ON DEL PEZZO ELLIPTIC VARIETIES OF DEGREE \( \leq 4 \)

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Abstract. Let \( Y \) be a del Pezzo variety of degree \( d \leq 4 \) and dimension \( n \geq 3 \), let \( H \) be an ample class such that \(-K_Y = (n-1)H\) and let \( Z \subset Y \) be a 0-dimensional subscheme of length \( d \) such that the subsystem of elements of \(|H|\) with base locus \( Z \) gives a rational morphism \( \pi_Z: Y \dashrightarrow \mathbb{P}^{n-1} \). Denote by \( \pi: X \to \mathbb{P}^{n-1} \) the elliptic fibration obtained by resolving the indeterminacy locus of \( \pi_Z \). Extending the results of [5] we study the geometry of the variety \( X \) and we prove that the Mordell-Weil group of \( \pi \) is finite if and only if the Cox ring of \( X \) is finitely generated.

Introduction

Let \( Y \) be a del Pezzo variety of dimension \( n \geq 3 \) and \( H \) an ample class such that \(-K_Y = (n-1)H\) and let \( d := H^n \) be the degree of \( Y \). We consider the rational map \( \pi_Z: Y \dashrightarrow \mathbb{P}^{n-1} \) associated to a linear series \( V \subset |H| \) of dimension \( n-1 \), having 0-dimensional base locus \( Z \). In what follows we say that the map \( \pi: X \to \mathbb{P}^{n-1} \), obtained by resolving the indeterminacy of \( \pi_Z \), is a del Pezzo elliptic fibration while \( X \) is a del Pezzo elliptic variety of degree \( d \).

In [3] the case of general \( V \) is considered in relation with the Morrison-Kawamata cone conjecture. In [5] the case \( \text{deg}(Y) = 3 \) has been studied, providing the Mordell-Weil groups of all the types of fibrations that can be obtained and proving that the group is finite if and only if the Cox ring of \( X \) is finitely generated.

In this paper we extend the results of [5] to del Pezzo elliptic varieties of degree \( \leq 4 \). Our first result is about the Mordell-Weil groups of the corresponding del Pezzo elliptic fibrations (the notation will be explained in Section 2).

Theorem 1. The Mordell-Weil groups of the del Pezzo elliptic fibrations of degree \( d \leq 4 \) and dimension \( n \geq 3 \) are the following:

| Degree | Type | MW(\( \pi \)) |
|--------|------|--------------|
| 1 \( X_1 \) | \( (0) \) | \( X_3, X_S \) |
| 2 \( X_{11} \) | \( \mathbb{Z} \) | \( X_4, X_3, X_40 \) |
| \( X_{SS} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( X_{41}, X_{30} \) |
| \( X_2 \) | \( (0) \) | \( X_{42} \) |
| 3 \( X_{111} \) | \( \mathbb{Z}^2 \) | \( X_{31}, X_{20}, X_{21} \) |
| \( X_{S11}, X_{12} \) | \( \mathbb{Z} \) | \( X_{43} \) |
| \( X_{SSS} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( X_{21}, X_{22}, (\mathbb{Z}/2\mathbb{Z})^2 \) |
| \( X_{S2} \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( X_{10}, X_{11} \) |

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Table 1: Mordell-Weil groups of del Pezzo elliptic fibrations

Our second result is about Cox rings of elliptic del Pezzo varieties.

Theorem 2. Let $X$ be a del Pezzo elliptic variety of degree $\leq 4$ and dimension $n \geq 3$. Then the following are equivalent:

1. the Cox ring of $X$ is finitely generated;
2. the Mordell-Weil group of $\pi: X \to \mathbb{P}^{n-1}$ is finite.

We prove Theorem 2 showing that any del Pezzo elliptic variety, whose corresponding elliptic fibration has finite Mordell-Weil group, is a Mori Dream Space and viceversa. Then we conclude by means of [6, Proposition 2.9]. The proof of our second theorem makes use of a detailed study of the structure of the moving and effective cones of elliptic del Pezzo varieties. In particular we prove the following.

Theorem 3. Let $\pi: X \to \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 4$ and having finite Mordell-Weil group. Then the effective cone $\text{Eff}(X)$ is generated by the vertical classes and the classes of sections, the cone $\text{Mov}(X)$ is the dual of $\text{Eff}(X)$ with respect to the bilinear form introduced in (1.2). The intersection graphs for the effective cones are given in the following table, where each vertex corresponds to a section or a vertical class $D$, the label in the vertex is $-\langle D, D \rangle$ and the number of edges connecting two vertices $D$ and $D'$ is $\langle D, D' \rangle$.

| deg = 1 | $X_1$ |
|---------|-------|
|         |   0   |
|         |   1   |

| deg = 2 | $X_{SS}$ | $X_2$ |
|---------|----------|-------|
|         |   1     |   6   |
|         |   1     |   1   |

| deg = 3 | $X_{SSS}$ | $X_{S2}$ | $X_3$ | $X_S$ |
|---------|-----------|---------|------|------|
|         |   1     |   2    |   2  |   2  |
|         |   1     |   1    |   1  |   1  |

| deg = 4 | $X_{13}$ | $X_{21}$ | $X_{22}$ | $X_{11}$ | $X_{10}$ |
|---------|----------|---------|----------|---------|---------|
|         |   2     |   2    |   2      |   2    |   2    |
|         |   1     |   1    |   1      |   1    |   1    |

Table 2. Intersection graphs for the effective cones.
The paper is structured as follows. In Section 1, we introduce del Pezzo elliptic fibrations and del Pezzo elliptic varieties and we define the bilinear form on the Picard group of such varieties. In Section 2, we study the geometry of these varieties and in the next section we use these results in order to classify the Mordell-Weil groups of the del Pezzo elliptic fibrations, their vertical classes and sections. Section 4 contains the description of the nef, effective and moving cones of del Pezzo elliptic varieties and moreover in the same section we prove Theorem 2. In the last section, we provide the Cox rings of the del Pezzo elliptic varieties whose fibration has finite Mordell-Weil group and having degree one, two and four (few examples), and a lemma about the Cox ring of the blow-up in one point of the complete intersection of two quadrics.

1. Del Pezzo elliptic varieties

Let $Y$ be a del Pezzo variety of dimension $n \geq 3$ such that $-K_Y = (n-1)H$, with $H$ ample and $d := H^n \leq 4$. It is well known (see for instance [2]) that the Picard group of $Y$ has rank one and it is generated by the class $H$. Let us recall the following. If $d = 1$ then $Y$ is a smooth hypersurface of degree six of the weighted projective space $\mathbb{P}(3,2,1,\ldots,1)$ and $H$ is the restriction of a degree one class of the ambient space. If $d = 2$ then $Y$ is a double cover of $\mathbb{P}^n$ branched along a smooth quartic hypersurface and $H$ is the pull-back of a hyperplane of $\mathbb{P}^n$. If $d \in \{3,4\}$ then $Y$ is a projectively normal subvariety of $\mathbb{P}^{n+d-2}$ and $H$ is the class of a hyperplane section.

Let us consider a $n-1$-dimensional sublinear system of $|H|$, whose base locus $Z$ has dimension zero and length $d$. In particular, if $d = 1$ we have $Z = V(x_3,\ldots,x_{n+2})$, if $d = 2$, $Z$ is preserved by the covering involution and if $d \in \{3,4\}$, $Z$ spans a linear subspace $\Lambda \subseteq \mathbb{P}^{n+d-2}$ of dimension $d-2$. Let us denote by $\pi_Z: Y \dasharrow \mathbb{P}^{n-1}$ the rational map defined by the given system and by $\pi: X \rightarrow \mathbb{P}^{n-1}$ the resolution of the indeterminacy of $\pi_Z$. The variety $X$ comes with two morphisms:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathbb{P}^{n-1} \\
\downarrow{\sigma} & \nearrow{\pi_Z} \\
Y & \end{array}
\]

where $\sigma$ is the composition of $d$ blowing-ups $\sigma_1,\ldots,\sigma_d$ at the points $q_1,\ldots,q_d$, respectively. Moreover, assuming that $\Lambda$ is not contained in the tangent space of $Y$ at any point of $Z$ when $d = 4$, the general fiber of $\pi$ is a smooth genus one curve, that is $\pi$ is an elliptic fibration.

In what follows, by abuse of notation, we use the same letter $H$ to denote the pull-back of $H$ via $\sigma$ while we denote by $E_i$ the pull-back of the exceptional divisor of $\sigma_i$, for $i \in \{1,\ldots,d\}$. Observe that some of the points $q_2,\ldots,q_d$ can lie on the exceptional divisor of one of the $\sigma_i$'s. Therefore $E_i$ can be either a $\mathbb{P}^{n-1}$ or the union of a $\mathbb{P}^{n-1}$ with some other components isomorphic to the projectivization $\mathbb{F}$ of the vector bundle $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. In any case, we can write

\[\text{Pic}(X) = \langle H, E_1, \ldots, E_d \rangle,\]
where, with abuse of notation, we are adopting the same symbols for the divisors and for their classes. We will also adopt the following notation

\[(1.1) \quad F := -\frac{1}{n-1}K_X.\]

Observe that \(F\) is the pull-back of a hyperplane section of \(\mathbb{P}^{n-1}\) via \(\pi\), so that \(F = H - \sum_{i=1}^{d} E_i\).

**Remark 1.1.** The map \(\sigma\) is a composition of blow-ups at points and we claim that we blow up at most one point on each exceptional divisor. Indeed, assume by contradiction that there is a prime divisor \(E\) which is the strict transform of an exceptional divisor blown up at two or more points. The preimage \(S\) of a general line \(\ell\) of \(\mathbb{P}^{n-1}\) via \(\pi\) is a rational elliptic surface with nef anticanonical class which contains a prime divisor \(E|_S\) of self-intersection \(<-2\), a contradiction.

### 1.1. A bilinear form on the Picard group.

Let now \(X\) be the blow-up of \(Y\) at \(r\) general points. Using the above notation for \(F\), we introduce a bilinear form on \(\text{Pic}(X)\) by setting

\[(1.2) \quad \langle A, B \rangle := F^{n-2} \cdot A \cdot B\]

for any two divisors \(A\) and \(B\) on \(X\). Thus the quadratic form \(q\) induced by the above linear form is hyperbolic and the matrix with respect to the basis \((H, E_1, \ldots, E_r)\) is diagonal with entries \(d, -1, \ldots, -1\). Since \(\langle F, F \rangle = d - r\), the sublattice \(F^\perp\) is negative definite if \(1 < r < d\) and it is negative semidefinite if \(r = d\). In the first case, a basis consists of the classes \(E_1 - E_2, \ldots, E_{r-1} - E_r\), while in the second case it consists of the above classes plus \(F\). These are roots lattices of type \(A_{r-1}\) and \(\tilde{A}_{d-1}\), respectively.

When \(r = d\) and the linear system \(|F|\) on the blow-up \(X\) induces the elliptic fibration \(\pi: X \to \mathbb{P}^{n-1}\), we observe that \(F^{n-2}\) is rationally equivalent to a smooth rational elliptic surface \(S\) which is the preimage via \(\pi\) of a line. Thus we have \(\langle A, B \rangle = A|_S \cdot B|_S\), where the right hand side is the intersection product in \(\text{Pic}(S)\).

**Proposition 1.2.** Let \(A\) and \(B\) be effective divisors of \(X\) with \(B\) a prime divisor. If \(\langle A, B \rangle < 0\) then \(B\) is contained in the stable base locus of \(|A|\).

**Proof.** Let \(\ell\) be a general line of \(\mathbb{P}^{n-1}\) and let \(S\) be the surface \(\pi^{-1}(\ell)\). According to the definition of the bilinear form we have \(A|_S \cdot B|_S < 0\). Being \(B\) prime and \(\ell\) general, the divisor \(B|_S\) of \(S\) is prime as well. Thus the linear series \(|A|_S\) contains \(B|_S\) into its base locus and the same holds for the linear series \(|A|\). Varying \(\ell\) we get the claim. \(\square\)

## 2. Types

In this section we are going to describe the possible types of del Pezzo elliptic varieties of degree \(d \leq 4\).

### 2.1. Degree one.

In this case \(Y\) is a degree 6 hypersurface of the \((n+1)\)-dimensional weighted projective space \(\mathbb{P}(3,2,1,\ldots,1)\). After applying a change of coordinates, we can assume that a defining equation for \(Y\) is

\[(2.1) \quad x_1^2 - x_2^3 + x_2 f_4 + f_6 = 0,\]
where \( f_t \) is a degree \( t \) homogeneous polynomial in \( x_3, \ldots, x_{n+2} \). The blow-up \( \sigma: X \to Y \) is centered at the point \( q = (1,1,0,\ldots,0) \in Y \) and the rational map \( Y \to \mathbb{P}^{n-1} \) is defined by \((x_1, \ldots, x_{n+2}) \mapsto (x_3, \ldots, x_{n+2})\).

2.2. Degree two. In this case \( Y \) is a double covering \( \varphi: Y \to \mathbb{P}^n \) branched along a smooth quartic hypersurface \( S \). In order to distinguish the different cases that can occur, we observe that the preimage of a line \( \ell \) through a point \( p := \varphi(q_1) \) is one of the following:

\[
\varphi^{-1}(\ell) = \begin{cases} 
\text{elliptic curve} & \text{if } |\ell \cap S| = 4 \\
\text{rational nodal curve} & \text{if } |\ell \cap S| = 3 \\
\text{union of two smooth rational curves} & \text{if } \ell \text{ is bitangent to } S.
\end{cases}
\]

Therefore we distinguish three different cases depending on the position of \( p \) with respect to \( S \) and on the dimension of the variety \( B \subseteq \mathbb{P}^n \) spanned by the bitangent to \( S \) passing through \( p \).

Case 1. The point \( p \) does not lie on \( S \) and \( B \) is not a hypersurface. In this case the preimage of \( p \) in the double covering \( Y \to \mathbb{P}^n \) consists of two distinct points \( q_1 \) and \( q_2 \). We denote by \( X_{11} \) the variety that we obtain by blowing up these two points.

Case 2. The point \( p \) does not lie on \( S \) and \( B \) is a hypersurface. In this case after a linear change of coordinates we can assume that \( p = (0, \ldots, 0, 1) \). An equation for \( Y \) has the following form

\[
x_{n+2}^2 = g + h^2,
\]

where \( g \in \mathbb{C}[x_1, \ldots, x_n] \) is a homogeneous polynomial of degree four such that \( V(g) \) is the cone spanned by the bitangents through \( p \), while \( h \in \mathbb{C}[x_1, \ldots, x_{n+1}] \) is a homogeneous polynomial of degree two such that \( h(p) \neq 0 \) and \( S = V(g + h^2) \). We denote by \( X_{SS} \) the variety obtained by blowing up the two distinct points \( q_1 \) and \( q_2 \) in the pre-image of \( p \).

Case 3. The point \( p \) lies on \( S \). In this situation \( B \) cannot be a hypersurface since otherwise \( S \) would be singular. In order to get an elliptic fibration we need to blow up the point \( q_1 := \varphi^{-1}(p) \) and the point on the exceptional divisor which is invariant with respect to the lifted involution. We denote by \( X_2 \) the variety that we obtain after the blowing-ups. In this case an equation for \( Y \) has the following form

\[
x_{n+2}^2 = x_n x_{n+1}^3 + f.
\]

where \( f \in \mathbb{C}[x_1, \ldots, x_{n+1}] \) is a homogeneous polynomial of degree four which does not contain monomials of degree \( \geq 3 \) in the variable \( x_{n+1} \). The point \( q_1 \) has coordinates \((0, \ldots, 0, 1, 0)\) and the tangent space to \( S \) at \( p \) is \( V(x_n) \).

2.3. Degree four. Let us first collect some facts about smooth complete intersections of two hyperquadrics \( Y := Q \cap Q' \subseteq \mathbb{P}^{n+2} \), for \( n \geq 3 \). Observe that any quadric in the pencil \( \Lambda \) generated by \( Q \) and \( Q' \) has rank at least \( n+2 \), since otherwise \( Y \) would not be smooth, and there are \( n+3 \) singular quadrics in the pencil, counting multiplicities. We claim that there are exactly \( n+3 \) quadrics of rank \( n+2 \) and their vertices are in general position in \( \mathbb{P}^{n+2} \). Indeed, let us suppose that either there are less than \( n+3 \) vertices or that they are not in general position.

In the former case the pencil of quadrics is tangent to the discriminant hypersurface at some point. Without loss of generality we can assume \( Q \) to be a cone of
vertex \( p = (1, 0, \ldots, 0) \) in diagonal form \( g \) and the pencil \( g + tg' \) is tangent to the discriminant hypersurface at \( t = 0 \). If the Hessian matrix of \( g \) is \( M \) and that of \( g' \) is \( M' \), the above tangency condition is equivalent to the vanishing of the following derivative

\[
\frac{d}{dt} \text{Det}(M + tM')|_{t=0}.
\]

Expanding the above derivative and using the fact that \( M \) is diagonal we see that the above is equivalent to \( m'_{11} = 0 \), that is \( p \in Q' \). This is not possible since it contradicts the smoothness of \( Y \).

In the latter case there exists a hyperplane \( H \subseteq \mathbb{P}^{n+2} \) containing all the vertices and if we restrict \( \Lambda \) to \( H \) we obtain a pencil \( \Lambda_H \) of quadrics in \( \mathbb{P}^{n+1} \), containing at least \( n+3 \) singular quadrics (counting multiplicities). Hence all the quadrics of \( \Lambda_H \) must be singular and by Bertini’s theorem their vertices are contained in the base locus of \( \Lambda_H \). This implies that all the vertices of these cones are in \( Y \) and this is a contradiction since they give singular points of \( Y \).

In what follows we will denote by \( Q_1, \ldots, Q_{n+3} \) the singular quadrics and by \( p_1, \ldots, p_{n+3} \) the corresponding vertices. By the above discussion, we can assume that \( p_i \) is the \( i \)-th fundamental point of \( \mathbb{P}^{n+2} \) for \( i = 1, \ldots, n+3 \), so that \( Q_1 \) and \( Q_2 \) are defined by diagonal forms. Moreover, after possibly rescaling the variables, we can assume the quadrics to be defined by the following polynomials

\[
(2.5) \quad x_2^2 - x_3^2 + \cdots + x_{n+3}^2, \quad x_1^2 - x_3^2 + \alpha_4 x_4^2 + \cdots + \alpha_{n+3} x_{n+3}^2
\]

respectively, where the coefficients \( \alpha_i \) are distinct and not in \( \{0, 1\} \). Let us now prove the following result that will be useful in the next section.

**Proposition 2.1.** Let \( q_1 \) and \( q_2 \) be two points of \( Y \), possibly infinitely closed. Then the conics of \( Y \) through these two points span a hypersurface of \( Y \) if and only if the line \( \langle q_1, q_2 \rangle \) passes through one of the vertices \( p_i \).

**Proof.** If \( p_i \) lies on the line \( \langle q_1, q_2 \rangle \), then this line is a generatrix of the cone \( Q_i \). We can write \( Y = Q_i \cap Q \), where \( Q \) is any other quadric of the pencil. We conclude observing that there exists an \((n-2)\)-dimensional family of planes of \( Q_i \) containing a generatrix and each of them intersects \( Q \) along a conic through the two fixed points.

Let us suppose now that the conics through \( q_1 \) and \( q_2 \) span a hypersurface \( S \), i.e. there exists an \((n-2)\)-dimensional family of such conics. Observe that when we have a conic \( C \) contained in \( Y \) then the plane \( \pi_C \) of the conic must be contained in one of the quadrics of the pencil \( \Lambda \) (since the generic quadric of the pencil cuts \( \pi_C \) along the curve \( C \), imposing to pass through one point of \( \pi_C \) not lying on \( C \) we get the whole plane). Therefore, under our hypotheses, we must have a quadric of the pencil containing an \((n-2)\)-dimensional family of planes sharing the line \( \langle q_1, q_2 \rangle \). Hence this quadric is a cone with vertex on that line. \( \square \)

Let us fix now a plane \( \Pi \subseteq \mathbb{P}^{n+2} \) and let us analyze the different types of del Pezzo elliptic varieties of degree four. By Proposition 2.1 the type depends not only on the number of points we blow up but also on the number of vertices \( p_i \) contained in the plane \( \Pi \). Hence we are going to use the symbol \( X_{kl} \) to denote the variety that we obtain by choosing a plane \( \Pi \) intersecting \( Y \) in \( k \) distinct points and containing \( l \) vertices.
Let us spend few words about the geometry of this construction and about the possible values of \( k \) and \( l \) for \( X_{k,l} \). We remark that we can write
\[
\Pi \cap Y = C \cap C'
\]
where \( C := Q \cap \Pi \) and \( C' := Q' \cap \Pi \) are two plane conics. We discuss four cases.

Case 1. If \( \Pi \) contains no vertices, then we have two smooth conics, whose intersection consists of \( k \) distinct points, for \( k \in \{1, 2, 3, 4\} \) and hence we get the types \( X_{40}, X_{30}, X_{20}, X_{10} \). Observe that when \( k = 2 \) we have two possibilities: either \( C \) and \( C' \) are tangent at their two intersection points \( q_1 \) and \( q_2 \), or they intersect transversally at \( q_2 \) and with multiplicity three at \( q_1 \).

Case 2. If \( \Pi \) contains one vertex \( p_i \), then we can suppose that \( C \) is a smooth conic while \( C' := \Pi \cap Q_i \) has (at least) a singular point at the vertex \( p_i \in \Pi \). The intersection of \( C \) and \( C' \) consists of \( k \) points, for \( k \in \{1, 2, 3, 4\} \) and we obtain the types \( X_{41}, X_{31}, X_{21}, X_{11} \). As before, when \( k = 2 \) we have two possibilities. Either \( C' \) is the union of two distinct lines and each of them is tangent to the conic \( C \), or \( C' \) is a double line (which means that \( \Pi \) is tangent to \( Q_i \)) intersecting \( C \) in two distinct points.

Case 3. If \( \Pi \) contains two vertices, then we can suppose that both \( C \) and \( C' \) are singular and they can not intersect at the vertices so that \( k \) can be either 1, 2 or 4. Moreover, when \( k = 1 \) we deduce that the plane \( \Pi \) is contained in the tangent space to \( Y \) at the only intersection point \( q_1 \). We are not going to consider this case since it does not give an elliptic fibration, being all the fibers singular rational curves. Hence we have only the two types \( X_{42} \) and \( X_{22} \).

Case 4. Finally, observe that if \( \Pi \) contains three vertices then it is fixed and it can intersect \( Y \) only at four distinct points (otherwise \( Y \) would be singular), giving case \( X_{43} \).

**Remark 2.2.** We provide here an example of defining equations for \( \Pi \) for each of the following five types:
\[
\begin{align*}
X_{13} : & \quad \Pi = V(x_1, x_5, \ldots, x_{n+3}) \\
X_{22} : & \quad \Pi = V(x_3 - x_4, x_5, \ldots, x_{n+3}) \\
X_{21} : & \quad \Pi = V(\sqrt{\alpha_4 + \alpha_5} \cdot x_2 - \sqrt{\alpha_4 + \alpha_5 - 2} \cdot x_3, x_4 - x_5, \ldots, x_{n+3}) \\
X_{11} : & \quad \Pi = V(x_1 - \alpha_4 x_2 + (\alpha_4 - 1) x_3, x_5, \ldots, x_{n+3}) \\
X_{10} : & \quad \Pi = V(2x_1 - (\alpha_4 + \alpha_5)x_2 + (\alpha_4 + \alpha_5 - 2)x_3, x_4 - x_5, x_6, \ldots, x_{n+3}),
\end{align*}
\]
where \( \alpha_4 + \alpha_5 \neq 0, 2 \) in cases \( X_{21} \) and \( X_{10} \).

**Remark 2.3.** We recall that if \( Y = Q \cap Q' \subseteq \mathbb{P}^{n+2} \) and \( n \geq 3 \), then through any point of \( Y \) we have at least one line of \( Y \). So let us fix a point \( q_i \in Y \) and a line \( \ell \) of \( Y \), passing through this point, and let us describe the fiber of \( \pi : X \rightarrow \mathbb{P}^{n-1} \) containing the strict transform of that line. The image of this fiber inside \( Y \) is the curve obtained by intersecting \( Y \) with the \( \mathbb{P}^3 \) spanned by the plane \( \Pi \) and the line \( \ell \). This can also be described as the base locus of the pencil of quadric surfaces obtained by restricting \( \Lambda \) to the \( \mathbb{P}^3 \) that we are considering. Observe that any time we have a vertex \( p_i \) in \( \Pi \), the intersection of \( Q_i \) with the \( \mathbb{P}^3 \) is a quadric cone containing a line not passing through \( p_i \). Hence it must be the union of two planes intersecting along a line passing through \( p_i \). Therefore, in Case 1 the image of the fiber inside \( Y \) is obtained by intersecting two smooth quadric surfaces sharing a line and hence it is the union of that line and a rational normal cubic, intersecting
in two points. In Case 2 the base locus is the intersection of a smooth quadric with a reducible one and then it is the union of two lines and a smooth conic. In Case 3 the base locus is the intersection of two reducible quadrics and hence it consists of four lines. Finally, in Case 4 we have the base locus of a pencil containing three reducible quadrics. Thus, after a possible renaming of the coordinates, the pencil has the form \((x_1^2 - x_2^2) + t(x_2^2 - x_3^2)\). All the quadrics in this pencil are singular at the point \(p = (1, 1, 1, 1)\) and the base locus of the pencil consists of four lines intersecting at the point \(p\). We remark that in this last case the corresponding fiber in \(X\) is the union of four rational components passing through one point and hence it is a type that does not appear in the Kodaira’s list of singular fibers for elliptic surfaces.

3. Mordell-Weil groups

The main result of this section is the proof of Theorem 1 but we postpone it to the end of the section and we begin by studying the vertical divisors of all the del Pezzo elliptic fibrations of degree \(d \leq 4\), that is divisors \(D\) such that \(\pi(D)\) is a hypersurface of \(\mathbb{P}^{n-1}\). If \(d = 1\), then the only vertical class is \(F\) since the rank of the subgroup of vertical divisors equals \(\text{rk Pic}(X) - 1 = 1\).

When \(d = 2\), recall that there is a double covering \(\varphi: Y \rightarrow \mathbb{P}^n\) branched along a smooth quartic hypersurface \(S\) and \(\sigma: X \rightarrow Y\) is the blow-up of \(Y\) at two points \(q_1, q_2\) exchanged by the covering involution. By (2.2), if \(D\) is a prime proper vertical divisor of \(X\) whose class is not a multiple of \(\mathbb{P}\), then either \(D\) is contained in the pull-back of an exceptional divisor of \(\sigma\), or \(\varphi(D)\) is covered by bitangent lines to \(S\). Therefore in case \(X_{11}\) there are no proper vertical divisors.

In case \(X_{SS}\) we have the two vertical classes \(2H - 4E_1, 2H - 4E_2\), and assuming that \(Y\) has the equation (2.3), they are the classes of the strict transforms of \(V(x_{n+2} - h)\) and \(V(x_{n+2} + h)\), respectively.

Finally, in case \(X_2\), \(E_1 - E_2\) and \(H - 2E_1\) are the only prime proper vertical divisors.

The case \(d = 3\) has already been studied in [5] and we refer to that paper for the classification of prime proper vertical divisors.

For \(d = 4\), if \(D\) is a prime proper vertical divisor, then \(D\) is strictly contained in the support of \(\pi^* \pi(D)\). Let \(\gamma\) be a general fiber of \(\pi\) over a point \(q \in \pi(D)\) and let us denote by \(C\) the image \(\sigma(\gamma)\) in \(Y\). Then either (i) \(C\) is an irreducible rational curve or (ii) it contains lines and/or conics.

In case (i), \(C\) is singular at one of the points \(q_i \in \Pi \cap Y\) and the union of these curves gives a prime proper divisor having class \(H - 2E_i - E_j\). In order to obtain the class of a fiber we have to add some prime proper vertical exceptional divisors of the form \(E_i - E_j\).

In case (ii), observe that by [3] through any point of \(Y\) there is only a \((n - 3)\)-dimensional family of lines and hence they can not fill up a divisor. Therefore the curve \(C\) must contain a conic through two points \(q_i\) and \(q_j\) of \(\Pi \cap Y\), possible infinitely near. By Proposition 2.1 the line \(\langle q_i, q_j \rangle\) passes through one vertex \(p_k\) and hence \(p_k \in \Pi\). In this case the class of one of the irreducible components of \(\pi^* \pi(D)\) is of the form \(H - 2E_i - 2E_j\). For instance, in case \(X_{31}\) we can write \(Y = Q_1 \cap Q\), where \(Q_1\) is a cone with vertex \(p_1 \in \Pi\) and \(Q\) is a smooth quadric. Furthermore, \(Q\) intersects \(\Pi\) along a smooth conic \(C\) while \(Q_1 \cap \Pi\) is the union of two generatrices and one of them is tangent to \(C\) at \(q_1\) while the other one intersects
C in \( q_3 \) and \( q_4 \). Therefore we have the vertical class \( H - 2E_1 - 2E_2 \) corresponding to the conics through \( q_1 \) and whose tangent line at \( q_1 \) is the line \( \langle q_1, p_1 \rangle \) and the class \( H - 2E_3 - 2E_4 \) corresponding to the conics through \( q_3 \) and \( q_4 \). Observe that the sum of these classes gives twice a fiber. Moreover, we also have the vertical class \( E_1 - E_2 \) sitting inside the exceptional locus and the class \( H - 2E_1 - E_3 - E_4 \) which is spanned by the union of the strict transforms of the singular rational quartic curves of \( Y \) obtained intersecting it with a hyperplane tangent to \( Y \) at \( q_1 \).

We summarise the above observations in the following

**Proposition 3.1.** Let \( \pi: X \to \mathbb{P}^{n-1} \) be a del Pezzo elliptic fibration of degree \( d \leq 4 \) and dimension \( n \geq 3 \). Then for each type the sections and the vertical divisors are as follows:

| Degree | Type | Sections | Proper prime vertical divisors |
|--------|------|----------|-------------------------------|
| 1      | \( X_1 \) | \( E_1 \) |                                |
| 2      | \( X_{11} \) | \( E_1, E_2 \) | \( 2H - 4E_1, 2H - 4E_2 \) |
|        | \( X_{SS} \) | \( E_1, E_2 \) | \( E_1 - E_2, H - 2E_1 \) |
|        | \( X_{2} \) | \( E_2 \) |                                |
| 3      | \( X_{111} \) | \( E_1, E_2, E_3 \) | \( H - 3E_1, 2H - 3E_2 - 3E_3 \) |
|        | \( X_{S11} \) | \( E_1, E_2, E_3 \) | \( H - 3E_1, H - 3E_2, H - 3E_3 \) |
|        | \( X_{SSS} \) | \( E_1, E_2, E_3 \) |                                |
|        | \( X_{12} \) | \( E_1, E_3 \) | \( E_2 - E_3, H - E_1 - 2E_2 \) |
|        | \( X_{S2} \) | \( E_1, E_3 \) | \( H - 3E_1, 2H - 3E_2 - 3E_3, E_2 - E_3, H - E_1 - 2E_2 \) |
|        | \( X_3 \) | \( E_3 \) | \( E_1 - E_2, E_2 - E_3, H - E_1 - E_2 \) |
|        | \( X_S \) | \( E_3 \) | \( E_1 - E_2, E_2 - E_3, H - 2E_1 - E_2, H - 3E_1, 2H - 3E_2 - 3E_3 \) |
| 4      | \( X_{40} \) | \( E_1, E_2, E_3, E_4 \) |                                |
|        | \( X_{41} \) | \( E_1, E_2, E_3, E_4 \) | \( H - 2E_1 - 2E_3, H - 2E_2 - 2E_4 \) |
|        | \( X_{42} \) | \( E_1, E_2, E_3, E_4 \) | \( H - 2E_1 - 2E_3, H - 2E_2 - 2E_4 \) |
|        | \( X_{43} \) | \( E_1, E_2, E_3, E_4 \) | \( H - 2E_1 - 2E_3, H - 2E_2 - 2E_4 \) |
|        | \( X_{32} \) | \( E_2, E_3, E_4 \) | \( E_1 - E_2, H - 2E_1 - E_3 - E_4 \) |
|        | \( X_{31} \) | \( E_2, E_3, E_4 \) | \( E_1 - E_2, H - 2E_1 - E_3 - E_4 \) |
|        | \( X_{20} \) | \( E_3, E_4 \) | \( E_1 - E_3, H - 2E_1 - E_2 - E_4, E_2 - E_4, H - E_1 - 2E_2 - E_3 \) |
|        | \( X_{21} \) | \( E_3, E_4 \) | \( E_1 - E_3, H - 2E_1 - E_2 - E_3, H - 2E_2 - E_4 \) |
|        | \( X_{22} \) | \( E_3, E_4 \) | \( H - 2E_1 - 2E_2, H - E_1 - 2E_2 - E_3 \) |
|        | \( X_{10} \) | \( E_4 \) | \( E_1 - E_2, E_2 - E_3, E_3 - E_4, H - 2E_1 - E_2 - E_3 \) |
|        | \( X_{11} \) | \( E_4 \) | \( E_1 - E_2, E_2 - E_3, E_3 - E_4, H - 2E_1 - E_2 - E_3 \) |

Table 3: Sections and vertical classes of del Pezzo elliptic fibrations with \( d \leq 4 \).

**Proof of Theorem 1.** Recall that the Mordell-Weil group of the elliptic fibration \( \pi \) is the group of rational sections of \( \pi \) or, equivalently, the group of \( K = \mathbb{C}(\mathbb{P}^{n-1}) \)-rational points \( X_0(K) \) of the generic fiber \( X_0 \) of \( \pi \) once we choose one of such points \( O \) as an origin for the group law. Let \( \mathcal{T} \) be the subgroup of \( \text{Pic}(X) \) generated by the classes of the vertical divisors and by the class of the section \( O \). There is an
exact sequence [7, Section 3.3]:

\[
0 \rightarrow \mathcal{T} \rightarrow \text{Pic}(X) \rightarrow X_\eta(K) \rightarrow 0.
\]

In degree \(d\), the Picard group of \(X\) is free of rank \(d + 1\), generated by the classes \(H, E_1, \ldots, E_d\). Observe that if \(F\) is defined as in (1.1), then \(\langle F, E_d \rangle \subseteq \mathcal{T}\) holds and by Proposition 3.1 and the sequence (3.1) we get the statement. 

4. Cones

The aim of this section is to provide a description of the nef, effective and moving cone of del Pezzo elliptic varieties. Moreover, we discuss the Mori chamber decomposition of the moving cones and we use this decomposition in order to prove Theorem 2.

4.1. The nef cones. Given a subset \(I\) of \(\{1, \ldots, d\}\), in what follows we denote by \(F_I\) the divisor \(H - \sum_{i \in I} E_i\).

**Theorem 4.1.** Let \(\pi: X \rightarrow \mathbb{P}^{n-1}\) be a del Pezzo elliptic fibration with \(n \geq 3\). Then the extremal rays of the nef cone \(\text{Nef}(X)\) are all the \(F_I\) such that \(I \subseteq \{1, \ldots, d\}\) and \(\langle F_I, V \rangle \geq 0\) for each exceptional vertical class \(V\).

**Proof.** Let us consider the subcone \(C\) of the Mori cone of \(X\) generated by the following classes:

- \(e_i\) such that \(E_i\) is a section;
- \(e_i - e_j\) such that \(E_i - E_j\) is a prime vertical divisor;
- \(h - e_i\) for each \(q_i \in Y\).

Let \(D := \alpha H - \sum_i m_i E_i\) be a class in the dual \(C^*\). Then we have the following inequalities: \(m_i \geq 0\) \(\forall i\), \(m_i \geq m_j\) if the point \(q_j\) lies on the exceptional divisor of the blowing-up at \(q_i\) and finally \(\alpha \geq m_i \forall i\). Let us write \(\{m_1, \ldots, m_d\} = \{\mu_1, \ldots, \mu_r\}\), where \(r \leq d\) and \(0 = \mu_0 \leq \mu_1 < \cdots < \mu_r\), and let us denote by \(I_i := \{j \mid m_j \geq \mu_i\}\), for each \(i = 1, \ldots, r\). Then we can write

\[
D = (\alpha - \mu_r)H + \sum_{i=1}^r (\mu_i - \mu_{i-1})F_{I_i},
\]

where the \(F_{I_i}\) are nef and their product with any effective \(E_j - E_k\) is non negative. In order to conclude the proof we need to show that these \(F_I\) are extremal rays of the nef cone.

Let us first suppose that \(X\) is obtained by blowing up \(d\) distinct points on \(Y\). In this case, we have to consider all the \(F_I\) as \(I\) varies in the subsets of \(\{1, \ldots, d\}\), and by induction on \(d\) it can be proved that they are vertices of a \(d\)-dimensional hyper-cube. In particular, they are extremal rays of the cone they generate.

In addition, we can also infer that no \(F_I\) lies in the convex hull of the remaining and hence the general case follows.

4.2. The effective and moving cones. We now restrict our attention to del Pezzo elliptic fibrations of degree \(d \leq 4\) and having finite Mordell-Weil group, proving Theorem 3.
Proof of Theorem 3. Let us consider, for each del Pezzo elliptic variety $X$ the cone $\mathcal{M}$ of $\text{Pic}_0(X)$ generated by the vertical classes and the sections of $\pi$. Let $\varrho_1, \ldots, \varrho_n$ be the extremal rays of $\mathcal{M}$. We have the following inclusions

$$\bigcap_{i=1}^n \text{cone}(\varrho_1, \ldots, \varrho_i, \ldots, \varrho_n) \subseteq \text{Mov}(X) \subseteq \text{Eff}(X) \subseteq \mathcal{M}^\vee,$$

where the first inclusion is due to the fact that each $\varrho_i$ is generated by a prime divisor, the second one is a consequence of Proposition 1.2 and the last one follows from $\mathcal{M} \subseteq \text{Eff}(X)$. The proof goes as follows. If the degree $d$ is at most three, then the Cox ring is known (Theorem 5.1 for degree one or two and [5] for degree three) and a direct computation shows that the rays of $\mathcal{M}^\vee$ are movable. When $d = 4$, observe that if $X$ is of type $X_{43}, X_{22}, X_{21}$, then the first cone and the last one in (4.1) are equal and the two assertions of the theorem follow. In the remaining cases, we are going to check that the rays of $\mathcal{M}^\vee$ are movable.

If $X$ is of type $X_{11}$, then the only class we have to check is $H - 2E_1$ (all the other rays of $\mathcal{M}^\vee$ are indeed nef classes). We are going to see that the base locus of the linear system $|H - 2E_1|$ has codimension two. Indeed, this linear system corresponds on $Y$ to the linear system of hyperplane sections containing the tangent space at $q_1$. When we blow up $q_1$, the strict transforms of these lines intersect $E_1$ along a subvariety of codimension two. Observe that the second point $q_2$ that we blow up do not lie on this subvariety, since otherwise the plane $\Pi$ would intersect $Y$ along a line. Then the base locus of $|H - 2E_1|$ can not be divisorial.

In case $X_{10}$ the only classes we have to check are $H - 2E_1$ and $3H - 4E_1 - 4E_2$. The first one can be done as in case $X_{11}$ while the second one can be obtained as the image of $H$ via the Geiser involution described in Subsection 4.3, and hence it is movable. \hfill \Box

As a consequence of Theorem 3, if $X$ is a del Pezzo elliptic variety of degree $d \leq 4$ such that the Mordell-Weil group of $\pi : X \to \mathbb{P}^{n-1}$ is finite, then the effective cone $\text{Eff}(X)$ can be read from Table 3. The graphs of the quadratic form on the primitive generators of the extremal rays of $\text{Eff}(X)$ are listed in Table 2.

Let us consider an example in which the Mordell-Weil group of the fibration is not finite and the moving cone is the union of infinitely many chambers.

When the elliptic fibration has degree $d = 2$ and type $X_{11}$, we have seen that the Mordell-Weil group is $\langle \sigma \rangle \cong \mathbb{Z}$. The action of $\sigma$ on the Picard group of $X$, with respect to the basis $B := (H - E_1 - E_2, E_2 - E_1, E_1)$, is given by the following matrix

$$\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.$$

The cone $\sigma^k(\text{Nef}(X))$ is generated by the classes corresponding to the columns of the matrix

$$\begin{pmatrix}
1 & k^2 + k + 1 & k^2 - k + 1 & 2k^2 + 1 \\
0 & k + 1 & k & 2k + 1 \\
0 & 1 & 1 & 2
\end{pmatrix},$$

Let us consider an example in which the Mordell-Weil group of the fibration is not finite and the moving cone is the union of infinitely many chambers.
with respect to the basis $B$. We claim that the classes $\sigma^k(H)$ are extremal rays of the moving cone and generate it, so that the following equality holds

$$\text{Mov}(X) = \bigcup_{k \in \mathbb{Z}} \sigma^k(\text{Nef}(X)).$$

First of all, observe that for each $k \in \mathbb{Z}$ the cones $\sigma^k(\text{Nef}(X))$ and $\sigma^{k+1}(\text{Nef}(X))$ share the two-dimensional face generated by $F$ and $\sigma^k(H - E_1) = \sigma^{k+1}(H - E_2)$. Moreover $\sigma^k(H) + \sigma^{k+1}(H) = 4\sigma^k(H - E_1)$, so that the union of the cones $\sigma^k(\text{Nef}(X))$ is a convex cone and the classes $\sigma^k(H - E_i)$, $i = 1, 2$, are on its boundary but they are not extremal rays. Now observe that the right hand side cone is contained in $\text{Mov}(X)$.

Finally, since the property of lying on the boundary of $\text{Mov}(X)$ is preserved by $\sigma^k$, we only have to prove that the two faces $\langle H, H - E_i \rangle$, for $i = 1, 2$, are on the boundary of the moving cone $\text{Mov}(X)$. We conclude observing that if we move outside from $\text{Nef}(X)$ along a direction orthogonal to the face $\langle H, H - E_i \rangle$ (respectively $\langle H, H - E_2 \rangle$) we obtain classes containing $E_2$ (respectively $E_1$) in the stable base locus.

### 4.3. Generalized Bertini and Geiser involutions.

We consider here a generalization of the classical Bertini and Geiser involutions to blow-ups of del Pezzo varieties. Let $Y$ be a degree $d \geq 3$ del Pezzo variety and let $Z \subseteq Y$ be a zero-dimensional subscheme such that $\dim(Z) = l(Z) - 1$ and the intersection of $d - 1$ general elements of $L_Z := |O_Y(1) \otimes \mathcal{I}_Z|$ is an elliptic curve. We denote by $\sigma: Y_Z \to Y$ the blow-up of $Y$ along $Z$ as in Section 1.

If $Z$ has length $l(Z) = d - 2$, then the general $(d - 2)$-dimensional linear space containing $Z$ intersects $Y \setminus Z$ at two distinct points. The birational involution obtained by exchanging these two points induces a birational involution $\sigma_G$ on the blow-up $Y_Z$ of $Y$ at $Z$. We call this $\sigma_G$ a generalized Geiser involution.

When $Z$ has length $l(Z) = d - 1$, denote by $F$ the divisor on $Y_Z$ defined as before. The base locus of the linear system $|F|$ consists of one point $q$ while $|2F|$ defines a morphism $\varphi$. Since $F^n = 1$, we have that $F^{n-1}$ is rationally equivalent to an elliptic curve $C$ passing through $q$ and the restriction $\varphi|_C$ is a double covering of a line passing through the point $p := \varphi(q)$. Hence the image $\varphi(Y_Z)$ is a cone $V$. If we denote by $E$ the exceptional divisor corresponding to the last blow-up of $\sigma$, we have that the restriction $\varphi|_E$ is the 2-veronese embedding $v_2$ of $\mathbb{P}^{n-1}$. We conclude that $V$ is a cone over $v_2(\mathbb{P}^{n-1})$ and $\varphi$ induces a birational involution $\sigma_B$ on $Y_Z$ that we call a generalized Bertini involution. We remark that if $X$ is the del Pezzo elliptic variety obtained by blowing up $Y_Z$ in $q$, then $\sigma_B$ induces on $X$ the hyperelliptic involution with respect to the origin given by the exceptional divisor.

**Remark 4.2.** If $Y$ has degree four and the line $\langle Z \rangle$ does not contain any vertex $p_i$, then the indeterminacy locus of the corresponding Geiser involution $\sigma_G$ has codimension two. Moreover, it lifts to an isomorphism in codimension one for the elliptic varieties of type $X_{21}$ and $X_{10}$. The action on the Picard group of $X$ in each case is given by the following matrices respectively.
To prove this, we first claim that the lifted birational map preserves the elliptic fibration $\pi$ and thus it is a flop. Indeed, if $f$ is a fibre of $\pi$ whose image $C$ in $Y$ is cut out by a three-dimensional linear space $L$ and we fix a point $y \in C$, then the plane spanned by $y$ and $\langle Z \rangle$ is contained in $L$ and thus it intersects $C$ at a fourth point, so that $\phi(f) = f$, which proves the claim. Since $\phi$ preserves the fibration $\pi$, its pull-back $\phi^*$ must preserve both the sets of horizontal and vertical divisors of $X$. A direct calculation shows that the representative matrix for $\phi^*$ in the basis $(H, E_1, \ldots, E_4)$ is one of the above in each case.

4.4. Mori chambers. Let $X$ be a del Pezzo elliptic variety of degree four with finite Mordell-Weil group. We provide here the Mori chamber decomposition of the moving cone $\text{Mov}(X)$ of $X$. In the following proposition, we will denote by $N$ the nef cone of $X$ and by

$$N_i := \text{cone} \{F_i : i \in I \text{ and } F_i \in N \} \cup \{H - 2E_i\}.$$ 

**Proposition 4.3.** Let $X$ be a del Pezzo elliptic variety of degree four such that the corresponding elliptic fibration has finite Mordell-Weil group. Then the Mori chamber decomposition of $\text{Mov}(X)$ is given in the following table.

| Type | Cones |
|------|-------|
| $X_{43}$ | $N$, $N_1$, $N_2$, $N_3$, $N_4$ |
| $X_{22}$ | $N$, $N_1$, $N_2$ |
| $X_{21}$ | $N$, $N_1$, $N_2$, $\sigma_{21}^*(N)$, $\sigma_{21}^*(N_1)$, $\sigma_{21}^*(N_2)$ |
| $X_{11}$ | $N$, $N_1$ |
| $X_{10}$ | $N$, $N_1$, $\sigma_{10}^*(N)$, $\sigma_{10}^*(N_1)$ |

**Proof.** Let $X \to X_i$ be the flop of the class $h - e_i$ of the strict transform $C$ of a line through the point $q_i \in Y$. Note that such a flop exists by [3]. We show that the nef cone of $X_i$ is $N_i$, and then observe that the union of the cones in the table given in the statement is $\text{Mov}(X)$ for each type. To prove the claim, we begin by showing that the primitive generators of the extremal rays of the cone $N_i$ are nef in $X_i$. Observe that each $F_i \in N_i$ is nef in both $X$ and $X_i$ since $F_i \cdot (h - e_i) = 0$ by our definition of $N_i$. Hence we only have to check that also $H - 2E_i$ is nef in $X_i$. Since $H - 2E_i$ is the pull-back of a class on the blow-up $\tilde{Y}$ of $Y$ at $q_i$, it is enough to prove the claim on $\tilde{Y}$. By Lemma 5.2 the Cox ring of $\tilde{Y}$ is finitely generated and the moving cone decomposes as follows:

$$\text{Mov}(\tilde{Y}) = \text{cone}(H, H - E_i) \cup \text{cone}(H - E_i, H - 2E_i).$$

Thus, after flopping $h - e_i$ the class $H - 2E_i$ becomes nef as claimed so that we have the inclusion $N_i \subseteq \text{Nef}(X_i)$. To prove that this is indeed an equality, we show that the extremal rays of the dual cone of $N_i$ are classes of effective curves of $X_i$. To this aim we make use of [3, Lemma 4.1] which asserts that if $\Gamma$ is a curve of $X$ which meets $C$ transversally at $k$ points, and no other effective curve of class $h - e_i$,

$$\sigma_{21} = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ -4 & -2 & -1 & 0 & 0 \\ -4 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_{10} = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ -4 & -1 & -2 & 0 & 0 \\ -4 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
then the flop image $\Gamma'$ of $\Gamma$ has class
\begin{equation}
[\Gamma'] = [\Gamma] + k[C].
\end{equation}
By a direct calculation, we see that the extremal rays of the dual cone of $N_i$ are the following (here we list only the case $i = 1$, being the remaining cases analogous):

| Type | Extremal rays of the Mori cone | Extremal rays of the dual cone of $N_1$ |
|------|-------------------------------|---------------------------------------|
| $X_{43}$ | $e_1, e_2, e_3, e_4$ $h - e_1, h - e_2, h - e_3, h - e_4$ | $-h + e_1, e_2, e_3, e_4$ $2h - e_1 - e_2, 2h - e_1 - e_3, 2h - e_1 - e_4$ |
| $X_{22}, X_{21}$ | $e_2, e_4$ $h - e_1, h - e_3$ $e_1 - e_2, e_3 - e_4$ | $-h + e_1, e_2, e_4$ $2h - e_1 - e_2, 2h - e_1 - e_3$ $e_3 - e_4$ |
| $X_{11}, X_{10}$ | $h - e_1, e_4, e_1 - e_2$ $e_2 - e_3, e_3 - e_4$ | $-h + e_1, e_4, e_2 - e_3, e_3 - e_4, 2h - e_1 - e_2$ |

For each type the curves having class $e_i$ or $e_i - e_{i+1}$, with $i > 1$, do not intersect any curve of class $h - e_1$ and hence by (4.2) their classes in the Mori cone of $X_1$ are the same. Assume that $\Gamma$ is an irreducible curve such that
\begin{equation}
[\Gamma] = 2h - e_1 - e_i.
\end{equation}
We can assume that $\Gamma$ is the strict transform of a smooth conic $C$ of $Y$ passing through $q_1$ and $q_i$, which is possibly infinitely near to $q_1$. The tangent line to $C$ at $q_1$ cannot be contained in $Y$ since otherwise the plane spanned by $C$ and this line would be contained into each quadric of the pencil and thus in $Y$. This gives a contradiction, since the line through $q_1$ and $q_i$ is not contained in $Y$ by hypothesis. Thus we conclude again by (4.2), proving the assertion for $X_{43}, X_{22}$ and $X_{11}$.

In case $X = X_{21}$ the chamber $\sigma_{21}^*(N)$ is the pull-back of the nef cone $N = \text{Nef}(X)$ via the flop $\sigma_{21}$. Since $\sigma_{21}$ is the generator of the Mordell-Weil group of $\pi$, we deduce that $\sigma_{21}(X)$ is an elliptic del Pezzo variety of the same type. Thus each chamber $\sigma_{21}^*(N_i)$, for $i = 1, 2$, is a flop image of $\sigma_{21}^*(N)$ exactly as $N_i$ is a flop image of $N$. In particular the chamber $\sigma_{21}^*(N_i)$ is generated by finitely many semiample classes of $\sigma_{21}^*(X_i)$.

Finally, in case $X = X_{10}$ we proceed as we did for $X_{21}$, considering $\sigma_{10}$ instead of $\sigma_{21}$.

\hfill $\square$

**Proof of Theorem 2.** By [5, Lemma 3.5] (1) implies (2), so let us suppose that the Mordell-Weil group of $\pi$ is finite. If $d = 1$ or 2, then we conclude by means of Theorem 5.1, while the case $d = 3$ has been proved in [5, Theorem 3.6]. Finally, when $d = 4$, we observe that by Proposition 4.3, if the Mordell-Weil group of the fibration is finite, then the moving cone $\text{Mov}(X)$ satisfies all the hypotheses of [6].

\hfill $\square$

5. **Cox rings**

In this section, we provide a presentation for the Cox rings of the elliptic del Pezzo varieties of degree $\leq 4$. We recall that given a normal projective variety $X$
with finitely generated picard group, its Cox ring \( \mathcal{R}(X) \) can be defined as (see \[1\])
\[
\mathcal{R}(X) = \bigoplus_{[D] \in \text{Pic}(X)} H^0(X, \mathcal{O}_X(D)).
\]
We apply \([4, \text{Algorithm 5.4}]\) and we will explain all the steps in the algorithm for the convenience of the reader. Let \( Y_1 \) be a smooth projective variety with finitely generated Cox ring \( R_1 \), which admits a presentation \( R_1 = \mathbb{C}[T_1, \ldots, T_{r_1}] / I_1 \). Note that \( R_1 \) is \( K_1 \)-graded, where \( K_1 = \text{Cl}(Y_1) \). Define \( \overline{Y}_1 = \text{Spec}(R_1) \) and let \( \overline{Y}_1 \subseteq \overline{Y}_1 \) be the characteristic space of \( Y_1 \) with characteristic map \( \pi : \overline{Y}_1 \to Y_1 \). Let \( q \in Y_1 \) be the point that we want to blow up. We have the following commutative diagram:

\[
\begin{array}{c}
p^{-1}(q) \\
\downarrow \\
\overline{Y}_1 \\
p \downarrow \\
Y_1 \\
\end{array}
\]

Let \( I \subseteq R_1 \) be the ideal of the closure of \( p^{-1}(q) \) in \( \overline{Y}_1 \), and let \( J \subseteq R_1 \) be the irrelevant ideal, i.e. the ideal of \( \overline{Y}_1 \setminus \overline{Y}_1 \). We choose a system of homogeneous generators \( f_1, \ldots, f_s \in R_1 \) which form a basis for the ideal \( I \) and such that
\[
(5.1) \quad f_i \in (T_d : J^\infty), \quad \forall i = 1, \ldots, s,
\]
where \( d_i \) is a positive integer. An ample class \([D]\) of \( Y_1 \) defines an embedding \( Y_1 \to Z_1 \) into a projective toric variety \( Z_1 \), whose Cox ring is the \( K_1 \)-graded polynomial ring \( \mathbb{C}[T_1, \ldots, T_{r_1}] \) and such that the class \([D]\) is ample on \( Z_1 \). We embed \( Z_1 \) into another toric variety \( W_1 \) via the following map
\[
(T_1, \ldots, T_{r_1}) \mapsto (T_1, \ldots, T_{r_1}, f_1, \ldots, f_s),
\]
where the Cox ring of \( W_1 \) is the \( K_1 \)-graded polynomial ring \( \mathbb{C}[T_1, \ldots, T_{r_1+s}] \), with \( \deg(T_{r_1+i}) = \deg(f_i) \) for any \( i \), and again \([D]\) is an ample class of \( W_1 \). Now we blow-up \( W_1 \) equivariantly along the orbit \( V(T_{r_1+1}, \ldots, T_{r_1+s}) \), obtaining the toric variety \( Z_2 \) whose Cox ring is the polynomial ring \( \mathbb{C}[T_1, \ldots, T_{r_2}] \), where \( r_2 = r_1 + s + 1 \), graded by the group \( K_2 := K_1 \oplus \mathbb{Z} \). Let \( Y'_2 \subseteq Z_2 \) be the strict transform of the variety \( Y_1 \) as shown in the following diagram:

\[
\begin{array}{c}
Y'_2 \\
\downarrow \\
Y_1 \\
\end{array} \quad \begin{array}{c} \\
\downarrow \quad \quad \downarrow \\
Z_1 \\
\end{array} \quad \begin{array}{c} \\
\downarrow \\
W_1 \\
\end{array}
\]

Observe that \( Y'_2 \) is a blow-up (possibly weighted) of \( Y_1 \) at \( q \), whose defining ideal is the following saturation
\[
(5.2) \quad I_2 = \langle (T_{r_1+i}T_{r_2}^{-d} - f_i : 1 \leq i \leq s) + I_1 \rangle : \langle T_{r_2} \rangle
\]
with respect to the variable \( T_{r_2} \). Let \( Y_2 \) be the classical blow-up of \( Y_1 \) at \( q \). To conclude that \( Y'_2 = Y_2 \) and that \( \mathbb{C}[T_1, \ldots, T_{r_2}] / I_2 \) is isomorphic to the Cox ring of
Y_2, we need to check the following inequality
\begin{equation}
\dim I_2 + \langle T_{r_2} \rangle > \dim I_2 + \langle T_{r_2}, T_{r} \rangle,
\end{equation}
where \( T_r \) is the product of all the \( T_i \)'s, for \( 1 \leq i \leq r_1 \), such that \( V(T_i) \) does not vanish identically at \( p^{-1}(q) \).

5.1. Degree one and two. In this subsection, we provide a presentation for the Cox rings of the del Pezzo elliptic varieties of degree at most two with finite Mordell-Weil group. Our main result is the following.

**Theorem 5.1.** Let \( \pi : X \to \mathbb{P}^{n-1} \) be a del Pezzo elliptic fibration of degree \( d \leq 2 \) having finite Mordell-Weil group. Then the Cox ring of \( X \) and its grading matrix are listed in the following table:

| Type | Cox ring | Grading matrix |
|------|----------|----------------|
| \( X_1 \) | \( \mathbb{C}[T_1, \ldots, T_{n+2}, S] / (T_1^2 - T_2^2 f_1 T_1 S + f_2 S^2) \) | \[
\begin{bmatrix}
3 & 2 & 1 & \cdots & 1 & 0 \\
0 & 0 & -1 & \cdots & -1 & 1
\end{bmatrix}
\]
| \( \hat{f}_1 := f_1(T_1, T_2, T_3 S, \ldots, T_{n+2} S) \) |
| \( X_{SS} \) | \( \mathbb{C}[T_1, \ldots, T_{n+2}, S_1, S_2] / (T_{n+2}^2 S_1^2 - T_{n+3} S_1 + 2 \cdot T_{n+2} T_{n+3} - 5) \) | \[
\begin{bmatrix}
1 & \cdots & 1 & 1 & 2 & 2 & 0 & 0 \\
-1 & \cdots & -1 & 0 & -4 & 0 & 1 & 0 \\
-1 & \cdots & -1 & 0 & 0 & -4 & 0 & 1
\end{bmatrix}
\]
| \( \hat{h} := h(T_1 S_1 S_2, \ldots, T_n S_1 S_2, T_{n+1}) \) |
| \( \hat{g} := g(T_1, T_n) \) |
| \( X_2 \) | \( \mathbb{C}[T_1, \ldots, T_{n+2}, S_1, S_2] / (T_1^2 - S_1^2 f_2 T_2 T_{n+1}) \) | \[
\begin{bmatrix}
1 & \cdots & 1 & 1 & 2 & 0 & 0 \\
-1 & \cdots & -1 & -2 & 0 & -1 & 1 & 0 \\
-1 & \cdots & -1 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\]
| \( \hat{f} := f_2(T_1 S_1 S_2^2, \ldots, T_{n-1} S_1 S_2 S_3^2, S_1^2 S_2^2, T_{n+1}) / S_1^2 S_2^2 \) |

**Proof.** In order to prove the case \( X_1 \), let \( Y_1 \) be the del Pezzo variety given by the polynomial (2.1) and let \( q \in Y_1 \) be the point of coordinates \((1, 1, 0, \ldots, 0)\). The ring \( R_1 \) equals \( \mathbb{C}[T_1, \ldots, T_{n+2}] / I_1 \), where \( I_1 \) is the principal ideal generated by the polynomial (2.1). We take \( I, J \subseteq R_1 \) as before and we choose the following homogenous elements \( f_1, \ldots, f_n \):

\[
T_3, \ldots, T_{n+2} \in I
\]
as in (5.1), i.e. all of them have \( d_i = 1 \). Observe that the saturated ideal (5.2) is

\[
I_2 = \langle T_{n+2+i} T_{2n+5} - f_i : 1 \leq i \leq n \rangle + I_1
\]
since, after applying the substitution \( T_{2+i} = T_{n+2+i} T_{2n+5} \) for each \( i = 1, \ldots, n + 2 \), the resulting polynomial \( T_1^2 - T_2^2 + T_2 f_1 T_{2n+5} + f_2^2 T_{2n+5} \) is not divisible by \( T_{2n+5} \). Finally, according to (5.3), we need to check that

\[
\dim I_2 + \langle T_{2n+5} \rangle > \dim I_2 + \langle T_{2n+5}, T_1 T_2 \rangle,
\]
and this is easily checked to hold, being $I_1$ a principal ideal. We conclude that the ring $\mathbb{C}[T_1, \ldots, T_{2n+5}] / I_2$ is isomorphic to the the Cox ring of the blow-up $X_1$ of $Y$ at $q$. After eliminating the fake linear relations and renaming the variables, we get the claimed presentation for the Cox ring.

We now prove the case $X_{SS}$. Let $Y_1$ be the del Pezzo variety given by the polynomial (2.3) and let $q \in Y_1$ be the point of coordinates $(0, \ldots, 0, 1, 1)$. The ring $R_1$ equals $\mathbb{C}[T_1, \ldots, T_{2n+2}] / I_1$, where $I_1$ is the principal ideal generated by the polynomial (2.3). We take $I, J \subseteq R_1$ as before and choose the following homogenous elements $f_1, \ldots, f_{n+1}$:

$$T_1, \ldots, T_n \in I, \quad T_{n+2} - h \in (I^4 : J^\infty)$$

as in (5.1), that is the first $n$ sections have $d_i = 1$, while $d_{n+1} = 4$. Observe that the ideal in (5.2) is

$$I_2 = \langle T_{n+2+i}^{d_i} - f_i : 1 \leq i \leq n + 1 \rangle + \langle T_{2n+3}^{2n+4} T_{2n+4}^4 + 2T_{2n+3}h' - g' \rangle,$$

where $h' = h(T_{n+3}T_{2n+4}, \ldots, T_{2n+2}T_{2n+4}, T_{n+1})$ and $g' = g(T_{n+3}, \ldots, T_{2n+2})$. According to (5.3), we can easily check that the following inequality holds:

$$\dim I_2 + \langle T_{2n+4} \rangle > \dim I_2 + \langle T_{2n+4}, T_{n+1}T_{n+2} \rangle.$$

Thus, after eliminating the fake linear relations from $I_2$ and renaming the variables, we can conclude that the Cox ring and the grading matrix of the blow-up $Y_2$ of $Y_1$ at $q$ are the following

$$R_2 = \frac{\mathbb{C}[T_1, \ldots, T_{n+2}, S]}{(T_{2n+2}^4 + 2h''T_{n+2} - g''')}$$

where $h'' = h(T_1S, \ldots, T_nS, T_{n+1})$ and $g'' = g(T_1, \ldots, T_n)$. The irrelevant ideal is $J_2 = \langle T_1, \ldots, T_n, T_{n+2} \rangle \cap (T_{n+1}, S)$. We now repeat the procedure blowing-up $Y_2$ at the point $q_2'$ which lies over $q_2 = (0, \ldots, 0, 1, -1) \in Y$. Recall that there is a $\mathbb{C}^*$-equivariant embedding of total coordinate spaces

$$\mathcal{Y}_1 \to \mathcal{Y}_2 \quad (T_1, \ldots, T_{n+2}) \to (T_1, \ldots, T_{n+1}, T_{n+2} - h, 1)$$

which induces the birational map $Y_1 \dasharrow Y_2$. The image of $q_2$ is the point of homogenous coordinates $q_2' = (0, \ldots, 0, 1, -2, 1)$. We choose the following homogenous elements $f_1, \ldots, f_{n+2}$:

$$T_1, \ldots, T_n, 2T_{n+1}^2 + T_{n+2}S^4 \in I, \quad T_{n+2}S^4 + 2h'' \in (I^4 : J^\infty)$$

as in (5.1), that is the first $n + 1$ sections have $d_i = 1$, while $d_{n+2} = 4$. The ideal in (5.2) is

$$I_3 = \langle T_{n+3+i}^{d_i} + f_i : 1 \leq i \leq n + 2 \rangle + \langle T_{2n+3}^2 S^4 + 2T_{2n+2}h''' - T_{2n+3}g''' \rangle,$$

where $h''' = h(T_{n+4}T_{2n+6}S, \ldots, T_{2n+3}T_{2n+6}S, T_{n+1})$ and $g''' = g(T_{n+4}, \ldots, T_{2n+3})$.

After eliminating the fake linear relations from the above ideal and renaming the variables, we get the statement for the case $X_{SS}$.

Finally, let us prove the case $X_2$. Let $Y_1$ be the del Pezzo variety given by the polynomial (2.4) and let $q \in Y_1$ be the point of coordinates $(0, \ldots, 0, 1, 0)$. The ring $R_1$ equals $\mathbb{C}[T_1, \ldots, T_{n+2}] / I_1$, where $I_1$ is the principal ideal generated by the polynomial (2.4). We take $I, J \subseteq R_1$ as before and choose the following homogenous elements $f_1, \ldots, f_{n+1}$:

$$T_1, \ldots, T_{n-1}, T_{n+2} \in I, \quad T_n \in (I^2 : J^\infty)$$
as in (5.1), that is the first $n$ sections have $d_i = 1$, while $d_{n+1} = 2$. Observe that the ideal in (5.2) is

$$I_2 = (T_{n+2+i}T_{2n+4}^d - f_i : 1 \leq i \leq n + 1) + (T_{2n+2}^2 - f' - T_{2n+3}T_{n+1}^3)$$

with $f' = T_{2n+4}^{-2}f(T_{n+3}T_{2n+4}, \ldots, T_{2n+1}T_{2n+4}, T_nT_{2n+4}, T_{n+1})$. According to (5.3) it can be easily checked that

$$\dim I_2 + \langle T_{2n+4} \rangle > \dim I_2 + \langle T_{2n+4}, T_{n+1} \rangle.$$

Thus, after eliminating the fake linear relations from $I_2$ and renaming the variables, we conclude that the Cox ring and the grading matrix of the blow-up $Y_2$ of $Y_1$ at $q$ are the following

$$R_2 = \frac{\mathbb{C}[T_1, \ldots, T_{n+3}]}{\langle T_{n+2}^2 - f'' - T_nT_{n+1}^3 \rangle} \quad \begin{bmatrix} 1 & \ldots & 1 & 1 & 1 & 2 & 0 \\ -1 & \ldots & -1 & -2 & 0 & -1 & 1 \end{bmatrix}$$

with $f'' = T_{n+3}^2f(T_1T_{n+3}, \ldots, T_{n+1}T_{n+3}, T_nT_{n+3}, T_{n+1})$. The irrelevant ideal of $R_2$ is $J_2 = (T_1, \ldots, T_{n+2}) \cap \langle T_n, S \rangle$. Now repeat the procedure by blowing up $Y_2$ at the point $q_2 = (0, \ldots, 0, 1, 1, 1, 0)$ which is the invariant point with respect to the lifted involution $(T_1, \ldots, T_{n+3}) \mapsto (T_1, \ldots, T_{n+1}, -T_{n+2}, T_{n+3})$, and it corresponds to the generator of the kernel of the differential $d\varphi_q$. We choose the following homogenous elements $f_1, \ldots, f_n$:

$$T_1, \ldots, T_{n-1}, T_{n+3} \in I,$$

as in (5.1), i.e. $d_i = 1$ for all the sections. The ideal in (5.2) is

$$J_3 = (T_{n+3+i}T_{2n+3}^d - f_i : 1 \leq i \leq n) + (T_{n+2}^2 - T_{n+3}^2\tilde{f} - T_nT_{n+1}^3)$$

where $\tilde{f} = T_{2n+3}^{-2}f''(T_{n+4}T_{2n+3}S, \ldots, T_{2n+3}T_{2n+3}S, T_{n+4}, T_{n+1})$. After eliminating the fake linear relations from the above ideal and renaming the variables, we obtain the statement for the case $X_2$. 

5.2. Degree four. In this last subsection, we first provide the following presentation for the Cox rings of the blowing-up of a del Pezzo variety of degree four at a point.

**Lemma 5.2.** Let $Y$ be a smooth complete intersection of two quadrics of $\mathbb{P}^{n+2}$. After possibly applying a linear change of coordinates, the ideal of $Y$ is generated by $x_2x_3 - x_1x_2 + f(x_4, \ldots, x_{n+3})$ and $x_2x_3 - x_1x_3 + g(x_4, \ldots, x_{n+3})$. The blow-up $\tilde{Y}$ of $Y$ at the point $q = (1, 0, \ldots, 0) \in Y$ has the following Cox ring and grading matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 0 \\ 0 & -2 & -2 & -1 & \ldots & -1 & 1 \end{bmatrix}$$

respectively, where $f = f(T_1, \ldots, T_{n+3})$ and $g = g(T_4, \ldots, T_{n+3})$.

*Proof.* After applying a linear change of coordinates, we can assume that $q$ is a point of $Y = Q \cap Q'$, where $Q$ is singular at $(1, 1, 0, 0, \ldots, 0)$ and $Q'$ is singular at $(1, 0, 1, 0, \ldots, 0)$, and that the tangent hyperplanes to $Q$ and $Q'$ at $q$ are $V(x_2)$ and $V(x_3)$, respectively. This proves the first claim.

To prove the second statement, we take $R_1$ to be $\mathbb{C}[T_1, \ldots, T_{n+3}]/I_1$, where $I_1$ is the ideal of $Y$, and we apply [4, Algorithm 5.4]. We take $I, J \subseteq R_1$ as before and choose the following homogenous elements $f_1, \ldots, f_{n+2}$:

$$T_4, \ldots, T_{n+3} \in I, \quad T_2, T_3 \in (I^2 : J^\infty)$$
as in (5.1), that is the first $n$ sections have $d_i = 1$, while $d_{n+1} = d_{n+2} = 2$. The ideal in (5.2) is

\[
I_2 = \langle T_{n+3}^{d_i} T_{2n+6}^{d_i} - f_i : 1 \leq i \leq n+2 \rangle + \langle T_{2n+4} T_{2n+5} T_{2n+6}^2 - T_{1} T_{2n+4} + \tilde{f}, T_{2n+5} T_{2n+6}^2 - T_{1} T_{2n+5} + \tilde{g} \rangle,
\]

where $\tilde{f} := f(T_{n+4}, \ldots, T_{2n+3})$ and $\tilde{g} := g(T_{n+4}, \ldots, T_{2n+3})$. According to (5.3) it can be easily checked that

\[
\dim I_2 + \langle T_{2n+6} \rangle > \dim I_2 + \langle T_{2n+6}, T_1 \rangle.
\]

After eliminating the fake linear relations from $I_2$ and renaming the variables, we get the second statement.

\[
\square
\]

We conclude with two examples of the computation of the Cox rings for the del Pezzo elliptic varieties of degree four and dimension three. We only report the final results, since the computations have been done with the same procedure as before.

**Case $X_{43}$:**

**Equations:**

\[
\begin{align*}
x_1^2 & = x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2, \\
x_1^2 & = x_2^2 + 2x_4^2 + x_5^2 + x_6^2.
\end{align*}
\]

**Cox ring $C[T_1, \ldots, T_{13}]/I$, where $I$ is generated by:**

\[
\begin{align*}
T_1^2 & = T_2^2 + 2T_3 T_6 - T_6 T_7, \\
T_2^2 & = 2T_3^2 - T_3 T_6 + T_6 T_7, \\
T_4 T_6 - T_5 T_7 & + T_7 T_9, \\
T_4 T_7 - T_5 T_8 & + T_8 T_9, \\
T_8 T_9 T_{12}^2 & - T_5 T_6 T_7 T_8, \\
T_8 T_9 T_{12}^2 & - T_5 T_6 T_7 T_8, \\
T_8 T_9 T_{12}^2 & - T_5 T_6 T_7 T_8, \\
T_8 T_9 T_{12}^2 & - T_5 T_6 T_7 T_8, \\
T_8 T_9 T_{12}^2 & - T_5 T_6 T_7 T_8.
\end{align*}
\]

**Degree matrix:**

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -2 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & -2 & 0 & -2 & -2 & 0 & 0 & 0 & 1 \\
-1 & -1 & 0 & 0 & -2 & 0 & -2 & -2 & -2 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Case $X_{22}$:

Equations:
\[
\begin{align*}
    x_2^2 &= x_3^3 + x_4^4 + x_5^5 + x_6^6, \\
    x_1^2 &= x_3^3 + 2x_4^4 + 3x_5^5 + 3x_6^6
\end{align*}
\]

Cox ring $\mathbb{C}[T_1, \ldots, T_{10}] / I$, where $I$ is generated by:
\[
\begin{align*}
    27T_2^2 - 2T_2^2 T_3^2 + 11T_2^2 T_4^2 T_5^2 T_6^2 - sT_4 T_5 T_7^2 T_8^2 + sT_4 T_6 T_7^2 T_8^2 T_9^2 - 4T_5 T_6 T_7^2 T_9^2 - \\
    2T_2^2 + 3T_2^2 + 3T_4 T_5 T_7^2 T_8^2 + 4T_5 T_6
\end{align*}
\]

Degree matrix:
\[
\begin{pmatrix}
    1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
    -1 & -1 & -1 & -2 & -2 & 0 & 1 & 0 & 0 & 0 \\
    -1 & -1 & -1 & -2 & 0 & -2 & 0 & 0 & 1 & 0 \\
    0 & -1 & -1 & 0 & -2 & 0 & -1 & 1 & 0 & 0 \\
    0 & -1 & -1 & 0 & 0 & -2 & 0 & 0 & -1 & 1
\end{pmatrix}
\]

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