Estimate of a Trigonometrical Sum Involving Naturals with Binary Decompositions of a Special Kind

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Abstract

Let $N_0$ be a class of natural numbers whose binary decompositions has even number of 1. We estimate of the sum $\sum_{n \in N_0, n \leq X} \exp(2\pi i \alpha n^2)$.

Key words: binary decomposition, estimate of a trigonometrical sum

1. Introduction

Consider the binary decomposition of a positive integer $n$:

$$n = \sum_{k=0}^{\infty} \varepsilon_k 2^k,$$

where $\varepsilon_k = 0, 1$ and $k = 0, 1, \ldots$

We split the set of positive integers into two nonintersecting classes as follows:

$$N_0 = \{n \in \mathbb{N}, \sum_{k=0}^{\infty} \varepsilon_k \equiv 0 \pmod{2}\}, \quad N_1 = \{n \in \mathbb{N}, \sum_{k=0}^{\infty} \varepsilon_k \equiv 1 \pmod{2}\}.$$

In 1968, A.O. Gel’fond [11] obtained the following theorem: for the number of integers $n$, $n \leq X$, satisfying the conditions $n \equiv l \pmod{m}$, $n \in N_j$ ($j = 0, 1$), the following asymptotic formula is valid:

$$T_j(X, l, m) = \frac{X}{2m} + O(X^\lambda), \quad (1)$$

where $m, l$ are any naturals and $\lambda = \frac{\ln 3}{\ln 4} = 0.7924818\ldots$

Suppose that

$$\varepsilon(n) = \begin{cases} 
1 & \text{for } n \in N_0, \\
-1 & \text{otherwise.}
\end{cases}$$

The proof of formula (1) is based on the estimate

$$|S(\alpha)| \ll X^\lambda$$

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of the trigonometrical sum

\[ S(\alpha) = \sum_{n \leq X} \varepsilon(n) e^{2\pi i n \alpha}, \]

which is valid for any real values of \( \alpha \).

Note that \( S(\alpha) \) is a linear sum.

In present paper we estimate the quadratic sum

\[ S_0(\alpha) = \sum_{n \leq X} \varepsilon(n) e^{2\pi i n \alpha^2}. \]

Sum of this type, as far as we know, isn’t yet been considered in mathematical literature.

Our main result is the following theorem.

**Theorem 1.** Suppose that \( X > 2, \alpha \in \mathbb{R} \). Then for any \( c > 0 \) there exist positive numbers \( A \) and \( B \) such that if

\[ \alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad |\theta| < 1, \quad \log^A X < q \leq X \log^{-B} X, \]

then the inequality

\[ |S_0(\alpha)| \ll X \log^{-c} X \]

holds.

Proof of the theorem 1 is based on the following lemmas.

**2. Lemmas**

**Lemma 1.** Suppose that \( Q \in \mathbb{N}, Q \leq X \). Then the following estimate holds:

\[ S_0(\alpha) \ll \frac{X}{\sqrt{Q}} + \left\{ \frac{X}{Q} \sum_{h=1}^{Q-1} |S(X-h, h, 2\alpha)| \right\}^{1/2}, \]

where \( S(Y, h, \beta) = \sum_{n \leq Y} \varepsilon(n) e(n+h) e(\beta n), e(x) = \exp\{2\pi i x\} \).

The proof of this lemma coincides with the proof of a well known lemma in van der Corput’s method.

**Lemma 2.** Suppose that \( \beta \in \mathbb{R}, h \in \mathbb{N}, 2 < h \leq \log^{c_1} X, \) where \( c_1 > 0 \) is a constant. Let \( N+1 \) be a number of binary digits of \( h \), \( X_{N+1} = X 2^{-N-1}, \beta_{N+1} = 2^{N+1} \beta. \)

Then the inequality

\[ |S(X, h, \beta)| \ll h^{\log_2 3} |S(X_{N+1}, 1, \beta_{N+1})| + h^{\log_2 3} |S(X_{N+1}, 2, \beta_{N+1})| + h^{\log_2 3} \]

holds.
Proof. Let \( h = 1w_Nw_{N-1} \ldots w_1w_0 \) be a binary expansion of \( h, w_j = 0, 1 \) (\( j = 0, 1, \ldots , N \)).

Define numbers \( h_j \) with the following binary expansions:

\[
    h_0 = h = 1w_Nw_{N-1} \ldots w_1w_0, \quad h_1 = 1w_Nw_{N-1} \ldots w_1, \quad h_2 = 1w_Nw_{N-1} \ldots w_2, \ldots , h_N = 1w_N.
\]

Define numbers \( s_j \) as follows: \( s_j = 1 - 2w_j \) (\( j = 0, \ldots , N \)).

Grouping summands over even and over odd \( n \) and using obvious formulae \( \varepsilon(2n) = \varepsilon(n), \varepsilon(2n + 1) = -\varepsilon(n) \) we have the following equalities

\[
    S(X, 2h_1, \beta) = (1 + \varepsilon(\beta))S(X2^{-1}, h_1, 2\beta) + O(1), \quad \text{(2)}
\]

\[
    S(X, 2h_1 + 1, \beta) = -S(X2^{-1}, h_1, 2\beta) - \varepsilon(\beta)S(X2^{-1}, h_1 + 1, 2\beta) + O(1). \quad \text{(3)}
\]

Consider the linear combination

\[
    x_jS(X_j, h_j, \beta_j) + y_jS(X_j, h_j + 1, \beta_j),
\]

where \( 0 \leq j \leq N, X_j = X2^{-j}, \beta_j = 2^j \beta \). By \( \text{(2)} \) and \( \text{(3)} \) we have

\[
    x_jS(X_j, h_j, \beta_j) + y_jS(X_j, h_j + 1, \beta_j) = x_{j+1}S(X_{j+1}, h_{j+1}, \beta_{j+1}) + y_{j+1}S(X_{j+1}, h_{j+1} + 1, \beta_{j+1}) + O(|x_j| + |y_j|), \]

where

\[
    x_{j+1} = (s_j + \frac{s_j + 1}{2}\varepsilon(\beta_j))x_j - \frac{s_j + 1}{2}y_j, \quad \text{(4)}
\]

\[
    y_{j+1} = \frac{s_j - 1}{2}\varepsilon(\beta_j)x_j - (\frac{s_j - 1}{2} + s_j\varepsilon(\beta_j))y_j. \quad \text{(5)}
\]

Define \( x_0 \) and \( y_0 \)

\[
    (x_0, y_0) = \begin{cases} (1, 0), & \text{if } h \text{ is even}, \\ (0, 1), & \text{if } h \text{ is odd}. \end{cases}
\]

Then we have

\[
    S(X, h, \beta) = x_{N+1}S(X_{N+1}, 1, \beta_{N+1}) + y_{N+1}S(X_{N+1}, 2, \beta_{N+1}) + O(\sum_{j=0}^{N}(|x_j| + |y_j|)).
\]

From the definition of \( x_0 \) and \( y_0 \) and from \( \text{(4)}, \text{(5)} \) it follows by induction that

\[
    |x_j| \leq 3^j, \quad |y_j| \leq 3^j
\]

for any \( j \geq 0 \). Thus we have

\[
    |S(X, h, \beta)| \ll h^{\log_2 3}|S(X_{N+1}, 1, \beta_{N+1})| + h^{\log_2 3}|S(X_{N+1}, 2, \beta_{N+1})| + h^{\log_2 3}.
\]

\( \square \)
Lemma 3. Suppose that $\gamma \in \mathbb{R}$, $j_0 \geq 1$, $Y > 2$. Then the inequality
\[
|S(Y, 1, \gamma)| + |S(Y, 2, \gamma)| \ll \log Y \sum^{j_0}_{j=1} \left| \sum_{n \leq Y2^{-j}} e(2^j \gamma n) \right| + Y2^{-j_0} + \log Y.
\]

Proof follows immediately from the equalities
\[
S(Y, 1, \gamma) = -\sum_{n \leq Y2^{-1}} e(2\gamma n) - e(\gamma)S(Y2^{-1}, 1, 2\gamma) + O(1),
\]
\[
S(Y, 2, \gamma) = S(Y2^{-1}, 1, 2\gamma) + e(\gamma)S(Y2^{-1}, 2, 2\gamma) + O(1).
\]

3. Proof of Theorem 1

First apply lemma 1 with $Q = \log^{2c} X$. Now it is sufficient to estimate sums $S(X-h, h, 2h\alpha)$ for $1 \leq h \leq Q - 1$.

By lemma 2 we arrive to the inequality
\[
|S(X-h, h, \beta)| \ll h^{\log_2 3} |S(X_{N+1}, 1, \beta_{N+1})| + h^{\log_2 3} |S(X_{N+1}, 2, \beta_{N+1})| + h^{\log_2 3}.
\]

Then by lemma 3 we have
\[
|S(X-h, h, 2h\alpha)| \ll h^{\log_2 3} \log^2 X j_0 \max_{1 \leq j \leq j_0} \left| \sum_{n \leq Y2^{-j}} e(2^j \beta_{N+1} n) \right| + X h^{\log_2 3} 2^{-j_0} + h^{\log_2 3} \log X,
\]

where $j_0$ satisfy to inequalities
\[
Q^{\log_2 3} j_0 2^{-j_0} \leq \log^{-2c} X < Q^{\log_2 3} (j_0 - 1) 2^{-j_0 + 1}.
\]

Since
\[
\left| \sum_{n \leq Y2^{-j}} e(2^j \beta_{N+1} n) \right| \ll \frac{1}{\| 2^j \beta_{N+1} \|^1},
\]
where $\| x \| = \min\{\{x\}, 1 - \{x\}\}$, we get the inequality
\[
|S(X-h, h, 2h\alpha)| \ll h^{\log_2 3} \log^2 X \max_{1 \leq j \leq j_0} \frac{1}{\| 2^j \beta_{N+1} \|^1} + X \log^{-2c} X,
\]

which holds for any $h$, $1 \leq h \leq Q - 1$.

Let $q$ satisfies the inequality $q > \log^A X$, where $A > 0$ is such that $2^{N+j_0+2h} < \frac{1}{10} \log^A X$. Then we have
\[
\| 2^j \beta_{N+1} \|^1 \geq \frac{1}{q} - \frac{2^{N+j_0+2h}q^2}{q^2} \geq \frac{0.9}{q}, \quad \| 2^j \beta_{N+1} \|^1 \leq \frac{10q}{9}.
\]

Let $q$ also satisfies the inequality $q \leq X \log^{-B} X$, where $B > 0$ is such that
\[
\frac{Q^{\log_2 3} j_0 \log^2 X}{\log^B X} \leq \log^{-2c} X.
\]
Then for any $h$, $1 \leq h \leq Q - 1$, the inequality

$$|S(X - h, h, 2ha)| \ll X \log^{-2c}$$

holds, and so

$$|S_0(\alpha)| \ll X \log^{-c}.$$

**Corollary 1.** Suppose that $X > 2, \alpha \in \mathbb{R}$. Then for any $c > 0$ there exist positive numbers $A$ and $B$ such that if

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad |\theta| < 1, \quad \log^A X < q \leq X \log^{-B} X,$$

then the inequality

$$|\sum_{n \leq X, \ n \in \mathbb{N}_0} e(\alpha n^2)| \ll X \log^{-c} X$$

holds.

**Proof.** By definition we have

$$\sum_{n \leq X, \ n \in \mathbb{N}_0} e(\alpha n^2) = \sum_{n \leq X} \frac{1 + \varepsilon(n)}{2} e(\alpha n^2) = \frac{1}{2} S_0(\alpha) + \frac{1}{2} S_1(\alpha),$$

where $S_1(\alpha) = \sum_{n \leq X} e(\alpha n^2)$.

Sum $S_0(\alpha)$ is estimated in theorem 1. Estimate $S_1(\alpha)$ in a standard way:

$$|S_1(\alpha)|^2 \ll X + \sum_{|h| \leq X} |\sum_{n \leq X} e(2hna)| \ll X + \sum_{|h| \leq 2X} \min(X, \frac{1}{\|ah\|}) \ll$$

$$\ll (\frac{X}{q} + 1)(X + q \log q) \ll X^2 \log^{-2c} X.$$

□

**References**

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