WIDTH ESTIMATE AND DOUBLY WARPED PRODUCT

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ABSTRACT. In this paper, we give an affirmative answer to Gromov’s conjecture ([3, Conjecture E]) by establishing an optimal Lipschitz lower bound for a class of smooth functions on orientable open 3-manifolds with uniformly positive sectional curvatures. For rigidity we show that the universal covering of the given manifold must be $\mathbb{R}^2 \times (-c, c)$ with doubly warped product metric if the optimal bound is attained. As a corollary, we also obtain a focal radius estimate for immersed toruses in 3-spheres with positive sectional curvatures.

1. Introduction

Let $M^n$ be an orientable manifold with non-empty boundary $\partial M$, which can be divided into two disjoint parts $\partial_+ M$ and $\partial_- M$ consisting of boundary components of $\partial M$. For any smooth metric $g$ on $M$, the width of $(M, g)$ is defined as

\begin{equation}
\text{width}(M, g) = \text{dist}_g(\partial_+ M, \partial_- M).
\end{equation}

In his paper [3], Gromov introduced the following definition.

Definition 1.1. Let $M_0 = T^{n-1} \times [-1, 1]$ and $\partial_\pm M_0 = T^{n-1} \times \{-1\}$. If $M$ admits a continuous map $f : (M, \partial_\pm M) \rightarrow (M_0, \partial_\pm M_0)$ with nonzero degree, then $M$ is called an overtorical band.

With this notion, he proved the following width estimate.

Theorem 1.2. For $2 \leq n \leq 8$, let $(M^n, g)$ be a smooth overtorical band with its scalar curvature $R(g) \geq n(n-1)\sigma^2$ for some $\sigma > 0$. Then

\begin{equation}
\text{width}(M, g) \leq \frac{2\pi}{n\sigma}.
\end{equation}

This estimate is optimal and it is believed that any band with the extreme width is isometric to an open torical band with warped product metric. That is, up to scaling the band must be $M = T^{n-1} \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ and $g = dt^2 + \cos^4 \left(\frac{n}{2} t\right) g_{\text{flat}}$, where $g_{\text{flat}}$ is an arbitrary flat metric on $T^{n-1}$.

For bands with positive sectional curvatures, he made the following conjecture:

Conjecture 1.3 ([3, Conjecture E]). Let $g$ be a smooth metric on $M = T^2 \times [-1, 1]$ with $\sec(g) \geq 1$. Then

\begin{equation}
\text{width}(M, g) \leq \frac{\pi}{2}.
\end{equation}

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The main purpose of this paper is to give an affirmative answer to this conjecture by establishing a Lipschitz lower bound for a class of smooth functions on orientable open 3-manifolds as in [4]. Apart from the inequality, we also prove a rigidity result for our Lipschitz constant estimate. It turns out that up to rescaling the equality forces the given manifold to be isometrically covered by an open manifold \( \tilde{M}_0 = \mathbb{R}^2 \times (-\frac{\pi}{4}, \frac{\pi}{4}) \) with the doubly warped product metric

\[
g = \left( dt^2 + \sin^2 \left( t + \frac{\pi}{4} \right) ds_1^2 \right) + \cos^2 \left( t + \frac{\pi}{4} \right) ds_2^2,
\]

where \( s_1 \) and \( s_2 \) are arc length of the real line \( \mathbb{R} \).

For an orientable open manifold \( \tilde{M}^3 \), we define

\[
F_{\tilde{M}} = \{ \phi \in C^\infty (\tilde{M}, (-1, 1)) : \phi \text{ is surjective and proper and } \phi^*[\text{pt}] \text{ is non-spherical} \}.
\]

Our main theorem is

**Theorem 1.4.** Assume \((\tilde{M}^3, g)\) is an open Riemannian manifold with uniformly positive sectional curvatures and non-empty \( F_{\tilde{M}} \). Then we have

\[
(\text{inf } \text{sec}(g))^{\frac{1}{2}} \cdot \text{Lip } \phi \geq \frac{4}{\pi}
\]

for any \( \phi \in F_{\tilde{M}} \). Up to rescaling the equality holds if and only if \((\tilde{M}, g)\) is isometric to \( \tilde{M}_0 / \Gamma \) for a lattice \( \Gamma \) of \( \mathbb{R}^2 \) and \( \phi \) is a multiple of the standard signed distance function.

Several corollaries follow from our main theorem. The first one is the desired width estimate for overtorical bands with positive sectional curvatures.

**Corollary 1.5.** Let \((M^3, g)\) be a smooth overtorical band with positive sectional curvatures. Then

\[
(\text{inf } \text{sec}(g))^{\frac{1}{2}} \cdot \text{width}(M, g) \leq \frac{\pi}{2}.
\]

The second one is a focal radius estimate for immersed toruses in 3-spheres with positive sectional curvatures. Recall that the focal radius of any immersed surface \( i : \Sigma \to M \) is defined as

\[
r_f(\Sigma, M) = \sup \{ r > 0 : \text{map } \exp_\nu : \nu_{\Sigma, r} \to M \text{ has no critical point} \},
\]

where \( \nu_{\Sigma} \) the total space of the pull-back normal bundle on \( \Sigma \) and

\[
\nu_{\Sigma, r} = \{ (x, v) \in \nu_{\Sigma} : |v| < r \}.
\]

We have

**Corollary 1.6.** Let \((M^3, g)\) be a 3-sphere with \( \text{sec}(g) \geq 1 \) and \( i : \Sigma \to M \) an immersed torus. Then the focal radius of \( \Sigma \) satisfies

\[
r_f(\Sigma, M) \leq \frac{\pi}{4}.
\]

The equality holds if and only if \( g \) has constant curvature 1 and \( i(\Sigma) \) is congruent to the Clifford torus.
Generalizations of these results in higher dimensions are interesting and we make the following conjectures:

**Conjecture 1.7.** Let \((M^n, g)\) be a \(n\)-sphere with \(\text{sec}(g) \geq 1\), where \(n = p + q + 1\) for \(p, q \in \mathbb{N}_+\). Suppose that \(i : \Sigma \approx \mathbb{S}^p \times \mathbb{S}^q \to M\) is an immersed hypersurface, then the focal radius of \(\Sigma\) satisfies

\[
rf(\Sigma, M) \leq \frac{\pi}{4},
\]

where the equality holds if and only if \(g\) has constant sectional curvature 1 and \(i(\Sigma)\) is congruent to the Clifford hypersurface \(\mathbb{S}^p(\frac{1}{\sqrt{2}}) \times \mathbb{S}^q(\frac{1}{\sqrt{2}})\).

With the name *Clifford band* for manifolds \(M_{p,q} = \mathbb{S}^p \times \mathbb{S}^q \times [-1, 1]\), we guess

**Conjecture 1.8.** If \((M_{p,q}, g)\) be a smooth Clifford band with its sectional curvature \(\text{sec}(g) \geq 1\), then

\[
\text{width}(M, g) \leq \frac{\pi}{2}.
\]

Now let us say some words on our proof for Theorem 1.4. Basically the proof follows the line of \(\mu\)-bubble method in [4] and the inequality (1.4) comes from a classical analysis on the second variation formula. However, the rigidity is very subtle due to the lack of compactness on open manifolds for minimizing \(\mu\)-bubbles. In this case, we follow a similar idea from [2] to fix our approximating minimizing \(\mu\)-bubbles, which is a key to make a limiting procedure possible. Once this is done, the rigidity result comes from a standard foliation argument from [11] (see also [9]). We emphasize that our idea can also be applied to establish a rigidity result for Theorem 1.2.

The rest part of this paper will be organized as follows. In section 2, we show the inequality (1.4). In section 3, we handle the rigidity case for our main theorem. In section 4, we give a proof for Corollary 1.5 and Corollary 1.6.

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**2. Proof for (1.4)**

In this section, \((\hat{M}, g)\) denotes an orientable open 3-manifold with nonempty \(\mathcal{F}_{\hat{M}}\). Let \(\phi\) be a fixed element in \(\mathcal{F}_{\hat{M}}\). For any smooth function \(h : (-T, T) \to \mathbb{R}\) with \(0 < T < 1\), we define the following functional

\[
A^h(\Omega) = H^2(\partial^*\Omega) - \int_{\hat{M}} (\chi_{\Omega} - \chi_{\Omega_0})h \circ \phi \, dH^3, \quad \Omega_0 = \{\phi < 0\},
\]

where \(\Omega\) is any Cacciopoli set of \(\hat{M}\) with reduced boundary \(\partial^*\Omega\) such that

\[
\Omega \Delta \Omega_0 \in D(h \circ \phi) = \{-T < \phi < T\}.
\]
and $\chi_\Omega$ is the characteristic function of region $\Omega$.

First we prove the following proposition which is fundamental to our proof.

**Proposition 2.1.** If $\pm T$ are regular values of $\phi$ and $h$ satisfies

\begin{equation}
\lim_{t \to -T} h(t) = +\infty \quad \text{and} \quad \lim_{t \to T} h(t) = -\infty,
\end{equation}

then there exists a smooth minimizer $\hat{\Omega}$ for $\mathcal{A}^h$ such that $\hat{\Omega} \Delta \Omega_0 \in \mathcal{D}(h \circ \phi)$.

**Proof.** Define

$$C(\hat{M}) = \{\text{Cacciopoli sets $\Omega$ in $\hat{M}$ such that $\Omega \Delta \Omega_0 \in \mathcal{D}(h \circ \phi)$}\}$$

and

$$I = \inf \{\mathcal{A}^h(\Omega) : \Omega \in C(\hat{M})\}.$$  

First we show $I > -\infty$. For any $s > 0$, denote

$$\Sigma^\pm_s = \{x \in \mathcal{D}(h \circ \phi) : \text{dist}(x, \phi^{-1}(\pm T)) = s\}.$$  

Since $\pm T$ are regular values of $\phi$, $\Sigma^\pm_s$ becomes a foliation around $\phi^{-1}(\pm T)$ when $s$ is small. From (2.2) we can assume $H^s_\phi \leq h \circ \phi$ and $H^s_+ \leq -h \circ \phi$ for $s \leq s_0$, where $s_0$ is a small positive constant and $H^s_\phi$ is the mean curvature of $\Sigma^\pm_s$ with respect to $\partial_s$. Let $\Omega^\pm_0$ be the region enclosed by $\Sigma^\pm_s$ and $\phi^{-1}(\pm T)$. Possibly decreasing the value of $s_0$, we can construct a smooth vector field $X$ such that $X = \partial s$ on $\Omega^\pm_0$. Notice that for any region $\Omega \in C(\hat{M})$ we have the following estimate

$$\mathcal{A}^h(\Omega \cup \Omega^-_{s_0} \setminus \Omega^+_0) - \mathcal{A}^h(\Omega) = \mathcal{H}^2(\partial\Omega^-_0 \setminus \Omega) - \mathcal{H}^2(\partial^s \Omega \cap \Omega^-_0) + \mathcal{H}^2(\partial\Omega^+_0 \cap \Omega)$$

$$- \mathcal{H}^2(\partial^s \Omega \cap \Omega^-_0) - \int_{\Omega^-_0 \setminus \Omega} h \circ \phi \, d\mathcal{H}^3_g + \int_{\Omega^+_0 \setminus \Omega} h \circ \phi \, d\mathcal{H}^3_g$$

$$\leq \mathcal{H}^2(\partial\Omega^-_0 \setminus \Omega) - \mathcal{H}^2(\partial^s \Omega \cap \Omega^-_0) + \mathcal{H}^2(\partial\Omega^+_0 \cap \Omega)$$

$$- \mathcal{H}^2(\partial^s \Omega \cap \Omega^-_0) - \int_{\Omega^-_0 \setminus \Omega} \text{div}_g X \, d\mathcal{H}^3_g - \int_{\Omega^+_0 \setminus \Omega} \text{div}_g X \, d\mathcal{H}^3_g$$

$$\leq 0,$$

since

$$\int_{\Omega^-_0 \setminus \Omega} \text{div}_g X \, d\mathcal{H}^3_g = \int_{\partial^s(\Omega^-_0 \setminus \Omega)} \langle X, \nu \rangle_g \, d\mathcal{H}^2_g \geq \mathcal{H}^2(\partial\Omega^-_0 \setminus \Omega) - \mathcal{H}^2(\partial^s \Omega \cap \Omega^-_0)$$

and

$$\int_{\Omega \cap \Omega^+_0} \text{div}_g X \, d\mathcal{H}^3_g = \int_{\partial^s(\Omega \cap \Omega^+_0)} \langle X, \nu \rangle_g \, d\mathcal{H}^2_g \geq \mathcal{H}^2(\partial\Omega^+_0 \cap \Omega) - \mathcal{H}^2(\partial^s \Omega \cap \Omega^+_0).$$

It follows

$$\mathcal{A}^h(\Omega) \geq \mathcal{A}^h(\Omega \cup \Omega^-_{s_0} \setminus \Omega^+_0) \geq -C \mathcal{H}^3(\mathcal{D}(h \circ \phi)), \quad \forall \Omega \in C(\hat{M}),$$

where $C$ is a universal constant such that $|h \circ \phi| \leq C$ on $\mathcal{D}(h \circ \phi) \setminus \Omega^-_0 \cup \Omega^+_0$. Hence $I > -\infty$.

Now we establish the existence of a smooth minimizer for $\mathcal{A}^h$ in $C(\hat{M})$. Let $\Omega_k$ be a sequence of regions in $C(\hat{M})$ such that $\mathcal{A}^h(\Omega_k) \to I$ as $k \to \infty$. According to the discussion above we can
assume $\Omega_k \Delta \Omega_0 \subset D(h \circ \phi) - \Omega_{-} \cup \Omega_{+}$. For $k$ large enough there holds

$$\mathcal{H}^2(\partial^* \Omega_k) \leq 1 + C\mathcal{H}^3(D(h \circ \phi)).$$

From the compactness of Caccipoli sets, after taking the limit of $\Omega_k$ we can obtain $\hat{\Omega} \in \mathcal{C}(\hat{M})$ with $A^h(\hat{\Omega}) = I$. The smoothness of $\partial \hat{\Omega}$ comes from the regularity theorem [5, Theorem 2.2].

Now we give the proof for (1.4).

Proof. Without loss of generality, we can assume $\inf \sec(g) = 1$ from rescaling. If (1.4) is not true, then there is a $\phi \in \mathcal{F}_{\hat{M}}$ with Lip $\phi < 4/\pi$. With $\beta < 1$ a constant to be determined later, we define

$$h = -2 \tan \left(\frac{\pi}{2\beta} t\right), \quad t \in (-\beta, \beta).$$

If $\pm \beta$ are regular values of $\phi$, then we can apply Proposition 2.1 to obtain a smooth minimizer $\hat{\Omega}$ for functional $A^h$. It follows from definition of $\mathcal{F}_{\hat{M}}$ that $\partial \hat{\Omega}$ has a component $\hat{\Sigma}$ with nonzero genus. For any smooth function $\psi$ on $\hat{\Sigma}$, we take a smooth vector field $X$ such that $X = \psi \nu$ on $\hat{\Sigma}$ and $X$ vanishes outside a small neighborhood of $\hat{\Sigma}$, where $\nu$ is the outward unit normal vector field on $\hat{\Sigma}$. Denote $\Phi_t$ to be the flow generated by $X$, then we have

$$\delta A^h(\psi) = \frac{d}{dt} \bigg|_{t=0} A^h(\Phi_t(\hat{\Omega})) = \int_{\hat{\Sigma}} (\hat{H} - h \circ \phi) \psi \, d\sigma_g = 0,$$

where $\hat{H}$ is the mean curvature of $\hat{\Sigma}$ with respect to $\nu$. Since $\psi$ is arbitrary, we see $\hat{H} = h \circ \phi$ on $\hat{\Sigma}$. The second variation formula yields

$$\delta^2 A^h(\psi, \psi) = \frac{d^2}{dt^2} \bigg|_{t=0} A^h(\Phi_t(\Omega_\infty)) = \int_{\hat{\Sigma}} |\nabla \psi|^2 - (\text{Ric}(\nu, \nu) + |A|^2 + \nu(h \circ \phi)) \psi^2 \, d\sigma_g \geq 0.$$

Now we deduce a contradiction from (2.3). Since the sectional curvature $\sec(g) \geq 1$, we have $\text{Ric}(\nu, \nu) \geq 2$ and $|A|^2 \geq (h \circ \phi)^2 - \hat{R} + 2$, where $\hat{R}$ is the scalar curvature of $\hat{\Sigma}$ with the induced metric. Notice also

$$(h \circ \phi)^2 + \nu(h \circ \phi) \geq 4 \tan^2 \left(\frac{\pi}{2\beta} t\right) - \pi \beta^{-1} \text{Lip} \phi \cos^{-2} \left(\frac{\pi}{2\beta} t\right).$$

From Srad’s theorem we can choose $\beta$ such that $\pm \beta$ are regular values of $\phi$ and $\pi \beta^{-1} \text{Lip} \phi < 4$. Taking the testing function $\psi$ to be identically one, we conclude from (2.3) that

$$4\pi \chi(\hat{\Sigma}) = \int_{\hat{\Sigma}} \hat{R} \, d\sigma_g \geq \int_{\hat{\Sigma}} 4 + (h \circ \phi)^2 + \nu(h \circ \phi) \, d\sigma_g > 0,$$

which leads to a contradiction. \hfill \Box

3. THE RIGIDITY CASE

In this section, $(\hat{M}, g)$ is assumed to be an orientable open 3-manifold with $\inf \sec(g) = 1$ such that there is a $\phi \in \mathcal{F}_{\hat{M}}$ satisfying Lip $\phi = 4/\pi$. We are going to show that $(\hat{M}, g)$ is isometric to the open manifold $\hat{M}_0/\Gamma$ for some lattice $\Gamma$ and $\phi = 4\rho/\pi$, where $\rho$ is the standard signed distance function with range $(-\pi/4, \pi/4)$. 

The first lemma provides appropriate functions for us to construct approximating minimizing \( \mu \)-bubbles.

**Lemma 3.1.** For \( \epsilon \in (0,1) \), there is a family of odd smooth functions \( h_\epsilon \) defined on \((-T_\epsilon, T_\epsilon)\) such that

- \( T_\epsilon \uparrow 1 \) as \( \epsilon \to 0 \);
- the derivative \( h_\epsilon' \) is negative everywhere;
- if \(|t| < 2/3\), then \( 4h_\epsilon'/\pi + h_\epsilon^2 < -4 \); if \( 2/3 < |t| < 1\), then \( 4h_\epsilon'/\pi + h_\epsilon^2 > -4 \).

**Proof.** The proof comes from an explicit construction. Define \( \alpha_\epsilon(t) = t + \epsilon \left( \sin\left(\frac{\pi}{2} t\right) - \frac{\pi}{4} t\right) \) and \( h_\epsilon(t) = -2 \tan\left(\frac{\pi}{2} \alpha_\epsilon(t)\right) \).

Clearly, \( h_\epsilon \) is an odd smooth function defined on \((-T_\epsilon, T_\epsilon)\), where \( T_\epsilon \) is the unique positive solution to the equation \( \alpha_\epsilon(t) = 1 \). The behavior of \( T_\epsilon \) comes from implicit function theorem applied to this equation. Through direct calculation, we have

\[
\frac{4}{\pi} h_\epsilon' + h_\epsilon^2 = -4 - \pi \epsilon \cos^{-2}(2\alpha_\epsilon) \left( 2 \cos\left(\frac{\pi}{2} t\right) - 1 \right) .
\]

The conclusion now follows from the fact \( 0 < \epsilon < 1 \). \( \square \)

From Sard’s theorem, there is a sequence \( \epsilon_k \to 0 \) such that \( \pm T_{\epsilon_k} \) are regular values of \( \phi \).

Let \( h_k = h_{\epsilon_k} \). Using Proposition 2.1 we can find a minimizer \( \hat{\Omega}_k \) for functional \( A^{h_k} \). Possibly passing to a subsequence, the sequence \( \hat{\Omega}_k \) converges in locally \( L^1 \) sense to a local minimizer \( \hat{\Omega} \) for functional \( A^h \) with \( h = -2 \tan(\pi t/2) \). That is, for any Cacciopoli set \( \Omega \) such that \( \Omega \Delta \hat{\Omega} \subseteq \hat{M} \) we have \( A^h_U(\Omega) \geq A^{h_k}_U(\hat{\Omega}) \) for any region \( U \) with compact closure and \( \Omega \Delta \hat{\Omega} \subseteq U \), where

\[
A^h_U(\Omega) = \mathcal{H}^2(\partial^* \Omega \cap U) - \int_{\Omega \cap U} h \circ \phi \, d\mathcal{H}^3 .
\]

The following lemma is well-known, but we give a proof for completeness.

**Lemma 3.2.** Denote \( \hat{\mu}_k \) and \( \hat{\mu} \) to be the Radon measure of \( \partial \hat{\Omega}_k \) and \( \partial \hat{\Omega} \) respectively. Then \( \hat{\mu}_k \) converges to \( \hat{\mu} \) after passing to a subsequence. That is, for any continuous function \( f \) with compact support there holds

\[
\lim_{k \to \infty} \int_{\hat{M}} f \, d\hat{\mu}_k = \int_{\hat{M}} f \, d\hat{\mu} .
\]

**Proof.** Let \( U \) be any region with compact closure in \( \hat{M} \). For any compact subset \( K \subseteq U \), we denote

\[
K_\delta = \{ x \in U : \text{dist}(x, K) < \delta \}, \quad \delta > 0 .
\]
There is a positive constant $\delta_0$ such that $K_\delta \subset U$ for any $\delta < \delta_0$. Since $H^2(\partial \tilde{\Omega} \cap U)$ are bounded and $H^3 \left( (\hat{\Omega}_k \cup \tilde{\Omega}) \cap U \right)$ converges to 0, for almost every $\delta$ we see

$$H^2(\partial \tilde{\Omega} \cap \partial K_\delta) = 0,$$

and

$$H^2 \left( (\hat{\Omega}_k \cup \tilde{\Omega}) \cap \partial K_\delta \right) \rightarrow 0, \quad k \rightarrow \infty.$$

Let $\Omega_k = (\hat{\Omega} \cap K_\delta) \cup (\hat{\Omega} - K_\delta)$ be a competitor for $\hat{\Omega}_k$. From $\mathcal{A}_{\tilde{\Omega}}^h(\hat{\Omega}_k) \geq \mathcal{A}_{\tilde{\Omega}}^h(\hat{\Omega}_k)$ we obtain

$$H^2(\partial^* \hat{\Omega} \cap U) \geq H^2(\partial \hat{\Omega}_k \cap U) + \int_U (\chi_{\hat{\Omega}_k} - \chi_{\hat{\Omega}_k}) h_k \circ \phi \, d\mathcal{H}^3_g.$$

Notice that

$$H^2(\partial^* \hat{\Omega} \cap U) = H^2(\partial \hat{\Omega}_k - K_\delta) + H^2(\partial \hat{\Omega}_k \cap K_\delta) + H^2(\partial \hat{\Omega}_k \cap \partial K_\delta) + H^2 \left( (\hat{\Omega}_k \cup \hat{\Omega}) \cap \partial K_\delta \right).$$

It follows from (3.3), (3.4) and $L^1$ convergence that

$$H^2(\partial^* \hat{\Omega} \cap U) \geq H^2(\partial \hat{\Omega}_k \cap K) - o(1), \quad k \rightarrow \infty.$$

Taking $k \rightarrow \infty$ and then $\delta \rightarrow 0$, we conclude

$$\hat{\mu}(K) \geq \limsup_{k \rightarrow \infty} \hat{\mu}_k(K), \quad \forall \ K \ \text{compact}.$$

From lower semi-continuity of perimeter, we also have

$$\hat{\mu}(U) \leq \liminf_{k \rightarrow \infty} \hat{\mu}_k(U), \quad \forall \ U \ \text{open}.$$

Now (3.2) comes from a standard approximation argument using (3.5) and (3.6).

The following proposition gives a description for boundary of locally minimizing $\mu$-bubble $\hat{\Omega}$.

**Proposition 3.3.** The boundary $\partial \hat{\Omega}$ has a torus component $\hat{\Sigma}$ contained in a level set of $\phi$. Furthermore, the induced metric of $\hat{\Sigma}$ is flat and $g$ has constant sectional curvature 1 at any point of $\hat{\Sigma}$. We also have $d\phi(\nu) = 4/\pi$, where $\nu$ is the outward unit normal vector field on $\hat{\Sigma}$.

**Proof.** From the definition of $\mathcal{F}_{\hat{M}}$ the boundary of $\hat{\Omega}_k$ has a component $\hat{\Sigma}_k$ with nonzero genus. It follows from the second variation formula that

$$0 \geq 4\pi \chi(\hat{\Sigma}_k) \geq \int_{\hat{\Sigma}_k} 4 + (h_k \circ \phi)^2 + \nu_k(h_k \circ \phi) \, d\sigma_g,$$

where $\nu_k$ is the outward unit normal vector field of $\hat{\Sigma}_k$. Combined with Lemma 3.1 surface $\hat{\Sigma}_k$ must have a non-empty intersection with the compact subset $K = \{-2/3 \leq \phi \leq 2/3\}$. Since $\hat{\Omega}_k$ is a local minimizer of $\mathcal{A}_h$ and $h_k$ converges to $h$ smoothly, for any region $U$ with compact closure we have $\text{area}(\hat{\Sigma}_k \cap U) \leq C$ for some universal constant $C$. From the curvature estimate for stable $h_k$-surfaces ([8 Theorem 3.6]) and elliptic estimates, $\hat{\Sigma}_k$ converges smoothly to a surface $\hat{\Sigma}$ with $\hat{\Sigma} \cap K \neq \emptyset$ in locally graphical sense after passing to a subsequence. It follows
from (3.1), (3.7) and the fact \( \text{Lip} \phi = \frac{4}{\pi} \) that
\[
\int_{\Sigma_k} \left| d\phi(\nu_k) - \frac{4}{\pi} \right| |h' \circ \phi| \, d\sigma_g \leq C \epsilon_k \text{area}(\hat{\Sigma}_k \cap K) \to 0, \quad k \to \infty.
\]
Denote \( \nu \) to be the limit of \( \nu_k \), then we have \( d\phi(\nu) = \frac{4}{\pi} \) on \( \hat{\Sigma} \), which yields \( \nabla_{\hat{\Sigma}} \phi = 0 \). Therefore surface \( \hat{\Sigma} \) has a closed component \( \hat{\Sigma}' \) contained in \( \phi^{-1}(t_0) \) with some \( t_0 \in [-\frac{2}{3}, \frac{2}{3}] \). From (3.2) we see \( \hat{\Sigma} \subset \partial \hat{\Omega} \) and hence \( \hat{\Sigma}' \) is a connected component of \( \partial \hat{\Omega} \). In particular, there is an open neighborhood \( V \) of \( \hat{\Sigma}' \) such that \( \hat{\Sigma} \cap V = \hat{\Sigma}' \). Using the same argument as in [6, Proposition B.1], combined with the connectedness of \( \hat{\Sigma}_k \), we conclude that \( \hat{\Sigma}_k \) is a graph over \( \hat{\Sigma}' \) for \( k \) sufficiently large. Hence \( \hat{\Sigma} = \hat{\Sigma}' \) is a surface with nonzero genus. Now we can run a standard analysis on the second variation formula to obtain desired consequences. Since
\[
0 \geq 4\pi \chi(\hat{\Sigma}) \geq \int_{\hat{\Sigma}} 4 + (h \circ \phi)^2 + \nu(h \circ \phi) \, d\sigma_g \geq 0,
\]
all the inequalities are in fact equality, which yields that \( \hat{\Sigma} \) is a torus, \( d\phi(\nu) = \frac{4}{\pi} \) and that \( g \) has constant sectional curvature 1 at any point of \( \hat{\Sigma} \). In this case, the Jacobi operator becomes \( -\Delta_{\hat{\Sigma}} - \hat{R} \) and the first eigenfunction is nonzero constant function due to the \( A^h \)-stability. It then follows that the induced metric of \( \hat{\Sigma} \) is flat.

From above proposition we can construct a foliation around \( \hat{\Sigma} \).

**Lemma 3.4.** There is a foliation \( \{\hat{\Sigma}_s\}_{-\epsilon<s<\epsilon} \) with \( \hat{\Sigma}_0 = \hat{\Sigma} \) such that
- each \( \hat{\Sigma}_s \) is a graph over \( \hat{\Sigma} \) with graph function \( \tilde{u}_s \) along outward unit normal vector field \( \nu \) such that
  \[
  \frac{\partial}{\partial s} \bigg|_{s=0} \tilde{u}_s = 1 \quad \text{and} \quad \int_{\hat{\Sigma}} \tilde{u}_s \, d\sigma_g = s;
  \]
- \( \tilde{H}_s - h \circ \phi \) is a constant function on \( \hat{\Sigma}_s \), where \( \tilde{H}_s \) is the mean curvature of \( \hat{\Sigma}_s \).

**Proof.** Let \( H_u \) be the mean curvature of the graph over \( \hat{\Sigma} \) with graph function \( u \) along outward unit normal vector field \( \nu \). We define
\[
\Psi : C^{2,\alpha}(\hat{\Sigma}) \to \check{C}^{\alpha}(\hat{\Sigma}) \times \mathbb{R}, \quad u \mapsto \left( H_u - h \circ \phi - \int_{\hat{\Sigma}} H_u - h \circ \phi \, d\sigma_g, \int_{\hat{\Sigma}} u \, d\sigma_g \right),
\]
where
\[
\check{C}^{\alpha}(\hat{\Sigma}) = \{ f \in C^{\alpha}(\hat{\Sigma}) : \int_{\hat{\Sigma}} f \, d\mu = 0 \}.
\]
The linearized operator of \( \Psi \) at \( u = 0 \) is
\[
D\Psi|_{u=0}(v) = \left( -\Delta_{\hat{\Sigma}} v, \int_{\hat{\Sigma}} v \, d\sigma_g \right).
\]
Clearly this operator is invertible and it follows from inverse function theorem that there is a family of functions \( \tilde{u}_s \) with \( -\epsilon < s < \epsilon \) such that \( H_{\tilde{u}_s} - h \circ \phi \) is constant function and
\[
\frac{\partial}{\partial s} \bigg|_{s=0} \tilde{u}_s = 1, \quad \int_{\hat{\Sigma}} \tilde{u}_s \, d\sigma_g = s.
\]
With a smaller \( \epsilon \), graphs \( \hat{\Sigma}_s \) with graph function \( \tilde{u}_s \) over \( \hat{\Sigma} \) give the desired foliation. \( \Box \)
The foliation provides us a family of local minimizers for functional $A^h$.

**Proposition 3.5.** Assume that $\partial \hat{\Omega}$ is disjoint to the region enclosed by $\hat{\Sigma}$ and $\hat{\Sigma}_s$. Let $\tilde{\Omega}_s$ be the region enclosed by $\partial \hat{\Omega} - \hat{\Sigma}$ and $\hat{\Sigma}_s$. Then $\tilde{\Omega}_s$ is still a local minimizer for $A^h$.

**Proof.** We only give a proof for $\tilde{\Sigma}_s$ with $s > 0$ since the argument is valid when $s < 0$ as well. First let us show that $\hat{H}_r - h \circ \phi$ is identically zero for any $0 < \tau < s$. Otherwise there is a $\tau$ such that $\hat{H}_r - h \circ \phi$ is positive since $\hat{\Omega}$ is a local minimizer for $A^h$. Assume that $\hat{\Sigma}$ is contained in $\phi^{-1}(t_0)$. We consider the following differential equation

$$\frac{4}{\pi} \tilde{h}_\delta' + \tilde{h}_\delta^2 = -4 + \delta, \quad \tilde{h}_\delta(t_0) = h(t_0) + \delta, \quad \delta > 0.$$ 

With $\delta$ chosen to be small enough, we have $\tilde{H}_0 < \tilde{h}_\delta \circ \phi$ on $\hat{\Sigma}$ and $\hat{H}_r > \tilde{h}_\delta \circ \phi$ on $\hat{\Sigma}_r$. Therefore, we can find a smooth minimizer $\tilde{\Omega}$ for $A^{h_\delta}$ in the region enclosed by $\hat{\Sigma}$ and $\hat{\Sigma}_r$ with $\hat{\Sigma} \subset \partial \tilde{\Omega}$. It follows from the second variation formula that

$$0 \geq 4\pi\chi(\tilde{\Sigma}) \geq \int_{\Sigma} 4 + (h \circ \phi)^2 + \nu(h \circ \phi) \, d\sigma_g \geq \delta \, \text{area}(\tilde{\Sigma}) > 0,$$

which yields a contradiction. Now it is quick to see that $\tilde{\Omega}_s$ is still a local minimizer for $A^h$. Denote $\Omega_r$ to be the region enclosed by $\hat{\Sigma}$ and $\hat{\Sigma}_r$. Notice that for any region $U$ with compact closure containing $\Omega_s$ there holds

$$A^h_t(\Omega_s) = A^h_t(\hat{\Omega}) + \text{area}(\hat{\Sigma}_s) - \text{area}(\hat{\Sigma}) - \int_{\Omega_s} h \circ \phi \, d\mu_g$$

$$= A^h_t(\hat{\Omega}) + \int_0^s d\tau \int_{\hat{\Sigma}_r} (\hat{H}_r - h \circ \phi) \tilde{f}_r \, d\sigma_g$$

$$= A^h_t(\hat{\Omega}),$$

where $\tilde{f}_r$ is the lapse function of $\hat{\Sigma}_r$ moving along the foliation. If $\Omega$ satisfies $\Omega \Delta \hat{\Omega}_s \subseteq U$, then

$$A^h_t(\Omega) \geq A^h_t(\hat{\Omega}) = A^h_t(\tilde{\Omega}_s).$$

Therefore we complete the proof. \(\square\)

Now we are ready to prove the rigidity.

**Proof.** First we show that $\hat{\Sigma}_s$ are $s$-equadistant surfaces to $\hat{\Sigma}$ contained in level sets of $\phi$. With the foliation we can write the metric as $g = \tilde{f}_s^2 ds^2 + \tilde{g}_s$. Since $\Omega_s$ is $A^h$-minimizing, $\Sigma_s$ satisfies the properties in Proposition 3.3. It follows from $A^h_t(\Omega_s) \equiv A^h_t(\hat{\Omega})$ and the second variation formula that $-\Delta \tilde{f}_s = 0$, which implies that $\tilde{f}_s$ is a constant function. Combined with (3.8), $\tilde{f}_s \equiv 1$ and $\hat{\Sigma}_s$ is $s$-equadistant surface to $\hat{\Sigma}$. Since $\nu(\nu) = 4/\pi$ and $\hat{\Sigma}$ is contained in level sets of $\phi$, so is $\hat{\Sigma}_s$. In particular, we know that $\{\hat{\Sigma}_s\}_{a < s < b}$ is a geodesic flow starting from $\hat{\Sigma}$. Extending it to a maximum solution $\{\hat{\Sigma}_s\}_{a < s < b}$, we are going to show

- $\hat{\Sigma}_s$ does not intersect $\partial \hat{\Omega} - \hat{\Sigma}$;
- $\hat{\Sigma}_s$ satisfies properties in Proposition 3.3.
Denote $\phi(s)$ to be the value of $\phi$ on $\tilde{\Sigma}_s$, then $a$ and $b$ satisfy

\begin{equation}
\lim_{s \to a^+} \phi(s) = -1 \quad \text{and} \quad \lim_{s \to b^-} \phi(s) = 1.
\end{equation}

We verify these one by one. Assume by contradiction that one slice $\tilde{\Sigma}_{s_0}$ has non-empty intersection with $\partial \tilde{\Omega} - \tilde{\Sigma}$, we can take $s_0$ to be the first such moment. Namely, for any $s$ between $0$ and $s_0$ surface $\tilde{\Sigma}_s$ has no intersection with $\partial \tilde{\Omega} - \tilde{\Sigma}$. For convenience we assume $s_0 > 0$. Define

$$S = \{0 < s < s_0 : \tilde{\Omega}_s \text{ is locally } A^h\text{-minimizing with } \tilde{\Omega}_s \text{ defined in Proposition 3.5}\}.$$ 

From previous discussion, $S$ is non-empty and relatively open in $(0, s_0)$. Notice that we have

$$\sup_{0 \leq s \leq s_0} \sup_{\tilde{\Sigma}_s} |\phi| \leq t_0 < 1$$

from the smoothness of $\phi$. Combined with Proposition 3.3, the principal curvatures of $\tilde{\Sigma}_s$ with $s \in S$ satisfies

$$\lambda_1 \lambda_2 = -1 \quad \text{and} \quad |\lambda_1 + \lambda_2| \leq C(t_0),$$

which implies $|A| \leq C(t_0)$. Furthermore, we have uniform area bound for $\tilde{\Sigma}_s$ with $s \in S$ due to the locally $A^h$-minimizing property. Therefore $\{\tilde{\Sigma}_s : s \in S\}$ is compact in $C^\infty$ topology and $S$ is relatively closed in $(0, s_0)$. This yields $S = (0, s_0)$. Combined with the compactness again, we know that $\tilde{\Omega}_{s_0}$ is $A^h$-minimizing and then smooth, which leads to a contradiction since $\tilde{\Omega}_{s_0}$ intersects other boundary components of $\tilde{\Omega}_{s_0}$. If $\tilde{\Sigma}_s$ has no intersection with $\partial \tilde{\Omega} - \tilde{\Sigma}$, then above argument actually tells us that all $\tilde{\Omega}_s$ are locally $A^h$-minimizing and $\tilde{\Sigma}_s$ satisfies the properties in Proposition 3.3. In particular, $\phi(s)$ is monotone increasing and the limits in (3.9) make sense. If (3.9) does not hold, then we can still obtain the compactness from the properness of $\phi$. Then the geodesic flow can be extended through endpoints, which is impossible.

From the connectedness of $M$ it follows $\tilde{M} = \tilde{\Sigma} \times (a, b)$. Let $\rho = \pi\phi/4$, then it is a smooth distance function with range $(-\pi/4, \pi/4)$ and $h \circ \phi = -2\tan(2\rho)$. Therefore we can write

$$\tilde{M} = \tilde{\Sigma} \times \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \quad \text{and} \quad g = d\rho^2 + \tilde{g}_\rho,$$

where $\tilde{g}_\rho$ are flat metrics on $\tilde{\Sigma}$ and $g$ has constant sectional curvature $1$. Denote $\tilde{\Sigma}_\rho = \tilde{\Sigma} \times \{\rho\}$, then $\tilde{\Sigma}_\rho$ has constant mean curvature $-2\tan(2\rho)$. Combined with the Gauss equation, we conclude that the principal curvatures of $\tilde{\Sigma}_\rho$ are

\begin{equation}
\lambda_1 = \cot \left(\rho + \frac{\pi}{4}\right) \quad \text{and} \quad \lambda_2 = -\tan \left(\rho + \frac{\pi}{4}\right).
\end{equation}

Now we lift the metric to the universal covering $\hat{M}_0 = \mathbb{R}^2 \times (-\pi/4, \pi/4)$, denoted by $g_0 = d\rho^2 + \tilde{g}_\rho$. Fixing a coordinate $(x, y)$ on $\mathbb{R}^2$ such that $\tilde{g}_\rho = dx^2 + dy^2$. From (3.10) the second fundamental form of $\{\rho = 0\}$ is

\begin{equation}
A = -\cos (2\theta(x, y)) \ dx^2 + 2 \sin (2\theta(x, y)) \ dxdy + \cos (2\theta(x, y)) \ dy^2.
\end{equation}

Since $g_0$ has constant curvature $1$, the Codazzi equation gives

$$\sin (2\theta(x, y)) \theta'_y(x, y) = \cos (2\theta(x, y)) \theta'_x(x, y).$$
and
\[-\sin(2\theta(x,y))\theta_y'(x,y) = \cos(2\theta(x,y))\theta_y'(x,y).\]

It follows that \(\theta(x,y)\) is a constant function and we can assume \(A = dx^2 - dy^2\) after a possible rotation. From the Jacobi equation we conclude
\[g_0 = d\rho^2 + 2\sin^2\left(\rho + \frac{\pi}{4}\right)dx^2 + 2\cos^2\left(\rho + \frac{\pi}{4}\right)dy^2.\]

This yields \(\tilde{M} = \tilde{M}_0/\Gamma\) for some lattice \(\Gamma\) and we complete the proof. \(\square\)

4. Proof of corollaries

In this section, we give a proof for Corollary 1.5 and Corollary 1.6. First let us show the following lemma:

**Lemma 4.1.** Assume \((M,g)\) is a smooth Riemannian manifold with \(\text{width}(M,g) > 2l\), then there exists a surjective smooth map \(\phi : (M,g) \rightarrow [-l,l]\) with \(\text{Lip } \phi < 1\) such that \(\phi^{-1}(-l) = \partial_- M\) and \(\phi^{-1}(l) = \partial_+ M\).

**Proof.** Let \(\rho_- (x) = \text{dist}(x,\partial_- M)\). The strict inequality for width implies that there is a small positive constant \(\epsilon\) such that \(\text{width}(M,g) > 2l + 4\epsilon\). Fix such an \(\epsilon\) and define
\[\phi_1 = \min\{\max\{\rho_- - l - 2\epsilon, -l - \epsilon\}, l + \epsilon\}.\]

Clearly \(\phi_1\) has the following properties:
- the Lipschitz constant \(\text{Lip } \phi_1 \leq 1\);
- \(\phi_1 \equiv -l - \epsilon\) around \(\partial_- M\) and \(\phi_1 \equiv l + \epsilon\) around \(\partial_+ M\).

Through a standard mollification procedure (for example refer to [7, Appendix]), for any positive constant \(\delta\) there is a smooth function \(\phi_2\) satisfying \(|d\phi_2| \leq 1 + \delta\) and the second property above. Let

\[\phi_3 = \frac{l}{l + \epsilon}\phi_2,\]

then we have
\[|d\phi_3| = \frac{l}{l + \epsilon}|d\phi_2| \leq \frac{l}{l + \epsilon}(1 + \delta).\]

With \(\delta\) prescribed small enough, we have \(|d\phi_3| \leq c < 1\) for some constant \(c\). The only thing that we need to guarantee now is \(\phi_3^{-1}(-l) = \partial_- M\) and \(\phi_3^{-1}(l) = \partial_+ M\). This can be done by passing to a perturbation of \(\phi_3\). In detail, we take a smooth function \(\eta\) vanishing at the boundary but positive at every interior point of \(M\) and define \(\phi = (1 - \epsilon'\eta)\phi_3\) with \(\epsilon'\) a positive constant. If \(\epsilon'\) is small enough, we can guarantee \(|d\phi| \leq (c + 1)/2 < 1\) and moreover \(\phi\) satisfies our requirements.

To see \(\phi^{-1}(l) = \partial_+ M\), it suffices to show \(\phi^{-1}(l) \subset \partial_+ M\) since the other direction is clear. For any point \(x\) with \(\phi(x) = l\), the only possibility is \(\phi_3(x) = l\) and \(\eta(x) = 0\), which yields \(x \in \partial_+ M\).

A similar argument gives \(\phi^{-1}(-l) \subset \partial_- M\). It follows from the construction that \(\phi : M \rightarrow [-l,l]\) is surjective. \(\square\)

**Proof for Corollary 1.5** We only need to prove the width estimate when \(\inf \text{sec}(g) > 0\). Without loss of generality, we assume \(\inf \text{sec}(g) = 1\). If the width estimate does not hold, we obtain
from Lemma 4.1 a surjective smooth function \( \phi : M \to [-\pi/4, \pi/4] \) with \( \text{Lip} \phi < 1 \) such that \( \phi^{-1}(-\pi/4) = \partial_- M \) and \( \phi^{-1}(\pi/4) = \partial_+ M \). Denote \( \tilde{M} \) to be the interior of \( M \) and \( \tilde{\phi} = 4\phi/\pi \), then \( \tilde{\phi} : \tilde{M} \to (-1, 1) \) is surjective and proper. Recall that \( M \) is an ovoidal band, \( \tilde{\phi}^*([\text{pt}]) \) must be non-spherical. Therefore \( \tilde{\phi} \) is an element in \( F_{\tilde{M}} \) with \( \text{Lip} \tilde{\phi} < 4/\pi \), which leads to a contradiction to (4.3).

**Proof for Corollary 1.6.** Pull back the metric on \( \nu_{\Sigma,rf} \), we obtain an open manifold \((\tilde{M}, \tilde{g})\) with \( \tilde{M} = T^2 \times (-rf, rf) \) and \( \text{sec}(\tilde{g}) \geq 1 \). Denote \( \rho \) to be the signed distance function and \( \phi = r_f^{-1} \rho \). It follows from (1.4) that \( r_f^{-1} \geq 4/\pi \), which is the desired focal radius estimate. If the equality holds, then \((\tilde{M}, \tilde{g})\) is the quotient \( M_0/\Gamma \) with \( \Gamma \) a lattice of \( \mathbb{R}^2 \). In particular, the focal radius is obtained along each unit normal vector of \( \Sigma \), which implies that the closure of the image \( \exp^\Sigma(\nu_{\Sigma,rf}) \) must be the entire \( M \). From the smoothness we know that \( g \) also has constant curvature \( 1 \). Notice that the immersed surface \( \Sigma \) is flat, the conclusion follows from local rigidity result [5, Corollary 3].

It is worth mention that a different proof for Corollary 1.6 is pointed out to the author by professor André Neves, which can also be used to verify Conjecture 1.7 in the case that either of \( p \) and \( q \) is 1. We present it below as the end of this section.

**An alternative proof for Corollary 1.6.** Assume otherwise the focal radius \( r_f > \pi/4 \), we are going to show that the principal curvatures of \( \Sigma \) have absolute value less than \( 1 \). Once this is done, the induced metric of \( \Sigma \) will have positive curvature, which contradicts to the Gauss-Bonnet formula. As before we pull back the metric onto \( \nu_{\Sigma,rf} \) to obtain an open manifold \((\tilde{M}, \tilde{g})\) with \( \tilde{M} = (-rf, rf) \) and \( \text{sec}(\tilde{g}) \geq 1 \). Denote \( \rho \) to be the signed distance function to \( \Sigma \). It follows from the Riccati equation that

\[
\partial_\rho \text{Hess} \rho + \text{Hess}^2 \rho = -R(\cdot, \partial_\rho, \partial_\rho, \cdot).
\]

Let \( \gamma : [0, rf) \to \tilde{M} \) be any normal geodesic starting from \( \Sigma \) with \( \gamma'(0) = \nabla \rho \) and \( \lambda(s) \) the least eigenvalue of \( \text{Hess} \rho \) at \( \gamma(s) \). Then \( \lambda(s) \) is locally Lipschitz and hence differentiable at almost every point. For all these points the function \( \lambda(s) \) satisfies \( \lambda'(s) + \lambda^2(s) \leq -1 \), which implies

\[
\lambda(s) \leq \frac{\lambda(0) - \tan s}{1 + \lambda(0) \tan s}, \quad \forall s \in [0, rf).
\]

Therefore the principal curvatures of \( \Sigma \) with respect to \( \nabla \rho \) is greater than \(-1 \). With the same argument to the signed distance function \(-\rho \), we obtain the desired estimate. If \( rf = \pi/4 \), then \( \Sigma \) is flat and the principal curvatures at all points of \( \Sigma \) are \(-1 \) and \( 1 \). Furthermore, (1.4) takes its equality and \( \Sigma_s \) has a principal curvature \(-\tan(s + \pi/4) \), where \( \Sigma_s = \{ \rho = s \} \). Denote \( \mu(s) \) to be the largest eigenvalue of \( \text{Hess} \rho \). It follows a similar argument that

\[
\mu(s) \leq \tan \left( \frac{\pi}{4} - s \right), \quad s \in [0, \pi/4).
\]

Applying the Gauss-Bonnet formula to \( \Sigma_s \), we obtain the equality for (1.2) as well. It follows from the Gauss and Ricatti equation that \( \tilde{g} \) has constant curvature \( 1 \) and so is \( g \). Now the proof is completed by [2, Corollary 3].
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