Rotated and scaled center of mass tomography for several particles

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The tomographic map of quantum state of a system with several degrees of freedom which depends on one random variable, analogous to center of mass considered in rotated and scaled reference frame in the phase space, is constructed. Time evolution equation of the tomogram for this map is given in explicit form. Properties of the map like the transition probabilities between the different states and relation to the star product formalism are elucidated. Example of multimode oscillator is considered in details. Symmetry properties in respect to identical particles permutations are discussed in the framework of proposed tomography scheme.

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I. INTRODUCTION

Recently\textsuperscript{1} a new formulation of quantum mechanics was suggested. This new formulation uses non-negatively defined probability distribution function to describe quantum states, called marginal distribution of tomogram. This function can be considered as an analog of known quasidistribution functions like nonnegative Husimi Q-function or Sudarshan-Glauber P-function. Tomographic approach was initially developed for one-mode systems, in this case the quantum state is described by the density matrix \( \rho(q,q’') \) or by symplectic tomogram \( w(X,\mu,\nu) \). Here density matrix is the function of 2 variables and tomogram is the function of 3 variables. Seeming overcompleteness of the tomographic description is balanced by the fact that quantum tomogram is a homogeneous function of 3 variables.

The state of the system with \( N \) degrees of freedom is described by density matrix \( \rho(q,q’’) \), the function of \( 2N \) variables. Then, what is the generalization of quantum tomogram, will it depend on \( 3N \) variables or on \( 2N + 1 \) variables? Such generalization was developed for the tomogram, depending on \( 3N \) variables (usual symplectic tomography) or by symplectic tomogram \( w(X,\mu,\nu) \). In this paper we propose the tomographic map with only one random variable (i.e., \( 2N + 1 \) variables totally) and discuss its properties in details.

Quantum tomography becomes popular nowadays for a number of reasons. It was developed not only for continuous variables like position but also for discrete spin variables. First, tomographic representation operates with the values, which are directly measured in experiments, for example non-classical and coherent states of light or matter optics experiments. Second, the tomogram is non-negative, and this must attract the attention of those who deal with computer simulations of quantum systems. Many problems in this field, e.g. for Path Integral Monte Carlo simulations, result from the use of the alternating-sign, or even complex, values to describe the quantum state (for example, sign problem in Fermi-systems simulations). The latter leads to difficulties with convergence which can be overcome by quantum tomography approach. These facts along with the tomography applications in quantum computations and entanglement (see, e.g., [3]), as well as in the theory of information and signal analysis [20], show that the development of convenient and simple tomographic map for the case of many particles (or, which is the same, many modes) is a task of great significance.

The paper is organized as follows. In Sec. II we present the definition of tomographic scheme with one random variable, elucidate some of its useful properties and discuss the physical meaning of the map. In Sec. III we derive the equations describing quantum evolution, stationary states, quantum transitions and rules for average values calculation for the proposed tomography map. Some examples of state description using the developed approach are given in Sec. IV and symmetry of the map with respect to particles permutations is discussed in Sec. V. The work is summarized in Sec. VI.

II. ONE RANDOM VARIABLE TOMOGRAPHY

A. Definition of the tomographic map

We begin with the one-dimensional (1D) case of a particle with continuous degree of freedom (in this paper we do not consider spin variables, but generalization of the formalism is straightforward). Quantum mechanics states that we know ‘everything’ about the system if we know density matrix. In practice, to obtain any information about the system we have to measure some quantities, for example coordinate \( q \) or momentum \( p \). It is also possible sometimes to measure an intermediate quantity, \( \mu q + \nu p \), where \( \mu, \nu \) are real parameters. Formally, this quantity (denote it \( X \)) is coordinate, measured in scaled and rotated reference frame in the phase space. It turns
out, that the distribution function of \( X \ (w(X, \mu, \nu)) \), measured for all sets of \( \mu, \nu \) gives complete quantum mechanical description of the system, in the sense that there is a unique correspondence between \( w(X, \mu, \nu) \) and density matrix (see, e.g. \ref{ref1} \ref{ref2} \ref{ref3} \ref{ref4} \ref{ref5}). Note that distribution function \( w(X, \mu, \nu) \) is equal to \( \langle \delta(X - \mu \hat{q} - \nu \hat{p}) \rangle \), where \( \langle \ldots \rangle \) is quantum mechanical average. Then there is, in principle, a possibility of complete experimental density matrix determination through the set of coordinate measurements.

When we deal with more than one particle and dimension we can consider individual \( X_j = \mu_j \hat{q}_j + \nu_j \hat{p}_j \) for every \( j \)-th degree of freedom. This results in the symplectic tomography representation \ref{ref1} \ref{ref1} \ref{ref1}. Here we are to show that it is enough to work with only one \( X = \sum_j X_j \). To do this, let us consider the generalization of \( w(X, \mu, \nu) = \langle \delta(X - \mu \hat{q} - \nu \hat{p}) \rangle \), where \( q, p \) and \( \mu, \nu \) becomes the vectors, their products become scalar products of vectors, while \( X \) remains a real number:

\[
w(X, \mu, \nu) = \left\langle \delta(X - \mu \hat{q} - \nu \hat{p}) \right\rangle \quad (1)
\]

Related problems were discussed in \ref{ref1} \ref{ref2}.

Throughout the paper designations are the following. We consider the system of \( N \) particles in \( d \) dimensions, the number of degrees of freedom is \( Nd \). Vectors are written as \( \vec{a} \), we use everywhere the vectors with \( Nd \) components, if the otherwise is not mentioned. Designation \( \vec{c} \) is used for the vector with all components equal to 1 (\( e_1 = 1 \)). Scalar product of vectors is designated \( a = \vec{b} \vec{c} \) (meaning \( a = \sum_j b_j c_j \), \( \vec{a} = \vec{b} \circ \vec{c} \) denotes the componentwise product of vectors (\( a_j = b_j c_j \)). The tomogram for usual symplectic scheme is designated as \( w_s(\vec{X}, \vec{\mu}, \vec{\nu}) \) (\( \vec{X}, \vec{\mu} \) and \( \vec{\nu} \) with \( Nd \) components each); the tomogram with one random variable is written as \( w(X, \mu, \nu) \). We also use Planck constant \( \hbar = 1 \) everywhere.

We can begin the construction of one-random-variable tomography representation from the known star product expressions. In the framework of star product formalism \ref{ref1} \ref{ref2} \ref{ref3} \ref{ref4} \ref{ref5}, every operator is replaced by the function (‘symbol’), depending on specific set of parameters \( y \), products of operators turn into ‘star products’ of corresponding symbols (star product problem was discussed also in Ref. \ref{ref2}). As a result, one deals with functions only, avoiding operators. For example, using a pair of operators \( \hat{D}(y), \hat{U}(y) \), we construct the connection between the symbols \( f_A(y) \) and operators \( \hat{A} \):

\[
f_A(y) = Tr(\hat{A}\hat{U}(y)),
\]

\[
\hat{A} = \int f_A(y)\hat{D}(y)dy,
\]

\[
\int Tr(\hat{D}(y)\hat{U}(y))dy = 1
\]

For \( y = \{X, \mu, \nu\} \) one can choose

\[
\hat{U}(y) = \delta \left( X - \mu \hat{q} - \nu \hat{p} \right),
\]

\[
\hat{D}(y) = (2\pi)^{-Nd} \exp \left[ i \left( X - \mu \hat{q} - \nu \hat{p} \right) \right],
\]

which defines the symbols (denote them \( w_A(X, \mu, \nu) \)) and star product:

\[
(w_A * w_B)(y) = \int w_A(y'')w_B(y')K(y'', y')dy'' dy'
\]

The kernel of star product \( K(y'', y', y) \) is expressed as follows:

\[
K(y'', y', y) = Tr[\hat{D}(y'')\hat{D}(y')\hat{U}(y)] = \int e^{-i(kX'' - X' - X)}\delta(\mu'' + \mu' - k\mu)\delta(\nu'' + \nu' - k\nu) \times e^{-i(\mu'' + \nu' - k\mu' - \nu) \int w(X, \mu, \nu)\delta_{D}(y, X, \mu, \nu)\delta_{D}(y', X, \mu, \nu)\delta_{D}(y'', X, \mu, \nu)}/(2\pi)^{Nd+1}
\]

For any operator \( \hat{A} \) we have \( \langle A \rangle = Tr(\hat{A}\hat{U}) \), therefore \( w_{\hat{A}} \)-symbol of density operator \( \hat{\rho} \) is the same as \( w(X, \mu, \nu) \) defined by Eq. \( (1) \). Density matrix in any representation is just a matrix element of density operator. Then, finally, we come to the unique correspondence (invertable map) between the tomogram \( w \) and density matrix mentioned above:

\[
w(X, \mu, \nu) = \left\langle \delta(X - \mu \hat{q} - \nu \hat{p}) \right\rangle,
\]

\[
\hat{\rho} = \int w(X, \mu, \nu)e^{i(X - \mu \hat{q} - \nu \hat{p})}dXd\mu d\nu
\]

Density matrix always can be reconstructed from the tomogram \( w \) using these equations, so one random variable tomogram describes quantum state completely. Note that now the state-describing function is nonnegative and depends on \( 2Nd + 1 \) variables in contrast to symplectic tomogram, depending on \( 3Nd \) variables.

B. Properties of \( w(X, \mu, \nu) \)

It is convenient to consider density matrix in coordinate representation and Wigner formulation of quantum mechanics \ref{ref1} to derive the properties and evolution equation for the tomogram \( w \). In the framework of Wigner formalism the state of the system is described by real Wigner function \( W(q, \mu) \) defined in phase space and connected with the density matrix as follows:

\[
W(q, \mu) = \int \rho(q' + \frac{\mu}{2} - \frac{\mu'}{2} - \frac{\mu''}{2})d\mu d\mu' d\mu''/(2\pi)^{Nd},
\]

\[
\rho(q', q'') = \int W(q' + \frac{\mu}{2} - \frac{\mu''}{2}, \mu)e^{i(\mu'q' - \mu''q'')}d\mu
\]

Using Eqs. \ref{eq:10} and Eqs. \ref{eq:11} we obtain:

\[
w(X, \mu, \nu) = \int W(q, \mu)e^{i(kX - \mu \hat{q} - \nu \hat{p})}dXd\mu d\nu/(2\pi)^{Nd},
\]

\[
W(q, \mu) = \int e^{-i(\mu \hat{q} + \nu \hat{p} - X)}w(X, \mu, \nu)dXd\mu d\nu/(2\pi)^{2Nd}
\]
Usual symplectic tomography map is developed in references [13] and [20]. The symplectic tomogram \( w_s(\vec{X}, \vec{\mu}, \vec{\nu}) \) and Wigner function are connected as follows:

\[
w_s(\vec{X}, \vec{\mu}, \vec{\nu}) = \int W(\vec{q}, \vec{p}) e^{-i k (\vec{X} - \vec{\mu} \cdot \vec{q} - \vec{\nu} \cdot \vec{p})} d^{2Nd}, \tag{15}
\]

\[
W(\vec{q}, \vec{p}) = \int e^{-i (\vec{q} \cdot \vec{p} - \vec{q} \cdot \vec{p})} w_s(\vec{X}, \vec{\mu}, \vec{\nu}) d\vec{X} d\vec{q} d\vec{p} \tag{16}
\]

Since the Wigner function is connected by invertible maps with both tomograms \( w \) and \( w_s \), it is obvious that they contain the same information about the quantum state. In fact one has

\[
w(X, \vec{\mu}, \vec{\nu}) = \int w_s(\vec{Y}, \vec{\mu}, \vec{\nu}) \delta(\vec{X} - \sum_{j=1}^{Nd} Y_j) d\vec{Y}, \tag{17}
\]

\[
w_s(\vec{X}, \vec{\mu}, \vec{\nu}) = \int w(Y, \vec{k} \cdot \vec{Y}, \vec{k} \cdot \vec{p}) e^{i(\vec{X} - \vec{k} \cdot \vec{Y})} d\vec{Y}. \tag{18}
\]

The Wigner function is normalized:

\[
\int W(\vec{q}, \vec{p}) d\vec{q} d\vec{p} = \int \rho(\vec{q} + \frac{i}{2} \vec{p}, \vec{q} - \frac{i}{2} \vec{p}) e^{-i \pi \vec{q} \cdot \vec{p}} d\vec{q} d\vec{p} = (2\pi)^{Nd} \tag{19}
\]

where we choose the normalization for density matrix \( Tr(\hat{\rho}) = 1 \). Then the tomogram \( w \) is normalized in \( X \) variable:

\[
\int w(X, \vec{\mu}, \vec{\nu}) dX = \int W(\vec{q}, \vec{p}) \delta(k)e^{i k (\vec{\mu} + \vec{\nu})} d\vec{q} d\vec{p} = (2\pi)^{Nd}. \tag{20}
\]

Although the tomogram depends on \( 2Nd + 1 \) variables, instead of \( 2Nd \) for density matrix, the completeness of physical description is the same for both formulations, due to the fact that the tomogram is a homogeneous function. Consider the definition [13] and multiply all variables in \( w \) by a real number \( \lambda \):

\[
w(\lambda X, \lambda \vec{\mu}, \lambda \vec{\nu}) = \int W(\vec{q}, \vec{p}) e^{-i \lambda k (X - \vec{\mu} \cdot \vec{q} - \vec{\nu} \cdot \vec{p})} d\vec{q} d\vec{p} = (2\pi)^{Nd} \tag{21}
\]

where we just made the change of variables \( \lambda k \to k \).

Property (21) make obvious the relations

\[
w(\lambda X, \vec{\mu}, \vec{\nu}) = |X|^{-1} w(1, \lambda \vec{\mu} / X, \lambda \vec{\nu} / X). \tag{22}
\]

For the pure state with wave function \( \Psi(\vec{q}) \) symplectic tomogram \( w_{ps} \) was expressed in terms of modulus squared of fractional Fourier transform of the wave function [13]. The tomogram \( w \) for pure state is given by

\[
|\Psi(\vec{q})|^2 \exp \left[ i \left( \frac{\vec{Y} \cdot \vec{q}}{\vec{\nu}} - \frac{\vec{q} \cdot \vec{Y}}{\vec{\mu}} \right) \right] d\vec{q}^2. \tag{23}
\]

### C. Physical meaning

We have defined the nonnegative function \( w(X, \vec{\mu}, \vec{\nu}) \) completely describing quantum state. For any set of \( \{ \vec{\mu}, \vec{\nu} \} \) it is normalized as a function of \( X \), therefore \( w(X, \vec{\mu}, \vec{\nu}) \) is the set of distribution functions of quantity \( X \). Then to know the quantum state completely one has to consider all sets of \( \{ \vec{\mu}, \vec{\nu} \} \) (in practice, moving with some step) and measure \( X = \vec{\mu} \cdot \vec{q} + \vec{\nu} \cdot \vec{p} \) many times for each set: this yields the distribution function \( w(X, \vec{\mu}, \vec{\nu}) \) for given set of \( \{ \vec{\mu}, \vec{\nu} \} \).

Looking at Eq. (22) we see that we even do not have to know \( w(X, \vec{\mu}, \vec{\nu}) \), the value of this function in some point in \( X \) for all \( \{ \vec{\mu}, \vec{\nu} \} \) is enough. This does not change the scheme of measurements, we still need to measure full distribution function of \( X \) for given \( \{ \vec{\mu}, \vec{\nu} \} \) (it is necessary to compare the values of distribution function in different points to be sure that statistical precision is good), but one has to store the smaller arrays of information.

Property (21) can be used in another way. If \( X \) is equal to \( \vec{\mu} \cdot \vec{q} + \vec{\nu} \cdot \vec{p} \), we can parameterize \( \{ \vec{\mu}, \vec{\nu} \} \) by \( \lambda \) and \( 2Nd - 1 \) angles (to use the spherical coordinates in the space of \( \{ \vec{\mu}, \vec{\nu} \} \)). Applying Eq. (21) we come to reduced tomogram with \( 2Nd \) variables and \( \{ \vec{\mu}, \vec{\nu} \} \) located on the sphere with radius equal unity in \( 2Nd \)-dimensional space. This new tomogram also completely describes the state and in some cases it can be convenient to use this one in measurements, because it is easier to sample \( 2Nd - 1 \) angle than \( 2Nd \)-dimensional space from \(-\infty \) to \( \infty \) (see, e.g., [20, 23]). On the other hand such formulation causes trouble with the derivation of evolution equations and arbitrary average values calculation.

The only remaining unclear point is the meaning of \( X = \vec{\mu} \cdot \vec{q} + \vec{\nu} \cdot \vec{p} \). It is the sum of positions measured in scaled and rotated reference frame in the phase space. But what does it mean physically? It is impossible to measure \( \vec{q} \) and \( \vec{p} \) simultaneously, but sometimes one can transform \( \vec{q} \) and \( \vec{p} \) into the form \( \vec{\mu} \cdot \vec{q} + \vec{\nu} \cdot \vec{p} \), for example mixing the signal beam with local oscillator field (in quantum optics, see [23] and references therein). Another scheme was proposed in [20], where \( \vec{q} \) and \( \vec{p} \) are mixed due to wave (electromagnetic or matter) propagation through a lens (or an analog of a lens in atomic optics). Taking into account the present development of science concerning controlling the Bose-condensates of atoms, this also can be a possible realm of tomography measurements. Bose-condensate is a coherent macroscopic state of many atoms and it is described by macroscopic wave function.

For example, one can mix two such waves (condensates of the same atoms), using the first as a signal wave and the second as local oscillator. Varying the phase difference of the condensates we sample different \( \vec{\mu}, \vec{\nu} \). Probably, the same can be done in superconductors (where the electrons of superconductivity also form the coherent macroscopic matter wave), using Josephson junctions.

If we somehow accomplished the scaling and rotation of reference frame in the phase space we can measure the set of positions in this reference frame \( X_j = \mu_j q_j + \)
\[ \mathbf{\nu} \mathbf{p} \] but it is enough to measure their sum, \( X = \mathbf{\mu} \mathbf{q} + \mathbf{\nu} \mathbf{p} \). It is analogous to the the position of center of mass measurement (the sum of coordinates of corresponding vector, to be more precise). Indeed, the center of mass position is

\[ X_{cm} = \sum_j m_j X_j / M = \sum_j m_j (\mathbf{\mu}_j q_j + \mathbf{\nu}_j p_j) / M, \tag{24} \]

where \( M = \sum_j m_j \) and \( m_j \) is the mass corresponding to \( j \)-th degree of freedom, and \( X_{cm} \) can be associated with \( X = \mathbf{\mu} \mathbf{q} + \mathbf{\nu} \mathbf{p} \) for some other set of \( \{\mathbf{\mu}, \mathbf{\nu}\} \). We sample all sets of \( \{\mathbf{\mu}, \mathbf{\nu}\} \) therefore it is enough to measure the center-of-mass position in each scaled and rotated reference frame.

Finally, we would like to make the following remark. The storage of arrays representing full density matrix or tomogram becomes impossible when the number of degrees of freedom growth. If we use some grid, the number of arrays elements is proportional to \( n^{Nd} \), where \( n \) is the number of grid steps. Increasing \( Nd \) we soon come to the situation when all data carriers in the world can not store corresponding arrays. And this is not necessary as the state of the system is uniquely determined by the one-particle density (through the density functional, see [11] and references therein). Then for many-particles systems description we can use reduced density matrices (one-body, two-body, etc.), and tomography map is constructed for them in the same way as for full density matrix. Then the situation with reference frame scaling and rotation is simplified because \( \mu \) and \( \nu \) are the same for all particles (if one-body density matrix is considered) and distribution functions are averaged over all particles.

### III. STATE TRANSFORMATIONS

#### A. Evolution equations

Let us discuss the evolution equation for tomogram \( w \). Begin with the most general evolution equation for density matrix:

\[ i \frac{\partial \rho(q', q'')}{\partial t} = [\hat{H}, \rho(q', q'')] \tag{25} \]

Here and throughout the paper we omit the dependence on time \( t \), but imply that all functions, describing the state (density matrix, Wigner function, tomogram) depend on time as parameter. We consider the Hamiltonian \( \hat{H} = \sum_i p_i^2 / (2m_i) + V(q) \). To derive the evolution equation for tomograms we consider the Moyal evolution equation for Wigner function [16, 13, 12, 11]:

\[ \frac{\partial W(q, p)}{\partial t} + \frac{\partial W(q, p)}{m \partial q'} + i \left[ V(q' + \frac{i}{2} \partial p') - V(q' - \frac{i}{2} \partial p') \right] W = 0, \tag{26} \]

where \( \mathbf{\vec{p}} / m \) means the vector with components \( p_i / m_i \) (the equation holds for the case of different masses for different particles and directions), the operators in the potential \( V \) designates the analytical expansion of the potential and use of the products of corresponding operators. This equation can be easily obtained applying the transform \( \{11\} \) to the Eq. \( \{25\} \).

To derive the evolution equation for tomogram one applies the transform \( \{13\} \) to evolution equation for Wigner function \( \{20\} \). Expanding the potential in Eq. \( \{20\} \), it is seen that we have to consider the transforms of the following quantities: \( qW, \partial W / \partial q', \mathbf{\vec{p}} W \) and \( \partial W / \partial \mathbf{\vec{p}} \). The transform \( \{13\} \) of \( qW \) is

\[ \int qW(q', \mathbf{\vec{p}}) \exp[-ik(X - \mathbf{\mu}q' - \mathbf{\nu}p')] \frac{dkdqdp}{(2\pi)^3} = \]

\[ -i \frac{\partial}{\partial \mathbf{\mu}} \int W(q, \mathbf{\vec{p}}) \exp[-ik(X - \mathbf{\mu}q - \mathbf{\nu}p')] \frac{dkdqdp}{(2\pi)^3} \tag{27} \]

Consider the operator \( (\partial / \partial X)^{-1} \), which gives the antiderivative of the function it works on. Then we have

\[ i e^{-ikX} = \frac{i}{k} \left( \frac{\partial}{\partial X} \right)^{-1} e^{-ikX} = \left( \frac{\partial}{\partial X} \right)^{-1} e^{-ikX}, \tag{28} \]

and Eq. \( \{27\} \) becomes

\[ \int qW(q', \mathbf{\vec{p}}) \exp[-ik(X - \mathbf{\mu}q' - \mathbf{\nu}p')] \frac{dkdqdp}{(2\pi)^3} = \]

\[ -\frac{\partial}{\partial \mathbf{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} w(X, \mathbf{\mu}, \mathbf{\nu}) \tag{29} \]

Using the same simple operations we obtain the rules of Eq. \( \{20\} \) terms transformation, which we formally designate as \( \rightarrow \mathbf{\rightarrow} \):

\[ qW(q, \mathbf{\vec{p}}) \rightarrow -\frac{\partial}{\partial \mathbf{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} w(X, \mathbf{\mu}, \mathbf{\nu}) \tag{30} \]

\[ \frac{\partial W(q, \mathbf{\vec{p}})}{\partial q'} \rightarrow \mathbf{\mu} \frac{\partial}{\partial X} w(X, \mathbf{\mu}, \mathbf{\nu}) \tag{31} \]

\[ \mathbf{\nu} W(q, \mathbf{\vec{p}}) \rightarrow -\frac{\partial}{\partial \mathbf{\nu}} \left( \frac{\partial}{\partial X} \right)^{-1} w(X, \mathbf{\mu}, \mathbf{\nu}) \tag{32} \]

\[ \frac{\partial W(q, \mathbf{\vec{p}})}{\partial \mathbf{\vec{p}}} \rightarrow \mathbf{\nu} \frac{\partial}{\partial X} w(X, \mathbf{\mu}, \mathbf{\nu}) \tag{33} \]

Successive application of rules \( \{30, 31, 32, 33\} \) allows us to transform all powers of \( q, \mathbf{\vec{p}} \) and corresponding derivatives in Eq. \( \{20\} \). As a result we obtain the evolution equation for one random variable quantum tomogram \( w \):

\[ \frac{\partial w}{\partial t} - \frac{\partial}{\partial \mathbf{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} V \left( -\frac{\partial}{\partial \mathbf{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} \frac{i}{2} \frac{\partial}{\partial \mathbf{\nu}} \right) - \]

\[ V \left( -\frac{\partial}{\partial \mathbf{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} - \frac{i}{2} \frac{\partial}{\partial \mathbf{\nu}} \right) \right] w = 0 \tag{34} \]
B. Stationary states

For the stationary state with definite energy we can turn from the time-dependent Schrödinger equation to the eigenvalue equation:

$$\hat{H}\tilde{\rho}_E = \tilde{\rho}_E \hat{H} = E\tilde{\rho}_E$$  \hspace{1cm} (35)

Applying the transform we obtain the following rules of transition from the equation for density matrix to equation for Wigner function:

$$\frac{\partial^2 \rho(q', q)}{\partial q'^2} \to \left( \frac{1}{4} \frac{\partial^2}{\partial q^2} - \frac{i\hbar}{2} \frac{\partial}{\partial q} \right) W(\tilde{q}, \tilde{p})$$

and recalling the connection of Wigner function with quantum states. Consider two states, designate them as $w_1$ and $w_2$. The probability of transition from state $a$ to state $b$ is $P_{ab} = Tr(\rho_a \rho_b) = \int \rho_a(\tilde{q}, \tilde{q}'') \rho_b(\tilde{q}'', \tilde{q}''') d\tilde{q} d\tilde{q}''$. In terms of the Wigner formalism this can be rewritten as

$$P_{ab} = (2\pi)^N \int F^W(a)(\tilde{q}, \tilde{p}) F^W(b)(\tilde{q}'', \tilde{p}'') d\tilde{q} d\tilde{q}'$$  \hspace{1cm} (38)

and recalling the connection of Wigner function with tomograms $w_1$ and $w_2$ we easily get the following expressions for $P_{ab}$ in tomography approach:

$$\int w_a(X, \mu, \nu) w_b(Y, -\nu, -\mu') e^{i(X+Y)} dX dY d\mu d\nu$$  \hspace{1cm} (39)

C. Quantum transitions

In general, there is a possibility of transition between the quantum states. Consider two states, designate them as $a$ and $b$. The probability of transition from state $a$ to state $b$ is $P_{ab} = Tr(\rho_a \rho_b) = \int \rho_a(\tilde{q}, \tilde{q}'') \rho_b(\tilde{q}'', \tilde{q}''') d\tilde{q} d\tilde{q}''$. In terms of the Wigner formalism this can be rewritten as

$$P_{ab} = (2\pi)^N \int F^W(a)(\tilde{q}, \tilde{p}) F^W(b)(\tilde{q}'', \tilde{p}'') d\tilde{q} d\tilde{q}''$$  \hspace{1cm} (38)

and recalling the connection of Wigner function with tomograms $w_1$ and $w_2$ we easily get the following expressions for $P_{ab}$ in tomography approach:

$$\int w_a(X, \mu, \nu) w_b(Y, -\nu, -\mu') e^{i(X+Y)} dX dY d\mu d\nu$$  \hspace{1cm} (39)

D. Tomographic map in temperature-dependent processes

The tomographic representation can be analogously introduced for the systems with temperature $T \neq 0$. In this case we consider 'imaginary time' $\beta = 1/T$ (measuring $T$ in units of energy). $\beta$ enters as a parameter in the density matrix, which is now defined by the equation

$$-\frac{\partial \rho(\tilde{q}', \tilde{q}'', \beta)}{\partial \beta} = \hat{H}_\beta \rho(\tilde{q}', \tilde{q}'', \beta),$$  \hspace{1cm} (40)

where index $\tilde{q}'$ in $\hat{H}_\beta$ shows that the Hamiltonian acts only on those variables.

Now the transition to the tomograms $w_1$ or $w_2$ is straightforward. We just use the same rules, as in the derivation of evolution equation and eigenvalue equation. Then the evolution equation in imaginary time $\beta$ for $w$ is given by

$$-\frac{\partial w}{\partial \beta} = \sum_{j=1}^{N\beta} \left[ \frac{1}{2m_j} \frac{\partial^2}{\partial \tilde{q}_j^2} \left( \frac{\partial}{\partial X} \right)^2 - \frac{1}{8m_j} \rho_j^2 \frac{\partial^2}{\partial X^2} \right] w +$$

$$Re \left( \frac{i}{2} \tilde{\beta} \frac{\partial}{\partial X} - \frac{\partial}{\partial \tilde{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} \right) w$$

$$Im \left( \frac{i}{2} \tilde{\beta} \frac{\partial}{\partial X} - \frac{\partial}{\partial \tilde{\mu}} \left( \frac{\partial}{\partial X} \right)^{-1} \right) w$$

(41)

Initial condition for Eq. is $\rho(\tilde{q}', \tilde{q}'', \beta = 0) = \delta(\tilde{q}' - \tilde{q}'')$. It corresponds to constant Wigner function (see Eq. [37]). Using Eq. [39] we see that the tomogram $w$ for $\beta = 0$ must have the delta-function form, equal zero everywhere, besides the point $\tilde{\mu}, \nu = 0$ and constant in $X$ direction in that point.

E. Average values calculation

Developing the 'center-of-mass' tomography formalism we must provide the rules of average values calculation to complete the picture. Using the density matrix to describe the state of the system we can obtain the average value of some operator $\hat{A}$ as

$$\langle A \rangle = Tr(\hat{\rho} \hat{A}),$$  \hspace{1cm} (42)

where we choose $Tr(\hat{\rho}) = 1$.

Here it is again convenient to begin with the Wigner-Moyal formulation of quantum mechanics. In its framework to calculate the average value one deals with the Weyl symbol $A^W(\tilde{q}, \tilde{p})$ of operator $A(\tilde{q}, \tilde{p})$ (see [43] for review):

$$\langle A \rangle = \int A^W(\tilde{q}, \tilde{p}) W(\tilde{q}, \tilde{p}) d\tilde{q} d\tilde{p},$$  \hspace{1cm} (43)

where the Weyl symbol is given by

$$A^W(\tilde{q}, \tilde{p}) = \int Tr(A(\tilde{q}, \tilde{p}) e^{i\xi \tilde{q} + i\eta \tilde{p}}) e^{-i\xi \tilde{q} - i\eta \tilde{p}} \frac{d\tilde{q} d\tilde{p}}{(2\pi)^{2N}}$$  \hspace{1cm} (44)
Expression for the average values in one random variable tomography formulation is obtained using the connection between $w$ and Wigner function [14]:

$$\langle A \rangle = \int e^{iX} w(X, \vec{\mu}, \vec{\nu}) A(\vec{\mu}, \vec{\nu}) dX d\vec{\mu} d\vec{\nu}, \quad (45)$$

$$A(\vec{\mu}, \vec{\nu}) = \int A^W(\vec{\eta}, \vec{p}) e^{-i(\vec{\eta} \vec{\nu} + \vec{p} \vec{\mu})} \frac{d\vec{\eta} d\vec{p}}{(2\pi)^{2Nd}}, \quad (46)$$

If considered operator depends on coordinates $\vec{q}$ or momenta $\vec{p}$ only, Weyl symbols have the same form as corresponding operators in coordinate or momentum representation. Operator $A(\vec{q})$ is $A(\vec{x})$ in $\vec{x}$-coordinate representation, then its Weyl symbol $A^W(\vec{q}, \vec{p})$ is equal to $A(\vec{q})$. The same is valid for momenta-dependent operator: $B(\vec{p})$ is $B(\vec{y})$ in $\vec{y}$-momentum representation, and $B^W(\vec{q}, \vec{p}) = B(\vec{p})$.

Consider an operator $A(\vec{q})$, depending on coordinates only. For momenta-dependent operators all equations are the same, provided $\mu$ is replaced by $\nu$, and vice versa, because the pairs $\vec{q}, \vec{\mu}$ and $\vec{p}, \vec{\nu}$ enter the equations connecting the tomogram $w$ with Wigner function symmetrically. Integration over $\vec{\nu}$ in Eq. (46) for operator $A(\vec{q})$ gives the delta-function $\delta(\vec{\nu})$. Then we have:

$$\langle A \rangle = \int A^W(\vec{q}) e^{-i(\vec{p} \vec{q} - \vec{\mu} \vec{x})} w(X, \vec{\mu}, \vec{\nu} = 0) \frac{dX d\vec{p} d\vec{q}}{(2\pi)^{2Nd}}, \quad (47)$$

It is often necessary to operate with the one-particle and one-dimension operators. Then, quite generally, we can consider an operator $A(\vec{q}_1)$. Corresponding average value is given by:

$$\langle A \rangle = \int A^W(X) w(X, \mu_1 = 1, \vec{\mu} = 0, 0) dX, \quad (48)$$

where $\vec{\mu}$ designates all $\mu_j$ except the specified $\mu_1$.

**IV. EXAMPLES**

In this section we introduce several examples of tomographic map for many-particles quantum states. For simplicity, here we do not regard symmetry over particles exchange. Permutations properties are considered in Sec. V.

**A. Gaussian states**

Quite simple is the case of pure state and wave function of Gaussian form. This can be the ground state of the system of independent oscillators, as well as coherent or squeezed states, or any many-dimensional Gaussian wave packet. Such wave packet can be created due to parametric excitation of multimode vacuum state of electromagnetic field [51], e.g., in the framework of nonstationary Casimir effect [52].

Consider the pure state with the wave function $\Psi(\vec{q}) = \prod_{j=1}^{Nd} \psi_j(q_j)$, where

$$\psi_j(q) = (A_j/\pi)^{1/4} e^{-A_j (q-x_j)^2 + i\nu_j q}, \quad (49)$$

The only mathematical fact we need here is that the Fourier transform of a Gaussian is Gaussian. Then, using Eq. (11) we immediately obtain the Wigner function as a product of $W_j(q_j, p_j)$, where

$$W_j(q, p) = e^{-A_j (q-x_j)^2} e^{-B_j (p-y_j)^2} (A_j B_j)^{1/2} / \pi, \quad (50)$$

and for states [49] $B_j = 1/A_j$. For the set of parameters $x_j, y_j, A, B$ applying Fourier transformation [10] to (51) we have:

$$w^{Gauss}(X, \vec{\mu}, \vec{\nu}) = e^{-(X-\vec{\mu} \vec{x}-\vec{\nu} \vec{y})^2 / C} / \sqrt{\pi C}, \quad (51)$$

where $C = \sum_{j=1}^{Nd} (\mu_j^2/A_j + \nu_j^2/B_j)$.

Thermal density matrix of independent oscillators is also Gaussian, but it is not a product of wave functions, as the state is not pure. Still it is a product of density matrices of individual oscillators (see, e.g. [52]):

$$\rho_j(q, q') = \sqrt{\frac{2A_j(B_j-1)}{\pi}} e^{-A_j(B_j(q^2 + q'^2) - 2qq')}, \quad (52)$$

where $A_j = m \omega_j / (2sh(\omega_j \beta))$ and $B = ch(\omega_j \beta)$. Omitting the straightforward calculations, we obtain the tomogram $w$ in the following form:

$$w^{(\beta)}(X, \vec{\mu}, \vec{\nu}) = e^{-X^2 / D} / \sqrt{\pi D}, \quad (53)$$

$$D = \sum_{j=1}^{Nd} \left( \frac{\mu_j^2}{2A_j(B_j-1)} + 2\nu_j^2 A_j(B_j+1) \right) \quad (54)$$

**B. Fock states**

The Fock states of light (the eigenstates in representation of photons number) correspond to ground or excited states of multimode oscillator. The state is labeled by vector $\vec{n}$ of integer numbers and wave function has the form:

$$\Psi(\vec{q}) = \prod_{j=1}^{Nd} \frac{e^{-\vec{q}_j^2 / 2 \omega_j}}{\pi^{1/4} \sqrt{2^{\gamma_j} n_j!}}, \quad (55)$$

where $H_m$ is the Hermite polynomial of $m$-th order. To obtain the tomogram for such state we use the following facts. First, coherent state of an oscillator is described by the Gaussian wave function and, correspondingly, by the Gaussian tomogram (see Eq. (51)). Coherent state is labeled by complex vector $\vec{a}$ of integer numbers and wave function has the form: $\Psi(\vec{q}) = \prod_{j=1}^{Nd} e^{-\vec{q}_j^2 / 2 \omega_j} / \pi^{1/4} \sqrt{2^{\gamma_j} n_j!}$, where_t2| 

where $H_m$ is the Hermite polynomial of $m$-th order.
is considered here) is expanded in the basis of Fock states as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$  \hspace{1cm} (56)

which is connected with the expression for generating function of Hermit polynomials:

$$e^{-\alpha^2+2\alpha q} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(q)$$  \hspace{1cm} (57)

Expanding the tomogram of coherent state in Hermit polynomials and wave function of coherent state in corresponding integral expression in wave functions of Fock states we have

$$w^\mu(X, \bar{\mu}, \bar{\nu}) = \int \delta(X - \sum_{j=1}^{N_d} X_j) \times \prod_{j=1}^{N_d} \frac{H_n^2}{2^{n_j} n_j!} \frac{X_j}{\sqrt{n_j^2 + \nu_j^2}} e^{(\bar{\nu}_j X_j - \bar{\mu}_j^2 X_j)} dX_j \hspace{1cm} (58)$$

For example, for $N = 2, d = 1$ and states with $n_1, n_2$ equal to 0 or 1 (denoted $(n_1, n_2)$) the tomograms $w(X, \mu_1, \mu_2, \nu_1, \nu_2)$ have the forms

$$w_2^{(0,0)} = \frac{\exp\left[-X^2/\pi\right]}{\sqrt{\pi C}}, \hspace{1cm} (59)$$

$$w_2^{(0,1)} = \frac{\sqrt{C_2}}{\pi C_1} \frac{2C_2 X^2 + C_1 C_2 + C_2^2 e^{-X^2/\pi}}{C_1 C_2} \times \hspace{1cm} (60)$$

$$w_2^{(1,1)} = \frac{4C_2^2 e^{-X^2/\pi}}{\sqrt{\pi C}} \times \left(\frac{X^4}{C_2^2} + \frac{X^2}{C_1 C_2} + \frac{C_2^2 - 4C_1 C_2}{C_1 C_2} + \frac{3}{4}\right), \hspace{1cm} (61)$$

where $C_1 = \mu_1^2 + \nu_1^2$, $C_2 = \mu_2^2 + \nu_2^2$ and $C = C_1 + C_2$.

V. SYMMETRY PROPERTIES WITH RESPECT TO PARTICLES PERMUTATIONS

Consideration of identical particles exchange imposes the restrictions concerning the possible form of the state-describing functions. In this section we discuss the corresponding properties of one-random-variable tomographic map (see for permutation symmetry properties of the symplectic tomogram).

Further we use the following notations. A vector without index $\tilde{a}$ has $Nd$ components, vector with index $\tilde{a}_j$ denotes the set of some values, corresponding to $j$-th particle, and consists of $d$ components. A vector $\tilde{a}$ denotes the collection of all components of $\tilde{a}_j$ except those that are specified in the same expression. For example, $\tilde{q}$ in the expression $\psi(q_j, \tilde{q})$ is the vector of all the coordinates, except the coordinates of the $j$-th particle.

For particles obeying Fermi or Bose statistics, we have the following symmetry properties concerning their permutations:

$$\rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') = \rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') = \rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') = \rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') = \pm \rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') \hspace{1cm} (62)$$

where the upper sign (‘+’) is for Bose systems, and lower sign (‘-’) is for Fermi systems. Note that ‘entire’ particles permutation (two particles exchange both $q$ and $q'$ variables) corresponds to sign conservation for both Fermi and Bose statistics:

$$\rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') = \rho(q_j, q_j', \tilde{q}; q_j', q_j', \tilde{q}'') \hspace{1cm} (63)$$

In the expressions for obtaining the Wigner function form density matrix and tomogram $w$ from Wigner function we can exchange the integration variables $(\tilde{a}_j \leftrightarrow \tilde{a}_i, \text{etc.})$, then we immediately have:

$$W(q_j, q_j', \tilde{q}; q_j', \tilde{q}'') = W(q_j, q_j', \tilde{q}; q_j', \tilde{q}''), \hspace{1cm} (64)$$

$$w(X; \tilde{\mu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j) \hspace{1cm} (65)$$

We see that there is no distinction between Fermi and Bose statistics when the particles exchange ‘entirely’, i.e. $q$ and $q'$ in the density matrix, $q$ and $p$ in Wigner function or $\mu, \nu$ in $w$ are permuted simultaneously. The distinction appears when not all the variables, corresponding to the considered particles, are permuted. When we use the density matrix, Fermi and Bose statistics differ only in the sign $\pm 1$, which appears after the permutation of either $q_j, q_j'$ or $q_j', q_j''$. For the Wigner function and tomogram this difference is expressed in far more complicated manner, through the integral transforms (see corresponding formulae for the symplectic tomography in [54]).

First, regard the permutation of $q_j, q_j'$ or $p_i, p_j$ for the Wigner function. Again exchanging the integration variables in (64) we come to

$$W(q_j, q_j', \tilde{q}; p_j', q_j', \tilde{q}) = W(q_i, q_i', \tilde{q}; p_i', q_i', \tilde{q}) \hspace{1cm} (66)$$

The same considerations lead us to the similar expression for $w$:

$$w(X; \tilde{\mu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j) = w(X; \tilde{\mu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j, \tilde{\nu}_j) \hspace{1cm} (67)$$

Then it is enough to develop the formulae for coordinate (Wigner function) or $\tilde{\mu}$ (tomogram) permutations only. Corresponding integral expressions has the following form:
\[ W(\vec{q}_i, \vec{q}_j, \vec{p}_i, \vec{p}_j) = \int K^W(\vec{x}_i, \vec{x}_j, \vec{y}_i, \vec{y}_j, \vec{q}_i, \vec{q}_j, \vec{p}_i, \vec{p}_j) W(\vec{x}_i, \vec{x}_j, \vec{y}_i, \vec{y}_j, \vec{p}_i, \vec{p}_j) d\vec{x}_i d\vec{x}_j d\vec{y}_i d\vec{y}_j, \]  

\[ w(X, \vec{p}_i, \vec{p}_j, \vec{q}_i, \vec{q}_j, \vec{v}_i, \vec{v}_j) = \int K(X, \vec{p}_i, \vec{p}_j, \vec{v}_i, \vec{v}_j; Y, \vec{\xi}_i, \vec{\xi}_j, \vec{\eta}_i, \vec{\eta}_j) w(Y, \vec{\xi}_i, \vec{\xi}_j, \vec{\eta}_i, \vec{\eta}_j, \vec{v}_i, \vec{v}_j) dY d\vec{\xi}_i d\vec{\xi}_j d\vec{\eta}_i d\vec{\eta}_j, \]  

and kernels are given by
\[ K^W(\vec{x}_i, \vec{x}_j, \vec{y}_i, \vec{y}_j, \vec{q}_i, \vec{q}_j, \vec{p}_i, \vec{p}_j) = \pm \left( \frac{4}{2\pi} \right)^d d(\vec{x}_i - \vec{x}_j - \vec{q}_i - \vec{q}_j) d(\vec{y}_i + \vec{y}_j - \vec{p}_i - \vec{p}_j) e^{\pm i(\vec{q}_i - \vec{q}_j)(\vec{y}_i - \vec{y}_j) + (\vec{x}_i - \vec{x}_j)(\vec{p}_i - \vec{p}_j)}, \]

\[ K(X, \vec{p}_i, \vec{p}_j, \vec{v}_i, \vec{v}_j; Y, \vec{\xi}_i, \vec{\xi}_j, \vec{\eta}_i, \vec{\eta}_j) = \pm \int \frac{|k|^{2d}}{(2\pi)^{d+1}} d|\vec{\xi}_i + \vec{\xi}_j - \vec{\mu}_i - \vec{\mu}_j| d|\vec{\eta}_i + \vec{\eta}_j - \vec{\nu}_i - \vec{\nu}_j| e^{-i(k(X-Y)-k^2/4)[(\vec{\mu}_i - \vec{\mu}_j)(\vec{\xi}_i - \vec{\xi}_j) + (\vec{\eta}_i - \vec{\eta}_j)(\vec{\nu}_i - \vec{\nu}_j)]} dk \]

VI. CONCLUSION

We studied in details the version of tomographic map of the density matrix and Wigner function for which the quantum state of multimode system is associated with probability distribution function. This function depends on one random variable X and 2Nd real parameters (real Nd-vectors \( \vec{\mu} \) and \( \vec{\nu} \)) and it determines the quantum state completely. It means that provided this probability distribution function is known one can reconstruct the Wigner function of the system state and corresponding density operator. The random variable X can be interpreted as the system ”center of mass” coordinate considered in specifically rotated and scaled reference frame in the complete phase space of the system. Real parameters (vectors \( \vec{\mu} \) and \( \vec{\nu} \)) determine this rotated and scaled reference frame.

Information contained in the introduced tomogram \( w \) is the same as that contained in the symplectic tomo-gram \( w_1 \), which depends on larger number of variables. It corresponds to the fact that the tomograms have high symmetry properties. By means of the symmetry operations one can reconstruct the dependence of the function on larger number of variables starting from initial function with smaller number of variables.

We have constructed the quantum evolution equations and energy level equations for the introduced ”center of mass” tomogram. Example of multimode oscillator and symmetry properties of the tomogram for identical particles (fermions and bosons) were discussed in details.

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[1] S. Mancini, V.I. Man’ko and P. Tombesi, Phys. Lett. A 213, 1 (1996); Found. Phys. 27, 801 (1997).
[2] J. Bertrand and P. Bertrand, Found. Phys. 17, 397 (1987).
[3] K. Vogel and H. Risken, Phys. Rev. A 40, 2847 (1989).
[4] K. Husimi, Proc. Phys. Math. Soc. Japan 22, 264 (1940).
[5] E.C.G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963).
[6] R.J. Glauber, Phys. Rev. Lett. 20, 84 (1963).
[7] L.D. Landau, Z. Phys. 45, 430 (1927).
[8] J. von Neumann, Mathematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932).
[9] S. Mancini, V.I. Man’ko and P. Tombesi, Quantum Semi-class. Opt. 7, 615 (1995).
[10] A. Wünsche, J. Modern Opt. 47, 33 (2000).
[11] V.I. Man’ko, L. Rosa and P. Vitale, Phys. Rev. A 57, 3291 (1998).
[12] G.M. D’Ariano, S. Mancini, V.I. Man’ko and P. Tombesi, J. Opt. B 8, 1017 (1996).
[13] O.V. Man’ko, V.I. Man’ko and G. Marmo, J.Phys.A 35, 699 (2002).
[14] V.V. Dodonov and V.I. Man’ko, Phys. Lett. A 229, 335 (1997).
[15] S. Weigert, Phys. Rev. Lett. 84, 802 (2000).
[16] V.I. Man’ko and O.V. Man’ko, JETP 85, 430 (1997).
[17] O.V. Man’ko, V.I. Man’ko and G. Marmo, Phys. Scr. 62, 446 (2000).
[18] U. Leonhardt, Phys. Rev. A 53, 2998 (1996).
[19] A.B. Klimov, O.V. Man’ko, V.I. Man’ko, Yu.F. Smirnov and V.N. Tolstoy, J. Phys. A 35, 6101 (2002).
[20] O. Castaños, R. López-Peña, M.A. Man’ko and V.I. Man’ko, J. Phys. A 36, 4677 (2003); J. Opt. B 5, 227 (2003).
[21] D.T. Smithey, M. Beck, M.G. Raymer and A. Faridani, Phys. Rev. Lett. 70, 1244 (1993).
[22] G.M. D’Ariano, L. Maccone and M. Paini, J. Opt. B 5, 77 (2003).
[23] S. Schiller, G. Breitenbach, S.F. Pereira, T. Muller and J. Mlynek, Phys. Rev. Lett. 77, 2933 (1996).
[24] D.G. Welsch, W. Vogel and T. Opatrny, in *Progress in Optics*, edited by E. Wolf, (Elsevier, Amsterdam, 1999).
[25] M. Beck, D. T. Smirhey, M. G. Raymer, and A. Faridani, Phys. Rev. Lett. **70**, 1244 (1993).
[26] M.G. Raymer, M. Beck and D.F. McAlister, Phys. Rev. Lett. **72**, 1137 (1994).
[27] M. G. Raymer, D. F. McAlister and U. Leonhardt, Phys. Rev. A **54**, 2397-2401 (1996).
[28] M. G. Raymer and A. C. Funk, Phys. Rev. A **61**, 015801 (2000).
[29] J. Ashburn, R. Cline, P. van der Burgt, W. Westerveld and J. Risley, Phys. Rev. A **41**, 2407 (1990).
[30] O. Carnal and J. Mlynek, Phys. Rev. Lett. **66**, 2689 (1991).
[31] D. W. Keith, C. R. Ekstrom, Q. A. Turchette and D. E. Pritchard, Phys. Rev. Lett. **66**, 2693 (1991).
[32] T. J. Dunn, I. A. Walmsley and S. Mukamel, Phys. Rev. Lett. **74**, 884 (1995).
[33] Yu.E. Lozovik, V.A. Sharapov and A.S. Arkhipov, to be published.
[34] V.A. Andreev and V.I. Man’ko, J. Opt. B **2**, 122 (2000).
[35] E. Wigner, Phys.Rev. **40**, 749 (1932).
[36] V.I. Man’ko, L. Rosa and P. Vitale, Phys. Lett. B **439**, 328 (1998).
[37] V.I. Man’ko and R. Mendes, Physica D **145**, 330 (2000).
[38] S. L. Stratonoich. JETP **4**, 891 (1957).
[39] J. M. Gracia-Bondiu, Phys. Rev. A **30**, 691 (1984).
[40] T. Curtright, D. Fairlie and C. Zachos, Phys. Rev. D **58**, 025002 (1998).
[41] C. Brif and A. Mann, Phys. Rev. A **59**, 971 (1999).
[42] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. **111**, 61 (1978).
[43] V.I. Man’ko and R.V. Mendes, Phys. Lett. A, **263**, 53 (1999).
[44] W. Kohn, Rev. Mod. Phys. **7**, 1253 (1998).
[45] J.E. Moyal, Proc. Camb. Phil. Soc. **45**, 99 (1949).
[46] V.I. Man’ko, [arXiv:quant-ph/9902079](https://arxiv.org/abs/quant-ph/9902079).
[47] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931).
[48] V.I. Tatarsky, Sov. Phys. Usp. **139**, 4, 587 (1983).
[49] H.W. Lee, Phys. Rep. **259**, 147 (1995).
[50] O. Man’ko and V.I. Man’ko, J. Russ. Laser Res. **18**, 407 (1997).
[51] V.V. Dodonov and V.I. Man’ko, *Invariants and Evolution of Nonstationary Quantum Systems* (Proc. P.N. Lebedev Physical Institute, vol. 183), (Nova Science, New York, 1987).
[52] V.V. Dodonov, in *Modern Nonlinear Optics (Advances in Chemical Physics, vol. 119)*, 2nd edn, edited by M.W. Evans (Wiley, New York, 2001).
[53] R.P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1972).
[54] V.I. Man’ko, L. Rosa and P. Vitale, J. Phys. A **36**, 255 (2003).