Strong Convergence of a Stochastic Rosenbrock-type Scheme for the Finite Element Discretization of Semilinear SPDEs Driven by Multiplicative and Additive Noise

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Abstract This paper aims to investigate the numerical approximation of a general second order parabolic stochastic partial differential equation (SPDE) driven by multiplicative and additive noise. Our main interest is on such SPDEs where the nonlinear part is stronger than the linear part, usually called stochastic dominated transport equations. Most standard numerical schemes lose their good stability properties on such equations, including the current linear implicit Euler method. We discretise the SPDE in space by the finite element method and propose a new scheme in time appropriate for such equations, called stochastic Rosenbrock-Type scheme, which is based on the local linearisation of the semi-discrete problem obtained after space discretisation. We provide a strong convergence of the new fully discrete scheme toward the exact solution for multiplicative and additive noise. Our convergence rates are in agreement with results in the literature. Numerical experiments to sustain our theoretical results are provided.

Keywords Rosenbrock-type scheme · Stochastic partial differential equations · Multiplicative & Additive noise · Strong convergence · Finite element method.

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1 Introduction

We consider numerical approximation of SPDE defined in $A \subset \mathbb{R}^d$, $d = 1, 2, 3$, with initial value and boundary conditions of the following type

$$dX(t) + [AX(t) + F(X(t))]dt = B(X(t))dW(t), \quad X(0) = X_0,$$

for all $t \in [0, T]$, on the Hilbert space $L^2(A)$, $T > 0$ the final time. $F$ and $B$ are nonlinear functions, $X_0$ is the initial data which is random, $A$ is a linear operator, unbounded, not necessary self-adjoint, and $-A$ is assumed to be a generator of a contraction semigroup $S(t) := e^{-tA}$, $t \geq 0$. The noise $W(t) = W(x, t)$ is a $Q$–Wiener process defined in a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$. The filtration is assumed to fulfill the usual conditions (see [34, Definition 2.1.11]). We assume that the noise can be represented as

$$W(x, t) = \sum_{i \in \mathbb{N}^d} \sqrt{q_i}e_i(x)\beta_i(t), \quad t \in [0, T],$$

where $q_i, e_i, i \in \mathbb{N}^d$ are respectively the eigenvalues and the eigenfunctions of the covariance operator $Q$, and $\beta_i$ are independent and identically distributed standard Brownian motions. Precise assumptions on $F$, $B$, $X_0$ and $A$ will be given in the next section to ensure the existence of the unique mild solution $X$ of (1), which has the following representation (see [32, 33, 34])

$$X(t) = S(t)X_0 - \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s),$$

for all $t \in [0, T]$. Equations of type (1) are used to model different real world phenomena in different fields such as biology, chemistry, physics etc [5, 35, 36].

In many cases explicit solutions of SPDEs are unknown, therefore numerical approximations are the only tool appropriate to approach them. Numerical approximation of SPDE of type (1) is therefore an active research area and has attracted a lot of attentions since two decades, see e.g. [11, 12, 16, 18, 22, 33, 35, 36, 43–46] and references therein. Due to the time step restriction of the explicit Euler method, linear implicit Euler method is used in many situations. Linear implicit Euler method has been investigated in the literature, see e.g. [18, 23, 40, 44] and the references therein. The resolvent operator $(I + \Delta tA_h)^{-1}$ plays a key role to stabilise the linear implicit Euler method, where $A_h$ is the discrete form of $A$ after space discretization. Such approach is justified when the linear operator $A$ is strong. Indeed, when $A$ is stronger than $F$, the operator $A$ drives the SPDE (1) and the good stability properties of the linear implicit Euler method and exponential integrators are guaranteed. In more concrete applications, the nonlinear function $F$ can be stronger. Typical examples are stochastic reaction equations with stiff reaction term.
For such equations, both linear implicit Euler method \cite{18,24,10,14} and exponential integrators \cite{11,22,43} behave like the standard explicit Euler method (see Section 2.3) and therefore lose their good stabilities properties. For such problems in the deterministic context, Exponential Rosenbrock-type \cite{10,39} methods and Rosenbrock-type \cite{28,29,39} methods were proved to be efficient. Recently, the exponential Rosenbrock method was extended to the case of stochastic partial differential equations \cite{26} and was proved to be very stable for stochastic reactive dominated transport equations. Since solving linear systems are more straightforward than computing the exponential of a matrix, it is important to develop alternative methods based on the resolution of linear systems, which may be more efficient if the appropriate preconditioners are used. In this paper, we propose a new scheme based on the combination of the Rosenbrock-type method and the linear implicit Euler method. The resulting numerical scheme is called Stochastic Rosenbrock-type scheme (SROS). The space discretization is done with the finite element method and the new scheme is based on the local linearization of the nonlinear part of the semi-discrete problem in space. The local linearization therefore weakens the nonlinear part of the drift function such that the linearized semi-discrete problem is driven by its new linear part, which changes at each time step. The standard linear implicit Euler method \cite{18,44} is applied at the end to the linearized semi-discrete problem. This combination gives our new scheme SROS. We analyze the strong convergence of the new fully discrete scheme toward the exact solution in the $L^2$-norm. The challenge here is that the resolvent of the operator $S_{h,\Delta t}(\omega)$ appearing in the numerical scheme \cite{11} is not constant as it changes at each time step. Furthermore the operator $S_{h,\Delta t}(\omega)$ is a random operator. We use some tools from \cite{26} and furthermore, we provide in Section 3.1 some stability estimates to handle the composition of the perturbed random resolvent operators, useful in our convergence analysis. The results indicate how the convergence orders depend on the regularity of the initial data and the noise. More precisely, we achieve the optimal convergence orders $O\left(h^{\beta} + \Delta t^{\min(\beta,1)/2}\right)$ for multiplicative noise and the optimal convergence orders $O\left(h^{\beta} + \Delta t^{3/2-\epsilon}\right)$ for additive noise, where $\beta$ is the regularity’s parameter of the noise (see Assumption 2) and $\epsilon > 0$ is an arbitrary number small enough.

The rest of this paper is organized as follows. Section 2 deals with the well posedness problem, the numerical scheme and the main results. In section 3 we provide some error estimates for the deterministic homogeneous problem as preparatory results and then we provide the proof of the main results. Section 4 provides some numerical experiments to sustain the theoretical findings.

2 Mathematical setting and main results

2.1 Main assumptions and well posedness problem

Let us define functional spaces, norms and notations that will be used in the rest of the paper. Let $(H, (.,.)_H, \|\|)$ be a separable Hilbert space. For all
$p \geq 2$ and for a Banach space $U$, we denote by $L^p(\Omega, U)$ the Banach space of all equivalence classes of $p$ integrable $U$-valued random variables. We denote by $L(U, H)$ the space of bounded linear mappings from $U$ to $H$ endowed with the usual operator norm $\| \cdot \|_{L(U, H)}$. By $\mathcal{L}_2(U, H) := HS(U, H)$, we denote the space of Hilbert-Schmidt operators from $U$ to $H$. We equip $\mathcal{L}_2(U, H)$ with the norm

$$\|l\|_{\mathcal{L}_2(U, H)} := \left( \sum_{i=1}^{\infty} \|l\psi_i\|^2 \right)^{1/2}, \quad l \in L_2(U, H),$$

where $(\psi_i)_{i=1}^{\infty}$ is an orthonormal basis of $U$. Note that (4) is independent of the orthonormal basis of $U$. For simplicity we use the notations $L(U, U) := L(U)$ and $L_2(U, U) := \mathcal{L}_2(U)$. It is well known that for all $l \in L(U, H)$ and $l_1 \in L_2(U)$, $ll_1 \in L_2(U, H)$ and $\|ll_1\|_{L_2(U, H)} \leq \|l\|_{L(U, H)} \|l_1\|_{L_2(U)}$. (5)

We assume that the covariance operator $Q : H \rightarrow H$ is positive and self-adjoint. Throughout this paper $W(t)$ is a $Q$-wiener process. The space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to $H$ is denoted by $L_2^0 := \mathcal{L}_2(Q^{1/2}(H), H) = HS(Q^{1/2}(H), H)$ with the corresponding norm $\| \cdot \|_{L_2^0}$ defined by

$$\|l\|_{L_2^0} := \|lQ^{1/2}\|_{HS} = \left( \sum_{i=1}^{\infty} \|lQ^{1/2}e_i\|^2 \right)^{1/2}, \quad l \in L_2^0,$$

where $(e_i)_{i=1}^{\infty}$ is an orthonormal basis of $H$. Note that (6) is independent of the orthonormal basis of $H$. In the rest of this paper, we take $H = L^2(A)$.

In order to ensure the existence and the uniqueness of solution of (1) and for the purpose of the convergence analysis, we make the following assumptions.

Assumption 1 [Linear operator A] $-A : \mathcal{D}(A) \subset H \rightarrow H$ is a generator of an analytic semigroup $S(t) = e^{-At}$.

Assumption 2 [Initial value $X_0$] Let $p \in [2, \infty)$. We assume that $X_0 \in L^p(\Omega, \mathcal{D}((A)^{\beta/2}))$, $0 < \beta \leq 2$.

As in the current literature for deterministic Rosenbrock-type methods, [28, 29] deterministic exponential Rosenbrock-type method [10, 27] and stochastic exponential Rosenbrock-type methods [26], we make the following assumption on the nonlinear drift term.

Assumption 3 [Nonlinear term $F$] We assume the nonlinear map $F : H \rightarrow H$ to be Fréchet differentiable with bounded derivative, i.e. there exists a constant $b > 0$ such that

$$\|F'(u)\|_{L(H)} \leq b, \quad u \in H.$$  (7)

Moreover, as in [28] Page 6 for deterministic Rosenbrock-type method, we assume that the resolvent set of $-A - F'(u)$ contains $[0, \infty]$ for all $u \in H$. 


Remark 1 Inequality (7) together with the mean value theorem show that there exists a constant $L > 0$ such that
\[ \|F(u) - F(v)\| \leq L\|u - v\|, \quad u, v \in H. \] (8)

As a consequence of (8), there exists a positive constant $C$ such that
\[ \|F(u)\| \leq \|F(0)\| + \|F(u) - F(0)\| \leq \|F(0)\| + L\|u\| \leq C(1 + \|u\|), \quad u \in H. \]

Remark 2 An illustrative example for which the resolvent set of $-A - F'(u)$ contains $[0, \infty[$ is obtained if we assume $A$ to be the generator of a contraction semigroup and the derivative of the nonlinear drift term $F$ to satisfy the following coercivity condition
\[ \langle F'(u)v, v \rangle_H \geq 0, \quad u, v \in H. \] (9)

In fact, it follows from (10) that $-F'(u)$ is an relatively $A$-bounded [1] Chapter III, Definition 2.1] and dissipative operator with $A$-bound $a_0 = 0$. Therefore, from [1] Chapter III, Theorem 2.7], it follows that $-A - F'(u)$ is a generator of a contraction semigroup. Hence $[0, \infty[ \subset \rho(-A - F'(u))$, for all $u \in H$.

Remark 3 The assumption that $[0, \infty[ \subset \rho(-A - F'(u))$ can be relaxed, but the drawback is that the resolvent set of the perturbed semigroup is smaller than that of the initial semigroup.

Following [22] Chapter 7] or [13][22][45] we make the following assumption on the diffusion term.

Assumption 4 [Diffusion term] We assume that the operator $B : H \to L^2_0$ satisfies the global Lipschitz condition, i.e. there exists a positive constant $C$ such that
\[ \|B(0)\|_{L^2_0} \leq C, \quad \|B(u) - B(v)\|_{L^2_0} \leq C\|u - v\|, \quad u, v \in H. \] (10)

As a consequence, it holds that
\[ \|B(u)\|_{L^2_0} \leq \|B(0)\|_{L^2_0} + \|B(u) - B(0)\|_{L^2_0} \leq \|B(0)\|_{L^2_0} + C\|u\| \leq L(1 + \|u\|), \quad u \in H. \] (11)

We equip $V_\alpha := D(A^{\alpha/2})$, $\alpha \in \mathbb{R}$ with the norm $\|v\|_\alpha := \|A^{\alpha/2}v\|$, for all $v \in H$. It is well known that $(V_\alpha, \|\cdot\|_\alpha)$ is a Banach space [9].

To establish our $L^p$ strong convergence result when dealing with multiplicative noise, we will also need the following further assumption on the diffusion term when $\beta \in [1, 2)$, which was also used in [13][19] to achieve optimal regularity order and in [13][22][26] to achieve optimal convergence order in space and time.

Assumption 5 We assume that there exists a positive constant $c > 0$ such that $B(D(A^{(\beta-1)/2})) \subset HS(Q^{1/2}(H), D(A^{(\beta-1)/2}))$ and $\|A^{(\beta-1)/2}B(v)\|_{L^2_0} \leq c(1 + \|v\|_{\beta-1})$ for all $v \in D(A^{(\beta-1)/2})$, where $\beta$ comes from Assumption [22].
Typical examples which fulfill Assumption 5 are stochastic reaction diffusion equations (see [13, Section 4]).

When dealing with additive noise (i.e. when $B = I$), the strong convergence proof will make use of the following assumption, also used in [20, 43, 44].

**Assumption 6** We assume that the covariance operator $Q$ satisfies the following estimate

$$\left\| A^{\frac{\beta}{2}} Q A^{\frac{1}{2}} \right\|_{L^2(H)} < C,$$

(12)

where $\beta$ comes from Assumption 2.

When dealing with additive noise, to achieve higher convergence order in time, we assume that the nonlinear function satisfies the following assumption, also used in [26, 43, 44].

**Assumption 7** The deterministic mapping $F : H \rightarrow H$ is twice differentiable and there exist two constants $L \geq 0$ and $\eta \in (0, 2)$ such that

$$\| F'(u)v \| \leq L \| v \|, \quad u, v \in H,$$

(13)

$$\| F''(u)(v_1, v_2) \| \leq L \| v_1 \| \| v_2 \|, \quad u, v_1, v_2 \in H.$$

(14)

Let us recall the following proposition which provides some smooth properties of the semigroup $S(t)$ generated by $-A$, that will be useful in the rest of the paper.

**Proposition 1** [Smoothing properties of the semigroup] [9] Let $\alpha > 0$, $\delta \geq 0$ and $0 \leq \gamma \leq 1$, then there exists a constant $C > 0$ such that

$$\| A^\delta S(t) \|_{L(H)} \leq Ct^{-\delta}, \quad t > 0,$$

$$\| A^{-\gamma} (I - S(t)) \|_{L(H)} \leq C t^\gamma, \quad t \geq 0$$

$$A^\delta S(t) = S(t)A^\delta, \quad \text{on} \quad D(A^\delta)$$

$$\| D_1^l S(t) v \|_\delta \leq C t^{-l-(\delta - \alpha)/2} \| v \|_\alpha, \quad t > 0, \quad v \in D(A^\alpha),$$

where $l = 0, 1$, and $D_1^l = \frac{d^l}{dt^l}$.

If $\delta \geq \gamma$ then $D(A^\delta) \subset D(A^\gamma)$.

**Theorem 8** [32, Theorem 7.2]

Let Assumption 1, Assumption 3 and Assumption 4 be satisfied. If $X_0$ is a $\mathcal{F}_0$-measurable $H$ valued random variable, then there exists a unique mild solution $X$ of the problem (4) represented by (5) and satisfying

$$\mathbb{P} \left[ \int_0^T \| X(s) \|^2 ds < \infty \right] = 1,$$

and for any $p \geq 2$, there exists a constant $C = C(p, T) > 0$ such that

$$\sup_{t \in [0, T]} \mathbb{E} \| X(t) \|^p \leq C (1 + \mathbb{E} \| X_0 \|^p).$$
2.2 Finite element discretization

In the rest of the paper, to simplify the presentation, we consider the linear operator $A$ to be of second-order. More precisely, we consider the SPDE to be a second-order semilinear parabolic which takes the following form

$$
\begin{align*}
dX(t, x) + [-\nabla \cdot (D \nabla X(t, x)) + q \cdot \nabla X(t, x)]dt \\
+ f(x, X(t, x))dt = b(x, X(t, x))dW(t, x), \quad x \in A, \quad t \in [0, T],
\end{align*}
$$

where the functions $f : \Lambda \times \mathbb{R} \to \mathbb{R}$ and $b : \Lambda \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable with globally bounded derivatives. In the abstract framework, the linear operator $A$ takes the form

$$
Au = -\sum_{i,j=1}^{d} \partial \frac{\partial u}{\partial x_i} \left( D_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{d} q_i(x) \frac{\partial u}{\partial x_i},
$$

where $D_{ij} \in L^\infty(\Lambda)$, $q_i \in L^\infty(\Lambda)$. We assume that there exists a positive constant $c_1 > 0$ such that

$$
\sum_{i,j=1}^{d} D_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad x \in \Omega.
$$

The functions $F : H \to H$ and $B : H \to HS(Q^{1/2}(H), H)$ are defined by

$$
(F(v))(x) = f(x, v(x)) \quad \text{and} \quad (B(v)u)(x) = b(x, v(x)) \cdot u(x),
$$

for all $x \in \Lambda$, $v \in H$, $u \in Q^{1/2}(H)$, with $H = L^2(\Lambda)$. For an appropriate family of eigenfunctions $(\varepsilon_i)$ such that $\sup_{i \in \mathbb{N}_d} \sup_{x \in \Lambda} \|\varepsilon_i(x)\| < \infty$, it is well known that the Nemystskii operator $F$ related to $f$ and the multiplication operator $B$ defined in (18) satisfy Assumption 3, Assumption 4 and Assumption 5. As in [4,22] we introduce two spaces $\mathbb{H}$ and $V$, such that $\mathbb{H} \subset V$, the two spaces depend on the boundary conditions and the domain of the operator $A$. For Dirichlet (or first-type) boundary conditions we take

$$
V = \mathbb{H} = H^1_0(\Lambda) = \{ v \in H^1(\Lambda) : v = 0 \quad \text{on} \quad \partial \Lambda \}.
$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition, we take $V = H^1(\Lambda)$

$$
\mathbb{H} = \{ v \in H^2(\Lambda) : \partial v/\partial v_A + \alpha_0 v = 0, \quad \text{on} \quad \partial \Lambda \}, \quad \alpha_0 \in \mathbb{R},
$$

where $\partial v/\partial v_A$ is the normal derivative of $v$ and $v_A$ is the exterior pointing normal $n = (n_i)$ to the boundary of $A$, given by

$$
\partial v/\partial v_A = \sum_{i,j=1}^{d} n_i(x)D_{ij}(x)\frac{\partial v}{\partial x_j}, \quad x \in \partial A.
$$
Using the Green’s formula and the boundary conditions, the corresponding bilinear form associated to $A$ is given by

$$a(u, v) = \int_{A} \left( \sum_{i,j=1}^{d} D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} q_i \frac{\partial u}{\partial x_i} v \right) \, dx, \quad u, v \in V,$$

for Dirichlet and Neumann boundary conditions, and

$$a(u, v) = \int_{A} \left( \sum_{i,j=1}^{d} D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} q_i \frac{\partial u}{\partial x_i} v \right) \, dx + \int_{\partial A} \alpha_0 uv \, dx, \quad u, v \in V,$$

for Robin boundary conditions. Using the Gårding’s inequality (see e.g. [36]), it holds that there exist two constants $c_0$ and $\lambda_0$ such that

$$a(v, v) \geq \lambda_0 \|v\|^2_{H^1(A)} - c_0 \|v\|^2, \quad \forall v \in V.$$  \hfill (19)

By adding and subtracting $c_0 X dt$ in both sides of (1), we have a new linear operator still denoted by $A$, and the corresponding bilinear form is also still denoted by $a$. Therefore, the following coercivity property holds

$$a(v, v) \geq \lambda_0 \|v\|^2, \quad v \in V.$$  \hfill (20)

Note that the expression of the nonlinear term $F$ has changed as we included the term $c_0 X$ in a new nonlinear term that we still denote by $F$. The coercivity property (20) implies that $-A$ is sectorial on $L^2(A)$, i.e. there exist $C_1, \theta \in (\frac{1}{2}, \pi)$ such that

$$\| (\lambda I + A)^{-1} \|_{L(L^2(A))} \leq \frac{C_1}{|\lambda|}, \quad \lambda \in S_\theta,$$  \hfill (21)

where $S_\theta = \{ \lambda \in \mathbb{C} : \lambda = \rho e^{i \phi}, \rho > 0, 0 \leq |\phi| \leq \theta \}$ (see [9]). The coercivity property (20) also implies that $A$ is a positive operator and its fractional powers are well defined for any $\alpha > 0$, by

$$A^\alpha = \begin{cases} A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tA} dt, \\ A^\alpha = (A^{-\alpha})^{-1}, \end{cases}$$  \hfill (23)

where $\Gamma(\alpha)$ is the Gamma function (see [9]). Let us now turn to the space discretization of our problem (1). We start by splitting the domain $A$ in finite triangles. Let $T_h$ be the triangulation with maximal length $h$ satisfying the usual regularity assumptions, and $V_h \subset V$ be the space of continuous functions
that are piecewise linear over the triangulation $T_h$. We consider the projection $P_h$ from $H = L^2(A)$ to $V_h$ defined for every $u \in H$ by
\[
\langle P_h u, \chi \rangle_H = \langle u, \chi \rangle_H, \quad \forall \chi \in V_h.
\]  
(24)

The discrete operator $A_h : V_h \rightarrow V_h$ is defined by
\[
\langle A_h \phi, \chi \rangle_H = \langle A \phi, \chi \rangle_H = a(\phi, \chi), \quad \forall \phi, \chi \in V_h,
\]  
(25)

Like $-A$, $-A_h$ is also a generator of a bounded analytic semigroup $S_h(t) := e^{-tA_h}$. Let $K$ be a constant satisfying
\[
\|S_h(t)\|_{L(H)} \leq K, \quad t \geq 0.
\]  
(26)

As any semigroup and its generator, $-A_h$ and $S_h(t)$ satisfy the smoothing properties of Proposition 1 with a uniform constant $C$ (i.e. independent of $h$). Following [24,21,22], we characterize the domain of the operator $A^{h/2}$, $1 \leq k \leq 2$ as follows:
\[
\mathcal{D}(A^{h/2}) = \mathbb{H} \cap H^{k}(A), \quad \text{(for Dirichlet boundary conditions),}
\]
\[
\mathcal{D}(A) = \mathbb{H}, \quad \mathcal{D}(A^{1/2}) = H^{1}(A), \quad \text{(for Robin boundary conditions).}
\]

The semi-discrete version of problem (1) consists to find $X^h(t) \in V_h$, $t \in (0,T]$ such that $X^h(0) = P_h X_0$ and
\[
dX^h(t) + [A_h X^h(t) + P_h F(X^h(t))]dt = P_h B(X^h(t))dW(t), \quad t \in (0,T].
\]  
(27)

The following lemma can be found in [26] Lemma 4 & Lemma 5. It provides the space and time regularities of the mild solution $X^h(t)$ of (28).

**Lemma 1** (i) For multiplicative noise, let Assumption 4 with $\beta \in [0,1)$, Assumption 5 and Assumption 6 be fulfilled, then the following estimates hold
\[
\left\|A^{\beta/2} X^h(t)\right\|_{L^p(\Omega,H)} \leq C \left( 1 + \left\|A^{\beta/2} X_0\right\|_{L^p(\Omega,H)} \right), \quad t \in [0,T],
\]  
(28)
\[
\left\|X^h(t_2) - X^h(t_1)\right\|_{L^p(\Omega,H)} \leq C|t_2 - t_1|^{\beta/2} \left( 1 + \left\|A^{\beta/2} X_0\right\|_{L^p(\Omega,H)} \right),
\]  
(29)

for all $t_1, t_2 \in [0,T]$. Moreover, if $\beta \in [1,2]$ and if Assumption 5 is fulfilled, then
\[
\left\|X^h(t_2) - X^h(t_1)\right\|_{L^p(\Omega,H)} \leq C|t_2 - t_1|^{\beta/2} \left( 1 + \left\|A^{\beta/2} X_0\right\|_{L^p(\Omega,H)} \right),
\]  
(30)

for all $t_1, t_2 \in [0,T]$.

(ii) For additive noise, let Assumption 4, Assumption 5, Assumption 6 and Assumption 7 be fulfilled with $\beta \in [0,2)$, then the following estimates hold
\[
\left\|A^{\beta/2} X^h(t)\right\|_{L^p(\Omega,H)} \leq C \left( 1 + \left\|A^{\beta/2} X_0\right\|_{L^p(\Omega,H)} \right), \quad t \in [0,T],
\]  
(31)
\[
\left\|X^h(t_2) - X^h(t_1)\right\|_{L^p(\Omega,H)} \leq C|t_2 - t_1|^{\min(\beta,1)/2} \left( 1 + \left\|A^{\beta/2} X_0\right\|_{L^p(\Omega,H)} \right),
\]  
(32)

for all $t_1, t_2 \in [0,T]$.

Here $C$ is a positive constant, independent of $h$, $t$, $t_1$ and $t_2$. 
Corollary 1

As a consequence of Lemma 1, it holds that
\[
\|X^h(t)\|_{L^2(\Omega,H)} \leq C, \quad \|F\left(X^h(t)\right)\|_{L^2(\Omega,H)} \leq C, \quad \|B\left(X^h(t)\right)\|_{L^2(\Omega,H)} \leq C,
\]
for all \( t \in [0,T] \).

2.3 Standard linear implicit Euler method and stability properties

Let us recall that the linear implicit Euler scheme applied to the semi-discrete problem (27) leads to

\[
Z^h_{m+1} = S^h_{\Delta t}Z^h_m + \Delta t S^h_{\Delta t}P_h F(Z^h_m) + S^h_{\Delta t}P_h B(Z^h_m),
\]

\[ S^h_{\Delta t} = (I + \Delta t A_h)^{-1}, \quad Z^h_0 = P_h X_0. \] (33)

If the linear operator \( A \) tends to the null operator, its corresponding discrete operator \( A_h \) tends to the null operator and \( S^h_{\Delta t} \) tends to the identity operator \( I \). In this case, the numerical scheme (33) and the standard exponential integrator method [22] become the unstable Euler-Maruyama scheme. See also [26, Section 2.3] for more details. For a simple illustration of the stability properties of such problems, let us consider the following deterministic linear differential equation

\[
y' = ay + cy, \quad a > 0, c < 0, \quad \text{such that} \quad c < -a. \] (35)

The linear implicit Euler method applied to (35) by considering \( F(y) = cy \) as the nonlinear part reads as

\[
y_{n+1} = \frac{1 + c\Delta t}{1 - a\Delta t} y_n, \quad n \geq 0.
\] (36)

The numerical scheme (36) is stable [30,39] if and only if \( \Delta t < \frac{1}{2a} \). Note that when \( a \) is small enough and \( |c| \) large enough, the numerical scheme (36) becomes the explicit Euler method and the stability region becomes very small. The Rosenbrock-type methods were proved to be efficient and were studied in [7,8,30] for ordinary differential equations. Applying the Rosenbrock-Euler method to the linear problem (35) reads as

\[
y_{n+1} = \frac{1}{1 - (a + c)\Delta t} y_n, \quad n \geq 0.
\] (37)

Note that (37) coincides with the full implicit method with \( F(y) = cy \). The Rosenbrock-Euler method (37) is unconditionally stable (A-stable). This demonstrates the strong stability property of the Rosenbrock-type methods for stiff problems. Authors of [28,29] extended the Rosenbrock-type methods to parabolic partial differential equations and the methods were proved to be efficient for solving transport equations in porous media [39]. The case of stiff stochastic partial differential equations is not yet studied to the best of our knowledge and will be the aim of this paper.

1 Think for instance of the Laplace operator \( A = \alpha \Delta \), with \( \alpha \to 0 \)
2.4 Novel fully discrete scheme and main results

Let us build a more stable scheme, robust when the operator $A$ tends to null. For the time discretization, we consider the one-step method which provides the numerical approximated solution $X^h_m$ of $X^h(t_m)$ at discrete time $t_m = m\Delta t$, $m = 0, \cdots, M$. The method is based on the continuous linearization of (27). More precisely we linearize (27) at each time step as follows

$$dX^h(t) + [A_h, X^h(t)] + J^h_m X^h(t)dt = G^h_m (X^h(t)) dt + P_h B (X^h(t)) dW(t),$$

for all $t_m \leq t \leq t_{m+1}$, where $J^h_m$ is the Fréchet derivative of $P_h F$ at $X^h_m$ and $G^h_m$ is the remainder at $X^h_m$. Both $J^h_m$ and $G^h_m$ are random functions and are defined for all $\omega \in \Omega$ by

$$J^h_m(\omega) := (P_h F)' (X^h_m(\omega)) = P_h F'(X^h_m(\omega)),
$$

$$G^h_m(\omega)(X^h(t)) := -P_h F(X^h(t)) + J^h_m(\omega) X^h(t).$$

Applying the linear implict Euler method to (38) gives the following fully discrete scheme, called linear implicit Rosenbrock Euler method

$$\begin{cases}
X^h_0 = P_h X_0, \\
X^h_{m+1} = S^m_{h, \Delta t} X^h_m + \Delta t S^m_{h, \Delta t} G^h_m (X^h_m) + S^m_{h, \Delta t} P_h B (X^h_m) \Delta W_m,
\end{cases}
$$

where $\Delta W_m$ and $S^m_{h, \Delta t}$ are defined respectively by

$$\Delta W_m := W_{m+1} - W_m,
$$

$$S^m_{h, \Delta t}(\omega) := (I + \Delta t A_h, m(\omega))^{-1},$$

and

$$A_h, m(\omega) := A_h + J^h_m(\omega), \quad \omega \in \Omega.
$$

In the numerical scheme (41), the resolvent operator in (42) is random and changes at each time step. Having the numerical method (41) in hand, our goal is to analyze its strong convergence toward the exact solution in the $L^2$ norm for multiplicative and additive noise.

Throughout this paper we take $t_m = m \Delta t \in [0, T]$, where $T = M \Delta t$ for $m, M \in \mathbb{N}, m \leq M$, $T$ is fixed, $C$ is a generic constant that may change from one place to another but is independent of both $\Delta t$ and $h$. The main results of this paper are formulated in the following theorems.

**Theorem 9** Let $X(t_m)$ and $X^h_m$ be respectively the mild solution given by (3) and the numerical approximation given by (41) at $t_m = m \Delta t$. Let Assumption 7 with $p = 2$, Assumption 8 and Assumption 9 be fulfilled.

(i) If $0 < \beta < 1$, then the following error estimate holds

$$\|X(t_m) - X^h_m\|_{L^2(\Omega, H)} \leq C \left(h^\beta + \Delta t^{\beta/2}\right).
$$

(ii) If $1 \leq \beta \leq 2$ and if Assumption 9 is fulfilled, then the following error estimate holds

$$\|X(t_m) - X^h_m\|_{L^2(\Omega, H)} \leq C \left(h^\beta + \Delta t^{1/2}\right).$$
Theorem 10 When dealing with additive noise (i.e. when \( B = I \)), let Assumption 1, Assumption 3 with \( p = 4 \), Assumption 2 and Assumption 7 be fulfilled. Then the following error estimate holds for the mild solution \( X(t) \) of (1) and the numerical approximation (41)

\[
\|X(t_m) - X_h(t_m)\|_{L^2(\Omega, H)} \leq C \left( h^\beta + \Delta t^{3/2} \epsilon \right). \tag{44}
\]

3 Proof of the main results

The proofs of the main results require some preparatory results.

3.1 Preparatory results

For non commutative operators \( H_j \) on a Banach space, we introduce the following notation, which will be used in the rest of the paper.

\[
\prod_{j=l}^k H_j = \begin{cases} 
H_k H_{k-1} \cdots H_l, & \text{if } k \geq l, \\
I, & \text{if } k < l.
\end{cases}
\tag{45}
\]

Lemma 2 Let Assumption 2 be fulfilled. Then for all \( \omega \in \Omega \) the following estimate holds

\[
\left\| \left( \prod_{j=1}^m e^{\Delta t A_h, j(\omega)} \right) A_h^{\gamma} \right\|_{L(H)} \leq C t^{\gamma-1}, \quad 0 \leq t \leq m, \quad 0 \leq \gamma < 1. \tag{46}
\]

Proof See [26, Lemma 10].

Lemma 3 For all \( m \in \mathbb{N} \) and all \( \omega \in \Omega \), the random linear operator \( A_h + J_h^m(\omega) \) is the generator of an analytic semigroup \( S_h^m(\omega)(t) := e^{(A_h + J_h^m(\omega))t} \) called random (or stochastic) perturbed semigroup and uniformly bounded on \([0, T]\), i.e there exists a positive constant \( C_1 \) independent of \( h, m, \Delta t \) and the sample \( \omega \) such that

\[
\left\| e^{(A_h + J_h^m(\omega))t} \right\|_{L(H)} \leq Ke^{Kt}, \quad t \geq 0
\leq C_1, \quad 0 \leq t \leq T.
\]

Proof See [26, Lemma 5].

The following lemma is an analogous of [24, (3.31)], but here our semigroup is not constant. In fact it is random and further its changes at each time step.

Lemma 4 Let Assumption 1 and Assumption 3 be fulfilled.
(i) For all $\alpha \in [0, 1]$, $n > 1$ and all $\omega \in \Omega$, it holds that
\[
\|A_h^\alpha (I + tA_h,j(\omega))^{-n}\|_{L(H)} \leq C((n-1)t)^{-\alpha} \leq C(nt)^{-\alpha}, \quad t > 0. \tag{47}
\]

(ii) For all $\alpha \in [0, 1)$ and $\omega \in \Omega$, it holds that
\[
\|A_h^\alpha (I + tA_h,j(\omega))^{-1}\|_{L(H)} \leq Ct^{-\alpha}, \quad t > 0. \tag{48}
\]

(iii) For all $n \in \mathbb{N}$, it holds that
\[
\|A_h (I + tA_h,j(\omega))^{-n}\|_{L(H)} \leq C, \quad t > 0. \tag{49}
\]

Proof Note that for all $n \geq 2$, $\frac{1}{2}nt \leq (n-1)t$. Therefore $((n-1)t)^{-\alpha} \leq C(nt)^{-\alpha}$. It remains to prove the first part of the estimate (47). Using the interpolation theory, we only need to prove that (47) holds for $\alpha = 0$ and $\alpha = 1$. Since $\frac{1}{2} > 0$ and the resolvent set of $-A_{h,j}$ contains $[0, \infty]$, it follows from [31, (5.23)] that
\[
(I + tA_{h,j}(\omega))^{-n} x = t^{-n} \left( \frac{1}{t} I + A_{h,j}(\omega) \right)^{-n} x
= \frac{t^{-n}}{(n-1)!} \int_0^\infty s^{n-1} e^{-\frac{1}{t}s} S_{h,j}(\omega)(s)xds, \quad x \in H. \tag{50}
\]

Taking the norm in both sides of (50) and using the uniformly boundedness of $S_{h,j}(\omega)$ (see Lemma 3) yields
\[
\|A_h (I + tA_{h,j}(\omega))^{-n} x\| \leq \frac{Ct^{-n}}{(n-1)!} \int_0^\infty s^{n-1} e^{-\frac{1}{t}s}\|x\|ds. \tag{51}
\]

Using the change of variable $u = \frac{1}{t}$ yields
\[
\|A_h (I + tA_{h,j}(\omega))^{-n} x\| \leq \frac{C}{(n-1)!} \int_0^\infty u^{n-1} e^{-u}\|x\|du \leq C\|x\|. \tag{52}
\]

This shows that (47) holds for $\alpha = 0$. Pre-multiplying both sides of (50) by $A_h$ yields
\[
A_h (I + tA_{h,j}(\omega))^{-n} x = \frac{t^{-n}}{(n-1)!} \int_0^\infty s^{n-1} e^{-\frac{1}{t}s} A_h S_{h,j}(\omega)(s)xds. \tag{53}
\]

Taking the norm in both sides of (53) and using [26, Lemma 9 (iii)] yields
\[
\|A_h (I + tA_{h,j}(\omega))^{-n} x\| \leq \frac{Ct^{-n}}{(n-1)!} \int_0^\infty s^{n-2} e^{-\frac{1}{t}s}\|x\|ds. \tag{54}
\]
Using the change of variable $u = \frac{t}{s}$ yields

$$
\left\| (I + tA_{h,j}(\omega))^{-n} \right\| \leq \frac{Ct^{-1}(n-1)!}{(n-1)!} \int_0^{\infty} u^{n-2} e^{-u} \|x\| du
\leq C (n-1)t^{-1} \|x\|.
$$

This proves that (47) holds for $\alpha = 1$, and the proof of (47) is complete by interpolation theory. The proofs of (48) and (49) follow from the integral equation (50).

The following lemma will be useful in our convergence analysis.

**Lemma 5** Let Assumption 1 and Assumption 3 be fulfilled.

(i) For all $\alpha \in (0,1]$ it holds that

$$
\left\| A_h^\alpha \left( \prod_{j=1}^m S_{h,\Delta t}(\omega) \right) \right\|_{L(H)} \leq Ct^{-\alpha}_{m-i+1}, \quad 0 \leq i \leq m \leq M,
$$

for all $0 \leq k \leq M$.

(ii) For all $\alpha_1, \alpha_2 \in [0,1)$ it holds that

$$
\left\| A_h^{\alpha_1} \left( \prod_{j=1}^m S_{h,\Delta t}(\omega) \right) A_h^{-\alpha_2} \right\|_{L(H)} \leq Ct^{-\alpha_1+\alpha_2}_{m-i+1}, \quad 0 \leq i \leq m \leq M,
$$

for all $0 \leq k \leq M$.

**Proof** Note that the proof for $i = m$ is straightforward from Lemma 4. We only concentrate on the case $i < m$.

(i) Using Lemma 4 it holds that

$$
\left\| A_h^{\alpha} (I + tA_{h,i}(\omega))^{m-i+1} \right\|_{L(H)} \leq C t^{-\alpha}_{m-i+1}.
$$

It remains to estimate $A_h^{\alpha} \Delta h_{m,i}(\omega)$, where

$$
\Delta h_{m,i}(\omega) := \prod_{j=1}^m S_{h,\Delta t}(\omega) - \left( S_{h,\Delta t}(\omega) \right)^{m-i+1}.
$$

One can easily check that the following identity holds

$$
(I + \Delta tA_{h,j+1}(\omega))^{-1} - (I + \Delta tA_{h,j+1}(\omega))^{-1} \\
= \Delta t (I + \Delta tA_{h,j+1}(\omega))^{-1} (A_{h,i}(\omega) - A_{h,j+1}(\omega)) (I + \Delta tA_{h,i}(\omega))^{-1} \\
= \Delta t (I + \Delta tA_{h,j+1}(\omega))^{-1} (J^h_i(\omega) - J^h_{j+1}(\omega)) (I + \Delta tA_{h,i}(\omega))^{-1}.
$$
Using the telescopic sum, it holds that

\[ \Delta_{m,i}^h(\omega) = \sum_{j=0}^{m-i-1} \left( \prod_{k=j+1}^{j+i+1} S_{h,\Delta t}(\omega) \right) (I + \Delta t A_{h,j+i+1}(\omega)) \]

\[ \left[ (I + \Delta t A_{h,j+i+1}(\omega))^{-1} - (I + \Delta t A_{h,i}(\omega))^{-1} \right] (I + \Delta t A_{h,i}(\omega))^{-j} \] \hfill (61)

Substituting the identity (60) in (61) yields

\[ \Delta_{m,i}^h(\omega) = \Delta t \sum_{j=0}^{m-i-1} \left( \prod_{k=j+1}^{j+i+1} S_{h,\Delta t}(\omega) \right) \left( J_{h,1}^i(\omega) - J_{h,1}^{j+i+1}(\omega) \right) (I + \Delta t A_{h,i}(\omega))^{-j-2} \]

\[ + \Delta t \sum_{j=0}^{m-i-1} \Delta_{m,j+i+1}^h(\omega) \left( J_{h,1}^i(\omega) - J_{h,1}^{j+i+1}(\omega) \right) (I + \Delta t A_{h,i}(\omega))^{-j-2}. \] \hfill (62)

Therefore we have

\[ A_h^\alpha \Delta_{m,i}^h(\omega) = \Delta t \sum_{j=0}^{m-i-1} A_h^\alpha (I + \Delta t A_{h,j+i+1}(\omega))^{-j-i} \left( J_{h,1}^i(\omega) - J_{h,1}^{j+i+1}(\omega) \right) (I + \Delta t A_{h,i}(\omega))^{-j-1} \]

\[ + \Delta t \sum_{j=0}^{m-i-1} A_h^\alpha \Delta_{m,j+i+1}^h(\omega) \left( J_{h,1}^i(\omega) - J_{h,1}^{j+i+1}(\omega) \right) (I + \Delta t A_{h,i}(\omega))^{-j-1}. \] \hfill (63)

Taking the norm in both sides of (63), using the triangle inequality and Lemma 3 yields

\[ \| A_h^\alpha \Delta_{m,i}^h(\omega) \|_{L(H)} \leq C \Delta t \sum_{j=0}^{m-i-1} \| \Delta_{m,j+i+1}^h(\omega) \|_{L(H)} \]

\[ + C \Delta t \sum_{j=0}^{m-i-1} \| A_h^\alpha \Delta_{m,j+i+1}^h(\omega) \|_{L(H)}. \] \hfill (64)

Using the notation

\[ \Gamma_{m-i} := A_h^\alpha \Delta_{m,i}^h, \] \hfill (65)

it follows from (64) that

\[ \| \Gamma_{m-i} \|_{L(H)} \leq C + C \Delta t \sum_{j=0}^{m-i-1} \| \Gamma_{m-j} \|_{L(H)} = C + C \Delta t \sum_{k=0}^{m-i-1} \| \Gamma_k \|_{L(H)}. \] \hfill (66)

Applying the discrete Gronwall’s lemma to (66) yields

\[ \| A_h^\alpha \Delta_{m,i}^h(\omega) \|_{L(H)} = \| \Gamma_{m-i} \|_{L(H)} \leq C. \] \hfill (67)

This completes the proof of (i).
(ii) Following the same lines as Lemma 4 we can show

\[ \left\| A_h^{\alpha_1} (I + \Delta t A_{h,i}(\omega))^{m-\alpha} A_h^{-\alpha_2} \right\|_{L(H)} \leq C \Delta t^{\alpha_1+\alpha_2}. \]  

(68)

It remains to bound \( A_h^{\alpha_1} \Delta m_{i+1}(\omega) A_h^{-\alpha_2} \), where \( \Delta m_{i+1}(\omega) \) is defined in (60). From (62), it holds that

\[
A_h^{\alpha_1} \Delta m_{i+1}(\omega) A_h^{-\alpha_2} = \Delta t \sum_{j=0}^{m-\alpha} A_h^{\alpha_1} (I + \Delta t A_{h,j+i+1}(\omega)) \left( j_h^i(\omega) - j_h^i(\omega) \right) (I + \Delta t A_{h,i}(\omega))^{\alpha-1} A_h^{-\alpha_2}
+ \Delta t \sum_{j=0}^{m-\alpha} A_h^{\alpha_1} \Delta m_{j+i+1}(\omega) \left( j_h^i(\omega) - j_h^i(\omega) \right) (I + \Delta t A_{h,i}(\omega))^{\alpha-1} A_h^{-\alpha_2}. \]

(69)

Taking the norm in both sides of (63), using the triangle inequality, Lemma 4 and Lemma 5 (i) yields

\[
\left\| A_h^{\alpha_1} \Delta m_{i+1}(\omega) A_h^{-\alpha_2} \right\|_{L(H)} \leq C \Delta t \sum_{j=0}^{m-\alpha} \left\| A_h^{\alpha_1} \Delta m_{j+i+1}(\omega) \right\|_{L(H)}
+ C \Delta t \sum_{j=0}^{m-\alpha} \left\| A_h^{\alpha_1} (I + \Delta t A_{h,j+i+1}(\omega))^{\alpha-1} \right\|_{L(H)}
\leq C \Delta t \sum_{j=0}^{m-\alpha} + C \Delta t \sum_{j=0}^{m-\alpha} \leq C.

(70)

This proves (ii) and the proof of the lemma is complete.

The following lemma will be useful in our convergence analysis.

**Lemma 6** (i) For all \( \alpha_1, \alpha_2 \in (0, 1) \), \( 0 \leq j \leq M \) and \( \omega \in \Omega \) the following estimate holds

\[ \left\| A_h^{-\alpha_1} e^{A_h(\omega) \Delta t} - S_{h,\Delta t}^j(\omega) A_h^{-\alpha_2} \right\|_{L(H)} \leq C \Delta t^{\alpha_1+\alpha_2}. \]

(71)

(ii) For all \( \alpha_1 \in [0, 1], \alpha_2 \in (0, 1), 0 \leq j \leq M \) and \( \omega \in \Omega \) the following estimate holds

\[ \left\| A_h^{\alpha_1} e^{A_h(\omega) \Delta t} - S_{h,\Delta t}^j(\omega) A_h^{-\alpha_2} \right\|_{L(H)} \leq C \Delta t^{\alpha_1+\alpha_2}. \]

(72)

**Proof** We prove only (71) since the proof of (72) is similar. Let us set

\[ K_j^{i,h}(\omega) := e^{A_h(\omega) \Delta t} - S_{h,\Delta t}^j(\omega). \]

(73)
We can easily check that

\[-K^j_{h,\Delta t}(\omega) = \int_0^{\Delta t} \frac{d}{ds} \left( (I + sA_{h,j}(\omega))^{-1} e^{-(\Delta t-s)A_{h,j}(\omega)} \right) ds\]

\[= \int_0^{\Delta t} sA^2_{h,j}(\omega) (I + sA_{h,j}(\omega))^{-2} e^{-(\Delta t-s)A_{h,j}(\omega)} ds\]

\[= \int_0^{\Delta t} sA_{h,j}(\omega) (I + sA_{h,j}(\omega))^{-2} A_{h,j}(\omega)e^{-(\Delta t-s)A_{h,j}(\omega)} ds\] (74)

From (74) it holds that

\[-A^{-\alpha_1}_h K^j_{h,\Delta t}(\omega)A^{-\alpha_2}_h\]

\[= \int_0^{\Delta t} sA^{-\alpha_1}_h A_{h,j}(\omega) (I + sA_{h,j}(\omega))^{-2} e^{-(\Delta t-s)A_{h,j}(\omega)} A^{-\alpha_2}_h ds. \] (75)

Taking the norm in both sides of (75) yields

\[\left\| -A^{-\alpha_1}_h K^j_{h,\Delta t}(\omega)A^{-\alpha_2}_h \right\|_{L(H)} \leq \int_0^{\Delta t} s \left\| A^{-\alpha_1}_h A_{h,j}(\omega) (I + sA_{h,j}(\omega))^{-2} \right\|_{L(H)} \times \left\| e^{-(\Delta t-s)A_{h,j}(\omega)} A^{-\alpha_2}_h \right\|_{L(H)} ds.\] (76)

Using the triangle inequality and Lemma 4 it holds that

\[\left\| A^{-\alpha_1}_h A_{h,j}(\omega) (I + sA_{h,j}(\omega))^{-2} \right\|_{L(H)} \leq \left\| A^{-\alpha_1}_h (I + sA_{h,j}(\omega))^{-1} \right\|_{L(H)} + \left\| A^{-\alpha_1}_h J^j_{h} (\omega) (I + sA_{h,j}(\omega))^{-2} \right\|_{L(H)}\]

\[\leq Cs^{-1+\alpha_1} + C\]

\[\leq Cs^{-1+\alpha_1}.\] (77)

Using triangle inequality and Lemma 9 (ii), it holds that

\[\left\| e^{-(\Delta t-s)A_{h,j}(\omega)} A^{-\alpha_2}_h \right\|_{L(H)} \leq \left\| e^{-(\Delta t-s)A_{h,j}(\omega)} A^{-\alpha_2}_h \right\|_{L(H)} + \left\| e^{-(\Delta t-s)A_{h,j}(\omega)} J^j_{h} A^{-\alpha_2}_h \right\|_{L(H)}\]

\[\leq C(\Delta t - s)^{-1+\alpha_2} + C\]

\[\leq C(\Delta t - s)^{-1+\alpha_2}.\] (78)

Substituting (78) and (77) in (76) yields

\[\left\| -A^{-\alpha_1}_h K^j_{h,\Delta t}(\omega)A^{-\alpha_2}_h \right\|_{L(H)} \leq C \int_0^{\Delta t} ss^{-1+\alpha_1}(\Delta t - s)^{-1+\alpha_2} ds\]

\[\leq C \Delta t^{\alpha_1+\alpha_2}.\] (79)

This completes the proof of (71). The proof of (72) is similar.
The following lemma can be found in [21].

Lemma 7 For all $\alpha_1, \alpha_2 > 0$ and $\alpha \in [0, 1]$, there exist two positive constants $C_{\alpha_1, \alpha_2}$ and $C_{\alpha, \alpha_2}$ such that

$$\Delta t \sum_{j=1}^{m} t_{m-j+1}^{1+\alpha_2} \leq C_{\alpha_1, \alpha_2} t_{m}^{1+\alpha_1+\alpha_2}, \quad (80)$$

$$\Delta t \sum_{j=1}^{m} t_{m-j+1}^{-\alpha} \leq C_{\alpha_2} t_{m}^{-\alpha+\alpha_2}. \quad (81)$$

Proof The proof of (80) follows by comparison with the integral

$$\int_{0}^{t} (t-s)^{-1+\alpha_1} s^{-1+\alpha_2} ds.$$  

The proof of (81) is a consequence of (80).

Lemma 8 Let $0 \leq \alpha < 2$ and let Assumption [7] be fulfilled.

(i) If $v \in D ((A^{\alpha/2}), \omega \in \Omega, 0 \leq i \leq M$, then the following estimate holds

$$\left\| \left( \prod_{j=i}^{m} e^{A_{h,j}(\omega)\Delta t} \right) P_h v - \left( \prod_{j=i}^{m} S_{h,\Delta t}^j(\omega) \right) P_h v \right\| \leq C \Delta t^{\alpha/2} \| v \|_\alpha. \quad (83)$$

(ii) For non-smooth data, i.e. for $v \in H$ and for all $\omega \in \Omega, 0 \leq i < m \leq M$, it holds that

$$\left\| \left( \prod_{j=i}^{m} e^{A_{h,j}(\omega)\Delta t} \right) P_h v - \left( \prod_{j=i}^{m} S_{h,\Delta t}^j(\omega) \right) P_h v \right\| \leq C \Delta t^{\alpha/2} t_{m-i}^{-\alpha/2} \| v \|. \quad (84)$$

(iii) For all $\alpha_1, \alpha_2 \in [0, 1)$ such that $\alpha_1 \leq \alpha_2$, $\omega \in \Omega$ and $0 \leq i < m \leq M$, it holds that

$$\left\| \left[ \left( \prod_{j=i}^{m} e^{A_{h,j}(\omega)\Delta t} \right) - \left( \prod_{j=i}^{m} S_{h,\Delta t}^j(\omega) \right) \right] A_{h,\Delta t}^{\alpha_1-\alpha_2} \right\|_{L(H)} \leq C \Delta t^{\alpha_2} t_{m-i}^{-\alpha_1}. \quad (85)$$

Proof (i) Using the telescopic identity, we have

$$\left( \prod_{j=i}^{m} e^{A_{h,j}(\omega)\Delta t} \right) P_h v - \left( \prod_{j=i}^{m} S_{h,\Delta t}^j(\omega) \right) P_h v = \sum_{k=1}^{m-i-1} \left( \prod_{j=i+k}^{m} e^{A_{h,j}(\omega)\Delta t} \right) \left( e^{A_{h,i+k-1}(\omega)\Delta t} - S_{h,\Delta t}^{i+k-1}(\omega) \right) \left( \prod_{j=i+k}^{m} S_{h,\Delta t}^j(\omega) \right) P_h v.$$  

(86)
Writing down the first and the last terms of (86) explicitly, we obtain

\[
\left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) P_h v - \left( \prod_{j=1}^{m} S_{h,j}^{\Delta t}(\omega) \right) P_h v = \left( e^{A_{h,m}(\omega) \Delta t} - S_{h,m}^{\Delta t}(\omega) \right) \left( \prod_{j=1}^{m-1} S_{h,j}^{\Delta t}(\omega) \right) P_h v + \left( \prod_{j=m+1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) \left( e^{A_{h,i}(\omega) \Delta t} - S_{h,i}^{\Delta t}(\omega) \right) P_h v + \sum_{k=2}^{m-1} \left( \prod_{j=i+k}^{m} e^{A_{h,j}(\omega) \Delta t} \right) \left( e^{A_{h,i+k-1}(\omega) \Delta t} - S_{h,i+k-1}^{\Delta t}(\omega) \right) \left( \prod_{j=i+k}^{m} S_{h,j}^{\Delta t}(\omega) \right) P_h v. \tag{87}
\]

Taking the norm in both sides of (87), inserting an appropriate power of \( A_h \) and using the triangle inequality yields

\[
\left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) P_h v - \left( \prod_{j=1}^{m} S_{h,j}^{\Delta t}(\omega) \right) P_h v \right\| \leq \left\| \left( e^{A_{h,m}(\omega) \Delta t} - S_{h,m}^{\Delta t}(\omega) \right) \left( \prod_{j=1}^{m-1} S_{h,j}^{\Delta t}(\omega) \right) A_h^{-\alpha/2} A_h^{n/2} P_h v \right\| + \left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) \left( e^{A_{h,i}(\omega) \Delta t} - S_{h,i}^{\Delta t}(\omega) \right) A_h^{-\alpha/2} A_h^{n/2} P_h v \right\| + \sum_{k=2}^{m-1} \left\| \left( \prod_{j=i+k}^{m} e^{A_{h,j}(\omega) \Delta t} \right) A_h^{-\alpha/2} \left( e^{A_{h,i+k-1}(\omega) \Delta t} - S_{h,i+k-1}^{\Delta t}(\omega) \right) A_h^{-\alpha/2} P_h v \right\| =: I_1 + I_2 + I_3. \tag{88}
\]

Using Lemma 6, Lemma 5 (ii) and [26] Lemma 1] yields

\[
I_1 \leq \left\| \left( e^{A_{h,m}(\omega) \Delta t} - S_{h,m}^{\Delta t}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)} \left\| \left( e^{A_{h,i}(\omega) \Delta t} - S_{h,i}^{\Delta t}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)} \left\| A_h^{n/2} P_h v \right\| \leq C \Delta^{\alpha/2} \| v \|_\alpha. \tag{89}
\]

Using Lemma 2, Lemma 6 and [26] Lemma 1] yields

\[
I_2 \leq \left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) \right\|_{L(H)} \left\| \left( e^{A_{h,i}(\omega) \Delta t} - S_{h,i}^{\Delta t}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)} \left\| A_h^{n/2} P_h v \right\| \leq C \Delta^{\alpha/2} \| v \|_\alpha. \tag{90}
\]
Using Lemma 2, Lemma 5, Lemma 6 (ii), Lemma 7 and [26, Lemma 1] yields

\[
I_3 \leq \sum_{k=2}^{m-i} \left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) A_h^{-\epsilon} \left( e^{A_{h,i+k}(\omega) \Delta t} - S_{h,i+k}^{i+k-1}(\omega) \right) A_h^{-\alpha/2 - \epsilon} \right\|_{L(H)} \\
\times \left\| A_h^{\alpha/2 + \epsilon} \left( \prod_{j=i}^{i+k-2} S_{h,j}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)} \\
\times \left\| A_h^{\alpha/2} \left( \prod_{j=i}^{i+k-2} S_{h,j}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)}
\]

\[
\leq C \sum_{k=2}^{m-i} t_{m+1-i-k}^{1+\alpha/2} t_{k-1}^{-\epsilon} \Delta t^{\alpha/2} \sum_{k=2}^{m-i} t_{m-i-k+1}^{1+\epsilon} t_{k-1}^{-\epsilon} \Delta t
\]

\[
\leq C \Delta t^{\alpha/2}. \tag{91}
\]

Substituting (91), (90) and (89) in (88) yields

\[
\left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) P_h v - \left( \prod_{j=i}^{m} S_{h,j}(\omega) \right) P_h v \right\| \leq C \Delta t^{\alpha/2} \|v\|_\alpha. \tag{92}
\]

This completes the proof of (i).

(ii) For non-smooth initial data, taking the norm in both sides of (87) and inserting an appropriate power of \(A_h\) yields

\[
\left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) P_h v - \left( \prod_{j=1}^{m} S_{h,j}(\omega) \right) P_h v \right\| \leq C \Delta t^{\alpha/2} \|v\|_\alpha. \tag{93}
\]
Using Lemma 5, Lemma 3 (i), Lemma 7 and Lemma 2 it follows that

\[
\left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) P_h v - \left( \prod_{j=1}^{m} S_{h,j}(\omega) \right) P_h v \right\| \\
\leq C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} \|v\| + C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} \|v\| \\
+ C \Delta t^{1-\epsilon} \sum_{k=2}^{m-1} \Delta t_{m-1-k+1}^{1+\epsilon} \|v\| \\
\leq C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} \|v\| + C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} \|v\| + C \Delta t^{1-\epsilon} t_{m-1}^{1+2\epsilon} \|v\| \\
\leq C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} \|v\|. \tag{94}
\]

(iii) Taking the norm in both sides of (94) and inserting an appropriate power of \( A_h \) yields

\[
\left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) - \left( \prod_{j=1}^{m} S_{h,j}(\omega) \right) \right\|_{L(H)} \\
\leq \left\| \left( e^{A_{h,m}(\omega) \Delta t} - S_{h,m}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)} \left\| A_h^{\alpha/2} \left( \prod_{j=1}^{m} S_{h,j}(\omega) \right) A_{h}^{-\alpha/2} \right\|_{L(H)} \\
+ \left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) A_h^{\alpha/2} \right\|_{L(H)} \left\| A_h^{\alpha/2} \left( e^{A_{h,m}(\omega) \Delta t} - S_{h,m}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)} \\
+ \sum_{k=2}^{m-1} \left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) A_h^{\alpha/2} \right\|_{L(H)} \\
\times \left\| A_h^{\alpha/2} \left( e^{A_{h,i+k-1}(\omega) \Delta t} - S_{h,i+k-1}(\omega) \right) A_h^{1+\epsilon} \right\|_{L(H)} \\
\times \left\| A_h^{1+\epsilon} \left( \prod_{j=1}^{m} S_{h,j}(\omega) \right) A_h^{-\alpha/2} \right\|_{L(H)}. \tag{95}
\]

Using Lemma 5, Lemma 3 (ii), Lemma 7 and Lemma 2 it follows from (95) that

\[
\left\| \left( \prod_{j=1}^{m} e^{A_{h,j}(\omega) \Delta t} \right) - \left( \prod_{j=1}^{m} S_{h,j}(\omega) \right) \right\|_{L(H)} \\
\leq C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} + C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} + C \Delta t^{\alpha/2} \sum_{k=2}^{m-1} \Delta t_{m-1-k+1}^{1+\epsilon} t_{k-1} \left| a_{2-\alpha/2} \right| \\
\leq C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} + C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} + C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2} \\
\leq C \Delta t^{\alpha/2} t_{m-1}^{-\alpha/2}. \tag{96}
\]

This completes the prove of (iii).
Lemma 9 (i) Let Assumption 6 be fulfilled. Then the following estimate holds
\[ \left\| (A_h)^{\frac{\beta}{2}} P_h Q_h^{\perp} \right\|_{L^2(H)} \leq C, \quad (97) \]
where \( \beta \) comes from Assumption 1.

(ii) Under Assumption 7, for all \( \omega \in \Omega \) and \( m \in \mathbb{N} \), the following estimates hold
\[ \left\| \left( G_{m}^h(\omega) \right)'(u)v \right\| \leq C \| u \| \| v \|, \quad u, v \in H, \quad (98) \]
\[ \left\| (A_h)^{-\frac{\eta}{2}} \left( G_{m}^h(\omega) \right)''(u)(v_1, v_2) \right\| \leq C \| v_1 \| \| v_2 \|, \quad s, t \in [0, T], \quad u, v_1, v_2 \in H, \quad (99) \]
where \( \eta \) comes from Assumption 7.

Proof The proof of (i) can be found in [20, Lemma 11] and the proof of (ii) can be found in [20, Lemma 12].

With the above preparation, we are now in position to prove our main results.

3.2 Proof of Theorem 9

Iterating the numerical solution (41) at \( t_m \) by replacing \( X^h_i, i = m-1, \cdots, 2, 1 \) by its expression only on the first appearance yields
\[ X^h_m = \left( \prod_{k=0}^{m-1} S_{h,\Delta t}^k \right) P_h X_0 + \Delta t S_{h,\Delta t}^{m-1} G_{m-1}^h(X^h_{m-1}) + S_{h,\Delta t}^{m-1} P_h B(X^h_{m-1}) \Delta W_{m-1} \]
\[ + \Delta t \sum_{i=2}^{m-1} \left( \prod_{j=m-i}^{m-1} S_{h,\Delta t}^j \right) C_{m-1}^h(X^h_{m-i}) \]
\[ + \sum_{i=2}^{m-1} \left( \prod_{j=m-i}^{m-1} S_{h,\Delta t}^j \right) P_h B(X^h_{m-i}) \Delta W_{m-i} \quad (100) \]

The mild solution associated at (38) at \( t_m \) is given by
\[ X^h(t_m) = e^{A_h \Delta t} X^h(t_m) + \int_{t_{m-1}}^{t_m} e^{A_h(t_m-s)} G_{m-1}^h(X^h(s)) ds \]
\[ + \int_{t_{m-1}}^{t_m} e^{A_h(t_m-s)} P_h B(X^h(s)) dW(s). \quad (101) \]
Iterating the mild solution (101) at time $t_m$ yields

$$X^h(t_m) = \left( \prod_{k=0}^{m-1} e^{A_{h,k} \Delta t} \right) P_h X_0 + \int_{t_{m-1}}^{t_m} e^{(t_m-s)A_{h,m-1}} G_{m-1}^h (X^h(s)) \, ds$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{A_{h,j} \Delta t} \right) e^{(t_{m-i}-s)A_{h,m-i-1}} G_{m-i}^h (X^h(s)) \, ds$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{A_{h,j} \Delta t} \right) e^{(t_{m-i}-s)A_{h,m-i-1}} P_h B (X^h(s)) \, dW(s)$$

$$+ \int_{t_{m-1}}^{t_m} e^{(t_m-s)A_{h,m-1}} P_h B (X^h(s)) \, dW(s). \quad (102)$$

Subtracting (102) and (100), taking the $L^2$ norm and using triangle inequality yields

$$\| X^h(t_m) - X^h_m \|_{L^2(\Omega,H)}^2 \leq 25 \sum_{i=0}^{4} \| I_i \|_{L^2(\Omega,H)}^2 \quad (103)$$

where

$$I_0 = \left( \prod_{j=0}^{m-1} e^{A_{h,j} \Delta t} \right) P_h X_0 - \left( \prod_{j=0}^{m-1} S_{h,j}^2 \right) P_h X_0, \quad (104)$$

$$I_1 = \int_{t_{m-1}}^{t_m} (e^{(t_m-s)A_{h,m-1}} G_{m-1}^h (X^h(s)) - G_{h,\Delta t}^{m-1} G_{m-1}^h (X^h_{m-1})) \, ds, \quad (105)$$

$$I_2 = \int_{t_{m-1}}^{t_m} (e^{(t_m-s)A_{h,m-1}} P_h B (X^h(s)) - S_{h,\Delta t}^{m-1} P_h B (X^h_{m-1})) \, dW(s),$$

$$I_3 = \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{A_{h,j} \Delta t} \right) e^{(t_{m-i}-s)A_{h,m-i-1}} G_{m-i}^h (X^h(s)) \, ds$$

$$- \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} S_{h,j}^2 \right) G_{m-i}^h (X^h_{m-i}) \, ds, \quad (106)$$

$$I_4 = \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{A_{h,j} \Delta t} \right) e^{(t_{m-i}-s)A_{h,m-i-1}} P_h B (X^h(s)) \, dW(s)$$

$$- \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} S_{h,j}^2 \right) P_h B (X^h_{m-i}) \, dW(s). \quad (107)$$

In the following sections we estimate $I_i$, $i = 0, \ldots, 4$ separately.
3.2.1 Estimation of $I_0$, $I_1$ and $I_2$

Using Lemma 5 (i) with $\alpha = \beta$, it holds that

$$
\|I_0\|_{L^2(\Omega, H)} \leq \left( \mathbb{E} \left[ \left\| \prod_{j=0}^{m-1} e^{A_{b,j} \Delta t} \right\| P_h X_0 - \left( \prod_{j=0}^{m-1} S^j_{b, \Delta t} \right) P_h X_0 \right]^2 \right)^{1/2}
\leq C \Delta t^{3/2} \left( \mathbb{E} \|X_0\|_2^2 \right)^{1/2} \leq C \Delta t^{3/2}.
$$

(108)

The term $I_1$ can be recast in three terms as follows.

$$
I_1 = \int_{t_{m-1}}^{t_m} e^{(t_{m-s})A_{b,m-1}} \left[ C^h_{m-1} (X^h(s)) - G^h_{m-1} (X^h(t_{m-1})) \right] ds
+ \int_{t_{m-1}}^{t_m} \left( e^{(t_{m-s})A_{b,m-1}} - S^{m-1}_{b, \Delta t} \right) G^h_{m-1} (X^h(t_{m-1})) ds
+ \int_{t_{m-1}}^{t_m} S^{m-1}_{b, \Delta t} \left[ C^h_{m-1} (X^h(t_{m-1})) - G^h_{m-1} (X^h(t_{m-1})) \right] ds.
$$

:= I_{11} + I_{12} + I_{13}.

(109)

Therefore

$$
\|I_1\|_{L^2(\Omega, H)} \leq \|I_{11}\|_{L^2(\Omega, H)} + \|I_{12}\|_{L^2(\Omega, H)} + \|I_{13}\|_{L^2(\Omega, H)}.
$$

(110)

Using Corollary 8 it yields

$$
\|I_{11}\|_{L^2(\Omega, H)} \leq C \int_{t_{m-1}}^{t_m} \|C^h_{m-1} (X^h(s))\|_{L^2(\Omega, H)} ds
+ \int_{t_{m-1}}^{t_m} \|G^h_{m-1} (X^h(t_{m-1}))\|_{L^2(\Omega, H)} ds
\leq C \int_{t_{m-1}}^{t_m} (1 + \|X_0\|_{L^2(\Omega, H)}) ds
\leq C \Delta t.
$$

(111)

Using Lemma 5 (i) with $\alpha = 0$ and Corollary 8 it holds that

$$
\|I_{12}\|_{L^2(\Omega, H)} \leq C \int_{t_{m-1}}^{t_m} \|G^h_{m-1} (X^h(t_{m-1}))\|_{L^2(\Omega, H)} ds
\leq C \int_{t_{m-1}}^{t_m} (1 + \|X_0\|_{L^2(\Omega, H)}) ds
\leq C \Delta t.
$$

(112)

Using Lemma 5 (i) with $\alpha = 0$ and Assumption 3 it holds that

$$
\|I_{13}\|_{L^2(\Omega, H)} \leq C \Delta t \|X^h(t_{m-1}) - X^h_{m-1}\|_{L^2(\Omega, H)}.
$$

(113)
Substituting (113), (112) and (111) in (110) yields
\[
\|II_1\|_{L^2(\Omega, H)} \leq C \Delta t + C \Delta t \|X^h(t_{m-1}) \rangle - X^h_{m-1}\|_{L^2(\Omega, H)},
\] (114)

We can recast as follows:
\[
II_2 = \int_{t_{m-1}}^{t_m} e^{(t_{m-s})A_h, m-1} \left( P_h B \left( X^h(s) \right) - P_h B \left( X^h(t_{m-1}) \right) \right) dW(s)
+ \int_{t_{m-1}}^{t_m} \left( e^{(t_{m-s})A_h, m-1} - S_{h, \Delta t}^{m-1} \right) P_h B \left( X^h(t_{m-1}) \right) dW(s)
+ \int_{t_{m-1}}^{t_m} S_{h, \Delta t}^{m-1} \left( P_h B \left( X^h(t_{m-1}) \right) \right) dW(s)
:= II_{21} + II_{22} + II_{23}.
\] (115)

Therefore
\[
\|II_2\|_{L^2(\Omega, H)}^2 \leq 9 \|II_{21}\|_{L^2(\Omega, H)}^2 + 9 \|II_{22}\|_{L^2(\Omega, H)}^2 + 9 \|II_{23}\|_{L^2(\Omega, H)}^2.
\] (116)

Using the Itô-isometry property, [26, Lemma 5], Assumption 3 and Lemma 1, it holds that
\[
\|II_{21}\|_{L^2(\Omega, H)}^2 = \int_{t_{m-1}}^{t_m} \left\| e^{(t_{m-s})A_h, m-1} \left( P_h B \left( X^h(s) \right) - P_h B \left( X^h(t_{m-1}) \right) \right) \right\|_{L^2(\Omega, H)}^2 ds
\leq C \int_{t_{m-1}}^{t_m} (s-t_{m-1})^{\min(\beta, 1)} ds
\leq C \Delta t^{\min(\beta+1, 2)}.
\] (117)

Using again the Itô-isometry property, Lemma 8(i) with \(\alpha = 0\) and Corollary 1 yields
\[
\|II_{22}\|_{L^2(\Omega, H)}^2 = \int_{t_{m-1}}^{t_m} \left\| e^{(t_{m-s})A_h, m-1} - S_{h, \Delta t}^{m-1} \right\|_{L^2(\Omega, H)}^2 ds
\leq C \int_{t_{m-1}}^{t_m} \left( 1 + \|X_0\|_{L^2(\Omega, H)}^2 \right) ds
\leq C \Delta t.
\] (118)

The Itô-isometry property together with Lemma 5(i) with \(\alpha = 0\) and Assumption 3 yields
\[
\|II_{23}\|_{L^2(\Omega, H)}^2 = \int_{t_{m-1}}^{t_m} \left\| S_{h, \Delta t}^{m-1} \left( P_h B \left( X^h(t_{m-1}) \right) \right) \right\|_{L^2(\Omega, H)}^2 ds
\leq C \Delta t \left\| X^h(t_{m-1}) \rangle - X^h_{m-1}\|_{L^2(\Omega, H)}^2.
\] (119)

Substituting (119), (118) and (117) in (116) yields
\[
\|II_2\|_{L^2(\Omega, H)}^2 \leq C \Delta t + C \Delta t \left\| X^h(t_{m-1}) \rangle - X^h_{m-1}\|_{L^2(\Omega, H)}^2.
\] (120)
3.2.2 Estimation of $I_{3}$

We can recast $I_{3}$ in four terms as follows:

\[
I_{3} = \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(t_{m-i} - s) A_{h,m-i-1}} - 1 \right) C_{m-i}^{h} \left( X^{h}(s) \right) ds \\
+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) C_{m-i}^{h} \left( X^{h}(s) - G_{m-i}^{h} \left( X^{h}(t_{m-i}) \right) \right) ds \\
+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( \prod_{j=m-i}^{m-1} S_{h,\Delta t}^{j} \right) G_{m-i}^{h} \left( X^{h}(s) \right) ds \\
+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( \prod_{j=m-i}^{m-1} S_{h,\Delta t}^{j} \right) G_{m-i}^{h} \left( X^{h}(t_{m-i}) \right) ds
\]

\[= I_{31} + I_{32} + I_{33} + I_{34}. \quad (121)\]

Therefore, we have

\[
\| I_{3} \|_{L^{2}(\Omega, H)} \leq \| I_{31} \|_{L^{2}(\Omega, H)} + \| I_{32} \|_{L^{2}(\Omega, H)} \\
+ \| I_{33} \|_{L^{2}(\Omega, H)} + \| I_{34} \|_{L^{2}(\Omega, H)}. \quad (122)\]

Inserting an appropriate power of $A_{h,m-i}$, using Lemma 2 with $\alpha = 0$ and Corollary 1 yields

\[
\| I_{31} \|_{L^{2}(\Omega, H)} \leq \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) A_{h}^{-1+\epsilon} \right\|_{L(H)}^{2} \right] ds \\
\times \left\| A_{h}^{-1+\epsilon} \left( e^{(t_{m-i} - s) A_{h,m-i-1}} - 1 \right) \right\|_{L(H)}^{2} \left\| G_{m-i}^{h} \left( X^{h}(s) \right) \right\| ds^{1/2} \\
\leq C \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( t_{i}^{-1+\epsilon} (t_{m-i} - s)^{-\epsilon} \right) ds \\
\leq C \Delta t^{1-\epsilon} \sum_{i=2}^{m-1} \Delta t \left( t_{i}^{-1+\epsilon} \right) \\
\leq C \Delta t^{1-\epsilon}. \quad (123)\]
Using triangle inequality, Lemma 2, Assumption 3 and Lemma 1 yields

\[ \|II_3\|_{L^2(\Omega, H)} \leq \sum_{i=2}^{m-1} \int_{t_{m-i}}^{t_{m-i-1}} E \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t \Lambda_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^j \right) \right\|_{L(H)}^2 \right] \times \| G_{m-i}^h (X^h(s)) - G_{m-i}^h (X^h(t_{m-i})) \|_{L^2(\Omega,H)}^2 \right]^{1/2} ds \]
\[ \leq C \sum_{i=2}^{m-1} \int_{t_{m-i}}^{t_{m-i-1}} \| X^h(s) - X^h(t_{m-i}) \|_{L^2(\Omega,H)} ds \]
\[ \leq C \sum_{i=2}^{m-1} \int_{t_{m-i}}^{t_{m-i-1}} (t_{m-i} - s)^{\min(\beta/2)} ds \]
\[ \leq C \Delta t^{\min(\beta/2)}. \quad (124) \]

Using triangle inequality, Lemma 3(ii) and Corollary 1 it holds that

\[ \|II_{33}\|_{L^2(\Omega, H)} \leq \sum_{i=2}^{m-1} \int_{t_{m-i}}^{t_{m-i-1}} E \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t \Lambda_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^j \right) \right\|_{L(H)}^2 \right] \times \| G_{m-i}^h (X^h(t_{m-i})) \|_{L^2(\Omega,H)}^2 \right]^{1/2} ds \]
\[ \leq C \sum_{i=2}^{m-1} \int_{t_{m-i}}^{t_{m-i-1}} t_{i-1}^{-\beta/2} \Delta t^{\beta/2} ds \leq C \Delta t^{\beta/2}. \quad (125) \]

Using Lemma 5(i) with \( \alpha = 0 \) and Assumption 3 yields

\[ \|II_{34}\|_{L^2(\Omega, H)} \leq \sum_{i=2}^{m-1} \int_{t_{m-i}}^{t_{m-i-1}} E \left[ \left\| \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^j \right) \right\|_{L(H)}^2 \right] \times \| (G_{m-i}^h (X^h(t_{m-i})) - G_{m-i}^h (X^h(t_{m-i-1})) \|_{L^2(\Omega,H)}^2 \right]^{1/2} ds \]
\[ \leq C \Delta t \sum_{i=2}^{m-1} \| X^h(t_{m-i}) - X^h(t_{m-i-1}) \|_{L^2(\Omega,H)} \]. \quad (126) \]

Substituting (124), (125), (123) and (124) in (122) yields

\[ \|II_4\|_{L^2(\Omega, H)} \leq C \Delta t^{\min(\beta/2)} + C \Delta t \sum_{i=2}^{m-1} \| X^h(t_{m-i}) - X^h(t_{m-i-1}) \|_{L^2(\Omega,H)} \}. \quad (127) \]
3.2.3 Estimation of $I_4$

We can recast $I_4$ in four terms as follows.

$$I_4 = \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(t_{m-i-1}-s)A_{h,m-i-1}} - 1 \right) P_h B \left( X^h(s) \right) dW(s)$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) (P_h B \left( X^h(s) \right) - P_h B \left( X^h(t_{m-i}) \right)) dW(s)$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( \prod_{j=m-i}^{m-1} S_{h,\Delta t}^j \right) P_h B \left( X^h(t_{m-i}) \right) dW(s)$$

$$:= I_{41} + I_{42} + I_{43} + I_{44}. \quad (128)$$

Therefore we have

$$\|I_4\|^2_{L^2(\Omega, H)} \leq 16 \|I_{41}\|^2_{L^2(\Omega, H)} + 16 \|I_{42}\|^2_{L^2(\Omega, H)}$$

$$+ 16 \|I_{43}\|^2_{L^2(\Omega, H)} + 16 \|I_{44}\|^2_{L^2(\Omega, H)}. \quad (129)$$

Using the Itô-isometry property, inserting an appropriate power of $A_h$, using Lemma 2 with $\alpha = \frac{1}{2}$ and Corollary 1 yields

$$\|I_{41}\|^2_{L^2(\Omega, H)}$$

$$= \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \mathbb{E} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(t_{m-i-1}-s)A_{h,m-i-1}} - 1 \right) P_h B \left( X^h(s) \right) \right]^2 \right] ds$$

$$\leq \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \mathbb{E} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right)^{\frac{1}{2}} \left( e^{(t_{m-i-1}-s)A_{h,m-i-1}} - 1 \right)^{\frac{1}{2}} \right]^2 \right] ds$$

$$\times \left\| A_h^{-\frac{1}{2}} \left( e^{(t_{m-i-1}-s)A_{h,m-i-1}} - 1 \right) P_h B \left( X^h(s) \right) \right\|^2_{L^2(H)}$$

$$\leq C \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( t_{m-i-s} \right)^{1+\delta} ds$$

$$\leq C \Delta t^{1-\varepsilon} \sum_{i=2}^{m-1} \Delta t \left( t_{m-i-s} \right)^{1+\delta} \leq C \Delta t^{1-\varepsilon}. \quad (130)$$
Using the Itô-isometry property, Lemma 2 with $\alpha = 0$, Assumption 3 and Lemma 1 yields

$$
\|HI_{2}\|_{L^2(\Omega, H)}^2 = \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( P_h B \left( X^h(t) \right) - P_h B \left( X^h(t_{m-i}) \right) \right) \right\|_{L^2}^2 \right] ds
\leq \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \right\|_{L(H)}^2 \left\| P_h B \left( X^h(t) \right) - P_h B \left( X^h(t_{m-i}) \right) \right\|_{L^2}^2 \right] ds
\leq C \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \| X^h(t) - X^h(t_{m-i}) \|_{L^2(\Omega, H)}^2 ds
\leq C \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} (t_{m-i} - s)^\gamma \| X^h(t) - X^h(t_{m-i}) \|_{H}^2 ds \leq C \Delta t^{\gamma\min(\beta, 1)}.
$$

Using the Itô-isometry property, Lemma 3(ii) with $\alpha = \frac{1-\gamma}{2}$ and Corollary 4 it holds that

$$
\|HI_{3}\|_{L^2(\Omega, H)}^2 = \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^{j} \right) \right\|_{L^2}^2 \left\| P_h B \left( X^h(t_{m-i}) \right) \right\|_{L^2}^2 \right] ds
\leq \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^{j} \right) \right\|_{L(H)}^2 \left\| P_h B \left( X^h(t_{m-i}) \right) \right\|_{L^2}^2 \right] ds
\leq C \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} t_{m-i}^{-s} \Delta t^{1-s} ds \leq C \Delta t^{1-s}.
$$

Using the Itô-isometry property, Lemma 5(i) with $\alpha = 0$ and Assumption 3 yields

$$
\|HI_{4}\|_{L^2(\Omega, H)}^2 = \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^{j} \right) \left( P_h B \left( X^h(t_{m-i}) \right) - P_h B \left( X^h(t_{m-i}) \right) \right) \right\|_{L^2}^2 \right] ds
\leq \sum_{i=2}^{m-1} \int_{t_{m-1-i}}^{t_{m-1-i-1}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} S_{h, \Delta t}^{j} \right) \right\|_{L[H]}^2 \left\| P_h B \left( X^h(t_{m-i}) \right) - P_h B \left( X^h(t_{m-i}) \right) \right\|_{L^2}^2 \right] ds
\leq C \Delta t \sum_{i=2}^{m-1} \| X^h(t_{m-i}) - X^h(t_{m-i}) \|_{L^2(\Omega, H)}^2.
$$

Substituting (133), (132), (131) and (130) in (129) yields

$$
\|HI_{4}\|_{L^2(\Omega, H)}^2 \leq C \Delta t^{\gamma\min(\beta, 1-\epsilon)} + C \Delta t \sum_{i=2}^{m-1} \| X^h(t_{m-i}) - X^h(t_{m-i}) \|_{L^2(\Omega, H)}^2.
$$
Substituting (134), (127), (120), (114) and (108) in (103) yields
\[
\|X^h(t_m) - X^h_m\|^2_{L^2(\Omega, H)} \leq C \Delta t^{\min(\beta, 1 - \epsilon)} + C \Delta t \sum_{i=1}^{m-1} \|X^h(t_i) - X^h_i\|^2_{L^2(\Omega, H)}. \tag{135}
\]
Applying the discrete Gronwall’s lemma to (135) yields
\[
\|X^h(t_m) - X^h_m\|_{L^2(\Omega, H)} \leq C \Delta t^{\min(\beta, 1 - \epsilon)} \tag{136}
\]
This completes the proof of Theorem 9.

3.3 Proof of Theorem 10

Let us recall that
\[
\|X^h(t_m) - X^h_m\|^2_{L^2(\Omega, H)} \leq 25 \sum_{i=0}^{4} \|III_i\|_{L^2(\Omega, H)}, \tag{137}
\]
where $III_0$ and $III_1$ are exactly the same as $II_0$ and $II_1$ respectively. Therefore from (108) and (114) we have
\[
\|III_0\|_{L^2(\Omega, H)} + \|III_1\|_{L^2(\Omega, H)} \leq C \Delta t^{\beta/2} + C \Delta t \|X^h(t_m-1) - X^h_{m-1}\|_{L^2(\Omega, H)}. \tag{138}
\]
It remains to re-estimate $III_3$ in order to achieve higher order. We also need to re-estimate the terms involving the noise $III_2$ and $III_4$, which are given below
\[
III_2 = \int_{t_{m-1}}^{t_m} \left( e^{(t_m-s)A_h, m-1} - e^{(t_m-s)A_h, m-1} S^h_{m-1} \right) P_h dW(s), \tag{139}
\]
\[
III_3 = \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) e^{(t_m-s)A_h, m-i-1} G^h_{m-i} (X^h(s)) ds
- \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} S^h_{j, \Delta t} \right) e^{(t_m-s)A_h, m-i-1} G^h_{m-i} (X^h_{m-i}) ds
\]
\[
III_4 = \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) e^{(t_m-s)A_h, m-i-1} P_h dW(s)
- \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} S^h_{j, \Delta t} \right) P_h dW(s). \tag{140}
\]
3.3.1 Estimation of $III_2$

We can split $III_2$ in two terms as follows:

$$III_2 = \int_{t_{m-1}}^{t_m} \left[ e^{(t_{m-s})A_{h,m-1}} - e^{\Delta t A_{h,m-1}} \right] P_h dW(s)$$

$$+ \int_{t_{m-1}}^{t_m} \left[ e^{\Delta t A_{h,m-1}} - S_{h,\Delta t}^{m-1} \right] P_h dW(s)$$

$$:= III_{21} + III_{22}. \quad (141)$$

Using the Itô-isometry property, Lemma 9 (i) and (ii) and Lemma 8 (i), it holds that

$$\|III_{21}\|_{L^2(I,H)}^2 = \int_{t_{m-1}}^{t_m} \mathbb{E} \left[ \left\| e^{(t_{m-s})A_{h,m-1}} - e^{(t_{m-t_{m-1})A_{h,m-1}}} \right\|^2_{\mathcal{L}_2(H)} \right] ds$$

$$\leq \int_{t_{m-1}}^{t_m} \mathbb{E} \left[ \left\| e^{(t_{m-s})A_{h,m-1}} \left( I - e^{(s-t_{m-1})A_{h,m-1}} \right) \frac{1}{A_{h}^{1/2}} \right\|^2_{\mathcal{L}_2(H)} \right] ds$$

$$\leq C \int_{t_{m-1}}^{t_m} \left( t_{m-s} \right)^{1+} \left( s-t_{m-1} \right)^{3+} ds \leq C \Delta t^3. \quad (142)$$

Applying again the Itô-isometry property, using Lemma 8 (i) and Lemma 9 (i) yields

$$\|III_{22}\|_{L^2(I,H)}^2 = \int_{t_{m-1}}^{t_m} \mathbb{E} \left[ \left\| e^{\Delta t A_{h,m-1}} - S_{h,\Delta t}^{m-1} \right\|^2_{\mathcal{L}_2(H)} \right] ds$$

$$\leq C \int_{t_{m-1}}^{t_m} \Delta t^{3+} \left\| \frac{1}{A_h^{1/2}} P_h Q_{h,\Delta t}^{1/2} \right\|_{\mathcal{L}_2(H)}^2 ds$$

$$\leq C \Delta t^3. \quad (143)$$

Substituting (143), (142) in (141) yields

$$\|III_2\|_{L^2(I,H)}^2 \leq 2 \|III_{21}\|_{L^2(I,H)}^2 + 2 \|III_{22}\|_{L^2(I,H)}^2 \leq C \Delta t^3. \quad (144)$$

3.3.2 Estimation of $III_3$

Since $III_3$ is the same as $I_3$, it follows from (122) that

$$III_3 = III_{31} + III_{32} + III_{33} + III_{34}. \quad (145)$$
where $III_{31}$, $III_{32}$, $III_{33}$ and $III_{34}$ are respectively $II_{31}$ $II_{32}$, $II_{33}$ and $II_{34}$. Therefore from (123), (125) and (126) we have

$$\|III_{31}\|_{L^2(\Omega, H)} + \|III_{33}\|_{L^2(\Omega, H)} + \|III_{34}\|_{L^2(\Omega, H)}$$

$$\leq C \Delta t^3 + C \Delta t \sum_{i=2}^{m-1} \|X^h(t_{m-i}) - X^h_{m-i}\|_{L^2(\Omega, H)}.$$ 

(146)

To achieve higher order we need to re-estimate $III_{32}$ by using Assumption\[7\]

Recall that $III_{32}$ is given by

$$III_{32} = \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) \left( C^h_{m-i} \left( X^h(s) - X^h_{m-i} \right) \right) ds.$$ 

(147)

Using the Taylor’s formula in Banach space yields

$$III_{32}$$

$$= \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) \left( e^{(s-t_{m-i-1})A_h, m-i} - 1 \right) X^h(t_{m-i}) ds$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) \left( \frac{d}{ds} X^h(t_{m-i}) \right) ds$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) \left( \frac{d^2}{ds^2} X^h(t_{m-i}) \right) ds$$

$$+ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) R^h_{m-i} ds$$

$$:= III_{32}^{(1)} + III_{32}^{(2)} + III_{32}^{(3)} + III_{32}^{(4)},$$ (148)

where

$$R^h_{m-i} := \int_0^1 \left( C^h_{m-i} \right)^{(n)} \left( X^h(t_{m-i}) \right) ds.$$ (149)

Inserting an appropriate power of $A_h$, using Lemma\[2\] and Corollary\[1\] it holds that

$$\|III_{32}^{(1)}\|_{L^2(\Omega, H)} \leq \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left[ E \left( \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_h, j} \right) A_h^{1-\epsilon} \right) \right]^2 ds$$

$$\times \left( A_h^{1+\epsilon} \left( e^{(s-t_{m-i-1})A_h, m-i} - 1 \right) \right) \|X^h(t_{m-i})\|_{L^2(\Omega, H)}^{1/2} ds$$

$$\leq C \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} t_i^{1+\epsilon} (s-t_{m-i-1})^{1-\epsilon} ds$$

$$\leq C \Delta t^{1-\epsilon} \sum_{i=2}^{m-1} t_i^{1+\epsilon} \Delta t \leq C \Delta t^{1-\epsilon}. \quad (150)$$
Using Lemma 3, Lemma 7 (ii) and Corollary 11 yields

\[
\|III_{32}\|_{L^2(\Omega,H)}^2 
\leq C \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \int_{t_{m-i-1}}^{s} e^{(s-\sigma)A_{h,m-i-1}} \left( G_{m-i}^h \right)(X^h(\sigma)) ds \] ds
\leq C \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} (s-t_{m-i-1}) ds
\leq C \Delta t. \tag{151}
\]

Since the expectation of the cross-product vanishes, using Itô-isometry property, triangle inequality, Hölder inequality, Lemma 2 yields

\[
\|III_{32}\|_{L^2(\Omega,H)}^2 
= E \left[ \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \left( \sum_{j=m-i}^{m-1} \int_{t_{m-i-1}}^{s} e^{\Delta t A_{h,j}} \right) \left( G_{m-i}^h \right)' \left( X^h(t_{m-i}) \right) \int_{t_{m-i-1}}^{s} e^{(s-\sigma)A_{h,m-i-1}} P_h dW(\sigma) ds \right]^2
\leq \Delta t \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} E \left[ \left( \sum_{j=m-i}^{m-1} \int_{t_{m-i-1}}^{s} e^{\Delta t A_{h,j}} \right) \left( G_{m-i}^h \right)' \left( X^h(t_{m-i}) \right) e^{(s-\sigma)A_{h,m-i-1}} P_h \|Q^h\|^2 \right] ds
\leq \Delta t \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} E \left[ \left( \sum_{j=m-i}^{m-1} \int_{t_{m-i-1}}^{s} e^{\Delta t A_{h,j}} \right) \left( G_{m-i}^h \right)' \left( X^h(t_{m-i}) \right) e^{(s-\sigma)A_{h,m-i-1}} P_h \|Q^h\|^2 \right] \|Q^h\|^2 ds, \tag{152}
\]

Using Lemma 9 yields

\[
E \left[ \left( G_{m-i}^h \right)' \left( X^h(t_{m-i}) \right) e^{(s-\sigma)A_{h,m-i-1}} P_h Q^h \right]^2_{L^2(H)}
= E \left[ \left( G_{m-i}^h \right)' \left( X^h(t_{m-i}) \right) e^{(s-\sigma)A_{h,m-i-1}} A_{h}^{A_{h}^{-1}} P_h Q^h \right]^2_{L^2(H)}
\leq E \left[ \left( G_{m-i}^h \right)' \left( X^h(t_{m-i}) \right) e^{(s-\sigma)A_{h,m-i-1}} A_{h}^{A_{h}^{-1}} P_h Q^h \right]^2_{L^2(H)}
\leq C(\sigma - \beta)^{\min(-1+\beta,0)}. \tag{153}
\]

Substituting (153) in (152) yields

\[
\|III_{32}\|_{L^2(\Omega,H)}^2 \leq C \Delta t \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \int_{t_{m-i-1}}^{s} (s-\sigma)^{\min(-1+\beta,0)} ds \]
\leq C \Delta t^{\min(1+\beta,2)}. \tag{154}
\]
Using Lemma 9 (ii) and Lemma 1 yields
\[
\left\| A^h R_{h,m-1}^h \right\|_{L^2(\Omega,H)} \leq C \left\| X^h(s) - X^h(t_{m-1}) \right\|_{L^2(\Omega,H)}^2 \\
\leq C \left\| X^h(s) - X^h(t_{m-1}) \right\|^2_{L^2(\Omega,H)} \\
\leq C \Delta t^{\min(\beta,1)}. \tag{155}
\]
Therefore we obtain the following estimate for \( III_{32}^{(4)} \)
\[
\| III_{32}^{(4)} \|_{L^2(\Omega,H)} \leq C \Delta t^{\min(\beta,1)} \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-1}} \mathbb{E} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) A^2 \right] ds \\
\leq C \Delta t^{\min(\beta,1)} \sum_{i=2}^{m-1} t_i^{-\frac{\beta}{2}} \Delta t \\
\leq C \Delta t^{\min(\beta,1)}. \tag{156}
\]
Substituting (156), (154), (151) and (150) in (148) yields
\[
\| III_{32} \|_{L^2(\Omega,H)} \leq \| III_{32}^{(1)} \|_{L^2(\Omega,H)} + \| III_{32}^{(2)} \|_{L^2(\Omega,H)} + \| III_{32}^{(3)} \|_{L^2(\Omega,H)} + \| III_{32}^{(4)} \|_{L^2(\Omega,H)} \\
\leq C \Delta t^{\beta/2-\epsilon}. \tag{157}
\]
Substituting (157) and (146) in (145) yields
\[
\| III_{4} \|_{L^2(\Omega,H)} \leq C \Delta t^{\beta/2-\epsilon}. \tag{158}
\]

3.3.3 Estimation of \( III_{4} \)

We can recast \( III_{4} \) in two terms as follows
\[
III_{4} = \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-1}} \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(t_{m-1}-s)A_{h,m-1}} - I \right) P_h dW(s) \\
+ \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-1}} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h,\Delta t}^j \right) \right] P_h dW(s) \\
:= III_{41} + III_{42}. \tag{159}
\]
Using the Itô-isometry property, Lemma 9 (i) and (iv), Lemma 2 and [26, Lemma 10] yields

\[
\|III_{1}\|^2_{L^2(0,T,L^2)} \\
\leq \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \mathbb{E} \left[ \left\| \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(t_{m-i} - s) A_{h,m-1}} - I \right) P_h Q_{h}^2 \right\|_{L^2(0,T,L^2)}^2 \right] ds \\
\leq C \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \mathbb{E} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) A_h^{-\beta} \left( e^{(t_{m-i} - s) A_{h,m-1}} - I \right) A_h^{-\beta + \epsilon} \right] ds \\
\leq C \Delta t^{-\beta - \epsilon} \sum_{i=2}^{m-1} t_{i-1}^{-1+\epsilon} \Delta t \leq C \Delta t^{1-\epsilon}. \quad (160)
\]

Using again the Itô-isometry property and Lemma 3 (i) yields

\[
\|III_{2}\|^2_{L^2(0,T,L^2)} \\
\leq \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \mathbb{E} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h,j}^i \right) \right] P_h Q_{h}^2 \left\|_{L^2(0,T,L^2)}^2 \right] ds \\
\leq C \sum_{i=2}^{m-1} \int_{t_{m-1}}^{t_{m-i}} \mathbb{E} \left[ \left( \prod_{j=m-i}^{m-1} e^{\Delta t A_{h,j}} \right) - \left( \prod_{j=m-i}^{m-1} S_{h,j}^i \right) \right] A_h^{-\frac{\beta-\epsilon}{2}} \left\|_{L^2(0,T,L^2)}^2 \right] ds \\
\leq C \Delta t^{1-\epsilon} \sum_{i=2}^{m-1} t_{i-1}^{-1+\epsilon} \Delta t \leq C \Delta t^{1-\epsilon}. \quad (161)
\]

If \( \beta < 1 \) then applying Lemma 3 (ii) yields

\[
\|III_{2}\|^2_{L^2(0,T,L^2)} \\
\leq C \Delta t^{1-\epsilon} \sum_{i=2}^{m-1} t_{i-1}^{-1+\epsilon} \Delta t \leq C \Delta t^{1-\epsilon}. \quad (162)
\]
If $\beta \geq 1$ then applying Lemma (iii) yields
\[
\|III_{42}\|_2^2 \leq C \sum_{i=2}^{m-1} \int_{t_{m-i-1}}^{t_{m-i}} \Delta t^{3-\epsilon} t_{i-1}^{-1+\epsilon} ds \leq C \Delta t^{3-\epsilon}.
\] (163)

Therefore for all $\beta \in [0, 2)$ it holds that
\[
\|III_{42}\|_2^2 \leq C \Delta t^{3-\epsilon}.
\] (164)

Substituting (164) and (160) in (159) yields
\[
\|III_{42}\|_2^2 \leq C \Delta t^{3-\epsilon}.
\] (165)

Substituting (165), (158), (144) and (138) in (137) yields
\[
\|X_h(t_m) - X_h(t_m)\|_2^2 \leq C \Delta t^{3/2-\epsilon}.
\] (166)

This completes the proof of Theorem 10.

4 Numerical Simulations

We consider the following stochastic reactive dominated advection diffusion reaction equation with constant diagonal diffusion tensor
\[
\begin{align*}
\begin{array}{c}
dX = \left[ \nabla \cdot (D \nabla X) - \nabla \cdot (q X) - \frac{10X}{X + 1} \right] dt + b(X)dW \quad \text{and} \\
D = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-1} \end{pmatrix}
\end{array}
\end{align*}
\] (167, 168)

with mixed Neumann-Dirichlet boundary conditions on $\Lambda = [0, L_1] \times [0, L_2]$. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x,y) : x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. The eigenfunctions $\{e_{i,j}\} = \{e^{(1)}_i \otimes e^{(2)}_j\}_{i,j \geq 0}$ of the covariance operator $Q$ are the same as for Laplace operator $\Delta$ with homogeneous boundary condition given by
\[
e^{(i)}_l(x) = \sqrt{\frac{1}{L_l}}, \quad e^{(i)}_l(x) = \sqrt{\frac{2}{L_l}} \cos \left( \frac{i \pi}{L_l} x \right), \quad i \in \mathbb{N},
\]
where $l \in \{1, 2\}$, $x \in \Lambda$. We assume that the noise can be represented as
\[
W(x,t) = \sum_{i,j \in \mathbb{N}^2} \sqrt{A_{i,j}} e^{(i)}_i(x) e^{(j)}_j(x) \beta_{i,j}(t),
\] (169)
where $\beta_{i,j}(t)$ are independent and identically distributed standard Brownian motions, $\lambda_{i,j}$, $(i,j) \in \mathbb{N}^2$ are the eigenvalues of $Q$, with
\[
\lambda_{i,j} = (i^2 + j^2)^{-(\beta+\epsilon)}, \quad \beta > 0,
\]
in the representation (169) for some small $\epsilon > 0$. For additive noise, we take $b(u) = 1$, so Assumption 6 is obviously satisfied for $\beta \in (0,2]$. For multiplicative noise, we take $b(u) = u$ in (18). Therefore, from [13, Section 4] it follows that the operators $B$ defined by (18) fulfill obviously Assumption 4 and Assumption 5. For both additive and multiplicative noise, the function $F(X) = -\frac{10X}{1+X}$ obviously satisfies the global Lipschitz condition in Assumption 3 and Assumption 7. We obtain the Darcy velocity field $q = (q_i)$ by solving the following system
\[
\nabla \cdot q = 0, \quad q = -\frac{k}{\mu} \nabla p,
\]
with Dirichlet boundary conditions on $\Gamma_D^1 = \{ 0, L_1 \} \times [0, L_2]$ and Neumann boundary conditions on $\Gamma_N^1 = (0, L_1) \times \{ 0, L_2 \}$ such that
\[
p = \begin{cases} 
1 & \text{in } \{ 0 \} \times [0, L_2] \\
0 & \text{in } \{ L_1 \} \times [0, L_2] 
\end{cases}
\]
and $-k \nabla p(x,t) \cdot n = 0$ in $\Gamma_N^2$. Note that $k$ is the permeability tensor. We use a random permeability field as in [38] and take $\mu = 10$. The finite volume method viewed as a finite element method (see [37]) is used to the advection and the finite element method is used for the remainder. In the legends of our graphs, we use the following notations
1. ’Rosenbrock-A-noise’ is using for graphs from our Rosenbrock scheme with additive noise.
2. ’Rosenbrock-M-noise’ is using for graphs from our Rosenbrock scheme with multiplicative noise.
3. ’Expo-Rosenbrock-A-noise’ is using for graphs stochastic exponential Rosenbrock scheme presented in [26] with additive noise.
4. ’Expo-Rosenbrock-M-noise’ is using for graphs stochastic exponential Rosenbrock scheme presented in [26] with multiplicative noise.

We take $L_1 = 2$ and $L_2 = 2$ and our reference solutions samples are numerical solutions using at time step of $\Delta t = 1/2048$. The errors are computed at the final time $T = 1$. The initial solution is $X_0 = 0$, so we can therefore expect high orders convergence, which depend only on the noise term. For both additive and multiplicative noise, we use $\beta = 2$ and $\epsilon = 10^{-1}$. In Figure (a) the graphs of strong errors versus the time steps are plotting for stochastic Rosenbrock scheme and exponential Rosenbrock. The orders of convergence is 0.59 (exponential Rosenbrock scheme) and 0.55 (Rosenbrock scheme) for multiplicative noise and 0.95 (Rosenbrock scheme) and 0.92 (Rosenbrock scheme) for
additive noise, which are close to 0.5 and 1 in our theoretical results in Theorem 9 and Theorem 10 respectively. The implementation of the stochastic Rosenbrock-type scheme is straightforward and only need the resolution of a linear system of equations at each time step. For efficiency, all linear systems are solved using the Matlab function bicgstab coupled with ILU(0) preconditioners with no fill-in. The ILU(0) are done on the deterministic part of the the matrix $A_{h,m}$, that is $(I - \Delta t A_h)$, at each time step. Figure 1(b) shows the mean of CPU time per sample versus the root mean square $L^2$ errors. As we can observe, the stochastic Rosenbrock scheme is well efficient than the stochastic exponential scheme, thanks to the preconditioners. Note that the matrix exponential functions are computed using Krylov subspace technique with dimension 10 in the stochastic Rosenbrock scheme.

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