AN UNFITTED FINITE ELEMENT METHOD WITH DIRECT EXTENSION STABILIZATION FOR TIME-HARMONIC MAXWELL PROBLEMS ON SMOOTH DOMAINS

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Abstract. We propose an unfitted finite element method for numerically solving the time-harmonic Maxwell equations on a smooth domain. The model problem involves a Lagrangian multiplier to relax the divergence constraint of the vector unknown. The embedded boundary of the domain is allowed to cut through the background mesh arbitrarily. The unfitted scheme is based on a mixed interior penalty formulation, where Nitsche penalty method is applied to enforce the boundary condition in a weak sense, and a penalty stabilization technique is adopted based on a local direct extension operator to ensure the stability for cut elements. We prove the inf-sup stability and obtain optimal convergence rates under the energy norm and the $L^2$ norm for both the vector unknown and the Lagrangian multiplier. Numerical examples in both two and three dimensions are presented to illustrate the accuracy of the method.

keywords: unfitted finite element method; direct extension; time-harmonic Maxwell equation;

1. Introduction

Maxwell equations describe the laws of macroscopic electric and magnetic phenomena, and have a wide range of applications in science and engineering fields like plasma physics, electrodynamics, antenna design, satellites and telecommunication. In this paper, we present and analyze an unfitted finite element method for time-harmonic Maxwell equations on smooth domains. In the literature, many finite element methods have been developed for solving the time-harmonic Maxwell equations, e.g. $H(\text{curl})$-conforming edge element methods [4, 11, 12, 15, 26, 35, 37, 38, 39, 48], discontinuous Galerkin methods [16, 27, 34, 40, 41, 42], and nodal type finite element methods [6, 14, 37].

The above mentioned methods are based on fitted meshes that cover the computational domain exactly. For complex geometries, it is not a trivial task to generate a high quality mesh to represent the domain accurately, especially in high dimensions. For problems with complex geometries, the unfitted finite element method has been a popular and successful tool because the geometry description is decoupled from the mesh generation [3, 17, 9, 33, 44, 19, 31, 22]. An important advance [23] is to combine with the Nitsche-type penalization to weakly impose the boundary condition or the interface condition, then the domain can be easily embedded into the unfitted mesh. This method, sometimes called Nitsche-XFEM or CutFEM, has been extensively applied in a variety of problems, see [5, 9, 25, 8, 36, 19] and the references therein. It is noticeable that for the penalty method, one difficulty is the small cuts appearing in elements that are cut by the embedded geometry, which may adversely affect the convergence of the numerical scheme [8]. There are two common ways to handle this issue: one is employing some stabilization mechanism, such as the ghost penalty [7], and the other one is applying some cell agglomeration algorithms [29, 8, 28]. To our best knowledge, there is no unfitted finite element method for solving the time-harmonic Maxwell problem.

In this contribution, we develop a finite element method for solving the time-harmonic Maxwell problem with unfitted meshes. The proposed method is based on a mixed interior penalty formulation with the Nitsche penalty method to enforce the boundary condition in a weak sense. In our method, we apply a local extension operator [47], together with the idea of the ghost penalty stabilization [7], to address the issue caused by the small cuts for both curl operator and Lagrangian multiplier. The method is shown to be stable in the sense that the
constants lying in the error estimates are independent of how the embedded geometry intersects the mesh. The local extension operator just directly extends the polynomial defined on the interior neighbouring element to the cut elements. The method is easily implemented and can achieve the high-order accuracy. For the mixed formulation, we prove the inf-sup stability for the form of the divergence constraint, following the ideas in [21]. The optimal convergence rates for both solution variables under the associated energy norm and the $L^2$ norm are obtained. In addition, we explore the relationship between the wave number $k$ and the constants appearing in the error bounds. We confirm the theoretical predictions and illustrate the accuracy in a series of numerical examples in both two and three dimensions.

The rest of this paper is as follows. In Section 2, we introduce the local extension operator and the associated penalty bilinear forms. We also give some basic properties that are fundamental in the numerical analysis. In Section 3, we define the mixed formulation and introduce an auxiliary formulation which is more suitable for analysing. In Section 4, we present the main theoretical results including the inf-sup stability, the continuity and the coercivity of bilinear forms and the final error estimates. The numerical performance of the proposed method is tested in Section 5. Finally, we make some concluding remarks in Section 6.

2. Preliminaries

Let $\Omega^* \subset \mathbb{R}^d (d = 2, 3)$ be a polygonal (polyhedral) domain, and we let $\Omega \subset \Omega^*$ be an open subdomain with a $C^2$-smooth boundary $\Gamma := \partial \Omega$. We denote by $T_h$ the background mesh on $\Omega^*$ into triangles (tetrahedrons). For any element $K \in T_h$, we denote by $h_K$ the diameter of $K$ and by $\rho_K$ the radius of the largest ball inscribed in $K$. The mesh size $h$ is given as $h := \max_{K \in T_h} h_K$. The mesh $T_h$ is assumed to be quasi-uniform in the sense that there exists a constant $C$ independent of $h$, such that for any element $K \in T_h^*$, there holds $h \leq C \rho_K$.

We introduce two sets related to the domain $\Omega$,

$$\mathcal{T}_h := \{ K \in T_h^* \mid K \cap \Omega \neq \emptyset \}, \quad \mathcal{T}_h^0 := \{ K \in \mathcal{T}_h \mid K \in \Omega \}.$$  

Here $\mathcal{T}_h$ is the computational mesh, which is the minimal subset of $T_h^*$ that entirely covers the domain $\Omega$, and $\mathcal{T}_h^0$ is a subset of $\mathcal{T}_h$ consisting of all interior elements located inside $\Omega$. We set their corresponding domains as

$$\Omega_h := \text{Int} \left( \bigcup_{K \in \mathcal{T}_h} K \right), \quad \Omega_h^0 := \text{Int} \left( \bigcup_{K \in \mathcal{T}_h^0} K \right),$$

and obviously, there holds $\Omega_h^0 \subset \Omega \subset \Omega_h$. We denote by $\mathcal{F}_h$ the collection of all $d-1$ dimensional faces in $\mathcal{T}_h$, and then $\mathcal{F}_h$ is decomposed into $\mathcal{F}_h = \mathcal{F}_h^e \cup \mathcal{F}_h^b$, where $\mathcal{F}_h^e$ and $\mathcal{F}_h^b$ are the sets of interior faces and boundary faces in $\mathcal{T}_h$, respectively. For any $f \in \mathcal{F}_h$, we define $h_f$ as the diameter of $f$. Moreover, we denote by $\mathcal{F}_h^o$ the set of all $d-1$ dimensional faces in $\mathcal{T}_h^0$ and, similarly, we decompose $\mathcal{F}_h^o$ into $\mathcal{F}_h^o = \mathcal{F}_h^{o,i} \cup \mathcal{F}_h^{o,b}$, where $\mathcal{F}_h^{o,i}$ and $\mathcal{F}_h^{o,b}$ consist of interior faces and boundary faces in $\mathcal{T}_h^0$, respectively. We define $\mathcal{T}_h^T$ and $\mathcal{F}_h^T$ as the collections of the elements and faces that are cut by the boundary $\Gamma$,

$$\mathcal{T}_h^T := \{ K \in \mathcal{T}_h \mid K \cap \Gamma \neq \emptyset \}, \quad \mathcal{F}_h^T := \{ f \in \mathcal{F}_h \mid f \cap \Gamma \neq \emptyset \}.$$  

It can readily seen that $\mathcal{T}_h^T = \mathcal{T}_h \setminus \mathcal{T}_h^0$ and $\mathcal{F}_h^T = \mathcal{F}_h \setminus \mathcal{F}_h^0$. For any element $K \in \mathcal{T}_h$ and any face $f \in \mathcal{F}_h$, we define $K^0 := K \cap \Omega$ and $f^0 := f \cap \Omega$ as their parts inside the domain $\Omega$. For any element $K \in \mathcal{T}_h^T$, we define $\Gamma_K = \Gamma \cap K$.

We make the following geometrical assumptions [23, 36, 46, 19, 21], which can be always fulfilled for the fine enough mesh, to ensure the embedded boundary $\Gamma$ is well-resolved.

**Assumption 1.** For any cut face $f \in \mathcal{F}_h^T$, $f$ is intersected by $\Gamma$ at most once.

**Assumption 2.** For any element $K \in \mathcal{T}_h^T$, we can assign an interior element $K^o \in \Delta(K) \cap \mathcal{T}_h^0$, where $\Delta(K) := \{ K' \in \mathcal{T}_h \mid \overline{K'} \cap K \neq \emptyset \}$ denotes the set of elements that touch $K$. 


By the quasi-uniformity of $T_h$, there exists a constant $C_\Delta$ independent of $h$ such that for any element $K$, the set $\Delta(K) \subset B(x_K, C_\Delta h_K)$, where $B(z, r)$ denotes the ball centered at the point $z$ with the radius $r$, and $x_K$ is the barycenter of the element $K$. In addition, we assume the open bounded domain $\Omega^*$ contains the union of all balls $B(x_K, C_\Delta h_K)(\forall K \in T_h)$.

We introduce the jump and average operators which are widely used in the discontinuous Galerkin framework. Let $f \in F^i_h$ be an interior face shared by two neighbouring elements $K^+$ and $K^-$, with the unit outward normal vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ on $f$, respectively. For any piecewise smooth scalar-valued function $v$ and any piecewise smooth vector-valued function $q$, the following jump operators $\{\cdot\}$ and average operators $\{\cdot\}$ are involved in our scheme:

$$[v] := \mathbf{n}^+ v^+|_f + \mathbf{n}^- v^-|_f,$$

$$\{v\} := \mathbf{n}^+ v^+|_f + \mathbf{n}^- v^-|_f$$

and

$$[\mathbf{n} \cdot q] := \mathbf{n}^+ \cdot q^+|_f + \mathbf{n}^- \cdot q^-|_f,$$

$$[\mathbf{n} \times q] := \mathbf{n}^+ \times q^+|_f + \mathbf{n}^- \times q^-|_f$$

and

$$\{\mathbf{n} \cdot q\} := \frac{1}{2} (\mathbf{n}^+ \cdot q^+|_f + \mathbf{n}^- \cdot q^-|_f),$$

$$\{\mathbf{n} \times q\} := \frac{1}{2} (\mathbf{n}^+ \times q^+|_f + \mathbf{n}^- \times q^-|_f),$$

where $v^+ := v|_{K^+}$, $v^- := v|_{K^-}$, $q^+ := q|_{K^+}$, $q^- := q|_{K^-}$. The jump operators $\{\cdot\}$ and the averages $\{\cdot\}$ on $\Gamma$ will also be used, and their definitions are modified as

$$[v]|_K := \mathbf{n} v|_K, \quad [\mathbf{n} \cdot q]|_K := \mathbf{n} \cdot q|_K,$$

$$[\mathbf{n} \times q]|_K := \mathbf{n} \times q|_K,$$

$$\{v\}|_K := v|_K, \quad \{\mathbf{n} \cdot q\}|_K := \mathbf{n} \cdot q|_K,$$

and

$$\{\mathbf{n} \times q\}|_K := \mathbf{n} \times q|_K,$$

for any element $K \in T^i_h$, where $\mathbf{n}$ is the unit outward normal on $\Gamma$.

In this paper, $C$ and $C^*$ with subscripts are denoted as generic constants which may differ between lines but are always independent of the mesh size $h$ and the location of the boundary $\Gamma$ relative to the mesh. For a bounded domain $\Omega$, we will follow the standard notations to the Sobolev spaces $L^2(\Omega)$, $H^s(\Omega)$, $L^2(\Omega)^d$, $H^s(\Omega)^d$ with the regular exponent $s \geq 0$, and their corresponding inner products, norms and seminorms. The spaces $H^s(\text{div}, \Omega)$ and $H^s(\text{curl}, \Omega)$ are also involved as well as their associated norms and semi-norms. We define $H^s_0(\text{curl}, \Omega)$ as the space of functions in $H^s(\text{curl}, \Omega)$ with vanishing tangential traces.

Next, we introduce the local extension operator $[47]$, which will be involved in the numerical scheme to ensure the stability of our method. Given an integer $m \geq 0$ and for element $K \in T_h$, the local operator $E_K$ is defined as

$$(1) \quad E_K : L^2(K) \to \mathbb{P}_m(B(x_K, C_\Delta h_K)), \quad (E_K v)|_K = \Pi_K v, \quad \forall v \in L^2(K),$$

where $\Pi_K$ is the $L^2$ projection operator from $L^2(K)$ to $\mathbb{P}_m(K)$. For $v \in L^2(K)$, $E_K$ extends its $L^2$ projection $\Pi_K v$ to define on the ball $B(x_K, C_\Delta h_K)$. Particularly for $v \in \mathbb{P}_m(K)$, $E_K v$ is just the direct extension of $\Pi_K v$ to the ball $B(x_K, C_\Delta h_K)$.

Let us give some basic properties of the operator $E_K$.

**Lemma 1.** There hold

$$(2) \quad \|E_K v\|_{L^2(B(x_K, C_\Delta h_K))} \leq C \|\Pi_K v\|_{L^2(K)}, \quad \forall v \in L^2(K), \quad \forall K \in T_h,$$

and

$$(3) \quad \|\nabla E_K v\|_{L^2(B(x_K, C_\Delta h_K))} \leq C \|\nabla v\|_{L^2(K)}, \quad \forall v \in \mathbb{P}_m(K), \quad \forall K \in T_h.$$

**Proof.** We mainly prove the $L^2$ estimate (2). From the quasi-uniformity of the mesh, the ball $B(x_K, C_\Delta h_K) \subset K$ and there exists a constant $C_1$ such that $h_K \leq C_1 \rho_K$. By the norm equivalence restricted on the space $\mathbb{P}_m(\cdot)$, we have that

$$\|q\|_{L^2(B(x_K, C_\Delta h_K))} \leq C \|q\|_{L^2(B(x_K, 1))}, \quad \forall q \in \mathbb{P}_m(B(x_K, C_\Delta)),$$

and

$$\|\nabla q\|_{L^2(B(x_K, C_\Delta h_K))} \leq C \|\nabla q\|_{L^2(B(x_K, 1))}, \quad \forall q \in \mathbb{P}_m(B(x_K, C_\Delta)).$$
Considering the affine mapping from the ball $B(x_K, 1)$ to $B(x_K, \rho_K)$, which simultaneously maps the ball $B(x_K, C \Delta C_1)$ to $B(x_K, C \Delta C_1 \rho_K)$, we conclude that
\[
\|E_K v\|_{L^2(B(x_K, C \Delta h K))} \leq \|E_K v\|_{L^2(B(x_K, C \Delta C_1 \rho_K))} \leq C\|\Pi_K v\|_{L^2(B(x_K, \rho_K))} \leq C\|\Pi_K v\|_{L^2(K)}.
\]
The estimates in (3) are then the direct consequences of (2). This completes the proof. \hfill \Box

The stability of the curl operator near the boundary is also guaranteed by the local operator. To this end, we extend the operator $E_K$ to vector-valued functions:
\[
E_K : L^2(K)^d \to \mathbb{P}_m(B(x_K, C \Delta h K))^d, \quad v \to E_K v, \quad \forall v \in L^2(K)^d,
\]
where $\Pi_K$ is still the $L^2$ projection from $L^2(K)^d$ to the polynomial space $\mathbb{P}_m(K)^d$. $E_K$ in (4) can be regarded as acting on vector-valued functions in a columnwise manner with (1). Hence, Lemma 1 also hold for vector-valued functions.

**Lemma 2.** There hold
\[
\|E_K v\|_{L^2(B(x_K, C \Delta h K))} \leq C\|\Pi_K v\|_{L^2(K)}, \quad \forall v \in L^2(K)^d, \quad \forall K \in \mathcal{T}_h,
\]
and
\[
\|\nabla \times E_K v\|_{L^2(B(x_K, C \Delta h K))} \leq C\|\nabla \times v\|_{L^2(K)}, \quad \forall v \in \mathbb{P}_m(K)^d \quad \forall K \in \mathcal{T}_h.
\]

In addition, for any piecewise smooth scalar(or vector)-valued function $q_h$, we simply write $E_K(q_h|\Delta)$ as $E_Kq_h$.

We define discontinuous/continuous piecewise polynomial spaces for the partition $\mathcal{T}_h$,
\[
Q_h^{m,d} := \{v_h \in L^2(\Omega_h) \mid v_h|\Delta \in \mathbb{P}_m(\Omega_h), \forall K \in \mathcal{T}_h\}, \quad Q_h^{m,c} := Q_h^{m,d} \cap H^1(\Omega_h),
\]
and we let the space $Q_h^m$ be either $Q_h^{m,d}$ or $Q_h^{m,c}$. We then define the penalty operator $j_h(\cdot, \cdot)$ based on $E_K$. The form $j_h(\cdot, \cdot)$ will be used to provide the stabilization for the scalar-valued unknown. We note that this method follows the idea of the ghost penalty method [7], which extends the control of the relevant norms from the interior domain to the entire computational domain [20, 7, 10].

We set $Q_h := Q_h^m + H^1(\Omega)$, and define
\[
j_h(p_h, q_h) := \sum_{K \in \mathcal{T}_h} \int_K (p_h - E_K \cdot p_h)(q_h - E_K \cdot q_h)dx, \quad \forall p_h, q_h \in Q_h,
\]
and introduce the corresponding seminorm $|\cdot|_j$ with
\[
|q_h|_j^2 := j_h(q_h, q_h), \quad \forall q_h \in Q_h.
\]
Then we show the following properties of $j_h(\cdot, \cdot)$, which are instrumental in the error estimation.

**Lemma 3.** There holds
\[
\|q_h\|_{L^2(\Omega_h^0)}^2 + j_h(q_h, q_h) \leq C\|q_h\|_{L^2(\Omega_h)}^2 \leq C(\|q_h\|_{L^2(\Omega_h^0)}^2 + j_h(q_h, q_h)), \quad \forall q_h \in Q_h^m.
\]

**Proof.** Applying the triangle inequality immediately gives $j_h(q_h, q_h) \leq C\|q_h\|_{L^2(\Omega_h)}^2$. By Lemma 1, for any $K \in \mathcal{T}_h$, we obtain that
\[
\|q_h\|_{L^2(K)} \leq C(\|q_h - E_K \cdot q_h\|_{L^2(K)} + \|E_K \cdot q_h\|_{L^2(K)}) \leq C(\|q_h - E_K \cdot q_h\|_{L^2(K)} + \|q_h\|_{L^2(K)}).
\]

Summation over all elements in $\mathcal{T}_h$ immediately gives the desired estimate, which completes the proof. \hfill \Box
Lemma 4. There holds

\[ |v|_j \leq Ch^s \|v\|_{H^{s}(\Omega^*)}, \quad \forall v \in H^l(\Omega^*), \]

where \( s = \min(t, m), t \geq 1. \)

Proof. For any element \( K \in T_h^\Gamma \), there exists \( p \in \mathbb{P}_m(B(x_K, c \Delta h_K)) \) such that

\[ \|v-p\|_{L^2(K)} \leq Ch^s \|v\|_{H^{s}(B(x_K, c \Delta h_K))}. \]

Therefore, we have that

\[ \|v-E_{K^\circ}v\|_{L^2(K)} \leq \|v-p\|_{L^2(K)} + \|p-E_{K^\circ}v\|_{L^2(K)} = \|v-p\|_{L^2(K)} + \|E_{K^\circ}(p-v)\|_{L^2(K)}, \]

and from Lemma 1, we find that

\[ \|E_{K^\circ}(p-v)\|_{L^2(K)} \leq C\|p-\Pi_{K^\circ}v\|_{L^2(K)} \leq C(\|v-p\|_{L^2(K)} + \|v-\Pi_{K^\circ}v\|_{L^2(K)}). \]

Combing the approximation properties of \( p \) and \( \Pi_{K^\circ}v \), we arrive at the estimate (9). This completes the proof. \( \square \)

For the vector-valued unknown, we still define the discontinuous/continuous piecewise polynomial spaces on the partition \( T_h \),

\[ V_h^{r, d} := \{ v_h \in L^2(\Omega)^d \mid v_h|_K \in \mathbb{P}_m(K)^d, \forall K \in T_h \}, \quad V_h^{r, c} := V_h^{r, c} \cap H^1(\Omega), \]

and we let \( V_h^r \) be either \( V_h^{r, d} \) or \( V_h^{r, c} \). For the space \( V_h := V_h^r + H^2(\Omega)^d \), we define the bilinear form \( s_h(\cdot, \cdot) \) as

\[ (g_h(u_h, v_h) := \sum_{K \in T_h^\Gamma} \int_K (\nabla \times (u_h - E_{K^\circ}u_h)) \cdot (\nabla \times (v_h - E_{K^\circ}v_h))dx, \quad \forall u_h, v_h \in V_h, \]

and define the corresponding seminorm \( |\cdot|_g \) as

\[ |v_h|_g^2 := g_h(v_h, v_h), \quad \forall v_h \in V_h. \]

We also give the following properties of \( g_h(\cdot, \cdot) \).

Lemma 5. There holds

\[ C \left( \sum_{K \in T_h} \|\nabla \times v_h\|_{L^2(K)}^2 + |v_h|_g^2 \right) \leq \sum_{K \in T_h} \|\nabla \times v_h\|_{L^2(K)}^2 \]

\[ \leq C \left( \sum_{K \in T_h} \|\nabla \times v_h\|_{L^2(K)}^2 + |v_h|_g^2 \right), \quad \forall v_h \in V_h. \]

Lemma 6. There holds

\[ |v|_g \leq Ch^s \|v\|_{H^{s+1}(\Omega^*)}, \quad \forall v \in H^l(\Omega^*), \]

where \( s = \min(t - 1, r), t \geq 2. \)

Lemma 5 and Lemma 6 also follow from the triangle inequality and Lemma 2; see proofs of Lemma 3 and Lemma 4.

We end this section by giving some basic results for unfitted methods. The first is the trace estimate on the curve [24, 28]:

Lemma 7. There exists a constant \( h_0 \), independent of \( h \), such that for any \( h \leq h_0 \), there holds

\[ \|w\|_{L^2(\Gamma_K)} \leq C \left( h_K^{-1} \|w\|_{L^2(K)}^2 + h_K \|w\|_{H^1(K)}^2 \right), \quad \forall w \in H^1(K), \quad \forall K \in T_h^\Gamma. \]

Hereafter, the condition \( h \leq h_0 \) is assumed to be always fulfilled. In the error estimation, we require the Sobolev extension theory [1]: there exists an extension operator \( E^* : H^s(\Omega) \to H^s(\Omega^*)(s \geq 0) \) such that

\[ (E^*w)|_\Omega = w, \quad \|E^*w\|_{H^q(\Omega^*)} \leq C\|w\|_{H^s(\Omega)}, \quad 0 \leq q \leq s, \quad \forall w \in H^s(\Omega). \]
3. Numerical Scheme to the Maxwell Problem in Smooth Domain

In this section, we are concerned with the time-harmonic Maxwell problem defined on the smooth domain \( \Omega \), which seeks the vector field \( \mathbf{u} \) and the scalar unknown (Lagrangian multiplier) \( p \) such that

\[
\nabla \times (\mu_r^{-1} \nabla \times \mathbf{u}) - k^2 \varepsilon_r \mathbf{u} - \varepsilon_r \nabla p = \mathbf{j}, \quad \text{in } \Omega,
\]

\[
\nabla \cdot (\varepsilon_r \mathbf{u}) = 0, \quad \text{in } \Omega,
\]

\[
p = 0, \quad \mathbf{n} \times \mathbf{u} = \mathbf{g}, \quad \text{on } \Gamma,
\]

where \( \mathbf{j} \) is the external source field and \( \mathbf{g} \) is the prescribed tangential trace. The magnetic permeability \( \mu_r \) and the electric permittivity \( \varepsilon_r \) are assumed to be \( C^2 \)-smooth and satisfy

\[
0 < \mu_* \leq \mu(\mathbf{x}) \leq \mu^*, \quad 0 < \varepsilon_* \leq \varepsilon(\mathbf{x}) \leq \varepsilon^*, \quad \forall \mathbf{x} \in \overline{\Omega}.
\]

We first show the well-posedness of the problem (15).

**Theorem 1.** Assume that \( k \) is not a Maxwell eigenvalue, then for \( \mathbf{j} \in L^2(\Omega)^d \) and the boundary data \( \mathbf{g} = 0 \), the problem (15) admits a unique solution \( (\mathbf{u}, p) \in H^2(\Omega)^d \times H^1(\Omega) \) such that

\[
\|\mathbf{u}\|_{H^2(\Omega)^d} \leq C_{\text{reg}} \|\mathbf{j}\|_{L^2(\Omega)^d}, \quad \|p\|_{H^1(\Omega)} \leq C\|\mathbf{j}\|_{L^2(\Omega)^d},
\]

where \( C_{\text{reg}} \) depends on \( k \).

**Proof.** From the Helmholtz decomposition [18], \( \mathbf{j} \in L^2(\Omega)^d \) can be decomposed as \( \mathbf{j} = \mathbf{J} + \nabla \mathbf{q} \), where \( \mathbf{J} \) is a divergence-free field and \( \mathbf{q} \in H^1_0(\Omega) \) is the solution to the problem \( \Delta \mathbf{q} = \nabla \cdot \mathbf{j} \) in \( \Omega \) such that \( \|\mathbf{q}\|_{H^1(\Omega)} \leq C\|\mathbf{j}\|_{L^2(\Omega)^d} \). Thanks to the orthogonality of the Helmholtz decomposition, \( p \) is the solution of the elliptic problem

\[
-\nabla \cdot (\varepsilon_r \nabla p) = -\Delta \mathbf{q}, \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma.
\]

By the condition (16), we know that \( p \in H^1_0(\Omega) \) with \( \|p\|_{H^1(\Omega)} \leq C\|\mathbf{q}\|_{H^1(\Omega)} \leq C\|\mathbf{j}\|_{L^2(\Omega)^d} \). Further, \( \mathbf{u} \) is the solution of the system

\[
\nabla \times (\mu_r^{-1} \nabla \times \mathbf{u}) - k^2 \varepsilon_r \mathbf{u} = \mathbf{J}, \quad \text{in } \Omega,
\]

\[
\nabla \cdot (\varepsilon_r \mathbf{u}) = 0, \quad \text{in } \Omega,
\]

\[
\mathbf{n} \times \mathbf{u} = \mathbf{g}, \quad \text{on } \Gamma.
\]

From [42, Proposition 1] and [18, Section 3], one has that \( \mathbf{u} \in H^1_0(\text{curl}, \Omega) \) with \( \|\mathbf{u}\|_{H^1(\text{curl},\Omega)} \leq C(k)\|\mathbf{j}\|_{L^2(\Omega)^d} \). In addition, the condition \( \nabla \cdot (\varepsilon_r \mathbf{u}) = 0 \) brings that \( \varepsilon \nabla \cdot \mathbf{u} = -\mathbf{u} \cdot \nabla \varepsilon \), together with (16), which implies \( \nabla \cdot \mathbf{u} \in H^1(\Omega) \). From [2, Corollary 2.15], we conclude that \( \mathbf{u} \in H^2(\Omega) \) with the estimate \( \|\mathbf{u}\|_{H^2(\Omega)^d} \leq C(k)\|\mathbf{j}\|_{L^2(\Omega)^d} \). The above results give the estimate (17) and complete the proof. \( \square \)

**Remark 1.** The regularity result in Theorem 1 may be invalid for the polygonal (polyhedral) domain because the embedding \( H^1_0(\text{curl}, \Omega) \cap H^1(\text{div}, \Omega) \hookrightarrow H^2(\Omega) \) requires the domain \( \Omega \) to be of class \( C^2 \) [2, 18].

**Remark 2.** Throughout this paper, the constants \( C \) and \( C_i \) are independent of \( k \) unless otherwise stated.

In this section, we propose a mixed numerical scheme for the Maxwell problem (15), which reads: seek \( (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h^m \) \((r \geq m + 1, m \geq 0)\) such that

\[
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - k^2(\mathbf{u}_h, \mathbf{v}_h) + g_h(\mathbf{u}_h, \mathbf{v}_h) = l_h(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h,
\]

\[
b_h(\mathbf{u}_h, q_h) - c_h(p_h, q_h) - j_h(p_h, q_h) = 0, \quad \forall q_h \in Q_h^m.
\]

The bilinear form \( a_h(\cdot, \cdot) \) is defined as

\[
a_h(\mathbf{u}_h, \mathbf{v}_h) := a_{h,0}(\mathbf{u}_h, \mathbf{v}_h) + a_{h,1}(\mathbf{u}_h, \mathbf{v}_h),
\]
Our error estimation is established in the case that \( \forall \epsilon \in [0,1] \). Theorem 2. Let \( h \leq h_F \) and let \( a_h(\cdot,\cdot) \) be defined with a sufficiently large \( \alpha \). The linear form \( l_h(\cdot) \) is defined as

\[
\begin{align*}
l_h(v_h) &:= \sum_{K \in T_h} \int_{K^0} j \cdot v_h \, \mathrm{d}x - \sum_{K \in T_h} \int_{\Gamma_K} (\mu^{-1} \nabla \times v_h) \cdot (\mu^{-1} \nabla \times v_h) \, \mathrm{d}s, \\
norm{u - u_h}_A + \norm{p - p_h}_C &\leq C_0 h^s (\norm{u}_{H^{s+1}(\Omega)} + \norm{p}_{H^s(\Omega)}), \\
\norm{u - u_h}_{L^2(\Omega)} &\leq C_1 h^{s+1} (\norm{u}_{H^{s+1}(\Omega)} + \norm{p}_{H^s(\Omega)}),
\end{align*}
\]

where

\[
a_{h,0}(u_h, v_h) := \sum_{K \in T_h} \int_{K^0} \mu^{-1}_r \nabla \times u_h \cdot (\nabla \times v_h) \, \mathrm{d}x + \sum_{K \in T_h} \int_{K^0} \nabla \cdot (\varepsilon_r u_h) \nabla \cdot (\varepsilon_r v_h) \, \mathrm{d}x
\]

\[
- \sum_{f \in F_h^\Gamma} \int_f \left( \{ \mu^{-1}_r \nabla \times u_h \} \cdot [n \times v_h] + \{ \mu^{-1}_r \nabla \times v_h \} \cdot [n \times u_h] \right) \, ds
\]

\[
- \sum_{K \in T_h} \int_{\Gamma_K} \left( \{ \mu^{-1}_r \nabla \times u_h \} \cdot [n \times v_h] + \{ \mu^{-1}_r \nabla \times v_h \} \cdot [n \times u_h] \right) \, ds,
\]

\[
a_{h,1}(u_h, v_h) := \sum_{f \in F_h^\Gamma} \int_f h_f^{-1} \left[ n \cdot (\varepsilon_r v_h) \right] \cdot [n \cdot (\varepsilon_r v_h)] \, ds
\]

\[
+ \sum_{K \in T_h} \int_{\Gamma_K} \left( \alpha h^{-1}_F [n \times u_h] \cdot [n \times v_h] \right) \, ds.
\]

for \( \forall u_h, v_h \in V_h \), where \( \alpha > 0 \) is the penalty parameter that will be specified later. The bilinear form \( b_h(\cdot,\cdot) \) is defined as

\[
b_h(v_h, p_h) := \sum_{K \in T_h} \int_{K^0} \nabla \cdot (\varepsilon_r v_h) p_h \, \mathrm{d}x - \sum_{f \in F_h^\Gamma} \int_f [n \cdot (\varepsilon_r v_h)] \{ p_h \} \, ds,
\]

for \( \forall v_h \in V_h, \forall p_h \in Q_h \). The bilinear form \( c_h(\cdot,\cdot) \) is defined as

\[
c_h(p_h, q_h) := \sum_{f \in F_h^\Gamma} \int_f h_f^{-1} [p_h] \cdot [q_h] \, ds + \sum_{K \in T_h} \int_{\Gamma_K} h_K[p_h] : [q_h] \, ds,
\]

for \( \forall p_h, q_h \in Q_h \). The forms \( a_{h,0}(\cdot,\cdot) \) and \( a_{h,1}(\cdot,\cdot) \) are defined to weakly impose the continuity condition and the boundary condition of the exact solution.

Remark 3. Our error estimation is established in the case that \( V_h^r = V_{h}^{r,d} \) and \( Q_h^m = Q_h^{m,d} \). We note that the numerical scheme also allows \( V_h^r = V_h^{r,c} \) and \( Q_h^m = Q_h^{m,c} \). For this case, the forms \( a_{h,0}(\cdot,\cdot), b_{h,0}(\cdot,\cdot), c_{h,0}(\cdot,\cdot) \) can be further simplified since \( [n \times v_h]_{F_h^r} = 0, [n \cdot (\varepsilon_r v_h)]_{F_h^r} = 0 \) for \( \forall v_h \in V_h \), and \( [q_h]_{F_h^r} = 0 \) for \( \forall q_h \in Q_h^m \).

We first present the error estimates of the numerical solution to (18).

Theorem 2. Let \( (u, p) \in H^{t+1}(\Omega) \times H^1(\Omega) \) be the exact solution to the problem (15), and let \( a_h(\cdot,\cdot) \) be defined with a sufficiently large \( \alpha \), and let \( (u_h, p_h) \in V_h^r \times Q_h^m \) be the numerical solution, then there exists a constant \( h_1 \) depending on \( C_{\text{reg}} \) such that for any \( h \leq h_1 \), there hold

\[
\|u - u_h\|_A + \|p - p_h\|_C \leq C_0 h^s (\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)}),
\]

\[
\|u - u_h\|_{L^2(\Omega)} \leq C_1 h^{s+1} (\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^s(\Omega)}),
\]

where \( s = \min(m + 1, t) \) and \( C_1 \) depends on \( C_{\text{reg}} \). The definitions of energy norms \( \| \cdot \|_A \) and \( \| \cdot \|_C \) are given in the next section.

The uniqueness of the problem (18) directly follows from the results in Theorem 2.

Corollary 1. Let \( a_h(\cdot,\cdot) \) be defined with a sufficiently large \( \alpha \), the mixed formulation (18) admits a unique solution provided \( h \leq h_1 \).
For the error estimation, we write (18) into the following equivalent formulation: seek \((u_h, p_h) \in V_h \times Q_h^m\) such that

\[
A_h(u_h, p_h; v_h, q_h) + B_h(v_h, q_h; p_h) = 0, \quad \forall q_h \in Q_h^m,
\]

where

\[
A_h(u_h, p_h; v_h, q_h) := a_h(u_h, v_h) + c_h(p_h, q_h),
\]

\[
B_h(v_h, q_h; p_h) := b_h(v_h, p_h) - c_h(p_h, q_h) - j_h(p_h, q_h).
\]

4. Error Estimate for the Mixed Formulation

In this section, we derive the error estimates by analysing the problem (23). We begin by introducing energy norms: for the space \(V_h\), we define the seminorm

\[
|v_h|_a^2 := \sum_{K \in T_h} ||\nabla \times v_h||^2_{L^2(K)} + \sum_{K \in T_h} ||\nabla \cdot (\varepsilon_r v_h)||^2_{L^2(K^0)} + \sum_{J \in F_h} h_f^{-1} [n \times v_h]^2_{L^2(f^0)}
\]

and norms:

\[
||v_h||_a^2 := |v_h|_a^2 + ||v_h||_{a, v_h}^2 + ||v_h||_{a, q_h}^2,
\]

\[
||v_h||_{\tilde{A}}^2 := ||v_h||_a^2 + \sum_{f \in F_h} h_f ||\nabla \times v_h||^2_{L^2(f^0)} + \sum_{K \in T_h} h_K ||\nabla \times v_h||^2_{L^2(\Gamma_K)},
\]

for \(\forall v_h \in V_h\). For the space \(Q_h\), we define the seminorm

\[
|q_h|_c^2 := \sum_{f \in F_h} h_f ||q_h||^2_{L^2(f^0)} + \sum_{K \in T_h} h_K ||q_h||^2_{L^2(\Gamma_K)} + ||q_h||_{f, \tilde{C}}^2,
\]

and norms

\[
||q_h||_c^2 := |q_h|_c^2 + ||q_h||_{c, v_h}^2, \quad ||q_h||_{\tilde{C}}^2 := ||q_h||_c^2 + \sum_{f \in F_h} h_f ||q_h||^2_{L^2(f^0)},
\]

for \(\forall q_h \in Q_h\). For \(\forall (v_h, q_h) \in V_h^r \times Q_h^m\), we define

\[
\|(v_h, q_h)||_{\tilde{W}}^2 := ||v_h||_{\tilde{A}}^2 + ||q_h||_{\tilde{C}}^2.
\]

From Lemma 5 and Lemma 3, we state the following equivalence results of above energy norms.

Lemma 8. There holds

\[
||v_h||_a \leq ||v_h||_{\tilde{A}} \leq C ||v_h||_a, \quad \forall v_h \in V_h^r.
\]

Proof. From the standard trace estimate, the estimate (13), and Lemma 5, we derive that

\[
\sum_{f \in F_h} h_f ||\nabla \times v_h||^2_{L^2(f^0)} \leq \sum_{f \in F_h} h_f ||\nabla \times v_h||^2_{L^2(f)} \leq C \sum_{K \in T_h} ||\nabla \times v_h||^2_{L^2(K)} \leq C ||v_h||_a^2,
\]

and

\[
\sum_{K \in T_h} h_K ||\nabla \times v_h||^2_{L^2(\Gamma_K)} \leq C \sum_{K \in T_h} ||\nabla \times v_h||^2_{L^2(K)} \leq C ||v_h||_a^2.
\]

The above estimates bring us the estimate (24). This completes the proof. \(\square\)
Lemma 9. There holds
\begin{equation}
\|q_h\|_c \leq \|q_h\|_C \leq C\|q_h\|_c, \quad \forall q_h \in Q^m_h.
\end{equation}

Proof. The proof is similar to the proof of Lemma 8. \hfill \Box

We show the equivalence between \( \| \cdot \|_a \) and \( \| \cdot \|_A \) restricted on the space \( V_h^\circ \), and also \( \| \cdot \|_C \) and \( \| \cdot \|_c \) are equivalent on the space \( Q^m_h \). The two norms \( \| \cdot \|_a \) and \( \| \cdot \|_c \) are more natural for the analysis when dealing with piecewise polynomial spaces \( V_h^\circ \) and \( Q^m_h \).

The derivation of the error estimation for the mixed formulation (23) is decoupled into several steps.

Inf-Sup Stability. We first prove an inf-sup stability for the bilinear form \( B_h(\cdot, \cdot) \), which is crucial for the mixed formulation. The proof requires some stability results of spaces defined on the interior domain \( \Omega_h^0 \). For the partition \( T_h \), we introduce the following piecewise polynomial spaces:
\begin{align*}
Q^m_{h,\circ} &= \{ q_h \in L^2(\Omega_h^0) \mid q_h \in P_m(K), \forall K \in T_h \}, \\
V^{r,\circ}_h &= \{ v_h \in L^2(\Omega_h^0)^d \mid v_h \in P_m(K)^d, \forall K \in T_h \},
\end{align*}

The inf-sup stability of our method is based on the following result:

Lemma 10. For \( m \geq 1 \), there holds
\begin{equation}
\sup_{v_h \in V^{r,\circ}_h \cap H^1(\Omega_h^0)^d} \int_{\Omega_h^0} \nabla \cdot (\varepsilon_r v_h) q_h \, dx \geq C \| q_h \|_{L^2(\Omega_h^0)}^2, \quad \forall q_h \in Q^m_{h,\circ}.
\end{equation}

The inf-sup condition for the divergence constraint on the interior domain \( \Omega_h^0 \) has been established in [21], and this estimate (26) is a modification of [21, Theorem 1]. The proof is included in Appendix A.

Now, we are ready to prove the inf-sup stability.

Theorem 3. There holds
\begin{equation}
\sup_{(\nu_h, r_h) \in V^{r,\circ}_h \times Q^m_h} B_h(\nu_h, r_h; q_h) \geq C \| q_h \|_C, \quad \forall q_h \in Q^m_h.
\end{equation}

Proof. We first prove the case \( m \geq 1 \). For fixed \( q_h \in Q^m_h \), from [30, Theorem 2.1], there exists \( q_{h,c} \in Q^{m,c}_{h} \) such that
\begin{equation}
\| q_h - q_{h,c} \|_{L^2(\Omega_h)}^2 \leq C \sum_{f \in F_h} h_f \| q_h \|_{L^2(f)}^2,
\end{equation}

and we let \( q_{h,\perp} := q_h - q_{h,c} \) and \( \tilde{q}_{h,c} := q_{h,c}|_{\Omega_h^c} \in Q^{m,c}_{h} \). From Lemma 10, there exists \( \tilde{v}_h \in V^{r,\circ}_h \cap H^1(\Omega_h^\circ) \) such that
\begin{equation}
\| \tilde{q}_{h,c} \|_{L^2(\Omega_h^\circ)} \| \tilde{v}_h \|_{H^1(\Omega_h^\circ)} \leq C \int_{\Omega_h^\circ} \nabla \cdot (\varepsilon_r \tilde{v}_h) \tilde{q}_{h,c} \, dx,
\end{equation}

and \( \tilde{v}_h \) is selected to satisfy \( \| \tilde{v}_h \|_{H^1(\Omega_h^\circ)} = \| \tilde{q}_{h,c} \|_{L^2(\Omega_h^\circ)} \). Since \( \tilde{v}_h \in H^1(\Omega_h^\circ) \), we extend it to the domain \( \Omega_h \) by zero and there holds
\begin{equation}
B_h(\tilde{v}_h, 0; q_{h,c}) = \int_{\Omega_h} \nabla \cdot (\varepsilon_r \tilde{v}_h) \tilde{q}_{h,c} \, dx \geq C \| \tilde{q}_{h,c} \|_{L^2(\Omega_h^\circ)}^2 = C \| q_{h,c} \|_{L^2(\Omega_h^\circ)}^2.
\end{equation}
Further, for any $\delta > 0$, we derive that
\[
B_h(\tilde{v}_h, 0; q_h) = \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot (\varepsilon_r \tilde{v}_h) q_h \, dx = -C\delta \|\tilde{v}_h\|_{H^1(\Omega_h)}^2 - \delta^{-1} \|q_h\|_{L^2(\Omega_h)}^2
\geq -C\delta \|q_h\|_{L^2(\Omega_h)}^2 - \delta^{-1} \|q_h\|_{L^2(\Omega_h)}^2.
\]
From (28), we have that
\[
B_h(0, -q_h; q_h) = \sum_{f \in \mathcal{F}_h} h_f \|q_h\|_{L^2(\Omega_f)}^2 + \|q_h\|_{L^2(\Omega_f)}^2 \geq C \|q_h\|_{L^2(\Omega_h)}^2 + \|q_h\|_{L^2(\Omega_f)}^2.
\]
We take $(v_h, r_h) := (\tilde{v}_h, -tq_h)$ with a parameter $t > 0$, we conclude that there exist constants $C_0, \ldots, C_3$ such that
\[
B_h(v_h, r_h; q_h) = B_h(v_h, 0; q_h) + B_h(0, -q_h; q_h)
\geq C_0 \|q_h\|_{L^2(\Omega_h)}^2 - C_2 \|q_h\|_{L^2(\Omega_f)}^2 - C_3 \delta^{-1} \|q_h\|_{L^2(\Omega_h)}^2 + t \|q_h\|_{L^2(\Omega_h)}^2 + t \|q_h\|_{L^2(\Omega_f)}^2.
\]
Selecting suitable parameters $\delta, t$ and by Lemma 3, we arrive at $B_h(v_h, r_h; q_h) \geq C \|q_h\|_{\mathcal{C}}^2 \geq C \|q_h\|_{\mathcal{C}}^2$. Moreover, we have that $\|v_h\|_{H^1(\Omega)} \leq \|v_h\|_{\mathcal{C}} + \|r_h\|_{\mathcal{C}} \leq C \|q_h\|_{L^2(\Omega_h)}^2 + C \|q_h\|_{L^2(\Omega_f)}^2 \leq C \|q_h\|_{\mathcal{C}}^2$. Collecting all above estimates gives the desired inf-sup estimate (27) for the case $m \geq 1$.

Then we prove the case $m = 0$. Since $\varepsilon_r$ is smooth, there exists $v \in H^1_0(\Omega)^d$ such that [37]
\[
\nabla \cdot (\varepsilon_r v) = q_h, \quad \text{in } \Omega, \quad v = 0, \quad \text{on } \Gamma,
\]
with $\|v\|_{H^1(\Omega)} \leq C \|q_h\|_{L^2(\Omega)}$. We extend $v$ by zero to the domain $\Omega_h$ and let $v_h \in V_h$ be its Scott-Zhang interpolant on the mesh $\mathcal{T}_h$. We have that
\[
B_h(v_h, 0; q_h) = \int_{\Omega} \nabla \cdot (\varepsilon_r v_h) q_h \, ds = \|q_h\|_{L^2(\Omega)}^2 + \int_{\Omega} \nabla \cdot (\varepsilon_r (v_h - v)) q_h \, dx.
\]
From the integration by parts and the approximation property of the interpolant, there holds
\[
\int_{\Omega} \nabla \cdot (\varepsilon_r (v_h - v)) q_h \, dx = \sum_{f \in \mathcal{F}_h} \int_{\partial f} (\varepsilon_r (v_h - v)) \cdot [q_h] \, ds + \sum_{K \in \mathcal{T}_h} \int_{\Gamma_K} (\varepsilon_r (v_h - v)) \cdot [q_h] \, ds
\geq -C_0 \|v\|_{H^1(\Omega)}^2 - C_1 \delta^{-1} \left( \sum_{f \in \mathcal{F}_h} h_f \|q_h\|_{L^2(\partial f)}^2 + \sum_{K \in \mathcal{T}_h} h_K \|q_h\|_{L^2(\Gamma_K)}^2 \right),
\]
for any $\delta > 0$. Similarly, we let $r_h := -tq_h$ and take suitable $\delta$ and $t$, we have that
\[
B_h(v_h, r_h; q_h) \geq C \|q_h\|_{\mathcal{C}}^2 \geq C \|q_h\|_{\mathcal{C}}^2.
\]
Clearly, one has that $\|v_h, r_h\|_{W} \leq C \|q_h\|_{\mathcal{C}}$. Combining all results leads to the inf-sup estimate (27), which completes the proof. \hfill \Box

**Continuity and Coercivity Properties.** Next, we show the continuity and the coercivity properties of the bilinear form $A_h(\cdot; \cdot)$ and $B_h(\cdot; \cdot)$.

**Lemma 11.** There hold
\[
|A_h(u_h, p_h; v_h, q_h)| \leq C \|\left(\!\begin{array}{c} u_h \\ p_h \end{array} \!\right)\| \|\left(\!\begin{array}{c} v_h \\ q_h \end{array} \!\right)\|, \quad \forall (u_h, p_h, (v_h, q_h)) \in V_h \times Q_h,
\]
and
\[
|B_h(v_h, r_h; q_h)| \leq C \|\left(\!\begin{array}{c} v_h \\ r_h \\ q_h \end{array} \!\right)\|, \quad \forall (v_h, r_h) \in V_h \times Q_h, \forall q_h \in Q_h.
\]

**Proof.** The proofs of the continuity results (30) and (31) are quite formal, directly following from the Cauchy-Schwarz inequality and the definitions (7), (10) and (19) - (21). \hfill \Box
Let us show the coercivity of the bilinear form $a_h(\cdot,\cdot)$.

**Lemma 12.** Let the bilinear form $a_h(\cdot,\cdot)$ be defined with a sufficiently large $\alpha > 0$, then there holds

\begin{equation}
 a_h(v_h, v_h) + g_h(v_h, v_h) \geq C |v_h|^2_{\alpha}, \quad \forall v_h \in V_h^r.
\end{equation}

**Proof.** From Lemma 5 and the trace estimate, there holds

\begin{equation}
 -2 \sum_{f \in F_h} \int_{\partial f} \{ \mu_r^{-1} \nabla \times v_h \} \cdot [n \times v_h] \, ds \\
 \geq - C_0 t \sum_{f \in F_h} h_f \| \{ \mu_r^{-1} \nabla \times v_h \} \|_{L^2(f)}^2 - C_1 t^{-1} \sum_{f \in F_h} h_f^{-1} \| [n \times v_h] \|_{L^2(f')}^2 \\
 \geq - C_0 t \sum_{K \in T_h} \| \nabla \times v_h \|_{L^2(K')}^2 + |v_h|_g^2 - C_1 t^{-1} \sum_{f \in F_h} h_f^{-1} \| [n \times v_h] \|_{L^2(f')}^2,
\end{equation}

for $\forall t > 0$. Similarly, from the trace estimate (13), we have that

\begin{equation}
 -2 \sum_{K \in T_h} \int_{\partial K} \{ \mu_r^{-1} \nabla \times v_h \} \cdot [n \times v_h] \, ds \\
 \geq - C_0 t \sum_{K \in T_h} \| \nabla \times v_h \|_{L^2(K')}^2 + |v_h|_g^2 - C_1 t^{-1} \sum_{K \in T_h} h_K^{-1} \| [n \times v_h] \|_{L^2(\Gamma_K)}^2,
\end{equation}

for $\forall t > 0$. Thus, we conclude that there exist constants $C_0, \ldots, C_3$ such that

$$
a_h(v_h, v_h) + g_h(v_h, v_h) \geq (C_0 - tC_1) \sum_{K \in T_h} \| \nabla \times v_h \|_{L^2(K')}^2 + \sum_{K \in T_h} \| \nabla \cdot (\varepsilon_r v_h) \|_{L^2(K')}^2 \\
+ (\alpha - C_2 t^{-1}) \sum_{f \in F_h} h_f^{-1} \| [n \times v_h] \|_{L^2(f')}^2 + \sum_{f \in F_h} \| [n \cdot (\varepsilon_r v_h)] \|_{L^2(f')}^2 \\
+ (\alpha - C_3 t^{-1}) \sum_{K \in T_h} h_K^{-1} \| [n \times v_h] \|_{L^2(\Gamma_K)}^2, \quad \forall t > 0.
$$

Taking a proper $t$ such that $C_0 - tC_1 > 0$ and choosing a sufficiently large $\alpha$, we obtain the estimate (32). This completes the proof. \qed

As a direct consequence of Lemma 12 and Lemma 8, we have that

\begin{equation}
 A_h(v_h, q_h; v_h, q_h) + g_h(v_h, v_h) \geq C (\| v_h \|_{A}^2 + |q_h|_A^2) - C k^2 \| v_h \|_{L^2(\Omega)}, \quad \forall (v_h, q_h) \in V_h \times Q^m.
\end{equation}

**Error estimates.** We are ready to derive the error bounds under both the energy norm and the $L^2$ norm for the problem (23). We first present the approximation results under energy norms. Here we assume the mesh size $h$ satisfies $hk \leq C$.

**Lemma 13.** For any $v \in H^{l+1}(\Omega)^d$ ($l \geq 1$), there exists $v_h \in V_h^r$ such that

\begin{equation}
 \| v - v_h \|_A \leq C \| v \|_{H^{l+1}(\Omega)}, \quad |v_h|_g \leq C \| v \|_{H^{l+1}(\Omega)},
\end{equation}

where $s = \min(t, r)$.

**Proof.** Let $v_h$ be the Lagrange interpolant of $E^*v$ into the space $V_h^r$, where $E^*$ is defined as (14). For any element $K \in T_h^r$, there exists $\tilde{v}_h \in P_r(B(x_{K^c}, \mathcal{C} \Delta h_{K^c}))^d$ such that

$$
 \| \nabla \times (E^*v - \tilde{v}_h) \|_{L^2(B(x_{K^c}, \mathcal{C} \Delta h_{K^c}))} \leq C \mathcal{H}^s_{K^c} \| E^*v \|_{H^{l+1}(B(x_{K^c}, \mathcal{C} \Delta h_{K^c}))}.
$$
The term $|v_h|^2$ can be bounded as in Lemma 6, i.e.

$$|v_h|^2 = \sum_{K \in T_h^k} \|\nabla \times v_h - \nabla \times E_K v_h\|^2_{L^2(K)}$$

$$\leq C \sum_{K \in T_h^k} (\|\nabla \times v_h - \nabla \times \tilde{v}_h\|^2_{L^2(K)} + \|\nabla \times \tilde{v}_h - \nabla \times E_K v_h\|^2_{L^2(K)})$$

$$\leq Ch^r \|E^s v\|_{H^{s+1}(\Omega^*)} + C \sum_{K \in T_h^k} \|\nabla \times \tilde{v}_h - \nabla \times v_h\|^2_{L^2(K^*_{\c}))}$$

$$\leq Ch^r \|E^s v\|_{H^{s+1}(\Omega^*)} \leq Ch^r \|v\|_{H^{s+1}(\Omega^*)}.$$

The approximation errors $\|v - v_h\|_a$ and $\|v - v_h\|_{L^2(\Omega)}$ can be bounded in a standard procedure, as the Sobolev extension operator (14), the approximation property of $v_h$ and the inequality $kh \leq C$. Consequently, the approximation estimate (34) is reached, which completes the proof.

**Lemma 14.** For any $q \in H^{s+1}(\Omega^*) (s \geq 0)$, there exists $q_h \in Q_h^m$ such that

$$\|q - q_h\|_{C} \leq Ch^s \|q\|_{H^{s+1}(\Omega)}, \quad |q_h|_j \leq Ch^s \|q\|_{H^{s+1}(\Omega)},$$

where $s = \min(t, m)$.

**Proof.** The proof is similar to that of Lemma 13. □

We then state the Galerkin orthogonality of $A_h(\cdot, \cdot)$ and $B_h(\cdot, \cdot)$.

**Lemma 15.** Let $(u, p) \in H^1(\Omega) \times H^1(\Omega)$ be the exact solution to the problem (15), and let $(u_h, p_h) \in V_h^r \times Q_h^m$ be the numerical solution, then there hold

$$A_h(u - u_h, p - p_h; v_h, q_h) + B_h(v_h, q_h; p - p_h) - k^2(u - u_h, v_h) = g_h(u_h, v_h), \quad \forall (v_h, q_h) \in V_h^r \times Q_h^m$$

$$B_h(u - u_h, p - p_h; q_h) = -j_h(p, q_h), \quad \forall q_h \in Q_h^m.$$

**Proof.** The equation (36) follows from the continuity condition of the exact solution and the definitions of $A_h(\cdot, \cdot)$ and $B_h(\cdot, \cdot)$. □

Based on Theorem 3, Lemma 11 and Lemma 12, we can prove the main results of Theorem 2, which are given in Theorems 4 and 5.

**Theorem 4.** Let $(u, p) \in H^{t+1}(\Omega) \times H^1(\Omega)(t \geq 1)$ be the exact solution to the problem (15), and let $a_h(\cdot, \cdot)$ be defined with a sufficiently large $\alpha$, and let $(u_h, p_h) \in V_h^r \times Q_h^m (r \geq m+1, m \geq 0)$ be the numerical solution, then there holds

$$\|u - u_h\|_A + \|p - p_h\|_C \leq Ch^s(\|u\|_{H^{s+1}(\Omega)} + \|p_h\|_{H^s(\Omega)}) + Ck\|u - u_h\|_{L^2(\Omega)},$$

where $s = \min(t, m+1)$.

**Proof.** We first take $(v_h, q_h) \in \text{Ker}(B_h) := \{(w_h, v_h) \in V_h^r \times Q_h^m | B_h(w_h, v_h; r_h) = 0, \forall r_h \in Q_h^m\}$. From the coercivity (33) and the orthogonality (36), we have that

$$C(\|v_h - u_h\|^2_A + \|q_h - p_h\|^2_C - k^2\|v_h - u_h\|^2_{L^2(\Omega)})$$

$$\leq A_h(v_h - u_h, q_h - p_h; v_h - u_h, q_h - p_h) + g_h(v_h - u_h, v_h - u_h)$$

$$\leq A_h(v_h - u_h, q_h - p_h; v_h - u_h, q_h - p_h) + B_h(v_h - u_h, q_h - p_h; p - r_h)$$

$$+ k^2(u - u_h, v_h - u_h) + g_h(v_h, v_h - u_h),$$

for $\forall r_h \in Q_h^m$. Applying the triangle inequality and the continuity results in Lemma 11 brings us that

$$\|v_h - u_h\|_A + \|q_h - p_h\|_C \leq C(\|(u - v_h, p - q_h)\|_W + \|p - r_h\|_C + k\|u - u_h\|_{L^2(\Omega)} + \|u\|_G).$$
Then we obtain
\begin{equation}
\|u - u_h\|_A + |p - p_h|_e \leq C \left( \inf_{(v_h, q_h) \in \text{Ker}(B_h)} \|u - v_h, p - q_h\|_W + k\|u - u_h\|_{L^2(\Omega)} + |u|_g \right).
\end{equation}

It remains to bound the error \( \inf_{(v_h, q_h) \in \text{Ker}(B_h)} \|u - v_h, p - q_h\|_W \). Fix \((v_h, q_h) \in \mathcal{V}_h \times Q_h^m\), and let \((w_h, r_h) \in \mathcal{V}_h \times Q_h^m\) be the solution of the problem
\begin{equation}
B_h(w_h, r_h; t_h) = B_h(u - v_h, p - q_h; t_h) + j_h(p, t_h), \quad \forall t_h \in Q_h^m.
\end{equation}
The inf-sup condition (27) ensures the existence of the solution \((w_h, r_h)\), which satisfies
\begin{equation}
\|w_h, r_h\|_W \leq C \sup_{t_h \in Q_h^m} \frac{B_h(u - v_h, p - q_h; t_h) + j_h(p, t_h)}{\|t_h\|_C} \leq C\left(\|u - v_h, p - q_h\|_W + |p|_j\right).
\end{equation}
Note that \(B_h(u, p; t_h) + j_h(p, t_h) = 0\), which implies \((w_h + v_h, r_h + q_h) \in \text{Ker}(B_h)\). Thus, we obtain
\begin{equation}
\|u - (w_h + v_h), p - (r_h + q_h)\|_W \leq C\left(\|u - v_h, p - p_h\|_W + |p|_j\right).
\end{equation}
Combining (38), we arrive at
\begin{equation}
\|u - u_h\|_A + |p - p_h|_e \leq C \left( \inf_{(v_h, q_h) \in \mathcal{V}_h \times Q_h^m} \|u - v_h, p - q_h\|_W + k\|u - u_h\|_{L^2(\Omega)} + |u|_g + |p|_j \right).
\end{equation}
Now we turn to the error \(\|p - p_h\|_C\). From the inf-sup stability (27), we obtain
\begin{equation}
\|p_h - q_h\|_C \leq C \sup_{(v_h, t_h) \in \mathcal{V}_h \times Q_h^m} \frac{B(v_h, t_h; q_h - p_h)}{\|v_h, t_h\|_W}, \quad \forall q_h \in Q_h^m.
\end{equation}
The Galerkin orthogonality (36) gives
\begin{equation}
B(v_h, t_h; q_h - p_h) = -A_h(u - u_h, p - p_h; v_h, t_h) - B_h(v_h, t_h; p - q_h) + k^2(u - u_h, v_h) + g_h(u_h, v_h)
\leq C\|v_h, t_h\|_W \left(\|u - u_h\|_A + |p - p_h|_e + C\|p_h - q_h\|_C \right) + C\|p_h - q_h\|_C \|v_h, t_h\|_W + k\left\|v_h, t_h\right\|_W \|u - u_h\|_{L^2(\Omega)} + g_h(u_h, v_h).
\end{equation}
For the last term, we have that
\begin{equation}
g_h(u_h, v_h) \leq (\|u - u_h\|_g + |u|_g)\|v_h\|_g \leq C(\|u - u_h\|_A + |u|_g)\|v_h, t_h\|_W.
\end{equation}
Combining the triangle inequality and all above estimates leads to
\begin{equation}
\|p - p_h\|_C \leq C\left(\|u - u_h\|_A + |p - p_h|_e + |u|_g + \inf_{q_h \in Q_h^m} \|p - q_h\|_C + k\|u - u_h\|_{L^2(\Omega)} \right).
\end{equation}
From (39) and the approximation estimates (34), (35) and (12), we immediately arrive at the desired estimate (37), which completes the proof. \(\square\)

**Theorem 5.** Under the conditions in Theorem 4, there holds
\begin{equation}
\|u - u_h\|_{L^2(\Omega)} \leq C_0 h(\|u - u_h\|_A + |p - p_h|_e) + C_1 h^{s+1}(\|u\|_{H^{s+1}(\Omega)} + |p|_{H^s(\Omega)}),
\end{equation}
where \(s = \min(t, m + 1)\) and the constants \(C_0, C_1\) depend on \(C_{\text{reg}}\).

**Proof.** We prove the \(L^2\) error estimate by the dual argument. Let \((z, \psi)\) be the solution of the problem
\begin{equation}
\nabla \times (\mu_r^{-1} \nabla \times z) - k^2 \varepsilon \varepsilon \nabla \psi = k^2(u - u_h), \quad \nabla \cdot (\varepsilon \varepsilon z) = 0, \quad \text{in } \Omega,
\end{equation}
\begin{equation}
\mathbf{n} \times z = 0, \quad \psi = 0, \quad \text{on } \Gamma.
\end{equation}
From Theorem 1, we know that \(z \in H^2(\Omega)\) and \(\psi \in H^1(\Omega)\) with
\begin{equation}
\|z\|_{H^2(\Omega)} \leq C_{\text{reg}}k^2\|u - u_h\|_{L^2(\Omega)}, \quad \|\psi\|_{H^1(\Omega)} \leq Ck^2\|u - u_h\|_{L^2(\Omega)}.
\end{equation}
Let \((z_h, \psi_h) \in V_h^r \times Q_h^m\) be their interpolation functions. Applying integration by parts and the Galerkin orthogonality (36), we see that

\[
k^2\|u - u_h\|_{L^2(\Omega)}^2 = A_h(z, \psi; u - u_h, p - p_h) + B_h(u - u_h, p - p_h; \psi) - k^2(u - u_h, z)
\]

\[
= A_h(u - u_h, p - p_h; z - z_h, \psi - \psi_h) - B_h(z_h, \psi_h; p - p_h) + g_h(u_h, z_h)
+ B_h(u - u_h, p - p_h; \psi - \psi_h) - k^2(u - u_h, z - z_h) - j_h(p, \psi_h).
\]

From Lemma 11 and the approximation property (34), there hold

\[
A_h(u - u_h, p - p_h; z - z_h, \psi - \psi_h) \leq Ch\|u - u_h, p - p_h\|_{W(\|z\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)}),}
\]

\[
B_h(u - u_h, p - p_h; \psi - \psi_h) \leq Ch\|u - u_h, p - p_h\|_{W(\|\psi\|_{H^1(\Omega)}).
\]

Note that \(B_h(z, \psi; p - p_h) = -j_h(\psi, p - p_h)\). We can further deduce that

\[
B_h(z_h, \psi_h; p - p_h) = B_h(z_h - z, \psi_h - \psi; p - p_h) - j_h(\psi, p - p_h)
\leq Ch(\|z\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)}))\|p - p_h\|_{C}.
\]

From the approximation estimates (34) and (35), we show that

\[
g_h(u_h, z_h) = g_h(u_h - u, z_h) + g_h(u, z_h) \leq C\|u - u_h\|_{A}|z_h|_{g} + |u|_{g}|z_h|_{g}
\leq Ch\|u - u_h\|_{A}|z|_{H^2(\Omega)} + Ch^{s+1}|u|_{H^s+1(\Omega)}|z|_{H^2(\Omega)},
\]

and

\[
j_h(p, \psi_h) \leq |p|_{j}|\psi_h|_{j} \leq Ch^{s+1}|p|_{H^s(\Omega)}|\psi|_{H^s(\Omega)}.
\]

Putting together all the above estimates yields the error estimate (41), which completes the proof. \(\square\)

Combining Theorem 4 and Theorem 5 leads to the desired error estimates in Theorem 2. Particularly, we point out that for the divergence-free source term \(j\), the solution \(p\) is just zero, and this case is also often encountered in practice [42]. For such a problem, \(p\) can be approximated by piecewise constant spaces, i.e. we can use the pair of spaces \(V_h^r \times Q_h^0(r \geq 1)\) in the numerical scheme. For the exact solution \((u, p)\) with \(u \in H^{s+1}(\Omega)^d\) and \(p = 0\), by estimates (39) and (40), the numerical solution \((u_h, p_h)\) from the discrete problem (18) has the following error estimates:

\[
\|u - u_h\|_{A} + \|p - p_h\|_{C} \leq C_0 h^s |u|_{H^{s+1}(\Omega)},
\]

\[
\|u - u_h\|_{L^2(\Omega)} \leq C_1 h^{s+1} |u|_{H^{s+1}(\Omega)},
\]

where \(C_1\) depends on \(C_{\text{reg}}\) and \(s = \min(t, r)\), provided by \(h \leq h_1\).

5. Numerical Results

In this section, a number of numerical tests in two and three dimensions are presented to show the numerical performance of the unfitted method. For all tests, the source term \(j\) and the boundary data \(g\) are taken accordingly from the exact solution \(u\). We assume that the coefficients \(\mu_r\) and \(\varepsilon_r\) correspond to the case of the vacuum, i.e. \(\mu_r = \varepsilon_r = 1\), and we also take \(k = 1\). The curved boundary \(\Gamma\) in each case is described by a level set function \(\phi\).

5.1. 2D Examples. The spaces \(V_h^r\) and \(Q_h^m\) for the two-dimensional case are selected to be the discontinuous piecewise polynomial spaces. The penalty parameter \(\alpha\) is taken as \(3(m + 1)^2 + 15\). We refer to [43, 13] for some discussions on the choice of the parameter.
Example 1. In the first example, we solve the Maxwell problem (15) defined in a circle centered at the origin with radius \( r = 0.7 \). The corresponding level set function \( \phi(x, y) \) is
\[
\phi(x, y) = x^2 + y^2 - r^2, \quad \forall (x, y) \in \mathbb{R}^2, \quad r = 0.7.
\]
The domain \( \Omega = \{(x, y) \in \mathbb{R}^2 \mid \phi(x, y) < 0\} \) and the background mesh \( T^*_h \) is taken to cover the squared domain \((-1, 1)^2\) (cf. Figure 1), with the mesh size \( h = 1/3, 1/6, 1/12, 1/24 \). The exact solution is chosen as
\[
\begin{align*}
\mathbf{u}(x, y) &= \begin{bmatrix} \cos(\pi x) \sin(\pi y) \\ -\sin(\pi x) \cos(\pi y) \end{bmatrix}, \\
p(x, y) &= x^2 + y^2 - r^2.
\end{align*}
\]

We solve this problem with the pair of spaces \( V^{m+1}_h \times Q^m_h \) (\( 0 \leq m \leq 2 \)). The numerical errors are reported in Table 1. It can be observed that for \( \mathbf{u} \) the errors under both the energy norm and the \( L^2 \) norm approach zero at the optimal convergence speed \( O(h^{m+1}) \) and \( O(h^{m+2}) \), respectively, and the error \( \| p - p_h \|_C \) also converges to zero with the optimal rate \( O(h^{m+1}) \). The numerical results are in perfect agreement with the theoretical estimate (22).

Table 1. The numerical errors of Example 1 in 2D.

| \( m \) | \( h \) | \( 1/3 \) | \( 1/6 \) | \( 1/12 \) | \( 1/24 \) | order |
|---|---|---|---|---|---|---|
| 1 | \( \| \mathbf{u} - \mathbf{u}_h \|_{L^2(\Omega)} \) | 1.528e-1 | 3.433e-2 | 7.307e-3 | 1.850e-3 | 1.99 |
| 1 | \( \| \mathbf{u} - \mathbf{u}_h \|_A \) | 1.476e-0 | 6.662e-1 | 3.158e-1 | 1.585e-1 | 0.99 |
| 1 | \( \| p - p_h \|_C \) | 1.553e-1 | 6.901e-2 | 2.741e-2 | 1.280e-2 | 1.10 |
| 2 | \( \| \mathbf{u} - \mathbf{u}_h \|_{L^2(\Omega)} \) | 3.402e-2 | 1.607e-3 | 1.871e-4 | 1.946e-5 | 3.26 |
| 2 | \( \| \mathbf{u} - \mathbf{u}_h \|_A \) | 4.067e-1 | 6.509e-1 | 1.491e-2 | 3.293e-3 | 2.18 |
| 3 | \( \| \mathbf{u} - \mathbf{u}_h \|_{L^2(\Omega)} \) | 3.553e-3 | 1.436e-4 | 6.665e-6 | 3.489e-7 | 4.26 |
| 3 | \( \| \mathbf{u} - \mathbf{u}_h \|_A \) | 7.159e-2 | 6.542e-3 | 6.096e-4 | 5.481e-5 | 3.48 |
| 3 | \( \| p - p_h \|_C \) | 2.061e-3 | 2.592e-4 | 2.595e-5 | 2.918e-6 | 3.15 |

Example 2. In this example, we consider the problem in a star-shaped domain [20] (see Figure 2), where the boundary \( \Gamma \) is governed by the following level set function in the polar coordinate \((r, \theta)\):
\[
\phi(r, \theta) = r - \frac{1}{2} - \frac{\sin(5\theta)}{7}.
\]
The problem is solved with a family of triangular meshes on the domain \( \Omega^* = (-1, 1)^2 \) with the mesh size \( h = 1/6, 1/12, 1/24, 1/48 \). The analytical solution is given by
\[
\begin{align*}
\mathbf{u}(x, y) &= \begin{bmatrix} e^x(y \cos(y) + \sin(y)) \\ e^x y \sin(y) \end{bmatrix}, \\
p(x, y) &= 0.
\end{align*}
\]
Since \( p = 0 \) in this case, the problem is approximated by the spaces \( V^{m+1}_h \times Q^0_h \) with the accuracy \( 0 \leq m \leq 2 \), as suggested in Section 4. The convergence history is shown in Table 2. The errors

![Figure 1](image-url)
\|u - u_{h}\|_{A} \text{ and } \|u - u_{h}\|_{L^2(\Omega)} \text{ decrease to zero at the optimal rates } O(h^{m+1}) \text{ and } O(h^{m+2}), \text{ respectively, which are well consistent with the estimate (42). Although } Q_{h}^{0} \text{ is the piecewise constant space, we still observe that the error } \|p - p_{h}\|_{C} \text{ tends to zero at the speed } O(h^{m+1}), \text{ which validates the estimate (42).}

**Table 2.** The numerical errors of Example 2 in 2D.

| m | \|u - u_{h}\|_{L^2(\Omega)} | \|u - u_{h}\|_{A} | \|p - p_{h}\|_{C} |
|---|-----------------|----------------|----------------|
| 1 | 1.069e-2 | 1.658e-1 | 1.665e-2 |
| 2 | 1.495e-4 | 4.942e-3 | 2.308e-4 |
| 3 | 4.120e-6 | 1.388e-4 | 1.763e-6 |

5.2. 3D Examples. The spaces \( V_{h}^{m+1} \) and \( Q_{h}^{m} \) in three dimensions are selected to be \( C^0 \) finite element spaces. The penalty parameter \( \alpha \) is taken as \( 3m^2 + 25 \).

**Example 3.** We first test a three-dimensional example by solving the problem defined in a sphere, which is centered at \((0.5, 0.5, 0.5)\) with radius \( r = 0.35 \), i.e. the corresponding level set function is

\[
\phi(x, y, z) = (x - 0.5)^2 + (y - 0.5)^2 + (z - 0.5)^2 - r^2, \quad r = 0.35.
\]

We employ a series of tetrahedral meshes on the cubic domain \((0, 1)^3\) to solve this problem; see Figure 3. The exact solution takes the form

\[
\begin{bmatrix}
\sin(\pi y) \sin(\pi z) \\
\sin(\pi x) \sin(\pi z) \\
\sin(\pi x) \sin(\pi y)
\end{bmatrix}, \quad p = (x - 0.5)^2 + (y - 0.5)^2 + (z - 0.5)^2 - 0.35^2.
\]

The numerical results are gathered in Table 3. One can observe that the numerical solution of the unfitted method still has the optimal rates under all error measurements in three dimensions, which are clearly in accordance with the theoretical estimates.

**Example 4.** In the last example, we consider the case that the boundary of the domain \( \Gamma \) is a smooth molecular surface of two atoms (see Figure 4), whose level set function reads [45, 32]:

\[
\phi(x, y, z) = (2.5(x - 0.5)^2 + 4(y - 0.5)^2 + (2.5(z - 0.5)^2 + 0.6)^2 - 3.5(4(y - 0.5))^2 - 0.6.
\]

\[
\|u - u_{h}\|_{A} \text{ and } \|u - u_{h}\|_{L^2(\Omega)} \text{ decrease to zero at the optimal rates } O(h^{m+1}) \text{ and } O(h^{m+2}), \text{ respectively, which are well consistent with the estimate (42). Although } Q_{h}^{0} \text{ is the piecewise constant space, we still observe that the error } \|p - p_{h}\|_{C} \text{ tends to zero at the speed } O(h^{m+1}), \text{ which validates the estimate (42).}
\]

**Figure 2.** The curved domain \( \Omega \) and the partition \( T_{h}^{*} \) of Example 2 in 2D.
We solve the problem (15) with the exact solution

\[
\mathbf{u} = \begin{bmatrix}
\cos(x) \sin(y) e^{2z} \\
\sin(x) \cos(y) e^{2z} \\
\sin(x) \sin(y) e^{2z}
\end{bmatrix}, \quad p = 0.
\]

For this case, we also adopt the approximation spaces \( V_h^{m+1} \times Q_h^0 (m = 0, 1) \) in the numerical scheme, and use a sequence of tetrahedral meshes with \( h = 1/4, 1/8, 1/16, 1/32 \). The results are collected in Table 4. The numerically detected convergence rates again confirm the theoretical predictions.

### 6. Conclusion

In this paper, we have developed an unfitted finite element method for the time-harmonic Maxwell equations on a smooth domain, based on a local extension operator and a ghost penalty technique. The unfitted mixed interior penalty scheme allows the curved boundary to intersect the background mesh arbitrarily, and is of optimal convergence rates under both the energy norm and the \( L^2 \) norm for all variables. A number of numerical results have confirmed the theoretical predictions.
The proof of Lemma 10 follows from the ideas in [21]. Define the scaled seminorms
\[ |q_h|_0^2 := \sum_{K \in T_h} h_K^2 \|\nabla q_h\|_{L^2(K)}^2, \quad \forall q_h \in Q_h^{n,o,c}. \]
\[ |q_h|_{e}^2 := \sum_{K \in T_h} h_K^2 \|\nabla q_h\|_{L^2(K)}^2 + \sum_{f \in F_h} h_f \|q_h\|_{L^2(f)}^2, \quad \forall q_h \in Q_h^m. \]

Let us first recall some existing results [21, Lemma 3 and Lemma 4]:

**Lemma 16.** For any \( q_h \in Q_h^{n,o,c} \), there exists \( \tilde{q}_h \in Q_h^m \) such that \( \tilde{q}_h|_{\Omega_h^0} = q_h \) and
\[
|\tilde{q}_h|_e \leq C|q_h|_0, \quad ||\tilde{q}_h||_{L^2(\Omega_h)} \leq C||q_h||_{L^2(\Omega_h^0)}.
\]

**Lemma 17.** For any \( \psi_h \in V_h^1 \cap H^1(\Omega_h)^d \), there exists a unique decomposition \( \psi_h = \psi_{h,1} + \psi_{h,2} \) such that \( \psi_{h,1} \in H^1_0(\Omega_h^0)^d, \) \( \text{supp}(\psi_{h,1}) = \Omega_h^0, \) \( \psi_{h,2} \in V_h^1 \cap H^1(\Omega_h)^d, \) and
\[
\sum_{K \in T_h} h_K^{-2} \|\psi_{h,2}\|_{L^2(K)}^2 \leq \sum_{K \in T_h} h_K^{-2} \|\psi_{h,1}\|_{L^2(K)}^2.
\]

We state the following inf-sup stability property:

**Lemma 18.** For \( m \geq 1 \), there holds
\[
\sup_{\psi_h \in V_h^{n,o,c} \cap H^1(\Omega_h)^d} \frac{\int_{\Omega_h^0} \nabla \cdot (\varepsilon_r \psi_h) q_h \, dx}{\|\psi_h\|_{H^1(\Omega_h^0)}} \geq C|q_h|_0, \quad \forall q_h \in Q_h^{n,o,c}.
\]

**Proof.** This result is a modification of [21, Assumption 3]. We assume that every element \( K \in T_h^o \) has an interior vertex. We denote by \( E_h^{0,i} \) the set of all interior edges in \( T_h^o \) (\( E_h^{0,i} = F_h^{0,i} \) in two dimensions). Let \( x_e \) be the midpoint of the edge \( e \). Then let \( \phi_e \) be the Lagrange basis function of the second-order \( C^0 \) space corresponding to the point \( x_e \). For any \( e \in E_h^{0,i} \), we have that \( \phi_e \in H^1_0(\Omega_h^0) \) and \( \phi_e(x_e) = 1 \) and \( \phi_e(x) \geq 0 (\forall x \in \Omega_h^0) \). We define \( \psi_h = -\sum_{e \in E_h^{0,i}} h_e^2 \phi_e(x)(t_e \cdot \nabla q_h(x))t_e \), where \( t_e \) is the unit tangential vector on \( e \). Since \( q_h \) is continuous on the domain \( \Omega_h^0 \), there holds \( \psi_h \in H^1_0(\Omega_h^0)^d \). Thus, applying the integration by parts yields that
\[
\int_{\Omega_h^0} \nabla \cdot (\varepsilon_r \psi_h) q_h \, dx = \sum_{e \in E_h^{0,i}} h_e^2 \int_{w(e)} \varepsilon_r \phi_e |t_e \cdot \nabla q_h|^2 \, dx \geq C \sum_{e \in E_h^{0,i}} h_e^2 \int_{w(e)} |\phi_e| |t_e \cdot \nabla q_h|^2 \, dx,
\]
where \( w(e) \) is the set of elements that have the edge \( e \). From the inverse inequality, we have that
\[
\sum_{e \in E_h^{0,i}} h_e^2 \int_{w(e)} |\phi_e| |t_e \cdot \nabla q_h|^2 \, dx \geq C \sum_{K \in T_h^o} h_K^2 \|\nabla q_h\|_{L^2(K)}^2 = C|q_h|_0^2.
\]
In addition, it is trivial to check $\|v_h\|_{H^1(\Omega_h^m)} \leq C|q_h|_o$, which gives the estimate (44). □

Now let us verify the estimate (26). Given $q_h \in Q_h^{m, \omega, c}$, we let $\tilde{q}_h \in Q_h^m$ be the corresponding piecewise polynomial function in Lemma 16. As (29), we let $v \in H^1_0(\Omega)$ be the solution to $\nabla \cdot (\varepsilon \cdot v) = \tilde{q}_h$ in $\Omega$, and extend $v$ to $\Omega_h$ by zero, and let $v_h \in V_h^\varepsilon$ be its Scott-Zhang interpolant. Decomposing $v_h = v_{h,1} + v_{h,2}$ as in Lemma 17, we further have that

$$\|\tilde{q}_h\|_{L^2(\Omega_h^m)} = (\nabla \cdot (\varepsilon \cdot v_{h,1}), \tilde{q}_h)_{L^2(\Omega_h^m)} + (\nabla \cdot (\varepsilon \cdot v_{h,2}), \tilde{q}_h)_{L^2(\Omega_h^m)} + (\nabla \cdot (\varepsilon \cdot (v - v_h)), \tilde{q}_h)_{L^2(\Omega_h^m)}.$$ 

Applying integration by parts, the approximation property of $v_h$, and (43), we get

$$(\nabla \cdot (\varepsilon \cdot (v - v_h)), \tilde{q}_h)_{L^2(\Omega_h^m)} \leq C\|q_h\|_{L^2(\Omega)} q_h|_o \leq C\|q_h\|_{L^2(\Omega_h^m)} q_h|_o.$$ 

Using the fact that $v_h = 0$ on $\partial \Omega_h$ and Lemma 17, we have that

$$(\nabla \cdot (\varepsilon \cdot v_{h,2}), \tilde{q}_h)_{L^2(\Omega_h^m)} = -(\varepsilon \cdot v_{h,2}, \nabla \tilde{q}_h)_{L^2(\Omega_h^m)} + \sum_{f \in T_h^k} \int_f (\varepsilon \cdot v_{h,2}) \cdot [\tilde{q}_h] ds \leq C\|q_h\|_{L^2(\Omega_h^m)} q_h|_o,$$

and by the trace estimate, we get that

$$\sum_{f \in T_h^k} \int_f (\varepsilon \cdot v_{h,2}) \cdot [\tilde{q}_h] ds \leq C\left( \sum_{K \in T_h} h_K^{-2}\|v_h\|_{L^2(K)}^2 + \|\nabla (\varepsilon \cdot v_{h,2})\|_{L^2(K)}^2 \right)^{1/2} q_h|_o \leq C\|v_h\|_{L^2(\Omega_h^m)} q_h|_o \leq C\|q_h\|_{L^2(\Omega_h^m)} q_h|_o.$$ 

Moreover, there hold $\|v_{h,1}\|_{H^1(\Omega_h^m)} \leq C\|v_h\|_{H^1(\Omega_h^m)} \leq C\|q_h\|_{L^2(\Omega_h^m)}$ and

$$(\nabla \cdot (\varepsilon \cdot v_{h,1}), \tilde{q}_h)_{L^2(\Omega_h^m)} \leq \|v_{h,1}\|_{H^1(\Omega_h^m)} \sup_{w_h \in V_h^{\varepsilon,c \cap H^1_0(\Omega_h^m)}} \int_{\Omega_h^m} \nabla \cdot (\varepsilon \cdot w_h) q_h dx / \|w_h\|_{H^1(\Omega_h^m)}.$$ 

Collecting all the above estimates implies that

$$\|q_h\|_{L^2(\Omega_h^m)} \leq C\left( \sup_{w_h \in V_h^{\varepsilon,c \cap H^1_0(\Omega_h^m)}} \int_{\Omega_h^m} \nabla \cdot (\varepsilon \cdot w_h) q_h dx / \|w_h\|_{H^1(\Omega_h^m)} + |q_h|_o \right),$$

which, together with Lemma 18, immediately yields the estimate (26). This completes the proof.

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