Growth of Sobolev norms for unbounded perturbations of the Laplacian on flat tori

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Abstract

We prove a $\langle t \rangle^\varepsilon$ bound on the growth of Sobolev norms for unbounded time dependent perturbations of the Laplacian on flat tori.

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1 Introduction

We consider the Schrödinger equation

$$i \partial_t \psi = H(t) \psi, \quad H(t) = -\Delta + V(t) \quad \psi \in L^2(T^d),$$

where $V(t)$ is a smooth family of time dependent self-adjoint unbounded pseudo-differential operators of order $m < 2$ and $T^d = \mathbb{R}^d / \Gamma$ is a torus with arbitrary periodicity lattice $\Gamma$. We prove a $\langle t \rangle^\varepsilon$ bound on the growth of the Sobolev norms of the solution. As far as we know this is the first result of this kind for unbounded perturbations of the Laplacian on a manifold different from a Zoll manifold.

We recall that previous results ensuring $\langle t \rangle^\varepsilon$ growth of Sobolev norms for time dependent Schrödinger equations on tori were proved in [Bon99a, Del10, BM19] for bounded perturbations of the Laplacian and they were based on the use of a lemma by Bourgain on the structure of the “resonant clusters” on a suitable lattice.

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On the contrary, our proof follows the general strategy of [BGMR20] and consists essentially of three steps. The first step consists in using a normal form approach to conjugate (1.1) to an equation which is a smoothing perturbation of a normal form equation; the second step consists in proving that, in the dynamics of the normal form equation, the Sobolev norms of the solutions are bounded; finally the third step consists in adding the remainder and using Duhamel formula and a standard interpolation result to get the \( \langle t \rangle^\varepsilon \) bound. Our main improvement with respect to [BGMR20] consists of the normal form theorem and of the analysis of the normal form dynamics, which are here obtained by extending to the time dependent case the theory of [BLM20a, BLM20b] (see also [PS10, PS12]).

To compare in more precise (and technical) way the present method to the one of [BGMR20], we start by recalling that the normal form theorems of [BGMR20] (see also [Mon17]) deal with perturbations of pseudo-differential operator with a trivial geography of the resonances. Precisely, in [BGMR20], the considered unperturbed operators are the quantization of a classical integrable system which is either one dimensional or isochronous\(^1\). As a consequence, for the systems considered in [BGMR20] and [Mon17], it is possible to develop a semiclassical normal form theory \textit{global} in the phase space, namely to conjugate the system to a new block diagonal operator which is just the average of the perturbation over the flow of the unperturbed operator.

On the contrary, for the Laplacian on the torus or on more general manifolds, the resonance relations fulfilled by the frequencies depend on the point of the phase space, therefore also the kind of normal form that one can construct has a different structure in different regions of the phase space, as in the proof of the classical Nekhoroshev’s theorem [Nek77, Nek79, Gio03]. So, in order to deal with a perturbation of the Laplacian on \( \mathbb{T}^d \), one has to develop a semiclassical averaging method adapted to the geometry of the resonances. Such a method was already developed in the time independent case in [PS10, PS12, BLM20a, BLM20b]. In these papers, using suitable cutoff functions, a time independent Schrödinger operator was conjugated to a normal form operator adapted to the different resonance relations fulfilled in the different regions of the phase space. Furthermore, in these papers the normal form operator was studied and it was shown to admit a block diagonal structure with blocks that were constructed in a quite explicit way. Here we just adapt to the time dependent case the results of [PS10, PS12, BLM20a, BLM20b]. We also extract from the proofs in [BLM20b] some more information which lead in a quite simple way to the proof of the main result of the present paper. We recall that the construction

\(^1\)namely the frequencies of its motions are independent of the point of the phase space
of \([\text{PS10, PS12, BLM20a, BLM20b}]\) is actually a quantization of the proof of Nekhoroshev theorem (both analytic and geometric part \([\text{BL21}]\)).

We conclude this introduction by recalling that the problem of understanding the flow of energy to high frequency modes in time dependent linear equations has a long history and has been investigated mainly in the context of time periodic or quasiperiodic perturbations of harmonic or anharmonic quantum oscillators. We first mention the works \([\text{How89, How92, Joy94, Nen97, BJ98, Joy96, DLS08}]\) in which perturbations of 1-d oscillators were studied and estimates on the transfer of energy between low and high frequency modes were given. Then, in the works \([\text{BG01, Bam17, Bam18, BM18}]\) the related problem of reducibility was studied in the 1-d case with unbounded perturbations. For the higher dimensional case only a few reducibility results are known. The first breakthrough result have been proved by Eliasson and Kuksin \([\text{EK10}]\), in the case of the Schrödinger equation on \(\mathbb{T}^d\) with a bounded analytic time-quasi periodic potential. In the case of unbounded potentials, we mention \([\text{BGMR18, BLM19, Mon19, FGN19}]\) which are based on an approach similar to that of \([\text{BGMR20}]\) and have in common the fact that the geography of the resonances is trivial in all the models considered.

We also recall that in some cases logarithmic bounds on the growth of Sobolev norms have been obtained \([\text{Bon99b, Wan08, FZ12}]\). We think that this kind of results can be obtained by developing the methods of the present paper and adding quantitative estimates for the analytic or Gevrey case.

Acknowledgments. We warmly thank Alberto Maspero who convinced us to try to apply the methods of \([\text{BLM20b}]\) to the problem of growth of Sobolev norms in an unbounded context: his encouragement was what convinced us to tackle this problem and finally led to this paper.

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2 Main result

Let \(e_1, e_2, \ldots, e_d\) be a basis of \(\Gamma\), namely

\[
\Gamma := \left\{ \sum_{i=1}^{d} k_i e_i : k_1, \ldots, k_d \in \mathbb{Z} \right\}.
\]  

(2.1)
By introducing in $\mathbb{R}^d$ the basis of the vectors $e_i$, one is reduced to an equation of the form
\begin{align*}
    i\partial_t \psi &= \tilde{H}(t) \psi = (-\Delta_g + \tilde{V}(t)) \psi, \quad \text{(2.2)} \\
    -\Delta_g &= - \sum_{A,B=1}^d g^{AB} \partial_A \partial_B, \quad \text{(2.3)}
\end{align*}
with p.b.c. on the standard torus $T^d := \mathbb{R}^d/(2\pi \mathbb{Z})^d$. Here $\{g^{AB}\}_{A,B=1}^d$ is the inverse of the matrix with elements
\begin{equation}
    g_{AB} := e_A \cdot e_B, \quad \text{(2.4)}
\end{equation}
namely it is defined by
\begin{equation}
    \sum_{B=1}^d g^{AB} g^{BC} = \delta_A^C. \quad \text{(2.6)}
\end{equation}

From now we will restrict our study to the equation (2.2) on $L^2(T^d)$, and in the following we will omit the checks. Thus we will consider a Schrödinger equation of the form
\begin{equation}
    i\partial_t \psi = H(t) \psi = (-\Delta_g + V(t)) \psi, \quad \psi \in L^2(T^d). \quad \text{(2.5)}
\end{equation}

In the following we will only deal with scalar products and norms with respect to the metric $g$. We will denote by
\begin{equation}
    (\xi; \eta) := \sum_{A,B=1}^d g^{AB} \xi_A \eta_B, \quad \|\xi\|^2 := (\xi; \xi) \quad \text{(2.6)}
\end{equation}
the scalar product and the norm of covectors with respect to this metric.

Finally, let $d\mu(x)$ be the volume form corresponding to $g$: for any $\psi \in L^2(T^d)$, we define its Fourier coefficients by
\begin{equation}
    \hat{\psi}_k = \frac{1}{\mu(T^d)} \int_{T^d} \psi(x) e^{-ik \cdot x} \, d\mu_g(x), \quad \forall k \in \mathbb{Z}^d,
\end{equation}
where $k \cdot x = \sum_{A=1}^d k_A x^A$ is the usual pairing between a covector $k$ and a vector $x$.

We now precisely define the class of symbols and of pseudo-differential operators we will use:

**Definition 2.1.** Fix $0 < \delta < 1$, and let $g$ be a flat Riemannian metric. Given $m \in \mathbb{R}$, we say that $v \in C^\infty(T^d \times \mathbb{R}^d; \mathbb{C})$ is a symbol of order $m$,
and we write \( v \in S^m \), if for any \( N_1, N_2 \in \mathbb{N} \) there exists a positive constant \( C_{N_1, N_2} = C_{N_1, N_2}(v) \) such that

\[
\sup_{x \in \mathbb{T}^d} \left| \partial_x^{N_1} \partial_\xi^{N_2} v(x, \xi) \right| \leq C_{N_1, N_2} \langle \xi \rangle^{m - \delta N_2} \quad \forall \xi \in \mathbb{R}^d,
\]

where \( \langle \xi \rangle := (1 + \| \xi \|^2)^{\frac{1}{2}} \).

**Definition 2.2.** Given \( m \in \mathbb{R} \) and a symbol \( v \in S^m \), we define the corresponding Weyl operator by

\[
V \psi(x) = \sum_{\xi \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \hat{v}_k \left( \xi + \frac{k}{2} \right) \hat{\psi}_\xi e^{i(\xi + k) \cdot x} \quad \forall \psi \in L^2(\mathbb{T}^d). \tag{2.8}
\]

A linear operator \( V \) is said to be a pseudo-differential operator of order \( m \) if there exists \( v \in S^m \) such that (2.8) holds; in such a case we write \( V = \text{Op}^W(v) \in OPS^m \).

**Remark 2.3.** For any \( m \in \mathbb{R} \), \( OPS^m \) endowed with the sequence of seminorms \( \{ | \cdot |_{N_1, N_2} \}_{N_1, N_2} \) given by \( |\text{Op}^W(v)|_{N_1, N_2} = C_{N_1, N_2}(v) \forall N_1, N_2 \in \mathbb{N} \), with \( C_{N_1, N_2}(v) \) defined as in (2.7), is a Fréchet space. Given \( V = \text{Op}^W(v) \in OPS^m \), we refer to \( \{ C_{N_1, N_2}(v) \}_{N_1, N_2} \) as the sequence of seminorms of \( V \).

**Remark 2.4.** It is well known that a pseudo-differential operator \( A = \text{Op}^W(a) \) is self-adjoint on \( L^2(\mathbb{T}^d) \) if and only if the corresponding symbol \( a \) is real valued.

For any \( \sigma \geq 0 \), we define the Sobolev space \( H^\sigma = H^\sigma(\mathbb{T}^d) \) as the completion of \( C^\infty(\mathbb{T}^d) \) with respect to the norm

\[
\| \psi \|_{2}^\sigma = \sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2\sigma} |\hat{\psi}_\xi|^2. \tag{2.9}
\]

Furthermore, for any \( \sigma_1, \sigma_2 \geq 0 \), we define \( \mathcal{B}(H^{\sigma_1}; H^{\sigma_2}) \) as the space of bounded linear operators from \( H^{\sigma_1} \) to \( H^{\sigma_2} \), and for any \( A \in \mathcal{B}(H^{\sigma_1}; H^{\sigma_2}) \) we define the standard operator norm as

\[
\| A \|_{\sigma_1, \sigma_2} = \sup_{u \in H^{\sigma_1}, \| u \|_{\sigma_1} = 1} \| Au \|_{\sigma_2}.
\]

If \( \sigma_1 = \sigma_2 \), we simply write \( \mathcal{B}(H^{\sigma_1}) \) instead of \( \mathcal{B}(H^{\sigma_1}; H^{\sigma_2}) \). We will consider the following classes of time dependent operators:
Definition 2.5. Let \( \{V(t)\}_{t \in \mathbb{R}} \) be a family of pseudo-differential operators of order \( m \). If \( V = Op^W(v) \in C^\infty(\mathbb{R}; OPS^m) \) and for any \( K, N_1, N_2 \in \mathbb{N} \) one has
\[
\sup_{t \in \mathbb{R}} |\partial^K V(t)|_{N_1, N_2} < \infty,
\]
we say \( V \in C^\infty_b(\mathbb{R}; OPS^m) \), or equivalently \( v \in C^\infty_b(\mathbb{R}; S^m) \).

With the above definitions, we recall that the following well known result holds:

Theorem 2.6 (Calderon-Vaillancourt). Let \( m \in \mathbb{R} \) and \( A \in C^\infty_b(\mathbb{R}; OPS^m) \). Then for any \( \sigma \in \mathbb{R} \) there exists a positive constant \( C \), depending only on \( \sigma \), on \( m \), on the metric \( g \) and on the sequence of seminorms of \( A \), such that
\[
\sup_{t \in \mathbb{R}} \|A(t)\|_{\sigma, \sigma-m} \leq C.
\]

Other than family of operators with a smooth dependence on time, we need to consider also operators which are only uniformly bounded in time. More precisely, we give the following definition.

Definition 2.7. Let \( \sigma_1, \sigma_2 \geq 0 \) and consider a \( t \)-dependent family of linear operators \( \{V(t)\}_{t \in \mathbb{R}} \subset B(H^{\sigma_1}; H^{\sigma_2}) \). We say that \( V \in L^\infty(\mathbb{R}; B(H^{\sigma_1}; H^{\sigma_2})) \) if
\[
\sup_{t \in \mathbb{R}} \|V(t)\|_{\sigma_1, \sigma_2} < +\infty.
\]

Remark 2.8. The reason why we consider such a weaker regularity with respect to time variable is that, given a one parameter family of self-adjoint operators \( G(t) \) with the smooth time regularity required in Definition 2.5, we will have to consider the family of unitary operators \( e^{\pm G(t)} \). These operators are bounded, but not differentiable, in time, as operators in \( B(H^\sigma) \). (See Lemma 3.4 below)

Given a time dependent family of self-adjoint operators \( A(t) \), consider the initial value problem
\[
i\partial_t \psi(t) = A(t) \psi(t), \quad \psi(s) = \psi.
\]

When the solution \( \psi(t) \) exists globally in time, for any \( t, s \in \mathbb{R} \) we denote by \( U_A(t, s) \) the evolution operator mapping \( \psi \in L^2(\mathbb{T}^d) \) into \( \psi(t) \).

We now are in position to state the main result of our paper.
Theorem 2.9. Let $H$ be as in (2.5) and $V \in C^\infty_b (\mathbb{R}; OPS^m)$ with $m < 2$. Then for any $\sigma \geq 0$ and for any initial datum $\psi \in H^\sigma$ there exists a unique global solution $\psi(t) := U(t,s) \psi \in H^\sigma$ of the initial value problem

$$i\partial_t \psi(t) = H(t) \psi(t), \quad \psi(s) = \psi.$$  \hspace{1cm} (2.11)

Furthermore, for any $\sigma > 0$ and $\varepsilon > 0$ there exists a positive constant $K_{\sigma,\varepsilon}$ such that for any $\psi \in H^\sigma$

$$\|UH(t,s) \psi\|_{\sigma} \leq K_{\sigma,\varepsilon} (t-s)^\varepsilon \|\psi\|_{\sigma} \quad \forall t, s \in \mathbb{R}. \hspace{1cm} (2.12)$$

The rest of the paper is dedicated to the proof of Theorem 2.9.

3 Normal form

3.1 Statement of the normal form result

We fix $\delta, \varepsilon, \tau > 0$ such that

$$0 < \varepsilon (\tau + 1) < \delta < 1, \quad \tau \geq d - 1, \quad \delta + d(d + \tau + 1)\varepsilon < 1; \hspace{1cm} (3.1)$$

We start by giving the following definition:

Definition 3.1 (Normal form operator). $Z = Op_W(z) \in C^\infty_b (\mathbb{R}; OPS^m)$ is said to be in normal form (with parameters $\delta, \varepsilon, \tau$) if its symbol $z(t,x,\xi) = \sum_{k \in \mathbb{Z}^d} \hat{z}_k(t,\xi)e^{ik \cdot x}$, satisfies

$$\hat{z}_k(t,\xi) \neq 0 \Rightarrow |(\xi; k)| \leq \langle \xi \rangle^\delta \|k\|^{-\tau} \text{ and } \|k\| \leq \langle \xi \rangle^\varepsilon$$

for any $k \neq 0$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^d$.

Theorem 3.2. Let $H$ be as in equation (2.5), with $V \in C^\infty_b (\mathbb{R}; OPS^m)$, $m < 2$, and assume that $V(t)$ is a family of self-adjoint operators. Take $\delta, \varepsilon, \tau$ fulfilling (3.1) and $m < 2\delta$. For any $N \in \mathbb{N}$ there exists a time dependent family $U_N(t)$ of unitary (in $L^2(\mathbb{T}^d)$) maps such that, if $\psi(t)$ solves (2.5), then $\phi(t)$ defined by $\psi(t) = U_N(t) \phi(t)$ solves

$$i\partial_t \phi(t) = \left(\tilde{H}^{(N)}(t) + R^{(N)}(t)\right) \phi(t), \hspace{1cm} (3.2)$$

where

$$\tilde{H}^{(N)}(t) = -\Delta_p + Z^{(N)}(t)$$

and the following properties hold
1. \( Z^{(N)} \in \mathcal{C}^\infty_b(\mathbb{R}; \text{OPS}^m) \) is in normal form, and \( Z^{(N)}(t) \) is a family of self-adjoint operators.

2. \( R^{(N)} \in \mathcal{C}^\infty_b(\mathbb{R}; \text{OPS}^{m-\delta N}) \), and the family \( R^{(N)}(t) \) is self-adjoint.

3. For any \( \sigma \geq 0 \), \( U_N, U_N^{-1} \in L^\infty(\mathbb{R}; \mathcal{B}(H^\sigma)) \).

Theorem 3.2 is a variant of Theorem 5.1 of [BLM20a] (see also Theorem 2.18 of [BLM20b] and Theorem 4.3 of [PS10]), which deals with the time independent case. More precisely, Theorem 5.1 of [BLM20a] provides the conjugation of an operator \( H = -\Delta_g + V \) with \( V \in \text{OPS}^m, m < 2 \) to an operator \( \tilde{H} + R \), where \( \tilde{H} \) is in normal form and \( R \) is regularizing. Here we adapt the construction performed therein to the time dependent case.

### 3.2 Lie transform

The conjugating maps \( U_N(t) \) that one looks for are compositions of maps of the form \( e^{iG(t)} \), where \( G(t) \) is a self-adjoint pseudo-differential operator for any \( t \in \mathbb{R} \). Thus the first information we need to provide is how the conjugation induced by the time dependent map \( e^{iG(t)} \) transforms the equation \( i\partial_t \psi(t) = H(t)\psi(t) \). This is the content of the following lemma, whose proof follows from a direct computation (see Lemma 3.2 of [Bam18] for more details and a complete proof).

**Lemma 3.3.** Let \( H(t) \) and \( G(t) \) be two smooth families of self-adjoint operators. Then \( \psi(t) \) solves the equation \( i\partial_t \psi(t) = H(t)\psi(t) \) if and only if \( \phi(t) = e^{iG(t)}\psi(t) \) solves \( i\partial_t \phi(t) = H^+(t)\phi(t) \), with

\[
H^+(t) = e^{iG(t)}H(t)e^{-iG(t)} - \int_0^1 e^{irG(t)}\partial_r(G(t))e^{-irG(t)} \, d\tau \quad \forall t \in \mathbb{R}.
\] (3.3)

We then recall the following Egorov type result, which ensures that \( e^{iG(t)}H(t)e^{-iG(t)} \) is still a family of time dependent pseudo-differential operators provided \( G(t) \) and \( H(t) \) are, and gives an expansion in pseudo-differential operators of decreasing order (for the proof, see Lemma 3.2 of [BGMR20]):

**Lemma 3.4 (Egorov type Theorem).** Let \( G \in \mathcal{C}^\infty_b(\mathbb{R}; \text{OPS}^\eta) \) with \( \eta < \delta \) and suppose that \( G(t) \) a self-adjoint family of operators. Then for any \( \tau \in [-1, 1] \) and for any \( t \in \mathbb{R} \) \( e^{irG(t)} \) is a unitary operator, and

\[
e^{irG} \in L^\infty(\mathbb{R}; \mathcal{B}(H^\sigma)) \quad \forall \sigma \geq 0.
\] (3.4)
Furthermore, given $m \in \mathbb{R}$ and $A \in \mathcal{C}_b^{\infty}(\mathbb{R}; OPS^m)$, for any $N \in \mathbb{N}$ one has $e^{i\tau G}A e^{-i\tau G} \in \mathcal{C}_b^{\infty}(\mathbb{R}; OPS^{m})$, with

$$e^{i\tau G}A e^{-i\tau G} = \sum_{j=1}^{N} \frac{(i\tau)^j \text{Ad}_j^j A}{j!} + \mathcal{C}_b^{\infty}(\mathbb{R}; OPS^{m+N(\eta-\delta)}) ,$$

(3.5)

where $\{\text{Ad}_j^j A\}_{j \in \mathbb{N}}$ is defined by

$$\text{Ad}_0^0 A = A, \quad \text{Ad}_j^j A = [G, \text{Ad}_j^j A] \quad \forall \ j \geq 1 .$$

**Remark 3.5.** Note that in (3.4) the family of operators $e^{i\tau G(t)}$ is only bounded with respect to $t$; concerning time derivatives, one has

$$\partial_t e^{i\tau G(t)} \in L^\infty(\mathbb{R}; B(H^{\sigma}; H^{\sigma-\alpha})) \quad \forall \alpha \in \mathbb{N}\setminus\{0\} .$$

### 3.3 Homological equation

We come to explain the algorithm leading to conjugate equation (2.5) to (3.2). Let us focus on the first step. One looks for a changes of variables of the form $\phi = e^{i\tau G(t)} \psi$, where $G \in \mathcal{C}_b^{\infty}(\mathbb{R}; OPS^\eta)$ for some $\eta < 1$ is a family of time dependent self-adjoint operators. By Lemma 3.3 and Lemma 3.4, if $\psi(t)$ solves (2.5), one has that $\phi(t)$ solves

$$i\partial_t \phi(t) = H^+(t)\phi(t) ,$$

(3.6)

with $H^+(t) = -\Delta_g + Z(t) - i[-\Delta_g; G(t)] + V(t) - \int_0^1 e^{i\tau G(t)} \partial_t (G(t)) e^{-i\tau G(t)} d\tau + R(t) \quad \forall t \in \mathbb{R} ,

and $R \in \mathcal{C}_b^{\infty}(\mathbb{R}; OPS^{m+\eta-\delta})$. Assume now that $\eta < \min\{m, 1\}$; then both the operators in the third line of (3.6) are to be considered remainder terms, in the sense that they are of lower order with respect to the initial perturbation $V(t)$. Thus one proceeds looking for a time dependent family of operators $G(t)$ with the above properties and such that

$$-i[-\Delta_g; G(t)] + V(t)$$

is in normal form, up to smoothing remainders. This is done following the construction in [BLM20a], which we now recall (see also [PS10, PS12, BLM20b]). One works at the level of symbols, namely one sets $G(t) = Op_W(g(t))$ and $V(t) = Op_W(v(t))$ and observes that

$$-i[-\Delta_g; G(t)] + V(t) = Op_W (\{||\xi||^2; g(t)\} + v(t)) .$$

(3.7)

Then one proceeds as follows. First of all, consider an even smooth cutoff function $\chi : \mathbb{R} \rightarrow [0, 1]$ with the property that $\chi(y) = 1$ for all $y$ with $|y| \leq \frac{1}{2}$ and $\chi(y) = 0$ for all $y$ with $|y| \geq 1$. 

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Definition 3.6. Given $\epsilon, \delta > 0$ and $\tau > d-1$ as in (3.1), define the following functions:

$$
\chi_k(\xi) = \chi \left( \frac{2\|k\|_{\tau}(\xi, k)}{\langle \xi \rangle^\delta} \right), \quad k \in \mathbb{Z}^d \setminus \{0\},
$$

$$
\tilde{\chi}_k(\xi) = \chi \left( \frac{\|k\|}{\langle \xi \rangle^\epsilon} \right), \quad k \in \mathbb{Z}^d \setminus \{0\}.
$$

Correspondingly, given a symbol $w \in C^\infty_b(\mathbb{R}; S^m)$, we decompose it as follows:

$$
w = \langle w \rangle + w^{(nr)} + w^{(res)} + w^{(S)},
$$

(3.8)

where $\langle w \rangle$ is the average symbol of $w$, namely

$$
\langle w \rangle(t, \xi) = \frac{1}{\mu(T^d)} \int_{T^d} w(t, x, \xi) \, d\mu(x)
$$

(3.9)

and

$$
w^{(res)}(t, x, \xi) = \sum_{k \neq 0} \hat{w}_k(t, \xi) \chi_k(\xi) \tilde{\chi}_k(\xi) e^{ik \cdot x},
$$

$$
w^{(nr)}(t, x, \xi) = \sum_{k \neq 0} \hat{w}_k(t, \xi) (1 - \chi_k(\xi)) \tilde{\chi}_k(\xi) e^{ik \cdot x},
$$

$$
w^{(S)}(t, x, \xi) = \sum_{k \neq 0} \hat{w}_k(t, \xi) (1 - \tilde{\chi}_k(\xi)) e^{ik \cdot x}.
$$

(3.10)

The following result was proven in [BLM20a]:

Lemma 3.7 (Lemma 5.6 of [BLM20a]). Let $w \in C^\infty_b(\mathbb{R}; S^m)$ for some $m \in \mathbb{R}$; then $\langle w \rangle, w^{(nr)}, w^{(res)} \in C^\infty_b(\mathbb{R}; S^m)$, and $w^{(S)} \in C^\infty_b(\mathbb{R}; S^{-\infty})$. Furthermore, if $w$ is real valued, also $w^{(res)}, w^{(nr)}, w^{(S)}$ and $\langle w \rangle$ are real valued.

Then, in order to reduce to normal form the contribution of (3.7), one is left to solve the equation

$$
\{\|\xi\|^2; g\} + v^{(nr)} = 0,
$$

(3.11)

which is dealt with in the following lemma:

Lemma 3.8 (Lemma 5.8 of [BLM20a]). Let $w \in C^\infty_b(\mathbb{R}; S^m)$ for some $m \in \mathbb{R}$; then

$$
g(t, x, \xi) := \sum_{k \in \mathbb{Z}^d} \frac{\hat{w}^{(nr)}_k(t, \xi)}{2 \langle \xi, k \rangle} e^{ik \cdot x}
$$

(3.12)

solves (3.11), and $g \in C^\infty_b(\mathbb{R}; S^{m-\delta})$. Furthermore, if $w$ is real valued, $g$ is real valued.
This leads to the proof of Theorem 3.2.

Proof of Theorem 3.2. The result is obtained arguing by induction. When 
$N = 0$, $H(t) = H_0(t)$ is of the form (3.2), with $Z^{(0)} = 0$ and $R_0(t) = V(t)$, 
thus the first step of the induction is immediately satisfied with $U_0 = \text{Id}$. Suppose that the thesis holds for some $N \in \mathbb{N}$: then one looks for a family of self-
adjoint pseudo-differential operators $G_N(t)$ such that, if $\phi(t) = e^{iG_N(t)}\phi_+(t)$ 
satisfies (3.2), then $\phi_+(t)$ satisfies (3.2) with $N$ replaced by $N + 1$. For all 
t \in \mathbb{R}$, let $r_N(t)$ be the symbol of $R_N(t)$ and consider for $r_N(t)$ the decompo-
sition (3.8), namely

$$r_N(t) = \langle r_N \rangle(t) + r_{N_{(\text{res})}}(t) + r_{N_{(\text{nr})}}(t) + r_{N_{(S)}}(t).$$

One then sets $G_N(t) = \text{Op}(g_N(t))$, where $g_N$ is the solution of equation 
(3.11) with $w_{(\text{nr})} = r_{N_{(\text{nr})}}$. By Lemma 3.8, $G_N \in C^\infty_\beta(\mathbb{R};\text{OPS}^{m - \delta(N + 1)})$, and 
$G_N(t)$ is self-adjoint for all $t \in \mathbb{R}$, since it has real valued symbol $g_N(t)$ and 
recalling Remark 2.4. Then by Lemma 3.3 and Lemma 3.4, one has that 
(3.6) holds, with $H_N(t) = -\Delta_g + Z_N(t) + R_N(t)$ instead of $H(t)$, $H_{N+1}(t)$ 
instead of $H^{+}(t)$ and $G_N(t)$ instead of $G(t)$, namely \forall t one has

$$H_{N+1}(t) = -\Delta_g + Z_{N+1}(t) + R_{N+1}(t),$$

with $R_{N+1}(t)$ a family of self-adjoint operators such that $R_{N+1} \in C^\infty_\beta(\mathbb{R};\text{OPS}^{m - \delta(N + 1)})$, 
and $Z_{N+1}(t) = Z_N(t) + \text{Op}(\langle r_N \rangle(t)) + \text{Op}(r_{N_{(\text{res})}}(t))$. Then the thesis follows setting 
$U_{N+1}(t) = e^{iG_N(t)}U_N(t)$ for all $t \in \mathbb{R}$ and recalling that, by Lemma 
3.4, if $e^{\pm iG_N} \in L^\infty(\mathbb{R};\mathcal{B}(H^\sigma))$ for any $\sigma \geq 0$. \hfill $\square$

4 Analysis of the normal form operator

In the present section we recall the results proven in [BLM20b] (see also 
[PS10, PS12]), which describe the structure of a self-adjoint operator of the form

$$\tilde{H}(t) = -\Delta_g + Z(t),$$

where $Z \in C^\infty_\beta(\mathbb{R};\text{OPS}^m)$ is in normal form. We start by a few definitions.

Definition 4.1. Let $E \subseteq \mathbb{Z}^d$. We call

$$\mathcal{E} := \text{span} \{ e^{i\xi \cdot x} \mid \xi \in E \} \subseteq L^2(\mathbb{T}^d)$$

subspace generated by $E$. 

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Definition 4.2. Given a subspace $\mathcal{E}$ of $L^2(\mathbb{T}^d)$ and a linear operator $A$ on $L^2(\mathbb{T}^d)$, we define $P_\mathcal{E}$ as the orthogonal projection operator on $\mathcal{E}$, and

$$A_\mathcal{E} := P_\mathcal{E}AP_\mathcal{E}.$$  \hfill (4.1)

Definition 4.3. A subgroup $M$ of $\mathbb{Z}^d$ is called a module if $\mathbb{Z}^d \cap \text{span}_\mathbb{R}M = M$. Here and below, $\text{span}_\mathbb{R}M$ is the subspace generated by taking linear combinations with real coefficients of elements of $M$. If $M \neq \{0\}$ and $M \neq \mathbb{Z}^d$, we say $M$ is a non trivial module.

Given a covector $\xi \in \mathbb{Z}^d$ and a module $M$, we need to decompose it in a component along $M$ and a component in the orthogonal direction. We denote

$$\xi = \xi_M + \xi_M \perp, \quad \xi_M \in \text{span}_\mathbb{R}M, \quad \xi_M \perp \in (\text{span}_\mathbb{R}M)^\perp.$$  

Our starting point is the following result, proven in [BLM20b]:

Theorem 4.4. [Theorem 2.18 of [BLM20b]] Let $\delta, \epsilon, \tau$ as in (3.1) be such that $0 < \delta_* = \delta + d(d + \tau + 1)\epsilon < 1$. For any modulus $M$ there exists a countable set $\mathcal{I}_M$ such that there exists a partition of $\mathbb{Z}^d$

$$\mathbb{Z}^d = \bigcup_{M \in \mathcal{I}_M} \bigcup_{i \in \mathcal{I}_M} W_{M,i},$$

where $M$ runs over modules of $\mathbb{Z}^d$, with the following properties: let $Z \in \mathcal{C}_0^\infty(\mathbb{R};\mathcal{O}PS^m)$ be in normal form with respect to $\delta, \epsilon, \tau$, with $Z(t)$ a family of self-adjoint operators; then for any $M \subseteq \mathbb{Z}^d$, and for any $i \in \mathcal{I}_M$ and for any $t \in \mathbb{R}$ the operator

$$\tilde{H}(t) = -\Delta_g + Z(t)$$

leaves invariant the subspace $W_{M,i} \subset L^2(\mathbb{T}^d)$ generated by $W_{M,i}$. Furthermore:

1. If $M = \{0\}$, then $\sharp W_{M,i} = 1 \forall i \in \mathcal{I}_M$, and $\tilde{H}_{W_{M,i}}(t)$ is a time dependent family of Fourier multipliers

2. If $M = \mathbb{Z}^d$, then $\sharp \mathcal{I}_M = 1$, and the corresponding block $W_{Z^d}$ has finite cardinality $n_*$, depending only on $g$, on the dimension $d$ and on the parameters $\delta, \epsilon, \tau$

3. If $M \subset \mathbb{Z}^d$ is a non trivial module, and $\xi, \xi' \in W_{M,i}$ for some $i \in \mathcal{I}_M$, then

$$\xi_M \perp = \xi'_M \perp.$$  

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4. There exists a positive constant $K$, depending only on $d$, $\delta, \epsilon, \tau$ and $g$, such that, for any non trivial module $M$ and for any $i \in \mathcal{I}_M$,

$$\xi \in W_{M,i} \Rightarrow \|\xi_M\| \leq K\langle \xi \rangle^\delta_\ast$$

(4.2)

Remark 4.5. Theorem 2.18 of [BLM20b] is stated and proved only for bounded pseudo-differential operators $V$, but since the partition $\{W_{M,\beta}\}_{M,\beta}$ ($\{W_{M,i}\}_{M,i}$ in the present notations) only depends on the unperturbed operator $-\Delta_g$, the extension is straightforward.

Remark 4.6. The sets $\{W_{M,i}\}_{M,i}$ are orthogonal one each other, both with respect to the $L^2(\mathbb{T}^d)$ and with respect to the $H^\sigma$ scalar products for any $\sigma > 0$.

Let $M \subset \mathbb{Z}^d$ be a non trivial module and let $i \in \mathcal{I}_M$. By Item 3. of Theorem 4.4, the quantity $\xi_M^\bot$ is constant along the block $W_{M,i}$. For this reason, given an arbitrary $\xi$ in $W_{M,i}$, we define

$$\ell_{M,i} := \langle \xi_M^\bot \rangle,$$

(4.3)

and we have that $\langle \xi_{M,i}' \rangle = \ell_{M,i}$ for any $\xi' \in W_{M,i}$. The following result immediately follows from Theorem 4.4.

Lemma 4.7. There exists a positive constant $C$, depending only on $d, \delta, \epsilon, \tau$ and $g$, such that for any non trivial module $M$ and any $i \in \mathcal{I}_M$

$$\|\xi_M\| \leq C\ell_{M,i} \quad \forall \xi \in W_{M,i}$$

(4.4)

Proof. Let $M$ be a non trivial module and let $i \in \mathcal{I}_M$. For any $\xi \in \mathbb{Z}^d$ one has

$$\langle \xi \rangle^2 = 1 + \|\xi\|^2 = 1 + \|\xi_M\|^2 + \|\xi_M^\bot\|^2 = \|\xi_M\|^2 + \ell_{M,i}^2.$$ 

Thus recalling that for any $x, y, a > 0$ one has $(x + y)^a \leq 2^a(x^a + y^a)$, by estimate (1.2) one also has

$$\|\xi_M\| \leq 2^{\frac{\delta_\ast}{\delta}} K \left(\|\xi_M\|^{\delta_\ast} + \ell_{M,i}^{\delta_\ast} \right) \quad \forall \xi \in W_{M,i}.$$ 

Then the thesis follows observing that $\delta_\ast < 1$ and arguing as in Lemma B.5 of [BLM20b].

Remark 4.8. By Lemma 4.7 and by Item 3. of Theorem 4.4 it follows that each block $W_{M,i}$ has finite cardinality, depending on $\ell_{M,i}$. However, such a quantity is not uniformly bounded as the couple $M, i$ varies.
5 Time evolution of Sobolev norms

In the present section we provide upper bounds on the growth in time of the Sobolev norms of the solutions of (2.5).

5.1 The normal form dynamics

Given $N \in \mathbb{N}$, we start with providing upper bounds on the Sobolev norms of the solutions of the system

$$i \partial_t \psi(t) = \tilde{H}^{(N)}(t)\psi(t),$$

with $\tilde{H}^{(N)}$ as in Theorem 3.2. Since, by Theorem 4.4, $\tilde{H}^{(N)}(t)$ is a family of block diagonal operators, it is sufficient to analyze the evolution operator of its restrictions $\tilde{H}^{(N)}_{W_{M,i}}(t)$ to the blocks $W_{M,i}$, which for brevity we simply denote as $U_{W_{M,i}}(t,s)$, instead of $U_{H^{(N)}_{W_{M,i}}}(t,s)$. We remark that the global well posedness of the normal form equation is basically a straightforward consequence of the block invariant structure of $\tilde{H}^{(N)}$. As a first consequence of Theorem 4.4, one immediately has the following:

Lemma 5.1. For any $\sigma > 0$ there exists a positive constant $K_\sigma$ such that the following holds:

1. For any $\psi \in W_{Z^d}$
   $$\|\psi\|_\sigma \leq K_\sigma\|\psi\|_0$$
   (5.2)

2. For any non trivial module $M$ of $Z^d$, for any $i \in I_M$, and for any $\psi \in W_{M,i}$,
   $$\|\psi\|_\sigma \leq K_\sigma\ell_{M,i}^\sigma\|\psi\|_0,$$
   $$\ell_{M,i}^\sigma\|\psi\|_0 \leq \|\psi\|_\sigma.$$
   (5.3)

Proof. Item 1. follows observing that, by Theorem 4.4, $W_{Z^d}$ has finite dimension $n_*$. Thus there exists a positive $R = R(n_*)$ such that

$$W_{Z^d} \subseteq \{\xi \in Z^d \mid \langle \xi \rangle \leq R\}.$$

As a consequence, for any $\psi \in W_{Z^d}$ one has

$$\|\psi\|_\sigma^2 = \sum_{\xi \in W_{Z^d}} \langle \xi \rangle^{2\sigma}|\hat{\psi}_\xi|^2 \leq R^{2\sigma}\|\psi\|_0^2.$$  

(5.4)

In order to prove Item 2., observe that for any $\psi \in W_{M,i}$ one has

$$\|\psi\|_\sigma^2 = \sum_{\xi \in W_{M,i}} (\|\xi_M\|^2 + \langle \xi_{M^\perp} \rangle^2)^\sigma |\hat{\psi}_\xi|^2 = \sum_{\xi \in W_{M,i}} (\|\xi_M\|^2 + \ell_{M,i}^2)^\sigma |\hat{\psi}_\xi|^2,$$

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with $\ell_{M,i}$ as in (4.3). Since by Lemma 4.7, there exists a positive constant $C$ independent of $M$ and $i$ such that $\|\xi_M\| \leq C\ell_{M,i}$, one also has
\[
\|\psi\|_2^2 \leq (C^2 + 1)^\sigma \ell_{M,i}^2 \sum_{\xi \in W_{M,i}} |\hat{\psi}_\xi|^2 = (C^2 + 1)^\sigma \ell_{M,i}^2 \|\psi\|_0^2,
\]
which gives the first of (5.3). To prove the second of (5.3), one simply observes that, by Lemma 4.7, for any $\psi \in W_{M,i}$ one has
\[
\ell_{M,i}^2 \|\psi\|_0^2 = \ell_{M,i}^2 \sum_{\xi \in W_{M,i}} |\hat{\psi}_\xi|^2 = \sum_{\xi \in W_{M,i}} \langle \xi_{M} \| \hat{\psi}_\xi \rangle^2.
\]
Then one has $\langle \xi_{M} \| \hat{\psi}_\xi \rangle \leq \langle \xi \| \hat{\psi}_\xi \rangle$, which immediately implies
\[
\ell_{M,i}^2 \|\psi\|_0^2 \leq \sum_{\xi \in W_{M,i}} \langle \xi \| \hat{\psi}_\xi \rangle^2 = \|\psi\|_2^2.
\]

The following Lemma is a consequence of Lemma 5.3 and of the self-adjointness of the operators $H^{(N)}_{W_{M,i}}$.

**Lemma 5.2.** For any $\sigma \geq 0$ there exists a positive constant $K_\sigma$ such that, for any $M \subseteq \mathbb{Z}^d$, $M \neq \{0\}$, $\forall i \in \mathcal{I}_M$, and $\forall \psi \in W_{M,i}$,
\[
\|U^{(N)}_{M,i}(t,s)\psi\|_\sigma \leq K_\sigma \|\psi\|_\sigma \quad \forall t, s \in \mathbb{R}.
\]  

**Proof.** Let $\psi \in W_{M,i}$. Since $H^{(N)}_{W_{M,i}}$ is self-adjoint, $C^\infty$-smooth in time and $W_{M,i}$ is a finite dimensional space the flow map $U^{(N)}_{M,i}(t,s) : W_{M,i} \to W_{M,i}$ is well defined for any $t, s \in \mathbb{R}$ and for any $\psi \in W_{M,i}$, the map $t \mapsto U(t,s)\psi$ is $C^\infty(\mathbb{R}, W_{M,i})$. Moreover
\[
\|U^{(N)}_{M,i}(t,s)\psi\|_0 = \|\psi\|_0 \quad \forall t, s \in \mathbb{R}.
\]
If $M$ is a non trivial module and $i \in \mathcal{I}_M$, by (5.3) one deduces that, for any $t, s \in \mathbb{R}$,
\[
\|U^{(N)}_{M,i}(t,s)\psi\|_\sigma \leq K_\sigma \ell_{M,i}^\sigma \|U^{(N)}_{M,i}(t,s)\psi\|_0 = K_\sigma \ell_{M,i}^\sigma \|\psi\|_0 \leq K_\sigma \|\psi\|_\sigma.
\]
In the case $M = \mathbb{Z}^d$, one uses (5.2) to deduce analogously that
\[
\|U^{(N)}_{\mathbb{Z}^d}(t,s)\psi\|_\sigma \leq K_\sigma \|\psi\|_\sigma \quad \forall t, s \in \mathbb{R}.
\]  

\[\square\]
Lemma 5.3 (Estimate on the normal form flow). For any $\sigma \geq 0$, the pseudo-PDE \eqref{5.1} is globally well posed in $H^\sigma$ and the corresponding flow satisfies the estimate $\|U_{H^{(N)}}(t,s)\psi\|_\sigma \leq C_\sigma \|\psi\|_\sigma$ for any $\psi \in H^\sigma$. Moreover the map $t \mapsto U_{H^{(N)}}(t,s)\psi$ is in $C(\mathbb{R}, H^\sigma) \cap C^1(\mathbb{R}, H^{\sigma-2})$.

Proof. Consider the equation \eqref{5.1} and recall that, by Theorem 4.4, one has $L^2(T^d) = \bigcup_{M,i} W_{M,i}$, and $H^{(N)}(t)$ leaves invariant all the spaces $W_{M,i}$. Thus, let $\psi \in L^2(T^d)$ and $\sigma \geq 0$: one has

$$
\|U_{H^{(N)}}(t,s)\psi\|_\sigma = \|U_{H^{(N)}}(t,s) \sum_{M,i} P_{W_{M,i}} \psi\|_\sigma
\leq \sum_{M,i} \|U_{H^{(N)}}(t,s) P_{W_{M,i}} \psi\|_\sigma.
$$

(5.6)

If $M = \{0\}$, by Item 1. of Theorem 4.4 for any $i \in \mathcal{I}_M$ $H^{(N)}_{W_{M,i}}(t)$ is a family of self-adjoint Fourier multipliers, thus

$$
\|U_{H^{(N)}}(t,s)\psi\|_\sigma = \|P_{W_{(0)i}} \psi\|_\sigma \quad \forall t, s \in \mathbb{R}.
$$

(5.7)

Thus, by (5.7) and applying Lemma 5.2 for all modules $M \neq \{0\}$, by (5.6) one deduces that there exists a positive constant $K$, depending on $\sigma$, such that

$$
\|U_{H^{(N)}}(t,s)\psi\|_\sigma \leq K^2 \sum_{M,i} \|P_{W_{M,i}} \psi\|_\sigma
\leq K^2 \|\psi\|_\sigma.
$$

(5.8)

The proof of the Lemma is then concluded.

\[\square\]

5.2 Assembling the blocks: proof of Theorem 2.9

We deduce Theorem 2.9 from the following couple of abstract results:

\textbf{Proposition 5.4.} Let $N \in \mathbb{N}$ and suppose that $H(t) = H_0(t) + R(t)$ is a time dependent family of self-adjoint operators such that

1. For all $\sigma \geq 0$ there exists a positive constant $K_\sigma$ such that

$$
\|U_{H_0}(t,s)\phi\|_\sigma \leq K_\sigma \|\phi\|_\sigma \quad \forall t, s \in \mathbb{R}, \quad \phi \in L^2(T^d),
$$

2. $R = Op^W(r) \in C^\infty_b(\mathbb{R}; OPS^{-N})$
Then there exists a unique global solution $\psi(t) = U_H(t, s) \psi \in H^\sigma$ of the initial value problem (2.11) with initial datum $\psi \in H^\sigma$, and the map $\psi : t \mapsto \psi(t)$ is in $C(\mathbb{R}; H^\sigma) \cap \mathcal{C}^1(\mathbb{R}; H^{\sigma-2})$. Furthermore there exists $K' \in \mathbb{R}^+$, depending only on $N$, on $K_\sigma$ and on the family of seminorms of $R$, such that for any $\psi \in L^2(\mathbb{T}^d)$ one has

$$\|U_H(t, s)\psi\|_N \leq K'(t-s)\|\psi\|_N \quad \forall t, s \in \mathbb{R}.$$ (5.9)

**Proof.** We start with proving global well posedness. As usual we find solutions of the pseudo-PDE, on a time interval $[s-T, s+T]$}

$$i\partial_t \psi = (H_0 + R)\psi, \quad \psi(s) = \psi_0$$

arguing by a standard fixed point argument on the map

$$\Phi(\psi) := U_{H_0}(s, t)\psi_0 + \int_s^t U_{H_0}(t, s')(-iR(s'))\psi(s') \, ds'$$

in a suitable ball $B_{\sigma}(\rho) := \{\psi \in C([s-T, s+T], H^\sigma) : \|\psi\|_{L^\infty(\mathbb{R}; H^\sigma)} \leq \rho\}$, where $\rho = \rho(\|\psi_0\|_{\sigma})$ and $T = T(\rho)$ have to be chosen appropriately. Moreover, a straightforward application of the Gronwall inequality, implies that

$$\|\psi(t)\|_{\sigma} \leq C_\sigma \exp\left(T\|R\|_{L^\infty(\mathbb{R}; \mathcal{B}(H^\sigma))}\right)\|\psi_0\|_{\sigma},$$

for a suitable positive constant $C_\sigma$. Hence $\psi(t)$ does never blow up in $[s - T, s + T]$ and hence it can be extended outside this interval. This implies that it is globally defined.

We proceed proving estimate (5.9). Let $\psi \in L^2(\mathbb{T}^d)$. One observes that

$$U_H(t, s)\psi = U_{H_0}(t, s)\psi + \int_s^t U_{H_0}(t, s')(-iR(s'))U_H(s', s)\psi \, ds',$$ (5.10)

which implies

$$\|U_H(t, s)\psi\|_N \leq K_N\|\psi\|_N + \int_s^t K_N\|R(s')\|_{N,0} \|U_H(s', s)\psi\|_0 \, ds'$$

$$= K_N\|\psi\|_N + \int_s^t K_N\|R(s')\|_{N,0} \|\psi\|_0 \, ds',$$

due to the fact that $H$ is a self-adjoint family. Thus, by Calderon Vaillancourt Theorem 2.6, one also has that there exists $C > 0$, depending on $N$ and on the family of seminorms of $R$, such that

$$\|U_H(t, s)\psi\|_N \leq K_N\|\psi\|_N + \int_s^t K_NC\|\psi\|_0 \, ds' \leq K_N\|\psi\|_N + K_NC(t-s)\|\psi\|_0.$$
Proposition 5.5. Given a family of self-adjoint operators $H(t)$, suppose that there exists $p > 0$ such that, for any positive integer $N \in \mathbb{N}$, there exists $K_{N,p} > 0$ such that

$$\|U_H(t, s)\psi\|_N \leq K_{N,p} \langle t - s \rangle^p \|\psi\|_N \quad \forall \psi \in H^N(\mathbb{T}^d).$$

Then for any $\sigma > 0$ and for any $\varepsilon > 0$ there exists $K'_{\sigma, \varepsilon} > 0$ such that

$$\|U_H(t, s)\psi\|_{\sigma} \leq K'_{\sigma, \varepsilon} \langle t - s \rangle^{\varepsilon} \|\psi\|_{\sigma} \quad \forall \psi \in H^\sigma(\mathbb{T}^d).$$

Proof. Let $\sigma' > 0$ and $N_{\sigma'} = \lfloor \sigma' \rfloor + 1$. Then $\sigma' = \theta N_{\sigma'}$ for some $\theta = \theta_{\sigma'} \in (0, 1)$, and some $N_{\sigma'} \in \mathbb{N}$. Thus standard interpolation theory implies that for any $t$ and $s \in \mathbb{R}$

$$\|U_H(t, s)\|_{\sigma', \sigma'}^\sigma = \|U_H(t, s)\|_{\theta N_{\sigma'}, \theta \sigma, N_{\sigma'}}^\theta \leq (\|U_H(t, s)\|_{N_{\sigma'}, N_{\sigma'}})^{\theta_{\sigma'}} (\|U_H(t, s)\|_{0,0})^{1-\theta_{\sigma'}} \leq K^0_{N_{\sigma'}, \theta} \langle t - s \rangle^{\theta_{\sigma'} p} \leq K^0_{N_{\sigma'}, p} \langle t - s \rangle^{\sigma_{\sigma'} p},$$

namely for any $\sigma' \geq 0$ there exists $C = C_{\sigma'}$ such that

$$\|U_H(t, s)\|_{\sigma', \sigma'} \leq C_{\sigma'} \langle t - s \rangle^{p \varepsilon}.$$ 

Let now $\sigma \geq 0$ and $\varepsilon > 0$ : one has $\sigma = \varepsilon \sigma'$, with $\sigma' = \varepsilon^{-1} \sigma$, thus arguing as above one obtains that

$$\|U_H(t, s)\|_{\sigma, \sigma} \leq (\|U_H(t, s)\|_{\varepsilon^{-1} \sigma, \varepsilon^{-1} \sigma})^\varepsilon \leq (C_{\varepsilon^{-1} \sigma})^\varepsilon \langle t - s \rangle^{\varepsilon \varepsilon}.$$

By the arbitrariness of $\varepsilon$, one obtains the thesis.

Proof of Theorem 2.9. For any $N \in \mathbb{N}$, we apply Theorem 5.3. By Lemma 5.4, one has

$$\|U_{\tilde{H}(N)}(t, s)\psi\|_{\sigma} \leq K \|\psi\|_{\sigma}, \quad \psi \in H^\sigma. \quad (5.11)$$

The latter estimate implies that $\tilde{H}(N)$ satisfies Hypothesis 1. of Proposition 5.4 and $R'(N) \in C_{\delta}^N (\mathbb{R}; OPS^{m-\delta N})$ satisfies Hypothesis 2. of Proposition 5.4 with $N$ replaced by $N' = [m - \delta N]$. Thus we deduce that

$$\|U_{\tilde{H}(N) + R(N)}(t, s)\psi\|_{N'} \leq K' \langle t - s \rangle \|\psi\|_{N'}, \quad \forall t, s \in \mathbb{R}, \quad \forall \psi \in L^2(\mathbb{T}^d).$$

By Theorem 5.2, $H$ is unitarily equivalent to $\tilde{H}(N) + R(N)$ via a map $U_N$ such that $U_N, U_N^* \in L^\infty (\mathbb{R}; B(H^N(\mathbb{T}^d)))$. Thus one also has

$$\|U_H(t, s)\psi\|_{N'} \leq K'' \langle t - s \rangle \|\psi\|_{N'}, \quad \forall t, s \in \mathbb{R}, \quad \forall \psi \in L^2(\mathbb{T}^d),$$

for a positive constant $K''$ which depends on $K'$ and on $\sup_{t \in \mathbb{R}} \|U_N(t)\|_{N', N'}$, $\sup_{t \in \mathbb{R}} \|U_N^*(t)\|_{N', N'}$. By the arbitrariness of $N$ and by Proposition 5.5, the thesis follows.
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