Resilience of Complex Networks to Random Breakdown

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Using Monte Carlo simulations we calculate $f_c$, the fraction of nodes which are randomly removed before global connectivity is lost, for networks with scale-free and bimodal degree distributions. Our results differ with the results predicted by an equation for $f_c$ proposed by Cohen, et al. We discuss the reasons for this disagreement and clarify the domain for which the proposed equation is valid.

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I. INTRODUCTION

It has been shown that random uncorrelated networks with degree distribution $P(k)$ lose global connectivity when

$$\kappa \equiv \frac{(k^2)}{\langle k \rangle} < 2. \quad (1)$$

As explained in Ref. [2, 4], random removal of a fraction $f$ of nodes from a network with degree distribution $P_0(k)$ results in a new degree distribution

$$P(k) = \sum_{k_0=k}^K P_0(k_0) \left( \frac{k_0}{k} \right)^f f^k_{k_0} (1-f)^{k-k_0}. \quad (2)$$

Using this degree distribution to calculate $\langle k \rangle$ and $\langle k^2 \rangle$ after random removal of sites it was determined [2, 4] that

$$f_c = 1 - \frac{1}{\kappa_0 - 1} \quad (3)$$

where $\kappa_0$ is the value of $\kappa$ computed from the original degree distribution, before the random removal. Equation (3) was observed to hold for a number of network types, including random networks that have a Poisson degree distribution, and was used in the analysis of scale-free networks that have power-law degree distributions [2, 4].

Using Monte-Carlo simulations we find that Eq. (3) does not hold for networks with (i) self-loops and multiple edges and/or (ii) high variance in $f_c$. We illustrate our findings using scale-free and bimodal networks and clarify the domains where Eq. (3) is valid.

II. MONTE CARLO SIMULATIONS

We create random networks having specified degree distributions using the method described in Ref. [1]. We then randomly delete nodes in the network and after each node is removed, we calculate $\kappa$. When $\kappa$ becomes less than 2 we record the number of nodes $i$ removed up to that point. This process is performed for many realizations of random graphs with a specified degree distribution and, for each graph, for many different realizations of the sequence of random node removals. The threshold $f_c$ is defined as

$$f_c = \frac{\langle i \rangle}{N} \quad (4)$$

where $\langle i \rangle$ is the average value of $i$.

III. SCALE-FREE NETWORKS

We study scale-free random networks with degree distribution

$$P(k) \sim k^{-\lambda} \quad [m \leq k \leq K]. \quad (5)$$

We choose the lower cutoff $m = 4$ and the upper cutoff $K = N$. In Figs. (a), (b) and (c), we show the dependence on $\lambda$ of $1 - f_c^{\text{MC}}$ obtained by the Monte Carlo simulations and compare it with $1 - f_c^{\text{th}}$ obtained theoretically from Eq. (3). The simulation results agree well with Eq. (3) for $\lambda > \lambda^*$, where $\lambda^* \approx 3$, and the agreement becomes better for increasing $N$. However, for $\lambda < \lambda^*$ there is significant disagreement, and the disagreement becomes larger as $N$ increases, as seen clearly Fig. (d) in which we plot the normalized difference

$$\Delta = \frac{f_c^{\text{th}} - f_c^{\text{MC}}}{f_c^{\text{MC}}} \quad (6)$$

The nonzero value of $\Delta$ has its root in the use of Eq. (2) to derive Eq. (3). Equation (2) is valid only if, in the original network, two conditions hold: (i) There are no self loops, i.e. all links from node $i$ are to distinct nodes $j$ with $j \neq i$ and (ii) there are no multiple links between $i$ and $j$. In graph theory networks satisfying these two conditions are called simple. If the original network is not simple, Eq. (2) must then be interpreted as operating on the original network but with self-loops and multiple links deleted. But this deletion changes the properties of the degree distribution. As seen in Figs. (a), (b), and (c) the cutoff is changed, and for large $N$, the slope of the tail of the distribution is modified. Also the degrees of adjacent nodes become correlated as seen in Fig. (5).
which shows the $\lambda$-dependence of the degree correlation

$$r \equiv \frac{1}{\sigma_q^2} \sum_{j,k} (e_{jk} - q_j q_k).$$

(7)

Here $e_{jk}$ is the joint probability of the remaining degrees of the two vertices at either end of a randomly chosen edge, $q_k$ is the probability of the remaining degree of a single vertex at the end of a randomly chosen edge, and

$$\sigma_q^2 \equiv \sum_k k^2 q_k - \left( \sum_k k q_k \right)^2.$$  (8)

Because of the degree correlations, Eq. (11) no longer applies and therefore Eq. (3) no longer holds. The similarity in appearance between Fig. 1(d) and Fig. 3 confirms that the nonzero correlations play a major role in the difference between $f_{eh}$ and $f_{eh}^h$.

We can explain the domain of validity of Eq. (3) as follows. It is known that for any desired random degree distribution, the networks created by such methods as those of Molloy-Reed or Chum-Lu create simple graphs only if $P(k) = 0$ for $k$ greater than the structural cutoff

$$K_s \equiv \sqrt{(k/N).}$$

(9)

It is also known that for scale-free networks the number of nodes with degree greater than the natural cutoff

$$K_c \equiv mN^{1/(\lambda - 1)}$$

(10)

is statistically insignificant. These two facts are sufficient to understand that Eq. (3) is valid for scale-free networks only if $\lambda > 3$ (in which case the natural cutoff $K_c$ results in nodes with degree $\gtrsim \sqrt{N}$ being statistically insignificant) or for $\lambda < 3$ if the maximum degree is less than the structural cutoff $K_s$.

\section*{IV. BIMODAL NETWORKS}

\subsection*{A. Star Networks}

First, we discuss a simple example with a bimodal degree distribution for which Eq. (3) fails. Consider a star network of $N$ nodes with degree distribution

$$P(k) = \begin{cases} (N-1)/N & [k = 1] \\ 1/N & [k = N-1] \end{cases}$$

(11)

and $P(k) = 0$ for all other values of $k$. If nodes are randomly removed, the criterion for losing global connectivity, $\kappa < 2$, is obtained when the single node with degree $N-1$, the hub node, is removed or when almost all of the degree $1$ nodes, the leaf nodes, are removed. The probability that almost all the leaf nodes are removed before the hub node is removed approaches 0 for large $N$. Let $i$ be the number of nodes which are removed before the hub node is removed. Since the removal is random, $i$ is uniformly distributed between $0$ and $N-1$ and, from Eq. (11), $f_c = 1/2$. On the other hand, Eq. (3) predicts $f_c = 1 - 1/(\kappa_0 - 1)$ which asymptotically approaches unity for large $N$.

As for the case of scale-free networks, we can understand this disagreement as a result of the presence of self loops. We can also use this star network example to identify another implicit assumption used in the derivation of Eq. (3), namely that

$$\langle i \rangle \equiv \langle i|\kappa(i) = 2 \rangle = \langle i|\kappa(i) = 2 \rangle$$

(12)

where $\kappa(i)$ is the value of $\kappa$ after the removal of $i$ nodes.

That is, we define $\langle i \rangle$ to be the average of $i$ such that in each random removal $\kappa(i) = 2$; the derivation of Eq. (3) assumes that $\langle i \rangle$ is equal to $i$ such that the average of $\kappa(i)$ over all random removals equals 2. Equation (12) will be true in the limit in which the variance $\langle (i - \langle i \rangle)^2 \rangle$ is zero. But when the variance becomes large as is the case for the star network, Eq. (12) may be not hold. Figure 4 illustrates graphically an example for which Eq. (12) does not hold because the variance in $i$ is large.

\subsection*{B. Other Bimodal Networks}

In order to study other bimodal networks, we extend the star network to networks with $q$ high degree hubs connected to the remaining nodes of degree one. For networks with average degree $\langle k \rangle$, the degree distribution is specified as

$$P(k) = \begin{cases} (N-q)/N & [k = k_2] \\ q/N & [k = k_1] \end{cases}$$

(13)

and $P(k) = 0$ for all other $k$. We first consider networks with $\langle k \rangle = 2$. In Fig. 5(a), for the distribution of Eqs. (13) and (14), we plot $1 - f_c$ as a function of $q$ for $N = 10^2$, $10^3$, $10^4$, and $10^5$. Also shown in Fig. 5(a) are plots for approximations $f_{eh}^{\text{high}}$ and $f_{eh}^{\text{low}}$ which we expect to be valid respectively for high and low values of $q$. We will use these approximations to determine how $f_c(q)$ scales and for which values of $q$ Eq. (3) is valid. The approximations are determined as follows:

(i) When $q \sim N$, i.e., the network is homogeneous we expect Eq. (3) to hold so $f_{eh}^{\text{high}} = 1 - 1/(\kappa_0 - 1)$.

(ii) For small $q$, the network loses global connectivity when all $q$ high degree nodes are removed. The
probability that all \( q \) high degree nodes are removed after the first \( i \) nodes of all types have been removed is

\[
g(q, N, i) = \frac{q^{(i-1)}}{N^{(N-1)}}. \tag{15}
\]

Here \( i \) is now the average number of nodes that must be removed before all \( q \) high degree nodes are removed. Then \( \langle i \rangle = \sum_{i=q}^{N} ig(q, N, i) \) and

\[
f_{c}^{\text{low}} = \frac{\langle i \rangle}{N} = \frac{\sum_{i=q}^{N} ig(q, N, i)}{N}. \tag{16}
\]

Note that \( f_{c}^{\text{low}} \) does not depend on \( \langle k \rangle \) since changing \( \langle k \rangle \) results simply in a different number of links between the high degree nodes; if our criterion for collapse is the removal of all high degree nodes, the number of links between them is irrelevant. As expected, the plots of \( f_{c}^{\text{low}} \) and \( f_{c}^{\text{high}} \) approximate the values of \( f_{c} \) for low and high values of \( q \), respectively.

In Fig. 5(b), we plot the the number of hubs, \( q^* \), for which the functions \( f_{c}^{\text{low}}(q) \) and \( f_{c}^{\text{high}}(q) \) intersect. We find that

\[
q^* \sim N^{0.5} \tag{17}
\]

Similar plots (see Fig. 3 for \( \langle k \rangle = 3 \) and \( \langle k \rangle = 4 \)) also exhibit scaling of \( q^* \) as \( N^{0.5} \) with only a change in the prefactor; the scaling is independent of \( \langle k \rangle \).

The simulation results suggest that \( q^* \) scales as \( \sqrt{N} \). We can show this to be the case by solving analytically for \( q^* \) for large \( N \) as follows: For general \( \langle k \rangle \), using the distribution in Eqs. (13), we find for \( N \gg q \gg 1 \)

\[
f_{c}^{\text{high}} = 1 - \frac{q}{\langle k \rangle - 1).N. \tag{18}
\]

For \( f_{c}^{\text{low}} \), the sum in Eq. (16) can be performed analytically, yielding

\[
f_{c}^{\text{low}} = \frac{\Gamma(N+2)(\Gamma(q+2) - \Gamma(q+1))}{N\Gamma(N+1)\Gamma(q+2)} \tag{19}
\]

for \( q > 0 \). For large \( N \),

\[
f_{c}^{\text{low}} = \frac{\Gamma(q+2) - \Gamma(q+1)}{\Gamma(q+2)}. \tag{20}
\]

To first order in \( 1/q \), Eq. (20) yields

\[
f_{c}^{\text{low}} = 1 - \frac{1}{q} + O\left(\frac{1}{q^2}\right). \tag{21}
\]

Equating Eqs. (21) and (18) we find

\[
q^* = \sqrt{\langle k \rangle - 1}\sqrt{N} \tag{22}
\]

consistent with the plot in Fig. 5(b) and Eq. (17). From the fact that \( q^* \) scales like \( \sqrt{N} \), we conclude that all characteristic values of \( f_c \) scale like \( \sqrt{N} \) with a prefactor dependent on \( \langle k \rangle \). In particular the value of \( q \) at which \( f_{c}^{\text{MC}} \) (found from Monte Carlo simulations) agrees to any desired degree with the value of \( f_{c}^{\text{th}} \) (from Eq. (3)) will scale with \( N \) in the same fashion in which \( q^* \) scales with \( N \), Eq. (17). For simplicity, we consider Eq.(3) to be valid for \( q > q^* \).

We now confirm that the variance in \( f_c \) is in fact small for values of \( q \) for which Eq. (12) holds. In Fig. 4(a), for \( N = 10^3 \) and \( q = 1, 5, 10, \) and 20, we plot \( P(1-f_c) \) vs. \( 1-f_c \). As expected, for \( q = 1 \) (star network) the distribution is uniform because there is an equal probability that the single high degree node will be removed at any value of \( q \). For the larger values of \( q \), the distributions \( P(1-f_c) \) develop a well-defined peak. To quantify the definition of these peaks, we plot in Fig. 7(b), the standard deviation of \( f_c \)

\[
\sigma = \frac{\sqrt{\langle i^2 \rangle - \langle i \rangle^2}}{N}. \tag{23}
\]

versus \( q \) for \( N = 10^2, 10^3, 10^4, \) and \( 10^5 \). Each of the plots has a large deviation at \( q = 1 \) and decrease to a local minimum, the position of which \( \tilde{q} \) increases with increasing \( N \). For \( q \) greater than the \( \tilde{q} \), the deviation is small and decreases with increasing \( N \). In Fig. 5(b) we plot \( q^* \) as a function of \( N \). We see that the values of these minima are essentially the same as the values of \( q^* \), the value of \( q \) above which Eq. (3) is valid. This is consistent with our understanding that Eq. (3) is valid when the variance is small.

### C. Domain of Validity

Since \( q \) and the degree of the hubs \( k_2 \) are related by Eq. (11), we can determine for what values of \( k_2 \) Eq. (12) is valid. Substituting Eq. (22) in Eq. (14) we find that Eq. (12) is valid when

\[
k_2 < \sqrt{\langle k \rangle - 1).N. \tag{24}
\]

Thus the criterion for Eq. (12) holding is essentially the same as the criterion discussed in Sec. III for the graph being simple. The bimodal networks we study here in which a relatively small number of nodes control the global connectivity of the network yield large variances in \( f_c \) for networks with a given number of nodes; in fact, for \( q = 1 \) the worst case variance is obtained. This suggests that the criterion of Eq. (24) may hold for all degree distributions as a requirement for a low variance in \( f_c \). If this is the case, we can use the requirement that \( P(k) = 0 \) for \( k \lesssim K_c \) as the criterion for both the network being simple and \( f_c \) having a small variance. Note, however, that while the criteria are similar, it is not true that the presence of self-loops and multiple edges implies that the distribution of \( f_c \) has a large variance; for example, the variance of \( f_c \) in scale-free networks is small even in the presence of self-loops and multiple edges, as seen in Fig. 8.
V. DISCUSSION AND SUMMARY

We have clarified the domain of validity of Eq. (3), a general equation for determining $f_c$, the fraction of nodes which must be randomly removed before global connectivity is lost. For Eq. (3) to be valid, (i) the highest degree of any nodes present in statistically significant numbers in a random network must be less than the structural cutoff $K_s \equiv \sqrt{\langle k \rangle N}$ and (ii) the variance of $f_c$ must be small. For bimodal networks the variance in $f_c$ is small when the hubs have degree less than $\sqrt{\langle k \rangle - 1}N$. That the bimodal networks we have studied represent a worst case for large variance suggests that in general the criterion that the network be simple is sufficient for Eq. (12) to hold. It is not clear if there is a deeper connection between these two criteria both of which scale as $\sqrt{N}$.

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[12] The notation $(x|y)$ should be read as “x such that condition y holds”.

\[ \text{Acknowledgments} \]

We thank S. Havlin for helpful discussions and ONR for support.
FIG. 1: For $N = 10^2, 10^3,$ and $10^4$ respectively in (a), (b) and (c), $1 - f_c$ versus $\lambda$. The solid line represents the results of Monte-Carlo simulations; the dashed line is the prediction of Eq. (3). (d) The difference $\Delta$ (see Eq. (6)) between the prediction of Eq. (3) and Monte-Carlo simulations for (from top to bottom) $N = 10^2, 10^3, 10^4$. Note that if we had used a larger value of the upper cutoff $K$, then $\Delta$ would decrease monotonically from $\lambda = 3$ to $\lambda = 1$ instead of having a minimum near $\lambda = 2$. 
FIG. 2: $P(k)$ versus $k$ for $N = 10^2, 10^3, 10^4$ in (a),(b) and (c) respectively. The solid line represents $P(k)$ after network construction using the Molloy-Reed method; the dashed line is the distribution after the removal of self-loops and multiple edges.
FIG. 3: Correlation $r$ as a function of $\lambda$ for (from top to bottom at left) $N = 10^2, 10^3$, and $10^4$ for distributions after removal of self-loops and multiple edges. Note that the correlation increases with $N$ for $\lambda \lesssim 3$ and decreases with $N$ for $\lambda \gtrsim 3$.

FIG. 4: Example illustrating case in which $\langle i|\kappa = 2 \rangle \neq \langle i|\langle \kappa \rangle = 2 \rangle$ for star network of 1 hub of degree 99 and 99 nodes of degree 1. Thin lines are $\kappa$ vs $i$, where $i$ denotes the number of the step at which a node is deleted, for cases in which the hub is deleted at step (from left to right) 1, 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100. The thick line is the average of the thin lines. Note that the value of $i$ at which the average crosses the horizontal line $\kappa = 2$ is much higher than 50, the average of the values of $i$ at which the thin lines cross the horizontal line $\kappa = 2$. 
FIG. 5: For $\langle k \rangle = 2$ and for (from left to right) $N = 10^2, 10^3, 10^4$ and $10^5$ (a) $1 - f_c$ vs. number of hubs $q$. The solid lines represent Monte-Carlo simulation results. Dashed lines(short) are approximation $f_c^{\text{low}}$; dashed lines(long) are approximation $f_c^{\text{high}}$. (b) Number of hubs, $q$ versus $N$. Squares represent characteristic values $q^*$ at which high and low $q$ approximations intersect. Triangles represent values of $q$ at which the standard deviation in $1 - f_c$ is minimal.

FIG. 6: (a) Number of hubs, $q^*$, at which approximations for low and high $q$ intersect vs. $N$. Squares, triangles and circles represent networks with $\langle k \rangle = 2, 3,$ and 4 respectively.
FIG. 7: (a) $P(1 - f_c)$, the probability distribution of $1 - f_c$ for $N = 10^3$ and $q = 1$ (dashed line) and (from left to right in order of increasing position of peaks) $q = 5, 10, \text{ and } 20$. (b) Standard deviation $\sigma$ versus $q$ for $N = 10^2, 10^3, 10^4$ and $10^5$ (from left to right in order of increasing length of the tails of the distributions). Note that the second peak in this plot which is most pronounced for smaller $N$ is an artifact of finite size.

FIG. 8: For random scale-free networks with $4 \leq k \leq N$, standard deviation $\sigma_{f_c}$ versus $\lambda$ for $N = 10^2, 10^3, 10^4$ and $10^5$ (from top to bottom).