A PROBABILISTIC APPROACH TO GENERALIZED ZECKENDORF DECOMPOSITIONS

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ABSTRACT. Zeckendorf’s Theorem states that every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers \( \{F_n : n \in \mathbb{N}\} \) (with \( F_1 = 1, F_2 = 2 \), etc.). This sum is known as the Zeckendorf decomposition. Similarly to base 2 decompositions of integers, a natural number has a Zeckendorf decomposition of length \( n \) if \( F_n \) is the largest summand in its decomposition. A natural probabilistic structure is obtained by considering the uniform measure \( Q_n \) on decompositions of length \( n \). This leads to questions on statistical properties of the decompositions. Lekkerkerker’s theorem provides the mean number of summands under \( Q_n \), and more refined results include laws of large numbers, central limit theorems, and asymptotic distribution of gaps between indices appearing in decompositions. When instead of Fibonacci numbers one considers sequences which are solutions to linear recurrences with positive integer coefficients, under natural conditions there exists a unique decomposition. Similarly to the Zeckendorf and base \( B \)-decompositions, one can define the length of the decomposition, and then study the statistical properties of the decompositions under the uniform measure on decompositions of length \( n \). All results mentioned above for the Fibonaccis have been extended to this setting. We introduce a unified probabilistic framework for this setting, based on the observation that the resulting distributions coincide with the distribution of a conditioned Markov chain. This allows us to obtain results on decompositions from analogous results for Markov chains. As an application, we reprove, sharpen and generalize existing results, as well as obtain some results which have not appeared before on the distribution of the maximal gap between indices of summands in the decompositions.

1. INTRODUCTION

There are many ways to generalize base \( B \) expansions of integers. One interesting choice is the Zeckendorf decomposition. If we define the Fibonacci numbers \( \{F_n\} \) by \( F_1 = 1, F_2 = 2 \) and \( F_{n+2} = F_{n+1} + F_n \), then every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers. This is known as Zeckendorf’s Theorem [Ze]. For integers \( m \in [F_n, F_{n+1}) \), using a continued fraction approach Lekkerkerker [Lek] proved that the average number of summands is \( n/(\varphi^2 + 1) \), with \( \varphi = \frac{1+\sqrt{5}}{2} \) the golden mean. The precise probabilistic meaning of “average” is the expectation with respect to the uniform measure on the decompositions of integers in \( [F_n, F_{n+1}) \), and then Zeckendorf’s theorem provides an asymptotic statement on a certain statistic under the sequence of uniform probability measures on decompositions of length \( n \), as \( n \to \infty \). Analogues hold for more general recurrences, such as linear recurrences with non-negative coefficients [Al, BCCSW, Day, GT, Ha, Ho, Ke, Len, MW1, MW2], generalizations where additionally the...
summands are allowed to be signed [DDKMU] [MWT], and $f$-decompositions (given a function $f : \mathbb{N} \to \mathbb{N}$, if $a_n$ is in the decomposition then we do not have $a_{n-1}, \ldots, a_{n-f(n)}$ in the decomposition) [DDKMMU]. The notion of a legal decomposition below generalizes the non-adjacency condition.

**Definition 1.1.** Given a length $L \in \mathbb{N}$ and coefficients $c_1, \ldots, c_L \in \mathbb{Z}_+$ with $c_1 c_L > 0$, the corresponding **positive linear recursion** is a sequence $1 = G_1, G_2, \ldots \in \mathbb{N}$ satisfying:

$$G_{n+1} = c_1 G_n + c_2 G_{n-1} + \cdots + c_n G_1 + 1, \quad n = 1, \ldots, L. \tag{1.1}$$

**Definition 1.2.** Given a positive linear recursion with coefficients $c_1, \ldots, c_L$, an integer $N$ has a **legal** decomposition of length $n \in \mathbb{N}$ if there exist $a_1 \in \mathbb{N}, a_2, \ldots, a_n \in \mathbb{Z}_+$, such that

$$N = \sum_{i=1}^{n} a_i G_{n+1-i}, \tag{1.2}$$

and

- $n < L$ and $a_i = c_i$ for $1 \leq i \leq n$; or
- there exists some $s \in \{1, \ldots, L\}$ such that
  $$a_1 = c_1, \quad a_2 = c_2, \ldots, \quad a_s-1 = c_{s-1}, \quad \text{and} \quad a_s < c_s,$$
  $$\left\{a_{s+1}, \ldots, a_{s+\ell} = 0 \text{ for some } \ell \geq 0, \right.$$
  $$\{b_i\}_{i=1}^{n-s-\ell} \text{ with } b_i = a_{s+\ell+i} \text{, is either legal or empty.} \tag{1.3}$$

**Theorem 1.3** (Generalized Zeckendorf Decomposition). **Consider a positive linear recurrence with coefficients $c_1, \ldots, c_L$. Then every $N \in \mathbb{N}$ has a unique legal decomposition.**

In the sequel we will fix a linear recurrence as in Definition 1.1. From Theorem 1.3 it follows that there’s a one-to-one correspondence between the set of integers in $[G_n, G_{n+1})$ through (1.2), where the integer $N$ is mapped to its legal decomposition $(a_1(N), \ldots, a_n(N))$. Let $Q_n$ denote the uniform distribution on the legal decompositions of integers in $[G_n, G_{n+1})$, and with this identification it is natural to consider $N$ and $a_1(N), \ldots, a_n(N)$ as random variables. In what follows, we denote expectation with respect to $Q_n$ by $E^Q_n$.

For $N \in [G_n, G_{n+1}), (1.2)$ can be rewritten as

$$N = G_{i_1(N)} + G_{i_2(N)} + \cdots + G_{i_k(N)}, \tag{1.4}$$

where $1 \leq i_1 \leq \cdots \leq i_k(N) \leq n$. The random variable $k(N)$ gives the number of summands in the generalized Zeckendorf decomposition, and was the main object of previous works. The first result was Lekkerkerker’s theorem on the asymptotic expectation of $k(N)$ when $G_n = F_n$. Here is its generalization to our setting.

**Theorem 1.4** (Generalized Lekkerkerker’s Theorem). **There exist constants $C_{\text{Lek}} > 0$ and $d$ such that

$$E^Q_n k(N) = C_{\text{Lek}} n + d + o(1), \quad \text{as } n \to \infty. \tag{1.5}$$**
Once the average number of summands has been determined, it is natural to investigate other and finer properties of the decompositions. Three natural questions concern the fluctuations in the number of summands \( k(N) \) about the mean, the distribution of gaps \( i_{j+1}(N) - i_j(N), j = 1, \ldots, k(N) - 1 \) between adjacent summands, and the length of the longest gap in a decomposition. For positive linear recurrences as in Theorem 1.3, the distribution of the number of summands converges to a Gaussian with computable mean and variance, both of order \( n \). There is an extensive literature on these results. See [DG, FGNPT, GTNP, LT, Ste1] for an analysis using techniques from ergodic theory and number theory, and [KKMW, MW1, MW2] for proofs via a combinatorial perspective. As before, all these are statements on the asymptotic behavior of certain statistics of generalized Zeckendorf decompositions of integers in \([G_n, G_{n+1})\) under the uniform measure, as \( n \to \infty \).

Results on the distribution of gaps between adjacent summands have recently been obtained by Beckwith, Bower, Gaudet, Insoft, Li, Miller and Tosteson [BBGILMT, BILMT]. They show that the distribution of gaps larger than the recurrence length converges to that of a geometric random variable whose parameter is the largest eigenvalue of the characteristic polynomial of the recurrence relation. For gaps smaller than the recurrence relation closed forms exist for special recurrences, though with enough work explicit formulas can be derived for any given relation. They also determine the distribution of the longest gap, and prove the behavior is similar to that of the length of the longest run of heads in a sequence of tosses of a possibly biased coin. Their proofs are a mix of combinatorics and a careful analysis of polynomials associated with the recurrence relations. The details become involved as some of the associated polynomials depend on the interval \([G_n, G_{n+1})\) under consideration.

The main goal of this paper is to obtain a Markov-chain description of the sequence \( Q_n \) rather than focusing on statements on asymptotic behavior of some explicit statistics. We will show that there exists a finite-state Markov chain such the distribution of the first \( n \) steps, conditioned to avoid a certain fixed subset of states is equal to \( Q_n \). This type of process is also known in the literature as the Doob transform or the \( h \)-transform of the Markov chain. We will show how to adapt classical results on finite-state Markov chains to our model, and will apply this to obtain a new proof for previously derived results mentioned above, as well as some new results. The focus in this paper is on laws of large numbers, central limit theorem and gap distribution (which could be viewed as renewal theory), but the message is that essentially any result on finite state Markov chains could be adapted to this new situation, and this includes large deviations, law of iterated logarithm, functional central limit theorem, etc.

The paper is organized as follows. In Section 2 we describe the Markovian model, and how to obtain large-time asymptotics for our model from that of the underlying Markov chain. In Section 3 we present the results additive functionals in a setting which includes our particular model, first by introducing the theoretical results in Section and then applying them to the generalized decompositions in Section 3.2. These results include sharp estimates on expectation, a law of large numbers and a central limit theorem. In Section 4 we then apply the results on additive functionals to the generalized Zeckendorf decompositions. Our treatment includes all additive functionals, and the number of summands, extensively studied in the past is just one example. As another example, consider the number of times a certain subset of digits (values attained by \( a_s \)) appears, and the
number of $s$ for which $a_s < c_s$, or even (pushing the ideas further), the number of gaps of a certain size. In Section 3 we treat the gap distribution as a consequence of the regenerative structure of the underlying Markov chain, and the analogy with Bernoulli trials.

2. Probabilistic Approach

We remind that throughout the discussion we assume that $L \in \mathbb{N}$ and the coefficients $c_1, \ldots, c_L \in \mathbb{Z}_+$ satisfy $c_1 c_L > 0$ as in Definition 1.1.

The main idea is to show that for a given $n \in \mathbb{Z}_+$, the uniform distribution on generalized Zeckendorf decompositions consisting of $n + 1$ digits (that is, the $(n + 1)$-th digit is non-vanishing and all higher digits are not present) coincides with the distribution of a certain conditioned Markov chain. This provides a unified framework for the model, which, in particular, gives rather easy access to many asymptotic results. We first define the Markov chain. Let $(X, Y) = ((X_n, Y_n) : n \in \mathbb{N})$ be the two-dimensional process with $X_n \in \{0, \ldots, \max_i c_i\}$ and $Y_n \in \{1, \ldots, L\}$. The idea is that $X_0, X_1, \ldots$ will be used to represent the coefficients $a_i$ in (1.2), while $Y_0, Y_1, \ldots$ will be used to keep track whether the $X_i$’s satisfy the condition (1.3). This will be explained below, after we finish describing our construction. Let $P$ denote the distribution under which this is an IID process, $(X_0, Y_0)$ being uniformly distributed over $\{0, \ldots, \max_i c_i\} \times \{1, \ldots, L\}$.

Definition 2.1. Suppose $L \in \mathbb{N}$ and $c_1, \ldots, c_L \in \mathbb{Z}_+$, $c_1 c_L > 0$ are the coefficients of a linear recursion. We say that the realization $((X_0, Y_0), (X_1, Y_1), \ldots)$ of the process $(X, Y)$ is legal with respect to the recursion if

1. $X_0 > 0$ and $Y_0 = 1$.
2. There exists a random variable $J \in \mathbb{Z}_+$ such that $X_J > 0$, $X_n = 0$ and $Y_n = 1$ for $n > J$.
3. For all $n \in \mathbb{N}$, either
   a. $X_n < c_{Y_n}$ and $Y_{n+1} = 1$.
   b. $X_n = c_{Y_n}$ and $Y_n = Y_{n+1} + 1$.

Note that condition 3b and the assumption that $Y_n \in \{1, \ldots, L\}$ for all $n$ implicitly mean that in a legal realization $X_n = c_{Y_n}$ only if $Y_n < L$.

The main observation is the following. Given a legal realization and letting (compare to (1.2))

$$N = \sum_{j=0}^{n} X_j G_{n-j+1},$$

then $(X_0, \ldots, X_n)$ is the legal decomposition of $N \in [G_{n+1}, G_{n+2}]$, according to Definition 1.2.

Let

$$\tau = \inf\{n \in \mathbb{Z}_+ : ((X_0, Y_0), (X_1, Y_1), \ldots, (X_n, Y_n)) \text{ does not extend to a legal realization}\}.$$  

(2.2)

With a slight abuse of notation, let $Q_n$ be the probability measure on the $\sigma$-algebra generated by $(X_0, Y_0), \ldots, (X_n, Y_n)$ defined through

$$Q_n(B) = P(B|\tau > n).$$

(2.3)

Since $P$ is uniform, $Q_n$ is uniform over all finite realizations $(X_0, Y_0), \ldots, (X_n, Y_n)$ that extend to legal realizations. Any such finite realization corresponds to a unique Zeckendorf decomposition of length $n + 1$ given in (2.1). Conversely, every integer with Zeckendorf decomposition of length
\( n + 1 \) corresponds to a unique finite realization \((X_0, Y_0), \ldots, (X_n, Y_n)\) extending to a legal realization. Therefore \( Q_n \) could be identified with the uniform distribution on generalized Zeckendorf decompositions of length \( n + 1 \).

We now define an auxiliary process that allows us to introduce ideas on conditioned Markov chains. The reason for doing that is the following: \( \tau \) is not a hitting or even stopping time for \((X, Y)\), as in order to determine whether \( \tau = n \), it is evident from Definition 2.1 3b. that on certain circumstances the value of \( Y_{n+1} \) is needed. Therefore, the probabilistic analysis of Markov chains through stopping times, and which is key to our approach, cannot be applied. To fix this, let \( Z_n = (X_n, Y_n, Y_{n+1}) \), and let \( Z = (Z_n : n \in \mathbb{Z}_+) \). Below we will write \( Z_n(1) \) for \( X_n \), \( Z_n(2) \) for \( Y_n \) and \( Z_n(3) \) for \( Y_{n+1} \). It is easy to see that \( \tau \) is a hitting time for \( Z \). Specifically, letting

\[
\mathcal{L} = \{(x, j, j') : (x < c_j \text{ and } j' = 1) \text{ or } (j < L \text{ and } x = c_j \text{ and } j' = j + 1)\};
\]

\[
\mathcal{L}_0 = \mathcal{L} \cap \{(x, 1, j') : x > 0\},
\]

then

\[
\tau = \begin{cases} 
0 & \text{if } Z_0 \notin \mathcal{L}_0 \\
\inf\{n : Z_n \notin \mathcal{L}\} & \text{otherwise.}
\end{cases}
\]

Under \( P, Z \) is a Markov chain. We abuse notation and denote its transition function by \( P \) as well. Since the measure \( P \) is uniform, it immediately follows that the restriction \( P_L \) of the transition function \( P \) to \( \mathcal{L} \times \mathcal{L} \) is an irreducible and aperiodic substochastic matrix. From the Perron-Frobenius theorem we know that \( P_L \) possesses a Perron root \( \lambda_c \in (0, 1) \) and corresponding left and right eigenfunctions, \( \nu_c \) and \( \varphi_c \), respectively, whose entries are strictly positive. We normalize them so that \( \varphi_c \) and \( \nu_c \varphi_c \) are probability measures. Let \( Q \) be a stochastic transition function on \( \mathcal{L} \times \mathcal{L} \) defined as follows:

\[
Q(z, z') = \frac{1}{\lambda_c \varphi_c(z)} P_L(z, z') \varphi_c(z').
\]

Observe that \( Q \) inherits irreducibility and being aperiodic from \( P_L \). As a result, \( Q \) is ergodic, and we denote its unique stationary distribution by \( \pi^Q \). Recall that from the definition of a stationary distribution, \( \pi^Q Q = \pi^Q \), if \( \pi^Q \) is considered as a row vector, and it immediately follows that

\[
\pi^Q(z) = \nu_c(z) \varphi_c(z).
\]

We also define the marginal of the first coordinate \( \pi^Q_1 \) by letting

\[
\pi^Q_1(x) = \sum_{b, b'} \pi^Q(x, b, b').
\]

Next we fix some notation. We write \( P_\mu \) for the distribution of the Markov chain \( Z \) under \( P \) with initial distribution \( \mu \), and \( E^P_\mu \) for the corresponding expectation. When \( \mu \) is a point mass \( \delta_z \), we abbreviate and use \( z \) as a subscript instead of the notationally correct but cumbersome \( \delta_z \). We also define the analogous expressions with \( Q \) instead of \( P \). Finally, if \( \mu \) is a probability distribution on \( \mathcal{L} \), we write \( E^P_\mu \) for the expectation associated with the process \( Z \) and initial distribution \( \mu \), and \( E^Q_\mu \) for expectation with respect to the probability \( \mu \), equal to the expectation of the marginal \( Z_0 \) under \( P_\mu \).

The following result identifies the uniform distribution \( Q^n \) with the distribution of the Markov chain \( Z \) under \( Q \).
\textbf{Theorem 2.2.} Let $f = f(Z_0, \ldots, Z_n)$ be a complex-valued random variable. Then

\[ E^{Q_n}(f) = \frac{E^Q_{\hat{\varphi}_c} \left( \frac{f}{\varphi_c(Z_n)} \right)}{E^Q_{\hat{\varphi}_c} \left( \frac{1}{\varphi_c(Z_n)} \right)}, \tag{2.9} \]

where $\hat{\varphi}_c$ is the probability measure given by $\varphi_c$ conditioned on $L_0$ in (2.4).

\textbf{Proof of Theorem 2.2.} Observe that if $z_0 \in L_0$ and $z_1, \ldots, z_n \in L$, then

\[ P_{z_0} \left( \prod_{j=0}^{n} \{ Z_j = z_j \}, \tau > n \right) = \prod_{j=0}^{n-1} P(z_j, z_{j+1}) \]

\[ = \lambda_c^n \prod_{j=0}^{n-1} \varphi_c(z_j) Q_z(z_j, z_{j+1}) \frac{1}{\varphi_c(z_{j+1})} \]

\[ = \lambda_c^n \varphi_c(z_0) Q_{z_0} \left( \prod_{j=0}^{n} \{ Z_j = z_j \} \right) \frac{1}{\varphi_c(z_n)}, \tag{2.10} \]

and otherwise $P_{z_0} \left( \prod_{j=0}^{n} \{ Z_j = z_j \}, \tau > n \right) = 0$. In particular, if $f = f(Z_0, \ldots, Z_n)$ is a complex valued random variable, then

\[ E^P(f, \tau > n) = \sum_{z_0 \in L_0} E^P(f, \tau > n, Z_0 = z_0) = \sum_{z_0 \in L_0} E^P(1_{\{Z_0 = z_0\}} f(z_0, \ldots, Z_n), \tau > n) \]

\[ = \sum_{z_0 \in L_0} P(Z_0 = z_0) E^P_{z_0}(f(Z_0, \ldots, Z_n), \tau > n) \]

\[ = \lambda_c^n \sum_{z_0 \in L_0} P(Z_0 = z_0) \varphi_c(z_0) E^Q_{z_0} \left( \frac{f}{\varphi_c(Z_n)} \right). \tag{2.11} \]

Since $P$ is uniform, it follows that $P(Z_0 = z_0)$ is constant on $L_0$, and the result follows. \hfill \Box

Next we consider limits. The following provides sufficient conditions under which $Q_n$ expectations and expectations with respect to $Q$ are asymptotically equivalent.

\textbf{Proposition 2.3.} Suppose that for $n \in \mathbb{Z}_+$, $f_n(Z_0, \ldots, Z_n)$ is a complex-valued random variable, and $(j_n : n \in \mathbb{Z}_+)$ is a subsequence of $\mathbb{Z}_+$ such that

\begin{enumerate}
\item $\min(j_n, n-j_n) \to \infty,$
\item $E^{Q_{\hat{\varphi}_c}} |f_n - f_{j_n}| \to 0.$
\end{enumerate}

Then

\[ |E^{Q_n} f_n - E^{\hat{\varphi}_c} f_n| = o(1) \max(|E^{Q_{\hat{\varphi}_c}}(f_n)|, 1). \tag{2.12} \]

\textbf{Proof.} Because of condition (2), we have

\[ E^{Q_n}(f_n) = \frac{E^{Q_{\hat{\varphi}_c}} \left( \frac{f_{j_n}}{\varphi_c(Z_n)} \right)}{E^{Q_{\hat{\varphi}_c}} \left( \frac{1}{\varphi_c(Z_n)} \right)} + o(1). \tag{2.13} \]

Then, by the Markov property,

\[ E^{Q_{\hat{\varphi}_c}} \left( \frac{f_{j_n}}{\varphi_c(Z_n)} \right) = E^{Q_{\hat{\varphi}_c}} \left( f_{j_n} E_{Z_n} \left( \frac{1}{\varphi_c(Z_n-j_n)} \right) \right). \tag{2.14} \]
The ergodicity of $Z$ under $Q$ and the fact that $n - j_n \to \infty$ guarantee that $E_{Z,j_n}^Q \left( \frac{1}{\varphi_c(Z_{n-j_n})} \right) = E_{\pi \nu_c} \frac{1}{\varphi_c} + o(1) = \| \nu_c \|_1 + o(1)$. Thus
\[
E_{Q_n}^Q(f_n) = \frac{\| \nu_c \|_1 + o(1) E_{Q_n}^Q(f_{j_n})}{\| \nu_c \|_1 + o(1)} + o(1)
= (1 + o(1)) E_{\varphi_c}^Q(f_{j_n}) + o(1) = (1 + o(1)) E_{\varphi_c}^Q(f_n) + o(1).
\] (2.15)

For applications, it would be useful to know more about $Q$. It turns out that the underlying structure is determined by the matrix $C$, which we now describe. Let $C$ be the $L \times L$ matrix given by $C = (C_{i,j})$, $C_{i,1} = c_i$ and $C_{i,i+1} = 1$, and all other entries equal to 0:
\[
C = \begin{pmatrix}
c_1 & 1 & 0 & \cdots \\
c_2 & 0 & 1 & 0 & \cdots \\
\vdots & 0 & \ddots & \ddots & \ddots \\
c_{L-1} & 0 & \cdots & 0 & 1 \\
c_L & 0 & \cdots & 0 & 0
\end{pmatrix}
\] (2.16)

Let $\lambda_C$ denote the Perron eigenvalue of $C$, $\varphi_C$ a corresponding positive right eigenvector and $\nu_C$ a corresponding left eigenvector. A straightforward computation gives the following.

**Lemma 2.4.** Let $C$ be as in (2.16). Then

1. The characteristic polynomial of $C$ is $\lambda^L - \sum_{j=1}^{L} c_j \lambda^{L-j}$.
2. Up to multiplicative constants: $\nu_C(b) = \lambda_C^{-b}$ and $\varphi_C(b') = \lambda_C^{b'} - \sum_{j=1}^{b'-1} c_j \lambda_C^{b'-j}$.

With this lemma we obtain a description of $Q$.

**Proposition 2.5.** Let $C$ be as in (2.16), and let $\lambda_c, \nu_c, \varphi_c$, respectively, be the Perron eigenvalue, and corresponding left and right eigenvalues for $P_L$, the restriction of the transition function $P$ to $L$, normalized so that $\varphi_c$ and $\nu_c \varphi_c$ are probability distributions. Then

1. $\lambda_c = \lambda_C / (\max c_{i+1})$.  
2. There exist positive constants $K_1, K_2$ such that $\varphi_c(a, b, b') = K_1 \varphi_C(b')$ and $\nu_c(a, b, b') = K_2 \nu_C(b)$. In particular, $\pi(a, b, b') = K_1 K_2 \nu_C(b) \varphi_C(b')$, and $K_1 K_2 = \frac{1}{\lambda_C \sum_{b=1}^{L} \nu_C(b) \varphi_C(b)}$.
3. $Q((a, b, b'), (a', b', b'')) = \frac{\varphi_c(b')}{\lambda_C \varphi_C(b')}$ for allowed transitions and is 0 otherwise. Furthermore, allowed transitions satisfy either of the following:
   (a) $b'' = 1$ and then the probability of the transition is $\frac{\varphi_c(b')}{\lambda_C \varphi_C(b')}$;
   (b) $b'' = b' + 1$ and then the probability of the transition is $1 - \frac{\varphi_c(b')}{\lambda_C \varphi_C(b')}$.

**Example 2.6.** For the standard Zeckendorf decomposition, we have

1. $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. In particular,
   (a) The characteristic polynomial is $\lambda^2 - \lambda - 1$, and $\lambda_C = \phi$, where $\phi$ is the golden ratio $\frac{1 + \sqrt{5}}{2}$.
   (b) $\nu_C(b) = \phi^{-b}$, and $\varphi_C(b') = \phi^{2-b'}$.
2. $L = \{(0, 1, 1), (0, 2, 1), (1, 1, 2)\}$. Identifying these states as 1, 2 and 3 in the order written, then
(a) \( Q = \begin{pmatrix}
\frac{1}{\phi} & 0 & 1 - \frac{1}{\phi} \\
0 & 0 & 1 - \frac{1}{\phi} \\
0 & 1 & 0
\end{pmatrix} \).

(b) \( \pi^Q(0,1,1) = \frac{\phi}{2+\phi}, \pi^Q(0,2,1) = \frac{1}{2+\phi}, \pi^Q(1,1,2) = \frac{1}{2+\phi}, \) and
\[
\pi^Q(0) = \frac{1+\phi}{2+\phi}, \quad \pi^Q(1) = \frac{1}{2+\phi}.
\]

(c) \( \varphi_c = \frac{1}{2\phi+1} (\phi, \phi, 1)^t \).

(d) \( \nu_c = \frac{1}{\phi+1} (2\phi + 1, \phi + 1, 2\phi + 1)^t \).

**Proof of Proposition 2.5**

1. The first part is a straightforward calculation.

2. Observe that for the row of \( P \) corresponding to transition from \((a, b, b')\), we have exactly \(|S_1| \times |S_2| = (\max c_i + 1)L\) allowed sites to transition to, and due to the choice of uniform distribution, all are of equal probability. As \( P \) is stochastic, its nonzero entries are equal to \( \gamma = \frac{1}{(\max c_i + 1)L} \). We first study the restriction \( P_L \) of \( P \) to \( L \times L \). Recall that the elements of \( L \) are of the form \((x, k, 1)\), where \( x < c_k \) or \((c_k, k, k + 1)\) where \( k = 1, \ldots, L - 1 \). For each \((a, b, b') \in L\), \( P_L \) has a corresponding row, listing all transitions from \((a, b, b')\). We will count the number of such non-zero entries according to the value of \( b' \). If \( b' \in \{1, \ldots, L-1\} \) then there are \( 1 + c_{b'} \) transitions: one to the site \((c_{b'}, b', b' + 1)\) and \( c_{b'} \) to \((x, b', 1)\) where \( x \in \{0, \ldots, c_{b'} - 1\} \). If \( b' = L \) then there are only \( c_L \) allowed transitions, all of which are of the second kind.

We define a function \( \varphi \) on \( L \) by letting \( \varphi(a, b, b') = \varphi_C(b') \). Fix \((a, b, b') \in A\). If \( b' < L \), then according to the allowed transitions listed above, we have

\[
P_L\varphi(a, b, b') = \gamma(C\varphi_C)(b') = \gamma\lambda_C\varphi(a, b, b').
\tag{2.17}
\]

Similarly, if \( b' = L \), then \( P_L\varphi(a, b, L) = \gamma_L\varphi_C(1) = \gamma\lambda_C\varphi(a, b, L) \). Thus \( \gamma\lambda_C = \lambda_c \), the Perron root for \( P_L \), and \( \varphi \) is a corresponding positive eigenvector. Next we want to find the corresponding left-eigenvector for \( P_L \). To do that, let \( D \) be the transpose of \( C \), and let \( \nu_C \) be a Perron eigenvector. Define \( \nu_C(a, b, b') := \nu_C(b) \). If \( b \in \{2, \ldots, L\} \), then there is exactly one allowed transition to it, that is from \((c_{b-1}, b-1, b)\). As a result, \( \nu_C P_L(a, b, b') = \gamma\nu_C(c_{b-1}, b-1, b) = \gamma(D\nu_C)(b) = \gamma\lambda_C\nu_C(a, b, b') \). Next, if \( b = 1 \), then the allowed transitions are from \((x, k, 1)\) where \( k = 1, \ldots, L \) and \( x \in \{0, \ldots, c_k - 1\} \). We obtain \( \nu_C P_L(a, 1, b') = \gamma \sum_{k=1}^{L} c_k\nu_C(k) = \gamma(D\nu_C)(1) = \gamma\lambda_C\nu_C(a, 1, b') \).
The formula for $\pi^Q$ follows directly from (2.7) and the preceding identities, while the formula for $K_1 K_2$ follows from the calculation below.

$$\sum_{a,b,b'} \pi^Q(a, b, b') = \sum_{a,b} \pi^Q(a, b, 1) + \sum_{a,b} \pi^Q(a, b, b + 1)$$

$$= K_1 K_2 \left( \sum_{b=1}^L c_b \nu_C(b) \varphi_C(1) + \sum_{b=1}^{L-1} \nu_C(b) \varphi_C(b + 1) \right)$$

$$= K_1 K_2 \sum_{b=1}^L \nu_C(b) (c_b \varphi_C(1) + \varphi_C(b + 1))$$

$$= K_1 K_2 \lambda_C \sum_{b=1}^L \nu_C(b) \varphi_C(b). \quad (2.18)$$

3. This follows from (2.6) and parts 1. and 2. \qed

3. ADDITIVE FUNCTIONALS

3.1. General Theory. In this section we will study some theoretical aspects of large-time behavior of additive functionals of an ergodic finite-state Markov chain, under a change of measure which generalizes the way $Q_n$ was obtained from $Q$. The assumptions in this section are the following:

**Definition 3.1.** Let $Z = (Z_n : n \in \mathbb{Z}_+)$ be an irreducible and aperiodic Markov chain on the finite state space $\mathcal{L}$ with transition function $Q$. Let $\varphi : \mathcal{L} \to (0, \infty)$ be a positive function, and let $\mu$ be a probability distribution on $\mathcal{L}$. For every $n \in \mathbb{Z}_+$, let $Q_n$ be a probability measure on $\sigma(Z_0, \ldots, Z_n)$ given by

$$Q_n(A) = \frac{E^Q_\mu \left( \frac{1}{\varphi(Z_n)} \right)}{E^Q_\mu \frac{1}{\varphi(Z_n)}}, \quad A \in \sigma(Z_0, \ldots, Z_n). \quad (3.1)$$

We will consider the behavior of additive functionals of the form $S_n = \sum_{j=0}^n g(Z_j)$ where $g : \mathcal{L} \to \mathbb{C}$ under $Q_n$ as $n \to \infty$. In the context of generalized Zeckendorf decompositions, an example for an additive functional is the number of, say, nonzero digits in the decomposition. In the next section, we show that gaps in the decomposition can be viewed as additive functionals of some Markov chain, so we can treat them with the same tools.

We need to fix some notation. Functions on $\mathcal{L}$ will interchangeably be viewed as column vectors. As an example, if $g$ is such a function then $Qg$ is to be identified as the function or, equivalently the column $f$ vector given by $f(z) = \sum_{z' \in \mathcal{L}} Q(z, z') g(z)$. We will write $hg$ for the product of such two functions, namely $hg$ is the function given by $(hg)(z) = h(z) g(z), \quad z \in \mathcal{L}$. In addition, $h(Qg)$ means the product of the function $h$ and the function $Qg$, not their scalar product.

Let $\pi^Q$ denote the stationary distribution for $Q$. Recall that $I - Q$ is invertible on the $Q$-invariant subspace of $V$, where $V = \{ g : E_{\pi^Q} g(z) = 0 \}$. We denote this inverse by $Q^\#$, and extend it to all functions by letting $Q^\# 1 = 0$. This is the only choice that guarantees that $Q$ and $Q^\#$ commute,
and $Q^\#$ is known as the group inverse of $Q$. It is well-known that
\[
\sum_{j=0}^\infty E^Q_z(g(Z_j) - E_{\pi Q}g) = (Q^\# g)(z). \tag{3.2}
\]

Our first result is the following.

**Theorem 3.2.** Let $g : \mathcal{L} \to \mathbb{C}$. Let $\tilde{g} = g - E_{\pi Q}g$, and $\tilde{S}_n = \sum_{j=0}^n \tilde{g}(Z_j)$. Then
\[
E^Q_{\mu} \tilde{S}_n = E^Q_{\mu} (Q^\# g) + \frac{E_{\pi Q} \tilde{g} (Q^\# \frac{1}{\varphi})}{E_{\pi Q} \frac{1}{\varphi}} + o(1) \tag{3.3}
\]
\[
E^Q_{\pi Q} \tilde{S}_n^2 = (n + 1) E_{\pi Q} (\tilde{g}((2Q^\# - I)\tilde{g})) + o(1) \quad \text{and} \quad E^Q_{\pi Q} \tilde{S}_n^2 = (1 + o(1)) E^Q_{\pi Q} \tilde{S}_n^2. \tag{3.4}
\]

**Proof.** We will first prove (3.3). From Theorem 2.2 with $f = \tilde{S}_n$, and the Markov property, we have that
\[
E^Q_{\mu} \frac{1}{\varphi(Z_n)} \times E^Q_{\pi Q} \tilde{S}_n = \sum_{j=0}^n E^Q_{\mu} \tilde{g}(Z_j) E_{Z_j} \frac{1}{\varphi(Z_{n-j})}
\]
\[
= \sum_{j=0}^n E^Q_{\mu} \tilde{g}(Z_j) \left( E^Q_{Z_j} \frac{1}{\varphi(Z_{n-j})} - E_{\pi Q} \frac{1}{\varphi} \right) \tag{3.5}
\]
\[
+ E_{\pi Q} \frac{1}{\varphi} \sum_{j=0}^n E^Q_{\mu} \tilde{g}(Z_j).
\]

By (3.2), (II) $\to E_{\mu} (Q^\# \tilde{g}) = E_{\mu} Q^\# g$, because $Q^\#$ maps constant function to 0. In order to estimate (I), we recall that from the exponential ergodicity of irreducible finite state Markov chains, there exists $\rho \in (0, 1)$ and $c_1 > 0$, such that for every function $h$ and $k \in \mathbb{Z}_+$,
\[
\sup_z |E^Q_z h(Z_k) - E_{\pi Q} h| \leq c_1 \|h\|_\infty \rho^k. \tag{3.6}
\]

Letting $h(z) = \frac{1}{\varphi(z)} - E_{\pi Q} \frac{1}{\varphi}$, we have that $E_{\pi Q} h = 0$. This allows us to rewrite (I) as $\sum_{j=0}^n E^Q_{\mu} \tilde{g}(Z_j) E_{Z_j} h(Z_{n-j})$. In order to estimate this sum, we break it into two parts. First
\[
\left| \sum_{j=0}^{[n/2]} E^Q_{\mu} \tilde{g}(Z_j) E_{Z_j} h(Z_{n-j}) \right| \leq \|\tilde{g}\|_\infty \sum_{j=0}^{[n/2]} \sup_z |E_z h(Z_{n-j})| \leq c_1 \|\tilde{g}\|_\infty \|h\|_\infty \rho^{n/2} n/2 \to 0,
\]
where the last inequality follows from (3.6). Next, let $h_k(z) = \tilde{g}(z) E_z h(Z_k)$. Then
\[
\sum_{j=[n/2]+1}^n E^Q_{\mu} \tilde{g}(Z_j) E_{Z_j} \frac{1}{\varphi(Z_{n-j})} = \sum_{j=[n/2]+1}^n E^Q_{\mu} h_{n-j}(Z_j). \tag{3.8}
\]

Applying (3.6) to each of the functions $h_k$, and observing that $\|h_k\|_\infty \leq \|\tilde{g}\|_\infty \|h\|_\infty$, it follows that for $j \geq [n/2] + 1$,
\[
|E^Q_{\mu} h_k(Z_j) - E_{\pi Q} h_k| \leq c_1 \|\tilde{g}\|_\infty \|h\|_\infty \rho^{n/2}. \tag{3.9}
\]
Also, since $\pi^Q$ is the stationary distribution for $Q$, we have that $E_{\pi^Q}h_k = E_{\pi^Q}^Q h_k(Z_j)$, and as a result
\[
\sum_{j=\lfloor n/2 \rfloor+1}^{n} \left( E_{\pi}^Q h_{n-j}(Z_j) - E_{\pi^Q}^Q h_{n-j}(Z_j) \right) \leq c_1 \|\tilde{g}\|_\infty \|h\|_\infty \rho^{n/2} n/2 \to 0. \tag{3.10}
\]
In addition, $E_{\pi^Q}^Q h_{n-j}(Z_j) = E_{\pi^Q}\tilde{g}(Z_0)\tilde{g}_{Z_0}(Z_{n-j})$, and therefore
\[
\sum_{j=\lfloor n/2 \rfloor+1}^{n} E_{\pi^Q}^Q h_{n-j}(Z_j) = \sum_{k=0}^{n-\lfloor n/2 \rfloor} E_{\pi^Q}(Z_0)\tilde{g}_{Z_0}(Z_k). \tag{3.11}
\]
Since by our choice $E_{\pi^Q}h = 0$, it follows from (3.2) that the righthand side is equal to $E_{\pi^Q}\tilde{g}(Q^#h) + o(1)$. As a result, $(I) = E_{\pi^Q}\tilde{g}(Q^#h) + o(1)$, completing the proof of (3.3).

We turn to proving (3.4). We first prove the first equality.
\[
E_{\pi^Q}^Q \left( S_n^2 \right) = \sum_{j=0}^{n} E_{\pi^Q}^Q \tilde{g}^2(Z_j) + 2 \sum_{0 \leq j < k \leq n} E_{\pi^Q}(X_j)\tilde{g}(X_k) = (n+1)E_{\pi^Q}^Q \tilde{g}^2 + 2 \sum_{0 \leq j < k \leq n} E_{\pi^Q}(X_0)E_{X_0}^Q\tilde{g}(X_{k-j})
\]
\[
= -(n+1)E_{\pi^Q}^Q \tilde{g}^2 + 2 \sum_{j=0}^{n} E_{\pi^Q}(X_0) \left( \sum_{k=0}^{n-j} E_{X_0}^Q\tilde{g}(X_k) \right)
\]
\[
= -(n+1)E_{\pi^Q}^Q \tilde{g}^2 + 2 \sum_{j=0}^{n} E_{\pi^Q}(X_0) \left( \sum_{k=0}^{n-j} E_{X_0}^Q\tilde{g}(X_k) \right) - 2 \sum_{j=0}^{n} E_{\pi^Q}(X_0) \left( \sum_{k>n-j} E_{X_0}^Q\tilde{g}(X_k) \right)
\]
\[
= (n+1)E_{\pi^Q}^Q (2Q^# - I)\tilde{g} + (\ast). \tag{3.12}
\]
Observe that by exponential ergodicity, (3.6), $|E_{\pi^Q}^Q \tilde{g}(X_k)| \leq c_1 \|\tilde{g}\|_\infty \rho^k$, uniformly over $z$, and so
\[
|\ast| \leq c_1 \|\tilde{g}\|_\infty \sum_{j=0}^{n} \rho^{n-j+1} 1 - \rho \leq c_1 \|\tilde{g}\|_\infty \frac{1}{(1 - \rho)^2} = O(1). \tag{3.13}
\]
This completes the proof of the first equality in (3.4). It remains to the asymptotic equivalence of $E_{Q_n}^Q \tilde{S}_n^2$ and $E_{\mu}^Q \tilde{S}_n^2$. This, again, follows from the exponential ergodicity, as we now explain. We have
\[
\tilde{S}_n^2 = \tilde{S}_m^2 + 2\tilde{S}_m (\tilde{S}_n - \tilde{S}_m) + (\tilde{S}_n - \tilde{S}_m)^2. \tag{3.14}
\]
From the Markov property and exponential ergodicity (3.6), it follows that
\[
|E_{\mu}(S_n - S_m)^2 - E_{\pi^Q}^Q \tilde{S}_n^2_{n-m}| \leq c_1 \|\tilde{g}\|_\infty n^2 \rho^m. \tag{3.15}
\]
Choose $m = c \ln n$ for $c = 4/\ln(1/\rho)$. It follows that righthand side tends to 0 as $n \to \infty$. In particular, $E_{\mu}(S_n - S_m)^2 \leq c_2 n$. Next, observe that $E_{\mu}^Q \tilde{S}_m^2 \leq \|\tilde{g}\|_\infty^2 n^2$, and by Cauchy-Schwarz,
\[ |E_\mu \tilde{S}_m (\tilde{S}_n - \tilde{S}_m)| \leq \sqrt{E_\mu \tilde{S}_m^2} \sqrt{E_\mu (\tilde{S}_n - \tilde{S}_m)^2} \leq c_3 m \sqrt{n}. \] In summary, for all \( n \) large enough,
\[ |E_\mu (\tilde{S}_m^2 + 2\tilde{S}_m (\tilde{S}_n - \tilde{S}_m))| \leq c_4 (\ln n)^2 \sqrt{n} \leq c_4 n^{3/4}. \] (3.16)
In particular,
\[ |E_\mu \tilde{S}_m^2 - E_{\pi \varphi} Q \tilde{S}_{n-m}^2| \leq c_4 n^{3/4}, \] (3.17)
so that
\[ E_\mu S_n^2 = (1 + o(1))n E_{\pi \varphi} \tilde{g} (2Q# - I) \tilde{g}, \] (3.18)
and the claim is proved.

We turn to laws of large numbers and central limit theorems for additive functionals.

**Theorem 3.3.** Under the same assumptions of Theorem 3.2 we have:

1. **Weak Law of Large Numbers:** For \( \epsilon > 0 \), \( \lim_{n \to \infty} Q_n \left( \frac{\tilde{S}_n}{n+1} \right) > \epsilon \) = 0.
2. **Central Limit Theorem:** \( Q_n \left( \frac{\tilde{S}_n}{\sqrt{n+1}} \leq x \right) \Rightarrow P(Y \leq x) \) where \( Y \sim N(0, \sigma^2) \), and \( \sigma^2 = E_{\pi \varphi} \tilde{g} (2Q# - I) \tilde{g} \).

**Proof of Theorem 3.3** The Weak Law of Large Numbers follows from Chebychev’s inequality and the asymptotic estimate for \( E_{\pi \varphi} Q \tilde{S}_n^2 \) given in Theorem 3.2.

\[ Q_n \left( \left| \frac{\tilde{S}_n}{n+1} \right| > \epsilon \right) \leq \frac{E_{\pi \varphi} Q \tilde{S}_n^2}{(n+1)^2 \epsilon^2} = \frac{E_{\pi \varphi} \tilde{g} (2Q# - I) \tilde{g}}{(n+1) \epsilon^2} \to 0, \text{ as } n \to \infty. \] (3.19)

We now prove the Central Limit Theorem. To do this we apply Proposition 2.3 with \( j_n = n - \lfloor \ln n \rfloor \) and
\[ f_n = \exp \left( \frac{i\theta}{\sqrt{n+1}} \tilde{S}_n \right). \] (3.20)
Observe that the choice of \( j_n \) guarantees that condition 1. in the proposition holds. Next,
\[ E_z |f_n - f_{j_n}| \leq E_z |1 - E_z |e^{i\theta \tilde{S}_n-j_n}| \leq \max_z \left( |1 - E_z \cos \left( \frac{\theta \tilde{S}_n-j_n}{\sqrt{n+1}} \right)| + |E_z \sin \left( \frac{\theta \tilde{S}_n-j_n}{\sqrt{n+1}} \right)| \right). \] (3.21)
Since \( |S_n-j_n| = O(\ln n) \), it follows from bounded convergence that \( \sup_z E_z |f_n - f_{j_n}| \to 0 \), and so condition 2. holds. Finally, we recall from the Central Limit Theorem for additive functionals of finite state Markov chains (e.g. [MW00], [BAN12], Theorem 5), that
\[ E_\mu (f_n) \to e^{-\frac{\sigma^2}{2}}, \] (3.22)
where \( \sigma^2 = \lim_{n \to \infty} \frac{1}{n+1} E_{\pi \varphi} Q \tilde{S}_n^2 \). The result now follows from Theorem 3.2.

### 3.2. Application to Zeckendorf

In this section we show how the results obtained in Section 3.1 apply to generalized Zeckendorf decompositions. In particular we will show that the generalized Lekkerkerker’s theorem (Theorem 3.4) and the corresponding Central Limit Theorem are special cases to Theorem 3.2.1 and Theorem 3.3.2. We will also carry out explicit computations for the standard Zeckendorf decomposition, where all quantities are easily computable.

In order to apply the results in the context of generalized Zeckendorf decomposition, in Definition 3.1 we identify \( L, Z \) and \( Q \) in the definition as the same quantities defined in Section 2 and
also set \( \varphi = \varphi_c \), and \( \mu = \bar{\varphi}_c \), where \( \varphi_c \) and \( \bar{\varphi}_c \) are as in Section 2. With these choices, the measure \( Q_n \) of Definition 3.1 coincides with \( Q_n \) of Section 2.

Recall \( k(N) \), the number of nonzero summands in the generalized Lekkerker decomposition of \( N \), defined in (1.4). Let \( g : \mathcal{L} \to \{0, 1\} \) be defined as \( g(x, j, j') = 1 \) if and only if \( x > 0 \). Then if \( N \in [G_{n+1}, G_{n+2}) \), the from (2.1) have that that \( k(N) = S_n \), where \( S_N \) is the additive functional \( S_n = \sum_{j=0}^{n} g(Z_j) \). Observe that \( E_{\pi_Q} g = 1 - \pi_1(0) \), and so \( \bar{g} = g - 1 + \pi_1(0) \). Furthermore, since \( \pi^Q(z) = \varphi_c(z)\nu_c(z) \), it follows that \( E_{\pi^Q} \frac{1}{\varphi_c} = \|\varphi\|_1 \). The following therefore follow immediately from Theorem 3.2 and Theorem 3.3.

**Corollary 3.4.** For generalized Zeckendorf decomposition:

1. **Generalized Lekkerkerer’s Theorem (Th. 3.4):**
   \[
   E^Q_n k(N) = C_{\text{Lek}}(n + 1) + d
   \]  
   where
   \[
   C_{\text{Lek}} = 1 - \pi_1(0), \quad d = E_{\varphi_c} \pi^#(1 - \delta) + \frac{E_{\varphi_c} (1 - \delta)(\pi_c^#)}{\|\varphi_c\|_1}
   \]  
   (3.24)

2. **Variance:**
   \[
   E^Q_n (k(N) - C_{\text{Lek}}(n + 1))^2 = (1 + o(1))(n + 1)^2
   \]
   where
   \[
   \sigma^2 = E_{\pi_Q} \bar{g}(2\pi^# - I)\bar{g}.
   \]  
   (3.26)

**Corollary 3.5.** For generalized Zeckendorf decomposition:

1. **Law of Large Numbers:**
   \[
   Q_n (|k(N) - C_{\text{Lek}}(n + 1)| > n\epsilon) \to 0.
   \]  
   (3.27)

2. **Central Limit Theorem:**
   \[
   Q_n \left( \frac{k(N) - C_{\text{Lek}}(n + 1)}{\sqrt{n + 1}} \in \cdot \right) \to N(0, \sigma^2)
   \]  
   (3.28)

where \( \sigma^2 \) is as in Corollary 3.4.

In the remainder of the section we compute all constants above for the standard Zeckendorf decomposition. First we need to compute \( \pi^# \).

**Example 3.6.** For the standard Zeckendorf decomposition,

\[ Q^# = \frac{1}{5} \begin{pmatrix} 5 - \phi & \phi - 4 & -1 \\ -\phi & \phi + 1 & -1 \\ 1 - 3\phi & 2\phi - 2 & \phi + 1 \end{pmatrix}, \]  
(3.29)

To prove the identity, recall the expressions for \( Q \) and \( \pi^Q \) computed in Example 2.6. Let \( A = I - Q \), and let \( v_1 = (0, 1, -1)^t \), \( v_2 = (1, 0, -\phi)^t \), and \( v_3 = (1, 1, 1)^t \). Then \( E_{\pi^Q} v_1 = E_{\pi^Q} v_2 = 0 \). Since \( v_1 \) and \( v_2 \) are linearly independent, it follows that they span the \( A \)-invariant space \( V = \{v : E_{\pi^Q} v = 0\} \). In addition \( Av_3 = 0 \). Letting \( q = 1 - \frac{1}{\lambda_c} = 1 - \frac{1}{\bar{\varphi}_c} \), a straightforward calculation shows that \( Av_1 = qv_2 + (1 + q)v_1 \), and \( Av_2 = v_2 \). Thus \( v_1 = qv_2 + (1 + q)Q^# v_1 \),
\[ Q^# \nu_2 = \nu_2 \text{ and } Q^# \nu_3 = 0. \text{ These determine } Q^#. \]

Also, from Example 2.6 we have that \( \pi_1(0) = \phi + 1, \varphi_c \) is a point mass, and \( \|\nu_c\|_1 = \frac{5\phi + 3}{\phi + 2} \). In addition, \( \pi^Q = \frac{1}{\phi + 2} (\phi, 1, 1)^t \), and \( \varphi_c = \frac{1}{3\phi + 1} (\phi, \phi, 1)^t \). Since also \( g = (0, 0, 1)^t \), we have \( Q^# g = \frac{1}{5} (1 - 3\phi, 2\phi - 2, \phi + 1)^t \), and \( Q^# \frac{1}{\varphi_c} = \frac{1}{5(\phi - 1)} (-1, -1, \phi + 1)^t \). As a result, we have the following.

**Example 3.7.** For the standard Zeckendorf:

\[ C_{Lek} = \frac{1}{\phi + 2} = \frac{5 - \sqrt{5}}{10}, \quad d = \frac{3}{5}. \] (3.30)

We finally compute \( \sigma^2 \). Clearly, \( \bar{g} = (0, 0, 1)^t - \frac{1}{2 + \phi} (1, 1, 1)^t = \frac{1}{2 + \phi} (-1, -1, 1 + \phi)^t \). It therefore follows that \( \bar{g} Q^# \bar{g} = \bar{g} Q^# (0, 0, 1)^t = \frac{1}{5} \left( \frac{1}{\phi + 2}, \frac{1}{2 + \phi}, (1 - \frac{1}{2 + \phi})(1 + \phi) \right)^t \), and so the expectation is equal to

\[ 2E_{\pi Q} \bar{g} Q^# \bar{g} = \frac{2}{5} \left( 1 + \phi + (1 + \phi)^2 \right) = \frac{2(\phi + 2)}{25}. \] (3.31)

Since \( \bar{g}^2 = \left( \frac{1}{(\phi + 2)^2}, \frac{1}{(\phi + 2)^2}, (\phi + 1)^2 \right)^t = \frac{1}{5(1 + \phi)} (1, 1, (1 + \phi)^2)^t \), it follows that

\[ E_{\pi Q} \bar{g}^2 = \frac{1}{5(1 + \phi)} \left( \frac{(\phi + 1) + (\phi + 1)^2}{\phi + 2} \right) = \frac{1}{5}. \] (3.32)

We therefore have

**Example 3.8.** For the standard Zeckendorf decomposition: \( \sigma^2 = \frac{2\phi - 1}{25} = \frac{\sqrt{5}}{25} \).

### 4. Gaps in Zeckendorf Decomposition

**4.1. Gap Distribution.** In this section we consider the asymptotic distribution of gaps between non-zero terms in the generalized Zeckendorf decomposition. This will be an application of our results on additive functionals from the previous section. We will first prove a statement on an “average” gap distribution, Theorem 4.1, and we will later prove convergence of empirical gap measures in probability, Theorem 4.2. Let us first define the notion of a gap. We work under the same assumptions and notation as in Section 2. Suppose that \( N \in \mathbb{N} \) admits a legal decomposition (2.1) with \( X_0 > 0 \). Note that \( X_j \) counts the repetitions of \( G_{n-j+1} \), and if repeating more than 1 times, we can view this as \( X_j - 1 \) gaps of size zero. If if \( X_j > 0 \), than we have a gap of size 1 or larger, the size of the gap equal to \( \min\{k \geq 1 : X_{j+k} > 0\} \). Let \( N_n(k) \) denote the number of gaps of size \( k \) in the first \( n \) digits, and let \( N_n = \sum_k N_n(k) \). We define the gap distribution \( \mu_n \) as a probability measure on \( \mathbb{Z}_+ \) given by

\[ \mu_n(k) = \frac{E^Q_n N_n(k)}{E^Q_n N_n}. \] (4.1)

To state the next theorem, let

\[ \nu(k) = \lambda_C^{-(k-1)}(1 - \lambda_C^{-1}) \] (4.2)

denote the probability density of a Geometric random variable with parameter \( \lambda_C^{-1} \). We have
**Theorem 4.1.** Let $H_1 = \{(0, b, 1) \in \mathcal{L}\}$ and $H_2 = \{(0, b+1, b+2) \in \mathcal{L} : c_b > 0, c_{b+1} = 0\}$. For $z = (0, b+1, b+2) \in H_2$ we let

$$r(b) = \max\{j : c_{b+j} = 0\},$$

$$\rho(b) = Q((0, b+r(b), b+r(b)+1), (0, b+r(b)+1, 1)) = \frac{\varphi_C(1)}{\lambda_C \varphi_C(b+r(b)+1)},$$

and

$$h(b, k) = \begin{cases} 0 & k < r(b) + 1 \\ 1 - \rho(b) & k = r(b) + 1 \\ \rho(b) \lambda_C^{(k-r(b)-2)}(1 - \lambda_C^{-1}) & k > r(b) + 1. \end{cases}$$

Then

(1) \(\lim_{n \to \infty} \frac{1}{n} E^{Q_n} N_n = M_{\pi_1}^Q\).

(2) \(\lim_{n \to \infty} \mu_n(k) = \begin{cases} 1 - \frac{1 - \pi_Q(0)}{M_{\pi_1}^Q} & k = 0 \\ \frac{1 - \pi_Q(0) - \pi_Q(H_1)(1 - \lambda_C^{-1}) - \sum_{z \in H_2} \pi_Q(z)(1 - \rho(z))}{M_{\pi_1}^Q} & k = 1. \end{cases}\)

(3) For \(k \geq 2\),

$$\lim_{n \to \infty} \mu_n(k) = \frac{\pi_Q(H_1) \nu(k - 1)}{M_{\pi_1}^Q} + \frac{\sum_{z \in H_2} \pi_Q(z)(h(z(2) - 1, k) - \rho(z(2)) \nu(k - 1))}{M_{\pi_1}^Q}.$$  

Since \(\sum_{k \geq 2} \nu(k - 1) = \sum_{k \geq 2} h(b, k) = 1\), it follows that the limit \(\lim_{n \to \infty} \mu_n(\cdot)\) is a probability measure, which we denote by \(\mu_\infty\). A simple argument shows that a stronger result holds. For \(n \in \mathbb{N}\), define the empirical gap distribution \(\hat{\mu}_n\) as a random measure on \(\mathbb{Z}_+\), defined by

$$\hat{\mu}_n(A) = \sum_{k \in A} N_n(k) / \max(N_n, 1).$$

We therefore have the following.

**Theorem 4.2.** For any \(A \subset \mathbb{Z}_+\) and \(\epsilon > 0\),

$$\lim_{n \to \infty} Q_n(\{\hat{\mu}_n(A) - \mu_\infty(A)\} > \epsilon) = 0.$$  

We comment that the expression for the limit in Theorem 4.1 is much simpler when \(c_j > 0\) for all \(j = 1, \ldots, L\). In this case \(H_2 = \emptyset\). For the standard Zeckendorf, we have the following.

**Example 4.3.** For the standard Zeckendorf decomposition, \(M_{\pi_1}^Q = \pi_1^Q(1) = \frac{1}{\phi+2}\) and \(\lambda_C = \phi\). Therefore

(1) \(\lim_{n \to \infty} \frac{1}{n} E^{Q_n} N_n = \frac{1}{\phi+2}\).

(2) \(\lim_{n \to \infty} \mu_n(k) = \begin{cases} 0 & k = 0, 1 \\ \phi^{-k} & k \geq 2. \end{cases}\).
When some of the coefficients are zero, then some gaps of size $\geq 2$ are forced by the recurrence relation, and taking this into account is the source of the lengthy expression in the theorem.

Example 4.4. Consider the recurrence relation with $L = 4$, $c_1 = 1$, $c_2 = c_3 = 0$, $c_4 = 2$. Then $\lambda_C$ is the largest (real) root of $\lambda^3(\lambda - 1) = 2$, $\lambda_C \approx 1.5437$. We have

$$h(k) = \begin{cases} 0 & k < 3 \\ \frac{1}{\lambda_C} & k = 3 \\ \frac{1}{2}\lambda_C^{-(k-4)}(1 - \lambda_C^{-1}) & k \geq 4 \end{cases} \quad (4.9)$$

and

$$\lim_{n \to \infty} \mu_n(k) = \begin{cases} 0 & k = 0 \\ 2 - \frac{\lambda_C^2 + 1}{3\lambda_C} & k = 1 \\ \nu(k - 1) + \frac{2\lambda_C - 1}{3\lambda_C}h(k) & k \geq 2. \end{cases} \quad (4.10)$$

In this example,

$$\mathcal{L} = \{z^1 = (0, 1, 1), z^2 = (1, 1, 2), z^3 = (0, 2, 3), z^4 = (0, 3, 4), z^5 = (0, 4, 1), z^6 = (1, 4, 1)\}. \quad (4.11)$$

There are no gaps of size 0 as the coefficients immediately show. Gaps of size 1 only appear in the form $(1, 4, 1)$ followed by $(1, 1, 2)$. Larger gaps can be formed as follows.

- Gaps of size $k \geq 2$ through a sequence of the form $(1, 4, 1), (0, 1, 1), \ldots, (1, 1, 2)$, with $(0, 1, 1)$ repeated $k - 1$ times.
- Gaps of size $k \geq 3$ through a sequence beginning with $(1, 1, 2), (0, 2, 3), (0, 3, 4)$, followed by $(1, 4, 1)$ if length is 3, or by $k - 3$ repetitions of $(0, 1, 1)$ followed by $(1, 1, 2)$ otherwise.

The larger gaps of the second type are forced by the recurrence, in the sense that the condition $c_2 = c_3 = 0$ implies $Q((1, 1, 2), (0, 2, 3)) = Q((0, 2, 3), (0, 3, 4)) = 1$, and so every time the sequence hits the state $(1, 1, 2)$, a gap of minimal size 3 occurs. Let us see how this is reflected in the formula. $H_1 = \{(0, 1, 1), (0, 4, 1)\}$ and $H_2 = \{(0, 2, 3)\}$. There's only one element in $H^2$ and therefore we omit the reference to $b$ in the functions $r, \rho, h$. So $r = 2, \rho = Q((0, 3, 4), (0, 4, 1)) = \frac{1}{2}$, and the expression for $h$ follows.

We now compute $\pi^Q$. Let $p = \pi^Q(z^2)$. Since $Q(z^2, z^3) = Q(z^3, z^4) = 1$, we have that $p = \pi^Q(z^3) = \pi^Q(z^4)$. Next, $Q(z^4, z^5) = Q(z^4, z^6) = \frac{1}{2}$, and so $\pi^Q(z^5) = \pi^Q(z^6) = p/2$. We also observe that

$$\pi^Q(z^1) = \pi^Q(z^1)\lambda_C^{-1} + \pi^Q(z^5)Q(z^5, z^1) + \pi^Q(z^6)Q(z^6, z^1) \quad (4.12)$$

Therefore, $\pi^Q(z^1) = \frac{p}{\lambda_C - 1}$. Now we have $1 = \frac{p}{\lambda_C - 1} + 4p$, so altogether, $p = \frac{\lambda_C - 1}{4\lambda_C - 3}$, and the expression for the limit of $\mu_n$ follow after some algebra.

**Proof of Theorem 4.1** For a real number $x$, let $x_+ = \max(x, 0)$. We begin with gaps of size 0:

$$E^{Q}_2 N_n(0) = \sum_{j=0}^{n-1} (Z_j(1) - 1)_+. \quad (4.13)$$

The ergodicity of $Z$ under $Q$ implies that

$$\lim_{n \to \infty} \frac{E^{Q}_2 N_n(0)}{n} = \sum_{z=(x,j,j')} \pi^Q(z)(x-1)_+ = M_{\pi^Q} - 1 + \pi^Q_1(0). \quad (4.14)$$
Before moving to gaps of larger size, we consider the total number of jumps. We have
\[
\frac{1}{n} E_z^Q \sum_{k \geq 1} N_n(k) = \frac{1}{n} E_z^Q \sum_{j=0}^{n-1} 1_{\{Z_j(0) > 0\}} \rightarrow 1 - \pi_1^Q(0),
\]
(4.15)
and so from (4.14), (4.15)
\[
\lim_{n \to \infty} \frac{1}{n} E_z^Q N_n = M_{\pi_1^Q}.
\]
(4.16)
We move to calculation of gaps of size \( \geq 2 \). We will treat gaps of size 1 last. Let \( k \geq 2 \). Then
\[
\frac{1}{n} E_z^Q N_n(k) = \frac{1}{n} E_z^Q \sum_{j=0}^{n-k} 1_{\{Z_j(1) > 0\}} \left( \prod_{\ell=1}^{k-1} 1_{\{Z_{j+\ell}(1) = 0\}} \right) 1_{\{Z_{j+k}(1) > 0\}}.
\]
(4.17)
Let \( B = \{(0, b, b') \in \mathcal{L}\} \). It therefore follows from the Markov property and ergodicity that
\[
\lim_{n \to \infty} \frac{1}{n} E_z^Q N_n(k) = \sum_{z^0 \in A} \pi^Q(z^0) f_B(z^0)
\]
(4.18)
where for \( D \subset \mathcal{L} \) we have
\[
f_D(z^0) = \left( \sum_{z^1 \in D, \ldots, z^{k-1} \in B} \prod_{\ell=1}^{k-1} Q(z^{\ell-1}, z^\ell) \right) Q(z^{k-1}, A).
\]
(4.19)
Letting
\[
B_0 = \{(0, 1, 1)\}
\]
\[
B_1 = \{(0, b + 1, 1) \in \mathcal{L} : b \geq 1, c_b > 0\},
\]
\[
B_2 = \{(0, b + 1, b + 2) \in \mathcal{L} : b \geq 1, c_b > 0, c_{b+1} = 0\}, \text{ and}
\]
\[
B_3 = \{(0, b + 1, 1) \in \mathcal{L} : b \geq 1, c_b = 0, c_{b+1} > 0\},
\]
(4.20)
we can write
\[
\sum_{z^0 \in A} \pi^Q(z^0) f_B(z^0) = \sum_{m=0}^{3} \sum_{z^0 \in A} \pi^Q(z^0) f_{B_m}(z^0).
\]
(4.21)
Note that \( \bigcup_{m=0}^{3} B_m = \{(0, b, b') \in \mathcal{L} : c_b \neq 0 \text{ or } c_{b'} \neq 0\} \), and so this union does not necessarily contain all elements \((0, b, b') \in \mathcal{L}\). However, it does contain all such elements which are accessible from \( A \) in one step (and more, whenever \( B_3 \) is not empty).

We now simplify the expression, beginning with the sum over \( B_1 \). It is important to observe that \( B_1 \) is the subset of states in \( B \) accessible in one step only from \( A \). In addition, if \( z^1 \in B_1 \), then it immediately follows that \( z^2 = \cdots = z^{k-1} = (0, 1, 1) \), and that allowed transition to \((0, 1, 1)\) always have probability \( \lambda_{C^{-1}} \). As a result, we have that
\[
f_{B_1}(z^0) = Q(z^0, z^1) \lambda_{C^{-1}(k-2)}(1 - \lambda_{C^{-1}}),
\]
(4.22)
and thus
\[
\sum_{z^0 \in A} \pi^Q(z^0) f_{B_1}(z^0) = \pi^Q(B_1) \nu(k - 1).
\]
(4.23)
Next we consider the sum over $B_0$, namely $z^1 = (0, 1, 1)$. Clearly:
\[
\sum_{x^0 \in A} \pi^Q(z^0)f_{(0,1,1)}(z^0) = \sum_{x^0 \in A} \pi^Q(z^0)f_{(0,1,1)}(z^0) - \sum_{x^0 \in B} \pi^Q(z^0)f_{(0,1,1)}(z^0).
\]

Since $(0, 1, 1)$ is accessible in one step either from $A$ or from states in $z \in B_0 \cup B_1 \cup B_3$ and for all such $z$, $Q(z, (0, 1, 1)) = \lambda_C^{-1}$, it follows that
\[
\sum_{x^0 \in A} \pi^Q(z^0)f_{(0,1,1)}(z^0) = (\pi^Q((0, 1, 1)) - \pi^Q(B_0 \cup B_1 \cup B_3)\lambda_C^{-1})\nu(k-1).
\]

Hence,
\[
\sum_{x^0 \in A} \pi^Q(z^0)f_{B_0 \cup B_1}(z^0) = \pi^Q(B_0 \cup B_1 \cup B_3)(1 - \lambda_C^{-1})\nu(k-1) - \pi^Q(B_3)\nu(k-1). \tag{4.25}\]

We now consider $z^1 \in B_2$. Suppose then that $z^0 \in A$ and $z^1 \in B_2$ and $Q(z^0, z^1) > 0$. Since $z^1 = (0, b+1, b+2)$, it follows that $z^0 = (c_b, b, b+1)$ and $c_b > 0$. Now if $c_{b+2} = 0$, then the only allowed transition form $z^1$ is to $z^2 = (0, b+2, b+3)$. Let $r = r(b)$ and $\rho = \rho(b)$ as defined in the statement of the theorem. Then $z^j = (0, b+j, b+j+1)$ for all $j = 1, \ldots, r$, and we conclude that $Q(z^j, z^{j+1}) = 1$ for $j = 0, \ldots, r$. We continue according the the following two cases.

1. $r > k - 1$. In this case $Q^k(z^0, A) = 0$.
2. $r \leq k - 1$. Then either
   - $r = k - 1$, in which case $Q^k(z^0, A) = Q((0, b+r, b+r+1), A) = 1 - \rho$; or
   - $1 < r \leq k - 2$, in which case $z^{r+1} = (0, b+r+1, 1)$ and $z^{r+l} = (0, 1, 1)$ for all $2 \leq l \leq k - 1 - r$. In particular, since $Q((0, 1, 1), A) = Q((0, b+r+1, 1), A) = 1 - \lambda_C^{-1}$, we have that
     \[
     Q^k(z^0, A) = Q((0, b+r, b+r+1), (0, b+r+1, 1))\nu(k-r-1). \tag{4.27}\]

The only allowed transitions from $(0, b+r, b+r+1)$ to $(x, b+r+1, 1)$ are to $x = 0, \ldots, c_{b+r+1} - 1$, all with equal transition probability. Since there are exactly $c_{b+r+1} - \delta_L(b+r+1)$ possible values for $x$, exactly one of which is with $x = 0$, letting $\rho(b) = \frac{1}{c_b - \delta_L(b)}$, we have
   \[
   Q^k(z^0, A) = \begin{cases} 
   0 & k < r(z) + 1 \\
   1 - \rho(b+r+1) & k = r(z) + 1 \\
   \rho(b+r+1)\nu(k-r-1) & k > r(z) + 1.
   \end{cases} \tag{4.28}\]

Summarizing the two cases, we conclude that
\[
\sum_{x^0 \in A} \pi^Q(z^0)f_{B_2}(z^0) = \sum_{z^1 \in B_2} h(z^1(2) - 1, k). \tag{4.29}\]

Next, when $z^0 \in A$, and $z^1 \in B_3$, then $Q(z^0, z^1) = 0$. Thus, we have proved
\[
\sum_{m=0}^{3} \pi^Q(z^0)f_{B_m}(z^0) = \left((1 - \lambda_C^{-1})\pi^Q(H_1) - \pi^Q(B_3)\right)\nu(k-1) + \sum_{z^0 = (0, b+1, b+2) \in B_2} \pi^Q(z^0)h(b-1, k). \tag{4.30}\]

Let $z' \in B_3$. Then there exists a unique $z^1 = (0, b+1, b+2) \in B_2$ such that $z^1 = (0, b, b+1)$, $z^2 = (0, b+2, b+3), \ldots, z^{r(b)} = (0, b+r(b), b+r(b)+1)$ and $z^{r(b)+1} = z'$. Since $Q(z^k, z^{k+1}) = 1$ for $k = 1, \ldots, r(b) - 1$, it easily follows that $\pi^Q(z') = \pi^Q(z^r)\rho(b) = \pi^Q(z^{r-1})\rho(b) = \ldots =
\[ \pi^Q(z^1) \rho(b). \] This shows that \( \pi^Q(B_3) = \sum_{\{z^1=(0,b+1,b+2)\in B_2\}} \pi^Q(z^1) \rho(b). \) Plugging this into the formula above, and noting that \( H_1 \) in the theorem is \( B_0 \cup B_1 \cup B_3 \) and \( H_2 \) in the theorem is \( B_2 \), we obtain

\[
\lim_{n \to \infty} \frac{1}{n} E_z^Q N_n(k) = \sum_{m=0}^{2} \pi^Q(z^0) f_{B_m}(z^0) = (1 - \lambda_C^{-1}) \pi^Q(H_1) \nu(k-1) + \sum_{z^0=(0,b+1,b+2)\in H_2} \pi^Q(z^0) (h(b - 1, k) - \rho(b) \nu(k-1)) \quad (4.31)
\]

We turn to gaps of size 1:

\[
\frac{1}{n} E_z^Q N_n(1) = \frac{1}{n} E_z^Q \sum_{j=0}^{n-1} 1\{Z_j(1)>0\} 1\{Z_{j+1}(1)>0\} = \frac{1}{n} \sum_{j=0}^{n-1} E^Q_z 1\{Z_j(1)>0\} E^Q_{Z_j} 1\{Z_1(1)>0\}, \quad (4.32)
\]

where in the second line we applied the Markov property. Let

\[ A = \{(x, b, b') \in L : x > 0\}. \quad (4.33) \]

Ergodicity of \( Z \) under \( Q \) then gives

\[
\lim_{n \to \infty} \frac{1}{n} E_z^Q N_n(1) = \sum_{z \in A} \pi^Q(z) Q(z, A) = \pi^Q(A) - \sum_{z \in A^c} \pi^Q(z) Q(z, A). \quad (4.34)
\]

Given \( z = (0, b, b') \in A^c \), exactly one of the following holds.

- \( c_b = 0, b' = b + 1, c_{b+1} = 0 \), and then \( Q(z, A) = 0 \).
- \( c_b = 0, b' = b + 1, c_{b+1} > 0 \). From the argument in the paragraph above (4.31), and since \( Q(z, A^c) = 1 - Q(z, A^c) \) we obtain that

\[
\sum_{\{z=(0,b,b+1)\in L : c_b=0, c_{b+1}=1\}} \pi^Q(z) Q(z, A) = \pi^Q(B_2) - \pi^Q(B_3) = \sum_{z^0=(0,b+1,b+2)\in H_2} \pi^Q(z^0) (1 - \rho(b)). \quad (4.35)
\]

- \( c_b > 0 \) and then \( b' = 1 \), equivalently, \( z \in H_1 \), in which case \( Q(z, A) = 1 - Q(z, (0,1,1)) = 1 - \lambda_C^{-1} \).

Summarizing,

\[
\lim_{n \to \infty} \frac{1}{n} E_z^Q N_n(1) = 1 - \pi^Q(0) - (1 - \lambda_C^{-1}) \pi^Q(H_1) - \left( \sum_{z^0=(0,b+1,b+2)\in H_2} \pi^Q(z^0) (1 - \rho(b)) \right). \quad (4.36)
\]

To finish the proof, we need to show that the results continue to hold when considering the measure \( Q^n \) instead of \( Q \). However, by the Markov property, the expectation under \( Q^n \) of \( N_n \), and \( N_n(k) \) are equal to the expectations of corresponding additive functionals. Therefore it follows
from Theorem 3.2 that the expectations of \( N_n(k) \) and \( N_n \) under \( Q_n \) are asymptotical equivalent to their expectations with respect to \( Q_{\varphi_c} \). The theorem now follows.

**Proof of Theorem 4.2**

We have

\[
\{ |\hat{\mu}_n(A) - \mu_\infty(A)| > \epsilon \} \subset \bigcup_{k \in A} \{ |\hat{\mu}_n(k) - \mu_\infty(k)| > \epsilon \} = \bigcup_{k \in A} \{ |N_n(k) - \mu_\infty(k)N_n| > \epsilon N_n \} \cup \{N_n = 0\}. \tag{4.37}
\]

Since \( Q_n(N_n = 0) = Q_n(Z_0 > 0, Z_1 = \ldots = Z_n = 0) \to 0 \), we can ignore the event \( \{N_n = 0\} \).

Now for every fixed \( k \in A \), we have

\[
\{ |N_n(k) - \mu_\infty(k)N_n| > \epsilon N_n \} \subset \{ |N_n(k) - \mu_\infty(k)E_{\pi Q}^Q N_n| > \epsilon/2 \} \cup \{N_n - E_{\pi Q}^Q N_n| > \epsilon/2 \}. \tag{4.38}
\]

Next observe that both \( N_n \) and \( N_n(k) \) are additive functionals for the process \( Z^k = (Z^k_n : k \in \mathbb{Z}_+) \), where \( Z^k_n = (Z_n, Z_{n+1}, \ldots, Z_{n+k}) \), and so we can consider \( N_n(k) \) and \( N_n \) as additive functionals of \( Z^k \). Letting \( \varphi'_c(z^0, z^1, \ldots, z^k) = \varphi_c(z^0) \) and \( \varphi'_c(z^0, \ldots, z^k) \), the distribution of \( Z_0, \ldots, Z_k \) under \( Q_{\varphi_c} \), then if as in Definition 3.1 we define

\[
Q'_{n,k}(A) = \frac{E_{\varphi_c}^Q \left( \frac{1}{\varphi_c(Z_n^k)} \right)}{E_{\varphi'_c}^Q \left( \frac{1}{\varphi'_c(Z_n^k)} \right)}, \tag{4.39}
\]

it follows that the restriction of \( Q'_{n,k} \) to events generated by \( Z_0, \ldots, Z_n \) coincides with \( Q_n \). In particular, the distribution of the additive functionals \( N_n \) and \( N_n(k) \) for \( Z^k \) under \( Q_{n,k'} \) coincides with their distribution under \( Q_n \). From the variance estimate (3.4) in Theorem 3.2 applied to these additive functionals under \( Q'_{n,k} \), we conclude that

\[
Q_n(\{ |N_n(k) - \mu_\infty(k)E_{\pi Q}^Q N_n| > \epsilon/2 \}) = O(n^{-1}) \quad \text{and} \quad Q_n(\{ |N_n - E_{\pi Q}^Q N_n| > \epsilon/2 \}) = O(n^{-1}). \tag{4.40}
\]

Therefore if \( A \) is finite, we obtain that

\[
\lim_{n \to \infty} Q_n( |\hat{\mu}_n(A) - \mu_\infty(A)| > \epsilon ) = 0. \tag{4.41}
\]

Now if \( A \) is infinite, letting \( A_M = A \cap \{0, \ldots, M\} \), we observe that

\[
|\hat{\mu}_n(A) - \mu_\infty(A)| = |\hat{\mu}_n(A_M) - \mu_\infty(A_M)| + \hat{\mu}_n(\{M+1, \ldots\}) + \mu_\infty(\{M+1, \ldots\}) \leq |\hat{\mu}_n(A_M) - \mu_\infty(A_M)| + \hat{\mu}(\{M+1, \ldots\}) + \mu_\infty(\{M+1, \ldots\}). \tag{4.42}
\]

Fix \( \epsilon \), and let \( M \) be such that \( \mu_\infty(\{M+1, \ldots\}) < \epsilon \). Thus for \( n \) large enough,

\[
\{ |\hat{\mu}_n(A) - \mu_\infty(A)| > 5\epsilon \} \subset \{ |\hat{\mu}_n(A_M) - \mu_\infty(A_M)| > 2\epsilon \} \cup \{ \hat{\mu}_n(\{M+1, \ldots\}) > 2\epsilon \}. \tag{4.43}
\]

The measure of the first event on the right-hand side tends to 0 as \( n \to \infty \) by (4.41). As for the second event, it is equal to the event \( \mu_\infty(\{0, \ldots, M\}) < 1 - 2\epsilon \). However, since, again by (4.41)

\[
\mu( |\hat{\mu}_n(\{0, \ldots, M\}) - \mu_\infty(\{M+1, \ldots, \})| > \epsilon/2 ) \to 0, \quad \text{it follows that} \quad Q_n( |\hat{\mu}_n(\{0, \ldots, M\}) > 1 - 3\epsilon/2 ) \to 0.
\]

But this event is \( Q_n( |\hat{\mu}_n(\{M+1, \ldots\}) < 3\epsilon/2 \) \), and so \( Q_n( |\hat{\mu}_n(\{M+1, \ldots\}) > 2\epsilon ) \) tends to 0 as well. The result now follows.  

\[\square\]
4.2. Maximal Gap. Next we consider the maximal gap \( M_n \), defined as
\[
M_n = \sup\{k \in \mathbb{Z}_+ : N_n(k) > 0\}. \tag{4.44}
\]

Although we can prove the results at the same level of generality as in the previous section, we prefer to keep the expressions cleaner and simpler, and will assume throughout this section that \( c_1, \ldots, c_L > 0 \).

Our analysis is based on a renewal structure we now describe. We refer to the gaps of length \( k \geq 2 \) as “long gaps”, and denote the lengths of the long gaps, indexed by order of appearance, by \((R_j : j \in \mathbb{N})\). Observe that any long gap is followed by a (possibly empty) sequence of length-0 and length-1 gaps, independent of \( k \), and is then followed again by an independent long gap. The number of the small gaps is bounded above by \((L - 1) + \sum_i (c_i - 1) = (\sum_i c_i) - 1\), as the first summand bounds the number of length-1 gaps and the second summand bounds the number of length 0-gaps. Let \( T_m \) denote the first time exactly \( m \) long gaps are completed, \( m(n) = \sup\{m : T_m \leq n\} \). Observe that a long gap is completed whenever the digit zero is followed by a nonzero digit. Therefore
\[
m(n) = \sum_{j=0}^{n-1} 1_0(Z_j(1))1_{\{Z_{j+1}(1) > 0\}}. \tag{4.45}
\]

From the Markov property,
\[
E^Q m(n) = \sum_{j=0}^{n-1} E^Q (1_0(Z_j(1))QZ_j(Z_1(1) > 0)), \tag{4.46}
\]

Letting \( A = \{z = (x, b, b') \in \mathcal{L} : x > 0\} \), and repeating a similar computation as in the proof of the case \( k = 1 \) in Theorem 4.1, it follows that
\[
\lim_{n \to \infty} \frac{1}{n} E^Q m(n) = \sum_{z \in A^c} \pi^Q(z)Q(z, A) = \pi^Q(A^c) - \sum_{z \in A} \pi^Q(z)Q(z, A)
= (1 - \pi^Q(0)) - (1 - \pi^Q(0)) + (1 - \lambda_C^{-1})\pi^Q(0), \tag{4.47}
\]
where the last equality follows from (4.34) and (4.36). Also, by the renewal theorem [Dur10, Theorem 2.4.6]
\[
\lim_{n \to \infty} \frac{m(n)}{n} = \alpha, \text{ Q-a.s.}, \tag{4.48}
\]
where \( \alpha = 1/E^Q T_1 \) and \( \rho \) is the uniform distribution on \( c_1 \) elements: \((x, 1, 1), 1 < x < c_1 \) and \((c_1, 1, 2)\). The limit above also holds in \( L^1(Q) \), as \( m(n) \leq n \). Consequently
\[
\alpha = \pi^Q_1(0) \left(1 - \frac{1}{\lambda_C}\right). \tag{4.49}
\]

To state our result we need to introduce some additional assumption. We say that a sequence \((n_k : k \in \mathbb{N})\) of natural numbers tending to \( \infty \) satisfies the spacing condition with respect to \( \alpha \) and \( q \) if
\[
\liminf_{k \to \infty} \inf_{z \in \mathbb{Z}_+} \left|\frac{\ln(n_k \alpha)}{\ln \frac{1}{q} - z}\right| > 0. \tag{4.50}
\]

Roughly speaking, this means that \( n_k \alpha \) is eventually uniformly far from integer powers of \( 1/q \) in some normalized sense.
Theorem 4.5. Assume $c_1 c_2 \cdots c_L > 0$. Then for every $k \in \mathbb{Z}$,
\[
\lim_{n \to \infty} Q_n \left( M_n \leq \left| \frac{\ln n \pi_1(0)(1 - \frac{1}{\lambda C})}{\ln \lambda C} \right| + k \right) = e^{-\lambda_C (k-2)},
\] (4.51)
when the limit is taken along any sequence satisfying the spacing condition with respect to $\alpha = \pi_1(0)(1 - \frac{1}{\lambda C})$ and $q = \frac{1}{\lambda C}$.

Example 4.6. For the standard Zeckendorf decomposition, $\lambda_C = \phi$ and $\pi_1(0) = \frac{\phi + 1}{\phi + 2}$. This gives:
\[
\lim_{n \to \infty} Q_n \left( M_n \leq \left| \frac{\ln n - \ln(\phi + 2)}{\ln \phi} \right| + k \right) = e^{-\phi^{-(k-2)}}.
\] (4.52)

Proof of Theorem 4.5. To prove the theorem, we need to recall some facts on the maximum of negative geometric random variables. Let $\Theta$ be a negative geometric random variable with parameter $p \in (0, 1)$. That is, for $k \in \mathbb{Z}_+$, $P(\Theta \geq k) = q^k$ where $q = 1 - p$. Let $G$ be negative geometric with parameter $p$. That is, $\Theta$ takes values in $\mathbb{Z}_+$, and $P(\Theta \geq k) = q^k$, where $q = 1 - p$. We denote this distribution by $\text{Geom}^-(p)$. Let $(\Theta_k : k \in \mathbb{N})$ be IID $\text{Geom}^-(p)$-distributed random variables, and let $M_m^\Theta = \max_{k \leq m} \Theta_k$. Then $P(M_m^\Theta \leq j) = (1 - q^j)^m$. For each $m \in \mathbb{N}$, let $\delta_m$ be chosen so that $\frac{\ln m \delta_m}{\ln 1/q} = \left\lfloor \frac{\ln m}{\ln 1/q} \right\rfloor$. Observe then that $\delta_m \in (q, 1]$. From this we obtain that for any $k \in \mathbb{Z}$,
\[
P \left( M_m^\Theta \leq \left| \frac{\ln m}{\ln 1/q} \right| + k \right) = \left( 1 - \frac{q^k}{m \delta_m} \right)^m \to e^{-q^k}.
\] (4.53)

We return to the proof. Fix some sequence satisfying the spacing condition. Abusing notation, we will refer to a generic element in the sequence as $n$. Observe that if we choose $\Theta_j = R_j - 2$, then $(\Theta_j : j \in \mathbb{N})$ is an IID sequence of $\text{Geom}^-(p)$ random variables with $p = 1 - \lambda_C^{-1}$. In particular, for every $m$,
\[
M_{T_m} = M_m^\Theta + 2.
\] (4.54)

Clearly $T_{m(n)} \leq n$, but also by the law of large numbers and (4.48)
\[
\frac{T_{m(n)}}{n} = \frac{T_{m(n)}}{m(n)} \times \frac{m(n)}{n} \to 1, \ Q\text{-a.s.}
\] (4.55)

From (4.48) we can find $\epsilon_n > 0$ with $\lim_{n \to \infty} \epsilon_n = 0$ and satisfying
\[
Q \left( \frac{m(n)}{n} \in [1 - \epsilon_n, 1 + \epsilon_n] \alpha \right) \to 1.
\] (4.56)

Observe then that
\[
Q \left( M_n \leq \left| \frac{\ln n \alpha}{\ln \frac{1}{q}} \right| + k \right) \geq Q \left( M_n \leq \left| \frac{\ln n \alpha}{\ln \frac{1}{q}} \right| + k, 0 < m(n) \leq (1 + \epsilon_n)n \alpha \right)
\geq Q \left( M_{(1+\epsilon_n)n \alpha}^\Theta \leq \left| \frac{\ln n \alpha}{\ln \frac{1}{q}} \right| + k - 2 \right) - Q(m(n))
\geq (1 + \epsilon_n)n \alpha - Q(m(n)).
\] (4.57)

The last two terms on the righthand side tend to 0. In addition, since $\ln(n(1+\epsilon_n)\alpha) - \ln(n) \to 0$, it follows from the spacing condition that for all $n$ large enough, $\left| \frac{(1+\epsilon_n)n \alpha}{\ln \frac{1}{q}} \right| = \left| \frac{\ln n \alpha}{\ln \frac{1}{q}} \right|$. It then
follows from (4.53) that
\[
\liminf_{n \to \infty} Q \left( M_n \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \right) \geq e^{-qk-2}. \tag{4.58}
\]

We turn to the upper bound.
\[
Q \left( M_n \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \right) \leq Q \left( M_n \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k - 2, m(n) \geq (1 - \epsilon_n)n\alpha \right)
+ Q \left( m(n) < (1 - \epsilon_n)n\alpha \right)
\leq Q \left( M_{\lfloor (1 - \epsilon_n)n\alpha \rfloor} \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \right) + o(1). \tag{4.59}
\]

The same argument as before shows that for \( n \) large enough, \( \left\lfloor \frac{\ln (1 - \epsilon_n)n\alpha}{\ln \frac{1}{q}} \right\rfloor = \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor \), and so
\[
\limsup_{n \to \infty} Q \left( M_n \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \right) \leq e^{-qk-2}. \tag{4.60}
\]

Summarizing,
\[
\lim_{n \to \infty} Q \left( M_n \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \right) = e^{-qk-2}. \tag{4.61}
\]

It remains to convert the result to \( Q_n \). Let \( A_n = \{ M_n \geq \lfloor \ln \ln n \rfloor \} \). Then \( Q(A_n) \to 1 \). Let \( b_n = \lfloor \ln \ln n \rfloor \). Then as \( n - b_n = n(1 + o(1)) \), we conclude that the sequence \( n - b_n \) also satisfies the spacing condition. Furthermore, for sufficiently large \( n \), \( \left\lfloor \frac{\ln (n - b_n)\alpha}{\ln 1/q} \right\rfloor = \left\lfloor \frac{\ln n\alpha}{\ln 1/q} \right\rfloor \). Thus, from (4.61)
\[
\lim_{n \to \infty} Q \left( M_{n-b_n} \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \right) = e^{-qk-2}. \tag{4.62}
\]

Letting \( B_n = \{ M_{n-b_n} \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k \} \), it follows from the Markov property and the ergodicity of \( Z \) that
\[
E^Q \left( 1_{B_n} \frac{1}{\varphi_c(Z_n)} \right) = E^Q \left( 1_{B_n} E_{X_{n-b_n}} \frac{1}{\varphi_c(X_{b_n})} \right)
= E^Q \left( 1_{B_n} E_{\pi Q} \frac{1}{\varphi_c} \right) + o(1) = Q(B_n) + o(1). \tag{4.63}
\]

Now
\[
Q \left( M_n \leq \left\lfloor \frac{\ln n\alpha}{\ln \frac{1}{q}} \right\rfloor + k, \frac{1}{\varphi_c(X_n)} \right) \leq Q \left( 1_{B_n}, E_{X_{n-b_n}}^{\pi Q} \frac{1}{\varphi_c(X_{b_n})} \right)
= Q(B_n) E_{\pi Q} \frac{1}{\varphi_c} + o(1), \tag{4.64}
\]

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and so
\[
\limsup_{n \to \infty} Q_n \left( M_n \leq \left\lfloor \frac{\ln n \alpha}{\ln 1/q} \right\rfloor + k \right) \leq e^{-q^{k-2}}. \tag{4.65}
\]

We turn to the lower bound. Observe that \( M_n > M_{n-b_n} \) only if one of the last \( b_n + 1 \) long gaps among the first \( m(n) \) is maximal. Fix \( c > 0 \), then for all \( n \) large enough, depending on \( c \) and on the event \( \{M_n > c \ln n\} \), those maximal gap among the last \( b_n + 1 \) long gaps begins before \( n - b_n \) (because otherwise it will have length at most \( b_n < c \ln n \)) and end after \( n - b_n \) (otherwise already included in \( M_{n-b_n} \)). That is,
\[
\{M_n > M_{n-b_n}\} \cap \{M_n > c \ln n\} \subset \{\max_{j=1, \ldots, m(n-b_n)+1} \Theta_j = \Theta_{m(n-b_n)+1}\}. \tag{4.66}
\]
Denote the event on the right-hand side by \( C_n \). We have that
\[
Q(C_n) \leq Q(C_n, m(n) \in ((1 - \epsilon)n\alpha, (1 + \epsilon)n\alpha)) + o(1)
\leq 2\epsilon n\alpha \times \frac{1}{(1 - \epsilon)n\alpha} + o(1) \to \frac{2\epsilon}{1 - \epsilon}. \tag{4.67}
\]

Since \( \epsilon \) is arbitrary, we conclude that \( Q(C_n) \to 0 \). Hence
\[
E^Q \left( M_n > \left\lfloor \frac{\ln n \alpha}{\ln 1/q} \right\rfloor + k, \frac{1}{\varphi_c(Z_n)} \right) \leq Q \left( M_{n-b_n} > \left\lfloor \frac{\ln n \alpha}{\ln 1/q} \right\rfloor + k, C_n^c, \frac{1}{\varphi_c(X_n)} \right) + Q(C_n)
\leq Q \left( M_{n-b_n} > \left\lfloor \frac{\ln (n - b_n)\alpha}{\ln 1/q} \right\rfloor + k, \frac{1}{\varphi_c(Z_n)} \right) + o(1). \tag{4.68}
\]

The remainder of the proof is identical to the argument presented in \( (4.63) \), with the obvious changes. This gives the lower bound
\[
\liminf_{n \to \infty} Q^n \left( M_n \leq \left\lfloor \frac{\ln n \alpha}{\ln 1/q} \right\rfloor + k \right) \geq e^{-q^{k-2}}, \tag{4.69}
\]
thus completing the proof.

\[\Box\]

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