Discrete boundary treatment for the shifted wave equation in second order form and related problems

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Abstract.

We present strongly stable semi-discrete finite difference approximations to the quarter space problem ($x > 0$, $t > 0$) for the first order in time, second order in space wave equation with a shift term. We consider space-like (pure outflow) and time-like boundaries, with either second or fourth order accuracy. These discrete boundary conditions suggest a general prescription for boundary conditions in finite difference codes approximating first order in time, second order in space hyperbolic problems, such as those that appear in numerical relativity. As an example we construct boundary conditions for the Nagy-Ortiz-Reula formulation of the Einstein equations coupled to a scalar field in spherical symmetry.

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1. Introduction

One of the major obstacles to obtaining long-term stability in numerical simulations of strongly gravitating systems, such as a binary black hole space-time, is the proper treatment of boundaries. In general, when hyperbolic formulations of Einstein’s equations based on space-like hypersurfaces are used, one can distinguish between two types of boundaries: inner and outer boundaries. Whereas the inner boundary is a space-like hypersurface introduced to excise the singularity from the computational domain, the outer boundary is an artificial time-like surface introduced because of limited computational resources.

If maximally dissipative boundary conditions are used with symmetrizable hyperbolic fully first order formulations of the Einstein equations [1, 2, 3, 4, 5] it is known how to construct stable schemes to high order of accuracy. (Another desirable property of boundary conditions in general relativity is that they are compatible with the constraints, but we do not consider this here.) With first order in time and second order in space formulations such as [6, 7], on the other hand, although the continuum problem is reasonably well understood [8, 9, 10, 11], much less is known about discretisations. The issue of constructing stable finite difference approximations of boundary conditions for first order in time and second order in space formulations of Einstein’s equations has not yet been investigated and it is the focus of this work. As a first step we consider...
the shifted wave equation in one space dimension and look at both second and fourth-order accuracy. A one-dimensional toy problem already captures many of the technical difficulties that arise in the higher dimensional case. For recent progress regarding fully first order or fully second order formulations, see [12] [13].

In Section 2 we review the continuum initial-boundary value problem for the shifted wave equation, both to fix the notation and to state the energy estimates that establish well-posedness of the continuum problem. Section 3 states our main results: a prescription for strongly stable finite differencing schemes for the quarter space problem \((x > 0, t > 0)\), with second and fourth order accuracy, and for the two cases where the boundary is time-like or space-like (outflow).

The main part of the paper is the proof of the stability and accuracy of these prescriptions at the semi-discrete level. Instead of using an energy method, we use the Laplace transform method, as described in Chapter 12 of [14]. In its final step our proof relies on plotting a function of a complex variable to show that it is bounded. In Section 4 we apply this method to the semi-discrete initial-boundary value problem for the shifted wave equation and prove strong stability, estimate (84), and convergence. Finally, we turn the semi-discrete scheme into a fully discrete one using the fourth-order Runge-Kutta method for integrating in time. In Section 5 we present numerical tests confirming the desired order of convergence of the resulting schemes for the wave equation. As an application, in Section 6 we consider the Nagy-Ortiz-Reula formulation of the Einstein equations in spherical symmetry, where the boundary conditions introduced in Section 3 are generalized to constraint-preserving boundary conditions.

For the second-order in space wave equation without a shift term, stable and accurate boundary conditions can be constructed using the summation by parts rule [15] [16]. The proof of stability and accuracy of these methods uses the existence of a discrete conserved energy which is the precise analogue of the continuum energy. In Appendix A we attempt to generalise these methods to the wave equation with a shift, but we fail. The reason is that in the presence of a shift, three separate summation by parts properties must be obeyed instead of one with vanishing shift, and this appears to overspecify the finite differencing scheme. This suggests to us that standard discrete energy methods are not suitable for the second-order in space wave equation with a shift, and by extension for other second-order in space hyperbolic systems.

For completeness, and for comparison with the wave equation, we give the analogous results for the advection equation in Appendix B. In Appendix C we briefly compare our results with those of [12].

2. The continuum initial-boundary value problem

The one-dimensional shifted wave equation consists of a system of coupled linear partial differential equations (PDEs) of the form

\[
\partial_t \phi = \beta \partial_x \phi + \Pi + F^\phi, \tag{1}
\]

\[
\partial_t \Pi = \beta \partial_x \Pi + \partial_x^2 \phi + F^\Pi, \tag{2}
\]

In Section 2 we review the continuum initial-boundary value problem for the shifted wave equation, both to fix the notation and to state the energy estimates that establish well-posedness of the continuum problem. Section 3 states our main results: a prescription for strongly stable finite differencing schemes for the quarter space problem \((x > 0, t > 0)\), with second and fourth order accuracy, and for the two cases where the boundary is time-like or space-like (outflow).

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Discrete boundary treatment for the shifted wave equation . . .

where $F^\phi(x,t)$ and $F^{\Pi}(x,t)$ are forcing terms and the parameter $\beta$, the shift, is assumed to be constant. The homogeneous version of system (12) is obtained from the one-dimensional wave equation, $\partial_t^2 \phi = \partial_x^2 \phi$, after a Galilean change of coordinates, $t = \tilde{t}$, $x = \tilde{x} - \beta \tilde{t}$ and the introduction of a new variable $\Pi = \partial_t \phi - \beta \partial_x \phi$. (See [17] and Appendix C for why this is better than using $\partial_t \phi$ in finite difference schemes.) At time $t = 0$ we prescribe initial data

$$\phi(x,0) = f^\phi(x), \quad \Pi(x,0) = f^{\Pi}(x).$$  

Introducing the vector valued function $u(x,t) = (\phi(x,t), \Pi(x,t), \partial_x \phi(x,t))^T$, with periodic boundary conditions on an interval $D$ the energy

$$\|u(\cdot,t)\|^2 \equiv \int_D [\phi^2 + \Pi^2 + (\partial_x \phi)^2] \, dx$$

satisfies the estimate

$$\|u(\cdot,t)\|^2 \leq K(t) \left( \|u(\cdot,0)\|^2 + \int_0^t \|F(\cdot,\tau)\|^2 \, d\tau \right),$$

where $F(x,t) = (F^\phi(x,t), F^{\Pi}(x,t), \partial_x F^\phi(x,t))^T$ and $K(t)$ is a function which is bounded on any finite time interval and does not depend on the initial data.

We now introduce a boundary at $x = 0$ and consider two “quarter space” problems (meaning $x > 0$ and $t > 0$): one for the pure outflow case ($\beta \geq 1$) and one for the time-like case ($|\beta| < 1$). No boundary condition is needed in the outflow case. In the time-like case we impose the Sommerfeld boundary condition

$$w_-(0,t) = g(t),$$

where $w_\pm \equiv \Pi \pm \partial_x \phi$ are the characteristic variables and $g$ is a freely specifiable function compatible with the initial data. To obtain an energy estimate we take a time derivative of the energy and use integration by parts to obtain the inequality

$$\frac{d}{dt} \|u(\cdot,t)\|^2 \leq -\beta \Pi^2 + 2 \Pi \partial_x \phi + \beta (\partial_x \phi)^2 |_{x=0} + \text{const} \left( \|u(\cdot,t)\|^2 + \|F(\cdot,t)\|^2 \right).$$

(8)

Rewriting the boundary term as

$$\frac{1}{2} [(1 - \beta) w_-^2 - (1 + \beta) w_+^2] |_{x=0},$$

which is negative definite in the outflow case and bounded by $\frac{1}{2} (1 - \beta) g^2$ in the time-like case, and integrating, gives the estimate showing strong well-posedness

$$\|u(\cdot,t)\|^2 \leq K(t) \left( \|f\|^2 + \int_0^t (\|F(\cdot,\tau)\|^2 + \delta |g(\tau)|^2) \, d\tau \right),$$

(10)

where $f(x) = u(x,0)$ and $\delta = 0$ for the outflow case and $\delta = 1$ in the time-like case.

Using the energy method we have proved the well-known fact that the initial-boundary value problem for the shifted wave equation is well-posed in the outflow case with no boundary condition, and in the time-like boundary case with a Sommerfeld boundary condition. In the remainder of this paper we investigate the stability of
finite difference discretisations of these two problems. We shall restrict our discussions to discretising the equations in space but not in time (the method of lines). This transforms the partial differential equation into a large coupled system of ordinary differential equations (the semi-discrete problem) which can be solved by a standard ODE integrator, such as fourth order Runge-Kutta.

3. The semi-discrete initial-boundary value problem: summary of results

In this Section we summarise our results regarding strong stability and convergence of schemes approximating the quarter space problem for the shifted wave equation. We introduce the grid \( x_j = jh \), with \( j = 0, 1, 2, \ldots \), and the grid functions \( \phi_j(t) \) and \( \Pi_j(t) \) approximating the continuum variables. In the interior, \( j \geq p/2 \) (with \( p = 2, 4 \) depending on the accuracy of the scheme), we use the standard centered minimal width discretization

\[
\frac{d}{dt} \phi_j = \beta D^{(1)} \phi_j + \Pi_j + F^\phi_j, \tag{11}
\]

\[
\frac{d}{dt} \Pi_j = \beta D^{(1)} \Pi_j + D^{(2)} \phi_j + F^\Pi_j, \tag{12}
\]

where \( D^{(1)} \) and \( D^{(2)} \) approximate the first and second derivatives, respectively. In the second order accurate case, these operators are

\[
D^{(1)} = D_0, \quad D^{(2)} = D_+ D_-, \tag{13}
\]

where \( D_+ u_j = (u_{j+1} - u_j)/h \), \( D_- u_j = (u_j - u_{j-1})/h \) and \( D_0 = (D_+ + D_-)/2 \). In the fourth order accurate case we use

\[
D^{(1)} = D_0 \left( 1 - \frac{h^2}{6} D_+ D_- \right), \tag{14}
\]

\[
D^{(2)} = D_+ D_- \left( 1 - \frac{h^2}{12} D_+ D_- \right). \tag{15}
\]

We know that the Cauchy problem (no boundaries) for (11,12) is stable for any value of the \( \beta \) in both second and fourth-order accuracy \cite{17}.

For the quarter space problem, it is convenient to introduce ghost points, that is, we assume that the interior equations hold for all \( j \geq 0 \) and provide numerical prescriptions for the grid functions at the grid points \( j = -p/2, \ldots, -1 \). Stable discrete boundary conditions are given in the following subsections. The proofs follow in Section 4.

For completeness, and for comparison with the second-order wave equation, we give second and fourth-order accurate boundary prescriptions for the advection equation in Appendix B.

3.1. Second order accuracy

3.1.1. Outflow boundary We start with the outflow case \( \beta > 1 \). The continuum problem does not require any boundary condition, but at the discrete level a special
prescription is needed. Third order extrapolation for $\phi_j$ and second order extrapolation for $\Pi_j$, namely
\begin{align}
h^3 D^3_+ \phi_{-1} &= 0, \\
h^2 D^2_+ \Pi_{-1} &= 0,
\end{align}
give strong stability and second order convergence. Interestingly, the minimum order of extrapolation required for second order convergence is not the same for the two grid functions $\phi_j$ and $\Pi_j$. The reason for this becomes clear in the convergence analysis of subsection 4.1.1. This is to be contrasted with the result for fully first order symmetrizable hyperbolic systems, in which second order extrapolation (or, equivalently, first order one-sided differencing) for the outgoing characteristic variables yields second order convergence.

3.1.2. Time-like boundary  When the shift satisfies $|\beta| < 1$, one of the two characteristic variables is entering the domain through the boundary. We seek discrete boundary conditions approximating
\begin{align}
\Pi(0, t) - \partial_x \phi(0, t) &= g(t),
\end{align}
which lead to strong stability and preserve the internal accuracy. This is achieved by populating the ghost point $j = -1$ using
\begin{align}
\Pi_0 - D_0 \phi_0 &= g, \\
h^2 D^2_+ \Pi_{-1} &= 0,
\end{align}
or, explicitly,
\begin{align}
\phi_{-1} &= \phi_1 + 2h(g - \Pi_0), \\
\Pi_{-1} &= 2\Pi_0 - \Pi_1.
\end{align}

3.2. Fourth order accuracy

3.2.1. Outflow boundary  In the outflow case, the extrapolation conditions
\begin{align}
h^5 D^5_+ \phi_{-1} &= 0, \\
h^5 D^5_+ \phi_{-2} &= 0, \\
h^4 D^4_+ \Pi_{-1} &= 0, \\
h^4 D^4_+ \Pi_{-2} &= 0,
\end{align}
lead to strong stability and fourth order convergence.

3.2.2. Time-like boundary  The prescriptions
\begin{align}
\Pi_0 - D^{(1)} \phi_0 &= g, \\
h^5 D^5_+ \phi_{-2} &= 0, \\
h^4 D^4_+ \Pi_{-1} &= 0, \\
h^4 D^4_+ \Pi_{-2} &= 0,
\end{align}
where $D^{(1)}$ is defined in (14), give strong stability and fourth order convergence. (Increasing the extrapolation order by one in both $\phi$ and $\Pi$ also gives fourth order convergence.) Explicitly solving (27) and (28) for $\phi_{-1}$ and $\phi_{-2}$ gives

$$
\phi_{-1} = 4(g - \Pi_0)h - \frac{10}{3} \phi_0 + 6\phi_1 - 2\phi_2 + \frac{1}{3} \phi_3, \\
\phi_{-2} = 20(g - \Pi_0)h - \frac{80}{3} \phi_0 + 40\phi_1 - 15\phi_2 + \frac{8}{3} \phi_3.
$$

(31) (32)

4. Proofs of strong stability and convergence

In this Section we use the Laplace transform method for difference approximations as described in Chapter 12 of [14] to prove strong stability for second and fourth-order accurate discretisations of the initial-boundary value problem for the shifted wave equation. In order to apply the theorems of that reference, which assume that hyperbolic problems are written in fully first order form (see equation (12.1.11) of [14]), we need to perform a discrete reduction to first order [18].

4.1. Second order accuracy

4.1.1. Outflow boundary  We consider the semi-discrete quarter space problem for the outflow case ($\beta > 1$)

$$
\frac{d}{dt} \phi_j = \beta D_0 \phi_j + \Pi_j + F^{\phi}_j, \\
\frac{d}{dt} \Pi_j = \beta D_0 \Pi_j + D_+ D_- \phi_j + F^{\Pi}_j, \\
\phi_j(0) = f_j^{\phi}, \\
\Pi_j(0) = f_j^{\Pi}, \\
h^{q_1+1} D_+^{q_1} \phi_{-1} = g^{\phi}, \\
h^{q_2} D_+^{q_2} \Pi_{-1} = g^{\Pi}, \\
\|\Pi\|_h^2 + \|D_{+}\phi\|_h^2 < \infty,
$$

(33) (34) (35) (36) (37) (38) (39)

where $j = 0, 1, 2, \ldots$, $\|u\|_h^2 = \sum_{j=0}^{\infty} |u_j|^2 h$, and $q_1$ and $q_2$ are non-negative integers. In practice one would choose $g^{\phi} = g^{\Pi} = 0$, but in the analysis that follows we will need the inhomogeneous case. A first order reduction of the problem obtained by introducing the grid function $X_j = D_{+}\phi_j$ gives

$$
\frac{d}{dt} \Pi_j = \beta D_0 \Pi_j + D_- X_j + F^{\Pi}_j, \\
\frac{d}{dt} X_j = \beta D_0 X_j + D_+ \Pi_j + F^{X}_j, \\
\Pi_j(0) = f_j^{\Pi}, \\
X_j(0) = f_j^{X}, \\
h^{q_1} D_+^{q_1} \Pi_{-1} = g^{\Pi}, \\
h^{q_2} D_+^{q_2} X_{-1} = g^{X},
$$

(40) (41) (42) (43) (44) (45)
where \( j = 0, 1, 2, \ldots \), \( F_j^X = D_j F_j^\phi \), \( f_j^X = D_j f_j^\phi \), and \( g^X = g^\phi / h \). The auxiliary constraint \( C_j \equiv X_j - D_j \phi_j \) satisfies the homogeneous system of ODEs
\[
\frac{d}{dt} C_j = \beta D_0 C_j, \quad j = 0, 1, 2, \ldots, \\
C_j(0) = 0, \\
h^{q_2} D_{j+1}^2 C_{-1} = 0,
\]
and therefore vanishes identically.

We want to show that the semi-discrete initial-boundary value problem \((40-45)\) is strongly stable and second order convergent if \( q_1, q_2 \geq 2 \). Since the evolution equation for \( \phi_j \) only involves lower order terms (\( D_0 \phi_j \) can be expressed as a combination of \( X_j \) and \( X_{j-1} \)), it can be ignored in the analysis that follows.

The proof of stability is divided into three steps. We first estimate the solution of the problem with \( F = 0 \) and \( f = 0 \) in terms of the boundary data by checking that the Kreiss condition, inequality \((64)\), is satisfied. We then estimate the solution of an auxiliary problem with modified homogeneous boundary conditions. Finally, we combine the two estimates to obtain the strong stability estimate, inequality \((83)\), for the original problem. In its final step the proof relies on plotting a function of a complex variable to show that it is bounded.

**Step 1.** By estimating the solution of the \( F = 0 \), \( f = 0 \) problem near the boundary using the Kreiss condition \((64)\) we obtain the estimate \((49)\). Let \( F = 0 \) and \( f = 0 \) in Eqs. \((40-45)\). We want to show that
\[
\| \Pi(t) \|_h^2 + \| X(t) \|_h^2 \leq \text{const} \int_0^t (|g^\Pi(\tau)|^2 + |g^X(\tau)|^2) \, d\tau.
\]
Observing that
\[
\frac{d}{dt} (\| \Pi(t) \|_h^2 + \| X(t) \|_h^2) = -\beta (\Pi_0 \Pi_{-1} + X_0 X_{-1}) - 2\Pi_0 X_{-1}
\]
\[
\leq \text{const} \sum_{j=-1}^0 (|\Pi_j|^2 + |X_j|^2),
\]
it is clear that if we can estimate the solution near the boundary (i.e., at \( j = -1, 0 \)) in terms of the boundary data, we recover the estimate \((49)\). We show that this is possible by explicitly solving the Laplace transformed problem
\[
\hat{s} \Pi_j = \beta(\hat{\Pi}_{j+1} - \hat{\Pi}_{j-1})/2 + \hat{X}_j - \hat{X}_{j-1},
\]
\[
\hat{s} \hat{X}_j = \beta(\hat{X}_{j+1} - \hat{X}_{j-1})/2 + \hat{\Pi}_{j+1} - \hat{\Pi}_j,
\]
\[
h^{q_2} D_X^2 \hat{\Pi}_{-1} = \hat{g}^\Pi,
\]
\[
h^{q_2} D_X^2 \hat{X}_{-1} = \hat{g}^X,
\]
\[
\| \hat{\Pi} \|_h^2 + \| \hat{X} \|_h^2 < \infty,
\]
where \( \hat{u}(s) = \int_0^{+\infty} e^{-st} u(t) \, dt \) and \( \hat{s} = sh \).
Eqs. (51) and (52) form a system of difference equations. The characteristic equation associated with it, obtained by looking for solutions of the form \( \hat{\Pi}_j = k^j \hat{\Pi}_0 \), \( \hat{X}_j = k^j \hat{X}_0 \), is

\[
\left( \frac{s - \beta}{2} (k - k^{-1}) \right)^2 - \frac{(k - 1)^2}{k} = 0,
\]

(56)
a polynomial of degree 4 in \( k \). For \( \text{Re}(s) > 0 \) there are no solutions with \( |k| = 1 \). If \( k = e^{i\xi} \) with \( \xi \in \mathbb{R} \) was a solution we would have \( \text{Re}(s) = 0 \), which is a contradiction. Observing that the roots are continuous functions of \( s \) and for large values of \(|s|\) we have \( s \simeq \beta/(2k) \pm 1/\sqrt{k} \), we conclude that for \( \text{Re}(s) > 0 \) there are two and only two solutions, \( k_1 \) and \( k_2 \), inside the unit circle. For \( s = 0 \) the four roots are

\[
k_{1,2} = \frac{2 - \beta^2 \pm 2i\sqrt{\beta^2 - 1}}{\beta^2}, \quad k_{3,4} = 1.
\]

(57)
Since, for small \(|s|\), \( k_{3,4} = 1 + (\beta \pm 1)^{-1} s + O(|s|^2) \), the roots \( k_1 \) and \( k_2 \) are those which are inside the unit circle for \( \text{Re}(s) > 0 \).

The general solution of the difference equation (51,52) satisfying \( ||\hat{\Pi}||_h^2 + ||\hat{X}||_h^2 < \infty \) can be written as

\[
\hat{\Pi}_j = \sigma_1 \hat{s}_1 k_1^j + \sigma_2 \hat{s}_2 k_2^j, \quad (58)
\]
\[
\hat{X}_j = \sigma_1 (k_1 - 1) k_1^j + \sigma_2 (k_2 - 1) k_2^j, \quad (59)
\]
where \( \hat{s}_{1,2} = s - \frac{\beta}{2} (k_{1,2} - k_{1,2}^{-1}) \). Note that since we will be constructing the explicit solution, the case \( k_1 = k_2 \) does not require special treatment. Inserting (58,59) into the boundary conditions (53,54) gives rise to a \( 2 \times 2 \) system in \( \sigma_1 \) and \( \sigma_2 \)

\[
\begin{align*}
\sigma_1 \hat{s}_1 (k_1 - 1)^{q_1} k_1^{-q_1 - 1} + \sigma_2 \hat{s}_2 (k_2 - 1)^{q_1} k_2^{-q_1 - 1} &= \hat{g}^\Pi, \\
\sigma_1 (k_1 - 1)^{q_2} k_1^{-q_2 - 1} + \sigma_2 (k_2 - 1)^{q_2} k_2^{-q_2 - 1} &= \hat{g}^X.
\end{align*}
\]

(60)
(61)
If the coefficient matrix \( C(s) \) multiplying \( (\sigma_1, \sigma_2)^T \) is non-singular for \( \text{Re}(s) > 0 \), we can solve for \( \sigma_1 \) and \( \sigma_2 \) and substitute into (58) and (59), and obtain an explicit solution of the Laplace transformed problem, which we write in the form

\[
\hat{\Pi}_j = \sum_{k=\Pi,X} c_{j,k}^\Pi \hat{g}^k, \quad (62)
\]
\[
\hat{X}_j = \sum_{k=\Pi,X} c_{j,k}^X \hat{g}^k. \quad (63)
\]
To verify the Kreiss condition, namely that

\[
\sum_{j=-1}^{0} \left( |\hat{\Pi}_j|^2 + |\hat{X}_j|^2 \right)^{1/2} \leq K(|\hat{g}^\Pi|^2 + |\hat{g}^X|^2), \quad (64)
\]
we numerically compute the coefficients \( c_{j,k}^\Pi, c_{j,k}^X \), for \( j = -1, 0 \) and \( k = X, \Pi, \) and plot the quantity

\[
N = \left( \sum_{j=-1,0, k=\Pi,X} \left( |c_{j,k}^\Pi|^2 + |c_{j,k}^X|^2 \right) \right)^{1/2}. \quad (65)
\]
We restrict our attention to the case \( q_1 = q_2 = 2 \) (same order of extrapolation for \( \Pi_j \) and \( X_j \)). Since for \( |\tilde{s}| \geq C_0 \) we have that \( |\hat{\Pi}_j|^2 + |\hat{X}_j|^2 \leq K_0(|\hat{g}_\Pi|^2 + |\hat{g}_X|^2) \), see Lemma 12.2.2 of \cite{14}, we only need to consider the compact set \( S = \{ \tilde{s} \in \mathbb{C} : |\tilde{s}| \leq C_0, \text{Re}(\tilde{s}) \geq 0 \} \). From the chain of inequalities

\[
\|\hat{\Pi}\|^2 + \|\hat{X}\|^2 = \frac{1}{|\tilde{s}|^2} \left( \|h\beta D_0 \hat{\Pi} + hD_- \hat{X}\|^2 + \|h\beta D_0 \hat{X} + hD_+ \hat{\Pi}\|^2 \right)
\]

\[
\leq \frac{2}{|\tilde{s}|^2} \left( \beta^2 \|hD_0 \hat{\Pi}\|^2 + \|hD_- \hat{X}\|^2 + \beta^2 \|hD_0 \hat{X}\|^2 + \|hD_+ \hat{\Pi}\|^2 \right)
\]

\[
\leq \frac{2}{|\tilde{s}|^2} \left[ \beta^2 \left( \|\hat{\Pi}\|^2 + \frac{1}{2} |\hat{\Pi}_{-1}|^2 h \right) + 4\|\hat{X}\|^2 + 2|\hat{X}_{-1}|^2 h + \beta^2 \left( \|\hat{X}\|^2 + \frac{1}{2} |\hat{X}_{-1}|^2 h \right) + 4\|\hat{\Pi}\|^2 \right]
\]

\[
\leq \frac{14}{|\tilde{s}|^2} (4 + \beta^2) \left( \|\hat{\Pi}\|^2 + \|\hat{X}\|^2 + |\hat{g}_\Pi|^2 h + |\hat{g}_X|^2 h \right),
\]

where in the last inequality we used the boundary conditions \( \{53\} \{54\} \), we see that for \( \beta = 2 \) we can take \( C_0 = 12 \). In figure \( \square \) we plot \( N(\tilde{s}) \) where \( \tilde{s} \in S \).

**Figure 1.** On the left we plot \( N(\tilde{s}) \) for \( \tilde{s} \in S \cap \{ \text{Im}(\tilde{s}) \geq 0 \} \). Since this function is symmetric across the real axis, we only need to display the region with non-negative imaginary part. The Kreiss condition is satisfied for the initial-boundary value problem \( \{53\} \{54\} \) with \( \beta = 2 \) and \( q_1 = q_2 = 2 \). The spike that appears at a point near \( \text{Re}(\tilde{s}) = 0 \) is bounded. This is illustrated on the right, where we have increased the resolution of the plot in a neighbourhood of this point. The function \( N(\tilde{s}) \) is continuous but not differentiable on \( \text{Re}(\tilde{s}) = 0 \) due to branch cuts in its definition.

Using Parseval’s relation and the fact that the solution at time \( t_1 \) does not depend on the boundary data at time \( t_2 > t_1 \), the Kreiss condition implies the estimate in
physical space

\[
\int_0^t \sum_{j=-1}^0 (|\Pi_j|^2 + |X_j|^2) \, d\tau \leq K \int_0^t (|g^\Pi|^2 + |g^X|^2) \, d\tau. \tag{66}
\]

Combining this with the integral of inequality (50) gives the desired estimate (49).

**Step 2:** We estimate the solution of a problem with modified homogeneous boundary conditions in terms of the initial data and forcing term, inequalities (74) and (76). So far we have shown that, for vanishing initial data and in the absence of a forcing term, the solution can be estimated in terms of the boundary data. We now consider the auxiliary problem

\[
\frac{d}{dt}\Pi_j = \beta D_0 \Pi_j + D_\cdot X_j + F^\Pi_j, \tag{67}
\]

\[
\frac{d}{dt}X_j = \beta D_0 X_j + D_\cdot \Pi_j + F^X_j, \tag{68}
\]

\[
\Pi_j(0) = f^\Pi_j, \quad X_j(0) = f^X_j, \tag{69}
\]

\[
\Pi_{-1} = \frac{2}{\beta} \left( \Pi_0 - \frac{2}{\beta} X_0 \right), \tag{70}
\]

\[
X_{-1} = \frac{2}{\beta} X_0, \tag{71}
\]

\[
\|\Pi\|^2_h + \|X\|^2_h < \infty, \tag{72}
\]

where the boundary conditions were chosen so that a direct estimate in physical space can be obtained. The estimate

\[
\frac{d}{dt} (\|\Pi\|^2_h + \|X\|^2_h) \leq -2(\|\Pi_0\|^2 + |X_0|^2) + \|\Pi\|^2_h + \|X\|^2_h + \|F^\Pi\|^2_h + \|F^X\|^2_h \tag{73}
\]

implies

\[
\|\Pi(t)\|^2_h + \|X(t)\|^2_h \leq \text{const} \left( \|f\|^2_h + \int_0^t \|F(\tau)\|^2_h \, d\tau \right) \tag{74}
\]

and

\[
\int_0^t (|\Pi_j|^2 + |X_j|^2) \, d\tau \leq \text{const} \left( \|f^\Pi\|^2_h + \|f^X\|^2_h + \int_0^t (\|F^\Pi\|^2_h + \|F^X\|^2_h) \, d\tau \right) \tag{75}
\]

for \( j = -1, 0 \). Since our interior scheme uses the same number of grid points in each side, one can show (Lemma 12.2.10 of [14]) that for every fixed \( j \)

\[
\int_0^\infty (|\Pi_j|^2 + |X_j|^2) \, dt \leq \text{const} \left( \|f\|^2_h + \int_0^\infty \|F(t)\|^2_h \, dt \right), \tag{76}
\]

where \( f = (f^\Pi, f^X)^T \) and \( F = (F^\Pi, F^X)^T \).

**Step 3:** Using the estimates of Steps 1 and 2 we derive the estimate (83) showing strong stability. If we denote by \( \Pi^a_j \) and \( X^a_j \) the solution of the auxiliary problem of Step
where $F$ the quantity 2, and by $\Pi_j$ and $X_j$ the solution of the original problem, we see that the differences $\bar{\Pi}_j = \Pi_j - \Pi_j^a$ and $\bar{X}_j = X_j - X_j^a$ satisfy
\begin{align}
\frac{d}{dt}\bar{\Pi}_j &= \beta D_0 \bar{\Pi}_j + D_- \bar{X}_j, \quad (77) \\
\frac{d}{dt}\bar{X}_j &= \beta D_0 \bar{X}_j + D_+ \bar{\Pi}_j, \quad (78) \\
\Pi_j(0) &= 0, \quad (79) \\
\bar{X}_j(0) &= 0, \quad (80) \\
h^{a_1} D_+^{a_1} \bar{\Pi}_{-1} = g^I - h^{a_1} D_+^{a_1} \bar{\Pi}_{-1}, \quad (81) \\
h^{a_2} D_+^{a_2} \bar{X}_{-1} = g^X - h^{a_2} D_+^{a_2} \bar{X}_{-1}. \quad (82)
\end{align}

Using (49), (76) and (74) we obtain the estimate
\begin{align}
\|\Pi(t)\|_h^2 + \|X(t)\|_h^2 &\leq 2(\|\bar{\Pi}(t)\|_h^2 + \|\bar{X}(t)\|_h^2 + \|\Pi^a(t)\|_h^2 + \|X^a(t)\|_h^2) \leq \quad (83) \\
&\quad \text{const} \left( \|f\|_h^2 + \int_0^t \left( \|F(\tau)\|_h^2 + |g(\tau)|^2 \right) d\tau \right),
\end{align}

where $g = (g^I, g^X)^T$. This inequality proves strong stability. Reintroducing the evolution equation for $\phi_j$, we have an estimate with respect to the $D_+$-norm
\begin{align}
\|\phi(t)\|_h^2 + \|\Pi(t)\|_h^2 + \|D_+ \phi(t)\|_h^2 &\leq \text{const} \left( \|f^\phi\|_h^2 + \|F^\Pi\|_h^2 + \|D_+ f^\phi\|_h^2 + \
\int_0^t \left( \|\phi^\phi(\tau)\|_h^2 + \|F^\Pi(\tau)\|_h^2 + \|D_+ F^\phi(\tau)\|_h^2 + |g^I(\tau)|^2 + |g^X(\tau)/h|^2 \right) d\tau \right). \quad (84)
\end{align}

**Proof of convergence:** Having shown strong stability it is straightforward to prove convergence. We only need an estimate for the error. Defining the errors $e_j^\phi(t) = \phi_j(t) - \phi(x_j,t)$, $e_j^\Pi(t) = \Pi_j(t) - \Pi(x_j,t)$ we see that they satisfy the initial-boundary value problem
\begin{align}
\frac{d}{dt} e_j^\phi &= \beta D_0 e_j^\phi + e_j^\Pi + F_j^\phi, \quad (85) \\
\frac{d}{dt} e_j^\Pi &= \beta D_0 e_j^\Pi + D_+ D_- e_j^\phi + F_j^\Pi, \quad (86) \\
e_j^\phi(0) &= 0, \quad (87) \\
e_j^\Pi(0) &= 0, \quad (88) \\
h^{a_1} D_+^{a_1} e_{-1}^\phi &= -h^{a_2+1} \phi(q_{a_2+1})(0,t) + O(h^{a_2+2}), \quad (89) \\
h^{a_1} D_+^{a_1} e_{-1}^\Pi &= -h^{a_2} \Pi(q_1)(0,t) + O(h^{a_1+1}), \quad (90)
\end{align}

where $F_j^\phi = O(h^3)$ and $F_j^\Pi = O(h^2)$. We perform a discrete reduction by introducing the quantity $e_j^X = D_+ e_j^\phi$ which satisfies
\begin{align}
\frac{d}{dt} e_j^X &= \beta D_0 e_j^X + D_+ e_j^\Pi + F_j^X, \quad (91) \\
e_j^X(0) &= 0, \quad (92) \\
h^{a_2} D_+^{a_2} e_{-1}^X &= -h^{a_2} \phi(q_{a_2+1})(0,t) + O(h^{a_2+1}). \quad (93)
\end{align}

Note that since $F_j^\phi = \beta(D_0 \phi(x_j,t) - \phi_x(x_j,t))$, we have $F_j^X = D_+ F_j^\phi = O(h^2)$. 


The strong stability estimate (84) applied to (86), (88), (90-93), guarantees that the scheme is second-order convergent, i.e.

\[
\left( \|e^{\phi}\|_h^2 + \|e^{\Pi}\|_h^2 + \|D_+e^{\phi}\|_h^2 \right)^{1/2} \leq O(h^2),
\]

provided that \(q_1, q_2 \geq 2\). In this analysis we have implicitly assumed that the initial data and forcing terms are exact. However, this assumption can be easily replaced by the requirement that the initial data errors for \(e^{\Pi}\) and \(e^{X}\) are of order \(h^2\) (note that this means that \(D_+e^{\phi} = O(h^2)\)) and that the errors in the forcing terms \(F^{\Pi}\) and \(F^{X}\) are of order \(h^2\). An immediate consequence of (94) is that we have convergence with respect to the discrete \(L_2\)-norm, \(\left( \|\phi\|^2_h + \|\Pi\|^2_h \right)^{1/2}\).

We have also studied the case \(q_1 = q_2 + 1 = 3\) (same order of extrapolation for \(\phi\) and \(\Pi\)). The Kreiss condition can be verified directly and the rest of the stability proof applies.

### 4.1.2. Time-like boundary

In this Subsection we prove stability for the shifted wave equation problem with a discretization of the Sommerfeld boundary condition, equation (18). The case with zero shift was discussed in Appendix A of [19] using the discrete energy method. Here we focus on the \(0 \neq |\beta| < 1\) case.

We consider the same semi-discrete evolution system (33-36) and (39) with boundary conditions

\[
\Pi_0 - D_0\phi_0 = g^X, \quad (95)
\]

\[
h^nD^n_+\Pi_{-1} = g^{\Pi}. \quad (96)
\]

In applications one would set \(g^X\) equal to the boundary data \(g(t)\) of the continuum problem and \(g^{\Pi} = 0\). The discrete reduction of (95) is

\[
\Pi_0 - \frac{1}{2}(X_0 + X_{-1}) = g^X, \quad (97)
\]

which implies the boundary condition \(C_{-1} = -C_0\) for the auxiliary constraint.

The proof of strong stability proceeds as in the outflow case. We only need to show that the Kreiss condition is satisfied. Inserting \(\Pi_j(t) = e^{st}k^j\tilde{\Pi}_0\) and \(X_j(t) = e^{st}k^j\tilde{X}_0\) into the scheme, and looking for non trivial solutions gives the characteristic equation (56). For small \(|\tilde{s}|\) the roots are

\[
k_1 = \frac{2 - \beta^2 - 2\sqrt{1 - \beta^2}}{\beta^2} + O(|\tilde{s}|), \quad (98)
\]

\[
k_2 = 1 + \frac{1}{\beta - 1}\tilde{s} + O(|\tilde{s}|^2), \quad (99)
\]

\[
k_3 = \frac{2 - \beta^2 + 2\sqrt{1 - \beta^2}}{\beta^2} + O(|\tilde{s}|), \quad (100)
\]

\[
k_4 = 1 + \frac{1}{\beta + 1}\tilde{s} + O(|\tilde{s}|^2). \quad (101)
\]

For \(\text{Re}(\tilde{s}) > 0\) the roots \(k_1\) and \(k_2\) have magnitude smaller than 1 and the remaining two have magnitude greater than 1. The requirement \(\|\tilde{\Pi}\|_h^2 + \|\tilde{X}\|_h^2 < \infty\) implies that
4.2. Fourth order accuracy

4.2.1. Outflow boundary

The fourth order accurate standard discretization of the shifted wave equation is given by

\[ \frac{D}{dt}^2 \phi_j = \beta D^{(1)} \phi_j + \Pi_j + F_j^\phi, \]

\[ \frac{D}{dt} \Pi_j = \beta D^{(1)} \Pi_j + D^{(2)} \phi_j + F_j^\Pi, \]

where \( j = 0, 1, 2, \ldots \) and

\[ D^{(1)} u_j \equiv D_0 \left( 1 - \frac{h^2}{6} D_+ D_- \right) u_j = \frac{(-u_{j+2} + 8 u_{j+1} - 8 u_{j-1} + u_{j-2})}{12 h}, \]

\[ D^{(2)} u_j \equiv D_+ D_- \left( 1 - \frac{h^2}{12} D_+ D_- \right) u_j \]

\[ = \frac{(-u_{j+2} + 16 u_{j+1} - 30 u_j + 16 u_{j-1} - u_{j-2})}{12 h^2}. \]

We consider the outflow case first (\( \beta > 1 \)), with the boundary conditions

\[ h^5 D_+^5 \phi_{-1} = g_1^\phi, \quad h^5 D_+^5 \phi_{-2} = g_2^\phi, \]

\[ h^4 D_+^4 \Pi_{-1} = g_1^\Pi, \quad h^4 D_+^4 \Pi_{-2} = g_2^\Pi. \]

As in the second order accurate case, we need to perform a discrete reduction to first order. For this purpose it is convenient to introduce a grid function \( \tilde{X}_j \), which is a suitable linear combination of \( D_+ \phi_j \) and \( D_- \phi_j \). The choice of the particular combination is determined by the following two observations. First, the operator \( D^{(2)} \) can be decomposed as

\[ D^{(2)} = \tilde{D}_+ \tilde{D}_-, \]

where \( \tilde{D}_\pm = (1 \mp a h D_\mp) D_\pm = (1 - \alpha) D_\pm + \alpha D_\mp \) and \( \alpha \) is a root of the quadratic equation \( \alpha^2 - \alpha - 1/12 = 0 \). Second, in the absence of boundaries, the following discrete energy,

\[ \| \Pi \|^2_h + \| D_\phi \|^2_h + \frac{h^2}{12} \| D_+ D_- \phi \|^2_h, \]
which is equivalent to \( \| \Pi \|^2_h + \| D_+ \phi \|^2_h \), is conserved and it can be written as:

\[
\| \Pi \|^2_h + \| \tilde{X} \|^2_h,
\]

where \( \tilde{X}_j = \tilde{D}_+ \phi_j \).

These two facts suggest introducing the discrete variable \( \tilde{X}_j \), leading to the (interior) discrete reduction

\[
\begin{align*}
\frac{d}{dt} \Pi_j &= \beta D^{(1)} \Pi_j + \tilde{D}_- \tilde{X}_j + F_j^\Pi, \\
\frac{d}{dt} \tilde{X}_j &= \beta D^{(1)} \tilde{X}_j + \tilde{D}_+ \Pi_j + F_j^X, \\
\Pi_j(0) &= f_j^\Pi, \\
\tilde{X}_j(0) &= f_j^X,
\end{align*}
\]

where \( F_j^X = \tilde{D}_+ F_j^\phi, f_j^X = \tilde{D}_+ f_j^\phi \).

We need to translate the extrapolation boundary conditions for \( \phi \) and \( \Pi \) in terms of boundary conditions for the new variables, \( \Pi \) and \( \tilde{X} \). To do this, we go back to the original system \(^{(105-106)}\) and eliminate the variables \( \phi_{-1}, \phi_{-2}, \Pi_{-1}, \) and \( \Pi_{-2} \) using the boundary conditions \(^{(109-110)}\). Defining \( \tilde{X}_j = \tilde{D}_+ \phi_j \) for \( j \geq 0 \) as before, where in \( X_0 \) the grid function \( \phi_{-1} \) is eliminated using \(^{(108)}\) but with \( g_0^\phi = 0 \), we can eliminate each occurrence of \( \phi_j \) in terms of \( \tilde{X}_j \). The evolution equations for the \( \tilde{X}_j \) variables can be computed by taking appropriate combinations of the evolution equations of the \( \phi_j \) variables. This leads to a semi-discrete problem for \( \Pi_j \) and \( \tilde{X}_j, j \geq 0 \). For \( j \geq 2 \) the equations have the form \(^{(114-115)}\), with the exception that

\[
F_2^X = \tilde{D}_+ F_2^\phi + \alpha \beta g_1^\phi / (12h^2).
\]

However, for \( j = 0,1 \) they are more complicated. To analyze the stability of the system we reintroduce ghost points. By setting the evolution equations near the boundary to be formally equal those at the interior, for \( j = 0,1 \), we obtain the prescriptions

\[
\begin{align*}
\Pi_{-1} &= 4\Pi_0 - 6\Pi_1 + 4\Pi_2 - 3\Pi_3 + g_1^\Pi + \frac{g_1^X}{\beta}, \\
\Pi_{-2} &= 10\Pi_0 - 20\Pi_1 + 15\Pi_2 - 4\Pi_3 + 4g_1^\Pi + g_2^\Pi + \frac{2(7 - 6\alpha)}{\beta} g_1^X, \\
\tilde{X}_{-1} &= 4\tilde{X}_0 - 6\tilde{X}_1 + 4\tilde{X}_2 - \tilde{X}_3 + (1 - 5\alpha) g_1^X + \alpha g_2^X, \\
\tilde{X}_{-2} &= \frac{-137 + 132\alpha}{12\alpha - 13} \tilde{X}_0 - \frac{-289 + 288\alpha}{12\alpha - 13} \tilde{X}_1 + \frac{-241 + 252\alpha}{12\alpha - 13} \tilde{X}_2 - \frac{-43 + 48\alpha}{12\alpha - 13} \tilde{X}_3 - \frac{12\alpha - 11}{12\alpha - 13} \tilde{X}_4 + \frac{12\alpha - 1}{12\alpha - 13} \tilde{X}_5 + \frac{12\alpha - 1}{\beta(12\alpha - 13)} \Pi_0 - \frac{48\alpha - 1}{\beta(12\alpha - 13)} \Pi_1 + \frac{72\alpha - 1}{\beta(12\alpha - 13)} \Pi_2 - \frac{48\alpha - 1}{\beta(12\alpha - 13)} \Pi_3 + \frac{12\alpha - 1}{\beta(12\alpha - 13)} \Pi_4 + \frac{2(6 - 6\alpha + 53\alpha^2 - 55\beta^2) g_1^X + \alpha - 2}{12\alpha - 13} g_1^X + \frac{12\alpha - 1}{\beta(12\alpha - 13)} g_1^\Pi.
\end{align*}
\]

\( \dagger \) To show that \( \| D_+ \phi_j - \alpha^j h D_+ D_- \phi_j \|^2_h = \| D_+ \phi \|^2_h + h^2/12 \| D_+ D_- \phi \|^2_h \) one can use the identity \( h D_+ D_- = D_+ - D_- \) and \( \alpha^2 - \alpha = -1/12 = 0 \).
where \( g_k^X = g_k^\phi / h \). The forcing terms near the boundary are \( F_1^X = \tilde{D}_+ F_1^\phi \) and \( F_0^X = \tilde{D}_+ F_0^\phi \) (assuming \( F_0^\phi \) to be defined via \( h^\phi \)).

We now need to show that the semi-discrete initial-boundary value problem (114) is strongly stable and fourth order convergent. Neglecting forcing terms, we have the estimate
\[
\frac{d}{dt} (\|\Pi\|_h^2 + \|\tilde{X}\|_h^2) = \frac{\beta}{6} (\tilde{X}_{-1}\tilde{X}_1 + \Pi_{-1}\Pi_1 + \tilde{X}_{-2}\tilde{X}_0 + \Pi_{-2}\Pi_0 - 8\Pi_0\Pi_{-1} - 8\tilde{X}_{-1}\tilde{X}_0) + 2(\alpha - 1)\tilde{X}_{-1}\Pi_0 - 2\alpha\Pi_{-1}\tilde{X}_0 \leq \text{const} \sum_{j=-2}^1 (|\Pi_j|^2 + |\tilde{X}_j|^2).
\]

As in the second order accurate case, we explicitly solve the Laplace transformed problem for vanishing initial data and no forcing term and write the solution as
\[
\tilde{X}_j = c^\Pi_{j1}\tilde{g}_1 + c^\Pi_{j2}\tilde{g}_2 + c^X_{j1}\tilde{g}_1^X + c^X_{j2}\tilde{g}_2^X,
\]
where
\[
\tilde{g}_1 = \left( \sum_{k=-2}^1 c^\Pi_{jk}c^\Pi_{kj} + c^\Pi_{jl}c^\Pi_{lj} + c^\Pi_{jk}c^\Pi_{kj} + c^\Pi_{jl}c^\Pi_{lj} \right)^{1/2}.
\]
We numerically verify the Kreiss condition by plotting the quantity
\[
N = \left( \sum_{k = k_1, k_2, x_1, x_2} (|c^\Pi_{jk}|^2 + |c^X_{jk}|^2) \right)^{1/2}
\]
inside the semi-disk \( S \) with \( C_0 = 30 \).

The modified homogeneous boundary conditions for the auxiliary problem are
\[
\tilde{X}_{-1} = \frac{1}{\beta} \left( -48\tilde{X}_0 - 6\tilde{X}_1 + \frac{72(\alpha - 1)}{\beta}\Pi_0 \right),
\]
\[
\tilde{X}_{-2} = -\frac{6}{\beta^3}(65\beta^2 + 144\alpha^2)\tilde{X}_0,
\]
\[
\Pi_{-1} = \frac{1}{\beta} \left( \frac{48 - \alpha}{\alpha}\Pi_0 - 6\Pi_1 - \frac{72\alpha}{\beta}\tilde{X}_0 \right),
\]
\[
\Pi_{-2} = -\frac{6}{\beta^3\alpha}(65\alpha - 128)\beta^2 + 12(\alpha - 1))\Pi_0 - \frac{96}{\beta\alpha}\Pi_1,
\]
and they give the estimate
\[
\frac{d}{dt} (\|\Pi\|_h^2 + \|\tilde{X}\|_h^2) \leq -\sum_{j=0}^1 (|\Pi_j|^2 + |\tilde{X}_j|^2) + \|\Pi\|_h^2 + \|\tilde{X}\|_h^2 + \|F^\Pi\|_h^2 + \|F^X\|_h^2.
\]
This implies inequalities (174) and (176) and strong stability with respect to the \( D_+ \) norm follows from the fact that \( \|\Pi\|_h^2 + \|\tilde{X}\|_h^2 \) is equivalent to \( \|\Pi\|_h^2 + \|D_+\phi\|_h^2 \). Convergence is a consequence of estimates for the error. However, due to the modified forcing term (118) we are only able to show that \( \|\Pi\|_h^2 + \|\tilde{X}\|_h^2 < O(h^7) \). Given that the numerical tests of Section 5 indicate fourth order convergence, we believe that this estimate is not optimal.
where \( D^{(1)} \) is given in (134). We introduce \( \bar{X}_j = \tilde{D}_+ \phi_j \) for \( j \geq 0 \), where \( \phi_{-1} \) in \( \bar{X}_0 = X_0 \) is given by system (131)-(132). Note that \( \bar{X}_0 \) contains also \( \Pi_0 \). A discrete reduction to first order gives the ghost-point prescriptions

\[
\Pi_0 = 4 \Phi_0 - 6 \Phi_1 - 4 \Phi_2 - 4 \Phi_3 + g_1^{\Pi} + \frac{4}{\beta} g_1^{\chi} + \frac{1}{3 \beta} g_2^{\chi},
\]

\[
\Pi_1 = 10 \Phi_0 - 20 \Phi_1 + 15 \Phi_2 - 4 \Phi_3 + 4 g_1^{\Pi} + 2 g_2^{\Pi} - 32 \frac{16 \alpha - 17}{\beta(10 \alpha - 9)} g_1^{\chi},
\]

where \( D^{(1)} \) is given in (131). We introduce \( \bar{X}_j = \tilde{D}_+ \phi_j \) for \( j \geq 0 \), where \( \phi_{-1} \) in \( \bar{X}_0 \) is given by system (131)-(132). Note that \( X_0 \) contains also \( \Pi_0 \). A discrete reduction to first order gives the ghost-point prescriptions

\[
\Pi_0 = 4 \Phi_0 - 6 \Phi_1 - 4 \Phi_2 - 4 \Phi_3 + g_1^{\Pi} + \frac{4}{\beta} g_1^{\chi} + \frac{1}{3 \beta} g_2^{\chi},
\]

\[
\Pi_1 = 10 \Phi_0 - 20 \Phi_1 + 15 \Phi_2 - 4 \Phi_3 + 4 g_1^{\Pi} + 2 g_2^{\Pi} - 32 \frac{16 \alpha - 17}{\beta(10 \alpha - 9)} g_1^{\chi},
\]

where \( g_1^{\chi} = g_1^{\phi} \) and \( g_2^{\chi} = g_2^{\phi}/h \). The forcing terms near the boundary are

\[
F_2^{\chi} = \tilde{D}_+ F_2^{\phi} + \frac{\alpha \beta}{36 h} (g_2^{\chi} - 12 g_1^{\chi}),
\]

\[
F_1^{\chi} = \tilde{D}_+ F_1^{\phi},
\]

and \( F_0^{\chi} \) is obtained from the definition of \( \bar{X}_0 \) with the replacements \( \phi_j \rightarrow F_j^{\phi}, \Pi_j \rightarrow F_j^{\Pi} \).

The rest of the proof proceeds as in the outflow case. That the scheme is convergent follows from estimates for the errors. As in the outflow case, we are only able to show
3.5th order convergence, although experiments suggest that the scheme is fourth order convergent.

5. The fully discrete initial-boundary value problem: numerical tests

The semi-discrete schemes we have considered here can be turned into fully discrete schemes using, for example, fourth order Runge-Kutta as a time integrator. In general, stability of the semi-discrete scheme does not guarantee that the fully discrete scheme is stable \[20, 21\]. We therefore check stability of the fully discrete scheme by performing numerical convergence tests.

The convergence tests are performed using the domain \(0 \leq x \leq L\), where \(x = 0\) is the physical boundary, and by monitoring the errors in the interval \(0 \leq x \leq 1\) at time \(t = 1\). The value of \(L\) is chosen large enough so that the numerical solution in the interval \(0 \leq x \leq 1\) for \(t \leq 1\) is numerically unaffected by the boundary at \(x = L\). We have used a Courant factor of 0.5 and resolutions ranging from \(h = 1/25\) to \(1/400\). The code is tested against the exact solution \(\phi(x, t) = f(-x + (1 - \beta)t), \Pi(x, t) = f'(-x + (1 - \beta)t)\), where \(f(x) = \sin(2\pi x)\).

As shown in table \[1\] we find good second or fourth-order convergence in the norm \((\|\Pi\|_2^2 + \|D_+\phi\|_2^2)^{1/2}\) over all resolutions. We focused on the \(\beta = 2\) and \(\beta = -1/5\) cases, but the schemes are convergent for other values of the shift parameter in the ranges \(\beta > 1\) and \(|\beta| < 1\). However, we noticed that in the time-like, fourth order accurate case, in order to avoid losing accuracy, the boundary data \(g\) had to be Taylor expanded at the intermediate time steps of the Runge-Kutta integrator. This issue is discussed in \[22\].

6. Application to the Nagy-Ortiz-Reula formulation of the Einstein equations in spherical symmetry

The main interest of the one dimensional shifted wave equation is as a toy model for second order in space formulations of three-dimensional general relativistic initial-boundary value problems. For such problems initial and boundary data cannot be freely specified. The initial data has to satisfy the constraints and the boundary conditions should be such that no constraint violation is injected. Whereas maximally dissipative boundary conditions involve only first derivatives across the boundary, constraint-preserving boundary conditions require second derivatives across the boundary.

We consider the Nagy-Ortiz-Reula (NOR) formulation \[23\] of the Einstein equations, restricted to spherical symmetry and coupled to a massless minimally coupled scalar field. In spherical symmetry the line element takes the form

\[
ds^2 = -\alpha^2 dt^2 + g_{rr}(dr + \beta dt)^2 + g_{\theta\theta}(d\theta^2 + \sin^2 \theta d\phi^2).
\] (139)

and the reduction of the NOR system to spherical symmetry is straightforward. However, if the origin is included in the domain, the definition of the auxiliary variable
Table 1. The discrete boundary conditions described in Section \ref{section3} give the expected order of convergence.

| Errors and convergence rates |
|-----------------------------|
| $N$ | $l_2$ | $q$ |
| (a) Second order accurate case, $\beta = 2$ | | |
| 25  | 7.35084 $10^{-1}$ | | |
| 50  | 1.83951 $10^{-1}$ | 1.9986 | |
| 100 | 4.60081 $10^{-2}$ | 1.9994 | |
| 200 | 1.15021 $10^{-2}$ | 2.0000 | |
| 400 | 2.87555 $10^{-3}$ | 2.0000 | |
| (b) Second order accurate case, $\beta = -1/5$ | | |
| 25  | 1.06042 $10^{-1}$ | | |
| 50  | 2.59231 $10^{-2}$ | 2.0323 | |
| 100 | 6.41559 $10^{-3}$ | 2.0146 | |
| 200 | 1.59673 $10^{-3}$ | 2.0065 | |
| 400 | 3.98366 $10^{-4}$ | 2.0030 | |
| (c) Fourth order accurate case, $\beta = 2$ | | |
| 25  | 9.70747 $10^{-3}$ | | |
| 50  | 6.10334 $10^{-4}$ | 3.9914 | |
| 100 | 3.82024 $10^{-5}$ | 3.9979 | |
| 200 | 2.38809 $10^{-6}$ | 3.9997 | |
| 400 | 1.49255 $10^{-7}$ | 4.0000 | |
| (d) Fourth order accurate case, $\beta = -1/5$ | | |
| 25  | 1.01955 $10^{-3}$ | | |
| 50  | 6.71790 $10^{-5}$ | 3.9238 | |
| 100 | 4.30128 $10^{-6}$ | 3.9652 | |
| 200 | 2.72064 $10^{-7}$ | 3.9827 | |
| 400 | 1.71025 $10^{-8}$ | 3.9917 | |

$f$ and the addition of its definition constraint to the right hand side of the evolution equations require consideration. We use the regularised dynamical variables $g_T \equiv r^{-2}g_{\theta\theta}$, $K_T \equiv r^{-2}K_{\theta\theta}$, and $f \equiv -2(\ln g_T)_r$, and instead of adding a multiple of $\mathcal{G}_r$ to the right-hand side of the evolution equation for $K_{rr}$, where $\mathcal{G} \equiv f + 2(\ln g_T)_r$ is the definition constraint for the auxiliary variable $f$, we add $\mathcal{G}_r - \mathcal{G}/r$. The regularity conditions are the origin are that $g_{rr}$, $g_T$, $K_{rr}$, $K_T$, $\phi$, $\Pi$ are even functions of $r$ satisfying $g_T |_{r=0} = g_{rr} |_{r=0}$, $K_T |_{r=0} = K_{rr} |_{r=0}$ and $f$ is an odd function of $r$.

With fixed densitised lapse, $\alpha = \sqrt{g_{rr} g_T Q}$, and fixed shift, the evolution equations are

$$
\partial_t g_{rr} = \beta \partial_r g_{rr} + 2 g_{rr} \partial_r \beta - 2 \alpha K_{rr},
$$

(140)
\[ \partial_t g_T = \beta \partial_r g_T + \frac{2\beta}{r} g_T - 2\alpha K_T, \] (141)

\[ \partial_t K_{rr} = \beta \partial_r K_{rr} - \frac{\alpha}{2g_{rr}} \partial_r^2 g_{rr} + \alpha \partial_r f + \frac{\alpha}{2g_{rr}} (\partial_r g_{rr})^2 + 2K_{rr} \partial_r \beta - \frac{2\alpha}{g_T Q} \partial_r g_T \partial_r, Q \] (142)

\[ \partial_t K_T = \beta \partial_r K_T - \frac{\alpha}{2g_{rr}} \partial_r^2 g_{rr} + \frac{2\beta}{r} K_T - \frac{3\alpha}{r g_{rr}} \partial_r g_T + \frac{\alpha}{r^2} \left( 1 - \frac{g_T}{g_{rr}} \right) + \frac{\alpha}{g_{rr}} K_r K_T \] (143)

\[ \partial_t f = \beta \partial_r f + f \partial_r \beta + \frac{4\beta}{r} \partial_r \beta + \frac{2\alpha}{r g_{rr}} K_T \partial_r g_{rr} + \frac{4\alpha}{g_T Q} K_T \partial_r Q - \frac{4\alpha}{r g_T} K_T \] (144)

\[ \partial_t \phi = \beta \partial_r \phi - \alpha \Pi, \] (145)

\[ \partial_t \Pi = \beta \partial_r \Pi - \frac{2\alpha}{g_{rr}} \partial_r g_T \partial_r \phi - \frac{2\alpha}{r g_{rr}} \partial_r \phi - \frac{\alpha}{Q g_{rr}} \partial_r Q \partial_r \phi + \alpha K_T, \] (146)

where \( K = K_{rr}/g_{rr} + 2K_T/g_T \).

The Hamiltonian and momentum constraints and the constraint defining \( f \) are

\[ C \equiv - \frac{\partial_r^2 g_T}{g_{rr} g_T} + \frac{\partial_r g_{rr} \partial_r g_T}{2g_{rr}^2 g_T} + \frac{1}{g_{rr}^2 g_T} - \frac{1}{r^2 g_{rr}} + \frac{\partial_r g_{rr}}{r^2 g_{rr} g_T} - \frac{3\partial_r g_T}{g_T g_{rr} g_T} + \frac{K_T^2}{g_T^2} + \frac{2K_{rr} K_T}{g_{rr} g_T} \] (147)

\[ + \left( \frac{\partial_r g_T}{g_{rr}^2} - 4\pi \left( \Pi^2 + \frac{\partial_r \phi)^2}{g_{rr}} \right) = 0, \]

\[ C_r \equiv -2 \frac{\partial_r K_T}{g_T} + \left( \frac{K_T^2}{g_T^2} + \frac{K_{rr}}{g_T g_{rr}} \right) \partial_r g_T + \frac{2}{r} \left( \frac{K_T}{g_{rr}} + \frac{K_T}{g_T} \right) - 8\pi \Pi \partial_r \phi = 0, \] (148)

\[ G \equiv f + \frac{2\partial_r g_T}{g_T} = 0. \] (149)

System (140)–(146) is symmetric hyperbolic with a symmetric hyperbolic constraint evolution system. The characteristic speeds and variables are

\[ \beta, \quad w^0 = f, \] (150)

\[ \beta \pm \frac{\alpha}{\sqrt{g_{rr}}}, \quad w^\pm_T = \partial_r g_T \mp 2\sqrt{g_{rr}} K_T, \] (151)

\[ \beta \pm \frac{\alpha}{\sqrt{g_{rr}}}, \quad w^\pm_{rr} = \partial_r g_{rr} \mp 2\sqrt{g_{rr}} K_{rr} - 2g_{rr} f, \] (152)

\[ \beta \pm \frac{\alpha}{\sqrt{g_{rr}}}, \quad w^\pm_\phi = \partial_r \phi \mp \sqrt{g_{rr}} \Pi. \] (153)

The characteristic constraint variables are

\[ C^0 = G, \] (154)

\[ C^\pm = C_r \mp \sqrt{g_{rr}} C - \left( \frac{2K_T}{g_T} + \frac{\partial_r g_T}{\sqrt{g_{rr} g_T}} \pm \frac{2}{r \sqrt{g_{rr}}} \right) G. \] (155)
Note that in spherical symmetry, some of the highest derivative terms that are present generically in 3D cancel, both in the main evolution equations and in the implied constraint evolution. In particular, the right-hand side of $\dot{C}$ and $\dot{C}_i$ does not contain $\mathcal{G}_{i,jk}$ terms. For this reason $\mathcal{G}$ appears undifferentiated in $C^\pm$, whereas in 3D its first derivatives would appear.

If we introduce an artificial outer boundary and assume that $0 < \beta < \alpha/\sqrt{\mathcal{g}}$ at the outer boundary, we have four incoming characteristic variables for the main system and two incoming characteristic constraints. The constraint preserving boundary conditions are

\begin{align}
  w_{rr}^+ &= g_{rr}^+, \\
  w_\phi^+ &= g_\phi^+, \\
  \mathcal{G} &= 0, \\
  C_r - \sqrt{\mathcal{g}} C &= 0.
\end{align}

where $g_{rr}^+$ and $g_\phi^+$ are two freely specifiable functions compatible with the initial data.

We generalize the discrete boundary conditions introduced in the first part of the paper in the following way. We extrapolate all fields using fifth order extrapolation and populate $g_{rr}$, and $\phi$ at the two ghost points, $r_{N+1}$ and $r_{N+2}$, by solving

\begin{align}
  D_0 \left( 1 - \frac{h^2}{6} D_+ D_- \right) g_{rr} - 2\sqrt{\mathcal{g}} K_{rr} - 2g_{rr} f &= g_{rr}^+, \quad (160) \\
  h^6 D_6 g_{rr} \big|_{N+2} &= 0, \\
  D_0 \left( 1 - \frac{h^2}{6} D_+ D_- \right) \phi - \sqrt{\mathcal{g}} \Pi &= g_\phi^+, \quad (162) \\
  h^6 D_6 \phi \big|_{N+2} &= 0, \quad (163)
\end{align}

where (160) and (162) are evaluated at $r_N$. We then solve (159) for $\partial^2_r g_T$ and use its fourth order accurate approximation, combined with sixth order extrapolation at $r_{N+2}$, to populate $g_T$ at the ghost zones (we are able to compute the first derivative of $g_T$ from the extrapolation and the first derivatives of $g_{rr}$ and $\phi$ from the two maximally dissipative boundary conditions above). Finally, we impose $\mathcal{G} = 0$ by giving data to $f$ at $r_N$ using

\begin{equation}
  f = -\frac{2}{g_T} D_0 \left( 1 - \frac{h^2}{6} D_+ D_- \right) g_T. \quad (164)
\end{equation}

We evolve Schwarzschild space-time in standard Kerr-Schild coordinates, obtaining the initial data and the fixed lapse and shift from the exact solution. We introduce a space-like (outflow) boundary to excise the singularity. At this boundary we use fifth order extrapolation for the extrinsic curvature, $f$ and $\Pi$ and sixth order for the 3-metric components and $\phi$. In order to obtain a non-vacuum solution, we inject a scalar field pulse through the outer boundary by using

\begin{equation}
  g_\phi^+ = A \sin^8(t - t_0), \quad t_0 < t < t_0 + \pi \quad (165)
\end{equation}

where $A = 0.02$ and $t_0 = 0.1$, and monitor the $L_2$-norm of the Hamiltonian constraint at times $t = 8M$, after the pulse has completely entered the domain, and $t = 16M$, after
the pulse has fallen into the black hole. See table 2. We obtain similar convergence rates for $C_r$ and $G$.

The domain extends from $r = 1.8M$ and $r = 11.8M$. We set $M = 1$, use a Courant factor of 0.1 and a dissipation parameter $\sigma = 0.05$. Dissipation is switched off at the last few grid points near the outer boundary, as we have observed a numerical instability when Kreiss-Oliger dissipation is used in combination with fifth and sixth order extrapolation. On the other hand, according to our tests a lower order of extrapolation only gives third order convergence for the constraints.

Table 2. We give initial data corresponding to a Schwarzschild black hole in Kerr-Schild coordinates, inject a scalar field pulse through the outer boundary and monitor the constraints. Here we give the discrete $L_2$-norm and convergence rate for the Hamiltonian constraint.

| $N$ | $l_2$   | $q$     |
|-----|---------|---------|
| (a) $t = 8M$ |
| 100 | $3.68824 \times 10^{-4}$ | 3.6358 |
| 200 | $2.96704 \times 10^{-5}$ | 4.4761 |
| 400 | $1.33308 \times 10^{-6}$ | 4.3550 |
| 800 | $6.51396 \times 10^{-8}$ |         |
| (b) $t = 16M$ |
| 100 | $9.88525 \times 10^{-5}$ | 3.9867 |
| 200 | $6.23527 \times 10^{-6}$ | 4.3909 |
| 400 | $2.97205 \times 10^{-7}$ | 4.4295 |
| 800 | $1.37924 \times 10^{-8}$ |         |

7. Conclusions

We have constructed second and fourth order accurate approximations of maximally dissipative boundary conditions for the shifted wave equation and have shown strong stability of the semi-discrete scheme using the Laplace transform method. Two important steps were obtaining an appropriate discrete reduction to first order, and verifying that the solution of the problem with trivial initial data and no forcing term can be estimated in terms of the boundary data (the Kreiss condition). The Kreiss condition was verified by plotting the function $N(\tilde{s})$ to show that it is bounded. We have also shown numerically that the fully discrete schemes obtained by integrating our semi-discrete systems with a fourth-order Runge-Kutta time integrator are second or fourth-order accurate.

Whereas semi-discrete approximations of one-dimensional first-order hyperbolic systems with constant coefficients can be transformed by a change of variables into a
set of advection equations, one for each characteristic variable, the same is not true for second-order systems\textsuperscript{§}. Therefore numerical methods for second-order in space systems need to be developed independently. Our results for the second-order in space wave equation suggest a general heuristic prescription for discretising boundary conditions in general second-order in space hyperbolic systems: \textit{Populate all necessary ghost zones by extrapolation, combined with centered difference approximations of the continuum boundary conditions at the boundary point}. We have given a simple example, the NOR system with scalar field matter in spherical symmetry, where this prescription gives a stable fourth-order accurate scheme.

The interior schemes discussed in this paper always use a minimal width centered discretization, even in the outflow case ($\beta > 1$). Despite reports on the benefits of the upwind (one-sided) discretization of the shift terms \cite{24, 25}, we find that, at least in the linear constant coefficient case, this is not necessary. Note that in contrast to an upwind scheme our semi-discrete central scheme is non-dissipative, as the Cauchy problem (without boundaries) admits a conserved energy.

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\textbf{Appendix A. The discrete energy method and summation by parts}

To derive continuum energy estimates one uses integration by parts to generate boundary terms which can be controlled by imposing suitable boundary conditions. To obtain similar estimate for the semi-discrete problem one can use difference operators satisfying the \textit{summation by parts} rule \cite{16}. Discretisations of $d^2/dx^2$ with that property have been constructed in \cite{15}.

In this Appendix, we attempt to use summation by parts methods to construct stable (and sufficiently accurate) schemes for the shifted wave equation. We are not successful, and try to explain why.

We adopt the notation of reference \cite{15} and definitions used there, and write grid functions as column vectors, so that for two grid functions $u_j$ and $v_j$

$$
(u, v) \equiv \sum_{j=0}^{N} u_j v_j \equiv u^T v. \quad (A.1)
$$

In contrast to the body of this paper, we assume that there are two boundaries in $x$, but the case with only one boundary can be obtained trivially by setting $N \rightarrow \infty$

\textsuperscript{§} As an example consider the system $d\phi_j/dt = \Pi_j$, $d\Pi_j/dt = D_+ D_- \phi_j$, whose characteristic variables in Fourier space are $\tilde{\Pi} \pm 2i/h \sin(\xi/2)\tilde{\phi}$. It is not possible to translate these variables back into physical space.
and ignoring all terms arising at the right boundary in what follows. A more general inner product on grid functions can be characterised by a positive definite symmetric \((N+1) \times (N+1)\) matrix \(H\),

\[
(u, v)_H \equiv u^T Hv, \quad H^T = H > 0.
\] (A.2)

Its continuum limit should be the \(L_2\) inner product. In a similar way, discrete derivative operators can be written as matrices.

In this notation, the semi-discrete shifted wave equation can be written as

\[
\begin{align*}
\frac{d}{dt} \phi &= \beta \hat{D}_1 \phi + \Pi, \\
\frac{d}{dt} \Pi &= \beta D_1 \Pi + D_2 \phi,
\end{align*}
\]

where \(\hat{D}_1\) and \(D_1\) are approximations to \(d/dx\) and \(D_2\) is an approximation to \(d^2/dx^2\). Similarly, we can write the discrete energy as

\[
E = \Pi^T H \Pi + \phi^T A \phi. \tag{A.3}
\]

where \(A^T = A\) and \(\phi^T A \phi\) represents \(\int (\partial_x \phi)^2 dx\). In particular, it should have the positivity properties

\[
\begin{align*}
\phi^T A \phi &\geq 0 \text{ for all } \phi, \\
\phi^T A \phi &= 0 \text{ if and only if } D_+ \phi_j = 0
\end{align*}
\] (A.4) (A.5)

We assume \(\beta > 1\) and look for (sufficiently accurate) difference operators and matrices \(H\) and \(A\) that give a discrete energy estimate. Taking a time derivative of the discrete energy (A.3), we have

\[
\frac{d}{dt} E = 2\beta (\Pi^T HD_1 \Pi + \phi^T A \hat{D}_1 \phi)
\] + 2\(\Pi^T HD_2 \phi + 2\Pi^T A \phi. \tag{A.6}
\]

The key point is that the requirement that this expression be a pure boundary term leads to only one summation by parts condition for \(\beta = 0\), but to three separate conditions for \(\beta \neq 0\), namely

\[
\begin{align*}
2\Pi^T HD_1 \Pi &= \Pi^2 |_{0}^{N}, \\
\Pi^T HD_2 \phi &= -\Pi^T A \phi + (\Pi S \phi) |_{0}^{N}, \\
2\phi^T A \hat{D}_1 \phi &= (S \phi)^2 |_{0}^{N}
\end{align*}
\] (A.7) (A.8) (A.9)

where \(S \phi\) is an approximation of \(\partial_x \phi\) at the boundaries.

We now assume that (in the terminology of [13]) \(D_1\) is a \textit{first derivative summation by parts operator}, meaning that

\[
HD_1 + (HD_1)^T = B, \tag{A.10}
\]

where \(B \equiv \text{diag}(-1,0,\ldots,0,1)\). We also assume that \(D_2\) is a \textit{symmetric second derivative summation by parts operator}, i.e., it satisfies

\[
HD_2 = (-A + BS). \tag{A.11}
\]
These two assumptions guarantee that (A.7) and (A.8) are satisfied. The third condition, equation (A.9), gives
\[ A \hat{D}_1 + (A \hat{D}_1)^T = S^T BS + M, \] (A.12)
where \( M = M^T \leq 0. \)

Assuming a minimal width second order accurate approximation in the interior, we have that \[ H = h \text{ diag}(1/2, 1, \ldots, 1, 1/2), \] (A.13)
\[ D_1 = (D_+, D_0, \ldots, D_0, D_-), \] (A.14)
\[ D_2 = (D_+^2, D_+ D_-, \ldots, D_+ D_-, D_-^2), \] (A.15)
\[ BS = (D_+ - h/2 D_+, 0, \ldots, 0, D_- - h/2 D_-), \] (A.16)
and \( \phi^T A \phi = \sum_{j=0}^{N-1} (D_+ \phi_j)^2 h. \) We see that the matrix \( A \) satisfies the desired properties.

We need to construct an operator \( \hat{D}_1 \) approximating \( \partial \) which coincides with \( D_0 \) at the interior. We consider
\[ (\hat{D}_1 \phi)_0 = a D_+ \phi_0 + (1-a) D_+ \phi_1, \] (A.17)
\[ (\hat{D}_1 \phi)_1 = b D_+ \phi_0 + (1-b) D_+ \phi_1 \] (A.18)
and find that if \( 2a = 8b = 3 \), condition (A.9) is satisfied. However, this approximation is not accurate enough: it is only first order convergent.

Alternatively, one could consider the modification
\[ (\hat{D}_1 \phi)_0 = \frac{3}{2} D_+ \phi_0 - \frac{1}{2} D_+ \phi_1, \] (A.19)
\[ (\hat{D}_1 \phi)_1 = D_0 \phi_1 + ah^2 D_+^3 \phi_0. \] (A.20)
We find that \( M \) in equation (A.12) is indefinite (the product of two of its eigenvalues is negative) unless \( a = 0 \), in which case it is semi-definite positive.

Clearly there is an infinity of other choices to make, and it is difficult to exhaust all possibilities. We have made a number of attempts aimed at obtaining summation by parts for the shifted wave equation, but we have not been able to construct operators and scalar products which give second order convergent schemes. We also note that a direct use of the operators of Appendix C.1 of [15], i.e., using \( D_+ \) and \( D_+^2 \) to approximate first and second derivatives at the boundary, gives rise to a first order reflection from the boundary. With the same settings used in Section 5 we carried out convergence tests for \( \beta = 2 \) to confirm this. See table A1.

**Appendix B. The advection equation**

In this Appendix we list a number of known results regarding the discrete boundary treatment for the advection equation and give a simple prescription for the inflow fourth order accurate case.
Discrete boundary treatment for the shifted wave equation . . .

**Table A1.** Using $D_+$ and $D_+^2$ to approximate first and second derivatives at the outflow boundary gives only first order accuracy. Here we used $\beta = 2$.

| $N$  | $l_2$             | $q$  |
|------|-------------------|------|
| 50   | 6.54655 $10^{-1}$ | 1.0630 |
| 100  | 3.13340 $10^{-1}$ | 1.0252 |
| 200  | 1.53950 $10^{-1}$ | 1.0065 |
| 400  | 7.66263 $10^{-2}$ |      |

**Appendix B.1. Second order accuracy**

The semi-discrete initial-boundary value problem with an outflow boundary,

\[
\frac{d}{dt} v_j = aD_0 v_j \quad j = 0, 1, 2, \ldots, \quad (B.1)
\]

\[
h^q D_+^q v_{-1} = 0 \quad (B.2)
\]

with $a > 0$ is strongly stable and second order convergent for $q \geq 2$. For an inflow boundary, $a < 0$, the scheme

\[
\frac{d}{dt} v_j = aD_0 v_j \quad j = 1, 2, \ldots \quad (B.3)
\]

\[
v_0 = g \quad (B.4)
\]

is strongly stable and second order convergent.

However, this last scheme is actually strongly stable for any $a$, even if we impose data on an outflow boundary. Moreover, if the redundant boundary data is second order accurate with respect to the continuum solution, the scheme is also second order convergent.

**Appendix B.2. Fourth order accuracy**

As shown in [14] the semi-discrete initial-boundary value problem

\[
\frac{d}{dt} v_j = aD_0 \left(1 - \frac{h^2}{6} D_+ D_- \right) v_j \quad j = 0, 1, 2, \ldots, \quad (B.5)
\]

\[
h^q D_+^q v_{-1} = h^q D_+^q v_{-2} = 0 \quad (B.6)
\]

with $q \geq 4$ and $a > 0$ is strongly stable and fourth order convergent.

In the inflow case ($a < 0$) the boundary conditions

\[
v_0 = g, \quad (B.7)
\]

\[
h^q D_+^q v_{-1} = 0 \quad (B.8)
\]

lead to strong stability for $q = 4$ or 5 and fourth order convergence is obtained provided that $g$ is fourth order accurate. Interestingly, for $q \geq 6$ the scheme is unstable. One
can show that for $a = -1$, $q = 6$, $g = 0$, the scheme admits solutions of the form $e^{st}f_j$, where $sh \approx 0.093 \pm 1.308i$.

Note that a set of boundary conditions which lead to strong stability for any value of $a$ (including the case where boundary conditions are imposed at an outflow boundary) is

$$v_{-1} = g_{-1}, \quad v_0 = g_0$$  \hspace{1cm} (B.9)

If, in addition, $g_{-1} = u(t, -h) + O(h^4)$ and $g_0 = u(t, 0) + O(h^4)$ the scheme is fourth order convergent [14].

**Appendix C. Direct second-order treatment of the shifted wave equation**

Eliminating $\Pi$ from (1-2), we obtain

$$\partial_t^2 \phi = 2\beta \partial_t \partial_x \phi + (1 - \beta^2) \partial_x^2 \phi.$$  \hspace{1cm} (C.1)

In [12], this equation is investigated as a toy model for the Einstein equations in the presence of a shift. The minimal width second-order accurate discretisation in space is

$$\partial_t^2 \phi = 2\beta \partial_t D_0 \phi + (1 - \beta^2) D_+ D_- \phi,$$  \hspace{1cm} (C.2)

but this is stable only for $|\beta| < 1$. Therefore in regions where $|\beta| \geq 1$, the discretisation

$$\partial_t^2 \phi = 2\beta \partial_t D_0 \phi + (D_+ D_- - \beta^2 D_0^2) \phi,$$  \hspace{1cm} (C.3)

is used instead, with a blending between the two algorithms in a transition region. The authors can then construct second-order accurate and stable boundary treatments for both outflow and time-like boundaries. They focus on Dirichlet and Neumann boundary conditions.

The system could be reduced to first order in time, second order in space form by introducing $\partial_t \phi$ as an auxiliary variable. On the level of a semi-discrete (continuous in time) system this change of variable is trivial. In particular, the difficulty for $|\beta| \geq 1$ and its resolution would be the same. The crucial difference is not that (C.1) is in second-order in time form but that $\Pi$ has been replaced by $\partial_t \phi$ in the equivalent first-order in time, second-order in space system.

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