Bianchi III and V Einstein metrics

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Abstract

We present diagonal Einstein metrics for Bianchi III and V, both for minkowskian and euclidean signatures and we show that the Einstein Bianchi III metrics have an integrable geodesic flow.
1 Introduction

As is clear from the book by Stephani et al [7], the field of exact solutions to Einstein’s field equations has been enriched substantially, and quite recently impressive progresses have taken place for ricci-flat metrics within the Bianchi B family. In [2] solutions were obtained for some non-diagonal Bianchi III metrics, and in [8] for the more general case of Bianchi VII$_h$.

The aim of this article is to obtain some new exact Einstein metrics, for both minkowskian and euclidean signatures. These metrics are obtained for the simplest Bianchi B metrics: the type III and the type V, under the simplifying assumption that they are diagonal with respect to the invariant 1-forms.

The content of this article is the following: in Section 2 we present background informations and the field equations for the Bianchi III metrics. In Section 3 we derive the Einstein metric and prove the integrability of the geodesic flow; quite remarkably this metric exhibits a fourth Killing vector. Then section 4 presents background informations and the field equations of the Bianchi V diagonal metrics. They are first solved, in Section 5, for the ricci-flat case to recover Joseph’s metric and its euclidean partner. The basic integration of the Einstein equations is given in Section 6. Due to the higher complexity of these metrics, which do involve elliptic functions, the explicit forms are given in Section 7 for the minkowskian signature and in Section 8 for the euclidean signature. Some conclusions are presented in Section 9. We give in the appendices more details on a curious form of de Sitter metric encountered in the analysis of the Bianchi V Einstein metrics and some technicalities related to elliptic functions.

2 Bianchi III metrics

The Bianchi III Lie algebra is defined as

\[
[\mathcal{L}_1, \mathcal{L}_2] = 0, \quad [\mathcal{L}_2, \mathcal{L}_3] = 0, \quad [\mathcal{L}_3, \mathcal{L}_1] = \mathcal{L}_3.
\]  

A representation by differential operators is

\[
\mathcal{L}_1 = \partial_x + z \partial_z, \quad \mathcal{L}_2 = \partial_y, \quad \mathcal{L}_3 = \partial_z,
\]

and the invariant Maurer-Cartan 1-forms are

\[
\sigma_1 = dx, \quad \sigma_2 = dy, \quad \sigma_3 = e^{-x} dz, \quad \Rightarrow \quad d\sigma_1 = d\sigma_2 = 0, \quad d\sigma_3 = \sigma_3 \wedge \sigma_1.
\]

We will look for diagonal metrics of the form

\[
g = \beta^2 \sigma_1^2 + \gamma^2 \sigma_2^2 + \delta^2 \sigma_3^2 + \epsilon \alpha^2 dt^2.
\]

2.1 Flat space

Before writing down the Einstein equations, it is interesting to look for flat space within our coordinates choice. An easy computation shows that it is given by

\[
g_0 = \sigma_1^2 + t^2(\sigma_1^2 + \sigma_3^2) - dt^2 = dy^2 + t^2 \frac{(dz^2 + dr^2)}{r^2} - dt^2, \quad r = e^x.
\]
This metric is *unique* and does exist only for the minkowskian signature. The flattening coordinates are

\[ x_1 = y, \quad x_2 = \frac{tz}{r}, \quad x_3 = \frac{t}{2r}(-1 + z^2 + r^2), \quad \tau = \frac{t}{2r}(1 + z^2 + r^2), \]

which gives

\[ g_0 = d\vec{r} \cdot d\vec{r} - d\tau^2, \quad \vec{r} = (x_1, x_2, x_3). \]

The appearance of \( \sigma_1^2 + \sigma_3^2 \) signals an extra symmetry

\[ \mathcal{L}_4 = z \partial_x + \frac{1}{2} (z^2 - e^{2x}) \partial_z, \quad (6) \]

which enlarges the infinitesimal isometries to the four dimensional Lie algebra\(^1\)

\[ [\mathcal{L}_3, \mathcal{L}_1] = \mathcal{L}_3, \quad [\mathcal{L}_1, \mathcal{L}_4] = \mathcal{L}_4, \quad [\mathcal{L}_3, \mathcal{L}_4] = \mathcal{L}_1. \quad (7) \]

### 2.2 The field equations

The Einstein equations\(^2\)

\[ \text{Ric}^\nu_\mu = \lambda \delta^\nu_\mu \]

give for the Bianchi III case

\[ (I) \quad \frac{\dot{\delta}}{\delta} = \frac{\dot{\beta}}{\beta}, \]

\[ (II) \quad \frac{\dot{\beta}}{\beta} + \frac{\dot{\beta}}{\beta} \left( \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\alpha}}{\alpha} \right) + \epsilon \left( \frac{1}{\beta^2} + \lambda \right) \alpha^2 = 0, \]

\[ (III) \quad \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\gamma}}{\gamma} \left( 2 \frac{\dot{\beta}}{\beta} - \frac{\dot{\alpha}}{\alpha} \right) + \epsilon \lambda \alpha^2 = 0, \]

\[ (IV) \quad \frac{\dot{\beta}^2}{\beta^2} + 2 \frac{\dot{\beta} \dot{\gamma}}{\beta \gamma} + \epsilon \left( \frac{1}{\beta^2} + \lambda \right) \alpha^2 = 0. \]

### 3 Einstein metrics and their geodesic flow

Let us now consider the \( \lambda \neq 0 \) case. Relations (I) and (II)-(IV) integrate up to

\[ \frac{\dot{\beta}}{\beta} = c \frac{\alpha \gamma}{\beta}, \quad c \in \mathbb{R}, \quad \delta = c_2 \beta, \quad c_2 \neq 0, \quad (8) \]

The coordinate choice

\[ \alpha = \frac{\beta}{\gamma} \quad \Longrightarrow \quad \beta = \beta_0 e^{c_1}, \quad \delta = c_2 \beta_0 e^{c_1}. \]

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1 We give only the non-vanishing commutators.
2 In our notations the spheres have positive curvature.
To determine $\gamma$ we have to use (III) which becomes

$$\frac{\ddot{\gamma}}{\gamma} + \frac{\dot{\gamma}^2}{\gamma^2} + c \frac{\dot{\gamma}^2}{\gamma} + \epsilon \lambda \beta_0^2 e^{2ct} \frac{e^{2ct}}{\gamma^2} = 0. \quad (9)$$

This equation does linearize in $\gamma^2$ to

$$(\ddot{\gamma}^2) + c (\dot{\gamma}^2) + 2\epsilon \lambda \beta_0^2 e^{2ct} = 0, \quad (10)$$

and the remaining relation (IV) becomes

$$c (\dot{\gamma}^2) + c^2 \gamma^2 + \epsilon (1 + \lambda \beta_0^2) = 0. \quad (11)$$

Let us organize the discussion according to the values of $c$.

### 3.1 Vanishing $c$

The relation (11) gives $\beta_0^2 = -1/\lambda$ and (10) is easily integrated to $\gamma^2 = \gamma_0 + \gamma_1 t + \epsilon t^2$. By a translation of $t$ we can set $\gamma_1 \to 0$ and by a rescaling of $z$ we can set $c_2 \to 1$, so we can write the metric

$$g = \frac{1}{|\lambda|} \left[ \sigma_1^2 + \sigma_3^2 + \gamma^2 \sigma_2^2 + \epsilon \frac{dt^2}{\gamma^2} \right], \quad \gamma^2 = \gamma_0 + \epsilon t^2, \quad \lambda < 0. \quad (12)$$

Let us notice that all the metrics will have $\lambda < 0$.

For the minkowskian signature we must have $\gamma_0 > 0$ and the change of variable $t = \sqrt{\gamma_0} \theta \tau$ transforms the metric into

$$g = \frac{1}{|\lambda|} \left[ \sigma_1^2 + \sigma_3^2 + \frac{1}{\cosh^2 \tau} \left[ \sigma_2^2 - d\tau^2 \right] \right]. \quad (13)$$

We get a decomposable space-time \cite[p. 554]{7} which is the product of two 2-dimensional Einstein metrics: $\mathbb{H}^2$ on the one hand with metric and isometries

$$g_0 = \sigma_1^2 + \sigma_3^2 = dx^2 + e^{-2x} dz^2, \quad \left\{ \begin{array}{l} L_1 = \partial_z, \\ L_2 = \partial_x + z \partial_z, \\ L_3 = z \partial_x + \frac{1}{2}(z^2 - e^{2x}) \partial_z, \end{array} \right. \quad (14)$$

and a Lorentzian 2-dimensional metric on the other hand with isometries

$$g_1 = \frac{dy^2 - d\tau^2}{\cosh^2 \tau}, \quad \left\{ \begin{array}{l} M_1 = \partial_y, \\ M_2 = e^{-y}(\sinh \tau \partial_y - \cosh \tau \partial_\tau), \\ M_3 = e^y(\sinh \tau \partial_y + \cosh \tau \partial_\tau). \end{array} \right. \quad (15)$$

In this very special case we have as many as 6 Killing vectors!

For the euclidean signature, according to the sign of $\gamma_0$ we have 3 cases:

$$\gamma_0 > 0 \quad g = \frac{1}{|\lambda|} \left[ \sigma_1^2 + \sigma_3^2 + \frac{1}{\cos^2 \tau} \left[ \sigma_2^2 + d\tau^2 \right] \right],$$

$$\gamma_0 = 0 \quad g = \frac{1}{|\lambda|} \left[ \sigma_1^2 + \sigma_3^2 + \frac{1}{\tau^2} \left[ \sigma_2^2 + d\tau^2 \right] \right], \quad \lambda < 0.$$  

$$\gamma_0 < 0 \quad g = \frac{1}{|\lambda|} \left[ \sigma_1^2 + \sigma_3^2 + \frac{1}{\sinh^2 \tau} \left[ \sigma_2^2 + d\tau^2 \right] \right],$$

We have again decomposable Einstein metrics made up of two copies of $\mathbb{H}^2$. 
3.2 Non-vanishing $c$

In this more general case we obtain

$$\Gamma^2 \equiv c_2^2 = -\epsilon + \gamma_1 e^{-ct} - \frac{\epsilon \lambda \beta_0^2}{3} e^{2ct}. \quad (16)$$

Taking as variable $s = \beta_0 e^{ct}$, and cleaning up the irrelevant parameters, we eventually obtain the Einstein metric

$$g = s^2(\gamma_1^2 + \gamma_2^2) + \frac{ds^2}{\Gamma^2}, \quad \Gamma^2 = -\epsilon + \frac{\gamma_0}{s} - \frac{\epsilon \lambda}{3} s^2. \quad (17)$$

This metric exhibits the extra Killing vector $\mathcal{L}_4$ defined in (6) but it is no longer decomposable.

3.3 Integrable geodesic flow

The Einstein metric that we have found is type D. If it were a vacuum metric, the existence of one Killing-Yano tensor and of one Killing-St"ackel tensor would follow from [10], [3]. Nevertheless, for the obvious tetrad, we found the following Killing-Yano tensor for metrics with both signatures

$$Y = s e_3 \wedge e_1. \quad (18)$$

Its square gives the Killing-Stackel tensor

$$S = s^2(e_1^2 + e_3^2). \quad (19)$$

Let consider, for the minkowskian signature (the euclidean case is similar), the geodesic flow induced by the Hamiltonian

$$H = \frac{1}{2} \left( \frac{1}{\Gamma^2} \Pi_y^2 + \frac{\Pi_x^2 + e^{2x} \Pi_z^2}{s^2} - \Gamma^2 \Pi_s^2 \right). \quad (20)$$

The KS tensor (19) gives for conserved quantity

$$Q = \Pi_x^2 + e^{2x} \Pi_z^2, \quad \{H, Q\} = 0. \quad (21)$$

It cannot be obtained from symmetrized tensor products of Killing vectors because their corresponding linear conserved quantities are

$$\tilde{\mathcal{L}}_1 = \Pi_y, \quad \tilde{\mathcal{L}}_2 = \Pi_z, \quad \tilde{\mathcal{L}}_3 = \Pi_x + z \Pi_z, \quad \tilde{\mathcal{L}}_4 = z \Pi_x + \frac{1}{2} \left( z^2 - e^{2x} \right) \Pi_z, \quad \{H, \tilde{\mathcal{L}}_i\} = 0.$$

The dynamical system with hamiltonian $H$ is therefore integrable, since it exhibits 4 independent conserved quantities: $H, Q, \Pi_y, \Pi_z$ in involution for the Poisson bracket. Writing the action as

$$S = E t + p \Pi_y + q \Pi_z + A(s) \quad (22)$$

we get for separated equation

$$\left( \frac{dA}{ds} \right)^2 = \frac{Q}{s^2 \Gamma^2} + \frac{p^2}{\Gamma^4} - \frac{2E}{\Gamma^2}. \quad (23)$$

Notice that since these metrics are Einstein, the “minimal quantization” discussed in [4] does preserve integrability at the quantum level and implies the separability of the Schrödinger equation.

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3 We follow the same terminology as in [9].
3.4 The ricci-flat limit

The coordinates used in (17) allow to take the $\lambda \to 0$ limit, giving

$$g_0 = s^2(\sigma_1^2 + \sigma_3^2) + \gamma^2 \sigma_2^2 + \epsilon \frac{ds^2}{\gamma^2}, \quad \gamma^2 = -\epsilon + \frac{\gamma_0}{s}. \quad (24)$$

Of course, this metric is certainly not new since it is type D: it must lie somewhere in Kinnersley analysis [6] of all ricci-flat minkowskian type D metrics. Obviously its geodesic flow is also integrable.

4 Bianchi V

In this case the Lie algebra is

$$[L_1, L_2] = L_2, \quad [L_2, L_3] = 0, \quad [L_3, L_1] = -L_3, \quad (25)$$

with the Killing vectors

$$L_1 = \partial_x - y\partial_y - z\partial_z, \quad L_2 = \partial_y, \quad L_3 = \partial_z, \quad (26)$$

and the invariant Maurer-Cartan 1-forms

$$\sigma_1 = dx, \quad \sigma_2 = e^x dy, \quad \sigma_3 = e^x dz, \quad \Rightarrow \quad d\sigma_1 = 0, \quad d\sigma_2 = \sigma_1 \wedge \sigma_2, \quad d\sigma_3 = \sigma_1 \wedge \sigma_3. \quad (27)$$

We will look again for a diagonal metric

$$g = \beta^2 \sigma_1^2 + \gamma^2 \sigma_2^2 + \delta^2 \sigma_3^2 + \epsilon \alpha^2 dt^2. \quad (28)$$

4.1 The flat space

Let us first determine the flat space Bianchi V metric. It is easy to check that it is given by

$$g_0 = t^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - dt^2 = t^2 \gamma - dt^2. \quad (29)$$

This metric is unique and does exist only with the minkowskian signature. The metric $\gamma$ is easily seen to be the Poincaré metric for $H^3$, since we can write

$$\gamma \equiv \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{dy^2 + dz^2 + d\rho^2}{\rho^2}, \quad \rho = e^{-x},$$

which has 6 Killing vectors. The flattening coordinates for (29) are

$$x_1 = \frac{ty}{\rho}, \quad x_2 = \frac{tz}{\rho}, \quad x_3 = \frac{t}{2\rho}(-1 + y^2 + z^2 + \rho^2), \quad \tau = \frac{t}{2\rho}(1 + y^2 + z^2 + \rho^2), \quad (30)$$

leading to

$$g_0 \equiv t^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - dt^2 = d\vec{r} \cdot d\vec{r} - d\tau^2, \quad \vec{r} = (x_1, x_2, x_3). \quad (31)$$
4.2 The field equations

The Einstein equations for Bianchi V are

\[
(I) \quad \frac{\ddot{\beta}}{\beta} + \frac{\dot{\beta}}{\beta} \left( \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} - \frac{\dot{\alpha}}{\alpha} \right) + \epsilon(2 + \lambda \beta^2) \frac{\alpha^2}{\beta^2} = 0,
\]

\[
(II) \quad \frac{\ddot{\gamma}}{\gamma} + \frac{\dot{\gamma}}{\gamma} \left( \frac{\dot{\beta}}{\beta} + \frac{\dot{\delta}}{\delta} - \frac{\dot{\alpha}}{\alpha} \right) + \epsilon(2 + \lambda \beta^2) \frac{\alpha^2}{\beta^2} = 0,
\]

\[
(III) \quad \frac{\ddot{\delta}}{\delta} + \frac{\dot{\delta}}{\delta} \left( \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\alpha}}{\alpha} \right) + \epsilon(2 + \lambda \beta^2) \frac{\alpha^2}{\beta^2} = 0,
\]

\[
(IV) \quad \frac{\ddot{\gamma}\dot{\gamma}}{\beta^2} + \frac{\dot{\gamma}\dot{\delta}}{\beta^2} + \frac{\dot{\beta}\dot{\delta}}{\beta^2} + \epsilon(3 + \lambda \beta^2) \frac{\alpha^2}{\beta^2} = 0,
\]

We will begin by the Ricci-flat case.

5 Ricci-flat metrics

The most general metric, due to Joseph [5], is well known, but as a warming up, let us present a new short derivation. Let us put \( \lambda = 0 \) in (32); the differences (I)-(II) and (III)-(I) integrate to

\[
\frac{\dot{\gamma}}{\gamma} - \frac{\dot{\gamma}}{\gamma} = c \frac{\alpha}{\beta \gamma \delta}, \quad \frac{\dot{\delta}}{\delta} - \frac{\dot{\beta}}{\beta} = c_2 \frac{\alpha}{\beta \gamma \delta},
\]

and (V) implies \( c_2 = -c \).

The coordinates choice

\[
\alpha = \beta \gamma \delta, \quad \gamma = \gamma_0 e^{ct} \beta, \quad \delta = \delta_0 e^{-ct} \beta.
\]

Let us notice that \( \gamma_0 \) (resp. \( \delta_0 \)) can be absorbed in a re-definition of the coordinate \( y \) (resp. \( z \)) appearing in \( \sigma_2 \) and \( \sigma_3 \), so we will take \( \gamma_0 = \delta_0 = 1 \) in what follows. This remark allows to write the metric

\[
g = \beta^2 \left( \sigma_1^2 + e^{2ct} \sigma_2^2 + e^{-2ct} \sigma_3^2 + \epsilon \beta^4 dt^2 \right).
\]

Relation (I) becomes

\[
D_t \left( \frac{\dot{\beta}}{\beta} \right) + 2\epsilon \beta^4 = 0, \quad \implies \quad \frac{\dot{\beta}^2}{\beta^2} + \epsilon \beta^4 = E.
\]

Then relation (IV) gives \( E = \frac{c^2}{3} \).

In the minkowskian case, we may have \( E = c = 0 \). This implies the relation \( dt^2 = d\beta / \beta^6 \), and using \( \beta = s \) as a new variable we recover the flat metric (29).

For \( c_1 \neq 0 \), using as a new variable \( u = \frac{\sqrt{3}}{c} \beta^2 \) we get

\[
\frac{du}{\sqrt{1 + u^2}} = \pm \frac{2c}{\sqrt{3}} dt, \quad \implies \quad u = \text{sh} \left[ 2c(t - t_0) / \sqrt{3} \right],
\]

(35)
and, setting $c = 1, t_0 = 0$, we have

$$g = \frac{1}{u} \left[ \sigma_1^2 + (u + \sqrt{1+u^2})^{-\frac{\sqrt{3}}{2}} \sigma_2^2 + \left( u + \sqrt{1+u^2} \right)^{\frac{\sqrt{3}}{2}} \sigma_3^2 - \frac{du^2}{4u^2(1+u^2)} \right], \quad u \in (0, +\infty).$$

Switching to the new variable $\tau$ we eventually obtain

$$sh(2\tau) = \frac{1}{u} \Rightarrow g = sh(2\tau) \left[ \sigma_1^2 + (th \tau)^{\frac{\sqrt{3}}{2}} \sigma_2^2 + \left( th \tau \right)^{-\frac{\sqrt{3}}{2}} \sigma_3^2 - d\tau^2 \right], \quad (36)$$

the standard form of the minkowskian Joseph metric. Due to the symmetric role played by $(\sigma_2, \sigma_3)$, the coefficients of $\sigma_2^2$ and of $\sigma_3^2$ may be interchanged, and this corresponds to the exchange $(c \leftrightarrow -c)$.

For the euclidean Joseph metric we get merely

$$g = \sin(2\tau) \left[ \sigma_1^2 + (\tan \tau)^{\frac{\sqrt{3}}{2}} \sigma_2^2 + \left( \tan \tau \right)^{-\frac{\sqrt{3}}{2}} \sigma_3^2 + d\tau^2 \right], \quad \tau \in (0, \pi/2), \quad (37)$$

and there is no special case $E = 0$.  

### 6 Einstein Bianchi V metrics

Let us consider a non-vanishing $\lambda$. The differences (I)-(II) and (I)-(III) integrate to

$$\frac{\dot{\gamma}}{\gamma} - \frac{\dot{\beta}}{\beta} = c \frac{\alpha}{\beta \gamma \delta}, \quad \frac{\dot{\delta}}{\delta} - \frac{\dot{\beta}}{\beta} = c_2 \frac{\alpha}{\beta \gamma \delta},$$

and (V) implies $c_2 = -c$.

The coordinates choice

$$\alpha = \beta \gamma \delta \quad \Rightarrow \quad \gamma = \gamma_0 e^{\epsilon t} \beta, \quad \delta = \delta_0 e^{-\epsilon t} \beta, \quad \alpha = \gamma_0 \delta_0 \beta^3. \quad (38)$$

By the same argument as for the ricci-flat case we may set $\gamma_0 = \delta_0 = 1$ and relation (I) becomes

$$D_t \left( \frac{\dot{\beta}}{\beta} \right) + \epsilon \beta^4(2 + \lambda \beta^2) = 0, \quad \Rightarrow \quad \frac{\dot{\beta}^2}{\beta^2} + \epsilon \beta^4(1 + \lambda \beta^2/3) = E. \quad (39)$$

Eventually relation (IV) gives $E = \epsilon^2/3 \geq 0$.

### 6.1 The special case $E = c = 0$

Relation (39) becomes

$$dt = \frac{d\beta}{\beta^3 \sqrt{-\epsilon - \epsilon \lambda \beta^2/3}}. \quad (40)$$

Taking $\beta \to s$ as a new variable, we get the metric

$$g = s^2 \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{ds^2}{1 + \lambda s^2}. \quad (41)$$
The minkowskian or euclidean character of the metric does depend solely on the range taken by the variable \( t \), and in the \( \lambda \to 0 \) limit we recover, as it should, the flat space metric \((29)\).

For \( \lambda > 0 \), we can have only a minkowskian metric. As already experienced with the special \( c = 0 \) case for Bianchi III, we may expect some higher symmetry and it is indeed the case! Defining \( \sqrt{\lambda} \frac{3}{3} s = \frac{2t}{1-t^2} \) we can write the metric:

\[
g^+_M = \frac{12}{\lambda} \frac{1}{(1-t^2)^2} \left( t^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - dt^2 \right),
\]
on which we recognize a symmetric space, since by using the flattening coordinates \((30)\), we have

\[
g^+_M = \frac{12}{\lambda} \frac{d \vec{r} \cdot d \vec{r} - d \tau^2}{(1 + \vec{r}^2 - \tau^2)^2}.
\]

Indeed, using the constrained coordinates

\[
z_0 = \frac{1 - \vec{r}^2 + \tau^2}{1 + \vec{r}^2 - \tau^2}, \quad \vec{z} = \frac{2\vec{r}}{1 + \vec{r}^2 - \tau^2}, \quad z_4 = \frac{2\tau}{1 + \vec{r}^2 - \tau^2}, \quad z_0^2 + \vec{z}^2 - z_4^2 = 1,
\]
we see that we end up with de Sitter metric

\[
g^+_M = \frac{3}{\lambda} \left( dz_0^2 + d\vec{z} \cdot d\vec{z} - d z_4^2 \right),
\]
and the isometry group enlarges to \( O(4,1) \). In some sense the metric \((41)\) is an exotic but simple way of writing de Sitter metric and some further details are gathered in Appendix A.

For \( \lambda < 0 \) we have, for Minkowskian signature, anti de Sitter metric

\[
g^-_M = \frac{12}{|\lambda| (1 + t^2)^2} \left( t^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - dt^2 \right).
\]

and a euclidean one

\[
g^-_E = \frac{3}{|\lambda|} \left[ \text{ch}^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + d \theta^2 \right],
\]
which is also a symmetric space but we were not able to put a name on it.

6.2 The general case \( E \neq 0 \)

In relation \((39)\), let us introduce as a new variable

\[
\rho = \frac{|c|}{\beta^2} > 0 \quad \Rightarrow \quad \frac{\rho \, d\rho}{\sqrt{P(\rho)}} = \pm \frac{2c \, dt}{\sqrt{3}}, \quad P(\rho) \equiv \rho (\rho^3 - 3 \epsilon \rho - \epsilon \lambda |c|),
\]
which gives for the metric

\[
g = \frac{|c|}{\rho} \left( \sigma_1^2 + \gamma^2 \sigma_2^2 + \frac{1}{\gamma^2} \sigma_3^2 + \frac{3}{4} \frac{d \rho^2}{P(\rho)} \right), \quad \gamma^2 \equiv e^{2|c|t}.
\]
Remark: Due to the symmetric role played by \((\sigma_2, \sigma_3)\), the coefficients of \(\sigma_2^2\) and of \(\sigma_3^2\) may be interchanged and this corresponds to the exchange \((c \leftrightarrow -c)\) or \((\gamma \leftrightarrow 1/\gamma)\). This means that if the metric (43) is Einstein, then

\[
g = \frac{|c|}{\rho} \left( \sigma_1^2 + \frac{1}{\gamma^2} \sigma_2^2 + \gamma^2 \sigma_3^2 + \frac{3}{4} \epsilon \frac{d\rho^2}{P(\rho)} \right),
\]

will be Einstein too. We will use this observation to get rid of the sign in relation (42) and to take \(c > 0\).

Let us observe that the integration of relation (42) will require the use of elliptic functions. The corresponding reductions are given in the appendix; using these results we get the final form of the metrics, according to their signature.

7 Minkowskian signature

In this case \(P(\rho) = \rho(\rho^3 + 3\rho + \lambda c)\) has, no matter what the value of \(c\) is, always 2 real and 2 complex conjugate roots (recall that we exclude \(\lambda = 0\)). So we fix \(c = 1\) and, to express most conveniently the roots of \(P\), we parametrize the Einstein constant according to \(\lambda = 2 \sinh(\theta), \quad \theta \in \mathbb{R} \setminus \{0\}\).

We will use now the results from appendix B to give the explicit form of the metric.

1. For \(\lambda < 0\):

In this case the roots are

\[
a = -2 \sinh(\theta/3) > b = 0, \quad a_1 = \sqrt{3} \cosh(\theta/3), \quad b_1 = \sinh(\theta/3),
\]

so we have

\[
\begin{align*}
A &= \sqrt{3 + 12 \sinh^2(\theta/3)}, \\
B &= \sqrt{3 + 4 \sinh^2(\theta/3)}, \\
k^2 &= \frac{(A + B)^2 - 4 \sinh^2(\theta/3)}{4AB}
\end{align*}
\]

and

\[
\text{sn} v_0 = \sqrt{\frac{2B}{A + B - 2 \sinh(\theta/3)}}.
\]

In formula (43) we have to transform \(d\rho\) into \(dv\) to get eventually

\[
g_M = \frac{1}{\rho} \left( \sigma_1^2 + \gamma^2 \sigma_2^2 + \frac{1}{\gamma^2} \sigma_3^2 - \frac{3}{AB} (dv)^2 \right), \quad v \in [0, v_0),
\]

where \(\rho\) and \(\gamma^2\) are given respectively by

\[
\rho = \frac{aB \text{cn}^2 v}{B \text{cn}^2 v - A \text{sn}^2 v \text{dn}^2 v},
\]

and by

\[
\gamma^2 = \left( e^{-\xi v} \frac{H(v_0 + v) \Theta_1(v_0 + v)}{H(v_0 - v) \Theta_1(v_0 - v)} \right) \sqrt{\xi}, \quad \xi = 2 \left( \frac{\Theta'(v_0)}{\Theta(v_0)} + \frac{H'_1(v_0)}{H_1(v_0)} \right).
\]

2. For $\lambda > 0$:

In this case the roots are

$$a = 0 > b = -2 \text{sh}(\theta/3), \quad a_1 = \sqrt{3} \text{ch}(\theta/3), \quad b_1 = \text{sh}(\theta/3),$$

so we have

$$\begin{cases} 
A = \sqrt{3 + 4 \sinh^2(\theta/3)}, \\
B = \sqrt{3 + 12 \sinh^2(\theta/3)}, \\
k^2 = \frac{(A + B)^2 - 4 \sinh^2(\theta/3)}{4AB}.
\end{cases}$$

The parameter $k^2$ remains unchanged while $A$ and $B$ are interchanged and $v_0$ becomes

$$\text{sn} v_0 = \sqrt{\frac{2B}{A + B + 2 \sinh(\theta/3)}}.$$ 

The metric is still given by (45), where now $\rho$ and $\gamma^2$ are respectively

$$\rho = \frac{|b|A \text{sn}^2 v \text{dn}^2 v}{B \text{cn}^2 v - A \text{sn}^2 v \text{dn}^2 v},$$

and by

$$\gamma^2 = \left( e^{-\xi v} \frac{H(v_0 + v) \Theta_1(v_0 + v)}{H(v_0 - v) \Theta_1(v_0 - v)} \right)^{\sqrt{3}}, \quad \xi = 2 \left( \frac{|b|}{AB} + \frac{\Theta'}{\Theta}(v_0) + \frac{H_1'}{H_1}(v_0) \right).$$

8 Euclidean signature

In this case $P(\rho) = \rho(\rho^3 - 3\rho - \lambda c)$. It has two real roots for $\lambda c \in (-\infty, -2) \cup (+2, +\infty)$, four real roots for $\lambda c \in [-2, 0) \cup (0, +2]$ and a double root for $\lambda c = \pm 2$. Since the parameter $c$ is free, we can collapse $(-\infty, 0) \cup (0, +\infty)$ to two points by taking $c = 2/|\lambda|$. In this case elliptic functions are no longer required, leading to simpler metrics.

We have to discuss two cases:

1. $\lambda < 0$:

We have $P(\rho) = \rho(\rho + 2)(\rho - 1)^2$ and

$$\frac{2c}{\sqrt{3}} \, dt = \frac{\rho d\rho}{|\rho - 1|\sqrt{\rho(\rho + 2)}}.$$

The change of variable $\rho = \frac{2s^2}{3 - s^2}$ simplifies to

$$2c \, dt = \frac{4s^2 \, ds}{(1 - s^2)(3 - s^2)}.$$ 

We obtain

$$\gamma^2 \equiv e^{\xi t} = \frac{1 + s}{|1 - s|} \left( \frac{\sqrt{3} - s}{\sqrt{3} + s} \right)^{\sqrt{3}},$$

(50)
and the Einstein metric

\[ g_E = \frac{(3 - s^2)}{|\lambda| s^2} \left( \sigma_1^2 + \gamma^2 \sigma_2^2 + \frac{1}{\gamma^2} \sigma_3^2 + \frac{ds^2}{(1 - s^2)^2} \right). \] (51)

In fact we have two metrics: the first one for \( s \in (0, 1) \), and the second one for \( s \in (1, \sqrt{3}) \).

2. \( \lambda > 0 \):

We have \( P(\rho) = \rho(\rho - 2)(\rho + 1)^2 \) and

\[ \frac{2c}{\sqrt{3}} \ dt = \frac{\rho \ d\rho}{(\rho + 1)\sqrt{\rho(\rho + 2)}} , \quad \rho > 2. \]

The change of variable \( \rho = \frac{2}{1 - s^2} \) simplifies to

\[ \frac{2c}{\sqrt{3}} \ dt = -\frac{4 \ ds}{(1 - s^2)(3 - s^2)} , \quad s \in (-1, +1). \]

Deleting the sign we obtain

\[ \gamma^2 \equiv e^{2ct} = \frac{\sqrt{3} - s}{\sqrt{3} + s} \left( \frac{1 + s}{1 - s} \right)^{\sqrt{3}} \] (52)

and the Einstein metric

\[ g_E = \frac{(1 - s^2)}{\lambda} \left[ \sigma_1^2 + \gamma^2 \sigma_2^2 + \frac{1}{\gamma^2} \sigma_3^2 + \frac{3 \ ds^2}{(3 - s^2)^2} \right]. \] (53)

### 9 Conclusion

We have obtained some new Einstein metrics for Bianchi III and V. For this last case the complexity of the results remains reasonable since we end up simply with elliptic functions and not Painlevé transcendents.

A very unusual “bifurcation” is observed: while in the minkowskian we need elliptic functions, in the euclidean one can dispense with them. This raises the following question: would it be possible, through clever changes, to get rid of the elliptic functions for all the Bianchi V Einstein metrics? Another question of interest is to what extent one could work out the more general Bianchi VI\(_h\) and Bianchi VII\(_h\) cases.

### Appendix

#### A De Sitter metric re-visited

We have shown that the metric

\[ g = s^2 \left( \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) - \frac{ds^2}{1 + \lambda s^2}, \] (54)
is, for $\lambda > 0$ de Sitter and for $\lambda < 0$ anti-de Sitter. We will discuss only de Sitter. Taking $\sh \theta = \sqrt{\frac{\lambda}{3}} s$ as a new variable the metric becomes
\[ g = \frac{3}{\lambda} \left( \sh^2 \theta (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - d\theta^2 \right), \quad \lambda > 0. \] (55)
This is quite a simple form for de Sitter, which could be useful in other applications. So we will examine the isometries.

Let us first observe that the three dimensional metric
\[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = dx^2 + e^{2x} (dy^2 + dz^2), \]
has 6 Killing vectors, shared by the metric (55). It is made up with 2 sub-algebras:
\[ \mathcal{A}_1 = \{ P_1, P_2, M_3 \}, \quad \mathcal{A}_2 = \{ Q_1, Q_2, L_3 \}. \] (56)
The first one is $e(2)$ ($M_3$ is a rotation)
\[ P_1 = \partial_y, \quad P_2 = \partial_z, \quad M_3 = -z \partial_y + y \partial_z, \]
\[ [M_3, P_1] = -P_2, \quad [M_3, P_2] = P_1, \quad [P_1, P_2] = 0, \] (57)
and the second one ($L_3$ mixes translation and dilatation)
\[ \left\{ \begin{array}{l}
Q_1 = y \partial_x + \frac{1}{2} \left( -y^2 + z^2 + e^{-2x} \right) \partial_y - yz \partial_z, \\
Q_2 = z \partial_x - yz \partial_y + \frac{1}{2} \left( y^2 - z^2 + e^{-2x} \right) \partial_z,
\end{array} \right. \]
\[ L_3 = \partial_x - y \partial_y - z \partial_z, \]
with
\[ [L_3, Q_1] = -Q_2, \quad [L_3, Q_2] = -Q_1, \quad [Q_1, Q_2] = 0. \] (59)
These 2 sub-algebras close up according to
\[ [M_3, Q_1] = -Q_2, \quad [M_3, Q_2] = Q_1, \]
\[ [L_3, P_1] = P_1, \quad [L_3, P_2] = P_2, \]
\[ [P_1, Q_1] = L_3, \quad [P_1, Q_2] = M_3, \]
\[ [P_2, Q_1] = -M_3, \quad [P_2, Q_2] = L_3, \]
\[ [M_3, L_3] = 0. \] (60)
We need 4 extra Killing vectors to get the 10 dimensional $o(4,1)$ Lie algebra for de Sitter metric. They are given by
\[ C_1 = e^x \left( \frac{1}{\th \theta} \frac{\partial}{\partial x} - \frac{\partial}{\partial \theta} \right), \]
\[ C_2 = ye^x \left( \frac{1}{\th \theta} \frac{\partial}{\partial x} - \frac{\partial}{\partial \theta} \right) + e^{-x} \frac{\partial}{\th \theta} \frac{\partial}{\partial y}, \]
\[ C_3 = ze^x \left( \frac{1}{\th \theta} \frac{\partial}{\partial x} - \frac{\partial}{\partial \theta} \right) + e^{-x} \frac{\partial}{\th \theta} \frac{\partial}{\partial z}, \]
\[ C_4 = \frac{y^2 + z^2}{2} e^x \left( -\frac{1}{\th \theta} \frac{\partial}{\partial x} + \frac{\partial}{\partial \theta} \right) + e^{-x} \frac{1}{\th \theta} \left( \frac{1}{2} \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - z \frac{\partial}{\partial z} + \frac{\th \theta}{2} \frac{\partial}{\partial \theta} \right). \] (61)
So despite the simple form of the metric, the isometries are quite awkward.

The remaining commutators (we give only the non-vanishing ones) are ordered as:

\[
\begin{align*}
[Q_1, C_1] &= C_2 & [Q_2, C_1] &= C_3 & [L_3, C_1] &= C_1 \\
[P_1, C_2] &= C_1 & [M_3, C_2] &= -C_3 & [Q_1, C_2] &= -C_4 \\
[P_2, C_3] &= C_1 & [M_3, C_3] &= C_2 & [Q_2, C_3] &= -C_4 \\
[P_1, C_4] &= -C_2 & [P_2, C_4] &= -C_3 & [L_3, C_4] &= -C_4
\end{align*}
\]

(62)

and

\[
\begin{align*}
[C_1, C_2] &= -P_1 & [C_1, C_3] &= -P_2 & [C_2, C_3] &= -M_3 \\
[C_2, C_4] &= -Q_1 & [C_3, C_4] &= -Q_2 & [C_1, C_4] &= -L_3
\end{align*}
\]

(63)

B Elliptic functions: some tools

There are plenty of books on elliptic function theory, but we used mainly the books by Byrd and Friedman [1] and by Whittaker and Watson [11]. We use Jacobi rather than Weierstrass notation for elliptic functions. Similarly we use earlier Jacobi notation for the theta functions which is best adapted to our purposes. They are related to the more symmetric notations used in [11] according to

\[
H(v) = \theta_1(w), \quad H_1(v) = \theta_2(w), \quad \Theta_1(v) = \theta_3(w), \quad \Theta(v) = \theta_4(w), \quad w = \frac{\pi v}{2K}.
\]

Let us start from the relation (42)

\[
\frac{2dt}{\sqrt{3}} = \frac{\rho d\rho}{\sqrt{P(\rho)}},
\]

(64)

If the quartic polynomial \( P(\rho) \) has 2 real roots, and therefore two complex conjugate ones, we will write it

\[
P(\rho) = (\rho - a)(\rho - b)[(\rho - b_1)^2 + a_1^2], \quad a > b.
\]

In this case, the positivity of \( \rho \) and \( P(\rho) \) requires \( \rho \geq a \). One defines

\[
A = \sqrt{(a - b_1)^2 + a_1^2} > B = \sqrt{(b - b_1)^2 + a_1^2}, \quad k^2 = \frac{(A + B)^2 - (a - b)^2}{4AB} < 1,
\]

where \( k^2 \) will be the parameter of the elliptic functions involved. Let us define the change of variable

\[
\begin{align*}
\text{sn}^2 v &= \frac{2B(\rho - a)}{D_+}, & \text{cn}^2 v &= \frac{D_-}{D_+}, & \text{dn}^2 v &= \frac{D_-}{2A(\rho - b)},
\end{align*}
\]

(65)

with

\[
D_\pm = A(\rho - b) \pm B(\rho - a) + (a - b)\sqrt{(\rho - b_1)^2 + a_1^2},
\]

(66)

and the parameters

\[
s_0 \equiv \text{sn} v_0 = \sqrt{\frac{2B}{A + B + a - b}} < 1, \quad s_1 \equiv \text{sn} v_1 = \sqrt{\frac{2B}{A + B - a + b}} > 1,
\]
for which the reader can check that \( v_1 = K + iK' + v_0 \).

The change of variable (65) transforms \( \rho \in [a, +\infty) \) into \( v \in [0, v_0) \subset [0, K_0) \). The inverse relation is

\[
\rho = \frac{aB c^2 - bA s^2 d^2}{B c^2 - A s^2 d^2}.
\]

Using

\[
\frac{\rho - a}{a - b} = \frac{A s^2 d^2}{B c^2 - A s^2 d^2}, \quad \frac{\rho - b}{a - b} = \frac{B c^2}{B c^2 - A s^2 d^2},
\]

\[
\sqrt{(\rho - b_1)^2 + a_1^2} = AB \frac{d^2 - c^2 + c^2 d^2}{B c^2 - A s^2 d^2},
\]

straightforward computations give

\[
\frac{d\rho}{\sqrt{P(\rho)}} = \frac{2}{\sqrt{AB}} dv.
\]

It remains to give the explicit form of \( \gamma^2 = e^{2t} \) as a function of \( v \) by integrating (64), which becomes now:

\[
\frac{2dt}{\sqrt{3}} = \frac{2}{\sqrt{AB}} \frac{aB c^2 - bA s^2 d^2}{B c^2 - A s^2 d^2} dv.
\]

The relation

\[
\frac{c_0^2}{s^2 - s_0^2} = -\frac{c_0}{2s_0d_0} \left( \frac{H'(v_0 - v)}{H}(v_0 - v) + \frac{H'(v_0 + v)}{H}(v_0 + v) - 2 \frac{\Theta'(v_0)}{\Theta}(v_0) \right),
\]

and a similar one, obtained by the substitution \( v_0 \rightarrow v_1 = K + iK' + v_0 \):

\[
\frac{c_1^2}{s^2 - s_1^2} = \frac{c_0}{2s_0d_0} \left( \frac{\Theta'(v_0 - v)}{\Theta_1}(v_0 - v) + \frac{\Theta'(v_0 + v)}{\Theta_1}(v_0 + v) - 2 \frac{H'(v_0)}{H_1}(v_0) \right),
\]

allow us to integrate up to

\[
\gamma^2 \equiv e^{2t} = \left( e^{-\xi v} \frac{H(v_0 + v)}{H(v_0 - v)} \frac{\Theta_1(v_0 + v)}{\Theta_1(v_0 - v)} \right)^{\sqrt{3}}, \quad \xi = 2 \left( -\frac{b}{\sqrt{AB}} + \frac{\Theta'(v_0)}{\Theta}(v_0) + \frac{H'(v_0)}{H_1}(v_0) \right).
\]

As the reader may notice, in [1][p. 135] a different change of variables is given, which differs from ours. It is

\[
\text{cn} u = \frac{(A - B)\rho - bA + aB}{(A + B)\rho - bA - aB}.
\]

As a consequence we get in the metric (43) the term

\[
-\frac{3}{4} \frac{d\rho^2}{P(\rho)} = -\frac{3}{4} \frac{dv^2}{AB} \left( \frac{du}{2} \right)^2.
\]

To avoid the 1/4 factor we have used a duplication transformation to switch to our variable by \( u = 2v \). Notice that in the limit \( \lambda \rightarrow 0 \) we have \( 3/AB \rightarrow 1 \).

\(^4\)From now on we will use the simplified notations \( s \equiv sn(v, k^2) \), \( c \equiv cn(v, k^2) \), \( d \equiv dn(v, k^2) \) as well as \( s_0 = sn v_0 \), \( s_1 = sn v_1 \) etc...
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