Near-horizon physics of an evaporating black hole: 
One-loop effects in the $\lambda\Phi^4$-theory

Slava Emelyanov

Institute for Theoretical Physics, 
Karlsruhe Institute of Technology (KIT), 
76131 Karlsruhe, Germany

Abstract

We study massless scalar theory with the quartic self-interacting term far away from and near to evaporating and spherically symmetric black hole. We propose a principle of how to define the physical notion of particle in curved space-time. Employing this definition, we compute one-loop correction to the self-energy and coupling constant of the scalar field near the horizon in the freely-falling frame. We find that the coupling constant becomes slightly stronger near to the horizon. We also find that the term in the 2-point function that is (partially) responsible for the black-hole evaporation corresponds to the sub-leading correction to the Feynman propagator whose pole structure is of the leading order.

Keywords: black holes, self-interacting scalar field, local renormalisation, quantum/vacuum fluctuations

*Electronic address: viacheslav.emelyanov@kit.edu
I. INTRODUCTION

The quantum physics of black holes is full of various paradoxes. One of these problems is related to the understanding of the particle production process induced by black holes. The created particles \( \{ \tilde{\psi}_i \} \) possess a thermal profile and turn out to appear in empty space, i.e. outside of the matter that has gravitationally collapsed into a black hole. The fact is that such a process is in apparent conflict with the unitary evolution in quantum theory [1].

In other words, the \( S \)-matrix between initial vacuum state and final thermal state cannot be unitary. If one demands the \( S \)-matrix be unitary, one comes to another problem - the firewall paradox [2], - i.e. the near-horizon region is populated by the high-energy particles of the type \( \{ \tilde{\psi}_i \} \). If this was correct, then this would imply the equivalence principle, being at the heart of general relativity, would not in general hold in quantum theory.

There exists a logically non-excludable interpretation of the black-hole evaporation (non-vanishing positive outgoing energy flux) which seems to be free of the mentioned inconsistencies. The quantum state remains empty outside of the collapsed matter, i.e. there are no physical particles in the locally Minkowski vacuum (we denote this vacuum as \( |\Omega\rangle \) in the following). In other words, the initial Hilbert space representation \( \mathcal{H} \) of the algebra \( \mathcal{A} \) of all field operators known in the Standard Model is still a physical representation even after the black hole has formed. This implies in particular that the spectrum of particles \( \{ |\tilde{\psi}_i\rangle \} \) as being elements of \( \mathcal{H} \) (assigned far from the event horizon to the irreducible representations of the Poincaré group \( \mathcal{P}^+ \); below we show how this can be done near the event horizon) from which the collapsing matter were composed are still physical excitations after the gravitational collapse.

The crucial question is then whether thermally distributed particles \( \{ |\tilde{\psi}_i\rangle \} \) are elements of \( \mathcal{H} \). To put it differently, the question is whether there exist projection operators \( \hat{P}_i \in \mathcal{A} \) which map \( \mathcal{H} \) on a one-dimensional subspace associated with each of \( |\tilde{\psi}_i\rangle \). The fact, however, is that these projections do not exist in \( \mathcal{A} \) and, hence, the particles \( \{ |\tilde{\psi}_i\rangle \} \) are elements of another Hilbert space \( \tilde{\mathcal{H}} \not\subset \mathcal{H} \). This can be shown by employing the argument that the splitting of the algebra \( \mathcal{A} \) into factor subalgebras does not lead to the factorization of the Hilbert space \( \mathcal{H} \) in quantum field theory (in contrast to quantum mechanics with finite dimensional Hilbert space representations, e.g., of a finite qubit system, which are all unitarily equivalent according to the Stone-von Neumann theorem). Thus, there does not exist a unitary \( S \)-matrix mapping elements of \( \mathcal{H} \) into elements of \( \tilde{\mathcal{H}} \) within the framework of local quantum field theory. Therefore, for the states \( \{ |\tilde{\psi}_i\rangle \} \) to be physically realisable as quanta of the Standard Model fields, there must occur a phase transition \( \mathcal{H} \to \tilde{\mathcal{H}} \) whenever a black hole forms. This scenario is physically unacceptable.

How can the black-hole evaporation then be understood? The evaporation process can still be accounted for within local quantum field theory in a consistent way without referring to the states \( \{ |\tilde{\psi}_i\rangle \} \). The non-trivial value of the renormalised stress tensor \( \langle \hat{T}^{\mu\nu} \rangle \) of a certain quantum field is assigned to the vacuum \( |\Omega\rangle \). In other words, the vacuum is gravitation-
ally “active” after the event horizon has formed (as \(\langle \hat{T}^\mu_\nu \rangle\) enters the Einstein equations). This occurs, because the field operators are sensitive (through their field equations) to the geometry of space-time and, hence, are modified not too far away from the horizon. The modification of the field operators results in the change of the vacuum stress tensor \(\langle \hat{T}^\mu_\nu \rangle\) (this explains the featurelessness of the outgoing energy flux as the state \(|\Omega\rangle\) is full of the featureless quantum fluctuations only). This tensor decreases as \(1/r^2\) far from the hole \([4, 5]\) and is thus practically zero sufficiently far away from the horizon.

An analogous effect occurs, e.g., in the Casimir set-up, where the (Minkowski) vacuum \(|\Omega\rangle\) also possesses a non-vanishing energy-momentum tensor. This is due to the modification of the electromagnetic operators \(\hat{E}_i\) and \(\hat{B}_i\) between the conducting plates. This leads to \(\langle \hat{E}_i \hat{E}_j \rangle \neq 0\) and \(\langle \hat{B}_i \hat{B}_j \rangle \neq 0\), although these vanish in the absence of the plates.

This analogy seems also to hold when one studies electromagnetic properties of the vacuum within quantum electrodynamics. The electromagnetic properties of the vacuum \(|\Omega\rangle\) are characterised by the electric permittivity \(\epsilon\) and the magnetic permeability \(\mu\) entering the Maxwell equations. It has been found within the Casimir set-up that the dispersion relation of low-energy photons is modified in-between the plates \([4]\). We have recently shown that a similar effect exists in the black-hole background \([7]\). Moreover, we have shown that photons acquire an effective mass decreasing as \(1/r\) and a point-like electric charge can be partially screened due to black holes \([8]\).

In this paper we study properties of the vacuum \(|\Omega\rangle\) in the framework of the massless \(\lambda\Phi^4\)-theory. The main purpose is to find out any physical inconsistencies related to the idea of having medium-like characteristics of the vacuum in the background of black holes. Specifically, we consider a self-interacting scalar model in the far-from- and near-horizon region of a large black hole. The one-loop correction to the self-energy and coupling constant are computed in the near-horizon region. As expected, we find that these corrections are suppressed as \((\lambda_p/r_H)^2\) near the event horizon, where \(\lambda_p\) is the de Broglie wavelength of the scalar particle and \(r_H\) is a size of the black-hole horizon.

The outline of this paper is as follows. In Sec. III we introduce the principle of how the physical particles should be defined in curved space-times. The basic idea is to employ a local Minkowski frame to identify a particle with a localised state as one has been successfully doing that in Minkowski space. The equivalence principle plays a crucial role in extending the particle notion to any space-time domain in which gravity is not too strong. This tacitly implies that the notion of particle is \textit{not} observer-dependent, i.e. covariant, as opposed to the common belief.

In Sec. IIII the self-interacting scalar field is considered in the background of an evaporating and spherically symmetric black hole of astrophysical mass, i.e. \(M \geq M_\odot\), where \(M_\odot\) is the solar mass. For these black holes, the local inertial frame might be of the size \(l_M \gtrsim 300(M/M_\odot)\) m. Thus, a particle detector of the size \(l_D \ll 30(M/M_\odot)\) m could be employed to study how particle scattering reactions quantitatively differ from the same re-
actions in the asymptotically flat region, i.e. in the region, where the influence of black holes can be ignored.

In Sec. [IV] we provide an argument why the firewall paradox does not exist. The non-existence of the unitary $S$-matrix relating the states \{|$\psi_i$\}\ with \{|$\tilde{\psi}_i$\}\ is on the contrary consistent with local quantum field theory, but does not imply the unitarity in the gravitational collapse is broken. The outstanding problem is how to take into account the backreaction of the quantum fields on the quantum state of the collapsed matter which is under the horizon.

In Sec. [V] the main results are summarised.

Throughout this paper the fundamental constants are set to $c = G = k_B = \hbar = 1$, unless stated otherwise.

II. LOCAL NOTION OF PARTICLE IN CURVED SPACE-TIME

We want to study certain scattering processes in the vicinity of the horizon of a large black hole formed through the gravitational collapse and compare these with observations in the far-from-horizon region. For this purpose it is first necessary to define a physical notion of particle in curved space-time. Bearing in mind the remarkable success of particle physics based on QFT, the notion of particle should be related to the pole structure of the Feynman propagator when computed in the local Minkowski frame. Therefore, the guiding principle should be based on reproducing the standard results of particle physics at any given point in the universe.

Particle physics formulated in Minkowski space and based on the Standard Model has successfully passed so far all tests in the particle colliders up to the energy scale 1 TeV. Certainly, there is physics beyond the Standard Model which is assumed to be associated with the theory itself (e.g., the neutrino oscillation implies that at least two among of three neutrino flavors are massive), rather than the modifications of the basic QFT principles.

However, the universe is globally non-flat. The observable part of the universe looks at cosmological scales ($100 \, \text{Mpc} \lesssim l \lesssim 3000 \, \text{Mpc}$) as de Sitter space due to dark energy (see, e.g., [9]). The universe becomes inhomogeneous and anisotropic at smaller scales due to dark matter, clusters of galaxies, galaxies and so on. At much smaller, but still macroscopic scales, the universe definitely looks as being nearly Minkowski space.

Moreover, earth is also a source of the non-trivial local curvature. If one introduces the normal Riemannian coordinates $y$, then the local geometry becomes flat:

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\lambda\rho\nu}(y) y^\lambda y^\rho + O(\nabla R y^3) \text{ with } |R y^2| \ll 1. \quad (1)$$

One usually employs the Minkowski-space approximation in order to describe various scattering processes in the particle colliders. Therefore, it is \textit{a posteriori} legitimate to use the Fourier-transform technique (with integration over the whole space-time) whenever one
restricts oneself to space-time regions with a size \( l \ll r(r/r_\oplus)^{1/2} \) (\( r_\oplus \approx 8.7 \text{ mm} \) is earth’s gravitational radius) for the reason explained below. Specifically, the size \( l_{\text{LHC}} \) of the LHC is about 27 km and we find \( R_\oplus(R_\oplus/r_\oplus)^{1/2} \approx 1.7 \times 10^8 \text{ m} \gg R_\oplus \gg l_{\text{LHC}} \), where \( R_\oplus \approx 6.4 \times 10^3 \text{ km} \) is earth’s radius. In other words, the gravitational influence of earth on scattering processes in the LHC can be safely ignored.

The integration over the whole space-time (as if it is infinitely large Minkowski space) in vertices in non-linear field theories does not entail any sort of inconsistencies, because particles are described by localized states in quantum field theory \[10\]. The particle state looks like the vacuum \( |\Omega\rangle \) for measurements performed outside of its support. This is in turn characterised by the size of particle’s wave packet. For instance, electron has a size which is about its Compton wavelength \( \lambda_e = \frac{\hbar}{m_e c} \approx 2.4 \times 10^{-12} \text{ m} \). Thus, it makes a physical sense to speak about the electron (at rest) as a localised object, whenever one restricts oneself to space-time regions of the size \( \lambda_e \ll l \ll l_c \), where \( l_c \) is a characteristic size of the curvature (e.g., \( l_c \approx 10^8 \text{ km} \) for earth and a hypothetical particle of the rest mass less than \( 10^{-17} \text{ eV} \) could not be understood as being localised).

In Minkowski space-time, the wave packet \( h_e(x) \) characterising the electron is a positive energy solution of the Dirac field equation. It possesses a non-vanishing support in the spatial region of the extent \( \lambda_e \). The notion of energy is defined with respect to the Minkowski time translation operator \( G = \partial_\tau \) whose integral curves are geodesics, i.e. it satisfies the geodesic equation \( \nabla G G = 0 \). Thus, one has

\[
    h_e(x) = \frac{1}{(2\pi)^{3/2}} \int d^4 p \theta(p_0) \delta(p^2 - m^2 e) e^{-ipx} h_e(p),
\]

where \( h_e(p) \) is a wave packet in momentum space and \( x_0 = \tau \). At scales \( l \ll l_c \) in curved space-time, the vector \( G \) approximately satisfies the Killing equation, i.e. \( l_c \nabla_{(\mu} G_{\nu)} \approx 0 \), so that it is one of the generators of the local Poincaré group. It turns out to be \( l_c \nabla_{(\mu} G_{\nu)} \approx O(1) \) at larger scales \( l \gtrsim l_c \), i.e. \( G \) is a local Killing vector.

Since the positive-energy packet \( h(x) \) is chosen with respect to \( G \) in particle physics and this choice is consistent with the observations performed so far on earth (freely-moving, rather than moving along any global Killing vector), we come to the following principle:

\textit{A physical particle corresponds to a covariant wave packet \( h(x) \) being a positive energy solution of the field equation with respect to a geodesic vector \( G \) determining the dynamics in the local Minkowski frame.}

The quantisation procedure of, e.g., a scalar non-interacting field \( \hat{\Phi}(x) \) is performed by expanding the field over the positive- and negative-frequency modes defined with respect to a certain time-like Killing vector \( K \):

\[
    \hat{\Phi}(x) = \int d\mu(p) \left( \phi_p(x) \hat{a}_p + \phi_p^*(x) \hat{a}_p^\dagger \right) = \hat{a}(x) + \hat{a}^\dagger(x),
\]

where \( K \phi_p = -i\omega \phi_p \) and \( K \phi_p^\dagger = +i\omega \phi_p^\dagger \) and \( d\mu(p) \) is some positive measure of integration/summation. Due to the linearity of the field equation, there are infinitely many ways
of choosing the modes $\phi_\mu(x)$. According to our principle, there is a preferred choice (unique up the local Lorentz transformation). This depends on a point in curved space-time, such that $K = G$ and, hence, only locally satisfies the Killing equation. In this case, $\hat{a}(h)$ is a creation operator of the physical particle from the vacuum $|\Omega\rangle$ around of a support $\sigma$ of the wave packet $h(x)$. However, one has

$$ \hat{a}(h) = i \int d\Sigma \sqrt{-g(x)} g^{\mu
u}(x) \left( \Phi_\mu(x) \nabla_\nu h(x) - h(x) \nabla_\nu \Phi_\mu(x) \right) $$

for an arbitrary choice of the mode functions, where $\Sigma$ is a Cauchy surface. Taking now into account the finite support of the wave packet $h(x)$ whose size is supposed to be much smaller than the characteristic curvature scale $l_c$, we obtain

$$ \hat{a}(h) = i \int_\sigma d^3y \left( \Phi_\mu(y) \partial_\mu h(y) - h(y) \partial_\mu \Phi_\mu(y) \right) \quad \text{with} \quad y_0 = \tau. $$

This is consistent with the Minkowski-space approximation one has been successfully employing in particle physics. The physical particle is thus given by

$$ |\psi\rangle = \hat{a}(h)|\Omega\rangle, $$

which is localised over the support of the wave packet $h(x)$ and, hence, is normalisable.

Thus, the equivalence principle plays an essential role in our proposal for defining the physical notion of particle at any space-time point of the universe, where the local curvature length is much larger than the particle size. Since these particles when considered in the local Minkowski frame are identical to Wigner’s ones, we can employ the standard methods of the Feynman rules and diagrams to describe their scattering reactions.

### III. SELF-INTERACTING SCALAR FIELD IN SCHWARZSCHILD SPACE

We shall consider a massless scalar field with the conformal coupling to gravity and the quartic self-interacting term in the background of an evaporating and spherically symmetric black hole. Specifically, the Lagrangian $\mathcal{L}$ is given by

$$ \mathcal{L} = -\frac{1}{2} \Phi \Box \Phi + \frac{1}{12} R \Phi^2 - \frac{\lambda}{4!} \Phi^4, $$

where $R$ is the Ricci scalar. The Ricci scalar vanishes in the Schwarzschild geometry which is described by the line element

$$ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2, \quad \text{where} \quad f(r) = 1 - \frac{r_H}{r}, $$

where $r_H = 2M$ is a size of the black-hole horizon of mass $M$. In the following we shall use the surface gravity $\kappa$ on the horizon which is defined as $\kappa \equiv \frac{1}{2} f'(r_H) = 1/2r_H$. 


A freely-falling frame in Schwarzschild space is characterised by an affine parameter $\tau$. This parameter corresponds to the Painlevé-Gullstrand time (see, e.g., [11] for a brief review). The line element (7) in the Painlevé-Gullstrand coordinates reads
\[ ds^2 = f(r)d\tau^2 - 2(1 - f(r))^{1/2}d\tau dr - dr^2 - r^2d\Omega^2. \] (8)

The time coordinate $\tau$ reduces to the standard Minkowski (M) time $t_M$ for $r \gg r_H$. That is also the case for the Schwarzschild (S) time $t_S$ for $r \to \infty$. However, the Painlevé-Gullstrand and Schwarzschild time considerably differ from each other in the near-horizon region. We shall establish the relation between these times at $r \sim r_H$ in what follows.

### A. Wightman function

We found in [8] the Wightman two-point function of the non-interacting ($\lambda = 0$), massless scalar field for the Unruh (U) state [12]. In this paper, we want to approximate the local Minkowski vacuum $|\Omega\rangle$ by the Unruh state in the far- and near-horizon region. This approximation is sufficiently accurate at $r \gg r_H$. In case of $r \sim r_H$, the approximation is still adequate up to some corrections (we shall come back to this issue below).

The 2-point function in the Unruh state reads
\[ W_U(x,x') = \tilde{W}_\beta(x,x') + \tilde{W}(x,x') \]
\[ = \int_0^{+\infty} d\omega \left( \frac{\cos(\omega \Delta t + i\frac{\omega}{2})}{4\pi \omega \sinh \left( \frac{\beta \omega}{2} \right)} \tilde{K}_\omega(x,x') + \frac{\exp(-i\omega \Delta t)}{4\pi \omega} \tilde{K}_\omega(x,x') \right), \] (9)

where $\beta = 1/T_H$ is the inverse Hawking temperature. The right arrow refers to the “out-going” modes, whereas the left one to the “ingoing” modes. The functions $\tilde{K}_\omega(x,x')$ and $\tilde{K}_\omega(x,x')$ are given by
\[ \tilde{K}_\omega(x,x') \approx \frac{\Delta^2(\rho) \sin(\omega \rho)}{4\pi \omega \rho(f(r)f(r'))^{1/2}} \left\{ 4\omega^2 - \left( \frac{f(r)f(r')}{rr'} \right) \Gamma_\omega, \quad r \sim r_H ; \right. \]
\[ \left. 4\omega^2 - \left( \frac{f(r)f(r')}{rr'} \right) \Gamma_\omega, \quad r \gg r_H ; \right\} \] (10)

and
\[ \tilde{K}_\omega(x,x') \approx \frac{\Delta^2(\rho) \sin(\omega \rho)}{4\pi \omega \rho(f(r)f(r'))^{1/2}} \left\{ (\frac{f(r)f(r')}{rr'}) \frac{\Gamma_\omega}{\Gamma_\omega}, \quad r \sim r_H ; \right. \]
\[ \left. 4\omega^2 - (\frac{f(r)f(r')}{rr'}) \frac{\Gamma_\omega}{\Gamma_\omega}, \quad r \gg r_H ; \right\} \] (11)

Note that a stationary observer, i.e. the observer moving along the Killing vector $\partial_t$, possesses a non-trivial acceleration, which asymptotically vanishes in the spatial infinity.
where \( \rho \equiv (2\sigma(x, x'))^{\frac{1}{2}} \), \( \sigma(x, x') \) is the three-dimensional geodetic interval for the ultra-static or optical metric \( \bar{g}_{\mu
u} = g_{\mu
u}/f(r) \), \( \bar{\Delta}(x, x') \) is the Van Vleck-Morette determinant and

\[
\Gamma_\omega \equiv \sum_{l=0}^{+\infty} (2l + 1)|B_{\omega l}|^2 \approx 27\omega^2M^2
\]

in the DeWitt approximation [14].

The origin of the “ingoing” and “outgoing” part of the Wightman function \( W_U(x, x') \) can be understood as follows. The scalar field operator \( \hat{\Phi}(x) \) is represented through a sum of the non-Hermitian operators, namely \( \hat{\Phi}(x) = \hat{a}(x) + \hat{a}^\dagger(x) \), where \( \hat{a}(x) \) annihilates the vacuum, i.e. \( \hat{a}(x)|\Omega\rangle = 0 \) (cf. Eq. (13)). This operator can in turn be represented as

\[
\hat{a}(x) = \hat{a}_\triangleright(x) + \hat{a}_\triangleleft(x),
\]

such that \( \hat{a}_\triangleleft(x)|\Omega\rangle = \hat{a}_\triangleright(x)|\Omega\rangle = 0 \), but \( \hat{a}_\triangleleft(x) \) has a non-vanishing support only for the advanced Finkelstein-Eddington time \( v < v_H \), while \( \hat{a}_\triangleright(x) \) possesses a non-vanishing support for \( v > v_H \), where \( v_H \) corresponds to the moment when the event horizon has formed (see, e.g., [14]).

Since the operators \( \hat{a}_\triangleleft(x) \) and \( \hat{a}_\triangleright(x) \) have non-intersecting supports, it generally holds

\[
[\hat{a}_\triangleleft(x), \hat{a}_\triangleright(x')] = [\hat{a}_\triangleleft(x), \hat{a}_\triangleright^\dagger(x')] = 0.
\]

The operator \( \hat{a}_\triangleleft(x) + \text{H.c.} \) is further split as follows [14]:

\[
\hat{a}_\triangleleft(x) + \text{H.c.} = \hat{b}(x) + \hat{c}(x) + \text{H.c.},
\]

where \( \hat{b}(x) \) and \( \hat{c}(x) \) have a non-vanishing support above and under the horizon, respectively, and \( \hat{b}(x)|\bar{\Omega}\rangle = \hat{c}(x)|\bar{\Omega}\rangle = 0 \), where \( |\bar{\Omega}\rangle \) is the Boulware vacuum [15]. These operators commute with each other as possessing non-intersecting supports:

\[
[\hat{b}(x), \hat{c}(x')] = [\hat{b}(x), \hat{c}^\dagger(x')] = 0.
\]

The scalar field operator expressed through these operators becomes

\[
\hat{\Phi}(x) = \hat{\Phi}_\triangleright(x) + \hat{\Phi}_\triangleleft(x) = \hat{\Phi}_\triangleright(x) + \hat{\Phi}_b(x) + \hat{\Phi}_c(x),
\]

and, hence, the Wightman function \( W_U(x, x') \) above the horizon (\( \hat{\Phi}_c(x) = 0 \)) is

\[
\langle \hat{\Phi}(x)\hat{\Phi}(x') \rangle = \langle \hat{\Phi}_\triangleright(x)\hat{\Phi}_\triangleright(x') \rangle + \langle \hat{\Phi}_b(x)\hat{\Phi}_b(x') \rangle + \langle \hat{\Phi}_c(x)\hat{\Phi}_c(x') \rangle
\]

\[
= \langle \hat{\Phi}_\triangleright(x)\hat{\Phi}_\triangleright(x') \rangle + \langle \hat{\Phi}_b(x)\hat{\Phi}_b(x') \rangle = \bar{W}(x, x') + \bar{W}_\beta(x, x').
\]

2 In Minkowski space this splitting can be done, e.g., with respect to the origin of the reference frame, such that \( \hat{a}_\triangleright(x) \) and \( \hat{a}_\triangleleft(x) \) have a support only on \( \Sigma_\triangleright \) and \( \Sigma_\triangleleft \), respectively, where \( \Sigma_\triangleright \) is a part of the Cauchy surface for \( x > 0 \), whereas \( \Sigma_\triangleleft \) for \( x < 0 \). The total Cauchy surface \( \Sigma \) in space is thus given by \( \Sigma_\triangleleft \cup \Sigma_\triangleright \).
Therefore, the “ingoing” part of the 2-point function is due to the operator \( \hat{\Phi}_b(x) \), while the “outgoing” one originates from the operator \( \hat{\Phi}_b(x) \). Note that the vacuum \( |\Omega\rangle \) responds to the action of the operator \( \hat{\Phi}_b(x) \) as a thermal state at the Hawking temperature \( T_H \) defined with respect to the Schwarzschild time \( t_S \), while as an empty state when probed by \( \hat{\Phi}_b(x) \).

In general, any polynomial composed of the operator \( \hat{\Phi}_b(x) \) probes \( |\Omega\rangle \) as an empty state, while composed of \( \hat{\Phi}_b(x) \) as if \( |\Omega\rangle \) is a mixed state. It is worth emphasising that the latter effect is due to the field operator \( \hat{\Phi}_b(x) \), rather than the vacuum state (see for an earlier version of this point \([16, 17]\)).

It is tempting to conclude that particles \( |\tilde{\psi}\rangle \) defined as

\[
|\tilde{\psi}\rangle = \hat{b}^\dagger(\tilde{h})|\Omega\rangle
\]

are thermally populated in the vacuum \( |\Omega\rangle \), where \( \tilde{h}(x) \) is a wave packet being a positive frequency solution of the field equation with respect to \( t_S \) with definite values of the orbital and magnetic numbers. However, one can show that \( \langle \tilde{\psi}|\Omega\rangle \) is identically zero, i.e. \( \langle \tilde{\psi}|\Omega\rangle = 0 \). Moreover, \( \langle \tilde{\Omega}|\Omega\rangle = 0 \) and, hence, the vacua \( |\Omega\rangle \) and \( |\tilde{\Omega}\rangle \) give unitarily inequivalent Hilbert space representations (\( \mathcal{H} \) and \( \tilde{\mathcal{H}} \), respectively) of the same operator algebra \( \mathcal{A} \). This means that the physical interpretation of the state \( |\Omega\rangle \) as a thermal state of the particles \( |\tilde{\psi}\rangle \) at the Hawking temperature \( T_H \) is not self-consistent, although most of the researchers take the contrary for granted. We shall come back to this issue below.

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3 This can be elucidated as follows. Consider a quantum-mechanical system of two non-interacting harmonic oscillators \( \{\hat{a}, \hat{a}^\dagger\} \) and \( \{\hat{b}, \hat{b}^\dagger\} \) of the same frequency \( \omega \) and with a ground state \( |0\rangle = |0\alpha\rangle \otimes |0\beta\rangle \), such that \( \langle 0|\hat{a}^\dagger \hat{a}|0\rangle = \langle 0|\hat{b}^\dagger \hat{b}|0\rangle = 0 \). Perform now a double squeezed transformation, i.e. \( \hat{\alpha} = \cosh \theta \hat{a} - \sinh \theta \hat{b}^\dagger \) and \( \hat{\beta} = \cosh \theta \hat{b} - \sinh \theta \hat{a}^\dagger \). It is straightforward to show that \( [\hat{\alpha}, \hat{\beta}] = [\hat{\alpha}, \hat{\beta}^\dagger] = 0 \), i.e. the oscillators \( \{\hat{\alpha}, \hat{\alpha}^\dagger\} \) and \( \{\hat{\beta}, \hat{\beta}^\dagger\} \) are independent. If one chooses the parameter \( \theta \) of the transformation to satisfy \( \tan h \theta = \exp(-\omega \beta/2) \), then one has, e.g., \( \langle 0|\hat{a}^\dagger \hat{a}|0\rangle = 1/(\exp(\beta \omega) - 1) = \text{tr}(\hat{\rho}_\beta \hat{a}^\dagger \hat{a}) \), where \( \hat{\rho}_\beta \) is a density matrix of inverse temperature \( \beta \). Thus, one may say that the ground state \( |0\rangle \) is a mixed state when probed by operators of the type \( \hat{\alpha} \). Thus, a pure state can sometimes respond as a mixed state due to the non-triviality of quantum operators. Note that a ground state \( |0\rangle \) annihilated by both \( \hat{\alpha} \) and \( \hat{\beta} \) (but not by \( \hat{a} \) or \( \hat{b} \)) can be mapped to \( |0\rangle \) by a unitary operator. It is an easy exercise to show that and is actually guaranteed by the Stone-von Neumann theorem. This is not anymore the case in quantum field theory (QFT), where this map is not unitarily implementable. This fact allows in particular to describe phase transitions (e.g. normal phase \( \leftrightarrow \) superconductive phase) in the framework of QFT, which is impossible in quantum mechanics. Note that, for that reason, toy models in the background of black holes based on qubits presuppose a physical realisation of the Hawking excitations. Therefore, this kind of the models cannot anyhow provide a resolution of the information loss problem, but can and does cause further confusions.
B. One-loop correction to self-energy

In general, it is hardly possible to obtain the full propagator \( G(x,x') \) in the non-linear theories. This exact propagator in the massless \( \lambda \Phi^4 \)-theory up to the 1-loop order satisfies

\[
\left( \Box + m_\phi^2 + O(\lambda^2) \right) G(x,x') = \frac{-i}{(-g(x))^{\frac{1}{2}}} \delta(x-x'),
\]

Expanding the propagator \( G(x,x') \) through the free propagator \( G_U(x,x') \) by employing the standard Feynman rules for this theory, one can obtain the effective mass of the scalar field at one-loop approximation. This can be pictorially expressed as follows

\[
m_\phi^2 G_U(x,x') = -\Box_x \left( \begin{array}{c}
\end{array} \right).
\]

To compute the effective mass of the scalar field, one thus needs to establish the Feynman propagator \( G_U(x,x') \) in the Unruh state that approximates the local Minkowski one \( |\Omega\rangle \). This vacuum state is supposed to be physically (unitarily) equivalent to the vacuum state before the black hole has formed. In other words, we do not expect the change of quantum physics (the change of the Hilbert space representation of the field operator algebra or the phase transition) as a result of the black-hole formation.

1. Far-horizon region: \( R \gg r_H \)

Far away from the black hole, i.e. \( R \gg r_H \), where \( R \) is the distance to the centre of the black hole, one can approximate geometry by Minkowski space. It implies that the geodetic interval for the optical and physical metric approximately coincide and read

\[
2\sigma(x,x') \approx 2\tilde{\sigma}(x,x') \approx (t-t')^2 - (x-x')^2,
\]

where \( x = (r \cos \theta \cos \phi, r \cos \theta \sin \phi, r \sin \theta) \). It should be noted that the Schwarzschild time \( t_S \) approaches the Painlevé-Gullstrand time \( \tau \) in the asymptotically flat region as

\[
t_S = \tau \left( 1 + O(r_H/R) \right),
\]

such that \( t_S \to \tau \) for \( R \to \infty \).

The 2-point function in the asymptotically flat region is

\[
W_U(x,x') \approx \left( 1 - \frac{27r_H^2}{16R^2} \right) W_M(x,x') + \frac{27r_H^2}{16R^2} W_M^\beta(x,x')
\]

for \( |x-x'| \ll R \), where \( W_M^\beta(x,x') \) is functionally given by the Minkowski two-point function at the inverse temperature \( \beta = 1/T_H \), which becomes \( W_M(x,x') \) for \( T_H \to 0 \). In the limit
$R \to \infty$, $W_U(x, x')$ reduces to the standard Minkowski correlator $W_M(x, x')$ and, hence, the influence of the black hole on local physics can be fully neglected.\(^4\)

The Feynman propagator $G_U(x, x')$ can be expressed through the commutator function and reads

\[
G_U(k, k') \approx \left( \frac{i}{k^2 + i\varepsilon} + 2\pi \frac{27r_H^2}{16R^2} \frac{\delta(k^2)}{\exp[|k_0|T_H] - 1} \right) \delta(k - k')
\]

in the momentum representation \(^8\). The frequency $k_0$ here is defined with respect to the Painlevé-Gullstrand time $\tau$. The second term on the right-hand side of (25) depends on the distance to the black hole and vanishes in the limit $R \to \infty$. Again, it means that local physics does not change in the asymptotic region, where the field propagator is insensitive to the black-hole properties.

Thus, one obtains

\[
\Box_x \left( \Box \right) = -\lambda \int dx_1 (-g(x_1))^{\frac{3}{2}} \Box_x G_U(x, x_1) G_U(x_1, x_1) G_U(x_1, x') \approx -\lambda \int \frac{d^3k}{(2\pi)^3 |k|} \left( \frac{1}{2} + \frac{27r_H^2}{16R^2} \frac{1}{\exp[|k|/T_H] - 1} \right) G_U(x, x').
\]

The integral over $k$ in (26) diverges unless one subtracts the first term in the parenthesis. In Minkowski space with a hot physical plasma, one can renormalise this by adding a mass counter-term to the Lagrangian density. This essentially implies that one removes this UV divergence by subtracting all terms which do not vanish in the limit of the vanishing plasma temperature. In the black-hole background, this can be accounted for the divergent Boulware contribution to the scalar mass.

Having got rid of the ultraviolet divergence in (26), we obtain

\[
\Box_x \left( \Box \right) \approx -\lambda \frac{\xi}{16\pi^2 R^2} G(x, x') \quad \text{with} \quad \xi \equiv \frac{9}{128}.
\]

The effective mass of the scalar field is thus finite and reads

\[
m_{\phi}^2 \approx \frac{\lambda \xi}{16\pi^2 R^2}.
\]

The scalar-field mass $m_{\phi}$ turns out not to depend on the black-hole mass $M$ in the leading order of the approximation. It appears to be the case, because (27) equals $T_H(r_H/R) \propto 1/R$ up to a numerical factor. We found a similar property of the one-loop correction to the photon self-energy in the case of evaporating black holes of mass $M \ll 10^{16}$ g in \(^8\).

\(^4\) It is not the case for eternal black holes. This does not serve a problem, because black holes of this type are not realistic. However, if a tiny black hole of this type could appear as a result of a quantum space-time fluctuation, then its influence on local physics cannot be neglected even at spatial infinity. For instance, photons acquire an effective thermal mass of the order of $\alpha^2 T_H$ for $T_H \gg m_e$ in QED, assuming such a black hole exists for a sufficiently large time interval \(^8\). These are certainly ruled out if the vacuum state is given by the Hartle-Hawking one.
2. Near-horizon region: $R \sim r_H$

The Wightman function $W_U(x,x')$ has a geometrical prefactor $1/f(r)$. Therefore, it is tempting to conclude that the loop diagrams are divergent on the horizon. For instance, the 1-loop correction to the self-energy of the scalar field seems to increase as $1/f(r)$ in the near-horizon region, while as $1/f^2(r)$ at 2-loop level. However, this does not happen to be the case if one studies this carefully in the freely-falling frame.

To analyse near-horizon physics, we introduce a local Minkowski frame at $R \sim r_H$ of a black hole of mass $M \gtrsim M_\odot$. The size of the local Minkowski frame is then about

$$l_M \gtrsim 300 \times (M/M_\odot) \text{ m.}$$ (29)

The local dynamics in this frame is set by the geodesic vector $G$, such that

$$G^\mu = \{1, 0, 0, 0\}$$

in the Painlevé-Gullstrand coordinates with the time coordinate $\tau$. This vector $G$ is one of the generators of the local Poincaré group whose irreducible representations correspond to the particle states.

The light-cone Kruskal-Szekeres coordinates $U$ and $V$ near the horizon behave as

$$U = \alpha \left(\tau - \tau_0 - 2\kappa(\tau - \tau_0)^2 + O((\tau - \tau_0)^3)\right),$$ (30a)

$$V = \left(e/\alpha\right) \left(2/\kappa + \tau - \tau_0 + O((\tau - \tau_0)^3)\right),$$ (30b)

where $e$ is the Euler number and $0 < \alpha \leq e^{1/2}$. The event horizon corresponds to $U_H = 0$ or $\tau = \tau_0 > 0$, while $V_H = 2e/(\alpha \kappa)$ holds at $\tau_0$. This reveals the geometrical meaning of the constant $\alpha$ in (30). Introducing $\tau' = (V + U)/2e^{1/2}$ and $x' = (V - U)/2e^{1/2}$, we obtain

$$ds^2 \approx d\tau'^2 - dx'^2 - dy^2 - dz^2$$ (31)

near the horizon, where $y^2 + z^2 = 4r_H^2 \tan^2(\theta/2)$, $z/y = \tan \phi$ and $y^2 + z^2 \ll r_H^2$. Employing the local Lorentz transformation with $v/c = (e - \alpha^2)/(e + \alpha^2)$, the time coordinate $\tau'$ can be transformed to the Painlevé-Gullstrand time $\tau$ (up to a translation).

On the other hand, the Schwarzschild metric in the near-horizon region can be approximated by the Rindler metric. Specifically, it holds

$$ds^2 \approx \kappa^2 \rho^2 dt^2 - d\rho^2 - dy^2 - dz^2,$$ (32)

where $\rho = \int dr/f^2(r) \approx (4r_H(r - r_H))^{1/2}$. This line element can be further transformed into Minkowski one via the diffeomorphism $\tau = \rho \sinh(\kappa t)$ and $x = \rho \cosh(\kappa t)$:

$$ds^2 \approx d\tau^2 - dx^2 - dy^2 - dz^2.$$ (33)

Note that we have tacitly introduced a new time $\tau$ which is up to the Lorentz transformation coincides with the Painlevé-Gullstrand time. Therefore, we have denoted both by the same symbol.
We now want to study local physics in this coordinate system. For this purpose we need to derive the Feynman propagator $G_U(x, x')$ in the freely-falling frame. For the reasons which become clear later on, we study first the vacuum expectation value of the operator $\hat{\Phi}^2(x)$.

**Wick squared operator $\hat{\Phi}^2(x)$ near horizon** Using the background-field method to compute the 1-loop contribution to the scalar field equation, we find

$$\left(\Box + \frac{\lambda}{2} \langle \hat{\Phi}^2(x) \rangle \right)\Phi(x) = 0,$$

where $\langle \hat{\Phi}^2(x) \rangle$ has been appropriately renormalised (see below). Thus, the effective scalar mass at 1-loop level is given by

$$m^2_{\Phi} = \frac{\lambda}{2} \langle \hat{\Phi}^2(x) \rangle.$$

The renormalised value of the Wick squared operator $\hat{\Phi}^2(x)$ in the Unruh vacuum was computed in [5]. This quantity turns out to be finite on the horizon and decreases as $1/R^2$ at the spatial infinity. Specifically, it holds that

$$\langle \hat{\Phi}^2(x) \rangle \approx \begin{cases} 
\left(\frac{1}{3} - 2\xi\right) T_H^2, & R \sim r_H, \\
\xi/8\pi^2 R^2, & R \gg r_H.
\end{cases}$$

This is in full agreement with our result obtained above for $R \gg r_H$ using the diagrammatic approach.

The finiteness of $\langle \hat{\Phi}^2(x) \rangle$ on the black-hole horizon seems a priori not to be guaranteed. Indeed, the Wick squared operator is defined in general as

$$\hat{\Phi}^2(x) = \lim_{x' \to x} \left(\hat{\Phi}(x)\hat{\Phi}(x') - H(x, x')\mathbb{1}\right),$$

where $H(x, x')$ is the Hadamard parametrix. It is a geometrical (state-independent) object and designed to subtract the ultraviolet divergences in the 2-point function only. In our case, the Hadamard parametrix is of the form

$$H(x, x') = -\frac{1}{8\pi^2\sigma(x, x')} + O(\sigma \ln \sigma),$$

where $\sigma(x, x')$ is the geodetic interval for the physical metric. It is worth mentioning that the term $\sigma \ln \sigma$ in $H(x, x')$ is fully responsible for the trace aka conformal anomaly [18, 19].

The parametrix can be expressed via the geodetic interval in the optical metric:

$$H(x, x') = -\frac{(f(x)f(x'))^{-\frac{1}{2}}}{8\pi^2\sigma(x, x')} + \frac{M^2 f^{-1}(r)}{48\pi^2 r^4} + O(\bar{\sigma} \ln \bar{\sigma}).$$

The first term in (39) coincides with the 2-point function $W_B(x, x')$ in the Boulware vacuum for $x \sim x'$, whereas the second term in (39) gives a non-vanishing value of $-\langle \hat{\Omega}|\hat{\Phi}^2(x)|\hat{\Omega} \rangle$ far away from the event horizon which disappears in the limit $M \to 0$ [5].
In the far-from-horizon region, the geodetic intervals \( \sigma(x, x') \) and \( \bar{\sigma}(x, x') \) go over to the Minkowski geodetic distance. These significantly differ from each other in the near-horizon region. We obtain that
\[
\sigma(x, x') = \sigma_0(x, x') + \frac{1}{24} \left( \frac{f'(r)^2 - 4\kappa^2}{f(r)} - \frac{f'(r')^2 - 4\kappa^2}{f(r')} \right)^{\frac{1}{2}} \sigma_0^2(x, x') + O(\sigma_0^3(x, x')),
\]
where we have taken into account that \( f'(r_H) = 1/r_H = 2\kappa \) and
\[
2\sigma_0(x, x') \approx \Delta x^2 - \Delta y^2 - \Delta z^2
\]
close to the horizon, while \( \bar{\sigma}(x, x') \) can be found in [17] and approaches in the limit \( R \to r_H \) to the geodetic interval of the static space with the hyperbolic spatial section. Substituting \( \sigma(x, x') \) into the Hadamard parametrix, we find that
\[
H(x, x') = -\frac{1}{8\pi^2\sigma_0(x, x')} - \frac{\kappa^2}{12\pi^2} + O(\sigma_0 \ln \sigma_0, \kappa^2 f(r)).
\]
This result and (10) with (11) at \( R \sim r_H \) in the local Minkowski frame in turn allow us to compute the Wightman function in the near-horizon region:
\[
W_U(x, x') \approx -\frac{1}{8\pi^2\sigma_0(x, x')} - \frac{\xi\kappa^2}{2\pi^2}.
\]
Thus, we reproduce the result (36) obtained in [5] by subtracting \( H(x, x) \) from \( W_U(x, x) \).

**Feynman propagator near horizon** The Feynman propagator can be expressed through the commutator function \( C(x, x') \) which is equal to \( W_U(x, x') - W_U(x', x) \). The commutator \( C(x, x') \) is insensitive to the time-independent term in the correlation function. There are at least two possibilities to deal with the second term in \( W_U(x, x') \): Either one needs to introduce an imaginary “temperature” \( \Theta \sim i\kappa \) or a discrete frequency spectrum \( \omega_n \sim \kappa n \). Since we expect merely a slight change of physics in the local inertial frame even near the horizon of a large black hole, we do not consider the possibility of the change of the continuous spectrum into the discrete one.

---

5 It is worth noticing that the finite terms in \( W_U(x, x') \) and \( H(x, x') \) in the coincidence limit \( x' \to x \) are separately regular on the event horizon \( r = r_H \) in the freely-falling frame. In the Schwarzschild frame, these parts of the Wightman function and the Hadamard parametrix increase as \( 1/f(r) \) for \( r \to r_H \), but their difference turns out to be non-singular at \( r = r_H \) as found in [3].

6 It is worth mentioning that the finite term in the coincidence limit \( x' \to x \) of the thermal two-point function of temperature \( T \) is given by \(+T^2/12\). However, the Wightman function in the near-horizon region given in (14) has a negative correction to the term \(-1/\sigma_0(x, x')\). This effectively corresponds to the imaginary “temperature”.

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The time-independent term in (43) is negligibly small, because we have been working in the regime \( \sigma_0 \kappa^2 \ll 1 \). To reproduce the value of \( \langle \Phi^2(x) \rangle \) through the tadpole diagram, we thus define

\[
G_U(p, p') \approx \left( \frac{i}{p^2 + i\varepsilon} + \frac{2\pi \delta(p^2)}{e^{|p_0|/\Theta} - 1} \right) \delta(p - p'),
\]

where we have introduced an effective imaginary “temperature”:

\[
\Theta_\varepsilon \equiv \varepsilon + \frac{i\kappa}{\pi} (6\xi)^{1/2} \quad \text{with} \quad \varepsilon \to +0.
\]

Since \( \langle \Phi^2(x) \rangle \propto G_U(x, x) \) from the tadpole diagram (see Eq. (26)), one needs to renormalise it by subtracting the “Hadamard propagator” at \( x = x' \). We define it as follows

\[
G_H(p, p') \approx \left( \frac{i}{p^2 + i\varepsilon} + \frac{2\pi \delta(p^2)}{e^{|p_0|/\theta} - 1} \right) \delta(p - p'), \quad \text{where} \quad \theta_\varepsilon \equiv \varepsilon + \frac{i\kappa}{\pi}.
\]

We can now reproduce the result of [5] for the Wick squared or the effective scalar mass if we renormalise the ultraviolet divergence of the tadpole diagram by subtracting \( G_H(x, x') \) from \( G_U(x, x') \). Specifically, we have

\[
m^2 \Phi = \frac{\lambda}{2} \left( G_U(x, x) - G_H(x, x) \right)
\]

\[
\approx \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \left( \frac{1}{e^{|k_0|/\Theta} - 1} - \frac{1}{e^{|k_0|/\theta} - 1} \right) = \frac{\lambda}{24} \left( \Theta^2 - \theta^2 \right) = \frac{\lambda}{2} \left( \frac{1}{3} - 2\xi \right) T_H^2.
\]

Note that both \( \Theta^2 \) and \( \theta^2 \) are negative, but the effective scalar mass \( m^2_\Phi \) is positive, because the absolute value of \( \theta \) is larger than that of \( \Theta \). It should be noted that the first term of the right-hand side in Eq. (47) is due to \( \theta \) that comes in turn from the Hadamard parametrix. It is a geometrical object and its contribution to \( m^2_\Phi \) is a result of the coordinate transformation from the local Rindler geometry into the local Minkowski geometry near the event horizon.

C. Local renormalisation scheme

Above we have found that the effective scalar mass is finite and coincides with (35) if we add the minus “Hadamard loop” to the Feynman loop. One may represent this subtraction pictorially as follows

\[
m^2_\Phi G(x, x') = -\Box_x \left( \quad + \quad + \quad O(\lambda^2) \right),
\]

where the dashed line corresponds to the minus Hadamard propagator, i.e. \(-G_H(x, x')\).

We want to propose a local renormalisation scheme in curved space-times. Specifically, we introduce a fictitious scalar field \( \phi(x) \) with a negative norm (a wrong sign in front of
the propagator) with the propagator being constructed from the Hadamard parametrix. We need also to introduce extra vertices in the Lagrangian $L$, namely

$$\frac{\lambda}{4!} \phi^4, \quad \frac{4\lambda'}{4!} \phi^3 \Phi, \quad \frac{6\lambda}{4!} \phi^2 \phi^2 \quad \text{and} \quad \frac{4\lambda'}{4!} \phi \Phi^3,$$

such those

$$\begin{align*}
\begin{array}{c}
\text{Figure 1a} \\
\text{Figure 1b}
\end{array}
\end{align*}
= -i\lambda, \quad (50a)$$

$$\begin{align*}
\begin{array}{c}
\text{Figure 1b} \\
\text{Figure 1c}
\end{array}
\end{align*}
= -i\lambda', \quad (50b)
$$

in the momentum representation. The ratio $\lambda'/\lambda$ will be fixed below in order to recover the standard result for the running coupling constant in the asymptotically flat region.

### D. One-loop correction to coupling constant

We now study how the coupling constant $\lambda$ changes in the near-horizon and asymptotically flat region at 1-loop level to further investigate the imprints of evaporating black holes in scattering processes. The four-point vertex function can be computed by functionally differentiating the path integral over the external current which is linearly coupled to the scalar field. The result of this method is by now standard and reads

$$\Gamma^{(4)}(x_i) = \begin{array}{c}
\text{Figure 2a} \\
\text{Figure 2b}
\end{array} + \frac{3}{2} \left( \begin{array}{c}
\text{Figure 3a} \\
\text{Figure 3b} \\
\text{Figure 3c}
\end{array} \right) + O(\lambda^3), \quad (51)
$$

where $i$ runs from 1 to 4.

#### 1. Near-horizon region: $R \sim r_H$

The first diagram in the momentum representation is given by $-i\lambda$, whereas the first 1-loop diagram in Eq. (51) is

$$\begin{align*}
\begin{array}{c}
\text{Figure 4a}
\end{array}
\end{align*}
= -i\lambda^2 \int \frac{d^4p}{(2\pi)^4} \left( \frac{i}{p^2 + i\varepsilon} \left( \begin{array}{c}
\frac{1}{(q - p)^2 + i\varepsilon}
\end{array} \right) + \frac{4\pi \delta(p^2)}{e^{p_0/\Theta} - 1} \frac{1}{(q - p)^2 + i\varepsilon} \right), \quad (52)
$$

where we have chosen the external momenta be non-exceptional ($k_0^i = 0, \mathbf{k}^i\mathbf{k}^j = q^2(\delta^{ij} - 1/4)$ for the four external legs, i.e. $i, j = 1, 2, 3, 4$ and $q = k_1 + k_2 = -k_3 - k_4$) not to have extra IR divergence of the logarithmic type that is due to the zero mass of the scalar field [20]. The origin of this divergence is the same as in quantum electrodynamics. Specifically, the zero photon mass leads to the non-negligible mutual influence of two charged particles even

\[ \text{Note that the Mandelshtam variables } s = (k_1 + k_2)^2, t = (k_1 + k_3)^2 \text{ and } u = (k_1 + k_4)^2 \text{ are all equal to } -q^2 \text{ for this choice of the external momenta. This explains the factor of 3 in Eq. (51).} \]
when these are at infinite distance from each other. This in turn entails the IR divergence of the S-matrix constructed from the asymptotic particle states.

The first integral in (52) can be evaluated by employing the standard technique of the dimensional regularisation [20]. The second integral can be simplified after the integration over the solid angle. The result reads

\[ \lambda' = \frac{i \lambda^2}{(4\pi)^2} \left( 2 \frac{2}{\epsilon} - \gamma + 2 - \ln \left( \frac{q^2}{4\pi \mu^2} \right) + 2 \int_0^{+\infty} dx \ln \left| \frac{1 + x}{1 - x} \frac{1}{e^{qx/2\Theta\epsilon} - 1} \right| \right) , \]  

(53)

where \( \epsilon \to 0 \) in our case, \( \gamma \) is the Euler constant and \( \mu \) is an arbitrary mass scale inherent to the dimensional regularisation. The same result holds for the third 1-loop diagram in (51) after the substitution \( \Theta \to \theta \). The second 1-loop diagram in (51) equals a quarter of the sum of the first and third 1-loop diagram if we set

\[ \lambda' = \frac{\lambda}{2\pi} . \]  

(54)

Then, after the \( \overline{\text{MS}} \) renormalisation, we obtain

\[ \lambda(q, r_H) = \lambda + \frac{3 \lambda^2}{32\pi^2} \left( \ln \left( \frac{q^2}{4\pi \mu^2} \right) - \int_0^{+\infty} dx \ln \left| \frac{1 + x}{1 - x} \left( \frac{1}{e^{qx/2\Theta\epsilon} - 1} + \frac{1}{e^{qx/2\theta\epsilon} - 1} \right) \right| \right) + O(\lambda^3) 
\]

\[ = \lambda + \frac{3 \lambda^2}{32\pi^2} \left( \ln \left( \frac{q^2}{4\pi \mu^2} \right) - \frac{\pi^2}{3q^2} (\Theta^2 + \theta^2) \right) + O(\lambda^3, \lambda^2 \kappa^4/q^4) \]  

(55)

in the regime \( q^2 \gg \kappa^2 \) which is consistent with our approximation.

Note that, whenever some loop correction depends on the external momenta, the UV divergence is not cancelled by the fictitious field. We could rearrange the local renormalisation scheme in a manner that the UV divergence of the diagram (52) is absent, but then the entire 1-loop correction to the coupling constant \( \lambda \) would not depend on the external momenta in the limit \( \kappa \to 0 \). This turns out to be in disagreement with the well-known result in particle physics, namely \( \lambda \) depends on the energy scale at which one is measuring the coupling constant. For this reason, one had to employ the dimensional regularisation (or any other standard regularisation) and the \( \overline{\text{MS}} \) renormalisation as well.

Substituting \( \Theta \) and \( \theta \) in Eq. (55), we find

\[ \lambda(q, r_H) \approx \lambda + \frac{3 \lambda^2}{32\pi^2} \left( \ln \left( \frac{q^2}{4\pi \mu^2} \right) + \left( \frac{1}{3} + 2\xi \right) \frac{1}{4r_H^2 q^2} \right) . \]  

(56)

Thus, the coupling constant \( \lambda \) becomes stronger at 1-loop level in the near-horizon region. It should be noted that the correction due to the Schwarzschild black hole is given by the second term in the parenthesis of Eq. (56). This is in turn composed of two contributions. One of these originates from the Hadamard subtraction, while another (that is proportional to \( \xi \)) comes from the correction to the propagator that is related to the black-hole evaporation.
2. **Far-horizon region: \( R \gg r_H \)**

Far away from the horizon, \( G_U(x, x') \) is given by \(^{25}\). The Hadamard parametrix \( H(x, x') \) takes the form at the leading order of the approximation as if there is no black hole, namely

\[
H(x, x') = -\frac{1}{8\pi^2 \sigma_0(x, x')} + O(\sigma_0 \ln \sigma_0, M^2/R^4)
\]

(57)

In this case, we find

\[
\lambda(q, R) \approx \lambda + \frac{3\lambda^2}{32\pi^2} \left( \ln \left( \frac{q^2}{4\pi \mu^2} \right) - \frac{27r_H^2}{16R^2} \int_0^\infty dx \ln \left| \frac{1 + x}{1 - x} \right| e^{xq/2T_H} \right) + O(\lambda^3)
\]

\[
= \lambda + \frac{3\lambda^2}{32\pi^2} \left( \ln \left( \frac{q^2}{4\pi \mu^2} \right) - \frac{\xi}{2R^2 q^4} \right) + O(\lambda^3, \lambda^2 \kappa^2/q^4 R^2)
\]

(58)

Thus, we reproduce the standard result known in particle physics far away \( (R \gg r_H) \) from an evaporating black hole. For a large, but fixed \( R \), the vacuum polarisation induced by the black hole slightly suppresses the coupling constant at one-loop approximation.

**IV. LOCAL PARTICLE PHYSICS**

In Sec. III, we have derived corrections to the self-energy and coupling constant at 1-loop level near to and far away from the event horizon. In this section, we study their physics.

1. **Particle physics: Near-horizon region**

We observe no physical particles in the locally Minkowski vacuum \( |\Omega\rangle \) near the event-horizon region. Indeed, one can speak about a massless particle as a localised object in the near-horizon region when its de Broglie wavelength \( \lambda_p \) is much smaller than the size of the event horizon \( r_H \), i.e. \( \lambda_p \ll r_H \). More precisely, \( \lambda_p \) must actually be much smaller than the size of a particle detector \( l_D \) which is in turn much smaller than the horizon size. In this case, the Wightman function \( W_U(x, x') \) approximately coincides with the two-point function as if there is no black hole plus a small correction of the order of \( (\lambda_p/r_H)^2 \ll (\lambda_p/l_D)^2 \ll 1 \).\(^8\) Therefore, the state \( |\Omega\rangle \) is not populated by the real particles. The same holds in the stationary frame near the event horizon as the notion of particle is covariant.

There has been recently argued that an in-falling observer should discover a firewall (a sort of cloud of the high-energy (blue-shifted) Hawking particles \( |\tilde{\psi}\rangle \)) in the near-horizon region.

\(^8\) To our knowledge, this correction which is due to the “ingoing” part of the correlation function has not been discussed in the literature. In the case of eternal black holes, this correction does not appear, because the decreasing (with the distance) parts of the “ingoing” and “outgoing” modes cancel each other.
If this is a real phenomenon, the equivalence principle does not hold, because the event horizon of evaporating black holes would then physically be a distinguishable set of space-time points. If so, the whole framework of general relativity which led to the notion of black hole would be not reliable. Importantly, no evidences have been found so far that the principle of equivalence does not hold.

The equivalence principle was sacrificed in favour of the unitarity. By the unitarity one should here understand the existence of the unitary $S$-matrix between the in-state and the thermal out-state. The in-state in our notations is identified with the vacuum $|\Omega\rangle$, while the out-state corresponds to the Boulware vacuum $|\tilde{\Omega}\rangle$. However, the mathematical subtlety is that the in-state can only formally be represented as a thermally populated state of the Hawking particles defined with respect to the out-state. The relation between the in-state and out-state is formal, because these do not define unitarily equivalent representations of the field operator algebra. Thus, this means that if one demands that there exists a unitary operator $\hat{S}$ that relates the in-state and the thermally populated out-state, then one comes to the idea of having the firewall near the event horizon. The problem is that the existence of the unitary operator $\hat{S}$ is not consistent with the principles of local quantum field theory as pointed out in 3.

We have found above that the Feynman propagator $G_U(x, x')$ in the near-horizon region is given by a thermal-like propagator with the imaginary “temperature” $\Theta$ given in (45). We interpret this small correction as being due to the modification of the field operator in the presence of black holes. A similar effect occurs in the Casimir set-up. Indeed, if one considers a wave packet of a photon of the de Broglie wavelength $\lambda_p \ll d$, where $d$ is a distance between the conducting plates, then the photon propagator turns out to be as a thermal-like one (for the modes in the perpendicular direction with respect to the plates) with an imaginary “temperature” $T_C = i/2d$ when localised far from the plates. The “thermal” term in this propagator is understood as being due to the boundary conditions satisfied by the electromagnetic operators or the vacuum fluctuations which are present in the Minkowski vacuum.

We want now to discuss the vacuum expectation value of the particle number operator.

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9 The very existence of $T_C$ in the Casimir effect can be envisaged from the Tomita-Takesaki theorem. Indeed, according to the theorem the operator algebra composed of the electromagnetic field operators in-between the conducting plates must satisfy the Kubo-Martin-Schwinger condition in the Minkowski vacuum with respect to a certain one parameter group of automorphism of the quantum operators. This group can in general be of a geometrical as well as non-geometrical origin. In the present case, it is a symmetry related with the periodicity of the operators in the spatial direction which is transferred to the periodicity in the Minkowski time with a real period, which corresponds to the imaginary “temperature”. In a private discussion with Bernard Kay I got to know about a representation of the Minkowski vacuum as an “imaginary-temperature state” in-between the conducting plates, which was found in.
This operator is defined as
\[ \hat{N}(h) = \hat{a}^\dagger(h)\hat{a}(h), \] (59)
where the wave packet \( h(x) \) is the same as \([2]\), but for the scalar particle and \( h(p) \) having a maximum near the momentum \( q \). Due to the thermal-like term of the imaginary “temperature” \( \Theta \) in \( G_U(x, x') \), the (imaginary) quantity \( \langle \hat{N}(h) \rangle \) does not vanish. However, it is in general true that the localised operators have a non-vanishing vacuum expectation value in the Minkowski vacuum due to its Reeh-Schlieder property \([10]\).\(^{10}\) The physical interpretation of this mathematical theorem is given in terms of the quantum fluctuations. One needs thus to subtract these or calibrate the particle detector \([2, 3]\). If we do the same in the near-horizon region, we obtain \( \langle \hat{N}(h) \rangle = 0 \).\(^{11}\)

2. **Particle physics: Far-horizon region**

The Feynman propagator \( G_U(x, x') \) far away from the black hole \( (R \gg r_H) \) is on the contrary given by the propagator with a thermal-like term at the Hawking temperature \( T_H \). This extra term vanishes as \( 1/R^2 \) at \( R \to \infty \) and can be assigned to the modification of the field operator in the presence of evaporating black holes.

Utilising the number operator introduced in Eq. \((59)\), one can define a number density operator. Its vacuum expectation value in \(|\Omega\rangle\) for the “outgoing” plane-wave modes within the frequency range from \( \omega \) to \( \omega + d\omega \) is given by
\[ dn_\omega = \langle \hat{n}_\omega \rangle \, d\mu_\omega, \quad \text{where} \quad \mu_\omega = \frac{\omega^2 d\omega}{2\pi^2}, \] (60)
is the standard measure of integration, and the distribution of the modes in the frequency interval \((\omega, \omega + d\omega)\) is given by
\[ \langle \hat{n}_\omega \rangle \approx \frac{1}{4\omega^2 R^2} \frac{\Gamma_\omega}{\omega^2/T_H - 1}, \] (61)
where \( \Gamma_\omega \) has been given in Eq. \((12)\). It is worth pointing out that the right-hand side of \((61)\) is fully due to \( \tilde{W}_\beta(x, x') \) or the field operator \( \hat{\Phi}_b(x) = \hat{b}(x) + \hat{b}^\dagger(x) \).

\(^{10}\) As a consequence of this property, a sufficiently sensitive thermometer should measure a non-zero “temperature” of the vacuum as being a local operator. For the same reason, the particle detector is excited all the time by “particles” in the vacuum. The “temperature” and “particles” of the vacuum are merely a quantum noise.

\(^{11}\) We do not consider the imaginary “temperature” \( \Theta \) as being of any fundamental meaning, rather than a footprint of our approach. We further study the near-horizon physics by employing quantum kinetic theory in \([24]\), wherein we derive the 2-point function in the local inertial frame at \( R \sim r_H \) without any reference to \( \Theta \).
The prefactor $1/4\omega^2R^2$ also appears in the effective covariant Wigner function $W_{\text{eff}}(x,p)$ playing a role of the phase-space distribution function in relativistic kinetic theory (see, e.g., [25]), namely

$$W_{\text{eff}}(x,k) \approx \frac{1}{(2\pi)^3} \frac{\Gamma_\omega}{4\omega^3R^2 e^{\omega/T_H} - 1} f_{\text{eff}}(x,k) \frac{\delta(\omega - k)}{\omega}, \quad (62)$$

where $\omega = k_0 = k = |k|$ with $k^i = (k,0,0)$ and the index $i$ runs over $\{r,\theta,\phi\}$. The effective function $f_{\text{eff}}(x,k)$ is known as the one-particle Wigner distribution. We reproduce the result (61) by employing the standard formula known in kinetic theory:

$$n = \int d^3p f_{\text{eff}}(x,p) = \int dn_\omega = \int d\mu_\omega \langle \hat{n}_\omega \rangle. \quad (63)$$

Moreover, we can compute the outward positive energy flux as found in [4, 5] as the second moment (with respect to the momentum) of the distribution function [25], namely

$$\int d^3 p p f_{\text{eff}}(x,p) = \frac{L}{4\pi R^2}, \quad \text{where} \quad L = \frac{1}{2\pi} \int d\omega \frac{\omega \Gamma_\omega}{e^{\omega/T_H} - 1} \quad (64)$$

is the luminosity. We further study local quantum physics near the event horizon by employing quantum kinetic theory in [24].

The equation (61), however, differs from the distribution of the Hawking modes [1]:

$$\langle \hat{n}_\omega \rangle_H = \frac{\Gamma_\omega}{e^{\omega/T_H} - 1}. \quad (65)$$

By now the equation (65) is a widely-accepted result in black-hole physics. This has been derived by computing the Bogolyubov coefficients relating $\hat{b}(x)$ with $\hat{a}_<(x)$ and $\hat{a}_<(x)$ [26]. The reason of the discrepancy is that the formula (61) is local and expected to be valid only in a volume of the size being much smaller than $R \gg r_H$, whereas (65) is a global result. Thus, the physical meaning of (61) and (65) is different. Specifically, the number of the locally plane-wave modes in a spherical shell of volume $dV = 4\pi R^2 dR$ for the fixed distance $R \gg r_H$ from the black hole is given by

$$dN = n dV = 4\pi n R^2 dR = 4\pi n R^2 (dR/d\tau) d\tau = 4\pi n R^2 d\tau, \quad (66)$$

where we have set $dR/d\tau = c = 1$. Therefore, we obtain the flux of these modes:

$$\dot{N} = \frac{1}{2\pi} \int d\omega \langle \hat{n}_\omega \rangle_H \quad (67)$$

(or the number of the modes per the radial distance $dR$, i.e. $dN/dR$). It is worth noticing that the number of these modes in a detector of volume $l_D \times l_D \times l_D$ drops out with the distance as $(l_D/R)^2$ for $R \gg r_H$. Analogously, we rederive a well-known result

$$\dot{E} = \frac{1}{2\pi} \int d\omega \omega \langle \hat{n}_\omega \rangle_H \quad (68)$$
from the local distribution \( \{6\} \) by computing the energy in the spherical shell of the volume \( dV = 4\pi R^2 dR \) or \( dV = 4\pi R^2 d\tau \). This implies that \( \{21\} \) and \( \{25\} \) are good approximations to the exact Wightman function and Feynman propagator whenever the conditions \( |r-r'| \ll R \) and \( r_H \ll R \) are satisfied.

All local observables are composed of the fundamental field operator \( \hat{\Phi}(x) \). For instance, the particle creation operator is given by Eq. \( \{4\} \) and this is employed to construct the particle number operator. The field operator \( \hat{\Phi}(x) \) can in turn be represented as the sum \( \hat{\Phi}_\geq(x) + \hat{\Phi}_b(x) \) (see Eq. \( \{17\} \)). As emphasised above, the Wigner distribution \( \mathcal{W}(x,k) \) is due to the part \( \hat{\Phi}_b(x) \) of the field operator \( \hat{\Phi}(x) \). In the asymptotically flat region, local physics is oblivious to the presence of black holes, because the Wigner distribution drops out as \( (r_H/R)^2 \) for \( R \to \infty \). This means that if we choose the standard wave packet \( h(x) \) in the spatial infinity \( (R \to \infty) \) as we have been doing that on earth when we study scattering processes in particle physics, we then find

\[
\hat{a}^\dagger(h) = \hat{a}^\dagger_\geq(h),
\]

where \( \hat{a}^\dagger_\geq(x) \) is the standard creation operator of the scalar particle in Minkowski space-time and the vacuum \(|\Omega\rangle\) in the spatial infinity “reduces” to the Minkowski vacuum (in the sense that the local operators probe \(|\Omega\rangle\) at \( R \to \infty \) as the Minkowski vacuum in Minkowski space-time), such that \( \hat{a}_\geq(h)|\Omega\rangle = 0 \). This implies that the effective density matrix\(^\dagger\) introduced in black-hole physics must actually decrease with the distance as \( (r_H/R)^2 \), such that the vacuum \(|\Omega\rangle\) is probed at \( R \to \infty \) as being pure, but as if it is mixed at finite \( R \gg r_H \). It implies that the formal representation of the vacuum \(|\Omega\rangle\) as the thermally populated state of the particles defined with respect to \(|\tilde{\Omega}\rangle\) in the spatial infinity is not a self-consistent picture of the black-hole evaporation.

Indeed, if we consider a wave packet \( h(p_n) \) (one of the elements of the countable set of orthonormalised functions) with a definite value of the momentum \( p_n \) localised in a ball of volume \( \sigma \sim (\lambda_p)^3 \) at the distance \( R \gg r_H \) from an evaporating black hole, then the vacuum expectation value of the operator \( \hat{N}(h) \) defined in Eq. \( \{52\} \) is given by

\[
\langle \hat{N}_{p_n} \rangle = \frac{27 r_H^2}{16 R^2} \frac{1}{e^{\omega_n/T_H - 1} - 1} \to 0 \quad \text{for} \quad R \to \infty,
\]

where \( \omega_n = |p_n| \). However, if we take a wave packet \( h(\omega_n |lm) \) which corresponds to a spherical wave of the frequency \( \omega_n \), the orbital number \( l \) and the magnetic number \( m \) localised around radial distance \( R \gg r_H \), we obtain

\[
\langle \hat{N}_{\omega_n lm} \rangle = \frac{\Gamma_{\omega_n}}{e^{\omega_n/T_H - 1}} \approx \frac{1}{4} \frac{27 r_H^2 \omega_n^2}{e^{\omega_n/T_H - 1}}.
\]

\(^\dagger\) Note that the description of the vacuum \(|\Omega\rangle\) in terms of the thermal density matrix comes from the assumption that the probes of this vacuum in the spatial infinity \( (R \to \infty) \) are performed only by the operators composed of \( \hat{b}(x) \) and \( \hat{b}^\dagger(x) \) \( \{4\} \). However, this turns out not to be the case, because \( \hat{\Phi}(x) \) reduces to \( \hat{\Phi}_\geq(x) \), rather than \( \hat{\Phi}_b(x) \) in the spatial infinity. The effective density matrix characterises the quantum fluctuations of the \( \hat{\Phi}_b \)-part of the field \( \hat{\Phi}(x) \), which is locally irrelevant at \( R \gg r_H \).
This does not depend on the radial distance $R$, because the support of the spherical shell scales as $R^2$. Therefore, the Hawking particles are associated with the spherical waves. The spherical waves are not localised in the angular directions. For this reason, one cannot understand these as localised excitations, which one can put, for instance, in a box of the one-cubic-meter size.

As shown above, the quantum operator $\hat{\Phi}(x)$ can be represented as $\hat{\Phi}(x) + \hat{\Phi}_b(x)$ above the horizon ($R > r_H$). The operators $\hat{\Phi}(x)$ and $\hat{\Phi}_b(x')$ commute with each other for any points $x$ and $x'$, i.e.

$$[\hat{\Phi}(x), \hat{\Phi}_b(x')] = 0 \quad \text{for} \quad \forall \ x, x'. \quad (72)$$

In the asymptotically flat region, the operator $\hat{\Phi}_b(x')$ drops out to zero as $r_H/R$. It seems this means that the matter outside of the hole is composed only of the operator $\hat{\Phi}(x)$ at $R \to \infty$ and cannot be used to directly discover the Hawking modes which are due to $\hat{\Phi}_b(x)$. It is still possible to discover these indirectly through its gravitational influence, because $\langle \hat{T}_{\mu}^{\nu} \rangle \neq 0$. Thus, these could be a sort of “the dark radiation”. Another argument in favour of this idea is the following: If we prepare a thermal gas in a small box at $R \to \infty$ and let it fall towards the horizon, then the energy-momentum tensor will be finite at $R = r_H$ within the box. This is not the case for the Hawking gas, because the energy-momentum tensor will diverge for any temperature $T \neq T_H$ on the horizon and the whole consideration becomes self-inconsistent (unless one starts to treat this gedankenexperiment at the level of non-perturbative quantum gravity). Therefore, one might conclude that “the Hawking matter” is decoupled from the normal matter. However, this statement makes sense only if one assumes that the “outgoing” modes correspond to real particles which can be literally used to prepare the Hawking gas heated up to any temperature $T$. This decoupling does not make any sense, if one understands these as virtual particles/vacuum fluctuations, because the fundamental field is $\hat{\Phi}(x)$, rather than $\hat{\Phi}(x)$ or $\hat{\Phi}_b(x)$ separately.

If we calibrate the particle detector (as made above at $R \sim r_H$) or subtract the vacuum contribution to the Wigner function at large, but fixed $R$, then we obtain $|\Omega\rangle$ is empty. It does not imply the vanishing energy-momentum tensor, i.e. $\langle \hat{T}_{\mu}^{\nu} \rangle \neq 0$. For the same reason, the Fourier transform of the Wightman function with respect to the time (this represents the Unruh-DeWitt detector) is also non-vanishing (although $|\Omega\rangle$ does not contain the physical particles), because this yields the frequency spectrum of the vacuum fluctuations (according to the Wiener-Khinchin theorem) as pointed out in [5].

Since we have been trying to interpret the black-hole evaporation as the vacuum would possess (inhomogeneous, but isotropic) medium-like properties (because this interpretation seems to be self-consistent and does not suffer from the absence of the unitary $S$-matrix as understood by many researchers as well as the firewall problem), it is of interest to be aware of any other physical examples when this kind of the viewpoint is fruitful. There exists at least one to our knowledge. Specifically, the propagation of photons in the Minkowski vacuum with a super-strong magnetic field $B \gtrsim \pi m_e^2/\alpha e$, where $\alpha$ is the fine structure
constant and $e$ the elementary charge) occurs as if the photons move through a magnetised physical plasma, i.e. in the plasma held at the external magnetic field $\mathbf{B}$.

V. CONCLUDING REMARKS

A. Particles in black-hole geometry

We have proposed a new, covariant definition of the notion of particle in curved space-time which is observer-independent. This definition is mostly motivated by the success of particle physics we have been testing on earth and is consistent with the particle creation effect in expanding universe [29] (see for a recent short review [30]).

We have found that the term in the 2-point function that is (partially) responsible for the black-hole evaporation is of the sub-leading order with respect to the term providing the correct pole structure of the Feynman propagator. Namely, this hierarchy of the terms is regulated by the ratio $(\lambda_p/r_H)^2$ near the horizon, where $\lambda_p$ is a de Broglie wavelength of the scalar particle of momentum $p$. The modes leading to this correction should thus be understood as vacuum fluctuations. In the far-horizon region, this suppression is even stronger: $(\lambda_p/R)^2$ for $R \gg r_H$. This implies that the black-hole evaporation originating in the near-horizon region cannot be understood as a local effect that agrees with [31].

The flux of the energy density changes its direction at the distance $R \sim 3M$ outside of the event horizon [32, 33]. Hence, it might imply that the sub-leading term in the propagator is associated with the flux of the negative energy density inside the black-hole horizon (see [24]). Thus, we come to a conclusion that the Hawking’s partner mode is a vacuum fluctuation. The same observation based on a different argument has been recently made in [34].

B. One-loop correction to self-energy

The tadpole diagram yields the effective mass of the self-interacting scalar field $\Phi(x)$. Employing the effective action approach for computing the 1-loop correction to the scalar field equation, one can obtain the well-known result $m_\Phi^2 = \frac{1}{2} \langle \hat{\Phi}^2(x) \rangle$. The Wick squared operator $\hat{\Phi}^2(x)$ in the Unruh vacuum was derived in [2]. Hence, the value of the effective scalar mass at one-loop approximation is a straightforwardly computable quantity.

Perhaps, the less trivial computation is to reproduce the result for $m_\Phi^2$ by applying Feynman’s method in the freely-falling frame. The non-trivial part of this computation is how to “properly” renormalise the ultraviolet divergence of the tadpole diagram. As a guideline, we have used the above mentioned findings for the Wick squared operator. This can be considered as an intermediate step to compute other scattering reactions with the radiative corrections.
C. One-loop correction to coupling constant

A presence of black holes entails the modification of quantum field propagators. This in turn leads to the non-trivial corrections to the self-energy and coupling constant in the loop expansion. The latter gets a loop contribution depending on the external momenta $q \gg \kappa$.

We have found that the 1-loop correction to $\lambda(q)$ in the near-horizon region $R \sim r_H$ is larger than that in the absence of the black hole, while this reduces to the standard result found in the Minkowski-space approximation in the asymptotically flat region $R \gg r_H$.\(^\text{13}\)

D. One-loop correction to vacuum energy

The energy-momentum tensor of the scalar field gets a correction due to the self-coupling. The one-loop contribution to the vacuum energy is given by the standard method of the perturbation theory, namely

$$\langle \Delta \hat{T}_{\mu\nu} \rangle = \frac{\lambda}{4!} \eta_{\mu\nu} \langle \hat{\Phi}^4 \rangle = \frac{\lambda}{8} \eta_{\mu\nu} \langle \hat{\Phi}^2 \rangle^2.$$ \hspace{1cm} (73)

The same result can be reproduced through taking into account vacuum bubbles at two-loop level, namely

$$\langle \Delta \hat{T}_{\mu\nu} \rangle = \frac{i}{8V_4} \eta_{\mu\nu} \left( \begin{array}{c} \phantom{+}2^2 \end{array} \right) = \frac{\lambda}{8} \eta_{\mu\nu} \left( G_U(x, x) - G_H(x, x) \right)^2 = \frac{\lambda}{8} \eta_{\mu\nu} \langle \hat{\Phi}^2 \rangle^2.$$ \hspace{1cm} (74)

according to Eqs. (35) and (47), where $V_4 = \int d^4x$ is a four-dimensional volume of a local Minkowski frame as if it is infinitely large.

Thus, we find that the 2-loop correction to the vacuum energy after having been renormalised is finite at $R \sim r_H$ in the freely-falling frame. In the asymptotically flat region, the 2-loop correction to the vacuum energy is of the order of $1/R^4$. This is much smaller than the 1-loop contribution that vanishes as $1/r_H^2 R^2$ in the limit $R \to \infty$ \cite{4,5}.

E. Local renormalisation scheme

A sort of ambiguity is inherent to the local renormalisation scheme we have proposed. One could choose the fictitious field $\phi(x)$ be anticommuting (instead of commuting) and with a

\(^{13}\) We have derived in \cite{24} the higher-order corrections to the 2-point function in $\Delta x$ up to the second order in both the far-horizon and near-horizon region. These corrections give a contribution to the running coupling constant $\lambda(q)$ of the order of $(1 + 3 \cos^2 \gamma) T_H^4/q^4$, where $\gamma$ is an angle between $q$ and $n = R/R$. This is much smaller than the leading order term found above in the regime $q \gg T_H$.\footnote{We have derived in \cite{24} the higher-order corrections to the 2-point function in $\Delta x$ up to the second order in both the far-horizon and near-horizon region. These corrections give a contribution to the running coupling constant $\lambda(q)$ of the order of $(1 + 3 \cos^2 \gamma) T_H^4/q^4$, where $\gamma$ is an angle between $q$ and $n = R/R$. This is much smaller than the leading order term found above in the regime $q \gg T_H$.}
positive norm (with the correct sign in front of the propagator). This would change only the 1-loop correction to the coupling constant which depends on the external momenta, while (48) and (74) are insensitive to these modifications. Our choice is, however, symmetric with respect to how the propagators $G_U(x,x')$ and $G_H(x,x')$ contribute to the coupling constant $\lambda$ (at least) at one-loop level.

In addition, the coupling constant $\lambda$ in the thermal state of temperature $T$ without introducing the fictitious field $\phi(x)$ would be

$$\lambda(q,T) = \lambda + \frac{3\lambda^2}{32\pi^2} \left( \ln \left( \frac{q^2}{4\pi\mu^2} \right) - \frac{2\pi^2}{3q^2} T^2 \right) + O(\lambda^3,\lambda^2T^4/q^4).$$

(75)

Thus, the temperature-dependent correction is double of what one finds if the fictitious field is taken into account. However, it will be the same result if we also heat the fictitious field up to the temperature $T$.

In the case of a massive $\lambda \Phi^4$-theory, the one-loop correction to the scalar self-energy after having been regularised by one of the standard methods (dimensional or Pauli-Villars regularisation) is non-vanishing after the subtraction of the UV-divergent term. However, the extra UV-divergent term ($\propto m^2 \ln \sigma_0(x,x')$) of $\langle \hat{\Phi}(x)\hat{\Phi}(x') \rangle$ in the limit $x' \to x$ is precisely cancelled by the same term in the Hadamard parametrix. Thus, the mass of the scalar field does not get a quantum correction at one-loop level in this renormalisation approach.

The UV-divergent part of the stress tensor in Minkowski space is associated with the vacuum bubbles. One usually ignores this in particle physics as these do not show up for scattering processes (where one measures the energy differences only). This is illegitimate when one takes gravity into account. One of the methods to compute the renormalised stress tensor is to subtract the Hadamard parametrix from the Wightman function (see, e.g., [19] for a brief review).

It is a well-known problem in standard electroweak theory, that the quantum/loop contributions to the self-energy of the Higgs field are (polynomially) divergent. That is the origin of the hierarchy problem. In this case, the one-loop self-energy term depends on the external momentum of the Higgs particle. If we do not introduce the fictitious field in the loop diagrams containing the external momenta, then we recover the standard results.

To summarise, we should either introduce the fictitious field only for the purpose to renormalise expectation values of local quantum operators in Hadamard states or demand that the fictitious field also appears in the loop diagrams depending on the external momenta. The consequences of the latter should however be investigated in detail to draw any decisive conclusions.

F. Locally Minkowski and Unruh vacuum

These vacua are indistinguishable in the far-from-horizon region at the leading order of the approximation. The deviation of the Unruh vacuum from the locally Minkowski one $|\Omega\rangle$
can be, however, established near the event horizon, taking into account that these vacua are characterised by the 2-point functions $W_U(x, x')$ and $W(x, x')$, respectively.

The singular part of $W_U(x, x')$ at $r \sim r_H$ is given by $\vec{W}_\beta(x, x')$ (see Eq. (9)). It is inversely proportional to $\cosh(\kappa \Delta t_S) - \cosh(\kappa (2\bar{\sigma}(x, x'))^{1/2})$, where $\kappa = \frac{1}{2} f'(r_H)$. This can be transformed to the Minkowski form $\sigma_0(x, x')$ following the procedure outlined in Sec. III B 2, where, e.g., the new time coordinate is approximately equal to $f^{1/2}(r) \sinh(\kappa t_S)/\kappa$.

The singular part of $W(x, x')$ is approximately given by $-1/(8\pi^2 \sigma(x, x'))$, where $\sigma(x, x')$ is a geodetic distance between the space-time points $x$ and $x'$. This can be expressed in terms of $\sigma_0(x, x') \propto \cosh(\kappa \Delta t_S) - \cosh(\kappa (2\bar{\sigma}(x, x'))^{1/2})$ as found in Eq. (40). However, in terms of the Riemann normal coordinates, $\sigma(x, x')$ acquires a Minkowski form as well, but at a fixed spatial point at $r \sim r_H$, the Riemann normal time reads

$$\Delta y^0 \approx \frac{f^{1/2}(r)}{2f'(r)} \sinh \left( \frac{1}{2} f'(r) \Delta t_S \right).$$

Thus, the Unruh vacuum differs from the locally Minkowski vacuum in the near-horizon region, because $\kappa = \frac{1}{2} f'(r_H)$ and $\frac{1}{2} f'(r)$ coincide in the limit $r \to r_H$ (implying $f(r) \to 0$) only. We shall determine $W(x, x')$ at any radial distance from the event horizon elsewhere.

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