Quantum hydrodynamics approach to the research of quantum effects and vorticity evolution in spin quantum plasmas

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Received May 3, 2013; Revised May 28, 2013; Accepted September 13, 2013; Published November 1, 2013

In this paper, we explain a magneto quantum hydrodynamics (MQHD) method for the study of the quantum evolution of a system of spinning fermions in an external electromagnetic field. The fundamental equations of microscopic quantum hydrodynamics (the momentum balance equation and the magnetic moment density equation) are derived from the many-particle microscopic Schrödinger equation with a spin–spin and Coulomb modified Hamiltonian. Using the developed approach, an extended vorticity evolution equation for the quantum spinning plasma is derived. The effects of the new spin forces and spin–spin interaction contributions on the motion of fermions, the evolution of the magnetic moment density, and vorticity generation are predicted. The influence of the intrinsic spin of electrons on whistler mode turbulence is investigated. The results can be used for theoretical studies of spinning many-particle systems, especially dense quantum plasmas in compact astrophysical objects, plasmas in semiconductors, and micro-mechanical systems, in quantum X-ray free-electron lasers.

Subject Index A01, A63, A64, I20, I74

1. Introduction

The spinning quantum fluid plasma is becoming of increasing current interest [1–7]. Hydrodynamics equations of a spinning fluid for the Pauli equation with quantum particle angular momentum spin were presented in the pioneering works of Takabayashi and Vigier [8–11]. The vector representation of a non-relativistic spinning particle leads to the appearance of new quantum effects that had been separated as nonlinear terms, which arises from the inhomogeneity of spin distribution. The extension of the interpretation to the developed approach was carried out in Refs. [12–16].

The quantum effects in plasma can be represented by three main quantum corrections. The first is a quantum force, the multiparticle quantum Bohm or Madelung potential, proportional to powers of \( \hbar \) and produced by density fluctuations [17,18]. The second is associated with the quantum particle angular momentum spin by the possible inhomogeneity of the external and spin magnetic fields. In the momentum balance equation this force appears through the magnetization energy [1]. The final force is associated only with the spin magnetic moment of the particle [8].

The most interesting and defining features of quantum spinning plasmas can be derived from the vorticity equation. It has been shown in Ref. [19] that the vorticity, constructed from the spin field of a quantum spinning plasma, combines with the classical generalized vorticity to yield a new grand generalized vorticity that obeys the standard vortex dynamics.
Astrophysics is also a rapidly growing field of research. It is important that the consequences of turbulent plasma movement in the solar photosphere lead to the generation of vorticity, while magnetic vortices are produced by magnetic tension. For example, magnetohydrodynamics (MHD) simulations of magnetoconvection have been used to analyze the generation of small-scale vortex motions in the solar photosphere. Using the vorticity equation, combined with G-band radiative diagnostics, it has been shown that two different types of photospheric vorticity, magnetic and non-magnetic, are generated in the domain [20]. The presence of vortex motions for astrophysics was developed in Refs. [21,22].

The extraction of coherent vortices out of turbulent flows has been applied to simulations of resistive drift-wave turbulence in magnetized plasma [23]. The quasi-hydrodynamic and quasi-adiabatic regimes have been investigated.

The formation and dynamics of dark solitons and vortices in quantum electron plasmas have been studied in Ref. [24]. A pair of equations comprising the nonlinear Schrödinger and Poisson system of equations, which conserves the number of electrons as well as their momentum and energy, was used. It has been shown that the gradient ‘free-energy’ contained in equilibrium spin vorticity can cause electromagnetic modes, in particular light waves [25].

The collective electron angular momentum spin effects in spinning quantum plasmas can be investigated using insights from quantum kinetic theory or some effective theory. We propose a method of quantum hydrodynamics that allows one to obtain a description of the collective effects in magnetized quantum plasmas in terms of functions in physical space. The fermion model was developed in Refs. [1,2,26,27]. The waves in a magnetized plasma with spin were studied in Ref. [26], exploring a new quantum hydrodynamics method of the generation wave in the plasma. The new formalism given in these references is used in this article to study vorticity evolution in a magnetized plasma with spin. A quantum mechanics description for systems of \( N \) interacting spinning particles is based upon the many-particle Schrödinger equation (MPSE) that specifies a wave function in a \( 3N \)-dimensional configuration space. As wave processes, processes of information transfer, and other spin transport processes occur in 3D physical space, it becomes necessary to turn to a mathematical method of physically observable values that are determined in a 3D physical space. To do this, we should derive the fundamental equations that determine the dynamics of functions of three variables, starting from MPSE. This problem has been solved with the creation of a many-particle quantum hydrodynamics (MPQHD) method.

In this article, to study vorticity and spin vortex effects we generalize and use the method of the many-particle quantum hydrodynamics MQHD approach. We derive a fundamental balance equation, a magnetic moment evolution equation, and a new vorticity dynamics equation for magnetized quantum plasmas.

### 2. Fundamental equations of fermion quantum hydrodynamics

In this section we derive a system of magneto quantum hydrodynamics (MQHD) equations for charged and neutral particles from the many-particle microscopic Schrödinger equation:

\[
    i\hbar \frac{\partial \psi_s(R,t)}{\partial t} = (\hat{H}\psi)_s(R,t),
\]

where \( R = (\vec{r}_1, \ldots, \vec{r}_N) \). We consider a system of \( N \) interacting fermions with equal masses \( m_j \), charged and proper magnetic moments in an external electromagnetic field. The state of the system of \( N \) fermions is determined by a wave function in the \( 3N \)-dimensional configuration space, which
is a rank-$N$ spinor:

$$\psi_s(R, t) = \psi_{s_1, s_2, \ldots, s_N}(\vec{r}_1, \ldots, \vec{r}_N, t).$$  \hfill (2.2)

The Hamiltonian has the form

$$\hat{H} = \sum_{j=1}^{N} \left( \frac{\hat{p}_j^2}{2m_j} + q_j \varphi_{j, \text{ext}} - \mu_j \hat{\sigma}^\alpha_j B^\alpha_{j, \text{ext}} \right) + \frac{1}{2} \sum_{j \neq k}^{N} q_j q_k G_{jk} - \frac{1}{2} \sum_{j \neq k, k}^{N} \mu_j^2 F^\alpha_{jk} \hat{\sigma}^\alpha_j \hat{\sigma}^\alpha_k,$$

\hfill (2.3)

where $\mu_j = g \mu_B / 2$, $\mu_B$ is the electron or positron magnetic moment and $\mu_B = q_e h / 2 m_e c$ is the Bohr magneton, $q_j$ stands for the electron charge $q_e = -e$ or the positron charge $q_p = e$, and $h$ is the Planck constant, $g \approx 2.0023193$. The covariant derivative operator is

$$\hat{D}_j^\alpha = -i \hbar \hat{\sigma}^\alpha_j = \frac{q_j}{c} A_j^\alpha,$$

\hfill (2.4)

where $\vec{A}_{\text{ext}}, \varphi_{j, \text{ext}}$ are the vector and scalar potentials of the external electromagnetic field.

The Green’s functions of the Coulomb and spin–spin interaction are

$$G_{jk} = \frac{1}{r_{jk}}, \quad F^\alpha_{jk} = 4\pi \delta_\alpha^\beta \delta(r_{jk}) + \partial^\alpha_j \partial^\beta_k \frac{1}{r_{jk}}.$$

\hfill (2.5)

The first step in the construction of the MQHD apparatus is to determine the concentration of particles in the neighborhood of $\vec{r}$ in a physical space. If we define the concentration of particles as the quantum average of the concentration operator in the coordinate representation $\hat{\rho} = \sum_j \delta(\vec{r} - \vec{r}_j)$, we obtain

$$\rho(\vec{r}, t) = \sum_S \int dR \sum_j^{N} \delta(\vec{r} - \vec{r}_j) \psi_s^+ (R, t) \psi_s(R, t).$$

\hfill (2.6)

Differentiation of $\rho(\vec{r}, t)$ with respect to time and application of the Schrödinger equation with Hamiltonian (2.3) leads to the continuity equation

$$\partial_t \rho(\vec{r}, t) + \vec{\nabla} \cdot \vec{j}(\vec{r}, t) = 0,$$

\hfill (2.7)

where the current density takes the form

$$j^\alpha(\vec{r}, t) = \sum_S \int dR \sum_j^{N} \delta(\vec{r} - \vec{r}_j) \frac{1}{2 m_j} (\hat{D}_j^\alpha \psi_s^+ (R, t) \psi_s(R, t) + \psi_s^+ (R, t) \hat{D}_j^\alpha \psi_s(R, t)).$$

\hfill (2.8)

The momentum balance equation can be derived by differentiating the current density (2.8) with respect to time:

$$\partial_t j^\alpha(\vec{r}, t) + \frac{1}{m} \partial_\beta \gamma^\alpha_{\beta}(\vec{r}, t) = \frac{q_e}{m} \rho(\vec{r}, t) E^\alpha_{\text{ext}}(\vec{r}, t)$$

$$+ \frac{q}{mc} \epsilon^{\alpha\beta\gamma} j_\beta(\vec{r}, t) B^\alpha_{\text{ext}}(\vec{r}, t) - \frac{1}{m} \int d\vec{r}' q^2 \partial_\alpha G(\vec{r}, \vec{r}') \rho_2(\vec{r}, \vec{r}', t)$$

$$+ \frac{1}{m} M^\alpha_\beta(\vec{r}, t) \partial_\beta B^\alpha_{\text{ext}}(\vec{r}, t) + \frac{1}{m} \int d\vec{r}' \partial_\alpha F^\alpha_{\beta\gamma}(\vec{r}, \vec{r}') \delta^\beta_{\gamma}(\vec{r}, \vec{r}', t).$$

\hfill (2.9)

$$\gamma^\alpha_{\beta}(\vec{r}, t) = \sum_S \int dR \sum_{j=1}^{N} \delta(\vec{r} - \vec{r}_j) \frac{1}{4m_j} (\psi_s^+(R, t) \hat{D}_j^\beta \psi_s(R, t)$$

$$+ (\hat{D}_j^\beta \psi_s(R, t))^+ \hat{D}_j^\beta \psi_s(R, t) + \text{h.c.})$$

\hfill (2.10)

represents the momentum current density tensor.
The momentum balance equation (2.9) contains the particle magnetic moment density [1]

\[ M^\alpha(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \mu_j \psi_s^{\dagger} \sigma_j^\alpha \psi_s. \]  

(2.11)

The Coulomb and spin–spin interactions between the particles are represented in Eq. (2.9) by the terms

\[ \rho_2(\vec{r}, \vec{r}', t) = \sum_s \int dR \sum_{j \neq k}^N \delta(\vec{r} - \vec{r}_j) \delta(\vec{r}' - \vec{r}_k) \psi_s^{\dagger}(R, t) \psi_s(R, t) \]  

(2.12)

is the two-particle probability density for the occurrence of two particles in the neighborhoods of the points \( \vec{r} \) and \( \vec{r}' \) normalized by \( N(N - 1) \), and the two-particle tensor of the magnetic moment density

\[ M^{\alpha\beta}(\vec{r}, \vec{r}', t) = \sum_s \int dR \sum_{j \neq k}^N \delta(\vec{r} - \vec{r}_j) \delta(\vec{r}' - \vec{r}_k) \mu_j \mu_k \psi_s^{\dagger} \sigma_j^\alpha \sigma_k^\beta \psi_s(R, t). \]  

(2.13)

Differentiation of \( M^\alpha \) with respect to time and application of the Schrödinger equation with Hamiltonian (2.3) leads to the magnetization equation. The equation representing the non-relativistic evolution of spin-1/2 motion takes the form

\[ \partial_t M^\alpha(\vec{r}, t) + \partial_\beta \zeta^{\alpha\beta}_M = \frac{2\mu}{\hbar} \epsilon^{\alpha\beta\gamma} M^\beta(\vec{r}, t) B_{\text{ext}}^\gamma(\vec{r}, t) \]

\[ + \frac{2\mu}{\hbar} \epsilon^{\alpha\beta\gamma} \int d\vec{r}' F^{\gamma3}(\vec{r}, \vec{r}') M^{\beta3}(\vec{r}, \vec{r}', t), \]  

(2.14)

where the tensor of the magnetic moment flux density is

\[ \zeta^{\alpha\beta}_M(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{\mu_j}{4m_j} (\psi_s^{\dagger} \sigma_j^\alpha \hat{D}_j^\beta \psi_s + (\sigma_j^\alpha \hat{D}_j^\beta \psi_s)) \psi_s(R, t). \]  

(2.15)

Equations (2.9) and (2.14) are not closed because they contain the two-particle functions (2.12) and (2.13) in terms of the interactions. For the determination of two-particle functions, we can obtain equations that will contain three-particle functions [1,2]. The complex expansion of (2.12) and (2.13) has the form of (A2) and (A3) (see Ref. [1]). However, we will use the self-consistent field approximation in this paper, when the correlation functions are equal to zero.

The spinning quantum magnetohydrodynamics should explain the vorticity evolution. The main idea of this paper was to create a hydrodynamics foundation for the vortex dynamic in the context of spinning quantum plasma. We use the MPQHD approach to receive the equations for the particle vorticity density, obeying the standard vortex dynamics. We determine the vorticity density vector of particles in the neighborhood of \( \vec{r} \) in a physical space as

\[ \Omega^\alpha(\vec{r}, t) = \sum_s \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{\epsilon^{\alpha\beta\gamma}}{2m_j} \hat{\gamma}_{j\beta} (\hat{D}_j^{\gamma3} \psi_s^{\dagger} \psi_s + \psi_s^{\dagger} \hat{D}_j^{\gamma3} \psi_s)(R, t). \]  

(2.16)

where we construct the vorticity density in terms of the wave function and denote the macroscopic vorticity density as \( \tilde{\Omega} = \vec{\nabla} \times \vec{j} \), as will be shown below. The classical generalized vorticity density \( \Omega \) can be defined as the curl of the current density. However, in Ref. [19] the ordinary vorticity of the plasma is proportional to the curl of the flow velocity of the fermions (vorticity has the
dimensions of the magnetic field). On the other hand, vorticity can be defined as the curl of velocity \( \vec{\omega} = \vec{\nabla} \times \vec{v} \) [20]. Our idea is that we use the definition (2.16) and MQHD method based upon the many-particle Schrödinger equation to derive the generalized dynamical equation for classical vorticity \( \vec{\omega} = \vec{\nabla} \times \vec{v} \) similar to Refs. [20] and [21] (see below), but containing information about interactions inside the fluid.

2.1. Velocity field

The velocity of the \( j \)th particle \( \vec{v}_j \) is determined by

\[
\vec{v}_j = \frac{1}{m_j}(\vec{\nabla}_j S - i\hbar\frac{\partial}{\partial t} \bar{\varphi}_j) - \frac{q_j}{m_j c} \vec{A}_j, \tag{2.17}
\]

The quantity \( \vec{v}_j(R, t) \) describes the current of probability connected with the motion of the \( j \)th particle; in general, \( \vec{v}_j(R, t) \) depends on the coordinates of all particles of the system \( R \), where \( R \) is the totality of the \( 3N \) coordinates of the \( N \) particles of the system \( R = (\vec{r}_1, \ldots, \vec{r}_N) \).

The \( S(R, t) \) value in Eq. (2.17) represents the phase of the wave function and, as the electron has spin, the wave function can now be expressed in the form

\[
\psi_s(R, t) = a(R, t)e^{iS/\hbar} \varphi(R, t), \tag{2.18}
\]

where \( \varphi \), normalized such that \( \varphi^+ \varphi = 1 \), is the new spinor, defined in the local frame of reference with the origin at the point \( \vec{r} \). The spinor gives the spin part of the wave function.

We substituted the wave function in the definition of the basic hydrodynamical quantities. Using the fact that the velocity field \( \vec{v} \) is the velocity of the local center of mass and is determined by

\[
\vec{j}(\vec{r}, t) = \rho(\vec{r}, t)\vec{v}(\vec{r}, t), \tag{2.19}
\]

the vorticity density field (2.16) and the momentum current density tensor (2.10) have the new form of

\[
\tilde{\Omega}^\alpha(\vec{r}, t) = (\vec{\nabla} \times j^\alpha)(\vec{r}, t), \tag{2.20}
\]

\[
\gamma^{\alpha\beta}(\vec{r}, t) = m\rho(\vec{r}, t)v^\alpha(\vec{r}, t)v^\beta(\vec{r}, t) + \psi^{\alpha\beta}(\vec{r}, t) + \Lambda^{\alpha\beta}(\vec{r}, t) + \tau^{\alpha\beta}(\vec{r}, t), \tag{2.21}
\]

where

\[
\psi^{\alpha\beta}(\vec{r}, t) = \sum_S \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j)a^2 m_j u^\alpha_j u^\beta_j \tag{2.22}
\]

is the well known kinetic pressure tensor. \( u^\alpha_j(\vec{r}, R, t) \) is a quantum equivalent of the thermal speed and \( u^\alpha_j(\vec{r}, R, t) = v^\alpha_j(R, t) - v(\vec{r}, t) \).

The tensor \( \Lambda^{\alpha\beta} \) is proportional to \( \hbar^2 \), has a purely quantum origin, and can therefore be interpreted as an additional quantum pressure

\[
\Lambda^{\alpha\beta}(\vec{r}, t) = -\sum_S \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j)a^2 \frac{\hbar^2}{2m_j} \frac{\partial^2 \ln a}{\partial x^\alpha_j \partial x^\beta_j}. \tag{2.23}
\]

The quantum tensor (2.25) is a quantity that can be rewritten in terms of concentration \( \rho \) in the approximation of noninteracting particles, using the definition \( \rho = \sum_S \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j)a^2(R, t) \)
as

$$\Lambda^{\alpha\beta}(\vec{r}, t) = -\frac{\hbar^2}{4m} \rho(\vec{r}, t) \partial^\alpha \partial^\beta (\ln \rho)(\vec{r}, t). \tag{2.24}$$

Using simple manipulation with the expression (2.24), we may substitute it for a large system of noninteracting particles; this tensor is

$$\Lambda^{\alpha\beta}(\vec{r}, t) = -\frac{\hbar^2}{4m} \partial^\alpha \partial^\beta \left( \frac{\rho(\vec{r}, t)}{\rho(\vec{r}, t)} \{ \partial^\beta \rho(\vec{r}, t) \} \right). \tag{2.25}$$

It should be explained that the tensor (2.25) arises as a consequence of the quantum Madelung potential and can be interpreted as an additional quantum pressure.

The tensor $\Upsilon^{\alpha\beta}_s$ appears in the theory as a result of representations of rotating electrons as an assembly of bodies continuously distributed in space. In the context of quantum hydrodynamics, the force due to a new spin stress inside the fluid takes the form

$$\Upsilon^{\alpha\beta}_s = -\frac{\hbar^2}{4m\mu^2} \partial^\alpha \partial^\beta \left( \frac{M^\gamma}{\rho} \right). \tag{2.26}$$

This new force emerges from the inhomogeneity of spin distribution and must be considered in the equation of motion, being of the order of $\hbar^2$.

On the other hand, after the presentation of the wave function in the exponential form, the tensor of the magnetic moment flux density takes the form

$$\Sigma^{\alpha\beta}_M(\vec{r}, t) = M^\alpha \nu^\beta(\vec{r}, t) + \gamma^{\alpha\beta}_s(\vec{r}, t), \tag{2.27}$$

where

$$\gamma^{\alpha\beta}_s(\vec{r}, t) = \sum_S \int dR \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \frac{\hbar^2}{4m} \left( \partial^\alpha \hat{D}_j^\beta \varphi + (\hat{\sigma}_j^\alpha \partial^\beta \varphi) + \varphi \right)(R, t). \tag{2.28}$$

differ from $\Sigma^{\alpha\beta}_M$ only by the change of the spinor $\psi$ in (2.15) to the spin part of the wave function. In the context of quantum hydrodynamics, we have the additional spin torque

$$\gamma^{\alpha\beta}_s(\vec{r}, t) = \frac{\hbar}{2m\mu} e^{\alpha\gamma\lambda} M^\gamma \partial^\beta \left( \frac{M^\lambda}{\rho} \right). \tag{2.29}$$

Note that, to simplify the problem, we consider that the thermal spin-interactions are neglected and the microscopic spin $s^\alpha_j = s^\alpha$ is equal to the macroscopic average $s^\alpha$. Taking the self-consistent field approximation, from (2.9) and (2.14) we have a set of MQHD equations for the electrons and positrons ($p = e, i$): the continuity equation, momentum balance equation, and magnetic moment density equation take the form

$$\partial_t \rho_p + \vec{v}_p(\rho_p \vec{v}_p) = 0, \tag{2.30}$$
\[ m_p \rho_p (\partial_t + \nu^\beta_p \partial_\beta) \tilde{v}_p = q_p \rho_p \tilde{E}_{\text{ext}} + \frac{1}{c} j_{pe} \times \tilde{B}_{\text{ext}} - \tilde{\nabla} \varphi_p + \frac{\hbar^2}{2m_p} \rho_p \tilde{\nabla} \left( \frac{\Delta \sqrt{\rho_p}}{\sqrt{\rho_p}} \right) + M_p^\beta \tilde{\nabla} B_{\text{ext}}^\beta + \frac{\hbar^2}{4m_p} \rho_p \tilde{\nabla} \left( M_p^\beta \rho_p \tilde{\nabla} \rho_p \tilde{\nabla} \rho_p \right) \]

\[ + \frac{\hbar^2}{4m_p^2} \partial_\beta \left\{ M_p^\beta \tilde{\nabla} \rho_p \tilde{\nabla} \left( \frac{M_p^\gamma}{\rho_p} \right) \right\} - \rho_p \tilde{\nabla} \int d\tilde{r}' q_p^2 \tilde{G}(\tilde{r}, \tilde{r}') \rho_p(\tilde{r}', t) \]

\[ + M_p \tilde{\nabla} \int d\tilde{r}' F^{\gamma \delta}(\tilde{r}, \tilde{r}') M_p^\delta(\tilde{r}', t), \tag{2.31} \]

\[(\partial_t + \nu^\beta_p \partial_\beta) \tilde{M}_p = \frac{2\mu_p}{\hbar} \tilde{M}_p \times \tilde{B}_{\text{ext}} + \frac{\hbar}{2m_p \mu_p} \partial_k \left\{ \tilde{M}_p \times \partial^k \left( \frac{\tilde{M}_p}{\rho_p} \right) \right\} \]

\[ + \frac{\hbar}{m_p} \epsilon^{\alpha \beta \gamma} M_p^\beta \int d\tilde{r}' F^{\gamma \delta}(\tilde{r}, \tilde{r}') M_p^\delta(\tilde{r}', t). \tag{2.32} \]

Let us discuss the physical significance of the terms on the right-hand side of the system of MQHD equations obtained above, (2.30)–(2.32). The first and second terms in Eq. (2.31) describe the well known interaction with the external electromagnetic field, where the first term represents the effect of the external electric field on the charge density and the second term is the Lorentz force field. The fourth term is a quantum force produced by density fluctuations, which has its origin in the so-called Madelung potential. The fifth term appears in the equation of motion (2.31) through the magnetization energy and depends on the spin or magnetic moment density of particles. The sixth term represents the self-force or magnetic moment density stress inside the electron or positron fluid. This spin self-force appears even in the absence of the electromagnetic fields and arises from the inhomogeneity of the magnetic moment density distribution. Other terms in (2.31) describe a force field that represents interactions between particles, namely the Coulomb interaction of charges and spin–spin interactions.

The second term in the equation of magnetic moment density motion (2.32) represents the additional magnetic moment density torque effect on the magnetic moment density evolution and tends to align spins parallel. It is important that the second term has a similar form to the contribution of exchange interaction in ferromagnetic media of isotropic cubic ferromagnetism.

Using the definition (2.16) and the Madelung decomposition (2.17) with the momentum balance dynamical equation (2.31), the hydrodynamics classical vorticity dynamical equation, where \( \tilde{\omega}_p = \tilde{\nabla} \times \tilde{v}_p \), hfs the form of

\[ \partial_t \tilde{\omega}_p = \tilde{\nabla} \times (\tilde{v}_p \times \tilde{\omega}_p) - \tilde{\nabla} \left( \frac{1}{m_p \rho_p} \right) \times \tilde{\nabla} \varphi_p + \frac{1}{m_p} \tilde{\nabla} \left( \frac{M_p k}{\rho_p} \right) \times \tilde{\nabla} B_{\text{ext}}^k + \frac{1}{cm_p} \tilde{\nabla} \]

\[ \times \left( \frac{1}{\rho_p} \tilde{j}_{pe} \times \tilde{B} - \frac{q_p^2}{m_p} \tilde{\nabla} \times \tilde{\nabla} \int d\tilde{r}' G(\tilde{r}, \tilde{r}') \rho_p(\tilde{r}', t) + \eta_p \frac{\hbar^2}{4 m_p^2} \tilde{\nabla} \left( \frac{M_p^\gamma}{\rho_p} \right) \right) \]

\[ \times \tilde{\nabla} \left\{ \frac{1}{\rho_p} \nabla_k \left( \rho_p \nabla^k \frac{M_p^\gamma}{\rho_p} \right) \right\} + \frac{1}{m_p} \tilde{\nabla} \left( \frac{M_p k}{\rho_p} \right) \times \tilde{\nabla} \int d\tilde{r}' F^{\gamma \delta}(\tilde{r}, \tilde{r}') M_p^\delta(\tilde{r}', t). \tag{2.33} \]

The vorticity evolution equation (2.33) shows the different physical factors associated with the generation of vorticity. The second term on the right-hand side of (2.33) is proportional to the gas pressure and is responsible for the hydrodynamic baroclinic vorticity generation of the classical vortex field. The third term represents the magnetic baroclinic vorticity and is associated with the anisotropic magnetic pressure effect. The fourth term contains information about the vorticity generated by the magnetic tension. The sixth term is associated with the magnetic vorticity generation, even in the absence of the magnetic field. The fifth and seventh terms characterize the effect of
Coulomb and spin–spin interactions on the vorticity evolution. Equation (2.33) contains the normal electron or positron current density \( \mathbf{j}_{pe} = q_p \rho_p \mathbf{v}_{pe} \), and the magnetic moment density \( \mathbf{M}_p = \rho_p \mathbf{\mu}_p \). The vorticity evolution equation (2.33) is a generalization of the classical vorticity equation that was presented in Refs. [19,20,28]. At first, Eq. (2.33) combines the erstwhile generalized classical vorticity, but, in contrast to Ref. [20], contains information about interactions inside the quantum vortical fluid and has been derived using the MQHD method.

Note that for a 3D system of particles the momentum balance equation (2.31), the magnetic density equation (2.32), and the vorticity evolution equation (2.33) may be written down in terms of the magnetic intensity of the field that is created by charges \( q_p \) and spins \( \mathbf{\mu}_p \) of the particle system

\[
m_p (\partial_t + \mathbf{v}_p \partial_\beta) \mathbf{\omega}_p = q_p \mathbf{\hat{E}} + \frac{q_p}{c} \mathbf{v}_{pe} \times \mathbf{B} - \frac{\mathbf{\nabla} \mathbf{\varphi}_p}{\rho_p} + \frac{\hbar^2}{2m_p} \mathbf{\nabla} \left( \frac{\Delta \sqrt{\rho_p}}{\sqrt{\rho_p}} \right) + \frac{2\mu_p}{\hbar} \mathbf{s}_p \times \mathbf{\nabla} \mathbf{B}_{\text{eff}},
\]

(2.34)

\[
(\partial_t + \mathbf{v}_p \partial_\beta) \mathbf{s}_p = \frac{2\mu_p}{\hbar} \mathbf{s}_p \times \mathbf{B}_{\text{eff}}
\]

(2.35)

and

\[
\partial_t \mathbf{\omega}_p = \mathbf{\nabla} \times (\mathbf{\nu}_p \times \mathbf{\omega}_p) - \frac{1}{m_p \rho_p} \mathbf{\nabla} \mathbf{\nu}_p + \frac{2\mu_p}{\hbar m_p} \mathbf{\nabla} \mathbf{s}_k \times \mathbf{\nabla} \mathbf{B}_{\text{eff}},
\]

(2.36)

where \( \mathbf{\omega}_p = \mathbf{\omega}_p + \frac{q_p}{m_p} \mathbf{B} \) is the generalized vorticity and the effective magnetic field \( \mathbf{B}_{\text{eff}} = \mathbf{\hat{B}} + \mathbf{\hat{B}}_{\text{in}} \) includes the total magnetic field and internal magnetic field \( \mathbf{\hat{B}}_{\text{in}} \):

\[
\mathbf{\hat{B}}_{\text{in}} = \frac{c}{q_p \rho_p} \mathbf{\nabla}_k (\rho_p \mathbf{\nabla}^k \mathbf{s}_p).
\]

(2.37)

The total magnetic field \( \mathbf{\hat{B}} \) consists of the field generated by the charge and the field generated by the spins. Ampère’s law including the magnetization spin current \( j_m = 2\mu / \hbar \mathbf{\nabla} \times (\rho \mathbf{s}) \) takes the form

\[
\mathbf{\nabla} \times \mathbf{B} = \frac{4\pi}{c} \sum_p \mathbf{j}_{pe} + \frac{8\pi \mu}{\hbar} \sum_p \mathbf{\nabla} \rho_p \times \mathbf{s}_p + \frac{8\pi \mu}{\hbar} \sum_p \rho_p \mathbf{\nabla} \times \mathbf{s}_p.
\]

(2.38)

We must note that the spin stress term (2.25) has a notably interesting nature, existing even in the absence of magnetic field, having only a quantum foundation, and arising out of the spin part of the wave function. Equation (2.36) was rewritten by separating the magnetic and non-magnetic terms [8], the effective magnetic field, and the last three terms produced by the magnetization vortex generation.

2.2. Whistler mode turbulence in magnetized plasmas

The nonlinear turbulent processes associated with electromagnetic waves in spinning plasmas have attracted interest. Nonlinear whistler mode turbulence has been studied in a magnetized plasma [29–32]. The authors focused on low-frequency (in comparison with the electron gyrofrequency) nonlinearly interacting electron whistlers and nonlinearly interacting Hall-magnetohydrodynamic (H-MHD) fluctuations in Ref. [29]. In this section we investigate the electron whistler wave properties based on extended 2D magnetohydrodynamic equations. However, we understand that the electron spin effect on the whistler wave dispersion typically requires a strong external magnetic field.

Two-dimensional turbulence has been studied in a magnetized plasma involving incompressible electrons and immobile ions. We consider that the electrons carry currents, while the immobile ions provide a neutralizing background to a quasi-neutral spinning plasma. Using the fact that the electron
Ampère’s law, normalized as $\vec{\omega} = \mu_0 e \vec{V} \times \vec{s} / h$, which is determined by the spin vector $\vec{s}$, we have, from Ampère’s law,

$$\vec{v}_e = -\frac{c}{4\pi \rho_0 \mu_0} \vec{V} \times \vec{B} - \frac{1}{m_e} \vec{V} \times \vec{s},$$  \hspace{1cm} (2.39)

where $\mu_0 = -g e h / 4 m_e c$, $m_e$ is the electron mass, and $\rho_0$ is the electron density. We take into account that the electron density is constant and the electron continuity equation (2.30) shows a divergence-less electron fluid velocity $\vec{V} \vec{v}_e = 0$.

All physical quantities are presented in the form of the sum of the equilibrium part and small perturbations $f = f_0 + f_1$:

$$\vec{B}(\vec{r}, t) = B_0 \vec{y} + \vec{B}_1(\vec{r}, t) + \ldots$$
$$\vec{s}_e(\vec{r}, t) = s_0 \vec{y} + \vec{s}_1(\vec{r}, t) + \ldots$$

$$\rho_e(\vec{r}, t) = \rho_0, \quad \vec{v}_e(\vec{r}, t) = \vec{v}_1(\vec{r}, t) + \ldots$$

(2.40)

where $B_0$ is the external uniform magnetic field directed along the axis $y$ and $s_0$ is the unperturbed spin vector.

In this case, we assume that linear excitations $f_1$ are proportional to $\exp(-i \omega t + i k \vec{s})$, where $\omega$ is the wave frequency and $k^2 = k_x^2 + k_y^2$. The 3D equation (2.36) closed by (2.35) becomes 2D by regarding the variation in the $z$-direction as ignorable or $\partial / \partial z = 0$ and using the separation of the total magnetic field into two scalar variables $\vec{B}_1 = \vec{z} \times \vec{V} \vec{\psi} + b_z^2$ [29].

We will assume propagation of the waves along an external magnetic field $B_0$ or $k = k_y$. A linearized set of equations (2.35) and (2.36) in this case gives us the dispersion equation

$$(1 + k^2)\omega_k^3 - (1 - \omega_\mu) k^2 \omega_\kappa^2 + (\omega_\mu \omega_g - \omega_\kappa^2) k^2 - \omega_\kappa \omega_k k^2 = 0,$$  \hspace{1cm} (2.41)

where the length and time scales are normalized respectively as $d_e = c / \omega pe$ and $\omega_c = e B_0 / m_e c$, $d_e$ is the electron skin depth or inertial length scale, $\omega_\kappa^2 = 4 \pi e^2 \rho_0 / m_e$ is the electron plasma frequency, $\omega_c$ is the electron cyclotron frequency, and $c$ is the speed of light. The other physical quantities are normalized as

$$\omega_k \rightarrow \omega_k / \omega_c, \quad k \rightarrow k d_e \quad \omega_g \rightarrow \omega_g / \omega_c, \quad \omega_\mu \rightarrow \omega_\mu / \omega_c;$$

where $\omega_g = g \omega_c / 2 + k^2 h / 2 m_e$ is the spin-precession frequency, which includes the internal magnetic field influence, $\omega_\mu = g^2 s_0 / 4 m_e d_e^2$ is a frequency that involves a spin correction due to the plasma magnetization current and appears even in the absence of the external magnetic field $B_0$, and $h$ is the reduced Planck constant. We use the fact that an unperturbed spin state $s_0 = h / 2$. This function appears as the solution of the spin evolution equation for spin quantum plasmas where the spin inertia and the spin-thermal coupling terms are neglected [433]. The temperature $T_\kappa$ is the Fermi electron temperature $T_F = h^2 (3 \pi^2 \rho_0)^{2/3} / 2 m_e$, where $k_B$ is the Boltzmann constant. A situation arises in which magnetization effects might be important in a regime of very strong magnetic field in which the external field strength approaches or exceeds the quantum critical magnetic field $B_0 \sim 4.4138 \times 10^{15} \text{~G}$ and highly dense plasmas $\rho_0 \sim 10^{30} \text{~1/cm}^3$. However, it can be assumed that the internal magnetic field inside the fluid, which is dependent on the gradient of the spin distribution (2.37), can tend to align neighboring spins parallel.

The effect of the frequency that involves the spin correction due to the plasma magnetization current is small $\omega_\mu < \omega_c$; the cubic expression (2.41) may be expanded to yield formulae $\omega_k$ in the...
The eigenmode for whistler waves obtained as solutions of the dispersion expression (2.42) is shown with respect to the normalized wave number and frequency for $B_0 = 5 \times 10^{12} \text{ G}$, $\rho_0 = 10^{28} \text{ 1/sm}^3$, and $m_e = 9.1093 \times 10^{-28} \text{ g}$. The blue and red lines represent the well known classical dispersion relation (see Ref. [29]) in the absence of spin magnetization, and the spin correction since the spin effect is appreciable.

$$\omega = k^2 \left( 1 - \omega_{\mu} \right) + \frac{\omega_{\mu} (\omega_k - 1)k^2}{(\omega_k + \omega_{\mu} - 1)k^2 + \omega_k},$$

(2.42)

and

$$\omega^2 = \frac{\omega_{\mu} (\omega_k - 1)k^2}{(\omega_k + \omega_{\mu} - 1)k^2 + \omega_k}.$$ 

(2.43)

The relation (2.42), shown in Fig. 1, expresses the dispersion of low-frequency whistler waves in the spinning quantum plasma using the model based on the 2D electromagnetic turbulence equation (2.36). The solution (2.43), shown in Fig. 2, expresses the dispersion of waves that emerge as a result of spin dynamics $\omega_k = g\omega_e/2 + k^2 \hbar/2m_e$, where the internal spin torque has been taken into account. The spectrum is divided by the electron inertial skin depth into two regions, a short-scale region $kd_e > 1$, $\omega_k \sim 1$ and a long-scale region $kd_e < 1$, $\omega_k \sim k^2$. 

Fig. 1. The eigenmode for whistler waves obtained as solutions of the dispersion expression (2.42) is shown with respect to the normalized wave number and frequency for $B_0 = 5 \times 10^{12} \text{ G}$, $\rho_0 = 10^{28} \text{ 1/sm}^3$, and $m_e = 9.1093 \times 10^{-28} \text{ g}$. The blue and red lines represent the well known classical dispersion relation (see Ref. [29]) in the absence of spin magnetization, and the spin correction since the spin effect is appreciable.

Fig. 2. The eigenmode for whistler waves obtained as solutions of the dispersion expression (2.42) is shown with respect to the normalized wave number and frequency for $B_0 = 5 \times 10^{12} \text{ G}$, $\rho_0 = 10^{28} \text{ 1/sm}^3$, and $m_e = 9.1093 \times 10^{-28} \text{ g}$. The blue and red lines represent the classical dispersion relation with the influence of spin torque, and the spin correction since the effects of spin current and magnetization energy are appreciable.
3. Conclusions

In this paper we have analyzed vorticity excitations caused by the magnetic moment density dynamics in systems of charged 1/2-spin particles. MQHD equations are a consequence of MPSE, in which particles’ interaction is directly taken into account. In our work we consider the Coulomb and spin–spin interactions. The system of MQHD equations we have constructed comprises equations of continuity, of the momentum balance, of the magnetic moment density evolution, and of the vorticity density dynamics. In our studies of wave processes we have used a self-consistent field approximation of the MQHD equations.

The equations we are interested in, determining the system dynamics, are the hydrodynamics equations for a spinning plasma. These equations (2.31) and (2.32) have an additional quantum contribution proportional to $\hbar^2$ and spin corrections, and additional magnetic moment stress and magnetic moment torque that have been derived in the absence of (thermal) fluctuation of the spin about the macroscopic average. However, in such a situation (thermal) effects on the spin might be important. The main objective of this paper was to construct an appropriate new generalized vorticity equation (2.36) for a spinning quantum plasma that contains magnetic and non-magnetic terms. The turbulent processes in plasmas have been investigated using the vorticity equation [19–21]. We have derived the vortex dynamic formulation of a spinning non-relativistic quantum plasma, using the method of magneto quantum hydrodynamics (MQHD). We have generalized the classical vorticity equation for a spinning quantum fluid plasma and derived the vorticity equation (2.33) in which particles’ interactions (Coulomb and spin–spin) are directly taken into account. It is important that the quantum Madelung potential does not contribute to the vorticity evolution.

Using MQHD equations, we have analyzed elementary excitations in various physical systems in a linear approximation. We have studied the influence of the intrinsic spin of electrons in the nonlinear whistler mode turbulence. Dispersion branches characterize new waves, one of which propagates below the electron cyclotron frequency (2.42), shown in Fig. 1, and one of which propagates above the spin-precession frequency due to the spin perturbations (2.43), shown in Fig. 2. This result has been derived for the incompressible electrons in the model based on the 2D vorticity equation. The spin effects are seen to be substantial in the very strong magnetic field of dense plasmas.

The investigation of this approach leads to knowledge of interesting spin effects of dense quantum plasmas in compact astrophysical objects, plasmas in semiconductors and micro-mechanical systems, and in quantum X-ray free-electron lasers.

Appendix

The terms representing Coulomb and spin–spin interactions in Eqs. (2.31), (2.32), and (2.33) lead to the appearance of the self-consistent electric field $E_{\text{int}}$ and the self-consistent magnetic field $B_{\text{spin}}$:

\begin{align}
\vec{\nabla} E_{\text{int}} &= 4\pi q \rho, \\
\vec{\nabla} \times \vec{E}_{\text{int}} &= 0,
\end{align}

\begin{align}
\vec{\nabla} B_{\text{spin}} &= 0, \\
\vec{\nabla} \times \vec{B}_{\text{spin}} &= 4\pi \vec{\nabla} \times \vec{M}.
\end{align}

The two-particle hydrodynamics functions can be produced using the self-consistent field method. The two-particle functions (2.12) and (2.13) have the ground expressions [1,2]

\begin{align}
\rho_2(\vec{r}, \vec{r}'; t) &= \rho(\vec{r}, t) \rho(\vec{r}', t) + \varrho(\vec{r}, \vec{r}', t), \\
M_2^\alpha(\vec{r}, \vec{r}'; t) &= M^\alpha(\vec{r}, t) M^\alpha(\vec{r}', t) + \chi^\alpha(\vec{r}, \vec{r}', t),
\end{align}

where $\varrho(\vec{r}, \vec{r}', t)$ and $\chi^\alpha(\vec{r}, \vec{r}', t)$ are the correlation functions. We must note that the two-particle functions are the functionals of the wave function $\psi(R, t)$. 

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