POLYNOMIAL SIEGEL DISKS ARE TYPICALLY JORDAN DOMAINS

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Abstract. We prove that for typical rotation numbers polynomial Siegel disks are Jordan domains with boundaries containing at least one critical point.

1. Introduction

Let $f$ be a holomorphic germ which has an indifferent fixed point. Suppose $f$ is linearizable at $p$. The maximal linearizable domain $D$ containing $p$ is called the Siegel disk of $f$ centered at $p$. The topology of the boundaries of Siegel disks is one of the important topics in complex dynamics. For general holomorphic germs, the boundary of the Siegel disk can be very complicated. Recently Chéritat constructed a holomorphic germ with a Siegel disk compactly contained in the germ such that the boundary of the Siegel disk is non-locally connected [2]. For rational maps, however, it has been believed that the Siegel disk is always a Jordan curve. This was conjectured by Douady and Sullivan in early 1980’s and is one of the dominant open problems in this area. The conjecture has been settled in the case that the rotation number of the Siegel disk is of bounded type. Here we say an irrational number $0 < \theta < 1$ is of bounded type if all the coefficients of its continued fraction have a finite upper bound.

Theorem 1 (Douady-Herman [7], Zakeri [24], Shishikura [20], Zhang [25]). All bounded type Siegel disks of rational maps are quasi-disks with at least one critical point on their boundaries.

An irrational number $0 < \theta < 1$ is said to be of Brjuno type if $\sum_n \frac{\log q_{n+1}}{q_n} < \infty$ where $q_n$ are the denominators of the convergents of $\theta$. By Brjuno-Yoccoz theorem, $P_\theta(z) = e^{2\pi i \theta} z + z^2$ has a Siegel disk at the origin if and only if $\theta$ is of Brjuno type. As a consequence of their renormalization theory for parabolic fixed points, Ino and Shishikura recently showed

Theorem 2 (Inou-Shishikura [10]). There is an $N > 1$ such that for any Brjuno number $\theta$, if all the coefficients of the continued fraction of $\theta$ are larger than $N$, then the boundary of the Siegel disk of $P_\theta(z) = e^{2\pi i \theta} z + z^2$ is a Jordan curve.

Note that the sets of both the bounded type and Ino-Shishikura type irrational numbers have zero Lebesgue measure.

In 2002 Petersen and Zakeri proved that for typical rotation numbers, a quadratic Siegel disk is a Jordan domain with a critical point on its boundary. To state their theorem more precisely, let us introduce another type of irrational numbers. Let $C > 0.$ Let $\Theta_C$ denote the set of all irrational numbers $0 < \theta < 1$ such that

\begin{equation}
\log a_n \leq C \sqrt{n}
\end{equation}

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holds for all \( n \geq 1 \) where \( a_1, a_2, \cdots \) are all the coefficients of the continued fraction of \( \theta \).

We say \( 0 < \theta < 1 \) is of David type if \( \theta \in \Theta_C \) for some \( C > 0 \). It is known that the set of David type irrational numbers in \([0, 1]\) has full Lebesgue measure \([11]\).

**Theorem 3** (Petersen-Zakeri [18]). Let \( 0 < \theta < 1 \) be a David type irrational number. Then the Siegel disk of \( P_\theta(z) = e^{2\pi i \theta} z + z^2 \) is a Jordan domain whose boundary contains the unique finite critical point of \( P_\theta \).

The main purpose of this paper is to generalize the above result to polynomial maps of all degrees.

**Main Theorem.** All David type Siegel disks of polynomial maps are Jordan domains with boundaries containing at least one of the critical points.

One of the fundamental tools in our proof is trans-qc surgery. This surgery technique was pioneered by Hainsinsky, who used it to transform an attracting basin into a parabolic basin, and then introduced to the study of Siegel disks by Petersen and Zakeri in [18]. Compared with qc surgery, the main difficulty in performing trans-qc surgery is to verify the integrability of certain degenerate Beltrami differentials. This often requires some delicate area estimates. In [18] the authors there used Petersen puzzles to obtain the desired estimate for the Douady-Ghys premodel. This idea was then adapted to some slightly more general situations in [27]. Up to now it is not known if the invariant Beltrami differential is always integrable for an arbitrarily given premodel. This is the essential challenge in generalizing Petersen-Zakeri’s theorem to polynomial maps of higher degrees.

The idea of our proof can be sketched as follows. Suppose \( C > 0 \) is a fixed constant and \( D \) is a David type Siegel disk of an arbitrary polynomial map with rotation number \( \theta \in \Theta_C \). By perturbing \( \theta \) we get a sequence of bounded type Siegel disks \( D_N \) with rotation numbers \( \theta_N \in \Theta_C \) such that \( \theta_N \to \theta \). By Shishikura’s theorem, each \( \partial D_N \) is a quasi-circle passing through at least one of the critical points. We shall see if \( \theta \) is not of bounded type, the qc constants of \( \partial D_N, N \geq 1 \), are not bounded. This means that the oscillations of these quasi-circles cannot be uniformly controlled with respect to the qc constants. Thus nothing could be obtained if we let \( N \) go to \( \infty \) at this point. The key of our proof is to find an appropriate way to measure the oscillations of these quasi-circles so that in this way, the oscillations can be uniformly controlled. To do this, we will introduce a sequence of oscillation functions. We prove that these oscillation functions are uniformly bounded for bounded type Siegel disks of a class of special polynomial maps with the rotation numbers belonging to \( \Theta_C \). This is the place where we use trans-qc surgery. We then show that for a bounded type Siegel disk of an arbitrary polynomial map with the rotation number belonging to \( \Theta_C \), the oscillation functions can be controlled, in certain sense, by those for the special ones. From this we derive that the oscillations of the sequence of quasi-circles are uniformly bounded. By passing to a subsequence if necessary, we may assume that the sequence of quasi-circles converge to some Jordan curve which passes through at least one of the critical points. This Jordan curve must be the boundary of the David type Siegel disk. This proves the Main Theorem.

The following is the organization of the paper.

In §2 we present a detailed outline of the proof. We first formulate a reduced version of the Main Theorem by introducing oscillation functions. We then state four key lemmas. The proofs of these four lemmas form the core part of the paper. Finally we prove the Reduced Main Theorem by assuming these four lemmas.

In §3 we prove that the Reduced Main Theorem implies the Main Theorem.
In §4 we prove the first key lemma. This lemma says that the oscillation of the boundaries of bounded type Siegel disks for a class of special polynomial maps, with rotation numbers belonging to $\Theta_C$, are uniformly bounded. The proof uses essentially the idea of trans-qc surgery.

In §5 we prove the third and fourth key lemmas. These two lemmas are used to construct a chain of slices in the parameter space. Each of these slices is an algebraic Riemann surface determined by a finite system of polynomial equations. We use this chain of Riemann surfaces to establish a connection between an arbitrary Siegel polynomial map to those special ones.

In §6 we establish a topological characterization of a class of polynomial maps with bounded type Siegel disks (cf. Theorem 2.1). This class of Siegel polynomial maps play a key role in this work. The proof of Theorem 2.1 contains most of the ingredients needed in the proof of the Key-Lemma 2. After the proof of Theorem 2.1, the Key-Lemma 2 follows by a little more effort. The Key-Lemma 2 is used to perturb certain Siegel polynomial map so that the resulted one can be embedded into an appropriate slice in the parameter space.

In §7, the Appendix of the paper, we present a list of basic properties about bounded type Siegel disks of polynomial maps. Among these the most important one is Shishikura’s theorem that all bounded type Siegel disks of polynomial maps are quasi-disks with qc constants depending only on the degree and the rotation number. From this we easily derive that when the rotation number is fixed, the boundary of bounded type Siegel disks moves continuously. This property plays an important role in our proof.

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2. Outline of the Proof

Fix an integer $d \geq 2$ and a David type irrational number $0 < \theta < 1$ throughout the paper. Let $[a_1, \ldots, a_n, \ldots]$ be the continued fraction of $\theta$. By definition we have $C > 0$ such that

$$\log a_n \leq C\sqrt{n}$$

for all $n \geq 1$. Let

$$P(z) = e^{2\pi i \theta}z + \alpha_2 z^2 + \cdots + \alpha_d z^d$$

with $\alpha_d \neq 0$. We want to show that the Siegel disk of $P$ centered at the origin is a Jordan domain with at least one critical point on its boundary.

For the above $C > 0$, recall that $\Theta_C$ denotes the set of all the irrational numbers $0 < \theta < 1$ such that $\log a_n \leq C\sqrt{n}$ for all $n \geq 1$. Let

$$\Theta_C^b \subset \Theta_C$$

denote the subset consisting of all the bounded type irrational numbers in $\Theta_C$. Let $\theta_N \in \Theta_C^b$, $N \geq 1$, be a sequence such that $\theta_N \to \theta$ as $N \to \infty$. Such sequence can be constructed in many ways. To fix the idea let us take

$$\theta_N = [a_1, \ldots, a_N, 1, 1, 1, \ldots].$$

For each $N \geq 1$, let

$$P_N(z) = e^{2\pi i \theta_N}z + \alpha_2 z^2 + \cdots + \alpha_d z^d.$$ 

Then $P_N$ converges to $P$ uniformly in any compact set of the complex plane. It follows that the critical sets of all $P_N$ are contained in a neighborhood of that of $P$, and therefore contained in a compact set of the plane. Let $D_N$ denote the Siegel disk of $P_N$ centered
at the origin. Since $\theta_N$ is of bounded type, by Shishikura’s theorem ([20], see also [25]), there is a critical point $c_N$ of $P_N$ and a $K_N > 1$ depending only on
\[
\sup_{1 \leq k \leq N} \{a_k\},
\]
such that $\partial D_N$ is a $K_N$-quasi-circle and passes through $c_N$. By taking a subsequence, we may assume that $c_N$ converges to some critical point $c$ of $P$. Note that $K_N \to \infty$ if $\sup_{k \geq 1} \{a_k\} = \infty$. Because otherwise, by taking a subsequence, $\partial D_N$ would converge to a quasi-circle passing through $c$. This quasi-circle must be the boundary of the Siegel disk of $P$ centered at the origin. But by a result of Petersen [17], the rotation number of such Siegel disk must be of bounded type. This is a contradiction.

Let
\[
Q = e^{-1}P(cz) \text{ and } Q_N(z) = e^{-1}N(c_Nz).
\]
Then 1 is a critical point of both $Q$ and $Q_N$. Let us still use $D_N$ to denote the Siegel disk of $Q_N$ centered at the origin. It follows that $1 \in \partial D_N$ for all $N \geq 1$.

The main task of our proof is to show that the sequence of curves $\partial D_N$ converge to a Jordan curve passing through the critical point 1. Since the quasiconformal constant $K_N$ is unbounded, we need to find an appropriate way to measure the oscillation of these curves such that in this way, the oscillation of the curves can be uniformly controlled. This is the key of the work. More precisely,

**Reduced Main Theorem.** There exist a pair of positive functions $\eta, \lambda : (0, 2] \to \mathbb{R}^+$ satisfying
\[
\lim_{\delta \to 0^+} \lambda(\delta) = \lim_{\delta \to 0^+} \eta(\delta) = 0
\]
such that for any $N \geq 1$, any pair of integers $k > m \geq 0$, and any pair of positive numbers $0 < \delta' < \delta$, if
\[
\delta' < |e^{2\pi i k \theta_N} - e^{2\pi i m \theta_N}| < \delta,
\]
then
\[
(3) \quad \eta(\delta') \leq |Q_N^k(1) - Q_N^m(1)| \leq \lambda(\delta).
\]

**Proposition 2.1.** The Reduced Main Theorem implies the Main Theorem.

The proof of Proposition 2.1 will be given in §3. The main task of the paper is to prove the Reduced Main Theorem. The proof is based on four key lemmas.

**Definition 2.1.** Let $0 < \alpha < 1$ be a bounded type irrational number. Let
\[
\Pi_{\alpha, d}^{geom}
\]
denote the class of all polynomial maps $f$ of degree $d$ such that
1. $f$ has a Siegel disk centered at the origin with rotation number $\alpha$,
2. the point 1 is a critical point of $f$,
3. all the finite critical points of $f$ are contained in the boundary of the Siegel disk.

We will give a topological characterization of a more general class of Siegel polynomial maps (cf. Theorem 2.1). We will see that each $f \in \Pi_{\alpha, d}^{geom}$ is uniquely determined by the $d - 2$ angles between 1 and all the other $d - 2$ critical points. The first key lemma states that the Reduced Main Theorem holds for these special Siegel polynomial maps. Recall that $\Theta_C^b$ is the subset of $\Theta_C$ consisting of all bounded type irrational numbers in $\Theta_C$. 
**Key-Lemma 1.** There exist a pair of positive functions $\eta_1, \lambda_1 : (0, 2] \to \mathbb{R}^+$ satisfying
\[
\lim_{\delta \to 0^+} \lambda_1(\delta) = \lim_{\delta \to 0^+} \eta_1(\delta) = 0,
\]
such that for any $f \in \bigcup_{\alpha \in \Theta_C^b} \Pi_{\text{geom}}^{\alpha,d}$, any pair of integers $k > m \geq 0$ and any pair of positive numbers $0 < \delta' \leq \delta$, the inequality
\[
\eta_1(\delta') \leq |f^k(1) - f^m(1)| \leq \lambda_1(\delta)
\]
holds.

The fundamental tool in the proof of Key-Lemma 1 is trans-qc surgery. The proof is based on Lemmas 2.1–2.3. Before we state the three lemmas, let us introduce some notations first. For any $f \in \bigcup_{\alpha \in \Theta_C^b} \Pi_{\text{geom}}^{\alpha,d}$, we will show that there is a Blaschke product $B_f$ of degree $(2d - 1)$ which models $f$. In particular, all the critical points of $B_f$, except $0$ and $\infty$, are contained in $\mathbb{T}$. Let $R_\alpha : z \mapsto e^{2\pi i \alpha}z$ be the rigid rotation given by $\alpha$ and $h_f : \mathbb{T} \to \mathbb{T}$ be the circle homeomorphism such that
\[
h_f \circ (B_f|_\mathbb{T}) \circ h_f^{-1} = R_\alpha.
\]
Since $\alpha$ is of bounded type, by Herman’s theorem, $h_f : \mathbb{T} \to \mathbb{T}$ is a quasisymmetric circle homeomorphism. Because the qc constants can not be uniformly controlled, instead of making a usual quasiconformal extension of $h_f$ and then performing a qc surgery, we will construct a David extension $H_f : \Delta \to \Delta$ of $h_f$ by adapting the idea in [18] and then perform a trans-qc surgery. Since $\alpha$ is of bounded type, the map $H_f$ obtained in this way is necessarily quasiconformal. The key point here is that we regard $H_f$ as a David homeomorphism when we measure its distortion.

**Lemma 2.1.** There exist $M, \beta > 0$ and $0 < \epsilon_0 < 1$ depending only on $C$ and $d$ such that for any $f \in \bigcup_{\alpha \in \Theta_C^b} \Pi_{\text{geom}}^{\alpha,d}$, the conjugation map $h_f : \mathbb{T} \to \mathbb{T}$ has a David extension $H_f : \Delta \to \Delta$ which fixes the origin and satisfies the following. For any $0 < \epsilon < \epsilon_0$, we have
\[
\text{area}\{ z \in \Delta \mid |\mu_{H_f}(z)| > 1 - \epsilon \} < Me^{-\beta}
\]
where area$(\cdot)$ denotes the area with Euclidean metric.

The key idea in the proof of Lemma 2.1 is that $h_f$ has a uniform saddle-node geometry. One of the key tools in our proof is Herman’s uniform estimate on the distortion of cross-ratios for the compact family
\[
\{ B_f \mid f \in \bigcup_{\alpha \in \Theta_C^b} \Pi_{\text{geom}}^{\alpha,d} \}.
\]
We would like to point out that the compactness of the family $(4)$ plays an essential role in the proof. The corresponding family in constructing Siegel disks of rational maps is not compact, even for quadratic rational maps. In this case Herman’s uniform estimate does not hold and the conjugation map $h_f$ does not have uniform saddle-node geometry. We believe Lemma 2.1 still holds in this general situation though we do not know how to prove it. This is the main difficulty to overcome if one wants to use the idea in this paper to generalize the Main Theorem to all rational maps.
Now let $H_f : \Delta \to \Delta$ be the David homeomorphism in Lemma 2.1. Define

$$(5) \quad \tilde{B}_f(z) = \begin{cases} B_f(z) & \text{for } z \in \hat{C} \setminus \Delta, \\ H_f^{-1} \circ R_0 \circ H_f(z) & \text{for } z \in \Delta. \end{cases}$$

Let $\mu_f$ denote the Beltrami differential on the whole plane which is obtained by pulling back $\mu_{H_f}$ through the iteration of $\tilde{B}_f$.

**Lemma 2.2.** There exist $\tilde{M}, \tilde{\beta} > 0$ and $0 < \tilde{\epsilon}_0 < 1$ depending only on $C$ and $d$ such that for any $f \in \bigcup_{\alpha \in \Theta_C^d} \Pi^\alpha_d$, we have

$$\text{area}\{z \in \mathbb{C} \mid |\mu_f(z)| > 1 - \epsilon\} < \tilde{M}e^{-\frac{\tilde{\beta}}{\epsilon}}$$

for any $0 < \epsilon < \tilde{\epsilon}_0$.

Lemma 2.2 asserts the uniform integrability of the invariant Beltrami differentials for all Blaschke premodels in $[4]$. As we mentioned before the main difficulty in performing a trans-qc surgery is to verify the integrability of certain degenerate Beltrami differential. The key idea used in the proof of Lemma 2.2 is a method developed in [27] which allows us to obtain the desired area estimate.

**Lemma 2.3** (Tukia, [21]). Let $\mathcal{F}$ denote the class of all $(\tilde{M}, \tilde{\beta}, \tilde{\epsilon}_0)$-David homeomorphisms of the plane to itself which fix 0 and 1. Then there exist three constants $\tilde{M}, \tilde{\beta} > 0$ and $0 < \tilde{\epsilon}_0 < 1$ depending only on $\tilde{M}, \tilde{\beta}$ and $\tilde{\epsilon}_0$ such that any sequence in $\mathcal{F}$ has a subsequence which converges uniformly to a $(M, \hat{\alpha}, \hat{\epsilon}_0)$-David homeomorphism of the plane which fixes 0 and 1.

The Key-Lemma 1 is a direct consequence of Lemmas 2.1 - 2.3 (cf. §4). We would like to point out that in the case of cubic polynomial maps, the Reduced Main Theorem follows from Key-Lemma 1. The following is the detailed argument.

Let $\alpha \in \Theta_C^d$, and let $\mathcal{P}_\alpha^d$ denote the space of all cubic polynomials which have a Siegel disk at the origin with rotation number $\alpha$ and a critical point at 1. Let $c$ denote the other critical point. Then $\mathcal{P}_\alpha^d$ is parameterized by $c$ and under this parametrization, $\mathcal{P}_\alpha^d$ is homeomorphic to the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For each $c \in \mathbb{C}^*$, let $f_c$ denote the corresponding cubic polynomial and $D_c$ denote the Siegel disk of $f_c$ centered at the origin. For any two integers $k > m \geq 0$, define

$$\sigma_{k,m}(c) = f_c^k(1) - f_c^m(1).$$

Since $f_c$ depends holomorphically on $c$ when $c$ varies in $\mathbb{C}^*$, $\sigma_{k,m}$ is a holomorphic function in $\mathbb{C}^*$. By a result of Zakeri (cf. §14 of [24]), there is a Jordan curve $\Gamma \subset \mathbb{C}^*$ such that

1. $\Gamma$ separates 0 and $\infty$, passes through 1 and is invariant under $c \to 1/c$,
2. for all $c$ belonging to the inside of $\Gamma$ and not equal to 0, $\partial D_c$ passes through the critical point $c$ only; for all $c$ belonging to the outside of $\Gamma$ and not equal to $\infty$, $\partial D_c$ passes through the critical point 1 only; for all $c$ belonging to $\Gamma$, $\partial D_c$ passes through both 1 and $c$.

For the $\alpha$ given and $d = 3$, let $\lambda_1$ and $\eta_1$ be the two positive functions in the Key-Lemma 1. Note that $\Gamma$ corresponds to the class $\Pi_{\text{geom}}^{\alpha,3}$ by the second assertion above. So for any pair of integers $k > m \geq 0$ and any pair of positive numbers $0 < \delta' \leq \delta$ satisfying

$$\delta' \leq |e^{2\pi i k \alpha} - e^{2\pi i m \alpha}| \leq \delta,$$

we have

$$\eta_1(\delta') \leq |\sigma_{k,m}(c)| \leq \lambda_1(\delta) \quad \text{for all } c \in \Gamma.$$ 

Since 1 belongs to $\partial D_c$ for $c$ belonging to the outside of $\Gamma$, $\sigma_{k,m}$ does not vanish in the outside of $\Gamma$. Noting that as $c \to \infty$, $f_c$ converges uniformly to a quadratic polynomial,
it follows that $\sigma_{k,m}$ has a removable singularity at infinity. This implies that $\sigma_{k,m}$ is a holomorphic function in the outside of $\Gamma$ and does not vanish. So both the maximal and minimal principles apply. It follows that $\sigma_{k,m}$ holds for all $c$ belonging to the outside of $\Gamma$.

The Reduced Main Theorem for cubic polynomials follows by taking $\lambda = \lambda_1$ and $\eta = \eta_1$.

The argument above, however, does not work for polynomial maps of degree $d \geq 4$. The reason is as follows. For each $\alpha \in \Theta_C$, a Siegel polynomial map in $\Pi_{\text{geom}}^{\alpha,d}$ is uniquely determined by the $d - 2$ angles between 1 and all the other $d - 2$ finite critical points. This means that there is a bijection between $\Pi_{\text{geom}}^{\alpha,d}$ and the set $S^1 \times \cdots \times S^1 / G_{d-2}$, where $G_{d-2}$ is the permutation group of order $d - 2$, and in this sense, the set $\Pi_{\text{geom}}^{\alpha,d}$ has real dimension $d - 2$. Thus for $d \geq 4$, the set $\Pi_{\text{geom}}^{\alpha,d}$ can not bound a domain in the parameter space $C^* \times \cdots \times C^*$. So for $d \geq 4$, we can not use maximal and minimal principles directly. To solve this problem, we will introduce certain slices in the parameter space. Each slice is an algebraic Riemann surface determined by a system of polynomial equations. We will apply the maximal and minimal principles successively on a chain of such slices, and finally prove that the oscillation of the boundary a Siegel disk for an arbitrary polynomial map, in certain sense, can be controlled by the oscillation of the boundaries of Siegel disks for those Siegel polynomial maps in the class $\Pi_{\text{geom}}^{\alpha,d}$. Since we have proved that the later can be uniformly controlled, the Reduced Main Theorem follows. The construction of these Riemann surfaces relies on the other three key lemmas. Before we state them, let us first introduce some notations.

**Definition 2.2.** Let $0 < \alpha < 1$ be an irrational number of bounded type and $d \geq 2$ be an integer. We use $\Sigma_{\text{geom}}^{\alpha,d}$ to denote the class of all degree $d$ polynomial maps with a Siegel disk $D$ of rotation number $\alpha$ such that each critical point either is attracted to some periodic attracting cycle, or eventually lands on a super-attracting periodic cycle, or belongs to the boundary of the Siegel disk.

It is clear that

$$\Pi_{\text{geom}}^{\alpha,d} \subset \Sigma_{\text{geom}}^{\alpha,d}.$$ 

In §6 we will establish a topological characterization of all the polynomial maps in $\Sigma_{\text{geom}}^{\alpha,d}$. Suppose $g \in \Sigma_{\text{geom}}^{\alpha,d}$ such that the boundary of the Siegel disk contains more than one critical point. The Key-Lemma 2 states that one can always perturb $g$ in $\Sigma_{\text{geom}}^{\alpha,d}$ so that after the perturbation all critical points in the boundary of the Siegel disk satisfy an orbit relation.

**Key-Lemma 2.** Let $g \in \Sigma_{\text{geom}}^{\alpha,d}$. Suppose $g$ has two or more distinct critical points on the boundary of the Siegel disk, say $c_1, \ldots, c_m$, where $m \geq 2$. Then for any $\epsilon > 0$, there is a $\tilde{g} \in \Sigma_{\text{geom}}^{\alpha,d}$ such that

1. $\|g - \tilde{g}\| < \epsilon$ where $\|g - \tilde{g}\|$ denotes the maximal absolute value of all the coefficients of $g - \tilde{g}$,
2. $\tilde{g}$ has exactly $m$ distinct critical points on the boundary of the Siegel disk, say $\tilde{c}_i$, $1 \leq i \leq m$, such that $|\tilde{c}_i - \tilde{c}_1| < \epsilon$ for all $1 \leq i \leq m$, 

3. there are integers $k_i$, $2 \leq i \leq m$, such that $\hat{g}^{k_i}(1) = \hat{c}_i$ for $2 \leq i \leq m$.

The proof of Key-Lemma 2 is based on Theorem 2.1 which gives a topological characterization of all the maps in $\Sigma^\alpha_{geom}$. The theorem is an extension of Thurston’s characterization theorem for post-critically finite rational maps. Before we state the theorem, let us introduce some terminologies first. A topological polynomial $f$ is a branched covering map of the sphere to itself and of finite degree such that $f^{-1}(\infty) = \{\infty\}$. Let $\mathcal{O} = \{x_1, \cdots, x_p\}$ be a periodic cycle of $f$ with period $p$. We say $\mathcal{O}$ is a holomorphic attracting cycle if (1) $f$ is holomorphic in an open neighborhood $U$ of $\mathcal{O}$, and (2) $|Df^p(x_1)| < 1$, and (3) $\mathcal{O}$ attracts at least one infinite critical orbit of $f$.

**Definition 2.3.** Let $0 < \alpha < 1$ be a bounded type irrational number. Let $\Sigma^\alpha_{top}$ denote the class of all topological polynomials of degree $d$ such that

1. $f$ has at least one critical point in $\mathbb{T}$,
2. $f|\Delta$ is the rigid rotation given by $z \to e^{2\pi i \alpha} z$,
3. any critical point of $f$ either is attracted to some holomorphic attracting cycle of $f$, or eventually lands on a periodic cycle containing some critical point, or belongs to $\mathbb{T}$.

Let $P_f$ denote the closure of the union of all critical orbits of $f$. We say a map $f \in \Sigma^\alpha_{top}$ is CLH-equivalent to a map $g \in \Sigma^\alpha_{geom}$ if there exist two homeomorphisms $\phi$ and $\psi$ of the plane to itself such that

1. $\phi|\Delta = \psi|\Delta$ is holomorphic,
2. for each holomorphic attracting cycle $\mathcal{O}$, there is an open neighborhood $U$ of $\mathcal{O}$ such that $\phi|U = \psi|U$ are holomorphic,
3. $\phi$ is isotopic to $\psi$ rel $P_f \cup \bigcup_i U_i$ where $U_i$s are open neighborhoods of all holomorphic attracting cycles,
4. $\phi \circ f = g \circ \psi$.

**Theorem 2.1.** A map $f \in \Sigma^\alpha_{top}$ is CLH-equivalent to a map $g \in \Sigma^\alpha_{geom}$ if and only if $f$ has no Thurston obstructions in the exterior of $\Delta$. Such $g$ if exists, must be unique up to a linear conjugation.

In §6 we will prove Theorem 2.1 and derive the Key-lemma 2 from it.

Before we state the other two key lemmas, let us present some preliminaries first. Let $0 < \alpha < 1$ be a bounded type irrational number. Let $\mathcal{P}_\alpha^d$ denote the class of all the polynomial maps $f$ such that

$$f(z) = e^{2\pi i \alpha} z + \alpha_2 z + \cdots + \alpha_d z^d$$

with $\alpha_d \neq 0$ and $f'(1) = 0$. Since $f$ has $d - 1$ critical points (counting by multiplicities) and at least one of them is contained in the boundary of the Siegel disk centered at the origin, there are at most $d - 2$ attracting periodic cycles. Note that such $f$ is uniquely determined by the set of its critical points. More precisely, for each $(d - 1)$-tuple

$$X = (c_1, \cdots, c_{d-1}), \ c_i \in \mathbb{C}^* \text{ for } 1 \leq i \leq d - 2, \text{ and } c_{d-1} = 1,$$

there is a unique $f \in \mathcal{P}_\alpha^d$ such that $X$ is the critical set of $f$. By a simple calculation, we have

$$f(z) = \sum_{i=1}^d a_i z^i$$

with

$$a_i = e^{2\pi i \alpha} \cdot \frac{(-1)^{i-1}}{i} \cdot \frac{Q_{d-i}(c_1, \cdots, c_{d-1})}{c_1 \cdots c_{d-1}}$$

(7)
where $Q_{d-i}$ is the degree-$(d-i)$ elementary polynomials of $c_1, \ldots, c_{d-1}$. Let us denote such $f$ by

$$f_{c_1, \ldots, c_{d-1}}$$

Let us recall some basic notions of algebraic functions. For more knowledge in this aspect, the reader may refer to [1]. Suppose $P(w, z)$ is an irreducible polynomial. Suppose $P$ has degree $m$ with respect to $w$. That is,

$$P(w, z) = b_0(z)w^m + b_1(z)w^{m-1} + \cdots + b_m(z)$$

where $b_i(z)$ are polynomials of $z$ and $b_0(z)$ is not identically zero. Let $R(z)$ be the resultant of $P(w, z)$ and $P_w(w, z)$. Let

$$Z = \{z \in \mathbb{C} \mid R(z) = 0 \text{ or } b_0(z) = 0 \}.$$ 

Then for any $z \in \mathbb{C} \setminus Z$, there are exactly $m$ distinct $w$ such that $P(w, z) = 0$. Thus the equation

$$P(w, z) = 0$$

determines a multi-valued analytic function $w = w(z)$ in the sense $P(w(z), z) = 0$. For each $z_0 \in Z$, there are two cases.

In the first case, $R(z_0) = 0$. Then there is a $w_0 \in \mathbb{C}$ such that $P(w_0, z_0) = 0$ and $P_w(w_0, z_0) = 0$. In this case, there is an integer $p \geq 2$ such that in a small neighborhood of $(w_0, z_0)$,

$$w(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^{k/p}.$$ 

In the second case, $b_0(z) = 0$. Then there are $l < m$ finite complex numbers $w$, counting by multiplicity, such that $P(w, z_0) = 0$. In this case, for $z$ near $z_0$ and $w$ near infinity,

$$w(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^{k/p}$$

where $p = m - l$ and $n \geq 1$ is some integer.

The set

$$\{(w, z) \in \mathbb{C}^2 \mid P(w, z) = 0, z \notin Z \}$$

consist of a Riemann surface embedded in $\mathbb{C}^2$. Note that for each $z_0 \in Z$, there may exist $w_0 \in \mathbb{C}$ such that $P(w_0, z_0) = 0$ and in a small neighborhood of $(w_0, z_0)$, $w(z)$ is an analytic function of $z$. After we add all such points to the above set, the resulted set is again an Riemann surface. We call this Riemann surface the algebraic Riemann surface determined by $P(w, z) = 0$. We call $Z$ the set of algebraic singularities.

Suppose now we have $l \geq 2$ irreducible polynomials $P_i(w_i, z)$ with $Z_i$ being the set of algebraic singularities of $P_i$. Then $Z = \bigcup_{1 \leq i \leq l} Z_i$ is a finite set. Suppose $(w_0', \ldots, w_0', z_0) \in \mathbb{C}^{l+1}$ such that $z_0 \notin Z$ and for each $1 \leq i \leq l$, $P_i(w_0', z_0) = 0$. Now let $z$ run through all possible paths which start from $z_0$ and do not pass through points in $Z$, we can simultaneously continue all $w_i$ to multi-valued analytic functions on $\mathbb{C} \setminus Z$. The set

$$\{(w_1(z), \ldots, w_l(z), z)\}$$

obtained in this way consist of a Riemann surface which is locally parameterized by $z$. The Riemann surface obtained in this way may have punctures $(w'_1, \ldots, w'_l, z'_0)$ with $z'_0 \in Z$. That is, each $w_i$ can be extended to an analytic function in a small neighborhood of $(w_i', z_0')$. Let us add all such punctures to the above set. The resulted set is again a Riemann surface locally parameterized by $z$. 


Let \( f \in \mathcal{P}_d^a \). The Key-Lemma 3 states one can always embed \( f \) in an algebraic Riemann surface in the parameter space

\[
\mathbb{C}^* \times \cdots \times \mathbb{C}^* \\
(\text{\textit{(d-2) copies}})
\]

such that all the periodic attracting cycles of \( f \) (if there are) are preserved with the same multipliers when moving \( f \) along the Riemann surface.

**Key-Lemma 3.** Let \( f \in \mathcal{P}_d^a \) and \( 1 \leq l \leq d - 3 \). Suppose \( f \) has \( l \) attracting cycles with multipliers \( t_1, \ldots, t_l \). Then there exist \( l \) irreducible polynomials

\[
P_i(x, y), \quad 1 \leq i \leq l,
\]

and a way to label the critical points of \( f \) as

\[
c_i^0, \ldots, c_d^{0}, c_{d-1}^0 = 1,
\]

such that for any \( (d-1) \)-tuple

\[
X = (w_1(c_{l+1}), \ldots, w_l(c_{l+1}), c_{l+1}, c_{l+1}^0, \ldots, c_{d-2}^0, 1)
\]

where \( w_i, 1 \leq i \leq l \), are algebraic functions of \( c_{l+1} \) determined by

\[
P_i(w_1, c_{l+1}) = 0 \text{ and } w_i(c_{l+1}) = c_i^0, \quad 1 \leq i \leq l,
\]

the polynomial \( f_X \) has \( l \) attracting cycles which depend holomorphically on \( c_{l+1} \) and has constant multiplies \( t_1, \ldots, t_l \).

Now consider an \( f \in \Sigma_{\text{geom}}^{d,a} \). Assume that \( f \) has \( l + 1 \) periodic attracting cycles and all the critical points in the boundary of the Siegel disk satisfy some orbit relations. The Key-Lemma 4 says by sacrificing one of the attracting periodic cycles, one can embed \( f \) in an algebraic Riemann surface in the parameter space such that when the parameters vary in this algebraic Riemann surface, the other \( l \) attracting cycles and their multipliers, and all the orbit relations among the critical points on the boundary of the Siegel disk are preserved.

**Key-Lemma 4.** Let \( f \in \Sigma_{\text{geom}}^{d,a} \) and \( 0 \leq l \leq d - 3 \). Suppose

1. \( f \) has \( l + 1 \) attracting cycles with multiplies \( t_1, \ldots, t_{l+1} \) (each of them attracts exactly one of the critical points),
2. \( f'(1) = 0 \) and 1 belongs to the boundary of the Siegel disk of \( f \),
3. there are other \( d-l-3 \) critical points, say \( c_1^0, \ldots, c_{d-l-3}^0 \), on the boundary of the Siegel disk, and \( d-l-3 \) integers \( k_i \geq 1 \), such that \( f^{k_i}(1) = c_i^0 \) for \( 1 \leq i \leq d-l-3 \).

Then there exist a way to label the critical points of \( f \) as

\[
c_i^0, \ldots, c_{d-2}^0, c_{d-1}^0 = 1
\]

and \( d-3 \) irreducible polynomials

\[
P_i(x, y), \quad 1 \leq i \leq d - 3,
\]

such that for any \( (d-1) \)-tuple

\[
X = (w_1(c_{d-2}), \ldots, w_{d-3}(c_{d-2}), c_{d-2}, 1)
\]

where \( w_i, 1 \leq i \leq d - 3 \), are algebraic functions of \( c_{d-2} \) determined by

\[
P_i(w_1, c_{d-2}) = 0 \text{ and } w_i(c_{d-2}) = c_i^0, \quad 1 \leq i \leq d - 3,
\]

the polynomial \( f_X \) has \( l \) attracting cycles which depend holomorphically on \( c_{d-2} \) and has constant multiplies \( t_1, \ldots, t_l \), and moreover, \( f_X^{k_i}(1) = w_i(c_{d-2}) \) for \( 1 \leq i \leq d-l-3 \).
In particular, when \( l = d - 3 \), there is only one critical point, which is 1, on the boundary of the Siegel disk and thus there are no orbit relations. In this case, only one periodic attracting cycles are preserved and with same multipliers.

The proofs of Key-Lemmas 3 and 4 will be given in §5. The main idea of the proofs is Euclidean algorithm. In the following we will prove the Reduced Main Theorem by assuming the four Key-Lemmas. It suffices to prove the following proposition which implies the Reduced Main Theorem as an immediate consequence.

**Proposition 2.2.** Let \( C > 0 \) and \( d \geq 2 \). Then there exist a pair of positive functions \( \eta, \lambda : (0, 2] \to \mathbb{R}^+ \) satisfying

\[
\lim_{\delta \to 0_+} \lambda(\delta) = \lim_{\delta \to 0_+} \eta(\delta) = 0,
\]

such that for any \( \alpha \in \Theta_C^b \), if \( f \) is a polynomial of degree \( d \) with a Siegel disk centered at the origin and a critical point at 1 such that the boundary of the Siegel disk contains the point 1, then for any pair of integers \( k > m \geq 0 \) and any pair of positive numbers \( 0 < \delta' \leq \delta \) satisfying \( \delta' \leq |e^{2\pi i k \alpha} - e^{2\pi i m \alpha}| \leq \delta \), the inequality

\[
\eta(\delta') \leq |f^k(1) - f^m(1)| \leq \lambda(\delta)
\]

holds.

For \( d = 2 \), Proposition 2.2 is equivalent to the Key-Lemma 1. Suppose \( d \geq 3 \) and assume that Proposition 2.2 holds for polynomials of degrees less than \( d \); that is, there exist a pair of positive functions \( \eta_0, \lambda_0 : (0, 2] \to \mathbb{R}^+ \) satisfying

\[
\lim_{\delta \to 0_+} \lambda_0(\delta) = \lim_{\delta \to 0_+} \eta_0(\delta) = 0,
\]

such that for any \( \alpha \in \Theta_C^b \), if \( f \) is a polynomial of degree less than \( d \) with a Siegel disk centered at the origin and a critical point at 1 such that the boundary of the Siegel disk contains the point 1, then for any pair of integers \( k > m \geq 0 \) and any pair of positive numbers \( 0 < \delta' \leq \delta \) satisfying \( \delta' \leq |e^{2\pi i k \alpha} - e^{2\pi i m \alpha}| \leq \delta \), the inequality

\[
\eta_0(\delta') \leq |f^k(1) - f^m(1)| \leq \lambda_0(\delta)
\]

holds. Let us prove this also holds for polynomials of degree \( d \). Proposition 2.2 then follows by induction.

Take an arbitrary small \( \epsilon > 0 \). Let \( \alpha \in \Theta_C^b \) and

\[
f(z) = e^{2\pi i \alpha} z + \alpha_2 z^2 + \cdots + \alpha_d z^d
\]

with \( \alpha_d \neq 0 \) and \( f'(1) = 0 \). Assume that 1 belongs to the boundary of the Siegel disk centered at the origin. The remaining proof is divided into three steps.

In the first step, we will construct a finite chain of Riemann surfaces in the parameter space which connects \( f \) to some polynomial map \( g \) such that the boundary of the Siegel disk of \( g \) centered at the origin contains only the critical point 1, and moreover, \( g \) has \( d - 2 \) periodic attracting cycles each of which attracts exactly one of the other critical points of \( g \). The main tool used in this step is Key-Lemma 3.

In the second step, we will construct a finite chain of Riemann surfaces in the parameter space to connect \( g \) to some polynomial map in the class \( \Pi_{\text{geom}}^{\alpha,d} \). The main tools used in this step are Key-Lemma 2 and Key-Lemma 4.

In the third step, we draw the conclusion.

**Step I.** Assume that the number of the periodic attracting cycles of \( f \) is less than \( d - 2 \). Otherwise, we go to Step II directly. Let us label the critical points of \( f \) as

\[
c_1^0, \ldots, c_{d-2}^0, c_{d-1}^0 = 1.
\]
Fix $c^0_1, \ldots, c^0_{d-2}$ and $c^0_{d-1} = 1$. We will repeat the following process at most $d - 2$ times. Each time we will get some polynomial map which has at least one more periodic attracting cycle.

Assume that $f$ has $l$ attracting periodic cycles with $0 \leq l \leq d - 3$. In the case $l = 0$, that is, $f$ has no periodic attracting cycles, we just embed $f$ into the one-parameter holomorphic family

$$h_{c^1} = f_{c_1, c^0_2, \ldots, c^0_{d-2}, 1}, \ c_1 \in \mathbb{C}^*.$$  

For the expression of $f_{c_1, c^0_2, \ldots, c^0_{d-2}, 1}$, see (75). Otherwise, we have $1 \leq l \leq d - 3$. Then by Key-Lemma 3, we can embed $f$ into the holomorphic family

$$h_{c^{l+1}} = f_{c_1, \ldots, c_l, c_{l+1}, c^0_{l+2}, \ldots, c^0_{d-2}, 1}$$

which is locally parameterized by $c_{l+1}$, where all $c_i = w_i(c_{l+1})$, $1 \leq i \leq l$, are algebraic functions of $c_{l+1}$ determined respectively by the $l$ irreducible polynomial equations

$$P_i(c_1, c_{l+1}) = 0, \ 1 \leq i \leq l.$$ 

Since $c_1, 1 \leq i \leq l$, are all multi-valued functions of $c_{l+1}$, the notation $h_{c^{l+1}}$ is a simplified notation and it just means that the family depends locally on $c_{l+1}$.

Let $k > m \geq 0$ be two integers. Recall that

$$\sigma_{k, m}(h_{c^{l+1}}) = h^{k}_{c^{l+1}}(1) - h^{m}_{c^{l+1}}(1).$$

The function $\sigma_{k, m}(h_{c^{l+1}})$ can be regarded as a single-valued holomorphic function on a finitely many sheeted Riemann surface $R$ branched over the $c_{l+1}$-punctured-plane ($\mathbb{C}^*$) with finitely many branch points, and on each sheet, there are finitely many punctures corresponding to the non-algebraic poles of the $l$ algebraic functions $c_i = w_i(c_{l+1})$, $1 \leq i \leq l$. Recall that a point $z$ is called an algebraic pole of an algebraic function if it is not only a pole but also a branch point. Let $S_1$ denote the set of all the poles (algebraic and non-algebraic) and $S_2$ the set of all the branch points which are not algebraic poles. Although each point in $R$ is determined by which sheet it lies in as well as its projection onto the $c_{l+1}$-punctured-plane, to simplify the notation, in the following discussion we just use its $c_{l+1}$-coordinate to denote the point when no confusion is caused. There are two cases.

In the first case, $h_{c^{l+1}}$ is $J$-stable at $c^{0}_{l+1}$. Let $U \subset R$ (In the case $l = 0$, $R = \mathbb{C}^*$) be the component in which $J_{h_{c^{l+1}}}$ moves holomorphically. This implies that the critical point $1$ always stays on the boundary of the Siegel disk for all $c^{l+1} \in U$ and thus the holomorphic function $\sigma_{k, m}(h^{\sigma_{l+1}}_{c^{l+1}})$ does not vanish in $U$. Take a sequence $c^{\sigma}_{l+1}$ in $U$ such that $|\sigma_{k, m}(h^{\sigma}_{c^{l+1}})|$ converges to the supremum. By taking a subsequence we may assume that the sequence $c^{\sigma}_{l+1}$ converges to some boundary point $c^{\sigma}_{l+1}$ of $U$ (the same argument works for the infimum for the function $\sigma_{k, m}(h_{c^{l+1}})$ does not vanish in $U$). Since for all the parameters in $U$, the boundary of the Siegel disk centered at the origin passes through the critical point $1$, by the second assertion of Lemma 7.4 the critical points of $h_{c^{l+1}}$ are uniformly bounded away from the origin for all the parameters in $U$. Thus we have the following three subcases.

Subcase I. $c^{\sigma}_{l+1} = \infty$ or $c^{\sigma}_{l+1} \in S_1$. By Lemma 7.4 there is some polynomial map $g$ of degree less than $d$ for which the boundary of the Siegel disk contains the critical point $1$ such that

$$|\sigma_{k, m}(f)| \leq \lim_{n \to \infty} |\sigma_{k, m}(h^{\sigma}_{c^{l+1}})| = |\sigma_{k, m}(g)| \leq \lambda_0(\delta).$$

the last inequality comes from our induction assumption.

Subcase II. $c^{\sigma}_{l+1}$ is a regular boundary point of $U$, that is, it is neither a pole nor a branch point. Then $h_{c^{l+1}}$ is not $J$-stable at $c^{\sigma}_{l+1}$. Thus by McMullen-Sullivan’s Theorem,
one can take a $c_{1+1}$, which can be arbitrarily close to $c_{1+1}^*$, such that $h_{c_{1+1}}$ has at least one more periodic attracting cycle than $h_{c_{1+1}^*}$. Let $D_{c_{1+1}^*}$ and $D_{c_{1+1}}$ denote respectively the Siegel disks of $h_{c_{1+1}^*}$ and $h_{c_{1+1}}$ centered at the origin. Note that it is possible that $1 \notin \partial D_{c_{1+1}}$. In this case, $\partial D_{c_{1+1}}$ contains some critical point $c$.

Since $1 \in \partial D_{c_{1+1}^*}$, we have $\text{diam}(D_{c_{1+1}^*}) \geq 1$. Since the boundary of the Siegel disk moves continuously by Lemma 7.4, by taking $c_{1+1}$ close enough to $c_{1+1}^*$ we can make sure that $\text{diam}(D_{c_{1+1}}) > 1$. Now from the second assertion of Lemma 7.4 we have some $L > 1$ depending only on $d$ such that

\begin{equation}
L \geq \text{diam}(D_{c_{1+1}}) \geq |c| \geq \text{diam}(D_{c_{1+1}^*})/L > 1/L.
\end{equation}

By taking $c_{1+1}$ close enough to $c_{1+1}^*$, we can make the $c$ arbitrarily close to $\partial D_{c_{1+1}^*}$, and thus by taking an appropriate integer $p \geq 0$, we can make $h_{c_{1+1}^*}^p(c)$ arbitrarily close to $1$. So for the given $\epsilon > 0$, by taking $c_{1+1}$ close enough to $c_{1+1}^*$, and letting $k' = k + p$ and $m' = m + p$, we have

$$
|\sigma_{k,m}(h_{c_{1+1}^*})| = |h_{c_{1+1}^*}^{k'}(1) - h_{c_{1+1}^*}^m(1)| \leq |h_{c_{1+1}^*}^{k'}(c) - h_{c_{1+1}^*}^m(c)| + \epsilon.
$$

Let

$$
h(z) = c^{-1} h_{c_{1+1}}(cz).
$$

Then the boundary of the Siegel disk of $h$ centered at the origin passes through the critical point $1$. Since $|c| \leq L$ by (10), we have

$$
|h_{c_{1+1}^*}^{k'}(c) - h_{c_{1+1}^*}^m(c)| \leq L \cdot |\sigma_{k',m'}(h)|.
$$

This means

$$
|\sigma_{k,m}(h_{c_{1+1}^*})| \leq L \cdot |\sigma_{k',m'}(h)| + \epsilon.
$$

We finally get

\begin{equation}
|\sigma_{k,m}(f)| \leq \lim_{n \to \infty} |\sigma_{k,m}(h_{c_{1+1}^*})| = |\sigma_{k,m}(h_{c_{1+1}^*})| \leq L \cdot |\sigma_{k',m'}(h)| + \epsilon.
\end{equation}

**Remark 2.1.** To get the lower bound of $|\sigma_{k,m}(f)|$, besides the minimal principle instead of the maximal principle for holomorphic functions, we need the last inequality of (10). In this case, (11) becomes $|\sigma_{k,m}(f)| > |\sigma_{k',m'}(h)|/L - \epsilon$.

Subcase III. $c_{1+1}^* \in S_2$. We may assume that $c_{1+1}^*$ is not an isolated boundary point. This is because otherwise, in a small punctured neighborhood of $c_{1+1}^*$, by using a coordinate transformation $c_{1+1} = \zeta^j + c_{1+1}^*$ where $j \geq 2$ is the branch degree at $c_{1+1}^*$,

$$
F(\zeta) = \sigma_{k,m}(h_{c_{1+1}^*})(\zeta^j + c_{1+1}^*)
$$

is a holomorphic function of $\zeta$ in an open punctured neighborhood of the origin. As $\zeta \to 0$, $F(\zeta)$ converges to $\sigma_{k,m}(h_{c_{1+1}^*})$ which is a finite complex number. Thus $\zeta = 0$ is a removable singularity of $F(\zeta)$. Since $|F(\zeta)|$ obtains maximal at $\zeta = 0$, it follows that $F(\zeta)$ is a constant in a small neighborhood of the origin. It follows that $\sigma_{k,m}(h_{c_{1+1}^*})$ is a constant in $U$. Since there are only finitely many branched points and poles, we can assume that $c_{1+1}^* = \infty$ or is a regular boundary point of $U$. Hence we go back to Subcase I or Subcase II and hence either (9) or (11) holds.

Now assume that $c_{1+1}^*$ is not an isolated point in $\partial U$. Since there are only finitely many poles and branch points, we can take a regular boundary point of $U$, say $c_{1+1}$, in an arbitrarily small neighborhood of $c_{1+1}^*$. Since the Julia set of $h_{c_{1+1}^*}$ moves holomorphically in $U$, the critical point $1$ belongs to the boundaries of the Siegel disks for all the parameters in $U$. This, together with Lemma 7.4 implies that for both $h_{c_{1+1}}$ and $h_{c_{1+1}^*}$,
the boundaries of the Siegel disks centered at the origin contain the critical point 1. For the given \( \epsilon > 0 \), by taking \( c_{l+1} \) close enough to \( c_{l+1}' \) we can make sure that

\[
|\sigma_{k,m}(h_{\tilde{c}_{l+1}})| < |\sigma_{k,m}(h_{\tilde{c}_{l+1}'}(h))| + \epsilon.
\]

Since \( c_{l+1} \) is a regular boundary point of \( U \), using the same argument as in Subcase II, we can find a polynomial map \( h \in \mathcal{P}^d_\alpha \) which has at least one more periodic attracting cycle than \( h_{\tilde{c}_{l+1}} \) and for which the boundary of the Siegel disk centered at the origin contains the critical point 1, and moreover, there are two integers \( k' > m' \geq 0 \) with \( k' - m' = k - m \) such that

\[
|\sigma_{k,m}(h_{\tilde{c}_{l+1}})| < L \cdot |\sigma_{k',m'}(h)| + \epsilon.
\]

So finally we have

(12) \[ |\sigma_{k,m}(f)| \leq |\sigma_{k,m}(h_{c_{l+1}})| < |\sigma_{k,m}(h_{\tilde{c}_{l+1}'})| + \epsilon < L \cdot |\sigma_{k',m'}(h)| + 2\epsilon. \]

In the second case, \( h_{c_{l+1}} \) is \( J \)-unstable at \( c_{l+1}' \). Note that in a small neighborhood all \( c_i, 1 \leq i \leq l \), are holomorphic functions of \( c_{l+1} \). Thus by regarding \( c_{l+1}' \) as \( c_{l+1} \) and using the same argument as in the Subcase II of the first case, we can find a polynomial map \( h \in \mathcal{P}^d_\alpha \) which has at least one more periodic attracting cycle than \( h_{c_{l+1}} \), and for which the boundary of the Siegel disk centered at the origin contains the critical point 1, and moreover, there are two integers \( k' > m' \geq 0 \) with \( k' - m' = k - m \) such that (11) holds.

Now we repeat the above process on \( h \). Since each time the number of the attracting periodic cycles is increased by at least one, there are two possibilities. The first one is that we get an inequality like (11) at some step, and therefore get

(13) \[ |\sigma_{k,m}(f)| \leq L \cdot (\cdot \cdot (L \cdot \lambda_0(\delta) + 2\epsilon) + \cdot \cdot \cdot ) + 2\epsilon, \]

where the number of the recursive steps is not greater than \( d - 3 \). The second one is that we get a pair of integers \( k' > m' \geq 0 \) with \( k' - m' = k - m \) and a polynomial map \( g \in \mathcal{P}^d_\alpha \) which has \( d - 2 \) periodic attracting cycles, each of which attracts a critical point, and moreover, the boundary of the Siegel disk centered at the origin contains the critical point 1, such that

(14) \[ |\sigma_{k,m}(f)| \leq L \cdot (\cdot \cdot (L \cdot |\sigma_{k',m'}(g)| + 2\epsilon) + \cdot \cdot \cdot ) + 2\epsilon, \]

where the number of the recursive steps is not greater than \( d - 2 \).

If the first possibility occurs, then we get (13) and we go to **Step III** directly. So suppose that the second possibility occurs and we get the above polynomial map \( g \) which satisfies (14).

**Step II.** Note that \( g \) has \( d - 2 \) attracting cycles and exactly one critical point, 1, on the boundary of the Siegel disk centered at the origin. We will repeat the following process by induction.

Suppose \( g \) has \( l + 1 \) attracting cycles and \( d - l - 2 \) critical points on the boundary of the Siegel disk which satisfy orbit relations as given in Key-Lemma 4. By Key-Lemma 4, we can embed \( g \) in a holomorphic family

\[ h_{c_{d-2}} = f_{c_1, \ldots, c_{d-3}, c_{d-2}, 1} \]

such that \( h_{c_{d-2}} \) has \( l \) attracting cycles with constant multipliers \( t_1, \ldots, t_l \), and all the orbit relations among the critical points on the boundary of the Siegel disk, are preserved, that is, \( h_{c_{d-2}}^{k_i}(1) = c_i, 1 \leq i \leq d - l - 3 \), where \( c_i, 1 \leq i \leq d - 3 \), are determined by \( d - 3 \) irreducible polynomial equations \( P_i(c_i, c_{d-2}) = 0 \). Note that we start from \( l = d - 3 \) and the point 1 is the only critical point on the boundary of the Siegel disk. So in the beginning there are no orbit relations among critical points on the boundary of the Siegel disk.
As before, $\sigma_{k',m'}(h_{c_{d-2}})$ is a holomorphic function defined on a finitely many sheeted Riemann surface $R$ branched over the punctured $c_{d-2}$-plane $(\mathbb{C}^*)$. As in Step I let $S_1$ denote the set of all the poles (including algebraic and non-algebraic poles) and $S_2$ the set of all the branch points which are not poles. Let $S = S_1 \cup S_2$. Then $S$ is a finite set.

Let $\Sigma \subset R \setminus S$ be the subset which contains all those points for which the boundary of the Siegel disk of $h_{c_{d-2}}$ contains exactly those $d - l - 2$ critical points, $c_1, \ldots, c_{d-l-3}, 1$, which satisfy the orbit relations. We claim that $\Sigma$ is an open set. Let us prove the claim. Suppose $c_{d-2} \in \Sigma$. Let $D_{c_{d-2}}$ denote the Siegel disk of $h_{c_{d-2}}$ centered at the origin. Then all the critical points of $h_{c_{d-2}}$ which do not belong to $\partial D_{c_{d-2}}$, have a positive distance from $\partial D_{c_{d-2}}$. By Lemma [23], in a small open neighborhood of $c_{d-2}$, all these critical points are still bounded away from $\partial D_{c_{d-2}}$. It suffices to show that the critical points, which satisfy the orbit relations, still belong to $\partial D_{c_{d-2}}$. Let us prove this by contradiction. Note that except 1, there are exactly $d - l - 3$ critical points, $c_1, \ldots, c_{d-3}$ on the boundary of the Siegel disk, and $d - l - 3$ integers, $1 \leq k_1 < \cdots < k_{d-l-3}$ such that $f^{k_i}(1) = c_i$, and moreover, there are $l$ attracting cycles with multipliers $t_1, \ldots, t_l$. If the claim were not true, then there would be a sequence $c_{d-2}^n \to c_{d-2}$ such that for $h_{c_{d-2}}^n$, at least one of the $d - l - 2$ critical points, $1, c_1^n, \ldots, c_{d-l-3}^n$, does not belong to $\partial D_{c_{d-2}}$. For the convenience, let us denote $c_0^n = 1$. By taking a subsequence, we may assume that there is a an $0 \leq i \leq d - l - 3$ such that $c_i^n \notin \partial D_{c_{d-2}},$ and $c_j^n \in \partial D_{c_{d-2}}$, for all $i + 1 \leq j \leq d - l - 3$. Consider the map

$$F_{c_{d-2}} = h_{c_{d-2}}^{k_i + 1 - k_i}.$$  

The map $F_{c_{d-2}}$ maps $c_j$ to $c_{j+1}$ and the local degree of $F_{c_{d-2}}$ at $c_j$ is equal to that of $h_{c_{d-2}}$ at $c_j$, which is two. This means there is a pair of Jordan domains $A$ and $B$ containing $c_j$ and $c_{j+1}$ respectively such that $F_{c_{d-2}} : A \to B$ is a branched covering map of degree two. Then for all $c_{d-2}^n$ close to $c_{d-2}$ enough, there is a pair of Jordan domains $A_n$ and $B_n$ with $A_n \to A$ and $B_n \to B$ such that $F_{c_{d-2}}^n : A_n \to B_n$ is a branched covering map of degree two. But on the other hand, when $c_{d-2}^n$ is close to $c_{d-2}$, $c_i^n$ is close to $c_{i+1}$ and thus belongs to $B_n$ and $c_j^n$ is close to $c_j$ and thus belongs to $A_n$. Let $z_n \in \partial D_{c_{d-2}}$ be the point such that $F_{c_{d-2}}^n(z_n) = c_i^n$. As $c_i^n \to c_i$, we must have $z_n \to c_i$. Thus as $n$ is large enough, $z_n$ belongs to $A_n$ also. This implies that the preimage of $c_i^n$ in $A_n$ under the map $F_{c_{d-2}}^n$, counting by multiplicities, is at least three. This is a contradiction. Thus $\Sigma$ is an open set and the claim has been proved.

Let $U$ denote the component of $\Sigma$ which contains the point $c_0^n$. Since for all $c_{d-2} \in U$, the point 1 belongs to the boundary of the Siegel disk on which $h_{c_{d-2}}$ is qc-conjugate to the irrational rotation $R_{\sigma}$. $\sigma_{k',m'}(h_{c_{d-2}})$ does not vanish in $U$. So both the maximal and minimal principles apply to $\sigma_{k',m'}(h_{c_{d-2}})$ in $U$. Take a sequence $c_{d-2}^n$ in $U$ such that $\sigma_{k',m'}(h_{c_{d-2}}^n)$ converges to its supremum (The same argument works for the infimum). By taking a subsequence, we may assume that $c_{d-2}^n$ converges to a point $c_{d-2}^* \in \partial U$.

Since for all $c_{d-2} \in U$, the boundary of the Siegel disk of $h_{c_{d-2}}$ centered at the origin passes through the critical point 1, by the second assertion of Lemma [23], all critical points of $h_{c_{d-2}}$ are uniformly bounded away from the origin. Thus we have the following subcases.

Subcase I. $c_{d-2}^* = \infty$ or $c_{d-2}^* \in S_1$. By Lemma [23] there is some polynomial map $h$ of degree less than $d$ for which the boundary of the Siegel disk contains the critical point 1 such that

$$|\sigma_{k',m'}(g)| \leq \lim_{n \to \infty} |\sigma_{k',m'}(h_c^n)| = |\sigma_{k',m'}(h)| \leq \lambda_0(\delta).$$ (15)
Subcase II. $c_{d-2}^*$ is a regular boundary point of $U$. We thus have

$$|\sigma_{k',m'}(g)| \leq |\sigma_{k',m'}(h_{c_{d-2}^*})|.$$  

We claim that $h_{c_{d-2}^*}$ has exactly one more critical point on the boundary of the Siegel disk, that is, there are $d - l - 1$ critical points on $\partial D_{c_{d-2}^*}$. This is because the boundary of the Siegel disk moves continuously by Lemma 7.3, and because all the $d - l - 2$ critical points $c_1^*, \ldots, c_{d-3}^*$, belong to $\partial D_{c_{d-2}^*}$, all the $d - l - 2$ critical points $c_1^*, \ldots, c_{d-1}^*$, 1 must belong to $h_{c_{d-2}^*}$ also. If there is no more critical point on $\partial D_{c_{d-2}^*}$, then $c_{d-2}^* \in \Sigma$. Because $\Sigma$ is open, there exists an open disk neighborhood of $c_{d-2}^*$, say $V$ in $\Sigma$. Since $c_{d-2}^* \rightarrow c_{d-2}^*$ and $c_{d-2}^* \in U$, it follows that $U \cup V$ is a connected and open subset of $\Sigma$. Since $c_{d-2}^* \in V \cap \partial U$, $U \cup V \notin U$. This contradicts the assumption that $U$ is a connected component of $\Sigma$. So there must be at least one more critical point in $\partial D_{c_{d-2}^*}$. Since $h_{c_{d-2}^*}$ has $l$ attracting cycles which attract at least $l$ critical points, it follows that $\partial D_{c_{d-2}^*}$ contains exactly one more critical point. The claim has been proved.

Subcase III. $c_{d-2}^* \in \mathbb{S}_2$. Then as in the Subcase III in Step I, by changing the coordinate $c_{d-2} = c_{d-2}^* + c_j^*$ for some appropriate integer $j$ we may assume that $c_{d-2}^*$ is not an isolated boundary point of $U$, for otherwise, we will go back to Subcases I or II. Thus we can choose a regular boundary point $\hat{c}_{d-2}$ arbitrarily close to $c_{d-2}^*$. By Lemma 7.4 we can take $\hat{c}_{d-2}$ close enough to $c_{d-2}^*$ such that

$$|\sigma_{k',m'}(g)| \leq |\sigma_{k',m'}(h_{\hat{c}_{d-2}^*})| < |\sigma_{k',m'}(h_{c_{d-2}^*})| + \epsilon.$$  

Using the same argument as in the Subcase II, $\partial D_{c_{d-2}^*}$ contains exactly one more critical point. That is, $\partial D_{c_{d-2}^*}$ contains $d - l - 1$ critical points, and has $l$ attracting periodic cycles, each of which attracts exactly one critical point.

Now let us summarize the above three subcases in Step II. Either the Subcase I happens, for which we will finish the Step II, or one of the Subcases II and Subcase III happens, for which we have a polynomial map, say $\tilde{g}$, having one more critical point on the boundary of the Siegel disk and $l$ attracting cycles each of which attracts exactly one of the other critical points, and moreover, $\tilde{g}$ satisfies either (16) or (17), thus we always have

$$|\sigma_{k',m'}(g)| < |\sigma_{k',m'}(\tilde{g})| + \epsilon.$$  

Now applying the Key-Lemma 2 to the map $\tilde{g}$, we get a polynomial map, say $\tilde{\tilde{g}} \in \mathcal{P}_d$, which can be arbitrarily close to $\tilde{g}$ such that all the critical points on the boundary of the Siegel disk satisfy orbit relations, that is, $\tilde{\tilde{g}}^i(1) = c_i$ for $1 \leq i \leq d - l - 2$; and moreover, $\tilde{\tilde{g}}$ has $l$ attracting cycles with the same multipliers as those of $\tilde{g}$. Because $\tilde{\tilde{g}}$ can be arbitrarily close to $\tilde{g}$, we may assume that $|\sigma_{k',m'}(\tilde{g})| < |\sigma_{k',m'}(\tilde{\tilde{g}})| + \epsilon$, and thus by (18) we have

$$|\sigma_{k',m'}(g)| < |\sigma_{k',m'}(\tilde{\tilde{g}})| + 2\epsilon.$$  

Note that $\tilde{\tilde{g}}$ has $l$ attracting cycles and $d - l - 1$ critical points, including the critical point 1, on the boundary of the Siegel disk. Now we repeat the above process for the polynomial map $\tilde{g}$ from the beginning of the Step II. Since each time the number of the critical points on the boundary of the Siegel disk is increased by one, after at most $d - 2$ steps, we can either have the Subcase I and get an inequality like (16), and therefore by (19) we have

$$|\sigma_{k',m'}(g)| < |\sigma_{k',m'}(h)| + 2(d - 2)\epsilon,$$

or after $(d - 2)$ steps we finally get a polynomial map $h \in \Pi_{\text{geom}}^{\alpha,d}$ such that

$$|\sigma_{k',m'}(g)| < |\sigma_{k',m'}(h)| + 2(d - 2)\epsilon - \lambda_1(\delta) - 2(d - 2)\epsilon.$$
The last inequality holds because $k' - m' = k - m$ implies
\[ |e^{2\pi i k'} - e^{2\pi i m'}| = |e^{2\pi i k\alpha} - e^{2\pi i m\alpha}| < \delta \]
and thus $|\sigma_{k',m'}(h)| \leq \lambda_1(\delta)$ by the Key-Lemma 1.

**Step III.** From (13), (14), (20) and (21) we have in all the cases
\[ |\sigma_{k,m}(f)| \leq L \cdot (L \cdot \cdots (L \cdot \max\{\lambda_0(\delta), \lambda_1(\delta)\} + 2\epsilon + \cdots + 2\epsilon) + 2\epsilon, \]
where the number of the recursive steps is 2d times. Since $\epsilon > 0$ is arbitrary, by letting $\epsilon \to 0$, we have
\[ |\sigma_{k,m}(f)| \leq L^{2d} \cdot \max\{\lambda_0(\delta), \lambda_1(\delta)\}. \]

Using the same argument and replacing the maximal principle by the minimal principle (cf. Remark 2.1), we get
\[ |\sigma_{k,m}(f)| \geq L^{-2d} \cdot \min\{\eta_0(\delta), \eta_1(\delta)\}. \]

Now define $\lambda, \eta : (0, 2] \to (0, +\infty)$ by setting
\[ \lambda(x) = L^{2d} \cdot \max\{\lambda_0(x), \lambda_1(x)\} \quad \text{and} \quad \eta(x) = L^{-2d} \cdot \min\{\eta_0(x), \eta_1(x)\}. \]

This completes the proof of Proposition 2.2. The Reduced Main Theorem follows immediately.

3. PROOF OF PROPOSITION 2.1

Let $F : C \to C$ be a continuous map. For any pair of integers $k > m \geq 0$, recall that
\[ \sigma_{k,m}(F) = F^k(1) - F^m(1). \]

Suppose $\lambda, \eta : (0, 1] \to \mathbb{R}^+$ are a pair of positive functions such that
\[ \lim_{\delta \to 0^+} \lambda(\delta) = \lim_{\delta \to 0^+} \eta(\delta) = 0. \]

**Lemma 3.1.** Let $0 < \theta < 1$ be an irrational number. Suppose for any pair of integers $k > m \geq 0$ and any pair of positive numbers $0 < \delta' < \delta$ satisfying
\[ \delta' < |e^{2\pi i k\theta} - e^{2\pi i m\theta}| < \delta, \]
we have
\[ \eta(\delta') \leq |\sigma_{k,m}(F)| \leq \lambda(\delta). \]

Then
\[ \Gamma = \left\{ F^k(1) \right\}_{k=0}^{\infty} \]
is a Jordan curve. Moreover, $F : \Gamma \to \Gamma$ is a topological circle homeomorphism with rotation number $\theta$.

**Proof.** Let $T$ denote the unit circle. Then $X = \{e^{2k\pi i\theta}\}_{k \geq 0}$ is a dense subset of $T$. Define a map $\phi : X \to C$ by setting $\phi(e^{2k\pi i\theta}) = F^k(1)$ for every $k \geq 0$. Then $\phi : X \to C$ is uniformly continuous by assumption. Thus $\phi$ can be uniquely extended to a continuous map from $T$ to $C$. Let us still denote the map by $\phi$. We claim that $\phi$ is injective.

Let us prove it by contradiction. Assume that $\phi(x) = \phi(y)$ for some $x \neq y \in T$. Let $\delta' = \frac{1}{2} |x - y|$. Since $X$ is dense in $T$ and $\phi : T \to C$ is continuous, we have two integers $k > m \geq 0$ such that
\begin{enumerate}
  \item $|e^{2k\pi i\theta} - x| < |x - y|/4,$
  \item $|e^{2m\pi i\theta} - y| < |x - y|/4,$
  \item $|\phi(e^{2k\pi i\theta}) - \phi(x)| < \eta(\delta')/2,$
  \item $|\phi(e^{2m\pi i\theta}) - \phi(y)| < \eta(\delta')/2.$
\end{enumerate}
From (1) and (2) we have $|e^{2k\pi i\theta} - e^{2m\pi i\theta}| > |x - y|/2 = \delta'$. From the assumption, we have $|\phi(e^{2k\pi i\theta}) - \phi(e^{2m\pi i\theta})| > \eta(\delta')$. But from (3), (4) and $\phi(x) = \phi(y)$, we have $|\phi(e^{2k\pi i\theta}) - \phi(e^{2m\pi i\theta})| < \eta(\delta')$. This is a contradiction. This implies that $\phi : \mathbb{T} \to \mathbb{C}$ is injective. Thus

$$\{F^k(1)\}_{k=0}^{\infty} = \phi(X) = \phi(T)$$

is a Jordan curve. This proves the first assertion.

To prove the second assertion, note that $F \circ \phi = \phi \circ R_\theta$ holds on $X$. Since $X$ is dense on $\mathbb{T}$, $\phi$ is continuous on $\mathbb{T}$ and $F$ is continuous on $\mathbb{C}$, the above equation still holds on $\mathbb{T}$. Since $\phi$ is injective on $\mathbb{T}$, it follows that $\phi^{-1} \circ F \circ \phi = R_\theta$ holds on $\mathbb{T}$. This proves the second assertion.

□

Let us now prove Proposition 2.1. Let $Q_N$ and $Q$ be the Siegel polynomial maps in (2). Let $D_N$ and $D$ be respectively the Siegel disks of $Q_N$ and $Q$ centered at the origin. Let $\lambda, \eta : [0, 2] \to \mathbb{R}^+$ be the pair of positive functions in the Reduced Main Theorem. Since $Q_N$ converges to $Q$ uniformly in any compact set of the plane, it follows for any pair of integers $k > m \geq 0$, $\sigma_{k,m}(Q_N)$ converges to $\sigma_{k,m}(Q)$. Since $\theta_N \to \theta$, thus for any pair of integers $k > m \geq 0$, if $\delta' < |e^{2k\pi i\theta} - e^{2m\pi i\theta}| \leq \delta$

for some $0 < \delta' < \delta \leq 2$, then $\delta' < |e^{2k\pi i\theta_N} - e^{2m\pi i\theta_N}| \leq \delta$

for all $N$ large enough. By the Reduced Main Theorem, we thus have $\eta(\delta') < |\sigma_{k,m}(Q_N)| \leq \lambda(\delta)$ for all $N$ large enough. Since $\sigma_{k,m}(Q_N)$ converges to $\sigma_{k,m}(Q)$ as $N \to \infty$, we have $\eta(\delta') \leq |\sigma_{k,m}(Q)| \leq \lambda(\delta)$.

By Lemma 3.1, $\Gamma = \{Q^k(1)\}_{k=0}^{\infty}$ is a Jordan curve contains the critical point 1. By the second assertion, $Q : Gamma \to \Gamma$ is topologically conjugate to the rigid rotation $R_\theta$. This implies that $\Gamma$ bounds a Siegel disk of rotation number $\theta$. It remains to show $\Gamma = \partial D$, i.e., the Siegel disk bounded by $\Gamma$ is the one which is centered at the origin.

To see this, note that $\partial D_N = \{Q^k(1)\}_{k=0}^{\infty}$. From the uniform oscillation condition one can easily derive that $\text{dist}_H(\partial D_N, \Gamma) \to 0$ as $N \to \infty$ where $\text{dist}_H(\cdot, \cdot)$ denotes the Hausdorff distance between two compact sets of the plane. This implies that $\Gamma$ encloses the origin in its inside. Thus the Siegel disk bounded by $\Gamma$ is $D$ and $\partial D = \Gamma$. This implies the Main Theorem.

4. PROOF OF THE KEY-LEMMA 1

4.1. Uniform real bounds. We shall identify the unit circle with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and give $\mathbb{T}$ the induced orientation. Let $f : \mathbb{T} \to \mathbb{T}$ be a homeomorphism. Let $(a, b, c, d)$ be a quadruple with $a < b < c < d < a + 1$. Define the cross-ratio

$$[a, b, c, d] = \frac{b - a}{d - c} \div \frac{c - a}{d - b}$$
and the cross-ratio distortion

\[ D(a, b, c, d, f) = \frac{[f(a), f(b), f(c), f(d)]}{[a, b, c, d]} . \]

Let \( d \geq 2 \) be an integer. Let

\[ \mathcal{H}_d = \{ g(z) = \lambda z^d \prod_{i=1}^{d-1} \frac{1 - \overline{a_i}z}{z - a_i}, \, 0 < |a_i| < 1, g|_I \text{ is a homeomorphism} \}. \]

**Lemma 4.1** (Herman, [9]). There is a \( 0 < C(d) < \infty \) depending only on \( d \) such that for any quadruple \( (a, b, c, d) \), any \( g \in \mathcal{H}_d \) and any integer \( m \geq 1 \), if all the intervals in the collection

\[ \{ f^i(I) \}_{i=0}^{m-1} \]

are disjoint with each other, then

\[ D(a, b, c, d, f^m) < C(d). \]

**Lemma 4.2** (Uniform power law). There exist constants \( \nu, C' > 1 \) such that for any \( g \in \mathcal{H}_d \), if \( c \in \mathbb{T} \) is a critical point of \( g \), then for any \( x, y \in \mathbb{T} \) with \( |x - c| \leq |y - c| \), we have

\[ \left| \frac{g(x) - g(c)}{g(y) - g(c)} \right| \leq C' \left| \frac{x - c}{y - c} \right|^\nu. \]

**Proof.** Since \( \mathcal{H}_d \) is compact, it suffices to prove the lemma in the case that \( y \) and \( x \) are both close to \( c \). Let \( I \) and \( J \) denote respectively the smaller arc intervals which connect \( y \) to \( c \), and \( x \) to \( c \). Then \( |J| \leq |I| \) by assumption. By Lemma 15 of [9], there exists an open neighborhood of \( \mathbb{T} \) depending only on \( d \) on which \( \mathcal{H}_d \) is a normal family. Thus there exist \( 0 < m < M \) and an \( 0 < r < 1 \) which depend only on \( d \) such that for any \( g \in \mathcal{H}_d \) there exists an open neighborhood \( U \) of \( \mathbb{T} \) which contains \( \{ z \mid 1 - r < |z| < 1 \} \) and a holomorphic function \( \phi \) defined on \( U \) such that

\[ g'(z) = \phi(z) \cdot (z - c)^l \cdot \prod_{i \in \Lambda} (z - c_i)^{l_i} \cdot \prod_{i \in \Theta} (z - c_i)^{l_i} \]

where

1. \( m \leq |\phi(z)| \leq M \) for all \( z \in U \),
2. \( \{ c \} \cup \{ c_i, i \in \Lambda \cup \Theta \} = \{ z \in U \mid g'(z) = 0 \} \),
3. \( \text{dist}(c_i, c) \geq 2|I| \) for all \( c_i \in \Lambda \) and \( \text{dist}(c_i, c) \geq 2|I| \) for all \( c_i \in \Theta \).

Note that except 0 and \( \infty \) \( g \) has \( 2(d-1) \) critical points, counting by multiplicities. So there exist constants \( 0 < \alpha(d), \beta(d) < 1 \) depending only on \( d \) and a sub-interval \( I_1 \subset I \) such that

i. \( |I_1| > \beta(d) \cdot |I| \),
ii. \( \text{dist}(c, I_1) > \beta(d) \cdot |I| \),
iii. for any \( c_i \in \Theta \), \( \text{dist}(c_i, I_1) > \alpha(d) \cdot |I| \).

For any \( c_i \in \Theta \) and \( z \in J \), it is clear that \( \text{dist}(c_i, z) \leq \text{dist}(c_i, c) + \text{dist}(c, z) \). Since \( \text{dist}(c_i, c) < 2|I| \) and \( \text{dist}(c, z) \leq |J| \leq |I| \), we have

\[ \sup_{z \in J} \text{dist}(c_i, z) < 3|I|. \]

For any \( c_i \in \Lambda \) and \( z \in J \), it is clear that \( \text{dist}(c_i, z) \leq \text{dist}(c_i, I_1) + \text{dist}(z, I_1) + |I_1| \).

Since \( \text{dist}(z, I_1) \leq \text{dist}(z, c) + \text{dist}(c, I_1) \leq |J| + |I| \leq 2|I| \) and \( \text{dist}(c_i, I_1) \geq \text{dist}(c_i, c) - \text{dist}(c_i, I_1) > \text{dist}(c_i, c) - |I| \geq |I| \), we have

\[ \sup_{z \in J} \text{dist}(c_i, z) < \text{dist}(c_i, I_1) + \text{dist}(z, I_1) + |I_1| \leq \text{dist}(c_i, I_1) + 3|I| < 4\text{dist}(c_i, I_1). \]
Now from the intermediate value theorem we have $\xi \in I_1$ and $\zeta \in J$ such that
\[
|g(I)| \geq |g(I_1)| = |g'(\xi)| \cdot |I_1| = \prod_{i \in \Lambda} |\xi - c_i^{(1)}| \cdots \prod_{i \in \Theta} |\xi - c_i^{(1)}| \cdot |I_1|,
\]
and
\[
|g(J)| = |g'(\zeta)| \cdot |J| = |\phi(\zeta)| \cdot |\xi - c_i^{(1)}| \cdots \prod_{i \in \Theta} |\xi - c_i^{(1)}| \cdot |J|.
\]
Since $g$ is of degree of $2d - 1$ we have $\sum_{i \in \Lambda} l_i \leq 2d - 2$ and $\sum_{i \in \Theta} l_i \leq 2d - 2$. This, together with (i-iii) and $\|\Phi\| \leq 2$, implies
\[
\frac{|g(J)|}{|g(I)|} \leq \frac{|J|^{1+1}}{|I|^{1+1}} \cdot 4^{2d-2} \cdot \left( \frac{3}{\kappa(d)} \right)^{2d-2}.
\]
Since $2 \leq l \leq 2d - 2$ the lemma follows by taking $\nu = 3$ and
\[
C' = \frac{M \cdot 4^{4d-4}}{m \cdot \eta(d)^{2d-1} \cdot \kappa(d)^{2d-2}}.
\]

Let $B \in H_d$ such that the rotation number of $B|\mathbb{T} : \mathbb{T} \to \mathbb{T}$ is an irrational number and the point 1 is a critical point of $B$. It is necessary that the local degree of $B$ at 1 is odd and not less than 3. Let $p_n/q_n$ denote the convergents of the rotation number. Let $x_i$ denote the point in $\mathbb{T}$ such that $B^i(x_i) = 1$. Let
\[
I_n = [1, x_{q_n}] \text{ and } I_{n+1} = [1, x_{q_{n+1}}].
\]
For $i \geq 0$, let $I_n^i$ denote the subintervals of $\mathbb{T}$ such that $B^i(I_n) = I_n$. Then the collections of the intervals
\[
\Pi_n(B) = \{ I_n^i, \ 0 \leq i \leq q_{n+1} - 1 \}; \ I_n^{i+1}, \ 0 \leq i \leq q_n - 1 \}
\]
form a partition of $\mathbb{T}$. We call $\Pi_n(B)$ the dynamical partition of level $n$. For $K > 1$ and $I, J \subset \mathbb{T}$, we say $I$ and $J$ are $K$-commensurable if $|J|/K < |I| / K|J|$.

**Theorem 4.1** (Herman-Swiatak’s uniform real bounds). There is a $1 < C(d) < \infty$ depending only on $d$ such that for any $B \in H_d$ for which $B|\mathbb{T} : \mathbb{T} \to \mathbb{T}$ is a circle homeomorphism with irrational rotation number and has a critical point at 1, we have

1. for any $x \in \mathbb{T}$ and all $n \geq 1$ the two intervals $[x, B^n(x)]$ and $[x, B^{-n}(x)]$ are $C(d)$-commensurable,
2. for all $n \geq 1$, any two adjacent intervals in the dynamical partition of level $n$ are $C(d)$-commensurable.

**Proof.** The first assertion is implied by Proposition 3.3 of [16]. The second assertion is implied by Proposition 3.3, Theorem 3.5 and Corollary 3.6 of [16]. We only need to notice that the constants guaranteed by Proposition 3.3, Theorem 3.5 and Corollary 3.6 of [16] depend only on the constants $C, C'$ and $\nu$ in the two assumptions of the Hypothesis 1 of [16]. By Lemmas 4.1 and 4.3 we can take these three constants depending only on $d$ so that the Hypothesis 1 of [16] is satisfied.

4.2. The family $S_d^\alpha$. Let $0 < \alpha < 1$ be an irrational number and $d \geq 2$ be an integer. Let $\Gamma$ be a Jordan curve and $f : \Gamma \to \Gamma$ be a circle homeomorphism with rotation number $\alpha$. Suppose there exists a homeomorphism $\phi : \Gamma \to \mathbb{T}$ such that
\[
f = \phi^{-1} \circ R_\alpha \circ \phi.
\]
For any two points $x, y \in \Gamma$ we define the *dynamical angle* between $x$ and $y$ to be the angle between $\phi(x)$ and $\phi(y)$ anticlockwise.
Lemma 4.3. For any tuple \((\alpha_1, \ldots, \alpha_{d-1})\) with \(0 \leq \alpha_i < 2\pi\) and \(\sum_{i=1}^{d-1} \alpha_i = 2\pi\), there exists a \(B \in \mathcal{H}_d\) such that

1. there are exactly \(d-1\) critical points \(c_1 = 1, c_2, \ldots, c_{d-1}\) in \(\mathbb{T}\), ordered anticlockwise and counted by multiplicities,
2. the dynamical angle from \(c_i\) to \(c_{i+1}\) anticlockwise is \(\alpha_i\) for \(1 \leq i \leq d-1\) (we identify \(c_1\) with \(c_d\)),
3. the rotation number of \(B|\mathbb{T} : \mathbb{T} \to \mathbb{T}\) is \(\alpha\).

Proof. It is a direct consequence of the proof of Theorem 2.1, cf. §6. \(\square\)

Let \(S^d_\alpha \subset \mathcal{H}_d\) denote the class of all the Blaschke products guaranteed by Lemma 4.3. In §6, we shall see that for any \(\alpha \in \Theta^b_C\), each \(f \in \Pi^d_{\text{geom}}\) is uniquely determined by the group of angles \((\alpha_1, \ldots, \alpha_{d-1})\) formed by the \(d-1\) critical points on the boundary of the Siegel disk. Thus for any \(f \in \Pi^d_{\text{geom}}\), there exists a \(B \in S^d_\alpha\) such that \(f\) is obtained from \(B\) through a qc surgery.

4.3. Uniform saddle node geometry. Most of the arguments and ideas in this subsection are adapted from the appendices of [6] and [18]. Let \(\alpha \in \Theta^b_C\) and \(B \in S^d_\alpha\). Then there is a circle homeomorphism \(h : \mathbb{T} \to \mathbb{T}\) such that

\[B|\mathbb{T} = h^{-1} \circ R_n \circ h.\]

The aim of this subsection is to show that the conjugation map \(h\) exhibits a uniform saddle node geometry which depends only on \(d\) and \(C\). Here we would like to point out that the proof relies essentially on the compactness property of the family \(S^d_\alpha\), and the property does not hold for the corresponding family for rational maps.

Let us recall some notations first. For \(i \geq 0\), we use \(x_i \in \mathbb{T}\) denote the point such that \(B^i(x_i) = 1\) and \(p_n/q_n\) denote the \(n\)-th convergent of \(\alpha\). For each \(n \geq 0\), define

\[Q_n = \{x_i \mid 0 \leq i < q_n\}.\]

Then \(Q_0 = \{1\}\). The following proposition is summarized from §6 of [18] where the circle homeomorphism is induced by the Douady-Ghys Blaschke model which contains exactly one (double) critical point at 1. Since it only involves the combinatorial information about the rotation number and is independent of the number of the critical points in \(\mathbb{T}\), it still holds for \(B \in S^d_\alpha\). Recall that \(\Pi_n(B)\) is the collection of intervals in the dynamical partition of level \(n\) defined in (27).

Proposition 4.1 (cf. §6 of [18]). Let \(0 \leq j < k < q_n\). Then \(x_j\) and \(x_k\) are adjacent in \(Q_n\) if and only if \(k = j + q_n-1\) and \(0 \leq j < q_n-1\), or \(k = j + q_n - q_n-1\) and \(0 \leq j < q_n-1\). In the former case we have

\[x_k, x_j] \cap Q_{n+1} = \{x_k, x_{k+q_n}, x_{k+2q_n}, \ldots, x_{k+(q_n-1)q_n}, x_j\},\]

and in the later case we have

\[x_j, x_k] \cap Q_{n+1} = \{x_j, x_{j+q_n}, x_{j+2q_n}, \ldots, x_{j+(q_n-1)q_n}, x_k\}.\]

Moreover, each interval in \(\mathbb{T} \setminus Q_n\) either is a single interval in \(\Pi_{n-1}(B)\), or is the union of two adjacent intervals in \(\Pi_{n-1}(B)\). In particular, any two adjacent intervals in \(\mathbb{T} \setminus Q_n\) are \(K\)-commensurable with \(K > 1\) being some constant depending only on \(d\).

The last assertion of Proposition 1.1 is implied by the second assertion of Lemma 4.1 that any two adjacent intervals in \(\Pi_{n-1}(B)\) are \(C(d)\)-commensurable with \(C(d) > 1\) being some constant depending only on \(d\).
When $d = 2$, $B$ is exactly the Douady-Ghys Blaschke model considered in [18]. In this case the point 1 is the unique critical point of $B$ in $T$. It is clear that in this case for every interval component $I$ of $T \setminus Q_n$, the map
\[ B^{kn} : I \to f^{kn}(I) \]
is a diffeomorphism.

For $d > 2$, any $B \in S^d$ has $(d-1)$ critical points in $T$, counting by multiplicities. If all these critical points collide into one critical point at 1, then for any interval component $I$ of $T \setminus Q_n$, the map $B^{kn} : I \to I$ is still a diffeomorphism. Otherwise, there are $1 \leq d' \leq d-2$ distinct critical points other than 1. Let us denote them by $c_i, 1 \leq i \leq d'$ and denote 1 by $c_0$. For any integer $k \geq 0$, let $c_i^k \in T$ denote the point such that $B^k(c_i^k) = c_i$. Let
\[ Q_n^i = \{c_i^k | 0 \leq k < q_n\}, 1 \leq i \leq d', n \geq 1. \]
Then $Q_n^i = Q_n = \{x_i | 0 \leq i < q_n\}$.

**Lemma 4.4.** Let $n \geq 1$. Then for each $1 \leq i \leq d'$, we have

1. in the case that $k = j + q_{n-1}$ and $0 \leq j < q_n - q_{n-1}$, the interval $(x_k, x_j)$ contains at most one point in $Q_n^i$,
2. in the case that $k = j + q_n - q_{n-1}$ and $0 \leq j < q_n - q_{n-1}$, the interval $(x_j, x_k)$ contains at most two points in $Q_n^i$.

**Proof.** Fix an $1 \leq i \leq d'$. By replacing $Q_n$ by $Q_n^i$ in Proposition 4.1 it follows that each component of $T \setminus Q_n^i$ either has the form $(c_j^m, c_i^l)$, where $m = l + q_{n-1}$ and $0 \leq l < q_n - q_{n-1}$, or has the form $(c_i^m, c_l^q)$ where $m = l + q_n - q_{n-1}$ and $0 \leq l < q_n - q_{n-1}$.

Suppose $k = j + q_{n-1}$ and $0 \leq j < q_n - q_{n-1}$. By the property of closest returns, $(x_k, x_j)$ can not contain any interval either of the form $(c_j^m, c_i^l)$ with $m = l + q_{n-1}$ and $0 \leq l < q_n - q_{n-1}$, or of the form $(c_i^m, c_l^q)$ with $m = l + q_n - q_{n-1}$ and $0 \leq l < q_n - q_{n-1}$. It follows that $(x_k, x_j)$ contains at most one point in $Q_n^i$. This proves the first assertion.

Suppose $k = j + q_n - q_{n-1}$ and $0 \leq j < q_n - q_{n-1}$. Again by the property of closest returns, the interval $(x_j, x_k)$ can contain at most one interval of the form $(c_j^m, c_i^l)$ with $m = l + q_{n-1}$ and $0 \leq l < q_n - q_{n-1}$ and can not contain any interval with the form $(c_i^m, c_l^q)$ with $m = l + q_n - q_{n-1}$ and $0 \leq l < q_n - q_{n-1}$. It follows that $(x_j, x_k)$ can contain at most two points in $Q_n^i$. This proves the second assertion.

Let $I$ be an interval component of $T \setminus Q_n$. Then the points in $Q_{n+1}$ divide $I$ into finitely many sub-intervals. By Theorem 4.1 any two such sub-intervals are $K$-commensurable for some constant $K > 1$ depending only on $d$. When the number of such sub-intervals are large, however, the ones which lie in the middle position are very small compared to the ones near the end. This is the so called “saddle-node” geometry. More precisely,

**Lemma 4.5** (cf. Theorem 6.6 of [18]). Let $d = 2$. Then there is a universal constant $K > 1$ such that for any interval component $I$ of $T \setminus Q_n$, if the points in $Q_{n+1}$ divide $I$ into sub-intervals
\[ I_1, \ldots, I_m, \]
then we have
\[ \frac{1}{K} \min\{|k|, |k-m+1|\} \leq |I_k| < K \cdot \frac{|I|}{\min\{|k|, |k-m+1|\}}. \]

Now suppose $d > 2$ and $\alpha \in \Theta_C$. Let $B \in S^d\alpha$. In the case that all the critical points of $B$ which belong to $T$ collide to 1, that is, $d' = 0$, the above lemma still holds for some constant $K > 1$ depending only on $d$. This can be derived by taking the limit in the following lemma, where we suppose $d' \geq 1$. 


Lemma 4.6 (Uniform Saddle Node Geometry). Let \( \alpha \in \Theta_k^p \). Then there is a \( K > 1 \) depending only on \( d \) such that the following holds. Suppose \( B \in S_d^d \) such that \( B \) has \( d' \geq 1 \) distinct critical points in \( T \) other than the critical point 1. Then for any component \( I \) of \( T \setminus Q_n \), if \( J \) is a component of

\[
I \setminus \bigcup_{i=1}^{d'} Q_n^i
\]

which contains at least one interval component of \( I \setminus Q_{n+1} \), we have

\[
\frac{1}{K} |I| < |J| \leq |I|.
\]

Moreover, if \( J_1, \cdots, J_m \) denote all the interval components of \( I \setminus Q_{n+1} \) contained in \( J \), labeled by order, then we have

\[
\frac{1}{K} \frac{|J|}{\min\{|k|,|k-m+1|\}^2} < |J_k| < K \cdot \frac{|J|}{\min\{|k|,|k-m+1|\}^2}.
\]

As in the proof of Lemma 4.5 (cf. [6]), the basic tool used in our proof of Lemma 4.6 is the Yoccoz’s almost parabolic lemma. Before we state this lemma we need to introduce a terminology first. Let \( n \) be a positive integer and \( I_1, \cdots, I_n \) be consecutive intervals on the line or circle. According to [9], by an almost parabolic map of length \( n \) and fundamental domains \( I_1, \cdots, I_n \), we mean a negative-Schwarzian diffeomorphism

\[
f: I_1 \cup \cdots \cup I_n \rightarrow I_2 \cup \cdots \cup I_{n+1}
\]

such that \( f(I_j) = I_{j+1} \). The basic geometric estimate on almost parabolic maps is

Lemma 4.7 (Yoccoz’s almost parabolic lemma, cf. [6]). Let \( f: I \rightarrow f(I) \) be an almost parabolic map of length \( n \) and fundamental domains \( I_j, 1 \leq j \leq n \). If \( |I_j| \geq \sigma \cdot |I| \) and \( |I_n| \geq \sigma \cdot |I| \) for some \( \sigma > 0 \), then

\[
\frac{1}{C_0 \min\{j, n-j\}^2} \leq |I_j| \leq C_0 \frac{|I|}{\min\{j, n-j\}^2}
\]

where \( C_0 > 1 \) is a constant depending only on \( \sigma \).

Let us begin the proof of Lemma 4.6. The first assertion of Lemma 4.6 is implied by the following lemma.

Lemma 4.8. Let \( I \) and \( J \) be the intervals in Lemma 4.6. Let \( Q_n^0 = Q_n \). Then for any interval component \( S \) of \( I \setminus Q_{n+1} \), if

\[
S \cap \bigcup_{i=1}^{d'} Q_n^i \neq \emptyset,
\]

then \( |S| > |I|/C \) where \( C > 1 \) is some constant depending only on \( d \).

Proof. The argument is standard. By Proposition 4.1 we have two cases: either \( I = (x_k, x_j) \) where \( k = j + q_n - 1 \) and \( 0 \leq j < q_n - q_{n-1} - 1 \), or \( I = (x_j, x_k) \) where \( k = j + q_n - q_{n-1} - 1 \) and \( 0 \leq j < q_{n-1} - 1 \). In the first case, by [28] \( S \) either has the form \((x_j+(l+1)q_n, x_k+(l+1)q_n)\) for some \( 0 \leq l \leq a_{n+1} - 2 \), or has the form \((x_j+(a_{n+1} - 1)q_n, x_k)\). In the second case, by [29] \( S \) either has the form \((x_j+(l+1)q_n, x_k)\) for some \( 0 \leq l \leq a_{n+1} - 1 \) or has the form \((x_j+(a_{n+1} - 1)q_n, x_k)\). Since the proofs of all these four subcases are similar to each other, let us only deal with the first subcase. With a minor changes of the proof, the reader shall easily prove the remaining three subcases.
Now we can suppose $I = (x_k, x_j)$ where $k = j + q_{n-1}$ and $0 \leq j < q_n - q_{n-1}$ and $S = (x_{k+lq_n}, x_{k+(l+1)q_n})$ for some $0 \leq l < q_{n+1} - 2$. Since $\mathcal{S} \cap \bigcup_{i=0}^{d'} Q_i \neq \emptyset$, there is a least integer $0 \leq t \leq q_n - 1$ and a point $x \in \mathcal{S}$ such that $B^t(x) = c_i$ for some $0 \leq i \leq d'$. For $k \geq 0$, recall that $c_i^{q_n}$ denote the point in $\mathbb{T}$ such that $B^k(c_i^{q_n}) = c_i$.

Consider the following two group of intervals

I. $[c_i, c_i^{q_n}], [c_i^{q_n}, c_i^{q_n-q_{n-1}}], [c_i^{q_n-q_{n-1}}, c_i^{q_n-2q_{n-1}}]$
and

II. $[c_i^{q_{n-1}}, c_i^{q_{n-1}-1}], [c_i^{q_{n-1}-1}, c_i], [c_i, c_i^{q_n}]$.

In the case that $q_n = q_{n-2} + q_{n-1}$, we replace the last interval in the first group by $[c_i^{q_{n-2}}, c_i^{q_{n-2}+q_n}]$. These intervals belong to the collection of the intervals of the dynamical partition of level $n - 1$ with respect to the critical point $c_i$, and moreover, they are adjacent to each other. By Theorem 4.1, these intervals are $C(d)$-commensurable with $C(d) > 1$ being some constant depending only on $d$. Thus the cross ratios of both $(c_i, c_i^{q_n}, c_i^{q_n-q_{n-1}}, c_i^{q_n-2q_{n-1}})$ and $(c_i^{q_{n-1}}, c_i^{q_{n-1}-1}, c_i, c_i^{q_n})$ have a lower bound $\kappa(d) > 0$ depending only on $d$. Pull back the two group of intervals by $B^{-t}$. We get the following two group of intervals

I'. $[x, B^{-q_n}(x)], [B^{-q_n}(x), B^{-q_n+q_{n-1}}(x)], [B^{-q_n+q_{n-1}}(x), B^{-q_n+2q_{n-1}}(x)]$
and

II'. $[B^{-2q_{n-1}}(x), B^{-q_n}(x)], [B^{-q_n}(x), x, B^{-q_n}(x)]$.

Since $0 \leq t < q_n$, the pull backs of each interval in I and II by $B^i, i = 0, 1, \cdots, t$, are disjoint. Thus the intersection multiplicity of the pull backs of each of the two groups is not greater than 3. Now by Lemma 4.1 it follows that the cross ratios of both

$(x, B^{-q_n}(x), B^{-q_n+q_{n-1}}(x), B^{-q_n+2q_{n-1}}(x))$

and

$(B^{-2q_{n-1}}(x), B^{-q_n}(x), x, B^{-q_n}(x))$

have a positive lower bound $\eta(d) > 0$ with $\eta(d) > 0$ being some constant depending only on $d$. This then implies that

$$\|x, B^{-q_n}(x)\| > \lambda(d) \times \|B^{-q_n}(x), B^{-q_n+q_{n-1}}(x)\| \text{ and } \|x, B^{-q_n}(x)\| > \lambda(d) \times \|B^{-q_{n-1}}(x), x\|$$

with $\lambda(d) > 0$ being some constant depending only on $d$.

By assumption we have $I = (x_k, x_j) = (B^{-q_{n-1}}(x), x_j)$ and $x \in I$. Thus we have

$I \subset [B^{-q_{n-1}}(x), x] \cup [x, B^{-q_n}(x)] \cup [B^{-q_n}(x), B^{-q_n+q_{n-1}}(x)]$.

This, together with (32), implies that

$$\|x, B^{-q_n}(x)\| > \frac{\lambda(d)}{2 + \lambda(d)} \cdot |I|.$$
Now let us prove the second assertion of Lemma 4.6. Let \( J_1, \ldots, J_m \) be the intervals in Lemma 4.6. Since any two adjacent interval components in \( T \setminus \mathbb{Q}_{n+1} \) are \( K \)-commensurable for some \( 1 < K < \infty \) depending only on \( d \), it suffices to assume that \( m \geq 4 \) and prove \( J_3, \ldots, J_{m-1} \) satisfies the uniform saddle node geometry described by \( \text{Lemma 4.9} \). Let us consider the diffeomorphism

\[
B^{q_n} : J_3 \cup \cdots \cup J_{m-1} \to J_2 \cup \cdots \cup J_{m-2}.
\]

From Lemma 4.7 we need only to check two conditions. The first one is to show that the two boundary sub-intervals, that is, \( J_3 \) and \( J_{m-1} \), are uniformly commensurable with the whole interval \( J_3 \cup \cdots \cup J_{m-1} \). The second one is to show that \( B^{q_n} \) has negative Schwarz derivative on \( J_3 \cup \cdots \cup J_{m-1} \). Since \( J \supset J_3 \cup \cdots \cup J_{m-1} \supset J_3 \cup J_{m-1} \), the following lemma implies the first condition.

**Lemma 4.9.** There exists a \( \sigma > 0 \) depending only on \( d \) such that \( |J_3| > \sigma \cdot |J| \) and \( |J_{m-1}| > \sigma \cdot |J| \).

**Proof.** Let us prove the first inequality only. The second one can be proved by the same argument. Since \( J_1, J_2 \) and \( J_3 \) are interval components in \( T \setminus \mathbb{Q}_{n+1} \) and adjacent to each other, \( J_3 \) is \( K \)-commensurable with \( J_i \) for some \( 1 < K < \infty \) depending only on \( d \). It suffices to prove that \( |J_1| > \sigma |J| \) for some \( \sigma > 0 \) depending only on \( d \). There are two cases. In the first case \( J_1 \) has a common boundary point with \( J_3 \). Then the boundary point must be a point in \( \bigcup_{i=0}^{d-1} \mathbb{Q}_{n} \). By Lemma 4.8 we have \( |J_1| > |I|/C > |J|/C \) for some \( C > 1 \) depending only on \( d \). In the second case, \( J_1 \) is adjacent to an interval component of \( I \setminus \mathbb{Q}_{n+1} \), say \( S \), which contains a boundary point of \( J_3 \). Again this boundary point must be a point in \( \bigcup_{i=0}^{d-1} \mathbb{Q}_{n} \). By Proposition 4.4 \( S \) and \( J_1 \) are \( K \)-commensurable for some \( K > 1 \) depending only on \( d \). On the other hand, by Lemma 4.8 we have \( |S| > |I|/C > |J|/C \) for some \( C > 1 \) depending only on \( d \). This implies that \( |J_1| > (KC)^{-1} \cdot |J| \). The same argument can be used to prove that \( |J_{m-1}| > \sigma \cdot |J| \) for some \( \sigma > 0 \) depending only on \( d \). This proves the lemma.

It remains to prove that \( B^{q_n} \) has negative Schwarz derivative on \( J_3 \cup \cdots \cup J_{m-1} \). Here when we talk about the Schwarz derivatives of the iterations of \( B \), we regard \( T \) as \( \mathbb{R}/\mathbb{Z} \) and \( B : T \to T \) as its lift \( \tilde{B} : \mathbb{R} \to \mathbb{R} \) and regard the intervals \( J_i \) in \( T \) as its lift \( \tilde{J}_i \) in \( \mathbb{R} \). In this way, \( B \) is real analytic in a strip neighborhood of \( \mathbb{R} \), and moreover, \( B(x+1) = B(x) + 1 \). To simplify the notation we still use \( B \) and \( J_i \) to denote these objects.

**Lemma 4.10.** There is a \( C > 1 \) depending only on \( d \) such that for any \( x \) and \( y \) in \( J_3 \cup \cdots \cup J_{m-1} \) and all \( 1 \leq k \leq q_n \), we have

\[
C^{-1} < \frac{DB^k(x)}{DB^k(y)} < C.
\]

**Proof.** It is known that the map \( B^{q_n} : J_3 \cup \cdots \cup J_m \to J_1 \cup \cdots \cup J_{m-1} \) is a diffeomorphism. That is to say, \( J_3 \cup \cdots \cup J_{m-1} \) contains no critical values of \( B^{q_n} \) in its interior. Since \( J_1 \) and \( J_{m-1} \) are \( K \)-commensurable with \( J_1 \cup \cdots \cup J_{m-1} \) with \( 1 < K < \infty \) being a constant depending only on \( d \) (cf. the proof of Lemma 4.9), there is a Jordan domain \( U \) in the punctured plane \( \mathbb{C} \setminus \{0\} \) such that \( U \cap T = J_1 \cup \cdots \cup J_{m-1} \) and the modulus of the annulus \( U \setminus J_2 \cup \cdots \cup J_{m-2} \) has a positive lower bound depending only on \( d \). Note that \( U \) does not intersect the critical values of \( B^{q_n} \). So \( B^{-q_n} \) can be holomorphically extended to a univalent function on \( U \) which maps \( J_1 \cup \cdots \cup J_{m-1} \) to \( J_2 \cup \cdots \cup J_m \). Let
be the component of \(B^{-q_n}(U)\) which contains \(J_2 \cup \cdots \cup J_m\). Then the modulus of \(V \setminus J_2 \cup \cdots \cup J_{m-1}\) is equal to that of \(U \setminus J_2 \cup \cdots \cup J_{m-2}\) and thus has a positive lower bound depending only on \(d\). It is clear that for every \(1 \leq k \leq q_n\), the map \(B^k\) is univalent in \(V\). The lemma then follows from the Koebe’s distortion theorem.

**Lemma 4.11.** There is an \(N \geq 1\) such that for any \(\alpha \in \Theta^0\), any \(B \in \mathcal{S}^\alpha_d\) and every \(n \geq N\), if \(J_i, 1 \leq i \leq m\), are the intervals in Lemma 4.6 then \(S(B^{q_n})(z) < 0\) for all \(z \in J_3 \cup \cdots \cup J_{m-1}\).

**Proof.** Recall that we regard \(B\) as a holomorphic function defined in a strip neighborhood of \(\mathbb{R}\) such that \(B : \mathbb{R} \to \mathbb{R}\) is a homeomorphism and \(B(x+1) = B(x) + 1\). Since \(\mathcal{S}^\alpha_d\) is compact, one can take a \(\xi > 0\) such that all \(B \in \mathcal{S}_d\) are holomorphic in

\[S = \{x + iy \mid -\xi < y < \xi\}.\]

In particular, \(B'(x)\) is a periodic function with period 1 and all the zeros of \(B'\) are contained in the real line. Since \(B'\) has at most \(d-1\) distinct zeros in each interval \([x, x+1)\), there is a \(0 < \kappa < 1\) such that for each \(B \in \mathcal{S}^\alpha_d\), we can find a \(t \in [0, 1)\) such that all zeros of \(B'\) in \([t - \kappa, t + 1 + \kappa]\) belong to \((t, t+1)\). By symmetry the order of \(B'\) at each zero is even and is not less than two. Let \(c_1, \cdots, c_{d-1}\) denote all the zeros of \(B'\) in \((t, t+1)\), counting by multiplicities, such that the order of \(B'\) at each \(c_i\) is exactly two. Let \(U = \{x + iy \mid t - \kappa < x < t + 1 + \kappa, -\xi < y < \xi\}\). For \(z \in U\), let

\[B'(z) = g(z) \cdot \prod_{1 \leq i \leq d-1} (z - c_i)^2.\]

Then \(g\) is a holomorphic function defined in \(U\). Let \(V = \{x + iy \mid t - \kappa/2 < x < t + 1 + \kappa/2, -\xi/2 < y < \xi/2\}\). Since \(\mathcal{S}_d^\alpha\) is compact, by a compactness argument it follows that there is a \(0 < \eta < 1\) depending only on \(d\) such that for all \(z \in V\), we have

\[i. \ |g(z)| \geq \eta, \]
\[ii. \ |g'(z)| < 1/\eta, \]
\[iii. \ |g''(z)| < 1/\eta. \]

Let \(x \in (t - \kappa/2, t + 1 + \kappa/2)\) and \(x \neq c_i, 1 \leq i \leq d-1\). Let

\[P(x) = \prod_{1 \leq i \leq d-1} (x - c_i).\]

Then \(B'(x) = P(x)^2 \cdot g(x)\). By direct calculations we have

\[B''(x) = 2P(x)P'(x)g(x) + P^2(x)g'(x)\]
\[= 2P^2(x)\Sigma(x)g(x) + P^2(x)g'(x) = P^2(x)(2\Sigma(x)g(x) + g'(x))\]

where

\[\Sigma(x) = \frac{P'(x)}{P(x)} = \sum_{1 \leq i \leq d-1} \frac{1}{x - c_i},\]

and

\[B'''(x) = 2P^2(x)\Sigma(x)(2\Sigma(x)g(x) + g'(x)) + P^2(x)(-2\sigma(x)g(x) + 2\Sigma(x)g'(x) + g''(x))\]

where

\[\sigma(x) = -\Sigma'(x) = \sum_{1 \leq i \leq d-1} \frac{1}{|x - c_i|^2}.\]

Then

\[\frac{B'''(x)}{B'(x)} = 4\Sigma^2(x) + 4\Sigma(x)\frac{g'(x)}{g(x)} - 2\sigma(x) + \frac{g''(x)}{g(x)}.\]
and
\[
\frac{B''(x)}{B'(x)} = 2\Sigma(x) + \frac{g'(x)}{g(x)}.
\]

From
\[
SB(x) = \frac{B''(x)}{B'(x)} - \frac{3}{2} \left( \frac{B''(x)}{B'(x)} \right)^2,
\]
we finally have
\[
SB(x) = -2\Sigma^2(x) - 2\Sigma(x) \frac{g'(x)}{g(x)} - 2\sigma(x) + \frac{g''(x)}{g(x)} - \frac{3}{2} \left( \frac{g'(x)}{g(x)} \right)^2.
\]

Recall that \( x \in (t - \kappa/2, t + 1 + \kappa/2) \) and \( x \neq c_i, 1 \leq i \leq d - 1 \). Let
\[
\delta = \min_{1 \leq i \leq d - 1} |x - c_i| = \operatorname{dist}(x, \Omega_B).
\]

From the properties (i), (ii) and (iii), it follows that there is a \( 0 < C < \infty \) depending only on \( d \) such that
\[
1. \quad -2\Sigma^2(x) < 0,
\]
\[
2. \quad | -2\Sigma(x) \frac{g'(x)}{g(x)} | \leq \frac{C}{5},
\]
\[
3. \quad -2\sigma(x) \leq -\frac{\kappa}{\beta},
\]
\[
4. \quad | \frac{g''(x)}{g(x)} - \frac{3}{2} \left( \frac{g'(x)}{g(x)} \right)^2 | < C.
\]

All of these implies that there is an \( \epsilon > 0 \) depending only on \( d \) such that whenever \( \operatorname{dist}(x, \Omega_B) < \epsilon \), one has
\[
SB(x) < -\frac{1}{\operatorname{dist}^2(x, \Omega_B)}.
\]

Now let us prove the lemma. Let \( \bar{J} = J_3 \cup \cdots \cup J_{m-1} \) and for \( i \geq 0 \) let \( \bar{J}_i = B^i(\bar{J}) \). Let \( x \in \bar{J} \) be an arbitrary point. Let us consider the sum
\[
SB^{\alpha_n}(x) \cdot |ar{J}|^2 = \sum_{j=0}^{\alpha_n-1} SB(B^j(x))(DB^j(x))^2 \cdot |ar{J}|^2.
\]

By Lemma 4.10, it follows that
\[
K_{1}^{-1} \cdot |ar{J}|^2 < (DB^j(x))^2 \cdot |ar{J}|^2 < K_1 \cdot |ar{J}|^2
\]
where \( K_1 > 1 \) is some constant depending only on \( d \).

Let \( U = \{ x | \operatorname{dist}(x, \Omega_B) < \epsilon \} \) and \( V = \{ x | \operatorname{dist}(x, \Omega_B) > \epsilon/2 \} \). By Theorem 4.11 there is an \( N > 0 \) depending only on \( d \) and \( \epsilon \) such that for any \( n \geq N \), any \( B \in S_d \) and any \( x \in \mathbb{T} \), the length of the arc interval \([B^{\alpha_n}(x), x]\) is less than \( \epsilon/2 \). Since \( \bar{J} \) is contained in some arc interval with the form \([B^{\alpha_n}(x), x]\), thus for \( n > N_1 \), each \( \bar{J}_j \) has length less then \( \epsilon/2 \). This implies that each \( \bar{J}_j, 0 \leq j < q_n \), is either contained in \( U \) or contained in \( V \). We split the sum in (37) into \( \Sigma_1 \) and \( \Sigma_2 \): \( \Sigma_1 \) consists of all the terms such that \( \bar{J}_j \) is contained in \( U \), and \( \Sigma_2 \) contains all the other terms.

Note that all the terms in \( \Sigma_3 \) is negative. Recall that in Lemma 4.10 the two boundary points of \( J \) belong to \( \bigcup_{i=0}^{d} Q_n^i \). Let \( J_{m+1} \) be the interval such that \( B^{\alpha_n}(J_{m+1}) = J_m \). From (4.11) it is easy to see that \( \overline{J_{m}} \cap \overline{J_{m+1}} \neq \emptyset \). Then there exists a \( 0 \leq j < q_n \) such that either \( B^j(J_{m-1}) \) or \( B^j(J_{m+1}) \cup B^j(J_m) \) contains a critical point \( c_i \) of \( B \). By Theorem 4.11 in either of the two cases, \( B^j(J_{m-1}) \), \( B^j(J_{m}) \) and \( |c_i^{\alpha_n}, c_i| \) are \( K(d) \)-commensurable with each other. Note that \( |c_i^{\alpha_n}, c_i| \) contains \( \bar{J}_j = B^j(\bar{J}) \). On the other hand, By Theorem 4.11...
$B^j(J_m)$ is $K(d)$-commensurable with $B^j(J_{m-1})$ which is contained in $\tilde{J}_j = B^j(\tilde{J})$. Here $K(d) > 1$ is some constant depending only on $d$. So for any $x \in \tilde{J}$,

$$\text{dist}(B^j(x), \Omega_{\tilde{J}}) \leq |\tilde{J}_j| + |B^j(J_m)| + |B^j(J_{m+1})| < K_2 \cdot |\tilde{J}_j|$$

where $K_2 > 1$ is some constant depending only on $d$. By taking $N$ larger if necessary we may assume that $K \cdot |\tilde{J}_j| < \epsilon$. Thus by (36) we have for such $j$,

$$SB(B^j(x)) < -\frac{1}{K_2^2 \cdot |\tilde{J}_j|^2}.$$ 

This implies that

$$\Sigma_1 < -\frac{1}{K_2^2}$$

provided that $n \geq N$.

On the other hand, for all $x \in V$, $|SB(x)| < C$ for some $0 < C < \infty$ depending only on $d$ and $\epsilon$. Since $\epsilon$ depends on $d$, such $C$ depends eventually on $d$. Since all $\tilde{J}_j$, $0 \leq j < q_n$, are disjoint, we have

$$\sum_{j=0}^{q_n-1} |\tilde{J}_j| < 2\pi.$$ 

By taking $N$ larger if necessary, we can make sure that $|\tilde{J}_j| < K_1^{-1} \cdot K_2^{-2} \cdot C^{-1}$ provided that $n \geq N$. Then for all $n \geq N$, we have

$$\Sigma_2 < C \cdot \sum_{j=0}^{q_n-1} K_1 \cdot |\tilde{J}_j|^2 \leq \frac{1}{K_2^2}.$$ 

This completes the proof of the lemma. 

Now apply Lemma 4.7 to the diffeomorphism $B^{q_n} : J_3 \cup \cdots \cup J_{m-1} \rightarrow J_2 \cup \cdots \cup J_{m-2}$. By Lemmas 4.9 and 4.11 it follows that two conditions in Lemma 4.7 are satisfied. The second assertion of Lemma 4.6 now follows from Lemma 4.7. This completes the proof of Lemma 4.6.

4.4. Constructing qc homeomorphisms between polygons. In this subsection we will introduce the key idea in the proof of Lemma 2.1. Let $\alpha \in \Theta_C$ and $B \in S_d^\alpha$. As in [18] in §4.5 we will give two ways to divide $\Delta$ into countably many polygons, one for the circle homeomorphism $B|T : \mathbb{T} \rightarrow \mathbb{T}$ and the other for the rigid rotation $R_{\alpha} : \mathbb{T} \rightarrow \mathbb{T}$. For each pair of corresponding polygons, we construct a qc homeomorphism between them so that the restriction of the homeomorphism to each edge of the polygon is linear. We then glue all these qc homeomorphisms along the edges of the polygons and get a desired David homeomorphism $H : \Delta \rightarrow \Delta$. Compared with the situation in [18], a slight difference arises here. For the Douady-Ghys’ Blaschke model $G$ used in [18], the bottom side of each polygon for the circle homeomorphism $G|T : \mathbb{T} \rightarrow \mathbb{T}$ consists of a single piece of saddle node geometry, while in our case, the bottom side of each polygon for the circle homeomorphism $B|T : \mathbb{T} \rightarrow \mathbb{T}$ may consist of several pieces of saddle node geometry. Our idea is to divide each polygon into finitely many pieces in an appropriate way so that the bottom side of each new one has one single saddle node geometry. In this way we can apply the idea in [18] to each of such new sub-polygons. The following lemma, which is essentially the Lemma 6.5 in [18], is the fundamental block in this construction.
Lemma 4.12 (cf. Lemma 6.5, [13]). Let $P$ and $Q$ be two unit squares. Let

$$X = \{x_1, x_2, \cdots, x_m\}$$

and

$$Y = \{y_1, y_2, \cdots, y_m\}$$

be two partitions of the two bottom sides of $P$ and $Q$, respectively. Then $P$ and $Q$ become into two polygons by adding the points in $X$ and $Y$ to the set of vertices of $P$ and $Q$ respectively. Suppose the partition $X$ has $C_0$-bounded saddle node geometry for some $C_0 > 1$, that is,

$$\frac{1}{C_0 \min\{i, m-i\}^2} \leq |x_i - x_{i+1}| \leq C_0 \frac{|x_i - x_m|}{\min\{i, m-i\}^2}, \quad 1 \leq i \leq m-1,$$

and $Y$ has the $C_1$-bounded linear geometry for some $C_1 > 1$, that is,

$$\frac{1}{C_1} \cdot \frac{|y_i - y_{m}|}{m} \leq |y_i - y_{i+1}| \leq C_1 \cdot \frac{|y_i - y_m|}{m}, \quad 1 \leq i \leq m-1.$$

Then there is a $K$-qc homeomorphism $F : P \rightarrow Q$ such that when restricted to the corresponding edges, $F$ is linear and

$$K < C \cdot (1 + (\log m)^2)$$

where $C > 1$ is a constant depending only on $C_0$ and $C_1$.

The following lemma, which is a generalized version of Lemma 4.12 is the key of the proof of Lemma 2.3.

Lemma 4.13. Let $P$ and $Q$ be two unit squares. Let

$$X = \{x_1, x_2, \cdots, x_m\}$$

and

$$Y = \{y_1, y_2, \cdots, y_m\}$$

be two partitions of the two bottom sides of $P$ and $Q$, respectively. Then $P$ and $Q$ become into two polygons by adding the points in $X$ and $Y$ to the set of vertices of $P$ and $Q$ respectively. Let $l \geq 1$.

Suppose the partition $X$ consists of $l$ pieces all of which have $C_0$-bounded saddle node geometry and are $C_0$-commensurable with each other, with $C_0 > 1$ being some constant, that is,

1. there exist

$$1 = m_0 < m_1 < m_2 < \cdots < m_{l-1} < m_l = m$$

such that

$$|x_0 - x_m|/C_0 \leq |x_{m_{i-1}} - x_{m_i}| \leq |x_0 - x_m|, \quad 1 \leq i \leq l,$$

and

2. for $0 \leq i \leq m-1$ and $m_i \leq j < m_{i+1}$,

$$\frac{1}{C_0} \cdot \frac{|x_{m_i} - x_{m_{i+1}}|}{\min\{j-m_i+1, m_{i+1}-j\}^2} \leq |x_j - x_{j+1}| \leq C_0 \cdot \frac{|x_{m_i} - x_{m_{i+1}}|}{\min\{j-m_i+1, m_{i+1}-j\}^2}.$$

Suppose in addition that $Y$ has $C_1$-bounded linear geometry for some $C_1 > 1$, that is,

$$\frac{1}{C_1} \cdot \frac{|y_1 - y_m|}{m} \leq |y_j - y_{j+1}| \leq C_1 \cdot \frac{|y_1 - y_m|}{m}, \quad 1 \leq j \leq m-1.$$

Then there is a $K$-qc homeomorphism $F : P \rightarrow Q$ such that when restricted to the corresponding edges, $F$ is a linear map and

$$K < C \cdot (1 + (\log m)^2)$$

where $C > 1$ is a constant depending only on $C_0$, $C_1$ and $l$. 
Proof. Let $A$ and $B$ denote the two vertices of the bottom side of $P$. Let $A'$ and $B'$ denote the two corresponding vertices of the bottom side of $Q$.

If $l = 1$, then the lemma is implied by Lemma 12.

Suppose $l \geq 2$ and the lemma holds for $l - 1$. Let us prove the lemma for $l$. Let us assume that $m_1 - m_0 \geq m_l - m_{l-1}$. The case that $m_1 - m_0 < m_l - m_{l-1}$ can be treated in the same way. Let

$$n = \left\lfloor \frac{m_l - m_0}{m_l - m_{l-1}} \right\rfloor + 16$$

where $\lfloor \cdot \rfloor$ denote the integer part of a number. Then $n \geq 18$.

Claim: There exist $K_1, C_2 > 1$ depending only on $C_0$, and $K_2, C_3 > 1$ depending only on $C_1$, and two group of points $x'_1, \ldots, x'_n$ and $y'_1, \ldots, y'_n$, such that

1. $x'_i = x_i = A$, $x'_{n-3} = x_{m_l-1}$, $x'_n = B$,
2. $x'_j$ lies in the interior of $P$ for all $2 \leq j \leq n - 4$ and $j = n - 2, n - 1$,
3. $y'_i = y_i = A'$, $y'_{n-3} = y_{m_l-1}$, $y'_n = y_{m_l} = B'$,
4. $y'_j$ lies in the interior of $Q$ for all $2 \leq j \leq n - 4$ and $j = n - 2, n - 1$,

and moreover, if $L$ and $L'$ are two polylines connecting $x'_1, \ldots, x'_n$, and $y'_1, \ldots, y'_n$ respectively, then

5. Let $L_1$ be the part of $L$ connecting $A$ and $x_{m_l-1}$, and $L_2$ be the remaining part of $L$. There exist a polyline $S$ between $L_1$ and the straight segment $[A, x_{m_l-1}]$ which consists of three straight segments and connects $A$ and $x_{m_l-1}$ such that the following properties hold. $P$ is divided by $L$ and $S$ into four polygons $P_1$, $P_2$, $P_3$, $P_4$, where $P_1$ is the top one, $P_2$ is the one at the right-lower corner, $P_3$ is the one bounded by $S$ and $L_1$, $P_4$ is the one bounded by $S$ and $[A, x_{m_l-1}]$. Moreover, for each $i = 1, 2, 3, 4$, there is a $K_1$-qc homeomorphism $\phi_i$ mapping $P_i$ to a polygon which is the standard unit square with the bottom side consisting of either a single piece or $(l - 1)$ pieces satisfying $C_2$-bounded saddle node geometry. More precisely, for $P_1$, $L$ is mapped to the bottom side which satisfies $C_2$-bounded saddle node geometry; for $P_2$, $[x_{m_l-1}, B]$ is mapped to the bottom side which satisfies $C_2$-bounded saddle-node geometry; for $P_3$, $L_1$ is mapped to the bottom side which satisfies $C_2$-bounded saddle-node geometry; for $P_4$, $[A, x_{m_l-1}]$ is mapped to the bottom side which consists of $(l - 1)$ pieces all of which satisfy $C_2$-bounded saddle-node geometry and are $C_2$-commensurable with each other. For each $P_i$, $1 \leq i \leq 4$, the map $\phi_i$ is linear on each edge of $P_i$ and maps each edge of $P_i$ to the corresponding edge of the polygon.

6. Let $L'_1$ be the part of $L'$ connecting $A'$ and $y_{m_l-1}$, and $L'_2$ be the remaining part of $L'$. There exists a polyline $S'$ between $L'_1$ and the straight segment $[A', y_{m_l-1}]$ which consists of three straight segments and connects $A'$ and $y_{m_l-1}$ such that the following properties hold. $Q$ is divided by $S'$ and $L'$ into four polygons $Q_1$, $Q_2$, $Q_3$, $Q_4$, where $Q_1$ is the top one, $Q_2$ is the one at the right-lower corner, $Q_3$ is the one bounded by $S'$ and $L'_1$, $Q_4$ is the one bounded by $S'$ and $[A', y_{m_l-1}]$. Moreover, for each $i$, there is a $K_2$-qc homeomorphism $\psi_i$ mapping $Q_i$ to a polygon which is the standard square with the bottom side satisfying $C_3$-bounded linear geometry. More precisely, for $Q_1$, $L'$ is mapped to the bottom side satisfying $C_3$-bounded linear geometry, for $Q_2$, $[y_{m_l-1}, B']$ is mapped to the bottom side satisfying $C_3$-bounded linear geometry; for $Q_3$, $L'_1$ is mapped to the bottom side satisfying $C_3$-bounded linear geometry, for $Q_4$, $[A', y_{m_l-1}]$ is mapped to the bottom side which satisfies $C_3$-bounded linear geometry. For each $Q_i$, $1 \leq i \leq 4$, the map is linear on each edge of $Q_i$ and maps each edge of $Q_i$ to the corresponding edge of the polygon.
Note that to the corresponding edge of $Q$ glue the maps $C_i$ on $K < K$ interior of $a/\times$ by $[\times n \times 1]$ so that the horizontal polyline $P_i \to \psi(\tau_i)$ and satisfying the following properties.

1. $K_0 < C_i \cdot (1 + (\log m)^2)$,
2. each edge of $\phi_i(P_i)$ is linearly mapped to the corresponding edge of $\psi_i(Q_i)$.

Note that $P_i$ is mapped to $Q_i$ by $\psi_i^{-1} \circ \sigma_i \circ \phi_i$ such that each edge of $P_i$ is linearly mapped to the corresponding edge of $Q_i$. Since $\phi_i$ is $K_1$-qc and $\psi_i$ is $K_2$-qc, thus $\psi_i^{-1} \circ \sigma_i \circ \phi_i$ is $K_1 \cdot K_2 \cdot K_0$-qc. Since $K_1$ and $K_2$ depend respectively on $C_0$ and $C_1$ and $K_0$ depends on $C_0$, $C_1$ and $l$, it follows that $K = K_1 \cdot K_2 \cdot K_0$ depends only on $C_0$, $C_1$ and $l$. We can now glue the maps $\psi_i^{-1} \circ \sigma_i \circ \phi_i$ along the edges of $P_i$ and get a $K$-qc map $F : P \to Q$ with $K < K_1 \cdot K_2 \cdot C_4 \cdot (1 + (\log m)^2)$. This proves the lemma by assuming the Claim.

Now let us prove the Claim. Let us first describe how to choose the points $x_i'$, $i = 0, \cdots, n$. Let $a = |[A, x_{m_{i-1}}]|$ and $b = |[x_{m_{i-1}}, B]|$. Let $x_0' = x_0 = A$, $x_{n-3}' = x_{m_{i-1}}$ and $x_n' = B$. Let $x_i'$ be the point in the interior of $P$ such that the length of $[x_0', x_i']$ is equal to $a/2$, and the angle formed by $[x_0', x_i']$ and $[A, B]$ is $\pi/3$. Let $x_{n-4}'$ be the point in the interior of $P$ such that the length of $[x_{n-4}', x_{n-3}']$ is equal to $a/2$ and the angle formed by $[x_{n-4}', x_{n-3}']$ and $[x_{m_{i-1}}, A]$ is equal to $\pi/3$. Then the straight segment $[x_1', x_{n-4}']$ has length $a/2$. There is an obvious way to insert points $x_2', \cdots, x_{n-5}'$ in the interior of $[x_1', x_{n-4}']$ so that the horizontal polyline $[x_1', \cdots, x_{n-4}']$ satisfies A-bounded saddle-node geometry with $A > 1$ being some universal constant.

Now let $x_n' - 2$ be the point in the interior of $P$ such that the length of $[x_{n-3}', x_{n-2}']$ is equal to $b/2$ and the angle formed by $[x_{n-3}', x_{n-2}']$ and $[x_{n-3}', B]$ is $\pi/3$. Let $x_{n-1}'$ be the point in the interior of $P$ such that the length of $[x_n', x_{n-1}']$ is equal to $b/2$ and the angle formed by $[x_n', x_{n-1}']$ and $[B, A]$ is equal to $\pi/3$. Then the length of the horizontal straight segment $[x_{n-2}', x_{n-1}']$ is $b/2$.

Now let us describe how to construct the polyline $S$. Let $s_0 = A$ and $s_3 = x_{m_{i-1}}$. Let $s_1$ be the point in the interior of $P$ such that the length of $[s_0, s_1]$ is equal to $a/3$ and the angle formed by $[s_0, s_1]$ and $[A, B]$ is $\pi/4$. Let $s_2$ be the point in the interior of $P$ such

![Figure 1. Divide $P$ and $Q$ into four polygons with one side satisfying saddle-node and linear geometry respectively.](image-url)
that the length of \([s_2, s_3]\) is equal to \(a/3\) and the angle formed by \([s_2, s_3]\) and \([x_{m_{i-1}}, A]\) is \(\pi/3\). Let \(S\) be the polyline which connects \(s_0, s_1, s_2\) and \(s_3\).

Let \(L_1\) be the part of \(L\) connecting \(x_0\) and \(x_{m_{i-1}}\) and \(L_2\) be the remaining part of \(L\). Then from the construction we see that \(L\) and \(S\) divide \(P\) into four polygons \(P_i, 1 \leq i \leq 4: P_1\) is the top one; \(P_2\) is the one bounded by \(L_2\) and \([x_{m_{i-1}}, B]\); \(P_3\) is the one bounded by \(L_1\) and \(S\); and \(P_4\) is the one bounded by \(S\) and \([A, x_{m_{i-1}}]\). Each of these polygons has four sides, the three of which are straight segments and the last one is a polyline which consists of either a single piece or multiple pieces satisfying \(C_2\)-saddle node geometry with \(C_2 > 1\) depending only on \(C_0\). From the construction of \(L\), the geometry of each \(P_i\) is well controlled and relies only on \(C_0\). From all of these one can easily see the existence of the constant \(K_1 > 0\) depending only on \(C_0\) and the desired \(K_1\)-qc homeomorphisms \(\phi_i, i = 1, 2, 3, 4\).

Let us now describe how to choose the points \(y'_i, i = 0, \ldots, n\). Let \(a' = ||[A', y_{m_{i-1}}]||\) and \(b' = ||[y_{m_{i-1}}, B']||. \) Let \(y'_0 = y_0 = A', \ y'_n = y_{m_{i-1}}\) and \(y'_n = B'\). Let \(n_1 = [n/3]\) and \(n_2 = [2n/3]\). Let \(y'_1\) be the point in the interior of \(Q\) such that the length of \([y'_0, y'_n]\) is \(\pi/3\), and the angle formed by \([y'_0, y'_n]\) and \([A', B']\) is \(\pi/3\). Let \(y'_2\) be the point in the interior of \(Q\) such that the length of \([y'_2, y'_{n-3}]\) is \(\pi/3\) and the angle formed by \([y'_2, y'_3]\) and \([y'_{m_{i-1}}, A']\) is \(\pi/3\). Then the straight segment \([y'_0, y'_1]\) has length \(\pi/2\). Now we insert points \(y'_1, \ldots, y'_{n_1-1}\) in the interior of \([y'_0, y'_1]\) and \(y'_{n_1+1}, \ldots, y'_{n_4-1}\) in the interior of \([y'_n', y'_n]\), so that

\[
||y'_i, y'_{i+1}|| = \frac{||y'_0, y'_n||}{n_1}, \quad 0 \leq i \leq n_1 - 1,
\]

\[
||y'_i, y'_{i+1}|| = \frac{||y'_1, y'_n||}{n_2 - n_1}, \quad n_1 \leq i \leq n_2 - 1,
\]

and

\[
||y'_i, y'_{i+1}|| = \frac{||y'_{n_2}, y'_{n_3}||}{n - 3 - n_2}, \quad n_2 \leq i \leq n - 4.
\]

Now let \(y'_{n_2}\) be the point in the interior of \(Q\) such that the length of \([y'_{n_2}, y'_n]\) is equal to \(b'/2\) and the angle formed by \([y'_{n_2}, y'_{n-2}]\) and \([y'_n, B']\) is \(\pi/3\). Let \(y'_{n-1}\) be the point in the interior of \(Q\) such that the length of \([y'_n, y'_{n-1}]\) is equal to \(b'/2\) and the angle formed by \([y'_n, y'_{n-3}]\) and \([B', A']\) is \(\pi/3\). Then the length of the horizontal straight segment \([y'_{n-2}, y'_{n-1}]\) is \(b'/2\).

The construction of \(S'\) is very similar to that of \(S\). Let \(s'_0 = A'\) and \(s'_1 = y_{m_{i-1}}\). Let \(s'_1\) be the point in the interior of \(Q\) such that the length of \([s'_0, s'_1]\) is \(\pi/3\) and the angle formed by \([s'_0, s'_1]\) and \([A', B']\) is \(\pi/4\). Let \(s'_2\) be the point in the interior of \(Q\) such that the length of \([s'_2, s'_3]\) is \(\pi/3\) and the angle formed by \([s'_2, s'_3]\) and \([y'_{m_{i-1}}, A']\) is \(\pi/3\). Let \(S'\) be the polyline which connects \(s'_0, s'_1, s'_2\) and \(s'_3\).

Let \(L'_1\) be the part of \(L'\) connecting \(y_0\) and \(y_{m_{i-1}}\) and \(L'_2\) be the remaining part of \(L'\). Then from the construction we see that \(L'\) and \(S'\) divide \(Q\) into four polygons \(Q_i, 1 \leq i \leq 4: Q_1\) is the top one; \(Q_2\) is the one bounded by \(L'_2\) and \([y_{m_{i-1}}, B']\); \(Q_3\) is the one bounded by \(L'_1\) and \(S'\); and \(Q_4\) is the one bounded by \(S'\) and \([A', y_{m_{i-1}}]\). Each of these polygons has four sides, the three of which are straight segments and the last one is a polyline satisfying \(C_3\)-bounded linear geometry with \(C_3 > 1\) being some constant depending only on \(C_1\). From the above construction, it follows that the geometry of each \(Q_i\) is well controlled. Now one can easily see the existence of the constant \(K_2 > 0\) such that for each \(i = 1, 2, 3, 4\), there exists a \(K_2\)-qc homeomorphism \(\psi_i\) which maps \(Q_i\) to a standard square with the bottom side satisfying \(C_3\)-bounded linear geometry.

This proves the Claim and completes the proof of the lemma.
4.5. **Proof of Lemma 4.13** We will use the same idea as used in [13] in constructing the David extension \( H : \Delta \rightarrow \Delta \) of the circle homeomorphism \( h : \mathbb{T} \rightarrow \mathbb{T} \). The difference in the situation here is that there are two or more critical points in the unit circle. So we will use Lemma 4.13 instead of 4.12 to control the dilatation of \( H \).

Let \( \alpha \in \Theta_C \) and \( f \in \Pi_{\text{geom}}^d \). Let \( B \in S_d^0 \) be the Blaschke product which models \( f \). Recall that \( x_i \in \mathbb{T} \) be the point such that \( f'(x_i) = 1 \). Let \( x'_i \in \mathbb{T} \) be the point such that \( R'_n(x'_i) = 1 \). Let \( p_n/q_n \) be the convergents of \( \alpha \) and

\[
Q_n = \{x_i \mid 0 \leq i < q_n\} \quad \text{and} \quad Q'_n = \{x'_i \mid 0 \leq i < q_n\}.
\]

Since \( \bigcup_{n} \Theta_C S_d^n \) is compact, there is an integer \( N_0 \geq 1 \) depending only on \( C \) and \( d \) such that for all \( n \geq N_0 \), \( d(x_i, x_j) < 1 \) and \( d(x'_i, x'_j) < 1 \) hold for any two adjacent points \( x_i \) and \( x_j \) in \( Q_n \) and any two adjacent points \( x'_i \) and \( x'_j \) in \( Q'_n \).

For each \( x_i \in Q_n \), let \( y_i \) be the point on the radial segment \([0, x_i]\) such that

\[
|y_i - x_i| = d(x_r, x_l)/2
\]

where \( x_r \) and \( x_l \) denote the two points immediately to the right and left of \( x_i \) in \( Q_n \).

**Definition 4.1** (Yoccoz’s cells). Let \( x_i \) and \( x_j \) be any two adjacent points in \( Q_n \). Connect \( y_i \) and \( y_j \) by a straight segment. Then the three straight segments \([x_i, y_i], [y_i, y_j], [x_j, y_j]\) and the arc segment \([x_i, x_j]\) bound a domain in \( \Delta \). We call the closure of this domain a **cell of level** \( n \). The segment \([y_i, y_j]\) is called the top side of the cell.

From the construction and the last assertion of Proposition 4.1, it is not difficult to see

**Lemma 4.14** (cf. Lemma 6.1, [27]). There exist \( C(d) > 1 \) and \( 0 < \gamma(d) < \sigma(d) < \pi \) depending only on \( d \) such that for any cell \( E \) with level \( n \geq N_0 \), the diameters of the four sides of the cell \( E \) are \( C(d) \)-commensurable with each other, and moreover, the angles formed by the top side and its two radial sides are within \( \gamma(d) \) and \( \sigma(d) \).

Let \( E \) be a cell of level \( n \). Let \( E_1, \ldots, E_m \) be all the cells of level \((n + 1)\) which are contained in \( E \). Then

\[
E \setminus \bigcup_{i=1}^m E_i
\]

is a polygon. Note that one side of such polygon is a polyline which is the union of the top sides of all the cells of level \((n + 1)\) which are contained in \( E \). Let us call such polygons of type I. By Lemma 4.14, Lemmas 4.4 and 4.6 we have \( K(d), C(d) > 1 \) depending only on \( d \) such that for any polygon of type I, there is a \( K(d) \)-qc homeomorphism \( \xi \) which maps the polygon homeomorphically onto the standard polygon, which is the unit square with the bottom sides consisting of one or several pieces satisfying \( C(d) \)-bounded saddle node geometry, and moreover, the diameter of each piece is \( C(d) \)-commensurable with the diameter of the unit square, and moreover, when restricted to each edge of the polygon, \( \xi \) is linear.

Now replacing \( Q_n \) by \( Q'_n \), and using the same construction as above, we can construct cells and polygons for \( R_n \). We call such polygons of type II. From [28], [29] in Proposition 4.1 and the above construction, it follows that there exists a universal constant \( K > 1 \) such that for any polygon of type II, there is a \( K \)-qc homeomorphism \( \sigma \) which maps the polygon homeomorphically onto the standard polygon which is the unit square.
with the bottom side being divided into several pieces with equal length, and moreover, when restricted to each edge of the polygon, \( \sigma \) is linear.

Let us now construct the David extension \( H : \Delta \to \Delta \). Suppose \( P \) is a polygon of type I and of level \( n \geq N_0 \). Let \( Q \) be the polygon of type II which corresponds to \( P \). Let \( \xi \) and \( \sigma \) be the qc-homeomorphisms described as above. Then \( \xi(P) \) and \( \sigma(Q) \) are respectively standard squares satisfying the conditions in Lemma 4.13 such that all the involved constants depends only on \( d \). By Lemma 4.13 there is a constant \( C(d) > 1 \) depending only on \( d \) and a qc map \( \tau : \xi(P) \to \sigma(Q) \) such that \( \tau \) maps each edge of \( \xi(P) \) linearly to the corresponding edge of \( \sigma(Q) \) and the qc constant of \( \tau \) is bounded by

\[
C(d) \cdot (1 + (\log a_{n+1})^2) < K(d, C) \cdot n
\]

where \( K(d, C) > 0 \) is some constant depending only on \( d \) and \( C \). Here we uses the arithmetic condition that \( \log a_n \leq C \sqrt{n} \). Now define

\[
\phi = \sigma^{-1} \circ \tau \circ \xi.
\]

Since the qc constants of \( \xi \) and \( \sigma \) are bounded by some constant depending only on \( d \), by increasing \( K(d, C) \) if necessary, we may assume that the qc constant of \( \phi \) is bounded by \( K(d, C) \cdot n \) for all polygons \( P \) of level \( n \geq N_0 \).

The complement of the union of all polygons \( P \) of type I in \( \Delta \) is a polygon which contains the origin in its interior whose boundary is the union of the top sides of all cells of level \( N_0 \) (for \( B \)). Let us denote this polygon by \( P_0 \). Similarly, the complement of the union of all polygons \( Q \) of type II in \( \Delta \) is a polygon which contains the origin in its interior whose boundary is the union of the top sides of all cells of level \( N_0 \) (for \( R_n \)). Let us denote this polygon by \( Q_0 \). Since \( \bigcup_{\alpha \in \theta_c} S^\alpha_d \) is compact, by increasing \( K(d, C) \) if necessary, we may assume that for all \( B \in S^\alpha_d \) there is a \( K(d, C) \)-qc homeomorphism \( \phi : P_0 \to Q_0 \) which maps each edge of \( P_0 \) linearly to the corresponding edge of \( Q_0 \), and maps the origin to itself. Now we can define \( H : \Delta \to \Delta \) by gluing all such \( \phi \) along the edges of all polygons \( P \).

Let us now prove the existence of the constants \( M, \alpha > 0 \) and \( 0 < \epsilon_0 < 1 \) so that Lemma 2.1 holds. Let

\[
\epsilon_0 = \frac{2}{1 + K(d, C) \cdot N_0}.
\]

For any \( 0 < \epsilon < \epsilon_0 \), let \( n > 0 \) be the least integer such that \( \epsilon > \frac{2}{1 + K(d, C) \cdot n} \). Then \( n \geq N_0 + 1 \). By a simple calculation, it follows that \( n > \frac{\eta_0(d, C)}{\epsilon} \) with \( \eta_0(d, C) > 0 \) being some constant depending only on \( d \) and \( C \). By the minimal property of \( n \) it follows that \( \epsilon \leq \frac{2}{1 + K(d, C) \cdot (n-1)} \). This means the following set

\[
\{ z \in \Delta \mid |\mu_H(z)| > 1 - \epsilon \}
\]

is contained in the union of all the cells of level \( n \). By Theorem 4.11 there exist \( C_1(d) > 1 \) and \( 0 < \delta(d) < 1 \) depending only on \( d \) such that the area of the union of all cells of level \( n \) is less that \( C_1(d) \cdot \delta^a(d) \). Thus the area of the above set is bounded by \( C_1(d) \delta^a(d) \).

Since

\[
C_1(d) \delta(d)^n = C_1(d) \cdot e^{-n \ln \frac{1}{\delta(d)}} < C_1(d) \cdot e^{-\frac{\eta_0(d, C)}{\epsilon} \ln \frac{1}{\delta(d)}},
\]

Lemma 2.1 follows by taking \( M = C_1(d) \) and \( \alpha = \eta_0(d, C) \cdot \ln \frac{1}{\delta(d)} \).

4.6. **Proof of Lemma 2.2**. Let \( H : \Delta \to \Delta \) be the David extension of \( h \) given in the last subsection. Let

\[
\mu_H = \frac{H \, dz}{H \, dz}
\]
be the Beltrami differential given by $H$. Let $\mu$ denote the Beltrami differential on the plane which is the pull back of $\mu_H$ by the iteration of $\hat{B}$.

Let $Y_n$ be the union of all the Yoccoz’s cells of level $n$. Then the outer boundary component of $Y_n$ is $T$, and the inner boundary component of $Y_n$ is the union of finitely many straight segments, and moreover,

$$Y_{N_0} \supset Y_{N_0+1} \supset \cdots \supset Y_{n} \supset Y_{n+1} \supset \cdots.$$ 

From the proof of Lemma 2.1, it follows that $\delta_0$ depending only on $\lambda$. Lemma 4.16 implies Lemma 2.2.

**Lemma 4.15.** Let $C > 0$ and $d \geq 2$. Then there exists a constant $1 < \lambda(d, C) < \infty$ depending only on $d$ and $C$ such that for any $\alpha \in \Theta_C$, any $B \in S_d^3$ and any $n \geq N_0$, the dilatation of $H$ in $\Delta \but Y_n$ is not greater than $\lambda(d, C) \cdot n$.

Define

$$X = \{ z \in \mathbb{C} \setminus \bigcup \mathbb{N} \mid B^k(z) \in \Delta \text{ for some } k \geq 1 \}.$$ 

For each $z \in X$, let $k_z \geq 1$ be the least positive integer such that $B^{k_z}(z) \in \Delta$. Define

$$X_n = \{ z \in X \mid B^{k_z}(z) \in Y_n \}.$$ 

**Lemma 4.16.** Let $d \geq 2$ be an integer and $C > 0$. Then there exist $C_0(d, C) > 1$ and $0 < \delta_0(d, C) < 1$ depending only on $d$ and $C$ such that for any $\alpha \in \Theta_C$ and $B \in S_d^3$, $\text{area}(X_n) < C_0(d, C) \cdot \delta_0(d, C)^n$ for all $n \geq N_0$.

**Proposition 4.2.** Lemma 4.16 implies Lemma 2.2.

**Proof.** By Theorem 4.1, there exist constants $C(d) > 0$ and $0 < \delta(d) < 1$ depending only on $d$ such that

$$\text{area}(Y_n) \leq C(d) \cdot \delta(d)^n$$

holds for all $n \geq N_0$. From this and Lemma 4.16 there exist $C_1(d, C) > 0$ and $0 < \delta_1(d, C) < 1$ depending only on $d$ and $C$ such that for all $n \geq N_0$,

$$\text{area}(X_n \cup Y_n) < C_1(d, C) \cdot \delta_1(d, C)^n.$$ 

Let $\lambda(d, C)$ be the constant in Lemma 4.15. Let $\epsilon_0 = \frac{1}{\lambda(d, C) \cdot N_0}$. For any $0 < \epsilon < \epsilon_0$, let $n \geq 0$ be the least integer such that $\epsilon > \frac{1}{\lambda(d, C) \cdot (n+1)}$. Then $n \geq N_0$. By the minimal property of $n$ we have $\epsilon \leq \frac{1}{\lambda(d, C) \cdot n}$. Then for $z$ with $|\mu(z)| > 1 - \epsilon$, the real dilatation of $\mu$ at $z$ is greater than $\lambda(d, C) \cdot n$. By Lemma 4.15, we have $z \in X_n \cup Y_n$. It follows that

$$\text{area}(\{ z \mid |\mu(z)| > 1 - \epsilon \}) < \text{area}(X_n \cup Y_n) < C_1(d, C) \cdot \delta_1(d, C)^n$$ 

$$< C_1(d, C) \cdot \delta_1(d, C)^{-1} \cdot \delta_1(d, C)^{\frac{1}{\lambda(d, C) \cdot n}} = C_1(d, C) \cdot \delta_1(d, C)^{-1} \cdot e^{-(1/\lambda(d, C)) \cdot (\ln \delta_1(d, C)^{-1}) / \epsilon}$$

The last inequality uses the fact that $n + 1 > 1/(\lambda(d, C) \cdot \epsilon)$. Lemma 2.2 then follows by taking $M = C_1(d, C) \cdot \delta_1(d, C)^{-1}$ and $\alpha = (1/\lambda(d, C)) \cdot (\ln \delta_1(d, C)^{-1})$. This completes the proof of Proposition 4.2.

**Lemma 4.17.** Let $d \geq 2$ be an integer and $C > 0$. Let $\alpha \in \Theta_C$. Then there exist $C_1(d, C) > 0$, $0 < \epsilon_1(d, C) < 1$, $0 < \delta_1(d, C) < 1$ and an integer $N_1(d, C) \geq N_0$ depending only on $d$ and $C$ such that for all $B \in S_d^3$,

$$\text{area}(X_{n+2}) \leq C_1(d, C) \cdot \epsilon_1(d, C)^n + \delta_1(d, C) \cdot \text{area}(X_n), \ \forall \ n > N_1(d, C).$$

**Proposition 4.3.** Lemma 4.17 implies Lemma 4.16.
Proof. To simplify notations, let us just use $C_1, \epsilon_1, \delta_1$ and $N_1$ to denote $C_1(d, C), \epsilon_1(d, C), \delta_1(d, C)$ and $N_1(d, C)$ respectively. We may assume that

$$\delta_1 > \epsilon_1^2$$

since otherwise we may replace $\delta_1$ by $(1 + \epsilon_1^2)/2$ and (39) still holds. Note that every $B$ in $S^d_\delta$ behaves like $z \mapsto z^d$ near infinity. By the compactness of $\bigcup_{n \in \Theta_\epsilon} S^d_\delta$, it follows that there is an $R > 1$ such that $|B(z)| > z$ for all $|z| \geq R$. This implies that $X_n$ is contained in the disk $\{|z| < R\}$ and thus

$$\text{area}(X_n) < \pi R^2$$

for all $n \geq 1$. Thus it suffices to prove Lemma 4.16 for $n \geq N_1 + 2$.

From (39) we have

$$\text{area}(X_{n+2}) + \frac{C_1}{\delta_1 - \epsilon_1^2} \cdot \epsilon_1^{n+2} \leq \delta_1 \cdot \text{area}(X_n) + \frac{C_1}{\delta_1 - \epsilon_1^2} \cdot \epsilon_1^n.$$ 

In the case that $n = N_1 + 2k$ for some $k \geq 1$, we have

$$\text{area}(X_n) \leq \delta_1^k (\text{area}(X_{N_1}) + \frac{C_1}{\delta_1 - \epsilon_1^2} \cdot \epsilon_1^{N_1}) = (\sqrt{\delta_1})^{-N_1} \cdot (\pi R^2 + \frac{C_1}{\delta_1 - \epsilon_1^2} \cdot \epsilon_1^{N_1})(\sqrt{\delta_1})^{n}.$$ 

In the case that $n = N_1 + 2k + 1$ for some $k \geq 1$, we have

$$\text{area}(X_n) \leq \delta_1^k (\text{area}(X_{N_1+1}) + \frac{C_1}{\delta_1 - \epsilon_1^2} \cdot \epsilon_1^{N_1+1}) = (\sqrt{\delta_1})^{-N_1-1} \cdot (\pi R^2 + \frac{C_1}{\delta_1 - \epsilon_1^2} \cdot \epsilon_1^{N_1+1})(\sqrt{\delta_1})^{n}.$$ 

This implies Lemma 4.16 Proposition 4.3 follows.

The remaining part of this section is devoted to the proof of Lemma 4.17. The formulation of Lemma 4.17 is strongly inspired by [18] where a similar area estimate was established by using Petersen’s puzzle construction for the Douady-Ghys Blaschke model. For $d \geq 3$ and $\alpha \in \Theta C$ every $B \in S^d_\delta$ has more than one critical point in $T$. These critical points may interact with each other if one wants to adapt Petersen’s puzzle construction to the Blaschke model $B$. To get around this problem, we will use an idea from [27]. This idea allows us to obtain the area estimate without using puzzle construction. Before we present the proof, let us introduce some notations and terminologies first. For $z \in \mathbb{C}$ and $r > 0$, let $B_r(z)$ denote the Euclidean disk with radius $r$ and center $z$.

Definition 4.2 (K-bounded geometry). Let $K > 1$ and $(U, V)$ be a pair of sets in $\mathbb{C}$ such that $V \subset U$. We say $(U, V)$ has $K$-bounded geometry if there exist $x \in V$ and $r > 0$ such that

$$B_r(x) \subset V \subset U \subset B_{Kr}(x).$$

The following lemma is a variant of Vitali’s covering lemma. For a proof, see [27].

Lemma 4.18 (cf. Lemma 2.1 of [27]). Let $K > 1$ and $L = 8K + 9$. Then for any finite family of pairs of measurable sets $\{(U_i, V_i)\}_{i \in \Lambda}$ all of which have $K$-bounded geometry, namely, for each $i \in \Lambda$, there exist $x_i \in V_i$ and $r_i > 0$ satisfying

$$B_{r_i}(x_i) \subset V_i \subset U_i \subset B_{Kr_i}(x_i),$$

there is a subfamily $\sigma_0$ of $\Lambda$ such that all $B_{r_j}(x_j), j \in \sigma_0$, are disjoint, and moreover,

$$\bigcup_{i \in \Lambda} U_i \subset \bigcup_{j \in \sigma_0} B_{Kr_j}(x_j).$$

In particular, we have

$$\text{area}(\bigcup_{i \in \Lambda} U_i) \leq L^2 \cdot \text{area}(\bigcup_{i \in \Lambda} V_i)$$

where $\text{area}(\cdot)$ denotes the area with respect to the Euclidean metric.
Recall $\Delta$ and $\mathbb{T}$ denote the unit disk and unit circle respectively. Let $\text{diam}(\cdot)$ and $\text{dist}(\cdot, \cdot)$ denote the diameter and distance with respect to the Euclidean metric. Let $\Omega = \mathbb{C} \setminus \overline{\Delta} = \{ z \in \mathbb{C} \mid |z| > 1 \}$. Then $\Omega$ is a hyperbolic Riemann surface.

**Definition 4.3.** Let $1 < K < \infty$ and $z \in X_{n+2}$. We say $z$ is associated to a $K$-admissible pair $(U, V)$ if $V \subset U \subset \Omega$ are two open topological disks such that $z \in U$ and

1. $V \subset X_n \setminus X_{n+2}$,
2. the pair $(U, V)$ has $K$-bounded geometry,
3. $\text{diam}(U) < K \cdot \text{dist}(U, \mathbb{T})$.

Let $I \subset \mathbb{T}$ be an open interval. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the punctured plane. Set

\begin{equation}
\Omega_I = \mathbb{C}^* \setminus (\mathbb{T} \setminus I).
\end{equation}

Then $\Omega_I$ is a hyperbolic Riemann surface. For $d > 0$, the hyperbolic neighborhood of $I$ is defined by

\begin{equation}
\Omega_d(I) = \{ z \in \Omega_I \mid d_{\Omega_I}(z, I) < d \}
\end{equation}

where $d_{\Omega_I}(\cdot, \cdot)$ denotes the hyperbolic distance in $\Omega_I$.

When $I$ is an open interval in the real line $\mathbb{R}$, the hyperbolic neighborhood of $I$ in the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus I)$ is bounded by two arc segments of Euclidean circles which are symmetric about $\mathbb{R}$, and moreover, the exterior angles $\sigma$ formed by the arcs and $\mathbb{R}$ are all equal and determined by the formula $d = \log \cot(\sigma/4)$. The next lemma says when $I$ is small enough, $\Omega_d(I)$ is very much like the hyperbolic neighborhood in the slit plane.

**Lemma 4.19** (cf. Lemma 2.2 of [27]). Let $d > 0$ be given. Then when $I$ is small, $\Omega_d(I)$ is a Jordan domain and is like the hyperbolic neighborhood of the slit plane in the following sense.

1. $\partial \Omega_d(I) = \gamma_{\text{int}} \cup \gamma_{\text{out}}$ where $\gamma_{\text{int}}$ and $\gamma_{\text{out}}$ are two real analytic curve segments both of which connect the two end points of $I$. Moreover, $\gamma_{\text{int}} \setminus \partial I \subset \Delta$ and $\gamma_{\text{out}} \setminus \partial I \subset \mathbb{C} \setminus \overline{\Delta}$;
2. $\gamma_{\text{int}}$ and $\gamma_{\text{out}}$ are symmetric about $\mathbb{T}$, and each of them is like an arc segment of some Euclidean circle;
3. Let $\sigma$ denote the exterior angles formed by $\gamma_{\text{int}}$ and $\mathbb{T}$, $\gamma_{\text{out}}$ and $\mathbb{T}$, all of which are the same, then $d = \log \cot(\sigma/4)$.

Note that $\Omega_d(I)$ is divided by $I$ into two parts: one is in the interior of $\Delta$ and the other one is in the exterior of $\Delta$. We only consider the part which is in the exterior of $\Delta$. Let $H_{\sigma}(I)$ denote this part. That is,

\begin{equation}
H_{\sigma}(I) = \{ z \in \Omega_d(I) \mid |z| > 1 \}
\end{equation}

where $\alpha$ is determined by the formula $d = \log \cot(\sigma/4)$.

Let

\begin{equation}
\Pi_n(B) = \{ I_n^i, 0 \leq i \leq q_{n+1} - 1; I_{n+1}^i, 0 \leq i \leq q_n - 1 \}
\end{equation}

be the collection of intervals defined in (27). Define

\begin{equation}
Z_n = \bigcup_{0 \leq i \leq q_{n+1} - 1} H_{\sigma}(I_n^i) \cup \bigcup_{0 \leq i \leq q_n - 1} H_{\sigma}(I_{n+1}^i).
\end{equation}

It is easy to see that $Z_n$ is the outer half of an open neighborhood of $\mathbb{T}$. See Figure 2 for an illustration. As a consequence of Theorem 4.14 and the fact that $\text{diam}(H_{\sigma}(I)) = O(|I|)$ (cf. Lemma 4.19), it follows that there exist $C(d) > 1$ and $0 < \epsilon(d) < 1$ depending only on $d$ such that for all $n \geq 1$ and $B \in \mathcal{S}_d^\alpha$ with $\alpha \in \Theta_C$,

\begin{equation}
\text{area}(Z_n) < C(d) \cdot \epsilon(d)^n.
\end{equation}
Let us first reduce Lemma 4.17 to Lemma 4.20.

**Lemma 4.20.** Let $C > 0$ and $d \geq 2$ be an integer. Then there exist $K > 1$ and $N_1 \geq N_0$ such that for all $n \geq N_1$ and $B \in S_{\alpha}^d$ with $\alpha \in \Theta_C$, if $z \in X_{n+2}$, then either $z \in Z_n$, or $z$ is associated to some $K$-admissible pair $(U, V)$.

**Proposition 4.4.** Lemma 4.20 implies Lemma 4.17.

**Proof.** From the proof of Proposition 4.3 we have an $R > 1$ such that for all $n \geq 1$ and $B \in S_{\alpha}^d$ with $\alpha \in \Theta_C$, we have

$$X_n \subset \{z \mid |z| \leq R\}.$$  

Note that $X_{n+2}$ is a Lebesgue measurable set. So for any $\eta > 0$, by a standard result in Lebesgue measure theory, there is a closed set $F \subset X_{n+2}$ such that

$$\text{area}(X_{n+2} \setminus F) < \eta.$$  

Since $Z_n$ is open and $F$ is a bounded and closed set, it follows that $F \setminus Z_n$ is a compact set.

For any point $x \in F \setminus Z_n$, by Lemma 4.20, $x$ is associated to some $K$-admissible pair $(U_i, V_i)$ for some uniform $1 < K < \infty$. This implies that the sets $U_i$s form an open cover of $F \setminus Z_n$. Since $F \setminus Z_n$ is a compact set, we have finitely many pairs $(U_i, V_i), i \in \Lambda$, such that

$$F \setminus Z_n \subset \bigcup_{i \in \Lambda} U_i.$$  

From (46) and (47), we have

$$\text{area}(X_{n+2}) \leq \text{area}(Z_n) + \text{area}(\bigcup_{i \in \Lambda} U_i) + \eta.$$  

By Lemma 4.18 we have

$$\text{area}(\bigcup_{i \in \Lambda} U_i) \leq \lambda(K) \cdot \text{area}(\bigcup_{i \in \Lambda} V_i).$$
where \( \lambda(K) = (8K + 9)^2 \). Since \( V_i \subset X_n \setminus X_{n+2} \) for all \( i \in \Lambda \), we have

\[
\text{area}(\bigcup_{i \in \Lambda} V_i) \leq \text{area}(X_n) - \text{area}(X_{n+2}) \text{.}
\]

This, together with (49), implies that

\[
\text{area}(\bigcup_{i \in \Lambda} U_i) \leq \lambda(K) \cdot (\text{area}(X_n) - \text{area}(X_{n+2})) \text{.}
\]

From (48) and (50) we get

\[
(1 + \lambda(K)) \cdot \text{area}(X_{n+2}) \leq \text{area}(Z_n) + \lambda(K) \cdot \text{area}(X_n) + \eta \text{.}
\]

From (45) and the above inequality we have

\[
\text{area}(X_{n+2}) \leq \frac{\lambda(K)}{1 + \lambda(K)} \cdot \text{area}(X_n) + \frac{C(d)}{1 + \lambda(K)} \cdot \varepsilon(d)^n + \frac{\eta}{1 + \lambda(K)} \text{.}
\]

Since \( \eta > 0 \) is arbitrary, by letting \( \eta \to 0 \), we get

\[
\text{area}(X_{n+2}) \leq \frac{\lambda(K)}{1 + \lambda(K)} \cdot \text{area}(X_n) + \frac{C(d)}{1 + \lambda(K)} \cdot \varepsilon(d)^n \text{.}
\]

The implies Lemma 4.17. The proof of Proposition 4.4 is completed.

\[\square\]

**Lemma 4.21.** Let \( 1 < L < \infty \). Then there is a \( 1 < K < \infty \) depending only on \( L \) such that for any \( B \in S^n_q \) with \( \alpha \in \Theta_C \), any \( z \in X_{n+2} \) and any integer \( m \geq 1 \), if \( B^i(z) \in \mathbb{C} \setminus \Xi \) for all \( 1 \leq i \leq m \) and \( \zeta = B^m(z) \) is associated to some \( L \)-admissible pair \((U_1, V_1)\), then \( z \) is associated to some \( K \)-admissible pair.

**Proof.** This is a direct consequence of the Koebe’s distortion theorem.

Note that there are exactly \( d - 1 \) Jordan curves in the outside of \( \Delta \), each of which is a pre-image of \( \mathbb{T} \) and is attached to \( \mathbb{T} \) at a critical point \( c \). We call these Jordan curves pre-circles. The following lemma says that these pre-circles satisfy a uniform cone geometry in a small neighborhood of \( c \). More precisely,

**Lemma 4.22.** There exists an \( \eta > 0 \) depending only on \( d \) such that the following property holds. Let \( B \in S^n_q \) with \( \alpha \in \Theta_C \). Let \( \Gamma \) be a pre-circle attached at some \( c \in \mathbb{T} \) and \( \Omega \) be the domain bounded by \( \Gamma \). Then \( \Gamma \cap \{ z \mid |z| = 1 + \eta \} \) consists of exactly two points, say \( a \) and \( b \). Let \( \Gamma_\eta = \{ z \in \Gamma \mid 1 < |z| < 1 + \eta \} \) and \( \Omega_\eta = \{ z \in \Omega \mid 1 < |z| < 1 + \eta \} \). We have

1. \( \Gamma_\eta \) is the union of two curve segments \( \Gamma_r \) and \( \Gamma_l \), where \( \Gamma_r \) is the one on the right connecting \( c \) to \( x \), and \( \Gamma_l \) is the one on the left connecting \( c \) to \( y \).
2. \( \Omega_\eta \) is a Jordan domain bounded by the union of \( \Gamma_r, S \) and \( \Gamma_l \), where \( S \) is the smaller arc in \( \{ z \mid |z| = 1 + \eta \} \) which connects \( a \) and \( b \).
3. Let \( v = B(c) \). Then \( B(\Gamma_r) = [B(x), v] \subset \mathbb{T} \) and \( B(\Gamma_l) = [v, B(y)] \subset \mathbb{T} \), where \( B(x) \) and \( B(y) \) are to the left and right of \( v \), respectively.
4. Let \( \Lambda_\eta \) be the cone in the exterior of \( \mathbb{T} \) spanned by two rays starting from \( c \) such that the exterior angles formed by \( \partial \Lambda \) and \( \mathbb{T} \) are both equal to \( \eta \cdot \pi \). Then \( \Omega_\eta \subset \Gamma_\eta \).

**Proof.** Since \( \mathcal{H}_d \) is compact, the existence of the constant \( \eta \) follows directly by a compactness argument such that the first three assertions hold. Let us prove by taking \( \eta > 0 \) smaller if necessary, the last assertion also holds. Suppose it were not true. Then there would be a point \( z \in \Gamma_r \) (or \( \Gamma_l \)) and a point \( w \in \mathbb{T} \) to the right (or left) of \( c \) such that \( \text{dist}(z, w)/\text{dist}(z, c) \) could be arbitrarily small. Let \( \Pi \) denote the set of all critical values. For a point or set \( X \) let \( X^* \) denote the symmetric image of \( X \) about \( \mathbb{T} \). Let \( \Xi = B^{-1}(\Pi \cup \{ B(z), B(w) \}) \). Since \( B(z) = B(z^*) \), \( z^* \in \Xi \) and \( \Xi \) is symmetric about \( \mathbb{T} \).
Then there would be a simple closed geodesic in \( \hat{\mathbb{C}} \setminus \Xi \) which separates \( \{z, w, z^*\} \) from \( \{0, c, \infty\} \) and the length of \( \gamma \) could be arbitrarily small provided that \( \text{dist}(z,w)/\text{dist}(z,c) \) is small enough. By symmetry \( \gamma^* \) is also a simple closed geodesic in \( \hat{\mathbb{C}} \setminus \xi \) with the length being equal to that of \( \gamma \). Since any two short simple closed geodesic are either identified with each other or disjoint with each other, when the length of \( \gamma \) is small enough, \( \gamma = \gamma^* \).

This means \( \gamma \) intersects \( I \) at two points, both of which are to the right of \( c \). Let us denote these two points by \( p \) and \( q \). Since the degree of \( B: \hat{\mathbb{C}} \setminus \Xi \to \hat{\mathbb{C}} \setminus (\Pi \cup \{B(z), B(w)\}) \) is a holomorphic covering map of degree \( 2d - 1 \), \( B(\gamma) \) is a simple closed geodesic in \( \hat{\mathbb{C}} \setminus (\Pi \cup \{B(z), B(w)\}) \) whose length could be arbitrarily small. In particular, \( B(\gamma) \) is also symmetric and thus intersects \( T \) at two points. The two points must be \( B(p) \) and \( B(q) \). Since \( B: \mathbb{T} \to \mathbb{T} \) is a homeomorphism, \( B(p) \) and \( B(q) \) must be to the right of \( v = B(c) \). But on the other, by the compactness of \( \mathcal{H}_d \), \( B \) is holomorphic in the interior of \( \gamma \) and maps the interior of \( \gamma \) to the interior of \( B(\gamma) \). Thus \( B(z) \) belongs to the interior of \( B(\gamma) \). Since \( B \) is symmetric, \( B(z) \) belongs to the smaller arc connecting \( B(p) \) and \( B(q) \). Since \( p \) and \( q \) are to the right of \( c \), \( B(z) \) is to the right of \( v \) also. This contradicts the fact that \( z \in \Gamma \) and \( B(z) \) is mapped to a point to the left of \( v = B(c) \).

Let us now begin the proof of Lemma 4.20. Let \( C > 0 \) and \( d \geq 2 \) be an integer. Let \( \alpha \in \Theta_C \) and \( B \in \mathcal{S}_d^\alpha \). It suffices to prove that there exist \( 1 < K < \infty \) and \( N_1 \geq N_0 \) depending only on \( d \) and \( C \) such that if \( z \in X_{n+2} \setminus Z_n \) for some \( n \geq N_1 \), then \( z \) is associated to some \( K \)-admissible pair \((U, V)\).

Recall that \( k_z \geq 1 \) is the least positive integer such that \( B^{k_z}(z) \in \Delta \). Since \( z \in X_{n+2} \), we have \( B^{k_z}(z) \in Y_{n+2} \). Let us denote

\[
\zeta = B^{k_z}(z), \quad 0 \leq l \leq k_z.
\]

Since \( z_0 = z \notin Z_n \), the set

\[
\Pi = \{k \in \mathbb{Z} | 0 < k < k_z \text{ and } B^k(z) \notin Z_n\}
\]

is not empty. It is clear that \( \Pi \) contains at most \( k_z \) elements and is thus a finite set. Let

\[
k_0 = \max_{k \in \Pi} \{k\}.
\]

Then \( 0 \leq k_0 \leq k_z - 1 \). Set

\[
\zeta = z_{k_0} \text{ and } \omega = B(\zeta) = z_{k_0 + 1}.
\]

By the definition of \( k_0 \), \( \zeta \notin Z_n \), and moreover,

\[
\omega \in Z_n \text{ if } k_0 = k_z - 1 \text{ and } \omega \in Y_{n+2} \text{ if } k_0 = k_z - 1.
\]

In the case that \( \omega \in Z_n \), there are \( d \) pre-images of \( \omega \) in the exterior of \( T \), and in the case that \( \omega \in Y_{n+2} \), there are \( d - 1 \) pre-images of \( \omega \) in the exterior of \( T \). Since \( \zeta \) belongs to the exterior of \( T \), thus \( \zeta \) is one of these pre-images. By Lemma 4.20, it suffices to prove that \( \zeta \) is associated to some \( L \)-admissible pair \((U, V)\) for some uniform \( 1 < L < \infty \) depending only on \( d \). The proof is divided into two cases.

Let \( I \) be the the arc interval in \( \Pi_{n-1}(B) \) defined by (27) such that either \( \omega \in \mathcal{H}_s(I) \) or \( \omega \) belongs to the cell \( E \) with \( I \subset \partial E \cap T \). By Proposition 1.1 in the later case \( \partial E \cap T \) is either equal to \( I \) or equal to the union of \( I \) and one of its adjacent intervals in \( \Pi_{n-1}(B) \). Let us first group the critical points such that the ones which are closed to each other are classified into one group.

Now take a large constant \( R = R(d) \) and fix it. The dependence of \( R \) on \( d \) will be seen in the following. Since there are at most \( d - 1 \) critical values in \( T \), their exist a constant \( 0 < \kappa = \kappa(R, d) < 1 \) depending only on \( R \) and \( d \) (thus eventually only on \( d \)) such that for any \( B \in \bigcup_{\alpha \in \Theta_C} \mathcal{S}_d^\alpha \), if \( \omega \) and \( I \) are chosen as above, then one can find \( \epsilon \) and \( \delta \) satisfying
1. $\kappa < \epsilon < \delta < 1$ and $\delta/\epsilon > R$.
2. for each critical value $v$ of $B$ in $\mathbb{T}$, we either have $\text{dist}(\omega, v) < \epsilon \cdot |I|$ or $\text{dist}(\omega, v) > \delta \cdot |I|$.

We have two cases.

Case I. There is a critical value, say $v$, such that $\text{dist}(\omega, v) < \epsilon \cdot |I|$. By the choice of $\epsilon$ and $\delta$, for each critical value $v$ of $B$ in $\mathbb{T}$, we either have $\text{dist}(\omega, v) < \epsilon \cdot |I|$ or $\text{dist}(\omega, v) > \delta \cdot |I|$. By the geometry of the cells (cf. Lemma 6.3 of [18] or Lemma 6.1 of [27]), we have a pair of Jordan domains $V_0 \subset U_0$ such that the following properties hold.

1. $V_0 \subset E \setminus Y_{n+2}$ is a Euclidean disk with $\text{diam}(V_0) \asymp |I|$,
2. $U_0$ contains the Euclidean disk $B_\omega(\delta \cdot |I|/2)$, in particular, $U_0$ contains $\omega$ and all critical values $v$ with $\text{dist}(\omega, v) < \epsilon \cdot |I|$ and moreover, $\text{diam}(U_0) \asymp |I|$,
3. $U_0$ has a Jordan neighborhood $W_0$ such that $U_0$ contains no critical values.

We need only to prove that these pre-images of $\omega$ in $\mathbb{T}$, for each critical value $\zeta \in \Omega$, the claim follows from Lemma 4.22. Otherwise, let $\Gamma_r$ and $\Gamma_l$ be the two arc segments described as in Lemma 4.22. Then $W_1$ must cross either $\Gamma_r$ or $\Gamma_l$. Without loss of generality we may assume that $W_1$ crosses $\Gamma_r$. By Lemma 4.22 $W_1$ can not cross $\Gamma_l$ nor $\mathbb{T}$, since otherwise $W_0 = B(W_1)$ will contain points in both left and right sides of $v$, and this would contradict the property (3) above.

In particular this implies that $W_1$ does not contain critical point and $B : W_1 \rightarrow W_0$ is a holomorphic isomorphism. We thus have $\text{mod}(W_1 \setminus U_1) = \text{mod}(W_0 \setminus U_0) \asymp 1$ and $\text{diam}(U_1) < L_2 \cdot \text{dist}(U_1, \mathbb{T})$ for some $0 < L_2 < \infty$ depending only on $d$. This proves the claim. Let $L = \max\{L_1, L_2\}$. Thus if $\zeta$ belongs to $U_1$, then $\zeta$ is associated to the $L$-admissible pair $(U_1, V_1)$ and Lemma 4.20 then follows from Lemma 4.21.

We have seen that for each pre-circle $\Gamma$ at $c$ with $\text{dist}(\omega, B(c)) > \delta \cdot |I|$, there is exactly one corresponding pair $(U_1, V_1)$. Suppose there are $0 \leq l \leq d-1$ critical points $c$, counting by multiplicities, such that $\text{dist}(\omega, B(c)) > \delta \cdot |I|$. Then there are $l$ $U_1$’s corresponding to these pre-circles each of which contains a pre-image of $\omega$. Now suppose $\zeta$ is not any of these $l$ pre-images of $\omega$ and let us prove that all the other pre-images of $\omega$ must belong to $Z_n$. As we mentioned before, if $\omega \not\in Z_n$, there are $d-l$ other pre-images of $\omega$ in the exterior of $\mathbb{T}$, and if $\omega \not\in Y_{n+2}$, there are $d-l-1$ other pre-images of $\omega$ in the exterior of $\mathbb{T}$.

Note that $U_0$ contains all the critical values $v$ in $\mathbb{T}$ with $\text{dist}(\omega, v) < \epsilon \cdot |I|$. So there is a component $U_1$ of $B^{-1}(U_0)$ which intersects $\mathbb{T}$ such that the map $B : U_1 \rightarrow U_0$ is of degree $2d - 2l - 1$. When $\omega \in Z_n$, $U_1$ contains $d-l$ pre-images of $\omega$ which are in the exterior of $\mathbb{T}$, and when $\omega \in Y_{n+2}$, $U_1$ contains $d-l-1$ pre-images of $\omega$ which are in the exterior of $\mathbb{T}$. This means all the remaining pre-images of $\omega$, which we are concerned about, are all contained in $U_1$. We need only to prove that these pre-images of $\omega$ must be contained in $Z_n$. To see this, let $\Pi$ denote the set of all the critical values. Because $\delta/\epsilon > R(d)$, by
taking $R$ large, we can make sure that there is a simple closed geodesic $\gamma$ in $\mathbb{C} \setminus \Pi \cup \partial I$ which can be arbitrarily short provided that $R$ is large enough such that $\gamma$ encloses $\omega$ and all those critical values with $\text{dist}(\omega, v) < \epsilon \cdot |I|$, and moreover, $\gamma$ intersects $I$ or one of two the adjacent intervals of $I$, say $I'$, in $\Pi_{n-1}(B)$. From the properties (1-3) above we may assume that $U_0$ contains $\gamma$. Let $J \subset T$ be the arc such that $B(J) = I$ (In the case that $\gamma$ intersects $I'$, let $J'$ be the arc such that $B(J') = I'$). Let $\eta$ be the pre-image of $\gamma$ which intersects $J$ (or $J'$). Then $\eta$ is a simple closed geodesic in $\mathbb{C} \setminus B^{-1}(\Pi \cup \partial I)$. Since $\gamma \subset U_0$, $\eta \subset U_1$. Since the covering degree of $B : \eta \to \gamma$ is not greater than $d$, $\eta$ can be arbitrarily short provided that $\gamma$ is short enough. Thus compared with $|J|$ (or $|J'|$), the Euclidean diameter of $\eta$ can be arbitrarily small provided that $\gamma$ is short enough. Note that the covering degree of $B : \eta \to \gamma$ is $2d - 2l - 1$, it follows that all the pre-images of $\omega$, which belong to the exterior of $\mathbb{T}$, are all contained in the interior of $\eta$. By the construction of $Z_n$, it follows that $Z_n$ contains a definite neighborhood of $J$ (or $J'$). Thus $Z_n$ contains $\eta$. This implies that all the remaining pre-images of $\omega$, which we are concerned about, belong to $Z_n$. This proves Lemma 4.20 in Case I.

Case II. $\text{dist}(\omega, v) > \delta |I|$ for all critical values $v$. In this case we may assume that $\omega \in Z_n$. This is because if $\omega \in Y_{n+2}$, then $\omega$ has exactly $d - 1$ pre-images which belong to the exterior of $\mathbb{T}$. As we have seen in the proof of Case I, each of the $d - 1$ pre-images corresponds to exactly one of the $d - 1$ pre-images of $\omega$ which is associated to a $L$-admissible pair $(U_1, V_1)$. Thus from now on we may assume that $\omega \in H_{\sigma}(I)$. It suffices to show that the last pre-image of $\omega$, which belong to the exterior of $\mathbb{T}$, either lies in $Z_n$, or is associated to some $L$-admissible pair $(U_1, V_1)$.

Subcase I of Case II: $I$ contains no critical values and $I \neq I_{n+1}^{n-1}$.

Let $J \subset T$ be the arc such that $B(J) = I$. Since $I$ contains no critical value and $B(1) \in I_{n+1}^{n-1}$, we have $I \neq I_{n+1}^{n-1}$. So $J$ is one of the intervals in the collection $\Pi_{n-1}(B)$. By Lemmas 4.14 and 4.19 we can construct a pair of Euclidean disk $V_0 \subset U_0$ with $\omega \in U_0$ such that $V_0 \subset E \setminus Y_{n+2}$ and

1. $U_0 \setminus \mathbb{D} \subset H_{\sigma}(I)$ (since $\omega_0 \in H_{\sigma}(I)$ and $\omega_0 \in U_0$, it follows that $U_0 \cap I \neq \emptyset$),
2. for any critical value $v$ of $B$, $\text{diam}(U_0) \prec \text{diam}(V_0) \prec |I|$,
3. $U_0$ has a Jordan neighborhood $W_0$ which contains no critical values and such that $W_0 \cap \mathbb{T}$ is a connected arc segment and $W_0 \setminus \overline{U_0} \asymp 1$.

Using the same argument as in the proof of Case I, it follows that there are $d - 1$ components of $B^{-1}(U_0)$ each of which intersects the interior of one of the $d - 1$ pre-circles, and that if $\zeta$ belong to one of these components, then $\zeta$ is associated to some $L$-admissible pair for some $0 < L < \infty$ depending only on $d$. Suppose $\zeta$ does not belong to any of these $d - 1$ components. Let $J \subset T$ be the arc such that $B(J) = I$. Since $U_0 \cap I \neq \emptyset$, there is a component of $B^{-1}(U_0)$, say $U_1$, such that $U_1 \cap J \neq \emptyset$. This implies this $U_1$ does not belong to the $d - 1$ ones we just mentioned since all the later ones do not intersect $T$, and thus the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is contained in $U_1$. Since $U_0 \setminus \mathbb{D} \subset H_{\sigma}(I)$, by Schwarz lemma, $U_1 \setminus \mathbb{D} \subset H_{\sigma}(J) \subset Z_n$. This implies that the last pre-images of $\omega$ is contained in $Z_n$. This proves Lemma 4.20 in the Subcase I of Case II.

Subcase II of Case II: $\text{dist}(\omega, T) < \epsilon |I|$.

Let $\omega^*$ be the symmetric image of $\omega$ with respect to $T$. Let $X$ be the set consisting of all critical values, $\omega, \omega^*$ and the two end points of $I$. Since $\text{dist}(\omega, \omega^*) < 2\epsilon |I|$ and $\epsilon > 0$ is small, we have a short and simple closed geodesic $\gamma$ in $\mathbb{C} \setminus X$ which intersects $I$ and enclose $\omega$ and $\omega^*$ in its inside, and $\gamma$ can be arbitrarily short provided that $\epsilon$ is taken small enough (Note that $\omega \in H_{\sigma}(I)$. By Lemma 4.19 when $\omega$ is close to one of the two end points of $I$, such $\gamma$ necessarily encloses this end point in its inside also). Let
$J \subset \mathbb{T}$ be the arc such that $B(J) = I$. The pre-image of $\gamma$ which intersects $J$, say $\eta$, is a short simple closed geodesic in $\mathbb{C} \setminus B^{-1}(X)$. Since $B : \eta \rightarrow \gamma$ is not greater than $2d - 1$, $\eta$ can be arbitrarily short provided that $\epsilon > 0$ is small enough. Since the two end points belong to $B^{-1}(X)$ and $\eta$ intersects $J$, this implies that the ratio $\text{diam}(\eta)/|J|$ can be arbitrarily small provided that $\epsilon > 0$ is small enough. Thus by taking $\epsilon > 0$ small, $\eta$ can be contained in an arbitrarily small neighborhood of $J$. Note that $\eta$ encloses a pre-image of $\omega$, say $\zeta_0$. Since $Z_n$ contains a definite neighborhood of $J$, it follows that $Z_n$ contains $\zeta_0$.

On the other hand, since $\text{dist}(\omega, v) > \delta|I|$ for all critical values $v$ in $\mathbb{T}$ and $\text{dist}(\omega, \mathbb{T}) < \epsilon|I|$, we can construct a Euclidean disk $V_0 \subset E \setminus Y_{n+2}$ and a Jordan domain $U_0$ containing the disks $V_0$ and $B|_{|I|/2}(\omega)$ such that the requirements (1), (2), and (3) in the proof of Subcase I of Case II are satisfied. Now consider the $d$ pre-images of $\omega$ which belong to the exterior of $\mathbb{T}$. As we have seen before, there are $d-1$ components of $B^{-1}(U_0)$ each of which intersects one of the $(d-1)$ pre-circles attached to $\mathbb{T}$ and contains exactly one pre-image of $\omega$, and each of these $d-1$ pre-images of $\omega$ is associated to some $L$-admissible pair for some $1 < L < \infty$ depending only on $d$. To check the last pre-image of $\omega$ which belongs to the exterior of $\mathbb{T}$, consider the component of $B^{-1}(U_0)$, say $U_1$, which intersects $\mathbb{T}$. By the construction of $U_0$, $U_0$ contains $\gamma$ and thus $U_1$ contains $\eta$. This implies that the $\zeta_0$, which belongs to the interior of $\eta$, belongs to $U_0$. Thus $\zeta_0$ is the last pre-image of $\omega$ in the exterior of $\mathbb{T}$. As we have seen in the last paragraph, $\zeta_0 \in Z_n$. This proves Lemma 4.20 in the Subcase II of Case II.

Subcase III of Case II: $\text{dist}(\omega, \mathbb{T}) \geq \epsilon|I|$ and $I$ contains at least one critical value or $I = I^{n,-1}_{n+1}$. Since $I^{n,-1}_{n+1}$ contains $B(1)$ and is adjacent to $I = I_{n+1}$ in the collection $\Pi_{n-1}$, either $I$ or one of its adjacent intervals in $\Pi_{n-1}$ contains at least one critical value of $B$. Then there exists a disk $V_0 \subset E \setminus Y_{n+2}$ and two Jordan domains, say $U_0$ and $U_0'$, such that

1. $\{\omega\} \cup V_0 \subset U_0 \cap U_0'$,
2. $\text{dist}(U_0, v) \sim \text{dist}(U_0', v) \sim \text{diam}(U_0) \approx \text{diam}(U_0') \approx |I|$ for every critical value $v \in \mathbb{T}$,
3. $U_0 \cup U_0'$ is a topological annulus and separates at least one critical value from $\infty$, that is, $U_0$ and $U_0'$ turn around one or more critical values in an opposite way,
4. $U_0$ and $U_0'$ have respectively Jordan neighborhoods $W_0$ and $W_0'$ both of which contain no critical values and such that both $W_0 \cap \mathbb{T}$ and $W_0' \cap \mathbb{T}$ are connected arc segments and $\text{mod}(W_0 \setminus \overline{U_0}) \approx \text{mod}(W_0' \setminus \overline{U_0'}) \approx 1$.

Here we would like to emphasize that the condition $\text{dist}(\omega, \mathbb{T}) \geq \epsilon|I|$ is needed to meet the requirement (4): because if $\omega$ is too close to $\mathbb{T}$, to satisfy (3) and $\text{mod}(W_0 \setminus \overline{U_0}) \approx \text{mod}(W_0' \setminus \overline{U_0'}) \approx 1$, one of $W_0 \cap \mathbb{T}$ and $W_0' \cap \mathbb{T}$ must contain at least two interval components which are separated by some critical value(s) and thus can not be a connected arc segment.

Let $c_i, 1 \leq i \leq d-1$ denote all the critical points in $\mathbb{T}$, counting by multiplicities, and let $c_i, 1 \leq i \leq l$ denote those ones such that the corresponding critical values $B(c_1), \ldots, B(c_l)$ are separated by $U_0 \cup U_0'$ from $\infty$. As we have seen before, for each $1 \leq i \leq d-1$, there is a component $U_1$ of $B^{-1}(U_0)$ intersecting the pre-circle $\Gamma$ attached to $c_i$, and moreover, $U_1$ contains a pre-image of $\omega$ which is associated to some $L$-admissible pair for some $1 < L < \infty$ depending only on $d$. Since each of these $d-1$ components does not contain critical points, they are disjoint from each other. Thus we have $d-1$ distinct pre-images of $\omega$ in the exterior of $\mathbb{T}$ each of which is associated to some $L$-admissible pair for some $1 < L < \infty$ depending only on $d$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$ depending only on $d$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$ depending only on $d$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$ depending only on $d$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$. To see how the last pre-image of $\omega$, which belongs to the exterior of $\mathbb{T}$, is also associated to some $L$-admissible pair for some $1 < L < \infty$.
depending only on \( d \), consider the components of \( B^{-1}(U_0') \) which intersects the pre-circles attatched at \( c_i, 1 \leq i \leq l \). Let \( \eta > 0 \) be the constant in Lemma 4.22. By the compactness of \( S_d \), we may take \( N_1 \) large enough such that for all \( n \geq N_1 \), the components of \( B^{-1}(U_0) \) and \( B^{-1}(U_0') \) which intersects the pre-circles attached to \( c_i, 1 \leq i \leq l \), are all contained in the annulus \( \{ z \mid 1 < |z| < 1 + \eta \} \). Without loss of generality, we may assume that \( B(c_i), 1 \leq i \leq l \), are ordered from left to the right, and are all to the right of \( U_0 \cap T \) and to the right of \( U_0' \cap \mathbb{T} \). 

Let \( \Gamma_i \) denote the pre-circle attached to \( c_i \). \( \Gamma_i \) and \( \Gamma_i' \) denote respectively the components of \( B^{-1}(U_0) \) and \( B^{-1}(U_0') \) which intersect \( \Gamma_i \). By Lemma 4.22 \( U_1' \) intersects \( \Gamma_i' \) and \( U_1'' \) intersects \( \Gamma_i^\ast \) for \( 1 \leq i \leq l \). Then for each \( 2 \leq i \leq l, U_1' \) and \( U_1'' \) share one pre-image of \( \omega \). But the pre-image of \( \omega \) which is contained in \( U_1'' \) is not contained in any of \( U_1' \) for \( 1 \leq i \leq l \). Note that this pre-image is not contained in any \( U_1' \) for \( l + 1 \leq i \leq d - 1 \). This is because otherwise, it would follow that \( U_0 \cup U_0' \) separates \( B(c_i) \) for some \( l + 1 \leq i \leq d - 1 \). This contradicts our assumption. So this is the last pre-image of \( \omega \) in the exterior of \( \mathbb{T} \). Let us denote it by \( \zeta \). Let \( V_1 \) be the component of \( B^{-1}(U_0) \) contained in \( U_1'' \). Then by using the same argument as before, there is an \( 1 < L < \infty \) depending only on \( d \) such that the pair \((U_1'', V_1)\) is an \( L \)-admissible pair to which \( \zeta \) is associated.

This completes the proof of Lemma 4.20. Lemma 2.2 thus follows.

4.7. Proof of the Key-Lemma 1.

Lemma 4.23. Let \( M, \beta > 0 \) and \( 0 < \epsilon_0 < 1 \). Let \( \Psi_{M, \beta, \epsilon_0} \) denote the family of all David homeomorphisms \( \chi : \mathbb{C} \to \mathbb{C} \) of the plane to itself which fix \( 0 \) and \( 1 \) and such that \( \mu_\chi \) is \((M, \beta, \epsilon_0)\)-integrable. Then there exist \( \vartheta, \iota : (0, 2] \to (0, \infty) \) such that

\[
\lim_{\delta \to 0^+} \vartheta(\delta) = \lim_{\delta \to 0} \iota(\delta) = 0
\]

and for any \( \phi \in \Psi_{M, \beta, \epsilon_0} \) and any two \( z_1, z_2 \in \mathbb{T} \) we have

\[
\iota(|z_1 - z_2|) \leq |\phi(z_1) - \phi(z_2)|, |\phi^{-1}(z_1) - \phi^{-1}(z_2)| \leq \vartheta(|z_1 - z_2|).
\]

Proof. Let

\[
\vartheta(\delta) = \max\{\sup |\phi(z_1) - \phi(z_2)|, \sup |\phi^{-1}(z_1) - \phi^{-1}(z_2)|\}
\]

and

\[
\iota(\delta) = \min\{\inf |\phi(z_1) - \phi(z_2)|, \inf |\phi^{-1}(z_1) - \phi^{-1}(z_2)|\}
\]

where \( \up\delta \) is set to be 0 when \( \sup \phi(z_1) - \phi(z_2) \) and \( \inf \phi^{-1}(z_1) - \phi^{-1}(z_2) \) are all contained in \( \Psi_{M, \beta, \epsilon_0} \) and all the pairs \( z_1, z_2 \in \mathbb{T} \) with \( |z_1 - z_2| \leq \delta \) and \( \inf \) is taken over all \( \phi \in \Psi_{M, \beta, \epsilon_0} \) and all the pairs \( z_1, z_2 \in \mathbb{T} \) with \( |z_1 - z_2| \geq \delta \). Then

\[
\lim_{\delta \to 0} \vartheta(\delta) = \lim_{\delta \to 0} \iota(\delta) = 0.
\]

Now let \( C > 1 \) and \( d \geq 2 \) be an integer. Let

\[
f \in \bigcup_{\alpha \in \Theta_C^b} \Pi_{\alpha, \text{geom}}^n.
\]

From the proof Theorem 2.1 (cf. §6), we shall see there is a Blaschke product

\[
B_f \in \bigcup_{\alpha \in \Theta_C} S_{\alpha}^n
\]

such that \( f \) can be obtained as the following. Let \( B_f | \mathbb{T} = h_f^{-1} \circ R_{\alpha} \in h_f \) where \( h_f \) is a circle homeomorphism with \( h_f(1) = 1 \). Let \( H_f \) be the David extension of \( h_f \) given in §4.5. Then by Lemma 2.1 \( \mu_{H_f} \) is uniformly integrable. Define \( \widehat{B_f} \) as in (5) Let \( \mu \) be the \( \widehat{B_f}\)-invariant Beltrami differential obtained by pulling back \( \mu_{H_f} \) through the iteration
of $\hat{B}_f$. By Lemma 2.2 $\mu$ is uniformly integrable. Let $\phi$ be the David homeomorphism of the complex plane to itself which fixes 0 and 1, and satisfies the Beltrami equation $\phi_z = \mu(z)\phi_z$. Using the same argument as in the proof of Lemma 5.5 of [18], it follows that $\phi \circ \hat{B}_f \circ \phi^{-1}$ is holomorphic and thus belong to
\[ \bigcup_{\alpha \in \Theta^b_{\text{geom}}} \Pi_{\alpha}^{d}. \]

As we shall see in the proof of Theorem 2.1 it turns out that
\[ f = \phi \circ \hat{B}_f \circ \phi^{-1}. \]

To see the existence of the functions $\lambda_1, \eta_1 : (0, 2] \to (0, \infty)$ in the Key-Lemma-1, let us extend $H_f$ to a plane homeomorphism of the plane by define $H(z) = H(z^*)^*$ for all $|z| > 1$, where $w^*$ denotes the symmetric image of $w$ about $T$. Since the dilation of $\mu_H$, is uniformly bounded near the origin and is degenerate only near the unit circle, it follows that the Beltrami coefficient of the new map is uniformly integrable in the whole plane. Let us still use $H_f$ denote the map. Thus there exist $M, \beta > 0$ and $0 < \epsilon_0 < 1$ depending only on $C$ and $d$ such that $H_f$ and $\phi$ are both $(M, \beta, \epsilon_0)$-integrable. For such $M, \beta$ and $\epsilon_0$, let $\vartheta$ and $\iota$ be the functions in Lemma 4.23. Then define $\lambda_1, \eta_1 : (0, 2] \to (0, \infty)$ by setting for any $\delta \in (0, 2]$
\[ \lambda_1(\delta) = \vartheta(\min\{\vartheta(\delta), 2\}) \quad \text{and} \quad \eta_1(\delta) = \iota(\min\{\iota(\delta), 2\}). \]

Let us prove the two functions satisfy the requirement in the Key-Lemma-1. Suppose $k > m > 0$ are two integers such that $|e^{2\pi ik\alpha} - e^{2\pi im\alpha}| < \delta$ for some $0 < \delta \leq 2$. Then
\[ |f^k(1) - f^m(1)| = |\phi \circ H_f^{-1} \circ R^k \circ H_f \circ \phi^{-1}(1) - \phi \circ H_f^{-1} \circ R^m \circ H_f \circ \phi^{-1}(1)|. \]

Since both $H_f$ and $\phi$ fix 1, it follows that
\[ |f^k(1) - f^m(1)| = |\phi \circ H_f^{-1}(e^{2\pi ik\alpha}) - \phi \circ H_f^{-1}(e^{2\pi im\alpha})|. \]

Since $H_f^{-1}(e^{2\pi ik\alpha}), H_f^{-1}(e^{2\pi im\alpha}) \in T$ and
\[ \min\{\iota(\delta, 2) \leq |H_f^{-1}(e^{2\pi ik\alpha}) - H_f^{-1}(e^{2\pi im\alpha})| \leq \min\{\vartheta(\delta), 2\}, \]

by Lemma 4.23 we have
\[ \iota(\min\{\iota(\delta, 2) \leq |H_f^{-1}(e^{2\pi ik\alpha}) - \phi \circ H_f^{-1}(e^{2\pi im\alpha})| \leq \vartheta(\min\{\vartheta(\delta), 2\}). \]

This completes the proof of the Key-Lemma-1.

5. PROOFS OF THE KEY-LEMMAS 3 AND 4

Let $m, l \geq 1$ be two integers and
\[ \zeta_0 = (w^0_1, \cdots, w^0_m, w^0_{m+1}, \cdots, w^0_{m+l}) \]
be a point in $\mathbb{C}^{m+l}$. Let $\xi_0 = (w^0_{m+1}, \cdots, w^0_{m+l})$ be the point in $\mathbb{C}^l$. Let
\[ Q_i(w_1, \cdots, w_m, w_{m+1}, \cdots, w_{m+l}), \quad 1 \leq i \leq m, \]
be polynomials of $m + l$ complex variables.

Suppose that there exist an open neighborhood $U$ of $\xi_0$ in $\mathbb{C}^l$ and $m$ holomorphic functions
\[ w_i = g_i(w_{m+1}, \cdots, w_{m+l}), \quad 1 \leq i \leq m, \]
defined in $U$ such that
\[ (1) \quad w^0_i = g_i(w^0_{m+1}, \cdots, w^0_{m+l}) \quad \text{for all} \quad 1 \leq i \leq m, \]

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(2) for all \((w_{m+1}, \cdots, w_{m+l}) \in U\) and \(1 \leq i \leq m\), we have
\[
Q_i(g_1(w_{m+1}, \cdots, w_{m+l}), \cdots, g_m(w_{m+1}, \cdots, w_{m+l}), w_{m+1}, \cdots, w_{m+l}) = 0,
\]
(3) there is an open neighborhood \(V\) of \((w_1^0, \cdots, w_m^0)\) in \(\mathbb{C}^m\) such that for any points \((w_{m+1}^1, \cdots, w_{m+l}^1) \in U\) and \((w_1^1, \cdots, w_m^1) \in V\), if
\[
Q_i(w_1^1, \cdots, w_m^1, w_{m+1}^1, \cdots, w_{m+l}^1) = 0, \quad 1 \leq i \leq m,
\]
then
\[
w_i^1 = g_i(w_{m+1}^1, \cdots, w_{m+l}^1), \quad 1 \leq i \leq m.
\]
We first prove that these \(g_i\) are determined by polynomial equations. More precisely,

**Lemma 5.1.** Suppose there exist holomorphic functions \(g_i, 1 \leq i \leq m\), defined in \(U\) such that the above three requirements are satisfied. Then there exist \(m\) irreducible polynomials \(P_i, 1 \leq i \leq m\), of \(m+1\) variables such that
\[
P_i(g_1, w_{m+1}, \cdots, w_{m+l}) = 0, \quad 1 \leq i \leq m.
\]

**Proof.** Let us start from the system of polynomial equations
\[
Q_i(w_1, \cdots, w_m, w_{m+1}, \cdots, w_{m+l}) = 0, \quad 1 \leq i \leq m.
\]
From the Condition (2) above, it follows that
\[
Q_i(g_1, \cdots, g_m, w_{m+1}, \cdots, w_{m+l}) = 0, \quad 1 \leq i \leq m,
\]
for all \((w_{m+1}, \cdots, w_{m+l}) \in U\). So if some \(Q_i\) is not irreducible, then we will have an irreducible factor of \(Q_i\), say \(S\), such that \(S(g_1, \cdots, g_m, w_{m+1}, \cdots, w_{m+l}) = 0\) for all \((w_{m+1}, \cdots, w_{m+l}) \in U\). If this is the case, we replace \(Q_i\) by \(S\) in the system. So we may assume that all \(Q_i, 1 \leq i \leq m\), are irreducible.

Let us first write all \(Q_i\) into polynomials of \(w_1\) with coefficients being polynomials of \(w_2, \cdots, w_{m+l}\). If some \(Q_i\) has a term with coefficient, say \(h(w_2, \cdots, w_m, w_{m+1}, w_{m+l})\), satisfying
\[
h(g_2, \cdots, g_m, w_{m+1}, \cdots, w_{m+l}) = 0
\]
for all \((w_{m+1}, \cdots, w_{m+l}) \in U\), then we add the equation
\[
h(w_2, \cdots, w_m, w_{m+1}, \cdots, w_{m+l}) = 0
\]
to the system, and at the same time, delete the corresponding term from \(Q_i\). In this way we get a new system of polynomial equations. By replacing a polynomial by one of its irreducible factors if necessary, we can make sure all the polynomials in the new system are still irreducible. Besides this, if one polynomial is the constant multiple of the other, we just remove one of them from the system. In this way we can also make sure the new system is not redundant. In particular, \(w_i = g_i(w_{m+1}, \cdots, w_{m+l}), 1 \leq i \leq m\), still satisfy the polynomial equations in the new system.

We claim that there is at least one polynomial in the new system which involves \(w_1\). Since otherwise the polynomial equations in the new system will hold in \(U\) by taking
\[
w_i = g_i(w_{m+1}, \cdots, w_{m+l}), \quad 2 \leq i \leq m,
\]
and taking \(w_1\) to be any function of \((w_{m+1}, \cdots, w_{m+l})\). But this contradicts with the uniqueness property assumed by Condition (3). The claim has been proved.

Now suppose \(Q_1\) is a polynomial in the new system which involves \(w_1\) and moreover, among all the polynomials in the new system which involve \(w_1\), the degree of \(Q_1\) with respect to \(w_1\) is the lowest. Assume there is some other polynomial in the system which also involves \(w_1\). Otherwise we go to the next step. Suppose \(Q_2\) is a polynomial which involves \(w_1\). We do the polynomial long division as follows,
\[
Q_2 = D \cdot Q_1 + R
\]
where $D$ and $R$ are polynomials of $w_1$ with coefficients being rational functions of $w_2, \ldots, w_{m+1}$, and moreover, the degree of $R$ with respect to $w_1$ is less than that of $Q$. Since the degree of $Q$ with respect to $w_1$ is the lowest, it follows that the degree of $R$ with respect to $w_1$ is less than that of $Q$. Since all $Q$ are irreducible and the system is not redundant, $R$ is not identically zero. Let $h(w_2, \ldots, w_{m+1})$ be the coefficient of the leading term of $Q$. From the process of polynomial long division, we know if $D = D'/D''$ and $R = R'/R''$ then $D'$ and $D''$ are the powers of $h(w_2, \ldots, w_{m+1})$. Since $h(g_2, \ldots, g_m, w_{m+1}, \ldots, w_{m+1})$ is not identically zero in $U$, both

$$D''(g_2, \ldots, g_m, w_{m+1}, \ldots, w_{m+1}) \text{ and } R''(g_2, \ldots, g_m, w_{m+1}, \ldots, w_{m+1})$$

are not identically zero in $U$. Now we replace $Q$ by $R'$ and get a new system of polynomial equations. Again repeat the procedure we previously used, we may assume that the polynomials in the new system are all irreducible and are not redundant. From (55) it follows that under the condition that $R'' \neq 0$ and $D'' \neq 0$, the two equations $R' = 0$ and $Q_1 = 0$ imply $Q_2 = 0$. Since $R''$ and $D''$ are not identically zero in $U$, if $w_i = \hat{g}_i(w_{m+1}, \ldots, w_{m+1}), 1 \leq i \leq m$, are a group of holomorphic functions in $U$ satisfying the polynomial equations in the new system and are close enough to $g_i, 1 \leq i \leq m$, such that $R''(g_1', \ldots, g_m', w_{m+1}, \ldots, w_{m+1})$ and $D''(g_1', \ldots, g_m', w_{m+1}, \ldots, w_{m+1})$ do not vanish in some open subset $W$ of $U$, then they also satisfy the polynomial equations in the original system in $W$ and thus in $U$ by the Uniqueness Theorem for holomorphic functions. By Condition (3), it follows that $g_i' = g_i$ for $1 \leq i \leq m$.

Note that after the above process, the total degree of the system with respect to $w_1$, that is, the sum of the degrees of all polynomials in the system with respect to $w_1$, is decreased at least by 1. So after finitely many steps, we have a system of irreducible polynomial equations such that

1. there is only one polynomial which involves $w_1$,
2. $w_i = g_i, 1 \leq i \leq m$, are a group of solutions of the system,
3. if $w_i = \hat{g}_i, 1 \leq i \leq m$, are a group of solutions of the system defined in some open subset in $U$, which are close enough to $g_i$, then $\hat{g}_i = g_i$ for all $1 \leq i \leq m$.

Now we claim that, except $Q_1$, there is at least one polynomial in the system which involves $w_2$. Since otherwise, we may let $\hat{g}_i = g_i$ for $3 \leq i \leq m$, and let $\hat{g}_2$ be any holomorphic function in $U$ which is close to $g_2$ but not identical with $g_2$, then solve $\hat{g}_2$ by $Q_1(g_1, g_2, g_3, \ldots, g_m, w_{m+1}, \ldots, w_{m+1}) = 0$ in some open subset of $U$ and get a solution $\hat{g}_1$ which can be arbitrarily close to $g_1$ provided that $\hat{g}_2$ is close enough to $g_2$. By the third property above we get $\hat{g}_2 = g_2$. This is a contradiction.

Now consider all the polynomial equations except $Q_1$ in the system. Let $Q_2$ be a polynomial in the system which involves $w_2$ and whose degree with respect to $w_2$ is the lowest. If except $Q_1$ and $Q_2$, there are no other polynomials which involve $w_2$, we go to the next step. Otherwise, choose a polynomial, say $Q_3$, which involves $w_2$. Then we repeat the process of the polynomial long division as in (55) for $Q_2$ and $Q_3$, with respect to $w_2$. In this way, after finitely many times we get a new system such that

1. except $Q_1$, there is only one polynomial $Q_2$ which involves $w_2$,
2. $w_i = g_i, 1 \leq i \leq m$, are a group of solutions of the system,
3. if $w_i = \hat{g}_i, 1 \leq i \leq m$, are a group of solutions of the system defined in some open subset in $U$, which are close enough to $g_i$, then $\hat{g}_i = g_i$ for all $1 \leq i \leq m$.

Repeating this procedure for $w_3, \ldots, w_m$. We finally get a system which consists of exactly $m$ irreducible polynomial equations, say

$$Q_1(w_1, \ldots, w_m, w_{m+1}, \ldots, w_{m+1}) = 0,$$

$$Q_2(w_2, \ldots, w_m, w_{m+1}, \ldots, w_{m+1}) = 0,$$
Then by relabeling Lemma 5.3.1, moreover, there are parameters \(w_1, \cdots, w_{m-1}\) as \(w_m\) and repeating the above process, we get an irreducible polynomial \(P_i\) of \(m + 1\) variables such that

\[P_i(g_i, w_{m+1}, \cdots, w_{m+i}) = 0, \quad 1 \leq i \leq m - 1.\]

The proof of Lemma 5.1 is completed. \(\square\)

5.1. **Proof of the Key-Lemma 3.** Recall that \(c_{d-1} = 1\). Let \(x_i\) be one of the points in the \(i\)-th periodic attracting cycle and \(p_i \geq 1\) be the period. Let \(t_i\) be the multiplier. Then we have

\[
f_{c_1^0, \cdots, c_{d-2}^0}(x_i) - x_i = 0 \quad \text{and} \quad Df_{c_1^0, \cdots, c_{d-2}^0}(x_i) - t_i = 0.
\]

Since the cycle is attracting, there exist open neighborhood \(U\) of \((c_1^0, \cdots, c_{d-2}^0)\) and open neighborhood \(V\) of \((x_i, t_i)\) and two holomorphic functions

\[x = g_1(c_1, \cdots, c_{d-2}) \quad \text{and} \quad t = g_2(c_1, \cdots, c_{d-2})\]

defined in \(U\) such that for all \((c_1, \cdots, c_{d-2}) \in U\) we have

\[
f_{c_1^0, \cdots, c_{d-2}^0}(x) - x = 0 \quad \text{and} \quad Df_{c_1^0, \cdots, c_{d-2}^0}(x) - t = 0,
\]

and moreover, for any \((c_1^0, \cdots, c_{d-2}^0) \in U\) and \((x', t') \in V\) satisfying the above two equations, we have

\[x' = g_1(c_1^0, \cdots, c_{d-2}^0) \quad \text{and} \quad t' = g_2(c_1^0, \cdots, c_{d-2}^0).
\]

Now apply Lemma 5.1 we have

**Lemma 5.2.** There is an open neighborhood \(U\) of \((c_1^0, \cdots, c_{d-2}^0)\) such that when the parameters \((c_1, \cdots, c_{d-2})\) moves in \(U\), the \(l\) attracting cycles moves analytically, and moreover, there are \(l\) irreducible polynomials \(Q_i, 1 \leq i \leq l,\) of \(d - 1\) variables, such that for each \(1 \leq i \leq l,\) the multiplier \(t_i\) is determined by the polynomial equation

\[Q_i(t_i, c_1, \cdots, c_{d-2}) = 0.
\]

**Lemma 5.3.** Let \(t_1, \cdots, t_l\) denote the multipliers of the \(l\) attracting cycles respectively. Then by relabeling \(c_1, \cdots, c_{d-2}\) if necessary, we have

\[
\left| \frac{\partial(t_1, \cdots, t_l)}{\partial(c_1, \cdots, c_{d-2})} \right|_{(c_1^0, \cdots, c_{d-2}^0)} \neq 0.
\]

Let us first prove the Key-Lemma 3 by assuming Lemma 5.3. By Implicit function Theorem and Lemma 5.2, there exist holomorphic functions \(c_i = g_i(t_1, \cdots, t_l, c_{l+1}, \cdots, c_{d-2}), 1 \leq i \leq l,\) such that

\[Q_i(t_i, g_1, \cdots, g_l, c_{l+1}, \cdots, c_{d-2}) = 0, \quad 1 \leq i \leq l,
\]

where \(Q_i, 1 \leq i \leq l,\) are the polynomials provided by Lemma 5.2. It is easy to verify that these \(g_i, 1 \leq i \leq l,\) satisfy the three conditions before Lemma 5.1. Applying Lemma 5.1 we have irreducible polynomials \(P_i, 1 \leq i \leq l,\) such that the functions \(c_i = g_i(t_1, \cdots, t_l, c_{l+1}, \cdots, c_{d-2}), 1 \leq i \leq l,\) are determined by the polynomial equations

\[P_i(c_i, t_1, \cdots, t_l, c_{l+1}, \cdots, c_{d-2}), \quad 1 \leq i \leq d - 2.
\]

Now fix \(t_1, \cdots, t_l\) and take \(c_i = c_i^0\) for all \(l + 2 \leq i \leq d - 2.\) Then the left hand of each of the above equations is a polynomial of two variables, \(c_i\) and \(c_{i+1}.\) Let us still denote it
by $P_i$. By choosing an appropriate irreducible factor if necessary, we may assume each $P_i$ is an irreducible polynomial of two variables. This implies the Key-Lemma 3.

Now let us prove Lemma 5.3 As we seen before we have a holomorphic map

$$\Phi : (c_1, \cdots, c_{l-1}, \cdots, c_{d-2}) \rightarrow (t_1, \cdots, t_l)$$

in a small open neighborhood $U$ of $(c_1^0, \cdots, c_{l-1}^0, \cdots, c_{d-2}^0)$. Let

$$t_i^0 = t_i(c_1^0, \cdots, c_{l-1}^0, c_{d-2}^0), \quad 1 \leq i \leq l.$$ 

It suffices to prove that there exists a small neighborhood $W$ of $(t_1^0, \cdots, t_l^0)$ and a homomorphic map $\Psi : W \rightarrow U$ such that

$$\Phi \circ \Psi = id.$$ 

It is clear that this will imply Lemma 5.3. Thus it remains to construct such $\Psi$. The construction is as follows. Take a point $x_i$, $1 \leq i \leq l$, in each attracting periodic cycle and suppose $p_i$ is the period of the $i$-th cycle. For each $x_i$, there is a Jordan domain $U_i$ such that

1. $\partial U_i$ is real analytic,
2. $f^{p_i} : U_i \rightarrow f^{p_i}(U_i)$ is conjugate to either $z \rightarrow t_i z$ when $0 < |t_i| < 1$ or to $z \rightarrow z^m$ for some $m \geq 2$ when $t_i = 0$.

Let $U^k_i = f^k(U_i)$ for $0 \leq k \leq p_i$. For each $1 \leq i \leq l$, the map

$$f : U^{p_i-1}_i \rightarrow U^{p_i}_i.$$

is either a holomorphic isomorphism or a $m$ to $1$ branched covering map with $f^{p_i-1}(x_i)$ being the only branched point. Let $\phi_i : \Delta \rightarrow U^{p_i-1}_i$ and $\psi_i : \Delta \rightarrow U^{p_i}_i$ be holomorphic isomorphisms with $\phi_i(0) = f^{p_i-1}(x_i), \phi_i'(0) > 0$, and $\psi_i(0) = x_i$. Then we can lift the map $f : U^{p_i-1}_i \rightarrow U^{p_i}_i$ to a holomorphic isomorphism $\Lambda_i : \Delta \rightarrow \Delta$ with $\Lambda_i(0) = 0$. It is clear that $\Lambda_i$ is either given by $z \rightarrow \lambda \cdot z$ for some $|\lambda| = 1$ or $z \rightarrow \lambda \cdot z^m$ for some $|\lambda| = 1$ and $m \geq 2$ being some integer. By choosing an appropriate argument of $\psi_i'(0)$, we can make $\lambda = 1$. That is, $\Lambda_i(z) = z$ or $z^m$ for some integer $m \geq 2$.

Then we have two cases. In the first case, $0 < |t_i| < 1$. Then for an $\epsilon > 0$ small we can define $\Lambda_{i,s_i} : \Delta \rightarrow \Delta$ for all $|s_i| < \epsilon$ by

$$\Lambda_{i,s_i}(z) = \begin{cases} 
(1 + s_i)\Lambda_i(z) & \text{for } |z| < 1/2, \\
(1 + 2s_i(1 - |z|))\Lambda_i(z) & \text{for } 1/2 \leq |z| < 1.
\end{cases}$$

In the second case, $t_i = 0$. Then for an $\epsilon > 0$ small we can define $\Lambda_{i,s_i} : \Delta \rightarrow \Delta$ for all $|s_i| < \epsilon$ by

$$\Lambda_{i,s_i}(z) = \begin{cases} 
\Lambda_i(z) + s_i & \text{for } |z| < 1/2, \\
\Lambda_i(z) + 2s_i(1 - |z|) & \text{for } 1/2 \leq |z| < 1.
\end{cases}$$

For $s = (s_1, \cdots, s_l)$, define a family of quasi-regular map $g_s$, $|s| = \sum_i |s_i| < \epsilon$, by

$$g_s(z) = \begin{cases} 
\psi_i \circ \Lambda_{i,s_i} \circ \phi_i^{-1} & \text{if } z \in U^{p_i-1}_i, \\
\tau_f & \text{otherwise}.
\end{cases}$$

Note that for each attracting periodic cycle of $f$, $g_s$ has an attracting periodic cycle nearby this cycle, and the multiplier of this cycle depends analytically on $s$ for $|s| < \epsilon$. Moreover, by a direct calculation, for $\epsilon > 0$ small, the multiplier of the $i$-th attracting periodic cycle of $g_s$ is an univalent function of $s_i$ for $|z_i| < \epsilon$. Now pulling back the standard complex structure by the iteration of $g_s$, one can easily obtain a $g_s$-invariant complex structure $\mu_s$ on the whole plane such that $\mu_s$ depends analytically on $s$. Let $\phi_s$ be the qc homeomorphism of the plane to itself which solves the Beltrami equation.
given by \( \mu_s \) and fixes 0 and 1. Then \( f_s = \phi_s \circ g_s \circ \phi_s^{-1} \) is a polynomial of degree \( d \).

Note that the multiplier of the \( i \)-th attracting periodic cycle for \( f_s \) is equal to that of the corresponding cycle for \( g_s \), and is thus a univalent function of \( s_i \), say \( \tau_i(s_i) \). The map \( \Psi \) can be defined as follows.

Let \( W = \{(\tau_1(s_1), \ldots, \tau_l(s_l)) \mid |s_i| < \epsilon \} \). Then for each \( t = (t_1, \ldots, t_l) \in W \), let \( s = \tau^{-1}(t) = (s_1, \ldots, s_l) \). Let \( c_1, \ldots, c_{d-2} \) be the critical points of \( f_s \). Then \( c_i, 1 \leq i \leq d-2 \), are holomorphic functions of \( s = (s_1, \ldots, s_l) \). This defines a holomorphic function \( \Psi : W \to U \). It is obvious that \( \Psi \circ \Psi = \text{id} \). The proof of the Key-Lemma 3 is completed.

5.2. Proof of the Key-Lemma 4. Suppose \( f \) has \( l+1 \) periodic attracting cycles with multipliers \( \theta_1^0, \ldots, \theta_{l+1}^0 \) with \( 0 \leq l \leq d-3 \) being some integer. For a small \( \epsilon > 0 \), using qc surgery as in the construction of \( \Psi \) in the proof of the Key-Lemma 3, we can construct a holomorphic family of polynomials \( f_{t+l} \) with \( |t_{l+1} - \theta_{l+1}^0| < \epsilon \), such that \( f_{t+l} \) preserves all the orbit relations and the multipliers of the first \( l \) attracting cycles. Let \( c_1, \ldots, c_{d-2} \) be the critical points of \( f_{t+l} \). Let \( x_1 \) be one of the points in the \( i \)-th attracting cycles, and \( t_i \) be the multiplier of the \( i \)-th attracting cycle. Then all \( c_i, 1 \leq i \leq d-2 \), and \( x_1, 1 \leq i \leq l+1 \), are holomorphic functions of \( t_{l+1} \) for \( |t_{l+1} - \theta_{l+1}^0| < \epsilon \). Moreover, \( c_1, \ldots, c_{d-2}, x_1, \ldots, x_{l+1}, t_{l+1} \), satisfy \( d-l+1 \) polynomial equations

\[
Q_j(c_1, \ldots, c_{d-2}, x_1, \ldots, x_{l+1}, t_{l+1}) = 0, \quad 1 \leq i \leq d+l-1.
\]

These polynomial equations come from the following \( d-l+1 \) relations.

\[
\begin{aligned}
&\{ f^p_i(x_i) = x_i, \quad 1 \leq i \leq l+1, \\
&D f^p_i(x_i) = t_i, \quad 1 \leq i \leq l+1, \\
&f^k_i(1) = c_i, \quad l+2 \leq i \leq d-2.
\end{aligned}
\]

Note that the multiplier of the \( l+1 \)-th attracting cycle of \( f_{t+l} \) depends analytically on \( c_1, \ldots, c_{d-2} \), that is, we have a holomorphic function \( \phi \) of \( d-2 \) variables such that

\[
\phi(c_1(t_{l+1}), \ldots, c_{d-2}(t_{l+1})) = t_{l+1}, \quad |t_{l+1} - \theta_{l+1}^0| < \epsilon.
\]

This implies that there is some \( 1 \leq i \leq d-2 \) with \( c_i'(t_{l+1}^0) \neq 0 \). Without loss of generality, let us assume that \( c_d'(t_{l+1}^0) \neq 0 \). This implies that \( c_d \) is a univalent function of \( t_{l+1} \) in a small neighborhood of \( \theta_{l+1}^0 \). Thus in a small neighborhood of \( \theta_{l+1}^0 \), all \( c_i, 1 \leq i \leq d-3 \), \( x_i, 1 \leq i \leq l+1 \), and \( t_{l+1} \) are holomorphic functions of \( c_d \), and satisfy the system of polynomial equations in (56). It is easy to check the first two conditions before Lemma 5.1. The third condition is guaranteed by the rigidity assertion of Theorem 2.4. Thus by Lemma 5.1 all \( c_i, 1 \leq i \leq d-3 \), are functions of \( c_d \) determined by some irreducible polynomial equation \( P(c_i, c_d) = 0 \). This completes the proof of the Key-Lemma 4.

6. TOPOLOGICAL CHARACTERIZATION OF THE MAPS IN \( \Sigma_{\text{geom}}^{n,d} \)

The purpose of this section is to prove Theorem 2.1. As a consequence we prove the Key-Lemma-2. Theorem 2.4 can be viewed as an extension of Thurston’s characterization theorem for post-critically finite rational maps. It may have independent interest to consider the topological characterization of more general rational maps, for instance, rational maps with Jordan Siegel disks for which the critical orbits are either eventually periodic, or attracted to some attracting or parabolic periodic cycles, or intersect the closures of the Siegel disks. But for the purpose of this work, we restrict our attention only to the maps in the class \( \Sigma_{\text{geom}}^{n,d} \), and this will be sufficient for our proof of the Key-Lemma-2.
6.1. Some Preliminaries. For readers' convenience, let us introduce some background knowledge for Thurston's characterization theorem for rational maps, especially, the extension of this theorem to sub-hyperbolic rational maps. The readers may refer to [3], [4], [8], [19] and [28] for more details in this aspect.

Let \( F : \hat{C} \to \hat{C} \) be a finitely branched covering map which preserves the orientation. Let \( \Omega_F = \{ z \in \hat{C} \mid \deg_z(F) \geq 2 \} \) and

\[
P_F = \bigcup_{k \geq 1} F^k(\Omega_F)
\]

be the critical set and the post-critical set of \( F \), respectively.

We say \( F \) is \textit{geometrically finite} if \( P_F \) is an infinite set but the accumulation set of \( P_F \) is a finite set. It is easy to check that each accumulation point of \( P_F \) is a period point of \( F \). We say a geometrically finite branched covering map \( F \) is \textit{sub-hyperbolic semi-rational} if each periodic cycle \( O \) in the accumulation set of \( P_F \) is holomorphically attracting, that is, there is an open neighborhood \( U \) of \( O \) such that \( F|_U \) is holomorphic, and moreover, \( |DF^p(x)| < 1 \) for each \( x \in O \) where \( p \geq 1 \) is the period of \( O \). We call \( U \) the \textit{holomorphic attracting basin} of \( O \). By taking \( U \) smaller if necessary, we can always take \( U \) to be the union of finitely many Jordan disks each of which contains exactly one point in the cycle, and moreover, for each component \( D \) of \( U \), \( F^p(D) \) is compactly contained in \( D \), and around each \( D \), there is an annulus \( A \) with the inner boundary component being equal to \( \partial D \) such that \( F^p \) maps \( A \) holomorphically holomorphic into \( D \) and \( A \cap P_F = \emptyset \).

We call each of such \( D \) a holomorphic disk, and the \( A \) surrounding it the protective annulus of \( D \). We may further assume that \( F^p : D \to D \) is holomorphically conjugate either to \( z \mapsto \lambda z \) for some \( 0 < |\lambda| < 1 \) or to \( z \mapsto z^m \) for some \( m \geq 2 \) (cf. §2 of [28]).

Let \( F \) be a sub-hyperbolic semi-rational branched covering map. Let \( \gamma \subset \hat{C} \setminus P_F \) be a simple closed curve. We say \( \gamma \) is \textit{non-peripheral} if each component of \( \hat{C} \setminus \gamma \) contains at least two points in \( P_F \). A \textit{multi-curve} \( \Gamma \) is a finite family of non-peripheral curves which are disjoint and non-homotopic to each other. We say \( \Gamma \) is \textit{stable} if for each \( \gamma_i \in \Gamma \), each non-peripheral component of \( F^{-1}(\gamma_i) \) is homotopic to some \( \gamma_j \) in \( \Gamma \).

Suppose \( \Gamma = \{ \gamma_1, \ldots, \gamma_n \} \) is a stable multi-curve. For \( 1 \leq i, j \leq n \), let \( \gamma_{i,j,\alpha} \) denote all the non-peripheral components of \( F^{-1}(\gamma_i) \) which are homotopic to \( \gamma_j \). Let \( d_{i,j,\alpha} \) be the covering degree of \( F: \gamma_{i,j,\alpha} \to \gamma_i \).

\[
a_{i,j} = \sum_{\alpha} \frac{1}{d_{i,j,\alpha}}.
\]

The matrix \( A = (a_{i,j}) \) is called Thurston linear transformation matrix. It is non-negative and thus has a maximal positive eigenvalue \( \lambda > 0 \). If \( \lambda \geq 1 \) we call \( \Gamma \) a Thurston obstruction.

**Definition 6.1.** Two sub-hyperbolic semi-rational maps \( F \) and \( G \) are called \textit{CLH-equivalent} (combinatorially and locally holomorphically equivalent) if there exist a pair of homeomorphisms of the sphere \( \phi, \psi : \hat{C} \to \hat{C} \) such that

1. \( \phi \circ F = G \circ \psi \),
2. For each holomorphic attracting cycle \( O \) in the accumulation set of \( P_F \), there is an open neighborhood \( U \) of \( O \), such that \( \phi|_U = \psi|_U \) and are holomorphic,
3. \( \phi \) is isotopic to \( \phi \) rel \( P_F \cup \cup_i U_i \).
The following is an extension of Thurston’s characterization theorem for post-critically finite rational maps to sub-hyperbolic rational maps.

**Theorem 6.1 (4 & 28).** Let $F$ be a semi-rational branched covering map. Then $F$ is CLH-equivalent to a sub-hyperbolic rational map $G$ if and only if $F$ has no Thurston obstructions.

Two different proofs of Theorem 6.1 are provided in [4] and [28], respectively. Let us sketch the idea of the proof in [28] as follows, which will be helpful in our later discussion. Let $T_F$ denote the Teichmüller space modeled on $(\hat{C}, P_F \cup \cup U_i)$. The branch covering $F$ induces an analytic map $\sigma_F : T_F \to T_F$. It turns out that the existence of the desired rational map $G$ is equivalent to the existence of a fixed point of $\sigma_F$. The proof in [28] is divided into two steps. In the first step it was proved that the non-existence of Thurston obstructions implies certain bounded geometry condition, which will be described later. In the second step it was proved that the bounded geometry implies the existence of a fixed point of $\sigma_F$.

Now let us describe the bounded geometry condition mentioned in the last paragraph. Let $\mu_0$ be the standard complex structure. Let $\mu_k$ be the complex structure which is the pull back of $\mu_0$ by $F^k$. Let $\phi_k : \hat{C} \to \hat{C}$ be the quasiconformal homeomorphism which solves the Beltrami equation given by $\mu_k$ and fixes 0, 1 and $\infty$. Consider the hyperbolic Riemann surface

$$X_k = \hat{C} \setminus \phi_k(P_F \cup \cup U_i).$$

For any non-peripheral curve $\gamma \subset \hat{C} \setminus (P_F \cup \cup U_i)$, there is a unique simple closed geodesic $\eta$ in $X_k$ which is homotopic to $\phi_k(\gamma)$. Let $[\mu_k]$ denote the Teichmüller class of $\mu_k$ in $T_F$ and $l_{[\mu_k]}(\gamma)$ the length of $\eta$ with respect to the hyperbolic metric in $X_k$. We say $\gamma$ is a $[\mu_k]$-geodesic if $\phi_k(\gamma)$ is a geodesic in $X_k$.

**Definition 6.2.** We say a sub-hyperbolic semi-rational map $F$ has bounded geometry if there exists a $\delta > 0$ such that for every non-peripheral curve $\gamma \subset \hat{C} \setminus (P_F \cup \cup U_i)$ and all $k \geq 0$, one has $l_{[\mu_k]}(\gamma) > \delta$.

**Theorem 6.2 (cf. 3, 19).** Let $F$ be a sub-hyperbolic semi-rational map. Then

1. there is a $\delta > 0$ such that for any non-peripheral curve $\gamma \subset \hat{C} \setminus (P_F \cup \cup U_i)$, either $l_{[\mu_k]}(\gamma) > \delta$ for all $k \geq 0$ or $l_{[\mu_k]}(\gamma) \to 0$ as $k \to 0$.
2. The multi-curve $\Gamma$ which represents the homotopy classes of all $\gamma$ such that $l_{[\mu_k]}(\gamma) \to 0$ as $k \to \infty$ is a Thurston obstruction. Such Thurston obstruction is called the canonical Thurston obstruction of $F$.
3. $F$ has a Thurston obstruction if and only if $F$ has a canonical Thurston obstruction.

### 6.2. Proof of Theorem 6.2

Let $f \in \Sigma_{\text{top}}^{a,d}$. If $f$ has a Thurston obstruction in the exterior of $\Delta$, by a result of McMullen (see Appendix B, [13]), $f$ cannot be CLH-equivalent to any $g \in \Sigma_{\text{geom}}^{a,d}$. In the following let us assume that $f$ has no Thurston obstructions in the exterior of $\Delta$. Let us first prove there is a $g \in \Sigma_{\text{geom}}^{a,d}$ such that $f$ is CLH-equivalent to any $g$. After that, we prove such $g$ must be uniqueness up to a linear conjugation. For $w \in \hat{C}$, let $w^*$ denote the symmetric image of $w$ about $\mathbb{T}$. Define

$$F(z) = \begin{cases} f(z) & \text{for } |z| \geq 1 \\ ([f(z^*)])^* & \text{for } |z| < 1. \end{cases}$$

Note that $F|\mathbb{T}$ is the rigid rotation $z \mapsto e^{2\pi i a}z$ and $F : \hat{C} \to \hat{C}$ is a branched covering map of degree $2d-1$ and is symmetric with respect to $\mathbb{T}$. Let $\alpha_n = p_n/q_n$ be the sequence
of convergents of $\alpha$. Then $\alpha_n \to \alpha$ as $n \to \infty$. By assumption, $1 \in \Omega_f$. It follows that $1 \in \Omega_F$. Let $\epsilon > 0$ and

$$H = \{ z \mid (1 + \epsilon)^{-1} < |z| < 1 + \epsilon \}.$$ 

By taking $\epsilon > 0$ small enough we may assume that

$$(H - T) \cap (\Omega_F \cup P_F) = \emptyset.$$ 

We can perturb $F$ in $H$ to get a sequence of sub-hyperbolic semi-rational maps $F_n : \hat{C} \to \hat{C}$ such that

1. $1 \in \Omega_F$, and $F_n(z^*) = [F_n(z)]^*$,
2. $F_n(z) = F(z)$ for all $z \notin H$,
3. $F_n(z) = e^{2\pi i n} z$ for all $z \in T$,
4. $(H - T) \cap (\Omega_F \cup P_F) = \emptyset$,
5. all the critical points of $F_n$ in $T$ belong to the forward orbit of 1, which is denoted as $\mathcal{O}_n$,
6. $F_n \to F$ uniformly with respect to the spherical metric.

Since topological branched coverings are quite robust, one can easily construct the sequence $\{F_n\}$ satisfying the above conditions. Since $F_n \to F$ uniformly with respect to the spherical metric by (6), the following property holds.

7. For all $n$ large enough, the degree of $F_n$ and the number of the critical points of $F_n$ in $T$, are respectively the same as those of $F$,
8. for each critical point $c$ of $F$, there is exactly one critical point of $F_n$, say $c_n$, such that $c_n \to c$ as $n \to \infty$, and the local degree of $F_n$ at $c_n$ is the same as that of $F$ at $c$.

**Lemma 6.1.** $F_n$ has no Thurston obstructions for all $n$ large enough.

**Proof.** Our proof is by contradiction. Suppose $F_n$ has a Thurston obstruction for some large $n$. By Proposition 6.2, $F_n$ would have a canonical Thurston obstruction $\Gamma$. By the definition of canonical Thurston obstruction, for any $\epsilon > 0$, there is a $k$ such that $\Gamma$ is the set of all $[\mu_k]$-geodesic with $l_{[\mu_k]}(\gamma) < \epsilon$. Since $F_n$ is symmetric about $T$, we may assume that $\Gamma$ is symmetric in the following sense: Suppose $\gamma \in \Gamma$. Then if $\gamma \cap T = \emptyset$, we have $\gamma^* \in \Gamma$, and if $\gamma \cap T \neq \emptyset$, $\gamma = \gamma^*$. Here $\gamma^*$ denote the symmetric image of $\gamma$ about $T$.

Claim 1: All curves in $\Gamma$ intersect $T$. Let us prove the claim by contradiction. Let $\Gamma' \subset \Gamma$ be the subset consisting of all the curves which do not intersect $T$ and assume that $\Gamma' \neq \emptyset$. Let $\Gamma'' = \Gamma \setminus \Gamma'$. Then any curve $\xi \in \Gamma''$ intersects $T$ and is thus symmetric about $T$. In particular, the set of points of $P_F$, contained in either component of $\hat{C} \setminus \xi$ is symmetric about $T$, and this implies that $\xi$ can not be homotopic to any curve which is disjoint with $T$. Now Let $\gamma \in \Gamma'$. Since $F_n$ maps $T$ to $T$, any non-peripheral component of $F_n^{-1}(\gamma)$ must be disjoint with $T$ and is homotopic to some element $\eta \in \Gamma$. Since we have just proved that $\gamma \notin \Gamma''$, it follows that $\eta \notin \Gamma'$. This implies that $\Gamma'$ is stable. Since $l_n(\gamma) \to 0$ for all $\gamma \in \Gamma'$, $\Gamma'$ is also a Thurston obstruction. Now we divide $\Gamma'$ into two groups of curves $\Gamma_1$ and $\Gamma_2$, where $\Gamma_1$ consists of all curves in the exterior of $\Delta$ and $\Gamma_2$ consists of all the curves in the interior of $\Delta$. Let us now show that both $\Gamma_1$ and $\Gamma_2$ are stable. By symmetry it suffices to prove that $\Gamma_2$ is stable. Suppose $\Gamma_2$ is not stable. Then $\Gamma_2$ would contains a curve $\gamma$ such that one of the non-peripheral components of $F_n^{-1}(\gamma)$, say $\eta$, is homotopic to some $\xi \in \Gamma_1$. Thus $\eta$ encloses at least two points of $P_F$ in the exterior of $\Delta$. By the construction of $F_n$ and $F$, these points must also belong to $P_F$ and thus $F$. Note that in the exterior of $\Delta$, $F_n$ behaves like $f$ and thus maps the inside of $\eta$ to the inside of $\gamma$. In particular, $F_n$ maps the points of $P_F$ contained in the inside of $\eta$ to the interior of $\Delta$. But this is a contradiction since the action of $F_n$ on
these points is the same as $f$ and thus maps these points to some points in the exterior of $\Delta$. This contradiction implies that $\Gamma_2$ is stable. By symmetry $\Gamma_1$ is stable also. Since $\Gamma' = \Gamma_1 \cup \Gamma_2$ is a Thurston obstruction, by symmetry and the fact that both $\Gamma_1$ and $\Gamma_2$ are stable, it follows that both $\Gamma_1$ and $\Gamma_2$ are Thurston obstructions of $F_n$. Since $F_n$ behaves like $f$ in the exterior of $\Delta$, it follows that $\Gamma_1$ is a Thurston obstruction of $f$. This contradicts the assumption that $f$ has no Thurston obstruction. So $\Gamma' = \emptyset$ and all the curves in $\Gamma$ intersect $T$. Claim 1 has been proved.

Now let $\mu_0$ be the standard complex structure. For $k \geq 1$ let $\mu_k$ be the pull back of $\mu_0$ by $F_n^k$. Let $\phi_k$ be the qc homeomorphism of the sphere which fixes $0$, $1$ and $\infty$ and solves the Beltrami equation given by $\mu_k$. Then for any $k \geq 1$,

$$G_{n,k} = \phi_{k-1} \circ F_n \circ \phi_k^{-1}$$

is a rational map of $2d - 1$. By symmetry $G_{n,k}$ is a Blaschke product.

Note that the restriction of each $G_{n,k}$ to $T$ is a circle homeomorphism. By a lemma of Herman (cf. §15 of [29]), there is a uniform $\kappa(d) > 0$ depending only on $d$ such that $\{G_{n,k}\}$ is a normal family of holomorphic functions in a $\kappa_0$-neighborhood of $T$. This implies

**Fact 1.** The sequence $\{G_{n,k}\}$ is equicontinuous in a $\kappa_0$-neighborhood of $T$.

Claim 2: Every $\gamma$ in $\Gamma$ encloses at least two points in $P_{T_n} \setminus \mathcal{O}_n$, which are symmetric about $T$. Let us prove the claim by contradiction. Assume the claim were not true. Then by symmetry, $\gamma$ would enclose no point in $P_{T_n} \setminus \mathcal{O}_n$. Since $\gamma$ is non-peripheral, it would enclose two points in $\mathcal{O}_n$. To get a contradiction, it suffices to show that there is a $d_0 > 0$ independent of $k$ such that for any two distinct points $x, y \in \mathcal{O}_n$ and any $k \geq 1$,

$$d(\phi_k(x), \phi_k(y)) > d_0,$$

where $d(w, w')$ denotes the Euclidean length of the smaller arc connecting $w$ and $w'$. This is because if $\gamma$ encloses two points $x$ and $y$ in $\mathcal{O}_n$, since $\phi_k$ is a plane homeomorphism, $\phi_k(\gamma)$ will enclose $\phi_k(x)$ and $\phi_k(y)$ and by (57) it follows that $l_{[\mu_k]}(\gamma)$ have a positive lower bound for all $k \geq 1$. This contradicts the assumption that $l_{[\mu_k]}(\gamma) \to 0$ as $k \to \infty$. So it suffices to prove (57) holds for some $d_0 > 0$ independent of $k$.

Before the proof of (57) let us recall that $d_{T_{F_n}}([\phi_k], [\phi_{k-1}]) \leq d_{T_{F_n}}([\phi_1], [\phi_0])$ for all $k \geq 1$ (The inequality comes from the fact that the Thurston pull back does not increase the Teichmüller distance, see [8] or [28]). Let $\delta = d_{T_{F_n}}([\phi_1], [\phi_0])$. Then by Proposition 7.2 of [8], we have

$$e^{-2\delta} \cdot l_{[\mu_k]}(\gamma) \leq l_{[\mu_k]}(\gamma) \leq e^{2\delta} \cdot l_{[\mu_k-1]}(\gamma).$$

Because all $\phi_k$ fix $0$, $1$ and $\infty$, the above inequality implies the following fact.

**Fact 2.** There is an $\epsilon > 0$ such that for any two integers $1 \leq k' \leq k'' < k' + q_n$, and any two points $a$ and $b$ in $\mathcal{O}_n$, if $d(\phi_{k'}(a), \phi_{k'}(b)) < \epsilon$, then $d(\phi_{k''}(a), \phi_{k''}(b)) < 2\pi/q_n$.

This is because if $d(\phi_k(a), \phi_k(b))$ is small, then we can find a non-peripheral curve $\gamma$ which encloses $a$ and $b$ such that $l_{[\mu_k]}(\gamma)$ is small (In the post-critically finite case, the existence of such $\gamma$ is obvious. In the sub-hyperbolic semi-rational case, we need to use the fact that for each holomorphic disk $D$, there is a protective annulus $A$ surrounding it such that for all $k \geq 1$, $\phi_k$ is conformal in the larger disk $\overline{D} \setminus A$. Thus by Koebe’s distortion theorem, we have $\text{diam}(\phi_k(D)) \leq \text{dist}(\phi_k(D), T)$). By (58) and $k' \leq k'' < k' + q_n$, we have $l_{[\mu_k]}(\gamma) < e^{-2\rho \cdot \delta} \cdot l_{[\mu_k]}(\gamma)$ is small also. This implies that $d(\phi_k(a), \phi_k(b))$ can be arbitrarily small provided that $d(\phi_k(a), \phi_k(b))$ is small enough.

Now let us go back to the proof of (57). It suffices to prove the existence of a $d_0 > 0$ independent of $k$ such that (57) holds for any two adjacent points $x$ and $y$ in $\mathcal{O}_n$ and any $k \geq q_n$. Let $N > q_n$ be an arbitrary integer and $x, y$ be any two adjacent points.
in $\mathcal{O}_n$. Since $\mathcal{O}_n$ is a periodic orbit of $F_n$ of period $q_n$, there is some $1 \leq l \leq q_n$ such that $d(\phi_N(F_n^l(x)), \phi_N(F_n^l(y))) \geq 2\pi/q_n$. Recall that by Fact 1 the sequence $\{G_{n,k}\}$ is equicontinuous in a $\kappa_0$-neighborhood of $T$. So for the $\epsilon > 0$ guaranteed by Fact 2, we have a $d_0 > 0$ independent of $k$ such that

$$d(\phi_{N-l}(F_n^l(x), \phi_{N-l}(F_n^l(y))) = d(G_{N-l} \circ G_{N-l+1} \circ \cdots \circ G_{N-1}(\phi_N(x)), G_{N-l} \circ G_{N-l+1} \circ \cdots \circ G_{N-1}(\phi_N(y))) < \epsilon$$

provided that $d(\phi_N(x), \phi_N(y)) \leq d_0$. This means the $d_0$ is the desired constant. This is because if $d(\phi_N(x), \phi_N(y)) \leq d_0$, then the above inequality implies

$$d(\phi_{N-l}(F_n^l(x), \phi_{N-l}(F_n^l(y)))) < \epsilon.$$

But by the Fact 2 we would have $d(\phi_N(F_n^l(x)), \phi_N(F_n^l(y))) < 2\pi/q_n$. This is a contradiction. This proves (57) and thus completes the proof of Claim 2.

For Claim 2, it follows that $\gamma$ encloses (at least) a pair of points in $P_{T^*} \setminus \mathcal{O}_n$, which are symmetric about $T$. Let $\mathcal{O}$ denote this pair of points as $z$ and $z^*$. Since $l_{\mu_k}(\gamma) \to 0$, it follows that $d(\phi_k(z), T) \to 0$ and $d(\phi_k(z^*), T) \to 0$ as $k \to \infty$. Note that the forward orbits of $z$ either eventually enters some holomorphic disk or lands on some periodic cycle.

In the first case there is an $l \geq 0$ such that $w = F_n^l(z)$ belongs to a holomorphic disk $D$ for some holomorphic attracting cycle $\mathcal{O}$. Let $\mathcal{O}^*$ be the symmetric image of $\mathcal{O}$ about $T$. By symmetry $\mathcal{O}^*$ is also a holomorphic attracting cycle of $F_n$. By symmetry $w^* = F_n^l(z^*)$ belongs to $D^*$, which is the symmetric image of $D$ about $T$ and is a holomorphic disk for $\mathcal{O}^*$. By symmetry $\mathcal{O}$ and $\mathcal{O}^*$ have the same period. By Fact 1, it follows that $d(\phi_k(w), T) \to 0$ and $d(\phi_k(w^*), T) \to 0$ as $k \to \infty$. Since $\text{diam} (\phi_k(D^*)) \leq \text{diam}(\phi_k(D), T) \leq d(\phi_k(w), T)$ and $\text{diam}(\phi_k(D^*)) \leq d(\phi_k(D^*), T) \leq d(\phi_k(w^*), T)$, so there is a $\eta \in \Gamma$ which encloses $D$ and $D^*$. Assume that the period of $\mathcal{O}$ and $\mathcal{O}^*$ is $p$. Let $I_l, l = 1, \cdots, q_n$, denote the components of $T \setminus \mathcal{O}_n$. For $N > p$, let $I_{i(N)}$ be the one such that

$$d(\phi_N(I_{i(N)}), \phi_N(w)) = \min_{1 \leq i \leq q_n} \{d(\phi_N(I_i), \phi_N(w))\}.$$
In the second case, there is an \( l \geq 0 \) such that \( w = F_n^l(z) \) belongs to a periodic cycle of period \( p \geq 1 \). The argument is the same as the one used in the proof of the first case, with \( D \) and \( D^* \) replaced by \( w \) and \( w^* \), respectively.

The proof of Lemma 6.1 is completed. \( \square \)

By Lemma 6.1 and Theorem 6.1, it follows that \( F_n \) is CLH-equivalent a rational map. Let \( \phi_n \) and \( \psi_n \) be a pair of homeomorphisms of the sphere which fix 0, 1 and \( \infty \) such that

1. \( \phi_n \circ F_n \circ \psi_n^{-1} \) is a rational maps, and
2. \( \phi_n \) is isotopic to \( \psi_n \) rel \( P_{F_n} \), moreover, for each holomorphic disk \( D_i \), \( \phi_n|D_i = \psi_n|D_i \) are holomorphic.

By symmetry of \( F_n \), \( \phi_n \) and \( \psi_n \) can be taken such that they are symmetric about \( T \). Let

\[
G_n = \phi_n \circ F_n \circ \psi_n^{-1}.
\]

Then \( G_n \) is a Blaschke product. Since all the zeros of \( G_n \), except the origin, belong to the exterior of the unit disk, we have

\[
G_n(z) = \lambda_n z^d \prod_{i=1}^{d-1} \frac{z - p_{n,i}}{1 - \overline{p_{n,i}}z}
\]

where \( |\lambda_n| = 1 \) and \( |p_{n,i}| > 1, 1 \leq i \leq d - 1 \).

**Lemma 6.2.** There exist \( 0 < r(d) < \sigma(d) < 1 < \kappa(d) < R(d) < \infty \) depending only on \( d \) such that for any \( z \) with \( |z| < R(d) \), \( G_n^k(z) \to \infty \) as \( k \to \infty \), and for any \( z \) with \( |z| < r(d) \), \( G_n^k(z) \to 0 \) as \( k \to \infty \). Moreover, we have

\[ \kappa(d) \leq |p_{n,i}| \leq R(d) \]

for all \( n \) large enough.

**Proof.** Note that all the finite poles of \( G_n \) belong to the interior of \( \Delta \) and except the origin, all the zeros of \( G_n \) belong to the exterior of \( \Delta \). By a lemma of Herman (cf. §15 of [24]), all the poles, and thus all the zeros by symmetry, are uniformly bounded away from \( T \). This implies the existence of \( \kappa(d) > 1 \) such that \( \kappa(d) \leq |p_{n,i}| \) holds for all \( 1 \leq i \leq d - 1 \) and all \( n \) large enough. So to prove the lemma, it suffices to show the existence of \( R(d) > 1 \) such that \( G_n^k(z) \to \infty \) for all \( |z| > R(d) \). Then the existence of the desired \( r(d) \) follows by symmetry. It is also clear that for such \( R(d) \) we must have \( |p_{n,i}| \leq R(d) \) for all \( 1 \leq i \leq d - 1 \) and all \( n \) large enough.

Now let

\[
\epsilon = (-1)^{d-1} \prod_{i=1}^{d-1} \frac{1}{|p_{n,i}|}.
\]

Then near infinity \( G_n(z) = \epsilon z^d + o(z^d) \). Take \( \eta \) such that \( \eta^{d-1} = \epsilon \). Consider the map

\[
\tilde{G}_n(z) = \eta \cdot G_n \left( \frac{z}{\eta} \right).
\]

Then \( \tilde{G}_n(z) = z^d + o(z^d) \) near infinity. Let \( \Phi(z) = z(1 + o(1)) \) be a holomorphic map in a neighborhood of infinity which conjugates \( \tilde{G}_n \) to \( z \mapsto z^d \). Since all critical orbits of \( G_n \), and thus of \( \tilde{G}_n \), are bounded, it follows that \( \Phi \) can be extended to a Riemann isomorphism from the immediate attracting basin of infinity for \( \tilde{G}_n \) to the exterior of the unit disk.

By Koebe’s 1/4-theorem, the immediate attracting basin of \( G_n \) at infinity contains the
exterior of \{z \mid |z| < 4\}. Go back to \(G_n\), it follows that the immediate attracting basin of \(G_n\) at infinity contains the exterior of \{z \mid |z| < 4/|\eta|\}, and in particular,

\(\text{(60)}\)

\[ |p_{n,i}| \leq \frac{4}{|\eta|}, \quad 1 \leq i \leq d - 1. \]

Now it suffices to prove that \(|\eta|\) has a positive lower bound depending only on \(d\). By the definition of \(\eta\), we have

\[ |\eta| \geq \frac{1}{\max_{1 \leq i \leq d - 1} |p_{n,i}|}. \]

So we need only to show that \(|p_{n,i}|, 1 \leq i \leq d - 1\), have an upper bound depending only on \(d\). To see this note that (60), together with the definition of \(\eta\), implies

\[ \min_{1 \leq i \leq d - 1} |p_{n,i}| \leq \max_{1 \leq i \leq d - 1} |p_{n,i}| \leq 4 \min_{1 \leq i \leq d - 1} |p_{n,i}|. \]

This implies that \(|p_{n,i}|, 1 \leq i \leq d - 1\), are all large provided that one of them is large. From (59), we see if all \(|p_{n,i}|, 1 \leq i \leq d - 1\), are large enough, then \(G_n\) is holomorphic in \(H = \{z \mid 1/2 < |z| < 2\}\), and moreover, \(G_n\) can be arbitrarily close to the linear map \(z \mapsto az\) in \(H\) for some complex number \(a\) with \(|a| = 1\) provided that these \(|p_{n,i}|\) are large enough. But this would imply \(G_n\) has no critical point in \(T\). This is a contradiction. This implies that all \(|p_{n,i}|\) must have an upper bound depending only on \(d\). The proof of Lemma 6.2 is completed.

By taking a subsequence, we may assume that \(G_n\) converges to \(G\) uniformly with respect to the spherical metric. Then \(G/T\) is a critical circle homeomorphism with rotation number \(\alpha\). So there is a \(\alpha\) circle homeomorphism \(h\) which conjugates \(G/T\) to the rigid rotation \(R_\alpha : z \mapsto e^{2\pi i \alpha} z\). Let \(\phi_n, \psi_n : \hat{C} \to \hat{C}\) be the pair of homeomorphisms defined before (59).

**Lemma 6.3.** \(\phi_n|T \to h\) and \(\psi_n|T \to h\) uniformly.

**Proof.** To simplify the notations, let us write \(\phi_n|T\) and \(\psi_n|T\) as \(\phi_n\) and \(\psi_n\) respectively. We claim that \(\phi_n\) and \(\psi_n\) converge uniformly. Let us first prove Lemma 6.3 by assuming the claim. Let \(\phi\) and \(\psi\) be the limit maps respectively. Since the convergence is uniform, it follows that both \(\phi\) and \(\psi\) are circle homeomorphisms. Since \(\phi_n|F_{G_n} = \psi_n|F_{G_n}\), for any \(k \geq 0\) we have \(\phi_n(F^k_{G_n}(1)) = \psi_n(F^k_{G_n}(1)) = G^k_{G_n}(1)\). Since \(F_n \to F\) and \(G_n \to G\) uniformly on \(T\), let \(n \to \infty\) we get \(\phi(F^k(1)) = \psi(F^k(1)) = G^k(1)\) for all \(k \geq 1\). This implies that \(\phi\) and \(\psi\) coincide on \(\{F^k(1) = e^{2\pi i k \alpha}\}_{k=0}^\infty\), which is a dense subset of \(T\). It follows that \(\phi = \psi\). Since \(\phi(1) = \psi(1) = h(1) = 1\), it follows that \(\phi = \psi = h\).

It remains to show that \(\phi_n\) and \(\psi_n\) converge uniformly. Let us do this only for \(\phi_n\) since the same argument will work for \(\psi_n\). By construction, the point 1 is a critical point for \(F\), \(G\), \(F_n\) and \(G_n\). For any \(N \geq 1\), since \(F_n \to F\) and \(G_n \to G\) uniformly, by taking \(n\) large enough, we can make sure the orbit segments

\[ \mathcal{O}_N(G) = \{1, G^1(1), \cdots, G^N(1)\} \]

and

\[ \mathcal{O}_N(G_n) = \{1, G^1_n(1), \cdots, G^N_n(1)\} \]

have the same order, and

\[ \mathcal{O}_N(F) = \{1, F^1(1), \cdots, F^N(1)\} \]

and

\[ \mathcal{O}_N(F_n) = \{1, F^1_n(1), \cdots, F^N_n(1)\} \]

have the same order. Since \(\phi_n\) and \(\psi_n\) are circle homeomorphisms and map \(\mathcal{O}_N(F_n)\) to \(\mathcal{O}_N(G_n)\) with the order being preserved, thus \(\mathcal{O}_N(F_n)\) and \(\mathcal{O}_N(G_n)\) have the same order. All of these implies that for any fixed \(N\), by taking \(n\) large enough, all the four orbit segments, \(\mathcal{O}_N(F), \mathcal{O}_N(F_n), \mathcal{O}_N(G)\) and \(\mathcal{O}_N(G_n)\), have the same order.
Let $\epsilon > 0$ be an arbitrary small number. Since $G\mid \mathbb{T}$ and $F\mid \mathbb{T}$ are circle homeomorphisms with irrational rotation numbers, taking $N = N(\epsilon)$ large enough we can make sure the length of each component of $\mathbb{T} \setminus \mathcal{O}_N(F)$ and $\mathbb{T} \setminus \mathcal{O}_N(G)$ is less than $\epsilon/4$. For such $N$, since $F_n \to F$ and $G_n \to G$ uniformly, there exists an $M' > 1$ such that for all $n \geq M'$, the length of each component of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$ and $\mathbb{T} \setminus \mathcal{O}_N(G_n)$ is less than $\epsilon/3$. For a component $I$ of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$, we use $\bar{I}$ and $\bar{I}'$ to denote the components of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$, which are adjacent to $I$ from the left and right respectively. Similarly, for a component $J$ of $\mathbb{T} \setminus \mathcal{O}_N(G_n)$, we use $J'$ and $J''$ to denote the components of $\mathbb{T} \setminus \mathcal{O}_N(G_n)$, which are adjacent to $J$ from the left and right respectively. For such $N$, since $F_n \to F$ and $G_n \to G$ uniformly on $\mathbb{T}$, there exists an $M > M'$ such that for all $m, n \geq M$, the following holds: For any component $I$ of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$ and any component $J$ of $\mathbb{T} \setminus \mathcal{O}_N(G_m)$, let $\bar{I}$ and $\bar{J}$ be the corresponding component of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$ and $\mathbb{T} \setminus \mathcal{O}_N(G_m)$ respectively, then

$$I \subset \bar{I} \cup \bar{I}' \cup \bar{I}$$

and

$$J \subset \bar{J} \cup \bar{J}' \cup \bar{J}''.$$  

(61)

For the above $M$, let $n, m > M$ be any two integers. Let $z \in \mathbb{T}$ be an arbitrary point and $I$ be a component of $\mathbb{T} \setminus \mathcal{O}_N(F_n)$ such that $z \in \bar{I}$. Since $I \subset \bar{I} \cup \bar{I}' \cup \bar{I}''$ by (61), we have $z \in \bar{I} \cup \bar{I}' \cup \bar{I}''$. Let $J = \phi_n(I)$ and $\bar{J} = \phi_m(\bar{I})$. It follows from (61) that

$$\phi_n(z) \in \bar{J} \subset \bar{J}'' \cup \bar{J} \cup \bar{J}''.$$  

and

$$\phi_m(z) \in \bar{J}'' \cup \bar{J} \cup \bar{J}''.$$  

Then $|\phi_n(z) - \phi_m(z)| \leq |\bar{J}| + |\bar{J}'| + |\bar{J}''| < \epsilon$. This proves that $\phi_n$ converges uniformly. The proof of Lemma 6.3 is completed. \hfill $\Box$

Recall that $D_i$s denote all the holomorphic disks of $F$ and $F_n$. Let

$$P = P_{F_n} \setminus (\mathbb{T} \cup \cup_i D_i) = P_F \setminus (\mathbb{T} \cup \cup_i D_i).$$

Then $P$ is a finite set.

**Lemma 6.4.** There exist a $K > 1$ such that for all $n \geq 1$,

1. $\phi_n(P \cup \cup_i D_i) \subset \{ z \mid 1/K < |z| < K \}$,
2. for every $D_i$, $\text{diam}(\phi_n(D_i)) > 1/K$,
3. for every two distinct points $z$ and $w$ in $P$, dist$(\phi_n(z), \phi_n(w)) > 1/K$,
4. for every point $z \in P$ and every $D_i$, dist$(\phi_n(z), \phi_n(D_i)) > 1/K$,
5. for every $D_i$, dist$(\phi_n(D_i), \mathbb{T}) > 1/K$,
6. for every two distinct $D_i$ and $D_j$, dist$(\phi_n(D_i), \phi_n(D_j)) > 1/K$,
7. for every point $z \in P_{F_n} \setminus \mathbb{T}$, dist$(\phi_n(z), \mathbb{T}) > 1/K$.

**Proof.** By Lemma 6.2 it follows that there is a $K > 1$ such that

$$\phi_n(P \cup \cup_i D_i) \subset \{ z \mid |z| < K \}.$$  

By symmetry we further have $\phi_n(P \cup \cup_i D_i) \subset \{ z \mid 1/K < |z| < K \}$. This proves (1).

Now let us prove (2). Let us fix a $D_i$. By symmetry we may assume that $D_i$ belongs to the exterior of $\Delta$. Note that $F_n[D_i] = F[D_i]$. Then $D_i$ contains a periodic point of $F_n$, say $x$. Suppose the period of $x$ is $p \geq 1$. There are two cases.

In the first case, $0 < |DF_n^p(x)| = |DF^p(x)| < 1$. Since $\phi_n$ and $\psi_n$ are holomorphic and identified with each other on all holomorphic disks, $F_n^p$ and $G_n^p$ are holomorphic conjugate on $D_i$. In particular, $0 < |DG_n^p(\phi_n(x))| = |DF^p(x)| < 1$. By Lemma 6.2 and a compactness argument, there is a $r > 0$ independent of $n$ such that $DG_n^p \neq 0$ in the disk $B_r(\phi_n(x))$. Now let $U_n = \phi_n(D_i)$. For $k \geq 0$, define $U_{k+1}$ to be the component of $G_n^{-p}(U_k)$ which contains $\phi_n(x)$. Then we have a sequence of increasing domains
let $l \geq 1$ be the least integer such that $U_l$ contains a critical point of $G^n_U$. Then $\text{diam}(U_l) \geq r$. If (2) were not true, then $\text{diam}(U_0)$ could be arbitrarily small for some $n$. But by Lemma 6.2 and a compactness argument, this would imply that $l$ could be arbitrarily large provided that $\text{diam}(U_0)$ is small enough. Go back to $F_n$, this means some critical point of $F_n$ goes through a long orbit segment before it enters $D_l$, and the length of this orbit segment can be arbitrarily large for some $n$. But this is impossible. Since $F_n$ is obtained by modifying $F$ only essentially on $T$, so there is an $L \geq 1$ such that for any critical point $c$ of $F_n$, either $F_n(c) \notin D_l$ for all $k \geq 0$, or there is some $0 \leq k \leq L$ such that $F_n^k(c) \in D_l$. This proves (2) in the first case.

In the second case, $|DF_n^k(x)| = |DF^k(x)| = 0$. By taking a subsequence, we may assume that $\phi_n(x) \to z$. It follows that $DG_n(z) = 0$. Since $G_n$ converges to $G$ uniformly in a small neighborhood of $z$, it follows that $\max_{w \in B_r(z)} |DG_n^p(w)| \to 0$ as $r \to 0$ and $n \to \infty$. Now suppose $\text{diam}(\phi_n(D_l)) \subset B_r(z)$ for some $r > 0$ small. Then

$$\frac{\text{diam}(G_n^p(\phi_n(D_l)))}{\text{diam}(\phi_n(D_l))} < \max_{w \in B_r(z)} |DG_n^p(w)|.$$

But on the other hand, since $G_n^p(\phi_n(D_l)) = \phi_n(F_n^p(D_l)) = \phi_n(F^p(D_l))$ and since $\phi_n$ is holomorphic in a larger domain containing $D_l$, by Koebe’s distortion theorem,

$$\frac{\text{diam}(G_n^p(\phi_n(D_l)))}{\text{diam}(\phi_n(D_l))} = \frac{\text{diam}(\phi_n(F^p(D_l)))}{\text{diam}(\phi_n(D_l))} > \eta \frac{\text{diam}(F^p(D_l))}{\text{diam}(D_l)}.$$

This implies that $\text{diam}(\phi_n(D_l))$ has a positive lower bound as $n \to \infty$. This completes the proof of (2).

We prove (3) by contradiction. Suppose (3) were not true. By taking a subsequence we may assume that there exist $x, y \in P$ such that $\text{dist}(\phi_n(x), \phi_n(y)) \to 0$ as $n \to \infty$. By definition of $\Sigma_{\text{cop}}^d$, there is an integer $k \geq 0$ such that $w = F_k(x)$ either belongs to a holomorphic disk $D_l$ or belongs to a periodic cycle containing a critical point, which is not a holomorphic attracting cycle and thus does not attract any other critical orbit.

In the first case, $w$ belongs to a holomorphic disk $D_l$. Noting that $\phi_n$ is holomorphic in a larger domain containing $D_l$, by Koebe’s distortion theorem, it follows that for any $z \in P_{F_n}$ and $z \neq w$, there is a $\delta > 0$ such that $\text{dist}(\phi_n(z), \phi_n(w)) > \delta$ for all $n \geq 1$. In the second case, $w$ belongs to some periodic cycle containing some critical point which does not attract any other critical orbit. Then $\phi_n(w)$ belongs to a super-attracting cycle of $G_n$ which does not attract any other critical orbit. By Lemma 6.2 and a compactness argument, it follows that there is an $r > 0$ such that the immediate attracting basin of this cycle contains the disk $B_r(\phi_n(w))$ for all $n \geq 1$. This implies that for all $z \in P_{F_n}$ with $z \neq w$, $\text{dist}(\phi_n(w), \phi_n(z)) > r$ for all $n \geq 1$. Now let $0 \leq l < k$ be the largest integer and $\zeta = F_l(x)$ such that there exists $\xi \in P_{F_n}$ with $\xi \neq \zeta$ and $\text{dist}(\phi_n(\xi), \phi_n(\zeta)) \to 0$ as $n \to \infty$. By the maximal property of $l$ we must have $F_n(\xi) = F_n(\zeta)$. By taking a subsequence we may assume that $\phi_n(\xi)$ and $\phi_n(\zeta)$ converge to a point, say $c$, and $\phi_n(F_n(\xi)) = \phi_n(F_n(\zeta))$ converges to a point, say $v$. Then $c$ must be a critical point of $G$ and $G(c) = v$. Since $G_n \to G$ uniformly, by taking a subsequence if necessary, there are critical points of $G_n$, say $c_1^n, \ldots, c_m^n$, near all of which converge to $c$ as $n \to \infty$. Again by taking a subsequence if necessary, we may assume $c_i^n = \phi_n(c_i)$ where $c_i, 1 \leq i \leq m$, are critical points of $F_n$ (also of $F$). Then $\text{dist}(\phi_n(F_n(c_i)), \phi_n(F_n(\zeta))) = \text{dist}(G_n(c_i^n), G_n(\phi_n(\zeta))) \to 0$. By the definition of $\zeta$, we have $F_n(c_i) = F_n(\zeta)$ for all $1 \leq i \leq m$. This implies that $G_n(c_i^n) = \phi_n(F_n(\zeta))$ for all $1 \leq i \leq m$. Note that $\phi_n(\zeta) \neq \phi_n(\xi)$ and $G_n(\phi_n(\zeta)) = G_n(\phi_n(\xi)) = \phi_n(F_n(\zeta))$. Since $G_n \to G$ uniformly in a neighborhood of $c$ and since $\phi_n(F_n(\zeta))$ converges to $v$, the number of the pre-images of $\phi_n(F_n(\zeta))$ in a small neighborhood of $c$ must be equal to
deg\_c G, the local degree of G at c. On the other hand, the above reasoning implies that this number is at least $\sum_{i=1}^{m} \deg_{c_i} G_n$. Since

$$\deg_{c} G - 1 = \sum_{i=1}^{m} (\deg_{c_i} G_n - 1)$$

it follows that $m = 1$ and $\deg_{c} G = \deg_{c_1} G_n$. But $c_i, \phi_n(\zeta)$ and $\phi_n(\xi)$ are all mapped to $\phi_n(F_n(\zeta))$ by $G_n$ and $\phi_n(\zeta) \neq \phi_n(\xi)$, so the number of the pre-images of $\phi_n(F_n(\zeta))$ in a small neighborhood of $c$ is at least $\deg_{c_1} G_n + 1 = \deg_{c} G + 1$. This is a contradiction. This completes the proof of (3).

Since each $\overline{D_i}$ has a protective annulus $A_i$ which does not contain points in $P_{F_n}$ and on which $\phi_n$ is holomorphic, by Koebe's distortion theorem, the thickness of $\phi_n(A_i)$ is $\geq \text{diam}(\phi_n(D_i))$. Since $\text{diam}(\phi_n(D_i)) > 1/K$ by (2), it follows that $\phi_n(A_i)$ has definite thickness. This implies (4), (5) and (6).

By symmetry it suffices to prove (7) for $z \in P_{F_n}$ and $|z| > 1$. Note that the forward orbits of all critical points of $G$ in the exterior of the unit disk converge to some attracting or super-attracting cycles of $G$ and are thus bounded away from the unit circle. Since for any $R > 1, G_n \to G$ uniformly in the compact set $\{z : 1 \leq |z| \leq R\}$, it follows that an attracting or super-attracting periodic cycle of $G_n$ in the exterior of the unit disk converges to the corresponding one for $G$, and the forward orbits of all critical points of $G_n$ in the exterior of the unit disk converge uniformly, with respect to $n$, to these attracting or super-attracting cycles of $G_n$. This implies that all the critical orbits of $G_n$ which belong to the exterior of the unit disk are uniformly bounded away from $T$. This proves (7) and thus completes the proof of Lemma 6.4.

Lemma 6.5. There exist a pair of homeomorphisms $\phi, \psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fix 0, 1 and $\infty$ such that

1. $\phi \circ F = G \circ \psi$,
2. $\phi$ is isotopic to $\psi$ rel $P_F \cup \cup_i \overline{D_i}$ where $D_i$'s are all the holomorphic disks of $F$, and $\phi|D_i = \psi|D_i$ is holomorphic for each holomorphic disk $D_i$.

Proof. By (1) and (2) of Lemma 6.4 and the fact that $\phi_n = \psi_n$ is holomorphic in a larger domain containing each $\overline{D_i}$, we may assume, by taking a subsequence if necessary, that $\phi_n = \psi_n$ converge uniformly to some univalent map on some domain compactly containing $D_i$. Let us denote the limit of $\phi_n(D_i)$ by $U_i$. Then by Lemmas 6.3 and 6.4 it follows that when restricted to the set $P_F \cup \cup_i \overline{D_i}$ (Note that $P_{F_n} \setminus T = P_F \setminus T$), $\phi_n$ and $\psi_n$ converge uniformly to some homeomorphism

\begin{equation}
\sigma : P_F \cup \cup_i \overline{D_i} \to P_G \cup \cup_i \overline{U_i}.
\end{equation}

Note that $\sigma|T = h$ where $h : T \to T$ is the circle homeomorphism in Lemma 6.5 and $\sigma$ is holomorphic in each $D_i$, and moreover,

\begin{equation}
\sigma \circ F|P_F \cup \cup_i \overline{D_i} = G \circ \sigma|P_F \cup \cup_i \overline{D_i}.
\end{equation}

The maps $\phi$ and $\psi$ are constructed by perturbing $\phi_n$ and $\psi_n$ for a large $n$. Before that, by Lemma 6.4 we may assume that there exists a $\delta > 0$ such that for every $n \geq 1$,

1. the closures of the $\delta$-neighborhoods of all points in $P$, the closures of the $\delta$-neighborhoods of all holomorphic disks $D_i$, and the closure of the $\delta$-neighborhood of $T$, are all disjoint with each other.

Since when restricted to $P_F \cup \cup_i \overline{D_i}$, $\phi_n = \psi_n$ converges uniformly to a homeomorphism

$\sigma : P_F \cup \cup_i \overline{D_i} \to P_G \cup \cup_i \overline{U_i}$, by deforming $\phi_n$ rel $P_{F_n}$ we may further assume

\begin{itemize}
\item[(62)]
\item[(63)]
\end{itemize}
2. When restricted to the closure of each of the above \( \delta \)-neighborhoods, \( \phi_n \) converges to a homeomorphism \( \chi \).

Since \( F_n \to F \) and \( G_n \to G \) uniformly with respect to the spherical metric, from \( \phi_n \circ F_n = G_n \circ \psi_n \) and the above assumption we made on \( \phi_n \), it follows that when restricted to a definite neighborhood of \( \partial \mathcal{U} \cup \bigcup \overline{D_i} \), \( \psi_n \) converge uniformly to some homeomorphism \( \tau \).

By the above assumption on \( \phi_n \), it follows that for any \( \epsilon > 0 \) small, there is an \( N \) such that for all \( n > N \), we can perturb \( \phi_n \) to construct a homeomorphism \( \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) which fixes 0, 1 and \( \infty \), such that

1. When restricted to the closure of each of the above \( \delta \)-neighborhoods, \( \phi = \chi \).
2. \( \text{dist}(\phi, \phi_n) < \epsilon. \)

Now let us show that when \( \epsilon > 0 \) is small enough and \( n \) is large enough, there exists a homeomorphism \( \psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that the two assertions in Lemma 6.5 hold. To see this, let \( \Pi \) denote the set of the critical values of \( F \). Take \( 0 < \rho < \delta \) such that all the disks, \( B_{\rho}(v), v \in \Pi \), are disjoint. Let

\[
X = \bigcup_{v \in \Pi} B_{\rho}(v) \text{ and } Y = F^{-1}(X).
\]

Note that \( Y \) is the union of finitely many Jordan domains whose closures are disjoint from each other. Since \( \psi \) is the only critical value of \( F \) contained in \( B_{\rho}(v) \), each of these Jordan domains either contains exactly one critical point of \( F \), which is mapped to \( v \) by \( F \), or contains no critical point of \( F \). In the following we will first define \( \psi \) on \( \hat{\mathbb{C}} \setminus Y \) and then extend it to \( Y \).

Now take an arbitrary \( x \in \hat{\mathbb{C}} \setminus Y \). Since the degree of \( G \) is \( 2d - 1 \), \( G^{-1}(\phi \circ F(x)) \) contains 2d - 1 points, counting by multiplicities. Define \( \psi(x) \) to be the one which is closest, with respect to the spherical metric, to \( \psi_n(x) \) among these 2d - 1 points. We need to explain that such definition does not cause any ambiguity. This comes from the following two observations. The first one is that when \( \epsilon > 0 \) is small and \( n \) is large, the set \( G^{-1}(\phi \circ F(x)) \) is close to the set \( G_n^{-1}(\phi_n \circ F_n(x)) \). The second one is that any two points in \( G^{-1}(\phi \circ F(x)) \) are uniformly bounded away from each other, because \( \phi \circ F(x) \) is bounded away from the set of the critical values of \( G \). Thus \( \psi \) can be well defined in \( \hat{\mathbb{C}} \setminus Y \). From the definition it is clear that on \( \hat{\mathbb{C}} \setminus Y \), \( \psi \) is locally homeomorphic and satisfies \( \phi \circ F = G \circ \psi \).

Let \( U \) be one of the Jordan components of \( Y \). Note that on \( \partial U \) \( \psi \) has been defined and satisfies \( \phi \circ F = G \circ \psi \). Then \( \psi(\partial U) \) is a component of \( G^{-1}(\phi \circ F(\partial U)) \). Since \( \phi \circ F(\partial U) \) contains no critical values of \( G \), \( \psi(\partial U) \) must be a Jordan curve. Let \( V \) be the Jordan domain bounded by this curve. Note that \( U \) and \( V \) do not depend on the choice of \( \epsilon \) and \( n \), and that \( \partial U \) contains no critical points of \( F \), and \( \partial V \) contains no critical point of \( G \). Let \( V_n = \psi_n(U) \). By taking \( \epsilon > 0 \) small and \( n \) large enough, \( \psi \) and \( \psi_n \) can be arbitrarily close to each other on \( \partial U \). Thus \( \partial V_n \) and \( \partial V \) can be arbitrarily close to each other. We have two cases.

In the first case, \( U \) contains no critical points of \( F \), and thus contains no critical points of \( F_n \) for all \( n \) large enough. This implies that \( V_n \) contains no critical points of \( G_n \) for all \( n \) large enough and thus \( V \) contains no critical points of \( G \). Then for any \( z \in U \), there is a unique point \( w \in V \) such that \( \phi(F(z)) = G(w) \). Define \( \psi(z) = w \). It is easy to see that \( \psi : U \to V \) is a homeomorphism and \( \phi \circ F = G \circ \psi \).

In the second case, \( U \) contains exactly one critical point of \( F \), say \( c \). Then \( U \) contains exactly one critical point of \( F_n \), say \( c_n \), which has the same local degree as \( c \) and \( c_n \to c \) as \( n \to \infty \). Thus \( V_n \) contains exactly one of the critical points of \( G_n \), \( \psi_n(c_n) \), which has the same local degree as that of \( c_n \). Since \( G_n \to G \) uniformly with respect to the
spherical metric, it follows that $V$ has exactly one critical point of $G$, which has the same degree as that of $\psi(c_n)$, and thus has the the same local degree as that of $c$. Then there is an obvious way to extend $\psi$ to $U$ such that $\psi: U \to V$ is a homeomorphism and $\phi \circ F = G \circ \psi$. In particular, $\psi$ maps the critical point of $F$ to a critical point of $G$.

From the construction we have $\phi \circ F = G \circ \psi$. Note that by taking $\epsilon > 0$ small and $n$ large, $\phi$ and $\phi_n$, $F$ and $F_n$, and $G$ and $G_n$ can be arbitrarily close to each other. From $\phi_n \circ F_n = G_n \circ \psi_n$ and $\phi \circ F = G \circ \psi$, it follows that $\psi$ and $\psi_n$ can be arbitrarily close to each other provided that $\epsilon > 0$ is small enough and $n$ is large enough. Since $\psi_n$ fixes $0$, $1$ and $\infty$, and since $\psi$ can be arbitrarily close to $\psi_n$ and maps critical points of $F$ to critical points of $G$, it follows that $\psi$ fixes $0$, $1$ and $\infty$ also. From the construction of $\psi$, we see that $\psi: \hat{C} \to \hat{C}$ is locally homeomorphic. Since $\hat{C}$ is compact, it follows that $\psi: \hat{C} \to \hat{C}$ is a homeomorphism.

Now let us prove the second assertion of Lemma 6.5. By the definition of $\phi$, it follows that $\phi|P_F \cup \bigcup_i D_i = \sigma$ where $\sigma$ is the map in (63). This, together with (63) and $\phi \circ F = G \circ \psi$, implies that $G \circ \sigma|P_F \cup \bigcup_i D_i = G \circ \psi|P_F \cup \bigcup_i D_i$. Since $\psi_n|P_F \cup \bigcup_i D_i \to \sigma$ and $\psi_n$ can be arbitrarily close to $\psi$ provided that $\epsilon > 0$ is small and $n$ is large, it follows that $\psi|P_F \cup \bigcup_i D_i = \sigma$. In particular, $\phi|D_i = \psi|D_i$ are holomorphic.

Now let us describe how to construct an isotopy between $\phi$ and $\psi \circ P_F \cup \bigcup_i D_i$. Recall that $P_F \cap \mathcal{T} = \mathcal{O}_n$ and $\phi_n|\mathcal{O}_n = \psi_n|\mathcal{O}_n$. Since all the interval components of $\mathcal{T} \setminus \mathcal{O}_n$ can be arbitrarily small provided that $n$ is large, we can deform $\psi_n$ in a small neighborhood of $\mathcal{T}$ to get a homeomorphism $\omega_n: \hat{C} \to \hat{C}$ such that

1. $\omega_n|\mathcal{T} = \phi_n|\mathcal{T}$,
2. $\psi_n$ is isotopic to $\omega_n$ rel $P_{F_n} \cup \bigcup_i D_i$.

It is clear that $\omega_n$ can be arbitrarily close to $\psi_n$ provided that $n$ is large enough. Since $\phi_n$ is isotopic to $\psi_n$ rel $P_{F_n} \cup \bigcup_i D_i$, it follows that $\phi_n$ is isotopic to $\omega_n$ rel $P_{F_n} \cup \bigcup_i D_i$. Let

$$H(t, \cdot), \ 0 \leq t \leq 1$$

be the isotopy between $\phi_n$ and $\omega_n$. Since $\phi_n|\mathcal{T} = \omega_n|\mathcal{T}$, $H$ can be constructed such that

$$H(t, \cdot)|\mathcal{T} = \omega_n|\mathcal{T} = \phi_n|\mathcal{T} \text{ for all } 0 \leq t \leq 1.$$ 

On the other hand, let $\xi: \hat{C} \to \hat{C}$ be a homeomorphism such that

$$\phi|P_F \cup \bigcup_i D_i = \xi \circ \phi_n|P_F \cup \bigcup_i D_i = \xi \circ \omega_n|P_F \cup \bigcup_i D_i.$$ 

Note that $P_F \setminus \mathcal{T} = P_F \setminus \mathcal{T}$ and $\omega_n|P_F \cup \bigcup_i D_i = \phi_n|P_F \cup \bigcup_i D_i$. So when restricted to $P_F \cup \bigcup_i D_i$, $\omega_n$ and $\phi_n$ converge to $\phi$. From this it follows that the homeomorphism $\xi$ can be chosen to be arbitrarily close to the identity map provided that $n$ is large enough. This implies that by taking $\epsilon > 0$ small enough and $n$ large enough, $\phi$ and $\xi \circ \phi_n$ can be arbitrarily close to each other, and are thus isotopic to each other rel $P_{F_n} \cup \bigcup_i D_i$. Note that

$$\xi \circ H(t, \cdot), \ 0 \leq t \leq 1,$$

is an isotopy between $\xi \circ \phi_n$ and $\xi \circ \omega_n$ rel $P_F \cup \bigcup_i D_i$. Since by taking $n$ large enough and $\epsilon > 0$ small enough, $\xi$ can be arbitrarily close to the identity, $\omega_n$ can be arbitrarily close to $\psi_n$, and $\psi_n$ can be arbitrarily close to $\psi$, it follows that $\xi \circ \omega_n$ can be arbitrarily close to $\psi$, and is thus isotopic to $\psi$ rel $P_F \cup \bigcup_i D_i$.

The proof of Lemma 6.5 is completed.

□

Now we can prove the existence part of Theorem 2.1 by performing qc surgery on $G$. The process is routine. Let $h: \mathcal{T} \to \mathcal{T}$ be the circle homeomorphism in Lemma 6.3. Let
$H : \Delta \to \Delta$ be the Douady-Earle extension of $h$. Define

$$\hat{G}(z) = \begin{cases} G(z) & \text{for } |z| \geq 1 \\ H \circ R_\alpha \circ H^{-1}(z) & \text{for } |z| < 1 \end{cases}$$

(64)

Let $\mu_0$ be the complex structure in $\Delta$ obtained by pulling back the standard complex structure in $\Delta$ by $H^{-1}$. We then pull back $\mu_0$ to the whole plane by the iteration of $\hat{G}$ and get a $\hat{G}$-invariant complex structure $\mu$. Let $\xi$ be the qc homeomorphism of the plane to itself which solves the Beltrami equation given by $\mu$ and fixes 1 and maps $H(0)$ to 0. It is clear that

$$g = \xi \circ \hat{G} \circ \xi^{-1}$$

is a Siegel polynomial map in $\Sigma^{\alpha,d}_{\text{geom}}$. Let us show that $f$ and $g$ are CLH-equivalent. Recall that $\phi \circ F = G \circ \psi$. Define $\hat{\phi} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $\hat{\phi}(z) = \phi(z)$ for $|z| \geq 1$ and $\hat{\phi}(z) = H(z)$ for $|z| < 1$. Similarly, define $\hat{\psi} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $\hat{\psi}(z) = \psi(z)$ for $|z| \geq 1$ and $\hat{\psi}(z) = H(z)$ for $|z| < 1$. The isotopy between $\phi$ and $\psi$ rel $P_f \cup \cup_{i=1} D_i$ induces an isotopy between $\hat{\phi}$ and $\hat{\psi}$ rel $P_f \cup \cup_{i=1} \overline{D_i}$, where the later union contains only the holomorphic disks in the outside of the unit disk. Let $\Omega_i, 1 \leq i \leq m$, denote all the components of $f^{-1}(\Delta)$ other than $\Delta$. By only change $\hat{\psi}$ in each $\Omega_i$, we can get a homeomorphism, say $\hat{\psi}$ such that

$$\hat{\phi} \circ f = \hat{G} \circ \hat{\psi}.$$ 

Since each $\Omega_i$ is a Jordan domain which does not intersect $P_f \cup \cup_{i=1} \overline{D_i}$, we can continuously deform $\hat{\psi}$ to $\hat{\psi}$ rel $P_f \cup \cup_{i=1} \overline{D_i}$. This implies that $\hat{\psi}$ is isotopic to $\hat{\psi}$, and is thus isotopic to $\hat{\phi}$ rel $P_f \cup \cup_{i=1} \overline{D_i}$. From (64) we get

$$\xi \circ \hat{\phi} \circ f = g \circ \xi \circ \hat{\psi}.$$ 

Note that on each holomorphic disk $D_i$ of $f$, $\hat{\phi} = \hat{\psi} = \hat{\psi}$ is holomorphic, and that in the attracting basin of each attracting cycle of $G$ which lies in the exterior of $\Delta$, $\mu = 0$ and thus $\xi$ is holomorphic. This implies that $\xi \circ \hat{\phi} = \xi \circ \hat{\psi}$ is holomorphic on each holomorphic disk $D_i$ of $f$. Since $\hat{\phi}$ is isotopic to $\hat{\psi}$ rel $P_f \cup \cup_{i=1} \overline{D_i}$, $\xi \circ \hat{\phi}$ is isotopic to $\xi \circ \hat{\psi}$ rel $P_f \cup \cup_{i=1} \overline{D_i}$. By the definition of $\xi$, $\hat{\phi}$ and $\hat{\psi}$, it follows that $\xi \circ \hat{\phi} \Delta = \xi \circ \hat{\psi} \Delta$ are holomorphic. All of these implies that $g$ is CLH-equivalent to $f$. This proves the existence part of Theorem 2.1.

Now it remains to prove that the $g$ is unique up to a linear conjugation. Let $\hat{G}$ be the modified Blaschke product defined in (64). Let $K_{\hat{G}}$ be the set of all points whose forward orbits under the iteration of $\hat{G}$ is bounded. Let $J_{\hat{G}} = \partial K_{\hat{G}}$. We call $J_{\hat{G}}$ the Julia set of $\hat{G}$. Let us first prove the uniqueness part of Theorem 2.1 by assuming the following lemma.

**Lemma 6.6.** The set $J_{\hat{G}}$ has zero Lebesgue measure.

Suppose $f$ is also CLH-equivalent to a Siegel polynomial $h \in \Sigma^{\alpha,d}_{\text{geom}}$. By the definition of CLH-equivalence and the fact that the Siegel disks of both $g$ and $h$ are quasi-circles(by Shishikura’s theorem), we have a pair of qc homeomorphisms $\phi_1, \phi_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

1. $\phi_1 \circ g = h \circ \phi_2$,
2. $\phi_1$ is isotopic to $\phi_2$ rel $P_g$ where $P_g$ is the post-critical set of $g$,
3. when restricted to the interior of the Siegel disk and an open neighborhood of each attracting periodic cycle of $g$, $\phi_1 = \phi_2$ are holomorphic.
Note that both $\phi_1$ and $\phi_2$ must map the center of the Siegel disk for $g$ to the center of the Siegel disk for $h$, and map the critical points of $g$ to those of $h$. So by a linear conjugation if necessary, we may assume that both $\phi_1$ and $\phi_2$ fix $0$, $1$, and $\infty$. Suppose $\phi_k : \mathbb{C} \to \mathbb{C}$ is a homeomorphism for $k \geq 2$ which is isotopic to $\phi_1 \text{ rel } P_g$. Since $\phi_1 \circ g = h \circ \phi_2$, we can define $\phi_{k+1}$ by lifting $\phi_k$ through

$$\phi_k \circ g = h \circ \phi_{k+1}. \tag{66}$$

It is clear that $\phi_{k+1}$ is isotopic to $\phi_2$ and is thus isotopic to $\phi_1 \text{ rel } P_g$. By induction we get a sequence of qc homeomorphisms $\hat{\phi}_k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fix $0$, $1$ and $\infty$ and satisfy the above equation. Note that the qc constants of each $\hat{\phi}_k$ is bounded by that of $\hat{\phi}_1$. Let $\mu_k$ be the Beltrami coefficient of $\hat{\phi}_k$. Since all the points in the Fatou set of $g$ is either contracted to some attracting cycle of $g$, or is eventually mapped to the interior of the Siegel disk of $g$, $\mu_k \to 0$ on the Fatou set of $g$. Since $J_g$ is the qc image of $G_{j_0}$, by Lemma 6.6 $J_g$ has zero Lebesgue measure. So $\mu_k \to 0 \text{ a.e.}$ This implies that $\hat{\phi}_k$ converges to some linear map. Letting $k \to \infty$ in (66), it follows that $g$ is linearly conjugate to $h$. This implies the uniqueness part of Theorem 2.1.

Now let us prove Lemma 5.6. The proof is by contradiction. Suppose $J_{\hat{G}}$ has positive Lebesgue measure. Let $z_0$ be a Lebesgue point of $J_{\hat{G}}$. For $n \geq 1$ let $z_n = g^n(z_0)$ (Note that $\hat{G} = G$ in the exterior of $\Delta$). By Proposition 1.14 of [12], $z_n$ accumulates to $\mathbb{T}$. The contradiction is derived by using an idea of McMullen [14]. We first show that there is a subsequence $n_j, j = 0, 1, \cdots$, and cones spanned at the critical points in $\mathbb{T}$ such that each $z_{n_j}$ belongs to one of these cones. Then for each $n_j$, we can take a small disk, say $B_j$, in the cone such that

1. $G(B_j) \subset \Delta$,
2. $\text{dist}(B_j, z_{n_j}) \leq \text{dist}(B_j, \mathbb{T}) \approx \text{diam}(B_j) \approx \text{dist}(z_{n_j}, \mathbb{T})$.

Let $P_{G_j}$ denote the post-critical set of $G$. From (2) we can take a Jordan domain $A_j$ which is disjoint with $P_{G_j}$ and contains both $B_j$ and $z_{n_j}$ such that

$$\text{diam}(A_j) \approx \text{dist}(A_j, \mathbb{T}).$$

Let $X$ be the unbounded component of $\hat{\mathbb{C}} \setminus P_{G_j}$. The above relation implies that

$$\text{diam}_X(A_j) < K$$

for some uniform constant $K > 0$, where $\text{diam}(\cdot)$ denotes the diameter with respect to the hyperbolic metric in $X$. Now we pull back $A_j$ along the orbit $z_0, \cdots, z_{n_j}$, and denote the component of $G^{-k}(A_j)$ which contains $z_{n_j-k}$ by $A_j^{n_j-k}$. Let $X_k$ be the unbounded component of $G^{-k}(X)$. Then $G^k : X_k \to X$ is a holomorphic covering map and $X_k \subset X$. Note that $A_j^{n_j-k} \subset X_k$. It follows that

$$\text{diam}_X(A_j^{n_j-k}) < \text{diam}_X(A_j^{n_j-k}) = \text{diam}_X(A_j) < K.$$

For each $1 \leq i \leq j$, since $A_j^{n_i}$ contains $z_{n_i}$ and $z_{n_i}$ is contained in some cone spanned at some critical point, from the above inequality it follows that $A_j^{n_i}$ is contained in a definite cone spanned at this critical point. Thus the inverse branch of $G^{-1}$ which maps $A_j^{n_i+1}$ to $A_j^{n_i}$ contracts the hyperbolic diameter by a definite factor $0 < \delta < 1$ (cf. Lemma 1.11 of [15] or Lemma 3.2 of [26]), that is,

$$\text{diam}_X(A_j^{n_i}) < \delta \cdot \text{diam}_X(A_j^{n_i+1}).$$

This implies that $\text{diam}_X(A_j^0) < \delta^j \cdot \text{diam}_X(A_j) < \delta^j \cdot K \to 0$ as $j \to \infty$. This implies that

$$\text{diam}(A_j^0) \to 0 \text{ as } j \to \infty.$$
where \( \text{diam}(\cdot) \) denotes the diameter with respect to the Euclidean metric. In addition, since \( \text{dist}(A_j, P_G) \sim \text{diam}(A_j) \), the distortion of \( G^{-n_j} \) on \( A_j \) is uniformly bounded. Let \( B_0 \) denote the component of \( G^{-n_j}(B) \) which is contained in \( A_j^0 \). We have

\[
\text{area}(B_0) \geq \text{diam}(A_j^0)^2.
\]

Since \( B \) and thus \( B_0 \) is disjoint from \( J_G \), we get a contradiction with the assumption that \( z_0 \) is a Lebesgue point of \( J_G \).

It remains to show the existence of the cones and the subsequence \( n_j \) satisfying the conditions in the last paragraph. The argument is similar to the one used in [26]. For each open arc \( I \subset T \), consider the space

\[
\Omega_I = \hat{C} \setminus (P_G \setminus I).
\]

Let \( d_{\Omega_I} (\cdot, \cdot) \) denote the hyperbolic distance in \( \Omega_I \) and for \( d_0 > 0 \), let

\[
\Omega_{d_0}(I) = \{ z \in \Omega_I \mid d_{\Omega_I}(z, I) < d_0 \}.
\]

As in Lemma 4.19 when \( I \) is small, \( \Omega_{d_0}(I) \) is like the hyperbolic neighborhood in the slit plane, that is, it is almost like the domain bounded two arcs of Euclidean circles which are symmetric about each other and such that the four exterior angles formed by the two arcs and \( T \) are all equal to \( \sigma \) with \( d = \ln \cot(\sigma/4) \). Since we are interested only in the dynamics in the outside of \( \Delta \), let us use \( H_{d_0}(I) \) to denote the part of \( \Omega_{d_0}(I) \) which belongs to the outside of the \( \Delta \), that is

\[
H_{d_0}(I) = \{ z \mid |z| > 1 \text{ and } z \in \Omega_{d_0}(I) \}.
\]

Then \( H_{d_0}(I) \) is bounded by \( I \) and some small arc \( S \). Take \( d_0 > 0 \) such that the two exterior angles formed by \( T \) and \( S \) are equal to \( (1 - \frac{1}{\pi(2d-1)}) \pi \). It follows that for any \( z \in T \setminus I \), if \( V \) is a cone spanned at \( z \) such that the angles formed by the two rays and \( T \) are equal to \( \frac{2\pi}{\pi(2d-1)} \), then \( V \) does not intersect \( H_{d_0}(I) \).

Now let \( h : T \to \hat{T} \) be the circle homeomorphism such that \( G|T = h^{-1} \circ R_\alpha \circ h \). For each \( z_n \), let \( I_n \subset I \) be the arc such that \( z_n \in H_{d_0}(I_n) \) and moreover, \( I_n \) is the smallest one in the following sense

\[
|h(I_n)| = \min \{ |h(I)| \mid I \subset T \text{ and } z_n \in H_{d_0}(I) \}.
\]

Since \( z_n \to T \), we have \( |I_n| \to 0 \) and thus

\[
|h(I_n)| \to 0 \text{ as } n \to \infty.
\]

So there is an increasing subsequence of integers, say \( m_j \), such that

\[
|h(m_j)| < |h(I_n)| \quad \text{for all } 1 \leq n < m_j.
\]

Let \( n_j = m_j - 1 \). We claim \( \{ n_j \} \) is the desired subsequence. Let us prove the claim. Since \( |I_{m_j}| \to 0 \), we may consider the sequence \( m_j \) from some very large \( j \) such that each \( T_{m_j} \) contains at most one critical value of \( G \). So these are two cases. In the first case, \( T_{m_j} \) contains no critical value. In the second case, \( T_{m_j} \) contains exactly one critical value.

In the first case, let \( J \subset T \) be the arc such that \( G(J) = I_{m_j} \). Let \( K \) be the component of \( G^{-1}(H_{d_0}(I_{m_j})) \) which is attached to \( T \). By Schwarz lemma it follows that \( K \subset H_{d_0}(J) \). By the minimal property of \( I_{m_j} \), it follows that \( z_{m_j-1} = z_{m_j} \notin H_{d_0}(J) \). This implies that \( z_{m_j} \) is near a critical value in \( T \), and \( H_{d_0}(J) \) is near some critical point \( c \) in \( T \). By the choice of \( d_0 \), \( H_{d_0}(J) \) belongs to the angle domain bounded by \( T \) and a ray starting from \( c \) such that the angle formed by the ray and \( T \) at \( c \) is equal to \( \frac{\pi}{3(2d-1)} \). Let \( m \geq 3 \) be the local degree of \( G \) at \( c \). Then \( m \leq 2d - 1 \). Now in a small neighborhood of \( c \), we may regard \( G \) approximately as the map \( z \mapsto \lambda \cdot (z - c)^m + v \) where \( \lambda \neq 0 \) is some constant.
and \( v = G(c) \). Let \( V \) be the cone spanned at \( c \) such that the exterior angles formed by the two rays of \( V \) and \( \mathbb{T} \) are equal to \( \frac{\pi}{2(2d-1)} \). Since there is some point \( w \in H_d(J) \) such that \( G(w) = G(z_{n_j}) \), it follows that \( z_{n_j} \) is contained in \( V \). In the second case, \( I_{m_j} \) contains exactly one critical value, and thus \( I_n \) contains exactly one critical point, say \( c \). Let \( w \in H_d(J) \) such that \( G(w) = G(z_{n_j}) \). If the angle between \( |c, z_{n_j}| \) and \( \pi \), we would have a \( J \) such that \( z_{n_j} \subset H_d(J) \) with

\[
|J| \ll ||c, z_{n_j}| \ll |c, w| \leq |I_{n_j}|.
\]

Since \( c \in I_{n_j} \) and \( h \) is quasi-symmetric, the above inequality would imply \( h(J) < h(I_{n_j}) \) (cf. Lemma 4.8 of [26]). This is a contradiction with the minimal property of \( I_{n_j} \). This implies that \( z_{n_j} \) must belong to a cone spanned at \( c \) such that the two rays of the cone form a definite angle with \( \mathbb{T} \). This proves Lemma 6.6. The proof of Theorem 2.1 is thus completed.

6.3. Proof of the Key-Lemma 2. Let \( g \in \Sigma_{\text{top}}^{a,d} \). Without loss of generality, let us assume that the Siegel disk \( D \) of \( g \) is centered at the origin and \( 1 \) is critical point of \( g \) which belongs to \( \partial D \). Let \( H : \Delta \to H \) be the holomorphic isomorphism such that \( H(0) = 0 \), \( H(1) = 1 \) and \( H \circ g \circ H^{-1} = R \) on \( \Delta \). Since \( \partial D \) is a quasi-circle, we can extend \( H \) to be a qc homeomorphism \( \Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) which fixes \( 0 \), \( 1 \) and \( \infty \), and moreover, \( \Phi \) is holomorphic in an open neighborhood of each periodic attracting cycle of \( g \). Define \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) by setting

\[
f(z) = \Phi \circ g \circ \Phi^{-1}(z), \quad \forall z \in \hat{\mathbb{C}}.
\]

Then \( f \in \Sigma^{a,d} \) and \( f \) is CLH-equivalent to \( g \). By Theorem 2.1, \( f \) has no Thurston obstructions in the exterior of \( \Delta \). Let \( c_1 = 1, c_2, \cdots, c_m \) denote all the critical points of \( f \) which belong to \( \mathbb{T} \). By perturbing \( f \) in a small neighborhood of the unit disk, we get a sequence \( f_n \in \Sigma^{a,d} \) such that each \( f_n \) has exactly \( m \) distinct critical points in \( \mathbb{T} \), \( c_1^n, c_2^n, \cdots, c_m^n \), \( 1 \leq i \leq m \), and moreover,

1. \( c_1^n = 1 \), and \( c_i^n \to c_i \) for \( 2 \leq i \leq m \),
2. the local degree of \( f_n \) at each \( c_i^n \) is the same as that of \( f \) at \( c_i \), \( 1 \leq i \leq m \),
3. there are integers \( k_{n,i}, 2 \leq i \leq m \), such that \( f_n^{k_{n,i}}(1) = e^{2\pi i k_{n,i}}c_i = c_i \),
4. \( f_n \to f \) uniformly.

Since in the exterior of \( \Delta \), \( f \) acts in the same was as \( f_n \), \( f_n \) has no Thurston obstruction in the exterior of \( \Delta \).

Now let \( F \) be the symmetrization of \( f \), that is, \( F(z) = f(z) \) for \( |z| \geq 1 \) and \( F(z) = \frac{f(z)}{z} \) for \( |z| < 1 \). For each \( f_n \), let \( F_n \) be the symmetrization of \( f_n \), that is, \( F_n(z) = f_n(z) \) for \( |z| \geq 1 \) and \( F_n(z) = [f_n(z)]^* \) for \( |z| < 1 \). Since \( f_n \to f \) uniformly, it follows that \( F_n \to F \) uniformly. By regarding the maps \( f_n \) and \( F_n \) as respectively the maps \( f \) and \( F \) in the proof of Theorem 2.1, we get a Blaschke product \( G_n \) such that \( F_n \) is CLH-equivalent to \( G_n \). That is, there is a pair of homeomorphisms \( \phi_n, \psi_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) which fix \( 0, 1 \) and \( \infty \) such that

1. \( \phi_n \circ F_n = G_n \circ \psi_n \),
2. \( \phi_n \) is isotopic to \( \psi_n \) rel \( P_{F_n} \cup \bigcup_i D_i \) where \( D_i \) are all the holomorphic disks of \( F_n \), and \( \phi_n | D_i = \psi_n | D_i \) are holomorphic for each holomorphic disk \( D_i \).

Here the maps \( f_n, F_n, G_n, \phi_n \) and \( \psi_n \) correspond respectively to the maps \( f, F, G, \phi \) and \( \psi \) in the proof of Theorem 2.1.

Note that each \( G_n \) has the form given by (69). Thus Lemma 6.2 still applies to the sequence \( \{G_n\} \). By taking a subsequence if necessary we may assume that \( G_n \to G \) uniformly where \( G \) is some Blaschke product of degree \( 2d - 1 \). In particular, when
restricted to $T$, $G$ is a critical circle homeomorphism with rotation number $\alpha$. Since $\alpha$ is of bounded type, there is a quasi-symmetric circle homeomorphism $h : T$ with $h(1) = 1$ and such that $h^{-1} \circ (G(T)) \circ h = R_\alpha$. Now we make a correspondence again. But this time, we correspond the objects $F$, $F_n$, $G$, $G_n$, $\phi_n$ and $\psi_n$ respectively to those, which have the same notations, that is, $F$, $F_n$, $G$, $G_n$, $\phi$ and $\psi$ in the proof of Theorem 2.1.

The only difference between the two situations is the following. Here the restriction of $F_n$ on $T$ is the irrational rotation $z \mapsto e^{2\pi i \alpha} z$, while in the proof of Theorem 2.1, the restriction of $F_n$ to $T$ is a rational rotation $z \mapsto e^{2\pi i \alpha_n} z$ with $\alpha_n = p_n/q_n \to \alpha$; here the restriction of $G_n$ on $T$ is qc conjugate to the irrational rotation $z \mapsto e^{2\pi i \alpha} z$, while in the proof of Theorem 2.1 the restriction of $G_n$ on $T$ has rational rotation number $\alpha_n = p_n/q_n$ and has a periodic orbit of period $q_n$. But all these difference does not affect the arguments in the proof of Lemmas 6.3, 6.4, 6.5 and 6.6. So the four lemmas still apply in the situation here. In particular, applying Lemma 6.5, we get a a pair of homeomorphisms $\phi, \psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fix $0, 1$ and $\infty$ such that

1. $\phi \circ F = G \circ \psi$,
2. $\phi$ is isotopic to $\psi$ rel $P_F \cup U \cup D_i$, where $D_i$s are all the holomorphic disks of $F$,
3. $\phi|D_i = \psi|D_i$ is holomorphic for each holomorphic disk $D_i$.

Then we construct the modified Blaschke product $\hat{G}$ as in (64). After a qc surgery on $G$, we get a Siegel polynomial $\xi \circ \hat{G} \circ \xi^{-1}$ given by (65). By the reasoning after the surgery, the Siegel polynomial $\xi \circ \hat{G} \circ \xi^{-1}$ is CLH-equivalent to $f$. By the rigidity assertion of Theorem 2.1 this polynomial must be the Siegel polynomial $g$.

On the other hand, for each $G_n$, we construct the modified Blaschke product $\hat{G}_n$ as in (64). After a qc surgery on $G_n$, we get a Siegel polynomial $\xi_n \circ \hat{G}_n \circ \xi_n^{-1}$ given by (65). Let $\mu_n$ be the Beltrami coefficient of $\xi_n$ and $\mu$ be the Beltrami coefficient of $\xi$. Since $G_n$ is contained in a compact family, by Herman’s theorem, there is a $0 < k < 1$ such that $||\mu||_\infty < k$ for all $n$. Since $G_n \to G$, by the construction of $\mu_n$ and $\mu$, it follows that $\mu_n \to \mu$ at all the points in $\hat{\mathbb{C}} \setminus J_G$. Since $J_G$ has zero Lebesgue measure, it follows that $\xi_n \to \xi$. Let $g_n = \xi_n \circ \hat{G}_n \circ \xi_n^{-1}$. Then $\{g_n\}$ is the desired sequence. This completes the proof of the Key-Lemma 2.

7. Appendix

Lemma 7.1. Let $f$ be a degree-$d$ polynomial map with a Siegel disk centered at the origin.

Then there exist $M > 1$ depending only on $d$ such that if $f$ has two critical points $c$ and $c'$ with $|c|/|c'| > M$, then there exist a pair of domains $U \Subset V$ containing the Siegel disk of $f$ centered at the origin such that the tuple $(U, V, f)$ is a polynomial-like map which is qc conjugate to a polynomial map $g$ of degree less than $d$. In particular, there exists a $L > 1$ which depends only on $d$ such that

$$\text{diam}(D) \leq L \cdot \min_{c \in \Omega_f} |c|$$

where $D$ is the Siegel disk of $f$ centered at the origin and $\Omega_f$ is the set of all critical points of $f$.

Proof. Let $c_1, \ldots, c_{d-1}$ be the critical points of $f$. Through a linear conjugation, we may assume that $1 = |c_{d-1}| \leq |c_{d-2}| \leq \cdots \leq |c_1|$.

Claim. For each $M_l > 1$, there exists an $M_{l+1} > M_l$ such that if

(67) $1 = |c_{d-1}| \leq |c_{d-2}| \leq \cdots \leq |c_1|$, then there exist domains $U \Subset V$ containing the origin such that $(f, U, V)$ is a polynomial map which is qc conjugate to some polynomial of degree $l + 1$. Let us prove the Claim
first. Recall that
\[ f(z) = f_{c_1, \ldots, c_{d-1}}(z) = \sum_{i=1}^{d} a_i z^i \]

with
\[ a_i = e^{2\pi i a} \cdot \left( \frac{(-1)^{i-1}}{i} \right) \cdot \frac{Q_{d-i}(c_1, \ldots, c_{d-1})}{c_1 \cdots c_{d-1}} \]

where \( Q_{d-i} \) is the degree-\((d-i)\) elementary polynomials of \( c_1, \ldots, c_{d-1} \). If (67) holds, a simple calculation shows that there is some constant
\[ C = C(d, M_l) > 1 \]
depending only on \( d \) and \( M_l \) such that by taking \( M_{l+1} \) large enough, the following inequalities hold.

(i) \( C^{-1} < |a_{l+1}| < C \),
(ii) \( |a_k| \leq C \) for \( 1 \leq k \leq l \),
(iii) \( |a_k| < d^l \cdot M_{l+1}^{-1} \) for \( l+2 \leq k \leq d \).

From (i) and (ii) it follows that by taking \( R > 0 \) large enough, we can make sure that
\[
\sum_{i=1}^{l} |a_i| |z|^i \ll |a_{l+1} z^{l+1}| \quad \text{for } |z| \geq \left( \frac{R}{2C} \right)^{1/(l+1)}.
\]

Fix such an \( R \). Then from (iii) it follows that if we take
\[ M_{l+1} > R \]
large enough, then we have
\[
\sum_{i=l+2}^{d} |a_i| |z|^i \ll |a_{l+1} z^{l+1}| \quad \text{for } |z| \leq |C \cdot R^{l+1}|.
\]

Now define a quasi-regular map \( F \) as follows.
\[
F(z) = \begin{cases} 
  f(z) \text{ for } |z| \leq R, \\
  f(z) - \frac{|z|-R}{a_{l+1} R^{l+1}} F(f(z) - a_{l+1} z^{l+1}), \text{ for } R < |z| < |a_{l+1} R^{l+1}| \\
  a_{l+1} z^{l+1} \text{ if } |z| \geq |a_{l+1} R^{l+1}|.
\end{cases}
\]

By definition, \( F \) is holomorphic for \( |z| < R \) and \( |z| > |a_{l+1} R^{l+1}| \). From (i-iii) and a simple calculation we get
\[
|F_z| \ll |F| \text{ for } R < |z| < |a_{l+1} R^{l+1}|.
\]

provided that \( R \) and \( M_{l+1} \) are large enough (the choice of \( M_{l+1} \) depends on \( R \)). This implies that the real dilatation of \( F \) in \( \{z \mid R < |z| < |a_{l+1} R^{l+1}| \} \) can be arbitrarily close to 1. Note that the forward orbit of any point \( z \) passes through the annulus \( \{z \mid R \leq |z| \leq |a_{l+1} R^{l+1}| \} \) at most two times. By a routine argument it follows that \( F \) is conjugate to a polynomial of degree \( l+1 \) through some \( K\)-qc homeomorphism where \( K > 1 \) can be arbitrarily close to 1 provided that \( R \) and \( M_{l+1} \) are chosen appropriately large.

Let \( V = \{z \mid |z| < R\} \) and \( U \) be the component of \( f^{-1}(V) \) which contains the origin. From (63), it follows that \( U \subset V \) and \( f : U \to V \) is of degree \( l+1 \) and thus \( (U, V, f) \) is a polynomial-like map (whose Julia set may not be connected). From the last paragraph it follows that \((U, V, f)\) is \( K\)-qc conjugate to some polynomial of degree \( l+1 \). This proves the Claim.
Now let \( M_1 = 1 \). By successively applying the Claim we get
\[ 1 = M_1 < M_2 < \cdots < M_{d-1}. \]

Let \( M = M_{d-1} \). Suppose \( 1 = |c_{d-2}| \leq \cdots \leq |c_1| \).

In the first case, \( |c_i| \leq M \) for all \( 1 \leq i \leq d - 2 \). The first assertion obviously holds in this case. Note that the absolute value of the leading coefficient of \( f \) is \( 1/|c_1| \cdots |c_{d-1}| \geq 1/M^{d-1} \), and that all the other coefficients of \( f \) are bounded above by some constant \( K(d, M) \) depending on \( d \) and \( M \). This implies the existence of an \( R(d, M) \) such that \( f \) is dominated by the leading term for all \( |z| > R(d, M) \). In particular, this implies that the diameter of the Siegel disk is not greater than \( R(d, M) \). Since \( M \) depends only on \( d \), the second assertion of the lemma follows.

In the second case, there is some \( c_i \) such that \( |c_i| > M \). Let \( 1 \leq l \leq d - 2 \) be the least integer such that \( |c_{d-l-1}| > M_{l+1} \). Then we have (27). The first assertion follows from the Claim. For the second assertion, let us go back to the proof of the Claim. We see the Siegel disk centered at the origin is contained in \( V \Delta \). The Siegel disk centered at the origin is a \( K \)-quasicircle. 

\[ \text{Let } 0 < \alpha < 1 \text{ be a bounded type irrational number and } d \geq 2 \text{ be an integer. Let } \mathcal{P}_d^\alpha \text{ denote the class of all the polynomial maps } f \text{ such that} \]
\[ f(z) = e^{2\pi i \alpha} z + \alpha_2 z + \cdots + \alpha_d z^d \]
with \( \alpha_d \neq 0 \) and \( f'(1) = 0 \). Let \( \Delta \) denote the unit disk and \( D \) denote the Siegel disk of \( f \) centered at the origin.

**Lemma 7.2** (Shishikura). There exists a \( K = K(\alpha, d) > 1 \) depending only on \( d \) and \( \alpha \) such that for any polynomial map \( f \in \mathcal{P}_d^\alpha \), if \( \phi : \Delta \to D \) is the holomorphic isomorphism such that \( \phi^{-1} \circ f \circ \phi(z) = e^{2\pi i \alpha} z + z \in \Delta \), then \( \phi \) can be extended to a \( K \)-qc homeomorphism of the plane. In particular, the boundary of the Siegel disk of \( f \) centered at the origin is a \( K \)-quasicircle.

For a proof, see [20] or [25].

**Lemma 7.3.** The boundaries of the Siegel disks of
\[ f \in \bigcup_{2 \leq l \leq d} \mathcal{P}_l^\alpha \]
at the origin moves continuously with respect to the Hausdorff metric on the spaces of non-empty compact sets of the plane and the topology of \( \bigcup_{2 \leq l \leq d} \mathcal{P}_l^\alpha \) is given by open-compact topology, that is, \( f_n \to f \) with respect to this topology means that \( f_n \) uniformly converges to \( f \) in any compact set of the plane.

**Proof.** Let \( f \in \bigcup_{2 \leq l \leq d} \mathcal{P}_l^\alpha \). Assume that \( f_n \to f \). Let \( D \) and \( D_n \) be respectively the Siegel disks of \( f \) and \( f_n \) centered at the origin. It suffices to prove that \( \partial D_g \) is close to \( \partial D_f \) with respect to the Hausdorff metric for all \( n \) large enough. It is known that both of them contains critical points.

We may assume that \( D \neq D_n \) since otherwise there is nothing to prove. Since both \( D \) and \( D_n \) contains the origin as an interior point there is a point \( w \in (D \cap \partial D_n) \cup (D_n \cap \partial D) \). Without loss of generality, let us assume that \( w \in D \cap \partial D_n \). Let \( \Gamma_w \subset D \) be the \( f \)-invariant curve containing \( w \). Let \( O_f(w) = \{f^k(w)\}_{k \geq 0} \) and \( O_g(w) = \{g^k(w)\}_{k \geq 0} \). Then
\( \mathcal{O}_f(w) \) and \( \mathcal{O}_g(w) \) are dense in \( \Gamma_w \) and \( \partial D_n \) respectively. For any integer \( m \geq 1 \), the two finite orbit segments
\[
\{ f^k(w), 0 \leq k \leq m \}; \quad \text{and} \quad \{ f^k(w), 0 \leq k \leq m \}
\]
can be arbitrarily close to each other provided that \( n \) is large enough. By Lemma 7.2 there exist two \( K(\alpha, d)\)-qc homeomorphisms of the plane which fix 0 and \( \infty \), say \( \phi \) and \( \psi \) such that \( \phi^{-1} \circ f \circ \phi(z) = e^{2\pi i \alpha}z \) and \( \psi^{-1} \circ g \circ \psi(z) = e^{2\pi i \alpha}z \) for all \( z \in \mathbb{D} \). \( \phi(\Gamma_f) = \Gamma_w \) and \( \psi(\Gamma_f) = \partial D_g \). Suppose \( w = \phi(z_0) \) for some \( z_0 \) with \( 0 < |z_0| < 1 \) and \( w = \psi(\zeta_0) \) for some \( \zeta_0 \in \mathbb{T} \). Then Each component of \( \Gamma_w \setminus \{ f^k(w), 0 \leq k \leq m \} \) is the \( \phi \)-image of a component of
\[
\{ z \mid |z| = |\zeta_0| \} \setminus \{ e^{2k\pi i \alpha} \cdot \zeta_0 \mid 0 \leq k \leq m \}
\]
and each component of \( \partial D_n \setminus \{ f^k(w), 0 \leq k \leq m \} \) is the \( \psi \)-image of a component of
\[
\mathbb{T} \setminus \{ e^{2k\pi i \alpha} \cdot \zeta_0 \mid 0 \leq k \leq m \}
\]
By the second assertion of Lemma 7.1, the Siegel disks of \( f \) and \( f_n \) are contained in some compact set of the plane. Since \( \phi \) and \( \psi \) fix 0 and \( \infty \) and are \( K \)-qc homeomorphisms of the plane for some \( K \) depending only on \( d \) and \( \alpha \), there exist \( C > 0 \) and \( 0 < \eta < 1 \) which are independent of \( n \) such that for any \( z_1, z_2 \) with \( |z_1|, |z_2| \leq 1 \) we have
\[
|\phi(z_1) - \phi(z_2)| < C \cdot |z_1 - z_2|^\eta \quad \text{and} \quad |\psi(z_1) - \psi(z_2)| < C \cdot |z_1 - z_2|^\eta.
\]
Since the components of
\[
\{ z \mid |z| = |\zeta_0| \} \setminus \{ e^{2k\pi i \alpha} \cdot \zeta_0 \mid 0 \leq k \leq m \}
\]
and
\[
\mathbb{T} \setminus \{ e^{2k\pi i \alpha} \cdot \zeta_0 \mid 0 \leq k \leq m \}
\]
can be arbitrarily small provided that \( m \) is large enough, it follows that \( \Gamma_w \) and \( \partial D_n \) can be arbitrarily to each other provided that \( n \) is large enough.

Now we claim that \( \Gamma_w \) is close to \( \partial D \) also provided that \( n \) is large enough. This is because if not, then by Lemma 7.2 it follows that the modulus of the annulus bounded by \( \partial D \) and \( \Gamma_w \) has a positive lower bound. But since \( \partial D_n \) is close to \( \Gamma_w \) and contains a critical point of \( f_n \), it follows that there is a critical point of \( f_n \) contained in \( D \) and is bounded away from \( \partial D \). But since \( f_n \to f \) uniformly in \( D \), there would be a critical point of \( f \) contained in \( D \). This is impossible. This completes the proof of Lemma 7.3 \( \square \)

**Lemma 7.4.** Suppose \( f_n \in \bigcup_{2 \leq i \leq d} \mathbb{P}_\alpha^i \) is a sequence such that for each \( f_n \), the boundary of the Siegel disk centered at the origin passes through the critical point 1. Then \( f_n \) has a subsequence \( f_{n_k} \) which converges to some \( f \in \bigcup_{2 \leq i \leq d} \mathbb{P}_\alpha^i \). Moreover, the Siegel disk of the limit polynomial map \( g \) centered at the origin also passes through the critical point 1, and for any \( k > m \geq 0 \), we have
\[
\lim_{n \to \infty} \sigma_{k,m}(f_n) = \sigma_{k,m}(g).
\]

**Proof.** By the second assertion of Lemma 7.1 the critical points of all \( f_n \) are uniformly bounded away from the origin; that is, there is a uniform \( L > 0 \) such that for each \( f_n \), the critical points of \( f_n \) are contained in the outside of the disk \( \{ z \mid |z| > L \} \). For each \( f_n \), let us label the critical points of \( f_n \) by
\[
c_n^1, \ldots, c_n^d, c_n^{d-2}, c_n^{d-1} = 1.
\]
By taking a subsequence, we get \( 1 \leq i_1 < \cdots < i_l \leq d-2 \) and \( d-1 - l \) such that for all \( 1 \leq j \leq l \),
\[
c_n^{i_j} \to \infty \quad \text{and} \quad n \to \infty,
\]
and \( \sigma_{k,m}(f_n) = \sigma_{k,m}(g) \). This completes the proof of Lemma 7.4. \( \square \)
and for all $k \neq i, 1 \leq j \leq l$ and $1 \leq k \leq d - 1$, we have

$$c_k^n \to c_k^*$$

where $c_k^*$ is some non-zero complex number. Let $g$ denote the polynomial of degree $d - l$ which has critical points at these $c_k^*, 1 \leq k \leq d - 1$ and $k \neq i, 1 \leq j \leq l$. It is clear that $f_n$ converges to $g$ uniformly in any compact set of the plane. This proves the first assertion of Lemma 7.4. Let us prove that the boundary of the Siegel disk of $g$ centered at the origin must also contain the critical point 1. Suppose this were not true. Then the critical point 1 is bounded away from the boundary of the Siegel disk $g$. Since $f_n \to g$, by Lemma 7.3, the boundary of the Siegel disk of $f_n$ centered at the origin can be arbitrarily close to the boundary of the Siegel disk of $g$ centered at the origin. This would imply for all $n$ large enough, the boundary of the Siegel disk of $f_n$ centered at the origin does not pass through the critical point 1. This is a contradiction. For $k > m \geq 0$ given, $f_n^k \to g^k$ and $f_n^m \to g^m$ uniformly in any compact set of the plane. Thus $\sigma_{k,m}(f_n) \to \sigma_{k,m}(g)$ as $n \to \infty$. This implies the second assertion. The proof of Lemma 7.4 is complete.

□

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