On Relating Edges in Graphs without Cycles of Length 4

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Abstract

An edge \( xy \) is relating in the graph \( G \) if there is an independent set \( S \), containing neither \( x \) nor \( y \), such that \( S \cup \{x\} \) and \( S \cup \{y\} \) are both maximal independent sets in \( G \). It is an \( \text{NP} \)-complete problem to decide whether an edge is relating \([1]\). We show that the problem remains \( \text{NP} \)-complete even for graphs without cycles of length 4 and 5. On the other hand, for graphs without cycles of length 4 and 6, the problem can be solved in polynomial time.

1 Introduction

Throughout this paper \( G = (V, E) \) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \).

Let \( S \subseteq V \) be a set of vertices, and let \( i \in \mathbb{N} \). Then

\[
N_i(S) = \{ w \in V | \min_{s \in S} d(w, s) = i \},
\]

where \( d(x, y) \) is the minimal number of edges required to construct a path between \( x \) and \( y \). If \( i \neq j \) then \( N_i(S) \cap N_j(S) = \emptyset \). If \( S = \{v\} \) for some \( v \in V \), then \( N_i(\{v\}) \) is abbreviated to \( N_i(v) \).

A set of vertices \( S \subseteq V \) is independent if for every \( x, y \in S \), \( x \) and \( y \) are not adjacent. It is clear that an empty set is independent. The independence number of \( G \), denoted by \( \alpha(G) \), is the cardinality of the maximum size independent set in the graph.

A graph is well-covered if every maximal independent set has the same cardinality, \( \alpha(G) \).

Let \( T \subseteq V \). Then \( S \) dominates \( T \) if \( S \cup N_1(S) \supseteq T \). If \( S \) and \( T \) are both empty, then \( N_1(S) = \emptyset \), and therefore \( S \) dominates \( T \). If \( S \) is a maximal independent set of \( G \), then it dominates the whole graph.

Two adjacent vertices, \( x \) and \( y \), in \( G \) are said to be related if there is an independent set \( S \), containing neither \( x \) nor \( y \), such that \( S \cup \{x\} \) and \( S \cup \{y\} \) are both maximal independent sets in the graph. If \( x \) and \( y \) are related, then \( xy \) is a relating edge. To decide whether an edge in an input graph is relating is an \( \text{NP} \)-complete problem \([1]\).

Theorem 1.1 \([1]\) The following problem is \( \text{NP} \)-complete:

Input: A graph \( G = (V, E) \), and an edge \( xy \in E \).
Question: Is \( xy \) a relating edge in \( G \)?
In [1], Brown, Nowakowski and Zverovich investigate well-covered graphs with no cycles of length 4. They denote the set of such graphs by $WC(\hat{C}_4)$, and prove the following.

**Theorem 1.2** [1] Let $G \in WC(\hat{C}_4)$. If $xy$ is an edge in $G$, but $x$ and $y$ are not related, then $G - xy$ is well-covered and $\alpha(G) = \alpha(G - xy)$.

In this paper we continue the investigation of the structure of graphs with no cycles of length 4. We denote the set of graphs without cycles of sizes $k$ and $l$ by $G(\hat{C}_k, \hat{C}_l)$. We prove that Theorem 1.1 holds even for the case, where the input graph does not contain cycles of length 4 and 5, i.e., $G \in G(\hat{C}_4, \hat{C}_5)$. On the other hand, if the input graph does not contain cycles of length 4 and 6, i.e., $G \in G(\hat{C}_4, \hat{C}_6)$, then the problem of identifying relating edges turns out to be polynomial.

The fact that identifying relating edges is NP-complete for the input restricted to $G(\hat{C}_4, \hat{C}_5)$ is important, because the analogous problem concerning well-covered graphs is known to be polynomial [5].

**Theorem 1.3** [2] The following problem can be solved in polynomial time:

Input: A graph $G \in G(\hat{C}_4, \hat{C}_5)$.

Question: Is $G$ well-covered?

## 2 Main Results

Let $X = \{x_1, ..., x_n\}$ be a set of 0-1 variables. We define the set of literals $L_X$ over $X$ by $L_X = \{x_i, \overline{x_i} : i = 1, ..., n\}$, where $\overline{x}$ is the negation of $x$. A truth assignment to $X$ is a mapping $t: X \rightarrow \{0, 1\}$ that assigns a value $t(x_i) \in \{0, 1\}$ to each variable $x_i \in X$. We extend $t$ to $L_X$ by putting $t(\overline{x}) = \overline{t(x_i)}$. A literal $l \in L_X$ is true under $t$ if $t(l) = 1$. A clause over $X$ is a conjunction of some literals of $L_X$. Let $C = \{c_1, ..., c_m\}$ be a set of clauses over $X$. A truth assignment $t$ to $X$ satisfies a clause $c_j \in C$ if $c_j$ involves at least one true literal under $t$.

SAT is a well-known NP-complete problem [6]. It is defined as follows.

Input: A set of variables $X = \{x_1, ..., x_n\}$, and a set of clauses $C = \{c_1, ..., c_m\}$ over $X$.

Question: Is there a truth assignment to $X$ which satisfies all clauses of $C$?

**Theorem 2.1** The following problem is NP-complete:

Input: A graph $G = (V, E) \in G(\hat{C}_4, \hat{C}_5)$, and an edge $xy \in E$.

Question: Is $xy$ a relating edge in $G$?

**Proof.** Clearly, the problem is in NP. We use a polynomial time reduction from SAT. Let $(X = \{x_1, ..., x_n\}, C = \{c_1, ..., c_m\})$ be an instance of SAT. We construct a graph $G = G_{X,C}$ as follows (see Figure 1).

The vertex set of $G$ contains:

- Two vertices, $x$ and $y$.
- A set $T = \{x_i, t_i, f_i : 1 \leq i \leq n\}$.
- A set $C = \{c_j : 1 \leq j \leq m\}$.
• A set \( I_T = \{ t_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m, x_i \text{ appears in } c_j \} \).
• A set \( I_F = \{ f_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m, \overline{x_i} \text{ appears in } c_j \} \).

The edge set of \( G \) contains:
• The edge \( xy \).
• All edges \( yx_i \), for \( 1 \leq i \leq n \).
• All triangles \( (x_i, t_i, f_i) \), for \( 1 \leq i \leq n \).
• An edge \( t_i f_{i,j} \), if \( x_i \text{ appears in } c_j \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).
• An edge \( f_i t_{i,j} \), if \( \overline{x_i} \text{ appears in } c_j \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).
• An edge \( t_{i,j} c_j \), if \( x_i \text{ appears in } c_j \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).
• An edge \( f_{i,j} c_j \), if \( x_i \text{ appears in } c_j \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).
• All edges \( xc_j \), for \( 1 \leq j \leq m \).

![Figure 1: The structure of the graph \( G_{X,C} \).](image)

The graph \( G \) does not contain cycles of length 4 and 5. We show that \( xy \) is a relating edge in \( G \) if and only if \( (X, C) \) has a satisfying truth assignment.

Let \( \Phi \) be a satisfying truth assignment for \( (X, C) \). Define \( S = \{ t_i, t_{i,j} : \Phi(x_i) = 1 \} \cup \{ f_i, f_{i,j} : \Phi(x_i) = 0 \} \). Clearly, \( S \) is independent. The fact that \( \Phi \) is a satisfying truth assignment implies that \( S \) dominates \( C \). Hence, \( S \cup \{ x \} \) and \( S \cup \{ y \} \) are both maximal independent sets in \( G \), and \( xy \) is a relating edge.
Conversely, assume \( xy \) is a relating edge. Let \( S \) be an independent set, such that \( S \cup \{x\} \) and \( S \cup \{y\} \) are both maximal independent sets in \( G \). Clearly, \( S \) does not contain vertices of \( C \cup \{x_1, \ldots, x_n\} \). Hence, for each \( 1 \leq i \leq n \) exactly one of \( t_i \) and \( f_i \) belongs to \( S \). If \( t_i \in S \) then \( t_{i,j} \in S \) for each possible \( j \). If \( f_i \in S \) then \( f_{i,j} \in S \) for each possible \( j \). Define a truth assignment \( \Phi \): If \( t_i \in S \) then \( \Phi(x_i) = 1 \), else \( \Phi(x_i) = 0 \), for every \( 1 \leq i \leq n \). The fact that \( C \) is dominated by \( S \) implies that every clause of \( C \) involves a true literal. Therefore, \( \Phi \) is a satisfying truth assignment for \((X, C)\).

**Theorem 2.2** The following problem can be solved in polynomial time:

**Input:** A graph \( G = (V, E) \in \mathcal{G}(\hat{C}_4, \hat{C}_6) \), and an edge \( xy \in E \).

**Question:** Is \( xy \) a relating edge in \( G \)?

**Proof.** For every \( v \in \{x, y\} \), let \( u = \{x, y\} - \{v\} \), and define:
\[
M_1(v) = N_1(v) \cap N_2(u), \quad M_2(v) = N_1(M_1(v)) - \{v\}.
\]

The vertices \( x \) and \( y \) are related if and only if there exists an independent set in \( M_2(x) \cup M_2(y) \) which dominates \( M_1(x) \cup M_1(y) \).

The fact that the graph does not contain cycles of length 6 implies the following 3 conclusions:

- There are no edges which connect vertices of \( M_2(x) \) with vertices of \( M_2(y) \).
- The set \( M_2(x) \cap M_2(y) \) is independent.
- There are no edges between \( M_2(x) \cap M_2(y) \) and other vertices of \( M_2(x) \cup M_2(y) \).

Hence, if \( S_x \subseteq M_2(x) \) and \( S_y \subseteq M_2(y) \) are independent, then \( S_x \cup S_y \) is independent, as well. Therefore, it is enough to prove that one can decide in polynomial time whether there exists an independent set in \( M_2(v) \) which dominates \( M_1(v) \), where \( v \in \{x, y\} \).

Let \( v \) be any vertex in \( \{x, y\} \). Every vertex of \( M_2(v) \) is adjacent to exactly one vertex of \( M_1(v) \), or otherwise the graph contains a \( C_6 \). Every connectivity component of \( M_2(v) \) contains at most 2 vertices, or otherwise the graph contains either a \( C_4 \) or a \( C_6 \). Let \( A_1, \ldots, A_k \) be the connectivity components of \( M_2(v) \).

Define a flow network \( F_v = \{G_F = (V_F, E_F), s \in V_F, t \in V_F, w : E_F \to R\} \) as follows.

Let \( V_F = M_1(v) \cup M_2(v) \cup \{a_1, \ldots, a_k, s, t\} \), where \( a_1, \ldots, a_k, s, t \) are new vertices, \( s \) and \( t \) are the source and sink of the network, respectively.

The directed edges \( E_F \) are:

- the directed edges from \( s \) to each vertex of \( M_1(v) \);
- all directed edges \( v_1v_2 \) s.t. \( v_1 \in M_1(v) \), \( v_2 \in M_2(v) \) and \( v_1v_2 \in E \);
- the directed edges \( aw_i \), for each \( 1 \leq i \leq k \) and for each \( v \in A_i \);
- the directed edges \( a_it \), for each \( 1 \leq i \leq k \).

Let \( w \equiv 1 \). Invoke any polynomial time algorithm for finding a maximum flow in the network, for example Ford and Fulkerson’s algorithm. Let \( S_v \) be the set of vertices in \( M_2(v) \) in which there is a positive flow. Clearly, \( S_v \) is independent. The maximality of \( S_v \) implies that \( |M_2(v) \cap N_1(S_v)| \geq |M_2(v) \cap N_1(S_v')| \), for any independent set \( S_v' \) of \( M_2(v) \).

Let us conclude the proof with the recognition algorithm for relating edges.
For each $v \in \{x, y\}$, build a flow network $F_v$ as described above, and find a maximum flow. Let $S_v$ be the set of vertices in $M_2(v)$ in which there is a positive flow. If $S_v$ does not dominate $M_1(v)$ the algorithm terminates announcing that $x$ and $y$ are not related. Otherwise, let $S$ be any maximal independent set of $G - \{x, y\}$ which contains $S_x \cup S_y$. Each of $S \cup \{x\}$ and $S \cup \{y\}$ is a maximal independent set of $G$, and $x$, $y$ are related.

This algorithm can be implemented in polynomial time: One iteration of Ford and Fulkerson’s algorithm includes:

- Updating the flow function. (In the first iteration the flow is equal to 0.)
- Constructing the residual graph.
- Finding an augmenting path, if exists. It is worth mentioning that the residual capacity of every augmenting path equals 1.

Each of the above can be implemented in $O(|V| + |E|)$ time. In each iteration the number of vertices in $M_2(v)$ with a positive flow increases by 1. Therefore, the number of iterations can not exceed $|V|$, and Ford and Fulkerson’s algorithm terminates in $O(|V|(|V| + |E|))$ time. Our algorithm invokes Ford and Fulkerson’s algorithm twice, and terminates in $O(|V|(|V| + |E|))$ time.

3 Conjectures

Our main conjecture reads as follows.

**Conjecture 3.1** For every integer $k \geq 7$, the following recognition problem is NP-complete.

**Input:** A graph $G = (V, E) \in \mathcal{G}(C_4, \hat{C}_k)$, and an edge $xy \in E$.

**Question:** Is $xy$ a relating edge in $G$?

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