We present a new application of affine Lie algebras to massive quantum field theory in 2 dimensions, by investigating the $q \rightarrow 1$ limit of the q-deformed affine $\widehat{sl}(2)$ symmetry of the sine-Gordon theory, this limit occurring at the free fermion point. We describe how radial quantization leads to a quasi-chiral factorization of the space of fields. The conserved charges which generate the affine Lie algebra split into two independent affine algebras on this factorized space, each with level 1, in the anti-periodic sector. The space of fields in the anti-periodic sector can be organized using level-1 highest weight representations, if one supplements the $\widehat{sl}(2)$ algebra with the usual local integrals of motion. Using the integrals of motion, a momentum space bosonization involving vertex operators is formulated. This leads to a new way of computing form-factors, as vacuum expectation values in momentum space.
1. Introduction

Algebraic methods are becoming increasingly important for solving quantum field theory non-perturbatively. This is especially true for the massless conformal field theories in two spacetime dimensions, which can be completely solved using their infinite non-abelian symmetries, such as the Virasoro\cite{Virasoro}, and the affine Lie algebra symmetries\cite{AffineLie}. In this paper we present a new application of affine Lie algebras to massive 2d quantum field theory.

Though massive integrable quantum field theories have previously been studied extensively, many of their important properties, such as their correlation functions, have proven to be beyond computation using the existing methods. The exact method of quantum inverse scattering\cite{QIS} provides an algebraic foundation for the Bethe ansatz. There the emphasis is on the infinite number of local integrals of motion $P_n$, which satisfy a trivial abelian algebra: $[P_n, P_m] = 0$. This lack of algebraic structure in these conserved quantities makes it impossible for example to obtain constraints on correlation functions from Ward identities. One is therefore led to the search for interesting non-abelian conserved charges. An integrable quantum field theory with massive particles in its spectrum is characterized by a factorizable S-matrix which must satisfy the Yang-Baxter equation\cite{Yang-Baxter}. Since genuine conserved charges must commute with the S-matrix, the interesting non-abelian symmetries can in principle be deduced on the mass shell by inspection of the known S-matrix.

The $q$-deformations of affine Lie algebras, i.e. quantum affine algebras, were originally invented to provide an algebraic characterization of known solutions of the Yang-Baxter equation\cite{QAffine}. If the solution to the Yang-Baxter equation is taken to be a physical S-matrix, then this characterization is precisely a symmetry criterion. In this way, the on-shell quantum affine symmetries of the S-matrix can be understood as the minimal symmetry that is strong enough to fix the S-matrix up to overall scalar factors. The quantum affine symmetry thus replaces the original bootstrap principles that led to the
S-matrix, since solutions of the symmetry equations automatically satisfy the Yang-Baxter equation. Quantum affine algebraic structures also exist in the quantum inverse scattering method; indeed this was the context in which they were originally discovered. However, we emphasize that the latter structures are not symmetries in the conventional sense and that the quantum affine symmetry of the S-matrix has a completely different and independent physical content. The quantum monodromy matrix has a smooth classical limit, whereas the quantum affine charges do not.

Understanding the full implications of the quantum affine symmetry requires an off-shell understanding of the symmetry, namely, the conserved currents should be explicitly constructed. Consider the sine-Gordon (SG) theory, which is the relevant example for this paper. The action is

\[ S = \frac{1}{4\pi} \int d^2z \left( \partial_z \phi \partial_{\bar{z}} \phi + 4\lambda \cos(\beta \phi) \right). \]  

(1.1)

In [7][8], explicit conserved currents were constructed for the 6 generators corresponding to the simple roots of the \( q - \widehat{sl}(2) \) affine Lie algebra, where \( q = \exp(-2\pi i/\beta^2) \). In this realization the central extension, or level, is zero, and the symmetry is actually a deformed loop algebra. The construction of these currents is purely quantum mechanical, and the resulting charges do not have a smooth classical (\( \beta \to 0 \)) limit. The S-matrix for the soliton scattering is the minimal solution to the quantum affine symmetry equations. This symmetry is somewhat exotic, in that the conserved charges have in general fractional Lorentz spin \( \pm(2/\beta^2 - 1) \). Though the S-matrix was known in this case, this method was used to determine the S-matrices in other models where it was not known, such as the affine Toda theories, and perturbed minimal conformal models.

Using the quantum affine symmetry to compute properties beyond the S-matrix in even the SG theory has proven to be difficult. We now believe this has been largely due to having no understanding of the role of affine Lie algebras in the massive theory which occurs at \( q = 1 \). The primary aim of this paper is to properly develop some new structures in the \( q = 1 \) theory, and more importantly, structures that can and will eventually be
$q$-deformed. We emphasize that the deformation parameter $q$ has nothing to do with perturbation away from the massless conformal limit. Thus, though one can formally $q$-deform the structures of conformal field theory by replacing affine Lie algebras with their $q$-analog, in doing this the connection of these constructions with massive quantum field theory is largely lost.

Using the quantum affine algebras, Frenkel and Reshetikhin formally defined and studied some $q$-analogs of conformal vacuum expectation values\cite{9}. These satisfy difference equations which can be viewed as $q$-deformations of the Knizhnik-Zamolodchikov equations which govern physical correlation functions of fields in conformal field theory. Smirnov used some aspects of this structure to reinterpret the basic form-factor axioms from quantum affine symmetry\cite{10}. (See also \cite{12}.) However a direct link between the results in \cite{9} and the physical form factors is missing. One of the difficulties encountered is that the Frenkel-Reshetikhin construction generally utilizes the infinite highest weight representations of non-zero level $k$, whereas the physical quantum affine symmetries studied thus far in quantum field theory all have zero level.

That the global quantum affine symmetry of the S-matrix necessarily has zero level is easily understood. The Hilbert space $\mathcal{H}_P$ of the theory has a multiparticle description, where states diagonalize the momentum operators $P_\mu$. At fixed particle number, $\mathcal{H}_P$ is a finite dimensional vector space depending on the continuous momentum parameters, such as rapidity. Given an algebra of conserved charges that commute with the particle number operator, its representation on $\mathcal{H}_P$ must be the direct sum of finite dimensional representations which are tensor products of the 1-particle representation. Only the level zero affine algebras, or loop algebras, have finite dimensional representations. In a specific physical realization, the loop parameter is related to the rapidity in a computable way.

\footnote{The symmetry considered in \cite{10} is actually the Yangian symmetry, which occurs at the $sl(2)$ invariant point of the SG theory at $\beta = \sqrt{2}$, and is reached in the limit $q \rightarrow -1$ of the quantum affine algebra. See \cite{11} \cite{7}.}
Finally, since the quantum affine symmetry charges commute with $P_\mu$, they can only relate states of the same energy and thus cannot generate the spectrum of $H_P$.

It is important to understand the significance of non-zero level quantum affine algebras in massive integrable field theory. Having such algebras would entail the application of the rich representation theory of these algebras. Physical reasoning provides an answer to this question. Instead of $H_P$, consider the space of fields. The action of fields evaluated at the origin of space-time on the vacuum $\Phi(0)|0\rangle$ defines a vector space $H_F$. The inner products of states in $H_F$ with states in $H_P$ are the form-factors. The space $H_F$ is a discrete space depending on no continuous parameters. Let $L$ be the generator of Lorentz boosts, or Euclidean rotations in the space-time plane. Each state in $H_F$ has a well-defined $L$ eigenvalue. The operators $L, P_\mu$ comprise the Poincaré algebra:

\[
[L, P_z] = P_z, \quad [L, P_\tau] = -P_\tau, \quad [P_z, P_\tau] = 0.
\]

Obviously the spaces $H_P$ and $H_F$ are not simultaneously diagonalizable. As we will see, the infinite dimensional representations of the non-zero level affine Lie algebras characterize the space $H_F$.

In this paper we deal with the $q - \hat{sl}(2)$ symmetry of the SG theory when $q = 1$. Studying this case serves the purpose of disentangling the conceptual issues from the technical complications which occur when $q \neq 1$. Fortunately, the point $q = 1$ occurs at $\hat{\beta} = 1$, which is just the free Dirac fermion point of the SG theory[13]. Nevertheless, this is far from an empty exercise. Though this theory is trivial from the fermionic description, form-factors and correlation functions of the fields $\exp(i\alpha \phi)$ for $\alpha \not\in \mathbb{Z}$ are not so trivial, since these fields are not simply expressed in terms of the free fermions. Surprisingly, the $\hat{sl}(2)$ symmetry of this free fermion theory has not been considered before.

We now summarize the main results of this paper. We first construct the full infinite set of conserved $\hat{sl}(2)$ charges directly in the free massive Dirac theory, and also the usual infinite number of abelian conserved charges $P_n$. Radial quantization is introduced as the
natural way to obtain operators which diagonalize $L$. This introduces a fermionic fock space description of $\mathcal{H}_F$. Furthermore, $\mathcal{H}_F$ factorizes into $\mathcal{H}_F^L \otimes \mathcal{H}_F^R$, where in the massless limit, $\mathcal{H}_F^L$ ($\mathcal{H}_F^R$) is the left (right) ‘moving’ space of states. Techniques are developed for studying this ‘quasi-chiral factorization’ in momentum space, including the definition of an operator product expansion in this space. This leads to a new and very simple way to derive the form factors of the fields $\exp(\pm i\phi/2)$ in the SG theory, as momentum space correlation functions. In section 5, we show how the conserved charges also factorize in their action on $\mathcal{H}_F^L \otimes \mathcal{H}_F^R$. This leads to two separate algebras $\hat{sl}(2)_L$ and $\hat{sl}(2)_R$, which each have level 1.

In the same fashion, the integrals of motion $P_n^{L,R}$ satisfy an infinite Heisenberg algebra. It is shown how the spectrum of $\mathcal{H}_F$ can be obtained by supplementing the $\hat{sl}(2)$ algebra with this Heisenberg algebra, the fields being organized into infinite highest weight modules. We use the integrals of motion to formulate an exact momentum space bosonization. In this operator formulation, non-trivial SG form-factors are computed as expectation values of vertex operators between level 1 highest weight states.

2. Affine $\hat{sl}(2)$ Symmetry of the Massive Dirac Fermion

The Dirac theory is a massive free field theory of charged fermions. Introducing the Dirac spinors $\Psi_{\pm} = \left( \begin{array}{c} \overline{\psi}_{\pm} \\ \psi_{\pm} \end{array} \right)$ of $U(1)$ charge $\pm 1$, with the appropriate choice of $\gamma$-matrices, the action reads

$$S = -\frac{1}{4\pi} \int dx dt \left( \overline{\psi}_- \partial_\tau \psi_+ + \psi_- \partial_\tau \overline{\psi}_+ + i \hat{m}(\psi_- \overline{\psi}_+ - \overline{\psi}_- \psi_+) \right). \quad (2.1)$$

We have continued to Euclidean space $t \to -it$, and $z, \overline{z}$ are the usual Euclidean light-cone coordinates:

$$z = \frac{t + ix}{2}, \quad \overline{z} = \frac{t - ix}{2}. \quad (2.2)$$

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2 In this paper the massless limit always corresponds to the ultraviolet conformal field theory, and ‘massless’ and ‘conformal’ will be used interchangeably.

3 $\gamma^0 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right)$, $\gamma^1 = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$. 

5
Canonical quantization gives
\[
\{\psi_+(x,t), \psi_-(y,t)\} = \{\bar{\psi}_+(x,t), \bar{\psi}_-(y,t)\} = 4\pi\delta(x-y), \quad \{\psi(x,t), \bar{\psi}(y,t)\} = 0. \tag{2.3}
\]

Generally, the conserved quantities follow from conserved currents \( J_\mu \):
\[
\partial_\tau J_z + \partial_z J_\tau = 0. \tag{2.4}
\]

The usual time-independent conserved charges are then
\[
Q = \frac{1}{4\pi} \int dx \left( J_z(x) + J_\tau(x) \right). \tag{2.5}
\]

In the sequel, we will often display the two components of conserved currents by writing
\[
Q = \int \frac{dz}{2\pi i} J_z - \int \frac{d\bar{z}}{2\pi i} J_\bar{z}, \tag{2.6}
\]
i.e. without specifying the contour of integration. The standard conserved charges in the Dirac theory are the \( U(1) \) charge \( T \),
\[
T = \int \frac{dz}{2\pi i} \psi_+\psi_- - \int \frac{d\bar{z}}{2\pi i} \bar{\psi}_+\bar{\psi}_-, \tag{2.7}
\]
and the Poincaré generators \( L, P_z, P_\bar{z} \). The operator \( L \) generates Euclidean rotations (Lorentz boosts), and \( P_z, P_\bar{z} \) generate space-time translations, where the Hamiltonian is \( P_z + P_\bar{z} \). On fields, \( P_z = \partial_z, P_\bar{z} = \partial_\bar{z} \).

The Dirac theory has an infinite number of additional conserved quantities, which we now construct. Though, as we will see, these can be constructed directly in the Dirac theory, it is interesting to see how some of them arise from the \( q \to 1 \) limit of the results in \([4]\), as the eventual goal is to go beyond \( q = 1 \). There, four non-local charges were constructed in the SG theory for any value of the coupling \( \hat{\beta} \). They were constructed using the methods of conformal perturbation theory developed generally by Zamolodchikov\([14]\).

Let \( \varphi^L, \varphi^R \) denote the quasi-chiral components of the SG field \( \phi \), such that in the massless ultraviolet limit
\[
\phi \xrightarrow{\hat{m} \to 0} \varphi^L(z) + \varphi^R(\bar{z}). \tag{2.8}
\]
The exact propagator in this limit is

\[ \langle 0|\phi(z,\overline{z})\phi(0)|0 \rangle = -\log(z \overline{z}/R^2), \quad (\hat{m} = 0) \quad (2.9) \]

where \( R \) is an infrared cutoff. At the free fermion point \( \hat{\beta} = 1 \), the non-local charges of \[ \] take the following form:

\[ Q_{\pm} = \int \frac{dz}{4\pi i} \exp(\pm i \varphi^L) - \lambda \int \frac{dz}{4\pi i} \exp(\pm i \varphi^L \mp i \varphi^R) \]

\[ \overline{Q}_{\pm} = \int \frac{dz}{4\pi i} \lambda \exp(\mp i \varphi^R \pm i \varphi^L) - \int \frac{dz}{4\pi i} \exp(\mp 2i \varphi^R). \quad (2.10) \]

All field operators are well-defined in the framework of conformal perturbation theory\[ \]
and are implicitly normal ordered.

It was noted in \[ \] that at the special values of the SG coupling constant \( \hat{\beta} = 1/n \), the non-local currents can be expressed in terms of the Thirring fermions. At \( \hat{\beta} = 1 \) the non-perturbative identification of the terms in the action is\[ ]

\[ \lambda : \cos(\phi) := -i\hat{m} (\psi_- \overline{\psi}_+ - \overline{\psi}_- \psi_+). \quad (2.12) \]

It is a simple matter to use the bosonized relations\[ \]

\[ \psi_{\pm} = \exp(\pm i \varphi^L), \quad \overline{\psi}_{\pm} = \exp(\mp i \varphi^R) \quad (2.13) \]

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\[ \]

4 Based on the work of Mandelstam\[ ] in \[ \] the following definition of the quasi-chiral components was proposed:

\[ \varphi^L(x, t) = \frac{1}{2} \left( \phi(x, t) + \int_{-\infty}^{x} dy \partial_t \phi(y, t) \right) \]

\[ \varphi^R(x, t) = \frac{1}{2} \left( \phi(x, t) - \int_{-\infty}^{x} dy \partial_t \phi(y, t) \right). \quad (2.11) \]

The conservation of the above non-local charges in a canonical equal-time framework was studied in \[ \] [16] [17], where it was found that the currents in this framework differ slightly from the expression obtained by literally substituting the expressions (2.11) into (2.10).
in the expressions (2.10) to yield:

\[
Q_\pm = \int \frac{dz}{4\pi i} (\psi_\pm \partial_z \psi_\pm) - \int \frac{dz}{4\pi i} (i\hat{m}\psi_\pm \psi_\pm)
\]
\[
\bar{Q}_\pm = \int \frac{dz}{4\pi i} (-i\hat{m}\psi_\pm \bar{\psi}_\pm) - \int \frac{dz}{4\pi i} (\psi_\pm \bar{\psi}_\pm) .
\]

(2.14)

In this fermionic description, one sees that the currents are local, and that they have the structure of the energy-momentum tensor, except that they carry \(U(1)\) charge \(\pm 2\). That the currents for the charges \(Q_\pm, \bar{Q}_\pm\) are conserved (eq. (2.4)) is now a simple consequence of the equations of motion:

\[
\partial_z \bar{\psi}_\pm = i\hat{m}\psi_\pm, \quad \partial_z \psi_\pm = -i\hat{m}\bar{\psi}_\pm.
\]

(2.15)

It is interesting to note that whereas the conservation of the charges (2.10) in the bosonic SG description is a purely quantum mechanical phenomenon (verification of their conservation by construction uses the operator product expansion, and furthermore, in the classical limit \(\hat{\beta} \to 0\) the charges are ill-defined due to the \(1/\hat{\beta}\)'s appearing in the exponentials), in the fermionic description the fermionic statistics introduces just enough quantum mechanics so that the conservation of the charges is now purely classical in origin.

Having understood the classical origin of the conservation of the above charges, it is now straightforward to find an infinite number of additional quantities which are also conserved as a simple consequence of the equations of motion (2.15). They are the following:

\[
\bar{Q}^\pm_{-n} = \frac{(-1)^{n+1}}{2} \left( \int \frac{dz}{2\pi i} (\psi_\pm \partial_z^n \psi_\pm) - \int \frac{dz}{2\pi i} (i\hat{m} \bar{\psi}_\pm \partial_z^{n-1} \psi_\pm) \right)
\]
\[
\bar{Q}^\pm_n = \frac{(-1)^{n+1}}{2} \left( \int \frac{dz}{2\pi i} (-i\hat{m} \psi_\pm \partial_z^{n-1} \bar{\psi}_\pm) - \int \frac{dz}{2\pi i} (\psi_\pm \bar{\psi}_\pm \partial_z^n \bar{\psi}_\pm) \right)
\]
\[
\alpha_{-n} = (-)^n \left( \int \frac{dz}{2\pi i} (\psi_+ \partial_z^n \psi_-) - \int \frac{dz}{2\pi i} (i\hat{m} \bar{\psi}_+ \partial_z^{n-1} \psi_-) \right)
\]
\[
\alpha_n = (-)^n \left( \int \frac{dz}{2\pi i} (-i\hat{m} \psi_+ \partial_z^{n-1} \psi_-) - \int \frac{dz}{2\pi i} (\psi_+ \bar{\psi}_- \partial_z^n \psi_-) \right),
\]

(2.16)

where \(n \geq 1\) is an integer.
The previous charges are identified as $Q_\pm = \tilde{Q}_\pm$ and $\overline{Q}_\pm = \tilde{Q}_\pm^{\dagger}$. The ordinary momentum operators are identified as

$$P_z = \alpha_{-1}, \quad P_{\overline{z}} = \alpha_1. \tag{2.17}$$

For convenience of notation we define

$$\alpha_0 \equiv T. \tag{2.18}$$

From the hermiticity properties $(\psi_\pm)\dagger = \psi_\mp$, $(\overline{\psi}_\pm)\dagger = \overline{\psi}_\mp$, one finds

$$\left(\tilde{Q}_n^{\pm}\right)\dagger = \tilde{Q}_n^{\mp}, \quad \alpha_n = \alpha_n. \tag{2.19}$$

In order to simplify the computation of the algebra of these charges, we translate them to on-shell momentum space. Introducing a rapidity variable $\theta$ which parameterizes on-shell momentum,

$$p_z(\theta) = \hat{m}e^\theta, \quad p_{\overline{z}}(\theta) = \hat{m}e^{-\theta}, \tag{2.20}$$

the conventional expansions are the following:

$$\psi_+(x,t) = i\sqrt{\hat{m}} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} e^{\theta/2} \left( c(\theta) e^{-i\theta \cdot x} - d(\theta) e^{i\theta \cdot x} \right)$$

$$\overline{\psi}_+(x,t) = \sqrt{\hat{m}} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} e^{-\theta/2} \left( c(\theta) e^{-i\theta \cdot x} + d(\theta) e^{i\theta \cdot x} \right), \tag{2.21}$$

with $\psi_- = \psi_\dagger_+, \overline{\psi}_- = \overline{\psi}_\dagger_+$. The momentum-space operators satisfy

$$\{d(\theta), d^\dagger(\theta')\} = \{c(\theta), c^\dagger(\theta')\} = 4\pi^2 \delta(\theta - \theta'). \tag{2.22}$$

To further simplify the computation, and also in anticipation of the following section, we combine creation and annihilation operators into a single operator as follows. Define the momentum space variable $u$ as

$$u \equiv e^\theta, \tag{2.23}$$
and let
\[ \hat{b}^+(u) = d^t(u), \quad \hat{b}^-(u) = c^t(u) \quad \text{for } u > 0 \] (2.24)
\[ \hat{b}^+(u) = ic(-u), \quad \hat{b}^-(u) = id(-u) \quad \text{for } u < 0. \]

These operators satisfy
\[ \{\hat{b}^+(u), \hat{b}^-(u')\} = 4\pi^2 i|u|\delta(u + u'), \quad \{\hat{b}^\pm(u), \hat{b}^\pm(u')\} = 0 \] (2.25)
\[ \left(\hat{b}^+(u)\right)^\dagger = -i\hat{b}^-(u). \]

Then,
\[ \Psi_\pm = \left(\begin{array}{c} \psi_\pm \\ \psi_\pm \end{array}\right) = \pm \sqrt{\frac{m}{2\pi i u}} \int_{-\infty}^{\infty} \frac{du}{2\pi i|u|} b^\pm(u) \left(\frac{1/\sqrt{u} - i\sqrt{u}}{u}\right) e^{iu + m\pi/2}. \] (2.26)

The charges now have the simple momentum space expressions:
\[ \tilde{Q}_n^\pm = \frac{\hat{m}^{\lfloor n \rfloor}}{4\pi} \int_{-\infty}^{\infty} \frac{du}{2\pi i|u|} u^{-n} \hat{b}^\pm(u)\hat{b}^\pm(-u) \] (2.27)
\[ \alpha_n = \frac{\hat{m}^{\lfloor n \rfloor}}{2\pi} \int_{-\infty}^{\infty} \frac{du}{2\pi i|u|} u^{-n} \hat{b}^+(u)\hat{b}^-(u), \]
for all \( n \). These expressions were derived by first expressing the charges (2.16) in the form (2.5), and then substituting the expressions (2.26). It is now straightforward to compute the commutation relations of these charges using (2.25). One finds that the structure of the resulting relations depends significantly on whether \( n \) is odd or even. First note that by making the redefinition \( u \to -u \) in (2.27), and using (2.25), one finds
\[ \tilde{Q}_n^\pm = (-1)^{n+1} \tilde{Q}_n^\pm \Rightarrow \tilde{Q}_m^\pm = 0 \quad \text{for } m \text{ even.} \]

Define
\[ P_n \equiv \alpha_n, \quad Q_n^\pm \equiv \tilde{Q}_n^\pm, \quad n \text{ odd} \] (2.28)
\[ T_n \equiv \alpha_n, \quad n \text{ even.} \]

Then one finds the following algebraic relations
\[ [P_n, P_m] = 0 \] (2.29a)
\[ [P_n, T_m] = [P_n, Q_m^\pm] = 0 \] (2.29b)
\[ [T_n, T_m] = 0 \] (2.29c)
\[ [T_n, Q_m^\pm] = \pm 2 \frac{\hat{m}^{\lfloor n \rfloor + |m|}}{\hat{m}^{\lfloor n \rfloor + |m|}} Q_{n+m}^\pm \] (2.29d)
\[ [Q_n^+, Q_m^-] = \frac{\hat{m}^{\lfloor n \rfloor + |m|}}{\hat{m}^{\lfloor n \rfloor + |m|}} T_{n+m} \] (2.29e).
It will be important to introduce the conserved Lorentz boost operator $L$; in momentum space one finds
\[ L = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{du}{2\pi i|u|} \hat{b}^+(u) u \partial_u \hat{b}^-(u). \] (2.30) eIIxxiv

The conserved charges have integer Lorentz spin:
\[ [L, \alpha_n] = -n \alpha_n, \quad [L, Q^\pm_n] = -n Q^\pm_n. \] (2.31) eIIxxv

We now interpret this algebraic structure. The $P_n$’s are the usual infinity of commuting integrals of motion with Lorentz spin equal to an odd integer, where $P_z = P_{-1}$, $P_\bar{z} = P_1$, and the hamiltonian is $P_1 + P_{-1}$. These were already known to exist at all values of the SG coupling constant. The additional charges $T_n, Q^\pm_n$ all commute with the $P_m$’s; for $m = \pm 1$ this is just the statement that they are all conserved. In the SG theory, the $T_n$’s are generalizations of the topological charge $T_0$.

The commutation relations of the $T_n, Q^\pm_n$ are the defining relations of the level 0 \( \hat{sl}(2) \) affine Lie algebra. Since this realization of affine Lie algebras is relatively unfamiliar in the physics literature in comparison with their realization in current algebra, we clarify this point\(^5\). The most economical definition of the affine Lie algebras involves only the finite number of generators for the simple roots, $e_i, f_i, h_i$, satisfying
\[
[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j \quad [e_i, f_j] = \delta_{ij}h_i
\]
\[
ad^{1-a_{ij}}e_i(e_j) = ad^{1-a_{ij}}f_i(f_j) = 0 \quad (i \neq j),
\] (2.32) eIIxxvi

where $a_{ij}$ is the generalized Cartan matrix. For $sl(2), i, j \in \{0, 1\}$, and $a = \left(\begin{array}{cc} 2 & -2 \\ -2 & 2 \end{array}\right)$.

The algebra (2.32) defines an infinite dimensional Lie algebra. Though conformal current algebra is not relevant for the problem we are considering, it is nevertheless useful to recall how the infinite algebra arises in that context. There one deals with an $sl(2)_L \otimes sl(2)_R$ invariant theory, with chiral currents $j^a(z), \bar{j}^i(\bar{z})$. Upon making the analytic conformal

\(^5\) For reviews of affine Lie algebras in mathematics and physics see [19][20].
transformation $z \rightarrow \log(z)$, the currents can be expanded in modes $j^n(z) = \sum_n j^a_n z^{-n-1}$, and similarly for $j^n$. Introducing the derivation $d$, these modes satisfy

$$[j^0_n, j^\pm_m] = \pm 2 j^\pm_{n+m}, \quad [j^0_n, j^0_m] = 2k n \delta_{n,-m}$$

$$[j^+_n, j^-_m] = j^0_{n+m} + kn \delta_{n,-m}$$

$$[d, j^a_n] = n j^a_n,$$

where the central extension $k$ is called the level. The algebras defined by (2.32) and (2.33) are identical. The simple root generators are given by

$$e_1 = j^+_0, \quad f_1 = j^-_0, \quad h_1 = j^0_0$$

$$e_0 = j^-_1, \quad f_0 = j^+_1, \quad h_0 = -j^0_0 + k.$$  

The algebra (2.33) cannot describe a symmetry algebra for the Dirac theory since it has an $sl(2)$ subalgebra generated by the Lorentz scalars $j^0$, whereas the Dirac theory has only a $U(1)$ global symmetry. What is relevant for the Dirac theory is actually a twisted $\hat{sl}(2)$ affine algebra. Consider the inner automorphism $\tau$ of $sl(2)$, $\tau(j^a_0) = e^{i\pi j^0_0/2} j^a e^{-i\pi j^0_0/2}$.

This $\tau$ can be used to construct an inner automorphism of the algebra $\hat{sl}(2)$ by the formula

$$t^a(v) = \sum_m t^a_m v^m = v^{j^0_0/2} j^a (v^2) v^{-j^0_0/2}.$$  

Thus, defining

$$t^\pm_n = j^\pm_{(n+1)/2}, \quad n \text{ odd}$$

$$t^0_n = j^0_{n/2} - \frac{k}{2} \delta_{n,0} \quad n \text{ even}$$

$$d' = 2d + j^0/2,$$

one finds

$$[t^0_n, t^0_m] = k n \delta_{n,-m}, \quad [t^0_n, t^\pm_m] = \pm 2 t^\pm_{n+m}$$

$$[t^+_n, t^-_m] = t^0_{n+m} + \frac{kn}{2} \delta_{n,-m}$$

$$[d', t^a_n] = n t^a_n.$$  

This twisted $\hat{sl}(2)$ algebra is of course isomorphic to (2.33) by construction. The simple root generators are now

$$e_1 = t^+_1, \quad f_1 = t^-_1, \quad h_1 = t^0_0 + \frac{k}{2}$$

$$e_0 = t^-_1, \quad f_0 = t^+_1, \quad h_0 = -t^0_0 + \frac{k}{2}.$$
In the mathematics literature, the algebra (2.33) is referred to as being in the homogeneous gradation, whereas (2.37) is in the principal gradation. Note that in going from the homogeneous to the principal gradation the $sl(2)$ global zero mode algebra is broken to $U(1)$ generated by $t^0_0$.

The algebra of the charges $T_n$, $Q_n^\pm$ is the same as for the twisted $\hat{sl}(2)$ algebra at level $k = 0$ if one absorbs the mass dependence into the definition of the charges: $t^\pm_n = Q_n^\pm \hat{m}^{-|n|}, t^0_n = T_n \hat{m}^{-|n|}$, or simply by a choice of units sets $\hat{m} = 1$. One also has the identification $d' = L$. This implies that in the massless limit $\hat{m} \to 0$ one doesn’t recover an $\hat{sl}(2)$ algebra. Rather, in the massless limit one obtains two decoupled Borel subalgebras generated by \{e_i, h_i\} and \{f_i, h_i\} respectively. Note further that in position space the charges (2.16) do not correspond to moments of a universal operator as in conformal field theory, however in momentum space they do.

The conserved charges have the following simple commutation relations with the operators $\hat{b}^\pm(u)$:

$$\left[\alpha_n, \hat{b}^\pm(u)\right] = (\pm)^{n+1} \hat{m}^{|n|} u^{-n} \hat{b}^\pm(u) \quad \left[Q_n^\pm, \hat{b}^\mp(u)\right] = \hat{m}^{|n|} u^{-n} \hat{b}^\pm(u)$$

$$\left[Q_n^\pm, \hat{b}^\pm(u)\right] = 0.$$

This implies that on the doublet of one particle states $(d^\dagger(\theta)|0\rangle, c^\dagger(\theta)|0\rangle)$, the charges have the following loop algebra representation

$$\alpha_n = \hat{m}^{|n|} u^{-n} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{n+1} \end{pmatrix}, \quad Q_n^+ = \hat{m}^{|n|} u^{-n} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q_n^- = \hat{m}^{|n|} u^{-n} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

From the equations (2.39) and (2.20) one finds the following action on the fermion fields:

$$[\alpha_n, \psi_\pm] = -i\hat{m}(\pm)^{n+1} \partial_z^{n-1} \bar{\psi}_\pm, \quad [\alpha_{-n}, \psi_\pm] = (\pm)^{n+1} \partial_z^n \psi_\pm$$

$$[\alpha_n, \bar{\psi}_\pm] = (\pm)^{n+1} \partial_z^n \bar{\psi}_\pm, \quad [\alpha_{-n}, \bar{\psi}_\pm] = i\hat{m}(\pm)^{n+1} \partial_z^{n-1} \psi_\pm$$

$$[Q_n^+, \psi_\mp] = i\hat{m} \partial_z^{n-1} \bar{\psi}_\pm, \quad [Q_n^-, \psi_\mp] = -\partial_z^n \psi_\mp$$

$$[Q_n^+, \bar{\psi}_\mp] = -\partial_z^n \bar{\psi}_\pm, \quad [Q_n^-, \bar{\psi}_\mp] = -i\hat{m} \partial_z^{n-1} \psi_\mp.$$  

(2.41)
for \( n \geq 0 \); all other commutators are zero.

The algebra \( (2.29) \), along with the above action on the fermion fields is sufficient to reconstruct the Dirac theory. Introducing the notation \( Q(\psi) = [Q, \psi] \), one has

\[
Q_1^+ (Q_{-1}^- (\psi_+)) = -\partial_z Q_1^+ (\psi_-) = -i\hat{m}\partial_z \psi_+.
\]

(2.42) eIxxxiv

On the other hand, since \( Q_1^+ (\psi_+) = 0 \),

\[
Q_1^+ (Q_{-1}^- (\psi_+)) = [Q_1^+, Q_{-1}^-] (\psi_+) = \hat{m}^2 T_0 (\psi_+) = \hat{m}^2 \psi_+.
\]

(2.43) eIxxxv

Comparing the last two equations, one finds the Dirac equation \( \partial_z \psi_+ = i\hat{m}\psi_+ \).

We now illustrate how Ward identities for the \( \hat{sl}(2) \) algebra fix the 2-point correlation functions of the fermions. Certainly, these correlation functions are known from far less sophisticated reasoning. We go through this exercise merely to confirm that it is indeed possible. All of the charges annihilate the vacuum, thus the Ward identities take the form

\[
\langle 0 | Q_1^+ (\psi_1(x_1)) \psi_2(x_2) | 0 \rangle + \langle 0 | \psi_1(x_1) Q_1^- (\psi_2(x_2)) | 0 \rangle = 0
\]

(2.44) eIxxxvi

for any conserved charge \( Q \). One has

\[
\langle 0 | Q_{-1}^- (\psi_+ (z, \overline{z})) Q_1^+ (\psi_- (0)) | 0 \rangle = -\partial_z \partial_{\overline{z}} \langle 0 | \psi_- (z, \overline{z}) \psi_+ (0) | 0 \rangle.
\]

(2.45) eIxxxvii

On the other hand, from the Ward identity one has that the LHS of the above equation equals

\[
-\langle 0 | \psi_+ (z, \overline{z}) Q_{-1}^- Q_1^+ (\psi_- (0)) | 0 \rangle = -\hat{m}^2 \langle 0 | \psi_+ (z, \overline{z}) \psi_- (0) | 0 \rangle.
\]

(2.46) eIxxxviii

Comparing \((2.45)\) and \((2.46)\), and using charge conjugation symmetry, one finds

\[
(\partial_z \partial_{\overline{z}} - \hat{m}^2) \langle 0 | \psi_- (z, \overline{z}) \psi_+ (0) | 0 \rangle = 0.
\]

(2.47) eIxxxix

Given the solution to \((2.47)\), the Dirac equation can then be used to find the other 2-point functions. The result is

\[
\langle 0 | \psi_- (z, \overline{z}) \psi_+ (0) | 0 \rangle = -2i\hat{m} K_0 (\hat{m}r)
\]

\[
\langle 0 | \psi_- (z, \overline{z}) \psi_+ (0) | 0 \rangle = 2\hat{m} \sqrt{\frac{z}{\overline{z}}} K_1 (\hat{m}r)
\]

(2.48) eIxxxx

\[
\langle 0 | \psi_- (z, \overline{z}) \psi_+ (0) | 0 \rangle = 2\hat{m} \sqrt{\frac{\overline{z}}{z}} K_1 (\hat{m}r),
\]
where $K_n$ is the standard modified Bessel function.

3. Quasi-Chiral Factorization of the Space of Fields

3.1 General Remarks

The origin of the quasi-chiral factorization we will introduce in the Dirac theory can be understood in a more general setting. In conventional approaches to free or interacting massive quantum field theory one constructs an asymptotic free particle fock space $\mathcal{H}_P$. These states diagonalize the momentum operators $P_z$, $P_\bar{z}$, whose eigenvalues are parameterized by the rapidity of the particles. On 1-particle states $P_z = \hat{m} u$, $P_\bar{z} = \hat{m} / u$. To study form-factors and correlation functions one considers the space of fields $\mathcal{H}_F$:

$$\mathcal{H}_F \equiv \{ |\Phi\rangle \equiv \Phi(z, \bar{z} = 0)|0\rangle \} ,$$

where $\Phi(z, \bar{z})$ is any field in the theory. The spaces $\mathcal{H}_P$ and $\mathcal{H}_F$ are completely different; the inner product of states in $\mathcal{H}_P$ with states in $\mathcal{H}_F$ are the form-factors, which are generally non-trivial.

Since $P_z, P_\bar{z}$ act as spacetime derivatives on fields, the states in $\mathcal{H}_F$ obviously do not diagonalize $P_z, P_\bar{z}$. Rather, since every field has well-defined properties under Euclidean rotations, the space $\mathcal{H}_F$ diagonalizes the Lorentz boost operator $L$:

$$L |\Phi\rangle = s |\Phi\rangle ,$$

where $s$ is the Lorentz spin of the field $\Phi$.

In relativistic quantum field theory, the standard equal-time, planar quantization (e.g. (2.3)) yields the space $\mathcal{H}_P$. To construct the space $\mathcal{H}_F$, one should consider instead radial quantization. Define the radial coordinates $(r, \varphi)$ as follows:

$$z = \frac{r}{2} e^{i\varphi} , \quad \bar{z} = \frac{r}{2} e^{-i\varphi} .$$

See [21] for the construction of the multiparticle form-factors in a variety of models, including the SG theory.
In radial quantization one treats the \( r \)-coordinate as a ‘time’, and \( \varphi \) as the ‘space’. Equal-
time commutation relations are specified along circles surrounding the origin. For example,
in the Dirac theory, rewriting the action in terms of the \( r, \varphi \) coordinates using

\[
\partial_z = e^{-i\varphi}(\partial_r - \frac{i}{r}\partial_\varphi), \quad \partial_{\bar{z}} = e^{i\varphi}(\partial_r + \frac{i}{r}\partial_\varphi),
\]

and applying canonical quantization yields

\[
\{\psi_+(r, \varphi), \psi_-(r, \varphi')\} = \frac{4\pi}{r} e^{i\varphi}\delta(\varphi - \varphi'), \quad \{\overline{\psi}_+(r, \varphi), \overline{\psi}_-(r, \varphi')\} = \frac{4\pi}{r} e^{-i\varphi}\delta(\varphi - \varphi').
\]

As a differential operator on the spacetime coordinates, \( L = -i\partial_\varphi \), thus the states
constructed in radial quantization are eigenstates of \( L \). The Poincaré algebra does not
provide any additional quantum numbers. In order to find additional quantum numbers
of the space \( \mathcal{H}_F \), we consider the following. In the usual planar quantization, the states
in \( \mathcal{H}_P \) are created by asymptotic ‘in’ or ‘out’ fields, in the sense of scattering theory.
The mass-shell condition arises from the Klein-Gordon equation satisfied by these fields:

\[
(\partial_z \partial_{\bar{z}} - \hat{m}^2)\Phi_{\text{in, out}} = 0.
\]

Consider now the analog of these asymptotic fields in radial
quantization. Defining the scaling operator \( D = r\partial_r \), the Klein-Gordon equation reads

\[
(D + L)(D - L)\Phi(r, \varphi) = \hat{m}^2 r^2 \Phi(r, \varphi).
\]

This implies

\[
(D + L)(D - L) \Phi(r = 0) = 0.
\]

Assuming the space \( \mathcal{H}_F \) can be constructed from the analog of these asymptotic fields, one
has the factorization

\[
\mathcal{H}_F = \mathcal{H}_F^L \otimes \mathcal{H}_F^R,
\]

where

\[
(D - L)|\mathcal{H}_F^L\rangle = (D + L)|\mathcal{H}_F^R\rangle = 0.
\]
Thus, the quasi-chiral factorization arises as the radial analog of the mass shell condition. In massive quantum field theory, $D$ is not a quantum conserved operator; nevertheless, every field $\Phi$ has a well-defined scaling dimension $\Delta_{\Phi}$, and we define $D|\Phi\rangle = \Delta_{\Phi}|\Phi\rangle$.

The manner in which the mass $\hat{m}$ enters the equation (3.6) indicates that the space $H_F$ can also be obtained by taking the $\hat{m} \to 0$ limit. More precisely, let $H_F^{(0)}$ denote the space $H_F$ modulo identifications resulting from the mass-dependent equations of motion. For example, in the Dirac theory, since $\partial_z \bar{\psi}_+ = \hat{m}\bar{\psi}_+$, the two fields $\partial_z \bar{\psi}_+$ and $\bar{\psi}_+$ are linearly related, and $\psi_+$ is defined to be in $H_F^{(0)}$ but $\partial_z \bar{\psi}_+$ is not. I.e. we define $H_F^{(0)}$ to be a linearly independent basis for the space $H_F$, where each field in $H_F^{(0)}$ has no explicit mass dependence. Then for $\Phi(0)|0\rangle \in H_F^{(0)}$,

$$\lim_{r \to 0} \Phi(r)|0\rangle = \lim_{\hat{m} \to 0} \lim_{r \to 0} \Phi(r)|0\rangle.$$  \hfill (3.10) eIIIx

The significance of the equation (3.10) is that it implies the structure of $H_F^{(0)}$ is identical to that in the $\hat{m} \to 0$ limit. In this limit $H_F^L$ and $H_F^R$ are the well-understood left and right ‘moving’ space of states in conformal field theory[1]. We emphasize however that the fields in $H_F^L$ ($H_F^R$) are obviously not solely functions of $z$ ($\bar{z}$) in the massive theory.

In the above discussion, there are some hidden assumptions about the renormalization group properties of the theory that we now clarify. The massive theories we have in mind can generally be formulated as perturbed conformal field theories [14], with the formal action

$$S = S_{CFT} + \sum_i \frac{\lambda_i}{2\pi} \int d^2z \ O_i(z, \bar{z}),$$  \hfill (3.11) eIIIx

where $O_i$ are relevant operators with scaling dimension $\leq 2$. We assume there is no wavefunction renormalization, so that the anomalous scaling dimension of a field is constant along the renormalization group trajectory. In the SG theory for example, this situation occurs since the theory may be renormalized by simply normal ordering the $\cos(\hat{\beta}\phi)$ potential and redefining $\lambda$[13]. This means that the theory cannot have any non-trivial
infra-red fixed points. Since the scaling dimension of a field is independent of the $\lambda^i$ under these assumptions, the scaling dimensions of fields and coupling constants $\lambda^i$ is fixed by the properties of the ultraviolet conformal field theory. The physical mass scale $\hat{m}$ is a function of the $\lambda^i$, and physical correlation functions are expressed in terms of $\hat{m}$. In summary, we are making the same basic assumptions as in [14], i.e. that the space of fields in the massive theory has the same structure as in the conformal field theory. Radial quantization provides a way of constructing the space $\mathcal{H}_F$ with these structures explicitly, with the $L$ and $D$ quantum numbers the same as in the conformal limit.

It will be important for us to understand the above statements in momentum space. As we will show by explicit construction, radial quantization leads to momentum space operators $\hat{\Phi}^L(u)$ and $\hat{\Phi}^R(u)$ with analytic expansions in the variable $u$. Since $P_z = \hat{m}u$, $P_\bar{z} = \hat{m}/u$, the limit $u \to \infty$ sets $P_\bar{z} = 0$, whereas $u \to 0$ sets $P_z = 0$. Thus the spaces $\mathcal{H}_{F}^{L,R}$ can be constructed in momentum space as follows:

$$\mathcal{H}_F^L = \left\{ \hat{\Phi}^L(u \to \infty)|0\rangle \right\}, \quad \mathcal{H}_F^R = \left\{ \hat{\Phi}^R(u \to 0)|0\rangle \right\}.$$  \hspace{1cm} (3.12) eIIxii

Other general aspects of this quasi-chiral structure will be developed by example.

### 3.2 Free Dirac Fermions

We now apply the above ideas to the free Dirac fermion theory. Some aspects of radial quantization in this situation were developed in [22][23].

The simplest way to formulate radial quantization is through analytic properties in the momentum space $u$-variable. The $\hat{b}^\pm(u)$ expansion for the fermion fields (2.20) is constructed to satisfy the equations of motion. In radial quantization, the equations of

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7 In the conformal limit, the usual Virasoro zero modes are

$$L_0 = \frac{D + L}{2}, \quad \bar{L}_0 = \frac{D - L}{2}.$$
motion are unchanged, only the canonical quantization procedure differs. This means that in radial quantization the expressions (2.26) are still valid with the appropriate modification of the integral over $u$. In radial quantization the angular variable $\varphi$ is treated as a space variable, thus the $u$-integral should be analytically continued to a circle surrounding the origin in the complex $u$ plane. Let us now suppose that $\tilde{b}^\pm(u)$ is single valued in the $u$ plane. The branch cuts due to $\sqrt{u}$ in (2.26) then imply that the fermion fields are anti-periodic: $\psi(r, \varphi + 2\pi i) = -\psi(r, \varphi)$. There are precisely two ways of avoiding the cut to close the $u$ contour, either toward $u = 0$ or $u = \infty$. We define the following prescription for analytic continuation of the $u$ integral in (2.26):

\[ \int_{-\infty}^{\infty} \frac{du}{2\pi i |u|} \tilde{b}^\pm(u) \to \left( \int_{C^L_\varphi} \frac{du}{2\pi i u} b^\pm(u) - \int_{C^R_\varphi} \frac{du}{2\pi i u} \tilde{b}^\pm(u) \right), \]  

where $C^L_\varphi, C^R_\varphi$ are contours depending on the angular direction $\varphi$ of the cut displayed in figures 1,2, and $b^\pm(u)$ and $\tilde{b}^\pm(u)$ are distinct operators defined explicitly below.

![Figure 1. The contour $C^L_\varphi$. The cut (wavy line) is oriented at an angle $\varphi$ from the negative $y$-axis. The circle is at $|u| = 1$.](image)

The choice of the contours $C^L_\varphi, C^R_\varphi$ is dictated by the following reasoning. In planar quantization, the $x$-component of momentum is $P_x = -\tilde{m}(u - 1/u)$. Thus left-moving particles with $P_x < 0$ have $u > 1$ whereas right moving particles have $P_x > 0$, $u < 1$. 

19
Thus the choice of $\mathcal{C}_L^\varphi$ versus $\mathcal{C}_R^\varphi$, i.e. $|u| > 1$ versus $|u| < 1$, corresponds to an analytic continuation of left versus right moving particles in planar quantization. This choice can also be viewed as dictated by (3.12). The precise orientation of the branch cut in figures 1,2 is chosen for later convenience.

The operators $b^\pm(u)$, $\overline{b}^\pm(u)$ have the following analytic expansions in $u$:

\[
b^\pm(u) = \pm i \sum_{\omega \in \mathbb{Z}} \Gamma(\frac{1}{2} - \omega) \hat{m}^\omega b^\pm_\omega u^\omega
\]

\[
\overline{b}^\pm(u) = \pm \sum_{\omega \in \mathbb{Z}} \Gamma(\frac{1}{2} - \omega) \hat{m}^\omega \overline{b}^\pm_\omega u^{-\omega}.
\]

(3.14) eIIIxiv

The Gamma functions in these expressions are included to simplify subsequent expressions.

The contour integrals can now be performed in (2.26) with the replacement (3.13). One needs

\[
e^{-i\omega \varphi} I_{\omega}(\hat{m} r) = -\int_{\mathcal{C}_L^\varphi} \frac{du}{2\pi i u} u^\omega e^{\hat{m}_zu + \hat{m}_z/u} e^{\hat{m}_zu + \hat{m}_z/u} u^{-\omega}
\]

(3.15a)

\[
e^{i\omega \varphi} I_{\omega}(\hat{m} r) = \int_{\mathcal{C}_L^\varphi} \frac{du}{2\pi i u} u^{-\omega} e^{\hat{m}_zu + \hat{m}_z/u}
\]

(3.15b),
where $I_\omega$ is the modified Bessel function$^8$. The result is

$$\left(\begin{array}{c} \psi_+ \\ \psi_- \end{array}\right) = \sum_{\omega \in \mathbb{Z}} b^\pm_\omega \Psi^{(a)}_{\gamma - \omega - 1/2} + b^\pm_\omega \Psi^{(a)}_{\gamma - \omega + 1/2}, \quad (3.16)$$

where

$$\Psi^{(a)}_{\gamma - \omega - 1/2} = \Gamma(\frac{1}{2} - \omega) \hat{m}^{\omega + 1/2} \begin{pmatrix} i e^{i(\frac{1}{2} - \omega)\phi} I^{\frac{1}{2} - \omega}(\hat{m} r) \\ e^{-i(\omega + \frac{1}{2})\phi} I_{\gamma + \frac{1}{2} - \omega}(\hat{m} r) \end{pmatrix},$$

$$\Psi^{(a)}_{\gamma - \omega + 1/2} = \Gamma(\frac{1}{2} - \omega) \hat{m}^{\omega + 1/2} \begin{pmatrix} e^{i(\frac{1}{2} + \omega)\phi} I^{\frac{1}{2} - \omega}(\hat{m} r) \\ -i e^{-i(\frac{1}{2} - \omega)\phi} I_{\gamma - \frac{1}{2} - \omega}(\hat{m} r) \end{pmatrix}. \quad (3.17)$$

(Henceforth the labels $a, p$ represent anti-periodic versus periodic.)

One can verify explicitly that the expressions $\Psi^{(a)}$ continue to satisfy the Dirac equations of motion $\Psi^{(a)}$ using the identities

$$r \partial_r I_\omega(\hat{m} r) \pm \omega I_\omega(\hat{m} r) = \hat{m} r I_{\omega \mp 1}(\hat{m} r). \quad (3.18)$$

In conventional approaches to radial quantization the expansions $\Psi^{(a)}$ are found directly as solutions to the radial Dirac equation.

Similar but not as elegant results apply to the periodic sector. Now one takes the combination $\sqrt{u} \hat{b}^\pm(u)$ to be single valued, which means that the expansions analogous to $\Psi^{(a)}$ are in half-integral powers of $u$. It is possible to define a prescription for analytic continuation as in (3.13), however the absence of a branch cut in this situation does not lead to as clear a distinction between left and right as for the anti-periodic sector. The result is that now

$$\int_{-\infty}^{\infty} \frac{du}{2\pi i u} \hat{b}^\pm(u) \to \left( \oint_{C^\leftrightarrow_<} \frac{du}{2\pi i u} \left( b^\pm_<(<u) + \hat{b}^\pm_<(<u) \right) + \oint_{C^\leftrightarrow_>^\leftrightarrow} \frac{du}{2\pi i u} \left( b^\pm_>(u) + \hat{b}^\pm_>(u) \right) \right). \quad (3.19)$$

$^8$ The formulas (3.15) are proven using

$$I_\omega(r) = \oint_{C_0} \frac{du}{2\pi i u} u^\omega e^{r(u + 1/u)/2},$$

where $C_0$ is the same as the contour $C_0^R$ rotated by an angle $\phi$ clockwise. To prove (3.15b) one makes the change of variable $u \to 1/u$ in (3.15a) and interchanges $z \leftrightarrow \bar{z}$. 

21
where
\[ b^\pm(u) = \pm i \sum_{\omega \leq -1/2} \Gamma\left(\frac{1}{4} - \omega\right) \hat{m}^\omega \ b^\pm_\omega u^\omega \pm \sum_{\omega \geq 1/2} \frac{2\pi}{\Gamma(\omega + \frac{1}{2})} (-1)^{\omega + 1/2} \hat{m}^\omega \ b^\pm_\omega u^\omega \]
\[ \equiv b^\pm_<(u) + b^\pm_>(u) \]
\[ \bar{b}^\pm(u) = \pm \sum_{\omega \leq -1/2} \Gamma\left(\frac{1}{4} - \omega\right) \hat{m}^\omega \ b^\pm_\omega u^{-\omega} \pm i \sum_{\omega \geq 1/2} \frac{2\pi}{\Gamma(\omega + \frac{1}{2})} (-1)^{1/2 - \omega} \hat{m}^\omega \ b^\pm_\omega u^{-\omega} \]
\[ \equiv \bar{b}^\pm_<(u) + \bar{b}^\pm_>(u). \]

The contour \( C_\leq \) is defined to be a closed contour on the unit circle, whereas \( C_\geq \) runs from 0 to \( \infty \) along a ray at an angle \( \varphi \) above the negative \( x \)-axis in the \( u \)-plane. Using the integrals
\[ e^{-in\varphi} \ I_n(\hat{m}r) = \int_{C_\leq} \frac{du}{2\pi i u} \ u^n e^{\hat{m}zu + \hat{m}\pi/u}, \]
\[ e^{-in\varphi} \ K_n(\hat{m}r) = i\pi(-)^n+1 \int_{C_\geq} \frac{du}{2\pi i u} \ u^n e^{\hat{m}zu + \hat{m}\pi/u}, \]
on one finds an expansion of the form (3.7) where now the sum runs over \( \omega \in \mathbb{Z} + 1/2 \), and the basis spinors differ from the expressions (3.8) only for \( \omega \geq 1/2 \):
\[ \Psi^{(p)}_{-\omega-1/2} = \frac{2\hat{m}^{\omega+1/2}}{\Gamma(\frac{1}{4} + \omega)} \begin{pmatrix} -i e^{(\frac{1}{4} - \omega)\varphi} K_{\omega - \frac{1}{4}}(\hat{m}r) \\ e^{-i(\omega + \frac{1}{2})\varphi} K_{\omega + \frac{1}{2}}(\hat{m}r) \end{pmatrix} \quad \omega \geq 1/2 \]
\[ \bar{\Psi}^{(p)}_{-\omega-1/2} = \frac{2\hat{m}^{\omega+1/2}}{\Gamma(\frac{1}{4} + \omega)} \begin{pmatrix} e^{i(\frac{1}{4} + \omega)\varphi} K_{\omega + \frac{1}{2}}(\hat{m}r) \\ i e^{i(\omega - \frac{1}{2})\varphi} K_{\omega - \frac{1}{2}}(\hat{m}r) \end{pmatrix} \quad \omega \geq 1/2. \]

In the above expansions the distinction between left and right is as clear as in the anti-periodic sector. Henceforth, unless otherwise indicated the operators \( b^\pm(u), \bar{b}^\pm(u) \) will be in the anti-periodic sector.

The commutation relations of the operators \( b^\pm_\omega, \bar{b}^\pm_\omega \) can be computed from (3.10) and the expansions (3.14) in either sector. We now derive these same commutation relations in a different way, one that emphasizes the momentum space operators. Since the expansion of the fermion fields follows from an analytic continuation of (2.26), the commutation relations of the \( b \) operators are simply related to the commutation relations of the \( \bar{b} \)'s given in (2.25). In the anti-periodic sector the relations (3.3) imply
\[ \{b^+(u), b^-(u')\} = 2\pi^2 i u' \delta(u + u'), \quad \{\bar{b}^+(u), \bar{b}^-(u')\} = -2\pi^2 i u' \delta(u + u') \]
\[ \{b(u), \bar{b}(u')\} = 0. \]
One has the formal identity

\[ 2\pi i \delta(u + u') = \frac{1}{u'} \sum_{n \in \mathbb{Z}} (-)^n \left( \frac{u}{u'} \right)^n. \tag{3.24} \]

This analytic \( \delta \)-function is constructed to satisfy

\[ \oint_0 du' \delta(u' - u)f(u') = f(u) \tag{3.25} \]

for any \( f(u) \) with a Laurent series expansion about \( u = 0 \). Inserting the expansions (3.14) into (3.23), and using the identity

\[ \Gamma\left(\frac{1}{2} - \omega\right)\Gamma\left(\frac{1}{2} + \omega\right) = \frac{\pi}{\cos(\pi\omega)}, \tag{3.26} \]

one finds

\[ \{b_\omega^+, b_{\omega'}^-\} = \delta_{\omega,-\omega'}, \quad \{\overline{b}_\omega^+, \overline{b}_{\omega'}^-\} = \delta_{\omega,-\omega'}, \quad \{b_\omega, \overline{b}_{\omega'}\} = 0. \tag{3.27} \]

These commutation relations also hold in the periodic sector. The operators \( b_\omega^+, \overline{b}_\omega^- \) have scaling dimension \( D = -\omega \), and Lorentz spin \( \pm \omega \):

\[ [L, b_\omega^+] = -\omega b_\omega^+, \quad [L, \overline{b}_\omega^-] = \omega \overline{b}_\omega^-, \tag{3.28} \]

in accordance with (3.9).

As we now explain, the fock spaces built with the fermionic oscillators correspond to the space of fields. Consider first the periodic sector. We define the physical vacuum as follows:

\[ b_\omega^+ |0\rangle = \overline{b}_\omega^- |0\rangle = 0, \quad \omega \geq 1/2 \tag{3.29} \]

\[ \langle 0 | b_\omega^+ = \langle 0 | \overline{b}_\omega^- = 0, \quad \omega \leq -1/2. \]

Define the left and right periodic fock spaces:

\[ \mathcal{H}_p^L = \{ b_{-\omega_1}^- b_{-\omega_2}^- \cdots b_{-\omega'_1}^- b_{-\omega'_2}^- \cdots |0\rangle \} \]

\[ \mathcal{H}_p^R = \{ \overline{b}_{-\omega_1}^+ \overline{b}_{-\omega_2}^+ \cdots \overline{b}_{-\omega'_1}^+ \overline{b}_{-\omega'_2}^+ \cdots |0\rangle \} \tag{3.30} \]
where $\omega, \omega' \geq 1/2$. Let us first illustrate the explicit connection with the space of fields by considering the fermion fields. One needs the asymptotic expansions of the Bessel functions

\begin{align*}
I_\omega(r) &= \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\omega + k + 1)} \left( \frac{r}{2} \right)^{\omega + 2k} \\
K_n(r) &= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left( \frac{r}{2} \right)^{2k-n} + (-1)^n \sum_{k=0}^{\infty} \frac{r^{n+2k}}{(n+k)!} (\log(r/2) - \psi(k+1/2) - \psi(n+k+1)/2),
\end{align*}

(3.31) eIIIxxvii

where $\psi(x) = \partial_x \log(\Gamma(x))$, and $n \geq 0$. Using this, one finds

\begin{align*}
\lim_{r \to 0} \psi_\pm(r, \varphi)|0\rangle &= \lim_{r \to 0} \sum_\omega \left( b_{\omega}^{' \pm} z^{-\omega-1/2} - \frac{i\hat{m}}{1/2 - \omega} b_{\omega}^{\pm} z^{1/2-\omega} \right) |0\rangle \\
&= b_{-1/2}^{\pm} |0\rangle.
\end{align*}

(3.32) eIIIxxviii

Similarly,

\begin{align*}
\lim_{r \to 0} \overline{\psi}_\pm(r, \varphi) |0\rangle &= b_{-1/2}^{' \pm} |0\rangle.
\end{align*}

(3.33) eIIIxxix

Following the previously introduced terminology, the fields $\psi_\pm, \overline{\psi}_\pm$ are in $\mathcal{H}_F^{(0)}$, and (3.10) holds. Indeed it is evident that the space $\mathcal{H}_F^{(0)}$ is identical to the space of fields in the massless limit from the expressions:

\begin{align*}
\left( \begin{array}{c}
\overline{\psi}_\pm \\
\psi_\pm
\end{array} \right) \xrightarrow{\hat{m} \to 0} \sum_\omega \left( \begin{array}{c}
b_{\omega}^{\pm} z^{-\omega-1/2} \\
b_{\omega}^{\pm} z^{1/2-\omega}
\end{array} \right).
\end{align*}

(3.34) eIIIxxx

The above equation is valid in either sector. The higher modes correspond to derivatives of the fermions:

\begin{align*}
\partial_0^m \psi_\pm(0)|0\rangle &= n! \ b_{-n-\frac{1}{2}}^{\pm} |0\rangle, \quad \partial_0^m \overline{\psi}_\pm(0)|0\rangle &= n! \ b_{-n-\frac{1}{2}}^{\pm} |0\rangle.
\end{align*}

(3.35) eIIIxxxi

Mixed derivative fields can always be simplified using the equations of motion to relate them linearly to the above states and are thus not in $\mathcal{H}_F^{(0)}$. For example

\begin{align*}
\partial_0^m \partial_0^{n+m} \psi_\pm(0)|0\rangle &= \hat{m}^{2m} n! \ b_{-n-\frac{1}{2}}^{\pm} |0\rangle.
\end{align*}

(3.36) eIIIxxxii

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9 For a review of the conformal field theory properties of free fermions and bosons used in this paper see [24].
Other states in $\mathcal{H}_p^{L,R}$ correspond to composite operators. For example, consider the $U(1)$ current $J_z = \psi_+ \psi_-$. One has

$$J_z(0)|0\rangle = b^+_\frac{1}{2} b^-_{-\frac{1}{2}} |0\rangle, \quad J_{\bar{z}}(0)|0\rangle = \overline{b}^+_{-\frac{1}{2}} \overline{b}^-_{\frac{1}{2}} |0\rangle. \quad (3.37)$$

We now turn to the anti-periodic sector. Due to the existence of the zero modes $b_0^\pm, \overline{b}_0^\pm$, the ‘vacuum’ in this sector is doubly degenerate for both left and right. Define these vacua as $| \pm \frac{1}{2} \rangle_L$ and $| \pm \frac{1}{2} \rangle_R$, characterized by

$$b_0^\pm | \mp \frac{1}{2} \rangle_L = | \pm \frac{1}{2} \rangle_L, \quad b_0^\mp | \pm \frac{1}{2} \rangle_L = 0, \quad b_n^\pm | \pm \frac{1}{2} \rangle_L = 0, \quad n \geq 1$$

$$\overline{b}_0^\pm | \mp \frac{1}{2} \rangle_R = | \pm \frac{1}{2} \rangle_R, \quad \overline{b}_0^\mp | \pm \frac{1}{2} \rangle_R = 0, \quad \overline{b}_n^\pm | \pm \frac{1}{2} \rangle_R = 0, \quad n \geq 1. \quad (3.38)$$

The vacua $| \pm \frac{1}{2} \rangle$ are defined by the inner products

$$L \langle \mp \frac{1}{2} | \pm \frac{1}{2} \rangle_L = R \langle \mp \frac{1}{2} | \pm \frac{1}{2} \rangle_R = 1. \quad (3.39)$$

These vacuum states have $U(1)$ charge $\pm 1/2$. The anti-periodic fock spaces are defined as

$$\mathcal{H}_a^L = \left\{ b^-_{-n_1} b^-_{-n_2} \cdots b^+_n b^+_{n'} \cdots | \pm \frac{1}{2} \rangle_L \right\}$$

$$\mathcal{H}_a^R = \left\{ \overline{b}^-_{-n_1} \overline{b}^-_{-n_2} \cdots b^+_n \overline{b}^+_{n'} \cdots | \pm \frac{1}{2} \rangle_R \right\}, \quad (3.40)$$

for $n, n' \geq 1$.

The fock states in $\mathcal{H}^{L,R}_{a \pm}$, including the vacuum states, should also be identified with fields. In order to identify fields corresponding to $| \pm \frac{1}{2} \rangle_{L,R}$, we first study the conformal limit. In this limit the Dirac theory is exactly equivalent to a massless scalar field $\phi$, which splits into chiral pieces $\phi = \varphi^L(z) + \varphi^R(\overline{z})$. From the bosonized expressions for the fermion fields (2.13), one finds the following operator product expansions:

$$\psi_{\pm}(z) e^{\mp i\varphi^L(0)/2} \sim \frac{1}{\sqrt{z}} e^{\pm i\varphi^L(0)/2} + \ldots \quad (3.41)$$

$$\overline{\psi}_{\pm}(\overline{z}) e^{\pm i\varphi^R(0)/2} \sim \frac{1}{\sqrt{z}} e^{\mp i\varphi^R(0)/2} + \ldots$$

Defining

$$e^{\pm i\varphi^L(0)/2} |0\rangle = | \pm \frac{1}{2} \rangle_L^{m=0}, \quad e^{\mp i\varphi^R(0)/2} |0\rangle = | \pm \frac{1}{2} \rangle_R^{m=0}, \quad (3.42)$$

25
one has
\[ b_0^\pm |\pm \frac{1}{2}\rangle_{L,R}^L = \oint \frac{dz}{2\pi i} \frac{1}{\sqrt{z}} \psi_\pm(z) e^{\mp i\varphi_L(0)/2} |0\rangle = |\pm \frac{1}{2}\rangle_{L,R}^L, \tag{3.43} \]
and similarly for $\overline{b}_0^\pm |\mp \frac{1}{2}\rangle_R$. From (3.41) one has the well-known result that the fields corresponding to the states $|\pm \frac{1}{2}\rangle_{L,R}^L$ in the conformal field theory are the analog of the Ising disorder and spin fields for this $U(1)$ theory.

It is essential in what we are doing that the states $|\pm \frac{1}{2}\rangle_{L,R}^L$ were precisely defined in the massive theory. Based on the above massless reasoning, we propose the following identification:
\[ e^{\pm i\varphi(0)/2} |0\rangle \propto (|\pm \frac{1}{2}\rangle_L \otimes |\mp \frac{1}{2}\rangle_R) \equiv |\pm \frac{1}{2}\rangle_L^R, \tag{3.44} \]
where $\varphi(z, \bar{z})$ is the local SG field. The correct interpretation of (3.44) is that it provides a non-perturbative definition of the LHS, and furthermore, a quasi-chiral factorization of this field. Note that with this identification, $e^{\pm i\varphi(0)/2}$ is $U(1)$ neutral. As in the conformal field theory, the states $|\pm \frac{1}{2}\rangle_{L,R}$ have the following Lorentz spin and dimension:
\[ L |\pm \frac{1}{2}\rangle_L = 1/8 |\pm \frac{1}{2}\rangle_L, \quad L |\pm \frac{1}{2}\rangle_R = -1/8 |\pm \frac{1}{2}\rangle_R \tag{3.45} \]
\[ D |\pm \frac{1}{2}\rangle_{L,R} = 1/8 |\pm \frac{1}{2}\rangle_{L,R}. \]
These formulas will be justified in the massive theory in section 6. The identification (3.44) implies that $e^{\pm i\varphi(0)/2}$ has scaling dimension $D = 1/4$ and is Lorentz spinless.

Other examples of field identifications are
\[ \overline{b}_{-1}^+ |\mp \frac{1}{2}\rangle_L = \partial_z \left( e^{i\varphi_L(0)/2} \right) |0\rangle \tag{3.46} \]
\[ b_{-1}^+ |\pm \frac{1}{2}\rangle_L = :\psi_+(0) e^{i\varphi_L(0)/2} : |0\rangle = e^{3i\varphi_L(0)/2} |0\rangle. \]
The normal ordering in the above equation means a regularized product of the two fields.

The spaces $\mathcal{H}_{L,R}^F$ can also be obtained directly from the $b^\pm(u), \overline{b}^\pm(u)$ operators. For example, in either sector one has
\[ \lim_{u \to \infty} (\hat{m} u)^\omega b^\pm(u) |\epsilon\rangle = \pm i \Gamma(\omega + \frac{1}{2}) b^\pm_{-\omega} |\epsilon\rangle, \]
\[ \lim_{u \to 0} \left( \frac{\hat{m}}{u} \right)^\omega \overline{b}^\pm(u) |\epsilon\rangle = \pm \Gamma(\omega + \frac{1}{2}) \overline{b}^\pm_{-\omega} |\epsilon\rangle, \tag{3.47} \]
where \( \epsilon = 0, \pm 1/2 \), depending on the sector. This illustrates eq. (3.12).

Correlation functions in the periodic sector are easily computed in this framework. For example,

\[
\langle 0 | \psi_-(z, \vec{x}) \psi_+(0) | 0 \rangle = \langle 0 | \psi_-(z, \vec{x}) b_{-\frac{1}{2}}^+ | 0 \rangle = \left( \Psi^{(p)}_{-1} \right)_2^2,
\]

where \( \left( \Psi^{(p)}_{-1} \right)_2 \) is the second component of the spinor \( \Psi^{(p)}_{-1} \). In this way one reproduces the results (2.48). Similarly one obtains correlation functions of the \( U(1) \) current:

\[
\langle 0 | J_z(z, \vec{x}) J_z(0) | 0 \rangle = 4\hat{m}^2 \frac{z}{z'} (K_1(\hat{m}r))^2
\]

(3.49)

Somewhat less trivial correlation functions are those of the fermions in the anti-periodic sector:

\[
\langle \frac{1}{2} | \psi_+(r, \varphi) \psi_-(r', \varphi') | -\frac{1}{2} \rangle_{\varphi=\varphi'=0} = \hat{m} \pi \left[ \sum_{n=1}^{\infty} (-)^n (I_{-n-\frac{1}{2}}(\hat{m}r)I_{n-\frac{1}{2}}(\hat{m}r') - I_{-n}(\hat{m}r)I_{n}(\hat{m}r')) \right] - I_{\frac{1}{2}}(\hat{m}r)I_{\frac{1}{2}}(\hat{m}r')
\]

(3.50)

4. Momentum Space Correlation Functions and Form Factors

The analytic properties of the operators \( b^\pm(u), \overline{b}^\pm(u) \) acquired in radial quantization lead to well defined vacuum expectation values of products of such operators. As we now show, these vacuum expectation values have a physical meaning as form-factors. The properly normalized one particle states with \( U(1) \) charge \( \pm 1 \) are

\[
|\uparrow, \theta \rangle = \frac{1}{2\pi} d^\dagger(\theta)|0\rangle, \quad |\downarrow, \theta \rangle = \frac{1}{2\pi} c^\dagger(\theta)|0\rangle
\]

(4.1)
Consider the following 2-particle form-factor in the usual planar quantization

\[
f_{\pm \frac{1}{2}}(\theta, \theta') = \langle 0 | e^{\pm i\phi(0)/2} | \uparrow, \theta; \downarrow, \theta' \rangle
\]

where as usual \( u = e^\theta \). This form-factor is well-defined in the SG theory, where \( \phi \) is the SG field and \( |\uparrow, \theta; \downarrow, \theta'\rangle \) is a 2-soliton state. Due to the expressions (2.26), this form factor enters into the computation of the the correlation function

\[
\langle 0 | e^{\pm i\phi(0)/2} \psi_\uparrow(z, \overline{z}) \psi_\downarrow(w, \overline{w}) | 0 \rangle.
\]

In radial quantization this correlation function is computed by analytically continuing the \( u \)-integrals in (2.26) to obtain an expression of the form (3.50). This implies that the form-factors are computable in radial quantization as follows:

\[
\langle 0 | e^{\pm i\phi(0)/2} \hat{b}^+(u)\hat{b}^-(u') | 0 \rangle = L \langle \mp \frac{i}{2} | \hat{b}^+(u)\hat{b}^-(u') | \mp \frac{i}{2} \rangle_L = R \langle \mp \frac{i}{2} | \hat{b}^+(u)\hat{b}^-(u') | \mp \frac{i}{2} \rangle_R.
\]

We now describe some techniques for computing vacuum expectation values such as (4.3). Based on the vacuum properties (3.38), we define the following normal ordering prescription:

\[
: b_n^\pm b_{-n}^\mp := -b_n^\pm b_{-n}^\mp, \quad : b_n^\pm b_{-n}^\mp := -b_n^\pm b_{-n}^\mp, \quad n > 0.
\]

To simplify the computation of vacuum expectation values, it proves useful to separate out the zero modes:

\[
b^\pm(u) = \pm i\sqrt{\pi} b_0^\pm + b^\pm_\bullet(u)
\]

One has

\[
b^\pm_\bullet(u) b^\mp_\bullet(u') =: b^\pm_\bullet(u) b^\mp_\bullet(u') = -\pi \frac{u}{u+u'}
\]

and

\[
\overline{b}^\pm(u) \overline{b}^\mp(u') =: \overline{b}^\pm(u) \overline{b}^\mp(u') = +\pi \frac{u'}{u+u'}.
\]
The Wick theorem may now be used with the pieces $b_\pm(u)$, $\tilde{b}_\pm(u)$. The two point functions are easily computed to be

$$L\langle -\frac{1}{2} \mid b^+(u) b^-(u') \mid + \frac{1}{2} \rangle_L = R\langle +\frac{1}{2} \mid \tilde{b}^+(u) \tilde{b}^-(u') \mid - \frac{1}{2} \rangle_R = \pi \frac{u'}{u + u'}.$$  \hspace{0.5cm} (4.7)

$$L\langle +\frac{1}{2} \mid b^+(u) b^-(u') \mid - \frac{1}{2} \rangle_L = R\langle -\frac{1}{2} \mid \tilde{b}^+(u) \tilde{b}^-(u') \mid + \frac{1}{2} \rangle_R = -\pi \frac{u}{u + u'}.

The form-factors $f_{\pm \frac{1}{2}}(\theta, \theta')$ as computed from (4.3) and (4.7) agree exactly with the known result\cite{25}\cite{21}. As we will show more easily in section 6, this result extends to the multiparticle form factors.

5. The Spectrum Generating Affine Lie Algebra

5.1 Quasi-Chiral Splitting of the Conserved Charges

The quasi-chiral factorization described above leads to some additional structures for the conserved charges. In radial quantization, given a conserved current $J_z$, $J_\pi$, the conserved charge is

$$Q = \frac{1}{4\pi} \int_{-\pi}^{\pi} r \, d\varphi \left( e^{i\varphi} J_z + e^{-i\varphi} J_\pi \right).$$  \hspace{0.5cm} (5.1)

All of the conserved charges constructed in section 2 can thereby be expressed in terms of the radial modes $b_{\omega}^\pm$, $\tilde{b}_{\omega}^\pm$. More specifically, define the inner product of two spinors $A = \left( \begin{array}{c} \pi \\ a \end{array} \right)$, $B = \left( \begin{array}{c} \tilde{b} \\ b \end{array} \right)$ as

$$(A, B) = \frac{1}{4\pi} \int_{-\pi}^{\pi} r \, d\varphi \left( e^{i\varphi} a \, b + e^{-i\varphi} \overline{a} \, \overline{b} \right).$$  \hspace{0.5cm} (5.2)

Using the Wronskian identities

$$I_\omega(r) \, K_{\omega+1}(r) + I_{\omega+1}(r) \, K_\omega(r) = \frac{1}{r}$$

$$I_\omega(r) \, I_{-\omega-1}(r) - I_{\omega+1}(r) \, I_{-\omega}(r) = -2 \frac{\sin(\pi\omega)}{\pi r},$$

one finds simple inner products among the spinors in the radial basis. In either sector:

$$(\Psi_\omega, \Psi_{-\omega'-1}) = (\overline{\Psi}_\omega, \overline{\Psi}_{-\omega'-1}) = \delta_{\omega,\omega'}, \quad (\Psi_\omega, \overline{\Psi}_\omega) = 0.$$  \hspace{0.5cm} (5.4)
The conserved charges can all be expressed using the above inner product, e.g.

\[
Q_{\pm n} = \frac{1}{2} (\Psi_{\pm}, \partial_z \Psi_{\pm}), \quad Q_{1 \pm} = \frac{1}{2} (\Psi_{\pm}, \partial_\tau \Psi_{\pm})
\]

\[
T_0 = : (\Psi_+, \Psi_-) :. \tag{5.5} \text{eVv}
\]

and are thus easily expressed in terms of the \( b, \bar{b} \)-modes.

It is more useful for us to work in momentum space, i.e. with the analogs of (2.27) in radial quantization. The proper analytic continuation of (2.27) is the following:

\[
\int_{-\infty}^{\infty} \frac{du}{2\pi i |u|} \hat{b}(u) \hat{b}(-u) \to 2 \int \frac{du}{2\pi i u} b(u) b(-u) - 2 \int \frac{du}{2\pi i u} \bar{b}(u) \bar{b}(-u). \tag{5.6} \text{eVvi}
\]

One can check explicitly that the prescription (5.6) is equivalent to the position space expressions (5.5). Thus, the conserved charges split into left and right pieces:

\[
Q_{n \pm} = Q_{n \pm}^L + Q_{-n \pm}^R
\]

\[
\alpha_n = \alpha_n^L + \alpha_n^R \tag{5.7} \text{eVvii}
\]

\[
L = L^L + L^R.
\]

The additional minus sign in the subscript \( -n \) of the right piece of the charges in comparison to the left piece is dictated by Lorentz covariance. The explicit expressions are as
follows:

$$\alpha_{n}^{L} = \frac{\hat{m}^{n}}{\pi} \left\{ \int \frac{du}{2\pi i u} u^{-n} : b^{+}(u) b^{-}(-u) : \right\} = \frac{\hat{m}^{n}}{\pi} \sum_{\omega} (-)^{n-\omega} \Gamma\left(\frac{1}{2} - \omega\right) \Gamma\left(\frac{1}{2} + \omega - n\right) : b^{+}_{\omega} b^{-}_{n-\omega} :$$

$$\alpha_{n}^{R} = -\frac{\hat{m}^{n}}{\pi} \left\{ \int \frac{du}{2\pi i u} u^{n} : \overline{b}^{+}(u) \overline{b}^{-}(-u) : \right\} = \frac{\hat{m}^{n}}{\pi} \sum_{\omega} (-)^{n+\omega} \Gamma\left(\frac{1}{2} + \omega\right) \Gamma\left(\frac{1}{2} - \omega - n\right) : \overline{b}^{+}_{\omega} \overline{b}^{-}_{n-\omega} :$$

$$Q_{n}^{\pm, L} = \frac{\hat{m}^{n}}{2\pi} \left\{ \int \frac{du}{2\pi i u} u^{-n} b^{\pm}(u) b^{\pm}(-u) \right\} = \frac{\hat{m}^{n}}{2\pi} \sum_{\omega} (-)^{n-\omega+1} \Gamma\left(\frac{1}{2} - \omega\right) \Gamma\left(\frac{1}{2} + \omega - n\right) b^{\pm}_{\omega} b^{\pm}_{n-\omega}$$

$$Q_{n}^{\pm, R} = -\frac{\hat{m}^{n}}{2\pi} \left\{ \int \frac{du}{2\pi i u} u^{n} \overline{b}^{\pm}(u) \overline{b}^{\pm}(-u) \right\} = \frac{\hat{m}^{n}}{2\pi} \sum_{\omega} (-)^{n-\omega+1} \Gamma\left(\frac{1}{2} + \omega\right) \Gamma\left(\frac{1}{2} - \omega - n\right) \overline{b}^{\pm}_{\omega} \overline{b}^{\pm}_{n-\omega}$$

$$L^{L} = \frac{1}{\pi} \int \frac{du}{2\pi i u} : b^{+}(u) u \partial_{u} b^{-}(-u) : = -\sum_{\omega} \omega : b^{+}_{\omega} b^{-}_{-\omega} :$$

$$L^{R} = -\frac{1}{\pi} \int \frac{du}{2\pi i u} : \overline{b}^{+}(u) u \partial_{u} \overline{b}^{-}(-u) : = \sum_{\omega} \omega : \overline{b}^{+}_{\omega} \overline{b}^{-}_{-\omega} :$$

The above expressions were derived in the anti-periodic sector, but similar (but not identical) expressions apply to the periodic sector. Namely, if one first expresses the product of Gamma-functions as a rational function of $\omega$ using (3.26), and then lets the sum run over $\omega \in \mathbb{Z} + 1/2$ while omitting terms that are singular one obtains the correct periodic sector expressions.

The operators we have introduced have the following Lorentz spin properties

\[
[L^{L}, b^{\pm}_{\omega}] = -\omega \ b^{\pm}_{\omega}, \quad [L^{R}, \overline{b}^{\pm}_{\omega}] = \omega \ \overline{b}^{\pm}_{\omega} \quad (5.9)_{eVx}
\]

\[
[L^{L}, O^{L}_{n}] = -n \ O^{L}_{n}, \quad [L^{R}, O^{R}_{n}] = n \ O^{R}_{n}, \quad (5.9)_{eVx}
\]

where $O^{L}_{n} = Q^{\pm, L}_{n}$, or $\alpha^{L}_{n}$, and similarly for $O^{R}_{n}$. The scaling operator is realized as

\[
D = L^{L} - L^{R} + \hat{m} \partial_{\hat{m}}, \quad (5.10)_{eVx}
\]

31
so that

\[ [D, b_\omega^\pm] = -\omega b_\omega^\pm, \quad [D, \overline{b}_\omega^\pm] = -\omega \overline{b}_\omega^\pm \]  \hspace{1cm} (5.11) eVxi

Here we see an important distinction with the situation encountered in conformal field theory, in that all the operators \( O_n^{L,R} \) have positive scaling dimension. Note that the relations (5.7),(5.9) reproduce the unsplit relations (2.31).

We now compute the commutation relations of the left and right components among themselves in the anti-periodic sector. Obviously, any left operator commutes with a right operator. This computation is difficult if one tries to use the integrated expressions in (5.8) due to the infinite sums, so we develop a different technique. More generally, let \( O_n^{L,R} \) have contour integral expressions in momentum space of the form

\[ O_n^L = \int \frac{du}{2\pi i u} u^{-n} O^L(u), \quad O_n^R = \int \frac{du}{2\pi i u} u^n O^R(u). \]  \hspace{1cm} (5.12) eVxii

The non-trivial contributions to \([O_n^L, O_m^L]\) arise from pole singularities in the product \( O^L(u)O^L(u') \). There is no translation invariance in \( u \)-space, so these poles can be at either \( u = \pm u' \), the poles being related to each other by ‘crossing symmetry’ \( \theta \to i\pi + \theta \). The operators \( O^L(u) \) and \( O^R(u) \) have a development in opposite powers of \( u \), which can be seen from (3.12), or for example in (3.14). Thus in the vacuum expectation value, \( O^L(u)O^L(u') \) is properly defined as an expansion in powers of \( u/u' \) for \( u < u' \), whereas \( O^R(u)O^R(u') \) is a well-defined expansion in powers of \( u'/u \) for \( u > u' \). Taking all of this into account, one has

\[ [O_n^L, O_m^L] = \int_{\pm u} \frac{du'}{2\pi i u'} \int_{0} \frac{du}{2\pi i u} (u')^{-m} u^{-n} O^L(u)O^L(u') \]  \hspace{1cm} (5.13) eVxiii

\[ [O_n^R, O_m^R] = \int_{\pm u} \frac{du'}{2\pi i u'} \int_{0} \frac{du}{2\pi i u} (u')^{m} u^{n} O^R(u)O^R(u'), \]

where in the first equation the \( u' \) contour is larger than the \( u \)-contour, and visa-versa for the second equation.

It is useful to introduce the concept of an operator product expansion in momentum space simply defined as a way of displaying the pole singularities in the product of the
momentum space operators. For the actual computation we will perform, we need the vacuum expectation values

\[ L(\pm \frac{1}{2}) : b^+(u)b^-(-u) : b^+(u')b^-(-u') : | \pm \frac{1}{2} \rangle_L = \pi^2 \frac{uu'}{(u-u')^2} \]

\[ L(\pm \frac{1}{2}) : b^+(u)b^-(u')b^-(u') : | \pm \frac{1}{2} \rangle_L = \pi^2 u^2 \left( \frac{1}{(u-u')^2} - \frac{1}{(u+u')^2} \right), \]

which are easily derived using the techniques of the last section. Using this, one finds the following operator product expansions:

\[ : b^+(u)b^-(-u) : : b^+(u')b^-(-u') : \sim \pi^2 \frac{uu'}{(u-u')^2} \]

\[ : b^+(u)b^-(-u) : (b^\pm(u')b^\pm(-u')) \sim \pm \left( \frac{\pi u}{u'-u} b^\pm(u)b^\mp(-u') + \frac{\pi u}{u+u'} b^\mp(u)b^\pm(u') \right) \]

\[ (b^+(u)b^+(u')) (b^-(u')b^-(u')) \sim \pi^2 u^2 \left( \frac{1}{(u-u')^2} - \frac{1}{(u+u')^2} \right) \]

\[ + \frac{\pi u}{u'-u} (b^+(u)b^-(-u') + b^+(u)b^-(u')) \]

\[ + \frac{\pi u}{u'+u} (b^+(u)b^-(-u') + b^+(u)b^-(-u')). \]  

(5.14) eVxiv

Define as before

\[ P_{n}^{L,R} = \alpha_{n}^{L,R}, \quad n \text{ odd}; \quad T_{n}^{L,R} = \alpha_{n}^{L,R}, \quad n \text{ even}. \]  

(5.16) eVxvi

Then using the above formulas one finds the following result:

\[ [P_{n}^{L}, P_{m}^{L}] = n \hat{m}^{2|n|} \delta_{n,-m} \]

\[ [P_{n}^{L}, T_{m}^{L}] = [P_{n}^{L}, Q_{m}^{\pm,L}] = 0 \]

\[ [T_{n}^{L}, T_{m}^{L}] = n \hat{m}^{2|n|} \delta_{n,-m} \]  

(5.17) eVxvii

\[ [T_{n}^{L}, Q_{m}^{\pm,L}] = \pm 2 \hat{m}^{2|n|+|m|} Q_{n+m}^{\pm,L} \]

\[ [Q_{n}^{+,L}, Q_{m}^{-,L}] = \frac{\hat{m}^{2|n|+|m|}}{\hat{m}^{2|n+m|}} T_{n+m}^{L} + \frac{n}{2} \hat{m}^{2|n|} \delta_{n,-m}. \]

In deriving the last equation above, we have defined

\[ : b_0^+ b_0^- : \equiv \frac{1}{2} (b_0^+ b_0^- - b_0^- b_0^+) = b_0^+ b_0^- - 1/2. \]  

(5.18) eVxviii
This shifts $T_0^L \rightarrow T_0^L + 1/2$, so that

$$T_0^L | \pm \frac{1}{2} \rangle_L = \pm \frac{1}{2} | \pm \frac{1}{2} \rangle_L. \quad (5.19)$$

This shifts the form of the central term from being proportional to the value $(n - 1)$ obtained from the contour integration to $n$.

The operators $T_n^L, Q_n^\pm L$ thus satisfy a level 1 $\hat{sl}(2)$ algebra, and the $P_n^L$ satisfy an infinite Heisenberg algebra. A similar computation yields precisely the same algebra for $P_n^R, T_n^R, Q_n^\pm R$, with the same level 1. The appearance of central terms is entirely consistent with the results of section 2. Namely, the unsplitted charges in (5.7) continue to satisfy the level 0 algebra $\hat{sl}(2)$ since the central terms cancel between left and right in the computation $[O_n, O_m] = [O_n^L, O_m^L] + [O_n^R, O_m^R]$. The above results are for the anti-periodic sector; the periodic sector will be considered at the end of this section.

### 5.2 Highest Weight Representations and the Space of Fields

Let us denote the $P_n$ extension of the algebra $\hat{sl}(2)$ defined in (5.17) as $\hat{sl}(2)$. As we have seen, the symmetry algebra factorizes into $\hat{sl}(2)_L \otimes \hat{sl}(2)_R$. We now show that the algebra $\hat{sl}(2)$ is a complete spectrum generating algebra for the anti-periodic sector, namely, that the complete spectrum of quasi-chirally factorized fields can be obtained from infinite highest weight representations of $\hat{sl}(2)$.

We first review the known structure of highest weight representations of the algebra $\hat{sl}(2)$ [19]. At level 1, there are only 2 highest weight states $|\Lambda_j\rangle$, $j = 0, 1/2$, satisfying

$$t_n^\pm |\Lambda_j\rangle = t_n^0 |\Lambda_j\rangle = 0, \quad n \geq 1$$

$$t_0^0 |\Lambda_0\rangle = -\frac{1}{2} |\Lambda_0\rangle, \quad t_0^0 |\Lambda_{1/2}\rangle = \frac{1}{2} |\Lambda_{1/2}\rangle. \quad (5.20)$$

Consider first the anti-periodic sector. Using the expressions in (5.8), one finds

$$Q_n^\pm L | \pm \frac{1}{2} \rangle_L = T_n^L | \pm \frac{1}{2} \rangle_L = P_n^L | \pm \frac{1}{2} \rangle_L = 0, \quad n \geq 1$$

$$T_0^L | \pm \frac{1}{2} \rangle_L = \pm \frac{1}{2} | \pm \frac{1}{2} \rangle_L. \quad (5.21)$$

We are using the twisted basis defined in section 2. The highest weight conditions below were obtained from the more familiar relations $j_n > 0 |\Lambda_j\rangle = 0$, $j_0^+ |\Lambda_j\rangle = 0$, $j_0^- |\Lambda_j\rangle = 2j |\Lambda_j\rangle$. 

34
Comparing with (5.20), one sees that the states $| \pm \frac{1}{2} \rangle_L$ are highest weight for the $\hat{sl}(2)$ subalgebra of $\hat{sl}(2)$, with the identification

$$| + \frac{1}{2} \rangle_L = | \Lambda_1 \rangle, \quad | - \frac{1}{2} \rangle_L = | \Lambda_0 \rangle. \quad (5.22)$$

We only need to proceed to a few more levels to get a clear picture. One finds

$$Q_{-1}^{\pm L} | \pm \frac{1}{2} \rangle_L = 0, \quad Q_{-1}^{\pm L} | \mp \frac{1}{2} \rangle_L = -\frac{1}{2} b_{-1}^{\pm} | \pm \frac{1}{2} \rangle_L. \quad (5.23)$$

The first equation is a ‘null-state’ condition. The second equation indicates that the spaces $\mathcal{H}_{a+}^L$ and $\mathcal{H}_{a-}^L$ mix under the action of $\hat{sl}(2)$. Another important point is that one cannot obtain all of the states in $\mathcal{H}_a^L \equiv \mathcal{H}_{a+}^L \oplus \mathcal{H}_{a-}^L$ from the action of $\hat{sl}(2)$ alone. One must also descend with the $P_n$’s. For example

$$P_{-1}^L | \pm \frac{1}{2} \rangle_L = \frac{1}{2} b_{-1}^{\pm} | \mp \frac{1}{2} \rangle_L, \quad (5.24)$$

and this is the only way to obtain the state on the RHS.

With the above qualitative picture, one can now proceed to precisely identify the spaces $\mathcal{H}_a^{L,R}$ with $\hat{sl}(2)$ modules. We do this by introducing generating functions, or characters, for these spaces. Let $q$ denote a formal parameter (not to be confused with the $q$-deformation parameter). Using results familiar from the known partition functions of free fermions\textsuperscript{[24]}, one has

$$\text{Tr}_{\mathcal{H}_a^{L \pm}} (q^L) = q^{1/8} \prod_{n=1}^{\infty} (1 + q^n)^2 \quad (5.25)$$

and

$$\text{Tr}_{\mathcal{H}_a^{R \pm}} (q^L) = q^{-1/8} \prod_{n=1}^{\infty} (1 + q^{-n})^2. \quad (5.25)$$

\textsuperscript{11} We remark that this is already rather different than the situation in conformal current algebra, where $L^L | \Lambda_0 \rangle = 0, \quad L^L | \Lambda_\frac{1}{2} \rangle = \frac{1}{4} | \Lambda_\frac{1}{2} \rangle$. Here the $L^L$ eigenvalues are both 1/8. It is the twist that makes this possible.
(The power of 2 comes from the fact that we have a complex fermion.) In computing the above trace over $\mathcal{H}^L (\mathcal{H}^R)$ only $L^L (L^R)$ matters in the sum $L = L^L + L^R$. Also,

$$\text{Tr}_{\mathcal{H}_a^L} (q^L) = \text{Tr}_{\mathcal{H}_a^L} (q^L) + \text{Tr}_{\mathcal{H}_a^L} (q^L),$$

and similarly for $\mathcal{H}_a^R$. We present explicit formulas for the left sector; identical results apply in the right sector with everywhere $L \rightarrow R$, $q \rightarrow q^{-1}$.

Define the $\widehat{sl}(2)_L$ modules $\widehat{V}_{\Lambda_j}^L$ as follows,

$$\widehat{V}_{\Lambda_0}^L = \left\{ Q_{-n_1}^L Q_{-n_2}^L \cdots T_{-n_1}^L T_{-n_2}^L \cdots P_{-n_1''}^L P_{-n_2''}^L \cdots | - \frac{1}{2} \rangle_L \right\}$$

$$\widehat{V}_{\Lambda_{\frac{1}{2}}}^L = \left\{ Q_{-n_1}^L Q_{-n_2}^L \cdots T_{-n_1}^L T_{-n_2}^L \cdots P_{-n_1''}^L P_{-n_2''}^L \cdots | + \frac{1}{2} \rangle_L \right\},$$

for $n, n', n'' \geq 1$. Define also

$$\widehat{\chi}_j^L \equiv \text{Tr}_{\widehat{V}_{\Lambda_j}^L} (q^L).$$

We will prove that

$$\text{Tr}_{\mathcal{H}_a^L} (q^L) = \widehat{\chi}_0^L + \widehat{\chi}_{\frac{1}{2}}^L.$$

The above result implies that we have the identification

$$\mathcal{H}_a^L = \widehat{V}_{\Lambda_0}^L \oplus \widehat{V}_{\Lambda_{\frac{1}{2}}}^L.$$

To compute $\widehat{\chi}_j^L$, note first that since the $P_n^L$'s commute with the $sl(2)_L$ subalgebra of $\widehat{sl}(2)_L$, one has the factorization

$$\widehat{\chi}_j^L = q^{1/8} \chi_P^L \chi_{t_w,L}^j,$$

where $\chi_P^L$ is the contribution from the Heisenberg algebra of the $P_n^L$. The latter contribution is easily seen to be

$$\chi_P^L = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})}.$$
The functions $\hat{\chi}^{t, L}$ are the twisted characters for the $\hat{sl}(2)$ algebra, which are defined and computed in the appendix. The result is

$$\hat{\chi}^{t, L}_0 = \hat{\chi}^{t, L}_{\frac{1}{2}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}}. \tag{5.33}$$

Putting these pieces together, to establish the main result (5.29), one only needs to prove

$$\prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} = \prod_{n=1}^{\infty} (1 + q^n). \tag{5.34}$$

A simple proof is as follows:

$$\prod_{n=1}^{\infty} (1 + q^n)(1 - q^{2n-1}) = \prod_{n=1}^{\infty} (1 + q^{2n})(1 + q^{2n-1})(1 - q^{2n-1})
= \prod_{n=1}^{\infty} (1 + q^{2n})(1 - q^{4n-2}) \tag{5.35}
= \prod_{n=1}^{\infty} \frac{(1 + q^{2n})(1 - q^{2n})}{(1 - q^{4n})} = 1.$$

We turn now to the periodic sector. The situation here is very different. First note that we have already exhausted the available level 1 highest weight representations for the anti-periodic sector. The true physical vacuum resides in the periodic sector, and is characterized by (3.29). The momentum operators $P_1, P_{-1}$, and thus the hamiltonian must annihilate the vacuum. Since the $\hat{sl}(2)$ charges commute with $P_1, P_{-1}$, they must also annihilate the vacuum, otherwise they would be spontaneously broken symmetries. Using the expressions in (5.8) specialized to the periodic sector, one indeed verifies

$$P_n^L |0\rangle = Q_n^{+,L} |0\rangle = T_n^L |0\rangle = 0, \quad \forall \ n, \tag{5.36}$$

and similarly for the right charges. This implies the $\hat{sl}(2)$ algebra must have level $k = 0$ in the periodic sector:

$$[Q_1^{+,L}, Q_{-1}^{-,L}] |0\rangle = \hat{m}^2 (T_0^L + k/2) |0\rangle = 0, \quad \Rightarrow \quad k = 0. \tag{5.37}$$

\[\text{Here we have another important distinction from the situation in conformal field theory, where, in the latter, the vacuum is highest weight. This is possible in conformal field theory since, after the conformal transformation } z \rightarrow \log z, \text{ one treats }(L - D)/2 \text{ and }(L + D)/2 \text{ as the right and left 'hamiltonians', and thus the above argument does not apply.}\]
Eq. (5.36) also implies
\[ [p^L_n, p^L_m] = 0, \quad \forall \ n, m \quad (5.38) \]
in this sector. The affine charges still have a meaningful action on the other states. For example
\[ Q^L_{-1} b^L_{-1/2} |0\rangle = - b^L_{-3/2} |0\rangle, \quad (5.39) \]
which says the same thing as (2.41).

6. Vertex Operators and Momentum Space Bosonization

In this section we will describe the role played by vertex operators. There exists an alternative ‘bosonic’ organization of \( \mathcal{H}^{L,R}_a \). Using the Jacobi triple product identity\[13\]
\[ \prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^{n-1} v^n)(1 - u^n v^{n-1}) = \sum_{m \in \mathbb{Z}} (-)^m u^{m(m-1)/2} v^{m(m+1)/2}, \quad (6.1) \]
one finds
\[ \text{Tr}_{\mathcal{H}^L_a} (q^L) = \sum_{\alpha \in \mathbb{Z} + 1/2} \frac{q^{\alpha^2/2}}{\prod_{n=1}^{\infty} (1 - q^n)}. \quad (6.2) \]
This suggests the following construction. Let \( Hb^L \ (Hb^R) \) denote the infinite Heisenberg algebra generated by \( \alpha^L_n \ (\alpha^R_n) \ \forall \ n \) in the anti-periodic sector (eq. (5.17)):
\[ [\alpha^L_n, \alpha^L_m] = n \hat{m}^{2|n|} \delta_{n,-m}, \quad [\alpha^R_n, \alpha^R_m] = n \hat{m}^{2|n|} \delta_{n,-m}. \quad (6.3) \]
Let us define highest weight states of \( Hb^{L,R} \) as satisfying
\[ \alpha^L_n |\alpha\rangle_L = 0, \quad \alpha^L_0 |\alpha\rangle_L = \alpha |\alpha\rangle_L, \quad n > 0 \quad (6.4) \]
\[ \alpha^R_n |\alpha\rangle_R = 0, \quad \alpha^R_0 |\alpha\rangle_R = \alpha |\alpha\rangle_R, \quad n > 0. \quad (6.4) \]
Let us further suppose
\[ L^L |\alpha\rangle_L = \alpha^2/2 |\alpha\rangle_L, \quad L^R |\alpha\rangle_R = -\alpha^2/2 |\alpha\rangle_R. \quad (6.5) \]

\[13\] One proof of this identity uses the \( \hat{sl}(2) \) algebra\[19\].
Construct modules \( V^L_\alpha \) as follows
\[
V^L_\alpha = \{ \alpha_{-n_1}^L \alpha_{-n_2}^L \cdots |\alpha\rangle_L \}, \quad n_i > 0,
\] (6.6) 

and similarly for \( V^R_\alpha \). Then
\[
\text{Tr}_{V^L_\alpha} (q^L) = \frac{q^{\alpha^2/2}}{\prod_{n=1}^\infty (1 - q^n)},
\] (6.7) 

and
\[
\text{Tr}_{H^L_\alpha} (q^L) = \sum_{\alpha \in \mathbb{Z} + 1/2} \text{Tr}_{V^L_\alpha} (q^L).
\] (6.8) 

This suggests that one can identify
\[
H^L_\alpha = \bigoplus_{\alpha \in \mathbb{Z} + 1/2} V^L_\alpha.
\] (6.9) 

In order to construct the states \(|\alpha\rangle_{L,R}\) explicitly, we define some momentum space vertex operators \( \tilde{\sigma}_\alpha^L(u) \) (\( \tilde{\sigma}_\alpha^R(u) \)) which act from \( H^L \to H^L \) (\( H^R \to H^R \)) and transform as follows with respect to \( Hb^{L,R} \):
\[
[\alpha^L_n, \tilde{\sigma}_\alpha^L(u)] = \alpha \hat{m}^{\leftarrow n} u^{-n} \tilde{\sigma}_\alpha^L(u)
\]
\[
[\alpha^R_n, \tilde{\sigma}_{-\alpha}^R(u)] = \alpha \hat{m}^{\leftarrow n} u^n \tilde{\sigma}_{-\alpha}^R(u) \quad n \neq 0
\]
\[
= -\alpha \tilde{\sigma}_{-\alpha}^R(u) \quad n = 0.
\] (6.10) 

Operators with these properties can be constructed from the \( \alpha_n \)'s. A requisite set of operators is given by
\[
\tilde{\sigma}_\alpha^L(u) = : e^{i\alpha \hat{\phi}_\alpha^L(u)} : \quad \tilde{\sigma}_{-\alpha}^R(u) = : e^{i\alpha \hat{\phi}_{-\alpha}^R(u)} :
\] (6.11) 

where
\[
-i \hat{\phi}_\alpha^L(u) = \sum_{n \neq 0} \tilde{m}^{\leftarrow n} \alpha^L_n \frac{u^n}{n} + \alpha^L_0 \log(u) - \tilde{\alpha}_0^L
\]
\[
-i \hat{\phi}_{-\alpha}^R(u) = \sum_{n \neq 0} \tilde{m}^{\leftarrow n} \alpha^R_n \frac{u^{-n}}{n} + \alpha^R_0 \log(u) + \tilde{\alpha}_0^R,
\] (6.12) 

and
\[
[\alpha^L_0, \tilde{\alpha}_0^L] = [\alpha^R_0, \tilde{\alpha}_0^R] = 1.
\] (6.13)
From (b.11), one sees that \( \hat{\phi}^{L,R} \) are dimensionless operators. In (6.11), normal ordering is taken with respect to the \( \alpha_n \)'s: \( : \alpha_n^L \alpha_{-n}^L : = \alpha_{-n}^L \alpha_n^L, \quad : \alpha_n^R \alpha_{-n}^R : = \alpha_{-n}^R \alpha_n^R, \quad n \geq 1. \) (6.14)

The vacua \( |\emptyset\rangle_{L,R} \) are defined to satisfy
\[
\alpha_n^L |\emptyset\rangle_L = \alpha_n^R |\emptyset\rangle_R = 0, \quad n \geq 0 \quad (6.15)
\]
\[\tilde{\alpha}_0^L |\emptyset\rangle_L, \quad \tilde{\alpha}_0^R |\emptyset\rangle_R \neq 0.\]

Note that \( |\emptyset\rangle \equiv |\emptyset\rangle_L \otimes |\emptyset\rangle_R \) does not correspond to the physical vacuum since it is not annihilated by \( P_1 \) and \( P_{-1} \). One has the following vacuum expectation values

\[
L \langle \emptyset | \hat{\phi}^L(u) \hat{\phi}^L(u') | \emptyset \rangle_L = - \log(1/u - 1/u') \quad (6.16)
\]
\[
R \langle \emptyset | \hat{\phi}^R(u) \hat{\phi}^R(u') | \emptyset \rangle_R = - \log(u - u')
\]

\[
L \langle \emptyset | \prod_i e^{i\alpha_i \hat{\phi}^L(u_i)} | \emptyset \rangle_L = \prod_i (1/u_i - 1/u_j)^{\alpha_i \alpha_j}
\]
\[
R \langle \emptyset | \prod_i e^{i\alpha_i \hat{\phi}^R(u_i)} | \emptyset \rangle_R = \prod_i (u_i - u_j)^{\alpha_i \alpha_j}. \quad (6.17)
\]

Based on the structure (3.12) we define the following states
\[
|\alpha\rangle_L = \tilde{\sigma}_\alpha^L(\infty) |\emptyset\rangle_L \quad (6.18)
\]
\[
|\alpha\rangle_R = \tilde{\sigma}_\alpha^R(0) |\emptyset\rangle_R.
\]

Conjugate states satisfying
\[
L \langle -\alpha | \alpha \rangle_L = R \langle -\alpha | \alpha \rangle_R = 1 \quad (6.19)
\]

can also be constructed:
\[
L \langle \alpha | = \lim_{u \to 0} u^{-\alpha^2} \langle \emptyset | \tilde{\sigma}_\alpha^L(u)
\]
\[
R \langle \alpha | = \lim_{u \to \infty} u^{\alpha^2} \langle \emptyset | \tilde{\sigma}_\alpha^R(u). \quad (6.20)
\]
The highest weight properties (6.4)(6.5) are a consequence of the defining relations (6.10):

$$\lim_{u \to \infty} \alpha_n \hat{\sigma}_n(u) \ket{\emptyset} = \lim_{u \to \infty} n^{-n} \hat{\sigma}_n(u) \ket{\emptyset} = 0 \quad n > 0$$

$$\alpha \quad n = 0,$$

and similarly for the right sector. Eq. (6.5) can be established by expressing $L^L (L^R)$ in terms of the operators $\hat{\phi}^L(u) (\hat{\phi}^R(u))$. It is simpler to derive the result from the vacuum expectation values

$$\braket{\emptyset | e^{-i\alpha \hat{\phi}^L(u)} e^{i\alpha \hat{\phi}^L(\infty)} | \emptyset} = u^{\alpha^2}$$

$$\braket{\emptyset | e^{-i\alpha \hat{\phi}^R(u)} e^{i\alpha \hat{\phi}^R(0)} | \emptyset} = u^{-\alpha^2}.$$  

Since as a differential operator in momentum space $L = u \partial_u$, the Lorentz spin of these vacuum expectation values is $\alpha^2$, $-\alpha^2$ respectively, and attributing $L = \pm \alpha^2/2$ to each operator yields (6.5).

The above construction provides an exact bosonization of the $b^\pm(u)$, $\overline{b}^\pm(u)$ operators in the anti-periodic sector, if one identifies

$$b^+(u) = \sqrt{\frac{\pi}{u}} : e^{i\hat{\phi}^L(u)} :, \quad b^-(u) = \sqrt{\frac{\pi}{u}} : e^{-i\hat{\phi}^L(u)} :$$

$$\overline{b}^+(u) = \sqrt{\frac{\pi}{u}} : e^{-i\hat{\phi}^R(u)} :, \quad \overline{b}^-(u) = -\sqrt{\frac{\pi}{u}} : e^{i\hat{\phi}^R(u)} :.$$  

One can easily check that this reproduces the 2-point functions (1.7).

The above above operator formalism can easily be used to compute the multiparticle form factors. Following the notation of section 4, the multiparticle form factors of the SG fields $\exp(\pm i\phi/2)$ are defined as follows:

$$f_{\pm \frac{1}{2}}(u_1, \ldots, u_{2n}) = \braket{0 | e^{\pm i\phi(0)/2} | u_1; \uparrow \ldots u_n; \downarrow u_{n+1} \ldots \downarrow u_{2n}}.$$  

As explained in section 4, in radial quantization these are computed as follows:

$$f_{\pm \frac{1}{2}}(u_1, \ldots, u_{2n}) = \frac{1}{(4\pi^2 i)^n} L \braket{\pm \frac{1}{2} | b^+(u_1) \ldots b^+(u_n) b^-(u_{n+1}) \ldots b^-(u_{2n}) | \pm \frac{1}{2}}_L$$

$$= \frac{1}{(4\pi^2 i)^n} R \braket{\pm \frac{1}{2} | \overline{b}^+(u_1) \ldots \overline{b}^+(u_n) \overline{b}^-(u_{n+1}) \ldots \overline{b}^-(u_{2n}) | \pm \frac{1}{2}}_R.$$
Using the above bosonized expressions, one finds

\[
 f_{\pm}^{\pm}(u_1, \ldots, u_{2n}) = \frac{1}{(4\pi i)^n} \sqrt{u_1 \cdots u_{2n}} \left( \prod_{i=1}^{n} \left( \frac{u_{i+n}}{u_i} \right)^{\pm 1/2} \right) \left( \prod_{i<j \leq n} (u_i - u_j) \right) \prod_{n+1 \leq i<j} (u_i - u_j) \prod_{r=1}^{n} \prod_{s=n+1}^{2n} \frac{1}{u_r + u_s}.
\]

Again these expressions agree with the known results, though they were originally computed using rather different methods [25], [21].

7. Concluding Remarks

As explained in the introduction, what began as the simple exercise of understanding the \( q \to 1 \) limit of the \( q - \hat{sl}(2) \) symmetry of the SG theory revealed a rich, previously unknown structure. By developing the role of undeformed affine Lie algebras in massive field theory, we have identified the proper structures that need to be deformed to understand massive integrable quantum field theories away from their free points. Though there are many important differences between the application of affine Lie algebras to massive field theory and conformal current algebra, many techniques developed in the context of conformal field theory are useful in momentum space for the massive theories. For example, we have shown how form-factors have virtually the same structure as conformal correlation functions.

We have shown how the field states \( \exp(\pm i\phi/2)|0\rangle \) are highest weight for the level 1 split \( \hat{sl}(2) \) symmetry. Whether it is possible to use the affine Lie algebra to provide a non-perturbative characterization of the space-time correlation functions for the fields \( \exp(\pm i\phi/2) \) is the most important question left open by our work.

Before discussing the extension of these results to \( q \neq 1 \), we remark that there are many other interesting extensions of this work that still involve only the undeformed affine Lie algebras since many other field theories are known to have \( q - \hat{sl}(2) \) symmetry and
they all generally have points in the coupling where \( q = 1 \). We have seen that the free fermion point of the SG theory is characterized by level 1 affine Lie algebra. The algebraic structures we have introduced can be formally extended to higher integer level \( k \), and the question arises as to what physical models are described by these structures. We believe the answer to this question is provided by the \( k \)-th fractional supersymmetric SG theory, defined and studied in [7]. This series of models corresponds to the SG model \( (k = 1) \), super SG \( (k = 2) \), and in general a system of \( Z_k \) parafermions interacting with a boson. These models also have points in the coupling constant where \( q = 1 \). However, whereas at \( k = 1 \) the theory can be mapped to a free one at this \( q \), for other \( k \) this is not the case: inspection of the S-matrices in [7] shows that the RSOS factor of the S-matrix remains non-trivial at \( q = 1 \). Consider also the generalization of our results to other groups. The \( \hat{G} \)-affine Toda theories have a \( q - \hat{G} \) symmetry[7], and also have points where \( q = 1 \). Again inspection of the S-matrices reveals that these are not free field theories, as the S-matrices of the multiplets of solitons reduces to the minimal solutions (the \( q - \hat{G} \) R-matrix pieces of the S-matrix are equal to the identity). This discussion leads us to define the notion of the massive \( \hat{G}_k \) \( q \)-free quantum field theory, which is completely characterized by a level \( k \) \( \hat{G} \) symmetry. Only the \( \widehat{sl}(2)_1 \) theory is genuinely free. The terminology ‘\( q \)-free’ refers to the fact that the \( R \)-matrix factors in the S-matrix which are characterized by \( q - \hat{G} \) symmetry are equal to the identity when \( q = 1 \).

Since we have developed the field theory using algebraic structures, it is relatively clear how to proceed to other values of the SG coupling \( \hat{\beta} \), or beyond \( q = 1 \), since the algebraic structures have mathematically well-defined \( q \)-deformations. Though many details need to be explicitly worked out, the strategy is well-defined. Here we outline the general aspects of this scheme that are evident generalizations of the above results. The analogs of the \( P_n \)’s are in principle known at all points in the SG theory. Though in [7] the \( q - \widehat{sl}(2) \) generators were only constructed for the simple roots, the remaining generators

\[14 \text{ Using the definitions in [7], this occurs at } \beta^2/8\pi = 1/(k(k+1)).\]
undoubtedly exist, so that the full set of \( \alpha_n \)'s exist. At the free fermion point, we were able to construct all of these charges in position space, but it is probably not necessary to do this in general. A momentum space construction of the higher integrals of motion is likely to be sufficient. The generalization of the operators \( \hat{b}^\pm (u) \) are evidently the Zamolodchikov-Faddeev operators \( \hat{Z}^\pm (u) \) which create asymptotic particles and satisfy S-matrix exchange relations. Under radial quantization these give rise to operators \( Z^\pm (u), \overline{Z}^\pm (u) \), which are the analog of the operators \( b^\pm (u), \overline{b}^\pm (u) \). Form-factors should be computable as vacuum expectation values of the \( Z(u) \) operators between level 1 highest weight states of \( q - \hat{sl}(2) \), and here one can borrow results from [9], after translating them to the principal gradation.

It should be possible to use the \( \alpha_n \)'s to construct a momentum space bosonization as we have done at \( q = 1 \). Vertex operator representations of quantum affine algebras have been developed extensively recently. Our work attributes a physical meaning to the ‘free bosons’ in these constructions as coming from the quasi-chirally split integrals of motion \( \alpha_n^{L,R} \) in a momentum space bosonization scheme.

Some results presented in this paper are reminiscent of some recent results in the context of integrable lattice models [26]. As shown there, in infinite volume, the finite quantum group symmetry of the lattice hamiltonian is extended to an affine quantum group. Exact affine quantum group symmetries can also exist in certain finite size chains [27]. The quantum affine symmetry generators act on each lattice site via a finite dimensional level 0 representation. It was conjectured in [26] that the Hilbert space on lattice sites could be replaced by a level 1 quantum affine representation tensored with a level \(-1\) representation.

It is very likely that our results are the natural continuum version of these lattice constructions, though a direct link has not been established since our construction is completely independent. We mention some obvious parallels, and some distinctions. Note that if in (5.7) one had replaced \( Q_n^\pm, \alpha_n^R \) with \( Q_n^\pm, \alpha_n^R \), then one would have obtained a level \(-1\) algebra in the right sector. In [26] the level \( \pm 1 \) modules arise when one considers the lattice Hilbert space on two semi-infinite lines, one extending to the left the other to the
right. The separation of these two spaces defines a distinguished point at the origin. In continuum radial quantization, \( r = 0 \) is a distinguished point, and the coordinate \( r \) defines a semi-infinite line. In [26] the lattice correlation functions algebraically have the structure of continuum form-factors, and they are expressed as traces over quantum affine modules. This is in contrast to the situation in our approach, where form-factors are not traces but instead vacuum expectation values. The work of [26] finds its origin in the corner transfer matrix methods, which is a lattice analog of constructing eigenstates of the Lorentz boost operator. The corner transfer matrix of the 8-vertex model at the free-fermion point was studied in [22], and this can probably be used to relate our work with the results in [26].

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Note added: After this work was completed, we learned from G. Sotkov that the algebra (2.29) is isomorphic to certain algebras of conserved charges found in [28] for the case of \( O(2) \) invariant free massive fermion theories.

Appendix A. Twisted Affine Characters

We review the aspects of affine Lie algebras needed to compute the twisted characters (5.33) [19].

In the basis \( j_n^\alpha \) for \( \widehat{sl}(2) \) (2.33), the Cartan subalgebra consists of \( j_0^0, d \) and \( k \), thus the roots are labeled by 3 numbers. A basis of roots is provided by \( \alpha = (\alpha', k, n), \delta = (0, 0, 1), \)
and \( \Lambda_0 = (0,1,0) \), where \( \alpha’ \) is the root of \( sl(2) \) with \( \alpha'^2 = 2 \). The weights \( \lambda \) may be expressed as \( \lambda = (\lambda’, k, n) \), with \( \lambda’ \) a weight of \( sl(2) \). A basis for \( \lambda’ \) is \( \lambda'_j = j\alpha’, j \in \mathbb{Z}/2 \).

Define the inner product of two roots as

\[
((\alpha’, k, n), (\beta’, l, m)) = (\alpha’, \beta’) + km + nl. \tag{A.1}
\]

The simple roots are \( \alpha_1 = (\alpha’, 0, 0) \) and \( \alpha_0 = (-\alpha’, 0, 1) \).

Highest weights \( \Lambda \) are characterized by \( (\Lambda, \alpha_i) = n_i \geq 0, \ i = 0, 1 \). The fundamental weights are \( \Lambda_{1/2} = (\lambda'_{1/2}, 1, 0) \) and \( \Lambda_0 \). The highest weight modules \( \hat{V}_{\Lambda} \) are defined as

\[
\hat{V}_{\Lambda} = \{ j_{-n_1} \cdots j_{-m_1} \cdots j_{-r_1} \cdots |\Lambda| \}, \quad n_i, m_i \geq 1, r_i \geq 0. \tag{A.2}
\]

Let \([\Lambda]\) denote the set of weights in \( \hat{V}_{\Lambda} \), and define \( \dim \hat{V}_{\Lambda} \) as the multiplicity of the weight \( \lambda \). Then the characters are defined as

\[
ch_{\hat{V}_{\Lambda}} = \sum_{\lambda \in [\Lambda]} \dim \hat{V}_{\Lambda} e^{\lambda}. \tag{A.3}
\]

Let

\[
e^{\lambda}(\hat{\alpha}) = e^{(\lambda, \hat{\alpha})}, \tag{A.4}
\]

where \( \hat{\alpha} \) parameterizes an arbitrary root:

\[
\hat{\alpha} = -2\pi i (z\alpha' + \tau \Lambda_0 + \mu \delta). \tag{A.5}
\]

Then the evaluated characters are defined as

\[
ch_{\hat{V}_{\Lambda}}(z, \tau, \mu) \equiv ch_{\hat{V}_{\Lambda}}(\hat{\alpha}). \tag{A.6}
\]

For level \( k = 1 \), the integrable highest weight states are \( \Lambda_0, \Lambda_{1/2} \), and the characters are explicitly given by the formula

\[
ch_{\hat{V}_{\Lambda_j}}(z, \tau, \mu) = e^{-2\pi i \mu} \sum_{n \in \mathbb{Z}} \frac{q^{n(n+2j)} e^{-4\pi i z(n+j)}}{\prod_{m \geq 1} (1 - q^n)}, \tag{A.7}
\]
where
\[ q \equiv e^{2\pi i \tau}. \]

In order to relate this result to the traces used in section 5, note that
\[ \text{ch}_{\hat{V}_\Lambda}(z, \tau, \mu = 0) = \text{Tr}_{\hat{V}_\Lambda} \left( q^{-d} e^{-2\pi i z} \right). \quad (A.8) \]

Recalling that in the twisted representation (2.37) \( d' = 2d + j_0^0 / 2 \), one finds
\[ \hat{\chi}^{tw}_j \equiv \text{Tr}_{\hat{V}_{\Lambda j}} \left( q^{-d'} \right) = \text{ch}_{\hat{V}_{\Lambda j}} (z = \tau / 2, \tau^2, \mu = 0). \quad (A.9) \]

It is useful to derive a different expression for the same twisted characters. The characters \( \hat{\chi}^{tw}_j \) are precisely the so-called principally specialized characters. The derivation \( d' \) satisfies \( (\alpha_i, d') = 1 \) for \( \alpha_i \) a simple root. Define
\[ F_1(e^\Lambda) = q^{(\Lambda, d')} \quad (A.10) \]

Then
\[ \text{Tr}_{\hat{V}_\Lambda} \left( q^{-d'} \right) = F_1 \left( e^\Lambda \text{ch}_{\hat{V}_\Lambda} \right). \quad (A.11) \]

For level 1 these were computed in [19]:
\[ \hat{\chi}^{tw}_0 = \hat{\chi}^{tw}_{1/2} = \prod_{n \geq 1} \frac{1}{1 - q^{2n-1}}. \quad (A.12) \]
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