RIEMANN ZETA FUNCTION

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Riemann Zeta function is one of the most studied transcendental functions, having in view its many applications in number theory, algebra, complex analysis, statistics, as well as in physics.

Another reason why this function has drawn so much attention is the celebrated Riemann conjecture regarding its non trivial zeros, which resisted proof or disproof until now.

There has been lately a lot of progress in the sense of proving its truthfulness.

It is known (see [1], page 215) that the Riemann function \( \zeta(s) \) is a meromorphic function in the complex plane having a single simple pole at \( s = 1 \) with the residue 1.

Since it is a transcendental function, \( s = \infty \) must be an essential isolated singularity.

We have proved that any meromorphic function in \( C \) defines a branched covering Riemann surface over its range which has infinitely many fundamental domains accumulating at infinity and only there, if \( \infty \) is an essential singularity of the function.

The idea of a proof of Riemann hypothesis is the following.

We divide first the complex plane into three regions \( R_k \), \( k = 1, 2, 3 \), where \( R_1 \) is the region above the pre-image by \( \zeta \) of the real axis passing through the non trivial zero \( s_1 = \sigma_1 + it_1 \) with the lowest positive value of \( t \).

The region \( R_2 \) is symmetric to \( R_1 \) with respect to the real axis and \( R_3 = C \setminus (R_1 \cup R_2) \).

The regions of interest are \( R_1 \) and \( R_2 \).

Next, we show that it is impossible to have non trivial zeros of \( \zeta \) with the same \( t \).

The pre-images of the real axis allow us to build fundamental domains in \( R_1 \) and \( R_2 \) describing the global mapping properties of \( \zeta \) in these regions and in particular showing that for an arbitrary non trivial zero \( \sigma_0 + it_0 \), we must have \( \sigma_0 = 1 - \sigma_0 \), i.e. \( \sigma_0 = 1/2 \), which is exactly the Riemann hypothesis.
The proof we are presenting does not make use of any computer imaging techniques, yet it uses some elementary geometric aspects of conformal mappings, which can be found in any classical book of complex analysis. We will only make reference to [1].

The representation formula

\[ \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{(e^z - 1)} dz \]  

where \( C \) is an infinite curve turning around the origin, which does not enclose any multiple of \( 2\pi i \), allows one to see that \( \zeta(-2m) = 0 \) for every positive integer \( m \) and there are no other zeros of \( \zeta \) on the real axis.

The Riemann conjecture is that all the other zeros of \( \zeta \) (the so-called non-trivial zeros) are on the critical line

\[ s = \frac{1}{2} + it, \quad t \in \mathbb{R}. \]

From the Laurent expansion of \( \zeta(s) \) for \( |s - 1| > 0 \)

\[ \zeta(s) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s - 1)^n, \]

where \( \gamma_n \) are the Stieltjes constants:

\[ \gamma_n = \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \left( \frac{\log k}{k} \right)^n - \frac{(\log m)^{n+1}}{(m+1)} \right] \]

one can see that:

i). \( \zeta(s) \) is an analytic function in \( C \setminus \{1\} \), having in \( s = 1 \) a simple pole with residue 1.

ii). \( \zeta(\overline{s}) = \overline{\zeta(s)} \).

We will use the functional equation (see [1], page 216)

\[ \zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s) \]

to study the non-trivial zeros of \( \zeta \).
Let us denote by \( s_0 = \sigma_0 + it_0 \) a complex number such that \( 1 - s_0 \) is a non trivial zero of \( \zeta \).

Since the poles of \( \Gamma \) are all negative integers, and the other factors in the right hand side of (4) do not have poles, the left hand side of (4) cancels for \( s = s_0 \).

Reciprocally, if \( \zeta(s_0) = 0 \), then the right hand side of (4) cancels for \( s = s_0 \),

which can happen only if \( \sin \frac{\pi s_0}{2} = 0 \), or \( \zeta(1 - s_0) = 0 \).

In the first case \( s_0 \) is a trivial zero and we can ignore it.

The conclusion is that \( s_0 \) is a non trivial zero of \( \zeta \) if and only if \( 1 - s_0 \) is a non trivial zero of \( \zeta \).

On the other hand, due to ii), \( s_0 \) is a non trivial zero of \( \zeta \) if and only if \( \overline{s_0} \) is a nontrivial zero of \( \zeta \).

Obviously, the points \( s_0 \) and \( 1 - \overline{s_0} \) are in the same half-plane and if they are in the upper half-plane, then \( \overline{s_0} \) and \( 1 - s_0 \) are in the lower half-plane and vice versa.

There are two alternatives: \( s_0 = 1 - \overline{s_0} \), or \( s_0 \neq 1 - \overline{s_0} \).

In the first case \( \sigma_0 + it_0 = 1 - \sigma_0 + it_0 \)

which implies \( \sigma_0 = 1/2 \), hence \( s_0 = \sigma_0 + it_0 \) is on the critical line.

In the second case we would have two distinct zeros with the same imaginary part.

Let us show that this is impossible.

The idea is to show that this assumption contradicts some global mapping properties of the function \( \zeta \).

Suppose that \( \zeta(\sigma_0 + it_0) = \zeta(1 - \sigma_0 + it_0) = 0 \) for some \( s_0 = \sigma_0 + it_0 \).

Obviously, we can assume that \( 0 < \sigma_0 < 1/2 \) and there is no \( \sigma'_0, \sigma_0 < \sigma'_0 < 1/2 \) with the same property.

We will make use of the pre-image by \( \zeta \) of the real axis from the \( z \)-plane.

Due to the symmetry with respect to the real axis of this pre-image, we only need to
deal with the region $R_1$. Every component of the pre-image is either a component of the pre-image of $(-\infty, 1)$, or a component of the pre-image of $(1, +\infty)$. We will use notations $\Gamma_k$ for the first type of component and $\Gamma_k'$ for the second type.

Every $\Gamma_k$ contains a unique non trivial zero of $\zeta$, while no zero belongs to any one of $\Gamma_k'$.

Let us show first that two consecutive curves $\Gamma_k$ and $\Gamma_{k+1}$ cannot intersect each other.

Indeed, if they intersect each other in two points $a$ and $b$, then since the arcs of $\Gamma_k$ and $\Gamma_{k+1}$ between $a$ and $b$ are both mapped by $\zeta$ on the same segment of the real axis, the domain between them would be mapped on the empty set, which is a contradiction.

Suppose that $\Gamma_k$ and $\Gamma_{k+1}$ intersect each other in a unique point $a$. Then in a neighborhood $V$ of $a$ we would have (see [1], page 133):

$$\zeta(s) = \zeta(a) + (s-a)^k g(s)$$

where $k$ is a positive integer and $g(s)$ is analytic in $V$, with $g(a) \neq 0$.

There are two arcs belonging to $\Gamma_k$ and respectively $\Gamma_{k+1}$, which are mapped by $\zeta$ on the same segment of the real axis, meaning that in fact $\Gamma_k$ and $\Gamma_{k+1}$ have infinitely many common points and we were brought again to a contradiction.

Obviously, the same is true for the curves $\Gamma_j'$. Moreover it is straightforward that no $\Gamma_k$ can intersect any $\Gamma_j'$ given the fact that $\zeta$ is a single valued function.

Lemma 1:
Between two consecutive curves $\Gamma_j'$ and $\Gamma_{j+1}'$ there is a unique curve $\Gamma_k$ such that $\lim_{\sigma \to +\infty} \zeta(\sigma + it) = 1$, where $\sigma + it \in \Gamma_k$.

Proof:
Let us denote by $\Omega_j$ the domain bounded by $\Gamma_j'$ and $\Gamma_{j+1}'$. The domain $\Omega_j$ is mapped by $\zeta$ onto the complex plane with a slit alongside the real axis from 1 to $+\infty$.

The mapping is not necessarily bijective.

If $z_0 \in (1, +\infty)$, then there is $s_j \in \Gamma_j'$, and $s_{j+1} \in \Gamma_{j+1}'$ such that $\zeta(s_j) = \zeta(s_{j+1}) = z_0$. 

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Let us connect $s_j$ and $s_{j+1}$ by an Jordan arc $\gamma$. Then $\zeta(\gamma)$ is a closed curve $C_\gamma$ in the $z$-plane, which can be the boundary of a domain $D$ or a path travelled twice in opposite directions, in which case $D = \emptyset$.

We need to show that $C_\gamma$ intersects again the real axis. Suppose that the contrary happens. Then $C_\gamma$ is contained in either the upper or in the lower half plane. It can be easily shown that in this case $\zeta$ maps conformally the half strip $S$ bounded by $\gamma$ and the branches of $\Gamma'_j$ and $\Gamma'_{j+1}$ corresponding to $s^- > +\infty$ onto $C \setminus D$ with a slit alongside the real axis from $z_0$ to $+\infty$.

This is impossible, since there is no zero of $\zeta$ in $S$.

Let us show that there cannot be more than one component $\Gamma_k$ between $\Gamma'_j$ and $\Gamma'_{j+1}$.

Indeed, suppose that there are more than one. Then, we can choose the strip between two of them and study its image by $\zeta$.

Taking this time $s_k \in \Gamma_k$ and $s_{k+1} \in \Gamma_{k+1}$ such that $\zeta(s_k) = \zeta(s_{k+1}) > 0$ and denoting by $\gamma$ a Jordan arc between them, we can repeat the previous argument for $\Gamma_k$ and $\Gamma_{k+1}$ and arrive at a contradiction.

These arguments do not exclude the possibility of having several components of the pre-image of $(-\infty, 1)$ in the strip $\Omega_j$ between two consecutive $\Gamma'_j$, such that all except one turn back after reaching a non trivial zero of $\zeta$.

We will call $\Omega_j$ an $m$-type strip if $m - 1$ such components exist. We have effectively found 1, 2, 3 and 4-type strips and there is no apparent reason to deny the existence of $m$-type strips, for an arbitrary $m$.

We denote by $\Gamma_{j,0}$ the component of the pre-image of $(-\infty, 1)$ situated in $\Omega_j$ and such that $\sigma + it \in \Gamma_{j,0}$ implies that $\lim_{\sigma \to \pm \infty} \zeta(\sigma + it) = 1$ and by $\Gamma_{j,k}$, $k \geq 1$ the other components, if they exist.

Lemma 2: The curves $\Gamma'_j$ and $\Gamma'_{j+1}$ are asymptotes to each other at both ends, i.e. the angles between their tangents at the points corresponding to the same $z$ tend to zero as
\[ z \rightarrow 1 \quad \text{and as} \quad z \rightarrow +\infty. \]

Proof:
It is known that if \( s \in \mathbb{R} \), then \( \zeta(s) \in \mathbb{R} \). As \( \Gamma'_j \) and the real axis are mapped by \( \zeta \) onto the real axis in the \( z \)-plane, their angle at the intersection point, which is \( \infty \), should be equal to the angle of their images, i.e. 0. This happens for both, \( \Gamma'_j \) and \( \Gamma'_{j+1} \), thus the angle between them at infinity is zero.

We notice that the same is true for the components \( \Gamma_{j,k} \) situated in \( \Omega_j \) of the pre-image of \( (-\infty, 1) \), as well as for the components situated in \( \Omega_j \) of the pre-image \( C(1) \) of unit circle.

The curve \( C(1) \) intersects every \( \Gamma_{j,k} \) in two points: one corresponding to \( z = -1 \) and the other denoted \( s_{j,k} \) such that \( \lim_{s \rightarrow s_{j,k}} \zeta(s) = 1 \) as \( s - s_{j,k}, s \in C(1) \).

Lemma 3:
Let \( \Gamma_{j,k} \) and \( \Gamma_{j,k'} = k \geq 0 \) be consecutive components of the pre-image of \( (-\infty, 1) \) situated in \( \Omega_j \). Then there are unbounded curves \( L_{j,k} \) and \( L_{j,k'} \) included in \( \Omega_j \) starting at \( s_{j,k} \), respectively at \( s_{j,k'} \) such that \( \zeta \) maps conformally the strip \( S_{k,k'} \) bordered by \( \Gamma_{j,k}, \Gamma_{j,k'}, L_{j,k} \) and \( L_{j,k'} \) onto the complex plane with a slit. If \( k = 0 \), or \( k' = 0 \), then there is no \( s_{j,0} \), and \( S_{0,k'} \) respectively \( S_{0,k} \) is bordered by \( \Gamma_{j,0}, L_{j,k'} \) and \( \Gamma_{j,k}, \) respectively \( \Gamma_{j,0}, L_{j,k} \) and \( \Gamma_{j,k'} \).

Proof:
Let \( r > 0 \) be small enough such that the components \( \gamma_1(r) \) and \( \gamma_2(r) \) of the pre-image by \( \zeta \) of the circle \( |z| = r \) are disjoint.
As \( r \rightarrow 1 \), the two curves expand. There are two possibilities:

a). \( \gamma_1(r) \) and \( \gamma_2(r) \) touch each other at a point \( s_0 \) as \( r = r_0 < 1 \).

b). For any \( r < 1 \), \( \gamma_1(r) \) and \( \gamma_2(r) \) are disjoint.

In the first case, let \( \gamma \) be a Jordan arc included in \( \Omega_j \) connecting \( s_0 \) and \( s_{j,k} \) and having in common with \( \gamma_1(r) \) and \( \gamma_2(r) \) only the point \( s_0 \).

The continuation over \( \zeta(\gamma) \) from \( s_0 \) in the opposite direction of \( \gamma \) produces an arc \( \gamma' \).
Let us denote by $L_{j,k}$ the union of $\gamma$ and $\gamma'$.
Since $\zeta(\gamma) = \zeta(\gamma')$, we have that $z = \zeta(s)$ travels on $\zeta(L_{j,k})$ from $z_0 = \zeta(s_0)$ to $z = 1$ twice in opposite directions when $s$ travels on $L_{j,k}$ from $s_{j,k}$ to $\infty$.

In the second case $\gamma_1(1)$ and $\gamma_2(1)$ meet each other in $s_{j,k}$ and the domain between them is the empty set, hence they have two infinite branches which coincide and this is $L_{j,k}$.

If we repeat this construction for $s'_{j,k}$ we obtain the curve $L'_{j,k}$ with similar properties.

The conformal mapping we were talking about is guaranteed by the boundary correspondence theorem.

**Lemma 4:**
It is impossible to have two distinct non trivial zeros of the form $s_1 = \sigma_0 + it_0$ and $s_2 = 1 - \sigma_0 + it_0$.

**Proof:**
Suppose that $\zeta(s_1) = \zeta(s_2) = 0$, where $s_1 = \sigma_0 + it_0$, $s_2 = 1 - \sigma_0 + it_0$, and $0 < \sigma_0 < 1/2$.

Obviously, we can assume that there is no $\sigma'_0$ such that $0 < \sigma_0 < \sigma'_0 < 1/2$.

There are two scenarios one can imagine:

a). $s_1 \in \Omega_j$, $s_2 \in \Omega_{j+1}$

b). $s_1$, $s_2 \in \Omega_j$.

In the first case $s_1$ and $s_2$ are separated by $\Gamma'_{j+1}$ and by Lemma 3 there are strips $S_k$ containing $s_k$, $k = 1, 2$ which are separated by $\Gamma'_{j+1}$ and are mapped conformally by $\zeta$ onto the complex plane with a slit. In the second case the strips $S_k$ are separated by a line $L_{j,k}$.

In both cases, the following computation is true:

\[ \frac{\partial \zeta'(\sigma + it_0)}{\partial \sigma} = \zeta'(\sigma + it_0) \]

and

\[ \frac{\partial \zeta'(1 - \sigma + it_0)}{\partial \sigma} = -\zeta'(1 - \sigma + it_0). \]

In particular, for $\sigma = 1/2$ we get

\[ \zeta'(\frac{1}{2} + it_0) = -\zeta'(\frac{1}{2} + it_0), \]

i.e. $\zeta'(\frac{1}{2} + it_0) = 0$. Thus $\frac{1}{2} + i\sigma_0$ is a branch point of $\zeta$.

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In the first case, it should be on $\Gamma'_{j+1}$, which is impossible, since $\zeta$ maps bijectively
$\Gamma'_{j+1}$ onto $(1, +\infty)$.

In the second case, let us define $\chi : S_{j,k} \to S_{j,k+1}$
by $\chi(s) = \zeta^{-1}_{|S_{j,k+1}} \circ \zeta(s)$ for every $s \in S_{j,k}$.
The function $\chi$ maps conformally the domain $S_{j,k}$ onto $S_{j,k+1}$.

Extended to the boundaries, $\zeta$ maps the common part of the boundary
of $S_{j,k}$ and $S_{j,k+1}$ onto itself.
Thus, $\frac{1}{2} + it_0$ and $s_0$ are both branch points for $\chi$, which is impossible,
and $s_0 = \frac{1}{2} + it_0$.

The function $\chi$ can be extended to $S_{j,k+1}$ by the formula $\chi(s) = \zeta^{-1}_{|S_{j,k}} \circ \zeta(s)$,
for every $s \in S_{j,k+1}$.
It is an analytic involution having the fixed point $\frac{1}{2} + it_0$.
We have also that $\zeta \circ \chi(s) = \zeta(s)$ for every $s \in S_{j,k} \cup S_{j,k+1}$.

The function $\zeta \circ \chi \circ \zeta^{-1}_{|S_{j,k}}(z)$ is the identity in the $z$-plane, hence its derivative,
which can be computed at every point $z \neq z_0 = \zeta(s_0)$ has a removable
singularity at $z_0$ and is identically equal to 1.
Thus, we cannot have $\chi'(s_0) = 0$, which is a contradiction.
This contradiction is due to the false assumption that $\sigma_0 + it_0$ and $1-\sigma_0 + it_0$
are both non trivial zeros of $\zeta$.

This proves completely the Riemann hypothesis.

Reference:
[1] Ahlfors, L.V., Complex Analysis, International Series in Pure and
Applied Mathematics, 1979.