EXISTENCE AND NON-DEGENERACY OF POSITIVE MULTI-BUBBLING SOLUTIONS TO CRITICAL ELLIPTIC SYSTEMS OF HAMILTONIAN TYPE

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ABSTRACT. This paper deals with the following critical elliptic systems of Hamiltonian type, which are variants of the critical Lane-Emden systems and analogous to the prescribed curvature problem:

\[
\begin{align*}
-\Delta u_1 &= K_1(y)u_1^p, \quad y \in \mathbb{R}^N, \\
-\Delta u_2 &= K_2(y)u_2^q, \quad y \in \mathbb{R}^N, \\
u_1, u_2 &> 0,
\end{align*}
\]

where \( N \geq 5, p, q \in (1, \infty) \) with \( \frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \), \( K_1(y) \) and \( K_2(y) \) are positive radial potentials.

At first, under suitable conditions on \( K_1, K_2 \) and the certain range of the exponents \( p, q \), we construct an unbounded sequence of non-radial positive vector solutions, whose energy can be made arbitrarily large. Moreover, we prove a type of non-degeneracy result by use of various Pohozaev identities, which is of great interest independently. The indefinite linear operator and strongly coupled nonlinearities make the Hamiltonian-type systems in stark contrast both to the systems of Gradient type and to the single critical elliptic equations in the study of the prescribed curvature problems. It is worth noting that, in higher-dimensional cases \( (N \geq 5) \), there have been no results on the existence of infinitely many bubbling solutions to critical elliptic systems, either of Hamiltonian or Gradient type.

1. Introduction and main results

In this paper, we are concerned with the multiplicity of solutions and its non-degeneracy property of the following elliptic system

\[
\begin{align*}
-\Delta u_1 &= K_1(y)u_1^p, \quad y \in \mathbb{R}^N, \\
-\Delta u_2 &= K_2(y)u_2^q, \quad y \in \mathbb{R}^N, \\
u_1, u_2 &> 0, \quad (u_1, u_2) \in \dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N),
\end{align*}
\]

where \( N \geq 5 \) and \( K_1, K_2 \in C(\mathbb{R}^N) \) are positive radial potentials, \( (p, q) \) is a pair of positive numbers lying on the critical hyperbola

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N},
\]

\( \dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \) and \( \dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N) \) are defined as the standard homogeneous Sobolev spaces. Without loss of generality, we may assume that \( p \leq \frac{N+2}{N-2} \leq q \).

Date: June 1, 2022.

Key words and phrases. Critical Lane-Emden systems; Multi-bubbling solutions; Reduction method; Pohozaev identities.
The standard Lane-Emden system
\[
\begin{align*}
-\Delta u_1 &= |u_2|^{p-1}u_2, & \text{in } \Omega, \\
-\Delta u_2 &= |u_1|^{q-1}u_1, & \text{in } \Omega, \\
u_1 = u_2 &= 0, & \text{on } \partial\Omega
\end{align*}
\]  
(1.3)
with a smooth bounded domain $\Omega \subset \mathbb{R}^N$ for $N \geq 3$ and $p, q \in (0, \infty)$ is a typical Hamiltonian-type strongly coupled elliptic systems, which have been a subject of intense interest and has a rich structure. Due to the fact that tools for analyzing a single equation cannot be used in a direct way to treat the systems, it seems that less attention has been paid to the existence of solutions for strongly indefinite systems and their qualitative properties. One of the first result about positive solutions of (1.3) appeared in [15] based on topological arguments. In [21], a variational argument relying on a linking theorem was used to show an existence result. In [10], the existence, positivity and uniqueness of ground state solutions for (1.3) was studied. One may also refer to [3, 37] and the surveys in [22]. It is well known that the system is strongly affected by the values of the couple $(p, q)$. The existence theory is associated with the critical hyperbola (1.2), which was introduced by [9] and [40]. According to [19, 26] and [6], we know that if $pq \neq 1$ and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N^2}{N^2}$, then problem (1.3) has a solution. While if the domain $\Omega$ is star-shaped and if $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N^2}{N^2}$, then (1.3) has no solutions.

Particularly for $\Omega = \mathbb{R}^N$, a positive ground state $(U, V)$ to the following system was found in [28],
\[
\begin{align*}
-\Delta U &= |V|^{p-1}V, & \text{in } \mathbb{R}^N, \\
-\Delta V &= |U|^{q-1}U, & \text{in } \mathbb{R}^N, \\
(U, V) &\in \dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N),
\end{align*}
\]  
(1.4)
where $N \geq 3$ and $(p, q)$ satisfy (1.2). By Sobolev embeddings, there holds that
\[
\begin{align*}
\dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) &\hookrightarrow \dot{W}^{1,p^*}(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N), \\
\dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N) &\hookrightarrow \dot{W}^{1,q^*}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N),
\end{align*}
\]
with
\[
\frac{1}{p^*} = \frac{p}{p+1} - \frac{1}{N}, \quad \frac{1}{q^*} = \frac{q}{q+1} - \frac{1}{N},
\]
and so the following energy functional is well-defined in $\dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N)$:
\[
I_0(u, v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} - \frac{1}{q+1} \int_{\mathbb{R}^N} |u|^{q+1}.
\]
According to [2], the ground state is radially symmetric and decreasing up to a suitable translation. Thanks to [26] and [41], the positive ground state $(U_{0,1}, V_{0,1})$ of (1.4) is unique with $U_{0,1}(0) = 1$ and the family of functions
\[
(U_{\xi,\lambda}(y), V_{\xi,\lambda}(y)) = (\lambda^{\frac{N}{p+1}} U_{0,1}(\lambda(y - \xi)), \lambda^{\frac{N}{q+1}} V_{0,1}(\lambda(y - \xi))
\]
for any $\lambda > 0, \xi \in \mathbb{R}^N$ also solves system (1.4). Sharp asymptotic behavior of the ground states to (1.4) (see [26]) and the non-degeneracy for (1.4) at each ground state (see [20]) play an important role to construct bubbling solutions especially using Lyapunov-Schmidt reduction methods.
Generally, in the literature, systems of the form
\[
\begin{cases}
-\Delta u = H_v(u, v), \\
-\Delta v = H_u(u, v)
\end{cases}
\]
with a Hamiltonian such as \( H(u, v) = \frac{|u|^{p+1}}{p+1} + \frac{|v|^{q+1}}{q+1} \) are usually referred to as elliptic systems of Hamiltonian type. It is also said strongly coupled in the sense that \( u \equiv 0 \) if and only if \( v \equiv 0 \). Another classical system is of Gradient type:
\[
\begin{cases}
-\Delta u = F_u(u, v), \\
-\Delta v = F_v(u, v)
\end{cases}
\]
with a functional \( F(u, v) = \frac{1}{2p}(\mu_1|u|^{2p} + \mu_2|v|^{2p} + 2\beta|u|^p|v|^p) \) for example. They are usually called nonlinear Schrödinger systems and have been the subject of extensive mathematical studies in recent years, for example, \([16, 30, 29, 31, 38, 39]\). We also refer to \([5, 12, 13, 23, 36, 35]\) for more references therein about systems with both critical and subcritical exponents.

Hamiltonian-type systems are significantly different from the Gradient type. Due to the indefinite property of the linear operator and the strongly coupled nonlinearity of Hamiltonian-type systems, the classical variational methods can not be used directly in the study of certain problems, such as the existence of infinitely many positive solutions. Moreover, in sharp contrast to the Gradient-type systems, it seems impossible that the strongly coupled systems have segregated vector solutions, whose components concentrate at different points respectively. Hence in this work, we apply finite-dimensional reduction method, combined with local Pohozaev identities, to construct infinitely many synchronized positive bubbling solutions with special symmetry. By “synchronized” we mean that the components of bubbling solutions to \((1.1)\) concentrate at the same points. It is also worth noting that, in higher-dimensional cases \((N \geq 5)\), there are no results on the existence of infinitely many concentrated solutions to critical elliptic systems, either of Hamiltonian or Gradient type.

System \((1.1)\) is analogous in form to the following scalar equation:
\[
\begin{cases}
-\Delta u = K(y)u^{\frac{N+2}{N-2}}, & u > 0, \ y \in \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]
which has been extensively studied, see \([1, 4, 7, 8, 11, 14, 17, 18, 24, 25, 32, 33, 34, 43]\) for example. In particular, Wei and Yan \([42]\) developed a technique to apply the reduction argument for non-singularly perturbed elliptic problems and constructed infinitely many solutions to problem \((1.5)\).

In this paper, we suppose \( N \geq 5 \) and assume that \( K_1 \) and \( K_2 \) are positive and radial satisfying the following conditions:

(K) There exists a constant \( r_0 > 0 \) such that for \( r \in (r_0 - \delta, r_0 + \delta) \),
\[
\begin{align*}
K_1(r) &= 1 - c_1|r - r_0|^{m_1} + O(|r - r_0|^{m_1 + \theta_1}), \\
K_2(r) &= 1 - c_2|r - r_0|^{m_2} + O(|r - r_0|^{m_2 + \theta_2}),
\end{align*}
\]
where \( c_1, c_2 > 0, \theta_1, \theta_2 > 0 \) are some constants, and \( m_1, m_2 \in [2, N - 2) \). Without loss of generality, we set \( m := \min\{m_1, m_2\} = m_1 \), and \( m < (2p - 1)(N - 2) - 8 \).
Moreover, to obtain sufficient decay of the bubbling solutions, we would require that the smaller exponent \( p \) on the critical hyperbola (1.2) should further satisfy

\[
\max \left\{ \frac{N + 1}{N - 2}, \frac{N + 8}{2(N - 2)}, \frac{N(N - 2)}{(N - 2)^2 - (N - 2 - m)} \right\} < p \leq \frac{N + 2}{N - 2},
\]

which precisely lead to the following assumptions:

\[
(P) \begin{cases} 
\frac{13}{6} < p \leq \frac{7}{3}, & \text{if } N = 5; \\
\max \left\{ \frac{N + 1}{N - 2}, \frac{N(N - 2)}{(N - 2)^2 - (N - 2 - m)} \right\} < p \leq \frac{N + 2}{N - 2}, & \text{if } N \geq 6.
\end{cases}
\]

Our main result in this paper can be stated as follows.

**Theorem 1.1.** Suppose \( N \geq 5 \) and \( p, q \) satisfy (1.2) and (P). If \( K_1(r), K_2(r) \) satisfy (K), then (1.1) has infinitely many non-radial positive solutions.

**Remark 1.2.** We make some supplementary explanations for the assumption (P). In fact, \( p > \frac{N + 8}{2(N - 2)} \) is equivalent to \((2p - 1)(N - 2) - 8 > 2\), which holds for \( p > \frac{N + 1}{N - 2} \) if \( N \geq 6 \). Thus, for \( N \geq 6 \), \( \max \left\{ \frac{N + 1}{N - 2}, \frac{N + 8}{2(N - 2)}, \frac{N(N - 2)}{(N - 2)^2 - (N - 2 - m)} \right\} = \max \left\{ \frac{N + 1}{N - 2}, \frac{N(N - 2)}{(N - 2)^2 - (N - 2 - m)} \right\} \); while for \( N = 5 \), it holds directly that \( \max \left\{ \frac{N + 1}{N - 2}, \frac{N + 8}{2(N - 2)}, \frac{N(N - 2)}{(N - 2)^2 - (N - 2 - m)} \right\} = \frac{13}{6} \). Moreover, note that when \( m \geq 3 \), \( \max \left\{ \frac{N + 1}{N - 2}, \frac{N(N - 2)}{(N - 2)^2 - (N - 2 - m)} \right\} = \frac{N + 1}{N - 2} \).

**Remark 1.3.** There are very few results about this critical elliptic systems of Hamiltonian type by use of the finite-dimensional reduction method except [27], where they construct families of blowing-up solutions to some Brezis-Nirenberg-type problem on smooth bounded domains. Different from finitely many multi-bubbling solutions studied in the bounded domain, the construction of infinitely many multi-bubbling solutions in the whole space \( \mathbb{R}^N \) requires relatively higher decay rate at infinity of the ground state solutions to the corresponding limit problems (see Lemma 1.4 below). Therefore, we require that the smaller exponent \( p \) should not be too small (larger than \( \frac{N + 1}{N - 2} \)). We expect that the condition (P) is almost sharp in the present results, although other problems involving the case \( p \leq \frac{N + 1}{N - 2} \) would be considered in our future work.

Before introducing the non-degeneracy result, we outline the main idea in the proof of Theorem 1.1. Let us fix a positive integer \( k \geq k_0 \), where \( k_0 \) is a large integer to be determined. Set

\[
\mu = \mu_k = k^{\frac{N - 2}{N - 2 - m}}
\]

to be the scaling parameter. Let \( 2^* = \frac{2N}{N - 2} \). Using the transformation

\[
u_1(y) \mapsto \mu^{-\frac{N}{q + 1}} u_1 \left( \frac{y}{\mu} \right), \quad u_2(y) \mapsto \mu^{-\frac{N}{p + 1}} u_2 \left( \frac{y}{\mu} \right),
\]
system (1.1) becomes

\[
\begin{align*}
-\Delta u_1 &= K_1 \left( \frac{y}{\mu} \right) u_2^p, \quad y \in \mathbb{R}^N, \\
-\Delta u_2 &= K_2 \left( \frac{y}{\mu} \right) u_1^q, \quad y \in \mathbb{R}^N, \\
u_1, u_2 &> 0, \quad (u_1, u_2) \in W^{2, \frac{p+1}{p}}(\mathbb{R}^N) \times W^{2, \frac{q+1}{q}}(\mathbb{R}^N).
\end{align*}
\]

(1.6)

Our proof requires the standard steps of the reduction procedure, where the following sharp asymptotic behavior and the non-degeneracy of the ground states for (1.4) play an important role.

Lemma 1.4. [26] Assume that \( p \leq \frac{N+2}{N-2} \). There exist some positive constants \( \alpha = \alpha_{N,p} \) and \( b = b_{N,p} \) depending only on \( N \) and \( p \) such that

\[
\lim_{r \to \infty} r^{N-2} V_{0,1}(r) = b;
\]

(1.7)

while

\[
\begin{align*}
\lim_{r \to \infty} r^{(N-2)p-2} U_{0,1}(r) &= a, \quad \text{if } p < \frac{N}{N-2}; \\
\lim_{r \to \infty} \frac{r^{N-2}}{\log r} U_{0,1}(r) &= a, \quad \text{if } p = \frac{N}{N-2}; \\
\lim_{r \to \infty} r^{N-2} U_{0,1}(r) &= a, \quad \text{if } p > \frac{N}{N-2}.
\end{align*}
\]

(1.8)

Furthermore, in the last case, we have \( b^p = \alpha ((N-2)p-2)(N-(N-2)p) \).

Lemma 1.5. [20] Set

\[
(\Psi^0_{0,1}, \Phi^0_{0,1}) = \left( y \cdot \nabla U_{0,1}, \frac{NU_{0,1}}{q+1}, y \cdot \nabla V_{0,1}, \frac{NV_{0,1}}{p+1} \right)
\]

and

\[
(\Psi^l_{0,1}, \Phi^l_{0,1}) = (\partial_l U_{0,1}, \partial_l V_{0,1}), \quad \text{for } l = 1, \ldots, N.
\]

Then the space of solutions to the linear system

\[
\begin{align*}
-\Delta \Psi &= p V_{0,1}^{p-1} \Phi, \quad \text{in } \mathbb{R}^N, \\
-\Delta \Phi &= q U_{0,1}^{q-1} \Psi, \quad \text{in } \mathbb{R}^N, \\
(\Psi, \Phi) &\in \dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N)
\end{align*}
\]

(1.9)

is spanned by

\[
\left\{ (\Psi^0_{0,1}, \Phi^0_{0,1}), (\Psi^1_{0,1}, \Phi^1_{0,1}), \ldots, (\Psi^N_{0,1}, \Phi^N_{0,1}) \right\}.
\]

Let \( y = (y', y'') \), \( y' \in \mathbb{R}^2, y'' \in \mathbb{R}^{N-2} \). We define

\[
H_8 = \left\{ (u_1, u_2) \in \dot{W}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \times \dot{W}^{2, \frac{q+1}{q}}(\mathbb{R}^N), u_i \text{ is even in } y_h, h = 2, \ldots, N, \\
u_i \left( r \cos \theta, r \sin \theta, y'' \right) = u_i \left( r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y'' \right), \quad i = 1, 2 \right\}.
\]

For any large integer \( k > 0 \), let

\[
x_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k,
\]

where \( \mu = \frac{1}{2}, \lambda = 2 \).
where 0 is the zero vector in $\mathbb{R}^{N-2}$. Set
\[
W_1(y) = W_{1,r}(y) = \sum_{j=1}^{k} U_{x_j,\lambda}(y), \quad W_2(y) = W_{2,r}(y) = \sum_{j=1}^{k} V_{x_j,\lambda}(y).
\]
Then both $W_1(y)$ and $W_2(y)$ are even in $y_h$, $h = 2, \ldots, N$, and
\[
W_i(r \cos \theta, r \sin \theta, y'') = W_i\left(r \cos\left(\theta + \frac{2\pi j}{k}\right), r \sin\left(\theta + \frac{2\pi j}{k}\right), y''\right),
\]
where $i = 1, 2, y = (y', y''), y' \in \mathbb{R}^2, y'' \in \mathbb{R}^{N-2}$.

In this paper, we assume $r \in \left[\tau_0 - \frac{1}{\mu^\theta}, \tau_0 + \frac{1}{\mu^\theta}\right]$ for some small $\tau > 0$, and $L_0 \leq \lambda \leq L_1$, for some constant $L_1 > L_0 > 0$. We will prove Theorem 1.1 by verifying the following result.

**Theorem 1.6.** Under the same assumptions as in Theorem 1.1, there exists an integer $k_0 > 0$ such that for any integer $k \geq k_0$, (1.6) has a positive solution $(u_{1,k}, u_{2,k})$ of the form
\[
u_{i,k} = W_{i,r_k}(y) + \varphi_{i,k}, \quad i = 1, 2,
\]
where $\varphi_{i,k} \in H_s, r_k \in [\tau_0 - \frac{1}{\mu^\theta}, \tau_0 + \frac{1}{\mu^\theta}]$ and as $k \to +\infty, \|\varphi_{i,k}\|_{L^\infty} \to 0$.

**Remark 1.7.** In contrast to the Gradient type systems, the strongly coupled characteristic make it more difficult to show the positivity of the solutions. We adopt some new idea to do it independently. Actually, we prove that the solutions obtained through the reduction scheme are positive vectors by establishing refined point-wise estimates and precise control of the errors (see Lemma 3.3).

Recall $\mu_k = k^{-\frac{N-2}{2}}$. To state our non-degeneracy result we return to the solutions $(v_{1,k}, v_{2,k})$ of the original problem (1.1) by taking
\[
v_{1,k}(y) = \mu_k^{\frac{N}{p+1}} u_{1,k}(\mu_k y), \quad v_{2,k}(y) = \mu_k^{\frac{N}{p+1}} u_{2,k}(\mu_k y).
\]
We introduce the rescaled norms $\|v_1\|_{*,1}$ and $\|v_2\|_{*,2}$ as follows:
\[
\|v_1\|_{*,1} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{\mu_k^{\frac{N}{p+1}}}{(1 + \mu_k |y - \bar{x}_j|)^{\frac{N-2}{2} + \gamma}} \right)^{-1} |v_1(y)|,
\]
\[
\|v_2\|_{*,2} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{\mu_k^{\frac{N}{p+1}}}{(1 + \mu_k |y - \bar{x}_j|)^{\frac{N-2}{2} + \gamma}} \right)^{-1} |v_2(y)|
\]
with $\bar{x}_j = \frac{x_j}{\mu_k}, \gamma = 1 + \bar{\eta}$ and $\bar{\eta} > 0$ small. Set
\[
W_{1,\tilde{r}_k}(y) = \sum_{j=1}^{k} U_{\bar{x}_j,\lambda_k}(y), \quad W_{2,\tilde{r}_k}(y) = \sum_{j=1}^{k} V_{\bar{x}_j,\lambda_k}(y)
\]
with $\tilde{r}_k = |\bar{x}_j|, \lambda_k = \lambda \mu_k$. Theorem 1.6 indeed gives a solution $(v_{1,k}, v_{2,k})$ to (1.1) of the form
\[
u_{i,k} = W_{i,\tilde{r}_k}(y) + \tilde{\varphi}_{i,k}, \quad i = 1, 2,
\]
where \( \bar{\varphi}_{i,k} \in H_s \), for some \( \bar{\theta} > 0 \) small, \( |\bar{r}_k - r_0| = O\left( \frac{1}{\mu^{1+\bar{\theta}}} \right) \), \( \mu_k = O\left( k^{\frac{N-2-\mu}{N}} \right) \) and as \( k \to +\infty \). Moreover,

\[
\|\bar{\varphi}_{1,k}\|_{s,1} + \|\bar{\varphi}_{2,k}\|_{s,2} = O\left( \frac{1}{\mu^{s+\bar{\theta}}} \right).
\]

The linearized operator \( Q_k \) related to \( (v_{1,k}, v_{2,k}) \) is defined by

\[
Q_k(\xi_1, \xi_2) = \left( -\Delta \xi_1 - pK_1(v_{p,1,k}^{p-1}) \xi_2, -\Delta \xi_2 - qK_2(v_{q,1,k}^{q-1}) \xi_1 \right).
\]
(1.12)

Another result of this present paper is the following:

**Theorem 1.8.** Under the assumptions in Theorem 1.6, we further suppose \( K_1 \) and \( K_2 \) satisfy that

\[
\Delta K_i - r(\Delta K_i + \frac{1}{2}(\Delta K_i)') \neq 0, \text{ at } r = r_0, \text{ for } i = 1, 2.
\]
(1.13)

If \( (\xi_1, \xi_2) \in H_s \) solves \( Q_k(\xi_1, \xi_2) = 0 \), then there holds \( (\xi_1, \xi_2) = 0 \).

**Remark 1.9.** The study of the non-degeneracy result is inspired by Guo-Musso-Peng-Yan [24], which is involved with finding new solutions whose shape is, at main order,

\[
\sum_{j=1}^{k} U_{x,\mu} + \sum_{j=1}^{n} U_{p,\lambda},
\]

where \( k \) and \( n \) are large integers,

\[
x_j = \left( r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0 \right), \quad j = 1, \ldots, k
\]

\[
p_j = \left( 0, 0, t \cos \frac{2(j-1)\pi}{n}, t \sin \frac{2(j-1)\pi}{n}, 0 \right), \quad j = 1, \ldots, n,
\]

\( r \) and \( t \) are close to \( r_0 \). Applying Theorem 1.8, we can also obtain new couple of concentrated solutions to (1.1) by gluing together bubbles with different concentration rates, whose concentrated points are distributed on the vertices of regular polygons on two disjoint coordinate planes.

**Remark 1.10.** The strong nonlinear characteristic of the Hamiltonian-type system makes our research much more difficult and nontrivial compared with the prescribed curvature problem. Some new techniques are needed in our work. Firstly, since the exponents \( p \) and \( q \) vary within some appropriate range on a critical hyperbola, we perform subtle decay estimates to cover a sharp range. Secondly, since the Hamiltonian-type systems possess indefinite linear operator, we choose some suitable workspace, which is not a Hilbert space even, to carry out the finite-dimensional reduction procedure. Thirdly, due to the strongly coupled nonlinearities, it seems feeble to use the common method to show the solutions of the system are positive vectors. Instead, we take some new ways to handle this difficulty when showing the vector-solutions positive. Finally, we will establish various local Pohozaev identities, which really makes all the difference in the proof of the existence and non-degeneracy of multi-bubbling solutions with some special symmetry.
This paper is organized as follows. In section 2, we perform the linear analysis and carry out the reduction procedure in suitable workspace with weighted maximum norm. In section 3, we solve the reduced finite-dimensional problem and prove Theorem 1.1. The non-degeneracy of the multi-bubbling solutions in Theorem 1.8 is obtained in section 4. Some delicate estimates are put in the appendix.

2. Finite-dimensional reduction

In view of the decay properties of the asymptotic solutions described in Lemma 1.4, in the case of $p > \frac{N}{N-2}$, we set

$$\|u\|_* = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\sigma}} \right)^{-1} |u(y)|,$$

$$\|f\|_{**} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\sigma+2}} \right)^{-1} |f(y)|,$$

where

$$\sigma = \frac{N-2}{2} + \tau$$

with $\tau = 1 + \bar{\eta}$ and $\bar{\eta} > 0$ small. Denote also $\|(u, v)\|_* = \|u\|_* + \|v\|_*.$

Let

$$Y_{j,1} = \frac{\partial U_{x_{j,\lambda}}}{\partial r}, \quad Y_{j,2} = \frac{\partial U_{x_{j,\lambda}}}{\partial \lambda}, \quad Z_{j,1} = \frac{\partial V_{x_{j,\lambda}}}{\partial r}, \quad Z_{j,2} = \frac{\partial V_{x_{j,\lambda}}}{\partial \lambda}.$$

First we consider the linear problem: for $(h_{1,k}, h_{2,k}) \in L^{\frac{q+1}{q}}(\mathbb{R}^N) \times L^{\frac{p+1}{p}}(\mathbb{R}^N)$

$$L_k(\varphi_{1,k}, \varphi_{2,k}) = (h_{1,k}, h_{2,k}) + \sum_{i=1}^{2} c_i \sum_{j=1}^{k} (pV_{x_{j,\lambda}}^{q-1}Y_{j,i}, qU_{x_{j,\lambda}}^{q-1}Y_{j,i}),$$

$$\langle (pV_{x_{j,\lambda}}^{q-1}Z_{j,i}, qU_{x_{j,\lambda}}^{q-1}Y_{j,i}), (\varphi_{1,k}, \varphi_{2,k}) \rangle = 0, \quad j = 1, \ldots, k, \quad i = 1, 2,$$

for some number $c_i$, where $\langle (u_1, u_2), (v_1, v_2) \rangle = \int_{\mathbb{R}^N} (u_1 v_1 + u_2 v_2)$,

$$L_k(\varphi_{1}, \varphi_{2}) = \left( -\Delta \varphi_1 - pK_1 \left( \frac{y}{\mu} \right) W_2^{q-1} \varphi_2, -\Delta \varphi_2 - qK_2 \left( \frac{y}{\mu} \right) W_1^{q-1} \varphi_1 \right).$$

Lemma 2.1. Assume that $(\varphi_{1,k}, \varphi_{2,k})$ solves (2.2). If $||(h_{1,k}, h_{2,k})||_{**} goes to zero as k goes to infinity, so does $||(\varphi_{1,k}, \varphi_{2,k})||_*.$

Proof. By contradiction, we assume that there exist $k \to \infty$, $(h_{1,k}, h_{2,k})$, $\lambda_k \in [L_1, L_2]$, $r_k \in \left[ r_0 - \frac{1}{\mu}, r_0 + \frac{1}{\mu} \right]$ and some $(\varphi_{1,k}, \varphi_{2,k})$ solving (2.2), with $||(h_{1,k}, h_{2,k})||_{**} \to 0$ and $||(\varphi_{1,k}, \varphi_{2,k})||_* \geq c' > 0$. We may assume that $||(\varphi_{1,k}, \varphi_{2,k})||_* = 1$. For simplicity, we drop the
and since \( \tau > 0 \), there exists some \( \theta > 0 \) such that

\[
\left| \int_{\mathbb{R}^N} \frac{p}{|y-z|^N} K_1 \left( \frac{z}{\mu} \right) W^{p-1}_2 \varphi_2(z) \, dz \right| 
\leq C \| \varphi_2 \|_{p} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N-2}} \right)^{p-1} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\theta}} \, dz 
\leq C \| \varphi_2 \|_{p} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\theta}}. 
\]

(2.5)

In fact, since \( p > \max \left\{ \frac{N+1}{N-2}, \frac{N(N-2)}{(N-2)^2 - (N-2 - m)} \right\} \), there exists \( \tau_1 \in \left[ \frac{N - 2 - m}{N - 2}, N - 2 - \frac{N}{p} \right) \) satisfying \( (N - 2)(p-1) - (p-1)\tau_1 - \tau_1 - 2 > 0 \), and then for \( z \in \Omega_i, i = 1, \ldots, k, \) and any \( \tau_1 \geq \frac{N - 2 - m}{N - 2} \), there exists some \( \theta > 0 \) such that

\[
\left( \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N-2}} \right)^{p-1} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\theta}} 
\leq C \frac{1}{(1 + |z - x_i|)^{(N-2)(p-1) - (p-1)\tau_1 + \frac{N-2}{2} + \tau - \tau_1} \leq C \frac{1}{(1 + |z - x_i|)^{\frac{N-2}{2} + \tau + 2}}. 
\]

Moreover, from Lemma B.2, there holds

\[
\left| \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} h_1(z) \, dz \right| 
\leq C \| h_1 \|_{**} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\theta+2}} \, dz 
\leq C \| h_1 \|_{**} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\theta}} 
\]

(2.6)

and since \( 0 < \theta < N - 2 \),

\[
\left| \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \sum_{j=1}^{k} V^{p-1}_{x_j, \lambda} Z_{j,i} \, dz \right| 
\]
\[ \leq C \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{p(N-2)}} \, dz \leq \frac{C}{(1 + |y - x_j|)^\sigma}. \quad (2.7) \]

Now, we estimate \( c_1 \) and \( c_2 \). Multiply the two equations in (2.2) by \( Z_{1,t} \) and \( Y_{1,t} \) respectively, and integrate to find that

\[ \sum_{i=1}^{2} \sum_{j=1}^{k} \left( \langle pV_{x_j,\lambda}Z_{j,i}, qU_{x_j,\lambda}Y_{j,i} \rangle, (Z_{1,t}, Y_{1,t}) \right) c_t = \langle L_k(\varphi_1, \varphi_2), (Z_{1,t}, Y_{1,t}) \rangle - \langle (h_1, h_2), (Z_{1,t}, Y_{1,t}) \rangle. \quad (2.8) \]

From Lemma B.1 and using \( \bar{\sigma} > 1 \), we get

\[ |\langle (h_1, h_2), (Z_{1,t}, Y_{1,t}) \rangle| \leq C \|(h_1, h_2)\|_{\ast\ast} \int_{\mathbb{R}^N} \frac{1}{(1 + |z - x_1|)^{N-2}} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{\sigma+2}} \, dz \]

\[ \leq C \|(h_1, h_2)\|_{\ast\ast}. \]

On the other hand,

\[ \langle L_k(\varphi_1, \varphi_2), (Z_{1,t}, Y_{1,t}) \rangle \]

\[ = \int_{\mathbb{R}^N} p\left(1 - K_1\left(\frac{y}{\mu}\right)\right)W_2^{p-1}Z_{1,t}\varphi_1 + p\left(V_{x_1,\lambda}^{p-1} - \left(\sum_{j=1}^{k} V_{x_j,\lambda}\right)^{p-1}\right)Z_{1,t}\varphi_1 \, dy \]

\[ + \int_{\mathbb{R}^N} q\left(1 - K_2\left(\frac{y}{\mu}\right)\right)W_1^{q-1}Y_{1,t}\varphi_2 + q\left(U_{x_1,\lambda}^{q-1} - \left(\sum_{j=1}^{k} U_{x_j,\lambda}\right)^{q-1}\right)Y_{1,t}\varphi_2 \, dy. \]

We have the following estimate which is put in Appendix B: with some \( \theta > 0 \),

\[ \|\langle L_k(\varphi_1, \varphi_2), (Z_{1,t}, Y_{1,t}) \rangle\| = O\left(\frac{1}{\mu^{\theta}}\right)\|(\varphi_1, \varphi_2)\|_{\ast}. \quad (2.9) \]

But, observe that, there exists some \( \bar{c} > 0 \) such that

\[ \sum_{j=1}^{k} \left( \langle pV_{x_j,\lambda}Z_{j,i}, qU_{x_j,\lambda}Y_{j,i} \rangle, (Z_{1,t}, Y_{1,t}) \right) = (\bar{c} + o(1))\delta_{ii}. \]

Therefore, from (2.8), we get that

\[ c_t = O\left(\frac{1}{\mu^{\theta}}\|(\varphi_1, \varphi_2)\|_{\ast} + \|(h_1, h_2)\|_{\ast\ast}\right). \quad (2.10) \]

Combining (2.5)-(2.10), we obtain that

\[ \|(\varphi_1, \varphi_2)\|_{\ast} \leq C \left(\|(h_1, h_2)\|_{\ast\ast} + \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\sigma+\theta}} \right). \quad (2.11) \]

Since \( \|(\varphi_1, \varphi_2)\|_{\ast} = 1 \), we get from (2.11) that there exists some \( R, a > 0 \) such that for some \( j \)

\[ \|(\varphi_1, \varphi_2)\|_{L^\infty(B_R(x_j))} \geq a > 0. \]

(2.12)
However, by transformation \((\varphi_1(y), \varphi_2(y)) = (\varphi_1(y - x_j), \varphi_2(y - x_j))\) converges uniformly in any compact set to a solution \((u, v)\) to

\[
\begin{cases}
-\Delta u - pV_{0,\lambda}^{p-1} v = 0, \\
-\Delta v - qU_{0,\lambda}^{q-1} u = 0,
\end{cases}
\] (2.13)

for some \(\lambda \in [L_1, L_2]\). Moreover, \((u, v)\) is perpendicular to the kernel of (2.13). Hence, 
\((u, v) = (0, 0)\), which is a contradiction to (2.12).

As a result of Lemma 2.1, we can prove the following result.

**Proposition 2.2.** There exist \(k_0 > 0\) and some constant \(C > 0\), independent of \(k\), such that for all \(k \geq k_0\) and all \((h_1, h_2) \in L^\infty \times L^\infty(\mathbb{R}^N)\) satisfying the assumptions in Lemma 2.1, the linear problem (2.2) has a unique solution \((\varphi_1, \varphi_2) \equiv L_k(h_1, h_2)\). Moreover, there hold that

\[
\|L_k(h_1, h_2)\|_* \leq C\|\(h_1, h_2)\|_{**}, \quad |c| \leq C\|\(h_1, h_2)\|_{**}.
\] (2.14)

Now we consider the following nonlinear problem

\[
\begin{align*}
-\Delta(W_1 + \varphi_1) &= K_1 \left(\frac{|y|}{\mu}\right)(W_2 + \varphi_2)^p + \sum_{t=1}^{2} c_t p \sum_{i=1}^{k} V_{x_i,\lambda}^{p-1} Z_{i,t}, \ y \in \mathbb{R}^N, \\
-\Delta(W_2 + \varphi_2) &= K_2 \left(\frac{|y|}{\mu}\right)(W_1 + \varphi_1)^q + \sum_{t=1}^{2} c_t q \sum_{i=1}^{k} U_{x_i,\lambda}^{q-1} Y_{i,t}, \ y \in \mathbb{R}^N, \\
(\varphi_1, \varphi_2) &\in H_s, \\
\langle (pV_{x_j,\lambda}^{p-1} Z_{j,t}, qU_{x_j,\lambda}^{q-1} Y_{j,t}), (\varphi_1, \varphi_2) \rangle &= 0, \ j = 1, \ldots, k, \ l = 1, 2.
\end{align*}
\] (2.15)

In this section, we are aimed to prove that

**Proposition 2.3.** There exists \(k_0 > 0\) and some constant \(C > 0\), independent of \(k\), such that for all \(k \geq k_0\), \(L_0 \leq \lambda \leq L_1\), \(|r - \mu r_0| \leq 1/\mu \hat{\theta}\), with \(\hat{\theta} > 0\) is a fixed small constant, problem (2.15) has a unique solution \((\varphi_1, \varphi_2) = (\varphi_1(r, \lambda), \varphi_2(r, \lambda))\) satisfying for some constant \(\theta > 0\),

\[
\|(\varphi_1, \varphi_2)\|_* \leq C \left(\frac{1}{\mu}\right)^{\frac{m+\theta}{2}+\theta}, \quad |c| \leq C \left(\frac{1}{\mu}\right)^{\frac{m+\theta}{2}+\theta}.
\] (2.16)

Rewrite problem (2.15) as

\[
\begin{align*}
L_k(\varphi_1, \varphi_2) &= R_k + N_k(\varphi_1, \varphi_2) + \sum_{t=1}^{2} c_t \left(p \sum_{i=1}^{k} V_{x_i,\lambda}^{p-1} Z_{i,t}, q \sum_{i=1}^{k} U_{x_i,\lambda}^{q-1} Y_{i,t}\right), \\
(\varphi_1, \varphi_2) &\in H_s, \\
\langle (pV_{x_j,\lambda}^{p-1} Z_{j,t}, qU_{x_j,\lambda}^{q-1} Y_{j,t}), (\varphi_1, \varphi_2) \rangle &= 0, \ j = 1, \ldots, k, \ l = 1, 2
\end{align*}
\] (2.17)

where operator \(L_k\) is defined in (2.3),

\[
N_k(\varphi_1, \varphi_2) = \left\{N_{1,k}(\varphi_2), N_{2,k}(\varphi_1)\right\}
\] (2.18)
with

\[
N_{1,k}(\varphi_2) = K_1 \left( \frac{y}{\mu} \right) \left( (W_2 + \varphi_2)^p - W_2^p - pW_2^{p-1}\varphi_2 \right),
\]

\[
N_{2,k}(\varphi_1) = K_2 \left( \frac{y}{\mu} \right) \left( (W_1 + \varphi_1)^q - W_1^q - qW_1^{q-1}\varphi_1 \right),
\]

and

\[
R_k = (R_{1,k}, R_{2,k}) = \left( K_1 \left( \frac{y}{\mu} \right) W_2^p - \sum_{j=1}^{k} V_{x_j, \lambda}^p, K_2 \left( \frac{y}{\mu} \right) W_1^q - \sum_{j=1}^{k} U_{x_j, \lambda}^q \right). \tag{2.19}
\]

In the following, we will use the contraction mapping theorem to show that there exists a unique solution to problem (2.17) in a set in which \( \| (\varphi_1, \varphi_2) \|_* \) is small. In order to do this, we first estimate \( N_k(\varphi_1, \varphi_2) \) and \( R_k \). Just as before, we may drop the subscript \( k \) for convenience.

**Lemma 2.4.** If \( N \geq 5 \), then

\[
\| N(\varphi_1, \varphi_2) \|_* \leq C \| (\varphi_1, \varphi_2) \|_{\min \{p, 2\}}.
\]

**Proof.** By definition of \( N = N_k \) in (2.18), we have

\[
|N_1(\varphi_2)| \leq \begin{cases} C|\varphi_2|^p, & \text{if } 1 < p \leq 2; \\ CW_2^{p-2}\varphi_2^2 + C|\varphi_2|^p, & \text{if } p > 2. \end{cases}
\]

For \( 1 < p \leq 2 \), by Hölder inequalities,

\[
|N_1(\varphi_2)| \leq \| \varphi_2 \|_*^p \left( \sum_{j=1}^{k} \frac{1}{1 + |y - x_j|} \right)^p 
\leq C \| \varphi_2 \|_*^p \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\tau}} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\tau}} \right)^{p-1} 
\leq C \| \varphi_2 \|_*^p \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\tau}},
\]

where \( \tau = \frac{N-2}{p-1} + \frac{N-2}{\tau} - \frac{N-2}{p-1} - \frac{N-2}{\tau} \geq \frac{N-2}{p-1} + \frac{N-2}{\tau} - \frac{N-2}{p-1} - \frac{N-2}{\tau} \) and \( \tau > \max \left\{ 1 + \frac{2}{p-1} - \frac{m}{N-2}, \frac{N-2-m}{N-2} \right\} \).

The case \( p > 2 \) and the other term \( N_2(\varphi_1) \) can be estimated in the same way. \( \square \)

Now, we estimate \( R_k \).

**Lemma 2.5.** Assume that \( |x_1| - \mu \bar{r}_0 \leq \frac{1}{\mu \bar{\theta}} \), where \( \bar{\theta} > 0 \) is a fixed small constant. Then, there exists some small \( \theta > 0 \) such that

\[
\| R_k \|_* \leq C \left( \frac{1}{\mu} \right)^{\frac{m}{2} + \theta}. \tag{2.21}
\]
Proof. Since \( p \leq q \) as we set, it suffices to deal with \( \| R_{1,k} \|_{**} \).

Recall by (2.19) that

\[
R_{1,k} = K_1 \left( \frac{y}{\mu} \right) W_2^p - \sum_{j=1}^{k} V_{x_j,\lambda}^p
\]

\[= K_1 \left( \frac{y}{\mu} \right) \left( W_2^p - \sum_{j=1}^{k} V_{x_j,\lambda}^p \right) + \sum_{j=1}^{k} V_{x_j,\lambda}^p \left( K_1 \left( \frac{y}{\mu} \right) - 1 \right) := J_1 + J_2. \tag{2.22}\]

By symmetry, we might as well assume that \( y \in \Omega_1 \). Then, \( |y - x_j| \geq |y - x_1| \). Therefore,

\[
|J_1| \leq C \left( \frac{1}{1 + |y - x_1|} \right)^{(p-1)(N-2)} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}}
\]

\[+ C \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^p. \tag{2.23}\]

By Lemma B.1, for any \( 0 < \alpha \leq \min\{(p-1)(N-2), N-2\} \), there holds that

\[
\frac{1}{(1 + |y - x_1|)^{(p-1)(N-2)}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \leq C \frac{1}{(1 + |y - x_1|)^{p(N-2)-\alpha}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^\alpha} \leq \frac{k^\alpha}{\mu^\alpha (1 + |y - x_1|)^{p(N-2)-\alpha}}. \tag{2.24}\]

Since \( p > \frac{N+1}{N-2} \), we can choose \( \frac{N-2}{2} < \alpha \leq (N-2)p - \frac{N+2}{2} - \tau \) to obtain that

\[
\frac{1}{(1 + |y - x_1|)^{(p-1)(N-2)}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \leq \frac{1}{\mu^{\frac{N+2}{2} + \theta} (1 + |y - x_1|)^{\frac{N-2}{2} + 2}}.
\]

Moreover, for \( y \in \Omega_1 \), Lemma B.1 implies that

\[
\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \leq \sum_{j=2}^{k} \frac{1}{(1 + |y - x_1|)^{\frac{N-2}{2}}} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2}}}
\]

\[\leq \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\frac{N-2}{2} - \left( \frac{p}{2} \right)^{\frac{N-2}{2} + \theta} - \frac{N-2}{2}}} \frac{1}{(1 + |y - x_1|)^{\frac{N-2}{2} + \left( \frac{p}{2} \right)^{\frac{N-2}{2} + \theta} - \frac{N-2}{2}}}.\]
Since for \( \tau = 1 + \bar{\eta} \) with \( \bar{\eta} > 0 \) small, \( p - \frac{N + 2}{2(N - 2)} - \frac{\tau}{N - 2} > \frac{1}{2} \), we see

\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^p \\
\leq C \left( \frac{\mu}{\mu^2 + \theta} \right)^p \frac{(\frac{N}{2} - \frac{1}{p} (\frac{N}{2} + \tau - \frac{N}{2}))}{(1 + |y - x_1|)^{\frac{N}{2} + \tau}} \leq \frac{C}{\mu^2 + \theta} \frac{1}{(1 + |y - x_1|)^{\frac{N}{2} + \tau}}.
\]

Hence, (2.23)-(2.25) imply that

\[
\|J_1\|_{**} \leq \frac{C}{\mu^2 + \theta}.
\] (2.26)

Now we estimate

\[
J_2 = \sum_{j=1}^{k} V_{x_j, \lambda}^p (K_1(\frac{y}{\mu}) - 1).
\]

First, for \( y \in \Omega_1 \) and \( j > 1 \), similar to (2.25), we obtain

\[
\left| \sum_{j=2}^{k} V_{x_j, \lambda}^p (K_1(\frac{y}{\mu}) - 1) \right| \leq \frac{C}{\mu^2 + \theta} \frac{1}{(1 + |y - x_1|)^{\frac{N}{2} + \tau}}.
\] (2.27)

Next, for \( y \in \Omega_1 \) and \( ||y| - \mu r_0| \geq \delta \mu \), where \( \delta > 0 \) is a fixed constant, we have \( ||y| - |x_1| | \geq \delta \mu \), and then

\[
\left| V_{x_1, \lambda}^p (K_1(\frac{y}{\mu}) - 1) \right| \leq \frac{C}{\mu^2 + \theta} \frac{1}{(1 + |y - x_1|)^{\frac{N}{2} + \tau}}.
\] (2.28)

While if \( y \in \Omega_1 \) and \( ||y| - \mu r_0| \leq \delta \mu \), we have that \( ||y| - |x_1| | \leq 2\delta \mu \) and

\[
K_1(\frac{y}{\mu}) - 1 \leq C \left| \frac{y}{\mu} - r_0 \right|^m \leq C \left( ||y| - |x_1||^m + ||x_1| - r_0| \right)^m \leq C \left( \left| y - x_1 \right|^m + \mu \right)^m.
\]

Since \( m < (2p - 1)(N - 2) - 8 \), we find \( p(N - 2) - \frac{N+2}{2} - \tau - \frac{m}{2} - \theta > 0 \). Thus

\[
\frac{||y| - |x_1||^m}{\mu^m} \frac{1}{(1 + |y - x_1|)^{p(N-2)}} \leq \frac{C}{\mu^2 + \theta} \frac{||y| - |x_1||^{m+\theta}}{(1 + |y - x_1|)^{p(N-2)}} \leq \frac{C}{\mu^2 + \theta} \frac{1}{(1 + |y - x_1|)^{\frac{N}{2} + \tau}}.
\]
Hence,
\[
|V_{x_1,\lambda}(K_1(\frac{y}{\mu}) - 1)| \leq \frac{C}{\mu^{\frac{m}{2} + \theta}} \frac{1}{(1 + |y - x_1|)^{\frac{N+2}{2} + \tau}}, \quad ||y|| - \mu r_0 \leq \delta \mu. \tag{2.29}
\]

From (2.27)-(2.29), we obtain
\[
\|J_2\|_{**} \leq \frac{C}{\mu^{\frac{m}{2} + \theta}}. \tag{2.30}
\]

Combing (2.26) and (2.30), we conclude the proof of (2.21).

\[\square\]

Now, we are ready to prove Proposition 2.3.

**Proof of Proposition 2.3:** Recall that \(\mu = k^{\frac{N-2}{2}}\). Let
\[
E = \left\{ (\varphi_1, \varphi_2) \in (C(\mathbb{R}^N))^2 \cap H_s, \| (\varphi_1, \varphi_2) \|_s \leq \frac{1}{\mu^{\frac{m}{2}}} \right\},
\]

\[
\int_{\mathbb{R}^N} (pV_{x_1,\lambda}^{-1}Z_{j,l}\varphi_1 + qU_{x_1,\lambda}^{-1}Y_{j,l}\varphi_2) = 0, \quad j = 1, \ldots, k, \quad l = 1, 2.
\]

Then we only need to solve
\[
(\varphi_1, \varphi_2) = A(\varphi_1, \varphi_2) := \mathbb{L}_k(N(\varphi_1, \varphi_2)) + \mathbb{L}_k(R_k)
\]
with \(\mathbb{L}_k\) defined as in Proposition 2.2. We will prove that \(A\) is a contraction map from \(E\) to \(E\).

First, by Proposition 2.2, Lemma 2.4 and Lemma 2.5, \(A\) maps \(E\) to \(E\) and
\[
\|A(\varphi_1, \varphi_2)\|_s \leq C\|N(\varphi_1, \varphi_2)\|_{**} + C\|R_k\|_{**}
\]
\[
\leq C\|N(\varphi_1, \varphi_2)\|_{1+\theta} + \frac{C}{\mu^{\frac{m}{2} + \theta}} \leq \frac{C}{\mu^{\frac{m}{2} + \theta}} \leq \frac{1}{\mu^{\frac{m}{2}}}.
\]

Next it is obvious that
\[
\|A(\bar{\varphi}_1, \bar{\varphi}_2) - A(\varphi_1, \varphi_2)\|_s \leq C\|N(\varphi_1, \varphi_2) - N(\bar{\varphi}_1, \bar{\varphi}_2)\|_{**}.
\]

Similar to the proof in (2.20),
\[
|N_1(\varphi_2) - N_1(\bar{\varphi}_2)| \leq C(\|\varphi_2\|_{p-1}^{p-1} + \|\bar{\varphi}_2\|_{p-1}^{p-1})\|\varphi_2 - \bar{\varphi}_2\|_s \left(\sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\theta}}\right)^p
\]
\[
\leq C(\|\varphi_2\|_{p-1}^{p-1} + \|\bar{\varphi}_2\|_{p-1}^{p-1})\|\varphi_2 - \bar{\varphi}_2\|_s \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\theta+2}}.
\]

Finally, \(A\) is a contraction map and it follows from the contraction mapping theorem that there exists a unique \((\varphi_1, \varphi_2) \in E\), such that
\[
(\varphi_1, \varphi_2) = A(\varphi_1, \varphi_2).
\]

Moreover, from Proposition 2.2 we get that for some \(\theta > 0\),
\[
\|(\varphi_1, \varphi_2)\|_s \leq C\left(\frac{1}{\mu}\right)^{\frac{m}{2} + \theta}.
\]

\[\square\]
3. Proof of the existence result

Let

\[ F(r, \lambda) = I(W_1 + \varphi_1, W_2 + \varphi_2), \]

where \( r = |x_1| \), \((\varphi_1, \varphi_2) \in H_s\) is obtained in Proposition 2.3, and

\[ I(u, v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \frac{1}{p+1} \int_{\mathbb{R}^N} K_1 \left( \frac{|y|}{\mu} \right) |v|^{p+1} - \frac{1}{q+1} \int_{\mathbb{R}^N} K_2 \left( \frac{|y|}{\mu} \right) |u|^{q+1}. \]

**Proposition 3.1.** We have

\[ F(r, \lambda) = I(W_1, W_2) + O \left( \frac{k}{\mu^{m+\theta}} \right) \]

\[ = k \left( A + \frac{\tilde{B}_1}{\lambda^{m_2} \mu_2} + \frac{\tilde{B}_2}{\lambda^{m_1} \mu_1} + (\frac{\tilde{B}_1}{\lambda^{m_1-2} \mu_1} + \frac{\tilde{B}_1}{\lambda^{m_2-2} \mu_2})(\mu_0 - r)^2 \right. \]

\[ \left. - \sum_{j=2}^{k} \frac{B_2}{\lambda^2 N_2 |x_j - x_1|^2} \right) + kO \left( \frac{1}{\mu^{m+\theta}} + (\frac{1}{\mu_1} + \frac{1}{\mu_2})|\mu r_0 - r|^3 \right), \]

where \( \theta > 0 \) is a fixed constant and \( \tilde{B}_1, \tilde{B}_2, \tilde{B}_1, \tilde{B}_2, B_2 \) are positive constants.

**Proof.** Since

\[ \left\langle (I'_u(W_1 + \varphi_1, W_2 + \varphi_2), I'_u(W_1 + \varphi_1, W_2 + \varphi_2)), (\varphi_1, \varphi_2) \right\rangle = 0, \forall (\varphi_1, \varphi_2) \in E, \]

there are \( t, s \in (0, 1) \) such that

\[ F(r, \lambda) \]

\[ = I(W_1, W_2) - \frac{1}{2} \left\langle D^2 I(W_1 + t \varphi_1, W_2 + s \varphi_2)(\varphi_1, \varphi_2), (\varphi_1, \varphi_2) \right\rangle \]

\[ = I(W_1, W_2) - \frac{1}{2} \int_{\mathbb{R}^N} \left( 2\nabla \varphi_1 \cdot \nabla \varphi_2 - qK_1(y) - pK_1(y) \right) (W_1 + t \varphi_1)^{q-1} \varphi_1^{2} - pK_1(y) (W_2 + s \varphi_2)^{p-1} \varphi_2^2 \]

\[ = I(W_1, W_2) + \frac{1}{2} \int_{\mathbb{R}^N} qK_1(y) ((W_1 + t \varphi_1)^{q-1} - W_1^{q-1}) \varphi_1^2 - (N_2(\varphi_1) + R_{2,k}) \varphi_1 \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N} pK_1(y) ((W_2 + s \varphi_2)^{p-1} - W_2^{p-1}) \varphi_2^2 - (N_1(\varphi_2) + R_{1,k}) \varphi_2. \]

Note that

\[ \int_{\mathbb{R}^N} (N_2(\varphi_1) + R_{2,k}) \varphi_1 + (N_1(\varphi_2) + R_{1,k}) \varphi_2 \]

\[ \leq C(\|N_K(\varphi_1, \varphi_2)\|_{s*} + \|R_k\|_{s*}) \|\varphi_1, \varphi_2\|_{s} \int_{\mathbb{R}^N} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{s}} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{s+2}} \]

\[ \leq \frac{Ck}{\mu^{m+\theta}}. \]

Therefore, we obtain

\[ F(r, \lambda) = I(W_1, W_2) + O \left( \frac{k}{\mu^{m+\theta}} \right), \]

and the result follows from Proposition A.1.
Consequently, there is a constant \( B_3 > 0 \) such that

\[
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \begin{cases} 
\frac{2}{(2|x_1|)^{N-2}} \sum_{j=2}^{k} \frac{1}{(\sin \left( \frac{(j-1)\pi}{k} \right))^{N-2}} + \frac{1}{(2|x_1|)^{N-2}}, & \text{if } k \text{ is even}, \\
\frac{2}{(2|x_1|)^{N-2}} \sum_{j=2}^{k} \frac{1}{(\sin \left( \frac{(j-1)\pi}{k} \right))^{N-2}}, & \text{if } k \text{ is odd}.
\end{cases}
\]

But

\[0 < c' \leq \frac{\sin \left( \frac{(j-1)\pi}{k} \right)}{\left( \frac{(j-1)\pi}{k} \right)} \leq c'', \quad j = 2, \ldots, \left[ \frac{k}{2} \right].\]

Consequently, there is a constant \( B_3 > 0 \) such that

\[
\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{N-2}} = \frac{B_3k^{N-2}}{|x_1|^{N-2}} + O\left( \frac{k}{|x_1|^{N-2}} \right).
\]

Therefore, for some \( B_4 > 0 \),

\[
F(r, \lambda) = k \left( A + \frac{B_1}{\lambda^{m_2} \mu^{m_2}} + \frac{B_2}{\lambda^{m_1} \mu^{m_1}} + \left( \frac{\tilde{B}_2}{\lambda^{m_2} \mu^{m_2}} + \frac{\tilde{B}_1}{\lambda^{m_1} \mu^{m_1}} \right)(\mu r_0 - r)^2 - \frac{B_4 k^{N-2}}{\lambda^{N-2} N_{N-2}} \right)
\]

\[+ kO\left( \frac{1}{\mu^{m+\theta}} + \left( \frac{1}{\mu^{m_1}} + \frac{1}{\mu^{m_2}} \right)|\mu r_0 - r|^3 \right),\]

\[
\frac{\partial F(r, \lambda)}{\partial \lambda} = k \left( - \frac{\tilde{B}_1 m_2}{\lambda^{m+1} \mu^{m_2}} - \frac{\tilde{B}_2 m_1}{\lambda^{m+1} \mu^{m_1}} + \frac{B_4(N-2)k^{N-2}}{\lambda^{N-2} N_{N-2}} \right)
\]

\[+ kO\left( \frac{1}{\mu^{m+\theta}} + \left( \frac{1}{\mu^{m_1}} + \frac{1}{\mu^{m_2}} \right)|\mu r_0 - r|^2 \right).
\]

If \( m = m_1 < m_2 \), we set \( \lambda_0 \) to be the solution of

\[- \frac{m \tilde{B}_2}{\lambda^{m+1}} + \frac{B_4(N-2)}{\lambda^{N-1} r_0^{N-2}} = 0,\]

while for the case of \( m = m_1 = m_2 \),

\[- \frac{m(\tilde{B}_2 + \tilde{B}_1)}{\lambda^{m+1}} + \frac{B_4(N-2)}{\lambda^{N-1} r_0^{N-2}} = 0.\]

Then

\[
\lambda_0 = \begin{cases} 
\left( \frac{B_4(N-2)}{m \tilde{B}_2 r_0^{N-2}} \right)^{\frac{1}{N-2-m}}, & m = m_1 < m_2, \\
\left( \frac{B_4(N-2)}{m(\tilde{B}_2 + \tilde{B}_1) r_0^{N-2}} \right)^{\frac{1}{N-2-m}}, & m = m_1 = m_2.
\end{cases}
\]

Define

\[D = \{(r, \lambda) : r \in \left[ r_0 \mu - \frac{1}{\mu^\theta}, r_0 \mu + \frac{1}{\mu^\theta} \right], \lambda \in \left[ \lambda_0 - \frac{1}{\mu^{2\theta}}, \lambda_0 + \frac{1}{\mu^{2\theta}} \right] \},\]
where $\bar{\theta} > 0$ is a small constant. For any $(r, \lambda) \in D$, we have $\frac{r}{\mu} = r_0 + O\left(\frac{1}{\mu^{1+\theta}}\right)$. Then,

$$r^{N-2} = \mu^{N-2}\left(r_0^{N-2} + O\left(\frac{1}{\mu^{1+\theta}}\right)\right).$$

We just deal with the case $m = m_1 < m_2$, since the other one can be handled similarly. For $(r, \lambda) \in D$,

$$F(r, \lambda) = k\left(A + \left(\frac{\bar{B}_2}{\lambda^m} - \frac{B_4}{\lambda^{N-2}r_0^{N-2}}\right)\frac{1}{\mu^m} + \frac{\tilde{B}_2}{\lambda^m-2\mu^m}(\mu r_0 - r)^2\right.$$

$$+ O\left(\frac{1}{\mu^{m+\theta}} + \frac{|\mu r_0 - r|^3}{\mu^m} + \frac{k}{\mu^{N-2}}\right),$$

(3.1)

$$\frac{\partial F(r, \lambda)}{\partial \lambda} = k\left(-\left(\frac{\tilde{B}_2 m}{\lambda^m} + \frac{B_4(N-2)}{\lambda^{N-2}r_0^{N-2}}\right)\frac{1}{\mu^m} + O\left(\frac{1}{\mu^{m+\theta}} + \frac{|\mu r_0 - r|^2}{\mu^m} + \frac{k}{\mu^{N-2}}\right)\right).$$

(3.2)

We define

$$\alpha_1 = k\left(-A - \left(\frac{\tilde{B}_2}{\lambda^m} - \frac{B_4}{\lambda^{N-2}r_0^{N-2}}\right)\frac{1}{\mu^m} - \frac{1}{\mu^{m+\theta}}\right), \quad \alpha_2 = k(-A + \eta),$$

where $\eta > 0$ is a small constant.

Let $\bar{F}(r, \lambda) = -F(r, \lambda), (r, \lambda) \in D$, where $\bar{\lambda} > 0$ is a small constant.

We define $\bar{F}^\alpha(r, \lambda) = \{(r, \lambda) \in D, \bar{F}(r, \lambda) \leq \alpha\}$.

Consider

$$\begin{cases}
\frac{dr}{dt} = -D_r \bar{F}, & t > 0, \\
\frac{d\lambda}{dt} = -D_\lambda \bar{F}, & t > 0, \\
(r, \lambda) \in \bar{F}^{\alpha_2}.
\end{cases}$$

Following the arguments used in [42], we can obtain

**Proposition 3.2.** The flow $(r(t), \lambda(t))$ does not leave $D$ before it reaches $F^{\alpha_1}$.

**Proof of Theorem 1.6:** Define

$$\Lambda = \left\{h: h(r, \lambda) = (h_1(r, \lambda), h_2(r, \lambda)) \in D, (r, \lambda) \in D, h(r, \lambda) = (r, \lambda), \text{if } |r - \mu r_0| = \frac{1}{\mu^\theta}\right\}.$$

Let

$$c = \inf_{h \in \Lambda} \max_{(r, \lambda) \in D} \bar{F}(h(r, \lambda)).$$

Proceeding as done in [42], we obtain that

(i) $\alpha_1 < c < \alpha_2$;

(ii) $\sup_{|r - \mu r_0| = \frac{1}{\mu^\theta}} \bar{F}(h(r, \lambda)) < \alpha_1, \forall h \in \Lambda$. 


Thus we conclude that $c$ is a critical value of $\bar{F}$.

To complete the proof of Theorem 1.6, it suffices to show that solution $(u_{1,k}, u_{2,k})$ of the form \( (1.10) \) is a positive vector solution, which can be deduced by the following result.

\[ \text{□} \]

**Lemma 3.3.** For any solution \( (\varphi_{1,k}, \varphi_{2,k}) \) to \( (2.15) \), with \( c_1 = c_2 = 0 \) and \( \| (\varphi_1, \varphi_2) \|_* \leq C \left( \frac{1}{\mu} \right)^{\frac{m+\sigma}{2}} \), there must hold further that
\[
|\varphi_i(y)| \leq \frac{1}{2} W_i(y), \quad i = 1, 2.
\]

**Proof.** Rewrite \( (2.15) \) with \( c_1 = c_2 = 0 \) as
\[
\varphi_1(y) = \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \left( pK_1 \left( \frac{z}{\mu} \right) W_2^{p-1} \varphi_2(z) + N_{1,k} \varphi_2(z) + R_{1,k}(z) \right) dz, \quad (3.3)
\]
\[
\varphi_2(y) = \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \left( qK_2 \left( \frac{z}{\mu} \right) W_1^{q-1} \varphi_1(z) + N_{2,k} \varphi_1(z) + R_{2,k}(z) \right) dz. \quad (3.4)
\]

**Step 1.** Firstly, we estimate
\[
R_{1,k} = K_1 \left( \frac{y}{\mu} \right) W_2^p - \sum_{j=1}^{k} V_{x_j,\lambda}^p
\]
\[
= K_1 \left( \frac{y}{\mu} \right) (W_2^p - \sum_{j=1}^{k} V_{x_j,\lambda}^p) + \sum_{j=1}^{k} V_{x_j,\lambda}^p (K_1 \left( \frac{y}{\mu} \right) - 1)
\]
\[
:= J_1 + J_2. \quad (3.5)
\]

By symmetry, we might as well assume that \( y \in \Omega_1 \). Then, \(|y - x_j| \geq |y - x_1|\). Therefore,
\[
|J_1| \leq \frac{C}{(1 + |y - x_1|)^{N-2}(p-1)} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)}}
\]
\[
+ C \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)}} \right)^p.
\]

By Lemma B.1, for any \( 0 < \tau_1 \leq \min\{(N-2)(p-1), N - 2\} \), there holds that
\[
\frac{1}{(1 + |y - x_1|)^{(N-2)(p-1)}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)}} \leq \frac{C}{(1 + |y - x_1|)^{p(N-2)-\tau_1}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\tau_1}}.
\]

Since \( p > \frac{N+1}{N-2} \), we can choose \( \frac{N-2-m}{N-2} < \tau_1 < p(N-2) - N \) to obtain that there exists some \( \sigma > 0, \epsilon_0 > 0 \) small such that
\[
\frac{1}{(1 + |y - x_1|)^{(N-2)(p-1)}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{(N-2)}} \leq \frac{k^{\tau_1}}{\mu^{\tau_1} (1 + |y - x_1|)^{p(N-2)-\tau_1}} \leq \frac{1}{\mu^{\sigma} (1 + |y - x_1|)^{N+\epsilon_0}}.
\]
Similarly, for \( y \in \Omega_1 \), Lemma B.1 gives that for \( \frac{N-2-m}{N-2} < \delta_1 < N - 2 - \frac{N}{p} \),
\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^p \leq \left( \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\sigma_1}(1 + |y - x_1|)^{N-2-\tau_1}} \right)^p \leq \left( \frac{k}{\mu} \right)^{p\tau_1} \frac{1}{(1 + |y - x_1|)^{(N-2-\tau_1)p}} \leq \frac{1}{\mu^\sigma} \frac{1}{(1 + |y - x_1|)^{N+\epsilon_0}}.
\]
Hence, there exists some \( \sigma > 0, \epsilon_0 > 0 \) small such that
\[
|J_1| \leq \frac{C}{\mu^\sigma} \frac{1}{(1 + |y - x_1|)^{N+\epsilon_0}}. \tag{3.6}
\]
Next we estimate
\[
J_2 = \sum_{j=1}^{k} V^p_{x_j, \lambda} \left( K_1 \left( \frac{y}{\mu} \right) - 1 \right).
\]
For \( y \in \Omega_1 \) and \( j > 1 \), by Lemma B.1, we obtain
\[
V^p_{x_j, \lambda} \leq \frac{C}{|x_j - x_1|^{\sigma_1}(1 + |y - x_1|)^{p(N-2)-\tau_1}},
\]
which implies that, if we choose \( \frac{N-2-m}{N-2} < \delta_1 < p(N-2) - N - \epsilon_0 \) there holds that
\[
\left| \sum_{j=2}^{k} V^p_{x_j, \lambda} \left( K_1 \left( \frac{y}{\mu} \right) - 1 \right) \right| \leq C \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\sigma_1}(1 + |y - x_1|)^{p(N-2)-\tau_1}} \leq \frac{C}{(1 + |y - x_1|)^{p(N-2)-\tau_1}} \left( \frac{k}{\mu} \right)^{\tau_1} \leq \frac{C}{\mu^\sigma} \frac{1}{(1 + |y - x_1|)^{N+\epsilon_0}}.
\]
For \( y \in \Omega_1 \) and \( ||y| - \mu r_0| \geq \delta \mu \), where \( \delta > 0 \) is a fixed constant, we have
\[
||y| - |x_1|| \geq \frac{\delta \mu}{2},
\]
and then for \( \delta_1 < p(N-2) - N - \epsilon_0 \),
\[
\left| V^p_{x_1, \lambda} \left( K_1 \left( \frac{y}{\mu} \right) - 1 \right) \right| \leq \frac{C}{\mu^{\tau_1}(1 + |y - x_1|)^{p(N-2)-\tau_1}}.
\]
While for \( y \in \Omega_1 \) and \( ||y| - \mu r_0| \leq \delta \mu \), we have that \( ||y| - |x_1|| \leq 2\delta \mu \) and
\[
\left| K_1 \left( \frac{y}{\mu} \right) - 1 \right| \leq C \left( \frac{|y|}{\mu} - r_0 \right)^{m_1} \leq \frac{C}{\mu^{m_1}} (||y| - |x_1||^{m_1} + ||x_1|-r_0\mu|^{m_1}) \leq \frac{C}{\mu^{m_1}} ||y| - |x_1||^{m_1} + \frac{C}{\mu^{m_1+\theta_1}}.
\]
As a result,\[
\frac{||y| - |x_1||^{m_1}}{\mu^{m_1}} \frac{1}{(1 + |y - x_1|)^{N+2}} \leq \frac{C}{\mu^{m_1}} \frac{||y| - |x_1||^{m_1-\tau_1}}{(1 + |y - x_1|)^{N+\epsilon_0}} \frac{||y| - |x_1||^{\tau_1}}{(1 + |y - x_1|)^{p(N-2)-N-\epsilon_0}}
\]
and then
\[
|J_2| \leq \frac{C}{\mu^\sigma} \frac{1}{(1 + |y - x_1|)^{N+\epsilon_0}}. \tag{3.7}
\]
Finally, (3.5)-(3.7) give that there exists some $\sigma > 0, \epsilon_0 > 0$ small such that for $y \in \Omega_1$

$$|R_{1,k}| \leq \frac{C}{\mu^\sigma} \left( \frac{1}{1 + |y - x_1|} \right)^{N - \epsilon_0},$$

which, by Lemma B.2, implies that

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} R_{1,k}(z) dz \leq \frac{C}{\mu^\sigma} \sum_{j=1}^{k} \left( \frac{1}{1 + |y - x_j|} \right)^{N-2}. \quad (3.8)$$

Similarly,

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} R_{2,k}(z) dz \leq \frac{C}{\mu^\sigma} \sum_{j=1}^{k} \left( \frac{1}{1 + |y - x_j|} \right)^{N-2}. \quad (3.9)$$

**Step 2.** For the nonlinearities, since $p > 1$, using Hölder inequalities, one has

$$\int_{\mathbb{R}^N} \frac{K_1(z)}{\mu} ((W_2 + \varphi_2)^p - W_2^p - pW_2^{p-1}\varphi_2)$$

$$\leq C\|\varphi_2\|^p \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^p$$

$$\leq \frac{C}{\mu^\sigma} \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \sum_{j=1}^{k} \left( \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \epsilon}} \right)^p$$

$$= \frac{C}{\mu^\sigma} \sum_{j=1}^{k} \left( \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \epsilon}} \right)^p. \quad (3.10)$$

where we see from (P), there exists $\epsilon > 0$ such that

$$\frac{p(N-2) + \tau}{p - 1} - \frac{(N-2) + 2 + \tau + \epsilon}{N - 2} > \frac{N - 2 - m}{N - 2}.$$

Similarly, for $\epsilon > 0$,

$$\int_{\mathbb{R}^N} \frac{K_2(z)}{\mu} ((W_1 + \varphi_1)^q - W_1^q - qW_1^{q-1}\varphi_1) \leq \frac{C}{\mu^\sigma} \sum_{j=1}^{k} \left( \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \epsilon}} \right)^q. \quad (3.11)$$

**Step 3.** The estimates for the linearities can be obtained with the same idea as Step 2:

$$\int_{\mathbb{R}^N} \frac{p}{|y - z|^{N-2}} K_1(z) W_2^{p-1}\varphi_2(z) dz \leq \frac{C}{\mu^\sigma} \sum_{j=1}^{k} \left( \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \epsilon}} \right)^p \quad (3.12)$$

and

$$\int_{\mathbb{R}^N} \frac{q}{|y - z|^{N-2}} K_2(z) W_1^{q-1}\varphi_1(z) dz \leq \frac{C}{\mu^\sigma} \sum_{j=1}^{k} \left( \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau + \epsilon}} \right)^q. \quad (3.13)$$
Substituting (3.8)-(3.13) into (3.3) and (3.4), we have that for \( i = 1, 2 \) and \( \epsilon > 0 \),

\[
|\varphi_i(y)| \leq \frac{C}{\mu^q} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2 + \tau + \epsilon}}.
\]

Since \( \frac{N}{2} + \tau + \epsilon > \frac{N-2}{2} + \tau \), applying Lemma B.2, we can continue this process iteratively until we obtain

\[
|\varphi_i(y)| \leq \frac{C}{\mu^q} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}}, \quad i = 1, 2.
\] (3.14)

As a consequence,

\[
|\varphi_i(y)| \leq \frac{1}{2} W_i(y),
\] (3.15)

which concludes Lemma 3.3.

\[\square\]

4. THE NON-DEGENERACY OF THE SOLUTIONS

4.1. Pohozaev identities. We first consider the following two systems

\[
\begin{cases}
-\Delta v_1 = K_1(y)v_2^\rho, \\
-\Delta v_2 = K_2(y)v_1^\tau,
\end{cases}
\] (4.1)

\[
\begin{cases}
-\Delta \xi_1 = pK_1(y)v_2^{\rho-1}\xi_2, \\
-\Delta \xi_2 = qK_2(y)v_1^{\tau-1}\xi_1.
\end{cases}
\] (4.2)

Assume that \( \Omega \) is a smooth domain in \( \mathbb{R}^N \).

Lemma 4.1. It holds that

\[
- \int_{\partial \Omega} \left( \frac{\partial v_1}{\partial \nu}\frac{\partial \xi_2}{\partial y_i} + \frac{\partial v_1}{\partial \nu}\frac{\partial \xi_2}{\partial y_j} + \frac{\partial v_2}{\partial \nu}\frac{\partial \xi_1}{\partial y_i} + \frac{\partial v_2}{\partial \nu}\frac{\partial \xi_1}{\partial y_j} \right) + \int_{\partial \Omega} \left( \langle \nabla v_1, \nabla \xi_2 \rangle v_i + \langle \nabla v_2, \nabla \xi_1 \rangle v_i \right)
\]

\[
- \int_{\partial \Omega} \left( K_1(y)v_2^\rho \xi_2 v_i + K_2(y)v_1^\tau \xi_1 v_i \right) = - \int_{\Omega} \left( \frac{\partial K_1(y)}{\partial y_i} v_2^\rho \xi_2 + \frac{\partial K_2(y)}{\partial y_i} v_1^\tau \xi_1 \right);
\] (4.3)

\[
\int_{\partial \Omega} \left( \frac{\partial v_1}{\partial \nu}\langle \nabla \xi_2, y - x_0 \rangle + \frac{\partial \xi_1}{\partial \nu}\langle \nabla v_2, y - x_0 \rangle + \frac{\partial v_2}{\partial \nu}\langle \nabla \xi_1, y - x_0 \rangle + \frac{\partial \xi_2}{\partial \nu}\langle \nabla v_1, y - x_0 \rangle \right)
\]

\[
- \langle \nabla v_1, \nabla \xi_2 \rangle \langle \nu, y - x_0 \rangle - \langle \nabla v_2, \nabla \xi_1 \rangle \langle \nu, y - x_0 \rangle
\]

\[
+ \int_{\partial \Omega} \left( K_1(y)v_2^\rho \xi_2 \langle \nu, y - x_0 \rangle + K_2(y)v_1^\tau \xi_1 \langle \nu, y - x_0 \rangle \right)
\]

\[
+ \int_{\partial \Omega} \left( \frac{N}{p + 1} \left( \xi_2 \frac{\partial v_1}{\partial \nu} + v_1 \frac{\partial \xi_2}{\partial \nu} \right) + \frac{N}{q + 1} \left( \xi_1 \frac{\partial v_2}{\partial \nu} + v_2 \frac{\partial \xi_1}{\partial \nu} \right) \right)
\]

\[
= \int_{\Omega} \left( \langle \nabla K_1, y - x_0 \rangle v_2^\rho \xi_2 + \langle \nabla K_2, y - x_0 \rangle v_1^\tau \xi_1 \right),
\]

where \( \nu \) is the outward unit normal of \( \partial \Omega \) at \( y \in \partial \Omega \), \( i = 1, \ldots, N \).
Proof. To show (4.3), we have
\[
\begin{align*}
\int_{\Omega} \left( -\Delta v_1 \frac{\partial \xi_2}{\partial y_i} - \Delta \xi_1 \frac{\partial v_2}{\partial y_i} - \Delta v_2 \frac{\partial \xi_1}{\partial y_i} - \Delta \xi_2 \frac{\partial v_1}{\partial y_i} \right) \\
= \int_{\Omega} \left( K_1(y)(v_2^p \frac{\partial \xi_2}{\partial y_i} + p v_2^{p-1} \xi_2 \frac{\partial v_2}{\partial y_i}) + K_2(y)(v_1^q \frac{\partial \xi_1}{\partial y_i} + q v_1^{q-1} \xi_1 \frac{\partial v_1}{\partial y_i}) \right) \tag{4.5}
\end{align*}
\]

The RHS of (4.5) implies
\[
\begin{align*}
\int_{\Omega} \left( K_1(y)(v_2^p \frac{\partial \xi_2}{\partial y_i} + p v_2^{p-1} \xi_2 \frac{\partial v_2}{\partial y_i}) + K_2(y)(v_1^q \frac{\partial \xi_1}{\partial y_i} + q v_1^{q-1} \xi_1 \frac{\partial v_1}{\partial y_i}) \right) \\
= \int_{\Omega} \left( K_1(y) \frac{\partial (v_2^p \xi_2)}{\partial y_i} + K_2(y) \frac{\partial (v_1^q \xi_1)}{\partial y_i} \right) \\
= - \int_{\partial \Omega} \left( v_2^p \xi_2 \frac{\partial K_1(y)}{\partial y_i} + v_1^q \xi_1 \frac{\partial K_2(y)}{\partial y_i} \right) + \int_{\partial \Omega} \left( K_1(y)v_2^p \xi_2 v_i + K_2(y)v_1^q \xi_1 v_i \right). \tag{4.6}
\end{align*}
\]

The LHS of (4.5) reads
\[
\begin{align*}
\int_{\Omega} \left( -\Delta v_1 \frac{\partial \xi_2}{\partial y_i} - \Delta \xi_1 \frac{\partial v_2}{\partial y_i} - \Delta v_2 \frac{\partial \xi_1}{\partial y_i} - \Delta \xi_2 \frac{\partial v_1}{\partial y_i} \right) \\
= \int_{\partial \Omega} \left( (\nabla v_1, \nabla \xi_2) v_i + (\nabla v_2, \nabla \xi_1) v_i \right) \tag{4.7} \\
- \int_{\partial \Omega} \left( \frac{\partial v_1}{\partial y} \frac{\partial \xi_2}{\partial y_i} + \frac{\partial v_1}{\partial y} \frac{\partial \xi_2}{\partial y_i} + \frac{\partial v_2}{\partial y} \frac{\partial \xi_1}{\partial y_i} + \frac{\partial v_2}{\partial y} \frac{\partial \xi_1}{\partial y_i} \right).
\end{align*}
\]
which combined with (4.6) gives (4.3).

Next, we prove (4.4). From the system (4.1), we have that
\[
\begin{align*}
\int_{\Omega} \left( -\Delta v_1 (\nabla \xi_2, y - x_0) - \Delta \xi_1 (\nabla v_2, y - x_0) - \Delta v_2 (\nabla \xi_1, y - x_0) - \Delta \xi_2 (\nabla v_1, y - x_0) \right) \\
= \int_{\Omega} \left( K_1(y)(v_2^p (\nabla \xi_2, y - x_0) + p v_2^{p-1} \xi_2 (\nabla v_2, y - x_0)) \right. \\
\left. + K_2(y)(v_1^q (\nabla \xi_1, y - x_0) + q v_1^{q-1} \xi_1 (\nabla v_1, y - x_0)) \right). \tag{4.8}
\end{align*}
\]

It is easy to see that
\[
\begin{align*}
\int_{\Omega} \left( K_1(y)(v_2^p (\nabla \xi_2, y - x_0) + p v_2^{p-1} \xi_2 (\nabla v_2, y - x_0)) \right. \\
+ K_2(y)(v_1^q (\nabla \xi_1, y - x_0) + q v_1^{q-1} \xi_1 (\nabla v_1, y - x_0)) \right) \\
= \int_{\Omega} \left( K_1(y)(\nabla (v_2^p \xi_2), y - x_0) + K_2(y)(\nabla (v_1^q \xi_1), y - x_0) \right) \tag{4.9} \\
= \int_{\partial \Omega} \left( K_1(y)v_2^p \xi_2 (\nu, y - x_0) + K_2(y)v_1^q \xi_1 (\nu, y - x_0) \right) \\
- \int_{\Omega} \left( v_2^p \xi_2 (\nabla K_1, y - x_0) + v_1^q \xi_1 (\nabla K_2, y - x_0) \right) - N \int_{\Omega} (K_1(y)v_2^p \xi_2 + K_2(y)v_1^q \xi_1).
\end{align*}
\]
Moreover,

\[
\int_{\Omega} \left( -\Delta v_1 \langle \nabla \xi_2, y - x_0 \rangle - \Delta \xi_1 \langle \nabla v_2, y - x_0 \rangle - \Delta v_2 \langle \nabla \xi_1, y - x_0 \rangle - \Delta \xi_2 \langle \nabla v_1, y - x_0 \rangle \right) \\
= -\int_{\partial \Omega} \frac{\partial v_1}{\partial \nu} \langle \nabla \xi_2, y - x_0 \rangle + \int_{\Omega} \frac{\partial v_1}{\partial y_j} \langle \frac{\partial \xi_2}{\partial y_j}, y - x_0 \rangle + \int_{\Omega} \langle \nabla v_1, \nabla \xi_2 \rangle \\
- \int_{\partial \Omega} \frac{\partial \xi_1}{\partial \nu} \langle \nabla v_2, y - x_0 \rangle + \int_{\Omega} \frac{\partial \xi_1}{\partial y_j} \langle \frac{\partial v_2}{\partial y_j}, y - x_0 \rangle + \int_{\Omega} \langle \nabla \xi_1, \nabla v_2 \rangle \\
- \int_{\partial \Omega} \frac{\partial v_2}{\partial \nu} \langle \nabla \xi_1, y - x_0 \rangle + \int_{\Omega} \frac{\partial v_2}{\partial y_j} \langle \frac{\partial \xi_1}{\partial y_j}, y - x_0 \rangle + \int_{\Omega} \langle \nabla v_2, \nabla \xi_1 \rangle \\
- \int_{\partial \Omega} \frac{\partial \xi_2}{\partial \nu} \langle \nabla v_1, y - x_0 \rangle + \int_{\Omega} \frac{\partial \xi_2}{\partial y_j} \langle \frac{\partial v_1}{\partial y_j}, y - x_0 \rangle + \int_{\Omega} \langle \nabla \xi_2, \nabla v_1 \rangle \\
= -\int_{\partial \Omega} \left( \frac{\partial v_1}{\partial \nu} \langle \nabla \xi_2, y - x_0 \rangle + \frac{\partial \xi_1}{\partial \nu} \langle \nabla v_2, y - x_0 \rangle + \frac{\partial v_2}{\partial \nu} \langle \nabla \xi_1, y - x_0 \rangle + \frac{\partial \xi_2}{\partial \nu} \langle \nabla v_1, y - x_0 \rangle \right) \\
+ \langle \nabla v_1, \nabla \xi_2 \rangle \langle \nu, y - x_0 \rangle + \langle \nabla v_2, \nabla \xi_1 \rangle \langle \nu, y - x_0 \rangle \\
- \int_{\Omega} (N - 2) (\langle \nabla v_1, \nabla \xi_2 \rangle + \langle \nabla v_2, \nabla \xi_1 \rangle).
\]

(4.10)

On the other hand, we multiply the first equation of (4.1) by $\frac{N}{p+1} \xi_2$, the second by $\frac{N}{q+1} \xi_1$, and multiply the first equation of (4.2) by $\frac{N}{p+1} v_2$, the second by $\frac{N}{q+1} v_1$. Adding them together and integrating on $\Omega$, it holds that

\[
\left( \frac{N}{q+1} + \frac{N}{p+1} \right) \int_{\Omega} ((\nabla v_1, \nabla \xi_2) + \nabla v_2, \nabla \xi_1)) \\
+ \int_{\partial \Omega} \left( \frac{N}{p+1} \xi_2 \frac{\partial v_1}{\partial \nu} + \frac{N}{q+1} \xi_1 \frac{\partial v_2}{\partial \nu} + \frac{N}{q+1} v_2 \frac{\partial \xi_1}{\partial \nu} + \frac{N}{p+1} v_1 \frac{\partial \xi_2}{\partial \nu} \right) \\
= \left( \frac{N}{p+1} + \frac{pN}{p+1} \right) \int_{\Omega} K_1(y)v_2^q \xi_2 + \left( \frac{N}{q+1} + \frac{qN}{q+1} \right) \int_{\Omega} K_2(y)v_1^q \xi_1 \\
= N \int_{\Omega} (K_1(y)v_2^q \xi_2 + K_2(y)v_1^q \xi_1).
\]

(4.11)

Noting that $\frac{N}{q+1} + \frac{N}{p+1} = N - 2$, (4.11) reads

\[
(N - 2) \int_{\Omega} ((\nabla v_1, \nabla \xi_2) + \nabla v_2, \nabla \xi_1)) \\
+ \int_{\partial \Omega} \left( \frac{N}{p+1} \xi_2 \frac{\partial v_1}{\partial \nu} + \frac{N}{q+1} \xi_1 \frac{\partial v_2}{\partial \nu} + \frac{N}{q+1} v_2 \frac{\partial \xi_1}{\partial \nu} + \frac{N}{p+1} v_1 \frac{\partial \xi_2}{\partial \nu} \right) \\
= N \int_{\Omega} (K_1(y)v_2^q \xi_2 + K_2(y)v_1^q \xi_1).
\]

(4.12)

Combining (4.9), (4.10) and (4.12), we get (4.4).
4.2. Some estimates on the bubbling solutions. Recall \( \bar{x}_j = \frac{x_j}{\mu_k}, \bar{r}_k = |\bar{x}_j| \) and

\[
v_{1,k}(y) = \frac{N}{\mu_k^{q+1}} u_{1,k}(\mu_k y), \quad v_{2,k}(y) = \frac{N}{\mu_k^{p+1}} u_{2,k}(\mu_k y).
\]

Define

\[
\|v_1\|_{*,1} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{N}{\mu_k^{q+1}} \frac{1}{(1 + \mu_k |y - \bar{x}_j|)^{q+1}} \right)^{-1} |v_1(y)|,
\]

\[
\|v_2\|_{*,2} = \sup_{y \in \mathbb{R}^N} \left( \sum_{j=1}^{k} \frac{N}{\mu_k^{p+1}} \frac{1}{(1 + \mu_k |y - \bar{x}_j|)^{p+1}} \right)^{-1} |v_2(y)|.
\]

Theorem 1.1 gives that for the multi-bubbling solutions \((v_{1,k}, v_{2,k})\) to system (1.1),

\[
|v_{1,k}(y)| \leq C \sum_{j=1}^{k} \frac{N}{\mu_k^{q+1}} \frac{1}{(1 + \mu_k |y - \bar{x}_j|)^{q+1}}, \quad |v_{2,k}(y)| \leq C \sum_{j=1}^{k} \frac{N}{\mu_k^{p+1}} \frac{1}{(1 + \mu_k |y - \bar{x}_j|)^{p+1}}.
\]

In fact, Lemma 3.3 further upgrades the estimates for \((v_{1,k}, v_{2,k})\) as stated in the following lemma, which is of importance in the study of the non-degeneracy result.

Lemma 4.2. There exists a constant \( C > 0 \) such that for all \( y \in \mathbb{R}^N \),

\[
|v_{1,k}(y)| \leq C \sum_{j=1}^{k} \frac{N}{\mu_k^{q+1}} \frac{1}{(1 + \mu_k |y - \bar{x}_j|)^{N-2}}, \quad |v_{2,k}(y)| \leq C \sum_{j=1}^{k} \frac{N}{\mu_k^{p+1}} \frac{1}{(1 + \mu_k |y - \bar{x}_j|)^{N-2}}.
\]

4.3. Non-degeneracy result. To show Theorem 1.8, we suppose by contradiction that there exist \( k_n \to +\infty \) such that \( \|\xi_{1,n}\|_{*,1} + \|\xi_{2,n}\|_{*,2} = 1 \), and by the definition (1.12),

\[
Q_k(\xi_{1,n}, \xi_{2,n}) = 0.
\]

Set

\[
\tilde{\xi}_{1,n}(y) = \frac{N}{\mu_{k_n}^{q+1}} \xi_{1,n}(\mu_{k_n}^{-1} y + x_{k_n,1}), \quad \tilde{\xi}_{2,n}(y) = \frac{N}{\mu_{k_n}^{p+1}} \xi_{2,n}(\mu_{k_n}^{-1} y + x_{k_n,1}).
\]

Lemma 4.3. It holds that

\[
\tilde{\xi}_{1,n} \to b_0 \Psi_0 + b_1 \Psi_1, \quad \tilde{\xi}_{2,n} \to b_0 \Phi_0 + b_1 \Phi_1, \quad \text{as} \ n \to +\infty
\]

uniformly in \( C^1(B_R(0)) \) for any \( R > 0 \), where \( b_0, b_1 \) are some constants, and

\[
\Psi_0 = \frac{\partial U_0}{\partial \mu} \bigg|_{\mu=1}, \quad \Psi_i = \frac{\partial U_i}{\partial y_i} \bigg|_{\mu=1}, \quad \Phi_0 = \frac{\partial V_0}{\partial \mu} \bigg|_{\mu=1}, \quad \Phi_i = \frac{\partial V_i}{\partial y_i}, \quad i = 1, \ldots, N.
\]

Proof. Since \( |\tilde{\xi}_{1,n}|, |\tilde{\xi}_{2,n}| \leq C \), we may assume that \( \tilde{\xi}_{1,n} \to \xi_1, \tilde{\xi}_{2,n} \to \xi_2 \) in \( C_{loc}(\mathbb{R}^N) \). Then we know that \((\xi_1, \xi_2)\) satisfies the system

\[
\begin{cases}
-\Delta \xi_1 - pV_{0,1}^{-1} \xi_2 = 0, \\
-\Delta \xi_2 - qU_{0,1}^{-1} \xi_1 = 0,
\end{cases}
\]
which combined with the non-degeneracy of \((U, V) = (U_{0,1}, V_{0,1})\) from Lemma 1.5, implies that

\[
\xi_1 = \sum_{i=0}^{N} b_i \Psi_i, \quad \xi_2 = \sum_{i=0}^{N} b_i \Phi_i.
\]

Since \(\xi_{1,n}, \xi_{2,n}\) are both even in \(y_i\) for \(i = 2, \ldots, N\), we obtain that \(b_i = 0, i = 2, \ldots, N\). □

Now we decompose

\[
\begin{align*}
\xi_{1,n} &= b_{0,n} \mu_{k_n} \sum_{j=1}^{k_n} \frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \sum_{j=1}^{k_n} \frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial r} + \xi_{1,*}^n, \\
\xi_{2,n} &= b_{0,n} \mu_{k_n} \sum_{j=1}^{k_n} \frac{\partial V_{x_{k_n,j},\mu_{k_n}}}{\partial \mu_{k_n}} + b_{1,n} \mu_{k_n}^{-1} \sum_{j=1}^{k_n} \frac{\partial V_{x_{k_n,j},\mu_{k_n}}}{\partial r} + \xi_{2,*}^n,
\end{align*}
\]

where \((\xi_{1,*}^n, \xi_{2,*}^n)\) satisfies that

\[
\left\langle \left( pV_{x_{k_n,j},\mu_{k_n}}^{p-1} \frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial \mu_{k_n}}, qU_{x_{k_n,j},\mu_{k_n}}^{q-1} \frac{\partial U_{x_{k_n,j},\mu_{k_n}}}{\partial r} \right), (\xi_{1,*}^n, \xi_{2,*}^n) \right\rangle = 0,
\]

\[
\left\langle \left( pV_{x_{k_n,j},\mu_{k_n}}^{p-1} \frac{\partial V_{x_{k_n,j},\mu_{k_n}}}{\partial r}, qU_{x_{k_n,j},\mu_{k_n}}^{q-1} \frac{\partial V_{x_{k_n,j},\mu_{k_n}}}{\partial r} \right), (\xi_{1,*}^n, \xi_{2,*}^n) \right\rangle = 0.
\]

As in the proof of Proposition 2.3 and Lemma 4.2, it is standard to obtain the following two lemmas.

**Lemma 4.4.** There exists a constant \(C > 0\) such that

\[
\|\xi_{1,*}^n\|_{*,1} \leq C \frac{1}{\mu_{k_n}^{\frac{1}{2} + \theta}}, \quad \|\xi_{2,*}^n\|_{*,2} \leq C \frac{1}{\mu_{k_n}^{\frac{1}{2} + \theta}}.
\]

**Lemma 4.5.** There exists a constant \(C > 0\) such that for all \(y \in \mathbb{R}^N\),

\[
\begin{align*}
|\xi_{1,n}(y)| &\leq C \sum_{j=1}^{k} \mu_{k_n}^N \left( 1 + \mu_{k_n} |y - \bar{x}_j| \right)^{N-2}, \\
|\xi_{2,n}(y)| &\leq C \sum_{j=1}^{k} \mu_{k_n}^N \left( 1 + \mu_{k_n} |y - \bar{x}_j| \right)^{N-2}.
\end{align*}
\]

To get a contradiction, we need the following result.

**Lemma 4.6.** For \(i = 1, 2\), if \(\Delta K_i - (\Delta K_i + \frac{1}{2}(\Delta K_1'))r \neq 0\) at \(r = r_0\), then we have \(\bar{\xi}_{i,n} \to 0\) uniformly in \(C^1(B_R(0))\) for any \(R > 0\).
Proof. We apply Lemma 4.1 in domain $\Omega_1$. Firstly, we consider (4.3)

$$-\int_{\partial \Omega_1} \left( \frac{\partial v_{1,k_n}}{\partial \nu} \frac{\partial \xi_{2,n}}{\partial y_1} + \frac{\partial v_{1,k_n}}{\partial y_1} \frac{\partial \xi_{2,n}}{\partial \nu} + \frac{\partial v_{2,k_n}}{\partial \nu} \frac{\partial \xi_{1,n}}{\partial y_1} + \frac{\partial v_{2,k_n}}{\partial y_1} \frac{\partial \xi_{1,n}}{\partial \nu} \right)$$

$$+ \int_{\partial \Omega_1} \left( (\nabla v_{1,k_n}, \nabla \xi_{2,n}) \nu_1 + (\nabla v_{2,k_n}, \nabla \xi_{1,n}) \nu_1 \right)$$

$$-\int_{\partial \Omega_1} \left( K_1(y)v_{2,k_n}^p \xi_{2,n} \nu_1 + K_2(y)v_{1,k_n}^q \xi_{1,n} \nu_1 \right)$$

$$= -\int_{\Omega_1} \left( \frac{\partial K_1(y)}{\partial y_1} v_{2,k_n}^p \xi_{2,n} + \frac{\partial K_2(y)}{\partial y_1} v_{1,k_n}^q \xi_{1,n} \right).$$

(4.13)

In view of the symmetry, there hold that

$$\frac{\partial v_{1,k_n}}{\partial \nu} = \frac{\partial v_{2,k_n}}{\partial \nu} = \frac{\partial \xi_{1,n}}{\partial \nu} = \frac{\partial \xi_{2,n}}{\partial \nu} = 0 \text{ on } \partial \Omega_1.$$

Thus, the LHS of (4.13)

$$= \int_{\partial \Omega_1} \left( (\nabla v_{1,k_n}, \nabla \xi_{2,n}) \nu_1 + (\nabla v_{2,k_n}, \nabla \xi_{1,n}) \nu_1 - K_1(y)v_{2,k_n}^p \xi_{2,n} \nu_1 - K_2(y)v_{1,k_n}^q \xi_{1,n} \nu_1 \right)$$

$$= -\sin \frac{\pi}{k_n} \int_{\partial \Omega_1} \left( (\nabla v_{1,k_n}, \nabla \xi_{2,n}) + (\nabla v_{2,k_n}, \nabla \xi_{1,n}) - K_1(y)v_{2,k_n}^p \xi_{2,n} - K_2(y)v_{1,k_n}^q \xi_{1,n} \right),$$

which, combined with (4.13), gives

$$-\int_{\Omega_1} \left( \frac{\partial K_1(y)}{\partial y_1} v_{2,k_n}^p \xi_{2,n} + \frac{\partial K_2(y)}{\partial y_1} v_{1,k_n}^q \xi_{1,n} \right)$$

$$= -\sin \frac{\pi}{k_n} \int_{\partial \Omega_1} \left( (\nabla v_{1,k_n}, \nabla \xi_{2,n}) + (\nabla v_{2,k_n}, \nabla \xi_{1,n}) - K_1(y)v_{2,k_n}^p \xi_{2,n} - K_2(y)v_{1,k_n}^q \xi_{1,n} \right).$$

(4.14)

Next, we apply (4.4) to deal with the left hand side of (4.14). Also by symmetry, it holds that

$$\int_{\partial \Omega_1} \left( K_1(y)v_{2,k_n}^p \xi_{2,n} \langle \nu, y - x_{k_n,1} \rangle + K_2(y)v_{1,k_n}^q \xi_{1,n} \langle \nu, y - x_{k_n,1} \rangle \right)$$

$$- (\nabla v_{1,k_n}, \nabla \xi_{2,n}) \langle \nu, y - x_{k_n,1} \rangle - (\nabla v_{2,k_n}, \nabla \xi_{1,n}) \langle \nu, y - x_{k_n,1} \rangle$$

$$= \int_{\Omega_1} \left( (\nabla K_1, y - x_{k_n,1}) v_{2,k_n}^p \xi_{2,n} + (\nabla K_2, y - x_{k_n,1}) v_{1,k_n}^q \xi_{1,n} \right).$$

(4.15)

Note that on $\partial \Omega_1$, it holds that $\langle \nu, y \rangle = 0$. Moreover, we have $\langle \nu, x_{k_n,1} \rangle = -\sin \frac{\pi}{k_n}$. Hence, (4.15) becomes

$$\sin \frac{\pi}{k_n} \int_{\partial \Omega_1} \left( K_1(y)v_{2,k_n}^p \xi_{2,n} + K_2(y)v_{1,k_n}^q \xi_{1,n} - (\nabla v_{1,k_n}, \nabla \xi_{2,n}) - (\nabla v_{2,k_n}, \nabla \xi_{1,n}) \right)$$

$$= \int_{\Omega_1} \left( (\nabla K_1, y - x_{k_n,1}) v_{2,k_n}^p \xi_{2,n} + (\nabla K_2, y - x_{k_n,1}) v_{1,k_n}^q \xi_{1,n} \right).$$

(4.16)
which, combined with (4.14), implies that

\[
- \int_{\Omega_1} \left( \frac{\partial K_1(y)}{\partial y_1} v_{2,k_n}^p \xi_{2,n} + \frac{\partial K_2(y)}{\partial y_1} v_{1,k_n}^q \xi_{1,n} \right)
= \int_{\Omega_1} \left( \langle \nabla K_1, y - x_{k_n,1} \rangle v_{2,k_n}^p \xi_{2,n} + \langle \nabla K_2, y - x_{k_n,1} \rangle v_{1,k_n}^q \xi_{1,n} \right).
\]

(4.17)

We observe by Lemma 4.4 that

\[
\int_{\Omega_1} \left( v_{2,k_n}^p \xi_{2,n} + v_{1,k_n}^q \xi_{1,n} \right)
= \int_{\Omega_{1,n}} \left( \left(\frac{\mu_{k_n}^{N+1}}{\mu_{k_n}^{N+1}} v_{2,k_n} (\mu_{k_n}^{-1} y + x_{k_n,1})\right)^p \xi_{2,n}(y) + \left(\frac{\mu_{k_n}^{N+1}}{\mu_{k_n}^{N+1}} v_{1,k_n} (\mu_{k_n}^{-1} y + x_{k_n,1})\right)^q \xi_{1,n}(y) \right)
= \int_{\mathbb{R}^N} \left( V^p (b_{0,n} \Phi_0 + b_{1,n} \Phi_1 + \mu_{k_n}^{N+1} \xi_{2,n}^{*} (\mu_{k_n}^{-1} y + x_{k_n,1}))
+ U^q (b_{0,n} \Psi_0 + b_{1,n} \Psi_1 + \mu_{k_n}^{N+1} \xi_{1,n}^{*} (\mu_{k_n}^{-1} y + x_{k_n,1})) \right) + O \left( \frac{1}{\mu_{k_n}^{4+\theta}} \right)
= O(\|\xi_{1,n}^{*}\|_{*,1} + \|\xi_{2,n}^{*}\|_{*,2}) + O \left( \frac{1}{\mu_{k_n}^{4+\theta}} \right) = O \left( \frac{1}{\mu_{k_n}^{4+\theta}} \right),
\]

(4.18)

where \(\Omega_{1,n} = \{ y : \mu_{k_n}^{-1} y + x_{k_n,1} \in \Omega_1 \}\).
Moreover, since \( \nabla K_i(x_{k,n}) = O(|x_{k,n}| - r_0) = O\left(\frac{1}{\mu_{k,n}}\right) \), we obtain

\[
\begin{align*}
&\int_{\Omega_t} \left( \frac{\partial K_1(y)}{\partial y_1} \nu_{2,k_n} \xi_{2,n} + \frac{\partial K_2(y)}{\partial y_1} \nu_{1,k_n} \xi_{1,n} \right) \\
= &\int_{\Omega_t} \left( \left( \frac{\partial K_1(x_{k,n})}{\partial y_1} \right) \nu_{2,k_n} \xi_{2,n} + \left( \frac{\partial K_2(x_{k,n})}{\partial y_1} \right) \nu_{1,k_n} \xi_{1,n} \right) + O\left(\frac{1}{\mu_{k,n}}\right) \\
= &\int_{\Omega_t} \left( \left( \frac{\partial K_1(x_{k,n})}{\partial y_1} \right) \nu_{2,k_n} \xi_{2,n} + \frac{1}{2} \left( \frac{\partial^2 K_1(x_{k,n})}{\partial y_1^2} (y - x_{k,n}, y - x_{k,n}) \right) \\
+ &O\left(|y - x_{k,n}|^3\right) \right) \nu_{2,k_n} \xi_{2,n} \\
+ &\left( \left( \frac{\partial K_2(x_{k,n})}{\partial y_1} \right) \nu_{1,k_n} \xi_{1,n} + \frac{1}{2} \left( \frac{\partial^2 K_2(x_{k,n})}{\partial y_1^2} (y - x_{k,n}, y - x_{k,n}) \right) \\
+ &O\left(|y - x_{k,n}|^3\right) \right) \nu_{1,k_n} \xi_{1,n} \right) + O\left(\frac{1}{\mu_{k,n}}\right) \\
= &\int_{\mathbb{R}^N} \left\{ V^p(b_{0,n} \Phi_0 + b_{1,n} \Phi_1) \left( \left( \frac{\partial K_1(x_{k,n})}{\partial y_1}, \frac{\partial K_2(x_{k,n})}{\partial y_1} \right) , y - x_{k,n} \right) \right) \\
+ &\left( U^q(b_{0,n} \Psi_0 + b_{1,n} \Psi_1) \left( \left( \frac{\partial K_1(x_{k,n})}{\partial y_1}, \frac{\partial K_2(x_{k,n})}{\partial y_1} \right) , y - x_{k,n} \right) \right) \right) \\
+ &O\left(\frac{1}{\mu_{k,n}} + \frac{1}{\mu_{k,n}^3}\right) \\
= &K_2^{\mu_{k,n}}(x_{k,n}) b_{1,n} \int_{\mathbb{R}^N} V^p \Phi_1 y_1 + \frac{\partial\Delta K_2(x_{k,n})}{\partial y_1^2} b_{0,n} \int_{\mathbb{R}^N} V^p \Phi_0 |y|^2 \\
+ &K_1^{\mu_{k,n}}(x_{k,n}) b_{1,n} \int_{\mathbb{R}^N} U^q \Psi_1 y_1 + \frac{\partial\Delta K_1(x_{k,n})}{\partial y_1^2} b_{0,n} \int_{\mathbb{R}^N} U^q \Psi_0 |y|^2 + O\left(\frac{1}{\mu_{k,n}^3} + \frac{1}{\mu_{k,n}^3}\right). \\
\end{align*}
\]

On the other hand, since as in the proof of (4.18),

\[
\begin{align*}
&\int_{\Omega_t} \left( \left( \nabla K_1(x_{k,n}), y - x_{k,n} \right) \nu_{2,k_n} \xi_{2,n} + \left( \nabla K_2(x_{k,n}), y - x_{k,n} \right) \nu_{1,k_n} \xi_{1,n} \right) \\
= &O\left(\frac{1}{\mu_{k,n}}\right),
\end{align*}
\]
thus,

\[
\int_{\Omega_1} \left( (\nabla K_1(y) - \nabla K_1(x_{k,n}), y - x_{k,n}) v_{2,k,n}^p \xi_{2,n} + (\nabla K_2(y), y - x_{k,n}) v_{1,k,n}^q \xi_{1,n} \right)
\]

\[
= \int_{\Omega_1} \left( (\nabla K_1(y) - \nabla K_1(x_{k,n}), y - x_{k,n}) v_{2,k,n}^p \xi_{2,n} + (\nabla K_2(y) - \nabla K_2(x_{k,n}), y - x_{k,n}) v_{1,k,n}^q \xi_{1,n} \right) + O\left(\frac{1}{\mu_{k_n}^{m+1+\theta}}\right)
\]

\[
= \int_{\Omega_1} \left( (\nabla^2 K_1(x_{k,n}), y - x_{k,n}) v_{2,k,n}^p \xi_{2,n} + (\nabla^2 K_2(x_{k,n}), y - x_{k,n}) v_{1,k,n}^q \xi_{1,n} \right) + O\left(\frac{1}{\mu_{k_n}^{m+1+\theta}} + \frac{1}{\mu_{k_n}^3}\right)
\]

(4.20)

\[
= \int_{\mathbb{R}^N} V^p \left(b_{0,n}\Phi_0 + b_{1,n}\Phi_1\right) \left(\nabla^2 K_1(x_{k,n}) y \mu_{k_n}, y \mu_{k_n}\right) N\mu_{k_n}^2 b_{0,n} \int_{\mathbb{R}^N} V^p \Phi_0 |y|^2 + \Delta K_1(x_{k,n}) \int_{\mathbb{R}^N} U^q \Psi_0 |y|^2
\]

\[
+ O\left(\frac{1}{\mu_{k_n}^{m+1+\theta}} + \frac{1}{\mu_{k_n}^3}\right).
\]

Combining (4.17), (4.19) and (4.20) together, we obtain that

\[
b_{1,n} = \frac{1}{\mu_{k_n} K_1^n(x_{k,n}) \int_{\mathbb{R}^N} U^q \Psi_1 y_1 + K_2^n(x_{k,n}) \int_{\mathbb{R}^N} V^p \Phi_1 y_1}
\]

\[
\times \left\{ \left( \Delta K_2(x_{k,n}) \frac{y}{N} + \frac{\partial \Delta K_2(x_{k,n})}{\partial y_1} \right) \int_{\mathbb{R}^N} V^p \Phi_0 |y|^2
\]

\[
+ \left( \Delta K_1(x_{k,n}) \frac{y}{N} + \frac{\partial \Delta K_1(x_{k,n})}{\partial y_1} \right) \int_{\mathbb{R}^N} U^q \Psi_0 |y|^2 \right\}
\]

\[
+ O\left(\frac{1}{\mu_{k_n}^{m+1+\theta}} + \frac{1}{\mu_{k_n}^3}\right).
\]

(4.21)

Next, from (1.6),

\[
\int_{\mathbb{R}^N} \left( (\nabla K_1(y), y) v_{2,k,n}^p \xi_{2,n} + (\nabla K_2(y), y) v_{1,k,n}^q \xi_{1,n} \right) = 0
\]

and so

\[
\int_{\Omega_1} \left( (\nabla K_1(y), y) v_{2,k,n}^p \xi_{2,n} + (\nabla K_2(y), y) v_{1,k,n}^q \xi_{1,n} \right) = 0. 
\]

(4.22)
On the other hand, just as proved in (4.18), we obtain

\[
\begin{align*}
&\int_{\Omega_1} \left( (\nabla K_1(x_{k,n}), y) v_{2,k_n}^p \xi_2, n + (\nabla K_2(x_{k,n}), y) v_{1,k_n}^q \xi_1, n \right) \\
&\quad = \int_{\Omega_1} \left( (\nabla K_1(x_{k,n}), y - x_{k,n}) v_{2,k_n}^p \xi_2, n + (\nabla K_2(x_{k,n}), y - x_{k,n}) v_{1,k_n}^q \xi_1, n \right) \\
&\quad + \int_{\Omega_1} \left( (\nabla K_1(x_{k,n}), x_{k,n}) v_{2,k_n}^p \xi_2, n + (\nabla K_2(x_{k,n}), x_{k,n}) v_{1,k_n}^q \xi_1, n \right) \\
&\quad = O\left( \frac{1}{\mu_{k_n}^{\frac{3}{2} + 1 + \theta}} \right).
\end{align*}
\]

From (4.21),

\[
\begin{align*}
&\int_{\Omega_1} \left( (\nabla K_1(y), y) v_{2,k_n}^p \xi_2, n + (\nabla K_2(y), y) v_{1,k_n}^q \xi_1, n \right) \\
&\quad = \int_{\Omega_1} \left( (\nabla K_1(y) - \nabla K_1(x_{k,n}, y), y) v_{2,k_n}^p \xi_2, n + (\nabla K_2(y) - \nabla K_2(x_{k,n}, y), y) v_{1,k_n}^q \xi_1, n \right) \\
&\quad + O\left( \frac{1}{\mu_{k_n}^{\frac{3}{2} + 1 + \theta}} \right)
\end{align*}
\]

\[
\begin{align*}
&= \int_{\mathbb{R}^N} \left( V^p(b_0, x_{k,n}) \Phi_0 + b_{1,n} \Phi_1 \right) \left( \nabla^2 K_1(x_{k,n}, y) \frac{y}{\mu_{k_n}} + x_{k,n} \right) \\
&\quad + U^q(b_0, x_{k,n}) \Psi_0 + b_{1,n} \Psi_1 \left( \nabla^2 K_2(x_{k,n}, y) \frac{y}{\mu_{k_n}} + x_{k,n} \right) \right) + O\left( \frac{1}{\mu_{k_n}^{\frac{3}{2} + 1 + \theta}} \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{\Delta K_1(x_{k,n})}{N \mu_{k_n}^2} b_{0,n} \int_{\mathbb{R}^N} V^p \Phi_0 |y|^2 + \frac{b_{1,n}}{\mu_{k_n}} \int_{\mathbb{R}^N} V^p \Phi_1 \nabla^2 K_1(x_{k,n}, y) \frac{y}{\mu_{k_n}} \\
&\quad + \frac{\Delta K_2(x_{k,n})}{N \mu_{k_n}^2} b_{0,n} \int_{\mathbb{R}^N} U^q \Psi_0 |y|^2 + \frac{b_{1,n}}{\mu_{k_n}} \int_{\mathbb{R}^N} U^q \Psi_1 \nabla^2 K_2(x_{k,n}, y) \frac{y}{\mu_{k_n}} \\
&\quad + O\left( \frac{1}{\mu_{k_n}^{\frac{3}{2} + 1 + \theta}} + \frac{1}{\mu_{k_n}^3} \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{\Delta K_1(x_{k,n})}{N \mu_{k_n}^2} b_{0,n} \int_{\mathbb{R}^N} V^p \Phi_0 |y|^2 + \frac{b_{1,n} K_1^p(x_{k,n}) |x_{k,n}|}{\mu_{k_n}} \int_{\mathbb{R}^N} V^p \Phi_1 y_1 \\
&\quad + \frac{\Delta K_2(x_{k,n})}{N \mu_{k_n}^2} b_{0,n} \int_{\mathbb{R}^N} U^q \Psi_0 |y|^2 + \frac{b_{1,n} K_2^q(x_{k,n}) |x_{k,n}|}{\mu_{k_n}} \int_{\mathbb{R}^N} U^q \Psi_1 y_1 + O\left( \frac{1}{\mu_{k_n}^{\frac{3}{2} + 1 + \theta}} + \frac{1}{\mu_{k_n}^3} \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{b_{0,n} |x_{k,n}|}{N \mu_{k_n}^2} \left( \int_{\mathbb{R}^N} V^p \Phi_0 |y|^2 \left( \Delta K_1(x_{k,n}) - r\left( \Delta K_1(x_{k,n}) + \frac{1}{2} \Delta K_1(x_{k,n}) \right) \right) \\
&\quad + \int_{\mathbb{R}^N} U^q \Psi_0 |y|^2 \left( \Delta K_2(x_{k,n}) - r\left( \Delta K_2(x_{k,n}) + \frac{1}{2} \Delta K_2(x_{k,n}) \right) \right) \right) \\
&\quad + O\left( \frac{1}{\mu_{k_n}^{\frac{3}{2} + 1 + \theta}} + \frac{1}{\mu_{k_n}^3} \right).
\end{align*}
\]
Since for \( i = 1, 2 \), \( \Delta K_i - r(\Delta K_i + \frac{1}{r}(\Delta K_i')) \neq 0 \) at \( r = r_0 \), combining (4.22) and (4.24), we obtain \( b_{0,n} = o(1) \), which, in view of (4.21), implies in turn \( b_{1,n} = o(1) \).

\[ \square \]

**Proof of Theorem 1.8.** Since \((\xi_{1,n}, \xi_{2,n})\) satisfies (4.2), we have

\[
\begin{align*}
\xi_{1,n}(y) &= p \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} K_1(z)v_{2,k_n}^{p-1}\xi_{2,n}(z)dz, \\
\xi_{2,n}(y) &= q \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} K_2(z)v_{1,k_n}^{q-1}\xi_{1,n}(z)dz.
\end{align*}
\]  

(4.25)

We estimate as in the proof of Lemma 4.2 to obtain that, for some \( \theta > 0 \),

\[
\left| \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} K_1(z)v_{2,k_n}^{p-1}\xi_{2,n}(z)dz \right| 
\leq C\|\xi_{2,n}\|_{*,2} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} K_1(z)|v_{2,k_n}|^{p-1} \sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau + \theta}} 
\leq C\|\xi_{2,n}\|_{*,2} \sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau + \theta}} 
\left| \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} K_2(z)v_{1,k_n}^{q-1}\xi_{1,n}(z)dz \right| 
\leq C\|\xi_{1,n}\|_{*,1} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} K_2(z)|v_{1,k_n}|^{q-1} \sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau + \theta}} 
\leq C\|\xi_{1,n}\|_{*,1} \sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau + \theta}}.
\]

As a result,

\[
\frac{|\xi_{1,n}(y)|}{\sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau}}} \leq C\|\xi_{2,n}\|_{*,2} \sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau + \theta}},
\]

\[
\frac{|\xi_{2,n}(y)|}{\sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau}}} \leq C\|\xi_{1,n}\|_{*,1} \sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau + \theta}}.
\]

From Lemma 4.6, \( \xi_{i,n} \to 0 \) in \( B_{R_{k_n}^{-1}}(x_{k_n,j}) \) and \( \sum_{i=1}^{2} \|\xi_{i,n}\|_{*,i} = 1 \), there hold that

\[
\frac{|\xi_{1,n}(y)|}{\sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau}}} \text{ and } \frac{|\xi_{2,n}(y)|}{\sum_{j=1}^{k_n} \frac{\mu_k^{N}}{(1 + \mu_k |y - \bar{x}|)^{\frac{N}{2} + \tau}}}
\]
attain their maximum in \( \mathbb{R}^N \setminus \bigcup_{j=1}^{k_n} B_{R_{k_n}}(x_{k_n,j}) \).

Therefore,
\[
\sum_{i=1}^{2} \|\xi_{i,n}\|_{*,i} \leq o(1) \sum_{i=1}^{2} \|\xi_{i,n}\|_{*,i},
\]
contradicting \( \sum_{i=1}^{2} \|\xi_{i,n}\|_{*,i} = 1 \).

\[\square\]

APPENDIX

A. Energy Expansion

Recall that
\[
I(u, v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v - \int_{\mathbb{R}^N} \left( \frac{1}{p+1} K_1 \left( \frac{|y|}{\mu} \right)|v|^{p+1} + \frac{1}{q+1} K_2 \left( \frac{|y|}{\mu} \right)|u|^{q+1} \right).
\]

In this section, we calculate \( I(W_1, W_2) \).

**Proposition A.1.**

\[
I(W_1, W_2) = k \left( A + \frac{\tilde{B}_1}{\lambda_2 \mu_2^2} + \frac{\tilde{B}_2}{\lambda_1 \mu_1^2} + \left( \frac{\tilde{B}_3}{\lambda_1 - 2 \mu_1} + \frac{\tilde{B}_1}{\lambda_2 - 2 \mu_2} \right)(\mu_0 - r)^2 \right.
\]

\[
- \sum_{j=2}^{k} \frac{B_j}{\lambda_j |x_j - x_1|^{N-2}} + kO \left( \frac{1}{\mu_0} + \frac{1}{\mu_1} \right) |\mu_0 - r|^3 \right),
\]

where \( r = |x_1| \), \( A = (1 - \frac{1}{q+1}) \int_{\mathbb{R}^N} U_{0,1}^{q+1} - \frac{1}{p+1} \int_{\mathbb{R}^N} V_{0,1}^{p+1} \), and \( \tilde{B}_i, \tilde{B}_1, B_i \) with \( i = 1, 2 \) are some positive constants.

**Proof.** By symmetry, we have that
\[
\int_{\mathbb{R}^N} \nabla W_1 \cdot \nabla W_2 = \sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x_i,\lambda}^{q+1} U_{x_j,\lambda} = k \left( \int_{\mathbb{R}^N} U_{0,1}^{q+1} + \sum_{j=2}^{k} \sum_{x_1}^{x_j} U_{x_i,\lambda}^{q+1} \right) \lambda_j |x_j - x_1|^{N-2} + O \left( \left( \frac{k}{\mu} \right)^{N-2} \right) \).
\]

If \( q \leq 3 \),
\[
\frac{1}{q+1} \int_{\mathbb{R}^N} K_2 \left( \frac{|y|}{\mu} \right)|W_1|^{q+1}
\]

\[
= \frac{k}{q+1} \int_{\Omega_1} \left( K_2 \left( \frac{|y|}{\mu} \right) \sum_{j=1}^{k} U_{x_j,\lambda}^{q+1} + (q+1) K_2 \left( \frac{|y|}{\mu} \right) \sum_{j=2}^{k} U_{x_1,\lambda}^{q+1} \right) ;
\]

while if \( q > 3 \),
\[
\frac{1}{q+1} \int_{\mathbb{R}^N} K_2 \left( \frac{|y|}{\mu} \right)|W_1|^{q+1}
\]

\[
= \frac{k}{q+1} \int_{\Omega_1} \left( K_2 \left( \frac{|y|}{\mu} \right) U_{x_1,\lambda}^{q+1} + (q+1) K_2 \left( \frac{|y|}{\mu} \right) \sum_{j=2}^{k} U_{x_1,\lambda}^{q+1} \right).
\]
Therefore, there hold that

\[ + O \left( U_{q+1}^{x_1,\lambda} \left( \sum_{i=2}^{k} U_{x_i,\lambda} \right)^{q+1} + U_{q+1}^{x_1,\lambda} \left( \sum_{i=2}^{k} U_{x_i,\lambda} \right)^2 \right) \].

Since in \( \mathbb{R}^N \setminus \Omega_1, |y - x_1| \geq \frac{c \mu k}{N}, \) there exists \( \frac{N - 2 - m}{N - 2} < \alpha < q(N - 2) - N \) (noting \( p, q > \frac{N + 1}{N - 2} \)) such that \( q(N - 2) - \theta - \alpha > N \). Then we estimate

\[
\int_{\mathbb{R}^N \setminus \Omega_1} U_{0,1}^{q+1} + \sum_{j=2}^{k} U_{x_j,\lambda}^{q+1} \leq C \left( \frac{k}{\mu} \right)^{N-2-\theta} \int_{\mathbb{R}^N \setminus \Omega_1} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{q(N-2)-(N-2+\theta)}} \frac{1}{(1 + |y - x_j|)^{N-2}} \\
+ C \left( \frac{k}{\mu} \right)^{N-2-\theta} \int_{\mathbb{R}^N \setminus \Omega_1} \frac{1}{(1 + |y - x_1|)^{q+1}(N-2)-(N-2+\theta)} \\
\leq C \left( \frac{k}{\mu} \right)^{N-2-\theta} \int_{\mathbb{R}^N \setminus \Omega_1} \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^\alpha} \frac{1}{(1 + |y - x_1|)^{q(N-2)-\theta-\alpha}} + \frac{1}{(1 + |y - x_j|)^{q(N-2)-\theta-\alpha}} \\
+ C \left( \frac{k}{\mu} \right)^{N-2-\theta} \int_{\mathbb{R}^N \setminus \Omega_1} \frac{1}{(1 + |y - x_1|)^{q+1}(N-2)-(N-2+\theta)} \\
= O \left( \left( \frac{k}{\mu} \right)^{N-2-\theta} \right) = O (\mu^{-m-\theta}).
\]

Therefore, there hold that

\[
\int_{\Omega_1} K_2 \left( \frac{|y|}{\mu} \right) U_{x_1,\lambda}^{q+1} \\
= \int_{\Omega_1} U_{x_1,\lambda}^{q+1} + \int_{\Omega_1} \left( K_2 \left( \frac{|y|}{\mu} \right) - 1 \right) U_{x_1,\lambda}^{q+1} \\
= \int_{\mathbb{R}^N} U_{0,1}^{q+1} - \frac{c_2}{\mu m_1} \int_{\Omega_1} |y - x_1|^{m_1} U_{0,1}^{q+1} + O (\mu^{-m-\theta}) \\
\leq \int_{\mathbb{R}^N} U_{0,1}^{q+1} - \frac{c_2}{\mu m_1 \lambda^{m_2}} \int_{\mathbb{R}^N} |y_1|^{m_1} U_{0,1}^{q+1} \\
- \frac{c_2}{\mu m_1 \lambda^{m_2}} \int_{\mathbb{R}^N} |y_1|^{m_1} U_{0,1}^{q+1} + O \left( \frac{|y_1|}{\mu} \right) + O (\mu^{-m-\theta}),
\]

and

\[
\int_{\Omega_1} K_2 \left( \frac{|y|}{\mu} \right) \sum_{j=2}^{k} U_{x_j,\lambda}^{q} U_{x_1,\lambda} = \int_{\mathbb{R}^N} \sum_{j=2}^{k} U_{x_j,\lambda}^{q} U_{x_1,\lambda} + \left( K_2 \left( \frac{|y|}{\mu} \right) - 1 \right) \sum_{j=2}^{k} U_{x_j,\lambda}^{q} U_{x_1,\lambda} + O (\mu^{-m-\theta}) \\
= \sum_{j=2}^{k} \frac{B_1}{\lambda^{N-2}|x_1 - x_j|^{N-2}} + O (\mu^{-m-\theta}).
\]

(A.3)
In view of the range of \( p \) and \( q \), we can take \( \frac{2(N - 2)}{q + 1} < \alpha < \min\{N - 2, 2(N - 2) - \frac{2N}{p + 1}\} = N - 2 \) (but \( \min\{N - 2, 2(N - 2) - \frac{2N}{p + 1}\} = N - 2 \)) to obtain

\[
\int_{\Omega_1} U_{x_1, \lambda}^{\frac{q + 1}{2}} \left( \sum_{j=2}^{k} U_{x_j, \lambda} \right) ^{\frac{q + 1}{2}} \leq \left( \frac{k}{\mu} \right) ^{\frac{2q + 1}{2}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y - x|)^{(N-2)(q+1)+\frac{(N-2-\alpha)(q+1)}{2}}} + O\left( \left( \frac{k}{\mu} \right) ^{N-2+\theta} \right)
\]

\[
= O\left( \left( \frac{k}{\mu} \right) ^{N-2+\theta} \right).
\]

Moreover, in the case \( q > 3 \), we could also estimate

\[
\int_{\Omega_1} U_{x_1, \lambda}^{q-1} \left( \sum_{j=2}^{k} U_{x_j, \lambda} \right) ^2 \leq \left( \frac{k}{\mu} \right) ^{2\alpha} \int_{\mathbb{R}^N} \frac{1}{(1 + |y - x|)^{(N-2)(q-1)+2(N-2-\alpha)}} + O\left( \left( \frac{k}{\mu} \right) ^{N-2+\theta} \right)
\]

\[
= O\left( \left( \frac{k}{\mu} \right) ^{N-2+\theta} \right).
\]

Thus,

\[
\int_{\mathbb{R}^N} K_2 \left( \frac{|y|}{\mu} \right) |W_1|^{q+1} = k \left( \int_{\mathbb{R}^N} U_{0,1}^{q+1} - \frac{c_2}{\mu m_1 \lambda^{m_1 - 2}} \int_{\mathbb{R}^N} \frac{1}{2} m_1 (m_1 - 1) |y_1|^{m_1 - 2} U_{0,1}^{2\ast} (\mu r_0 - |x_1|)^2 \right)
\]

\[
- \sum_{j=2}^{k} \frac{(q + 1)B_1}{\lambda^{N-2} |x_1 - x_j|^{N-2}} + O(\mu^{-m-\theta})
\]

Similar estimates hold for \( W_2 \),

\[
\int_{\mathbb{R}^N} K_1 \left( \frac{|y|}{\mu} \right) |W_2|^{p+1} = k \left( \int_{\mathbb{R}^N} V_{0,1}^{p+1} - \frac{c_1}{\mu m_2 \lambda^{m_2 - 2}} \int_{\mathbb{R}^N} |y_1|^{m_2} V_{0,1}^{p+1} \right)
\]

\[
- \sum_{j=2}^{k} \frac{(p + 1)B_2}{\lambda^{N-2} |x_1 - x_j|^{N-2}} + O(\mu^{-m-\theta})
\]

Combining the above estimates together, we conclude (A.1).
Proposition A.2.

\[ \frac{\partial I(W_1, W_2)}{\partial \lambda} = k \left( - \frac{\tilde{B}_1 m_2}{\lambda m_2 + 1} - \frac{\tilde{B}_2 m_1}{\lambda m_1 + 1} + \sum_{j=2}^{k} \lambda^{N-1} \frac{B_2(N - 2)}{x_j - x_1}^{N-\tau} \right) + kO \left( \frac{1}{\mu^{m+\theta}} + \frac{1}{\mu^{\rho}} |\mu r_0 - r|^2 \right), \]

where \( \tilde{B}_i, B_i \) with \( i = 1, 2 \) are the same positive constants in Proposition A.1.

The proof of this proposition is similar to Proposition A.1 and we omit it.

B. Some technical estimates

Now we first give the following known result which are useful in the previous sections.

Lemma B.1 (Lemma B.1, [42]). For any constant \( 0 < \sigma \leq \min\{\alpha, \beta\} \), there is a constant \( C > 0 \), such that

\[ \frac{1}{(1 + |y - x_i|)^{\alpha}} \frac{1}{(1 + |y - x_j|)^{\beta}} \leq \frac{C}{|x_i - x_j|^\sigma} \left( \frac{1}{(1 + |y - x_i|)^{\alpha + \beta - \sigma}} + \frac{1}{(1 + |y - x_j|)^{\alpha + \beta - \sigma}} \right). \]

Just by the similar arguments as that of Lemma B.2 in [42], we have

Lemma B.2. For any constant \( \sigma > 0, \sigma \neq N - 2 \), there exists a constant \( C > 0 \), such that

\[ \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z|)^{2+\sigma}} dz \leq C (1 + |y|)^{\min\{\sigma, N-2\} }. \]

Finally we give the proof of (2.9).

Proof of (2.9). First let us recall that \( \tilde{\sigma} = \frac{N-2}{2} + \tau \)

\[ \langle L_k(\varphi_1, \varphi_2), (Z_{1,I}, Y_{1,I}) \rangle = \langle L_k(Z_{1,I}, Y_{1,I}), (\varphi_1, \varphi_2) \rangle \]

\[ = \int_{\mathbb{R}^N} p \left( 1 - K_1(\frac{y}{\mu}) \right) W_2^{p-1} Z_{1,I} \varphi_1 + p \left( V_{x_1, \lambda}^{p-1} - \left( \sum_{j=1}^{k} V_{x_j, \lambda} \right)^{p-1} \right) Z_{1,I} \varphi_1 dy \]

\[ + \int_{\mathbb{R}^N} q \left( 1 - K_2(\frac{y}{\mu}) \right) W_1^{q-1} Y_{1,I} \varphi_2 + q \left( U_{x_1, \lambda}^{q-1} - \left( \sum_{j=1}^{k} U_{x_j, \lambda} \right)^{q-1} \right) Y_{1,I} \varphi_2 dy \]

We show that, with some \( \theta > 0 \),

\[ |\langle L_k(\varphi_1, \varphi_2), (Y_{1,I}, Z_{1,I}) \rangle| = O\left( \frac{1}{\mu^{\theta}} \right) \|(\varphi_1, \varphi_2)\|_{*}. \] (C.1)

In fact, it suffices to show that

\[ (i) \int_{\mathbb{R}^N} \left( 1 - K_2(\frac{y}{\mu}) \right) W_1^{q-1} Y_{1,I} \varphi_2 dy \leq C \frac{\mu^\theta}{\mu^{\theta}} \|\varphi_2\|_{*}; \]

\[ (ii) \int_{\mathbb{R}^N} (U_{x_1, \lambda}^{q-1} - W_1^{q-1}) Y_{1,I} \varphi_2 dy \leq C \frac{\mu^\theta}{\mu^{\theta}} \|\varphi_2\|_{*}, \]

since the counterpart terms corresponding to \( W_2 \) can be handled similarly. Moreover, one can refer to [42] for similar proof for (i). We just prove (ii) in two cases respectively: \( q - 1 \leq 1 \) and \( q - 1 > 1 \).
First, if \( q - 1 \leq 1 \), then we have

\[
|U_{x_1}^{q-1} - W_1^{q-1}| \leq \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^{q-1}.
\]

Since \( q, p > \frac{N+1}{N-2} \), we obtain \((N-2)(q-1) + (N-2) + \tau > N + 1 + \tau\) and \(q(\bar{\tau} - \tau_1) + N - 2 > \frac{3N}{2} - \frac{3}{2} + \tau - \tau_1\) with some \( \tau_1 > \frac{N - 2 - m}{N - 2} \). Thus, there exist some \( \alpha \in \left[ \frac{N - 2 - m}{N - 2}, q(N - 2) + \bar{\tau} \right), \beta \in \left[ \frac{N - 2 - m}{N - 2}, q(N - 2) + \bar{\tau} \right], \) and \( \theta, \bar{\theta}, \bar{\theta} > 0 \), such that

\[
\int_{\mathbb{R}^N} (U_{x_1}^{q-1} - W_1^{q-1}) Y_{1,\ell} \varphi_2 \, dy
\]

\[
\leq C \| \varphi_2 \|_* \int_{\mathbb{R}^N} \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^{q-1} \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^2} \right) \sigma \, dy
\]

\[
\leq C \| \varphi_2 \|_* \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)} \right)^q \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^2} \right) \sigma \, dy
\]

\[
+ \int_{\mathbb{R}^N} \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N+\theta}} \right) \left( \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N+\theta}} \right) \sigma \, dy
\]

\[
\leq C \frac{\mu^q}{\mu^q} \| \varphi_2 \|_*.
\]

On the other hand, if \( q - 1 > 1 \), then we have

\[
|U_{x_1}^{q-1} - W_1^{q-1}|
\]

\[
\leq C \left[ \frac{1}{(1 + |y - x_1|)^{(N-2)(q-1)}} \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \bar{\tau} + \frac{1}{(1 + |y - x_1|)^{N-2}} \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^{q-2} + \left( \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right)^{q-1} \right].
\]

Thus, following the method used when dealing with the case \( p - 1 \leq 1 \), it is not difficult to show that there exists some \( \theta > 0 \) such that

\[
\int_{\mathbb{R}^N} (U_{x_1}^{q-1} - W_1^{q-1}) Y_{1,\ell} \varphi_2 \, dy \leq C \frac{\mu^q}{\mu^q} \| \varphi_2 \|_*.
\]
Similar estimates hold to give
\[ \int_{\mathbb{R}^N} \left( V^{p-1}_{x_1, \lambda} - W^{p-1}_2 \right) Z_{1, \beta} \varphi_1 \, dy \leq \frac{C}{\mu \theta} \| \varphi_1 \|_r. \]

\[ \square \]

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