A wavelet uncertainty principle in quantum calculus

Sabrine Arfaoui

Laboratory of Algebra, Number Theory and Nonlinear Analysis, LR18ES15, Department of Mathematics, Faculty of Sciences, 5000 Monastir, Tunisia.
& Department of Mathematics, Faculty of Sciences, University of Tabuk, Saudi Arabia.

Maryam G. Alshehri

Laboratory of Algebra, Number Theory and Nonlinear Analysis, LR18ES15, Department of Mathematics, Faculty of Sciences, 5000 Monastir, Tunisia.
& Department of Mathematics, Faculty of Sciences, University of Tabuk, Saudi Arabia.

Anouar Ben Mabrouk

Department of Mathematics, Higher Institute of Applied Mathematics and Computer Science, Street of Assad Ibn Alfourat, 3100 Kairouan, Tunisia.
& Laboratory of Algebra, Number Theory and Nonlinear Analysis, LR18ES15, Department of Mathematics, Faculty of Sciences, 5000 Monastir, Tunisia.
& Department of Mathematics, Faculty of Sciences, University of Tabuk, Saudi Arabia.

Abstract

In the present paper, a new uncertainty principle is derived for the generalized q-Bessel wavelet transform studied earlier in [55]. In this paper, an uncertainty principle associated with wavelet transforms in the q-calculus framework has been established. A two-parameters extension of the classical Bessel operator is applied to generate a wavelet function which is exploited next to explore a wavelet uncertainty principle already in the q-calculus framework.

Key words: q-calculus; q-Bessel function; q-wavelets; q-uncertainty principle.
PACS: 42C40; 33C10.

* Corresponding author.

Email addresses: sabrine.arfaoui@issatm.rnu.tn (Sabrine Arfaoui),
1 Introduction

Wavelet analysis also called Wavelet theory has attracted much attention recently in different domains such as mathematics, quantum physics, electrical engineering, seismic geology. Interchanges between these fields during the last ten years have led to many new wavelet applications in astronomy, acoustics, nuclear engineering, sub-band coding, signal and image processing, neurophysiology, music, magnetic resonance imaging, speech discrimination, optics, fractals, turbulence, earthquake-prediction, radar, human vision, and pure mathematics applications such as solving partial differential equations.

Like Fourier analysis wavelet analysis deals with expansions of functions in terms of a set of basis functions. Unlike Fourier analysis, wavelet analysis expands functions not in terms of trigonometric polynomials but in terms of wavelets which are a family of functions obtained from one function known as the mother wavelet, by translations and dilations. This tool permits the representation of $L^2$-functions in a basis well localized in time and in frequency. The fundamental idea behind wavelets is to analyze according to scale. Indeed, some researchers in the wavelet field feel that, by using wavelets, one is adopting a whole new mindset or perspective in processing data.

Hence, wavelets are special functions characterized by special properties that may not be satisfied by other functions.

Bessel functions form an important class of special functions and are applied almost everywhere in mathematical physics. The Bessel function was the result of Bessels study of a problem of Kepler for determining the motion of three bodies moving under mutual gravitation. In 1824, he incorporated Bessel functions in a study of planetary perturbations where the Bessel functions appear as coefficients in a series expansion of the indirect perturbation of a planet, that is the motion of the Sun caused by the perturbing body. It was likely Lagrange’s work on elliptical orbits that first suggested to Bessel to work on the Bessel functions. They are also known as cylindrical functions, or cylindrical harmonics, because of their strong link to the solutions of the Laplace equation in cylindrical coordinates.

One of the interesting fields of extensions of Fourier and wavelet analyses is the so-called $q$-theory which is an important sub-field in harmonic analysis and which provides some discrete and/or some refinement of continuous harmonic analysis in sub-spaces such as $\mathbb{R}_q$ composed of the discrete grid $\pm q^n$, $n \in \mathbb{Z}$, $q \in (0, 1)$. Recall that for all $x \in \mathbb{R}^*$ there exists a unique $n \in \mathbb{Z}$ such that $q^{n+1} < |x| \leq q^n$ which guarantees some density of the set $\mathbb{R}_q$ in $\mathbb{R}$.

mgalshehri@ut.edu.sa (Maryam G. Alshehri), anouar.benmabrouk@fsm.rnu.tn (Anouar Ben Mabrouk).
Many special functions have been shown to admit generalizations to a base \( q \), and are usually reported as \( q \)-special functions. Interest in such \( q \)-functions is motivated by the recent and increasing relevance of \( q \)-analysis in exactly solvable models in statistical mechanics. Like ordinary special functions, \( q \)-analogues satisfy second order \( q \)-differential equations and various identities of recurrence relations. Basic analogues of Bessel function have been introduced by Jackson and Swarttow as \( q \)-generalizations of the power series expansions.

In the present context, We aim to apply the generalized \( q \)-Bessel and associated \( q \)-wavelets introduced in the context of \( q \)-theory to develop new uncertainty principle based on wavelets in the framework of \( q \) (quantum)-calculus. We serve of the new \( q \)-wavelets developed recently in [55] to develop a new \( q \)-wavelet uncertainty principle.

The present paper is organized as follows. In section 2, a brief review focusing on the last developments in \( q \)-wavelets’ theory is developed. Section 3 is concerned to a review on the uncertainty principle. In section 4, we present our main results. We precisely introduce the new generalized \( q \)-wavelet uncertainty principle. Section 5 is concerned with the proofs of our main results. Finally section 6 presents our conclusion.

2 \( q \)-calculus toolkit

The present section aims to recall the basic tools in \( q \)-calculus. For \( 0 < q < 1 \), denote

\[ \mathbb{R}_q = \{ \pm q^n, n \in \mathbb{Z} \}, \quad \mathbb{\bar{R}}_q = \mathbb{R}_q \cup \{0\}, \quad \mathbb{R}_q^+ = \{ q^n, n \in \mathbb{Z} \} \text{ and } \mathbb{\bar{R}}_q^+ = \mathbb{R}_q^+ \cup \{0\}. \]

On \( \mathbb{\bar{R}}_q^+ \), the \( q \)-Jackson integrals on \([a, b]\) \((a \leq b \text{ in } \mathbb{\bar{R}}_q^+)\) is defined by

\[ \int_a^b f(x) d_q x = (1 - q) \sum_{n \in \mathbb{Z}} q^n (bf(q^n) - af(q^n)) \]

and for \([0, +\infty)\) it is defined by

\[ \int_0^\infty f(x) d_q x = (1 - q) \sum_{n \in \mathbb{Z}} f(q^n) q^n \]

provided that the sums converge absolutely. On \([a, +\infty)\), we have to apply an analogue of Chasle’s rule and thus get

\[ \int_a^{+\infty} f(x) d_q x = \int_0^{+\infty} f(x) d_q x - \int_0^a f(x) d_q x. \]
This leads next to define the functional space
\[ \mathcal{L}_{q,p,v}(\mathbb{R}_q) = \{ f : \mathbb{R}_q \to \mathbb{C}, \text{ even ; } \| f \|_{q,p,v} < \infty \}, \]
where \( \| . \|_{q,p,v} \) is the norm defined analogously by
\[ \| f \|_{q,p,v} = \left[ \int_0^\infty |f(x)|^p x^{2|v|+1} d_q x \right]^\frac{1}{p}. \]
where \( v \) is a fixed real parameter such that \( 2v > -1 \). Denote next, \( C^0_q(\mathbb{R}_q^+) \) the space of functions defined on \( \mathbb{R}_q^+ \), continuous at 0 and vanishing at \( +\infty \), equipped with uniform norm
\[ \| f \|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)| < \infty. \]
Finally, \( C^b_q(\mathbb{R}_q^+) \) designates the space of functions that are continuous at 0 and bounded on \( \mathbb{R}_q^+ \). Recall here that \( q^n \to \infty \) as \( n \to -\infty \) in \( \mathbb{Z} \). The \( q \)-derivative of a function \( f \in \mathcal{L}_{q,p,v}(\mathbb{R}_q^+) \) is defined by
\[ D_q f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ f'(0), & \text{else.} \end{cases} \]
In \( q \)-theory, we may also have an analogue of the change variables theorem. Particularly,
\[ \int_0^\infty f(t)t^{2|v|+1} d_q t = \frac{1}{a^{2v+2}} \int_0^\infty f(\frac{x}{a})x^{2v+1} d_q x. \] (1)
The \( q \)-shifted factorials are defined by
\[ (a,q)_0 = 1, \quad (a,q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a,q)_\infty = \prod_{k=0}^{+\infty} (1 - aq^k). \]

3 A literature review on the uncertainty principle

The uncertainty principle has been introduced at the beginning of the early 20th years of the last century by Heisenberg, and it takes consequently the name of Heisenberg uncertainty principle ([22,23]. Mathematically speaking, the uncertainty principle formula states that a non-zero function and its Fourier transform cannot be both sharply localized. It is thus a consequence of Fourier transform. It mathematically explains the functional infinite dimensionality property (See [10]). In physics, the uncertainty principle states that the determination of positions by performing measurement on the system disturbs it.
sufficiently to make the determination of momentum imprecise and vice-versa.
In quantum mechanics, it may be explained by the existence of a fundamental
limit to the precision with which it is possible to simultaneously know the
position and the momentum of the particle. The uncertainty principle inter-
acts noadays with many fields such as pure mathematics, physics, engineering,
communication, quantum mechanics.

The literature on the uncertainty principle is sure wide, and studies investigat-
ing such concept are multidisciplinary. Exploitations and also generalizations
of the uncertainty principle have been developed. The recent variants are those
based on wavelet theory. Our present work lies in theis last topic, and con-
stitutes a contribution in the uncertainty principle based on Bessel theory,
wavelet theory and quantum calculus.

In [2] established a Pitt’s and Beckner logarithmic uncertainty inequalities
using Fourier transform. In [3,30], Fourier-Riemann–Liouville operator has
been applied to derive an entropy based uncertainty principle and a Heisen-
berg–Pauli–Weyl inequality. Rachdi and collaborators derived two types of
uncertainty principle such as Heisenberg-Pauli-Weyl and Beurling-Hörmander
in [47,48], by applying Fourier and spherical mean operators.

In wavelet theory framework, Rachdi and Mehrzi have derived a Heisenberg’s
uncertainty principle by combining the continuous wavelet transform with
spherical mean operators [52]. Extensions of this result have been developed
in [53,54]. A continuous shearlet uncertainty principle has been investi-
gated in [9] provided with the computation of some minimizers of such uncertain-
ty principle.

As an extension of wavelet theory, and harmonic analysis in general, Clifford
wavelet analysis has taken place especially in the last decades. The Heisen-
berg uncertainty principle and its different variants and extensions have been
investigated in the large field of Clifford analysis. In [13], the authors investig-
gated a quaternionic linear transform and combined it with quaternion Fourier
transform to derive different uncertainty principles such as Heisenberg-Weyls,
Hardys, Beurlings and logarithmic ones. El-Houi et al established in [14] some
Clifford uncertainty inequalities such as Hausdorf-Young, and Donoh-Stark
uncertainty principles.

Feichtinger and Grochenig applied the Heisenberg uncertainty principle dut
to Gabor wavelets and short time Fourier tranform for the control of signals
[15]. Hitzer al introduced in [25] a general formula for the uncertainty prin-
ciple based on the local Clifford geometric algebra wavelet transform. In [26]
the same author derived a new directional uncertainty principle for quater-
nion valued functions based on Clifford quaternion Fourier transform. In [28],
Clifford-Fourier transform has been applied to establish an uncertainty prin-
ciple in the case of Clifford wavelets. Such a principle has been shown to be useful in signal processing. See also [27,36,37,38,39,40,41,42,43,44] for the same team of researchers and for eventual extensions of their works.

Ma and Zhao applied quaternion ridgelet transform and curvelet transform to derive an associated uncertainty principles [34]. In [33] a quaternionic linear canonical transform has been applied to derived an uncertainty principle for vector-valued signals. The authors showed that Gaussian quaternion signals are the only minimizers for the uncertainty principle formula.

In [45], a hybrid method combing continuous wavelet transform and spherical mean operators has been used for both Donoho-Stark and Benedicks-type uncertainty principles. Mejjaoli et al proved in [46] an inversion formula due to the Dunkl Gabor transform. Many versions of the uncertainty principle has been shown such as Heisenberg, Donoho–Stark’s, local Cowling–Price’s, and Faris-Price’s uncertainty principles.

In [60] a stronger uncertainty principles has been derived in terms of covariance and absolute covariance based on Fourier transform for vector-valued signals. Yang and Kou in [61] extended the uncertainty principle to the case of hypercomplex signals by means of the linear canonical transform. Besides, they proved that Gaussian signals are the only minimizers.

In $q$-theory, uncertainty principle has been the subject of several studies, in both Fourier and wavelet frameworks. In [18], the authors developed a general $q$-Heisenberg uncertainty principle for the $q$-Fourier-cosine/sine and the $q$-Bessel-Fourier transform. In [19], a heisenberg-Weyl type uncertainty principle has been shown for a $q$-Dunkl transform. Besides, a sharp $q$-Bessel-Dunkl uncertainty principle has been proved in [20] generalizing the case of basic Bessel transform. In [21], the $q$-Dunkl Transform on the Real Line has been applied to derive an $L_p$-version of the Hardy Uncertainty Principle.

In [29], Weinstein operator has been applied to introduce a new wavelet transform, which is used by the next to derive a Heisenberg uncertainty principle. Nemri in [50] investigated an extension of the Donoho-Stark’s uncertainty principle for a special class of Fourier multiplier operators. Ogawa et al ([51]) applied the Boltzmann entropy functional to derive a Heisenberg-type uncertainty principle. Saoudi et al ([56]) proved a Heisenberg-Pauli-Weyl uncertainty principle, and Donho-Stark’s uncertainty principle associated to the Weinstein $L_2$-multiplier operators.

The uncertainty principle in its classical or original form may be explained by the following result, which will be the starting step to our extension in the present work.

**Theorem 3.1** ([6,7,59]) Let $A$ and $B$ be two self-adjoint operators on a
Hilbert space \( X \) with domains \( \mathcal{D}(A) \) and \( \mathcal{D}(B) \) respectively and consider their commutator \( [A, B] = AB - BA \). Then

\[
\|Af\|_2 \|Bf\|_2 \geq \frac{1}{2} |<[A, B] f, f >|, \forall f \in \mathcal{D}([A, B]). \tag{2}
\]

Consider next the special case

\[ A_k f(x) = x_k f(x) \quad \text{and} \quad B_k f(x) = \partial_{x_k} f(x), \quad k = 1, 2, \ldots, n. \]

By applying Theorem 3.1 above, we get the following result.

**Corollary 3.1** ([6,7])

\[
\|A_k f\|_2 \|B_k f\|_2 \geq \frac{1}{2} |<[A_k, B_k] f, f >|.
\]

Furthermore,

\[
\|x_k f\|_2 \|\xi_k \hat{f}\|_2 \geq \frac{1}{2} \|f\|_2^2. \tag{3}
\]

For more backgrounds on the uncertainty principle, its variants, Fourier and wavelet transforms on the Euclidean space \( \mathbb{R}^n \), the readers may be referred also to [32], [49], [57], [58], [61].

**4 Main results**

In [55], a general two-parameters \( q \)-wavelet analysis has been developed generalizing thus quasi all the previous versions of \( q \)-wavelets. The idea is based on [12], where a two-parameters \( q \)-theory leading to a two-parameters \( q \)-Bessel function has been introduced provided with suitable associated Fourier transform. The authors in [55] exploited such a \( q \)-version of the Bessel function to construct \( q \)-wavelets and develop an associated \( q \)-wavelet analysis. The parameter \( v \) in the classical \( q \)-Bessel theory is replaced by a couple of parameters \( v = (\alpha, \beta) \in \mathbb{R}^2, \alpha + \beta > -1 \), for which a modified functional space is associated with suitable integrating measure. For \( 1 \leq p < \infty \), we denote

\[
\mathcal{L}_{q, p, v}(\mathbb{R}^n_+) = \left\{ f : \|f\|_{q, p, v} = \left[ \int_0^\infty |f(x)|^p x^{2|v|+1} d_q x \right]^{1/p} < \infty \right\}.
\]

where \( |v| = \alpha + \beta \). The generalized \( q \)-Bessel operator is defined by

\[
\tilde{\Delta}_{q, v} f(x) = \frac{f(q^{-1} x) - (q^{2\alpha} + q^{2\beta}) f(x) + q^{2\alpha+2\beta} f(q x)}{x^2}, \quad \forall x \neq 0.
\]
The normalized $q$-Bessel function is given by

\[
j_{\alpha}(x, q^2) = \sum_{n \geq 0} (-1)^n \frac{q^{n(n+1)}}{(q^{2n+2}, q^2)_n (q^2, q^2)_n} x^{2n}.
\]

We denote finally $\tilde{j}_{q,v}(x, q^2)$ the modified version of $q$-Bessel function is

\[
\tilde{j}_{q,v}(x, q^2) = x^{-2\beta} j_{\alpha-\beta}(q^{-\beta} x, q^2).
\]

More details may be found in [12,55]. The generalized $q$-Bessel Fourier transform $F_{q,v}$ is defined by

\[
F_{q,v} f(x) = c_{q,v} \int_{0}^{\infty} f(t) \tilde{j}_{q,v}(tx, q^2) t^{2|v|+1} dt, \quad \forall f \in L_{q,p,v}(\mathbb{R}^+_q),
\]

where $c_{q,v}$ is a suitable constant. This has induced a generalized $q$-Bessel wavelet already as an even function $\Psi \in L_{q,2,v}(\mathbb{R}^+_q)$ satisfying an analogue admissibility condition as for the existing cases with suitable and necessary modifications. The associated translation operator has been already defined in [12] by

\[
T_{q,x}^v f(y) = c_{q,v} \int_{0}^{\infty} F_{q,v} f(t) \tilde{j}_{q,v}(yt, q^2) \tilde{j}_{q,v}(xt, q^2) t^{2|v|+1} dt.
\]

**Definition 4.1** [55] A generalized $q$-Bessel wavelet is an even function $\Psi \in L_{q,2,v}(\mathbb{R}^+_q)$ satisfying the following admissibility condition:

\[
C_{v,\Psi} = \int_{0}^{\infty} |F_{q,v} \Psi(a)|^2 \frac{da}{a} < \infty.
\]

The continuous generalized $q$-Bessel wavelet transform of a function $f \in L_{q,2,v}(\mathbb{R}^+_q)$ at the scale $a \in \mathbb{R}^+_q$ and the position $b \in \mathbb{R}^+_q$ is defined by

\[
C_{q,v}^w(f)(a, b) = c_{q,v} \int_{0}^{\infty} f(x) \Psi_{(a,b),v}(x) x^{2|v|+1} dx, \quad \forall a \in \mathbb{R}^+_q, \forall b \in \mathbb{R}^+_q,
\]

where

\[
\Psi_{(a,b),v}(x) = \sqrt{a} T_{q,b}^v (\Psi_a) \quad \text{and} \quad \Psi_a(x) = \frac{1}{a^{2|v|+2}} \Psi\left(\frac{x}{a}\right).
\]

In the present work, we will exploit such wavelets to prove a $q$-variant of the uncertainty principle relative to the $q$-wavelet transform developed in [55].

Let $\Psi$ be a generalized $q$-Bessel wavelet in $L_{q,2,v}(\mathbb{R}^+_q)$ and $f \in L_{q,2,v}$. Denote for $R$ and $S$ the operators defined on $L_{q,2,v}(\mathbb{R}^+_q)$ by

\[
R f(a, b) = b C_{q,v}^w(f)(a, b) \quad \text{and} \quad S f(t) = t F_{q,v} f(t).
\]

8
Denote also \( d_q(a, b) = b^{2|v|+1} \frac{d_q a d_q b}{a^2} \).

We prove a general two-parameters \( q \)-wavelet uncertainty principle, we consider the general case of \( q \)-wavelets developed in [55]. The following \( q \)-wavelet uncertainty principle holds.

**Theorem 4.1** Bessel wavelet uncertainty principle

\[
\|f\|_{q,2,v}^2 \leq K_{q,v} \left( \int_0^\infty \int_0^\infty |\mathcal{R}f(a, b)|^2 d_q(a, b) \right) \left( \int_0^\infty |\mathcal{S}f(t)|^2 t^{2|v|+1} d_q t \right)
\]

for some constant \( C_{q,v} > 0 \) depending on \( q \) and \( v = (\alpha, \beta) \).

5 Proof of main results

To prove this result we need the following lemma.

**Lemma 5.1**

\[
\int_0^\infty \int_0^\infty \left| b \mathcal{F}_{q,v} C_{q,v}(f)(a, b) \right|^2 \frac{d_q a d_q b}{a^2} = K_{1,q,v} \left\| \xi \mathcal{F}_{q,v} f(\xi) \right\|_2^2.
\]

**Proof.** Let us express firstly the Fourier transform \( \mathcal{F}_{q,v} \psi_{a,b,v} \) by means of \( \mathcal{F}_{q,v} \psi \).

So, denote for simplicity \( \Lambda_{q,v}(t, s, \xi, b) = \tilde{j}_{q,v}(ts, q^2) \tilde{j}_{q,v}(bs, q^2) \tilde{j}_{q,v}(t\xi, q^2) \).

Observe that

\[
\mathcal{F}_{q,v} \psi_{a,b,v}(\xi) = c_{q,v} \int_0^\infty \Psi_{a,b,v}(t) \tilde{j}_{q,v}(t\xi, q^2) t^{2|v|+1} d_q t
\]

\[
= c_{q,v} \sqrt{\alpha} \int_0^\infty \int_0^\infty \mathcal{F}_{q,v} \psi(t, \xi, q^2, s^2) \Lambda_{q,v}(t, s, \xi, b) (st)^{2|v|+1} d_q s d_q t
\]

\[
= c_{q,v} \sqrt{\alpha} \int_0^\infty \int_0^\infty \mathcal{F}_{q,v} \psi(\alpha s, \xi) \tilde{j}_{q,v}(bs, q^2) s^{2|v|+1} d_q s
\]

\[
= \sqrt{\alpha} \int_0^\infty \mathcal{F}_{q,v} \psi(a \xi) \tilde{j}_{q,v}(b\xi, q^2).
\]

As a consequence, we get

\[
C_{q,v}(f)(a, b) = \sqrt{\alpha} \mathcal{F}_{q,v}^{-1} \left[ \mathcal{F}_{q,v} \psi(a \cdot) \mathcal{F}_{q,v} f(.) \right] (b)
\]
and
\[ \mathcal{F}_{q,v}^{w}C_{q,v}(f)(a,b) = \sqrt{a} [\mathcal{F}_{q,v}^{w}(a) \mathcal{F}_{q,v}(.)] (b). \]

Therefore,
\[
\int_0^\infty \left| b \mathcal{F}_{q,v}^{w}C_{q,v}(f)(a,b) \right|^2 \, dq_a \, dq_b \\
= \int_0^\infty \left| b \sqrt{a} [\mathcal{F}_{q,v}^{w}(a) \mathcal{F}_{q,v}(.)] (b) \right|^2 \, dq_a \, dq_b \\
= \int_0^\infty \left| \sqrt{a} \mathcal{F}_{q,v}^{w}(a) \right|^2 |b \mathcal{F}_{q,v}(f)|^2 \, dq_a \, dq_b.
\]

It follows that
\[
\int_0^\infty \int_0^\infty \left| b \mathcal{F}_{q,v}^{w}C_{q,v}(f)(a,b) \right|^2 \, dq_a \, dq_b \frac{d_q a d_q b}{a^2} \\
= \int_0^\infty \int_0^\infty \left| \sqrt{a} \mathcal{F}_{q,v}^{w}(a) \right|^2 |b \mathcal{F}_{q,v}(f)|^2 \, dq_a \, dq_b \frac{d_q a d_q b}{a^2} \\
= C_{v,\Psi} \int_0^\infty |b \mathcal{F}_{q,v}(f)|^2 \, dq_b \\
= C_{v,\Psi} \| \xi \mathcal{F}_{q,v}(\xi) \|_{q,v}^2.
\]

For the next, we will use the following result which shows a Plancherel/Parseval type rule for the generalized \( q \)-Bessel wavelet transform (see [55]).

**Lemma 5.2** [55] Let \( \Psi \) be a generalized \( q \)-Bessel wavelet in \( \mathcal{L}_{q,2,v}(\mathbb{R}^+_q) \). Then we have, \( \forall f \in \mathcal{L}_{q,2,v}(\mathbb{R}^+_q) \),
\[
\frac{1}{C_{v,\Psi}} \int_0^\infty \int_0^\infty |C_{q,v}(f)(a,b)|^2 b^{2|v|+1} \frac{d_q a d_q b}{a^2} = \| f \|_{q,v}^2.
\]

**Proof.** We have
\[
q^{4|v|+2} \int_0^\infty \int_0^\infty |C_{q,v}(f)(a,b)|^2 b^{2|v|+1} \frac{d_q a d_q b}{a^2} \\
= q^{4|v|+2} \int_0^\infty \left( \int_0^\infty |\mathcal{F}_{q,v}(f)(x)|^2 |\mathcal{F}_{q,v}(\overline{\Psi})|^2 (x)^2 x^{2|v|+1} \, dq_x \right) \frac{d_q a}{a} \\
= \int_0^\infty |\mathcal{F}_{q,v}(f)(x)|^2 \left( |\mathcal{F}_{q,v}(\Psi)(ax)|^2 \frac{d_q a}{a} \right) x^{2|v|+1} \, dq_x \\
= C_{v,\Psi} \int_0^\infty |\mathcal{F}_{q,v}(f)(x)|^2 x^{2|v|+1} \, dq_x \\
= C_{v,\Psi} \| f \|_{q,v}^2.
\]

**Proof of Theorem 4.1.** Observe firstly that
\[
\| bC_{q,v}(f)(a,b) \|_{q,v} \| b\mathcal{F}_{q,v}C_{q,v}(f)(a,b) \|_2 \geq \frac{1}{2} \| C_{q,v}(f)(a,b) \|_2^2.
\]
Next, it is easy to see that
\[ \| bC_{q,\Psi}^w(f)(a,b) \|_{q,2,v} = \int_0^\infty \int_0^\infty |\mathcal{R}f(a,b)|^2 d_q(a,b). \]

On the other hand, from Lemma 5.1, we get
\[ \int_0^\infty |\mathbf{S}f(t)|^2 t^{2|v|+1} d_q t = \| b\mathcal{F}_{q,v}C_{q,\Psi}^w(f)(a,b) \|_2. \]
Therefore, we obtain
\[ \left( \int_0^\infty \int_0^\infty |\mathcal{R}f(a,b)|^2 d_q(a,b) \right) \left( \int_0^\infty |\mathbf{S}f(t)|^2 t^{2|v|+1} d_q t \right) \geq \frac{1}{2} \| C_{q,\Psi}^w(f)(a,b) \|_2^2. \]

So, finally, from Lemma 5.2, we observe that the \( q \)-wavelet transform is isometric (modulo a normalizing constant, ([55])), we get
\[ \left( \int_0^\infty \int_0^\infty |\mathcal{R}f(a,b)|^2 d_q(a,b) \right) \left( \int_0^\infty |\mathbf{S}f(t)|^2 t^{2|v|+1} d_q t \right) \geq K_{q,\Psi}^w \| f \|_{q,2,v}^2. \]

6 Conclusion

In this paper, an uncertainty principle associated to wavelet transforms in the \( q \)-calculus framework has been established. A two-parameters extension of the classical Bessel operator is applied to generate a wavelet function which is exploited next to explore a wavelet uncertainty principle already in the \( q \)-calculus framework. The result joins and extended many works in the same topic such as [11,12,55]

References

[1] A. Abouelaz, R. Daher and L. El Mehdi, Harmonic Analysis associated with the generalized \( q \)-Bessel operator, International Journal of Analysis and Applications ISSN 2291-8639, 10 (2016), pp. 17-23.

[2] Amri, B., Rachdi, L.T.: Beckner logarithmic uncertainty principle for the Riemann-Liouville operator. International Journal of Mathematics, 24(09), 1350070. doi:10.1142/s0129167x13500705 (2013)

[3] Amri, B., Rachdi, L.T.: Uncertainty Principle in Terms of Entropy for the Riemann–Liouville Operator. Bulletin of the Malaysian Mathematical Sciences Society, 39(1), 457–481. doi:10.1007/s40840-015-0121-5 (2015)
[4] M. H. Annaby and Z. S. Mansour, q-Fractional Calculus and Equations. Lecture Notes in Mathematics 2056, Editors: J.-M. Morel and B. Teissier, Springer 2012.

[5] A. Aral, V. Gupta and R. P. Agarwal, Applications of q-calculus in operator theory, Springer, New York, 2013.

[6] Banouh, H., Ben Mabrouk, A. and Kesri, M., Clifford Wavelet Transform and the Uncertainty Principle. Adv. Appl. Clifford Algebras 29, 106 2019, 23 pages. https://doi.org/10.1007/s00006-019-1026-4.

[7] Banouh, H., Ben Mabrouk, A., A sharp Clifford wavelet Heisenberg-type uncertainty principle. Journal of Mathematical Physics 61(9), 093502 (2020); https://doi.org/10.1063/5.0015989

[8] S. Bouaziz, The q-Bessel wavelet packets, Advances in Analysis, 1(1) (2016), pp. 27-39.

[9] Dahkle, S., Kutyniok, G., Maass, P., Sagiv, C., Stark, H.-G., Teschke, G.: The uncertainty principle associated with the continuous shearlet transform. International Journal of Wavelets, Multiresolution and Information Processing, 6(2), 157–181 (2008)

[10] S. Das, Assumptions in Quantum Mechanics. International Journal of Theoretical and Mathematical Physics 2013, 3(2): 53-68, DOI: 10.5923/j.ijtmp.20130302.02.

[11] L. Dhaouadi, On the q-Bessel Fourier transform, Bulletin of mathematical analysis and applications, 5(2) (2013), pp. 42-60.

[12] L. Dhaouadi and M. Hleili, Generalized q-Bessel operator, Bulletin of mathematical analysis and applications, 7(1) (2015), pp. 20-37.

[13] El Haoui, Y., Fahlaoui, S., Hitzer, E.: Generalized Uncertainty Principles associated with the Quaternionic Offset Linear Canonical Transform. arXiv:1807.04068v2 [math.CA], 15 pages (2019)

[14] El Haoui, Y., Fahlaoui, S.: Donoho-Stark’s Uncertainty Principles in Real Clifford Algebras. arXiv:1902.08465v1 [math.CA], 9 pages (2019)

[15] Feichtinger, H.G., Gröchenig, K.: Gabor Wavelets and the Heisenberg Group: Gabor Expansions and Short Time Fourier Transform from the Group Theoretical Point of View. Wavelets, 359–397. doi:10.1016/b978-0-12-174590-5.50018-6 (1992).

[16] A. Fitouhi, N. Bettaibi and W. Binous, Inversion Formulas For The q-Riemann-Liouville and q-Weyl Transforms Using Wavelets, Fractional calculus and Applied Analysis, 10(4) (2007), pp. 327-342.

[17] A. Fitouhi, K. Trimeche and J. L. Lions, Transmutation operators and generalized continuous wavelets, Preprint, Faculty of Science of Tunis (1995).

12
[18] A. Fitouhi, N. Bettaibi, R. H. Bettaieb, and W. Binous, On Some Inequalities of Uncertainty Principles Type in Quantum Calculus. International Journal of Mathematics and Mathematical Sciences, Volume 2008, Article ID 465909, 13 pages, DOI:10.1155/2008/465909.

[19] A. Fitouhi, F. Nouri, and S. Guesmi, on heisenberg and local uncertainty principles for the q-dunkl transform. J. of inequalities in pure and applied mathematics, Volume 10 (2009), Issue 2, Article 42, 10 pages.

[20] A. Fitouhi, N. Bettaibi, W. Binous, and H. Ben Elmonser, An uncertainty principle for the basic Bessel transform. Ramanujan J. 18 (2009), 171–182. DOI 10.1007/s11139-007-9117-6.

[21] A. Fitouhi, F. Nouri, and A. Safraoui, An $L_p$ Version of Hardy Uncertainty Principle for the $q$-Dunkl Transform on the Real Line. Int. Journal of Math. Analysis, 4(6) (2010), 249-257.

[22] Heisenberg, W.: Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. Zeitschrift fur Physik 43, 172–198 (1927).

[23] W. Heisenberg. Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. In Original Scientific Papers Wissenschaftliche Originalarbeiten, 478–504, Springer Berlin Heidelberg (1985)

[24] Hitzer, E.: New Developments in Clifford Fourier Transforms, in N. E. Mastorakis, P. M. Pardalos, R. P. Agarwal, L. Kocinac (eds.), Advances in Applied and Pure Mathematics, Proceedings of the 2014 International Conference on Pure Mathematics, Applied Mathematics, Computational Methods (PMAMCM 2014), Santorini Island, Greece, July 17-21, 2014, Mathematics and Computers in Science and Engineering Series 29, 19-25 (2014)

[25] Hitzer, E.: Clifford (Geometric) Algebra Wavelet Transform, in V. Skala and D. Hildenbrand (eds.), Proc. of GraVisMa 2009, 02-04 Sep. 2009, Plzen, Czech Republic, pp. 94-101 (2009)

[26] Hitzer, E.: Directional Uncertainty Principle for Quaternion Fourier Transform. Adv. appl. Clifford alg. 20, 271–284. DOI: 10.1007/s00006-009-0175-2 (2010)

[27] Hitzer, E., Mawardi, B.: Uncertainty Principle for the Clifford-Geometric Algebra $\mathcal{C}_{43,0}$ based on Clifford Fourier Transform. arXiv:1306.2089v1 [math.RA] 4 pages (2013)

[28] Hitzer, E.: Tutorial on Fourier Transformations and Wavelet Transformations in Clifford Geometric Algebra, in K. Tachibana (ed.), Lecture notes of the International Workshop for Computational Science with Geometric Algebra (FCSGA2007), Nagoya University, Japan, 14-21 Feb. 2007, pp. 65-87 (2007)

[29] K. Hleili, Continuous wavelet transform and uncertainty principle related to the Weinstein operator. Integral Transforms and Special Functions, 29(4) (2018), 252–268 DOI:10.1080/10652469.2018.1428581.

[30] K. Hleili, S. Omri, L. T. Rachdi, Uncertainty principle for the Riemann-Liouville operator. CUBO A Mathematical Journal 13(03), 91–115 (2011)
[31] F. H. Jackson, The application of basic numbers to Bessel’s and Legendre’s functions, Proc. London Math. Soc. (2) 2 (1905) 192-220.

[32] Jorgensen, P., Tian, F.: Non-commutative analysis. World Scientific Publishing Company (2017)

[33] Kou, K.I., Ou, J.-Y., Morais, J.: On Uncertainty Principle for Quaternionic Linear Canonical Transform. Abstract and Applied Analysis, Article ID 725952, 14 pages, http://dx.doi.org/10.1155/2013/725952 (2013)

[34] Ma, G., Zhao, J.: Quaternion Ridgelet Transform and Curvelet Transform. Adv. Appl. Clifford Algebras 28:80, https://doi.org/10.1007/s00006-018-0897-0 (2018)

[35] Mawardi, B.: Construction of Quaternion-Valued Wavelets, Matematika 26(1), 107-114 (2010)

[36] Mawardi, B., Ryuichi, A.: A Simplified Proof of Uncertainty Principle for Quaternion Linear Canonical Transform. Abstract and Applied Analysis, Article ID 5874930, 11 pages. http://dx.doi.org/10.1155/2016/5874930 (2016)

[37] Mawardi, B., Ashino, R.: Logarithmic uncertainty principle for quaternion linear canonical transform. Proceedings of the 2016 International Conference on Wavelet Analysis and Pattern Recognition, Jeju, South Korea, 10-13 July, 6 pages (2016)

[38] Mawardi, B., Ashino, R.: A Variation on Uncertainty Principle and Logarithmic Uncertainty Principle for Continuous Quaternion Wavelet Transforms. Abstract and Applied Analysis, Article ID 3795120, 11 pages, https://doi.org/10.1155/2017/3795120 (2017)

[39] Mawardi, B., Hitzer, E.: Clifford Algebra $Cl(3,0)$-valued Wavelets and Uncertainty Inequality for Clifford Gabor Wavelet Transformation, Preprints of Meeting of the Japan Society for Industrial and Applied Mathematics, ISSN: 1345-3378, Tsukuba University, 16-18 Sep. 2006, Tsukuba, Japan, pp. 64-65 (2006)

[40] Mawardi, B., Hitzer, E.: Clifford algebra $Cl(3,0)$-valued wavelet transformation, Clifford wavelet uncertainty inequality and Clifford Gabor wavelets. International Journal of Wavelets, Multiresolution and Information Processing 5(6), 997–1019 (2007)

[41] Mawardi, B., Hitzer, E.: Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$. Adv. appl. Clifford alg. 16, 41–61. DOI 10.1007/s00006-006-0003-x (2006)

[42] Mawardi, B., Hitzer, E.: Clifford Fourier Transform on Multivector Fields and Uncertainty Principles for Dimensions $n = 2(mod4)$ and $n = 3(mod4)$. Adv. appl. Clifford alg. 18, 715–736. DOI: 10.1007/s00006-008-0098-3 (2008)

[43] Mawardi, B., Hitzer, E., Hayashi, A., Ashino, R.: An Uncertainty Principle for Quaternion Fourier Transform, Computer & Mathematics with Applications 56, 2398-2410 (2008)
[44] Mawardi, B., Ashino, R., Vaillancourt, R.: Two-dimensional quaternion wavelet transform. Applied Mathematics and Computation 218, 10–21 (2011)

[45] Mejjaoli, H., Ben Hamadi, N., Omri, S.: Localization operators, time frequency concentration and quantitative-type uncertainty for the continuous wavelet transform associated with spherical mean operator. International Journal of Wavelets, Multiresolution and Information Processing Vol. 17(04), ID 1950022, 28 pages. https://doi.org/10.1142/S021969131950022X (2019)

[46] Mejjaoli, H., Sraieb N., and Trimèche K., Inversion theorem and quantitative uncertainty principles for the Dunkl Gabor transform on \( \mathbb{R}^d \). J. Pseudo-Differ. Oper. Appl. (2019) 10:883–913. DOI:10.1007/s11868-019-00276-4.

[47] Msehli, N., Rachdi, L.T.: Heisenberg-Pauli-Weyl Uncertainty Principle for the Spherical Mean Operator. Mediterranean Journal of Mathematics, 7(2), 169–194. doi:10.1007/s00009-010-0044-1 (2010)

[48] Msehli, N., Rachdi, L.T.: Beurling-Hörmander uncertainty principle for the spherical mean operator 10(2), Article 38, 22 pages (2009)

[49] Nagata, K., Nakamura, T: Violation of Heisenberg's Uncertainty Principle. Open Access Library Journal, 2, e1797. http://dx.doi.org/10.4236/oalib.1101797 (2015)

[50] A. Nemri, q-Donoho-Stark’s uncertainty principle and q-Tikhonov regularization problem. U.P.B. Sci. Bull., Series A, 81(1) (2019), 81-92.

[51] T. Ogawa and K. Seraku, Logarithmic sobolev and shannon’s inequalities and an application to the uncertainty principle. Communications on pure and applied analysis, volume 17, number 4, july 2018 pp. 1651-1669. DOI:10.3934/cpaa.2018079.

[52] Rachdi, L.T., Meherzi, F.: Continuous Wavelet Transform and Uncertainty Principle Related to the Spherical Mean Operator. Mediterranean Journal of Mathematics, 14(1). doi:10.1007/s00009-016-0834-1 (2016)

[53] Rachdi, L.T., Amri, B., Hammami, A.: Uncertainty principles and time frequency analysis related to the Riemann–Liouville operator. Annali Dell’Università Di Ferrara. doi:10.1007/s11565-018-0311-9 (2018)

[54] Rachdi, L.T., Herch, H.: Uncertainty principles for continuous wavelet transforms related to the Riemann–Liouville operator. Ricerche Di Matematica, 66(2), 553–578. doi:10.1007/s11587-017-0320-5 (2017)

[55] I. Rezgui and A. Ben Mabrouk, Some Generalized q-Bessel type Wavelets and associated transforms, Anal. Theory Appl, 34(1) (2017), pp. 1-15.

[56] A. Saoudi and I. A. Kallel, \( L_2 \)-uncertainty principle for the Weinstein-multiplier operators. International Journal of Analysis and Applications, 17(1) (2019), 64-75, DOI:10.28924/2291-8639-17-2019-64.
[57] Sen, D.: The uncertainty relations in quantum mechanics. Current Science 107(2), 203–218 (2018)

[58] Stabnikov, P.A.: Geometric Interpretation of the Uncertainty Principle. Natural Science 11(5), 146-148 (2019)

[59] Weyl, H.: The theory of groups and quantum mechanics. Dover, New York, second edition (1950)

[60] Yang, Y., Dang, P., Qian, T.: Stronger uncertainty principles for hypercomplex signals, Complex Variables and Elliptic Equations, 60(12), 1696–1711, DOI: 10.1080/17476933.2015.1041938 (2015)

[61] Yang, Y., Kou, K.I.: Uncertainty principles for hypercomplex signals in the linear canonical transform domains. Signal Processing 95, 67–75 (2014)