Could dark matter be a natural consequence of a dynamical universe?

Zhi-Wei Wang and $^{a,b}$ Samuel L. Braunstein$^b$

$^a$College of Physics, Jilin University, Changchun, 130012, People’s Republic of China

$^b$Computer Science, University of York, York YO10 5GH, United Kingdom

E-mail: zhiweiwang.phy@gmail.com, sam.braunstein@york.ac.uk

ABSTRACT: We construct the gravitating mass of an isolated composite system on asymptotically-flat spacetimes within conventional general relativity and investigate when this quantity is well defined. For stationary spacetimes, this quantity is known to exactly equal the physical (ADM) mass. However, it remains an open question whether these two masses are equal in the absence of a timelike Killing vector. This is especially apropos since our universe has an ‘origin’ and hence no such Killing vector. Further, if these masses failed to agree then composite systems could behave as if they had a ‘dark component,’ whose gravitating mass would not equal the physical mass-energy present. The existence of such an apparent discrepancy is indeed ubiquitous in galaxies and galaxy clusters, though currently it is attributed to the presence of dark matter. We conclude that the theoretical question of the relation between these masses for dynamical spacetimes is ripe for attention.

KEYWORDS: Dark matter, Classical Theories of Gravity, Differential and Algebraic Geometry, Space-Time Symmetries
1 Introduction

Our understanding of general relativity has been largely predicated on the behavior of stationary spacetimes. Such spacetimes admit a timelike Killing vector whose algebraic properties allow for significant simplifications. For example, since 1978 it has been known that the Komar mass and the physical (ADM) mass are equal for stationary spacetimes [1]. Indeed, for such spacetimes the Komar mass may be interpreted as the mass producing gravitational attraction assuming an asymptotically Newtonian inverse-square law. Thus, for more than forty years, it has been taken as fact in general relativistic systems, that the gravitating mass and the physical mass are identical. But does this result really hold for dynamical spacetimes? Indeed, in the absence of a timelike Killing vector the Komar mass is not even considered a legitimate well-defined measure of mass for dynamical spacetimes [2]. However, the untested assumption of the equality between the gravitating and physical masses for such generally dynamical composite systems lies at the heart of the dark matter paradigm.

Here we begin to address this foundational issue by formulating the gravitating mass for arbitrary systems in generally dynamical spacetimes. We show that this quantity is in fact well-defined for a broad class of asymptotically-flat dynamical spacetimes (i.e., where no Killing vector exists). This analysis therefore allows us to precisely formulate for the first time the question of whether conventional general relativity really does predict the equivalence between gravitating mass and physical mass-energy for dynamical composite systems.

2 Gravitating mass for dynamical spacetimes

Consider an arbitrary smooth foliation of spacetime into a family of spacelike hypersurfaces. Such a foliation singles out a unique unit timelike covector field $\hat{T}_\mu$ whose kernel is the tangent space on each hypersurface [3]. These covectors are dual to a vector field $\hat{T}^\mu$ which may be interpreted as 4-velocities of a uniquely specified family of canonical observers with regard to that family of hypersurfaces. We will use the 4-acceleration

$$a^\mu = \hat{T}^\nu \hat{T}_\nu, \quad (2.1)$$
of these canonical observers to ‘measure’ the gravitating mass-energy within a closed surface, \( S \), on any specific hypersurface \( \Sigma \) within the family, by

\[
M_{\text{Grav}} = \frac{1}{4\pi} \int_S \mathcal{N} \hat{a}^\mu \hat{N}_\mu dA = \frac{1}{4\pi} \int_S \mathcal{N} \hat{T}^\mu_{\nu,\mu} \hat{T}^\nu_{\mu} \hat{N}_\mu dA.
\]  
(2.2)

In the stationary setting, Eq. (2.2) is called the Komar mass [4]. Here, the inclusion of the lapse function, \( \mathcal{N} \), is to guarantee that energies/forces are quantified with regard to the values that would be seen at spatial infinity. In addition, here \( dA \) is the area element on \( S \) and \( \hat{N}_\mu \) is the outward pointing unit normal to \( S \) tangent to the hypersurface \( \Sigma \). For example, for the Schwarzschild metric of a black hole of mass \( M \), this expression Eq. (2.2) yields

\[
M_{\text{Grav}} = M,
\]  
(2.3)

for any \( S \) containing the black hole.

Of course, for \( M_{\text{Grav}} \) to be physically meaningful, we would require that the expression given in Eq. (2.2) be independent of the choice of surface \( S \). Let us now explore the implications of this constraint.

\[
\frac{1}{4\pi} \int_S \mathcal{N} \hat{T}^\mu_{\nu,\mu} \hat{T}^\nu_{\mu} \hat{N}_\mu dA
\]
\[
= \frac{1}{4\pi} \int_S (\mathcal{N} \hat{T}^{[\mu,\nu]} \hat{N}_\mu \hat{T}_\nu dA
\]
\[
= \frac{1}{4\pi} \int_{S'} \mathcal{N} \hat{T}^\mu_{\nu,\mu} \hat{T}^\nu_{\mu} \hat{N}_\mu dA - \frac{1}{4\pi} \int_{\Delta \Sigma} (\mathcal{N} \hat{T}^{[\mu,\nu]} \hat{N}_\mu \hat{T}_\nu dV;
\]  
(2.4)

where \( S' \) is a new closed surface on the hypersurface \( \Sigma \), \( \Delta \Sigma \) is a portion of the hypersurface bounded at either end by \( S \) and \( S' \) (see Fig. 1), and \( dV \) is the volume element on \( \Sigma \). We use Lemma 1 for the first step and Stokes’ theorem for the final step.

**Lemma 1:** For a spacelike hypersurface \( \Sigma \) with tangent vector \( \hat{N}_\mu \) and unit normal \( \hat{T}^\mu \) we have

\[
\mathcal{N} \hat{T}^\mu_{\nu,\mu} \hat{T}^\nu_{\mu} \hat{N}_\mu = (\mathcal{N} \hat{T}^{[\mu,\nu]} \hat{N}_\mu \hat{T}_\nu
\]  
(2.5)

where \( \mathcal{N} \) is the lapse function.

**Proof:** The foliation structure guarantees that our covector takes the form \( \hat{T}_\mu \propto t_\mu \), where \( t \) is a function on the spacetime that labels individual members of the foliation. Thus, \( t \) corresponds to a ‘time’ coordinate [5]. Normalizing \( \hat{T}_\mu \), and orienting it to be future directed, then gives \( \hat{T}_\mu = -\mathcal{N} t_\mu = -\mathcal{N} t_{\nu,\mu} \) and hence

\[
\hat{T}_{\mu,\nu} = -(\mathcal{N} t_{\nu,\mu}) = -\mathcal{N}_\nu t_{\mu,\nu} - \mathcal{N} t_{\nu,\mu}
\]
\[
= \mathcal{N} \hat{T}_\mu - \mathcal{N} t_{\nu,\mu}
\]
\[
= \frac{1}{\mathcal{N}} \mathcal{N}_\mu \hat{T}_\mu + \mathcal{N}(\frac{1}{\mathcal{N}} \hat{T}_\nu)_{\mu}
\]
\[
= \frac{1}{\mathcal{N}} \mathcal{N}_\mu \hat{T}_\mu - \frac{1}{\mathcal{N}} \mathcal{N}_\nu \hat{T}_\nu + \hat{T}_{\nu,\mu},
\]  
(2.6)
For $M_{\text{Grav}}$ to be well-defined, it must be independent of the choice of boundary $S$. Here the portion of hypersurface $\Delta \Sigma$ is bounded by $S$ and $S'$.

where since $t$ is a scalar we have used $t_{\mu\nu} = t_{\nu\mu}$ in the first line. Note that $\hat{T}_{[\mu,\nu]} = \hat{T}_{[\mu,\nu]}/\hat{N}$ follows directly from Eq. (2.6); a form which is consistent with the existence of our foliation [3, 5, 6].

From Eq. (2.6), the 4-acceleration $a_{\mu}$ may be written as

$$a_{\mu} \equiv \hat{T}_{\mu,\nu} \hat{T}^{\nu} = \frac{1}{\hat{N}} \hat{N}_{\nu} \hat{T}^{\nu} \hat{T}_{\mu} - \frac{1}{\hat{N}} \hat{N}_{\mu} \hat{T}^{\nu} \hat{T}_{\nu} + \hat{T}_{\nu,\mu} \hat{T}^{\nu} = \frac{1}{\hat{N}} \hat{N}_{\nu} \hat{T}^{\nu} \hat{T}_{\mu} + \frac{1}{\hat{N}} \hat{N}_{\mu} = \frac{1}{\hat{N}} \hat{N}_{\mu} h_{\nu}^{\nu},$$

(2.7)

where we have used $\hat{T}_{\nu,\mu} \hat{T}^{\nu} = 0$ in the second line and $h_{\mu\nu} \equiv g_{\mu\nu} + \hat{T}_{\mu} \hat{T}_{\nu}$ in the last step.

Finally, from Eq. (2.7), $(N \hat{T}_{[\mu,\nu]} \hat{T}^{\nu} \hat{N}^{\mu})$ may be simplified as

$$\begin{align*}
(N \hat{T}_{[\mu,\nu]} \hat{T}^{\nu} \hat{N}^{\mu}) &= \frac{1}{2} (N \hat{T}_{\mu,\nu} \hat{T}^{\nu} \hat{N}^{\mu} - N \hat{T}_{\nu,\mu} \hat{T}^{\nu} \hat{N}^{\mu}) \\
&= \frac{1}{2} N \hat{N}_{\nu} \hat{T}^{\nu} \hat{T}_{\mu} + \frac{1}{2} N \hat{T}_{\mu} \hat{T}^{\nu} \hat{N}^{\mu} - \frac{1}{2} N \hat{N}_{\mu} \hat{T}^{\nu} \hat{N}^{\mu} - \frac{1}{2} N \hat{T}_{\nu,\mu} \hat{T}^{\nu} \hat{N}^{\mu} \\
&= \frac{1}{2} N \hat{T}_{\mu,\nu} \hat{T}^{\nu} \hat{N}^{\mu} - \frac{1}{2} N \hat{N}_{\mu} \hat{T}^{\nu} \hat{N}^{\mu} \\
&= N \hat{T}_{\mu,\nu} \hat{T}^{\nu} \hat{N}^{\mu},
\end{align*}$$

(2.8)

where we have used $\hat{T}_{\nu,\mu} \hat{T}^{\nu} = 0$ in the third line, and Eq. (2.7) in the second-last line.

This completes the proof of Lemma 1.

From Eq. (2.4) we know that if we want $M_{\text{Grav}}$ to be independent of the choice of $S$, we require that the volume integral between these two boundaries must vanish, i.e.,

$$\frac{1}{4\pi} \int_{\Delta \Sigma} (N \hat{T}^{[\mu]} \hat{T}^{\nu})_{\mu} \hat{T}_{\nu} dV = 0.$$

(2.9)
To simplify this condition, we use:

**Lemma 2:** For a vector field $\xi^\mu$ (although $\xi^\mu = N\hat{T}^\mu$ is used for this paper), we have

$$
\xi_{[\beta;\alpha]\nu} + \frac{1}{2}(\mathcal{L}_\xi g_{\nu\alpha})_{;\beta} - \frac{1}{2}(\mathcal{L}_\xi g_{\beta\nu})_{;\alpha} = R_{\mu\nu\alpha\beta}\xi^\mu, \tag{2.10}
$$

where $\mathcal{L}_\xi$ is the Lie derivative along $\xi^\mu$.

**Proof:** Since $\xi_{\nu;\alpha\beta} - \xi_{\nu;\beta\alpha} = R_{\mu\nu\alpha\beta}\xi^\mu$, the left-hand-side of Eq. (2.10) may be simplified as

$$
= \xi_{[\beta;\alpha]\nu} + \xi_{(\nu\alpha);\beta} - \xi_{(\beta\nu);\alpha}
= \frac{1}{2}\left(\xi_{\beta;\alpha\nu} - \xi_{\alpha;\beta\nu} + \xi_{\nu;\alpha\beta} + \xi_{\alpha;\nu\beta} - \xi_{\beta;\nu\alpha} - \xi_{\nu;\beta\alpha}\right)
= \frac{1}{2}\left(\xi_{\beta;\alpha\nu} - \xi_{\beta;\nu\alpha}\right) + \frac{1}{2}\left(\xi_{\nu;\alpha\beta} - \xi_{\nu;\beta\alpha}\right) + \frac{1}{2}\left(\xi_{\alpha;\nu\beta} - \xi_{\alpha;\beta\nu}\right)
= \frac{1}{2}\left(R_{\mu\beta\alpha\nu}\xi^\mu + R_{\mu\nu\alpha\beta}\xi^\mu + R_{\mu\nu\beta\alpha}\xi^\mu\right)
= \frac{1}{2}\left(R_{\mu\nu\alpha\beta}\xi^\mu - R_{\mu\nu\beta\alpha}\xi^\mu\right) = R_{\mu\nu\alpha\beta}\xi^\mu, \tag{2.11}
$$

where the Bianchi identity, $R_{\mu\alpha\nu\beta} + R_{\mu\nu\beta\alpha} + R_{\mu\beta\alpha\nu} = 0$, is used in the fourth line to obtain the fifth line.

This completes the proof of Lemma 2.

Contracting $\nu$ with $\beta$ in Eq. (2.10) and replacing $\alpha$ by $\nu$ yields

$$
\xi_{[\mu;\nu]}^{\nu} = R_{\mu\nu}\xi^{\mu} - \frac{1}{2}(\mathcal{L}_\xi g_{\mu\nu})^{\mu} + \frac{1}{2}(\mathcal{L}_\xi g_{\alpha\beta};\nu)g^{\alpha\beta}. \tag{2.12}
$$

**Corollary:** When $\mathcal{N}\hat{T}^\mu$ is a Killing vector (and by construction, orthogonal to the hypersurface containing the boundary $S$) then $M_{\text{Grav}}$ is independent of variations of $S$ through any matter-free region of space.

This follows straightforwardly by taking $\xi^\mu = \mathcal{N}\hat{T}^\mu$, since from Eq. (2.12) the conditions of the corollary immediately imply that the condition in Eq. (2.9) is exactly satisfied (since all Lie derivatives, $\mathcal{L}_\xi$, vanish when $\xi^\mu$ is Killing). Indeed, this is the standard text-book result for stationary spacetimes [2]. To go beyond this for dynamical spacetimes we make use of:

**Lemma 3:** For $\xi^\mu = \mathcal{N}\hat{T}^\mu$ we have

$$
\frac{1}{2}\left[(\mathcal{L}_\xi g_{\alpha\beta};\nu)g^{\alpha\beta} - (\mathcal{L}_\xi g_{\mu\nu})^{\mu}\right] \hat{T}^\nu = \mathcal{N} (K_{\nu}\hat{T}^\nu + K_{\mu}\hat{T}^\mu), \tag{2.13}
$$

where $K_{\mu\nu} = h^{\alpha}_{\mu}h_{\nu,\beta}\hat{T}^{(\alpha;\beta)}$ is the extrinsic curvature and $K = g^{\mu\nu}K_{\mu\nu} = h^{\mu\nu}K_{\mu\nu} = \hat{T}^{\mu}_{\mu}$ is its trace.
Proof: Taking $\xi^\mu = N\hat{T}^\mu$ we find the first term of the Lemma may be rewritten as

\[
\frac{1}{2}(\xi^\mu g_{\alpha\beta} \mu g^{\alpha\beta} = (N\hat{T}_{(\alpha)};\beta)_{\nu} g^{\alpha\beta}
= (N_{\alpha} \hat{T}_\beta + N\hat{T}_{(\alpha)} g^{\alpha\beta})_{\nu}
= (N_{\alpha} \hat{T}_\alpha + N(-\hat{T}^\alpha \hat{T}_\beta + h^{\alpha\beta})_{(\alpha)};\beta)_{\nu}
= (N_{\mu} \hat{T}^\mu + NK)_{\nu},
\]
(2.14)

In a similar manner we find

\[
\frac{1}{2}(\xi^\mu g_{\alpha\mu})^\nu = -(N\hat{T}_{(\mu)};\nu)^\mu
= (-N\hat{T}_{(\mu)} - N_{(\mu)} \hat{T}_{\nu})^\mu
= (-N\hat{T}_{(\alpha)} (-\hat{T}^\alpha \hat{T}_\mu + h^{\alpha\mu}(-\hat{T}^\beta \hat{T}_\nu + h^{\beta\nu}) - N_{(\mu)} \hat{T}_{\nu})^\mu
= (N\hat{T}_{(\mu) a_\nu}) - N_{(\mu)} \hat{T}_{\nu} - NK_{\mu
\nu})^\mu
= (N\hat{T}_{(\mu) a_\nu}) - N(a_\nu \hat{T}_{\mu}) + N_{(\mu)} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu} - NK_{\mu
\nu})^\mu
= (N_{\lambda} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu} - NK_{\mu
\nu})^\mu,
\]
(2.15)

where we use $a_\mu \hat{T}^\mu = 0$ and $a_\mu h^{\mu\nu} = a_\nu$ to obtain the fourth line, and $N_{\mu
\nu} = N a_\mu - N_{\nu} \hat{T}^\nu \hat{T}_\mu$

from Eq. (2.7) to obtain the fifth line.

Now combining Eqs. (2.14) and (2.15), we have

\[
\frac{1}{2}[(\xi^\mu g_{\alpha\beta} \mu g^{\alpha\beta} - (\xi^\mu g_{\alpha\mu})^\nu] \hat{T}^\nu
= (N_{\mu} \hat{T}^\mu + NK)_{\nu} \hat{T}^\nu + (N_{\lambda} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu} - NK_{\mu
\nu})^\mu \hat{T}^\nu
= (N_{\mu} \hat{T}^\mu)_{\nu} \hat{T}^\nu + (NK)_{\nu} \hat{T}^\nu + (N_{\lambda} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu} + N_{\lambda} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu})^\mu \hat{T}^\nu - (NK_{\mu
\nu})^\mu \hat{T}^\nu
= (NK)_{\nu} \hat{T}^\nu + N_{\lambda} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu} - (NK_{\mu
\nu})^\mu \hat{T}^\nu
= (NK)_{\nu} \hat{T}^\nu - N_{\lambda} \hat{T}^{\alpha\beta} \hat{T}_{(\mu) \nu} - (NK_{\mu
\nu})^\mu \hat{T}^\nu
= NK_{\mu\nu} \hat{T}^\nu - NK_{\mu\nu}^\mu \hat{T}^\nu
= NK_{\mu\nu} \hat{T}^\nu - NK_{\mu\nu} \hat{T}^\nu + NK_{\mu\nu} \hat{T}^\nu
= NK_{\mu\nu} \hat{T}^\nu + NK_{\mu\nu} \hat{T}^\nu
\]
(2.16)

where we use $K = \hat{T}^\mu_{\mu}$ and $K_{\mu\nu} \hat{T}^\nu = 0$.

This completes the proof of Lemma 3. \qed

We may now state our main result:

**Theorem:** On asymptotically matter-free, $T_{\mu\nu} = o(1/r^3)$, asymptotically-flat dynamical spacetimes

\[
M_{\text{Grav}} \equiv \lim_{S \to 0} \frac{1}{4\pi} \int_{S} N \hat{T}^\mu_{\mu} \hat{T}^\nu \hat{N}_\mu dA,
\]
(2.17)

provides a well-defined measure of gravitating mass on the hypersurface within the closed surface $S$ as it is taken to spatial infinity, $i^0$, provided $K_{\mu\nu} \hat{T}^\nu = o(1/r^3)$. (Here, $r$ is the
asymptotic radial coordinate and $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r)$, $g_{\mu\nu,\sigma} = O(1/r^2)$, etc. Recall the little-o notation $f(r) = o(1/r^n)$ means $\lim_{r \to \infty} r^n f(r) = 0$.

**Proof:** In order to prove our theorem, we must show that the limit is independent of the choice of boundary, $S$, as it limits to spatial infinity. In order to do this, we need only show that volume term in Eq. (2.4) vanishes asymptotically. Using the integrand expressed as Eq. (2.12) and Lemma 3, we may write this volume term as

$$\frac{1}{4\pi} \int_{\Delta \Sigma} (N \hat{T}^{[\mu]}_{\mu} \hat{T}^{\nu}_{\nu} + R_{\mu\nu} \hat{T}^{\mu}_{\mu} + K_{\mu\nu} K^{\mu\nu}) dV.$$

For an asymptotically matter-free spacetime with $T_{\mu\nu} = o(1/r^3)$, we see that $R_{\mu\nu} = o(1/r^3)$ so this term will not contribute asymptotically. Further, $K_{\mu\nu} = O(1/r^2)$ so $K_{\mu\nu} K^{\mu\nu} = O(1/r^4)$ and this term too will make no asymptotic contribution. This only leaves the term $K_{\mu\nu} \hat{T}^{\mu}_{\mu}$. Ordinarily the asymptotically-flat constraint would imply that $K_{\mu\nu} = O(1/r^2)$ and so an asymptotic contribution would be possible from this remaining term. However, if we modestly strengthen this constraint to be $K_{\mu\nu} \hat{T}^{\mu}_{\mu} = o(1/r^3)$, the integral in Eq. (2.18) now vanishes asymptotically and consequently, Stokes’ theorem and Eq. (2.4) guarantee that $M_{\text{Grav}}$ is asymptotically independent of the choice of boundary as it is taken to spatial infinity.

This completes the proof for our theorem.

We emphasize that at no stage has the existence of a Killing vector of any kind been assumed to obtain this result. Further, our asymptotic conditions are only required to hold at spatial infinity, $i^0$. In particular, we place no constraints on the behavior at null infinity. Thus, our dynamical composite system may potentially be extremely violent on the interior of the spacelike hypersurface, $\Sigma$, for example, generating significant gravitational radiation.

Thus, we have shown that the gravitating mass is well-defined even for a broad class of asymptotically-flat dynamical spacetimes. Note that the gravitating mass, $M_{\text{Grav}}$, is evaluated on a closed surface $S$ at spatial infinity $i^0$ and is generally only well defined when evaluated there (just as is the case with the ADM mass). As already noted, the gravitating and ADM mass are known to be equal for stationary spacetimes, though their relationship is far less clear in the dynamical setting.

The mathematical form of $M_{\text{Grav}}$ in Eq. (2.17) is identical to that of the Bondi mass [7] except that the latter is evaluated on a null hypersurface at null infinity. When the metric under consideration may be reduced to Bondi form at null infinity the Bondi mass gives a measure of the energy contained in null (e.g., gravitational) radiation. Further, assuming such an asymptotic-Bondi form and in the absence of gravitational radiation, the infinite past limit of the Bondi mass is known to equal the ADM mass [8]. Thus, one might be tempted to take the position that it is likely that the gravitating mass evaluated on a spacelike hypersurface and the ADM mass are identical even in the dynamical setting. However, i) the gravitating mass may be well defined even when the Bondi mass is not (since the assumed form of the asymptotic metric at null infinity may not hold) and ii) the
relation between the Bondi mass and ADM mass is known to break down for dynamical spacetimes [8]. If anything, one might instead expect a ‘mass anomaly’ whereby for dynamical spacetimes the gravitating mass and ADM mass generally do not agree. We now explore potential implications for dark matter were such a mass anomaly to exist.

3 Possible implications for dark matter

The orbital motion of stars within their host galaxy and even the motion of galaxies within galaxy clusters cannot be understood without some extra source of gravitational attraction other than that due to the visible matter. This extra attraction is widely accepted to be caused by ‘dark matter,’ with the current prevailing model being that of cold dark matter [9].

However, the failure to directly detect the particles responsible for dark matter has led to alternative theories to explain these unexpected gravitational phenomena. The most successful of these is known as modified Newtonian dynamics (MOND) and its variations [10]. Such modifications to Newtonian dynamics are challenged by scenarios involving collisions between clusters of galaxies, such as in the Bullet Cluster [11]. More recent evidence suggests that any modifications to general relativity capable of explaining dark matter would be inconsistent with large-scale cosmological observations [12].

Here we suggest a third option is worth investigating: In particular we ask, within conventional general relativity, whether the signals for dark matter might be understood in terms of the normative behavior of dynamical composite systems. For dynamical spacetimes there is currently no proof that the gravitating mass of a composite system equals the amount of physical mass there. Although these two masses are known to be equal for stationary spacetimes, this result does not apply to our own expanding universe, where stationarity is ruled out. A breakdown of equality between these masses would signal a ‘mass anomaly’ which may partly explain the observed discrepant motion of stars and galaxies.

Unlike the physical (ADM) mass which is based on the Hamiltonian formalism, the concept of gravitating mass has previously been limited to stationary spacetimes where it reduces to the Komar mass [2]. The major obstacle to extending this work has been that the Komar mass requires a timelike Killing vector which is by definition absent in dynamical spacetimes. In the stationary setting, the timelike Killing vector field yields the 4-velocities for a family of ‘static’ observers, which may be used to determine the gravitating mass on hypersurfaces orthogonal to this Killing field [2]. Here we invert this reasoning and determine the gravitating mass measured by canonical observers moving orthogonally to a family of spacelike hypersurfaces. We find that even in the absence of a Killing field, this approach leads to a well-defined measure of gravitating mass even for a broad class of dynamical asymptotically-flat spacetimes. Thus, our work is the first to generalize this concept to dynamical spacetimes. Further, at least on the scale of galaxies, galaxy clusters and even super clusters, where the signature of dark matter is unmistakable, the universe may be well approximated as flat [13]. The remaining open question is whether or not, for such dynamical spacetimes, the gravitating and physical masses are in agreement.
References

[1] Beig, R., Arnowitt-Deser-Misner energy and $g_{00}$, Physics Letters A, 69(3), 153-155 (1978).

[2] Wald, R. M., General Relativity (Univ. Chicago Press, Chicago & London, 1984) p.288, p.293, p.336, & p.432

[3] Frobenius, G., Über das Pfaffsche Problem, Journal für die Reine und Angewandte Mathematik, 1877(82), 230-315 (1877).

[4] Komar, A., Covariant conservation laws in general relativity, Physical Review 113, 934-936 (1959).

[5] Natário, J., Mathematical Relativity, arXiv:2003.02855v1 p.153.

[6] Poisson, E., A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics (Cambridge Univ. Press, Cambridge, 2004), p.38.

[7] Tamburino, L. A., and J. H. Winicour, Gravitational fields in finite and conformal Bondi frames, Physical Review 150, 1039-1053 (1966).

[8] Zhang, X., On the relation between ADM and Bondi energy–momenta, Advances in Theoretical and Mathematical Physics 10 261-282 (2006).

[9] Bertone G, Hooper D., History of dark matter. Reviews of Modern Physics 90(4) 045002 (2018).

[10] Milgrom M., A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. The Astrophysical Journal 270, 365-370 (1983).

[11] Markevitch, M., et al., Direct constraints on the dark matter self-interaction cross-section from the merging galaxy cluster 1E0657-56, The Astrophysical Journal 606, 819–824 (2004).

[12] Pardo, K. & Spergel, D. N., What is the price of abandoning dark matter? Cosmological constraints on alternative gravity theories Physical Review Letters, to appear.

[13] Dawson, K. S., et al., The baryon oscillation spectroscopy survey of SDSS-III, The Astrophysical Journal 145, 1-41 (2013).