Proton Electromagnetic Form Factor Ratios at Low $Q^2$

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We study the ratio $R \equiv \mu G_E(Q^2)/G_M(Q^2)$ of the proton at very small values of $Q^2$. Radii commonly associated with these form factors are not moments of charge or magnetization densities. We show that the form factor $F_2$ is correctly interpretable as the two-dimensional Fourier transformation of a magnetization density. A relationship between the measurable ratio and moments of true charge and magnetization densities is derived. We find that existing measurements show that the magnetization density extends further than the charge density, in contrast with expectations based on the measured reduction of $R$ as $Q^2$ increases.

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Electromagnetic form factors of the proton and neutron (nucleon) are probability amplitudes that the nucleon can absorb a given amount of momentum and remain in the ground state, and therefore should determine the nucleon charge and magnetization densities. Much experimental technique, effort and ingenuity has been used recently to measure these quantities [1, 2].

The text-book interpretation of electromagnetic form factors, $G_E, G_M$, explained in [3], is that their Fourier transforms are measurements of charge and magnetization densities, and conventional wisdom relates the charge and magnetization mean square radii to the slopes of $G_{E,M}$ at $Q^2 = 0$. However, this interpretation is not correct because the wave functions of the initial and final nucleons have different momenta and therefore differ, invalidating a probability or density interpretation. A proper charge density is related to the matrix element of an absolute square of a field operator.

Here we show that the proper magnetization density is the two-dimensional Fourier transform of the $F_2$ form factor. We use this and the result that the charge density is the two-dimensional Fourier transform of the $F_1$ form factor [3, 4, 5, 6] to show that the magnetization density of the proton extends significantly further than its charge density. This is surprising because the observed rapid decrease of the ratio of electric to magnetic form factors with increasing values of $Q^2$ [1, 2] might lead one to conclude that the charge radius is larger than the magnetization radius.

Form factors are matrix elements of the electromagnetic current operator $J^\mu(x')$ in units of the proton charge:

$$\langle p', \lambda' | J^\mu(0)| p, \lambda \rangle = \bar{u}(p', \lambda') \left( \gamma^\mu F_1(Q^2) + \frac{\sigma^{\mu\alpha}}{2M} q_\alpha F_2(Q^2) \right) u(p, \lambda),$$

where the momentum transfer $q_\alpha = p'_\alpha - p_\alpha$ is taken as space-like, so that $Q^2 \equiv -q^2 > 0$. The nucleon polarization states are those of definite light-cone helicities $\lambda, \lambda' [7]$. The normalization is such that $F_1(0)$ is the nucleon charge, and $F_2(0)$ is the proton anomalous magnetic moment $\kappa = 1.79$. The Sachs form factors are $G_E(Q^2) \equiv F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2), \ G_M(Q^2) \equiv F_1(Q^2) + F_2(Q^2)$. Any probability or density inter-
preparation of $G_E$ is spoiled by a non-zero value of $Q^2$, no matter how small. Nevertheless, $G_E, G_M$ are experimentally accessible so we define effective (*) square radii $R_E^2, R_M^2$ such that for small values of $Q^2$: 
$G_E(Q^2) \approx 1 - \frac{Q^2}{6} R_E^2 \quad G_M(Q^2) \approx \mu (1 - \frac{Q^2}{6} R_M^2), \quad$ where for the proton $\mu = 2.79$. Thus the accurately measurable [8] ratio

$$
\mu G_E(Q^2)/G_M(Q^2) \approx 1 + \frac{Q^2}{6} (R_E^2 - R_M^2). \quad (2)
$$

The form factor $F_1$ is a two-dimensional Fourier transform of the true charge density $\rho(b)$, where $b$ is the distance from the transverse center of mass position irrespective of the longitudinal momentum [3]-[6], with

$$
F_1(Q^2 = q^2) = \int d^2 b \rho(b) e^{-i q \cdot b}. \quad (3)
$$

At small values of $q^2$,

$$
F_1(Q^2 = q^2) \approx 1 - \frac{Q^2}{4} \langle b^2 \rangle_{Ch} \quad (4)
$$

where $\langle b^2 \rangle_{Ch}$ is the second moment of $\rho(b)$.

We now derive a similar interpretation for $F_2$ in terms of a magnetization density, starting with the relation that $\mu^2 \cdot B$ is the matrix element of $\mathbf{J} \cdot \mathbf{A}$ in a definite state, $|X\rangle$. Take the rest-frame magnetic field to be a constant vector in the 1 (or $b_x$) direction, and the corresponding vector potential as $\mathbf{A} = 2 b_b \mathbf{\hat{z}}$. Then consider the system in a frame in which the plus component of the momentum approaches infinity. The anomalous magnetic moment may be extracted by taking:

$$
|X\rangle \equiv \frac{1}{\sqrt{2}} \left( |p^+, R = 0, +\rangle + |p^+, R = 0, -\rangle \right), \quad (5)
$$

where $|p^+, R = 0, +\rangle$ represents a transversely localized state of definite $P^+$ and light-cone helicity. The state $|X\rangle$ [4, 10] may be interpreted as that of a transversely polarized target, up to relativistic corrections caused by the transverse localization of the wave packet [11]. The anomalous magnetic moment $\mu_a$ [12] is then given by

$$
\mu_a = \frac{\langle X | \int dx^- d^2 b_y \bar{q}(x^-, b) \gamma^+ q(x^-, b) | X\rangle}{\langle X | X\rangle}. \quad (6)
$$

Use translational invariance to obtain:

$$
\mu_a = \frac{\langle X | \int d^2 b_y q_+(0, b) q_-(0, b) | X\rangle}{\langle X | X\rangle}, \quad (7)
$$

where $q_-(x^-, b)$ is a quark-field operator, and $q_+ = \gamma^0 \gamma^+ q$. This matrix element of a quark density operator is closely related to the Burkardt’s [4, 10] impact parameter GPD $q_X(x = 0)$:

$$
q_X(x, b) \equiv \langle X | \int dx^- \frac{1}{4\pi} q_+^0 (0, b) q_-(x^-, b) e^{i p^+ x^-} | X\rangle = \frac{1}{2 p^+} \langle \mathcal{H}_q (x, \xi = 0, b) - \frac{1}{2 M} \frac{\partial}{\partial b_y} E_q (x, \xi = 0, b) \rangle, \quad (8)
$$

where $\mathcal{H}_q$ and $E_q$ are two-dimensional Fourier transforms of the GPDs $H_q, E_q$ [13]. Integration of Eq. (8) over $x$ sets $x^-$ to zero, so that Eq. (7) can be re-expressed (after integration by parts) as

$$
\mu_a = \frac{1}{2 p^+} \int d^2 b \int dx E_q(x, \xi = 0, b). \quad (9)
$$

But the integral of $E_q$ over $x$ is just the two-dimensional Fourier transform of $2p^+ F_2$, so that

$$
\mu_a = \int d^2 b \rho_M(b), \quad (10)
$$

where

$$
\rho_M(b) = \int \frac{d^2 q}{(2\pi)^2} F_2(t = -q^2) e^{i q \cdot b} \quad (11)
$$

The subscript $M$ denotes that this density generates the anomalous magnetic moment, is properly a true magnetization density, and is distinct from $\rho(b)$. It is also possible to consider the quantity $-\int dx b_y \frac{\partial}{\partial b_y} E_q(x, 0, b)$ as the magnetization density. However, this definition would depend on the choice of the $x$ axis as the direction of the magnetic field. A true intrinsic quantity should not depend on such a choice, so we use the form of Eqs. (9, 10).

For small values of $q^2$:

$$
F_2(Q^2 = q^2) \approx \kappa \left( 1 - \frac{Q^2}{4} \langle b^2 \rangle_M \right), \quad (12)
$$

where $\langle b^2 \rangle_M$ is the second moment of $\rho_M(b)$. Use the definitions of the Sachs form factors, Eq. (2) and the expansions Eqs. (11, 12) to relate the true moments to the effective square radii so that

$$
\langle b^2 \rangle_M - \langle b^2 \rangle_{Ch} = \frac{\mu^2}{\kappa 3} (R_E^2 - R_M^2) + \frac{\mu}{M^2} \quad (13)
$$

The low $Q^2$ measurement of the form factor ratio determines also the difference of true moments $\langle b^2 \rangle_M - \langle b^2 \rangle_M$. The result is
We fix the value of $R_M^*$ from a new state-of-art determination \cite{22}, $R_M^* = 0.778(29)$ fm and plot the data as a function of the parameter $\frac{Q^2}{6} R_M^*$. We find

$$\langle b^2 \rangle_M - \langle b^2 \rangle_{ch} = 0.10960 \pm 0.00678 \text{ fm}^2, \quad (14)$$

which is about 12% smaller than the contribution of the magnetic moment Foldy term. Thus the difference $\frac{2}{3} \kappa (R_M^* - R_E^*) = -0.0139 \pm 0.00678 \text{ fm}^2$ presently has the opposite sign of the result for the difference of the true moments of the distribution, indicating the need to base interpretations on the true moments. Note also that $(R_M^* - R_E^*)$ is determined to an accuracy of only about 50%. Improving the accuracy can only be achieved by using very small values of $Q^2$, for which no high precision polarization transfer results exist. Two-photon effect corrected cross section measurements of the proton form factor ratio at very low $Q^2$, however, are consistent with the ratio $R = 1$ \cite{26}, corresponding to $R_{M^*} = R_{E^*}$, in rough agreement with our results.

Figure 2 shows several model calculations and fits for the form factor ratio \cite{21} \cite{24} \cite{28} \cite{29} \cite{30} \cite{31} \cite{32} \cite{33}, which are seen to vary greatly. Improved experiments \cite{8} would be able to distinguish these diverse approaches, and more fundamentally, better determine the value of $(R_M^* - R_E^*)$. We also use a linear fit, at small values of $Q^2$, to the results of various calculations and some global fits. These are shown in Fig. 3. While there is significant variation, all agree with our result Eq. (14).

Our result that the magnetization density extends further than the charge density is consistent with the
failure of the spin of the quarks to account for the angular momentum of the proton [34], and the likelihood importance of quark orbital angular momentum (OAM). This is because quarks carrying OAM, and therefore much magnetization-generating current, are located away from the center. For example, consider the pion cloud, which dominates the proton’s exterior, as a source of OAM. The pion cloud is more influential for magnetic properties than for electric ones (e.g. Refs. [35, 36]), and causes a proton magnetization radius that is larger than the charge radius.

To reiterate: our model independent result is that the magnetization density of the proton extends further than its charge density. A natural interpretation involves the orbital angular momentum carried by quarks. Future experimental measurements of the ratio of the proton’s electromagnetic form factors would render the present results more precise.

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