CONFORMAL UPPER BOUNDS FOR THE VOLUME SPECTRUM

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ABSTRACT. In this paper, we prove upper bounds for the volume spectrum of a Riemannian manifold that depend only on the volume, dimension and a conformal invariant.

1. Introduction

Let \((M^n, g)\) be a closed Riemannian manifold of dimension \(n \geq 2\). In [1], Almgren proved that the space of mod 2 relative cycles \(Z_{n-1}(M; \mathbb{Z}_2)\) is weakly homotopic to \(\mathbb{R}P^\infty\); see also [21, §2.5]. By performing a min-max procedure, Gromov [9] defined the volume spectrum of \(M\), which is a sequence of non-decreasing positive numbers

\[0 < \omega_1(M, g) \leq \omega_2(M, g) \leq \cdots \leq \omega_k(M, g) \to \infty,\]

depending only on \(M\) and \(g\). Moreover, Gromov [9] also showed that for each \(g\), \(\omega_k\) grows like \(k^n\); see also Guth [14].

For closed Riemannian surfaces (i.e. \(n = 2\)), Y. Liokumovich [20] bounded all of the volume spectrum using the genus of the surfaces. In this paper, we generalize these results and prove conformal upper bounds for all of the volume spectrum of closed Riemannian manifolds.

Theorem 1.1. There exists a constant \(C = C(n)\) such that for any \(n\)-dimensional closed Riemannian manifold \((M, g)\), we have

\[\omega_k(M, g) \leq C|M|^{\frac{n-1}{n}} \max\{k^n, \text{MCV}(M, g)^{\frac{1}{n}}\} \cdot\]

Here \(\Sigma\) is denoted as the \(H^m\)-measure with respect to \(g'\) for any \(m\)-dimensional submanifold \(\Sigma\) of \(M\) and

\[\text{MCV}(M, g) := \inf\{|M|_{g_0} : g_0 \text{ is a metric conformal to } g \text{ and } \text{Ric}_{g_0}(M) \geq -(n-1)\},\]

which is called the min-conformal volume of \(M\); c.f. [7, Definition 1.2; 15, §1]. For simplicity, we use \(|g|\) to denote the collection of Riemannian metrics that are conformal to \(g\).

Remark 1.2. We make several remarks here:

1. Note that by the uniformization theorem, \(\text{MCV}(M) \leq 2\gamma\) if \(M\) is a closed surface of genus \(\gamma\). Then Theorem 1.1 is exactly the same with [20].

2. From the proof, the estimates also hold for compact domains \(N \subset M\) in Theorem 1.1, i.e. for any \(g_0 \in [g]\) with \(\text{Ric}_{g_0}(M) \geq -(n-1)\) and \(N \subset M\),

\[\omega_k(N, g) \leq C|N|^{\frac{n-1}{n}} \max\{k^n, |N|^{\frac{1}{n}}_{g_0}\} \cdot\]
(3) Our theorem is sharp in some sense. In general cases (not in a conformal class), L. Guth [14, Section 5] gave a counterexample to this question for the first width in the volume spectrum. In other words, a closed oriented Riemannian \( n \)-manifold may have volume 1 and arbitrarily large \( \omega_1(M, g) \).

We point out that Glynn-Adey-Liokumovich [7] proved conformal upper bounds for the first width in the volume spectrum (i.e. the case of \( k = 1 \)), which will be used in this paper. For closed Riemannian manifolds with non-negative Ricci curvature, the uniform upper bounds for the volume spectrum was proved by Glynn-Adey-Liokumovich [7] and Sabourau [31]. Observe that \( \text{MCV}(M, g) = 0 \) provided that there exists \( g_0 \in [g] \) with \( \text{Ric}_{g_0}(M) \geq 0 \). Hence we have the following corollary.

**Corollary 1.3.** Let \((M, g)\) be a closed Riemannian manifold and there exists \( g_0 \in [g] \) with \( \text{Ric}_{g_0}(M) \geq 0 \). There exists a constant \( C = C(n) \) such that

\[
\omega_k(M, g) \leq C |M|^{\frac{n-1}{n}} k^\frac{1}{n}.
\]

To understand the volume spectrum, Gromov [10, Remark 8.4] had an insightful idea that many properties of the eigenvalues of the Laplacian operators have analogs for the volume spectrum. Furthermore, Gromov conjectured that the volume spectrum \( \{\omega_k(M, g)\}_{k \in \mathbb{N}} \) satisfy a Weyl’s law, which has been fully proved by Liokumovich-Maques-Neves [21]. For Laplacian operators, Korevaar [17] proved the upper bounds for the Neumann eigenvalues of Riemannian manifolds which are conformal to a manifold with non-negative Ricci curvature. Later, Hassannezhad [15] obtained the conformal upper bounds for the eigenvalues of the Laplacian in the conformal class of compact Riemannian manifolds. Our Theorem 1.1 and Corollary 1.3 are volume spectrum analogs of the results of Hassannezhad [15] and Korevaar [17], respectively.

We refer to [4, 6, 19] for the estimates of the Laplacian operators and [3, 22, 23, 29] for some developments of sweepouts by cycles.

Due to the development of min-max theory by Almgren [1, 2], Pitts [30], Schoen-Simon [32] and Marques-Neves [24], the volume spectrum bounds give information about finding minimal hypersurfaces in closed Riemannian manifolds; see [16,25,27,33,34]. In particular, using the Multiplicity One Theorem proven by X. Zhou [35] (see also [5]), Marques-Neves [26] proved that in any closed Riemannian manifold \( M \) of dimension \( 3 \leq n \leq 7 \), for generic metrics \( g \), there exists a sequence of embedded minimal hypersurfaces \( \{\Sigma_k\} \) such that

\[
\omega_k(M, g) = \mathcal{H}^{n-1}(\Sigma_k) \quad \text{and} \quad \text{index}(\Sigma_k) = k.
\]

Then our Theorem 1.1 gives a conformal upper bound for these embedded minimal hypersurfaces.

**Idea of the proof.** Let \((M, g)\) be a closed Riemannian manifold and \( g_0 \in [g] \) such that \( \text{Ric}_{g_0}(M) \geq -(n-1) \). Denote by \( B^0(r) \) the geodesic ball in \( M \) of radius \( r \) and center \( p \) with respect to \( g_0 \). For simplicity, we use \(|\cdot|\) to denote \( |\cdot|_{g_0} \).

We first recall the construction of \( k \)-sweepouts by Gromov [9] and Guth [14, Section 5], where they proved that if a closed manifold \( M \) is divided into a collection of open domains \( \{V_j\} \), then

\[
\omega_k(M, g) \leq \left| \bigcup_j \partial V_j \right| + k \max_j \omega_1(V_j, g).
\]
Then the challenge is to divide $M$ into suitable domains for each $k$. Without loss of
generality, we assume that $|M| = |M|_{g_0}$. By the work of Glynn-Adey-Liokumovich [7],
it suffices to consider $k \geq |M|_{g_0}$ and $k > 10^n$. We now fix $k$ and let $\alpha = |M|/k$ and
$r = \alpha^{1/2}/C$. The aim is to bound $\omega_k(M, g)$ by $C|M|^{1/2}k^{1/2}$.

In the first step, we subdivide $M$ into domains $\{D_j\}_{j=1}^{m+1}$ ($m \leq k - 1$) such that $|D_j|_{g_0} < 1$
for $1 \leq j \leq m$, $\sum |\partial D_j| \sim k^{1/2}$ and $|B_0^0(p) \cap D_{m+1}| \leq \alpha$ for all $p \in M$. This can be done
inductively by taking $B_0^0(p)$ such that its $g$-volume is larger than $\alpha$. Then the length-area
method also enables us to control $|\partial D_j|$; see Claim 2 for details.

The next step is to subdivide $D_{m+1}$. To do this, we always take $B_0^0(p)$ that has the largest
area in the remaining part with respect to $g$. Then the length-area method allows us to
find a domain $V_i$ between $B_0^0(p)$ and $B_0^0(q)$ such that its boundary has a desired bound.
The difficulty here is that $|B_{4r}^0(p)|_{g_0}$ is used to bound $|\partial V_i|$. And these balls of radius $4r$
will intersect each other. To overcome this, we proved that for each point $x \in D_{m+1}$, the
number of $V_i$ that contains $x$ is bounded by a uniform constant depending only on $n$. Then
using the Hölder’s inequality, we obtain the desired covers for $D_{m+1}$.

Finally, we are going to subdivide $D_j$ for $1 \leq j \leq m$. One of the key ingredients is the
isoperimetric inequality developed by Glynn-Adey-Liokumovich [7] Theorem 3.4] (see also
Theorem 4.1, which allows us to subdivide $D_j$ into two parts. Repeating this process, we
finally subdivide $D_j$ into $\{U_i\}_i$ until each small domain has $g$-volume bounded by $|M|/k$.
Then using the estimates for the first width in the volume spectrum in [7] (see also Theorem
2.2 for compact domains), $k\omega_1(U_i^j, g)$ is naturally bounded by $k^{1/2}$. It remains to bound the
boundary of $U_i^j$ that lies in $\text{Int} D_j$, which are exactly the isoperimetric hypersurfaces in
Theorem 4.1 In Subsection 2.4 a general way will be developed to study this kind of tree
decomposition; see Proposition 2.4 for details. We would like to emphasize that $|D_j| < 1$ is
crucial to have the desired bounds in this part.

Outline. This paper is organized as follows. Section 2 includes some results that will be
used in this paper and an upper bound for the tree decomposition. In Section 3 and 4
we provide the details to subdivide the conformally thin and thick domains, respectively.
Finally, Section 5 is devoted to prove the main theorem. We also give more details of the
proof of Theorem 2.2 in Appendix A.

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2. Preliminary

2.1. Notations. In this paper, $(M^n, g)$ is always a closed Riemannian manifold with di-

mension $n$ and $N$ is a compact domain in $M$ with piecewise smooth boundary.

We now recall the formulation in [21]. Let $(N, \partial N, g) \subset \mathbb{R}^L$ be a compact Riemannian

manifold with piecewise smooth boundary. Let $\mathcal{R}_k(N; \mathbb{Z}_2)$ (resp. $\mathcal{R}_k(\partial N; \mathbb{Z}_2)$) be the space

of $k$-dimensional rectifiable currents in $\mathbb{R}^L$ with coefficients in $\mathbb{Z}_2$ which are supported in

$N$ (resp. $\partial N$). Denote by $\mathcal{M}$ the mass norm. Let

$$Z_k(N, \partial N; \mathbb{Z}_2) := \{ T \in \mathcal{R}_k(N; \mathbb{Z}_2) : \text{spt}(\partial T) \subset \partial N \}.$$ (2.1)

We say that two elements $S_1, S_2 \in Z_k(N, \partial N; \mathbb{Z}_2)$ are equivalent if $S_1 - S_2 \in \mathcal{R}_k(\partial N; \mathbb{Z}_2)$.

Denote by $Z_k(M, \partial N; \mathbb{Z}_2)$ the space of all such equivalence classes. The mass and flat norms
for any $\tau \in Z_k(N, \partial N; \mathbb{Z}_2)$ are defined by
\[ M(\tau) := \inf \{ M(S) : S \in \tau \} \quad \text{and} \quad F(\tau) := \inf \{ F(S) : S \in \tau \}. \]
The support of $\tau \in Z_k(M, \partial M; \mathbb{Z}_2)$ is defined by
\[ spt(\tau) := \bigcap_{S \in \tau} spt(S). \]
Let $X$ be a finite dimensional simplicial complex. Given $p \in \mathbb{N}$, a continuous map in the flat topology $\Phi : X \to Z_n(N, \partial N; \mathbb{Z}_2)$ is called a $k$-sweepout if the $k$-th cup power of $\lambda = \Phi^*(\bar{\lambda})$ is non-zero in $H^k(X; \mathbb{Z}_2)$ where $0 \neq \bar{\lambda} \in H^1(Z_n(N, \partial N; \mathbb{Z}_2); \mathbb{Z}_2) \cong \mathbb{Z}_2$. Denote by $P_k(N)$ the set of all $k$-sweepouts that are continuous in the flat topology and have no concentration of mass (§3.7), i.e.
\[ \lim_{r \to 0} \sup \{ M(\Phi(x) \cup B_r(q)) : x \in X, q \in M \} = 0. \]
In [25] and [21], the $k$-width of codimension one is defined as
\[ \omega_k(N, g) := \inf_{\Phi \in P_k} \sup \{ M(\Phi(x)) : x \in \text{dom}(\Phi) \}. \]
{\omega_k(N, g)} are also called the volume spectrum.

Remark 2.1. In this paper, we used the integer rectifiable currents, which is the same with [18]. However, the formulations are equivalent to that in [21]; see [11, Proposition 3.2] for details.

### 2.2. Conformal bounds for the first width

In [7], Glynn-Adey and Liokumovich proved the uniform bound of the first width for all closed manifolds. With minor modification, their arguments can be applied for compact domains. Such a uniform bound will be used in this paper later.

Let $g$ and $g_0$ be two Riemannian metrics on $M$. For any $m$-dimensional submanifold $\Sigma$ of $M$, we use $|\Sigma|$ and $|\Sigma|_{g_0}$ to denote the $\mathcal{H}^m$-measure with respect to $g$ and $g_0$.

**Theorem 2.2** (Glynn-Adey-Liokumovich [7]). Let $N$ be a compact domain of a closed Riemannian manifold $(M, g)$ with dimension $n$. Let $g_0$ be another metric on $M$ which is conformal to $g$ and $\text{Ric}_{g_0}(M) \geq -1$. There exists a constant $K$ depending only on the dimension of $N$ such that
\[ \omega_1(N, g) \leq K \cdot \frac{1}{|N|} \cdot (1 + |N|^{\frac{2}{n}}). \]

For completeness, we sketch the idea of the proof here and give more details in Appendix A.

We first handle the case that $N$ has smooth boundary. Following the steps in [7], we decompose the domain with small volume into small pieces so that the argument in [21, Proposition 2.3] can be applied, and then we use the inductive method in [7, Theorem 5.1].

To decompose $D \subset N$ with small volume, we will cut the part intersecting $\partial N$. Then the regularity theory [28, Theorem 4.7] (see also [11, Theorem 4.7]) for the free boundary minimizing problem is used. In order to show that such a minimizing hypersurface does not intersect a smaller ball, we employ the monotonicity formula in [12, Theorem 3.4].

Finally, for compact domain with piecewise smooth boundary, we can take a tubular neighborhood $U$ with $|U| \leq 2|N|$ and $|U|_{g_0} \leq 2|N|_{g_0}$ and $U$ has smooth boundary. Then the desired inequality follows from $\omega_1(N, g) \leq \omega_1(U, g)$. 
2.3. The length-area method. Let \((M, g)\) be a closed Riemannian manifold and \(N\) be a compact domain with piecewise smooth boundary. Let \(g_0\) be a metric on \(M\) which is conformal to \(g\) and \(\text{Ric}_{g_0}(M) \geq -(n-1)\). Denote by \(\nabla\) and \(\nabla^0\) the Levi-Civita connection with respect to \(g_0\) and \(g\). For any compact domain \(D \subset M\), denote by

\[
\mathcal{N}^0_r(A) := \{x \in M : \text{dist}_{g_0}(x, A) \leq r\},
\]

where \(\text{dist}_{g_0}(\cdot, \cdot)\) is the distance with respect to \(g_0\). Recall that \(|\Sigma|\) and \(|\Sigma|_{g_0}\) are used to denote the \(\mathcal{H}^m\)-measure with respect to \(g\) and \(g_0\) if \(\Sigma\) is a \(m\)-dimensional submanifold of \(M\).

The following inequality is from the well-known length-area method (see \cite{7} Theorem 3.4; \cite{8} §5; \cite{20} Lemma 4.1) and will be used in this paper.

**Proposition 2.3.** For any compact domain \(D \subset N\) and \(r > 0\), there exists a compact domain \(V\) of \(N\) satisfying \(D \subset V \subset \mathcal{N}^0_r(D)\) and

\[
|\partial U \cap \text{Int} N| \leq (1/r) \cdot |N \cap \mathcal{N}^0_r(D) \setminus D| g_0 \cdot |N \cap \mathcal{N}^0_r(D) \setminus D|^{\frac{n}{n-1}}.
\]

**Proof.** We present the proof in \cite{7} Theorem 3.4] here. For \(x \in M\), denote by

\[
f(x) = \text{dist}_{g_0}(x, D).
\]

By the co-area formula,

\[
\int_0^r |f^{-1}(t) \cap \text{Int} N| dt = \int_{f^{-1}(0, r) \cap N} |\nabla f| d\mathcal{H}^n(g)
\]

\[
\leq \left( \int_{f^{-1}(0, r) \cap N} |\nabla f|^n d\mathcal{H}^n(g) \right)^\frac{1}{n} \cdot |f^{-1}(0, r) \cap N|^{\frac{n-1}{n}}
\]

\[
= |f^{-1}(0, r) \cap N|^{\frac{1}{g_0}} \cdot |f^{-1}(0, r) \cap N|^{\frac{n-1}{n}}.
\]

Here the last equality follows from \(|\nabla f|^n d\mathcal{H}^n(g) = |\nabla^0 f|^n d\mathcal{H}^n(g_0)|\). Note that

\[
f^{-1}(0, r) = \mathcal{N}^0_r(D) \setminus D.
\]

Hence Proposition 2.3 is proved. \hfill \Box

2.4. Tree decomposition. Let \(\alpha = \overline{\alpha_1 \alpha_2 \cdots \alpha_m}\) be an ordered binary array with \(\alpha_j \in \{0, 1\}\). Then we define \(|\alpha| = m\). For two binary arrays \(\alpha\) and \(\beta\), we say \(\alpha \preceq \beta\) if \(\alpha_j = \beta_j\) for all \(j \leq |\alpha|\). We say \(\Lambda\) is an admissible tree provided the following holds:

- if \(\alpha \in \Lambda\), then \(\beta \in \Lambda\) for any \(\beta \preceq \alpha\);
- \(\overline{\alpha} \in \Lambda\) if and only if \(\alpha \overline{1} \in \Lambda\);

Denote by \(\partial \Lambda = \{\alpha \in \Lambda : \text{if } \beta \in \Lambda \text{ with } \alpha \preceq \beta, \text{ then } \beta = \alpha\}\).

Let \(\Lambda\) be an admissible tree and \(\lambda \in (0, 1/2]\). For any positive real number \(X \geq 1\), we say a sequence of real numbers \(\{X_\alpha\}\) is a \((\Lambda, \lambda)\)-decomposition if

- \(X = X_0 + X_1\) and \(X_i > \lambda X\) for \(i \in \{0, 1\}\);
- \(X_\overline{\alpha} = \overline{X_\alpha} + X_\overline{\alpha}\) and \(X_\overline{\alpha} \geq \lambda X_\alpha\) for all \(\alpha \in \Lambda \setminus \partial \Lambda\) and \(i \in \{0, 1\}\);
- \(X_\alpha \geq 1\) for all \(\alpha \in \Lambda\).

In this section, \(\lambda \in (0, 1/2]\) is a constant. Let

\[
\tilde{\lambda} = [\lambda^{\frac{n-1}{n}} + (1 - \lambda)^{\frac{n-1}{n}} - 1]^{-1}.
\]
Then for any \( t \in [\lambda, 1 - \lambda] \), we have
\[
(2.3) \quad \tilde{\lambda} \cdot \left( t \frac{\lambda}{\alpha} - (1 - t) \frac{\lambda}{\alpha} - 1 \right) \geq 1.
\]
For any \( X \geq 1 \), we define
\[
\mathcal{N}(X) := \sup \left\{ X^{\frac{n-1}{n}} + \sum_{\alpha \in \Lambda} X^{\frac{n-1}{n}} : \{X_{\alpha}\} \right\} \text{ is a } (\Lambda, \lambda)-\text{decomposition for some admissible tree } \Lambda.
\]

The main result in this subsection is that \( \mathcal{N}(X) \) has linear growth.

**Proposition 2.4.** For any \( X \geq 1 \), we have
\[
\mathcal{N}(X) + \tilde{\lambda}X^{\frac{n-1}{n}} \leq (1 + \tilde{\lambda})X.
\]

**Proof.** For any \( (\Lambda, \lambda)-\text{decomposition } \{X_{\alpha}\} \), we have that
\[
X^{\frac{n-1}{n}} + \sum_{\alpha \in \Lambda} X^{\frac{n-1}{n}} \leq X^{\frac{n-1}{n}} + \mathcal{N}(X_0) + \mathcal{N}(X_1) \leq X^{\frac{n-1}{n}} + \sup_{\lambda \leq t \leq 1-\lambda} (\mathcal{N}(tX) + \mathcal{N}((1-t)X)).
\]
This implies that
\[
\mathcal{N}(X) \leq X^{\frac{n-1}{n}} + \sup_{\lambda \leq t \leq 1-\lambda} (\mathcal{N}(tX) + \mathcal{N}((1-t)X)).
\]
Denote by \( \tilde{\mathcal{N}}(X) = \mathcal{N}(X) + \bar{\lambda}X^{\frac{n-1}{n}} \). Then
\[
(2.4) \quad \tilde{\mathcal{N}}(X) \leq \sup_{\lambda \leq t \leq 1-\lambda} (\tilde{\mathcal{N}}(tX) + \tilde{\mathcal{N}}((1-t)X)),
\]
where we used the fact (2.3). For any \( X \in [1,2] \), we have \( \tilde{\mathcal{N}}(X) = X^{\frac{n-1}{n}} + \bar{\lambda}X^{\frac{n-1}{n}} \leq (1 + \bar{\lambda})X \). Now we prove the inequality inductively. Suppose that it holds true for \( X < Y \) \((Y \geq 2)\). Then for any \( X \in [Y, Y + \lambda] \) and \( t \in [\lambda, 1 - \lambda] \), we have
\[
tX \leq (1 - \lambda)(Y + \lambda) \leq Y - \lambda.
\]
Hence
\[
\tilde{\mathcal{N}}(tX) \leq (1 + \bar{\lambda})tX \quad \text{and} \quad \tilde{\mathcal{N}}((1-t)X) \leq (1 + \bar{\lambda})(1-t)X,
\]
Together with (2.4), we conclude that
\[
\tilde{\mathcal{N}}(X) \leq \sup_{\lambda \leq t \leq 1-\lambda} [(1 + \bar{\lambda})tX + (1 + \bar{\lambda})(1-t)X] = (1 + \bar{\lambda})X.
\]
This finishes the proof of Proposition 2.4. \( \square \)

3. Dividing conformally thin domains

Let \((M, g)\) be a closed Riemannian manifold and \(g_0 \in [g] \) with \(\text{Ric}_{g_0}(M) \geq -(n-1)\). Denote by \(B^0_r(p)\) the geodesic ball in \((M, g_0)\) with center \(p\) and radius \(r\). In this section, we divide the compact domains \(N\) that geodesic balls in \((M, g_0)\) of radius \(r\) satisfying
\[
|B^0_r(p) \cap N| \leq \alpha, \quad \forall p \in M,
\]
where \(r\) and \(\alpha\) are given constants. This kind of domains are called to be conformally thin.

Denote by \(v(r, n)\) the volume of the geodesic ball in an \(n\)-dimensional hyperbolic manifold (with sectional curvature \(-1\) ). Denote by
\[
C(r) = \max_{0 \leq t \leq r} \left\{ 1 + \left[ \frac{v(9r/2, n)}{v(r/2, n)} \right] \right\}.
\]
Then in any complete Riemannian manifold with $\text{Ric} \geq -(n-1)$, every geodesic ball of radius $4s$ can be covered by $C(r)$ many balls of radius $s$ for all $s \in (0, r]$. Note that $C(r)$ is a constant depending only on $r$ and $n$; c.f. [3] Example 2.1.

Let $C_0 = C_0(n)$ be the constant such that for $r < 10$,

$$v(r, n) \leq C_0 r^n.$$ 

By the classical Bishop-Gromov inequality, a geodesic ball with radius $r < 10$ in a Riemannian manifold with $\text{Ric} \geq -(n-1)$ has the volume bounded by $C_0 r^n$ from above. Let $K$ be the constant in Theorem 2.2.

**Lemma 3.1.** Let $N$ be a compact domain with (possibly empty) piecewise smooth boundary in some closed Riemannian manifold $(M, g)$. Suppose that there exist $\alpha > 0$ and $r \in (0, 1)$ satisfying

$$|B_r^0(p) \cap N| \leq \alpha$$

for all $p \in M$. Then $N$ can be divided into finitely many open domains $\{V_j\}_{j=1}^L$ by $\partial \cup V_j$ satisfying

$$\left| \bigcup_{j=1}^L \partial V_j \cap \text{Int} N \right| \leq (C_1/r) \cdot |N|^{1/2} \cdot |N|^{\alpha/2};$$

$$\omega_1(V_j, g) \leq C_1 \alpha^{\frac{n-1}{n}} \quad \text{for} \quad 1 \leq j \leq L,$$

where $C_1 = C(r/2)C(r) + (4C_0 + 1)K \cdot C(r)$.

**Proof.** Since $\text{Ric}_{g_0}(M) \geq -(n-1)$ and $r < 1$, then we have

$$|B_{4r}^0(p)|_{g_0} \leq C_0 (4r)^n.$$ 

Now we construct $\{V_j\}$ inductively. Let $V_0 = \emptyset$. Suppose we have $V_0, \ldots, V_j$ and $M \setminus \bigcup_{i=1}^j \overline{V}_i \neq \emptyset$. Then we take $p_{j+1} \in M \setminus \bigcup_{i=0}^j \overline{V}_i$ such that for all $p \in M$,

$$|N \cap B_r^0(p_{j+1}) \setminus \bigcup_{i=0}^j V_i| \geq |N \cap B_r^0(p) \setminus \bigcup_{i=0}^j V_i|.$$ 

Note that $B_{4r}^0(p_{j+1})$ is covered by $C(r)$ many balls of radius $r$. It follows that

$$\left| N \cap B_{4r}^0(p_{j+1}) \setminus \bigcup_{i=0}^j V_i \right| \leq C(r) \left| N \cap B_{4r}^0(p_{j+1}) \setminus \bigcup_{i=0}^j V_i \right| \leq C(r) \alpha.$$ 

Then by Proposition 2.3, we take $V_{j+1}$ satisfying

$$B_{3r}^0(p_{j+1}) \cap N \setminus \bigcup_{i=0}^j V_i \subset V_{j+1} \subset B_{4r}^0(p_{j+1}) \cap N \setminus \bigcup_{i=0}^j V_i$$

and

$$\left| \partial V_{j+1} \cap \text{Int}(N \setminus \bigcup_{i=0}^j V_i) \right| \leq \frac{1}{r} \cdot |B_{4r}^0(p_{j+1}) \cap N|^{\alpha/2} \cdot |B_{4r}^0(p_{j+1}) \cap N|^{1/2};$$

By Theorem 2.2,

$$\omega_1(V_{j+1}, g) \leq K |V_{j+1}|^{\alpha/2} \left(1 \cdot \left| B_{4r}^0(p_{j+1}) \right|_{g_0}^{1/2} \right) \leq (4C_0 + 1)K \cdot C(r) \alpha^{\frac{n-1}{n}}.$$
Here we used (3.4) and (3.3) in the last inequality. Observe that \( p_{j+1} \notin B_{2r}(p_i) \) for \( i \leq j \), which implies that 
\[
B_r^0(p_{j+1}) \cap B_r^0(p_i) = \emptyset, \quad 1 \leq i \leq j.
\]
Then there exists \( L \geq 1 \) such that 
\[
N = \bigcup_{j=1}^L V_j.
\]
It remains to prove that these open sets satisfy our requirements. We first prove that every \( x \in M \) is contained in at most \( C(r/2) \cdot C(2r) \) many \( V_j \). Namely, if \( x \in V_j \), then \( B_r^0(p_j) \subset B_{2r}^0(x) \). Now let 
\[
J(x) = \#\{V_j : 1 \leq j \leq L \text{ and } B_r^0(p_j) \subset B_{2r}^0(x)\}.
\]
Note that \( B_{2r}^0(x) \) can be covered by \( C(r/2)C(2r) \) many balls \( \{B_{r/2}^0(z_i)\} \) in \( M \). By taking \( z_j \) such that \( p_j \in B_{r/2}^0(z_j) \), then we have \( B_{r/2}^0(z_j) \subset B_r^0(p_j) \). Thus \( J(x) \leq C(r/2)C(2r) \).

By (3.5), we have 
\[
\left| \bigcup_{j=1}^L \partial V_j \cap \text{Int} N \right| = \sum_{j=0}^{L-1} \left| \partial V_{j+1} \cap \text{Int} (N \setminus \bigcup_{i=0}^j V_i) \right|
\leq \sum_{j=1}^L \frac{1}{r} \cdot |B_{4r}^0(p_j) \cap N|^{\frac{n-1}{n}} \cdot |B_{4r}^0(p_j) \cap N|_{g_0}^{\frac{1}{n}}
\leq \frac{1}{r} \cdot \left( \sum_{j=1}^L |B_{4r}^0(p_j) \cap N| \right)^{\frac{n-1}{n}} \cdot \left( \sum_{j=1}^m |B_{4r}^0(p_j) \cap N|_{g_0} \right)^{\frac{1}{n}}
\leq C(r/2)C(r) \cdot \frac{1}{r} \cdot |N|^{\frac{n-1}{n}} \cdot |N|_{g_0}^{\frac{1}{n}}.
\]
Together with (3.6), Lemma 3.1 follows by taking \( C_1 = C(r/2)C(r) + (4C_0 + 1)K \cdot C(r) \).

\[\Box\]

4. Dividing conformally thick domains

Let \((M, g)\) be a closed manifold and \( g_0 \in [g] \) such that \( \text{Ric}_{g_0}(M) \geq -(n-1) \). Let \( N \) be a compact domain in \( M \) with piecewise smooth boundary. In this section, we estimates the volume spectrum of the domains with small \( g_0 \)-volume.

We first recall the isoperimetric inequality developed by Glynn-Adey-Liokumovich in [7], which is a consequence of the length-area method.

**Theorem 4.1** ([7] Theorem 3.4). There exists a constant \( c(n) \) such that the following holds: Let \( U \subset M \) be an open subset. There exists an \((n-1)\)-submanifold \( \Sigma \subset U \) subdividing \( U \) into two open sets \( U_1 \) and \( U_2 \) such that \( |U_1| \geq 25^{-n}|U| \) and \( |\Sigma| \leq c(n) \max\{1, |U|_{g_0}^{\frac{1}{2}} \}|U|_{g_0}^{-\frac{n-1}{n}} \).

Now we are ready to prove the main result of this section.

**Theorem 4.2.** There exists \( C_2 = C_2(n) \) satisfying the following: for every positive integer \( k \), each closed \( n \)-dimensional Riemannian manifold \((M, g)\) and compact domain \( N \subset M \)
with $|N|_{g_0} \leq 1$, there exists a collection of compact domains \( \{U_j\} \) such that \( N = \bigcup \overline{U}_j \) and
\[
|\bigcup_j \partial U_j \cap (\text{Int} N)| + k \max_j \omega_1(U_j, g) \leq C_2 |N|^{\frac{n-1}{n}} k^\frac{1}{n}.
\]
(4.1)

As a corollary, \( \omega_k(N, g) \leq C_2 |N|^{\frac{n-1}{n}} k^\frac{1}{n} \).

Proof. Without loss of generality, we assume that \( |N| = 1 \). Let \( K > 1 \) be the constant in Theorem 2.2. Then every compact domain \( N' \subset M \) satisfies
\[
\omega_1(N', g) \leq K |N'|^{\frac{n-1}{n}}.
\]
(4.2)

Now let \( k > 50^n \). Then by Theorem 4.1, there exists an \( (n-1) \)-submanifold \( \Sigma \subset M \) subdividing \( N \) into two open sets \( M_0 \) and \( M_1 \) such that \( |N_1| \geq |N_0| \geq 1/25^n \) and \( |\Sigma| \leq c(n) \max \{1, |N|^{\frac{1}{50}} \} = c(n) \).

Let \( N_\alpha \) be a compact domain of \( N \), where \( \alpha = i_1 i_2 \cdots i_{|\alpha|} \) and \( i_j \in \{0, 1\} \). If \( |N_\alpha| \geq 50^n/k \), then using Theorem 4.1 again, there exists an \( (n-1) \)-submanifold \( \Sigma_\alpha \) subdividing \( N_\alpha \) into two open sets \( N_{\alpha 0} \) and \( N_{\alpha 1} \) such that \( |N_{\alpha 1}| g \geq |N_{\alpha 0}| \geq |N_\alpha|/25^n \), and
\[
|\Sigma_\alpha| \leq c(n)|N_\alpha|^{\frac{1}{n}} \max \{1, |N_\alpha|^{\frac{1}{50}} \} = c(n)|N_\alpha|^{\frac{1}{n}}.
\]
(4.3)

Note that we always have \( k|N_\alpha| \geq k|N_{\alpha 0}| \geq k|N_\alpha|/25^n \geq 2^n \).

Denote by \( \Lambda \) the collection of \( \alpha \) appeared in the previous process. Then \( \Lambda \) is an admissible tree (see Subsection 2.4). Recall that
\[
\partial \Lambda = \{ \alpha \in \Lambda : \overline{\alpha 0} \notin \Lambda \}.
\]

Then we have
\[
N = \bigcup_{\alpha \in \partial \Lambda} N_\alpha,
\]
where \( |N_\alpha| < 50^n/k \) and \( \text{Int} N_\alpha \cap \text{Int} N_\beta = \emptyset \) for any \( \alpha \neq \beta \in \partial \Lambda \).

Now we define \( \{U_j\} \) as \( \{N_\alpha\}_{\alpha \in \partial \Lambda} \). Then we prove that such a collection of domains satisfy our requirements. Denote by \( k_\alpha = k|N_\alpha| \). Note that \( |N_\alpha| < 50^n/k \). Then for each \( \alpha \in \partial \Lambda \),
\[
k_\alpha = k|N_\alpha| < 50^n.
\]

By (4.2), we have
\[
k_\alpha \omega_1(N_\alpha, g) < 50^n \omega_1(N_\alpha, g) \leq 50^n K |N_\alpha|^{\frac{n-1}{n}},
\]
which implies that for all \( \alpha \in \partial \Lambda \),
\[
k \omega_1(N_\alpha, g) \leq k/k_\alpha \cdot 50^n K |N_\alpha|^{\frac{n-1}{n}} = k^{\frac{1}{n}}/k_\alpha \cdot 50^n K \cdot (k_\alpha)^{\frac{n-1}{n}} \leq 50^n K \cdot k^{\frac{1}{n}}.
\]

Here in the equality, we used \( k|N_\alpha| = k_\alpha \).

Claim 1. There exists \( K_1(n) \) depending only on \( n \) such that
\[
\left| \bigcup_{\alpha \in \partial \Lambda} \partial N_\alpha \cap \text{Int} N \right| \leq K_1(n) k^{\frac{1}{n}}.
\]
(4.4)

Proof of Claim 1. Note that \( \{k_\alpha\} \) is a \( (1/50^n, \Lambda) \)-decomposition. Then by Proposition 2.4 (by letting \( \lambda = 1/50^n \)),
\[
k^{\frac{n-1}{n}} + \sum_{\alpha \in \Lambda \setminus \partial \Lambda} k_\alpha^{\frac{n-1}{n}} \leq (1 + \lambda)k,
\]
(4.5)
where \( \tilde{\lambda} \) is defined by (2.3). Then we have
\[
\left| \bigcup_{\alpha \in \partial \Lambda} \partial N_{\alpha} \cap \text{Int} N \right| = |\Sigma| + \sum_{\alpha \in \Lambda \setminus \partial \Lambda} |\Sigma_{\alpha}|
\leq c(n) \left( |N|^{\frac{n-1}{n}} + \sum_{\alpha \in \Lambda \setminus \partial \Lambda} |N_{\alpha}|^{\frac{n-1}{n}} \right)
\leq c(n) k^{\frac{n-1}{n}} \left( \frac{k^{\frac{n-1}{n}}}{n} + \sum_{\alpha \in \Lambda \setminus \partial \Lambda} k_{\alpha}^{\frac{n-1}{n}} \right)
\leq 2c(n) \cdot k^{\frac{1}{n}} \left( 1 + \tilde{\lambda} \right).
\]
Here the first inequality is from (4.3); the last one follows from (4.5).

Let
\[
K_1(n) = 2c(n)(1 + \tilde{\lambda}).
\]
Then Claim 1 is proved.

Recall that \( \{ U_j \} \) are exactly \( \{ N_{\alpha} \}_{\alpha \in \partial \Lambda} \). Using Claim 1 and (4.4), we obtain
\[
|\bigcup_j \partial U_j \cap \text{Int} N| + k \max_j \omega_1(U_j, g) \leq (50^n K + K_1(n)) k^{\frac{1}{n}}.
\]
This is the desired inequality by letting \( C_2 = 50^n K + K_1(n) \). \( \square \)

5. THE CONFORMAL UPPER BOUNDS

In this section, we prove the conformal upper bounds for the volume spectrum. We will first divide the manifold into conformally thin and thick domains and then Lemma 3.1 and Theorem 4.2 can be applied respectively.

Recall that \( |\cdot| \) and \( |\cdot|_{g_0} \) are denoted as the Hausdorff measure with respect to \( g \) and \( g_0 \). The following result is equivalent to Theorem 1.1.

**Theorem 5.1.** There exists a constant \( C_3 = C_3(n) \) such that for any \( n \)-dimensional closed Riemannian manifold \( (M, g) \), we have
\[
\omega_k(M, g) \leq C_3|M|^{\frac{n-1}{n}} \max \{ k^{\frac{1}{n}}, |M|_{g_0}^{\frac{1}{n}} \},
\]
where \( g_0 \) is conformal to \( g \) and \( \text{Ric}_{g_0}(M) \geq -(n-1) \).

**Proof.** Without loss of generality, we assume that \( |M| := |M|_g = |M|_{g_0} \). For any \( k > 100^n \), define
\[
r_k = \frac{1}{4} \cdot \left( \frac{|M|}{2kC_0C(1)} \right)^{\frac{1}{n}}, \quad \text{and} \quad \alpha_k = \frac{|M|}{k}.
\]
Denote by
\[
\bar{k} = \left[ \frac{|M|}{2C(1)} \right] + 1.
\]
Then for any \( k \geq \bar{k} \), we have \( r_k < 1/4 \).

**Claim 2.** There exist \( m(\leq k - 1) \) many domains \( \{ D_j \}_{j=1}^m \) such that
- \( |D_j|_{g_0} < 1 \) and \( |\cup \partial D_j| \leq 4C_0C(1)|M|^{\frac{n-1}{n}} \cdot k^{\frac{1}{n}} \); 
- \( |B_{r_k}(p) \setminus \cup_{j=1}^m D_j| < \alpha_k \) for all \( p \in M \).
Proof of Claim 2. Let $D_0 = \emptyset$. Then we construct $\{D_j\}$ inductively. Suppose we have $D_0, \ldots, D_j$. If $|B_{r_k}^0(p) \setminus \bigcup_{i=1}^j D_i| < \alpha_k$ for all $p \in M$, then we just let $m = j$. Otherwise, take $p_{j+1}$ such that for all $p \in M$,

$$\left| B_{r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i \right| \geq \left| B_{r_k}^0(p) \setminus \bigcup_{i=0}^j D_i \right|.$$ 

Clearly, $p_{j+1} \notin B_{2r_k}^0(p_i)$ for all $i \leq j$ and $|B_{r_k}^0(p_{j+1}) \setminus \bigcup_{i=1}^j D_i| \geq \alpha_k$. Note that $B_{3r_k}^0(p_{j+1})$ is covered by $C(r_k)$ many balls of radius $r_k$. Thus we have

$$(5.1) \quad \left| B_{4r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i \right| \leq C(r_k) \left| B_{r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i \right|.$$

Since $r_k < 1$, we have that

$$(5.2) \quad \left| B_{4r_k}^0(p_{j+1}) \right|_{g_0} \leq C_0 \cdot (4r_k)^n.$$ 

Then by Proposition 2.8, we can take $D_{j+1}$ satisfying

$$B_{3r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i \subset D_{j+1} \subset B_{4r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i$$

and

$$\left| \partial D_{j+1} \cap \text{Int}(M \setminus \bigcup_{i=0}^j D_i) \right| \leq \frac{1}{r_k} \cdot \left| B_{4r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i \right|_{g_0} \cdot \left| B_{4r_k}^0(p_{j+1}) \setminus \bigcup_{i=0}^j D_i \right|^{\frac{n-1}{n}}.$$ 

Here in the last inequality, we used (5.1) and (5.2). Then there exists an integer $m \geq 0$ such that after $m$ many steps, we have $\{D_j\}_{j=1}^m$ such that for all $p \in M$,

$$\left| B_{r_k}^0(p) \setminus \bigcup_{j=0}^m D_j \right| < \alpha_k.$$ 

This gives that these domains $\{D_j\}$ satisfy the second item.

Now we are going to verify that these domains satisfy the first requirement. From the fact of $|D_j|_{g} > \alpha_k = 1/k$, we conclude that $m \leq k - 1$. Recall that $D_j \subset B_{4r_k}^0(p_j)$. Then we have

$$|D_j|_{g_0} \leq |B_{4r_k}^0(p_j)|_{g_0} \leq C_0(4r_k)^n < 1.$$
Moreover,

\[
\left| \bigcup_{j=1}^{m} \partial D_j \right| = \sum_{j=0}^{m-1} \left| \partial D_{j+1} \cap \text{Int}(M \setminus \bigcup_{i=0}^{j} D_i) \right|
\]

\[
\leq 4C_0C(r_k) \sum_{j=0}^{m-1} \left| B^0_{r_k}(p_{j+1}) \setminus \bigcup_{i=0}^{j} D_i \right|^{n-1 \over n}
\]

\[
\leq 4C_0C(r_k) \cdot m^{1 \over n} \cdot \left( \sum_{j=0}^{m-1} \left| B^0_{r_k}(p_{j+1}) \right|^{n-1 \over n} \right) \leq 4C_0C(r_k) \cdot m^{1 \over n} \cdot k^{1 \over n} \leq 4C_0C(1) |M|^{n-1 \over n} \cdot k^{1 \over n}.
\]

Here the first inequality is from (5.3); we used the Hölder’s inequality in the second one; the third one follows from the fact of \( B^0_{r_k}(p_i) \cap B^0_{r_k}(p_j) = \emptyset \) for \( i \neq j \); for the last one, we used \( r_k \leq 1 \). So far, Claim 2 is proved. \( \square \)

Denote by \( D_{m+1} = M \setminus \bigcup_{j=1}^{m} D_j \) and \( k_j = k |D_j|/|M| \) for all \( 1 \leq j \leq (m + 1) \). Note that \( |D_j|_{\Omega_0} \leq 1 \). Then by Theorem 4.2 (using \( k = [k_j] + 1 \) and \( N = D_j \) there), for each \( 1 \leq j \leq m \), there exists a finite cover \( \{ U_{1,j} \}_{i} \) of \( D_j \) such that

\[
(5.4) \quad \left| \bigcup_{i} \partial U_{1,j} \cap \text{Int}(D_j) \right| + k_j \max_{i} \omega_1(U_{1,j}, g) \leq C_2 |D_j|^{n-1 \over n} \left( 1 + [k_j] \right)^{1 \over n} \leq 2C_2 |D_j|^{n-1 \over n} k_j^{1 \over n},
\]

which also implies

\[
(5.5) \quad k \max_{i} \omega_1(U_{1,j}, g) \leq k_j \cdot 2C_2 \left( k_j \cdot |M| \right)^{n-1 \over n} k_j^{1 \over n} = 2C_2 |M|^{n-1 \over n} k^{1 \over n}.
\]

Note that \( |B^0_{r_k}(p) \cap D_{m+1}| \leq \alpha_k \) for each \( p \in D_{m+1} \). Applying Lemma 3.1 (\( \alpha = \alpha_k \) and \( r = r_k \)), \( D_{m+1} \) can be subdivided into disjoint open sets \( \{ V_j \} \) by \( \bigcup_{j=1}^{L} \partial V_j \) satisfying the following:

\[
(5.6) \quad \left| \bigcup_{j=1}^{L} \partial V_j \cap \text{Int} D_{m+1} \right| \leq (C_4/r_k) \cdot |D_{m+1}|^{1 \over n} \cdot |D_{m+1}|^{n-1 \over n} \cdot |D_{m+1}|^{n-1 \over n}.
\]

\[
(5.7) \quad \omega_1(V_j, g) \leq C_4 \alpha_k^{n \over n} \quad \text{for} \quad 1 \leq j \leq L.
\]

Here \( C_4 = 5C_0(K + C(1/2))C(1) > C(r_k/2)C(r_k) + (4C_0 + 1)K \cdot C(r_k) \).
Note that $M$ is covered by $\{D_j\}_{j=1}^{m+1}$. Hence $M$ is subdivided into $\bigcup_{j=1}^m (U_j^i) \cup \{V_i\}_{i=1}^L$. Then by Gromov \[9\] and Guth \[14\] (see also \[7\] Proof of Theorem 7.1),

$$\omega_k(M, g) \leq \sum_{j=1}^m \sum_{i} |\partial U_i^j \cap \text{Int} D_j| + \left| \bigcup_{j=1}^m \partial D_j \right| + \left| \bigcup_{j=1}^L \partial V_j \cap \text{Int} D_{m+1} \right| + k \max_{1 \leq j \leq L} \omega_1(V_j, g) + k \max_{1 \leq j \leq L} \omega_1(V_j, g)$$

$$\leq \sum_{j=1}^m 2C_2 |D_j| \frac{n-1}{n} k_j^\frac{1}{n} + 4C_0 C(1) |M| \frac{n-1}{n} k_0^\frac{1}{n} + (C_4 / r_k) \cdot |D_{m+1}|_g \cdot |D_{m+1}|_g \frac{n-1}{n} k_0^\frac{1}{n} +$$

$$+ 2C_2 |M| \frac{n-1}{n} k_0^\frac{1}{n} + C_4 a_k \frac{n-1}{n} \cdot k$$

$$\leq 2C_2 \left( \sum_{j=1}^m |D_j| \right) \frac{n-1}{n} \left( \sum_{j=1}^m k_j \right)^\frac{1}{n} + 4C_0 C(1) |M| \frac{n-1}{n} k_0^\frac{1}{n} + 8C_0 C(1) C_4 \cdot |M| \frac{n-1}{n} k_0^\frac{1}{n} +$$

$$+ (2C_2 + C_4) |M| \frac{n-1}{n} k_0^\frac{1}{n} +$$

$$\leq (4C_2 + 13C_0 C(1) C_4) |M|_g \frac{n-1}{n} k_0^\frac{1}{n}.$$

Here the second inequality is from \[5.4\], \[5.6\], \[5.5\] and \[5.7\] and Claim 2 in the third inequality, we used the Hölder’s inequality for the first item, and the fact $|D_{m+1}|_g \leq |M|_g = |M|_g$ for the third item. Then we conclude that for any $k \geq \tilde{k}$,

$$\omega_k(M, g) \leq C_3 |M| \frac{n-1}{n} k_0^\frac{1}{n},$$

where $C_3 := 4C_2 + 13C_0 C(1) C_4$.

If $\tilde{k} = 1$, then we are done. Otherwise, it remains to estimate $\omega_k(M, g)$ for $k < \tilde{k}$. Note that in this case,

$$\tilde{k} \leq \frac{2|M|}{2C(1)} \leq |M| = |M|_g.$$

Then by \(5.8\),

$$\omega_{\tilde{k}}(M, g) \leq C_3 |M| \frac{n-1}{n} k_0^\frac{1}{n} \leq C_3 |M| \frac{n-1}{n} |M|_g \frac{1}{n},$$

Recall that $\omega_k(M, g) \leq \omega_k(M, g)$ for $1 \leq k \leq \tilde{k}$. Thus we conclude that for all $k \geq 1$,

$$\omega_k(M, g) \leq C_3 |M| \frac{n-1}{n} (k_0^\frac{1}{n} + |M|_g \frac{1}{n}).$$

\[\square\]

**Appendix A. Proof of Theorem 2.2**

**Proof of Theorem 2.2** We follows the steps given by Glynn-Adey and Liokumovich in \[7\], where they proved this theorem for $N = M$. Here we give the outline and point out some necessary modifications.

Suppose that $N$ has smooth boundary. For any $\epsilon_0 \in (0, 1)$, take $\tilde{r}(M, N, \epsilon_0)$ such that:

- for every $x \in \partial N$, we have that $B_r(x)$ is $(1 + \epsilon_0)$-bilipschitz diffeomorphic to the Euclidean ball of radius $r$ and $B_r(x) \cap N$ is mapped onto a half-ball under the diffeomorphism. Denote by $B_r^+(x) = B_r(x) \cap N$;
- the monotonicity formula \[12\] Theorem 3.4] holds.
From now on, we fix some $\epsilon_0 < 1$.

**Step 1:** Suppose that $N$ has smooth boundary. There exists $\epsilon = \epsilon(M,N,\bar{r})$ satisfying the following: for any domain $D \subset N$ with $|D| < \epsilon$, there exists a collection of domains $D(=:D_0) \supset D_1 \supset D_2 \supset \cdots \supset D_m$ satisfying

- $D_m \subset \text{Int}N$;
- $|\partial D_j \cap \text{Int}N| \geq |\partial D_{j+1} \cap \text{Int}N|$ for $0 \leq j \leq m - 1$;
- for $0 \leq j \leq m - 1$, $D_j \setminus D_{j+1}$ is contained in some ball of radius $\bar{r}$ and center $x \in N$;

**Proof of Step 1.** Suppose that $x \in \partial D_j \cap \partial N$, now we construct $D_{j+1} \subset D_j$. By the co-area formula, we can find $r' \in (3\bar{r}/4, \bar{r}/4)$ such that $\partial D_j \cap \text{Int}N$ is transverse to $\partial B_{r'}(x)$ and

$$|D_j \cap \partial B_{r'}(x)| \leq (8/\bar{r}) \cdot |D_j \cap B_{r'}(x)|.$$  

Denote by $S = [D_j \cap \partial B_{r'}(x)]$. Let $T$ be the minimizing current $T$ among all $T' \in \mathcal{Z}_{n-1}(B_{\bar{r}'}(x), \partial B_{\bar{r}'}(x); \mathbb{R})$ with $\text{spt}(\partial T' - \partial S) \subset \partial N$. Then by the regularity theory [28, Theorem 4.7] (see also [11, Theorem 4.7]), $T$ is induced by a free boundary hypersurface $\Sigma$ with $(n-8)$-dimensional singular set. By taking $\epsilon$ small enough, from the monotonicity formula [12, Theorem 3.4], $\Sigma \cap \partial N \cap B_{\bar{r}/2}(x) = \emptyset$. Using the monotonicity formula again, $\Sigma \cap B_{\bar{r}/4}(x) = \emptyset$. Note that by the isoperimetric choice [18], there exists $V \subset B_{\bar{r}'}(x)$ such that $\partial [V] = T - S$ and the volume of $V$ is small. Hence $V$ does not contain $B_{\bar{r}/4}^+(x)$. Together with the fact of $\partial V$ does not intersect $B_{\bar{r}/4}^+(x)$, we conclude that $V \cap B_{\bar{r}/4}^+(x) = \emptyset$.

Now we define

$$D_{j+1} = D_j \cap (N \setminus (B_{\bar{r}'}(x) \setminus V)).$$

Clearly, $D_j \setminus D_{j+1}$ is contained in $B_{\bar{r}'}^+(x)$. Note that $T$ is minimizing in $B_{\bar{r}'}^+(x)$. Then it is minimizing in $B_{\bar{r}'}^+(x) \setminus V$, i.e.

$$|\Sigma \cap D_j| \leq |\partial D_j \cap \text{Int}(B_{\bar{r}'}^+(x) \setminus V)|.$$

This implies

$$|\partial D_j \cap \text{Int}N| - |\partial D_{j+1} \cap \text{Int}N| = |\partial D_j \cap \text{Int}(B_{\bar{r}'}^+(x) \setminus V)| - |\Sigma \cap D_j| \geq 0.$$  

Thus Step 1 is completed. \hfill \Box

**Step 2:** Suppose that $N$ has smooth boundary. There exist constants $\beta_1 = \beta_1(n)$ and $\epsilon = \epsilon(M,N,\bar{r})$ such that for any domain $D \subset N$ with $|D| \leq \epsilon$, the following bound holds:

\[ \omega_1(D,g) \leq \beta_1 |D| \frac{n-1}{n} + |\partial D \cap \text{Int}N|. \]  

(A.1)

**Proof of Step 2.** Let $\{D_j\}_{j=1}^m$ be the domains constructed in Step 1. Then repeating the process inside $N$ (see also [7, Proposition 4.3]), there exists $D_m \supset D_{m+1} \supset \cdots \supset D_L$ such that

- $|\partial D_j \cap \text{Int}N| \geq |\partial D_{j+1} \cap \text{Int}N|$ for $m \leq j \leq L - 1$;
- for $m \leq j \leq L$, $D_j \setminus D_{j+1}$ is contained in some ball of radius $\bar{r}$ and center $x \in N$, where $D_{L+1} = \emptyset$;
By [13], there exists $\beta_1 = \beta_1(n)$ such that for $0 \leq j \leq L$,
\begin{equation}
\omega_1(D_j \setminus D_{j+1}, g) \leq \beta_1 |D_j \setminus D_{j+1}|^{\frac{n-1}{n}}.
\end{equation}

Now let $\Phi_j$ be a sweepout of $D_j \setminus D_{j+1}$ having no concentration of mass. Then there exist lifting maps $\tilde{\Phi}_j : [0, 1] \rightarrow \mathcal{C}(D_j \setminus D_{j+1})$ such that
\[ \partial \circ \tilde{\Phi}_j = \Phi_j \quad \text{for} \quad 0 \leq j \leq L. \]

Without loss of generality, we assume that $\tilde{\Phi}_j(0) = 0$, $\tilde{\Phi}_j(1) = [D_j \setminus D_{j+1}]$. By [7, Proposition 2.3], we can construct a sweepout of $D$ as follows: we first define $\tilde{\Phi} : [0, 1] \rightarrow \mathcal{C}(D)$ by
\[ \tilde{\Phi}(t) = \tilde{\Phi}_{L-j}((L+1)(t - \frac{j}{L+1})) + [D_{L+1-j}] \quad \text{for} \quad \frac{j}{L+1} \leq t \leq \frac{j+1}{L+1}. \]

Then $\Phi = \partial \circ \tilde{\Phi}$ is the desired sweepout, which has no concentration of mass. Such a construction gives that
\[ \omega_1(D, g) \leq \max_{0 \leq j \leq L} \{ \omega_1(D_j \setminus D_{j+1}, g) + |\partial D_j \setminus \partial D| \}. \]

Together with (A.2), we have
\[ \omega_1(D, g) \leq \beta_1 |D|^{\frac{n-1}{n}} + |\partial D \cap N|. \]

\[ \square \]

**Step 3:** Suppose that $N$ has smooth boundary. There exists $\beta_2 = \beta_2(n)$ such that for any domain $D \subset N$, the following bound holds
\begin{equation}
\omega_1(D, g) \leq \beta_2 \cdot (1 + |D|_{\beta_1(n)}) |D|^{\frac{n-1}{n}} + 2|\partial D \cap \text{Int} N|.
\end{equation}

**Proof of Step 3.** We use the argument in [7, Theorem 5.1]. Let $\epsilon_1 = 25^{-n} \cdot \epsilon$. Take $\beta_2(n) = \beta_1(n) + 3c(n) \cdot \left[1 - (1 - 25^{-n})^{\frac{n-1}{n}}\right]$. Here $c(n)$ is the constant in [7, Lemma 3.4]. It follows that
\begin{equation}
\left[1 - (1 - 25^{-n})^{\frac{n-1}{n}}\right] \beta_2(n) \geq 3c(n).
\end{equation}

By Step 2, for $k \leq 25^n$, (A.3) holds for $D$ with $|D| \leq k\epsilon_1$. We proceed by induction on $k$.

Suppose the inequality holds for compact domains with volume at most $k\epsilon$. Then for any $D \subset N$ with $k\epsilon_1 < |D| \leq (k+1)\epsilon_1$. By Theorem 4.1, there exists a hypersurface $\Sigma$ subdividing $D$ into $D_0$ and $D_1$ such that $|D_j| \leq (1 - 25^{-n})|D|$ (for $j = 0, 1$) and
\begin{equation}
|\Sigma| \leq c(n) |D|^{\frac{n-1}{n}} (1 + |D|_{\beta_1(n)}).
\end{equation}

Then using the construction of sweepouts in Step 2, we have
\begin{equation}
\omega_1(D, g) \leq \max_{j \in \{0, 1\}} \{ \omega_1(D_j, g) + |\partial D_j \setminus \partial D| \}.
\end{equation}

Note that for $j = 0, 1$,
\[ |D_j| \leq (1 - 25^{-n})|D| \leq |D| - \epsilon_1 < (k+1)\epsilon_1 - \epsilon_1 < k\epsilon_1. \]
Hence by the assumption,
\[ \omega_1(D_j, g) \leq \beta_2 \cdot (1 + |D_j|_{g_0})|D_j|^{\frac{n-1}{n}} + 2|\partial D_j \cap \Int N| \]
\[ \leq \beta_2 \cdot (1 + |D_j|_{g_0})|D_j|^{\frac{n-1}{n}} \cdot (1 - 25^{-\frac{n-1}{n}}) + 2|\partial D \cap \Int N| + 2|\Sigma| \]
\[ \leq (\beta_2 - 3c(n))(1 + |D_j|_{g_0})|D_j|^{\frac{n-1}{n}} + 2|\partial D \cap \Int N| + 2|\Sigma| \]
\[ \leq \beta_2 \cdot (1 + |D_j|_{g_0})|D_j|^{\frac{n-1}{n}} + 2|\partial D \cap \Int N| - |\Sigma|. \]

Here the third inequality is from (A.4) and we used (A.5) in the last one. Then together with (A.6), we conclude that
\[ \omega_1(D, g) \leq \beta_2 \cdot (1 + |D|_{g_0})|D|^{\frac{n-1}{n}} + 2|\partial D \cap \Int N|. \]

This finishes Step 3.

\[ \square \]

**Step 4:** We prove the theorem for general compact domain \( N \) (having piecewise smooth boundary).

**Proof of Step 4.** Now let \( N \) be a compact domain with piecewise smooth boundary. Then we have a tubular neighborhood \( U \) of \( N \) such that \( U \) has smooth boundary and \( |U|_{g_0} \leq 2|N|_{g_0} \) and \( |U| \leq 2|N| \). Then by Step 3,
\[ \omega_1(U, g) \leq \beta_2 \cdot (1 + |U|_{g_0})|U|^{\frac{n-1}{n}} \leq 2\beta_2 \cdot (1 + |N|_{g_0})|N|^{\frac{n-1}{n}}. \]

Then the desired inequality follows from
\[ \omega_1(N, g) \leq \omega_1(U, g) \]
if we take \( K = 2\beta_2(n) \).

So far, Theorem 2.2 is proved.

\[ \square \]

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