New relations between spinor and scalar one-loop effective Lagrangians in constant background fields

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(Dated: June 14, 2011)

Simple new relations are presented between the one-loop effective Lagrangians of spinor and scalar particles in constant curvature background fields, both electromagnetic and gravitational. These relations go beyond the well-known cases for self-dual background fields.

I. INTRODUCTION

The QED one-loop effective Lagrangian was obtained for the first time by Heisenberg who considered the effect that the Dirac sea would have on the dynamics of electromagnetic fields, and summarized this effect as a correction to the classical Maxwell Lagrangian [1]. The effective Lagrangian corresponding to scalar QED was obtained shortly afterwards by Weisskopf [2]. The effect of the quantum vacuum, encoded in the effective Lagrangian, accounts for physical phenomena such as pair-production from vacuum, light-light scattering and vacuum birefringence, among others [3]. Using the proper-time representation [4], and using Schwinger’s choice of units, we write both effective Lagrangians in modern notation [5]:

\[ L_{\text{spinor}}(E, B) = \frac{1}{8\pi^2} \int_0^{\infty} ds \, \frac{s^2}{e^{-m^2 s}} \left\{ EBs^2 \cot(Es) \coth(Bs) - 1 - \frac{s^2}{3} (B^2 - E^2) \right\}, \]

\( (I.1) \)

\[ L_{\text{scalar}}(E, B) = \frac{1}{16\pi^2} \int_0^{\infty} ds \, \frac{s^2}{e^{-m^2 s}} \left\{ \frac{EBs^2 \sin(Es) \sinh(Bs)}{\sinh^2(Bs)} - 1 + \frac{s^2}{6} (B^2 - E^2) \right\}. \]

\( (I.2) \)

These expressions are the first (one-loop) quantum corrections to the classical Maxwell Lagrangian. We notice a similar structure in both expressions. Both parametric integrals contain three types of terms. The main term involves trigonometric and hyperbolic functions. We also have the ”−1” term inside the curly brackets which corresponds to a subtraction of the free-field \((E = B = 0)\) Lagrangian and ensures the vanishing of the full expression in the absence of a background field. And in each case, the last term, which is proportional to the classical Maxwell Lagrangian, corresponds to charge renormalization [1, 2, 4].

Looking at both spinor and scalar Lagrangians, we notice a similar structure in terms of certain trigonometric functions. In fact, simple relations between the spinor and scalar effective Lagrangians occur for a self-dual background field, or a field of definite helicity [6, 7]. In Minkowski space this means \(E = \pm iB\), and by inspection of (I.1) and (I.2) we find

\[ L_{\text{spinor}}(\pm iB, B) = -2 L_{\text{scalar}}(\pm iB, B), \]

\( (I.3) \)

as follows from the trigonometric identity \(\coth^2(x) = 1 + 1/\sinh^2(x)\).

This relation reflects the isospectrality of the Dirac and Klein-Gordon operators for a self-dual background gauge field: it is a consequence [8] of the self-duality of the background that apart from zero-modes, the Dirac operator has exactly the same spectrum as the corresponding Klein-Gordon operator, with a multiplicity factor of 4. Since these one-loop effective Lagrangians can be expressed in terms of logarithms of the determinants of the Dirac and Klein-Gordon operators, this means that, due to the isospectrality, when the background is self-dual, we have

\[ L_{\text{spinor}}(\pm iB, B) = -2 L_{\text{scalar}}(\pm iB, B) + \frac{1}{2} \left( \frac{eB}{2\pi} \right)^2 \ln \left( \frac{m^2}{\mu^2} \right), \]

\( (I.4) \)

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where \( N_0 = \left( \frac{\mu^2}{\pi} \right)^2 \) is the zero-mode number density. When we renormalize on-shell (i.e., \( \mu^2 = m^2 \)) we recover (1.3). An example of a consequence of this relation is that the \( \mathcal{N} = 2 \) SUSY effective Lagrangian vanishes at one-loop:

\[
\mathcal{L}_{N=2 \text{ SUSY}}(\pm iB, B) = \mathcal{L}_{\text{spinor}}(\pm iB, B) + 2 \mathcal{L}_{\text{scalar}}(\pm iB, B) = 0. \tag{1.5}
\]

This kind of relation has been exploited in order to calculate the effective action, for a fermion in an instanton background [10].

In this paper we present new relations between \( \mathcal{L}_{\text{spinor}} \) and \( \mathcal{L}_{\text{scalar}} \) that apply when the background is not self-dual.

**II. SPINOR/SCALAR RELATIONS IN QED**

In this section we use trigonometric identities to derive spinor/scalar relations for three physically interesting types of background configurations, including a purely magnetic background field, a purely electric field (where Schwinger pair production is possible) and the general case of a constant electromagnetic field.

**A. The case of a purely magnetic background**

For a constant magnetic field background of strength \( B \), the spinor and scalar effective Lagrangians are:

\[
\mathcal{L}_{\text{spinor}}(B) = -\frac{B}{8\pi^2} \int_0^\infty ds \frac{ds}{s^2} e^{-m^2s} \left( \coth Bs - \frac{1}{Bs} - \frac{B s}{3} \right), \tag{II.1}
\]

\[
\mathcal{L}_{\text{scalar}}(B) = \frac{B}{16\pi^2} \int_0^\infty ds \frac{ds}{s^2} e^{-m^2s} \left( \frac{1}{\sinh Bs} - \frac{1}{Bs} + \frac{B s}{6} \right). \tag{II.2}
\]

The simple trigonometric identity

\[
\left( \coth s - \frac{1}{s} - \frac{s}{3} \right) = \left( \coth 2s - \frac{1}{2s} - \frac{2s}{3} \right) + \left( \frac{1}{\sinh 2s} - \frac{1}{2s} + \frac{2s}{6} \right), \tag{II.3}
\]

reveals a relation between the spinor and scalar Lagrangians:

\[
\mathcal{L}_{\text{spinor}}(B) = \frac{1}{2} \mathcal{L}_{\text{spinor}}(2B) - \mathcal{L}_{\text{scalar}}(2B). \tag{II.4}
\]

Iterating this relation \( N \) times, we find

\[
\mathcal{L}_{\text{spinor}}(B) = \frac{1}{2} \left( \frac{1}{2} \mathcal{L}_{\text{spinor}}(4B) - \mathcal{L}_{\text{scalar}}(4B) \right) - \mathcal{L}_{\text{scalar}}(2B)
\]

\[\vdots\]

\[
= \frac{1}{2^N} \mathcal{L}_{\text{spinor}}(2^N B) - \frac{N}{2} \mathcal{L}_{\text{scalar}}(2^1 B). \tag{II.5}
\]

Rescaling \( B \) by \( 2^{-N} \), we can write this as:

\[
\mathcal{L}_{\text{spinor}}(B) = 2^N \mathcal{L}_{\text{spinor}}(2^{-N} B) + \sum_{l=1}^N 2^l \mathcal{L}_{\text{scalar}}(2^{1-l} B). \tag{II.6}
\]

From the perturbative expansion of the Lagrangians it is clear that the leading term in the weak-field limit gives \( \mathcal{L}(B) \propto B^4 \). Therefore, we can take the \( N \to \infty \) limit and relate the spinor effective Lagrangian to an infinite sum of scalar effective Lagrangians:

\[
\mathcal{L}_{\text{spinor}}(B) = \sum_{l=1}^\infty 2^l \mathcal{L}_{\text{scalar}}(2^{1-l} B). \tag{II.6}
\]
This is a completely new relation between spinor and scalar effective Lagrangians.

This raises the question of convergence of this series. Let us first reformulate the relation in terms of functional determinants, we have

\[
\Delta_{\text{spinor}}(B) = \prod_{l=1}^{\infty} (\Delta_{\text{scalar}}(2^{1-l}B))^{-2^l}.
\]  

(II.9)

This relation is illustrated in Figure 1 where we approximate the spinor determinant in terms of a finite product of scalar determinants. Note that just three terms already provide a good approximation. The lower curve shows the exact spinor determinant (solid) while the upper curve (dot-dashes) is the scalar determinant.

We can thus write (II.6) as

\[
\Delta_{\text{spinor}}(B) = \prod_{l=1}^{\infty} (\Delta_{\text{scalar}}(2^{1-l}B))^{-2^l}.
\]

(II.9)

As \( B \to \infty \), by definition, we have [14]

\[
\mathcal{L}_{\text{spinor}}(B) \sim \beta_{\text{spinor}} \frac{B^2}{2} \ln B,
\]

(II.10)

\[
\mathcal{L}_{\text{scalar}}(B) \sim \beta_{\text{scalar}} \frac{B^2}{2} \ln B,
\]

(II.11)
where $\beta_{\text{spinor}}$ and $\beta_{\text{scalar}}$ are the coefficients of the one-loop $\beta$-function for spinor and scalar QED, respectively. In the case of a constant magnetic background this gives $\beta_{\text{spinor}} = \frac{1}{12\pi}$ and $\beta_{\text{scalar}} = \frac{1}{48\pi}$. The strong-field limit of the spinor/scalar relation is easily obtained from (II.6), yielding

$$
\beta_{\text{spinor}} \frac{B^2}{2} \ln B \sim \beta_{\text{scalar}} \sum_{l=1}^{\infty} 2^l \left( \frac{2^1 l B}{2} \right)^2 \ln(2^{1-l} B).
$$

(II.12)

The leading large $B$ behavior implies the correct relation between the one-loop $\beta$-function coefficients:

$$
\beta_{\text{scalar}} = \frac{1}{4} \beta_{\text{spinor}}.
$$

(II.13)

B. The case of a purely electric background

The expressions for the spinor and scalar effective Lagrangians under a constant electric field are

$$
\mathcal{L}_{\text{spinor}}(E) = -\frac{E}{8\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-m^2 s} \left( \cot Es - \frac{1}{Es} - \frac{Es}{3} \right),
$$

(II.14)

$$
\mathcal{L}_{\text{scalar}}(E) = \frac{E}{16\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-m^2 s} \left( \frac{1}{\sin Es} - \frac{1}{Es} + \frac{Es}{6} \right).
$$

(II.15)

As we see, the electric field case is obtained from the constant magnetic field case by the replacement $B \to iE$, so the same argument leads to the same relation:

$$
\mathcal{L}_{\text{spinor}}(E) = \frac{1}{2} \mathcal{L}_{\text{spinor}}(2E) - \mathcal{L}_{\text{scalar}}(2E).
$$

(II.16)

Iterating this basic relation (II.16), we obtain, as before

$$
\mathcal{L}_{\text{spinor}}(E) = \sum_{l=1}^{\infty} 2^l \mathcal{L}_{\text{scalar}}(2^{1-l} E).
$$

(II.17)

Physically, however, there is a big difference between the electric and magnetic cases, as the electric background leads to pair production from vacuum, which is encoded in the imaginary part of $\mathcal{L}$. The pair production rate is obtained after integrating over the proper-time, and gathering the contributions from the poles of $\cot (Es)$ and $1/\sin (Es)$:

$$
\text{Im} \mathcal{L}_{\text{spinor}}(E) = \frac{E^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left[ \frac{-m^2 \pi n}{E} \right]
$$

(II.18)

$$
\text{Im} \mathcal{L}_{\text{scalar}}(E) = \frac{E^2}{16\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \exp \left[ \frac{-m^2 \pi n}{E} \right]
$$

(II.19)

Thus it is easy to see that:

$$
\frac{1}{2} \text{Im} \mathcal{L}_{\text{spinor}}(2E) - \text{Im} \mathcal{L}_{\text{scalar}}(2E) = \frac{(2E)^2}{16\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^{n-1}}{n^2} \exp \left[ \frac{-m^2 \pi n}{(2E)} \right]
$$

$$
= \frac{(2E)^2}{8\pi^3} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{1}{n^2} \exp \left[ \frac{-m^2 \pi n}{(2E)} \right]
$$

$$
= \frac{E^2}{8\pi^3} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{1}{n^2} \exp \left[ \frac{-m^2 \pi n}{E} \right]
$$

$$
= \text{Im} \mathcal{L}_{\text{spinor}}(E)
$$

(II.20)

which is consistent with (II.10). The alternating sign in the imaginary part of the scalar action has the effect of cancelling the odd terms of the spinor action and the resulting expression is identifiable as another spinor Lagrangian.
C. The case of a general constant electromagnetic background

For a general constant field, the spinor and scalar Lagrangians are given by (I.1) and (I.2). We find the basic doubling relation:

\[ L_{\text{scalar}}(2E, 2B) = L_{\text{spinor}}(2E, B) - \frac{1}{2} L_{\text{spinor}}(2E, 2B) \]
\[ - 2 L_{\text{spinor}}(E, B) + L_{\text{spinor}}(E, 2B), \]

which follows immediately from the trigonometric identity

\[ \frac{1}{\sin(x)\sinh(y)} = \cot(x)\coth(y) - \cot(x/2)\coth(y) \]
\[ + \cot(x/2)\coth(y/2) - \cot(x)\coth(y/2), \]

Note that in addition to the trigonometric functions appearing in this identity, each of the Lagrangians (I.1) and (I.2) contains renormalization terms. Remarkably, when we write both sides of (II.21), these terms coincide with (II.22), as required.

An interesting consequence of (II.21) is that it expresses the SUSY combination purely in terms of spinor Lagrangians:

\[ \mathcal{L}_{\text{N=2 SUSY}}(E, B) \equiv \mathcal{L}_{\text{spinor}}(E, B) + 2 \mathcal{L}_{\text{scalar}}(E, B) \]
\[ = 2 \mathcal{L}_{\text{spinor}}(E, B/2) - 4 \mathcal{L}_{\text{spinor}}(E/2, B/2) + 2 \mathcal{L}_{\text{spinor}}(E/2, B). \]

III. ZETA FUNCTION AND GAMMA FUNCTION REPRESENTATIONS

Using different representations of the effective action is a way to reveal further spinor-scalar relations. Other methods for computing functional determinants involve zeta functions. Given an operator \( \mathcal{O} \), its \( \zeta \)-function is defined as

\[ \zeta(s) = \text{tr}(\mathcal{O}^{-s}) = \sum_{\lambda} \lambda^{-s}. \]

This allows one to write the functional determinant as:

\[ \text{det} \mathcal{O} = \exp[-\zeta'(0)] \]

The spinor Euler-Heisenberg effective action in a constant magnetic background may be written in terms of the Hurwitz \( \zeta \)-function defined as

\[ \zeta_H(s, z) = \sum_{n=0}^{\infty} (n+z)^{-s}. \]

Like the Riemann zeta function \( \zeta_R(s) \equiv \zeta_H(s, 1) \), the Hurwitz zeta function has an analytic continuation throughout the entire complex \( s \) plane with only a pole at \( s = 1 \). In the case of a magnetic background the eigenvalues of the Dirac operator are

\[ \lambda_n^\pm = m^2 + k_{\perp}^2 + eB(2n + 1 \pm 1), \quad n = 0, 1, \ldots \]

where the \( \pm \) refers to different spin components and \( k_{\perp} \) is the transverse momentum. The corresponding \( \zeta \)-function is

\[ \zeta_{\text{spinor}}(s) = \frac{eB}{2\pi} \sum_{n=0}^{\infty} \sum_{\pm} \int \frac{d^2 k_{\perp}}{(2\pi)^2} \left( \frac{m^2 + k_{\perp}^2 + eB(2n + 1 \pm 1)}{\mu^2} \right)^{-s} \]
\[ = \frac{m^4}{4\pi^2} \left( \frac{eB}{m^2} \right)^2 \frac{2\zeta_H(s, m^2/2eB)}{s-1} \left[ 2\zeta_H(s-1, m^2/2eB) - 2\zeta_H(s-1, m^2/2eB)^{-s} \right] \]
where the scale \( \mu \) is introduced to make the eigenvalues dimensionless. Using on-shell renormalization \( (\mu = m) \) and subtracting the zero-field contribution the one-loop effective Lagrangian is obtained:

\[
\mathcal{L}_{\text{spinor}} = \frac{eB}{2\pi^2} \left\{ \zeta_H \left( -1, \frac{m^2}{2eB} \right) + \zeta_H \left( -1, \frac{m^2}{2eB} \right) \ln \left( \frac{m^2}{2eB} \right) - \frac{1}{12} + \frac{1}{4} \left( \frac{m^2}{2eB} \right)^2 \right\}.
\]  

(III.5)

We find useful to express the one-loop effective Lagrangian in terms of the gamma function \([5]\). The Hurwitz \( \zeta \)-function and the logarithm of the gamma function are related in the following way:

\[
\zeta_H'(-1, z) = \zeta'(-1) - \frac{z}{2} \ln(2\pi) - \frac{z}{2} (1 - z) + \int_0^z \ln \Gamma(x) dx,
\]  

(III.6)

substituting this in (III.5) yields

\[
\mathcal{L}_{\text{spinor}} = \frac{(eB)^2}{2\pi^2} \left\{ -\frac{1}{12} + \zeta'(-1) - \frac{m^2}{4eB} + \frac{3}{4} \left( \frac{m^2}{2eB} \right)^2 \ln(2\pi) \right. \\
+ \left. \left[ -\frac{1}{12} + \frac{m^2}{4eB} + \frac{1}{2} \left( \frac{m^2}{2eB} \right)^2 \right] \ln \left( \frac{m^2}{2eB} \right) + \int_0^{\frac{m^2}{2eB}} \ln \Gamma(x) dx \right\}.
\]  

(III.7)

A similar expression holds for scalar QED

\[
\mathcal{L}_{\text{scalar}} = \frac{(eB)^2}{4\pi^2} \left\{ \frac{5}{4} \left( \frac{m^2}{2eB} \right)^2 + \left[ \frac{1}{24} - \frac{1}{2} \left( \frac{m^2}{2eB} \right)^2 \right] \ln \left( \frac{m^2}{2eB} \right) \right. \\
- \frac{1}{2} \zeta'(-1) - \frac{\ln 2}{24} + \int_0^{\frac{m^2}{2eB}} \ln \Gamma(x + \frac{1}{2}) dx \right\}.
\]  

(III.8)

We can derive the relation (II.4) from the properties of the gamma function. Note that

\[
\mathcal{L}_{\text{spinor}} = \frac{(eB)^2}{2\pi^2} \int_0^{\frac{m^2}{2eB}} \ln \Gamma(x) dx + (\text{other terms}),
\]  

(III.9)

while

\[
\mathcal{L}_{\text{scalar}} = -\frac{(eB)^2}{4\pi^2} \int_0^{\frac{m^2}{2eB}} \ln \Gamma(x + 1/2) dx + (\text{other terms}).
\]  

(III.10)

In this form, the basic doubling relation (II.4) results from the following duplication formula

\[
\Gamma(x) \Gamma(x + 1/2) = 2^{1-2x} e^{-\zeta'(0)} \Gamma(2x).
\]  

(III.11)

The spinor/scalar relation (II.6) is derived by iteration, as before.

### IV. SPINOR/SCALAR RELATIONS IN CURVED SPACETIME

So far we have shown how one can find relations between the spinor and scalar effective Lagrangians using different representations. In particular we have used the representation of the effective action in terms of \( \ln \Gamma(x) \). The digamma function is defined as

\[
\psi(x) \equiv \frac{d}{dx} \ln \Gamma(x).
\]  

(IV.1)

This type of function is common in the effective actions of different gauge-theories.

We now turn our attention to the effective Lagrangians of spinor and scalar particles in a curved two-dimensional AdS space. In this scenario the Dirac and Klein-Gordon operators get modified by gauge terms that account for the
curvature of space. Let us write the spinor and scalar effective Lagrangians for a particle in a two-dimensional AdS following [15] and [16].

For a spin-$\frac{1}{2}$ particle the coincident propagator is obtained from the trace of the inverse Dirac operator

$$\frac{\partial \mathcal{L}}{\partial m^2} = -\text{Tr} \left( \frac{1}{\nabla - m} \right)_{_{\text{AdS}}} = -\lambda^{\frac{d+1}{2}} \frac{2^d \Gamma(1 - \frac{d}{2}) \Gamma(\frac{d}{2} + \sqrt{m^2/\lambda})}{(4\pi)^{\frac{d}{2}} \Gamma(1 - \frac{d}{2} + \sqrt{m^2/\lambda})},$$  \hspace{1cm} (IV.2)

where $\lambda$ is given by the Ricci scalar and represents the curvature of space and $d$ is the number of space-time dimensions. Expanding around $d = 2$ we get

$$-\text{Tr} \left( \frac{1}{\nabla - m} \right)_{_{\text{AdS}_2}} = \frac{m}{\pi(d-2)} + m \left[ \psi \left( \sqrt{\frac{m^2}{\lambda}} \right) + \frac{1}{2} \ln \left( \frac{\lambda}{2\pi} \right) + \frac{\gamma}{2} \right] + \frac{\sqrt{\lambda}}{2\pi} + \mathcal{O}(d-2).$$

Thus the effective Lagrangian is

$$\mathcal{L}^{_{\text{spinor}}} (\sqrt{m^2/\lambda}) = \frac{1}{\pi} \int dm \, m \psi(\sqrt{m^2/\lambda}),$$  \hspace{1cm} (IV.3)

after rescaling we get

$$\mathcal{L}^{_{\text{spinor}}} (\sqrt{m^2/\lambda}) = \frac{\lambda}{\pi} \int_{0}^{\sqrt{m^2/\lambda}} dy \, y \psi(y).$$  \hspace{1cm} (IV.4)

In the case of the scalar effective Lagrangian we have the following operator

$$\mathcal{H} = -\Box - m^2,$$  \hspace{1cm} (IV.5)

where $\Box = \frac{1}{\sqrt{-g}} \partial_{\mu} (g^{{\mu}{\nu}} \sqrt{-g} \partial_{\nu})$. In the case $d = 2$ the coincident propagator is

$$\frac{i}{2} \mathcal{G}(x,x) = \frac{1}{4\pi(n-2)} + \frac{1}{8\pi} \left[ \ln \left( \frac{4\pi \Lambda}{\lambda} \right) - \gamma - 2 \psi \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda}} \right) \right]$$  \hspace{1cm} (IV.6)

and the corresponding effective Lagrangian is

$$\mathcal{L}^{_{\text{scalar}}} (\sqrt{m^2/\lambda}) = \frac{1}{4\pi} \int dm^2 \, \psi \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4m^2}{\lambda}} \right)$$  \hspace{1cm} (IV.7)

or after changing the variable

$$\mathcal{L}^{_{\text{scalar}}} (\sqrt{m^2/\lambda}) = \frac{\lambda}{4\pi} \int_{1/2}^{\sqrt{m^2/\lambda} + 1/4} dy \, y \psi \left( y + \frac{1}{2} \right).$$  \hspace{1cm} (IV.8)

We use the following identity to connect the spinor and scalar effective actions.

$$\psi(x + 1/2) + \psi(x) - 2\psi(2x) + 2 \ln 2 = 0$$  \hspace{1cm} (IV.9)

Substituting this equation into (IV.8) we get

$$\mathcal{L}^{_{\text{scalar}}} (\sqrt{m^2/\lambda}) = -\frac{\lambda}{4\pi} \int_{1/2}^{\sqrt{m^2/\lambda} + 1/4} dy \, y \psi(y) + \frac{\lambda}{4\pi} \int_{1/2}^{\sqrt{m^2/\lambda} + 1/4} dy \, (2y) \psi(2y)$$

$$- \frac{\lambda(2 \ln 2)}{\pi} \int_{1/2}^{\sqrt{m^2/\lambda} + 1/4} y \, dy.$$  \hspace{1cm} (IV.10)

Rescaling the integration variable in the first two terms we write the right-hand side in terms of spinor Lagrangians :

$$\mathcal{L}^{_{\text{scalar}}} (\sqrt{m^2/\lambda}) = -\frac{1}{4} \cdot \frac{\lambda}{\lambda'} \mathcal{L}^{_{\text{spinor}}} (\sqrt{m^2/\lambda'}) + \frac{1}{8} \frac{\lambda}{\lambda'} \mathcal{L}^{_{\text{spinor}}} (\sqrt{m^2/\lambda''}) - \left( \frac{\ln 2}{4\pi} \right) m^2,$$  \hspace{1cm} (IV.11)
where
\[
\lambda' = \frac{\lambda}{1 + \frac{\lambda}{4m^2}}
\] (IV.12)
and
\[
\lambda'' = \frac{1}{4} \left( \frac{\lambda}{1 + \frac{\lambda}{4m^2}} \right)
\] (IV.13)

We write the last expression as
\[
\mathcal{L}_{\text{scalar}}^{\text{AdS}_2}(\sqrt{m^2/\lambda}) = -\frac{1}{4} \cdot \frac{\lambda}{\lambda'} \mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(\sqrt{m^2/\lambda'}) + \frac{1}{2} \cdot \frac{\lambda}{\lambda'} \mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(2\sqrt{m^2/\lambda'}) - \left( \frac{\ln 2}{4\pi} \right) m^2,
\] (IV.14)

Since \( \frac{\lambda'}{\lambda} = 1 - \frac{\lambda'}{4m^2} \) we can write the basic relation as
\[
\mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(x) = 2 \mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(2x) - 4 \left( 1 - \frac{1}{4x^2} \right) \mathcal{L}_{\text{scalar}}^{\text{AdS}_2}(x \left( 1 - \frac{1}{4x^2} \right)^{\frac{1}{2}}),
\] (IV.15)

where \( x = \sqrt{m^2/\lambda} \) and we have omitted the mass term.

Iterating this relation \( N \) times we obtain
\[
\mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(x) = 2^{N+1} \mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(2^{N+1}x) - \sum_{n=0}^{N} 2^N 4 \left( 1 - \frac{1}{4(2^n x)^2} \right) \mathcal{L}_{\text{scalar}}^{\text{AdS}_2}(2^n x \left( 1 - \frac{1}{4(2^n x)^2} \right)^{\frac{1}{2}})
\] (IV.16)

Rescaling \( x \) by \( 2^{-(N+1)} \), we obtain
\[
\mathcal{L}_{\text{spinor}}^{\text{AdS}_2}(x) = 2 \sum_{l=0}^{N} 2^{-l} \left( 1 - \frac{1}{2^{-2l} x^2} \right) \mathcal{L}_{\text{scalar}}^{\text{AdS}_2}(2^{-(l+1)} x \left( 1 - \frac{1}{2^{-2l} x^2} \right)^{\frac{1}{2}}).
\] (IV.17)

V. CONCLUSION

In this paper I have presented simple new relations between the spinor and scalar one-loop effective Lagrangians in both electromagnetic and gravitational backgrounds. Since the effective action is the generating function of scattering amplitudes, these relations may be used to relate the low energy limits of such scattering amplitudes.

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