Intermediate mass scales in the non-supersymmetric $SO(10)$ grand unification: a reappraisal

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The constraints of gauge unification on intermediate mass scales in non-supersymmetric $SO(10)$ scenarios are systematically discussed. With respect to the existing reference studies we include the $U(1)$ gauge mixing renormalization at the one- and two-loop level, and reassess the two-loop beta-coefficients. We evaluate the effects of additional Higgs multiplets required at intermediate stages by a realistic mass spectrum, and update the discussion to the present day data. On the basis of the obtained results, $SO(10)$ breaking patterns with up to two intermediate mass scales are discussed for potential relevance and model predictivity.

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I. INTRODUCTION

Understanding theoretically the patterns of masses and mixings of ordinary fermions is one of the long aimed goals in particle physics. Of the 56 parameters in the Standard Model (SM) Yukawa sector (including Majorana neutrinos) only 22 can be measured at low energy and just 17 have been determined from the experiment. Grand Unified Theories (GUTs), by enforcing stringent relations among the different particle sectors and by reducing the degeneracy in the parameter space, do provide a powerful tool for addressing the multiplicity of matter states and the detailed structure of the Yukawa sector.

Appealing candidates for realistic GUTs are models based on the $SO(10)$ gauge group \[1\]. All the known SM fermions plus three right-handed neutrinos fit into three copies of the 16-dimensional spinorial representation of $SO(10)$, thus providing a rationale for the SM hypercharge structure. The model also provides a natural explanation for the sub-eV light neutrino masses via the seesaw mechanism \[2, 3\].

The purpose of this paper is to review the constraints enforced by gauge unification on the intermediate mass scales in the non-supersymmetric $SO(10)$ GUTs, a needed preliminary step for assessing the structure of the multitude of the different breaking patterns before entering the details of a specific model. Eventually, our goal is to envisage and examine scenarios potentially relevant for the understanding of the low energy matter spectrum. In particular those setups that, albeit non-supersymmetric, may exhibit a predictivity comparable to that of the minimal supersymmetric $SO(10)$, scrutinized at length in the last few years \[4\].

The most recent discussion of fermion masses and mixings in non-supersymmetric $SO(10)$ GUTs was given in Ref. \[5\]. The authors focussed only on renormalizable models (i.e. without the spinorial $16_H$ in the Higgs sector) with combinations of $10_H$ and $120_H$ or $126_H$ driving the Yukawa interactions. Particular attention is paid to the leptonic sector and the mechanism of generation of neutrino masses via see-saw.

The constraints imposed by the absolute neutrino mass scale on the position of the $B-L$ threshold, together with the proton decay bound on the unification scale $M_U$, provide a discriminating tool among the many $SO(10)$ scenarios and the corresponding breaking patterns. These were studied at length in the eighties and early nineties, and detailed surveys of two- and three-step $SO(10)$ breaking chains (one and two intermediate thresholds respectively) are found in Refs. \[6, 8, 9\].

We perform a systematic survey of $SO(10)$ unification with two intermediate stages. In addition to updating the analysis to present day data, this reappraisal is motivated by (a) the absence of $U(1)$ mixing in previous studies, both at one- and two-loops in the gauge coupling renormalization, (b) the need for additional Higgs multiplets at some intermediate stages, and (c) a reassessment of the two-loop beta coefficients reported in the literature.

The outcome of our study is the emergence of sizeably different features in some of the breaking patterns as compared to the existing results. This
allows us to rescue previously excluded scenarios. All that before considering the effects of threshold corrections \[9\], that are unambiguously assessed only when the details of a specific model are worked out.

It is remarkable that the chains corresponding to the minimal SO(10) setup with the smallest Higgs representations \((10, 45)_H\) and \(126_H\), or \(210_H\) in the renormalizable case) and the smallest number of parameters in the Higgs potential, are still viable. The complexity of this non-supersymmetric scenario is comparable to that of the minimal supersymmetric SO(10) model, what makes it worth of detailed consideration.

In Sect. III we set the framework of the analysis. Sect. III provides a collection of the tools needed for a two-loop study of grand unification. The results of the numerical study are reported and scrutinized in Sect. IV. Perspectives for further progress are discussed in Sect. V. Finally, the relevant one- and two-loop \(\beta\)-coefficients are detailed in Appendix A.

II. THREE-STEP SO(10) BREAKING CHAINS

The relevant \(SO(10) \rightarrow G2 \rightarrow G1 \rightarrow SM\) symmetry breaking chains with two intermediate gauge groups \(G2\) and \(G1\) are listed in Table II. Effective two-step chains are obtained by identifying two of the high-energy scales, paying attention to the possible deviations from minimality of the scalar content in the remaining intermediate stage (this we shall discuss in Sect. IV B).

For the purpose of comparison we follow closely the notation of ref. [5], where \(P\) denotes the unbroken D-parity [12]. For each step the Higgs representation responsible for the breaking is given.

The breakdown of the lower intermediate symmetry \(G1\) to the SM gauge group is driven either by the 16- or 126-dimensional Higgs multiplets \(16_H\) or \(126_H\). An important feature of the scenarios with \(126_H\) is the fact that in such a case a potentially realistic \(SO(10)\) Yukawa sector can be constructed already at the renormalizable level. Together with \(10_H\) all the effective Dirac Yukawa couplings as well as the Majorana mass matrices at the SM level emerge from the contractions of the matter bilinears \(16_F \cdot 16_F\) with \(126_H\) or with \(126_H \cdot 10_H / \Lambda\), where \(\Lambda\) denotes the scale (above \(M_{GUT}\)) at which the effective dimension five Yukawa couplings arise.

The Higgs transforming as 10 under \(SO(10)\) may carry in general extra quantum numbers of a complex representation of some additional symmetry (a discussion on the implementation of a Peccei-Quinn \(U(1)_{PQ}\) symmetry in this scenario is given in Ref. [5]). In this case it is sufficient to consider only two complex symmetric matrices \(Y_{10}\) and \(Y_{126}\) at the renormalizable \(SO(10)\) level, namely

\[
16_F (Y_{10} 10_H + Y_{126} 126_H) 16_F , \tag{1}
\]

that govern all the effective Yukawa couplings at lower energies. Such scenarios are rather constrained and hence their detailed numerical studies are well motivated.

D-parity is a discrete symmetry acting as charge conjugation in a left-right symmetric context [12], and as that it plays the role of a left-right symmetry (it enforces for instance equal left and right gauge couplings). \(SO(10)\) invariance then implies exact D-parity (because D belongs to the \(SO(10)\) Lie algebra). D-parity may be spontaneously broken by D-odd Pati-Salam (PS) singlets contained in 210 or 45 Higgs representations. Its breaking can therefore be decoupled from the \(SU(2)_L\) breaking, allowing for different left and right gauge couplings.

The possibility of decoupling the D-parity breaking from the scale of right-handed interactions is a cosmologically relevant issue. On the one hand baryon asymmetry cannot arise in a left-right symmetric \((g_L = g_R)\) universe [14]. On the other hand, the spontaneous breaking of a discrete symmetry, such as D-parity, creates domain walls that, if massive enough (i.e. for intermediate mass scales) do not disappear, overclosing the universe [12]. These potential problems may be overcome either by confining D-parity at the GUT scale or by invoking infla-

| Chain | G2 | G1 |
|-------|----|----|
| I:    | \(\{(2L2R4c)\}_{210}\) \(\rightarrow\) \(\{2L2R1x3c\}_{45}\) | \(\{2L2R1x3c\}_{45}\) |
| II:   | \(\{(2L2R4cP)\}_{54}\) \(\rightarrow\) \(\{2L2R1x3cP\}_{210}\) | \(\{2L2R1x3c\}_{45}\) |
| III:  | \(\{(2L2R4cP)\}_{54}\) \(\rightarrow\) \(\{2L2R1x3c\}_{45}\) | \(\{2L2R1x3c\}_{45}\) |
| IV:   | \(\{(2L2R4c)\}_{210}\) \(\rightarrow\) \(\{2L1R4c\}_{45}\) | \(\{2L1R4c\}_{45}\) |
| V:    | \(\{(2L2R4c)\}_{54}\) \(\rightarrow\) \(\{2L1R4c\}_{45}\) | \(\{2L1R4c\}_{45}\) |
| VI:   | \(\{(2L2R4c)\}_{54}\) \(\rightarrow\) \(\{2L1R4c\}_{45}\) | \(\{2L1R4c\}_{45}\) |
| VII:  | \(\{(2L2R4cP)\}_{54}\) \(\rightarrow\) \(\{2L2R1x3c\}_{45}\) | \(\{2L1R1x3c\}_{45}\) |
| VIII: | \(\{(2L2R1x3c)\}_{45}\) \(\rightarrow\) \(\{2L1R1x3c\}_{45}\) | \(\{2L1R1x3c\}_{45}\) |
| IX:   | \(\{(2L2R4cP)\}_{210}\) \(\rightarrow\) \(\{2L1R1x3c\}_{45}\) | \(\{2L1R1x3c\}_{45}\) |
| X:    | \(\{(2L2R4c)\}_{210}\) \(\rightarrow\) \(\{2L1R1x3c\}_{45}\) | \(\{2L1R1x3c\}_{45}\) |
| XI:   | \(\{(2L2R4c)\}_{54}\) \(\rightarrow\) \(\{2L1R1x3c\}_{45}\) | \(\{2L1R1x3c\}_{45}\) |
| XII:  | \(\{(2L2R4cP)\}_{210}\) \(\rightarrow\) \(\{2L1R1x3c\}_{45}\) | \(\{2L1R1x3c\}_{45}\) |

TABLE I: Relevant SO(10) symmetry breaking chains via two intermediate gauge groups G1 and G2. For each step the representation of the Higgs multiplet (in SO(10) notation) responsible for the breaking is given. The breaking to the SM group 1\(y_2\)\(L_3c\) is obtained via a 16 or 126 Higgs representation. The naming and ordering of the gauge groups follows the notation of ref. [5].
tion. The latter solution implies that domain walls are formed above the reheating temperature, enforcing a lower bound on the D-parity breaking scale of $10^{12}$ GeV. Realistic SO(10) breaking patterns must therefore include this constraint.

### A. The extended survival hypothesis

Throughout all three stages of running we assume that the scalar spectrum obeys the so called extended survival hypothesis (ESH) \[12\] which requires that *at every stage of the symmetry breaking chain only those scalars are present that develop a vacuum expectation value (VEV) at the current or the subsequent levels of the spontaneous symmetry breaking*. ESH is equivalent to the requirement of the minimal number of fine-tunings to be imposed on the scalar potential \[17\] so that all the symmetry breaking steps are performed at the desired scales.

On the technical side one should identify all the Higgs multiplets needed by the breaking pattern under consideration and keep them according to the gauge symmetry down to the scale of their VEVs. This typically pulls down a large number of scalars in scenarios where $\Sigma_{126}^{R}$ provides the $B-L$ breakdown.

On the other hand, one must take into account that the role of $\Sigma_{126}^{R}$ is twofold: in addition to triggering the $G_1$ breaking it plays a relevant role in the Yukawa sector (Eq. \[11\]) where it provides the necessary breaking of the down quark - charged lepton mass degeneracy. For this to work one needs a reasonably large admixture of the $\Sigma_{126}^{R}$ component in the effective electroweak doublets. Since $(2, 2, 1)_{10}$ can mix with $(2, 2, 15)_{126}$ only below the Pati-Salam breaking scale, both fields must be present at the Pati-Salam level (otherwise the scalar doublet mass matrix does not provide large enough components of both these multiplets in the light Higgs fields).

Note that the same argument applies also to the $2L_{1R}4C$ intermediate stage when one must retain the doublet component of $\Sigma_{126}^{R}$, namely $(2, 2, 15)_{126}$ in order for it to eventually admix with $(2, 2, 1)_{10}$ in the light Higgs sector. On the other hand, at the $2L_{2R}1X_{3c}$ and $2L_{1R}1X_{3c}$ stages, the (minimal) survival of only one combination of the $\phi_{10}$ and $\phi_{126}$ scalar doublets (see Table \[11\]) is compatible with the Yukawa sector constraints because the degeneracy between the quark and lepton spectra has already been smeared-out by the Pati-Salam breakdown.

In summary, potentially realistic renormalizable Yukawa textures in settings with well-separated SO(10) and Pati-Salam breaking scales call for an additional fine tuning in the Higgs sector. In the scenarios with $\Sigma_{126}^{R}$, the $10_H$ bidoublet $(2, 2, 1)_{10}$, included in Refs \[6, 7, 8, 9\], must be paired at the $2L_{2R}1X_{3c}$ scale with an extra $(2, 2, 15)_{126}$ scalar bidoublet (or $(2, 2, 1, 1)_{10}$ with $(2, 2, 15)_{126}$ scalar bidoublet at the $2L_{1R}4C$ stage). This can affect the running of the gauge couplings in chains I, II, III, V, VI, VII, X, XI and XII.

For the sake of comparison with previous studies \[6, 7, 8, 9\] we shall not include the $\phi_{126}$ multiplets in the first part of the analysis. Rather, we shall comment on their relevance for gauge unification in Sect. \[IVC\]

| Surviving Higgs multiplets in SO(10) subgroups | SO(10) | $\{2L_{1R}4C\}$ | $\{2L_{2R}1X_{3c}\}$ | $\{2L_{1R}1X_{3c}\}$ | Notation |
|-----------------------------------------------|--------|------------------|---------------------|---------------------|---------|
| 10   | $(2, +\frac{1}{2}, 1)$ | $(2, 2, 1)$ | $(2, 2, 0, 1)$ | $(2, +\frac{1}{2}, 0, 1)$ | $\phi_{10}$ |
| $\overline{16}$ | $(1, +\frac{1}{2}, 1)$ | $(1, 2, 4)$ | $(1, 2, \frac{-1}{2}, 1)$ | $(1, +\frac{1}{2}, \frac{-1}{2}, 1)$ | $\phi_{126}$ |
| $\overline{16}$ | $(2, +\frac{1}{2}, 15)$ | $(2, 2, 15)$ | $(2, 2, 0, 1)$ | $(2, +\frac{1}{2}, 0, 1)$ | $\phi_{126}$ |
| $\overline{126}$ | $(1, 1, 10)$ | $(1, 3, 10)$ | $(1, 3, -1, 1)$ | $(1, 1, -1, 1)$ | $\Delta_{126}^{R}$ |
| $\overline{126}$ | $(3, 1, 10)$ | $(3, 1, 1, 1)$ | | | $\Delta_{126}^{L}$ |
| 45  | $(1, 0, 15)$ | $(1, 1, 15)$ | | | $\Lambda_{45}$ |
| 210 | $(1, 1, 15)$ | | | | $\Lambda_{210}$ |
| 45  | $(1, 3, 1)$ | $(1, 3, 0, 1)$ | | | $\Sigma_{45}$ |
| 45  | $(3, 1, 1)$ | $(3, 1, 0, 1)$ | | | $\Sigma_{45}$ |
| 210 | $(1, 3, 15)$ | | | | $\sigma_{210}^{R}$ |
| 210 | $(3, 1, 15)$ | | | | $\sigma_{210}^{L}$ |

**TABLE II:** Scalar multiplets contributing to the running of the gauge couplings for a given SO(10) subgroup according to minimal fine tuning. The survival of $\phi_{126}$ (not required by minimalism) is needed by a realistic leptonic mass spectrum, as discussed in the text (in the $2L_{2R}1X_{3c}$ and $2L_{1R}1X_{3c}$ stages only one linear combination of $\phi_{10}$ and $\phi_{126}$ remains). The $U(1)_{X}$ charge is given, up to a factor $\sqrt{3/2}$, by $(B-L)/2$ (the latter is reported in the table). For the naming of the Higgs multiplets we follow the notation of Ref. \[12\] with the addition of $\phi_{126}$. When the D-parity ($P$) is unbroken the particle content must be left-right symmetric. D-parity may be broken via P-odd Pati-Salam singlets in $45_{R}$ or $210_{R}$. 

III. TWO-LOOP GAUGE RENORMALIZATION GROUP EQUATIONS

In this section we report, in order to fix a consistent notation, the two-loop renormalization group equations (RGEs) for the gauge couplings. We consider a gauge group of the form $U(1) \otimes \ldots \otimes U(1)_N \otimes G_1 \otimes \ldots \otimes G_{N'}$, where $G_i$ are simple groups.

A. The non-abelian sector

Let us focus first on the non-abelian sector corresponding to $G_1 \otimes \ldots \otimes G_{N'}$ and defer the full treatment of the effects due to the extra $U(1)$ factors to section III.B. Defining $t = \log(\mu/\mu_0)$ we write

$$\frac{dg}{dt} = g_p \beta_p$$

where $p = 1, ..., N'$ is the gauge group label. Neglecting for the time being the abelian components, the $\beta$-functions for the $G_1 \otimes \ldots \otimes G_{N'}$ gauge couplings read at two-loop level \[18, 19, 20, 21\]:

$$g_p = \frac{g^2}{(4\pi)^2} \left\{ -\frac{11}{3} C_2(G_p) + \frac{4}{3} \kappa S_2(F_p) + \frac{1}{3} \eta S_2(S_p) \right. \right.$$

$$- \frac{2\kappa}{(4\pi)^2} Y_4(F_p) + \frac{2\kappa^2}{(4\pi)^2} \left[ \frac{34}{3} C_2(G_p)^2 \right]$$

$$+ \left\{ \frac{4C_2(F_p) + \frac{20}{3} C_2(G_p)}{3C_2(G_p)} \right\} \kappa S_2(F_p)$$

$$+ \left\{ \frac{4C_2(S_p) + \frac{2}{3} C_2(G_p)}{3C_2(G_p)} \right\} \eta S_2(S_p)$$

$$\left. + \frac{g_2^2}{(4\pi)^2} \left( \kappa C_2(F_q) S_2(F_p) + \eta C_2(S_q) S_2(S_p) \right) \right\},$$

where $\kappa = 1, \frac{1}{2}$ for Dirac and Weyl fermions respectively. Correspondingly, $\eta = 1, \frac{1}{2}$ for complex and real scalar fields. The sum over $q \neq p$ corresponding to contributions to $\beta_p$ from the other gauge sectors labelled by $q$ is understood. Given a fermion $F$ or a scalar $S$ field that transforms according to the representation $R = R_1 \otimes \ldots \otimes R_{N'}$, where $R_p$ is an irreducible representation of the group $G_p$ of dimension $d(R_p)$, the factor $S_2(R_p)$ is defined by

$$S_2(R_p) \equiv T(R_p) \frac{d(R)}{d(R_p)},$$

where $T(R_p)$ is the Dynkin index of the representation $R_p$. The corresponding Casimir eigenvalue is then given by

$$C_2(R_p) d(R_p) = T(R_p) d(G_p),$$

while the other terms stand for the two-loop corrections, including that induced by Yukawa interactions. The latter is accounted for in terms of a factor

$$Y_4(F_p) = \frac{1}{d(G_p)} \text{Tr} \left[ C_2(F_p) Y Y^\dagger \right],$$

where the “general” Yukawa coupling

$$Y^{abc} \bar{\psi}_a \psi_b h_c + \text{h.c.}$$

includes family as well as group indices. The coupling in Eq. (6) is written in terms of four-component Weyl spinors $\psi_{a,b}$ and a scalar field $h_c$ (be complex or real). The trace includes the sum over all relevant fermion and scalar fields.

B. The abelian couplings and $U(1)$ mixing

In order to include the abelian contributions to Eq. (3) at two loops and the one- and two-loop effects of $U(1)$ mixing \[22\], let us write the most general interaction of $N$ abelian gauge bosons $A^\mu_b$ and a set of Weyl fermions $\psi_f$ as

$$\bar{\psi}_f \gamma_\mu Q^\dagger_f \psi_f g_{rb} A^\mu_b.$$  \[8\]

The gauge coupling constants $g_{rb}$, $r, b = 1, ..., N$, couple $A^\mu_b$ to the fermionic current $J^\mu_f = \bar{\psi}_f \gamma_\mu Q^\dagger_f \psi_f$. The $N \times N$ gauge coupling matrix $g_{rb}$ can be diagonalized by two independent rotations: one acting on the $U(1)$ charges $Q^\dagger_f$ and the other on the gauge boson fields $A^\mu_b$. For a given choice of the charges, $g_{rb}$ can be set in a triangular form ($g_{rb} = 0$ for $r > b$) by the gauge boson rotation. The resulting $N(N+1)/2$ entries are observable couplings.

Since $F^\alpha_{\mu\nu}$ in the abelian case is itself gauge invariant, the most general kinetic part of the lagrangian reads at the renormalizable level

$$-\frac{1}{4} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} - \frac{1}{4} \xi_{ab} F^a_{\mu\nu} F^{b\mu\nu},$$

where $a \neq b$ and $|\xi_{ab}| < 1$. A non-orthogonal rotation of the fields $A^\mu_a$ may be performed to set the gauge kinetic term in a canonical diagonal form. Any further orthogonal rotation of the gauge fields will preserve this form. Then, the renormalization prescription may be conveniently chosen to maintain at each scale the kinetic terms canonical and diagonal on-shell while renormalizing accordingly the gauge coupling matrix $g_{rb}$ \[23\]. Thus, even if at

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1 Alternatively one may work with off-diagonal kinetic terms while keeping the gauge interactions diagonal \[23\].
one scale $g_{ab}$ is diagonal, in general non-zero off-diagonal entries are generated by renormalization effects. One shows [24] that in the case the abelian gauge couplings are at a given scale diagonal and equal (i.e. there is a $U(1)$ unification), there may exist a (scale independent) gauge field basis such that the abelian interactions remain to all orders diagonal along the RGE trajectory.

In general, the renormalization of the abelian part of the gauge interactions is determined by

$$\frac{dg_{ab}}{dt} = g_{ra} \beta_{ab}, \quad (10)$$

where, as a consequence of gauge invariance,

$$\beta_{ab} = \frac{d}{dt} \left( \log Z^3 \right)_{ab}, \quad (11)$$

with $Z^3$ denoting the gauge-boson wave-function renormalization matrix. In order to further simplify the notation it is convenient to introduce the “reduced” couplings [24]

$$\tilde{g}_{ab} = Q^2 \gamma_{ab}, \quad (12)$$

that evolve according to

$$\frac{d\tilde{g}_{ab}}{dt} = \tilde{g}_{ra} \tilde{\beta}_{ab}. \quad (13)$$

The index $k$ labels the fields (fermions and scalars) that carry $U(1)$ charges.

In terms of the reduced couplings the $\beta$-function that governs the $U(1)$ running up to two loops is given by [18, 19, 20]

$$\beta_{ab} = \frac{1}{(4\pi)^2} \left\{ \frac{4}{3} \kappa g_{fa} g_{fb} Y + \frac{1}{3} \eta g_{sa} g_{sb} - \frac{2\kappa}{(4\pi)^2} \text{Tr} \left[ g_{fa} g_{fb} YY^\dagger \right] + \frac{4}{(4\pi)^2} \left[ \kappa \left( g_{fa} g_{fb} g_{fc} + g_{fa} g_{fb} g_{sd} C_2(F_q) \right) + \eta \left( g_{sa} g_{sb} g_{sc} + g_{sa} g_{sb} g_{sd} C_2(S_q) \right) \right] \right\}, \quad (14)$$

where repeated indices are summed over, labelling fermions ($f$), scalars ($s$) and $U(1)$ gauge groups ($c$). The terms proportional to the quadratic Casimir $C_2(R_p)$ represent the two-loop contributions of the non abelian components $G_a$ of the gauge group to the $U(1)$ gauge coupling renormalization.

Correspondingly, using the notation of Eq. (12), an additional two-loop term that represents the renormalization of the non abelian gauge couplings induced at two loops by the $U(1)$ gauge fields is to be added to Eq. (3), namely

$$\Delta \beta_p = \frac{g_p^2}{(4\pi)^4} \left[ \kappa g_{fa}^2 S_2(F_p) + \eta g_{sc}^2 S_2(S_p) \right]. \quad (15)$$

In Eqs. (14)-(15), we use the abbreviation $f \equiv F_p$ and $s \equiv S_p$ and, as before, $\kappa = 1, \frac{1}{2}$ for Dirac and Weyl fermions, while $\eta = 1, \frac{1}{2}$ for complex and real scalar fields respectively.

### C. Some notation

When at most one $U(1)$ factor is present, and neglecting the Yukawa contributions, the two-loop RGEs can be conveniently written as

$$\frac{d\alpha_i^{-1}}{dt} = -\frac{a_i}{2\pi} - \frac{b_{ij}}{8\pi^2} \alpha_j, \quad (16)$$

where $a_i = g_i^2/4\pi$. The $\beta$-coefficients $a_i$ and $b_{ij}$ for the relevant $SO(10)$ chains are given in Appendix A.

Substituting the one-loop solution for $\alpha_i$ into the right-hand side of Eq. (16) one obtains

$$\alpha_i^{-1}(t) - \alpha_i^{-1}(0) = -\frac{a_i}{2\pi} t + \frac{b_{ij}}{4\pi} \log \left( 1 - \omega_j t \right), \quad (17)$$

where $\omega_j = a_j \alpha_j(0)/(2\pi)$ and $b_{ij} = b_{ij}/a_j$. The analytic solution in (17) holds at two loops (for $\omega_j t < 1$) up to higher order effects. A sample of the rescaled $\beta$-coefficients $\tilde{b}_{ij}$ is given, for the purpose of comparison with previous results, in Appendix A.

We shall conveniently write the $\beta$-function in Eq. (14), that governs the abelian mixing, as

$$\beta_{ab} = \frac{1}{(4\pi)^2} g_{ra} \gamma_{sr} g_{rb}, \quad (18)$$

where $\gamma_{sr}$ include both one- and two-loop contributions. Analogously, the non-abelian beta function in Eq. (3), including the $U(1)$ contribution in Eq. (15), is conveniently written as

$$\beta_p = \frac{g_p^2}{(4\pi)^2} \gamma_p. \quad (19)$$

The $\gamma_p$ functions for the $SO(10)$ breaking chains considered in this work are reported in Appendix A.1

Finally, the Yukawa term in Eq. (6), and correspondingly in Eq. (14), can be written as

$$Y_4(F_p) = y_{ik} \text{Tr} \left( Y_k Y_k^\dagger \right), \quad (20)$$

where $Y_k$ are the “standard” $3 \times 3$ Yukawa matrices in the family space labelled by the flavour index $k$. The trace is taken over family indices and $k$ is summed over the different Yukawa terms present at each stage of $SO(10)$ breaking. The coefficients $y_{ik}$ are given explicitly in Appendix A.2.
D. One-loop matching

The matching conditions between effective theories in the framework of dimensional regularization have been derived in \cite{21,28}. Let us consider first a simple gauge group $G$ spontaneously broken into subgroups $G_p$. Neglecting terms involving logarithms of mass ratios which are expected to be sub-leading (massive states clustered near the threshold\(^3\)), the one-loop matching for the gauge couplings can be written as

$$\alpha_p^{-1} = \frac{C_2(G_p)}{12\pi} \alpha_G^{-1} - \frac{C_2(G)}{12\pi} .$$  \hspace{1cm} (21)

Let us turn to the case when several non-abelian simple groups $G_p$ (and at most one $U(1)_X$) spontaneously break whilst preserving a $U(1)_Y$ charge. The conserved $U(1)$ generator $T_Y$ can be written in terms of the relevant generators of the various Cartan subalgebras (and of the consistently normalized $T_X$) as

$$T_Y = p_i T_i ,$$  \hspace{1cm} (22)

where $\sum p_i^2 = 1$, and $i$ runs over the relevant $p$ (and $X$) indices. The matching condition is then given by

$$\alpha_Y^{-1} = \sum_i p_i^2 \left( \alpha_i^{-1} - \frac{C_2(G_i)}{12\pi} \right) ,$$  \hspace{1cm} (23)

where for $i = X$, if present, $C_2 = 0$.

Consider now the breaking of $N$ copies of $U(1)$ gauge factors to a subset of $M$ elements $U(1)$ (with $M < N$). Denoting by $T_n$ ($n = 1, \ldots, N$) and by $\tilde{T}_m$ ($m = 1, \ldots, M$) their properly normalized generators we have

$$\tilde{T}_m = P_{mn} T_n$$  \hspace{1cm} (24)

with the orthogonality condition $P_{mn} P_{m'n} = \delta_{mn'}$. Let us denote by $g_{na}$ ($n, a = 1, \ldots, N$) and by $g_{mb}$ ($m, b = 1, \ldots, M$) the matrices of abelian gauge couplings above and below the breaking scale respectively. By writing the abelian gauge boson mass matrix in the broken vacuum and by identifying the massless states, we find the following matching condition

$$(gg^T)^{-1} = P (gg^T)^{-1} P^T .$$  \hspace{1cm} (25)

Notice that Eq. (25) depends on the chosen basis for the $U(1)$ charges (via $P$) but it is invariant under orthogonal rotations of the gauge boson fields

$$gO^T g = gg^T .$$

The massless gauge bosons $\tilde{A}_m$ are given in terms of $A_n^\mu$ by

$$\tilde{A}_m^\mu = \left[ gT \left( g^{-1} \right)^T \right]_{mn} A_n^\mu ,$$  \hspace{1cm} (26)

where $m = 1, \ldots, M$ and $n = 1, \ldots, N$.

The general case of a gauge group $U(1)_1 \otimes \ldots \otimes U(1)_{N} \otimes G_1 \otimes \ldots \otimes G_{N'}$ spontaneously broken to $U(1)_{1} \otimes \ldots \otimes U(1)_{M}$ with $M \leq N + N'$ is taken care of by replacing $(gg^T)^{-1}$ in Eq. (25) with the block-diagonal $(N + N') \times (N + N')$ matrix

$$(GG^T)^{-1} = \text{Diag} \left[ \left( gg^T \right)^{-1} g g^2 - \frac{C_2(G_p)}{48\pi^2} \right]$$  \hspace{1cm} (27)

thus providing, together with the extended Eq. (21) and Eq. (26), a generalization of Eq. (25).

IV. NUMERICAL RESULTS

At one-loop, and in absence of the $U(1)$ mixing, the gauge RGEs are not coupled and the unification constraints can be studied analytically. When two-loop effects are included (or at one-loop more than one $U(1)$ factor is present) there is no closed solution and one must solve the system of coupled equations, matching all stages between the weak and unification scales, numerically. On the other hand (when no $U(1)$ mixing is there) one may take advantage of the analytic formula in Eq. (17). The latter turns out to provide, for the cases here studied, a very good approximation to the numerical solution. The discrepancies with the numerical integration do not generally exceed the $10^{-3}$ level.

We perform a scan over the relevant breaking scales $M_1$, $M_2$ and $M_3$ and the value of the grand unified coupling ($O_G$) and impose the matching with the SM gauge couplings at the $M_Z$ scale requiring a precision at the per mil level. This is achieved by minimizing the parameter

$$\delta = \sqrt{\sum_{i=1}^{3} \left( \frac{\alpha_i^{\text{th}} - \alpha_i}{\alpha_i} \right)^2} ,$$  \hspace{1cm} (28)

where $\alpha_i$ denote the experimental values at $M_Z$ and $\alpha_i^{\text{th}}$ are the renormalized couplings obtained from unification.

The input values for the (consistently normalized) gauge SM couplings at the scale $M_Z = 91.19$ GeV are \cite{20}

$$\begin{align*}
\alpha_1 &= 0.016946 \pm 0.000006 , \\
\alpha_2 &= 0.033812 \pm 0.000021 , \\
\alpha_3 &= 0.1176 \pm 0.0020 ,
\end{align*}$$  \hspace{1cm} (29)

corresponding to the electroweak scale parameters

$$\begin{align*}
\cos \theta_W &= 0.8055 \pm 0.0009 , \\
\cos \phi &= 0.23119 \pm 0.00014 ,
\end{align*}$$  \hspace{1cm} (30)

\(^3\) An early discussion of thresholds effects in SO(10) GUT is found in \cite{10}.
All these data refer to the modified minimally subtracted (\(\overline{\text{MS}}\)) quantities at the \(M_Z\) scale.

For \(\alpha_{1,2}\) we shall consider only the central values while we resort to scanning over the whole \(3\sigma\) domain for \(\alpha_3\) when a stable solution is not found.

The results, i.e. the positions of the intermediate scales \(M_1, M_2\) and \(M_U\) shall be reported in terms of decadic logarithms of their values in units of GeV, i.e. \(n_1 = \log_{10}(M_1/\text{GeV})\), \(n_2 = \log_{10}(M_2/\text{GeV})\), \(n_U = \log_{10}(M_U/\text{GeV})\). In particular, \(n_U\), \(n_2\) are given as functions of \(n_1\) for each breaking pattern and for different approximations in the loop expansion. Each of the breaking patterns is further supplemented by the relevant range of the values of \(\alpha_U\).

### A. \(U(1)_R \times U(1)_X\) mixing

The chains VIII to XII require consideration of the mixing between the two \(U(1)\) factors. While \(U(1)_R\) and \(U(1)_X\) do emerge with canonical diagonal kinetic terms, being the remnants of the breaking of non-abelian groups, the corresponding gauge couplings are at the onset different in size. In general, no scale independent\footnote{The lack of abelian gauge mixing in Ref. \([5]\) was first observed in Ref. \([20]\).} orthogonal rotations of charges and gauge fields exist that diagonalize the gauge interactions to all orders along the RGE trajectories. According to the discussion in Sect. III, off-diagonal gauge couplings to all orders along the RGE trajectories. Each of the breaking patterns is further supplemented by the relevant range of the values of \(\alpha_U\).

\[
Q^Y = \sqrt{\frac{3}{5}} Q^R + \sqrt{\frac{2}{5}} Q^X, \tag{31}
\]

where

\[
Q^R = I_{3R} \quad \text{and} \quad Q^X = \sqrt{\frac{3}{2}} \left( B - \frac{L}{2} \right). \tag{32}
\]

The matching of the gauge couplings is then obtained from Eq. (25)

\[
g_Y^{-2} = P (gg^T)^{-1} P^T, \tag{33}
\]

with

\[
P = \left( \begin{array}{cc} \sqrt{\frac{3}{5}} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} & -\sqrt{\frac{3}{5}} \end{array} \right) \tag{34}
\]

and

\[
g = \left( \begin{array}{cc} g_{RR} & g_{RX} \\ g_{XR} & g_{XX} \end{array} \right). \tag{35}
\]

When neglecting the off-diagonal terms, Eq. (35) reproduces the matching condition used in Refs. \([7,8,9]\). For all other cases, in which only one \(U(1)\) factor is present, the matching relations can be read off directly from Eq. (21) and Eq. (24).

### B. Two-loop results (purely gauge)

The results of the numerical analysis are organized as follows: Fig. 1 and Fig. 2 show the values of \(n_U\) and \(n_2\) as functions of \(n_1\) for the pure gauge running (i.e. no Yukawa interactions), in the \(126_H\) and \(16_H\) case respectively. The differences between the patterns for the \(126_H\) and \(16_H\) setups depend on the substantially different scalar content. The shape and size of the various contributions (one-loop, with and without \(U(1)\) mixing, and two-loops) are compared in each figure. The dissection of the RGE results shown in the figures allows us to compare our results with those of Refs. \([6,7,8,9]\).

Table \(\text{III}^{*}\) shows the two-loop values of \(\alpha_U^{-1}\) in the allowed region for \(n_1\). The contributions of the additional \(\phi^{16}\) multiplets, and the Yukawa terms are discussed separately in Sect. IV C and Sect. IV D respectively. With the exception of a few singular cases detailed therein, these effects turn out to be generally subdominant.

As already mentioned in the introduction, two-loop precision in a GUT scenario makes sense once (one-loop) thresholds effects are coherently taken into account, as their effect may become comparable if not larger than the two loop itself (the argument becomes stronger as the number of intermediate scales increases). On the other hand, there is no control on the spectrum unless a specific model is studied in details. The purpose of this work is to set the stage for such a study by reassessing and updating the general constraints and patterns that SO(10) grand unification enforces on the spread of intermediate scales.

The one and two-loop \(\beta\)-coefficients used in the present study are reported in Appendix A. Table \(\text{IX}\) in the appendix shows the reduced \(b_{ij}\) coefficients for those cases where we are at variance with Ref. \([7]\).

One of the largest effects in the comparison with Refs. \([6,7,8,9]\) emerges at one-loop and it is due to the implementation of the \((U(1)\) gauge mixing when \(U(1)_R \otimes U(1)_X\) appears as an intermediate stage of the SO(10) breaking\footnote{4 The lack of abelian gauge mixing in Ref. \([5]\) was first observed in Ref. \([20]\).}. This affects chains VIII to XII, and it exhibits itself in the exact (one-loop) flatness of \(n_2\), \(n_U\) and \(\alpha_U\) as functions of \(n_1\).

The rationale for such a behaviour is quite simple.
FIG. 1: The values of $n_U$ (red/upper branches) and $n_2$ (blue/lower branches) are shown as functions of $n_1$ for the pure gauge running in the $\text{SU}(16)$ case. The bold black line bounds the region $n_1 \leq n_2$. From chains Ia to VIIa the short-dashed lines represent the result of one-loop running while the solid ones correspond to the two-loop solutions. For chains VIIIa to XIIa the short-dashed lines represent the one-loop results without the $U(1)_Y \otimes U(1)_R$ mixing, the long-dashed lines account for the complete one-loop results, while the solid lines represent the two-loop solutions. The scalar content at each stage corresponds to that considered in Ref. [9], namely to that reported in Table II without the $\phi^{126}$ multiplets. For chains I to VII the two-step $SO(10)$ breaking consistent with minimal fine tuning is recovered in the $n_2 \to n_U$ limit. No solution is found for chain Xa.
When considering the gauge coupling renormalization in the $2L1R1Xc$ stage, no effect at one-loop appears in the non-abelian $\beta$-functions due to the abelian gauge fields. On the other hand, the Higgs fields surviving at the $2L1R1Xc$ stage, responsible for the breaking to $1Y2L3c$, are (by construction) SM singlets. Since the SM one-loop $\beta$-functions are not affected by their presence, the solution found for $n_2$, $n_U$, and $\alpha_U$ in the $n_1 = n_2$ case holds for $n_1 < n_2$ as well. Only by performing correctly the mixed $1R1X$ gauge running and the consistent matching with $1Y$ one recovers the expected $n_1$ flatness of the GUT solution.

In this respect, it is interesting to notice that the
The absence of \( U(1) \) mixing in Refs. [6, 7, 8, 9] makes the argument for the actual possibility of a light (observable) \( U(1)_R \) gauge boson an "approximate" statement (based on the approximate flatness of the solution).

One expects this feature to break at two-loops. The \( SU(2)_L \) and \( SU(3)_c \) \( \beta \)-functions are affected at two-loops directly by the abelian gauge bosons via Eq. (15) (the Higgs multiplets that are responsible for the \( U(1)_R \otimes U(1)_X \) breaking do not enter through the Yukawa interactions). The net effect on the non-abelian gauge running is related to the difference between the contribution of the \( U(1)_R \) and \( U(1)_X \) gauge bosons and that of the standard hypercharge. We checked that such a difference is always a small fraction (below 10%) of the typical two-loop contributions to the \( SU(2)_L \) and \( SU(3)_c \) \( \beta \)-functions. As a consequence, the \( n_1 \) flatness of the GUT solution is at a very high accuracy (10\(^{-3}\)) preserved at two-loops as well, as the inspection of the relevant chains in Figs. 12 shows.

Still at one-loop we find a sharp disagreement in the \( n_1 \) range of chain XIa, with respect to the result of Ref. [9]. The authors find \( n_1 < 5.3 \), while strictly following their procedure and assumptions we find \( n_1 < 10.2 \) (the updated one- and two-loop results are given in Fig. 1k). As we shall see, this difference brings chain XIa back among the potentially realistic ones.

As far as two-loop effects are at stakes, their relevance is generally related to the length of the running involving the largest non-abelian groups. On the other hand, there are chains where \( n_2 \) and \( n_{U^1} \) have a strong dependence on \( n_1 \) (we will refer to them as "unstable" chains) and where two-loop corrections affect substantially the one-loop results. Evident examples of such unstable chains are Ia, IVa, Va, IVb, and VIIb. In particular, in chain Va the two-loop effects flip the slopes of \( n_2 \) and \( n_{U^1} \), that implies a sharp change in the allowed region for \( n_1 \). It is clear that when dealing with these breaking chains any statement about their viability should account for the details of the thresholds in the given model.

In chains VIII to XII (where the second intermediate stage is \( 2L_1R_1X_3 \)), two-loop effects are mild and exhibit the common behaviour of lowering the GUT scale (\( n_{U^1} \)) while raising (with the exception of Xb and XIa,b) the largest intermediate scale (\( n_2 \)). The mildness of two-loop corrections (no more that one would a-priori expect) is strictly related to the (\( n_1 \)) flatness of the GUT solution discussed before.

Worth mentioning are the limits \( n_2 \sim n_{U^1} \) and \( n_1 \sim n_2 \). While the former is equivalent to neglecting the first stage \( G2 \) to and reducing effectively the three breaking steps to just two (namely \( SO(10) \rightarrow G1 \rightarrow SM \)) with a minimal fine tuning in the scalar sector, care must be taken of the latter. One may naively expect that the chains with the same \( G2 \) should exhibit for \( n_1 \sim n_2 \) the same numerical behavior (\( SO(10) \rightarrow G2 \rightarrow SM \)), thus clustering the chains (I,V,X), (II,III,VI,VII,XI) and (IV,IX). On the other hand, one must recall that the existence of \( G1 \) and its breaking remain encoded in the \( G2 \) stage through the Higgs scalars that are responsible for the \( G2 \rightarrow G1 \) breaking. This is why the chains with the same \( G2 \) are not in general equivalent in the \( n_1 \sim n_2 \) limit. The numerical features of the degenerate patterns (with \( n_2 \sim n_{U^1} \)) can be crosschecked among the different chains by direct inspection of Figs. 12 and Table III.

In any discussion of viability of the various scenarios the main attention is paid to the constraints emerging from the proton decay. In non supersymmetric GUTs, this process is mediated by baryon number violating gauge interactions, inducing at low energies a set of effective dimension 6 operators that conserve \( B-L \). In the \( SO(10) \) scenarios we consider here, such gauge bosons are integrated out at the unification scale, and therefore proton decay constrains \( n_{U^1} \) from below. The present experimental limit \( \tau_p(p \rightarrow e^+\pi^0) > 1.6 \times 10^{33} \) years [25] implies

\[
\left( \frac{\alpha_{U^1}^{-1}}{45} \right) 10^{2(n_{U^1} - 15)} > 5.2 ,
\]

that, for \( \alpha_{U^1}^{-1} = 45 \) yields \( n_{U^1} > 15.4 \). Taking the results in Figs. 12 and Table III at face value the chains VIa,b, XIa,b, XIIa,b, Vb and VIIb should be excluded from realistic considerations.

On the other hand, one must recall that once a specific model is scrutinized in detail there can be large threshold corrections in the matching [10, 11, 12], that can easily move the unification scale
by a few orders of magnitude (in both directions). In particular, as a consequence of the spontaneous breaking of accidental would-be global symmetries of the scalar potential, pseudo-Goldstone modes (with masses further suppressed with respect to the expected threshold range) may appear in the scalar spectrum, leading to potentially large RGE effects \cite{[30]}. Therefore, we shall follow a conservative approach in interpreting the limits on the intermediate scales coming from a simple threshold clustering. These limits, albeit useful for a preliminary survey, may not be sharply used to exclude marginal but otherwise well motivated scenarios.

Below the scale of the $B - L$ breaking, processes that violate separately the barion or the lepton numbers emerge. In particular, $\Delta B = 2$ effective interactions give rise to the phenomenon of neutron oscillations and below the eV, one expects values (VEVs) that give a Majorana mass in the low energy theory.

$B - L \Delta$ about $10^{12}$ years, which translates into a bound on the neutron oscillation time $\tau_{n-\bar{n}} > 10^5$ sec. Analogous limits come from direct reactor oscillations experiments. This sets a lower bound on the scale of $\Delta B = 2$ non-supersymmetric (dimension 9) operators that varies from 10 to 300 TeV depending on model couplings. Thus, neutron-antineutron oscillations probe scales far below the unification scale. In a supersymmetric context the presence of $\Delta B = 2$ dimension 7 operators softens the dependence on the $B - L$ scale and for the present bounds the typical limit goes up to about $10^7$ GeV.

Far more reaching in scale sensitivity are the $\Delta L = 2$ neutrino masses emerging from the see-saw mechanism. At the $B - L$ breaking scale the $\Delta^{126}_R$ (\(\delta_R^{15}\)) scalars acquire $\Delta L = 2$ ($\Delta L = 1$) vacuum expectation values (VEVs) that give a Majorana mass to the right-handed neutrinos. Once the latter are integrated out, dimension 5 operators of the form $\nu_L^c \nu_L H H^2$ generate light Majorana neutrino states in the low energy theory.

In the type-I seesaw, the neutrino mass matrix $m_\nu$ is proportional to $Y_N M_R^{-1} Y_N^T v^2$ where the largest entry in the Yukawa couplings is typically of the order of the top quark one and $M_R \sim M_1$. Given a neutrino mass above the limit obtained from astrophysical neutrino oscillations and below the eV, one infers a (loose) range $10^{12}$ GeV $< M_1 < 10^{14}$ GeV. It is interesting to note that the lower bound pairs with the cosmological limit on the D-parity breaking scale (see Sect. II). In the scalar-triplet induced (type-II) seesaw the evidence of the neutrino mass entails a lower bound on the VEV of the heavy $SU(2)_L$ triplet in $T_{26H}$ (or in $\nu_L^c \nu_L H H^2$). This translates into an upper bound on the mass of the triplet that depends on the structure of the relevant Yukawa coupling. If both type-I as well as type-II contribute to the light neutrino mass, the lower bound on the $M_1$ scale may then be weakened by the interplay between the two contributions. Once again this can be quantitatively assessed only when the vacuum of the model is fully investigated.

Finally, it is worth noting that if the $B - L$ breakdown is driven by $T_{26\mu}$, the elementary triplets couple to the Majorana currents at the renormalizable level and $m_\nu$ is directly sensitive to the position of the $G1 \rightarrow SM$ threshold $M_1$. On the other hand, the $\alpha_1$-dependence of $m_\nu$ is loosened in the b-type of chains due to the non-renormalizable nature of the relevant effective operator $16_L 16_F T_{26H} T_{26H}/\Lambda$, where the effective scale $\Lambda > M_U$ accounts for an extra suppression.

With these considerations at hand, the constraints from proton decay and the see-saw neutrino scale favor the chains II, III and VII, which all share $2_L 4_R A_C P$ in the first $SO(10)$ breaking stage \cite{[31]}. On the other hand, our results rescue from oblivion other potentially interesting scenarios that, as we shall expand upon shortly, are worth of in depth consideration. In all cases, the bounds on the $B - L$ scale enforced by the see-saw neutrino mass exhausts the possibility of observable $U(1)_R$ gauge bosons.

C. The $\phi^{126}$ Higgs multiplets

As mentioned in Sect. II A in order to ensure a rich enough Yukawa sector in realistic models there may be the need to keep more than one $SU(2)_L$ Higgs doublet at intermediate scales, albeit at the price of an extra fine-tuning. A typical example is the case of a relatively low Pati-Salam breaking scale where one needs at least a pair of $SU(2)_L \otimes SU(2)_R$ bidoublets with different $SU(4)_C$ quantum numbers to transfer the information about the PS breakdown into the matter sector. Such additional Higgs multiplets are those labelled by $\phi^{126}$ in Table III.

Table IV shows the effects of including $\phi^{126}$ at the $SU(4)_C$ stages of the relevant breaking chains. The two-loop results at the extreme values of the intermediate scales, with and without the $\phi^{126}$ multiplet, are compared. In the latter case the complete functional dependence among the scales is given in Fig. I. Degenerate patterns with only one effective intermediate stage are easily crosschecked among the different chains in Table IV.

In most of the cases, the numerical results do not exhibit a sizeable dependence on the additional $(2, 2, 15)_{\text{T26}}$ (or $(2, +\frac{1}{2}, 15)_{\text{T26}}$) scalar multiplets. The reason can be read off Table X in Appendix A and it rests on an accidental approximate coincidence of the $\phi^{126}$ contributions to the $SU(4)_C$ and $SU(2)_{L,R}$ one-loop beta coefficients (the same argument applies to the $2_L 1_R 4_C$ case).

Considering for instance the $2_L 2_R 4_C$ stage, one
In particular, the inclusion of $\phi$ are most sensitive to variations of the $\beta$ and maximum values allowed for $n$ flips at two-loops the slopes of $n$ most sensitive to variations of the $\beta$-functions. Generally the effects on the intermediate scales are below the percent level, with the exception of chains Ia and Va that are most sensitive to variations of the $\beta$-functions.

D. Yukawa terms

The effects of the Yukawa couplings can be at leading order approximated by constant negative shifts of the one-loop $a_i$ coefficients $a_i \rightarrow a_i' = a_i + \Delta a_i$ obtains $\Delta a_4 = \frac{1}{3} \times 4 \times T_2(15) = \frac{16}{3}$, and $\Delta a_2 = \frac{1}{3} \times 30 \times T_2(2) = 5$, that only slightly affects the value of $\alpha_U$ (when the PS scale is low enough), but has generally a negligible effect on the intermediate scales.

An exception to this argument is observed in chains Ia and Va that, due to their $n_{2, U}(n_1)$ slopes, are most sensitive to variations of the $\beta$-coefficients. In particular, the inclusion of $\phi^{126}$ in the Ia chain flips at two-loops the slopes of $n_2$ and $n_U$ so that the limit $n_2 = n_U$ (i.e. no G2 stage) is obtained for the maximal value of $n_1$ (while the same happens for the minimum $n_1$ if there is no $\phi^{126}$).

Fig. 3 shows three template cases where the $\phi^{126}$ effects are visible. The highly unstable Chain Ia shows, as noticed earlier, the largest effects. In chain Va the effects of $\phi^{126}$ are moderate. Chain VII is the only "stable" chain that exhibits visible effects on the intermediate scales. This is due to the presence of two full-fledged PS stages.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Chain & $n_1$ & $n_2$ & $n_U$ & $\alpha_U^{-1}$ \\
\hline
Ia & [9.50, 10.0] & [16.2, 10.0] & [16.2, 17.0] & [45.5, 46.4] \\
 & [8.00, 9.50] & [10.4, 16.2] & [18.0, 16.2] & [30.6, 45.5] \\
IIa & [10.5, 13.7] & [15.4, 13.7] & [15.4, 15.1] & [43.7, 40.8] \\
 & [10.5, 13.7] & [15.4, 13.7] & [15.4, 15.1] & [43.7, 37.6] \\
IIIa & [9.50, 13.7] & [16.2, 13.7] & [16.2, 15.1] & [45.5, 40.8] \\
 & [9.50, 13.7] & [16.2, 13.7] & [16.2, 15.1] & [45.5, 37.6] \\
Va & [11.0, 11.4] & [11.0, 14.4] & [15.9, 14.4] & [45.4, 44.1] \\
 & [10.1, 11.2] & [10.1, 14.5] & [16.5, 14.5] & [32.5, 40.8] \\
VIa & [11.4, 13.7] & [14.4, 13.7] & [14.4, 14.9] & [44.1, 41.0] \\
 & [11.2, 13.7] & [14.5, 13.7] & [14.5, 14.9] & [40.8, 38.1] \\
VIIa & [11.3, 13.7] & [15.9, 13.7] & [15.9, 14.9] & [45.4, 41.1] \\
 & [10.5, 13.7] & [16.5, 13.7] & [16.5, 15.0] & [33.3, 38.1] \\
XIIa & [8.00, 9.50] & [10.4, 16.2] & [18.0, 16.2] & [30.6, 45.5] \\
 & [9.50, 10.0] & [10.5, 16.2] & [18.0, 16.2] & [30.6, 45.5] \\
\hline
\end{tabular}
\caption{Impact of the additional multiplet $\phi^{126}$ (second line of each chain) on those chains that contain the gauge groups $2_4 H 4 C$ as intermediate stages, and whose breaking to the SM is obtained via a $126_H$ representation. The values of $n_2$, $n_U$ and $\alpha_{U}^{-1}$ are showed for the minimum and maximum values allowed for $n_1$ by the two-loop analysis. Generally the effects on the intermediate scales are below the percent level, with the exception of chains Ia and Va that are most sensitive to variations of the $\beta$-functions.

\begin{equation}
\Delta a_i = -\frac{1}{(4\pi)^2} y_{ik} \text{Tr} Y_k Y_k^\dagger.
\end{equation}

The impact of $\Delta a_i$ on the position of the unification scale and the value of the unified coupling can be simply estimated by considering the running induced by the Yukawa couplings from a scale $t$ up to the unification point ($t = 0$). The one-loop result for the change of the intersection of the curves corresponding to $\alpha_i^{-1}(t)$ and $\alpha_j^{-1}(t)$ reads (at the leading order in $\Delta a_i$):

\begin{equation}
\Delta t_U = \frac{2\pi}{(a_i - a_j)^2} \Delta a_j \left[ \alpha_j^{-1}(t) - \alpha_i^{-1}(t) \right] + \ldots
\end{equation}
\[
\Delta a_U^{-1} = \frac{1}{2} \left[ \Delta a_i + \Delta a_j - \frac{(a_i + a_j)(\Delta a_i - \Delta a_j)}{a_i - a_j} \right] \\
\times \left[ \alpha_j^{-1}(t) - \alpha_i^{-1}(t) \right] + \ldots
\]  
(39)

for any \( i \neq j \). For simplicity we have neglected the changes in the \( a_i \) coefficients due to crossing intermediate thresholds. It is clear that for a common change \( \Delta a_i = \Delta a_j \) the unification scale is not affected, while a net effect remains on \( \alpha_U^{-1} \). In all cases, the leading contribution is always proportional to \( \alpha_j^{-1}(t) - \alpha_i^{-1}(t) \) (this holds exactly for \( \Delta t_U \)).

In order to assess quantitatively such effects we shall consider first the SM stage that accounts for a large part of the running in all realistic chains. The case of a low \( n_1 \) scale leads, as we explain in the following, to comparably smaller effects. The impact of the Yukawa interactions on the gauge RGEs is readily estimated assuming only the up-type Yukawa contribution to be sizeable and constant, namely \( \text{Tr} Y_U Y_U^\dagger \sim 1 \). This yields \( \Delta a_1 \sim -6 \times 10^{-3} y_{1U} \), where the values of the \( y_{1U} \) coefficients are given in Table XI. For \( i = 1 \) and \( j = 2 \) one obtains \( \Delta a_1 \sim -1.1 \times 10^{-2} \) and \( \Delta a_2 \sim -0.9 \times 10^{-2} \), respectively. Since \( a_3^{2SM} = \frac{19}{10} \) and \( a_2^{SM} = -\frac{19}{6} \), the first term in (39) dominates and one finds \( \Delta \alpha_U^{-1} \sim 0.04 \). For a typical value of \( \Delta a_U^{-1} \sim 40 \) this translates into \( \Delta \alpha_U^{-1}/\alpha_U^{-1} \sim 0.1 \%. \) The impact on \( t_U \) is indeed tiny, namely \( \Delta t_U \sim -1 \times 10^{-2} \). In both cases the estimated effect agrees to high accuracy with the actual numerical behavior we observe.

The effects of the Yukawa interactions emerging at intermediate scales (or of a non-negligible \( \text{Tr} Y_D Y_D^\dagger \)) in a two Higgs doublet setting with large tan \( \beta \) can be analogously accounted for. As a matter of fact, in the \( SO(10) \) type of models \( \text{Tr} Y_N Y_N^\dagger \sim \text{Tr} Y_U Y_U^\dagger \) due to the common origin of \( Y_U \) and \( Y_N \). The unified structure of the Yukawa sector yields therefore homogeneous \( \Delta a_i \) factors (see the equality of the \( y_{ik} \) in Table XI). This provides the observed large suppression of the Yukawa effects on threshold scales and unification compared to typical two-loop gauge contributions.

In summary, the two-loop RGE effects due to Yukawa couplings on the magnitude of the unification scale (and intermediate thresholds) and the value of the GUT gauge coupling turn out to be very small. Typically we observe negative shifts at the per-mil level in both \( n_U \) and \( \alpha_U \), with no relevant impact on the gauge-mediated proton decay rate.

E. The privilege of being minimal

With all the information at hand we can finally approach an assessment of the viability of the various scenarios. As we have argued at length, we cannot discard a marginal unification setup without a detailed information on the fine threshold structure.

Obtaining this piece of information involves the study of the vacuum of the model, and for \( SO(10) \) GUTs this is in general a most challenging task. In this respect supersymmetry helps: the superpotential is holomorphic and the couplings in the renormalizable case are limited to at most cubic terms; the physical vacuum is constrained by GUT-scale \( F \)- and \( D \)-flatness and supersymmetry may be exploited to studying the fermionic rather than the scalar spectra.

It is not surprising that for non-supersymmetric \( SO(10) \), only a few detailed studies of the Higgs potential and the related threshold effects (see for instance Refs. \[32, 33, 34, 35, 36\]) are available. In view of all this and of the intrinsic predictivity related to minimality, the relevance of carefully scrutinizing the simplest scenarios is hardly overstressed.

The most economical \( SO(10) \) Higgs sector includes the adjoint \( 45_H \), that provides the breaking of the GUT symmetry, either \( 10_H \) or \( 126_H \), responsible for the subsequent \( B - L \) breaking, and \( 10_H \), participating to the electroweak symmetry breaking. The latter is needed together with \( 16_H \) or \( 126_H \) in order to obtain realistic patterns for the fermionic masses and mixing. Due to the properties of the adjoint representation this scenario exhibits a minimal number of parameters in the Higgs potential. In the current notation such a minimal non-supersymmetric \( SO(10) \) (MSO10) GUT corresponds to the chains VIII and XII.

From this point of view, it is quite intriguing that our analysis of the gauge unification constraints improves the standing of these chains (for XIIa dramatically) with respect to existing studies. In particular, considering the renormalizable cases (126H), we find for \( \text{chain VIIIa, } n_1 \leq 9.1, n_U = 16.2 \) and \( \alpha_U^{-1} = 45.4 \) (to be compared to \( n_1 \leq 7.7 \) given in Ref. \[9\]). This is due to the combination of the updated weak scale data and two loop running effects. For \( \text{chain XIIa, } n_1 \leq 10.8, n_U = 14.6 \) and \( \alpha_U^{-1} = 44.1 \), showing a dramatic (and pathological) change from \( n_1 \leq 5.3 \) obtained in \[9\]. Our result sets the \( B - L \) scale nearby the needed scale for realistic light neutrino masses.

We observe non-negligible two-loop effects for the chains VIIIB and XIIb (16H) as well. For chain VIIIB we obtain \( n_1 \leq 10.5, n_U = 16.2 \) and \( \alpha_U^{-1} = 45.6 \) (that lifts the \( B - L \) scale while preserving \( n_U \) well above the proton decay bound Eq. \[46\]). A similar
shift in $n_1$ is observed in chain XIIb where we find $n_1 \leq 12.5$, $n_U = 14.8$ and $\alpha^{-1}_U = 44.3$. As we have already stressed one should not too readily discard $n_U = 14.8$ as being incompatible with the proton decay bound. We have verified that reasonable GUT threshold patterns exist that easily lift $n_U$ above the experimental bound. For all these chains D-parity is broken at the GUT scale thus avoiding any cosmological issues (see the discussion in Sect. [1]).

As remarked in Sect. [1] the limit $n_1 = n_2$ leads to an effective two-step $SO(10) \rightarrow G2 \rightarrow SM$ scenario with a non-minimal set of surviving scalars at the G2 stage. As a consequence, the unification setup for the MSO10 can be recovered (with the needed minimal fine tuning) by considering the limit $n_2 = n_U$ in those chains among I to VII where G1 is either $2_L \otimes 2_R \otimes 1_X 3_c$ or $2_L \otimes 1_R 4_C$ (see Table II). From inspection of Figs. [2] and of Table III one reads the following results: for $SO(10) \rightarrow 2_L 2_R 1_X 3_c \rightarrow SM$ we find

$$n_1 = 9.5, \ n_U = 16.2 \ \text{and} \ \alpha^{-1}_U = 45.5, \ \text{in case a}$$

and

$$n_1 = 10.8, \ n_U = 16.2 \ \text{and} \ \alpha^{-1}_U = 45.7, \ \text{in case b},$$

while for $SO(10) \rightarrow 2_L 1_R 4_C \rightarrow SM$

$$n_1 = 11.4, \ n_U = 14.4 \ \text{and} \ \alpha^{-1}_U = 44.1, \ \text{in case a}$$

and

$$n_1 = 12.6, \ n_U = 14.6 \ \text{and} \ \alpha^{-1}_U = 44.3, \ \text{in case b}.$$  

We observe that the patterns are quite similar to those of the non-minimal setups obtained from chains VIII and XII in the $n_1 = n_2$ limit. Adding the $\phi^{126}$ multiplet, as required by a realistic matter spectrum in case a, does not modify the scalar content in the $2_L 2_R 1_X 3_c$ case: only one linear combination of the $10_H$ and $\overline{126}_H$ bidoublets (see Table II) is allowed by minimal fine tuning. On the other hand, in the $2_L 1_R 4_C$ case, the only sizeable effect is a shift on the unified coupling constant, namely $\alpha^{-1}_U = 40.7$ (see the discussion in Sect. [1]).

In summary, in view of realistic thresholds effects at the GUT (and $B-L$) scale and of a modest fine tuning in the see-saw neutrino mass, we consider both scenarios worth of a detailed investigation.

V. OUTLOOK

We presented an updated and systematic two-loop discussion of non-supersymmetric $SO(10)$ gauge unification with two (and one) intermediate scales. We completed and corrected existing analyses by including a thorough discussion of $U(1)$ mixing, which affects the gauge running already at the one-loop level in a number of interesting $SO(10)$ breaking chains. We assessed the relevance of additional Higgs multiplets, needed at some of the intermediate stages in order to reproduce a realistic fermionic mass spectrum. Finally, we found and fixed several discrepancies in the two-loop $\beta$-coefficients.

The updated results have a non-negligible impact on the values of the unification and $B-L$ scales (as well as on the value of the unified gauge coupling). This is due to the combined effects of the one-loop dynamics corresponding to the $U(1)$ gauge mixing and of the two-loop RGE contributions.

We discussed the viability of the different $SO(10)$ scenarios on the basis of proton decay and the see-saw induced neutrino mass. We were lead to focus our attention on the minimal $SO(10)$ setup, emerging from a balance of minimal dimensionality Higgs representations and a minimal number of parameters in the scalar potential. Such a scenario invokes, in addition to a complex $10_H$, one adjoint $45_H$ together with one $\overline{126}_H$ or $\overline{10}_H$ at the effective level.

Although the updated values of the unification or $B-L$ scales are in some of the setups still conflicting with the experimental requirements, they are close enough that reasonable spreads in the GUT thresholds (or a moderate fine tuning in the neutrino mass matrix) can easily restore the agreement. This may entail the detailed study of the scalar potential of the model beyond the tree approximation, that is a rather non-trivial task. Nevertheless, the appeal of minimalism (with supersymmetry confined to the Planck scale) motivates us to pursue this study.

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APPENDIX A: ONE- AND TWO-LOOP $\beta$-COEFFICIENTS

In this appendix we report the one- and two-loop $\beta$-coefficients used in the numerical analysis. The calculation of the $U(1)$ mixing coefficients and of the Yukawa contributions to the gauge coupling renormalization is detailed in Apps. [A1] and [A2] respectively.
| Chain | $a_j$ | $b_{ij}$ | Chain | $a_j$ | $b_{ij}$ |
|-------|-------|---------|-------|-------|---------|
| Ia    | $(-3, \frac{11}{3}, -7)$ | $\begin{pmatrix} 8 & 3 \\ 3 & 156 \end{pmatrix}$ | Ib    | $(-3, -\frac{7}{3}, -\frac{26}{3})$ | $\begin{pmatrix} 8 & 3 \\ 3 & 156 \end{pmatrix}$ |
| IIa   | $(\frac{11}{3}, \frac{11}{3}, -4)$ | $\begin{pmatrix} 584 & 3 \\ 3 & 153 \end{pmatrix}$ | IIb   | $(-\frac{7}{3}, -\frac{7}{3}, -\frac{26}{3})$ | $\begin{pmatrix} 584 & 3 \\ 3 & 153 \end{pmatrix}$ |
| IIIa  | $(\frac{11}{3}, \frac{11}{3}, -4)$ | $\begin{pmatrix} 584 & 3 \\ 3 & 153 \end{pmatrix}$ | IIIb  | $(-\frac{7}{3}, -\frac{7}{3}, -\frac{26}{3})$ | $\begin{pmatrix} 584 & 3 \\ 3 & 153 \end{pmatrix}$ |
| IVa   | $(\frac{7}{3}, -\frac{7}{3}, 7, -7)$ | $\begin{pmatrix} 80 & 3 \\ 3 & 153 \end{pmatrix}$ | IVb   | $(-\frac{27}{7}, -\frac{27}{7}, 9, -7)$ | $\begin{pmatrix} 80 & 3 \\ 3 & 153 \end{pmatrix}$ |
| Va    | $(-3, -\frac{22}{4})$ | $\begin{pmatrix} 8 & 3 \\ 3 & 204 \end{pmatrix}$ | Vb    | $(-3, -\frac{31}{3})$ | $\begin{pmatrix} 8 & 3 \\ 3 & 204 \end{pmatrix}$ |
| VIa   | $(4, 4, -\frac{14}{3})$ | $\begin{pmatrix} 204 & 3 \\ 3 & 153 \end{pmatrix}$ | VIb   | $(-2, -2, -10)$ | $\begin{pmatrix} 204 & 3 \\ 3 & 153 \end{pmatrix}$ |
| VIIIa | $(\frac{11}{3}, \frac{11}{3}, -\frac{14}{3})$ | $\begin{pmatrix} 204 & 3 \\ 3 & 153 \end{pmatrix}$ | VIIIb | $(-\frac{7}{3}, -\frac{7}{3}, -10)$ | $\begin{pmatrix} 204 & 3 \\ 3 & 153 \end{pmatrix}$ |
| VIIIa | $(-3, -2, \frac{11}{3}, -7)$ | $\begin{pmatrix} 8 & 3 \\ 3 & 153 \end{pmatrix}$ | VIIIb | $(-3, -\frac{7}{3}, \frac{17}{3}, -7)$ | $\begin{pmatrix} 8 & 3 \\ 3 & 153 \end{pmatrix}$ |
| IXa   | $(-2, -2, 7, -7)$ | $\begin{pmatrix} 36 & 3 \\ 12 & 153 \end{pmatrix}$ | IXb   | $(-\frac{7}{3}, -\frac{7}{3}, -7)$ | $\begin{pmatrix} 36 & 3 \\ 12 & 153 \end{pmatrix}$ |
| Xa    | $(-3, \frac{26}{3}, -\frac{17}{3})$ | $\begin{pmatrix} 8 & 3 \\ 3 & 153 \end{pmatrix}$ | Xb    | $(-3, \frac{8}{3}, -\frac{25}{3})$ | $\begin{pmatrix} 8 & 3 \\ 3 & 153 \end{pmatrix}$ |
| XIa   | $(\frac{26}{3}, \frac{26}{3}, -\frac{2}{3})$ | $\begin{pmatrix} 1004 & 3 \\ 3 & 153 \end{pmatrix}$ | XIb   | $(\frac{8}{3}, \frac{8}{3}, -6)$ | $\begin{pmatrix} 1004 & 3 \\ 3 & 153 \end{pmatrix}$ |
| XIIa  | $(-\frac{12}{3}, -\frac{12}{3}, -9)$ | $\begin{pmatrix} 45 & 3 \\ 45 & 153 \end{pmatrix}$ | XIIb  | $(-\frac{12}{3}, -\frac{12}{3}, -9)$ | $\begin{pmatrix} 45 & 3 \\ 45 & 153 \end{pmatrix}$ |

**TABLE V:** The $a_i$ and $b_{ij}$ coefficients due to pure gauge interactions are reported for the G2 chains with $1_{26}H$ (left) and $16_H$ (right) respectively. The two-loop contributions induced by Yukawa couplings are given in Appendix $\Lambda_3$.2
The TABLE VII: The $a_i$ and $b_{ij}$ coefficients due to gauge interactions are reported for the G1 chains I to VII with $\bar{126}_H$ (left) and $\bar{15}_H$ (right) respectively. The two-loop contributions induced by Yukawa couplings are given in Appendix A2.

| Chain   | $a_i$ | $b_{ij}$ | Chain   | $a_i$ | $b_{ij}$ |
|---------|-------|----------|---------|-------|----------|
| Ia      | $(-3, -\frac{7}{3}, \frac{12}{7}, -7)$ | $(8, 3, \frac{4}{7})$ | Ib      | $(-3, -\frac{17}{6}, \frac{17}{7}, -7)$ | $(8, 3, \frac{4}{7})$ |
| Ha      | $(-\frac{7}{3}, -\frac{7}{3}, \frac{12}{7}, -7)$ | $(8, 3, \frac{4}{7})$ | Hb      | $(-\frac{17}{6}, -\frac{17}{7}, -7)$ | $(8, 3, \frac{4}{7})$ |
| IIIa    | $(-3, -\frac{7}{3}, \frac{12}{7}, -7)$ | $(8, 3, \frac{4}{7})$ | IIIb    | $(-3, -\frac{17}{6}, \frac{17}{7}, -7)$ | $(8, 3, \frac{4}{7})$ |
| IVa     | $(-3, -\frac{7}{3}, \frac{12}{7}, -7)$ | $(8, 3, \frac{4}{7})$ | IVb     | $(-3, -\frac{17}{6}, \frac{17}{7}, -7)$ | $(8, 3, \frac{4}{7})$ |
| Va      | $(-\frac{19}{6}, \frac{15}{7}, -\frac{29}{3})$ | $(8, 3, \frac{4}{7})$ | Vb      | $(-\frac{19}{6}, \frac{9}{7}, -\frac{21}{2})$ | $(8, 3, \frac{4}{7})$ |
| VIa     | $(-\frac{19}{6}, \frac{15}{7}, -\frac{29}{3})$ | $(8, 3, \frac{4}{7})$ | VIb     | $(-\frac{19}{6}, \frac{9}{7}, -\frac{21}{2})$ | $(8, 3, \frac{4}{7})$ |
| VIIa    | $(-3, -\frac{7}{3}, \frac{12}{7}, -7)$ | $(8, 3, \frac{4}{7})$ | VIIb    | $(-3, -\frac{7}{3}, \frac{31}{7}, -7)$ | $(8, 3, \frac{4}{7})$ |

TABLE VII: The $a_i$ and $b_{ij}$ coefficients due to purely gauge interactions for the G1 chains VIII to XII are reported. For comparison with previous studies the $\beta$-coefficients are given neglecting systematically one- and two-loops $U(1)$ mixing effects (while all diagonal $U(1)$ contributions to abelian and non-abelian gauge coupling renormalization are included). The complete (and correct) treatment of $U(1)$ mixing is detailed in Appendix A1.
We report those cases where disagreement is found between abelian and non-abelian gauge group factors. Contributions are neglected and no $U(1)$ mixing is included in the SM $(M_1 \rightarrow M_2)$ stages.

| Chain   | $a_i$ | $b_{ij}$ |
|---------|-------|----------|
| All     | $\left( \frac{41}{10}, \frac{19}{10}, -7 \right)$ | $\left( \begin{array}{ccc} 199 & 27 & 44 \\ 9 & 11 & 9 \\ 10 & 2 & -26 \end{array} \right)$ |

TABLE VIII: The $a_i$ and $b_{ij}$ coefficients are given for the $1y2L3c$ (SM) gauge running. The scalar sector includes one Higgs doublet.

| Chain   | $b_{ij}$ | Eq. in Ref. [7] |
|---------|----------|-----------------|
| All/SM  | $\left( \begin{array}{ccc} 199 & 27 & 44 \\ 9 & 11 & 9 \\ 10 & 2 & -26 \end{array} \right)$ | A7 |
| VIIIa/G1| $\left( \begin{array}{ccc} 25 & 0 & 0 \\ 1 & 3 & 3 \\ 27 & 17 & 26 \end{array} \right)$ | A10 |
| VIIa/G2 | $\left( \begin{array}{ccc} 61 & 0 & 0 \\ 1 & 3 & 3 \\ 27 & 17 & 26 \end{array} \right)$ | A13 |
| Ia/G2   | $\left( \begin{array}{ccc} -3 & 0 & 0 \\ 27 & 17 & 26 \end{array} \right)$ | A14 |
| Va/G1   | $\left( \begin{array}{ccc} -45 & 15 & -120 \\ 19 & 9 & 101 \\ 19 & 9 & 101 \end{array} \right)$ | A15 |
| XIIa/G2 | $\left( \begin{array}{ccc} -35 & 15 & -41 \\ 19 & 9 & 101 \\ 19 & 9 & 101 \end{array} \right)$ | A18 |

TABLE IX: The rescaled two-loop $\beta$-coefficients $\tilde{b}_{ij}$ computed in this paper are shown together with the corresponding equations in Ref. [7]. For the purpose of comparison Yukawa contributions are neglected and no $U(1)$ mixing is included in chain VIIa/G1. Care must be taken of the different ordering between abelian and non-abelian gauge group factors in Ref. [7]. We report those cases where disagreement is found in some of the entries, while we fully agree with the Eqs. A9, A11 and A16.

| $\phi^{126}$ | $a_i$ | $b_{ij}$ |
|--------------|-------|----------|
| (2, 2, 15)   | $(\frac{5}{5}, \frac{5}{5}, \frac{5}{5})$ | $(\frac{65}{19}, \frac{45}{11}, 120)$ |
| (2, + $\frac{1}{2}$, 15) | $(\frac{5}{5}, \frac{5}{5}, \frac{5}{5})$ | $(\frac{65}{19}, \frac{45}{11}, 120)$ |

TABLE X: One- and two-loop additional contributions to the $\beta$-coefficients related to the presence of the $\phi^{126}$ scalar multiplets in the $2L2R^4$ (top) and $2L1R^4$ (bottom) stages.
1. Beta-functions with $U(1)$ mixing

The basic building blocks of the one- and two-loop $\beta$-functions for the abelian couplings with $U(1)$ mixing, c.f. Eqs. (14)–(15), can be conveniently written as

$$ g_{AA}g_{BB} = g_{AA}\Gamma^{(1)}_{AA}g_{BB} \quad \text{(A1)} $$

and

$$ g_{AB}g_{BB}g_{ab}^2 = g_{AA}\Gamma^{(2)}_{AB}g_{BB} \quad \text{(A2)} $$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are functions of the abelian charges $Q^a_k$ and, at two loops, also of the gauge couplings. In the case of interest, i.e. for two abelian charges $U(1)_A$ and $U(1)_B$, one obtains

$$ \Gamma^{(1)}_{AA} = (Q^A_k)^2, \quad \Gamma^{(1)}_{AB} = Q^A_kQ^B_k, \quad \Gamma^{(1)}_{BB} = (Q^B_k)^2, $$

$$ \Gamma^{(2)}_{AA} = (Q^4_k)^2(Q^A_k)^4(g^2_{AA} + g^2_{AB}) + 2(Q^4_k)^2Q^B_k(g_{AA}g_{BA} + g_{AB}g_{BB}) + (Q^4_k)^2(Q^B_k)^2(g^2_{BA} + g^2_{BB}), $$

$$ \Gamma^{(2)}_{AB} = (Q^4_k)^2Q^B_k(g^2_{AA} + g^2_{AB}) + 2(Q^4_k)^2Q^B_k(g_{AA}g_{BA} + g_{AB}g_{BB}) + Q^4_k(Q^B_k)^2(g^2_{BA} + g^2_{BB}), $$

$$ \Gamma^{(2)}_{BB} = (Q^4_k)^2(Q^B_k)^2(g^2_{AA} + g^2_{AB}) + 2Q^4_k(Q^B_k)^2(g_{AA}g_{BA} + g_{AB}g_{BB}) + (Q^4_k)^2(g^2_{BA} + g^2_{BB}). $$

All other contributions in Eq. (14) and Eq. (15) can be easily obtained from Eqs. (A3)–(A4) by including the appropriate group factors. It is worth mentioning that for complete $SO(10)$ multiplets, $(Q^A_k)^n(Q^B_k)^n = 0$ for $n$ and $m$ odd (with $n + m = 2$ at one-loop and $n + m = 4$ at two-loop level).

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*Chains VIII-XII with $\mathbf{T}_{26}^H$ in the Higgs sector:

$$ \gamma_{RR} = \frac{9}{2} + \frac{1}{(4\pi)^2} \left[ \frac{15}{2}(g^2_{RR} + g^2_{RX}) - 4\sqrt{6}(g_{RR}g_{XR} + g_{RX}g_{XX}) + \frac{15}{2}(g^2_{XR} + g^2_{XX}) + \frac{3}{2}g^2_L + 12g^2_c \right], $$

$$ \gamma_{RX} = \gamma_{XR} = -\frac{1}{\sqrt{6}} + \frac{1}{(4\pi)^2} \left[ -2\sqrt{6}(g^2_{RR} + g^2_{RX}) + 15(g_{RR}g_{XR} + g_{RX}g_{XX}) - 3\sqrt{6}(g^2_{XR} + g^2_{XX}) \right], $$

$$ \gamma_{XX} = \frac{9}{2} + \frac{1}{(4\pi)^2} \left[ \frac{15}{2}(g^2_{RR} + g^2_{RX}) - 6\sqrt{6}(g_{RR}g_{XR} + g_{RX}g_{XX}) + \frac{25}{2}(g^2_{XR} + g^2_{XX}) + \frac{9}{2}g^2_L + 4g^2_c \right], $$

$$ \gamma_{LL} = -\frac{19}{6} + \frac{1}{(4\pi)^2} \left[ \frac{1}{2}(g^2_{RR} + g^2_{RX}) + \frac{3}{2}(g^2_{XR} + g^2_{XX}) + \frac{35}{6}g^2_L + 12g^2_c \right], $$

$$ \gamma_{cc} = -7 + \frac{1}{(4\pi)^2} \left[ \frac{3}{2}(g^2_{RR} + g^2_{RX}) + \frac{1}{2}(g^2_{XR} + g^2_{XX}) + \frac{9}{2}g^2_L - 26g^2_c \right]; $$

*Chains VIII-XII with $\mathbf{T}_{16}^H$ in the Higgs sector:
\[\gamma_{RR} = \frac{17}{4} + \frac{1}{(4\pi)^2} \left[ \frac{15}{4} (g_{RR}^2 + g_{RX}^2) - \frac{1}{2} \sqrt{\frac{3}{2}} (g_{RR} + g_{RX}) + \frac{15}{8} (g_{XR}^2 + g_{XX}^2) + \frac{3}{2} g_L^2 + 12 g_c^2 \right],\]

\[\gamma_{RX} = \gamma_{XR} = -\frac{1}{4 \sqrt{6}} + \frac{1}{(4\pi)^2} \left[ -\frac{1}{4} \sqrt{\frac{3}{2}} (g_{RR} + g_{RX}) + \frac{15}{4} (g_{RR} + g_{RX}) - \frac{3}{8} \sqrt{\frac{3}{2}} (g_{XR}^2 + g_{XX}^2) \right],\]

\[\gamma_{XX} = \frac{33}{8} + \frac{1}{(4\pi)^2} \left[ \frac{15}{8} (g_{RR}^2 + g_{RX}^2) - \frac{3}{4} \sqrt{\frac{3}{2}} (g_{RR} + g_{RX}) + \frac{65}{16} (g_{XR}^2 + g_{XX}^2) + \frac{9}{2} g_L^2 + 4 g_c^2 \right],\]

\[\gamma_l = -\frac{19}{6} + \frac{1}{(4\pi)^2} \left[ \frac{1}{2} (g_{RR}^2 + g_{RX}^2) + \frac{3}{2} (g_{XR}^2 + g_{XX}^2) + \frac{35}{6} g_L^2 + 12 g_c^2 \right],\]

\[\gamma_c = -7 + \frac{1}{(4\pi)^2} \left[ \frac{3}{2} (g_{RR}^2 + g_{RX}^2) + \frac{1}{2} (g_{XR}^2 + g_{XX}^2) + \frac{9}{2} g_L^2 - 26 g_c^2 \right].\]

By setting \(\gamma_{XX} = \gamma_{RX} = 0\) and \(g_{XR} = g_{RX} = 0\) in Eqs. (A5)–(A6) one obtains the one- and two-loop \(\beta\)-coefficients in the diagonal approximation, as reported in Table VII. The latter are used in Figs. 1–2 for the only purpose of exhibiting the effect of the abelian mixing in the gauge coupling renormalization.

2. Yukawa contributions

The Yukawa couplings enter the gauge \(\beta\)-functions first at the two-loop level, c.f. Eq. (5) and Eq. (13). Since the notation adopted in Eqs. (6)–(7) is rather concise we shall detail the structure of Eq. (6), paying particular attention to the calculation of the \(y_{pk}\) coefficients in Eq. (20).

The trace on the RHS of Eq. (5) is taken over all indices of the fields entering the Yukawa interaction in Eq. (7). Considering for instance the up-quark Yukawa sector of the SM the term \(\bar{Q}_L Y_U U_R \tilde{h} + h.c.\) (with \(\tilde{h} = i \sigma_2 h^*\)) can be explicitly written as

\[Y_{U}^{ab} \varepsilon^{jl} \delta_3 j \bar{Q}_{L}^{i} U_{R}^{bj} \tilde{h}^i + h.c.,\]

where \(\{a,b\}\) and \(\{i,j\}\) label flavour, \(SU(3)_c\) and \(SU(2)_L\) indices respectively, while \(\delta_n\) denotes the \(n\)-dimensional Kronecker \(\delta\) symbol. Thus, the Yukawa coupling entering Eq. (6) is a 6-dimensional object with the index structure \(Y_{U}^{ab} \varepsilon^{kl} \delta_3 j\). The contribution of Eq. (A7) to the three \(y_{pu}\) coefficients (conveniently separated into two terms corresponding to the fermionic representations \(Q_L\) and \(U_R\)) can then be written as

\[y_{pU} = \frac{1}{d(C_p)} \left[ C_{(2)}^{(p)} (Q_L) + C_{(2)}^{(p)} (U_R) \right] \times \sum_{ab,ij,kl} Y_{U}^{ab} \varepsilon^{jl} \delta_3 j \epsilon_{ij} Y_{U}^{ab} \varepsilon^{kl} \delta_3 l.\]
| \( G_p \) | \( y_{ph} \) | \( k \) | Gauge structure | Higgs representation | Tensor \( \Delta \) | \( \text{Tr}[\Delta^\dagger] \) |
|---|---|---|---|---|---|---|
| 1Y | \( \left( \frac{47}{3} \frac{1}{2} \frac{3}{2} \right) \) | U | \( \overline{Q}_{Lk} U_{R}^i h_i \) | \( h^i : (+\frac{1}{2}, 2, 1) \) | \( \epsilon^{ik} \delta_{3i}^j \) | 6 |
| 2L | \( \left( \frac{4}{3} \frac{1}{2} \frac{3}{2} \right) \) | D | \( \overline{Q}_{Lk} D_{R}^i h_i \) | \( h^i : (+\frac{1}{2}, 2, 1) \) | \( \delta_{2i}^k \delta_{3i}^j \) | 6 |
| 3c | \( \left( 2 \ 2 \ 0 \right) \) | E | \( L_{Lk} E_{R}^i h_i \) | \( \delta_{2i}^k \) | 2 |

| 2L | \( \left( \frac{3}{2} \frac{3}{2} \frac{3}{2} \right) \) | U | \( \overline{Q}_{Lk} U_{R}^i h_i \) | \( h^i : (+\frac{1}{2}, 2, 1) \) | \( \epsilon^{ik} \delta_{3i}^j \) | 6 |
| 1RX | \( \frac{1}{2} \sqrt{3} \sqrt{3} \frac{1}{2} \frac{1}{2} \) | D | \( \overline{Q}_{Lk} D_{R}^i h_i \) | \( h^i : (+\frac{1}{2}, 2, 1) \) | \( \epsilon^{ik} \delta_{3i}^j \) | 6 |
| 1XR | \( \frac{1}{2} \sqrt{3} \sqrt{3} \frac{1}{2} \frac{1}{2} \) | N | \( L_{Lk} N_{R} h_i \) | \( h^i : (+\frac{1}{2}, 2, 1) \) | \( \epsilon^{ik} \delta_{3i}^j \) | 6 |
| 1XX | \( \frac{1}{2} \sqrt{3} \sqrt{3} \frac{1}{2} \frac{1}{2} \) | E | \( L_{Lk} E_{R} h_i \) | \( \delta_{2i}^k \) | 2 |
| 3c | \( \left( 2 \ 2 \ 0 \ 0 \right) \) | Q | \( Q_{Lij}^k Q_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 0, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 12 |
| 1X | \( \left( 3 \ 1 \right) \) | L | \( L_{Lij}^l L_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 0, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \) | 4 |
| 3c | \( \left( 4 \ 0 \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |

| 2L | \( \left( 2 \ 2 \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |
| 1X | \( \left( 4 \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |
| 4C | \( \left( 4 \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |
| 2L | \( \left( \frac{15}{4} \frac{15}{4} \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |
| 1X | \( \left( \frac{15}{4} \frac{15}{4} \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |
| 4C | \( \left( \frac{15}{4} \frac{15}{4} \right) \) | F | \( F_{Lij}^k F_{Lj}^{*m} \phi^{in} \) | \( \phi^{in} : (2, 2, 1) \) | \( \epsilon_{ik} \epsilon_{mn} \delta_{3i}^j \) | 16 |

**TABLE XI:** The two-loop Yukawa contributions to the gauge sector \( \beta \)-functions in Eq. (20) are detailed. The index \( p \) in \( y_{ph} \) labels the gauge groups while \( k \) refers to flavour. In addition to the Higgs bi-doublet from the 10-dimensional representation (whose components are denoted according to the relevant gauge symmetry by \( h \) and \( \phi \)) extra bi-doublet components in \( \overline{26}_H \) (denoted by \( H \) and \( \Phi \)) survives from unification down to the Pati-Salam breaking scale as required by a realistic SM fermionic spectrum. The \( T_a \) factors are the generators of \( SU(4)_C \) in the standard normalization. As a consequence of minimal fine tuning, only one linear combination of 10\(_H \) and \( \overline{26}_H \) doublets survives below the \( SU(4)_C \) scale. The \( U(1)_{R,X} \) mixing in the case \( 2L_{1R}1Xc \) is explicitly displayed.
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