Structure and asymptotics for Catalan numbers modulo primes using automata

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12th January 2017

Abstract

Let $C_n$ be the $n$th Catalan number. We show that the asymptotic density of the set \( \{n : C_n \equiv 0 \mod p\} \) is 1 for all primes $p$. We also show that if $n = p^k - 1$ then $C_n \equiv -1 \mod p$. Finally we show that if $n \equiv \{p^k+1, p^k+3, ..., p - 2\} \mod p$ then $p$ divides $C_n$. All results are obtained using the automata method of Rowland and Yassawi.

1 Introduction

The Catalan numbers are defined by

$$C_n := \frac{1}{n+1} \binom{2n}{n}.$$ 

There has been much work in recent years and also going back to Kummer [7] on analysing the Catalan numbers modulo primes and prime powers. Deutsch and Sagan [3] provided a complete characterisation of Catalan numbers modulo 3. A characterisation of the Catalan numbers modulo 2 dates back to Kummer. Eu, Liu and Yeh [4] provided a complete characterisation of Catalan numbers modulo 4 and 8. This was extended by Liu and Yeh [9] to a complete characterisation modulo 8, 16 and 64. This result was restated in a more compact form by Kauers, Krattenthaler and Müller in [5] by representing the generating function of $C_n$ as a polynomial involving a special function. The polynomial for $C_n$ modulo 4096 was also calculated. A method for extracting the coefficients of the generating function (i.e. $C_n$ modulo a prime power) was provided. Given the complexity of the polynomials (the polynomial for the 4096 case takes a page and a half to write down) the computation would need to be done by computer. Krattenthaler and Müller [6] used a similar method to examine $C_n$ modulo powers of 3. They wrote down the polynomial for the generating function of $C_n$ modulo 9 and 27 and thereby generalised the mod 3 result of [3]. The article
by Lin [8] discussed the possible values of the odd Catalan numbers modulo $2^k$ and Chen and Jiang [2] dealt with the possible values of the Catalan numbers modulo prime powers. Rowland and Yassawi [10] investigated $C_n$ in the general setting of automatic sequences. The values of $C_n$ (as well as other sequences) modulo prime powers can be computed via automata. Rowland and Yassawi provided algorithms for creating the relevant automata. They established a full characterisation of $C_n$ modulo $\{2, 4, 8, 16, 3, 5\}$ in terms of automata. They also extended previous work by establishing forbidden residues for $C_n$ modulo $\{32, 64, 128, 256, 512\}$. In theory the automata can be constructed for any prime power but computing power and memory quickly becomes a barrier.

We will use Rowland and Yassawi’s automata to establish asymptotic densities of $C_n$ modulo primes. We will also make note of some structure results that appear from an examination of the relevant state diagrams of the automata. Asymptotic densities for $C_n$ modulo $2^k$ and 3 are available from [1]. In this paper we will be discussing primes $p \geq 5$.

Here, the asymptotic density of a subset $S$ of $\mathbb{N}$ is defined to be

$$\lim_{N \to \infty} \frac{1}{N} \# \{ n \in S : n \leq N \}$$

if the limit exists, where $\#S$ is the number of elements in a set $S$.

In particular, we will show that the asymptotic density of the set

$$S_p(0) = \{ n : C_n \equiv 0 \mod p \}$$

is 1 for all primes $p$. For $p \in \{2, 3\}$ this result was shown in [1]. We will also show that if $n \equiv \{ \frac{p+1}{2}, \frac{p+3}{2}, ..., p-2 \} \mod p$ then $p$ divides $C_n$. Finally, we will show that if $n = p^k - 1$ for prime $p$ then $C_n \equiv -1 \mod p$. A stronger result has been known for $p = 2$ since Kummer. Namely,

**Theorem 1.** For all $n \geq 0$, $C_n$ is odd if and only if $n = 2^k - 1$ for some $k \geq 0$.

The ”only if” part does not apply for primes greater than 2. For $p \in \{3, 5\}$ the result is proved by Rowland and Yassawi in [10].

For a number $p$, we write the base $p$ expansion of a number $n$ as

$$[n]_p = \langle n_r n_{r-1} ... n_1 n_0 \rangle$$

where $n_i \in [0, p-1]$ and

$$n = n_r p^r + n_{r-1} p^{r-1} + ... + n_1 p + n_0.$$
2 Background on automata for $C_n \mod p$

Rowland and Yassawi showed in [10] that the behaviour of sequences such as $C_n \mod p$ can be studied by the use of finite state automata. The automaton has a finite number of states and rules for transitioning from one state to another. In the form described in [10] (algorithm 1) each state $s$ is represented by a polynomial in 2 variables $x$ and $y$. Each state has a value obtained by evaluating the polynomial at $x = 0$ and $y = 0$. All calculations are made modulo $p$. For the Catalan case the initial state $s_1$ is represented by the polynomial

$$R(x, y) = y(1 - 2xy - 2xy^2). \quad (2)$$

New states are constructed by applying the Cartier operator $\Lambda_{d,d}$ to the polynomials

$$s_i \ast Q(x, y)^{p-1}$$
for $d \in \{0, 1, ..., p-1\}$ where $\{s_i\}$ are the already calculated states and the polynomial $Q$ is defined by

$$Q(x, y) = x(y + 1)^2 - 1. \quad (3)$$

The Cartier operator is a linear map on polynomials defined by

$$\Lambda_{d_1,d_2}(\sum_{m,n \geq 0} a_{m,n} x^m y^n) = \sum_{m,n \geq 0} a_{pm+d_1, pn+d_2} x^m y^n.$$

Since the Cartier operator maintains or reduces the degree of the polynomial and there are only finitely many polynomials modulo $p$ of each degree, all states of the automaton are obtained within a known finite time. It will be seen later that the automaton has at most $p + 3$ states. If

$$\Lambda_{d,d}(s \ast Q^{p-1}) = t$$
for states (i.e. polynomials) $s$ and $t$ then the transition from state $s$ to state $t$ under the input $d$ is part of the automaton.

To calculate $C_n \mod p$, $n$ is first represented in base $p$. The base $p$ digits of $n$ are fed into the automaton starting with the least significant digit. The automaton starts at the initial state $s_1$ and transitions to a new state as each digit is fed into it. The value of the final state after all $n$’s digits have been used is equal to $C_n \mod p$. Refer to [10] for more details.

In the remainder of this article we will provide details of the automata for a general prime $p \geq 5$. We will provide the polynomials and values for the states and the transitions between states. States are listed as $s_1$, $s_2$, ... . Transitions, when provided, will be in the form $(s, j) \rightarrow t$ which means that if the automaton is in state $s$ and

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receives digit $j$ then it will move to state $t$. We will call a state $s$ a **loop** state if all transitions from $s$ go to $s$ itself, i.e. $(s, j) \rightarrow s$ for all choices of $j$.

States and transitions are represented visually in the form of a directed graph. For example, figure 1 represents an automaton which moves from state $s_1$ to state $s_2$ when it receives the digit 3. It also moves from state $s_2$ to state $s_2$ (i.e. loops) if it is in state $s_2$ and receives a digit 4.

3 Preliminary calculations

Before we start constructing the automata it will be convenient to first precompute $\Lambda_{d, d}(s(x, y) \ast Q(x, y)^{p-1})$ for various choices of the polynomial $s$. The relevant results are contained in table 1. When reading the table note that $\binom{n}{m} = 0$ for $m < 0$. We will go through a few of the calculations from table 1.

Firstly, the polynomial $Q^{p-1}$ can be written as

\[
Q^{p-1}(x, y) = (x(y + 1)^2 - 1)^{p-1}
\]

\[
= \sum_{k=0}^{p-1} \binom{p-1}{k} x^k (y + 1)^{2k} (-1)^{p-1-k}
\]

\[
= \sum_{k=0}^{p-1} \sum_{l=0}^{2k} \binom{p-1}{k} \binom{2k}{l} (-1)^k x^k y^l
\]

(4)

\[
= \sum_{k=0}^{p-1} \sum_{l=0}^{2k} a_{k,l} x^k y^l
\]

where

\[
a_{k,l} = \binom{p-1}{k} \binom{2k}{l} (-1)^k
\]

(5)

for $0 \leq k \leq p - 1$ and $0 \leq l \leq 2k$.

Then, for $r \geq 0$ and $t \geq 0$, 

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Figure 1: Example of transition from state $s_1$ to state $s_2$ and a loop.
| State $s$ | $\Lambda_{d,d}(s \times Q^{p-1})$ |
|----------|----------------------------------|
| $1$      | $\binom{2d}{d}$ for $0 \leq d \leq p - 1$ |
| $y$      | $\binom{2d}{d-1}$ for $0 \leq d \leq p - 2$  
|          | $y + 1$ for $d = p - 1$ |
| $xy$     | $\binom{2d-2}{d-1}$ for $0 \leq d \leq p - 1$ |
| $xy^2$   | $xy(y + 1)$ for $d = 0$  
|          | $\binom{2d-2}{d-2}$ for $1 \leq d \leq p - 1$ |
| $xy^3$   | $-2xy(y + 1)$ for $d = 0$  
|          | $\binom{2d-2}{d-3}$ for $1 \leq d \leq p - 2$  
|          | $y + 1$ for $d = p - 1$ |

Table 1: Table of values of $\Lambda_{d,d}(s \times Q^{p-1})$
\[ \Lambda_{d,d}(x^r y^t Q(x, y)^{p-1}) = \Lambda_{d,d}(\sum_{k=0}^{p-1} \sum_{l=0}^{2k} a_{k,l} x^{k+r} y^{l+t}) \]
\[ = \Lambda_{d,d}(\sum_{k} \sum_{l} b_{k,l} x^{k} y^{l}) \]

where
\[ b_{k,l} = a_{k-r,l-t} = \binom{p-1}{k-r} \binom{2(k-r)}{l-t} (-1)^{k-r} \]
and the indices \( k \) and \( l \) in \( b_{k,l} \) satisfy
\[ r \leq k \leq p - 1 + r ; t \leq l \leq 2(k-r) + t \leq 2p - 1 + t. \]

So,
\[ \Lambda_{d,d}(x^r y^t Q(x, y)^{p-1}) = \sum_{k} \sum_{l} b_{pk+d,pl+d} x^{k} y^{l}. \]

and the indices \( k \) and \( l \) in (8) satisfy
\[ r \leq pk + d \leq p - 1 + r ; t \leq pl + d \leq 2(pk + d - r) + t \leq 2p - 1 + t. \]

We first compute \( \Lambda_{d,d}(Q(x, y)^{p-1}) \) (i.e. \( r = 0 \) and \( t = 0 \) in (8)). When \( r = 0 \) and \( t = 0 \) the only choice for \( k \) satisfying the bounds in (9) is \( k = 0 \). This then leaves \( l = 0 \) as the only choice for \( l \) satisfying (9). Then,
\[ \Lambda_{d,d}(Q(x, y)^{p-1}) = b_{d,d} = a_{d,d} = \binom{p-1}{d} \binom{2d}{d} (-1)^{d} \]
\[ \equiv \binom{2d}{d} \mod p \]

since
\[ \binom{p-1}{d} \equiv (-1)^{d} \mod p. \]

This gives the first line of table 1. Note that \( \binom{0}{0} = 1 \) and \( \binom{2d}{d} \equiv 0 \mod p \) for \( d \geq \frac{p+1}{2} \).

We next calculate \( \Lambda_{d,d}(yQ(x, y)^{p-1}) \), (i.e. \( r = 0 \) and \( t = 1 \) in (8)). We have,
\[ \Lambda_{d,d}(yQ(x,y)^{p-1}) = \sum_k \sum_l b_{pk+d,pl+d} x^k y^l. \]  

(10)

where \( b_{k,l} = a_{k,l-1} \) for \( 0 \leq pk + d \leq p - 1 \) and \( 1 \leq pl + d \leq 2(pk + d) + 1 \). If \( d = 0 \), \( k \) must be 0 but there are then no choices for \( l \) which fit the required bounds. Therefore, \( \Lambda_{0,0}(yQ(x,y)^{p-1}) = 0 \).

If \( 1 \leq d \leq p - 2 \) then the only suitable choice for \( \{k, l\} \) is \( k = 0 \) and \( l = 0 \). In this case

\[ \Lambda_{d,d}(yQ(x,y)^{p-1}) = b_{d,d} = a_{d,d-1} \]

\[ = \binom{p-1}{d} \binom{2d}{d-1} (-1)^d \equiv \binom{2d}{d-1} \mod p. \]

We note that \( \binom{2d}{d-1} \equiv 0 \mod p \) for \( d \geq \frac{p+1}{2} \).

If \( d = p - 1 \), there are two choices for \( k \) and \( l \) that satisfy the bounds in equation (10). These are \( k = 0 \), \( l = 0 \) and \( k = 0 \), \( l = 1 \). Therefore,

\[ \Lambda_{p-1,p-1}(yQ(x,y)^{p-1}) = b_{p-1,p-1} + b_{p-1,2p-1} y \]

\[ = \binom{p-1}{p-1} \binom{2p-2}{p-2} (-1)^{p-1} + \binom{p-1}{p-1} \binom{2p-2}{p-2} (-1)^{p-1} y. \]

But \( \binom{2p-2}{p-2} \equiv 1 \mod p \) so

\[ \Lambda_{p-1,p-1}(yQ(x,y)^{p-1}) = y + 1. \]

We next calculate \( \Lambda_{d,d}(x yQ(x,y)^{p-1}) \), (i.e. \( r = 1 \) and \( t = 1 \) in (8) ). We have,

\[ \Lambda_{d,d}(x yQ(x,y)^{p-1}) = \sum_k \sum_l b_{pk+d,pl+d} x^k y^l. \]

and from (9) \( k \) and \( l \) satisfy

\[ 1 \leq pk + d \leq p ; \ 1 \leq pl + d \leq 2(pk + d - 1) + 1 \leq 2p. \]

If \( d = 0 \) there is only one choice for \( \{k, l\} \) namely \( k = 1 \) and \( l = 1 \). In this case
\[ \Lambda_{0,0}(x y Q(x, y)^{p-1}) = b_{p,p} x y \]
\[ = \left( \frac{p - 1}{p - 1} \right) \left( \frac{2p - 2}{p - 1} \right) (-1)^{p-1} x y \equiv 0 \mod p. \]

For \( 1 \leq d \leq p - 1 \) the only suitable choice for \( \{k, l\} \) is \( k = 0 \) and \( l = 0 \). So

\[ \Lambda_{d,d}(x y Q(x, y)^{p-1}) = b_{d,d} \]
\[ = \left( \frac{p - 1}{d - 1} \right) \left( \frac{2d - 2}{d - 1} \right) (-1)^{d-1} \equiv \left( \frac{2d - 2}{d - 1} \right) \mod p. \]

We note that \( \left( \frac{2d - 2}{d - 1} \right) \equiv 0 \mod p \) for \( d \geq \frac{p + 3}{2} \).

For brevity we will skip the calculation of \( \Lambda_{d,d}(x y^2 Q(x, y)^{p-1}) \) and finally calculate \( \Lambda_{d,d}(x y^3 Q(x, y)^{p-1}) \) (i.e. \( r = 1 \) and \( t = 3 \) in (8)). We have,

\[ \Lambda_{d,d}(x y^3 Q(x, y)^{p-1}) = \sum_k \sum_l b_{pk+d,pl+d} x^k y^l. \]

and from (9) \( k \) and \( l \) satisfy

\[ 1 \leq pk + d \leq p \; ; \; 3 \leq pl + d \leq 2(pk + d - 1) + 3 \leq 2p. \]  
\[ (11) \]

When \( d = 0 \), \( k \) must be 1 and \( l \) can be 1 or 2. Therefore,

\[ \Lambda_{0,0}(x y^3 Q(x, y)^{p-1}) = b_{p,p} xy + b_{p,2p} xy^2 \]
\[ = \left( \frac{p - 1}{p - 1} \right) \left( \frac{2p - 2}{p - 3} \right) (-1)^{p-1} xy + \left( \frac{p - 1}{p - 1} \right) \left( \frac{2p - 2}{2p - 3} \right) (-1)^{p-1} xy^2 \]
\[ = -2xy - 2xy^2 \]

since \( \left( \frac{2p - 2}{p - 3} \right) \equiv -2 \mod p \) and \( \left( \frac{2p - 2}{2p - 3} \right) \equiv -2 \mod p. \)

For \( d \in \{1, 2\} \) there are no suitable choices for \( k \) and \( l \) in (11) so

\[ \Lambda_{d,d}(x y^3 Q(x, y)^{p-1}) = 0. \]
for $d \in \{1,2\}$. For $3 \leq d \leq p - 2$ the only suitable choice for $k$ in (11) is $k = 0$. Then $l$ must also be 0. Therefore, for $3 \leq d \leq p - 2$,

$$\Lambda_{d,d}(x^2 Q(x,y)^{p-1}) = b_{d,d} = \binom{p-1}{d-1} \binom{2d-2}{d-3} (-1)^{d-1}$$

$$= \binom{2d-2}{d-3}$$

since $\binom{p-1}{d-1} = (-1)^{d-1} \mod p$. Note that $\binom{2d-2}{d-3} = 0$ for $d = 1$ and $d = 2$.

When $d = p - 1$, $k = 0$, $l = 0$ and $k = 0$, $l = 1$ both satisfy the bounds in (11). So,

$$\Lambda_{p-1,p-1}(x^2 Q(x,y)^{p-1}) = b_{p-1,p-1} + b_{p-1,2p-1} y$$

$$= \binom{p-1}{p-2} \binom{2p-4}{p-4} (-1)^{p-2} + \binom{p-1}{p-2} \binom{2p-4}{2p-4} (-1)^{p-2} y$$

$$\equiv y + 1 \mod p$$

since $\binom{2p-4}{p-4} \equiv 1 \mod p$.

4 Constructing the automata for $C_n \mod p$

In this section we will describe the states and transitions of the automata for $C_n \mod p$. These are summarised in Table 2. For given $d : 0 \leq d \leq p - 1$ and state $s \in \{s_1, s_2, 1, -(y+1)\}$ the table gives the state equal to

$$\Lambda_{d,d}(s \ast Q^{p-1})$$

The transition $(s, d) \rightarrow \Lambda_{d,d}(s \ast Q^{p-1})$ is then part of the automata.

Figures 2 and 3 provide an alternative pictorial summary of the automata for $C_n \mod p$. We have broken the state diagram into 2 figures to improve clarity. In the figures the group of states labelled $C$ represents all states which consist of a constant non-zero polynomial.
Figure 2: Partial state diagram for $C_n \mod p$. 
Figure 3: Partial state diagram for $C_n \mod p$. 
Table 2: Table of states and transitions.

The calculation of the states will rely on the data contained in table 1. As mentioned earlier, the initial state $s_1$ for the automata is the polynomial defined in equation 2. The second state $s_2$ is then given by

$$s_2 = \Lambda_{0,0}(s_1 * Q(x, y)^{p-1})$$

$$= \Lambda_{0,0}(y(1 - 2xy - 2xy^2) * Q(x, y)^{p-1})$$

$$= 0 - 2(xy(y + 1)) - 2(-2xy(y + 1))$$

$$= 2xy(y + 1)$$

For $1 \leq d \leq p - 2$ we have

$$\Lambda_{d,d}(s_1 * Q(x, y)^{p-1})$$

$$= \Lambda_{d,d}(y(1 - 2xy - 2xy^2) * Q(x, y)^{p-1})$$

$$= \binom{2d}{d-1} - 2\binom{2d-2}{d-2} - 2\binom{2d-2}{d-3}$$

$$= \binom{2d}{d-1} - 2\binom{2d-1}{d-2}$$

$$= \frac{(2d - 1)!}{(d - 1)!(d + 1)!}\binom{2d - 2(d - 2)}{d - 2}$$
\[ = \frac{1}{d} \binom{2d}{d-1} \]

where we have used the identity

\[
\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}.
\]  \tag{12}

So these states are constant polynomials. Note that \(\binom{2d}{d-1} \equiv 0 \mod p\) for \(d \geq \frac{p+1}{2}\).

For \(d = p - 1\) we have

\[
\Lambda_{p-1,p-1}(y(1 - 2xy - 2xy^2) * Q(x,y)^{p-1})
\]

\[
= y + 1 - 2 \left( \binom{2p-4}{p-3} \right) - 2(y + 1)
\]

\[
= -(y + 1)
\]

since \(\binom{2p-4}{p-3} \equiv 0 \mod p\).

The transitions for the state \(s_2 = 2xy(y+1)\) can be worked out similarly. Using table 1, we have for \(d = 0\),

\[
\Lambda_{0,0}(2xy(y+1) * Q^{p-1}) = 2xy(y+1) + 2\left\langle -\frac{2}{1} \right\rangle = s_2.
\]

For \(1 \leq d \leq p - 1\),

\[
\Lambda_{d,d}(2xy(y+1) * Q^{p-1}) = 2\left( \binom{2d-2}{d-2} \right) + 2\left( \binom{2d-2}{d-1} \right)
\]

\[
= 2\left( \binom{2d-1}{d-1} \right) = 2\left( \binom{2d-1}{d} \right)
\]

where we have again used (12).

The transitions for the constant state with value \(1\) can be taken straight from table 1, noting that \(\binom{2d}{d} \equiv 0 \mod p\) for \(d \geq \frac{p+1}{2}\).

We finally consider the transitions for the state represented by the polynomial \(- (y + 1)\). For \(0 \leq d \leq p - 2\),

\[
\Lambda_{d,d}(- (y + 1)) = -\left( \binom{2d}{d} \right) - \left( \binom{2d}{d} \right) = -\left( \binom{2d+1}{d} \right).
\]

For \(d = p - 1\),
\[
\Lambda_{p-1,p-1}(-(y + 1)) = -(y + 1) - \binom{2p - 2}{p - 1} = -(y + 1)
\]

since \(\binom{2p - 2}{p - 1} \equiv 0 \mod p\).

Transitions for the constant states which have value other than 1 can be obtained from the transitions for the state 1 using the linearity of the Cartier operator.

### 5 Conclusions

Table 2 shows that, under the algorithm we have used, the automata for \(C_n \mod p\) contains at most \(p + 3\) states. The 3 polynomials \(y(1 - 2xy - 2x y^2), 2xy(y + 1)\) and 
\(- (y + 1)\) are always states of the automata. The other states are residues mod \(p\). It is possible that all residues modulo \(p\) always appear as states. In order to show that all residues appear it is sufficient to show that the set

\[
\left\{ \binom{2d}{d} : 0 \leq d \leq \frac{p - 1}{2} \right\}
\]

generates \((\mathbb{Z}_p^\times)^\times\). Forbidden residues do exist for \(C_n \mod p^k\) for some primes \(p\) and \(k > 1\) as discussed in [10].

Table 2 and figures 2 and 3 provide a clear view of the the working of \(C_n \mod p\). Firstly, it is obvious that if \(n = p^k - 1\) and so has a base \(p\) representation in the form

\[
[n]_p = \langle p - 1, p - 1, \ldots, p - 1, p - 1 \rangle
\]

then \(C_n \equiv -1 \mod p\).

Secondly, the zero state is a loop state. Once the state path of a number reaches the zero state it cannot transition to any other state.

Thirdly, if the base \(p\) representation of \(n\) contains 1 or more digits from the set \(\{\frac{p + 1}{2}, \frac{p + 3}{2}, \ldots, p - 2\}\) then \(C_n \equiv 0 \mod p\) since

\[
\Lambda_{d,d}(sQ^{p-1}) = 0
\]

for all states \(s\) when \(d \in \{\frac{p + 1}{2}, \frac{p + 3}{2}, \ldots, p - 2\}\). Since the set of base-\(p\) numbers which have no digits from the set \(\{\frac{p + 1}{2}, \frac{p + 3}{2}, \ldots, p - 2\}\) has asymptotic density 0, it follows that the set \(S_p(0)\) has asymptotic density 1 for all \(p\).
Fourthly, as a consequence of the point above, if \( n \equiv \{ \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-2 \} \mod p \) then \( p \) divides \( C_n \). For \( p = 5 \) this can be seen from the state diagram in [10].

Finally, none of the constants contained within the group of states \( C \) in figures 2 and 3 can be 0. Therefore the criteria that \( n \) must satisfy in order that \( p \mid C_n \) can be deduced from the figures.

6 Acknowledgement

We would like to thank Eric Rowland for introducing us to automata as a tool for examining Catalan numbers.

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