g_j(b, \mu) = \sum_{\gamma}^{\ast} q^{\frac{1}{2} \sum_{i \in B} \gamma_i (\gamma_i - 1) - \gamma_s} q^{\sum_{i \in B} H(b \otimes i) \gamma_i} \binom{j}{\gamma}_q, \]
where \( s = n \) if \( b \succ 0 \) and \( s = 1 \) if \( b \prec 0 \). When \( b = 0 \), either choice \( s = n \) or \( s = 1 \) is valid. The sum \( \sum_{\gamma} \) extends over \( \gamma = (\gamma_i)_{i \in B} \in (\mathbb{Z}_{\geq 0})^{2n+1} \) such that
\[ \gamma_i - \gamma_s = \mu_i \text{ for } i = 1, \ldots, n, \quad \sum_{i \in B} \gamma_i = j. \]

**3.3** \((D_1^{(1)}, B(\Lambda_1))\) case

The level 1 perfect crystal \( B = B(\Lambda_1) = \{1, 2, \ldots, n, \pi, \ldots, \pi\} \) is depicted by the crystal graph.
Elements of $B$ have the weights
\[
wt(b) = -wt(\overline{b}) = \begin{cases} 
\Lambda_b - \Lambda_{b-1} & b \neq 2, n-1 \\
\Lambda_2 - \Lambda_1 - \Lambda_0 & b = 2 \\
\Lambda_n + \Lambda_{n-1} - \Lambda_{n-2} & b = n-1.
\end{cases}
\]

We introduce an order on $B$ by
\[1 \prec \cdots \prec n-1 \prec n \prec n-1 \prec \cdots \prec 1.\]

There is no order between $n$ and $\pi$. The energy function is given by
\[
H(b \otimes b') = \begin{cases} 
0 & \text{if } b \prec b', \\
1 & \text{if } b \succeq b'.
\end{cases}
\]

with the exceptions:
\[
H(n \otimes \pi) = H(\pi \otimes n) = 0, \quad H(1 \otimes 1) = -1.
\]

There are 4 level 1 dominant integral weights $(P^+_{cl})_1 = \{\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n\}$. Due to the Dynkin diagram symmetry it suffices to consider $\lambda = \Lambda_0$. Then the result in [3] reads
\[
d = 2n - 2, \quad \lambda_j = \Lambda_{\epsilon_j}, \quad \overline{b}_j = \begin{cases} 
1 & j : \text{even}
\end{cases},
\]
\[
B_a^{(j)} = \begin{cases} 
\overline{b}_j, 2, \ldots, a + 1 & 1 \leq a \leq n - 2 \\
\overline{b}_j, 2, \ldots, n-1, \pi & a = n - 1 \\
\overline{b}_j, 2, \ldots, n, \pi, n-1, \ldots, 2n-1-a & n \leq a \leq 2n - 3
\end{cases},
\]
\[
i_a^{(j)} = \begin{cases} 
\epsilon_1 - j & a = 1, 2n - 2 \\
\min(a, 2n - 1 - a) & a \neq 1, n - 1, 2n - 2 \\
2n - 1 - a & a = n - 1.
\end{cases}
\]

One can also make another choice of $B_a^{(j)}$ and $i_a^{(j)}$ as
\[
B_a^{(j)} = \begin{cases} 
\overline{b}_j, 2, \ldots, a + 1 & 1 \leq a \leq n - 1 \\
\overline{b}_j, 2, \ldots, n, \pi, n-1, \ldots, 2n-1-a & n \leq a \leq 2n - 3
\end{cases},
\]
\[
i_a^{(j)} = \begin{cases} 
\epsilon_1 - j & a = 1, 2n - 2 \\
\min(a, 2n - 1 - a) & a \neq 1, n - 1, 2n - 2 \\
\min(a, 2n - 1 - a) & a = n.
\end{cases}
\]

In any case, $B_0^{(j)} = \{\overline{b}_j\}$ and $B_{d}^{(j)} = B$ hold. Let us parametrize the level 0 elements $\mu \in P_{cl}$ by $(\mu_i)_{i=1}^n \in \mathbb{Z}^n$ as
\[
\mu = (\mu_1 - \mu_2)\Lambda_0 + (\mu_1 - \mu_2)\Lambda_1 + \cdots + (\mu_{n-1} - \mu_n)\Lambda_{n-1} + (\mu_{n-1} + \mu_n)\Lambda_n.
\]
Proposition 3.3 (cf. [7]) For $j \in \mathbb{Z}_{\geq 0}$, $b \in B$ and the above $\mu \in P_{cl}$, we have

$$g_j(b, \mu) = \sum_{\gamma} q^{\frac{1}{2} \sum_{i \in B} \gamma_i (\gamma_i - 1)} \sum_{i \in B} H(b \otimes i) \gamma_i \left[ j \right]_q,$$

where $s = n$ if $b \in \{\pi, \ldots, \overline{1}\}$ and $s = 1$ if $b \in \{1, \ldots, n\}$. The sum $\sum_{\gamma}$ extends over $\gamma = (\gamma_i)_{i \in B} \in (\mathbb{Z}_{\geq 0})^{2n}$ such that

$$\gamma_i - \gamma_j = \mu_i \quad \text{for} \quad i = 1, \ldots, n, \quad \sum_{i \in B} \gamma_i = j.$$

This is a very similar form to the $B_n^{(1)}$ case.

3.4 $(A_{2n-1}^{(2)}, B(\Lambda_1))$ case

The level 1 perfect crystal $B = B(\Lambda_1) = \{1, 2, \ldots, n, \pi, \ldots, \overline{1}\}$ is depicted by the crystal graph

Elements of $B$ have the weights

$$\text{wt}(b) = -\text{wt}(b) = \begin{cases} 
\Lambda_b - \Lambda_{b-1} & b \neq 2 \\
\Lambda_2 - \Lambda_1 - \Lambda_0 & b = 2.
\end{cases}$$

We introduce an order on $B$ by

$$1 \prec \cdots \prec n \prec \pi \prec \cdots \prec \overline{1}.$$  

The energy function is given by

$$H(b \otimes b') = \begin{cases} 
0 & \text{if} \ b \prec b' \\
1 & \text{if} \ b \succeq b',
\end{cases}$$

with the exception:

$$H(1 \otimes \overline{1}) = -1.$$  

There are 2 level 1 dominant integral weights $(P_{cl}^+)_1 = \{\Lambda_0, \Lambda_1\}$. Due to the Dynkin diagram symmetry it suffices to consider $\lambda = \Lambda_0$. Then the result in [8]
reads
\[ d = 2n - 1, \quad \lambda_j = \Lambda_{\epsilon_j}, \quad b_j = \begin{cases} 1 & j : \text{even} \\ \frac{1}{2} & j : \text{odd} \end{cases}, \]
\[ B_a^{(j)} = \begin{cases} \{b_j, 2, \ldots, a + 1\} & 1 \leq a \leq n - 1 \\ \{b_j, 2, \ldots, n, n - 1, \ldots, 2n - a\} & n \leq a \leq 2n - 2 \end{cases}, \]
\[ j_a^{(j)} = \begin{cases} 1 & a = 1, 2n - 1 \\ \min(a, 2n - a) & 2 \leq a \leq 2n - 2. \end{cases} \]
\[ B_0^{(j)} = \{b_j\} \text{ and } B_d^{(j)} = B \text{ hold. Let us parametrize the level 0 elements } \mu \in P_{cl} \text{ by } (\mu_i)_{i=1}^{n} \in \mathbb{Z}^{n} \text{ as} \]
\[ \mu = (-\mu_1 - \mu_2) \Lambda_0 + (\mu_1 - \mu_2) \Lambda_1 + \cdots + (\mu_{n-1} - \mu_n) \Lambda_{n-1} + \mu_n \Lambda_n. \]

Proposition 3.4 (cf. [20]) For \( j \in \mathbb{Z}_{\geq 0}, b \in B \) and the above \( \mu \in P_{cl} \), we have
\[ g_j(b, \mu) = \sum_{*} \gamma Q \left( \frac{(q^2)^{\gamma_1 + \gamma_2}(q)_j}{(q^2)^{\gamma_1}(q^{\gamma_2})^{\gamma_1 + \gamma_2} \prod_{i=2}^{\gamma_1}(q^{\gamma_1})(q^{\gamma_2})} \right) G(b, \mu, \gamma), \]
\[ Q = \frac{1}{2} \sum_{i \in B} \gamma_i (\gamma_i - 1) - \gamma_1 \gamma_2 + \sum_{i \in B} H(b \otimes i) \gamma_i, \]
\[ G(b, \mu, \gamma) = \begin{cases} 1 & \text{if } b = 1 \text{ or } \top \\ \frac{1}{q^{\gamma_1 + \gamma_2} + q^{\gamma_1 + \gamma_2}} & \text{otherwise}. \end{cases} \]
The sum \( \sum_{*} \gamma \) extends over \( \gamma = (\gamma_i)_{i \in B} \in (\mathbb{Z}_{\geq 0})^{2n} \) such that
\[ \gamma_i - \gamma_2 = \mu_i \text{ for } i = 1, \ldots, n, \quad \sum_{i \in B} \gamma_i = j. \]

3.5 \( (A^{(2)}_{2n}, B(0) \oplus B(\Lambda_1)) \) case

For a technical reason, we take the opposite ordering for the labeling of vertices of the Dynkin diagram from [8]. The level 1 perfect crystal \( B = B(0) \oplus B(\Lambda_1) = \{1, 2, \ldots, n, 0, \pi, \ldots, \top\} \) is depicted by the crystal graph

```
  1 ——— 2 ——— 2 ——— n-2 ——— n-1 ——— n ——— n
  \downarrow         \downarrow
  0 ——— \top ——— 2 ——— n-2 ——— n-1 ——— n ——— n
  \downarrow         \downarrow
  \pi ——— n ——— n ——— n ——— n ——— n ——— n
```

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Elements of $B$ have the weights
\[
wt(b) = -wt(\overline{b}) = \begin{cases}
\Lambda_b - \Lambda_{b-1} & 1 \leq b \leq n - 1 \\
2\Lambda_n - \Lambda_{n-1} & b = n
\end{cases}
\]
\[
wt(0) = 0.
\]

We introduce an order $\prec$ on $B$ by
\[
1 \prec \cdots \prec n \prec 0 \prec n \prec \cdots \prec 1.
\]
The energy function is given by
\[
H(b \otimes b') = \begin{cases}
0 & if \ b \prec b' \\
1 & if \ b \trianglerighteq b',
\end{cases}
\]
with the exception:
\[
H(0 \otimes 0) = 0.
\]

There is a unique level 1 dominant integral weight $(P_{cl})_1 = \{\Lambda_n\}$. Thus we set $\lambda = \Lambda_n$, for which the result in [3] reads
\[
d = 2n, \quad \lambda_j = \Lambda_n, \quad b_j = 0,
\]
\[
B_j^{(a)} = \begin{cases}
\{0, \pi, \pi - 1, \ldots, \pi + 1 - a\} & 1 \leq a \leq n \\
\{0, \pi, \pi - 1, \ldots, \pi, 1, 2, \ldots, a - n\} & n + 1 \leq a \leq 2n
\end{cases}
\]
\[
i_a^{(j)} = |n + 1 - a| \quad 1 \leq a \leq 2n.
\]

Let us parametrize the level 0 elements $\mu \in P_{cl}$ by $(\mu_i)_{i=1}^n \in \mathbb{Z}^n$ as
\[
\mu = -\mu_1\Lambda_0 + (\mu_1 - \mu_2)\Lambda_1 + \cdots + (\mu_{n-1} - \mu_n)\Lambda_{n-1} + 2\mu_n\Lambda_n.
\]

**Proposition 3.5** (cf. [20]) For $j \in \mathbb{Z}_{\geq 0}, b \in B$ and the above $\mu \in P_{cl}$, we have
\[
g_j(b, \mu) = \sum_{\gamma} q^{\frac{1}{2} \sum_{i \in B, i \neq \mu} \gamma_i (\gamma_i - 1) + \sum_{i \in B} H(b \otimes i) \gamma_i} j_{\gamma} q,
\]
where the sum $\sum_{\gamma}$ extends over $\gamma = (\gamma_i)_{i \in B} \in (\mathbb{Z}_{\geq 0})^{2n + 1}$ such that
\[
\gamma_i - \gamma_{i+1} = \mu_i \text{ for } i = 1, \ldots, n, \quad \sum_{i \in B} \gamma_i = j.
\]
3.6 \((D_{n+1}^{(2)}, B(0) \oplus B(\Lambda_1))\) Case

The level 1 perfect crystal \(B = B(0) \oplus B(\Lambda_1) = \{1, 2, \ldots, n, 0, \pi, \ldots, \top, \phi\}\) is depicted by the crystal graph

![Crystal Graph]

Elements of \(B\) have the weights

\[
wt(b) = -wt(\bar{b}) = \begin{cases} 
\Lambda_1 - 2\Lambda_0 & b = 1 \\
\Lambda_b - \Lambda_{b-1} & 2 \leq b \leq n - 1 \\
2\Lambda_n - \Lambda_{n-1} & b = n 
\end{cases}
\]

\[
wt(0) = wt(\phi) = 0.
\]

We introduce an order \(\prec\) on \(B \setminus \{\phi\}\) by

\[1 \prec \cdots \prec n \prec 0 \prec \pi \prec \cdots \prec \top.\]

The energy function is given by

\[
H(b \otimes b') = \begin{cases} 
0 & \text{if } b \prec b' \text{ or } (b, b') = (0, 0), (\phi, \phi) \\
1 & \text{if one and only one of } b \text{ and } b' \text{ is } \phi \\
2 & \text{if } b \succeq b' \text{ and } (b, b') \neq (0, 0), (\phi, \phi).
\end{cases}
\]

There are 2 level 1 dominant integral weights \((P_{cl}^+)\) \(= \{\Lambda_0, \Lambda_n\}\). Due to the Dynkin diagram symmetry it suffices to consider \(\lambda = \Lambda_0\). Then the result in [3] reads

\[
d = 2n, \quad \lambda_j = \Lambda_0, \quad \overline{b}_j = \phi,
\]

\[
B_{a}^{(j)} = \begin{cases} 
\{\phi, 1, 2, \ldots, a\} & 0 \leq a \leq n \\
\{\phi, 1, \ldots, n, 0, \pi, n-1, \ldots, 2n+1-a\} & n+1 \leq a \leq 2n
\end{cases}
\]

\[
i_a^{(j)} = \min(a-1, 2n+1-a), \quad 1 \leq a \leq 2n.
\]

Let us parametrize the level 0 elements \(\mu \in P_{cl}\) by \((\mu_i)_{i=1}^{n} \in \mathbb{Z}^n\) as

\[
\mu = -2\mu_1 \Lambda_0 + (\mu_1 - \mu_2) \Lambda_1 + \cdots + (\mu_{n-1} - \mu_n) \Lambda_{n-1} + 2\mu_n \Lambda_n.
\]
Proposition 3.6 For $j \in \mathbb{Z}_{\geq 0}, b \in B$ and the above $\mu \in P_{\lambda}$, we have

$$g_j(b, \mu) = \sum_{\gamma}^* q^{\sum_{i \in B \setminus \{0\}} \gamma_i(\gamma_i - 1) - \sum_{i \in B} H(b \otimes i) \gamma_i} \left[ \gamma_j \right] q^2,$$

where the sum $\sum_{\gamma}^*$ extends over $\gamma = (\gamma_i)_{i \in B} \in (\mathbb{Z}_{\geq 0})^{2n+2}$ such that

$$\gamma_i - \gamma_i = \mu_i \quad \text{for } i = 1, \ldots, n, \quad \sum_{i \in B} \gamma_i = j.$$

4 Discussion

We have shown that various characters can be viewed in a unified way as the 1dsums under the path realization of Demazure crystals. Our picture is summarized in Table 1 in the end of section 2. It is yet another task to actually evaluate these 1dsums. In this paper it has been done in section 3 for level 1 cases of the unrestricted 1dsum $g_j$. Substitution of them into Proposition 2.7 generates formulæ also for $X_j$ and $\overrightarrow{X}_j$. As seen explicitly there, the results necessarily involve alternating signs from the Weyl group signature. Such formulæ are sometimes called bosonic. In this respect it is interesting also to seek fermionic formulæ. By this one roughly means those series or polynomials which are free of signs, admit a quasi-particle interpretation or have an origin in string hypotheses in the Bethe ansatz, etc. Formulæ with such features have been explored extensively for several cases of $\overrightarrow{X}_j$ and $X_j$ in our Table 1 by many authors. See for example [14] and references therein. On the other hand relatively fewer fermionic formulæ seem known or even conjectured for $g_j$. A possible reason for this is that $g_j$ does not correspond to a counting of highest weight vectors as opposed to $\overrightarrow{X}_j$ and $X_j$. We hope to discuss this point further and higher level cases in near future.

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