QUIVERS WITH POTENTIALS ASSOCIATED TO TRIANGULATIONS OF
CLOSED SURFACES WITH AT MOST TWO PUNCTURES

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Abstract. We tackle the classification problem of non-degenerate potentials for quivers arising from triangulations of surfaces in the cases left open by Geiss-Labardini-Schröer. Namely, for once-punctured closed surfaces of positive genus, we show that the quiver of any triangulation admits infinitely many non-degenerate potentials that are pairwise not weakly right-equivalent; we do so by showing that the potentials obtained by adding the 3-cycles coming from triangles and a fixed power of the cycle surrounding the puncture are well behaved under flips and QP-mutations. For twice-punctured closed surfaces of positive genus, we prove that the quiver of any triangulation admits exactly one non-degenerate potential up to weak right-equivalence, thus confirming the veracity of a conjecture of the aforementioned authors.

Contents

1. Introduction 1
2. Preliminaries 3
3. Once-punctured surfaces 10
4. Twice-punctured surfaces 13
Acknowledgements 18
References 18

1. Introduction

Albeit technical in nature, the problem of classifying all non-degenerate potentials on a given 2-acyclic quiver is relevant in different interesting, seemingly unrelated, contexts. In cluster algebra theory, having only one weak right-equivalence class means, very roughly speaking, that Derksen-Weyman-Zelevinsky’s representation-theoretic approach to the corresponding cluster algebra can be performed in essentially only one way.

The classification problem of non-degenerate potentials plays a role also in algebraic geometry and in symplectic geometry (more precisely, in the subjects of Bridgeland stability conditions and Fukaya categories). In [1, Theorem 9.9], the uniqueness of non-degenerate potentials on the quivers arising from positive genus closed surfaces with at least three punctures is used by Tom Bridgeland and Ivan Smith to prove that there is a short exact sequence

$$1 \longrightarrow \text{Sph}_\Delta(D(\Sigma, M)) \longrightarrow \text{Aut}_\Delta(D(\Sigma, M)) \longrightarrow \text{MCG}^\pm(\Sigma, M) \longrightarrow 1,$$

Date: August 25, 2020.

2010 Mathematics Subject Classification. Primary 16P10, 16G20; Secondary 13F60, 57N05, 05E99.

Key words and phrases. Surface, marked points, punctures, triangulation, flip, quiver, potential, mutation, non-degenerate potential.
where $D(\Sigma, M)$ is the 3-Calabi-Yau triangulated category associated to $(\Sigma, M)$, defined as the full subcategory that the dg-modules with finite-dimensional cohomology determine inside the derived category of the Ginzburg dg-algebra of the quiver with potential of any tagged triangulation of $(\Sigma, M)$, the group $\text{Aut}_\Delta(D(\Sigma, M))$ is the quotient of the group of auto-equivalences of $D(\Sigma, M)$ that preserve the distinguished connected component $\text{Tilt}_\Delta(D(\Sigma, M))$ by the subgroup of auto-equivalences that act trivially on $\text{Tilt}_\Delta(D(\Sigma, M))$. $\text{Sph}_\Delta(D(\Sigma, M))$ is the subgroup of $\text{Aut}_\Delta(D(\Sigma, M))$ generated by (the quotient images of) the twist functors at the simple objects of an heart $A \in \text{Tilt}_\Delta(D(\Sigma, M))$, and $\text{MCG}^\pm(\Sigma, M) = \text{MCG}(\Sigma, M) \rtimes \mathbb{Z}/2\mathbb{Z}$ is the signed mapping class group.

In [15, Theorem 1.1], Ivan Smith shows that if $(\Sigma, M)$ is a positive genus closed surface with at least three punctures (i.e., $|M| \geq 3$), then there is a linear fully faithful embedding of the 3-Calabi-Yau triangulated category $D(\Sigma, M)$ into a Fukaya category of a 3-fold that fibers over $\Sigma$ (with poles of a quadratic differential removed from $\Sigma$). He explains that the reason behind the hypothesis $|M| \geq 3$ in [15, Theorem 1.1] arises from the fact that for positive genus closed surfaces with at least three punctures, the quiver of any triangulation has exactly one non-degenerate potential up to weak right-equivalence (a fact shown by Geiss-Labardini-Schr"{o}er [4]). See [15, Sections 1.3 and 2.2].

Together with results from his work [1] with Bridgeland, the embeddings from the previous paragraph allow Smith to obtain non-trivial computations of spaces of stability conditions on Fukaya categories of symplectic six-manifolds.

In this paper we prove the following:

**Theorem 1.1.**

1. For once-punctured closed surfaces of positive genus, the quiver of any triangulation admits infinitely many non-degenerate potentials that are pairwise not weakly right-equivalent, provided the underlying field has characteristic zero.

2. For twice-punctured closed surfaces of positive genus, the quiver of any triangulation admits exactly one non-degenerate potential up to weak right-equivalence, provided the underlying field is algebraically closed.

Let $(\Sigma, M)$ be a once-punctured closed surface, $n$ a positive integer and $x \in K$ any scalar. For a triangulation $\tau$ of $(\Sigma, M)$ let $S(\tau, x, n)$ be the potential obtained by adding the 3-cycles of $Q(\tau)$ arising from triangles of $\tau$ and the $x$-multiple of the $n^{th}$ power of the cycle of $Q(\tau)$ that runs around the puncture of $(\Sigma, M)$. The following result of independent interest plays a central role in our proof of part (1) of Theorem 1.1.

**Theorem 1.2.** Let $(\Sigma, M)$ be a once-punctured closed surface. If $\tau$ and $\sigma$ are triangulations of $(\Sigma, M)$ related by the flip of an arc $k \in \tau$, then the quivers with potential $(Q(\tau), S(\tau, x, n))$ and $(Q(\sigma), S(\sigma, x, n))$ are related by the mutation of quivers with potential $\mu_k$.

That the quivers associated to triangulations of once-punctured closed surfaces of positive genus admit more than one weak-right-equivalence class of non-degenerate potentials has always been expected, since the works [2] of Derksen-Weyman-Zelevinsky and [7, 8] of Labardini exhibit non-degenerate potentials for the Markov quiver that are not weakly right-equivalent.

In [4, Theorem 8.4], Geiss-Labardini-Schr"{o}er proved that every quiver associated to some triangulation of a positive-genus closed surface with at least three punctures admits exactly one

1. That $D(\Sigma, M)$ is independent of the tagged triangulation used follows after combining results of Keller-Yang [6] and Labardini [9], see [10, Section 5].
2. The Markov quiver arises as the quiver associated to any triangulation of the once-punctured torus.
weak-right-equivalence class of non-degenerate potentials, and conjectured that the same result holds in the case of two punctures. The reason why their proof fails for twice-punctured closed surface is that these do not admit triangulations all of whose arcs connect distinct punctures. The fact that such triangulations do exist for closed surfaces with at least three punctures plays an essential role in the proof of [4, Theorem 8.4].

The structure of the paper is straightforward: in Section 2 we prove a few facts (some of them quite technical) about the form of cycles and non-degenerate potentials for quivers arising from combinatorially nice triangulations of surfaces with empty boundary (see conditions (2.1) and (2.2)). Section 3 is devoted to proving part (1) of Theorem 1.1 whereas Section 4 is devoted to showing part (2).

2. Preliminaries

Let $K$ be any field. For a quiver $Q$, the vertex span is the $K$-algebra $R$ defined as the $K$-vector space with basis $\{e_j \mid j \in Q_0\}$, with multiplication defined as the $K$-bilinear extension of the rule
\[ e_i e_j := \delta_{i,j} e_j \quad \text{for all } i, j \in Q_0, \]
where $\delta_{i,j} \in K$ is the Kronecker delta of $i$ and $j$. Thus, $R$ is (a $K$-algebra isomorphic to) $K^{Q_0}$ with both sum and multiplication defined componentwise. The complete path algebra of $Q$ is the $K$-vector space
\[ K\langle\langle Q\rangle\rangle := \prod_{\ell \in \mathbb{Z}_{>0}} A^{(\ell)}, \]
where $A^{(0)} := R$, and for $\ell > 0$, $A^{(\ell)}$ is the $K$-vector space with basis all the paths of length $\ell$ on $Q$. The multiplication of $K\langle\langle Q\rangle\rangle$ is defined in terms of the concatenation of paths.

The vertex span $R$ is obviously a subring of $K\langle\langle Q\rangle\rangle$ (actually, a $K$-subalgebra), but it is often not a central subring. Despite this, any ring automorphism $\varphi : K\langle\langle Q\rangle\rangle \to K\langle\langle Q\rangle\rangle$ such that $\varphi|_R = 1_R$ will be said to be an $R$-algebra automorphism of $K\langle\langle Q\rangle\rangle$.

Definition 2.1. Let $Q$ be a quiver and $S, W \in K\langle\langle Q\rangle\rangle$ be potentials on $Q$. We will say that:

1. two cycles $a_1 \cdots a_{\ell_1}$ and $b_1 \cdots b_{m}$ on $Q$ are rotationally equivalent if $a_1 \cdots a_{\ell_1} = b_1 \cdots b_{m}$ or $a_1 \cdots a_{\ell_1} = b_k \cdots b_m b_1 \cdots b_{k-1}$ for some $k \in \{2, \ldots, m\}$;
2. $S$ and $W$ are rotationally disjoint if no cycle appearing in $S$ is rotationally equivalent to a cycle appearing in $W$;
3. $S$ and $W$ are cyclically equivalent if, with respect to the $m$-adic topology of $K\langle\langle Q\rangle\rangle$, the element $S - W$ belongs to the topological closure of the vector subspace of $K\langle\langle Q\rangle\rangle$ spanned by all elements of the form $a_1 \cdots a_{\ell_1} - a_2 \cdots a_{\ell_1} a_1$ with $a_1 \cdots a_{\ell_1}$ running through the set of all cycles on $Q$; notation: $S \sim_{\text{cyc}} W$;
4. $S$ and $W$ are right-equivalent if there exists a right equivalence from $S$ to $W$, i.e., an $R$-algebra automorphism $\varphi : K\langle\langle Q\rangle\rangle \to K\langle\langle Q\rangle\rangle$ that acts as the identity on the set of idempotents $\{e_j \mid j \in Q_0\}$ and satisfies $\varphi(S) \sim_{\text{cyc}} W$; notation: $S \sim_{\text{r.e.}} W$;
5. $S$ and $W$ are weakly right-equivalent if $S$ and $\lambda W$ are right-equivalent for some non-zero scalar $\lambda \in K$.

Throughout the paper, $(\Sigma, \mathbb{M})$ will be a punctured closed surface of positive genus. That is, $\Sigma$ will be a compact, connected, oriented two-dimensional real differentiable manifold with positive genus and empty boundary, and $\mathbb{M}$ will be a non-empty finite subset of $\Sigma$. 
It is very easy to show that there exists at least one triangulation $\tau$ of $(\Sigma, M)$ such that

\begin{align*}
(2.1) & \quad \text{every puncture has valency at least 4 with respect to } \tau; \\
(2.2) & \quad \text{for any two arcs } i \text{ and } j \text{ of } \tau, \text{ the quiver } Q(\tau) \text{ has at most one arrow from } j \text{ to } i.
\end{align*}

Throughout the paper, we will permanently suppose that $\tau$ satisfies (2.1) and (2.2).

Following Laidaki [11] we define two maps $f, g : Q(\tau)_1 \to Q(\tau)_1$ as follows. Each triangle $\Delta$ of $\tau$ gives rise to an oriented 3-cycle $\alpha_\Delta \beta_\Delta \gamma_\Delta$ on $Q(\tau)$. We set $f(\alpha_\Delta) = \gamma_\Delta$, $f(\beta_\Delta) = \alpha_\Delta$ and $f(\gamma_\Delta) = \beta_\Delta$. Now, given any arrow $\alpha$ of $Q(\tau)$, the quiver $Q(\tau)$ has exactly two arrows starting at the terminal vertex of $\alpha$. One of these two arrows is $f(\alpha)$. We define $g(\alpha)$ to be the other arrow.

Note that the map $f$ (resp. $g$) splits the arrow set of $Q(\tau)$ into $f$-orbits (resp. $g$-orbits). The set of $f$-orbits is in one-to-one correspondence with the set of cycles of $\tau$. All $f$-orbits have exactly three elements. The set of $g$-orbits is in one-to-one correspondence with the set of punctures of $(\Sigma, M)$. For every arrow $\alpha$ of $Q(\tau)$, we denote by $m_\alpha$ the size of the $g$-orbit of $\alpha$ ($m_\alpha \geq 4$ by (2.1)). Note that \( g^m = a \cdot g \cdot g \cdot \ldots \cdot g \cdot g \) is a cycle surrounding the puncture $p$ corresponding to the $g$-orbit of $\alpha$, we denote this cycle as $G(\alpha)$ or $G(p)$. Whereas, for every arrow $\beta$ of $Q(\tau)$ and any non-negative integer $r$, we use the notation $G(r, \beta)$ to denote the path $g^{-1}(\beta)g^{-2}(\beta)\ldots g(\beta)\beta$. Similarly, we use the notation $F(r, \beta)$ to denote the path $f^{-1}(\beta)f^{-2}(\beta)\ldots f(\beta)\beta$.

Let $x = (x_p)_{p \in M}$ be a choice of a non-zero scalar $x_p \in K$ for each puncture $p \in M$. For ideal triangulations which satisfy (2.1) and (2.2) the potential $S(\tau, x)$ defined by the second author [9] takes a simple form, namely,

$$S(\tau, x) = T(\tau) + \sum_{p \in F} x_p G(p),$$

with $T(\tau) \sim_{\text{cyc}} \sum_{\alpha \in F} (f^2(\alpha)f(\alpha)\alpha)$ for any fixed set $\Gamma$ containing exactly one arrow from each triangle of $\tau$.

**Lemma 2.2 (Types of cycles).** Let $(\Sigma, M)$ be a punctured surface with empty boundary, and let $\tau$ be a triangulation of $(\Sigma, M)$ that satisfies (2.1) and (2.2). Then every cycle in $Q(\tau)$ is rotationally equivalent to a cycle of one of the following types:

- (f-cycles) $(f^2(\alpha)f(\alpha)\alpha)^n$ for some $n \geq 1$;
- (g-cycles) $(g^m = a \cdot g \cdot g \cdot \ldots \cdot g \cdot g)(\beta)^n$ for some $n \geq 1$;
- (fg-cycles) $f^2(a)f(\alpha)\lambda$ for some arrow $\alpha$ and some path $\lambda$, such that $\lambda = g^{-1}f(\alpha)\lambda'$ with $\lambda'$ of positive length.

**Proof.** Let $\xi = \alpha_1 \ldots \alpha_r$ be any cycle on $Q(\tau)$. Denote $\alpha_{r+1} = \alpha_1$, and notice that for every $\ell = 1, \ldots, r$, we have either $\alpha_\ell = f(\alpha_{\ell+1})$ or $\alpha_\ell = g(\alpha_{\ell+1})$. Let $s_\xi$ be the length-$r$ sequence of $fs$ and $gs$ that has an $f$ at the $\ell$th place if $\alpha_\ell = f(\alpha_{\ell+1})$ and a $g$ otherwise.

If $s_\xi$ consists only of $fs$, then $\xi$ is rotationally equivalent to $(f^2(\alpha)f(\alpha)\alpha)^n$ for some arrow $\alpha$ and some $n \geq 1$. Furthermore, if $s_\xi$ consists only of $gs$, then $\xi$ is rotationally equivalent to $(g^m = a \cdot g \cdot g \cdot \ldots \cdot g \cdot g)(\beta)^n$ for some arrow $\beta$ and some $n \geq 1$. Therefore, if $s_\xi$ involves only $fs$ or only $gs$, then $\xi$ is an f-cycle or a g-cycle.

Suppose that at least one $f$ and at least one $g$ appear in $s_\xi$. Rotating $\xi$ if necessary, we can assume that $s_\xi$ starts with an $f$ followed by a $g$, i.e., $s_\xi = (f, g, \ldots)$. This means that if we set $a := f^{-1}(a_2)$, then $\alpha_1 = f(\alpha_2)$, $\alpha_2 = f(a)$ and $\alpha_3 = g^{-1}f(a)$. By (2.2) $a$ is the only arrow in $Q(\tau)_1$ such that $a_1 \alpha_2 a$ is a cycle. Since $\alpha_3 = g^{-1}f(a) \neq a$, this implies $\xi = f^2(a)f(a)g^{-1}f(a)\lambda'$ with $\lambda'$ of positive length. \(\Box\)
Remark 2.3. As in the case of cycles, every path falls within exactly one of three types of paths: $f$-paths, $g$-paths, and $fg$-paths.

By Lemma 2.2 up to cyclical equivalence we can write every potential $S$ in $Q(\tau)$ as $S = S_f + S_g + S_{fg}$, where

$$S_f = \sum_{\triangle} \sum_{n=1}^{\infty} z_{\triangle,n} (f^2(\alpha_{\triangle}) f(\alpha_{\triangle}) \alpha_{\triangle})^n,$$

$$S_g = \sum_{p \in \mathbb{R}} \sum_{n=1}^{\infty} \nu_{p,n} (G(p))^n,$$

$$S_{fg} = \sum_{\alpha \in Q(\tau)_1} f^2(a) f(a) \omega_a,$$

with each $z_{\triangle,n}, \nu_{p,n} \in K$, and $\omega_a$ a possibly infinite linear combination of paths of the form $g^{-1}f(a)\lambda'$ for each $a \in Q(\tau)$.

Lemma 2.4. Let $(\Sigma, \mathbb{M})$ be a punctured surface with empty boundary, and let $\tau$ be a triangulation of $(\Sigma, \mathbb{M})$ that satisfies (2.1) and (2.2). Every non-degenerate potential $S$ on $Q(\tau)$ is right-equivalent to a potential of the form $T(\tau) + U$ for some $U$ rotationally disjoint from $T(\tau)$.

Proof. By (2.2) the hypotheses of [4, Corollary 2.5] are satisfied, so if $S$ is a non-degenerate potential, then every $f$-cycle $f^2(\alpha)f(\alpha)\alpha$ appears in $S$. So,

$$S \sim_{cyc} \sum_{\triangle} z_{\triangle,1} f^2(\alpha_{\triangle}) f(\alpha_{\triangle}) \alpha_{\triangle} + U',$$

with all $z_{\triangle,1} \neq 0$ and $U'$ rotationally disjoint from $T(\tau)$.

We define an $R$-algebra automorphism $\varphi : K\langle Q(\tau) \rangle \to K\langle Q(\tau) \rangle$ by means of the rule

$$\varphi(\alpha_{\triangle}) = \frac{1}{z_{\triangle,1}} \alpha_{\triangle}.$$

We see that $\varphi(S) \sim_{cyc} T(\tau) + U$, for some potential $U$ rotationally disjoint from $T(\tau)$. \qed

Lemma 2.5 (Replacing $f$-potentials and $fg$-potentials by longer ones). Let $(\Sigma, \mathbb{M})$ be a punctured surface with empty boundary, and let $\tau$ be a triangulation of $(\Sigma, \mathbb{M})$ that satisfies (2.1) and (2.2). Let $\phi$ be one of the symbols $f$ and $fg$, and let $\nu$ be the other symbol, so that $\{\phi, \nu\} = \{f, fg\}$ as sets of symbols. If $W, A \in K\langle Q(\tau) \rangle$ are potentials rotationally disjoint from $T(\tau)$, and if $A_{\phi} \neq 0$, then there exists a potential $B \in K\langle Q(\tau) \rangle$ which is rotationally disjoint from $T(\tau)$ and satisfies the following four conditions:

$$\begin{align*}
\text{short}(B_{\phi}) &> \text{short}(A_{\phi}); \\
\text{short}(B_{g}) &\geq \min(\text{short}(A_{g}), \text{short}(A_{\phi}) + 1); \\
\text{short}(B_{\nu}) &\geq \min(\text{short}(A_{\nu}), \text{short}(A_{\phi}) + 1); \\
(Q(\tau), T(\tau) + W + A) &\sim_{r.e.} (Q(\tau), T(\tau) + W + B).
\end{align*}$$

Proof. Let us deal with the case $\phi = f$. Write

$$A_f = \sum_{\triangle} \sum_{n \geq \text{short}(A_f)} z_{\triangle,n} (f^2(\alpha_{\triangle}) f(\alpha_{\triangle}) \alpha_{\triangle})^n.$$
and define an $R$-algebra homomorphism $\varphi : K\langle \langle Q(\tau) \rangle \rangle \rightarrow K\langle \langle Q(\tau) \rangle \rangle$ by means of the rule
\[
\varphi(\alpha_\Delta) = \alpha_\Delta - \sum_{n \geq \text{short}(A_f)} z_{\Delta,n} \omega_n (f^2(\alpha_\Delta)f(\alpha_\Delta)\alpha_\Delta)^{n-1}.
\]
Then $\varphi$ is a unitriangular automorphism of depth $\text{short}(A_f) - 3$, and
\[
\varphi(T(\tau) + W + A) = T(\tau) - A_f + W + A + (\varphi(W + A) - (W + A)).
\]
Consequently, if we set $B = A_g + A_{fg} + (\varphi(W + A) - (W + A))$, then:

- $\varphi(T(\tau) + W + A) = T(\tau) + W + B$;
- $\text{short}(\varphi(W + A) - (W + A)) \geq \text{depth}(\varphi) + \text{short}(W + A) \geq \text{short}(A_f) - 3 + 4 = \text{short}(A_f) + 1$;
- $\text{short}(B_f) = \text{short}((\varphi(W + A) - (W + A))_f) \geq \text{short}(A_f) + 1$;
- $\text{short}(B_g) \geq \min(\text{short}(A_g), \text{short}((\varphi(W + A) - (W + A))_g) \geq \min(\text{short}(A_g), \text{short}(A_f) + 1)$; and
- $\text{short}(B_{fg}) \geq \min(\text{short}(A_{fg}), \text{short}((\varphi(W + A) - (W + A))_{fg})) \geq \min(\text{short}(A_{fg}), \text{short}(A_f) + 1)$.

Now we deal with the case $\phi = f g$. Write
\[
A_{fg} = \sum_{a \in Q(\tau)_1} f^2(a)\omega_f(a) \omega_a,
\]
with $\omega_a \in \epsilon_{h(a)}K\langle \langle Q(\tau) \rangle \rangle \epsilon_{t(a)}$ for each $a \in Q(\tau)_1$ and define an $R$-algebra homomorphism $\varphi : K\langle \langle Q(\tau) \rangle \rangle \rightarrow K\langle \langle Q(\tau) \rangle \rangle$ by means of the rule $\varphi(a) = a - \omega_a$ for $a \in Q(\tau)_1$. Then $\varphi$ is a unitriangular automorphism of depth $\text{short}(A_{fg}) - 3$, and
\[
\varphi(T(\tau) + W + A) = \sum_{\Delta} \left( f^2(\alpha_\Delta) - \omega_f(a_\Delta) \right) \left( f(\alpha_\Delta) - \omega_f(a_\Delta) \right) (\alpha_\Delta - \omega_{a_\Delta})
+ W + A + (\varphi(W + A) - (W + A))
\sim_{\text{cyc}} T(\tau) + W + A_f + A_g + (\varphi(W + A) - (W + A))
+ \sum_{a \in Q(\tau)_1} f^2(a)\omega_f(a) \omega_a - \sum_{\Delta} \omega_f(a_\Delta) \omega_f(a_\Delta) \omega_{a_\Delta}.
\]
Consequently, if we set $B = A_f + A_g + (\varphi(W + A) - (W + A)) + \sum_{a \in Q(\tau)_1} f^2(a)\omega_f(a) \omega_a - \sum_{\Delta} \omega_f(a_\Delta) \omega_f(a_\Delta) \omega_{a_\Delta}$, then:

- $\varphi(T(\tau) + W + A) \sim_{\text{cyc}} T(\tau) + W + B$;
- $\text{short}(\varphi(W + A) - (W + A)) \geq \text{depth}(\varphi) + \text{short}(W + A) \geq \text{short}(A_{fg}) - 3 + 4 = \text{short}(A_{fg}) + 1$ (since $\text{short}(A_{fg}) \geq 4$);
- $\text{short} \left( \sum_{a \in Q(\tau)_1} f^2(a)\omega_f(a) \omega_a \right) \geq 2 \text{short}(A_{fg}) - 3 \geq \text{short}(A_{fg}) + 4 - 3 = \text{short}(A_{fg}) + 1$;
- $\text{short} \left( \sum_{\Delta} \omega_f(a_\Delta) \omega_f(a_\Delta) \omega_{a_\Delta} \right) \geq 3 \text{short}(A_{fg}) - 6 \geq \text{short}(A_{fg}) + 8 - 6 \geq \text{short}(A_{fg}) + 1$;
- $\text{short}(B_{fg}) \geq \min(\text{short}(\varphi(W + A) - (W + A)), \text{short} \left( \sum_{a \in Q(\tau)_1} f^2(a)\omega_f(a) \omega_a \right))$
  , $\text{short} \left( \sum_{\Delta} \omega_f(a_\Delta) \omega_f(a_\Delta) \omega_{a_\Delta} \right) \geq \text{short}(A_{fg}) + 1$;
- $\text{short}(B_g) \geq \min(\text{short}(A_g), \text{short}(\varphi(W + A) - (W + A)), \text{short} \left( \sum_{a \in Q(\tau)_1} f^2(a)\omega_f(a) \omega_a \right))$
  , $\text{short} \left( \sum_{\Delta} \omega_f(a_\Delta) \omega_f(a_\Delta) \omega_{a_\Delta} \right) \geq \min(\text{short}(A_g), \text{short}(A_{fg}) + 1)$; and
We set in the claim are satisfied. For the fifth property, note that if $n$

Fix a positive integer $n$

We shall produce the three sequences ($\phi_n$, $\omega_n$) satisfying (2.1) and (2.2). If $U, V, W \in K\langle\langle Q(\tau)\rangle\rangle$ are potentials rotationally disjoint from $T(\tau)$, then there exist a unimodular automorphism $\varphi : K\langle\langle Q(\tau)\rangle\rangle \to K\langle\langle Q(\tau)\rangle\rangle$ of depth at least short($U$) - 3 and a potential $W \in K\langle\langle Q(\tau)\rangle\rangle$ involving only positive powers of $g$-cycles, such that short($W$) $\geq$ short($U$) and $\varphi$ is a right-equivalence ($Q(\tau), T(\tau) + Z + U) \to (Q(\tau), T(\tau) + Z + W$).

**Proof.** Set $W_0 = U_g$ and $U_0 = U - U_g$. We obviously have short($U_0$), short($W_0$) $\geq$ short($U$).

**Claim 1.** There exist sequences $(U_n)_{n \geq 1}$ and $(W_n)_{n \geq 1}$ of potentials on the quiver $Q(\tau)$, and a sequence $(\varphi_n)_{n \geq 1}$ of unimodular automorphisms of $K\langle\langle Q(\tau)\rangle\rangle$, such that the following properties are satisfied for every $n \geq 1$:

- $\varphi_n$ is a right-equivalence $(Q(\tau), T(\tau) + Z + U_n - 1 + W_n - 1) \to (Q(\tau), T(\tau) + Z + U_n + W_n)$;
- depth($\varphi_n$) $= \text{short}(U_{n-1}) - 3$;
- each of $U_n$ and $W_n$ is rotationally disjoint from $T(\tau)$, $U_n$ does not involve powers of $g$-cycles and $W_n$ involves only powers of $g$-cycles;
- short($W_n - W_{n-1}$) $\geq$ short($U_{n-1}$) $+ 1$;
- short($U_{n+1}$) $\geq$ short($U_{n-1}$) $+ 1$.

**Proof of Claim 1.** We shall produce the three sequences $(U_n)_{n \geq 1}$, $(W_n)_{n \geq 1}$ and $(\varphi_n)_{n \geq 1}$ recursively. Fix a positive integer $n$. If $U_{n-1} = 0$, we set $U_n$ to be $U_{n-1}$, $W_n$ to be $V_{n-1}$ and $\varphi_n$ to be the identity of $K\langle\langle Q(\tau)\rangle\rangle$. Otherwise, let $\phi_{n-1}, \nu_{n-1} \in \{f, fg\}$ be symbols such that $\{\phi_{n-1}, \nu_{n-1}\} = \{f, fg\}$ and short($\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle$) $\leq$ short($\langle\langle U_{n-1}\nu_{n-1}\rangle\rangle$). By the proof of Lemma 2.5, there exist a potential $V_n \in K\langle\langle Q(\tau)\rangle\rangle$ rotationally disjoint from $T(\tau)$ and a unimodular automorphism $\varphi_n : K\langle\langle Q(\tau)\rangle\rangle \to K\langle\langle Q(\tau)\rangle\rangle$ such that

- depth($\varphi_n$) $= \text{short}(U_{n-1}) - 3$;
- $\varphi_n$ is a right-equivalence $(Q(\tau), T(\tau) + Z + W_{n-1} + U_n - 1) \to (Q(\tau), T(\tau) + Z + W_n - 1 + V_n)$;
- short($\langle\langle V_n\rangle\rangle$) $\geq$ min(short($\langle\langle U_{n-1}\rangle\rangle$), short($\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle$) $+ 1$);
- short($\langle\langle W_n\rangle\rangle$) $\geq$ min(short($\langle\langle U_{n-1}\rangle\rangle$), short($\langle\langle U_{n-1}\nu_{n-1}\rangle\rangle$), short($\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle$) $+ 1$).

We set $U_n = V_n - (V_n)g$ and $W_n = W_{n-1} + (V_n)g$. It is clear that the first four properties stated in the claim are satisfied. For the fifth property, note that if $\phi_n = \phi_{n-1}$, then short($\langle\langle U_n\rangle\rangle$) $> \text{short}(\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle)$, whereas if $\phi_n \neq \phi_{n-1}$, then $\text{short}(\langle\langle U_{n+1}\phi_n\rangle\rangle) > \text{short}(\langle\langle U_n\rangle\rangle) > \text{short}(\langle\langle U_{n-1}\rangle\rangle)$ and

$$\text{short}(\langle\langle U_{n+1}\phi_n\rangle\rangle) \geq \text{min}(\text{short}(\langle\langle U_n\phi_{n-1}\rangle\rangle), \text{short}(\langle\langle U_n\phi_n\rangle\rangle) + 1) \geq \text{min}(\text{short}(\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle) + 1, \text{min}(\text{short}(\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle) + 1 + 1) > \text{short}(\langle\langle U_{n-1}\phi_{n-1}\rangle\rangle).$$

These facts, together with the observation that for each $n \geq 0$ we have short($\langle\langle U_n\phi_n\rangle\rangle) = \text{short}(U_n)$, allow us to deduce that short($U_{n+1}$) $\geq$ short($U_{n-1}$) $+ 1$ for all $n \geq 1$.

From the claim, we see that

$$\lim_{n \to \infty} \text{short}(U_n) = \infty, \quad \lim_{n \to \infty} \text{short}(W_n - W_{n-1}) = \infty \quad \text{and} \quad \lim_{n \to \infty} \text{depth}(\varphi_n) = \infty.$$
Hence, if we set \( W = \lim_{n \to \infty} W_n \), then \( \varphi := \lim_{n \to \infty} \varphi_n \circ \ldots \circ \varphi_1 \) is a right-equivalence \( (Q(\tau), T(\tau) + Z + U) \to (Q(\tau), T(\tau) + Z + W) \). Proposition 2.6 follows.

\[ \square \]

\textbf{Lemma 2.7.} Let \((\Sigma, M)\) be a punctured surface with empty boundary, let \( \tau \) be a triangulation of \((\Sigma, M)\) satisfying (2.1) and (2.2), and let \( x = (x_p)_{p \in M} \) be any choice of non-zero scalars. Suppose that \( m \) and \( t \) are positive integers and \( U, W \in K\langle Q(\tau) \rangle \) are potentials rotationally disjoint from \( S(\tau, x) \) that satisfy the following properties:

1. \( \text{short}(U) \geq m; \)
2. \( \text{short}(W) - 3 > m; \)
3. \( W = \lambda f(a) a G(t, g^{-t}(a)) c \) for some non-zero scalar \( \lambda \in K \), some arrow \( a \) and some path \( c \).

Then there exists a unitriangular \( R \)-algebra automorphism \( \zeta : K\langle Q(\tau) \rangle \to K\langle Q(\tau) \rangle \) of depth \( \text{short}(W) - 3 \) that serves as a right-equivalence between the QPs \((Q(\tau), S(\tau, x) + U + W)\) and \((Q(\tau), S(\tau, x) + U + U' + W')\) for some potentials \( U', W' \in K\langle Q(\tau) \rangle \) that satisfy:

1. \( \text{short}(U') > m; \)
2. \( \text{short}(W') > \text{short}(W); \)
3. \( W' = \lambda' f(b) b G(t - 1, g^{-(t-1)}(b)) c' \) for some non-zero scalar \( \lambda' \), some arrow \( b \) and some path \( c' \).

\textbf{Proof.} Let \( \zeta : K\langle Q(\tau) \rangle \to K\langle Q(\tau) \rangle \) be the \( R \)-algebra homomorphism given by the rule

\[ \zeta(f^{-1}(a)) = f^{-1}(a) - \lambda G(t, g^{-t}(a)) c. \]

Since \( \tau \) satisfies (2.2), \( \text{short}(W) - 3 \) is a positive integer by Lemma 2.2 and hence \( \zeta \) is actually a unitriangular automorphism of \( K\langle Q(\tau) \rangle \). The depth of \( \zeta \) is obviously \( \text{short}(W) - 3 \).

The arrow \( f^{-1}(a) \) connects two arcs of \( \tau \). Let \( p_{f^{-1}(a)} \) be the puncture at which these arcs are incident. Direct computation shows that

\[
\zeta(S(\tau, x) + U + W) \sim_{\text{cyc}} S(\tau, x) - W - \lambda x_{p_{f^{-1}(a)}} G(m_{f^{-1}(a)} - 1, g f^{-1}(a)) G(t, g^{-t}(a)) c
\]
\[
+ U + W + (\zeta(U + W) - (U + W))
\]
\[
= S(\tau, x) - \lambda x_{p_{f^{-1}(a)}} G(m_{f^{-1}(a)} - 2, g^2 f^{-1}(a)) g f^{-1}(a) g^{-1}(a) G(t - 1, g^{-t}(a)) c
\]
\[
+ U + (\zeta(U + W) - (U + W))
\]
\[
\sim_{\text{cyc}} S(\tau, x) - \lambda x_{p_{f^{-1}(a)}} g f^{-1}(a) g^{-1}(a) G(t - 1, g^{-t}(a)) c G(m_{f^{-1}(a)} - 2, g^2 f^{-1}(a))
\]
\[
+ U + (\zeta(U + W) - (U + W)).
\]

So, the lemma follows if we remember that \( g f^{-1}(a) = f g^{-1}(a) \) and set

\[
U' := \zeta(U + W) - (U + W),
\]
\[
\lambda' := -\lambda x_{p_{f^{-1}(a)}},
\]
\[
b := g^{-1}(a),
\]
\[
c' := c G(m_{f^{-1}(a)} - 2, g^2 f^{-1}(a))
\]

and \( W' := \lambda' f(b) b G(t - 1, g^{-(t-1)}(b)) c' \).
Indeed, property (3) is obviously satisfied, whereas the inequalities short\((U) \geq m\), depth\((\zeta) > 0\) and \(2\text{short}(W) - 3 > m\) imply that
\[
\text{short}(U') \geq \min(\text{short}(\zeta(U) - U), \text{short}(\zeta(W) - W)) \\
\geq \min(\text{depth}(\zeta) + \text{short}(U), \text{depth}(\zeta) + \text{short}(W)) \\
= \min(\text{depth}(\zeta) + \text{short}(U), 2\text{short}(W) - 3) \\
> m.
\]
Furthermore, we also have
\[
\text{short}(W') = m_{t-1(a)} - 2 + \text{short}(W) - 1 > \text{short}(W),
\]
where the inequality follows from the fact that \(\tau\) satisfies (2.1).

**Corollary 2.8** (Replacing certain cycles by sums of long \(g\)-cycles). Under the same hypotheses of Lemma 2.7, if the path \(c\) is assumed to be an arrow, then there exists a unitriangular \(R\)-algebra automorphism \(\Pi : K\langle\langle Q(\tau)\rangle\rangle \rightarrow K\langle\langle Q(\tau)\rangle\rangle\) of depth at least \(\min(m - 3, \text{short}(W) - 3)\) that serves as a right-equivalence between the QPs \((Q(\tau), S(\tau, x) + U + W)\) and \((Q(\tau), S(\tau, x) + U + \xi)\) for some potential \(\xi\) that involves only positive powers of \(g\)-cycles and satisfies \(\text{short}(\xi) > m\).

**Proof.** This corollary follows from an inductive use of Lemma 2.7. Set \(U_0 = U, W_0 = W, a_0 = a, c_0 = c\) and \(\lambda_0 = \lambda\). Using Lemma 2.7 we obtain a unitriangular automorphism \(\zeta_1 : K\langle\langle Q(\tau)\rangle\rangle \rightarrow K\langle\langle Q(\tau)\rangle\rangle\), potentials \(Z_1, W_1 \in K\langle\langle Q(\tau)\rangle\rangle\), an arrow \(a_1\), a path \(c_1\) and a non-zero scalar \(\lambda_1\), such that:

1. \(\text{depth}(\zeta_1) = \text{short}(W_0) - 3\);
2. \(\zeta_1\) is a right-equivalence \((Q(\tau), S(\tau, x) + U_0 + W_0) \rightarrow (Q(\tau), S(\tau, x) + U_0 + Z_1 + W_1)\);
3. \(\text{short}(Z_1) > m\) and \(\text{short}(W_1) \geq \text{short}(W_0) + 1\);
4. \(W_1 = \lambda_1 f(a_1) a_1 G(t - 1, g^{-1}(a_1)) c_1\).

Setting \(U_1 = U_0 + Z_1\), we see that \(U_1, W_1, a_1, c_1\) and \(\lambda_1\) satisfy the hypotheses of Lemma 2.7 for the integers \(m\) and \(t - 1\).

Assuming that for \(i \in \{0, \ldots, t - 1\}\) we have \(U_i, W_i, a_i, c_i\) and \(\lambda_i\) satisfying the hypotheses of Lemma 2.7 for the integers \(m\) and \(t - i\), we can produce a unitriangular automorphism \(\zeta_{i+1} : K\langle\langle Q(\tau)\rangle\rangle \rightarrow K\langle\langle Q(\tau)\rangle\rangle\), potentials \(Z_{i+1}, W_{i+1} \in K\langle\langle Q(\tau)\rangle\rangle\), an arrow \(a_{i+1}\), a path \(c_{i+1}\) and a non-zero scalar \(\lambda_{i+1}\), such that:

1. \(\text{depth}(\zeta_{i+1}) = \text{short}(W_i) - 3\);
2. \(\zeta_{i+1}\) is a right-equivalence \((Q(\tau), S(\tau, x) + U_i + W_i) \rightarrow (Q(\tau), S(\tau, x) + U_i + Z_{i+1} + W_{i+1})\);
3. \(\text{short}(Z_{i+1}) > m\) and \(\text{short}(W_{i+1}) \geq \text{short}(W_i) + 1\);
4. \(W_{i+1} = \lambda_{i+1} f(a_{i+1}) a_{i+1} G(t - i - 1, g^{-1}(a_{i+1})) c_{i+1}\).

Setting \(U_{i+1} = U_i + Z_{i+1}\), we see that \(U_{i+1}, W_{i+1}, a_{i+1}, c_{i+1}\) and \(\lambda_{i+1}\) satisfy the hypotheses of Lemma 2.7 for the integers \(m\) and \(t - (i + 1)\).

The composition \(\zeta = \zeta_t \circ \zeta_{t-1} \circ \ldots \circ \zeta_1\) is a unitriangular automorphism of \(K\langle\langle Q(\tau)\rangle\rangle\) that has depth at least \(\text{short}(W) - 3\) and serves as a right-equivalence \((Q(\tau), S(\tau, x) + U + W) \rightarrow (Q(\tau), S(\tau, x) + U + W_i)\). Notice that \(U_t = U + \sum_{i=1}^t Z_i\), that \(\text{short}\left(\sum_{i=1}^t Z_i\right) > m\), and that \(\text{short}(W_t) \geq \text{short}(W) + t = 2\text{short}(W) - 3 > m\).

By Proposition 2.6, there exists a unitriangular automorphism \(\varphi : K\langle\langle Q(\tau)\rangle\rangle \rightarrow K\langle\langle Q(\tau)\rangle\rangle\) of depth greater than \(m - 3\) that makes \((Q(\tau), S(\tau, x) + U + \sum_{i=1}^t Z_i + W_t)\) right-equivalent to \((Q(\tau), S(\tau, x) + U + \xi)\) for some potential \(\xi \in K\langle\langle Q(\tau)\rangle\rangle\) that involves only powers of \(g\)-cycles and satisfies \(\text{short}(\xi) \geq \text{short}\left(\sum_{i=1}^t Z_i + W_t\right) > m\).
From the two previous paragraphs we deduce that the automorphism $\Pi := \varphi \circ \zeta$ satisfies the desired conclusion of Corollary 2.8.

3. Once-punctured surfaces

In [7] and [9], the second author showed that the potentials $S(\tau, x)$ are well behaved with respect to flips and mutations, in the sense that if two triangulations are related by a flip, then the associated QPs are related by the corresponding QP-mutation. In this section, we show that for once-punctured closed surfaces the same result is true for a wider class of potentials. Namely, given a triangulation $\tau$ of a once-punctured closed surface of positive genus $(\Sigma, \mathbb{M})$, a scalar $x \neq 0$ and a positive integer $n$, we define a potential $S(\tau, x, n)$ as

$$S(\tau, x, n) = T(\tau) + xG(p)^n,$$

where $p$ is the only puncture in $(\Sigma, \mathbb{M})$.

**Theorem 3.1.** Let $(\Sigma, \mathbb{M})$ be a once-punctured closed surface of positive genus, $n$ be any positive integer, let $x \in \mathbb{K}$ be any scalar. If $\tau$ and $\sigma$ are triangulations of $(\Sigma, \mathbb{M})$ that are related by the flip of an arc $k \in \tau$, then the QPs $\mu_k(Q(\tau), S(\tau, x, n))$ and $(Q(\sigma), S(\sigma, x, n))$ are right-equivalent.

**Proof.** Let $a_i, b_i, c_i, i = 1, 2$, be the arrows in the two triangles with one side $k$ as in the figure below.

![Figure 1. The two triangles with one side k.](image)

Up to rotation we can write $G(p) = a_1Aa_2B$. Notice that $b_2c_1, b_1c_2$ are factors of $G(p)$, but $b_1c_1$ and $b_2c_2$ are not. The potential $\tilde{\mu}_k(S(\tau, x, n))$ is cyclically equivalent to

$$\begin{align*}
T(\tau) + x([a_1Aa_2B])^n + c_1^*b_2^*[b_2c_1] + c_2^*b_1^*[b_1c_2] + c_1^*b_1^*[b_1c_1] + c_2^*b_2^*[b_2c_2] &= \\
T(\sigma) + a_1[b_1c_1] + a_2[b_2c_2] + c_1^*b_1^*[b_1c_1] + c_2^*b_2^*[b_2c_2] + x(a_1[A]a_2[B])^n,
\end{align*}$$

where the paths $[A], [B]$ are the result of replacing $b_2c_1, b_1c_2$ in $A, B$ by $[b_2c_1], [b_1c_2]$, respectively.
We define $R$-algebra homomorphisms $\varphi_1, \varphi_2 : K \langle \langle Q(\tau) \rangle \rangle \to K \langle \langle Q(\tau) \rangle \rangle$ by means of the rules

$$\varphi_1(a_1) = a_1 - c_1^a b_1^e;$$

$$\varphi_2([b_1 c_1]) = [b_1 c_1] - x \sum_{j=0}^{n-1} (-1)^j [A] a_2 [B] ((a_1 - c_1^a b_1^e) [A] a_2 [B])^{n-j-1} (c_1^a b_1^e [A] a_2 [B])^j.$$

Applying $\varphi_1$ to $\widetilde{\mu}_k(S(\tau, x, n))$ we get

$$\varphi_1(\widetilde{\mu}_k(S(\tau, x, n))) \sim_{cyc} T(\sigma) + a_1 [b_1 c_1] + a_2 [b_2 c_2] + c_2^a b_2^e [b_2 c_2] + x((a_1 - c_1^a b_1^e) [A] a_2 [B])^n$$

$$\sim_{cyc} T(\sigma) + a_1 [b_1 c_1] + a_2 [b_2 c_2] + c_2^a b_2^e [b_2 c_2] + x(-1)^n (c_1^a b_1^e [A] a_2 [B])^n$$

$$+ x \sum_{j=0}^{n-1} (-1)^j a_1 [A] a_2 [B] ((a_1 - c_1^a b_1^e) [A] a_2 [B])^{n-j-1} (c_1^a b_1^e [A] a_2 [B])^j.$$

The potential $\varphi_2 \varphi_1(\widetilde{\mu}_k(S(\tau, x, n)))$ is cyclically equivalent to

$$\varphi_2 \varphi_1(\widetilde{\mu}_k(S(\tau, x, n))) \sim_{cyc} T(\sigma) + a_1 [b_1 c_1] + a_2 [b_2 c_2] + c_2^a b_2^e [b_2 c_2] + x(-1)^n (c_1^a b_1^e [A] a_2 [B])^n.$$

In an analogous way, we define $R$-algebra homomorphisms $\varphi_3, \varphi_4 : K \langle \langle Q(\tau) \rangle \rangle \to K \langle \langle Q(\tau) \rangle \rangle$ by means of the rules

$$\varphi_3(a_2) = a_2 - c_2^a b_2^e;$$

$$\varphi_4([b_2 c_2]) = [b_2 c_2] - x(-1)^n \sum_{j=0}^{n-1} (-1)^j [B] c_1^a b_1^e [A] ((a_2 - c_2^a b_2^e) [B] c_1^a b_1^e [A])^{n-j-1} (c_2^a b_2^e [B] c_1^a b_1^e [A])^j.$$

We obtain

$$\varphi_4 \varphi_3 \varphi_2 \varphi_1(\widetilde{\mu}_k(S(\tau, x, n))) \sim_{cyc} T(\sigma) + a_1 [b_1 c_1] + a_2 [b_2 c_2] + x(c_1^a b_1^e [A] c_2^a b_2^e [B])^n$$

$$\sim_{cyc} S(\sigma, x, n) + a_1 [b_1 c_1] + a_2 [b_2 c_2].$$

Therefore, the QPs $\mu_k(Q(\tau), S(\tau, x, n))$ and $(Q(\sigma), S(\sigma, x, n))$ are right-equivalent. \qed

**Remark 3.2.**

1. For once-punctured closed surfaces, Theorem 3.1 constitutes a generalization of the second author’s [7, Theorem 30] and [9, Theorem 8.1].

2. It was observed by Ladkani [12, Proposition 3.1] that the proof of [7, Theorem 30] can be applied without change to produce a proof of Theorem 3.1 above for $x = 0$. 

**Figure 2.** The cycle on $Q(\tau)$ and $Q(\sigma)$ surrounding the puncture.
Therefore, dim

Remark 3.5.

Corollary 3.4.

\[ \dim_K(\mathcal{P}(Q(\tau), S(\tau, x, n))) < \infty \quad \text{and} \quad \lim_{n \to \infty} \dim_K(\mathcal{P}(Q(\tau), S(\tau, x, n))) = \infty. \]

Proof. For the proof of finite-dimensionality we follow ideas suggested by Ladkani in his proof of \textbf{[11] Proposition 4.2], whereas our proof that the limits of the dimensions is \infty follows ideas that appear in the first author’s Master thesis.}

First, note that when we compute the cyclic derivative of \( S(\tau, x, n) \) with respect to an arrow \( \alpha \), we get

\[ (3.1) \quad \partial_\alpha(S(\tau, x, n)) = f^2(\alpha)f(\alpha) + xnG(nm_\alpha - 1, g(\alpha)). \]

So, \( f^2(\alpha)f(\alpha) \) and \( -xnG(nm_\alpha - 1, g(\alpha)) \) become equal in the Jacobian algebra \( \mathcal{P}(Q(\tau), S(\tau, x, n)) \).

Every \( fg \)-path of length three has the form \( f^2(\alpha)f(\alpha)g^{-1}f(\alpha) \) or \( gf^2(\alpha)f(\alpha)f(\alpha) \) for some arrow \( \alpha \), and it is hence equal to \( -xnG(nm_\alpha - 1, g(\alpha))g^{-1}f(\alpha) = -xnG(nm_\alpha - 3, g^2(\alpha))g^2(\alpha)g^{-1}f(\alpha) \) or \( -xnf^2(\alpha)G(nm_\alpha - 1, g(\alpha)) = -xnG(nm_\alpha - 3, g(\alpha)) \) in \( \mathcal{P}(Q(\tau), S(\tau, x, n)) \). Thus every \( fg \)-path of length three is equal in \( \mathcal{P}(Q(\tau), S(\tau, x, n)) \) to another \( fg \)-path of length greater than three. In the same vein, an easy inductive argument shows that, in the Jacobian algebra, every \( fg \)-path is to an arbitrarily long \( fg \)-path, and therefore equal to \( 0 \in \mathcal{P}(Q(\tau), S(\tau, x, n)) \).

Any \( f \)-path \( F(r, f(\beta)) = F(r - 2, \beta)f^2(\beta)f(\beta) \) of length \( r \) greater than three, is equal to the \( fg \)-path \( -xnF(r - 2, \beta)G(nm_\beta - 1, g(\beta)) \) in \( \mathcal{P}(Q(\tau), S(\tau, x, n)) \), and in this way, to \( 0 \). Furthermore, any \( g \)-path of the form \( G(r, g(\beta)) = G(r - nm_\beta + 1, \beta)G(nm_\beta - 1, g(\beta)) \) with length greater than \( nm_\beta \), is equal to the \( fg \)-path \( x^{-1}n^{-1}G(r - nm_\beta + 1, \beta)f^2(\beta)f(\beta) \), hence equal to \( 0 \) in the Jacobian algebra. Notice that here, we have used that \( K \) is a field of characteristic zero.

Thus far, we have shown that every path of length greater than \( nm_\alpha \) is equivalent to \( 0 \) in the Jacobian algebra \( \mathcal{P}(Q(\tau), S(\tau, x, n)) \), and therefore the latter has finite dimension.

On the other hand, as the cyclic derivative of \( S(\tau, x, n) \) with respect to any arrow \( \alpha \) is equal to the sum of an \( f \)-path of length two and a scalar multiple of a \( g \)-path of length \( nm_\alpha - 1 \) \textbf{[3.1]}, and since no \( g \)-path is a multiple of any \( f \)-path of length greater than one, we conclude that for any \( a, b \in K \langle Q(\tau) \rangle \), no \( g \)-path of length smaller than \( nm_\alpha - 1 \) appears in the expression of the element \( a\partial_\alpha(S(\tau, x, n))b \) as a possibly infinite sum of paths on the quiver \( Q(\tau) \). From this, it follows that no finite linear combination of \( g \)-paths of lengths smaller than \( nm_\alpha - 1 \) can be written as a limit of finite sums of elements of the form \( a\partial_\alpha(S(\tau, x, n))b \), i.e., the set of \( g \)-paths of length smaller than \( nm_\alpha - 1 \) is linearly independent in the Jacobian algebra \( \mathcal{P}(Q(\tau), S(\tau, x, n)) \). Therefore, \( \dim_K(\mathcal{P}(Q(\tau), S(\tau, x, n))) \geq nm_\alpha - 2. \)

\textbf{Corollary 3.4.} Over a field of characteristic zero, the quiver of any triangulation of a once-punctured closed surface of positive genus admits infinitely many non-degenerate potentials up to weak right-equivalence.

\textbf{Remark 3.5.} (1) In the case of the once-punctured torus, Proposition \textbf{3.3} was proved by the first author in his Master thesis \textbf{[5]}.
(2) In his Undergraduate thesis [13], the third author has computed an actual $K$-vector space basis of $P(Q(\tau), S(\tau, x, n))$ for each $n \geq 1$, showing in particular that different values of $n$ never yield Jacobian algebras with the same dimension. This implies that different values of $n$ always yield potentials that are not weakly right-equivalent.

4. Twice-punctured surfaces

In this section we prove part (2) of Theorem 1.1, namely:

**Theorem 4.1.** Let $(\Sigma, M)$ be a twice-punctured closed surface of positive genus, and let $\tau$ be any (tagged) triangulation of $(\Sigma, M)$. Over an algebraically closed field, any two non-degenerate potentials on the quiver $Q(\tau)$ are weakly right-equivalent.

Since any two ideal triangulations of $(\Sigma, M)$ are related by a finite sequence of flips (see [14]), the first paragraphs of the proof of [1] Lemma 8.5 imply that the mere exhibition of a single triangulation $\tau$ of $(\Sigma, M)$, with $Q(\tau)$ having only one weak right equivalence class of non-degenerate potentials, suffices in order to prove Theorem 4.1.

**Example 4.2.** Figure 3 sketches a triangulation $\tau$ of a positive-genus twice-punctured surface with empty boundary. The triangulation is easily seen to satisfy (2.1) and (2.2). Note that the puncture $p$ has valency $8g$ and the other puncture $q$ has valency $4g$.

![Figure 3](image-url)  

**Figure 3.** A triangulation $\tau$ of a twice-punctured closed surface $(\Sigma, M)$ of positive-genus.
Lemma 4.3. Let $(\Sigma, M)$ be a twice-punctured closed surface of positive genus, and let $\tau$ be the triangulation of $(\Sigma, M)$ depicted in Figure 3. If $V \in K\langle\langle Q(\tau)\rangle\rangle$ is a potential involving only $\geq 2$-powers of $g$-cycles, then $(Q(\tau), S(\tau, x) + V)$ is right-equivalent to $(Q(\tau), S(\tau, x))$ for any choice $x = (x_p, x_q)$ of non-zero scalars.

Proof. Let $g$ be the genus of $(\Sigma, M)$. Then

$$V \sim_{\text{cyc}} \sum_{n=2}^{\infty} \nu_{p,n} G(p))^n + \sum_{n=2}^{\infty} \nu_{q,n} G(q))^n$$

for some scalars $\nu_{p,n}$ and $\nu_{q,n}$ for $n \geq 2$. Note that $\text{short}(V) \geq 2 \text{val}_r(g) = 8g$.

Claim 2. There exist a sequence $(V_m)_{m=8g}^{\infty}$ of potentials on $Q(\tau)$, and a sequence $(\varphi_m)_{m=8g}^{\infty}$ of untriangular $R$-algebra automorphisms of $K\langle\langle Q(\tau)\rangle\rangle$, satisfying the following properties:

1. $V_{8g} = V$;
2. $\lim_{m \to \infty} \text{depth}(\varphi_m) = \infty$;
3. for every $m \geq 8g$:
   a. $\varphi_m$ is a right-equivalence $(Q(\tau), S(\tau, x) + V_m) \to (Q(\tau), S(\tau, x) + V_{m+1})$;
   b. $V_m$ involves only $\geq 2$-powers of $g$-cycles;
   c. $\text{short}(V_m) \geq m$.

Proof of Claim 2. Start by setting $V_{8g} = V$. Let $a_p$ (resp. $a_q$) be an arrow lying in the $g$-orbit that surrounds $p$ (resp. $q$). Suppose that for a fixed value of $m \geq 8g$ we have already defined a potential $V_m$ involving only $\geq 2$-powers of $g$-cycles and satisfying $\text{short}(V_m) \geq m$. We shall use $V_m$ to define $V_{m+1}$ and $\varphi_m$. Write:

$$V_m \sim_{\text{cyc}} \sum_{n=2}^{\infty} \lambda_{p,n} G(a_p))^n + \sum_{n=2}^{\infty} \lambda_{q,n} G(a_q))^n$$

Figure 4. The associated quiver $Q(\tau)$ to the triangulation $\tau$. 
with $\lambda_{p,n}, \lambda_{q,n} \in K$ for $n \geq 2$. Set $r_{p,m}$ (resp. $r_{q,m}$) to be the first value of $n$ for which $\lambda_{p,n} \neq 0$ (resp. $\lambda_{q,n} \neq 0$) if such an $n$ exists, and $\infty$ if such an $n$ does not exist. Note that short($V_m$) = \min(8gr_{p,m}, 4gr_{q,n}) \geq 8g.

Define an $R$-algebra homomorphism $\Upsilon_{p,m} : K \langle \langle Q(\tau) \rangle \rangle \rightarrow K \langle \langle Q(\tau) \rangle \rangle$ by means of the rule

$$\Upsilon_{p,m} : a_p \mapsto a_p - \frac{\lambda_{p,r_{p,m}}}{x_p} a_p (G(a_p))^{r_{p,m}-1}. $$

Since $r_{p,m} - 1 > 0$, $\Upsilon_{p,n}$ is a unitriangular automorphism, its depth is $8g(r_{p,m} - 1)$. Direct computation shows that

$$\Upsilon_{p,m}(S(\tau, x) + V_m) \sim_{\text{cyc}} S(\tau, x) + U + W,$$

where

$$U = -\lambda_{p,r_{p,m}}(G(a_p))^{r_{p,m}} + \Upsilon_{p,m} \left( \sum_{n=r_{p,m}}^{\infty} \lambda_{p,n}(G(a_p))^n \right) + \sum_{n=r_{q,m}}^{\infty} \lambda_{q,n}(G(a_q))^n,$$

$$W = -\frac{\lambda_{p,r_{p,m}}}{x_p} f(a_p) a_p (G(a_p))^{r_{p,m}-1} f^2(a_p).$$

Note that short($U$) $\geq m$ and $2\text{short}(W) - 3 = 2*8g(r_{p,m} - 1) + 3 \geq 8gr_{p,m} + 3 > 8gr_{p,m} \geq m$. So, applying Corollary 2.8 we see that there exists a unitriangular $R$-algebra automorphism $\Pi_{p,m}$ of $K \langle \langle Q(\tau) \rangle \rangle$ that has depth at least $\min(m - 3, 8g(r_{p,m} - 1))$ and serves as a right-equivalence between $S(\tau, x) + U + W$ and $S(\tau, x) + U + \xi$ for some potential $\xi$ that involves only positive powers of $g$-cycles and satisfies short($\xi$) $> m \geq 8g$. These last inequalities imply that, actually, $\xi$ involves only $\geq 2$-powers of $g$-cycles.

Now, we can definitely write

$$(4.1) \quad U \sim_{\text{cyc}} \sum_{n=r_{p,m}+1}^{\infty} \kappa_{p,n}(G(a_p))^n + \sum_{n=r_{q,m}}^{\infty} \lambda_{q,n}(G(a_q))^n$$

for some scalars $\kappa_{p,n} \in K$. Define an $R$-algebra homomorphism $\Upsilon_{q,m} : K \langle \langle Q(\tau) \rangle \rangle \rightarrow K \langle \langle Q(\tau) \rangle \rangle$ by means of the rule

$$\Upsilon_{q,m} : a_q \mapsto a_q - \frac{\lambda_{q,r_{q,m}}}{x_q} a_q (G(a_q))^{r_{q,m}-1}.$$ 

Since $r_{q,m} - 1 > 0$, $\Upsilon_{q,m}$ is a unitriangular automorphism, its depth is $4g(r_{q,m} - 1)$. Direct computation shows that

$$\Upsilon_{q,m}(S(\tau, x) + U + \xi) \sim_{\text{cyc}} S(\tau, x) + U' + W',$$

where

$$U' = -\lambda_{q,r_{q,m}}(G(a_q))^{r_{q,m}} + \sum_{n=r_{p,m}+1}^{\infty} \kappa_{p,n}(G(a_p))^n + \Upsilon_{q,n} \left( \sum_{n=r_{q,m}}^{\infty} \lambda_{q,n}(G(a_q))^n \right) + \Upsilon_{q,m}(\xi),$$

$$W' = -\frac{\lambda_{q,r_{q,m}}}{x_q} f(a_q) a_q (G(a_q))^{r_{q,m}-1} f^2(a_q).$$

Note that short($U'$) $> m$ and $2\text{short}(W') - 3 = 2*4g(r_{q,m} - 1) + 3 \geq 4gr_{q,m} + 3 > 4gr_{q,m} \geq m$. So, applying Corollary 2.8 we see that there exists a unitriangular $R$-algebra automorphism $\Pi_{q,m}$ of $K \langle \langle Q(\tau) \rangle \rangle$ that has depth at least $\min(m - 3, 4g(r_{q,m} - 1))$ and serves as a right-equivalence between $S(\tau, x) + U' + W'$ and $S(\tau, x) + U' + \xi'$ for some potential $\xi'$ that involves only positive
powers of $g$-cycles and satisfies $\text{short}(\xi') > m \geq 8g$. These last inequalities imply that, actually, $\xi'$ involves only $\geq 2$-powers of $g$-cycles.

It is clear that $U'$ involves only positive powers of $g$-cycles; these powers are actually greater than 1 because $\text{short}(U') > m \geq 8g$. So, if we set $V_{m+1} = U' + \xi'$ and $\varphi_m = \Pi_{q,m} \Upsilon_{q,m} \Pi_{p,m} \Upsilon_{p,m}$, we see that $\varphi_m$ is a right-equivalence $(Q(\tau), S(\tau, x) + V_m) \to (Q(\tau), S(\tau, x) + V_{m+1})$, that $V_{m+1}$ involves only $\geq 2$-powers of $g$-cycles, and that $\text{short}(V_{m+1}) \geq m + 1$.

From the previous paragraph we deduce that the sequences $(V_m)_{m \geq 8g}$ and $(\varphi_m)_{m \geq 8g}$ satisfy the third condition stated in Claim 2. Moreover, since $m \leq \text{short}(V_m) = \min(8gr_{p,m}, 4gr_{q,m})$ for every $m \geq 8g$, we deduce that $\lim_{m \to \infty} r_{p,m} = \infty = \lim_{m \to \infty} r_{q,m}$. This and the inequalities

$$\text{depth}(\varphi_m) \geq \min(\text{depth}(\Pi_{q,m}), \text{depth}(\Upsilon_{q,m}), \text{depth}(\Pi_{p,m}), \text{depth}(\Upsilon_{p,m}))$$

$$\geq \min(m - 3, 4g(r_{q,m} - 1)), 4g(r_{q,m} - 1), \min(m - 3, 8g(r_{p,m} - 1)), B(r_{p,m} - 1))$$

imply that $\lim_{m \to \infty} \text{depth}(\varphi_m) = \infty$.

Our Claim 2 is proved. \hfill \square

Lemma 4.3 follows from an obvious combination of Claim 2 and [9, Lemma 2.4]. \hfill \square

**Proposition 4.4.** Let $(\Sigma, \mathbb{M})$ be a twice-punctured closed surface of positive genus, and let $\tau$ be the triangulation of $(\Sigma, \mathbb{M})$ depicted in Figure 3. If $W \in K(\|Q(\tau)\|)$ is a potential that involves only positive powers of $g$-cycles and such that $(Q(\tau), T(\tau) + W)$ is a non-degenerate QP, then $W$ involves each of the $g$-cycles that arise from the two punctures $p$ and $q$ of $(\Sigma, \mathbb{M})$, that is, $T(\tau) + W = S(\tau, x) + V$ for some choice $x = (x_p, x_q)$ of non-zero scalars and some potential $V$ involving only $\geq 2$-powers of $g$-cycles.

**Proof.** With the notation of Figures 3 and 4, let us write

$$W = y a_1 a_2 \ldots a_{4g} + A + z \left( \prod_{j=0}^{g-1} b_{4(g-j)} c_{4(g-j)} - 2b_{4(g-j)} - 3c_{4(g-j)} - 1b_{4(g-j)} - 2c_{4(g-j)} b_{4(g-j)} - 1c_{4(g-j)} - 3 \right)$$

$$+ B, \text{ with }$$

$$A = \sum_{n=2}^{\infty} y_n (a_1 a_2 \ldots a_{4g})^n \text{ and }$$

$$B = \sum_{n=2}^{\infty} z_n \left( \prod_{j=0}^{g-1} b_{4(g-j)} c_{4(g-j)} - 2b_{4(g-j)} - 3c_{4(g-j)} - 1b_{4(g-j)} - 2c_{4(g-j)} b_{4(g-j)} - 1c_{4(g-j)} - 3 \right)^n .$$

If we set $I = \{ 2g + 1, 2g + 2, \ldots, 6g - 1, 6g \}$, then $(Q(\tau), T(\tau) + W)$ and $I$ satisfy the hypotheses of [41, Proposition 2.4], and we deduce that $y \neq 0$.

Note that for every $k \in \{1, \ldots, 2g - 1\}$, the quiver $\tilde{\mu}_k \tilde{\mu}_{k-1} \ldots \tilde{\mu}_2 \tilde{\mu}_1(Q(\tau))$ does not have 2-cycles incident to the vertex labelled $k + 1$. Therefore, the QP $\mu_2 \mu_2 \mu_{2g} \ldots \mu_2 \mu_1(Q(\tau), T(\tau) + W)$ is right-equivalent to the reduced part of the QP $\tilde{\mu}_2 \tilde{\mu}_2 \mu_{2g} \ldots \tilde{\mu}_2 \mu_1(Q(\tau), T(\tau) + W)$, whose underlying
quiver and potential are \( \tilde{\mu}_{2g} \ldots \tilde{\mu}_1(Q(\tau)) \) and

\[
\tilde{\mu}_{2g} \ldots \tilde{\mu}_1(T(\tau) + W) = \left( \sum_{j=1}^{4g} a_j b_j c_j \right) + ya_1 \ldots a_{4g} + A + [B] \\
+ z \left( \prod_{j=0}^{g-1} [b_4(g-j) c_4(g-j)-2][b_4(g-j)-3c_4(g-j)-1][b_4(g-j)-2c_4(g-j)][b_4(g-j)-1c_4(g-j)-3] \right) \\
+ \left( \sum_{j=1}^{2g} c_j^* b_j^*[b_j c_j] + c_j^* b_j^*[b_j c_j + 2] + c_j^* b_j^*[b_j c_j + 2] \right).
\]

Consider the QP \( (\tilde{\mu}_{2g} \ldots \tilde{\mu}_1(Q(\tau)), S) \), where

\[
S = \left( \sum_{j=1}^{4g} a_j b_j c_j \right) + ya_1 \ldots a_{4g} + A + \left( \sum_{j=1}^{2g} c_j^* b_j^*[b_j c_j] + c_j^* b_j^*[b_j c_j + 2] \right),
\]

and let \((Q, S)\) be its reduced part, computed according to the limit process with which Derksen-Weyman-Zelevinsky \cite{2} Theorem 4.6\) prove their Splitting Theorem. Note the presence of the sum \( \sum_{j=1}^{4g} a_j b_j c_j \) in \( S \). Then \( Q = Q(\sigma) \), where \( \sigma \) is a triangulation that can be obtained from \( \tau \) by applying an orientation-preserving homeomorphism of \((\Sigma, M)\) that exchanges \( p \) and \( q \) (thus \( \tau \) and \( \sigma \) have the same shape, sketched in Figure 3; see also Example 4.5 below). Moreover, since no arrow of the form \( a_j \) or \( b_j c_j \) appears in any of the terms of the potential \( W' \)

\[
W' := z \left( \prod_{j=0}^{g-1} [b_4(g-j) c_4(g-j)-2][b_4(g-j)-3c_4(g-j)-1][b_4(g-j)-2c_4(g-j)][b_4(g-j)-1c_4(g-j)-3] \right) + [B] \\
+ \left( \sum_{j=1}^{2g} c_j^* b_j^*[b_j c_j + 2] + c_j^* b_j^*[b_j c_j] \right),
\]

the QP \( (Q(\sigma), S + W') \) is a reduced part of \((\tilde{\mu}_{2g} \ldots \tilde{\mu}_1(Q(\tau)), \tilde{\mu}_{2g} \ldots \tilde{\mu}_1(T(\tau) + W))\) and hence is (right-equivalent to) the mutation \( \mu_{2g} \ldots \mu_1(Q(\tau), T(\tau) + W) \). Furthermore, from the fact that no arrow of the form \( b_j c_j \) with \( j \neq \ell \) appears in any of the terms of \( S \) we deduce that the coefficient in \( S \) of any of the rotations of the cycle

\[
\prod_{j=0}^{g-1} [b_4(g-j) c_4(g-j)-2][b_4(g-j)-3c_4(g-j)-1][b_4(g-j)-2c_4(g-j)][b_4(g-j)-1c_4(g-j)-3]
\]

is 0. Therefore, the coefficient of this cycle in \( S + W' \) is 0 (and its proper rotations do not appear).

The non-degeneracy of \((Q(\tau), T(\tau) + W)\) implies the non-degeneracy of \((Q(\sigma), S + W')\). Furthermore, it is easy to see that if we set \( I = \{2g+1, 2g+2, \ldots, 6g-1, 6g\} \), then \((Q(\sigma), S + W')\) and \( I \) satisfy the hypotheses of \cite{4} Proposition 2.4\), from which we deduce that \( z \neq 0 \). This finishes the proof of Proposition 4.4.

\[\square\]

**Example 4.5.** Figure 5 sketches the flip sequence in the proof of Proposition 4.4 in the case of a twice-punctured torus. Note that the first and last triangulations have the same shape.
Proof of Theorem 4.1. Let \((\Sigma, M)\) be a twice-punctured closed surface of positive genus, and let \(\tau\) be a triangulation of \((\Sigma, M)\) satisfying (2.1) and (2.2). By Lemma 2.4, every non-degenerate potential on \(Q(\tau)\) is right-equivalent to a potential of the form \(T(\tau) + U\) for some \(U\) which is rotationally disjoint from \(T(\tau)\). By Proposition 2.6, \(T(\tau) + U\) is right-equivalent to \(T(\tau) + W\) for some potential that involves only positive powers of \(g\)-cycles. Theorem 4.1 now follows from Proposition 4.4, Lemma 4.3 and [4, Lemma 8.5]. □

Acknowledgements

We thank Christof Geiss and Jan Schröer for many helpful discussions.

The three authors were supported by the second author’s grant PAPIIT-IA102215. The first two authors were supported by the second author’s grant CONACyT-238754 as well. DLF received support from a Cátedra Marcos Moshinsky and the grant PAPIIT-IN112519.

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