Spinor field in Bianchi type-IX space-time

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Within the scope of Bianchi type-IX we have studied the role of spinor field in the evolution of the Universe. It is found that unlike the diagonal Bianchi models in this case the components of energy-momentum tensor of spinor field along the principal axis are not the same, i.e. $T_1^1 \neq T_2^2 \neq T_3^3$, even in absence of spinor field nonlinearity. The presence of non-trivial non-diagonal components of energy-momentum tensor of the spinor field imposes severe restrictions both on geometry of space-time and on the spinor field itself. As a result the space-time turns out to be either locally rotationally symmetric or isotropic. In this paper we considered the Bianchi type-IX space-time both for a trivial $b$, that corresponds to standard BIX and the one with a non-trivial $b$. It was found that a positive $\lambda_1$ gives rise to an oscillatory mode of expansion, while a trivial $\lambda_1$ leads to rapid expansion at the early stage of evolution.

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I. INTRODUCTION

Nonlinear spinor fields play a significant role in explaining the evolution of the Universe at different stages. It was shown by a number of authors that the introduction of nonlinear spinor field into the system can (i) give rise to a singularity-free Universe; (ii) accelerate the isotropization process of initially anisotropic space-time and (iii) generate late time acceleration of space-time expansion [1–17]. Moreover, it can simulate different types of dark energy and perfect fluid [18–22]. Recently it was also found that the presence of non-diagonal components of energy-momentum tensor of the spinor field imposes severe restrictions to the space-time geometry as well [23–26].

In this paper we plan to extend our previous study to Bianchi type-IX cosmological model. One of the reasons to consider this model is familiar solutions like the FRW Universe with positive curvature, the de-sitter Universe, the Taub-Nut solutions etc. are of Bianchi type-IX space-times. It should be noted that due to its importance many authors have studied the evolution of the Universe within the scope of a Bianchi type-IX model. Bali et. al. have studied the Bianchi type-IX string cosmological models filled with bulk viscous fluid [27, 28], whereas such a model for perfect fluid was investigated by Tyagi et. al. in [29]. Analogous system with a time varying \( \Lambda \)-term was studied in [30]. A scalar tensor theory of gravitation within the framework of Bianchi type-IX was studied by Reddy and Naidu [31].

II. BASIC EQUATION

Let us consider the spinor field Lagrangian in the form

\[
L = \frac{i}{2} \left[ \bar{\psi} \gamma^{\mu} \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma^{\mu} \psi \right] - m_{sp} \bar{\psi} \psi - F, \tag{2.1}
\]

where the nonlinear term \( F \) describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field. We consider the case when \( F = F(K) \) with \( K \) taking one of the followings values \( \{I, J, I + J, I - J\} \). By virtue of Fierz theorem this comes out to be the most general form of spinor field nonlinearity.

Here \( \nabla_{\mu} \) covariant derivative of the spinor field having the form

\[
\nabla_{\mu} \psi = \partial_{\mu} \psi - \Gamma_{\mu} \psi, \tag{2.2a}
\]

\[
\nabla_{\mu} \bar{\psi} = \partial_{\mu} \bar{\psi} + \bar{\psi} \Gamma_{\mu}, \tag{2.2b}
\]

where \( \Gamma_{\mu} \) is the spinor affine connection defined as

\[
\Gamma_{\mu} = \frac{1}{4} \gamma_{a} \gamma^{\nu} \partial_{\mu} e^{(a)}_{\nu} - \frac{1}{4} \gamma_{\rho} \gamma^{\nu} \Gamma_{\mu}^{\rho}, \tag{2.3}
\]

where \( \gamma_{a} \) are the Dirac matrices in flat space-time, \( \gamma_{\nu} \) are the Dirac matrices in curved space-time, \( e^{(a)}_{\nu} \) are the tetrad and \( \Gamma_{\mu}^{\rho} \) are the Christoffel symbols.

The energy momentum tensor of the spinor field is given by

\[
T_{\mu}^{\rho} = \frac{i}{4} g^{\rho \nu} \left( \bar{\psi} \gamma_{\mu} \nabla_{\nu} \psi + \bar{\psi} \gamma_{\nu} \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma_{\nu} \psi - \nabla_{\nu} \bar{\psi} \gamma_{\mu} \psi \right) - \delta_{\mu}^{\rho} L
\]

\[
= \frac{i}{4} g^{\rho \nu} \left( \bar{\psi} \gamma_{\mu} \partial_{\nu} \psi + \bar{\psi} \gamma_{\nu} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma_{\nu} \psi - \partial_{\nu} \bar{\psi} \gamma_{\mu} \psi \right)
\]

\[
- \frac{i}{4} g^{\rho \nu} \bar{\psi} \left( \gamma_{\mu} \Gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} + \gamma_{\nu} \Gamma_{\mu} + \Gamma_{\mu} \gamma_{\nu} \right) \psi - \delta_{\mu}^{\rho} L. \tag{2.4}
\]

Bianchi type IX space-time \( (BIX) \) is given by
\[ ds^2 = dt^2 - a_1^2(t)dx_1^2 - [h^2(x_3)a_1^2(t) + f^2(x_3)a_2^2(t)]dx_2^2 - a_3^2dx_3^2 + 2a_1^2(t)h(x_3)dx_1dx_2, \]  
\[ (2.5) \]

with \( a_1, a_2, a_3 \) being the functions of \( t \). Here \( h \) and \( f \) are the functions of \( x_3 \). Here we consider the Bianchi type \( IX \) space-time, which imposes the following restriction of \( f \), namely

\[ \delta = -\frac{f''}{f} = 1 \Rightarrow f'' + f = 0. \]  
\[ (2.6) \]

It should be noted that it is customary to assume \( f(x_3) = \sin(x_3) \) and \( h(x_3) = \cos(x_3) \). We don’t write these concrete expressions for the functions \( f(x_3) \) and \( h(x_3) \) here, but do it later in due course.

To find the spinor affine connection \((2.3)\) we have to know the tetrad corresponding to the metric \((2.5)\). Exploiting the well known relation

\[ g_{\mu\nu} = \epsilon^{(a)}_{\mu} \epsilon^{(b)}_{\nu} \eta_{ab}, \]  
\[ (2.7) \]

we choose the tetrad corresponding to \((2.5)\) as follows:

\[ \epsilon^{(0)} = 1, \quad \epsilon^{(1)} = a_1, \quad \epsilon^{(2)} = a_2 f, \quad \epsilon^{(3)} = a_3, \quad \epsilon^{(1)} = -a_1 h. \]  
\[ (2.8) \]

From

\[ \gamma^\mu = \epsilon^{(a)}_{\mu} \bar{\gamma}^a, \quad \gamma^\nu = \epsilon^{(b)}_{\nu} \bar{\gamma}^b, \]  
\[ (2.9) \]

such that

\[ \epsilon^{(a)}_{\mu} \epsilon^{(b)}_{\nu} = \delta^a_b, \quad \epsilon^{(a)}_{\mu} \epsilon^{(a)}_{\nu} = \delta^\mu_\nu, \]  
\[ (2.10) \]

one now finds

\[ \bar{\gamma}^0 = \bar{\gamma}_0 = \bar{\gamma}^0, \quad \bar{\gamma}^1 = a_1 \bar{\gamma}_1 = -a_1 \bar{\gamma}^1, \]
\[ \bar{\gamma}^2 = -a_1 h \bar{\gamma}_1 + a_2 f \bar{\gamma}_2 = a_1 h \bar{\gamma}^1 - a_2 f \bar{\gamma}^2, \quad \bar{\gamma}^3 = a_3 \bar{\gamma}_3 = -a_3 \bar{\gamma}^3, \]  
\[ (2.11) \]

where we take into account that

\[ \bar{\gamma}^0 = \bar{\gamma}_0, \quad \bar{\gamma}^1 = -\bar{\gamma}_1, \quad \bar{\gamma}^2 = -\bar{\gamma}_2, \quad \bar{\gamma}^3 = -\bar{\gamma}_3. \]

Using the laws of raising and lowering the indices one also finds

\[ \gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \frac{1}{a_1} \bar{\gamma}^1 + \frac{h}{a_2 f} \bar{\gamma}^2, \quad \gamma^2 = \frac{1}{a_2 f} \bar{\gamma}^2, \quad \gamma^3 = \frac{1}{a_3} \bar{\gamma}^3. \]  
\[ (2.12) \]

Now we are ready to compute spinor affine connections using \((2.3)\): \n
\[ \Gamma_1 = \frac{1}{2} a_1 \bar{\gamma}^1 \bar{\gamma}^0 - \frac{1}{4} a_2^2 h f \bar{\gamma}^2 \bar{\gamma}^3, \]  
\[ (2.13a) \]
\[ \Gamma_2 = \frac{1}{2} f a_2 \bar{\gamma}^2 \bar{\gamma}^2 - \frac{1}{2} h a_1 \bar{\gamma}^1 \bar{\gamma}^0 - \frac{1}{4} a_1 h \bar{\gamma}^1 \bar{\gamma}^3 - \frac{1}{2} a_2 f \bar{\gamma}^2 \bar{\gamma}^3 + \frac{1}{4} a_2^2 h f \bar{\gamma}^2 \bar{\gamma}^3, \]  
\[ (2.13b) \]
\[ \Gamma_3 = \frac{1}{2} a_3 \bar{\gamma}^3 \bar{\gamma}^0 + \frac{1}{4} a_1 h \bar{\gamma}^1 \bar{\gamma}^2, \]  
\[ (2.13c) \]
\[ \Gamma_0 = 0. \]  
\[ (2.13d) \]
Then one finds

\[
\gamma^\mu \Gamma_\mu = -\frac{1}{2} \dot{V} \gamma^0 - \frac{i}{4 a_2 a_3 f} \bar{\gamma}^5 \gamma^0 - \frac{1}{2} \frac{f'}{a_3 f} \bar{\gamma}^3,
\]

(2.14a)

\[
\Gamma_\mu \gamma^\mu = \frac{1}{2} \dot{V} \gamma^0 - \frac{i}{4 a_2 a_3 f} \bar{\gamma}^5 \gamma^0 + \frac{1}{2} \frac{f'}{a_3 f} \bar{\gamma}^3,
\]

(2.14b)

where we introduce the volume scale

\[
V = a_1 a_2 a_3
\]

(2.15)

and \(\bar{\gamma}^5 = -i \bar{\gamma}^0 \bar{\gamma}^1 \bar{\gamma}^2 \bar{\gamma}^3\).

The spinor field equations corresponding to the Lagrangian (2.1) has the form

\[
\bar{i} \bar{\gamma}^\mu \nabla_\mu \psi - m_{sp} \psi - \mathcal{D} \psi - i \mathcal{G} \bar{\gamma}^5 \psi = 0,
\]

(2.16a)

\[
\bar{i} \nabla_\mu \bar{\psi} \gamma^\mu + m_{sp} \bar{\psi} + \mathcal{D} \bar{\psi} + i \mathcal{G} \bar{\psi} \bar{\gamma}^5 = 0,
\]

(2.16b)

where \(\mathcal{D} = 2SF_K K_J\) and \(\mathcal{G} = 2PF_K K_J\).

In view of (2.14a) and (2.14b) the system (2.16) can be rewritten as

\[
\bar{i} \bar{\gamma}^0 \psi + \frac{i}{2} \frac{\dot{V}}{V} \gamma^0 \psi + \frac{1}{4 a_2 a_3 f} \bar{\gamma}^5 \gamma^0 \psi + \frac{i}{2} \frac{f'}{a_3 f} \bar{\gamma}^3 \psi - [m_{sp} + \mathcal{D}] \psi - i \mathcal{G} \bar{\gamma}^5 \psi = 0,
\]

(2.17a)

\[
\bar{i} \bar{\gamma}^0 + \frac{i}{2} \frac{\dot{V}}{V} \bar{\gamma}^0 - \frac{1}{4 a_2 a_3 f} \bar{\gamma}^5 \gamma^0 + \frac{i}{2} \frac{f'}{a_3 f} \bar{\gamma}^3 + [m_{sp} + \mathcal{D}] \bar{\psi} + i \mathcal{G} \bar{\psi} \bar{\gamma}^5 = 0.
\]

(2.17b)

In view of (2.16) the spinor field Lagrangian can be written as

\[
L = \frac{i}{2} [\bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi] - m_{sp} \bar{\psi} \gamma^0 \psi - F(K)
\]

\[
= \frac{i}{2} [\bar{\psi} \gamma^0 \nabla_0 \psi - m_{sp} \psi] - \frac{i}{2} [\nabla_0 \bar{\psi} \gamma^0 \psi + m_{sp} \bar{\psi} \gamma^0 \psi - F(K)]
\]

\[
= 2(\mathcal{F}_I + \mathcal{F}_J) - F = 2KF_K - F(K),
\]

(2.18)

from (2.4) one finds the following components of energy-momentum tensor
From (2.19) we see that the spinor field distribution along the main axis is anisotropic, i.e. $T_1^1 \neq T_2^2 \neq T_3^3$ and these components do not vanish even in absence of spinor field nonlinearity.

The components of Einstein tensor corresponding to (2.5) are
From (2.20) one finds the following relations between its components:

\[
G_1^1 = - \left( \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} \right) + \frac{a_1^2 h^2}{a_2^2 a_3 f^2} \left( \frac{3 h'^2}{4 h^2} + \frac{h''}{2 h} - \frac{1}{2 h f} \right) + \frac{1}{a_3} \frac{f''}{f}, \tag{2.20a}
\]

\[
G_1^2 = \frac{1}{2} \frac{a_1^2 h}{a_2^2 a_3 f^2} \left( \frac{h''}{h} - \frac{h' f'}{h f} \right), \tag{2.20b}
\]

\[
G_1^3 = h \left( \frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_1 \dot{a}_1 + \ddot{a}_2 \dot{a}_3 - \dot{a}_1 \ddot{a}_3}{a_2 a_3} \right) + \frac{h}{a_3^2} \left( \frac{1}{h} \frac{h''}{2 - \frac{f''}{f} - \frac{1}{h f}} \right)
+ \frac{a_1^2 h^3}{a_2^2 a_3^2 f^2} \left( \frac{1}{h} \frac{f''}{2 h f} - \frac{1}{h} \frac{f''}{h f} + \frac{h^2}{h^2} \right), \tag{2.20c}
\]

\[
G_2^2 = - \left( \frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1 \dot{a}_1 + \ddot{a}_3 \dot{a}_1}{a_3 a_1} \right) - \frac{1}{2} \frac{a_1^2 h^2}{a_2^2 a_3^2 f^2} \left( \frac{h''}{h} - \frac{h' f'}{h f} + \frac{1}{h^2} \right), \tag{2.20d}
\]

\[
G_3^2 = - \left( \frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2 \dot{a}_2 + \ddot{a}_1 \dot{a}_2}{a_1 a_2} \right) - \frac{1}{4} \frac{a_1^2 h^2}{a_2^2 a_3^2 f^2} \frac{h^2}{h^2}, \tag{2.20e}
\]

\[
G_0^1 = \left( \frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_3}{a_3} \right) \frac{f'}{f}, \tag{2.20f}
\]

\[
G_0^2 = - \frac{1}{a_3} \left( \frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_3}{a_3} \right) \frac{f'}{f}, \tag{2.20g}
\]

\[
G_0^3 = - \left( \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_2 \dot{a}_3 + \ddot{a}_3 \dot{a}_2}{a_2 a_3} \right) + \frac{1}{4} \frac{a_1^2 h^2}{a_2^2 a_3^2 f^2} \frac{h^2}{h^2} + \frac{1}{a_3} \frac{f''}{f}. \tag{2.20h}
\]

From (2.20) one finds the following relations between its components:

\[
G_1^1 = h \left( G_2^2 - G_1^1 \right) + \left( h^2 + \frac{a_2^2 f^2}{a_1^2} \right) G_1^2. \tag{2.21}
\]

From (2.19) it can be shown that

\[
T_2^1 = h \left( T_2^2 - T_1^1 \right) + \left( h^2 + \frac{a_2^2 f^2}{a_1^2} \right) T_1^2. \tag{2.22}
\]

Moreover from (2.19) it follows that

\[
T_2^3 = \frac{a_2^2 f^2}{a_3^2} T_3^2 - h T_3^3, \tag{2.23}
\]

\[
T_3^1 = h T_3^2 + \frac{a_2^2 T_3^3}{a_1^2}, \tag{2.24}
\]

\[
T_0^1 = h T_0^2 - \frac{1}{a_1^2} T_0^1, \tag{2.25}
\]

and

\[
T_0^0 = -\frac{a_2^2 f^2 T_0^2}{T_0^1} - h T_0^0. \tag{2.26}
\]

Then the system of Einstein equations

\[
G_\nu^\mu = -\kappa T_\nu^\mu, \tag{2.27}
\]
on account of linearly dependent components takes the form

\[
\left( \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_4}{a_4} + \frac{\dot{a}_5}{a_5} \right) - \frac{1}{2} \frac{a_2^2 h^2}{a_2^2 a_3^2 f^2} \left( \frac{h'' - h'}{h^2} + \frac{3 h'^2}{2 h^2} \right) - \frac{1}{a_3^2} f' = \kappa \left[ (F(K) - 2KF_K) + \frac{1}{4 a_2 f} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A^3 + \frac{1}{4 a_2 a_3 f} \left( \frac{f'}{f} - \frac{h'}{h} \right) A^0 \right], \tag{2.28a}
\]

\[
\left( \frac{\dot{a}_3}{a_3} + \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_4}{a_4} \right) - \frac{1}{2} \frac{a_2^2 h^2}{a_2^2 a_3^2 f^2} \left( \frac{h'' - h'}{h^2} + \frac{h'^2}{2 h^2} \right) = \kappa \left[ (F(K) - 2KF_K) + \frac{1}{4 a_2 f} \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A^3 - \frac{1}{4 a_2 a_3 f} \left( \frac{f'}{f} - \frac{h'}{h} \right) A^0 \right], \tag{2.28b}
\]

\[
\left( \frac{\dot{a}_4}{a_4} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) - \frac{1}{4} \frac{a_2 h'^2}{a_2^2 a_3^2 f^2} - \frac{1}{a_3^2} f' = \kappa \left[ m_{sp} S + F(K) - \frac{1}{4 a_2 a_3 f} A^0 \right], \tag{2.28c}
\]

\[
\frac{1}{2} \frac{a_2^2 h^2}{a_2^2 a_3^2 f^2} \left( \frac{h'' - h'}{h} - h' \frac{a_3}{a_1} \right) = -\kappa \frac{1}{4 a_2 f} \left[ \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A^3 + \frac{a_1 f'}{a_3 f} A^0 \right], \tag{2.28d}
\]

\[
\left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) \frac{f'}{f} = 0, \tag{2.28e}
\]

\[
0 = \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} \right) A^1, \tag{2.28f}
\]

\[
0 = \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) A^2, \tag{2.28g}
\]

\[
0 = \left[ \frac{a_1 h'}{a_3 f} A^1 + f' A^2 \right], \tag{2.28h}
\]

\[
0 = \left( 1 + \frac{a_1}{a_2 f} \right) \frac{f'}{f} A^1. \tag{2.28i}
\]

Then in view of \( \frac{f'}{f} \neq 0 \) from (2.28i) we find \( \left( \frac{\dot{a}_1}{a_2} - \frac{\dot{a}_3}{a_3} \right) = 0 \). On the other hand for same reason (2.28j) yields \( A^1 = 0 \), whereas inserting \( A^1 = 0 \) into (2.28i) we obtain \( A^2 = 0 \). Thus in this case from (2.28i) - (2.28j) we have

\[
A^1 = 0, \quad A^2 = 0, \quad \left( \frac{\dot{a}_1}{a_2} - \frac{\dot{a}_3}{a_3} \right) = 0. \tag{2.29}
\]

In view of \( A^2 = 0 \) the equation (2.28h) yields two possibilities:

\[
\left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) \neq 0, \tag{2.30}
\]

which means the model is rotationally symmetric, or

\[
\left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) = 0. \tag{2.31}
\]
which means the model is isotropic. Note that in case of diagonal energy-momentum tensor we have strictly locally rotationally symmetric space-time [32].

It should be noted that for the diagonal Bianchi models the volume scale $V$ plays crucial role in the evolution of the Universe. $V$ plays important role for non-diagonal Bianchi models too. So let us now write the euqtion for $V$. Summation of $(2.28a)$, $(2.28b)$, $(2.28c)$ and 3 times $(2.28d)$ gives

$$
\ddot{V} - \frac{1}{4} \frac{a_1^2}{a_2^2 a_3^2} \frac{h'^2}{f^2} - \frac{1}{2a_3^2} \frac{f''}{f} = \frac{3}{2} \left[ m_{sp} S + 2 (F - K F_K) - \frac{1}{2} \frac{a_1}{a_2 a_3} \frac{h'}{f} A^0 \right].
$$

(2.32)

As one sees, to find the solution of $(2.32)$ one need to know $f$, $h$, $A_0$, the spinor field nonlinearity as well as the metric functions, or at least their expressions in terms of $V$.

To find the metric functions in terms of $V$ we use the proportionality condition. In doing so let us compute the expansion and shear corresponding to metric $(2.5)$. Let the four-velocity is given by $u^\mu = (1, 0, 0, 0)$. Then for the expansion we have

$$
\vartheta = u^\mu;_\mu = u^\mu;_\mu + \Gamma^\mu_{\mu\alpha} u^\alpha = \Gamma^\mu_{\mu0}
= \Gamma^1_{10} + \Gamma^2_{30} + \Gamma^3_{30} = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} = \frac{\dot{V}}{V}.
$$

(2.33)

The shear is given by

$$
\sigma_{\alpha\beta} = \frac{1}{2} \left( u^\alpha P^\beta_\nu + u^\beta P^\nu_\alpha \right) - \frac{1}{3} \vartheta P_{\alpha\beta}.
$$

(2.34)

Taking into account that $u^\mu = u^\nu g_{\nu\mu} = u^0 g_{0\mu} = (1, 0, 0, 0)$, $u^\alpha;_\nu = u^\alpha;_\nu - \Gamma^\mu_{\alpha\nu} u^\mu = -\Gamma^0_{\alpha\nu}$, $P_{\alpha\beta} = g_{\alpha\beta} - u^\alpha u_{\beta}$ and $P^\alpha_\beta = \delta^\alpha_\beta - u^\alpha u_\beta$ from $(2.34)$ we find

$$
\sigma_{\alpha\beta} = \frac{1}{2} \left( -\Gamma^0_{\alpha\nu} P^\nu_\beta - \Gamma^0_{\beta\nu} P^\nu_\alpha \right) - \frac{1}{3} \vartheta P_{\alpha\beta}.
$$

(2.35)

Using $(2.35)$ we find

$$
\sigma^1_1 = \frac{\dot{a}_1}{a_1} - \frac{1}{3} \vartheta,
$$

(2.36a)

$$
\sigma^2_2 = \frac{\dot{a}_2}{a_2} - \frac{1}{3} \vartheta,
$$

(2.36b)

$$
\sigma^3_3 = \frac{\dot{a}_3}{a_3} - \frac{1}{3} \vartheta,
$$

(2.36c)

$$
\sigma^2_1 = \h \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right),
$$

(2.36d)

$$
\sigma^1_2 = 0.
$$

(2.36e)

Thus we see that the diagonal components of shear tensor $\sigma^\beta_{\alpha}$ does not depend on $f$ and $h$.

Let us assume the proportionality condition

$$
\sigma^1_1 = q_1 \vartheta, \quad q_1 = \text{const.}
$$

(2.37)

which after inserting $(2.36a)$ and $(2.33)$ gives

$$
\frac{\dot{a}_1}{a_1} = \left( q_1 + \frac{1}{3} \right) \vartheta.
$$

(2.38)
In view of \( \vartheta = \dot{V}/V \) from (2.38) in one hand we find

\[
\frac{\dot{a}_1}{a_1} = \left( q_1 + \frac{1}{3} \right) \frac{\dot{V}}{V},
\]

with

\[
a_1 = q_2 V^{q_1 + 1/3}, \quad q_2 = \text{const.}
\]

(2.40)

On the other hand from (2.28f) one finds

\[
a_2 = q_3 a_3, \quad q_3 = \text{const.}
\]

(2.41)

Then from (2.15) one finds

\[
a_3 = \frac{1}{\sqrt{q_2 q_3}} V^{1/3-q_1/2}.
\]

(2.42)

So finally we can write the expressions for metric functions in terms of \( V \) as

\[
a_1 = X_i V^{Y_i}, \quad \prod_{i=1}^{3} X_i = 1, \quad \sum_{i=1}^{3} Y_i = 0.
\]

(2.43)

In this concrete case we have \( X_1 = q_2, \quad X_2 = \sqrt{q_3/q_2}, \quad X_3 = 1/\sqrt{q_2 q_3}, \quad Y_1 = q_1 + 1/3, \) and \( Y_2 = Y_3 = 1/3 - q_1/2 \).

From (2.43) one finds that \( a_1/(a_2 a_3) = q_2^2 V^{2q_1-1/3} \). Then the equation for \( V \) (2.32) can be rewritten as

\[
\frac{\dot{V}}{V} - \frac{1}{4} q_2^4 V^{4q_1-2/3} \left( \frac{h'}{f} \right)^2 - \frac{1}{2} q_2 q_3 V^{q_1-2/3} \frac{f''}{f} = \frac{3\kappa}{2} \left[ m_0 S + 2 (F - K F K) - \frac{1}{2} q_2^2 V^{2q_1-1/3} \frac{h'}{f} A_0 \right],
\]

(2.44)

From the spinor field equations (2.16) we find that the bilinear forms of the spinor field in this case obey the following system of equations:

\[
S_0 + \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} P_0 + 4 P F K J A_0 = 0,
\]

(2.45a)

\[
P_0 + \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} S_0 - 2 \left[ m_0 + 2 S F K J \right] A_0 = 0,
\]

(2.45b)

\[
A_0^3 + \frac{1}{2} \frac{f'}{a_3 f} A_0^3 + 2 \left[ m_0 + 2 S F K J \right] P_0 + 4 P F K J S_0 = 0,
\]

(2.45c)

\[
A_0^3 + \frac{1}{2} \frac{f'}{a_3 f} A_0^3 = 0.
\]

(2.45d)

From (2.45) it can be easily shown that

\[
P_0^2 - S_0^2 + (A_0^3)^2 = \text{const.},
\]

(2.46)

On the other hand from Fierz theorem we have

\[
I_A = (A_0^3)^2 - (A_1^3)^2 - (A_2^3)^2 = (A_3^3)^2 = - (S_0^2 + P_0^2),
\]

(2.47)

Now taking into account that \( A_1 = 0 \) and \( A_2 = 0 \) from (2.47) one finds

\[
(A_0^3)^2 - (A_3^3)^2 = - (S_0^2 + P_0^2),
\]

(2.48)
Then inserting (2.48) into (2.46) one finds
\[ S = \frac{V_0}{V}, \quad V_0 = \text{const.} \] (2.49)

It should be emphasized that in case of diagonal Bianchi space-time we obtain the expression (2.49) only when the spinor field nonlinearity depends on \( K = I = S^2 \). Thus we see that in case of BIX space-time \( S = V_0/V \) independent to our choice of \( K \).

Let us now see, what happens if \( K \) takes any of the following expressions \( \{ J, I + J, I - J \} \). In case of diagonal Bianchi models exact expressions were found for massless spinor field only. So here we consider massless spinor field. Then in case of \( K = J \) from (2.45b) we find
\[
\dot{P}_0 + \frac{1}{2} q_2^2 V^{2q_1 - 1/3} \frac{h'}{f} V_0 = 0. \] (2.50)

From (2.50) one can formally express \( K = J = P^2 \) in terms of \( V \).
In case of \( K = I + J \) we have
\[
\dot{S}_0 + \frac{1}{2} q_2^2 V^{2q_1 - 1/3} \frac{h'}{f} P_0 + 4PFKA_0^0 = 0, \] (2.51a)
\[
\dot{P}_0 + \frac{1}{2} q_2^2 V^{2q_1 - 1/3} \frac{h'}{f} S_0 - 4SFKA_0^0 = 0. \] (2.51b)

Summation of (2.51a) multiplied by \( S_0 \) and (2.51b) multiplied by \( P_0 \) gives
\[
(S_0 \dot{S}_0 + P_0 \dot{P}_0) + q_2^2 V^{2q_1 - 1/3} \frac{h'}{f} S_0 P_0 = 0. \] (2.52)

Further in view of \( S_0 = V_0 \) from (2.52) one finds equation for \( P_0 \) analogous to (2.50). Knowing \( J = P^2 \) we obtain the expression for \( K = I + J \) in terms of \( V \).
Finally, for \( K = I - J \) we have
\[
\dot{S}_0 + \frac{1}{2} q_2^2 V^{2q_1 - 1/3} \frac{h'}{f} P_0 - 4PFKA_0^0 = 0, \] (2.53a)
\[
\dot{P}_0 + \frac{1}{2} q_2^2 V^{2q_1 - 1/3} \frac{h'}{f} S_0 - 4SFKA_0^0 = 0. \] (2.53b)

Subtraction of (2.53b) multiplied by \( P_0 \) from (2.53a) multiplied by \( S_0 \) gives
\[
(S_0 \dot{S}_0 - P_0 \dot{P}_0) = 0, \] (2.54)
with the solution
\[
K = (I - J) = (S^2 - P^2) = \frac{V_0^2}{V^2}. \] (2.55)

Here it is interesting note that in absence of spinor field nonlinearity the system (2.45) takes the form
\[
\dot{S}_0 + \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} P_0 = 0, \] (2.56a)
\[
\dot{P}_0 + \frac{1}{2} \frac{a_1 h'}{a_2 a_3 f} S_0 - 2m_{sp} A_0^0 = 0, \] (2.56b)
\[
\dot{A}_0^0 + \frac{1}{2} \frac{f'}{a_3 f} A_0^3 + 2m_{sp} P_0 = 0, \] (2.56c)
\[
\dot{A}_0^3 + \frac{1}{2} \frac{f'}{a_3 f} A_0^0 = 0. \] (2.56d)
One can easily find that this system too allows the first integral (2.46). Moreover, in case of massless spinor field we find

\[ P_0^2 - S_0^2 = \text{const.}, \quad (A_0^0)^2 - (A_0^3)^2 = \text{const}. \]  

(2.57)

Now in view of (2.29) and (2.49) Eq. (2.57) can be rewritten as

\[ P_0^2 - S_0^2 = \text{const.}, \quad -(P_0^2 + S_0^2) = \text{const}. \]  

(2.58)

which gives

\[ S_0 = \text{const.}, \quad R_0 = \text{const.} \]  

(2.59)

Hence the behavior of invariants, constructed from bilinear spinor forms categorically differs from that we obtain for diagonal Bianchi models.

Let us now recall that the functions \( f(x_3) \) can be concretize using the restriction imposed on it, namely, from the (2.6) one finds the following solutions for \( f = \sin(x_3) \) and \( f = \cos(x_3) \). Following a number of authors we choose the nonlinearity to be the function of \( x \) determined from (2.28e). In doing so we go back to (2.28e) which can be rewritten as

\[ \frac{1}{f} \left( h'' - \frac{f'}{f} h' \right) = -\frac{\kappa a_2 a_3^2}{2 a_1^2} \left[ \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A_0^3 + \frac{a_1 f'}{a_3 f} A_0^3 \right] , \]  

(2.60)

In view of (2.45d) this equations can be written as

\[ \frac{1}{f} \left( h'' - \frac{f'}{f} h' \right) = -\frac{\kappa a_2 a_3^2}{2 a_1^2} \left[ \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A_0^3 - 2a_1 A_0^3 \right] , \]  

(2.61)

Now the left hand side of (2.61) depends of \( x_3 \) only, while the right hand side depends on only \( t \). So we can finally write the following system

\[ \frac{1}{f} \left( h'' - \frac{f'}{f} h' \right) = b, \]  

(2.62a)

\[ \frac{\kappa a_2 a_3^2}{2 a_1^2} \left[ \left( \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1} \right) A_0^3 - 2a_1 A_0^3 \right] = -b, \]  

(2.62b)

From the foregoing equations in view of \( f = \sin(x_3) \) and (2.43) we finally obtain

\[ h'' + \cot(x_3) h' = b \sin(x_3), \]  

(2.63a)

\[ \dot{A}_0^3 + \frac{3q_1}{4q_2} A_0^3 V^{-(q_1+4)/3} \dot{V} = \frac{\kappa}{\gamma} \sqrt{q_2^5 q_3^5 V^{5q_1/2 - 2/3}}. \]  

(2.63b)

So finally in view of (2.45d) equation (2.44) \( V \) can be rewritten as

\[ \ddot{V} - \frac{3\kappa}{2} \sqrt{\frac{q_2^2 h'}{q_3 f'' A_0^3 V^{3q_1/2}} - \frac{1}{4} q_2^4 V^{4q_1+1/3} \left( \frac{h'}{f} \right)^2} - \frac{1}{2} q_2 q_3 V^{q_1+1/3} f'' \]  

\[ = \frac{3\kappa}{2} \left[ m_{sp} S + 2(F - K F) \right] V, \]  

(2.64)

To find the solution to the equation (2.64) we have to give the concrete form of spinor field nonlinearity. Following some previous papers, we choose the nonlinearity to be the function of \( S \) only, having the form

\[ F = \sum_k \lambda_k I^{n_k} = \sum_k \lambda_k S^{2n_k}. \]  

(2.65)
Recently, this type of nonlinearity was considered in a number of papers [23–26]. In what follows, we thoroughly study the cases with trivial and non-tribal $b$.

**Case 1.** $b = 0$

Here we consider the simplest possible case setting $b = 0$. This case corresponds to the diagonal energy-momentum tensor.

From (2.6) we have $f = \sin(x_3)$. Solving (2.62a) in this case one finds $h = \cos(x_3)$. It should be emphasized that this form of Bianchi type-IX metric with $f = \sin(x_3)$ and $h = \cos(x_3)$ is generally considered in literature. In this case for $A_0^3$ from (2.63b) we also find

$$\frac{\dot{A}_0^3}{A_0^3} + \frac{3q_1}{4q_2} V^{-(q_1+4/3)} = 0,$$

**which with the solution**

$$A_0^3 = C_1 \exp \left[ \frac{9q_1}{4q_2(3q_1+1)} V^{-(q_1+1/3)} \right].$$

Now on account of $S = V_0/V$ equation (2.64) together with (2.63b) can be written as

$$\dot{V} = Y,$$

$$\dot{Y} = -\frac{3\kappa}{2} \sqrt{q_3^3/q_3 \tan(x_3)} V^{3q_1/2} \Phi_1(V,A_0^3,Y) + \Phi_2(V,A_0^3,Y),$$

$$\dot{A}_0^3 = \Phi_1(V,A_0^3,Y).$$

In what follows, we solve the foregoing equation numerically. For this reason we first rewrite it as a system of equations in the following way:

For simplicity we consider only three terms of the sum. We set $n_k = n_0 : 1 - 2n_0 = 0$ which gives $n_0 = 1/2$. In this case the corresponding term can be added with the mass term. We assume that $q_1$ is a positive quantity, so that $4q_1 + 1/3$ is positive too. For the nonlinear term to be dominant at large time, we set $n_k = n_1 : 1 - 2n_1 > 4q_1 + 1/3$, i.e., $n_1 < 1/3 - 2q_1$. And finally, for the nonlinear term to be dominant at the early stage we set $n_k = n_2 : 1 - 2n_2 < 0$, i.e., $n_2 > 1/2$. Since we are interested in qualitative picture of evolution, let us set $q_2 = 1$, $q_3 = 1$ and $\kappa = 1$. We also assume $V_0 = 1$. Then we have

$$\Phi_2(V,A_0^3,Y) = \frac{1}{4} V^{4q_1+1/3} - \frac{1}{2} V^{q_1+1/3}$$

$$+ \frac{3}{2} \left[ (m_{sp} + \lambda_0) + 2\lambda_1 (1-n_1) V^{1-2n_1} + 2\lambda_2 (1-n_2) V^{1-2n_2} \right].$$

We set $m_{sp} = 1$ and $l_0 = 2$. As far as $q_1$, $n_1$ and $n_2$ are concerned, in line of our previous discussions we choose them in such a way that the power of nonlinear term in the equations become integer. We have also studied the case for some different values, but they didn’t give any principally different picture. We choose $q_1 = 2/3$, $n_1 = -3/2 < 1/3 - 2q - 1 = -1$ and $n_2 = 3/2 > 1/2$. It
should be noted that we have taken some others value for $q_1$ such as $q_1 = -1$, but it does not give qualitatively different picture. We have also set $x_3 = [0, \pi] = k\pi/5$ with step $k = 0..5$. Finally we have considered time span $[0, 2]$ with step size 0.001. Here we consider different values of $\lambda_i$ both positive and negative. We choose the initial values for $V(0) = 1$, $Y(0) = \dot{V}(0) = 0.1$, and $A_0^3(0) = 1$, respectively.

In Figs. 1 and 2 we have plotted the phase diagram of $[V, \dot{V}, A_0^3]$ for both positive and negative $\lambda_2$, respectively. In both cases $\lambda_1 = 1$. Analogical picture was found for $\lambda_2 = 0$ and $\lambda_1 = 1$.

In Figs. 3 and 4 evolution of $V$ corresponding to Figs. 1 and 2 are demonstrated. As one sees, in both cases we have oscillatory mode of expansion.

In Figs. 5 and 6 we have illustrated the evolution of $V$ for $\lambda_1 = 0$ and $\lambda_2 = 0$ (linear spinor field) and for $\lambda_1 = 0$ and $\lambda_2 = 1$, respectively. This shows that in case of $\lambda_1 = 0$ we have a rapid expansion of $V$ at a very early stage.

In the Figures each color corresponds to a concrete value of $x_3 = k\pi/5$, namely, red, green, yellow, blue, magenta and black color corresponds to $k = 0, 1, 2, 3, 4, 5$.

![Phase diagram](image_url)

**FIG. 1.** Phase diagram of $[V, \dot{V}, A_0^3]$ in case of $\lambda_1 = 1$ and $\lambda_2 = 1$.

**Case 2.** $b \neq 0$

Let us consider the case with non-trivial $b$. Inserting $f = \sin(x_3)$ into (2.62a) one finds

$$h(x_3) = b \left( \sin(x_3) - x_3 \cos(x_3) \right) - c_1 \cos(x_3) + c_2,$$

where $c_1 = \text{const.}$, $c_2 = \text{const.}$ (2.69)

The equations (2.64) together with (2.63b) in this case can be written as

$$\dot{V} = Y, \quad (2.70a)$$

$$\dot{Y} = \frac{3\kappa}{2} \sqrt{\frac{q_0^3}{q_3 f'}} V^{3q_1/2} F_1(V, A_0^3, Y) + F_2(V, A_0^3, Y), \quad (2.70b)$$

$$\dot{A}_0^3 = F_1(V, A_0^3, Y). \quad (2.70c)$$
where we denote

$$
\Phi_1(V, A_0^3, Y) = -\frac{3q_1}{4q_2} A_0^3 V^{-(q_1+4/3)} Y + \frac{b}{\kappa} \sqrt{5 \frac{q_5 q_1}{q_2 q_3}} V^{5q_1/2-2/3},
$$

$$
\Phi_2(V, A_0^3, Y) = \frac{1}{4} q_2^4 V^{4q_1+1/3} \left( \frac{h'}{f} \right)^2 + \frac{1}{2} q_2 q_3 V^{q_1+1/3} \frac{f''}{f}
+ \frac{3\kappa}{2} \left[ (m_{sp} + \lambda_0) + 2\lambda_1 (1-n_1) V^{1-2n_1} + 2\lambda_2 (1-n_2) V^{1-2n_2} \right].
$$

In what follows, we solve the foregoing system numerically for $b = 1$ with all other parameters taking same value as corresponding cases for a trivial $b$.

In Figs. 7 and 8 we have plotted the phase diagram of $[V, \dot{V}, A_0^3]$ for both positive and negative $\lambda_2$, respectively. In both cases $\lambda_1 = 1$. Analogical picture was found for $\lambda_2 = 0$ and $\lambda_1 = 1$. As one sees, while in case of a trivial $b$ we have a focus like phase diagram, in this case with non-zero $b$ the phase diagram is a spiral.

In Figs. 9 and 10 evolution of $V$ corresponding Figs. 7 and 8 are demonstrated. As one sees, in both cases we have oscillatory mode of expansion.

In Figs. 11 and 12 we have illustrated the evolution of $V$ for $\lambda_1 = 0$ and $\lambda_2 = 0$ (linear spinor field) and for $\lambda_1 = 0$ and $\lambda_2 = 1$, respectively. This shows that in case of $\lambda_1 = 0$ we have a rapid expansion of $V$ at a very early stage.

**III. CONCLUSION**

Within the scope of Bianchi type-IX cosmological model we have studied the role of spinor field in the evolution of the Universe. It is found that unlike the diagonal Bianchi models in this
case the components of energy-momentum tensor of the spinor field along the principal axis are not the same, i.e. $T^1_1 \neq T^2_2 \neq T^3_3$, even in absence of spinor field nonlinearity. The presence of nontrivial non-diagonal components of energy-momentum tensor of the spinor field imposes severe restrictions both on geometry of space-time and on the spinor field itself. As a result the space-time turns out to be either locally rotationally symmetric or isotropic. In this paper we considered the Bianchi type-IX space-time both for a trivial $b$, that corresponds to standard BIX and the one with a non-trivial $b$. It was found that a positive $\lambda_1$ gives rise to an oscillatory mode of expansion, while a trivial $\lambda_1$ leads to rapid expansion at the early stage of evolution.

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FIG. 4. Evolution of $V$ in case of $\lambda_1 = 1$ and $\lambda_2 = -0.1$.

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FIG. 5. Evolution of $V$ in case of $\lambda_1 = 0$ and $\lambda_2 = 0$.

FIG. 6. Evolution of $V$ in case of $\lambda_1 = 0$ and $\lambda_2 = 1$. 
FIG. 7. Phase diagram of $[V, \dot{V}, A^3_0]$ in case of $\lambda_1 = 1$ and $\lambda_2 = 1$.

FIG. 8. Phase diagram of $[V, \dot{V}, A^3_0]$ in case of $\lambda_1 = 1$ and $\lambda_2 = -0.1$. 
FIG. 9. Evolution of $V$ in case of $\lambda_1 = 1$ and $\lambda_2 = 1$.

FIG. 10. Evolution of $V$ in case of $\lambda_1 = 1$ and $\lambda_2 = -0.1$. 

Spinor field in Bianchi type-IX space-time
FIG. 11. Evolution of $V$ in case of $\lambda_1 = 0$ and $\lambda_2 = 0$.

FIG. 12. Evolution of $V$ in case of $\lambda_1 = 0$ and $\lambda_2 = 1$. 