Relaxing and Virializing a Dark Matter Halo

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ABSTRACT

Navarro, Frenk, and White have suggested that the density profiles of simulated dark matter halos have a “universal” shape so that a given halo can be characterized by a single free parameter which fixes its mass. In this paper, we revisit the spherical infall model in the hope of recognizing in detail the existence and origin of any such universality. A system of particles is followed from linear perturbation, through first shell crossing, then through an accretion or infall phase, and finally to virialization. During the accretion phase, the system relaxes through a combination of phase mixing, phase space instability, and moderate violent relation. It is driven quickly, by the flow of mass through its surface, toward self-similar evolution. The self-similar solution plays its usual rôle of intermediate attractor and can be recognized from a virial-type theorem in scaled variables and from our numerical simulations. The transition to final equilibrium state once infall has ceased is relatively gentle, an observation which leads to an approximate form for the distribution function of the final system. The infall phase fixes the density profile in intermediate regions of the halo to be close to $r^{-2}$. We make contact with the standard hierarchical clustering scenario and explain how modifications of the self-similar infall model might lead to density profiles in agreement with those found in numerical simulations.

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1. Introduction

Navarro, Frenk and White (1996) have summarized the results of their dissipationless cosmological clustering simulations in terms of a ‘universal’ shape for the density profile of dark halos. This profile (henceforth the NFW profile),

\[ \rho(r) = \frac{M_s}{r (r + R_s)^2}, \]  

is characterized by an \( r^{-1} \) central cusp and an outer region where the density falls off faster than that of an isothermal sphere. Good fits are obtained using Eq. (1) for halos that range in mass from \( 3 \times 10^{11} M_\odot \) (dwarf galaxies) to \( 3 \times 10^{15} M_\odot \) (rich galaxy clusters). In addition, the results show a strong correlation between the constants \( R_s \) and \( M_s \) so that there is essentially a single free parameter (Navarro, Frenk, & White 1996). The possible existence of a universal profile suggests that the structure of collapsed objects can be understood from simple physical arguments.

The earliest attempts to understand cosmological structure were based on the spherical radial infall model (e.g., Gunn & Gott 1972; Henriksen & De Robertis 1980; Fillmore & Goldreich 1984, hereafter FG; Bertshinger 1985, hereafter B85; Hoffman & Shaham 1985; White & Zaritsky 1992) in which a primordial density perturbation, assumed to be spherically symmetric and smooth, slowly accretes matter from the cosmic background. If this initial perturbation is also scale-free (i.e., \( \delta \rho \propto r^{-\epsilon} \), see e.g., FG) with a velocity distribution corresponding to an unperturbed Hubble law, the structure that develops will be self-similar in the sense that the distribution function at one time can be obtained from that at another time by rescaling the phase space coordinates \( r \) and \( v_r \). One way to determine the scaling law is to note that at each time \( t \) there is a single mass shell that is just beginning to break away from the expansion and fall in towards the center. The radius at which this occurs defines a function of time, \( r_{ta} \propto t^\delta \) where

\[ \delta = \frac{2}{3} + \frac{2}{3\epsilon} , \]  

suggesting that the appropriate radial coordinate for the self-similar solution is \( X \equiv r/r_{ta} \) (FG and B85).

Each FG and B85 solution represents a single trajectory in phase space, albeit one that describes multiple velocity streams in the inner parts of the system. An alternative approach (Henriksen & Widrow 1997, hereafter HW), based directly on the collisionless Boltzmann equation (CBE), treats the distribution function as continuous in the scaled phase-space variables. In this picture, the single-trajectory solutions of FG and B85 represent a subset of the characteristic curves of the CBE. Numerical simulations of systems that begin from a “cold start” (i.e., single-stream distribution function) show a rapid transition to a distribution function that is continuous, the transition being facilitated by an instability (HW). The form of the scaling in the CBE (dictated by \( \delta \)) is equivalent to the similarity ‘class’ in the sense of Carter and Henriksen.
This is determined by the logarithmic derivative of $r_{ta}$ with respect to future turn-round time at the epoch of first shell crossing, and is given by Eq. (2).

Self-similarity would seem to be an appealing feature for a model that is to explain universal characteristics in cosmological structures of the type described by Navarro, Frenk, & White (1996). This is particularly so given the usual rôle of self-similarity as an attractor. However, the self-similar infall model (SSIM) has been, by and large, dismissed as a paradigm for structure formation. First, the SSIM does not include angular momentum which likely plays a key role in determining the density profile, at least in the central regions. Second, structure formation is thought to develop by way of hierarchical clustering: Small mass objects form first and merge with one another to create systems of ever-increasing size. This process is neither smooth nor spherically symmetric. Moreover, the solutions found by FG and B85 describe an eternal infall of matter and say nothing about how a system might enter or exit such a self-similar state. In the standard scenario for structure formation, gravitational collapse begins only after the Universe has become matter dominated. Moreover, at late times (e.g., as for galaxies today), the evolution of a system is dominated by infrequent mergers with comparable-sized objects. Therefore at best, self-similarity will arise as an intermediate phase in the evolution of a system. (This is of course always the case with self-similar behaviour.) Finally, it is generally believed that the SSIM predicts a power-law density profile for “virialized” halos, $(\rho(r) \sim r^{-\mu} \text{ with } 2 < \mu < 9/4}$ where $\mu$ depends on $\epsilon$) in contrast with what is found in the simulations (cf. Eq. (1)).

In this paper we argue instead that the SSIM provides a natural framework for understanding cosmological structure formation. A simple shell code is used to treat spherically symmetric collisionless particles (i.e., spherical shells) on radial orbits. The particles are followed from linear perturbation through first shell crossing, then through the recently recognized self-similar relaxation phase, and into eventual virialization which occurs after the cessation of infall. Many features of the self-similar solutions discovered by FG and B85 are present in this intermediate phase though much of the fine-grained phase-space structure characteristic of these solutions is washed out. We maintain that there is a close connection between the SSIM and semi-analytic models of structure formation based more directly on hierarchical clustering (e.g., Lacey & Cole 1993). In addition we will show that simple modifications of the SSIM lead to density profiles in agreement with results from N-body simulations. An advantage of this approach is that it provides a connection, through a simple semi-analytic model, between initial conditions and the distribution function and density profile of the final virialized system.

In a hierarchical clustering universe, the progenitors of present day nonlinear structures are small-amplitude primordial density fluctuations. Bond et al. (1991) and Lacey & Cole (1993) have developed an analytic model for hierarchical structure formation that relates the initial power spectrum to the mass and formation time of dark matter halos. In their formalism, the linear density field at some early time, $t_i$, is smoothed on various mass scales. Objects that will collapse by some later time, $t_{coll}$, are identified at the earlier epoch as regions where the mean density is above a certain threshold. This threshold is estimated using the spherical top-hat model, i.e., the perturbation within a given radius is modeled as a region of constant density. Of course in the
spherical top-hat model, all parts of the perturbation collapse to the center at the same time and it is therefore necessary to make an ad hoc assumption about what happens after collapse. Lacey & Cole (1993) assume that at \( t_{\text{coll}} \) the system reaches virial equilibrium with a radius equal to half its maximum or turnaround radius. Given this assumption, the mean density within a collapsed object at a particular epoch turns out to be roughly \( \sim 200 \) times the background density at that time. It is now common practice to define the ‘virial radius’, \( r_v \), as the radius of a sphere that contains an overdense region whose mean density is \( v \) times the background density. The claim is that the mass within this sphere has, more or less, reached a final equilibrium state. \( r_v \) is used, for example, to identify virialized halos in N-body simulations (e.g., Cole & Lacey 1996, Navarro, Frenk, & White 1996).

Let us see how the SSIM might improve this picture. If the initial perturbation spectrum is scale-free (\( \langle |\delta_k|^2 \rangle \propto k^n \)) the RMS mass fluctuation will be a power-law function of radius, \( \langle \delta_R(x)^2 \rangle^{1/2} \propto R^{-(n+3)/2} \). In this expression, \( R \) is the radius of the window function used in computing the mass fluctuation, \( x \) is the position vector about which this window function is centered, and \( \langle \ldots \rangle \) indicates an average over \( x \). This suggests an initial density perturbation with \( \delta \rho_i \propto r^{-\epsilon} \) (\( \epsilon = (n+3)/2 \)) as the most appropriate “toy-model” for understanding structure formation via hierarchical clustering and allows us to write the similarity class \( \delta \) and final density profile index \( \mu \) in terms of the spectral index \( n \):

\[
\delta = \frac{2}{3} \left( \frac{n+5}{n+3} \right) ;
\]

\[
\mu = \begin{cases} 
3 \left( \frac{n+3}{n+5} \right) & n > 1 \\
2 & n \leq 1 
\end{cases}
\]

(3)

(4)

Indeed the results obtained from the SSIM are consistent with those found in Lacey & Cole (1993). Their ad hoc virialization radius \( r_v \) corresponds to an \( X = \) constant \( \left( r = \text{constant} \times t^\delta \right) \) surface and the expression for the mass within this surface, as a function of \( t \), leads to their result for the mass-formation time relationship. While the “scaled mass” within \( r_v \) approaches a constant, the physical mass increases as \( t^{2/\epsilon} = t^{35-2} \). A mass flux through the system boundary is required but this is built into the self-similar solution. The SSIM avoids the singular nature of the collapse in the spherical top-hat model and thereby affords insight into the virialization process and the ultimate fate of the halo once the mass flux has ceased.

But the question remains as to the relevance of the SSIM to halo formation via hierarchical clustering. Recently, Syer and White (1997) (see, also Nusser and Sheth, 1997) have suggested that the universality seen in the simulations reflects a balance between two opposing processes, dynamical friction and tidal stripping, that act on a small object as it merges with a larger one. Dynamical friction will bring a satellite to the centre of a parent halo but only if the satellite has not been tidally disrupted. Whether or not the satellite reaches the centre intact depends on the density profile of the parent. For a steep density profile, the satellite is disrupted before it reaches
the centre so that its material is spread throughout the halo, thus softening the density law. Conversely, if the density profile of the parent halo is relatively flat, the satellite reaches the centre largely intact and so boosts the density there. In this way, dynamical friction and tidal dissipation act as negative feedback mechanisms driving the density profile toward a universal shape (Syer & White 1997). (For an alternative discussion in which the universality of dark halo density profiles does not depend crucially on hierarchical merging, see Huss, Jain, & Steinmetz 1998.)

A similar situation arises in the SSIM (FG, Moutarde et al. 1995). Suppose the initial density perturbation is relatively flat (here flat and steep are with respect to a $\rho \propto r^{-2}$ density law). The system evolves toward a universal profile which, in the intermediate regions of the halo, is $\propto r^{-2}$ and is a self-similar attractor which we discuss in detail below. As in the hierarchical scenario, the central regions are dominated by material that has fallen in recently (FG) since the binding energy, $GM(r)/r$, is an increasing function of $r$. By contrast, steep initial density profiles evolve stably as one of a 1-parameter continuum of self-similar solutions. The parameter is the quantity $\delta$ introduced above (related ultimately to $n$) and the density profile is the same as that of the perturbation at first shell crossing. The $\rho \propto r^{-2}$ attractor is actually the flattest of these self-similar profiles. We can say then that the universality predicted by the SSIM is ‘one-sided’. It is worth mentioning however that the density laws for the stably evolving systems in the SSIM vary only from $r^{-2}$ to $r^{-9/4}$ for $n \in \{1, 3\}$.

As discussed above, a major criticism of the SSIM is that the density profile it predicts for collapsed objects does not agree with the results from N-body simulations. The power-law index in the NFW profile, for example, varies smoothly from $-1$ in the core to $-3$ in the envelope, in contrast with what is found in the similarity solutions discussed above. There is some controversy over what the true density law within the inner regions of simulated halos is. Moore et al. (1997) find that mass resolution and force softening have a significant effect on the density profiles of collapsed objects in collisionless N-body simulations. As the mass and force resolution is increased the density profile in the central region becomes steeper. Even with 3 million particles per halo, the results have not converged to a unique density profile. Moore et al. (1997) attribute their results to the issue of dynamic range in the clustering hierarchy: With better resolution, smaller halos would collapse earlier causing the density profile to steepen. A similar situation arises in the SSIM. As noted above, the solutions derived by FG and B85 correspond to an eternal infall of matter on a single trajectory. If instead, collapse begins at a finite time, there would be relatively fewer particles in the inner regions leading to a shallower density law in the center. In the real Universe, the size of the first objects to form is set by the horizon size at matter-radiation equality while in the simulations it is set by the force and mass resolution of the experiment.

As noted above, the NFW profile is noticeably steeper than $r^{-2}$ in the outer regions of simulated halos. We propose that this region forms after the primary accretion phase when the evolution of a system becomes dominated by major mergers. The system is still accreting mass during this phase, but not at a rate sufficient to maintain self-similar growth. The infalling material can be treated as test particles and naturally forms an $r^{-3}$ outer halo.
In Section 2 we write the basic equations describing collisionless radial dynamics (CBE, mass conservation and Euler equations, virialization condition) in scaled variables. We use these variables to follow numerically a collisionless spherical cosmological perturbation from linear perturbation to a self-similar infall phase (steady-state in scaled variables) and finally to a true virialized state which arises once the stream of particles falling into the perturbation is shut off. The simulations are presented in Section 3. Section 4 presents a more detailed discussion of the transition from infall phase to virialized isolated system. A summary and conclusions are given in Section 5.

2. ‘Virialization’ in Scaled Variables

We begin this section by writing the CB and Poisson equations in scaled variables. Standard manipulations (see, for example Binney & Tremaine 1987) lead to the usual mass conservation and Euler equations as well as the virial theorem in these variables. This formulation is inspired by the analysis of self-similarity given by Carter & Henriksen (1991) (see also Henriksen (1997) for a more physical exposition and Henriksen & Widrow (1995, 1997) for relevant examples). A self-similar state is identified as one that is independent of $T$. $X$ and $Y$ will then correspond to the self-similar or scale-invariant variables. Therein lies the interest in this formulation, since the emergence of a time-independent state in these variables allows one to recognize self-similarity of the type found in FG, B85, Moutarde et al. 1995, HW, and others.

The CB and Poisson equations for a spherically symmetric system are

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \left( \frac{j^2}{r^3} - \frac{\partial \Phi}{\partial r} \right) \frac{\partial f}{\partial v_r} = 0 ,$$

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = 4\pi^2 \int f(r,v_r,j^2) \, dv_r \, dj^2$$

where $f$ is the phase space mass density and $\Phi$ is the ‘mean field’ gravitational potential. (We are considering scales much smaller than the Hubble length.) For purely radial orbits it is convenient to introduce the canonical distribution function $F(t,r,v_r)$ where $f \equiv (4\pi^2)^{-1} \delta(j^2)F$. The equations then become

$$\frac{\partial F}{\partial t} + v_r \frac{\partial F}{\partial r} - \frac{\partial \Phi}{\partial r} \frac{\partial F}{\partial v_r} = 0 ,$$

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = G \int F \, dv_r .$$

The next step is to introduce the scaled phase-space variables

$$X \equiv e^{-\delta T} r$$

(9)
\[ Y \equiv e^{(1-\delta)T} v_r \] (10)

and

\[ T \equiv \ln(t/t_i) , \] (11)

along with the appropriately scaled distribution function and potential:

\[ \mathcal{F}(X,Y,T) = Ge^{(1-\delta)T} F(r,v_r,t) \] (12)

\[ \Psi(X,Y,T) = e^{2(1-\delta)T} \Phi(r,v_r,t) . \] (13)

The CB and Poisson equations become

\[ \frac{\partial \mathcal{F}}{\partial T} + (\delta - 1) \mathcal{F} + (Y - \delta X) \frac{\partial \mathcal{F}}{\partial X} - \left( (\delta - 1) Y + \frac{\partial \Psi}{\partial X} \right) \frac{\partial \mathcal{F}}{\partial Y} = 0 \] (14)

\[ \frac{\partial}{\partial X} \left( X^2 \frac{\partial \Psi}{\partial X} \right) = \int \mathcal{F} dY . \] (15)

Integrating Eq. (14) over \( Y \) yields the continuity equation:

\[ \frac{\partial S}{\partial T} + (3\delta - 2) S + \frac{\partial}{\partial X} \left( S \left( Y - \delta X \right) \right) = 0 \] (16)

where

\[ S \equiv \int dY \mathcal{F} ; \quad S\Psi \equiv \int dY Y \mathcal{F} . \] (17)

\( S \) is essentially the scaled mass per unit radius and is related to the physical density:

\[ \rho \equiv \int d^3v f = \exp(-2T)S/4\pi GX^2. \]

The mass conservation equation is obtained by integrating Eq. (16) over \( X \). (Note that we have assumed \( S \to 0 \) at \( X \to 0 \).) We find

\[ \frac{dM}{dT} + (3\delta - 2) M + \{ S \left( Y - \delta X \right) \}_{X=X_s} = 0 \] (18)

where \( X_s \) defines the (spherical) boundary of the system and

\[ M \equiv \int_0^{X_s} S dX = \int_0^{X_s} dX \int_{-\infty}^{\infty} dY \mathcal{F} \] (19)
is the scaled mass within this boundary, related to the physical mass $M$ by

$$M = e^{(3\delta - 2)T} \mathcal{M} = \left( \frac{t}{t_i} \right)^{(3\delta - 2)} \mathcal{M}.$$  \hfill (20)

For an isolated system, $S \to 0$ for $X \to \infty$. Taking $X_s \to \infty$ we find (from (18)) $\mathcal{M} \propto e^{-(3\delta - 2)T}$ which states the obvious: the mass of an isolated system is constant.

In a cosmological setting, the distribution of particles extends to $X = \infty$. A particularly simple example of a self-similar model is the matter-dominated Einstein-de Sitter universe which is described in terms of our scaled variables as follows:

$$\mathcal{F} = \frac{2\rho_i}{3} X^2 \delta_D \left( Y - \frac{2X}{3} \right)$$

$$S = \frac{2\rho_i}{3} X^2 \left( \text{equivalently, } \rho = \frac{1}{6\pi G t^2} \right)$$

$$\bar{Y} = \frac{2X}{3} \left( \text{equivalently, } v_r = \frac{2r}{3t^\delta} \right)$$

where $\delta_D$ is the Dirac delta function. The similarity class is characterized by $\delta = 2/3$ with $X$ being the usual comoving radius. However, by adding a power-law perturbation to this background density, one changes the similarity class and obtains the models considered in FG and B85 as well as in the present work.

A self-similar state requires by definition that the quantities $S$, $\mathcal{M}$, and $\bar{Y}$ be independent of $T$. We conclude from Eq. (18) that the surface term $\left\{ S \left( \bar{Y} - \delta X \right) \right\}_{X=X_s}$ must also be independent of $T \text{ and } \text{nonzero}$, unless $\delta = 2/3$. The continuing infall thus maintains the self-similarity.

Ultimately in these calculations the similarity class, $\delta$, is set by the density profile of the initial perturbation. Thus for $\delta \rho_i \propto r^{-\epsilon}$, at the epoch of first shell crossing the initial shell label or ‘effective radius’ of a shell (see e.g. Henriksen 1989) that will turn around at time $t$ is asymptotically proportional to $t^{2/3 + 2/3\epsilon}$. This establishes a fundamental relation between the time and space scaling in the subsequent infall (we have set the time scaling equal to unity) which suggests that we set $\delta = 2/3 + 2/3\epsilon$. The requirement for the system boundary to be fixed at $X_s$ in the self-similar state, as found above, implies that the current boundary in physical coordinates varies as $t^\delta$. This boundary is approximately the current turn-round radius $r_{ta}$ which then varies also as $t^\delta$ in agreement with the arguments of FG and B85.

In the general self-similar state we have, from Eq. (18),

$$\mathcal{M} = (3\delta - 2)^{-1} \left\{ S \left( \delta X - \bar{Y} \right) \right\}_{X=X_s}$$

$$\text{(24)}$$
in agreement with the semi-analytic results of B85 (cf. his Eq.(4.5)). Eq. (24), together with Eq. (20), imply that $M \propto t^{(3\delta - 2)/2} \propto t^{2/\epsilon}$. In other words, the physical mass of the system increases with time due to a mass flux through the system boundary. To make the connection with previous work on hierarchical clustering, consider initial conditions in which the power spectrum is a pure power law, $\langle |\delta_k|^2 \rangle \propto k^n$. The RMS mass fluctuation as a function of scale (essentially the radius of a spherical window function) is $\delta M/M \propto R^{-(n+3)/2}$. Let us suppose that our initial spherical perturbation has the same dependence on radius as the RMS mass fluctuation derived from Gaussian random fields, i.e., $\epsilon = (n + 3)/2$. With this identification, we can relate the mass $M$ of a relaxing system to the time $t$, or equivalently redshift $z$. We find $1 + z \propto M^{(n+3)/6}$, in agreement with Lacey & Cole (1993)

Multiplying Eq. (14) by $Y$ and integrating over $Y$ leads to the Jeans equation in scaled variables:

$$
\frac{\partial (SY)}{\partial T} + 3(\delta - 1)SY - \delta X \frac{\partial (SY)}{\partial X} + \frac{\partial (SY^2)}{\partial X} + SY \frac{\partial \Psi}{\partial X} = 0 .
$$

(25)

where

$$
S Y^2 \equiv \int dY Y^2 F .
$$

(26)

A virial theorem in scaled variables is obtained by multiplying Eq.(23) by $X$ and integrating over $X$:

$$
\frac{1}{2} \frac{d^2 I}{dT^2} + \frac{5}{2} \frac{dI}{dT} + \frac{(5\delta - 2)(5\delta - 3)}{2} I + (5\delta - 3) A - \frac{dA}{dT} - 2K - W = 0
$$

(27)

where

$$
K \equiv \frac{1}{2} \int_0^{X_S} SY^2 dX \quad \mathcal{W} \equiv - \int_0^{X_S} SY \frac{\partial \Psi}{\partial X} dX \quad I = \int X^2 S dX
$$

(28)

are the scaled total kinetic and potential energies and moment of inertia of the system respectively and

$$
A \equiv \left\{ SX \left( Y^2 - \delta XY \right) \right\}_{X = X_S}
$$

(29)

is a surface term. In the self-similar state we have

$$
\frac{2K}{\mathcal{W}} - 1 = \frac{(5\delta - 2)(5\delta - 3)}{2} \frac{I}{\mathcal{W}} + (5\delta - 3) \frac{A}{\mathcal{W}} .
$$

(30)
This virial theorem is satisfied for a system that is undergoing self-similar infall. The left-hand side is reminiscent of the usual virial theorem (e.g., Binney & Tremaine, 1987). The first of the two terms on the right-hand side comes from the inherent time-dependence of the infall solution and is analogous to the terms found in cosmology when working in comoving coordinates. The second term is due to the infall of matter through the system boundary. We can of course rewrite this equation in terms of more familiar quantities such as the total kinetic energy, \[ K = 2\pi \int \rho v_r^2 r^2 dr = K \exp(5\delta - 4) T. \] The result has essentially the same form, i.e., \( 2K/W = \text{constant} \) and suggests that we consider the self-similar infall phase as a type of time-dependent virial equilibrium.

3. Numerical Simulations

The FG and B85 solutions describe an eternal system in which self-similarity is exact. The profile of the initial density perturbation is used to determine the appropriate scaling relations but is otherwise not part of the solution, i.e., the self-similar solutions do not follow a system from initial perturbation to self-similar phase. Nor, for that matter, do they shed light on the ultimate state of the system once infall has ceased. The simulations described in this section are designed to address these issues.

A simple shell model is used to follow the evolution of spherically symmetric density perturbations in an Einstein-de Sitter universe. This model describes a system of collisionless, self-gravitating “particles” each of which represents a spherical shell of matter. The scale of the perturbations is assumed to be small as compared with the Hubble length. The equations of motion for a shell of radius \( r(t) \) and radial velocity \( v_r(t) \) are therefore given by Newtonian dynamics:

\[
\frac{dr}{dt} = v_r, \quad \frac{dv_r}{dt} = -\frac{GM(r,t)}{r^2}.
\] (31)

At some initial time \( t_i \), the unperturbed Hubble-flow is described by a constant background density, \( \rho_0(t_i) = 1/6\pi G t_i^2 \), and a velocity field \( v_r(t_i) = 2r(t_i)/3t_i \). We introduce a cut-off in the initial mass distribution at a finite radius \( r_0 \). This cut-off will allow us to follow the evolution from infall phase to isolated system. The mass of a given shell is chosen to be proportional to the square of its radius at time \( t = t_i \): i.e., \( m = 2r^2(t_i)r_0/3NGt_i^2 \) where \( N \) is the number of shells. In other words, at \( t = t_i \), the total mass in the system is divided into shells of uniform thickness \( r_0/N \). By partitioning the mass in this way, we improve the resolution of the simulation during the early stages of collapse but at the cost of having a system which is modelled by unequal masses.

The initial conditions that lead to the self-similar solutions of FG and B85 assume a density profile that is a power-law function of radius: \( \delta \rho_i \propto r^{-\epsilon} \). As discussed above, this choice can be justified for models where the spectrum of density perturbations is a scale-free function of wavenumber. However there are various reasons to expect that actual perturbations will be
modified at small scales. First, even if the primordial perturbation spectrum is scale-free, physical processes (e.g., neutrino free-streaming, radiation damping) will suppress small-scale fluctuations. Second, the connection between the primordial power spectrum and expected rms mass excess is valid only for linear perturbations and breaks down near nonlinear density peaks. There are additional effects, peculiar to N-body simulations, related to finite mass and spatial resolution, which also damp small scale fluctuations.

These considerations suggest an initial density perturbation of the form

\[ \rho(r, t_i) = \rho_b(t_i) (1 + \Delta(r, t_i)) \] (32)

where

\[ \Delta(r, t_i) = \begin{cases} A \left( 1 - B \left( \frac{r}{r_c} \right)^2 \right) & r < r_c \\ A (1 - B \left( \frac{r}{r_c} \right)^{-\epsilon}) & r \geq r_c \end{cases} \] (33)

We set \( B = 5\epsilon / (3\epsilon + 6) \) so that, for \( r \geq r_c \), the mass interior to \( r \) is what it would have been had the density profile been a strict power-law function of radius. (This choice is made to maintain a close connection with the FG solutions. Other forms for the initial density profile are possible. See, for example Hoffman & Shaham 1985; Bardeen, Bond, Kaiser, & Szalay 1986; Ryden & Gunn, 1987; and Ryden, 1988.)

The desired initial density perturbation is achieved by displacing each shell by an amount \( \delta r(r_i) \) where

\[ \frac{\delta r(r_i)}{r_i} = -\frac{1}{r_i^3} \int_0^{r_i} dr r^2 \Delta(r, t_i) \] (34)

A modified force law \( \left( \frac{d v_r}{d t} = -GM(r, t) r / (r^2 + \epsilon^2)^{3/2} \right) \) is used to handle the dynamics of shells as they pass through the origin.

As discussed in Section II, the natural variables for a system of this type are \( X, Y \) and \( T \). We therefore perform our simulations in these variables. Eqs. (31) become:

\[ \frac{dX}{dT} = Y - \delta X \quad \frac{dY}{dT} = (1 - \delta) Y - \frac{M(X, T)}{X^2} \] (35)

where \( M = \exp(2 - 3\delta) \times (\text{mass interior to } X) \). These are just the equations one would write to describe the characteristic curves of the CBE (see HW and Eq. (14)). At the initial time \( t_i, T = 0, r = X, \) and \( v_r = Y \).

Gravitational collapse simulations are performed for a variety of parameters. In Figure 1 we show the evolution of a system of 16,000 particles in \((X, Y)\) phase space. The density profile of
the initial perturbation is given by Eqs. (32) and (33) with $\epsilon = 2.5$. This implies a similarity class characterized by $\delta = 14/15$. The panels a-d in Figure 1 show the system at $T = 2.7, 8.0, 13.3$ and $18.7$ or equivalently $t/t_i = 14, 3.0 \times 10^3, 6.2 \times 10^5$, and $1.3 \times 10^8$ respectively. If the final frame is taken as the present epoch, then the initial frame corresponds to a redshift of $z_i = 2.6 \times 10^5$. As expected, the size of the system in scaled variables is roughly constant. Of course, the physical size of the system grows with time, increasing by panel d, over its initial size, by a factor $\exp(\delta T) = 3.7 \times 10^7$. Actually, by this stage, the stream of infalling particles has ceased and the system, now an isolated one, maintains a fixed size in $r$ but shrinks in $X$. This simulation demonstrates the advantage of scaled variables for studying systems that grow from “scale-free” initial conditions. In the usual physical coordinates, the size and mass of the system grows by many orders of magnitude whereas the system is nearly time-independent in scaled variables. Of course the variables immediately lose their advantage once the supply of particles is shut off.

The phase-space plots in Figure 1 are somewhat misleading in that the shells, each represented by a single point, have masses that range from $\rho_i (R/N)^3$ (initially the innermost shell) to $\rho_i R^2 (R/N)$ (initially the outermost shell). We have therefore repeated the experiment but with the mass distributed equally among the shells. The results for $T = 13.3$ and $18.7$ (final two frames of Figure 1) are shown in Figure 2. While these provide what is at least visually a more accurate representation of the phase-space distribution function, the simulations used to produce Figure 1 clearly have better coverage of phase space, and in particular, are better able to follow the early stages of collapse.

The semi-analytic solutions of FG and B85 are for an eternal self-similar collapsing system and are described by a semi-infinite spiral in phase space. In contrast, our system evolves for a finite period of time. Moreover there is an inner region ($r_i < r_c$) in the initial perturbation where the density profile deviates from a pure power-law. We can see these effects in the phase space plots of Figure 2 where there appear to be relatively few particles with both $X$ and $Y$ small. This region of phase space corresponds to the most tightly bound particles, i.e., ones with $r_i \to 0$. FG have shown that the similarity class determines the density profile of the collapsing system. For $\delta < 1$ ($\epsilon > 2$) the final density profile is $\rho \propto r^{-\mu}$ where $\mu = 2/\delta$. For $\delta > 1$, the final density profile is $\rho \propto r^{-2}$. Figures 3a, b, and c show the evolution of the density profile with time for the simulation in Figure 1 as well as two other cases $\delta \rho_i \propto r^{-2}$ and $\delta \rho_i \propto r^{-3/2}$. The agreement with the predictions of FG is quite good. These predictions have also been confirmed by Moutarde et al. (1995) in a rather different set of numerical experiments.

The density law in the outer regions of the system ($-2 \lesssim \log_{10} X \lesssim -1.2$) is somewhat steeper than $r^{-\mu}$, a result which is also evident in the semi-analytic solutions of FG (see their Figure 12). This region of the system is composed of particles that have passed through the origin only a few times and is therefore not fully relaxed. We see here an example of a system whose evolution is self-similar but whose density profile is not a pure power-law function of radius.

As noted above, in N-body simulations, finite resolution effects introduce an effective short distance cut-off in the perturbation spectrum as well as the development of small-scale structure.
The behaviour of the density profile as a function of softening length $\varepsilon$ (Figure 4) is reminiscent of the results found by Moore et al. (1997). Like Navarro, Frenk & White (1996), they find that the density profile for a dark matter halo is flatter than $r^{-2}$ in the central regions. However the size of the region shrinks as one improves the resolution of the simulations. Our results provide further evidence of this effect.

The effects of a softened force law are also evident in the phase-space plots of Figures 1 and 2 where the maximum velocities attained by particles as $X \to 0$ approaches a constant. Were it not for force softening, this maximum velocity would increase as $X \to 0$.

In Figures 5a-d, we show, as a function of logarithmic time $T$, the “size” of the system, its scaled mass and energy, and the ratio $2K/W$. The system is defined as the region $X < X_s$ where $X_s$ is the smallest radius that includes all particles that have passed through the origin at least once. These plots show rather dramatically the speed with which the system reaches the self-similar state. At $T \simeq 15$, infall ceases and the system shrinks in physical variables while remaining more or less stationary in scaled variables. Figure 5d is perhaps the most interesting. We see that the ratio $2K/W (= 2\mathcal{K}/W)$ is constant, though different from 1, during the infall phase. The results are consistent with Eq. (30) with the terms on the right-hand side being small, but non-negligible. Once infall ceases, this ratio settles quickly to 1, the value expected for an isolated, virialized system.

4. Analysis

4.1. Self-similar to Virial Transition

In Section 2 we learned that a system undergoing self-similar infall, when followed in scaled variables, is “stationary”. These results, together with our numerical simulations, provide clues as to how a system makes its transition from infall phase to isolated state. To the extent that the two terms on the right-hand side of Eq. (30) are negligible, this transition is “smooth” in that when the flux of particles through the system boundary is “turned off”, the system finds itself in a virialized state that is stationary in the usual sense (i.e., with respect to the physical variables $r$ and $v_r$).

In this section, we explore the transition from infall phase to isolated system by attempting to match the SSIM distribution function to one that describes a system in equilibrium. The analysis leads to a particular choice for the equilibrium configuration, namely time-independent self-similar solutions that correspond to the so-called power-law models discussed elsewhere (Evans 1994, Henriksen & Widrow 1995).

Consider a system that is undergoing self-similar infall. In this state, the scaled distribution function and potential are independent of $T$ and therefore satisfy the following constraint equations (cf. (14))

$$
(\delta - 1) F + (Y - \delta X) \frac{\partial F}{\partial X} - \left( (\delta - 1) Y + \frac{\partial \Psi}{\partial X} \right) \frac{\partial F}{\partial Y} = 0
$$

(36)
and
\[ \frac{\partial \Psi}{\partial T} = 0. \] (37)

If we impose the additional constraint that at the instant infall ceases, the system is also time-independent then:
\[ \frac{\partial F}{\partial t} = 0 ; \quad \frac{\partial \Phi}{\partial t} = 0. \] (38)

These latter conditions can be written in terms of the scaled variables as:
\[ \frac{\partial F}{\partial T} + (\delta - 1) F - \delta X \frac{\partial F}{\partial X} - (\delta - 1) Y \frac{\partial F}{\partial Y} = 0 \] (39)

and
\[ \frac{\partial \Psi}{\partial T} + 2(\delta - 1) \Psi - \delta X \frac{\partial \Psi}{\partial X} = 0 . \] (40)

Combining Eqs. (36)-(40) leads to the following results:
\[ F = F_0 |E|^{1/2}. \] (41)
\[ \Psi = \Psi_0 X^{2(\delta - 1)/\delta} \] (42)

and
\[ S = \frac{2(3\delta - 2)(\delta - 1)}{\delta^2} \Psi_0 X^{2(\delta - 1)/\delta}, \] (43)

where \( E \equiv Y^2/2 + \Psi(X) \). These equations provide an analytic solution to the Poisson-CBE pair where the density and potential are power-law functions of the radial coordinate \( X \). The solutions are valid for \( 2/3 < \delta < 1 \). (For \( \delta \geq 1 \), there is no natural cut-off for the velocity integral which does not then converge. For \( \delta < 2/3 \) the mass within a finite radius is infinite.)

It is interesting to note that the physical distribution function has the same functional form as in Eq. (11): \( F = F_0 |E|^{1/2} \) where \( E \equiv v_r^2/2 + \Phi(r) \) is the usual particle energy. The fact that \( F \) depends on \( v_r \) and \( r \) only through \( E \) follows from the condition that at the instant infall ceases, the system is time-independent (the Jeans theorem). The specific form found, the \( E^{1/2} \) law, is a result of the constraint that the system be self-similar during the infall phase and continuously into the virialized state.
Our conjecture is that Eqs. (41) and (42) describe the system up to the instant when infall ceases and that after this, the distribution function is given by \( F(E) = F_0|E|^{1/2} \). This conjecture is subject to the following caveats: 1) The solution is reached asymptotically and in general will apply to the most tightly bound particles. 2) The system must match onto the cosmological background beyond its boundary and therefore the potential and density must change near \( X_s \); 3) The system does undergo some readjustment after infall cease, e.g., \( 2K/W \) does change, though not by very much.

Eqs. (42) and (43) might, by continuity, be taken to imply that during the infall phase the physical potential and density, \( \Phi \) and \( \rho \), are time-independent. However, during the infall phase, the system itself is not time-independent: Quite to the contrary, the size of the relaxed region is increasing as \( t^\delta \). The new material that falls in does not change the density profile in the already-relaxed region and so in this sense the system is steady. Thus it is not surprising that the \( E^{1/2} \) solutions have been encountered in studies of equilibrium solutions of the CBE (Evans 1994, Henriksen & Widrow 1995). In Henriksen & Widrow (1995), we surveyed all self-similar steady-state solutions with radial orbits in spherical symmetry. These were found to be the very same power-law solutions found here (cf. Eqs. (2.20) and (2.22) in that paper with Eqs. (42) and (43)). Note that \( \delta_{HW95} = \delta/(\delta - 1) \) so that the allowed range for \( \delta \) found here corresponds to \( \delta_{HW95} < -2 \). The \( E^{1/2} \) law for the distribution function was also discussed there. Indeed, one can show, using the standard solution to Abel’s integral equation (e.g., Binney and Tremaine, 1987) that any power-law density profile for a system of particles on radial orbits leads to this form for the distribution function.

Figure 6 gives the distribution function \( F(\mathcal{E}) \) for our simulated halo. We have chosen a time-frame close to the point where the last particles are falling in. As expected, the agreement with the \( E^{1/2} \) law is best for the most tightly bound particles.

### 4.2. Negative Temperature Models for the Distribution Function

It is of fundamental interest in astrophysics and cosmology to understand complicated systems like galaxies and clusters in terms of simple distribution functions. For an isolated system in equilibrium, the distribution function will depend only on the integrals of motion. Merritt, Tremaine, and Johnstone (1989) have investigated models based on the following functional form for the distribution function:

\[
    f = f(E, j^2) = \begin{cases} 
        A(-E)^{3/2}e^{a(j^2+2r_a^2)} & E < 0 \\
        0 & E \geq 0 
    \end{cases} 
\]  

(44)

where \( A, a, r_a \) are positive parameters. \( a > 0 \) corresponds to “negative temperature” and represents the essential distinction between these models and those considered by Stiavelli and Bertin (1986). Figure 7 shows this distribution function in \((r, v_r)\) phase space. We note that the effect of the exponential is primarily to “empty” the central region of phase space corresponding
to what would be the most tightly bound particles. It is not clear whether this is a physically plausible equilibrium configuration in light of the radial instability discovered by Hénon (1973) (see also Barnes, Goodman, & Hut 1986).

Eq. (44) was arrived by considering a crude model of violent relaxation (Lynden-Bell 1967). In particular, the energy dependence assumes that the relaxation process redistributes the energies of particles as they pass through regions of rapidly varying gravitational potential (Tremaine 1987; Merritt, Tremaine, and Johnstone 1989). However, the process of structure formation in hierarchical clustering models may proceed through relaxation that is violent enough to wash out the intricate phase space structure of the FG and B85 solutions but not so violent as to remove all memory of the initial particle ordering. To illustrate this last point, we plot initial vs. final energies for the particles in the simulation (Figure 8). While there is a certain amount of scatter due to the instability, the correlation is clear: particles remember where they were in the initial perturbation. This point was discussed in the context of full 3-dimensional N-body simulations by Quinn and Zurek (1988).

We propose an alternative distribution function, motivated by the results of the previous section, and found by replacing the \((-E)^{3/2}\) factor in Eq. (44) with \((-E)^{1/2}\). We restrict our discussion to strictly radial orbits and consider a distribution function of the form

\[
f(r, v) = \begin{cases} 
A (-E)^{1/2} e^{aE} \delta (J^2) & E < 0 \\
0 & E \geq 0
\end{cases}
\] (45)

The analysis of this family of models is straightforward and the details are left for the appendix. The distribution function in \((r, v_r)\) phase space is shown in Figure 9. Note that the distribution function does decrease as \(E \to -\infty\) though not as dramatically as in the model described by Eq. (44). Our conjecture is that Eq. (45) also describes, at least approximately, the distribution function during the self-similar phase, provided we substitute \(E\) for \(E\). This would allow for arguments similar to those found in the previous section where the system finds itself in an equilibrium configuration once infall ceases.

As noted above, the effect of the exponential is to “empty” the central region of phase space. Physically, this might arise because the initial density profile is not strictly a power-law function of radius. In Eq. (33), for example, the density as \(r \to 0\) is less than what it would be for a pure power-law profile. This manifests itself in slightly fewer particles in the central regions of phase space in the final collapsed object, as can be seen in Figure 2. In addition, the mild form of violent relaxation that occurs in our model may also lead to a cut-off at large negative energies in the distribution function.

Given an empty center in phase space, one can show, by methods similar to those in the appendix, that \(\Psi(X) \propto (-\log(X))^{2/3}\) and thus \(\rho t^2 \propto (-\log X)^{-1/3} X^{-2}\). Our proposed form for the distribution function, Eq. (45), requires only a weak logarithmic correction to the inverse square law (the time dependence depends on \(\delta\)).
4.3. $\delta = 1$: Boundary of Self-similar Behaviour

To complete our discussion of the SSIM we turn to initially flat profiles ($\delta > 1; \epsilon < 2$) and to the transition case ($\delta = 1; \epsilon = 2$). It is clear, both from our simulations and from the work of FG, that there is no strictly self-similar evolution for these initial conditions. This can be understood in a variety of ways. Consider first the phase diagram for the characteristic curves of the self-similar CBE. It is useful to define an effective potential, $\Psi_{\text{eff}} \equiv \Psi + \delta(\delta - 1)X^2/2$ (see HW). For $\delta < 1$ (Figure 2 of HW) there is a saddle point singularity at $Y = \delta X$ and $d\Psi/dX_{\text{eff}} = 0$, i.e., simultaneously a turning point and an extremum in the effective potential. However, when $\delta > 1$, the effective potential increases monotonically and there are no singular points at finite $X$. Since it is the appropriate pair of separatrices of the saddle point that form an effective boundary to the relaxed region of phase space, the $\delta > 1$ case corresponds to the situation where the region of relaxation extends to infinity. Infalling particles continuously perturb the inner regions of the halo. This last comment can be understood from the observation that the potential per unit mass increases outward for $\delta > 1$. The implication is that in this case the system is $T$-dependent in our scaled variables. Our simulations as well as others (e.g. Moutarde et al., 1995) show however that the system evolves toward the $r^{-2}$ profile, the boundary of the so-called steep cases.

The $\delta = 1$ case has $X = r/t$ and is an example of the general class of self-similarity called ‘homothetic’ (e.g. Carter and Henriksen, 1991). It has been studied in the context of isothermal Newtonian collapse and in spherically-symmetric General Relativistic collapse. It is recognized here, in the context of the CBE-Poisson system, for the first time.

Following HW we consider the characteristic curves for the CBE-Poisson system. Along a given characteristic,

$$\frac{dY}{dX} = \frac{1}{X - Y} \frac{d\Psi}{dX}$$  \hspace{1cm} (46)$$

and

$$\mathcal{F} = \text{constant}$$  \hspace{1cm} (47)$$

(cf. Eq. (14) and (15) with $\delta = 1$ and $\partial/\partial T = 0$). It is a simplifying property of this case that $\mathcal{F}$ is constant on a characteristic.

We label characteristic curves by the continuous variable $\zeta$ so that $\mathcal{F} = \mathcal{F}(\zeta)$ and $Y = Y(X, \zeta)$. The right-hand side of the Poisson equation can then be transformed to an integral over $\zeta$:

$$\frac{d}{dX} \left( X^2 \frac{d\Psi}{dX} \right) = \int \mathcal{F}(\zeta) \frac{\partial Y}{\partial \zeta} d\zeta .$$  \hspace{1cm} (48)$$

Eqs. (46) and (48) pose a rather intractable integro-differential problem. Fortunately the self-similar nature of this case suggests an approximate form for the density profile and potential...
(i.e., $\rho \propto r^{-2}$ and $\Psi = \Psi_o \ln X$) which can be used as an initial guess in an iterative attack on the problem. With this choice for the potential, the equation for a characteristic curve reduces to

$$\frac{dY}{dX} = \frac{\Psi_o}{X(X-Y)}. \quad (49)$$

This equation may be integrated to give the family of characteristics implicitly:

$$\frac{1}{x} = e^{\frac{y^2}{2}} \left( \frac{1}{\zeta} - \text{erf}(y) \right), \quad (50)$$

where $X \equiv x\sqrt{2\Psi_o\pi}$ and $Y \equiv y\sqrt{2\Psi_o}$. The iteration procedure proceeds by taking the above solution for $Y(X, \zeta)$ and using it to perform the integration in the Poisson equation.

Examples from this family of curves are shown in Figure 10. This figure is reminiscent of the FG and B85 solutions. However each of those solutions represents a single-stream trajectory in phase space. Conversely, each curve in Figure 10 is regarded as an independent characteristic in the solution to the CB and Poisson equations, at least to this order of approximation. This represents an essential departure from the FG and B85 approach wherein each ‘loop’ is populated sequentially from the same stream of particles so that $\mathcal{F}$ is non-zero along a one-dimensional curve in phase space. In our approach phase space is populated not only by this orderly ‘phase mixing’, but also through the HW instability. This allows the phase-space density to take up its most general variation as allowed by the characteristic curves of the CBE. This is perhaps the best definition of a ‘relaxed’ system.

Still this last argument does not explain why the system with a flat initial profile is driven toward the self-similar “boundary” rather than away from it. This question is of course very general and is not unrelated to the question as to why a violently relaxing, isolated, collisionless system eventually virializes. Some clues are provided by Tremaine, Hénon and Lynden-Bell (1986) and references therein. In particular they show that equilibrium is always attained as the maximum of an H-function. The H-function is some (unfortunately undefined) integral over a functional of the distribution function.

We propose that there exists a similar H-theorem that describes the onset of self-similarity. It is, however, necessary to consider a more general integrands than those considered by Tremaine, Hénon and Lynden-Bell (1986). The extension to more general integrands is by way of the Lyapounov functional idea (Berryman, 1980). Essentially in the present context, this reduces to finding a functional of the form

$$\mathcal{H} = -\int h(\mathcal{F}, X, Y) \, dX \, dY, \quad (51)$$

that increases throughout the collisionless evolution, and that is a finite constant in the self-similar state. Because the transformation to scaled variables leads to a transformation of the physical
quantities by various powers of $e^T$, we must in general require that $H$ vary more rapidly than an exponential, i.e., that it reach the self-similar state in a finite time. This last requirement precludes candidates like $-\mathcal{M}$ which otherwise have the properties of a Lyapounov functional. Such a choice would, in fact, give no information since the logarithmic derivative of the functional is then constant and there is no real approach to the self-similar state.

One possible functional is

\[ H = -\frac{1}{2} \int \mathcal{F} (Y - \delta X)^2 dX dY, \quad (52) \]

provided that the integral is restricted to the region of phase space for which $(Y - \delta X)(\partial \Psi_{\text{eff}} / \partial X) > 0$. This last condition essentially restricts the integration to the region of phase space where $v_r > \delta r / t$ since $\partial \Psi_{\text{eff}} / \partial X > 0$, at least for the cosmological cases considered here. The condition also means that particles that are still taking part in the expansion (i.e., those particles at large $X$) are excluded from the integration since these have $v_r = 2r / 3t$. $H$ is clearly constant in the self-similar state since $\partial T \mathcal{F} = 0$. However, time-dependence in scaled variables (as well as physical variables) arises as the “spiral pattern” of the FG and B85 solution is filled in. This implies that the corresponding function in physical variables, $H \equiv H \exp (5\delta - 4)T$ is not a constant (which would render $H$ trivial) so long as particles are falling in. One can show explicitly that $\partial T H > 0$ since infall increases the number of outward going particles and $\delta r / t$ decreases with time. We see that the system is driven to the self-similar state by the continuing infall of new particles. Once the source of particles is removed, self-similar infall gives way to self-similar equilibrium.

It thus seems possible that generalized Lyapounov functionals exist whose monotonic increase measure the approach to self-similarity just as the H-functional of Tremaine, Hénon and Lynden-Bell (1986) for the approach to equilibrium. In fact similar arguments may be carried out for the steep profiles ($\delta < 1$) using an integrand $h = \frac{1}{2} \mathcal{F} X^2$ and carrying the integration over the inward going particles. In this case the system remembers the self-similarity in the ultimately steady state.

### 4.4. Keplerian Halos

A second example that can be treated analytically is that of test particles on radial orbits outside a fixed mass. Such a situation arises when accretion continues after the bulk of the system mass has fallen in. We expect that this ‘Keplerian’ set of particles will pursue a self-similar evolution with similarity class $\delta = 2/3$ since now the effective remembered constant is $GM$ where $M$ is the mass that accumulates during the primary accretion phase (see e.g. Henriksen & Widrow 1995). This reveals immediately the possible interest since now the power law part of the self-similar evolution will have a density profile $\rho \propto r^{-2/\delta}$ or $r^{-3}$ and hence might explain the outer steep part of the NFW profile.
The characteristics of the CBE delineate the shape of the eventual relaxed region in phase space. The topology of the phase diagram is typical of steep self-similar evolution. The equation of the characteristics in scaled variables is found from Eq. (8) of HW by setting $\delta = 2/3$ and $\Phi = \Psi = -M/X$. The family of curves has the anticipated saddle point at $X_c^3 = 9M/2$ and $Y_c = 2X_c^3/3$. The discussion is simplest in terms of the variables $u \equiv (Y - X/3)/(X_c/3)$, and $x \equiv X/X_c$. Then the family of characteristics is given by

$$\frac{du}{dx} = \frac{2(x - 1/x^2)}{(u - x)}. \quad (53)$$

A sample of characteristic curves is shown in Figure 11. Figure 11a is constructed in terms of the variables $u$ and $x$. The critical curves are not drawn explicitly, but it is relatively easy to identify their slopes ($1/3$ and $-1/2$) as the limiting curves passing through $(1, 1)$ (the ‘x’ in the diagram). These limiting curves connect respectively the stable nodes (lower left to upper right) and unstable nodes (upper left to lower right) that exist at infinity. Figure 11b shows the same curves, but drawn in terms of the variables used in other plots in the paper.

We expect that the only curves to be populated are those in the left part of the figure (gravitationally bound particles) together with an accretion stream which joins the node at the upper right. There is of course an inner boundary for these types of particles corresponding to the outer edge of the main system. Roughly speaking, the radius of the transition region will correspond to the parameter $R_s$ in the NFW profile.

5. Summary and Conclusions

N-body simulations of collisionless, self-gravitating matter, such as those by Navarro, Frenk, and White (1996) and Moore et al. (1997), have provided some tantalizing results, in particular suggesting that nonlinear structure in the Universe possesses certain scale-invariant or self-similar characteristics. These results are perhaps not surprising given that most simulations assume an Einstein-de Sitter Universe and an at least approximately scale-free initial spectrum of density perturbations. With these assumptions, there is only one characteristic scale in the Universe, the expansion rate. This observation has been exploited by Press and Schecter (1974), Bower (1991), Lacey and Cole (1993), and others to formulate an analytic model for the evolution of nonlinear structure within the framework of the hierarchical clustering scenario. However, these models are limited in scope. While they provide predictions for distribution of collapsed objects as a function of mass, they say little about the dynamical process by which a system relaxes to a virialized (or quasi-virialized) state.

The SSIM is, in some respects, complementary to a Press-Schecter type formalism. The model begins with highly idealized initial conditions (strictly power-law initial density profile; no angular momentum) but allows one to follow in detail the complete evolution of a system. Analytic calculations and numerical simulations provide clues as to the density profile and distribution
function of the system both during the infall phase and in the final equilibrium state.

Our results can be summarized as follows:

- Soon after gravitational collapse begins, the system, driven by infall of mass through its boundary, enters a period of self-similar evolution. The system quickly virializes once infall has ceased. It is likely that the final virialized state is one of the stationary self-similar family discussed by Evans (1994) and Henriksen & Widrow (1995) with the self-similar ‘class’ $\delta$ remembered from the dynamic self-similar infall phase.

- During the infall phase, the system relaxes through a combination of phase mixing, phase space instability, and moderate violent relaxation. However, relaxation does not completely randomize particle energies, i.e., particles maintain some memory of their initial state.

- As the system relaxes, the single trajectory of the FG and B85-type solution is transformed into a continuous distribution in phase space. The subsequent cycles of the single trajectory are now regarded as characteristic curves of the smooth distribution function. We have used this approach to study two similarity classes that are important to our arguments; $\delta = 1, 2/3$.

- The self-similar phase may be recognized as “stationary” in the appropriate scaled variables. By following the development of a perturbation in these variables, we can achieve a remarkable dynamic range in our simulations. The size of the system is always a fixed fraction of the turn-around radius. In addition, the relation for the mass of the system as a function of time is in agreement with that found by Lacey and Cole (1993).

- During the self-similar phase, the system obeys a virial condition $2K/W = \text{constant}$. The constant differs from the usual value 1 (though by only about 10%) due to mass flux through the system boundary and the time-dependent nature of the infall solution.

- From the observation that the transition from infall phase to isolated state is relatively gentle, follows the conclusion that the distribution function for the final object has the form $F \propto (-E)^{1/2}$. In a realistic system, we expect that the distribution function will eventually decrease at large negative energy and so we include an exponential negative temperature factor. This leads to a form for the distribution function that is similar to the ones proposed by Stiavelli & Bertin (1985) and Merritt, Tremaine, and Johnstone (1989) and reminiscent of the Fermi-type function proposed by Lynden-Bell (1967).

- The SSIM predicts an effectively universal profile $\rho \propto r^{-\mu}$ with $\mu \simeq 2$ for the intermediate region of a dark matter halo. The system does remember the initial profile for $\epsilon > 2$ (the so-called steep cases) but the effects on the final density profile are relatively minor. For flat initial profiles, the system is driven towards the limiting self-similar profile ($\delta = 1$) by accretion of spherical shells with ever-increasing binding energy, a process reminiscent of that considered by Syer and White (1997). In this sense, the density profile is a one-sided attractor.
Because of finite lifetime effects, the inevitable breaking of scale invariance at the centre of the initial perturbation, and the presence of angular momentum, one can expect a physical (although ill-defined) flattening near the centre of the relaxed halo. In addition, the simulations themselves are sensitive to resolution effects in this region and so the actual flattening may be less pronounced than in the NFW profile.

- The outer parts of the halo, still in a self-similar relaxation phase, can be quite steep either because the particles there are not full relaxed or because they are essentially in Keplerian orbits about the bulk of the halo mass. In this latter case, the power law is $r^{-3}$ in accordance with the NFW profile and corresponding to the similarity class $\delta = 2/3$.

- In our simulations, the outer edge of the initial mass distribution, $r_0$, determines ultimately the radius of the transition region between $r^{-2}$ and $r^{-3}$ behaviour ($R_s$ in the NFW profile). Likewise, the total mass in the simulation fixes $M_s$.

- The evolution of the initially flat systems towards a limiting self-similar state during the infall phase recalls the evolution of an isolated thermodynamic system towards a maximum entropy state. Here we propose Lyapounov functions that are maximized in the self-similar state and which are monotonic under collisionless evolution of the distribution function during continued infall. The existence of such functions tends to strengthen our belief in the ‘universality’ of the SSIM.

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Appendix

In this appendix we discuss equilibrium models for spherically symmetric systems of collisionless particles on completely radial orbits. In particular, we consider the following functional form for the distribution function:

\[
    f(r, v) = \begin{cases} 
        A (-E)^{\alpha} e^{aE} \delta (j^2) & E < 0 \\
        0 & E \geq 0 
    \end{cases} 
\]

where \(A\), \(a\), and \(\alpha\) are positive constants. In the negative temperature models of Merritt, Tremaine, and Johnstone (1989), \(\alpha = 3/2\). (The precursor to these models, studied by Stiavelli and Bertin (1985), also had \(\alpha = 3/2\) but with \(a < 0\), i.e., “positive temperature). On the other hand, our analysis suggests \(\alpha = 1/2\), at least in the relaxed region.

We begin with the Poisson equation. Following Merritt, Tremaine, and Johnstone (1989), we define the auxiliary potential, \(W = -a \Phi\) and change integration variables from \(v_r\) to \(x\) where \(x^2 \equiv -aE = W - av_r^2\). The Poisson equation then becomes:

\[
    \frac{d}{dr} r^2 \frac{d}{dr} W = -\frac{g(W)}{\gamma} 
\]

where \(\gamma \equiv a^{n-1/2}/2\pi GA\) and

\[
    g(W) \equiv 2^{5/2} \pi \int_0^{\sqrt{W}} \frac{dx x^{2\alpha+1} e^{-x^2}}{\sqrt{W-x^2}} 
\]

At this stage, Merritt, Tremaine, Johnstone (1989) introduce the ‘homology invariant functions’ \(z = W\) and \(y = -rdW/dr\) (Chandrasekhar 1939) thereby reducing the Poisson equation to a first order ODE. This technique does not seem appropriate for \(\alpha = 1/2\) and we choose instead to solve Eq. (2) directly.

It is convenient to change variables from \(r\) to \(s \equiv -\ln(r)\),

\[
    \frac{d^2 W}{ds^2} - \frac{dW}{ds} = -\frac{g(W)}{\gamma}, \quad (4)
\]

and integrate from large \(r\) toward the origin, i.e., \(s = s_i\) to \(s = s_f\) where \(s_i < 0\) (e.g., \(s_i = -15\)) and \(s_f > 0\) (e.g., \(s_f = 15\)). At large \(r\), \(W \ll 0\) and \(g(W) = \sqrt{2\pi^2 W/\gamma}\). Asymptotic solutions are easily found:

\[
    W = W_+ e^{\beta_+ s} + W_- e^{\beta_- s} \quad (5)
\]

where \(\beta_{\pm} = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4\sqrt{2\pi^2/\gamma}} \right)\). At large \(r\) (\(s \to -\infty\)) the \(\beta_-\) solution dominates and corresponds to power-law behaviour for the potential: \(\Phi \propto r^{-\beta_-}\). In general, initial conditions at
finite $s$ must include an admixture of both solutions in order to satisfy the appropriate boundary conditions at the origin, namely that the mass be positive definite but not include a point mass at $r = 0$.

We use standard numerical techniques to find the potential. With the potential in hand, the distribution function, $F(r, v_r)$ is easily computed leading the gray-scale plots in Figure 7 and 9.
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Figure Captions

Figure 1: Evolution of a system of spherical shells in $(X, Y)$-phase space. The initial density perturbation is a power law function of radius $\delta \rho_i \propto r^{-2.5}$ except near the origin (see text). Initial velocities follow the unperturbed Hubble flow. The panels a-d correspond to logarithmic times $T = 2.7, 8.0, 13.3$ and $18.7$ respectively.

Figure 2: Same as Figure 1 (only $T = 13.3$ and 18.7 are shown) but with equal mass particles.

Figure 3: Density as a function of $X$ for 5 different choices of $T$ with $T$ increasing from bottom to top. We plot $\rho r^3 \propto XS \exp (3\delta - 2)T$. (If instead we were to plot $XS$ the curves would lie on top of one another.) The initial density profiles are given by $\delta \rho_i \propto r^{-\epsilon}$ with $\epsilon = 2.5, 2.0, \text{and } 1.5$ for Figures 3a, 3b, and 3c respectively. Dashed lines are the predictions from FG, for $\epsilon = 2.5$, $XS \propto X^{6/7}$ and for $\epsilon = 2.0$ and 1.5, $XS \propto r^1$.

Figure 4: Density law for different values of the softening length $\epsilon$ at two different times. The solid curves are for $\epsilon = 0.0005$; dotted curves are for $\epsilon = 0.0025$; dashed curves are for $\epsilon = 0.01$.

Figure 5: System-wide quantities as a function of $T$. Note that infall ceases at $T = 15$. Panel a gives the scaled size of the system, $X_s$, defined in the text. Panels b and c give the total scaled mass and energy within $X_s$. Panel d gives the virial ratio $2K/W = 2\mathcal{K}/\mathcal{W}$.

Figure 6: $F(E)$ vs $E$: the distribution of particles as a function of energy,

Figure 7: Gray-scale plots of the distribution function in $r, v_r$ phase space for the so-called negative temperature distribution function of Merritt, Tremaine, and Johnstone (1989).

Figure 8: Scatter plot of initial vs. final particle energies for the same simulation as shown in Figure 1

Figure 9 Same as Figure 7 but for the $|E|^{1/2} \exp (aE)$ distribution function suggested in this work.

Figure 10: Semi-analytic solutions for the characteristic curves of the CB and Poisson system with $\delta = 1$.

Figure 11: Same as Figure 10 but for $\delta = 2/3$. 
$\delta \rho_{\text{init}} \propto r^{-3/2}$
