Uniform Logical Characterizations of Testing Equivalences for Nondeterministic, Probabilistic and Markovian Processes

Marco Bernardo

University of Urbino – Italy

© March 2009
Modal Logics for Behavioral Equivalences

- Behavioral equivalences establish whether computing systems possess the same behavioral properties.

- The specific set of properties that are preserved depends on the specific behavioral equivalence (bisimulation, testing, trace, ...).

- These properties can usually be characterized by means of a modal logic.

- Notable example: Hennessy-Milner logic (HML) and bisimilarity.
- Comparative study conducted in 2006 (operator set, quantitative info):
Sets of Processes

- Minimal syntax generating all finite-state processes without silent moves.

- Nondeterministic processes:

  \[
  P ::= 0 \mid a . P \mid P + P \mid A
  \]

- Probabilistic processes \textit{(generative)}:

  \[
  P ::= 0 \mid \sum_{i \in I} <a_i, p_i>.P_i \mid A \quad p_i \in \mathbb{R}_{[0,1]}, \sum_{i \in I} p_i = 1
  \]

- Markovian processes \textit{(generative)}:

  \[
  P ::= 0 \mid <a, \lambda>.P \mid P + P \mid A \quad \lambda \in \mathbb{R}_{>0}
  \]
Sets of Tests

• Introduction of a success state $s$.

• Nondeterministic tests:

\[
T ::= s | T' \\
T' ::= a \cdot T | T' + T'
\]

• Reactive tests ($w \in \mathbb{R}_{>0}$):

\[
T ::= s | T' \\
T' ::= <a, *_w>.T | T' + T'
\]
Testing Equivalences

- Nondeterministic testing equivalence: \( P_1 \sim_{NT} P_2 \) if and only if for all nondeterministic tests \( T \)

\[
\begin{align*}
P_1 \text{ may pass } T & \iff P_2 \text{ may pass } T \\
P_1 \text{ must pass } T & \iff P_2 \text{ must pass } T
\end{align*}
\]

- Probabilistic testing equivalence: \( P_1 \sim_{PT} P_2 \) if and only if for all reactive tests \( T \)

\[
\text{prob}(\text{SC}(P_1, T)) = \text{prob}(\text{SC}(P_2, T))
\]

- Markovian testing equivalence: \( P_1 \sim_{MT} P_2 \) if and only if for all reactive tests \( T \) and sequences \( \theta \in (\mathbb{R}_{>0})^* \) of average amounts of time

\[
\text{prob}(\text{SC}_{\leq \theta}(P_1, T)) = \text{prob}(\text{SC}_{\leq \theta}(P_2, T))
\]
New Modal Characterization of $\sim_{NT}$

- Modal language syntax:

\[
\begin{align*}
\phi & ::= \text{true} | \phi' \\
\phi' & ::= \langle a \rangle \phi | \phi' \lor \phi'
\end{align*}
\]

- Actions initially occurring in a formula:

\[
\begin{align*}
\text{init}(\text{true}) & = \emptyset \\
\text{init}(\phi_1 \lor \phi_2) & = \text{init}(\phi_1) \cup \text{init}(\phi_2) \\
\text{init}(\langle a \rangle \phi) & = \{ a \}
\end{align*}
\]
- **May-satisfy relation:**

| $P$ | $\models_{\text{may}} \phi_1 \lor \phi_2$ if $P \models_{\text{may}} \phi_1$ or $P \models_{\text{may}} \phi_2$
|------------------|------------------|
| $P$ | $\models_{\text{may}} \langle a \rangle \phi$ if there exists $P'$ such that $P \xrightarrow{a} N P'$ and $P' \models_{\text{may}} \phi$

- **Must-satisfy relation:**

| $P$ | $\models_{\text{must}} \phi_1 \lor \phi_2$ if $\text{init}(P) \cap (\text{init}(\phi_1) \cup \text{init}(\phi_2)) \neq \emptyset$
|------------------|------------------|
| | and $\text{init}(P) \cap \text{init}(\phi_1) \neq \emptyset$ implies $P \models_{\text{must}} \phi_1$
| | and $\text{init}(P) \cap \text{init}(\phi_2) \neq \emptyset$ implies $P \models_{\text{must}} \phi_2$
| $P$ | $\models_{\text{must}} \langle a \rangle \phi$ if there exists $P'$ such that $P \xrightarrow{a} N P'$ and each such $P' \models_{\text{must}} \phi$
• Non-standard interpretation of disjunction and diamond (must case):

○ Should it be \( P \models \text{must } \phi_1 \lor \phi_2 \) if \( P \models \text{must } \phi_1 \) or \( P \models \text{must } \phi_2 \), then we would have \( a.0 + b.0 \models \text{must } \langle a \rangle \text{true} \lor \langle b \rangle \langle c \rangle \text{true} \) because \( a.0 + b.0 \models \text{must } \langle a \rangle \text{true} \) . . .

○ . . . but it is not the case that \( a.0 + b.0 \) must pass \( a.s + b.c.s \).

○ Should it be \( P \models \text{must } \langle a \rangle \phi \) if for all \( P' \) whenever \( P \xrightarrow{a} P' \) then \( P' \models \text{must } \phi \), then we would have \( 0 \models \text{must } \langle a \rangle \text{true} \) because there is no \( P' \) reachable from \( 0 \) via \( a \) . . .

○ . . . but it is not the case that \( 0 \) must pass \( a.s \).
New Modal Characterization of $\sim_{PT}$

- $\phi_1 \lor \phi_2$ now obeys $\text{init}(\phi_1) \cap \text{init}(\phi_2) = \emptyset$.
- Quantitative interpretation: $\llbracket \phi \rrbracket_{PT}(P) = 0$ for $\phi \not\equiv \text{true}$ whenever $\text{init}(P) \cap \text{init}(\phi) = \emptyset$, otherwise

\[
\begin{align*}
\llbracket \text{true} \rrbracket_{PT}(P) &= 1 \\
\llbracket \phi_1 \lor \phi_2 \rrbracket_{PT}(P) &= p_1 \cdot \llbracket \phi_1 \rrbracket_{PT}(P) + p_2 \cdot \llbracket \phi_2 \rrbracket_{PT}(P) \\
\llbracket (a)\phi \rrbracket_{PT}(P) &= \sum_{a, p} \frac{p}{\text{prob}_c(P|\{a\})} \cdot \llbracket \phi \rrbracket_{PT}(P')
\end{align*}
\]

where (avoiding an overestimate):

$p_j = \frac{\text{prob}_c(P|\text{init}(\phi_j))}{\text{prob}_c(P|\text{init}(\phi_1 \lor \phi_2))}$
New Modal Characterization of $\sim_{MT}$

- $\phi_1 \lor \phi_2$ again obeys $\text{init}(\phi_1) \cap \text{init}(\phi_2) = \emptyset$.

- Quantitative interpretation: $\llbracket \phi \rrbracket_{MT}(P, \theta) = 0$ for $\phi \not\equiv \text{true}$ whenever $\text{init}(P) \cap \text{init}(\phi) = \emptyset$ or $\theta = \varepsilon$, otherwise

\[
\llbracket \text{true} \rrbracket_{MT}(P, \theta) = 1 \\
\llbracket \phi_1 \lor \phi_2 \rrbracket_{MT}(P, t \circ \theta) = p_1 \cdot \llbracket \phi_1 \rrbracket_{MT}(P, t_1 \circ \theta) + p_2 \cdot \llbracket \phi_2 \rrbracket_{MT}(P, t_2 \circ \theta) \\
\llbracket \langle a \rangle \phi \rrbracket_{MT}(P, t \circ \theta) = \begin{cases} 
\sum_{P \xrightarrow{a, \lambda}_{MT} P'} \frac{\lambda}{\text{rate}_c(P|\{a\})} \cdot \llbracket \phi \rrbracket_{MT}(P', \theta) & \text{if } \frac{1}{\text{rate}_c(P|\{a\})} \leq t \\
0 & \text{if } \frac{1}{\text{rate}_c(P|\{a\})} > t
\end{cases}
\]

where (avoiding an overestimate/underestimate):

\[
p_j = \frac{\text{rate}_c(P|\text{init}(\phi_j))}{\text{rate}_c(P|\text{init}(\phi_1 \lor \phi_2))} \\
t_j = t + \left( \frac{1}{\text{rate}_c(P|\text{init}(\phi_j))} - \frac{1}{\text{rate}_c(P|\text{init}(\phi_1 \lor \phi_2))} \right)
\]
|        | Markovian | Probabilistic | Nondeterministic |
|--------|-----------|---------------|------------------|
| Bisimulation | true \(\langle a \rangle\) | true \(\langle a \rangle\) | true \(\langle a \rangle\) |
| Testing  | true \(\langle a \rangle\) | true \(\langle a \rangle\) | true \(\langle a \rangle\) |
| Trace    | true \(\langle a \rangle\) | true \(\langle a \rangle\) | true \(\langle a \rangle\) |