GROUP OBJECTS AND INTERNAL CATEGORIES

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ABSTRACT. Algebraic structures such as monoids, groups, and categories can be formulated within a category using commutative diagrams. In many common categories these reduce to familiar cases. In particular, group objects in \( \text{Grp} \) are abelian groups, while internal categories in \( \text{Grp} \) are equivalent both to group objects in \( \text{Cat} \) and to crossed modules of groups. In this exposition we give an elementary introduction to some of the key concepts in this area.

This expository essay was written in the winter of 1999-2000, early in the course of my PhD research, and has since been updated with supplementary references. I hope you will find it useful. I am indebted to my supervisor, Professor Tim Porter, for his help in preparing this article.

1. GROUPS WITHIN A CATEGORY

Let \( \mathcal{C} \) be a category with finite products. For this it is necessary and sufficient that \( \mathcal{C} \) have pairwise products (i.e. for any 2 objects \( C, D \in \text{Ob}(\mathcal{C}) \), there is a product \( C \times D \)) and a terminal object, which we shall denote by 1. Examples of suitable categories include \( \text{Set} \), \( \text{Grp} \), \( \text{Top} \) and \( \text{Ab} \).

Let \( G \) be an object of \( \mathcal{C} \). Then \( G \times G \) is also an object of \( \mathcal{C} \). Suppose we can find a morphism \( m : G \times G \to G \) such that the diagram

\[
\begin{array}{ccc}
G \times G \times G & \xrightarrow{id_G \times m} & G \times G \\
\downarrow{m \times id_G} & & \downarrow{m} \\
G \times G & \xrightarrow{m} & G
\end{array}
\]

commutes. Now, \( m \) is a binary operation and if we temporarily think in terms of elements we see that the diagram ensures that the operation is associative. For example, take \( \mathcal{C} = \text{Set} \); then \( G \) is a set and \((id_G \times m)(a, b, c) := (a, bc)\) for \( a, b, c \in G \), etc. We may take this diagram as a general definition of associativity, even for categories in which the objects do not have elements. Therefore \( \langle G, m \rangle \) gives us an abstract semigroup in \( \mathcal{C} \). It is often convenient, in categories...
such as \textbf{Set}, to think in terms of the elements and their images under the given morphisms\footnote{Sometimes it is actually more confusing to think in terms of elements, for example when dealing with opposite categories.}, but in fact the definitions are much more general.

Suppose we also have a morphism \( e : 1 \to G \) from the terminal object to \( G \), such that

\[
1 \times G \xrightarrow{e \times id_G} G \times G \xrightarrow{id_G \times e} G \times 1
\]

commutes\footnote{Note that the isomorphisms \( 1 \times G \cong G \cong G \times 1 \) follow automatically from the definition of the product and the fact that \( 1 \) is a terminal object (so there is a unique arrow \( G \to 1 \)).}. Then \( e \) acts as an identity for \( G \). For example, in \textbf{Set}, the terminal object \( 1 \) is a singleton and so \( e \) picks out an element of \( G \) which acts as the identity under the “multiplication” \( m \).

Hence an object \( G \) with morphisms \( m, e \) as above can be thought of as a monoid in \( \mathcal{C} \). In fact we can define (see MacLane [10, section 3.6]) a monoid in \( \mathcal{C} \) to be precisely a triple \( \langle G, m, e \rangle \) such that the above diagrams commute. In the case of \textbf{Set}, the monoids thus given correspond exactly to monoids in the classical, i.e. non-categorical, sense. It is clear that, given any classical monoid (written multiplicatively), we can define \( m \) by \( m(x, y) := xy \) and if \( 1 \) denotes the identity element, we define \( e : \star \mapsto 1 \) (taking \( 1 = \{\star\} \)). Conversely, given any categorical monoid in \textbf{Set}, the morphism \( m \) gives a well-defined associative binary operation for which \( e(\star) \) is an identity.

To get from here to groups it remains to define inverses. Again we must do this using morphisms, as the objects in our category need not have elements. Let \( i \) be a morphism \( G \to G \) such that

\[
G \xrightarrow{\Delta} G \times G \xrightarrow{i \times id_G} G \times G \xrightarrow{m} G
\]

commutes, where \( \Delta : G \to G \times G \) is the diagonal morphism (i.e. \( p\Delta = q\Delta = id_G \), where \( p, q \) are the projections from the product to its components).

In \textbf{Set}, if \( g \in G \) then \( \Delta(g) = (g, g) \) and the commutativity of the diagram gives \( m(i(g), g) = e(\star) \), so \( i(g) \) is a left inverse for \( g \). We can also replace \( i \times id_G \) in the diagram with \( id_G \times i \), which would give \( i(g) \) as a right inverse for \( g \). Thus to get a two-sided inverse for each element in \( G \), we may stipulate that \( i \) should exist so that both diagrams commute.
We are now in a position to define a **group object** in $\mathcal{C}$ to be an ordered quadruple $\langle G, m, e, i \rangle$ with morphisms defined as above. In $\mathbf{Set}$, the group objects are just groups in the classical sense, as we have seen. In $\mathbf{Top}$, each set $G$ is a topological space and all maps are continuous, so the group objects are topological groups. In the category of differentiable manifolds, the group objects are Lie groups. In $\mathbf{Grp}$, the objects are themselves groups; in this case, the group objects are abelian groups. Why?

Firstly, every abelian group is a group object in $\mathbf{Grp}$. Suppose $A$ is an abelian group. Then it consists of a set $A$ together with an associative, commutative binary operation with an identity and inverses. We can view the binary operation as a map $m : A \times A \to A$ and the identity can be considered as a map $e : 1 \to A$ which selects the identity element; similarly there is a map $i : A \to A$ which takes each element in $A$ to its inverse. These maps then clearly satisfy the diagrams given above, so that $\langle A, m, e, i \rangle$ is a group object in $\mathbf{Set}$. However, $A$ is a group, so $A$ is an object of $\mathbf{Grp}$. It remains to show that $m, e, i$ are group homomorphisms and hence morphisms in $\mathbf{Grp}$. Now $A \times A$ is a direct product of groups so its multiplication is defined componentwise. Writing $\otimes$ for the multiplication in $A \times A$ we thus have

$$(x, y) \otimes (x', y') := (m(x, x'), m(y, y')).$$

We now get

$$m((x, y) \otimes (x', y')) = m(m(x, x'), m(y, y')) = (x + x') + (y + y') = (x + y) + (x' + y') = m(m(x, y), m(x', y')).$$

Note that here we switched notation, writing $m(x, y) = x + y$, so that the associativity and commutativity could be visualised more easily. Comparing the left and right hand ends of this chain of equalities shows that $m$ is a homomorphism, as required. The other two are, fortunately, easier to write down. We have $e : \{\star\} \to A$; now $\{\star\}$ is the trivial group. Thus we get $e(\star \star) = e(\star) = id_A = m(id_A, id_A) = m(e(\star), e(\star))$ and hence $e$ is a homomorphism from the trivial group. Finally, $i : A \to A$. We have (again writing additively for convenience) $i(a + b) = -(a + b) = (-a) + (-b) = i(a) + i(b)$, so this too is a homomorphism. Since these maps are all homomorphisms, they exist in $\mathbf{Grp}$ and hence we get that $\langle A, m, e, i \rangle$ is a group object in $\mathbf{Grp}$, as required.

Conversely, suppose we have a group object $\langle G, m, e, i \rangle$ in $\mathbf{Grp}$. We must show that $G$ is an abelian group. Now $G$, being a group, already has an operation which we shall write as $x \cdot y$. This is associative, with inverses and an identity,

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The morphisms are smooth maps of class $C^\infty$, i.e. they are infinitely differentiable.
but not necessarily commutative. We have a second operation defined by the homomorphism \( m \), which we shall write as \( x \ast y := m(x, y) \). \( G \) has an identity, denoted\(^4 \) \( 1_G \), under \( \cdot \); it also has an identity under \( \ast \) given by \( e : 1 \to G \). Since \( e \) is a homomorphism and \( \ast \) is the identity element of the group \( \{ \ast \} \), we deduce that \( e(\ast) = 1_G \). Now, for \( x, y, z, w \in G \), we have \( m(x, y) \cdot m(z, w) = m((x, y) \otimes (z, w)) \), since \( m \) is a homomorphism. But, by definition of the product \( \otimes \) in \( G \times G \), this is just \( m(xz, yw) \). Hence, switching notation, we get the interchange law:

\[
(x \ast y) \cdot (z \ast w) = (x \cdot z) \ast (y \cdot w).
\]

Using this, we get:

\[
x \cdot y = (x \cdot 1_G) \cdot (1_G \ast y) = (x \cdot 1_G) \ast (1_G \cdot y) = x \ast y \\
= (1_G \cdot x) \ast (y \cdot 1_G) = (1_G \ast y) \cdot (x \cdot 1_G) = y \cdot x.
\]

Hence \( \langle G, \cdot \rangle \) is abelian, as required. Note also that the first line shows that the operation given by \( m \) is necessarily the same as the group operation already defined on \( G \).

**Group Actions.** Having formulated groups within \( \text{Set} \), we can also formulate group actions by means of suitable diagrams. Let \( \langle G, m, e, i \rangle \) be a group and \( X \) a set. Then a (left) action of \( G \) on \( X \) is given by a map \( n : G \times X \to X \) such that the following diagrams commute:

\[
\begin{array}{c}
G \times G \times X \\
m \times \text{id}_X \downarrow \\
G \times X \\
\downarrow n \\
G \times X \\
\end{array}
\]

\[
\begin{array}{c}
1 \times X \\
\varepsilon \times \text{id}_X \downarrow \\
G \times X \\
\downarrow n \\
X \\
\end{array}
\]

In the usual notation for (left) group actions, the first of these diagrams gives \( g(g'x) = (gg')x \), while the second gives \( 1_Gx = x \). Hence the action so defined is just a group action in the usual sense and indeed the two sorts of action are entirely equivalent.

A similar construction can be made for group objects in other categories. For instance in \( \text{Top} \), we get continuous actions of topological groups on spaces. The definition of an action did not make use of the group inverses, so we can talk

\(^4\text{Note that we are using } 1_G \text{ to denote the identity element in the group } G \text{ and } \text{id}_G \text{ to denote the identity homomorphism } G \to G. \text{ Similar notation is applied throughout.}\)
about more general monoid actions using the same definition. One case worth
mentioning is in the category $\text{Ab}$, where a monoid $\langle M, m, e \rangle$ relative to the tensor
product, $\otimes$, and the “terminal object” $\mathbb{Z}$ gives us a ring$^5$. In this case the action
of $M$ on an abelian group $A$ makes $A$ into a left $M$-module.

Right actions of groups (and monoids) can be defined similarly with a map
$X \times G \to X$. These are not intrinsically different from left actions, but the
notation of right actions (for example, $x^g$) is sometimes more natural.

2. Other Structures

In the same way that we have defined groups, we could define other structures
such as rings and lattices within a category by giving suitable morphisms and
commutative diagrams. This can, in principle, be done for any algebraic structure
that is defined in terms of operations satisfying specified properties expressed by
equations.

Also, we can dualize the construction of group objects by reversing all the
arrows in the given diagrams and replacing products by coproducts and terminal
objects by initial objects. This gives us an operation $w : C \to C \amalg C$ such that
the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{w} & C \amalg C \\
\downarrow{w} & & \downarrow{w \amalg \text{id}_C} \\
C \amalg C & \xrightarrow{\text{id}_C \amalg w} & C \amalg C \amalg C \\
\end{array}
$$

commutes (giving “co-associativity”). Similar analogues exist to the other dia-
grams given earlier, noting that we now have $\eta : C \to I$ (where $I$ is the initial
object of $C$). In place of the diagonal morphism in the inverses diagram, we use
a morphism $\nabla : C \amalg C \to C$ which maps both copies of an element in the co-
product to the same element in $C$. This definition gives us a cogroup object in $C$
(see Rotman [16, chapter 11] for more details). In a similar way we could dualize
any of the other algebraic structures within a category, to get comonoids, corings,
colattices, etc.

These dual constructions do not seem to correspond to any standard algebraic
structures with which I am familiar (although presumably a cogroup in $\text{Set}^{\text{op}}$

$^5$Note that this is not the monoid object defined above, since $\otimes$ is not the product (defined as a
limit) in $\text{Ab}$, nor is $\mathbb{Z}$ a terminal object. However $\langle \text{Ab}, \otimes, \mathbb{Z} \rangle$ forms a monoidal category, so the
given diagrams work with $\times$, $1$ replaced by $\otimes$, $\mathbb{Z}$ respectively. See MacLane [10, introduction, ch. 7]
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would correspond to a group in \textbf{Set}, etc.). They do find uses in modern mathematics, however. A quick search of the MathSciNet database\textsuperscript{6} reveals a number of papers on cogroups and corings (although only one mentioning colattices). Putting together coalgebra and algebra structures, along with some further conditions yields Hopf Algebras, which in turn can be extended to give quantum groups [11].

3. \textsc{Internal Categories}

Just as groups and other algebraic structures can be defined in a category by giving suitable objects and morphisms, we can sometimes build categories within a category. Let \( \mathcal{C} \) be a finitely complete category (in fact, it is sufficient for \( \mathcal{C} \) to have pullbacks). Suppose there are objects \( A, O \in \text{Ob}(\mathcal{C}) \) and morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{s} & O \\
\text{ } & \equiv & \text{ } \\
\text{ } & \equiv & \text{ } \\
\downarrow{t} & & \downarrow{e}
\end{array}
\]

such that \( se = id_O = te \). We can consider \( A \) as a collection of directed edges, \( O \) as a collection of vertices and the maps \( s, t, e \) as giving respectively the source and target vertices of each edge and a loop at each vertex. Thus \( \langle A, O, s, t, e \rangle \) specifies an internal reflexive directed graph (or “digraph”) in \( \mathcal{C} \).

Recall that every (small) category has an underlying digraph and conversely a reflexive digraph can be turned into a category by considering vertices as objects and edges as morphisms and specifying a composition (this may necessitate adding further edges). The loops given by the reflexive property become identity morphisms on each object. Similarly we now seek to extend our internal digraph in \( \mathcal{C} \) to an internal category by defining a suitable composition.

We can form the pullback square:

\[
\begin{array}{ccc}
A \times_s A & \xrightarrow{p_2} & A \\
\downarrow{p_1} & & \downarrow{s} \\
A & \xrightarrow{t} & O
\end{array}
\]

The pullback object \( A \times_s A \) can be considered as the collection of all composable pairs of morphisms. In a category such as \textbf{Set} or \textbf{Grp} where the objects have elements and the product is an ordered set of elements (possibly with some algebraic structure imposed) we get \( A \times_s A = \{(f, g) \in A \times A : tf = sg\} \). In order to form a category (with objects \( O \) and morphisms \( A \)) inside \( \mathcal{C} \), we need to define a

\textsuperscript{6}http://klymene.mpim-bonn.mpg.de/mathscinet/
composition \( m : A_t \times_s A \to A \) which is associative and respects identities; note in particular that \( m \) is also a morphism in \( C \).

I shall formulate internal categories for a category \( C \) in which the objects contain elements we can work with, although in principle they could be formulated for more general categories. In particular, we can describe pullbacks in terms of their elements, as above. Define the morphism \( m \) as in the previous paragraph so that \( t(m(f, g)) := tg, \, s(m(f, g)) := sf \). It will be convenient to write \( g \circ f \) for \( m(f, g) \). Now, using the maps \( tm : A_t \times_s A \to O \) and \( s : A \to O \), we can form the pullback object \((A_t \times_s A)_t \times_s A\), which in terms of elements is the set \( \{(f, g, h) \in A \times A \times A : tf = sg, \, t(g \circ f) = sh\} \). Similarly we can form the pullback object \( A_t \times_s (A_t \times_s A) = \{(f, g, h) \in A \times A \times A : tf = s(h \circ g), \, tg = sh\} \). Since \( s(h \circ g) = sg \) and \( t(g \circ f) = tg \) we deduce that these two pullback objects are in fact equal. Hence we can form the diagram

\[
\begin{array}{ccc}
A_t \times_s A & \xrightarrow{m \times id_A} & A_t \times_s A \\
\downarrow{id_A \times m} & & \downarrow{m} \\
A_t \times_s A & \xrightarrow{m} & A
\end{array}
\]

To get associativity of composition we now just require that this diagram commute.

Our final requirement for a category is that the composition respect identities. The morphism \( e : O \to A \) selects the identity arrow \( id_x \) for each object \( x \) in \( O \). If \( f \in A \) with \( sf = x, \, tf = y \) then we need \( id_y \circ f = f = f \circ id_x \). Now, using the composite map \( te = id_O \) we can form the pullback \( O_{idO} \times_s A = \{(x, f) : x = sf\} \). Similarly with \( se = id_O \) we get \( A_t \times_{idO} O = \{(f, y) : tf = y\} \). These pullbacks have obvious projections onto \( A \), namely \( p : O_{idO} \times_s A \to A \) with \( p(x, f) = f \) and \( q : A_t \times_{idO} O \to A \) with \( q(f, y) = f \). These are clearly bijective, since we can define inverses \( p^{-1}(f) := (sf, f), \, q^{-1}(f) := (f, tf) \). Then \( pp^{-1}(f) = f = id_A(f) \) and \( p^{-1}p(x, f) = (sf, f) = (x, f) = id_O \times_{idO} A \), so \( p \) is an isomorphism; similarly for \( q \). Using these maps we express our requirement for identities in the commutativity of the following diagram:

\[
\begin{array}{ccc}
O_{idO} \times_s A & \xrightarrow{e \times id_A} & A \times A & \xleftarrow{id_A \times e} & A_t \times_{idO} A \\
\downarrow{p} & \cong & \downarrow{m} & \cong & \downarrow{q} \\
A & & A & & A
\end{array}
\]

Thus an internal category in \( C \) is defined to be a sextuple \( C = \langle A, O, s, t, e, m \rangle \) where \( A, O \) are objects (giving respectively the morphisms and the objects of the internal category) and \( s, t, e, m \) are morphisms satisfying the given diagrams.
These represent the category axioms, hence the internal categories in \textbf{Set} are just ordinary small categories. In \textbf{Grp}, an internal category is a small category in which both the objects and the morphisms form groups and all the structure maps are homomorphisms. Suppose \( A \) has multiplication \( \mu_A \) and \( O \) has multiplication \( \mu_O \). Then for \( s \) to be a homomorphism means that the following diagram commutes:

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\mu_A} & A \\
\downarrow{s \times s} & & \downarrow{s} \\
O \times O & \xrightarrow{\mu_O} & O
\end{array}
\]

and similarly for the other morphisms. We can define a multiplication on \( C \) as \( \mu : C \times C \to C \) with \( \mu = \mu_O \) on objects and \( \mu = \mu_A \) on arrows. Then it is straightforward to check that \( \mu \) is a functor on \( C \). Similarly, since both \( O \) and \( A \) are groups, these have multiplicative identities and inverses and so we can build functors \( \varepsilon : 0 \to C \) (where \( 0 \) is the terminal object in \textbf{Cat}, i.e. the one-object discrete category) and \( \iota : C \to C \) which pick out respectively an identity object and arrow and inverses for multiplication. But \( C \) is a small category, so it is an object of \textbf{Cat} and the functors are all morphisms of \textbf{Cat}, hence from our internal category we have constructed a group object\(^7\) in \textbf{Cat}. Conversely, given a group object, \( G \), in \textbf{Cat} we have sufficient data to reconstruct \( G \) as an internal category in \textbf{Grp} (essentially the above process in reverse). Hence internal categories in \textbf{Grp} are equivalent to group objects in \textbf{Cat}.

\textbf{Interlude.} Suppose \( A, O \) are groups with homomorphisms \( A \xrightarrow{s} O \) such that \( se = id_O \). Then \( s \) is an epimorphism since if \( a, b : O \to B \) with \( as = bs \), then \( ase = bse \Rightarrow a = b \). We call \( s \) a split epimorphism and \( e \) a splitting of \( s \) (note that \( e \) is itself a monomorphism, by much the same proof). This definition does not use any specific properties of groups or homomorphisms, so it is valid in any category.

Given two groups \( C, G \) with a left \( G \)-action on \( C \), we can form a group \( C \rtimes G = \{(c, g) : c \in C, g \in G\} \) with multiplication \((c, g) \cdot (c', g') := (cc', gg')\). This is like the direct product of \( C \) and \( G \), except that the component of multiplication in \( C \) is “twisted” by the \( G \)-action. The group \( C \rtimes G \) is called the \textit{semidirect product} of \( C \) by \( G \).

Now let us return to the split epimorphism \( s : A \to O \) and its splitting \( e \). For each \( a \in A \), we can write \( a = ke(x) \), where \( k = a(es(a))^{-1} \in \text{Ker} \, s \) and \( x =
Suppose \( a' = k'e(x') \). Then \( aa' = ke(x)k'e(x') = ke(x)k'(e(x))^{-1}e(x)e(x') \). We can define an action of \( O \) on \( \text{Ker } s \) by \( \ast k := e(x)k(e(x))^{-1} \) (since \( e \) is a homomorphism, we can of course rewrite \( (e(x))^{-1} \) as \( e(x^{-1}) \)). We can then write \( aa' = k^2k'e(xx') \). Note that unless brackets indicate otherwise, \( x \) acts only on the symbol to its immediate right (and similarly for morphisms). There is a map \( \phi : A \to \text{Ker } s \rtimes O, \phi(ke(x)) = (k, x) \). Now \( \phi(aa') = \phi(k^2k'e(xx')) = (k^2k', xx') = (k, x)(k', x') = \phi(a)\phi(a') \), so \( \phi \) is a homomorphism. Also, there is an obvious inverse \( \phi^{-1} : \text{Ker } s \rtimes O \to A, \phi^{-1}(k, x) := ke(x) \), which is also a homomorphism. Hence \( \phi \) is an isomorphism and we have established that \( A \cong \text{Ker } s \rtimes O \).

Let \( \partial : C \to G \) be a morphism of groups. Then \( (C, G, \partial) \) is a crossed module (of groups) if there is a (left) \( G \)-action on \( C \) such that the following two properties hold for all \( c, d \in C, g \in G \):

\[
\begin{align*}
(1) \quad & \partial gc = g\partial cg^{-1} \\
(2) \quad & \partial cd = cdc^{-1}
\end{align*}
\]

The first property is known as equivariance of \( \partial \) with respect to the action (see [6, p. 220]) and the second is called the Peiffer identity (see [13, p. 250]). For example, if \( N \triangleleft G \) then the inclusion \( N \hookrightarrow G \) gives a crossed module with the trivial action. Further generic examples of crossed modules may be found in [4]. An immediate consequence of the crossed module axioms is that \( \text{Ker } \partial \) is abelian, while the image is a normal subgroup of \( G \).

**Crossed Modules and Internal Categories.** We shall show that crossed modules of groups are equivalent to internal categories in \( \text{Grp} \). In other words, given any crossed module, we can construct an internal category and vice versa.

Let \( (C, G, \partial) \) be a crossed module. Since \( G \) acts on \( C \), we can form the semidirect product \( C \rtimes G \) as defined above and define maps \( s, t : C \rtimes G \to G \) and \( e : G \to C \rtimes G \) by \( s(c, g) := g, t(c, g) := \partial cg \) and \( e(g) := (1_C, g) \). Then \( s \) is clearly a homomorphism. Also,

\[
\begin{align*}
t((e, g) \cdot (d, h)) &= t((e^q d, gh)) = \partial (e^q d) gh = \partial cg \partial d g^{-1} gh \\
&= \partial cg \partial d h = t(c, g) t(d, h)
\end{align*}
\]

and \( e(gh) = (1_C, gh) = (1_C, g) \cdot (1_C, h) = e(g) \cdot e(h) \) so \( t \) and \( e \) are also homomorphisms (for \( e \) we implicitly used the fact that the action of \( G \) on \( C \) determines a map \( G \to \text{Aut}(C) \) and hence \( s \circ t = 1_C \), \( \forall g \in G \)). Furthermore, for

\[\text{Note that neither of these names for the crossed module axioms is consistently applied in the literature. Most authors seem content just to number the axioms.}\]
all $g \in G$, $se(g) = s(1_C, g) = g = id_G(g)$ and $te(g) = t(1_C, g) = \partial(1_C)g = g$
so $se = te = id_G$, i.e. $s$ and $t$ are split epimorphisms with common splitting $e$.

We have thus constructed an internal (reflexive) digraph $C \rtimes G \xymatrix{\ar[r] & \ar[l] & \ar[r] & \ar[l]} G$. The elements can be pictured as follows:

$g \xymatrix{\ar@(ul,ur)[]^{(c,g)} \ar[r] & \partial cg}$

Vertices correspond to elements of $G$ and edges to elements of $C \rtimes G$. The source and target vertices of a given edge $(c, g)$ are given by $s(c, g)$ and $t(c, g)$ respectively and $e(g)$ gives a loop on vertex $g$. There is an obvious composition of edges:

$g \xymatrix{\ar@(ul,ur)[]^{(c,g)} \ar[r] & \partial cg \ar[r]^{(c',\partial cg)} & \partial c'\partial cg}$

Thus we define $(c', \partial cg) \circ (c, g) := (c'c, g)$ (note that $s(c', \partial cg) = \partial cg = t(c, g)$).

Since $\partial$ is a homomorphism, we have $\partial(c'c) = \partial c'\partial c$ as required.

To get an internal category we now just require that $\circ$ be a homomorphism. In other words, we need

(1) $((c', \partial cg) \cdot (d', \partial dh)) \circ ((c, g) \cdot (d, h)) = ((c', \partial cg) \circ (c, g)) \cdot ((d', \partial dh) \circ (d, h))$

This is the familiar interchange law (see page 4). Evaluating the two sides separately, we get:

$$
\text{LHS} = (c' \partial cg \partial dh) \circ (c' \partial d, gh) = (c' \partial c \partial d, gh) = (c' \partial c \partial d, gh)
$$

and

$$
\text{RHS} = (c'c, g) \cdot (d' d, h) = (c'c, d' d, gh)
$$

whence equality. Therefore $\circ$ is indeed a homomorphism and so we have constructed an internal category in $\textbf{Grp}$.

Conversely, suppose that we have an internal category $\langle A \xymatrix{\ar[r] & \ar[l] & \ar[r] & \ar[l]} O, \circ \rangle$. We have seen that $A \cong \text{Ker } s \rtimes O$ with $O$ acting on $\text{Ker } s$ by $\cdot k := e(x)ke(x^{-1})$ for $x \in O, k \in \text{Ker } s$. Objects of the category are the elements of $O$ while morphisms are of the form $(k, x)$ with $k \in \text{Ker } s, x \in O$. The maps $s, t$ give respectively the source and target objects of each morphism (note that the $x$ in $(k, x)$ is effectively a label to show the source of the morphism), while $e$ gives the identity arrow for each object.
Define $\partial : \text{Ker } s \to O$ to be the restriction of $t$ to $\text{Ker } s$, i.e. $\partial = t|_{\text{Ker } s}$. Then $\partial$ is automatically a homomorphism, since $t$ is one. For any $k \in \text{Ker } s$, $x \in O$, we have $\partial(t(k)) = t(e(x)ke(x^{-1}))$ by definition. But $t$ is a homomorphism so this is the same as $te(x)tke(x^{-1}) = xt(k)x^{-1}$ (since $te = id_O = x\partial k x^{-1}$). Hence $\partial$ is equivariant with respect to the action. It remains to verify the Peiffer identity. We know that composition is a morphism and hence with the multiplication in $\text{Ker } s \ltimes O$ it satisfies the interchange law, i.e. we have:

$$((k', \partial(k)x) \cdot (l', \partial(l)y)) \circ ((k, x) \cdot (l, y)) = ((k', \partial(k)x) \circ (k, x)) \cdot ((l', \partial(l)y) \circ (l, y)).$$

Evaluating the two sides of this equation gives:

\begin{align*}
\text{LHS} &= (k'\partial(k)x'l, \partial(k)x'l) \circ (k^2l, xy) \\
&= (k'\partial(k)x'l, xy)
\end{align*}

(this composition is defined since $\partial(k^2l)xy = \partial k \partial(l)xy = \partial kx \partial(l)x^{-1}xy = \partial kx \partial(l)y$ by equivariance) and

\begin{align*}
\text{RHS} &= (k'k, x) \cdot (l'l, y) = (k'k^2(l'l), xy).
\end{align*}

Since the two sides are equal, we know that their first components must be equal. So we have

\begin{align*}
k'\partial(k)x'l &= k'\partial(k)x'l \\
&= k'k^2(l'l) \\
&= k'k^2l'k^{-1}l.
\end{align*}

Cancelling on both sides and writing $m = x'l \in \text{Ker } s$, we get $\partial m = kmk^{-1}$, which is the Peiffer identity as required. Hence $\langle \text{Ker } s, O, \partial \rangle$ is a crossed module with the action arising from the semidirect product $\text{Ker } s \rtimes O$.

We have shown that crossed modules of groups are equivalent to internal categories in $\text{Grp}$. We saw earlier that these are in turn equivalent to group objects in $\text{Cat}$ and hence have arrived at a result proved by Brown and Spencer [3] in the 1970s, namely that crossed modules of groups are equivalent to group objects in $\text{Cat}$ (they did not go via internal categories of groups but went directly between the two using a functor). In fact, they went slightly further and showed that if a category is equipped with a group structure it must in fact be a groupoid and hence the group objects in $\text{Cat}$ are the same as the group objects in $\text{Grpd}$, the category of groupoids. Thus they proved the equivalence of crossed modules of groups and group objects in the category of groupoids.

Since doing this work I have gained access to the new edition of MacLane’s seminal volume [10], which now includes some material on internal categories as
well as on group objects and other internal algebraic structures. The reader may also find it useful to consult the following sources, which treat various aspects of this area [1, 2, 5, 7–9, 12, 14, 15].

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