We address two-player combinatorial games whose graph of positions is a directed Galton–Watson tree. We consider normal and misère rules (where a player who cannot move loses or wins, respectively), as well as an “escape game” in which one designated player loses if either player cannot move. We study phase transitions for the probability of a draw or escape under optimal play, as the offspring distribution varies. Across a range of natural cases, we find that the transitions are continuous for the normal and misère games but discontinuous for the escape game; we also exhibit examples where these properties fail to hold. We connect the nature of the phase transitions to the length of the game under optimal play. We establish inequalities between the different games. For instance, the draw probability is no smaller in the misère game than in the normal game.

KEYWORDS
Branching process, combinatorial game, phase transition, random game

1 INTRODUCTION

Game theory naturally often focuses on carefully chosen games for which interesting mathematical analysis is possible. What can be said about games in the wild? One approach to this question is to consider games whose rules are typical, that is, chosen at random, although known to the players. In this article, we consider rules arising from random trees.

We consider combinatorial games whose positions and moves are described by a directed acyclic graph $\mathcal{G}$. A token is located at some vertex of $\mathcal{G}$, and the two players take turns to move it along a
directed edge to a new vertex. In the **normal game**, a player **loses** the game if they cannot move (i.e., if the token is at a vertex with out-degree zero), and the other player **wins**.

We are interested in optimal play. Thus, a **strategy** for a particular player is a map that assigns a legal move for that player (where one exists) to every vertex. For a given starting vertex for the token, a strategy is **winning** if it yields a win for that player, no matter what strategy the other player uses. Fix a starting vertex. If \( G \) is finite, then it is easily seen that exactly one player has a winning strategy; we then say that the game is a **win** for that player (and a **loss** for the other). More interestingly, if \( G \) is infinite, then it is possible that neither player has a winning strategy, in which case we say that the game is a **draw**.

We also consider two other rules for determining the game outcome. In the **miseré game**, a player **wins** if they cannot move. In the **escape game**, the two players have distinct goals. One designated player, called **Stopper**, wins if either player is ever unable to move, in which case the other player, **Escaper**, loses. If Stopper has no winning strategy then the game is said to be a win for Escaper.

In a sense there is no loss of generality in assuming that \( G \) is a directed tree: if not, every game position may be augmented with a record of the sequence of moves that led to it; these augmented positions then form a tree.

We focus on Galton–Watson trees. Thus, consider a Galton–Watson branching process with offspring probability mass function \( p = (p_0, p_1, \ldots) \), and let \( G = T \) be the graph whose vertices are the individuals of the process, and with directed edges from parents to children. We emphasize that although the graph is random, it is assumed known to be known to both players when they decide on their strategies.

Let \( N = N(p) \) be the probability that the normal game is a win for the first player (\( N \) stands for “Next”), let \( P = P(p) \) be the probability that it is a win for the second player (\( P \) stands for “Previous”), and let \( D = 1 - N - P \) be the probability that it is a draw. Let \( \tilde{N}, \tilde{P}, \tilde{D} \) be the analogous probabilities for the misère game. For the escape game, let \( S^{(1)} \) (respectively, \( S^{(2)} \)) be the probability that the stopper wins assuming the stopper has the first (respectively, second) move. Similarly let \( E^{(1)} = 1 - S^{(2)} \) and \( E^{(2)} = 1 - S^{(1)} \) be the win probabilities for the escaper when moving first or second, respectively.

It is well known that the Galton–Watson process exhibits a phase transition: the process survives (i.e., \( T \) is infinite) with positive probability if and only if \( \mu > 1 \) (or \( p_1 = 1 \)), where \( \mu := \sum_i p_i \) is the mean of the offspring distribution. However, survival is not sufficient for the existence of a draw—intuitively, that requires not just an infinite path, but an infinite path that neither player can profitably deviate from. Indeed, we will find that the draw and escape probabilities \( D, \tilde{D}, E^{(1)}, E^{(2)} \) undergo phase transitions as \( p \) is varied, but typically not at the same location as the survival phase transition.

The model can be analyzed in terms of generating functions. Let \( G(x) = G_p(x) := \sum_{i=0}^{\infty} p_i x^i \) be the generating function of the offspring distribution. It is also convenient to define the functions \( F = 1 - G \) and \( H = 1 - G + p_0 \). We denote iterates of functions by superscripts: \( F^2(x) = (F \circ F)(x) = F(F(x)) \), and so on. Let \( \text{FP}(f) = \text{FP}_{[0,1]}(f) := \{ x \in [0,1] : f(x) = x \} \) denote the set of fixed points of a function \( f \) in the interval \( [0,1] \).

**Theorem 1** (Fixed points). For the normal, misère, and escape games played on a Galton–Watson tree with offspring distribution \( p \), we have:

(i) \( D = \max \text{FP}(F^2) - \min \text{FP}(F^2) \); \( N = \min \text{FP}(F^2) \); \( P = 1 - \max \text{FP}(F^2) \);

(ii) \( \tilde{D} = \max \text{FP}(H^2) - \min \text{FP}(H^2) \); \( \tilde{N} = \min \text{FP}(H^2) \); \( \tilde{P} = 1 - \max \text{FP}(H^2) \);

(iii) \( E^{(1)} = \max \text{FP}(F \circ H) \); \( E^{(2)} = 1 - \min \text{FP}(H \circ F) \).

Note for instance that \( D > 0 \) if and only if \( F^2 \) has multiple fixed points in \( [0,1] \).
Next we examine how the three games are related to each other. It turns out that several inequalities hold. Some are obvious, others more surprising. In the following, $a, b \leq c$ means that $a \leq c$ and $b \leq c$.

**Theorem 2** (Inequalities). *For a Galton–Watson process with any fixed offspring distribution, we have:*

(i) $N, \tilde{N} \leq S^{(1)}$; $P, \tilde{P} \leq S^{(2)}$;
(ii) $S^{(2)} \leq S^{(1)}$ (equivalently $E^{(2)} \leq E^{(1)}$); $\tilde{P} \leq \tilde{N}$;
(iii) $\tilde{P} \leq P, N$; $D \leq \tilde{D}$.

Besides these inequalities and those implied by them, no other inequalities between pairs of the 10 outcome probabilities hold in general.

The inequalities given in Theorem 2 are illustrated in Figure 1. The classification into parts (i)–(iii) reflects different types of argument. The inequalities in (i) follow from simple implications that hold on any directed acyclic graph; for example, if the first player can force the game to end after an odd number of moves then she can of course force it to end. Those in (ii) come from strategy-stealing arguments involving the (distributional) homogeneity of the Galton–Watson tree: if the first player opens with a random move then the resulting position has the same law as before. The inequalities in (iii) are proved by analytic methods, and we lack intuitive explanations for them. The last inequality is perhaps the most striking: draws are at least as likely in the misère game as in the normal game.

Now we describe some examples of phase transitions that arise as the offspring distribution is varied.

**Proposition 3** (Examples).

(i) Binary branching. Let $(p_0, p_1, p_2) = (1 - t, 0, t)$ for $t \in [0, 1]$, and note that the probability of survival is positive if and only if $t > 1/2$. The normal game draw probability $D$ has a phase transition at $t_n := \sqrt{3}/2 \approx 0.866 \ldots$, in the sense that $D > 0$ if and only if $t > t_n$. The transition is continuous: $D \to 0$ as $t \downarrow t_n$. Similarly, the misère draw probability $\tilde{D}$ has a continuous phase transition at $t_m := 3/4$. In contrast, the escape game has a discontinuous phase transition at $t_e := 3/2^{5/3} \approx 0.945 \ldots$: $E^{(1)}$ is positive if and only if $E^{(2)}$ is positive, which happens if and only if $t \geq t_e$. In fact $E^{(1)} = 2^{4/3}/3 \approx 0.840 \ldots$ at $t = t_e$.

(ii) Poisson offspring. Let the offspring distribution be Poisson with mean $\lambda$, and note that the survival probability is positive if and only if $\lambda > 1$. The normal and misère games have continuous phase transitions at $\lambda_n = e$ and $\lambda_m = 2.103 \ldots$, respectively (where the latter is the solution of $\lambda = e^{\lambda(1-e^{-1})}$); the draw probability is positive if and only if $\lambda$ exceeds the respective threshold. The escape game has a discontinuous phase transition at some $\lambda_e$ (which we estimate numerically as $3.319 \ldots$).

(iii) Geometric offspring. Let $p_i = (1 - \alpha)\alpha^i$ for $i \geq 0$. The draw and escape probabilities $D, \tilde{D}, E^{(1)}, E^{(2)}$ are zero for all $\alpha \in (0, 1)$. 

![FIGURE 1](image-url) The inequalities proved in Theorem 2. An arrow from $y$ to $x$ indicates $x \leq y$
FIGURE 2  The function $F^2(x) - x$ for the binary branching distribution of Proposition 3(i). The roots of this function are the elements of $FP(F^2)$. On the left, $p = 0.85$ (subcritical for the normal game)—the function has a unique root and the probability of a draw in the normal game is 0. On the left, $p = 0.89$—the function has three roots and the probability of a draw is the distance from the smallest to the largest root. As $p$ passes through the critical point, the two new roots emerge continuously from the existing root. For a contrasting example with a discontinuous phase transition, see Figure 4 in Section 5.

Note that the draw probability $D$ is not in general monotone in the offspring distribution: the geometric distribution in (iii) stochastically dominates the binary branching distribution in (i) if $\alpha$ is small enough as a function of $t$, but the former has $D = 0$ while the latter has $D > 0$ (for suitable $t$). Similar remarks apply to $\tilde{D}$, $E^{(1)}$, and $E^{(2)}$.

See Figure 2 for an illustration of how Theorem 1(i) applies in the binary branching case of Proposition 3(i).

Theorem 1 enables the games to be analyzed for many other offspring distributions: the outcome probabilities are given in terms of solutions of equations (although not always as closed-form...
expressions). Another interesting case is the Binomial\((n,p)\) distribution, under which \(\mathcal{T}\) can be viewed as the percolation cluster on a regular tree. Here the normal game has a continuous phase transition, with draws if and only if \(p > (n + 1)^{\ast - 1}/n^\ast\). (See Proposition 12 in Section 5.)

We typically find that phase transitions are continuous for the normal and misère games and discontinuous for the escape game, as in the above examples. However, we can concoct examples with the opposite behavior, as well as more exotic phase transitions, as follows.

**Proposition 4** (Exotic examples). For each of (i)–(iii) below there exists a continuous family \(\{p(t) : t \in (0,1)\}\) of offspring distributions, of uniformly bounded support and satisfying \(p_1(t) < 1\), with the given properties.

(i) The normal game has a discontinuous phase transition: there exists \(t^* \in (0,1)\) such that \(D = 0\) for \(t < t^*\) while \(D \in (0,1)\) for \(t \geq t^*\).

(ii) The normal game has two phase transitions: there exist \(0 < t^- < t^+ < 1\) such that \(D\) increases continuously from 0 to positive values at \(t^-\), and jumps discontinuously from one positive value to another at \(t^+\).

(iii) The escape game has a continuous phase transition: there exists \(t_e \in (0,1)\) such that \(E^{(1)} = 0\) for \(t \leq t_e\), while \(E^{(1)} \in (0,1)\) for \(t > t_e\), and \(E^{(1)}\) is a continuous function of \(t\).

Notwithstanding the above examples, the next result establishes some general patterns concerning the nature of phase transitions. In particular, for certain simple families of distributions, phase transitions are indeed continuous for the normal and misère games but discontinuous for the escape game. To make the statements precise, we need two different metrics on offspring distributions \(p = (p_0, p_1, \ldots )\).

Let \(M_0\) be the space \(\{p : p_i \geq 0, \sum_i p_i = 1\}\) of all offspring distributions, with the \(\ell^1\) metric \(d_0(p,q) : = \sum_i |p_i - q_i|\). Let \(M_1\) be the space \(\{p \in M_0 : \sum_i ip_i < \infty\}\) of distributions with finite mean \(\mu\), with the metric \(d_1(p,q) : = \sum_i |p_i - q_i|\).

**Theorem 5** (Phase transitions). Consider a Galton–Watson process with offspring distribution \(p\).

(i) The probabilities \(N, P, \tilde{P}, S^{(1)}, S^{(2)}\) are lower semicontinuous as functions of \(p\) with respect to \(d_0\). Hence, \(D\) is upper semicontinuous, and \(N\) and \(P\) are continuous on \(\{p : D = 0\}\), and similarly for the misère game.

(ii) The probabilities \(D\) and \(\tilde{D}\) are continuous with respect to \(d_0\) on the set of distributions \(p\) supported on \(\{0,1,2\}\) and satisfying \(0 < p_0 < 1\).

(iii) The set \(\{p : E^{(1)} > 0\}\) contains \(\{p \in M_1 : \mu p_1 > 1\}\) and is closed with respect to \(d_1\) in \(\{p \in M_1 : \mu p_1 < 1\}\). We have \(E^{(1)} > 0\) if and only if \(E^{(2)} > 0\).

Part (iii) above deserves some explanation. The condition \(\mu p_1 > 1\) corresponds to a particularly simple explanation for an Escaper win: there is a supercritical branching process on which Escaper can always leave Stopper with exactly one legal move. (See Proposition 14 and its proof in Section 6 for more details.) The result says essentially that a continuous transition between \(E^{(1)} > 0\) and \(E^{(1)} = 0\) can occur only where the above criterion is the sole explanation for escapes, that is, when the transition occurs as a result of crossing the boundary of the region \(\mu p_1 > 1\). Elsewhere, the escape region is closed and thus includes its critical surface.

It would be desirable to find more general conditions under which the conclusion of part (ii) holds (although Proposition 4 shows that it does not hold in full generality). What is the largest \(k\) for which it holds for all distributions with support \(\{0, \ldots , k\}\)? Can it be established for some broader class of “reasonable distributions”?
Finally, we investigate further the topology of the region of distributions giving positive draw probability, and the nature of the phase transitions which can occur, by considering quantities related to the length of the game.

Consider the normal or misère game. We define the length of the game with optimal play, denoted by $T$, as follows. Suppose that the game is a win for one of the players. Then $T$ is the number of turns in the game (i.e., the distance from the root to the leaf where the game ends) if the winning player tries to win the game as quickly as possible while the losing player tries to prolong it as much as possible. Equivalently, $T$ is the smallest $n$ such that some player has a strategy that ensures a win in $n$ turns or fewer. (From a simple compactness result, Proposition 7, such an $n$ exists if the game is not a draw.) If the game is a draw with optimal play, define $T = \infty$.

Next, say that a vertex $v$ is enforceable if each player has a strategy that guarantees that either they do not lose, or that the game passes through $v$. Let $T^*$ be the supremum of the depths of all enforceable vertices. If the game is a draw, then trivially any vertex $v$ is enforceable, since both players have strategies that guarantee not to lose, and $T^* = \infty$. On the other hand, if one player has a winning strategy, then $T^*$ is finite, and we have the following interpretation: although the other player is destined to lose eventually, they can control the path of the game for the first $T^*$ turns, unless the opponent is willing to give up the win. Note that $T^* \leq T$.

**Theorem 6** (Length of the game). Consider the normal or misère game on a Galton–Watson tree, with offspring distribution $\mathbf{p}$. Write $B$ for the set of offspring distributions such that the probability of a draw is 0, $\partial B$ for its boundary in $(M_0, d_0)$, and $B^0$ for its interior.

(i) If $\mathbf{p} \in B^0$, then $\mathbb{E}T < \infty$ and $\mathbb{E}T^* < \infty$.
(ii) If $\mathbf{p} \in \partial B \cap B$, then $\mathbb{E}T = \infty$ and $\mathbb{E}T^* = \infty$.
(iii) Along any sequence of offspring distributions in $B$ converging in $(M_0, d_0)$ to a distribution in $\partial B \cap B$, we have $\mathbb{E}T \to \infty$ and $\mathbb{E}T^* \to \infty$.
(iv) Along any sequence of offspring distributions in $B$ converging in $(M_0, d_0)$ to a distribution in $\partial B \setminus B$, we have $\mathbb{E}T \to \infty$.

The set $\partial B \cap B$ is the set of distributions in the boundary of $B$ which have draw probability 0; hence we may interpret $\partial B \cap B$ as the set of “continuous phase transition” points, and similarly the set $\partial B \setminus B$ as the set of “discontinuous phase transition” points.

In parts (iii) and (iv) of Theorem 6, we see that $\mathbb{E}T$ blows up as we approach the boundary of $B$, and that $\mathbb{E}T^*$ blows up if we approach a continuous phase transition point. It would be convenient to complete the result with the statement that $\mathbb{E}T^*$ does not blow up at a discontinuous phase transition point. However, such a statement is not true without further qualification. For the case of the normal game, let $x^*$ be the unique fixed point in $[0, 1]$ of the function $F$. During the proof of Theorem 6, we show that $\mathbb{E}T^* \to \infty$ precisely if at the limit point, $F'(x^*) = -1$. At a continuous phase transition point (where new fixed points of the function $F^2$ emerge smoothly from the fixed point $x^*$), we will show that indeed $F'(x^*) = -1$. At discontinuous phase transition points (where new fixed points of $F^2$ are created away from $x^*$), it is not generally the case that $F'(x^*) = -1$. However, it can occur that $F'(x^*) = -1$; we could loosely interpret such cases by saying that a continuous phase transition is occurring, but it is masked by a simultaneously occurring discontinuous phase transition. (For the case of the misère game, replace the function $F$ by the function $H$ throughout.)

Accordingly, we conjecture that the correct completion of the result in Theorem 6(iii)–(iv) is as follows: $\mathbb{E}T^*$ stays bounded if the limit distribution is in $\partial B \setminus (\partial B \cap B)$ (a phase transition point which is separated from the set of continuous phase transition points), and $\mathbb{E}T^* \to \infty$ if the limit distribution is any other point in $\partial B$. However, we do not have a proof of this statement.
It is instructive to compare the phase transitions considered here with those involving other properties of branching processes. First let $\mathcal{A}$ be the set of offspring distributions for which the branching process dies out with probability 1 (i.e., the probability that an infinite path exists is 0), and let $p^*$ be the degenerate distribution with $p_1^* = 1$ and $p_k^* = 0$ for $k \neq 1$. Then $\mathcal{A} \cup \{p^*\}$ is closed as a subset of $M_0$ (it is well known that $\mathcal{A}$ consists precisely of those distributions with mean less than or equal to 1, except for $p^*$). Along a sequence of distributions in $\mathcal{A}$ converging to a point in $\mathcal{A}$, the expected length of the longest path in the tree goes to $\infty$ (see Lemma 18, and the remark after its proof), and the probability of extinction is continuous at the boundary of $\mathcal{A}$ (except at $p^*$).

On the other hand, consider the event that the tree of the branching process contains a complete infinite binary tree, rooted at the root of the branching process. Let $\mathcal{B}$ be the set of offspring distributions for which this event has probability 0. Now it is possible to show that the set $\mathcal{B}$ is open as a subset of $M_1$. (We do not write the proof here, but we observe that a closely related property involving the 3-core of sparse random graphs which converge locally to a branching process is studied extensively by Janson [10]). Hence within $M_1$, the phase transitions at the boundary of $\mathcal{B}$ are discontinuous; for example, it was shown by Dekking [5] that for the particular case of a Poisson($\lambda$) offspring distribution, the probability of existence of such a binary subtree is 0 for $\lambda < \lambda_c \approx 3.35$, and jumps to around 0.535 at $\lambda_c$.

In contrast to the previous two paragraphs, we see that for the case of the draw probability, the set $\mathcal{M}$ considered in Theorem 6 is neither open nor closed. Along a sequence of distributions converging to a distribution in $\partial B \cap B$, we see a continuous phase transition as in the case of survival/extinction of a branching process. Indeed, in the proof we show that the collection of all enforceable vertices gives a subtree of the game tree which is a two-type Galton–Watson process, which itself approaches criticality (for survival/extinction) at the phase transition point. In this case, we can explain the emergence of draws by the divergence to infinity of the length of the path to an enforceable vertex, along which the losing player can steer the game. On the other hand, the case of a discontinuous phase transition is much more similar to that observed for the set $\mathcal{B}$ defined in terms of the occurrence of a binary tree within a branching process; here it seems that the emergence of draws cannot be explained in terms of a single path, but intrinsically involves a more complicated branching structure.

1.1 Background and related work

Two recent articles by the current authors together with Basu and Wästlund [1] and with Marcovici [9] address these games and their variants on other structured random graphs.

Specifically, [9] considers the normal game and a variant of the misère game on percolation clusters of oriented Euclidean lattices. Using probabilistic cellular automata and hard core models, it is proved that draws occur in dimensions $d \geq 3$ and greater (on certain lattices) if the percolation parameter is large enough, but not in dimension 2. Many questions remain unresolved, such as monotonicity of the draw probability in the percolation parameter (which would imply uniqueness of the phase transition for $d \geq 3$).

On the other hand, [1] is concerned with percolation on unoriented lattices. The normal game as defined earlier is less interesting on an undirected graph, since (unless the starting vertex has no neighbor) either player can draw by immediately reversing every move of the other player. We therefore consider a different extension of the rules, in which the token is forbidden to ever revisit a vertex, giving a game that we call Trap. (On a tree, Trap and the normal game are clearly equivalent). For percolation on Euclidean lattices in any dimension $d \geq 2$, it is unknown whether Trap has draws for some nontrivial percolation parameter. Simulation evidence tends to support a negative answer for $d = 2$, while analogy with the directed case might suggest a positive answer for $d \geq 3$. The article [1] uses connections with maximal matchings and bootstrap percolation to establish finite scaling results.
on a biased percolation model where vertices have two different occupation parameters according to their parity, thus favoring one player.

Compared with the cases discussed above, the recursive structure of Galton–Watson trees allows a considerably deeper analysis. Two special cases of the normal game have been partially analyzed before: the phase transition for the Binomial$(2, p)$ offspring was found in the PhD thesis of one of the current authors [7]. The case of the Poisson offspring family is closely related to the analysis of the Karp–Sipser algorithm used to find large matchings or independent sets of a graph, introduced by Karp and Sipser in [12]. For the case of Erdős–Rényi random graphs $G(n, \lambda/n)$ they identified a phase transition at $\lambda = e$ corresponding to that noted in Theorem 3, and dubbed it the “$e$-phenomenon”; the link to games is not described explicitly but the choice of notation and terminology makes clear that the authors were aware of it.

We mention some recent papers particularly closely related to the current study. In [14], Martin and Stański consider minimax recursions defined on Galton–Watson trees with no leaves, truncated at some depth $k$. Terminal values at the level-$k$ nodes are drawn independently from some common distribution. Such recursions give the value of a general class of two-player combinatorial games; the behavior of the value associated with the root is studied as $k \to \infty$. Johnson, Podder and Skerman [11] study a wider class of recursions on supercritical Galton–Watson trees, with a particular focus on cases where the one-level generating-function recursion has multiple fixed points. Broutin, Devroye and Fraiman [3] study related questions for minimax functions and more general recursions, in the case of Galton–Watson trees conditioned to have a given number of vertices.

Other work on combinatorial games in random settings includes the study of positional games (such as Maker–Breaker games) on random graphs, for example, [2,6,15], and [4] which deals with matching games played on random point sets, with an intimate connection to Gale–Shapley stable marriage. In another direction, [8] uses certain games as tools for proving statements involving second-order logic on random trees, and [16] uses a game in the analysis of optimization problems in a random setting. One striking observation from all these examples is that games, by their competitive nature, often automatically tease out and magnify some of the most interesting and subtle structural properties of random systems.

## 2 Recursions and Compactness

In this section, we give the basic recursions underlying the analysis of the games. First consider the normal game on any directed acyclic graph with vertex set $V$, and let $\mathcal{N}$ be the set of vertices $v$ for which the game is a next-player win if the token is started at $v$. Similarly define $\mathcal{P}$ and $\mathcal{D}$ to be the sets of vertices from which the game is a previous-player win and a draw, respectively (so that $(\mathcal{N}, \mathcal{P}, \mathcal{D})$ is a partition of $V$). In the case of the Galton–Watson tree with offspring distribution $\mathbf{p}$ we have $\mathcal{N} = N(\mathbf{p}) = \mathbb{P}(o \in \mathcal{N})$, and similarly for $\mathcal{P}$ and $\mathcal{D}$. Let $v$ be a vertex and let $\Gamma = \Gamma(v)$ be its out-neighborhood, that is, the set of end-vertices of the edges leading from $v$. By considering the first move, it is immediate that the following relations hold.

\begin{align*}
v \in \mathcal{N} & \iff \Gamma \cap \mathcal{P} \neq \emptyset; \\
v \in \mathcal{P} & \iff \Gamma \subseteq \mathcal{N}; \\
v \in \mathcal{D} & \iff \Gamma \cap \mathcal{P} = \emptyset \text{ but } \Gamma \cap \mathcal{D} \neq \emptyset.
\end{align*}

(1)

Similar relations hold for the other games. However, these relations are not in general sufficient to determine the sets. For example, consider the normal game on a singly infinite path directed toward
proof. Consider first the normal game. Let \( \mathcal{N}_n \) be the set of starting vertices from which the Next player has a winning strategy that guarantees a win after strictly fewer than \( n \) moves (counting the moves of both players). Similarly, let \( \mathcal{P}_n \) be the set of vertices from which the Previous player can guarantee a win in fewer than \( n \) moves. In particular, we have \( \mathcal{N}_0 = \mathcal{P}_0 = \emptyset \). Let \( D_n = V \setminus (\mathcal{N}_n \cup \mathcal{P}_n) \). This may be interpreted as the set of starting vertices from which the game is a draw, under the convention that we declare the game a draw whenever it lasts for \( n \) moves. By considering the first move again, we have for \( n \geq 0 \),

\[
\begin{align*}
v \in \mathcal{N}_{n+1} & \iff \Gamma \cap \mathcal{P}_n \neq \emptyset; \\
v \in \mathcal{P}_{n+1} & \iff \Gamma \subseteq \mathcal{N}_n.
\end{align*}
\]

(It is easy to deduce that \( \mathcal{N}_1 = \emptyset \), while \( \mathcal{N}_{2k} = \mathcal{N}_{2k+1} \) and \( \mathcal{P}_{2k-1} = \mathcal{P}_{2k} \) for \( k \geq 1 \).)

Similarly, let \( \mathcal{N}_n, \mathcal{P}_n \) be the sets of starting vertices from which the misère game is a Next player win, a Previous player win, and a draw, respectively. Let \( \mathcal{N}_n, \mathcal{P}_n \) be the sets from which the relevant player can guarantee to win in fewer than \( n \) moves, and let \( \mathcal{D}_n = V \setminus (\mathcal{N}_n \cup \mathcal{P}_n) \). Then we have

\[
\begin{align*}
v \in \mathcal{N}_{n+1} & \iff \Gamma \cap \mathcal{P}_n \neq \emptyset \text{ or } \Gamma = \emptyset; \\
v \in \mathcal{P}_{n+1} & \iff \Gamma \subseteq \mathcal{N}_n \text{ and } \Gamma \neq \emptyset.
\end{align*}
\]

For the escape game, let \( S^{(1)}_n, S^{(2)}_n \) be the sets from which Stopper wins, when he has the first move and the second move, respectively, and let \( E^{(1)}_n, E^{(2)}_n \) be the sets where Escaper wins, when moving first and second, respectively. Let \( S^{(1)}_n, S^{(2)}_n \) be the sets from which Stopper can win in fewer than \( n \) moves, and let \( E^{(1)}_n = V \setminus S^{(2)}_n \) and \( E^{(2)}_n = V \setminus S^{(1)}_n \). (These are Escaper’s winning sets if we declare Escaper the winner after the \( n \)th move). We have

\[
\begin{align*}
v \in S^{(1)}_{n+1} & \iff \Gamma \cap S^{(2)}_n \neq \emptyset \text{ or } \Gamma = \emptyset; \\
v \in S^{(2)}_{n+1} & \iff \Gamma \subseteq S^{(1)}_n.
\end{align*}
\]

To use the above relations, we need the following simple but important fact: if a player can win (or, in the escape game, if Stopper can win), then they can guarantee to do so within some finite number of moves which they can specify in advance. This follows from compactness arguments going back to [13]. For the reader’s convenience, we include a proof.

**Proposition 7** (Compactness). Let \( G \) be a directed acyclic graph with all out-degrees finite. We have \( \mathcal{N} = \bigcup_{n=0}^\infty \mathcal{N}_n \), and similarly for each of \( \mathcal{P}, \mathcal{N}, \mathcal{P}, S^{(1)}, S^{(2)} \).

**Proof.** Consider first the normal game. Let \( \mathcal{N}' := \mathcal{N} \setminus \bigcup_{n=0}^\infty \mathcal{N}_n \) and \( \mathcal{P}' := \mathcal{P} \setminus \bigcup_{n=0}^\infty \mathcal{P}_n \) be the sets from which the relevant player can win, but cannot guarantee to do so within any finite number of moves. We must show that \( \mathcal{N}' \neq \mathcal{P}' \neq \emptyset \).

If \( v \in \mathcal{N}' \) then the out-neighborhood \( \Gamma(v) \) contains some vertex in \( \mathcal{P}' \), but not in \( \mathcal{P} \setminus \mathcal{P}' \) (otherwise the Next player could win in finitely many moves). If \( v \in \mathcal{P}' \) then all vertices of \( \Gamma(v) \) lie in \( \mathcal{N}' \), and we claim that at least one of them lies in \( \mathcal{N}' \). Indeed, if not, then for each \( w \in \Gamma(v) \) there exists some \( m(w) \) such that \( w \in \mathcal{N}_{m(w)} \). But then \( M := \max \{ m(w) : w \in \Gamma(v) \} \) is finite, and so \( v \in \mathcal{P}_M \), a contradiction.
We now claim that from any vertex in \( \mathcal{N}' \), the Previous player has a strategy that guarantees a draw or better. Indeed, if the Next player is foolish enough to move to a vertex in \( \mathcal{N}' \cup D \) then the Previous player simply plays to win or draw as usual. If the Next player instead moves to a vertex in \( \mathcal{P}' \) then the Previous player replies by moving again to a vertex in \( \mathcal{N}' \). The same strategy allows the Next player to draw from any vertex in \( \mathcal{P}' \). Hence, there are no such vertices.

For the misère game, we can reduce to the normal game on a modified graph: from each vertex of out-degree 0 we add a single outgoing edge to a new vertex of out-degree 0. We now appeal to the normal game case already proved.

For the escape game, we can reduce to the normal game on a different modified graph. Fix a starting vertex \( u \) and suppose that Stopper moves first. First, split each vertex \( v \) into two copies \( v_0 \) and \( v_1 \) to indicate whether it is reached after an even or odd number of moves. Let the token start at \( u_0 \). Split edge \((v, w)\) into two edges \((v_0, w_1)\) and \((v_1, w_0)\). The resulting graph is bipartite. Finally, for any \( v \) with out-degree 0, add an outgoing edge from \( v_0 \). The case when Stopper moves second is handled similarly, except that in the final step we instead add the outgoing edge to \( v_1 \).

The finite out-degree assumption in the last result is needed. For instance, if \( G \) is a tree consisting of outgoing paths of every even length 2, 4, 6, \ldots emanating from a root \( o \), then the Previous player wins, but the Next player can make the game arbitrarily long.

### 3 Generating Functions and Fixed Points

We next prove Theorem 1. From now on we specialize to the case \( G = \mathcal{T} \), the Galton–Watson tree with offspring distribution \( p = (p_0, p_1, \ldots) \). Recall that we write \( N = N(p) = \mathbb{P}(o \in \mathcal{N}) \), and similarly for \( P, D, \tilde{N}, \tilde{P}, \tilde{D}, S^{(1)}, S^{(2)}, E^{(1)}, E^{(2)} \). Recall the sets \( \mathcal{N}_n \), and so on defined in the previous section. Define the associated probabilities \( N_n : = \mathbb{P}(o \in \mathcal{N}_n) \), and so on.

On a tree, these probabilities may be interpreted as follows. Let \( \mathcal{T}_n \) be the finite subgraph of \( \mathcal{T} \) induced by the set of vertices of depth \( n \) (i.e., distance from \( o \)) at most \( n \). Consider the normal game played on \( \mathcal{T}_n \), but declared to be a draw if the token ever reaches depth \( n \). The outcome of this game may be computed by assigning all depth-\( n \) vertices of \( \mathcal{T}_n \) to \( D \), and then using the recurrence (2) to classify the other vertices. Then \( N_n \) is the probability that the Next player wins starting from \( o \), and similarly for \( P_n \) and \( D_n \). Similarly, let \( \tilde{N}_n, \tilde{P}_n, \tilde{D}_n \) be the outcome probabilities for the misère game on \( \mathcal{T}_n \) where we declare a draw at depth \( n \). For the escape game, declare vertices at depth \( n \) to be wins for the escaper; then \( S_n^{(1)}, S_n^{(2)}, E_n^{(1)}, E_n^{(2)} \) are the relevant outcome probabilities.

**Corollary 8** (Truncation and limits). For any offspring distribution \( p \), with the above notation, we have \( N = \lim_{n \to \infty} N_n \), and similarly for \( P, D, \tilde{N}, \tilde{P}, \tilde{D}, S^{(1)}, S^{(2)}, E^{(1)}, E^{(2)} \).

**Proof.** By Proposition 7, we have \( N_n \not\to N \) as \( n \to \infty \). Similarly, \( P_n \not\to P \). (In fact, since the first player can only win in an odd number of moves, we have \( N_{2k} = N_{2k+1} \) for all integers \( k \geq 0 \), and similarly \( P_{2k+1} = P_{2k+2} \).) Since \( D = 1 - P - N \) and \( D_n = 1 - P_n + N_n \), we have \( D_n \setminus D \). The same argument works for the misère game. Similarly, for the escape game we get \( S_n^{(j)} \not\to S_n^{(j)} \) for \( j = 1, 2 \), but \( E^{(1)} = 1 - S^{(2)} \) and \( E^{(2)} = 1 - S^{(1)} \).}

Recall from Section 1 that we define the generating function \( G(x) = G_p(x) : = p_0 + p_1 x + p_2 x^2 + \ldots \), which is a continuous, increasing, convex function from \([0, 1]\) to \([0, 1]\). We will use repeatedly the fact that if some property holds for the tree with probability \( x \), then the probability that it holds for all the subtrees rooted at children of the root is \( G(x) \).
Recall that we also define the functions \( F(x) := 1 - G(x) \) and \( H(x) := 1 - G(x) + p_0 \), which are decreasing and concave.

**Proof of Theorem 1.** First consider the normal game. Corollary 8 gives \((N, P, D) = \lim_{n \to \infty} (N_n, P_n, D_n)\). We apply the recursion (2) at the root \( o \), noting that there is an independent copy of \( T \) rooted at each child. We obtain that for \( n \geq 0 \),

\[
P_{n+1} = G(N_n); \quad 1 - N_{n+1} = G(1 - P_n).
\]

This implies that \( N_{n+2} = F^2(N_n) \) and \( 1 - P_{n+2} = F^2(1 - P_n) \). Note also that \( N_0 = P_0 = 0 \). Therefore, since \( F^2 \) is increasing and continuous,

\[
N = \lim_{n \to \infty} F^{2n}(0) = \min FP(F^2); \quad 1 - P = \lim_{n \to \infty} F^{2n}(1) = \max FP(F^2).
\]

Hence,

\[
D = 1 - N - P = \max FP(F^2) - \min FP(F^2).
\]

The arguments for the other games are similar. For the misère game, the recursion (3) gives \( \tilde{N}_{n+2} = H^2(\tilde{N}_n) \) and \( 1 - \tilde{P}_{n+2} = H^2(1 - \tilde{P}_n) \), so that

\[
\tilde{N} = \lim_{n \to \infty} H^{2n}(0) = \min FP(H^2); \quad 1 - \tilde{P} = \lim_{n \to \infty} H^{2n}(1) = \max FP(H^2);
\]

\[
\tilde{D} = 1 - \tilde{N} - \tilde{P} = \max FP(H^2) - \min FP(H^2).
\]

For the escape game, (4) gives \( S_{n+1} = H(E_n) \) and \( E_{n+1} = F(S_n) \), so that

\[
S^{(1)} = \lim_{n \to \infty} (H \circ F)^n(0) = \min FP(H \circ F); \quad E^{(1)} = \lim_{n \to \infty} (F \circ H)^n(1) = \max FP(F \circ H);
\]

\[
S^{(2)} = 1 - E^{(1)}, \quad E^{(2)} = 1 - S^{(1)}. \tag{5}
\]

We note a sense in which the escape game is intermediate between the other two games: its outcome probabilities arise from *alternately* iterating the two functions \( F \) and \( H \) that govern the others. For later use, we note the following relations between outcome probabilities of the games on the full tree.

**Corollary 9.** For any offspring distribution we have:

\[
1 - P = F(N); \quad N = F(1 - P);
\]

\[
1 - \tilde{P} = H(\tilde{N}); \quad \tilde{N} = H(1 - \tilde{P});
\]

\[
S^{(1)} = H(E^{(1)}); \quad E^{(1)} = F(S^{(1)}).
\]

**Proof.** These can be deduced either by taking limits as \( n \to \infty \) of the corresponding recurrences in the above proof, or by directly applying (1) and its analogs for the other games. \( \blacksquare \)
4 | INEQUALITIES

In this section, we prove the inequalities of Theorem 2. The fact that no other inequalities hold in general is proved in Section 6.

**Proof of Theorem 2(i).** As remarked earlier, these inequalities of probabilities reflect inclusions that hold more generally. Specifically, for the games on any directed acyclic graph $G$, we have

$$\mathcal{N} \subseteq S^{(1)}; \quad \tilde{\mathcal{N}} \subseteq S^{(1)}; \quad \mathcal{P} \subseteq S^{(2)}; \quad \tilde{\mathcal{P}} \subseteq S^{(2)}.$$

Indeed, the starting vertex lies in $\mathcal{N}$ if and only if the first player can ensure that the game reaches a vertex of out-degree zero after an odd number of steps; similarly the vertex lies in $\tilde{\mathcal{N}}$ if and only if the first player can ensure that the game reaches a vertex of out-degree zero after an even number of steps. In either case, Stopper (if playing first) can win the escape game by using the same strategy. This gives the first two inclusions. Similarly, considering the second player gives the last two inclusions.

**Proof of Theorem 2(ii).** We show that $S^{(2)} \leq S^{(1)}$ and $\tilde{P} \leq \tilde{N}$ for any Galton–Watson tree $T$. Consider the escape game, and suppose Stopper has the first move. We propose a partial strategy for Stopper. If the root has no children, Stopper wins immediately. If the root has one or more children, let Stopper move to a child chosen uniformly at random (without looking at the remainder of the tree). The rest of the game is then played in a subtree with the same law as $T$, with Stopper moving second. This yields

$$S^{(1)} \geq p_0 + (1 - p_0)S^{(2)} \geq S^{(2)}.$$

Since $E^{(1)} = 1 - S^{(2)}$ and $E^{(2)} = 1 - S^{(1)}$, this gives equivalently $E^{(1)} \geq E^{(2)}$.

In the misère game, the first player can follow the same strategy, to give

$$\tilde{N} \geq p_0 + (1 - p_0)\tilde{P} \geq \tilde{P}.$$

Moving on to the more interesting inequalities in Theorem 2(iii), we start with some lemmas.

**Lemma 10.** Consider any offspring distribution. We have $H'(x) \geq -1$ for all $x \leq \tilde{N}$. If $1 - \tilde{P} > \tilde{N}$ (i.e., if $\tilde{D} > 0$) then $H'(x) \leq -1$ for all $x \geq 1 - \tilde{P}$.

**Proof.** If $p_0 \in \{0, 1\}$ then the lemma is easy to check. Therefore assume that $0 < p_0 < 1$. Since $H$ is concave, it is enough to check the values of $H'$ at $\tilde{N}$ and $1 - \tilde{P}$. Recall from the proof of Theorem 1 that $\tilde{N}$ is the smallest fixed point of $H^2$ in $[0, 1]$, and $1 - \tilde{P}$ is the largest fixed point. Recall also that $\lim_{n \to \infty} H^{2^n}(0) = \tilde{N}$. We claim that the sequence $(H^{2^n}(0))_{n \geq 0}$ is strictly increasing. Indeed, we have $H^2(0) > 0$, and we can apply the strictly increasing function $H^2$ repeatedly to both sides.

Suppose first that $H^2$ has only one fixed point. Then $H$ has the same fixed point, that is, $H(\tilde{N}) = \tilde{N}$.

Suppose for a contradiction that $H'(\tilde{N}) < -1$. The idea is that $\tilde{N}$ is an unstable fixed point for $H$ under iteration. More precisely, since $H$ is continuous and concave, we have for some $\epsilon > 0$ that $H'(x) < -1$ for all $x > \tilde{N} - \epsilon$. Since $H^{2^n}(0)$ is strictly increasing with limit $\tilde{N}$, we have $\tilde{N} - \epsilon < H^{2m}(0) < \tilde{N}$ for some $m$. But then the assumption on $H'$ gives that the next two iterations move the iterate further from $\tilde{N}$, that is,

$$\tilde{N} - H^{2m+2}(0) > H^{2m+1}(0) - \tilde{N} > \tilde{N} - H^{2m}(0),$$

contradicting that $H^{2^n}(0)$ is increasing.
On the other hand, if \( H^2 \) has more than one fixed point, then \( \tilde{N} \) and \( 1 - \tilde{P} \) are a two-cycle of \( H \)—that is, \( H(\tilde{N}) = 1 - \tilde{P} \) and \( H(1 - \tilde{P}) = \tilde{N} \)—with \( \tilde{N} < 1 - \tilde{P} \). Now consider the square \([\tilde{N}, 1 - \tilde{P}]^2\). The graph of the function \( H \) passes through the top-left and bottom-right corners of this square. Since \( H \) is concave, it follows that \( H'(1 - \tilde{P}) \leq -1 \) and \( H'(\tilde{N}) \geq -1 \) as required.

\[ \square \]

**Lemma 11.** For any offspring distribution, \( \tilde{P}_n \leq P_n \) for all \( n \geq 0 \).

**Proof.** The result is true for \( n = 0, 1 \), since \( P_0 = \tilde{P}_0 = \tilde{P}_1 = 0 \) and \( P_1 = p_0 \). So it will be enough to show that \( \tilde{P}_n \leq P_n \) implies \( \tilde{P}_{n+2} \leq P_{n+2} \).

So suppose that \( \tilde{P}_n \leq P_n \). Then, using the recurrences in the proof of Theorem 1, and the fact that \( F^2 \) is increasing,

\[
P_{n+2} - \tilde{P}_{n+2} = (1 - \tilde{P}_{n+2}) - (1 - P_{n+2})
= H^2(1 - \tilde{P}_n) - F^2(1 - P_n)
\geq H^2(1 - \tilde{P}_n) - F^2(1 - \tilde{P}_n)
= H[H(1 - \tilde{P}_n)] - H[H(1 - \tilde{P}_n) - p_0] + p_0.
\]

Since \( H \) is concave, the last expression is at least

\[
p_0H'[H(1 - \tilde{P}_n)] + p_0.
\]

Now \( 1 - \tilde{P}_n \leq 1 - \tilde{P} \) and so \( H(1 - \tilde{P}_n) \not\leq H(1 - \tilde{P}) = \tilde{N} \). In particular \( H(1 - \tilde{P}_n) \leq \tilde{N} \), and so by Lemma 10, \( H'(H(1 - \tilde{P}_n)) \geq -1 \). Hence \( P_{n+2} - \tilde{P}_{n+2} \geq 0 \) as required.

\[ \square \]

**Proof of Theorem 2(iii).** The inequality \( \tilde{P} \leq P \) follows immediately from Lemma 11 and Corollary 8.

For the inequality \( D \leq \tilde{D} \), it will similarly be enough to prove that \( D_n \leq \tilde{D}_n \) for all \( n \). Again we proceed by induction. We have \( D_0 = \tilde{D}_0 = 1 \). Suppose that \( D_n \leq \tilde{D}_n \). From Lemma 11, we have \( \tilde{P}_n \leq P_n \). Then, since \( F \) is decreasing and concave, and \( F \) and \( H \) differ by a constant, and using the recurrences from the proof of Theorem 1,

\[
D_{n+1} = 1 - P_{n+1} - N_{n+1}
= F(N_n) - F(1 - P_n)
= F(1 - P_n - D_n) - F(1 - P_n)
\leq F(1 - \tilde{P}_n - \tilde{D}_n) - F(1 - \tilde{P}_n)
= H(1 - \tilde{P}_n - \tilde{D}_n) - H(1 - \tilde{P}_n)
= \tilde{D}_{n+1},
\]

completing the induction.

Finally, we will show that \( \tilde{P} \leq N \) by considering two cases. First suppose that \( \tilde{D} > 0 \). Then by Lemma 10, we have \( H'(x) \leq -1 \) for all \( x \geq 1 - \tilde{P} \). Since \( F \) and \( H \) differ by a constant, also \( F'(x) \leq -1 \) for all \( x \geq 1 - \tilde{P} \). Since \( F(1) = 0 \), it follows that

\[
F(1 - \tilde{P}) \geq \tilde{P}.
\]

(6)

As proved above, we have \( \tilde{P} \leq P \). Since \( F \) is decreasing, this gives \( F(1 - \tilde{P}) \leq F(1 - P) = N \). Combining this with (6) gives \( \tilde{P} \leq N \) as required.
Now suppose instead that $\tilde{D} = 0$. Since $D \leq \tilde{D}$ we have also $D = 0$. Then $N = 1 - p$ is a fixed point of $F$, and $\tilde{N} = 1 - \tilde{p}$ is a fixed point of $H$. Since $\tilde{P} \leq P$ from above, we have $N \leq 1 - \tilde{P}$. The functions $F$ and $H$ differ by a constant, and both are concave and decreasing, so

$$H'(y) \leq F'(x) \leq 0 \quad \text{for all } x \in (0, N) \text{ and } y \in (1 - \tilde{P}, 1).$$

Comparing the lengths of the intervals $(0, N)$ and $(1 - \tilde{P}, 1)$, this implies that

$$\tilde{P} \leq N \quad \text{or} \quad H(1) - H(1 - \tilde{P}) \leq F(N) - F(0). \quad (7)$$

In the former case, we are done. For the latter case note that

$$H(1) - H(1 - \tilde{P}) = p_0 - (1 - \tilde{P}) = \tilde{P} + p_0 - 1;$$

$$F(N) - F(0) = N - (1 - p_0) = N + p_0 - 1.$$

Substituting into (7) gives $\tilde{P} \leq N$ in the latter case also. \blackslug

5 \ EXAMPLES

In this section, we use Theorem 1 to prove Propositions 3 and 4.

Proof of Proposition 3(i) - binary branching. Recall that $(p_0, p_1, p_2) = (1 - p, 0, p)$, so each individual has either 0 or 2 children. It turns out that in this example all relevant quantities can be computed explicitly. We have $G(x) = 1 - p + px^2$, $F(x) = p(1 - x^2)$ and $H(x) = 1 - px^2$. We treat the three games separately.

Normal game. Theorem 1 gives the draw probability $D$ in terms of the fixed points of $F^2$, that is, the zeros of $F^2(x) - x$. See Figure 2 for graphs of this function. We have the factorization into two quadratics

$$F^2(x) - x = (p - x - px^2)(1 - p^2 - px + p^2x),$$

where the first factor equals $F(x) - x$. Viewed as a function of $x$, the first factor has exactly one zero, at $x_0$ say, in $[0, 1]$ for all $p \in [0, 1]$. The second factor has two distinct zeros $x_- < x_+$ in $[0, 1]$ if and only if its discriminant $p^2(4p^2 - 3)$ is positive, that is, when $p > p_d := \sqrt{3}/2$. Moreover, we have $x_- < x_0 < x_+$ for $p > p_d$, while at $p = p_d$, all three roots coincide, and the function has a stationary point of inflection on the axis. (These last facts can be seen without further computation: if $x_-$ is a fixed point of $F^2$ satisfying $x_- < x_0$, then $x_+ := F(x_-)$ is also a fixed point, and since $F$ is strictly decreasing we have $x_0 < x_+$. Moreover, the roots of a quadratic vary continuously with its coefficients.) Therefore, by Theorem 1 we have $D = 0$ for $p \leq p_d$, and $D = x_+ - x_-$ for $p > p_d$, giving the claimed continuous phase transition.

Misère game. The analysis is similar. We have the factorization

$$H^2(x) - x = (1 - x - px^2)(1 - p - px - p^2x^2),$$

where the first factor is $H(x) - x$. The first factor has exactly one zero, at $x_0$ say, and the second factor has two further zeros at $x_- < x_0 < x_+$ if and only if $p > p_m := 3/4$. For the same reasons as before, the transition is continuous.
The function $F(H(x)) - x$ for the binary branching distribution, for the three values $p = 0.935, 0.945, 0.955$. The probability of escape is the largest root. The lowest curve has its only root at 0. At the critical point (middle curve) a new root appears and the probability of escape jumps to a positive value. Above the critical point (upper curve) the function has three roots.

**Escape game.** Theorem 1 gives $S^{(1)} = \min FP(H \circ F)$. See Figure 3. We have

$$H \circ F(x) - x = (1 - x)[1 - p^3(1 - x)(1 + x)^2].$$

There is always a zero at $x = 1$. On $[0, 1]$, the function $(1 - x)(1 + x)^2$ has maximum $32/27$ at $x = 1/3$. Therefore, there are two additional zeros if $(32/27)p^3 \leq 1$, that is, if $p \geq p_e := 3/2$. The two additional zeros are strictly less than 1, and coincide at $x = 1/3$ when $p = p_e$. Thus, $S^{(1)}$ equals 1 for $p < p_e$, and jumps to $1/3$ at $p = p_e$, giving the claimed behavior for $E^{(2)} = 1 - S^{(1)}$. Corollary 9 gives that $E^{(1)} > 0$ if and only if $E^{(2)} > 0$.

**Proof of Proposition 3(ii) - Poisson.** The offspring distribution is Poisson($\lambda$). Thus, we have $G(x) = e^{-\lambda(1-x)}$, $F(x) = 1 - e^{-\lambda(1-x)}$ and $H(x) = 1 + e^{-\lambda}(1 - e^{-\lambda x})$. We will find that the behavior of the three games is qualitatively identical to that in the binary branching case considered above, but that not all quantities can be computed explicitly.

**Normal game.** By Theorem 1, we are interested in the fixed points of $F^2$. Differentiating $F^2(x) - x$ twice with respect to $x$, we find that its first derivative has exactly one turning point, a maximum at $x^* := 1 - (\log \lambda) / \lambda$, at which the first derivative equals $\lambda / e - 1$. We deduce that when $\lambda \leq e$ the function $F^2(x) - x$ is strictly decreasing on $[0, 1]$, and thus has exactly one zero in $[0, 1]$. When $\lambda > e$, the function $F^2(x) - x$ has two turning points, a local minimum followed by a local maximum. Therefore, it has at most three zeros. We claim that it has exactly three. To check this, note first that $F$ itself always has exactly one fixed point in $[0, 1]$, say $x_0$, which satisfies $\lambda = [\log(1 - x_0)]/(1 - x_0)$. This $x_0$ is also a fixed point of $F^2$. To show that $F^2(x) - x$ has three zeros it suffices to show that its derivative is positive at $x_0$, which is equivalent to showing $|F'(x_0)| > 1$. But $F'$ is negative and strictly decreasing in $x$, and equals $-1$ precisely at $x = x^*$ (as defined above). Now $x_0$ is strictly
increasing as a function of $\lambda$, while $x^*$ is strictly decreasing as a function of $\lambda$. Therefore, they coincide at exactly one $\lambda$, which is easily checked to be $\lambda = e$. Therefore, we have $F'(x_0) < -1$ if and only if $\lambda > e$, as required.

At the critical point $\lambda = e$, the function $F^2(x) - x$ has a stationary point of inflection on the axis at $x_0$. By Theorem 1, $D$ is the distance between the zeros, which is continuous in $\lambda$, and equals 0 if and only if $\lambda \leq e$.

**Misère game.** The analysis and behavior are similar to the normal game, except that the critical point now has no closed-form expression. The derivative of $H^2(x) - x$ has its maximum at $x^* = 1 - (\log \lambda)/\lambda$, at which the first derivative is $\lambda e^{-1 + \lambda e^{-\lambda}} - 1$. This is positive if and only if $\lambda > \lambda_m$, where $\lambda_m = 2.103 \ldots$ is the solution of $\log \lambda + \lambda e^{-\lambda} = 1$. Thus, the function $H^2(x) - x$ has one zero for $\lambda \leq \lambda_m$. Again, $H'(x^*) = -1$ for all $\lambda$, and $x^*$ is increasing in $\lambda$, while the fixed point $x_0$ of $F$ is decreasing (by implicit differentiation), with $x_0 = x^*$ at $\lambda = \lambda_m$. By the same argument as before, this gives that $H'(x^*) > -1$ if and only if $\lambda > \lambda_m$, hence $H^2$ has three fixed points if and only if $\lambda = \lambda_m$. And as before, at the critical point $\lambda_m$, the function $H^2(x) - x$ has a stationary point of inflection on the axis. We deduce from Theorem 1 that $\tilde{D}$ is continuous in $\lambda$, and equals 0 if and only if $\lambda \leq \lambda_m$.

**Escape game.** The proof of Theorem 1 gives that $S^{(1)}$ is the minimum fixed point of $H \circ F$. This function always has a fixed point at $x = 1$. Since $H \circ F(x) = F^2(x) + e^{-\lambda}$, we can make use of the previous analysis of $F^2$. For $\lambda < e$ the function $H \circ F(x) - x$ is decreasing and therefore has exactly one zero. At $\lambda = e$ a stationary point of inflection appears, but now it is strictly above the axis. For all $\lambda > e$, the function has a local minimum followed by a local maximum. For $\lambda$ sufficiently close to $e$, the value of the function at its local minimum is strictly positive. But for $\lambda$ sufficiently large, it is easy to check that the value at the local minimum is negative, and so $H \circ F(x) - x$ has three zeros. Moreover, we claim that the value of the function at the local minimum is strictly decreasing as a function of $\lambda$ on $(e, \infty)$, so that it is negative if and only if $\lambda > \lambda_e$ for some critical point $\lambda_e > e$. To check this, it suffices to show that the function $H \circ F(x) - x$ never has partial derivative with respect to $x$ and $\lambda$ simultaneously. In fact, some algebra shows that the difference between the two derivatives is never zero. Finally, observe that $H \circ F(x) - x$ is decreasing in a neighborhood of 1, so the locations of other zeros are bounded away from 1. Hence $E^{(2)} = 1 - S^{(1)}$ undergoes a discontinuous phase transition at $\lambda = \lambda_e$ from 0 to a positive value, is positive at the critical point, and is continuous elsewhere.

Numerically, we find $\lambda_e \approx 3.3185$. Corollary 9 shows that $E^{(1)} > 0$ if and only if $E^{(2)} > 0$.

**Proof of Proposition 3(iii) - Geometric.** Let $\alpha \in (0, 1)$ and let $p_k = (1 - \alpha)\alpha^k$ for $k \geq 0$. Then $G(x) = (1 - \alpha)/(1 - x\alpha)$. It is straightforward to show that the functions $F^2(x) - x$, $H^2(x) - x$, and $H \circ F(x) - x$ are all strictly decreasing on $(0, 1)$. Therefore, by Theorem 1, the probabilities of draws and escapes are zero.

As promised in Section 1, we next analyze the normal game for a Binomial offspring distribution.

**Proposition 12.** Let the offspring distribution be Binomial($n, p$), where $n \geq 2$. The normal game has a phase transition at $p_c := (n + 1)^{(n-1)/n} n^\alpha$; we have $D > 0$ if and only if $p > p_c$.

**Proof.** The situation is qualitatively identical to the binary branching and Poisson cases, with the proof being a matter of calculus. We have $F(x) = F_p(x) = 1 - (1 - p + p^2)x$. Let $x^* = x^*(p)$ be the unique fixed point of $F$ in [0, 1], which is also a fixed point of $F^2$. Note that $F^2$ has derivative $(F'(x^*))^2$ at $x^*$. The critical point $p_c := (n + 1)^{(n-1)/n} n^\alpha$ is the unique $p$ for which $F'(x^*)(p) = -1$. To prove this, we can solve the simultaneous equations $F_p(x) = x$ and $F'_p(x) = -1$ by dividing the first by the second, solving for $x$ in terms of $p$ and substituting into the second. Since the solution is unique, and by considering
the endpoints \( p = 0, 1 \), continuity shows that \( F'_p(x^*(p)) \leq -1 \) for \( p \leq p_c \) and \( F'_p(x^*(p)) > -1 \) for \( p > p_c \). In the latter case, this implies that \( F^2 \) has multiple fixed points in \([0, 1]\). It remains to show that \( F^2 \) has only one fixed point in the former case. If on the contrary there were multiple fixed points then \( F^2 \) would need to have at least four turning points, and thus its derivative would need to have at least three turning points (as in Figure 4). But we can write

\[
\frac{d}{dx} F^2(x) = F'(F(x))F'(x) = n^2 p^2 [(1 - p r^p) r]^{n-1},
\]

where \( r = 1 - p + px; \) and \((1 - pr^p)r\) is a unimodal function of \( r \) (and hence of \( x \)).

We now turn to the exotic examples of Proposition 4.

Proof of Proposition 4(i). Let

\[
G(x) = (1 - t) + t(0.5x^2 + 0.5x^{10});
\]

see Figure 4. There is a discontinuous phase transition at \( t_c \approx 0.9791 \).

For \( t < t_c \), the equation \( F^2(x) = x \) has a single solution. For \( t \) just smaller than \( t_c \), we have \( N = 1 - P \approx 0.7133 \), and \( D = 0 \).

At \( t = t_c \), new solutions to \( F^2(x) = x \) appear, at \( x^- \approx 0.264 \) and \( x^+ \approx 0.945 \). So the probability of a draw jumps from 0 to \( x^+ - x^- \approx 0.687 \). At \( t_c \) itself the equation has three solutions, with those at \( x^- \) and \( x^+ \) being repeated roots, while above \( t_c \) the equation has five solutions.

We remark that it is even possible for the draw probability \( D \) to jump from 0 to 1 as shown by the example \((p_0, p_1, p_2, p_3) = (\epsilon, 2/3, 0, 1 - \epsilon)\) discussed at the end of the final proof in this section.

Proof of Proposition 4(ii). Let

\[
G(x) = (1 - t) + t(0.15x + 0.85x^{20});
\]

see Figure 5. There are two phase transition points \( t^- \approx 0.9877 \) and \( t^+ \approx 0.99219 \). When \( t \leq t^- \), the equation \( F^2(x) = x \) has a single solution, and there are no draws. At \( t = t^- \) we see a continuous phase transition into a region where the equation has three solutions and draws occur; on \([t^-, t^+]\) the probability of a draw increases continuously. Just below \( t^+ \) we have \( N \approx 0.774 \), \( P \approx 0.149 \), \( D \approx 0.077 \).

At \( t^+ \) there is a discontinuous phase transition, and for \( t = t^+ \) we have \( N \approx 0.285 \), \( P \approx 0.020 \), \( D \approx 0.695 \). For \( t > t^+ \) there are seven solutions to \( F^2(x) = x \).

Proof of Proposition 4(iii). Let

\[
G(x) = \left(\frac{1}{18} - \epsilon\right) + \frac{2}{3}x + \left(\frac{5}{18} + \epsilon\right)x^3.
\]

Note that \( p_1 \mu > 1 \) if and only if \( \epsilon > 0 \). At \( \epsilon = 0 \), the probability of escape is 0, but Proposition 14 tells us that for \( \epsilon > 0 \) the probability of escape must be positive. The function \( F(H(x)) - x \) has \( x = 0 \) as its only root for \( \epsilon \leq 0 \), but as \( \epsilon \) becomes positive, the derivative of \( F(H(x)) - x \) at \( x = 0 \) moves from negative to positive, and a second root emerges continuously from 0. That is, \( E^{(1)} > 0 \) for all \( \epsilon > 0 \), with \( E^{(1)} \to 0 \) as \( \epsilon \to 0 \).
FIGURE 4 The function \(F(F(x)) - x\) for the family of distributions for Proposition 4(i), with \(p = 0.976\) on the left and \(p = 0.9791\) on the right. On the left, the function has a unique root and the probability of a draw is 0. At the critical point, two new (double) roots appear and the probability of a draw jumps to a positive value. Above the critical point the function has five roots.

6 | CONTINUITY

In this section, we prove Theorem 5, and we complete the proof of Theorem 2 by showing that no further inequalities hold.

Proof of Theorem 5(i). Recall that \(N = N(p)\) is the increasing limit as \(n \to \infty\) of \(N_2n = H^{2n}(0)\). But the latter is a continuous function of \(p\) with respect to \(d_0\) for each \(n\). Therefore, \(N\) is a lower semicontinuous function of \(p\). The same argument gives lower semicontinuity of \(P, \tilde{N}, \tilde{P}, S^{(1)}, S^{(2)}\). Then \(N + P\) is also lower semicontinuous, so \(D = 1 - N - P\) is upper semicontinuous. On \(\{p : D = 0\}\) we have \(N = 1 - P\),
so \( N \) is upper and lower semicontinuous, hence continuous. The same arguments apply to the misère game.

The following simple observations will be useful for the proof of part (ii).

**Lemma 13** (Roots in pairs). Let \( p \) be any offspring distribution with \( 0 < p_0 < 1 \). There is a unique fixed point \( x^* \) of \( F \) in \([0, 1]\). Besides \( x^* \), all other fixed points of \( F^2 \) in \([0, 1]\) can be partitioned into pairs of the form \( \{x, F(x)\} \). If \( p \) is finitely supported (so that \( F^2 \) is a polynomial) and one element of such a pair is a repeated root of \( F^2(x) - x \), then so is the other.

**Proof.** First note that \( F(x) - x \) is positive at 0, negative at 1, and strictly decreasing on \([0, 1]\), so \( F \) has a unique fixed point \( x^* \) in \([0, 1]\). Clearly \( x^* \) is also a fixed point of \( F^2 \). If \( x \) is any fixed point of \( F^2 \) then so is \( F(x) \), and if \( x \neq x^* \) then \( F(x) \neq F(x^*) = x^* \). Moreover if \( x \in [0, 1] \) then \( F(x) \in [0, 1] \). Finally, the derivative of \( F^2(x) - x \) is \( F'(F(x))F'(x) - 1 = \Delta(x) \), say. If \( x \) is a repeated root of \( F^2(x) - x \) then \( \Delta(x) = 0 \), but this implies that \( \Delta(F(x)) = F'(x)F'(F(x)) - 1 = 0 \) also.

**Proof of Theorem 5(ii).** We prove continuity of \( D \); the proof for \( \tilde{D} \) is essentially identical. Let \( Q := \{ p = (p_0, p_1, p_2) : \sum_i p_i = 1, \ p_0 \in (0, 1) \} \) be the relevant set of distributions. Recall from Theorem 1 that \( D \) is the difference between the largest and smallest fixed points of \( F^2 \) (i.e., roots of \( F^2(x) - x \)) in \([0, 1]\).

Suppose for a contradiction that \( D \) is not continuous at \( p \in Q \), so that there exists a continuous family \( (p(t))_{t \in [0, 1]} \) in \( Q \) with \( p(t) \to p(0) = p \) but \( D(p(t)) \to D(p) \) as \( t \to 0 \). The complex roots of a polynomial vary continuously with its coefficients (possibly becoming or ceasing to be coincident, and going off to or arriving from infinity). Therefore, either some root of \( F^2_{p(t)}(x) - x \) must enter the interval \([0, 1]\) at \( t = 0 \), or some complex root must become real.

The first possibility is ruled out because \( F^2_p(0) = F(1 - p_0) \neq 0 \) and \( F^2_p(1) = 1 - p_0 \neq 1 \), so the polynomial \( F^2_p(x) - x \) does not have roots at 0 or 1. Turning to the second possibility, since
the polynomial $F^2(x) - x$ has real coefficients, any nonreal roots come in conjugate pairs, so such a pair must become coincident and real at $t = 0$, so $F_p^2(x) - x$ has a repeated root in $[0, 1]$. Now recall Lemma 13, and note that the special root $x^* = x^*(p(t))$ varies continuously with $t$. It is possible for two roots to become coincident and real and simultaneously coincide with $x^*$ (as indeed happens in many cases), but this would not account for the discontinuity in $D$. Hence, the polynomial $F_p^2(x) - x$ must have a repeated root in $[0, 1]$ that is not at $x^*$. But then by Lemma 13 it must have another repeated root in $[0, 1]$. Hence, there are at least 5 roots in $[0, 1]$, counted with multiplicity. (Essentially, the picture must resemble Figure 4.) But $G_p$ is (at most) a quadratic, so $F_p^2(x) - x$ is (at most) a quartic, which is a contradiction.

We break the proof of Theorem 5(iii) into parts.

**Proposition 14** (Forcing strategy). Consider the escape game, and let $\mu$ be the mean of the offspring distribution $p$. If $p_1 \mu > 1$, then $E^{(1)} > 0$.

**Proof.** We give two explanations, one analytic and one in terms of the game. First, $E^{(1)}$ is the largest solution in $[0, 1]$ of $F(H(x)) - x = 0$. The function $F(H(x))$ is continuous on $[0, 1]$, with $F(H(0)) = 0$ and $F(H(1)) - 1 < 0$. Further one can calculate that the derivative of $F(H(x)) - x$ at $x = 0$ is $p_1 \mu - 1$. Hence if $p_1 \mu > 1$, there must be a solution to $F(H(x)) - x = 0$ somewhere in $(0, 1)$, and hence $E^{(1)} > 0$.

For the alternative argument, consider the set of paths in the tree $T$, starting at the root, with the property that every vertex at odd depth on the path has precisely one child. The union of these paths is a subtree $T'$ containing the root. Each odd-depth vertex of $T'$ has exactly two neighbors: its parent and its unique child. Let $T''$ be the tree obtained by removing every odd-depth vertex from $T'$ and joining its parent directly to itself. Now $T''$ is a Galton–Watson tree; its offspring distribution is the original distribution $p$ thinned by $p_1$ (i.e., conditional on a random variable $M$ distributed according to $p$, the number of offspring of a vertex is Binomial($M$, $p_1$)), with mean $\mu p_1$. If $T''$ is infinite, then Escaper can win the escape game on the original tree $T$, provided he moves first, by always playing in $T'$, so that Stopper never has any choice. Hence, $E^{(1)}$ is at least as large as the survival probability of the Galton–Watson tree $T''$; in particular, if $\mu p_1 > 1$ then $E^{(1)} > 0$.

**Proposition 15** (Perturbation). Let $S := \{p : E^{(1)} = 0\}$ be the set of distributions with zero probability of an Escaper win. If $p \in S$ with $p_1 \mu < 1$, then $S$ also contains a neighborhood of $p$ in the metric space $(M_1, d_1)$.

**Proof.** A distribution is in $S$ if and only if there is no root of $F(H(x)) - x$ in $(0, 1]$. (There is always a root at 0.) Let $p \in S$ with $p_1 \mu < 1$.

The derivative of $F(H(x)) - x$ is $F'(H(x))H'(x) - 1$, which equals $p_1 \mu - 1$ at $x = 0$. By continuity of the generating function, the derivative converges to $p_1 \mu - 1$ as $x \downarrow 0$. Let $\tilde{p} \in S$ be another distribution with corresponding functions $\tilde{F}$ and $\tilde{H}$. Then we have $|F(x) - \tilde{F}(x)| \leq d_1(p, \tilde{p})$ and $|F'(x) - \tilde{F}'(x)| \leq d_1(p, \tilde{p})$ for all $x \in [0, 1]$, and similarly for $H$ and $\tilde{H}$.

Putting these facts together, for any $\epsilon > 0$, there exist $u$ and $\delta_1$ such that if $x \in [0, u]$ and $d_1(p, \tilde{p}) < \delta_1$ then

$$|\tilde{F}'(\tilde{H}(x))\tilde{H}'(x) - \mu_1 p| < \epsilon.$$ 

Hence by choosing $\epsilon$ small enough, we have that the derivative of $\tilde{F}(\tilde{H}(x)) - x$ is negative on all of $[0, u]$. Since $\tilde{F}(\tilde{H}(0)) - 0 = 0$, it follows that $\tilde{F}(\tilde{H}(x)) - x$ has no roots in $(0, u]$.
Now $F(H(x)) - x$ is negative on all of $[u, 1]$, and so (by uniform continuity on closed intervals) is bounded away from 0 on that interval. We have $|\hat{F}(\hat{H}(x)) - F(H(x))| \leq d_1(p, \hat{p})^2$. So we can find $\delta > 0$ such that if $d_1(p, \hat{p}) < \delta_2$ then $\hat{F}(\hat{H}(x)) - x$ has no roots on $[u, 1]$.

Taking $\delta = \min(\delta_1, \delta_2)$, we find that if $d_1(p, \hat{p}) < \delta$ then $\hat{p} \in S$, as required.

**Proof of Theorem 5(iii).** The main claims are immediate from Propositions 14 and 15. To see that $E^{(1)} > 0$ if and only if $E^{(2)} > 0$, note that from (5) and Corollary 9, $E^{(2)} = 1 - H(E^{(1)})$, and that $H$ is strictly decreasing with $H(0) = 1$.

We now complete the proof of Theorem 2. The claimed inequalities were proved in Section 4. Here we give examples to show that further inequalities hold in general.

**Proof of Theorem 2** (counterexamples). We will give examples that rule out any inequality not listed in or implied by Theorem 2(i)–(iii) (see Figure 1 for an illustration).

It will be enough to show that all of the following are possible:

$$S^{(1)} < D; \quad S^{(1)} < E^{(2)}; \quad E^{(1)} < \tilde{P}; \quad D < \tilde{P}; \quad E^{(1)} < D; \quad \tilde{D} < E^{(2)};$$

and that any inequality between any pair of $N, \tilde{N}, P,$ and $S^{(2)}$ is possible except for $S^{(2)} < P$.

We start by giving examples for the possibilities in (8) in turn.

In any case where $p_0 = 0$, we have $S^{(1)} = 0$ and $D = E^{(2)} = 1$, so that the first two cases in (8) hold.

In any nontrivial case where the tree is finite with probability 1, for example, $p_0 = p_1 = 1/2$, we have $E^{(1)} = \tilde{D} = 0$ and $\tilde{P} > 0$, giving the third and fourth cases of (8).

Next consider the case of binary branching in Theorem 1(i) with $p = p_2$ between $\sqrt{3}/2 = 0.866 \ldots$ and $3/2^{5/3} = 0.945 \ldots$. We have $E^{(1)} = 0$ while $D > 0$, so that the fifth case of (8) is possible.

Finally, consider the example $(p_0, p_1, p_2, p_3) = (\epsilon, \frac{2}{3}, 0, \frac{1}{3} - \epsilon)$. For $\epsilon > 0$ this has $D = \tilde{D} = 0$, but for sufficiently small $\epsilon > 0$ we have $p_1 \mu > 1$, and therefore $E^{(2)} > 0$ by Theorem 5. This gives the last case of (8).

We turn to the comparisons between $N, \tilde{N}, P,$ and $S^{(2)}$.

In the trivial case $p_0 = 1$, we have $N = 0$ while $\tilde{N} = P = S^{(2)} = 1$, so that

$$N < \tilde{N}, P, S^{(2)}.$$  \hfill (9)

For the remaining cases, we need a little more work.

Returning to binary branching, take an extreme case with $p_0 = \epsilon$ and $p_2 = 1 - \epsilon$, and $\epsilon \to 0$. Then considering the first three levels of the tree, we get

$$N = 2\epsilon + O(\epsilon^2)$$

$$S^{(2)} = \epsilon + 9\epsilon^2 + O(\epsilon^3)$$

$$P = \epsilon + 4\epsilon^2 + O(\epsilon^3)$$

$$\tilde{N} = \epsilon + 2\epsilon^2 + O(\epsilon^3),$$

so that for small enough $\epsilon$,

$$\tilde{N} < P < S^{(2)} < N.$$  \hfill (10)

Finally, suppose $p_0 = p_1 = 1/K$ and $p_{K^3} = 1 - 2/K$, where $K$ is a large integer. As $K \to \infty$, the following event occurs with probability tending to 1: the root has a child with out-degree 1, and the
unique child of that child is a leaf. The first player can win the misère game by making the first move
to such a child, and so $N \to 1$.

The Escaper can also win with high probability when playing first, by following the strategy in
the proof of Proposition 14 earlier in this section. As $K \to \infty$, the offspring distribution of the tree
$T''$ defined in that proof eventually dominates any distribution that has finite support and that puts
positive weight at 0, so the survival probability tends to 1. Hence, $E^{(1)} \to 1$ and so $S^{(2)} \to 0$. Since from
Theorem 2(ii), we always have $P \leq S^{(2)}$, we obtain that for large enough $K$,

$$P \leq S^{(2)} < \bar{N}. \quad (11)$$

Putting together (9)–(11), we obtain that any inequality between any pair of $N, \bar{N}, P,$ and $S^{(2)}$ is
possible, except for $S^{(2)} < P$. This completes the proof of Theorem 2. □

7 | LENGTH OF THE GAME

In this section, we prove Theorem 6. Initially, we write the proof for the case of the normal game, and
indicate the analogous argument for the case of the misère game at the end.

The function $F$ is strictly decreasing with $F(0) > 0$ and $F(1) = 0$, so has a unique fixed point. We
begin by considering the derivative of $F$ and related functions at this fixed point.

**Lemma 16.** Let $x^*$ be the unique fixed point of $F$.

(a) If $p \in B$, then $F'(x^*) \leq -1$.

More precisely:

(b) If $p \in B'$, then $F'(x^*) < -1$;

(c) If $p \in B \cap \partial B$, then $F'(x^*) = -1$.

Note that since $F(x) = 1 - G(x)$, we have $F'(x) = -G'(x)$. We can also rewrite $F'(x^*)$ in terms of the
function $F^2(x) - x$ which we plotted for example in Figures 2 and 4. Writing $\Delta(x) = \frac{d}{dx}(F^2(x) - x)$ as
in the proof of Lemma 13, we have $\Delta(x^*) = F'(F(x^*))F'(x^*) - 1 = F'(x^*)^2 - 1$. Hence

$$F'(x^*) < -1 \iff \Delta(x^*) > 0; \quad (12)$$

$$F'(x^*) = -1 \iff \Delta(x^*) = 0.$$ 

Before proving Lemma 16, we note a useful technical property:

**Lemma 17.** Let $x^* \in (1/2, 1)$. Then there is an offspring distribution $\hat{p}$ with generating function $\hat{G}$
satisfying $\hat{G}(x^*) = 1 - x^*$ and $\hat{G}'(x^*) > 1$.

**Proof.** We have

$$x^* > 1 - x^*, \quad (13)$$

but $(x^*)^k < 1 - x^*$ for sufficiently large $k$. Hence, there is some $k \geq 2$ such that

$$(x^*)^{k-1} \geq 1 - x^* \quad (14)$$
and 

\[(x^*)^k < 1 - x^*.
\] (15)

Then from (13) and (15), for some \(q \in (0, 1)\), the generating function \(\hat{G}(x) = (1 - q)x + qx^k\) (corresponding to the distribution \(\hat{p}\) with \(\hat{p}_1 = 1 - q\) and \(\hat{p}_k = q\)) has \(\hat{G}(x^*) = 1 - x^*\).

Also (14) gives

\[x^* \geq 1 - (x^*)^{k-1},\]

so that

\[(x^*)^{k-1} \geq \left[1 - (x^*)^{k-1}\right]^{k-1} > 1 - (k-1)(x^*)^{k-1}, \]

and so \(k(x^*)^{k-1} > 1\). Then \(\hat{G}'(x^*) = 1 - q + qk(x^*)^{k-1} > 1\), as required.

\[\blacksquare\]

**Proof of Lemma 16.** The proof of part (a) is very easy. Note that the function \(F^2(x) - x\) is positive at \(x = 0\), is negative at \(x = 1\), and is zero at \(x = x^*\). If in addition \(F'(x^*) > -1\) then by (12), \(F^2(x) - x\) crosses from negative to positive at \(x = x^*\), and so must have at least one fixed point in \((0, x^*)\) and another in \((x^*, 1)\). Then by Theorem 1, \(D > 0\). Hence if \(p \in B\) (i.e., if \(D = 0\)) we must indeed have \(F'(x^*) \leq -1\).

For part (b), suppose indeed that \(F'(x^*) = -1\), that is, \(G'(x^*) = 1\). We will show that \(p\) is not in \(B^c\), by showing that there are points of \(B^c\) arbitrarily close to \(p\).

First note that we must have \(x^* > 1/2\) (excluding the trivial case \(G(x) \equiv x\), that is, \(p_1 = 1\), where \(x^* = 1/2\), since by strict convexity of \(G\),

\[1 = G(1) \]
\[> G(x^*) + (1 - x^*)G'(x^*) \]
\[= 1 - x^* + (1 - x^*) \times 1 \]
\[= 2(1 - x^*).\]

So from Lemma 17, there is an offspring distribution \(\hat{p}\) whose generating function \(\hat{G}\) has \(\hat{G}(x^*) = 1 - x^*\) and \(\hat{G}'(x^*) > 1\). Then for any \(\epsilon > 0\), the distribution \(p_\epsilon := (1 - \epsilon)p + \epsilon\hat{p}\) with generating function

\[G_\epsilon(x) = (1 - \epsilon)G(x) + \epsilon\hat{G}(x)\] (16)

also has \(G_\epsilon(x^*) = 1 - x^*\) and \(G_\epsilon'(x^*) > 1\). Hence by part (a), for all \(\epsilon, p_\epsilon \notin B\). But since \(p_\epsilon\) is arbitrarily close to \(p\) in \(M_0\), we have that \(p \notin B^0\), as required for part (b).

Finally for part (c), suppose that \(p \in B\) with \(F'(x^*) > -1\). We need to show that \(p \in B^0\), that is, that all distributions in some neighborhood of \(p\) in \(M_0\) also have no draws.

The function \(F(F(x)) - x\) has a unique zero at \(x^*\), and has derivative \(\Delta(x) = F'(F(x))F'(x) - 1\) which is continuous on \((0, 1)\) with \(\Delta(x^*) < 0\), as at (12). Hence for some \(\epsilon > 0\),

\[
\frac{d}{dx}(F(F(x)) - x) < -\epsilon \text{ for all } x \in [x^* - \epsilon, x^* + \epsilon].
\] (17)
Also $F(F(x)) - x$ is a continuous function and so attains its bounds on any closed interval; hence for some $\delta > 0$,

$$
|F(F(x)) - x| > \delta \text{ for all } x \in [0, x^* - \epsilon/2] \cup [x^* + \epsilon/2, 1].
$$

(18)

We want to show that properties like (17) and (18) continue to hold if we perturb $p$ slightly. We note the following properties:

(i) $F$ is uniformly continuous on $[0, 1]$.

(ii) For any $x$, the quantity $F(x)$ is continuous as a function of $p$, uniformly in $x$; specifically, for all $p, \tilde{p}$, and $x$,

$$
|F_p(x) - F_{\tilde{p}}(x)| \leq d_0(p, \tilde{p}).
$$

Combining (i) and (ii) with (18), it follows that whenever $d_0(p, \tilde{p})$ is sufficiently small, (18) again holds with $F$ replaced by $F_p$ and $\delta$ by $\delta/2$.

Continuing, note that:

(iii) The function $F$ maps $[x^* - \epsilon/2, x^* + \epsilon/2]$ to some $[a, b]$ with $0 < a < b < 1$.

(iv) $F'$ is uniformly continuous on $[0, z]$, for any $z < 1$; specifically, for all $0 < x < y$,

$$
|F'(x) - F'(y)| \leq \sum_{n=2}^{\infty} n(y^{n-1} - x^{n-1}) \leq |y - x| \sum_{n=2}^{\infty} n^2 x^{n-2}.
$$

(v) For any given $x$, $F'(x)$ is continuous as a function of $p'$; specifically, for all $p, p'$, and $x$,

$$
|F'_p(x) - F'_p(x)| \leq d_0(p, \tilde{p}) \sum_{n=2}^{\infty} nx^{n-1}.
$$

Combining (i)–(v) with (17), and using $\frac{d}{dx} (F(F(x)) - x) = F'(F(x))F'(x) - 1$, it follows that whenever $d_0(p, \tilde{p})$ is sufficiently small, (17) holds with $F$ replaced by $F_p$ and $\epsilon$ replaced by $\epsilon/2$ throughout.

The new versions of (17) and (18) thus obtained then guarantee that for all $\tilde{p}$ in some neighborhood of $p$ in $M_0$, the function $F^2$ has no fixed point outside $[x^* - \epsilon/2, x^* + \epsilon/2]$ and has at most one fixed point inside that interval. Hence by Theorem 1, the game with distribution $p'$ has no draws. This shows that $p$ is in the interior of $B$, as required for (c).

Proof of Theorem 6(i). We wish to show that if $p \in B^o$, then $\mathbb{E}T < \infty$ (and then certainly $\mathbb{E}T^* < \infty$ also, since $T^* \leq T$).

Note that $\mathbb{P}(T > n)$ is the probability that neither player can force a win within $n$ moves, which is $D_n$. Hence

$$
\mathbb{E}T = \sum_{n=0}^{\infty} \mathbb{P}(T > n) = \sum_{n=0}^{\infty} D_n = \sum_{n=0}^{\infty} [(1 - P_n) - N_n].
$$

(19)

Any game won by the first player has odd length, and any game won by the second player has even length. Then as in the proof of Theorem 1, we have

$$
1 - P_{2k-1} = 1 - P_{2k} = F^{2k}(1) \text{ and } N_{2k} = N_{2k+1} = F^{2k}(0).
$$

Since $p \in B$, Theorem 1 gives that $F^2$ has a unique fixed point which is $x^*$, and since $p \in B^o$, Lemma 16 gives that $|F'(x^*)| < 1$. So both $1 - P_n$ and $N_n$ converge exponentially quickly to $x^*$. Hence, the sequence $D_n = 1 - P_n - N_n$ has finite sum, and (19) gives $\mathbb{E}T < \infty$ as required.

$\blacksquare$
We use the following well-known fact.

**Lemma 18.** A Galton–Watson process with mean offspring size 1 has infinite expected depth.

**Proof.** Let \( V \) be the depth of the process, and \( a_n = \Pr(V \geq n) \) the probability that the process survives at least to depth \( n \), so that \( \mathbb{E}V = \sum a_n \). Then for example by conditioning on the number of children of the root in a standard way, we have \( a_{n+1} = 1 - G(1 - a_n) \).

Note that \( G'(1) = 1 \), so that as \( x \uparrow 1 \), Taylor’s theorem gives

\[
G(x) = 1 - (1 - x) + O(1 - x)^2,
\]

and as \( y \downarrow 0 \),

\[
1 - G(1 - y) - y = O(y^2).
\]

As \( n \to \infty \) we have \( a_n \downarrow 0 \), and so

\[
a_{n+1} - a_n = O(a_n^2).
\]

In particular, for some constant \( c \), for large enough \( n \) (say \( n \geq n_0 \)),

\[
a_{n+1} > (1 - ca_n)a_n.
\]

(20)

Since \( a_n \to 0 \), it follows from (20) that \( \prod_{n=n_0}^{\infty} (1 - ca_n) = 0 \). But this is equivalent to \( \sum a_n = \infty \). Hence \( \mathbb{E}V = \infty \), as required.

**Remark.** Suppose we have a sequence of offspring distributions converging in \((M_0, d_0)\) to a limit with mean offspring size 1. The argument in the proof of Lemma 18 shows that the expected depth tends to infinity (since all the probabilities \( \Pr(V \geq n) \) are continuous as functions of the offspring distribution).

**Proof of Theorem 6(ii).** We assume \( p \in B \cap \partial B \). So the probability of a draw is 0, and \( N \), the probability of a first-player win, is equal to \( x^* \), the unique fixed point of \( F \). Also, by Lemma 16, \( G'(N) = -F'(N) = 1 \).

We may mark each node of the tree as an \( \mathcal{N} \)-node (a first-player win), or a \( \mathcal{P} \)-node (a second-player win). The root is an \( \mathcal{N} \)-node with probability \( N \) and a \( \mathcal{P} \)-node with probability \( P = 1 - N \).

With these marks we can see the tree as a two-type branching process. Each \( \mathcal{P} \)-node has only \( \mathcal{N} \)-type children. Conditioned on being an \( \mathcal{P} \)-node, it has a number of children with probability mass function \( \tilde{p}_k \), \( k \geq 0 \) given by \( \tilde{p}_k = p_k N^k / P \), with mean

\[
\tilde{\mu} = \sum k p_k N^k / P = NG'(N) / P.
\]

(21)

Each \( \mathcal{N} \)-node has at least one \( \mathcal{P} \)-type child. Conditional on being a \( \mathcal{N} \)-node, the probability that it has precisely one \( \mathcal{P} \)-type child is \( \tilde{p}_1 \) given by

\[
\tilde{p}_1 = \sum_{j=1}^{\infty} p_j P N^{j-1} / N = PG'(N) / N.
\]

(22)

Now we define a reduced subtree. Call an \( \mathcal{N} \)-node bad if it is the child of another \( \mathcal{N} \)-node. (Such a node is never part of an optimal line of play.) Call a \( \mathcal{P} \)-node bad if it is the child of a
node which has another \( P \)-type child. (The winning player can guarantee to win without visiting this node.)

Remove all the bad nodes, and consider the reduced tree consisting of all those nodes still connected to the root. A node \( v \) is in the reduced tree if the player without a winning strategy can guarantee either not to lose, or to visit \( v \) (as in the definition of the quantity \( T^* \)). In particular, \( T^* \) is the height of this reduced tree.

The reduced tree is a two-type Galton–Watson process. Suppose that the root is a \( P \)-node. Then all the nodes at even levels are \( P \)-nodes, and all the nodes at odd levels are \( \mathcal{N} \)-nodes. The expected number of grandchildren of the root in this reduced tree is the product \( \tilde{\mu}\tilde{p}_1 \) of (21) and (22); this product is \( G'(N)^2 \) which equals 1. If we consider only the nodes at even levels, we obtain a simple Galton–Watson process, with mean offspring size 1, and hence (by Lemma 18) with infinite expected height. This gives \( \mathbb{E}T^* = \infty \) as required.

\[ \square \]

**Proof of Theorem 6(iii) and (iv).** As in (19), \( \mathbb{E}T = \sum_{n \geq 0} D_n \), where \( D_n \) is the probability that neither player can force a win within \( n \) moves.

Suppose that \( \{p^{(m)}_n, m \geq 1\} \) is a sequence of offspring distributions in \( B \) converging in \((M_0, d_0)\) as \( m \to \infty \) to a distribution \( p^{(\infty)} \in \partial B \). Write \( E^{(m)} \) and \( E^{(\infty)} \) for expectations in the models corresponding to \( p^{(m)} \) and \( p^{(\infty)} \), respectively, and similarly \( D^{(m)}_n \) and \( D^{(\infty)}_n \) for the draw probabilities.

We have \( E^{(\infty)}T = \sum_{n = 0}^{\infty} D^{(\infty)}_n = \infty \) (this follows from Theorem 6(ii) in the case \( p^{(\infty)} \in B \cap \partial B \), and from the fact that \( D^{(\infty)}_n \to 0 \) in the case \( p^{(\infty)} \in B \setminus \partial B \)).

For any fixed \( n \), we have \( D^{(m)}_n \to D^{(\infty)}_n \) as \( m \to \infty \), since the distribution of the first \( n \) levels of the tree under \( p^{(m)} \) converges in total variation distance to the distribution under \( p^{(\infty)} \). So \( \lim_{m \to \infty} \sum_{n = 0}^{\infty} D^{(m)}_n \geq \sum_{n = 0}^{K} D^{(\infty)}_n \) for any \( K \). We can make this lower bound arbitrarily large by taking \( K \) large enough, since \( \sum_{n = 0}^{\infty} D^{(\infty)}_n = \infty \). So indeed \( E^{(m)}T = \sum_{n = 0}^{\infty} D^{(m)}_n \to \infty \) as \( m \to \infty \), as required.

Finally, suppose that the limiting distribution \( p^{(\infty)} \) is in \( B \cap \partial B \). To show that the mean of \( T^* \) tends to infinity, we apply a similar argument but now to the reduced tree constructed in the proof of Theorem 6(ii) above.

Write \( x^*(p) \) for the unique fixed point of the function \( F_p \). We have \( x^*(p^{(m)}) \to x^*(p^{(\infty)}) \) as \( m \to \infty \) (since each of \( F_{p^{(m)}} \) and \( F_{p^{(\infty)}} \) is continuous and strictly decreasing, and \( F_{p^{(m)}}(x) \to F_{p^{(\infty)}}(x) \) as \( m \to \infty \) uniformly over \( x \in [0, 1] \)). So the first-player and second-player win probabilities \( N = x^* \) and \( P = 1 - x^* \) also converge to their values under \( p^{(\infty)} \) (since the draw probability \( D \) is 0 in each case).

It follows that for any \( n \), the distribution of the first \( n \) levels of the reduced tree under \( p^{(m)} \) converges in total variation distance as \( m \to \infty \) to the distribution under \( p^{(\infty)} \). Recall that \( T^* \) is the height of this reduced tree. Under the limit distribution, we have \( \mathbb{E}^{(\infty)}T^* = \sum_{n \geq 0} \mathbb{P}^{(\infty)}(T^* > n) = \infty \), by Theorem 6(ii), and by an analogous argument to the one above for \( \mathbb{E}T \), we obtain \( \mathbb{E}^{(m)}T^* \to \infty \) as \( m \to \infty \) as required.

\[ \square \]

We have completed the proof of Theorem 6 for the case of the normal game. One can prove the result for the misère case in an entirely similar way, which we indicate briefly.

Let \( \tilde{x}^* \) be the unique fixed point of the function \( H \). For the misère case, we have analogous criteria to those in Lemma 16 with \( x^* \) replaced by \( \tilde{x}^* \) (note that \( H' \equiv F' \)).

In the proof of Lemma 16, we relied on the fact that \( x^* > 1/2 \) in order to apply Lemma 17. Since \( H \) and \( F \) are both decreasing functions, and \( H \geq F \), we have \( \tilde{x}^* > x^* \), so again \( \tilde{x}^* > 1/2 \). Just as at (16), if we have a distribution \( p \) with generating function \( G \) such that \( H(\tilde{x}^*) = 1 - G(\tilde{x}^*) + G(0) = \tilde{x}^* \) and \( H'(\tilde{x}^*) = -1 \), we can obtain a distribution \( \tilde{p} \) which is arbitrarily close to \( p \) in \( M_0 \), with generating function \( G' \) such that \( H'_\epsilon(\tilde{x}^*) = 1 - G'_\epsilon(\tilde{x}^*) + G'_\epsilon(0) = \tilde{x}^* \), and \( H'_\epsilon(\tilde{x}^*) < -1 \). The rest of the argument goes through identically, with \( \tilde{x}^* \) replacing \( x^* \) and \( H \) replacing \( F \) throughout.
For the proof of Theorem 6(ii) in the misère case, we again consider a two-type Galton–Watson tree, where each node is an $\tilde{\mathcal{N}}$-node (a first-player win for the misère game) or a $\tilde{\mathcal{P}}$-node (a second-player win for the misère game). The root is an $\tilde{\mathcal{N}}$-node with probability $\tilde{N} = \tilde{x}^*$ and a $\tilde{\mathcal{P}}$-node with probability $\tilde{P} = 1 - \tilde{x}^*$.

Again each $\tilde{\mathcal{P}}$-node has only $\tilde{\mathcal{N}}$-type children. Conditional on being a $\tilde{\mathcal{P}}$-node, it has a number of children with mean $\tilde{N}G'(\tilde{N})/\tilde{P}$ just as in (21). In the misère case, each $\tilde{\mathcal{N}}$-node either has at least one $\tilde{\mathcal{P}}$-type child, or has no children at all. Just as in (22), conditional on being a $\tilde{\mathcal{N}}$-node, the probability of having precisely one $\tilde{\mathcal{P}}$-type child is $\tilde{P}G'(\tilde{N})/\tilde{N}$. The product of these two quantities is $G'(\tilde{N})^2$ which again equals 1. The rest of the proof is entirely analogous.

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