Effect of linear lumping on controllability and observability

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Running head: Lumping, controllability and observability

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Abstract

The effect of linear lumping, linear transformation to reduce the number of state variables on controllability and observability of linear differential equations has been studied. Controllability of the original system implies the controllability of the lumped system. Examples taken from reaction kinetics illustrate our results.

KEY WORDS

completely controllable, lumping matrix, compartmental system, M-matrix

MSC

80A30, 93B17, 34A30

1 Introduction

Dealing with modeling of real questions the large number of variables is generally a problem. To have a model which can easily be treated, one possible way is to reduce the number of variables by a method called lumping [9]. By controllability of a system we mean that it can be brought from any position to any other position in a finite amount of time. On the other hand, observability of a system means that we can determine the initial state of the system from the knowledge of an input-output pair over a certain period of time.

In this paper we study the effect of linear lumping on such properties of the system as controllability and observability and apply the results to compartmental systems. Our main result is that a controllable compartmental system remains controllable under lumping.

2 Basic notions of lumping and control theory

In this section we collect the basis of the mathematical theory of controllability [2, page 16], observability [2, page 26] and lumping [9, page 1534]. Before turning to the formal definitions we mention that controllability means that any prescribed concentration can be attained using an appropriate control input, observability has the meaning that one can reconstruct the history of the process when knowing the present concentration composition.

Let $n, r, p \in \mathbb{N}, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}, C \in \mathbb{R}^{p \times n}$ and let us investigate:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$
$$y(t) = Cx(t) \quad (2)$$
with \( x(t) \) in \( \mathbb{R}^n \) as the dependent variable of the linear differential equations at time \( t \) in \( \mathbb{R} \) and \( u(t) \) in \( \mathbb{R}^r \) denoting the bounded and measurable control function and \( y \) with values in \( \mathbb{R}^p \) the observation function.

The fundamental definitions and the Kalman rank conditions below can be found in any textbook on linear system theory, e.g. in [2].

**Definition 1** A linear system with a state-space description given by (1) is said to be completely controllable if, starting from any position \( x_0 \) in \( \mathbb{R}^n \) at any initial time \( t_0 \), the state vector \( x \) can be brought to any other position \( x_1 \) in \( \mathbb{R}^n \) in a finite amount of time by some control function \( u \).

**Theorem 1** A linear system described by (1) is completely controllable if and only if the \( n \times rn \) matrix
\[
W_{AB} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}
\]
has rank \( n \).

Let \( t_0, t_1 \in \mathbb{R}; t_0 < t_1 \).

**Definition 2** A linear system with the state-space description (1)–(2) has the observability property on an interval \((t_0, t_1)\), if any input-output pair \( (u(t), y(t)) \), \( t_0 \leq t \leq t_1 \), uniquely determines the initial state \( x(t_0) \). Furthermore (1)–(2) is said to be observable at an initial time \( t_0 \) if it has the observability property on some interval \((t_0, t_1)\) where \( t_1 > t_0 \). It is said to be completely observable if it is observable at every initial time \( t_0 \).

**Theorem 2** A linear system described by (1)–(2) is completely observable if and only if the \( n \times pn \) matrix
\[
V_{CA} = \begin{bmatrix} C^\top & A^\top C^\top & \cdots & (A^\top)^{n-1}C^\top \end{bmatrix}
\]
has rank \( n \).

Let us consider
\[
\dot{x}(t) = A^\top x(t) + C^\top v(t) \quad \text{(3)}
\]
\[
z(t) = B^\top x(t) \quad \text{(4)}
\]
with \( v(t) \) in \( \mathbb{R}^p \) denoting the bounded and measurable control function and \( z(t) \) in \( \mathbb{R}^r \) the observation function.

**Theorem 3** The linear systems (1)–(2) and (3)–(4) respectively described above are dual to each other in the sense that (1)–(2) is completely observable if and only if (3) is completely controllable and (1) is completely controllable if and only if (3), (4) is completely observable [2, page 32].

Let us introduce a slightly modified notion of lumpability [7, page 1262].

**Definition 3** Suppose \( l \in \mathbb{N}, l \leq n, \) and \( M \in \mathbb{R}^{l \times n} \), \( \text{rank}(M) = l \). If for all solutions \( x(t) \) to (1)
\[
\dot{x}(t) := M x(t) \quad \text{(5)}
\]
obeys a differential equation

\[ \dot{x}(t) = \hat{A}x(t) + \hat{u}(t) \]  

(6)

with \( \hat{A} \in \mathbb{R}^{l \times l} \), and \( \hat{u}(t) := MBu(t) \) then (1) is said to be exactly lumpable to (6) by \( M \). The pair of matrices \( M \) and \( \hat{A} \) is sometimes referred to as a lumping scheme.

Remark 1 It can be shown that (1) is exactly lumpable to (6) by \( M \) if and only if there exists \( \hat{A} \in \mathbb{R}^{l \times l} \), such that \( \hat{A}M = MA \), so we get the lumping scheme consisting of \( M \) and \( \hat{A} = MAM^\top(MM^\top)^{-1} \) [9]. We note that the inverse \( (MM^\top)^{-1} \in \mathbb{R}^{l \times l} \) exists, since \( \text{rank}(M) = l \) and \( M^\top(MM^\top)^{-1} \) is a generalized inverse of \( M \) satisfying \( MM^\top(MM^\top)^{-1} = I_{l \times l} \) with \( I_{l \times l} \) being the \( l \)-identity matrix. It can be proved that there exists a matrix \( \hat{A} \) such that \( \hat{A}M = MA \) if

\[
M = N \begin{bmatrix} f_1^\top \\ \vdots \\ f_l^\top \end{bmatrix}
\]

where \( f_i \) (\( i = 1, \ldots, l \)) are any independent, real eigenvectors of the matrix \( A \) and the matrix \( N \in \mathbb{R}^{l \times l} \) is nonsingular [1, page 117].

3 Fundamentals

A typical chemical reaction step is written schematically as

\[ \text{Reactant} \rightarrow \text{Product}. \]  

(7)

The reactant and product are each called a complex, which is a set of elements with associated coefficients. The elements that make up complexes are called species, and can be anything that participates in a reaction, typically a chemical element, a molecule, or an electron. The species that are on the left side of the equation are used up, and those on the right are created when the reaction step occurs. The coefficient that a species takes indicates what proportion of it is created or used in the reaction, and by convention it is always a non-negative integer. The goal of chemical reaction theory is to monitor how the concentration of each species changes over time [9, page 1539]. The fundamental constituent of a chemical reaction is the species, the concentration of which we are interested in monitoring. A complex is a sum of species with integer coefficients, and a reaction is a pair of complexes with an ordering (to distinguish between products and reactants). Also a subclass of the class of chemical processes can be modeled as compartmental systems [4], [8]. A compartmental system consists of several compartments with more or less homogeneous amounts of material. The
compartments interact by processes of transport and diffusion. The dynamics of a compartmental system is derived from mass balance considerations. The mathematical theory of compartmental systems is of major importance to pharmacokineticists, physiologists [8]. Sometimes it is useful to reduce a model to get a new one with a lower dimension. The technique’s name is lumping, i.e. reduction of the number of variables by grouping them via a linear or nonlinear function. Very often problems in the physical, and social sciences can be reduced to problems involving matrices which, due to certain constraints, have some special structure [1, page 132].

4 Effect of lumping on controllability and observability

Our main result is expressed in the following statement.

Theorem 4 Let us assume that (1) is exactly lumpable to (6) by $M$ and the linear system (1) is completely controllable, then the linear system (6) is also completely controllable.

Proof. Let (1) be completely controllable then according to Theorem 1 we know that $\operatorname{rank}(W_{AB}) = n$. Furthermore, using the fact that $\hat{A}M = MA, \ldots, \hat{A}^{l-1}M = MA^{l-1}$ we get

$$\hat{W}_{AB} := [MB | \hat{A}MB | \ldots | (\hat{A})^{l-1}MB] = M[B | AB | \ldots | A^{l-1}B].$$

This implies that (6) is completely controllable if and only if the $l \times rl$ matrix

$$M[B | AB | \ldots | A^{l-1}B]$$

has rank $l$ by Theorem 1. Let us assume that the rank of $\hat{W}_{AB}$ is less than $l$, then there is a nonzero vector $b \in \mathbb{R}^l$ with $b^T \hat{W}_{AB} = 0 \in \mathbb{R}^{rl}$ or equivalently $b^T MB = b^T \hat{A}MB = \ldots = b^T (\hat{A})^{l-1}MB = 0 \in \mathbb{R}^r$. Application of the Cayley–Hamilton Theorem now gives

$$(\hat{A})^l = \gamma_1(\hat{A})^{l-1} + \gamma_2(\hat{A})^{l-2} + \ldots + \gamma_l I_{l \times l}$$

where $I_{l \times l}$ represents the $l \times l$ unit matrix and $\gamma_1, \ldots, \gamma_l$ are suitable constants, so $b^T (\hat{A})^l MB = 0 \in \mathbb{R}^r$. With induction we can derive that $b^T (\hat{A})^{l+j} MB = 0 \in \mathbb{R}^r$ for $(j = 1, 2, \ldots, n - 1 - l)$ also, thus

$$b^T [MB | \hat{A}MB | \ldots | (\hat{A})^{l-1}MB] = b^T M[B | AB | \ldots | A^{l-1}B] = 0 \in \mathbb{R}^{rn}.$$ 

Since $\operatorname{rank}(M) = l$, therefore $b^T M \neq 0 \in \mathbb{R}^n$ thus we get that the rows of matrix $W_{AB} = [B | AB | \ldots | A^{l-1}B]$ are linearly dependent, which is a contradiction. Consequently matrix $\hat{W}_{AB}$ has to have full rank and hence (6) is completely controllable.
Remark 2 The complete controllability of system (6) does not imply the complete controllability of system (1). Since even if the rank of \( \hat{W}_{AB} \) is \( l \), the rank of \( W_{AB} \) can be less than \( n \). A concrete example will also be given below on page 10.

Remark 3 In the case of \( l = n \) we get that \( M \) is a non-singular \( n \times n \) matrix, thus

\[
\text{rank}(\hat{W}_{AB}) = \text{rank}(M[B \mid AB \mid \ldots \mid A^{l-1}B]) = \text{rank}(M[B \mid AB \mid \ldots \mid A^{n-1}B]) = \text{rank}(W_{AB}).
\]

Therefore if (1) is completely controllable then (6) is also completely controllable and vice versa.

Remark 4 Let us assume that (1) is exactly lumpable to (6) by \( M \) then (3) is exactly lumpable to

\[
\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + MC^\top v(t)
\]

by \( M \), where \( \hat{A} \in \mathbb{R}^{l \times l} \) such that \( MA^\top = \hat{A}M \), so \( \hat{A} = MA^\top M^\top (MM^\top)^{-1} \). If we assume that (1), (2) is completely observable then (3) is completely controllable by Theorem 3 so (8) is also completely controllable by the application of Theorem 4.

Remark 5 The complete controllability of system (8) does not imply the complete observability of system (1)–(2). The controllability matrix of system (8) is

\[
\begin{bmatrix}
MC^\top & \hat{A}MC^\top & \ldots & (A^\top)^{l-1}MC^\top \\
(A^\top)^{l-1}MC^\top & \ldots & \ldots & \ldots
\end{bmatrix}
\]

where we have used \( \hat{A}M = MA^\top \), \( \hat{A}^{l-1}M = M(A^\top)^{l-1} \). Even if it is of has rank \( l \), the rank of \( V_{CA} \) can be less then \( n \). A concrete example will also be given below on page 11.

5 Examples

In this section we will give examples to illustrate our results.

Let us consider the following chemical reaction, a special case of a compartmental system [3, page 3]:

\[
X_1 \xrightarrow{k} X_2 \quad X_2 \xrightarrow{k} X_1 \quad X_2 \xrightarrow{k} X_3 \quad X_3 \xrightarrow{k} X_2
\]

where \( X_i \ (i = 1, 2, 3) \) is the \( i \)th chemical component or species and the positive real number \( k \) is the uniform reaction rate constant. The example may be degenerate from the point of view of kinetics, however it may be considered as a three stage approximation of diffusion in a tube. We are interested in the time evolution of the quantities of chemical components. If we assume that the physical circumstances are ideal, i.e. in the given
reaction the temperature, the pressure, and the volume of the vessel are constant we can build up the mass action-type model of the reaction (9):

\[
\begin{align*}
\dot{x}_1(t) &= -kx_1(t) + kx_2(t) \\
\dot{x}_2(t) &= kx_1(t) - 2kx_2(t) + kx_3(t) \\
\dot{x}_3(t) &= kx_2(t) - kx_3(t)
\end{align*}
\] (10)

where \(x_i(t) \ (i = 1, 2, 3)\) is to be interpreted as the concentration of the species \(X_i\) at time \(t\). Equation (10) is said to be the induced kinetic differential equation of (9).

To construct the lumping matrix \(M\) we use the following facts: if every element of the matrix \(M = N \begin{bmatrix} f_1^T \\ \vdots \\ f_l^T \end{bmatrix}\) is nonnegative, moreover for every row of \(M\) there exists an element from that row which is the only nonzero element of its column, then the lumped system of the induced kinetic differential equation of a reaction is also a induced kinetic differential equation of a reaction [5].

Using this and the fact that the independent, real eigenvectors of

\[
A = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}
\]

are

\[
f_1^T = [1, 1, 1], \quad f_2^T = [1, 0, -1], \quad f_3^T = [1, -2, 1]
\]

set

\[
M := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}
\]

in all the examples of this section. In this case the new variables are

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_2 + 2x_3 \end{bmatrix}
\]

and the lumped system is

\[
\begin{align*}
\dot{x}_1 &= -\frac{k}{2}\dot{x}_1 + \frac{k}{2}\dot{x}_2 \\
\dot{x}_2 &= \frac{k}{2}\dot{x}_1 - \frac{k}{2}\dot{x}_2
\end{align*}
\]

which is the induced kinetic differential equation of the reaction \(\hat{X}_1 \xrightarrow{k/2} \hat{X}_2 \xrightarrow{k/2} \hat{X}_1\). (11)
Let us remark that the new variables can be considered as groups of the old ones measured together, since they are nonnegative linear combinations of the old ones.

We shall use the notion and properties of an M-matrix below.

**Definition 4** Let $n \in \mathbb{N}$. If the matrix $A_n \in \mathbb{R}^{n \times n}$ can be expressed in the form

$$A_n = -s(I_n - T_n),$$

where $s \in \mathbb{R}^+$, $T_n \in \mathbb{R}^{n \times n}$ is a nonnegative, irreducible and symmetric matrix with the spectral radius $\rho$ obeying

$$\rho(T_n) = \max_{h=1,2,\ldots,n} |\lambda_h| \leq 1,$$

where $\lambda_h \in \mathbb{R}$ are eigenvalues of $T_n$, then $A_n$ is an M-matrix.

Since both the coefficient matrix of the original system and that of the lumped one are of the form (12):

$$A = \begin{bmatrix}
-k & k & 0 \\
k & -2k & k \\
0 & k & -k
\end{bmatrix} = -2k \left( \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix} \right),$$

$$\tilde{A} = \begin{bmatrix}
-k & k \\
k & -k
\end{bmatrix} = -2k \left( \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} - \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix} \right),$$

where $k > 0$ and $\rho(T_{3 \times 3}) = \max[1, |1|, \cdots, -\frac{1}{2}] = 1$, and $\rho(T_{2 \times 2}) = \max[0, |1|] = 1$, thus both matrices is an M-matrix. A similar construction can be obtained for an even number of species as follows.

First, let $n = 4\theta + 2$ ($\theta = 0, 1, \ldots$) and let us consider the following chemical reaction:

$$\begin{align*}
X_1 \overset{k}{\rightarrow} X_2 & \quad X_2 \overset{k}{\rightarrow} X_1 \\
X_2 \overset{k}{\rightarrow} X_3 & \quad X_3 \overset{k}{\rightarrow} X_2 \\
& \quad \ldots \\
X_{4\theta + 1} \overset{k}{\rightarrow} X_{4\theta + 2} & \quad X_{4\theta + 2} \overset{k}{\rightarrow} X_{4\theta + 1}
\end{align*}$$

(13)

then two independent, real eigenvectors of the matrix

$$A_{4\theta + 2} = \begin{bmatrix}
-k & k & 0 & \ldots & 0 & 0 \\
k & -2k & k & \ldots & 0 & 0 \\
0 & k & -2k & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -2k & k \\
0 & 0 & 0 & \ldots & k & -k
\end{bmatrix}$$

are

$$f_1^T = [1, 1, 1, \ldots, 1, 1], \quad f_2^T = [-1, 1, -1, \ldots, -1, 1]$$
so if we take
\[ M := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & 1 & -1 & \ldots & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 & \ldots & 2 & 0 \\ 2 & 0 & 0 & 2 & \ldots & 2 & 0 \end{bmatrix}, \]
then we obtain
\[ A_2 = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \]
and the lumped system is again (11).

Second, let \( n = 4\theta \) (\( \theta = 1, 2, \ldots \)), then two independent, real eigenvectors of the matrix \( A_{4\theta} \) are
\[ f_1^T = [1, 1, 1, 1, \ldots, 1, 1], \quad f_2^T = [-1, 1, 1, -1, \ldots, 1, -1] \]
so if we take
\[ M := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ -1 & 1 & 1 & -1 & \ldots & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 & \ldots & 2 & 0 \\ 2 & 0 & 0 & 2 & \ldots & 2 & 0 \end{bmatrix}, \]
then we get \( A_2 \) and the lumped system is again (11). We note that \( A_n, n = 4\theta, \) or \( n = 4\theta + 2 \) (\( \theta = 1, 2, \ldots \)), is again of the form (12), with \( s = 2k > 0 \) and
\[ T_{n \times n} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \ldots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} \end{bmatrix} > 0, \]
and \( \lambda_h = \cos\left(\frac{(h-1)\pi}{n}\right), \ h = 1, 2, \ldots, n \) are eigenvalues of \( T_{n \times n} \) \[ \text{[1 page 177]: and } \varrho(T_{n \times n}) = | \cos\left(\frac{\eta\pi}{2n}\right) | = 1 \text{ is the spectral radius of } T_{n \times n}, \text{ thus } A_n \text{ in an M-matrix in this case too. Furthermore, the eigenvalues of } A_n \text{ are given by } -2k(1 - \lambda_h) = -4k(\sin(\frac{h-1)\pi}{2n}))^2, \ h = 1, 2, \ldots, n, \text{ so they are real and negative, and zero, and any independent, real eigenvectors of } A_n \text{ are } f_{1q} = \frac{1}{\sqrt{n}}, \ f_{qq} = \frac{2}{n} \cos\left(\frac{(2q-1)(\eta-1)\pi}{2n}\right), \ \eta = 2, 3, \ldots, n, \ q = 1, 2, \ldots, n. \]

5.1 Examples of the effect of lumping on controllability

5.1.1 Controllable original—controllable lumped system

First, let us consider the differential equation
\[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \]
then

\[
W_{AB} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -k & k & 0 & 2k^2 & -3k^2 & k^2 \\ 0 & 1 & 0 & k & -2k & k & -3k^2 & 6k^2 & -3k^2 \\ 0 & 0 & 1 & 0 & k & -k & k^2 & -3k^2 & 2k^2 \end{bmatrix}.
\]

This matrix has rank 3, therefore this system is completely controllable. The lumped system is

\[
\frac{d}{dt} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

So \( \dot{W}_{AB} = [MB \mid \dot{AMB}] = \begin{bmatrix} 2 & 1 & 0 & -k & 0 & k \\ 0 & 1 & 2 & k & 0 & -k \end{bmatrix} \) which has rank 2, therefore in this case the lumped system is also completely controllable in accordance with Theorem 4.

5.1.2 Not controllable original—controllable lumped system

Second, let us consider the control differential equation

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

Since in this case the matrix \( W_{AB} = \begin{bmatrix} 1 & 1 & 0 & -k & 0 & 0 & k^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & k & 0 & 0 & -k^2 & 0 \end{bmatrix} \) has only rank 2, thus this system is not completely controllable. At the same time the lumped system is

\[
\frac{d}{dt} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -k & k \\ k & -2k \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 0 \\ 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

Therefore \( \dot{W}_{AB} = \begin{bmatrix} 3 & 2 & 0 & 0 & -2k & 0 \\ 3 & -2 & 0 & 0 & 2k & 0 \end{bmatrix} \) and this matrix has rank 2, so the lumped system is still completely controllable in this case.

5.2 Examples for the effect of lumping on observability

5.2.1 Observable original—observable lumped system

First, let us consider the observation system

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

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i.e. the observation matrix \( C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \). So \( V_{CA} = [C^T | A^T C^T] \)

\[
(A^T)^2 C^T = \begin{bmatrix} 1 & 0 & 0 & k & -k^2 & -2k^2 \\ 1 & 1 & -k & -k & 3k^2 & 3k^2 \\ 0 & 1 & k & 0 & -2k^2 & -k^2 \end{bmatrix}
\]

which has rank 3, hence this system is completely observable by Theorem 2. The lumped system is characterized by \( \tilde{A} = \begin{bmatrix} -k^2 \\ k \\ k \\ 0 \\ -k \end{bmatrix} \) and \( M C^T = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 3 & -k \end{bmatrix} \) by Remark 4.

So \( [MC^T | \tilde{A}MC^T] = \begin{bmatrix} 3 & 1 & -k & k \\ 1 & 3 & k & -k \end{bmatrix} \) which has rank 2, therefore in this case the lumped system is completely observable.

### 5.2.2 Not observable original—observable lumped system

Second, let us consider the observation system:

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

i.e. now \( C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \). Since in this case the matrix

\[
V_{CA} = \begin{bmatrix} 2 & 0 & -k & k & -k^2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & k & -k & -k^2 \end{bmatrix}
\]

which has rank only 2, thus this system is not completely observable. At the same time the lumped system is \( \tilde{A} = \begin{bmatrix} -k^2 \\ k \\ k \\ 0 \\ -k \end{bmatrix} \) and \( MC^T = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \).

Therefore \( [MC^T | \tilde{A}MC^T] = \begin{bmatrix} 5 & 1 & -2k & 2k \\ 1 & 5 & -2k & 2k \end{bmatrix} \) and this matrix has rank 2, so the lumped system is completely observable in this case.

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### Discussion and outlook

One of the major questions connected with lumping (with this special technique to reduce the number of variables) is: are the qualitative properties of
the lumped and of the original system connected? This problem has been investigated in a more general setting in ??; here we add a new statement: suppose we lump a completely controllable system of $n$ compartment. Then the lumped system will also be completely controllable. Previously, [6] investigated a similar problem: local observability and local controllability of reactions. [3] mainly concentrates on symbolic lumping of compartmental systems.

Possible further topics are: the effect of nonlinear lumping on local controllability and observability.

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