GERBE PATCHING AND A MAYER-VIETORIS SEQUENCE OVER ARITHMETIC CURVES

BASTIAN HAASE

ABSTRACT. We discuss patching techniques and local-global principles for gerbes over arithmetic curves. Our patching setup is that introduced by Harbater, Hartmann and Krashen (cf. [HH10], [HHK15a]). Our results for gerbes can be viewed as a 2-categorical analogue on their results for torsors. Along the way, we also discuss bitorsor patching and local-global principles for bitorsors. As an application of these results, we obtain a Mayer-Vietoris sequence with respect to patches for non-abelian hypercohomology sets with values in the crossed module $G \to \text{Aut}(G)$ for $G$ a linear algebraic group. Using local-global principles for gerbes, we also prove local-global principles for points on homogeneous spaces under linear algebraic groups $H$ that are special (e.g. $\text{SL}_n$ and $\text{Sp}_{2n}$) for certain kind of stabilizers.

1. INTRODUCTION

Starting with the famous Hasse principle for quadratic forms over number fields, local-global principles for algebro-geometric objects over number fields have been studied extensively. In [HH10], Harbater, Hartmann and Krashen introduced patching techniques for function fields over arithmetic curves. These patching techniques in turn allowed the authors to deduce local-global principles for quadratic forms and central simple algebras over these fields.

Given a function field of an arithmetic curve, the patching setup consists of a finite set of fields $F_i$ over $F$. The inverse limit of these fields with respect to certain inclusions is $F$. The definition of the fields $F_i$ is deduced from a geometric ”covering” of the special fiber of the curve. Let $\mathcal{F}$ denote the system of fields $\{F_i\}$. When we say that patching holds for a certain class of objects over $\mathcal{F}$, we mean that the datum of such an object over $F$ is equivalent to the datum of a collection of compatible objects over the fields $F_i$. We say that the local global principle holds if triviality of an object over $F$ is equivalent to triviality on the level of $F_i$ for all $i$.

In [HHK14] and [HHK15a], Harbater, Hartmann and Krashen prove patching results for torsors under linear algebraic groups and Galois cohomology groups. In both of these papers, the authors use their patching results to characterize when local-global principles hold.

In this paper, we extend these results to the case of gerbes over arithmetic curves. We prove patching results for certain gerbes and use these results to give a criterion for when a local-global principle for gerbes hold. Using this criterion, we then prove that the local-global principle holds for gerbes banded by certain linear algebraic groups.

Note that for a sheaf of abelian groups $A$ defined over $F$, its second cohomology group classifies gerbes banded by $A$.

2010 Mathematics Subject Classification: Primary: 14H25, 18G50. Secondary: 14M17.

Key words and phrases: gerbes, non-abelian cohomology, local-global, patching, Mayer-Vietoris, homogeneous spaces.
Gerbes can not only be thought of as a 2-categorical analogue of torsors, but also arise naturally in the context of algebraic geometry. For instance, if $X$ is a homogeneous space under a group sheaf $H$, then the quotient stack $[X/H]$ that describes relative morphisms of principal homogeneous spaces under $H$ to $X$ is a gerbe (whose band is determined by the stabilizers of geometric points of $X$).

Our main result on gerbe patching is the following theorem.

**Theorem A** (cf. Theorem 5.7). Let $L$ be a linear algebraic band over $F$. Assume either

- $\text{char}(k) = 0$ or
- $\text{char}(k) = p > 0$ and the center of $L$ has finite order not divisible by $\text{char}(k)$ and $G$-bitorsor factorization holds for $\mathcal{F}$ over every cover $Z \to Y$.

Then, gerbe patching holds over $\mathcal{F}$ for $L$-banded gerbes.

Our approach to gerbe patching relies on a semi-cocyclic description of gerbes in terms of bitorsors introduced by Breen in [Bre90] and [Bre91].

Recall that a $(G, H)$-bitorsor $P$ is a left $G$- and right $H$-torsors such that the left $G$- and the right $H$-action commute. In the case $G = H$ we simply speak of a $G$-bitorsor. The trivial $G$-bitorsor is $G$ with the natural right and left action.

In view of the semi-cocyclic description, we prove gerbe patching by proving that patching holds for bitorsors. Our results on patching gerbes and bitorsors allow a natural interpretation in terms of hypercohomology with values in the crossed module $G \to \text{Aut}(G)$.

**Corollary B** (Mayer-Vietoris of non-abelian hypercohomology over curves, Corollary 5.8). Let $G$ be a linear algebraic group defined over $F$ and let $L$ denote the associated band. Under the assumption of Theorem A, there is an exact sequence of pointed sets

\[
\begin{align*}
H^{-1}(F, G \to \text{Aut}(G)) &\to \prod_{i \in I,} H^{-1}(F_i, G \to \text{Aut}(G)) \to \prod_{k \in K,} H^{-1}(F_k, G \to \text{Aut}(G)) \\
\quad \to H^0(F, G \to \text{Aut}(G)) &\to \prod_{i \in I,} H^0(F_i, G \to \text{Aut}(G)) \to \prod_{k \in K,} H^0(F_k, G \to \text{Aut}(G)) \\
\quad \to H^1(F, G \to \text{Aut}(G)) &\to \prod_{i \in I,} H^1(F_i, G \to \text{Aut}(G)) \to \prod_{k \in K,} H^1(F_k, G \to \text{Aut}(G))
\end{align*}
\]

Note that this result is analogous to the Mayer-Vietoris sequences obtained in [HHK14] and [HHK15a].

By use of the Mayer-Vietoris sequence, we can prove that the local-global principle for $G$-bitorsors holds if and only if the center $Z(G)$ of $G$ satisfies simultaneous factorization over $\mathcal{F}$ (see Section 2 for details). In particular, using a result of Harbater, Hartmann and Krashen in [HHK09], we prove that local-global principle for $G$-bitorsors holds if $Z(G)$ is rational and connected.

The Mayer-Vietoris sequence also allows us to prove that the local-global principle for $G$-gerbes is equivalent to factorization for $G$-bitorsors (see Section 3.1 for details). We prove that bitorsor factorization (and thus local-global principle) holds for various groups such as $\text{SL}_1(D)$ where $D$ is a central simple algebra over $F$.

As an application of the theory, we use local-global principles for gerbes to deduce local-global principles for certain homogeneous spaces. Recall that a linear algebraic group $H$ over $F$ is special if $H^1(K, H)$ is trivial for every field extension $K/F$. The groups $\text{SL}_n$ and $\text{Sp}_{2n}$ are examples of special groups. Let now $X$ be a homogeneous space under a special group $H$ and assume that the stabilizer $G$ is defined over $F$. 
Theorem C (cf. Theorem 5.20). Let $H$ be special and let $X$ be homogeneous space under $H$. Assume that the stabilizer $G$ of a geometric point of $X$ is defined over $F$. If $\text{char}(k) = 0$, the local-global principle for $X$ holds if and only if bitorsor factorization holds for $G$. If $\text{char}(k) = p > 0$, assume that

- $p \nmid |Z(G)| < \infty$,
- $G$-bitorsor factorization holds for $\mathcal{F}$ over every cover $Z \to F$.

Then, the local-global principle holds for $X$ over $\mathcal{F}$.

Using our results on bitorsor factorization, we then show that local-global principle for $X$ holds for certain type of $G$, e.g. if $G$ is finite and simple.

The paper is structured as follows. In the second section we review the patching setup introduced by Harbater, Hartmann and Krashen and describe the main results concerning patching over arithmetic curves.

In the following two sections, we discuss bitorsor and gerbe patching over an arbitrary inverse factorization system of fields and relate it to patching of torsors.

In the third section, we utilize a description of bitorsors introduced by Breen in [Bre90] to reduce bitorsor patching to torsor patching. This then allows us to deduce the first two rows of the Mayer-Vietoris sequence in hypercohomology. From this sequence, we can then show that local-global principle for $G$-bitorsors with respect to patches holds if and only if the center of $G$ satisfies factorization.

In the fourth section, we recall a semi-cocyclic description of gerbes in terms of bitorsors. We also recall the notion of a band and prove the following patching results for its second non-abelian cohomology set.

Proposition D. 5.6 Let $L$ be a band over $F$ such that its center $Z(L)$ is a linear algebraic group over $F$ with finite order coprime to $\text{char}(k)$. Then, if $\text{H}^2(F, L) \neq 0$, patching holds for $\text{H}^2(\bullet, L)$ over $\mathcal{F}$.

Using our results on bitorsor patching, we can prove that gerbe patching holds under a further technical assumption (cf. Theorem 4.19). This allows us in turn to complete the construction of the Mayer-Vietoris sequence above. Finally, we relate the local-global principle for gerbes to factorization of bitorsors.

In the fifth section, we then specialize to patching over arithmetic curves and prove our main results for patching bitorsors and gerbes in this setting. We also prove that bitorsor factorization and local-global principle for gerbes hold for various interesting linear algebraic groups. We conclude the section by applying our results on local-global principles for gerbes to deduce our results on local-global principles for points on homogeneous spaces.

In the appendix, we review hypercohomology with values in crossed modules as developed by Breen in [Bre90] and Borovoi in [Bor92].

1.1. Acknowledgments. The author would like to thank Max Lieblich for various helpful conversations. The author is partially supported by National Science Foundation grants DMS-1401319 and DMS-1463882.

2. Patching and Local-Global Principle

2.1. Patching of vector spaces. In this section, we will recall patching results from Harbater, Hartman and Krashen (compare [HH10], [HHK15a] and [HHK15b]). We will adapt their notation.
Throughout this section, let $\mathcal{F} = \{ F_i \}_{i \in I}$ denote a finite inverse system of fields with inclusions as morphisms. Let $F$ denote $\lim_{i \in I} \mathcal{F}$. All sheaves will be sheaves in the big étale topology over $(\text{Sch}/F)$ or $(\text{Sch}/F_i)$ for $i \in I$.

**Definition 2.1.** A factorization inverse system over a field $F$ is a finite inverse system of fields such that

1. $F$ is the inverse limit.
2. The index set $I$ can be partitioned as $I = I_v \sqcup I_e$ such that:
   - (a) For any $k \in I_e$ there are exactly two elements $i, j \in I_v$ such that $i, j > k$.
   - (b) These are the only relations.

For each index $k \in I_e$, fix a labeling $l_k, r_k$ for the two elements in $I_e$ with $l_k, r_k > k$. Then, let $S_l$ denote the set of triples $(l_k, r_k, k)$.

Given a factorization inverse system, one can associate to it a (multi-)graph $\Gamma$. Its vertices are the elements of $I_v$, where the edges come from $I_e$ (explaining the subscripts). Given $k \in I_e$, the corresponding edge connects the vertices $i, j \in I_v$ iff $(i, j, k) \in S_l$. Note that $\Gamma$ is connected, as the inverse limit $F$ would otherwise admit zero-divisors. We will sometimes specialize to the case where $\Gamma$ is a tree.

**Example 2.2.** A basic example of a factorization inverse system is given by fields $F \subset F_1, F_2 \subset F_0$ such that $F = F_1 \cap F_2$. Pictorially, we get

$$
\begin{array}{c}
F_0 \\
\downarrow \\
F_1 \\
\downarrow \\
F_2 \\
\downarrow \\
F \end{array}
$$

Note that in this case $I_e = \{0\}, I_v = \{1, 2\}$ and $S_l = \{(1, 2, 0)\}$. The corresponding graph $\Gamma$ is a tree.

**Definition 2.3.** A vector space patching problem $\mathcal{V}' = (\{ V_i \}_{i \in I_v}, \{ v_k \}_{k \in I_e})$ for a factorization inverse system $\mathcal{F}$ is given by finite dimensional $F_i$ vector spaces $V_i$ together with $F_k$ isomorphisms $v_k : V_i \otimes_{F_i} F_k \rightarrow V_j \otimes_{F_j} F_k$ whenever $(i, j, k) \in S_l$.

A morphism of patching problems $\mathcal{V}' \rightarrow \mathcal{V}''$ is a collection of morphisms $V_i \rightarrow V_i'$ for all $i \in I_v$ that are compatible with the morphisms $v_k, v_k'$.

The category of vector space patching problems is denoted by $\text{PP}(\mathcal{F})$. By construction, we have $\text{PP}(\mathcal{F}) \simeq \prod_{(i,j,k) \in S_l} \text{VECT}(F_i) \times_{\text{VECT}(F_k)} \text{VECT}(F_j)$. If $A/F$ is a finite product of finite separable field extensions, let $\text{PP}(\mathcal{F}_{\text{A}/F})$ denote the category of free module patching problems: objects are collections $(\{ M_i \}_{i \in I_v}, \{ v_k \}_{k \in I_e})$ where $M_i$ is a free module over $A_i = F_i \otimes_F A$ of finite rank and $v_k : M_i|_{A_k} \rightarrow M_j|_{A_k}$ are isomorphisms of $A_k$ modules.

Note that we have a canonical functor

$$\beta : \text{VECT}(F) \rightarrow \text{PP}(\mathcal{F})$$

where $\text{VECT}(F)$ is the category of finite dimensional vector spaces over $F$.

**Definition 2.4.** A solution to a vector space patching problem $\mathcal{V}'$ is a $F$ vector space $V$ such that $\beta(V)$ is isomorphic to $\mathcal{V}'$.

If $\beta$ is an equivalence of categories, then every patching problem has a solution that is unique up to isomorphism.
Definition 2.5. A linear algebraic group $G$ over a field $K$ is a smooth affine group scheme of finite type.

Definition 2.6. A linear algebraic group $G$ satisfies simultaneous factorization over $\mathcal{F}$ if for any collection of elements $a_k \in G(F_k)$ with $k \in I$, there are elements $a_i \in G(F_i)$ for all $i \in I_i$ such that $a_k = a_r^{-1}a_l \in G(F_k)$ for all $(l, r, k) \in S_I$.

In the case where $G = \text{GL}_n$, we simply say that simultaneous factorization holds over $\mathcal{F}$.

The following result provides the basis of many patching results obtained by Harbater, Hartman and Krashen.

Proposition 2.7 ([HH10, Proposition 2.1]). The functor $\beta : \text{VECT}(F) \to \text{PP}(\mathcal{F})$ is an equivalence of categories if and only if simultaneous factorization holds over $\mathcal{F}$.

If $A = \prod_{i=1}^n L_i$ is a finite product of finite separable field extensions of $F$, then we denote by $\mathcal{F}_A$ the inverse system $\{A_i := A \otimes_F F_i\}_{i \in I}$. It is not necessarily an inverse factorization system but we have $A = \lim \mathcal{F}_A$. Let $\text{Mod}(A)$ denote the category of free modules of finite rank over $A$. Let $\text{PP}(\mathcal{F}_A)$ denote the category of patching problems of the form $\{ (M_i)_{i \in I_i}, \{v_k\}_{k \in I_k} \}$ where $M_i$ is a free $A_i$-module of finite rank and $v_k : M_i|_{A_k} \to M_j|_{A_k}$ is an isomorphism of $A_k$-modules for $(i, j, k) \in S_f$. Note that we have a natural functor $\hat{\beta} : \text{Mod}(A) \to \mathcal{P}(\mathcal{F}_A)$.

The following proposition is a slight variant of [HHK15b, Lemma 2.2.7].

Proposition 2.8. Assume that patching holds over $\mathcal{F}$, i.e. that the functor $\beta : \text{VECT}(F) \to \text{PP}(\mathcal{F})$ is an equivalence. Let $A = \prod_{i=1}^n L_i$ be a product of finitely many finite separable field extensions. Then, patching holds over $\mathcal{F}_A$, i.e. the natural functor $\hat{\beta} : \text{Mod}(A) \to \text{PP}(\mathcal{F}_A)$ is an equivalence.

Proof. Given a patching problem $\{ (M_i)_{i \in I_i}, \{v_k\}_{k \in I_k} \}$ in $\text{PP}(\mathcal{F}_A)$, note that $M_i$ is a finite dimensional vector space over $F_i$ for all $i \in I_i$. Also, $v_k$ is an isomorphism of $F_k$ vector spaces. Hence, by assumption, there is an $F$-vector space $M$ together with $F$-vector space isomorphisms $\phi_i : M|_{F_i} \to M_i$ for all $i \in I_i$, that are compatible with $v_k$. The $A_i$-module structure of $M_i$ is equivalent to the datum of a morphism $\alpha_i : A_i \to \text{End}_F(M_i)$ of $F_i$-vector spaces. As the $v_k$ are $A_k$ module morphisms, they are compatible with $\alpha_k$. Hence, as the functor $\hat{\beta}$ is full, there is a morphism $\alpha : A \to \text{End}_F(M)$ of $F$ vector spaces. The resulting $A$-module $M$ solves the patching problem. This shows that $\hat{\beta}$ is essentially surjective. As every morphism of $A_i$-modules is in particular a morphism of $F_i$-vector spaces, it follows that $\hat{\beta}$ is faithful. Given two $A$-modules $M, N$ and a morphism $\hat{\beta}(M) \to \hat{\beta}(N)$, note that we can lift it to an $F$-vector space morphism $\gamma : M \to N$. As the the image of this morphism in $\text{PP}(\mathcal{F}_A)$ commutes with the $A_i$-action and as $\hat{\beta}$ is faithful, it follows that $\gamma$ is an $A$-module morphism. $\square$

2.2. Patching of torsors and Galois cohomology. Fix a field $F$ and an inverse factorization system $\mathcal{F}$ over $F$. Let $G$ be a group sheaf in the big étale site over $F$.

Definition 2.9. A left principal homogeneous space $T$ over $F$ under a $F$-group scheme $G$ is a $F$-scheme $T$ together with a left $G$ action $G \times F T \to T$ such that the induced morphism $G \times F T \to T \times F T$ given by $(g, t) \mapsto (g \cdot t, t)$ is an isomorphism.

A morphism of left principal $G$-homogeneous spaces $T, T'$ over $F$ is a $F$-morphism $T \to T'$ that is $G$-equivariant.
**Remark 2.10.** When we speak of a principal homogeneous space, we mean left principal homogeneous space unless explicitly stated otherwise.

**Definition 2.11.** A left étale $G$-torsor $T$ over $F$ is an étale sheaf with a left action of the sheaf $h_G$ such that:

1. For all $K$-schemes $Y$ there is an étale cover $\{Y_i \to Y\}$ such that $T(Y_i) \neq \emptyset$.
2. The map $h_G \times T \to T \times T$ given by $(g, t) \mapsto (g \cdot t, t)$ is an isomorphism.

A morphism of $G$-torsors $T, T'$ over $K$ is a morphism of sheaves $T \to T'$ which is $G$-equivariant.

**Remark 2.12.** If $G$ is affine over the base field, then the notions of (left) principal homogeneous space and (left) torsor coincide, i.e. the category of (left) principal $G$-homogeneous spaces over $F$ is equivalent to the category of (left) $G$ torsors over $F$ (compare Proposition 4.5.6 in [Ols16]).

Let $\text{TORSOR}(G)(F)$ denote the category of $G$-torsors over $F$. Let $\text{TPP}(G)(\mathcal{F})$ denote the category of torsor patching problems, i.e. the category with objects $\{(T_i)_{i \in I}, \{v_k\}_{k \in L}\}$ where $T_i$ is a $G_i$-torsor over $F_i$ and $v_k : T_i|_{F_k} \to T_j|_{F_k}$ is an isomorphism of $G_k$-torsors for $(i, j, k) \in S_j$. A morphism of torsor patching problems

$$\left(\{T_i\}_{i \in I}, \{v_k\}_{k \in L}\right) \to \left(\{T'_i\}_{i \in I}, \{v'_k\}_{k \in L}\right)$$

is a collection of $G_i$-torsor morphisms $T_i \to T'_i$ compatible with $v_k$ and $v'_k$.

As in the case of vector spaces, there is a natural functor

$$\beta_G' : \text{TORSOR}(G)(F) \to \text{TPP}(G)(\mathcal{F}).$$

where $\text{TPP}(G)(\mathcal{F})$ is defined analogously to $\text{PP}(\mathcal{F})$.

**Theorem 2.13** ([HHK15a, Theorem 2.3]). Let $G$ be a linear algebraic group over $F$. If the natural functor $\beta : \text{VECT}(F) \to \text{PP}(\mathcal{F})$ is an equivalence of categories, then so is the functor $\beta_G' : \text{TORSOR}(G)(F) \to \text{TPP}(G)(\mathcal{F})$.

The following proposition is a slight variant of [HHK15b, Theorem 2.2.4(c)(iii)].

**Proposition 2.14.** Let $A = \prod_{r=1}^n L_r$ be a product of finitely many finite separable field extensions $L_r/F$. Let $G$ be a linear algebraic group over $A$ (i.e. $G|_{L_r}$ is a linear algebraic group over $L_r$). If the functor $\beta : \text{VECT}(F) \to \text{PP}(\mathcal{F})$ is an equivalence of categories, then so is the functor $\beta_G' : \text{TORSOR}(G)(A) \to \text{TPP}(G)(\mathcal{F}_A)$.

**Proof.** By Proposition 2.8, patching holds for free modules of finite rank over $\mathcal{F}_A$. Hence, $\text{GL}_n$ satisfies factorization over $\mathcal{F}_A$, cf. Theorem 2.7. The proof is now verbatim to the proof of [HHK15a, Theorem 2.3].

Let $G$ be an abelian algebraic group over $F$ and let $H^n(F, G)$ denote the $n$-th Galois cohomology group. For every $(i, j, k) \in S_j$, we have a map $H^2(F_i, G) \times H^2(F_j, G) \to H^2(F_k, G)$ given by $(\alpha_i, \alpha_j) \mapsto \alpha_i|_{F_k} \alpha_j|_{F_k}^{-1}$. The collection of these maps gives a map $\prod_{i \in I_i} H^2(F_i, G) \to \prod_{k \in L_k} H^2(F_k, G)$.

**Definition 2.15.** We say that patching holds for $H^2(\cdot, G)$ over $\mathcal{F}$ if the sequence

$$H^2(F, G) \to \prod_{i \in I_i} H^2(F_i, G) \to \prod_{k \in L_k} H^2(F_k, G)$$

is exact.
2.3. **Patching over arithmetic curves.** We will now recall a patching setup for function fields of arithmetic curves obtained by Harbater, Hartmann and Krashen in [HHK14]. Following the notation of [HHK14], let \( T \) be a complete discretely valued ring with field of fraction \( K \), uniformizer \( t \) and residue field \( k \). Let \( \tilde{X} \) be a projective, integral and normal \( T \)-curve with function field \( F \) and let \( X \) denote its closed fiber.

For any closed point \( p \in X \), let \( \hat{\mathcal{O}}_{\tilde{X},p} \) denote the completion of the local ring \( \mathcal{O}_{\tilde{X},p} \) at its maximal ideal and let \( F_p \) denote the fraction field of \( \hat{\mathcal{O}}_{\tilde{X},p} \). For a subset \( U \subset X \), that is contained in an irreducible component of \( X \) and does not meet other components, let \( R_U \) denote the subring of \( F \) of rational functions regular on \( U \). Let \( \hat{R}_U \) denote its \( t \)-adic completion and \( F_U \) denote the fraction field of \( \hat{R}_U \). For each branch of \( X \) at a closed point \( P \), i.e., for each height one prime \( b \) of \( \hat{\mathcal{O}}_{\tilde{X},p} \) that contains \( t \), let \( \hat{R}_b \) be the completion of \( \hat{\mathcal{O}}_{\tilde{X},p} \) at \( b \) and let \( F_b \) denote its fraction field.

Let \( \mathcal{P} \subset X \) be a non-empty set of closed points of \( X \) including all points where distinct irreducible components of \( X \) meet and all closed points where \( X \) is not unibranched. This implies that \( X \setminus \mathcal{P} \) is a disjoint union of finitely many irreducible affine \( k \) curves. The set of these curves will be denoted by \( \mathcal{U} \).

Let \( p \in \mathcal{P} \) and \( U \in \mathcal{U} \) be chosen such that \( p \) is contained in the closure of \( U \). Then, the ideal defining \( U \) induces an ideal in \( \mathcal{O}_{\tilde{X},p} \). The branches of \( U \) at \( p \) are the height one prime ideals in \( R_p \) containing said induced ideal. Let \( \mathcal{B} \) denote the set of all branches. Note that we have inclusions \( F \subset F_a , F_p \subset F_b \), whenever \( b \) is a branch corresponding to \( U \) and \( p \).

These fields now form a finite inverse factorization system \( \mathcal{F} \). With the notation from Section 2, we have \( I_v = \mathcal{P} \cup \mathcal{U} \) and \( I_v = \mathcal{B} \).

The system just described allows patching for \( G \)-torsors if \( G \) is a linear algebraic group over \( F \).

**Theorem 2.16** ([HH10, Theorem 6.4], [HH15a, Theorem 2.3], [HH15b, Proposition 3.2.1], [HHK14, Theorem 3.1.1]). *The inverse limit of \( \mathcal{F} \) is \( F \). Furthermore, patching holds for finite dimensional vector spaces over \( \mathcal{F} \), every linear algebraic group \( G \) over \( F \) is separably factorizable over \( \mathcal{F} \) and \( G \)-torsor patching holds over \( \mathcal{F} \).*

Once patching for torsors hold, one can describe local-global principle for torsors in terms of simultaneous factorization. We say that \( G \)-torsors satisfy the local-global principle with respect to \( \mathcal{F} \) if for any \( G \)-torsor \( P \), we have that \( P_i \simeq G_i \) for all \( i \in I_v \) if and only if \( P \simeq G \) (here, \( G \) denotes the trivial \( G \) torsor).

**Theorem 2.17** ([HH15a, Theorem 3.5]). *Let \( G \) be a linear algebraic group. Then, local-global principle for \( G \)-torsors holds if and only if \( G \) satisfies simultaneous factorization over \( \mathcal{F} \).*

Let us also recall a result describing when rational linear algebraic groups satisfy factorization.

**Theorem 2.18** ([HH15a, Corollary 6.5]). *Let \( G \) be a rational linear algebraic group. Then, \( G \) satisfies simultaneous factorization over \( \mathcal{F} \) if and only if \( G \) is connected or \( \Gamma \) is a tree.*

**Theorem 2.19** ([HHK14, Theorem 3.1.3]). *Let \( G \) be an abelian linear algebraic group. If \( \text{char}(k) = p > 0 \), assume furthermore that \( p \nmid |G| < \infty \). Then, for any \( n \geq 0 \), patching holds for \( \text{H}^n(\odot, G) \) over \( \mathcal{F} \).*
3. Bitorsors

We will later see that bitorsor patching naturally occurs in the context of gerbe patching. Thus, in this section, we will prove that bitorsor patching holds whenever torsor patching does.

3.1. Generalities. Before we define bitorsors, let us fix some notation. If $Y$ and $X$ are $F$-schemes for some field $F$, let $X_Y = X \times_{\text{Spec}(F)} Y$. We will also often write $F$ instead of $\text{Spec}(F)$.

Even though most of what we discuss here holds over any site, we will restrict ourselves to the special case of the big étale site of $F$. So, all sheaves in this section are sheaves over $F$ in the étale topology.

Definition 3.1. Let $G,H$ be group sheaves. Then, a $(G,H)$-bitorsor $T$ over $F$ is a sheaf over $F$ with a left action of $G$ making it a left $G$-torsor and a right $H$ action making it a right $H$-torsor such that these actions commute.

Let $\text{BITORSOR}(G,H)(K)$ denote the category of $(G,H)$-bitorsors over $F$.

Definition 3.2. Let $P$ be a $(G,H)$-bitorsor and let $P'$ be a $(H,G')$-bitorsor for group sheaves $G,G',H$. Then, we define a $(G,G')$-bitorsor $P \wedge^H P'$ as the sheafification of the presheaf $U \mapsto P(U) \times P'(U)/\sim$, where $(p,p') \sim (ph,hp')$ for $p \in P(U), p' \in P'(U)$ and $h \in H(U)$. It inherits its $G$ action from $P$ and its $G'$ action from $P'$. We say that $P \wedge^H P'$ is the wedge product of $P$ and $P'$.

It follows that the wedge product defines a group structure on the set of isomorphism classes of $G$-bitorsors. The identity element is given by the class of $G$ and the inverse of the class of a $G$-bitorsor $P$ is given by the class of $P^{\text{op}}$. We will later relate local-global principles of gerbes to a factorization of bitorsors over a finite inverse factorization system $\mathcal{F}$.

Definition 3.3. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a finite inverse factorization system and let $G$ be a group sheaf over $F = \varprojlim \mathcal{F}$. We say that $G$-bitorsor factorization holds over $\mathcal{F}$ if for any collection $\{P_k\}_{k \in I}$ of $G_k$-bitorsors, there are $G$-bitorsors $\{P_i\}_{i \in I}$, such that $P_i|_{F_k} \wedge^{G_k} P_j|_{F_k} \cong P_k$ holds whenever $(i,j,k) \in S_I$.

We will now recall a description of a $(G,H)$-bitorsor $P$ as a left $G$-torsor together with a $G$-equivariant morphism $P \to \text{Isom}(H,G)$. This description was first given by Breen, cf. [Bre90, Section 2] or [BM05, Section 1.4]. Details and proofs to the statements made below can be found there for the case $G = H$. The proofs of the case $G \neq H$ go through verbatim. Let $P$ be a $(G,H)$-bitorsor. Let $p$ be a section of $P$ over some $U \in (\text{Sch}/F)$. Define an isomorphism of group sheaves

$$u_p: H|_U \to G|_U$$

via

$$p|_V h = u_p(V)(h)p|_V$$

for $V \in (\text{Sch}/U)$ and $h \in H|_U(V)$.

Let $p'$ be another section over the same $U$. There is some $\gamma \in G(U)$ such that $p' = \gamma p$. Then, we have

$$u'_p = i_\gamma u_p$$

where $i_\gamma$ is conjugation $(g \mapsto \gamma g \gamma^{-1})$. 
Thus, we obtain an isomorphism of group sheaves
\[ u : P \longrightarrow \text{Isom}(H, G) \]
\[ p \mapsto u_p \]
that is equivariant with respect to the conjugation map
\[ i : G \longrightarrow \text{Aut}(G). \]

This process is reversible.

**Lemma 3.4** (cf. [Bre90, Lemma 2.5]). For a left G-torsor P, the following are equivalent:

1. the data of a right H action making it a bitorsor,
2. an morphism of sheaves \( u : P \longrightarrow \text{Isom}(H, G) \) equivariant with respect to \( i : G \rightarrow \text{Aut}(G) \).

The following Lemma continues this equivalence with respect to morphisms.

**Lemma 3.5** (cf. [BM05, Section 1.4.3]). Let \( f : P \rightarrow P' \) be a morphism of left G-torsors. Assume that \( P \) and \( P' \) are \( (G,H) \)-bitorsors and let \( u, u' \) denote the equivariant morphism of sheaves discussed in Lemma 3.4. Then, the following are equivalent:

1. \( f \) is a morphism of bitorsors
2. \( u = u' f \)

If we fix a cover \( Y \rightarrow F \) of \( F \) and a section \( p \in P(Y) \), we can reinterpret the two lemmas above:

**Lemma 3.6** (cf. [Bre90, Proposition 2.5] and [BM05, Section 1.4.5.]). Let \( (P, p) \) be a left G-torsor with a section \( p \) over a cover \( Y \rightarrow F \). Then, the following data are equivalent

1. A right \( H \)-action on \( P \) making it a \( (G,H) \)-bitorsor
2. a sheaf isomorphism \( u_p : H|_Y \rightarrow G|_Y \)

Let now \( (P', p') \) be another such tuple where \( p' \) is also a section over \( Y \). Assume that both \( P \) and \( P' \) are \( (G,H) \)-bitorsors. Let \( f : P \rightarrow P' \) be a morphism of left G-torsors. Let \( g \in G(Y) \) be such that \( f(p) = gp' \) holds. Then, the following are equivalent

1. \( f \) is a morphism of bitorsors
2. \( u_p = igu'_p \)

### 3.2. Patching for bitorsors.

In this section, fix a base field \( F \) and let \( \mathcal{C} = (\text{Sch}/F) \) equipped with the big étale topology.

**Lemma 3.7.** Let \( K \) be a finite separable extension of \( F \) and let \( \mathcal{X} \) denote the corresponding inverse system obtained from base change. Let \( G, H \) be linear algebraic groups over \( K \). Given isomorphism of group schemes \( u_i : G|_{K_i} \rightarrow H|_{K_i} \) for all \( i \in I_0 \) such that

\[ u_i|_{K_k} = u_j|_{K_k} \]

whenever \( (i, j, k) \in S_I \), there is an isomorphism of group schemes \( u : G|_K \rightarrow H_K \) satisfying \( u|_{K_i} = u_i \) for all \( i \in I \).

**Proof.** Let \( A, B \) be \( K \)-algebras such that \( G|_K = \text{Spec}(A) \) and \( H|_K = \text{Spec}(B) \). Then, the \( u_i \) induce ring isomorphisms \( f_i : B_i \rightarrow A_i \). Fix some \( b \in B \). Consider the elements \( f_i(b) \) for \( i \in I \). We have \( f_i(b) = f_j(b) \in A_k \) for \( (i,j,k) \in S_I \) by assumption. Hence, the elements \( f_i(b) \) determine a unique element \( f(b) \) in \( A \) which define a morphism \( f : B \rightarrow A \). It is clear that this is a ring isomorphism. Hence, we get an isomorphism of schemes \( u : G \rightarrow H \). For \( u \) to be compatible with the group structure, we need \( f \) to be compatible with Hopf
Let $\mathcal{P}$ be an inverse system of fields with limit $F$. Let $\text{BPP}(G,H)(\mathcal{P})$ denote the category of $(G,H)$-bitorsor patching problems over $\mathcal{P}$, defined analogously to $\text{TPP}(G)(F)$. There is a natural functor 
$$\beta''_{(G,H)}: \text{BITORSOR}(G,H)(F) \to \text{BPP}(G,H)(\mathcal{P}).$$

We recall the natural functor 
$$\beta'_G: \text{TORSOR}(G)(F) \to \text{TPP}(G)(\mathcal{P}).$$

We will now see that we can patch bitorsors whenever we can patch torsors.

**Theorem 3.8.** Assume that patching holds for $G$-torsors over $\mathcal{P}$. Then, patching holds for $(G,H)$-bitorsors, i.e. if $\beta'_G$ is an equivalence of categories, then so is $\beta''_{(G,H)}$.

**Proof.** We need to prove essential surjectivity and fully faithfulness. Let us start with essential surjectivity.

Fix some $\mathcal{P} \in \text{BPP}(G,H)(\mathcal{P})$. We have a commuting diagram of functors

$$
\begin{array}{ccc}
\text{BITORSOR}(G,H)(F) & \xrightarrow{\beta''_{(G,H)}} & \text{BPP}(G,H)(\mathcal{P}) \\
\downarrow & & \downarrow \\
\text{TORSOR}(G)(F) & \xrightarrow{\beta'_G} & \text{TPP}(G)(\mathcal{P}),
\end{array}
$$

where the vertical functors are the forgetful ones.

By assumption $\beta'_G$ is essentially surjective, so there is some left $G$-torsor $P$ defined over $F$ together with isomorphisms $\phi_i: P|_{F_i} \to P_i$ for all $i \in I$, that are compatible with the morphisms $v_k$. Let $K/F$ be a finite separable field extension, such that $P(K) \neq \emptyset$. Fix some $p_0 \in P(K)$. Let $K_i := F_i \otimes_F K$ for all $i \in I$. Set $p_i = \phi_i(K_i)(p_0|_{K_i})$, which defines a trivializing family $\{p_i\}_{i \in I}$. Note that, by construction, $v_k(p_i|_{K_k}) = p_j|_{K_k}$.

By Lemma 3.6, we get sheaf isomorphisms $u_{p_i}: H|_{K_i} \to G|_{K_i}$ for all $i \in I$. Recall that this morphism is defined over $V \to K_k$ via $p_i|_V.h = u_{p_i}(V)(h).p_i|_V$ for $h \in H(V)$. We claim that $u_{p_i}|_{K_k} = u_{p_j}|_{K_k}$ for all $(i,j,k) \in S_I$. This follows from

$$u_{p_i}(V)(h).p_j|_V = u_{p_i}(h).v_k(V)(p_i|_V) = v_k(V)(u_{p_i}(V)(h).p_i|_V) = v_k(p_i|_V.h) = v_k(p_i|_V).h = p_k.h = u_{p_j}(V)(h).p_j|_V.$$ 

By Lemma 3.7, we get a global isomorphism $u_{p_0}: G \to H$. By Lemma 3.6, this in turn equips $P$ with a $(G,H)$-bitorsor structure.

In order to show that $\beta''_{(G,H)}(P)$ is isomorphic to the given bitorsor patching problem $\mathcal{P}$, it is enough to show that the morphisms $\phi_i$ are in fact morphisms of bitorsors. By Lemma 3.6, this is equivalent to checking that

$$u_{p_0}|_{K_i} = u_{p_i}.$$
for all $i \in I_v$ (note that $\phi_i(p_0) = p_i$). But, this is clear by construction of $u_{p_0}$. Hence, $\beta''_{(G,H)}$ is essentially surjective.

Let us now show that $\beta''_{(G,H)}$ is fully faithful. It is clearly faithful as $\beta'_G$ is faithful. So, we only need to prove that it is full. Fix two $(G,H)$-bitorsors $P, P'$ over $F$ and let $\beta''_{(G,H)}(P) = P$ and $\beta''_{(G,H)}(P') = P'$. Let $\alpha: \mathcal{P} \to \mathcal{P}'$ be a morphism in BPP$(G,H)(\mathcal{F})$. Note that $\alpha$ is also a morphism in TPP$(G)(\mathcal{F})$. Hence, there is a morphism $\alpha: P \to P'$ of left $G$-torsors inducing the morphisms $\alpha_i: P_i \to P'_i$. A straightforward check shows that $\alpha$ is a morphism of bitorsors. 

In the context of gerbe patching, we will have to patch bitorsors over covers of $F$. These covers are formed by finite products of finite separable field extensions. We thus need to extend our patching results to this setup.

**Corollary 3.9.** Let $A$ be a finite product of finite separable field extensions of $F$ and let $G,H$ be linear algebraic groups over $A$. If vector space patching holds over $\mathcal{F}$ then $(G,H)$-bitorsor patching holds over $\mathcal{F}_A$.

**Proof.** Follows from Theorem 3.8 and Proposition 2.14. (Note that the proof of Theorem 3.8 goes through verbatim if we replace $\mathcal{F}$ by $\mathcal{F}_A$.)

### 3.3. A Mayer-Vietoris sequence and a local-global principle for bitorsors.

Using patching for bitorsors, we can construct the start of our Mayer-Vietoris sequence, which will be extended in section 4.4. Fix a group sheaf $G$ and let $Z$ denote its center.

We say that $G$-bitorsors satisfy local-global principle over $\mathcal{F}$ if for any $G$-bitorsor $T$, we have that $T \simeq G$ if and only if $T_i \simeq G_i$ for all $i \in I_v$.

Recall that automorphisms of the trivial $G$-bitorsor $G$ can be identified with $Z(F)$. Furthermore, note that $H^{-1}(F,G \to \text{Aut}(G)) = Z(F)$ and that $H^0(F,G \to \text{Aut}(G))$ classifies bitorsors up to isomorphism (cf. Appendix A).

Note that we have maps

$$H^0(F_i, G \to \text{Aut}(G)) \times H^0(F_j, G \to \text{Aut}(G)) \to \prod_{i \in I_k} H^0(F_k, G \to \text{Aut}(G))$$

for $(i, j, k) \in S_I$ defined via $(P_i, P_j) \mapsto P_i \wedge^G P_j^{op}$. This induces a map

$$\prod_{i \in I_v} H^0(F_i, G \to \text{Aut}(G)) \to \prod_{i \in I_k} H^0(F_k, G \to \text{Aut}(G)).$$

**Lemma 3.10.** There is a map of pointed sets

$$\prod_{i \in I_v} H^{-1}(F_i, G \to \text{Aut}(G)) \to H^0(F, G \to \text{Aut}(G))$$

**Proof.** We can define the map as follows: Given elements $e_k \in Z(F_k)$, consider the $G$-bitorsor patching problem ($\{G_i\}, \{e_k\}$). By Theorem 3.8, there is a $G$-bitorsor $P$ over $F$ in the essential preimage of the patching problem. Map $(e_k)_k$ to the equivalence class of $P$. Note that this is independent of the choice of $P$. 

**Theorem 3.11** (Mayer-Vietoris for non-abelian hypercohomology (1)). Assume that $G$-bitorsor patching holds over $\mathcal{F}$. Then, there is an exact sequence

$$1 \to H^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i \in I_v} H^{-1}(F_i, G \to \text{Aut}(G)) \to \prod_{k \in I_k} H^{-1}(F_k, G \to \text{Aut}(G)) \to H^0(F, G \to \text{Aut}(G)) \to \prod_{i \in I_v} H^0(F_i, G \to \text{Aut}(G)) \to \prod_{k \in I_k} H^0(F_k, G \to \text{Aut}(G))$$
Proof. Exactness in the first row follows from the description $H^{-1}(F, G \to \text{Aut}(G)) = Z(F)$ and the fact that $\mathcal{F}$ is a factorization inverse system with limit $F$. Exactness at $\prod_{i \in I} H^0(F_i, G \to \text{Aut}(G))$ follows from Theorem 3.8. Finally, exactness at $H^0(F, G \to \text{Aut}(G))$ also follows from Theorem 3.8 and the fact that the automorphism group of $G$ as a bitorsor is $Z$.

The sequence above allows us to deduce a local-global principle for bitorsors with respect to patches.

Corollary 3.12. Assume that $G$-bitorsor patching holds over $\mathcal{F}$. Then, $G$-bitorsors satisfy local-global principle over $\mathcal{F}$ iff $Z(G)$ satisfies factorization over $\mathcal{F}$.

4. Gerbes

The goal of this section is to prove our main theorem on gerbe patching.

4.1. Preliminaries. Let us again fix the big étale site $(\text{Sch}/F)_{\text{ét}}$ of the scheme $\text{Spec}(F)$ for some fixed field $F$. As before, we will often just write $F$ to denote $\text{Spec}(F)$. Note that covers of $F$ can all be taken to be of the form $Y \to F$ for a single arrow where $Y$ is a finite product of finite separable field extensions of $F$. We will use this throughout this section as it will simplify notation.

We will first recall the definition of a gerbe as a special kind of stack. Our main reference for gerbes is [Gir71].

Definition 4.1. Let $\mathcal{G} \to (\text{Sch}/F)_{\text{ét}}$ be a stack. We say that $\mathcal{G}$ is a gerbe if the following two conditions are satisfied:

1. For all $X \in (\text{Sch}/F)_{\text{ét}}$, there is a cover $Y \to X$ such that $\mathcal{G}(Y) \neq \emptyset$
2. For any $x, x' \in \mathcal{G}(X)$, there is a cover $Y \to X$ such that the pullbacks of $x$ and $x'$ are isomorphic in $\mathcal{G}(Y)$.

Example 4.2. Let $G$ be a sheaf of groups over $(\text{Sch}/F)_{\text{ét}}$. Then, the classifying stack $BG$ is a gerbe. Its fiber over $Z \to F$ is given by $BG(Z) = \text{TORSOR}(G)(Z)$.

Definition 4.3. We say that a gerbe $\mathcal{G} \to (\text{Sch}/F)_{\text{ét}}$ is trivial (or neutral) if $\mathcal{G}(F) \neq \emptyset$.

Example 4.4. The gerbe $BG$ is trivial as the trivial left $G$-torsor $G$ is in $BG(F)$.

Remark 4.5. Let $x, y \in \mathcal{G}(Z)$ for some $Z \in (\text{Sch}/F)_{\text{ét}}$. The sheaf $\text{Isom}_\mathcal{G}(x, y)$ on $(\text{Sch}/Z)_{\text{ét}}$ is a $(\text{Aut}_Z(y), \text{Aut}_Z(x))$-bitorsor.

Recall that a 2-category $\mathcal{C}$ consists of a class of objects $\text{obj}(\mathcal{C})$ together with categories $\text{mor}(A, B)$ for $A, B \in \text{obj}(\mathcal{C})$ and composition functors $\text{mor}(A, B) \times \text{mor}(B, C) \to \text{mor}(A, C)$ satisfying the usual identities. Additionally, we require $\text{mor}(A, A)$ to contain an identity element with respect to composition. Similarly, for $f \in \text{mor}(A, B)$, we require there to be an identity element in $\text{mor}(f, f)$. We will call an object in $\text{mor}(A, B)$ a 1-morphism and a morphism in $\text{mor}(A, B)$ a 2-morphism.

A typical example of a 2-category is the category of categories, where for two categories $\mathcal{A}$, $\mathcal{B}$, the category $\text{mor}(\mathcal{A}, \mathcal{B})$ has functors as objects and natural transformations as morphisms.

As stacks are in particular categories, we consider the 2-category of stacks over some site as the category with objects being stacks, 1-morphisms being functors of fibered categories and 2-morphisms being natural transformations of functors.

Similarly, we will consider the 2-category of gerbes in this section and will later define the 2-category of gerbe patching problems.
Definition 4.6. A 1-morphism of gerbes $\mathcal{G} \to \mathcal{G}'$ over $(\text{Sch}/F)_{\text{ét}}$ is a morphism of stacks $\mathcal{G} \to \mathcal{G}'$ over $(\text{Sch}/F)_{\text{ét}}$.

A 2-morphism of gerbes is a natural transformation of functors.

By Gerbes$(F)$, we denote the 2-category of gerbes with 1-morphisms given by equivalences and 2-morphisms given by natural isomorphisms of functors.

The category $\text{mor}(BG, BH)$ in Gerbes$(F)$ is equivalent to the category of $(G, H)$-bitorsors.

Theorem 4.7 ([Gir71, Giraud]). Let $P$ be a $(H, G)$-bitorsor over $F$. Then, the functor

$$BG \to BH$$

$$(S, T) \mapsto (S, P_S \wedge^G S, T)$$

is an equivalence of categories. Furthermore, any equivalence between $BG$ and $BH$ is of this form.

Remark 4.8. Given two equivalences $BG \to BH$ and $BH \to BK$ for group sheaves $G, H$ and $F$, we obtain an $(H, G)$-bitorsor $P$ and an $(K, H)$-bitorsor $P'$. The composition $BG \to BH \to BK$ then corresponds to $P' \wedge^H P$.

4.1.1. A semi-cocyclic description of a gerbe. We will now follow sections 2.3-2.6 of [Bre91] to introduce a cocyclic description of a gerbe $\mathcal{G} \to (\text{Sch}/F)_{\text{ét}}$.

Let $Y \to F$ be a cover such that there is an element $y \in \mathcal{G}(Y)$. Let $G = \text{Aut}_Y(y)$ denote the sheaf of automorphisms of $y$ over $(\text{Sch}/Y)$. The choice of an object $y$ defines an equivalence

$$\Phi: \mathcal{G}|_Y \longrightarrow BG$$

defined on fibers over $f: Z \to Y$ via

$$\Phi(Z): \mathcal{G}(Z) \longrightarrow \text{TORSOR}(G)(Z)$$

$$z \mapsto \text{Isom}_\mathcal{G}(z, f^* y, z)$$

where $\text{Isom}_\mathcal{G}(z, f^* y)$ is a $G|_Z$-torsor by the natural action on the left.

We fix the following notation: Let $\text{pr}_i: Y^2 = Y \times_F Y \to Y$ denote the natural projections on the $i$-th factor for $i = 1, 2$. We will denote by $G_i = \text{pr}_i^* G$ the pullbacks of $G$ to $Y^2$. Let $\text{pr}_{ij}: Y^3 \to Y^2$ denote the natural projections on the $i$-th and $j$-th component for $1 \leq i < j \leq 3$. We will denote by $G^i$ the pullbacks of $G$ to $Y^3$ along the natural projections $Y^3 \to Y$. Finally, let $\text{pr}_{ijk}: Y^4 \to Y^3$ denote the natural projection on the $i$-th, $j$-th and $k$-th component for $1 \leq i < j < k \leq 4$. Let $G(i)$ denote the pullbacks of $G$ to $Y^4$ along the natural projections $Y^4 \to Y$. Note that $\Phi$ induces an equivalence

$$\varphi = \text{pr}_1^* \Phi \circ (\text{pr}_2^* \Phi)^{-1}: BG_2 \longrightarrow BG_1.$$
where $Y^4 = Y \times_F Y \times_F Y \times_F Y$.

The gerbe $G$ is completely determined by the tuple $(G, \varphi, \psi)$.

We will now reinterpret the above description in terms of bitorsors. Note that the equivalence $\varphi$ can be identified with the $(G_2, G_1)$-bitorsor $E = \text{Isom}(\text{pr}_2^*y, \text{pr}_1^*y)$. Then, the isomorphism $\psi$ corresponds to an isomorphism of $(G^1, G^3)$-bitorsors

$$\text{pr}_1^*E \wedge^{G_2} \text{pr}_2^*E \longrightarrow \text{pr}_3^*E$$

where $G^i$ denotes the pullback of $G$ along $Y \times_F Y \times_F Y \rightarrow Y$ on the $i$-th component.

Finally, the compatibility condition translates to the commutativity of the diagram

$$\begin{array}{ccc}
\text{pr}_1^*E \wedge^{G_2} \text{pr}_2^*E \wedge^{G_3} \text{pr}_3^*E & \longrightarrow & \text{pr}_3^*E \wedge^{G_3} \text{pr}_4^*E \\
\downarrow & & \downarrow \\
\text{pr}_1^*E \wedge^{G_2} \text{pr}_2^*E & \longrightarrow & \text{pr}_3^*E
\end{array}$$

where the arrows are induced by the various pullbacks of $\psi$ and $G_{(i)}$ denotes the pullback of $G$ along $Y^4 \rightarrow Y$ onto the $i$-th component. So, a gerbe can be described by the triple $(G, E, \psi)$ and we will henceforth go back and forth between the categorical and the cocyclic description. We call the triple $(G, E, \psi)$ the semi-cocyclic description of $G$.

Let us now turn to morphisms of gerbes.

Let $\rho : G \rightarrow G'$ be a morphism of gerbes over a field $F$. Let $K/F$ be a finite separable extension such that there are $x \in G(K)$ and $y \in G'(K)$. After possibly replacing $K$ by a further finite separable extension, we may assume that $\rho(x)$ and $y$ are isomorphic in $G'(K)$. Using the cocyclic description, we get $G = (G, E, \psi)$ and $G' = (G', E', \psi')$ where we choose the descriptions induced by $x$ and $y$. In particular, $\text{Aut}(x) = G$ and $\text{Aut}(y) = G'$.

Note that $\rho$ induces a map

$$\rho' : G \longrightarrow G'$$

and a $\rho'|_{K \times_F K}$-equivariant map

$$\alpha : E \rightarrow E'.$$

Note that $\alpha$ is compatible with $\psi$ and $\psi'$. Conversely, given a map $\alpha$ compatible with $\psi$ and $\psi'$, one gets an induced morphism of gerbes $G \rightarrow G'$.

**Lemma 4.9** ([Bre91, Section 2.6]). Let $\rho : G \rightarrow G'$ be a morphism of gerbes. Assume that there is a finite separable cover $K/F$ such that $G(K) \neq G'(K)$. Fix some $x \in G(K)$ and $y \in G'(K)$. Assume that there is a morphism $\rho(K)(x) \rightarrow y$ in $G'(K)$. Rewrite $G = (\text{Aut}(x), E, \psi)$ and $G' = (\text{Aut}(y), E', \psi')$ using the construction described above coming from $x$ and $y$. Then, $\rho$ induces an equivariant isomorphism of bitorsors $E \rightarrow E'$ that is compatible with $\psi$ and $\psi'$. Conversely, given an isomorphism $E \rightarrow E'$ compatible with the gluing data, one can construct a morphism of gerbes.
4.2. Bands and patching of non-abelian $H^2$. In this section, we will continue to work over the big étale site of schemes over $F$. Before we can patch gerbes, we will first investigate when we can patch equivalence classes of gerbes, i.e. elements in the non-abelian second cohomology set of a band.

Given a gerbe $\mathcal{G}$, one can associate to it a band $L$. Let us quickly review the definition of a band. For more details, we refer again to [Gir71], compare also the appendix of [DM82].

Let $H, G$ be group sheaves and $G^{\text{ad}} = G/Z$, where $Z$ is the center of $G$. Then, $G^{\text{ad}}$ acts on $\text{Isom}(H, G)$ via conjugation. Let $\text{Isex}(H, G)$ denote the quotient of $\text{Isom}(H, G)$ by the $G^{\text{ad}}$-action. Then, a band $L$ consists of a triple $(Y, G, \phi)$ where $Y$ is a cover of $F$, $G$ is a group sheaf defined over $Y$ and $\phi \in \text{Isex}(G_1, G_2)$ where $G_1 = \text{pr}_1^* G$ for $\text{pr}_1: Y \times_F Y \to Y$. We require $\phi$ to satisfy the cocycle condition $\text{pr}_{31}^* \phi = \text{pr}_{32}^* \phi \circ \text{pr}_{21}^* \phi$ over $Y \times_F Y \times_F Y$. Thus, a band is a descent datum for a group sheaf over a cover $Y \to F$ modulo inner automorphisms. Every group sheaf over $F$ defines a band with trivial gluing datum. Furthermore, any abelian band (i.e. $G$ is abelian) is induced by a group sheaf over $F$ as the datum of a band in this case just gives descent datum since inner automorphisms of $G$ are all trivial. The center $Z(L)$ of band $L = (Y, G, \phi)$ is defined as $(Y, Z, \phi_Z)$ and we identify it with the group sheaf over $F$ determined by the descent datum of the band. For a cover $f: Y' \to Y$, we identify the bands $(Y, G, \phi)$ and $(Y', f^* G, f^* \phi)$. An isomorphism between two bands $(Y, H, \phi) \to (Y, G, \tau)$ is given by an element $g \in \text{Isex}(H, G)$ compatible with $\phi$ and $\tau$.

We say that a band $L = (Y, G, \phi)$ is linear algebraic, if $G$ is a linear algebraic group, i.e. if $Y = \text{Spec}(R)$ for $R = \prod_i L_i$ where $L_i$ is a finite separable field extension, $G_{L_i}$ is a linear algebraic group.

Given a gerbe $\mathcal{G}$, we can define an associated band. Pick a cover $Y \to F$ and an object $x \in \mathcal{G}(Y)$. Let $G = \text{Aut}(x)$ be the sheaf of automorphisms of $x$. Let $x_i = \text{pr}_1^* x$ denote the two pullbacks along the projections $\text{pr}_1: Y \times F Y \to Y$. By the definition of gerbes, there is a cover $U \to Y \times_F Y$ such that $x_1|_U$ and $x_2|_U$ are isomorphic in $\mathcal{G}(U)$. An isomorphism $f: x_1|_U \to x_2|_U$ defines an isomorphism $\lambda_f: G_1|_U \to G_2|_U$. If $g: x_1|_U \to x_2|_U$ is another isomorphism, then $\lambda_f, \lambda_g$ differ by an inner automorphism of $G_2$. Thus, there is a well defined element $\lambda \in \text{Isex}(G_1, G_2)(U)$. An easy calculation shows that the pullbacks of $\lambda_f$ on $U \times_Y \times_F Y U$ differ by an inner automorphism. Hence, the pullbacks of $\lambda$ agree on $U \times_Y \times_F Y U$ and we thus obtain an element $\lambda' \in \text{Isex}(G_1, G_2)(Y \times F Y)$ whose restriction to $U$ equals $\lambda$. It is not hard to see that $\lambda'$ satisfies the cocycle condition and thus $(Y, G, \lambda')$ defines a band. Given a gerbe, the associated band is unique up to unique isomorphism and we denote it by $\text{Band}(\mathcal{G})$. Note that a morphism $\mathcal{G} \to \mathcal{G}'$ induces a map $\text{Band}(\mathcal{G}) \to \text{Band}(\mathcal{G}')$. Given a band $L$, a gerbe banded by $L$ is a tuple $(\mathcal{G}, \theta)$ where $\mathcal{G}$ is a gerbe and $\theta: \text{Band}(\mathcal{G}) \to L$ is an isomorphism of bands. We often suppress $\theta$ from the notation.

Given two gerbes $\mathcal{G}, \mathcal{G}'$ banded by $L$, an $L$-morphism of gerbes is a morphism $\alpha: \mathcal{G} \to \mathcal{G}'$ that is compatible with the morphisms $\theta$ and $\theta'$, i.e. the diagram

\[\begin{array}{ccc}
\text{Band}(\mathcal{G}) & \xrightarrow{\alpha} & \text{Band}(\mathcal{G}') \\
\theta & & \theta' \\
L & \longleftarrow & L'
\end{array}\]

commutes. Any morphism of $L$-gerbes is an equivalence.

Let $H^2(F, L)$ denote the set of $L$-equivalence classes of gerbes banded by $L$ (this means that we only consider equivalences of $L$ gerbes that are compatible with the band $L$). If $L$ is abelian coming from the group sheaf $A$ over $F$, then this definition coincides with the usual definition of $H^2(F, A)$ as a Galois cohomology group.
Furthermore, there is a remarkable relation between $H^2(F, L)$ and $H^2(F, Z(L))$.

**Theorem 4.10** ([Gir71, Theorem 3.3.3]). If $H^2(F, L)$ is not empty, then it is a principal homogeneous space under $H^2(F, Z(L))$.

Harbater, Hartmann and Krashen have proved patching for Galois cohomology groups of abelian algebraic groups over arithmetic curves under some mild compatibility assumptions between the characteristic of $F$ and the order of the group (compare [HHK14]). Using the above theorem, we can deduce patching for non-abelian Galois cohomology over arithmetic curves whenever the center of the band admits patching. We will pursue this in Section 5. For the purpose of gerbe patching, let us note this immediate consequence.

We define patching for $H^2(\mathcal{o}, L)$ over a finite inverse factorization system $\mathcal{F}$ analogously to the case of patching for Galois cohomology, see Section 2.2. Note that for $(i, j, k) \in S_I$, we have two maps $H^2(F_i, L) \times H^2(F_j, L) \to H^2(F_k, L)$ given by restriction of the first and the second factor respectively. Hence, we have two maps $\prod_{i \in I_v} H^2(F_i, L) \to \prod_{k \in I_v} H^2(F_k, L)$.

**Definition 4.11.** Let $\mathcal{F}$ be a finite inverse factorization system with inverse limit $F$. For a band $L$ over $F$, we say that patching holds for $H^2(\mathcal{o}, L)$ over $\mathcal{F}$ if the following sequence is an equalizer diagram

$$H^2(F, L) \to \prod_{i \in I_v} H^2(F_i, L) \to \prod_{k \in I_v} H^2(F_k, L).$$

**Proposition 4.12.** Let $\mathcal{F}$ be a finite inverse system for fields with inverse limit $F$ and let $L$ be a band over $F$. If patching holds for $H^2(\mathcal{o}, Z(L))$ over $\mathcal{F}$ and $H^2(F, L) \neq \emptyset$, then patching holds for $H^2(\mathcal{o}, L)$ over $\mathcal{F}$.

**Proof.** By assumption, there is a class $\alpha \in H^2(F, L)$. Given a patching problem $\{\beta_i\}_{i \in I_v}$ with $\beta_i \in H^2(F_i, L)$, there are elements $\gamma_i \in H^2(F_i, Z(L))$ such that $\beta_i = \gamma_i \cdot \alpha|_{F_i}$ by Theorem 4.10. As the action of $H^2(F, Z(L))$ on $H^2(F, L)$ is simply transitive, it follows that $\{\gamma_i\}_{i \in I_v}$ defines a patching problem for $H^2(\mathcal{o}, Z(L))$. By assumption, there is $\gamma \in H^2(F, Z(L))$ such that $\gamma|_{F_i} = \gamma_i$ for all $i \in I_v$. The element $\beta := \gamma \cdot \alpha$ solves the patching problem $\{\beta_i\}_{i \in I_v}$. □

4.3. **Patching gerbes.** Let $\mathcal{F}$ be a factorization inverse system of fields with inverse limit $F$. Let $L$ be a band over $F$. Let $\text{Gerbes}(F, L)$ denote the 2-category of $L$-banded gerbes $\mathcal{G}$ over $F$. Here, morphisms are given by equivalences of gerbes and 2-morphisms are given by natural isomorphisms. Let $\text{GPP}(\mathcal{F}, L)$ denote the 2-category of $L$-gerbe patching problems, i.e. an object consists of a collection of gerbes $L$-banded gerbes $\mathcal{G}_i$ over $F_i$ together with $L$-equivalences $\sigma_k : \mathcal{G}_i|_{F_k} \to \mathcal{G}_j|_{F_k}$ for $(i, j, k) \in S_I$.

A 1-Morphisms $\left(\{\mathcal{G}_i\}, \{\sigma_k\}\right) \xrightarrow{(\alpha_f)} \left(\{\mathcal{G}_i'\}, \{\sigma_k'\}\right)$ is given by a collection of equivalences of gerbes $\alpha_i : \mathcal{G}_i \to \mathcal{G}_i'$ and natural isomorphisms $f_k : \sigma_k' \circ \alpha_i|_{F_k} \Rightarrow \alpha_j|_{F_k} \circ \sigma_k$, pictorially:

$$
\begin{array}{c}
\mathcal{G}_i|_{F_k} \\
\downarrow \sigma_k \\
\mathcal{G}_j|_{F_k}
\end{array}
\xrightarrow{
\begin{array}{c}
\alpha_i|_{F_k}
\end{array}
}
\begin{array}{c}
\mathcal{G}_i'|_{F_k} \\
\downarrow f_k \\
\mathcal{G}_j'|_{F_k}
\end{array}$$

Composition of morphisms is given by composing equivalences of gerbes and by horizontally composing the natural isomorphisms.
Given two morphisms \((\alpha, f), (\beta, g) : \{\mathcal{G}_i\}, \{\sigma_k\} \to (\{\mathcal{G}_i'\}, \{\sigma'_k\})\), a 2-morphism \(\psi = (\{\psi_i\})\) is given by a collection of natural isomorphisms

\[\begin{array}{c}
\xymatrix{
\mathcal{G}_i \ar[r]^{\psi} & \mathcal{G}_i' \\
\beta_i \ar[u]^\alpha & \beta_i' \ar[u]_{\beta_i} 
}\end{array}\]

such that the diagram of natural transformations commutes:

\[\begin{array}{c}
\xymatrix{
\mathcal{G}_i |_{F_k} \ar[r]^{\alpha_i} \ar[d]_{\sigma_k} & \mathcal{G}_i' |_{F_k} \ar[d]_{\sigma'_k} \\
\mathcal{G}_j |_{F_k} \ar[r]_{\sigma_j} \ar[d]_{f_k} & \mathcal{G}_j' |_{F_k} \\
\mathcal{G}_j |_{F_k} \ar[r]_{\alpha_j} & \mathcal{G}_j' |_{F_k} 
}\end{array}\]

i.e.

\[\begin{array}{c}
\xymatrix{
\sigma'_k \circ \alpha_i |_{F_k} \ar[r]^{f_k} & \alpha_j |_{F_k} \circ \sigma_k \\
\psi_i' \ar[u]^\psi & \psi_j' \ar[u]_{\psi_j} \\
\sigma'_k \circ \beta_i |_{F_k} \ar[r]^{g_k} & \beta_j |_{F_k} \circ \sigma_k
}\end{array}\]

commutes. Here \(\psi'_j\) is the natural isomorphism induced by \(\psi_j\) and \(\sigma'_k\). Composition of 2-morphisms is given by vertical composition of the various natural transformations.

**Lemma 4.13.** Every 1-morphism in \(\text{GPP}(\mathcal{F}, L)\) admits a quasi-inverse

**Proof.** Let \((\alpha, f) : (\{\mathcal{G}_i\}, \{\sigma_k\}) \to (\{\mathcal{G}_i'\}, \{\sigma'_k\})\) be a 1-morphism. Let \(\beta_i\) denote a quasi-inverse of \(\alpha_i\). Then, we get natural isomorphisms \(g_k : \beta_j |_{F_k} \circ \sigma'_k \to \sigma_k \circ \beta_i |_{F_k}\) defined via

\[\beta_j |_{F_k} \circ \sigma'_k \Rightarrow \beta_j |_{F_k} \circ \sigma'_k \circ \alpha_i |_{F_k} \circ \beta_i |_{F_k} \Rightarrow \beta_j |_{F_k} \circ \alpha_j |_{F_k} \circ \sigma_k \circ \beta_i |_{F_k} \Rightarrow \sigma_k \circ \beta_i |_{F_k}.\]

Here, the first and last natural transformation are induced by fixed natural isomorphisms \(\phi_i : \alpha_i \circ \beta_i \Rightarrow \text{id}\), while the middle arrow is induced by \(f_k\). Hence, we can define a 1-morphism \((\beta, g) : (\{\mathcal{G}_i'\}, \{\sigma'_k\}) \to (\{\mathcal{G}_i\}, \{\sigma_k\})\). It remains to check that \((\alpha, f) \circ (\beta, g)\) and \((\beta, g) \circ (\alpha, f)\) are 2-isomorphic to the identity morphism. We will prove that \((\alpha, f) \circ (\beta, g)\) is isomorphic to the identity, the other case is analogous. For this, we need to give a collection of 2-isomorphisms \(\psi_i : \alpha_i \circ \beta_i \Rightarrow \text{id}\) that are compatible with \(g_k\) and \(f_k\). It is tedious but straightforward to check that the choice \(\psi_i = \phi_i\) works. \(\square\)

Note that there is a natural functor of 2-categories

\[\beta'''_L : \text{Gerbes}(F, L) \to \text{GPP}(\mathcal{F}, L)\]

induced by base change. We say that patching holds for L-gerbes over \(\mathcal{F}\) if \(\beta'''_L\) is an equivalence.

In order to prove our main result, we need the following elementary lemma from the general theory of stacks. Given two stacks \(\mathcal{X}\) and \(\mathcal{Y}\) be stacks in groupoids over \((\text{Sch}/F)\), let \(\{U \to X\}\) be a cover in \(\mathcal{C}\).

Let \(\text{pr}_i : U \times_X U \to U\) and \(\text{pr}_{ij} : U \times_X U \times_X U \to U \times_X U\) denote the usual projections.
Lemma 4.14. The following data are equivalent:

(1) a morphism of stacks $\mathcal{X} \to \mathcal{Y}$

(2) a morphism of stacks $\alpha: \mathcal{X} \mid_U \to \mathcal{Y} \mid_U$ together with a natural transformation $\psi: \text{pr}_1^* \alpha \to \text{pr}_2^* \alpha$ such that the diagram

\[
\begin{array}{c}
\text{pr}_{12}^* \text{pr}_1^* \alpha \\
\downarrow \text{pr}_{12}^* \\
\text{pr}_{12}^* \text{pr}_1^* \alpha \rightarrow \text{pr}_{12}^* \text{pr}_2^* \alpha \\
\end{array}
\]

commutes.

Proof. This immediately follows from the fact that the category of morphisms $\mathcal{X} \to \mathcal{Y}$ is a stack. \qed

Definition 4.15. Let $(\{\mathcal{G}_i\}, \{v_k\})$ be a gerbe patching problem. We say that $(\{\mathcal{G}_i\}, \{v_k\})$ has property $D$ if there is a cover $Z \to F$ such that $\mathcal{G}_i(Z_i) \neq \emptyset$ for all $i \in I$, and that there are elements $x_i \in \mathcal{G}_i(Z_i)$ such that $v_k(x_i|Z_k)$ is isomorphic to $x_j|Z_k$ for all $(i, j, k) \in S_I$.

Proposition 4.16. Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Let $(\{\mathcal{G}_i\}, \{v_k\})$ be an $L$-gerbe patching problem with property $D$. Then, if patching holds for $G$-torsors, there is $\mathcal{G} \in \text{Gerbes}(F, L)$ such that $\beta''''(\mathcal{G}) \simeq (\{\mathcal{G}_i\}, \{v_k\})$.

Proof. Since the gerbe patching problem has property $D$, there is a cover $Z \to F$ and elements $x_i \in \mathcal{G}_i(Z_i)$ such that $v_k(x_i|Z_k) \simeq x_j|Z_k$ for $(i, j, k) \in S_I$. We may assume without loss of generality that $Z \to F$ factors through $Y \to F$ as we could replace $Z$ by $Z \times F Y$. Hence, we may assume $L = (Z, G, \psi)$ and that we can patch $G$-torsors over $Z$. Let $\text{pr}_j: Z \times F Z \to Z$ denote the natural projection for $j = 1, 2$. Let $G_j = \text{pr}_j^* G$.

Let $A = Z \times F Z$ and note that patching holds for $(G_1, G_2)$-bitorsors over $\mathcal{F}_A$ by assumption and Corollary 3.9.

By use of the cocyclic description of gerbes and their morphisms, we will show that the given gerbe patching problem induces a bitorsor patching problem in BPP($G_2, G_1$)(\mathcal{F}_A). Describing the gerbes $\mathcal{G}_i$ with respect to $Z_i$ as a cocycle, we get tuples $(G_i|Z_i, P_i, \psi_i)$ where $P_i$ are $(G_1, G_2)$-bitorsors over $A_i$. The equivalences of the various $\mathcal{G}_i|Z_i$ with the $\mathcal{G}_j|Z_i$ for $(i, j, k) \in S_I$ translate to isomorphisms $P_i|A_k \to P_j|A_k$ by Lemma 4.9. Hence, we get an element in in BPP($G_2, G_1$)(\mathcal{F}_A).

Thus, there is a $(G_2, G_1)$-bitorsor $P$ defined over $A = Z \times F Z$. The morphisms $\psi_i$ from the cocyclic description of $G_i$ glue together by bitorsor patching to give a global isomorphism $\psi: \text{pr}_{12} \wedge G^2 P_{23} \to P_{13}$ of $(G^1, G^3)$-bitorsors. Also, again by bitorsor patching, the morphism $\psi$ satisfies the coherence condition. Hence, we get a cocycle $(G, P, \psi)$ defining an $L$-gerbe $\mathcal{G}$ in Gerbes($F, L$). By construction, we obtain a 1-morphism $\beta''''(\mathcal{G}) \to (\{\mathcal{G}_i\}, \{v_k\})$. \qed

Proposition 4.17. Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Assume that patching holds for $G$-torsors. Then, the functor $\beta''''$ is essentially surjective on 1-morphisms.

Proof. Let $\mathcal{G}, \mathcal{G}' \in \text{Gerbes}(F, L)$ and let $(\{\mathcal{G}_i\}, \{\sigma_k\})$ and $(\{\mathcal{G}'_i\}, \{\sigma'_k\})$ denote the images in GPP($\mathcal{F}, L$). Given a morphism $(\{\alpha_i\}, \{f_k\}): (\{\mathcal{G}_i\}, \{\sigma_k\}) \to (\{\mathcal{G}'_i\}, \{\sigma'_k\})$, we want to construct a morphism $\mathcal{G} \to \mathcal{G}'$ whose image in GPP($\mathcal{F}, L$) is isomorphic to $(\alpha, f)$.

Let $Z \to F$ be a cover such that $\mathcal{G}(Z) \neq \emptyset \neq \mathcal{G}'(Z)$. Thus, $\mathcal{G}|Z, \mathcal{G}'|Z \simeq BG_k$, and therefore $(\{\mathcal{G}_i\}, \{\sigma_k\})|Z = (\{BG_i\}, \{\sigma_k|Z_k\})$ and we can identify $\sigma_k|Z_k$ with the trivial bitorsor $G_k$. The same conclusion holds for $\mathcal{G}'$ and $(\{\mathcal{G}'_i\}, \{\sigma'_k\})$. Therefore, over $Z$, $\alpha_i$ corresponds to
a $G$-bitorsor $P_i$ by Theorem 4.7. Over $Z_k$, the natural transformation $f_k$ corresponds to the diagram

$$
\begin{array}{ccc}
BG_i|_{Z_k} & \xrightarrow{P_i} & BG_i|_{Z_k} \\
G_k \downarrow & \searrow f_k & \downarrow G_k \\
BG_j|_{Z_k} & \xrightarrow{P_j} & BG_j|_{Z_k}
\end{array}
$$

and thus corresponds to an isomorphism of $G_k$-bitorsors $P_i|_{Z_k} \to P_j|_{Z_k}$. By Theorem 3.8, we get a $G$-bitorsor $P$ defined over $Z$ together with isomorphisms $\phi_i : P|_{Z_k} \to P_i$. This bitorsor in turn defines a morphism $\alpha : \mathcal{G}|_Z \to \mathcal{G'}|_Z$. We claim that it actually descends to a morphism $\mathcal{G} \to \mathcal{G}'$. According to Lemma 4.14, we need an isomorphism of functors $\psi : pr^*_1 \alpha \to pr^*_2 \alpha$ such that

$$
\begin{array}{ccc}
\text{pr}_{12}^* pr^*_1 \alpha & \xrightarrow{\text{pr}_{12}^* \psi} & \text{pr}_{12}^* pr^*_2 \alpha \\
\text{pr}_{13}^* pr^*_1 \alpha & \xrightarrow{\text{pr}_{13}^* \psi} & \text{pr}_{13}^* pr^*_2 \alpha
\end{array}
$$

commutes. In terms of bitorsors, this means that we need an isomorphism of bitorsors $\psi : \text{pr}^*_1 P \to \text{pr}^*_2 P$ making the analogous diagram commute. Such a morphism clearly exists for each $P_i$ as these bitorsors come from morphisms defined over $F_i$. Furthermore, as these morphisms are compatible with the gluing data, these isomorphisms glue to give a global $\psi : \text{pr}^*_1 P \to \text{pr}^*_2 P$ by Theorem 3.8. Hence, we get a morphism of gerbes $\mathcal{G} \to \mathcal{G}'$ and it is easy to see that its image is isomorphic to $(\{\alpha_i\}, \{f_k\})$.

**Proposition 4.18.** Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Assume that patching holds for $G$-torsors. Then, the functor $\beta''_L$ is fully faithful on 2-morphisms.

**Proof.** Fix two 1-morphisms of gerbes $\alpha, \beta : \mathcal{G} \to \mathcal{G}'$ and let

$$(\{\alpha_i\}, \{f_k\}), (\{\beta_i\}, \{g_k\}) : ((\mathcal{G}_i), \{\alpha_i\}) \to ((\mathcal{G'}_i), \{\beta_i\})$$

denote their images in $\text{GPP}(\mathcal{F}, L)$.

Given two 2-morphisms $\psi, \psi' : \alpha \to \beta$ whose image in $\text{GPP}(\mathcal{F}, L)$ are the same, we want to prove that $\psi = \psi'$. It is enough to show this after base change, i.e. to prove $\psi|_Z = \psi'|_Z$. Thus, we may assume $\mathcal{G} = \mathcal{G}' = BG$. Thus, $\alpha$ and $\beta$ correspond to $G$-torsors $P, Q$ and $\psi, \psi'$ are bitorsor isomorphisms $P \to Q$. We then obtain $\psi = \psi'$ immediately from Theorem 3.8.

Finally, we need to check fullness. Given a 2-morphism

$$(\{\psi_i\}) : ((\{\alpha_i\}, \{f_k\}) \to ((\{\beta_i\}, \{g_k\}))$$

we first base change to $Z$. Then, $\psi_i$ correspond to bitorsor isomorphisms and the compatibility condition for 2-morphisms ensures that these isomorphisms glue. Hence, we get $\psi'_i : \alpha|_Z \to \beta|_Z$ by Theorem 3.8. It remains to show that this morphism descends to a morphism $\psi : \alpha \to \beta$. This follows from arguments analogous to the argument to lift the 1-morphism in the proof of Proposition 4.17.

**Theorem 4.19.** Let $L = (Y, G, \psi)$ for some cover $Y \to F$. Assume that any $L$-gerbe patching problem $(\{\mathcal{G}_i\}, \{v_k\})$ has property D. If patching holds for $G$-torsors, then it also holds for $L$-gerbes, i.e. if the functor $\beta'_G$ is an equivalence of 1-categories, then $\beta''_L$ is an equivalence of 2-categories.
Proof. By Proposition 4.16 and assumption, $\beta''''_L$ is essentially surjective on objects. By Proposition 4.17 $\beta''''_L$ is essentially surjective on 1-morphisms and by Proposition 4.18, $\beta''''_L$ is fully faithful on 2-morphisms. Hence, $\beta''''_L$ is an equivalence.

Remark 4.20. Let $L$ be a band and assume that $L$-gerbe patching holds over $\mathcal{F}$. Let $\mathcal{G}, \mathcal{G}'$ be $L$-gerbes over $\mathcal{F}$ such that $\beta''''_{L, \mathcal{F}}(\mathcal{G})$ and $\beta''''_{L, \mathcal{F}}(\mathcal{G}')$ are isomorphic. Then, there is an equivalence $\mathcal{G} \simeq \mathcal{G}'$, unique up to unique isomorphism, as $\beta''''_{L, \mathcal{F}}$ is an equivalence of 2-categories.

We will now conclude this section with a sufficient condition for the technical assumption of the above theorem to be satisfied.

Proposition 4.21. Let $L = (Y, G, \psi)$ be a band over $F$. Assume that patching holds for $H^2(\bullet, L)$ and that bitorsor factorization holds for any cover $Z \to Y$. Then, any $L$-gerbe patching has property $D$.

Proof. Let $\alpha_i \in H^2(F_i, L_i)$ denote the equivalence class of $\mathcal{G}_i$. Then, we have that $\alpha_i = \alpha_j \in H^2(F_k, L_k)$ by assumption. Hence, there is some $\alpha \in H^2(F, L)$ inducing $\alpha_i$ for all $i \in I_v$. Let $\mathcal{G}$ be an $L$-gerbe representing $\alpha$ and let $Z \to Y$ be a cover such that $\mathcal{G}(Z) \neq \emptyset$. Then, $\mathcal{G}_i(Z_i) \neq \emptyset$ by construction. Fix equivalences $\mathcal{G}_i \to BG_i$. Then, for $(i, j, k) \in S_I$, we have a morphism

$$
\begin{array}{ccc}
BG_i & \to & BG_j \\
\downarrow & & \uparrow \\
\mathcal{G}_i & \xrightarrow{\sigma_k} & \mathcal{G}_j
\end{array}
$$

defined over $Z_k$. Let $P$ be the $G_k$-bitorsor corresponding to the composition $BG_i \to BG_j$. By assumption, we have there is a $G_i$-bitorsor $P_i$ over $Z_i$ and a $G_j$-bitorsor $P_j$ over $Z_j$ such that $P \simeq P_i \wedge G_k P_j$ over $Z_k$. Let $x_i \in \mathcal{G}_i(Z_i)$ correspond to $P_i^{\text{op}}$ with respect to the chosen equivalence $\mathcal{G}_i \to BG_i$ and let $x_j$ correspond to $P_j$. Then, it is straightforward to check that $\sigma_k(x_i|Z_k)$ and $x_j|Z_k$ are isomorphic in $\mathcal{G}_j(Z_k)$. □

Corollary 4.22. Under the assumptions of Proposition 4.21, $L$-gerbe patching holds over $\mathcal{F}$.

4.4. A Mayer-Vietoris sequence and a local-global principle for gerbes. In this section, we will fix a finite inverse factorization system $\mathcal{F}$ with inverse limit $F$. We will also fix a group sheaf $G$ defined over $F$ for which $G$-torsor patching holds. Let $L$ be the band induced by $G$.

Definition 4.23. We say that $G$-gerbes satisfy the local-global principle with respect to $\mathcal{F}$ if for any $G$-gerbe $\mathcal{G}$ over $F$,

$$
\mathcal{G}_i \simeq BG|_{F_i}
$$

for all $i \in I_v$ implies

$$
\mathcal{G} \simeq BG.
$$

Recall that $G$-bitorsors over $F$ are classified by $H^0(F, G \to \text{Aut}(G))$ and $G$ gerbes are classified by $H^1(F, G \to \text{Aut}(G))$ (cf. Appendix A).

Note that we obtain two maps

$$
\prod_{i \in I_v} H^1(F_i, G \to \text{Aut}(G)) \Rightarrow \prod_{k \in I_v} H^1(F_k, G \to \text{Aut}(G)).
$$

via base change.
Lemma 4.24. There is a map of pointed sets
\[ \prod_{k \in I_k} H^0(F_k, G \rightarrow \text{Aut}(G)) \rightarrow H^1(F, G \rightarrow \text{Aut}(G)) \]

Proof. We can define the map as follows: Given $G$-torsors $P_k$ over $F_k$, consider the $G$-gerbe patching problem ($\{BG_{F_i}\}_i, \{P_k\}_k$). By Theorem 4.19, there is a $G$-gerbe $\mathcal{G}$ over $F$ in the essential preimage of the patching problem. We map the equivalence class of $(P_k)_k$ to the equivalence class $\mathcal{G}$. Note that this is independent of the choice of $\mathcal{G}$ (cf. Remark 4.20).

We can put these maps in a Mayer-Vietoris type sequence.

Theorem 4.25 (Mayer-Vietoris for non-abelian hypercohomology (2)). Assume patching holds for $G$-torsors and $G$-gerbes over $\mathcal{P}$. Then, there is an exact sequence
\[
1 \rightarrow H^{-1}(F, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{i \in I_i} H^{-1}(F_i, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{k \in I_k} H^{-1}(F_k, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{i \in I_i} H^0(F_i, G \rightarrow \text{Aut}(G))
\]
\[
\rightarrow H^0(F, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{i \in I_i} H^0(F_i, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{k \in I_k} H^0(F_k, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{i \in I_i} H^1(F_i, G \rightarrow \text{Aut}(G)) \rightarrow \prod_{k \in I_k} H^1(F_k, G \rightarrow \text{Aut}(G))
\]

Proof. The exactness in the first two rows is the content of Theorem 3.11. The exactness at $\prod_{i \in I_i} H^1(F_i, G \rightarrow \text{Aut}(G))$ follows from gerbe patching (Theorem 4.19). The exactness at $H^1(F, G \rightarrow \text{Aut}(G))$ follows immediately from Theorem 4.7.

From this exact sequence, we can deduce a necessary and sufficient criterion for the local-global principle for gerbes to hold in terms of bitorsor factorization.

Theorem 4.26. Assume that gerbe patching holds for $L$-gerbes. Then, $L$-gerbes satisfy a local-global principle with respect to patches if and only $G$ satisfies bitorsor factorization.

Remark 4.27. These results are analogous to the results in [HHK15a] concerning local-global principles for $G$-torsors. They prove that local-global principle for $H^1(F, G)$ is equivalent to factorization of $\prod_k H^0(F_k, G)$. In other words, local-global principle for $G$-torsors is equivalent to $G$ satisfying factorization.

5. Patching over arithmetic curves

We will now apply the results on bitorsor and gerbe patching in the case of patching over arithmetic curves, (cf. Section 2.3). Let $T$ be a complete discretely valued ring with field of fraction $K$, uniformizer $t$ and residue field $k$. Let $\tilde{X}$ be a projective, integral and normal $T$-curve with function field $F$ and let $X$ denote its closed fiber. For a non-empty set of closed points $\mathcal{P} \subset X$ including all points where distinct irreducible components meet, let $\mathcal{F}$ denote a finite inverse factorization system with inverse limit $F$ as introduced in Section 2.3.

5.1. Patching and local-global principle for bitorsors. Let $G, H$ be a linear algebraic groups over $F$.

Recall the natural functor
\[ \hat{\beta}''_{(G,H)} : \text{BITORSOR}(G, H)(F) \rightarrow \text{BPP}(G, H)(\mathcal{F}) . \]
introduced in Section 3.2.
Theorem 5.1. Let \(G, H\) be linear algebraic groups over \(F\). Then, patching holds for \((G, H)\)-bitorsors over \(\mathcal{F}\), i.e. the functor \(\beta''_{(G, H)}\) is an equivalence.

Proof. Follows from Theorem 2.16 and Theorem 3.8. \(\square\)

As a corollary, we immediately obtain a criterion for when \(G\)-bitorsors satisfy a local-global principle.

Corollary 5.2. The local-global principle for \(G\)-bitorsors holds over \(F\) iff \(Z(G)\) satisfies factorization over \(F\).

Proof. Follows from Corollary 3.12 and Theorem 5.1. \(\square\)

Theorem 5.3. Let \(G\) be a linear algebraic group whose center \(Z\) is rational. Then, \(G\)-bitorsors satisfy local-global principle over \(F\) iff \(Z\) is connected or \(\Gamma\) is a tree.

Proof. In [HHK15a, Corollary 6.5], Harbater, Hartmann and Krashen proved that a rational linear algebraic group \(H\) satisfies factorization over \(F\) iff \(H\) is connected or \(\Gamma\) is a tree. Hence, the result follows from Corollary 5.2. \(\square\)

5.2. A Mayer-Vietoris sequence and gerbe patching over curves. We will now investigate when gerbe patching holds over arithmetic curves. Let us first collect some results related to the technical assumption in Theorem 4.19.

Theorem 5.4 (\([CTHH^{+} 17]\)). Assume that \(\text{char}(K) = 0 = \text{char}(k)\). Let \((L_i)_{i \in I}\) be a collection of finite separable field extensions \(L_i/F_i\). Then, there is a finite separable field extension \(E/F\) such that \(E_i\) dominates \(L_i\).

Corollary 5.5. Assume that \(\text{char}(k) = 0\). Then, every \(L\)-gerbe patching problem \((\{G_i\}_{i \in I}, \{\nu_k\}_{k \in I_e})\) over \(\mathcal{F}\) has property D.

Proof. Pick covers \(Z_i \to F_i\) for \(i \in I_v\) and \(z_i \in G_i(Z_i)\). Note that \(Z_i\) is a product of finite separable field extensions of \(F_i\). By Theorem 5.4, there is a cover \(Z' \to F\) such that \(Z'_i\) dominates \(Z_i\).

While \(\nu_k(x_i|Z'_i)\) and \(x_j|Z'_i\) may not be isomorphic in \(G_j(Z'_i)\), they are locally isomorphic, so there are covers \(Y_k \to Z'_k\) such \(\nu_k(x_i|Y_k)\) and \(x_j|Y_k\) are isomorphic. Again using Theorem 5.4, there is a cover \(Y \to F\) dominating \(Z \to F\) such that \(Y'_k \to F\) dominates \(Y_k \to F_k\).

Then, the choice \(Y' \to F\) and \(x_i|Y'\) proves the claim. \(\square\)

In the case where \(\text{char}(k) = p\), we want to use Proposition 4.21 to prove that every gerbe patching problem has property D. The next proposition proves that this is true under some mild assumptions.

Proposition 5.6. Let \(L\) be a band over \(F\) such that \(Z(L)\) is a linear algebraic group over \(F\) with finite order not divided by \(\text{char}(k)\). Then, if \(H^2(F, L) \neq 0\), patching holds for \(H^2(\bullet, L)\) over \(\mathcal{F}\).

Proof. By [HHK14, Theorem 3.1.3], patching holds for \(H^2(\bullet, Z(L))\). Thus, the conclusion follows by Proposition 4.12. \(\square\)

We are now ready to state our main result on gerbe patching over arithmetic curves.

Theorem 5.7. Let \(L = (Y, G, \psi)\) be a band over \(F\) with \(G\) being a linear algebraic group over \(Y\). Assume either
Theorem 5.7, there is an exact sequence of pointed sets.

\[ \text{char}(k) = 0 \text{ or} \]

\[ \text{char}(k) = p > 0 \text{ and } Z(L) \text{ has finite order not divisible by char}(k) \text{ and } G\text{-bitorsor} \]

factorization holds over \( \mathcal{F}_Z \) for every cover \( Z \to Y \).

Then, gerbe patching holds over \( \mathcal{F} \), i.e. the functor \( \beta_L'' \) is a 2-equivalence.

**Proof.** If \( \text{char}(k) = 0 \), this follows from Theorem 2.16, Theorem 4.19 and Theorem 5.5. If \( \text{char}(k) = p \), then it follows from Theorem 2.16, Theorem 4.19 and Proposition 5.6. \( \square \)

**Corollary 5.8** (Mayer-Vietoris of non-abelian hypercohomology over curves). Let \( G \) be a linear algebraic group defined over \( F \) and let \( L \) denote the associated band. Under the assumption of Theorem 5.7, there is an exact sequence of pointed sets

\[
1 \to H^{-1}(F, G \to \text{Aut}(G)) \to \prod_{i \in I} H^{-1}(F_i, G \to \text{Aut}(G)) \to \prod_{k \in I_e} H^{-1}(F_k, G \to \text{Aut}(G)) \\
\to H^0(F, G \to \text{Aut}(G)) \to \prod_{i \in I_e} H^0(F_i, G \to \text{Aut}(G)) \to \prod_{k \in I_e} H^0(F_k, G \to \text{Aut}(G)) \\
\to H^1(F, G \to \text{Aut}(G)) \to \prod_{i \in I_e} H^1(F_i, G \to \text{Aut}(G)) \to \prod_{k \in I_e} H^1(F_k, G \to \text{Aut}(G))
\]

**Proof.** Follows from Theorem 4.25 and Theorem 5.7. \( \square \)

### 5.3. Factorization of bitorsors over curves

We will now investigate which group schemes \( G \) over \( F \) admit \( G \)-bitorsor factorization over \( \mathcal{F} \). Let \( \mathcal{F} \) be indexed by \( I = I_r \sqcup I_e \) with associated graph \( \Gamma \) (cf. Section 2 for details). The short exact sequence \( 1 \to (1 \to \text{Aut}(G)) \to (G \to \text{Aut}(G)) \to (G \to 1) \to 1 \) of crossed modules induces the long exact sequence

\[
1 \to H^0(Z(G)) \to H^0(G) \to H^0(\text{Aut}(G)) \\
\to H^0(G \to \text{Aut}(G)) \to H^1(G) \to H^1(\text{Aut}(G)) \to H^1(G \to \text{Aut}(G))
\]

(compare Example A.4).

We will first consider the case of finite constant group schemes with trivial center.

**Theorem 5.9.** Assume that \( G \) is a finite constant group scheme over \( F \) with trivial center and that \( \Gamma \) is a tree. Then, \( G \) satisfies bitorsor factorization over \( \mathcal{F} \).

**Proof.** Given a collection \( \{P_b\}_{b \in \mathcal{B}} \) with \( P_b \) a \( G \)-bitorsor over \( F_b \), we need to show that there are \( \{P_U\}_{U \in \mathcal{W}} \) and \( \{P_p\}_{p \in \mathcal{P}} \) such that \( P_U \) is a \( G \)-bitorsor over \( F_U \), \( P_p \) is a \( G \)-bitorsors over \( F_p \) and \( P_U|_{F_p} \rtimes^G P_p|_{F_b} \simeq F_b \) whenever \( b \) is a branch at \( U \) and \( p \).

By Theorem A.5, this is equivalent to showing that

\[
\prod_{U \in \mathcal{W}} H^0(F_U, G \to \text{Aut}(G)) \times \prod_{p \in \mathcal{P}} H^0(F_p, G \to \text{Aut}(G)) \to \prod_{b \in \mathcal{B}} H^0(F_b, G \to \text{Aut}(G))
\]

is surjective. Since \( Z(G) = \{e\} \), the sequence

\[
1 \to G \to \text{Aut}(G) \to \text{Aut}(G)/G \to 1
\]

is exact. Since \( G \) is constant, so is \( \text{Aut}(G) \) and the long exact sequence associated to the short exact sequence above reads

\[
1 \to H^1(F_b, G) \to H^1(F_b, \text{Aut}(G)) \to \ldots
\]
for any $b \in \mathcal{B}$. Since $H^1(F_b, G) \to H^1(F_b, \text{Aut}(G))$ is injective, it follows from sequence (1) that $H^0(F_b, \text{Aut}(G)) \to H^0(F_b, G \to \text{Aut}(G))$ is surjective. It is thus enough to show that

$$
\prod_{U \in \mathcal{U}} H^0(F_U, \text{Aut}(G)) \times \prod_{p \in \mathcal{P}} H^0(F_p, \text{Aut}(G)) \to \prod_{b \in \mathcal{B}} H^0(F_b, \text{Aut}(G))
$$

is surjective, i.e. that $\text{Aut}(G)$ satisfies factorization. Since $\text{Aut}(G)$ is also a finite, constant group scheme, it satisfies factorization by assumption and Theorem 2.18.

**Examples 5.10.** Examples of group schemes satisfying the assumptions of theorem 5.9 include $S_n$ for any $n > 2$, $A_n$ for $n > 3$ and any finite, nonabelian simple group.

**Theorem 5.11.** Let $G$ be an algebraic group over $F$ such that the natural map $G \to \text{Aut}(G)$ is an isomorphism. If $G$ satisfies factorization over $\mathcal{F}$, then $G$ satisfies bitorsor factorization over $\mathcal{F}$.

**Proof.** Using the assumption, we can see that the map $H^0(F, G) = H^0(F, \text{Aut}(G)) \to H^0(F, G \to \text{Aut}(G))$ is surjective. Hence, the claim follows immediately. □

**Examples 5.12.** Let $G$ be semisimple adjoint. Then, $G \to \text{Aut}(G)$ is an isomorphism if $G$ is of type $A_1$, $B_n$, $C_n$, $E_7$, $E_8$, $F_4$, and $G_2$ (compare 24.A and [KMTR98, Proposition 25.15]).

Thus, if $G$ satisfies factorization over $\mathcal{F}$, then it also satisfies bitorsor factorization.

Let now $G$ be a semisimple group whose adjoint group admits no outer automorphism. Then, we have a short exact sequence of crossed modules (see Appendix A for the definition and see Corollary 25.17 in [KMTR98] for exactness):

$$1 \to (Z \to 1) \to (G \to \text{Aut}(G)) \to (G/Z \to \text{Aut}(G/Z)) \to 1$$

We thus obtain the following exact sequence of hypercohomology groups.

$$
\begin{array}{cccc}
1 & \to & H^0(F, Z) & \to H^0(F, Z) \to 1 \\
& & \downarrow \quad & \\
& & H^1(F, Z) \to H^0(F, G \to \text{Aut}(G)) \to H^0(F, G/Z \to \text{Aut}(G/Z)) \\
& & \downarrow \quad & \\
& & H^2(F, Z) \to H^1(F, G \to \text{Aut}(G)) \\
\end{array}
$$

**Lemma 5.13.** Let $G$ be a semisimple group whose adjoint group $G/Z$ admits no outer automorphisms. Then, the map

$$H^0(F, G/Z \to \text{Aut}(G/Z)) \to H^2(F, Z)$$

from the exact sequence (2) is the zero map.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
H^1(F, G/Z) & \to & H^2(F, Z) \\
\downarrow \quad & \delta & \\
H^0(F, G/Z \to \text{Aut}(G/Z)) & \to & H^2(F, Z).
\end{array}
$$
Note that this diagram does not commute: a cocycle \((f, \lambda) \in Z^0(F, G \to \text{Aut}(G))\) maps to \(\lambda(\delta(f))\) as opposed to \(\delta(f)\). However, as \(\delta(f) = 0\) implies \(\lambda(\delta(f)) = 0\), it is enough to show that

\[
H^0(F, G/Z \to \text{Aut}(G/Z)) \to H^1(F, G/Z)
\]
is the zero map.

From the exact sequence (1), we obtain that

\[
H^0(F, G/Z \to \text{Aut}(G/Z)) \to H^1(F, G/Z) \to H^1(F, \text{Aut}(G/Z))
\]
is exact. But, as \(G/Z\) is isomorphic to \(\text{Aut}(G/Z)\) by assumption, the claim follows. \(\square\)

**Theorem 5.14.** Let \(G\) be a semisimple group such that \(G/Z\) admits no outer automorphisms. If \(G/Z\) and \(Z\) satisfy bitorse factorization over \(\mathcal{F}\), then so does \(G\).

**Proof.** Let us start with \(\alpha \in H^0(F_b, G \to \text{Aut}(G))\). Let \(\beta\) denote its image in \(H^1(F_b, G/Z \to \text{Aut}(G/Z))\). By assumption, there are \(\beta_U \in H^0(F_U, G/Z \to \text{Aut}(G/Z))\) and \(\beta_p \in H^0(F_p, G/Z \to \text{Aut}(G/Z))\) such that \(\beta_U \beta_p = \beta \in H^0(F_b, G/Z \to \text{Aut}(G/Z))\). By Sequence (2) and Lemma 5.13, we can lift \(\beta_U\) and \(\beta_p\) to elements \(\tilde{\alpha}_U\) and \(\tilde{\alpha}_p\) in \(H^0(F_U, G \to \text{Aut}(G))\) and \(H^0(F_p, G \to \text{Aut}(G))\) respectively. Consider the element \(\tilde{\alpha} = \tilde{\alpha}_U^{-1} \tilde{\alpha}_p^{-1} \in H^0(F_b, G \to \text{Aut}(G))\). By construction, \(\tilde{\alpha}\) comes from an element \(\gamma \in H^1(F_b, Z)\). By assumption, there are \(\gamma_U \in H^1(F_U, Z)\) and \(\gamma_p \in H^1(F_p, Z)\) satisfying \(\gamma_U \gamma_p = \gamma \in H^1(F_b, Z)\). Consider the elements \(\alpha_U = \gamma_U \tilde{\alpha}_U \in H^0(F_U, G \to \text{Aut}(G))\) and \(\alpha_p = \gamma_p \tilde{\alpha}_p \in H^0(F_p, G \to \text{Aut}(G))\). Then, \(\alpha = \alpha_U \alpha_p\). This follows from

\[
\alpha_U^{-1} \alpha \alpha_p^{-1} = \gamma_U^{-1} \tilde{\alpha}_U^{-1} \tilde{\alpha}_p^{-1} \gamma_p^{-1} = \gamma_U^{-1} \gamma_p^{-1} = 1.
\]

\(\square\)

We will now turn our attention to groups of type \(A_n\). For this, let \(D\) be a central simple algebra over \(F\).

**Lemma 5.15.** The group \(\text{Aut}(\text{SL}_1(D))\) satisfies factorization over \(\mathcal{F}\) if \(\Gamma\) is a tree.

**Proof.** Note that we have a short exact sequence

\[
1 \to \text{PGL}_1(D)(K) \to \text{Aut}(\text{SL}_1(D))(K) \to \mathbb{Z}/2\mathbb{Z} \to 1
\]

for any field \(K/F\). The result now follows from noting that \(\text{PGL}_1(D)\) and \(\mathbb{Z}/2\mathbb{Z}\) satisfy factorization over \(\mathcal{F}\) as they are both rational and \(\Gamma\) is a tree (see Theorem 2.18). \(\square\)

**Proposition 5.16.** Let \(D\) be a central simple algebra over \(K\). Then, the map of pointed sets

\[
H^1(K, \text{SL}_1(D)) \to H^1(K, \text{Aut}(\text{SL}_1(D)))
\]

has trivial image. In particular, the map \(H^0(\text{SL}_1(D) \to \text{Aut}(\text{SL}_1(D)))) \to H^1(\text{SL}_1(D))\) is surjective.

**Proof.** Note that a piece of sequence (1) for \(G = \text{SL}_1(D)\) reads

\[
H^0(\text{Aut}(\text{SL}_1(D))) \to H^0(\text{SL}_1(D) \to \text{Aut}(\text{SL}_1(D)))) \to H^1(\text{SL}_1(D)) \to H^1(\text{Aut}(\text{SL}_1(D))).
\]
Thus, injectivity of $H^1(K, SL_1(D) \to H^1(K, Aut(SL_1(D))))$ implies surjectivity of $H^0(SL_1(D) \to Aut(SL_1(D))) \to H^1(K, SL_1(D))$. Note that the map $SL_1(D) \to Aut(SL_1(D))$ factors through $SL_1(D) \to GL_1(D)$. Hence, $H^1(K, SL_1(D) \to H^1(K, Aut(SL_1(D))))$ factors through $H^1(K, SL_1(D) \to H^1(K, GL_1(D))$. The claim thus follows from $H^1(K, GL_1(D)) = 0$. \hfill \Box

**Theorem 5.17.** Let $D$ be a a central simple algebra over $F$ and let $\Gamma$ be a tree. Then, $SL_1(D)$ satisfies bitorsor factorization.

**Proof.** Fix a collection of

$$\alpha_b \in H^0(F_b, SL_1(D) \to Aut(SL_1(D)))$$

for $b \in \mathcal{B}$ and let $\beta_b$ denote their images in $H^1(F_b, SL_1(D))$ (along sequence 3) when $b$ corresponds to a branch at $p$. Recall that

$$H^1(\mathcal{F}_b, SL_1(D)) = F_b^*/Nrd(D_b)$$

holds. Let $n$ be the index of $D$. Since $F_b$ is a completion of $F_p$ at a discrete valuation, the map $F_p^* \to F_b^*/Nrd(D_b^n)$ is surjective. As $F_p^* \to F_b^*/Nrd(D_b^n)$ is surjective whenever $b$ is a branch at $p$. Thus, by weak approximation, there are $\beta_p \in H^1(F_p, SL_1(D))$ for all $p \in \mathcal{P}$ such that $\beta_p = \beta_b \in H^1(\mathcal{F}_b, SL_1(D))$ whenever $b$ is a branch at $p$.

By Proposition 5.16, there are $\alpha_p \in H^0(F_p, SL_1(D) \to Aut(SL_1(D)))$ mapping onto $\beta_p$ for all $p \in \mathcal{P}$. Let now $b$ be a branch at $p$ and $U$. Then, $\alpha_b \alpha_p^{-1}$ maps onto 0 in $H^1(F_b, SL_1(D))$ by construction. Hence, there is $\nu_p \in H^0(F_b, Aut(SL_1(D)))$ mapping onto $\alpha_b \alpha_p^{-1}$. By Lemma 5.15, there exist $\nu_U \in H^0(F_U, Aut(SL_1(D)))$ and $\nu_p \in H^0(F_p, Aut(SL_1(D)))$ such that their product in $H^0(F_b, Aut(SL_1(D)))$ is $\nu_b$ for all $(U, p, b) \in S_I$.

Let $\tau_U$ and $\tau_p$ denote the images of $\nu_U$ and $\nu_p$ in $H^0(F_U, SL_1(D) \to Aut(SL_1(D)))$ and $H^0(F_p, SL_1(D) \to Aut(SL_1(D)))$ respectively. Then, the collection of $\{\tau_U\}_{U \in \mathcal{W}}$ and $\{\tau_p \alpha_p\}_{p \in \mathcal{P}}$ give a factorization of $\{\alpha_b\}_{b \in \mathcal{B}}$. \hfill \Box

**5.4. Local-global principles for gerbes.** Building on our results on bitorsor factorization, we now use Theorem 4.26 to obtain local-global principles for gerbes.

**Theorem 5.18.** Let $G$ be a linear algebraic group over $F$ with center $Z$. Then, the local global principle for $G$-gerbes with respect to patching holds if

- $\text{char}(k) = 0$ and one of the following hold:
  - $\Gamma$ is a tree and $G$ is a finite constant group scheme with trivial center,
  - $G$ is connected, rational, semisimple, adjoint of type $A_1, B_n, C_n, E_7, E_8, F_4$ or $G_4$,
  - $G$ is semisimple such that $G/Z$ admits no outer automorphism and $Z, G/Z$ satisfy bitorsor factorization,
  - $\Gamma$ is a tree and $G = SL_1(D)$ where $D$ is a central simple algebra over $F$,
- $\text{char}(k) = p > 0$, $Z$ has finite order not divided by $\text{char}(k)$ and $G$ and $\Gamma$ are as in the case of $\text{char}(k) = 0$.

**Proof.** All results use Theorem 4.19 and Theorem 4.26. The results follow (in order) from Theorem 5.9, Example 5.12 and [HHK09, Theorem 3.6], Theorem 5.14, and Theorem 5.17. \hfill \Box
5.5. Local-global principle for homogeneous spaces. In this section, we will apply our local-global principle for gerbes to deduce local-global principles for points on homogeneous spaces.

Let \( H \) be a linear algebraic group and let \( X \) be a homogeneous space under \( H \) over \( F \). Assume that the stabilizer of a geometric point \( x \in X(F_{\text{sep}}) \) is isomorphic to \( G|_{F_{\text{sep}}} \) for some linear algebraic group \( G \subseteq H \) defined over \( F \). Consider the quotient stack \([X/H]\). Objects of this stack are given by tuples \((Y,P,\phi)\) where \( Y \) is an \( F \)-scheme, \( P \) is a principal homogeneous space under \( H \) over \( Y \) and \( \phi : P \to X|_Y \) is an \( H \)-equivariant map. The stack \([X/H]\) is a \( G \)-gerbe and it is trivial if and only if there is a \( H \)-equivariant map \( P \to X \) where \( P \) is principal homogeneous space under \( H \) over \( F \) (cf. [Bor93, Remark 7.7.1]).

Let \( Y \) be an \( F \)-scheme. If \( X(Y) \neq \emptyset \), pick a point \( x \in X(Y) \). This defines an \( H \)-equivariant morphism \( \phi_x : H|_Y \to X|_Y \) which gives an element \((Y,H|_Y,\phi_x) \in [X/H](Y)\). Hence, \( X(Y) \neq \emptyset \) implies \([X/H](Y) \neq \emptyset \). Assume now that \( H \) is special (i.e. \( H^1(K,H) = \{e\} \) for all field extensions \( K/F \)). Let \( K/F \) be a field extension. If \([X/H](K) \neq \emptyset \), then there is a principal homogeneous space \( P \) under \( H|_K \) over \( K \) together with an \( H \)-equivariant map \( \phi : P \to X \). As \( H \) is special, \( P \) admits a point \( p \in P(K) \). Hence, \( \phi(p) \in X(K) \). Thus, \([X/H](K) \neq \emptyset \) implies \( X(K) \neq \emptyset \).

Definition 5.19. We say that the local-global principle holds for \( X \) over \( \mathcal{F} \) if \( X(F_i) \neq \emptyset \) for all \( i \in I_\mathcal{F} \) implies \( X(F) \neq \emptyset \).

Theorem 5.20. Let \( H \) be special and let \( X \) be a homogeneous space under \( H \). Let \( G \subseteq H \) be a linear algebraic group over \( F \) such that \( G|_{F_{\text{sep}}} \) is isomorphic to the stabilizer of a geometric point of \( X \). If \( \text{char}(k) = 0 \), the local-global principle for \( X \) holds if and only if bitorsor factorization holds for \( G \).

If \( \text{char}(k) = p > 0 \), assume that

- \( p \nmid |Z(G)| < \infty \),
- \( G \)-bitorsor factorization holds over \( \mathcal{F}_Z \) for every cover \( Z \to F \).

Then, the local-global principle holds for \( X \) over \( \mathcal{F} \).

Proof. Since \( H \) is special, we have \( X(F) \neq \emptyset \) iff \([X/H](F) \neq \emptyset \) as well as \( X(F_i) \neq \emptyset \) iff \([X/H](F_i) \neq \emptyset \) by the discussion above. Hence, the local-global principle for \( X \) holds iff it holds for \([X/H]\). The result follows from Theorem 4.26 and Theorem 5.7. \( \square \)

Corollary 5.21. Let \( H \) be special (e.g. \( H = SL_n \) or \( H = Sp_{2n} \)) and let \( X \) be a homogeneous space under \( H \). Let \( G \subseteq H \) be a linear algebraic group over \( F \) such that \( G|_{F_{\text{sep}}} \) is isomorphic to the stabilizer of a geometric point of \( X \).

- If \( \text{char}(k) = 0 \), assume that one of the following holds:
  - \( \Gamma \) is a tree and \( G \) is a finite constant group scheme with trivial center,
  - \( G \) is connected, rational, semisimple, adjoint of type \( A_1, B_n, C_n, E_7, E_8, F_4 \) or \( G_2 \),
  - \( G \) is semisimple such that \( G/Z \) admits no outer automorphism and \( Z, G/Z \) satisfy bitorsor factorization,
  - \( \Gamma \) is a tree and \( G = SL_1(D) \) where \( D \) is a central simple algebra over \( F \),
- If \( \text{char}(k) = p > 0 \), assume that \( Z(G) \) has finite order not divided by \( \text{char}(k) \) and \( G \) and \( \Gamma \) are as in the case of \( \text{char}(k) = 0 \).

Then, the local-global principle holds for \( X \) over \( \mathcal{F} \).

Proof. Follows from Theorem 5.18 and Theorem 5.20. \( \square \)
APPENDIX A. NONABELIAN HYPERCOHOMOLOGY

Throughout this section, let $F$ be a field and $\Gamma$ its absolute Galois group. Furthermore, let $H, G$ be algebraic groups over $F$. We will review here basics from nonabelian hypercohomology. Nonabelian hypercohomology was introduced by Breen ([Bre90]) and Borovoi ([Bor98], [Bor92]). We refer to these papers for the definition of $H^i(F, H \to G)$ for $i = -1, 0, 1$. We will write $H^i(G)$ and $H^i(G \to H)$ for $H^i(F, G)$ and $H^i(F, G \to H)$ respectively.

Definition A.1. A crossed module over $K$ is a morphism $\rho : G \to H$ of algebraic groups over $K$ together with a left action $\alpha : H \times G \to G$ such that

- $\rho(g)^{g'} = g^g (g')^{-1}$ for $g, g' \in G$ and
- $\rho(hg) = h \rho(g) h^{-1}$ for $g \in G$ and $h \in H$

holds.

Given a crossed module $\rho : G \to H$, a $\Gamma$ action on $\rho$ consists of actions of $\Gamma$ on $G$ and $H$ satisfying

$$\rho^{(\sigma g)} = \sigma \rho(g)$$

and

$$\sigma^{(h g)} = \sigma^h (\sigma g)$$

for $g \in G, h \in H$ and $\sigma \in \Gamma$.

Examples A.2. We will mostly use the following two examples.

- Let $G$ be any algebraic group, then $1 \to G$ is a crossed module.
- For any algebraic group $G$, $G \xrightarrow{\text{Int}} \text{Aut}(G)$ is a crossed module.

Proposition A.3 ([Bor98, Section 3.4.2]). Let

$$1 \to (G_1 \to H_1) \xrightarrow{i} (G_2 \to H_2) \xrightarrow{j} (G_3 \to H_3) \to 1$$

be an exact sequence of complexes of $\Gamma$-groups where $i$ is an embedding of crossed modules with $\Gamma$ action. Then, there is an exact sequence of pointed sets

$$1 \to H^{-1}(G_1 \to H_1) \to H^{-1}(G_2 \to H_2) \to H^{-1}(G_3 \to H_3)$$

\[ \to H^0(G_1 \to H_1) \to H^0(G_2 \to H_2) \to H^0(G_3 \to H_3) \]

\[ \to H^1(G_1 \to H_1) \to H^1(G_2 \to H_2) \]

Example A.4. We will mostly use Proposition A.3 for the following short exact sequence:

$$1 \to (1 \to \text{Aut}(G)) \xrightarrow{i} (G \to \text{Aut}(G)) \xrightarrow{j} (G \to 1) \to 1$$

where all maps occurring are either the identity or trivial. Then, the corresponding long exact sequence simplifies to the following sequence (cf. [Bre90, Section 4.2.3]).
We will now describe characterizations of $H_i(G \to \text{Aut}(G))$ for $i = -1, 0, 1$. Recall that $H^{-1}(G \to \text{Aut}(G)) = \ker(\alpha)^G$ and hence

$$H^{-1}(F, G \to \text{Aut}(G)) = Z(G)(F)$$

where $Z(G)$ is the center of $G$.

As mentioned in section 3.6, $H^0(G \to \text{Aut}(G))$ classifies $G$-bitorsors.

**Proposition A.5** ([Bre90, Theorem 4.5]). There is a natural isomorphism

$$H^0(F, G \to \text{Aut}(G)) \simeq \{\text{Isomorphism classes of } G\text{-bitorsors over } F\}.$$

Unlike torsors, a $G$-bitorsor may not be trivial (i.e. isomorphic to $G$ as a bitorsor) even when it admits a point. This phenomenon motivates the next definition.

**Definition A.6.** Let $\alpha \in H^0(G \to \text{Aut}(G))$. We say that $\alpha$ is **neutral** if a bitorsor representing $\alpha$ admits a point over $F$.

We now turn our attention to $H^1(G \to \text{Aut}(G))$.

**Proposition A.7** ([Bre90, Theorem 4.5]). There is a natural isomorphism

$$H^1(F, G \to \text{Aut}(G)) \simeq \{\text{Equivalence classes of } G\text{-gerbes over } F\}.$$

**Definition A.8.** Let $\alpha \in H^1(G \to \text{Aut}(G))$. We say that $\alpha$ is **neutral** if a corresponding gerbe (and thus every corresponding bitorsor) admits a point over $F$.

**REFERENCES**

[BM05] Lawrence Breen and William Messing. Differential geometry of gerbes. *Advances in Mathematics*, 198(2 SPEC. ISS.):732–846, 2005.

[Bor92] Mikhail V Borovoi. Non-abelian hypercohomology of a group with coefficients in a crossed module, and Galois cohomology. *Preprint*, pages 1–20, 1992.

[Bor93] Mikhail V Borovoi. Abelianization of the second nonabelian galois cohomology. *Duke Mathematical Journal*, 72(1):217–239, 1993.

[Bor95] Mikhail Borovoi. The Brauer-Manin obstruction for homogeneous spaces with connected or abelian stabilizer. *Journal fuer die reine und angewandte Mathematik*, 473:181–194, 1995.

[Bor98] Mikhail V Borovoi. Abelian galois cohomology of reductive groups. *Memoirs of the American Mathematical Society*, 626:1–50, 1998.

[Bre90] Lawrence Breen. Bitorseur et Cohomologie Non Abeliene. In *The Grothendieck Festschrift* I, pages 401–476. Birkhauser, 1 edition, 1990.

[Bre91] Lawrence Breen. Tannakian Categories. In *Proceedings of Symposia in Pure Mathematics*, pages 337–376. American Mathematical Society, volume 55, edition, 1991.

[Bre09] Lawrence Breen. Notes on 1- and 2-gerbes. In J.C. Baez May and J.P., editors, *Towards Higher Categories*, pages 193–235. Springer, the ima vo edition, 2009.

[CTHH+17] Jean-Louis Colliot-Thélène, David Harbater, Julia Hartmann, Daniel Krashen, Raman Parimala, and Venapally Suresh. Local-global principles for zero-cycles on homogeneous spaces over arithmetic function fields. *preprint*, 2017.
[DM82] P. Deligne and James S. Milne. Tannakian Categories. *Hodge cycles, motives, and Shimura varieties*, 900:101–228, 1982.

[GA12] Cristian D. Gonzalez-Aviles. Quasi-abelian crossed modules and nonabelian cohomology. *Journal of Algebra*, 369:235–255, 2012.

[Gir71] J Giraud. *Cohomologie non abélienne*. Springer, 1971.

[HH10] David Harbater and Julia Hartmann. Patching over fields. *Israel Journal of Mathematics*, 176(1):61–107, oct 2010.

[HHK09] David Harbater, Julia Hartmann, and Daniel Krashen. Applications of patching to quadratic forms and central simple algebras. *Inventiones Mathematicae*, 178(2):231–263, sep 2009.

[HHK14] David Harbater, Julia Hartmann, and Daniel Krashen. Local-global principles for Galois cohomology. *Commentarii Mathematici Helvetici*, 89(1):215–253, aug 2014.

[HHK15a] David Harbater, Julia Hartmann, and Daniel Krashen. Local-global principles for torsors over arithmetic curves. *American Journal of Mathematics*, 137(6):1559–1612, aug 2015.

[HHK15b] David Harbater, Julia Hartmann, and Daniel Krashen. Refinements to patching and applications to field invariants. *International Mathematics Research Notices*, 20:10399–10450., 2015.

[KMTR98] Max-albert Knus, Alexander Merkurjev, J. P. Tignol, and Markus Rost. *The Book of Involutions*. American Mathematica Society Colloquium Publications, 1998.

[Ols16] Martin Olsson. *Algebraic spaces and stacks*. American Mathematical Society, 2016.