Fractional exclusion statistics: the method for describing interacting particle systems as ideal gases

Dragoș-Victor Anghel

Department of Theoretical Physics, Horia Hulubei National Institute for Physics and Nuclear Engineering, 30 Reactorului Street, PO Box MG-6, Măgurele, Jud. Ilfov, Romania
E-mail: dragos@theory nipne.ro

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Abstract
I show that if the total energy of a system of interacting particles may be written as a sum of quasiparticle energies, then the system of quasiparticles can be viewed, in general, as an ideal gas with fractional exclusion statistics (FES). The general method for calculating the FES parameters is also provided. The interacting particle system cannot be described as an ideal gas of Bose and Fermi quasiparticles except in trivial situations.

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(Some figures may appear in color only in the online journal)

1. Introduction
A fractional exclusion statistics (FES) system [1, 2] consists of a countable number of finite-dimensional species. I will count the species using the indexes i and j. Each species contains $G_i$ single-particle states and $N_j$ particles. The FES character consists in the dependence of the dimensions of all the other species change according to $\delta N_i = \alpha_{ij} \delta N_j$, for any i and j. The parameters $\alpha_{ij}$ are called the FES parameters. In practice, the species may be different types of particles that coexist in the same system, quasiparticle excitations in the lowest Landau level in the fractional quantum Hall effect [1, 2], excitations [1] or motifs of spins in spin chains [3, 4], elementary volumes obtained by coarse graining in the phase space of a system [5–8] and so on.

The thermodynamics and statistical mechanics of systems was calculated mainly by Wu [2] and Isakov [5], but some amendments have recently been introduced [6–11]. In this paper, I shall use the ansatz

$$\alpha_{ij} = G_i a_{ij} + a_i \delta_{ij},$$

which applies quite generally to quasicontinuous systems [8].

The equilibrium distribution of particles in the species may be calculated in two equivalent formulations: I call them the Bose and Fermi perspectives [8].

In the Bose perspective I use $G_i$ to denote the number of available single-particle states of species $i$. In such a case, the number of microconfigurations in which we can find the system is $W = \prod_i [(G_i + N_i - 1)/[N_i!(G_i - 1)!]]$. If we add a small perturbation, $\delta N_i$, to the particle distribution, $W$ becomes

$$W_B = \prod_i \left[\frac{G_i + N_i - 1 + (1-a_i) \delta N_i - \sum_j \alpha_{ij} \delta N_j}{(N_i + \delta N_i)!(G_i - 1 - a_i \delta N_i - \sum_j \alpha_{ij} \delta N_j)!}\right].$$

In the Fermi perspective, $T_i = G_i + N_i - 1$ and $W_F = \prod_i [T_i!/[(N_i!(T_i - N_i)!)]$. At the variations $\delta N_i$,
we obtain
\[
W_\text{F} = \prod_i \left\{ \left( T_i - a_i \delta N_i - G_i \sum_j a_j \delta N_j \right)! \right\} \frac{1}{\left( T_i - N_i - (1 + a_i) \delta N_i - T_i \sum_j a_j \delta N_j \right)! (N_i + \delta N_i)!} \tag{4}
\]
and the equations for the equilibrium particle populations, \( f_i = N_i / T_i \), are \[8\]
\[
\beta(\mu_i - \bar{\epsilon}_i) + \ln \left[ \frac{1 - f_i}{f_i} \right]^{1-a_i} = - \sum_j T_j a_{ji} \ln[1 - f_j]. \tag{5}
\]

I will show below that systems of interacting particles may be described as ideal FES systems. For interacting bosons, more natural is the Bose perspective, whereas for interacting fermions it is more convenient to employ the Fermi perspective.

Equations (3) and (5) can be readily transformed into integral equations in the quasicontinuous case. If, instead of the index \( i \) (or \( j \)), we introduce the quasicontinuous variable \( i \) (or \( j \)—which may be a multidimensional variable, like the quasimomentum, or a one-dimensional (1D) variable, like the quasienergy—of density of states (DOS) \( \sigma(\bar{\epsilon}) \), then equations (3) and (5) become \[8\]
\[
\beta(\mu_i - \bar{\epsilon}_i) + \ln \left[ \frac{1 + b_i^{1-a_i}}{b_i} \right] = \int \sigma(j) \ln[1 + b_j] \delta_{ij} \tag{6}
\]
and
\[
\beta(\mu_i - \bar{\epsilon}_i) + \ln \left[ \frac{1 - f_i}{f_i} \right] = \int \sigma(j) \ln[1 - f_j] \delta_{ij} \tag{7}
\]
respectively.

The thermodynamics of different FES systems have been calculated by several authors (see, e.g., \[2, 5, 8, 12–22\]).

2. The quasiparticles

2.1. The ideal gas description

Suppose that we have a system of interacting particles that we wish to describe as an ideal gas of quasiparticles. For this, we introduce the quasiparticle energies, \( \{\bar{\epsilon}_i\} \), which we want to satisfy three conditions, specific to ideal gases:

1. \( E(\{\bar{\epsilon}_i\}) = \sum_i n_i \bar{\epsilon}_i \), \( \tag{8a} \)
2. the energies \( \{\bar{\epsilon}_i\} \) are well defined and therefore independent of the set of occupation numbers, \( \{n_i\} \), \( \tag{8b} \)
3. the equilibrium populations, \( \langle n_i \rangle (T, \mu, \bar{\epsilon}_i) \), are functions of only \( T, \mu \) and \( \bar{\epsilon}_i \) and are independent of the populations. \( \tag{8c} \)

The heat capacity of any system (in units of \( k_B \)) is
\[
\frac{C_V}{k_B} = \left( \frac{\partial U}{\partial T} \right)_N = \left( \frac{\partial U}{\partial \mu} \right)_T \left( \frac{\partial N}{\partial \mu} \right)_T^{-1} \tag{9}
\]
If conditions (8) are satisfied, then
\[
\left( \frac{\partial U}{\partial (k_B T)} \right)_\mu = \sum_i \bar{\epsilon}_i \left( \frac{\partial \langle n_i \rangle}{\partial (k_B T)} \right)_\mu, \tag{10a}
\]
\[
\left( \frac{\partial U}{\partial \mu} \right)_T = \sum_i \bar{\epsilon}_i \left( \frac{\partial \langle n_i \rangle}{\partial \mu} \right)_T, \tag{10b}
\]
\[
\left( \frac{\partial N}{\partial \mu} \right)_T = \sum_i \left( \frac{\partial \langle n_i \rangle}{\partial \mu} \right)_T, \tag{10c}
\]
where we have used the notation \( U(T, \mu) \equiv \langle E \rangle_T,\mu \) for the internal energy of the system, at temperature \( T \) and chemical potential \( \mu \).

For an ideal gas of fermions in the quasicontinuous limit and with a DOS of the form \( \sigma(\bar{\epsilon}) \equiv C \bar{\epsilon}^2 \), with \( C \) and \( s \) being two constants \[23, 24\], the total particle number and the internal energy are
\[11a\]
\[11b\]
respectively, where the function \( \text{Li}_n \) is the polylogarithmic function of order \( n \) \[25, 26\].

Using equation (10) we calculate the specific heat (also in units of \( k_B \)):
\[
\frac{c_V}{k_B} \equiv \frac{C_V}{k_B N} = \left[ (s + 1)(s + 2) \frac{\text{Li}_{s+1}(\bar{\epsilon}^2)-\text{Li}_{s+2}(\bar{\epsilon}^2)}{\text{Li}_{s+1}(\bar{\epsilon}^2)} \right], \tag{11c}
\]
In the low-temperature limit \( \beta \mu \gg 1 \) and in the lowest orders of approximation, equations (11) become
\[12a\]
\[12b\]
\[12c\]
where equation (12a) defines the Fermi energy, \( \bar{\epsilon}_F = [(s + 1)N/C]^{\frac{1}{s+1}} \), and equation (12c) may be put into the low-temperature universal form \[19, 27\]
\[
\frac{C_V}{k_B} \equiv \frac{N C_V}{k_B} = \frac{\pi^2}{3} k_B T \tilde{\sigma}(\bar{\epsilon}_F). \tag{13}
\]


2.2. The quasiparticle gas

The gases of quasiparticles used for describing systems of interacting particles do not satisfy, in general, conditions (8). An example shown in [28] is Landau’s Fermi liquid theory (FLT). The quasiparticle energy, $\tilde{\epsilon}_i$, depends on the occupation of the other quasiparticle states, i.e., $\tilde{\epsilon}_i \equiv \tilde{\epsilon}_i ([n_j])$, where $\{n_j\}$ denotes the set of all occupation numbers. In such a case condition (8b) is not satisfied and through $\tilde{\epsilon}_i$, the population $\{n_i\}(T, \mu, \tilde{\epsilon}_i)$ depends on the populations of the other quasiparticle levels, violating condition (8c) as well. Moreover, in the FLT the sum of the energies of the quasiparticles is not equal to the total energy of the system, violating also condition (8a) (see [28]).

Let us now see how the heat capacity of the system can be calculated. For this I will suppose that condition (8a) is true; otherwise the gas of quasiparticles may not be used for this purpose [28].

If $\tilde{\epsilon}_i$ is a function of $\{n_j\}$, then equations (10a) and (10b) are not valid because $\tilde{\epsilon}_i$ varies with $T$ and $\mu$:

$$\frac{\partial \tilde{\epsilon}_i(T, \mu)}{\partial (k_B T)} = \sum_j \frac{\partial \tilde{\epsilon}_i(T, \mu) \partial (n_j)(T, \mu)}{\partial (k_B T)} .$$  \hfill (14a)

$$\frac{\partial \tilde{\epsilon}_i(T, \mu)}{\partial \mu} = \sum_j \frac{\partial \tilde{\epsilon}_i(T, \mu) \partial (n_j)(T, \mu)}{\partial \mu} .$$  \hfill (14b)

On the other hand, the derivatives of the populations are given by the equations

$$\frac{\partial (n_j)(T, \mu)}{\partial (k_B T)} = \frac{\partial (n_j)(T, \mu, \tilde{\epsilon}_i)}{\partial (k_B T)} + \frac{\partial (n_j)(T, \mu, \tilde{\epsilon}_i) \partial \tilde{\epsilon}_i(T, \mu)}{\partial (k_B T)} .$$  \hfill (15a)

$$\frac{\partial (n_j)(T, \mu)}{\partial \mu} = \frac{\partial (n_j)(T, \mu, \tilde{\epsilon}_i)}{\partial \mu} + \frac{\partial (n_j)(T, \mu, \tilde{\epsilon}_i) \partial \tilde{\epsilon}_i(T, \mu)}{\partial \mu} .$$  \hfill (15b)

Plugging equations (14) into (15) we obtain a self-consistent system of equations for the variation of the occupation numbers with $T$ and $\mu$.

Using the solutions to equation (15) together with equation (14), we can calculate $\partial U/\partial (k_B T)$, $\partial U/\partial \mu$, $\partial N/\partial (k_B T)$ and $\partial N/\partial \mu$ to finally obtain the heat capacity (9), where equations (10a) and (10b) become

$$\left( \frac{\partial U}{\partial (k_B T)} \right)_T = \sum_i \left[ \tilde{\epsilon}_i \left( \frac{\partial (n_j)}{\partial (k_B T)} \right)_T \right] .$$  \hfill (16a)

$$\left( \frac{\partial U}{\partial \mu} \right)_T = \sum_i \left[ \tilde{\epsilon}_i \left( \frac{\partial (n_j)}{\partial \mu} \right)_T \right] .$$  \hfill (16b)

3. The ideal FES gas

Assuming that we choose the quasiparticle energies, $\tilde{\epsilon}_i$, in such a way that condition (8a) is satisfied, let us see how we can also satisfy conditions (8b) and (8c).

The principle of the method is given in [7, 29, 30]. I change to the quasicontinuous description, $i \rightarrow i_1$, and I assume for simplicity that $i$ is a 1D quantity (e.g. the energy of the free particles, $\epsilon$ [7, 29, 30]). The quasiparticle energy, $\tilde{\epsilon}_i([n_j])$, is then a functional of the occupation numbers, $\{n_j\}$, or for the equilibrium distribution, it is a functional of the populations, $\tilde{\epsilon}_i([n_j])$.

I choose $i$ and $\tilde{\epsilon}_i$ in such a way that if $i \leq j$, then $\tilde{\epsilon}_i([n_k]) \lesssim \tilde{\epsilon}_i([n_k])$. In this way I establish a bijective correspondence between $i$ and $\tilde{\epsilon}_i$, which one may invert and write $\tilde{\epsilon}_i([n_j])$ or $\tilde{\epsilon}_i([n_j])$.

If $\sigma (i)$ is the DOS in the variable $i$, then

$$\tilde{\sigma} (\tilde{\epsilon}) = \sigma (i) \left| \frac{d \tilde{\epsilon}}{d \epsilon} \right| \equiv \sigma (i) \left| \frac{d \tilde{\epsilon}^{-1}}{d \epsilon} \right| ,$$  \hfill (17)

where $\tilde{\sigma} (\tilde{\epsilon})$ is a functional of $\{n_j\}$. With the aid of $\sigma (i)$ and $\tilde{\sigma} (\tilde{\epsilon})$ we define the particle densities, $\rho (i) = \sigma (i) [n_j]$ and $\tilde{\rho} (\tilde{\epsilon}) = \tilde{\sigma} (\tilde{\epsilon}) [n_j]$.

Now I transform the quasiparticle gas into an ideal gas by simply changing the perspective: the usual perspective is to view the quantum numbers of the ideal gas, $i$, as fixed and the quasiparticle energies, $\tilde{\epsilon}_i([n_j])$, as functionals of the populations (figure 1(a)). When we invert the perspective, we view the quasiparticle energies, $\tilde{\epsilon}_i$, as fixed and the free particle quantum numbers as functionals of the populations, $i [\tilde{n}_j]$ (figure 1(b)).
From relation (17) and since $\sigma(i)$ is a quantity that is fixed by the properties of the single-particle states, $\tilde{\sigma}(\tilde{\epsilon})$ becomes a functional of the particle density, $\tilde{\rho}(\tilde{\epsilon})$. This property ensures FES.

To show this I coarse-grain both the axes, $i$ and $\tilde{\epsilon}$. Each interval, $[m_i, m_{i+1}]$ on the $i$-axis or $[\epsilon_m, \epsilon_{m+1}]$ on the $\tilde{\epsilon}$-axis, represents a species, with $G_m = \sigma(m_i)(m_{i+1} - m_i) = \tilde{\sigma}(\epsilon_m)(\epsilon_{m+1} - \epsilon_m)$ and $N_m = \rho(m_i)(m_{i+1} - m_i) = \tilde{\rho}(\epsilon_m) \times (\epsilon_{m+1} - \epsilon_m)$ (assuming that the intervals are small enough to use the linear approximation); I use the letters $m$ and $n$ to designate the species.

In the FES perspective, the insertion of $\delta N_m$ particles in the species $m$ causes a change of the interval $[m_i, m_{i+1}]$ on the $i$-axis by

$$\delta N_m \left[ \frac{\delta n_{i+1}}{\delta \rho(\epsilon_m)} - \frac{\delta n_i}{\delta \rho(\epsilon_m)} \right] = \delta N_m (m_{i+1} - m_i) \frac{d}{d \tilde{\epsilon}} \left[ \frac{\delta \rho(\epsilon_m)}{\delta \rho(\epsilon_m)} \right].$$

(18)

where by $\delta n_i/\delta \rho(\epsilon_m)$ I denote the functional derivative, which I assume to be analytic.

A change in the interval $[m_i, m_{i+1}]$ leads to a change in the number of states in the species $n$ by

$$\delta G_n = \delta N_m \left[ \frac{\delta n_{i+1}}{\delta \rho(\epsilon_m)} - \frac{\delta n_i}{\delta \rho(\epsilon_m)} \right] \frac{d}{d \tilde{\epsilon}} \left[ \frac{\delta \rho(\epsilon_m)}{\delta \rho(\epsilon_m)} \right] \sigma(i_n).$$

$$= \delta N_m (m_{i+1} - m_i) \frac{d}{d \tilde{\epsilon}} \left[ \frac{\delta \rho(\epsilon_m)}{\delta \rho(\epsilon_m)} \right] \sigma(i_n).$$

(19)

Mapping back $i$ onto $\tilde{\epsilon}$, I can express the change in the number of states in the species $n$ as

$$\delta G_n = \delta N_m (\epsilon_{m+1} - \epsilon_m) \frac{d}{d \epsilon} \left[ \frac{\delta \sigma(\tilde{\epsilon})}{\delta \rho(\epsilon_m)} \right] \equiv -\alpha_{\epsilon_m} \delta N_m.$$  

(20)

where I denote by $\delta \epsilon_m/\delta \rho(\epsilon_m)$ the functional derivative of $\epsilon_m$ with respect to the variation of $\delta \rho(\epsilon_m)$, where $i$ is fixed—this calculation trick does not change the noninteracting character of the quasiparticles.

The last part of equation (20) gives us the FES parameter, $\alpha_{\epsilon_m}$. We note that $\alpha_{\epsilon_m} \propto (\epsilon_{m+1} - \epsilon_m) = G_m/\sigma(\tilde{\epsilon})$, so it is proportional to the dimension of the species that it acts upon, in accordance with the ansatz (1) [6, 8] and with the general FES rules of [9].

If the function $[\delta \epsilon_m/\delta \rho(\epsilon_m)]\sigma(\epsilon_m)$ is singular at some point, $\tilde{\epsilon}_n$, then at that point $\alpha_{\epsilon_m} = 0$ will not be proportional to the dimension of the species $G_m$ and might even disobey the ansatz (1) proposed in the introduction, but it should still obey the rules of [9]. Such a case was discussed in [7, 29], in relation to the FLT.

Once the FES parameters are known, the thermodynamics follows according to the formalism outlined in the introduction.

4. Conclusions

I showed that if a system of interacting particles is described as a gas of quasiparticles of energies, $\tilde{\epsilon}(n_i)$, such that $E(n_i) = \sum n_i \tilde{\epsilon}_i$ (condition (8a)), then the gas of quasiparticles may be viewed as an ideal gas that obeys FES. If condition (8a) is not satisfied, as it happens with Landau’s quasiparticles in the Fermi liquid theory, then the quasiparticle gas cannot be used for the calculation of the thermodynamic properties of the original interacting particle gas, except in some trivial cases.

The general method for calculating the FES parameters of the quasiparticle gas is also provided.

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