Conformal tensors via Lovelock gravity

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Abstract
Constructs from conformal geometry are important in low dimensional gravity models, while in higher dimensions the higher curvature interactions of Lovelock gravity are similarly prominent. Considering conformal invariance in the context of Lovelock gravity leads to natural, higher curvature generalizations of the Weyl, Schouten, Cotton and Bach tensors, with properties that straightforwardly extend those of their familiar counterparts. As a first application, we introduce a new set of conformally invariant gravity theories in \( D = 4k \) dimensions, based on the squares of the higher curvature Weyl tensors.

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1. Introduction

In this paper we introduce conformally invariant tensors, and related quantities, associated with the higher curvature interaction terms of Lovelock gravity theories [1]. This study can be motivated in the following way. The standard constructs of conformal geometry play key roles in a number of interesting gravity models. For example, the gravitational Chern–Simons interaction in \( D = 3 \) topologically massive gravity [2] contributes a term to the field equations that is proportional to the Cotton tensor, whose vanishing is a necessary and sufficient condition for conformal flatness. A second example in \( D = 3 \) is new massive gravity [3], where the key higher curvature interaction involves the Schouten tensor (see e.g. the discussion in [4]), which characterizes the difference between the Riemann and Weyl tensors. In \( D = 4 \) renewed attention has been drawn [5] to conformal \((\text{Weyl})^2\) gravity, where the equations of motion are proportional to the Bach tensor, which vanishes on all metrics that are conformally related to an Einstein metric.

Lovelock theories are extensions of general relativity to dimensions greater than 4 that include a certain limited set of higher curvature interactions. Lovelock theories are distinguished amongst the much larger class of all higher curvature theories by having field equations that depend only on powers of the Riemann tensor, and not on its derivatives. Importantly, Lovelock theories also have constant curvature vacua with ghost-free excitation spectra [6], although not all vacua share this property [7]. Furthermore, many properties of black hole solutions to Lovelock theories, including explicit solutions in the simplest
nontrivial cases, are accessible through analytic means [7–13] (see the reviews [14–16] for further references). These favorable properties of Lovelock theories have led to their use as testing grounds in studies of a wide variety of physical phenomena, such as brane world models (see [14] and references therein), holographic fluid dynamics (see the review [17] and references therein), applications of gauge/gravity duality to condensed matter systems [18–23] and entanglement entropy [24–26].

There is substantial interest to find gravitational models in higher dimensions that realize at least some of the key features of the lower dimensional models mentioned in the first paragraph. For example, certain properties of \( D = 3 \) new massive gravity have been successfully extended to higher dimensions in the ‘quasi-topological gravity’ models of [27–29], while a \( D = 6 \) conformal gravity theory was constructed in [30] that has all conformally Einstein metrics as solutions, as in \( D = 4 \) (Weyl)\(^2\) gravity. The (Weyl)\(^2\) interaction term is also a key ingredient in critical gravity models [31, 32], which fine-tune higher curvature couplings to make the added gravitational degrees of physically acceptable.

Given the extensive use made of conformal tensors in low dimensional gravity models, and the similar importance of Lovelock interactions in higher dimensions, it makes sense to try to combine the two and obtain new tools for gravitational model building. We will see that such a combination, in fact, arises quite naturally. The \( k \)th order Lovelock interaction term has associated with it a conformally invariant tensor first introduced by Kulkarni in the early 1970’s [33], which we call the Weyl\(^{(k)}\) tensor, that is likewise \( k \)th order in the curvature. We will see that Schouten\(^{(k)}\) and Cotton\(^{(k)}\) tensors can naturally be included in this construction as well. All have properties that straightforwardly generalize those of their familiar \( k = 1 \) counterparts. As a first application of these new tools, we explore the properties of a new conformal (Weyl)\(^{(k)}\)\(^2\) gravity theory in \( D = 4k \), which has the vanishing of a Bach\(^{(k)}\) tensor as its equation of motion and is solved by a generalized class of Einstein\(^{(k)}\) metrics.

The paper is organized as follows. In section 2, we present the higher curvature interactions of Lovelock gravity theories in terms of Riemann\(^{(k)}\) tensors. These are anti-symmetrized products of Riemann tensors that satisfy symmetries and Bianchi identities that naturally extend those satisfied by the Riemann tensor itself. Section 3 reviews the definitions, important properties, and inter-relations between certain basic constructs from conformal geometry, the Weyl, Schouten and Cotton tensors. Section 4 presents key features of \( D = 4 \) conformal (Weyl)\(^2\) gravity, including the statement of the equations of motion in terms of the Bach tensor. In section 5, we define (following [33]) the higher curvature Weyl\(^{(k)}\) tensor as the trace free piece of the Riemann\(^{(k)}\) tensor and study its properties, along with those of the associated Shouten\(^{(k)}\) and Cotton\(^{(k)}\) tensors. In section 6, we present (Weyl)\(^{(k)}\)\(^2\) gravity which is conformally invariant in \( D = 4k \), and has solutions that generalize the Einstein metrics of the \( k = 1 \) case. Finally, section 7 provides brief concluding remarks and directions for future investigation.

2. Lovelock gravity and Riemann\(^{(k)}\) tensors

In Lovelock gravity [1] a particular set of higher curvature interactions terms is added to general relativity. The action is given by

\[
S = \int d^Dx \sqrt{-g} \sum_{k=0}^{k_n} a_k \mathcal{R}^{(k)}
\]

with the scalar \( \mathcal{R}^{(k)} \) being \( k \)th order in the curvature. In particular, one has \( \mathcal{R}^{(0)} = 1 \) giving the cosmological term, \( \mathcal{R}^{(1)} = R \) the Einstein term, and \( \mathcal{R}^{(2)} = \frac{1}{6} (R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) \) the
Gauss–Bonnet term. The scalars $\mathcal{R}^{(k)}$ are dimensionally continued Euler densities [34]. The integral of $\mathcal{R}^{(k)}$ over a closed manifold in $D = 2k$ is proportional to its topologically invariant Euler characteristic. It contributes to the dynamics of Lovelock gravity only when ‘dimensionally continued’ to $D > 2k$. This fixes the upper bound $k_D = ([D - 1]/2)$ on the interaction terms contributing to the action. Finally, the constants $a_k$ appearing in (1) are couplings having dimensions $k^{D-k}$ respectively.

In the case $k = 1$, the scalar curvature can be written as a double trace of the Riemann tensor $R = R_{ab}^{\ \ \ \ \ b}$. A similar expression exists for all the scalars $\mathcal{R}^{(k)}$ with $k \geq 1$, in terms of a totally anti-symmetrized product of $k$ Riemann tensors, which we will call the Riemann tensor, given by

$$\mathcal{R}^{(k)}_{[a_1...a_k]b_1...b_2} \equiv R_{[a_1b_1]}^{\ c_{d_1}c_{d_2}...\ [a_kb_k]}.$$

The Riemann tensor is simply the Riemann tensor itself. General properties of these tensors were studied in [33] and their importance in Lovelock gravity was explored in [35] (see also [36, 37]). The Riemann tensor vanishes identically for $D < 2k$, as a consequence of the anti-symmetrizations in (2). For $D \geq 2k$ its symmetries straightforwardly generalize those of the Riemann tensor,

$$\mathcal{R}^{(k)}_{[a_1...a_2b_1...b_2]} = \mathcal{R}^{(k)}_{[a_1...a_2b_1...b_2]} = \mathcal{R}^{(k)}_{[a_1...a_2][b_1...b_2]} = \mathcal{R}^{(k)}_{b_1...b_2a_1...a_2}$$

and it likewise satisfies algebraic and differential Bianchi identities analogous to the $k = 1$ case,

$$\nabla_v \mathcal{R}^{(k)}_{[a_1...a_2b_1]} b_2...b_a = 0,$$

One can take the simultaneous trace of a Riemann tensor over $2k - n$ pairs of indices with $0 \leq n \leq 2k$ giving the quantities $\mathcal{R}^{(k)}_{[a_1...a_n]b_1...b_2} = \mathcal{R}^{(k)}_{[a_1...a_n]c_1...c_{2k-n}}$. The scalar Lovelock interactions $\mathcal{R}^{(k)}$ in (1) are obtained by contracting over all $2k$ sets of indices. The equations of motion obtained by varying the Lovelock action (1) can be written as a sum

$$\sum_{k=0}^{2D-2} a_k G^{(k)b}_a b = 0,$$

where the Einstein-like at $k$th order in the curvature is given by $G^{(k)b}_a = k \mathcal{R}^{(k)b} - (1/2) \delta^{(k)} b_a R^{(k)}$. The conservation law $\nabla_v G^{(k)a}_b = 0$ for these tensors follows from contracting $2k$ pairs of indices in the Bianchi identity (5) for the Riemann tensor.

Riemann tensors have interesting properties in ‘relatively’ low dimensions’ [35]. It is clear from (2) that in $D = 2k$ the Riemann tensor has only a single independent component and is therefore fully determined by the scalar $\mathcal{R}^{(k)}$. It is similarly determined in terms of traces (over fewer indices) for all $D < 4k$. In particular, in $D = 2k + 1$ which is the lowest dimension where the interaction $\mathcal{R}^{(k)}$ is nontrivial, the Riemann tensor is determined by the Ricci-like tensor $\mathcal{R}^{(k)}_{\ a b}$. In this case, it is interesting to consider the ‘pure’ $k$th order Lovelock theory such that only $a_k \neq 0$. The equations of motion for this theory are $\mathcal{R}^{(k)b}_a = 0$, which in turn implies that the Riemann tensor vanishes. For $k = 1$, this reduces to the well known behavior of Einstein gravity in $D = 3$, where all solutions to the equations of motion are locally flat [38]. We can say that all solutions to pure $k$th order Lovelock gravity in $D = 2k + 1$ are Riemann flat or simply ‘$k$-flat’.

There turn out to be interesting $k$-flat spacetimes that are not flat [35]. The static spherically symmetric solutions to pure $k$th order Lovelock gravity in $D = 2k + 1$ are characterized by missing solid angle

$$d^2 s^2_{2k+1} = -dt^2 + dr^2 + r^2 d\Omega^2_{2k-1}.$$
Weyl, Schouten and Cotton tensors

We are interested in finding higher curvature, Lovelock analogues of the conformally invariant, trace free Weyl tensor and certain related geometric constructs, namely the Schouten, Cotton and Bach tensors. In this section we will review the definitions and basic properties of these tensors given in terms of the Schouten tensor

\[ S_a^b = \frac{1}{D-2} \left( R_a^b - \frac{1}{2(D-1)} g_a^d R_d^b \right). \]  

The Weyl tensor shares the symmetries of the Riemann tensor, \( W_{abcd} = W_{[abcd]} = W_{[abc]d] = W_{[abcd]} = W_{cdab} \), and also satisfies the algebraic Bianchi identity \( W_{[abc]d] = 0 \).

From the differential Bianchi identity \( \nabla_{\left(\alpha\right)} R_{\left(\beta\right)\gamma}^{\left(\delta\right)} = 0 \) it follows that the Weyl tensor satisfies \( \nabla_{\left(\alpha\right)} W_{\left(\beta\right)\gamma}^{\left(\delta\right)} = -4\nabla_{\left(\alpha\right)} S_{\left(\beta\right)\gamma}^{\left(\delta\right)} \). Further relations are obtained by taking traces. Contracting one pair of indices gives a formula for the divergence of the Weyl tensor \( \nabla_a W_{ab}^{cd} = (D-3)C_{ab}^{\cd} - 2C_{[a]d} \delta_{b}^{d} \) where the Cotton tensor is defined as the curl of the Schouten tensor

\[ C_{ab}^{d} = 2\nabla_{\left(\alpha\right)} S_{\left(\beta\right)b}^{\left(\delta\right)} \].

Further contracting a second pair of indices shows that the Cotton tensor is traceless \( C_{ab}^b = 0 \). Taking this into account, the divergence of the Weyl tensor then reduces to

\[ \nabla_a W_{ab}^{cd} = (D-3)C_{ab}^{d}. \]

The Cotton tensor itself can be shown to be divergenceless \( \nabla_a C_{ab}^{d} = 0 \). As noted in the introduction, the Schouten and Cotton tensors play important roles in three dimensional massive gravity theories [2, 3].

The Weyl tensor is defined only for \( D \geq 3 \), as one sees from the expression for the Schouten tensor. Moreover, in \( D = 3 \) the Weyl tensor vanishes identically. This can be seen from considering the relation \( \delta_{\left(\alpha\right)\beta\gamma}^{\left(\delta\right)} R_{\left(\alpha\right)\beta\gamma}^{\left(\delta\right)} W_{\left(\alpha\right)\gamma}^{\left(\beta\right)d} = (1/6)W_{\left(\alpha\right)\beta\gamma}^{\left(\delta\right)d} \) which is valid in any dimension, following from the tracelessness of the Weyl tensor. In \( D = 3 \) the left hand side necessarily vanishes due to the anti-symmetrization over four coordinate indices, giving the result\(^1\). The fact that the curvature tensor has no trace free piece in \( D \leq 3 \) is due respectively to the absence of curvature in \( D = 1 \), and that the Riemann tensor is determined by its traces in \( D = 1 \) and \( D = 2 \).

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1. The multi-index delta symbol, which we will make extensive use of below, denotes an anti-symmetrized product of Kronecker delta functions with overall unit strength, so that \( \delta_{a_1 \ldots a_k} = \delta_{b_1 \ldots b_k} = \delta_{c_1 \ldots c_k} = \delta_{d_1 \ldots d_k} = \delta_{e_1 \ldots e_k} \).

2. This is an example of the method given by Lovelock in [39] for proving what he called ‘dimension dependent identities’.

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Class. Quantum Grav. 30 (2013) 195006
An efficient way to demonstrate the conformal invariance of the Weyl tensor is presented in [33]. Consider a tensor $A_{abcd}$ with the algebraic symmetries of the Riemann tensor

$$A_{abcd} = A[a \ b \ c \ d] = A_{[ab}[cd] = A_{cd}[ab]$$

and denote its traces by $A^c = A_{ab}^\ c$ and $A = A^\ a$. Its tracefree part is given by

$$A^{(t)cd} = A_{abcd} - \frac{4}{D - 2} \delta^{[c}[A_{b]}^{d]} + \frac{2}{(D - 1)(D - 2)} \delta^{cd} A,$$

which reproduces the Weyl tensor if $A_{abcd} = R_{abcd}$. If we take a tensor with the pure trace form $A_{abcd} = \delta^{[c}[A_{b]}^{d]}$ where $A_{ab} = \Lambda_{ab}$, then this satisfies the algebraic symmetries of the Riemann tensor, and one finds that its traceless part $A^{(t)cd}$ vanishes identically. Now consider also the Riemann tensor of a conformally related metric $\tilde{g}_{ab} = e^{2f} g_{ab}$. This is related to the Riemann tensor of the original metric according to

$$\tilde{R}_{ab}^{cd} = e^{-2f} \left( R_{ab}^{cd} + \delta^{[c}[\Lambda_{b]}^{d]} \right)$$

where $\Lambda_a^b = 4\nabla_a \nabla^b f + 4(\nabla_a f)\nabla^b f - 2\delta^b_a (\nabla_c f)\nabla^c f$ and one sees that $\Lambda_{ab}$ is symmetric. By the reasoning presented above, the second term on the right-hand side of (12) will not contribute to the Weyl tensor for the transformed metric, which is therefore given by

$$\bar{W}_{ab}^{cd} = e^{-2f} W_{ab}^{cd}.$$

The conformal transformation of the Cotton tensor, derived in the appendix, is given by

$$\bar{C}_{ab}^c = e^{-2f}(C_{ab}^c - W_{ab}^{de} \nabla_d f).$$

One sees that the Cotton tensor is conformally invariant if the Weyl tensor vanishes. This property underlies the special significance of the Cotton tensor in $D = 3$ where the Weyl tensor vanishes identically.

The Weyl tensor vanishes for flat spacetime and therefore also in spacetimes that are conformally flat. It follows that $W_{ab}^{cd} = 0$ is a necessary condition for a spacetime to be conformally flat. In dimensions $D \geq 4$, it can be shown that this condition is also a sufficient. Lower dimensions are special cases. As we have seen, the Weyl tensor vanishes identically in $D = 3$. However, it then follows from (14) that the Cotton tensor is conformally invariant, and it can be shown that $C_{ab}^c = 0$ is both a necessary and sufficient condition for conformal flatness in $D = 3$. It is well known that all metrics are locally conformally flat in $D = 2$, while all metrics are flat in $D = 1$.

4. Conformal (Weyl)$^2$ gravity in $D = 4$

There is a unique conformally invariant gravity theory in $D = 4$ with action given by the square of the Weyl tensor

$$S = \int d^4 x \sqrt{-g} W_{ab}^{cd} W_{cd}^{ab}.$$

The equations of motion for this theory may be written as $B_a^b = 0$ where

$$B_a^b = (\nabla^a \nabla^b f + \frac{1}{4} R^c_{ab}) W_{cd} W_{cd}^{bc}$$

is the Bach tensor, which is symmetric, traceless and transforms as $\tilde{B}_a^b = e^{-4f} B_a^b$ under a conformal transformation $\tilde{g}_{ab} = e^{2f} g_{ab}$. It follows that if a given metric satisfies the equations of motion, then so do conformally related metrics. Writing the Bach tensor in the form given in (16) requires a quadratic identity for the Weyl tensor which holds only in $D = 4$

$$W_{cd}^{be} W_{ae}^{cd} = \frac{1}{2} \delta_{[e}[W_{cd}^{\ c]} W_{ef}^{\ d]}$$

(17)
This is another example of a dimension dependent identity [39]. It is proved by considering the quantity $\delta_{ab}W_{cd}W_{ef}W_{gh} = \frac{1}{5} \left( \frac{\delta_{a}W_{cd}W_{ef}W_{gh} - 4W_{ac}^eW_{cd}^h}{} \right)$ which vanishes identically in $D = 4$ due to the anti-symmetrization over five coordinate indices on the left-hand side.

It is straightforward to check that the equations of motion $B_a^b = 0$ are solved by any metric having $R_{ab} = \alpha g_{ab}$ with $\alpha$ constant. These are known as Einstein metrics. First one sees that with this form for the Ricci tensor the contraction of conformal tensors in the last section, all extend to the general case of Riemann tensors. The symmetry and other properties of the Riemann tensor, which entered the discussion of the Weyl tensor, which vanishes by definition. The remaining term in (16) involves the divergence of the Weyl tensor and can be rewritten in terms of the Cotton tensor via equation (10). However, the Cotton tensor (9) vanishes for Einstein metrics, completing the proof. It has been shown recently [5] for asymptotically deSitter spacetimes (or Euclidean asymptotically AdS spaces) that the set of Einstein metrics may be selected out of the full set of solutions to conformal gravity by means of boundary conditions in the asymptotic region. Non-conformally-Einstein solutions to conformal gravity in $D = 4$ have also recently been studied [41].

5. Lovelock counterparts of the Weyl, Schouten and Cotton tensors

We now consider conformal tensors related to the higher curvature Lovelock interactions.

The symmetry and other properties of the Riemann tensor, which entered the discussion of conformal tensors in the last section, all extend to the general case of Riemann tensors with $k \geq 1$. It is therefore natural to continue the Riemann tensors of the Weyl, Schouten and Cotton tensors. We will see that the Weyl tensors with $k \geq 1$ are conformally invariant and that one can further define at each higher curvature order Schouten tensors, Cotton tensors and Bach tensors that have properties and inter-relations analogous to their $k = 1$ counterparts.

The Weyl tensors were first constructed by Kulkarni in [33]. Kulkarni began by considering tensors $A_{a_1\ldots a_n}^{b_1\ldots b_k}$ that have the algebraic symmetries in (3) characteristic of the Riemann tensors. The associated trace free tensor then has the form

$$A_{a_1\ldots a_n}^{b_1\ldots b_k} = \sum_{p=1}^{n} \alpha_p A_{[a_1\ldots a_p]}^{b_1\ldots b_k} A_{a_{p+1}\ldots a_n}$$

(18)

where $A_{a_1\ldots a_n}^{b_1\ldots b_k} \equiv A_{a_1\ldots a_p'}^{b_1\ldots b_k} A_{a_{p+1}\ldots a_n}^{a_{p+1}'} \cdots a_{n-1}'$ with $0 \leq m < n$ are (multiple) traces of the original tensor. By requiring that the trace of $A_{a_1\ldots a_n}^{b_1\ldots b_k}$ should vanish, one finds that the coefficients in (18) are given recursively by the formula

$$\alpha_{p+1} = - \frac{(n-p)^2}{(p+1)(D+p-2(n-1))} \alpha_p$$

(19)

with $\alpha_0 \equiv 1$. In order to establish the conformal invariance of the Weyl tensor, defined below, it is important to note that if a tensor $A_{a_1\ldots a_n}^{b_1\ldots b_k}$ has the pure trace form

$$A_{a_1\ldots a_n}^{b_1\ldots b_k} = \delta^{b_1\ldots b_k}_{[a_1\ldots a_n]}$$

(20)

then its trace free part $A_{a_1\ldots a_n}^{b_1\ldots b_k}$ necessarily vanishes. The Weyl tensor [33] is now defined as the traceless part of the Riemann tensor, so that one has

$$\mathcal{W}_{a_1\ldots a_2}^{b_1\ldots b_2} = R_{a_1\ldots a_2}^{b_1\ldots b_2}$$

(21)

and

$$\mathcal{W}_{a_1\ldots a_2}^{a_3\ldots a_2} = \mathcal{W}_{a_1\ldots a_2}^{b_1\ldots b_2} + \sum_{p=1}^{2k} \alpha_p \delta^{b_1\ldots b_k}_{[a_1\ldots a_p]} \mathcal{W}_{a_{p+1}\ldots a_2}^{a_3\ldots a_2}$$

(22)

3 See also the earlier related results in [40].
To understand the properties of the Weyl\(^{(k)}\) tensors, it will be helpful to evaluate the recursive formula (19) for the coefficients \(\alpha_p\) to obtain the explicit expressions

\[
\alpha_p = \frac{(2k)!}{(2k-p)!} \frac{(-1)^p (D - (4k - 1))!}{p!(D - (4k - p - 1))!}.
\]

For \(k = 1\) these formulas correctly reproduce the ordinary Weyl tensor. Some basic properties of the Weyl\(^{(k)}\) tensors are as follows. Singular factors in the coefficients \(\alpha_p\) indicate that the Weyl\(^{(k)}\) tensor is defined only for \(D \geq 4k - 1\). Moreover, one can show that the Weyl\(^{(k)}\) tensor vanishes identically in \(D = 4k - 1\) by considering the relation

\[
\delta_{b_1...b_2c_1...c_2} W^{(k)}_{a_1...a_2} d_{a_1...a_2} = \frac{(2k)!}{4k!} W^{(k)}_{a_1...a_2} b_1...b_2
\]

which holds in any dimension. The left hand side, however, vanishes identically in \(D = 4k - 1\) due to the anti-symmetrization over a total of 4\(k\) coordinate indices, leading to the result. Hence the Weyl\(^{(k)}\) tensor is non-trivial only for \(D \geq 4k\). This result also follows from the observations in [35] that the Riemann\(^{(4)}\) vanishes identically for \(D < 2k\), is fully determined by its traces for \(2k \leq D < 4k\), and hence has a non-trivial tracefree part only in dimensions \(D \geq 4k\).

The conformal properties of Weyl\(^{(k)}\) tensors were also established in [33]. It follows from the definition of the Riemann\(^{(k)}\) tensor (2) together with the conformal transformation of the Riemann tensor in (12), that the action of a conformal transformation on the Riemann\(^{(k)}\) is given by

\[
\tilde{R}^{(k)}_{a_1...a_2} b_1...b_2 = e^{-2k/2} (R^{(k)}_{a_1...a_2} b_1...b_2 + \delta^{[b_1}_{a_1} \Lambda_{a_2...a_2]} b_2...b_2)
\]

where \(\Lambda_{a_1...a_2...b_1...b_2...} \) is a tensor having the symmetries

\[
\Lambda_{a_1...a_2...b_1...b_2...} = \Lambda_{[a_1...a_2...]} b_1...b_2... = \Lambda_{a_1...a_2...[b_1...b_2...] = \Lambda_{b_1...b_2...a_1...a_2...}
\]

whose precise form will not be needed. It then follows from an argument given above that the second term on the right-hand side of (25) does not contribute to the trace free, transformed Weyl\(^{(k)}\) tensor, and that therefore one has

\[
\tilde{W}^{(k)}_{a_1...a_2} b_1...b_2 = e^{-2k/2} W^{(k)}_{a_1...a_2} b_1...b_2.
\]

We see then that the Weyl\(^{(k)}\) tensors constitute an interesting class of conformally invariant tensors in higher dimensions that are naturally associated with the geometric constructs of Lovelock gravity theories.

We now continue the investigation of higher order conformal tensors beyond the point where [33] leaves off, introducing Schouten\(^{(k)}\) and Cotton\(^{(k)}\) tensors that also generalize the \(k = 1\) case. It will follow from the Bianchi identity (5) for the Riemann\(^{(k)}\) tensor, that the Weyl\(^{(k)}\), Schouten\(^{(k)}\) and Cotton\(^{(k)}\) tensors with \(k > 1\) are inter-related in the same manner as their ordinary \(k = 1\) counterparts. We may begin by writing the Weyl\(^{(k)}\) tensor as

\[
\mathcal{Y}^{(k)}_{a_1...a_2} b_1...b_2 = R^{(k)}_{a_1...a_2} b_1...b_2 - (2k)^2 \delta^{[b_1}_{a_1} S_{a_2...a_2]} b_2...b_2
\]

in terms of the Schouten\(^{(k)}\) tensor

\[
S^{(k)}_{a_1...a_2...} b_1...b_2... = \frac{1}{(2k)^2} \sum_{p=0}^{2k-1} \alpha_p \delta^{[b_1}_{a_1} R^{(k)}_{a_2...a_2]} b_{p+1...b_2...1}
\]

where the coefficients are those given in (23). The Schouten\(^{(k)}\) tensors have the symmetries

\[
S^{(k)}_{a_1...a_2...} b_1...b_2...1 = S^{(k)}_{a_1...a_2...} b_1...b_2...1 = S^{(k)}_{a_1...a_2...} b_1...b_2...1 = S^{(k)}_{b_1...b_2...a_1...a_2...}
\]

7
The Cotton\(^{(k)}\) tensor can now be defined in analogy with equation (9) as the curl of the Schouten\(^{(k)}\) tensor
\[
C^{(k)}_{\alpha_1\ldots\alpha_k} = 2k \nabla_{[\alpha_1} c_{\alpha_2\ldots\alpha_k]}^{(k)} = 2k \nabla_{[\alpha_1} b_{\alpha_2\ldots\alpha_k]}^{(k)}. \tag{31}
\]

Due to the Bianchi identity (5) for the Riemann\(^{(k)}\) tensor the curl of the Weyl\(^{(k)}\) tensor is determined by the Cotton\(^{(k)}\) tensor
\[
\nabla_{\alpha_1} W_{\alpha_2\ldots\alpha_k}^{(k)} = -2k C_{\alpha_1\alpha_2}^{(k)} [b_{\alpha_3\ldots\alpha_k} - \delta_{\alpha_3\ldots\alpha_k}].
\]

Contracting one set of upper and lower indices then leads to a relation for the divergence of the Cotton\(^{(k)}\) tensor
\[
\nabla_{\alpha_1} W_{\alpha_2\ldots\alpha_k}^{(k)} c_{\alpha_1\ldots\alpha_k} = (D - (4k - 1)) C_{\alpha_1\alpha_2}^{(k)} b_{\alpha_3\ldots\alpha_k} - (2k)(2k - 1) c_{\alpha_1\alpha_2}^{(k)} c_{[b_{\alpha_3\ldots\alpha_k} - \delta_{\alpha_3\ldots\alpha_k}]} \tag{32}
\]

The trace of the Cotton\(^{(k)}\) tensor, which comes into the second term on the right hand side, can ultimately be seen to vanish by taking further traces of equation (32) giving
\[
\nabla_{\alpha_1} W_{\alpha_2\ldots\alpha_k}^{(k)} c_{\alpha_1\ldots\alpha_k} = (D - (4k - 1)) C_{\alpha_1\alpha_2}^{(k)} b_{\alpha_3\ldots\alpha_k} - (2k)(2k - 1) c_{\alpha_1\alpha_2}^{(k)} c_{[b_{\alpha_3\ldots\alpha_k} - \delta_{\alpha_3\ldots\alpha_k}]} \tag{33}
\]

The factor of \(D - (4k - 1)\) on the right hand side ensures the consistency of the Cotton\(^{(k)}\) tensor being non-vanishing in \(D = 4k - 1\), while the Weyl\(^{(k)}\) tensor vanishes identically, but the Cotton tensor can be non-vanishing. Finally, the conformal transformation of the Cotton\(^{(k)}\) tensor is found in the appendix to be given by
\[
\tilde{C}_{\alpha_1\alpha_2}^{(k)} b_{\alpha_3\ldots\alpha_k} = e^{-2k f} C_{\alpha_1\alpha_2}^{(k)} b_{\alpha_3\ldots\alpha_k} - W_{\alpha_1\alpha_2}^{(k)} b_{\alpha_3\ldots\alpha_k} \nabla_{\alpha_1} f \tag{34}
\]

which generalizes equation (14). We see from this that the Cotton\(^{(k)}\) tensor is conformally invariant if the Weyl\(^{(k)}\) tensor vanishes. In particular, this will be the case in \(D = 4k - 1\) where the Weyl\(^{(k)}\) tensor vanishes identically.

The property of conformal flatness is of considerable interest in both physical and mathematical contexts. As noted above \(W_{\alpha_1\alpha_2}^{(k)} = 0\) is a necessary and sufficient condition for conformal flatness in dimensions \(D \geq 4\), while the condition \(b_{\alpha_3\ldots\alpha_k} = 0\) plays a similar role in \(D = 3\). We have defined the property of \(k\)-flatness above as the vanishing of the Riemann\(^{(k)}\) tensor and discussed some examples of spacetimes that are \(k\)-flat for \(k > 1\), without being flat.

With corresponding ‘conformal \(k\)-flat’ tensors now in hand, we can speculate about conditions for ‘conformal \(k\)-flatness’, whether a given spacetime may be related to a \(k\)-flat one via a conformal transformation. It is obvious that vanishing of the Weyl\(^{(k)}\) tensor is a necessary condition for \(k\)-flatness in dimension \(D \geq 4\). It is natural to conjecture that this is also a sufficient condition. A similar conjecture regarding conformal \(k\)-flatness in \(D = 4k - 1\) and the vanishing of the Cotton\(^{(k)}\) tensor seems natural as well.

The analogue of \(D = 2\), where all geometries are locally conformally flat, appears to be \(D = 2k\). In this case the Riemann\(^{(k)}\) tensor has a single nontrivial component, and one can envision this as being set to zero by the single degree of freedom in a conformal transformation. The range of dimensions \(2k < D < 4k - 1\), however, is empty in the \(k = 1\) case, and there is no obvious conjecture to make regarding necessary and sufficient conditions for conformal \(k\)-flatness for dimensions in this range. The Riemann\(^{(k)}\) tensor is determined by its traces in these dimensions, but none of the conformal tensors introduced above are defined. It is intriguing to consider generalizing the proofs regarding conformal flatness for \(D = 2, D = 3\) and \(D \geq 4\) to the conformally \(k\)-flat case with \(k > 1\), but we will not pursue this here.

Finally, it is useful to note that the Weyl\(^{(k)}\) tensor may be alternatively expressed in terms of the ordinary Weyl tensor by considering the anti-symmetrized product
\[
W_{\alpha_1\alpha_2 \ldots \alpha_k}^{(k)} = W_{[\alpha_1 \alpha_2] \ldots \alpha_k}. \tag{35}
\]

This quantity is not itself tracefree. However, it differs from the Riemann\(^{(k)}\) tensor only by a pure trace term and hence its tracefree part will be identical to the Weyl\(^{(k)}\) tensor.
6. Conformal (Weyl\(^{(k)}\))\(^2\) gravity in \(D = 4k\)

There is no longer a unique conformally invariant gravitational action in higher dimensions \([42]\). A unique theory was singled out in \(D = 6\), from the full three parameter family of conformal gravities, by additionally requiring that it admit all conformally Einstein spacetimes as solutions \([30]\). Such a procedure could be carried out in still higher dimension as well, although there is no guarantee that this would produce a unique result. Here, we will present a different set of higher dimensional conformal gravity theories, distinguished by their close association with Lovelock theories.

In section 3 we discussed (Weyl)\(^2\) gravity, which is the unique conformally invariant gravity theory in \(D = 4\). Similar conformal gravity theories may be constructed in \(D = 4k\) dimensions using the square of the Weyl\(^{(k)}\) tensors, with actions given by

\[
S = \int d^{4k}x \sqrt{-g} \mathcal{W}_{a_1...a_{2k}}^{(k)} b_{1...b_{2k}} \mathcal{W}_{b_1...b_{2k}}^{(k)} a_{1...a_{2k}}.\]

(36)

Under a conformal transformation \(\tilde{g}_{ab} = e^{2f} g_{ab}\), the two factors of the Weyl\(^{(k)}\) tensor in (36) will together pick up a factor of \(e^{4k\bar{f}}\), while the volume element in \(D = 4k\) transforms by a compensating factor of \(e^{4k\bar{f}}\), yielding a conformally invariant action. One finds that the equation of motion for this theory can be written in terms of the Bach\(^{(k)}\) tensor,

\[
\mathcal{B}_{a}^{(k)b} = \left( R_{b_1...b_{2k}} - d_{b_1...b_{2k}} \nabla d_{b_{2k-1}} \nabla_{b_{2k-1}} + \frac{k}{2} R_{b_1...b_{2k-1}} d_{b_{2k-1}} \right) \mathcal{W}_{a_1...a_{2k}}^{(k)} b_{c_1...c_{2k}}\]

(37)

which reduces for \(k = 1\) to the ordinary Bach tensor (16). Note that the expression for \(\mathcal{B}_{a}^{(k)b}\) which one obtains by lowering an index will not be manifestly symmetric. Strictly speaking, we have shown that the equation of motion is that the symmetric part of the Bach\(^{(k)}\) tensor should vanish. In the \(k = 1\) case, it can be shown that the Bach tensor is, in fact, symmetric (see e.g. \([43]\)) and that the equation of motion for \(D = 4\) (Weyl)\(^2\) gravity is simply \(\mathcal{B}_{a}^{(k)b} = 0\). For \(k > 1\), we similarly expect, but have not yet been able to show, that the Bach\(^{(k)}\) tensor is symmetric and that the equations of motion of \(D = 4k\) (Weyl\(^{(1)}\))\(^2\) gravity are simply \(\mathcal{B}_{a}^{(k)b} = 0\). In the following, we will assume that this is the case.

Writing the equation of motion in this form requires using a quadratic identity for the Weyl\(^{(k)}\) tensor that holds in \(D = 4k\), which is obtained by considering the

\[
\delta_{a_1...a_{2k}b_1...b_{2k}}^{c_1...c_{2k}} \mathcal{W}_{a_1...a_{2k}}^{(k)} b_{1...b_{2k}} \mathcal{W}_{c_1...c_{2k}}^{(k)} d_{1...d_{2k}}.
\]

(38)

On one hand, this quantity vanishes identically in \(D = 4k\). On the other hand we can evaluate it explicitly in all dimensions, and thereby arrive at the identity

\[
\mathcal{W}_{a_1...a_{2k}}^{(k)} b_{1...b_{2k}} \mathcal{W}_{c_1...c_{2k}}^{(k)} d_{1...d_{2k}} = 4k \mathcal{W}_{a_1...a_{2k}}^{(k)} b_{1...b_{2k}} \mathcal{W}_{c_1...c_{2k}}^{(k)} b_{1...b_{2k}}
\]

(39)

generalizing the quadratic identity (17) for the Weyl tensor in \(D = 4\).

We have seen that all Einstein metrics solve the equations of motion of (Weyl)\(^2\) gravity in \(D = 4\). The generalization of this statement to all values of \(k \geq 1\) is straightforward. The Ricci tensor for an Einstein metric has the constant form \(R_{ab} = \alpha g_{ab}\). To obtain solutions to \(\mathcal{B}_{a}^{(k)b} = 0\), the correct generalization is to take the constant form for the first trace of the Riemann\(^{(k)}\) tensor

\[
R_{a_1...a_{2k-1}}^{(k)} b_{1...b_{2k-1}} = \alpha g_{a_1...a_{2k-1}}^{b_1...b_{2k-1}}.
\]

(40)

The term in (37) involving the contraction of this tensor with the Weyl\(^{(k)}\) tensor will then be proportional to a multiple trace of the Weyl\(^{(k)}\) tensor, which vanishes by definition. The remaining term in (37) can be rewritten in terms of the Cotton\(^{(k)}\) tensor, which is seen to vanish by noting that (40) implies that the Schouten\(^{(k)}\) tensor has a similar constant form. With the perspective
of \((\text{Weyl}^{(k)})^2\) in \(D = 4k\) in mind, it seems reasonable to call spacetimes satisfying the condition (40) Einstein\((^{(k)})\) metrics. They are generalizations of ordinary Einstein spacetimes in the sense that the first trace of the relevant \(4k\)th order curvature tensor has a constant form.

It will be interesting to explore the solutions to \((\text{Weyl}^{(k)})^2\) gravity in \(D = 4k\) further, including both the space of Einstein\((^{(k)})\) metrics and potential solutions that are not of this form, such as those found in [41] for \(k = 1\). A general result follows from writing the action (36) in terms of contractions of \(2k\) powers of the ordinary Weyl tensor by expressing the Weyl\((^{(k)})\) tensor as the tracefree part of the anti-symmetrized product (35). One then sees that \((\text{Weyl}^{(k)})^2\) gravity in \(D = 4k\) falls within the class of theories considered in [44] for which Birkhoff’s theorem has been shown to hold.

7. Conclusions

We have seen that the geometric structures underlying the higher curvature interactions of Lovelock gravity also provide higher curvature analogues of the tensors of conformal geometry; the Weyl, Schouten, Cotton and Bach tensors. We have explored the properties and inter-relations of the higher curvature Weyl\((^{(k)})\), Schouten\((^{(k)})\), Cotton\((^{(k)})\) and Bach\((^{(k)})\) tensors, and seen that, despite the presence of a cumbersome number of indices, these are straightforward generalizations of familiar results. Our hope is that this set of conformal\((^{(k)})\) tensors will provide new tools for gravitational model building. We have given a first such application by presenting the \(D = 4k\) conformal \((\text{Weyl}^{(k)})^2\) gravity models in section 6. From a physics perspective, some next steps would be to investigate what roles conformal\((^{(k)})\) tensors can play in contexts such as quasi-topological gravity [27–29] and critical gravity models [31, 32]. From a mathematical perspective, there are potentially interesting questions raised in the text relating to \(k\)-flatness and conformal \(k\)-flatness to follow up on.

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Appendix. Conformal transformation of Cotton tensor

The transformation law (14) for the Cotton tensor under a conformal rescaling of the metric can be obtained as follows. The action of the covariant derivative for the conformally rescaled metric on a vector field \(v^a\) is related to the original covariant derivative according to

\[
\tilde{\nabla}_a v^c = \nabla_a v^c + \lambda_{abc} v^b \quad \text{where} \quad \lambda_{abc} = (\nabla_a f) \delta^c_b + (\nabla_b f) \delta^c_a - g_{ab} (\nabla^c f). \]

Note that equation (10) can be equivalently rewritten as

\[
C_{ab}^{\quad d} = -\frac{3}{D-3} \nabla_d [W_{bc}]^{\quad dc}. \quad (A.1)
\]

Writing the transformed Cotton tensor in this way in terms of the transformed Weyl tensor and covariant derivative operator, it then follows that

\[
\tilde{C}_{ab}^{\quad d} = -\frac{3}{D-3} \{ \nabla_d [\tilde{W}_{bc}]^{\quad dc} + 2\lambda_{e[d}^{\quad [c} \tilde{W}_{bc]}^{\quad e]} \}. \quad (A.2)
\]

It also follows straightforwardly from the transformation law for the Weyl tensor that

\[
\nabla_d [\tilde{W}_{bc}]^{\quad dc} = -\frac{1}{2} e^{-2f} ((D-3)C_{ab}^{\quad d} + 2W_{ab}^{\quad dc} \nabla_d f) \quad (A.3)
\]
while making use of the algebraic Bianchi identity $W_{[abc]}^d = 0$ for the Weyl tensor and plugging
in the expression for $\kappa_{ab}^c$ yields
\[
2\kappa_{c[a}^e \tilde W_{e]b}^d = \frac{D - 1}{3} e^{-2f} W_{ab}^{cd} \nabla^c f.
\] (A.4)
These combine to give the result
\[
\tilde C_{a}^b = e^{-2f} (C_{ab}^c - W_{ab}^{cd} \nabla^c f).
\] (A.5)
One sees that if the Weyl tensor vanishes, as it does identically in $D = 3$ or in conformally flat
spacetimes in $D \geq 4$, then the Cotton tensor is conformally invariant. The computation in the
general case proceeds in a similar manner, making use of the algebraic Bianchi identity for the Weyl\[24\] tensor and yielding equation (34).

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