A solution selection problem with small symmetric stable perturbations

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Abstract

The zero-noise limit of differential equations with singular coefficients is investigated for the first time in the case when the noise is an $\alpha$-stable process. It is proved that extremal solutions are selected and the respective probability of selection is computed. For this purpose an exit time problem from the half-line, which is of interest in its own right, is formulated and studied by means of a suitable decomposition in small and large jumps adapted to the singular drift.

Keywords: stochastic differential equations, singular drifts, zero-noise limit, Peano phenomena, non-uniqueness, $\alpha$-stable process, persistence probabilities, exit problem, selection of solutions.

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1 Introduction

The zero-noise limit of a stochastic differential equation, with drift vector field $b$ and a Wiener process $W$, say of the form

$$X_t^\varepsilon = x_0 + \int_0^t b(X_s^\varepsilon) \, ds + \varepsilon W_t, \quad t \geq 0, \varepsilon > 0,$$

is a classical subject of probability, see for instance [10]. When the limit deterministic equation

$$X_t = x_0 + \int_0^t b(X_s) \, ds, \quad t \geq 0,$$
is well posed, usually one has $X_t^\varepsilon \to X_t$ a.s. and typical relevant questions are the speed of convergence and large deviations. On the contrary, when the Cauchy problem \((1.2)\) has more than one solution, the first question concerns the selection, namely which solutions of \((1.2)\) are selected in the limit and with which probability. This selection problem is still poorly understood and we aim to contribute with the investigation of the case when the noise is an $\alpha$-stable process.

The case treated until now in the literature is the noise of Wiener type. All known quantitative results are restricted to equations in dimension one. The breakthrough on the subject was due to Bafico and Baldi \([1]\) who solved the selection problem for very general drift $b$ having one point $x_0$ of singularity. The paradigmatic example of $b$ to test the theory is

$$b(x) = \begin{cases} B^+ |x|^{\beta^+} & \text{for } x \geq 0 \\ -B^- |x|^{\beta^-} & \text{for } x < 0. \end{cases}$$

(1.3)

where $B^\pm > 0$, $\beta^\pm \in (0,1)$; the deterministic equation \((1.2)\) with $x_0 = 0$ has infinitely many solutions, which are equal to zero on $[0, \infty)$ or on some interval $[0, t_0]$ (possibly $t_0 = 0$) and then, on $[t_0, \infty)$, they are equal either to $C^+ (t-t_0)^{-\frac{1}{1-\beta^+}}$ or to $-C^- (t-t_0)^{-\frac{1}{1-\beta^-}}$, with $C^\pm$ given in \((3.6)\) of Section 3 a central role will be played by the two extremal solution,

$$x^\pm = \pm C^\pm t^{\frac{1}{1-\beta^\pm}}.$$

The article \([1]\) completely solves the selection problem for this and more general examples, making use of explicit computations on the differential equations satisfied by suitable exit time probabilities; such equations are elliptic PDEs, in general, so they are explicitly solvable only in dimension one (except for particular cases). The final result is that the law $P^W_\varepsilon$, on $C([0,T];\mathbb{R})$, of the unique solution $X_t^\varepsilon$ of equation \((1.1)\) with $x_0 = 0$ and $b$ as in \((1.3)\), satisfies

$$P^W_\varepsilon \overset{w}{\to} p^+ \delta_{x^+} + p^- \delta_{x^-},$$

where $p^- = 1 - p^+$ and

$$p^+ = \begin{cases} 1 & \text{if } \beta^+ < \beta^- \\ \frac{(B^-)^{-\frac{1}{1+\beta^-}}}{(B^+)^{-\frac{1}{1+\beta^+}} + (B^-)^{-\frac{1}{1+\beta^-}}} & \text{if } \beta^+ = \beta^- =: \beta \\ 0 & \text{if } \beta^+ > \beta^- \end{cases}$$

(1.4)

This or part of this result was re-proved later on using other approaches, not based on elliptic PDEs but only on tools of stochastic analysis and dynamical arguments, see \([3,27]\). These investigations are also motivated by the fact that in dimension greater than one the elliptic PDE approach is not possible.
The aim of this paper is to investigate these questions when the Wiener process $W$ is replaced by a general pure-jump $\alpha$-stable process $L$. This process satisfies for any $a > 0$ the following self-similarity condition $(L_{at})_{t \geq 0} \overset{d}{=} (a^{\frac{-\alpha}{\alpha}}L_t + \gamma_0 t)_{t \geq 0}$, for a drift $\gamma_0 \in \mathbb{R}$ which accounts for the asymmetry of the law of $L$. The stochastic differential equation, then, takes the form

$$X_t^\varepsilon = x_0 + \int_0^t b(X_s^\varepsilon) \, ds + \varepsilon L_t, \quad t \geq 0, \varepsilon > 0. \quad (1.5)$$

Here explicit solution of the elliptic equations for exit time probabilities are not feasible and thus it is again an example where we need to understand the problem with new tools and ideas. This feature is similar to the theory of asymptotic first exit times for equations with regular coefficients and small noise, see [7, 13, 15, 21] for recent progresses in the case of Lévy noise. This requires a careful understanding of the role of small and large jumps, which is conceptually new and interesting; technically the more demanding part is the estimate of the Laplace transform of the exit times. Some ingredients are also inspired by [5].

The main result is the following theorem.

**Theorem 1.** If $\alpha > 1 - (\beta^+ \wedge \beta^-)$ and $\beta^+ \neq \beta^-$, then for any $T > 0$

$$P_{L}^\varepsilon \overset{w}{\rightarrow} p^+ \delta_{x^+} + p^- \delta_{x^-}$$

where $P_{L}^\varepsilon$ is the law, on Skorohod space $\mathbb{D}([0,T];\mathbb{R})$, of the unique solution $X_t^\varepsilon$ of equation (1.5) with $x_0 = 0$ and $p^+, p^- = 1 - p^+$ are given by (1.4).

The time interval where this convergence takes place can be chosen to be any bounded interval $[0,T]$, but with a suitable reformulation of the result it may also be an interval which increases like $[0, \varepsilon^{-\theta^*}]$, for suitable $\theta^* > 0$, see the technical statements below; this is a novelty compared with the literature on the Brownian case.

The condition $\alpha > 1 - (\beta^+ \wedge \beta^-)$ appears naturally in the investigation of the local behavior of $X^\varepsilon$ close to the origin. It states that there exists a time scale $t^\varepsilon$ below which, the solution behaves mainly “noise-like”, while for scales larger than $t^\varepsilon$ the drift takes over irresistibly. The condition ensures that this critical time scale tends to 0.

For this purpose we study an asymptotic first exit problem for the strong solution $X^\varepsilon$ of (1.5) from a half-line. This is a problem in its own right. The proof of this result yields an asymptotic lower bound of $X^\varepsilon$ for times beyond the occurrence of the first “large” jump in an appropriate sense as stated in Corollary [7]. Before such first large jump, that is on a time scale up to $\varepsilon^{-\theta^*}$ however, the system exhibits the mentioned behavior similar to a Brownian perturbation. Among the other
technical novelties, there is the use of the linearized system in order to show that excursions away from the origin are large enough.

It is well-known in the literature [7, 14, 16] in the case of systems of stable fixed points or attractors perturbed by a stable perturbation $\varepsilon L$, that the critical time scale is given by $\varepsilon^{-\alpha}$. The following exit time problem establishes that the critical time scale is larger than $\varepsilon^{-\alpha}$.

**Theorem 2.** For any $\beta^+ \in (0, 1)$ and $\alpha \in (0, 2)$ there is a monotonically increasing, continuous functions $\delta^+: (0, 1) \to (0, 1)$ of polynomial order with $\delta_+ \to 0$ as $\varepsilon \to 0$ such that the first exit time

$$
\tau^{x,\varepsilon,-} := \inf\{t > 0 \mid X^{\varepsilon,x}_t \leq \delta^+_\varepsilon\}
$$

of the solution $X^{x,\varepsilon}$ of (1.5) satisfies for any function $m_\varepsilon \to \infty$ with $\limsup_{\varepsilon \to 0} m_\varepsilon \varepsilon^\alpha < \infty$

$$
\lim_{\varepsilon \to 0} \sup_{x \geq 3\delta^+_\varepsilon} \mathbb{P}(\tau^{x,\varepsilon,-} \leq m_\varepsilon) = 0.
$$

The article is structured as follows. After a brief set of notations, we show the previously first exit result of Theorem 2 in Section 3. This is carried out for initial values which may approach 0 as a function of $\varepsilon$, however only sufficiently slowly, as $\varepsilon \to 0$. Section 4 zooms into the behavior of the solution in a space-time box of short temporal and spatial scales around the origin and determines the exit probabilities to each spatial side of the box with the help of the self-similarity of the driving Lévy noise. In Section 5 it is shown that an unstable linearized intermediate regime stabilizes the exit direction from the small environment of the origin and rapidly enhances the solution until it reaches the area of initial values for the regime in Section 3. In Section 6 we prove a slightly stronger result, which implies Theorem 1.

## 2 Preliminaries

For the following notation we refer to Sato [25]. A Lévy process $L$ with values in the real line over a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a stochastic process $L = (L_t)_{t \geq 0}$ starting in $0 \in \mathbb{R}$ with independent and identically distributed increments.

The Lévy-Khintchine formula establishes the following representation of the characteristic function of the marginal law of the Lévy process $Z$. There exists a drift $\gamma \in \mathbb{R}$, $\sigma > 0$ and a $\sigma$-finite Borel measure $\nu$ on $\mathbb{R}$, the so-called Lévy measure, satisfying

$$
\nu\{0\} = 0, \quad \text{and} \quad \int_{\mathbb{R}} (1 \wedge |u|)^2 \nu(du) < \infty, \quad (2.1)
$$
such that for any $t \geq 0$ the characteristic function reads

$$\mathbb{E}[e^{izL_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R},$$

$$\psi(z) = i\gamma z - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{izy} - 1 - izy1\{|y| \leq 1\})\nu(dz). \quad (2.2)$$

The triplet $(\gamma, \sigma, \nu)$ determines the process $L$ in law uniquely.

A symmetric $\alpha$-stable process $L$ in law for $\alpha \in (0, 2)$ is a Lévy process with canonical triplet $(0, 0, \nu)$, where $\nu$ is given as

$$\nu(dy) = \frac{c}{y^{\alpha+1}}1\{y < 0\} + \frac{c}{y^{\alpha+1}}1\{y > 0\}, \quad (2.3)$$

with $c > 0$.

A symmetric $\alpha$-stable processes $L$ in law satisfies the following self-similarity property. Given the Lévy measure associated to $\nu$ for any $a > 0$

$$(L_{at})_{t \geq 0} \overset{d}{=} (a^\frac{1}{\alpha} L_t)_{t \geq 0}. \quad (2.4)$$

For details consult [25], Section 8 and 14.

The Lévy-Itô decomposition [25], Theorem 19.2, yields the pathwise representation

$$L_t = \int_{0}^{t} \int_{0}^{\min\{|y| \leq 1\}} y(N(dsdy) - ds\nu(dy)) + \int_{0}^{t} \int_{|y| > 1} yN(dsdy) \quad \text{for all } t \geq 0, \mathbb{P} \text{-a.s.,} \quad (2.5)$$

where $N([0, t] \times B, \omega) = \#\{s \in [0, t] \in \mathbb{R} \mid (s, \Delta L_t(\omega)) \in B\}$ for $t \geq 0$, $B \in \mathcal{B}(\mathbb{R})$ and $\omega \in \Omega$, is the Poisson random measure associated to $dt \otimes \nu$.

**Proposition 3.** Let $\beta^+, \beta^- \in (0, 1)$, $B^+, B^- > 0$ and $L$ be a pure jump $\alpha$-stable process with $\alpha \geq 1 - (\beta^+ \wedge \beta^-)$ over a given filtered probability space given by (2.5). Then equation (1.5) with these coefficients has a unique strong solution, which satisfies the strong Markov property.

The result is given in Tanaka [26].

### 3 An exit problem from the half-line: Proof of Theorem 2

#### 3.1 Proof of Theorem 2

The proof of Theorem 2 is structured in four parts. After the technical preparation and two essential observations we derive the main recursion. In the last part we conclude.
1) Setting and notation: Let us denote \( u(t;x) := X^{x,0}_t \) for convenience. The first observation is the following. Let \( \delta > 0 \) and \( x \in \mathbb{R} \) an initial value with \( |x| > \delta \). Then \( b|_{ \mathbb{R} \setminus [-\delta,\delta] } \) satisfies global Lipschitz and growth conditions, such that there exists a unique strong local solution, which lives until to the stopping time

\[
\tau^{x,\varepsilon,\delta} := \inf\{ t > 0 \mid X^{x,\varepsilon}_t \in [-\delta,\delta] \}.
\]

Here the Lipschitz constant depends essentially on \( \delta \) and explodes as \( \delta \downarrow 0 \). As usually in this situation, we divide the process \( L = \eta^\varepsilon + \xi^\varepsilon \) by a \( \varepsilon \)-dependent threshold \( \varepsilon^{-\rho} \), where \( \rho \in (0,1) \) is a parameter to be made precise in the sequel. More precisely the compound Poisson process with

\[
\eta^\varepsilon_t = \sum_{i=1}^{\infty} W_i \mathbf{1}\{T_i \leq t\}
\]

with arrival times \( T_i = \sum_{j=1}^{i} t_i \), where \( t_i \) i.i.d. waiting times and i.i.d. “large” jump increments \( (W_i)_{i\in\mathbb{N}} \) with the conditional law

\[
W_i \sim \frac{1}{\lambda_\varepsilon} \nu(\cdot \cap (\mathbb{R} \setminus [-\varepsilon^{-\rho},\varepsilon^{-\rho}])), \tag{3.1}
\]

\[
t_i \sim \text{EXP}(\lambda_\varepsilon) \quad \text{for } i \in \mathbb{N}, \tag{3.2}
\]

where

\[
\lambda_\varepsilon = \nu(\mathbb{R} \setminus [-\varepsilon^{-\rho},\varepsilon^{-\rho}]) = 2 \int_{\varepsilon^{-\rho}}^{\infty} \frac{dy}{y^{\alpha+1}} = \frac{2}{\alpha} \varepsilon^{-\rho}, \tag{3.3}
\]

and the remaining semi-martingale

\[
\xi^\varepsilon = L - \eta^\varepsilon \tag{3.4}
\]

with uniformly bounded jumps, which implies the existence of exponential moments. Let us denote by \( Y^{x,\varepsilon} \) the solution of

\[
Y^{x,\varepsilon}_t = x + \int_0^t b(Y^{x,\varepsilon}_s) \, ds + \varepsilon \xi^\varepsilon_t, \tag{3.5}
\]

which exists uniquely under the same conditions as does \( X^{x,\varepsilon} \). For \( \delta > 0 \) we fix the notation

\[
D^{+}_\delta := (\delta,\infty).
\]

For a function \( \delta : (0,1) \to (0,1) \) with \( \delta_\varepsilon \downarrow 0 \) to be specified later we fix

\[
\tau^{x,\varepsilon,\delta_-} := \inf\{ t > 0 \mid X^{x,\varepsilon}_t \notin D^{+}_{\delta_\varepsilon} \}.
\]

2) Two observations: The following observations reveal the first exit mechanism.
2.1) Up to the first large jump, the deterministic solutions travel sufficiently far: Separation of variables yields the explicit representation for \( t \geq t' \) and \( x \geq 0 \)

\[
    u(t; t', x) = \left( B(1 - \beta)(t - t') + x^{1 - \beta} \right)^{\frac{1}{1 - \beta}}.
\]

(3.6)

Hence for \( z \geq x \) and \( t' = 0 \), we obtain

\[
    P(u(T_1; x) \geq z) = P\left( \left( B(1 - \beta)T_1 + x^{1 - \beta} \right)^{\frac{1}{1 - \beta}} \geq z \right)
    = P\left( T_1 \geq \frac{z^{1 - \beta} - x^{1 - \beta}}{B(1 - \beta)} \right)
    = \exp\left( -\left( z^{1 - \beta} - x^{1 - \beta} \right) \frac{\lambda_\varepsilon}{B(1 - \beta)} \right)
    = P(Z \geq z \mid Z \geq x).
\]

This is the tail of the distribution function of a Weibull distributed random variable \( Z \) with shape parameter \( 1 - \beta \) and scaling parameter

\[
    \left( \frac{\lambda_\varepsilon}{B(1 - \beta)} \right)^{\frac{1}{1 - \beta}} = \frac{\varepsilon^{\frac{\alpha}{1 - \beta}}}{B(1 - \beta)^{\frac{1}{1 - \beta}}}
\]

conditioned on the event \( \{ Z \geq x \} \). We define for \( \Gamma > 1 \) such that \( \Gamma < \frac{1}{1 - \beta} \) and

\[
    \gamma_\varepsilon := \left( \lambda_\varepsilon \Gamma^{\frac{1}{1 - \beta}} - (3\delta_\varepsilon)^{1 - \beta} \right)^{\frac{1}{1 - \beta}} \approx \varepsilon^{\frac{\alpha}{\Gamma^{1 - \beta}}}.
\]

Hence

\[
    \lim_{\varepsilon \to 0^+} \sup_{x \in \mathbb{D}_{3\delta_\varepsilon}} P(u(T_1; x) \geq \gamma_\varepsilon) \to 1,
\]

(3.7)

and

\[
    \sup_{3\delta_\varepsilon \leq x \leq \gamma_\varepsilon} P(u(T_1; x) \leq 2\gamma_\varepsilon) \approx \varepsilon^{1 - \frac{\alpha}{\Gamma^{1 - \beta}}}.
\]

2.2) Control the deviation of the small jump solution from the deterministic solution:

For each \( \rho \in (0, 1) \) there are functions \( \delta_\varepsilon : (0, 1) \to (0, 1) \), \( r_\varepsilon : (0, 1) \to (0, \infty) \) such that

\[
    \varepsilon^{\alpha \rho} r_\varepsilon \to \infty \quad \text{and} \quad \frac{\delta_\varepsilon}{\varepsilon^{1 - \rho} r_\varepsilon} \to \infty.
\]

Put in other terms the first result means \( r_\varepsilon \geq \varepsilon^{\frac{1}{\alpha \rho}} \). We define

\[
    r_\varepsilon := \left| \frac{\ln(\varepsilon)}{\varepsilon^{\alpha \rho}} \right|^2.
\]

(3.8)

For the second expression we have

\[
    \infty \leftarrow \frac{\delta_\varepsilon}{\varepsilon^{1 - \rho} r_\varepsilon} = \frac{\delta_\varepsilon}{\varepsilon^{1 - \rho - \alpha \rho} \varepsilon^{\alpha \rho} r_\varepsilon}.
\]

(3.9)
Therefore a necessary condition for (3.9) to be satisfied is $\delta_\varepsilon \gtrsim \varepsilon^{1-\rho(1+\alpha)}$. We define

$$\delta_\varepsilon := \varepsilon^{1-\rho(1+\alpha)}|\ln(\varepsilon)|^4.$$  

(3.10)

For the right-hand side to tend to 0 is equivalent to

$$\rho < \frac{1}{\alpha + 1}.$$  

(3.11)

In particular for all $\alpha \in (0, 2)$

$$\alpha \rho < \frac{\alpha}{1+\alpha} < \frac{2}{3} < 1.$$  

(3.12)

Since $\xi^\varepsilon$ has exponential moments we can compensate it

$$\tilde{\xi}_t^\varepsilon := \xi_t^\varepsilon - tE[\xi_t^\varepsilon].$$

It is a direct consequence of Lemma 2.1 in [16], which treats the same situation, that for any $c > 0$

$$P(\sup_{t \in [0,r^\varepsilon]} |\varepsilon\tilde{\xi}_t^\varepsilon| > c) \leq \exp\left(-\frac{c}{\varepsilon^{1-\rho(r^\varepsilon)}}\right).$$  

(3.13)

A small direct calculation or Lemma 3.1 in [16] yields that there is constant $h_1 > 0$ such that

$$|E[\varepsilon\xi_t^\varepsilon]| \leq h_1\varepsilon^{1-\rho}.$$  

The choice of $r^\varepsilon$ in (3.8) and $\rho$ in (3.11) we obtain that

$$h_1 r^\varepsilon \varepsilon^{1-\rho} = \varepsilon^{1-(\alpha+1)\rho}|\ln(\varepsilon)|^2 \leq \varepsilon e^{1-(\alpha+1)\rho}|\ln(\varepsilon)|^4 = \delta_\varepsilon.$$  

(3.14)

Hence for any $\varepsilon > 0$ sufficiently small we have $|r^\varepsilon E[\varepsilon\xi_t^\varepsilon]| \leq \delta_\varepsilon$ and infer

$$P(\sup_{t \in [0,T_1]} |\varepsilon\xi_t^\varepsilon| > c) = P(\sup_{t \in [0,T_1]} |\varepsilon\tilde{\xi}_t^\varepsilon| > c) \leq P(\sup_{t \in [0,r^\varepsilon]} |\varepsilon\tilde{\xi}_t^\varepsilon| > c) + P(T_1 > r^\varepsilon) \leq \exp\left(-\frac{c}{\varepsilon^{1-\rho(r^\varepsilon)}}\right) + \exp(-\varepsilon^{\alpha\rho} r^\varepsilon).$$  

(3.15)

Denote $V_{t,x}^\varepsilon = Y_{t,x}^\varepsilon - c - \varepsilon\xi_t^\varepsilon$. The monotonicity of $b$ on $(0, \infty)$ yields on the events $\{t \in [0, T_1]\}$ and $\{\sup_{t \in [0, T_1]} |\varepsilon\xi_t^\varepsilon| \leq 2c\}$ that

$$V_{t,x}^\varepsilon = x - c + \int_0^t b(V_{s,x}^\varepsilon + \varepsilon\xi_t^\varepsilon)ds \geq x - c + \int_0^t b(V_{s,x}^\varepsilon)ds.$$
By (3.14) we may set \( c = \delta \varepsilon \) we obtain

\[
V_t^x \geq x - \delta \varepsilon + \int_0^t b(V_s^x)ds \quad t \in [0,T_1].
\]

Hence an elementary comparison argument implies under these assumptions

\[
V_t^x \geq u(t; x - \delta \varepsilon), \quad \text{for all } t \in [0,T_1], \quad x \geq \delta \varepsilon.
\]

In particular in the preceding setting we take the supremum over all \( x \geq 4\delta \varepsilon \) and obtain

\[
\sup_{x \in D_{3\delta \varepsilon}^+} \mathbb{P}\left( \sup_{t \in [0,T_1]} (Y_t^x - (u(t; x - \delta \varepsilon) - \delta \varepsilon)) < 0 \right)
\]

\[
\leq \mathbb{P}\left( \sup_{t \in [0,T_1]} |\varepsilon \xi| > \delta \varepsilon \right) \leq \exp(-\frac{\delta \varepsilon}{\varepsilon^{1-\rho} \varepsilon}) + \exp(-\varepsilon^{\alpha \rho} \varepsilon) = 2\varepsilon^2. \tag{3.16}
\]

With the identical reasoning we obtain

\[
\sup_{x \geq (i-1)\gamma \varepsilon} \mathbb{P}\left( \sup_{t \in [0,T_1]} (Y_t^x - (u(t; x - \delta \varepsilon) - \delta \varepsilon)) < 0 \right) \leq \mathbb{P}\left( \sup_{t \in [0,T_1]} |\varepsilon \xi| > i\gamma \varepsilon \right)
\]

\[
\leq \exp\left(-\frac{i\gamma \varepsilon}{2\varepsilon^{1-\rho} \varepsilon}\right) + \exp\left(-\frac{i\varepsilon^{\alpha \rho} \varepsilon}{2}\right). \tag{3.17}
\]

Remark 3.1. In the light of the observations 2.1) and 2.2) it is clear that the exit behavior is mainly determined by the behavior of the large jumps \( \varepsilon W_i \).

3) Estimate of the Laplace transform of the exit time: We estimate the Laplace transform of the first exit time. Let \( \theta > 0 \). Then

\[
\sup_{x \in D_{3\delta \varepsilon}^+} \mathbb{E}\left[ e^{-\varepsilon^\alpha \rho T_{x,\varepsilon,\varepsilon}} \right] = \sum_{k=1}^{\infty} \mathbb{E}\left[ e^{-\varepsilon^\alpha \rho T_{x,\varepsilon,\varepsilon}} 1\{T_{x,\varepsilon,\varepsilon} \in (T_{k-1},T_k]\} \right]
\]

\[
\leq \sum_{k=1}^{\infty} \mathbb{E}\left[ e^{-\varepsilon^\alpha \rho T_{k-1}} 1\{T_{x,\varepsilon,\varepsilon} \in (T_{k-1},T_k]\} \right]
\]

\[
\leq \sum_{k=1}^{\infty} \mathbb{E}\left[ e^{-\varepsilon^\alpha \rho T_{k-1}} 1\{T_{x,\varepsilon,\varepsilon} \in (T_{k-1},T_k]\} \right] + \sum_{k=1}^{\infty} \mathbb{E}\left[ e^{-\varepsilon^\alpha \rho T_{k-1}} 1\{T_{x,\varepsilon,\varepsilon} \in (T_{k-1},T_k]\} \right]
\]

\[
=: \mathcal{I}_1(k) + \mathcal{I}_2 =: \mathcal{I}_1 + \mathcal{I}_2.
\]

3.1) The infinite remainder: For the second sum we obtain

\[
\mathcal{I}_2 = \sum_{k=n_\varepsilon}^{\infty} \left( 1 + \frac{\theta \varepsilon^{\alpha} \rho}{\lambda \varepsilon} \right)^k \leq \sum_{k=n_\varepsilon}^{\infty} e^{k \ln \left( 1 + \frac{\theta \varepsilon^{\alpha} \rho}{\lambda \varepsilon} \right)} \leq \sum_{k=n_\varepsilon}^{\infty} e^{-k \frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon}} = e^{-n_\varepsilon \frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon}} \frac{1}{1 - e^{\frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon}}}
\]

\[
\approx \varepsilon \frac{e^{-n_\varepsilon \frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon}}}{\frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon}} = e^{-n_\varepsilon \frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon} \ln \left( \frac{2 \theta \varepsilon^{\alpha}}{\lambda \varepsilon} \right)} =: S_1(\varepsilon). \tag{3.18}
\]
In order to get $S_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, we need the asymptotics
\[ n_\varepsilon \varepsilon^{\alpha(1-\rho)} + \ln(\varepsilon) \to \infty, \quad (3.19) \]
or for simplicity
\[ n_\varepsilon \gtrsim \frac{1}{\varepsilon^{\alpha(1-\rho)}} + |\ln(\varepsilon)|. \]
If we define
\[ n_\varepsilon := \frac{|\ln(\varepsilon)|^2}{\varepsilon^{\alpha(1-\rho)}}, \quad (3.20) \]
we obtain
\[ S_1(\varepsilon) \approx \varepsilon^{2+\alpha(1-\rho)} \to 0 \quad \text{as } \varepsilon \to 0. \]

3.2) **Estimate of the main sum:** The rest of the proof is devoted to estimate $\sum_{k=0}^{n_\varepsilon} I_1(k)$. We define the following events for $y \in D_{\delta_\varepsilon}^+$ and $s, t \geq 0$ by
\[ A_{t,s,y}^- := \{ X_r^\varepsilon \circ \theta_s(y) \in D_{\delta_\varepsilon}^+ \text{ for all } r \in [0,t] \}, \]
\[ B_{t,s,y}^- := \{ X_r^\varepsilon \circ \theta_s(y) \in D_{\delta_\varepsilon}^+ \text{ for all } r \in [0,t) \text{ and } X_t^\varepsilon \circ \theta_s(y) / \notin D_{\delta_\varepsilon}^+ \}. \]
Recall the waiting times $t_k := T_k - T_{k-1}$ and exploit the decomposition
\[ \{ \tau_{x,z}^{-} \in (T_{k-1}, T_k) \} = \bigcap_{i=1}^{k-1} A_{t_i,T_{i-1},X_{T_{i-1}}^\varepsilon}^- \cap \left( \bigcup_{t \in (0,t_k]} B_{t,T_{k-1}}^- \right). \]
3.2.1) Derivation of the recursion for the idealized exit from an unstable point \(0\): We estimate \(I_1(k)\) with the help of the strong Markov property

\[
I_1(k) = \sup_{x \in D_{3\delta_\varepsilon}^+} E \left[ \prod_{i=1}^{k-1} e^{-\theta \lambda_i t_i} 1 \left( A_{T_i-1,T_i-1,x}^- \right) \right] \\
= \sup_{y \geq \gamma_\varepsilon} E \left[ \prod_{i=1}^{k-1} e^{-\theta \lambda_i t_i} 1 \left( A_{T_i-1,T_i-1,y}^- \right) \right] \\
\leq \sup_{y \sim \gamma_\varepsilon} E \left[ e^{-\theta \lambda_Y T_{k-1}} 1 \left( A_{T_{k-1},0,y}^- \right) \right],
\]

where we recall that \(\gamma_\varepsilon = (\lambda_\varepsilon^{-\frac{1}{2}} - (3\delta_\varepsilon)^{1-\beta})^{-\frac{1}{1-\beta}}\). Taking a closer look we may identify the preceding inequality as the recursive estimate

\[
\sup_{x \in D_{3\delta_\varepsilon}^+} E \left[ e^{-\theta \lambda_Y T_{k-1}} 1 \left( \tau_{x,\varepsilon,-} \in (T_{k-1}, T_K) \right) \right] \\
\leq \sup_{y \sim \gamma_\varepsilon} E \left[ e^{-\theta \lambda_Y T_{k-2}} 1 \left( \tau_{y,\varepsilon,-} \in (T_{k-2}, T_{k-1}) \right) \right] \\
\times \sup_{y \sim \gamma_\varepsilon} E \left[ e^{-\theta \lambda_Y T_{1}} 1 \left( A_{T_{1},0,y}^- \right) \right] \left( \sup_{t \in [0,T_1]} (Y_{t,Y,\varepsilon,1} - (u(t; y - \delta_\varepsilon) - \delta_\varepsilon)) \geq 0 \right) \\
+ \sup_{y \sim \gamma_\varepsilon} E \left[ e^{-\theta \lambda_Y T_{1}} 1 \left( A_{T_{1},0,y}^- \right) \right] \left( \sup_{t \in [0,T_1]} (Y_{t,Y,\varepsilon,1} - (u(t; y - \delta_\varepsilon) - \delta_\varepsilon)) < 0 \right) \\
+ \sup_{y \sim \gamma_\varepsilon} E \left[ e^{-\theta \lambda_Y T_{1}} 1 \left( A_{T_{1},0,y}^- \right) \right] \left( u(T_1; y - \delta_\varepsilon) - \delta_\varepsilon + \varepsilon W_1 \leq \lambda_\varepsilon^{-\frac{1}{1-\beta}} \right). \tag{3.21}
\]
The same reasoning yields for all \( 2 \leq i \leq k \) the recursive inequality
\[
\sup_{x \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_{k-1}} I \{r_{x,\varepsilon,-} \in (T_{k-1}, T_k)\} \right] \\
\leq \sup_{y \geq T_k} \mathbb{E} \left[ e^{-\theta \lambda_c T_{k-2}} I \{r_{y,\varepsilon,-} \in (T_{k-2}, T_{k-1})\} \right] \\
\cdot \sup_{y \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{0 \leq y \leq y_{i,0,\gamma} \} \right] \\
+ \sup_{y \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{y_{i,0,\gamma} < y \leq y_{i,1,\gamma} \} \right] \\
+ \sup_{y \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{y_{i,1,\gamma} < y \leq y_{i,2,\gamma} \} \right].
\]

Hence solving the recursion we obtain
\[
\mathcal{I}_1(k) \leq \prod_{j=1}^{k-1} \sup_{y \geq (j-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{0 \leq y \leq y_{j,0,\gamma} \} \right] \\
\cdot \sup_{y \geq (k-1)\gamma} \mathbb{P} (\tau_{y,\varepsilon,-} \in (0, T_1]) \\
+ \sum_{i=1}^{k-2} \sup_{y \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{y_{i,0,\gamma} < y \leq y_{i,1,\gamma} \} \right] \\
+ \sum_{i=1}^{k-2} \sup_{y \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{y_{i,1,\gamma} < y \leq y_{i,2,\gamma} \} \right].
\text{(3.22)}
\]

3.2.2) Estimate of the second sum of the recursion (3.22): By (3.13) and (3.17) there exists \( \varepsilon_0 \in (0, 1) \) such that for \( \varepsilon \in (0, \varepsilon_0) \)
\[
\sum_{k=1}^{n_\varepsilon} \sum_{i=1}^{k-2} \sup_{y \in (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{y_{i,0,\gamma} < y \leq y_{i,1,\gamma} \} \right] \\
\leq n_\varepsilon \sum_{i=1}^{\infty} \sup_{y \geq (i-1)\gamma} \mathbb{E} \left[ e^{-\theta \lambda_c T_1} I \{y_{i,0,\gamma} < y \leq y_{i,1,\gamma} \} \right] \\
\leq 2n_\varepsilon (\exp(-\frac{\delta_\varepsilon}{\varepsilon^{1-\rho} - \varepsilon^{1-\rho} T}) + \exp(-\varepsilon^{1-\rho} T)) =: S_2(\varepsilon) \leq 0,
\]

with the convention \( \sum^{-1} = 0 \). We determine the order of \( S_2 \)
\[
n_\varepsilon (\exp(-\frac{\delta_\varepsilon}{\varepsilon^{1-\rho} T}) + \exp(-\varepsilon^{1-\rho} T)) \\
= |\ln(\varepsilon)|^2 \varepsilon^{-\alpha(1-\rho)} \exp(-\varepsilon^{1-\rho(1+\alpha)} |\ln(\varepsilon)|^4) + |\ln(\varepsilon)|^2 \varepsilon^{-\alpha(1-\rho)} \exp(-\varepsilon^{1-\rho} |\ln(\varepsilon)|^2 \varepsilon^{\alpha(1-\rho)}) \\
= 2|\ln(\varepsilon)|^2 \varepsilon^{2-\alpha + \alpha\rho}.
\]

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3.2.3) Estimate of the third sum in the recursion (3.22): For $i = 0$ and $0 < \varepsilon \leq \varepsilon_0$ we perform the core calculation of the article. The idea is the following: $X_t^{x, \varepsilon} \geq_{\varepsilon} u(t; x - \delta) - 2\delta + \varepsilon W_1 \{ t = T_1 \}$ for all $t \in [0, T_1]$. For small $\varepsilon$ and $3\delta \leq x \leq \gamma_\varepsilon$ the solution $u(T_1, x - \delta) - \delta$ escapes sufficiently far away from $x$, that is $u(T_1, x - \delta) - \delta \geq 2\gamma_\varepsilon$, such that the probability that $u(T_1, x - \delta) - \delta + \varepsilon W_1 < \gamma_\varepsilon$ decays sufficiently fast.

3.2.3.1) Estimate of the backbone decomposition of the first exit event: Due to the independence of $T_1$ and $W_1$ we may calculate for $\gamma_\varepsilon(x) = \frac{(2\gamma_\varepsilon + \delta)^{1-\beta} - (x - \delta)^{1-\beta}}{B(1-\beta)}$

$$\sup_{3\delta < x \leq \gamma_\varepsilon} \mathbb{E} \left[ e^{-\theta \lambda \varepsilon T_1} \mathbf{1}(u(T_1; x - \delta) - \delta + \varepsilon W_1 \leq \gamma_\varepsilon) \right]$$

$$\leq \sup_{3\delta < x \leq \gamma_\varepsilon} \mathbb{E} \left[ e^{-\theta \lambda \varepsilon T_1} (u(T_1; x - \delta) - \delta + \varepsilon W_1 \leq \gamma_\varepsilon) \mathbf{1}(u(T_1; x - \delta) > 2\gamma_\varepsilon + \delta) \right]$$

$$+ \sup_{3\delta < x \leq \gamma_\varepsilon} \mathbb{P}(u(T_1; x - \delta) \leq 2\gamma_\varepsilon + \delta)$$

$$= \sup_{3\delta < x \leq \gamma_\varepsilon} \int_{\gamma_\varepsilon(x)}^\infty \mathbb{P}(u(t; x - \delta) - \delta + \varepsilon W_1 \leq \gamma_\varepsilon) \lambda e^{-\lambda \varepsilon t} dt + \sup_{3\delta < x \leq \gamma_\varepsilon} \mathbb{P}(u(T_1; x - \delta) \leq 2\gamma_\varepsilon + \delta)$$

(3.23)

The second term is known from (3.7) and tends to 0. It remains to calculate the first one.

$$\sup_{3\delta < x \leq \gamma_\varepsilon} \int_{\gamma_\varepsilon(x)}^\infty \mathbb{P}(u(t; x - \delta) - \delta + \varepsilon W_1 \leq \gamma_\varepsilon) \lambda e^{-\lambda \varepsilon t} dt$$

$$= \sup_{3\delta < x \leq \gamma_\varepsilon} \int_{\gamma_\varepsilon(x)}^\infty \nu((-\infty, \frac{1}{\varepsilon}(\gamma_\varepsilon - (u(t; x - \delta) - \delta)))] e^{-\lambda \varepsilon t} dt$$

$$= \sup_{3\delta < x \leq \gamma_\varepsilon} \int_{\gamma_\varepsilon(x)}^\infty \nu((-\infty, \frac{1}{\varepsilon}(\gamma_\varepsilon + \delta - (B(1-\beta)t + (x - \delta)^{1-\beta})^{\frac{1}{1-\beta}})))] e^{-\lambda \varepsilon t} dt$$

$$= \sup_{3\delta < x \leq \gamma_\varepsilon} \frac{\alpha \varepsilon^\alpha}{4 \lambda \varepsilon} \frac{1}{\gamma_\varepsilon(x)} \int_{\gamma_\varepsilon(x)}^\infty \frac{1}{(B(1-\beta)t + (x - \delta)^{1-\beta})^{\frac{1}{1-\beta}} - (\gamma_\varepsilon + \delta)} \lambda e^{-\lambda \varepsilon t} dt$$

$$\leq \frac{\alpha \varepsilon^\alpha}{2 \lambda \varepsilon} \frac{1}{\gamma_\varepsilon(x)}.$$

(3.24)

In the last step we have used the fact that the integrand is monotonically decreasing in the variable $t$ and weight $c^-$ of of the negative branch of the Lévy measure. The term

$$\frac{\varepsilon^\alpha}{\lambda \varepsilon} \frac{1}{\gamma_\varepsilon(x)} \approx_{\varepsilon} \varepsilon^{\alpha(1-\rho) + \frac{\alpha^2}{2(1-\beta)}}.$$
converges to 0 as $\varepsilon \to 0$. This gives an estimate for the last term in (3.21). The last term in (3.22) deals with initial values $(i-1)\gamma_\varepsilon < x \leq i\gamma_\varepsilon$. We obtain for
\begin{equation}
\gamma_\varepsilon^*(i, x) := \frac{(i+1)\gamma_\varepsilon + \delta_\varepsilon}{B(1-\beta)}
\end{equation}
with the analogous calculations the following estimate
\begin{align*}
&\sup_{(i-1)\gamma_\varepsilon < x \leq i\gamma_\varepsilon} \mathbb{E}\left[ e^{-\theta \lambda_\varepsilon T_1} \mathbf{1}\left( u(T_1; x - \delta_\varepsilon) - \delta_\varepsilon + \varepsilon W_1 \leq \gamma_\varepsilon \right) \right] \\
&= \sup_{(i-1)\gamma_\varepsilon < x \leq i\gamma_\varepsilon} \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \int_0^\infty \frac{1}{\gamma_\varepsilon^*(i, x)} ((B(1-\beta)t + (x - \delta_\varepsilon)^{1-\beta} - (\gamma_\varepsilon - \delta_\varepsilon)^{1-\beta})^{1/(1-\beta)} - (\gamma_\varepsilon - \delta_\varepsilon)^{1/(1-\beta)})^{1/(1-\beta) - \lambda_\varepsilon e^{-\lambda_\varepsilon t} dt} \\
&\quad + \sup_{(i-1)\gamma_\varepsilon < x \leq i\gamma_\varepsilon} \mathbb{P}(u(T_1; x - \delta_\varepsilon) \leq (i+1)\gamma_\varepsilon + \delta_\varepsilon) \\
&\leq \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \gamma_\varepsilon^{\alpha i^\alpha} + \sup_{(i-1)\gamma_\varepsilon < x \leq i\gamma_\varepsilon} \mathbb{P}(u(T_1; x - \delta_\varepsilon) \leq (i+1)\gamma_\varepsilon + \delta_\varepsilon) \tag{3.25}
\end{align*}
Combining the estimates (3.23), (3.24) and (3.25) we obtain for any $C > 1$
\begin{align*}
\sum_{i=1}^{k-2} &\sup_{y \geq (i-1)\gamma_\varepsilon \vee 3\delta_\varepsilon} \mathbb{E}\left[ e^{-\theta \lambda_\varepsilon T_1} \mathbf{1}\left( u(T_1; y - \delta_\varepsilon) - \delta_\varepsilon + \varepsilon W_1 \leq \lambda_\varepsilon^{-\frac{1}{(1-\beta)}} \right) \right] \\
&= \sum_{i=1}^{k-2} \sup_{j \geq i} \sup_{(j-1)\gamma_\varepsilon \vee 3\delta_\varepsilon < y \leq j\gamma_\varepsilon} \mathbb{E}\left[ e^{-\theta \lambda_\varepsilon T_1} \mathbf{1}\left( u(T_1; y - \delta_\varepsilon) - \delta_\varepsilon + \varepsilon W_1 \leq \lambda_\varepsilon^{-\frac{1}{(1-\beta)}} \right) \right] \\
&\leq \varepsilon \sum_{j \geq i} \left( \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \gamma_\varepsilon^{\alpha j^\alpha} + \sup_{(j-1)\gamma_\varepsilon < x \leq j\gamma_\varepsilon} \mathbb{P}(u(T_1; x - \delta_\varepsilon) \leq (j+1)\gamma_\varepsilon + \delta_\varepsilon) \right) \\
&\leq \varepsilon \sum_{i=1}^{k-2} \left( \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \gamma_\varepsilon^{\alpha i^\alpha} + C(1 - \exp(-[(i+1)1-\beta - i^{1-\beta} \gamma_\varepsilon^{1-\beta} \lambda_\varepsilon]) \right) \\
&\leq \sum_{i=1}^{k-2} \left( \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \gamma_\varepsilon^{\alpha i^\alpha} + C(i+1)^{1-\beta} - i^{1-\beta} \gamma_\varepsilon^{1-\beta} \lambda_\varepsilon \right) \\
&\leq \sum_{i=1}^{k-2} \left( \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \gamma_\varepsilon^{\alpha i^\alpha} + \frac{C \gamma_\varepsilon^{1-\beta} \lambda_\varepsilon}{i^\beta} \right) \\
&= \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} \sum_{i=1}^{k-2} \frac{1}{i^\alpha} + \frac{C \gamma_\varepsilon^{1-\beta} \lambda_\varepsilon}{i^\beta} \\
&\leq C \frac{\alpha c^- \varepsilon_\alpha}{2 \lambda_\varepsilon} k^{1-\alpha} + \frac{C \gamma_\varepsilon^{1-\beta} \lambda_\varepsilon}{k^\beta}. \tag{3.26}
\end{align*}
Hence we may sum up
\[
\sup_{3\delta \leq x \leq \gamma} \mathbb{E}\left[e^{-\theta \lambda_{\varepsilon} T_1 \mathbf{1}} \left(u(T_1; x - \delta) - \delta + \varepsilon W_1 \leq \gamma \right)\right]
\]
\begin{align*}
&+ \sum_{k=2}^{n} \sum_{i=1}^{k-2} \sup_{y > (i-1)\gamma \vee 3\delta} \mathbb{E}\left[e^{-\theta \lambda_{\varepsilon} T_1 \mathbf{1}} \left(u(T_1; y - \delta) - \delta + \varepsilon W_1 \leq \lambda^{1-\rho} \right)\right] \\
&\lesssim \varepsilon^\alpha \frac{1}{\gamma^\gamma} + \lambda^1 \frac{C - \alpha \varepsilon^\alpha}{2 \lambda^\gamma} (n\varepsilon)^{2-\alpha} + \frac{C}{B(1-\beta)} \lambda^{1-\beta} (n\varepsilon)^{2-\beta} =: S_3(\varepsilon) \quad (3.27)
\end{align*}

**3.2.3.2) Conditions on parameters in order to establish the convergence \( S_3(\varepsilon) \to 0 \):**

- We check the order of the second to last expression on the right-hand side

\[
\varepsilon^{\alpha(1-\rho(1-1/(1-\beta)))} \varepsilon^\alpha \approx \varepsilon^{\alpha(1-\rho(1-1/(1-\beta))) - \alpha(2-\alpha)(1-\rho)} |\ln(\varepsilon)|^{2(2-\alpha)}
\]

\[
= \varepsilon^{\alpha((1-\rho(1-1/(1-\beta))) - (2-\alpha)(1-\rho))} |\ln(\varepsilon)|^{2(2-\alpha)}.
\]

The essential sign of the exponent hence is given as the sign of

\[
(1-\rho) + \frac{\rho\alpha}{\Gamma(1-\beta)} - (2-\alpha)(1-\rho) = (\alpha-1)(1-\rho) + \frac{\rho\alpha}{\Gamma(1-\beta)}. \quad (3.28)
\]

- For \( 1 \leq \alpha < 2 \) the sign is positive, since all terms are nonnegative and the last term is positive.

- For \( 0 < \alpha < 1 \) we calculate that the positivity of \((3.28)\)

\[
0 < -(1-\alpha)(1-\rho) + \frac{\rho\alpha}{2(1-\beta)} = -(1-\alpha) + \rho\left[\frac{\alpha}{2(1-\beta)} + (1-\alpha)\right]
\]

is equivalent to

\[
\rho_0(\alpha, \beta) := \frac{\Gamma(1-\alpha)(1-\beta)}{\Gamma(1-\alpha)(1-\beta) + \alpha} < \rho
\]

where the right-hand side is strictly less than 1. Hence in this case the sign is positive if we choose \( \rho_0 < \rho < 1 \).

- For the second expression on the right-hand side we obtain

\[
\gamma^{1-\beta} \lambda \varepsilon (n\varepsilon)^{2-\beta} \approx \varepsilon^{-(1-\beta)(1-\rho) + \alpha} \varepsilon^{-(1-\rho)(2-\beta)} |\ln(\varepsilon)|^{2-\beta} = \varepsilon^{\alpha\rho(1-1/\Gamma) - \alpha(1-\rho)(1-\beta)} |\ln(\varepsilon)|^{2-\beta}.
\]

The positivity of the exponent depends on the sign of

\[
0 < (1 - \frac{1}{\Gamma}) \rho - (1-\rho)(1-\beta) = \rho((1 - \frac{1}{\Gamma}) + (1 - \beta)) - (1 - \beta),
\]

which is equivalent to

\[
\rho > \frac{(1 - \frac{1}{\Gamma})(1-\beta)}{(1 - \frac{1}{\Gamma})(1-\beta) + 1} =: \rho_1(\beta).
\]

Since \( \rho_1(\beta) < 1 \) for all \( \rho_1 < \rho < 1 \) the second exponent is also positive.
3.2.3.3) Verify the compatibility of the choice of convergent parameters: We check that the parameters $\beta$ and $\alpha$ are compatible with $\rho < \frac{1}{1+\alpha}$ in (3.11), which ensures that $\delta_\varepsilon \to 0$, as $\varepsilon \to 0$. The first convergence in (3.27) yields

$$\rho_0 = \frac{\Gamma(1-\alpha)(1-\beta)}{\Gamma(1-\alpha)(1-\beta) + \alpha} < \frac{1}{1+\alpha} \iff \frac{\Gamma - 1 - \Gamma \beta}{\Gamma(1-\beta)} < \alpha,$$

where the left hand side is negative, since $\Gamma < \frac{1}{1-\beta}$. Hence it does not impose any additional restriction on $\alpha$. The second condition yields

$$\rho_1 = \frac{(1-\frac{1}{\tau})(1-\beta)}{(1-\frac{1}{\tau})(1-\beta) + 1} < \frac{1}{1+\alpha} \iff \frac{1}{(1-\frac{1}{\tau})(1-\beta)} > \alpha,$$

In order to get rid of any restriction on $\alpha$ we calculate

$$2 \leq \frac{1}{(1-\frac{1}{\tau})(1-\beta)} \iff \Gamma \leq \frac{2(1-\beta)}{2(1-\beta) - 1}.$$

We can always choose

$$\Gamma := \frac{1}{2} \left(1 + \frac{1}{2} \left( \frac{1}{1-\beta} + \frac{2(1-\beta)}{2(1-\beta) - 1} \right) \right) \quad (3.29)$$

$$\rho := \frac{1}{2} \left( \rho_1(\beta) + \frac{1}{1+\alpha} \right). \quad (3.30)$$

satisfying all conditions required before.

3.2.4) Estimate of the first sum of the recursion (3.22): It remains to estimate the expression

$$\sum_{n \in \mathbb{N}} \prod_{k=1}^{n-1} \sup_{j \geq (j-1)\gamma \vee 3\delta_\varepsilon} \mathbb{E} \left[ e^{-\theta \lambda T_1} 1 \left( A_{T_1,0,0}^\varepsilon \right) 1 \left\{ \sup_{t \in [0,T_1]} \left( Y_{t,\varepsilon,1}^\gamma - (u(t;y - \delta_\varepsilon) - \delta_\varepsilon) \right) \geq 0 \right\} \right]$$

$$\cdot \sup_{j \geq (k-1)\gamma \vee 3\delta_\varepsilon} \mathbb{P} \left( \tau_{\varepsilon,1} \in (0,T_1) \right).$$

3.2.4.1) We estimate the factors one by one: For $j \geq 2$

$$\sup_{j \geq (j-1)\gamma \vee 3\delta_\varepsilon} \mathbb{E} \left[ 1 \left( A_{T_1,0,0}^\varepsilon \right) 1 \left\{ \inf_{t \in [0,T_1]} \left( Y_{t,\varepsilon,1}^\gamma - (u(t;y - \delta_\varepsilon) - \delta_\varepsilon) \right) \geq 0 \right\} \right]$$

$$\leq \varepsilon - (1-C) \mathbb{P}(\varepsilon W_1 < -(j-1)\gamma_\varepsilon)$$

$$= 1 - \frac{(1-C)}{2} \left( \frac{\varepsilon}{(j-1)\gamma_\varepsilon} \right)^{\alpha \rho} \quad (3.31)$$
and for $j = 1$

$$\sup_{y \in D^+_{3\varepsilon}} \mathbb{E} \left[ 1 \left( A_{T_1,0,y} \right) 1 \{ \inf_{t \in [0,T_1]} (Y^\varepsilon_{y,t} - (u(t; y - \delta \varepsilon) - \delta \varepsilon)) \geq 0 \} \right]$$

$$\leq \sup_{y \in D^+_{3\varepsilon}} \mathbb{E} \left[ 1 \left( A_{T_1,0,y} \right) 1 \{ \inf_{t \in [0,T_1]} (Y^\varepsilon_{y,t} - (u(t; y - \delta \varepsilon) - \delta \varepsilon)) \geq 0 \} 1 \{ u(T_1, y - \delta \varepsilon) \geq 2 \gamma \varepsilon \} \right]$$

$$+ \sup_{y \in D^+_{3\varepsilon}} \mathbb{P}(u(t; y) \leq 2 \gamma \varepsilon + \delta \varepsilon)$$

$$\lesssim \varepsilon \left( 1 - (1-C) \mathbb{P}(\varepsilon W_1 < -\gamma \varepsilon) + C \lambda^{1-\frac{1}{\alpha}} \right)$$

$$\lesssim \varepsilon \left( 1 - \frac{(1-C)}{2} \left( \frac{\varepsilon}{\gamma \varepsilon} \right)^{\alpha \rho} + C \varepsilon^{\alpha \rho (1 + \frac{1}{\alpha})} \right). \quad (3.32)$$

We estimate for $k \geq 2$ with the help of (3.16)

$$\sup_{y \geq (k-1) \gamma \varepsilon} \mathbb{P} \left( \tau_{x,\varepsilon, -} \in (0, T_1) \right)$$

$$\leq \mathbb{P} \left( W_1 < -(k-1) \frac{\gamma \varepsilon}{\varepsilon} \right) + \sup_{y \geq (k-1) \gamma \varepsilon} \mathbb{P} \left( \sup_{t \in [0,T_1]} (Y^\varepsilon_{y,t} - (u(t; y) - \delta \varepsilon)) > 0 \right)$$

$$\leq \frac{1}{2} \left( \frac{\varepsilon}{\gamma \varepsilon} \right)^{\alpha \rho} \frac{1}{(k-1)^{\alpha \rho}} + \sup_{y \in D^+_{3\varepsilon}} \mathbb{P} \left( \sup_{t \in [0,T_1]} (Y^\varepsilon_{y,t} - (u(t; y) - \delta \varepsilon)) > 0 \right) \quad (3.33)$$

$$\leq \frac{1}{2} \left( \frac{\varepsilon}{\gamma \varepsilon} \right)^{\alpha \rho} \frac{1}{(k-1)^{\alpha \rho}} + 2 \varepsilon^2 \quad (3.34)$$

whereas for $k = 1$

$$\sup_{y \in D^+_{3\varepsilon}} \mathbb{P} \left( \tau_{x,\varepsilon, -} \in (0, T_1) \right)$$

$$\leq \frac{1}{2} \left( \frac{\varepsilon}{\gamma \varepsilon} \right)^{\alpha \rho} + \sup_{y \in D^+_{3\varepsilon}} \mathbb{P} \left( \sup_{t \in [0,T_1]} (Y^\varepsilon_{y,t} - (u(t; y) - \delta \varepsilon)) > 0 \right) + \sup_{y \in D^+_{3\varepsilon}} \mathbb{P}(u(T_1, y) \leq \gamma \varepsilon + \delta \varepsilon)$$

$$\leq \frac{1}{2} \left( \frac{\varepsilon}{\gamma \varepsilon} \right)^{\alpha \rho} + \sup_{y \in D^+_{3\varepsilon}} \mathbb{P} \left( \sup_{t \in [0,T_1]} (Y^\varepsilon_{y,t} - (u(t; y) - \delta \varepsilon)) > 0 \right) + \frac{2}{B(1-\beta)} \lambda^{1-\frac{1}{\alpha}} \varepsilon \quad (3.35)$$

where the last term is known from (3.7).
3.2.4.2) Estimate of the entire sum: Collecting the previous (3.31), (3.32), (3.34), (3.35) and for the small noise estimate (3.17) together with (3.26) we continue

\[ \sum_{k=1}^{n_k} \prod_{j=1}^{k-1} \sup_{y \in (j-1)\gamma \vee 3\delta} \mathbb{E}[e^{-\theta \lambda T_1} 1(A_{T_1,y}) 1 \{ \sup_{t \in [0,T_1]} (Y_t^{y,\varepsilon,1} - (u(t;y;\delta) - \delta)) \geq 0 \}] \]

\[ \cdot \sup_{y \in (k-1)\gamma \vee 3\delta} \mathbb{P}\left( \tau^{y,\varepsilon} \in (0,T_1) \right) \leq \frac{1}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} + \sup_{y \in D_{3\delta}} \mathbb{P}\left( \sup_{t \in [0,T_1]} (Y_t^{y,\varepsilon,1} - (u(t;y) - \delta)) > 0 \right) + \frac{2}{B(1-\beta)} \lambda_1^{1-\frac{1}{\alpha}} \]

\[ + \frac{1}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \sum_{k=1}^{n_k} \left( 1 - \frac{1 - C}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \right)^{k-1} \frac{1}{k^{1+\rho}} + C \varepsilon^2 \sum_{k=2}^{n_k} \left( 1 - \frac{1 - C}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \right)^{k-2} \frac{1}{k^{1+\rho}}. \]

We identify

\[ \frac{1}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \sum_{k=1}^{n_k} \left( 1 - \frac{1 - C}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \right)^{k-1} \frac{1}{k^{1+\rho}} \lesssim \varepsilon^\kappa \text{Li}_{\alpha \rho} \left( 1 - \frac{1 - C}{2} \varepsilon^\kappa \right), \]

where

\[ \kappa = \alpha \rho \left( 1 + \frac{\alpha \rho}{\Gamma(1-\beta)} \right) \]

and Li\_a(x) = \( \sum_{k=1}^{\infty} \frac{x^k}{k^a} \) is the polylogarithm function with parameter \( a \in \mathbb{R} \) and \( x \in (0,1) \), a well-known analytic extension of the logarithm. Recall that \( \alpha \rho < \frac{1}{1+\alpha} < 1 \) due to (3.12). By the following representation [17], Section 25.12, for \( a \neq \mathbb{N} \) and \( 0 < x < 1 \), given by

\[ \text{Li}_a(x) = \Gamma(1-a)(\ln(x))^a - \sum_{n=0}^{\infty} \zeta(a-n) \frac{(\ln(x))^n}{n!}, \quad (3.36) \]

we obtain that for \( a \in (0,1) \)

\[ \lim_{x \to 1} \text{Li}_a(x)/(1-x)^{a-1} = \Gamma(1 - a). \]

Hence there is \( C > 0 \) such that for \( \varepsilon \in (0,\varepsilon_0) \) sufficiently small

\[ \frac{1}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \sum_{k=1}^{n_k} \left( 1 - \frac{1 - C}{2} \left( \frac{\varepsilon}{\gamma} \right)^{\alpha \rho} \right)^{k-1} \frac{1}{k^{1+\rho}} \lesssim \varepsilon^\kappa \text{Li}_{\alpha \rho} \left( 1 - \frac{1 - C}{2} \varepsilon^\kappa \right) \]

\[ \lesssim C \varepsilon^\kappa \varepsilon^{-\kappa(1-\alpha \rho)} = \varepsilon^{\kappa \alpha \rho} = S_4(\varepsilon) \lesssim 0. \]
The same polylogarithmic asymptotics is carried out for
\[ C\varepsilon^2 \sum_{k=2}^{\infty} \left( 1 - \frac{1 - C}{2} \left( \frac{\varepsilon}{\gamma \varepsilon} \right)^{\alpha \rho} \right) \frac{1}{k^{\alpha \rho}} \]
\[ \leq C\varepsilon^2 \text{Li}_{\alpha \rho} \left( 1 - \frac{1 - C}{2} \varepsilon^\kappa \right) \]
\[ \leq C\varepsilon^2 \varepsilon^{-(1-\alpha \rho)} = \varepsilon^{2-\kappa - \alpha \rho} = S_5(\varepsilon) \downarrow 0, \]

since due to \( \Gamma(1 - \beta) < 1 \)
\[ 2 - (\frac{\alpha \rho}{\Gamma(1 - \beta)} + 1)(\alpha \rho - 1) \geq 2 - (\alpha \rho - 1)(\alpha \rho + 1) = 2 - (\alpha \rho^2 - 1) = 3 - (\alpha \rho)^2 > 0. \]

4) Estimate of the exit probabilities: For any \( m > 0 \)
\[ \sup_{y \in D_{\delta \varepsilon}^+} \mathbb{P}(\tau_{y,\varepsilon} \leq m) = \sup_{y \in D_{\delta \varepsilon}^+} \mathbb{P}(e^{-\theta \varepsilon \alpha \tau_{y,\varepsilon}} \geq e^{\theta \varepsilon \alpha m}) \]
\[ \leq \sup_{y \in D_{\delta \varepsilon}^+} \mathbb{E} \left[ e^{-\theta \varepsilon \alpha \tau_{y,\varepsilon}} \right] e^{\theta \varepsilon \alpha m} \]
\[ \leq C(S_1(\varepsilon) + S_2(\varepsilon) + S_3(\varepsilon) + S_4(\varepsilon) + S_5(\varepsilon)) e^{\theta \varepsilon \alpha m}. \]

Replacing \( m \) by \( m_\varepsilon \) with \( \lim \sup_{\varepsilon \to 0} m_\varepsilon \varepsilon^{\alpha} < \infty \) we obtain
\[ \sup_{y \in D_{3\delta \varepsilon}^+} \mathbb{P}(\tau_{y,\varepsilon} \leq m_\varepsilon) \lesssim \varepsilon S(\varepsilon) \to 0. \]
The function \( S \) can be chosen to be a monotonic function. This finishes the proof.

3.2 Consequences of the first exit result

Corollary 4. Let the assumptions of the last theorem be satisfied and \( \rho \) being chosen according to \( 3.30 \) and \( \lim \sup_{\varepsilon \to 0} m_\varepsilon \varepsilon^{\alpha} < \infty \). Construct recursively
\[ U_t^{x,\varepsilon,1} := \left( u(t; x - \delta \varepsilon) - \delta \varepsilon + W_1(t = T_1) \right) \land \gamma \varepsilon, \quad t \in [0, T_1] \]
\[ U_t^{x,\varepsilon,n+1} := \left( u(t; x - T_n; U_t^{x,\varepsilon,n} - \delta \varepsilon) - \delta \varepsilon + W_{n+1}(t = T_{n+1} - T_n) \right) \land \gamma \varepsilon, \quad t \in (0, T_{n+1} - T_n] \]
\[ Z_t^{x,\varepsilon} := \sum_{n=1}^{\infty} U_t^{x,\varepsilon,n} 1\{ t \in (T_n, T_{n+1}] \}, \quad t \geq 0. \]
where the arrival times \( T_n \) of the large jump increments \( W_n \) are defined in \( 3.1, 3.2 \) and \( 3.3 \).
Then
\[ \lim \inf_{\varepsilon \to 0} \inf_{x \geq 3\delta \varepsilon} \mathbb{P}(\sup_{t \in [0, m_\varepsilon]} X_t^{x,\varepsilon} - Z_t^{x,\varepsilon} \geq 0) = 1 \]
This is nothing but a reformulation of the proof of Theorem 2. The process we compare \( X^{\varepsilon,x} \) to the deterministic solution \( u(\cdot; x) \), starting in \( x \) with large heavy-tailed jump increments \((T_n^\varepsilon, W_n^\varepsilon \land \gamma_\varepsilon)\), where the increments \( W_n^\varepsilon \) are cut-off from below by a value \( \gamma_\varepsilon \). The choice of \( \gamma_\varepsilon \) has to satisfy two things: First, the deterministic trajectory has to overcome it during the waiting time \( T_n^\varepsilon - T_{n+1}^\varepsilon \) with a probability tending to 1. Second, for larger and larger initial values \( i\gamma_\varepsilon < x \leq (i+1)\gamma_\varepsilon \), the probability that \( u(t, x) + \varepsilon W_i \leq \gamma_\varepsilon \) has to decrease for growing \( i \) and decreasing \( \varepsilon \) with a sufficiently large.

**Corollary 5.** Let the assumptions of Theorem 2 be satisfied and \( \delta_\varepsilon \) being chosen according to (3.10). Then for any \( m : (0, 1) \to (0, \infty) \) satisfying \( \lim_{\varepsilon \to 0} m_\varepsilon^{\alpha \rho} = 0 \) we have

\[
\liminf_{\varepsilon \to 0} \inf_{x \geq \delta_\varepsilon} \mathbb{P}(\sup_{t \in [0, m_\varepsilon]} X^{\varepsilon}_t - x^+_t \geq -\delta_\varepsilon) = 1.
\]

**Proof.** We keep the notation of the proof of Corollary 4. First we obtain by a comparison argument that for all \( x \geq \delta_\varepsilon \)

\[
u(t; x) \geq x^+_t \quad t \geq 0.
\]

Secondly we observe that \( U^{\varepsilon,1}_t = u(t; x) \) for \( t < T_1 \) and \( \mathbb{P}(T_1 \geq m_\varepsilon) = e^{-m_\varepsilon \lambda_\varepsilon} \approx e^{-m_\varepsilon \varepsilon^{\alpha \rho}} \to 1 \). Hence combining these findings with inequality (3.16) we obtain

\[
\liminf_{\varepsilon \to 0} \inf_{x \geq \delta_\varepsilon} \mathbb{P}(\sup_{t \in [0, m_\varepsilon]} X^{\varepsilon}_t - x^+_t > -\delta_\varepsilon) = 0.
\]

\( \square \)

**Lemma 6.** Let the assumptions of the Theorem 2 be satisfied and \( \delta_\varepsilon \) being chosen according to (3.10). Then there is a function \( \tilde{m} : (0, 1) \to (0, \infty) \) satisfying \( \lim_{\varepsilon \to 0} \tilde{m}_\varepsilon \delta_\varepsilon^{\beta} = 0 \) such that for any function \( \Delta : (0, 1) \to (0, 1) \) with \( \lim_{\varepsilon \to 0} \Delta_\varepsilon = 0 \) and \( \lim_{\varepsilon \to 0} 3\delta_\varepsilon / \Delta_\varepsilon = 0 \) we have

\[
\liminf_{\varepsilon \to 0} \inf_{3\delta_\varepsilon \leq x \leq \Delta_\varepsilon} \mathbb{P}(\sup_{t \in [0, \tilde{m}_\varepsilon]} X^{\varepsilon}_t - x^+_t < \delta_\varepsilon^{\beta} / \Delta_\varepsilon^{1-\beta}) = 1.
\]

**Proof.** First choose \( \rho \) we choose according to (3.30) in the proof of Theorem 2 and fix for the moment \( 3\delta_\varepsilon \leq x \leq \Delta_\varepsilon \). Recall for \( t \in [0, T_1] \) the notation

\[
X^{\varepsilon,x}_t = Y^{\varepsilon,x}_t + \varepsilon W_1 \mathbb{1}\{t = T_1\}
\]

and

\[
V^{\varepsilon,x}_t = Y^{\varepsilon,x}_t - \varepsilon \xi^{\varepsilon}_t.
\]

The subadditivity of \( b(y) = B|y|^\beta \) on \((0, \infty)\) yields on the events \( \{t < T_1\} \) and \( \{\sup_{t \in [0, T_1]} |\varepsilon \xi^{\varepsilon}_s| \leq \delta_\varepsilon\} \) that

\[
V^{\varepsilon,x}_t \leq x + \int_0^t b(V^{\varepsilon,x}_s)ds + B\delta_\varepsilon^\beta \varepsilon^{\gamma} \leq \Delta_\varepsilon + B\delta_\varepsilon^\beta \tilde{m}_\varepsilon + \int_0^t b(V^{\varepsilon,x}_s)ds,
\]
where $\delta_{\varepsilon} := \delta_{\varepsilon}^{\frac{1}{2}\beta} \wedge \varepsilon$ with $\delta_{\varepsilon} = \varepsilon^{1-\rho(1+\alpha)}|\ln(\varepsilon)|^4$ in (3.10) and $r_{\varepsilon} = \varepsilon^{-\alpha\rho}|\ln(\varepsilon)|^2$ defined in (3.8).

Then Bihari’s inequality [20], Theorem 8.3, implies

$$
\sup_{t \in [0, \tilde{m}_\varepsilon]} V_{t}^{\varepsilon, x} - x_t^+ \\
\leq \sup_{t \in [0, \tilde{m}_\varepsilon]} \left[ \left( (1 - \beta)Bt + (\Delta_{\varepsilon} + B\delta_{\varepsilon}^{\beta} \tilde{m}_{\varepsilon})^{1-\beta} \right)^{\frac{1}{1-\beta}} - ((1 - \beta)Bt)^{\frac{1}{1-\beta}} \right] \\
\leq \left[ \left( (1 - \beta)B\tilde{m}_{\varepsilon} + (\Delta_{\varepsilon} + B\delta_{\varepsilon}^{\beta} \tilde{m}_{\varepsilon})^{1-\beta} \right)^{\frac{1}{1-\beta}} - ((1 - \beta)B\tilde{m}_{\varepsilon})^{1-\beta} \right] \\
\leq 2^{\frac{1}{1-\beta}} (\Delta_{\varepsilon} + B\delta_{\varepsilon}^{\beta} \tilde{m}_{\varepsilon})^{1-\beta} \to 0.
$$

Note that the bound of the right-hand side is of order

$$(\Delta_{\varepsilon} + B\delta_{\varepsilon}^{\beta} \tilde{m}_{\varepsilon})^{1-\beta} \lesssim \delta_{\varepsilon}^{\beta(1-\beta)} \vee \Delta_{\varepsilon}^{1-\beta}.$$

We finish the proof by

$$
\limsup_{\varepsilon \to 0} \sup_{3\delta_{\varepsilon} \leq x \leq \Delta_{\varepsilon}} \mathbb{P}(X_{t}^{\varepsilon, x} - x_t^+ > \delta_{\varepsilon}^{\frac{\beta(1-\beta)}{2}} \vee \Delta_{\varepsilon}^{1-\beta}) \\
\leq \limsup_{\varepsilon \to 0} \left( 1 - \mathbb{P}(T_1 > r_{\varepsilon}) - \mathbb{P}( \sup_{t \in [0,r_{\varepsilon}]} |\varepsilon^{\frac{1}{3}}| > \delta_{\varepsilon}) \right) = 0.
$$

\[\square\]

We obtain the main result of this section as a combination of Corollary 5 and Lemma 6.

**Corollary 7.** Let the assumptions of the Theorem 2 be satisfied and $\delta_{\varepsilon}$ chosen as in (3.10). Then for any $\Delta: (0,1) \to (0,1)$ monotonically increasing with $\lim_{\varepsilon \to 0} \Delta_{\varepsilon} = 0$ and $\limsup_{\varepsilon \to 0} 3\delta_{\varepsilon}/\Delta_{\varepsilon} \leq 1$ there exists $\theta^* > 0$ such that

$$
\lim_{\varepsilon \to 0^+} \sup_{3\delta_{\varepsilon} \leq x \leq \Delta_{\varepsilon}} \mathbb{P}( \sup_{t \in [0,\varepsilon^{\theta^*}]} |X_{t}^{\varepsilon, x} - x_t^+| > \delta_{\varepsilon}^{\frac{\beta(1-\beta)}{2}} \vee \Delta_{\varepsilon}^{1-\beta}) = 0.
$$

### 4 The solution leaves a small environment of the origin in a short time

Let us denote by $(X_t)_{t \geq 0}$ the strong solution $(X_{t}^{\varepsilon, 0})_{t \geq 0}$ of system (1.5) with initial value $x = 0$. In addition we stipulate for $r_1, r_2 > 0$

$$
\tau_{\varepsilon}(r_1, r_2) := \inf \{ t > 0 : X_t \leq -r_1 \text{ or } X_t \geq r_2 \}.
$$

and abbreviate for convenience $\tau_{r_1, r_2} = \tau_{\varepsilon}(r_1, r_2)$. 

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4.1 When the noise strength meets the non-linear impact: space-time transition points

Proposition 8. For

\[ \alpha > 1 - (\beta^+ \land \beta^-) \]

and any \( \vartheta \in (0, 1] \) there is a family of monotonically increasing functions \( \Theta^+_{\varepsilon, \vartheta}, \Theta^-_{\varepsilon, \vartheta}, t_{\varepsilon, \vartheta} : (0, 1) \rightarrow (0, 1) \) with \( \lim_{\varepsilon \rightarrow 0^+} \Theta^+_{\varepsilon, \vartheta} = \lim_{\varepsilon \rightarrow 0^+} \Theta^-_{\varepsilon, \vartheta} = \lim_{\varepsilon \rightarrow 0^+} t_{\varepsilon, \vartheta} = 0 \), such that for any function \( \hat{t}_{\varepsilon, \vartheta} : (0, 1) \rightarrow (0, \infty) \) satisfying \( \lim_{\varepsilon \rightarrow 0^+} \hat{t}_{\varepsilon, \vartheta} / t_{\varepsilon, \vartheta} = +\infty \) we have

\[ \lim_{\varepsilon \rightarrow 0^+} \mathbb{P} \left( \tau_{\Theta^-_{\varepsilon, \vartheta}, \Theta^+_{\varepsilon, \vartheta}} > \hat{t}_{\varepsilon, \vartheta} \right) = 0. \]

We omit the iteration argument by Markov property. The key result is the following.

Lemma 9. Under the previous assumptions and

\[ \alpha > 1 - (\beta^+ \land \beta^-) \]

and \( \vartheta \in (0, 1] \) we have the following statement. There is a family of monotonically increasing functions \( \Theta^+_{\varepsilon}, \Theta^-_{\varepsilon}, t_{\varepsilon} : (0, 1) \rightarrow (0, 1) \) with \( \lim_{\varepsilon \rightarrow 0^+} \Theta^+_{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} \Theta^-_{\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} t_{\varepsilon} = 0 \), such that we have

\[ \lim_{\varepsilon \rightarrow 0^+} \mathbb{P} \left( \tau_{\Theta^-_{\varepsilon}, \Theta^+_{\varepsilon}} > t_{\varepsilon} \right) < 1, \]

For notational convenience we will immediately the dependence on \( \vartheta \), whenever possible.

Remark 10. The parameter \( \vartheta \in (0, 1] \) is a purely technical device, it will turn out in the next section that if \( \beta^+ = \beta^- \) it cannot be chosen to be 1 but only arbitrarily close to 1. In any other case it will be set equal 1.

Proof. For the convenience of notation we will fix \( \vartheta \in (0, 1] \) and drop the respective subscript in the sequel. Assume there are \( \Theta^+_{\varepsilon}, \Theta^-_{\varepsilon}, t_{\varepsilon} \) as in the statement of the lemma and let us abbreviate for convenience \( \chi = \tau_{\Theta^-_{\varepsilon}, \Theta^+_{\varepsilon}} \). The definition of the event \( \{ \chi > t_{\varepsilon} \} \) implies

\[ -\Theta^-_{\varepsilon} \leq X_t \leq \Theta^+_{\varepsilon} \quad \forall t \in [0, t_{\varepsilon}]. \]

Therefore, we infer from the event \( \{ \chi > t_{\varepsilon} \} \) for \( t \in [0, t_{\varepsilon}] \) that

\[ \varepsilon L_t = X_t - \int_0^t b(X_s)ds \]

\[ \leq X_t + B^- \int_0^t (X_s)\beta^- ds \]

\[ \leq \Theta^+_{\varepsilon} + B^- t_{\varepsilon} (\Theta^-_{\varepsilon})^{\beta^-}. \]
Analogously we obtain
\[ \varepsilon L_t \geq -\Theta^-_\varepsilon - B^+ t \varepsilon (\Theta^+\varepsilon)^{\beta^+}. \]

If we now impose that the nonlinear term is asymptotically smaller, that is for instance \( \Theta^+_\varepsilon t^{1-\vartheta} \) than the boundary \( \Theta \)
\[ B^+ t \varepsilon (\Theta^+_\varepsilon)^{\beta^+} = \Theta^- t^{1-\vartheta} \]
\[ B^- t \varepsilon (\Theta^-\varepsilon)^{\beta^-} = \Theta^+ t^{1-\vartheta} \quad (4.2) \]
it follows
\[ -(1 + t^{1-\vartheta}) \Theta^- \leq \varepsilon L_t \leq (1 + t^{1-\vartheta}) \Theta^+, \quad t \in [0, t\varepsilon], \]
and in particular \(-(1 + t^{1-\vartheta}) \Theta^- \leq \varepsilon L_t \leq (1 + t^{1-\vartheta}) \Theta^+ \). As a first case we may assume that \( \Theta^+ / \Theta^- \to 0 \) as \( \varepsilon \to 0 \). If we stipulate for \( \vartheta \in (0, 1) \)
\[ \Theta^\vartheta = \frac{\frac{1}{\varepsilon t^{\vartheta}}}{1 + t^{1-\vartheta}} \quad (4.3) \]
this yields
\[ P\left( - (1 + t^{1-\vartheta}) \Theta^- \leq \varepsilon L_t \leq (1 + t^{1-\vartheta}) \Theta^+, \quad t \varepsilon \right) \]
\[ P\left( -(1 + t^{1-\vartheta}) \frac{\Theta^-}{\Theta^+} \leq t^{1-\vartheta} \leq (1 + t^{1-\vartheta}) \frac{\Theta^+}{\Theta^+} \right) \]
\[ = P\left( -(1 + t^{1-\vartheta}) \frac{\Theta^-}{\Theta^+} \leq L_1 \leq (1 + t^{1-\vartheta}) \frac{\Theta^+}{\Theta^+} \right) \]
\[ = P\left( -\frac{\Theta^-}{\Theta^+} \leq L_1 \leq 1 \right) \xrightarrow{\varepsilon \to 0} P\left( -\infty < L_1 \leq 1 \right) < 1. \]

As long as \( \lim_{\varepsilon \to 0} t\varepsilon = 0 \). The proof concludes with the following calculation which shows that for any exponent \( \alpha \in (0, 2) \), any powers \( \beta^+, \beta^- \in (0, 1) \) satisfying \( \alpha \geq 1 - (\beta^+ + \beta^-) \) and \( \varepsilon \in (0, 1) \) the system (4.2) together either with (4.3) as a unique solution \( (\Theta^\vartheta, \Theta^-\varepsilon, \Theta^-\varepsilon, \theta t\varepsilon)_{\varepsilon, \theta \in (0, 1]} \) with \( \lim_{\varepsilon \to 0} t\varepsilon = 0 \) for any \( \vartheta \in (0, 1) \).

We solve the equations for \( t\varepsilon, \Theta^\vartheta, \Theta^\vartheta \) and \( \Theta^-\varepsilon \) and start with the system (4.2) which implies by reinsertion
\[ \Theta^- = t^{1-\vartheta} B^+(\Theta^\vartheta)^{\beta^+} \]
\[ = t^{1-\vartheta} B^+(t^{1-\vartheta} B^- (\Theta^-)^{\beta^-})^{\beta^+} \]
\[ = B^+(B^-)^{\beta^+} t^{1+\beta^+}(\Theta^-)^{\beta^+\beta^-}, \]
\[
(\Theta^{-}_\epsilon)^{1-\beta^+\beta^-} = B^+(B^-)^{\beta^+} t^{\beta(1+\beta^+)}_\epsilon
\]

\[
\iff \Theta^{-}_\epsilon = \left( B^+(B^-)^{\beta^+} t^{\beta(1+\beta^+)}_\epsilon \right)^{1-\beta^+\beta^-} = \left( B^+ \right)^{1-\beta^+\beta^-} \left( B^- \right)^{1-\beta^+\beta^-} \frac{\beta^+}{t^{\beta(1+\beta^+)}_\epsilon}
\]

and by symmetry
\[
(\Theta^{+}_\epsilon)^{1-\beta^+\beta^-} = B^- (B^+)^{\beta^+} t^{\beta(1+\beta^-)}_\epsilon
\]

\[
\iff \Theta^{+}_\epsilon = \left( B^- (B^+)^{\beta^+} t^{\beta(1+\beta^-)}_\epsilon \right)^{1-\beta^+\beta^-} = \left( B^- \right)^{1-\beta^+\beta^-} \left( B^+ \right)^{1-\beta^+\beta^-} \frac{\beta^-}{t^{\beta(1+\beta^-)}_\epsilon}
\]

Denote by \(\beta^o := \beta^+ \land \beta^-\) and \(\beta^s := \beta^+ \lor \beta^-\). The last two formulas yield
\[
\Theta^o_\epsilon := \Theta^{+}_\epsilon \lor \Theta^{-}_\epsilon = (B^o)^{1-\beta^o} \left( B^o \right)^{\beta^o} t^{\beta(1+\beta^o)}_\epsilon
\]
\[
\Theta^o_\epsilon := \Theta^{+}_\epsilon \land \Theta^{-}_\epsilon = (B^o)^{1-\beta^o} \left( B^o \right)^{\beta^o} t^{\beta(1+\beta^o)}_\epsilon
\]

As a consequence, we obtain for \(\beta^o < \beta^s\)
\[
\lim_{\epsilon \to 0+} \Theta^o_\epsilon / \Theta^o_\epsilon = 0. \tag{44}
\]

and for \(\beta = \beta^s = \beta^o\)
\[
\frac{\Theta^o_\epsilon}{\Theta^o_\epsilon} = \left( B^o \right)^{1-\beta^o} \left( B^o \right)^{\beta^o} t^{\beta(1+\beta^o)}_\epsilon = \left( B^o \right)^{-1-\beta^o}. \tag{45}
\]

We complement the system (4.2) by equation (4.3). Plugging in directly yields
\[
\iff \epsilon = (B^o)^{1-\beta^o} \left( B^o \right)^{\beta^o} t^{\beta(1+\beta^o)}_\epsilon \frac{\beta(1+\beta^o)}{1-\beta^o} \frac{\beta^o}{1}. \tag{46}
\]

We examine the exponent
\[
\frac{\beta(1+\beta^o)}{1-\beta^o} - \frac{1}{\alpha} = \frac{\vartheta \alpha (1+\beta^o) - 1 + \beta^o \beta^o}{\alpha (1-\beta^o \beta^o)} = \frac{\vartheta \alpha - 1 + \beta^o (\vartheta \alpha + \beta^o)}{\alpha (1-\beta^o \beta^o)} \geq \frac{\vartheta \alpha - 1 + \beta^o}{\alpha (1-\beta^o \beta^o)} > 0,
\]

since \(\vartheta \alpha + \beta^o > 1\) and therefore \(\vartheta \alpha + \beta^o > 1\) we have
\[
\epsilon = (B^o)^{1-\beta^o} \left( B^o \right)^{\beta^o} \left( B^o \right)^{1-\beta^o} \frac{\vartheta \alpha - 1 + \beta^o (\vartheta \alpha + \beta^o)}{\alpha (1-\beta^o \beta^o)} \]

\[
\iff t_\epsilon = \frac{\left( B^o \right)^{1-\beta^o} \left( B^o \right)^{\beta^o} \left( B^o \right)^{1-\beta^o} \frac{\vartheta \alpha - 1 + \beta^o (\vartheta \alpha + \beta^o)}{\alpha (1-\beta^o \beta^o)}}{\left( B^o \right)^{1-\beta^o} \left( B^o \right)^{\beta^o} \left( B^o \right)^{1-\beta^o} \frac{\vartheta \alpha - 1 + \beta^o (\vartheta \alpha + \beta^o)}{\alpha (1-\beta^o \beta^o)}}. \tag{46}
\]

We obtain
\[
\Theta^{+}_\epsilon = \left( B^- \right)^{1-\beta^o \beta^o} \left( B^+ \right)^{\beta^o} \left( t^{\beta(1+\beta^-)}_\epsilon \right)^{\beta^o} \frac{\beta(1+\beta^-)}{1-\beta^o \beta^o} \]
\[
= \frac{\left( B^- \right)^{1-\beta^o \beta^o} \left( B^+ \right)^{\beta^o} \left( t^{\beta(1+\beta^-)}_\epsilon \right)^{\beta^o} \frac{\beta(1+\beta^-)}{1-\beta^o \beta^o}}{\left( B^o \right)^{1-\beta^o \beta^o} \left( B^o \right)^{\beta^o} \left( t^{\beta(1+\beta^-)}_\epsilon \right)^{\beta^o} \frac{\beta(1+\beta^-)}{1-\beta^o \beta^o}}.
\]
and
\[ \Theta^{-}_\varepsilon = \frac{(B^+)_{\alpha+\beta^*} (B^-)_{\alpha+\beta^-}}{(B^0)_{\alpha+\beta^*} - 1 + \beta^* (\alpha+\beta^-)} \cdot \varepsilon^{\alpha + \beta^* - 1 + \beta^* (\alpha+\beta^- - 1)}. \]

These calculations establish the existence and uniqueness of \((\Theta^+_{\varepsilon, \theta}, \Theta^-_{\varepsilon, \theta}, t_{\varepsilon, \theta}, \varepsilon, \theta \in (0,1])\) as claimed in the statement of Lemma 9.

\[ \square \]

4.2 The exit locations from a neighborhood of the origin

For a fixed parameter \(\theta\) fixed and we denote by \(\chi := \chi_{\varepsilon} := \tau \Theta^+_{\varepsilon, \theta}, \Theta^-_{\varepsilon, \theta}\) as defined in (4.1) and \((\Theta^+_{\varepsilon, \theta}, t_{\varepsilon, \theta}, \varepsilon, \theta \in (0,1])\) defined by Definition 13 and Lemma 9. In this subsection we determine the asymptotic probabilities

\[ \Pr(X^\varepsilon_{\chi} \geq \Theta^+_\varepsilon) \quad \text{and} \quad \Pr(X^\varepsilon_{\chi} \leq -\Theta^-_{\varepsilon}) \]

in the limit of small \(\varepsilon\).

**Proposition 11.** For \(\alpha > 1 - (\beta^* \land \beta^0)\) and \(\beta^+ \neq \beta^-\) we have

\[ \Pr(X^\varepsilon_{\chi} \geq \Theta^+_\varepsilon) = \begin{cases} 1 & \beta^+ < \beta^-, \\ 0 & \beta^+ > \beta^- \end{cases}, \quad (4.7) \]

4.2.1 Close to the transition points the non-linear impact of the noise remains sub-critical

This section controls that there is no exit by non-linear impact of the noise. We decompose \(X^\varepsilon\) into the sum of \(V^\varepsilon\) and \(\varepsilon L\), where \(V^\varepsilon_t := X^\varepsilon_t - \varepsilon L_t\). It satisfies

\[ V^\varepsilon_t = \int_0^t b(V^\varepsilon_s + \varepsilon L_s)ds, \quad t \geq 0. \]

**Lemma 12.** Assume \(\beta^+ > \beta^-\) the parametrized family of functions \((\Theta_{\varepsilon,1}^+, \Theta_{\varepsilon,1}^-, t_{\varepsilon,1}, \varepsilon, \theta \in (0,1])\) determined in Definition 13. Then there exists \(g > 0\) such that for \(\hat{t}_{\varepsilon} := t_{\varepsilon} |\ln(\varepsilon)|, \varepsilon \in (0,1)\) we have

\[ \Pr(\sup_{t_{\varepsilon} \in [0, \hat{t}_{\varepsilon}]} (V^\varepsilon_t)_+ > \Theta^+_{\varepsilon, \varepsilon^g}) \to 0. \]

**Proof.** As in the previous lemma the self-similarity

\[ \sup_{t_{\varepsilon} \in [0, \hat{t}_{\varepsilon}]} (\varepsilon L_{t_{\varepsilon}})^{\beta^*} \overset{d}{\rightarrow} \varepsilon^{\beta^*} t_{\varepsilon}^{\beta^*} (L_{t_{\varepsilon}})^{\beta^*} \]
yields

\[ \mathbb{P}\left( \sup_{t \in [0, \varepsilon]} (\varepsilon L_t)_{+}^{\beta^{*}} > \Theta_{\varepsilon}^{*} \right) \leq \mathbb{P}\left( \sup_{t \in [0, \varepsilon]} \varepsilon^{\beta^{*}} t_{\varepsilon}^{2} \right) (L_{1})_{+}^{\beta^{*}} > \Theta_{\varepsilon}^{*} \varepsilon^{g} \) 

We check whether

\[ \varepsilon^{\beta^{*}} t_{\varepsilon}^{\frac{\alpha + \beta^{*}}{\alpha} - \frac{1 + \beta^{*}}{1 - \beta^{*}}} \to 0, \quad \text{as } \varepsilon \to 0. \]

Check the exponent

\[ \beta^{*} + \frac{\alpha (1 - \beta^{*} \beta^{*})}{\alpha + \beta^{*} - 1 + \beta^{*} (1 + \beta^{*} - 1)} (\frac{\alpha + \beta^{*}}{\alpha} - \frac{1 + \beta^{*}}{1 - \beta^{*}}) \]

\[ = \frac{\beta^{*} (\alpha + \beta^{*} - 1 + \beta^{*} (1 + \beta^{*} - 1)) + (\alpha + \beta^{*}) (1 - \beta^{*} \beta^{*}) - \alpha (1 + \beta^{*})}{\alpha + \beta^{*} - 1 + \beta^{*} (1 + \beta^{*} - 1)} \]

By assumption the denominator is positive. The numerator behaves as

\[ \beta^{*} (\alpha + \beta^{*} - 1 + \beta^{*} (1 + \beta^{*} - 1)) + (\alpha + \beta^{*}) (1 - \beta^{*} \beta^{*}) - \alpha (1 + \beta^{*}) \]

\[ = \alpha \beta^{*} + (\beta^{*})^{2} - \beta^{*} + (\alpha \beta^{*})^{2} + \beta^{*} (\beta^{*})^{2} - (\beta^{*})^{2} + \alpha + \beta^{*} - \alpha \beta^{*} - \beta^{*} \beta^{*} - \alpha - \alpha \beta^{*} \]

\[ = \alpha \beta^{*} + \alpha (\beta^{*})^{2} - \alpha \beta^{*} \beta^{*} - \alpha \beta^{*} \]

\[ = \alpha (\beta^{*} - \beta^{*}) + \alpha \beta^{*} (\beta^{*} - \beta^{*}) > 0. \]

We set \(2g\) equal to the expression in (4.8).

**Definition 13.** Let \(\alpha \in (0, 2)\) and \(\beta^{\pm}, \beta^{-} \in (0, 1)\) given satisfying \(\alpha > 1 - (\beta^{+} \wedge \beta^{-})\). For any \(\alpha\)-stable noise \(L\), we define the family \((\Theta_{\varepsilon}^{+}, \Theta_{\varepsilon}^{-}, t_{\varepsilon})_{\varepsilon \in (0, 1)} := (\Theta_{\varepsilon}^{+}, \Theta_{\varepsilon}^{-}, t_{\varepsilon}, \Theta_{\varepsilon}^{*})_{\varepsilon \in (0, 1)}\) defined in the proof of Lemma [2] with

\[ \theta^{*} = \begin{cases} \frac{1}{2} (1 + \frac{1 - \beta^{*}}{\alpha}) & \text{if } \beta^{*} = \beta^{*}, \\ 1 & \text{else.} \end{cases} \] (4.9)

**4.2.2 The spatial exit probabilities from the space-time box of the transition points**

In the sequel we determine \(\lim_{\varepsilon \to 0^+} \mathbb{P}(\varepsilon L_{\varepsilon} \geq \Theta_{\varepsilon}^{+})\). This exit problem will be mainly treated in the spirit of the Brownian case as for instance in the book of Revuz and Yor [24]. Denote for \(\kappa \in \mathbb{R}\) and \(\varepsilon \in (0, 1)\) the jump time

\[ \tau_{\kappa}(\varepsilon) := \inf\{t > 0 \mid |\Delta L_{t}| > \varepsilon^{-\kappa}\}. \]

The appropriate choice of \(\kappa \in \mathbb{R}\) allows to give an estimate for the first exit problem of \(\varepsilon L\) from \([-\Theta_{\varepsilon}^{+}, \Theta_{\varepsilon}^{+}]\) in the sense of Revuz and Yor, since \(|\varepsilon \Delta \xi_{\varepsilon}^{*}| \leq C \varepsilon^{1-\kappa} \leq \Theta_{\varepsilon}^{*}\). This means the jump to exit the interval is small in comparison to the boundary and vanishes in the limit of small \(\varepsilon\). \(\kappa\) should
verify two properties. First it has to ensure that jumps beyond the threshold $\varepsilon^\kappa$ occur after $t_\varepsilon$, with a probability mass which tends to 1. More precisely, since

$$\nu(B_{\varepsilon^{-\kappa}}^c(0)) = 2 \int_{\varepsilon^{-\kappa}}^\infty \frac{dy}{y^{\alpha+1}} = \frac{2}{\alpha} y^{-\alpha}|_{\varepsilon^{-\kappa}}^{\infty} = \frac{2}{\alpha} \varepsilon^{\kappa\alpha}$$

we impose on $\kappa$ that

$$\mathbb{P}(\tau_\varepsilon(\varepsilon) > t_\varepsilon) = \exp\left(-\frac{2}{\alpha} \varepsilon^{\kappa\alpha} t_\varepsilon\right) \to 1,$$ as $\varepsilon \to 0$. \hfill (4.10)

This is satisfied if

$$\kappa\alpha > \frac{-\alpha(1 - \beta^0 \beta^*)}{\vartheta \alpha + \beta^* - 1 + \beta^* (\vartheta \alpha + \beta^* - 1)}.$$ \hfill (4.11)

As a second crucial feature we need

$$\varepsilon^{1-\kappa}/\Theta^- \to 0, \quad \text{as } \varepsilon \to 0^+.$$ \hfill (4.12)

This imposes

$$1 - \kappa > \frac{\vartheta \alpha (1 + \beta^*)}{\vartheta \alpha + \beta^* - 1 + \beta^* (\vartheta \alpha + \beta^* - 1)}.$$ \hfill (4.13)

We verify that the conditions (4.11) and (4.12) reading

$$\frac{\vartheta \alpha (1 + \beta^*)}{\vartheta \alpha + \beta^* - 1 + \beta^* (\vartheta \alpha + \beta^* - 1)} - 1 < -\kappa < \frac{(1 - \beta^0 \beta^*)}{\vartheta \alpha + \beta^* - 1 + \beta^* (\vartheta \alpha + \beta^* - 1)}$$ \hfill (4.14)

can be satisfied simultaneously. On the common the denominator we have to verify the positivity of the enumerators' difference

$$1 - \beta^0 \beta^* - \vartheta \alpha (1 + \beta^*) + (\vartheta \alpha + \beta^* - 1 + \beta^* (\vartheta \alpha + \beta^* - 1))$$

$$= -\beta^0 \beta^* - \vartheta \alpha - \vartheta \alpha \beta^0 + \vartheta \alpha + \beta^* \vartheta \alpha + \beta^* \beta^0 - \beta^*$$

$$= \vartheta \alpha (\beta^* - \beta^0) > 0.$$

For $\vartheta \in (0, 1)$ we may fix

$$\kappa = \frac{1 - \beta^0 \beta^*}{\vartheta \alpha + \beta^* - 1 + \beta^* (\vartheta \alpha + \beta^* - 1)},$$ \hfill (4.15)

Going back to the Poisson random measure representation of $L$ we then have for any $T > 0$

$$L_t = \int_0^t \int_{|y| \leq 1} y(N(dsdy) - d\nu(dy)) + \int_0^t \int_{|y| > 1} y N(dsdy)$$

$$= \int_0^t \int_{|y| \leq \varepsilon^{-\kappa}} y \tilde{N}(dsdy) \quad \mathbb{P}(\cdot | \tau_\varepsilon(\varepsilon) \geq T) - \text{a.s. for all } t \in [0, T].$$ \hfill (4.16)
The first summand is given as the Lévy martingale \((\xi_t^\kappa)_{t \geq 0}\),

\[
\xi_t^\kappa = \int_0^t \int_{|y| \leq \varepsilon^{-\kappa}} y \tilde{N}(dsdy), \quad t \geq 0.
\]

We define the for \(r^+, r^- > 0\) and \(\varepsilon > 0\) the hitting times of \(\mathbb{R} \setminus (-\Theta^{-}_\varepsilon, \Theta^{+}_\varepsilon)\)

\[
\sigma^+_r := \inf\{t > 0 \mid \varepsilon \xi_t^\kappa \geq r^+\},
\]

\[
\sigma^-_r := \inf\{t > 0 \mid \varepsilon \xi_t^\kappa \leq -r^-\},
\]

\[
\sigma_{r^+, r^-} := \sigma^+_r \wedge \sigma^-_r.
\]

(4.17)

**Lemma 14.** Under these assumptions and \(\beta^+ > \beta^-\) we obtain

\[
\limsup_{\varepsilon \to 0} \mathbb{P}(\sigma^+_r < \sigma^-_r) \leq \limsup_{\varepsilon \to 0} \frac{\Theta^{-}_\varepsilon + \varepsilon^{1-\kappa}}{\Theta^{+}_\varepsilon + \Theta^{-}_\varepsilon + \varepsilon^{1-\kappa}} = 0.
\]

**Proof.** The definition of the exit times and the choice of the jump size yields the estimates

\[
\varepsilon \xi^\kappa_{\sigma^-} \geq -(r^- + \varepsilon^{1-\kappa}) \quad \text{and} \quad \varepsilon \xi^\kappa_{\sigma^-} < -r^- \quad \text{a.s. on the event } \{\sigma^- \leq n\}.
\]

For \(r_1, r_2 > 0\) and \(n \in \mathbb{N}\) given we fix

\[
\sigma^{+ n} := \sigma^+_r \wedge n,
\]

\[
\sigma^{- n} := \sigma^-_r \wedge n,
\]

\[
\sigma^n := \sigma^{+ n} \wedge \sigma^{- n}.
\]

Applying the optional stopping theorem we obtain

\[
0 = \mathbb{E}[\varepsilon \xi^\kappa_{\sigma^n}]
\]

\[
= \mathbb{E}[\varepsilon \xi^\kappa_{\sigma^n}(1\{\sigma^{+ n} < \sigma^{- n}\} + 1\{\sigma^{+ n} \geq \sigma^{- n}\})]
\]

\[
= \mathbb{E}[\varepsilon \xi^\kappa_{\sigma^{+ n}} 1\{\sigma^{+ n} < \sigma^{- n}\} + \varepsilon \xi^\kappa_{\sigma^{- n}} 1\{\sigma^{+ n} \geq \sigma^{- n}\}]
\]

we may estimate

\[
0 = \mathbb{E}[\varepsilon \xi^\kappa_{\sigma^{+ n}} 1\{\sigma^{+ n} < \sigma^{- n}\} + \varepsilon \xi^\kappa_{\sigma^{- n}} 1\{\sigma^{+ n} \geq \sigma^{- n}\}]
\]

\[
\geq r^+ \mathbb{P}(\{\sigma^+_r < \sigma^-_r\} \cap \{\sigma_\varepsilon \leq n\}) - (r^- + \varepsilon^{1-\kappa}) \mathbb{P}(\sigma^-_r \leq \sigma^+_r).
\]

Letting \(n\) tend to \(\infty\) we obtain

\[
0 \geq r^+ \mathbb{P}(\sigma^+_r < \sigma^-_r) - (r^- + \varepsilon^{1-\kappa})(1 - \mathbb{P}(\sigma^+_r < \sigma^-_r)) \quad (4.18)
\]
and eventually
\[ P(\sigma^+_{\Theta^\varepsilon_+} < \sigma^-_{\Theta^\varepsilon_-}) \lesssim \frac{r^- + \varepsilon^{1-\kappa}}{r^+ + r^- + \varepsilon^{1-\kappa}}, \]  
(4.19)

The choice of \( \kappa \) now entails that \( r^+ \) replaced by \( \Theta^+_{\varepsilon} = \Theta^*_\varepsilon \) leads to
\[ \varepsilon^{1-\kappa} \lesssim \Theta^+_{\varepsilon} = C^+\varepsilon^{\frac{\beta(1+\beta^-)}{2(\alpha+\beta^+\alpha+\beta^-)-1}}, \]
and analogously for \( r^- \) being replaced by \( \Theta^-_{\varepsilon} \) eventually leading to the desired result
\[ P(\sigma^+_{\Theta^\varepsilon_+} < \sigma^-_{\Theta^\varepsilon_-}) \lesssim \frac{\Theta^-_{\varepsilon} + \varepsilon^{1-\kappa}}{\Theta^+_{\varepsilon} + \Theta^-_{\varepsilon} + \varepsilon^{1-\kappa}} \to 0, \quad \text{as } \varepsilon \to 0. \]
(4.20)

**Proof.** of Proposition 11

Without loss of generality, due to \( \beta^+ > \beta^- \) we may collect Lemma 12, Lemma 14, equation (4.10) and Proposition 8, which altogether guarantee the existence of a constant \( g > 0 \) such that
\[ \varepsilon \to 0+. \]
5 The linearized dynamics enhances the regime close to the origin

We already know by Section 4 Corollary 7 that for initial values $x \geq -3\delta_\varepsilon$ the law $\mathbb{P} \circ X^{\varepsilon,x} \to \delta_{x^+}$ uniformly on larger and larger time scales. Section 4.2 establishes that for any family of functions $(\Theta^+_\varepsilon, \Theta^-_\varepsilon, t_\varepsilon)_{\varepsilon \in (0,1]}$ defined by Definition 13 and initial values $x \in (-\Theta^-_\varepsilon, \Theta^+_\varepsilon)$ the solution $X^{\varepsilon,x}$ exits the interval $(-\Theta^-_\varepsilon, \Theta^+_\varepsilon)$ in time $\tilde{t}_\varepsilon$ almost surely as long as $\lim_{\varepsilon \to 0} \tilde{t}_\varepsilon / t_\varepsilon \to 0$. In order to fill the gap between the scales of initial values

$$\Theta^\pm_\varepsilon = \varepsilon^{\alpha + \rho \varepsilon^{1 - \alpha}} \leq 3\varepsilon^{1 - \rho^{1+\alpha}} = 3\delta^\pm_\varepsilon,$$

we consider the linearized dynamics.

The main result of this section tells us that with a probability tending to 1, the solution exits on the outer boundary of $[-3\delta^-_\varepsilon, -\Theta^-_\varepsilon] \cup [\Theta^+_\varepsilon, 3\delta^+_\varepsilon]$. We treat each subinterval individually with out loss of generality $[\Theta^+_\varepsilon, 4\delta^+_\varepsilon]$.

**Lemma 15.** Let $[t_0, t_1) \subset \mathbb{R}$ and $v, v_0, \phi : [t_0, t_0) \to \mathbb{R}$, where $v$ und $v_0$ measurable and locally bounded functions and $\phi \in L^1([t_0, t_1), \mathbb{R})$ with $\phi \geq 0$. Then for almost all $t \in [t_0, t_1)$

$$v(t) \geq v_0(t) + \int_{t_0}^{t} \phi(s)v(s)ds$$

implies for almost all $t \in [t_0, t_1)$

$$v(t) \geq v_0(t) + \int_{t_0}^{t} v_0(s)\phi(s)\exp\left(\int_{s}^{t} \phi(r)dr\right)ds.$$

For $\varepsilon > 0$ and $x \in [\Psi_0, \Psi_1]$ denote

$$v^x(\varepsilon) := \inf\{t > 0 \mid X^{\varepsilon,x}_t \geq \Psi_1\}.$$

**Proposition 16.** Without loss of generality we consider $\beta = \beta^c = \beta^+ \leq \beta^-$ and the family of functions $(\Theta^+_\varepsilon, \Theta^-_\varepsilon, t_\varepsilon)_{\varepsilon \in (0,1]}$ defined by Definition 13. Then there is an increasing, continuous function $s^n : (0, 1) \to (0, 1)$ with $s^n_\varepsilon \to 0$ for any fixed $n \in \mathbb{N}$ as $\varepsilon \to 0$, such that

$$\lim_{\varepsilon \to 0} \sup_{x \geq \Psi_{0,\varepsilon}} \mathbb{P}(v^x(\varepsilon) > s_\varepsilon) = 0.$$

**Proof.** We start with setting $\Psi_{0,\varepsilon} = \Theta^c_\varepsilon$ and introduce the time $s_\varepsilon$ with $s_\varepsilon \to 0$, as $\varepsilon \to 0$, and $\Psi_{1,\varepsilon} \geq \Psi_{0,\varepsilon}$, with $\Psi_{1,\varepsilon} \to 0$, as $\varepsilon \to 0$, which both will be determined below. The proof consists in the establishment of an appropriate choice of a parameter $\pi_1 \in \mathbb{R}$. For $\pi_1 \in \mathbb{R}$ given we denote the time

$$\tau_{\pi_1} = \tau_{\pi_1}(\varepsilon) := \inf\{t > 0 \mid |\Delta L_t| > \varepsilon^{-\pi_1}\}.$$
We write shorthand $\beta, B$ for $\beta^+, B^+$. Then analogously to (4.10) we have
\[
\varepsilon L_t = \varepsilon \xi_t^\pi
\]
for all $t \in [0, s_\varepsilon]$ and $\mathbb{P}(\cdot \mid \tau_{\pi_1} > s_\varepsilon) - \text{ a.s.}$

Hence for all $\omega \in \{\tau_{\pi_1} > s_\varepsilon\} \cap \{\sup_{t \in [0, s_\varepsilon]} |\varepsilon \xi_t^\pi| \leq \frac{B}{2} \Psi_{0, \varepsilon} s_\varepsilon\}$ we have $\mathbb{P}(\cdot \mid \tau_{\pi_1} > s_\varepsilon) - \text{ a.s.}$ for $t \in [0, s_\varepsilon]$ that
\[
X_t^{\varepsilon, x} = x + \int_0^t b(X_s^{\varepsilon, x})ds + \varepsilon L_t = x + \int_0^t b(X_s^{\varepsilon, x})ds + \varepsilon \xi_t^\pi
\]
\[
\geq \Psi_{0, \varepsilon} + B \int_0^t \left[ \frac{\Psi_{0, \varepsilon}^\beta}{2} + (X_s^{\varepsilon, x} - \Psi_{0, \varepsilon}) \frac{\Psi_{1, \varepsilon}^\beta - \Psi_{0, \varepsilon}^\beta}{\Psi_{1, \varepsilon} - \Psi_{0, \varepsilon}} \right] ds + \varepsilon \xi_t^\pi
\]
\[
\geq \Psi_{0, \varepsilon} + B \int_0^t \frac{\Psi_{0, \varepsilon}^\beta}{2} + (X_s^{\varepsilon, x} - \Psi_{0, \varepsilon}) \frac{\Psi_{1, \varepsilon}^\beta - \Psi_{0, \varepsilon}^\beta}{\Psi_{1, \varepsilon} - \Psi_{0, \varepsilon}} ds.
\]

Hence for $W_{1,t} := W_{1,t}^{\varepsilon, x} := X_t^{\varepsilon, x} - \Psi_{0, \varepsilon}$ and $t \geq 0$ we have
\[
W_t \geq \frac{B}{2} \int_0^t \left[ \psi_0^\beta + W_t \Psi_1^\beta - \psi_0^\beta \right] ds
\]
\[
\geq \frac{B}{2} \Psi_{0, \varepsilon}^\beta t + B \int_0^t \psi_0^\beta \Psi_1^\beta - \psi_0^\beta ds
\]
\[
\geq \frac{B}{2} \Psi_{0, \varepsilon}^\beta t + \frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \int_0^t W_t ds.
\]

A classical non-autonomous Gronwall inequality from below yields
\[
W_t \geq \frac{B}{2} \Psi_{0, \varepsilon}^\beta t + \frac{B}{2} \Psi_{0, \varepsilon}^\beta \exp \left( \frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \right) \int_0^t s \exp \left( -\frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \right) ds
\]
and by direct calculation
\[
W_t \geq \frac{B}{2} \Psi_{0, \varepsilon}^\beta t + \frac{B}{2} \Psi_{0, \varepsilon}^\beta \Psi_{1, \varepsilon}^{2(1-\beta)} \exp \left( \frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \right) \left( 1 - (1 + \frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta}) \exp \left( -\frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \right) \right).
\]

We set $s_\varepsilon = \frac{2}{B} \Psi_{1, \varepsilon}^{1-\beta}$. This choice yields for any $C > 0$ a constant $\varepsilon_0 \in (0, 1)$ such that $0 < \varepsilon \leq \varepsilon_0$
\[
(1 + \frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta}) \exp \left( -\frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \right) \leq C.
\]

Therefore for $\varepsilon \in (0, \varepsilon_0]$
\[
X_{s_\varepsilon}^{\varepsilon, x} \geq \Psi_{0, \varepsilon} + \frac{B}{2} \Psi_{0, \varepsilon} s_\varepsilon
\]
\[
+ \Psi_{1, \varepsilon}^{2(1-\beta)} \Psi_{0, \varepsilon} \left[ \frac{2}{B} (1 - C) \right] \exp \left( \frac{B}{2} \Psi_{1, \varepsilon}^{1-\beta} \right)
\]
\[
\geq \frac{1 - C}{B} \Psi_{1, \varepsilon}^{2(1-\beta)} \Psi_{0, \varepsilon} \exp \left( \frac{-\Psi_{1, \varepsilon}^{1-\beta}}{B} \right) \geq \Psi_{1, \varepsilon}. 
\]
We hence obtain

\[
\mathbb{P}(v^{1,x}_\varepsilon > s_\varepsilon) \leq \mathbb{P}(\sup_{t \in [0, s_\varepsilon]} |\varepsilon \xi^{\pi_1}(t)| > \frac{B}{2} \Psi^\beta_{0, \varepsilon} s_\varepsilon) + 1 - \mathbb{P}(\pi_1 > s_\varepsilon) \\
\leq \exp(-\frac{B}{2} \frac{\Psi^\beta_{0, \varepsilon}}{s_{1-\pi_1}}) + 1 - \exp(-\frac{2}{B} e^{\alpha_\pi_1} \Psi^\beta_{1, \varepsilon} (1-\beta)) \cdot (5.1)
\]

In order to conclude we determine \( \pi_1 \) and \( \Psi_{1, \varepsilon} \) such that the last two terms in (5.1) tend to 0. For further use we note that for \( \alpha > 1 \) we have

\[
\beta_0 \ln_{\varepsilon}(\Psi_{0, \varepsilon}) - 1 = \frac{\vartheta \alpha \beta_0 (1 + \beta^*) - (\vartheta \alpha + \beta^* - 1 + \beta^*(\alpha + \beta_0 - 1))}{\vartheta \alpha + \beta^* - 1 + \beta^*(\alpha + \beta_0 - 1)} < 0,
\]

since the denominator is positive by \( \vartheta \alpha > 1 - \beta_0 \) and

\[
\vartheta \alpha \beta_0 + \vartheta \alpha \beta_0 \beta^* - \vartheta \alpha - \beta^* + 1 - \vartheta \alpha \beta^* - \beta_0 \beta^* + \beta^* \\
= \vartheta \alpha \beta_0 + \vartheta \alpha \beta_0 \beta^* - \vartheta \alpha + 1 - \vartheta \alpha \beta^* - \beta_0 \beta^* \\
= \alpha \vartheta (\beta_0 - \beta^*) + \beta_0 \beta^*(\vartheta \alpha - 1) - (\vartheta \alpha - 1) \\
= \alpha \vartheta (\beta_0 - \beta^*) - (1 - \beta_0 \beta^*)(\vartheta \alpha - 1) < 0.
\]

The right-hand side of (5.1) yields

\[
- \pi_1 > \beta \ln_{\varepsilon}(\Psi_{0, \varepsilon}) - 1 \quad (5.2) \\
- \pi_1 < \frac{(1 - \beta)}{2 \alpha} \ln_{\varepsilon}(\Psi_{1, \varepsilon}). \quad (5.3)
\]

First note that (5.2) is satisfied for any \( \pi_1 < 0 \) since we impose that \( \ln_{\varepsilon}(\Psi_{0, \varepsilon}) > 0 \), while inequality (5.3) represents a restriction on \( \ln_{\varepsilon}(\Psi_{1, \varepsilon}) \), which can be circumvented for \( - \pi_1 \) small enough.

In this case we only have to take into account (5.2) and (5.3). We can hence may choose the desired quantities

\[
\Psi_{1, \varepsilon} := 3\delta_{\varepsilon}^{-\frac{1}{2}} \quad (\Leftrightarrow \ln_{\varepsilon}(\Psi_{1, \varepsilon}) = \frac{1}{2} \ln_{\varepsilon}(3\delta_{\varepsilon})) \\
(-\pi_1) := \frac{(1 - \beta) \gamma}{2} \ln_{\varepsilon}(\Psi_{1, \varepsilon}). \quad (5.4)
\]
6 The solution selection problem: Proof of Theorem 1

In this section we prove a slightly stronger statement than Theorem 1. In the sequel we collect all tailor-made partial result of this article. Let \( \alpha, \beta^+, \beta^- \) with \( \alpha > 1 - (\beta^+ \land \beta^-) \) be given.

In Corollary 1 the time scale \( \tilde{m}_\varepsilon \) is bounded by \( \delta_\varepsilon^{-\frac{\beta^2}{2}} \). By definition (3.10) there is \( \theta^* > 0 \) such that \( \varepsilon^{-\theta^*} / \delta_\varepsilon^{-\frac{\beta^2}{2}} \to 0 \) as \( \varepsilon \to 0 \). Recall \( (\Theta^+_\varepsilon, \Theta^-\varepsilon, t_\varepsilon)_{\varepsilon \in (0,1]} \) defined by Definition 13 and Lemma 9 and the respective hitting times as defined by (4.1)

\[
\tau_{\Theta^+_\varepsilon, \Theta^-\varepsilon}(\varepsilon, x) = \inf\{t > 0 \mid X_t^{\varepsilon, x} < -\Theta^-\varepsilon \text{ or } X_t^{\varepsilon, x} \geq \Theta^+_\varepsilon\}
\]

\[
\sigma_{\delta^+_\varepsilon, \delta^-\varepsilon}(\varepsilon, x) = \inf\{t > 0 \mid X_t^{\varepsilon, x} < -3\delta^-\varepsilon \text{ or } X_t^{\varepsilon, x} > 3\delta^+_\varepsilon\},
\]

where we dropped the dependence on \( \vartheta \). Fix a time scale \( \hat{t}_\varepsilon = t_\varepsilon / |\ln(\varepsilon)| \) chosen according to Proposition 8 with respect to \( t_\varepsilon \) and \( \bar{s}_\varepsilon = s_\varepsilon / |\ln(\varepsilon)| \) with respect to \( s_\varepsilon \) determined by Proposition 16. Furthermore we recall the exponents \( \kappa < 0 \) defined in (4.13) and \( \pi < 0 \) defined in (??) and \( \tau_\kappa \) and \( \tau_\pi \) for

\[
\tau_c = \inf\{t > 0 \mid |\Delta L_t| > \varepsilon^{-c}\}, \quad c \in \mathbb{R}.
\]

Since all other dependencies are clear we shall write shorthand \( \chi = \tau_{\Theta^+_\varepsilon, \Theta^-\varepsilon}(\varepsilon, 0) \) and \( \sigma^c = \sigma_{\delta^+_\varepsilon, \delta^-\varepsilon}(\varepsilon, x) \) and \( X^x = X^{\varepsilon, x} \). The solution \( X^{\varepsilon, x} \) will be denoted by \( X^x \). We use the strong Markov property of \( X^x \) to control the exit from the neighborhood \((-\Theta^-\varepsilon, \Theta^+_\varepsilon)\) of the origin. For all \( \varepsilon \) sufficiently small such that \( \hat{t}_\varepsilon \leq \varepsilon^{-\theta^*} \) and any \( f \) a bounded, uniformly continuous function we have

\[
\mathbb{E}[f((X_0^x)_{t \in [0,\varepsilon^{-\theta^*}]})]
\]

\[
= \mathbb{E}[\mathbb{E}[f((X_t^x)_{t \in [0,\varepsilon^{-\theta^*}]})\mathbf{1}\{\chi \leq \hat{t}_\varepsilon\}|\mathcal{F}_\chi]]
\]

\[
+ \mathbb{P} (\chi > \hat{t}_\varepsilon) + \mathbb{P}(\tau_\kappa < t_\varepsilon)
\]

\[
\leq \mathbb{P} (X_\chi^0 > \Theta^+_\varepsilon) \sup_{\Theta^+_\varepsilon < \xi \leq \Theta^+_\varepsilon + \varepsilon^{1-\kappa}} \mathbb{E}[f((X_t^x)_{t \in [0,\varepsilon^{-\theta^*}]})|\chi \leq \hat{t}_\varepsilon]
\]

\[
+ \mathbb{P} (X_\chi^0 < -\Theta^-\varepsilon) \sup_{-\Theta^-\varepsilon - \varepsilon^{1-\kappa} < \xi \leq -\Theta^-\varepsilon} \mathbb{E}[f((X_t^x)_{t \in [0,\varepsilon^{-\theta^*}]})|\chi \leq \hat{t}_\varepsilon]
\]

\[
+ \mathbb{P}(\chi > \hat{t}_\varepsilon) + \mathbb{P}(\tau_\kappa < t_\varepsilon).
\]

Proposition 11 and Definition 13 establish that the probability \( \mathbb{P}(X_\chi^0 > \Theta^+_\varepsilon) \) tends to \( p^+ \) as \( \varepsilon \to 0 \) as given the statement of Theorem 1. The penultimate term tends to 0 due to Proposition 8 and the last one due to relation (4.10). In the following we first consider the positive branch. Since
κ < 0 and we have ε^{1−κ}3δ_ε^+ → 0 as ε → 0 in addition. Hence for ε sufficiently small

\[
\sup_{\Theta_ε^+ \leq x < 3δ_ε^+} \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*} - \chi)]1\{X^t \leq \hat{t}_ε\}) \leq \sup_{\Theta_ε^+ \leq x < 3δ_ε^+} \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*}])] \leq \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*} - \sigma^ε)]1\{\sigma^ε \leq \hat{t}_ε\}) + \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*}])]
\]

where the next-to-last and the last term tend to 0 by Proposition \[16\] as ε → 0. We continue with the strong Markov property

\[
\sup_{\Theta_ε^+ \leq x < 3δ_ε^+} \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*}]]) \leq \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*} - \sigma^ε)]1\{\sigma^ε \leq \hat{t}_ε\}) + \mathbb{P}(\hat{t}_π \leq \hat{t}_ε)
\]

Again since π < 0 we have that ε^{1−π}δ_ε^+ → 0 as ε → 0. First let f be uniformly continuous with respect to D([0, ∞); R) equipped with the uniform norm. We denote by Ξ the module of uniform continuity of f. For Δ_ε = 3δ_ε^+ + ε^{1−π} in the statement of Corollary \[7\] we have

\[
\sup_{3δ_ε^+ \leq x \leq 3δ_ε^+ + ε^{1−π}} \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*}])]
\]

\[
\leq \sup_{3δ_ε^+ \leq x \leq 3δ_ε^+ + ε^{1−π}} \mathbb{E}[f((X^t)_t \in [0, \varepsilon^{-θ^*}]][X^x_t - x^+_t] \leq (δ_ε^{+\frac{1}{2}}(\varepsilon^{-θ^*}))]
\]

\[
+ \|f\|_\infty \sup_{3δ_ε^+ \leq x \leq 3δ_ε^+ + ε^{1−π}} \mathbb{P}(\sup_{t \in [0, \varepsilon^{-θ^*}]} |X^x_t - x^+_t| > (δ_ε^{+\frac{1}{2}}(\varepsilon^{-θ^*}))]
\]

Corollary \[7\] yields that the last term converges to 0. The negative branch is treated analogously. For the case of general case of f not uniformly continuous, we define the cutoff function \(f_m(x) := f(x)1\{-m \leq x \leq m}\), which is uniformly continuous and finally send m to infinity, which is justified by the Beppo-Levi theorem.
We prove the lower bound. Let $f$ be uniformly continuous.

\[
\mathbb{E}[f((X^0_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})] \\
\geq \mathbb{E}[f((X^0_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})1\{\chi \leq \hat{\xi}_\varepsilon\} + 1\{\chi > \hat{\xi}_\varepsilon\}] \\
= \mathbb{E}[\mathbb{E}[f((X^0_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})1\{\chi \leq \hat{\xi}_\varepsilon\}1\{X^0_{\chi} \geq \Theta^+_\varepsilon\} + 1\{X^0_{\chi} \leq -\Theta^-_\varepsilon\}) | \mathcal{F}_\chi]] \\
\geq \mathbb{P}(X^0_{\chi} \geq \Theta^+_\varepsilon) \sup_{x \leq \Theta^+_\varepsilon} \mathbb{E}[f((X^{x,-\varepsilon - \chi})_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})] \\
+ \mathbb{P}(X^0_{\chi} \geq -\Theta^-_\varepsilon) \sup_{x \leq -\Theta^-_\varepsilon} \mathbb{E}[f((X^0_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})]
\]

Application of Proposition 11 and Definition 13 ensure again that $\lim_{\varepsilon \to 0} \mathbb{P}(X^0_{\chi} \geq \Theta^+_\varepsilon) = p^+$. We may continue with the positive branch

\[
\sup_{x \geq \Theta^+_\varepsilon} \mathbb{E}[f((X^x_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})] \\
\geq \sup_{x \geq \Theta^+_\varepsilon} \mathbb{E}[f((X^{x,-\varepsilon - \chi})_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})1\{X^{x,-\varepsilon - \chi} \geq 3\delta^+_\varepsilon\}1\{\sigma^x \leq s_{\varepsilon}\}] \\
\geq \sup_{x \geq 3\delta^+_\varepsilon} \mathbb{E}[f((X^x_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]})1\{\sup_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]|X^x_t - x^+_t| \leq \delta^+_\varepsilon\} \beta^{1-\beta^+} \} \\
\geq f((x^+_t)_{t \in [0, \varepsilon - \hat{\theta}_\varepsilon^+]}) - \Xi((\delta^+_\varepsilon)^{\beta^{1-\beta^+} \frac{1}{2}}).
\]

The negative branch is treated analogously. For a function $f$ not uniformly continuous we use the same truncation argument as before. This proves the desired result.

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