Dark energy in Horndeski theories after GW170817: A review

Ryotaro Kase and Shinji Tsujikawa
Department of Physics, Faculty of Science, Tokyo University of Science,
1-3, Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
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The gravitational-wave event GW170817 from a binary neutron star merger together with the electromagnetic counterpart showed that the speed of gravitational waves $c_t$ is very close to that of light for the redshift $z < 0.009$. This places tight constraints on dark energy models constructed in the framework of modified gravitational theories. We review models of the late-time cosmic acceleration in scalar-tensor theories with second-order equations of motion (dubbed Horndeski theories) by paying particular attention to the evolution of dark energy equation of state and observables relevant to the cosmic growth history. We provide a gauge-ready formulation of scalar perturbations in full Horndeski theories and estimate observables associated with the evolution of large-scale structures, cosmic microwave background, and weak lensing by employing a so-called quasi-static approximation for the modes deep inside the sound horizon.

In light of the recent observational bound of $c_t$, we also classify surviving dark energy models into four classes depending on different structure-formation patterns and discuss how they can be observationally distinguished from each other. In particular, the nonminimally coupled theories in which the scalar field $\phi$ has a coupling with the Ricci scalar $R$ of the form $G_4(\phi)R$, including $f(R)$ gravity, can be tightly constrained not only from the cosmic expansion and growth histories but also from the variation of screened gravitational couplings. The cross correlation of integrated Sachs-Wolfe signal with galaxy distributions can be a key observable for placing bounds on the relative ratio of cubic Galileon density to total dark energy density. The dawn of gravitational-wave astronomy will open up a new window to constrain nonminimally coupled theories further by the modified luminosity distance of tensor perturbations.

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I. INTRODUCTION

The late-time cosmic acceleration was first discovered in 1998 from the observations of distant supernovae type Ia (SN Ia) [1, 2]. The source for this phenomenon, which was dubbed dark energy [3], occupies about 70% of today’s energy density of the Universe. The existence of dark energy has been independently confirmed from the observation data of Cosmic Microwave Background (CMB) [4–8] and baryon acoustic oscillations (BAO) [9]. Despite the tremendous observational and theoretical progress over the past two decades [10–25], the origin of dark energy has not been identified yet.

A key quantity describing the property of dark energy is the equation of state (EOS) \( w_{DE} = \frac{P_{DE}}{\rho_{DE}} \), where \( \rho_{DE} \) and \( P_{DE} \) are the dark energy density and pressure respectively. If we use the EOS parametrization of the form \( w_{DE}(a) = w_0 + (1 - a)w_a \) [26, 27], where \( w_0, w_a \) are constants and \( a \) is a scale factor (with today’s value \( a = 1 \)), the joint data analysis of Planck 2018 combined with the SN Ia and BAO data placed the bounds \( w_0 = -0.961 \pm 0.077 \) and \( w_a = -0.28^{+0.33}_{-0.27} \) at 68% confidence level (CL) [8]. This shows the overall consistency with the EOS \( w_{DE} = -1 \), but the time variation of \( w_{DE} \) around \( -1 \) is also allowed from the data. We also note that the parametrization \( w_{DE}(a) = w_0 + (1 - a)w_a \) is not necessarily versatile to accommodate models with a fast varying dark energy EOS [28–31] or models in which \( w_{DE} \) has an extremum [32, 33].

In the theoretical side, the simplest candidate for dark energy is the cosmological constant \( \Lambda \) characterized by the EOS \( w_{DE} = -1 \). If the cosmological constant originates from the vacuum energy in particle physics, however, the theoretically predicted value is enormously larger than the observed dark energy scale [34–36]. There is a local theory of the vacuum energy sequestering in which quantum vacuum energy is cancelled by an auxiliary four-form field [37] (see also Ref. [38] for a nonlocal version). In this formulation, what is left after the vacuum energy sequestering is a radiatively stable residual cosmological constant \( \Delta \Lambda \). The value of \( \Delta \Lambda \) is not uniquely fixed by the underlying theory, but it should be measured to match with observations. To explain today’s cosmic acceleration, it is fixed as \( \Delta \Lambda \approx 10^{-12} \text{ eV}^4 \).

If the cosmological constant problem is solved in such a way that the residual vacuum energy completely vanishes or it is much smaller than today’s energy density of the Universe, we need to find an alternative mechanism for explaining the origin of dark energy. There are dynamical models of dark energy in which \( w_{DE} \) changes in time. The representative example is a minimally coupled scalar field \( \phi \) with a potential energy \( V(\phi) \), which is dubbed quintessence [39–48]. The quintessence EOS varies in the region \( w_{DE} > -1 \). While the “freezing” quintessence models [49] in which the deviation of \( w_{DE} \) from \(-1\) has been large by today are in strong tension with observations, the
“thawing” models in which $w_{\text{DE}}$ starts to deviate from $-1$ at late times have been consistent with the data for today’s EOS $w_{\text{DE}}(a = 1) \lesssim -0.7$ [50].

We also have other minimally coupled scalar-field theories dubbed k-essence in which the Lagrangian $G_2$ is a general function of $\phi$ and its derivative $X = -\partial^\mu \partial_\mu \phi/2$ [51–54]. One of the examples of k-essence is the dilatonic ghost condensate model $G_2 = -X + c e^{\lambda \phi} X^2$ [55–56] (c and $\lambda$ are constants), in which case the typical evolution of $w_{\text{DE}}$ is similar to that in thawing quintessence. It is also possible to construct unified models of dark energy and dark matter in terms of a purely kinetic Lagrangian $G_2(X)$ [57–58].

There are also theories in which the scalar field is nonminimally coupled to the Ricci scalar $R$ in the form $G_4(\phi)R$ [59]. One of the representative examples is Brans-Dicke (BD) theory [60] with a scalar potential $V(\phi)$. This theory is given by the Lagrangian $L = \phi R/2 + \omega_{\text{BD}} X\phi - V(\phi)$, where $\omega_{\text{BD}}$ is a constant called the BD parameter. The $f(R)$ gravity whose equations of motion are derived under the variation of metric tensor $g_{\mu\nu}$ [61–63] corresponds to the special case of BD theory with $\omega_{\text{BD}} = 0$ [64–65]. The application of $f(R)$ gravity to the late-time cosmic acceleration has been extensively performed in the literature [66–74]. In this case, it is possible to realize the dark energy EOS smaller than $-1$ without having ghosts [70–73]. Moreover, the growth rate of matter perturbations is larger than that in General Relativity (GR) [74–76]. Hence it is possible to observationally distinguish $f(R)$ gravity models from quintessence and k-essence.

The construction of dark energy models in BD theories with the scalar potential is also possible [77–78], in which case observational signatures are different depending on the BD parameter. In these models, the propagation of fifth forces can be suppressed in local regions of the Universe [70–79,83] under the chameleon mechanism [84–85].

There are other modified gravity theories dubbed Galileons [86–88] containing scalar derivative self-interactions and nonminimal couplings to gravity. In the limit of Minkowski space, the equations of motion following from the Lagrangian of covariant Galileons are invariant under the Galilean shift $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$. The cubic Galileon of the form $X^3\phi$ arises in the Dvali-Gabadadze-Porrati braneworld model due to the mixture between longitudinal and transverse gravitons [89] and also in the Dirac-Born-Infeld decoupling theory with bulk Lovelock invariants [90]. This derivative self-interaction can suppress the propagation of fifth forces in local regions of the Universe [91–106] under the Vainshtein mechanism [107], while modifying the gravitational interaction at cosmological distances [108]. For covariant Galileons including quartic and quintic Lagrangians, there exist self-accelerating de Sitter attractors responsible for the late-time cosmic acceleration [109–111]. The self-accelerating solution of full covariant Galileons preceded by a tracker solution with $w_{\text{DE}} = -2$ is in tension with observational data of SN Ia, CMB, and BAO [112–118]. However, this is not necessarily the case for covariant Galileons with a linear potential [119] or the model in which an additional term $X^2$ is present to the Galileon Lagrangian [120].

The aforementioned theories belong to a subclass of more general scalar-tensor theories—dubbed Horndeski theories [121]. In 2011, Deffayet et al. [122] derived the action of most general scalar-tensor theories with second-order equations of motion after the generalizations of covariant Galileons. Kobayashi et al. [123] showed that the corresponding action is equivalent to that derived by Horndeski in 1974 [124] (see also Ref. [125]). The application of Horndeski theories to the late-time cosmic acceleration was extensively performed in the literature [126–140]. Since different modified gravity theories predict different background expansion and cosmic growth histories, it is possible to distinguish between dark energy models in Horndeski theories from the observations of SN Ia, CMB, BAO, large-scale structures, and weak lensing.

The recent detection of gravitational waves (GWs) by GW170817 [141] from a binary neutron star merger together with the gamma-ray burst GRB 170817A [142] constrained the propagation speed $c_t$ of GWs, as [143]

$$-3 \times 10^{-15} \leq c_t - 1 \leq 7 \times 10^{-16},$$

for the redshift $z \leq 0.009$. If we strictly demand that $c_t$ and do not allow any tuning among functions in Horndeski theories, the Lagrangian needs to be of the form $L = G_2(\phi, X) + G_4(\phi, X)\Box \phi + G_4(\phi) R$ [144–150]. This includes the theories such as quintessence, k-essence, $f(R)$ gravity, BD theories, and cubic Galileons, but the theories with quartic-order derivative and quintic-order couplings do not belong to this class. Now, we are entering the era in which dark energy models in Horndeski theories can be tightly constrained from observations.

In this article, we review the application of Horndeski theories to cosmology and discuss observational signatures of surviving dark energy models. After reviewing Horndeski theories in Sec. [1] we devote Sec. [II] for deriving the background equations of motion and the second-order action of tensor perturbations on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime in the presence of matter. In Sec. [IV] we obtain the second-order action of scalar perturbations and their equations of motion in full Horndeski theories without fixing any gauge conditions. We note that this gauge-ready formulation was carried out in a subclass of Horndeski theories [151] and in scalar-vector-tensor theories [152]. The formalism developed in Ref. [152] encompasses both Horndeski gravity and generalized Proca theories [153–156] as specific cases.

In Sec. [V] we derive conditions for the absence of ghost and Laplacian instabilities of scalar perturbations in the small-scale limit by choosing three different gauges and show that these conditions are independent of the gauge
The functions where \( g \) is no Ostrogradski instability \([157, 158]\) associated with the Hamiltonian unbounded from below. Most general scalar-tensor theories with second-order equations of motion. Due to the second-order property, there reflects the fact that the gravity sector has two tensor polarized degrees of freedom. Horndeski theories \([121]\) are the cubic derivative interactions. This is convenient to distinguish between surviving dark energy models in current and future observational data. We conclude in Sec. \( \text{XI} \).

Throughout the review, we adopt the metric signature \((-, +, +, +)\). We also use the natural unit in which the speed of light \( c \), the reduced Planck constant \( \hbar \), the reduced Planck constant \( \hbar \), and the Boltzmann constant \( k_B \) are equivalent to 1. The reduced Planck mass \( M_{\text{pl}} \) is related to the Newton gravitational constant \( G \), as \( M_{\text{pl}} = 1/\sqrt{8\pi G} \).

II. HORNDESKI THEORIES

The theories containing a scalar field \( \phi \) coupled to gravity are generally called scalar-tensor theories \([59]\). This reflects the fact that the gravity sector has two tensor polarized degrees of freedom. Horndeski theories \([121]\) are the most general scalar-tensor theories with second-order equations of motion. Due to the second-order property, there is no Ostrogradski instability \([157, 158]\) associated with the Hamiltonian unbounded from below.

Horndeski theories are given by the action \([123]\)

\[
S_{\text{H}} = \int d^4x \sqrt{-g} L, \tag{2.1}
\]

where \( g \) is a determinant of the metric tensor \( g_{\mu\nu} \), and

\[
L = G_2(\phi, X) + G_3(\phi, X) \Box \phi + G_4(\phi, X) R + G_{4,X}(\phi, X) \left[ (\Box \phi)^2 - (\nabla_{\mu} \nabla_{\nu} \phi)(\nabla^{\mu} \nabla^{\nu} \phi) \right] + G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{1}{6} G_{5,X}(\phi, X) \left[ (\Box \phi)^3 - 3(\Box \phi)(\nabla_{\mu} \nabla_{\nu} \phi)(\nabla^{\mu} \nabla^{\nu} \phi) + 2(\nabla^{\mu} \nabla_{\alpha} \phi)(\nabla^{\alpha} \nabla_{\beta} \phi)(\nabla^{\beta} \nabla_{\mu} \phi) \right]. \tag{2.2}
\]

Here, the symbol \( \nabla_{\mu} \) stands for the covariant derivative operator with \( \Box \equiv \nabla^{\mu} \nabla_{\mu} \), \( R \) is the Ricci scalar, \( G_{\mu\nu} \) is the Einstein tensor, and

\[
X \equiv -\frac{1}{2} \nabla^\mu \phi \nabla_{\mu} \phi. \tag{2.3}
\]

The functions \( G_{2,3,4,5} \) depend on \( \phi \) and \( X \), with \( G_{i,\phi} \equiv \partial G_i / \partial \phi \) and \( G_{i,X} \equiv \partial G_i / \partial X \). Originally, Horndeski derived the Lagrangian of scalar-tensor theories with second-order equations of motion in a form different from Eq. \((2.2)\) \([121]\), but their equivalence was explicitly shown in Ref. \([123]\). Depending on the papers \([123, 126, 137, 159]\), different signs and notations were used for the quantities \( G_3(\phi, X) \) and \( X \), so we summarize them in Appendix \( \text{A} \) to avoid confusion. Below, we list the theories within the framework of the action \((2.1)\).

- (1) Quintessence and k-essence

  K-essence \([31, 54]\) is given by the functions

  \[
  G_2 = G_2(\phi, X), \quad G_3 = 0, \quad G_4 = \frac{M_{\text{pl}}^2}{2}, \quad G_5 = 0. \tag{2.4}
  \]

  Quintessence \([39, 48]\) corresponds to the particular choice:

  \[
  G_2 = X - V(\phi), \tag{2.5}
  \]

  where \( V(\phi) \) is the potential of \( \phi \).

- (2) BD theory

  In BD theory \([60]\) with the scalar potential \( V(\phi) \), we have

  \[
  G_2 = M_{\text{pl}} \omega_{\text{BD}}(\phi) X - V(\phi), \quad G_3 = 0, \quad G_4 = \frac{1}{2} M_{\text{pl}} \phi, \quad G_5 = 0. \tag{2.6}
  \]

  In the limit that \( \omega_{\text{BD}} \to \infty \), we recover GR with a quintessence scalar field. We note that there are more general nonminimally coupled theories given by the couplings \( G_2 = \omega(\phi) X - V(\phi), G_3 = 0, G_4 = F(\phi), G_5 = 0 \) \([160, 169]\). Since the basic structure of such theories is similar to that in BD theories, we do not discuss them in this review.
(3) $f(R)$ gravity

The action of $f(R)$ gravity \[61\,63\] is given by

$$S_H = \int d^4x \sqrt{-g} \frac{M_{\text{pl}}^2}{2} f(R),$$

(2.7)

where $f(R)$ is an arbitrary function of $R$. The metric $f(R)$ gravity, which corresponds to the variation of (2.7) with respect to $g_{\mu
u}$, can be accommodated by the Lagrangian (2.2) for the choice

$$G_2 = -\frac{M_{\text{pl}}^2}{2} (RF - f), \quad G_3 = 0, \quad G_4 = \frac{M_{\text{pl}}^2}{2} F, \quad G_5 = 0,$$

(2.8)

where $F(R) \equiv \partial f/\partial R$. In this case, the scalar degree of freedom $\phi = M_{\text{pl}} F(R)$ arises from the gravity sector. Comparing Eq. (2.6) with Eq. (2.8), it follows that metric $f(R)$ gravity is equivalent to BD theory with $\omega_{\text{BD}} = 0$ and the scalar potential $V = (M_{\text{pl}}/2)(RF - f)$ \[61\,63\].

(4) Covariant Galileons

In original Galileons \[86\], the field equations of motion are invariant under the shift $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$ in Minkowski spacetime \[86\]. In curved spacetime, the Lagrangian of covariant Galileons \[87\] is constructed to keep the equations of motion up to second order, while recovering the Galilean shift symmetry in the Minkowski limit. Covariant Galileons are characterized by the functions

$$G_2 = \beta_1 X - m^3 \phi, \quad G_3 = \beta_3 X, \quad G_4 = \frac{M_{\text{pl}}^2}{2} + \beta_4 X^2, \quad G_5 = \beta_5 X^2,$$

(2.9)

where $\beta_{1,3,4,5}$ and $m$ are constants. In absence of the linear potential $V(\phi) = m^3 \phi$, there exists a self-accelerating de Sitter solution satisfying $X = \text{constant}$ \[110\,111\,170\,171\].

(5) Derivative couplings

There is also a derivative coupling theory in which the scalar field couples to the Einstein tensor of the form $G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi$ \[172\,173\]. This corresponds to the choice

$$G_2 = X - V(\phi), \quad G_3 = 0, \quad G_4 = \frac{M_{\text{pl}}^2}{2}, \quad G_5 = c\phi,$$

(2.10)

where $V(\phi)$ is a scalar potential and $c$ is a constant. Integrating the term $c\phi G_{\mu\nu} \nabla^\mu \nabla^\nu \phi$ by parts, it is equivalent to $-c G_{\mu\nu} \nabla^\mu \nabla^\nu \phi + \text{boundary term}$.

(6) Gauss-Bonnet couplings

One can consider a coupling of the form $\xi(\phi) G$ \[174\], where $\xi(\phi)$ is a function of $\phi$ and $G$ is the Gauss-Bonnet curvature invariant defined by

$$G = R^2 - 4R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}.$$

(2.11)

The theories given by the action \[175\,178\]

$$S_H = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} R + X - V(\phi) + \xi(\phi) G \right],$$

(2.12)

can be accommodated in the framework of Horndeski theories for the choice \[123\]

$$G_2 = X - V(\phi) + 8 \xi^{(4)}(\phi) X^2 (3 - \ln X), \quad G_3 = -4 \xi^{(3)}(\phi) X (7 - 3 \ln X),$$

$$G_4 = \frac{M_{\text{pl}}^2}{2} + 4 \xi^{(2)}(\phi) X (2 - \ln X), \quad G_5 = -4 \xi^{(1)}(\phi) \ln X,$$

(2.13)

where $\xi^{(n)}(\phi) \equiv \partial^n \xi(\phi)/\partial \phi^n$. 
• (7) $f(\mathcal{G})$ gravity

There is also modified gravitational theories given by the action $^{[179]}$ $^{[180]}$

$$S_{H} = \int d^{4}x \sqrt{-g} \left[ \frac{M_{P}^{2}}{2} R + f(\mathcal{G}) \right], \quad (2.14)$$

where $f$ is a function of $\mathcal{G}$. Since the Lagrangian $f(\mathcal{G})$ is equivalent to $f(\phi) + f,\phi(\mathcal{G} - \phi)$, the action (2.14) belongs to a subclass of Horndeski theories with the couplings

$$G_{2} = f(\phi) - f,\phi \phi + 8f^{(5)}(\phi)X^{2} (3 - \ln X), \quad G_{3} = -4f^{(4)}(\phi)X (7 - 3 \ln X),$$

$$G_{4} = \frac{M_{P}^{2}}{2} + 4f^{(3)}(\phi)X (2 - \ln X), \quad G_{5} = -4f^{(2)}(\phi)\ln X. \quad (2.15)$$

• (8) Kinetic braidings and its extensions

There are theories given by the Lagrangian

$$L = G_{2}(\phi, X) + G_{3}(\phi, X)\Box \phi + G_{4}(\phi)R. \quad (2.16)$$

As we will show later, the Lagrangian (2.16) corresponds to most general Horndeski theories with the tensor propagation speed $c_{t}$ equivalent to 1. The kinetic braiding scenario $^{[181]}$ $^{[182]}$ corresponds to the minimally coupled case, i.e., $G_{4} = M_{P}^{2}/2$. The cubic Galileon given with $L = \beta_{1}X - m^{3}\phi + \beta_{3}X \Box \phi + (M_{P}^{2}/2)R$ belongs to a subclass of kinetic braidings. The dark energy scenario given by $L = \beta_{1}X + \beta_{2}X^{2} + \beta_{3}X \Box \phi + (M_{P}^{2}/2)R$ $^{[120]}$ is also in the framework of kinetic braidings. In presence of the nonminimal coupling $G_{4}(\phi)R$, it is known that a self-accelerating solution characterized by $\dot{\phi}/\phi = \text{constant}$ exists for the model $L = \omega(\phi/M_{P})^{1-n}X + \left(\lambda/\mu^{3}\right)(\phi/M_{P})^{-n}X \Box \phi + (M_{P}^{2}/2)(\phi/M_{P})^{3-n}R$ with $2 \leq n \leq 3$ $^{[183]}$ $^{[186]}$. There is also a nonminimally coupled model given by $L = \beta_{1} (1 - 6Q^{2}) e^{-2Q\phi/M_{P}}X - m^{3}\phi + \beta_{3}X \Box \phi + (M_{P}^{2}/2)e^{-2Q\phi/M_{P}}R$ $^{[119]}$ $^{[187]}$, which recovers the minimally coupled cubic Galileon in the limit $Q \to 0$.

Thus, Horndeski theories can accommodate a wide variety of scalar-tensor theories with second-order equations of motion. Except for quintessence and k-essence, the scalar field has derivative self-interactions and nonminimal/derivative couplings to gravity. In such cases, we need to confirm whether there are neither ghost nor Laplacian instabilities. In Secs. III E and V, we will address this issue after deriving the second-order actions of tensor and scalar perturbations on the flat FLRW background in full Horndeski theories. The same analysis also allows one to identify the speed of gravitational waves on the isotropic cosmological background. Under the observational bound $^{[171]}$, this result can be used to pin down viable Horndeski theories relevant to the late-time cosmic acceleration. In Sec. VI D, we classify surviving dark energy models into four classes depending on their observational signatures.

To discuss the dynamics of cosmic acceleration preceded by the matter-dominated epoch, we take into account a matter perfect fluid minimally coupled to gravity. The vector perturbations are nondynamical in Horndeski theories, so we only need to consider the evolution of scalar and tensor perturbations. The perfect fluid without the vector degree of freedom can be described by the Schutz-Sorkin action $^{[188]}$ $^{[189]}$.

$$S_{m} = - \int d^{4}x \left[ \sqrt{-g} \rho_{m}(n) + J^{\mu} \partial_{\mu} \ell \right], \quad (2.17)$$

where $\rho_{m}$ is a fluid density, $\ell$ is a scalar quantity, and $J^{\mu}$ is a four vector associated with the fluid number density $n$, as

$$n = \sqrt{\frac{J^{\mu} J^{\nu} g_{\mu\nu}}{g}}. \quad (2.18)$$

Varying the action (2.17) with respect to $J^{\mu}$, we obtain

$$u_{\mu} \equiv \frac{J_{\mu}}{n \sqrt{-g}} = \frac{\partial_{\mu} \ell}{\rho_{m,n}}, \quad (2.19)$$

where $u_{\mu}$ corresponds to a normalized four velocity, and $\rho_{m,n} \equiv \partial \rho_{m}/\partial n$.

In this review, we focus on the theories given by the sum of (2.1) and (2.17), i.e.,

$$S = S_{H} + S_{m}. \quad (2.20)$$

After deriving the background and perturbation equations of motion, they can be applied to subclasses of Horndeski theories listed above.
III. FLRW BACKGROUND AND TENSOR PERTURBATIONS

A. Background equations of motion

Let us consider the flat FLRW spacetime given by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t) (dx^i dx^i), \quad (3.1)$$

where $N(t)$ is a lapse and $a(t)$ is a scale factor. The lapse is introduced here for deriving the Friedmann equation, but we finally set $N = 1$ after the variation of $S$. The scalar field $\phi$ depends on the cosmic time $t$ alone on the background. For the matter sector, the temporal component of $J^\mu$ in Eq. (2.18) is given by

$$J^0 = n_0 a^3, \quad (3.2)$$

where $n_0$ is the background value of $n$. Then, the Schutz-Sorkin action (2.17) reduces to

$$S_m = - \int d^4 x \, a^3 \left( N \rho_m + n_0 \dot{\ell} \right). \quad (3.3)$$

Varying this matter action with respect to $\ell$, it follows that

$$\mathcal{N}_0 \equiv n_0 a^3 = \text{constant}, \quad (3.4)$$

where $\mathcal{N}_0$ corresponds to the total conserved fluid number.

Now, we compute the action (2.20) on the spacetime metric (3.1) and vary it with respect to $N$, $a$ and $\phi$. Setting $N = 1$ in the end, we obtain the background equations of motion:

$$6G_4 H^2 + G_2 - \dot{\phi}^2 G_{2,X} + \dot{\phi}^2 \left( 3H \phi G_{3,X} - G_{3,\phi} \right) + 6H \phi \left( G_{4,\phi} + \dot{\phi}^2 G_{4,X,\phi} - 2H \phi G_{4,X} - H \phi^3 G_{4,X,X} \right)$$

$$+ \dot{H}^2 \phi^2 \left( 9G_{5,\phi} + 3\dot{\phi}^2 G_{5,X,\phi} - 5H \phi G_{5,X} - H \phi^3 G_{5,X,X} \right) = \rho_m, \quad \text{(3.5)}$$

$$2q_4 \dot{H} - D_6 \ddot{\phi} + D_7 \dot{\phi} = -\rho_m - P_m, \quad \text{(3.6)}$$

$$3D_6 \dot{H} + 2D_1 \ddot{\phi} + 3D_7 H - D_5 = 0, \quad \text{(3.7)}$$

where $H \equiv \dot{a}/a$ is the Hubble expansion rate, and $P_m$ is the matter pressure defined by $P_m \equiv -n_0 \dot{\ell} - \rho_m$. Substituting $J_0 = -n_0 a^3$ into Eq. (2.19), the temporal component $u_0$ of four velocity is equivalent to $-1$ and hence $\dot{\ell} = -\rho_{m,n}$. Thus, the matter pressure can be written as

$$P_m = n_0 \rho_{m,n} - \rho_m. \quad \text{(3.8)}$$

From Eq. (3.4), the fluid number density obeys the differential equation $q_0 + 3Hn_0 = 0$. On using the property $\dot{\rho}_m = \rho_{m,n} \dot{n}_0$ and the relation (3.8), the conservation of total fluid number translates to

$$\dot{\rho}_m + 3H (\rho_m + P_m) = 0. \quad \text{(3.9)}$$

We note that this continuity equation also follows from Eqs. (3.5)-(3.7).

The quantity $q_4$ in Eq. (3.6) is defined by

$$q_4 = 2G_4 - 2\ddot{\phi}^2 G_{4,X} + \ddot{\phi}^2 G_{5,\phi} - H \phi^3 G_{5,X}, \quad \text{(3.10)}$$

which is associated with the no-ghost condition of tensor perturbations discussed later in Sec. III B. The definitions of coefficients $D_{1,5,6,7}$ appearing in Eqs. (3.6) and (3.7) are presented in Appendix B. As we will show in Sec. IV, they also appear as coefficients in the second-order action of scalar perturbations. We note that Eq. (3.6) has been derived by varying the action with respect to $a$ and then eliminating the term $G_2$ by using Eq. (3.5).

Solving Eqs. (3.6)-(3.7) for $\dot{H}$ and $\ddot{\phi}$, it follows that

$$\dot{H} = -\frac{1}{q_4} \left[ 2D_1 D_7 \dot{\phi} - D_6 (D_5 - 3D_7 H) + 2D_1 (\rho_m + P_m) \right], \quad \text{(3.11)}$$

$$\ddot{\phi} = \frac{1}{q_4} \left[ 3D_6 D_7 \dot{\phi} + 2(D_5 - 3D_7 H) q_4 + 3D_6 (\rho_m + P_m) \right], \quad \text{(3.12)}$$
where
\[ q_s \equiv 4D_1 q_t + 3D_6^2. \]  
(3.13)

To avoid the divergences on the right hand sides of Eqs. (3.11) and (3.12), we require that
\[ q_s \neq 0. \]  
(3.14)

As we will show later, the determinant \( q_s \) is related to the no-ghost condition of scalar perturbations. The no-ghost condition corresponds to \( q_s > 0 \), in which case \( \dot{H} \) and \( \ddot{\phi} \) can remain finite.

In the presence of matter, the condition for the cosmic acceleration is given by
\[ w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2} < -\frac{1}{3}. \]  
(3.15)

We can express Eqs. (3.5) and (3.6) in the forms
\[ 3M_{\text{pl}}^2 H^2 = \rho_{\text{DE}} + \rho_m, \]  
(3.16)
\[ 2M_{\text{pl}}^2 \dot{H} = -\rho_{\text{DE}} - P_{\text{DE}} - \rho_m - P_m, \]  
(3.17)
where the density \( \rho_{\text{DE}} \) and pressure \( P_{\text{DE}} \) of the "dark" component are given, respectively, by
\[ \rho_{\text{DE}} = 3H^2 (M_{\text{pl}}^2 - 2G_4) - G_2 + \phi G_{2,X} + \phi^2 (3H \dot{G}_{3,X} - G_{3,\phi}) - 6H \dot{\phi} (G_{4,\phi} + \phi^2 G_{4,X\phi}) - 2H \dot{\phi} G_{4,X} - H \phi^3 G_{4,XX} \]  
(3.18)
\[ P_{\text{DE}} = 2 (q_t - M_{\text{pl}}^2) \dot{H} - D_6 \ddot{\phi} + D_7 \dot{\phi} - \rho_{\text{DE}}. \]  
(3.19)

We define the dark energy EOS, as
\[ w_{\text{DE}} \equiv \frac{P_{\text{DE}}}{\rho_{\text{DE}}} = -1 + \frac{2(q_t - M_{\text{pl}}^2) \dot{H} - D_6 \ddot{\phi} + D_7 \dot{\phi}}{\rho_{\text{DE}}}, \]  
(3.20)
which is different from the effective EOS (3.15) due to the presence of additional matter (dark matter, baryons, radiation). The necessary condition for the late-time cosmic acceleration is given by \( w_{\text{DE}} < -1/3 \), but we caution that this is not a sufficient condition. In quintessence and k-essence we have \( q_t = M_{\text{pl}}^2 \), so the time variation of \( \phi \) leads to the deviation of \( w_{\text{DE}} \) from \(-1\). In other theories listed in Sec. [1] the quantity \( q_t \) generally differs from \( M_{\text{pl}}^2 \), so the term \( 2(q_t - M_{\text{pl}}^2) \dot{H} \) in Eq. (3.20) also contributes to the additional deviation of \( w_{\text{DE}} \) from \(-1\). Since the evolution of \( w_{\text{DE}} \) is different depending on dark energy models, it is possible to distinguish between them from the observations of SN Ia, CMB, and BAO.

### B. Tensor perturbations

The speed of gravitational waves on the flat FLRW background can be derived by expanding the action (2.20) up to second order in tensor perturbations \( h_{ij} \). As a by-product, we can identify conditions for the absence of ghost and Laplacian instabilities in the tensor sector. The perturbed line element, which contains traceless and divergence-free tensor perturbations obeying the conditions \( h_{ii} = 0 \) and \( \partial_i h_{ij} = 0 \), is given by
\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} + h_{ij})dx^i dx^j. \]  
(3.21)

Without loss of generality, we can choose nonvanishing components of \( h_{ij} \) in the forms
\[ h_{11} = h_1(t,z), \quad h_{22} = -h_1(t,z), \quad h_{12} = h_{21} = h_2(t,z), \]  
(3.22)
where \( h_1 \) and \( h_2 \) characterize the two polarization states.

Expanding the Horndeski action (2.1) up to second order in perturbations and integrating it by parts, the quadratic action \( S_{(2)}^t \) contains the time derivative \( h_i^2 \), the spatial derivative \( (\partial h_i)^2 \), and the mass term \( h_i^2 \) (where \( i = 1, 2 \)). The second-order action of tensor perturbations arising from the matter action (2.17) can be written in the form
\[ (S_{(2)}^t)_i = -\int dt d^3x \left[ (\sqrt{-g})^{(2)} \rho_m + \sqrt{-g} \rho_m \delta n \right], \]  
(3.23)
where $(\sqrt{-g})^{(2)} = -a^3(h_t^2 + h_k^2)/2$ and $\delta n = n_0(h_t^2 + h_k^2)/2$ with $\sqrt{-g} = a^3$. Then, Eq. (3.23) reduces to

$$(S^{(2)}_m)_t = -\int dt d^3x \sum_{i=1}^{2} \frac{1}{2} a^3 P_m h_i^2 , \quad (3.24)$$

where $P_m$ is the matter pressure given by Eq. (3.18). Using the background Eqs. (3.5) and (3.6), the terms proportional to $h_k^2$ identically vanish from the total second-order action $S^{(2)} = (S^{(2)}_H)_t + (S^{(2)}_m)_t$. Finally, we can express $S^{(2)}_t$ in a compact form:

$$S^{(2)}_t = \int dt d^3x \sum_{i=1}^{2} \frac{a^3}{4} q_t \left[ h_i^2 - \frac{c_i^2}{a^2} (\partial h_i)^2 \right] , \quad (3.25)$$

where $q_t$ was already introduced in Eq. (3.10), and $c_i^2$ is the tensor propagation speed squared given by

$$c_i^2 = \frac{1}{q_t} \left( 2G_4 - \phi^2 G_{5,\phi} - \phi^2 \phi_{5,X} \right) . \quad (3.26)$$

To avoid the ghost and Laplacian instabilities, we require that

$$q_t > 0 , \quad c_i^2 > 0 . \quad (3.27, 3.28)$$

Varying the action (3.25) with respect to $h_i$, we obtain the tensor perturbation equation of motion in Fourier space, as

$$\ddot{h}_i + \left( 3H + \frac{\dot{q}_t}{q_t} \right) \dot{h}_i + c_i^2 \frac{k^2}{a^2} h_i = 0 , \quad (3.29)$$

where $k$ is a coming wavenumber. The time variation of $q_t$ and the deviation of $c_i^2$ from 1 lead to the modified evolution of $h_i$ compared to that in GR.

The observational bound (1.1) of $c_t$ places tight constraints on surviving dark energy models. To realize the exact value $c_t^2 = 1$, we require the following condition

$$2G_{4,X} - 2G_{5,\phi} + \left( H \dot{\phi} - \ddot{\phi} \right) G_{5,X} = 0 . \quad (3.30)$$

If we do not allow any tuning among functions, the dependence of $G_4$ on $X$ and $G_5$ on $\phi, X$ is forbidden. Then, the Horndeski Lagrangian is restricted to be of the form \[144\] \[150\]

$$L = G_2(\phi, X) + G_3(\phi, X) \Box \phi + G_4(\phi) R . \quad (3.31)$$

Among the theories listed in Sec. II, the theories (5), (6), (7) lead to $c_t^2$ different from 1. In particular, as long as the derivative and Gauss-Bonnet couplings contribute to the late-time cosmological dynamics, the deviation of $c_t^2$ from 1 is large and hence such couplings are excluded from the bound (1.1) as a source for dark energy \[190\].

Quintessence and k-essence, BD theory, and $f(R)$ gravity give rise to the exact value $c_t^2 = 1$, so they automatically satisfy the bound (1.1). The quartic and quintic Galileon couplings $G_4 = \beta_4 X^2$ and $G_5 = \beta_5 X^2$ lead to the deviation of $c_t^2$ from 1, but the Galileon Lagrangian up to the cubic interaction $G_3 = \beta_3 X$ is allowed (which includes the model with the additional term $G_2 = \beta_2 X^2$ to the cubic Galileon Lagrangian \[120\]). The kinetic braidings \[181\] \[182\] and its extensions \[119\] \[183\] \[187\] are also consistent with the bound (1.1).

Even if $c_t^2$ is constrained to be close to 1, there is another modification to Eq. (3.29) arising from the time variation of $q_t$. This leads to the modified luminosity distance of GWs relative to that of electromagnetic signals \[192\] \[193\]. Since $q_t = 2G_4(\phi)$ for the Lagrangian (3.31), it is possible to distinguish between nonminimally and minimally coupled theories from the luminosity distance measurements of GWs together with the electromagnetic counterpart. After the accumulation of GW events in future, the luminosity distance can be a key observable to probe the existence of nonminimal couplings.

---

1 Besides this problem, matter perturbations in $f(G)$ gravity are subject to strong instabilities attributed to the negative sound speed squared \[191\], so $f(G)$ cosmological models are not cosmologically viable.
For the Lagrangian \([3.31]\), we have

\[
D_6 = -\dot{\phi}^2 G_{3,X} - 2G_{4,\phi} \quad \text{and} \quad D_7 = \dot{\phi}(G_{2,X} + 2G_{3,\phi} + 2G_{4,\phi}) - H(2G_{4,\phi} + 3\dot{\phi}^2 G_{3,X}),
\]

so the dark energy EOS \([3.20]\) reduces to

\[
w_{\text{DE}} = -1 + \frac{2(2G_4 - M_{\text{pl}}^2)\dot{H} + (\dot{\phi}^2 G_{3,X} + 2G_{4,\phi})\dot{\phi} + [\dot{\phi}(G_{2,X} + 2G_{3,\phi} + 2G_{4,\phi}) - H(2G_{4,\phi} + 3\dot{\phi}^2 G_{3,X})]\dot{\phi}}{3H^2(M_{\text{pl}}^2 - 2G_4) - 2G_2 + \dot{\phi}^2 G_{2,X} - \dot{\phi}^2 (3H\dot{\phi} G_{3,X} - G_{3,\phi}) - 6H\phi G_{4,\phi}}.
\]  

(3.32)

In Secs. \([\text{VII-X}]\), we will study the evolution of \(w_{\text{DE}}\) for concrete models of the late-time cosmic acceleration in the framework of Horndeski theories.

### IV. GAUGE-READY FORMULATION OF SCALAR PERTURBATIONS

In this section, we derive the second-order action of scalar perturbations in full Horndeski theories without fixing gauge conditions. The resulting linear perturbation equations of motion are written in a gauge-ready form \([151, 152]\), so that one can choose convenient gauges depending on the problems at hand. For generality, we do not restrict the Horndeski Lagrangian to the form \([3.31]\).

We begin with the linearly perturbed line-element given by \([195–199]\)

\[
ds^2 = -(1 + 2\alpha)dt^2 + 2\partial_i\chi dt dx^i + a^2(t)\left[(1 + 2\zeta)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j,
\]

(4.1)

where \(\alpha, \chi, \zeta, E\) are scalar metric perturbations, and the symbol \(\partial\) stands for the partial derivative \(\partial/\partial x^i\). We do not take into account vector perturbations, as they are nondynamical in scalar-tensor theories. The scalar field is decomposed into the form

\[
\phi = \bar{\phi}(t) + \delta\phi(t, x^i),
\]

(4.2)

where \(\bar{\phi}(t)\) and \(\delta\phi(t, x^i)\) are the background and perturbed values, respectively. In the following, we omit the bar from the background quantities.

#### A. Second-order matter action

We first expand the Schutz-Sorkin action \([2.17]\) up to second order in perturbations. We decompose the temporal and spatial components of \(J^\mu\) into the background and perturbed parts, as \([\text{200, 201}]\)

\[
J^0 = \mathcal{N}_0 + \delta J, \quad J^i = \frac{1}{a^2(t)}\delta^{ik}\partial_k \delta j,
\]

(4.3)

where \(\mathcal{N}_0 = n_0 a^3\) is the conserved background fluid number defined by Eq. \([3.4]\), and \(\delta J, \delta j\) are scalar perturbations. The spatial component of four velocity can be expressed as \(u_i = -\partial_i v\), where \(v\) is the velocity potential. On using Eq. \([2.19]\), the scalar quantity \(\ell\) contains the perturbation \(-\rho_{m,n} v\). Then, \(\ell\) can be decomposed as

\[
\ell = -\int^\ell \rho_{m,n}(\bar{t})d\bar{t} - \rho_{m,n} v,
\]

(4.4)

where the first contribution to the right hand side corresponds the background quantity. Defining the matter density perturbation

\[
\delta \rho_m \equiv \frac{\rho_{m,n}}{a^3} \left[\delta J - \mathcal{N}_0 (3\zeta + \partial^2 E)\right],
\]

(4.5)

where \(\partial^2 E = (\partial_i E)(\partial_i E)\) (the same latin subscripts are summed over), the perturbation of fluid number density \(n\), up to second order, is expressed as

\[
\delta n = \frac{\delta \rho_m}{\rho_{m,n}} - \frac{(\mathcal{N}_0 \partial_X + \partial \delta j)^2}{2\mathcal{N}_0 a^5} - \frac{(3\zeta + \partial^2 E)\delta \rho_m}{\rho_{m,n}} - \frac{\mathcal{N}_0 (\zeta + \partial^2 E)(3\zeta - \partial^2 E)}{2a^3}.
\]

(4.6)

Since \(\delta n\) reduces to \(\delta \rho_m/\rho_{m,n}\) at linear order, this confirms the consistency of the definition \((4.5)\).
Now, we are ready for expanding the Schütz-Sorkin action \( 2.17 \) up to quadratic order in scalar perturbations. This manipulation gives the second-order matter action \[152]\:

\[
(S_n^{(2)})_s = \int dt d^3x \, a^3 \left[ \frac{\rho_{m,n}}{2a^2n_0} (\partial^2 j^2 + \frac{\rho_{m,n}}{a^2} (\partial \chi + \partial v)(\partial \delta j) + (\dot{v} - 3Hc_m^2 v - \alpha) \delta \rho_m - \frac{c_m^2}{2n_0\rho_{m,n}} \delta \rho_m^2 + \frac{\rho_m}{2} \alpha^2 \\
+ \frac{n_0\rho_{m,n} - \rho_m}{2} \left( \left( \partial \chi \right)^2 (\zeta + \partial^2 E)(3\zeta - \partial^2 E) + (3\zeta + \partial^2 E) \left\{ n_0\rho_{m,n}(\dot{v} - 3Hc_m^2 v) - \rho_m\alpha \right\} \right) \right], \tag{4.7}
\]

where \( c_m^2 \) is the matter sound speed squared given by

\[
c_m^2 = \frac{P_m}{\rho_{m,n}} = \frac{n_0\rho_{m,n,n}}{\rho_{m,n}}. \tag{4.8}
\]

Variation of the action \( 4.7 \) with respect to \( \delta j \) leads to

\[
\partial \delta j = -a^3n_0(\partial v + \partial \chi). \tag{4.9}
\]

Substituting the relation \( 4.9 \) into Eq. \( 4.7 \), we obtain

\[
(S_n^{(2)})_s = \int dt d^3x \, a^3 \left[ (\dot{v} - 3Hc_m^2 v - \alpha) \delta \rho_m - \frac{c_m^2}{2(\rho_m + P_m)} \delta \rho_m^2 \left\{ (\partial v)^2 + 2\partial v \partial \chi \right\} - \frac{\rho_m + P_m}{2a^2} (\partial \chi)^2 + \frac{\rho_m}{2} \alpha^2 \\
+ \frac{P_m}{2} (\zeta + \partial^2 E)(3\zeta - \partial^2 E) + (3\zeta + \partial^2 E) \left\{ (\rho_m + P_m)(\dot{v} - 3Hc_m^2 v) - \rho_m\alpha \right\} \right], \tag{4.10}
\]

where we used Eq. \( 3.8 \).

### B. Full second-order scalar action and linear perturbation equations of motion

We also expand the Horndeski action \( 2.1 \) up to second order in scalar perturbations and take the sum with Eq. \( 4.10 \). Using the background Eq. \( 3.5 \), the term \( \rho_m\alpha^2/2 \) in Eq. \( 4.10 \) is cancelled by a part of contributions proportional to \( \alpha^2 \) arising from the Horndeski action. After the integration by parts, we can write the full second-order action of \( S \) in the gauge-ready form

\[
S_n^{(2)} = \int dt d^3x \left( L_{\text{flat}} + L_\zeta + L_E \right), \tag{4.11}
\]

where

\[
L_{\text{flat}} = a^3 \left[ D_1 \dot{\phi}^2 + D_2 \left( \frac{\partial \delta \phi}{\partial \phi} \right)^2 + D_3 \dot{\phi} \dot{\phi} + \left( D_4 \dot{\phi} + D_5 \dot{\phi} + D_6 \frac{\partial^2 \delta \phi}{\partial \phi^2} \right) \alpha - \left( D_6 \dot{\phi} - D_7 \dot{\phi} \right) \frac{\partial^2 \chi}{\partial \phi^2} \right. \\
+ \left( \dot{\phi} D_6 - 2Hq_1 \right) \alpha \frac{\partial^2 \chi}{\partial \phi^2} + \left( \frac{\partial^2 D_1 + 3H\dot{\phi} D_6 - 3H^2 q_1}{\partial \phi^2} \right) \alpha^2 \\
+ \left( \rho_m + P_m \right) \left( \frac{\partial^2 \chi}{\partial \phi^2} - v \dot{\phi} \rho_m - 3H(1 + c_m^2)v \delta \rho_m - \frac{1}{2}(\rho_m + P_m) \frac{\partial \dot{\chi}}{\partial \phi^2} \right), \tag{4.12}
\]

\[
L_\zeta = a^3 \left[ 3D_6 \dot{\phi} + 3D_2 \dot{\phi} - 3 \left( \dot{\phi} D_6 - 2Hq_1 \right) \alpha - 3(\rho_m + P_m)v + 2q_1 \frac{\partial^2 \chi}{\partial \phi^2} \right] \dot{\zeta} - 3q_1 \dot{\zeta}^2 \\
- \left( B_1 \dot{\phi} + 2q_1 \alpha \right) \frac{\partial^2 \zeta}{\partial \phi^2} + q_1^2 \left( \frac{\partial \zeta}{\partial \phi} \right)^2, \tag{4.13}
\]

\[
L_E = a^3 \left[ 2q_2 \zeta + 2B_2 \dot{\zeta} - D_6 \dot{\phi} - B_5 \dot{\phi} + B_4 \dot{\phi} + \frac{1}{a^2} \frac{d}{dt} \left\{ a^3 \left( \dot{\phi} D_6 - 2Hq_1 \right) \alpha \right\} + (\rho_m + P_m)(\dot{v} - 3Hc_m^2 v) \right] \delta^2 E, \tag{4.14}
\]

where the explicit forms of coefficients \( D_1,...,7 \) are presented in Appendix B. The quantities \( q_1 \) and \( c_m^2 \) are given by Eqs. \( 3.10 \) and \( 3.26 \), respectively. The coefficients \( B_{1,2,3,4} \) in Eqs. \( 4.13 \) and \( 4.14 \) can be expressed by using other quantities, as

\[
B_1 = \frac{2}{\phi} \left[ \dot{q}_1 + (1 - c_1^2)Hq_1 \right], \quad B_2 = \dot{q}_1 + 3Hq_1, \quad B_3 = \dot{D}_6 + 3HD_6 - D_7, \quad B_4 = \dot{D}_7 + 3HD_7. \tag{4.15}
\]
The Lagrangian $L_{\text{flat}}$ is present for the flat gauge $\zeta = 0 = E$ \cite{202}, while the other two Lagrangians $L_\zeta$ and $L_E$ arise in the presence of metric perturbations $\zeta$ and $E$. We note that, in scalar-vector-tensor theories, the generalized version of the action \cite{152} was derived in Ref. \cite{152}.

Since there are no kinetic terms for the variables $\alpha, \chi, v, E$ in Eqs. \cite{112,114}, they correspond to nondynamical perturbations. Varying the second-order action \cite{411} with respect to $\alpha, \chi, v, E$, their constraint equations in Fourier space are given, respectively, by

$$\mathcal{E}_\alpha \equiv D_4 \dot{\delta} \phi - 3 \left( \dot{\phi} D_6 - 2 H q_t \right) \dot{\zeta} + D_3 \delta \phi + 2 \left( \dot{\phi}^2 D_1 + 3 H \dot{\phi} D_6 - 3 H^2 q_t \right) \alpha$$

$$+ \frac{k^2}{a^2} \left[ 2 q_t \zeta - \left( \dot{\phi} D_6 - 2 H q_t \right) \left( \chi - a^2 \dot{E} \right) - D_6 \delta \phi \right] - \delta \rho_m = 0 ,$$

(4.16)

$$\mathcal{E}_\chi \equiv D_6 \dot{\delta} \phi - 2 q_t \dot{\zeta} - D_7 \delta \phi - \left( \dot{\phi} D_6 - 2 H q_t \right) \alpha - \left( \rho_m + P_m \right) v = 0 ,$$

(4.17)

$$\mathcal{E}_v \equiv \delta \rho_m + 3 H \left( 1 + c_m^2 \right) \delta \rho_m + 3 \left( \rho_m + P_m \right) \zeta + \frac{k^2}{a^2} \left( \rho_m + P_m \right) \left( v + \chi - a^2 \dot{E} \right) = 0 ,$$

(4.18)

$$\mathcal{E}_E \equiv 2 q_t \dot{\zeta} + 2 D_4 \dot{\phi} - B_2 \delta \phi + B_4 \delta \phi + \frac{1}{a^2} \frac{d}{dt} \left( a^3 \left( \dot{\phi} D_6 - 2 H q_t \right) \alpha \right) + \left( \rho_m + P_m \right) \left( \dot{v} - 3 H c_m^2 v \right) = 0 ,$$

(4.19)

where $k$ is a comoving wavenumber. Variations of the action \cite{411} with respect to the remaining variables $\delta \phi, \delta \rho_m, \zeta$ lead to

$$\mathcal{E}_{\delta \phi} \equiv \dot{Z} + 3 H Z + 3 D_9 \dot{\zeta} - 2 D_5 \delta \phi - D_5 \alpha - \frac{k^2}{a^2} \left( 2 D_2 \delta \phi - D_6 \alpha - D_7 \chi + B_1 \dot{\zeta} - a^2 B_4 \right) = 0 ,$$

(4.20)

$$\mathcal{E}_{\delta \rho_m} \equiv \dot{W} + 3 H W + \left( \rho_m + P_m \right) \left( \dot{v} - 3 H c_m^2 v \right) + \frac{k^2}{a^2} \left( 2 q_t \alpha + 2 q_t c_m^2 \zeta + B_3 \delta \phi \right) = 0 ,$$

(4.21)

where

$$Z \equiv 2 D_1 \delta \phi + 3 D_6 \dot{\zeta} + D_4 \alpha + \frac{k^2}{a^2} \left[ D_6 \chi - a^2 \left( \dot{D}_6 \dot{E} + D_7 \right) \right] ,$$

(4.23)

$$W \equiv 2 q_t \dot{\zeta} - D_5 \delta \phi + D_7 \delta \phi + \left( \dot{\phi} D_6 - 2 H q_t \right) \alpha + \frac{2 k^2}{3 a^2} q_t \left( \chi - a^2 \dot{E} \right) .$$

(4.24)

By combining Eq. \cite{419} with Eq. \cite{422}, we can eliminate the time derivatives $\dot{\zeta}$ and $\dot{\delta} \phi$. This manipulation leads to

$$q_t \left( \alpha + \dot{\chi} + c_m^2 \zeta + H \chi - a^2 \dot{E} - 3 a^2 \dot{H} \dot{E} \right) + \dot{q}_t \left( \chi - a^2 \dot{E} \right) + \frac{B_3}{2} \delta \phi = 0 .$$

(4.25)

The perturbation equations of motion \cite{416}-\cite{422} and \cite{425} are written in the gauge-ready form, such that they can be used for arbitrary gauge choices.

The consistency of the above perturbation equations can be confirmed in terms of Bianchi identities. For this purpose, we resort to the following properties:

$$2 \delta^2 D_2 = -2 H \left[ \dot{q}_t + H \left( 1 - c_m^2 \right) q_t \right] - \dot{\phi} \left( \dot{D}_6 + H D_6 + D_7 \right) ,$$

(4.26)

$$2 \dot{\phi} D_3 = \frac{1}{a^3} \frac{d}{dt} \left( a^3 D_5 \right) - \frac{3 H}{a^3} \frac{d}{dt} \left( a^3 D_7 \right) ,$$

(4.27)

$$D_4 = -2 \dot{\phi} D_1 - 3 H D_6 .$$

(4.28)

Using also the background Eqs. \cite{3.6}, \cite{3.7} and \cite{3.9}, we find that the perturbation equations \cite{416}-\cite{422} satisfy the two particular relations:

$$\frac{1}{a^3} \frac{d}{dt} \left( a^3 \mathcal{E}_\alpha \right) - 3 H \mathcal{E}_\zeta - \dot{\phi} \mathcal{E}_{\delta \phi} + \frac{k^2}{a^2} \mathcal{E}_\chi + 3 H \left( \rho_m + P_m \right) \mathcal{E}_{\delta \rho_m} + \mathcal{E}_v = 0 ,$$

(4.29)

$$\mathcal{E}_E - \frac{1}{a^3} \frac{d}{dt} \left( a^3 \mathcal{E}_\alpha \right) = 0 ,$$

(4.30)

which correspond to the temporal and spatial components of Bianchi identities, respectively. Thus, we have confirmed the consistency of Eqs. \cite{416}--\cite{422} with Bianchi identities.
C. Gauge issues

Since there are residual gauge degrees of freedom, we can fix the gauges in the perturbation equations derived in Sec. IV B. We discuss the issues of gauge transformations, gauge-invariant variables, and gauge choices. Let us consider a scalar infinitesimal gauge transformation from one coordinate $x^\mu = (t, x^i)$ to another coordinate $\tilde{x}^\mu = (\tilde{t}, \tilde{x}^i)$, as

$$\tilde{t} = t + \xi^0, \quad \tilde{x}^i = x^i + \delta^i j \partial_j \xi,$$

(4.31)

where $\xi^0$ and $\xi$ are scalar quantities. Then, the metric perturbations are subject to the transformations:

$$\tilde{\alpha} = \alpha - \dot{\xi}^0, \quad \tilde{\chi} = \chi + \xi^0 - a^2 \ddot{\xi}, \quad \tilde{\zeta} = \zeta - H \xi^0, \quad \tilde{E} = E - \xi,$$

(4.32)

while the perturbations associated with the scalar field and matter transform as

$$\tilde{\delta \phi} = \delta \phi - \dot{\phi} \xi^0, \quad \tilde{\delta \rho_m} = \delta \rho_m - \dot{\rho}_m \xi^0, \quad \tilde{\delta \rho} = \delta \rho - \dot{\phi} \delta \phi,$$

(4.33)

One can construct a family of gauge-invariant variables whose forms are unchanged under the transformation (4.31). From Eq. (4.32), the Bardeen gravitational potentials defined by

$$\Psi = \alpha + \frac{d}{dt} \left( \chi - a^2 \dot{E} \right), \quad \Phi = \zeta + H \left( \chi - a^2 \dot{E} \right)$$

(4.34)

are gauge-invariant. On using the properties (4.33) as well, there are the following gauge-invariant combinations:

$$\delta \phi_t = \delta \phi - \frac{\dot{\phi}}{H} \zeta, \quad \delta \phi_N = \delta \phi + \frac{\dot{\phi}}{H} \left( \chi - a^2 \dot{E} \right),$$

$$\delta \rho_t = \delta \rho_m - \frac{\dot{\rho}_m}{H} \zeta, \quad \delta \rho_N = \delta \rho_m - \frac{\dot{\rho}_m}{\phi} \delta \phi,$$

$$\delta m = \frac{\delta \rho_m}{\rho_m} + 3H \left( 1 + \frac{P_m}{\rho_m} \right) v, \quad \mathcal{R} = \zeta - \frac{H}{\phi} \delta \phi, \quad \mathcal{B} = Hv - \zeta.$$

(4.35)

The Mukhanov-Sasaki variable $\delta \phi_t$ is related to the gauge-invariant curvature perturbation $\mathcal{R}$, as

$$\mathcal{R} = -\frac{H}{\phi} \delta \phi_t.$$

(4.36)

The residual gauge degrees of freedom can be removed by fixing $\xi^0$ and $\xi$ in the transformation (4.33). There are several gauge conditions commonly used in the literature:

$$\delta \phi = 0, \quad E = 0, \quad \text{(Unitary gauge)},$$

(4.37)

$$\zeta = 0, \quad E = 0, \quad \text{(Flat gauge)},$$

(4.38)

$$\chi = 0, \quad E = 0, \quad \text{(Newtonian gauge)}.$$

(4.39)

In the so-called synchronous gauge characterized by $\alpha = 0$ and $\chi = 0$, the transformation scalar $\xi^0$ is not unambiguously fixed and hence this gauge is not chosen in the following. In Sec. V we derive no-ghost conditions and the speed of scalar perturbations in full Horndeski theories by taking the three gauge conditions (4.37)-(4.39). Independent of the gauge choices, we are dealing with the same physics.

D. Perturbation equations expressed in terms of gauge-invariant variables

We rewrite the perturbation equations of motion by using the gauge-invariant variables $\Psi, \Phi, \delta \phi_N, \mathcal{B}$ without fixing gauge conditions. The perturbations $\alpha, \zeta, \delta \phi$ can be expressed in terms of $\Psi, \Phi, \delta \phi_N$ as well as $E, \chi$ and their time derivatives. On using the background Eqs. (3.6), (3.7), (3.9) with Eqs. (4.26)-(4.28), the terms containing $E, \chi$ and their time derivatives identically vanish.
First of all, the perturbation Eqs. (4.20) and (4.22), which correspond to the equations of motion for \( \delta \phi \) and \( \Phi \) respectively, can be expressed as

\[
2D_1 \delta \ddot{\phi}_N + 2(\dot{D}_1 + 3H D_1) \delta \dot{\phi}_N + \left( M^2_{\phi} - 2D_2 \frac{k^2}{a^2} \right) \delta \phi_N + 3D_\theta \dot{\Phi} + 3(\dot{D}_6 + 3HD_6 + D_7) \Phi - (\dot{\phi} D_1 + 3HD_6) \dot{\Phi} \\
- B_1 \frac{k^2}{a^2} \dot{\Phi} - \left[ 2\phi (\dot{D}_1 + 3HD_1) + 2D_5 + 3H (\dot{D}_6 + 3HD_6 - D_7) - D_6 \frac{k^2}{a^2} \right] \Psi = 0 ,
\]

(4.40)

\[
-2\dot{q}_t \ddot{\Phi} - \left( 2\dot{q}_t + 6Hq_t + \frac{\rho_m + P_m}{H} \right) \dot{\Phi} + (2Hq_t - \dot{\phi} D_6) \dot{\Phi} + \left[ (\rho_m + P_m) \left( \frac{\dot{H}}{H^2} + 3\epsilon_m \right) - \frac{2q_t c_s^2 k^2}{3a^2} \right] \Phi \\
- \left[ \phi (\dot{D}_6 + 3HD_6 + D_7) - 2H(\dot{q}_t + 3Hq_t) + \rho_m + P_m + \frac{2q_t k^2}{3a^2} \right] \Psi - \left( \rho_m + P_m \right) \left[ \frac{\dot{K}}{H} - \left( \frac{\dot{H}}{H^2} + 3\epsilon_m \right) \frac{B}{\dot{\Phi}} \right] \\
+ D_6 \delta \ddot{\phi}_N + (\dot{D}_6 + 3HD_6 - D_7) \delta \dot{\phi}_N - \left( \dot{D}_7 + 3HD_7 + \frac{B_1 k^2}{3a^2} \right) \delta \phi_N = 0 ,
\]

(4.41)

where

\[
M^2_{\phi} \equiv -2D_3 .
\]

(4.42)

The quantity \( M^2_{\phi} \) corresponds to the mass squared of scalar-field perturbation \( \delta \phi \). In quintessence with the Lagrangian \( G_2 = X - V(\phi) \), the term \( G_2,\phi/2 \) in \( D_4 \) gives rise to the contribution \( V,\phi \delta \phi_N \) to Eq. (4.40). For the models in which \( M^2_{\phi} \) exceeds the order of \( H^2 \) in the past (like \( f(R) \) models of the late-time cosmic acceleration [70, 74]), the condition \( M^2_{\phi} > 0 \) is required for avoiding the tachyonic instability [70, 72, 207, 208]. Equations (4.40) and (4.41) can be closed by solving them for \( \dot{\phi} \delta \phi_N \) and \( \dot{\Phi} \).

The remaining perturbation equations of motion can be compactly expressed by using the following dimensionless variables:

\[
\alpha_M \equiv \frac{\dot{q}_t}{H q_t} , \quad \alpha_B \equiv -\frac{\dot{\phi} D_6}{2H q_t} , \quad \alpha_K \equiv \frac{2\dot{\phi}^2 D_1}{H^2 q_t} ,
\]

(4.43)

and

\[
\delta \varphi_N \equiv \frac{H}{\phi} \delta \phi_N , \quad h \equiv \frac{\dot{H}}{H^2} , \quad \dot{\Omega}_m \equiv \frac{\rho_m}{3H^2 q_t} , \quad w_m \equiv \frac{P_m}{\rho_m} , \quad K \equiv \frac{k}{aH} , \quad \mathcal{N} \equiv \ln a .
\]

(4.44)

The quantity \( \alpha_B \) introduced by Bellini and Sawicki [137], which is denoted as \( \alpha_B^{(BS)} \), is related to our \( \alpha_B \) according to

\[
\alpha_B^{(BS)} = -\frac{1}{2} \alpha_B .
\]

(4.45)

Our notations of \( \alpha_M \), \( \alpha_B \), and \( \alpha_K \) match with those used in Ref. [159] in the context of effective field theory of dark energy [209, 217].

From Eq. (4.28), the coefficient \( D_4 \) is written as \( D_4 = H^2 q_t (6\alpha_B - \alpha_K)/\dot{\phi} \). We also solve Eqs. (3.6) and (3.7) for \( D_5 \) and \( D_7 \) to express them in terms of \( \alpha_B \), \( \alpha_K \), and other dimensionless variables given in Eq. (4.44). When we take the time derivative of \( \delta m \) in Eq. (4.35), we use Eq. (3.9) and the relation \( c^2_m = P_m/\rho_m \). Then, Eqs. (4.16)-(4.18), (4.21) and (4.25) reduce, respectively, to

\[
6\alpha_B - \alpha_K) \delta \varphi_N' + 6(1 + \alpha_B) \Phi' + \left[ h(\alpha_K - 12\alpha_B - 6) - 9(1 + w_m)\dot{\Omega}_m + 2\alpha_B K^2 \right] \delta \varphi_N \\
+ \left[ 9(1 + w_m)\dot{\Omega}_m + 2K^2 \right] \Phi + (\alpha_K - 12\alpha_B - 6)\Psi - 3\dot{\Omega}_m \delta_m + 9(1 + w_m)\dot{\Omega}_m B = 0 ,
\]

(4.46)

\[
\alpha_B \delta \varphi_N' + \Phi' - \left[ h(1 + \alpha_B) + 3(1 + w_m)\dot{\Omega}_m \right] \delta \varphi_N - (1 + \alpha_B)\Psi + \frac{3}{2}(1 + w_m)\dot{\Omega}_m (\Phi + B) = 0 ,
\]

(4.47)

\[
\delta m' - 3(1 + w_m)B' + (1 + w_m)K^2 (\Phi + B) + 3(c^2_m - w_m) \delta m = 0 ,
\]

(4.48)

\[
\Phi' + B' - h(\Phi + B) - \Psi - \frac{c^2_m}{1 + w_m} \delta m = 0 ,
\]

(4.49)

\[
\Psi + c^2_m (1 - c^2_i + \alpha_M) \delta \varphi_N = 0 ,
\]

(4.50)

\[\text{The model } f(R) = R - \beta/R^n \quad (n > 0) \text{ proposed in Refs. [60, 69] leads to } M^2_{\phi} < 0 \text{, so it is ruled out by the tachyonic instability of scalar perturbations.}\]
where a prime represents a derivative with respect to $\mathcal{N}$. Since $\mathcal{R} = \Phi - \delta \varphi_N$, Eq. (4.50) can be also expressed as

$$\Psi + (1 + \alpha_M) \Phi + (c^2 - 1 - \alpha_M) \mathcal{R} = 0.$$  \hspace{1cm} (4.51)

The relation (4.51) shows that the time-dependence of $q_0$ and the deviation of $c^2$ from 1 lead to the difference between $-\Psi$ and $\Phi$. Under the bound (1.1), the gravitational slip ($-\Psi \neq \Phi$) mostly arises from the time variation of $q_0$.

We solve Eqs. (4.46)-(4.49) for $\Phi', \delta \varphi_N, \delta_m', B'$ by using Eq. (4.50) to eliminate the perturbation $\Psi$. Then, we obtain

$$\Phi' = -(c^2 - b_2 + 4\alpha_B b_1 K^2) \Phi - (1 - c^2 + \alpha_M - h + b_2 + 4\alpha_B b_1 K^2) \delta \varphi_N + 6\alpha_B b_1 \tilde{\Omega}_m \delta_m + b_2 B,$$  \hspace{1cm} (4.52)

$$\delta \varphi_N' = -(c^2 + b_3 - 4\alpha_B b_1 K^2) \Phi - (1 - c^2 + \alpha_M - h - b_3 - 4\alpha_B b_1 K^2) \delta \varphi_N + 6\alpha_B \tilde{\Omega}_m \delta_m - b_3 B,$$  \hspace{1cm} (4.53)

$$\delta_m' = \frac{3(1 + w_m) \mathcal{K}^2}{2\alpha_B b_1 - \frac{1}{3}}, \hspace{1cm} B' = (h - b_2 + 4\alpha_B b_1 K^2) \Phi - (h - b_2 - 4\alpha_B b_1 K^2) \delta \varphi_N + \left(\frac{c^2}{1 + w_m} - 6\alpha_B b_1 \tilde{\Omega}_m\right) \delta_m + (h - b_2) B,$$  \hspace{1cm} (4.54)

where $b_1, b_2, b_3$ are dimensionless quantities defined by

$$b_1 \equiv \frac{1}{2(\alpha_B + 6\alpha_G)} , \quad b_2 \equiv \frac{3}{2}(1 + w_m)(12\alpha_B^2 b_1 - 1) \tilde{\Omega}_m, \quad b_3 \equiv 18(1 + w_m)\alpha_B b_1 \tilde{\Omega}_m .$$  \hspace{1cm} (4.56)

The evolution of $\Phi, \delta \varphi_N, \delta_m, B$, and $\Psi$ is known by integrating Eqs. (4.52)-(4.55) with (4.50) together with the background Eqs. (3.5), (3.9), (3.11), (3.12). In GR without the scalar field $\varphi$, the viable models of late-time cosmic acceleration are usually constructed to recover the behavior close to GR in the asymptotic past. In such cases, the initial conditions of perturbations can be chosen to satisfy $\Phi' \simeq 0$, $\delta \varphi_N' \simeq 0$, and $\delta_m' \simeq \delta_m$ in the early matter era. Hence $\Phi, \delta \varphi_N, B$ and $\Psi$ can be expressed in terms of $\delta_m$ by using Eqs. (4.52)-(4.54) and (4.50).

The choice of these initial conditions amounts to neglecting the oscillating mode of $\delta \varphi_N$ induced by a heavy mass term $M_\phi$ larger than $H$. The large-field mass arises for $f(R)$ dark energy models in the asymptotic past, so in such cases, there is a fine-tuning problem of initial conditions for avoiding the dominance of the oscillating mode over the mode induced by $\delta_m$. This problem does not arise for dark energy models in which $M_\phi$ does not exceed the order $H$ in the past (like k-essence and Galileons).

V. STABILITY CONDITIONS IN THE SMALL-SCALE LIMIT

By using the scalar perturbation equations of motion obtained in Sec. IV, we derive conditions for the absence of ghost and Laplacian instabilities in the small-scale limit. We choose the three different gauges (4.37)-(4.39) and show that these stability conditions are independent of the choice of gauges.

A. Unitary gauge

We begin with the unitary gauge (4.37), under which the dynamical perturbations are given by $\mathcal{R} = \zeta$ and $\delta \rho_\alpha = \delta \rho_m$. We solve Eqs. (4.16), (4.17), (4.18) for $\alpha, \chi, v$, and substitute them into Eq. (4.11). After the integration by parts, the quadratic action in Fourier space is expressed in the form

$$S^{(2)} = \int dt d^3x a^3 \left( \dot{\mathcal{K}} \mathcal{K} - \frac{k^2}{a^2} \dot{\mathcal{K}} \mathcal{K} - \dot{\mathcal{M}} \mathcal{M} - \dot{\mathcal{B}} \mathcal{B} \right),$$  \hspace{1cm} (5.1)

where $\mathcal{K}, \mathcal{G}, \mathcal{M}, \mathcal{B}$ are $2 \times 2$ matrices, and

$$\dot{\mathcal{X}} = (\mathcal{R}, \delta \rho_\alpha / k).$$  \hspace{1cm} (5.2)

The leading-order contributions to $\mathcal{M}$ and $\mathcal{B}$ are of order $k^0$. 

In the small-scale limit \((k \to \infty)\), the non-vanishing components of matrices \(K\) and \(G\) are give by

\[
K_{11}^{(u)} = \frac{\dot{q}_s^2 q_s}{(2Hq_t - \phi D_6)^2}, \quad K_{22}^{(u)} = \frac{a^2}{2(\rho_m + P_m)},
\]

\[
G_{11}^{(u)} = -q_s c_s^2 - \frac{\rho_m + P_m}{2Hq_t - \phi D_6} F_1 + \frac{1}{a} \frac{d}{dt} (a F_1), \quad G_{22}^{(u)} = \frac{a^2 c_m^2}{2(\rho_m + P_m)},
\]

where \(q_s\) is defined by Eq. (3.13), and

\[
F_1 = \frac{2q_t^2}{2Hq_t - \phi D_6}.
\]

Since there are no off-diagonal components in \(K\) and \(G\), the matter perturbation \(\delta \rho_u\) is decoupled from the other field \(\mathcal{R}\). For the matter sector, the conditions for the absence of ghost and Laplacian instabilities correspond to \(\rho_m + P_m > 0\) and \(c_m^2 > 0\). For the perturbation \(\mathcal{R}\), the ghost does not arise for

\[
q_s^{(u)} \equiv K_{11}^{(u)} = \frac{\dot{q}_s^2 q_s}{(2Hq_t - \phi D_6)^2} > 0.
\]

Since the absence of tensor ghosts requires that \(q_t > 0\), the condition \((5.5)\) translates to

\[
q_s > 0,
\]

under which the denominators of Eqs. (3.11) and (3.12) do not cross 0.

Taking the small-scale limit in Eq. (5.1), we obtain the dispersion relation of the form

\[
\det (c_s^2 K - G) = 0,
\]

where \(c_s\) is the propagation speed of scalar perturbations. One of the solutions is the matter propagation speed squared \(c_m^2\), while the other solution is \(c_s^2 = G_{11}^{(u)}/q_s^{(u)}\). On using Eqs. (3.6), (4.15), and (4.26) to solve for \(\ddot{\phi}, \dot{q}_t, \) and \(D_6\), the latter solution can be simply expressed as

\[
c_s^2 = \frac{G_{11}^{(u)}}{q_s^{(u)}} = -\frac{c_m^2 D_0^2 + 2B_1 D_6 + 4q_t D_2}{q_s}.
\]

The small-scale Laplacian instability can be avoided for

\[
c_s^2 > 0.
\]

In sections after VII we will compute \(c_s^2\) for concrete dark energy models in the framework of Horndeski theories.

### B. Flat gauge

In the flat gauge \((4.38)\), the dynamical scalar perturbations are given by the matrix \(\dot{\delta \chi} = (\delta \dot{\phi}_t, \delta \dot{\phi}_\rho/k)\). Solving Eqs. (4.16), (4.17), (4.18) for \(\alpha, \chi, v\) and substituting them into Eq. (4.11), the second-order action reduces to the same form as Eq. (5.1) after the integration by parts. In the small-scale limit, the matrix components \(K_{22}^{(f)}\) and \(G_{22}^{(f)}\) are identical to those in the unitary gauge. The other nonvanishing matrix components are given by

\[
q_s^{(f)} = K_{11}^{(f)} = \frac{H^2 q_t q_s}{(2Hq_t - \phi D_6)^2}, \quad G_{11}^{(f)} = -D_2 + \frac{D_6 D_7 - (\rho_m + P_m) F_2}{2Hq_t - \phi D_6} + \frac{1}{a} \frac{d}{dt} (a F_2),
\]

where

\[
F_2 = \frac{D_0^2}{2(2Hq_t - \phi D_6)}.
\]

Note that we eliminated the term \(D_4\) by using the relation (4.28). As long as there is no tensor ghost, the condition for avoiding the scalar ghost again corresponds to \(q_s > 0\).

On using Eqs. (3.6) and (4.26), it follows that

\[
\frac{q_s^{(f)}}{q_s^{(u)}} = \frac{G_{11}^{(f)}}{G_{11}^{(u)}} = \frac{H^2}{\phi^2}.
\]

Hence the scalar propagation speed squared \(c_s^2 = G_{11}^{(f)}/q_s^{(f)}\) in the flat gauge is equivalent to that in the unitary gauge.
C. Newtonian gauge

Let us finally consider the Newtonian gauge (4.39), under which $\Phi = \zeta$, $\delta\phi_N = \delta\phi$, and $\delta\rho_N = \delta\rho_m$. We first solve Eqs. (4.16) and (4.18) for nondynamical perturbations $\alpha$ and $v$, and substitute them into Eq. (4.11). Then, the terms proportional to $\Phi^2$, $\delta\phi_N^2$, and $\delta\phi\dot{\phi}_N$ appear in the second-order action. Apparently, this looks as if the two fields $\Phi$ and $\delta\phi_N$ were dynamical, but the second-order action in the small-scale limit is factorized as

$$S^{(2)}_s = \int dt d^3x a^3 \left[ \frac{3q_s}{4\{3H^2q_t - \dot{\phi}(D_1(\phi + 3HD_0))\}} \left(\dot{\phi}\Phi - H\delta\phi_N\right)^2 + \frac{a^2}{2(\rho_m + P_m)} \delta\rho_N^2 + \cdots \right], \quad (5.13)$$

where the abbreviation corresponds to terms without containing the product of first-order time derivatives of perturbed quantities. If we consider the combination

$$\mathcal{R} = \Phi - \frac{H}{\dot{\phi}}\delta\phi_N, \quad \text{or} \quad \delta\phi_t = \delta\phi_N - \frac{\dot{\phi}}{H}\Phi, \quad (5.14)$$

then the action (5.13) can be expressed in terms of $\dot{R}^2$ or $\dot{\delta}\phi_t^2$ in addition to $\dot{\delta}\rho_N^2$, without any other products of first-order time derivatives of perturbations. In other words, $\mathcal{R}$ or $\delta\phi_t$ corresponds to the dynamical perturbation besides the matter perturbation $\delta\rho_N$.

If we choose $\mathcal{R}$ as a dynamical perturbation, the terms in the abbreviation of Eq. (5.13) contain nondynamical variables like $\delta\phi_N$ and $\delta\phi_N$, which can be eliminated by using Eqs. (4.17) and (4.25). After the integration by parts, the second-order action (5.13) reduces to the same form as Eq. (5.1) with dynamical perturbations $\dot{X}^t = (\mathcal{R}, \delta\rho_N/k)$. In the small-scale limit, the components of matrices $K$ and $G$ are identical to those in the unitary gauge given in Eq. (5.3).

If the combination $\delta\phi_t$ is chosen as a dynamical perturbation besides $\delta\rho_N$, the nondynamical variables like $\Phi$ and $\Phi$ can be eliminated from the action (5.13) by using Eqs. (4.17) and (4.25). In the small-scale limit, the resulting second-order action is expressed in the form (5.1) with the same matrix components of $K$ and $G$ as those in the flat gauge.

Thus, the conditions for the absence of ghost and Laplacian instabilities in the Newtonian gauge are equivalent to those in the unitary and flat gauges.

D. Summary

We have shown that the quantities $q_s^{(u)}$ and $q_s^{(f)}$ contain the common term $q_s$. The positivity of this term, i.e.,

$$q_s = 4D_1q_t + 3D_0^2 > 0, \quad (5.15)$$

is the condition for the absence of scalar ghosts for any gauge choices. The small-scale Laplacian instability can be avoided as long as the gauge-invariant sound speed squared is positive, i.e.,

$$c_s^2 = \frac{G^{(u)}_{11}}{q_s^{(u)}} = \frac{G^{(f)}_{11}}{q_s^{(f)}} = -\frac{c_t^2D_0^2 + 2B_1D_6 + 4q_tD_2}{q_s} \geq 0. \quad (5.16)$$

Depending on the problems at hand, we can choose most convenient gauges.

We caution that the perturbations $\delta\phi_t$ and $\delta\rho_t$ contain the term $\dot{H}$ in their denominators, so they are not well defined at $H = 0$. In this case, it apparently looks possible to regulate the combination $\zeta/H$ by choosing the flat gauge ($\zeta = 0$). However, the problem arises for the gravitational potential $\Phi$ and the curvature perturbation $\mathcal{R}$, both of which exactly vanish at $H = 0$ for the gauge choice $\zeta = 0$. This suggests that, for the cosmological evolution crossing $H = 0$ (such as the bouncing cosmology), it is not appropriate to choose the flat gauge [152]. In the bouncing Universe, the quantity $q_s^{(f)}$ vanishes at the bounce ($H = 0$). Apparently, this looks the appearance of a strong coupling problem, but it simply comes from the inappropriate gauge choice for this problem. The real strong coupling problem arises when the quantity $q_s$ in Eq. (5.15) approaches 0, in which case the background Eqs. (3.11) and (3.12) exhibit divergences. In the unitary gauge, $\mathcal{R}$ and $\delta\rho_t$ corresponds to dynamical perturbations, but they are not well defined at $\phi = 0$. For this gauge choice, the problems similar to those mentioned above arise for the case in which $\phi$ crosses 0. If neither $H$ nor $\phi$ vanishes during the cosmological evolution, we can choose any gauge among (4.37)-(4.39).
VI. GROWTH OF LARGE-SCALE STRUCTURES

The dark energy EOS $w_{\text{DE}}$ introduced in Sec. III A is a key quantity to distinguish between different dark energy models at the background level. In modified gravity theories, the gravitational coupling with the matter sector is different from that in GR. In this case, the growth rate of matter perturbations and the evolution of gravitational potentials are subject to modifications. Then, we can distinguish between different models of the late-time cosmic acceleration from the observations of large-scale structures, weak lensing, and CMB.

In this section, we study observables associated with the evolution of linear cosmological perturbations in full Horndeski theories given by the action (2.20). By using the linear perturbation equations of motion derived in Sec. IV, it is possible to estimate the effective gravitational coupling with matter perturbations on scales relevant to the growth of large-scale structures in the gauge-invariant way. We also discuss the gravitational coupling around local objects screened by nonlinear interactions.

A. Observable quantities

Since we are interested in the evolution of perturbations after the end of the radiation-dominated epoch, we consider nonrelativistic matter satisfying

$$P_m = 0, \quad \dot{c}_m^2 = 0,$$

for the action $\mathcal{S}_m$. Taking the time derivative of Eq. (4.18) and using Eq. (4.21), the gauge-invariant matter density contrast $\delta_m = \delta \rho_m / \rho_m + 3Hv$ obeys

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi = 3\left(\dot{B} + 2HB\right),$$

(6.2)

where $\Psi$ and $B$ are gauge-invariant perturbations defined in Eqs. (4.34) and (4.35). We relate the gravitational potential $\Psi$ with the density contrast $\delta_m$ through the modified Poisson equation

$$\frac{k^2}{a^2}\Psi = -4\pi G\mu \rho_m \delta_m,$$

(6.3)

where

$$\mu = \frac{G_{\text{eff}}}{G}.$$

(6.4)

The quantity $\mu$ characterizes the ratio between the effective gravitational coupling $G_{\text{eff}}$ and the Newton constant $G = 1/(8\pi M_{\text{pl}}^2)$. In Sec. VI B, we will derive an explicit form of $\mu$ in full Horndeski theories by using a so-called quasi-static approximation for perturbations deep inside the sound horizon.

We also define the gravitational slip parameter $\eta$ and the effective gravitational potential $\psi_{\text{eff}}$ relevant to the light bending in weak lensing and CMB observations, as [218, 219]

$$\eta = -\frac{\Phi}{\Psi}, \quad \psi_{\text{eff}} = \Phi - \Psi.$$

(6.5)

From Eqs. (6.3) and (6.5), it follows that

$$\frac{k^2}{a^2}\psi_{\text{eff}} = 8\pi G\Sigma \rho_m \delta_m,$$

(6.6)

where

$$\Sigma = \frac{1 + \eta G_{\text{eff}}}{2G}.$$

(6.7)

The dimensional ratios $\mu$ and $\Sigma$ are two key quantities characterizing the linear growth of $\delta_m$ as well as $\Psi, \psi_{\text{eff}}$. 


To confront dark energy models in the framework of Horndeski theories with the observations of large-scale structures and weak lensing, we would like to derive analytic expressions of $\mu$ and $\Sigma$ for the perturbations deep inside the sound horizon. In doing so, we exploit the scalar perturbation equations of motion expressed in terms of gauge-invariant variables. As we will see below, we do not need to fix gauges for the derivation of $\mu$ and $\Sigma$.

In Sec. 17 we showed that the curvature perturbation $R$ (or equivalently $\delta\phi = -\dot{\phi}/H$) and the matter perturbation are the dynamical degrees of freedom. Let us derive the closed-form equation of motion for $R$ in the small-scale limit. First of all, we replace $\Phi$ with $R$ and weak lensing, we would like to derive analytic expressions of $\Sigma$ for the perturbations deep inside the sound horizon. As we will see below, we do not need to fix gauges for the derivation of $\mu$ and $\Sigma$.

In the following, we apply the quasi-static approximation to the scalar perturbation equations of motion. First of all, we replace $\Phi$ with $R$ and weak lensing, we would like to derive analytic expressions of $\Sigma$ for the perturbations deep inside the sound horizon. As we will see below, we do not need to fix gauges for the derivation of $\mu$ and $\Sigma$.

B. Quasi-static approximation deep inside the sound horizon

In the so-called quasi-static approximation, the dominant contributions to the perturbation equations are regarded as those containing the terms $\delta_m$ and $k^2/a^2$, without any time derivatives of metric perturbations [14, 76, 93, 163, 220, 221]. Provided that $c_s^2$ is not very close to zero, the quasi-static approximation is sufficiently accurate for sub-horizon perturbations in dark energy models where the mass $M_\phi$ associated with the field perturbation $\delta\phi_N$ is at most of order $H$. This is generally the case for a nearly massless scalar field like quintessence, k-essence, and Galileons [108]. One exception is $f(R)$ models of the late-time cosmic acceleration, in which case the scalar mass $M_\phi$ becomes larger than $H$ as we go back to the past. Then, the oscillating mode of field perturbations cannot be ignored relative to the matter-induced mode in the asymptotic past [71, 72]. In other words, in $f(R)$ gravity, we need a fine tuning for initial conditions of perturbations such that $|\rho_m| \gg |\rho_h|$.

In the following, we apply the quasi-static approximation to the scalar perturbation equations of motion. First of all, we remind that there is the relation (4.50) among two gravitational potentials $\Psi$ and $\Phi$. Applying the quasi-static approximation to Eq. (4.46), it follows that

$$2\dot{q}_t k^2 \frac{\alpha_B \delta\phi_N}{a^2} \simeq \rho_m \delta_m.$$  \hfill (6.10)

In Eq. (4.40), there exists the mass term $M_\phi^2 \delta\phi_N$ of the scalar-field perturbation. In viable models of the late-time cosmic acceleration based on $f(R)$ gravity [70, 74] and BD theories with the field potential [77, 78], the mass of $\delta\phi_N$ tends to be large in the early cosmological epoch. Taking into account its contribution and using the quasi-static approximation in Eq. (4.40), we obtain

$$M_\phi^2 \delta\phi_N + \frac{k^2}{a^2} (D_6 \Phi - B_1 \Phi - 2D_2 \delta\phi_N) \simeq 0.$$  \hfill (6.11)

From Eq. (5.8), the coefficient $D_2$ can be expressed as $D_2 = -(q_t c_s^2 + c_s^2 D_6^2 + 2B_1 D_6)/(4q_t)$. We also note that $B_1$ defined in Eq. (4.15) is written as $B_1 = 2Hq_t(1 - c_t^2 + \alpha_M)/\phi$. Then, Eq. (6.11) can be expressed as

$$\frac{k^2}{a^2} \left[ \alpha_B \Phi + (1 - c_t^2 + \alpha_M) \Phi \right] - \frac{k^2}{a^2} \frac{\alpha_B \left( c_t^2 (2 + \alpha_B) - 2(1 + \alpha_M) \right) + \phi^2 \left( c_s^2 k^2 q_s + 2M_\phi^2 a^2 q_t \right)}{4a^2 H^2 q_t} \delta\phi_N \simeq 0,$$  \hfill (6.12)

where we used $D_6 = -2Hq_t\alpha_B/\phi$. We also note that $B_1$ defined in Eq. (4.15) is written as $B_1 = 2Hq_t(1 - c_t^2 + \alpha_M)/\phi$. Then, Eq. (6.11) can be expressed as

$$\frac{k^2}{a^2} \left[ \alpha_B \Phi + (1 - c_t^2 + \alpha_M) \Phi \right] - \frac{k^2}{a^2} \frac{\alpha_B \left( c_t^2 (2 + \alpha_B) - 2(1 + \alpha_M) \right) + \phi^2 \left( c_s^2 k^2 q_s + 2M_\phi^2 a^2 q_t \right)}{4a^2 H^2 q_t} \delta\phi_N \simeq 0,$$  \hfill (6.12)

where we used $D_6 = -2Hq_t\alpha_B/\phi$. We also note that $B_1$ defined in Eq. (4.15) is written as $B_1 = 2Hq_t(1 - c_t^2 + \alpha_M)/\phi$. Then, Eq. (6.11) can be expressed as

$$\frac{k^2}{a^2} \left[ \alpha_B \Phi + (1 - c_t^2 + \alpha_M) \Phi \right] - \frac{k^2}{a^2} \frac{\alpha_B \left( c_t^2 (2 + \alpha_B) - 2(1 + \alpha_M) \right) + \phi^2 \left( c_s^2 k^2 q_s + 2M_\phi^2 a^2 q_t \right)}{4a^2 H^2 q_t} \delta\phi_N \simeq 0,$$  \hfill (6.12)

where we used $D_6 = -2Hq_t\alpha_B/\phi$.
Solving Eqs. (6.10) and (6.12) for $\Psi, \Phi$, and $\delta \varphi_N$, we obtain

$$\Psi = -\frac{1}{2\Delta_2} \left( \Delta_1^2 + \frac{c^2_1 \Delta_2}{c^2_1 q_t} \right) \frac{\alpha_B}{k^2 \rho_m \delta_m}, \quad \Phi = \frac{1}{2\Delta_2} \left( \frac{\alpha_B \Delta_1 + \Delta_2}{\eta} \right) \frac{\alpha_B}{k^2 \rho_m \delta_m}, \quad \delta \varphi_N = -\frac{\Delta_1}{2\Delta_2} \frac{a^2}{k^2 \rho_m \delta_m},$$

(6.13)

where

$$\Delta_1 \equiv c_1^2(1 + \alpha_B) - 1 - \alpha_M, \quad \Delta_2 \equiv \frac{c_2^2 q_t c_2^2}{4H^2 q_t} \left( 1 + \frac{2a^2M^2q_t}{c^2_2 k^2 q_s} \right).$$

(6.14)

Then the quantities $\mu, \eta, \Sigma$ defined by Eqs. (6.4), (6.5), and (6.7) reduce to

$$\mu = \frac{c^2_1}{8\pi G q_t} \left( 1 + \frac{q_t \Delta_1}{c^2_1 \Delta_2} \right), \quad \eta = \frac{q_t \alpha_B \Delta_1 + \Delta_2}{q_t \Delta_1^2 + c^2_1 \Delta_2}, \quad \Sigma = \frac{1}{16\pi G q_t} \left[ 1 + \frac{q_t(\alpha_B + \Delta_1)\Delta_1}{(1 + c^2_2)\Delta_2} \right].$$

(6.15), (6.16), (6.17)

The effective gravitational coupling $\mu = G_{\text{eff}} / G$ is composed of the following two contributions:

$$\mu_t = \frac{c^2_1}{8\pi G q_t}, \quad \xi_s = \frac{q_t \Delta_1^2}{c^2_1 \Delta_2}. \quad \mu$$

(6.18)

The term $\mu_t$ arises from the modification of gravity in the tensor sector. Under the no-ghost and stability conditions $q_t > 0$ and $c^2_1 > 0$, we have $\mu_t > 0$. The term $\xi_s$ quantifies the interaction between the scalar field $\phi$ and matter. For the field mass squared $M^2_\phi > 0$, the quantity $\xi_s$ is positive under the no-ghost conditions $q_t > 0, q_s > 0$ and the stability conditions $c^2_1 > 0, c^2_s > 0$. Thus, the scalar-matter interaction in full Horndeski theories is attractive under theoretically consistent conditions [120][222]. Since $\mu, \eta, \Sigma$ given in Eqs. (6.15), (6.16), (6.17) have been derived without fixing any gauge conditions, they are gauge-invariant quantities for the modes deep inside the sound horizon.

The modification to the effective gravitational coupling manifests itself in the “massless” regime in which the condition $M^2_\phi \lesssim (c^2_1 k^2 / a^2)(q_t/q_s)$ holds. For $M^2_\phi \lesssim H^2$ with $q_s/q_t = \mathcal{O}(1)$, this condition is satisfied for perturbations inside the sound horizon ($H^2 \lesssim c^2_1 k^2 / a^2$). Taking the massless limit $M^2_\phi \to 0$ in $\Delta_2$, the $k$ dependence disappears in the expressions of $\mu, \eta, \Sigma$. In Ref. [120], the present authors derived these quantities by choosing the unitary gauge. Since there is the relation $q_s^{(a)} / q_s = \dot{\phi}^2 / [4H^2 q_t (1 + \alpha_B)^2]$ from Eq. (5.5), the massless limits of Eqs. (6.15) and (6.16) yield

$$\mu = \frac{c^2_1}{8\pi G q_t} \left[ 1 + \frac{q_t \Delta_1^2}{q_s^{(a)} c^2_1 c^2_2 (1 + \alpha_B)^2} \right], \quad \eta = \frac{q_t \alpha_B \Delta_1 + q_s^{(a)} c_1^2 (1 + \alpha_B)^2}{q_t \Delta_1^2 + q_s^{(a)} c_1^2 c_2^2 (1 + \alpha_B)^2},$$

(6.19)

which coincide with Eqs. (3.33) and (3.34) of Ref. [120], respectively, after replacing $q_t$ with $2Q_t$.

The GW170817 event [141] placed the tight bound (1.1) on the tensor propagation speed $c_t$. Setting $c^2_t = 1$ in Eqs. (6.15)-(6.17), it follows that

$$\mu = \frac{1}{8\pi G q_t} \left[ 1 + \frac{q_t(\alpha_B - \alpha_M)^2}{\Delta_2} \right], \quad \eta = 1 + \frac{q_t \alpha_M(\alpha_B - \alpha_M)}{\Delta_2 + q_t(\alpha_B - \alpha_M)^2}, \quad \Sigma = \frac{1}{8\pi G q_t} \left[ 1 + \frac{q_t(2\alpha_B - \alpha_M)(\alpha_B - \alpha_M)}{2\Delta_2} \right],$$

(6.20), (6.21), (6.22)

where we used the property $\Delta_1 = \alpha_B - \alpha_M$. The nonvanishing values of $\alpha_B$ and $\alpha_M$ satisfying $\alpha_B \neq \alpha_M$ lead to the enhancement of the gravitational interaction with matter, such that $\mu > 1/(8\pi G q_t)$. The gravitational slip ($\eta \neq 1$) arises for the theories with $\alpha_M \neq 0$ and $\alpha_B \neq \alpha_M$. Under the no-ghost and stability conditions, the quantity $\Sigma$ is larger than $1/(8\pi G q_t)$ for $(2\alpha_B - \alpha_M)(\alpha_B - \alpha_M) > 0$.

From Eq. (4.49), the perturbation $\mathcal{B}$ is at most of order $\Psi$. Then the terms on the right-hand side of Eq. (6.2) is at most of order $H^2\Psi$, so they can be ignored compared to those on its left-hand side for the perturbations deep inside the Hubble radius. On using Eq. (6.3), the matter density contrast approximately obeys

$$\delta_m + 2H\delta_m - 4\pi G \mu \rho_m \delta_m \simeq 0.$$

(6.23)
For a given model the quantity $\mu$ is known from Eq. [6.20], so we can integrate Eq. [6.23] to solve for $\delta_m$. The matter power spectrum and the growth rate of matter perturbations can be constrained from the measurements of galaxy clusterings [223, 224] and redshift-space distortions [225, 226], so it is possible to distinguish between different dark energy models. The evolution of two gravitational potentials is also determined from Eqs. [6.3] and [6.6]. This information can be used to place further constraints on dark energy models from the observations of CMB [71, 78] and weak lensing [231, 232].

C. Screened gravitational coupling

We have shown that Horndeski theories generally give rise to modifications to the gravitational interaction for scales relevant to the growth of large-scale structures and weak lensing. In local regions of the Universe, the fifth force induced by the scalar-matter interaction needs to be small for the consistency with solar-system tests of gravity. There are several mechanisms to suppress the propagation of fifth forces in regions of the high density: (i) chameleon mechanism [84], and (ii) Vainshtein mechanism [107].

The chameleon mechanism can be at work for a scalar potential whose mass is different depending on the matter densities in the surrounding environment. If the effective scalar mass is sufficiently large in regions of the high density, the coupling between the field and matter can be suppressed by having a thin shell inside a spherically symmetric body (see Ref. [85] for detail). In $f(R)$ gravity or BD theories with a scalar potential, it is possible to design functional forms of $f(R)$ or the potential $V(\phi)$ to realize the large mass squared $M_\phi^2$ for increasing $R$, while realizing the late-time cosmic acceleration by the potential of a light scalar field [70–73, 77, 78]. Cosmologically, $M_\phi^2$ decreases in time, so there is a transition from the “massive” regime $M_\phi^2 \gg (c_s^2 k^2/a^2)(q_s/q_i)$ to the “massless” regime $M_\phi^2 \ll (c_s^2 k^2/a^2)(q_s/q_i)$. In the massive limit $M_\phi^2 \to \infty$, we have $\xi_s \to 0$ and $\mu \to c_s^2/(8\pi G q_i)$, by reflecting the fact that the scalar degree of freedom does not propagate. This is the regime in which the chameleon mechanism is at work in the local region whose matter density $\rho_m$ is much higher than today’s critical cosmological density $\rho_c$.

The Vainshtein mechanism operates around local sources in the presence of nonlinear scalar derivative interactions. One of the representative examples is the cubic Galileon given by the Lagrangian $X \Box \phi$. This nonlinear interaction leads to the decoupling of the field $\phi$ from matter within a radius $r_V$, called the Vainshtein radius (see Refs. [99, 100, 106] for detail). For the Sun, the Vainshtein radius can be of order $10^{20}$ cm, which is much larger than the solar-system scale. On scales relevant to the growth of large-scale structures ($\gtrsim 10^{24}$ cm), the cubic Galileon modifies the effective gravitational coupling $\mu$, while, in the solar system, the scalar-matter interaction term $\xi_s = q_s \Delta_1^4/(c_s^2 \Delta_2)$ in Eq. [6.15] is much smaller than 1.

In the rest of this section, we focus the case in which $c_s^2$ is close to 1, so that $\mu, \eta, \Sigma$ are given by Eq. [6.20]–[6.22] on scales relevant to the growth of large-scale structures. If the screening of fifth forces occurs efficiently around local sources, the screened gravitational coupling $G_{sc}$ is given by [233]

$$G_{sc}(t) = \frac{1}{8\pi q_i(t)} = \frac{1}{16\pi} \left[ G_4 - \frac{1}{2} \phi^2 G_{4,X} + \frac{1}{2} \phi^2 G_{5,\phi} - \frac{1}{2} H \phi^3 G_{5,X} \right]^{-1}. \quad (6.24)$$

If we strictly demand that $c_s^2 = 1$, we have $G_{sc}(t) = 1/[16\pi G_4(\phi)]$. Since we live in a screened environment, today’s value of Eq. [6.24], i.e., $G_{sc}(t_0) = 1/(8\pi q_i(t_0))$ should be close to the Newton gravitational constant $G$, such that

$$q_i(t_0) \simeq \frac{1}{8\pi G}. \quad (6.25)$$

The quantity $\mu_t$ reduces to

$$\mu_t(t) \simeq \frac{q_i(t_0)}{q_i(t)}. \quad (6.26)$$

If $q_i(t) < q_i(t_0)$ in the past, we have $\mu_t > 1$ and hence $\mu > 1$ under the conditions $q_s > 0, q_i > 0, c_s^2 > 0$, and $M_\phi^2 > 0$. In this case, the effective gravitational coupling $G_{eff}$ is larger than $G$ for scales relevant to the linear growth of large-scale structures. In the opposite case, $q_i(t) > q_i(t_0)$, $\mu_t$ is smaller than 1. This is the necessary condition for realizing $G_{eff} < G$, but it is not sufficient due to the existence of the positive term $\xi_s = q_s (\alpha_B - \alpha_M)^2/\Delta_2$ in Eq. [6.20]. In other words, even if $q_i(t) > q_i(t_0)$ in the past, the scalar-matter interaction $\xi_s$ can lead to $G_{eff}$ larger than $G$.

There are bounds on the variation of the gravitational coupling constrained from Lunar Laser Ranging experiments [187, 224]. In the screened environment, the experimental bound corresponds to $|G_{sc}/G_{sc}| < 0.02 H_0$, where $H_0$ is today’s value of the Hubble parameter. On using Eq. [6.24], this bound translates to

$$|\alpha_M(t_0)| < 0.02. \quad (6.27)$$
Assuming that the quantity $\alpha_M$ is nearly constant around today, we obtain $q_\mu(t) = q_\mu(t_0)e^{(t-t_0)H_0\alpha_M(t_0)}$ and hence $\mu(t) \simeq e^{-(t-t_0)H_0\alpha_M(t_0)}$. The difference of $\mu(t)$ from 1 over the cosmological time scale ($t - t_0 \sim 1/H_0$) is of order $\alpha_M(t_0)$, so the modification to $\mu = G_{\text{eff}}/G$ arising from the tensor contribution $\mu(t)$ is suppressed under the bound \((6.27)\). The scalar-matter contribution $\xi_\mu = q_\mu(\alpha_B - \alpha_M)^2/\Delta_2$ is the main source for modifying the gravitational interaction on scales relevant to the growth of large-scale structures. The two quantities $\alpha_B$ and $\alpha_M$ play important roles for the evolution of gravitational potentials $\Psi$ and $\Phi$.

\section{Classification of surviving Horndeski theories in terms of $\mu$ and $\Sigma$}

In the following, we focus on Horndeski theories given by the Lagrangian \((6.31)\), i.e., those satisfying the condition $c_\parallel^2 = 1$. Then, the quantities $\alpha_B$ and $\alpha_M$ reduce, respectively, to

$$\alpha_B = \frac{2\dot{\phi}G_{4,\phi} + \dot{\phi}^3G_{3,\phi}}{4HG_4}, \quad \alpha_M = \frac{\dot{\phi}G_{4,\phi}}{HG_4},$$

\(\text{(6.28)}\)

with $q_\mu = 2G_4 > 0$. Depending on the values of $\alpha_B$ and $\alpha_M$, the surviving theories can be classified into the following four classes.

- **(A) $G_2 = G_2(\phi, X)$, $G_3 = 0$, $G_4 = M^2_{\text{pl}}/2$.**

  This class accommodates both quintessence and k-essence. Since $\alpha_B = 0 = \alpha_M$ in Eqs. \((6.20)-(6.22)\), it follows that

  $$\mu = 1, \quad \eta = 1, \quad \Sigma = 1.$$  

  \(\text{(6.29)}\)

  Hence $G_{\text{eff}}$ is equivalent to $G$ without the gravitational slip.

- **(B) $G_2 = G_2(\phi, X)$, $G_3 = 0$, $G_4 = G_4(\phi)$.**

  This class includes metric $f(R)$ gravity and BD theories. Since there is the specific relation

  $$\alpha_B = \frac{\alpha_M}{2},$$

  we have

  $$\mu = \frac{1}{16\pi G G_4} \left(1 + \frac{G_4\alpha_M^2}{2\Delta_2}\right), \quad \eta = \frac{2\Delta_2 - G_4\alpha_M^2}{2\Delta_2 + G_4\alpha_M^2}, \quad \Sigma = \frac{1}{16\pi G G_4}.$$  

  \(\text{(6.31)}\)

  The nonminimal coupling $G_4(\phi)$ enhances the gravitational interaction with matter ($\xi_\mu = G_4\alpha_M^2/(2\Delta_2) > 0$). There is the difference between $\mu$ and $\Sigma$, so the Newtonian gravitational potential $\Psi$ and the weak lensing potential $\psi_{\text{eff}}$ evolve in different ways.

- **(C) $G_2 = G_2(\phi, X)$, $G_3 = G_3(\phi, X)$, $G_4 = M^2_{\text{pl}}/2$.**

  The theories of this class are known as kinetic braidings, which accommodate cubic Galileons as a specific case. Since

  $$\alpha_B = \frac{\dot{\phi}^3G_{3,\phi}}{2HM_{\text{pl}}}, \quad \alpha_M = 0,$$

  we obtain

  $$\mu = \Sigma = 1 + \frac{M^2_{\text{pl}}\alpha_B^2}{\Delta_2}, \quad \eta = 1.$$  

  \(\text{(6.33)}\)

  Unlike the case (B) there is no gravitational slip. The cubic derivative coupling $G_3(X)$ enhances the two gravitational potentials $-\Psi$ and $\Phi$ in the same manner.

- **(D) $G_2 = G_2(\phi, X)$, $G_3 = G_3(\phi, X)$, $G_4 = G_4(\phi)$.**

  This is the most general case including kinetic braidings and its extensions. The relation between $\alpha_B$ and $\alpha_M$ is

  $$\alpha_B - \frac{\alpha_M}{2} = \frac{\dot{\phi}^3G_{3,\phi}}{4HG_4}.$$  

  \(\text{(6.34)}\)
The original kinetic braiding scenario \cite{181} corresponds to \( G_4 = M_{\text{pl}}^2/2 \), in which case \( \mu, \eta, \Sigma \) are of the same forms as those given in Eq. (6.33).

From Eqs. (6.20) and (6.22), the difference between \( \mu \) and \( \Sigma \) is

\[
\mu - \Sigma = -\frac{\alpha_M(\alpha_B - \alpha_M)}{16\pi G G_4^2}.
\]  

For the theories with \( \alpha_M = 0 \) (including the class (C)) or \( \alpha_B = \alpha_M, \mu \) is equivalent to \( \Sigma \). The case \( \alpha_B = \alpha_M \) is special in that Eqs. (6.20)-(6.22) reduce to \( \mu = \Sigma = 1/(16\pi G G_4) \) and \( \eta = 1 \).

In subsequent sections, we will discuss observational signatures for concrete dark energy models which belong to the classes (A), (B), (C), (D) in more detail.

\[ \text{VII. CLASS (A): QUINTESSENCE AND K-ESSENCE} \]

Quintessence and k-essence belong to the class (A) given by the Lagrangian

\[
L = G_2(\phi, X) + \frac{M_{\text{pl}}^2}{2} R,
\]  

which is within the framework of GR. From Eq. (3.32), the dark energy EOS of k-essence yields

\[
w_{\text{DE}} = -\frac{G_2}{G_2 - \phi^2 G_{2,X}}.
\]  

The quantities \( \alpha_K, \alpha_B, \) and \( \alpha_M \) are

\[
\alpha_K = \frac{\dot{\phi}^2}{H^2 q_t} \left( G_{2,X} + \phi^2 G_{2,XX} \right), \quad \alpha_B = 0, \quad \alpha_M = 0,
\]  

with \( q_t = M_{\text{pl}}^2 \) > 0. From Eqs. (5.15) and (5.16), we obtain

\[
q_s = 2M_{\text{pl}}^2 \left( G_{2,X} + \phi^2 G_{2,XX} \right), \quad c_s^2 = \frac{G_{2,X}}{G_{2,X} + \phi^2 G_{2,XX}}.
\]  

The ghost and Laplacian instabilities of scalar perturbations are absent for

\[
G_{2,X} + \phi^2 G_{2,XX} > 0, \quad \text{and} \quad G_{2,X} > 0.
\]  

The background Eqs. (3.5), (3.11), and (3.12) reduce, respectively, to

\[
3M_{\text{pl}}^2 H^2 = -G_2 + \phi^2 G_{2,X} + \rho_m,
\]

\[
2M_{\text{pl}}^2 \dot{H} = -\dot{\phi}^2 G_{2,X} - \rho_m - P_m,
\]

\[
\ddot{\phi} = -3H \dot{\phi} G_{2,X} + \phi^2 G_{2,XX} - G_{2,\phi}.
\]  

Provided that the field energy density \( \rho_{\text{DE}} = -G_2 + \phi^2 G_{2,X} \) is positive, the k-essence EOS (7.2) is in the range

\[
w_{\text{DE}} > -1,
\]  

under the second condition of (7.5). The evolution of \( w_{\text{DE}} \) is different depending on the potential of quintessence and the form of k-essence Lagrangian.

On using Eq. (7.3), the quantities (6.20) and (6.22) reduce, respectively, to

\[
\mu = 1, \quad \Sigma = 1,
\]  

independent of the models of quintessence and k-essence. Although this situation is degenerate, the evolution of matter perturbations and gravitational potentials is affected by the difference of the background cosmology \cite{235}. Moreover, as we will see in the following, the scalar sound speed squared \( c_s^2 \) in k-essence differs from that in quintessence. Hence there are still possibilities for distinguishing between models of quintessence and k-essence.
A. Quintessence

Quintessence is given by the function

\[ G_2(\phi, X) = X - V(\phi), \]  

(7.11)

under which Eq. (7.2) reduces to

\[ w_{DE} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} > -1. \]  

(7.12)

The cosmological constant \( \Lambda \) corresponds to the limit \( \dot{\phi}^2/2 \to 0 \) and \( V(\phi) \to \Lambda \), i.e., \( w_{DE} \to -1 \). For quintessence, the quantities given in Eq. (7.4) reduce to \( q_s = 2M_{pl}^2 \) and \( c_s^2 = 1 \), so the conditions for the absence of ghost and Laplacian instabilities are trivially satisfied.

To study the dark energy dynamics, we introduce the following dimensionless quantities:

\[ \Omega_{DE} = \frac{\rho_{DE}}{3M_{pl}^2H^2}, \quad \Omega_m = \frac{\rho_m}{3M_{pl}^2H^2}, \quad w_m = \frac{p_m}{\rho_m}, \]  

(7.13)

where \( \rho_{DE} = \dot{\phi}^2/2 + V(\phi) \). For the matter sector, we consider nonrelativistic matter characterized by the constant equation of state \( w_m \), close to +0. The Hamiltonian constraint \( \dot{\phi}^2 \) gives the relation \( \Omega_{DE} + \Omega_m = 1 \). On using Eqs. (7.7) and (7.8), we obtain the following differential equations:

\[ w'_{DE} = (w_{DE} - 1)\sqrt{3(1 + w_{DE})} \left[ \sqrt{3(1 + w_{DE})} - \lambda \sqrt{\Omega_{DE}} \right], \]  

(7.14)

\[ \Omega_{DE}' = 3(w_m - w_{DE})\Omega_{DE}(1 - \Omega_{DE}), \]  

(7.15)

\[ \lambda' = -\sqrt{3(1 + w_{DE})}\Omega_{DE}(\Gamma - 1)\lambda^2, \]  

(7.16)

where a prime represents a derivative with respect to \( N = \ln a \), and

\[ \lambda \equiv -\frac{M_{pl}V_{,\phi}}{V}, \quad \Gamma \equiv \frac{VV_{,\phi\phi}}{V_{,\phi}^2}. \]  

(7.17)

From Eq. (7.14), besides the trivial case \( w_{DE} \simeq 1 \), there are two important situations in which \( w_{DE} \) stays nearly constant:

1. Thawing quintessence

In case (1), \( w_{DE} \) is close to \(-1\), but the deviation of \( w_{DE} \) from \(-1\) still occurs at late times. From Eq. (7.15), we find that \( \Omega_{DE} \) increases according to \( \Omega_{DE}' \simeq 3(1 + w_m)\Omega_{DE}(1 - \Omega_{DE}) > 0 \). Eventually, the growth of \( \Omega_{DE} \) in Eq. (7.14) leads to the variation of \( w_{DE} \), such that \( w_{DE}' \simeq 2\lambda\sqrt{3(1 + w_{DE})}\Omega_{DE} \). This belongs to the class of thawing quintessence in which the scalar field is nearly frozen by the Hubble friction in the early cosmological epoch and it starts to evolve only recently. The representative potential of this class is the one arising from the pseudo-Nambu-Goldstone (PNGB) boson:

\[ V(\phi) = \mu^4 \left[ 1 + \cos \left( \frac{\phi}{f} \right) \right], \]  

(7.18)

where \( \mu \) and \( f \) are constants characterizing the energy scale and the mass scale of spontaneous symmetry breaking, respectively. The axion can be the candidate for the ultra-light PNGB boson. When a global U(1) symmetry is spontaneously broken, the axion appears as an angular massless field \( \phi \) with an expectation value \( \phi = f_\star e^{i\phi}/f_\star \) of a complex scalar at a scale \( f_\star \).

If the field mass squared \( |m_\phi^2| \simeq V_{,\phi}\phi \) around the potential maximum is smaller than \( H^2 \), the field is stuck there with the dark energy EOS \( w_{DE} \simeq -1 \). After \( H^2 \) drops below \( |m_\phi^2| \), \( w_{DE} \) starts to increase from \(-1\). For \( |m_\phi^2| \)

\[ \text{The case } w_{DE} \simeq 1 \text{ is irrelevant to the late-time cosmic acceleration, as Eq. (7.15) shows that } \Omega_{DE} \text{ decreases for } 0 < \Omega_{DE} < 1. \]
of order $H_0^2$ (i.e., $\mu \approx \sqrt{H_0 M_{\text{pl}}} \approx 10^{-3}$ eV with the mass scale $f \approx M_{\text{pl}}$), the growth of $w_{\text{DE}}$ occurs at low redshifts. The radiative correction, which is proportional to $\mu^4$, does not give rise to explicit symmetry breaking terms. Then, the small mass term $|m_{\phi}| \approx \mu^2/M_{\text{pl}} \approx 10^{-33}$ eV relevant to the late-time cosmic acceleration can be protected against radiative corrections. There are several interesting attempts for explaining the small mass scale $\mu$ of order $10^{-3}$ eV in the context of supersymmetric theories \[241-246\].

In case (a) of Fig. 1, we plot one example for the evolution of $w_{\text{DE}}$ in terms of $z + 1 = 1/a$, where $z$ is the redshift (with today's value $z = 0$). In this case, $w_{\text{DE}}$ starts to deviate from $-1$ around the redshift $z \lesssim 2$, with today's value $w_{\text{DE}}(z = 0) \approx -0.7$. The likelihood analysis using CMB shift parameters measured by WMAP7 \[247\] combined with SN Ia and BAO data placed the bound $w_{\text{DE}}(z = 0) < -0.695$ (95 % CL) with the quintessence prior $w_{\text{DE}} > -1$ \[50\]. An updated data analysis based on the Planck 2015+SN Ia+BAO data provided the bound $w_{\text{DE}}(z = 0) < -0.473$ (95 % CL) \[248\]. This difference is mostly attributed to the fact that today’s Hubble parameter $H_0$ constrained from the Planck data \[7, 8\] favors a lower value than that constrained from the WMAP7 data \[247\]. The bound of $H_0$ derived by the Planck team has been in tension with direct measurements of $H_0$ at low redshifts \[249\]. This is the main reason why the analysis of Ref. \[248\] provided a more conservative bound on $w_{\text{DE}}(z = 0)$ than that obtained in Ref. \[50\].

**FIG. 1.** Examples for the evolution of $w_{\text{DE}}$ versus $z + 1$ in (a) thawing quintessence, (b) scaling freezing quintessence, and (c) tracking freezing quintessence. The present epoch is identified by the condition $\Omega_{\text{DE}}(z = 0) = 0.68$.

2. **Freezing quintessence**

In case (2), the field density parameter obeys the particular relation

$$\Omega_{\text{DE}} = \frac{3(1 + w_{\text{DE}})}{\lambda^2}.$$  \hspace{1cm} (7.19)

If $w_{\text{DE}} = w_m$, then $\Omega_{\text{DE}} = 3(1 + w_m)/\lambda^2 = \text{constant}$ and hence $\lambda$ is constant. This is known as a scaling solution \[46, 250\], along which the ratio $\Omega_{\text{DE}}/\Omega_m$ does not vary in time. This can be realized by the exponential potential $V(\phi) = V_0 e^{-\lambda \phi/M_{\text{pl}}}$, where $V_0$ is a constant. Since $\Gamma = 1$ in this case, Eq. (7.16) is trivially satisfied. The scaling solution has a nice feature in that the field density is not much smaller than the background density in the past, which can be compatible with the energy scale relevant to particle physics. Hence it can alleviate the coincidence problem of dark energy (see Refs. \[56, 254, 255\] for more general models allowing for scaling solutions). However, since $w_{\text{DE}} = w_m \approx 0$ for nonrelativistic matter, the scaling solution is not appropriate to be used for the cosmic
acceleration. If the single exponential potential is modified at late times to slow down the evolution of \( \phi \), it is possible to realize a scaling matter-dominated epoch followed by the accelerated expansion \[254, 258\]. We call this model

**scaling freezing quintessence** \[254\].

The typical example of scaling freezing quintessence is given by the potential \[254\]

\[
V(\phi) = V_1 e^{-\lambda_1 \phi/M_{Pl}} + V_2 e^{-\lambda_2 \phi/M_{Pl}},
\]

where \( \lambda_1 \) and \( V_i \) \((i = 1, 2)\) are constants. We consider the case in which the slopes \( \lambda_1 \) and \( \lambda_2 \) are in the ranges \( \lambda_1 > O(1) \) and \( \lambda_2 < O(1) \). In the early cosmological epoch, the steep exponential potential \( V_1 e^{-\lambda_1 \phi/M_{Pl}} \) dominates over the other potential \( V_2 e^{-\lambda_2 \phi/M_{Pl}} \), which is followed by the scaling matter era \((w_{DE} = 0 \text{ and } \Omega_{DE} = 3/\lambda_1^2)\). From the big bang nucleosynthesis there is the bound \( \Omega_{DE} < 0.045 \) in the radiation era \[259\], which translates to \( \lambda_1 > 9.4 \). The Universe finally enters the epoch of cosmic acceleration after the second potential \( V_2 e^{-\lambda_2 \phi/M_{Pl}} \) in Eq. \(7.20\) starts to contribute to the cosmological dynamics. The fixed point relevant to the late-time cosmic acceleration satisfies \( \sqrt{3(1 + w_{DE})} = \lambda \sqrt{\Omega_{DE}} \), \( \Omega_{DE} = 1 \), \( \Gamma = 1 \), with \( \lambda = \lambda_2 \) in Eqs. \(7.14\)-\(7.16\) so that

\[
w_{DE} = -1 + \frac{\lambda_2^2}{3}.
\]

The necessary condition for the cosmic acceleration is given by \( \lambda_2^2 < 2 \), under which the fixed point is a stable attractor \[14, 250\]. In case (b) of Fig. 1, we plot the evolution of \( w_{DE} \) for \( \lambda_1 = 50 \) and \( \lambda_2 = 0.1 \) with today's field density parameter \( \Omega_{DE}(z = 0) = 0.68 \). We observe that \( w_{DE} \) is close to 0 in the scaling matter era and it approaches the asymptotic value \(7.21\).

In order to confront scaling freezing quintessence with observations, it is convenient to quantify the evolution of \( w_{DE} \) in terms of the transition scale factor \( a_t \) and the thickness of transition to the freezing regime driven by the potential \( V_2 e^{-\lambda_2 \phi/M_{Pl}} \). In Ref. \[50\], it was shown that the change of \( w_{DE} \) from the scaling matter era to the accelerated attractor with \( \lambda_2^2 
lessapprox 1 \) can be well approximate by the parametrization \[261\]:

\[
w_{DE} = -1 + \frac{1}{1 + (a_t/a_{t0})^{1/\tau}},
\]

with \( \tau \simeq 0.33 \). The joint likelihood analysis based on the Planck 2015 data combined with SN Ia and BAO data showed that the transition scale factor \( a_t \) is constrained to be \( a_t < 0.11 \) (95\% CL) \[248\] (which updated the bound \( a_t < 0.23 \) (95\% CL) derived in Ref. \[50\]). This translates to the transition redshift \( z_t > 8.1 \), so the scaling matter era needs to end at quite early time. This is mostly attributed to the fact that, since the sound speed squared \( c_s^2 \) of quintessence is equivalent to 1, the perturbation of scaling quintessence hardly contributes to gravitational potentials. This slows down the growth of structures and leads to a large early Integrated-Sachs-Wolfe (ISW) effect \[248\]. For decreasing \( z_t \) in the range \( z_t < 8 \), the CMB angular power spectrum is subject to stronger modification.

There is the other class of freezing models dubbed **tracking freezing quintessence**. In this case, the slope \( \lambda \) decreases in time with \( w_{DE} \) nearly constant, so that \( \Omega_{DE} \) in Eq. \(7.19\) grows in time. From Eq. \(7.16\), the condition for decreasing \( \lambda \) translates to

\[
\Gamma > 1.
\]

Since \( \Omega_{DE} \) increases, Eq. \(7.15\) shows that \( w_{DE} < w_m \) for this tracker solution. This property is different from the scaling solution along which \( w_{DE} = w_m \) and \( \Omega_{DE} = \text{constant} \). Taking the \( \mathcal{N} \) derivative of Eq. \(7.19\), it follows that \( \Omega_{DE}/\Omega_{DE} = -2\lambda'/\lambda \). Substituting Eqs. \(7.15\), \(7.16\), and \(7.19\) into this relation under the condition that \( \Omega_{DE} \ll 1 \), the dark energy EOS in the matter era can be estimated as \[202\]

\[
w_{DE} \simeq w(0) \equiv -\frac{2(\Gamma - 1)}{2\Gamma - 1},
\]

where we used \( w_m \simeq 0 \). As long as \( \Gamma \) evolves slowly, \( w_{DE} \) stays nearly constant at high redshifts \( (z \gg 1) \). The tracker solution can be realized by the inverse power-law potential given by

\[
V(\phi) = M^{4+p} \phi^{-p},
\]

where \( M \) and \( p \) are positive constants. This potential gives the value \( \Gamma = 1 + 1/p = \text{constant} > 1 \), so it satisfies the tracking condition \(7.23\). The tracker EOS \(7.24\) reduces to

\[
w(0) = -\frac{2}{p + 2}.
\]
For \( p \) closer to 0, \( w_{(0)} \) approaches the cosmological constant value \(-1\).

As \( \Omega_{DE} \) grows to the order of 0.1, \( w_{DE} \) starts to deviate from the value [7.24], see case (c) in Fig. 1. It is possible to estimate the deviation \( \delta w_{DE} \) from \( w_{(0)} \) by dealing with \( \Delta \) as a perturbation to the 0-th order solution [7.24]. The leading-order solution to \( \Omega_{DE} \) can be derived by substituting \( w_{DE} \approx w_{(0)} \) into Eq. (7.15) and integrating it with respect to \( a \), such that

\[
\Omega_{DE}(a) = \Omega_{DE}^{(0)} \left[ \Omega_{DE}^{(0)} + a^{3w_{(0)}} \left( 1 - \Omega_{DE}^{(0)} \right) \right]^{-1},
\]

where \( \Omega_{DE}^{(0)} \) is today’s value of \( \Omega_{DE} \). Plugging Eq. (7.27) into Eq. (7.14) and integrating the linear perturbation equation of \( \delta w_{DE} \) with respect to \( a \), it follows that [263]

\[
w_{DE}(a) = w_{(0)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} w_{(0)} (1 - w_{(0)}^2)}{1 - (n + 1) w_{(0)} + 2n(n + 1) w_{(0)}^2} \left( \Omega_{DE}(a) \right)^n.
\]

As shown in Refs. [50, 263], the iterative formula (7.28) up to the order \( n = 3 \) exhibits fairly good agreement with the numerically integrated solution. Under the quintessence prior \( w_{(0)} > -1 \), the likelihood analysis using the iterative solution (7.28) put the bounds \( w_{(0)} < -0.923 \) and \( 0.675 < \Omega_{DE}^{(0)} < 0.703 \) (95 % CL) from the observational data of Planck 2015, SN Ia, and CMB [248]. For the potential (7.25), this bound translates to \( p > 0.17 \). For example, the tracker solution (c) plotted in Fig. 1 corresponds to \( p = 1 \), which is strongly disfavored by the data. Thus, the tracker solutions arising from the inverse power-law potential with positive integer powers \( (p \geq 1) \) areobservationally excluded.

### B. k-essence

The Lagrangian of k-essence is a general function of \( \phi \) and \( X \). For example, the tachyon field given by the Lagrangian \( G_2 = - V(\phi) \sqrt{-\det (g_{\mu\nu} + \nabla_\mu \phi \nabla^\nu \phi)} \), where \( V(\phi) \) is a tachyon potential, living on a non-BPS D3-brane [264, 266]. There is also the so-called Dirac-Born-Infeld (DBI) scenario in which the movement of a probe D3-brane along the radial direction in the AdS5 spacetime is described by the action \( P = - f(\phi)^{-1} \sqrt{1 - 2 f(\phi) X + f(\phi)^{-1} - V(\phi)} \) [267, 268]. In the original setup of string theory, however, these two scenarios do not account for the late-time cosmic acceleration. Allowing for the freedom to modify the functions \( V(\phi) \) and \( f(\phi) \), there are possibilities for applying such theories to dark energy [269, 274]. In both tachyon and DBI theories, we require the existence of scalar potential \( V(\phi) \) for driving the cosmic acceleration (as in quintessence).

There are k-essence scenarios in which nonlinear terms in \( X \) play crucial roles for the late-time cosmological dynamics [52, 57]. In the following, we review two typical k-essence theories of this type and discuss their observational signatures.

#### 1. Ghost condensate

If the scalar field has a negative kinetic energy \(-X\), this is typically a sign for the appearance of scalar ghosts. However, the existence of an additional term \( X^2 \) to \(-X\) can allow a possibility for avoiding the ghost [55]. The so-called dilatonic ghost condensate [56] given by the Lagrangian

\[
G_2(\phi, X) = -X + e^{\lambda \phi/M_\text{pl}} \frac{X^2}{M^4}
\]

belongs to such a class (where \( \lambda \) and \( M \) are positive constants). In the limit \( \lambda \to 0 \), there exists a de Sitter solution at \( X = M^4/2 \). As we will see below, the exponential term \( e^{\lambda \phi/M_\text{pl}} \) in Eq. (7.29) leads to the deviation from the de Sitter solution.

To discuss the cosmological dynamics of dilatonic ghost condensate, we introduce the dimensionless quantities:

\[
x_1 \equiv \frac{\dot{\phi}}{\sqrt{6H M_\text{pl}}} , \quad x_2 \equiv \frac{\dot{\phi}^2 e^{\lambda \phi/M_\text{pl}}}{2M^4}.
\]

From Eq. (7.4), the quantities \( q_s \) and \( c_8^2 \) are

\[
q_s = 2 (6x_2 - 1) M_\text{pl}^2 , \quad c_8^2 = \frac{2x_2 - 1}{6x_2 - 1}.
\]
which are both positive for \( x_2 > 1/2 \). The dark energy density parameter and its EOS are given, respectively, by

\[
\Omega_{\text{DE}} = x_2^2 (-1 + 3x_2), \quad w_{\text{DE}} = \frac{1 - x_2}{1 - 3x_2}.
\] (7.32)

The necessary condition for the cosmic acceleration \((-1 < w_{\text{DE}} < -1/3)\) translates to \(1/2 < x_2 < 2/3\), in which regime \(0 < c_s^2 < 1/9\) [56]. On using the background Eqs. (3.11) and (3.12), we obtain the differential equations for \(w_{\text{DE}}\) and \(\Omega_{\text{DE}}\), as

\[
w'_{\text{DE}} = \frac{(1 - 3w_{\text{DE}})(1 - w_{\text{DE}})[\sqrt{3}(1 - 3w_{\text{DE}})\Omega_{\text{DE}} \lambda - 3(1 + w_{\text{DE}})]}{5 - 3w_{\text{DE}}},
\]

\[
\Omega_{\text{DE}}' = 3(w_m - w_{\text{DE}})\Omega_{\text{DE}}(1 - \Omega_{\text{DE}}),
\] (7.33) (7.34)

where Eq. (7.34) is the same as Eq. (7.15) derived for quintessence. From Eq. (7.33), there are cases in which \(w_{\text{DE}}\) stays nearly constant. Among them, the fixed point corresponding to \(w_{\text{DE}} < -1/3\) is characterized by

\[
w_{\text{DE}} = -1 + \sqrt{\left(\frac{\lambda^2 \Omega_{\text{DE}}}{2}\right)^2 + \frac{4}{3}\lambda^2\Omega_{\text{DE}} - \frac{\lambda^2}{2}\Omega_{\text{DE}}}.
\] (7.35)

From Eq. (7.34), we have three fixed points satisfying \(w_{\text{DE}} = w_m, \Omega_{\text{DE}} = 0,\) and \(\Omega_{\text{DE}} = 1\). The first case corresponds to the scaling solution, but this is not responsible for dark energy unless the form of \(G_2(\phi, X)\) is appropriately modified at late times. The second fixed point (\(\Omega_{\text{DE}} = 0\)) is relevant to the early matter era, during which \(w_{\text{DE}}\) is close to \(-1\). The third fixed point (\(\Omega_{\text{DE}} = 1\)) is associated with the late-time cosmic acceleration with \(w_{\text{DE}} = -1 + \sqrt{\lambda^2/4 + 4X^2/3} - \lambda^2/2 > -1\). Then, the evolution of \(w_{\text{DE}}\) in the diatonic ghost condensate is similar to that in thawing quintessence discussed in Sec. VII A 1. Provided that \(w_{\text{DE}}\) does not significantly deviate from \(-1\), the integrated solution to Eq. (7.34) in the intermediate regime \(0 < \Omega_{\text{DE}} < 1\) can be derived by setting \(w_m \simeq 0\) and \(w_{\text{DE}} \simeq -1\), such that

\[
\Omega_{\text{DE}}(a) = a^3\Omega_{\text{DE}}^{(0)} \left[1 - \Omega_{\text{DE}}^{(0)} + a^3\Omega_{\text{DE}}^{(0)}\right]^{-1}.
\] (7.36)

Substituting Eq. (7.36) into Eq. (7.35), we obtain the approximate solution to \(w_{\text{DE}}\) as a function of \(a\). If we adopt the observational bound \(w_{\text{DE}}(a = 1) \lesssim -0.7\) with \(\Omega_{\text{DE}}^{(0)} = 0.68\), the parameter \(\lambda\) is constrained to be \(\lambda \lesssim 0.36\).

We note that, in the diatonic ghost condensate, \(c_s^2\) is initially close to \(+0\) and then it starts to deviate from \(+0\) only recently. In this case, the k-essence scalar with \(c_s^2 \simeq +0\) can work as a part of dark matter because of additional gravitational clusterings. This property is different from that in quintessence where \(c_s^2\) is always equivalent to \(1\), so there is a possibility for distinguishing between thawing quintessence and diatonic ghost condensate at the level of perturbations.

2. **K-essence as unified dark energy and dark matter**

There is a unified k-essence model of dark energy and dark matter given by [57]

\[
G_2(X) = -b_0 + b_2 (X - X_0)^2,
\] (7.37)

where \(b_0, b_2, X_0\) are positive constants. In this case, the k-essence pressure and density are \(P = -b_0 + b_2 (X - X_0)^2\) and \(\rho = b_0 + b_2(X - X_0)(3X + X_0)\), respectively, with the propagation speed squared:

\[
c_s^2 = \frac{X - X_0}{3X - X_0}.
\] (7.38)

As long as \(X\) stays around \(X_0\), \(c_s^2\) is close to \(0\). Substituting \(X = X_0 [1 + \epsilon(t)]\) with \(0 < \epsilon \ll 1\) into Eq. (7.8), we obtain the integrated solution \(\epsilon(t) = \epsilon_1 (a_1/a)^3\), where \(\epsilon_1\) and \(a_1\) are positive constants. Then, the sound speed squared (7.38) approximately reduces to

\[
c_s^2 \approx \frac{\epsilon(t)}{2} = \frac{\epsilon_1}{2} \left(\frac{a_1}{a}\right)^3,
\] (7.39)
which always stays in the region $0 < c_s^2 \ll 1$. Around $X = X_0$, we have $P \approx -b_0$ and $\rho \approx b_0 + 4b_2X_0(X - X_0)$, so the k-essence EOS, $w = P/\rho$, reads

$$w \approx -\left[1 + \frac{4b_2}{b_0}X_0^2\epsilon_1\left(\frac{a_1}{a}\right)^3\right]^{-1}.$$  

(7.40)

In the early matter era, the k-essence field behaves as dark matter with $w \approx 0$. Since $w$ approaches $-1$ at late times due to the dominance of the term $-b_0$ in Eq. (7.37), the same field behaves as dark energy. Thus, the k-essence Lagrangian (7.37) provides the unified description of dark energy and dark matter with $0 < c_s^2 \ll 1$. We note that the purely k-essence with the Lagrangian $G = G_2(X)$ is equivalent to a barotropic perfect fluid on the cosmological background $[275, 278]$.

In Refs. [58, 279], the authors proposed several unified models of dark matter and dark energy different from (7.37). In particular, there is a model in which $c_s^2$ starts to evolve from a value close to 0 and approaches an asymptotic value $c_s^2$. In such a case, the likelihood analysis based on the galaxy-ISW correlation data showed that $c_s^2$ is constrained to be smaller than $c_s^2 \lesssim 9 \times 10^{-3}$ [280].

VIII. CLASS (B): $f(R)$ GRAVITY AND BRANS-DICKE THEORIES

The theories of class (B) include BD theories and $f(R)$ gravity as specific cases. The Lagrangian of BD theories with a scalar potential $V(\phi)$ is given by

$$L = (1 - 6Q^2) e^{-2Q\phi/M_{pl}^2}X - V(\phi) + \frac{M_{pl}^2}{2} e^{-2Q\phi/M_{pl}^2} R,$$

(8.1)

where $Q$ is a constant. In terms of the redefined dimensionless field

$$\chi = e^{-2Q\phi/M_{pl}^2},$$

(8.2)

the Lagrangian (8.1) is equivalent to

$$L = -\frac{M_{pl}^2\omega_{BD}}{2\chi} g^{\mu\nu}\nabla_\mu \chi \nabla_\nu \chi - V(\chi) + \frac{M_{pl}^2}{2} \chi R,$$

(8.3)

where the BD parameter $\omega_{BD}$ is related to the constant $Q$, as

$$3 + 2\omega_{BD} = \frac{1}{2Q^2}.$$  

(8.4)

The original BD theory [60] is written in the form (8.3) without the scalar potential ($V = 0$).

The $f(R)$ gravity, which is given by the Lagrangian $L = M_{pl}^2 f(R)/2$, is equivalent to the scalar-tensor theory with

$$L = M_{pl}^2[f(\varphi) + f'_{\varphi}(R - \varphi)]/2,$$

where $\varphi$ is a scalar quantity. Varying the latter Lagrangian with respect to $\varphi$, we obtain $f_{\varphi\varphi}(R - \varphi) = 0$ and hence $\varphi = R$ for $f_{\varphi\varphi} \neq 0$. By defining $\chi = f_{\varphi} = f_R$, the Lagrangian of $f(R)$ gravity reduces to

$$L = \frac{M_{pl}^2}{2} \chi R - V(\chi), \quad \text{where} \quad V(\chi) = \frac{M_{pl}^2}{2} [\chi \varphi(\chi) - f(\varphi(\chi))].$$  

(8.5)

Comparing this Lagrangian with Eq. (8.3), it follows that $f(R)$ gravity corresponds to BD theory with $\omega_{BD} = 0$ [64, 65]. Under the conformal transformation $\tilde{g}_{\mu\nu} = \chi g_{\mu\nu}$, the Lagrangian (8.5) is transformed to that in the Einstein frame with a canonical scalar field $[79, 207, 208, 251]$

$$\phi = \sqrt{\frac{3}{2} M_{pl}} \ln f_R,$$

(8.6)

which corresponds to the field $\phi$ appearing in the Jordan-frame Lagrangian (8.1) of BD theory. On using the correspondence $\chi = e^{-2Q\phi/M_{pl}^2} = f_R$, the constant $Q$ in metric $f(R)$ gravity, which plays the role of couplings between the field $\phi$ and matter in the Einstein frame [77, 252], is given by

$$Q = -\frac{1}{\sqrt{6}}.$$  

(8.7)
In BD theory with an arbitrary BD parameter $\omega_{BD}$, the modification of gravity is significant for $|Q| > \mathcal{O}(0.1)$. In the limit that $Q \to 0$, i.e., $\omega_{BD} \to \infty$, the Lagrangian \[8.1\] recovers the minimally coupled quintessence in GR. If the scalar potential $V(\phi)$ is absent, gravitational experiments in the solar system constrain the BD parameter to be $\omega_{BD} > 40000 \ [28, 30]$, which translates to $|Q| < 2.5 \times 10^{-3}$. In the presence of $V(\phi)$, as in the case of $f(R)$ gravity, it is possible to suppress the fifth force induced by the scalar-matter interaction in the solar system under the chameleon mechanism even for $|Q| > \mathcal{O}(0.1) \ [70] \ [79] \ [83]$. Indeed, the viable dark energy models in $f(R)$ gravity \[70] \ [74] and BD theory \[77] \ [78] are constructed to have a large mass in regions of the high density, while the field mass is light on cosmological scales such that the effective potential of $\phi$ drives the late-time cosmic acceleration.

For the Lagrangian \[8.1\], the background Eqs. \[3.5\] and \[3.12\] reduce, respectively, to

\[3M_{pl}^2 H^2 = \frac{1}{2} (1 - 6Q^2) \dot{\phi}^2 + 6QM_{pl} H \dot{\phi} + (\rho_m + V) e^{2Q \phi/M_{pl}}, \tag{8.8} \]

\[\ddot{\phi} + 3H \dot{\phi} + V_{,\phi} e^{2Q \phi/M_{pl}} + \frac{Q}{M_{pl} (\rho_m - 3P_m + 4V) e^{2Q \phi/M_{pl}} - 2\dot{\phi}^2} = 0, \tag{8.9} \]

where, for the derivation of Eq. \[8.9\], we used Eq. \[8.8\] to eliminate $H^2$. From Eq. \[8.9\], the coupling $Q$ can induce a local minimum even for a runaway potential $V(\phi)$. If the scalar field is nearly frozen around the local minimum during the matter era characterized by $\rho_m \gg \{P_m, V\}$, it follows that

\[M_{pl} V_{,\phi} + Q \rho_m \simeq 0. \tag{8.10} \]

If $Q > 0$ (or $Q < 0$), then the solution to Eq. \[8.10\] exists for $V_{,\phi} < 0$ (or $V_{,\phi} > 0$). Since $\rho_m$ decreases in time, the field evolves slowly along the instantaneous minima determined by Eq. \[8.10\]. The scalar potential $V$ is responsible for the late-time cosmic acceleration. After $V$ dominates over the matter density $\rho_m$, there exists a de Sitter solution characterized by

\[M_{pl} V_{,\phi} + 4QV = 0. \tag{8.11} \]

From the above discussion, the field $\phi$ is almost frozen apart from the transient period from the matter era to the de Sitter solution. In other words, the deviation of $w_{DE}$ from $-1$ starts to occur around today and $w_{DE}$ finally approaches the asymptotic value $-1$. We note that, with decreasing $\rho_m$, the effective mass around the potential minimum induced by the coupling $Q$ gets smaller. In the early Universe, the field is nearly frozen due to the heavy mass associated with the large matter density $\rho_m$. In local regions of the Universe where the density $\rho_m$ is much larger than today’s critical density $\rho_c$, the similar suppression of the propagation of fifth forces can occur under the operation of the chameleon mechanism \[84] \ [85].

We also compute quantities relevant to the evolution of linear cosmological perturbations for the Lagrangian \[8.1\]. Since $q_s = M_{pl}^2 e^{-2Q \phi/M_{pl}}$, the no-ghost condition of tensor perturbations is automatically satisfied. The quantities $\alpha_K$, $\alpha_B$, and $\alpha_M$ are given by

\[\alpha_K = \frac{(1 - 6Q^2) \dot{\phi}^2}{M_{pl}^2 H^2}, \quad \alpha_B = \frac{\alpha_M}{2} = -\frac{Q \dot{\phi}}{M_{pl} H}. \tag{8.12} \]

From Eqs. \[5.15\] and \[5.16\], it follows that

\[q_s = 2 e^{-4Q \phi/M_{pl}} M_{pl}^2, \quad e_s^2 = 1, \tag{8.13} \]

so there are neither ghost nor Laplacian instabilities of scalar perturbations in BD theories. The dark energy EOS \[3.32\] reduces to

\[w_{DE} = -1 - \frac{4M_{pl}^2 (1 - e^{-2Q \phi/M_{pl}})}{1 - 6(1 - 2Q^2) e^{-2Q \phi/M_{pl}} \dot{\phi}^2 + 2V + 6M_{pl}^2 H^2 (1 - e^{-2Q \phi/M_{pl}}) + 12M_{pl} Q e^{-2Q \phi/M_{pl}} \dot{\phi}}. \tag{8.14} \]

Existence of the nonvanishing coupling $Q$ allows the possibility for realizing $w_{DE} < -1 \ [70] \ [73] \ [75]$, without having ghost and Laplacian instabilities. Applying the bound \[6.27\] to $\alpha_M$ in Eq. \[8.12\], the field time derivative today (denoted as $\dot{\phi}_0$) is constrained to be

\[\left| \frac{Q \dot{\phi}_0}{M_{pl} H_0} \right| < 0.01, \tag{8.15} \]

which limits the large deviation of $w_{DE}$ from $-1$ at low redshifts.
From Eqs. (6.20) and (6.22), we obtain
\[
\mu = e^{2Q \phi/M_{\text{pl}}} \frac{1 + 2Q^2 + F_M(k)}{1 + F_M(k)}, \quad \Sigma = e^{2Q \phi/M_{\text{pl}}},
\] (8.16)
where
\[
F_M(k) = \frac{a^2 M^2 e^{2Q \phi/M_{\text{pl}}}}{k^2}.
\] (8.17)
As we will see below, the viable dark energy models in \(f(R)\) gravity and BD theory have been constructed to have a growing mass \(M_0\) in the asymptotic past with the scalar field nearly frozen around \(\phi \simeq 0\). Since \(M_0^2\) becomes much larger than \(H^2\), the quantity \(F_M(k)\) can exceed the order 1 even for perturbations deep inside the Hubble radius \((k^2 \gg a^2 H^2)\). Taking the limit \(F_M(k) \rightarrow \infty\) in Eq. (8.16) in this massive regime, it follows that \(\mu \simeq e^{2Q \phi/M_{\text{pl}}} \simeq 1\). Then, the evolution of linear perturbations is similar to that in GR for the same background expansion history.

Since \(M_0^2\) gradually decreases in time, there is an instant at which \(F_M(k)\) crosses 1. For larger \(k\), the entry to the regime \(F_M(k) < 1\) occurs earlier. Taking the limit \(F_M(k) \rightarrow 0\) in Eq. (8.16), we obtain
\[
\mu \simeq (1 + 2Q^2) \Sigma, \quad \Sigma = e^{2Q \phi/M_{\text{pl}}},
\] (8.18)
This is the regime in which the modification of gravity manifests itself on scales relevant to the linear growth of large-scale structures. In particular the coupling \(Q\) leads to the enhancement of \(G_{\text{eff}}\), so the growth rate of matter perturbations gets larger than that in quintessence and k-essence. This allows the possibility for distinguishing \(f(R)\) gravity and BD theories from dark energy models in the framework of GR.

### A. \(f(R)\) gravity

In \(f(R)\) gravity, the examples of models relevant to the late-time cosmic acceleration are given by [70, 71]

(i) \(f(R) = R - \lambda R_0 \left(\frac{R}{R_0}\right)^{2n} + 1\),
(ii) \(f(R) = R - \lambda R_0 \left[1 - \left(1 + \frac{R^2}{R_0^2}\right)^{-n}\right]\),
\] (8.19) \(8.20\)
where \(\lambda, R_0, n\) are positive constants. These models satisfy \(f(R = 0) = 0\), so the cosmological constant disappears in the limit of flat spacetime. In the high-curvature regime satisfying \(R \gg R_0\), they have the asymptotic behavior
\[
f(R) \simeq R - \lambda R_0 \left[1 - \left(\frac{R}{R_0}\right)^{2n}\right],
\] (8.21)
so that they approach the \(\Lambda\)CDM model \((f(R) = R - \lambda R_0)\). We note that the model \(f(R) = R - \lambda R_0 \tanh(R/R_0)\) [73] also has the same asymptotic behavior. The deviation from GR manifests itself after \(R\) decreases to the order of \(R_0\).

On using the asymptotic form (8.21), the scalar field \(\phi\) defined by Eq. (8.6), which is called the scalaron [63], is expressed as
\[
\phi \simeq \sqrt{\frac{2}{3}} M_{\text{pl}} \ln \left[1 - 2n \lambda \left(\frac{R}{R_0}\right)^{-2n+1}\right] < 0,
\] (8.22)
which approaches 0 as \(R \rightarrow \infty\). In the regime \(R \gg R_0\), the scalar potential (8.5), i.e., \(V = (M_{\text{pl}}^2/2)(f_{,\phi}R - f)\), reduces to
\[
V(\phi) \simeq \frac{\lambda R_0 M_{\text{pl}}^2}{2} \left[1 - \frac{2n + 1}{(2n \lambda)^{2n/(2n+1)}} \left(1 - e^{\sqrt{2/3} \phi/M_{\text{pl}}} \right)^{2n/(2n+1)}\right],
\] (8.23)
which approaches the constant \(\lambda R_0 M_{\text{pl}}^2/2\) for \(R \rightarrow \infty\). Since the potential (8.23) satisfies \(V_{,\phi} > 0\) for \(\phi < 0\), the solutions to Eqs. (8.10) and (8.11) exist for the above \(f(R)\) models. The field mass squared \(M_\phi^2 = V_{,\phi\phi}\) increases for larger \(R\), with the divergence \(M_\phi^2 \rightarrow \infty\) for \(R \rightarrow \infty\). In terms of \(f(R)\), the mass squared can be expressed as
\[
M_\phi^2 = \frac{R f_{,R}}{3m} (1 + m),
\] (8.24)
where the quantity $m \equiv R f_{RR}/f_R$ characterizes the deviation from the $Λ$CDM model [284]. The increase of $M_0^2$ in the asymptotic past is attributed to the decrease of $f_{RR} = 2 \lambda x (2n + 1) R_0^1 (R/R_0)^{-2n-2}$ toward 0. Since $f_{RR}$ is positive for the models (8.19) and (8.20), the mass squared (8.24) is positive and hence there is no tachyonic instability. Unlike the $Λ$CDM model in which $f_{RR}$ is exactly 0, the above $f(R)$ models give rise to a large mass in the asymptotic past due to the small deviation of $f_{RR}$ from 0. In the regime $m \ll 1$ the mass squared (8.24) can be estimated as $M_0^2 \approx R f_{RR}/(3m) \approx H^2/m \gg H^2$, where we used the properties $R = 6(2H^2 + H) \approx 3H^2$ and $f_{RR} \approx 1$ in the matter era.

Unless the initial conditions of $φ$ are carefully chosen to be close to the instantaneous minimum given by Eq. (8.10), the scalar field oscillates due to the heavy mass in the early cosmological epoch [71, 73]. Then, the system can even access the curvature singularity at $φ = 0$ [285]. This is the general problem of late-time $f(R)$ cosmic acceleration models. For relativistic stars in $f(R)$ gravity, the similar problem also arises as a fine-tuning of boundary conditions [280, 281]. Provided that the initial conditions are fine-tuned such that the scalar field is very close to the position determined by Eq. (8.11), the field evolves slowly along the instantaneous minima with decreasing $ρ_m$.

In the regime $R \gg R_0$, the field stays in the region around $φ = 0$ and hence the potential $V(φ)$ is nearly constant: $V(φ) \approx \lambda R_0 M_0^2/2$. In this region, $w_{DE}$ is close to $-1$. After $R$ decreases to the order $R_0$, the field $φ$ tends to be away from 0, so the deviation of $w_{DE}$ from $-1$ starts to occur by today [75]. Finally, the solution reaches the de Sitter fixed point satisfying Eq. (8.11).

In $f(R)$ gravity, the condition (8.11) is equivalent to [208, 284]

$$R f_{RR} = 2f_R .$$

(8.25)

The stability analysis around this fixed point shows that the de Sitter point satisfying (8.25) is stable if the parameter $m = R f_{RR}/f_R$ is in the range [284, 290, 291]

$$0 < m \leq 1 .$$

(8.26)

In model (i), the conditions (8.25) and (8.26) translate, respectively, to

$$\lambda x_1^{2n-1} (2 + 2x_1^{2n} - 2n) = (1 + x_1^{2n})^2 ,$$

(8.27)

$$2x_1^{2n} - 2(2n + 2)/(2n - 1)x_1^{2n} + (2n - 1)(2n - 2) \geq 0 ,$$

(8.28)

where $x_1 \equiv R_1/R_0$, and $R_3$ is the value of $R$ at the de Sitter point. For given $n$, the condition (8.28) gives a lower bound on the parameter $λ$. If $n = 1$, we have $x_1 \geq \sqrt{3}$ and $λ \geq 8\sqrt{3}/9$.

The de Sitter fixed point corresponds to a stable spiral for $0 < m < 16/25$ [284], in which case the dark energy EOS approaches the asymptotic value $-1$ with oscillations around $w_{DE} = -1$. The phantom EOS $w_{DE} < -1$ can be realized by today without having ghost instabilities [70, 73, 75, 292]. This property is different from that in quintessence and k-essence where $w_{DE}$ is always in the range $w_{DE} > -1$. We note that, under the bound (8.15), the variation of $w_{DE}$ given by Eq. (8.14) is limited around today, such that $|w_{DE} + 1| < O(0.01)$. At the background level, the $f(R)$ models (8.19) and (8.20) are consistent with the observational data [293, 294].

For $m \ll 1$, the mass squared (8.24) in the matter era can be estimated as $M_0^2 \approx H^2/m$, so the function (8.17) is approximately given by $F_M(k) \approx a^2 H^2/(mk^2)$. This means that the entry from the massive regime ($F_M(k) > 1$) to the massless regime ($F_M(k) < 1$) occurs at

$$m \approx \left(\frac{aH}{k}\right)^2 .$$

(8.29)

For the models (8.19) and (8.20), the quantity $m$ grows in time during the matter era, while $(aH/k)^2$ decreases. Extrapolating the relation (8.29) up to the present epoch ($z = 0$), the perturbations relevant to the linear growth of large-scale structures ($k \lesssim 300H_0$) enter the massless regime by today under the criterion

$$m_0 \gtrsim 10^{-5} ,$$

(8.30)

where $m_0$ is today’s value of $m$. Since $μ$ and $Σ$ are close to 1 in the massive regime ($F_M(k) > 1$), the evolution of perturbations is similar to that in GR, i.e., $δ_m \propto t^{2/3}$ and $ψ_{eff} = -2Ψ = 2Φ = \text{constant}$ during the matter dominance. After the entry to the regime $F_M(k) < 1$, the quantities $μ$ and $Σ$ are given by Eq. (8.18) with $Q = -1/\sqrt{6}$. Solving Eq. (6.23) with Eqs. (6.3) and (6.6), the perturbations evolve as [71, 73]

$$δ_m \propto t^{(\sqrt{33} - 1)/6} , \quad ψ_{eff} = -\frac{3}{2}Ψ = 3Φ \propto t^{(\sqrt{33} - 5)/6} ,$$

(8.31)
during the matter dominance. The growth rates of matter perturbations and gravitational potentials are enhanced compared to those in GR. For \( k \gtrsim 300H_0 \), the nonlinear effects on the evolution of linear perturbations can be important, see Refs. [296, 302] for N-body simulations in \( f(R) \) gravity.

The modified growth of perturbations affects the observables associated with the power spectra of galaxy clusterings, CMB, and weak lensing. In the left panel of Fig. [2] we show the CMB angular temperature power spectrum for three different \( f(R) \) models and the ΛCDM model. The background expansion history is fixed to be the same as that in the ΛCDM model, but the evolution of perturbations is qualitatively similar to that in the \( f(R) \) models [8.19] and [8.20]. We define the following quantity [70]

\[
B = m\frac{H\dot{R}}{\dot{R}},
\]

(8.32)

whose today’s value is denoted as \( B_0 \). We note that \( B \) is of the same order as \( m \). For increasing \( B_0 \), the CMB power spectrum exhibits stronger amplification relative to that in the ΛCDM model. This is attributed to the late-time ISW effect induced by the enhanced gravitational potential \( \psi_{\text{eff}} \). As we see in the middle panel of Fig. [2] for increasing \( B_0 \), the weak lensing power spectrum is also subject to stronger enhancement. In the right panel, we can also confirm that the modified growth of \( \delta_m \) leads to the amplification of the matter power spectrum.

![Graphs showing the CMB angular temperature power spectrum, the lensing power spectrum, and the linear matter power spectrum](image)

**FIG. 2.** The CMB angular temperature power spectrum (left), the lensing power spectrum (middle), and the linear matter power spectrum (right) in \( f(R) \) models of the late-time cosmic acceleration with the ΛCDM background. We plot theoretical predictions of three \( f(R) \) models \( (B_0 = 5, 1, 0.1) \) besides that in the ΛCDM model. Taken from Ref. [604].

For the \( f(R) \) model [8.19], the likelihood analysis based on the data of CMB, galaxy clusterings, weak gravitational lensing combined with the SN Ia, BAO, and \( H_0 \) data showed that the parameter \( B_0 \) is constrained to be [303]

\[
B_0 < 1.1 \times 10^{-3},
\]

(8.33)

at 95 % CL. The similar bounds were also derived in Refs. [132, 133] and [304] by taking the effective field theory approach and the designer approach to \( f(R) \) gravity, respectively. In the designer approach, the time-dependence of \( \omega_{\text{DE}} \) is a priori assumed, under which the functional form of \( f(R) \) is determined. For designer \( f(R) \) models with constant \( \omega_{\text{DE}} \), the joint data analysis based on CMB, BAO, and lensing data placed the bounds \(|\omega_{\text{DE}} + 1| < 0.002 \) and \( B_0 < 0.006 \) at 95 % CL [304].

We also note that there are bounds on the value of \(|f_{,R} - 1| \) constrained from the suppression of fifth forces in dense environments. For the model [8.19], the screening of fifth forces in the Milky Way and dwarf galaxies put the bounds \(|f_{,R0} - 1| \lesssim 10^{-5} \) [70] and \(|f_{,R0} - 1| \lesssim 10^{-7} \) [305], respectively, at 95 % CL, where \( f_{,R0} \) is today’s value of \( f_{,R} \).

These bounds show that there have been no observational signatures for the \( f(R) \) modification of gravity. The crucial point is that the gravitational coupling with matter is large in the massless regime \( F_M(k) < 1 \), such that \( \mu = (4/3)e^{-2\phi/(\sqrt{5}M_\nu)} \). To avoid the large enhancement of \( \delta_m \), the perturbations need to be in the massive regime \( F_M(k) > 1 \) until recently. In this case, however, the scalar field hardly evolves by today, so the \( f(R) \) models are not be distinguished from the ΛCDM model both at the levels of background and perturbations.
B. BD theory with a scalar potential

The BD theory with a light scalar (satisfying $M_\phi^2 \lesssim H^2$) can be consistent with local gravity constraints for the coupling $|Q| < 2.5 \times 10^{-3}$. In this case, however, it is difficult to distinguish BD theory from quintessence. If $|Q| \gtrsim 10^{-3}$, then the potential needs to be designed to have a large mass in regions of the high density for suppressing the propagation of fifth forces. The potential of this type is given by [77]

$$V(\phi) = V_0 \left[ 1 - C(1 - e^{-2Q\phi/M_{\text{pl}}})^p \right], \quad (8.34)$$

where $V_0 > 0$, $0 < C < 1$, $0 < p < 1$ are constants. This is the generalization of the potential [8.23] in $f(R)$ gravity to arbitrary couplings $Q$.

The background cosmological dynamics in BD theory with [8.34] is similar to that in $f(R)$ models explained in Sec. VIII A. As long as $|\phi/M_{\text{pl}}| \ll 1$ in the matter era, the field slowly evolves along the instantaneous minimum at $\phi = \phi_m$ determined by the condition (8.10), i.e.,

$$\phi_m \approx M_{\text{pl}} \left( \frac{2V_0 p C}{\rho_m} \right)^{1/(1-p)}. \quad (8.35)$$

In the limit $\rho_m \to \infty$, $\phi_m$ approaches 0 with the divergent mass squared $M_\phi^2 = V_{,\phi\phi}$. Hence the fine-tuning problem of initial conditions mentioned in Sec. VIII A also persists in this case. During the early matter era, the variation of $\phi$ is large in the early matter era, the perturbations relevant to the linear growth of large-scale structures are initially in the massive region $\mathcal{F}_\text{M}(k) > 1$. After the entry to the massless regime $\mathcal{F}_\text{M}(k) < 1$, the quantities $\mu$ and $\Sigma$ approach the values given by Eq. (8.18). Solving Eq. (6.23) with Eqs. (6.3) and (6.6) in the massless regime, the evolution of perturbations during the matter-dominated epoch yields

$$\delta_m \propto t \left( \frac{\sqrt{25+48Q^2} - 1}{6} \right)^{1/2}, \quad \psi_{\text{eff}} = -\frac{2}{1+2Q^2} \Psi = \frac{2}{1-2Q^2} \Phi \propto t \left( \frac{\sqrt{25+48Q^2} - 5}{6} \right)^{1/2}. \quad (8.36)$$

For the coupling $|Q| > 1/\sqrt{6}$, the growth rate of $\delta_m$ is larger than that in $f(R)$ gravity. This is also the case for the gravitational potentials $\psi_{\text{eff}}, \Psi, \Phi$. Then, the observational constraints on BD theory with $|Q| > 1/\sqrt{6}$ should be more stringent than those on $f(R)$ models. As we explained in Sec. VIII A in $f(R)$ gravity, it is hard to observationally distinguish BD theory with the potential from the LCDM model.

If the chameleon mechanism works in local regions of the Universe, there is the bound arising from the violation of equivalence principle in the solar system [85, 283]. For the potential (8.34), this bound translates to [77]

$$p > 1 - \frac{5}{13.8 - \log_{10}|Q|}. \quad (8.37)$$

For $|Q| = 0.1$ and $|Q| = 0.01$, we have $p > 0.66$ and $p > 0.68$, respectively. It may be of interest to study whether BD theory with $10^{-3} \lesssim |Q| \lesssim 0.1$ can leave some observational signatures different from those in the LCDM model.

If we transform the action (8.1) to that in the Einstein frame under the conformal transformation, the scalar field $\phi$ has a direct coupling $Q$ with nonrelativistic matter (dark matter and baryons) [85, 208]. Instead, we may start from the Einstein frame action by assuming that the field $\phi$ is coupled to dark matter alone [306]. Then, the solar-system constraints are evaded due to the absence of interactions between baryons and the scalar field. In this coupled quintessence scenario with the inverse power-law potential $V(\phi) = V_0 \phi^{-p}$ ($p > 0$), the Planck CMB data placed the upper bound $|Q| < 0.066$ at 95% CL [307]. Combined with the BAO and weak lensing data, there is a peak around $|Q| = 0.04$ for the marginalized distribution of $Q$ [307, 308]. The statistically strong observational evidence for supporting the nonvanishing coupling $Q$ has been still lacking, but the future high-precision observations may clarify whether the coupling of order $|Q| = \mathcal{O}(0.01)$ is really favored from the data.

IX. CLASS (C): KINETIC BRAIDINGS

The Lagrangian of class (C), which is known as kinetic braidings, is expressed in the form

$$L = G_2(\phi, X) + G_3(\phi, X)\Box\phi + \frac{M_{\text{pl}}^2}{2} R. \quad (9.1)$$
Since \( q_t = M^2_{\text{pl}} \), the no-ghost condition of tensor perturbations is satisfied. The parameter \( \alpha_M \) vanishes due to the absence of nonminimal couplings \( G_4(\phi) \). Then, the above theories automatically evade the bound (6.27).

Unlike \( f(R) \) gravity, we do not need to consider a heavy mass of \( \phi \) in dense regions to suppress the propagation of fifth forces, so we will focus on the case in which the field mass squared \( M^2_\phi = -G_{2,\phi\phi} \) does not exceed the order of \( H^2 \). Provided that the ratio \( q_t/q_s \) is not much different from 1, the term 2\( a^2M^2_\phi q_t/(c_s^2k^2q_s) \) in the expression of \( \Delta_2 \) of Eq. (6.14) is much smaller than 1 for the modes deep inside the sound horizon. Substituting \( \Delta_2 \simeq \phi^a q_s c_s^2/(4M^2_{\text{pl}}H^2) \) into Eq. (6.33), it follows that

\[
\mu = \Sigma \simeq 1 + \frac{4M^4_{\text{pl}}H^2\alpha_B^2}{\phi^2q_sc_s^2},
\]

which does not have the scale-dependence. The evolution of linear perturbations is determined by the product \( 4M^4_{\text{pl}}H^2\alpha_B^2/(\phi^2q_sc_s^2) \). For smaller \( 4M^4_{\text{pl}}H^2\alpha_B^2/(\phi^2q_sc_s^2) \), the larger modification of gravity arises on scales relevant to the growth of large-scale structures. Due to the absence of nonminimal couplings in the Lagrangian (9.1), the typical solution around a spherically symmetric and static body corresponds to \( \phi = \text{constant} \) [99, 100], so that the propagation of fifth forces is suppressed in local regions of the Universe.

### A. Concrete dark energy models

We present models of the late-time cosmic acceleration which belong to the Lagrangian (9.1). The cubic covariant Galileon [87] corresponds to

Model (C1): \[ L = \beta_1X - m^3\phi + \beta_3X\Box\phi + \frac{M^2_{\text{pl}}}{2}R, \]

where \( \beta_1, \beta_3, m \) are constants. In the limit of Minkowski spacetime, the equations of motion following from the Lagrangian (9.3) respects the Galilean symmetry \( \partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu \).

In absence of the linear linear potential \( V(\phi) = m^3\phi \), it is known that there exists a tracker solution for the full Galileon Lagrangian containing quartic and quintic couplings [110, 111]. The dark energy EOS along the tracker is given by \( w_{\text{DE}} = -2 \) during the matter era. This is followed by the approach to the self-accelerating de Sitter solution characterized by \( \dot{X} = \text{constant} \). The tracker solution is disfavored from the joint data analysis of SNIa, CMB, and BAO [112] because of the large deviation of \( w_{\text{DE}} \) from \(-1\) in the matter era. Even in the case where the solutions approach the tracker at late times, the covariant Galileon without the linear potential is in strong tension with observational data [113, 114] due to a very different structure formation pattern compared to that in the \( \Lambda \)CDM model [108].

The likelihood analysis performed in Refs. [115, 118] assumed that the Galileon potential is absent. In presence of the linear potential \( V(\phi) = m^3\phi \), we do not have the self-accelerating solution with \( X = \text{constant} \), but the late-time cosmic acceleration can be driven by \( V(\phi) \) (see also Ref. [309] for other potentials). After the field \( \phi \) enters the region \( V(\phi) < 0 \), the Universe starts to enter the collapsing stage. Existence of the linear potential can modify the dynamics of background and perturbations, so it is worthy of studying the compatibility of model (9.3) with observations. In Sec. X, we will discuss the cosmological dynamics of such a model by taking into account a more general nonminimal coupling \( G_4(\phi) \).

Instead of the linear potential \( V(\phi) = m^3\phi \), we can consider the quadratic kinetic term \( X^2 \). This theory is described by the Lagrangian [120]:

Model (C2): \[ L = \beta_1X + \beta_2X^2 + \beta_3X\Box\phi + \frac{M^2_{\text{pl}}}{2}R, \]

where \( \beta_1, \beta_2, \beta_3 \) are constants. This can be regarded as the ghost condensate model with the cubic Galileon term. In absence of the term \( \beta_2X^2 \), the solutions tend to approach the tracker solution mentioned above, which is observationally excluded. Moreover, the dominance of the cubic Galileon term \( \beta_3X\Box\phi \) at low redshifts typically gives rise to the galaxy-ISW anti-correlation, which is also disfavored from the data [117]. For the model (9.4), the new term \( \beta_2X^2 \) prevents the approach to the tracker solution, so that the deviation of \( w_{\text{DE}} \) from \(-1\) is smaller than that in the tracker case. Moreover, the cubic Galileon term can be subdominant to the contribution \( G_2 = \beta_1X + \beta_2X^2 \), so it is expected that the galaxy-ISW anti-correlation does not necessarily occur. In Sec. [XIX] we study the cosmological dynamics and observational signatures for the model (9.4).
B. Cubic Galileon with ghost condensate

As an example of class (C), we review the cosmology of model (C2) given by the Lagrangian (9.4). In theories beyond Horndeski gravity [110, 111], it is possible to introduce the X field density parameter (3.16) gives the relation \(\Omega_{\phi} = -\frac{\beta_1 \dot{\phi}^2}{6 M_{\text{pl}}^2 H^2} \), \(\Omega_{\phi 2} = \frac{\beta_2 \dot{\phi}^4}{4 M_{\text{pl}}^2 H^2} \), \(\Omega_{\phi 3} = -\frac{\beta_3 \phi^3}{M_{\text{pl}}^2 H} \), \(\Omega_r = \frac{\rho_r}{3 M_{\text{pl}}^2 H^2} \), (9.5)

which correspond to the density parameters arising from \(\beta_1 X, \beta_2 X^2, \beta_3 X \Box \phi \), and radiation, respectively. The Hamiltonian constraint (3.16) gives the relation

\[
\Omega_m \equiv \frac{\rho_m}{3 M_{\text{pl}}^2 H^2} = 1 - \Omega_{\text{DE}} - \Omega_r , \quad \Omega_{\text{DE}} \equiv \Omega_{\phi 1} + \Omega_{\phi 2} + \Omega_{\phi 3} .
\] (9.6)

The density parameters obey the differential equations:

\[
\dot{\Omega}_{\phi 1} = 2 \Omega_{\phi 1} (\epsilon_{\phi} - h) , \quad \dot{\Omega}_{\phi 2} = 2 \Omega_{\phi 2} (2 \epsilon_{\phi} - h) , \quad \dot{\Omega}_{\phi 3} = \Omega_{\phi 3} (3 \epsilon_{\phi} - h) , \quad \dot{\Omega}_{r} = -2 \Omega_{r} (2 + h) ,
\] (9.7)

where, from Eqs. (3.11) and (3.12), the quantities \(h \equiv \ddot{H}/H^2\) and \(\epsilon_{\phi} \equiv \dddot{\phi}/(H \dot{\phi})\) are given, respectively, by

\[
h = -\frac{2(3 \Omega_{\phi 1} + \Omega_{\phi 2} + \Omega_{r} + 3)(\Omega_{\phi 1} + 2 \Omega_{\phi 2}) + 2 \Omega_{\phi 3} (6 \Omega_{\phi 1} + 3 \Omega_{\phi 2} + \Omega_{r} + 3) + 3 \Omega_{\phi 3}^2}{4 (\Omega_{\phi 1} + 2 \Omega_{\phi 2} + \Omega_{\phi 3}) + \Omega_{\phi 3}^2} ,
\] (9.8)

\[
\epsilon_{\phi} = -\frac{4(3 \Omega_{\phi 1} + 2 \Omega_{\phi 2}) - \Omega_{\phi 3} (3 \Omega_{\phi 1} + \Omega_{\phi 2} + \Omega_{r} - 3)}{4 (\Omega_{\phi 1} + 2 \Omega_{\phi 2} + \Omega_{\phi 3}) + \Omega_{\phi 3}^2} .
\] (9.9)

The dark energy EOS defined by Eq. (3.20) yields

\[
w_{\text{DE}} = \frac{3 \Omega_{\phi 1} + \Omega_{\phi 2} - \epsilon_{\phi} \Omega_{\phi 3}}{3(\Omega_{\phi 1} + \Omega_{\phi 2} + \Omega_{\phi 3})} .
\] (9.10)

For the dynamical system given by Eq. (9.7), there exists a self-accelerating de Sitter solution satisfying \(\dot{\phi} = \text{constant}\) and \(H = \text{constant}\), in which case \(\epsilon_{\phi} = 0, h = 0, \) and \(\Omega_r = 0\). From Eqs. (9.8) and (9.9), we obtain the following two relations on the de Sitter solution:

\[
\Omega_{\phi 1} = -2 + \frac{1}{2} \Omega_{\phi 3} , \quad \Omega_{\phi 2} = 3 - \frac{3}{2} \Omega_{\phi 3} ,
\] (9.11)

under which the dark energy EOS (9.10) reduces to \(w_{\text{DE}} = -1\). As shown in Ref. [120], this de Sitter solution is always a stable attractor. In the limit \(\Omega_{\phi 3} \to 0\), we have \(\Omega_{\phi 1} = -2 \) and \(\Omega_{\phi 2} = 3\), so that \(\beta_1 = -\beta_2 \phi^2\). Indeed, this is equivalent to the de Sitter solution in ghost condensate corresponding to \(G_{2,X} = 0\) in Eq. (7.2). As in the case of ghost condensate, we consider the couplings \(\beta_1 < 0 \) and \(\beta_2 > 0\) in the following discussion.

In the radiation and deep matter eras, we first consider the case in which \(\Omega_{\phi 3}\) dominates over \(\Omega_{\phi 1}\) and \(\Omega_{\phi 2}\). In these epochs, the field density parameters are smaller than the order 1, so we obtain the approximate relations \(h \simeq -(\Omega_r + 3)/2\) and \(\epsilon_{\phi} \simeq (\Omega_r - 3)/4\) from Eqs. (9.8) and (9.9). Substituting them into Eq. (9.10), it follows that

\[
w_{\text{DE}} \simeq \frac{1}{4} - \frac{1}{12} \Omega_r ,
\] (9.12)

and hence \(w_{\text{DE}} \simeq 1/6\) in the radiation era and \(w_{\text{DE}} \simeq 1/4\) in the matter era. This behavior can be confirmed in the numerical simulation of Fig. 3. Since \(h \simeq -3/2\) and \(\epsilon_{\phi} \simeq -3/4\) during the matter dominance, we can integrate Eq. (9.7) to give \(\Omega_{\phi 1} \propto a^{3/2}, \Omega_{\phi 2} \propto a^0\), and \(\Omega_{\phi 3} \propto a^{-3/4}\). Then, the density parameters \(\Omega_{\phi 1}\) and \(\Omega_{\phi 2}\) eventually catch up with \(\Omega_{\phi 3}\). After this catch up, the evolution of \(w_{\text{DE}}\) is no longer described by Eq. (9.12). There is a transient period in which \(w_{\text{DE}}\) enters the region \(w_{\text{DE}} < -1\). The solutions finally approach the de Sitter fixed point characterized by Eq. (9.11).
The present epoch \((z = 0)\) corresponds to \(\Omega_{\text{DE}} = 0.68\) and \(\Omega_r \approx 10^{-4}\). (Right) Evolution of \(\mu (= \Sigma)\) corresponding to the cases (a), (b), (c), (d) plotted in the left panel.

For smaller initial conditions of \(\Omega_{\phi 2}\), the minimum value of \(w_{\text{DE}}\) reached during the transient period tends to decrease toward \(-2\), see the left panel of Fig. 3. This can be understood as follows. Taking the limit \(\Omega_{\phi 2} \to 0\) in Eq. (9.7), the quantity \(y \equiv 2\Omega_{\phi 1} + \Omega_{\phi 3}\) obeys the differential equation

\[
y' = y \frac{12\Omega_{\phi 1}^2 + \Omega_{\phi 1}(4\Omega_r - 12 + 27\Omega_{\phi 3}) + \Omega_{\phi 3}(5\Omega_r - 3 + 3\Omega_{\phi 3})}{4\Omega_{\phi 1} + 4\Omega_{\phi 3} + \Omega_{\phi 3}^2},
\]

which admits the solution \(y = 0\), i.e., \(\Omega_{\phi 3} = -2\Omega_{\phi 1}\). Substituting this relation and Eq. (9.9) into Eq. (9.10), it follows that

\[
w_{\text{DE}} = -\frac{6 + \Omega_r}{3(1 - \Omega_{\phi 1})}.
\]

This is the tracker solution \([110, 111]\) along which \(w_{\text{DE}}\) evolves as \(-7/3\) (radiation era) \(\to -2\) (matter era) \(\to -1\) (de Sitter era satisfying \(\Omega_{\phi 1} = -1\) and \(\Omega_{\phi 3} = 2\)). As we already mentioned, the tracker is in tension with the observational data \([112]\) due to the large deviation of \(w_{\text{DE}}\) from \(-1\). Since \(\Omega_{\phi 2} \neq 0\) in the present model, the dominance of \(\Omega_{\phi 2}\) over \(\Omega_{\phi 3}\) prevents the approach to the tracker. For larger initial values of \(\Omega_{\phi 2}\), the deviation of \(w_{\text{DE}}\) from \(-1\) tends to be smaller. Moreover, the approach to \(w_{\text{DE}} = -1\) occurs at higher redshifts. Thus, the existence of the term \(\beta_2 X^2\) allows an interesting possibility for realizing the phantom equation of state \((w_{\text{DE}} < -1)\) which can be compatible with observations.

The quantities \(q_{s}\) and \(c_{s}^2\) can be expressed, respectively, as

\[
q_{s} = \frac{3M_{\text{pl}}^4 H^2}{\dot{\phi}^2} (4\Omega_{\phi 1} + 8\Omega_{\phi 2} + 4\Omega_{\phi 3} + \Omega_{\phi 3}^2),
\]

\[
c_{s}^2 = \frac{12\Omega_{\phi 1} + 8\Omega_{\phi 2} + 4(\epsilon_{\phi} + 2)\Omega_{\phi 3} - \Omega_{\phi 3}^2}{3(4\Omega_{\phi 1} + 8\Omega_{\phi 2} + 4\Omega_{\phi 3} + \Omega_{\phi 3}^2)}.
\]

The conditions \(q_s > 0\) and \(c_s^2 > 0\) hold for all the cases shown in Fig. 3, so there are no ghost and Laplacian instabilities of scalar perturbations. Substituting Eq. (9.15) and \(\alpha_5 = -\Omega_{\phi 3}/2\) into Eq. (9.2), we obtain

\[
\mu = \Sigma \simeq 1 + \frac{\Omega_{\phi 3}^2}{12\Omega_{\phi 1} + 8\Omega_{\phi 2} + 4(\epsilon_{\phi} + 2)\Omega_{\phi 3} - \Omega_{\phi 3}^2},
\]
so that the deviations of $\mu$ and $\Sigma$ arise from the cubic coupling. Using the relations \([9.11]\) on the de Sitter fixed point, it follows that

$$\mu = \Sigma \simeq 1 + \frac{\Omega_{\phi3}}{2 - \Omega_{\phi3}} . \tag{9.17}$$

In the right panel of Fig. 3, the evolution of $\mu$ is plotted for four different cases corresponding to those in the left panel. Solving the perturbation equations of motion numerically, we compute $\mu$ and $\Sigma$ according to their definitions given in Eqs. (6.3) and (6.6). We confirmed that the analytic estimation (9.16), which is derived under the quasi-static approximation, is sufficiently accurate for the modes deep inside the sound horizon. In the deep matter era, $\mu$ is close to 1 in all these cases, but the difference arises at low redshifts depending on the values of $\Omega_{\phi3}$. For the cases (a), (b), (c), (d), the numerical values of $\Omega_{\phi3}$ today are given, respectively, by $2.3 \times 10^{-2}$, $5.6 \times 10^{-2}$, $2.0 \times 10^{-1}$, $5.1 \times 10^{-1}$, so the deviation of $\mu$ from 1 tends to be more significant for larger $\Omega_{\phi3}$. This property is also consistent with the estimation (9.17) on the de Sitter solution.

For the cases in which $\Omega_{\phi3}$ gives the large contribution to the field density today (say $\Omega_{\phi3}(z = 0) \gg \mathcal{O}(0.1)$), the model may be tightly constrained from the galaxy-ISW correlation data. Moreover, in such cases, the deviation of $\alpha_{DE}$ from $-1$ at low redshifts tends to be significant. It will be of interest to place observational constraints on today’s value of $\Omega_{\phi3}$ by performing the joint likelihood analysis including the galaxy-ISW correlation data.

X. CLASS (D): NONMINIMALLY COUPLED SCALAR WITH CUBIC DERIVATIVE INTERACTIONS

The class (D) is the most general Horndeski theories given by the Lagrangian (3.31) in which $c_\phi^2$ is exactly equivalent to 1. In this case, the quantities $\alpha_B$ and $\alpha_M$ give rise to nonvanishing contributions to the second terms in Eqs. (6.20) and (6.22). The difference between $\mu$ and $\Sigma$ is given by

$$\mu - \Sigma = -\frac{2M_{\text{pl}}^2 H^2 q_s \alpha_M (\alpha_B - \alpha_M)}{\phi^2 q_s c_s^2}, \tag{10.1}$$

which means that the nonminimal coupling $G_A(\phi) R$ generally leads to $\mu \neq \Sigma$ due to the nonvanishing $\alpha_M$. This property is different from that in the models of classes (A) and (C).

In Sec. VII, we showed that it is difficult to observationally distinguish $f(R)$ gravity and BD theory with a scalar potential from the $\Lambda$CDM model. In such class (B) theories, the field potential needs to be designed to have a large mass in regions of the high density for the chameleon mechanism to work. There is yet another mechanism dubbed the Vainshtein mechanism [107] in which field nonlinear self-interactions (like the cubic Galileon) can suppress the propagation of fifth forces in local regions of the Universe. In such cases, the background cosmological dynamics and the evolution of perturbations are different from those in class (B).

In class (D) the quantity $\Sigma$ can grow rapidly at low redshifts, while this is not generally the case for class (B) in which $\Sigma = 1/(16\pi G G_4)$. This allows the possibility for distinguishing the models between classes (B) and (D) from the observations of galaxy-ISW correlations further. The class (D) is categorized as Horndeski theories containing a nonminimally coupled light scalar field with cubic derivative interactions.

A. Concrete dark energy models

One of the theories in class (D) is known as generalized BD theories [186] given by the Lagrangian

$$\text{Model (D1): } L = \omega \left(\frac{\phi}{M_{\text{pl}}} \right)^{1-n} X + \frac{\lambda}{\mu^3} \left(\frac{\phi}{M_{\text{pl}}} \right)^{-n} X \Box \phi + \frac{M_{\text{pl}}^2}{2} \left(\frac{\phi}{M_{\text{pl}}} \right)^{3-n} R , \tag{10.2}$$

where $\mu$ ($>0$) is a constant having a dimension of mass, and $n, \lambda, \omega$ are dimensionless constants. The original BD theory [60] without the cubic coupling corresponds to the power $n = 2$. In presence of the cubic derivative interaction, there is a de Sitter solution characterized by $\dot{\phi}/(H\phi) = \text{constant}$ even without the scalar potential. The case $n = 2$ was first studied in Ref. [183] (see also Refs. [184] [185]) and then it was generalized to the more general case with $n \neq 2$ in Ref. [186]. The power $n$ is constrained to be

$$2 \leq n \leq 3 , \tag{10.3}$$

to satisfy theoretical consistent conditions throughout the cosmic expansion history [186]. In model (D1), the cubic derivative coupling is the main source for the late-time cosmic acceleration. For $n = 2$, it was shown in Ref. [184]
that the galaxy-ISW anti-correlation tends to be stronger for larger $|\omega|$. For $n \neq 2$, the galaxy-ISW correlation power spectrum was not computed yet, but it is expected that the anti-correlation can still persist in the case in which the cubic interaction provides the dominant contribution to the dark energy density. It remains to be seen to put observational constraints on model (D1) by using the galaxy-ISW correlation data.

There is also the nonminimally coupled cubic Galileon model given by the Lagrangian

\[
\text{Model (D2): } L = (1 - 6Q^2) e^{-2Q(\phi - \phi_0)/M_{\text{pl}}} X - m^3 \phi + \beta_3 X \Box \phi + \frac{M_{\text{pl}}^2}{2} e^{-2Q(\phi - \phi_0)/M_{\text{pl}}} R, \tag{10.4}
\]

where $Q, \phi_0, m, \beta_3$ are constants. In the limit $Q \to 0$, Eq. (10.4) recovers the Lagrangian (9.3) of model (C1), i.e., cubic Galileons with $\beta_1 = 1$. In high-density regions, the cubic Galileon term can suppress the fifth force induced by the nonminimal coupling $e^{-2Q(\phi - \phi_0)/M_{\text{pl}}} R$. The cosmological dynamics of model (D2) was studied in Ref. [119] in a more general context beyond the Horndeski domain. In this model, the main source for the late-time cosmic acceleration is the linear potential $V(\phi) = m^3 \phi$ rather than the cubic Galileon term $\beta_3 X \Box \phi$. Then, unlike model (D1), the dominance of cubic interactions over $V(\phi)$ does not typically occur at low redshifts. In Sec. IX B, we discuss the cosmological dynamics and observational signatures of model (D2).

We note that it is also possible to take into account the nonminimal coupling for model (C2) introduced in Sec. IX. It remains to be seen to explore whether such a model is consistent with cosmological and local gravity constraints.

### B. Cosmology in nonminimally coupled cubic Galileons

To study the cosmological dynamics in model (D2) given by the Lagrangian (10.4), we introduce the following dimensionless quantities:

\[
x_1 \equiv \frac{\dot{\phi}}{\sqrt{6} M_{\text{pl}} H}, \quad \Omega_{\phi_2} \equiv \frac{m^3 \phi}{3 M_{\text{pl}}^2 H^2 F}, \quad \Omega_{\phi_3} \equiv -\frac{\beta_3 \dot{\phi}^3}{M_{\text{pl}}^2 H F}, \quad \Omega_r \equiv \frac{\rho_r}{3 M_{\text{pl}}^2 H^2 F}, \quad \lambda \equiv -\frac{M_{\text{pl}}^2}{\phi}, \tag{10.5}
\]

where $F \equiv e^{-2Q(\phi - \phi_0)/M_{\text{pl}}}$. The effective gravitational coupling in a screened environment is given by $G_{\text{eff}} = G/F$, so we consider the case in which the field value $\phi$ today is equivalent to $\phi_0$, i.e., $F(z = 0) = 1$. Taking into account the radiation and nonrelativistic matter as in the analysis of Sec. IX B, the Hamiltonian constraint (3.16) leads to

\[
\Omega_m \equiv \frac{\rho_m}{3 M_{\text{pl}}^2 H^2 F} = 1 - (1 - 6Q^2) x_1^2 - 2\sqrt{6} Q x_1 - \Omega_{\phi_2} - \Omega_{\phi_3} - \Omega_r. \tag{10.6}
\]

The quantities defined by Eq. (10.5) satisfy the differential equations:

\[
x'_1 = x_1 (\epsilon_\phi - h), \quad \Omega'_{\phi_2} = \Omega_{\phi_2} \left[\sqrt{6} (2Q - \lambda) x_1 - 2h\right], \quad \Omega'_{\phi_3} = \Omega_{\phi_3} \left(2\sqrt{6} Q x_1 + 3\epsilon_\phi - h\right), \quad \Omega'_r = 2\Omega_r \left(\sqrt{6} Q x_1 - 2h\right), \quad \lambda' = \sqrt{6} \lambda^2 x_1, \tag{10.7}
\]

where the quantities $h = \dot{H}/H^2$ and $\epsilon_\phi = \ddot{\phi}/(H \dot{\phi})$ are given, respectively, by

\[
h = -[\Omega_{\phi_3}(6 + 2\Omega_r - 6\Omega_{\phi_2} + 2\Omega_{\phi_3}) + \sqrt{6} \Omega_{\phi_3}(2Q - \lambda) \Omega_{\phi_2}] x_1 + 2(3 + \Omega_r - 3\Omega_{\phi_2} + 6\Omega_{\phi_3} - 6\lambda Q \Omega_{\phi_2} + 6Q^2(1 - \Omega_r + 3\Omega_{\phi_2} - 2\Omega_{\phi_3})) x_1^2 + 2\sqrt{6} \Omega_r (6Q^2 - 1)(\Omega_{\phi_3} - 2) x_1^3 + 6(12Q^4 - 8Q^2 + 1)x_1^4]/D, \tag{10.8}
\]

\[
\epsilon_\phi = [\sqrt{6} \Omega_{\phi_3}(\Omega_r - 3\Omega_{\phi_2} - 3) + 12Q(\Omega_r - 1 - 3\Omega_{\phi_2} - 2\Omega_{\phi_3}) + \lambda Q \Omega_{\phi_2}] x_1 + 3\sqrt{6}(\Omega_{\phi_3} - 4 + 2Q^2(4 + \Omega_{\phi_3})) x_1^2 + 12Q(5 - 6Q^2)x_1^3]/(\sqrt{6} D), \tag{10.9}
\]

with

\[
D = 4x_1^4 + 4\Omega_{\phi_3} + 4\sqrt{6} Q x_1 \Omega_{\phi_3} + \Omega_{\phi_3}^2. \tag{10.10}
\]

The dark energy EOS defined by Eq. (3.20) can be expressed as

\[
w_{\text{DE}} = \frac{3 + 2h - [3 + 2h + 3(1 + 2Q^2)x_1^2 - 3\Omega_{\phi_2} - \epsilon_\phi \Omega_{\phi_3} - 2\sqrt{6}Q x_1(2 + \epsilon_\phi)]F}{3 - [3 - \Omega_{\phi_2} - \Omega_{\phi_3} + (6Q^2 - 1)x_1^2 - 2\sqrt{6}Q x_1]} F, \tag{10.11}
\]

\[\text{In Ref. [119], the dark energy equation of state was defined in a different way such that Eqs. (3.16) and (3.17) are expressed to contain the terms } 3M_{\text{pl}}^2 F_0 H^2 \text{ and } 2M_{\text{pl}}^2 F_0 \tilde{H} \text{ on the left hand sides of them, respectively, where } F_0 \text{ is today’s value of } F. \text{ The definition of } w_{\text{DE}} \text{ in this review corresponds to the choice } F_0 = 1, \text{ which should be more suitable to constrain the model from observations.}
Let us consider the case in which the cubic Galileon density parameter $\Omega_{\phi3}$ (assumed to be positive) dominates over the other field density parameters in the early cosmological epoch, except for $Qx_1$, such that $1 \gg \Omega_{\phi3} \gg \{x_1^2, \Omega_{\phi2}\}$. Then, Eqs. (10.8) and (10.9) approximately reduce to

$$h \simeq -\frac{3}{2} - \frac{\Omega_r}{2}, \quad \epsilon_\phi \simeq -\frac{3}{4} + \frac{\Omega_r}{4} - \frac{\sqrt{6}}{2} \frac{Qx_1}{\Omega_{\phi3}}.$$  

(10.12)

Depending on the initial ratio $Qx_1/\Omega_{\phi3}$, there are two qualitatively different cases for the evolution of $x_1$, $\Omega_{\phi2}$, and $\Omega_{\phi3}$. If $|Qx_1| \lesssim \Omega_{\phi3}$, then we have $h \simeq -2$ and $\epsilon_\phi \simeq -1/2$ during the radiation dominance, so we obtain the integrated solutions $x_1^2 \propto a^3$, $\Omega_{\phi2} \propto a^4$, and $\Omega_{\phi3} \propto a^{1/2}$. In this case, even if $\Omega_{\phi3}$ initially dominates over the other field densities, $\Omega_{\phi2}$ quickly catches up with $\Omega_{\phi3}$. After this catch up, both $x_1^2$ and $\Omega_{\phi3}$ become much smaller than $\Omega_{\phi2}$, so the linear potential $V(\phi) = m^2\phi$ gives the dominant contribution to the scalar-field dynamics.

The other case corresponds to the initial conditions satisfying $|Qx_1| \gg \Omega_{\phi3}$. Since there is the deviation of $\Omega_r$ from 1 even during the radiation-dominated epoch, it can happen that the last term in $\epsilon_\phi$ of Eq. (10.12) exceeds the order 1. For $Qx_1/\Omega_{\phi3} < 0$, $\epsilon_\phi$ can be positive during the radiation era. In the simulation of Fig. 4, for example, the numerical value of $\epsilon_\phi$ at the redshift $z \approx 10^6$ is around 0.5, in which case the field density parameters evolve as $x_1^2 \propto a^3$, $\Omega_{\phi2} \propto a^4$, and $\Omega_{\phi3} \propto a^{3.5}$. Then, even if $x_1^2 < \Omega_{\phi2}, \Omega_{\phi3}$ initially, the Universe enters the stage in which $x_1^2$ dominates over $\Omega_{\phi2}$ and $\Omega_{\phi3}$, see Fig. 4. In this case, after the radiation dominance, the solutions temporally approach the so-called $\phi$-matter-dominated epoch ($\phi$MDE) characterized by the fixed point

$$(x_1, \Omega_{\phi2}, \Omega_{\phi3}) = \left(-\frac{\sqrt{6}Q}{3(1-2Q^2)}, 0, 0\right), \quad \text{with} \quad w_{\text{eff}} = \frac{4Q^2}{3(1-2Q^2)}, \quad \text{and} \quad \Omega_m = \frac{3-2Q^2}{3(1-2Q^2)^2}. \quad (10.13)$$

For $Q > 0$ satisfying the condition $Q^2 < 1/2$, the existence of $\phi$MDE requires that $x_1 < 0$, i.e., $\phi < 0$. In this case, the field rolls down along the linear potential $V(\phi) = m^2\phi$ with $m > 0$. The cosmic acceleration occurs in the region $\phi > 0$, so the existence of $\phi$MDE requires that $\lambda = -M_{\phi3}/\phi < 0$. Alternatively, as in Ref. [119], we can consider the cases $Q < 0, x_1 > 0, \lambda > 0$.

In the left panel of Fig. 4, we can confirm that $x_1^2$ temporally approaches a constant determined by Eq. (10.13). As we observe in the right panel of Fig. 4, the effective EOS $w_{\text{eff}}$ starts to evolve from the value close to 1/3 and temporally approaches the $\phi$MDE value $4Q^2/[3(1-2Q^2)]$. After the linear potential $V(\phi) = m^2\phi$ dominates over the other field densities, the Universe eventually enters the stage of cosmic acceleration ($w_{\text{eff}} < -1/3$). In Fig. 4, we also show the evolution of $w_{DE}$ given by Eq. (10.11), whose value during the matter era is slightly larger than 0. Compared to $w_{\text{eff}}$, the transition of $w_{DE}$ to the value around −1 occurs earlier.

In comparison to model (C2) discussed in Sec. X.B, the contribution of $\Omega_{\phi3}$ to the total field density tends to be negligible at an earlier cosmological epoch. This is attributed to the fact that $\Omega_{\phi2}$ in model (D2) evolves as $\Omega_{\phi2} \propto a^3$ during the matter era, whereas, in model (C2), the density parameter $\Omega_{\phi2}$ associated with the Lagrangian $\beta_2X^2$ evolves much more slowly ($\Omega_{\phi2} \propto a^0$). In the latter case, the cubic Galileon density can dominate over the other field densities at later cosmological epochs, so the dark energy EOS enters the region $w_{DE} < -1$ by temporally approaching the tracker ($w_{DE} = -2$). This behavior does not occur for model (D2), in which $w_{DE}$ typically stays in the region $w_{DE} > -1$, see the right panel of Fig. 4. Hence these two models can be observationally distinguished from each other even at the background level. In model (D2), $\Omega_{\phi3}$ is usually much smaller than 1 today, so it is expected that the galaxy-ISW correlation data do not put tight constraints on the model parameters.

The quantities $\alpha_M$ and $\alpha_B$, reduce, respectively, to

$$\alpha_M = -2\sqrt{6}Qx_1, \quad \alpha_B = -\sqrt{6}Qx_1 - \frac{\Omega_{\phi3}}{2}. \quad (10.14)$$

After $\Omega_{\phi3}$ becomes much smaller than $|Qx_1|$, the simple relation $\alpha_M \simeq 2\alpha_B$ holds. Indeed, this property can be confirmed in the left panel of Fig. 5. We recall that there is the bound (6.27) on $\alpha_M(z = 0)$ arising from Laser Laser Ranging experiments. In the numerical simulation on the left panel of Fig. 5, which corresponds to $Q = 0.02$, today’s value of $\alpha_M$ is $7.0 \times 10^{-3}$ and hence the bound (6.27) is satisfied. For the initial conditions similar to those used in Fig. 4, the coupling needs to be in the range $|Q| \lesssim 0.04$ for the consistency with (6.27).

The quantities associated with the stability of scalar perturbations are given by

$$q_s = \frac{M_{pl}^2 F^2(4x_1^2 + 4\Omega_{\phi3} + \Omega_{\phi3}^2 + 4\sqrt{6}Qx_1\Omega_{\phi3})}{2x_1^2}, \quad c_s^2 = \frac{4x_1^2 + 4(2 + \epsilon_\phi)\Omega_{\phi3} - \Omega_{\phi3}^2 + 4\sqrt{6}Qx_1\Omega_{\phi3}}{3(4x_1^2 + 4\Omega_{\phi3} + \Omega_{\phi3}^2 + 4\sqrt{6}Qx_1\Omega_{\phi3})}.$$  

(10.15)

At the late cosmological epoch in which the contribution of $\Omega_{\phi3}$ is suppressed relative to the terms containing $x_1$ in Eq. (10.15), it follows that $q_s \simeq 2M_{pl}^2 F^2$ and $c_s^2 \simeq 1$. In the numerical simulations of Figs. 4 and 5, we confirmed that
both $q_3$ and $\epsilon_3^2$ are positive from the radiation era to today. On using Eqs. (6.20)-(6.22) together with the property $M_\phi^2 = 0$ and ignoring the contribution $\Omega_{\phi_3}$ at late times, we obtain

$$\mu \simeq \frac{1}{F} \left( 1 + 2Q^2 \right), \quad \eta \simeq \frac{1 - 2Q^2}{1 + 2Q^2}, \quad \Sigma \simeq \frac{1}{F}. \quad (10.16)$$

For $Q > 0$, the quantity $F = e^{-2Q(\phi - \phi_0)}/M_\phi$ is smaller than 1 in the past ($\phi > \phi_0$). Since $Q/\lambda < 0$ for the existence of $\phi MDE$, both $\mu$ and $\Sigma$ are larger than 1. Since $Q^2 = 4 \times 10^{-4} \ll 1$ in the numerical simulation of Fig. 5, the deviation of $\mu$ from 1 mostly arises from the term $1/F$ ($> 1$). This property is different from that in $f(R)$ gravity, in which the large coupling $Q^2 = 1/6$ gives rise to the strong gravitational interaction with matter. Since $\mu - \Sigma \simeq 2Q^2/F$, the difference between $\mu$ and $\Sigma$ is much smaller than 1, see Fig. 5.

Let us estimate the evolution of $\delta_m$ during the $\phi MDE$. Substituting $\mu \simeq (1 + 2Q^2)/F$ into Eq. (6.23), it follows that

$$\frac{d^2\delta_m}{dN^2} + \frac{1}{2} \left( 1 - 3w_{\text{eff}} \right) \frac{d\delta_m}{dN} - \frac{3}{2} \left( 1 + 2Q^2 \right) \Omega_m \delta_m \simeq 0. \quad (10.17)$$

Using the values of $w_{\text{eff}}$ and $\Omega_m$ given in Eq. (10.13), the growing-mode solution to Eq. (10.17) for $Q^2 < 1/2$ is

$$\delta_m \propto a^{(1+2Q^2)/(1-2Q^2)}. \quad (10.18)$$

In the limit $Q \to 0$, we recover the standard evolution $\delta_m \propto a$ during the matter era. Although the coupling $Q$ leads to the faster growth of $\delta_m$, we recall that there exists the upper bound $|Q| < O(0.01)$ from (6.27). In the right panel of Fig. 5, we plot the evolution of $\delta'_m/\delta_m$ for $Q = 0.02$ together with that in the $\Lambda$CDM model. From Eq. (10.18), the value of $\delta'_m/\delta_m$ during the $\phi MDE$ is estimated as $\delta'_m/\delta_m \simeq 1.0016$, so it is hardly distinguishable from that in the $\Lambda$CDM model. This property can be confirmed in the numerical simulation of Fig. 5 at high redshifts. Indeed, even at low redshifts, the evolution of $\delta'_m/\delta_m$ is almost the same as that in the $\Lambda$CDM model. As we observe in the left panel of Fig. 5, the deviation of $\mu$ from 1 is at most of order 0.01 and hence the growth of $\delta_m$ is hardly affected by the modification of gravity.

The gravitational potentials $-\Psi$ and $\psi_{\text{eff}}$ also evolve in the similar way to those in the $\Lambda$CDM model with a small gravitational slip, i.e., $1 - \eta \simeq 4Q^2/(1 + 2Q^2) \ll 1$ for $|Q| < O(0.01)$. Although the model (D2) does not leave distinguished observational signatures for the cosmic growth history, the background evolution of $w_{\text{DE}}$ can be clearly distinguished from other (extended) Galileon models like (C2).
XI. CONCLUSIONS

Two decades have passed after the first discovery of late-time cosmic acceleration, but we did not identify the source for this phenomenon yet. Although the cosmological constant $\Lambda$ is the simplest candidate for dark energy, it is still a challenging problem to explain the very small value of $\Lambda$ consistent with today’s dark energy scale. Moreover, in the $\Lambda$CDM model, the values of $H_0$ and $\sigma_8(z=0)$ constrained from the Planck CMB data have been in tension with those measured in low-redshift observations. If there are other possibilities to explain the cosmic acceleration, it is worthwhile to construct theoretically consistent dark energy models and explore their observational signatures to distinguish them from the $\Lambda$CDM model.

In dynamical dark energy models, there are usually additional propagating degrees of freedom to those appearing in standard model of particle physics. A scalar field $\phi$, which is compatible with the isotropic and homogenous cosmological background, is one of the most natural candidates for such a new degree of freedom. If the scalar field has direct couplings to the gravity sector, Horndeski theories are known as the most general scalar-tensor theories with second-order equations of motion. As listed in Sec. II, Horndeski theories accommodate a wide variety of dark energy models proposed in the literature.

In Sec. IV, we derived the linear scalar perturbation equations of motion in full Horndeski theories without fixing the source for cosmic acceleration. A different choice of the starting point, with a different Lagrangian, would give a different evolution of the perturbations. The gravitational wave event GW170817 together with the gamma-ray burst GRB 170817A placed the tight observational bound on $c_t^2$. If we strictly demand that $c_t^2 = 1$, the Lagrangian of Horndeski theories is constrained to be of the form $\mathcal{L}_4 \supset G_4 \supset \beta_4 X^2$. In extended versions of Horndeski gravity, such as Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories and DHOST theories, there are models with $c_t^2 = 1$ even in presence of quartic-order derivative couplings. In this review, we did not explore the dark energy cosmology beyond the domain of Horndeski theories.

In Sec. IV, we derived the linear scalar perturbation equations of motion in full Horndeski theories without fixing the source for cosmic acceleration. A different choice of the starting point, with a different Lagrangian, would give a different evolution of the perturbations. The gravitational wave event GW170817 together with the gamma-ray burst GRB 170817A placed the tight observational bound on $c_t^2$. If we strictly demand that $c_t^2 = 1$, the Lagrangian of Horndeski theories is constrained to be of the form $\mathcal{L}_4 \supset G_4 \supset \beta_4 X^2$. In extended versions of Horndeski gravity, such as Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories and DHOST theories, there are models with $c_t^2 = 1$ even in presence of quartic-order derivative couplings. In this review, we did not explore the dark energy cosmology beyond the domain of Horndeski theories.

Finally, we should mention that, even in presence of the quartic Galileon term $G_4 \supset \beta_4 X^2$, there are cases in which the deviation of $c_t^2$ from 1 at low redshifts is so small that the bound $|c_t^2 - 1|$ can be consistently satisfied. It remains to be seen whether the model with such a tiny value of $|c_t^2 - 1|$ can survive in occurrence of further GW events with electromagnetic counterparts.
gauge conditions. This gauge-ready formulation is versatile in that any convenient gauges can be chosen depending on the problems at hand. The perturbation equations expressed in terms of gauge-invariant variables are directly applicable to the study of cosmic growth history (including initial conditions of perturbations). In Sec. [X], we identified conditions for the absence of scalar ghost and Laplacian instabilities in the small-scale limit by choosing three different gauges and showed that they are given by Eqs. (5.15) and (5.16) independent of the gauge choices.

In Sec. [X], we introduced two quantities \( \mu \) and \( \Sigma \) associated with the linear growth of nonrelativistic matter perturbations and the weak lensing gravitational potential. We computed them under the quasi-static approximation for the modes deep inside the sound horizon, see Eqs. (6.15) and (6.17). Under the stability conditions of tensor and scalar perturbations, the existence of scalar degree of freedom always enhances the gravitational coupling with matter, such that \( \mu > c_t^2/(8\pi G \dot{q}_t) \). Taking the limit \( c_t^2 \to 1 \), we showed that \( \mu \) and \( \Sigma \) can be expressed as Eqs. (6.20) and (6.22) in terms of two parameters \( \alpha_M \) and \( \alpha_B \) defined in Eq. (4.43). We classified surviving theories of the late-time cosmic acceleration into four classes according to these two parameters. In Sec. [X], we also showed that today’s value of \( \alpha_M \) is constrained as \( |\alpha_M(t_0)| < 0.02 \) from the Lunar Laser Ranging experiments measuring the variation of effective gravitational coupling in screened environments. In particular, this information can be used to constrain nonminimally coupled theories with the coupling \( G_4(\phi)R \).

In Secs. [X, X], we reviewed observational signatures of four classes of Horndeski theories with the exact value \( c_t^2 = 1 \) by paying particular attention to the evolution of dark energy EOS and the cosmic growth history. We summarize the main points in the following.

- Class (A): This is categorized as the theories with \( G_3 = 0 \) and \( G_4 = M_{pl}^2/2 \). In this case, we have \( \alpha_B = 0 = \alpha_M \) and hence \( \mu = 1 = \Sigma \). Quintessence and k-essence belong to this class.

Depending on the evolution of \( w_{DE} \) (which is in the range \( w_{DE} > -1 \) under stability conditions), we can classify quintessence into three subclasses: (a) thawing, (b) scaling, and (c) tracking, see Fig. [X]. The thawing quintessence is consistent with observations for \( w_{DE}(z = 0) \lesssim -0.7 \). The dark energy EOS in scaling quintessence realized by the potential (7.20) can be well approximated by Eqs. (7.22) for \( \lambda_5^2 \ll 1 \). The transition scale factor is constrained to be \( a_s < 0.11 \) (95% CL) to avoid the slow down of growth of structures induced by the scaling scalar field with \( c_t^2 = 1 \). In tracking quintessence with the inverse power-law potential (7.25), the power \( p \) is constrained to be \( p < 0.17 \) (95% CL) and hence positive integers \( p \geq 1 \) are excluded.

In k-essence, the ghost condensate model (7.29) predicts the evolution of \( w_{DE} \) consistent with observations for \( \lambda \lesssim 0.36 \). The k-essence Lagrangian (7.37) gives rise to the unified description of dark energy and dark matter with the scalar sound speed \( c_s^2 \) much smaller than 1.

- Class (B): Compared to class (A), a nonminimal coupling of the form \( G_4(\phi)R \) is present in class (B). In this case, there is the particular relation \( \alpha_B = \alpha_M/2 \neq 0 \). The interaction between the scalar field and matter enhances the gravitational coupling \( \mu = G_{eff}/G \), while this is not the case for \( \Sigma \). The \( f(R) \) gravity and BD theory with the scalar potential belong to this class.

In \( f(R) \) gravity, the models need to be constructed to suppress the propagation of fifth forces in regions of the high density, in which case the deviation of \( w_{DE} \) from \(-1\) occurs at low redshifts. After the perturbations enter the massless regime, the large coupling \( |Q| = 1/\sqrt{6} \) leads to the strong amplification of matter density contrast and gravitational potentials. For the consistency with CMB, BAO, and weak lensing data, the dark energy EOS \( w_{DE} \) and the deviation parameter \( B \) from the \( \Lambda \)CDM model need to be in the ranges \( |w_{DE} + 1| < 0.002 \) and \( B(z = 0) < 0.006 \) at 95% CL. The similar tight constraints on the deviation from the \( \Lambda \)CDM model also persist in BD theories with the potential (8.34) and the coupling \( |Q| \sim \mathcal{O}(0.1) \).

- Class (C): In this class, the cubic coupling \( G_3(\phi, X)\Box \phi \) is added to the Lagrangian of class (A). Since \( \alpha_M = 0 \), there is no gravitational slip \( \eta = -\dot{\Phi}/\Phi = 1 \) with \( \mu = \Sigma > 1 \). The bound (6.27) from the Lunar Laser Ranging experiments is trivially satisfied. In Sec. [X], we presented two models of this class, including the cubic Galileon with a linear potential [model (C1)].

In model (C2) given by the Lagrangian (9.4), the existence of an additional term \( \beta_2 X^2 \) to \( \beta_1 X \) gives rise to a self-accelerating de Sitter attractor \( (X = \text{constant}) \). If the dominance of this term over the cubic Galileon occurs at a later cosmological epoch, the deviations of \( w_{DE} \) from \(-1\) and \( \mu \) from 1 tend to be more significant, see Fig. [X]. The model (C2) allows an interesting possibility for realizing \( w_{DE} < -1 \) without having ghosts, while modifying the large-scale structure growth at low redshifts. We expect that the galaxy-ISW correlation data can provide an upper bound on today’s density parameter \( \Omega_{c5} \) of cubic Galileons.

- Class (D): This class corresponds to the most general Lagrangian in Horndeski theories with \( c_t^2 \) exactly equivalent to 1. The difference from class (C) is that the nonminimal coupling \( G_4(\phi)R \) is present. In model (D1) given by the Lagrangian (10.2), the cubic coupling \( G_3(\phi, X) \) is the main source for the late-time cosmic acceleration,
but in such cases the model typically predicts the galaxy-ISW anti-correlation which can be incompatible with observational data. In model (D2), i.e., nonminimally coupled cubic Galileons, there exists the φMDE followed by the cosmic acceleration driven by the linear potential \( V(\phi) = m^3 \phi \). In this case, the cubic Galileon density tends to be subdominant to the total field density at early cosmological epochs, so the dark energy density typically stays in the region \( \omega_{\text{DE}} > -1 \). From the bound (4.27), the coupling \( Q \) is constrained to be \(|Q| < \mathcal{O}(0.01)\), in which case the cosmic growth pattern is not much different from that in GR. However, it is possible to distinguish between model (D2) and uncoupled model (C2) due to different cosmic expansion histories and galaxy-ISW correlations.

We have thus shown that the surviving dark energy models after the GW170817 event predict different observational signatures which can be probed in current and future observational data. The dawn of GW astrophysics opened up new possibilities for measuring today’s Hubble parameter \( H_0 \) and the luminosity distance of GWs [141, 315]. In the concordance cosmological model there has been a tension of \( H_0 \) between CMB and low-redshift measurements, so new GW constraints on \( H_0 \) may give us a hint on whether we should go beyond the ΛCDM model or not. We also note that the GW luminosity distance is modified by the nonminimal coupling \( G_4(\phi)R \), so the accumulation of GW events will provide important information for the viability of models in classes (B) and (D). The models in classes (A) and (C) can be distinguished further by exploring whether the future observations favor the region \( \omega_{\text{DE}} > -1 \) or \( \omega_{\text{DE}} < -1 \). We hope that upcoming observational data will allow us to distinguish between four classes of dark energy models and that the origin of late-time cosmic acceleration is eventually identified.

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**Appendix A: Correspondence of notations and stability conditions with other papers**

| Cubic Lagrangian | Definition of X | Tensor no-ghost condition | Scalar no-ghost condition |
|-------------------|----------------|--------------------------|--------------------------|
| This paper        | \(+G_3(\phi, X)\square \phi\) | \(X = -\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi\) | \(q_t\) | Unitary gauge: \(q_s^{(u)}\) Flat gauge: \(q_s^{(f)}\) |
| Kobayashi et al.  | \(-G_3(\phi, X)\square \phi\) | \(X = -\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi\) | \(G_T = q_t\) | \(G_S = q_s^{(u)}\) |
| De Felice and Tsujikawa | \(-G_3(\phi, X)\square \phi\) | \(X = \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi\) | \(Q_T = \frac{q_t}{4}\) | \(Q_S = q_s^{(u)}\) |
| Bellini and Sawicki | \(-G_3(\phi, X)\square \phi\) | \(X = -\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi\) | \(M_2 = q_t\) | \(Q_S = q_s^{(u)}\) |
| Gleyzes et al.    | \(+G_3(\phi, X)\square \phi\) | \(X = \nabla^\mu \phi \nabla_\mu \phi\) | \(M_2 \left( \alpha_K + 6\alpha_B^2 \right) \left( 1 + \alpha_B \right)^2 = 2q_s^{(u)}\) |

**TABLE I. Notations and stability conditions in other papers.**

The cosmology in Horndeski theories has been extensively studied in the literature, but different notations were used depending on the papers. In Table I, we summarize the notation and sign convention for some quantities adopted in four other papers [123, 126, 137, 159]. The quantities \( q_t \), \( q_s^{(u)} \), \( q_s^{(f)} \) are defined, respectively, by Eqs. (3.10) and (5.5), (3.10). In terms of the dimensionless parameters \( \alpha_M \), \( \alpha_B \), \( \alpha_K \) introduced in Eq. (4.43), we can express \( q_s^{(u)} \), \( q_s^{(f)} \), \( q_s \) in the forms:

\[
q_s^{(u)} = \frac{q_t \left( \alpha_K + 6\alpha_B^2 \right)}{2(1 + \alpha_B)^2}, \quad q_s^{(f)} = \frac{H^2 q_t \left( \alpha_K + 6\alpha_B^2 \right)}{2\dot{\phi}^2(1 + \alpha_B)^2}, \quad q_s = \frac{2H^2 q_t^2 \left( \alpha_K + 6\alpha_B^2 \right)}{\dot{\phi}^2}.
\]
As we mentioned in Eq. (4.45), our definition of $\alpha_B$ is different from that used by Bellini and Sawicki [37], as $\alpha_B = -\alpha_{B(\beta)}^{(\text{BS})}/2$.

It may be also convenient to notice that the variables $w_1, w_2, w_3, w_4$ introduced in Ref. [36] are related to our $q_1, c_2^2, D_1, D_6$, as

$$w_1 = q_1, \quad w_2 = 2Hq_1 - \dot{\phi}D_6, \quad w_3 = 3\dot{\phi}^2D_1 - 9H(\dot{H}q_1 - \dot{\phi}D_6), \quad w_4 = q_1c_2^2. \quad (A2)$$

Appendix B: Coefficients in the second-order action of scalar perturbations

The coefficients $D_1, \ldots, 7$ appearing in the background Eqs. (3.6), (3.7) and the second-order action (4.11) of scalar perturbations are given by

$$D_1 = H^3\dot{\phi} \left(3G_{5,XX} + \frac{7}{2}\dot{\phi}^2G_{5,XXX} + \frac{1}{2}\dot{\phi}^4G_{5,XXXX} \right) + 3H^2 \left[G_{4,XX} - G_{5,\phi} + \dot{\phi}^2 \left(4G_{4,XX} - \frac{5}{2}G_{5,\phi} \right) \right] + \dot{\phi}^4 \left(G_{4,XXX} - \frac{1}{2}G_{5,XX,\phi} \right) - 3H\dot{\phi} \left[G_{3,XX} + 3G_{4,XX,\phi} + \dot{\phi}^2 \left(\frac{1}{2}G_{3,XX} + G_{4,XX,\phi} \right) \right] + \frac{1}{2} \left[2G_{2,XX} + 2G_{3,\phi} + \dot{\phi}^2 \left(G_{2,XX} + G_{3,XX,\phi} \right) \right],$$

$$D_2 = \frac{2H^4}{45} \left[2G_{4,XX} - G_{5,\phi} + \dot{\phi}^2 \left(2G_{4,XX} - G_{5,\phi} \right) + H\dot{\phi}(2G_{5,\phi} + \dot{\phi}^2G_{5,XX}) \right] \dot{H} + \left[G_{3,XX} + 3G_{4,XX,\phi} + \dot{\phi}^2 \left(\frac{1}{2}G_{3,XX} + G_{4,XX,\phi} \right) - 2H\dot{\phi}(3G_{4,XX} - 2G_{5,\phi}) \right] - H^2 \left[2G_{5,XX} - G_{5,\phi} + \frac{5}{2}\dot{\phi}^2G_{5,XX} + \frac{1}{2}\dot{\phi}^4G_{5,XXX} \right] \dot{\phi} - H^3\dot{\phi} \left(2G_{5,XX} + \dot{\phi}^2G_{5,XXX} \right) - H^2 \left[3(2G_{4,XX} - G_{5,\phi}) + 5\dot{\phi}^2 \left(G_{4,XX} - \frac{1}{2}G_{5,\phi} \right) + \frac{1}{2}\dot{\phi}^4G_{5,XX,\phi} \right] + 2H\dot{\phi}(G_{3,XX} + G_{4,XX,\phi}) - H^3\dot{\phi}(2G_{4,XX,\phi} - G_{5,XX,\phi}) + \dot{\phi}^2 \left(\frac{1}{2}G_{3,XX,\phi} + G_{4,XX,\phi} \right) - G_{3,\phi} - \frac{1}{2}G_{2,XX} \right],$$

$$D_3 = 3 \left[G_{4,XX,\phi} + \dot{\phi}^2 \left(\frac{1}{2}G_{3,XX,\phi} + G_{4,XX,\phi} \right) - 2H\dot{\phi}(G_{4,XX,\phi} - G_{5,XX,\phi}) \right] - H^2\dot{\phi} \left[2G_{5,XX,\phi} - G_{5,XX,\phi} + \frac{1}{2}\dot{\phi}^2G_{5,XX,\phi} \right] \dot{H} \right] - \left[\frac{1}{2}G_{2,XX} + 2G_{3,\phi} + \frac{1}{2}\dot{\phi}^2(3G_{2,XX,\phi} + G_{3,XX,\phi}) - 3H\dot{\phi}(G_{3,XX,\phi} + 3G_{4,XX,\phi}) \right] - 3H\dot{\phi} \left[\frac{1}{2}G_{3,XX,\phi} + G_{4,XX,\phi} \right] + 3H^2(G_{4,XX,\phi} - G_{5,XXX}) + 3H^2\dot{\phi} \left(4G_{4,XX,\phi} - \frac{5}{2}G_{5,XX,\phi} \right) \right] - \frac{1}{2}G_{2,XX} + 3G_{3,\phi} - \frac{1}{2}\dot{\phi}^2 \left(\frac{3}{2}G_{2,XX} + 3G_{4,XX,\phi} + \frac{1}{2}G_{5,XX,\phi} \right) + \dot{\phi}^4 \left(G_{4,XX,\phi} - \frac{1}{2}G_{5,XX,\phi} \right) \dot{\phi} - \frac{3}{2}H^4\dot{\phi} \left(3G_{5,XX,\phi} + \dot{\phi}^2G_{5,XXX,\phi} \right) - H^3\dot{\phi} \left[9(G_{4,XX,\phi} - G_{5,XX,\phi}) + \dot{\phi}^2 \left(9G_{4,XX,\phi} - \frac{7}{2}G_{5,XX,\phi} \right) + \frac{1}{2}\dot{\phi}^4G_{5,XXX,\phi} \right] + 3H^2 \left[2G_{4,XX,\phi} + \dot{\phi}^2 \left(\frac{3}{2}G_{3,XX,\phi} + 3G_{4,XX,\phi} + \frac{1}{2}G_{5,XX,\phi} \right) + \frac{1}{2}\dot{\phi}^2G_{2,XX,\phi} - G_{5,XX,\phi} \right] + \dot{\phi}^4 \left(2G_{4,XX,\phi} - G_{5,XX,\phi} \right) \dot{\phi} - \frac{1}{2}\dot{\phi}^2 \left(G_{2,XX} + G_{3,XX} \right) - \frac{1}{2}\dot{\phi}^2(G_{2,XX} + G_{3,XX,\phi} - \phi(G_{2,XX} + 2G_{3,\phi}) \right],$$

$$D_4 = -H^3\dot{\phi} \left(15G_{5,XX} + 10\dot{\phi}^2G_{5,XXX} + \dot{\phi}^4G_{5,XXXX} \right) - 3H^2\dot{\phi} \left[6(G_{5,XX} - G_{5,\phi}) + \dot{\phi}^2 \left(12G_{4,XXX} - 7G_{5,\phi} \right) + 3H \left[2G_{4,\phi} + \dot{\phi}^2 \left(3G_{3,XX} + 8G_{4,XX,\phi} \right) \right] \right] + \dot{\phi}^3 \left(3G_{3,XX} + 2G_{4,XX,\phi} \right) - \dot{\phi} \left(2G_{4,XX} + G_{5,XX,\phi} \right) - \dot{\phi}(G_{2,XX} + 2G_{3,\phi}) \right],$$

$$D_5 = -H^3\dot{\phi} \left(5G_{5,XX} + \dot{\phi}^2G_{5,XXX} + 3H^2 \left[2G_{4,\phi} - \dot{\phi} \left(4G_{4,XX,\phi} - 3G_{5,\phi} \right) - \dot{\phi}^3 \left(2G_{4,XX,\phi} - G_{5,\phi} \right) \right] \right]$$
\[ +3H\dot{\phi}\left[2G_{4,\phi\phi} + \dot{\phi}^2(G_{3,\phi\phi} + 2G_{4,\phi\phi\phi})\right] - \dot{\phi}^2(G_{2,\phi} + G_{3,\phi\phi}) + G_{2,\phi}, \]
\[ D_6 = H^2\dot{\phi}^2\left(3G_{5,\phi\phi} + \dot{\phi}^2G_{5,\phi\phi\phi}\right) + 2H\dot{\phi}\left[2G_{4,\phi} - G_{5,\phi\phi}\right] + \dot{\phi}^2\left(2G_{4,\phi\phi} - G_{5,\phi\phi\phi}\right) - \dot{\phi}^2\left(G_{3,\phi\phi} + 2G_{4,\phi\phi\phi}\right) - 2G_{4,\phi}, \]
\[ D_7 = H^3\dot{\phi}^2\left(3G_{5,\phi\phi} + \dot{\phi}^2G_{5,\phi\phi\phi}\right) + 2H^2\dot{\phi}\left[3G_{4,\phi} - G_{5,\phi\phi}\right] + \dot{\phi}^2\left(3G_{4,\phi\phi} - 2G_{5,\phi\phi\phi}\right) - H\left[2G_{4,\phi} + \dot{\phi}^2\left(3G_{3,\phi\phi} + 10G_{4,\phi\phi\phi} - 2G_{5,\phi\phi\phi}\right)\right] + \dot{\phi}\left(G_{2,\phi} + G_{3,\phi\phi} + 2G_{4,\phi\phi\phi}\right). \]  

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