**C*-ALGEBRA-VALUED-SYMBOL PSEUDODIFFERENTIAL OPERATORS: ABSTRACT CHARACTERIZATIONS**

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**Abstract.** Given a separable unital C*-algebra $C$ with norm $\| \cdot \|$, let $E_n$ denote the Banach-space completion of the $C$-valued Schwartz space on $\mathbb{R}^n$ with norm $\| f \|_2 = \| \langle f, f \rangle \|^{1/2}$, $\langle f, g \rangle = \int f(x)^* g(x) dx$. The assignment of the pseudodifferential operator $A = a(x, D)$ with $C$-valued symbol $a(x, \xi)$ to each smooth function with bounded derivatives $a \in B^C(\mathbb{R}^2n)$ defines an injective mapping $O$, from $B^C(\mathbb{R}^2n)$ to the set $\mathcal{H}$ of all operators with smooth orbit under the canonical action of the Heisenberg group on the algebra of all adjointable operators on the Hilbert module $E_n$. In this paper, we construct a left-inverse $S$ for $O$ and prove that $S$ is injective if $C$ is commutative. This generalizes Cordes’ description [2] of $\mathcal{H}$ in the scalar case. Combined with previous results of the second-named author, our main theorem implies that, given a skew-symmetric $n \times n$ matrix $J$, and if $C$ is commutative, then any $A \in \mathcal{H}$ which commutes with every pseudodifferential operator with symbol $F(x+J\xi), F \in B^C(\mathbb{R}^n)$, is a pseudodifferential operator with symbol $G(x-J\xi)$, for some $G \in B^C(\mathbb{R}^n)$. That was conjectured by Rieffel.

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1. **Introduction**

Let $C$ be a separable unital C*-algebra with norm $\| \cdot \|$, and let $S^C(\mathbb{R}^n)$ denote the set of all $C$-valued smooth functions on $\mathbb{R}^n$ which, together with all their derivatives, are bounded by arbitrary negative powers of $|x|, x \in \mathbb{R}^n$. We equip it with the $C$-valued inner-product

$$\langle f, g \rangle = \int f(x)^* g(x) dx,$$

which induces the norm $\| f \|_2 = \| \langle f, f \rangle \|^{1/2}$, and denote by $E_n$ its Banach-space completion with this norm. The inner product $\langle \cdot, \cdot \rangle$ turns $E_n$ into a Hilbert module $\mathcal{H}$. The set of all (bounded) adjointable operators on $E_n$ is denoted $B^*(E_n)$.

Let $B^C(\mathbb{R}^{2n})$ denote the set of all smooth bounded functions from $\mathbb{R}^{2n}$ to $C$ whose derivatives of arbitrary order are also bounded. For each $a$ in $B^C(\mathbb{R}^{2n})$, a linear mapping from $S^C(\mathbb{R}^n)$ to itself is defined by the formula

$$\langle Au \rangle(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi,$$

where $\hat{u}$ denotes the Fourier transform,

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-iy \cdot \xi} u(y) dy.$$
As usual, we denote $A = a(x,D)$. This operator extends to an element of $B^*(E_n)$
whose norm satisfies the following estimate. There exists a constant $k > 0$
depending only on $n$ such that

$$||A|| \leq k \sup \{ ||\partial_x^\alpha \partial_{\xi}^\beta a(x,\xi)||; (x,\xi) \in \mathbb{R}^{2n} \text{ and } \alpha, \beta \leq (1, \cdots, 1) \}.$$  

This generalization of the Calderón-Vaillancourt Theorem \cite{1} was proven by Merklen \cite{7,8}, following ideas of Hwang \cite{4} and Seiler \cite{11}. The case of $a(x,\xi) = F(x + J\xi)$, where $F \in B^C(\mathbb{R}^n)$
and $J$ is an $n \times n$ skew-symmetric matrix, had been proven earlier by Rieffel \cite{10} Corollary 4.7.

The estimate \cite{2} implies that the mapping

$$\mathbb{R}^{2n} \ni (z,\zeta) \mapsto A_{z,\zeta} = T_{-z}M_{-\zeta}AM_{\zeta}T_{z} \in B^*(E_n)$$

is smooth (i.e., $C^\infty$ with respect to the norm topology), where $T_z$ and $M_\zeta$ are defined by $T_zu(x) = u(x-z)$ and $M_\zeta u(x) = e^{i\zeta \cdot x}u(x)$, $u \in S^C(\mathbb{R}^n)$. That follows just like in the scalar case \cite{3} Chapter 8.

**Definition 1.** We call Heisenberg smooth an operator $A \in B^*(E_n)$ for which the mapping \cite{3} is smooth, and denote by $\mathcal{H}$ the set of all such operators.

The elements of $\mathcal{H}$ are the smooth vectors for the action of the Heisenberg group
on $B^*(E_n)$ given by the same formula as the standard one in the scalar case (i.e.,
when $C$ is the algebra $\mathbb{C}$ of complex numbers and then $E_n = L^2(\mathbb{R}^n)$, and we denote
$S^C(\mathbb{R}^n)$ and $B^C(\mathbb{R}^n)$ by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$, respectively).

We therefore have a mapping

$$O : B^C(\mathbb{R}^{2n}) \rightarrow \mathcal{H}$$

$$a \mapsto O(a) = a(x, D).$$

In the scalar case, it is well-known (this can be proven by a Schwartz-kernel ar-
ument) that if a pseudodifferential operator as in \cite{1} vanishes on $\mathcal{S}(\mathbb{R}^n)$, then $a$
must be zero. Let us show that this implies that $O$ is injective for arbitrary $C$.

Given any complex-valued function $u$ defined on $\mathbb{R}^n$, we denote by $\tilde{u} : \mathbb{R}^n \rightarrow C$
the function defined by

$$\tilde{u}(x) = u(x)1_C,$$

where $1_C$ denotes the identity of $C$. If $O(a) = 0$, the fact that $O(a)\tilde{u} = 0$
for every $u \in \mathcal{S}(\mathbb{R}^n)$ and the injectivity of $O$ in the scalar case imply that $(x,\xi) \mapsto \rho(a(x,\xi))$
vanishes identically, for every $\rho \in C^*$, the (Banach-space) dual of $C$. We then get
$a = 0$, as we wanted.

Our results in this paper can now be summarized in the following theorem,
proven in Sections 2 and 8.

**Theorem 1.** Let $C$ be a unital separable $C^*$-algebra. There exists a linear mapping
$S : \mathcal{H} \rightarrow B^C(\mathbb{R}^{2n})$ such that $S \circ O$ is the identity operator. If $C$ is commutative,
then $S$ is injective.

Since an injective left-inverse is an inverse, we get:

**Corollary 1.** If $C$ is commutative and an operator $A \in B^*(E_n)$ is given, then the mapping defined in \cite{3} is smooth if and only if $A = a(x, D)$ for some $a \in B^C(\mathbb{R}^{2n})$.

Theorem \cite{1} and Corollary \cite{1} were proven by Cordes \cite{2} in the scalar case. His construction \cite{3} Chapter 8] of the left-inverse $S$ works also in the general case, if only
one is careful enough to avoid mentioning trace-class or Hilbert-Schmidt operators.
That is what we show in Section 2. His proof that $S$ is injective, however, strongly
depends on the fact that, when $C = \mathbb{C}$, $E_n = L^2(\mathbb{R}^n)$ is a Hilbert space. In the
general commutative case, the lack of an orthonormal basis in $E_n$ can be bypassed
by still reducing the problem to copies of $L^2(\mathbb{R}^n)$, as shown at the beginning of
Section 3. After this reduction, we are then able to follow the steps of Cordes’ proof.
Crucial for this strategy, our Lemma 1 is essentially [9, Lemma 2.4] specialized to
commutative $C^*$-algebras. In Section 4 we explain how Theorem 1 implies, in the
commutative case, an abstract characterization, conjectured by Rieffel [10], of a
certain class of $C^*$-algebra-valued-symbol pseudodifferential operators.

The assumption of separability of $C$ is needed to justify several results about
vector-valued integration (see [8, Apêndice], for example), which are used without
further comments throughout the text.

2. Left Inverse for $O$

Given $f$ and $g$ functions from $\mathbb{R}^n$ to $X$ ($X$ will be either $C$ or $\mathbb{C}$), let $f \otimes g :$
$\mathbb{R}^{2n} \to X$ be defined by

$$f \otimes g(x, y) = f(x)g(y).$$

(6)

Given a vector space $V$ we denote by $V \otimes V$ the algebraic tensor product of $V$ by
itself. In case the elements of $V$ are functions from $\mathbb{R}^n$ to $X$, $V \otimes V$ is isomorphic
to the linear span of all function as in (6) with $f$ and $g$ in $V$.

**Lemma 1.** Given $A \in B^*(E_n)$ mapping $S^C(\mathbb{R}^n)$ to itself, there exists a unique
operator $A \otimes I \in B^*(E_{2n})$ such that, for all $f \in S^C(\mathbb{R}^n)$,

$$(A \otimes I)(f \otimes g) = Af \otimes g.$$  

(7)

**Proof:** Let $L^2(\mathbb{R}^n; C)$ denote the set of equivalence classes (for the equality almost
everywhere equivalence) of Borel measurable functions $f : \mathbb{R}^n \to C$ such that

$$\int \|f(x)\|^2 dx < \infty.$$  

and let $\|f\|_{L^2}$ denote the square root of the integral above. $L^2(\mathbb{R}^n; C)$ equipped
with $\| \cdot \|_{L^2}$ is a Banach space, containing $S^C(\mathbb{R}^n)$ as a dense subspace. It follows
from the inequality

$$\|f\|_2 \leq \|f\|_{L^2}, \text{ for all } f \in S^C(\mathbb{R}^n),$$

that $L^2(\mathbb{R}^n; C)$ embeds in $E_n$ as a $\| \cdot \|_2$-dense subspace.

Let $S_n$ denote the set of all simple measurable functions from $\mathbb{R}^n$ to $C$. It
takes an elementary but messy argument to show that $S_n \otimes S_n$ is $\| \cdot \|_{L^2}$-dense
in $S_{2n}$, which is dense in $L^2(\mathbb{R}^n; C)$. Since $S_n$ is dense in $L^2(\mathbb{R}^n; C)$, it follows
that $L^2(\mathbb{R}^n; C) \otimes L^2(\mathbb{R}^n; C)$ is dense in $L^2(\mathbb{R}^{2n}; C)$. Since $S^C(\mathbb{R}^n)$ is dense in
$L^2(\mathbb{R}^n; C)$, it follows that $S^C(\mathbb{R}^n) \otimes S^C(\mathbb{R}^n)$ is $\| \cdot \|_{L^2}$-dense in $L^2(\mathbb{R}^{2n}; C)$, hence
it is also $\| \cdot \|_{L^2}$-dense in $E_{2n}$.

Let $\phi : C \to B^*(E_n)$ be given by left multiplication on $S^C(\mathbb{R}^n)$, and denote by
$E_n \otimes_{\phi} E_n$ the interior tensor product (given by $\phi$) as defined in [5 page 41]. The
fact that $S^C(\mathbb{R}^n) \otimes S^C(\mathbb{R}^n)$ is dense in $E_{2n}$ allows us to identify $E_n \otimes_{\phi} E_n$ with
$E_{2n}$ (notice that the space $N$ in [5 Proposition 4.5] consists only of 0 in this case).
Given $A \in \mathcal{B}^\ast(E_n)$, it now follows from the more general result around \cite{5} (4.6) that there exists a unique $A \otimes I \in \mathcal{B}^\ast(E_{2n})$ such that $A \otimes I(f \otimes g) = Af \otimes g$ for all $f \otimes g \in E_n \otimes E_n$. In particular, we get \cite{7} for all $f$ and $g$ in $\mathcal{S}^\mathcal{C}(\mathbb{R}^n)$. That \cite{7} uniquely determines $A \otimes I$ also follows from the fact that $\mathcal{S}^\mathcal{C}(\mathbb{R}^n) \otimes \mathcal{S}^\mathcal{C}(\mathbb{R}^n)$ is dense in $E_{2n}$.

Let us denote by $\gamma_1(t)$ and $\gamma_2(t)$, respectively, the fundamental solutions of $(\partial_t + 1)$ and $(\partial_t + i)^2$ given by:

$$
\gamma_1(t) = \begin{cases} e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}
$$

and

$$
\gamma_2(t) = \begin{cases} te^{-t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}.
$$

We then define $u$ and $v$ in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by

$$(8) \quad v(\xi, \eta) = \gamma_1(\xi - \eta)/(1 + i\xi)^2$$

and

$$(9) \quad u(x, \eta) = (1 + \partial_\eta)[(1 - i\eta)^2\gamma_2(-\eta)e^{ix\eta}].$$

The following lemma can be proven exactly like in the scalar case \cite{3} Section 8.3]

**Lemma 2.** If $a$ and $b$ in $\mathcal{B}^\mathcal{C}(\mathbb{R}^2)$ are such that $(1 + \partial_x)^2(1 + \partial_\zeta)^2a(z, \zeta) = b(z, \zeta)$, then we have, for all $(z, \zeta) \in \mathbb{R}^2$,

$$(10) \quad a(z, \zeta) = \int_{\mathbb{R}^3} u(x, \eta)e^{ix\zeta}b(x + z, \xi + \zeta)v(\xi, \eta)d\xi d\eta.$$
For each \( l \in \mathbb{N} \), define \( c_l(z, \zeta) = \sqrt{2\pi} (\hat{u}, (B_{z, \zeta} F^* \otimes I) \hat{v}_l \rangle \), where \( v_l \) is the sequence given by Lemma 3 and \( B_{z, \zeta} \) is what one gets in (11), making \( A = O(a) \). Since, for every \( f \in \mathcal{S}(\mathbb{R}^2) \),

\[
\| \int [c_l(z, \zeta) - c(z, \zeta)] f(z, \zeta) \, dz \, d\zeta \| \leq \| u \|_{L^2} \cdot |B| \cdot \| v - v_l \|_{L^2} \cdot \int |f(z, \zeta)| \, dz \, d\zeta \to 0,
\]
as \( l \to \infty \), it is enough to show that

\[
\lim_{l \to \infty} \int [c_l(z, \zeta) - a(z, \zeta)] f(z, \zeta) \, dz \, d\zeta = 0.
\]

It follows from (2) that \( B = O(b) \), for \( b(x, \xi) = (1 + \partial_x)^2 (1 + \partial_\xi)^2 a(x, \xi) \). We then get \( B_{z, \zeta} = O(b_{z, \zeta}) \), for \( b_{z, \zeta}(x, \xi) = b(x + z, \xi + \zeta) \). Hence, if \( \varphi \) and \( \psi \) belong to \( \mathcal{S}(\mathbb{R}) \), then

\[
[(B_{z, \zeta} F^* \otimes I) (\varphi \otimes \psi)](x, \eta) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} b(x + z, \xi + \zeta) \varphi(\xi) \psi(\eta) \, d\xi.
\]

Using that \( v_l \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n) \), we then get

\[
c_l(z, \zeta) = \int_{\mathbb{R}^n} u(x, \xi) e^{ix\xi} b(x + z, \xi + \zeta) v_l(\xi, \eta) \, dx \, d\eta.
\]

By Lemma 2 we then have

\[
\int [c_l(z, \zeta) - a(z, \zeta)] f(z, \zeta) \, dz \, d\zeta = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u(x, \eta) e^{ix\xi} b(x + z, \xi + \zeta) (v(\xi, \eta) - v_l(\xi, \eta)) \, dx \, d\eta \, dz \, d\zeta.
\]

Since \( (x, \xi, \eta) \mapsto \overline{u(x, \eta)} (v(\xi, \eta) - v_l(\xi, \eta)) \) belongs to \( L^1(\mathbb{R}^3) \), we may interchange the order of integration and obtain that the above expression is bounded by

\[
\sup_{x, \xi} \| b(x, \xi) \| \cdot \| f \|_{L^1} \cdot \int_{\mathbb{R}^3} \| u(x, \eta) \| \cdot \| v(\xi, \eta) - v_l(\xi, \eta) \| \, dx \, d\eta \, dz \, d\zeta,
\]

which tends to zero, by Lemma 3 as we wanted.

This proves that \( S \) is a left-inverse for \( O \) when \( n = 1 \). We now comment on some of the changes needed to extend these definitions and proof for arbitrary \( n \). We have to replace \( u \) and \( v \), respectively, by \( u_n(x, \eta) = u(x_1, \eta_1) \cdots u(x_n, \eta_n) \) and \( v_n(\xi, \eta) = v(\xi_1, \eta_1) \cdots v(\xi_n, \eta_n) \). In the definitions on \( S \) and \( c_l \), we replace \( \sqrt{2\pi} \) by \( (2\pi)^{n/2} \), \( \langle \cdot, \cdot \rangle \) denotes the inner product of \( E_{2n} \) and \( F \in \mathcal{B}^*(E_n) \). The new \( B_{z, \zeta} \) is defined by

\[
B_{z, \zeta} = \prod_{j=1}^n (1 + \partial_{x_j})^2 (1 + \partial_{\xi_j})^2 |A_{z, \zeta}|.
\]

The integral in Lemma 2 is now an integral over \( \mathbb{R}^{3n} \) and the equality in (10) holds for all \( (z, \zeta) \in \mathbb{R}^{2n} \). The integral in Lemma 3 is also over \( \mathbb{R}^{3n} \), and \( v_l \) belongs to \( \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n) \).
3. Commutative Case

In this section, we assume that $C$ is equal to $C(\Omega)$, the algebra of continuous functions on a Hausdorff compact topological space $\Omega$. For each $\lambda \in \Omega$ and each $f \in \mathcal{S}^C(\mathbb{R}^n)$, we define $V_\lambda f \in \mathcal{S}(\mathbb{R}^n)$ by

$$(V_\lambda f)(x) = \langle f(x) \rangle(\lambda), \quad x \in \mathbb{R}^n.$$ 

$V_\lambda$ extends to a continuous linear mapping $V_\lambda : E_n \rightarrow L^2(\mathbb{R}^n)$, with $\|V_\lambda\| \leq 1$.

**Lemma 4.** Let there be given $T \in B^*(E_n)$, $f \in E_n$ and $\lambda \in \Omega$. If $V_\lambda f = 0$, then $V_\lambda Tf = 0$.

**Proof:** The equality $\langle V_\lambda g, V_\lambda g \rangle_{L^2(\mathbb{R}^n)} = \langle g, g \rangle(\lambda)$ holds for all $g \in \mathcal{S}^C(\mathbb{R}^n)$; hence also for all $g \in E_n$. We then have:

$$\sqrt{(f, f)(\lambda) \sqrt{(T^* Tf, T^* Tf)(\lambda)}} \leq \sqrt{(V_\lambda f, V_\lambda f)_{L^2(\mathbb{R}^n)}} \sqrt{(V_\lambda T^* Tf, V_\lambda T^* Tf)_{L^2(\mathbb{R}^n)}}.$$ 

This implies our claim. \hfill \Box

Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\tilde{\varphi} \in \mathcal{S}^C(\mathbb{R}^n)$ be defined by $[\tilde{\varphi}(x)](\lambda) = \varphi(x)$, for all $\lambda \in \Omega$ and all $x \in \mathbb{R}^n$. It is obvious that $V_\lambda \tilde{\varphi} = \varphi$. Given $T \in B^*(E_n)$ and $\lambda \in \Omega$, let $T_\lambda$ denote the unique linear mapping defined by the requirement that the diagram

$$\begin{array}{ccc}
\mathcal{S}(\mathbb{R}^n) & \xrightarrow{T_\lambda} & L^2(\mathbb{R}^n) \\
\downarrow V_\lambda & & \downarrow V_\lambda \\
\mathcal{S}^C(\mathbb{R}^n) & \xrightarrow{T} & E_n
\end{array}$$

commutes. This is well defined by Lemma 4 and because the left vertical arrow in the above diagram is surjective.

**Lemma 5.** For each $T \in B^*(E_n)$ and each $\lambda \in \Omega$, $T_\lambda$ extends to a bounded operator on $L^2(\mathbb{R}^n)$. Moreover, we have

$$\|T\| = \sup \{\|T_\lambda\|; \lambda \in \Omega\}.$$ 

**Proof:** Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, let $\tilde{\varphi}$ denote the element of $E_n$ defined after Lemma 4. We have:

$$\|T_\lambda \varphi\|_{L^2(\mathbb{R}^n)} = \|V_\lambda T \tilde{\varphi}\|_{L^2(\mathbb{R}^n)} \leq \|T \tilde{\varphi}\| \leq \|T\| \cdot \|\tilde{\varphi}\|_{L^2(\mathbb{R}^n)}.$$ 

This implies that $T_\lambda$ extends to a bounded operator on $L^2(\mathbb{R}^n)$ with norm bounded by $\|T\|$.

Let $M$ denote the right-hand side of (14). For each $\lambda \in \Omega$ and each $f \in \mathcal{S}^C(\mathbb{R}^n)$, using Lemma 4 and the first statement in its proof, we get:

$$\langle (Tf, Tf)(\lambda) \rangle = \|V_\lambda Tf, V_\lambda Tf\|_{L^2(\mathbb{R}^n)} \leq \|T_\lambda \varphi\| \cdot \|\varphi\|_{L^2(\mathbb{R}^n)} \leq M\|f\|_{L^2(\mathbb{R}^n)} \leq M\|f\|_2.$$ 

Taking the supremum in $\lambda$ on the left, we get $\|Tf\|_2 \leq M\|f\|_2$. \hfill \Box

Our goal in this Section is to prove that the mapping $S$ defined in the previous section is injective for $C = C(\Omega)$. This will finish the proof of Theorem 1.

Given $A \in \mathcal{H}$ such that $SA = 0$, we want to show that $A = 0$. In view of the following lemma, it suffices to show that $B = 0$, where $B = B_{0,0} \{B_{\epsilon, \zeta} \text{ as defined}}$
Lemma 6. If \( Y \in \mathcal{H}, Y_{z, \zeta} = T_{-z} M_{-\zeta} Y M_{\zeta} T_{z} \) (\( z, \zeta \in \mathbb{R}^n \)), and either \((1 + \partial_x)Y_{z, \zeta} \equiv 0 \) or \((1 + \partial_{\zeta})Y_{z, \zeta} \equiv 0 \) for some \( j \), then \( Y = 0 \).

By Lemma 5 in order to prove that \( B = 0 \), it suffices to show that \( B_{\lambda} = 0 \) for each \( \lambda \in \Omega \). For \( z \) and \( \zeta \) in \( \mathbb{R}^n \), define \( E_{z, \zeta} = M_{\zeta} T_{z} \). We then have \( B_{z, \zeta} = E_{z, \zeta}^* B E_{z, \zeta} \). Using that \( E_{z, \zeta} F^* = e^{iz \zeta} F^* E_{\zeta, -z} \), we may rewrite equation \( SA = 0 \) as

\[
e^{iz \zeta} ((E_{z, \zeta} \otimes I) \tilde{u}_n, (B F^* E_{\zeta, -z} \otimes I) \tilde{v}_n) = 0, \quad \text{for all } (z, \zeta).
\]

Evaluating this equation at \( \lambda \) gives:

\[
e^{iz \zeta} ((E_{z, \zeta} \otimes I) u_n, (B_{\lambda} F^* E_{\zeta, -z} \otimes I) v_n)_{L^2(\mathbb{R}^n)} = 0, \quad \text{for all } (z, \zeta).
\]

For a fixed \( \varphi \in \mathcal{C}_0^\infty (\mathbb{R}^{2n}) \) to be chosen soon, and for each bounded operator \( D \) on \( L^2(\mathbb{R}^n) \), define

\[
\Xi(D) = \int \varphi(z, \zeta) e^{iz \zeta} ((E_{z, \zeta} \otimes I) u_n, (D F^* E_{\zeta, -z} \otimes I) v_n)_{L^2(\mathbb{R}^n)} \, dz \, d\zeta.
\]

In case \( D \) is finite-rank, and hence we may take \( b^1, \ldots, b^k, c^1, \ldots, c^k \) in \( L^2(\mathbb{R}^n) \) such that, for all \( f \in L^2(\mathbb{R}^n) \),

\[
D F^* f = \sum_{j=1}^k b^j \langle c^j, f \rangle_{L^2(\mathbb{R}^n)},
\]

we have: \( \Xi(D) = 0 \).

Making the change of variables \( x - z = z', \xi - \zeta = \zeta' \) on the inner triple integral above, we get:

\[
\Xi(D) = \sum_{j=1}^k \int b^j (x) \bar{c}^j (\xi) \int \int e^{i z' \zeta} \varphi(x, \zeta') e^{-i z' \xi} \tilde{u}_n(x - z', \eta) e^{-i z' \zeta} v_n(\xi - \zeta, \eta) \, dx \, d\xi \, d\eta \, dx.
\]

For arbitrary \( \chi \) and \( \psi \) in \( \mathcal{C}_0^\infty (\mathbb{R}^n) \), let \( \varphi \) be defined by

\[
(1 + \partial_x)^2 (1 + \partial_{\zeta})^2 e^{i z \zeta} \bar{\chi}(-x) \psi(-\xi) = \varphi^2(x, \xi), \quad \varphi(x, \xi) = \varphi^2(-x, -\xi).
\]

Using the higher dimensional version of Lemma 2 mentioned at the end of Section 2, the right side of (17) becomes:

\[
\sum_{j=1}^k \int b^j (x) \bar{c}^j (\xi) \bar{\chi} (x) \psi(\xi) \, dx \, d\xi = \langle \chi, D F^* \psi \rangle_{L^2(\mathbb{R}^n)}.
\]

This shows that, for this choice of \( \varphi \),

\[
\Xi(D) = \langle \chi, D F^* \psi \rangle_{L^2(\mathbb{R}^n)},
\]

whenever \( D \) has finite rank.

Let \( \{ \phi_1, \phi_2, \ldots \} \) be an orthonormal basis of \( L^2(\mathbb{R}^n) \). For each positive integer \( j \), let \( P_j \) denote the orthogonal projection onto the span of \( \{ \phi_1, \ldots, \phi_j \} \). Cordes proved (\cite{Cordes} Chapter 8), between equations (3.27) and (3.29)) that, for any bounded
operator \( T \) on \( L^2(\mathbb{R}^n) \), one has \( \lim_{j \to \infty} \Xi(P_j TP_j) = \Xi(T) \). Applying this to \( T = B_\lambda \) and using (15), we get
\[
\Xi(B_\lambda) = \lim_{j \to \infty} \Xi(P_j B_\lambda P_j) = \lim_{j \to \infty} \langle \chi, P_j B_\lambda P_j F^* \psi \rangle_{L^2(\mathbb{R}^n)} = \langle \chi, B_\lambda F^* \psi \rangle_{L^2(\mathbb{R}^n)}.
\]
By (15), the left-hand side of this equality vanishes. Since \( \chi \) and \( \psi \) are arbitrary test functions, this shows that \( B_\lambda = 0 \). This finishes the proof of Theorem 1 (recall our remarks before and after the statement of Lemma 3).

4. Rieffel’s Conjecture

Given a skew-symmetric \( n \times n \) matrix \( J \) and \( F \in \mathcal{B}^C(\mathbb{R}^n) \) (i.e., \( F : \mathbb{R}^n \to C \) is smooth and, together with all its derivatives, is bounded), let us denote by \( L_F \) the pseudodifferential operator \( a(x, D) \in \mathcal{B}^s(E_n) \) with symbol \( a(x, \xi) = F(x + J\xi) \).

At the end of Chapter 4 in [10], Rieffel made a conjecture that may be rephrased as follows: any operator \( A \in \mathcal{B}^s(E_n) \) that is Heisenberg-smooth and commutes with every operator of the form \( R_G = b(x, D) \), where \( b(x, \xi) = G(x - J\xi) \) with \( G \in \mathcal{B}^C(\mathbb{R}^n) \), is of the form \( A = L_F \) for some \( F \in \mathcal{B}^C(\mathbb{R}^n) \).

Using Cordes characterization of the Heisenberg-smooth operators in the scalar case, we have shown [3] that Rieffel’s conjecture is true when \( C = \mathbb{C} \). It has been further proven by the second-named author [7] that Rieffel’s conjecture is true for any separable \( C^* \)-algebra \( C \) for which the operator \( O \) defined in (11) is a bijection.

Under this assumption, a result actually stronger than what was conjectured by Rieffel was proven in [7] Theorem 3.5: To get \( A = L_F \) for some \( F \in \mathcal{B}^C(\mathbb{R}^n) \), one only needs to require that a given \( A \in \mathcal{B}^s(E_n) \) is “translation-smooth” (i.e., the mapping \( \mathbb{R}^n \ni z \mapsto T_z AT_z \in \mathcal{B}^s(E_n) \) is smooth) and commutes with every \( R_G \) with \( G \in \mathcal{S}^C(\mathbb{R}^n) \). Combining this result with our Theorem 1 we then get:

**Theorem 2.** Let \( C \) be a unital commutative separable \( C^* \)-algebra. If a given \( A \in \mathcal{B}^s(E_n) \) is translation-smooth and commutes with every \( R_G, G \in \mathcal{S}^C(\mathbb{R}^n) \), then \( A = L_F \) for some \( F \in \mathcal{B}^C(\mathbb{R}^n) \).

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