Constrained Optimization Involving Nonconvex $\ell_p$ Norms: 
Optimality Conditions, Algorithm and Convergence

Hao Wang∗, Yining Gao†, Jiashan Wang‡ and Hongying Liu§

Abstract

This paper investigates the optimality conditions for characterizing the local minimizers of the constrained optimization problems involving an $\ell_p$ norm ($0 < p < 1$), which may appear in either the objective or the constraint. This kind of problem has strong applicability to a wide range of areas since the $\ell_p$ norm can promote sparse solutions. However, the nonsmooth and non-Lipschitz nature of the $\ell_p$ norm makes these problems difficult to analyze and solve. We derived the first-order necessary conditions and the sequential optimality conditions under various constraint qualifications. We extend the iteratively reweighted algorithms for solving the unconstrained $\ell_p$ norm regularized problems to constrained cases and show the sequential optimality conditions are satisfied by this algorithm. Global convergence is derived and the performance of the proposed algorithm is exhibited by numerical experiments.

1 Introduction

Sparse regularization problems have attracted considerable attentions over the past decades, which have numerous applications in the areas including compressed sensing [1], biomedical engineering [2,3], sensor selection [4] and portfolio management [5]. This is because sparse solutions usually lead to better generalization performance and robustness. A common approach of promoting sparsity in the solution is to involve a sparse-inducing term such as the $\ell_1$ norm or the $\ell_0$ norm of the variables either in the objective or in the constraint. In recent years, nonconvex and/or non-Lipschitz sparsity inducing terms such as the $\ell_p$ (quasi-)norm ($0 < p < 1$) are shown [6] to have preferable performance in many situations. In the past decade, many works focus on designing and analyzing the algorithms for solving the unconstrained $\ell_p$ regularized problems [7,8,9,10,11,12]. However, when it comes to the constrained cases, there are few works despite of its wider applicability. We list two examples of the constrained optimization problems involving an $\ell_p$ norm.

∗School of Information Science and Technology, ShanghaiTech University, Shanghai, China (E-mail: wang-haol@shanghaitech.edu.cn).
†School of Information Science and Technology, ShanghaiTech University, Shanghai, China (E-mail: gaoyn@shanghaitech.edu.cn).
‡Department of Mathematics, University of Washington, Seattle, WA, USA (E-mail: jsw1119@gmail.com).
§School of Mathematical Sciences, Beihang University, Beijing, China (E-mail: liuhongying@buaa.edu.cn).

arXiv:2110.14127v2 [math.OC] 15 Feb 2022
Example 1 Consider the cloud radio access network (Cloud-RAN) power consumption problem \cite{13,14}, which solves a group sparse problem to induce the group sparsity for the beamformers to guide the remote radio head (RRH) selection. This group sparse problem is addressed by minimizing the mixed $\ell_p/\ell_2$-norm with $p \in (0, 1]$, yielding the following problem

$$\min_v \sum_{l=1}^{L} \frac{\rho_l}{z_l} \|\tilde{v}_l\|_2^p$$

subject to

$$\frac{\sum_{i \neq k} \|h_k^H v_i\|_2^2 + \sigma_k^2}{\gamma_k} \leq \frac{1}{\gamma_k} \Re(h_k^H v_k)$$

$$\|\tilde{v}_l\|_2 \leq \sqrt{P_l}, \ l = 1, \cdots, L, k = 1, \cdots, K.$$ 

Here the Cloud-RAN architecture of this model has $L$ RRHs and $K$ single-antenna Mobile Users (MUs), where the $l$-th RRH is equipped with $N_l$ antennas. $v_{ik} \in \mathbb{C}^{N_l}$ is the transmit beamforming vector from the $l$-th RRH to the $k$-th user with the group structure of transmit vectors $\tilde{v}_l = [v_{1l}^T, \cdots, v_{Kl}^T]^T \in \mathbb{C}^{K N_l \times 1}$. Denote the relative fronthaul link power consumption by $\rho_l$, and the inefficient of drain efficiency of the radio frequency power amplifier by $z_l$. The channel propagation between user $k$ and RRH $l$ is denoted as $h_{lk} \in \mathbb{C}^{N_l}$. $P_l$ is the maximum transmit power of the $l$-th RRH. $\sigma_k$ is the noise at MU $k$. $\gamma = (\gamma_1, \cdots, \gamma_K)^T$ is the target signal-to-interference-plus-noise ratio (SINR).

Example 2. (The $\ell_p$-constrained sparse coding) In the context of sparse coding \cite{15}, the task is to reconstruct the unknown sparse code word $\bar{x} \in \mathbb{R}^n$ from the linear measurements $y = Ax + \epsilon$, where $y \in \mathbb{R}^m$ represents the data with $m$ features, $\epsilon \in \mathbb{R}^m$ denotes the noise vector, and $A$ corresponds to the fixed dictionary that consists of $n$ atoms with respect to its columns. This problem can be formulated as

$$\min_{x \in \mathbb{R}^n} \|Ax - y\|_2^2 \quad \text{s.t.} \quad \|x\|_p^p \leq \theta,$$  \hspace{1cm} (1.1)

where the $\ell_p$ ball constraint is to induce sparsity in the code word.

In this paper, we consider the following two general forms of constrained nonlinear optimization with $\ell_p$ norms. The first one is the constrained $\ell_p$ regularized problem, meaning the $\ell_p$ norm appears in the objective as a penalty,

$$\min F(x) := f_0(x) + \lambda \|x\|_p^p \quad \text{s.t.} \quad f_j(x) \leq 0, \ \forall j \in \mathcal{I}; \ f_j(x) = 0, \ \forall j \in \mathcal{E}. \quad (\mathcal{P}_1)$$

The second one has the $\ell_p$ norm in the constraint and requires it to be smaller than a prescribed value $\theta > 0$,

$$\min f_0(x) \quad \text{s.t.} \quad \|x\|_p^p \leq \theta; \ f_j(x) \leq 0, \ \forall j \in \mathcal{I}; \ f_j(x) = 0, \ \forall j \in \mathcal{E}. \quad (\mathcal{P}_2)$$

Here, $f_j : \mathbb{R}^n \to \mathbb{R}, j \in \{0\} \cup \mathcal{I} \cup \mathcal{E}$ are continuously differentiable on $\mathbb{R}^n$ and $\|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ with $p \in (0, 1]$. The positive $\lambda$ is the given regularization parameter and $\theta$ is referred to as the radius of $\ell_p$-ball.

Despite of the advantages of nonconvex $\ell_p$ norm in promoting sparse solutions, problems of the forms (\mathcal{P}_1) and (\mathcal{P}_2) are generally not easy to handle. This is largely due to the nonconvex and
non-Lipschitz nature of the $\ell_p$ norm which makes it difficult to characterize the optimal solutions. In particular, verifiable optimality conditions are often difficult to derive, leaving it an obstacle for designing efficient numerical algorithms. For example, for ($P_1$), many researchers [16, 17, 14, 11] tend to approximate the $\ell_p$ term by Lipschitz continuous functions and then solve for an approximate solution. As for ($P_2$), to the best of our knowledge, not much has been done except the special case that only the $\ell_p$ ball constraint presents in the problem [18], meaning the projection onto the $\ell_p$ ball.

1.1 Literature review

The optimality conditions of the unconstrained and inequality constrained versions of ($P_1$) were studied in [19], which is the immediate predecessor of our work. They derived the first-order and second-order necessary conditions by assuming the “extended” linear independence constraint qualification (ELICQ) is satisfied by ($P_1$), meaning the LICQ is satisfied at the local minimizer in the subspace consisting of the nonzero variables. They also stated that the second-order optimality conditions can be derived by considering the reduced problems after fixing the zero components at a stationary point. In [20], Bian and Chen derived a first-order necessary optimality condition using the theory of the generalized directional derivative, which is also closely related to our work. In particular, for the case that the constraints are all linear, Gabriel Haeser et al. [12] articulated first- and second-order necessary optimality conditions for this problem based on the perturbed problem and the limits of perturbation. Sufficient conditions for the $\epsilon$-perturbed stationary points are also presented. As for ($P_2$), [18] derived optimality conditions for the special case where only $\ell_p$ ball constraint exists using the concept of generalized Fréchet normal cone. To the best of our knowledge, there has been no study on the optimality conditions for more general cases of this problem.

1.2 Contributions

In this paper, we are interested in deriving the optimality conditions to characterize the local solutions of ($P_1$) and ($P_2$) under different constraint qualifications (CQ). First of all, we analyze the basic properties of the $\ell_p$ norm and the $\ell_p$ norm ball. We derive the regular and general subgradients of the $\ell_p$ norm and the regular and general normal of the $\ell_p$ norm ball, which indicate the $\ell_p$ norm is subdifferentially regular and the $\ell_p$ ball is Clarke regular. For ($P_1$) and ($P_2$), we derive the Karush-Kuhn-Tucker (KKT) conditions and discuss the constraint qualifications that ensure that the KKT conditions are satisfied at a local minimizer. For ($P_2$), we believe this is the first result.

Recently, Andreani et al. [21] introduced the sequential optimality conditions, namely, the approximate KKT (AKKT) conditions for constrained smooth optimization problems, which is commonly satisfied by many algorithms. They also proposed the Cone-Continuity Property (CCP), under which the AKKT conditions imply the KKT conditions. This is widely believed to be one of the weakest qualification under which KKT conditions hold at local minimizer. We also define the sequential optimality conditions for ($P_1$) and ($P_2$) and explore the constraint qualification under
which the sequential conditions imply KKT conditions. We believe these are much stronger results than existing ones.

To demonstrate the applicability of the proposed sequential optimality conditions, we extend the well-known iteratively reweighted algorithms for solving unconstrained $\ell_p$-regularized problem to general constrained cases and show that those conditions are satisfied at the limit points of the generated iterates. Therefore, under the proposed constraint qualification, the limit points satisfy the KKT conditions.

1.3 Notation and preliminary

We use 0 as the vector filled with all zeros of appropriate size. For $\mathcal{N} \subset \{1, \ldots, n\}$ and $x \in \mathbb{R}^n$, let $\mathbb{R}^{[\mathcal{N}]}$ be the reduced subspace of $\mathbb{R}^n$ that consists of the components $x_i, i \in \mathcal{N}$, and denote $x_{\mathcal{N}} \in \mathbb{R}^{[\mathcal{N}]}$ as the subvector of $x$ containing the elements $x_i, i \in \mathcal{N}$. Let $\mathcal{Z} = \{1, \ldots, n\} \setminus \mathcal{N}$. For a differentiable $f : \mathbb{R}^n \to \mathbb{R}$, let $\nabla_{\mathcal{N}} f(x)$ be the vector consisting of $\nabla_i f(x), i \in \mathcal{N}$. In $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, define the set

$$ \Gamma = \{x \mid f_j(x) \leq 0, \forall j \in \mathcal{I}; \ f_j(x) = 0, \forall j \in \mathcal{E}\}, $$

and the index set of active inequalities by $\mathcal{A}(x) = \{j \mid f_j(x) = 0, \ j \in \mathcal{I}\}$. The sets of zeros and nonzeros in $x \in \mathbb{R}^n$ are defined as

$$ \mathcal{Z}(x) = \{i \mid x_i = 0\} \quad \text{and} \quad \mathcal{N}(x) = \{i \mid x_i \neq 0\}. $$

For simplicity, we use shorthands $\bar{\mathcal{N}} = \mathcal{N}(\bar{x}), \bar{\mathcal{Z}} = \mathcal{Z}(\bar{x})$ and $\mathcal{N}^k = \mathcal{N}(x^k), \mathcal{Z}^k = \mathcal{Z}(x^k)$.

For $x \in \mathbb{R}^n$, define $\mathcal{B}(x; \delta) = \{z \mid \|z - x\|_2 \leq \delta\}$. For any cone $K \subset \mathbb{R}^n$, the polar of $K$ is defined to be the cone

$$ K^* := \{v \in \mathbb{R}^n \mid \langle v, w \rangle \leq 0 \text{ for all } w \in K\}. $$

For $C \subset \mathbb{R}^n$, its horizon cone is defined by

$$ C^\infty = \begin{cases} \{x \mid \exists x' \in C, \lambda' \searrow 0, \text{ with } \lambda' x' \to x\} & \text{when } C \neq \emptyset, \\ \{0\} & \text{when } C = \emptyset. \end{cases} $$

Another operation on $C$ is the smallest cone containing $C$, namely the positive hull of $C$, which is defined as $\text{pos } C = \{0\} \cup \{\lambda x \mid x \in C, \lambda > 0\}$. A vector $w \in \mathbb{R}^n$ is tangent to a set $C \subset \mathbb{R}^n$ at $\bar{x} \in C$, written $w \in T_C(\bar{x})$, if $(x' - \bar{x})/\tau' \to w$ for some $x' \to \bar{x}, \tau' \searrow 0$. The interior and the boundary of a set $C \subset \mathbb{R}^n$ is denoted as $\text{int } C$ and $\partial C$, respectively.

**Definition 1.1** ([22] Definition 6.3, 6.4). (a) Let $C \subset \mathbb{R}^n$ and $\bar{x} \in C$. A vector $v$ is a regular normal to $C$ at $\bar{x}$, written $v \in \hat{\mathcal{N}}_C(\bar{x})$, if

$$ \limsup_{x \to \bar{x}, x \neq \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0. $$
It is a (general) normal to \( C \) at \( \bar{x} \), written \( v \in N_C(\bar{x}) \), if there are sequences \( x^\nu \rightharpoonup \bar{x} \) and \( v^\nu \to v \) with \( v^\nu \in \overset{\sim}{N}_C(x^\nu) \). We call \( \overset{\sim}{N}_C(\bar{x}) \) the regular normal cone and \( N_C(\bar{x}) \) the normal cone to \( C \).

(b) A set \( C \subseteq \mathbb{R}^n \) is Clarke regular at \( \bar{x} \in C \) if it is locally closed at \( \bar{x} \) and \( N_C(\bar{x}) = \overset{\sim}{N}_C(\bar{x}) \).

For a nonempty convex \( C \subseteq \mathbb{R}^n \) and \( \bar{x} \in C \), \( \overset{\sim}{N}_C(\bar{x}) = N_C(\bar{x}) = \{ v \mid \langle v, z - \bar{x} \rangle \leq 0, \text{ for all } z \in C \} \).

**Definition 1.2** ([22] Definition 8.3). Consider a function \( f : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{ +\infty \} \) and \( f(\bar{x}) < \infty \). For a vector \( v \in \mathbb{R}^n \), one says that

(a) \( v \) is a regular subgradient of \( f \) at \( \bar{x} \), written \( v \in \partial f(\bar{x}) \), if
\[
f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(|x - \bar{x}|);
\]

(b) \( v \) is a (general) subgradient of \( f \) at \( \bar{x} \), written \( v \in \partial f(\bar{x}) \), if there are sequence \( x^\nu \rightharpoonup \bar{x} \) and \( v^\nu \in \overset{\sim}{\partial} f(x^\nu) \) with \( v^\nu \rightarrow v \); 

(c) \( v \) is a horizon subgradient of \( f \) at \( \bar{x} \), written \( v \in \partial \infty f(\bar{x}) \), if there are sequence \( x^\nu \rightharpoonup \bar{x} \) and \( v^\nu \in \overset{\sim}{\partial} f(x^\nu) \) with \( \lambda^\nu v^\nu \rightarrow v \) for some sequence \( \lambda^\nu \downarrow 0 \).

For \( f : \mathbb{R}^n \to \mathbb{R} \), the epigraph of \( f \) is the set \( \text{epi} f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x) \} \).

**Definition 1.3** ([22] Definition 7.25). A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called subdifferentially regular at \( \bar{x} \) if \( f(\bar{x}) \) is finite and \( \text{epi} f \) is Clarke regular at \( (\bar{x}, f(\bar{x})) \) as a subset of \( \mathbb{R}^n \times \mathbb{R} \).

## 2 First-order necessary optimality conditions

In this section, we present the first-order necessary optimality conditions for \((P_1)\) and \((P_2)\). Before proceeding to the optimality conditions, we provide some basic properties.

### 2.1 Basic Properties

Denote \( \phi(x) = \|x\|_p^p \) and the \( \ell_p \) norm ball \( \Theta := \{ x \in \mathbb{R}^n \mid \phi(x) \leq \theta \} \). In this subsection, we provide basic properties about \( \phi(x) \) and \( \Theta \). In particular, we derive regular and general subgradients of \( \phi \) and the regular and the general normal cones of \( \Theta \), and then show that the \( \phi \) is subdifferentially regular and \( \Theta \) is Clarke regular on \( \mathbb{R}^n \).

The regular, general, and horizon subgradients of \( \phi \) can be calculated as follows.
Theorem 2.1. For any \( \bar{x} \in \mathbb{R}^n \), it holds that
\[
\partial \phi(\bar{x}) = \tilde{\partial} \phi(\bar{x}) = \{ v \in \mathbb{R}^n | v_j = \text{sign}(x_j)p|\bar{x}_j|^{p-1}, \ j \in \bar{N} \}, 
\] (2.1)
\[
[\tilde{\partial} \phi(\bar{x})]^\infty = \partial^\infty \phi(\bar{x}) = \{ v \in \mathbb{R}^n | v_j = 0, \ j \in \bar{N} \}. 
\] (2.2)

Therefore, \( \phi \) is subdifferentially regular at every \( x \in \mathbb{R}^n \).

Proof. We first consider \( | \cdot |^p \) on \( \mathbb{R} \). If \( \bar{x} \neq 0 \), then \( \partial|\bar{x}|^p = \tilde{\partial}|\bar{x}|^p = \{ \nabla|\bar{x}|^p \} = \{ \text{sign}(\bar{x})p|\bar{x}|^{p-1} \}. \)

On the other hand, \( \lim_{x \to 0, x \neq 0} \frac{|x|^p}{|x|} = +\infty \), implying \( \liminf_{x \to 0, x \neq 0} \frac{|x|^p - |v(x-0)|}{|x|} \geq 0 \) for any \( v \in \mathbb{R} \). Therefore, if \( \bar{x} = 0 \), it follows from the definition of regular subgradient that \( \mathbb{R} \subset \tilde{\partial}|0|^p \subset \partial|0|^p \subset \mathbb{R} \). Hence, \( \tilde{\partial}|0|^p = \partial|0|^p = \mathbb{R} \). By [22] Proposition 10.5, we have
\[
\tilde{\partial} \phi(\bar{x}) = \partial \phi(\bar{x}) = \partial|\bar{x}_1|^p \times \ldots \times \partial|\bar{x}_n|^p = \{ v \in \mathbb{R}^n | v_j = \text{sign}(\bar{x}_j)p|\bar{x}_j|^{p-1}, \ j \in \bar{N} \}. 
\]

This proves (2.1).

By the definition of horizon cone and (2.1), it is obvious that \( [\tilde{\partial} \phi(\bar{x})]^\infty = \{ v \in \mathbb{R}^n | v_j = 0, \ j \in \bar{N} \} \). We next prove \( \partial^\infty \phi(\bar{x}) = [\tilde{\partial} \phi(\bar{x})]^\infty \).

For any \( v \in [\tilde{\partial} \phi(\bar{x})]^\infty \) and \( \{ \lambda^v \} \searrow 0 \), we can select sequence \( x^v \to \bar{x} \) such that \( \mathcal{N}(x^v) = \bar{N} \). Let \( v^v_j = v_j / \lambda^v \). From (2.1), this means \( v^v \in \tilde{\partial} \phi(x^v) \) and \( \lambda^v v^v \to v \). Therefore, \( v \in \partial^\infty \phi(\bar{x}) \).

On the other hand, for \( x^v \) sufficiently close to \( \bar{x} \), it holds that \( \bar{N} \subset \mathcal{N}(x^v) \). Therefore, by (2.1), \( v^v_j = \text{sign}(\bar{x}_j)p|\bar{x}_j|^{p-1}, j \in \bar{N} \) for any \( v^v \in \tilde{\partial} \phi(x^v) \). Hence, for any sequence \( \{ \lambda^v \} \searrow 0 \), \( \lambda^v v^v_j \to 0 \), \( j \in \bar{N} \), it holds that \( v_j = 0, j \in \bar{N} \) for any \( v \in \partial^\infty \phi(\bar{x}) \), or, equivalently, \( \partial^\infty \phi(\bar{x}) \subset [\tilde{\partial} \phi(\bar{x})]^\infty \).

Overall, we have shown that \( \partial^\infty \phi(\bar{x}) = [\tilde{\partial} \phi(\bar{x})]^\infty \).

It then follows from [22] Corollary 8.11 that \( \phi \) is subdifferentially regular at any \( x \in \mathbb{R}^n \). \( \square \)

The regular and general normal vectors can be calculated as follows.

Theorem 2.2. For any \( \bar{x} \in \Theta \), \( N_{\Theta}(\bar{x}) = \text{pos } \partial \phi(\bar{x}) \cup \partial^\infty \phi(\bar{x}) \), i.e.,
\[
N_{\Theta}(\bar{x}) = \begin{cases} 
\{ v \in \mathbb{R}^n | v_j = \lambda \text{sign}(\bar{x}_j)p|\bar{x}_j|^{p-1}, j \in \bar{N}; \ \lambda \geq 0 \} & \text{if } \bar{x} \in \partial \Theta, \\
\{ 0 \} & \text{if } \bar{x} \in \text{int } \Theta. 
\end{cases} 
\] (2.3)

Furthermore, \( \Theta \) is Clarke regular at any \( \bar{x} \in \Theta \), i.e., \( \tilde{N}_{\Theta}(\bar{x}) = N_{\Theta}(\bar{x}) \).

Proof. We only prove the case that \( \bar{x} \in \partial \Theta \) since the other is trivial. We have \( \bar{x} \neq 0 \) and \( 0 \notin \partial \phi(\bar{x}) \).

Together with Theorem 2.1 and [22] Proposition 10.3, it holds that
\[
\tilde{N}_{\Theta}(\bar{x}) = N_{\Theta}(\bar{x}) = \text{pos } \partial \phi(\bar{x}) \cup \partial^\infty \phi(\bar{x}) 
\]
and \( \Theta \) is Clarke regular at \( \bar{x} \). By the definition of pos and (2.1),
\[
\text{pos } \partial \phi(\bar{x}) = \{ 0 \} \cup \{ \lambda v \in \mathbb{R}^n | v_j = \text{sign}(\bar{x}_j)p|\bar{x}_j|^{p-1}, j \notin \bar{N}; \ \lambda > 0 \}. 
\]
Therefore, it holds that
\[ N_{\Theta}(\bar{x}) = \text{pos} \partial \phi(\bar{x}) \cup \partial^\infty \phi(\bar{x}) = \{ v \in \mathbb{R}^n \mid v_j = \lambda \text{sign}(\bar{x}_j) \| \bar{x} \|_p^{p-1}, j \in \bar{N}; \lambda \geq 0 \}. \]

From [22, Theorem 8.15], we have the following first-order necessary condition for \((P_1)\). For \((P_2)\), we only focus on the local minimizers \(\bar{x}\) on the boundary of the \(\ell_p\) ball, i.e., \(\|\bar{x}\|_p = \theta\); otherwise, the characterization of local minimizers reverts to the case of traditional constrained nonlinear problems.

**Theorem 2.3.** Suppose \(f_0\) is differentiable over \(\Gamma\). The following statements hold true.

(a) If \(\partial^\infty \phi(\bar{x})\) contains no vector \(v \neq 0\) such that \(-v \in N_{\Gamma}(\bar{x})\), then a necessary condition for \(\bar{x}\) to be locally optimal for \((P_1)\) is
\[ 0 \in \nabla f_0(\bar{x}) + \lambda \partial \phi(\bar{x}) + N_{\Gamma}(\bar{x}). \]  

(b) Suppose that \(\bar{x}\) is a local minimizer of \((P_2)\) with \(\bar{x} \in \partial \Theta\). Then
\[ -\nabla f_0(\bar{x}) \in \tilde{N}_{\Theta \cap \Gamma}(\bar{x}) \subset N_{\Theta \cap \Gamma}(\bar{x}). \]

Figure 1: A counterexample for Theorem 2.3. For \((P_1)\), the contour of \(\ell_p\) \((p = 0.5)\) regularization problem with \(F(x) = (x_1 + 1)^2 + \|x\|_p^p\) and \(\Gamma = \{(x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\}\).

Obviously, to find an optimal solution, we can only focus on the top semicircle of the given ball, so that the original problem is equivalent to
\[ \min (x_1 + 1)^2 + \sqrt{x_1} + \sqrt{1 - \sqrt{-x_1^2 + 2x_1}} \text{ s.t. } 0 \leq x_1 \leq 2 \]
The derivative over the domain is always positive, therefore $x^* = (0, 1)$ is a global minimizer. However, there exist $v \in \{v \in \mathbb{R}^2 \mid \nu_1 > 0, \nu_2 = 0\}$, such that $\partial^\infty(\bar{x}) \ni v \neq 0$ and $-v \in N_\Gamma(\bar{x})$. In this case, one can see (2.4) does not hold at $x^*$.

### 2.2 Optimality conditions for $(\mathcal{P}_1)$

To make condition (2.4) for $(\mathcal{P}_1)$ informative, we need to clarify when $-v \in N_\Gamma(\bar{x})$ happens and how to calculate the elements in $N_\Gamma(\bar{x})$. For this purpose, we define the following extended Mangasarian-Fromovitz constraint qualification (EMFCQ). The EMFCQ holds at $x$ if the subvectors $\nabla_N f_j(\bar{x})$, $j \in \mathcal{E} \cup \bar{A}$ are linearly independent and there exists $d \in \mathbb{R}^{|N|}$ such that

$$\langle \nabla_N f_j(\bar{x}), d \rangle = 0, \quad j \in \mathcal{E} \quad \text{and} \quad \langle \nabla_N f_j(\bar{x}), d \rangle < 0, \quad j \in \bar{A}. \quad (2.6)$$

Obviously, the EMFCQ is a weaker condition than the ELICQ proposed in [19]. Moreover, if the EMFCQ holds at $\bar{x} \in \Gamma$ for $\Gamma$, then the MFCQ holds naturally true at $\bar{x}$ for $\Gamma$; letting

$$\Lambda_\Gamma(\bar{x}) = \{ \sum_{j \in \mathcal{E} \cup \bar{A}} y_j \nabla f_j(\bar{x}) \mid y_j \geq 0, j \in \bar{A} \}, \quad (2.7)$$

we have from [22] Theorem 6.14 that $\Gamma$ is regular at $\bar{x}$ and $N_\Gamma(\bar{x}) = \Lambda_\Gamma(\bar{x})$.

**Theorem 2.4.** (a) Suppose the EMFCQ is satisfied at $\bar{x} \in \Gamma$ for $\Gamma$. Then $\partial^\infty \phi(\bar{x})$ contains no vector $v \neq 0$ such that $-v \in N_\Gamma(\bar{x})$. Furthermore, a necessary condition for $\bar{x}$ to be locally optimal for $(\mathcal{P}_1)$ is that there exist $y_j$, $j \in \mathcal{E}$ and $y_j \geq 0$, $j \in A$ such that

$$\nabla_i f_0(\bar{x}) + \lambda \text{sign}(\bar{x}_i) |\bar{x}_i|^{p-1} + \sum_{j \in \mathcal{E} \cup \bar{A}} y_j \nabla f_j(\bar{x}) = 0, \quad i \in \bar{N}. \quad (2.8)$$

(b) Suppose $\Gamma$ is closed and convex in $(\mathcal{P}_1)$. If $\bar{x}$ is a local minimizer of $(\mathcal{P}_1)$ and $\partial^\infty \phi(\bar{x})$ contains no vector $v \neq 0$ such that $-v \in N_\Gamma(\bar{x})$, then it holds that

$$\nabla_i f_0(\bar{x}) + \lambda \text{sign}(\bar{x}_i) |\bar{x}_i|^{p-1} + v_i = 0, \quad i \in \bar{N}; \quad v \in N_\Gamma(\bar{x}). \quad (2.9)$$

**Proof.** (a) Assume by contradiction that there exists nonzero $v \in \partial^\infty \phi(\bar{x})$ such that $-v \in N_\Gamma(\bar{x})$; then it follows from Theorem 2.1 and (2.7) that

$$\sum_{j \in \mathcal{E} \cup \bar{A}} y_j \nabla_N f_j(\bar{x}) = -v_N = 0; \quad \sum_{j \in \mathcal{E} \cup \bar{A}} y_j \nabla \bar{x}_j f_j(\bar{x}) = -v_{\bar{x}} \neq 0; \quad y_j \geq 0, j \in \bar{A}. \quad (2.10)$$

Since EMFCQ holds true at $\bar{x} \in \Gamma$, the dual form [23] of condition (2.6) tells that $y_j = 0, j \in \mathcal{E} \cup \bar{A}$ is the unique solution of the system

$$\sum_{j \in \mathcal{E} \cup \bar{A}} y_j \nabla f_j(\bar{x}) = 0, \quad y_j \geq 0, j \in \bar{A}. \quad \text{It follows that } \sum_{j \in \mathcal{E} \cup \bar{A}} y_j \nabla \bar{x}_j f_j(\bar{x}) = 0, \text{contradicting (2.10). Therefore, for any nonzero } v \in \partial^\infty \phi(\bar{x}), -v \notin N_\Gamma(\bar{x}). \text{From Theorem 2.3, at a local optimal solution } \bar{x} \text{ of } (\mathcal{P}_1), (2.8) \text{ is satisfied.}$$

(b) This is trivially true from Theorem 2.3.
We call the conditions (2.8) the Karush-Kuhn-Tucker (KKT) conditions for \((\mathcal{P}_1)\). Using the notation of \(\Lambda_{\Gamma}\), it can also be equivalently written as
\[
-\nabla f_0(\bar{x}) \in \lambda \partial \phi(\bar{x}) + \Lambda_{\Gamma}(\bar{x}).
\] (2.11)

### 2.3 Optimality conditions for \((\mathcal{P}_2)\)

We also consider other verifiable forms of condition (2.5) if some constraint qualification is satisfied at \(\bar{x}\). For \(\bar{x} \in \Theta \cap \Gamma\), define the extended linearized cone \(\mathcal{Y}_{\Theta \cap \Gamma}(\bar{x})\) as:
\[
\mathcal{Y}_{\Theta \cap \Gamma}(\bar{x}) := \{d \in \mathbb{R}^n \mid \langle v, d \rangle \leq 0, \forall v \in \partial \phi(\bar{x}); \langle \nabla f_j(\bar{x}), d \rangle = 0, j \in \mathcal{E}; \langle \nabla f_j(\bar{x}), d \rangle \leq 0, j \in \bar{\mathcal{A}}\}. 
\]

Obviously,
\[
\mathcal{N}_\Theta(\bar{x}) + \Lambda_{\Gamma}(\bar{x}) = \mathcal{Y}_{\Theta \cap \Gamma}(\bar{x})^* \\
= \{v \in \mathbb{R}^n \mid v_i = y_{0p} \text{sign}(\bar{x}_i)|\bar{x}_i|^{p-1} + \sum_{j \in \mathcal{A} \cup \mathcal{E}} y_j \nabla_i f_j(\bar{x}), i \in \bar{\mathcal{N}}; y_j \geq 0, j \in \{0\} \cup \bar{\mathcal{A}}\}.
\]

It follows from [22, Theorem 6.14] that \(\mathcal{Y}_{\Theta \cap \Gamma}(\bar{x})^* \subset \hat{\mathcal{N}}_{\Theta \cap \Gamma}(\bar{x})\). Hence, we have the following result.

**Proposition 2.5.** For \(\bar{x} \in \Gamma \cap \Theta\) with \(\bar{x} \in \partial \Theta\), \(\mathcal{Y}_{\Theta \cap \Gamma}(\bar{x}) \subset \hat{\mathcal{N}}_{\Theta \cap \Gamma}(\bar{x})\). Therefore, if \(-\nabla f_0(\bar{x}) \in \mathcal{Y}_{\Theta \cap \Gamma}(\bar{x})^*\), meaning that there exist \(y_j \geq 0, j \in \{0\} \cup \bar{\mathcal{A}}\) such that
\[
\nabla f_0(\bar{x}) + y_{0p} \text{sign}(\bar{x}_i)|\bar{x}_i|^{p-1} + \sum_{j \in \mathcal{A} \cup \mathcal{E}} y_j \nabla_i f_j(\bar{x}) = 0, i \in \bar{\mathcal{N}},
\]
then the first-order necessary condition (2.5) is satisfied at \(\bar{x}\).

The EMFCQ for \((\mathcal{P}_2)\) holds at \(\bar{x} \in \partial \Theta \cap \Gamma\) if the subvectors \(\text{sign}(x_{\mathcal{N}})|x_{\mathcal{N}}|^{p-1}, \nabla_{\mathcal{N}} f_j(\bar{x}), j \in \mathcal{E} \cup \bar{\mathcal{A}}\) are linearly independent and there exists \(d \in \mathbb{R}^{|\mathcal{N}|}\) such that
\[
\langle \text{sign}(x_{\mathcal{N}})|x_{\mathcal{N}}|^{p-1}, d \rangle < 0, \langle \nabla_{\mathcal{N}} f_j(\bar{x}), d \rangle = 0, j \in \mathcal{E} \quad \text{and} \quad \langle \nabla_{\mathcal{N}} f_j(\bar{x}), d \rangle < 0, j \in \bar{\mathcal{A}}. \quad (2.12)
\]

Equivalently, the dual form of EMFCQ for \((\mathcal{P}_2)\) holds at \(\bar{x} \in \Theta \cap \Gamma\) if \(y_j = 0, j \in \{0\} \cup \bar{\mathcal{A}}\) is the unique solution of
\[
\{y_{0p} \text{sign}(\bar{x}_i)|\bar{x}_i|^{p-1} + \sum_{j \in \mathcal{A} \cup \mathcal{E}} y_j \nabla_i f_j(\bar{x}) = 0, i \in \bar{\mathcal{A}}; y_j \geq 0, j \in \{0\} \cup \bar{\mathcal{A}}\}.
\]

We now state the necessary optimality conditions for \((\mathcal{P}_2)\).

**Theorem 2.6.** Suppose \(\bar{x} \in \Gamma \cap \Theta\) with \(\bar{x} \in \partial \Theta\) is local optimal for \((\mathcal{P}_2)\).

(a) If the EMFCQ holds at \(\bar{x}\), then there exist \(y_j, j \in \mathcal{E}\) and \(y_j \geq 0, j \in \{0\} \cup \bar{\mathcal{A}}\), such that
\[
\nabla_i f_0(\bar{x}) + y_{0p} \text{sign}(\bar{x}_i)|\bar{x}_i|^{p-1} + \sum_{j \in \mathcal{A} \cup \mathcal{E}} y_j \nabla_i f_j(\bar{x}) = 0, i \in \bar{\mathcal{N}}. \quad (2.13)
\]
(b) Suppose \( \Gamma \) is closed and convex. If \( v = 0 \) is the only vector such that \( v \in N_\Theta(x) \) and \(-v \in N_\Gamma(x) \), then \( N_\Theta \cap N_\Gamma = \emptyset \). Therefore, if \( x \) is local optimal for \((\mathcal{P}_2)\), then

\[-\nabla_i f(x) + \rho \text{sign}(x_i) |x_i|^{p-1} + v_i = 0, i \in \mathcal{N}; \ y_0 \geq 0; \ v \in N_\Gamma(x).\]

(c) Suppose \( \Gamma = \mathbb{R}^n \). If \( x \) is local optimal for \((\mathcal{P}_2)\), then there exists \( y_0 \geq 0 \) such that

\[-\nabla_i f(x) + y_0 \text{sign}(x_i) |x_i|^{p-1} = 0, i \in \mathcal{N}.

**Proof.** (a) From [23, Theorem 6.14], if the EMFCQ holds for \((\mathcal{P}_2)\) at \( x \in \Gamma \cap \Theta \) with \( x \in \partial \Theta \), then \( \Theta \cap \Gamma \) is regular at \( x \) and

\[\hat{N}_{\Theta \cap \Gamma}(x) = \gamma_{\Theta \cap \Gamma}(x).\]

By Theorem 2.3, (a) is true.

(b) By [22, Theorem 6.42], \( N_{\Theta \cap \Gamma}(x) = N_\Theta(x) + N_\Gamma(x) \). Therefore, if \( x \) is local optimal, then \( x \in \hat{N}_{\Theta \cap \Gamma}(x) \subset N_{\Theta \cap \Gamma}(x) = N_\Theta(x) + N_\Gamma(x) \).

(c) Trivial by (b). \( \square \)

We call the conditions \((2.13)\) the Karush-Kuhn-Tucker (KKT) conditions for \((\mathcal{P}_2)\). Using the notation of \( \Lambda_{\Gamma} \), it can also be equivalently written as

\[-\nabla f_0(x) \in N_\Theta(x) + \Lambda_{\Gamma}(x).\]

**3 First-order sequential optimality condition**

In this section, we study the sequential optimality conditions under the approximate Karush-Kuhn-Tucker (AKKT) conditions, which are defined as follows.

**Definition 3.1.** (i) For \((\mathcal{P}_1)\), we say that \( x \in \Gamma \) satisfies the AKKT if there exist \( \{x^\nu\} \subset \mathbb{R}^n \), \( \{y_0^\nu\} \subset \mathbb{R}, j \in \mathcal{E} \cup \mathcal{A} \) such that \( \lim_{\nu \to \infty} x^\nu = x, y_0^\nu \geq 0, j \in \mathcal{A} \) and

\[\lim_{\nu \to \infty} \nabla_i f(x^\nu) + \rho \text{sign}(x_i^\nu) |x_i^\nu|^{p-1} + \sum_{j \in \mathcal{E} \cup \mathcal{A}} y_j^\nu \nabla_j f_j(x^\nu) = 0, \forall i \in \mathcal{N}.

(ii) For \((\mathcal{P}_2)\), we say that \( x \in \Theta \cap \Gamma \) with \( x \in \partial \Theta \) satisfies the AKKT if there exist \( \{x^\nu\} \subset \mathbb{R}^n \), \( \{y_0^\nu\} \subset \mathbb{R}, j \in \{0\} \cup \mathcal{E} \cup \mathcal{A} \) such that \( \lim_{\nu \to \infty} x^\nu = x, y_0^\nu \geq 0, j \in \{0\} \cup \mathcal{A} \) and

\[\lim_{\nu \to \infty} \nabla_i f(x^\nu) + \rho \text{sign}(x_i^\nu) |x_i^\nu|^{p-1} + \sum_{j \in \mathcal{E} \cup \mathcal{A}} y_j^\nu \nabla_j f_j(x^\nu) = 0, \forall i \in \mathcal{N}.

Next we provide properties of \((\mathcal{P}_1)\) and \((\mathcal{P}_2)\) under which the AKKT implies the KKT. This property is named the extended cone-continuity property (ECCP), which is defined as follows.
Definition 3.2. (a) We say that $\bar{x} \in \Gamma$ satisfies the ECCP for $\mathcal{P}_1$ if the set-valued mapping \( \lambda \partial \phi(x) + \Lambda_\Gamma(x) \) is outer semicontinuous at $\bar{x}$, that is, 
\[
\limsup_{x' \to \bar{x}} [\lambda \partial \phi(x') + \Lambda_\Gamma(x')] \subset \lambda \partial \phi(\bar{x}) + \Lambda_\Gamma(\bar{x}).
\]

If $\Gamma$ is a closed and convex set, $\Lambda_\Gamma$ can be replaced by $N_\Gamma$.

(b) Similarly, we say $\bar{x} \in \Theta \cap \Gamma$ satisfies the ECCP for $\mathcal{P}_2$ if the set-valued mapping $\Lambda_{\Theta \cap \Gamma}(x)$ is outer semicontinuous at $\bar{x}$, that is, 
\[
\limsup_{x' \to \bar{x}} [N_\Theta(x') + \Lambda_\Gamma(x')] \subset N_\Theta(\bar{x}) + \Lambda_\Gamma(\bar{x}).
\]

If $\Gamma$ is a closed and convex set, $\Lambda_\Gamma$ can be replaced by $N_\Gamma$.

In the following theorem we show that if ECCP holds, the AKKT implies the KKT.

Theorem 3.3. (a) For $\mathcal{P}_1$, suppose the AKKT holds at $\bar{x} \in \Gamma$. If the ECCP holds at $\bar{x}$, then the KKT condition (2.11) are satisfied at $\bar{x}$.

(b) For $\mathcal{P}_2$, suppose the AKKT holds at $\bar{x} \in \partial \Theta \cap \Gamma$. If the ECCP holds at $\bar{x}$, then the KKT condition (2.15) are satisfied at $\bar{x}$.

Proof. (a) First of all, $\bar{N} \subset N(x')$ for $x'$ sufficiently close to $\bar{x}$. Since the AKKT condition holds at $\bar{x}$, there exist $\{x'\} \to \bar{x}$ and $\{w'\} \subset \mathbb{R}^n$ such that $\nabla f(x') + w' \to 0$, where $w' \in \lambda \partial \phi(x') + \Lambda_\Gamma(x')$. Taking limits and using the continuity of the gradient of $f$ near $\bar{x}$, we obtain
\[
-\nabla f(\bar{x}) = \lim_{\nu \to \infty} w' \in \limsup_{\nu \to \infty} [\lambda \partial \phi(x') + \Lambda_\Gamma(x')] \subset \limsup_{x' \to \bar{x}} [\lambda \partial \phi(x) + \Lambda_\Gamma(x)] \subset \lambda \partial \phi(\bar{x}) + \Lambda_\Gamma(\bar{x}),
\]
where the last inclusion follows from the ECCP. Therefore, $-\nabla f(\bar{x}) \in \lambda \partial \phi(\bar{x}) + \Lambda_\Gamma(\bar{x})$.

(b) Similarly, since the AKKT condition holds at $\bar{x}$, there exist $\{x'\} \to \bar{x}$ and $\{w'\} \subset \mathbb{R}^n$ such that $\nabla f(x') + w' \to 0$, where $w' \in N_\Theta(x') + \Lambda_\Gamma(x')$. Taking limits and using the continuity of the gradient of $f$ near $\bar{x}$, we obtain
\[
-\nabla f(\bar{x}) = \lim_{\nu \to \infty} w' \in \limsup_{\nu \to \infty} N_\Theta(x') + \Lambda_\Gamma(x') \subset \limsup_{x' \to \bar{x}} N_\Theta(x) + \Lambda_\Gamma(x) \subset N_\Theta(\bar{x}) + \Lambda_\Gamma(\bar{x}),
\]
where the last inclusion follows from the ECCP. Therefore, $-\nabla f(\bar{x}) \in N_\Theta(\bar{x}) + \Lambda_\Gamma(\bar{x})$. \qed

We discuss the cases when the ECCP holds true. For $\mathcal{P}_1$, we have the following results.

Proposition 3.4. For $\mathcal{P}_1$, the ECCP holds true for any of the following cases.

(a) $\Lambda_\Gamma$ is outer semicontinuous at $\bar{x}$ and $\partial^\infty \phi(\bar{x})$ contains no vector $v \neq 0$ such that $-v \in \Lambda_\Gamma(\bar{x})$.

(b) The EMFCQ is satisfied at $\bar{x}$.
(c) $\Gamma$ is a closed and convex set and $\partial^\infty \phi(\bar{x})$ contains no vector $v \neq 0$ such that $-v \in N_\Gamma(\bar{x})$.

Proof. (a) Let $\bar{w}$ be an element of $\limsup_{x \to \bar{x}} [\lambda \partial \phi(x) + \Lambda_\Gamma(x)]$, so there are sequences $\{x^\nu\}, \{w^\nu\}, \{u^\nu\}$ such that $x^\nu \to \bar{x}$, $w^\nu \to \bar{w}$ and

$$w^\nu = u^\nu + \sum_{j \in \bar{\mathcal{E}} \cup \bar{\mathcal{A}}} y_j \nabla_i f_j(x^\nu) \quad (3.1)$$

with $u^\nu \in \partial \lambda \phi(x^\nu)$ and $y_j \in \mathbb{R}_+, j \in \bar{\mathcal{A}}$.

If $\{y_j^\nu\}, j \in \mathcal{E} \cup \bar{\mathcal{A}}$ are bounded, then they all have limits $\bar{y}_j, j \in \mathcal{E} \cup \bar{\mathcal{A}}$; moreover, $\{u^\nu\}$ is also bounded and $\bar{u} := \lim_{\nu \to \infty} u^\nu \in \lambda \partial \phi(\bar{x})$ (possibly taking limits on a convergent subsequence) due to the outer semicontinuity of $\partial \phi$. By possibly extracting an convergent subsequence, we have

$$v^\nu + \sum_{j \in \mathcal{E} \cup \bar{\mathcal{A}}} y_j^\nu \nabla_i f_j(x^\nu) \to \bar{u} + \sum_{j \in \mathcal{E} \cup \bar{\mathcal{A}}} \bar{y}_j \nabla_i f_j(\bar{x}) \in \lambda \partial \phi(\bar{x}) + \Lambda_\Gamma(\bar{x}),$$

meaning $\lambda \partial \phi + \Lambda_\Gamma$ is outer semicontinuous at $\bar{x}$.

If $\{y_j^\nu\}, j \in \mathcal{E} \cup \bar{\mathcal{A}}$ are unbounded, letting $M^\nu = \max\{|y_j^\nu|, j \in \mathcal{E} \cup \bar{\mathcal{A}}\}$. Dividing [3.1] by $M^\nu$, we arrive at

$$\frac{w^\nu}{M^\nu} = \frac{u^\nu}{M^\nu} + \sum_{j \in \mathcal{E} \cup \bar{\mathcal{A}}} \frac{y_j^\nu}{M^\nu} \nabla_i f_j(x^\nu) \quad (3.2)$$

Since $\max\{y_j^\nu/M^\nu, j \in \{0\} \cup \mathcal{E} \cup \bar{\mathcal{A}}\} = 1$ for all $\nu$, they have nonzero limit point $\tilde{y}_j, j \in \mathcal{E} \cup \bar{\mathcal{A}}$. Moreover, $\bar{u} := \lim_{\nu \to \infty} u^\nu \in \lambda \partial \phi(\bar{x})$, we can extract a convergent subsequence. Thus, taking limits above, we get

$$\lambda \partial \phi(\bar{x}) \ni \bar{u} = - \sum_{j \in \mathcal{E} \cup \bar{\mathcal{A}}} \tilde{y}_j \nabla_i f_j(\bar{x}) \in \Lambda_\Gamma(\bar{x}),$$

a contradiction.

(b) If the EMFCQ holds at $\bar{x}$, then $\partial^\infty \phi(\bar{x})$ contains no vector $v \neq 0$ such that $-v \in \Lambda_\Gamma(\bar{x})$ and $\Lambda_\Gamma = N_\Gamma$ by Theorem 2.4(a). This reverts to (a).

(c) Let $\bar{w}$ be an element of $\limsup_{x \to \bar{x}} [\lambda \partial \phi(x) + N_\Gamma(x)]$, so there are sequences $\{x^\nu\}, \{w^\nu\}, \{u^\nu\}, \{v^\nu\}$ such that $x^\nu \to \bar{x}$, $w^\nu \to \bar{w}$ and

$$w^\nu = u^\nu + v^\nu \quad (3.2)$$

with $u^\nu \in \partial \lambda \phi(x^\nu)$ and $v^\nu \in N_\Gamma(x^\nu)$.

If $\{v^\nu\}$ are bounded, then $\{u^\nu\}$ and $\{v^\nu\}$ all have limits $\bar{u}$ and $\bar{v}$. Moreover, $\bar{u} := \lim_{\nu \to \infty} u^\nu \in \lambda \partial \phi(\bar{x})$ and $\bar{v} \in N_\Gamma(\bar{x})$ (possibly taking limits on a convergent subsequence) due to the outer semicontinuity of $\partial \phi$ and $N_\Gamma$. By possibly extracting an convergent subsequence, we have

$$u^\nu + v^\nu \to \bar{u} + \bar{v} \in \lambda \partial \phi(\bar{x}) + \Lambda_\Gamma(\bar{x}),$$

meaning $\lambda \partial \phi + \Lambda_\Gamma$ is outer semicontinuous at $\bar{x}$. 

12
If \( \{v^\nu\} \) are unbounded, letting \( M^\nu = \|v^\nu\| \). Dividing (3.1) by \( M^\nu \), we arrive at

\[
\frac{w^\nu}{M^\nu} = \frac{u^\nu}{M^\nu} + \frac{v^\nu}{M^\nu}.
\]

Since \( \frac{v^\nu}{M^\nu} = 1 \) for all \( \nu \), it has nonzero limit point \( \tilde{v} \in N_\Gamma(\bar{x}) \) due to the outer semicontinuity of \( N_\Gamma \). Moreover, \( \bar{u} := \lim_{\nu \to \infty} \frac{w^\nu}{M^\nu} \in \lambda \partial^\infty \phi(\bar{x}) \), we can extract a convergent subsequence. Thus, taking limits above, we get

\[
\lambda \partial^\infty \phi(\bar{x}) \ni \bar{u} = -\tilde{v} \in N_\Gamma(\bar{x}),
\]

a contradiction. \( \square \)

As for \( (\mathcal{P}_2) \), we have the following results.

**Proposition 3.5.** For \( (\mathcal{P}_2) \), the ECCP holds true for any of the following cases.

(a) \( \Lambda_\Gamma \) is outer semicontinuous at \( \bar{x} \) and the only solution for \( v_1 + v_2 = 0 \) with \( v_1 \in N_\Theta(\bar{x}) \) and \( v_2 \in \Lambda_\Gamma(\bar{x}) \) is \( v_1 = v_2 = 0 \).

(b) The EMFCQ is satisfied at \( \bar{x} \).

(c) \( \Gamma \) is a closed and convex set and and the only solution for \( v_1 + v_2 = 0 \) with \( v_1 \in N_\Theta(\bar{x}) \) and \( v_2 \in N_\Gamma(\bar{x}) \) is \( v_1 = v_2 = 0 \).

**Proof.** (a) The proof is similar to the argument for Proposition 3.4(a) by replacing the role of \( u^\nu \in \lambda \partial \phi(\bar{x}) \) with \( y_0^\nu \text{sign}(x^\nu_N) |x^\nu_N|^{p-1} \in N_\Theta(\bar{x}) \) (\( x^\nu \) is selected sufficiently close to \( \bar{x} \) so that \( \bar{N} \subset N(x^\nu) \)) and considering the boundedness of \( \{y_j^\nu\}, j \in \{0\} \cup \mathcal{E} \cup \bar{\mathcal{A}} \) instead of \( \{y_j^\nu\}, j \in \mathcal{E} \cup \bar{\mathcal{A}} \). Therefore, we skip the details of the proof.

(b) The EMFCQ for \( (\mathcal{P}_2) \) is equivalent to saying that the only solution for \( v_1 + v_2 = 0 \) with \( v_1 \in N_\Theta(\bar{x}) \) and \( v_2 \in \Lambda_\Gamma(\bar{x}) \) is \( v_1 = v_2 = 0 \). Moreover, notice that the EMFCQ for \( (\mathcal{P}_2) \) implies the EMFCQ for \( (\mathcal{P}_1) \). Therefore, we have from [22, Theorem 6.14] that \( \Gamma \) is regular at \( \bar{x} \) and \( N_\Gamma(\bar{x}) = \Lambda_\Gamma(\bar{x}) \). This case then reverts to (a).

(c) The proof is similar to the argument for (3.4)(c) by replacing the role of \( \lambda \partial \phi \) with \( N_\Theta \); therefore it is skipped. \( \square \)

### 4 Convergence analysis using AKKT

The research on algorithms for solving general constrained problems involving \( \ell_p \) norms is so far limited. To the best of our knowledge, only simple cases such as linearly constrained or convex set constrained cases have been studied. When solving for general constrained problems with \( \ell_p \) norm, many works focus on the reformulations where the constraint violation is penalized in the objective [17,24]. We now extend the existing algorithms for solving unconstrained cases to general
constrained cases and prove the global convergence by showing that AKKT conditions discussed in the previous section are satisfied at the limit point.

We extend the framework of iteratively reweighted \( \ell_1 \) methods for solving unconstrained \( \ell_p \) regularization problems [11] to constrained cases [12],

\[
\min F(x) := f_0(x) + \lambda \|x\|_p^p \quad \text{s.t.} \quad f_j(x) \leq 0, \ \forall j \in I; \quad f_j(x) = 0, \ \forall j \in E. \tag{4.1}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is continuously Lipschitz differentiable with constant \( L_f \geq 0 \). Here \( \Gamma \) is assumed to be closed and convex.

We first formulated a smooth approximation \( F(x; \epsilon) \) of \( F(x) \)

\[
F(x; \epsilon) := f_0(x) + \lambda \sum_{i=1}^n (|x_i| + \epsilon_i)^p,
\]

where \( \epsilon \in \mathbb{R}^n_+ \). At \( k \)th iteration, a convex local model to approximate \( F(x; \epsilon) \) is constructed

\[
G(x; x^k, \epsilon^k) := Q_k(x) + \lambda \sum_{i=1}^n w(x_i^k, \epsilon_i^k)|x_i|
\]

where the weights are given by \( w(x_i^k, \epsilon_i^k) = p(|x_i^k| + \epsilon_i^k)^{p-1} \) and \( Q_k(x) \) represents a local approximation mode to \( f \) at \( x^k \) and is generally assumed to be smooth and convex.

**Algorithm 1** General framework of iteratively reweighted \( \ell_1 \) approach

**Input:** \( \alpha \in (0, 1), \epsilon^0 \in \mathbb{R}^n_+ \) and \( x^0 \).

1. Initialization: Set \( k = 0 \).
2. repeat
   3. (Reweighing) Compute \( w^k = p(|x_i^k| + \epsilon_i^k)^{p-1}, i = 1, \cdots, n \).
   4. (Solving the Subproblem) \( x^{k+1} \leftarrow \arg \min_{x \in \Gamma} \{Q_k(x) + \lambda \sum_{i=1}^n w_i^k|x_i|\} \).
   5. (Update \( \epsilon \)) Set \( \epsilon^{k+1} \in (0, \alpha \epsilon^k) \).
   6. Set \( k \leftarrow k + 1 \) and go to step 3.
3. until Convergence

As in [11], we make the following assumptions about the choice of \( (x^0, \epsilon^0) \) and \( Q_k(\cdot) \).

**Assumption 4.1.** The initial point \((x^0, \epsilon^0)\) and local model \( Q_k(\cdot) \) are such that

(i) The level set \( L(F^0) := \{x \mid F(x) \leq F^0\} \) is bounded where \( F^0 := F(x^0; \epsilon^0) \).

(ii) For all \( k \in \mathbb{N}, \nabla Q_k(x^k) = \nabla f(x^k), Q_k(\cdot) \) is strongly convex with constant \( M > L_f/2 > 0 \).

First, we show that \( F(x, \epsilon) \) is monotonically decreasing over the iterates \((x^k, \epsilon^k)\). Define the following two terms

\[
\Delta F(x^{k+1}, \epsilon^{k+1}) := F(x^k, \epsilon^k) - F(x^{k+1}, \epsilon^{k+1})
\]

\[
\Delta G(x^{k+1}; x^k, \epsilon^k) := G(x^k; x^k, \epsilon^k) - G(x^{k+1}; x^k, \epsilon^k),
\]

and use the shorthands \( W^k := \text{diag} \left( w_1^k, \cdots, w_n^k \right) \).
Proposition 4.2. Suppose Assumption 4.1 holds. Let \( \{(x^k, \epsilon^k)\} \) be the sequence generated by Algorithm 1. It follows that \( F(x, \epsilon) \) is monotonically decreasing over \( \{(x^k, \epsilon^k)\} \) and

\[
(M - \frac{L_f}{2}) \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2_2 \leq F(x^0, \epsilon^0) - F(x^k, \epsilon^k). \tag{4.2}
\]

Hence, \( \lim_{k \to \infty} \|x^{k+1} - x^k\|_2 = 0 \).

Proof. From the same argument as the proof for [11, Proposition 1], we have

\[
\Delta F(x^{k+1}, \epsilon^{k+1}) \geq \Delta G(x^{k+1}; x^k, \epsilon^k) + (M - \frac{L_f}{2}) \|x^k - x^{k+1}\|^2_2. \tag{4.3}
\]

Assumption 4.1 implies the subproblem solution \( x^{k+1} \) satisfies the optimality condition

\[
\nabla Q_k(x^{k+1}) + \lambda W_k y^{k+1} + z^{k+1} = 0 \tag{4.4}
\]

with \( z^{k+1} \in N_T(x^{k+1}) \) and \( y^{k+1} \in \partial \|x^{k+1}\|_1 \). Hence,

\[
\Delta G(x^{k+1}; x^k, \epsilon^k) = Q_k(x^k) - Q_k(x^{k+1}) + \lambda \sum_{i=1}^n w_i^k (|x_i^k| - |x_i^{k+1}|)
\]

\[
\geq \langle \nabla Q_k(x^{k+1}), x^k - x^{k+1}\rangle + \frac{M}{2} \|x^{k+1} - x^k\|^2_2 + \lambda \sum_{i=1}^n w_i^k y_i^{k+1} (x_i^k - x_i^{k+1}) + z_i^{k+1} (x_i^k - x_i^{k+1})
\]

\[
= \langle \nabla Q_k(x^{k+1}) + \lambda W_k y^{k+1} + z^{k+1}, x^k - x^{k+1}\rangle + \frac{M}{2} \|x^{k+1} - x^k\|^2_2
\]

\[
= \frac{M}{2} \|x^{k+1} - x^k\|^2_2,
\]

where the inequality is by Assumption 4.1, the convexity of \( |\cdot| \) and the definition of normal cone, and the last equality is by (4.4). Combining (4.3) and (4.5), we have

\[
\Delta F(x^{k+1}, \epsilon^{k+1}) \geq (M - \frac{L_f}{2}) \|x^k - x^{k+1}\|^2_2. \tag{4.6}
\]

Replacing \( k \) with \( t \) and summing up from \( t = 0 \) to \( k - 1 \), we have

\[
\sum_{t=0}^{k-1} (F(x^t, \epsilon^t) - F(x^{t+1}, \epsilon^{t+1})) \geq (M - \frac{L_f}{2}) \sum_{t=0}^{k-1} \|x^t - x^{t+1}\|^2_2,
\]

completing the proof of (4.2).

Now we prove that every cluster point \( \bar{x} \) of \( \{x^k\} \) generated by Algorithm 1 satisfied the AKKT. As a result, it satisfies the first-order necessary optimality condition of (4.1) by Theorem 3.3 if the CCP condition holds at \( \bar{x} \).
Theorem 4.3. Suppose \( \{x^k\} \) is the sequence generated by Algorithm 1 for solving (P). It holds that every cluster point of \( \{x^k\} \) satisfies the AKKT condition.

Proof. Let \( \bar{x} \) be a cluster point of \( \{x^k\} \) with subsequence \( \{x^k\}_S \rightarrow \bar{x} \). By Proposition 4.2, \( \{x^{k+1}\}_S \rightarrow \bar{x} \). From the optimality condition of the subproblem of Algorithm 1, we have

\[
\nabla_i f(x^k) + M(x^k_i - x_i^k) + \lambda p(|x^k_i| + \epsilon_k^i)p^{-1}\text{sign}(x^k_i) + z^{k+1}_i = 0, \quad \forall i \in N^{k+1},
\]

where \( z^{k+1} \in N^k(x^{k+1}) \). For any \( i \in N^{k+1} \),

\[
|\nabla_i f(x^{k+1}) + \lambda p|x^{k+1}_i|p^{-1}\text{sign}(x^{k+1}_i) + z^{k+1}_i| \\
= |(\nabla_i f(x^{k+1}) - \nabla_i f(x^k)) + \lambda p\text{sign}(x^{k+1}_i)(|x^{k+1}_i|p^{-1} - (|x^k_i| + \epsilon_k^i)p^{-1}) - M(x^{k+1}_i - x^k_i)| \\
\leq (L_f - M)|x^{k+1}_i - x^k_i| + \lambda p(1 - p)|x^k_i|p^{-2}|x^{k+1}_i| - |x^k_i| - \epsilon_k^i|
\]

where \( \dot{x}_k^i \) is between \( |x^{k+1}_i|p^{-1} \) and \((|x^k_i| + \epsilon_k^i)p^{-1}\). For sufficiently large \( k \), \( |\dot{x}_k^i|p^{-2} = \delta > 0 \), \( N(\bar{x}) \subset N(x^k) \) and \( N(\bar{x}) \subset N(x^{k+1}) \). Therefore, for any \( i \in N(\bar{x}) \),

\[
|\nabla_i f(x^{k+1}) + \lambda p|x^{k+1}_i|p^{-1}\text{sign}(x^{k+1}_i) + z^{k+1}_i| \\
\leq (L_f - M)|x^{k+1}_i - x^k_i| + \lambda p(1 - p)\delta p^{-2}|x^{k+1}_i - x^k_i| + \lambda p(1 - p)\delta p^{-2}\epsilon_k^i \rightarrow 0
\]
as \( k \rightarrow \infty \). This completes the proof. \( \square \)

5 Numerical experiment

In this section, we design numerical experiments to demonstrate the performance of Algorithm 1 for solving (P) with application in portfolio management, where \( f \) can include various loss functions such as the variance of portfolio or tracking error of index tracking. Here we use the commonly seen Markowitz mean-variance model to predict an optimal portfolio. Specifically, we only consider the shorting-prohibited Markowitz model and assume the optimal Lagrangian multiplier associated with the mean constraint is known as \( \eta \) which can be chosen accordingly to a reasonable expected return rate.

In the experiment, we collected historical daily stock price data to obtain \( R \) and \( m \) in S&P 500 index from Yahoo finance\footnote{finance.yahoo.com} which spans from 01/01/2013 to 31/12/2013. We do not include any company unless it is traded on the market at least 90% of the trading days during the data period, nor do any company not listed on the market for the entire timescale. The total list has 471 companies by 251 trading days. We choose \( \eta = 0.001 \) and \( \lambda = 0.001 \), and recast the Markowitz model with no-shorting constraint as a linear equality constrained optimization problem with \( \ell_\frac{1}{2} \) regularizer,

\[
\begin{align*}
\min & \quad \frac{1}{2}x^TRx - \eta\mu^Tx + \lambda\|x\|^{\frac{1}{2}} \frac{1}{2} \\
\text{s.t.} & \quad e^Tx = 1, \\
& \quad x \geq 0
\end{align*}
\]

(5.1)
where $R \in \mathbb{R}^{n \times n}$ is the estimated covariance matrix of the portfolio, and $\eta \mu$ is the estimated return vector with $\eta > 0$ and $\mu \in \mathbb{R}^n$.

In Algorithm 1, we use $Q_k(x) = (Rx^k - \eta \mu)^T x + \frac{\beta}{2} \|x - x^k\|_2^2$. The parameters are selected as $\alpha = 0.998$, $\beta = 1.1L_f$, $\epsilon^0 = 0.001$ and the initial point is $x^0 = \frac{1}{n} e$ so that it is feasible. As mentioned in the last section, AKKT conditions are satisfied at the clustering point and the ECCP holds true by Proposition 3.4(c) since $\Gamma$ here is a closed and convex set. Therefore, from Theorem 3.3(a), the KKT condition of (5.1) is satisfied at the clustering point generated by the algorithm.

After solving each subproblem, we obtain a primal feasible iterate $x^{k+1}$ and dual iterate $\nu^k$ satisfying the optimality condition for the subproblem

$$(R_i x^k - \eta \mu)^T x^{k+1} + \beta (x_i^{k+1} - x_i^k) + \lambda w_i^k + \nu^k = 0, \quad i \in \mathcal{N}(x^{k+1}),$$

(5.2)

where $R_i$ is the $i$th row of $R$. For any primal-dual pair $(x, \nu)$ with $x \geq 0$, we can define the following metric to measure the optimality residual at $(x, \nu)$

$$\alpha(x, \nu) := \sum_{i \in \mathcal{N}(x)} |x^T R_i - c_i + \lambda per_i^{p-1} + \nu|,$$

(5.3)

where $\nu$ is the dual variable associated with the simplex constraint.

![Figure 2: The residual $\alpha(x^k, \nu^k)$ generated by Algorithm 1](image)

We plot the evolution of $\alpha(x^k, \nu^k)$ over iterations in Figure 2, which steadily decreases to 0. Figure 3 shows the number of nonzero components of the portfolio versus the regularization parameter $\lambda$ by fixing $\eta = 0.001$. We can see that larger $\lambda$ can yield sparser solutions.

For out-of-sample testing, we collected historical daily stock price data in S&P 500 index from Yahoo finance, which spans from 01/01/2014 to 31/03/2014. Figure 4 shows the Sharpe ratios of our $\ell_p$-norm regularized portfolio. The sparse portfolios are more implementable due to the transaction costs or physical limitations reasons. Our results indicate that an intermediate sparsity (around 10) portfolio may have the best Sharpe ratio in both in-sample or out-of-sample performance.
Figure 3: Portfolio sparsity (nonzero components) for different $\lambda$.

Figure 4: Portfolio Sharpe ratios for different sparsity: in-sample and out-of-sample.
References

[1] R. Chartrand, “Exact reconstruction of sparse signals via nonconvex minimization,” IEEE Signal Processing Letters, vol. 14, no. 10, pp. 707–710, 2007.

[2] P. Xu, Y. Tian, H. Chen, and D. Yao, “$\ell_p$ norm iterative sparse solution for EEG source localization,” IEEE Transactions on Biomedical Engineering, vol. 54, pp. 400–409, 2007.

[3] Z. Lin, Z. Huijuan, M. Wenjuan, J. Jingying, Z. Limin, L. Jiao, G. Feng, and Z. Zhongxing, “Resolution and noise performance of sparse view X-ray CT reconstruction via $\ell_p$-norm regularization,” Physica Medica, vol. 52, pp. 72–80, 2018.

[4] A. A. Ababneh, “Density-based sensor selection for RSS target localization,” IEEE Sensors Journal, vol. PP, pp. 1–1, 2018.

[5] C. Chen, X. Li, C. Tolman, S. Wang, and Y. Ye, “Sparse portfolio selection via quasi-norm regularization,” arXiv preprint arXiv:1312.6350, 2013.

[6] S. Oymak, B. Recht, and M. Soltanolkotabi, “Sharp time–data tradeoffs for linear inverse problems,” IEEE Transactions on Information Theory, vol. 64, no. 6, pp. 4129–4158, 2017.

[7] Z. B. Xu, Z. Hai, W. Yao, X. Y. Chang, and L. Yong, “$\ell_{\frac{1}{2}}$ regularization,” Science China Information Sciences, vol. 53, no. 6, pp. 1159–1169, 2010.

[8] D. Ge, X. Jiang, and Y. Ye, “A note on the complexity of $\ell_p$ minimization,” Mathematical programming, vol. 129, no. 2, pp. 285–299, 2011.

[9] Q. Ling, Z. Wen, and W. Yin, “Decentralized jointly sparse optimization by reweighted minimization,” Signal Processing, IEEE Transactions on, 2013.

[10] H. Wang, D.-H. Li, X.-J. Zhang, and L. Wu, “Optimality conditions for the constrained $\ell_p$-regularization,” Optimization, vol. 64, no. 10, pp. 2183–2197, 2015.

[11] H. Wang, H. Zeng, J. Wang, and Q. Wu, “Relating $\ell_p$ regularization and reweighted $\ell_1$ regularization,” Optimization Letters, pp. 1–22, 2021.

[12] G. Haeser, H. Liu, and Y. Ye, “Optimality condition and complexity analysis for linearly-constrained optimization without differentiability on the boundary,” Mathematical Programming, vol. 178, no. 1, pp. 263–299, 2019.

[13] Y. Shi, J. Zhang, and K. B. Letaief, “Group sparse beamforming for green Cloud-RAN,” IEEE Transactions on Wireless Communications, vol. 13, no. 5, pp. 2809–2823, 2014.

[14] H. Wang, F. Zhang, Q. Wu, Y. Hu, and Y. Shi, “Nonconvex and nonsmooth sparse optimization via adaptively iterative reweighted methods,” ArXiv, vol. abs/1810.10167, 2018.

[15] M. Thom, M. Rapp, and G. Palm, “Efficient dictionary learning with sparseness-enforcing projections,” International Journal of Computer Vision, vol. 114, no. 2, 2015.

[16] X. Chen, F. Xu, and Y. Ye, “Lower bound theory of nonzero entries in solutions of $\ell_2 - \ell_p$ minimization,” SIAM Journal on Scientific Computing, vol. 32, no. 5, pp. 2832–2852, 2010.
[17] Z. Lu, “Iterative reweighted minimization methods for $\ell_p$ regularized unconstrained nonlinear programming,” *Mathematical Programming*, vol. 147, no. 1-2, pp. 277–307, 2014.

[18] X. Yang, J. Wang, and H. Wang, “Towards an efficient approach for the nonconvex $\ell_p$-ball projection: algorithm and analysis,” *arXiv preprint arXiv:2101.01350*, 2021.

[19] H. Wang, D.-H. Li, X.-J. Zhang, and L. Wu, “Optimality conditions for the constrained $l_p$-regularization,” *Optimization*, vol. 64, no. 10, pp. 2183–2197, 2015.

[20] W. Bian and X. Chen, “Optimality and complexity for constrained optimization problems with nonconvex regularization,” *Mathematics of Operations Research*, vol. 42, no. 4, pp. 1063–1084, 2017.

[21] R. Andreani, J. M. Martínez, A. Ramos, and P. J. Silva, “A cone-continuity constraint qualification and algorithmic consequences,” *SIAM Journal on Optimization*, vol. 26, no. 1, pp. 96–110, 2016.

[22] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*. Springer Science & Business Media, 2009, vol. 317.

[23] M. V. Solodov, “Constraint qualifications,” *Wiley Encyclopedia of Operations Research and Management Science*, 2010.

[24] R. Chartrand and W. Yin, “Iteratively reweighted algorithms for compressive sensing,” in *2008 IEEE International Conference on Acoustics, Speech and Signal Processing*. IEEE, 2008, pp. 3869–3872.