Hamilton-Jacobi Theorems for Regular Reducible Hamiltonian Systems on a Cotangent Bundle

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Abstract: In this paper, we first prove a geometric Hamilton-Jacobi theorem for Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic structure under a weaker condition. Then we generalize the result to regular reducible Hamiltonian system with symmetry by using the reduced symplectic structure, and obtain the Hamilton-Jacobi theorems for regular point reduced Hamiltonian system and regular orbit reduced Hamiltonian system. As an application of the theoretical results, we consider the regular point reducible Hamiltonian system on a Lie group, and give the Hamilton-Jacobi theorem and Lie-Poisson Hamilton-Jacobi equation of the system. In particular, we show the Lie-Poisson Hamilton-Jacobi equations of rigid body and heavy top on the rotation group SO(3) and on the Euclidean group SE(3), respectively.

Keywords: Hamilton-Jacobi theorem, symplectic structure, momentum map, regular point reduction, regular orbit reduction.

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1 Introduction

Symmetry is a general phenomenon in the natural world, but it is widely used in the study of mathematics and mechanics. The reduction theory for mechanical system with symmetry has its origin in the classical work of Euler, Lagrange, Hamilton, Jacobi, Routh, Liouville and Poincaré and its modern geometric formulation in the general context of symplectic manifolds and equivariant momentum maps is developed by Meyer, Marsden and Weinstein; see Abraham and Marsden [1] or Marsden and Weinstein [17] and Meyer [19]. The main goal of reduction theory in mechanics is to use conservation laws and the associated symmetries to reduce the number of dimensions of a mechanical system required to be described. So, such reduction theory is regarded as a useful tool for simplifying and studying concrete mechanical systems. Hamiltonian reduction theory is one of the most active subjects in the study of modern analytical mechanics and applied mathematics, in which a lot of deep and beautiful results have been obtained, see the studies given by Abraham and Marsden [1], Arnold [2], Marsden et al [12, 13, 15, 17], Ortega and Ratiu [21], Libermann and Marle [11], León and Rodrigues [10] etc. on regular point reduction and regular orbit reduction, singular point reduction and singular orbit reduction, optimal reduction and reduction by stages for Hamiltonian systems and so on; and there is still much to be done in this subject.

At the same time, we note also that the well-known Hamilton-Jacobi theory is an important part of classical mechanics. On the one hand, it provides a characterization of the generating functions of certain time-dependent canonical transformations, such that a given Hamiltonian system in such a form that its solutions are extremely easy to find by reduction to the equilibrium, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [15]. On the other hand, it is possible in many cases that Hamilton-Jacobi theory provides an immediate way to integrate the equation of motion of system, even when the problem of Hamiltonian system itself has not been or cannot be solved completely. In addition, the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the development of numerical integrators that preserve the symplectic structure and in the study of stochastic dynamical systems, see Ge and Marsden [5], Marsden and West [18] and Lázaro-Camí and Ortega [7]. For these reasons it is described as a useful tools in the study of Hamiltonian system theory, and has been extensively developed in past many years and become one of the most active subjects in the study of modern applied mathematics and analytical mechanics, which absorbed a lot of researchers to pour into it and a lot of deep and beautiful results have been obtained, see Cariñena et al [3] and [4], Iglesias et al [6], León et al [8, 9] Ohsawa and Bloch [20], for more details.

Now, it is a natural problem how to study the Hamilton-Jacobi theory for a variety of reduced Hamiltonian systems by combining with reduction theory and Hamilton-Jacobi theory of Hamiltonian systems. This is goal of our research. The main contributions in this paper is given as follows. (1) We prove a geometric Hamilton-Jacobi theorem for Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic structure under a weaker condition; (2) We generalize the above result to regular reducible Hamiltonian system with symmetry, and obtain the Hamilton-Jacobi theorems for regular point reduced Hamiltonian system and regular orbit reduced Hamiltonian system, see Theorem 3.3 and 4.3, by using the reduced symplectic structure; (3) As an application, we give the Lie-Poisson Hamilton-Jacobi equation of the regular point reducible Hamiltonian system on a Lie group, and show the Hamilton-Jacobi equations of rigid body and heavy top, respectively. In general, we know that it is not easy
to find the solutions of Hamilton’s equation. But, if we can get a solution of Hamilton-Jacobi equation of the Hamiltonian system, by using the relationship between Hamilton’s equation and Hamilton-Jacobi equation, it is easy to give a special solution of Hamilton’s equation. Thus, it is very important to give explicitly the Hamilton-Jacobi equation of a Hamiltonian system.

A brief of outline of this paper is as follows. In the second section, we first prove a key lemma, which is obtained by a careful modification for the corresponding result of Abraham and Marsden in [1], then give a geometric version of Hamilton-Jacobi theorem of Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic structure under a weaker condition. From the third section we begin to discuss the regular reducible Hamiltonian systems with symmetry by combining with the Hamilton-Jacobi theory and regular symplectic reduction theory. The regular point and regular orbit reducible Hamiltonian systems with symmetry are considered respectively in the third section and the fourth section, and give the Hamilton-Jacobi theorems of regular point and regular orbit reduced Hamiltonian systems by using the reduced symplectic structure. As the applications of the theoretical results, in fifth section, we consider the regular point reducible Hamiltonian system on a Lie group, and give the Hamilton-Jacobi theorem and Lie-Poisson Hamilton-Jacobi equation of the system. In particular, we show the Lie-Poisson Hamilton-Jacobi equations of rigid body and heavy top on the rotation group SO(3) and on the Euclidean group SE(3), respectively. These research work develop the reduction and Hamilton-Jacobi theory of Hamiltonian systems with symmetry and make us have much deeper understanding and recognition for the structure of Hamiltonian systems.

2 Geometric Hamilton-Jacobi Theorem of Hamiltonian System

In this section, we first review briefly some basic facts about Hamilton-Jacobi theory, and state our idea to study the problem in this paper. Then we prove a key lemma, which is obtained by a careful modification for the corresponding result of Abraham and Marsden in [1]. This lemma is an important tool for the proofs of geometric Hamilton-Jacobi theorems of Hamiltonian system and regular reducible Hamiltonian system with symmetry. Finally, we give a geometric version of Hamilton-Jacobi theorem of Hamiltonian system on the cotangent bundle of a configuration manifold, by using the symplectic structure under a weaker condition. We shall follow the notations and conventions introduced in Abraham and Marsden [1], Marsden and Ratiu [15], Ortega and Ratiu [21], and Marsden et al [16]. In this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions.

It is well-known that Hamilton-Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which state that the integral of Lagrangian of a system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with a canonical symplectic form $\omega$ and the projection $\pi_Q : T^*Q \to Q$.

**Theorem 2.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W : Q \to \mathbb{R}$ is a given function. Then the following two assertions are equivalent:

(i) For every curve $\sigma : \mathbb{R} \to Q$ satisfying $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$. 

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(ii) \( W \) satisfies the Hamilton-Jacobi equation \( H(q^i, \frac{\partial W}{\partial q^i}) = E \), where \( E \) is a constant.

It is worthy of note that if we take that \( \gamma = dW \) in the above theorem, then \( \gamma \) is a closed one-form on \( Q \), and the equation \( d(H \cdot dW) = 0 \) is equivalent to the Hamilton-Jacobi equation \( H(q^i, \frac{\partial W}{\partial q^i}) = E \), where \( E \) is a constant. This result is used the formulation of a geometric version of Hamilton-Jacobi theorem for Hamiltonian system, see Cariñena et al [3] and Iglesias et al [6]. On the other hand, this result is developed in the context of time-dependent Hamiltonian system by Marsden and Ratiu in [15]. The Hamilton-Jacobi equation may be regarded as a nonlinear partial differential equation for some generating function \( S \), and the problem becomes how to choose a time-dependent canonical transformation \( \Psi : T^*Q \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R} \), which transforms the dynamical vector field of a time-dependent Hamiltonian system to equilibrium, such that the generating function \( S \) of \( \Psi \) satisfies the time-dependent Hamilton-Jacobi equation. In particular, for the time-independent Hamiltonian system, we may look for a symplectic map as the canonical transformation. This work offers an important idea that one can use the dynamical vector field of a Hamiltonian system to describe Hamilton-Jacobi equation. Moreover, assume that \( \gamma : Q \rightarrow T^*Q \) is a closed one-form on \( Q \), and define that \( X^\gamma_H = T\pi_Q \cdot X_H \cdot \gamma \), where \( X_H \) is the dynamical vector field of Hamiltonian system \( (T^*Q, \omega, H) \). Then the fact that \( X^\gamma_H \) and \( X_H \) are \( \gamma \)-related, that is, \( T\gamma \cdot X^\gamma_H = X_H \cdot \gamma \) is equivalent that \( d(H \cdot \gamma) = 0 \), which is given in Cariñena et al [3] and Iglesias et al [6].

Motivated by the above research work, we hope to use the dynamical vector field of the regular reduced Hamiltonian system to describe the Hamilton-Jacobi theory for regular reducible Hamiltonian system with symmetry in this paper. In order to do this, we need first to give a geometric version of Hamilton-Jacobi theorem of Hamiltonian system on the cotangent bundle of a configuration manifold. Thus, in the following we first give an important notion and prove a key lemma, which is an important tool for the proof of geometric Hamilton-Jacobi theorem of Hamiltonian system.

Let \( Q \) be a smooth manifold and \( TQ \) the tangent bundle, \( T^*Q \) the cotangent bundle with a canonical symplectic form \( \omega \), and the projection \( \pi_Q : T^*Q \rightarrow Q \) induces the maps \( \pi_Q^*: T^*Q \rightarrow T^*\ast T^*Q \), and \( T\pi_Q : TT^*Q \rightarrow TQ \). Denote by \( \pi_Q^*\omega \) the induced symplectic form on \( TT^*Q \) from the canonical symplectic form \( \omega \) on \( T^*Q \). Assume that \( \gamma : Q \rightarrow T^*Q \) is an one-form on \( Q \) and \( \gamma^*: T^*T^*Q \rightarrow T^*Q \). If \( \gamma^* \) is symplectic, then for any \( v, w \in TT^*Q \), we have \( \gamma^*\pi_Q^*\omega(v, w) = \gamma^*\omega(T\pi_Q(v), T\pi_Q(w)) \). If \( \gamma \) is closed, then \( d\gamma(x, y) = 0 \), \( \forall x, y \in TQ \). In the following we give a bit weak notion.

**Definition 2.2** One-form \( \gamma \) is called to be closed with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), if for any \( v, w \in TT^*Q \), we have \( d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0 \).

From this definition we know that, if \( \gamma \) is a closed one-form, then it must be closed with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \). But conversely, if \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), it may not be closed. We can prove a general result as follows.

**Proposition 2.3** Assume that \( \gamma : Q \rightarrow T^*Q \) is an one-form on \( Q \) and it is not closed. we define the subset \( N \subset TQ \), satisfying that for any \( x, y \in N \), \( d\gamma(x, y) \neq 0 \). Denote by \( \text{Ker}(T\pi_Q) = \{ u \in TT^*Q | T\pi_Q(u) = 0 \} \), and \( \gamma : TQ \rightarrow TT^*Q \). If \( T\gamma(N) \subset \text{Ker}(T\pi_Q) \), then \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \).

**Proof:** In fact, for any \( v, w \in TT^*Q \), if \( T\pi_Q(v) \notin N \), or \( T\pi_Q(w) \notin N \), then by the definition of \( N \), we know that \( d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0 \); If \( T\pi_Q(v) \in N \), and \( T\pi_Q(w) \in N \),...
from the condition $T\gamma(N) \subset \text{Ker}(T\pi_Q)$, we know that $T\pi_Q \cdot T\gamma \cdot T\pi_Q(v) = T\pi_Q(v) = 0,$ and $T\pi_Q \cdot T\gamma \cdot T\pi_Q(w) = T\pi_Q(w) = 0,$ and hence $\textbf{d}\gamma(T\pi_Q(v), T\pi_Q(w)) = 0.$ Thus, for any $v, w \in TT^*Q,$ we have always that $\textbf{d}\gamma(T\pi_Q(v), T\pi_Q(w)) = 0,$ that is, $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ.$

Now, we prove the following Lemma 2.4. It is worthy of note that this lemma is obtained by a careful modification for the corresponding result of Abraham and Marsden in [1].

**Lemma 2.4** Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$, and $\gamma^* : T^*T^*Q \rightarrow T^*Q$ is symplectic, and $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ.$ Then we have that

(i) for any $v, w \in TT^*Q,$ $\pi_Q^*\omega(v, w) = -\textbf{d}\gamma(T\pi_Q(v), T\pi_Q(w));$

(ii) for any $v, w \in TT^*Q,$ $\pi_Q^*\omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \pi_Q^*\omega(v, w - T(\gamma \cdot \pi_Q) \cdot w);$

(iii) $T\gamma : TQ \rightarrow TT^*Q$ is injective map with respect to $T\pi_Q : TT^*Q \rightarrow TQ,$ that is, for any $v, w \in TT^*Q,$ $T\gamma(T\pi_Q(v)) \neq T\gamma(T\pi_Q(w)),$ when $T\pi_Q(v) \neq T\pi_Q(w).$

**Proof:** We first prove the conclusion (i). Since $\omega$ is the canonical symplectic form on $T^*Q$, we know that there is an unique canonical one-form $\theta$, such that $\omega = -\textbf{d}\theta.$ From the Proposition 3.2.11 in Abraham and Marsden [1], we have that for one-form $\gamma : Q \rightarrow T^*Q,$ $\pi^*\omega = \gamma.$ By using the assumption that $\gamma^* : T^*T^*Q \rightarrow T^*Q$ is symplectic, we can obtain that

$$\gamma^*\pi_Q^*\omega(v, w) = \gamma^*\omega(T\pi_Q(v), T\pi_Q(w)) = \gamma^*(-\textbf{d}\theta)(T\pi_Q(v), T\pi_Q(w))$$

$$= -\textbf{d}(\gamma^*\theta)(T\pi_Q(v), T\pi_Q(w)) = -\textbf{d}\gamma(T\pi_Q(v), T\pi_Q(w)).$$

It follows that the conclusion (i) holds.

Next, we prove the conclusion (ii). For any $v, w \in TT^*Q,$ note that $v - T(\gamma \cdot \pi_Q) \cdot v$ is vertical, because

$$T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v) = T\pi_Q(v) - T(\pi_Q \cdot \gamma \cdot \pi_Q) \cdot v = T\pi_Q(v) - T\pi_Q(v) = 0,$$

where we used the relation $\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q.$ Thus,

$$\pi_Q^*\omega(v - T(\gamma \cdot \pi_Q) \cdot v, w - T(\gamma \cdot \pi_Q) \cdot w) = \omega(T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v), T\pi_Q(w - T(\gamma \cdot \pi_Q) \cdot w)) = 0,$$

and hence,

$$\pi_Q^*\omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \pi_Q^*\omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) + \pi_Q^*\omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w).$$

However, the second term on the right-hand side vanishes, that is,

$$\pi_Q^*\omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w) = \gamma^*\pi_Q^*\omega(T\pi_Q(v), T\pi_Q(w)) = -\textbf{d}\gamma(T\pi_Q(v), T\pi_Q(w)) = 0,$$

where we used the conclusion (i) and the assumption that $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ.$ It follows that the conclusion (ii) holds.

Finally, we prove the conclusion (iii). In fact, we can prove that if $\gamma$ is closed one-form on $Q,$ then $T\gamma : TQ \rightarrow TT^*Q$ is injective. We take a local coordinates $q^1, i = 1, \cdots, n = \text{dim}Q,$ on $Q,$ and assume that $\gamma = \sum_{i=1}^n \gamma_i(q) dq^i.$ Then $\textbf{d}\gamma = \frac{1}{2} \sum_{i<j} \left(\frac{\partial \gamma_j}{\partial q^i} - \frac{\partial \gamma_i}{\partial q^j}\right) dq^i \wedge dq^j.$ Since $\gamma$ is closed one-form on $Q,$ we have that $\frac{\partial \gamma_j}{\partial q^i} = \frac{\partial \gamma_i}{\partial q^j},$ $i \neq j,$ $i, j = 1, \cdots, n.$ Notice that $\gamma : Q \rightarrow T^*Q,$ $\gamma(q) = (q^i, \gamma_i(q)),$ and $T\gamma = (\frac{\partial}{\partial q^i}, \sum_{j=1}^n \frac{\partial \gamma_i}{\partial q^j} \frac{\partial}{\partial q^j}),$ and $T\gamma - T\gamma = (\frac{\partial}{\partial q^i} - \frac{\partial}{\partial q^j} + \frac{\partial \gamma_i}{\partial q^j} \frac{\partial}{\partial q^j} - \frac{\partial \gamma_j}{\partial q^i} \frac{\partial}{\partial q^i}),$ hence, $T\gamma - T\gamma \neq T\gamma, i \neq j.$ Thus, $T\gamma : TQ \rightarrow TT^*Q$ is injective. In the same way, if $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ,$ then
for any \(v, w \in TT^*Q\), \(T\gamma(T\pi_Q(v)) \neq T\gamma(T\pi_Q(w))\), when \(T\pi_Q(v) \neq T\pi_Q(w)\), and hence \(T\gamma : TQ \to TT^*Q\) is injective with respect to \(T\pi_Q : TT^*Q \to TQ\). 

Now, for any given Hamiltonian system \((T^*Q, \omega, H)\), by using the above Lemma 2.4, we can prove the following geometric Hamilton-Jacobi theorem for the Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-1.

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{\gamma} & Q \\
\pi_Q^* \downarrow & & \pi_Q \uparrow \\
T^*T^*Q & \xrightarrow{\gamma^*} & T^*Q \\
\end{array}
\]

**Theorem 2.5 (Hamilton-Jacobi Theorem of Hamiltonian System)** For a Hamiltonian system \((T^*Q, \omega, H)\), assume that \(\gamma : Q \to T^*Q\) is a one-form on \(Q\), and \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic, and \(\gamma\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\), and \(X_H = T\pi_Q \cdot X_H \cdot \gamma\), where \(X_H\) is the dynamical vector field of Hamiltonian system \((T^*Q, \omega, H)\). Then the following two assertions are equivalent:

(i) \(X_H^\gamma\) and \(X_H\) are \(\gamma\)-related, that is, \(T\gamma \cdot X_H^\gamma = X_H \cdot \gamma\);

(ii) \(X_{H, \gamma} = 0\), or \(X_{H, \gamma} = X_H^\gamma\).

Where the equation that \(X_{H, \gamma} = 0\), is another version of Hamilton-Jacobi equation for Hamiltonian system \((T^*Q, \omega, H)\), by comparing with \(d(H \cdot \gamma) = 0\), and \(\gamma\) is called a solution of the Hamilton-Jacobi equation.

**Proof:** We first prove that (i) implies (ii). We take that \(v = X_H \cdot \gamma \in TT^*Q\), for any \(w \in TT^*Q\), from Lemma 2.4(ii) we have that

\[
\pi_Q^*\omega(T\gamma \cdot X_H^\gamma, w) = \pi_Q^*\omega(T(\gamma \cdot \pi_Q) \cdot X_H \cdot \gamma, w) = \pi_Q^*\omega(X_H \cdot \gamma, w - T(\gamma \cdot \pi_Q) \cdot w)
\]

By assuming (i), \(T\gamma \cdot X_H^\gamma = X_H \cdot \gamma\), we can obtain that \(\pi_Q^*\omega(X_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w) = 0\). On the other hand, note that \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic, and \(X_H \cdot \gamma = T\gamma \cdot X_{H, \gamma}\), where \(X_{H, \gamma} \in TQ\) is the Hamiltonian vector field of function \(H \cdot \gamma : Q \to \mathbb{R}\). Then we have that

\[
0 = \pi_Q^*\omega(X_H \cdot \gamma, T(\gamma \cdot \pi_Q) \cdot w) = \pi_Q^*\omega(T\gamma \cdot X_{H, \gamma}, T\gamma \cdot (T\pi_Q \cdot w))
\]

It follows that either \(X_{H, \gamma} = 0\), or there is someone \(\tilde{v} \in TT^*Q\), such that \(X_{H, \gamma} = T\pi_Q \cdot \tilde{v}\). But from assuming (i), \(T\gamma \cdot X_H^\gamma = X_H \cdot \gamma = T\gamma \cdot X_{H, \gamma}\), and Lemma 2.4(iii), \(T\gamma : TQ \to TT^*Q\) is injective with respect to \(T\pi_Q : TT^*Q \to TQ\), hence, \(X_{H, \gamma} = X_H^\gamma\). Thus, (i) implies (ii).

Conversely, since \(\pi_Q^*\omega\) is nondegenerate, the proof that (ii) implies (i) follows from these arguments in the same way. 

**Remark 2.6** From the Proposition 2.3 we know that the condition that \(\gamma\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\) is weaker than the condition that \(\gamma\) is closed on \(Q\), that is, \(d\gamma(x, y) = 0, \forall x, y \in TQ\). When \(\gamma\) is closed on \(Q\), in this case Theorem 2.5 holds trivially, except for \(X_{H, \gamma} = X_H^\gamma\). Thus, we can also get the result given in Cariñena et al [3] and Iglesias et al [6] (but there is not the condition that \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic in Cariñena et al [3] and
Iglesias et al. [6]). It is worthy of note that when \(X_H \cdot \gamma \in \text{Ker}(T\pi_Q)\), then \(X_H^\gamma = T\pi_Q \cdot X_H \cdot \gamma = 0\), in this case we have the equation that \(X_H \cdot \gamma = X_H^\gamma = 0\), this is just Hamilton-Jacobi equation. Thus, the equation \(X_H \cdot \gamma = X_H^\gamma\) is called a non-homogeneous Hamilton-Jacobi equation.

In the following we shall generalize the above result to regular point and regular orbit reducible Hamiltonian systems with symmetry, and give a variety of Hamilton-Jacobi theorems for reduced Hamiltonian systems.

3 Hamilton-Jacobi Theorem of Regular Point Reduced Hamiltonian System

Let \(Q\) be a smooth manifold and \(T^*Q\) its cotangent bundle with the symplectic form \(\omega\). Let \(\Phi : G \times Q \rightarrow Q\) be a smooth left action of the Lie group \(G\) on \(Q\), which is free and proper. Then the cotangent lifted left action \(\Phi^* : G \times T^*Q \rightarrow T^*Q\) is symplectic, free and proper, and admits a \(\text{Ad}^*\)-equivariant momentum map \(J : T^*Q \rightarrow g^*\), where \(g\) is a Lie algebra of \(G\) and \(g^*\) is the dual of \(g\). Let \(\mu \in g^*\) be a regular value of \(J\) and denote by \(G_\mu\) the isotropy subgroup of the coadjoint \(G\)-action at the point \(\mu \in g^*\), which is defined by \(G_\mu = \{g \in G | \text{Ad}^*_g \mu = \mu\}\). Since \(G_\mu \subset G\) acts freely and properly on \(Q\) and on \(T^*Q\), then \(Q_\mu = Q/G_\mu\) is a smooth manifold and that the canonical projection \(\rho_\mu : Q \rightarrow Q_\mu\) is a surjective submersion. It follows that \(G_\mu\) acts also freely and properly on \(J^{-1}(\mu)\), so that the space \((T^*Q)_\mu = J^{-1}(\mu)/G_\mu\) is a symplectic manifold with symplectic form \(\omega_\mu\) uniquely characterized by the relation

\[
\pi^*_\mu \omega_\mu = i^*_\mu \omega. \tag{3.1}
\]

The map \(i_\mu : J^{-1}(\mu) \rightarrow T^*Q\) is the inclusion and \(\pi_\mu : J^{-1}(\mu) \rightarrow (T^*Q)_\mu\) is the projection. The pair \(((T^*Q)_\mu, \omega_\mu)\) is called Marsden-Weinstein reduced space of \((T^*Q, \omega)\) at \(\mu\).

Remark 3.1 If \((T^*Q, \omega)\) is a connected symplectic manifold, and \(J : T^*Q \rightarrow g^*\) is a non-equivariant momentum map with a non-equivariance group one-cocycle \(\sigma : G \rightarrow g^*\), which is defined by \(\sigma(g) := J(g \cdot z) - \text{Ad}^{-1}_g J(z)\), where \(g \in G\) and \(z \in T^*Q\). Then we know that \(\sigma\) produces a new affine action \(\Theta : G \times g^* \rightarrow g^*\) defined by \(\Theta(g, \mu) := \text{Ad}^{-1}_g \mu + \sigma(g)\), where \(\mu \in g^*\), with respect to which the given momentum map \(J\) is equivariant. Assume that \(G\) acts freely and properly on \(T^*Q\), and \(G_\mu\) denotes the isotropy subgroup of \(\mu \in g^*\) relative to this affine action \(\Theta\) and \(\mu\) is a regular value of \(J\). Then the quotient space \((T^*Q)_\mu = J^{-1}(\mu)/G_\mu\) is also a symplectic manifold with symplectic form \(\omega_\mu\) uniquely characterized by (3.1), see Ortega and Ratiu [21].

Let \(H : T^*Q \rightarrow \mathbb{R}\) be a \(G\)-invariant Hamiltonian, the flow \(F_\mu\) of the Hamiltonian vector field \(X_H\) leaves the connected components of \(J^{-1}(\mu)\) invariant and commutes with the \(G\)-action, so it induces a flow \(F^\mu_t\) on \((T^*Q)_\mu\), defined by \(F^\mu_t : \pi_\mu = \pi_\mu \cdot F^\mu_t \cdot i_\mu\), and the vector field \(X_{h_\mu}\) generated by the flow \(F^\mu_t\) on \(((T^*Q)_\mu, \omega_\mu)\) is Hamiltonian with the associated regular point reduced Hamiltonian function \(h_\mu : (T^*Q)_\mu \rightarrow \mathbb{R}\) defined by \(h_\mu : \pi_\mu = H \cdot i_\mu\), and the Hamiltonian vector fields \(X_H\) and \(X_{h_\mu}\) are \(\pi_\mu\)-related. Thus, we can define a regular point reducible Hamiltonian system as follows.

Definition 3.2 (Regular Point Reducible Hamiltonian System) A 4-tuple \((T^*Q, G, \omega, H)\), where the Hamiltonian \(H : T^*Q \rightarrow \mathbb{R}\) is \(G\)-invariant, is called a regular point reducible Hamiltonian system, if there exists a point \(\mu \in g^*\), which is a regular value of the momentum map \(J\), such that the regular point reduced system, that is, the 3-tuple \(((T^*Q)_\mu, \omega_\mu, h_\mu)\), where \((T^*Q)_\mu = J^{-1}(\mu)/G_\mu\),
\[ \pi^*_\mu \omega = i^*_\mu \omega, \ h_\mu \cdot \pi_\mu = H \cdot i_\mu, \] is a Hamiltonian system, which is also called Marsden-Weinstein reduced Hamiltonian system. Where \(((T^*Q)_\mu, \omega_\mu)\) is Marsden-Weinstein reduced space, the function \(h_\mu : (T^*Q)_\mu \to \mathbb{R}\) is called the reduced Hamiltonian.

For the regular point reducible Hamiltonian system \((T^*Q, G, \omega, H)\), we can prove the following Hamilton-Jacobi theorem for regular point reduced Hamiltonian system \(((T^*Q)_\mu, \omega_\mu, h_\mu)\). For convenience, the maps involved in the following theorem and its proof are shown in Diagram-2.

![Diagram-2](image)

**Theorem 3.3 (Hamilton-Jacobi Theorem of Regular Point Reduced Hamiltonian System)** For a regular point reducible Hamiltonian system \((T^*Q, G, \omega, H)\), assume that \(\gamma : \mathbb{R} \to T^*Q\) is an one-form on \(Q\), and \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic, and \(\gamma\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\), and \(X_H = T\pi_Q \cdot X_H \cdot \gamma\), where \(X_H\) is the dynamical vector field of Hamiltonian system with symmetry \((T^*Q, G, \omega, H)\). Moreover, assume that \(\mu \in \mathfrak{g}^*\) is the regular reducible point of the Hamiltonian system, and \(\text{Im}(\gamma) \subset J^{-1}(\mu)\), and it is \(G_\mu\)-invariant, and \(\bar{\gamma} = \pi_\mu(\gamma) : Q \to (T^*Q)_\mu\). Then the following two assertions are equivalent:

(i) \(X_H^{\gamma} \) and \(X_{h_\mu} \) are \(\bar{\gamma}\)-related, that is, \(T\bar{\gamma} \cdot X_H^{\gamma} = X_{h_\mu} \cdot \bar{\gamma}\), where \(X_{h_\mu}\) is the dynamical vector field of Marsden-Weinstein reduced Hamiltonian system \(((T^*Q)_\mu, \omega_\mu, h_\mu)\);

(ii) \(X_{h_\mu} \cdot \bar{\gamma} = 0\), or \(X_{h_\mu} \cdot \bar{\gamma} = X_H^{\gamma}\).

Where the equation that \(X_{h_\mu} \cdot \bar{\gamma} = 0\), is the Hamilton-Jacobi equation for Marsden-Weinstein reduced Hamiltonian system \(((T^*Q)_\mu, \omega_\mu, h_\mu)\), and \(\bar{\gamma}\) is called a solution of the Hamilton-Jacobi equation.

**Proof:** We first prove that (i) implies (ii). By using the reduced symplectic form \(\omega_\mu\), note that \(\text{Im}(\gamma) \subset J^{-1}(\mu)\), and it is \(G_\mu\)-invariant, in this case \(\pi^*_\mu \omega = i^*_\mu \omega = \pi^*_Q \omega\), along \(\text{Im}(\gamma)\). Thus, we take that \(v = X_H \cdot \gamma \in TT^*Q\), and for any \(w \in TT^*Q\), and \(T\pi_\mu \cdot w \neq 0\), from Lemma 2.4(ii) we have that

\[
\omega_\mu(T\bar{\gamma} \cdot X_H^{\gamma}, T\pi_\mu \cdot w) = \omega_\mu(T(\pi_\mu \cdot \gamma) \cdot X_H^{\gamma}, T\pi_\mu \cdot w) = \pi^*_\mu \omega_\mu(T\gamma \cdot X_H^{\gamma}, w)
\]

where we used that \(T\pi_\mu(X_H) = X_{h_\mu}\). By assuming (i), \(T\bar{\gamma} \cdot X_H^{\gamma} = X_{h_\mu} \cdot \bar{\gamma}\), we can obtain that \(\omega_\mu(X_{h_\mu} \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_\mu \cdot w) = 0\). On the other hand, note that \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic, and
\[ \tilde{\gamma}^* = \gamma^* \cdot \pi^*_\mu : T^*(T^*Q)_\mu \to T^*Q \] is also symplectic along \( \text{Im}(\gamma) \), and hence \( X_{h^\mu \cdot \tilde{\gamma}} = T\tilde{\gamma}^* \cdot X_{h^\mu \cdot \gamma} \), where \( X_{h^\mu \cdot \gamma} \in TQ \) is the Hamiltonian vector field of function \( h^\mu \cdot \gamma : Q \to \mathbb{R} \). Then we have that

\[
0 = \omega(\mu, X_{h^\mu \cdot \tilde{\gamma}}, T\tilde{\gamma} \cdot T\pi_Q \cdot w) = \mu(T\tilde{\gamma} \cdot X_{h^\mu \cdot \gamma}, T\tilde{\gamma} \cdot T\pi_Q \cdot w) = \tilde{\gamma}^* \cdot \pi^*_\mu \omega(X_{h^\mu \cdot \gamma}, T\pi_Q \cdot w) = \tilde{\gamma}^* \cdot \pi^*_\mu \omega(X_{h^\mu \cdot \gamma}, T\pi_Q \cdot w) = -d\gamma(X_{h^\mu \cdot \gamma}, T\pi_Q \cdot w).
\]

It follows that either \( X_{h^\mu \cdot \gamma} = 0 \), or there is someone \( \tilde{v} \in TT^*Q \), such that \( X_{h^\mu \cdot \gamma} = T\pi_Q \cdot \tilde{v} \). But from assuming (i), \( T\tilde{\gamma}^* \cdot X_{\hat{\gamma}} = X_{h^\mu \cdot \tilde{\gamma}} \), and \( \text{Lemma 2.4(iii)} \), \( T\tilde{\gamma} : TT^*Q \to TQ \) is injective with respect to \( T\pi_Q : TT^*Q \to TQ \), and hence \( T\tilde{\gamma} = T\pi_Q \cdot \gamma \), is injective with respect to \( T\pi_Q : TT^*Q \to TQ \), and \( X_{h^\mu \cdot \gamma} = \hat{X}^\gamma \). Thus, (i) implies (ii).

Conversely, from the above arguments we have that for any \( w \in TT^*Q \), and \( T\pi_\mu \cdot w \neq 0 \), then

\[
\omega(\mu, T\tilde{\gamma} \cdot X^\gamma_H, T\pi_\mu \cdot w) = \omega(\mu, X_{h^\mu \cdot \tilde{\gamma}}, T\pi_\mu \cdot w) = d\gamma(X_{h^\mu \cdot \gamma}, T\pi_Q \cdot w).
\]

Thus, since \( \omega(\mu) \) is nondegenerate, the proof that (ii) implies (i) follows in the same way.

**Remark 3.4** If \( (T^*Q, \omega) \) is a connected symplectic manifold, and \( J : T^*Q \to g^* \) is a non-equivariant momentum map with a non-equivariance group one-cocycle \( \sigma : G \to g^* \), in this case, for a given regular point reducible Hamiltonian system \( (T^*Q, G, \omega, H) \), we can also prove the Hamilton-Jacobi theorem for regular point reduced Hamiltonian system \( (T^*Q)_{\mu}, \omega(\mu), h^\mu \) by using the above way, where the reduced space \( ((T^*Q)_{\mu}, \omega(\mu)) \) is determined by the affine action given in Remark 3.1.

For a regular point reducible Hamiltonian system \( (T^*Q, G, \omega, H) \), we know that the dynamical vector fields \( X_H \) and \( X_{h^\mu} \) are \( \pi_\mu \)-related, that is, \( X_{h^\mu} \cdot \pi_\mu = T\pi_\mu \cdot X_H \cdot i_\mu \). Then we can prove the following Theorem 3.5, which states the relationship between the solutions of Hamilton-Jacobi equations and Marsden-Weinstein reduction.

**Theorem 3.5** For a regular point reducible Hamiltonian system \( (T^*Q, G, \omega, H) \), assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( \gamma^* : T^*T^*Q \to T^*Q \) is symplectic, and \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \). Moreover, assume that \( \mu \in g^* \) is the regular reducible point of the Hamiltonian system, and \( \text{Im}(\gamma) \subset J^{-1}(\mu) \), and it is \( G^\mu \)-invariant, and \( \tilde{\gamma} = \pi_\mu(\gamma) : Q \to (T^*Q)_{\mu} \). Then \( \tilde{\gamma} \) is a solution of the Hamilton-Jacobi equation for the Hamiltonian system with symmetry \( (T^*Q, G, \omega, H) \) if and only if \( \tilde{\gamma} \) is a solution of the Hamilton-Jacobi equation of Marsden-Weinstein reduced Hamiltonian system \( ((T^*Q)_{\mu}, \omega(\mu), h^\mu) \).

**Proof:** In fact, from the proof of Theorem 2.5, we know that \( T\gamma \cdot X^\gamma_H = X^\gamma_H \cdot \gamma = T\gamma^* \cdot X^\gamma_H \), and from the proof of Theorem 3.3, we have that \( T\tilde{\gamma}^* \cdot X^\gamma_H = X_{h^\mu \cdot \tilde{\gamma}} \), \( T\tilde{\gamma}^* \cdot X_{\hat{\gamma}} = T\tilde{\gamma}^* \cdot X_{h^\mu \cdot \gamma} \). Note that both maps \( T\gamma : TQ \to TT^*Q \) and \( T\tilde{\gamma} : TQ \to T(T^*Q)_{\mu} \) are injective with respect to \( T\pi_Q : TT^*Q \to TQ \). Thus, \( X_{h^\mu \cdot \gamma} = 0 \), \( \iff \) \( X^\gamma_H = 0 \), \( \iff \) \( X_{h^\mu \cdot \gamma} = 0 \). It follows that the conclusion of Theorem 3.5 holds.

### 4 Hamilton-Jacobi Theorem of Regular Orbit Reduced Hamiltonian System

Let \( \Phi : G \times Q \to Q \) be a smooth left action of the Lie group \( G \) on \( Q \), which is free and proper. Then the cotangent lifted left action \( \Phi^{\mathcal{T}^*} : G \times T^*Q \to T^*Q \) is symplectic, free and proper, and admits an \( \text{Ad}^* \)-equivariant momentum map \( J : T^*Q \to g^* \). Assume that \( \mu \in g^* \) is a regular value.
of the momentum map $J$ and $O_\mu = G \cdot \mu \subset g^*$ is the $G$-orbit of the coadjoint $G$-action through the point $\mu$. Since $G$ acts freely, properly and symplectically on $T^*Q$, then the quotient space $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G$ is a regular quotient symplectic manifold with the symplectic form $\omega_{O_\mu}$ uniquely characterized by the relation

$$i^*_{O_\mu} \omega = \pi^*_{O_\mu} \omega_{O_\mu} + J^*_{O_\mu} \omega^+_\mu,$$

(4.1)

where $J_{O_\mu}$ is the restriction of the momentum map $J$ to $J^{-1}(O_\mu)$, that is, $J_{O_\mu} = J \cdot i_{O_\mu}$ and $\omega^+_\mu$ is the $+$-symplectic structure on the orbit $O_\mu$ given by

$$\omega^+_\mu(\nu)(\xi, \eta) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in O_\mu, \xi, \eta \in g.$$

(4.2)

The maps $i_{O_\mu} : J^{-1}(O_\mu) \to T^*Q$ and $\pi_{O_\mu} : J^{-1}(O_\mu) \to (T^*Q)_{O_\mu}$ are natural injection and the projection, respectively. The pair $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is called the symplectic orbit reduced space of $(T^*Q, \omega)$.

**Remark 4.1** If $(T^*Q, \omega)$ is a connected symplectic manifold, and $J : T^*Q \to g^*$ is a non-equivariant momentum map with a non-equivariance group one-cocycle $\sigma : G \to g^*$, which is defined by $\sigma(g) := J(g \cdot z) - \text{Ad}^*_g \cdot J(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that $\sigma$ produces a new affine action $\Theta : G \times g^* \to g^*$ defined by $\Theta(g, \mu) := \text{Ad}^*_g \cdot \mu + \sigma(g)$, where $\mu \in g^*$, with respect to which the given momentum map $J$ is equivariant. Assume that $G$ acts freely and properly on $T^*Q$, and $O_\mu = G \cdot \mu \subset g^*$ denotes the $G$-orbit of the point $\mu \in g^*$ with respect to this affine action $\Theta$, and $\mu$ is a regular value of $J$. Then the quotient space $((T^*Q)_{O_\mu} = J^{-1}(O_\mu))/G$ is also a symplectic manifold with symplectic form $\omega_{O_\mu}$ uniquely characterized by (4.1), see Ortega and Ratiu [21].

Let $H : T^*Q \to \mathbb{R}$ be a $G$-invariant Hamiltonian, the flow $F_t$ of the Hamiltonian vector field $X_H$ leaves the connected components of $J^{-1}(O_\mu)$ invariant and commutes with the $G$-action, so it induces a flow $f^O_t$ on $(T^*Q)_{O_\mu}$, defined by $f^O_t \cdot \pi_{O_\mu} = \pi_{O_\mu} \cdot F_t \cdot i_{O_\mu}$, and the vector field $X_{h_{O_\mu}}$ generated by the flow $f^O_t$ on $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is Hamiltonian with the associated regular orbit reduced Hamiltonian function $h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}$ defined by $h_{O_\mu} \cdot \pi_{O_\mu} = H \cdot i_{O_\mu}$ and the Hamiltonian vector fields $X_H$ and $X_{h_{O_\mu}}$ are $\pi_{O_\mu}$-related. Thus, we can define a regular orbit reducible Hamiltonian system as follows.

**Definition 4.2** (Regular Orbit Reducible Hamiltonian System) A 4-tuple $(T^*Q, G, \omega, H)$, where the Hamiltonian $H : T^*Q \to \mathbb{R}$ is $G$-invariant, is called a regular orbit reducible Hamiltonian system, if there exists an orbit $O_\mu$, $\mu \in g^*$, where $\mu$ is a regular value of the momentum map $J$, such that the regular orbit reduced system, that is, the 3-tuple $((T^*Q)_{O_\mu}, \pi_{O_\mu}, h_{O_\mu})$, where $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G$, $\pi_{O_\mu} \omega_{O_\mu} = i_{O_\mu} \omega - J_{O_\mu}^* \omega^+_\mu$, $h_{O_\mu} \cdot \pi_{O_\mu} = H \cdot i_{O_\mu}$, is a Hamiltonian system. Where $((T^*Q)_{O_\mu}, \omega_{O_\mu})$ is the regular orbit reduced space, and the function $h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}$ is called the regular orbit reduced Hamiltonian.

For the regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, H)$, we can prove the following Hamilton-Jacobi theorem for regular orbit reduced Hamiltonian system $((T^*Q)_{O_\mu}, \omega_{O_\mu}, h_{O_\mu})$. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-3.
Theorem 4.3 (Hamilton-Jacobi Theorem of Regular Orbit Reduced Hamiltonian System) For a regular orbit reducible Hamiltonian system \((T^*Q, G, \omega, H)\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic, and \(\gamma\) is closed with respect to \(T\pi_Q : TT^*Q \to TQ\), and \(X_H = T\pi_Q \cdot X_H \cdot \gamma\), where \(X_H\) is the dynamical vector field of Hamiltonian system with symmetry \((T^*Q, G, \omega, H)\). Moreover, assume that \(\mathcal{O}_\mu, \mu \in \mathfrak{g}^*\), is the regular reducible orbit of the Hamiltonian system, and \(\text{Im}(\gamma) \subseteq J^{-1}(\mu)\), and it is \(G\)-invariant, \(\bar{\gamma} = \pi_{\mathcal{O}_\mu}(\gamma) : Q \to (T^*Q)_{\mathcal{O}_\mu}\). Then the following two assertions are equivalent:

(i) \(X_H^\gamma\) and \(X_{h_{\mathcal{O}_\mu}}\) are \(\bar{\gamma}\)-related, that is, \(T\bar{\gamma} \cdot X_H^\gamma = X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma}\), where \(X_{h_{\mathcal{O}_\mu}}\) is the dynamical vector field of regular orbit reduced Hamiltonian system \(((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu})\);

(ii) \(X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma} = 0\), or \(X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma} = X_H^\gamma\).

Where the equation that \(X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma} = 0\), is the Hamilton-Jacobi equation for regular orbit reduced Hamiltonian system \(((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu})\), and \(\bar{\gamma}\) is called a solution of the Hamilton-Jacobi equation.

Proof: Note that the symplectic orbit reduced space \((T^*Q)_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G \cong J^{-1}(\mu)/G \times \mathcal{O}_\mu\), with the symplectic form \(\omega_{\mathcal{O}_\mu}\) uniquely characterized by the relation \(i_{\omega_{\mathcal{O}_\mu}} \omega = \pi^*_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu} + J_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu}\). Since \(\text{Im}(\gamma) \subseteq J^{-1}(\mu)\), and it is \(G\)-invariant, in this case for any \(V \in TQ\), and \(w \in TT^*Q\), we have that \(J_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu}(T\gamma \cdot V, w) = 0\), and hence \(\pi^*_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu} = i_{\omega_{\mathcal{O}_\mu}} \omega = \pi^*_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu}\). In the following we first prove that (i) implies (ii). We take that \(v = X_H \cdot \gamma \in TT^*Q\), and for any \(w \in TT^*Q\), and \(T\pi_{\mathcal{O}_\mu} \cdot w \neq 0\), from Lemma 2.4(ii) we have that

\[
\omega_{\mathcal{O}_\mu}(T\bar{\gamma} \cdot X_H^\gamma, T\pi_{\mathcal{O}_\mu} \cdot w) = \omega_{\mathcal{O}_\mu}(T(\pi_{\mathcal{O}_\mu} \cdot \gamma) \cdot X_H^\gamma, T\pi_{\mathcal{O}_\mu} \cdot w) = \pi^*_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu}(T\gamma \cdot X_H^\gamma, w)
\]

where we used that \(T\pi_{\mathcal{O}_\mu} \cdot X_H = X_{h_{\mathcal{O}_\mu}}\). By assuming (i), \(T\bar{\gamma} \cdot X_H^\gamma = X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma}\), we can obtain that \(\omega_{\mathcal{O}_\mu}(X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_{\mathcal{O}_\mu} \cdot w) = 0\). On the other hand, note that \(\gamma^* : T^*T^*Q \to T^*Q\) is symplectic, and \(\bar{\gamma}^* = \gamma^* \cdot \pi^*_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to T^*Q\) is also symplectic along \(\text{Im}(\gamma)\), and hence \(X_{h_{\mathcal{O}_\mu}} \cdot \bar{\gamma} = T\bar{\gamma} \cdot X_{h_{\mathcal{O}_\mu}} \cdot \gamma\), where \(X_{h_{\mathcal{O}_\mu}} \cdot \gamma \in TQ\) is the Hamiltonian vector field of function
\( h_{O_{\mu}} \cdot \tilde{\gamma} : Q \to \mathbb{R} \). Then we have that

\[
0 = \omega_{O_{\mu}}(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\tilde{\gamma} \cdot T\pi_{Q} \cdot w) = \omega_{O_{\mu}}(T\tilde{\gamma} \cdot X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\tilde{\gamma} \cdot T\pi_{Q} \cdot w)
\]

\[
= \tilde{\gamma}^* \omega_{O_{\mu}}(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\pi_{Q} \cdot w) = \gamma^* \cdot \pi_{O_{\mu}}^* \omega_{O_{\mu}}(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\pi_{Q} \cdot w)
\]

\[
= \gamma^* \cdot \pi_{O_{\mu}}^* \omega(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\pi_{Q} \cdot w) = -d\gamma(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\pi_{Q} \cdot w).
\]

It follows that either \( X_{h_{O_{\mu}}} \cdot \tilde{\gamma} = 0 \), or there is someone \( \tilde{v} \in TT^*Q \), such that \( X_{h_{O_{\mu}}} \cdot \tilde{\gamma} = T\pi_{Q} \cdot \tilde{v} \).

But from assuming (i), \( T\tilde{\gamma} \cdot X_{\gamma} = X_{h_{O_{\mu}}} \cdot \tilde{\gamma} = T\tilde{\gamma} \cdot X_{h_{O_{\mu}}} \cdot \tilde{\gamma} \), and Lemma 2.4(iii), \( T\tilde{\gamma} : TQ \to TT^*Q \) is injective with respect to \( T\pi_{Q} : TT^*Q \to TQ \), and hence \( T\tilde{\gamma} = T\pi_{O_{\mu}} \cdot T\gamma \), is injective with respect to \( T\pi_{Q} : TT^*Q \to TQ \), and \( X_{h_{O_{\mu}}} \cdot \tilde{\gamma} = X_{\gamma} \). Thus, (i) implies (ii).

Conversely, from the above arguments we have that for any \( w \in TT^*Q \), and \( T\pi_{O_{\mu}} \cdot w \neq 0 \), then

\[
\omega_{O_{\mu}}(T\tilde{\gamma} \cdot X_{\gamma}, T\pi_{O_{\mu}} \cdot w) = \omega_{O_{\mu}}(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\pi_{O_{\mu}} \cdot w) = d\gamma(X_{h_{O_{\mu}}} \cdot \tilde{\gamma}, T\pi_{Q} \cdot w).
\]

Thus, since \( \omega_{O_{\mu}} \) is nondegenerate, the proof that (ii) implies (i) follows in the same way. ■

Remark 4.4 If \((T^*Q, \omega)\) is a connected symplectic manifold, and \( J : T^*Q \to g^* \) is a non-equivariant momentum map with a non-equivariance group one-cocycle \( \sigma : G \to g^* \), in this case, for a given regular orbit reducible Hamiltonian system \((T^*Q, G, \omega, H)\), we can also prove the Hamilton-Jacobi theorem for regular orbit reduced Hamiltonian system \([(T^*Q)_{O_{\mu}}, \omega_{O_{\mu}}, h_{O_{\mu}}]\) by using the above same way, where the reduced space \([(T^*Q)_{O_{\mu}}, \omega_{O_{\mu}}]\) is determined by the affine action given in Remark 4.1.

For a regular orbit reducible Hamiltonian system \((T^*Q, G, \omega, H)\), we know that the dynamical vector fields \( X_H \) and \( X_{h_{O_{\mu}}} \) are \( \pi_{O_{\mu}} \)-related, that is, \( X_{h_{O_{\mu}}} \cdot \pi_{O_{\mu}} = T\pi_{O_{\mu}} \cdot X_H \cdot i_{O_{\mu}} \). Then we can prove the following Theorem 4.5, which states the relationship between the solutions of Hamilton-Jacobi equations and regular orbit reduction.

Theorem 4.5 For a regular orbit reducible Hamiltonian system \((T^*Q, G, \omega, H)\), assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( \gamma^* : T^*T^*Q \to T^*Q \) is symplectic, and \( \gamma \) is closed with respect to \( T\pi_{Q} : TT^*Q \to TQ \). Moreover, assume that \( O_{\mu} \), \( \mu \in g^* \), is the regular reducible orbit of the Hamiltonian system, and \( \text{Im}(\gamma) \subset J^{-1}(\mu) \), and it is \( G \)-invariant, and \( \tilde{\gamma} = \pi_{O_{\mu}}(\gamma) : Q \to (T^*Q)_{O_{\mu}} \). Then \( \gamma \) is a solution of the Hamilton-Jacobi equation for the Hamiltonian system with symmetry \((T^*Q, G, \omega, H)\) if and only if \( \tilde{\gamma} \) is a solution of the Hamilton-Jacobi equation for regular orbit reduced Hamiltonian system \([(T^*Q)_{O_{\mu}}, \omega_{O_{\mu}}, h_{O_{\mu}}]\).

Proof: In fact, from the proof of Theorem 2.5, we know that \( T\gamma \cdot X_{H} \cdot \gamma = X_{H} \cdot \gamma = T\gamma \cdot X_{\gamma} \), and from the proof of Theorem 4.3, we have that \( T\tilde{\gamma} \cdot X_{\gamma} = X_{h_{O_{\mu}}} \cdot \tilde{\gamma} = T\tilde{\gamma} \cdot X_{h_{O_{\mu}}} \cdot \tilde{\gamma} \). Note that both maps \( T\gamma : TQ \to TT^*Q \) and \( T\tilde{\gamma} : TQ \to T(T^*Q)_{O_{\mu}} \) are injective with respect to \( T\pi_{Q} : TT^*Q \to TQ \). Thus, \( X_{H} \cdot \gamma = 0 \), \( \Leftrightarrow X_{\gamma} = 0 \), \( \Leftrightarrow X_{h_{O_{\mu}}} \cdot \tilde{\gamma} = 0 \). It follows that the conclusion of Theorem 4.5 holds. ■

5 Applications

In this section, as an application of the above theoretical results, we consider the regular point reducible Hamiltonian system on a Lie group, and give the Hamilton-Jacobi theorem and Lie-Poisson Hamilton-Jacobi equation of the system. In particular, we show the Lie-Poisson Hamilton-Jacobi equations of rigid body and heavy top on the rotation group SO(3) and on the Euclidean group SE(3), respectively. We shall follow the notations and conventions introduced in Marsden et al [14], Marsden and Ratiu [15], Ortega and Ratiu [21], and Marsden et al [16].
5.1 Lie-Poisson Hamilton-Jacobi Equation

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $T^*G$ its cotangent bundle with the canonical symplectic form $\omega$. A Hamiltonian system on $G$ is a 3-tuple $(T^*G, \omega, H)$, where the function $H : T^*G \to \mathbb{R}$ is a Hamiltonian, and has the associated Hamiltonian vector field $X_H$. At first, for the Lie group $G$, the left and right translation on $G$, defined by the map $L_g : G \to G$, $h \mapsto gh$, and $R_g : G \to G$, $h \mapsto hg$, for someone $g \in G$, induce the left and right action of $G$ on itself. Let $I_g : G \to G$: $I_g(h) = ghg^{-1} = L_g \cdot R_{g^{-1}}(h)$, for $g, h \in G$, be the inner automorphism on $G$. The adjoing representation of a Lie group $G$ is defined by $Ad_g = T_e I_g = T_g^{-1} L_g \cdot T_e R_{g^{-1}} : g \mapsto g$. The coadjoint representation is given by $Ad_{g^{-1}}^* : g^* \to g^*$, where $Ad_{g^{-1}}^*$ is the dual of the linear map $Ad_{g^{-1}}$, defined by $(Ad_{g^{-1}}^*(\mu), \xi) = (\mu, Ad_{g^{-1}}(\xi))$, where $\mu \in g^*$, $\xi \in \mathfrak{g}$ and $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Since the coadjoint representation $Ad_{g^{-1}}^* : g^* \to g^*$ can induce a left coadjoint action of $G$ on $g^*$, the coadjoint orbit $O_\mu$ of this action through $\mu \in g^*$ is the subset of $g^*$ defined by $O_\mu := \{ Ad_{g^{-1}}^*(\mu) \in g^* | g \in G \}$, and $O_\mu$ is an immersed submanifold of $g^*$. We know that $g^*$ is a Poisson manifold with respect to the $(\pm)$-Lie-Poisson bracket $(\cdot, \cdot)_\pm$ defined by

$$\{ f, g \}_\pm(\mu) := \pm \langle \mu, [\delta f/\delta \mu, \delta g/\delta \mu] \rangle, \quad \forall f, g \in C^\infty(\mathfrak{g}^*), \quad \mu \in \mathfrak{g}^*, \quad (5.1)$$

where the element $\delta f/\delta \mu \in \mathfrak{g}$ is defined by the equality $< v, \delta f/\delta \mu > := Df(\mu) \cdot v$, for any $v \in \mathfrak{g}^*$, see Marsden and Ratiu [15]. Thus, for the coadjoint orbit $O_\mu$, $\mu \in g^*$, the orbit symplectic structure can be defined by

$$\omega_{O_\mu}^\pm(\nu)(ad_\mu^*(\nu), ad_\mu^*(\nu)) = \pm \langle \nu, [\xi, \eta] \rangle, \quad \forall \xi, \eta \in \mathfrak{g}, \quad \nu \in O_\mu \subset g^*, \quad (5.2)$$

which are coincide with the restriction of the Lie-Poisson brackets on $g^*$ to the coadjoint orbit $O_\mu$. From the Symplectic Stratification theorem we know that a finite dimensional Poisson manifold is the disjoint union of its symplectic leaves, and its each symplectic leaf is an injective immersed Poisson submanifold whose induced Poisson structure is symplectic. In consequence, when $g^*$ is endowed one of the Lie Poisson structures $(\cdot, \cdot)_\pm$, the symplectic leaves of the Poisson manifolds $(g^*, (\cdot, \cdot)_\pm)$ coincide with the connected components of the orbits of the elements in $g^*$ under the coadjoint action. From Abraham and Marsden [1], we know that the coadjoint orbit $(O_\mu, \omega_{O_\mu}^\pm)$, $\mu \in g^*$, is symplectically diffeomorphic to a regular point reduced space $((T^*G)_\mu, \omega_\mu)$ of $T^*G$.

We now identify $T^*G$ and $G \times g^*$ by using the left translation. In fact, the map $\lambda : T^*G \to G \times g^*$, $\lambda(\alpha_g) := (g, (T_e L_g)^* \alpha_g)$, for any $\alpha_g \in T_g^* G$, which defines a vector bundle isomorphism usually referred to as the left trivialization of $T^*G$. In the same way, we can also identify tangent bundle $TG$ and $G \times g$ by using the left translation. In consequence, we can consider the Lagrangian $L(g, \xi) : TG \cong G \times g \to \mathbb{R}$, which is usual the kinetic minus the potential energy of the system, where $(g, \xi) \in G \times g$, and $\xi \in \mathfrak{g}$, regarded as the velocity of system. If we introduce the conjugate momentum $p_i = \partial L/\partial \dot{q}_i$, $i = 1, \cdots, n$, $n = dimG$, and by the Legendre transformation $FL : TG \cong G \times g \to T^*G \cong G \times g^*$, $(g^i, \dot{q}^i) \rightarrow (g^i, p_i)$, we have the Hamiltonian $H(g, p) : T^*G \cong G \times g^* \to \mathbb{R}$ given by

$$H(g^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}_i - L(g^i, \dot{q}_i). \quad (5.3)$$

If the Hamiltonian $H(g, p) : T^*G \cong G \times g \to \mathbb{R}$ is left cotangent lifted $G$-action invariant, for $\mu \in g^*$ we have the associated reduced Hamiltonian $h_\mu : (T^*G)_\mu \cong O_\mu \to \mathbb{R}$, defined by
\[ h_\mu \cdot \pi_\mu = H \cdot i_\mu. \] By the \((\pm)\)-Lie-Poisson brackets on \(\mathfrak{g}^*\) and the symplectic structure on the coadjoint orbit \(\mathcal{O}_\mu\), we have the reduced Hamiltonian vector field \(X_{h_\mu}\) given by

\[ X_{h_\mu}(\nu) = \mp \text{ad}^{*}_{h_\mu/\delta_\mu} \nu, \quad \forall \nu \in \mathcal{O}_\mu. \quad (5.4) \]

See Marsden and Ratiu [15]. Thus, the 4-tuple \((T^*G, G, \omega, H)\) is a regular point reducible Hamiltonian system on Lie group \(G\), where the Hamiltonian \(H: T^*G \to \mathbb{R}\) is left cotangent lifted \(G\)-action invariant. For a point \(\mu \in \mathfrak{g}^*\), the regular value of the momentum map \(J_L : T^*G \to \mathfrak{g}^*\), the Marsden-Weinstein reduced Hamiltonian system is \((\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}, h_\mu)\). Moreover, assume that \(\gamma : G \to T^*G\) is an one-form on \(G\), and \(\gamma^* : T^*T^*G \to T^*G\) is symplectic, and \(\gamma\) is closed with respect to \(T\pi_G : TT^*G \to TG\), and \(\text{Im}(\gamma) \subset J_{L^{-1}}(\mu)\), and it is \(G_\mu\)-invariant, and \(\check{\gamma} = \pi_\mu(\gamma) : G \to \mathcal{O}_\mu\). By using the same way in the proof of Hamilton-Jacobi theorem for the Marsden-Weinstein reduced Hamiltonian system, we can prove the following theorem.

**Theorem 5.1** For the regular point reducible Hamiltonian system \((T^*G, G, \omega, H)\) on Lie group \(G\), assume that \(\gamma : G \to T^*G\) is an one-form on \(G\), and \(\gamma^* : T^*T^*G \to T^*G\) is symplectic, and \(\gamma\) is closed with respect to \(T\pi_G : TT^*G \to TG\), and \(X_H^\gamma = T\pi_G \cdot X_H : \gamma\), where \(X_H\) is the dynamical vector field of Hamiltonian system with symmetry \((T^*G, G, \omega, H)\). Moreover, assume that \(\mu \in \mathfrak{g}^*\) is the regular reducible point of the Hamiltonian system, and \(\text{Im}(\gamma) \subset J_{L^{-1}}(\mu)\), and it is \(G_\mu\)-invariant, and \(\check{\gamma} = \pi_\mu(\gamma) : G \to \mathcal{O}_\mu\). Then the following two assertions are equivalent: (i) \(X_H^\gamma\) and \(X_{h_\mu}\) are \(\gamma\)-related, where \(X_{h_\mu}\) is the dynamical vector field of Marsden-Weinstein reduced Hamiltonian system \((\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}, h_\mu)\); (ii) \(X_{h_\mu, \check{\gamma}} = 0\), or \(X_{h_\mu, \check{\gamma}} = X_H^\gamma\). Moreover, \(\gamma\) is a solution of the Hamilton-Jacobi equation \(X_{H, \check{\gamma}} = 0\), if and only if \(\check{\gamma}\) is a solution of the Hamilton-Jacobi equation \(X_{h_\mu, \check{\gamma}} = 0\).

Note that the symplectic structure on the coadjoint orbit \(\mathcal{O}_\mu\) is induced by the \((\pm)\)-Lie-Poisson brackets on \(\mathfrak{g}^*\), then the Hamilton-Jacobi equation \(X_{h_\mu, \check{\gamma}} = 0\) for Marsden-Weinstein reduced Hamiltonian system \((\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}, h_\mu)\) is also called Lie-Poisson Hamilton-Jacobi equation. See Marsden and Ratiu [15], and Ge and Marsden [5].

### 5.2 Hamilton-Jacobi Equation of Rigid Body

In the following we regard the rigid body as a regular point reducible Hamiltonian system on the rotation group \(SO(3)\), and give its Lie-Poisson Hamilton-Jacobi equation. Note that our description of the motion and the equations of rigid body in this subsection follows some of the notations and conventions in Marsden and Ratiu [15], Marsden [12].

It is well known that, usually, the configuration space for a 3-dimensional rigid body moving freely in space is \(SE(3)\), the six dimension group of Euclidean (rigid) transformations of three dimensional space \(\mathbb{R}^3\), that is, all possible rotations and translations. If translations are ignored and only rotations are considered, then the configuration space \(Q\) is \(SO(3)\), consists of all orthogonal linear transformations of Euclidean 3-space to itself, which have determinant one. Its Lie algebra, denoted \(\mathfrak{so}(3)\), consists of all \(3 \times 3\) skew matrices. By using the isomorphism \(\gamma : \mathbb{R}^3 \to \mathfrak{so}(3)\) defined by

\[
(\Omega_1, \Omega_2, \Omega_3) = \Omega \to \hat{\Omega} = \begin{bmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{bmatrix},
\]

we can identify the Lie algebra \((\mathfrak{so}(3), [\cdot, \cdot])\) with \((\mathbb{R}^3, \times)\) and the Lie algebra bracket \([\cdot, \cdot]\) on \(\mathfrak{so}(3)\) with the cross product \(\times\) of vectors in \(\mathbb{R}^3\). Denote by \(\mathfrak{so}^*\) the dual of the Lie algebra \(\mathfrak{so}(3)\),
and we also identify \( so^*(3) \) with \( \mathbb{R}^3 \) by pairing the Euclidean inner product. Since the functional derivative of a function defined on \( \mathbb{R}^3 \) is equal to the usual gradient of the function, from (5.1) we know that the Lie-Poisson bracket on \( so^*(3) \) take the form

\[
\{f, g\}_\pm(\Pi) = \pm \Pi \cdot (\nabla_{\Pi} f \times \nabla_{\Pi} g), \quad \forall f, g \in C^\infty(so^*(3)), \quad \Pi \in so^*(3).
\]

The phase space of a rigid body is the cotangent bundle \( T^*G = T^*SO(3) \cong SO(3) \times so^*(3) \), with the canonical symplectic form. Assume that Lie group \( G = SO(3) \) acts freely and properly by the left translation on \( SO(3) \), then the action of \( SO(3) \) on the phase space \( T^*SO(3) \) is by cotangent lift of left translation at the identity, that is, \( \Phi : SO(3) \times T^*SO(3) \cong SO(3) \times SO(3) \times so^*(3) \rightarrow SO(3) \times so^*(3) \), given by \( \Phi(B, (A, \Pi)) = (BA, \Pi) \), for any \( A, B \in SO(3) \), \( \Pi \in so^*(3) \), which is also free and proper, and admits an associated Ad*-equivariant momentum map \( J : T^*SO(3) \rightarrow so^*(3) \) for the left \( SO(3) \) action. If \( \Pi \in so^*(3) \) is a regular value of \( J \), then the regular point reduced space \( (T^*SO(3))_\Pi = J^{-1}(\Pi)/SO(3)_\Pi \) is symplectically diffeomorphic to the coadjoint orbit \( \mathcal{O}_\Pi \subset so^*(3) \).

Let \( I \) be the moment of inertia tensor computed with respect to a body fixed frame, which, in a principal body frame, we may represent by the diagonal matrix \( \text{diag} (I_1, I_2, I_3) \). Let \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \) be the vector of angular velocities computed with respect to the axes fixed in the body and \( (\Omega_1, \Omega_2, \Omega_3) \in so(3) \). Consider the Lagrangian \( L(A, \Omega) : TSO(3) \cong SO(3) \times so^*(3) \rightarrow \mathbb{R} \), which is the total kinetic energy of the rigid body, given by

\[
L(A, \Omega) = \frac{1}{2} (\Omega \cdot \Omega) = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2),
\]

where \( A \in SO(3) \), \( (\Omega_1, \Omega_2, \Omega_3) \in so(3) \). If we introduce the conjugate angular momentum

\[
\Pi_i = \frac{\partial L}{\partial \Omega_i} = I_i \Omega_i, \quad i = 1, 2, 3,
\]

which is also computed with respect to a body fixed frame, and by the Legendre transformation \( \mathcal{F}L : TSO(3) \cong SO(3) \times so^*(3) \rightarrow T^*SO(3) \cong SO(3) \times so^*(3) \), \( (A, \Omega) \rightarrow (\Pi, \Omega) \), where \( \Pi = (\Pi_1, \Pi_2, \Pi_3) \in so^*(3) \), we have the Hamiltonian \( H(A, \Pi) : T^*SO(3) \cong SO(3) \times so^*(3) \rightarrow \mathbb{R} \) given by

\[
H(A, \Pi) = \Omega \cdot \Pi - L(A, \Omega)
\]

\[
= I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 - \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2) = \frac{1}{2} \left( \frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3} \right).
\]

From the above expression of the Hamiltonian, we know that \( H(A, \Pi) \) is invariant under the left \( SO(3) \)-action \( \Phi : SO(3) \times T^*SO(3) \rightarrow T^*SO(3) \). For the case \( \Pi_0 = \mu \in so^*(3) \) is a regular value of \( J \), we have the reduced Hamiltonian \( h_{\mathcal{O}_\mu}(\Pi) : \mathcal{O}_\mu \subset so^*(3) \rightarrow \mathbb{R} \) given by \( h_{\mathcal{O}_\mu}(\Pi) = H(A, \Pi)|_{\mathcal{O}_\mu} \). From the Lie-Poisson bracket on \( \mathfrak{g}^* \), we can get the rigid body Poisson bracket on \( so^*(3) \), that is, for \( F, K : so^*(3) \rightarrow \mathbb{R} \), we have that \( \{F, K\} \|_{so^*(3)} = -\pi : (\nabla F \times \nabla K) \). In particular, for \( F_{\mathcal{O}_\mu}, K_{\mathcal{O}_\mu} : \mathcal{O}_\mu \rightarrow \mathbb{R} \), we have that \( \omega_{\mathcal{O}_\mu}(X_{F_{\mathcal{O}_\mu}}, X_{K_{\mathcal{O}_\mu}}) = \{F_{\mathcal{O}_\mu}, K_{\mathcal{O}_\mu}\} \|_{so^*(3)} \). Moreover, for reduced Hamiltonian \( h_{\mathcal{O}_\mu}(\Pi) : \mathcal{O}_\mu \rightarrow \mathbb{R} \), we have the Hamiltonian vector field \( X_{h_{\mathcal{O}_\mu}}(K_{\mathcal{O}_\mu}) = \{K_{\mathcal{O}_\mu}, h_{\mathcal{O}_\mu}\} \|_{so^*(3)} \). Assume that \( \gamma : SO(3) \rightarrow T^*SO(3) \) is an one-form on \( SO(3) \), and \( \gamma^* : T^*T^*SO(3) \rightarrow T^*SO(3) \) is symplectic, and \( \gamma \) is closed with respect to \( T^*\pi_{SO(3)} : TT^*SO(3) \rightarrow TSO(3) \), and \( \text{Im}(\gamma) \subset \mathcal{J}^{-1}(\mu) \), and it is \( SO(3)_\mu \)-invariant, and \( \gamma = \pi_{\mu}(\gamma) : SO(3) \rightarrow \mathcal{O}_\mu \). Denote by \( \tilde{\gamma}(A) = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)(A) \in \mathcal{O}_\mu(\subset so^*(3)) \), then \( h_{\mathcal{O}_\mu} \cdot \tilde{\gamma} : SO(3) \rightarrow \mathbb{R} \) is given by

\[
h_{\mathcal{O}_\mu} \cdot \tilde{\gamma}(A) = H(A, \tilde{\gamma}(A))|_{\mathcal{O}_\mu} = \frac{1}{2} \left( \frac{\tilde{\gamma}_1^2}{I_1} + \frac{\tilde{\gamma}_2^2}{I_2} + \frac{\tilde{\gamma}_3^2}{I_3} \right),
\]
and the vector field
\[ X_{h\mathcal{O}_\mu} \cdot \tilde{\gamma}(\Pi) = \{ \Pi, h\mathcal{O}_\mu \cdot \tilde{\gamma}(A) \} \}_{\mathcal{O}_\mu} = -\Pi \cdot (\nabla_\Pi \nabla_\Pi (h\mathcal{O}_\mu \cdot \tilde{\gamma})) = -\Pi \cdot (\nabla_\Pi (h\mathcal{O}_\mu \cdot \tilde{\gamma}) \times \Pi) \]
\[ = (\Pi_1, \Pi_2, \Pi_3) \times \left( \frac{\tilde{\gamma}_1}{I_1}, \frac{\tilde{\gamma}_2}{I_2}, \frac{\tilde{\gamma}_3}{I_3} \right) \]
\[ = \left( \frac{I_2 \Pi_2 \tilde{\gamma}_3 - I_3 \Pi_3 \tilde{\gamma}_2}{I_2 I_3}, \frac{I_3 \Pi_3 \tilde{\gamma}_1 - I_1 \Pi_1 \tilde{\gamma}_3}{I_3 I_1}, \frac{I_1 \Pi_1 \tilde{\gamma}_2 - I_2 \Pi_2 \tilde{\gamma}_1}{I_1 I_2} \right), \]

since \( \nabla_\Pi \Pi = 1 \), and \( \nabla_\Pi h\mathcal{O}_\mu \cdot \tilde{\gamma} = \tilde{\gamma}_j/I_j, \ j = 1, 2, 3 \). Thus, the Lie-Poisson Hamilton-Jacobi equation of rigid body is given by

\[ \begin{aligned}
I_2 \Pi_2 \tilde{\gamma}_3 - I_3 \Pi_3 \tilde{\gamma}_2 &= 0, \\
I_3 \Pi_3 \tilde{\gamma}_1 - I_1 \Pi_1 \tilde{\gamma}_3 &= 0, \\
I_1 \Pi_1 \tilde{\gamma}_2 - I_2 \Pi_2 \tilde{\gamma}_1 &= 0.
\end{aligned} \] (5.6)

To sum up the above discussion, we have the following proposition.

**Proposition 5.2** The 4-tuple \((T^*SO(3), SO(3), \omega, H)\) is a regular point reducible Hamiltonian system. For a point \( \mu \in \mathfrak{so}^*(3) \), the regular value of the momentum map \( J : T^*SO(3) \to \mathfrak{so}^*(3) \), the Marsden-Weinstein reduced system is the 3-tuple \((\mathcal{O}_\mu, \omega_{\mathcal{O}_\mu}, h\mathcal{O}_\mu)\), where \( \mathcal{O}_\mu \subset \mathfrak{so}^*(3) \) is the coadjoint orbit, \( \omega_{\mathcal{O}_\mu} \) is orbit symplectic form on \( \mathcal{O}_\mu \), which is induced by the rigid body Poisson bracket on \( \mathfrak{so}^*(3) \), \( h\mathcal{O}_\mu(\Pi) = H(A, \Pi) \mid \mathcal{O}_\mu \). Assume that \( \gamma : SO(3) \to T^*SO(3) \) is an one-form on \( SO(3) \), and \( \gamma^* : T^*T^*SO(3) \to T^*SO(3) \) is symplectic, and \( \gamma \) is closed with respect to \( T\pi_{SO(3)} : TT^*SO(3) \to TSO(3) \), and \( \text{Im}(\gamma) \subset J^{-1}(\mu) \), and it is \( SO(3) \mu \)-invariant, and \( \tilde{\gamma} = \pi_\mu(\gamma) : SO(3) \to \mathcal{O}_\mu \). Then \( \tilde{\gamma} \) is a solution of either Lie-Poisson Hamilton-Jacobi equation of rigid body given by (5.6), or the equation \( X_{h\mathcal{O}_\mu} \cdot \tilde{\gamma} = X_{H}^\gamma \), if and only if \( X_{H}^\gamma \) and \( X_{h\mathcal{O}_\mu} \) are \( \tilde{\gamma} \)-related, where \( X_{H}^\gamma = T\pi_{SO(3)} \cdot X_{H} \cdot \gamma \).

### 5.3 Hamilton-Jacobi Equation of Heavy Top

In the following we regard the heavy top as a regular point reducible Hamiltonian system on the Euclidean group \( SE(3) \), and give its Lie-Poisson Hamilton-Jacobi equation. Note that our description of the motion and the equations of heavy top follows some of the notations and conventions in Marsden and Ratiu [15], Marsden [12].

We know that a heavy top is by definition a rigid body with a fixed point in \( \mathbb{R}^3 \) and moving in gravitational field. Usually, exception of the singular point, its physical phase space is \( T^*SO(3) \) and the symmetry group is \( S^1 \), regarded as rotations about the z-axis, the axis of gravity, this is because gravity breaks the symmetry and the system is no longer \( SO(3) \) invariant. By the semidirect product reduction theorem, see Marsden et al [13] and [16], we show that the reduction of \( T^*SO(3) \) by \( S^1 \) gives a space which is symplectically diffeomorphic to the reduced space obtained by the reduction of \( T^*SE(3) \) by left action of \( SE(3) \), that is the coadjoint orbit \( \mathcal{O}_{(\mu, a)} \subset \mathfrak{so}^*(3) \cong T^*SE(3)/SE(3) \). In fact, in this case, we can identify the phase space \( T^*SO(3) \) with the reduction of the cotangent bundle of the special Euclidean group \( SE(3) = SO(3) \mathbb{S} \mathbb{R}^3 \) by the Euclidean translation subgroup \( \mathbb{R}^3 \) and identifies the symmetry group \( S^1 \) with isotropy group \( G_a = \{ A \in SO(3) \mid Aa = a \} = S^1 \), which is Abelian and \( (G_a)_{\mu_a} = G_a = S^1 \), \( \forall \mu_a \in \mathfrak{g}_a^* \), where \( a \) is a vector aligned with the direction of gravity and where \( SO(3) \) acts on \( \mathbb{R}^3 \) in the standard way.
Now we consider the cotangent bundle $T^*G = T^*\text{SE}(3) \cong \text{SE}(3) \times \mathfrak{se}^*(3)$, with the canonical symplectic form. Assume that Lie group $G = \text{SE}(3)$ acts freely and properly by the left translation on $\text{SE}(3)$, then the action of $\text{SE}(3)$ on the phase space $T^*\text{SE}(3)$ is by cotangent lift of left translation at the identity, that is, $\Phi : \text{SE}(3) \times T^*\text{SE}(3) \cong \text{SE}(3) \times \text{SE}(3) \times \mathfrak{se}^*(3) \rightarrow \text{SE}(3) \times \mathfrak{se}^*(3)$, given by $\Phi((B, w), (A, v, \Pi, w)) = (BA, v, \Pi, w)$, for any $A, B \in \text{SO}(3)$, $\Pi \in \mathfrak{so}^*(3)$, $u, v, w \in \mathbb{R}^3$, which is also free and proper, and admits an associated $\text{Ad}$-equivariant momentum map $J : T^*\text{SE}(3) \rightarrow \mathfrak{se}^*(3)$ for the left $\text{SE}(3)$ action. If $(\Pi, w) \in \mathfrak{se}^*(3)$ is a regular value of $J$, then the regular point reduced space $(T^*\text{SE}(3))_{(\Pi, w)} = J^{-1}(\Pi, w)/\text{SE}(3)_{(\Pi, w)}$ is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\Pi, w)} \subset \mathfrak{se}^*(3)$.

Let $I = \text{diag}(I_1, I_2, I_3)$ be the moment of inertia of the heavy top in the body-fixed frame, which in principal body frame. Let $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ be the vector of heavy top angular velocities computed with respect to the axes fixed in the body and $(\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$. Let $\Gamma$ be the unit vector viewed by an observer moving with the body, $m$ be that total mass of the system, $g$ be the magnitude of the gravitational acceleration, $\chi$ be the unit vector on the line connecting the origin $O$ to the center of mass of the system, and $h$ be the length of this segment.

Consider the Lagrangian $L(A, v, \Omega, \Gamma) : T^*\text{SE}(3) \cong \text{SE}(3) \times \mathfrak{se}^*(3) \rightarrow \mathbb{R}$, which is the total kinetic minus potential energy of the heavy top, given by

$$L(A, v, \Omega, \Gamma) = \frac{1}{2} \langle \Omega, \Omega \rangle - mgh\Gamma \cdot \chi = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) - mgh\Gamma \cdot \chi,$$

where $(A, v) \in \text{SE}(3)$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathfrak{so}(3)$, $\Gamma \in \mathbb{R}^3$. If we introduce the conjugate angular momentum $\Pi_i = \frac{\partial L}{\partial \Omega_i} = I_i\Omega_i$, $i = 1, 2, 3$, and by the Legendre transformation $FL : T^*\text{SE}(3) \cong \text{SE}(3) \times \mathfrak{se}^*(3) \rightarrow T^*\text{SE}(3) \cong \text{SE}(3) \times \mathfrak{se}^*(3)$, $(A, v, \Omega, \Gamma) \rightarrow (A, v, \Pi, \Gamma)$, where $\Pi = (\Pi_1, \Pi_2, \Pi_3) \in \mathfrak{so}^*(3)$, we have the Hamiltonian $H(A, v, \Pi, \Gamma) : T^*\text{SE}(3) \cong \text{SE}(3) \times \mathfrak{se}^*(3) \rightarrow \mathbb{R}$ given by

$$H(A, v, \Pi, \Gamma) = \Omega \cdot \Pi - L(A, \Omega, \Gamma)$$

$$= I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 - \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) + mgh\Gamma \cdot \chi$$

$$= \frac{1}{2}(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}) + mgh\Gamma \cdot \chi.$$

From the above expression of the Hamiltonian, we know that $H(A, v, \Pi, \Gamma)$ is invariant under the left $\text{SE}(3)$-action $\Phi : \text{SE}(3) \times T^*\text{SE}(3) \rightarrow T^*\text{SE}(3)$. For the case $(\Pi_0, \Gamma_0) = (\mu, a) \in \mathfrak{se}^*(3)$ is a regular value of $J$, we have the reduced Hamiltonian $h_{\mathcal{O}_{(\mu, a)}}(\Pi, \Gamma) : \mathcal{O}_{(\mu, a)}(\subset \mathfrak{se}^*(3)) \rightarrow \mathbb{R}$ given by $h_{\mathcal{O}_{(\mu, a)}}(\Pi, \Gamma) = H(A, v, \Pi, \Gamma)|_{\mathcal{O}_{(\mu, a)}}$. From the semidirect product Poisson bracket, see Marsden et al. [13], we can get the heavy top Poisson bracket on $\mathfrak{se}^*(3)$, that is, for $F, K : \mathfrak{se}^*(3) \rightarrow \mathbb{R}$, we have that

$$(F, K)_-(\Pi, \Gamma) = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Gamma K - \nabla_\Pi K \times \nabla_\Gamma F). \tag{5.7}$$

In particular, for $F_{\mathcal{O}_{(\mu, a)}}, K_{\mathcal{O}_{(\mu, a)}} : \mathcal{O}_{(\mu, a)} \rightarrow \mathbb{R}$, we have that

$$\omega_{\mathcal{O}_{(\mu, a)}}(X_{F_{\mathcal{O}_{(\mu, a)}}}, X_{K_{\mathcal{O}_{(\mu, a)}}}) = (F_{\mathcal{O}_{(\mu, a)}}, K_{\mathcal{O}_{(\mu, a)}})_{\mathcal{O}_{(\mu, a)}}.$$

Moreover, for reduced Hamiltonian $h_{\mathcal{O}_{(\mu, a)}}(\Pi, \Gamma) : \mathcal{O}_{(\mu, a)} \rightarrow \mathbb{R}$, we have the Hamiltonian vector field $X_{h_{\mathcal{O}_{(\mu, a)}}}(K_{\mathcal{O}_{(\mu, a)}}) = \{K_{\mathcal{O}_{(\mu, a)}}, h_{\mathcal{O}_{(\mu, a)}}\}_{\mathcal{O}_{(\mu, a)}}$. Assume that $\gamma : \text{SE}(3) \rightarrow T^*\text{SE}(3)$ is an one-form on $\text{SE}(3)$, and $\gamma^* : T^*T^*\text{SE}(3) \rightarrow T^*\text{SE}(3)$ is symplectic, and $\gamma$ is closed with respect to
For a point \( (\mu, a) \in \mathfrak{se}^*(3) \), the regular value of the momentum map \( \mu, a \in \mathfrak{se}^*(3) \), the Marsden-Weinstein reduced system is 3-tuple \((O_{(\mu, a)}, \omega_{O_{(\mu, a)}}, h_{O_{(\mu, a)}})\), where \(O_{(\mu, a)} \subset \mathfrak{se}^*(3)\) is the coadjoint orbit, \(\omega_{O_{(\mu, a)}}\) is orbit symplectic form on \(O_{(\mu, a)}\), which is induced by the heavy top Poisson bracket on \(\mathfrak{se}^*(3)\), \(h_{O_{(\mu, a)}}(\Pi, \Gamma) = H(A, v, \Pi, \Gamma)|_{O_{(\mu, a)}}\). Assume that \(\gamma : SE(3) \to T^*SE(3)\) is an one-form on \(SE(3)\), and \(\gamma^* : T^*T^*SE(3) \to T^*SE(3)\) is symplectic, and \(\gamma\) is closed with respect to \(T\pi_{SE(3)}\). Then \(\gamma\) is a solution of either Lie-Poisson Hamilton-Jacobi equation of heavy top given by (5.8), or the equation \(X_{h_{O_{(\mu, a)} - \gamma}} = X_H\), if and only if \(X_H\) and \(X_{h_{O_{(\mu, a)}}}\) are \(\gamma\)-related, where \(X_H = T\pi_{SE(3)} \cdot X_H \cdot \gamma\).
It is well known that the theory of controlled mechanical systems has formed an important subject in recent years. Its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system, in a way that helps both analysis and design. Thus, it is natural to study controlled mechanical systems by combining with the analysis of dynamic systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. In particular, we note that in Marsden et al [16], the authors studied regular reduction theory of controlled Hamiltonian systems with symplectic structure and symmetry, as an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions. Wang in [23] generalized the work in [16] to study the singular reduction theory of regular controlled Hamiltonian systems, and Wang and Zhang in [26] generalized the work in [16] to study optimal reduction theory of controlled Hamiltonian systems with Poisson structure and symmetry by using optimal momentum map and reduced Poisson tensor (or reduced symplectic form), and Ratiu and Wang in [22] studied the Poisson reduction of controlled Hamiltonian system by controllability distribution. Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is natural idea to extend the Hamilton-Jacobi theory to the (regular) controlled Hamiltonian system and its a variety of reduced systems, and it is also possible to describe the relationship between the CH-equivalence for controlled Hamiltonian systems and the solutions of corresponding Hamilton-Jacobi equations. Wang in [24,25] studied this work and applied to give explicitly the motion equation and Hamilton-Jacobi equation of reduced spacecraft-rotor system on a symplectic leaf by calculation in detail, which show the effect on controls in regular symplectic reduction and Hamilton-Jacobi theory.

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