EXOTIC EXPANSIONS AND PATHOLOGICAL PROPERTIES OF 
ζ-FUNCTIONS ON CONIC MANIFOLDS

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ABSTRACT. We give a complete classification and present new exotic phenomena of the meromorphic structure of ζ-functions associated to general self-adjoint extensions of Laplace-type operators over conic manifolds. We show that the meromorphic extensions of these ζ-functions have, in general, countably many logarithmic branch cuts on the nonpositive real axis and unusual locations of poles with arbitrarily large multiplicity. The corresponding heat kernel and resolvent trace expansions also exhibit exotic behaviors with logarithmic terms of arbitrary positive and negative multiplicity. We also give a precise algebraic-combinatorial formula to compute the coefficients of the leading order terms of the singularities.

1. INTRODUCTION

In this paper we give a complete classification of the meromorphic structure of ζ-functions associated to conic manifolds; that is, general self-adjoint extensions of Laplace-type operators on conic manifolds introduced by Cheeger [10, 12]. In particular, we prove that such ζ-functions exhibit pathological meromorphic properties. Before giving a synopsis of these pathological properties, recall that the ζ-function ζ(s, ∆) of a Laplacian ∆ over a smooth closed manifold has a meromorphic extension to all of C with only simple poles at s = n−k2/∈ −N0 with n the dimension of the manifold and k ∈ N0 := {0, 1, 2, . . .} [33, 53, 62, 65]. The situation is completely different for conic manifolds. We show that the ζ-function associated to a general self-adjoint extension of a Laplace-type operator on a conic manifold has, as a general rule (except for very special cases, e.g. the Friedrichs extension), in addition to the singularities at s = n−k2/∈ −N0 for k ∈ N0, the following properties:

(1) It can have countably many poles of arbitrarily high multiplicity at “unusual” locations on the negative real axis; that is, at points not of the form s = n−k2/∈ −N0 for k ∈ N0.
(2) It can have countably many logarithmic singularities at “unusual” locations.
(3) The singularities in (1) and (2) can occur for the same ζ-function and at the same “unusual” locations. Moreover, we also give an elementary and explicit algebraic-combinatorial recipe to compute the exact locations and leading coefficients of the “unusual” poles and logarithmic singularities.

In fact, the explicit computation of these exotic singularities is so straightforward (see Section 2.2) that for low dimensions we can find the structure of zeta functions quickly. We also remark that one can always conjure up artificial zeta functions having (1) and (2), but for natural (geometric) zeta functions, properties (1) and (2) seem to have no parallels in the differential geometry literature.

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1.1. A simple example. Here is a surprising, and completely natural, example of a $\zeta$-function which has no meromorphic extension to all of $\mathbb{C}$. We first review conic manifolds. Let $M$ be an $n$-dimensional compact manifold with boundary $\Gamma$ and let $g$ be a smooth Riemannian metric on $M \setminus \Gamma$. We assume that near $\Gamma$ there is a collared neighborhood

$$ U \cong [0, \varepsilon)_r \times \Gamma, $$

where $\varepsilon > 0$ and the metric $g$ is of product type $dr^2 + r^2 h$ with $h$ a metric over $\Gamma$. Such a metric is called a conic metric and $M$ is called a conic manifold, ideas introduced by Cheeger [10] [12] (cf. [51]). Using a Liouville transformation over the collar $U$ as in [5], we can identify $L^2(M, dg)$ with $L^2(M, drdh)$ and the scalar Laplacian $\Delta_g$ can be identified with

$$ (1.1) \quad (\Delta_g)|_U = -\partial^2_r + \frac{1}{r^2} Ar, \quad \text{with} \quad A_r = \Delta_{\Gamma} + \left(1 - \frac{n}{2}\right)\left(1 + \frac{n-1}{2}\right), $$

where $\Delta_{\Gamma}$ is the Laplacian over $\Gamma$. Notice that $A_r \geq -\frac{1}{4}$ because the function $x(1 + x)$ has the minimum value $-\frac{1}{4}$ (when $x = -\frac{1}{2}$). Let us now assume that $n = 2$ so that

$$ \Delta_g|_U = -\partial^2_r + \frac{1}{r^2} Ar, \quad \text{with} \quad A_r = \Delta_{\Gamma} - \frac{1}{4}. $$

We remark that the term $-\frac{1}{4}$ can be considered a “singular potential,” and such Laplacians and their self-adjoint extensions have been studied by physicists since the 70’s [7, 15, 21, 58]. Note that $\Delta_{\Gamma}$ always has the eigenvalue 0. Let us assume that 0 is the only eigenvalue (counting multiplicity) of $\Delta_{\Gamma}$ in the interval $[0, 1)$; this is the case for the Euclidean Laplacian on a punctured region in $\mathbb{R}^2$, for in this case, $\Delta_{\Gamma}$ is just the Laplacian on the unit circle which has eigenvalues $\{k^2 \mid k \in \mathbb{Z}\}$. Then $A_r$ has exactly one eigenvalue in the interval $[-\frac{1}{4}, \frac{1}{4}]$, the eigenvalue $-\frac{1}{4}$, and (see Section 4) $\Delta_g$ has many different self-adjoint extensions, each of which is parameterized by an angle $\theta \in [0, \pi)$ (cf. [39] [33] [25]). It turns out that $\theta = \frac{\pi}{2}$ corresponds to the so-called Friedrichs extension [5], and any extension has a discrete spectrum [45]. Consider any one of the extensions, say $\Delta_{\theta}$ with $\theta \in [0, \pi)$, and form the corresponding $\zeta$-function

$$ \zeta(s, \Delta_{\theta}) := \sum_{\mu_j \neq 0} \frac{1}{\mu_j^s}, $$

where the $\mu_j$’s are the eigenvalues of $\Delta_{\theta}$. The surprising fact is that the meromorphic extension of every such $\zeta$-function corresponding to an angle $\theta \in [0, \pi)$ except $\theta = \frac{\pi}{2}$ (the Friedrich’s extension) has a logarithmic singularity at $s = 0$. More precisely, as a consequence of Theorem 2.4 for $\theta \neq \frac{\pi}{2}$, we can write

$$ (1.2) \quad \zeta(s, \Delta_{\theta}) = \zeta_{\text{reg}}(s, \Delta_{\theta}) - \frac{\sin(\pi s)}{\pi} e^{-2\kappa s} \log s, $$

where $\kappa = \log 2 - \gamma - \tan \theta$ with $\gamma$ the Euler-Mascheroni constant and $\zeta_{\text{reg}}(s, \Delta_{\theta})$ has a meromorphic extension over $\mathbb{C}$ with the “regular” simple poles at the well-known values $s = \frac{1}{2}, -k$ for $k \in \mathbb{N}_0$.

1.2. Operators on conic manifolds. Brüning and Seeley’s regular singular operators [5] generalize the example (1.1) of the Laplacian on a conic manifold as follows. Let $M$ be an $n$-dimensional compact manifold with boundary $\Gamma$ and we assume that near $\Gamma$ there is a collared neighborhood $U$ such that

$$ U \cong [0, \varepsilon)_r \times \Gamma, $$
where \( \varepsilon > 0 \) and the metric of \( M \) is of product type \( dr^2 + h \) with \( h \) a metric over \( \Gamma \). Let \( E \) be a Hermitian vector bundle over \( M \) and let \( \Delta \) be a second order regular singular operator acting on \( C_0^\infty(M \setminus \Gamma, E) \); this means that \( \Delta \) is an elliptic symmetric nonnegative second order differential operator such that the restriction of \( \Delta \) to \( U \) has the form

\[
-\partial_r^2 + \frac{1}{r^2} A, \tag{1.3}
\]

where \( A \) is a Laplace-type operator over \( \Gamma \) with \( A \geq -\frac{1}{4} \). (The condition \( A \geq -\frac{1}{4} \) is necessary otherwise \( \Delta \) is not bounded below \([5, 8]\).) Laplacians on forms and squares of Dirac operators on conic manifolds \([5, 10, 12, 13, 36, 44, 50]\) are examples of second order regular singular operators. We can also deal with the case when \( M \) has boundary components up to which \( \Delta \) is smooth; at such components, put local boundary conditions.

As shown by Mooers in \([55, \text{Prop. 2.4}]\) (cf. Gil and Mendoza \([25]\)), the self-adjoint extensions of \( \Delta \) are parameterized by Lagrangian subspaces in the eigenspaces of \( A \) with eigenvalues in the interval \([-\frac{1}{4}, \frac{3}{4}]\). To describe these extensions, denote by

\[
\begin{align*}
-\frac{1}{4} &= \lambda_1 = \lambda_2 = \cdots = \lambda_{q_0} < \lambda_{q_0+1} \leq \lambda_{q_0+2} \leq \cdots \leq \lambda_{q_0+q_1}, \\
\end{align*}
\]

the spectrum of \( A \) in \([-\frac{1}{4}, \frac{3}{4}]\) and by \( \{\phi_\ell\} \) the associated eigenvectors, and define

\[
V \cong \bigoplus_{-\frac{1}{4} < \lambda_\ell < \frac{3}{4}} E_\ell \oplus E_\ell \cong \mathbb{C}^{2q}
\]

with \( E_\ell := \text{span}\{\phi_\ell\} \) and \( q = q_0 + q_1 \);

see Section 3 for a more precise description of \( V \). We can endow \( V \) with a symplectic structure as described in Section 3. Then the self-adjoint extensions of \( \Delta \) are in a one-to-one correspondence to the Lagrangian subspaces in \( V \). Given a Lagrangian subspace \( L \) in \( V \), we denote by \( \Delta_L \) the self-adjoint extension corresponding to \( L \).

One of the natural questions for a given self-adjoint extension \( \Delta_L \) is whether the \( \zeta \)-function of \( \Delta_L \), \( \zeta(s, \Delta_L) \), would have a meromorphic extension over \( \mathbb{C} \) and if so, what the pole structure is. Here the \( \zeta \)-function of \( \Delta_L \) is defined by

\[
\zeta(s, \Delta_L) = \sum_{\mu_j \neq 0} \frac{1}{\mu_j^s}
\]

for \( \Re s \gg 0 \) where \( \mu_j \)'s are the eigenvalues of \( \Delta_L \). The \( \zeta \)-function has been studied in many papers for the Friedrich's extension or with the condition \( A \geq \frac{3}{4} \), which implies that \( \Delta \) is essentially self-adjoint \([3, 4, 5, 8, 9, 11, 12, 22, 23, 46, 49, 50, 65]\).

For scalable extensions (extensions for which the domain is invariant under \( r \mapsto cr \)), the \( \zeta \)-function has been studied in \([45]\). There are no “unusual” phenomena with these cases. Finally, for arbitrary self-adjoint extensions with \( A \geq -\frac{1}{4} \), the \( \zeta \)-function has been studied by Falomir, Muschietti and Pisani \([19]\) (see also \([18, 20]\)) for one-dimensional Laplace-type operators over \([0, 1]\) and by Mooers \([55]\) who was the first to study the general case of operators over manifolds and who noticed the presence of “unusual” poles. However, the works \([19, 55]\) only imply the existence of simple “unusual” poles and do not imply the existence of poles of arbitrary order nor of logarithmic singularities of the \( \zeta \)-function.

We now outline this paper. We begin in Section 2 by giving the statement of our main result, Theorem 2.1, and we also illustrate the ease of applying the main
result by giving examples; in particular, we re-derive the main result of [19] and we show that poles of arbitrary order and countably many logarithmic singularities show up even for simple situations. We also show how our theorem simplifies when we make assumptions on the self-adjoint extensions and we present corresponding resolvent and heat kernel expansions. Mooers [55] description of self-adjoint extensions as Lagrangian subspaces plays a key rôle in the proof of our main result and because of this reason, in Section 3 we briefly review this important topic. The main technical task of the proof of our main theorem is the explicit form of the parametrix of the resolvent of $\Delta_L$ near the boundary. This is handled by solving model problems explicitly over a finite interval employing Theorem 6.1, which is a new representation of the resolvent in terms of an implicit eigenvalue equation, and the contour integration method [36, 37, 38]. The presentation and details of the solutions of the model problems are given in Sections 4, 5 and 6. The results for the model problems and a parametrix construction then enable us to prove all the theorems listed in Section 2 below. This is described in Section 7.

2. Statement and examples of results

2.1. Statement of main result. Fix a Lagrangian $L \subset V$ and hence a self-adjoint extension $\Delta_L$ of $\Delta$ (we use the notation from Section 1.2). In Section 3 we show that $L$ can be described by $q \times q$ matrices $A$ and $B$ having the property that the rank of the $q \times 2q$ matrix $\begin{pmatrix} A & B \\ \text{Id}_{q_0} & 0 \end{pmatrix}$ is $q$ and $A' B^*$ is self-adjoint where $A'$ is the matrix $A$ with the first $q_0$ columns multiplied by $-1$ (conversely, any such $A$ and $B$ define a Lagrangian). Before stating the main result which describes the exact structure of $\zeta(s, \Delta_L)$, we apply a straightforward three-step algorithm to $A$ and $B$ that we need for the statement.

**Step 1:** First, we define the function

$$p(x, y) := \det \begin{pmatrix} x \text{Id}_{q_0} & 0 & 0 & 0 \\ 0 & \tau_1 y^{2\nu_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \tau_{q_1} y^{2\nu_{q_1}} \end{pmatrix},$$

where $\text{Id}_k$ denotes the $k \times k$ identity matrix and where

$$\nu_j := \sqrt{\lambda_{q_0+j} + \frac{1}{4}}, \quad \tau_j = 2^{2\nu_j} \frac{\Gamma(1 + \nu_j)}{\Gamma(1 - \nu_j)}, \quad j = 1, \ldots, q_1.$$

For specific $A$ and $B$, $p(x, y)$ is explicitly computable “by hand”; we shall give some examples in Subsection 2.2. Expanding the determinant using one’s favorite method, we can write $p(x, y)$ as a “polynomial”

$$p(x, y) = \sum a_{j\alpha} x^j y^{2\alpha},$$

where the $\alpha$’s are linear combinations of $\nu_1, \ldots, \nu_{q_1}$ and the $a_{j\alpha}$’s are constants. Let $\alpha_0$ be the smallest of all $\alpha$’s with $a_{j\alpha} \neq 0$ and let $j_0$ be the smallest of all $j$’s amongst the $a_{j\alpha_0} \neq 0$. Then factoring out the term $a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0}$ in $p(x, y)$ we can write $p(x, y)$ in the form

$$p(x, y) = a_{j_0\alpha_0} x^{j_0} y^{2\alpha_0} \left(1 + \sum b_{k\beta} x^k y^{2\beta}\right)$$

for some constants $b_{k\beta}$ (equal to $a_{k\alpha}/a_{j_0\alpha_0}$).
Step 2: Second, using the power series \( \log(1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k} z^k \) with \( z = \sum b_{k,\beta} x^k y^{2\beta} \) for a sufficiently small \( |z| \), we can write
\[
(2.3) \quad \log \left( 1 + \sum b_{k,\beta} x^k y^{2\beta} \right) = \sum c_{\ell,\xi} x^\ell y^{2\xi}
\]
for some constants \( c_{\ell,\xi} \). We emphasize that for specific \( A \) and \( B \), all the coefficients \( c_{\ell,\xi} \) are explicitly computable “by hand” when \( q \) is small (see the examples in Subsection 2.2) and easily with a computer for \( q \) large. With a little thought, one can see that the \( \xi \)’s appearing in (2.3) are nonnegative, countable and approach \( +\infty \) unless \( \beta = 0 \) is the only \( \beta \) occurring in (2.3), in which case \( \xi = 0 \) occurs in (2.3). Also the \( \ell \)'s with \( c_{\ell,\xi} \neq 0 \) for a fixed \( \xi \) are bounded below.

Step 3: Third, for each \( \xi \) appearing in (2.3), define
\[
(2.4) \quad p_\xi := \min \{ \ell \leq 0 \mid c_{\ell,\xi} \neq 0 \} \quad \text{and} \quad \ell_\xi := \min \{ \ell > 0 \mid c_{\ell,\xi} \neq 0 \},
\]
when these numbers are actually defined, that is, whenever the sets \( \{ \ell \leq 0 \mid c_{\ell,\xi} \neq 0 \} \) and \( \{ \ell > 0 \mid c_{\ell,\xi} \neq 0 \} \), respectively, are nonempty. We now define
\[
(2.5) \quad \mathcal{P} := \{ \xi \mid p_\xi \text{ is defined} \} \quad \text{and} \quad \mathcal{L} := \{ \xi \mid \ell_\xi \text{ is defined} \}.
\]

The following theorem is our main result.

**Theorem 2.1.** Let \( L \subset V \) be an arbitrary Lagrangian subspace of \( V \) and define \( \mathcal{P} \) and \( \mathcal{L} \) as in (2.5) from the matrices \( A \) and \( B \) defining \( L \). Then the \( \xi \)-function \( \zeta(s, \Delta_L) \) extends from \( \mathbb{R}s > \frac{1}{2} \) to a holomorphic function on \( \mathbb{C} \setminus (-\infty, 0] \). Moreover, \( \zeta(s, \Delta_L) \) can be written in the form
\[
\zeta(s, \Delta_L) = \zeta_{\text{reg}}(s, \Delta_L) + \zeta_{\text{sing}}(s, \Delta_L),
\]
where \( \zeta_{\text{reg}}(s, \Delta_L) \) has possible “regular” poles at the “usual” locations \( s = \frac{n-k}{2} \notin -\mathbb{N}_0 \) for \( k \in \mathbb{N}_0 \) and at \( s = 0 \) if \( \dim \Gamma > 0 \), and where \( \zeta_{\text{sing}}(s, \Delta_L) \) has the following expansion:
\[
(2.6) \quad \zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \left\{ (j_0 - q_0) e^{-2s(\log 2 - \gamma)} \log s + \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{(s + \xi)^{p_\xi + 1}} \right. \\
\left. + \sum_{\xi \in \mathcal{L}} g_\xi(s) \log(s + \xi) \right\},
\]
where \( j_0 \) appears in (2.4) and \( f_\xi(s) \) and \( g_\xi(s) \) are entire functions of \( s \) such that
\[
f_\xi(-\xi) = (-1)^{p_\xi+1} c_{p_\xi} \xi \left| \frac{p_\xi!}{2^{p_\xi}} \right|
\]
and near \( s = -\xi \),
\[
g_\xi(s) = \begin{cases} c_{\ell_\xi,0} \frac{2^{\ell_\xi}}{(\ell_\xi - 1)!} s^{\ell_\xi} + O(s^{\ell_\xi+1}) & \text{if } \xi = 0, \\ -c_{\ell_\xi,\xi} \frac{2^{\ell_\xi}}{(\ell_\xi - 1)!} (s + \xi)^{\ell_\xi - 1} + O((s + \xi)^{\ell_\xi}) & \text{if } \xi > 0. \end{cases}
\]

**Remark 2.2.** The zeta function \( \zeta_{\text{reg}}(s, \Delta_L) \) will only have possible poles at \( s = \frac{n-k}{2} \notin -\mathbb{N}_0 \) in the case that \( \Gamma \) is the only boundary component of \( M \) and the residue of \( \zeta_{\text{reg}}(s, \Delta_L) \) at \( s = 0 \) is given by \( \text{Res}_{s=0} \zeta_{\text{reg}}(s, \Delta_L) = -\frac{1}{2}\text{Res}_{s=-\frac{1}{2}} \zeta(s, A_\Gamma) \); in particular, this vanishes if \( \zeta(s, A_\Gamma) \) is in fact analytic at \( s = -\frac{1}{2} \). Later in Theorems 2.4 and 2.7 we shall present corresponding resolvent and heat kernel expansions.
In general, \( \mathcal{L} \) may contain the origin, so that the logarithm part, which has the branch cut at \( s = 0 \), is given by

\[
\sum_{\varepsilon \in \mathcal{V}, \xi \leq N} \frac{f_{\xi}(s)}{(s + \xi)^{\nu + 1}} + \sum_{\varepsilon \in \mathcal{L}, \xi \leq N} g_{\xi}(s) \log(s + \xi) \] + F_N(s),
\]

where \( F_N(s) \) is holomorphic for \( \Re s \geq -N \).

2.2. Examples. Via examples we show the ease and efficiency at which Theorem 2.1 computes the exact meromorphic structure of \( \zeta_{\text{sing}}(s, \Delta_L) \) (note that \( \zeta_{\text{reg}}(s, \Delta_L) \) is “uninteresting,” which is why we focus on \( \zeta_{\text{sing}}(s, \Delta_L) \)).

Example 1. (Taken from [19].) The paper by Falomir et al. [19] (along with Mooers’ [55]) is in many ways the inspiration for our paper and is the very first paper to find explicit formulas for the “unusual” poles of Laplacians; cf. [20] for the infinite interval and [18] for squares of \( 2 \times 2 \) systems. [19] studies the operator

\[
\Delta = -\frac{d^2}{dx^2} + \frac{1}{r^2} \lambda \quad \text{over} \quad [0,1]
\]

with \(-\frac{1}{4} \leq \lambda < \frac{3}{4}\). In this case, \( V = \mathbb{C}^2 \), therefore Lagrangians \( L \subset \mathbb{C}^2 \) are determined by \( 1 \times 1 \) matrices (numbers) \( A = \alpha \) and \( B = \beta \). Fix such an \((\alpha, \beta)\); we shall determine the strange singularity structure of \( \zeta(s, \Delta_L) \). Let us assume that \(-\frac{1}{4} < \lambda < \frac{3}{4}\) so there are no \(-\frac{1}{4}\) eigenvalues (we will come back to the \( \lambda = -\frac{1}{4} \) in a moment). Then with \( \nu := \sqrt{\lambda + \frac{1}{4}} \) and \( \tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \),

\[
p(x, y) := \det \begin{pmatrix} \alpha & \beta \\ \tau y^{2\nu} & 1 \end{pmatrix} = \alpha - \beta \tau y^{2\nu} = \alpha \left( 1 - \frac{\tau \beta}{\alpha} y^{2\nu} \right),
\]

where we assume that \( \alpha, \beta \neq 0 \) (the \( \alpha = 0 \) or \( \beta = 0 \) cases can be handled easily), and we write \( p(x, y) \) as in [22]. Forming the power series [23], we see that

\[
\log \left( 1 - \frac{\tau \beta}{\alpha} y^{2\nu} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( -\frac{\tau \beta}{\alpha} y^{2\nu} \right)^k = \sum_{k=1}^{\infty} c_{0,\nu k} x^0 y^{2\nu k},
\]

where \( c_{0,\nu k} = -\frac{1}{k} \left( \frac{\tau \beta}{\alpha} \right)^k \). Using the definitions [24] and [25] for \( p_\xi, \xi, \mathcal{P}, \) and \( \mathcal{L} \), we immediately see that \( \ell_{\nu k} \) never exists so \( \mathcal{L} = \emptyset \), while

\[
p_{\nu k} = \min \{ \ell \leq 0 \mid \ell_{\nu k} \neq 0 \} = 0, \quad \mathcal{P} = \{ \nu k \mid k \in \mathbb{N} \}.
\]

Therefore, by Theorem 2.1

\[
\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} \frac{f_k(s)}{s + \nu k}.
\]
with \( f_k(s) \) an entire function of \( s \) such that \( f_k(-\nu k) = \nu \left( \frac{\tau \beta}{\alpha} \right)^k \). In particular, \( \zeta_{\text{sing}}(s, \Delta_L) \) has possible poles at each \( s = -\nu k \) with the residue equal to

\[
\text{Res}_{s=-\nu k} \zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi(-\nu k))}{\pi} \nu \left( \frac{\tau \beta}{\alpha} \right)^k = -\frac{\nu \sin(\pi \nu k)}{\pi} \left( \frac{\tau \beta}{\alpha} \right)^k,
\]

which is the main result of [19] (see equation (7.11) of loc. cit.).

Assume now that \( \lambda = -\frac{1}{4} \). In this case,

\[
p(x, y) := \det \begin{pmatrix} \alpha & \beta \\ x & 1 \end{pmatrix} = \alpha - \beta x = \alpha \left( 1 - \frac{\beta}{\alpha} x \right),
\]

where we assume that \( \alpha, \beta \neq 0 \) (the \( \alpha = 0 \) or \( \beta = 0 \) cases can be handled easily), and we write \( p(x, y) \) as in [22]. Forming the power series [23], we see that

\[
\log \left( 1 - \frac{\beta}{\alpha} x \right) = \sum_{\ell=1}^{\infty} c_{\ell,0} x^\ell y^{2\ell-1}, \quad c_{\ell,0} = -\frac{1}{\ell} \left( \frac{\beta}{\alpha} \right)^\ell.
\]

Using the definitions [24] and [26] for \( p_\xi, \ell_\xi \) (there is only one "\( \xi \)" in the present situation, \( \xi = 0 \)), \( \mathcal{P} \), and \( \mathcal{L} \), we immediately see that \( p_0 \) never exists so \( \mathcal{P} = \emptyset \), while

\[
\ell_0 = \min \{ \ell > 0 \mid c_{\ell,0} \neq 0 \} = 1, \quad \mathcal{L} = \{ 0 \}.
\]

Therefore, by Theorem [21],

\[
\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \left\{ -e^{-2s(\log 2 - \gamma)} \log s + g_0(s) \log s \right\},
\]

\( g_0(s) \) is an entire function of \( s \) such that \( g_0(s) = O(s) \). Hence \( \zeta(s, \Delta_L) \) has a genuine logarithmic singularity at \( s = 0 \). This corrects unfortunate errors from the beautiful paper [19] (and [53]), which states that \( \zeta(s, \Delta_L) \) has the "usual" meromorphic structure.\(^1\) When \( \beta = 0 \), one can easily see that we still have a logarithmic singularity at \( s = 0 \), and when \( \alpha = 0 \), one can easily check that there is only the "regular" part \( \zeta_{\text{reg}}(s, \Delta_L) \) and no "singular" part; in fact, the case \( \alpha = 0 \) corresponds to the Friedrichs extension (see [5]); thus we can see that \( \zeta(s, \Delta_L) \) has a logarithmic singularity for all extensions except the Friedrichs.

**Example 2.** (The Laplacian on \( \mathbb{R}^2 \)) If \( \Delta \) is the Laplacian on a compact region in \( \mathbb{R}^2 \), then as we saw before in Section [14] \( A_1 \) has a \(-\frac{1}{4} \) eigenvalue of multiplicity one and no eigenvalues in \([-\frac{1}{4}, \frac{3}{4}] \). Therefore, the exact same argument we used in the \( \lambda = -\frac{1}{4} \) case of the previous example shows that \( \zeta(s, \Delta_L) \) has a logarithmic singularity for all extensions except the Friedrichs.

**Example 3.** Consider now the case of a regular singular operator \( \Delta \) over a compact manifold and suppose that \( A^\tau \) has two eigenvalues in \([-\frac{1}{4}, \frac{3}{4}] \), the eigenvalue \(-\frac{1}{4} \) and another eigenvalue \(-\frac{1}{4} < \lambda < \frac{3}{4} \), both of multiplicity one. In this case, \( V = \mathbb{C}^4 \), therefore Lagrangians \( L \subset \mathbb{C}^4 \) are determined by \( 2 \times 2 \) matrices \( A \) and \( B \). Consider the specific examples

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \text{Id}.
\]

\(^1\)The error in [19] occurs in equation (A13) where certain antiderivatives (specifically, \( xN_1(x) \) and \( x^2N_1(x)^2 \)) were accidentally set equal to zero at \( x = 0 \).
Then with \( \nu := \sqrt{\lambda + \frac{1}{4}} \) and \( \tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \), we have

\[
p(x, y) := \det \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ x & 0 & 1 & 0 \\ 0 & \tau y^{2\nu} & 0 & 1 \end{pmatrix} = 1 + \tau x y^{2\nu}.
\]

Forming the power series \( \log(1 + \tau x y^{2\nu}) \), we see that

\[
\log(1 + \tau x y^{2\nu}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\tau x y^{2\nu})^k = \sum_{k=1}^{\infty} c_{k,\nu k} x^k y^{2\nu k},
\]

where \( c_{k,\nu k} = (-1)^{k-1} \frac{\tau^k}{k} \). Using the definitions \( \nu_k \) and \( \ell_k \) for \( p_\xi, \ell_\xi, \mathcal{P}, \) and \( \mathcal{L} \), we immediately see that \( p_{\nu k} \) never exists so \( \mathcal{P} = \emptyset \), while

\[
\ell_{\nu k} = \min\{ \ell > 0 \mid c_{\ell,\nu k} \neq 0 \} = k, \quad \mathcal{L} = \{ \nu k \mid k \in \mathbb{N} \}.
\]

Therefore, by Theorem 2.1,

\[
\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \left\{ -e^{-2\pi s (\log 2 - \gamma)} \log s + \sum_{k=1}^{\infty} g_k(s) \log(s + \nu k) \right\},
\]

with \( g_k(s) \) an entire function of \( s \) such that near \( s = -\nu k \),

\[
g_k(s) = (-1)^k \frac{x^{\nu k}}{(k-1)!} (s + \nu k)^{k-1} + O((s + \nu k)^k).
\]

In particular, \( \zeta_{\text{sing}}(s, \Delta_L) \) has countably many logarithmic singularities!

**Example 4.** With the same situation as considered in Example 3, consider

\[
A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.
\]

Then with \( \nu := \sqrt{\lambda + \frac{1}{4}} \) and \( \tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \), we have

\[
p(x, y) := \det \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ x & 0 & 1 & 0 \\ 0 & \tau y^{2\nu} & 0 & 1 \end{pmatrix} = x - \tau x y^{2\nu} = x \left( 1 - \tau x^{-1} y^{2\nu} \right).
\]

Forming the power series \( \log(1 + \tau x^{-1} y^{2\nu}) \), we see that

\[
\log(1 + \tau x^{-1} y^{2\nu}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( -\tau x^{-1} y^{2\nu} \right)^k = \sum_{k=1}^{\infty} c_{-k,\nu k} x^{-k} y^{2\nu k},
\]

where \( c_{-k,\nu k} = -\frac{x^k}{k} \). Using the definitions \( \nu_k \) and \( \ell_k \) for \( p_\xi, \ell_\xi, \mathcal{P}, \) and \( \mathcal{L} \), we immediately see that \( \ell_{\nu k} \) never exists so \( \mathcal{L} = \emptyset \), while

\[
p_{\nu k} = \min\{ \ell \leq 0 \mid c_{-\ell,\nu k} \neq 0 \} = -k, \quad \mathcal{P} = \{ \nu k \mid k \in \mathbb{N} \}.
\]

Therefore, by Theorem 2.1,

\[
\zeta_{\text{sing}}(s, \Delta_L) = \frac{\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} \frac{f_k(s)}{(s + \nu k)^{k+1}}
\]

with \( f_k(s) \) an entire function of \( s \) such that \( f_k(-\nu k) = (-1)^k \frac{k^{\frac{1}{2}}}{k^{\frac{1}{2} + 1}} \). In particular, \( \zeta_{\text{sing}}(s, \Delta_L) \) has poles of arbitrarily large order!
Example 5. Consider one last example, the case of a regular singular operator $\Delta$ over a compact manifold such that $A$ has three eigenvalues in $[-\frac{1}{4}, \frac{3}{4})$, the eigenvalue $-\frac{1}{4}$ with multiplicity two and another eigenvalue $-\frac{1}{4} < \lambda < \frac{3}{4}$ of multiplicity one. In this case, $V = \mathbb{C}^6$ and Lagrangians $L \subset \mathbb{C}^6$ are determined by $3 \times 3$ matrices $A$ and $B$. Consider the specific example

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \text{Id}.$$ 

Then with $\nu := \sqrt{\lambda + \frac{1}{4}}$ and $\tau := 2^{2\nu} \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}$, we have

$$p(x, y) := \det \begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ x & 0 & 0 & 1 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 1 \\ 0 & 0 & \tau y^{2\nu} & 0 & 0 & 1 \end{pmatrix} = -x + \tau y^{2\nu} - \tau x^2 y^{2\nu} = -(1 + (x - x^{-1})\tau y^{2\nu}).$$

Forming the power series (2.7), we see that

$$(2.7) \quad \log \left( 1 + (x - x^{-1})\tau y^{2\nu} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \tau^k (x - x^{-1})^k y^{2\nu k}$$

$$= \sum_{k=1}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j-1}}{k} \tau^k \binom{k}{j} x^{2j} y^{2k} = \sum_{\ell, k} c_{\ell, \nu k} x^\ell y^{2\nu k},$$

where for each $k$, $\ell$ runs from $-k$ to $k$. Using the definitions (2.4) and (2.5) for $p_\xi$, $\ell_\xi$, $P$, and $L$, we immediately see that

$$p_{\nu k} = \min\{\ell \leq 0 \mid c_{\ell, \nu k} \neq 0\} = -k,$$

$$\ell_{\nu k} = \min\{\ell > 0 \mid c_{\ell, \nu k} \neq 0\} = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even}, \end{cases}$$

and $P = L = \{\nu k \mid k \in \mathbb{N}\}$. Therefore, by Theorem 2.1

$$(2.8) \quad \zeta_{\text{sing}}(s, \Delta L) = \frac{\sin(\pi s)}{\pi} \left\{ -e^{-2s(\log 2 - \gamma)} \log s + \sum_{k=1}^{\infty} \frac{f_k(s)}{(s + \nu k)^{k+1}} \right. \left. + \sum_{k=1}^{\infty} g_k(s) \log(s + \nu k) \right\},$$

where $f_k(s)$ and $g_k(s)$ are entire functions of $s$ such that $f_k(-\nu k) = (-1)^k \frac{\tau^k k!}{2^\nu}$ and

$$g_k(s) = 2\nu(-1)^{m+1} \tau^k \binom{k}{m+1} \times \begin{cases} 1 + \mathcal{O}(s + \nu k) & \text{if } k = 2m + 1 \text{ is odd}, \\ 2(s + \nu k) + \mathcal{O}((s + \nu k)^2) & \text{if } k = 2m \text{ is even}. \end{cases}$$

In particular, $\zeta_{\text{sing}}(s, \Delta L)$ has poles of arbitrarily high orders and in addition to a logarithmic singularity at the origin, countably many logarithmic singularities at the same locations of the poles!
2.3. Special Lagrangian subspaces. Theorem 2.3 simplifies considerably when the conditions given by the Lagrangian $L$ over $-\frac{1}{2}$ eigenspaces are separated from the conditions over $\lambda$ eigenspaces with $\lambda$ in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We shall call a Lagrangian subspace $L \subset V$ decomposable if $L = L_0 \oplus L_1$ where $L_0$ is an arbitrary Lagrangian subspace of $\bigoplus_{\lambda_i=\frac{1}{2}}^n E_i \oplus E_t$ and $L_1$ is an arbitrary Lagrangian subspace of $\bigoplus_{-\frac{1}{2}<\lambda_i<\frac{1}{2}} E_i \oplus E_t$. As described in Proposition 2.4, the Lagrangian subspace $L_0$ is determined by two $q_0 \times q_0$ matrices $A_0, B_0$ where $q_0 = \dim L_0$, that is, the multiplicity of the eigenvalues $\lambda_t = -\frac{1}{2}$. Similarly, the Lagrangian subspace $L_1$ is determined by two $q_1 \times q_1$ matrices $A_1, B_1$ where $q_1 = \dim L_1$, that is, the multiplicity of the eigenvalues $\lambda_t$ with $-\frac{1}{2} < \lambda_t < \frac{1}{2}$.

Two polynomials which are explicitly determined by the matrices $A_0, B_0$ and $A_1, B_1$ play peculiar rôles in the statement of our result. First, consider the polynomial in (2.1) (and (2.2)) using $A_1$ and $B_1$ in place of $A$ and $B$:

$$p_0(z) := \det \left( \begin{array}{cc} A_0 & B_0 \\ \Id_{q_0} & \log(2 - \gamma - z)\Id_{q_0} \end{array} \right).$$

Using the definition of determinant, it is easy to see that $p_0(z)$ is a polynomial of degree at most $q_0$ in $z$. Since the degree of $p'_0(z)$ is one less than the degree of $p_0(z)$, we can write

$$p'_0(z) = \sum_{k=1}^{\infty} \frac{\beta_k}{z^k}, \quad \beta_k \in \mathbb{C},$$

where the series on the right is absolutely convergent for $|z|$ sufficiently large. Second, consider the polynomial in (2.1) (and (2.2)) using $A_1$ and $B_1$ in place of $A$ and $B$:

$$p_1(y) := \det \left( \begin{array}{ccc} A_1 & \cdots & B_1 \\ \cdots & 0 & \cdots \\ 0 & \cdots & \Id_{q_1} \end{array} \right) = a_{\alpha_0} y^{2\alpha_0} \left( 1 + \sum b_{\beta} y^{2\beta} \right),$$

where the $\beta$'s are positive. Then as in (2.3), write

$$\log \left( 1 + \sum b_{\beta} y^{2\beta} \right) = \sum c_{\xi} y^{2\xi},$$

and let $\mathcal{P} := \{\xi \mid c_{\xi} \neq 0\}$. Then Theorem 2.3 simplifies to

**Theorem 2.3.** For an arbitrary decomposable Lagrangian $L \subset V$, the $\zeta$-function $\zeta(s, \Delta_L)$ has the following form:

$$\zeta(s, \Delta_L) = \zeta_{\text{reg}}(s, \Delta_L) + \zeta_{\text{sing}}(s, \Delta_L),$$

where $\zeta_{\text{reg}}(s, \Delta_L)$ has the “regular” poles at the “usual” locations $s = \frac{2k}{\pi} \notin \mathbb{N}_0$ for $k \in \mathbb{N}_0$ and at $s = 0$ if $\dim \Gamma > 0$, and where $\zeta_{\text{sing}}(s, \Delta_L)$ has the following expansion:

$$\zeta_{\text{sing}}(s, \Delta_L) = -\frac{\sin(\pi s)}{\pi} f(s) \log s + \frac{\sin(\pi s)}{\pi} \sum_{\xi \in \mathcal{P}} f_{\xi}(s) \frac{\xi}{s + \xi},$$

where $f(s)$ is the entire function defined explicitly by $f(s) = \sum_{k=1}^{\infty} \beta_k \frac{(-2s)^{k-1}}{(k-1)!}$, and the $f_{\xi}(s)$'s are entire functions of $s$ such that

$$f_{\xi}(-\xi) = -c_\xi \xi.$$
For certain types of Lagrangians, the formula for $f(s)$ becomes very simple. We shall call the Lagrangian $L_0$ split-type if it can be written as $\bigoplus_{\lambda} -$ $\frac{1}{2} L_{\ell}$ with $L_{\ell}$ a Lagrangian subspace of $E_{\ell} \oplus E_{\ell}$. As explained in Proposition 2.4, each component $L_{\ell}$ of $L_0$ is determined by an angle $\theta_{\ell} \in [0, \pi)$. Moreover, in this case, the coefficients $\beta_k$ in (2.15) are given by (as follows from Corollary 5.4)

$$\beta_k = \sum_{\ell \neq \frac{\pi}{2}} \kappa_{\ell}^{k-1}, \quad k = 1, 2, 3, \ldots \quad (\text{for split-type } L_0)$$

where $\kappa_{\ell} = \log 2 - \gamma - \tan \theta_{\ell}$ with $\theta_{\ell}$ the angle defining $L_{\ell}$ in $L_0$ and $\gamma$ is the Euler-Mascheroni constant. Then when $L_0$ is of split-type, we have

**Theorem 2.4.** For a decomposable Lagrangian $L \subset V$ such that $L_0$ is of split-type, $\zeta(s, \Delta_L)$ has the form as in (2.16) with $f(s) = \sum_{\theta_{\ell} \neq \frac{\pi}{2}} e^{-2s\kappa_{\ell}}$ in (2.15).

**Example 6.** Consider the case when $A_{r}$ has exactly one eigenvalue in $[-\frac{1}{4}, \frac{3}{2})$, the eigenvalue $-1$. Then Theorem 2.3 shows the result stated in (1.2).

2.4. Unusual resolvent and heat kernel expansions. Besides establishing exotic $\zeta$-expansions, we also derive equally exotic resolvent and heat kernel expansions.

**Theorem 2.5.** Let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis and choose $N \in \mathbb{N}$ with $N \geq \frac{\pi}{2}$. Then for an arbitrary Lagrangian $L$, as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have

$$\text{Tr}(\Delta_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} a_k (-\lambda)^{\frac{N-k}{2}} - N^{-1} + b(-\lambda)^{-N-1} \log(-\lambda)$$

$$+ \frac{1}{N!} \frac{d^N}{d\lambda^N} \left\{ \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\gamma)} \right\}$$

$$- \frac{1}{N!} \frac{d^{N+1}}{d\lambda^{N+1}} \left\{ \sum_{\ell} c_{\ell \lambda} (-\lambda)^{-\ell} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\}$$

where the $a_k$ and $b$ coefficients are independent of $L$, the $c_{\ell \lambda}$’s are the coefficients in (2.15), and $\gamma = \log 2 - \gamma$.

From the explicit formula (2.14) and from the binomial theorem for $\ell > 0$:

$$(2\gamma - \log(-\lambda))^{-\ell} = (-\log(-\lambda))^{-\ell} \left( 1 - \frac{2\gamma}{\log(-\lambda)} \right)^{-\ell}$$

$$= (-\log(-\lambda))^{-\ell} \sum_{j=0}^{\infty} \left( \begin{array}{c} -\ell \\ j \end{array} \right) \frac{(2\gamma)^j}{(\log(-\lambda))^j},$$

it is obvious that when $A_{r}$ has $-\frac{1}{4}$ eigenvalues, the resolvent trace expansion has, in general, $\log(-\lambda)$ terms of arbitrarily high multiplicity and inverse powers $(\log(-\lambda))^{-1}$ with infinite multiplicity! This phenomenon is new and even for pseudodifferential operators on compact manifolds, with or without boundary, and even conic, “regular” (not inverse powers of) $\log(-\lambda)$ terms occur with at most multiplicity two $22, 23, 24, 29, 30, 48, 49$. See $26, 43, 59$ for studies of resolvents for closed extensions of general cone operators in the sense of Schulze $60$. Here is a concrete example illustrating this discussion:
Example 7. For the self-adjoint extension $\Delta_L$ considered in Example 6 from the explicit formula (2.7), we immediately get

$$\text{Tr}(\Delta_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} a_k (-\lambda)^{\frac{n-k}{2} - N-1} + b (-\lambda)^{-N-1} \log(-\lambda)$$

$$+ \frac{1}{N!} \left( -\frac{1}{\lambda (\log(-\lambda) - 2\gamma)} \right)$$

$$- \frac{1}{N!} \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^{j-k}(1-j)^{j-1}}{k} \tau^k \left( \begin{array}{c} k \\ j \end{array} \right) (-\lambda)^{-2\nu k} (2\gamma - \log(-\lambda))^{k-2j} \right) .$$

In this very simple example, we see unusual powers $(-\lambda)^{-2\nu k - N-1}$ (after taking $N + 1$ derivatives) and log terms $\log(-\lambda)$ of arbitrarily high multiplicity (each unusual power $(-\lambda)^{-2\nu k - N-1}$ with a log term of highest power $\log(-\lambda)^k$), and inverse powers $(\log(-\lambda))^{-1}$ with infinite multiplicity because of the formula (2.17).

When $L$ is decomposable, the last two terms in (2.10) can be made very explicit.

Theorem 2.6. Let $\Lambda \subset \mathbb{C}$ be any sector (solid angle) not intersecting the positive real axis and choose $N \in \mathbb{N}$ with $N \geq \frac{n}{2}$. Then for an arbitrary decomposable Lagrangian $L$, as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have

(2.18)

$$\text{Tr}(\Delta_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} a_k (-\lambda)^{\frac{n-k}{2} - N-1} + b (-\lambda)^{-N-1} \log(-\lambda)$$

$$- \frac{1}{N!} \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{\xi} (-\lambda)^{-\xi} \right) + \frac{1}{N!} \left( \sum_{k=1}^{\infty} \frac{2^{k-1} \beta_k}{(\log(-\lambda))^k} \right) ,$$

where the $a_k$ and $b$ coefficients are independent of $L$, the $\beta_k$’s are the coefficients in (2.10), and the $c_{\xi}$’s are the coefficients in (2.12).

In the case when $L_0$ is of split-type, the second-to-last term in (2.18) can be made even more explicit because of the formula (2.10) for $\beta_k$. We also prove a corresponding heat kernel expansion.

Theorem 2.7. For an arbitrary Lagrangian $L$, the heat kernel $e^{-t\Delta_L}$ has the following trace expansion as $t \to 0$:

$$\text{Tr}(e^{-t\Delta_L}) \sim \sum_{k=0}^{\infty} a_k t^{-\frac{n+k}{2}} + b \log t + \sum_{k=0}^{\infty} b_k (\log t)^{-1-k}$$

$$+ \sum_{\xi \notin \mathbb{N}_0} \sum_{k=0}^{\left| p_{\xi} \right| + 1} c_{\xi k} t^{\xi} (\log t)^k + \sum_{\xi \notin \mathbb{N}_0} \sum_{k=0}^{\infty} d_{\xi k} t^{\xi} (\log t)^{-\xi-k} ,$$

with $c_{10} = 0$ and $c_{\xi \notin \mathbb{N}_0} = 0$ for $\xi \notin \mathbb{N}_0$.

Thus, the heat trace expansion, in general, has powers of $\log t$ with finite multiplicity and inverse powers $(\log t)^{-1}$ with infinite multiplicity. The $c_{\xi k}$ and $d_{\xi k}$ coefficients can be expressed in terms of the coefficients in the resolvent expansion (2.10) but not so explicitly. For decomposable Lagrangians we have
Theorem 2.8. For an arbitrary decomposable Lagrangian $L$, the heat kernel $e^{-t\Delta_L}$ has the following trace expansion as $t \to 0$:

$$
\text{Tr}(e^{-t\Delta_L}) \sim \sum_{k=0}^{\infty} \tilde{a}_k t^{-\frac{n+k}{2}} + \tilde{b} \log t + \sum_{\xi \in \mathcal{D}} \tilde{\varepsilon}_\xi t^\xi + \sum_{k=1}^{\infty} \tilde{d}_k (\log t)^{-k}.
$$

3. HERMITIAN SYMPLECTIC FORMALISM OF SELF-ADJOINT EXTENSIONS

To orient the reader to the various terminologies used throughout this paper, in this section we briefly review the classical theory of the Hermitian forms characterization of self-adjoint extensions. For more on this viewpoint, see [32, 33] and see [17] for applications of self-adjoint extensions to quantum physics.

3.1. The maximal domain. An elegant observation due to Gelfand around 1971 (according to Novikov [56, p. 1]) to analyze self-adjoint extensions of $\Delta$ is to find “maximal” domains $\mathcal{D} \subset \text{dom}_{\text{max}}(\Delta)$ that make the Hermitian quadratic form

$$
\langle \Delta \phi, \psi \rangle - \langle \phi, \Delta \psi \rangle, \quad \phi, \psi \in \mathcal{D},
$$

vanish. The use of Hermitian forms as a tool to analyze self-adjoint extensions of (one-dimensional) regular singular operators goes back at least to Kochubei in the late 70’s [59] (cf. [40, 41]) and Pavlov [57] in the late 80’s. Mooers [55] (cf. [11, 12, 50], and Gil and Mendoza [25] for general cone operators) used this technique to give a complete description of the self-adjoint extensions of the regular singular operator $\Delta$ in terms of Lagrangian subspaces in the eigenspaces of $A_F$ with eigenvalues in the interval $[-\frac{1}{2}, \frac{1}{2}]$. To describe these extensions, we first describe $\text{dom}_{\text{max}}(\Delta)$. Recall that $\Delta_{|\mathcal{U}} = -\partial_r^2 + \frac{1}{r^2} A_F$ (see [33]), where $\mathcal{U} \cong [0, \epsilon)_r \times \Gamma$ denotes a tubular neighborhood of $\Gamma$ and $A_F$ is a Laplace-type operator over $\Gamma$ such that $A_F \geq -\frac{1}{4}$. Recall that

$$
\frac{-1}{4} = \lambda_1 = \lambda_2 = \cdots = \lambda_{q_0} < \lambda_{q_0+1} \leq \lambda_{q_0+2} \leq \cdots \leq \lambda_{q_0+q_1},
$$

denotes the eigenvalues of $A_F$ in $[-\frac{1}{4}, \frac{1}{2}]$ with corresponding orthonormal eigenvectors $\{\phi_\ell\}$. Then, as shown by Cheeger [11, 12] (cf. Mooers [55]), we have

Proposition 3.1. A section $\phi \in L^2(M, E)$ is in

$$
\text{dom}_{\text{max}}(\Delta) := \{ \phi \in L^2(M, E) \mid \Delta \phi \in L^2(M, E) \},
$$

where “$\Delta \phi \in L^2(M, E)$” is in the distributional sense, if and only if $\phi$ is in $H^2$ away from the boundary $\Gamma$, and near $\Gamma$ we can write

$$
\phi = \sum_{\ell=1}^{q_0} \left\{ c^+_\ell (\phi) r^{\nu_\ell} \phi_\ell + c^-_\ell (\phi) r^{\nu_\ell} \log r \phi_\ell \right\}
$$

$$
+ \sum_{\ell=1}^{q_1} \left\{ \frac{c_{q_0+\ell}(\phi)}{\sqrt{2 \nu_\ell}} r^{\nu_\ell} \phi_{q_0+\ell} + \frac{c_{q_0+\ell}(\phi)}{\sqrt{2 \nu_\ell}} r^{\nu_\ell} \phi_{q_0+\ell} + \frac{\tilde{c}_{q_0+\ell}(\phi)}{\sqrt{2 \nu_\ell}} r^{\nu_\ell} \phi_{q_0+\ell} \right\} + \tilde{\phi},
$$

where the $c^\pm_\ell(\phi)'s$ are constants, $\nu_\ell := \sqrt{\lambda_{q_0+\ell} + \frac{1}{4}} > 0$, $\phi \in H^2$ and $\tilde{\phi} = O(r^{\frac{3}{2}})$. 


3.2. **Self-adjoint extensions.** Given a domain $\mathcal{D} \subset \text{dom}_{\text{max}}(\Delta)$, we say that

$$\Delta_{\mathcal{D}} := \Delta : \mathcal{D} \to L^2(M,E)$$

is **self-adjoint** if

$$\{ \psi \in \text{dom}_{\text{max}}(\Delta) \mid \langle \Delta \phi, \psi \rangle = \langle \phi, \Delta \psi \rangle \quad \text{for all} \quad \phi \in \mathcal{D} \} = \mathcal{D}. $$

Simply put: $\Delta$ is “maximally” symmetric on $\mathcal{D}$ in the sense that $\Delta$ is symmetric on $\mathcal{D}$ and adding any elements to $\mathcal{D}$ will destroy this symmetry. In this sense, $\mathcal{D}$ is a “maximal” domain that makes the Hermitian form $\langle \Delta \phi, \psi \rangle - \langle \phi, \Delta \psi \rangle$ vanish.

Now, an integration by parts argument (see [55, Prop. 2.4]) shows that

$$\langle \Delta \phi, \psi \rangle - \langle \phi, \Delta \psi \rangle = -\sum_{\ell=0}^{q} \left( c^{+}_\ell(\phi)c^{-}_\ell(\psi) - c^{-}_\ell(\phi)c^{+}_\ell(\psi) \right),$$

for all $\phi, \psi \in \text{dom}_{\text{max}}(\Delta)$.

To put this in a symplectic framework, we define

$$\psi^\pm_\ell := \begin{cases} r^{\frac{1}{2} + \nu_\ell} \phi_\ell & \text{for } 0 \leq \nu_\ell < 1, \\ r^{\frac{1}{2} \log r \phi_\ell} & \text{for } \nu_\ell = 0, \end{cases}$$

and define

$$V := \bigoplus_{-\frac{1}{4} \leq \lambda < \frac{1}{4}} E^+_{\lambda} \oplus E^-_{\lambda}$$

where $E^\pm_{\lambda} := \langle \psi^\pm_\ell \rangle$ with $\langle \psi^\pm_\ell \rangle := \text{span}_\mathbb{C} \{ \psi^\pm_\ell \}$. We endow $V$ with the symplectic structure $\omega : V \times V \to \mathbb{C}$ defined by

$$\omega(\psi^+_\ell, \psi^-_j) = \begin{cases} +1 & \text{when } \nu_\ell = 0, \\ \pm 1 & \text{when } \nu_\ell \neq 0; \end{cases} \quad \omega(\psi^+_\ell, \psi^+_j) = 0 \quad \text{otherwise},$$

and extending to $V \times V$ linearly in the first factor and conjugate linearly in the second factor. Then according to [55] and [56], we have (cf. [55, Prop. 2.4]):

**Lemma 3.2.** $\omega : V \times V \to \mathbb{C}$ is a Hermitian symplectic form and for any $\phi, \psi \in \text{dom}_{\text{max}}(\Delta)$, we have

$$\langle \Delta \phi, \psi \rangle - \langle \phi, \Delta \psi \rangle = \omega(\vec{\phi}, \vec{\psi})$$

where $\vec{\phi} := \sum \{ c^+_\ell(\phi)\psi^+_\ell + c^-_\ell(\phi)\psi^-_\ell \} \in V$ and $\vec{\psi}$ is defined similarly.

A subspace $L \subset V$ is called Lagrangian if

$$\{ w \in V \mid \omega(v, w) = 0 \quad \text{for all} \quad v \in L \} = L.$$

We can now prove:

**Theorem 3.3.** Self-adjoint extensions of $\Delta$ are in one-to-one correspondence with Lagrangian subspaces of $V$ in the sense that given any Lagrangian subspace $L \subset V$, defining

$$\text{dom}_L(\Delta) := \{ \phi \in \text{dom}_{\text{max}}(\Delta) \mid \vec{\phi} \in L \},$$

the operator

$$\Delta_L := \Delta : \text{dom}_L(\Delta) \to L^2(M,E)$$

is self-adjoint and any self-adjoint extension of $\Delta$ is of the form $\Delta_L$ for some Lagrangian $L \subset V$. 
Proposition 3.4. A subset $\mathcal{D}$ is self-adjoint if and only if
\begin{equation}
\omega(\vec{\phi}, \vec{\psi}) = 0 \quad \text{for all} \quad \vec{\phi} \in \mathcal{D} \iff \vec{\psi} \in \mathcal{D}.
\end{equation}

Suppose that $\mathcal{D}$ is self-adjoint and define $L := \{ \vec{\phi} \in V \mid \vec{\phi} \in \mathcal{D} \}$; we shall prove that $L$ is Lagrangian. Let $w \in L$ and choose $\vec{\psi} \in \mathcal{D}$ such that $\vec{\psi} = w$. Then by (3.4), $\omega(\vec{\phi}, w) = 0$ for all $\vec{\phi} \in \mathcal{D}$. Therefore, $\omega(v, w) = 0$ for all $v \in L$. Conversely, let $w \in V$ and assume that $\omega(v, w) = 0$ for all $v \in L$. Choose $\vec{\psi} \in \text{dom}_{\text{max}}(\Delta)$ such that $\vec{\psi} = w$; this can always be done, for if $w = \sum \{ c_+^\ast \psi_+^r + c_-^\ast \psi_-^r \}$, then
\begin{equation}
\vec{\psi} := \rho \left( \sum \{ c_+^\ast \psi_+^r + c_-^\ast \psi_-^r \} \right)
\end{equation}
will do, where $\rho \in C^\infty(M)$ is supported in the tubular neighborhood $U$ and equals 1 near $r = 0$. Then $\omega(v, w) = 0$ for all $v \in L$ implies that $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\vec{\phi} \in \mathcal{D}$, which by (3.4), implies that $\vec{\psi} \in \mathcal{D}$, which further implies that $w = \vec{\psi} \in L$.

Now let $L \subset V$ be Lagrangian; we shall prove that $\Delta_L$ is self-adjoint, that is, (3.3) holds. Let $\vec{\psi} \in \text{dom}_L(\Delta)$. Then, since $L$ is Lagrangian, we automatically have $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\vec{\phi} \in \text{dom}_L(\Delta)$. Conversely, let $\vec{\psi} \in \text{dom}_L(\Delta)$ and assume that $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\vec{\phi} \in \text{dom}_L(\Delta)$. By the construction (3.4), given any $v \in L$ we can find a $\vec{\phi} \in \text{dom}_{\text{max}}(\Delta)$ such that $\vec{\phi} = v$. Therefore, $\omega(\vec{\phi}, \vec{\psi}) = 0$ for all $\vec{\phi} \in \text{dom}_L(\Delta)$ implies that $\omega(v, \vec{\psi}) = 0$ for all $v \in L$, which by the Lagrangian condition on $L$, implies that $\vec{\psi} \in L$. This shows that $\vec{\psi} \in \text{dom}_L(\Delta)$ and our proof is complete.

3.3. Characterizations of Lagrangian subspaces. Kostrykin and Schrader [12, Lem. 2.2] characterize all Lagrangian subspaces in complex Euclidean space.

Proposition 3.4. A subset $L \subset \mathbb{C}^{2k}$ is Lagrangian (with respect to the standard Euclidean symplectic form) if and only if it can be described by a system of equations
\begin{equation}
L = \{ \vec{c} \in \mathbb{C}^{2k} \mid (A \ B) \vec{c} = 0 \} \subset \mathbb{C}^{2k},
\end{equation}
where $A$ and $B$ are $k \times k$ matrices such that the rank of $(A \ B)$ is $k$ and $AB^* = BA^*$.

As seen in (3.3), $V$ can be identified with $\mathbb{C}^{2q} = \mathbb{C}^{2q_0} \times \mathbb{C}^{2q_1}$, where $q = q_0 + q_1$, with minus the standard symplectic form on the $\mathbb{C}^{2q_0}$ factor and the standard symplectic form on the $\mathbb{C}^{2q_1}$ factor. Using this fact, we prove

Corollary 3.5. A Lagrangian subspace $L \subset V$ can be characterized by $q \times q$ matrices $A$ and $B$ via
\begin{equation}
L \cong \{ \vec{c} \in \mathbb{C}^{2q} \mid (A \ B) \vec{c} = 0 \} \subset \mathbb{C}^{2q},
\end{equation}
where $(A \ B)$ has rank $q$ and $A'B'^*$ is self-adjoint where $A'$ is the matrix $A$ with the first $q_0$ columns of $A$ multiplied by $-1$.

Proof. Define $T : \mathbb{C}^{2q} \to \mathbb{C}^{2q}$ by
\begin{equation}
T(v_1, \ldots, v_{q_0}, v_{q_0+1}, \ldots, v_{2q}) = (-v_1, \ldots, -v_{q_0}, v_{q_0+1}, \ldots, v_{2q}).
\end{equation}
Then using the definition of $\omega$ in (3.4), it follows that
\begin{equation}
\omega(v, w) = \omega_E(Tv, Tw) \quad \text{for all} \quad v, w \in \mathbb{C}^{2q},
\end{equation}
where we identify \( V \) with \( \mathbb{C}^{2q} \) and \( \omega_E \) is the standard Euclidean symplectic form on \( \mathbb{C}^{2q} \). Therefore,
\[
\{ \tilde{c} \in \mathbb{C}^{2q} | (A \cdot B) \tilde{c} = 0 \} \text{ is Lag. in } (\mathbb{C}^{2q}, \omega) \quad \iff \quad \{ T\tilde{c} \in \mathbb{C}^{2q} | (A \cdot B) \tilde{c} = 0 \} \text{ is Lag. in } (\mathbb{C}^{2q}, \omega_E) \quad \iff \quad \{ \tilde{c} \in \mathbb{C}^{2q} | (A' \cdot B) \tilde{c} = 0 \} \text{ is Lag. in } (\mathbb{C}^{2q}, \omega_E),
\]
which holds, by Proposition 3.3, if and only if \( A'B^* \) is self-adjoint. \( \square \)

As seen in the formula (3.1) of Proposition 3.1, the \( \lambda_\ell = -\frac{1}{4} \) eigenvalues of \( A_\Gamma \) and the \( -\frac{1}{4} < \lambda \ell < \frac{1}{4} \) eigenvalues of \( A_\Gamma \) give rise to rather distinct components of \( \text{dom}_{\max}(\Delta) \). For this reason, it is natural to separate Lagrangian subspaces of \( L \) into \( \lambda_\ell = -\frac{1}{4} \) components and \( -\frac{1}{4} < \lambda \ell < \frac{1}{4} \) components. With this discussion in mind, we call a Lagrangian subspace \( L \subset V \) \emph{decomposable} if \( L = L_0 \oplus L_1 \) where \( L_0 \) is an arbitrary Lagrangian subspace of \( \bigoplus_{\lambda_\ell = -\frac{1}{4}} E_\ell^+ \oplus E_\ell^- \) and \( L_1 \) is an arbitrary Lagrangian subspace of \( \bigoplus_{-\frac{1}{4} < \lambda_\ell < \frac{1}{4}} E_\ell^+ \oplus E_\ell^- \).

The characterization of all such \( L_0, L_1 \) follows from Proposition 3.3.

\begin{corollary} \textbf{Theorem 3.6.} \textit{The components} \( L_0 \) \textit{and} \( L_1 \) \textit{of a decomposable Lagrangian} \( L = L_0 \oplus L_1 \subset V \) \textit{can be characterized by matrices} \( (A_0 \ B_0) \) \textit{(with} \( A_0 \text{ and} B_0 \) \textit{q} \times \textit{q} \textit{matrices) and} \( (A_1 \ B_1) \) \textit{(with} \( A_1 \text{ and} B_1 \) \textit{q} × \textit{q} \textit{matrices)} \textit{via} \[
L_0 \cong \{ \tilde{c} \in \mathbb{C}^{2q_0} | (A_0 \ B_0) \tilde{c} = 0 \} \subset \mathbb{C}^{2q_0},
\]
\textit{and} \[
L_1 \cong \{ \tilde{c} \in \mathbb{C}^{2q_1} | (A_1 \ B_1) \tilde{c} = 0 \} \subset \mathbb{C}^{2q_1},
\]
\textit{where the matrix} \( (A_0 \ B_0) \) \textit{has rank} \( q_0 \) \textit{and} \( A_0 B_0^* = B_0 A_0^* \), \textit{and} \( (A_1 \ B_1) \) \textit{has rank} \( q_1 \) \textit{and} \( A_1 B_1^* = B_1 A_1^* \). \end{corollary}

In the Introduction we discussed split-type Lagrangians. Here, we say that the Lagrangian \( L_0 \subset \bigoplus_{\lambda_\ell = -\frac{1}{4}} E_\ell^+ \oplus E_\ell^- \) is of \textit{split-type} if it can be written as \( \bigoplus_{\lambda_\ell = -\frac{1}{4}} L_\ell \) with \( L_\ell \) a Lagrangian subspace of \( E_\ell^+ \oplus E_\ell^- \). In the following proposition, we characterize all such Lagrangians \( L_\ell \).

\begin{proposition} \textbf{Proposition 3.7.} \textit{L} \subset \mathbb{C}^2 \textit{is Lagrangian if and only if} \( L = L_\theta \) \textit{for some} \( \theta \in [0, \pi] \) \textit{where} \[
L_\theta = \{(x, y) \in \mathbb{C}^2 | \cos \theta x + \sin \theta y = 0 \}.
\end{proposition}

\textit{Proof.} According to Proposition 3.3 we have \( L = \{(x, y) \in \mathbb{C}^2 | a x + b y = 0 \} \subset \mathbb{C}^2 \), where \( a, b \in \mathbb{C} \) with \( a \overline{b} = \pi b \); that is, \( a \overline{b} \in \mathbb{R} \). If \( b = 0 \), then \( L_\ell \cong \{(x, y) \in \mathbb{C}^2 | 1 \cdot x + 0 \cdot y = 0 \} \), and we can take \( \theta = 0 \). If \( b \neq 0 \), then multiplying \( a x + b y = 0 \) by \( \overline{b} \) we can write \( L = \{(x, y) \in \mathbb{C}^2 | a \overline{b} x + |b|^2 y = 0 \} = \{(x, y) \in \mathbb{C}^2 | \alpha x + \beta y = 0 \} \), where \[
\alpha = \frac{a \overline{b}}{\sqrt{(a \overline{b})^2 + (|b|^2)^2}}, \quad \beta = \frac{|b|}{\sqrt{(a \overline{b})^2 + (|b|^2)^2}}.
\]

Note that \( \alpha, \beta \in \mathbb{R} \) and \( \alpha^2 + \beta^2 = 1 \). It follows that \( (\alpha, \beta) = (\cos \theta, \sin \theta) \) for some \( \theta \in \mathbb{R} \). If \( \theta \in (\pi, 2\pi] \), then we can replace \( (\alpha, \beta) \) by \( (-\alpha, -\beta) \) to ensure that \( \theta \in [0, \pi) \). This completes our proof. \( \square \)
4. The model problems

For the rest of this paper, unless stated otherwise, we fix an arbitrary Lagrangian $L$ in $V$. In this section we analyze the eigenvalue equation for the model problem.

4.1. The model operator. In the last section we saw that only the eigenvalues of $A_{T}$ in the interval $\left[-\frac{1}{q}, \frac{3}{q}\right]$ are involved in the various self-adjoint extensions of $\Delta$. For this reason, in this section as a first step to prove our main results we shall analyze the projection of

$$\Delta|_{d} = -\frac{d^{2}}{dr^{2}} + \frac{1}{r^{2}}A_{T}$$

onto the eigenspaces of $A_{T}$ with eigenvalues in $\left[-\frac{1}{q}, \frac{3}{q}\right]$. Recall from Corollary 3.5 with $q = q_{0} + q_{1}$, that the Lagrangian $L$ can be identified with the null space of a $q \times 2q$ matrix $(A \ B)$ of full rank with $A$ and $B$ $q \times q$ matrices such that $A^{*}B$ is self-adjoint. Then writing $A_{T}$ as a diagonal matrix with respect to its eigenfunctions with eigenvalues in $\left[-\frac{1}{q}, \frac{3}{q}\right]$, we shall consider the operator

$$\mathcal{L} := -\frac{d^{2}}{dr^{2}} + \frac{1}{r^{2}}A \quad \text{over} \quad [0, R],$$

where $R > 0$ is arbitrary, but fixed, and $A$ is the $q \times q$ matrix

$$A = \begin{pmatrix} -\frac{1}{q}I_{q_{0}} & \lambda_{q_{0}+1} & 0 & \cdots & 0 \\ 0 & \lambda_{q_{0}+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \lambda_{q_{0}+q_{1}} \end{pmatrix}.$$ 

We put Dirichlet conditions at the right end $r = R$ of the interval $[0, R]$. Then according to Proposition 4.1 we have

**Proposition 4.1.** $\phi \in \mathcal{D}_{\max}$, the maximal domain of $\mathcal{L}$, if and only if $\phi(R) = 0$ and $\phi$ has the following form:

$$\phi = \sum_{\ell = 1}^{q_{0}} \left\{ c_{\ell}(\phi) r^{\ell} e_{\ell} + c_{q_{0}+\ell}(\phi) r^{\ell} \log r e_{\ell} \right\}$$

$$+ \sum_{\ell = 1}^{q_{1}} \left\{ c_{q_{0}+\ell}(\phi) r^{\nu_{\ell}+\frac{1}{2}} e_{q_{0}+\ell} + c_{q_{0}+q_{1}+\ell}(\phi) r^{-\nu_{\ell}+\frac{1}{2}} e_{q_{0}+\ell} \right\} + \tilde{\phi},$$

where $\nu_{j} := \sqrt{\lambda_{q_{0}+j} + \frac{1}{q}} > 0$, $e_{\ell}$ is the column vector with 1 in the $\ell$-th slot and 0’s elsewhere, the $c_{\ell}(\phi)$’s are constants, and the $\tilde{\phi}$ is continuously differentiable on $[0, R]$ such that $\phi(r) = \mathcal{O}(r^{\frac{1}{2}})$ and $\tilde{\phi}'(r) = \mathcal{O}(r^{-\frac{1}{2}})$ near $r = 0$, and $\mathcal{L}\tilde{\phi} \in L^{2}([0, R], \mathbb{C}^{q})$.

We dropped the factors $\frac{1}{\sqrt{\nu_{\ell}}}$, which appear in the statement of Proposition 3.1 from the terms in (4.1) for $\phi$. Then as a consequence of Theorem 3.8 we know that

$$\mathcal{L}_{L} : \mathcal{D}_{L} \rightarrow L^{2}([0, R], \mathbb{C}^{q})$$

is self-adjoint, $\mathcal{D}_{L} = \{ \phi \in \mathcal{D}_{\max} \mid \tilde{\phi} \in L \}$, where $\phi$ has the form in (4.1) with $\tilde{\phi} = (c_{1}(\phi), c_{2}(\phi), \ldots, c_{2q}(\phi))^{t}$. In terms of the matrices $A$ and $B$, we can also write

$$\mathcal{D}_{L} = \{ \phi \in \mathcal{D}_{\max} \mid (A \ B) \tilde{\phi} = 0 \}.$$
4.2. Eigenvalue equation. To analyze the $\zeta$-function of $\mathcal{L}_L$, we derive an equation for the eigenvalues of $\mathcal{L}_L$. For this, we first find solutions to the equation

$$(\mathcal{L}_L - \mu^2) \phi = 0.$$ 

As the reader can easily check, this is just a system of Bessel equations as described in [1, p. 362], whose solution (after judiciously choosing the constants for later convenience) can be taken to be of the form

$$(4.3) \quad \phi = \sum_{\ell=1}^{q_0} \left\{ c_{\ell}(\phi) r^{1/2} J_0(\mu r) e_\ell + c_{q+\ell}(\phi) r^{1/2} \tilde{J}_0(\mu r) e_\ell \right\}$$

$$+ \sum_{\ell=1}^{q_1} \left\{ 2^{\nu}(1 + \nu_\ell)c_{q_0+\ell}(\phi) \mu^{-\nu_\ell} r^{1/2} J_\nu(\mu r) e_{q_0+\ell} + 2^{-\nu_\ell}(1 - \nu_\ell)c_{q+q_0+\ell}(\phi) \mu^{\nu_\ell} r^{1/2} J_{-\nu}(\mu r) e_{q_0+\ell} \right\},$$

where $J_\nu(z)$ denotes the Bessel function of the first kind and

$$(4.4) \quad \tilde{J}_0(\mu r) := \frac{\pi}{2} Y_0(\mu r) - (\log \mu - \log 2 + \gamma) J_0(\mu r),$$

with $Y_0(z)$ the Bessel function of the second kind. For notational convenience let us introduce $J_+(\mu R) = J_0(\mu R)$ and $J_-(\mu R) = \tilde{J}_0(\mu R)$.

Define $q \times q$ matrices $J_+(\mu), J_- (\mu)$ by

$$J_+(\mu) := \begin{pmatrix} J_{\pm}(\mu R) & 1_{d \times g_0}^T \\ 0 & 2^{\nu_1}(1 + \nu_1) \mu^{\nu_1} J_{\pm}(\mu R) & \cdots & 0 \\ 0 & \cdots & 0 & 2^{\nu_q}(1 + \nu_q) \mu^{\nu_q} J_{\pm}(\mu R) \\ 0 & \cdots & 0 & 2^{\nu_q}(1 + \nu_q) \mu^{\nu_q} J_{\pm}(\mu R) \end{pmatrix}.$$ 

In the following proposition, we determine an eigenvalue equation for the $\mu$’s.

**Proposition 4.2.** $\mu^2$ is an eigenvalue of $\mathcal{L}_L$ if and only if

$$F(\mu) := \det \begin{pmatrix} A & B \\ J_+(\mu) & J_- (\mu) \end{pmatrix} = 0.$$ 

**Proof.** Imposing the Dirichlet condition at $r = R$ on $\phi$ of the form (4.3), we obtain

$$c_{\ell}(\phi) J_0(\mu R) + c_{q+\ell}(\phi) \tilde{J}_0(\mu R) = 0, \quad \ell = 1, \ldots, q_0,$$

and

$$2^{\nu_\ell}(1 + \nu_\ell)c_{q_0+\ell}(\phi) \mu^{-\nu_\ell} J_\nu(\mu R) + 2^{-\nu_\ell}(1 - \nu_\ell)c_{q+q_0+\ell}(\phi) \mu^{\nu_\ell} J_{-\nu}(\mu R) = 0, \quad \ell = 1, \ldots, q_1.$$ 

We can summarize these two equations as

$$(4.5) \quad (J_+(\mu) \quad J_- (\mu)) \vec{\phi} = 0,$$

where $\vec{\phi} = (c_1(\phi), c_2(\phi), \ldots, c_{2q}(\phi))^T$. Now recall that [1, p. 360]

$$(4.6) \quad z^{-\nu} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{k+2k} k! \Gamma(v + k + 1)} \sim \frac{1}{2^v \Gamma(1 + v)} \left( 1 - \frac{z^2}{4(1 + v)} \right) + \cdots,$$

and

$$(4.7) \quad \frac{\pi}{2} Y_0(z) = \left( \log z - \log 2 + \gamma \right) J_0(z) - \sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{2} z^2)^k}{(k!)^2},$$

where $H_k$ denotes the $k$th harmonic number.
where $H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. Combining (4.1), (4.3), with $v = 0$, and (4.4), we get

$$J_0(\mu r) = (log r) J_0(\mu r) - \sum_{k=1}^{\infty} \frac{H_k(-\frac{1}{2}(\mu r)^2)^k}{(k!)} = log r + O(r).$$

From (4.3), (4.6) and (4.8), it follows that

$$\phi \sim \sum_{\ell=1}^{q_0} \left\{ c_\ell(\phi) r^\ell e_\ell + c_{q_0+\ell}(\phi) r^{\ell+1} log r e_\ell \right\}$$

$$+ \sum_{\ell=1}^{q_1} \left\{ c_{q_0+\ell}(\phi) r^{\ell+1} e_{q_0+\ell} + c_{q_0+q_0+\ell}(\phi) r^{\ell+1} e_{q_0+q_0+\ell} \right\} \text{ near } r = 0.$$

In particular, by (4.2), $\tilde{\phi}$ in $\mathcal{D}_L$ satisfies $(A \ B) \tilde{\phi} = 0$, and therefore, in view of (4.6), we conclude that

$$\left( \begin{array}{cc} A & B \\ J_+(\mu) & J_-(\mu) \end{array} \right) \tilde{\phi} = 0.$$

For nontrivial $\tilde{\phi}$, this equation can hold if and only if the matrix in front of $\tilde{\phi}$ is singular. This completes our proof. \hfill \Box

4.3. Asymptotics of $F(\mu)$. In order to find relevant properties of the resolvent, the heat-trace and the $\zeta$-function, we shall need the asymptotics of $F(\mu)$ as $|\mu| \to \infty$.

**Proposition 4.3.** Let $\mathcal{Y} \subset \mathbb{C}$ be a sector (closed angle) in the right-half plane. Then as $|x| \to \infty$ with $x \in \mathcal{Y}$, we have

$$F(ix) \sim (2\pi R)^{-\frac{\sigma}{2}} 2^{\frac{\sigma}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1 - \nu_j) x^{|\nu| - \frac{\sigma}{2}} e^{\sigma R (\gamma - \log x)^{\sigma_0} \times}$$

$$p \left( (\gamma - \log x)^{-1}, x^{-1} \right) \left( 1 + O(x^{-1}) \right),$$

where $\gamma = \log 2 - \gamma$, $p(x,y)$ is given in (2.3), $O(x^{-1})$ is a power series in $x^{-1}$, and

$$\frac{d}{dx} \log F(ix) \sim q R + \frac{q_0 - j_0}{x (log x - \gamma)}$$

$$+ \sum c_{\ell \xi} x^{-2\xi - 1} \left\{ \ell(\gamma - log x)^{-\ell} - 2\xi(\gamma - log x)^{-\ell} \right\} + O(x^{-1}),$$

with the same meaning for $O(x^{-1})$ and where the $c_{\ell \xi}$'s are the constants in (2.3).

**Proof.** Using the identity $(iz)^{-\nu} J_\nu(iz) = z^{-\nu} I_\nu(z)$, where $I_\nu(z)$ is the modified Bessel function of the first kind, we can write $F(ix) = det \left( \begin{array}{cc} A & B \\ J_+(ix) & J_-(ix) \end{array} \right)$

where (we use the notation $I_{\pm 0}(xR) = J_{\pm 0}(ixR)$)

$$J_{\pm}(ix) = \begin{pmatrix} I_{\pm 0}(xR) Id_{q_0} & 0 & \cdots & 0 \\ 0 & 2^{\pm \nu_1} \Gamma(1 + \nu_1) x^{\pm \nu_1} I_{\pm \nu_1}(xR) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^{\pm \nu_1} \Gamma(1 + \nu_1) x^{\pm \nu_1} I_{\pm \nu_1}(xR) \end{pmatrix}.$$


Factoring out $2^{-\nu_j} \Gamma(1 - \nu_j) x^{\nu_j} I_{-\nu_j}(xR)$ from the $(q + q_j + j)$-th row of the matrix

\[
\begin{pmatrix}
A & B \\
J_+(ix) & J_-(ix)
\end{pmatrix}
\]

we obtain

\[
(4.11) \quad F(ix) = \rho \prod_{j=1}^{q_j} x^{\nu_j} I_{-\nu_j}(xR) \times \det \begin{pmatrix} I_0(xR) & 0 & A(x) & J_0(ixR) & 0 \\ 0 & A(x) & \tilde{J}_0(ixR) & 0 & \text{Id}_{q_0} \end{pmatrix},
\]

where $\rho = \prod_{j=1}^{q_j} 2^{-\nu_j} \Gamma(1 - \nu_j)$ and

\[
A(x) = \begin{pmatrix}
\tau_1 x^{-2\nu_1} & 0 & \cdots & 0 \\
0 & \tau_2 x^{-2\nu_2} & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & \tau_{q_1} x^{-2\nu_{q_1}}
\end{pmatrix},
\]

with $\tau_j = 2^{2\nu_j} \frac{\Gamma(1+\nu_j)}{\Gamma(1-\nu_j)}$. In order to find the asymptotics of $F(ix)$ in \cite{KLP}, we shall determine the asymptotics of $A(x)$ and then of $\tilde{J}_0(x)$. To determine the asymptotics of $A(x)$, we recall (see \cite[p. 377]{KLP}) that as $z \to \infty$ with $z \in \mathcal{U}$, we have

\[
(4.12) \quad I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left( 1 - \frac{4\nu^2 - 1}{8z} + \mathcal{O}(z^{-2}) \right) \sim \frac{e^z}{\sqrt{2\pi z}} \left( 1 + \mathcal{O}\left(\frac{1}{z}\right) \right),
\]

where $\mathcal{O}\left(\frac{1}{z}\right)$ is a power series in $\frac{1}{z}$ and where only $v^2$'s occur in $\mathcal{O}\left(\frac{1}{z}\right)$. In particular, as $x \to \infty$ with $x \in \mathcal{U}$, we have $\frac{I_{\nu_j}(xR)}{I_{-\nu_j}(xR)} \sim 1$. Therefore,

\[
(4.13) \quad A(x) \sim \begin{pmatrix}
\tau_1 x^{-2\nu_1} & 0 & \cdots & 0 \\
0 & \tau_2 x^{-2\nu_2} & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & \tau_{q_1} x^{-2\nu_{q_1}}
\end{pmatrix}.
\]

To determine the asymptotics of $\tilde{J}_0(ix)$, note that $J_0(ix) = I_0(z)$ and

\[
\frac{\pi}{2} Y_0(ix) = \left( \log(ix) - \log 2 + \gamma \right) J_0(ix) - \sum_{k=1}^{\infty} \frac{H_k \left( \frac{1}{4} (ix)^2 \right)^k}{(k!)^2}
\]

\[
= \left( \log z + i \frac{\pi}{2} - \log 2 + \gamma \right) I_0(z) - \sum_{k=1}^{\infty} \frac{H_k \left( \frac{1}{4} z^2 \right)^k}{(k!)^2} = i \frac{\pi}{2} I_0(z) - K_0(z),
\]

where

\[
K_0(z) := - \left( \log z - \log 2 + \gamma \right) I_0(z) + \sum_{k=1}^{\infty} \frac{H_k \left( \frac{1}{4} z^2 \right)^k}{(k!)^2}
\]

is the modified Bessel function of the second kind. Thus, we can write

\[
\tilde{J}_0(ixR) = \frac{\pi}{2} Y_0(ixR) - (\log(ix) - \log 2 + \gamma) J_0(ixR)
\]

\[
= \frac{\pi}{2} I_0(xR) - K_0(xR) - \left( i \frac{\pi}{2} + \log x - \log 2 + \gamma \right) I_0(xR)
\]

\[
= \left( - (\log x - \tilde{\gamma}) I_0(xR) - K_0(xR) \right).
\]
By [1, p. 378], $K_0(x)$ is exponentially decaying as $x \to \infty$ in $\mathcal{Y}$, so

$$\tilde{J}_0(ixR) = -(\log x - \gamma)I_0(xR) - K_0(xR) \sim (\gamma - \log x)I_0(xR).$$

Summarizing our work so far, we see from (4.11) that

$$F(ix) \sim \rho x^{[\nu]} \prod_{j=1}^{q_1} I_{-\nu_j}(xR) \times \det \begin{pmatrix} A & 0 & (\gamma - \log x)I_0(xR) \Id_{q_0} \\ 0 & A(x) & 0 \\ 0 & 0 & \Id_{q_1} \end{pmatrix},$$

where $\rho = \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1 - \nu_j)$, $|\nu| = \nu_1 + \cdots + \nu_{q_1}$, and $A(x)$ satisfies (4.13). Now factoring out $(\gamma - \log x)I_0(xR)$ from $F(ix)$ and using the definition of $p(x,y)$ in (2.1) (with “$x$” replaced with $(\gamma - \log x)^{-1}$ and “$y$” replaced with $x^{-1}$), we obtain

$$F(ix) \sim \rho x^{[\nu]} \prod_{j=1}^{q_1} I_{-\nu_j}(xR) I_0(xR)^{q_0} (\gamma - \log x)^{q_0} p \left( (\gamma - \log x)^{-1}, x^{-1} \right).$$

In view of the asymptotics (4.12) for $I_0(z)$, we get

$$F(ix) \sim \rho x^{[\nu]} \left( (2\pi)^{-\frac{1}{2}} (xR)^{-\frac{1}{2}} e^{xR} \right)^{q_0+q_1} (\gamma - \log x)^{q_0} \times \left( \frac{\gamma}{\log x} \right)^{\frac{1}{2}} (1 + O(x^{-1})),

$$

which is equivalent to

$$F(ix) \sim (2\pi R)^{-\frac{1}{2}} \prod_{j=1}^{q_1} 2^{-\nu_j} \Gamma(1 - \nu_j) x^{[\nu]}(\gamma - \log x)^{q_0} \times \left( \frac{\gamma}{\log x} \right)^{\frac{1}{2}} (1 + O(x^{-1})),

$$

and the proof of our first asymptotic formula is complete. To prove our second formula, recall from (2.3) that

$$p(x,y) = a_{2\alpha_0} x^{2\alpha_0} \left( 1 + \sum_{k=1} b_{k\beta} x^k y^{2\beta} \right),$$

so that (with “$x$” replaced with $(\gamma - \log x)^{-1}$ and “$y$” replaced with $x^{-1}$)

$$F(ix) \sim \text{const.} \times x^{[\nu]}(\gamma - \log x)^{q_0-j_0} \times \left( (1 + \sum_{k=1} b_{k\beta} \gamma - \log x)^{-k} x^{-2\beta} \right) \left( 1 + O(x^{-1}) \right).$$

As in (2.6), \( \log \left( 1 + \sum b_{k\beta} x^k y^{2\beta} \right) = \sum c_{\xi \ell} x^\ell y^{2\xi} \) so taking the logarithm of $F(ix)$ we see that

$$\log F(ix) \sim \text{const.} + qxR + \left( [\nu] - \frac{q}{2} - 2\alpha_0 \right) \log x + \left( q_0 - j_0 \right) \log(\gamma - \log x) + \sum_{\xi,\ell} c_{\xi \ell} (\gamma - \log x)^{-\ell} x^{-2\xi} + O(x^{-1}),$$

and taking the derivative of both sides completes our proof. \( \square \)
4.4. The log terms only case. Suppose that \( q_1 = 0 \) so that the only eigenvalues of \( A_1 \) in the critical interval \([\mathcal{C}, \frac{3}{4}]\) are the \(-\frac{1}{4}\) eigenvalues. In this case, we shall denote \( A \) by \( A_0 \) and \( B \) by \( B_0 \) so that

\[
F(\mu) = \det \left( A_0 \begin{pmatrix} J_0(\mu R) \Id_{q_0} & B_0 \Id_{q_0} \end{pmatrix} \right).
\]

Recall from Section 2.3 (see (2.9)) the polynomial

\[
p_0(z) := \det \left( \begin{pmatrix} A_0 \Id_{q_0} & B_0 \Id_{q_0} \end{pmatrix} \right),
\]

which is a polynomial in the complex variable \( z \in \mathbb{C} \) of degree at most \( q_0 \). Then we can write

\[
\frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^{\infty} \frac{\beta_k}{z^k},
\]

where the series is absolutely convergent for \(|z| \) sufficiently large and where \( \beta_1 = \deg p_0 \). In the case that \( q_1 = 0 \), Proposition 4.3 can be written as follows.

**Proposition 4.4.** Suppose that \( q_1 = 0 \) and let \( \mathcal{T} \subseteq \mathbb{C} \) be a sector (closed angle) in the right-half plane. Then as \(|x| \to \infty \) with \( x \in \mathcal{T} \), we have

\[
(4.14) \quad F(ix) \sim (2\pi xR)^{-\frac{3}{4}} e^{q_0 xR} p_0(\log x) \left( 1 + \mathcal{O}(x^{-1}) \right),
\]

where \( \mathcal{O}(x^{-1}) \) is a power series in \( x^{-1} \), and

\[
(4.15) \quad \frac{d}{dx} \log F(ix) \sim \sum \frac{\beta_k}{x(\log x)^k} + q_0 R - \frac{q_0}{2x} + \mathcal{O}(x^{-2}),
\]

where \( \mathcal{O}(x^{-2}) \) is a power series in \( x^{-1} \) starting from \( x^{-2} \).

**Proof.** A direct application of (4.10) in Proposition 4.3 with \( q_1 = 0 \) gives

\[
F(ix) \sim (2\pi xR)^{-\frac{3}{4}} e^{q_0 xR} (\gamma - \log x)^{q_0} p_0(\gamma - \log x)^{-1} \left( 1 + \mathcal{O}(x^{-1}) \right),
\]

where \( \mathcal{O}(x^{-1}) \) is a power series in \( x^{-1} \) and

\[
p(x) := \det \left( \begin{pmatrix} A_0 \Id_{q_0} & B_0 \Id_{q_0} \end{pmatrix} \right).
\]

By definition of \( p_0 \), we have \((\gamma - \log x)^{q_0} p_0(\gamma - \log x)^{-1} = p_0(\log x)\). This proves (4.14) and then taking the logarithmic derivative of (4.14) gives (4.15). \( \square \)

5. The zeta function for the model problems

Working with the fixed Lagrangian \( L \), we now analyze the zeta function of \( \mathcal{L}_L \), which by the Argument Principle [10, p. 123] is given by

\[
\zeta(s, \mathcal{L}_L) = \frac{1}{2\pi i} \int_{\mathcal{C}} \mu^{-2s} \frac{d}{d\mu} \log F(\mu) d\mu = \frac{1}{2\pi i} \int_{\mathcal{C}} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu,
\]

where \( \mathcal{C} \) is a contour in the plane shown in Figure 1. Here we used that \( \mu^2 \) is an eigenvalue of \( \mathcal{L}_L \) if and only if \( \mu \) is a zero of \( F(\mu) \). By Proposition 4.3 the zeta function \( \zeta(s, \mathcal{L}_L) \) is well-defined for \( \Re s > \frac{1}{2} \).
5.1. A basic lemma. In order to determine the exact structure of the analytic continuation of \( \zeta(s, \mathcal{L}_L) \), we need the following fundamental result.

**Lemma 5.1.** Let \( c \) be a constant and let \( |t| \) be sufficiently large so that \( \log x > c \) for \( x \geq |t| \). Then for any \( k \geq 0 \) we can write

\[
\int_{|t|}^{\infty} x^{-2s-2\xi-1} (c - \log x)^k dx \equiv \sum_{j=0}^{k} \frac{\sigma_{kj}}{(s + \xi)^{j+1}},
\]

modulo an entire function, where \( \sigma_{kj} = (-1)^k \binom{k}{j} \frac{j!}{2^{j+1}} (-c)^{k-j} \), and for any \( k > 0 \) we can write

\[
\int_{|t|}^{\infty} x^{-2s-2\xi-1} (c - \log x)^{-k} dx \equiv \frac{2^{-1}}{(s+\xi)^{k-1}} \sum_{j=0}^{k} (-j)! \left( \frac{1}{s+j+1} \right) (s+\xi)^{k-j} e^{-2(s+\xi)c} \log(s+\xi)
\]

modulo an entire function.

**Proof.** Replacing \( s \) by \( s - \xi \), we can assume that \( \xi = 0 \) from the start. To analyze the first integral we first expand \((c - \log x)^k\) using the binomial theorem:

\[
\int_{|t|}^{\infty} x^{-2s-1} (c - \log x)^k dx = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (-c)^{k-j} \int_{|t|}^{\infty} x^{-2s-1} (\log x)^j dx.
\]

Thus, we are left to prove that

\[
\int_{|t|}^{\infty} x^{-2s-1} (\log x)^j dx = \frac{j!}{2j+1} \cdot \frac{1}{s+j+1}
\]

modulo an entire function. However, since the integral \( \int_{|t|}^{\infty} x^{-2s-1} (\log x)^j dx \) is entire, we can assume that the lower limit of the integral in (5.3) is 1. Now taking \( j \) derivatives of both sides of the equality \( \int_{1}^{\infty} x^{-2s-1} dx = \frac{1}{2s} \) with respect to \( s \), we obtain

\[
(-2)^j \int_{1}^{\infty} x^{-2s-1} (\log x)^j dx = \frac{(-1)^j j!}{2s^{j+1}},
\]
which proves (5.3).

To prove the second claim in this proposition, we make the change of variables

\[ y = 2s(\log x - c) \text{ or } x = e^{c/y^{2s}}, \]

and obtain

\[
\int_{|t|}^{\infty} x^{-2s-1} (c - \log x)^{-k} dx = (-1)^k e^{-2sc} (2s)^{k-1} \int_{2sc}^{\infty} e^{-y} dy. 
\]

where \( C := \log |t| - c. \) Recall that the exponential integral is defined by (see [11] p. 228 or [27] Sec. 8.2)

\[
\text{Ei}_k(z) := \int_1^{\infty} e^{-zu} \frac{du}{u^k} = z^{-k} \int_z^{\infty} e^{-y} \frac{dy}{y^k}.
\]

Therefore,

\[
(5.4) \quad \int_{|t|} x^{-2s-1} (c - \log x)^{-k} dx = \frac{(-1)^k e^{-2sc}}{C^{k-1}} \text{Ei}_k(2sC).
\]

Also from [11] p. 229] or [27] p. 877], we have

\[
\text{Ei}_k(z) = \frac{(-z)^{k-1}}{(k-1)!} \left( - \log z + \psi(k) \right) - \sum_{j=0, j \neq k-1}^{\infty} \frac{(-z)^{j-1}}{(j-k+1) j!},
\]

where \( \psi(1) := -\gamma \) and \( \psi(k) := -\gamma + \sum_{j=1}^{k-1} \frac{1}{j} \) for \( k > 1. \) Hence,

\[
\text{Ei}_k(2sC) = C^{k-1} \frac{(-2s)^{k-1}}{(k-1)!} \left\{ -\log(2sC) + \psi(k) \right\} - \sum_{j=0, j \neq k-1}^{\infty} \frac{(-2sC)^{j-1}}{(j-k+1) j!}.
\]

Replacing this into (5.4) and simplifying, we obtain

\[
\int_{|t|} x^{-2s-1} (c - \log x)^{-k} dx = \frac{(2s)^{k-1}}{(k-1)!} e^{-2sc} \log s
\]

modulo an entire function. This completes our proof. \( \square \)

5.2. The \( \zeta \)-function. We now prove the “model problem version” of Theorem 2.1 via the contour integration method [30] [37] [38].

**Proposition 5.2.** Let \( L \subset V \) be an arbitrary Lagrangian subspace of \( \mathbb{C}^{2g} \) and define \( \mathscr{P} \) and \( \mathcal{L} \) as in (2.24) from the matrices \( A \) and \( B \) defining \( L. \) Then the \( \zeta \)-function \( \zeta(s, \mathcal{L}_L) \) extends from \( \mathbb{R}s > \frac{1}{2} \) to \( \mathbb{C} \) \( \setminus \) \( (-\infty, 0] \). Moreover, \( \zeta(s, \mathcal{L}_L) \) can be written in the form

\[
\zeta(s, \mathcal{L}_L) = \zeta_{\text{reg}}(s, \mathcal{L}_L) + \zeta_{\text{sing}}(s, \mathcal{L}_L),
\]

where \( \zeta_{\text{reg}}(s, \mathcal{L}_L) \) has the “regular” poles at the “usual” locations \( s = \frac{1}{2} - k \) for \( k \in \mathbb{N}_0 \), and where \( \zeta_{\text{sing}}(s, \mathcal{L}_L) \) has the following expansion:

\[
\zeta_{\text{sing}}(s, \mathcal{L}_L) = \frac{\sin(\pi s)}{\pi} \left\{ (j_0 - q_0) e^{-2s(\log 2-\gamma)} \log s + \sum_{\xi \in \mathscr{P}} \frac{f_\xi(s)}{(s+\xi)|p_\xi|+1} \right.
\]

\[
+ \sum_{\xi \in \mathscr{X}} g_\xi(s) \log(s + \xi) \left\},
\]

where \( j_0 \) appears in (2.24) and \( f_\xi(s) \) and \( g_\xi(s) \) are entire functions of \( s \) such that

\[
f_\xi(-\xi) = (-1)^{|p_\xi|+1} e^{p_\xi \xi} \left| p_\xi \right| \left| \frac{p_\xi}{2 |p_\xi|} \right|
\]
and

\[ g_\xi(s) = \begin{cases} c_{\ell,0} \frac{2^{s_0}}{\Gamma(s_0 + 1)} s^{s_0} + O(s^{s_0 + 1}) & \text{if } \xi = 0, \\ -c_{\ell,\xi} \frac{s^{s_0}}{\Gamma(s_0 - 1)} (s + \xi)^{s_0 - 1} + O((s + \xi)^{s_0}) & \text{if } \xi > 0, \end{cases} \]

where the \( c_{\ell,\xi} \)’s are the coefficients in \( \mu^\xi \).

Proof. With Figure 1 in mind, we write

\[ \int_C = - \int_{C_1}^{0 + \infty} + \int_{C_1}^{0 - \infty} + \int_{C_t}, \]

where \( C_t \) is the curvy part of \( C \) from \( t \) to \( -t \), and second, using that

\[ i^{-2s} = (e^{i\pi/2})^{-2s} = e^{-i\pi s} \quad \text{and} \quad (-i)^{-2s} = (e^{-i\pi/2})^{-2s} = e^{i\pi s}, \]

we obtain the integral

\[
\zeta(s, \mathcal{L}_L) = \frac{1}{2\pi i} \int_C \mu^{-2s} \frac{d}{d\mu} \log F(\mu) d\mu
\]

\[
= \frac{1}{2\pi i} \left\{ - \int_{[t]} (ix)^{-2s} \frac{d}{dx} \log F(ix) dx + \int_{[t]} (-ix)^{-2s} \frac{d}{dx} \log F(-ix) dx \right\}
\]

\[
= \frac{1}{2\pi i} \left( -e^{-i\pi s} + e^{i\pi s} \right) \int_{[t]} x^{-2s} \frac{d}{dx} \log F(ix) dx + \frac{1}{2\pi i} \int_{C_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu,
\]

or,

\[ (5.5) \quad \zeta(s, \mathcal{L}_L) = \frac{\sin(\pi s)}{\pi} \int_{[t]} x^{-2s} \frac{d}{dx} \log F(ix) dx + \frac{1}{2\pi i} \int_{C_t} \mu^{-2s} \frac{F'(\mu)}{F(\mu)} d\mu, \]

a formula that will be analyzed in a moment. The second integral here is over a bounded contour so is an entire function of \( s \in \mathbb{C} \), so we are left to analyze the analytic properties of the first integral in \((5.5)\). To do so, recall the asymptotics \((4.10)\) in Proposition 4.3, which states that for \( x \to \infty \) we have

\[ (5.6) \quad \frac{d}{dx} \log F(ix) \sim \frac{q_0 - j_0}{x(\log x - \tilde{\gamma})} + G_1(x) + G_2(x) + G_3(x), \]

where \( \tilde{\gamma} = \log 2 - \gamma \), \( G_3(x) \) is a power series in \( x^{-1} \) starting with the constant term \( qR \),

\[ G_1(x) := \sum_\xi \sum_{\ell \leq 0} c_{\ell,\xi} x^{-2\xi - 1} \left\{ \ell(\tilde{\gamma} - \log x)^{-\ell - 1} - 2\xi(\tilde{\gamma} - \log x)^{-\ell} \right\}, \]

and

\[ G_2(x) := \sum_\xi \sum_{\ell > 0} c_{\ell,\xi} x^{-2\xi - 1} \left\{ \ell(\tilde{\gamma} - \log x)^{-\ell - 1} - 2\xi(\tilde{\gamma} - \log x)^{-\ell} \right\}. \]

Since

\[
\frac{\sin(\pi s)}{\pi} \left. \int_{[t]} x^{-2s - k} dx \right|_{x = |t|} = \frac{\sin(\pi s)}{\pi} \left. \frac{x^{-2s - k + 1}}{-2s - k + 1} \right|_{x = |t|} = \frac{\sin(\pi s)}{\pi} |t|^{-2s - k + 1},
\]

which has poles at \( s = \frac{1-k}{2} \) for \( s \notin \mathbb{Z} \), it follows that

\[ (5.7) \quad \frac{\sin(\pi s)}{\pi} \left. \int_{[t]} x^{-2s} \frac{d}{dx} \log G_3(ix) dx \right|_{x = |t|} = \frac{\sin(\pi s)}{\pi} |t|^{-2s - k + 1},\]
will contribute to the function \( \zeta_{\text{reg}}(s, L_L) \) in the statement of this proposition. Setting \( \xi = 0 \) and \( k = 1 \) in equation (5.1) in Lemma 4.1 we see that

\[
\int_{|t|}^\infty x^{-2s} \frac{q_0 - j_0}{x(\log x - \gamma)} \, dx \equiv -(q_0 - j_0)e^{-2\gamma \log s}
\]

modulo an entire function, which gives us the first term in \( \zeta_{\text{sing}}(s, L_L) \).

We now analyze \( \int_{|t|}^\infty x^{-2s} G_2(x) \, dx \). To do so, we apply equation (5.2) term-by-term to

\[
\int_{|t|}^\infty x^{-2s} G_2(x) \, dx = \int_{|t|}^\infty x^{-2s} \sum_{\ell > 0} c_\ell x^{-2\xi - 1} \left\{ \ell (\bar{\gamma} - \log x)^{-\ell - 1} - 2\xi (\gamma - \log x)^{-\ell} \right\} \, dx,
\]

and we see that, modulo an entire function,

\[
\int_{|t|}^\infty x^{-2s} G_2(x) \, dx = \sum_{\xi} \sum_{\ell > 0} c_\ell x^{-2\xi - 1} \left\{ \frac{x^2 \xi}{\ell!} e^{-2(s+\xi)\gamma} (s+\xi)^\ell \log(s+\xi) + \frac{\xi^2 \ell}{(\ell - 1)!} e^{-2(s+\xi)\gamma} (s+\xi)^{\ell-1} \log(s+\xi) \right\}
\]

\[
= \sum_{\xi} \left( e^{-2(s+\xi)\gamma} \sum_{\ell > 0} c_\ell \left\{ \frac{x^2 \xi}{\ell!} (s+\xi)^\ell - \frac{\xi^2 \ell}{(\ell - 1)!} (s+\xi)^{\ell-1} \right\} \right) \log(s+\xi),
\]

which can be written in the form \( \sum_\xi g_\xi(s) \log(s + \xi) \) where

\[
g_\xi(s) = e^{-2(s+\xi)\gamma} \sum_{\ell > 0} c_\ell \left\{ \frac{x^2 \xi}{\ell!} (s+\xi)^\ell - \frac{\xi^2 \ell}{(\ell - 1)!} (s+\xi)^{\ell-1} \right\}.
\]

From this explicit formula for \( g_\xi(s) \), we see that

\[
g_\xi(s) = \begin{cases} c_{\ell_0,0} \frac{x^{2\ell_0}}{(s+\xi)^{\ell_0}} S^{\ell_0} + O(s^{\ell_0+1}) & \text{if } \xi = 0, \\
-ct_\ell x e^{2\xi x/(\ell - 1)!} (s+\xi)^{\ell - 1} + O((s+\xi)^{\ell+1}) & \text{if } \xi > 0,
\end{cases}
\]

where we recall that \( \ell_\xi := \min\{\ell > 0 \mid c_\ell \xi \neq 0\} \).

We now analyze \( \int_{|t|}^\infty x^{-2s} G_1(x) \, dx \). With \( \sigma_{\ell \xi} = (\sigma^0_\ell)^{\ell_\xi} (-\gamma)^{\ell - \ell_\xi} \), from equation (5.1) in Lemma 4.1 we can write, modulo an entire function,

\[
(5.8) \int_{|t|}^\infty x^{-2s} \left( \sum_{\ell \leq 0} c_\ell x^{-2\xi - 1} \left\{ \ell (\gamma - \log x)^{-\ell - 1} - 2\xi (\gamma - \log x)^{-\ell} \right\} \right) \, dx
\]

\[
= \sum_{\ell \leq 0} c_\ell x^{-2\xi - 1} \left\{ \sum_{j=0}^{\ell} \frac{2\xi \sigma_{\ell|\ell-j}}{(s+\xi)^{j+1}} - \sum_{j=0}^{\ell} \frac{2\xi \sigma_{\ell|\ell-j}}{(s+\xi)^{j+1}} \right\} = \frac{f_\xi(s)}{(s+\xi)^{|p_\xi|+1}},
\]

where

\[
p_\xi := \min\{\ell \leq 0 \mid c_\ell \xi \neq 0\} \implies |p_\xi| = \max\{|\ell| \mid \ell \leq 0 \text{ and } c_\ell \xi \neq 0\}
\]

and

\[
f_\xi(s) := \sum_{\ell \leq 0} c_\ell x^{-2\xi - 1} \left\{ \sum_{j=0}^{\ell} \frac{2\xi \sigma_{\ell|\ell-j}}{(s+\xi)^{j+1}} - \sum_{j=0}^{\ell} \frac{2\xi \sigma_{\ell|\ell-j}}{(s+\xi)^{j+1}} \right\}.
\]
is entire. It follows that
\[
\int_{|t|}^\infty x^{-2s} G_1(x) \, dx = \\
\int_{|t|}^\infty x^{-2s} \sum_{\xi, \ell \leq 0} c_{\xi \ell} x^{-2\xi-1} \left\{ (\gamma - \log x)^{-\ell-1} - 2\xi (\gamma - \log x)^{-\ell} \right\} \, dx \\
= \sum_{\xi} \frac{f_\xi(s)}{(s + \xi)|p_\xi|^{1+1}}.
\]

Moreover, from the above explicit formula for \( f_\xi(s) \) we see that
\[
f_\xi(-\xi) = -2c_{\xi\ell} \xi\sigma_{|p_\xi|, |p_\xi|} = -2c_{\xi\ell} \xi(-1)^{|p_\xi|} \frac{|p_\xi|}{2|p_\xi|+1} = (-1)^{|p_\xi|+1} c_{\xi\ell} \xi \frac{|p_\xi|}{2|p_\xi|}.
\]
This completes our proof. \( \square \)

### 5.3. The decomposable case

Suppose now that \( L = L_0 \oplus L_1 \) is decomposable where as in Corollary 5.3, \( L_0 \) is given by \( q_0 \times q_0 \) matrices \( A_0 \) and \( B_0 \) and \( L_1 \) is given by \( q_1 \times q_1 \) matrices \( A_1 \) and \( B_1 \). Let us recall the polynomial \( p_0(z) \) introduced in (2.9) in Section 2.3 and consider the following result.

**Lemma 5.3.** For \(|t| \) sufficiently large so that \( p_0(\log x) \) has no zeros for \( x \geq |t| \), we can write
\[
\int_{|t|}^\infty x^{-2s} \frac{p_0'(\log x)}{xp_0(\log x)} \, dx = -f(s) \log s
\]
modulo an entire function, where \( f(s) \) is the entire function given explicitly by
\[
f(s) = \sum_{k=1}^\infty \beta_k \frac{(-2s)^{k-1}}{(k-1)!}
\]
with the \( \beta_k \)'s the coefficients of the expansion of \( \frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^\infty \beta_k \frac{z^k}{(k-1)!} \) in (2.10).

**Proof.** Using the expansion \( \frac{p_0'(z)}{p_0(z)} = \sum_{k=1}^\infty \beta_k \frac{z^k}{(k-1)!} \), we can write
\[
(5.9) \quad \int_{|t|}^\infty x^{-2s} \frac{p_0'(\log x)}{xp_0(\log x)} \, dx = \sum_{k=1}^\infty \beta_k \int_{|t|}^\infty x^{-2s} \frac{1}{x (\log x)^k} \, dx.
\]
To analyze this integral we put \( \xi = c = 0 \) in formula (5.2) from Lemma 5.4 to see
\[
\int_{|t|}^\infty x^{-2s} \frac{1}{x (\log x)^k} \, dx \equiv -\frac{(-2)^{k-1}}{(k-1)!} s^{k-1} \log s = -\frac{(-2s)^{k-1}}{(k-1)!} \log s,
\]
modulo an entire function. Replacing this formula into (5.9) and simplifying, we obtain our result. \( \square \)

Let us apply this theorem to the case when \( q_0 = 1 \). In this case, by Proposition 5.5, we have \( A_0 = \cos \theta \) and \( B_0 = \sin \theta \) for an angle \( \theta \in [0, \pi) \), therefore
\[
p_0(z) := \det \begin{pmatrix} \cos \theta & \sin \theta \\ 1 & (\gamma - z) \end{pmatrix} = -\cos \theta \cdot z + \gamma \cdot \cos \theta - \sin \theta.
\]
Hence, with \( \kappa := \gamma - \tan \theta = \log 2 - \gamma - \tan \theta \), we have
\[
(5.10) \quad \frac{p_0'(z)}{p_0(z)} = \frac{-\cos \theta}{-\cos \theta \cdot z + \gamma \cdot \cos \theta - \sin \theta} = \frac{1}{z - \gamma + \tan \theta} = \sum_{k=1}^\infty \kappa^{k-1} z^k.
\]
Thus, $\beta_k = \kappa^{k-1}$, so
\[
f(s) = \sum_{k=1}^\infty \beta_k \frac{(-2s)^{k-1}}{(k-1)!} = \sum_{k=1}^\infty \frac{(-2s\kappa)^{k-1}}{(k-1)!} = e^{-2s\kappa}.
\]
Therefore, Lemma 5.3 reduces to

**Corollary 5.4.** Suppose that $q_0 = 1$, $q_1 = 0$ and $\theta \neq \frac{\pi}{2}$. Then the coefficients $\beta_k$ in the expansion $\frac{p_0(x)}{p_0(x)}$ are given by $\beta_k = \kappa^{k-1}$. In particular, for $|t|$ sufficiently large so that $p_0(\log x)$ has no zeros for $x \geq |t|$, we can write
\[
\int_{|t|}^\infty x^{-2s} \frac{p_0'(\log x)}{xp_0(\log x)} \, dx = -e^{-2s\kappa} \log s
\]
modulo an entire function, where $\kappa = \log 2 - \gamma - \tan \theta$.

From (2.11) and (2.12) in Section 2.3 let us recall that the polynomial $p_1(y)$ has the expression
\[
p_1(y) = a_{\alpha_0} y^{2\alpha_0} \left(1 + \sum_{\beta} b_\beta y^{2\beta}\right),
\]
where the $\beta$'s are positive and
\[
(5.11) \quad \log \left(1 + \sum_{\beta} b_\beta y^{2\beta}\right) = \sum_{\xi} c_\xi y^{2\xi},
\]
and let $\mathcal{P} := \{\xi \mid c_\xi \neq 0\}$. We now prove the model problem versions of Theorems 2.3 and 2.4.

**Proposition 5.5.** For an arbitrary decomposable Lagrangian $L \subset V$, the $\zeta$-function $\zeta(s, \mathcal{L}_L)$ has the following form:
\[
\zeta(s, \mathcal{L}_L) = \zeta_{\text{reg}}(s, \mathcal{L}_L) + \zeta_{\text{sing}}(s, \mathcal{L}_L)
\]
where $\zeta_{\text{reg}}(s, \mathcal{L})$ has the “regular” poles at the “usual” locations $s = \frac{1}{2} - k$ for $k \in \mathbb{N}_0$, and where $\zeta_{\text{sing}}(s, \mathcal{L}_L)$ has the following expansion:
\[
\zeta_{\text{sing}}(s, \mathcal{L}_L) = -\frac{\sin(\pi s)}{\pi} f(s) \log s + \frac{\sin(\pi s)}{\pi} \sum_{\xi \in \mathcal{P}} \frac{f_\xi(s)}{s + \xi},
\]
where $f(s)$ is the entire function defined explicitly by $f(s) = \sum_{k=1}^\infty \beta_k \frac{(-2s)^{k-1}}{(k-1)!}$, and the $f_\xi(s)$'s are entire functions such that $f_\xi(-\xi) = -c_\xi \xi$ with the $c_\xi$'s the coefficients in (5.11).

**Proof.** Since $L = L_0 \oplus L_1$ is decomposable, it follows that
\[
\zeta(s, \mathcal{L}_L) = \zeta(s, \mathcal{L}_L_0) + \zeta(s, \mathcal{L}_L_1),
\]
where $\mathcal{L}_L_0$ is the operator $\mathcal{L}_L$ restricted to the $-\frac{1}{4}$ eigenspaces of $A$ and $\mathcal{L}_L_1$ is the operator $\mathcal{L}_L$ restricted to the eigenspaces of $A$ in $\left(-\frac{1}{4}, \frac{3}{4}\right)$. From (5.1), we can observe that $p_\xi = 0$ for any $\xi$ and $\mathcal{L} = \emptyset$ for the operator $\mathcal{L}_L_1$. Hence, there are only terms $G_1(x)$ with $\ell = 0$ and $G_3(x)$ in $\mathcal{L}_L_1$, that is,
\[
(5.12) \quad \frac{d}{dx} \log F(ix) \sim \sum_{k=0}^\infty b_k x^{-k} + \sum_{\xi} -2\xi c_\xi x^{-2\xi-1}
\]
where $c_\xi$'s are the coefficients in (5.11). It follows from the proof of Proposition 4.2 in particular, (5.8) with $\ell = 0$ that
\[
\zeta(s, \mathcal{L}_L_1) = \zeta_{\text{reg}}(s, \mathcal{L}_L_1) + \zeta_{\text{sing}}(s, \mathcal{L}_L_1)
\]
where \( \zeta_{\text{reg}}(s, L_{L_1}) \) has poles at \( s = \frac{1}{2} - k \) for \( k \in \mathbb{N}_0 \) and
\[
\zeta_{\text{sing}}(s, L_{L_1}) = \frac{\sin(\pi s)}{\pi} \sum_{\xi \in \mathbb{Z} \setminus \{0\}} \frac{f_{\xi}(s)}{s + \xi},
\]
where the \( f_{\xi}(s) \)'s are entire functions of \( s \) such that \( f_{\xi}(-\xi) = -e^{\xi} \).

Thus, it remains to analyze \( \zeta(s, L_{L_0}) \). To do so, we follow the proof of Proposition 4.2 up to equation 5.6, for \( |t| \gg 0 \) we can write
\[
\zeta(s, L_{L_0}) = \frac{\sin(\pi s)}{\pi} \int_{|t|}^{\infty} x^{-2s} \frac{d}{dx} \log F_0(ix) \, dx
\]
modulo an entire function, where \( F_0(\mu) \) is the function \( F(\mu) \) defined in Proposition 4.2 in the case that \( q_1 = 0, A = A_0, \) and \( B = B_0 \). By (4.15) in Proposition 4.4, we have
\[
\frac{d}{dx} \log F_0(ix) \sim \frac{p_0'(x \log x)}{x p_0(x \log x)} + G(x),
\]
where \( G(x) \) is a power series in \( x^{-1} \) starting with a constant term. Just as we noticed in (5.7) for \( G_3(x) \) (which has the same asymptotics as \( G(x) \)) in the proof of Proposition 4.4, the integral \( \frac{\sin(\pi s)}{\pi} \int_{|t|}^{\infty} x^{-2s} \log F_0(ix) \, dx \) will contribute to the function \( \zeta_{\text{reg}}(s, L_{L_0}) \) in the statement of this proposition. Finally, invoking Lemma 5.3,
\[
\int_{|t|}^{\infty} x^{-2s} \frac{p_0'(x \log x)}{x p_0(x \log x)} \, dx = -f(s) \log s
\]
modulo an entire function, where \( f(s) = \sum_{k=1}^{\infty} \beta_k \frac{(2s)^{-k-1}}{(k-1)!} \), completes the proof. \( \Box \)

Corollary 5.6. Suppose that \( q_0 = 1, q_1 = 0 \) and \( \theta \neq \frac{\pi}{2} \). Then the zeta function \( \zeta(s, \Delta_{\theta}) \) can be written in the form
\[
\zeta(s, \Delta_{\theta}) = -\frac{\sin(\pi s)}{\pi} e^{-2\kappa s} \log s + \zeta_{\theta}(s),
\]
where \( \kappa = \log 2 - \gamma - \tan \theta \) and \( \zeta_{\theta}(s) \) extends from \( \mathbb{R} s > \frac{1}{2} \) to a holomorphic function on \( \mathbb{C} \) with poles at \( s = \frac{1}{2} - k \) for \( k \in \mathbb{N}_0 \). In particular, \( \zeta(s, \Delta_{\theta}) \) has \( s = 0 \) as a logarithmic branch point. In the case that \( \theta = \frac{\pi}{2} \), the \( \zeta \)-function \( \zeta(s, \Delta_{\theta}) \) has the properties of \( \zeta_{\theta}(s) \).

The last statement for \( \theta = \frac{\pi}{2} \) in Corollary 5.6 follows from the results in [19].

6. The resolvent and heat kernel for the model problems

In this section we analyze the resolvent and heat kernel expansions for the model problems, which will be of great use for the general case.

6.1. The resolvent. Using the new contour \( \mathcal{C} \) shown in Figure 2, we see that if \( \{\mu_j^2\} \) denote the eigenvalues of \( L_L \), then by an application of the Argument Principle, we have
\[
2 \text{Tr}(L_L + x^2)^{-1} = 2 \sum_{j=1}^{\infty} \frac{1}{\mu_j^2 + x^2} = \frac{1}{2\pi i} \int_{\gamma} (\mu^2 + x^2)^{-1} \frac{d}{d\mu} \log F(\mu) \, d\mu,
\]
where \( |x|^2 \) is larger than the absolute value of the negative eigenvalues of \( L_L \) (if one exists). The factor of 2 on the left hand side is a result of all eigenvalues being
enclosed twice. Using this formula we can express the trace of the resolvent in terms of $F(ix)$ in the following important and remarkable theorem.

**Theorem 6.1.** We have

$$2x \text{Tr}(L_L + x^2)^{-1} = \frac{d}{dx} \log F(ix).$$

for all complex $x \in \mathbb{C}$ for which either (and hence both) sides make sense.

**Proof.** Deforming the contour as in Figure 3 and using Cauchy’s formula, we obtain

$$2 \text{Tr}(L_L + x^2)^{-1} = \frac{1}{2\pi i} \int_\gamma \frac{F'(\mu)}{(\mu - ix)(\mu + ix)} \frac{1}{F(\mu)} d\mu$$

$$= -\frac{1}{2ix} \frac{F'(ix)}{F(ix)} - \frac{1}{-2ix} \frac{F'(-ix)}{F(-ix)} = \frac{i}{xF(ix)} = \frac{1}{x} \frac{d}{dx} \log F(ix),$$

where we used the fact that $F(\mu)$ is an even function of $\mu$. Indeed, to see this observe that, by definition, $F(\mu)$ is expressed in terms of $\mu^v J_v(\mu R)$ with appropriate $v$'s and the function $\tilde{J}_0(\mu R)$, which are even functions by (4.4), (4.6) and (4.7). This proves that $2x \text{Tr}(L_L + x^2)^{-1} = \frac{d}{dx} \log F(ix)$ at least when $x$ is real and $x \gg 0$. However, by analytic continuation, both sides must still be equal for all complex $x$ for which both sides are defined. 

\[\square\]
Proposition 6.4. Let \( L \subset V \) be an arbitrary Lagrangian subspace of \( \mathbb{C}^{2q} \) and let \( \Lambda \subset \mathbb{C} \) be any sector (solid angle) not intersecting the positive real axis. Then for an arbitrary decomposable Lagrangian we have

\[
\text{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k(-\lambda)^{-\frac{k}{2}} + \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\gamma)}
\]

\[\text{+} \quad - \frac{d}{d\lambda} \left\{ \sum_{\xi} 2^\xi c_\xi (-\lambda)^{-\xi} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\},\]

where the \( a_k \) coefficients are independent of \( L \) and the \( c_\xi \)'s are given in (2.1).

Proof. By Proposition 6.3 (see equation (4.10)), we have

\[
\frac{d}{dx} \log F(ix) \sim \sum_{k=0}^{\infty} b_k x^{-k} + \frac{q_0 - j_0}{x(\log x - \gamma)}
\]

\[\text{+} \quad + \sum c_\xi x^{-2\xi-1} \left\{ \ell(\gamma - \log x)^{-\ell-1} - 2\xi(\gamma - \log x)^{-\ell} \right\},\]

for some coefficients \( b_k \) that, by the proof of Proposition 6.3, are independent of \( L \). Therefore, by Theorem 6.4 we have

\[
2x \text{Tr}(\mathcal{L}_L + x^2)^{-1} \sim \sum_{k=0}^{\infty} b_k x^{-k} + \frac{q_0 - j_0}{x(\log x - \gamma)}
\]

\[\text{+} \quad + \sum c_\xi x^{-2\xi-1} \left\{ \ell(\gamma - \log x)^{-\ell-1} - 2\xi(\gamma - \log x)^{-\ell} \right\}.
\]

Dividing by \( 2x \) and then setting \( x = (-\lambda)^{\frac{1}{2}} \), with \( a_k = (1/2) b_{k-1} \) we obtain

\[
\text{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k(-\lambda)^{-\frac{k}{2}} + \frac{q_0 - j_0}{(-\lambda)(\log(-\lambda) - 2\gamma)}
\]

\[\text{+} \quad - \frac{d}{d\lambda} \left\{ \sum_{\xi} c_\xi (-\lambda)^{-\xi} \left( \log(-\lambda) - \frac{1}{2} \right)^{-\ell} \right\},\]

which is equivalent to our desired result. \( \square \)

For decomposable \( L \), we have

Proposition 6.4. Let \( \Lambda \subset \mathbb{C} \) be any sector (solid angle) not intersecting the positive real axis. Then for an arbitrary decomposable Lagrangian \( L \), as \( |\lambda| \to \infty \) with \( \lambda \in \Lambda \) we have

\[
\text{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k(-\lambda)^{-\frac{k}{2}}
\]

\[\text{+} \quad - \frac{d}{d\lambda} \left\{ \sum_{\xi \in \mathcal{P}} c_\xi (-\lambda)^{-\xi} \right\} + \left\{ (-\lambda)^{-1} \sum_{k=1}^{\infty} 2^{k-1} \beta_k \right\},\]

where the \( a_k \) coefficients are independent of \( L \), the \( c_\xi \)'s are the coefficients in (2.1), and the \( \beta_k \)'s are the coefficients in (2.10).
Figure 4. The contour $C_h$.

Proof. Since $L = L_0 \oplus L_1$ is decomposable, it follows that

$$(\mathcal{L}_L - \lambda)^{-1} = (\mathcal{L}_{L_0} - \lambda)^{-1} + (\mathcal{L}_{L_1} - \lambda)^{-1}$$

where $\mathcal{L}_{L_0}$ is the operator $\mathcal{L}_L$ restricted to the $-\frac{1}{2}$ eigenspaces of $A$ and $\mathcal{L}_{L_1}$ is the operator $\mathcal{L}_L$ restricted to the eigenspaces of $A$ in $(-\frac{1}{2}, \frac{1}{2})$. For the operator $\mathcal{L}_{L_1}$, recall equation (5.12),

$$\frac{d}{dx} \log F(ix) \sim \infty \sum_{k=0}^{\infty} b_k x^{-k} + \sum_{\xi} -2\xi c_\xi x^{-2\xi - 1}.$$  

Combining this with Theorem 6.1, we have

$$\text{Tr}(\mathcal{L}_{L_1} - \lambda)^{-1} \sim \sum_{k=1}^{\infty} d_k (-\lambda)^{-\frac{k}{2}} - \frac{d}{d\lambda} \left\{ \sum_{\xi} c_\xi (-\lambda)^{-\xi} \right\}$$

where the $d_k$ coefficients are independent of $L_1$ and the $c_\xi$'s are the coefficients in (2.12) or (5.11). Let $F_0(\mu)$ denote the function $F(\mu)$ in Proposition 4.2 in the case that $q_1 = 0$, $A = A_0$, and $B = B_0$ where $(A_0 - B_0)$ defines $L_0$. Then just as in the proof of Proposition 6.3 in conjunction with Proposition 5.2 (see (4.11)),

$$\frac{d}{dx} \log F_0(ix) \sim \sum_{k=1}^{\infty} \beta_k x^{-\frac{k}{2} + i} + \sum_{k=0}^{\infty} \varepsilon_k x^{-k},$$

where the $\beta_k$'s are the coefficients in (2.10), using again Theorem 6.1, we obtain

$$\text{Tr}(\mathcal{L}_{L_0} - \lambda)^{-1} \sim \sum_{k=1}^{\infty} f_k (-\lambda)^{-\frac{k}{2}} + \left\{ (-\lambda)^{-1} \sum_{k=1}^{\infty} \frac{2^{k-1} \beta_k}{(\log(-\lambda))^k} \right\}.$$  

Combining this with $\text{Tr}(\mathcal{L}_{L_1} - \lambda)^{-1}$ analyzed just before completes our proof. □

6.2. The heat kernel. Now we consider the asymptotics of the trace of $e^{-t\mathcal{L}_L}$ as $t \to 0$. For this, we use

$$\text{Tr}(e^{-t\mathcal{L}_L}) = \frac{i}{2\pi} \int_{C_h} e^{-t\lambda} \text{Tr}(\mathcal{L}_L - \lambda)^{-1} d\lambda$$

where $C_h$ is a counter-clockwise contour in the plane surrounding eigenvalues of $\mathcal{L}_L$; see Figure 4. Then the small-time asymptotics of the heat trace is determined by the large-spectral parameter asymptotics of $\text{Tr}(\mathcal{L}_L - \lambda)^{-1}$ as we will see in the following proposition.
Proposition 6.5. For an arbitrary Lagrangian subspace \( L \subset \mathbb{C}^{2q} \), as \( t \to 0 \) we have
\[
\text{Tr}(e^{-tC_L}) \sim \sum_{k=0}^{\infty} a_k \frac{t^{-\frac{1+k}{2}}}{k!} + \sum_{k=0}^{\infty} b_k (\log t)^{-1-k} + \sum_{\xi \in \mathcal{P}} \sum_{k=0}^{\infty} \tilde{c}_{\xi k} t^\xi (\log t)^{k} + \sum_{\xi \in \mathcal{Q}} \sum_{k=0}^{\infty} \tilde{d}_{\xi k} t^\xi (\log t)^{-\xi-k},
\]
with \( \tilde{c}_{10} = 0 \) and \( \tilde{c}_{\xi}|_{|\mathcal{P}|+1} = 0 \) for \( \xi \notin \mathbb{N}_0 \).

Proof. By Proposition 6.3 we have
\[
\text{Tr}(\mathcal{L}_L - \lambda)^{-1} \sim \sum_{k=1}^{\infty} a_k (-\lambda)^{-\frac{1}{2}} + \frac{a_0 - j_0}{(\log(-\lambda) - 2\gamma)} - \frac{d}{d\lambda} \left\{ \sum_{\ell \in \mathcal{P}} 2^\ell c_{\ell \xi} (-\lambda)^{-\xi} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\}
\]
as \( |\lambda| \to \infty \) with \( \lambda \) in a sector not intersecting the positive real axis. We use (6.1) for each term on the right hand side. For the first term, making the change of variables \( \lambda \mapsto t^{-1}\lambda \),
\[
\int_{Ci_h} e^{-t\lambda} (-\lambda)^{-\frac{1}{2}} d\lambda = t^{-\frac{1}{2}} \int_{Ci_h} e^{-\lambda} (-\lambda)^{-\frac{1}{2}} d\lambda.
\]
The integral part depends on \( t \) via \( tCi_h \), but is smooth at \( t = 0 \). Hence, the first part \( \sum_{k=1}^{\infty} a_k (-\lambda)^{-\frac{1}{2}} \) contributes
\[
\sum_{k=0}^{\infty} a_k t^{-\frac{1+k}{2}}.
\]
For the third term on the right hand side of (6.2), using integration by parts, we have
\[
\int_{Ci_h} e^{-t\lambda} \frac{d}{d\lambda} \left\{ (-\lambda)^{-\xi} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\} d\lambda
\]
\[
= t \int_{Ci_h} e^{-t\lambda} \left\{ (-\lambda)^{-\xi} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\} d\lambda.
\]
Deforming \( Ci_h \) to the real line, we find
\[
t \int_{Ci_h} e^{-t\lambda} \left\{ (-\lambda)^{-\xi} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\} d\lambda
\]
\[
= t \left( \int_{-\infty}^{1} e^{-tx} \left( -(x + i0) \right)^{-\xi} \left( 2\gamma - \log(-(x + i0)) \right)^{-\ell} dx \right.
\]
\[
+ \int_{1}^{\infty} e^{-tx} \left( -(x - i0) \right)^{-\xi} \left( 2\gamma - \log(-(x - i0)) \right)^{-\ell} dx + h(t) \;
\]
\[
= t \left( e^{-i\pi \xi} \int_{1}^{\infty} e^{-tx} x^{-\xi} \left( 2\gamma - \log x - i\pi \right)^{-\ell} dx - e^{i\pi \xi} \int_{1}^{\infty} e^{-tx} x^{-\xi} \left( 2\gamma - \log x + i\pi \right)^{-\ell} dx + h(t) \right),
\]
where $h(t)$ is a smooth function at $t = 0$. Since for any complex number $z$ we have $i(z - \bar{z}) = -2\Im z$, we see that modulo a term that is a smooth function of $t$ at $t = 0$,
\[ \frac{i}{2\pi} \int_{C_h} e^{-tx} \frac{d}{d\lambda} \left\{ (-\lambda)^{-\xi} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\} d\lambda = -\frac{t}{\pi} \ell(t), \]
where
\[ \ell(t) = e^{-i\pi \xi} \int_{1}^{\infty} e^{-tx} x^{-\xi} (2\gamma - \log x - i\pi)^{-\ell} dx; \]
we shall compute the asymptotics of $\ell(t)$ as $t \to 0$. To do so, let $j \geq \xi > j - 1, j \in \mathbb{N}_0$; observe that the $j$-th derivative $\ell^{(j)}(t)$ of $\ell(t)$ is given by
\[ \ell^{(j)}(t) = e^{-i\pi \xi} (-1)^j \int_{1}^{\infty} e^{-tx} x^{j-\xi} (2\gamma - \log x - i\pi)^{-\ell} dx. \]
Note that $x^{j-\xi} (2\gamma - \log x - i\pi)^{-\ell}$ is integrable near $x = 0$, so we can write
\[ \ell^{(j)}(t) = e^{-i\pi \xi} (-1)^j (f(t) + g(t)) \]
with
\[ f(t) := \int_{0}^{\infty} e^{-tx} x^{j-\xi} (2\gamma - \log x - i\pi)^{-\ell} dx \]
and
\[ g(t) = -\int_{0}^{1} e^{-tx} x^{j-\xi} (2\gamma - \log x - i\pi)^{-\ell} dx. \]
Note that $g(t)$ is smooth at $t = 0$. We will now determine the asymptotics of $f(t)$ near $t = 0$. To this end, we make the change of variables $x \mapsto t^{-1}x$:
\[ f(t) = t^{\xi-j-1} \int_{0}^{\infty} e^{-x} x^{j-\xi} (2\gamma - \log x + \log t - i\pi)^{-\ell} dx. \]
We need to consider two cases; $\ell \leq 0$ and $\ell > 0$. For $\ell \leq 0$ we use the binomial expansion to find
\[ f(t) = t^{\xi-j-1} \sum_{k=0}^{\lfloor |\ell| \rfloor} \binom{\lfloor |\ell| \rfloor}{k} (\log t)^{|\ell| - k} \int_{0}^{\infty} e^{-x} x^{j-\xi} (2\gamma - \log x - i\pi)^{k} dx \]
\[ = t^{\xi-j-1} \sum_{k=0}^{\lfloor |\ell| \rfloor} c_{\xi,k,\ell}(\log t)^{k}, \]
with suitable coefficients $c_{k,\ell}$. For $\ell > 0$ we first write
\[ f(t) = t^{\xi-j-1} (\log t)^{-\ell} \int_{0}^{\infty} e^{-x} x^{j-\xi} \left( 1 - \frac{\log x + i\pi - 2\gamma}{\log t} \right)^{-\ell} dx. \]
Since \((1 - r)^{-1} = \sum_{k=0}^{N} r^k + r^{N+1}(1 - r)^{-1}\) for any \(N \in \mathbb{N}\), we see that for any \(N \in \mathbb{N}\),
\[(1 - r)^{-\ell} = \sum_{k=0}^{N-\ell+1} a_{k,\ell} r^k + r^{N-\ell+2} \sum_{k=0}^{\ell-1} b_{k,\ell} \frac{r^k}{(1 - r)^{k+1}}.\]

For \(f(t)\) this implies
\[f(t) = t^{\ell-j-1}(log t)^{-\ell} \sum_{k=0}^{N-\ell+1} a_{k,\ell} (log t)^{-k} \int_{0}^{\infty} e^{-x} x^{j-\ell} (log x + i\pi - 2\gamma)^k dx + t^{\ell-j-1}(log t)^{-\ell} (log t)^{-(N-\ell+2)} \sum_{k=0}^{\ell-1} b_{k,\ell} (log t)^{-k} \int_{0}^{\infty} e^{-x} x^{j-\ell} (log x + i\pi - 2\gamma)^{N-\ell+2+k} (1 - \frac{log x + i\pi - 2\gamma}{log t})^{k+1} dx.\]

The last integral is bounded as \(t \to 0\); as \(N \in \mathbb{N}\) was arbitrary, we conclude for \(\ell > 0\)
\[f(t) \sim t^{\ell-j-1} \sum_{k=0}^{\infty} a_{\xi,k,\ell} (log t)^{-k-\ell}.\]

In summary: for \(\ell \leq 0\) we have shown
\[(6.5) \quad \ell^{(j)}(t) \sim e^{-i\pi \xi} t^{\ell-j-1} \sum_{k=0}^{\ell} c_{\xi,k,\ell} (log t)^k + \sum_{k=0}^{\infty} \gamma_{\xi,k,\ell} t^k,\]
while for \(\ell > 0\), we found
\[(6.6) \quad \ell^{(j)}(t) \sim e^{-i\pi \xi} t^{\ell-j-1} \sum_{k=0}^{\infty} \tilde{a}_{\xi,k,\ell} (log t)^{-k-\ell} + \sum_{k=0}^{\infty} \tilde{\gamma}_{\xi,k,\ell} t^k.\]

In order to find the small-\(t\) asymptotics of \(\ell(t)\) we need to integrate \(j\) times. Using \[27\], equation 2.722,
\[\int t^n (log t)^m dt = t^{n+1} \sum_{m=0}^{m} (-1)^k (m+1) \cdots (m-k+1) (log t)^{m-k} (n+1)^{k+1},\]
for \(n \neq -1, m \neq -1\), and \[27\], equation 2.724, in the form
\[\int \frac{t^n}{(log t)^m} dt = \frac{t^{n+1}}{(n+1)(log t)^m} + \frac{m}{n+1} \int \frac{t^n}{(log t)^{m+1}} dt,\]
for \(n \neq -1, m \neq 0\), in addition
\[\int t^{-1}(log t)^{-1} dt = \log |log t| , \quad \int t^{-1}(log t)^{-k-1} dt = -\frac{1}{k} (log t)^{-k} \quad \text{for} \quad k \neq 0,\]
we obtain for \(\ell \leq 0, \xi \notin \mathbb{N}_0\), that
\[(6.7) \quad \Im(\ell(t)) \sim t^{\ell-1} \sum_{k=0}^{\infty} c_{\xi,k,\ell} (log t)^k + \sum_{k=0}^{\infty} \gamma_{\xi,k,\ell} t^k,\]
whereas for \(\xi \in \mathbb{N}_0\) the first summation extends up to \(|l|+1\) and \(c'_{1,0,\ell} = 0\).

For \(l > 0\) the answer reads
\[(6.8) \quad \Im(\ell(t)) \sim t^{\ell-1} \sum_{k=0}^{\infty} \tilde{c}_{\xi,k,\ell} (log t)^{-k-\ell} + \sum_{k=0}^{\infty} \tilde{\gamma}_{\xi,k,\ell} t^k.\]
Contributions from the second term in (6.2) are found from (6.8) with \( \xi = 1 \) and \( \ell = 1 \),
\[
\sum_{k=0}^{\infty} \tilde{c}_k (\log t)^{-k-1} + \sum_{k=0}^{\infty} \tilde{\gamma}_k t^k.
\]
(6.9)
Combining (6.3), (6.7), (6.8) and (6.9) completes the proof. \( \square \)

Using Proposition 6.4 and repeating the proof of Proposition 6.5, we have Proposition 6.6. For an arbitrary decomposable Lagrangian \( L \), the heat kernel \( e^{-tL} \) has the following trace expansion as \( t \to 0 \):
\[
\text{Tr}(e^{-tL}) \sim \sum_{k=0}^{\infty} \tilde{a}_k t^{-1+k} + \sum_{\xi \in \mathcal{P}} \tilde{c}_\xi t^\xi + \sum_{k=1}^{\infty} \tilde{d}_k (\log t)^{-k}.
\]

7. PROOFS OF THE MAIN THEOREMS

We now prove our main results starting with the resolvent expansion.

7.1. The resolvent expansion — Theorems 2.5 and 2.6. We work under the assumptions of Theorem 2.5, so \( \Lambda \subset \mathbb{C} \) denotes a sector not intersecting the positive real axis and \( L \subset V \) denotes a given, but arbitrary, Lagrangian subspace of \( V \). We cut the manifold \( M \) at the hypersurface \( r = R \) in the collar \([0, \varepsilon) \times \Gamma \) with \( 0 < R < \varepsilon \), giving a decomposition
\[
M = X \cup Y,
\]
where \( X = [0, R]_r \times \Gamma \) and \( Y \) is a manifold with a collared neighborhood \((R, \varepsilon) \times \Gamma \) near its boundary, which we identify with \( \Gamma \). Let \( \Delta_Y \) denote the restriction of \( \Delta \) to \( Y \) with the Dirichlet condition at \( r = R \) and let \( \Delta_{X,L} \) denote the restriction of \( \Delta \) to \( X \):
\[
\Delta_{X,L} := -\partial^2_r + \frac{1}{r^2} A_{\Gamma}
\]
with domain the restriction of \( \text{dom}(\Delta_L) \) to \( X \) and with the Dirichlet condition at \( r = R \). It is well-known that the Schwartz kernel of the resolvent \( (\Delta_Y - \lambda)^{-1} \), where \((y, y') \in Y \times Y\), is a smooth function of \((y, y')\) and vanishes to infinite order as \( |\lambda| \to \infty \) with \( \lambda \in \Lambda \) as long as \( y \neq y' \) (see for instance [64]). In the following lemma we prove a similar statement for the operator \( \Delta_{X,L} \) on the generalized cone.

Lemma 7.1. If \( \varphi, \psi \in C^\infty(X) \) have disjoint supports, then for any differential operator \( P \) that vanishes near \( \partial M \), the operator
\[
\varphi P(\Delta_{X,L} - \lambda)^{-1} \psi
\]
is a trace-class operator that vanishes, with all derivatives, to infinite order (in the trace-class norm) as \( |\lambda| \to \infty \) with \( \lambda \in \Lambda \).

Proof. If we prove this theorem for \( \overline{\psi(\Delta_{X,L} - \lambda)^{-1} P \varphi} \), then taking adjoints we get our theorem. Hence, we just have to prove the corresponding statement for \( \varphi(\Delta_{X,L} - \lambda)^{-1} P \psi \), where \( P \psi \) is the operator \( f \mapsto P(\psi f) \). We prove this lemma using the heat kernel \( e^{-t\Delta_{X,L}} \), whose structure is found in [55]. To this end, observe that
\[
(\Delta_{X,L} - \lambda)^{-1} = \int_0^1 e^t \lambda e^{-t\Delta_{X,L}} dt + e^\lambda (\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}}.
\]
Then
\[ \varphi(\Delta_{X,L} - \lambda)^{-1} P \psi = \int_0^1 e^{t\lambda} \varphi e^{-t\Delta_{X,L}} P \psi \, dt + e^{\lambda} \varphi(\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}} P \psi. \]

Assume for the moment that $\Lambda \subset \mathbb{R}$ that $X,L$ class and ($\Delta$ operator in (7.1) decays exponentially in the trace-class operator as $|\lambda| \to \infty$ with all derivatives, in the trace-class operators as $\lambda \in \mathbb{R}$ in the left-half plane (so that $|\lambda| \to \infty$ with $\lambda \in \Lambda$). Therefore, since the trace-class operators form an ideal within the bounded operators, the operator

\[ (\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}} P \]

is of trace-class and it decays like $|\lambda|^{-1}$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Hence,

\[ e^{\lambda}(\Delta_{X,L} - \lambda)^{-1} e^{-\Delta_{X,L}} P \]

decays exponentially, with all derivatives, in the trace-class operators as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. The supports of $\lambda \in \Lambda$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. By the main theorem of [55] (see also Theorem 4.1 of loc. cit.), since the bounded operators, the operator

\[ \varphi e^{-t\Delta_{X,L}} P \psi \]

is a trace-class operator that vanishes to infinite order at $t = 0$ (within the trace-class operators). Therefore, the operator $\int_0^1 e^{t\lambda} \varphi e^{-t\Delta_{X,L}} P \psi \, dt$ in (7.1) decays exponentially in the trace-class operators as $|\lambda| \to \infty$ with $\lambda \in \Lambda$.

Summarizing: We have proved our theorem when $\Lambda \subset \mathbb{C}$ is contained entirely in the left-half plane (so that $\Re \lambda \to -\infty$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$). Our proof is finished once we finish the cases when $\Lambda \subset \mathbb{C}$ is contained entirely in the upper-half plane and lower-half plane; for concreteness, let us focus on the upper-half plane. Then we can fix a complex number $a \in \mathbb{C}$ with positive real part (and positive imaginary part) such that $a \cdot \Lambda \subset \mathbb{C}$ is entirely contained in the left-half plane. Then one can construct the heat kernel $e^{-t\Delta_{X,L}}$ (cf. [52], p. 282–284) which has the same trace-class properties as $e^{-t\Delta_{X,L}}$ as described in [55], Th. 4.1. Now we proceed as above: Just as we wrote (7.1), one can check that

(7.2) \[ \varphi(\Delta_{X,L} - \lambda)^{-1} P \psi = a \int_0^1 e^{t\lambda} \varphi e^{-ta\Delta_{X,L}} P \psi \, dt + e^{a\lambda} \varphi(\Delta_{X,L} - \lambda)^{-1} e^{-a\Delta_{X,L}} P \psi. \]

By the choice of $a$, note that as $|\lambda| \to \infty$ with $\lambda \in \Lambda$, we have $\Re(a\lambda) \to -\infty$ as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Therefore, analyzing (7.2) by repeating the argument we used in the previous paragraph to analyze (7.1) proves our lemma in the case when $\Lambda \subset \mathbb{C}$ is contained entirely in the upper-half plane. 

Let us fix $0 < a < R$ and $R < b < \varepsilon$, and define

\[ M_0 := [a, b] \times \Gamma, \quad M_1 := [a, R] \times \Gamma, \quad M_2 := [R, b] \times \Gamma. \]

For $j = 0, 1, 2$, let $\Delta_j$ denote the Laplacian on $M_j$ with the Dirichlet boundary condition at the boundaries of $M_j$; see Figure 5. The importance of the operators $\Delta_0, \Delta_1, \Delta_2$ is that they are smooth (not singular) Laplace-type operators on compact manifolds with boundary with local boundary conditions, the properties of which are completely understood in [63, 64]. The idea to prove Theorem 2 in the case when $\Lambda \subset \mathbb{C}$ is contained entirely in the upper-half plane.
Lemma 7.2. The differences of resolvents

\[ S(\lambda) := (\Delta_L - \lambda)^{-1} - (\Delta_{X,L} - \lambda)^{-1} - (\Delta_Y - \lambda)^{-1} - \left( (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} \right) \]

is trace-class and vanishes, with all derivatives, to infinite order (in the trace-class norm) as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\).

Proof. Let \(g(r) \in C^\infty(\mathbb{R})\) be a non-decreasing function such that \(g(r) = 0\) for \(r \leq 1/4\) and \(g(r) = 1\) for \(r \geq 3/4\). For real numbers \(\alpha < \beta\), we define \(g_{\alpha,\beta}(r) := g\left( \frac{r - \alpha}{\beta - \alpha} \right)\). The main properties of \(g_{\alpha,\beta}\) we will use below are that \(g_{\alpha,\beta}(r) = 0\) on a neighborhood of \(\{r \leq \alpha\}\) and \(g_{\alpha,\beta}(r) = 1\) on a neighborhood of \(\{r \geq \beta\}\). Let us choose real numbers \(a_1, a_2, b_1, b_2\) such that

\[ a < a_1 < a_2 < R < b_1 < b_2 < b. \]

We define

\[ \psi_1(r) := 1 - g_{a_1,a_2}(r), \quad \psi_2(r) := g_{b_1,b_2}(r), \quad \psi_0(r) := 1 - \psi_1(r) - \psi_2(r), \]

and

\[ \varphi_1(r) := 1 - g_{a_2,R}(r), \quad \varphi_2(r) := g_{R,b_1}(r), \quad \varphi_0(r) := 1 - g_{a,a_1}(r) - g_{b_2,b}(r). \]

The functions \(\{\psi_i\}, \{\varphi_i\}\) extend either by 0 or 1 to define smooth functions on all of \(M\) and \(\{\psi_i\}\) forms a partition of unity of \(M\) such that \(\varphi_i = 1\) on \(\text{supp}(\psi_i)\).

Now to prove this lemma, we first claim that each of the following equalities holds modulo a trace-class operator vanishing to infinite order as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\):

\[ (\Delta_L - \lambda)^{-1} = \varphi_1(\Delta_{X,L} - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_Y - \lambda)^{-1}\psi_2, \]

(7.3) \[(\Delta_{X,L} - \lambda)^{-1} = \varphi_1(\Delta_{X,L} - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0, \]

\[(\Delta_Y - \lambda)^{-1} = \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_Y - \lambda)^{-1}\psi_2. \]

For instance, let us verify the first claim in (7.3); the other claims are verified using a similar argument. Define

\[ Q(\lambda) := \varphi_1(\Delta_{X,L} - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_Y - \lambda)^{-1}\psi_2. \]

Then observe that \((\Delta_L - \lambda)Q(\lambda) = \text{Id} + K(\lambda)\) where

\[ K(\lambda) = [(\Delta_L - \lambda), \varphi_1](\Delta_{X,L} - \lambda)^{-1}\psi_1 \]

\[ + [(\Delta_L - \lambda), \varphi_0](\Delta_0 - \lambda)^{-1}\psi_0 + [(\Delta_L - \lambda), \varphi_2](\Delta_Y - \lambda)^{-1}\psi_2, \]
where \([,]\) denotes the “commutator”. Now, because the supports of

\[[(\Delta_L - \lambda), \varphi_i] = -[\partial_r^2, \varphi_i] = -(\varphi''_r + 2\varphi'_r \partial_r)\]

and \(\psi_i\) are disjoint, it follows that each of the three operators making up \(K(\lambda)\) is trace-class and vanishes to infinite order (in the trace-class norm) as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\); indeed, this statement for \([(\Delta_L - \lambda), \varphi_i][(\Delta_{X,L} - \lambda)^{-1}\psi_1\] follows from Lemma 7.1 and the statements for \([(\Delta_L - \lambda), \varphi_0][\Delta_0 - \lambda)^{-1}\psi_0\) and \([(\Delta_L - \lambda), \varphi_2][\Delta_Y - \lambda)^{-1}\psi_2\) are well-known (see e.g. [64]). Therefore, \(K(\lambda)\) is trace-class and vanishes to infinite order (in the trace-class norm) as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\). Now applying \((\Delta_L - \lambda)^{-1}\) to both sides of \((\Delta_L - \lambda)Q(\lambda) = \text{Id} + K(\lambda)\), we obtain

\[(\Delta_L - \lambda)^{-1} = Q(\lambda) - (\Delta_L - \lambda)^{-1}K(\lambda),\]

which establishes our claim for the first equality in (7.4). A similar argument works to prove that the other equalities in (7.4) hold modulo trace-class with infinite decay (with all derivatives as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\)). From (7.4), it follows that modulo trace-class with infinite decay,

\[(\Delta_L - \lambda)^{-1} - (\Delta_{X,L} - \lambda)^{-1} - (\Delta_Y - \lambda)^{-1}
\]

\[= \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0.\]

On the other hand, very similar arguments used to establish (7.4) show that modulo trace-class with infinite decay,

\[(\Delta_0 - \lambda)^{-1} = \varphi_1(\Delta_1 - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_0 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_2 - \lambda)^{-1}\psi_2,\]

\[(\Delta_1 - \lambda)^{-1} = \varphi_1(\Delta_1 - \lambda)^{-1}\psi_1 + \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0,\]

\[(\Delta_2 - \lambda)^{-1} = \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0 + \varphi_2(\Delta_2 - \lambda)^{-1}\psi_2.\]

Combining these identities we can write, modulo trace-class with infinite decay,

\[\varphi_1(\Delta_1 - \lambda)^{-1}\psi_1 - \varphi_0(\Delta_1 - \lambda)^{-1}\psi_0 - \varphi_0(\Delta_2 - \lambda)^{-1}\psi_0.\]

Comparing this with (7.4) completes the proof of our lemma. \(\square\)

Using our standard notation, let \(\{\lambda_\ell\}\) denote the set of all eigenvalues of \(A_\Gamma\) and let \(E_\ell\) denote the span of the \(\lambda_\ell\)-th eigenvector. Let \(\Pi\) and \(\Pi^\perp\) denote, respectively, the orthogonal projections of \(L^2(\Gamma, E_\Gamma)\) onto \(W := \bigoplus_{-\frac{1}{2} \leq \lambda_\ell < \frac{1}{2}} E_\ell\) and \(W^\perp\). Using the isometry between

\[L^2([0,1] \times \Gamma, E) \cong L^2([0,1], L^2(\Gamma, E_\Gamma)),\]

where \(E_\Gamma := E|_{\Gamma}\), we obtain the corresponding projections on \(L^2([0,1] \times \Gamma, E)\), which we denote by the same notations \(\Pi\) and \(\Pi^\perp\). Let \(L_L\) denote the model operator introduced in Section 3 (specifically, Section 4.1), and define

\[\Delta'_L := -\partial_r^2 + \frac{1}{r^2} A'_r, \quad \text{where } A'_r := \begin{cases} \frac{4}{A_\Gamma} & \text{over } W \\ A_\Gamma & \text{over } W^\perp \end{cases}.\]

**Proposition 7.3.** We have

\[(\Delta_L - \lambda)^{-1} = \Pi(L_L - \lambda)^{-1}\Pi + \Pi^\perp(\Delta'_L - \lambda)^{-1}\Pi^\perp + (\Delta_Y - \lambda)^{-1} + (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + S(\lambda)\]
where $S(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$.

Proof. Observe that

$$(\Delta_{X,L} - \lambda)^{-1} = \Pi(\Delta_{X,L} - \lambda)^{-1}\Pi + \Pi^\perp(\Delta_{X,L} - \lambda)^{-1}\Pi^\perp,$$

since $\Delta_{X,L}$ preserves $W$ and $W^\perp$, and

$$\Pi(\Delta_{X,L} - \lambda)^{-1}\Pi = \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi.$$

Also observe that

$$\Pi^\perp(\Delta_{X,L} - \lambda)^{-1}\Pi^\perp = \Pi^\perp(\Delta_X' - \lambda)^{-1}\Pi^\perp.$$

Hence,

$$(\Delta_{X,L} - \lambda)^{-1} = \Pi(\mathcal{L}_L - \lambda)^{-1}\Pi + \Pi^\perp(\Delta_X' - \lambda)^{-1}\Pi^\perp.$$

Now solving for $(\Delta_L - \lambda)^{-1}$ in Lemma 7.2 we obtain

$$(\Delta_L - \lambda)^{-1} = (\Delta_{X,L} - \lambda)^{-1} + (\Delta_Y - \lambda)^{-1} + (\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + S(\lambda)$$

where $S(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. This completes our proof. $\square$

We can now prove Theorem 2.5. Let $N \geq \frac{n}{2}$ with $n = \dim M$. Then taking $N$ derivatives of both sides of the preceding equality we see that

$$(\Delta_L - \lambda)^{-N-1} = \Pi(\mathcal{L}_L - \lambda)^{-N-1}\Pi + \Pi^\perp(\Delta_X' - \lambda)^{-N-1}\Pi^\perp$$

$$+ (\Delta_Y - \lambda)^{-N-1} + (\Delta_0 - \lambda)^{-N-1} - (\Delta_1 - \lambda)^{-N-1} - (\Delta_2 - \lambda)^{-N-1} + S^{(N)}(\lambda),$$

where $S^{(N)}(\lambda) = \frac{d^N}{d\lambda^N}S(\lambda)$. We now analyze each term on the right. First, taking $N$ derivatives in the asymptotic expression of Proposition 6.3 we know that as $|\lambda| \to \infty$ with $\lambda \in \Lambda$ we have

$$(7.6) \quad \text{Tr}(\mathcal{L}_L - \lambda)^{-N-1} \sim \sum_{k=0}^{\infty} b_k(-\lambda)^{-\frac{1}{2}-N-1} + \frac{1}{N!} \frac{d^N}{d\lambda^N} \left\{ \frac{q_{0} - j_{0}}{(-\lambda)(\log(-\lambda) - 2\bar{\gamma})} \right\}$$

$$- \frac{1}{N!} \frac{d^{N+1}}{d\lambda^{N+1}} \left\{ \sum_{\ell} 2^\ell c_{\xi}(\lambda)^{-\ell} (2\bar{\gamma} - \log(-\lambda))^{-\ell} \right\}.$$ 

It follows from 59 that the operator $\Pi^\perp(\Delta_X' - \lambda)^{-N-1}\Pi^\perp$ is trace-class and

$$(7.7) \quad \text{Tr}(\Pi^\perp(\Delta_X' - \lambda)^{-N-1}\Pi^\perp) \sim \sum_{k=0}^{\infty} c_k(-\lambda)^{-\frac{n-k}{2} - N - 1} + b(-\lambda)^{-N-1} \log(-\lambda);$$

in principle (see 52, Ch. 7), one can derive this resolvent expansion with a lot of work from the corresponding heat kernel expansion 53 54 55 56. From the work of
Seeley [64], we also know that each of \((\Delta_Z - \lambda)^{-N-1}\), where \(Z = Y, 0, 1, 2\), is trace class, and

\[
(7.8) \quad \text{Tr}((\Delta_Z - \lambda)^{-N-1}) \sim \sum_{k=0}^\infty c_{Z,k} (-\lambda)^{-n-k-N-1}.
\]

Finally, we know that \(\mathcal{S}(\lambda)^{(N)}\) is trace-class and vanishes, with all derivatives, to infinite order as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\). In conclusion, in view of the expression (7.6) and our discussions around (7.7) and (7.8), we see that \((\Delta_L - \lambda)^{-N-1}\) is trace-class, and

\[
\text{Tr}(\Delta_L - \lambda)^{-N-1} \sim \sum_{k=0}^\infty a_k (-\lambda)^{-n-k-N-1} + b (-\lambda)^{-N-1} \log(-\lambda)
\]

\[
+ \frac{1}{N!} \frac{d^N}{d\lambda^N} \left\{ \left( \frac{q_0 - j_0}{(-\lambda) \log(-\lambda) - 2\gamma} \right) \right\}
\]

\[
- \frac{1}{N!} \frac{d^{N+1}}{d\lambda^{N+1}} \left\{ \sum 2^\ell c_{\lambda,\xi} (-\lambda)^{-\ell} \left( 2\gamma - \log(-\lambda) \right)^{-\ell} \right\}.
\]

This completes the proof of Theorem 2.5. Theorem 2.6 is established by replacing the trace expansion (7.5) with the trace expansion found in Proposition 6.2.

7.2. Proofs of Theorems 2.1, 2.3, 2.4. We now prove the \(\zeta\)-function theorem. We start off with Proposition 7.2, which states that

\[
(\Delta_L - \lambda)^{-1} = \Pi(\mathcal{L}_L - \lambda)^{-1} \Pi + \Pi^+(\Delta_X' - \lambda)^{-1} \Pi^+ + (\Delta_Y - \lambda)^{-1} + \mathcal{S}(\lambda)
\]

where \(\mathcal{S}(\lambda)\) is trace-class and vanishes, with all derivatives, to infinite order as \(|\lambda| \to \infty\) with \(\lambda \in \Lambda\). Therefore, by the definition of the \(\zeta\)-function:

\[
\zeta(s, \Delta_L) := \text{Tr}(\Delta_L^{-s} P_0), \quad \text{Tr}(\Delta_L^{-s} P_0) = \sum_{\lambda_{\ell} < 0} \lambda_{\ell}^{-s} + \int_{\Re\lambda = \delta} \lambda^{-s}(\Delta_L - \lambda)^{-1} d\lambda,
\]

where \(P_0\) is the orthogonal projection onto \(\ker \Delta_L\) and \(\delta > 0\) is any positive number sufficiently small so that the spectrum of \(\Delta_L\) intersected with \([0, \delta]\) is empty, it follows that

\[
(7.9) \quad \zeta(s, \Delta_L) \equiv \zeta(s, \mathcal{L}_L) + \zeta(s, \Delta_X') + \zeta(s, \Delta_Y) + \zeta(s, \Delta_0) - \zeta(s, \Delta_1) - \zeta(s, \Delta_2)
\]

modulo an entire function. The \(\zeta\)-function \(\zeta(s, \mathcal{L}_L)\) is studied thoroughly in Proposition 7.2. Also, by the standard relation between the asymptotics of the resolvent and the poles of the \(\zeta\)-function (see e.g. [31]) it follows from the resolvent expansions (7.7) and (7.8) that \(\zeta(s, \Delta_X'), \zeta(s, \Delta_Y), \zeta(s, \Delta_0), \zeta(s, \Delta_1), \) and \(\zeta(s, \Delta_2)\) have the “regular” poles at the “usual” locations \(s = \frac{n-k}{2} \notin \mathbb{N}_0\) for \(k \in \mathbb{N}_0\) and, only for \(\zeta(s, \Delta_X')\), at \(s = 0\) if \(\dim \Gamma > 0\). These facts together with Proposition 7.2 prove Theorem 2.4. Note that Theorems 2.3 and 2.4 follow from applying Proposition 7.2 and Corollary 7.4 to \(\zeta(s, \mathcal{L}_L)\) in (7.1).

7.3. Proof of Theorems 2.7 and 2.8. Finally, it remains to prove the heat expansion. As with the proof for the \(\zeta\)-function, we start off with Proposition 7.3:

\[
(\Delta_L - \lambda)^{-1} = \Pi(\mathcal{L}_L - \lambda)^{-1} \Pi + \Pi^+(\Delta_X' - \lambda)^{-1} \Pi^+ + (\Delta_Y - \lambda)^{-1} + \mathcal{S}(\lambda)
\]

\[
(\Delta_0 - \lambda)^{-1} - (\Delta_1 - \lambda)^{-1} - (\Delta_2 - \lambda)^{-1} + \mathcal{S}(\lambda)
\]
where $S(\lambda)$ is trace-class and vanishes, with all derivatives, to infinite order as $|\lambda| \to \infty$ with $\lambda \in \Lambda$. Then by the definition of the heat operator:

$$e^{-\Delta L} := \frac{i}{2\pi} \int_{C_h} e^{-t\lambda}(\Delta L - \lambda)^{-1} d\lambda,$$

where $C_h$ is a contour as in Figure 4, it follows that

$$e^{-t\Delta L} = \Pi e^{-t\Delta_L} \Pi + \Pi e^{-t\Delta_L^\perp} \Pi + e^{-t\Delta_L^\perp} + e^{-t\Delta_0} - e^{-t\Delta_1} - e^{-t\Delta_2} + \mathcal{T}(t)$$

where $\mathcal{T}(t)$ is trace-class and smooth at $t = 0$. Hence,

$$\text{Tr}(e^{-t\Delta_L}) = \text{Tr}(e^{-t\Delta_L^\perp}) + \text{Tr}(e^{-t\Delta_L^0}) + \text{Tr}(e^{-t\Delta_L^1}) + \text{Tr}(e^{-t\Delta_L^2})$$

modulo a function that is smooth at $t = 0$. The heat trace $\text{Tr}(e^{-t\Delta_L})$ is studied thoroughly in Proposition 6.5 and for decomposable Lagrangians in Proposition 6.6. Also, by the standard relation between the asymptotics of the resolvent and the heat trace expansion (see e.g. [31] or Section 6.2) it follows from the resolvent expansions (7.7) and (7.8) that $\text{Tr}(e^{-t\Delta_L^0}), \text{Tr}(e^{-t\Delta_L^1}), \text{Tr}(e^{-t\Delta_L^2})$ and $\text{Tr}(e^{-t\Delta_L^\perp})$ have the “regular” expansion except that $\text{Tr}(e^{-t\Delta_L^\perp})$ may have a log $t$ term if dim $\Gamma > 0$. These facts together with Propositions 6.5 and 6.6 prove Theorems 2.7 and 2.8.

References

1. M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., New York, 1992, Reprint of the 1972 edition.
2. N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, New York, 1992.
3. M. Bordag, S. Dowker, and K. Kirsten, *Heat-kernels and functional determinants on the generalized cone*, Comm. Math. Phys. 182, no. 2 (1996), 371–393.
4. M. Bordag, E. Elizalde, B. Geyer, and K. Kirsten, *Zeta function determinant of the Laplace operator on the D-dimensional ball*, Comm. Math. Phys. 179, no. 1 (1996), 215–234.
5. J. Brüning and R. Seeley, *The resolvent expansion for second order regular singular operators*, J. Funct. Anal. 73, no. 2 (1987), 369–429.
6. , *An index theorem for first order regular singular operators*, Amer. J. Math. 110 (1988), 659–714.
7. W. Bulla and F. Gesztesy, *Deficiency indices and singular boundary conditions in quantum mechanics*, J. Math. Phys. 26 (1985), no. 10, 2520–2528.
8. C. Callias, *The heat equation with singular coefficients. I. Operators of the form $-d^2/dx^2 + \kappa/x^2$ in dimension 1*, Comm. Math. Phys. 88, no. 3 (1983), 357–385.
9. C. Callias, *The resolvent and the heat kernel for some singular boundary problems*, Comm. Partial Differential Equations 13 (1988), no. 9, 1113–1155.
10. J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. 76 (1979), no. 5, 2103–2106.
11. , *Hodge theory of complex cones*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque 101 (1983), 118–134.
12. , *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. 18 no. 4 (1983), 575–657.
13. A. W. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. 289, no.1 (1985), 1–40.
14. G. Cognola and S. Zerbini, *Zeta-function on a generalized cone*, Lett. Math. Phys. 42 (1997), no. 1, 95–101.
15. S. A. Coon and B. R. Holstein, *Anomalies in quantum mechanics: the 1/r^2 potential*, Amer. J. Phys. 70 (2002), no. 5, 513–519.

16. J. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1978.

17. P. Exner and P. Řebka (eds), *Applications of self-adjoint extensions in quantum physics*, proceedings of a conference held at the Laboratory of Theoretical Physics, JINR, Dubna, USSR, September 29-October 1, 1987, Springer-Verlag, Berlin; New York, 1989.

18. H. Falomir, M. A. Muschiotti, P. A. G. Pisani and R. T. Seeley, *Unusual poles of the ζ-functions for some regular singular differential operators*, J. Phys. A 36, no. 39 (2003), 9991–10010.

19. H. Falomir, M. A. Muschiotti and P. A. G. Pisani, *On the resolvent and spectral functions of a second order differential operator with a regular singularity*, J. Math. Phys. 45, no. 12 (2004), 4560–4577.

20. W. M. Frank, D. J. Land, and R. M. Spector, *Singular potentials*, Rev. Modern Phys. 43 (1971), no. 1, 36–98.

21. J.B. Gil, *Full asymptotic expansion of the heat trace for non-self-adjoint elliptic cone operators*, Math. Nachr. 250 (2003), 25–57.

22. J.B. Gil and P. Loya, *On the noncommutative residue and the heat trace expansion on conic manifolds*, Manuscripta Math. 109 (2002), no. 3, 309–327.

23. K. Kirsten, *Spectral functions in mathematics and physics*, Chapman & Hall/CRC Press, Boca Raton, 2001.

24. K. Kirsten and A. J. McKane, *Functional determinants by contour integration methods*, CRC Press, 2001.

25. K. Kirsten and A. J. McKane, *Functional determinants for general Sturm-Liouville problems*, J. Phys. A 37, no. 16 (2004), 4649–4670.

26. A. N. Kochubei, *Extensions of a positive definite symmetric operator*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (1979), no. 3, 168–171, 237.

27. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, sixth ed., Academic Press Inc., San Diego, CA, 2000, Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.

28. P. Greiner, *An asymptotic expansion for the heat equation*, Arch. Rational Mech. Anal. 41, (197, 1971), 163–218.

29. G. Grubb and L. Hansen, *Complex powers of resolvents of pseudodifferential operators*, Comm. Partial Differential Equations 27 (2002), no. 11-12, 2333–2361.

30. G. Grubb and R. T. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer operators*, Invent. Math. 121 (1995), 481–529.

31. G. Grubb and R. T. Seeley, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, Jour. Geom. Anal. 6 (1996), 31–77.

32. M. Harmer, *Hermitian symplectic geometry and extension theory*, J. Phys. A 33 (2000), no. 50, 9193–9203.

33. M. Harmer, *Hermitian symplectic geometry and the factorization of the scattering matrix on graphs*, J. Phys. A 33 (2000), no. 49, 9015–9032.

34. S. W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Comm. Math. Phys. 55 (1977), no. 2, 133–148.

35. Bernard S. Kay and Urban M. Studer, *Boundary conditions for quantum mechanics on cones and fields around cosmic strings*, Comm. Math. Phys. 139 (1991), no. 1, 103–139.

36. K. Kirsten, *Spectral functions in mathematics and physics*, Chapman & Hall/CRC Press, Boca Raton, 2001.
42. V. Kostrykin and R. Schrader, *Kirchhoff’s rule for quantum wires*, J. Phys. A **32** (1999), no. 4, 595–630.
43. T. Krainer, *Resolvents of elliptic boundary problems on conic manifolds*, preprint 2005.
44. A. Legrand and S. Moroianu, *On the $L^p$ index of spin Dirac operators on conical manifolds*, preprint, 2004.
45. M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.
46. __________, *Determinants of regular singular Sturm-Liouville operators*, Math. Nachr. **194** (1998), 139–170.
47. M. Lesch and J. Tolksdorf, *On the determinant of one-dimensional elliptic boundary value problems*, Commun. Math. Phys. **193**, no. 3 (1998), 643–660.
48. P. Loya, *The structure of the resolvent of elliptic pseudodifferential operators*, J. Funct. Anal. **184** (2001), no. 1, 77–135.
49. __________, *On the resolvent of differential operators on conic manifolds*, Comm. Anal. Geom. **10** (2002), no. 5, 877–934.
50. P. Loya, P. McDonald and J. Park, *Zeta Regularized Determinants for Conic Manifolds*, Preprint (2005), [http://www.math.binghamton.edu/paul/papers/](http://www.math.binghamton.edu/paul/papers/).
51. R. Mazzeo and R.B. Melrose, *Hodge theory of stratified varieties*, Lecture notes for 18.157, Spring 1997.
52. R.B. Melrose, *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, 1993.
53. S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canadian J. Math. **1** (1949), 242–256.
54. S. Minakshisundaram, *A generalization of Epstein zeta function*, Canadian J. Math. **1** (1949), 320–327.
55. E. Mooers, *Heat kernel asymptotics on manifolds with conic singularities*, J. Anal. Math. **78**, (1999) 1–36.
56. S. P. Novikov, *Schrödinger operators on graphs and symplectic geometry*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 397–413.
57. B. S. Pavlov, *The theory of extensions, and explicitly solvable models*, Uspekhi Mat. Nauk **42**, no. 6 (1987), 99–131.
58. Charles Radin, *Some remarks on the evolution of a Schrödinger particle in an attractive $1/r^2$ potential*, J. Mathematical Phys. **16** (1975), 544–547.
59. E. Schrohe and J. Seiler, *The resolvent of closed extensions of cone differential operators*, to appear in Canadian Journal of Mathematics.
60. B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland Publishing Co., Amsterdam, 1991.
61. R. Seeley, *Heat kernel expansions in the case of conic singularities*, Proceedings of the Second International Winter School on Mathematical Methods in Physics (Londrina, 2002), vol. 18, 2003, pp. 2197–2203.
62. __________, *Complex powers of an elliptic operator*, A.M.S. Symp. Pure Math. **10** (1967), 288-307.
63. __________, *Singular integrals and boundary value problems*, Amer. J. Math. **88** (1966), 781–809.
64. __________, *The resolvent of an elliptic boundary problem*, Amer. J. Math. **91** (1969), 889–920.
65. M. Sprefalic, *Zeta function and regularized determinant on a disc and on a cone*, J. Geom. Phys. **54** (2005), no. 3, 55–71.
66. H. Weyl, *A supplementary note to “A generalization of Epstein zeta function”* Canadian J. Math. **1** (1949), 326–327.

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