AN INVERSE PROBLEM FOR THE MAGNETIC SCHRÖDINGER OPERATOR ON A HALF SPACE WITH PARTIAL DATA

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Abstract. In this paper we prove uniqueness for an inverse boundary value problem for the magnetic Schrödinger equation in a half space, with partial data. We prove that the curl of the magnetic potential \( A \), when \( A \in W^{1,\infty}_{\text{comp}}(\mathbb{R}^3, \mathbb{R}^3) \), and the electric potential \( q \in L^\infty_{\text{comp}}(\mathbb{R}^3, \mathbb{C}) \) are uniquely determined by the knowledge of the Dirichlet-to-Neumann map on parts of the boundary of the half space.

1. Introduction

We consider a magnetic Schrödinger operator \( L_{A,q} \), defined by

\[
L_{A,q} := \sum_{j=1}^{3} \left( -i \partial_j + A_j(x) \right)^2 + q(x),
\]

on the half space \( \mathbb{R}^3_- := \{ x \in \mathbb{R}^3 : x_3 < 0 \} \). We assume that

\[
A \in W^{1,\infty}_{\text{comp}}(\mathbb{R}^3, \mathbb{R}^3), \quad q \in L^\infty_{\text{comp}}(\mathbb{R}^3, \mathbb{C}), \quad \text{Im} \ q \leq 0.
\]

Here \( W^{1,\infty}_{\text{comp}}(\mathbb{R}^3, \mathbb{R}^3) := \{ A|_{\mathbb{R}^3_-} : A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3), \text{supp}(A) \subset \mathbb{R}^3 \text{ compact} \} \)

and similarly, we define \( L^\infty_{\text{comp}}(\mathbb{R}^3, \mathbb{C}) := \{ q \in L^\infty(\mathbb{R}^3, \mathbb{C}) : \text{supp}(q) \subset \mathbb{R}^3 \text{ compact} \} \).

Consider the Dirichlet problem

\[
( L_{A,q} - k^2 ) u = 0 \quad \text{in} \ \mathbb{R}^3_-,
\]

\[
u|_{\partial \mathbb{R}^3_-} = f,
\]

where \( k > 0 \) is fixed and \( f \in H_{\text{comp}}^{3/2}(\partial \mathbb{R}^3) \). Here we also require that the solution \( u \) should satisfy a boundary condition at infinity, which will be the Sommerfeld radiation condition

\[
\lim_{|x| \to \infty} \frac{|x|}{|x|^2} \left( \frac{\partial u(x)}{\partial |x|} - iku(x) \right) = 0.
\]

Solutions satisfying this condition are called outgoing or radiating solutions. We will also occasionally use the term incoming solution. This refers to a solution of (1.3) that satisfies (1.4), when the factor \(-ik\) is replaced by \(ik\). The existence and uniqueness of a solution \( u \in H^2_{\text{loc}}(\mathbb{R}^3) \) to the problem (1.3) and (1.4) is proven.
in [19]. This allows us to define the so called Dirichlet to Neumann map $\Lambda_{A,q}$.
(DN-map for short), $\Lambda_{A,q} : H^{3/2}_{\text{comp}}(\partial \mathbb{R}^3) \to H^{1/2}_{\text{loc}}(\partial \mathbb{R}^3)$ as
$$f \mapsto (\partial_n + i A \cdot n) u|_{\partial \mathbb{R}^3},$$
where $u$ is the solution of the Dirichlet problem (1.3), (1.4) and $f$ is the value of $u$. Here $n = (0, 0, 1)$ is the unit outer normal to the boundary $\partial \mathbb{R}^3$.

The inverse problem is to investigate if the DN-map uniquely determines the potentials $A$ and $q$ in $\mathbb{R}^3$. It turns out that the DN-map does not in general uniquely determine $A$. This is due to the gauge invariance of the DN-map, which was first noticed by [22].

Lemma 1.1. Let $A \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ and $q \in L^{\infty}(\mathbb{R}^3)$. Then

(i) For all $\psi \in C^{1,1}(\mathbb{R}^3, \mathbb{R})$ we have
$$e^{-i A \cdot \psi} = \Lambda A + \nabla \psi, q.$$

(ii) There exists $\psi \in C^{1,1}(\mathbb{R}^3, \mathbb{R})$ with $\psi|_{\{x_3=0\}} = 0$, for which
$$\Lambda_{A,q} = \Lambda A + \nabla \psi, q$$
and $(A + \nabla \psi)|_{\{x_3=0\}} = (A_1, A_2, 0)$.

Proof. See [19]. □

Part (ii) of this Lemma shows that $\Lambda_{A,q}$ cannot uniquely determine $A$, since we can change a potential by a gauge transformation without changing the DN-map. The DN-map does however carry enough information to determine $\nabla \times A$, which is the magnetic field in the context of electrodynamics.

When considering a pair of magnetic potentials $A_j$, $j = 1, 2$, we use the notation $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$ for the component functions. Furthermore we let $N \subset \mathbb{R}^3$ be a relatively open and bounded set, for which
$$\bigcup_{j=1,2} \text{supp}(A_j) \cup \text{supp}(q_j) \subset N,$$
and for which $\partial \text{N}$ piecewise $C^2$ and $\mathbb{R}^3 \setminus \text{N}$ is connected.

We now state the main result of this paper, which generalizes the corresponding results of [13], obtained in the case of the Schrödinger operator without a magnetic potential.

Theorem 1.2. Let $A_j \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ and $q_j \in L^{\infty}(\mathbb{R}^3, \mathbb{C})$ be such that
$$\text{Im} q_j \leq 0, j = 1, 2.$$ Let $\Gamma_1, \Gamma_2 \subset \partial \mathbb{R}^3_+$ be open sets such that
$$(\partial \mathbb{R}^3_+ \setminus \text{N}) \cap \Gamma_j \neq \emptyset, j = 1, 2.$$
Then if
$$\Lambda_{A_1,q_1}(f)|_{\Gamma_1} = \Lambda_{A_2,q_2}(f)|_{\Gamma_1},$$
for all $f \in H^{3/2}_{\text{comp}}(\partial \mathbb{R}^3)$, $\text{supp}(f) \subset \Gamma_2$, then
$$\nabla \times A_1 = \nabla \times A_2 \quad \text{and} \quad q_1 = q_2 \quad \text{in} \quad \mathbb{R}^3.$$

We would like to emphasize that in Theorem 1.2 the set $\Gamma_1$, where measurements are performed, and the set $\Gamma_2$, where the data is supported, can be taken arbitrarily small, provided that [13] holds. The result of Theorem 1.2 pertains therefore to the inverse problems with partial data. Such problems are important from the point of view of applications, since in practice, performing measurements on the entire boundary could be impossible, due to limitations in resources or obstructions from obstacles.
The first uniqueness result, in the context of inverse boundary value problems for the magnetic Schrödinger operator on a bounded domain, was obtained by Sun in [22], under a smallness condition on $A$. Nakamura, Sun and Uhlmann proved the uniqueness without any smallness condition in [17], assuming that $A \in C^2$. Tolmasky extended this result to $C^1$ magnetic potentials in [23], and Panchenko to some less regular but small magnetic potentials in [18]. Salo proved uniqueness for Dini continuous magnetic potentials in [20]. The most recent result is given by Krupchyk and Uhlmann in [12], where uniqueness is proved for $L^\infty$ magnetic potentials. In all of these works, the inverse boundary value problem with full data was considered.

In [6], Eskin and Ralston obtained a uniqueness result for the closely related inverse scattering problem, assuming the exponential decay of the potentials. The partial data problem in the magnetic case was considered by Dos Santos Ferreira, Kenig, Sjöstrand and Uhlmann in [5] and by Chung in [4].

The inverse problem for the half space geometry, without a magnetic potential was examined by Cheney and Isaacson in [2]. The uniqueness for this problem in the case of compactly supported electric potentials was proved by Lassas, Cheney and Uhlmann in [13], assuming that the supports do not come close to the boundary of the half space. The result of Theorem 1.2 is therefore already a generalization of the work [13], even in the absence of magnetic potentials. Li and Uhlmann proved uniqueness for the closely related infinite slab geometry with $A = 0$, in [16]. Krupchyk, Lassas and Uhlmann did this for the magnetic case in [11]. In both of these works, the reflection argument of Isakov [8] played an important role. The uniqueness problem for the magnetic potentials in the slab and half space geometries has also been studied in a recent paper by Li [15]. The half space results in [15] differ from the ones given in this work, by concerning the more general matrix valued Schrödinger equation and by assuming $C^6$ regularity on the magnetic potential.

The half space is perhaps the simplest example of an unbounded region with an unbounded boundary. It is of special interest in many applications, such as geophysics, ocean acoustics, and optical tomography, since it provides a simple model for semi infinite geometries. We would like to mention that the magnetic Schrödinger equation is closely related to the diffusion approximation of the photon transport equation, used in optical tomography [1].

The paper is organized as follows. Section 2 contains a review of the construction of complex geometric optics solutions for magnetic Schrödinger operators with Lipschitz continuous magnetic potentials. In section 3 we derive the central integral identity. The proof of Theorem 1.2 is contained in sections 4 and 5. The appendix contains an extension of Green’s second formula and a statement of the unique continuation principle for easy reference.

2. Complex geometric optics solutions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^\infty$-boundary, and let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$, $q \in L^\infty(\Omega, \mathbb{C})$. The task of this subsection is to review the construction of complex geometric optics solutions for the magnetic Schrödinger equation,

\begin{equation}
L_{A,q}u = 0 \quad \text{in} \quad \Omega.
\end{equation}

A complex geometric optics solution to (2.1) is a solution of the form

\begin{equation}
u(x, \zeta; h) = e^{x \cdot \zeta/h}(a(x, \zeta; h) + r(x, \zeta; h)),
\end{equation}

where $\zeta \in \mathbb{C}^3$, $\zeta \cdot \zeta = 0$, $a$ is a smooth amplitude, $r$ is a remainder, and $h > 0$ is a small parameter.
In the case when $A \in C^2(\overline{\Omega})$ and $q \in L^\infty(\Omega)$, such solutions were constructed in \cite{5} using the method of Carleman estimates, and the construction was extended to the case of less regular potentials in \cite{9}, see also \cite{11}.

Let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$. The fundamental role in the construction of complex geometric optics solutions is played by the following Carleman estimate,

$$h \|u\|_{H^1_{\text{loc}}(\Omega)} \leq C \|e^{\varphi/h} h^2 L_{A,q} e^{-\varphi/h} u\|_{L^2(\Omega)}, \tag{2.3}$$

valid for all $u \in C_0^\infty(\Omega)$ and $0 < h \leq h_0$, which was proved in \cite{5} and \cite{9}. Here $\|u\|_{H^1_{\text{loc}}(\Omega)} = \|u\|_{L^2(\Omega)} + \|h \nabla u\|_{L^2(\Omega)}$.

Based on the estimate \textup{(2.3)}, the following solvability result was established in \cite{9, Proposition 4.3}.

**Proposition 2.1.** Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$, $q \in L^\infty(\Omega, \mathbb{C})$, $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$ and $\varphi(x) = \alpha \cdot x$. Then there is $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, and any $f \in L^2(\Omega)$, the equation

$$e^{\varphi/h} h^2 L_{A,q} e^{-\varphi/h} u = f \quad \text{in} \quad \Omega,$$

has a solution $u \in H^1(\Omega)$ with

$$\|u\|_{H^1_{\text{loc}}(\Omega)} \leq \frac{C}{h} \|f\|_{L^2(\Omega)}.$$  

Our basic strategy in constructing solutions of the form \textup{(2.2)} is to write \textup{(2.1)}, as

$$L_{\zeta} r = -L_{\zeta} a,$$  

where $L_{\zeta} := e^{-x \cdot \zeta/h} h^2 L_{A,q} e^{x \cdot \zeta/h}$. Then we first search for a suitable $a$, after which we will get $r$ by Proposition \textup{2.1}. We must however take care in choosing $a$ and the way it depends on $h$, since we need later that $\|r\|_{H^1_{\text{loc}}(\Omega)} \to 0$, sufficiently fast as $h \to 0$. We need $a$ also to be smooth enough. This will be handled as in \cite{9}.

We extend $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ to a Lipschitz vector field, compactly supported in $\Omega$, where $\Omega \subset \mathbb{R}^3$ is an open bounded set such that $\Omega \subset \subset \Omega$. We consider the mollification $A^\varepsilon := A * \psi_\varepsilon \in C_0^\infty(\Omega, \mathbb{R}^3)$. Here $\varepsilon > 0$ is small and $\psi_\varepsilon(x) = \varepsilon^{-3} \psi(x/\varepsilon)$ is the usual mollifier with $\psi \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \psi \leq 1$, and $\int \psi dx = 1$. We write $A^\varepsilon = A - A^\varepsilon$. Notice that we have the following estimates for $A^\varepsilon$,

$$\|A^\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon), \quad \|\partial^\alpha A^\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon^{-|\alpha|}) \quad \text{for all} \quad \alpha,$$

as $\varepsilon \to 0$.

We shall work with a complex $\zeta = \zeta_0 + \zeta_1$ depending slightly on $h$, for which

$$\zeta \cdot \zeta = 0, \quad \zeta_0 := \alpha + i \beta, \quad \alpha, \beta \in S^2, \quad \alpha \cdot \beta = 0,$$

$\zeta_0$ independent of $h$ and $\zeta_1 = O(h)$, as $h \to 0$.

By expanding the conjugated operator we write the right hand side of \textup{(2.4)} as

$$L_{\zeta} a = (-h^2 \Delta - 2i(-i\zeta_0 + hA) \cdot h\nabla - 2\zeta_1 \cdot h\nabla + h^2 A^2 \tag{2.7}$$

$$- 2ih\zeta_0 \cdot (A^2 + A^\varepsilon) - 2ih\zeta_1 \cdot A - ih^2 \nabla \cdot A + h^2 q) a.$$

Now we want $a$ to be such that this expression decays more rapidly than $O(h)$, as $h \to 0$.

Consider the operator in \textup{(2.7)}, ignoring for the time being $a$ and its possible dependence on $h$. We would like to eliminate from this operator the terms that are of first order in $h$. Notice first that $\zeta_1 = O(h)$ and that we can control $\|A^\varepsilon\|_{L^\infty(\Omega)}$.
with \( h \), if we choose \( \epsilon \) to be dependent on \( h \). Then in an attempt to eliminate first order terms in \( h \), it is natural to search for an \( a \) for which

\[
(2.8) \quad \zeta_0 \cdot \nabla a = -i\zeta_0 \cdot A^2 a, \quad \text{in} \ \Omega.
\]

We will look for a solution of the form \( a = e^\Phi \). The above equation becomes then

\[
(2.9) \quad \zeta_0 \cdot \nabla \Phi = -i\zeta_0 \cdot A^2, \quad \text{in} \ \Omega.
\]

Pick a \( \gamma \in S^2 \), such that \( \gamma \perp \{\alpha, \beta\} \).

Next we consider the above equation in coordinates \( y \), associated with the basis \( \{\alpha, \beta, \gamma\} \). Let \( T \) be the coordinate transform \( y = Tx := (x \cdot \alpha, x \cdot \beta, x \cdot \gamma) \). Using the chain rule and the fact that \( T^{-1} = T^* \), one gets that

\[
(2.8) \quad \nabla(\Phi \circ T^{-1})(Tx) = T[\nabla \Phi(x)]^*.
\]

We therefore have that

\[
(1, i, 0) \cdot \nabla(\Phi \circ T^{-1})(Tx) = (1, i, 0) \cdot T[\nabla \Phi(x)]^*
\]

\[
= (\alpha \cdot \nabla + i\beta \cdot \nabla)\Phi(x)
\]

\[
= \zeta_0 \cdot \nabla \Phi(x).
\]

Equation \((2.8)\) gives hence the \( \partial \)-equation

\[
(2.10) \quad 2\partial_z \cdot (\Phi \circ T^{-1})(y) = -i\zeta_0 \cdot (A^2 \circ T^{-1})(y),
\]

where \( \partial_z = (\partial_{y_1} + i\partial_{y_2})/2 \). We will solve this using the Cauchy operator

\[
N^{-1} f(x) := \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{1}{s_1 + is_2} f(x - (s_1, s_2, 0)) ds_1 ds_2,
\]

which is an inverse for the \( \partial \)-operator, \( N := (\partial_{y_1} + i\partial_{y_2})/2 \) (see e.g. \cite{[21]} Theorem 1.2.2). We will need the following straightforward continuity result for the Cauchy operator.

**Lemma 2.2.** Let \( r > 0 \) and \( f \in W^{k,\infty}(\mathbb{R}^3) \), \( k \geq 0 \) and assume that \( \text{supp}(f) \subset B(0, r) \). Then

\[
\|N^{-1} f\|_{W^{k,\infty}(\mathbb{R}^3)} \leq C_k \|f\|_{W^{k,\infty}(\mathbb{R}^3)}
\]

for some constant \( C_k > 0 \). If \( f \in C_0(\mathbb{R}^3) \), then \( N^{-1} f \in C(\mathbb{R}^3) \).

**Proof.** See e.g. \cite{[21]}. \( \square \)

Returning to \((2.10)\), we get that \( \Phi = \frac{1}{2} N^{-1}(-i\zeta_0 \cdot (A^2 \circ T^{-1})) \circ T \). More explicitly we have

\[
(2.11) \quad \Phi(x, \zeta_0; \epsilon) = \frac{1}{2\pi} \int_{\mathbb{R}^2} -i\zeta_0 \cdot A^2(x - T^{-1}(s_1, s_2, 0)) ds_1 ds_2,
\]

where \( T^{-1}(s_1, s_2, 0) = s_1\alpha + s_2\beta \). We have thus found a solution \( a = e^\Phi \) to equation \((2.8)\). We will choose \( \epsilon \) so that it depends on \( h \), which implies that \( a \) will depend on \( h \). In order to determine how the norm of \( r \) will depend on \( h \) and also for later estimates, we will need to see how \( \|\partial^\alpha a\|_{L^\infty} \) depends on \( h \). Lemma \(2.2\) and estimate \((2.9)\) imply the following result.

**Lemma 2.3.** Equation \((2.8)\) has a solution \( a \in C^\infty(\overline{\Omega}) \) satisfying the estimates

\[
(2.12) \quad \|\partial^\alpha a\|_{L^\infty(\Omega)} \leq C_\alpha \epsilon^{-|\alpha|} \quad \text{for all} \ \alpha.
\]

\(^1\)Here \( T^* \) is the transpose of \( T \).
We can now write the $L^\infty(\Omega)$ norm of (2.7) as
\[
\|L_\zeta a\|_{L^\infty(\Omega)} = \| - h^2L_{A,a}a + 2ih\zeta_0 \cdot A^\dagger a + 2\zeta_1 \cdot h\nabla a + 2ih\zeta_1 \cdot Aa\|_{L^\infty(\Omega)}.
\]
Using (2.10), (2.12) and the fact that $\zeta_1 = O(h)$ we have that
\[
\|L_\zeta a\|_{L^\infty(\Omega)} = O(h^2 e^{-2} + h^e).
\]
Choosing $\epsilon = h^{1/3}$, gives finally $\|L_\zeta a\|_{L^\infty(\Omega)} = O(h^{4/3})$, as $h \to 0$.

Finally to solve (2.4) for $r$, we rewrite it as
\[
e^{-x \cdot \text{Re} \zeta/h} L_{A,a} e^{x \cdot \text{Re} \zeta/h} (e^{ix \cdot \text{Im} \zeta/h} r) = -e^{ix \cdot \text{Im} \zeta/h} L_\zeta a.
\]
If we replace $e^{ix \cdot \text{Im} \zeta/h} r$ by $\tilde{r}$, then the solvability result Proposition 2.1 shows that we can find a solution $\tilde{r}$, so that a solution $r$ to (2.13) is given by $r = e^{-ix \cdot \text{Im} \zeta/h} \tilde{r}$.

To get a norm estimate for $r$, notice that for the right hand side of (2.13) we have
\[
\| e^{ix \cdot \text{Im} \zeta/h} L_\zeta a \|_{L^\infty(\Omega)} = O(h^{4/3}),
\]
as $h \to 0$. The solvability result 2.1 gives then that
\[
\| \tilde{r} \|_{H^1_{x1}(\Omega)} = O(h^{1/3}),
\]
as $h \to 0$, which implies that $\| r \|_{H^1_{x1}(\Omega)} = O(h^{1/3})$, as $h \to 0$.

Thus we have obtained the following existence result for complex geometric optics solutions.

**Proposition 2.4.** Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ and $q \in L^\infty(\Omega, \mathbb{C})$. Then for $h > 0$ small enough, there exist solutions $u \in H^1(\Omega)$, of the equation
\[
L_{A,a} u = 0 \quad \text{in} \quad \Omega,
\]
that are of the form
\[
u(x, \zeta; h) = e^{x \cdot \zeta/h} (a(x, \zeta; h) + r(x, \zeta; h)),
\]
where $\zeta \in \mathbb{C}^3$, is of the form given by (2.10), $a \in C^\infty(\overline{\Omega})$ solves the equation (2.8), and where $a$ and $r$ satisfy the estimates
\[
\| \partial^\alpha a \|_{L^\infty(\Omega)} \leq C_\alpha h^{-|\alpha|/3} \quad \text{and} \quad \| r \|_{H^1_{x1}(\Omega)} = O(h^{1/3}).
\]

\[\square\]

**Remark 2.5.** In the sequel, we need complex geometric optics solutions belonging to $H^2(\Omega)$. To obtain such solutions, let $\Omega' \supset \Omega$ be a bounded domain with smooth boundary, and let us extend $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ and $q \in L^\infty(\Omega)$ to $W^{1,\infty}(\Omega', \mathbb{R}^3)$ and $L^\infty(\Omega')$-functions, respectively. By elliptic regularity, the complex geometric optics solutions, constructed on $\Omega'$, according to Proposition 2.4, belong to $H^2(\Omega)$.

**Remark 2.6.** Recall that $\Phi = \frac{1}{2} N^{-1}(-i(\alpha + i\beta) \cdot (A^\dagger \circ T^{-1})) \circ T$. Lemma 2.2 implies that $N^{-1} : C_0(\Omega) \to C(\Omega)$ is continuous. The estimates (2.8) show that $A^\dagger$ is $A$ uniformly on $\Omega$. It follows that, if we define $\Phi^0 := \frac{1}{2} N^{-1}(-i(\alpha + i\beta) \cdot (A \circ T^{-1})) \circ T$, then $\Phi^0$ solves the equation
\[
(2.14) \quad \zeta_0 \cdot \nabla \Phi^0 = -i\zeta_0 \cdot A \quad \text{in} \quad \Omega,
\]
and satisfies
\[
\| \Phi(x, \zeta_0; h^{1/3}) - \Phi^0 \|_{L^\infty(\Omega)} \to 0, \quad h \to 0.
\]
Remark 2.7. We shall later use a slightly more general form for the amplitude $a$ in the complex geometric optics solutions. Namely we suppose that $a = ge^\Phi$, where $g \in C^\infty(\Omega)$, is such that

$$\zeta_0 \cdot \nabla g = 0.$$  

(2.15)

This means that $g$ is holomorphic in a plane spanned by $\alpha$ and $\beta$. Notice also that by picking $a = ge^\Phi$, we get by (2.8) that

$$\zeta_0 \cdot g \nabla \Phi = -i\zeta_0 \cdot gA^\#,$$

in place of (2.9). But the $\Phi$ solving (2.9) also solves the above equation. Hence we can use the same argument to obtain the $\Phi$ for the above equation, as earlier.

We thus obtain CGO solutions of the form

$$u = e^{x \cdot \zeta/h}(ge^\Phi + r),$$

where $\Phi$ solves (2.13).

Notice also that setting $a = ge^\Phi$ does not affect the norm estimates on $a$ in Proposition 2.4, since $g$ does not depend on $h$.

3. An integral identity

One central step in the ideas that are used in proving uniqueness results for inverse boundary value problems, is to derive an integral equation that expresses $L^2$ orthogonality between the product of two solutions $u_1$ and $u_2$, and the difference of two potentials $q_1$ and $q_2$, see [24]. One shows that

$$\int (q_1 - q_2)u_1u_2 = 0,$$

provided that the DN-maps for $q_1$ and $q_2$ are equal.

A similar thing will be done in this subsection, for the magnetic case. The integral equation, is however more involved in the case of a magnetic potential and will not by itself be interpreted as an orthogonality relation. We will be considering the integral equation in conjunction with solutions that depend on a small positive parameter $h$. In the later sections we will see that in the limit $h \to 0$, we obtain a criterion for the curl being zero.

It will be convenient to set

$$l := \partial \mathbb{R}^3_+ \cap \mathbb{N},$$

Recall that we assume that

$$(\partial \mathbb{R}^3_+ \cap \mathbb{N}) \cap \Gamma_j \neq \emptyset, \quad j = 1, 2.$$  

We can thus choose $\tilde{\Gamma}_j$, such that

$$\tilde{\Gamma}_j \subset \Gamma_j, \quad \tilde{\Gamma}_j \subset \partial \mathbb{R}^3_+ \cap \mathbb{N}, \quad j = 1, 2.$$  

Then it follows from (1.6) that

$$(3.1) \quad \Lambda_{A_1,q_1}(f)|_{\tilde{\Gamma}_1} = \Lambda_{A_2,q_2}(f)|_{\Gamma_1},$$

for any $f \in H^{3/2}(\partial \mathbb{R}^3_+)$, supp$(f) \subset \tilde{\Gamma}_2$. In order to prove Theorem 1.2 we shall only use the data (3.1), which turns out to be enough to determine the magnetic field and the electric potential.

We now begin deriving the integral identity. We assume that $A_j, q_j$ and $\Gamma_j$ are as in Theorem 1.2 so that (3.1) also applies.

Let $u_1 \in H^2_0(\mathbb{R}^3_+)$ be the radiating solution to

$$(L_{A_1,q_1} - k^2)u_1 = 0 \text{ in } \mathbb{R}^3_+,$$

$$u_1|_{\partial \mathbb{R}^3_+} = f,$$
Thus, we obtain that
\[ u \in L^2(\mathbb{R}^3), \quad v|_{\mathbb{R}^3} = f. \]

Define \( w := v - u_1 \). Then
\[
(L_{A_2,q_2} - k^2)w = 2i(A_2 - A_1) \cdot \nabla u_1 + i \nabla \cdot (A_2 - A_1)u_1 + (A_1^2 - A_2^2)u_1 + (q_1 - q_2)u_1.
\]

(3.2)

It follows from (3.1) that
\[
(\partial_n + iA_1 \cdot n)u_1|_{\Gamma_1} = (\partial_n + iA_2 \cdot n)v|_{\Gamma_1}.
\]

By Lemma 1.1 we may and shall assume that \( A_1 \cdot n = A_2 \cdot n = 0 \) on \( \partial \mathbb{R}^3 \), so that \( \partial_n w = 0 \) on \( \tilde{\Gamma}_1 \). We also conclude from (3.2) that \( w \) satisfies the Helmholtz equation
\[
(-\Delta - k^2)w = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{N}.
\]

As \( w|_{\Gamma_1} = \partial_n w|_{\tilde{\Gamma}_1} = 0 \), by unique continuation, we get that \( w = 0 \) in \( \mathbb{R}^3 \setminus \overline{N} \) (see Theorem 6.3 and Corollary 6.4 in the appendix). Since \( w \in H^2_{\text{loc}}(\mathbb{R}^3) \), we have
\[
w = \partial_n w = 0 \quad \text{on} \quad \partial N \cap \mathbb{R}^3.
\]

Let \( u_2 \in H^2(N) \) be a solution to \( (L_{A_2,q_2} - k^2)u_2 = 0 \) in \( N \). Then by Green’s formula, we get
\[
((L_{A_2,q_2} - k^2)w, u_2)_{L^2(N)} = (w, (L_{A_2,q_2} - k^2)u_2)_{L^2(N)} = -((\partial_n + iA_2 \cdot n)w, u_2)_{L^2(\partial N)} + (w, (\partial_n + iA_2 \cdot n)u_2)_{L^2(\partial N)} = -\int_{\partial N} (\partial_n w, u_2).\]

Assuming that
\[
u_2 = 0 \quad \text{on} \quad \Gamma_1,
\]
we conclude that
\[
((L_{A_2,q_2} - k^2)w, u_2)_{L^2(N)} = 0.
\]

Using equation (3.2), we may write this as follows,
\[
\int_N (2i(A_2 - A_1) \cdot (\nabla u_1) \overline{u_2} + i \nabla \cdot (A_2 - A_1)u_1 \overline{u_2}) \, dx + \int_N (A_1^2 - A_2^2)u_1 \overline{u_2} \, dx = 0.
\]

Using again the fact that \( (A_2 - A_1) \cdot n = 0 \) on \( \partial N \) and an integration by parts, we get
\[
i \int_N \nabla \cdot (A_2 - A_1)u_1 \overline{u_2} \, dx = -i \int_N (A_2 - A_1) \cdot (\nabla u_1 \overline{u_2} + u_1 \nabla \overline{u_2}) \, dx.
\]

Thus, we obtain that
\[
\int_N i(A_2 - A_1) \cdot (\nabla u_1 \overline{u_2} - u_1 \nabla \overline{u_2}) \, dx + \int_N (A_1^2 - A_2^2)u_1 \overline{u_2} \, dx = 0,
\]

(3.4)

where \( u_1 \in W_1^1(\mathbb{R}^3) \) and \( u_2 \in W_2^2(N) \). Here
\[
W_1(\mathbb{R}^3) := \{ u \in H^1_{\text{loc}}(\mathbb{R}^3) : (L_{A_1,q_1} - k^2)u = 0 \text{ in } \mathbb{R}^3, \supp(u_1|_{\partial \mathbb{R}^3}) \subset \tilde{\Gamma}_2, \text{ u radiating} \},
\]
Suppose that

\[ W_2^2(N) := \{ u \in H^2(N) : (L_{A_\varphi} - k^2)u = 0 \text{ in } N, u|_{\partial N} = 0 \}. \]

We shall next extend the integral identity (3.3) to a richer class of solutions to the magnetic Schrödinger equation. To that end, let us introduce the following space of solutions,

\[ W_1(N) := \{ u \in H^2(N) : (L_{A_1, q_1} - k^2)u = 0 \text{ in } N, u|_{\partial N} = 0 \}. \]

The following Runge type approximation result is similar to those found in [8], [16] and [11].

**Lemma 3.1.** The space \( V := W_1(\mathbb{R}^3)|_N \) is dense in \( W_1(N) \) in the \( L^2(N) \)-topology.

**Proof.** Suppose that \( V \) is not dense in \( W_1(N) \). First notice that \( \text{span}(V) = V \) so that \( \overline{V} \) is a linear subspace of \( L^2(N) \). Since \( V \) is not dense in \( W_1(N) \), we have a vector \( u_0 \in W_1(N) \) such that \( u_0 \notin \overline{V} \). We can decompose \( u_0 \) as \( u_0 = a + b \), where \( a \in \overline{V}, b \in \overline{V}^\perp \) and \( b \neq 0 \). Let \( T \) be the linear functional on \( L^2(N) \), defined by

\[ T(x) := \text{proj}_{\overline{V}^\perp}(x)/\|b\|_{L^2}, \]

where \( \text{proj}_{\overline{V}^\perp} \) is the orthogonal projection to \( \overline{V}^\perp \). Now clearly \( \|T(u_0)\|_{L^2} = 1 \) and \( T|_V = 0 \).

By the Riesz representation theorem, there is \( g_T \in L^2(N) \) that corresponds to \( T \). Extend \( g_T \) by zero to the complement of \( N \) in \( \mathbb{R}^3 \). Let \( U \in H^2_{\text{loc}}(\mathbb{R}^3) \) be the incoming solution to

\begin{align*}
(L_{A_1, \varphi} - k^2)U &= g_T \quad \text{in } \mathbb{R}^3, \\
U|_{\partial \mathbb{R}^3} &= 0.
\end{align*}

The existence of such a solution is proved in [15].

Now let \( u \in W_1(\mathbb{R}^3) \). Then because \( T|_V = 0 \) and \( \text{supp}(g_T) \subset N \), we get by the Green's formula of Lemma 5.2 that

\[ 0 = (u, g_T)_{L^2(\mathbb{R}^3)} = (u, (L_{A_1, \varphi} - k^2)U)_{L^2(\mathbb{R}^3)} \]

\[ = ((L_{A_1, q_1} - k^2)u, U)_{L^2(\mathbb{R}^3)} - (u, \partial_n + iA_1 \cdot n)U)_{L^2(\mathbb{R}^3)} + ((\partial_\nu + iA_1 \cdot n)u, U)_{L^2(\mathbb{R}^3)} - (u, \partial_n U)_{L^2(\mathbb{R}^3)}. \]

Since the boundary condition \( u|_{\partial \mathbb{R}^3} \) can be chosen arbitrarily from \( C^\infty(\overline{\Gamma}_2) \), we get that \( \partial_\nu U|_{\partial \mathbb{R}^3} = 0 \). Since \( U|_{\partial \mathbb{R}^3} = 0 \), we apply the unique continuation principle to conclude that \( U|_{\mathbb{R}^3 \setminus N} = 0 \). As \( U \in H^2_{\text{loc}}(\mathbb{R}^3) \), we have

\[ U|_{\partial N \cap \mathbb{R}^3} = \partial_\nu U|_{\partial N \cap \mathbb{R}^3} = 0. \]

Now applying Green’s formula and doing the same computation as above for \( u_0 \) and \( N \) instead of \( u \) yields

\begin{align*}
(u_0, g_T)_{L^2(N)} &= (u_0, (L_{A_1, \varphi} - k^2)U)_{L^2(N)} \\
&= ((L_{A_1, q_1} - k^2)u_0, U)_{L^2(N)} - (u_0, \partial_n + iA_1 \cdot n)U)_{L^2(\partial N)} + ((\partial_\nu + iA_1 \cdot n)u_0, U)_{L^2(\partial N)} - (u_0, \partial_n U)_{L^2(\partial N)} = 0.
\end{align*}

Here we have used that \( u_0|_{\partial} = 0 \). It follows that \( T(u_0) = 0 \). This contradiction completes the proof. \( \square \)
Since \((A_2 - A_1)\cdot n = 0\) on \(\partial N\), we can rewrite (3.4) in the following form,
\[
- \int_N u_1 i \nabla \cdot ((A_2 - A_1) \bar{u_2}) dx - \int_N i (A_2 - A_1) \cdot (u_1 \nabla \bar{u_2}) dx \\
+ \int_N (A_2^2 - A_2^2 + q_1 - q_2) u_1 \bar{u_2} dx = 0.
\]
Hence, an application of Lemma 5.1 implies that the integral identity (3.3) is valid for any \(u_1 \in W_1(N)\) and any \(u_2 \in W_2^*(N)\).

We summarize the discussion in this subsection in the following result.

**Proposition 3.2.** Assume that \(A_j, q_j\) and \(\Gamma_j, j = 1, 2\) are as in Theorem 1.2 and that the DN-maps satisfy
\[
\Lambda_{A_1, q_1}(f)|\Gamma_1 = \Lambda_{A_2, q_2}(f)|\Gamma_1,
\]
for any \(f \in H^{3/2}_{comp}(\partial \mathbb{R}^3), \text{supp}(f) \subset \Gamma_2\). Then
\[
\int_N i (A_2 - A_1) \cdot (\nabla u_1 \bar{u_2} - u_1 \nabla \bar{u_2}) dx \\
+ \int_N (A_2^2 - A_2^2 + q_1 - q_2) u_1 \bar{u_2} dx = 0,
\]
for any \(u_1 \in W_1(N)\) and any \(u_2 \in W_2^*(N)\).

\[\square\]

**Remark 3.3.** Notice that the proof of Proposition 3.2 only uses the assumption (3.1), which follows from (3.3). Proposition 3.2 holds therefore also under the weaker assumption (3.1).
We need to extend the potentials $A_j$ and $q_j$, $j = 1, 2$, to $B_\pm$. For the component functions $A_{j,1}$, $A_{j,2}$, and $q_j$, we do an even extension, and for $A_{j,3}$, we do an odd extension, i.e., for $j = 1, 2$ we set,

$$
\tilde{A}_{j,k}(x) = \begin{cases} 
A_{j,k}(x), & x_3 < 0, \\
A_{j,k}(\bar{x}), & x_3 > 0, 
\end{cases} \quad k = 1, 2,

\tilde{A}_{j,3}(x) = \begin{cases} 
A_{j,3}(x), & x_3 < 0, \\
-A_{j,3}(\bar{x}), & x_3 > 0, 
\end{cases}

\tilde{q}_j(x) = \begin{cases} 
q_j(x), & x_3 < 0, \\
q_j(\bar{x}), & x_3 > 0,
\end{cases}

$$where $\bar{x} := (x_1, x_2, -x_3)$. By our assumptions, $A_{j,3}|_{x_3=0} = 0$, from which it follows that $\tilde{A}_j \in W^{1,\infty}(B)$ and $\tilde{q}_j \in L^\infty(B)$, $j = 1, 2$.

We can now by Proposition 2.4 and Remark 2.5 pick complex geometric optics solutions $\tilde{u}_1$ in $H^2(B)$,

$$
\tilde{u}_1(x, \zeta_1; h) = e^{x \cdot \zeta_1/h}(e^{\Phi_1(x, \gamma_1 + i\gamma_2, h)} + r_1(x, \zeta_1; h))
$$

of the equation $(L_{\tilde{A}_1, \tilde{q}_1} - k^2)\tilde{u}_1 = 0$ in $B$, where $\Phi_1 \in C^\infty(B)$. By Remark 2.6 $\Phi_1 \to \Phi_1^0$ in the $L^\infty$-norm as $h \to 0$, where $\Phi_1^0$ solves the equation

$$
(\gamma_1 + i\gamma_2) \cdot \nabla \Phi_1^0 = -i(\gamma_1 + i\gamma_2) \cdot \tilde{A}_1 \quad \text{in} \quad B.
$$

To obtain a function that is zero on the plane $x_3 = 0$, we set

$$
u_1(x) := \tilde{u}_1(x) - \tilde{u}_1(\bar{x}), \quad x \in B. \cup l_B.
$$

Then it is easy to check that the restriction $u_1|_N \in W_2^1(N)$.

We can similarly pick by Proposition 2.4 and Remark 2.5 complex geometric optics solutions $\tilde{u}_2$ in $H^2(B)$,

$$
\tilde{u}_2(x, \zeta_2; h) = e^{x \cdot \zeta_2/h}(e^{\Phi_2(x, -\gamma_1 + i\gamma_2, h)} + r_2(x, \zeta_2; h))
$$

of the equation $(L_{\tilde{A}_2, \tilde{q}_2} - k^2)\tilde{u}_2 = 0$ in $B$, where $\Phi_2 \in C^\infty(B)$. By Remark 2.6 $\Phi_2 \to \Phi_2^0$ in the $L^\infty$-norm as $h \to 0$, where $\Phi_2^0$ solves the equation

$$
(-\gamma_1 + i\gamma_2) \cdot \nabla \Phi_2^0 = -i(-\gamma_1 + i\gamma_2) \cdot \tilde{A}_2 \quad \text{in} \quad B.
$$

To obtain a function that is zero on the plane $x_3 = 0$, we set

$$
u_2(x) := \tilde{u}_2(x) - \tilde{u}_2(\bar{x}), \quad x \in B. \cup l_B.
$$

Then it is easy to check that the restriction $u_2|_N \in W_2^1(N)$.

The next step is to substitute the complex geometric optics solutions $u_1$ and $u_2$, given by (3.6) and (4.0), respectively, into the integral identity (3.0). This will be done in the Lemma below. We will use the abbreviations $P_1(x) := e^{\Phi_1(x)} + r_1(x)$ and $P_2(x) := e^{\Phi_2(x)} + r_2(x)$, so that

$$
u_1(x) = e^{x \cdot \zeta_1/h}P_1(x) - e^{\bar{x} \cdot \zeta_1/h}P_1(\bar{x}),
\quad u_2(x) = e^{x \cdot \zeta_2/h}P_2(x) - e^{\bar{x} \cdot \zeta_2/h}P_2(\bar{x}).
$$

For future references, it will be convenient to compute the product of the phases that occur in the terms $\nu_1\nu_2, \nabla u_1\nu_2$ and $u_1 \nabla \nu_2$,

$$
e^{x \cdot \zeta_1/h}e^{x \cdot \zeta_2/h} = e^{ix \cdot \xi},
\quad e^{\bar{x} \cdot \zeta_1/h}e^{\bar{x} \cdot \zeta_2/h} = e^{i\bar{x} \cdot \xi},
$$

$$
e^{ix \cdot \zeta_1/h}e^{i(0,0,-2\xi_3)} \cdot \zeta_1/h = e^{ix \cdot \xi_+ - 2\gamma_1,3\xi_3/h},
\quad e^{i\bar{x} \cdot \xi_+ - 2\gamma_1,3\xi_3/h},
$$

(4.7)
where

\[ \gamma_j = (\gamma_{j,1}, \gamma_{j,2}, \gamma_{j,3}), \quad j = 1, 2 \]

and

\[ \xi_{\pm} = \left( \xi_1, \xi_2, \pm \frac{2}{h} \sqrt{1 - \frac{h^2|\xi|^2}{4} \gamma_{2,3}} \right). \]

We restrict the choices of \( \gamma_1 \) and \( \gamma_2 \), by assuming that

\[ \gamma_{1,3} = 0 \quad \text{and} \quad \gamma_{2,3} \neq 0. \tag{4.8} \]

We need these conditions for the proof of the next Lemma. The first condition makes the above phases purely imaginary, which avoids exponential growth of the terms, as \( h \to 0 \). The second condition implies that \( |\xi_{\pm}| \to \infty \) as \( h \to 0 \). This will be needed since we will use the Riemann-Lebesgue Lemma to eliminate unwanted imaginary exponentials.

Finally it will also be convenient to explicitly state the following norm estimates, which follow from Proposition 2.4

\[ \| e^{\Phi_j} \|_{L^\infty} = O(1), \quad \| \nabla e^{\Phi_j} \|_{L^\infty} = O(h^{-1/3}), \]

\[ \| r_j \|_{L^2} = O(h^{1/3}), \quad \| \nabla r_j \|_{L^2} = O(h^{-2/3}), \quad j = 1, 2, \]

as \( h \to 0 \).

Lemma 4.1. We have

\[ (\gamma_1 + i\gamma_2) \cdot \int_{B} (\bar{A}_2 - \bar{A}_1) e^{ix \cdot \xi_{\pm}} e^{\Phi_1 + \bar{\Phi}_2} dx = 0, \tag{4.10} \]

where \( \gamma_1, \gamma_2 \) and \( \xi \) satisfy (4.1) and (4.8).

Proof. We will prove the statement by multiplying the integral equation (3.6) of Proposition 3.2 by \( h \), when \( u_1 \) and \( u_2 \) are given by (4.4) and (4.6), and then take the limit as \( h \to 0 \).

To begin with notice that we may integrate over \( B \) in (3.6), since

supp(\( A_j \)), supp(q_j) \subset N \subset B^c

and \( u_j \) are defined in \( B \), when \( j = 1, 2 \). We first show that for the second term in (3.6) we have

\[ h \int_{B_{\pm}} (A_1^2 - A_2^2 + q_1 - q_2) u_1 \overline{u}_2 dx \to 0, \tag{4.11} \]

as \( h \to 0 \). Using the phase computations (4.7) we get that

\[ u_1 \overline{u}_2 = e^{ix \cdot \xi} P_1(x)\overline{P}_2(x) - e^{ix \cdot \xi} P_1(x)\overline{P}_2(\bar{x}) + e^{i\xi \cdot \bar{x}} P_1(\bar{x})\overline{P}_2(x) + e^{-i\xi \cdot x} P_1(\bar{x})\overline{P}_2(\bar{x}). \]

This is multiplied by an \( L^\infty \) function in (4.11). Since we restricted the choice of \( \gamma_1 \) to make the exponents purely imaginary, we see easily using the estimates (4.9) that (4.11) holds.

Equation (3.6) multiplied by \( h \), is thus reduced in the limit to

\[ \lim_{h \to 0} \left( h \int_{B_{\pm}} i(A_2 - A_1) \cdot \nabla u_1 \overline{u}_2 dx - h \int_{B_{\pm}} i(A_2 - A_1) \cdot u_1 \nabla \overline{u}_2 dx \right) = 0. \tag{4.12} \]
We will proceed by examining the first term. Using \((4.14)\), we write \(\nabla u_1 \bar{u}_2\) as
\[
\nabla u_1 \bar{u}_2 = \frac{\zeta_i}{h} (e^{ix \xi} P_1(x) \overline{P_2(x)} - e^{ix \xi + \nabla} P_1(x) \overline{P_2(x)})
+ e^{ix \xi} \nabla P_1(x) \overline{P_2(x)} - e^{ix \xi + \nabla} P_1(x) \overline{P_2(x)}
- \frac{\zeta_i}{h} (e^{ix \xi - \nabla} P_1(\bar{x}) \overline{P_2(x)} - e^{i\xi \bar{x}} P_1(\bar{x}) \overline{P_2(\bar{x})})
- e^{ix \xi - \nabla} P_1(\bar{x}) \overline{P_2(x)} + e^{i\xi \bar{x}} \nabla P_1(\bar{x}) \overline{P_2(\bar{x})},
\]
where \(\zeta_i := (\zeta_{i,1}, \zeta_{i,2}, -\zeta_{i,3})\), \(j = 1, 2\). The terms of the product that do not contain
the factor \(1/h\), result in integrals similar to the one in \((4.11)\). One sees similarly
using estimates \((4.9)\), zero in the limit. The above limit is thus equal to
\[
\lim_{h \to 0} \int_{B_2} i(A_2 - A_1) \cdot (\zeta_i e^{ix \xi} P_1(x) \overline{P_2(x)} - \zeta_i e^{ix \xi + \nabla} P_1(x) \overline{P_2(x)})
- \zeta_i e^{ix \xi + \nabla} P_1(x) \overline{P_2(x)} + \zeta_i e^{i\xi \bar{x}} P_1(\bar{x}) \overline{P_2(\bar{x})}) dx.
\]
Now we use the Riemann-Lebesgue Lemma to conclude that the terms with exponents containing \(\xi_+\) and \(\xi_-\) are zero in the limit. To see this, notice that by
Remark 2.6, we see that
\[
\lim_{h \to 0} \int_{B_2} \left| \zeta_i e^{ix \xi} P_1(x) \overline{P_2(x)} \right| dx = 0.
\]
Estimates \((4.9)\) show that \(|r_i| \leq C h^{1/3}\). Hence \(|P_i| \leq h^{1/3}\), for
some \(C > 0\), when \(h\) is small enough.

The first term in \((4.12)\) is therefore
\[
\lim_{h \to 0} \int_{B_2} i(A_2 - A_1) \cdot (\zeta_i e^{ix \xi} P_1(x) \overline{P_2(x)} + \zeta_i e^{ix \xi + \nabla} P_1(x) \overline{P_2(x)}) dx
\]
as \(h \to 0\). The terms containing \(r_i\) in the products of \(P_1\) and \(P_2\) are, because of
\((4.9)\), zero in the limit. The above limit is thus equal to
\[
\lim_{h \to 0} \int_{B_2} i(A_2 - A_1) \cdot (\zeta_i e^{ix \xi} e^{\Phi_1(x)} + \zeta_i e^{ix \xi + \nabla} e^{\Phi_1(x)} + \zeta_i e^{i\xi \bar{x}} e^{\Phi_1(\bar{x})} + \zeta_i e^{ix \xi + \nabla} e^{\Phi_1(\bar{x})} dx.
\]
Finally we split the integral and do a change of variable in the second term and arrive at the expression
\[
(4.13) \quad \lim_{h \to 0} \int_{B_2} i(A_2 - A_1) \cdot \zeta_i e^{ix \xi} e^{\Phi_1(x)} + \Phi_2(x) dx,
\]
for the first term of \((4.12)\).

Returning to the second term in \((4.12)\), containing \(u_1 \bar{w}_2\). This is of the same form as the first one. By doing the above derivation by simply exchanging the roles of
\(u_1\) and \(\bar{w}_2\), we similarly see that the second term becomes
\[
(4.14) \quad \lim_{h \to 0} \int_{B_2} i(A_2 - A_1) \cdot \zeta_i e^{ix \xi} e^{\Phi_1(x)} + \Phi_2(x) dx.
\]
Now \(\zeta_i \to (\gamma_1 + i\gamma_2)\) and \(\zeta_i \to (\gamma_1 + i\gamma_2)\), as \(h \to 0\). Thus by using \((4.13)\) with
\((4.14)\), we can rewrite \((4.12)\) as
\[
\lim_{h \to 0} \int_{B_2} i(A_2 - A_1) \cdot \zeta_i e^{ix \xi} e^{\Phi_1(x)} + \Phi_2(x) dx = \int_{B_2} i(A_2 - A_1) \cdot (\gamma_1 + i\gamma_2) e^{ix \xi} e^{\Phi_1(x)} + \Phi_2(x) dx = 0.
\]
The next Proposition shows that \((4.10)\) holds even when the exponential function depending on \(\Phi^0_i, i = 1, 2\) is removed. The argument follows [5] closely. We will give details for the convenience of the reader.

**Proposition 4.2.** The equality \((4.10)\) implies that

\[
(4.15) \quad (\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1)e^{ix \cdot \xi} dx = 0,
\]

for \(\gamma_1, \gamma_2\) and \(\xi\) which satisfy \((4.1)\) and \((4.8)\).

**Proof.** By \((4.3)\) and \((4.5)\) we have that

\[
(4.16) \quad (\gamma_1 + i\gamma_2) \cdot \nabla (\Phi^0_1 + \Phi^0_2) = -i(\gamma_1 + i\gamma_2) \cdot (\tilde{A}_1 - \tilde{A}_2) \text{ in } B.
\]

Remark 2.7 furthermore implies that the amplitude \(e^{\Phi_1}\) in the definition of \(u_1\) can be replaced by \(ge^{\Phi_1}\), if \(g \in C^\infty(B)\) is a solution of

\[
(4.17) \quad (\gamma_1 + i\gamma_2) \cdot \nabla g = 0 \text{ in } B.
\]

Let \(\Psi(x) := \Phi^0_1(x) + \Phi^0_2(x)\). Then instead of \((4.10)\) we can write,

\[
(4.18) \quad (\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1)ge^{ix \cdot \xi} e^{\Phi(x)} dx = 0.
\]

We conclude from \((4.16)\) that

\[
(\gamma_1 + i\gamma_2) \cdot \nabla e^\Psi = -i(\gamma_1 + i\gamma_2) \cdot (g \nabla e^\Psi),
\]

and therefore, we get

\[
(4.19) \quad \int_B ge^{ix \cdot \xi}(\gamma_1 + i\gamma_2) \cdot \nabla e^\Psi dx = 0,
\]

for all \(g\) satisfying \((4.17)\).

We pick a \(\gamma_3\), with \(|\gamma_3| = 1\), so that we obtain an orthonormal basis \(\{\gamma_1, \gamma_2, \gamma_3\}\). Let \(T\) be the coordinate transform into this basis, i.e. \(y = Tx = (x \cdot \gamma_1, x \cdot \gamma_2, x \cdot \gamma_3)\). Set \(z = y_1 + iy_2\), so that \(\partial_z = (\partial_{y_1} + i\partial_{y_2})/2\) and

\[
(\gamma_1 + i\gamma_2) \cdot \nabla = 2 \partial_z.
\]

Rewriting \((4.18)\) using this and a change of variable given by \(T\) we have

\[
\int_{TB} ge^{iy \cdot \xi} \partial_y e^\Psi dy = 0,
\]

for all \(g\) satisfying \((4.17)\).

Notice that \(y \cdot \xi = y_3\xi_3\), since \(\xi\) is in the \(y\)-coordinates of the form \((0, 0, \xi_3)\). The above integral is therefore a Fourier transform w.r.t. \(\xi_3\). Let \(g \in C^\infty(TB)\) satisfy \(\partial_z g = 0\) and be independent of \(y_3\). Then taking the inverse Fourier transform we write

\[
0 = \int_{Ty_3} g \partial_{y_1} e^\Psi dy_1 dy_2 = \int_{Ty_3} \partial_z (ge^\Psi) dy_1 dy_2,
\]

where \(Ty_3 := TB \cap \Pi_{y_3}\) and \(\Pi_{y_3} = \{(y_1, y_2, y_3) : (y_1, y_2) \in \mathbb{R}^2\}\). Notice that the boundary of \(Ty_3\) is smooth. Multiplying the above by \(2i\) and using Stokes’ theorem
we get that
\[ 0 = 2i \int_{T_{y_3}} \partial_z (ge^\Psi) dy_1 dy_2 \]
\[ = \int_{T_{y_3}} \nabla \times (ge^\Psi, ige^\Psi, 0) \cdot ndy_1 dy_2 \]
\[ = \int_{\partial T_{y_3}} (ge^\Psi, ige^\Psi, 0) \cdot dl \]
\[ = \int_{\partial T_{y_3}} ge^\Psi dz, \]
(4.19)
for all holomorphic functions \( g \in C^\infty(T_{y_3}) \).

Next we shall show that (4.19) implies that there exists a nowhere vanishing holomorphic function \( F \in C(T_{y_3}) \) such that
\[ (4.20) \quad F|_{\partial T_{y_3}} = e^\Psi|_{\partial T_{y_3}}. \]

To this end, we define \( F \) to be
\[ F(z) = \frac{1}{2\pi i} \int_{\partial T_{y_3}} \frac{e^\Psi(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus \partial T_{y_3}. \]
The function \( F \) is holomorphic away from \( \partial T_{y_3} \). As \( e^\Psi \) is Lipschitz, we know because of the Plemelj-Sokhotski-Privalov formula (see e.g. \([10]\)), that
\[ \lim_{z \to z_0, z \in T_{y_3}} F(z) - \lim_{z \to z_0, z \not\in T_{y_3}} F(z) = e^\Psi(z_0), \quad z_0 \in \partial T_{y_3}. \]
(4.21)

Now the function \( \zeta \mapsto (\zeta - z)^{-1} \) is holomorphic on \( T_{y_3} \) when \( z \not\in T_{y_3} \). By choosing \( g(\zeta) = (\zeta - z)^{-1} \) in (4.19), we get therefore that \( F(z) = 0 \), when \( z \not\in T_{y_3} \). Hence, the second limit in (4.21) vanishes, and therefore, \( F \) is holomorphic function on \( T_{y_3} \), such that (4.20) holds.

Next we show that \( F \) is non-vanishing in \( T_{y_3} \). When doing so, let \( \partial T_{y_3} \) be parametrized by \( z = \gamma(t) \), and \( N \) be the number of zeros of \( F \) in \( T_{y_3} \). Then by the argument principle, we get
\[ N = \frac{1}{2\pi i} \int_{\gamma} \frac{F'(z)}{F(z)} dz = \frac{1}{2\pi i} \int_{F \circ \gamma} \frac{1}{\zeta} d\zeta = \frac{1}{2\pi i} \int_{e^{\Psi \circ \gamma}} \frac{1}{\zeta} d\zeta = 0. \]
To see that the last integral is zero, notice that this the winding number of the path \( e^{\Psi \circ \gamma} \). And that \( e^{\Psi \circ \gamma} \) is homotopic to the constant contour \( \{1\} \), with the homotopy given by \( e^{s\Psi \circ \gamma(t)} \), \( s \in [0,1] \).

Next, since \( F \) is a non-vanishing holomorphic function on \( T_{y_3} \) and \( T_{y_3} \) is simply connected, it admits a holomorphic logarithm. Hence, (4.20) implies that
\[ (\log F)|_{\partial T_{y_3}} = \Psi|_{\partial T_{y_3}}. \]
Because \( \log F = \Psi \) is continuous on \( \partial T_{y_3} \), we have by the Cauchy theory,
\[ \int_{\partial T_{y_3}} g \Psi dz = \int_{\partial T_{y_3}} g \log F dz = 0, \]
where \( g \in C^\infty(T_{y_3}) \) is an arbitrary function such that \( \partial_z g = 0 \). Using Stokes’ formula as in (4.19) allows us to write this as
\[ \int_{T_{y_3}} g \partial_z \Psi dy_1 dy_2 = 0. \]
Taking the Fourier transform with respect to \( y_3 \), we get
\[ \int_{T(B)} e^{iy \cdot \xi} g \partial_z \Psi dy = 0, \]
for all \( \xi = (0,0,\xi_3), \xi_3 \in \mathbb{R} \). Hence, returning back to the \( x \) variable, we obtain that
\[
(\gamma_1 + i\gamma_2) \cdot \int_B e^{ix \cdot \xi} g(x) \nabla \Psi(x) dx = 0,
\]
where \( g \in C^\infty(\mathbb{R}^3) \) is such that \( (\gamma_1 + i\gamma_2) \cdot \nabla g = 0 \) in \( B \).

Using \( 4.16 \), we finally get
\[
(\gamma_1 + i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) g(x) e^{ix \cdot \xi} dx = 0.
\]
Setting \( g = 1 \), we obtain \( 4.15 \). \( \square \)

By replacing the vector \( \gamma_2 \) by \( -\gamma_2 \) in \( 4.16 \), we see that
\[
(\gamma_1 - i\gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) e^{ix \cdot \xi} dx = 0.
\]
Hence, \( 4.13 \) and \( 4.23 \) imply that
\[
(\gamma_1 - \gamma_2) \cdot \int_B (\tilde{A}_2 - \tilde{A}_1) e^{ix \cdot \xi} dx = 0,
\]
for all \( \gamma \in \text{span}\{\gamma_1, \gamma_2\} \) and all \( \xi \in \mathbb{R}^3 \) such that \( 4.11 \) and \( 4.3 \) hold.

In the proof of the next Proposition we see that \( 4.24 \) is actually a condition for having \( \nabla \times (\tilde{A}_1 - \tilde{A}_2) = 0 \). This is therefore the last step in proving that the DN-map determines the curl of the magnetic potential.

**Proposition 4.3.** It follows from \( 4.24 \) that
\[
\nabla \times \tilde{A}_1 = \nabla \times \tilde{A}_2 \quad \text{in} \quad B.
\]

**Proof.** Assume that \( \xi \in \mathbb{R}^3 \) is not on the line \( L := (0,0,t), t \in \mathbb{R} \). Then the vectors \( \gamma_1 \) and \( \gamma_2 \) given by
\[
\tilde{\gamma}_1 := (-\xi_2, \xi_1, 0), \quad \gamma_1 := \tilde{\gamma}_1/|\tilde{\gamma}_1|,
\]
\[
\tilde{\gamma}_2 := \xi \times \gamma_1, \quad \gamma_2 := \tilde{\gamma}_2/|\tilde{\gamma}_2|,
\]
where \( \xi \times \gamma_1 \) stands for the cross product, satisfy \( 4.5 \) and \( 4.1 \). Thus, for any vector \( \xi \in \mathbb{R}^3 \setminus L, \) \( 4.21 \) says that
\[
\gamma \cdot v(\xi) = 0, \quad v(\xi) := \tilde{A}_2 \chi(\xi) - \tilde{A}_1 \chi(\xi),
\]
for all \( \gamma \in \text{span}\{\gamma_1, \gamma_2\} \). Here \( \chi \) is the characteristic function of the set \( B \). For any vector \( \xi \in \mathbb{R}^3 \), we have the following decomposition,
\[
v(\xi) = v_\parallel(\xi) + v_{\perp}(\xi),
\]
where \( \text{Re} v_\parallel(\xi), \text{Im} v_\parallel(\xi) \) are multiples of \( \xi, \) and \( \text{Re} v_{\perp}(\xi), \text{Im} v_{\perp}(\xi) \) are orthogonal to \( \xi \). Now we have \( \text{Re} v_{\perp}(\xi), \text{Im} v_{\perp}(\xi) \in \text{span}\{\gamma_1, \gamma_2\}, \) and therefore, it follows from \( 4.22 \) that \( v_{\perp}(\xi) = 0, \) for all \( \xi \in \mathbb{R}^3 \setminus L \).

Hence, \( v(\xi) = \alpha(\xi) \xi, \) so that that
\[
\xi \times v(\xi) = 0,
\]
for all \( \xi \in \mathbb{R}^3 \setminus L, \) and thus, everywhere, by the analyticity of the Fourier transform. Taking the inverse Fourier transform, we obtain \( 4.26 \). \( \square \)
5. Determining the Electric Potential

In order to complete the proof of Theorem 1.2 we need to show that the electric potential is also determined by the DN-map. Again we assume that $A_j, q_j$ and $\Gamma_j$, $j = 1, 2$ are as in Theorem 1.2 and that the DN-maps satisfy (1.6), and hence (3.1).

Since $B$ is simply connected, it follows from the Helmholtz decomposition of $\tilde{A}_1 - \tilde{A}_2$ and (4.25) that there exists $\psi \in C^{1,1}(\overline{B})$ with $\psi = 0$ near $\partial B$ such that

$$\tilde{A}_1 = \tilde{A}_2 + \nabla \psi \quad \text{in} \quad B.$$  

We extend $\psi$ to a function of class $C^{1,1}$ on all of $\mathbb{R}^3$ such that $\psi = 0$ on $\mathbb{R}^3 \setminus \overline{B}$. Then

$$\tilde{A}_1 = \tilde{A}_2 + \nabla \psi \quad \text{in} \quad \mathbb{R}^3.$$  

Since $\tilde{\Gamma}_j \subset \mathbb{R}^3 \setminus \overline{N}$, $j = 1, 2$, and $\mathbb{R}^3 \setminus \overline{N}$ is connected, we can assume that $\psi = 0$ on $\mathbb{R}^3 \setminus \overline{N}$ and hence, we have that $\psi = 0$ on $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$. It follows then from Lemma 1.21 part (i) and (3.1) that for all $f$ with supp$(f) \subset \tilde{\Gamma}_2$,

$$\Lambda A_{1,q_1}(f)|_{\tilde{\Gamma}_1} = \Lambda A_{2,q_2}(f)|_{\tilde{\Gamma}_1} = \Lambda A_{2} + \nabla \psi \cdot q_2(f)|_{\tilde{\Gamma}_1} = \Lambda A_{1,q_1}(f)|_{\tilde{\Gamma}_1}.$$  

Therefore, we can apply Proposition 5.2 with $A_1 = A_2$ and get

$$\int_N (q_1 - q_2) u_1 \nabla \overline{\gamma} dx = 0,$$

for all $u_1 \in W_1(N)$ and $u_2 \in W_2^*(N)$.

Choosing in (5.1) $u_1$ and $u_2$ as the complex geometric optics solutions, given by (4.4) and (4.26), passing to $B_\epsilon$, and letting $\epsilon \to 0$, we have

$$\int_B (\tilde{q}_1 - \tilde{q}_2) e^{ix \cdot \xi} e^{\Phi^0_1 + \Phi^2_2} dx = 0.$$  

By Remark 2.7 $e^{\Phi^0}$ in the definition (4.4) of $u_1$ can be replaced by $ge^{\Phi^2}$ if $g \in C^\infty(\overline{B})$ is a solution of

$$(\gamma_1 + i\gamma_2) \cdot \nabla g = 0 \quad \text{in} \quad B.$$  

Then (5.2) can be replaced by

$$\int_B (\tilde{q}_1 - \tilde{q}_2) g(x) e^{ix \cdot \xi} e^{\Phi^0_1 + \Phi^2_2} dx = 0.$$  

Now (4.8) has the form,

$$(\gamma_1 + i\gamma_2) \cdot \nabla (\Phi^0_1 + \Phi^2_2) = 0 \quad \text{in} \quad B,$$

since we have that $\tilde{A}_1 = \tilde{A}_2$. Thus, we can take $g = e^{-(\Phi^0_1 + \Phi^2_2)}$ and obtain that

$$\int_B (\tilde{q}_1 - \tilde{q}_2) e^{ix \cdot \xi} dx = 0,$$

for all $\xi \in \mathbb{R}^3$ such that there exist $\gamma_1, \gamma_2 \in \mathbb{R}^3$, satisfying (1.1) and (1.8). Since for any $\xi \in \mathbb{R}^3$ not of the form $\xi = (0, 0, \xi_3)$, the vectors, given by (4.26), satisfy (1.1) and (1.8), we conclude that (5.3) holds for all $\xi \in \mathbb{R}^3$ except those of the form $\xi = (0, 0, \xi_3)$, and therefore, by analyticity of the Fourier transform, for all $\xi \in \mathbb{R}^3$. Hence, $q_1 = q_2$ in $B_\epsilon$. This completes the proof of Theorem 1.2.
6. Appendix

6.1. Magnetic Green’s formulas. Let us first recall, following [1], the standard Green formula applied to the magnetic Schrödinger operator.

Lemma 6.1. Suppose that $\Omega \subset \mathbb{R}^3$ is open and bounded, with piecewise $C^1$ boundary. Let $A \in W^{1,\infty}(\Omega, \mathbb{R}^3)$ and $q \in L^\infty(\Omega)$. Then we have,

$$
(L_{A,q} u, v)_{L^2(\Omega)} - (u, L_{A,q} v)_{L^2(\Omega)} = (u, (\partial_n + i A \cdot n) v)_{L^2(\Omega^\partial)} - ((\partial_n + i A \cdot n) u, v)_{L^2(\Omega^\partial)},
$$

for all $u, v \in H^1(\Omega)$, with $\Delta u, \Delta v \in L^2(\Omega)$, where $n$ is the exterior unit normal to $\partial \Omega$.

We shall also need a version of the above result where $\Omega$ is replaced by $\mathbb{R}^3$. To be precise, let $A \in W_{comp}^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$, $q \in L^\infty(\mathbb{R}^3)$, and let $u \in H^2_{loc}(\mathbb{R}^3)$ be such that

$$
(L_{A,q} - k^2) u = 0 \quad \text{in} \quad \mathbb{R}^3,
$$

supp$(u|_{\partial \mathbb{R}^3})$ is compact, and $u$ is outgoing. Assume also that $v \in H^2_{loc}(\mathbb{R}^3)$ satisfies

$$
(L_{A,q} - k^2) v \in L^2_{comp}(\mathbb{R}^3),
$$

supp$(v|_{\partial \mathbb{R}^3})$ is compact, and $v$ is incoming.

Lemma 6.2. With $u$ and $v$ as above, we have

$$
((L_{A,q} - k^2) u, v)_{L^2(\mathbb{R}^3)} - (u, (L_{A,q} - k^2) v)_{L^2(\mathbb{R}^3)} = (u, (\partial_n + i A \cdot n) v)_{L^2(\partial \mathbb{R}^3)} - ((\partial_n + i A \cdot n) u, v)_{L^2(\partial \mathbb{R}^3)}.
$$

Proof. Let $B_R := B(x_0, R)$ be an open ball in $\mathbb{R}^3$ of radius $R$, and choose $R > 0$ large enough so that

$$
supp(A), \text{supp}(q) \subset B_R.
$$

Set $\Omega = \mathbb{R}^3 \cap B_R$. By Lemma 6.1 we know that

$$
((L_{A,q} - k^2) u, v)_{L^2(\Omega)} - (u, (L_{A,q} - k^2) v)_{L^2(\Omega)} = (u, (\partial_n + i A \cdot n) v)_{L^2(\Omega^\partial)} - ((\partial_n + i A \cdot n) u, v)_{L^2(\Omega^\partial)}.
$$

Thus, to obtain (6.1) we need to show that

$$
\int_{\partial B_R \cap \mathbb{R}^3} (\overline{u \omega_n} v - \partial_n u \overline{v}) dS_R \to 0, \quad R \to \infty.
$$

Let us rewrite the left hand side of the above as follows,

$$
\int_{\partial B_R \cap \mathbb{R}^3} (\overline{\partial_n \overline{\nu}} - ik\overline{\nu}) u dS_R - \int_{\partial B_R \cap \mathbb{R}^3} (\overline{\partial_n u} - iku \overline{\nu}) dS_R.
$$

We show that first term goes to zero as $R \to \infty$. The second term can be handled in the same way. Applying Cauchy-Schwarz gives

$$
\left| \int_{\partial B_R \cap \mathbb{R}^3} (\overline{\partial_n \overline{\nu}} - ik\overline{\nu}) u dS_R \right|^2 \leq \int_{\partial B_R \cap \mathbb{R}^3} |\overline{\partial_n \overline{\nu}} - ik\overline{\nu}|^2 dS_R \int_{\partial B_R \cap \mathbb{R}^3} |u|^2 dS_R.
$$

Here the first integral goes to zero, since $\overline{\partial_n \overline{\nu}} = \partial_n v + ikv$ and $|\partial_n v + ikv|^2$ is $O(1/r^2)$ as $r = |x| \to \infty$, since $v$ is incoming. We conclude the proof by showing that the second integral is bounded as $R \to \infty$. 

To this end we let $R_2 > R_1$, where $R_1$ is such that
\[ \text{supp}(A), \text{supp}(q) \subset B(x_0, R_1), \]
and set $B_j := B(x_0, R_j)$, $j = 1, 2$ and $U := (B_2 \setminus B_1) \cap \mathbb{R}^3$. We multiply the Sommerfeld condition (1.4) by its complex conjugate and integrate over $\partial B_2 \cap \mathbb{R}^3_+$, which gives
\[
\int_{\partial B_2 \cap \mathbb{R}^3_+} (k^2 |u|^2 + |\partial_n u|^2 + 2k \text{Im}(u \partial_n \overline{u}))dS
\]
(6.3)
as $R_2 \to \infty$ and where $n$ is the outer unit normal vector to $B_2$.

By Green’s formula we have on the other hand that
\[
\int_{\partial U} \partial_n u \partial_n u - \partial U u = \int_{\partial U} \Delta u - \Delta u = 0.
\]
(6.4)
We may assume that $u|_{\partial U \cap \partial \mathbb{R}^2_+} = 0$, since $\text{supp}(u|_{\partial \mathbb{R}^2_+})$ is compact. We can thus write (6.4) as
\[
\int_{\partial B_2 \cap \mathbb{R}^3_+} \text{Im}(u \partial_n \overline{u}) = \int_{\partial B_1 \cap \mathbb{R}^3_+} \text{Im}(u \partial_n \overline{u}).
\]
(6.5)
But this implies that the $\int |u|^2$ and $\int |\partial_n u|^2$ terms stay bounded in the limit (6.3).

\section*{6.2. The unique continuation principle.}
In this work we make heavy use of the so called unique continuation principle. The unique continuation principle can be seen as an extension of the familiar property that an analytic function that is zero on some open set is identically zero.

Let $\Omega \subset \mathbb{R}^n$ be an open connected set, and let
\[ Pu = \sum_{i,j=1}^{n} a_{ij}(x) \partial_i \partial_j u + \sum_i b_i(x) \partial_i u + c(x)u. \]
Here $a_{ij} \in C^1(\overline{\Omega})$ are real-valued, $a_{ij} = a_{ji}$, and there is $C > 0$ so that
\[ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq C|\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n. \]
Furthermore, $b_i \in L^\infty(\Omega, \mathbb{C})$ and $c \in L^\infty(\Omega, \mathbb{C})$. We have the following result, see [3] and [14].

\textbf{Theorem 6.3.} Let $u \in H^2_{loc}(\Omega)$ be such that $Pu = 0$ in $\Omega$. Let $\omega \subset \Omega$ be open non-empty. If $u = 0$ on $\omega$, then $u$ vanishes identically in $\Omega$.

\textbf{Corollary 6.4.} Assume that $\partial \Omega$ is of class $C^2$. Let $\Gamma \subset \partial \Omega$ be open non-empty. Let $u \in H^2(\Omega)$ be such that $Pu = 0$ in $\Omega$. Assume that
\[ u = \mathcal{B}_\nu u = 0 \quad \text{on} \quad \Gamma. \]
Here $\mathcal{B}_\nu u$ is the conormal derivative of $u$, given by
\[ \mathcal{B}_\nu u = \sum_{i,j=1}^{n} \nu_i (a_{ij} \partial_j u)|_{\partial \Omega} \in H^{1/2}(\partial \Omega). \]
Then $u$ vanishes identically in $\Omega$. 
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