EXISTENCE AND LOCAL UNIQUENESS OF BUBBLING SOLUTIONS FOR POLY-HARMONIC EQUATIONS WITH CRITICAL GROWTH

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Abstract. We consider the following poly-harmonic equations with critical exponents:

\[-(\Delta)^m u = K(y) u^{\frac{N+2m}{N-2m}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N, \]

(0.1)

where $N > 2m+2$, $m \in \mathbb{N}_+$, $K(y)$ is positive and periodic in its first $k$ variables $(y_1, \ldots, y_k)$, $1 \leq k < \frac{N-2m}{2}$. Under some conditions on $K(y)$ near its critical point, we prove not only that problem (0.1) admits solutions with infinitely many bubbles, but also that the bubbling solutions obtained in our existence result are locally unique. This local uniqueness result implies that some bubbling solutions preserve the symmetry of the scalar curvature $K(y)$.

1. Introduction

We consider the following poly-harmonic equations with critical exponent:

\[-(\Delta)^m u = K(y) u^{\frac{N+2m}{N-2m}}, \quad u > 0 \quad \text{in} \quad \mathbb{R}^N, \]

(P)

where $N > 2m+2$, $m \in \mathbb{N}_+$, and $K(y)$ is a bounded positive smooth function.

When $m = 1$, problem (P) is the prescribed scalar curvature problem in $\mathbb{R}^N$. It is also well-known that a solution of the following problem:

\[
\begin{cases}
-\Delta u = K(y) u^{\frac{N+2}{N-2}}, & u > 0 \quad \text{in} \quad \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]

(1.1)

solves the prescribed scalar curvature problem on $S^N$.

From the Pohozaev identity, it is easy to see that problem (1.1) does not always admit a solution. We are interested in the sufficient conditions on $K(y)$, under which (1.1) admits a solution. In the last three decades, there have been considerable interests in the existence and multiplicity of solutions for problem (1.1) under some suitable assumptions on the function $K(y)$. See for example, [1], [7], [21], [36] and the references therein. When $K(y)$ is positive and periodic, by gluing approximation solutions into genuine solutions which concentrate at some isolated maximum points of the function $K(y)$, Li [21], [22], [23] proved that (1.1) has infinitely many multi-bubbling solutions for $N \geq 3$ (see [36] for the more
general results). When \( K(y) \) is a positive radial function with a strict local maximum at \( |x| = r_0 > 0 \) and satisfies

\[
K(r) = K(r_0) - c_0 |r - r_0|^\beta + O(|r - r_0|)^{\beta + \theta}, \quad r \in (r_0 - \delta, r_0 + \delta),
\]

for some constants \( c_0 > 0, \beta \in [2, N - 2) \), and small constants \( \theta > 0, \delta > 0 \), Wei and Yan \([35]\) constructed solutions for (1.1) with large number of bubbles concentrating near the sphere \( |x| = r_0 \) for \( N \geq 5 \). Very recently, Li, Wei and Xu \([26]\) proved the existence of solutions with infinitely many bubbles for problem (P), where the centers of the bubbles can be placed on all the \( k \)-dimensional lattice points with \( k < \frac{N - 2}{2} \). Moreover, they showed that the dimension restriction is optimal. For other related problems with critical exponents, we refer to \([1],[2],[5],[7],[13],[21],[24],[27],[28],[33]\) and references therein.

In recent years, the poly-harmonic operators have found considerable interest. For instance, when \( m = 2 \), problem (P) is related to the Paneitz operator, which was introduced by Paneitz \([30]\) for smooth 4 dimensional Riemannian manifolds and was generalized by Branson \([6]\) to smooth \( N \) dimensional Riemannian manifolds. We refer the reader to the papers \([10],[3],[4],[15],[16],[18],[19],[20],[31],[32]\), and the references therein, for various existence results on the poly-harmonic operators and related problems. One can see from these papers that the poly-harmonic operator presents new and challenging features compared with the Laplace operator. To the best of our knowledge, not much is obtained for the existence and the properties of bubbling solutions for elliptic problems involving poly-harmonic operators and critical exponents.

The aim of this paper is two-fold. Firstly, we will construct solutions with infinitely many bubbles for problem (P) under some more reasonable conditions than those in \([26]\). Secondly, we will study the properties of the bubbling solutions for problem (P), especially, the periodicity property of these bubbling solutions. The problem for the symmetry of the bubbling solutions is independently interesting and is harder to study than the existence. Obviously it can not be solved by the methods of moving plane. Instead, we will attack it by studying the local uniqueness of a sequence of bubbling solutions via various Pohozaev identities. Note that for general integer \( m > 0 \), it is impossible to estimate each term appearing in the Pohozaev identities. Thus, a better understanding of the Pohozaev identities is essential in the proof of our local uniqueness result, which, we believe, will be very useful in the study of other related problems.

We assume that \( K(x) \) satisfies the following conditions:

- \( (A_1) \) \( 0 \leq \inf_{\mathbb{R}^N} K(x) < \sup_{\mathbb{R}^N} K(x) < \infty \);
- \( (A_2) \) \( K \in C^1(\mathbb{R}^N) \) is 1-periodic in its first \( k \) variables;
- \( (A_3) \) 0 is a critical point of \( K \) and there exists some real numbers \( \beta \in (N - 2m, N) \) such that for all \( |x| \) small, it holds

\[
K(x) = K(0) + \sum_{i=1}^{N} a_i |x_i|^\beta + R(x),
\]
where \( K(0) > 0, a_i \neq 0, \sum_{i=1}^{N} a_i < 0, R(x) \) is \( C^{[\beta],1}(\mathbb{R}^n) \) near 0 and \( \sum_{s=0}^{[\beta]} |\nabla^s R(x)||x|^{-\beta+s} = O(|x|^\theta) \) for some \( \theta > 0 \) as \( x \) tends to 0, where \( C^{[\beta],1} \) means that up to \([\beta]\) derivatives are Lipschitz functions, \([\beta]\) denotes the integer part of \( \beta \), \( \nabla^s \) denotes all the partial derivatives of order \( s \).

To state the main results of this paper, we need to introduce some notations first. For any integer \( k \in [1, N] \), we define \( k \)-dimensional lattice by:

\[
Q_k := \{ \text{all the integer points in } \mathbb{R}^k \times \{0\} \subset \mathbb{R}^N \}, \quad \text{where } 0 \in \mathbb{R}^{N-k}.
\]

In this paper, we always assume that \( k < \frac{N-2m}{2} \). Take any sequence of integers \( \tilde{P}_j \in Q_k \), satisfying \( \tilde{P}_i \neq \tilde{P}_j \) for \( i \neq j \). It is easy to check

\[
\sum_{i \neq j} \frac{1}{|\tilde{P}_i - \tilde{P}_j|^{\tau}} < +\infty, \quad \forall j,
\]

where \( \tau = \frac{N-2m}{2} - \vartheta \), \( \vartheta > 0 \) is a fixed small constant. With this choice of \( \vartheta \), we have \( \tau > k \).

The condition we impose on the choice of the points \( \tilde{P}_j \in Q_k \) is the following:

\[
\max_j \sum_{i \neq j} \frac{1}{|\tilde{P}_i - \tilde{P}_j|^{\tau}} \leq C \min_j \sum_{i \neq j} \frac{1}{|\tilde{P}_i - \tilde{P}_j|^{\tau}} < +\infty. \tag{1.2}
\]

Let \( L > 0 \) be a large integer. Denote \( P_j = \tilde{P}_j L \). We are going to construct a bubbling solution, concentrating at \( P_j, j = 1, 2, \ldots \). For this purpose, we take \( x_{j,L} \), which is close to \( P_j, \mu_{j,L} > 0 \) large, and define

\[
U_{x_{j,L}, \mu_{j,L}}(y) = \tilde{C}_m \cdot \frac{\mu_{j,L}^{\frac{N-2m}{2}}}{(1 + \mu_{j,L}^{2}|y - x_{j,L}|^2)^{\frac{N-2m}{4}}}.
\]

with \( \tilde{C}_m = \left( \frac{\Gamma^{m-1}}{\Gamma^{m-1+m(N+2h)}} \right)^{\frac{N-2m}{4m}} \). Note that \( U_{0,1} \) is the unique solution (up to a translation and a scaling) to the problem (see [34]):

\[
(-\Delta)^m u = u^{\frac{N+2m}{N-2m}}, \quad u > 0 \quad \text{in } \mathbb{R}^N.
\]

Similar to [26], we define the following norms:

\[
\| \phi \|_* = \sup_{y \in \mathbb{R}^N} \left( \sigma(y) \sum_{j=1}^{\infty} \frac{\mu_{j,L}^{\frac{N-2m}{2}}}{(1 + \mu_{j,L}^{2}|y - x_{j,L}|^2)^{\frac{N-2m}{4}}} \right)^{-1} |\phi(y)|, \tag{1.3}
\]

\[
\| f \|_{**} = \sup_{y \in \mathbb{R}^N} \left( \sigma(y) \sum_{j=1}^{\infty} \frac{\mu_{j,L}^{\frac{N+2m}{2}}}{(1 + \mu_{j,L}^{2}|y - x_{j,L}|^2)^{\frac{N+2m}{4}}} \right)^{-1} |f(y)|. \tag{1.4}
\]

where \( \sigma(y) = \min\{1, \min_{i=1}^{\infty} (1 + \mu_{i,L}^{2}|y - x_{i,L}|^2)^{-1}\} \), and \( \tau > k \) is the same constant as in (1.2).

In the following of this paper, we will also use the same notation \( \| \cdot \|_* \) and \( \| \cdot \|_{**} \) if the sum in (1.3) or (1.4) is from 1 to \( n \).

Our first result is the following.
Theorem 1.1. Suppose that \( K \) satisfies the conditions \((A_1), (A_2) \) and \((A_3)\). Assume that \( N > 2m + 2, 1 \leq k < \frac{N - 2m}{2} \), and the sequence \( P_j = \bar{P}_j L \) satisfies (1.2). Then, problem \((P)\) has a solution \( u_L \), satisfying

\[
\| u_L - \sum_{i=1}^{\infty} U_{x_i,L,\mu_i,L} \| = o_L(1), \tag{1.5}
\]

for some \( x_{i,L} \) and \( \mu_{i,L} \), with

\[
x_{i,L} = P_i + o_L(1) \tag{1.6}
\]

and

\[
\mu_{i,L} L^{-\frac{N - 2m}{N + 2m}} = O(1), \tag{1.7}
\]

where \( o_L(1) \to 0 \) as \( L \to +\infty \).

Note that (1.7) implies

\[
\mu_{0} \leq \mu_{i,L} \leq \mu_{0}', \quad \forall \ i, j, \tag{1.8}
\]

for some constants \( \mu_{0}' > \mu_{0} > 0 \) which are independent of \( L \).

To discuss the symmetry properties of the solutions obtained in Theorem 1.1, we proceed with the following local uniqueness result for the bubbling solutions of \((P)\).

Theorem 1.2. Under the same assumptions as in Theorem 1.1, if \( u^{(1)}_L \) and \( u^{(2)}_L \) are two sequence of solutions of problem \((P)\), which satisfy (1.5), (1.6) and (1.8), then \( u^{(1)}_L = u^{(2)}_L \) provided \( L > 0 \) is large enough.

Local uniqueness is an important topic in the study of elliptic partial equations. Theorem 1.2 can be used to study the properties of the bubbling solutions. A direct consequence of this result is the following periodicity property of the solutions.

Theorem 1.3. Under the same assumption as in Theorem 1.1, if \( \{ \bar{P}_j : j = 1, 2, \ldots \} = Q_k, \) and \( u_L \) is a solution of \((P)\), which satisfies (1.5), (1.6) and (1.8), then \( u_L \) is \( L \)-periodic in \( y_j, j = 1, 2, \ldots, k \), provided \( L > 0 \) is large enough.

The construction of bubbling solutions for \((P)\) is somewhat standard. So in this paper, we will be a bit sketchy in the proof of Theorem 1.1. Our main contribution to the existence result is to find a more suitable condition (1.2) for \( \bar{P}_j \) in order to construct a bubbling solution blowing at the given set \( \{ P_j = L \bar{P}_j : j = 1, 2, \ldots \} \). Condition (1.2) shows that it is not necessary to take all the lattice point \( Q_k \) to form the set \( \{ \bar{P}_j : j = 1, 2, \ldots \} \). Of course, if we take all the points in \( Q_k \), then (1.2) holds.

In order to prove the local uniqueness result, we will use various kinds of local Pohozaev identities. Note that in the case of \( m = 1 \) (studied in [14]), each integral appearing in the local Pohozaev identities can be calculated or estimated. However, we can not follow the same procedure as in [14] for general integer \( m > 0 \), because the number of the integrals appearing in the local Pohozaev identities approaches to infinity as \( m \to +\infty \). To make
the things even worse, it seems impossible to give a precise local Pohozaev identities for general $m$. Thus, a better understanding of all those local Pohozaev identities plays an essential role in the proof of Theorem 1.2. See the discussions in Section 3.

Note that in [33, 14], the center of the bubbles lies in a one dimensional space. When the center of the bubbles lies in a $k$ dimensional space, technical difficulties occur in the study of both the existence and the local uniqueness of the bubbling solutions. For the existence, these difficulties were overcome in [26] by introducing the weight function $\sigma$ in the norms defined in (1.3) and (1.4). For the local uniqueness, it seems that the key lemma in [14] does not hold anymore if $k$ is large. So new estimates need to be developed to deal with this case.

Let point out that we can replace $(A_2)$ by the following condition:

$$K \in C^1(\mathbb{R}^N) \text{ is } l_i\text{-periodic in } y_i, i = 1, \cdots, k, \text{ for some } l_i > 0.$$ 

Under this new condition, we just need to define $Q_k$ as follows:

$$Q_k := \{ (j_1 l_1, \cdots, j_k l_k, 0) \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^N, \text{ for all integers } j_i, i = 1, \cdots, k \}.$$

The paper is organized as follows. In section 2, we study the existence of bubbling solutions. Section 3 is devoted to the discussion of the local uniqueness and periodicity of a sequence of bubbling solutions. In Appendix A, some basic estimates are proved, while in Appendix B, we compute the formula for the asymptotic energy expansion. Some estimates of the error term are given in Appendix C. In Appendix D, we give some basic lemmas in algebra, which will be used in the proof of our existence and local uniqueness results.

2. Existence of solutions with infinitely many bubbles

In this section, we will prove Theorem 1.1. To this end, we firstly construct a bubbling solutions blowing-up at finite points $\{P_1, \cdots, P_n\}$. Throughout this paper, we define $m^* = \frac{2N}{N-2m}$.

**Theorem 2.1.** Suppose that $K$ satisfies the conditions $(A_1), (A_2)$ and $(A_3)$. Assume that $N > 2m + 2$, $1 \leq k < \frac{N-2m}{2}$, and the sequence $\{\tilde{P}_j : j = 1, \cdots, n\}$ satisfies (1.2), where the constant $C$ is independent of $n$. Then, problem $(P)$ admits a solution $u_{n,L}$, satisfying

$$||u_{n,L} - \sum_{i=1}^{n} U_{x_{n,i,L}, \mu_{n,i,L}}||_* = o_L(1),$$

for some $x_{n,i,L}$ and $\mu_{n,i,L}$, with $x_{n,i,L} = P_i + o_L(1)$ and $\mu_{n,i,L} L^{\frac{N-2m}{\beta-N+2m}} \leq C$, where $C > 0$ is independent of $n$, and $o_L(1) \to 0$ uniformly in $n$ as $L \to +\infty$.

2.1. Linearization and finite dimensional reduction. For $i = 1, \cdots, n$, let $x_i \in B_1(P_i)$, $\mu_i > 0$. Set $x = (x_1, \cdots, x_n)$ and $\mu = (\mu_1, \cdots, \mu_n)$ satisfying $0 < \mu_0 < \mu_i/\mu_j < \infty$. 

So new estimates need to be developed to deal with this case.

Let point out that we can replace $(A_2)$ by the following condition:

$$K \in C^1(\mathbb{R}^N) \text{ is } l_i\text{-periodic in } y_i, i = 1, \cdots, k, \text{ for some } l_i > 0.$$ 

Under this new condition, we just need to define $Q_k$ as follows:

$$Q_k := \{ (j_1 l_1, \cdots, j_k l_k, 0) \in \mathbb{R}^k \times \{0\} \subset \mathbb{R}^N, \text{ for all integers } j_i, i = 1, \cdots, k \}.$$
\( \mu_0 < +\infty \). For simplicity, we denote \( U_{x_i, \mu_i}(x) \) by \( U_i(x) \), \( i = 1, 2, \ldots, n \). Let

\[
\begin{align*}
Z_{i,j} &= \xi(y - x_i) \frac{\partial U_i}{\partial x_{i,j}}, \quad Z_{i,N+1} = \xi(y - x_i) \frac{\partial U_i}{\partial \mu_i}, \quad j = 1, \ldots, N, \quad i = 1, \ldots, n,
\end{align*}
\]

where \( \xi \in C_0^\infty(B_2(0)) \) satisfying \( \xi(y) = \xi(|y|), \xi = 1 \) in \( B_1(0) \), and \( \xi = 0 \) in \( \mathbb{R}^N \setminus B_2(0) \).

The purpose of using the cut-off function \( \xi \) above is just to make the calculations simpler.

We define the function spaces \( \mathbf{X} \) and \( \mathbf{Y} \) as follows: \( \phi \in \mathbf{X} \) if \( \|\phi\|_* < +\infty \), while \( f \in \mathbf{Y} \) if \( \|f\|_{**} < +\infty \). Set

\[
\mathbf{H}_n := \left\{ \phi : \phi \in \mathbf{X}, \int_{\mathbb{R}^N} \phi U_i^{4m} Z_{i,j} = 0, \ i = 1, \ldots, n, \ j = 1, \ldots, N+1 \right\}. \tag{2.2}
\]

Let \( W_n(y) = \sum_{i=1}^n U_i(y), \phi \in \mathbf{H}_n \). We want to find a solution of the form \( W_n(x) + \phi(x) \) for \( (P) \) with \( \|\phi\|_* \) small. To achieve this goal, we first prove that for fixed \((x, \mu)\), there exists a smooth function \( \phi \in \mathbf{H}_n \), such that

\[
(-\Delta)^m(W_n(x) + \phi(x)) - K(y)(W_n(x) + \phi(x))^{m^* - 1} = \sum_{i=1}^n \sum_{j=1}^{N+1} c_{ij} U_i^{m^* - 2} Z_{i,j} \tag{2.3}
\]

for some constants \( c_{ij} \). Then, we show the existence of \((x, \mu)\), such that

\[
\int_{\mathbb{R}^N} (-\Delta)^m(W_n(x) + \phi(x)) Z_{i,j} - \int_{\mathbb{R}^N} K(y)(W_n(x) + \phi(x))^{m^* - 1} Z_{i,j} = 0. \tag{2.4}
\]

With this \((x, \mu)\), it is easy to prove that all \( c_{ij} \) must be zero.

**Part I: The Reduction.** In this part, for fixed \((x, \mu)\), we find \( \phi(x, \mu) \), such that \( \|\phi\|_* \) is \( C^1 \) in \((x, \mu)\) and \( \|\phi\|_* \) holds. In fact, we will use the contraction mapping theorem to prove the following result.

**Proposition 2.2.** Under the assumptions of Theorem 1.1. If \( L > 0 \) is sufficiently large, \( \|\phi\|_* \) admits a unique solution \( \phi = \phi(x, \mu) \) in \( \mathbf{H}_n \) such that \( \|\phi\|_* \leq \frac{C}{\bar{\mu} \min(N + 2m/r, \beta - r+1)} \), where \( \bar{\mu} = \min(\mu_1, \ldots, \mu_n) \). Moreover \( \|\phi\|_* \) is \( C^1 \) in \((x, \mu)\).

Firstly, we consider the following linear problem:

\[
(-\Delta)^m \phi - (m^* - 1) K(y) W_n^{m^* - 2} \phi = h + \sum_{i=1}^n \sum_{j=1}^{N+1} c_{ij} U_i^{m^* - 2} Z_{i,j}, \tag{2.5}
\]

for some constants \( c_{ij} \), where \( h \) is a function in \( \mathbf{Y} \).

**Lemma 2.3.** Suppose that \( \phi \) solves \( \|\phi\|_* \leq C \|h\|_{**}, \) for some constant \( C > 0 \), independent of \( n \).
Proof. We can write
\[ \phi(y) = (m^* - 1) \int_{\mathbb{R}^N} \frac{C_m}{|z - y|^{N-2m}} K(z) W_n^{m^*-2}(z) \phi(z) \, dz \]
\[ + \int_{\mathbb{R}^N} \frac{C_m}{|z - y|^{N-2m}} \left( h(z) + \sum_{i=1}^{N} \sum_{j=1}^{N+1} c_{ij} U_i^{m^*-2}(z) Z_{i,j}(z) \right) \, dz. \] (2.6)

Using Lemma A.2, we have
\[ \int_{\mathbb{R}^N} \frac{C_m |h(z)|}{|z - y|^{N-2m}} \, dz \leq C \|h\|_{**} \int_{\mathbb{R}^N} \frac{\sigma(z)}{|y - z|^{N-2m}} \sum_{j} \frac{\mu_j^{N+2m}}{(1 + \mu_j |y - x_j|)^{N+2m} + \tau} \, dz \]
\[ \leq C \|h\|_{**} \sigma(y) \sum_{j} \frac{\mu_j}{(1 + \mu_j |y - x_j|)^{N+2m} + \tau}, \] (2.7)
and
\[ \left| \int_{\mathbb{R}^N} \frac{U_i^{m^*-2} Z_{i,j}}{|y - z|^{N-2m}} \, dz \right| \leq C \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2m}} \frac{\mu_i^{\alpha(j) + N+2m}}{(1 + \mu_i |y - x_i|)^{N+2m} + \tau}, \] (2.8)
where \( \alpha(j) = 1, \, j = 1, \ldots, N, \, \alpha(N + 1) = -1. \)

To estimate \( c_{ij}, \) we use (2.5) to find
\[ -c_{ij} \int_{\mathbb{R}^N} U_i^{m^*-2} Z_{i,j}^2 = \int_{\mathbb{R}^N} ((m^* - 1) K(y) W_n^{m^*-2} \phi + h) Z_{i,j}. \] (2.9)
But
\[ \left| \int_{\mathbb{R}^N} h Z_{i,j} \right| \leq C \mu_i^{\alpha(i)} \|h\|_{**}, \] (2.10)
and (see (A.19))
\[ \int_{\mathbb{R}^N} K(y) W_n^{m^*-2} \phi Z_{i,j} \]
\[ = \int_{\mathbb{R}^N} K(y) (W_n^{m^*-2} - U_i^{m^*-2}) \phi Z_{i,j} + \int_{\mathbb{R}^N} (K(y) - 1) U_i^{m^*-2} \phi Z_{i,j} + \int_{\mathbb{R}^N} U_i^{m^*-2} \phi Z_{i,j} \]
\[ = \mu_i^{\alpha(i)} \|\phi\|_{**} \left( \frac{1}{\min_{\mu_i} (N+2mN-2m-\tau, \beta)} + \frac{1}{\mu_i^{N-2m} L^{N-2m}} \right), \] (2.11)
which, together with (2.9) and (2.10), gives
\[ |c_{ij}| \leq C (\|h\|_{**} + o(1) \|\phi\|_{**}) \mu_i^{-\alpha(j)}. \] (2.12)
Combining Lemma A.4 and (2.7)–(2.12), we are led to
\[
\left| \phi(y) \right| \left( \sigma(y) \sum_i \frac{N-2m}{\mu_i \left( 1 + \mu_i |y - x_i| \right)^{N-2m+\tau}} \right)^{-1} \\
\leq C \left( \| h \|_{\ast \ast} + o(1) \| \phi \|_\ast + \sum_j \frac{1}{(1+\mu_j |z-x_j|)^{N-2m+\tau}} \| \phi \|_\ast \right).
\]
(2.13)

We can finish the proof of this lemma by using (2.13) as in [35].

**Proof of Proposition 2.2.** Let \( P \) be the operator defined as follows:
\[
P f = f + \sum_{i=1}^n \sum_{j=1}^{N+1} c_{ij} U_{i}^{m^* - 2} Z_{i,j}, \quad f \in Y,
\]
where \( c_{ij} \) are chosen such that \( \int_{\mathbb{R}^N} Z_{i,j} P f = 0 \). Then it is easy to check that
\[
\| P f \|_{\ast \ast} \leq C \| f \|_{\ast \ast}.
\]
In view of Lemma 2.3, by the Fredholm alternative theorem, for any \( h \in Y \), (2.5) has a unique solution \( Ah \in H_n \).

Equation (2.3) is equivalent to
\[
\phi = A[P(N(\phi))] + A[PL], \quad \phi \in H_n,
\]
(2.14)
where
\[
N(\phi) = K(y) \left( (W_n + \phi)^{m^* - 1} - W_n^{m^* - 1} - (m^* - 1)W_n^{m^* - 2}\phi \right),
\]
(2.15)
and
\[
l_L = K(y)W_n^{m^* - 1} - \sum_{j=1}^n U_j^{m^* - 1}.
\]
(2.16)

Then, we can use the contraction mapping theorem as in [35] to prove that for large \( L > 0 \), (2.14) has a solution \( \phi \in H_n \), satisfying
\[
\| \phi \|_\ast \leq C \| l_L \|_{\ast \ast}.
\]
Using Lemma A.6 we obtain the estimate for \( \| \phi \|_\ast \).

**Part II: The Finite Dimensional Problems.** Note that for any \( \gamma > 1 \), we have \( (1 + t)^\gamma - 1 - \gamma t = O(t^2) \) for all \( t \in \mathbb{R} \) if \( \gamma \leq 2 \); and \( |(1 + t)^\gamma - 1 - \gamma t| \leq C(t^2 + |t|^\gamma) \) for all \( t \in \mathbb{R} \) if \( \gamma > 2 \). So, we can deduce
\[
\int_{\mathbb{R}^N} (-\Delta)^m (W_n(x) + \phi(x)) Z_{i,j} - \int_{\mathbb{R}^N} K(y)(W_n(x) + \phi(x))_{+}^{m^* - 1} Z_{i,j} \\
= \int_{\mathbb{R}^N} (-\Delta)^m W_n(x) Z_{i,j} - \int_{\mathbb{R}^N} K(y)W_n(x)^{m^* - 1} Z_{i,j} \\
+ (m^* - 1) \int_{\mathbb{R}^N} K(y)W_n^{m^* - 2} Z_{i,j} \phi + \mu_i^{(j)} O \left( (\mu_i^{(j)} ||\phi||_\ast)^2 \right). 
\]
(2.17)
It follows from Lemmas 3.2 and 3.3 Proposition 2.2, (2.11) and (2.17) that (2.4) is equivalent to
\[ x_j - P_j = O\left(\frac{1}{\bar{\mu}^2}\right), \tag{2.18} \]
and
\[ \sum_{i \neq j} \frac{C_4}{(\mu_i \mu_j)^{N-2m} |P_i - P_j|^{N-2m}} - \frac{C_3}{\mu_j^{\beta}} = O\left(\frac{1}{\bar{\mu}^{\beta+1}}\right), \quad j = 1, \ldots, n, \tag{2.19} \]
where \( \bar{\mu} = \min_i \mu_i \).

2.2. Proof of the existence theorems.

Proof of Theorem 2.1. We need to solve (2.18) and (2.19). Note that
\[ \frac{1}{|P_i - P_j|^{N-2m}} = \frac{1}{|\tilde{P}_i - \tilde{P}_j|^{N-2m} L^{N-2m} L^{-2m}} := d_{ij}, \quad i \neq j. \]
So, we can use Lemma D.1 to obtain the result. \( \square \)

Proof of Theorem 1.1. The proof of Theorem 1.1 follows from Theorem 2.1 by a limiting argument, since we can easily shown that for any fixed \( L > 0 \) large, there exists some constant \( C = C(L) \), independent of \( n \), such that
\[ u_n(x) \leq C(L), \quad \forall \ x \in \mathbb{R}^N, \tag{2.20} \]
By elliptic estimate, for any \( R > 0 \), there exists a constant \( C_2 = C_2(L) \) independent of \( n \), such that \( \|u_n(x)\|_{C^{2m}(B_R)} \leq C_2(L), \ \forall \ n = 1, \ldots, \), which implies that (up to a subsequence, still denoted by \( u_n \)) \( u_n \to u \) in \( C^{2m}_{loc}(\mathbb{R}^N) \), satisfying \((-\Delta)^m u = K(x) u^{N+2m}_{N-2m} \) in \( \mathbb{R}^N \). Noticing that \( u \) decays at direction \( y_N \), we can deduce from the potential theory for elliptic equations that
\[ u(x) = \int_{\mathbb{R}^N} \frac{K(y)}{|y - x|^{N-2m}} u(\beta)^{N+2m}_{N-2m}, \]
which also implies \( u > 0 \). \( \square \)

3. Local uniqueness and periodicity

In this section, we study the local uniqueness of the bubbling solutions for \((P)\). We assume that conditions \((A_1)-(A_3)\) hold. Suppose that \( u_L^{(1)} \) and \( u_L^{(2)} \) are two sequence of solutions of \((P)\), which satisfy (1.5), (1.6) and (1.8). We will prove that \( u_L^{(1)} = u_L^{(2)} \) provided \( L > 0 \) is large enough.
3.1. **Pohozaev type identities.** Suppose that \( u \) and \( v \) are two smooth functions in a given bounded domain \( \Omega \). In the section, we study the following two bi-linear functionals:

\[
L_{1,i}(u, v) = \int_{\Omega} \left( (-\Delta)^m u \frac{\partial v}{\partial y_i} + (-\Delta)^m v \frac{\partial u}{\partial y_i} \right), \quad i = 1, 2, \ldots, N, \tag{3.1}
\]

\[
L_2(u, v) = \int_{\Omega} \left( (-\Delta)^m u \langle y - x, \nabla v \rangle + (-\Delta)^m v \langle y - x, \nabla u \rangle \right). \tag{3.2}
\]

**Proposition 3.1.** For any integer \( m > 0 \), there exists a function \( f_{m,i}(u, v) \), such that

\[
L_{1,i}(u, v) = \int_{\partial \Omega} f_{m,i}(u, v). \tag{3.3}
\]

Moreover, \( f_{m,i}(u, v) \) has the following form:

\[
f_{m,i}(u, v) = \sum_{j=1}^{2m-1} l_{j,i}(\nabla^j u, \nabla^{2m-j} v), \tag{3.4}
\]

where \( l_{j,i}(\nabla^j u, \nabla^{2m-j} v) \) is bi-linear in \( \nabla^j u \) and \( \nabla^{2m-j} v \).

**Proof.** For \( m = 1 \), we use the integration by parts to find

\[
\begin{align*}
L_{1,i}(u, v) &= -\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial y_i} + \frac{\partial v}{\partial \nu} \frac{\partial u}{\partial y_i} \right) + \int_{\Omega} \left( \frac{\partial u}{\partial y_j} \frac{\partial^2 v}{\partial y_j \partial y_i} + \frac{\partial v}{\partial y_j} \frac{\partial^2 u}{\partial y_j \partial y_i} \right) \\
&= -\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial y_i} + \frac{\partial v}{\partial \nu} \frac{\partial u}{\partial y_i} \right) + \int_{\Omega} \langle \nabla u, \nabla v \rangle \nu_i.
\end{align*}
\]

For any integer \( m > 1 \), we have

\[
\begin{align*}
\int_{\Omega} \left( (-\Delta)^m u \frac{\partial v}{\partial y_i} + (-\Delta)^m v \frac{\partial u}{\partial y_i} \right) \\
= -\int_{\partial \Omega} \left( \frac{\partial (-\Delta)^{m-1} u}{\partial \nu} \frac{\partial v}{\partial y_i} + \frac{\partial (-\Delta)^{m-1} v}{\partial \nu} \frac{\partial u}{\partial y_i} \right) \\
+ \int_{\Omega} \left( \frac{\partial (-\Delta)^{m-1} u}{\partial y_j} \frac{\partial^2 v}{\partial y_j \partial y_i} + \frac{\partial (-\Delta)^{m-1} v}{\partial y_j} \frac{\partial^2 u}{\partial y_j \partial y_i} \right) \\
= -\int_{\partial \Omega} \left( \frac{\partial (-\Delta)^{m-1} u}{\partial \nu} \frac{\partial v}{\partial y_i} + \frac{\partial (-\Delta)^{m-1} v}{\partial \nu} \frac{\partial u}{\partial y_i} \right) \\
+ \int_{\Omega} \left( (-\Delta)^{m-1} u \langle \nu, \nabla v \rangle + (-\Delta)^{m-1} v \langle \nu, \nabla u \rangle \right) \\
- \int_{\Omega} \left( (-\Delta)^{m-1} u \frac{\partial \Delta v}{\partial y_i} + (-\Delta)^{m-1} v \frac{\partial \Delta u}{\partial y_i} \right).
\end{align*}
\]

If \( m = 2 \), then the last term in (3.6) becomes

\[
- \int_{\Omega} \left( (-\Delta u) \frac{\partial \Delta v}{\partial y_i} + (-\Delta v) \frac{\partial \Delta u}{\partial y_i} \right) = \int_{\partial \Omega} \Delta u \Delta v \nu_i.
\]
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which gives

\[
\int_{\Omega} \left( (-\Delta)^2 \frac{\partial v}{\partial y_i} + (-\Delta)^2 \frac{\partial u}{\partial y_i} \right)
= - \int_{\partial\Omega} \left( \frac{\partial (-\Delta u)}{\partial y_i} \frac{\partial v}{\partial y_i} + \frac{\partial (-\Delta v)}{\partial y_i} \frac{\partial u}{\partial y_i} \right)
+ \int_{\partial\Omega} \left( (-\Delta u) \langle \nu, \nabla \frac{\partial v}{\partial y_i} \rangle + (-\Delta v) \langle \nu, \nabla \frac{\partial u}{\partial y_i} \rangle \right) + \int_{\partial\Omega} \Delta u \Delta v \nu_i.
\]

(3.7)

We assume that the conclusion holds up to \(m-1\), \(m \geq 3\). Take \(u_1 = -\Delta u\) and \(v_1 = -\Delta v\). Then

\[
- \int_{\Omega} \left( (-\Delta)^{m-1} u \frac{\partial \Delta v}{\partial y_i} + (-\Delta)^{m-1} v \frac{\partial \Delta u}{\partial y_i} \right) = \int_{\Omega} \left( (-\Delta)^{m-2} u_1 \frac{\partial v_1}{\partial y_i} + (-\Delta)^{m-2} v_1 \frac{\partial u_1}{\partial y_i} \right).
\]

(3.8)

Thus, by using the induction assumption, we can conclude that the result is true for any \(m\). □

**Remark 3.2.** From the the proof of Proposition 3.1, we can see that if \(\Omega\) is a ball centered at \(x\), \(u\) and \(v\) are functions of \(|y - x|\), then there exists a function \(\tilde{f}_m(r)\), such that \(f_{m,i}(u,v) = \tilde{f}_m(|y - x|)\nu_i\). As a result, \(\int_{\partial B_d(x)} f_{m,i}(u,v) = 0\).

**Proposition 3.3.** For any integer \(m > 0\), there exists a function \(g_m(u,v)\), such that

\[
L_2(u,v) = \int_{\partial\Omega} g_m(u,v) - \frac{N - 2m}{2} \int_{\Omega} \left( v(-\Delta)^m u + u(-\Delta)^m v \right).
\]

Moreover, \(g_m(u,v)\) has the following form:

\[
g_m(u,v) = \sum_{j=1}^{2m-1} \tilde{l}_j(y - x, \nabla^j u, \nabla^{2m-j} v) + \sum_{j=0}^{2m-1} \tilde{l}_j(\nabla^j u, \nabla^{2m-j-1} v),
\]

where \(\tilde{l}_j(y - x, \nabla^j u, \nabla^{2m-j} v)\) and \(\tilde{l}_j(\nabla^j u, \nabla^{2m-j-1} v)\) are linear in each component.

**Proof.** If \(m = 1\), then using integration by parts, we obtain

\[
\int_{\Omega} \left( (-\Delta u) \langle y - x, \nabla v \rangle + (-\Delta v) \langle y - x, \nabla u \rangle \right)
= - \int_{\partial\Omega} \left( \frac{\partial u}{\partial y} \langle y - x, \nabla v \rangle + \frac{\partial v}{\partial y} \langle y - x, \nabla u \rangle \right) + \int_{\partial\Omega} \langle y - x, v \rangle \langle \nabla u, \nabla v \rangle
- \frac{N - 2}{2} \int_{\partial\Omega} \left( \frac{\partial u}{\partial y} v + \frac{\partial v}{\partial y} u \right) + \frac{N - 2}{2} \int_{\Omega} (v \Delta u + u \Delta v).
\]

(3.11)
For any integer $m > 1$, we have

$$
\int_{\Omega} ((-\Delta)^m u \langle y - x, \nabla v \rangle + (-\Delta)^m v \langle y - x, \nabla u \rangle)
= - \int_{\partial \Omega} \left( \frac{\partial((-\Delta)^{m-1} u)}{\partial \nu} \langle y - x, \nabla v \rangle + \frac{\partial((-\Delta)^{m-1} v)}{\partial \nu} \langle y - x, \nabla u \rangle \right)
+ \int_{\Omega} \left( \frac{\partial((-\Delta)^{m-1} u)}{\partial y_j} \frac{\partial \nabla v}{\partial y_j} + \frac{\partial((-\Delta)^{m-1} v)}{\partial y_j} \frac{\partial \nabla u}{\partial y_j} \right)
+ \int_{\partial \Omega} \left( \frac{\partial((-\Delta)^{m-1} u \langle y - x, \nabla v \rangle + \partial((-\Delta)^{m-1} v \langle y - x, \nabla u \rangle)}{\partial \nu} \right)
= - \int_{\partial \Omega} \left( \frac{\partial((-\Delta)^{m-1} u}{\partial \nu} \langle y - x, \nabla v \rangle + \frac{\partial((-\Delta)^{m-1} v}{\partial \nu} \langle y - x, \nabla u \rangle \right)
+ \int_{\partial \Omega} \left( \frac{\partial((-\Delta)^{m-1} u \langle y - x, \nabla v \rangle + \partial((-\Delta)^{m-1} v \langle y - x, \nabla u \rangle)}{\partial \nu} \right)
+ \int_{\partial \Omega} \left( \frac{\partial((-\Delta)^{m-1} u \partial \nabla v}{\partial \nu} + \frac{\partial((-\Delta)^{m-1} v \partial \nabla u}{\partial \nu} \right)
+ \int_{\partial \Omega} \langle y - x, \nabla \Delta v \rangle \Delta u - \frac{N - 4}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \Delta v - u \frac{\partial \Delta v}{\partial \nu} + \frac{\partial v}{\partial \nu} \Delta u - v \frac{\partial \Delta u}{\partial \nu} \right)$$

Letting $m = 2$ in (3.12), we obtain

$$
\int_{\Omega} ((-\Delta)^2 u \langle y - x, \nabla v \rangle + (-\Delta)^2 v \langle y - x, \nabla u \rangle)
= - \int_{\partial \Omega} \left( \frac{\partial((-\Delta) u}{\partial \nu} \langle y - x, \nabla v \rangle + \frac{\partial((-\Delta) v}{\partial \nu} \langle y - x, \nabla u \rangle \right)
+ \int_{\partial \Omega} \left( \frac{\partial((-\Delta) u \langle y - x, \nabla v \rangle + \partial((-\Delta) v \langle y - x, \nabla u \rangle)}{\partial \nu} \right)
+ \int_{\partial \Omega} \left( \frac{\partial((-\Delta) u \partial \nabla v}{\partial \nu} + \frac{\partial((-\Delta) v \partial \nabla u}{\partial \nu} \right)
+ \int_{\partial \Omega} \langle y - x, \nabla \Delta v \rangle \Delta u - \frac{N - 4}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \Delta v - u \frac{\partial \Delta v}{\partial \nu} + \frac{\partial v}{\partial \nu} \Delta u - v \frac{\partial \Delta u}{\partial \nu} \right)$$

$$- \frac{N - 4}{2} \int_{\Omega} \left( v((-\Delta)^2 u + u((-\Delta)^2 v), \right).$$
since
\[
4 \int_\Omega \Delta u \Delta v + \int_\Omega \langle y - x, \nabla (\Delta u \Delta v) \rangle = \int_{\partial \Omega} \langle y - x, \nu \rangle \Delta u \Delta v - (N - 4) \int_\Omega \Delta u \Delta v
\]
\[
= \int_{\partial \Omega} \langle y - x, \nu \rangle \Delta u \Delta v - \frac{N - 4}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \Delta v - u \frac{\partial \Delta v}{\partial \nu} + \frac{\partial v}{\partial \nu} \Delta u - v \frac{\partial \Delta u}{\partial \nu} \right)
\]
\[
- \frac{N - 4}{2} \int_\Omega (v(\Delta)^2 u + u(\Delta)^2 v).
\]

For any integer \( m > 2 \), we assume that the result is true for any integer up to \( m - 1 \). First, we have
\[
-2 \int_\Omega \left( (-\Delta)^{m-1} u \Delta v + (-\Delta)^{m-1} v \Delta u \right)
\]
\[
= -2 \int_{\partial \Omega} \left( (-\Delta)^{m-1} u \frac{\partial v}{\partial \nu} - (-\Delta)^{m-1} \frac{\partial u}{\partial \nu} v + (-\Delta)^{m-1} v \frac{\partial u}{\partial \nu} - (-\Delta)^{m-1} \frac{\partial v}{\partial \nu} u \right) \tag{3.14}
\]
\[
+ 2 \int_\Omega \left( (\Delta)^m u)v + (\Delta)^m v)u \right).
\]
Moreover, by the induction assumption, we obtain by using the integration by parts,
\[
- \int_\Omega \left( (-\Delta)^{m-1} u \langle y - x, \nabla \Delta v \rangle + (-\Delta)^{m-1} v \langle y - x, \nabla \Delta u \rangle \right)
\]
\[
= \int_{\partial \Omega} g_{m-2}(\Delta u, -\Delta v) - \frac{N - 2(m - 2)}{2} \int_\Omega \left( \Delta v)(-\Delta)^{m-1} u + (-\Delta u)(-\Delta)^{m-1} v \right)
\]
\[
= \int_{\partial \Omega} g_m(u, v) - \frac{N - 2(m - 2)}{2} \int_\Omega \left( v(\Delta)^m u + u(\Delta)^m v \right) \tag{3.15}
\]
Hence, the result for any \( m \) follows from (3.12), (3.14) and (3.15).

\[\square\]

**Remark 3.4.** From (3.14) and (3.15), we can find the formula for \( \tilde{l}_{2m-1}(\nabla^{2m-1} u, v) \):
\[
\tilde{l}_{2m-1}(\nabla^{2m-1} u, v) = -\frac{N - 2m}{2} \int_{\partial \Omega} \frac{\partial (-\Delta)^{m-1} u}{\partial \nu} v. \tag{3.16}
\]

### 3.2. The bubbling solutions

Let \( u_L = \sum_{i=1}^\infty U_{x_i,L,\mu_i,L} + \omega_L \) be a solution of \((P)\), which satisfies (1.3), (1.6) and (1.8). In this section, we will estimate \( \mu_{j,L} \) and \( |x_j,L - P_j| \). We will use various Pohozaev identities to achieve this.

Using Propositions 3.1 and 3.3 we can obtain the following two Pohozaev identities:
\[
\frac{1}{2} \int_{\partial B_i(x_j,L)} f_m, i(u_L, u_L) = \frac{1}{m^*} \int_{\partial B_i(x_j,L)} K(y) u_L^{m^*}, \nu_i - \frac{1}{m^*} \int_{B_i(x_j,L)} \frac{\partial K(y)}{\partial y_i} u_L^{m^*}, \tag{3.17}
\]
and
\[ \frac{1}{2} \int_{\partial B_\delta(x_j, L)} \partial_B g_m(u_L, u_L) \]
\[ = \frac{1}{m^*} \int_{\partial B_\delta(x_j, L)} K(y) u_L^{m^*} \langle y - x_j, L, \nu \rangle - \frac{1}{m^*} \int_{B_\delta(x_j, L)} \langle \nabla K(y), y - x_j, L \rangle u_L^{m^*}, \]
where \( \nu \) is the outward unit normal of \( \partial B_\delta(x_j, L) \). We will estimate each term in (3.17) and (3.18).

We denote \( \mu_L = \max_j \mu_{j, L} \). Note that for \( y \in \partial B_\delta(x_j, L) \),
\[ U_{x_j, L, \mu_{j, L}} = \frac{C_m}{\mu_{j, L}} \sum_{i=0}^{m-1} \frac{\alpha_i}{\mu_{j, L}^2 |y - x_j, L|^{N-2m+2i}} + O\left( \frac{1}{\mu_{j, L}^{N-2m}} \right), \]
where \( \alpha_0 = 1 \), and \( \alpha_i \neq 0 \) is a constant, \( i = 1, \ldots, m - 1 \).

Let
\[ V_{j, L}(y) = \frac{C_m}{\mu_{j, L}^2} \sum_{i=0}^{m-1} \frac{\alpha_i}{\mu_{j, L}^2 |y - x_j, L|^{N-2m+2i}}. \]
Then, we have
\[ U_{x_j, L, \mu_{j, L}} = V_{j, L}(y) + O\left( \frac{1}{\mu_{j, L}^2} \right), \quad y \in \partial B_\delta(x_j, L). \]

Since
\[ (-\Delta)^{m-i} \frac{1}{|y - x_j, L|^{N-2(m-i)}} = 0, \quad \text{in} \ \mathbb{R}^N \setminus \{x_j, L\}, \]
we have
\[ (-\Delta)^{m-i} \frac{\alpha_i}{\mu_{j, L}^2 |y - x_j, L|^{N-2m+2i}} = 0, \quad \text{in} \ \mathbb{R}^N \setminus \{x_j, L\}. \]

So on \( \partial B_\delta(x_j, L) \), the leading term \( V_{j, L}(y) \) of \( u_L \) is an \( m \)-harmonic function. From this observation, by using Propositions 3.1 and 3.3, the estimates of the surface integrals on \( \partial B_\delta(x_j, L) \) in the left hand side of (3.17) and (3.18) for \( V_{j, L}(y) \) can be reduced to the estimates of the surface integrals on \( \partial B_\theta(x_j, L) \) for any small number \( \theta > 0 \), which can be done because we know the singular behavior at \( x_j, L \) of the function \( V_{j, L}(y) \).

**Lemma 3.5.** Relation (3.17) is equivalent to
\[ \int_{B_\delta(x_j, L)} \frac{\partial K(y)}{\partial y_i} U_{x_j, L, \mu_{j, L}}^{m^*} = O\left( \frac{1}{\mu_{j, L}^N} + \frac{1}{\mu_{j, L}^{N-2m} L^{N-2m}} + \frac{\max_i |x_i, L - P_i|^\beta}{\mu_{j, L}^{N-2m+\gamma}} \right) \]
\[ + \frac{\max_i |x_i, L - P_i|^{2\beta}}{\mu_{j, L}^{2\gamma}} + \max_i |x_i, L - P_i|^{m^*\beta}. \]

**Proof.** It follows from (3.21) that
\[ \int_{\partial B_\delta(x_j, L)} K(y) u_L^{m^*} \nu_i = O\left( \frac{1}{\mu_{j, L}^N} \right). \]
To estimate the left hand side of (3.17), noting that $V_{j,L}(y)$ is a function of $|y - x_{j,L}|$, we use Remark 3.2 to obtain

\[ \text{LHS of (3.17)} = \int_{B_b(x_{j,L})} \left( \sum_{i \neq j} U_{x_{j,L} - x_j, L} + \sum_{i \neq j} U_{x_j, L - x_i} \right)^{m^*} = O \left( \frac{1}{\mu_L^{-N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} \right) \]

Thus, (3.23) follows from (3.24), (3.25) and (3.26).

\[ (3.25) \]

**Lemma 3.6.** Relation (3.18) is equivalent to

\[ \frac{1}{\mu_j^L} - B \sum_{i \neq j} \frac{1}{\mu_i^L} \frac{1}{\mu_j^L} |x_{i,L} - x_{j,L}|^{N-2m} \]

\[ = O \left( \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \max_i |x_{i,L} - P_i| + \frac{1}{\mu_L^2} \right) \]

\[ (3.26) \]

where $B > 0$ is a constant.

**Proof.** We first estimate the left hand side of (3.18). We have

\[ u_L(y) = V_{j,L}(y) + \sum_{i \neq j} \frac{\tilde{C}_m}{\mu_i^L |y - x_{i,L}|^{N-2m}} \]

\[ + O \left( \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} + \frac{1}{\mu_L^{N-2m} L^{N-2m}} \right), \quad y \in \partial B_b(x_{j,L}). \]

\[ (3.27) \]

Let

\[ Q_{j,L}(y) = V_{j,L}(y) + \sum_{i \neq j} \frac{\tilde{C}_m}{\mu_i^L |y - x_{i,L}|^{N-2m}} =: V_{j,L}(y) + R_{j,L}(y). \]

\[ (3.28) \]
We have
\[
\int_{\partial B_\delta(x_j, L)} g_m(u_L, u_L) = \int_{\partial B_\delta(x_j, L)} g_m(Q_{j, L}, Q_{j, L}) + O\left( \frac{1}{\mu^N_L} + \frac{1}{\mu^{N-2m+2}_L} \frac{1}{L^{N-2m+2}} + \frac{\| \omega_L \|_\ast}{\mu^{N-2m+2}_L} + \frac{\| \omega_L \|^2_\ast}{\mu^{2r}_L} \right).
\]
(3.30)

Note that $R_{j, L}(y)$ is bounded on $\partial B_\delta(x_j, L)$. Now we compute
\[
\int_{\partial B_\delta(x_j, L)} g_m(Q_{j, L}, Q_{j, L}) = \int_{\partial B_\delta(x_j, L)} g_m(V_{j, L}, V_{j, L}) + 2 \int_{\partial B_\delta(x_j, L)} g_m(V_{j, L}, R_{j, L}) + \int_{\partial B_\delta(x_j, L)} g_m(R_{j, L}, R_{j, L})
\]
(3.31)
\[
= \int_{\partial B_\delta(x_j, L)} g_m(V_{j, L}, V_{j, L}) + 2 \int_{\partial B_\delta(x_j, L)} g_m(V_{j, L}, R_{j, L}) + O\left( \frac{1}{\mu^{N-2m}_L} \right).
\]
Moreover, from the definition of $V_{j, L}$, we get
\[
\int_{\partial B_\delta(x_j, L)} g_m(V_{j, L}, R_{j, L}) = \frac{\tilde{C}_m}{\mu^{N-2m}_L} \int_{\partial B_\delta(x_j, L)} g_m(\frac{1}{|y - x_{j, L}|^{N-2m}}, R_{j, L}) + O\left( \frac{1}{\mu^{N-2m}_L} \right).
\]
(3.32)

Note that both $\frac{1}{|y - x_{j, L}|^{N-2m}}$ and $R_{j, L}$ are $m$-harmonic in $\Omega = B_\delta(x_j, L) \setminus B_\theta(x_j, L)$, where $\theta > 0$ is any small constant. We apply Proposition 3.3 in $\Omega$ to obtain
\[
\int_{\partial B_\delta(x_j, L)} g_m(\frac{1}{|y - x_{j, L}|^{N-2m}}, R_{j, L}) = \int_{\partial B_\theta(x_j, L)} g_m(\frac{1}{|y - x_{j, L}|^{N-2m}}, R_{j, L}).
\]
(3.33)

Since $R_{j, L}$ and its derivatives are bounded on $\partial B_\theta(x_j, L)$, we find that the term in (3.10) satisfies
\[
|\tilde{I}_l(y - x_{j, L}, \nabla^l \frac{1}{|y - x_{j, L}|^{N-2m}}, \nabla^{2m-l} R_{j, L})| \leq \frac{C}{|y - x_{j, L}|^{N-2}}, \quad l = 0, \ldots, 2m - 1.
\]

As a result, as $\theta \to 0$,
\[
\int_{\partial B_\theta(x_j, L)} \tilde{I}_l(y - x_{j, L}, \nabla^l \frac{1}{|y - x_{j, L}|^{N-2m}}, \nabla^{2m-l} R_{j, L}) \to 0, \quad l = 0, \ldots, 2m - 1.
\]
(3.34)

For $l = 0, \ldots, 2m - 2$, the other terms in (3.10) satisfies
\[
|\tilde{I}_l(\nabla^l \frac{1}{|y - x_{j, L}|^{N-2m}}, \nabla^{2m-l-1} R_{j, L})| \leq \frac{C}{|y - x_{j, L}|^{N-2}}.
\]

Therefore,
\[
\int_{\partial B_\theta(x_j, L)} \tilde{I}_l(\nabla^l \frac{1}{|y - x_{j, L}|^{N-2m}}, \nabla^{2m-l-1} R_{j, L}) \to 0, \quad l = 0, \ldots, 2m - 2, \text{ as } \theta \to 0.
\]
(3.35)
Using (3.16), we have
\[
\int_{\partial B_\theta(x_j, L)} \tilde{l}_{2m-1} \big(\nabla^{2m-1} \frac{1}{|y - x_j, L|^{N-2m}}, R_{j, L} \big) = - \frac{N - 2m}{2} \int_{\partial B_\theta(x_j, L)} \partial \big[ (-\Delta)^{m-1} \frac{1}{|y - x_j, L|^{N-2m}} \big] R_{j, L}. \tag{3.36}
\]

The function \((-\Delta)^{m-1} \frac{1}{|y - x_j, L|^{N-2m}}\) depends on \(|y - x_j, L|\) only, which satisfies
\[
-\Delta \big[ (-\Delta)^{m-1} \frac{1}{|y - x_j, L|^{N-2m}} \big] = c_m \delta_{x_j, L},
\]
for some constant \(c_m > 0\). As a result, there is a constant \(c'_m > 0\), such that
\[
(-\Delta)^{m-1} \frac{1}{|y - x_j, L|^{N-2m}} = \frac{c'_m}{|y - x_j, L|^{N-2}}.
\]

This, together with (3.36) gives
\[
\int_{\partial B_\theta(x_j, L)} \tilde{l}_0 \big(\nabla^{2m-1} \frac{1}{|y - x_j, L|^{N-2m}}, R_{j, L} \big) = \frac{(N - 2)(N - 2m)}{2} \int_{\partial B_\theta(x_j, L)} R_{j, L} \frac{c'_m}{|y - x_j, L|^{N-2}}
\]
\[
= \sum_{i \neq j} \frac{B_m}{\mu_{i, L}^{2m - 2} |x_{j, L} - x_i, L|^{N-2m}} (1 + o_\theta(1)), \tag{3.37}
\]
where \(B_m > 0\) is a constant.

Combining (3.32)–(3.37), we conclude the existence of some constant \(B'_m > 0\) such that
\[
2 \int_{\partial B_\theta(x_j, L)} g_m(V_{j, L}, R_{j, L})
\]
\[
= \sum_{i \neq j} \frac{B'_m}{\mu_{i, L}^{2m - 2} \mu_{j, L}^{N-2m} |x_{j, L} - x_i, L|^{N-2m}} + O \big( \frac{1}{\mu_{L}^{N-2m + 2} |L|^{N-2m}} \big). \tag{3.38}
\]

We are now to estimate
\[
\int_{\partial B_\theta(x_j, L)} g_m(V_{j, L}, V_{j, L})
\]
\[
= \tilde{c}_m^{\, 2} \sum_{\mu_{j, L}^{N-2m}} \sum_{h=0}^{m-1} \sum_{k=0}^{m-1} \frac{\alpha_h \alpha_k}{\mu_{j, L}^{2h} \mu_{j, L}^{2k}} \int_{\partial B_\theta(x_j, L)} g_m \frac{1}{|y - x_j, L|^{N-2m + 2h}} \frac{1}{|y - x_j, L|^{N-2m + 2k}}. \tag{3.39}
\]

Since \(\frac{1}{|y - x_j, L|^{N-2m + 2k}}\) and \(\frac{1}{|y - x_j, L|^{N-2m + 2k}}\) are \(m\)-harmonic in \(B_\theta(x_j, L) \setminus B_\theta(x_j, L)\), we can use Proposition 3.3 to obtain
\[
\int_{\partial B_\theta(x_j, L)} g_m \frac{1}{|y - x_j, L|^{N-2m + 2h}} \frac{1}{|y - x_j, L|^{N-2m + 2k}}
\]
\[
= \int_{\partial B_\theta(x_j, L)} g_m \frac{1}{|y - x_j, L|^{N-2m + 2h}} \frac{1}{|y - x_j, L|^{N-2m + 2k}}. \tag{3.40}
\]
On the other hand, we have
\[
\begin{align*}
g_m &\left(\frac{1}{|y - x_j,L|^{N-2m+2n}} \frac{1}{|y - x_j,L|^{N-2m+2k}}\right) \\
&= \sum_{l=1}^{2m-1} \bar{l}_l (y - x_j,L, \nabla^l \frac{1}{|y - x_j,L|^{N-2m+2n}}, \nabla^{2m-l} \frac{1}{|y - x_j,L|^{N-2m+2k}}) \\
&+ \sum_{l=0}^{2m-1} \bar{l}_l (\nabla^l \frac{1}{|y - x_j,L|^{N-2m+2n}}, \nabla^{2m-l-1} \frac{1}{|y - x_j,L|^{N-2m+2k}}) \\
&= \frac{1}{|y - x_j,L|^{2N-2m+2(h+k)-1}} \bar{f}_{h,k}(\omega), \quad \omega \in \mathbb{S}^{N-1},
\end{align*}
\]
where \(\bar{f}_{h,k}\) is some function defined on \(\mathbb{S}^{N-1}\). Thus (3.40) and (3.41) yield
\[
\int_{\partial B_\delta(x_j,L)} g_m \left(\frac{1}{|y - x_j,L|^{N-2m+2n}} \frac{1}{|y - x_j,L|^{N-2m+2k}}\right) = \frac{1}{\theta^{N-2m+2(h+k)}} \int_{\mathbb{S}^{N-1}} \bar{f}_{h,k}(\omega). \tag{3.42}
\]
Since the left hand side of (3.42) is finite and \(N - 2m + 2(h + k) > 0\), we conclude \(\int_{\mathbb{S}^{N-1}} \bar{f}_{h,k}(\omega) = 0\), which gives
\[
\int_{\partial B_\delta(x_j,L)} g_m \left(\frac{1}{|y - x_j,L|^{N-2m+2n}} \frac{1}{|y - x_j,L|^{N-2m+2k}}\right) = 0. \tag{3.43}
\]
Therefore, we have proved
\[
\int_{\partial B_\delta(x_j,L)} g_m (V_{j,L}, V_{j,L}) = 0. \tag{3.44}
\]
Inserting (3.38) and (3.44) into (3.31), we obtain
\[
\int_{\partial B_\delta(x_j,L)} g_m (Q_{j,L}, Q_{j,L}) = \sum_{i \neq j} \frac{B'_m}{\mu_{j,L}^{N-2m} \mu_{i,L}^{N-2m}} |x_{j,L} - x_{i,L}|^{N-2m} + O\left(\frac{1}{\mu_L^{N-2m+2L N-2m} + \frac{1}{\mu_L^{N-2m} L^2(N-2m)}}\right). \tag{3.45}
\]
From (3.30) and (3.45), we get
\[
\text{LHS of (3.18)} = \sum_{i \neq j} \frac{B'_m}{\mu_{j,L}^{N-2m} \mu_{i,L}^{N-2m}} |x_{j,L} - x_{i,L}|^{N-2m} \\
+ O\left(\frac{1}{\mu_L^{N-2m} L N-2m} + \frac{1}{\mu_L^{N-2m+2L N-2m} + \frac{1}{\mu_L^{N-2m+2L N-2m}}}\right) + \frac{\max_i |x_{i,L} - P_1|^\beta}{\mu_L^{N-2m} L^2(N-2m)}. \tag{3.46}
\]
We now estimate the right-hand side of (3.18). Firstly, we have
\[
\int_{\partial B_{\delta}(x_{j,L})} K(y)u_{L}^{m^*} \langle y - x_{j,L}, \nu \rangle = O\left(\frac{1}{\mu_{L}^N}\right).
\] (3.47)

On the other hand,
\[
-\frac{1}{m^*} \int_{B_{\delta}(x_{j,L})} \langle \nabla K(y), y - x_{j,L} \rangle u_{L}^{m^*} = -\beta \sum_{i=1}^{N} a_{i} |y_{i} - x_{j,i,L}|^{\beta} u_{L}^{m^*}
\]
\[
+ O\left(\frac{\max_{i} |x_{i,L} - P_{i}|}{\mu_{L}^{\beta - 1}} + \frac{\max_{i} |x_{i,L} - P_{i}|^{\beta - 1}}{\mu_{L}^{\beta}} + \frac{\max_{i} |x_{i,L} - P_{i}|^{m^*\beta}}{\mu_{L}^{m^*\beta - \tau}} + \frac{1}{\mu_{j,L}^{\beta + 1}}\right).
\] (3.48)

Thus, the desired result follows from (3.46), (3.47) and (3.48).

\[\square\]

Proposition 3.7. It holds \(|x_{j,L} - P_{j}| = O\left(\frac{1}{\mu_{j,L}^N}\right)\).

Proof. With the same arguments as those of [14], we can verify the estimates \(\mu_{L} |x_{j,L} - P_{j}| \leq C\) and \(\frac{\mu_{L}^{\beta}}{\mu_{L}^{N - 2m}} \leq C\) for some \(0 < C < \infty\). Hence we may assume \(\mu_{L} (x_{j,L} - P_{j}) \to x_{0}\), which implies that
\[
\int_{B_{L}(x_{j,L})} \frac{\partial K(y)}{\partial y_{i}} u_{L}^{m^*} y_{i, x_{j,L}, \mu_{j,L}} = a_{i} \beta \int_{B_{p_{j,L}}(0)} |\mu_{j,L}^{-1} y_{i} + x_{j,i,L} - P_{j,i}|^{\beta - 2} (\mu_{j,L}^{-1} y_{i} + x_{j,i,L} - P_{j,i}) u_{0,1}^{m^*} + O\left(\frac{1}{\mu_{j,L}^{\beta}}\right)
\] (3.49)
\[
= -a_{i} \beta \mu_{j,L}^{-\beta - 1} \int_{B_{p_{j,L}}(0)} |y_{i} + x_{0,i}|^{\beta - 2} (y_{i} + x_{0,i}) u_{0,1}^{m^*} + o(1) + O\left(\frac{1}{\mu_{j,L}^{\beta}}\right).
\]

It follows from (3.23) and (3.49) that
\[
\int_{B_{p_{j,L}}(0)} |y_{i} + x_{0,i}|^{\beta - 2} (y_{i} + x_{0,i}) u_{0,1}^{m^*} = o(1),
\] (3.50)
which yields \(x_{0} = 0\).
Noting that \( \int_{B_{y(i,\mu)}} |y_i|^{-2} y_i U_{0,1}^{m*} = 0 \), we obtain
\[
\int_{B_{\delta(x,i,j)}} \frac{\partial K(y)}{\partial y_i} U_{x,i,j,\mu}^{m*} = \left( \frac{a_i}{\mu_{j,L}} \right)^{\beta - 1} \int_{B_{\delta(x,i,j)}} \frac{v|y|^{-2} U_{0,1}^{m*} \mu_{j,L}}{x,i,j,\mu} |x,i,j,\mu - P_j,i| + O\left( \frac{1}{\mu_{j,L}} \right),
\]
which, together with Lemma 3.5, implies \( \mu_{j,L}(x,j,i,L - P_{j,i}) = O\left( \frac{1}{\mu_{j,L}} \right) \). Therefore \( |x,j,L - P_j| = O\left( \frac{1}{\mu_{j,L}} \right) \).

**Proposition 3.8.** It holds \( \mu_{j,L} = L_{\beta - N + 2m}^{N - 2m} \left( \bar{B}_j + O\left( \frac{1}{L_{\beta - N + 2m}} \right) \right) \), for some constant \( \bar{B}_j > 0 \).

**Proof.** Noting that \( |x,j,L - P_j| = O\left( \frac{1}{\mu_{j,L}} \right) \), we find
\[
\frac{1}{|x,j,L - x,j,L|^{N-2m}} = \frac{1}{(|P_i - P_j| + O\left( \frac{1}{\mu_{L'}} \right))^{N-2m}} = \frac{1}{|\bar{P}_i - \bar{P}_j|^{N-2m}} \left( 1 + O\left( \frac{1}{|\bar{P}_i - \bar{P}_j| L^{\mu_{L'}}} \right) \right).
\]
As a result, we see that \( 3.27 \) is equivalent to
\[
\frac{1}{\mu_{j,L}} = B \sum_{i \neq j} \frac{1}{\mu_{i,L}^2} \frac{1}{\mu_{j,L}^2} \frac{1}{\mu_{i,L}^{N-2m}} |\bar{P}_i - \bar{P}_j|^{N-2m} + O\left( \frac{1}{\mu_{L} L^{N-2m}} \right).
\]
Since we assume that \( 0 < \mu_0 < \mu_{j,L} \leq \mu' \), we can easily deduce from \( 3.52 \) that
\[
0 < c_1 L^{N-2m} \leq \mu_{i,L} \leq c_1' L^{N-2m}.
\]
Let \( \frac{1}{\mu_{j,L}^{N-2m}} = \frac{a_i}{L^{(N-2m)^2}} \). Then, \( 0 < c_0 \leq a_{j,L} \leq c_1 < +\infty \), and
\[
a_{j,L}^2 = B \sum_{i \neq j} \frac{a_i}{|\bar{P}_i - \bar{P}_j|^{N-2m}} + O\left( \frac{1}{\mu_{j,L}} \right),
\]
where \( \kappa = \frac{\beta - N - 2m}{N - 2m} > 1 \). Hence, from Lemma D.2, we obtain \( a_{j,L} = a_j + O\left( \frac{1}{\mu_{j,L}} \right) \) for some \( a_j > 0 \).

### 3.3. Local uniqueness

Suppose that problem (P) have two different solutions \( u_L^{(1)} \) and \( u_L^{(2)} \), which blow up at \( P_j, j = 1, 2, \ldots \). For \( k = 1, 2 \), we use \( x_{j,L}^{(k)} \) and \( \mu_{j,L}^{(k)} \) to denote the center and the height of the bubbles appearing in \( u_L^{(k)} \), respectively.

Let
\[
\eta_L = \frac{u_L^{(1)} - u_L^{(2)}}{\|u_L^{(1)} - u_L^{(2)}\|_*},
\]
Then, \( \eta_L \) satisfies \( \|\eta_L\|_* = 1 \) and
\[
(-\Delta)^m \eta_L = f(y, u_L^{(1)}, u_L^{(2)}),
\]
where
\[ f(y, u_L^{(1)}, u_L^{(2)}) = \frac{1}{\|u_L^{(1)} - u_L^{(2)}\|_*} K(y)((u_L^{(1)})^{m^*} - 1 - (u_L^{(2)})^{m^*}) \]  
(3.55)

Write
\[ f(y, u_L^{(1)}, u_L^{(2)}) = K(y)c_L(y)\eta_L(y), \]  
(3.56)

Thus, we have proved
\[ c_L(y) = (m^* - 1) \int_0^1 (tu_L^{(1)}(y) + (1-t)u_L^{(2)}(y))^{m^*-2} \, dt. \]  
(3.57)

It follows from Propositions \ref{prop:3.7} and \ref{prop:3.8} that
\[ U_{x_i,L^{(1)}} - U_{x_i,L^{(2)}} = O\left(\left|\frac{\sum_{i = 1}^{\infty} U_{x_i,L^{(1)}} - U_{x_i,L^{(2)}}}{\mu_L} + \left|\omega_L^{(1)}\right| + \left|\omega_L^{(2)}\right|\right|L_{\infty}^{m^*-2} \right), \]  
(3.58)

which gives
\[ u_L^{(1)} - u_L^{(2)} = O\left(\frac{1}{\mu_L} \sum_{i = 1}^{\infty} U_{x_i,L^{(1)}} + \left|\omega_L^{(1)}\right| + \left|\omega_L^{(2)}\right|\right), \]  
(3.59)

Thus, we have proved
\[ c_L(y) = (m^* - 1)U^{m^*-2}_{x_i,L^{(1)}} \]  
(3.56)

Using the Hölder inequality, noting that \(\|\omega_L^{(i)}\|_* = o(1)\), we can deduce
\[ |f(z, u_L^{(1)}, u_L^{(2)})| \leq CW^{m^*-2}_{L,x^{(1)},\mu^{(1)}}|\eta_L(z)| + C(|\omega_L^{(1)}(z)|^{m^*-2} + |\omega_L^{(2)}(z)|^{m^*-2})|\eta_L(z)| \]  
(3.60)

\[ \leq CW^{m^*-2}_{L,x^{(1)},\mu^{(1)}}|\eta_L(z)| + o(1)\|\omega_L\|_*|\eta_L(z)| + o(1)\|\omega_L\|_*|\eta_L(z)| \]  
(3.61)

Similar to the proof of Lemma \ref{lem:A.4}, we deduce from (3.60) that
\[ |\eta_L(y)| \left(\sigma(y) \sum_{i = 1}^{\infty} \frac{(\mu_i^{(1)})^{N-2m}}{(1 + \mu_i^{(1)}|y - x_i^{(1)}|)^{N-2m + \tau}}\right)^{-1} \]  
\[ \leq C\left(o(1)\|\eta_L\|_* + \sum_{j = 1}^{\infty} \frac{1}{(1 + \mu_{j,L}^{(1)}|z - x_j^{(1)}|)^{N-2m + \tau}}\right) \|\eta_L\|_* \]  
(3.61)
To obtain a contradiction, we just need to show that $|\eta_L(y)| = o(1)$ in $\cup_j B_{R(\mu_{j,L}^{-1})^{-1}}(x_{j,L}^{(1)})$, which will be achieved by using the Pohozaev identities in the small ball $B_d(x_{j,L}^{(1)})$.

Let
\[
\tilde{\eta}_{L,j}(y) = \left( \frac{1}{\mu_{j,L}} \right)^{\frac{N-2m}{2}} \eta_L \left( \frac{1}{\mu_{j,L}} y + x_{j,L}^{(1)} \right).
\]

\[
(3.62)
\]

**Lemma 3.9.** It holds
\[
\tilde{\eta}_{L,j}(y) \rightarrow \sum_{k=0}^{N} b_{j,k} \psi_k(y), \quad \text{as } L \rightarrow \infty,
\]

uniformly in $C^m(B_R(0))$ for any $R > 0$, where $b_{j,k}$, $k = 0, \cdots, N$, are some constants, and
\[
\psi_0 = \left. \frac{\partial U_{0,\lambda}}{\partial \lambda} \right|_{\lambda=1}, \quad \psi_j = \left. \frac{\partial U_{0,1}}{\partial y_j} \right|, \quad j = 1, \cdots, N.
\]

\[
(3.63)
\]

**Proof.** In view of $|\tilde{\eta}_{L,j}| \leq C$ in any compact subset of $\mathbb{R}^N$, we may assume that $\tilde{\eta}_{L,j} \rightarrow \xi_j$ in $C_{loc}(\mathbb{R}^N)$. Then it follows from the elliptic regularity theory and (3.54) and (3.59) that $\xi_j$ satisfies
\[
(-\Delta)^m \xi_j = (m^* - 1) U_{m^* - 2} U_{0,1} \xi_j, \quad \text{in } \mathbb{R}^N,
\]

which combining with the non-degeneracy of $U_{0,1}$ gives
\[
\xi_j = \sum_{k=0}^{N} b_{j,k} \psi_k.
\]

\[
(3.65)
\]

\[
(3.66)
\]

Let $G(y, x) = C_m |y - x|^{2m-N}$ be the corresponding Green’s function of $(-\Delta)^m$ in $\mathbb{R}^N$.

**Lemma 3.10.** We have the following estimate:
\[
\eta_L(x) = \sum_{j=1}^{\infty} \sum_{|\alpha|=0}^{2m-1} A_{j,L,\alpha} \partial^\alpha G(x_{j,L}^{(1)}, x) + O\left( \frac{1}{\mu_L^{\frac{N}{2m}} - \theta} \right)
\]
\[
:= \sum_{j=1}^{\infty} F_{j,m,L}(x) + O\left( \frac{1}{\mu_L^{\frac{N}{2m}} - \theta} \right), \quad \text{in } C^{2m-1} \left( \mathbb{R}^N \setminus \bigcup_{j=1}^{\infty} B_{2\sigma}(x_{j,L}^{(1)}) \right),
\]

where $\sigma > 0$ is any small constant, $\partial^\alpha G(y, x) = \frac{\partial^\alpha G(y, x)}{\partial y_{i_1} \cdots \partial y_{i_N}}$, $\alpha = (\alpha_{i_1}, \cdots, \alpha_{i_N})$, and the constants $A_{j,L,\alpha}$ satisfy the following estimates:
\[
A_{j,L,0} = \int_{B_{2\sigma}(x_{j,L}^{(1)})} f(y, u_L^{(1)}(y), u_L^{(2)}(y)) dy + o\left( \frac{1}{\mu_L^{\frac{N}{2m}}} \right),
\]

\[
(3.68)
\]

\[
A_{j,L,\alpha} = O\left( \frac{1}{\mu_L^{\frac{N}{2m} + |\alpha|}} \right), \quad |\alpha| \geq 1.
\]

\[
(3.69)
\]
Proof. Denote \( f^*_L(y) = f(y, u^{(1)}_L(y), u^{(2)}_L(y)) \). We have

\[
\eta_L(x) = \int_{\mathbb{R}^N} G(y, x) f^*_L(y) \, dy \\
= \sum_{j=1}^{\infty} \int_{B_\sigma(x^{(1)}_{j,L})} G(y, x) f^*_L(y) \, dy + \int_{\mathbb{R}^N \setminus \bigcup_j B_\sigma(x^{(1)}_{j,L})} G(y, x) f^*_L(y) \, dy \\
= \sum_{j=1}^{\infty} \sum_{|\alpha|=0} A_{j,L,\alpha} \partial^\alpha G(x^{(1)}_{j,L}, x) \\
+ \sum_{j=1}^{\infty} O \left( \int_{B_\sigma(x^{(1)}_{j,L})} |y - x^{(1)}_{j,L}|^{2m} |f^*_L(y)| \, dy \right) + \int_{\mathbb{R}^N \setminus \bigcup_j B_\sigma(x^{(1)}_{j,L})} G(y, x) f^*_L(y) \, dy. \tag{3.70}
\]

For \( y \in \mathbb{R}^N \setminus \bigcup_j B_\sigma(x^{(1)}_{j,L}) \), noting that \( \tau = \frac{N-2m}{2} - \theta \) for \( \theta > 0 \) small, similarly to (3.60) and (A.6), we find

\[
|f^*_L(y)| \leq C \mu_L^{-\tau} \left( \frac{1}{\mu_L^{2m}} + (\mu_L^{-\tau} \|\omega_L^{(1)}\|)^{m^*-2} \right) \sum_{j=1}^{\infty} \int_{\mathbb{R}^N \setminus \bigcup_j B_\sigma(x^{(1)}_{j,L})} G(y, x) f^*_L(y) \, dy \tag{3.71}
\]

Thus, we have

\[
\int_{\mathbb{R}^N \setminus \bigcup_j B_\sigma(x^{(1)}_{j,L})} G(y, x) f^*_L(y) \, dy \leq C \mu_L^{-\frac{N+2m}{2}+\theta} \sum_{j=1}^{\infty} \frac{1}{|x - x^{(1)}_{j,L}|^{N-2m+\tau}} \leq \frac{C}{\mu_L^{\frac{N+2m}{2}-\theta}}. \tag{3.72}
\]

Similarly, by Lemma (A.3)

\[
\int_{B_\sigma(x^{(1)}_{j,L})} |y - x^{(1)}_{j,L}|^{2m} |f^*_L| \leq C \int_{B_\sigma(x^{(1)}_{j,L})} \frac{(\mu_{j,L}^{(1)})^{N+2m}}{1 + (\mu_{j,L}^{(1)} |y - x^{(1)}_{j,L}|)^{N-2m+\theta}(m^*-1)} \frac{1}{|y - x^{(1)}_{j,L}|^{N-\theta(m^*-1)}} \leq \frac{C}{\mu_L^{\frac{N+2m}{2} - (m^*-1)\theta}}.
\]
Inserting (3.72) and (3.71) into (3.70), we obtain (3.67). Similarly, we can prove that (3.67) holds in $C^{2m-1}(\mathbb{R}^N \setminus \bigcup_{j=1}^{\infty} B_{2\sigma}(x_{j,L}))$. It remains to estimate $A_{j,L,\alpha}$.

$$A_{j,L,0} = \int_{B_{\sigma}(x_{j,L})} f(y, u_L^{(1)}(y), u_L^{(2)}(y)) \, dy$$

$$= \frac{1}{(\mu_{j,L})^{N-2m}} \int_{B_R(0)} \frac{1}{(\mu_{j,L})^{2m}} f_L^*(\frac{1}{\mu_{j,L}}) y + x_{j,L}^{(1)} \, dy$$

$$+ \frac{1}{(\mu_{j,L})^{N-2m}} O \left( \int_{B_{\mu_{j,L}}(0) \setminus B_R(0)} \frac{1}{|y|^{(N-2m-\theta)(m-1)}} \, dy \right)$$

$$= \frac{1}{(\mu_{j,L})^{N-2m}} (m - 1) \int_{\mathbb{R}^N} U_0^{2^*(m) - 2} \sum_{k=0}^{N} b_{j,k} \psi_k + o(1)) = o\left( \frac{1}{\mu_L^{2m}} \right).$$

If $|\alpha| \geq 1$, then

$$|A_{j,L,\alpha}| \leq C \int_{B_{\sigma}(x_{j,L})} |y - x_{j,L}^{(1)}|^{1|\alpha|} |f(y, u_L^{(1)}(y), u_L^{(2)}(y))| = O\left( \frac{1}{\mu_L^{2m+|\alpha|}} \right).$$

Using (3.7) and (3.18), we can deduce the following identities:

$$\int_{\partial B_d(x_{j,L})} f_m, i(\eta_L, u_L^{(1)}) + \int_{\partial B_d(x_{j,L})} f_m, i(u_L^{(2)}, \eta_L)$$

$$= \int_{\partial B_d(x_{j,L})} K(y) C_L(y) \eta_L \nu_i \int_{B_d(x_{j,L})} \frac{\partial K(y)}{\partial y_i} C_L(y) \eta_L,$$

and

$$\int_{\partial B_d(x_{j,L})} g_m(\eta_L, u_L^{(1)}) + \int_{\partial B_d(x_{j,L})} g_m(u_L^{(2)}, \eta_L)$$

$$= \int_{\partial B_d(x_{j,L})} K(x) C_L(y) \eta_L \langle y - x_{j,L}^{(1)}, \nu \rangle \int_{B_d(x_{j,L})} \langle \nabla K(y), y - x_{j,L}^{(1)} \rangle C_L(y) \eta_L,$$

where $C_L(y) = T_0^{1_0}(tu_L^{(1)} + (1 - t)u_L^{(2)})^{m+1} \, dt$ and $d > 0$ is a small constant.

Similar to (3.59), we can deduce

$$C_L(y) = U_{m-1}^{x_{j,L}^{(1)}, \mu_{j,L}} + O\left( \left( \frac{1}{\mu_L} U_{x_{j,L}^{(1)}, \mu_{j,L}} + \frac{1}{\mu_L^{2m}} \frac{L^{N-2m}}{L^{N-2m}} + |\omega_L^{(1)}| + |\omega_L^{(2)}| \right)^{m-1} \right), \quad y \in B_d(x_{j,L}).$$
To estimate the boundary terms in (3.75) and (3.76), we need the following estimates which can be deduced from Proposition C.1 and Lemma 3.10

\[ u_L^{(k)}(y) = \frac{C_m}{\mu_L^{(k)}} \sum_{i=0}^{m-1} \frac{\alpha_i}{\mu_L^{(k)} 2^i |y - x_j^{(k)}|^{N-2m+2i}} + O\left(\frac{1}{\mu_L^{N-2m}} + \frac{1}{\mu_L^{N-2m}}\right) \]

\[ = V_L^{(k)} + O\left(\frac{1}{\mu_L^{N-2m}} + \frac{1}{\mu_L^{N-2m}}\right), \quad y \in \partial B_d(x_j^{(k)}), \]

and

\[ \eta_L(y) = F_{j,m,L}(y) + O\left(\frac{1}{\mu_L^{N-2m}}\right) + o\left(\frac{1}{\mu_L^{N-2m}}\right), \quad y \in \partial B_d(x_j^{(1)}). \]

**Proof of Theorem 1.2.** **Step 1.** We prove \( b_{j,k} = 0, k = 1, \cdots, N. \) We need to estimate each term in (3.75). From (3.77), we obtain

\[ \int_{\partial B_d(x_j^{(1)})} K(y) C_L(y) \eta_L \nu_i = O\left(\frac{1}{\mu_L^{N}}\right), \]

and

\[ - \int_{B_d(x_j^{(1)})} \frac{\partial K(y)}{\partial y_i} C_L(y) \eta_L \]

\[ = - \frac{\beta a_i}{(\mu_L^{(1)})^{\beta-1}} \int_{B_d(x_j^{(1)})^{(0)}} |y_i|^{\beta-2} y_i C_L\left(\frac{1}{\mu_L^{(1)}} y + x_j^{(1)}\eta\right) + O\left(\frac{1}{\mu_L^{N}}\right) \]

\[ = - \frac{\beta a_i}{(\mu_L^{(1)})^{\beta-1}} \left( \int_{B_d(0)} |y_i|^{\beta-2} y_i U_{0,1}^{m-1} \sum_{k=0}^{N} b_{j,k} \psi_k + o(1) \right) + O\left(\frac{1}{\mu_L^{N}}\right) \]

\[ = - \frac{\beta a_i}{(\mu_L^{(1)})^{\beta-1}} \left( b_{j,i} \int_{\mathbb{R}^N} |y_i|^{\beta-2} y_i U_{0,1}^{m-1} \psi_i + o(1) \right) + O\left(\frac{1}{\mu_L^{N}}\right). \]

Combining (3.81) and (3.81), we are led to

\[ \text{RHS of (3.75)} = - \frac{\beta a_i}{(\mu_L^{(1)})^{\beta-1}} \left( b_{j,i} \int_{\mathbb{R}^N} |y_i|^{\beta-2} y_i U_{0,1}^{m-1} \psi_i + o(1) \right) + O\left(\frac{1}{\mu_L^{N}}\right). \]

To estimate the left hand side of (3.75), using (3.78), we have

\[ \text{LHS of (3.75)} = \int_{\partial B_d(x_j^{(1)})} f_{m,i}(F_{j,m,L}, V_L^{(1)}) + \int_{\partial B_d(x_j^{(1)})} f_{m,i}(V_L^{(2)}, F_{j,m,L}) + o\left(\frac{1}{\mu_L^{N-2m}} + \frac{1}{\mu_L^{N-2m}}\right). \]

Now we claim that
\[
\int_{\partial B_d(x_{j,L}^{(1)})} f_{m,i}(F_{j,m,L}, V_L^{(1)}) = 0, \quad \int_{\partial B_d(x_{j,L}^{(1)})} f_{m,i}(V_L^{(2)}, F_{j,m,L}) = 0.
\] (3.84)

This gives
\[
\text{LHS of (3.75)} = o\left(\frac{1}{\mu_{L}^{N-2m}L^{N-2m}}\right) + O\left(\frac{1}{\mu_{L}^{N}}\right).
\] (3.85)

Hence, (3.82) and (3.85) imply \( b_{j,i} = 0, \ i = 1, \ldots, N. \)

It remains to prove (3.84). We just prove the second integral in (3.84) is zero. Note that this integral is a linear combination of the following integrals
\[
\int_{\partial B_d(x_{j,L}^{(1)})} f_{m,i} \left( \frac{1}{|y - x_{j,L}^{(2)}|^{N-2m+2i}} \right).
\] (3.86)

So, we just need to prove that the integral defined in (3.86) is zero.

Since \( \frac{1}{|y - x_{j,L}^{(2)}|^{N-2m+2i}} \) and \( \partial^a G(x_{j,L}^{(2)}, x) \) are \( m \)-harmonic in \( B_d(x_{j,L}^{(1)}) \setminus B_{\theta}(x_{j,L}^{(2)}) \), by Proposition 3.1, we find
\[
\int_{\partial B_d(x_{j,L}^{(1)})} f_{m,i} \left( \frac{1}{|y - x_{j,L}^{(2)}|^{N-2m+2i}} \right) = 0.
\] (3.88)

**Step 2.** We prove \( b_{j,0} = 0. \) It is easy to deduce

RHS of (3.76)
\[
= -\frac{\beta}{(\mu_{j,L}^{(1)})^\beta} \left( \int_{\mathbb{R}^N} \sum_{i=1}^{N} a_i |y_i|^\beta U_{0,1}^{m-1} \sum_{k=0}^{N} b_{j,k} \psi_k + o(1) \right) + O\left(\frac{1}{\mu_{L}^{N}}\right)
\] (3.89)

\[
= -\frac{\beta}{N(\mu_{j,L}^{(1)})^\beta} \left( b_{j,0} \int_{\mathbb{R}^N} |y|^\beta U_{0,1}^{m-1} \psi_0 + o(1) \right) + O\left(\frac{1}{\mu_{L}^{N}}\right).
\]

It follows from (3.78) and (3.79) that

LHS of (3.76)
\[
= \int_{\partial B_d(x_{j,L}^{(1)})} g_m(F_{j,m,L}, V_L^{(1)}) + \int_{\partial B_d(x_{j,L}^{(1)})} g_m(V_L^{(2)}, F_{j,m,L})
+ o\left(\frac{1}{\mu_{L}^{N-2m}L^{N-2m}}\right) + O\left(\frac{1}{\mu_{L}^{N}}\right).
\] (3.90)
Similar to the proof of (3.84) in Step 1, we can prove
\[ \int_{\partial B_d(x_j,L)} g_m(F_{j,m,L}, V^{(1)}_L) = 0, \quad \int_{\partial B_d(x_j,L)} g_m(V^{(2)}_L, F_{j,m,L}) = 0. \] (3.91)
Therefore
\[ \text{LHS of (3.76)} = o\left(\frac{1}{\mu^{N-2m}_{L} L^{N-2m}}\right) + O\left(\frac{1}{\mu^{N}_{L}}\right). \] (3.92)
Combining (3.89) and (3.92), we are led to
\[ b_{j,0} \int_{\mathbb{R}^N} |y|^\beta U_{0,1}^{m^*-1} \psi_0 = o(1). \] (3.93)
This gives \( b_{j,0} = 0. \)

**Proof of Theorem 1.3.** To prove that \( u_L \) is periodic in \( y_1 \), we let \( v_L(y) = u_L(y_1 - L, y_2, \cdots, y_N). \) Then, \( v_L \) is a bubbling solution whose blow-up set is the same as that of \( u_L \). By the local uniqueness, \( v_L = u_L \). Similarly, we can prove that \( u_L \) is periodic in \( y_j, j = 2, \cdots, k \). □

**Appendix A. Some basic estimates**

In this section, we give some technical lemmas. Throughout Appendixes A, B, C and D, we will use the same notations as before and we also use the same \( C \) to denote different constants unless otherwise stated. The proof of the following two Lemmas can be found in [35].

**Lemma A.1.** Let \( x_i, x_j \in \mathbb{R}^N, x_i \neq x_j, i \neq j \), it holds
\[
\frac{1}{(1 + |y - x_i|)^\alpha (1 + |y - x_j|)^\beta} \leq \frac{C}{(1 + |x_i - x_j|)^\sigma (1 + |y - x_j|)\alpha + 1} \leq \frac{1}{(1 + |y - x_i|)\alpha + 1},
\]
where \( \alpha \) and \( \beta \) are some positive constants, \( 0 < \sigma \leq \min(\alpha, \beta) \).

**Lemma A.2.** For any constant \( 0 < \sigma < N - 2m \), there exists a constant \( C = C(N, \sigma) > 1 \) such that
\[
\int_{\mathbb{R}^N} \frac{dz}{|y - z|^{N-2m} (1 + |z|)^{2m+\sigma}} \leq \frac{C}{(1 + |y|)^\sigma}.
\]
Set
\[ \Omega_i := \{ y \in \mathbb{R}^N \text{ such that } |y - x^i| \leq |y - x^j|, \text{ for all } j \neq i \}. \]

**Lemma A.3.** For any \( \theta > k \), there exists a constant \( C \), such that
\[
\sum_j \frac{1}{(1 + \mu_j |y - x_j|)^\theta} \leq \frac{C}{(1 + \mu_i |y - x_i|)^\theta}, \quad y \in B_i := B_1(x_i). \] (A.1)
Proof. For \( y \in B_1(x_i) \), it holds \( |y - x_j| \geq \frac{1}{2}|x_i - x_j| \). As a result,

\[
\sum_{j \neq i} \frac{1}{(1 + \mu_j|y - x_j|)^\theta} \leq \sum_{j \neq i} \frac{C}{(\mu_j|x_i - x_j|)^\theta} \leq \frac{C}{(\mu_i L)^\theta} \leq \frac{C}{(1 + \mu_i|y - x_i|)^\theta}.
\]

\[\square\]

The following lemma is the main ingredient in the discussion of the existence and the local uniqueness of bubbling solutions blowing-up at \( k \)-dimensional lattice for \( k \geq 1 \).

**Lemma A.4.** Suppose \( N > 2m + 2 \), \( 1 \leq k < \frac{N-2m}{2} \) and denote \( \bar{\mu} = \min\{\mu_1, \cdots, \mu_n\} \).

Then there exists \( \bar{\theta} > 0 \) small, such that

\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2m}} W_n^{4m} (z) \sigma(z) \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j|z - x_j|)^{N-2m+\bar{\theta}}} dz \\
\leq C\sigma(y) \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j|y - x_j|)^{N-2m+\bar{\theta}+\bar{\theta}}} + C \frac{\mu_j^{N-2m}}{(1 + \mu_j|y - x_j|)^{N-2m+\bar{\theta}}}. \tag{A.2}
\]

Proof. By Lemma A.3, if \( z \in B_1(x_i) \), we have

\[
W_n^{4m} (z) \leq C \frac{\mu_i^{N-2m+2m}}{(1 + \mu_i|z - x_i|)^{N-2m+4m+\tau}}. \tag{A.2}
\]

By Lemma A.2, we have

\[
\int_{\Omega_i \cap B_1} \frac{1}{|y - z|^{N-2m}} W_n^{4m} (z) \sigma(z) \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j|z - x_j|)^{N-2m+\tau}} dz \\
\leq C \int_{\Omega_i \cap B_1} \frac{1}{|y - z|^{N-2m}} \frac{\mu_i^{N-2m+2m}}{(1 + \mu_i|z - x_i|)^{N-2m+4m+\tau}} \left(1 + \frac{\mu_i|z - x_i|}{\mu_i}^{\tau-1}\right) dz \\
\leq \frac{C \mu_i^{N-2m+1-\tau}}{(1 + \mu_i|y - x_i|)^{\min(\frac{N-2m}{2}+2m+1,N-2m)}} \\
\leq \frac{C \mu_i^{N-2m+1-\tau}}{(1 + \mu_i|y - x_i|)^{\frac{N-2m}{2}+\bar{\theta}+\bar{\theta}}}. \tag{A.3}
\]
Similarly, we have

\[
\int_{\Omega_i \cap B_i} \frac{1}{|y-z|^{N-2m}} W_n^{4m}(z) \sigma(z) \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j |z - x_j|)^{N-2m}} \, dz \\
\leq C \int_{\Omega_i \cap B_i} \frac{1}{|y-z|^{N-2m}} \frac{\mu_i^{N-2m+2m}}{(1 + \mu_i |z - x_i|)^{N-2m+2m}} \\
\leq \frac{C \mu_i^{N-2m}}{(1 + \mu_i |y - x_i|)^{N-2m+2m}} \tag{A.4}
\]

which, together with (A.3), gives

\[
\int_{\Omega_i \cap B_i} \frac{1}{|y-z|^{N-2m}} W_n^{4m}(z) \sigma(z) \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j |z - x_j|)^{N-2m}} \, dz \\
\leq \frac{C \sigma(y)}{(1 + |y - x_i|)^{N-2m+2m}} \tag{A.5}
\]

We have the following inequality:

\[
\left( \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j |y - x_j|)^{N-2m+2m}} \right)^{m^*-1} \\
\leq C \sum_j \frac{\mu_j^{N+2m}}{(1 + \mu_j |y - x_j|)^{N+2m+\tau}} \left( \sum_j \frac{1}{(1 + \mu_j |y - x_j|)^\tau} \right)^{4m/(N-2m)} \tag{A.6}
\]

If \( z \in \mathbb{R}^N \setminus \bigcup_j B_1(x_j) \), then

\[
W_n(z) = O \left( \frac{1}{\mu^0} \right) \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j |z - x_j|)^{N-2m+\tau}}.
\]

Thus, we have

\[
W_n^{4m} \sum_j \frac{\mu_j^{N-2m}}{(1 + \mu_j |z - x_j|)^{N-2m+\tau}} \leq C \left( \frac{\mu_j^{N-2m}}{(1 + \mu_j |z - x_j|)^{N-2m+\tau}} \right)^{m^*-1} \\
\leq \frac{C}{\mu^0} \sum_j \frac{\mu_j^{N+2m}}{(1 + \mu_j |z - x_j|)^{N+2m+\tau}}, \quad \forall z \in \mathbb{R}^N \setminus \bigcup_j B_1(x_j). \tag{A.7}
\]
Then, the result follows from Lemma A.2.

Set

$$W_{L,x,\mu} = \sum_{i=1}^{\infty} U_{x_i,\mu_i,L}.$$  \hfill (A.8)

Define

$$N(\omega_L) = K(y)\left((W_{L,x,\mu} + \omega_L)^{m^*-1} - W_{L,x,\mu}^{m^*-1} - (m^* - 1)W_{L,x,\mu}^{m^*-2}\omega_L\right),$$  \hfill (A.9)

and

$$l_L = K(y)W_{L,x,\mu}^{m^*-1} - \sum_{j=1}^{\infty} U_{x_{j,L},\mu_{j,L}}^{m^*-1}.$$  \hfill (A.10)

We now estimate $N(\omega_L)$ and $l_L$.

**Lemma A.5.** If $N > 2m + 2$, then

$$\|N(\omega_L)\|_{\gamma} \leq C\|\omega_L\|_{\gamma}^{\min(m^*-1,2)}.$$  

**Proof.** We have

$$|N(\omega_L)| \leq \begin{cases} C|\omega_L|^{m^*-1}, & N \geq 6m; \\ CW_{L,x,\mu}^{m^*-3}\omega_L^2 + C|\omega_L|^{m^*-1}, & N < 6m. \end{cases}$$

Since $\tau = \frac{N-2m}{2} - \vartheta$, $k < \frac{N-2m}{2}$ and $\vartheta > 0$ is small, we find that if $N < 6m$, then

$$W_{L,x,\mu}^{m^*-3} \leq C\left(\sum_{j}^{\mu_{j,L}} \frac{\mu_{j,L}}{2} \frac{\mu_{j,L} - 1}{2^{m^*-2}}\right)^{m^*-3}.$$  

As a result,

$$|N(\omega_L)| \leq C\sigma(z)\left(\sum_{j}^{\mu_{j,L}} \frac{\mu_{j,L}}{2} \frac{\mu_{j,L} - 1}{2^{m^*-2}}\right)^{m^*-1}\|\omega_L\|_{\gamma}^{\min(m^*-1,2)}.$$  

The result follows (A.6).

**Lemma A.6.** If $N > 2m + 2$, then

$$\|l_L\|_{\gamma} \leq C\left(\frac{1}{\mu_{L}^{\min(N-2m)\beta-\gamma+1}} + \max_{i} |x_{i,L} - P_{i}|^{\beta}\right),$$

where $\mu_{L} = \min_{j} \mu_{j,L}$.

**Proof.** We have

$$l_L = K(y)\left(W_{L,x,\mu}^{m^*-1} - \sum_{j} U_{x_{j,L},\mu_{j,L}}^{m^*-1}\right) + (K(y) - 1)\sum_{j} U_{x_{j,L},\mu_{j,L}}^{m^*-1} := J_1 + J_2.$$
Assume \( y \in \Omega_i \). Then,

\[
|J_1| \leq C \frac{\mu_{i,L}^{2m}}{(1 + \mu_{i,L}|y - x_{i,L}|)^{4m}} \sum_{j \neq i} \frac{\frac{N-2m}{\mu_{j,L}^2}}{(1 + \mu_{j,L}|y - x_{j,L}|)^{N-2m}} + C \left( \sum_{j \neq i} \frac{\frac{N-2m}{\mu_{j,L}^2}}{(1 + \mu_{j,L}|y - x_{j,L}|)^{N-2m}} \right)^{m^* - 1} := J_{11} + J_{12}.
\]

(A.11)

Hence, we obtain

\[
||J_{11}||_{**} \leq \frac{C}{(\mu_{L,L})^{\frac{N+2m}{2}} - \tau}.
\]

(A.14)

Similarly, we can also prove

\[
||J_{12}||_{**} \leq \frac{C}{(\mu_{L,L})^{\frac{N+2m}{2}} - \tau}.
\]

(A.15)

Thus, (A.14) and (A.15) yield

\[
||J_1||_{**} \leq \frac{C}{(\mu_{L,L})^{\frac{N+2m}{2}} - \tau}.
\]

(A.16)

Now, we estimate \( J_2 \). Similar to the proof of (A.16), we have

\[
\sum_{j \neq i} U_{x_{j,L},\mu_{j,L}}^{m^* - 1} \leq \frac{C \sigma(y)^{\frac{N+2m}{2}}}{(1 + \mu_{L,L}|y - x_{i,L}|)^{\frac{N+2m}{2} + \tau}} \frac{1}{(\mu_{L,L})^{\frac{N+2m}{2} - \tau}}, \quad y \in \Omega_i.
\]

(A.17)

We also have

\[
|K(y) - 1| U_{x_{i,L},\mu_{i,L}}^{m^* - 1} \leq \frac{C \mu_{i,L}^{\frac{N+2m}{2}}}{(1 + \mu_{L,L}|y - x_{i,L}|)^{\frac{N+2m}{2} + \tau}} \frac{1}{(\mu_{L,L})^{\frac{N+2m}{2} - \tau}}, \quad |y - x_{i,L}| \geq 1.
\]

(A.18)

If \( |y - x_{i,L}| \leq 1 \),

\[
|K(y) - 1| U_{x_{i,L},\mu_{i,L}}^{m^* - 1} \leq C|y - P_i|^\beta U_{x_{i,L},\mu_{i,L}}^{m^* - 1} \leq \frac{C \sigma(y)^{\frac{N+2m}{2}}}{(1 + \mu_{L,L}|y - x_{i,L}|)^{\frac{N+2m}{2} + \tau}} \frac{1}{(\mu_{L,L})^{\frac{N+2m}{2} - \tau}} \left( |x_{i,L} - P_i|^\beta + |y - x_{i,L}|^\beta \right),
\]

(A.19)
Noting that for any $\tau_1 \in \{1, \frac{N-2m}{2}\}$, we have

$$\frac{|y - x_{i,L}|^\beta}{(1 + \mu_{i,L}|y - x_{i,L}|)^{N+2m-\tau_1}} \leq \begin{cases} \frac{C}{\mu_{i,L}}, & \text{if } \beta = \frac{N+2m}{2} + \tau_1 \leq 0; \\ \frac{C}{N+2m-\tau_1}, & \text{if } \beta = \frac{N+2m}{2} + \tau_1 > 0. \end{cases}$$

Hence, we have proved

$$\|J_2\|_{**} \leq \frac{C}{\mu_L^{2m-\tau_1}} + C \max_i |x_{i,L} - P_i|^\beta. \quad (A.20)$$

\section*{Appendix B. Asymptotic energy expansion}

\textbf{Lemma B.1.} There are constants $C_1 \neq 0$ and $C_2 > 0$, such that

$$\int_{\mathbb{R}^N} K(y) U_i^{m-1} \frac{\partial U_i}{\partial x_{i,j}} = \frac{a_j C_1}{\mu_{i-1}^2} (P_{i,j} - x_{i,j}) + O\left(\frac{1}{\mu_{i}^{\beta-1+\theta}} + |x_i - P_i|^\beta\right), \quad (B.1)$$

$$\int_{\mathbb{R}^N} K(y) U_i^{m-1} \frac{\partial U_i}{\partial \mu_i} = -C_2 \sum_{j=1}^{N} a_j \frac{1}{\mu_{i}^{\beta+1}} + O\left(\frac{1}{\mu_{i}^{\beta+1+\theta}} + \frac{|x_i - P_i|^\beta}{\mu_{i}^2}\right). \quad (B.2)$$

\textbf{Proof.} We first prove (B.1). We have

$$\int_{\mathbb{R}^N} K(y) U_i^{m-1} \frac{\partial U_i}{\partial x_{i,j}} = \frac{1}{m^s} \int_{\mathbb{R}^N} \frac{\partial K(y)}{\partial y_j} U_i^{m}$$

$$= \frac{\beta}{m^s} \int_{\mathbb{R}^N} a_j \mu_i^{-1} y_j + P_{i,j} - x_{i,j} |\beta-2 (\mu_i^{-1} y_j + P_{i,j} - x_{i,j}) U_0^{m}$$

$$+ O\left(\frac{1}{\mu_{i}^{\beta-1+\theta}} + |x_i - P_i|^\beta\right)$$

$$= \frac{a_j C_1}{\mu_{i-1}^2} (P_{i,j} - x_{i,j}) + O\left(\frac{1}{\mu_{i}^{\beta-1+\theta}} + |x_i - P_i|^\beta\right).$$

Now we prove (B.2). Using

$$\int_{\mathbb{R}^N} U_i^{m-1} \frac{\partial U_i}{\partial \mu_i} = 0,$$

we find

$$\int_{\mathbb{R}^N} K(y) U_i^{m-1} \frac{\partial U_i}{\partial \mu_i} = \int_{\mathbb{R}^N} \sum_{j=1}^{N} a_j |y_j - P_{i,j}|^{\beta} U_0^{m-1} \frac{\partial U_i}{\partial \mu_i} + O\left(\frac{1}{\mu_{i}^{\beta+1+\theta}}\right)$$

$$= \frac{1}{m^s} \int_{\mathbb{R}^N} \frac{\partial}{\partial \mu_i} \sum_{j=1}^{N} a_j \mu_i^{-1} y_j + x_{i,j} - P_{i,j} |\beta U_0^{m} + O\left(\frac{1}{\mu_{i}^{\beta+1+\theta}}\right)$$

$$= -C_2 \sum_{j=1}^{N} a_j \frac{1}{\mu_{i}^{\beta+1}} + O\left(\frac{1}{\mu_{i}^{\beta+1+\theta}} + \frac{|x_i - P_i|^\beta}{\mu_{i}^2}\right).$$
We complete the proof of Lemma B.1.

**Lemma B.2.** Set \( W(x) = \sum_i U_i \). Here the sum can be from 1 to \( n \), or from 1 to infinity. Then for \( j = 1, \ldots, N \),

\[
\int_{\mathbb{R}^N} (-\Delta)^m W Z_{i,j} dy - \int_{\mathbb{R}^N} K(y) W^{m^*-1} Z_{i,j} dy
= \frac{a_j C_1}{\mu_i^{\beta - 2}} (P_{i,j} - x_i) + O\left( \frac{1}{\mu_i^{\beta - 1 + \theta}} + \frac{1}{\mu_i^{N - 2m + 1}} \right).
\]

**Proof.** We have

\[
\int_{\mathbb{R}^N} (-\Delta)^m W Z_{i,j} dy = \int_{\mathbb{R}^N} \sum_l U_l^{m^*-1} Z_{i,j} dy = O\left( \frac{1}{\mu^N L^{N + 2m + 1}} \right) \quad (B.3)
\]

On the other hand,

\[
\int_{\mathbb{R}^N} K(y) W^{m^*-1} Z_{i,j} dy - \int_{\mathbb{R}^N} K(y) U_i^{m^*-1} Z_{i,j} dy
= (m^* - 1) \int_{\mathbb{R}^N} K(y) U_i^{m^*-2} Z_{i,j} \sum_{l \neq i} U_l + O \left( \int_{\mathbb{R}^N} |Z_{i,j}| \left( \sum_{l \neq i} U_l \right)^{m^*-1} \right) \quad (B.4)
\]

\[= O\left( \frac{1}{\mu^{N - 2m L - N - 2m + 1}} \right). \]

Combining (B.3), (B.4) and Lemma B.1, we obtain the desired result in Lemma B.2.

**Lemma B.3.** There exists some constant \( C > 0 \) independent of \( i, j, n \), such that

\[
\int_{\mathbb{R}^N} (-\Delta)^m W Z_{i,N+1} dy - \int_{\mathbb{R}^N} K(y) W^{m^*-1} Z_{i,N+1} dy
= \frac{C_2}{\mu_i^{\beta + 1}} \frac{a_j}{\mu_i^{\beta + 1}} + \sum_{l \neq i} \frac{C_4}{\mu_i (\mu_i \mu_l)^{N - 2m}} |x_i - x_l|^{N - 2m}
+ O \left( \frac{1}{\mu_i^{\beta + 1 + \theta}} + \frac{1}{\mu_i^{N - 2m + 1 + \theta}} \right).
\]

**Proof.** Similar to (B.4), we can deduce

\[
\int_{\mathbb{R}^N} K(y) W^{m^*-1} Z_{i,N+1} dy - \int_{\mathbb{R}^N} K(y) U_i^{m^*-1} Z_{i,N+1} dy
= (m^* - 1) \int_{\mathbb{R}^N} K(y) U_i^{m^*-2} Z_{i,N+1} \sum_{l \neq i} U_l + O \left( \frac{1}{\mu^N L^{N + 2m + 1}} \right)
\]

\[= (m^* - 1) \int_{\mathbb{R}^N} U_i^{m^*-2} Z_{i,N+1} \sum_{l \neq i} U_l\]

\[+ O \left( \frac{1}{\mu_i^{\beta + 1 + \theta}} + \frac{1}{\mu_i^{N - 2m + 1 + \theta}} \right), \]

\]
which, together with Lemma B.1, gives the result. □

Appendix C. Estimate of the error term

Let \( u_L \) be a solution of (P) with the form

\[
W_{L,x,\mu} = \sum_{j=1}^{+\infty} U_{x_j,L,\mu_j,L},
\]

(C.1)
satisfying (1.5), (1.6) and (1.8). In this section, we will estimate the error term \( \omega_L \).

It is easy to see that \( \omega_L \) satisfies the following equation:

\[
(-\Delta)^m \omega_L - (m^*-1)K(y)W_{L,x,\mu}^{m^*-2}\omega_L = N(\omega_L) + l_L,
\]

where \( l_L \) and \( N(\omega_L) \) are defined in (A.9) and (A.10) respectively.

By assumption (1.5), we have \( \|\omega_L\|_* \to 0 \) as \( L \to +\infty \).

Note that in the decomposition (C.1), we do not assume that \( \omega_L \in H_n \). See (2.2) for the definition of \( H_n \). Let us point out that \( x_j,L \) and \( \tilde{\mu}_j,L \) may not be a maximum point and the maximum value of \( u_L \) in \( B_\delta(x_j,L) \), respectively. Let \( \bar{x}_j,L \) be such that \( u_L(\bar{x}_j,L) = \max_{B_\delta(x_j,L)} u_L \) := \( \tilde{C}_m\tilde{\mu}_j,L \). From (1.5), we can deduce that as \( L \to +\infty \),

\[
\bar{\mu}_j,L = \mu_j,L(1 + o_L(1)), \quad \mu_j,L(\bar{x}_j,L - x_j,L) = o_L(1).
\]

As a result, we have

\[
U_{x_j,L,\bar{\mu}_j,L} - U_{\bar{x}_j,L,\bar{\mu}_j,L} = O(1)U_{\bar{x}_j,L,\bar{\mu}_j,L} = o_L(1)U_{\bar{x}_j,L,\bar{\mu}_j,L},
\]

and

\[
|\omega_L(x)| = o_L(1) \sum_{j=-\infty}^{+\infty} \frac{\tilde{\mu}_j,L^{2m}}{(1 + \tilde{\mu}_j,L|y - \bar{x}_j,L|)^{N-2m + \tau}}.
\]

So, we find that in (C.1), \( x_j,L \) and \( \mu_j,L \) can be replaced by \( \bar{x}_j,L \) and \( \bar{\mu}_j,L \) respectively. For simplicity, in the following, we still use \( x_j,L \) and \( \mu_j,L \) to denote \( \bar{x}_j,L \) and \( \bar{\mu}_j,L \) respectively. Thus in (C.1), it holds

\[
|\omega_L(x_j,L)| = O\left(\frac{1}{\mu_j,L^{2m}}\right), \quad |\nabla \omega_L(x_j,L)| = O\left(\frac{1}{\mu_j,L^{2m}}\right).
\]

(C.3)

In the following, we will use (C.2) to estimate \( \omega_L \).

Proposition C.1. It holds

\[
\|\omega_L\|_* \leq \frac{C}{\min\left(\frac{N+2m}{2} - \tau, \beta - \tau + 1\right)} + C \max_i |x_{i,L} - P_i|^\beta.
\]
Proof. In the existence part, we already know that

\[
|\omega_L(y)| \left(\sigma(y) \sum_i \left(1 + \frac{\mu_i r^{N-2m}}{\mu_i L} \right)^{\frac{N-2m}{2} + \tau} \right)^{-1} \\
\leq C \left( \|\omega_L\|_{\ast}^{\min(m^* - 1, 2)} + \|l_L\|_{\ast\ast} \right) + C \|\omega_L\|_{\ast} \sum_i \frac{1}{(1 + \mu_i L |y - x_i|) \frac{N-2m}{2} + \tau},
\]

(C.4)

where \( \tilde{\theta} > 0 \) is a constant.

Suppose that there is \( y \in \mathbb{R}^N \setminus \cup_{j=1}^\infty B_{R \mu_j L} (x_j, L) \) for some large \( R > 0 \), such that \( \|\omega\|_{\ast} \) is achieved at \( y \). Then, (C.4) gives

\[
\|\omega_L\|_{\ast} \leq C \left( \|\omega_L\|_{\ast}^{\min(m^* - 1, 2)} + \|l_L\|_{\ast\ast} \right) + o_R(1) \|\omega_L\|_{\ast}.
\]

(C.5)

Since \( \|\omega_L\|_{\ast} \to 0 \) as \( L \to +\infty \), we obtain

\[
\|\omega_L\|_{\ast} \leq C \|l_L\|_{\ast\ast}.
\]

(C.6)

Suppose that \( \|\omega\|_{\ast} \) is achieved at \( y \in B_{R \mu_j L} (x_j, L) \). Let

\[
\tilde{\omega}_L(y) = \frac{1}{\mu_j L} \omega_L(y, j, L),
\]

and

\[
\|\tilde{\omega}_L\|_{\ast} = \frac{\sigma(\mu_j^{-1} y + x_j, L)}{\sum_i \left(1 + \frac{\mu_i r^{N-2m}}{\mu_i L} \right)^{\frac{N-2m}{2} + \tau} \mu_j L} \mu_j L \omega_L(y, j, L) \left(\frac{N-2m}{2} + \tau\right)^{-1} \left|\tilde{\omega}_L(y)\right|.
\]

Then \( \|\tilde{\omega}_L\|_{\ast} \) is achieved at some \( y \in B_R(0) \).

Suppose that

\[
\|\omega_L\|_{\ast} \geq N_L \left( \frac{1}{\min \{ \frac{\mu_i r^{N-2m}}{\mu_i L} + 1 \}} \mu_j L \frac{N-2m}{2} + \tau \sum_i \frac{N-2m}{2} \right) + C \max \left| x_{i, L} - P_i \right|^\beta,
\]

for some \( N_L \to \infty \). Then as \( L \to +\infty \), \( \eta_L = \frac{\tilde{\omega}_L}{\|\tilde{\omega}_L\|_{\ast}} \) converges to \( \eta \neq 0 \), which satisfies

\[
(\Delta)^m \eta - (m^* - 1) U_{0,1}^{m^* - 2} \eta = 0, \quad \text{in} \ \mathbb{R}^N
\]

since \( \|l_L\|_{\ast\ast} \leq C \frac{\eta L}{N_L} \to 0 \). This gives

\[
\eta = \alpha_0 \left. \frac{\partial U_{0,\mu}}{\partial \mu} \right|_{\mu=1} + \sum_{j=1}^N \alpha_j \left. \frac{\partial U_{0,1}}{\partial x_j} \right|,
\]

for some constant \( \alpha_j \).

On the other hand, we have

\[
\tilde{\omega}_L(0) = \frac{N-2m}{2} \omega_L(x_j, L) = O \left( \frac{1}{\mu_j L^{N-2m} L^N} \right).
\]
\[ \nabla \omega_L(0) = \mu_j \frac{N - 2m + 2}{\mu L} \nabla \omega_L(x_j, L) = O\left(\frac{1}{\mu L^{N - 2m + 1}}\right). \]

So, we find \( \eta(0) = 0 \) and \( \nabla \eta(0) = 0 \), which implies \( \alpha_0 = \alpha_1 = \cdots = \alpha_N = 0 \). This is a contradiction.

\[ \square \]

**Corollary C.2.** For any \( \delta > 0 \), we have
\[
\| \omega_L(x) \|_{C^{2m-1}(B_\delta(x_j, L) \setminus B_{\frac{1}{4}\delta}(x_j, L))} \leq \frac{C}{\mu L} \left( \frac{1}{\mu L} \right)^{N - 2m + \tau} \]
\[
\leq \frac{C}{\mu L} \left( \min \left( \frac{N - 2m}{2}, \frac{\tau - \beta + 1}{\tau - \beta} + 1 \right) \right) + \max_i \| x_{i,L} - P_i \|^\beta, \quad x \in B_{\delta}(x_j, L) \setminus B_{\frac{1}{4}\delta}(x_j, L).
\]

On the other hand, from [C.2], using the \( L^p \) estimates, we can deduce that for any \( p > 1 \),
\[
\| \omega_L \|_{W^{2m,p}(B_\delta(x_j, L) \setminus B_{\frac{1}{4}\delta}(x_j, L))} \leq C \| \omega_L \|_{L^\infty(B_\delta(x_j, L) \setminus B_{\frac{1}{4}\delta}(x_j, L))}
\]
\[
+ C \| (m^* - 1)K(y)W_{L,x,\mu}^{m-2}\omega_L + N(\omega_L) + l_L \|_{L^\infty(B_{\delta}(x_j, L) \setminus B_{\frac{1}{4}\delta}(x_j, L))} \leq C \| \omega_L \|_{L^\infty(B_{\delta}(x_j, L) \setminus B_{\frac{1}{4}\delta}(x_j, L))} + C \left( \| N(\omega_L) \|_{L^\infty} + \| l_L \|_{L^\infty} \right) \frac{1}{\mu^\tau}.
\]

The result follows from Lemmas A.5 and A.6.

\[ \square \]

**Appendix D. Some basic lemmas**

For any integer \( n \geq 2 \), consider the following equations:

\[
|a_j|^\beta - a_j - \sum_{i=1}^n d_{ij}a_i = 0, \quad j = 1, \ldots, n,
\]

where \( \beta > 1 \) is a constant, \( d_{ij} \) satisfies \( d_{ii} = 0, \ i = 1, \ldots, k, \ d_{ij} > 0, \ d_{ij} = d_{ji}, \ i \neq j \), and \( c_1 \geq \max_{i,j} \sum_{i=1}^n d_{ij} \geq \min_{i,j} \sum_{i=1}^n d_{ij} \geq c_0 > 0 \). It is easy to see that if \( a = (a_1, \ldots, a_k) \) satisfies [D.1], then \( a \) is a critical point of the function defined as

\[ F(x) = \frac{1}{\beta + 1} \sum_{j=1}^n |x_j|^{\beta + 1} - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n d_{ij}x_i x_j. \]

It is easy to show that \( \min_{x \in \mathbb{R}^n} F(x) < 0 \) is achieved at \( a \neq 0 \). Moreover, from \( d_{ij} \geq 0 \), we can assume \( a_j \geq 0, \ j = 1, \ldots, n \). Using [D.1], if \( a_j = 0 \) for some \( j \), then \( a_i = 0 \) for all
Let $i \neq j$. Moreover, from (D.1), we find

$$
(\max_j a_j)^{\beta-1} \leq \max_j \sum_{i=1}^n d_{ji}, \quad (\min_j a_j)^{\beta-1} \geq \min_j \sum_{i=1}^n d_{ji} \geq c_0 > 0. \quad (D.3)
$$

We now prove the following result.

**Lemma D.1.** The solution of (D.1) is unique if $a_j > 0, j = 1, \cdots, n$. Moreover, if we define the linear operator $A$ as follows:

$$
(AX)_j = \beta a_j^{\beta-1} x_j - \sum_{i=1}^n d_{ji} x_i, \quad j = 1, \cdots, n. \quad (D.4)
$$

Then $\|AX\| \geq c'\|X\|$ for some $c' > 0$, where the norm for $X$ is defined as $\|X\| = \max_j |x_j|$.

**Proof.** This lemma was proved in [14]. For the readers’ convenience, we give its proof here.

Suppose that (D.1) has two solutions $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, $a_j > 0$, $b_j > 0, j = 1, \cdots, n$. Let $T = \max_j \frac{a_j - b_j}{a_j}$. We have two possibilities. (i) $T \leq 0$, (ii) $T > 0$.

If $T \leq 0$, then $a_j \leq b_j$ for all $j = 1, \cdots, n$. So we can define $T_1 = \sup_j \frac{t_j}{a_j}, t_j = b_j - a_j \geq 0$. Then $b_j^\beta \geq a_j^\beta + \beta a_j^{\beta-1} (b_j - a_j)$. So

$$
\beta a_j^{\beta-1} (b_j - a_j) \leq \sum_{i=1}^n d_{ji} (b_i - a_i) \leq T_1 \sum_{i=1}^n d_{ji} a_i = T_1 a_j^\beta,
$$

which implies $\beta T_1 \leq T_1$. So $T_1 = 0$.

If $T > 0$, then there is a $j$, such that $\frac{a_j - b_j}{a_j} = T > 0$. As a result,

$$
\beta a_j^{\beta-1} (a_j - b_j) \leq \sum_{i=1}^n d_{ji} (a_i - b_i) \leq T \sum_{i=1}^n d_{ji} a_i = T a_j^\beta,
$$

which implies $\beta T \leq T$. So $T = 0$. This is a contradiction.

To prove the last part, for any $X$ with $\|X\| = 1$, we let $x = \sup_j \frac{|x_j|}{a_j}$. Then

$$
|\sum_{i=1}^n d_{ji} x_i| \leq x \sum_{i=1}^n d_{ji} a_i = xa_j^\beta.
$$

As a result,

$$
|(AX)_j| \geq \beta a_j^{\beta-1} |x_j| - xa_j^\beta = a_j^\beta (\beta \frac{|x_j|}{a_j} - x).
$$

Since $\beta > 1$, we can choose $j$, such that

$$
|(AX)_j| \geq a_j^\beta (\beta \frac{|x_j|}{a_j} - x) \geq c' > 0.
$$

□
Now we consider
\[ a_j^\beta - \sum_{i=1}^n d_{ji}a_i = 0, \quad j = 1, \ldots, \]  
(D.5)

Using Lemma D.1 and (D.5), we can easily prove the following result.

**Lemma D.2.** Equation (D.5) has a unique solution \( a_j > 0, \quad j = 1, \ldots, \). Moreover, if we define the linear operator \( A \) as follows:

\[
(AX)_j = \beta a_j^{\beta - 1} x_j - \sum_{i=1}^\infty d_{ji}x_i, \quad j = 1, \ldots, \]

(D.6)

Then \( \|AX\| \geq c'\|X\| \) for some \( c' > 0 \), where the norm for \( X \) is defined as \( \|X\| = \max_j |x_j| \).

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