Zeta Functions of $\mathbb{F}_1$-buildings

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Introduction

It has been observed by several authors that many formulae in the geometry of Bruhat-Tits buildings over a non-archimedean field of residue cardinality $q$ do still make sense for $q = 1$, in which case they coincide with the analogous formulae on the corresponding spherical geometry. Jacques Tits asked in [Tit57], whether the explanation of this phenomenon might be the existence of a “field of one element” $\mathbb{F}_1$ such that for a Chevalley group $G$ the group $G(\mathbb{F}_1)$ equals the Weyl group of $G$. In the first decade of the new millenium, various approaches to the elusive “field” $\mathbb{F}_1$ have been suggested, see [KOW03, Sou04, Dei05, Har07, TV09, CC10]. In the middle between the geometry of the full Bruhat-Tits building and the spherical geometry, which may be considered as the local geometry of an apartment, there is the geometry of a single apartment and its affine Weyl group. If the field of one element is the analogue of the residue field of a non-archimedean field, then the geometry of a single apartment should correspond to a “non-archimedean field in characteristic one”. In order to fix terms we shall write $\mathbb{Q}_1$. It appears that the approach via monoids of [Dei05] yields an explanation whereby $\mathbb{Q}_1$ is the infinite cyclic group and one can describe the building of $\text{PGL}_n(\mathbb{Q}_1)$ in terms of lattices in perfect analogy to the case of a non-archimedean field.

Another strand of investigations, which is connected to $\mathbb{F}_1$-theory in this paper, is the theory of generalized Ihara zeta functions. The Ihara zeta

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function for a finite graph is defined as an Euler product over closed cycles in the graph. It turns out to be a polynomial and can be expressed in terms of the characteristic function of the adjacency operator, this latter fact being known under the name Ihara formula. For higher dimensional buildings, the question for a generalized Ihara formula is still open. For the case of the group PGL$_3$, see [KLV10]. In Section 4, we present a formula of this type for $\mathbb{F}_1$-buildings.

1 The building of PGL$_n(\mathbb{Q}_1)$

According to the philosophy of [Dei05] we will denote the trivial monoid of one element by $\mathbb{F}_1 = \{1\}$. Further we write $\mathbb{Z}_1 = \{1, \tau, \tau^2, \ldots\}$ for the free monoid of one generator $\tau$ and $\mathbb{Q}_1 = \{\ldots, \tau^{-1}, 1, \tau, \ldots\}$ for its quotient group. A module of a given monoid $A$ is a set with an $A$-action. The category of $A$-modules contains direct sums, these turn out to be disjoint unions of modules. For a given natural number $n$ consider the $\mathbb{Q}_1$-module $V = V_n = \bigoplus_{j=1}^n \mathbb{Q}_1 = \bigsqcup_{j=1}^n \mathbb{Q}_1$. A lattice in $V$ is a $\mathbb{Z}_1$-submodule $L$ of $V$ with the property that $\mathbb{Q}_1L = V$. Two lattices $L, L'$ are homothetic if there exists $\alpha \in \mathbb{Q}_1$ with $L' = \alpha L$.

The group GL$_n(\mathbb{Q}_1)$ is by definition the automorphism group of the $\mathbb{Q}_1$-module $V = \bigoplus_{j=1}^n \mathbb{Q}_1$. Each such automorphism permutes the copies of $\mathbb{Q}_1$ that make up $V$ and multiplies the inhabitants of each copy by a scalar in $\mathbb{Q}_1$. The structure of this group is

$$\text{GL}_n(\mathbb{Q}_1) \cong \mathbb{Q}_1^n \rtimes \text{Per}(n),$$

where Per$(n)$ denotes the permutation group in $n$ letters. Its center is the subgroup GL$_1(\mathbb{Q}_1) \cong \mathbb{Q}_1$ embedded diagonally. The group PGL$_n(\mathbb{Q}_1)$ is defined to be

$$\text{PGL}_n(\mathbb{Q}_1) = \text{GL}_n(\mathbb{Q}_1) / \text{GL}_1(\mathbb{Q}_1).$$

One way to picture GL$_n(\mathbb{Q}_1)$ is to consider all $n \times n$ matrices with exactly one non-zero entry in every row and column and this entry be in $\mathbb{Q}_1$. Then PGL$_n(\mathbb{Q}_1)$ consist of homothety classes of such matrices.

The building $\mathcal{B}$ of PGL$_n(\mathbb{Q}_1)$ is the $(n - 1)$-dimensional building defined as follows. The set of vertices is the set of all homothety classes of lattices in $V_n$. For $2 \leq k \leq n - 1$, distinct vertices $[L_0], \ldots, [L_k]$ form the vertices of a $k$-dimensional face if, after adjusting the order, one has representatives
satisfying $L_0 \supset L_1 \supset \cdots \supset L_k \supset \tau L_0$. Besides the mere geometry of being a building, this lattice description of $B$ adds more features, like the order of the vertices of a face which is determined up to a cyclic permutation. First of all, there is a standard chamber $C_0$ given by the vertices

$L_0 = \langle e_1, \ldots, e_{n-2}, e_{n-1}, e_n \rangle,$

$L_1 = \langle e_1, \ldots, e_{n-2}, e_{n-1}, \tau e_n \rangle,$

$L_2 = \langle e_1, \ldots, e_{n-2}, \tau e_{n-1}, \tau e_n \rangle,$

$\vdots$

$L_n = \langle e_1, \tau e_2, \ldots, \tau e_{n-2}, \tau e_{n-1}, \tau e_n \rangle.$

Here $e_i$ stands the element 1 in the $i$-th copy of $\mathbb{Q}_i$ in $V$. A general lattice in $V$ can be written as

$L = \langle \tau^{c_1} e_1, \ldots, \tau^{c_n} e_n \rangle$

for some integers $c_1, \ldots, c_n$. We define its type to be $c_1 + \cdots + c_n \mod n$. Note that the group $(1, \ldots, 1) \rtimes \text{Per}(n)$ is the stabilizer of $L_0$.

Observe that under this construction the building of $\text{PGL}_n(\mathbb{Q}_1)$ is exactly isomorphic to any apartment of the building attached to $\text{PGL}_n(\mathbb{Q}_p)$ for a prime number $p$ and the group $\text{PGL}_n(\mathbb{Q}_1)$ becomes the affine Weyl group of $\text{PGL}_n(\mathbb{Q}_p)$.

2 The trace formula for $\text{PGL}_n(\mathbb{Q}_1)$

Let $\Delta \subset \mathbb{Z}^n$ denote the subgroup spanned by the element $(1, \ldots, 1)$. We write $G = \text{PGL}_n(\mathbb{Q}_1) \cong (\mathbb{Z}^n/\Delta) \rtimes \text{Per}(n) = \Lambda \rtimes \text{Per}(n)$. The permutation subgroup $\text{Per}(n)$ by $K$. Let $\Gamma \subset G$ be a subgroup of finite index. The trace formula [DE09] for the pair $(G, \Gamma)$ says that for any $f \in \ell^1(G)$ one has

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \text{tr} \pi(f) = \sum_{[\gamma]} \#(\Gamma \gamma \backslash G) O_\gamma(f),$$

where $L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi) \pi$ is the decomposition into irreducibles, the sum on the right hand side runs over all conjugacy classes $[\gamma]$ in $\Gamma$, the groups $G_\gamma$ and $\Gamma_\gamma$ are the centralizers of $\gamma$ in $G$ and $\Gamma$, and

$$O_\gamma(f) = \sum_{x \in G/G_\gamma} f(x^{-1} \gamma x).$$
Proposition 2.2. The unitary dual \( \hat{G} \) of \( G \) is the set of all representations \( \pi_{\chi, \sigma} \), where \( \chi \) runs through a set of representatives of \( \Lambda / K \) and \( \sigma \) runs 
through $\hat{K}_\chi$. Note that as special case we have $\chi = 1$ in which case $K_\chi = K$, so $\hat{K}$ is in a canonical way a subset of $\hat{G}$. Every $\pi \in \hat{G}$ is finite-dimensional.

Proof. The space $V_{\chi, \sigma}$ is the space of $L^2$-sections of the homogeneous vector bundle on the $K$-orbit of $\chi$ given by the pair $(\chi, \sigma)$. These make up the unitary dual of the semi-direct product $G$.

The group $K \cong \text{Per}(n)$ acts on $\Lambda^+_R \cong \mathbb{R}^n/\Delta(\mathbb{R})$ and the closed cone

$$\Lambda^+_R = \{[x_1, \ldots, x_n] \in \Lambda_R : x_1 \geq x_2 \geq \cdots \geq x_n\}$$

is a set of representatives of $\Lambda_R/K$. Define elements of the dual space,

$$\alpha_1(x) = n!(x_1 - x_2), \ldots, \alpha_{n-1}(x) = n!(x_{n-1} - x_n).$$

Then $\Lambda^+_R$ is the set of all $x \in \Lambda_R$ with $\alpha_1(x) \geq 0, \ldots, \alpha_{n-1}(x) \geq 0$. Any subset $S \subset \{1, \ldots, n-1\}$ defines a face of the cone $\Lambda^+_R$ by

$$\Lambda^+_S = \{x \in \Lambda^+_R : \alpha_j(x) = 0 \iff j \in S\}.$$  

The cone $\Lambda^+_S$ is the disjoint union of its faces, in particular, $\Lambda^+_R$ is the open interior of $\Lambda^+_R$ and $\Lambda^+_\{1, \ldots, n-1\}$ is the point 0.

Lemma 2.3. The set $G^+_R \cap (\Lambda^+_R \times K)$ contains a set of representatives for $G_R$ modulo conjugation. If $(a, k), (a', k') \in G^+_R \cap (\Lambda^+_R \times K)$ are conjugate, then $a = a'$ and $k' = pkp^{-1}$ for some $p \in K$ with $p(a) = a$.

Therefore, there exists unique conjugation-invariant functions $l_1, \ldots, l_{n-1}$ on $G_R$ such that

$$l_j(v, k) = \alpha_j(v), \quad \text{if } (v, k) \in G^+_R \cap (\Lambda^+_R \times K).$$

The functions $l_1, \ldots, l_{n-1}$ are integer-valued on the subgroup $G$.

Proof. First, $G^+_R$ is a set of representatives with respect to $\Lambda_R$-conjugation, which is $K$-stable. As every element of $G^+_R$ is $K$-conjugate to an element of $\Lambda^+_R \times K$, the first claim follows. Now let $(a, k), (a', k') \in G^+_R \cap (\Lambda^+_R \times K)$ be conjugate, say $a'.k.) = (v, p)(a, k)(v, p)^{-1}$. Then

$$(a', k') = (p(a) + v - pkp^{-1}(v), pkp^{-1}).$$

Since $k(a) = a$, it follows $k'(p(a)) = p(a)$, i.e., $p(a) \in \text{Eig}(k', 1)$. As $a' \in \text{Eig}(k', 1)$ we get $v - pkp^{-1}(v) = v - k'(v) \in \text{Eig}(k', 1)$. This can only be if
\( v - k'(v) = 0 \), so \( a' = p(a) \). But as \( a, a' \) are both in \( \Lambda^+_R \), it follows \( a = a' \) as claimed.

Finally, we need to show that \( l_j(x) \) is integral for \( x \in G \). Write \( x = (v, p) \) with \( v \in \mathbb{Z}^n \). Then there exists \( a \in \mathbb{R}^n \) such that \( v + (1 - p)(a) \in \text{Eig}(p, 1) \).

We have to show that this vector lies in \( \frac{1}{m} \mathbb{Z}^n \). For this write \( p \) a product of disjoint cycles to reduce to the case of \( p \) being one cycle, say \( p = (1, 2, \ldots, k) \) for some \( k \leq n \). Then if \( v = (1, 0, \ldots, 0) \), one finds \( a \in \mathbb{R}^n \) with

\[
 v - (1 - p)(a) = \frac{1}{k} (1, \ldots, 1, 0, \ldots, 0) \in \text{Eig}(p, 1).
\]

From this the claim follows. \( \square \)

For \( u \in \mathbb{C}^{n-1} \) and \( x \in G \) we write

\[
 u^l(x) = u_1^l(x) \cdots u_{n-1}^l(x).
\]

for \( u \in \mathbb{C}^{n-1} \) let

\[
 ||u||_{\text{max}} = \max(|u_1|, \ldots, |u_{n-1}|).
\]

**Theorem 2.4** (Several variable Selberg type zeta function). The infinite sum

\[
 S_\Gamma(u) = \sum_{[\gamma]} \#(\Gamma \gamma \backslash G_\gamma) u^l(\gamma)
\]

converges locally uniformly for \( ||u||_{\text{max}} < 1 \) to a rational function in \( u \). There exist \( p_1, \ldots, p_k \in \mathbb{T}^{n-1} \) and a polynomial \( Q(u) \) such that

\[
 S_\Gamma(u) = \frac{Q(u)}{\prod_{i=1}^k \prod_{j=1}^{n-1} (u_j - p_{i,j})}.
\]

**Proof.** Let \( R \subset G \) be any set of representatives of \( G \) modulo conjugation. Define a function \( f_u \) on \( G \) by

\[
 f_u(x) = \begin{cases} 
 u^l(x) & x \in R, \\
 0 & x \notin R.
\end{cases}
\]

Here we use the common convention that \( 0^0 = 1 \).

**Lemma 2.5.** If \( ||u||_{\text{max}} = \max(|u_1|, \ldots, |u_{n-1}|) < 1 \), then \( f_u \in \ell^1(G) \).
Proof. By Lemma 2.3 it suffices to show that we have
\[ \sum_{x \in \frac{1}{n} \Lambda} |f_u(x)| < \infty, \]
where \( \frac{1}{n} \Lambda = \frac{1}{n} \mathbb{Z}^n / \Delta \). Modulo the diagonal \( \Delta \), every \( x \in \mathbb{R}^n \) can be assumed to have \( x_n = 0 \). Then the sum is
\[ \sum_{x \in \mathbb{Z}^n / \Lambda} |u_1|^{x_1} \cdots |u_{n-2}|^{x_{n-2}} |u_{n-1}|^{x_{n-1}}. \]
If all \( |u_j| \) are \( \leq q \) for some \( 0 < q < 1 \), then each summand is less that \( q^{x_1} \).

So we have to show that for \( q < 1 \) one has \( \sum_{k=0}^{\infty} c_k q^k < \infty \), where \( c_k \) is the number of tuples of integers with \( k \geq x_2 \geq \cdots \geq x_{n-1} \geq 0 \) which is \( \leq (k + 1)^{n-1} \), whence the claim. \( \square \)

We want to plug \( f_u \) into the trace formula. As either side of the trace formula is invariant under conjugation, neither depends on the choice of \( R \). We give a special choice for our computations.

Every subset \( S \subset \{1, \ldots, n-1\} \) defines a partition \( n = n_1 + \cdots + n_r \) given by
\( \Lambda^+_S = \{(t_1 e(n_1), \ldots, t_r e(n_r)) \in \mathbb{R}^n / \Delta : t_1 > t_2 > \cdots > t_r\} \),
where \( e(m) = (1, \ldots, 1) \in \mathbb{R}^m \). For \( x \in \Lambda^+_S \) we write \( x = [t_1, \ldots, t_r]_S \) for these coordinates. We define
\[ \Lambda^+_S \frac{1}{n} \mathbb{Z} \times R_S \]

Let \( K_S \) denote the pointwise stabilizer of \( \Lambda^+_S \) in \( K \) and fix a set \( R_S \subset K_S \) of representatives of \( K_S \) modulo conjugation.

If \( E \subset F \) is an extension of groups, the \( F \)-conjugacy class of an element \( e \) of \( E \) intersected with \( E \), may decompose into several \( E \)-conjugacy classes. In the following Lemma we show that this is not the case for the extension \( G \subset G_{\mathbb{R}} \).

Lemma 2.6. Every \( x \in G \) is in \( G_{\mathbb{R}} \) conjugate to a unique element of
\[ \bigcup_S \Lambda^+_S \frac{1}{n} \mathbb{Z} \times R_S. \]

Every element of this set is \( G_{\mathbb{R}} \)-conjugate to an element of \( G \). If \( x, y \in G \) are \( G_{\mathbb{R}} \)-conjugate, then they are \( G \)-conjugate.
Proof. The first two statements are proven similar to Lemma 2.3. For the third suppose \( x = (a, k) \) and \( y = (a', k') \) are \( G_{\mathbb{R}} \)-conjugate, i.e., there exists \( (v, p) \in G_{\mathbb{R}} \) with \( (a', k') = (v + p(a) - pkp^{-1}(v), pkp^{-1}) \). Then \( v - k'(v) \) lies in the intersection of \( \text{Eig}(k', 1)^\perp \) and \( \mathbb{Z}^n \). Writing \( p \) as product of disjoint cycles as in the proof of Lemma 2.3 one sees that there exists \( w \in \mathbb{Z}^n \) such that \( v - k'(v) = w - k'(w) \). \( \square \)

Let now \( \pi \in \hat{G} \). Then \( \pi \) is finite-dimensional and

\[
\text{tr} \, \pi(f_u) = \sum_{x \in R} u^{l(x)} \text{tr} \, \pi(x).
\]

Since the functions \( l_1, \ldots, l_{n-1} \) are defined on \( G_{\mathbb{R}} \) and are conjugation-invariant, we may, in the computation assume that \( R \) is equal to the set in Lemma 2.6 although this set is not contained in \( G \). In the expression \( \pi(x) \) for \( x \in R \setminus G \) we then replace \( x \) with any \( G_{\mathbb{R}} \)-conjugate inside \( G \). We then compute

\[
\text{tr} \, \pi(f_u) = \sum_S \sum_{a \in \Lambda^+_S \frac{1}{\mathbb{Z}}} u^{l(a)} \sum_{k \in R_S} \text{tr} \, \pi(ak).
\]

Let \( V_\pi = V_{\pi,1} \oplus \cdots \oplus V_{\pi,m} \) be the decomposition into \( \Lambda \)-eigenspaces, i.e., each \( a \in \Lambda \) acts on \( V_{\pi,j} \) as multiplication by \( \lambda_j^a \) for some character \( \Lambda \to \mathbb{T} \); \( a \mapsto \lambda_j^a \), where \( \mathbb{T} \) is the circle group, i.e., the set of complex numbers of absolute value one. Then \( \text{tr} \, \pi(f_u) \) equals

\[
\sum_S \sum_{j=1}^m \sum_{a \in \Lambda^+_S \frac{1}{\mathbb{Z}}} u^{l(a)} \lambda_j^a \sum_{k \in R_S} \text{tr} \, (\pi(k)|V_{\pi,j})_{=\mu_j}.
\]

Lemma 2.7. Let \( V \) denote a \( \mathbb{Q} \) vector space of dimension \( r \in \mathbb{N} \). Let \( V_{\mathbb{R}} = V \otimes \mathbb{R} \) and let \( C \subset V_{\mathbb{R}} \) be an open rational sharp cone with \( r \) sides, i.e., its closure \( \overline{C} \) does not contain a line and there exist \( \alpha_1, \ldots, \alpha_r \in \text{Hom}(V, \mathbb{Q}) \) such that

\[
C = \{ v \in V_{\mathbb{R}} : \alpha_1(v) > 0, \ldots, \alpha_r(v) > 0 \}.
\]

Let \( \Sigma \subset V \) be a lattice, i.e., a finitely generated subgroup which spans \( V \). Then there exists a finite subset \( S \subset \Sigma \) and \( a_1, \ldots, a_r \in \Sigma \) such that \( C \cap \Sigma \) is the set of all \( v \in V \) of the form

\[
v = v_0 + k_1a_1 + \cdots + k.ra_r,
\]

where \( v_0 \in S \) and \( k_1, \ldots, k_r \in \mathbb{N}_0 \). The vector \( v_0 \) and the numbers \( k_1, \ldots, k_r \in \mathbb{N}_0 \) are uniquely determined by \( v \).
Proof. For \( j = 1, \ldots, r \) let \( a_j \in \Sigma \) be the unique element such that \( \alpha_i(a_j) = 0 \) for \( i \neq j \) and \( \alpha_j(a_j) > 0 \) and minimal. Then \( a_1, \ldots, a_r \) is a basis of \( V \) inside \( \Sigma \), hence it generates a sublattice \( \Sigma' \subset \Sigma \). Let \( S \) be a set of representatives of \( \Sigma/\Sigma' \) which may be chosen such that each \( v_0 \in S \) lies in \( C \), but for every \( j = 1, \ldots, r \) the vector \( v_0 - a_j \) lies outside \( C \). It is clear that every \( v \) of the form given in the lemma is in \( C \cap \Sigma \).

For the converse, let \( v \in C \cap \Sigma \). Then there are uniquely determined \( v_0, k_1, \ldots, k_r \in \mathbb{Z} \) such that \( v = v_0 + k_1a_1 + \cdots + k_ra_r \). We have to show that \( k_1, \ldots, k_r \geq 0 \). Assume that \( k_j < 0 \). Then

\[
0 < \alpha_j(v) = \alpha_j(v_0) + k_j\alpha_j(a_j) \leq \alpha_j(v_0) - \alpha_j(a_j) = \alpha_j(v_0 - a_j)
\]

and the latter is \( \leq 0 \), as \( v_0 - a_j \) lies outside \( C \), a contradiction! \( \Box \)

We apply this lemma to \( V \) being the \( \mathbb{Q} \)-span of \( \Lambda \cap \Lambda_S^+ \) and \( C = \Lambda_S^+ \). We find that \( \text{tr} \pi(f_u) \) equals

\[
\sum S \sum_{j=1}^m \mu_j \sum_{a_0 \in F} \sum_{k_1, \ldots, k_r = 0}^{\infty} u^l(a_0 + k_1a_1 + \cdots + k_ra_r) \lambda_j^{a_0 + k_1a_1 + \cdots + k_ra_r}
\]

\[
= \sum S \sum_{j=1}^m \mu_j \sum_{a_0 \in F} \sum_{l(a_0)} u^l(a_0) \lambda_j^{a_0} \frac{1}{1 - \lambda_j^{a_0} u^{l(a_1)}} \cdots \frac{1}{1 - \lambda_j^{a_r} u^{l(a_r)}}
\]

where in the last row all sums are finite. We have shown

**Lemma 2.8.** For each \( \pi \in \hat{G} \), the map \( u \mapsto \text{tr} \pi(f_u) \), defined for small \( u \), is a rational function in \( u \).

We now finish the proof of the theorem. For \( ||u||_{\text{max}} < 1 \) the function \( f_u \) goes into the trace formula. This in particular means that the sum

\[
\sum_{[\gamma]} \#(\Gamma \gamma \backslash G) \mathcal{O}_\gamma(f_u) = S_\Gamma(u)
\]

converges locally uniformly. As the quotient \( \Gamma \backslash G \) is finite, the space \( L^2(\Gamma \backslash G) \) is finite-dimensional, so the sum \( \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \text{tr} \pi(f_u) \) is a finite sum, i.e., the coefficient \( N_\Gamma(\pi) \) vanishes for almost all \( \pi \). So \( S_\Gamma(u) \) is a finite sum of rational functions of the form in Lemma 2.8. As the representation \( \pi \) is unitary, the complex numbers \( \lambda_1, \ldots, \lambda_m \) in the lemma are all in \( \mathbb{T} \). The proof of the theorem is finished. \( \Box \)
3 Geometric interpretation

In this section we assume $\Gamma$ to be torsion-free. It follows that $\Gamma$ is the fundamental group of $\Gamma \backslash B$, where $B \cong \mathbb{R}^{n-1}$ is the building of $\text{PGL}_n(\mathbb{Q})$. Then each conjugacy class $[\gamma]$ gives a homotopy class of loops in $\Gamma \backslash B$, where a loop is a continuous map $S^1 \to B$. The euclidean structure makes $B$ and $\Gamma \backslash B$ a Riemannian manifold, where the geodesics in $B$ are straight lines.

Lemma 3.1. Every loop on $\Gamma \backslash B$ is homotopic to a closed geodesic.

Proof. Any loop is homotopic to the loop given by an element $\gamma$ of $\Gamma$. As $\Gamma$ is torsion-free and discrete, $\gamma$ fixes no point in $B$. The set $P_\gamma = \{x \in B : d(x, \gamma x) is minimal\}$ is a union of straight lines on which $\gamma$ acts by a translation. Each of these lines defines a closed geodesic.

The group $\Gamma$ is called a translation group, if $\Gamma \subset \Lambda$. Since $\Lambda$ has finite index in $G$, every $\Gamma$ contains a finite-index translation subgroup.

We say that a geodesic in $B$ or $\Gamma \backslash B$ is a rational geodesic, if it is contained in the 1-skeleton $B_1$ or $(\Gamma \backslash B)_1$.

Lemma 3.2. If the homotopy class attached to a given class $[\gamma]$ contains a rational geodesic then one has $l_j(\gamma) = 0$ for all but one $j \in \{1, \ldots, n-1\}$.

Conversely, if $l_j(\gamma) = 0$ for all but one $j \in \{1, \ldots, n-1\}$, then the homotopy class attached to some power $\gamma^k$ of $\gamma$ contains a rational geodesic. The minimal number $k$ as above is $\leq n$ and if $\Gamma$ is a translation group, one always has $k = 1$.

Proof. Suppose that the homotopy class given by $[\gamma]$ contains a rational geodesic $c$ in $\Gamma \backslash B$. This means that there exists a lift $\tilde{c} \in B_1$ with $\gamma \tilde{c}$ and $\gamma$ induces a translation on $\tilde{c}$. Since $\gamma$ preserves the affine structure on $B$, we may choose the origin in a vertex on the rational geodesic $\tilde{c}$, or, which amounts to the same, conjugate $\gamma$ by an element on $\Lambda$. Then $\gamma$ maps $x \in B$ to $Fx + t$, where $F(t) = t$, so $\gamma \in G_\mathbb{R}^+$. Conjugating by some $k \in K$, we may assume $t \in \Lambda_\mathbb{R}^+$. As $\tilde{c}$ is rational, $t$ lies in a 1-dimensional face of $\Lambda_\mathbb{R}^+$, which is equivalent to saying $l_j(\gamma) = 0$ for all but one $j$.

For the converse direction, assume $\gamma x = F(x) + t$ with $F(t) = t$. Let $k$ be the order of $F$ on $K = \text{Per}(n)$, then $k$ is a divisor of $n$ and $\gamma^k$ is a translation. As $t$ lies in a one-dimensional face of $\Lambda_\mathbb{R}^+$, $\gamma$ closes a rational geodesic. 

4 An Ihara type formula

The Ihara zeta function of a finite, \(q+1\) regular graph \(X\) is defined as the infinite product

\[
Z_X(u) = \prod_c (1 - u^{l(c)}),
\]

where the product runs over all backtrackingless closed cycles \(c\) and \(l(c)\) denotes the length. The Ihara formula asserts that

\[
Z_X(u) = \frac{\det(1 - Au + qu^2)}{(1 - u^2)^\chi},
\]

where \(A\) is the adjacency operator and \(\chi\) the Euler-characteristic. It has been proven in ascending order of generality in [Iha66, Has89, Sun86, Bas92]. For higher dimensional buildings, the question for a generalized Ihara formula is still open. For the \(\text{PGL}_3\)-case see [KLW10].

Fix the following set of generators of \(\Lambda\),

\[
S = \{ [a_1, \cdots, a_n] \in \Lambda, \max_{1 \leq i, j \leq n} \{|a_i - a_j|\} = 1 \}.
\]

The Cayley graph \(X\) of \((\Lambda, S)\) is a \((2^n - 2)\)-regular infinite graph naturally embedded into the Euclidean space \(\mathbb{R}^n\), which happens to coincide with the 1-skeleton \(B_1\) of the building attached to \(\text{PGL}_n(\mathbb{Q}_1)\) of Section 1.

Now for \(a = [a_1, \cdots, a_n] \in \Lambda\), define its type \(\tau(a)\) as \(a_1 + \cdots + a_n \mod n\). Fix a subgroup \(\Gamma\) of \(\Lambda\) of finite index \(N\). Then the quotient of \(X\) by \(\Gamma\), denoted by \(X_\Gamma\), is the Cayley graph \((\Lambda/\Gamma, S)\) of \(N\) vertices; it can be considered a subset of the \((n-1)\)-dimensional torus \(\Lambda \otimes \mathbb{R}/\Gamma \cong \mathbb{R}^{n-1}/\Gamma\). We shall assume that \(\Gamma\) only contains type zero elements so that the type is well-defined on \(\Lambda/\Gamma\). For a function \(f : \Lambda/\Gamma \to \mathbb{C}\), define

\[
A_i f(g\Gamma) = \sum_{s \in S, \tau(s) = i} f(s g\Gamma),
\]

call the type \(i\) adjacency operator of \(X_\Gamma\). Note that \(A = A_1 + \cdots + A_{n-1}\) is indeed the adjacency operator of the graph \(X_\Gamma\).

The graph zeta function of \(X\) is given by

\[
Z(X_\Gamma, u) = \prod_{[c]} (1 - u^{l(c)})
\]
where \([c]\) runs through all equivalence classes of primitive tailless backtrackless closed paths in \(X\). Moreover, the infinite product \(Z(X\Gamma, u)\), convergent for small \(u \in \mathbb{C}\), actually converges to a polynomial and it can be expressed as

\[
Z(X\Gamma, u) = \frac{\det(I_N - Au + (2^n - 3)u^2I_N)}{(1 - u^2)^{\chi(X\Gamma)}}
\]

where \(I_N\) is the \(N \times N\) identity matrix and \(\chi(X\Gamma)\) is the Euler characteristic of \(X\Gamma\). In this paper, we study a different kind of zeta function which encodes more information about the underlying space \(\mathbb{R}^n/\Gamma\).

A path \((v_0, \ldots, v_n)\) on the graph \(X\) is called a positive geodesic if it is geodesic in \(\mathbb{R}^n\) and \(\tau(v_{i+1}) = \tau(v_i) + 1\) for \(i\). A closed path \(c\) in \(X\Gamma\) is a positive geodesic if its lifting in \(X\) is a positive geodesic; \(c\) is primitive if \(p\) is not equal to a shorter path repeated several times. Two closed paths in \(X\Gamma\) are equivalent if one can be obtained by the other by changing the starting vertex. Denote the equivalence class of a closed path \(c\) by \([c]\). In this paper, we shall consider the following zeta function on \(X\Gamma\)

\[
Z_+(u) = Z_+(X\Gamma, u) = \prod_{[c]} (1 - u^{l(c)})
\]

where \([c]\) runs through all equivalence class of primitive positive closed geodesic in \(X\).

First, we will show that

\begin{theorem}
The infinite product \(Z_+(X\Gamma, u)\) converges to a polynomial which can be expressed as

\[
Z_+(X\Gamma, u) = \det(I_N - A_1u + \cdots + (-1)^{n-1}A_{n-1}u^{n-1} + (-1)^n u^n I_N).
\]

\end{theorem}

\begin{remark}
As the Euler characteristic of the torus is zero, we can rewrite the above equation as

\[
Z_+(X\Gamma, u) = \frac{\det(I_N - A_1u + \cdots + (-1)^{n-1}A_{n-1}u^{n-1} + (-1)^n u^n I_N)}{(1 - u^n)^{\chi(\mathbb{R}^n/\Gamma)}}.
\]

Now given a character \(\rho : \Lambda \to \mathbb{C}^\times\), define the Satake parameters of \(\rho\) to be \(\rho_j = \rho(e_j)\), where \(e_1, \ldots, e_n\) is the standard basis of \(\mathbb{Z}^n\). Then \(\rho_1 \cdots \rho_n = 1\) and we define the Langlands L-function of \(\rho\) to be

\[
L(\rho, u) = \prod_{j=1}^{n} (1 - \rho_j u).
\]
Finally, we define the $L$-function of $\Lambda/\Gamma$ as

$$L(\Lambda/\Gamma, u) = \prod_{\rho \in \hat{\Lambda}/\Gamma} L(\rho, u).$$

Note that there is no multiplicity showing up as an exponent to the factor $L(\rho, u)$ as, since $\Lambda/\Gamma$ is an abelian group, characters do not come with multiplicities other than one.

**Theorem 4.2.** $Z_+(X_\Gamma, u) = L(\Lambda/\Gamma, u)$.

**Proof of Theorem 4.1 and 4.2.** Let $s_i$ be an element of $S$ which has a representative in $\mathbb{Z}^n$ with all coordinates equal to zero except the $i$-th coordinate equal to 1. Set $S_0 = \{s_1, \cdots, s_n\}$, then

$$S = \left\{ \sum_{s \in S'} s : S' \text{ is a proper subset of } S_0 \right\}.$$

Given a vertex $g + \Gamma$ in $X_\Gamma$, each positive geodesic has the form

$$g + \Gamma \to (s_i + g) + \Gamma \to (2s_i + g) + \Gamma \to \cdots .$$

and it is primitive if its length is equal to the order of $s_i$ in $\Lambda/\Gamma$. Denote this order by $m_i$, then the contribution of such kind of positive closed geodesics is given by

$$(1 - u^{m_i})^{X_{\Gamma}} = \det (I_N - \lambda(s_i)).$$

Here $\lambda$ is the regular representation of $\Lambda/\Gamma$. We conclude that

$$Z_+(X_\Gamma, u) = \prod_{i=1}^{n} \det(I_N - \lambda(s_i)u).$$

The right hand side is clearly equal to $L(\Lambda/\Gamma, u)$. On the other hand,

$$A_i = \sum_{S' \subset S_0, |S'| = i} \lambda \left( \sum_{s \in S'} s \right) = \sum_{S' \subset S_0, |S'| = i} \prod_{s \in S'} \lambda(s),$$

so that

$$I_N - A_1 u + \cdots + (-1)^{n-1} A_{n-1} u^{n-1} + (-1)^n u^n I_N = \prod_{i=1}^{n} (I_N - \lambda(s_i)u),$$

which completes the proof of the two theorems. \qed
5 Comparison

We shall now compare the different types of zeta functions in the following theorem.

Theorem 5.1. If \( \Gamma \) is a translation group, then

\[
S_\Gamma(x, 0, \ldots, 0) = (n - 1)! \frac{Z'_+}{Z_+}(x).
\]

Proof. Let \( N \) be the index of \( \Gamma \) in \( \Lambda \). Then \( N \) equals the number of vertices in \( \Gamma \backslash \mathcal{B} \). As \( \Gamma \subset \Lambda \) we have for every \( \gamma \in \Gamma \) that \( G_\gamma = \Lambda \rtimes K_\gamma \) and so \( \#(\Gamma \backslash G_\gamma) = N \# K_\gamma \). In the sum \( S_\Gamma(u) = N \sum_{[\gamma]} \# K_\gamma u^{l(\gamma)} \) we find that if \( u = (x, 0, \ldots, 0) \) there will only survive those summands with \( l_2(\gamma) = \cdots = l_{n-1}(\gamma) = 0 \), i.e., those \( \gamma \) which are in \( G \) conjugate to an element of the form \( (c, 0, \ldots, 0) \) for some \( c > 0 \). The \( K \)-centralizer \( K_\gamma \) of such an element is isomorphic to \( \text{Per}(n-1) \), hence \( \# K_\gamma = (n-1)! \). Such a \( \gamma \) closes a geodesic \( c \) in the 1-skeleton and the number of vertices in that geodesic equals \( l(\gamma_0) \), where \( \gamma_0 \) is the underlying primitive element. The union of all geodesics inside the 1-skeleton of \( \Gamma \backslash \mathcal{B} \) which are homotopic to \( c \) contains all vertices of \( \Gamma \backslash \mathcal{B} \), hence, if \( k \) is their number, one has \( N = kl(\gamma_0) \), or

\[
S_\Gamma(x, 0, \ldots, 0) = (n - 1)! \sum_c l(c_0) x^{l(c)},
\]

where the sum runs over all positive closed geodesics in \( \Gamma \backslash \mathcal{B} \). On the other hand one has

\[
\frac{Z'_+}{Z_+}(x) = \left( \log \left( \prod_{c_0} (1 - x^{l(c_0)}) \right) \right)' = \left( \sum_{c_0} \sum_{j=1}^{\infty} x^{l(c_0)j} \right)' = \sum_{c_0} l(c_0) \sum_{j=1}^{\infty} x^{l(c_0)j} = \sum_c l(c_0) x^{l(c)}.
\]

\[ \square \]

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