Abstract. We prove that there is a product on the Hochschild and cyclic chain complex of a homotopy Gerstenhaber algebra. By restricting to the special case of algebra of Hochschild cochains (the so called deformation complex), we obtain operations on cyclic homology of associative algebras.

1 Introduction.

The goal of this article is to relate recent developments in cyclic homology theory [3] and the theory of operads and homotopical algebra [6,8], and hence to provide a general framework to define and study operations in cyclic homology theory. The link here is the bar construction.

In [4], P. Deligne conjectured that the Hochschild cochain complex of an associative algebra, also called the deformation complex of the algebra, has a natural structure of an algebra over a singular chain operad of the little squares operad. This conjecture is now proved by M. Kontsevich [16]. It should be noted, however, that the results of the present paper in no way depends on the geometric form of this conjecture. In fact we do not use the conjecture in its original geometric form. A closely related statement recently proved by Gerstenhaber and Voronov [6] and Getzler and Jones [8] states that the deformation complex of an associative algebra has a natural structure of a homotopy Gerstenhaber algebra, also called homotopy $G$ algebra. This result completes Gerstenhaber’s earlier work in [5] in the sense that it reveals the full structure of higher homotopies in the deformation complex. In a sense, this result shows that there is a natural ”quantum group” structure on the bar construction of the deformation complex. It is this algebraic version of the conjecture that is most useful to construct the operations.

In fact, we found it more conceptual to go beyond the deformation complex for associative algebras and define certain operations on the Hochschild and cyclic complexes of homotopy $G$ algebras (Theorem 8 below). As an application, by specializing to the homotopy $G$ algebra structure of the deformation complex, we obtain operations similar to those constructed by Nest and Tsygan in their study of algebraic index theorems [12].

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In view of increasing importance of homotopy $G$ algebras and the theory of operads in general in mathematics and mathematical physics (see, for example, [6,8,11,15] and references therein), we hope that theorem 8, and specially its method of proof, which is non-computational and lends itself to generalizations to algebras over operads, will prove useful in applications of noncommutative geometry and cyclic homology.

This paper is organized as follows. In Sect. 2 we recall the notion of homotopy $G$ algebra and specially its formulation in terms of the bar construction from [6]. In Sect. 3 we define operations on Hochschild and cyclic complex of homotopy $G$ algebras. A central tool here is the notion of $X$ complex and its refinements due to Cuntz and Quillen [3]. By a result of Quillen [14], cyclic and Hochschild chain complexes appear as the $X$ complex of the bar construction. From this point of view operations on cyclic and Hochschild complex of homotopy $G$ algebras are predicted by Kunneth formulas for the $X$ complex of differential graded coalgebras. Sect. 4 is mainly devoted to deriving explicit formulas in the context of Connes’ $b, B$ bicomplex and also specializing to the case of deformation complex.

I am much obliged to Maxim Kontsevich for a very informative communication on Deligne’s conjecture.

2 Homotopy Gerstenhaber algebras.

This section is based on [6]. In an attempt to make the paper as self contained as possible, we have reproduced the proofs of the main statements. Let $V = \bigoplus_{i \in \mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded linear space. We use $|x|$ to denote the degree of a homogeneous element $x \in V$. A brace algebra structure on $V$ is given by a collection of linear homogeneous maps of degree zero, indexed by $n \geq 0$,

$$V \otimes V^\otimes n \to V$$

$$x \otimes x_1 \otimes \cdots \otimes x_n \mapsto x\{x_1, \cdots, x_n\},$$

such that, for all $m, n$, the following higher pre-Jacobi identities are satisfied:

$$x\{x_1, \cdots, x_m\}\{y_1, \cdots, y_n\} = \sum_{0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_m \leq n} (-1)^\epsilon x\{y_1, \cdots, y_{i_1}, x_1\{y_{i_1+1}, \cdots, y_{j_1}\}, \cdots, x_m\{y_{i_m+1}, \cdots, y_{j_m}\}, \cdots, y_n\},$$

where $x_i$ and $y_j$ are homogeneous elements and $\epsilon = \sum_{p=1}^{m}(|x_p| \sum_{q=1}^{i_q} |y_q|)$. The degree zero assumption simply means that $|x_1, \cdots, x_n| = |x| + \sum |x_i|$. We also assume that for $n = 0$ the resulting map $x \mapsto x\{\}$ : $V \to V$ is the identity.

For example, as a consequence of (1), one checks that the bracket

$$[x, y] := x\{y\} - (-1)^{|x||y|} y\{x\}$$

defines a graded Lie algebra structure on $V$. Indeed, putting $m = n = 1$ in (1), one obtains

$$x\{y\}\{z\} - x\{y\{z\}\} = x\{y, z\} + (-1)^{|y||z|} x\{z, y\},$$

which measures the failure of the operation $(x, y) \mapsto x\{y\}$ to be associative. From this the graded Jacobi identity easily follows.
A brace algebra structure on $V$ has a particularly simple interpretation in terms of the tensor coalgebra of $V[1]$. Let $V[1]$ be the desuspension of $V$ defined by $V[1]_n = V_{n+1}$. Let

$$T(V[1]) = \bigoplus_{n \geq 0} (V[1])^\otimes n$$

be the tensor coalgebra of $V[1]$ with its coproduct

$$\Delta : T(V[1]) \to T(V[1]) \otimes T(V[1])$$

$$\Delta(x_1, \cdots, x_n) = \sum_{i=0}^n (x_1, \cdots, x_i) \otimes (x_{i+1}, \cdots, x_n),$$

where we have denoted a tensor $x_1 \otimes \cdots \otimes x_n$ in $T(V[1])$ by $(x_1, \cdots, x_n)$.

Note that $T(V[1])$ is bigraded. Its horizontal grading is denoted by $\deg$ and defined by $\deg (x_1, \cdots, x_n) = n$, and its vertical grading, denoted $|d|$, is given by $|(x_1, \cdots, x_n)|_d = \sum |x_i| - n$. The total grading is hence given by $|(x_1, \cdots, x_n)| = \sum_i |x_i|$. In the definition of $T(V[1])$ we are implicitly assuming that we are working with the total complex which is $\mathbb{Z}$-graded.

A linear homogeneous (with respect to the total grading) map of total degree zero,

$$\cup : T(V[1]) \otimes T(V[1]) \to T(V[1]),$$

is called left increasing if $\deg(\alpha \cup \beta) \geq \deg(\alpha)$. We always assume that $\cup$ is counital.

**Lemma 1.** There is a natural $1-1$ correspondence between brace algebra structures on $V$ and left increasing bialgebra structures on $T(V[1])$.

**Proof.** Since $T(V[1])$ is the free coalgebra generated by the desuspension $V[1]$, we have a natural $1-1$ correspondence between coalgebra morphisms

$$\cup : T(V[1]) \otimes T(V[1]) \to T(V[1])$$

and linear maps

$$m : T(V[1]) \otimes T(V[1]) \to V[1].$$

Given $m$, the $n$-th component of $\cup$ is defined by

$$\cup_n = m^\otimes n \circ \tilde{\Delta}^{(n-1)},$$

where $\tilde{\Delta}^{(n)}$ denotes the $n-th$ iteration of the coproduct $\tilde{\Delta}$ of the coalgebra $T(V[1]) \otimes T(V[1])$. Conversely, given $\cup$, $m$ is just the degree one component of $\cup$.

It is clear that $\cup$ is left increasing iff, for $n \geq 2$, $m|V[1]|^{\otimes n} \otimes T(V[1]) = 0$. In this case, let us denote the map $m : V[1] \otimes T(V[1]) \to V[1]$ by $x\{x_1, \cdots, x_n\}$. Then, using (2), we get an explicit formula for the coalgebra map $\cup : T(V[1]) \otimes T(V[1]) \to T(V[1])$. It is given by

$$\cup(x_1, \cdots, x_m) \cup (y_1, \cdots, y_n) = \sum_{0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_n \leq n} (-1)^{\varepsilon} \cdot x_{i_1} \cdots x_{i_m} \partial y_{j_1} \cdots y_{j_n}$$
\[(y_1, \ldots, y_i, x_1\{y_{i+1}, \ldots, y_{j_1}\}, \ldots, x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, \ldots, y_n),\]

where \(\varepsilon\) is the same as in (1).

It remains to check that \(\cup\) is associative iff the braces satisfy the higher pre-Jacobi identities (1). Assume \(\cup\) is associative. Then in particular we have \((\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)\), where \(\alpha = x, \beta = (x_1, \ldots, x_m)\) and \(\gamma = (y_1, \ldots, y_n)\). Taking the degree one component of both sides, one obtains (1).

Conversely, assume the braces satisfy the pre-Jacobi identities. It is easy to see that both maps \(\cup \otimes 1\) and \(1 \otimes \cup : T(V[1]) \otimes T(V[1]) \otimes T(V[1]) \rightarrow T(V[1])\) are coalgebra maps. By the universal property of \(T(V[1])\), the two maps are the same provided the degree one components of them coincide. One then checks that this is equivalent to the brace identity (1). The theorem is proved.

In the rest of this paper we only consider left increasing multiplications on \(T(V[1])\). Here is an example of a brace algebra. This example is due to Getzler [7]. Let \(A\) be a linear space and let \(V_n = \text{Hom}(A^\otimes n, A)\). One defines a brace algebra structure on \(V[1]\) by setting

\[
x\{x_1, \ldots, x_m\}(a_1, \ldots, a_n) = \sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} (-1)^\varepsilon \\
x(a_1, \ldots, x_1(a_{i_1+1}, \ldots, a_{i_1+d_1}), \ldots, x_m(a_{i_m+1}, \ldots, a_{i_m+d_m}), \ldots, a_n),
\]

where \(d_i = |x_i| + 1, n = 1 + |x| + \sum |x_i|\) and \(\varepsilon = \sum_{p=1}^m |x_p|i_p\). Checking (1) is straightforward.

A homotopy \(G\) algebra (\(G\) stands for Gerstenhaber) is a differential graded (DG) associative algebra equipped with a system of "higher homotopies" so that its cohomology is a graded poisson algebra. More precisely, Let \((V, \delta)\) be a DG algebra where we assume the differential has degree +1. A homotopy \(G\) algebra structure on \(V\) is given by a brace algebra structure on the desuspension \(V[1]\) of \(V\) such that the following axioms are satisfied:

\[
(x_1x_2\{y_1, \ldots, y_n\}) = \sum_{k=0}^n (-1)^\varepsilon x_1\{y_1, \ldots, y_k\}x_2\{y_{k+1}, \ldots, y_n\},
\]

where \(\varepsilon = (|x_2| - 1)(|y_1| + \cdots + |y_k| - k)\), and

\[
\delta(x\{x_1, \ldots, x_{n+1}\}) - \delta x\{x_1, \ldots, x_{n+1}\} - \\
(-1)^{|x|-1} \sum_{i=1}^{n+1} (-1)^\varepsilon x\{x_1, \ldots, \delta x_i, \ldots, x_{n+1}\} = \\
-(-1)^{(|x|-1)(|x_1|-1)} x_1.x\{x_2, \ldots, x_{n+1}\} + \\
(-1)^{|x|-1} \sum_{i=1}^n (-1)^{i+n} x\{x_1, \ldots, x_ix_{i+1}, \ldots, x_{n+1}\} \\
-x\{x_1, \ldots, x_n\}.x_{n+1}
\]
where \( \epsilon = \sum_{k=1}^{i-1} |x_k| \).

A large class of homotopy \( G \) algebras are constructed as follows [6]. Let \( V \) be a brace algebra and \( m \in V_1 \) a degree one element such that \( m\{m\} = 0 \). One defines a DG algebra structure on \( V[-1] \) by defining a differential and a product by

\[
\delta x = (-1)^{|x|} [m, x] \quad \quad xy = m\{x, y\}
\]

Using the brace relations (1), one then checks that the axioms of a homotopy \( G \) algebra are satisfied.

The axioms of a homotopy \( G \) algebra structure on \( V \) can be conceptually encoded in terms of the bar construction \( BV \). Let us describe this correspondence. We first need a definition.

Let \( (V, \delta) \) be a DG algebra where we assume the differential has degree +1. Recall that the bar construction of \( V \), denoted \( BV \), is a differential graded coalgebra whose underlying coalgebra is the tensor coalgebra \( T(V[2]) \) and its differential is the total differential \( b' + \delta \).

The individual differentials \( b', \delta : BV \to BV \) are defined by

\[
b'(x_1, \ldots, x_n) = (-1)^n \sum_{i=1}^{n-1} (-1)^{i-1} (x_1, \ldots, x_i x_{i+1}, \ldots, x_n),
\]

\[
\delta(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{|x_1| + \cdots + |x_{i-1}|} (x_1, \ldots, \delta x_i, \ldots, x_n).
\]

Note that both \( b' \) and \( \delta \) have total degree +1. Also note that the total degree of \( \alpha = (x_1, \ldots, x_n) \in BV \) is given by \( |\alpha| = \sum_i |x_i| - n \).

Next recall that a DG bialgebra is by definition a bialgebra object in the abelian tensor category of DG linear spaces (cochain complexes). In particular the differential of a DG-bialgebra is simultaneously a derivation and a coderivation.

**Lemma 2.** Let \( V \) be a DG algebra. Then there is a natural 1-1 correspondence between homotopy \( G \) algebra structures on \( V \) and DG bialgebra structures on the bar construction \( BV \).

**Proof.** By the above lemma, we have a natural 1-1 correspondence between brace algebra structures on \( V[1] \) and bialgebra structures on \( BV \). So all that we need to prove is that the axioms of homotopy \( G \) algebras (5, 6) are equivalent to differential \( b' + \delta \) being a derivation of \( BV \). That is for all \( \alpha, \beta \in BV \),

\[
(b' + \delta)(\alpha \cup \beta) = (b' + \delta)\alpha \cup \beta + (-1)^{|\alpha|} \alpha \cup (b' + \delta)\beta
\]

Now, since both \( b' \) and \( d \) are coderivations of \( BV \) and \( \cup \) is a coalgebra map, it follows that (8) holds if and only if the degree one components of both sides coincide. Note that the only possible contributions to degree one components are from the following two choices: \( \alpha = x, \beta = (x_1, \ldots, x_n) \) and \( \alpha = (x_1, x_2), \beta = (y_1, \ldots, y_m) \). Computing the first order terms in the expansions, we find that (8) is equivalent to (5, 6). The lemma is proved.
Given a homotopy $G$ algebra $V$, let $H(V)$ denote the cohomology of the complex $(V, \delta)$. The product and the Lie bracket in $V$, being compatible with the differential $\delta$, descend to $H(V)$ and define an associative product and a Lie algebra structure on $H(V)$. Moreover, the homotopy formulas in (5) and (6) can be used to show that the associative product in $V$ is, up to homotopy, graded commutative and the Lie bracket is, again up to homotopy, a derivation with respect to the associative product. It thus follows that $H(V)$ is a graded poisson algebra, also known as a Gerstenhaber algebra ($G$ algebra). This simply means that the product in cohomology is graded commutative and the Lie bracket is a derivation with respect to the product.

Examples of graded poisson algebras and homotopy $G$ algebras abound in algebraic topology, Geometry and mathematical physics. By a classical result of F. Cohen the cohomology groups of configuration spaces is a universal model for graded poisson algebras in the sense that any graded poisson algebra is an algebra over the latter as an operad. concrete examples of $G$ algebras include the semi-infinte cohomology of string theory [11], the algebra of polyvector fields on a manifold, Koszul complex of Lie algebras and more generally the deformation cohomology of any associative algebra, to be discussed in more detail in the next paragraph. Examples of homotopy $G$ algebra structures that are just emerging include the homotopy $G$ algebra structure of topological field theory [11], the homotopy $G$ algebra structure on singular cochains on a topological space [6] and finally the deformation complex of associative algebras which we describe next. This structure also appears in the recent work of M. Kontsevich on deformation quantization of Poisson manifolds [15].

Let $A$ be an associative algebra and let $C(A, A)$ denote the deformation complex of $A$. This is the standard complex that calculates the Hochschild cohomology $H^\bullet(A, A)$. We have $C^n(A, A) = Hom(A \otimes A^n, A)$. It thus follows from (4) that there is a brace algebra structure defined on $C(A, A)$[1]. Let $m : A \otimes A \rightarrow A$ be the multiplication of $A$. One has $m\{m\} = 0$, which is equivalent to associativity of $m$. It is easy to check that the differential and the product induced on $C(A, A)$ by (7) coincide, respectively, with the Hochschild coboundary and the cup product on $C(A, A)$. One thus obtains a homotopy $G$ algebra structure on $C(A, A)$, first discovered by Gerstenhaber and Voronov in [6]. As we will see in the next section, this homotopy $G$ algebra structure is at the heart of operations on Hochschild and cyclic homology.

3 Operations on homotopy $G$ algebras.

Let $C$ and $D$ be $DG$ coalgebras and let $A$ be an algebra. Our goal in this section is to show that any morphism of $DG$ coalgebras $C \otimes D \rightarrow BA$ induces a natural morphism of supercomplexes

$$\hat{X}(C) \otimes \hat{X}(D) \rightarrow \hat{X}(BA),$$

where $X$ is the $X$ complex functor of Cuntz and Quillen. We will then apply this result to the structure map of a homotopy $G$ algebra to construct operations on the cyclic and Hochschild homology of homotopy $G$ algebras.

Note that, in general, there is no natural map $X(C) \otimes X(D) \rightarrow X(C \otimes D)$; otherwise defining (9) would be a trivial matter. This is simply because $X$ only captures homological
information up to dimension one. Instead, we obtain (9) as a composition
\[ \hat{X}(C) \otimes \hat{X}(D) \longrightarrow \hat{X}^2(C \otimes D) \longrightarrow \hat{X}^2(BA) \longrightarrow \hat{X}(BA), \]
where \( X^2 \) is a certain refinement of \( X \) to capture degree 2 homology classes. Despite the fact that there is no natural transformation \( X^2 \longrightarrow X \), we can however make use of the fact that the underlying coalgebra of \( BA \) is free and show that \( \hat{X}(BA) \) is a deformation retract of \( \hat{X}^2(BA) \). This gives the last map in the above sequence. The first map is simply the \( DG \) coalgebra analogue of a map constructed by M. Puschnigg in his study of Kunneth formulas in cyclic homology [13].

We need to adopt some basic definitions and constructions from [9] to our \( DG \) coalgebra set up. Let \( C \) be a \( DG \) coalgebra and let \((\Omega C, d)\) denote the \( DG \) coalgebra of universal codifferential forms over \( C \). Let \( \eta : C \longrightarrow k \) be the counit of \( C \). We have \( \Omega^n C = C \otimes C^{\otimes n} \), where \( \bar{C} = \text{Ker}\eta \). Let \( b : \Omega^\bullet C \longrightarrow \Omega^{\bullet +1} C \) be the analogue of the Hochschild boundary operator and let \( N \) be the number operator which multiplies a differential form by its degree. Let \( \Omega^{\text{norm}} C = \ker \{(b + dN)^2 : \Omega C \longrightarrow \Omega C\} \).

Equipped with the differential \( b + dN \) and with its natural \( \mathbb{Z}/2 \) grading, \((\Omega^{\text{norm}} C, b + dN)\) can be regarded as a supercomplex. There is a decreasing filtration \( \{F^n \Omega^{\text{norm}} C\}_{n \geq 2} \) on \( \Omega^{\text{norm}} C \), where \( F^n \) consists of forms of degree at least \( n \). The successive quotient complexes \( \Omega^{\text{norm}} C / F^n \) approximate the normalized cyclic bicomplex for \( DG \) coalgebras. We need only the first two quotients, denoted by \( X(C) \) and \( X^2(C) \). These are the supercomplexes
\[
\begin{align*}
X(C) : & \quad C \xleftarrow{b_d} \Omega^1 C_2, \\
X^2(C) : & \quad C \bigoplus \Omega^2 C_2 \xleftarrow{b+2d} \dot{\Omega}^1 C,
\end{align*}
\]
where \( \dot{\Omega}^1 C \) denotes the cocommutator subspace and \( \dot{\Omega}^1 C = \Omega^{\text{norm}, 1} C \). Note that \( \Omega^1 C_2 \subset \dot{\Omega}^1 C \).

We use \( \partial_1 \) (resp. \( \partial_2 \)) to denote the horizontal (resp. vertical) differentials in \( X(C) \) and \( X^2(C) \). We are mostly interested in the total complexes of these bicomplexes which we denote by \( \hat{X}(C) \) and \( \hat{X}^2(C) \). We express an even, or odd, element of \( \hat{X}(C) \) as \( \omega_0 + \omega_1 \), where \( \omega_0 \in C \) and \( \omega_1 \in \Omega^1 C_2 \). Similarly we write \( \omega_0 + \omega_2 + \omega_1 \) to denote an even, or odd, element of \( \hat{X}^2(C) \). Note that we have a natural morphism of supercomplexes
\[
I : \hat{X}(C) \longrightarrow \hat{X}^2(C),
\]
obtained from the inclusions \( C \longrightarrow C \oplus \Omega^2 C_2 \) and \( \Omega^1 C_2 \longrightarrow \dot{\Omega}^1 C \).

In general, there is no natural map \( \hat{X}^2(C) \longrightarrow \hat{X}(C) \). However, we would like to show that if \( C = BA \) is the bar construction, then \( I \) is a homotopy equivalence and find an explicit homotopy inverse \( R : \hat{X}^2(C) \longrightarrow \hat{X}(C) \). The easiest way to find \( R \) is to apply homological perturbation theory. Indeed a simple version of the so called perturbation lemma which we recall now is enough for our purpose.
Recall that a (super)complex \( (L, \partial_1) \) is a special deformation retract of a (super)complex \( (M, \partial_1) \) if there are chain maps
\[
L \xrightarrow{i} M \xrightarrow{r} L
\]
and a homotopy \( h : M \rightarrow M \), of odd degree, such that \( ri = 1_L \), \( ir = 1_M + \partial_1 h + h \partial_1 \) and \( hi = 0 \). In particular \( i \) is a homotopy equivalence and \( r \) is a homotopy inverse to \( i \). Let us perturb the differentials to \( \partial_1 + \partial_2 \) and assume that \( \partial_2 i = i \partial_2 \). It is natural to ask if \( (L, \partial_1 + \partial_2) \) remains a deformation retract of \( (M, \partial_1 + \partial_2) \). It is not difficult to show that this is indeed the case, provided the operator \( K = \sum_{n \geq 0} (\partial_2 h)^n \) can be rigorously defined. In this case one shows that the chain maps
\[
L \xrightarrow{I} M \xrightarrow{R} L
\]
and the homotopy \( H : M \rightarrow M \) defined by \( I = i, R = rK \) and \( H = hK \), provide a special deformation retract of \( (M, \partial_1 + \partial_2) \) to \( (L, \partial_1 + \partial_2) \). In our applications \( K \) will be a finite sum.

Let \( C = BA \) be the bar construction of an algebra \( A \) with its counit \( \eta : BA \rightarrow k \) and let \( BA = \ker \eta = \bigoplus_{n \geq 1} A^\otimes n \). We have \( \Omega^1 BA = BA \otimes BA \simeq BA \otimes A \otimes BA \), \( \Omega^2 BA \simeq A \otimes BA \), and \( \Omega^3 BA = BA \otimes BA \otimes BA \). We fix a left inverse \( \theta : \Omega^1 BA \rightarrow \Omega^1 BA_2 \) for the inclusion \( \Omega^1 BA_2 \rightarrow \Omega^1 BA \), defined by
\[
\theta(\alpha \otimes a \otimes \beta) = \eta(\alpha)a \otimes \beta.
\]

¿From [9], one knows that one can use connections to construct homotopy operators with good algebraic properties for the Hochschild and cyclic complex of DG coalgebras. Let us define an operator \( \nabla : \Omega^2 BA \rightarrow \Omega^1 BA \), which is supported on \( BA \otimes BA \otimes A \), by the formula
\[
\nabla(\beta \otimes \alpha \otimes a) = \alpha \otimes a \otimes \beta,
\]
where \( \beta \in BA, \alpha \in BA \), and \( a \in A \). Define an odd operator \( h' : \hat{X}^2(BA) \rightarrow \hat{X}^2(BA) \) via the formula
\[
h'(\omega_0 + \omega_2 + \omega_1) = \nabla \omega_2.
\]
Also define even operators \( r' : \hat{X}^2(BA) \rightarrow \hat{X}(BA) \) and \( i' : \hat{X}(BA) \rightarrow \hat{X}^2(BA) \) via the formulas
\[
r'(\omega_0 + \omega_2 + \omega_1) = \omega_0 + \theta \omega_1,
\]
\[
i'(\omega_0 + \omega_2 + \omega_1) = \omega_0 + \omega_1,
\]
and \( i' = I \).

**Lemma 3.** \((r', h', i')\) is a special deformation retract of \((\hat{X}^2(BA), b)\) to \((\hat{X}(BA), b)\).

**Proof.** The relations \( r'i' = 1 \) and \( h'i' = 0 \) are easy to verify. The relation \( i'r' = 1 + bh' + h'b \) amounts to \( \theta \omega_1 = \omega_2 + \omega_1 + b\nabla \omega_2 + \nabla b \omega_1 \). This is equivalent to showing that, for all \( \omega_1 \in \Omega^1 BA \) and \( \omega_2 \in \Omega^2 BA_2 \),
\[
\omega_1 + \nabla b \omega_1 = \theta \omega_1
\]
\[
\omega_2 + b \nabla \omega_2 = 0
\]
While it is possible to prove these relations by a direct computation, it is perhaps more instructive to prove the corresponding dual statements for the tensor coalgebra \( T_A \). In this case the connection \( \nabla : \Omega^1 T_A \to \Omega^2 T_A \) is given by \( \nabla(\alpha \otimes a \otimes \beta) = \beta d\alpha da \). To prove the first relation let \( \omega_1 = \alpha \otimes a \otimes \beta \). We have

\[
b \nabla \omega_1 = b(\beta d\alpha da) = -b(da\beta da) = [da\beta, d\alpha] = da\beta a - ada\beta.
\]

And hence

\[
b \nabla \omega_1 + \omega_1 = da\beta a = \theta \omega_1.
\]

To prove the second relation, let \( \omega_2 = a_0 da_1 da_2 \). We have

\[
\nabla b \omega_2 = -\nabla[a_0 da_1, a_2] = -\nabla(a_0 da_1 a_2 - a_2 a_0 da_1) = -a_2 da_0 da_1 + d(a_2 a_0) da_1 = da_2 a_0 da_1 = -a_0 da_1 da_2 = -\omega_2.
\]

The lemma is proved.

To pass from the \( b \)-complex to \( \partial_1 \)-complex, we compute the operator \( k : \hat{X}^2 \to \hat{X}^2 \). It is given by

\[
k(\omega_0 + \omega_2 + \omega_1) = (\omega_0 + d\nabla \omega_2) + \omega_2 + \omega_1.
\]

Invoking the perturbation lemma, let us now define the operators \( h, r \) and \( i \) by the formulas:

\[
h(\omega_0 + \omega_2 + \omega_1) = \nabla \omega_2
\]

\[
r(\omega_0 + \omega_2 + \omega_1) = (\omega_0 + d\nabla \omega_2) + \theta \omega_1,
\]

and \( i = I \).

**Lemma 4.** \( (r, h, i) \) is a special deformation retract of \((\hat{X}^2(BA), \partial_1)\) to \((\hat{X}(BA), \partial_1)\).

We use the perturbation lemma once again to pass from the \( \partial_1 \)-complex to \( \partial_1 + \partial_2 \)-complex. The operator \( K \) is now given by

\[
K(\omega_0 + \omega_2 + \omega_1) = \omega_0 + \omega_2 + (\omega_1 + \partial_2 \nabla \omega_2).
\]

Let us define the operators \( R \) and \( H \) by

\[
R(\omega_0 + \omega_2 + \omega_1) = (\omega_0 + d\nabla \omega_2) + \theta (\omega_1 + \partial_2 \nabla \omega_2)
\]

\[
H(\omega_0 + \omega_2 + \omega_1) = \nabla \omega_2.
\]
Proposition 5. \((R, H, I)\) is a special deformation retract of \(\hat{X}^2(BA)\) to \(\hat{X}(BA)\).

Although we won’t need it in this paper, we note that the above proposition and its proof remain valid in the more general case where \(A\) is a DG algebra.

In his study of Kunneth formulas in cyclic homology \([13]\), M. Puschnigg constructed a natural map \(X^2(A \otimes B) \longrightarrow X(A) \otimes X(B)\), where \(A\) and \(B\) are algebras and the tensor product of supercomplexes is understood in the right hand side. This map lifts Connes’ external product in cyclic homology \([1]\) to the level of chains in the \(X\) complex. It is given by

\[
a_0b_0 \mapsto a_0 \otimes b_0 \\
a_0b_0d(a_1b_1) \mapsto \frac{1}{2}a_0da_1 \otimes [b_0, b_1]_+ + \frac{1}{2}[a_0, a_1]_+ \otimes b_0db_1 \\
a_0b_0d(a_1b_1)d(a_2b_2) \mapsto \frac{1}{2}a_0da_1a_2 \otimes b_0b_1db_2 - \frac{1}{2}a_0a_1a_2 \otimes b_0db_1b_2 \\
= \frac{1}{2}a_0d(a_1a_2) \otimes b_0b_1db_2 - \frac{1}{2}a_0a_1a_2 \otimes b_0db(b_1b_2),
\]

where \([a, b]_+ = ab + ba\) and, to keep the notation simple, we have supressed the tensor product sign on the left hand side. This map is functorial and can be dualized to a DG coalgebra context to define a morphism of supercomplexes

\[
p : \hat{X}(C) \otimes \hat{X}(D) \longrightarrow \hat{X}^2(C \otimes D),
\]

where \(C\) and \(D\) are DG coalgebras. Let \(p_{i,j} = p|X_i(C) \otimes X_j(D)\). We have

\[
p_{0,0} = id \\
p_{0,1} = \frac{1}{2}R_{2,3} \circ (1 \otimes (\Delta + R_{1,2}\Delta)) \\
p_{1,0} = \frac{1}{2}R_{2,3} \circ ((\Delta + R_{1,2}\Delta) \otimes 1) \\
p_{1,1} = \frac{1}{2}R \circ (1 \otimes \Delta \otimes \Delta \otimes 1 - \Delta \otimes 1 \otimes 1 \otimes \Delta).
\]

In the above formulas \(R_{i,j}\) denotes the signed exchange between \(i\) and \(j\) factors in a tensor product and \(R = R_{4,5}R_{3,5}R_{2,3}\). This observation, coupled with the above proposition, proves the following

Theorem 6. Let \(C\) and \(D\) be DG coalgebras and let \(A\) be an algebra. Then any morphism of DG coalgebras \(C \otimes D \longrightarrow BA\) induces a morphism of supercomplexes

\[
\hat{X}(C) \otimes \hat{X}(D) \longrightarrow \hat{X}(BA).
\]

In the remainder of this section we will apply this result to homotopy \(G\) algebras. So let \(V = \bigoplus_{i \geq 0} V_i\) be a homotopy \(G\) algebra with structure map \(\cup : BV \otimes BV \longrightarrow BV\). Using
the inclusion \( V_0 \to V \) and the surjection \( V \to V_0 \), we obtain a morphism of coalgebras \( \cup_1 : BV \otimes BV_0 \to BV_0 \). It is given by

\[
(x_1, \ldots, x_m) \cup_1 (a_1, \ldots, a_n) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_m \leq n} (-1)^\epsilon \\
(a_1, \ldots, a_{i_1}, x_1\{a_{i_1+1}, \ldots, a_{i_1+d_1}\}, \ldots, x_m\{a_{i_m+1}, \ldots, a_{i_m+d_m}\}, \ldots, a_n),
\]

where, \( \epsilon = \sum_{p=1}^m (|x_p| - 1)i_p \) and \( d_i = |x_i| \). In the special case where \( V = C^*(A, A) \), the braces in (11) are given by \( x_{i_k}\{a_{i_{k+1}}, \ldots, a_{i_k+d_k}\} = x_{i_k}(a_{i_{k+1}} \cdots a_{i_k+d_k}) \).

**Lemma 7.** \( \cup_1 \) is a morphism of DG coalgebras.

**Proof.** Note that the surjection \( \pi : BV \to BV_0 \) is a DG coalgebra map. For \( \alpha \in BV \) and \( \beta \in BV_0 \), we have

\[
b'(\alpha \cup_1 \beta) = b'\pi(\alpha \cup \beta) = \pi(b' + \delta)(\alpha \cup \beta) \\
= \pi[(b' + \delta)\alpha \cup \beta + (-1)^{|\alpha|}\alpha \cup (b' + \delta)\beta] \\
= (b' + \delta)\alpha \cup_1 \beta + (-1)^{|\alpha|}[\pi(\alpha \cup b'\beta) + \pi(\alpha \cup \delta\beta)] \\
= (b' + \delta)\alpha \cup_1 \beta + (-1)^{|\alpha|}\alpha \cup_1 b'\beta,
\]

since \( \pi(\alpha \cup \delta\beta) = 0 \) as \( \delta\beta \) has degree one. The lemma is proved.

Let

\[
P : \hat{X}(BV) \otimes \hat{X}(BV_0) \to \hat{X}^2(BV_0)
\]

denote the composition \( P = \cup_1 p \). Now we can apply theorem 6 to obtain

**Theorem 8.** Let \( V \) be a homotopy \( G \) algebra. Then there are natural maps of supercomplexes

\[
\hat{X}(BV) \otimes \hat{X}(BV) \to \hat{X}(BV),
\]

\[
\hat{X}(BV) \otimes \hat{X}(BV_0) \to \hat{X}(BV_0).
\]

We believe, although have not checked it, that the first product is homotopy associative and in fact there should exist a full structure of higher homotopies in the sense of \( A_\infty \)-algebras. The same should be true on the corresponding pairing between homologies.

### 4 Higher Operations on Cyclic Bicomplex.

Our goal in this section is to find explicit formulas for the second map in theorem 8 and to relate it to the Hochschild and cyclic homology of the homotopy \( G \) algebra \( V \). In the last part we apply our formulas to a very special homotopy \( G \) algebra, namely the deformation complex of an algebra, to obtain higher homotopy formulas in the cyclic and \( b, B \) complex of the algebra. The computations in this section are based on results and ideas from [9].

In [12], Nest and Tsygan have defined two different types of operations on cyclic homology. On one hand, they have defined an action of the \( b, B \) complex of the deformation complex (as a \( DG \) algebra) of an algebra \( A \) on the \( b, B \) complex of \( A \). This corresponds
to the pairing (14) below, though we do not know to what extent the explicit formulas match. As is explained in [12], this operation is very general and in particular yields Cartan homotopy formulas for the action of higher Hochschild cochains on the $b, \nabla$ complex. Secondly, they have defined an action of the Chevalley-Eilenberg complex of the deformation complex (as a DG Lie algebra) on the $b, \nabla$ complex of $A$. Most probably, this operation too is a consequence of the homotopy $G$ algebra structure of the deformation complex by a similar pattern as we derived (14) from theorem 8.

Let $\Omega_0 = (x_1, \cdots, x_m) \in BV$, $\Omega_1 = y_0 \otimes (y_1, \cdots, y_n) \in \Omega^1 BV_2, \omega_0 = (a_1, \cdots, a_p) \in BV_0$ and $\omega_1 = b_0 \otimes (b_1, \cdots, b_q) \in \Omega^1 BV_0$. Let $\eta_0 + \eta_1 \in \hat{X}(BV_0)$ be the image of $(\Omega_0 + \Omega_1) \otimes (\omega_0 + \omega_1)$ under the second map in Theorem 8. Using formula (10) for the retraction $R$, we have

$$\eta_0 = P(\Omega_0 \otimes \omega_0) + d\nabla P(\Omega_1 \otimes \omega_1)$$

$$\eta_1 = \theta P(\Omega_0 \otimes \omega_1 + \Omega_1 \otimes \omega_0) + \theta \partial_2 \nabla P(\Omega_1 \otimes \omega_1).$$

Note that

$$P(\Omega_0 \otimes \omega_0) = (x_1, \cdots, x_m) \cup_1 (a_1, \cdots, a_p).$$

To simplify the notation, we resort to the following convention. It is better to consider the points $y_0, y_1, \cdots, y_n$ as located in the clockwise order on the circle. Let $\pi_k$ denote the set of all partitions of these points on the circle into $k$ intervals. We allow one or several of these intervals to be empty, in which case they represent 1. For example, for $n = 2$, $\pi_1$ has 3 elements while $\pi_2$ has 12 elements. We denote an element of $\pi_2$ by a pair $(\alpha, \beta)$, and similarly for elements of $\pi_k$. It is also convenient to write $X(Y)$ for $X \cup_1 Y$.

To compute the other components of $\eta_0$ and $\eta_1$, first we have to find the image of $\Omega_1$ under the inclusion $\Omega^1 BV_2 \longrightarrow \Omega^1 BV$. We have

$$y_0 \otimes (y_1, \cdots, y_n) \mapsto \sum_{i=0, j=0}^{n, n-i} (-1)^{e(y_{i+1}, \cdots, y_{i+j})} \otimes (y_{i+j+1}, \cdots, y_i)$$

$$= \sum_{(\alpha, \beta) \in \beta, y_0 \in \beta} (-1)^{|\alpha||\beta|} \alpha \otimes \beta$$

and similarly for $\omega_1$. Now we have

$$d\nabla P(\Omega_1 \otimes \omega_1) = \frac{1}{2} \sum_{i=0}^{n} \sum_{(\alpha, \beta, y_0) \in (\alpha', \beta', y')} \sum_{b_0 \in \gamma'} \pm(\beta(\beta'), y_0(\gamma'), \alpha(\alpha'))$$

$$- \frac{1}{2} \sum_{(\alpha, \beta, y_0) \in (\alpha', \beta', \gamma')} \sum_{b_0 \in \gamma'} \pm(\beta(\beta'), y_0(\gamma'), \alpha(\alpha')).$$

The signs can be easily made explicit. Next we compute the contribution of $\Omega_0 \otimes \omega_1 + \Omega_1 \otimes \omega_0$ to $\eta_1$. First note that $\theta(\alpha \otimes \beta) \neq 0$ only if $\alpha = 1$ or $\alpha \in V_0$. In this case we have

$$\theta(a_0 \otimes (a_1, \cdots, a_n)) = a_0 \otimes (a_1, \cdots, a_n),$$

$$\theta(1 \otimes (a_1, \cdots, a_n)) = -a_1 \otimes (a_2, \cdots, a_n).$$
Using the above information plus our formulas for $P$, we obtain
\[
\theta P(\Omega_0 \otimes \omega_1) = -\sum_{(\alpha')} (x_1, \cdots, x_m)(\alpha') \\
+ \frac{1}{2} \sum_{(\alpha', \beta') \beta_0 \in \beta'} \pm (x_1(\alpha'), (x_2, \cdots, x_m)(\beta')) \\
+ \frac{1}{2} \sum_{(\alpha', \beta') \beta_0 \in \beta'} \pm (x_m(\alpha'), (x_1, \cdots, x_{m-1})(\beta')).
\]

Similarly we have
\[
\theta P(\Omega_1 \otimes \omega_0) = \frac{1}{2} \sum_{i=1}^{n} \sum_{(y_i, \beta)} \pm (y_i(a_1, \cdots, a_{d_i}), \beta(a_{d_i+1}, \cdots, a_p)) \\
- \sum_{(\beta)} \beta(a_1, \cdots, a_p) \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{(y_i, \beta)} \pm (y_i(a_{p-d_i}, \cdots, a_p), \beta(a_1, \cdots, a_{p-d_i-1}))
\]
where $d_i = |y_i|$.

Finally let us compute the contribution of $\Omega_1 \otimes \omega_1$ to $\eta_1$. Using the formulas for the connection $\nabla$ and for the induced differential $\partial_2$ we note that $\theta \partial_2 \nabla(\beta \otimes \alpha \otimes \gamma) \neq 0$ only if $\alpha \in V_0$ and $\gamma \in V_0$. In this case we have
\[
\theta \partial_2 \nabla(\beta \otimes \alpha \otimes \gamma) = \alpha \gamma \otimes \beta.
\]

From this we obtain
\[
\theta \partial_2 \nabla P(\Omega_1 \otimes \omega_1) = \frac{1}{2} \sum_{(\alpha', \beta', \gamma', \beta_0 \in \gamma')} \pm (y_0(\beta')y_1(\gamma'), (y_2, \cdots, y_n)(\alpha')) \\
- \frac{1}{2} \sum_{(\alpha', \beta', \gamma', \beta_0 \in \beta')} \pm (y_n(\beta')y_0(\gamma'), (y_1, \cdots y_{n-1})(\alpha')).
\]

We define the periodic cyclic homology of a DG algebra $V = \bigoplus_{i \geq 0} V_i$ to be the homology of the supercomplex $\hat{X}(BV)$. In the special case when $V = A$ is an algebra, the bicomplex $X(BA)$ has been shown by Quillen [14] to be isomorphic, up to a shift in the vertical direction, to the cyclic bicomplex of $A$. The same argument works in the DG case. Let $\mathcal{C}(V)$ denote the total complex of the cyclic bicomplex of $V$. We have
\[
\mathcal{C}_{ev}(V) = \mathcal{C}_{odd}(V) = \prod_{n \geq 0} C_n(V),
\]
where $C_n(V) = V^{\otimes (n+1)}$. Theorem 8 can now be reformulated as
Theorem 9. Let $V$ be a homotopy $G$ algebra. Then there is a natural morphism of supercomplexes

$$\mathcal{C}(V) \otimes \mathcal{C}(V_0) \longrightarrow \mathcal{C}(V_0).$$

Finally we turn to Connes’ $b, B$ bicomplex and the analogue of our formulas in that context. This is important because in many applications of cyclic homology and noncommutative geometry [1, 2] the $b, B$ bicomplex appears in a natural way. Using an explicit homotopy equivalence between the cyclic and $b, B$ bicomplexes, we can transform theorem 9 into a morphism between $b, B$ complexes. One obtains, however, simpler formulas if one restricts to the normalized $b, B$ complex.

Let $(V, \delta)$ be a unital DG algebra and let $\mathcal{B}(V)$ denote its $b, B$ complex. We have

$$\mathcal{B}(V)_{ev} = \prod_{n \geq 0} C_{2n}(V), \quad \mathcal{B}(V)_{odd} = \prod_{n \geq 0} C_{2n+1}(V).$$

The differential is given by $b + B + \delta$, where $B : C_\bullet(V) \longrightarrow C_{\bullet+1}(V)$ is Connes’ boundary operator. Let $\bar{V} = V/k$ and $\bar{C}_n(V) = V \otimes \bar{V}^\otimes n$. The normalized $b, B$ complex of $V$, denoted $\bar{\mathcal{B}}(V)$, is defined similarly except that we replace $C_n(V)$ by $\bar{C}_n(V)$.

For $n \geq 0$, let $\lambda : C_n(V) \longrightarrow C_n(V)$ denote the cyclic shift operator, let $N$ be the corresponding norm operator and let $s : C_n(V) \longrightarrow C_{n+1}(V)$ be defined by $s(v_0, \ldots, v_n) = (1, v_0, \ldots, v_n)$. Recall the morphisms of complexes

$$I : \mathcal{B}(V) \longrightarrow \mathcal{C}(V), \quad J : \mathcal{C}(V) \longrightarrow \mathcal{B}(V)$$

defined by

$$I = 1 + sN, \quad J = 1 + s(1 - \lambda).$$

It is known that the operators $I$ and $J$ are homotopy inverse to each other [10].

Using the chain maps $I$ and $J$ it is clear that we can transform theorem 9 into a morphism of complexes

$$\mathcal{B}(V) \otimes \mathcal{B}(V_0) \longrightarrow \mathcal{B}(V_0).$$

We specialize to the case where $V = C(A, A)$ is the deformation complex of a unital algebra $A$. Note that in this case $V_0 = C^0(A, A) = A$. A cochain $\phi \in C^n(A, A)$ is said to be normalized if $\phi(a_1, \ldots, a_n) = 0$ whenever $a_i = 1$ for some $i$. Let $C_{\text{norm}}(A, A)$ denote the subcomplex of normalized cochains. It is easy to check that $C_{\text{norm}}(A, A)$ is indeed a sub DG algebra of the DG algebra $C(A, A)$. Hence we have an inclusion $\mathcal{B}(C_{\text{norm}}(A, A)) \longrightarrow \mathcal{B}(C(A, A))$, and a morphism of supercomplexes

$$\mathcal{B}(C_{\text{norm}}(A, A)) \otimes \mathcal{B}(A) \longrightarrow \mathcal{B}(A).$$

Now our explicit formulas show that the above map descends to define a morphism of supercomplexes

$$\mathcal{B}(C_{\text{norm}}(A, A)) \otimes \bar{\mathcal{B}}(A) \longrightarrow \bar{\mathcal{B}}(A).$$

We denote this map as well as the one in theorem 9 by $\cup$. We obtain explicit formulas for $\cup$ as follows. Let $D = (D_0, \ldots, D_m) \in \mathcal{B}(C_{\text{norm}}(A, A))$ and $a = (a_0, \ldots, a_n) \in \mathcal{B}(A)$.
Let \( ID = \Omega_0 + \Omega_1 = s \mathcal{N}D + D \) and \( Ia = \omega_0 + \omega_1 = s \mathcal{N}a + a \). Also let \( ID \cup Ia = \eta_0 + \eta_1 \).

We have
\[
D \cup a = J(ID \cup Ia) = J(\eta_0 + \eta_1) = s(1 - \lambda)\eta_0 + \eta_1.
\]

Now we have
\[
s(1 - \lambda)\eta_0 = s(1 - \lambda)(P(s \mathcal{N}D \otimes s \mathcal{N}a) + d\nabla P(D \otimes a)).
\]

Because of our normalization conditions we have
\[
s(1 - \lambda)P(s \mathcal{N}D \otimes s \mathcal{N}a) = 0.
\]

Using (12) we get
\[
s(1 - \lambda)\eta_0 = s(1 - \lambda)d\nabla P(D \otimes a)
= \frac{1}{2} \sum_{i=0}^{m} \sum_{(\alpha, \beta, D_i)} \sum_{D_0 \in (\beta, D_i)} \sum_{a_0 \in \gamma'} \pm (\beta(\beta'), D_i(\gamma'), \alpha(\alpha'))
- \frac{1}{2} \sum_{(\alpha, \beta, D_0)} \sum_{(\alpha', \beta', \gamma')} \pm (\beta(\beta'), D_0(\gamma'), \alpha(\alpha')).
\]

Similarly we have
\[
\eta_1 = \theta P(s \mathcal{N}D \otimes \alpha + D \otimes s \mathcal{N}a) + \theta \partial_2 \nabla P(D \otimes a).
\]

**Lemma 10.** We have \( \theta P(D \otimes s \mathcal{N}a) = 0 \) and \( \theta P(s \mathcal{N}D \otimes a) = 0 \).

Using the above lemma and (13), we get
\[
\eta_1 = \theta \partial_2 \nabla P(D \otimes a)
= \frac{1}{2} \sum_{(\alpha', \beta', \gamma')} \pm (D_0(\beta')D_1(\gamma'), (D_2, \ldots, D_n)(\alpha'))
- \frac{1}{2} \sum_{(\alpha', \beta', \gamma')} \pm (D_n(\beta')D_0(\gamma'), (D_1, \ldots, D_{n-1})(\alpha')).
\]

Note that because of homogeneity we can drop the unpleasant factor of \( \frac{1}{2} \) from the formulas. In the unnormalized case, however, this can not be done.
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