Parry measure and the topological entropy of chaotic repellers embedded within chaotic attractors

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Abstract

We study the topological entropy of chaotic repellers formed by those points in a given chaotic attractor that never visit some small forbidden hole-region in the phase space. The hole is a set of points in the phase space that have a sequence $\alpha = (\alpha_0 \alpha_1 \ldots \alpha_{l-1})$ as the first $l$ letters in their itineraries. We point out that the difference between the topological entropies of the attractor and the embedded repeller is for most choices of $\alpha$ approximately equal to the Parry measure corresponding to $\alpha$, $\mu_P(\alpha)$. When the hole encompasses a point of a short periodic orbit, the entropy difference is significantly smaller than $\mu_P(\alpha)$. This discrepancy is described by the formula which relates the length of the short periodic orbit, the Parry measure $\mu_P(\alpha)$, and the topological entropies of the two chaotic sets.

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I. INTRODUCTION

Besides local instability, the main ingredient of chaotic motion is orbit complexity [1,4]. The topological entropy measures orbit complexity by looking at all possible motions without regard to their likelihood [1,4]. Within the field of communicating with chaos, the topological entropy of chaotic repellers embedded within chaotic attractors is an important issue for the analysis of trade-off between the channel capacity and noise resistance [6,14].

Motivated by this issue, we investigate the following problem. We consider an unimodal chaotic map on the interval, \( f(x) : I \rightarrow I \), with the chaotic attractor \( A \subset I \). Imagine a small region \( H \subset I \), referred to as the hole, located somewhere on the chaotic attractor. There is a set of points on \( A \) yielding trajectories that never enter the hole \( H \). This set of points is a chaotic repeller (call it \( R_H \)) embedded within the attractor \( A \).

The repeller \( R \) governs the dynamics of the transiently chaotic map with a hole \( f_H(x) : I \setminus H \rightarrow I \). The points that map into the hole under the dynamics of the map \( f_H \) are no longer considered. Besides the field of communicating with chaos [6,10], such maps with the hole(s) arise in the study of conditionally invariant measures [14], and the average lifetimes of chaotic transients preceding controlled periodic motion [15]. Here we are interested in the difference between the topological entropies of the attractor and the repeller for holes \( H \) that are small in size.

The critical point \( x_c \) of the unimodal map \( f \) defines the generating partition. Every point \( x \in I \) is assigned an infinite sequence of symbols \( \omega(x) \in \Sigma = \{0,1\}^\infty \) by the following prescription: if \( f^i(x) < x_c \) (\( f^i(x) > x_c \)), then \( \omega_i(x) = L(R, \text{respectively}) \); if \( f^i(x) = x_c \), then \( \omega_i(x) = C \). This definition includes the present state of the system. The symbols \( \omega_i \) are called letters from the alphabet \( A = \{L,C,R\} \), and the infinite sequence \( \omega(x) \) is called the itinerary of the point \( x \). Let \( \Sigma_A \subset \Sigma \) denote the set of itineraries corresponding to the trajectories on the attractor \( A \) (repeller \( R_H \), respectively). The dynamics on the attractor \( A \) (repeller \( R_H \)) is symbolically represented by the topological dynamical system \( (\Sigma_A,\sigma) \) \( ([\Sigma_{R_H},\sigma], \text{respectively}) \), where \( \sigma \) denotes the shift operation \([1,2,17]\)

\[
\sigma(\omega_0\omega_1\omega_2\ldots) = (\omega_1\omega_2\omega_3\ldots).
\]

The topological entropy of the attractor \( A \) (repeller \( R \)), denoted by \( h_A \) (\( h_R \), respectively), is identical to the topological entropy of \( (\Sigma_A,\sigma) \) \( ([\Sigma_{R_H},\sigma], \text{respectively}) \). Since the sets \( A \) and \( R_H \) are invariant under \( f \), \( \Sigma_A \) and \( \Sigma_{R_H} \) are shift-invariant.

We will simplify the analysis to allow for analytical calculations by special tailoring of the hole \( H \). Let \( H_a \equiv H_a \) be the set of all points \( x \in I \) that have the sequence \( \alpha = (\alpha_0\alpha_1\ldots\alpha_{l-1}) \) as the first \( l \) letters in their itineraries \( \alpha_i \in \{L,R\} \) (see Ref. [18] for equivalent tailoring in the context of controlling chaos). The endpoints of the interval \( H_a \) are preimages of the critical point. The itinerary of some point \( x \) contains the sequence \( (\alpha_0\alpha_1\ldots\alpha_{l-1}) \) if and only if \( x \) eventually visits the hole \( H_a \). Hence, the set \( \Sigma_{R_H} \) is obtained from the set \( \Sigma_A \) by extracting from it all itineraries that contain the sequence \( \alpha \).

Among all measures with support in \( \Sigma_A \) that are shift-invariant, Parry measure of the dynamical system \( (\Sigma_A,\sigma) \) maximize the value of entropy \([1,9] \); the entropy of \( (\Sigma_A,\sigma) \) with respect to the Parry measure \( \mu_p \) is equal to the topological entropy \([1] \). If the Parry measure contained within the cylinder \( C_{\alpha}^l = \{\omega \in \Sigma | \omega_i = \alpha_i \text{ for } i = 0,1,\ldots,l-1\} \) is larger, the
sequence $\alpha$ more frequently appears within itineraries $\omega \in \Sigma_A$. Hence, it is reasonable to expect that the entropy difference $\Delta_h(\alpha) := h_A - h_R(\alpha)$ will be larger for larger values of $\mu_P(C_\alpha^l)$. From now on, we abbreviate $\mu_P(\alpha) := \mu_P(C_\alpha^l)$.

In this manuscript we study the correlation between the Parry measure $\mu_P(\alpha)$, and the entropy difference $\Delta_h(\alpha)$, for various choices of $H_\alpha$.

We will numerically demonstrate that if the hole $H_\alpha$ is not visited by a short periodic orbit, then $\Delta_h(\alpha)$ is approximately equal to the Parry measure $\mu_P(\alpha)$. By using this observation, it will be demonstrated, analytically and numerically, that when $H_\alpha$ is visited by a short periodic orbit, then $\Delta_h(\alpha)$ is significantly smaller than $\mu_P(\alpha)$. The ratio, $\mu_P(\alpha)/\Delta_h(\alpha)$, which characterizes this discrepancy, can be expressed in terms of the topological entropy $h_A$, and the length (call it $p$) of the shortest periodic orbit visiting $H_\alpha$.

$$\frac{\mu_P(\alpha)}{\Delta_h(\alpha)} \approx \frac{1}{1 - e^{-ph_A}}. \quad (2)$$

Formula (2) is derived by using the following auxiliary result. Suppose that $x$ is a periodic orbit of prime period $p$, and $x_i = \omega_i(x)$, for $i = 0, 1, \ldots, l - 1$. For $l \gg 1$, the entropy difference $\Delta_h(\alpha)$, and $\mu_P(\alpha)$ decrease exponentially fast with the increase of $l$. The characteristic exponent is shown to be approximately equal to the topological entropy $h_A$.

**II. NUMERICAL AND THEORETICAL ANALYSIS OF THE ENTROPY DIFFERENCE AND THE PARRY MEASURE**

For the illustrations of the dependence of $\Delta_h(\alpha)$ against the position of $H_\alpha$ on the attractor, it is inconvenient to draw $\Delta_h(\alpha)$ against the mid-point of the hole $H_\alpha$. Namely, the holes $H_\alpha$ vary in size for different unimodal maps. Furthermore, the intervals $I$ may differ for different maps. For these reasons, it is convenient to introduce the topological mid-point of the hole $H_\alpha$. To a sequence $\alpha = (\alpha_0 \ldots \alpha_{l-1})$, we assign a number

$$t(\alpha) = \sum_{i=0}^{l-1} \frac{\gamma_i}{2^{i+1}} + \frac{1}{2^{l+1}}, \quad (3)$$

where $\gamma_i = 0(1)$ if the number of Rs in the sequence $(\alpha_0 \alpha_1 \ldots \alpha_i)$ is even (odd, respectively). Given a sequence $\alpha$, $t(\alpha)$ is identical for all unimodal maps, and it is called the topological mid-point of $H_\alpha$ since it preserves the spatial ordering of holes: if $t(\alpha') < t(\alpha'')$, then $H_{\alpha'}$ is to the left of $H_{\alpha''}$. \[20]\]

In order to illustrate the main result of this paper, consider the attractor $A = [0, 1]$ of the tent map $T_a(x) = a(1 - |1 - 2x|)/2$ for $a = 2$. The topological entropy is $h_A = \ln 2$. Every sequence is admissible by the systems dynamics, i.e., $(\Sigma_A, \sigma)$ is a shift of full type on two symbols \[22\]. The Parry measure corresponding to any sequence of length $l$ is $1/2^l$, i.e., $\mu_P(\alpha) = 1/2^l$. Figure \[1\](a) displays $\Delta_h(\alpha)$ and $\mu_P(\alpha)$, while Fig. \[1\](b) displays the ratio $\mu_P(\alpha)/\Delta_h(\alpha)$ against $t(\alpha)$. We observe that for almost every choice of $\alpha$, $\Delta_h(\alpha) \simeq \mu_P(\alpha)$.

However, when the hole $H_\alpha$ is visited by a short periodic orbit of period $p$ smaller than some critical value $l_c \sim 3 - 4$, the quantity $\mu_P(\alpha)$ is significantly larger than $\Delta_h(\alpha)$.

Consider the map $T_a(x)$ for $a = 1.72208 \ldots$, with the topological entropy $h_A = \ln a = 0.543535 \ldots$. The dynamics is pruned, that is, there are sequences inadmissible by the
dynamics of the map $T_{1.72208\ldots}(x)$, and $(\Sigma_A, \sigma)$ is a shift of finite type on two symbols. Fig. 2 illustrates $\mu_P(\alpha), \Delta h(\alpha)$, and their ratio against $t(\alpha)$ for all admissible sequences of length $l$. The same observations as in Fig. 4 can be made.

After performing a number of numerical experiments equivalent to the ones illustrated in Figs. 4 and 2, we make the following conjecture: If $H_\alpha$ is not visited by a short periodic orbit, then $\mu_P(\alpha) \simeq \Delta h(\alpha)$. By using this conjecture, we can analytically describe the discrepancy between $\mu_P(\alpha)$ and $\Delta h(\alpha)$ which occurs when $H_\alpha$ encompasses a short periodic orbit.

At first, we must study the dependence of $\Delta h(\alpha)$ and $\mu_P(\alpha)$ on the length of the sequence $\alpha$. We perform the following numerical experiment. Consider a point $x \in A$, which lies on a periodic orbit of prime period $p$. The itinerary $\omega(x) = \pi^\infty$ is defined by the sequence of $p$ letters, $\pi = (\pi_0 \pi_1 \ldots \pi_{p-1})$. By choosing

$$\alpha = (\pi_0 \pi_1 \ldots \pi_{p-1} \pi_0 \pi_1 \ldots \pi_{p-1} \pi_0 \pi_1 \ldots \pi_{r-1})$$

where $r = l \text{ div } p$, and $r' = l \mod p$, the hole $H_\alpha$ encompasses the point $x$ for every $l = rp + r'$. (The size of the hole $H_\alpha$ decreases with the increase of $l$.) Fig. 3 illustrates the dependence of the difference $\Delta h(\alpha)$, and the ratio $\mu_P(\alpha)/\Delta h(\alpha)$ on the length $l$ of the sequence $\alpha$; $\alpha$ is constructed from $\pi = (R), (RL)$, and $(LLR)$ [see Eq. (4)]. For this illustration, we have utilized the map $T_a(x)$ for $a = 2$, and $a = 1.72208\ldots$

We observe the following. The difference $\Delta h(\alpha)$ decreases approximately exponentially fast with the increase of $l$. The points in Figs. 3(a), and 3(b) are fitted to the functional dependence $\Delta h(\alpha) = A_0 e^{-A_1 l}$. The fitted values of the exponent $A_1$ are shown in Table 4. The characteristic exponent $A_1$ is approximately equal to the topological entropy $h_A$. In some cases, the graph $\Delta h(\alpha)$ against $l$ shows flat plateaus, as is illustrated in Fig. 3(b). From Fig. 3(c) we observe that the ratio $\mu_P(\alpha)/\Delta h(\alpha)$ is independent of $l$ for $l$ large enough. In other words, both $\Delta h(\alpha)$ and $\mu_P(\alpha)$ are observed to have the same (up to the multiplicative constant) functional dependence on $l$, $\Delta h(\alpha) \sim \mu_P(\alpha) \sim e^{-h_A l}$.

In order to explain these observations, we utilize the topological zeta function [2]. Let $\alpha$ and $\beta$ denote two sequences of length $l$ and $L \gg l$, respectively, which are constructed as prescribed in Eq. (4), so that $\beta_i = \alpha_i$ for $i = 0, 1, \ldots, l - 1$. These sequences define the embedded repellers $R^{(\alpha)}$ and $R^{(\beta)}$. Let $q$ denote the length of the shortest periodic orbit(s) that visits the hole $H_\alpha$, and which does not visit the hole $H_\beta \subset H_\alpha$. This orbit is a part of the repeller $R^{(\beta)}$, but it does not belong to $R^{(\alpha)}$. The topological entropies of $R^{(\alpha)}$ and $R^{(\beta)}$ can be calculated from the smallest positive zeros of their corresponding topological zeta functions [2]:

$$1/\zeta^{(\alpha)}_{\text{top}} = 1 - \sum_{n=1}^{q-1} c_n z^n - \sum_{n=q}^{\infty} c_n^{(\alpha)} z^n,$$  \hspace{1cm} (5)

and

$$1/\zeta^{(\beta)}_{\text{top}} = 1 - \sum_{n=1}^{q-1} c_n z^n - \sum_{n=q}^{\infty} c_n^{(\beta)} z^n.$$ \hspace{1cm} (6)

Only the coefficients of order $q$ and higher differ since the periodic orbits within the repellers $R^{(\alpha)}$ and $R^{(\beta)}$ that are of prime period $q - 1$ and smaller coincide [2]. For $H_\alpha$, and $H_\beta$ as
constructed above, it can be shown that \( q > (r - 1)p \), and most likely \( q \leq (r + 1)p \), that is, \( q \sim l = rp + r' \) (see Appendix A).

Let us define the function \( \Delta_\xi(z) := 1/\zeta^{(\beta)} - 1/\zeta^{(\alpha)} \), and the quantity \( \epsilon_z := z^{(\alpha)} - z^{(\beta)} \), where \( z^{(\alpha)} \) and \( z^{(\beta)} \) denote the smallest positive zeros of the functions \( 1/c^{(\alpha)} \) and \( 1/c^{(\beta)} \), respectively. The topological entropies of the repellers \( R^{(\alpha)} \), and \( R^{(\beta)} \) are \( h^{(\alpha)} = -\ln z^{(\alpha)} \), and \( h^{(\beta)} = -\ln z^{(\beta)} \), respectively. If \( l \) is large enough, then \( \eta = \epsilon_z/z^{(\beta)} \ll 1 \). This approximation leads to

\[
h^{(\beta)} - h^{(\alpha)} = \ln \frac{z^{(\alpha)}}{z^{(\beta)}} = \frac{\epsilon_z}{z^{(\beta)}} + \mathcal{O}(\eta^2), \tag{7}
\]

and

\[
\Delta_\xi(z^{(\beta)}) = 0 - 1/\zeta^{(\alpha)}(z^{(\beta)}) = -1/\zeta^{(\alpha)}(z^{(\alpha)}) - \epsilon_z = z^{(\beta)} \frac{d[1/\zeta^{(\alpha)}(z)]}{dz} \bigg|_{z=z^{(\alpha)}} \frac{\epsilon_z}{z^{(\beta)}} + \mathcal{O}(\eta^2). \tag{8}
\]

On the other hand, from Eqs. (5) and (6)

\[
\Delta_\xi(z^{(\beta)}) = \sum_{n=q}^{\infty} [c_n^{(\alpha)} - c_n^{(\beta)}] [z^{(\beta)}]^n = [z^{(\beta)}]^q \sum_{n=0}^{\infty} [c_{n+q}^{(\alpha)} - c_{n+q}^{(\beta)}] [z^{(\beta)}]^n. \tag{9}
\]

In the limit \( L \to \infty \), \( h^{(\beta)} = h_A \), i.e., \( z^{(\beta)} = e^{-h_A} \). By neglecting the terms of order \( \eta^2 \), and higher, from Eqs. (7), (8), and (9) we obtain

\[
\ln \Delta_h(\alpha) \approx -gh_A + \ln \left| \sum_{n=0}^{\infty} [c_{n+q}^{(\alpha)} - c_{n+q}^{(\beta)}] [z^{(\beta)}]^n \right| - \ln \left| z^{(\beta)} \frac{d[1/\zeta^{(\alpha)}(z)]}{dz} \bigg|_{z=z^{(\alpha)}} \right|. \tag{10}
\]

If the sequence \( \alpha \) is long enough, the first term on the right hand side (r.h.s.) of Eq. (10), \( -gh_A \sim -h_A \), is much larger than the second, and the third term. This explains the observed functional dependence \( \Delta_h(\alpha) \sim e^{-lh_A} \). The second term on the r.h.s. can introduce small oscillations superimposed on the global behavior \( \Delta_h(\alpha) \sim e^{-lh_A} \) [e.g., see open circles in Fig. 3(a)]. The third term is practically independent of \( l \) for large enough \( l \). Eq. (10) is consistent with the scaling of \( h_A \) with the size of the hole that is reported in Ref. [2] for the tent, \( 2x \mod 1 \), and the logistic map \( 4z(1-z) \).

The dependence of \( \Delta_h(\alpha) \) on \( l \) is equivalent to the problem of calculating the topological entropy of some attractor with infinite Markov partition by using the set of finite pruning rules. By using the set of finite pruning rules, one actually calculates the topological entropy of the embedded repeller [2]. If all periodic orbits up to the length \( l \) are included in the calculation of the topological entropy of the attractor, then the difference between the
topological entropies of the embedded repeller and the attractor is of order $\sim e^{-lh_A}$ (see the discussion in Ref. [4]).

In Refs. [4, 11] the entropy $h_R$ has been studied as a function of the size of the hole $H$. It has been demonstrated that $h_R$ is a devil’s staircase like function with flat plateaus. In the case considered here, the size of the hole is not a continuous parameter since it enters the calculations via the length $l$ of the sequence $\alpha$. The flat plateaus in the graph $\Delta h(\alpha)$ vs. $l$ [see Fig. 3(b)] are a fingerprint of the plateaus observed and explained in Refs. [7–10] in a more general settings.

If $(\Sigma_A, \sigma)$ is $n$-step topological Markov chain, the explanation of the functional dependence $\mu_P(\alpha) \sim e^{-lh_A}$ follows immediately from the definition of the Parry measure (see Appendix B, and Ref. [5], p. 175). For a more general class of topological dynamical systems $(\Sigma_A, \sigma)$, which naturally arise from one dimensional maps, the functional dependence $\mu_P(\alpha) \sim e^{-lh_A}$ is explained by using the following property of the map $T_a(x)$. Consider an unimodal map $f$ with $h_A > 0$, and the corresponding topological dynamical system $(\Sigma_A, \sigma)$. The map $T_a(x)$ with $a = e^{h_A}$ corresponds to the same topological dynamical system $(\Sigma_A, \sigma)$ (see Ref. [1], p. 514.). Let $h_{\mu_N}$ denote the entropy of $T_a(x)$ with respect to the naturally invariant measure $\mu_N$. The entropy $h_{\mu_N}$ is equal to the Lyapunov exponent $\lambda = \ln a$ [2, 3], i.e., for the map $T_a(x)$ we have $h_{\mu_N} \equiv \lambda \equiv h_A = \ln a$. Since $h_A$ is the entropy with respect to the maximal entropy measure (i.e. the Parry measure) coincides with the naturally invariant measure $\mu_N$. Therefore, $\mu_P(\alpha) \equiv \mu_N(H_\alpha)$. The length of the interval $H_\alpha$ is roughly $\sim 1/a^l$. From the scaling of $\mu_N$ it follows that [3],

$$\mu_P(\alpha) \equiv \mu_N(H_\alpha) \sim 1/a^l = e^{-lh_A}. \quad (11)$$

Equivalent analysis demonstrates that Parry measure coincides with the naturally invariant measure on the chaotic repeller of the properly chosen gap-tent map (see. Sec. 2 in Ref. [4]).

By using two points from the previous discussion: (i) The fact that $\mu_P(\alpha) \sim e^{-lh_A}$, and (ii) the observation that for $H_\alpha$ which does not encompass a short periodic orbit $\Delta h(\alpha) \simeq \mu_P(\alpha)$, we can explain the discrepancy between $\mu_P(\alpha)$ and $\Delta h(\alpha)$ which occurs when $H_\alpha$ is visited by a short periodic orbit. We will derive Eq. (2) for $H_\alpha$ located on a fixed point, i.e., for $\alpha = (R^l, p = 1$. The generalization for the case when $H_\alpha$ encompasses a point on a short periodic orbit of prime period $p = 2, 3, \ldots, l_c$, is straightforward.

The first preimages of the cylinder $C^{l+1}_{(LR^l)}$ are the cylinders $C^{l+1}_{(LR^l)}$ and $C^{l+1}_{(RR^l)} = C^{l+1}_{(RR^l)}$. From the invariance of Parry measure it follows that

$$\mu_P(R^l) = \mu_P(LR^l) + \mu_P(R^{l+1}). \quad (12)$$

Since $\mu_P(R^{l+1})/\mu_P(R^l) = e^{-h_A}$,

$$\mu_P(R^l) \simeq (1 - e^{-h_A})^{-1} \mu_P(LR^l). \quad (13)$$

Let us compare $\Delta h(R^l)$, and $\Delta h(LR^l)$. Let $P_{(R^l)} \ [P_{(LR^l)}]$ denote the set of all periodic orbits within the attractor $A$ which contain the sequence $(R^l)$ ($(LR^l)$, respectively). It is easy to see that every orbit within $P_{(R^l)}$, except the fixed point $(R^\infty)$ itself, contains the sequence $(LR^l)$. Thus, the two sets differ by a single periodic orbit. Since the topological entropy
quantitatively expresses the exponential growth of periodic orbits, it follows that \( h_R(R^l) \) and \( h_R(LR^l) \) are identical, i.e.,

\[
\Delta_h(R^l) = \Delta_h(LR^l). \tag{14}
\]

By combining Eqs. (13) and (14), and using the observation \( \mu_p(LR^l) \simeq \Delta_h(LR^l) \), we immediately obtain Eq. (2) for the case \( p = 1 \). To generalize for the case \( p = 2, 3, \ldots \), one has to consider the \( p \)th order preimages of the cylinder \( C^l_\alpha \) to obtain equations equivalent to (13) and (14). Fig. 3(c) confirms the validity of the approximations that lead to Eq. (2). These approximations work better for larger values of \( l \) [see Fig. 3(c)].

III. CONCLUSION

We have studied the topological entropy of an embedded chaotic repeller \( R_H \), defined as the set of points on a given attractor \( A \) that do not visit a small region (hole) in the phase space \( H_\alpha \). The hole \( H_\alpha \) is a set of points that have a sequence \( \alpha = (\alpha_0 \alpha_1 \ldots \alpha_{l-1}) \) as the first \( l \) letters in their itineraries. It has been shown that the difference between the topological entropies of the attractor and the repeller, denoted by \( \Delta_h(\alpha) \), is for most choices of \( \alpha \) approximately equal to the Parry measure corresponding to \( \alpha \), i.e., \( \Delta_h(\alpha) \simeq \mu_P(\alpha) \). Significant deviation from this global behavior occurs when the itinerary of some short periodic orbit contains the sequence \( \alpha \). The formula which relates the length of the short periodic orbit, the Parry measure \( \mu_p(\alpha) \), and the entropy difference \( \Delta_h(\alpha) \) has been derived [see Eq. (2)].

IV. APPENDIX A

Consider a periodic orbit \( \omega^{(\pi)} = \pi^\infty \in \Sigma_A \) of prime period \( p \); it is defined by the sequence of \( p \) letters, \( \pi = (\pi_0 \pi_1 \ldots \pi_{p-1}) \). Let \( \alpha \) and \( \beta \) denote two sequences of length \( l \) and \( L \gg l \), respectively, that are constructed from \( \pi \) [see Eq. (4)]. We assume that \( r = (l \mod p) \geq 3 \). Clearly, \( \omega^{(\pi)} \in C^l_\alpha \). Let \( \omega^{(v)} = v^{\infty} \in \Sigma_A \) denote a periodic orbit of prime period \( q \), defined by the sequence \( v = (v_0 v_1 \ldots v_{q-1}) \), such that \( \omega^{(v)} \neq \omega^{(\pi)} \), and \( \omega^{(v)} \in C^l_\alpha \). \( \omega^{(v)} \in C^l_\alpha \) means that \( \omega_i^{(v)} = \omega_i^{(\pi)} \), for \( i = 0, 1, \ldots, l \). In this appendix, we demonstrate that \( q \geq (r - 1)p \).

1. Suppose that \( p \leq q \leq (r - 1)p \). We define integers \( m = q \mod p \) and \( n = q \mod p \) \( (q = mp + n) \). For \( n = 0, v = \pi^m \); this implies \( \omega^{(v)} = \omega^{(\pi)} \), and we are in contradiction. For \( n > 0 \), the comparison of \( \alpha \) constructed form \( \pi \) and \( v \) yields \( \pi_{(n+i)} \mod p = \alpha_{q+i} = v_i = \pi_i \) for \( i = 0, 1, \ldots, p - 1 \), and this contradicts the assumption that \( \omega^{(\pi)} \) is of prime period \( p \).

2. Suppose that \( q < p \), and define integers \( m = p \mod q \), \( n = p \mod q \), \( n > 0 \) \( (p = mq + n) \). Since \( \omega^{(v)} \in C^l_\alpha \) implies \( v_{(n+i)} \mod q = \alpha_{p+i} = \pi_i = v_i \) for \( i = 0, 1, \ldots, q - 1 \), \( \omega^{(v)} \) can not be of prime period \( q \).

Thus, we have demonstrated that \( q \geq (r - 1)p \). Since periodic orbits are dense in \( \Sigma_A \) [1], for a given sequence \( \alpha \) constructed from \( \pi \), there most likely exists a periodic orbit \( \omega^{(v)} \) of prime period \( q \), such that \( (r - 1)p < q \leq (r + 1)p \), and \( \omega^{(v)} \in C^l_\alpha \). For large enough \( L \ll 1 \), this orbit is not within the cylinder \( C^l_\beta \). Therefore, the length \( q \) of the shortest periodic orbit that visits the hole \( H_\alpha \), and which does not visit the hole \( H_\beta \) is \( q > (r - 1)p \), and most likely \( q \leq (r + 1)p \).
V. APPENDIX B

In this Appendix, we recall the definition of the Parry measure for \( n \)-step topological Markov chain \([7,19]\). From this definition the explanation of the functional dependence \( \mu_P(\alpha) \sim e^{-lh_A} \) follows immediately.

The \( 2^n \times 2^n \) matrix \( C \) which determines all admissible transitions is defined as follows \([7]\). The matrix element \( C_{(i_0i_1...i_{n-1}), (j_0j_1...j_{n-1})} \), where \( i_k, j_k \in \{L, R\} \), is equal to 1 if \( j_k = i_{k+1} \) for \( k = 0, 1, \ldots, n - 2 \), and if \( (i_1...i_{n-1}j_{n-1}) \) is an admissible sequence. Otherwise, \( C_{(i_0i_1...i_{n-1}), (j_0j_1...j_{n-1})} = 0 \). [The indices of the matrix elements are not integers, but sequences of \( Ls \) and \( Rs \). By identifying \( L (R) \) with 0 (1, respectively), the index sequence \( (i_0i_1...i_{n-1}) \) can be thought of as an integer \( \in [0, 2^n - 1] \) represented in the binary system.]

The largest eigenvalue of the matrix \( C \) is identical to \( e^{h_A} \) \([1,2]\). Let \( y \), and \( w \) denote the eigenvectors of the matrix \( C \) and its transpose \( C^T \), respectively, corresponding to the eigenvalue \( e^{h_A} \); these eigenvectors are normalized so that

\[
\sum_{(i_0i_1...i_{n-1})} w(i_0i_1...i_{n-1}) y(i_0i_1...i_{n-1}) = 1, \quad i_k \in \{L, R\}. \tag{15}
\]

To find the Parry measure, another \( 2^n \times 2^n \) matrix (call it \( P \)) is constructed (see Ref. \([11]\). p. 175.),

\[
P_{(i_0i_1...i_{n-1}), (j_0j_1...j_{n-1})} = \frac{e^{-h_A} C_{(i_0i_1...i_{n-1}), (j_0j_1...j_{n-1})}}{w(i_0i_1...i_{n-1})} \frac{w(j_0j_1...j_{n-1})}{w(i_0i_1...i_{n-1})}. \tag{16}
\]

The Parry measure corresponding to the sequence \((i_0i_1...i_{n-1})\) is \([11]\)

\[
\mu_P(i_0i_1...i_{n-1}) = w(i_0i_1...i_{n-1}) y(i_0i_1...i_{n-1}). \tag{17}
\]

The Parry measure corresponding to a sequence \( \alpha = (\alpha_0\alpha_1...\alpha_{l-1}) \), of length \( l \geq n \) is \([11]\)

\[
\mu_P(\alpha) = P_{(\alpha_0\alpha_1...\alpha_{n-1}), (\alpha_1\alpha_2...\alpha_n)} P_{(\alpha_1\alpha_2...\alpha_n), (\alpha_2\alpha_3...\alpha_{n+1})} \cdots P_{(\alpha_{l-n}\alpha_{l-n+1}...\alpha_{l-2}), (\alpha_{l-n}\alpha_{l-n+1}...\alpha_{l-1})} \mu_P(\alpha_{l-n}\alpha_{l-n+1}...\alpha_{l-1}) \tag{18}
\]

\[
= e^{-(l-n)h_A} w(\alpha_0\alpha_1...\alpha_{n-1}) y(\alpha_{l-n}\alpha_{l-n+1}...\alpha_{l-1}).
\]

The last equality follows from the assumption \( \mu_P(\alpha) \neq 0 \). If \( \alpha \) is defined as in Eq. \([11]\), from Eq. \([18]\) it immediately follows that \( \mu_P(\alpha) \sim e^{-lh_A} \) with possible oscillations of period \( p \).
REFERENCES

[1] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, (Cambridge University Press, Cambridge, 1995).

[2] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Classical and Quantum Chaos*, www.nbi.dk/ChaosBook/, Niels Bohr Institute (Copenhagen 2001).

[3] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).

[4] R. L. Adler, A. G. Konheim, and M. H. McAndrew, Trans. Am. Math. Soc. 114, 309 (1965).

[5] P. Cvitanović, G. H. Gunaratne, I. Procaccia, Phys. Rev. A 38, 1503 (1988).

[6] W. Słomczyński, J. Kwapień, and K. Życzkowski, Chaos 10, 180 (2000).

[7] E. Bollt, Y.-C. Lai, and C. Grebogi, Phys. Rev. Lett. 79, 3787 (1997); E. Bollt and Y.-C. Lai, Phys. Rev. E 58, 1724 (1998); Y.-C. Lai, E. Bollt, C. Grebogi, Phys. Lett. A 255, 75-81 (1999); M. Dolnik, E. Bollt, Chaos 8(3), 702-710 (1998).

[8] J. Jacobs and E. Ott, B. R. Hunt, Phys. Rev. E 57, 6577 (1998).

[9] K. Życzkowski and E. M. Bollt, Physica D 132, 392 (1999).

[10] Li-Yi Hsu, Phys. Lett. A 271, 252 (2000).

[11] E. Bollt, T. Stanford, Y.-C. Lai, K. Życzkowski, Phys. Rev. Lett. 85, 3524 (2000).

[12] P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Berkhauser, Boston, 1980)

[13] J. Milnor and W. Thurston, *On Iterated Maps of the Interval*, in *Dynamical Systems, College Park*, 1986-1987, Lec. Notes in Math. 1342 (Springer, Berlin, 1988).

[14] N. Chernov, R. Markarian, and S. Troubetzkoy, Ergod. Th. and Dynam. Sys. 18, 1049 (1998); P. Collet, S. Martinez and B. Schmitt, Nonlinearity 7, 1437 (1994).

[15] H. Buljan and V. Paar, Phys. Rev. E 63, 066205 (2001); V. Paar and H. Buljan, Phys. Rev. E 62, 4869 (2000).

[16] V. Paar and N. Pavin, Phys. Rev. E 55, 4112 (1997); V. Paar and N. Pavin, Phys. Lett. A 235, 139 (1997).

[17] K. T. Hansen, PhD. thesis, University of Oslo (1993).

[18] C. M. Place and D. K. Arrowsmith, Phys. Rev. E 61, 1357 (2000); ibid 61, 1369 (2000).

[19] W. Parry, Trans. Am. Math. Soc. 112, 55 (1964); ibid 122, 368 (1966).

[20] Formula (3) is an adaptation of an equivalent formula [see e.g., Eqs. (2.3) and (2.4) in Ref. [6], and also Refs. [6,7]] that define the ordering of itineraries, which preserves the ordering of corresponding points for unimodal maps.

[21] Although the alphabet is \( \mathcal{A} = \{L, C, R\} \), there is only one itinerary within \( \Sigma_{\mathcal{A}} \) that begins with the letter \( \mathcal{C} \), namely the kneading sequence \([1]\). Therefore, in the case of unimodal one-dimensional maps, we are effectively dealing with the binary alphabet (eg., see Ref. [6]).

[22] M. Artin, and B. Mazur, Annals. Math. 81 82 (1965).

[23] T. Bohr, and D. Rand, Physica D 25, 387 (1987).
FIGURES

FIG. 1. (a) The difference $\Delta_h(\alpha)$ (solid line) and the Parry measure $\mu_P(\alpha)$ (dashed line) against $t(\alpha)$. (b) The ratio $\mu_P(\alpha)/\Delta_h(\alpha)$ against $t(\alpha)$. The integers above the peaks denote the length of the shortest periodic orbit visiting $H_\alpha$. The length of the sequence $\alpha$ is $l = 8$.

FIG. 2. (a) The Parry measure $\mu_P(\alpha)$ corresponding to the allowed sequences of length $l = 8$. (b) The difference $\Delta_h(\alpha)$, and (c) the ratio $\mu_P(\alpha)/\Delta_h(\alpha)$ against the topological mid-point of $H_\alpha$. The integers above the peaks denote the length of the shortest periodic orbit visiting $H_\alpha$.

FIG. 3. (a) The difference $\Delta_h(\alpha)$ against the length $l$ of the sequence $\alpha$. The lines represent the least square fit to the functional dependence $\Delta_h(\alpha) = A_0 e^{-A_1 l}$. (b) The same as figure (a). (c) The ratio $\Delta_h(\alpha)/\mu_P(\alpha)$ against $l$. The lines represent the quantity $(1 - e^{-\mu_A})^{-1}$ [see Eq. (2)]. Closed squares correspond to the map $T_a(x)$ for $a = 2$, and $\alpha = (RR\ldots R) = (R^l)$; closed circles correspond to $a = 2$, and $\alpha = (RLRL\ldots)$; open squares correspond to $a = 1.72208\ldots$, and $\alpha = (R^l)$; open circles correspond to $a = 1.72208\ldots$, and $\alpha = (RLRL\ldots)$; open diamonds correspond to $a = 1.72208\ldots$, and $\alpha = (LLRLLR\ldots)$. 
TABLE I. The fitted values of the parameter $A_1$ [$\Delta_h(\alpha) = A_0 e^{-l A_1}$] for the map $T_\alpha(x)$, and sequences $\alpha$ constructed from $\pi$ [see Eq. (4)].

| $\alpha$ | $\pi$ | $A_1$ | $h_A$ |
|----------|-------|-------|-------|
| 2.0      | $(R)$ | 0.6958| 0.6931... |
|          | $(RL)$| 0.6968|        |
| 1.72208... | $(R)$ | 0.550 |       |
|          | $(RL)$| 0.565 | 0.5435... |
|          | $(LLR)$| 0.533|       |
\[ \Delta_h(\alpha) \]

(a)

\[ \mu_p(\alpha)/\Delta_h(\alpha) \]

(b) plateau

(c) plateau

\[ l \]

6 8 10 12 14 16