MAGNETOCONDUCTANCE OSCILLATIONS IN METALLIC RINGS &
DECOHERENCE DUE TO ELECTRON-ELECTRON INTERACTION

Christophe Texier†,‡ and Gilles Montambaux‡
†Laboratoire de Physique Théorique et Modèles Statistiques, UMR 8626, CNRS,Université Paris-Sud,
F-91405 Orsay Cedex, France.
‡Laboratoire de Physique des Solides, UMR 8502, CNRS, Université Paris-Sud, F-91405 Orsay.

We study weak localization in chains of metallic rings. We show that nonlocality of quantum
transport can drastically affect the behaviour of the harmonics of magnetoconductance oscil-
lations. Two different geometries are considered : the case of rings separated by long wires
compared to the phase coherence length and the case of contacted rings. In a second part we
discuss the role of decoherence due to electron-electron interaction in these two geometries.

1 Introduction

At low temperature, quantum interferences of reversed electronic trajectories are responsible
for a small reduction of the averaged conductivity called the weak localization (WL) correction.
This correction is a manifestation of quantum coherence which is always limited over a certain
length scale, named the phase coherence length $L_\phi$. A way to extract this important length
scale in experiments is to use the magnetic field sensitivity of the WL. For example the WL
correction of an infinitely long wire of rectangular section of width $W$ and area $S$ submitted to
a perpendicular magnetic field $B$ is

$$\langle \Delta \sigma \rangle = -\frac{2e^2}{h} S \left[ \frac{1}{L_\phi^2} + \frac{1}{3} \left( \frac{eB}{h} \right)^2 \right]^{-1/2}$$ (in the following we
will forget the $1/S$ factor). The width of the magnetoconductance (MC) curve provides a direct
determination of $L_\phi$.

![Figure 1: Chains of rings. If we consider the regime $b \gg L_\phi \gg L$ the rings can be considered as independent in
case (a) but not in case (b).](image)

Another possibility to extract phase coherence length is to study arrays of rings whose
MC present oscillations as a function of the flux $\phi$ per ring with period $\frac{\Phi_0}{2}$ (AAS), observed in
many experiments. In order to extract the phase coherence length from AAS oscillations,
a precise theoretical prediction for the behaviour of the AAS harmonics with the phase coherence
length is needed. Harmonics of the oscillations are defined as

$$\langle \Delta \sigma(\phi) \rangle = \sum_n \Delta \sigma_n e^{4\pi i n \phi/\Phi_0}.$$ A
well-known expression has been derived in Ref. for an isolated ring of perimeter $L$ :

$$\Delta \sigma_n = -\frac{2e^2}{h} L_\phi e^{-|n|L/L_\phi}$$

However, in a real experiment where the ring is connected to wires, or embedded in a larger
network, this expression can only be relevant in the regime $L_\phi \ll L$. At the lowest tempera-
tures, when $L_\phi \gtrsim L$, the AAS harmonics are strongly affected by the surrounding wires since
trajectories can expand outside the ring over distances larger than the perimeter. It is the aim
of this paper to discuss the behaviour of AAS harmonics in chains of rings when $L_\phi \gtrsim L$. We
will consider two cases represented on figure : in the first situation the rings are separated
by a distance \( b \gg L_\phi \) and can therefore be considered as independent (however the connecting wires will affect the AAS harmonics). In the second case rings are in contact and harmonics can involve trajectories winding around several neighbouring rings. In section \[3\] we will see that when electron-electron interaction is the dominant process for decoherence, eq. \[1\] cannot be used even in the regime \( L \ll L_\phi \).

## 2 Nonlocality of quantum transport in chain of rings

We consider an array of rings all pierced by the same flux \( \phi \). The \( n \)-th harmonic of the WL correction at a given point \( x \) of a network can be expressed as

\[
\Delta \sigma_n(x) = -\frac{2e^2 D}{\pi} \int_0^\infty \text{d}t \mathcal{P}_n(x, x; t) e^{-t/\tau_\phi}
\]

(2)

where \( D \) is the diffusion constant and \( \tau_\phi = L_\phi^2 / D \) the phase coherence time. The factor 2 stands for spin degeneracy. \( \mathcal{P}_n(x, x; t) \) is the probability that a particle diffusing into the network comes back to its initial point \( x \) in a time \( t \), after having encircled a flux \( n\phi \). For example, in an isolated ring \( \mathcal{P}_n(x, x; t) = \frac{1}{\sqrt{\pi D t}} e^{-(nL)^2/4Dt} \) which immediately gives eq. \[1\]. Except in translation invariant systems, \( \Delta \sigma_n(x) \) depends on \( x \) and expression \[2\] must be averaged over the network in a proper way described in Ref. \[7\].

**A ring with two arms.**— The case of a ring connected to two arms has been studied in detail in Ref. \[5\] where it has been shown that \( \mathcal{P}_n(x, x; t) \approx \frac{\sqrt{L^2 / 2(Dt)^{3/2}} \Psi(\frac{n \sqrt{L} \phi}{L \sqrt{2Dt}})}{\sqrt{\pi D t}} \) for time scales \( t \gg L^2 / D \) with \( x \) inside the ring (the precise form of the dimensionless function \( \Psi(\xi) \) is inessential for the present discussion). Compared with the isolated ring case, where the typical number of winding scales with time as \( n_t \sim t^{1/2} \), diffusion around the ring is slowed down as \( n_t \sim t^{1/4} \) due to the time spent in the arms. As a consequence the harmonics of the conductance of a ring are given by \[5\]:

\[
\Delta \sigma_n \propto L^{3/2} e^{-|n| \sqrt{2L/L_\phi}}
\]

(note that the scaling \( n \sim L_\phi^{1/2} \) is analogous to the scaling of winding with time \( t \)).

**The chain of distant rings.**— The same argument holds for the chain of rings separated by a distance \( b \gg L_\phi \) (figure \[1\]a). In this case, averaging properly \( \Delta \sigma_n(x) \) inside the chain of \( N_r \) rings, one finds that the harmonics of the dimensionless conductance read:

\[
\Delta g_n \approx \frac{N_r L^{1/2} L_\phi^{3/2}}{\sqrt{2(N_r + 1)b^2}} e^{-|n| \sqrt{2L/L_\phi}} \quad \text{for} \quad b \gg L_\phi \gg L
\]

(3)

**The chain of attached rings.**— If we now consider the network of figure \[1\]b, we can show that the probability reads \[9\]

\[
\mathcal{P}_n(x, x; t) \approx \frac{L^{1/2}}{8\pi D t} e^{-(nL)^2/4Dt} \quad \text{for} \quad t \gg L^2 / D.
\]

The AAS harmonics are given in this case by \[9\] :

\[
\Delta g_n \approx -\frac{1}{N_r} \frac{2}{\pi} \ln(2L_\phi / |n| L) + b_n \quad \text{for} \quad L_\phi / L \gg |n| \tag{4}
\]

\[
\approx -\frac{1}{N_r} \frac{2}{\pi} \frac{e^{-|n| L/L_\phi}}{\sqrt{|n| L/L_\phi}} \quad \text{for} \quad |n| \gg L_\phi / L \gg 1 \tag{5}
\]

where \( b_n \) depends weakly on \( n \) (\( b_\infty = -\zeta \), the Euler constant).

## 3 Decoherence due to electron-electron interaction

The above results rely on the fact that, in eq. \[2\], the long times have been cut off with an exponential damping \( e^{-t/\tau_\phi} \). However it has been shown recently that this simple modelization
does not account correctly for the decoherence due to electron-electron interaction, which is the dominant one at low temperature [3]. In this case, an alternative description was proposed by Al’tshuler, Aronov & Khmel’nitskii (AAK) [12], but it is only recently that the consequences for AAS oscillations have been understood [13-15].

The model of AAK.-- The length scale characterizing the efficiency of electron-electron interaction to suppress phase coherence in wires is known as the Nyquist length \( L_N = (v_0 D/\nu)^{1/3} \) where \( \nu_0 \) is the density of states, \( D \) the diffusion constant and \( \nu \) the temperature (\( h = k_B = 1 \)).

In the model of AAK, the random phase accumulated by an electron moving in the fluctuating electric potential due to other electrons is included in the calculation of the WL. The pair of reversed interfering trajectories picks a phase \( e^{i\Phi[C]} \), where \( C \) designates a closed diffusive trajectory, and the harmonics of WL are given by

\[
\Delta \sigma_n \sim -\sum_C \langle e^{i\Phi[C]} \rangle_V = -\sum_C e^{-\frac{1}{2} \langle \Phi[C]^2 \rangle_V} \tag{6}
\]

The sum runs over all closed trajectories with winding \( n \) (a proper formulation of eq. (6) requires a path integral). Gaussian fluctuations of the electric potential are given by the fluctuation-dissipation theorem \( \langle V(\vec{r}, t)V(\vec{r}', t') \rangle_V = \frac{2e^2}{\sigma_0^2} T \delta(t - t') P_d(\vec{r}, \vec{r}') \) (written here in the classical limit \( T \ll \omega \)), where \( \sigma_0 \) is the classical Drude conductivity. \( P_d \) is solution of the diffusion equation \( -\Delta P_d(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \) and therefore depends on the topology of the system. Then \[14\]

\[
\langle \Phi[x(\tau)]^2 \rangle_V = \frac{4D}{L_N^3} \int_0^t d\tau \left[ P_d(x(\tau), x(\tau)) - P_d(x(\tau), x(t - \tau)) \right] \tag{7}
\]

where \( C \equiv (x(\tau), 0 \leq t \leq t | x(0) = x(t) \) is a closed diffusive path. The crucial point is that the simple exponential damping of eq. (2) is replaced in eq. (6) by a functional of the trajectory \( e^{-\frac{1}{2} \langle \Phi[C]^2 \rangle_V} \). Therefore decoherence is now network-dependent and \( a \text{ priori} \) sensitive to the nature of trajectories (in particular whether they do enclose a magnetic flux or not).

The limit \( L_N \ll L.-- \) The model described above was applied to the case of a single ring [13-14,15]. The result for an isolated ring is relevant to describe arrays of rings in the limit \( L_N \ll L \) where winding trajectories hardly exit from a ring, which makes rings independent from each other. For the chain of distant rings (figure 1a) we have

\[
\Delta g_n \sim -\frac{N_r L}{\left((N_r + 1)b\right)^2} L_N e^{-|n| \frac{\pi}{8}(L/L_N)^{3/2}} \sim \frac{e^{-n L^{3/2} T^{1/2}}}{T^{1/2}} \quad \text{for } L_N \ll L \ll b \tag{8}
\]

(for the case of the chain of rings in contact (figure 1b), \( (N_r + 1)b \) in the denominator is replaced by \( N_r L/4 \)). Whereas the time characterizing efficiency of electron-electron interaction to suppress phase coherence in a wire is the Nyquist time \( \tau_N = L^2 / D \propto T^{-2/3} \), it was shown in Refs. [13-14] that the behaviour (8) is related to a new time scale characterizing decoherence for winding trajectories: \( \tau_c = \tau_N^{3/2} / \tau_L^{1/2} \propto T^{-1} \), where \( \tau_L = L^2 / D \) is the Thouless time of the ring.

The chain of distant rings.-- If we consider a ring connected to long arms, winding trajectories spend most of the time in the arms [8] and the decoherence mostly occurs in the arms. Therefore decoherence occurs on a time scale \( \tau_N \), like in a wire. The function \( \langle e^{i\Phi[C]} \rangle_V \) for a wire was

\[\text{Footnote text here:}\]

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studied in Ref.\textsuperscript{16}. Using this result and the winding properties recalled in section \textsuperscript{2} leads to

\[ \Delta g_n \sim -\frac{N_r L^{1/2} L_N^{3/2}}{[N_r + 1] b]} \]

\[ \sim -\frac{N_r L^{1/2} L_N^{3/2}}{[N_r + 1] b]} \left( \frac{n^2 L}{L_N} \right)^{7/12} e^{-\kappa_2 |n| \sqrt{L/L_N}} \sim e^{-n L^{1/2} T^{1/6}} \]

\[ \frac{T^{11/36}}{N} \]

for \( n^2 \ll L_N/L \)

(9)

\[ \sim -\frac{N_r L^{1/2} L_N^{3/2}}{[N_r + 1] b]} \left( \frac{n^2 L}{L_N} \right)^{7/12} e^{-\kappa_2 |n| \sqrt{L/L_N}} \sim e^{-n L^{1/2} T^{1/6}} \]

\[ \frac{T^{11/36}}{N} \]

for \( n^2 \gg L_N/L \)

(10)

where \( \kappa_2 = \sqrt{2} |u_1|^{1/4} \approx 1.421 \).

**The chain of attached rings.**—In this case, the nature of decoherence was shown to be closely related to the one of a wire since diffusion along the chain is reminiscent of a 1d diffusion and again occurs on time scale \( \tau_N \): \n
\[ \Delta g_n \sim -\frac{2}{N_r \pi} \ln(L_N/|n| L) + \text{cste} \]

for \( |n| \ll L_N/L \)

(11)

\[ \sim -\frac{1}{N_r |u_1|^{3/2}} e^{-\kappa_3 |n| L/L_N} \sim e^{-n L T^{1/3}} \]

for \( |n| \gg L_N/L \)

(12)

where \( \kappa_3 = 2^{-1/3} |u_1|^{1/2} \approx 0.801 \).

4 Conclusion

We have considered networks of connected rings, made of weakly disordered wires. We have first shown that geometrical effects can strongly modify the exponential behaviour of AAS harmonics well-known for an isolated ring, since trajectories can now explore the network around each ring. In the second part we have shown that decoherence due to electron-electron interaction is sensitive to geometry, a second reason that modifies the simple AAS result. An interesting experiment would be to compare precisely AAS oscillations for the two networks of figure \textsuperscript{11} in the low temperature regime \( L_N \gg L \).

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