Large time asymptotics for a cubic nonlinear Schrödinger system in one space dimension, II

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Abstract: This is a sequel to the paper “Large time asymptotics for a cubic nonlinear Schrödinger system in one space dimension” by the same authors. We continue to study the Cauchy problem for the two-component system of cubic nonlinear Schrödinger equations in one space dimension. We provide criteria for large time decay or non-decay in $L^2$ of the small amplitude solutions in terms of the Fourier transforms of the initial data.

Key Words: Nonlinear Schrödinger system, large time behavior, decay/non-decay in $L^2$.

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1 Introduction

This is a sequel to the paper [2]. We continue to study the Cauchy problem for

\[
\begin{align*}
\mathcal{L}u_1 &= -i|u_2|^2 u_1, \\
\mathcal{L}u_2 &= -i|u_1|^2 u_2,
\end{align*}
\]

with

\[
u_j(0, x) = \varphi_j^0(x), \quad x \in \mathbb{R}, \quad j = 1, 2,
\]

where $i = \sqrt{-1}$, $\mathcal{L} = i\partial_t + (1/2)\partial_x^2$, and $\varphi^0 = (\varphi_1^0(x), \varphi_2^0(x))$ is a given $C^2$-valued function of $x \in \mathbb{R}$. The following result has been obtained in [2]. As in [2], $H^{s,\sigma}$ stands for the $L^2$-based weighted Sobolev space of order $s$, $\sigma$. We also write $H^s$ for $H^{s,0}$.

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Proposition 1.1 ([2]). Suppose that \( \varphi^0 = (\varphi^0_1, \varphi^0_2) \in H^2 \cap H^{1,1} \) and \( \|\varphi^0\|_{H^2 \cap H^{1,1}} \) is suitably small. Let \( u = (u_1, u_2) \in C([0, \infty); H^2 \cap H^{1,1}) \) be the solution to (1.1)–(1.2). Then there exists \( \varphi^+ = (\varphi^+_1, \varphi^+_2) \in L^\infty \) such that

\[
\lim_{t \to +\infty} \| u_j(t) - U(t) \varphi^+_j \|_{L^2} = 0, \quad j = 1, 2,
\]

where \( U(t) = \exp(i \frac{t^2}{2} \partial_x^2) \). Moreover we have

\[
\hat{\varphi}^+_1(\xi) \cdot \hat{\varphi}^+_2(\xi) = 0 \quad (1.4)
\]

for each \( \xi \in \mathbb{R} \), where \( \hat{\varphi} = \mathcal{F}\varphi \) denotes the Fourier transform of a function \( \varphi \).

As we have mentioned in [2], the most important thing in Proposition 1.1 is the relation (1.4) which does not appear in the usual short-range situation. The purpose of this sequel is to investigate it in more detail. For this purpose, let us recall the following proposition also shown in [2].

Proposition 1.2 ([2]). We put \( \varphi^+_j = \lim_{t \to +\infty} U(-t)u_j(t) \) in \( L^2 \), \( j = 1, 2 \), for the global solution \( u = (u_1, u_2) \) to (1.1)–(1.2), whose existence is guaranteed by Proposition 1.1. There exists a function \( m : \mathbb{R} \to \mathbb{R} \) such that the following holds for each \( \xi \in \mathbb{R} \):

- \( m(\xi) > 0 \) implies \( \hat{\varphi}^+_1(\xi) \neq 0 \) and \( \hat{\varphi}^+_2(\xi) = 0 \);
- \( m(\xi) < 0 \) implies \( \hat{\varphi}^+_1(\xi) = 0 \) and \( \hat{\varphi}^+_2(\xi) \neq 0 \);
- \( m(\xi) = 0 \) implies \( \hat{\varphi}^+_1(\xi) = \hat{\varphi}^+_2(\xi) = 0 \).

Note that (1.2) is an immediate consequence of Proposition 1.2. In other words, Proposition 1.2 is more precise than (1.4), and the function \( m(\xi) \) plays an important role in it. This indicates that better understanding of \( m(\xi) \) will bring us more precise information on the scattering state \( \varphi^+ \).

Our aim in the present paper is to specify the leading term of \( m(\xi) \) for sufficiently small initial data. This will allow us to find criteria for \( L^2 \) decay/non-decay of each component of the solutions to (1.1).

2 The leading term of \( m(\xi) \) in the small amplitude limit

In what follows, we put a small parameter \( \varepsilon \) in front of the initial data to distinguish information on the amplitude from the others, that is, we replace the initial condition (1.2) by

\[
u_j(0, x) = \varepsilon \psi_j(x), \quad j = 1, 2,
\]

where \( \psi_j \in H^2 \cap H^{1,1} \) is independent of \( \varepsilon \). Our main theorem reads as follows.
**Theorem 2.1.** Let $m$ be the function given in Proposition 1.2 with the initial condition \((1.2)\) replaced by \((2.1)\). We have
\[
m(\xi) = \varepsilon^2 (|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2) + O(\varepsilon^4)
\]
as $\varepsilon \to +0$ uniformly in $\xi \in \mathbb{R}$.

As a consequence of Theorem 2.1, we have the following criteria for (non-)triviality of the scattering state $\varphi^+ = (\varphi_1^+, \varphi_2^+)$ for \((1.1)-(2.1)\).

**Corollary 2.1.** Assume that there exist points $\xi^* \in \mathbb{R}$ and $\xi_* \in \mathbb{R}$ such that
\[
|\hat{\psi}_1(\xi^*)| > |\hat{\psi}_2(\xi^*)|,
\]
and
\[
|\hat{\psi}_1(\xi_*)| < |\hat{\psi}_2(\xi_*)|,
\]
respectively. Then, for sufficiently small $\varepsilon$, we have $\|\varphi_1^+\|_{L^2} > 0$ and $\|\varphi_2^+\|_{L^2} > 0$.

**Corollary 2.2.** Assume that
\[
|\hat{\psi}_1(\xi)| > |\hat{\psi}_2(\xi)|
\]
for all $\xi \in \mathbb{R}$. Then, for sufficiently small $\varepsilon$, $\varphi_2^+$ vanishes almost everywhere on $\mathbb{R}$, while $\|\varphi_1^+\|_{L^2} > 0$.

It follows from \((1.3)\) and Corollary 2.1 that both $u_1(t)$ and $u_2(t)$ behave like non-trivial free solutions as $t \to +\infty$. In particular, we see that $L^2$ decay does not occur for $u_1(t)$ and $u_2(t)$ under \((2.2)\) and \((2.3)\). To the contrary, Corollary 2.2 tells us that only the second component $u_2(t)$ is dissipated as $t \to \infty$ in the sense of $L^2$ under \((2.4)\). We emphasize again that such phenomena do not occur in the usual short-range settings. In this sense, the dynamics for the system \((1.1)\) is much more delicate than that for the single Schrödinger equation with dissipative cubic nonlinear terms.

## 3 Proofs

This section is devoted to the proofs of Theorem 2.1 and its corollaries. In what follows, we will denote various positive constants by the same letter $C$, which may vary from one line to another.

### 3.1 Proof of Theorem 2.1

We set $\alpha_j(t, \xi) = \mathcal{F}[\mathcal{U}(t)u_j(t, \cdot)](\xi)$ for the solution $u = (u_1, u_2)$ to \((1.1)-(2.1)\). According to [2], we have in fact the following expression for $m(\xi)$ in Proposition 1.2
\[
m(\xi) = |\alpha_1(2, \xi)|^2 - |\alpha_2(2, \xi)|^2 + \int_2^\infty \rho(\tau, \xi)d\tau,
\]
(3.1)
where
\[
\rho(t, \xi) = 2 \text{Re} \left[ \alpha_1(t, \xi)R_1(t, \xi) - \alpha_2(t, \xi)R_2(t, \xi) \right],
\]
\[
R_1 = \frac{1}{t} |\alpha_2|^2 \alpha_1 - \mathcal{F}U(-t) |u_2|^2 u_1,
\]
\[
R_2 = \frac{1}{t} |\alpha_1|^2 \alpha_2 - \mathcal{F}U(-t) |u_1|^2 u_2.
\]

From the argument of Section 3 in [2], we already know the following estimate for \( \rho \):
\[
\int_2^\infty |\rho(\tau, \xi)| d\tau \leq C \varepsilon^4 \langle \xi \rangle^{-2}.
\] (3.2)

It follows from (3.1) and (3.2) that
\[
\sup_{\xi \in \mathbb{R}} |m(\xi) - (|\alpha_1(2, \xi)|^2 - |\alpha_2(2, \xi)|^2)| \leq C \varepsilon^4.
\]

Therefore, it suffices to prove the following lemma.

**Lemma 3.1.** For \( j = 1, 2 \), we have
\[
\alpha_j(2, \xi) = \varepsilon \hat{\psi}_j(\xi) + O(\varepsilon^3)
\]
as \( \varepsilon \to +0 \), uniformly in \( \xi \in \mathbb{R} \).

**Proof.** First we recall the estimates which have been shown in [3] (or [1]):
\[
\|u(t)\|_{L^2} + \|\mathcal{J}u(t)\|_{L^2} \leq C \varepsilon (1 + t)^{\gamma},
\]
\[
\|u(t)\|_{L^\infty} \leq C \varepsilon (1 + t)^{-1/2},
\]
where \( \gamma \in (0, 1/12) \) and \( \mathcal{J} = x + it \partial_x \). We set \( N_1(u) = |u_2|^2 u_1 \) and \( N_2(u) = |u_1|^2 u_2 \). Then it follows from the relations \( \partial_t \mathcal{U}(-t)u_j = -\mathcal{U}(-t)N_j(u) \), \( \mathcal{U}(t)x\mathcal{U}(-t) = \mathcal{J} \) and the Sobolev embedding that
\[
\sup_{\xi \in \mathbb{R}} |\alpha_j(2, \xi) - \varepsilon \hat{\psi}_j(\xi)| \leq C \|\mathcal{U}(-2)u_j(2, \cdot) - u_j(0, \cdot)\|_{H^{0.1}}
\]
\[
\leq C \int_0^2 \|\mathcal{U}(-\tau)N_j(u(\tau))\|_{H^{0.1}} d\tau
\]
\[
\leq C \int_0^2 \|u(\tau)\|_{L^\infty} \left( \|u(\tau)\|_{L^2} + \|\mathcal{J}u(\tau)\|_{L^2} \right) d\tau
\]
\[
\leq C \varepsilon^3.
\]
3.2 Proof of Corollary 2.1

We put \( V = \{ \xi \in \mathbb{R} \mid |\hat{\psi}_1(\xi)| > |\hat{\psi}_2(\xi)| \} \). By (2.2), we see that \( V \) is a non-empty open set. Now we take \( r > 0 \) so small that the closed interval \( I = [\xi^*-r, \xi^*+r] \) is included in \( V \), and we put
\[
C_1 = \min_{\xi \in I} (|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2).
\]
Then we have \( C_1 > 0 \), and Theorem 2.1 gives us
\[
m(\xi) \geq C_1 \varepsilon^2 - C \varepsilon^4 > 0
\]
for \( \xi \in I \), if \( \varepsilon > 0 \) is small enough. By Proposition 1.2, we have \( \hat{\varphi}_1(\xi) \neq 0 \) for \( \xi \in I \). Therefore we obtain
\[
\|\varphi_1^+\|_{L^2} \geq \|\hat{\varphi}_1^+\|_{L^2(I)} > 0.
\]
Similarly, (2.3) yields \( \|\varphi_2^+\|_{L^2} > 0 \).

3.3 Proof of Corollary 2.2

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a cut-off function satisfying \( \chi(\xi) = 1 \) (\( |\xi| \leq 1 \)) and \( \chi(\xi) = 0 \) (\( |\xi| \geq 2 \)). For given \( \delta > 0 \), we can choose \( q \geq 1 \) so large that \( \|(1 - \chi_q)\hat{\varphi}_2^+\|_{L^2} < \delta \), where \( \chi_q(\xi) = \chi(\xi/q) \). With this \( q \), we put
\[
C_2 = \min_{|\xi| \leq 2q} (|\hat{\psi}_1(\xi)|^2 - |\hat{\psi}_2(\xi)|^2).
\]
Then we have \( C_2 > 0 \), because of (2.4). So it follows from Theorem 2.1 that
\[
m(\xi) \geq C_2 \varepsilon^2 - C \varepsilon^4 > 0
\]
for \( |\xi| \leq 2q \), if \( \varepsilon > 0 \) is small enough. By Proposition 1.2, we deduce that \( \chi_q(\xi)\hat{\varphi}_2^+(\xi) = 0 \) for all \( \xi \in \mathbb{R} \). Therefore
\[
\|\varphi_2^+\|_{L^2} = \|(1 - \chi_q)\hat{\varphi}_2^+\|_{L^2} < \delta.
\]
Since \( \delta \) can be taken arbitrarily small, this means that \( \varphi_2^+ \) vanishes almost everywhere on \( \mathbb{R} \).

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