SUBCONVEX EQUIDISTRIBUTION OF CUSP FORMS: REDUCTION TO EISENSTEIN OBSERVABLES

PAUL D. NELSON

ABSTRACT. Let \( \pi \) traverse a sequence of cuspidal automorphic representations of \( \text{GL}_2 \) with large prime level, unramified central character and bounded infinity type. For \( G \in \{ \text{GL}_1, \text{PGL}_2 \} \), let \( H(G) \) denote the assertion that subconvexity holds for \( G \)-twists of the adjoint \( L \)-function of \( \pi \), with polynomial dependence upon the conductor of the twist. We show that \( H(\text{GL}_1) \) implies \( H(\text{PGL}_2) \).

In geometric terms, \( H(\text{PGL}_2) \) corresponds roughly to an instance of arithmetic quantum unique ergodicity with a power savings in the error term, \( H(\text{GL}_1) \) to the special case in which the relevant sequence of measures is tested against an Eisenstein series.

CONTENTS

1. Introduction 2
1.1. Statement of main result 3
1.2. Overview of the proof 5
Acknowledgements 7
2. Local preliminaries 8
2.1. Groups, measures, and general notation 8
2.2. Weil representation 9
2.3. Induced representations 10
2.4. Unramified representations and vectors 13
2.5. Whittaker models 13
2.6. Local Waldspurger packets 13
2.7. Complementary series 13
2.8. Newvectors 14
2.9. Sobolev norms 15
2.10. Change of polarization 15
2.11. Bounds for matrix coefficients and varia 15
2.12. Local triple product periods on the general linear group 17
2.13. Local triple product periods on the metaplectic group 17
2.14. Linearizing local triple product periods on the metaplectic group 17
2.15. Lower bounds at uninteresting places 22
2.16. Upper bounds at uninteresting places 25
2.17. Estimates at the interesting place 25
2.18. Sobolev–type bounds for twisting isometries 27
3. Global preliminaries 27

Date: December 13, 2018.
2010 Mathematics Subject Classification. Primary 11F70; Secondary 11F27, 58J51.
1. Introduction

In the 1980’s and 1990’s, it was discovered that the subconvexity problems\(^1\) for the families of automorphic \(L\)-functions

\[
L(\pi, \frac{1}{2}) \quad (\tau \text{ on } \text{GL}_2 \text{ varying}, \pi \text{ on } \text{GL}_2)
\]

\[
L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) \quad (\tau \text{ on } \text{PGL}_2 \text{ varying})
\]

were intimately related to fundamental arithmetical equidistribution problems, concerning in the first case (1.1) the distribution of integral points on spheres, representations of integers by ternary quadratic forms, Heegner points and closed geodesics on the modular surface, and so on (see e.g. [22, 30, 29] and references), and in the second case (1.2) the limiting mass distribution of automorphic forms ("arithmetic quantum unique ergodicity", see e.g. [44, 22, 18, 39, 46] and references). Motivated in part by such applications, fundamental and increasingly robust methods for attacking those problems were introduced and developed by many authors. A capstone of this progress was the solution by Michel–Venkatesh [31] in 2009 of the general case of the subconvexity problem for (1.1). One crucial step towards [31] was Michel’s observation [28] that the subconvexity problem for (1.1) may be reduced to the corresponding problem for \(L(\tau \otimes \chi, 1/2) \) (\(\tau \text{ on } \text{GL}_2 \text{ fixed}, \chi \text{ on } \text{GL}_1 \text{ varying}).

\(^1\)A family of \(L\)-functions \(L(\pi, s)\) attached to a family \(\mathcal{F}\) of parameters \(\pi \in \mathcal{F}\) with analytic conductors \(C(\pi) \in \mathbb{R}_{>1}\) is said to satisfy a subconvex bound if there are fixed quantities \(\delta = \delta(\mathcal{F}) > 0\) and \(c = c(\mathcal{F}) > 0\) so that \(|L(\pi, \frac{1}{2})| \leq cC(\pi)^{1/4-\delta}\) for all \(\pi \in \mathcal{F};\) the subconvexity problem for a given family consists of establishing a subconvex bound (see e.g. [22], [29], [31]). In this article, \(L(\pi, s)\) always refers to the finite part of an \(L\)-function, omitting \(\Gamma\)-factors at \(\infty.\)
By contrast, and despite sustained interest, the subconvexity problem for (1.2) has seen no progress in non-dihedral cases until very recently. (The case of (1.2) in which \( \pi \) is dihedral reduces to (1.1), as exploited by Sarnak [45] in one of the first works on cases of (1.1) in which both factors are cuspidal.)

Under important local assumptions (roughly "prime level aspect"), the main result of this article reduces the subconvexity problem for (1.2) to the corresponding problem for

\[
L(\pi \otimes \tau, \frac{1}{2}) \quad (\chi \text{ on GL}_1 \text{ fixed}, \pi \text{ on GL}_2 \text{ varying}).
\]  

(1.3)

In view of the factorizations\(^2\)

\[
L(\pi \otimes \tau, \frac{1}{2}) = L(\tau, \frac{1}{2})L(\ad(\pi) \otimes \tau, \frac{1}{2})
\]  

(1.4)

\[
L(\pi \otimes \chi, \frac{1}{2}) = L(\chi, \frac{1}{2})L(\ad(\pi) \otimes \chi, \frac{1}{2}),
\]  

(1.5)

the latter problem reduces further to that for

\[
L(\ad(\pi) \otimes \chi, \frac{1}{2}) \quad (\chi \text{ on GL}_1 \text{ fixed}, \pi \text{ on GL}_2 \text{ varying}).
\]  

(1.6)

Our result thus mildly strengthens that indicated in the abstract.

Essential motivation for this work came from a talk by R. Munshi at ETH Zurich in May 2015, where he announced a proof of a subconvex bound for (1.6) in a specific aspect (\( \pi \) corresponding to a holomorphic form of large prime level over \( \mathbb{Q} \)); a detailed draft [32] of that proof has been available since April 2017. The families (1.2) specialize to (1.6) upon restricting \( \tau \) to be an Eisenstein series. The motivating applications indicated above require also the cuspidal case of (1.2). That case should now follow from the reduction established here, leading to subconvex bounds for (1.2) and hence strong quantitative forms of arithmetic quantum unique ergodicity in the prime level aspect. We have proposed to describe such applications jointly with Munshi once our respective contributions have been finalized. An intriguing open problem is to what extent these methods may be generalized to other aspects; we indicate some of the challenges associated with doing so at the end of this paper.

### 1.1. Statement of main result

Fix a number field \( F \); all results are new already when \( F = \mathbb{Q} \). Let \( q \) traverse a sequence of finite primes in \( F \), with norms tending off to \( \infty \). We assume given for each such \( q \) a cuspidal automorphic representation \( \pi \) of \( \text{GL}_2(\mathbb{A}) \) (by convention, \( \pi \) is unitary) satisfying the following assumptions:

(i) The local component \( \pi_q \) is a twist of the special representation. Equivalently, some twist of \( \pi_q \) has unramified central character and conductor \( q \).

(ii) \( \pi \) is "essentially unramified away from \( q \)" in the sense that \( \prod_{p \neq q} C(\pi_p) = \text{norm}(q)^{o(1)} \), where the product is over all places \( p \), \( C(\cdots) \) denotes the analytic conductor, and \( o(1) \) denotes a quantity tending to zero as the norm of \( q \) tends to \( \infty \).

\(^2\)The reader might ask whether such identities are known to hold for the ramified Euler factors at finite places. We believe this question may be answered affirmatively (e.g., by defining those local factors using known cases of local Langlands, and noting that this definition is compatible with the period formulas that we cite, which indeed depend only upon the unramified Euler factors). On the other hand, the bad Euler factors at finite places are irrelevant to our purposes, since bounds towards Ramanujan show that their presence has no effect on the subconvexity problem (see §4.2, item 3). The reader is thus free to replace every \( L(\cdots) \) in this article with \( L^{(S)}(\cdots) \), where \( S \) denotes the set of bad places and \( L^{(S)}(\cdots) \) the Euler product obtained by omitting factors from such places.
These assumptions may seem artificial, but we show in §5 that they are essential to a sufficiently restricted form of our method. Informally, they mean that (some twist of) \( \pi \) has essentially unramified central character and level \( \approx q \). (Twisting matters little here, since it does not change the quantities (1.2) and (1.3).) For example, we might take \( F = \mathbb{Q} \), so that \( q \) corresponds to a prime number \( p \), and take \( \pi \) corresponding to a normalized weight two newform on \( \Gamma_0(p) \).

Let \( \tau \) be a cuspidal automorphic representation of \( \text{PGL}_2(\mathbb{A}) \). We allow \( \tau \), like \( \pi \), to depend upon the varying prime \( q \), but our results are nontrivial only when this dependence is mild. For technical convenience, we impose the following local assumption:

(PS) Each local component of \( \tau \) belongs to the principal series.

For instance, this assumption is satisfied if \( \tau \) corresponds to a spherical Maass form on \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \), which is the case relevant for the basic “configuration space” forms of quantum unique ergodicity. This assumption could likely be removed with further work orthogonal to the primary novelty of this paper (see the conjecture of §2.15.1).

We will show that a subconvex bound for \( L(\pi \otimes \pi \otimes \chi, \frac{1}{2}) \) in the \( \pi \)-aspect with polynomial dependence upon the Hecke character \( \chi \) implies a subconvex bound for \( L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) \) in the \( \pi \)-aspect with polynomial dependence upon \( \tau \). More precisely, we impose the following hypothesis, inspired by Munshi’s work [32]:

(H) For some fixed \( c, A \geq 0, \delta > 0 \) and all unitary characters \( \chi \) of \( \mathbb{A}^\times / F^\times \),

\[
|L(\pi \otimes \pi \otimes \chi, \frac{1}{2})| \leq c C(\pi \otimes \pi \otimes \chi)^{1/4 - \delta} C(\chi)^A.
\]

Here and henceforth “fixed” means “independent of \( q \).” We will derive from this the following conclusion:

(SC) For some fixed \( c, A \geq 0 \) and \( \delta > 0 \),

\[
|L(\text{ad}(\pi) \otimes \tau, \frac{1}{2})| \leq c C(\text{ad}(\pi) \otimes \tau)^{1/4 - \delta} C(\tau)^A.
\]

**Theorem 1.** Under the stated assumptions, (H) implies (SC).

We formulate this slightly more precisely in §4.1.4.

Remark 1. As indicated above, Theorem 1 is already new in the special case that

- \( F = \mathbb{Q} \),
- \( \pi \) corresponds to a weight two cuspidal \( L^2 \)-normalized newform \( \varphi \) on \( \Gamma_0(p) \) for some large prime \( p \), and
- \( \tau \) corresponds to an essentially fixed Maass cusp form \( \Psi \) on \( \text{SL}_2(\mathbb{Z}) \).

(In this informal discussion, we refer to certain quantities \( X \) as “essentially fixed”: this means all estimates are required to depend “polynomially” upon such quantities in a sense which should be clear in each case.) Under the period-to-\( L \)-value dictionary (see for instance [39, §1] and references), the informal content of our result in this special case is that “subconvex bounds” for \( \langle \varphi(z), E(z) \varphi(\ell^2 z) \rangle \), where

- \( \ell \in \mathbb{Z}_{\geq 1} \) is essentially fixed, and
- \( E \) is an essentially fixed unitary Eisenstein series on \( \Gamma_0(\ell^2) \) of parameter \( 1/2 + it \), completed with the factor \( \xi(1 + 2it) \),

imply “subconvex bounds” (with weaker exponents) for \( \langle \varphi, \Psi \varphi \rangle \). An unconditional logarithmic savings for the latter was obtained in [34] using the Holowinsky–Soundararajan method [18].

Our result may thus be interpreted roughly as follows: to establish equidistribution with a power savings as \( p \to \infty \) of the sequence of probability measures
\( \mu_\varphi \) on \( SL_2(\mathbb{Z}) \backslash \mathbb{H} \) obtained by pushforward of the \( L^2 \)-mass of \( \varphi \) (cf. [34]), it suffices to prove a power savings estimate for the quantities \( \mu_\varphi(E) \) when \( E \) is a fixed Eisenstein series, together with mild generalizations of such quantities.

Our result is not the first concerning equidistribution on arithmetic quotients to exhibit a distinguished role played by Eisenstein series. Earlier works exhibiting this role include

1. Lindenstrauss’s [26] on arithmetic quantum unique ergodicity (AQUE) for Maass forms on \( SL_2(\mathbb{Z}) \backslash \mathbb{H} \), through the (implicit) role played by Eisenstein series in conditionally ruling out “escape of mass” prior to the unconditional results of Soundararajan [48];

2. Holowinsky–Soundararajan’s [18] on AQUE for holomorphic forms on \( SL_2(\mathbb{Z}) \backslash \mathbb{H} \) in that the additional smoothing implemented by [16, Thm 1.1] is necessary only when testing against Eisenstein series;

3. Einsiedler, Lindenstrauss, Michel and Venkatesh’s [7] on Duke’s theorem for cubic fields, through the use of Eisenstein series to establish a priori bounds concerning tightness and positivity of entropy.

4. Ghosh, Reznikov and Sarnak’s [12] concerning nodal domains of Maass forms, through its invocation of AQUE for \( SL_2(\mathbb{Z}) \backslash \mathbb{H} \) tested (exclusively) against Eisenstein series.

All of these works point to a distinguished role played by Eisenstein series in arithmetic equidistribution problems. The precise implication observed in this article seems particularly direct, striking, and counterintuitive. For instance, it shows also that the Eisenstein case of (prime level aspect) AQUE on congruence covers of \( SL_2(\mathbb{Z}) \backslash \mathbb{H} \) controls the general case on compact quotients \( \Gamma \backslash \mathbb{H} \) attached to non-split quaternion algebras.

Remark 2. In the context of Remark 1, it remains an open problem to obtain analogous savings on compact arithmetic quotients attached to non-split quaternion algebras (see e.g. [35, §2]); those would likely follow from the proof of Theorem 1 and strong enough logarithmic savings over the trivial bound in the Eisenstein case on congruence covers of \( SL_2(\mathbb{Z}) \backslash \mathbb{H} \) (e.g., the Eisenstein case of [39, Thm 2] for \( q \) prime but with \( \delta_2 \) large enough).

Remark 3. The dependence of the quantities \( \delta, A \) in (SC) upon those in (H) is effective. We do not explicate it here.

1.2. Overview of the proof. Over the past couple decades, many people have attempted to estimate \( L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) \) or the closely related quantity \( L(\pi \otimes \bar{\pi} \otimes \tau, \frac{1}{2}) \) (see (1.4)) by embedding \( \pi \) in a family \( F \) and trying to estimate some (possibly amplification-weighted) moment, such as the mean value

\[
S(F) := \sum_{\pi' \in F} L(\text{ad}(\pi') \otimes \tau, \frac{1}{2}). \tag{1.7}
\]

It is known that each summand \( L(\text{ad}(\pi') \otimes \tau, \frac{1}{2}) \) is nonnegative, hence that \( L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) \leq S(F) \). In the context of Remark 1, one might take for \( F \) the set of weight two normalized newforms on \( \Gamma_0(p) \); a Lindelöf-consistent estimate

\[
S(F) \ll p^{1+\varepsilon} \tag{1.8}
\]

would then recover the convexity bound for \( L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) \). In general, one might hope to derive a subconvex bound from a sharp mean value estimate over a sufficiently small (possibly amplification-weighted) family.
Although this approach has succeeded spectacularly for superficially similar problems (see e.g. [6, 14] and references), it remains a fantasy in the present setting: the estimates for $L(\text{ad}(\pi) \otimes \pi, \frac{1}{2})$ achieved this way fail even to approach the convexity bound (compare with [27, 23], for instance). A proof of (1.8) (let alone its amplified variant) seems inaccessible by existing techniques (cf. [21]).

Alternatively, one could try to estimate $L(\pi \otimes \pi \otimes \pi, \frac{1}{2})$ by using the triple product formula to relate it to a period integral $\langle \varphi_1, \varphi_2 \varphi_3 \rangle$ on $[\text{PGL}_2] := \text{PGL}_2(F) \setminus \text{PGL}_2(\mathcal{A}_F)$, with unit vectors $\varphi_1, \varphi_2, \varphi_3 \in \pi$ and $\varphi_3 \in \tau$, and applying the technique developed by Michel–Venkatesh [31] in their resolution of the subconvexity problem for $\text{GL}_2$. That technique proceeds via the Cauchy–Schwarz inequality followed by a spectral expansion on $L^2([\text{PGL}_2])$ (together with an “amplification” step that we elide here):

$$\langle \varphi_1, \varphi_2 \varphi_3 \rangle^2 \leq \langle \varphi_2 \varphi_3, \varphi_2 \varphi_3 \rangle = \langle \varphi_2 \rangle^2 |\varphi_3|^2 = \int_{\psi} \langle \varphi_2 \rangle^2 \langle \psi, |\varphi_3|^2 \rangle. \quad (1.9)$$

One encounters in such an attempt the unfortunate circularity that one cannot adequately estimate the contribution from $\psi \in \tau$ to the RHS of (1.9) without already knowing a subconvex bound for the quantity $L(\pi \otimes \pi \otimes \pi, \frac{1}{2})$ of primary interest.

The strategy pursued in this article is similar to that of Michel–Venkatesh, but with triple product integrals on $\text{PGL}_2(\mathcal{A}_F)$ replaced by Shimura–type integrals on the metaplectic double cover of $\text{SL}_2(\mathcal{A}_F)$. A circularity similar to that noted above arises in this approach; we manage to break it, as discussed below, by employing a crucial observation made in [37].

Turning to details, we use a period formula of Qiu [42, Thm 4.5] and local estimates to derive an integral representation

$$\frac{L(\text{ad}(\pi) \otimes \pi, \frac{1}{2})}{C(\text{ad}(\pi) \otimes \pi)^{1/4+o(1)}} = \langle \varphi_1 \varphi_2, \varphi_3 \rangle^2, \quad (1.10)$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product on $[\text{SL}_2] := \text{SL}_2(F) \setminus \text{SL}_2(\mathcal{A}_F)$, $\varphi_1$ belongs to $\pi$, $\varphi_2$ is an elementary theta function defined on the metaplectic double cover of $\text{SL}_2(\mathcal{A}_F)$, and $\varphi_3$ belongs to a suitable Waldspurger lift $\theta(\tau)$ of $\tau$; moreover, $||\varphi_i|| = 1$ for $i = 1, 2, 3$. (The proof of (1.10) uses the Shimizu correspondence; the idea of applying it here arose as a natural continuation of the works [35, 36, 38].)

We summarize how these vectors vary (see §2.17 for details):

- $\varphi_1$ is essentially a newvector in the varying representation $\pi$.
- $\varphi_2$ is essentially fixed.
- $\varphi_3$ is a varying vector in the essentially fixed representation $\theta(\tau)$, given in the “line model at $q$” by an $L^2$-normalized multiple of the characteristic function of the maximal ideal.

By Cauchy–Schwarz, the RHS of (1.10) is bounded by $||\varphi_1 \varphi_2||^2 = \langle |\varphi_1|^2, |\varphi_2|^2 \rangle$. We consider the spectral expansion of the latter inner product (regularized via [37]):

$$\langle |\varphi_1|^2, |\varphi_2|^2 \rangle = \langle |\varphi_1|^2, 1 \rangle \langle 1, |\varphi_2|^2 \rangle + \sum_{\psi} \langle |\varphi_1|^2, \psi \rangle \langle \psi, |\varphi_2|^2 \rangle + \text{(CSC)}, \quad (1.11)$$

where $\psi$ traverses an orthonormal basis for the space of cusp forms on $[\text{SL}_2]$ and (CSC) denotes the contribution of the continuous spectrum; it is an integral of $\langle |\varphi_1|^2, E \rangle \langle E, |\varphi_2|^2 \rangle$ for some Eisenstein series $E$. We estimate the RHS of (1.11) as follows:
Hypothesis (H) may be seen to imply an adequate estimate for (CSC).

(The fact that $\varphi_2$ is essentially fixed forces $\langle E, |\varphi_2|^2 \rangle$ to decay rapidly with respect to the parameters of $E$, so it suffices to bound the individual factors $\langle |\varphi_1|^2, E \rangle$, which are related to the $L$-values (1.3) via Rankin–Selberg theory.)

(ii) $|\varphi_2|^2$ is orthogonal to cusp forms [37], so the sum over $\psi$ vanishes.

(iii) We “amplify away” the contribution of the constant function, as in [31].

The proof is then complete.

We highlight three reasons why our approach may have been unanticipated:

1. The observation (ii) that $\langle |\varphi_2|^2, \psi \rangle = 0$ for cusp forms $\psi$ is the deus ex machina needed to break the apparent circularity of the argument: by the triple product formula, a bound for $\langle |\varphi_1|^2, \psi \rangle$ with $\psi \in \tau$ amounts to a bound for $L(\pi \otimes \pi \otimes \tau, \frac{1}{2})$, hence (by the factorization (1.4)) to one for $L(ad(\pi) \otimes \tau, \frac{1}{2})$.

2. The analytic theory of period integrals on $\text{PGL}_2$ was developed substantially and applied to the subconvexity problem in work of Bernstein–Reznikov [1], Venkatesh [53], Michel–Venkatesh [31], and others. We initiate here an analogous theory on the metaplectic double cover of $\text{SL}_2$, which we have attempted to develop robustly so as to be of general use. Before doing so, it was not obvious that an approach as indicated above should exist.

3. The strategy indicated following (1.7) may be implemented in our setting by applying Cauchy–Schwarz following (1.10) to the vector $\varphi_1 \in \pi$ belonging to the varying representation, as in the work of Michel–Venkatesh [31], Iwaniec–Michel [21], and many others. We instead apply Cauchy–Schwarz to $\varphi_3$; this amounts to embedding the fixed representation $\tau$ (rather than the varying representation $\pi$) into an (implicit) family. In this respect, our basic strategy is counter-traditional.

A similarly counter-traditional application of Cauchy–Schwarz may be seen in the method of Bykovsky [4] and its generalization by Fouvry–Kowalski–Michel [8], where an algebraically-twisted sums of the Fourier coefficients of a fixed $\text{GL}_2$ automorphic form are estimated by averaging over a varying family containing that form. It seems likely to us that some of Munshi’s recent arguments (e.g., those in [33]) may also be understood from this perspective.

Remark 4. The restriction that $\pi$ have “essentially prime level and trivial central character” seems serious at the moment. One could just as easily treat squarefree levels (i.e., allowing multiple independent varying primes $q$), but any further extension would be interesting. An analogue of Theorem 1 for archimedean, depth, or even “prime-squared level” aspects remains open.

The precise source of the present limitation of our method remains poorly understood by us. The issue is roughly that for aspects other than those pursued here, there do not seem to exist vectors for which (1.10) holds and for which the strategy indicated following (1.11) succeeds (see §5). We remain hopeful that a viable further extension of the strategy exists.

Acknowledgements. We thank Ph. Michel for many helpful and extended discussions concerning the work [31]. We gratefully acknowledge the support of NSF grant OISE-1064866 and SNF grant SNF-137488 during some of the work leading
2. Local preliminaries

The inputs to this section are:

- a local field $F$ not of characteristic 2, thus $F$ is a finite extension of either \(\mathbb{R}\) or \(\mathbb{Q}_p\) or (for \(p \neq 2\)) \(\mathbb{F}_p(t)\);
- a nontrivial unitary character \(\psi : F \to \mathbb{C}^{(1)} := \{z \in \mathbb{C}^\times : |z| = 1\}\); and
- some Haar measures on the groups \(G \in \{\text{GL}_1(F) = F^\times, \text{SL}_2(F), \text{PGL}_2(F)\}\).

We equip \(F\) with the \(\psi\)-self dual Haar measure \(dx\). We denote by \(|.| : F \to \mathbb{R}_{\geq 0}\) the normalized absolute value, so that \(d(cx) = |c| \, dx\).

2.1. Groups, measures, and general notation.

2.1.1. The metaplectic group. Denote by \(\text{Mp}_2(F)\) the metaplectic double cover of \(\text{SL}_2(F)\), defined using Kubota cocycles as the set of all pairs \((\sigma, \zeta) \in \text{SL}_2(F) \times \{\pm 1\}\) with the multiplication law \((\sigma_1, \zeta_1)(\sigma_2, \zeta_2) = (\sigma_1\sigma_2, \zeta_1\zeta_2c(\sigma_1, \sigma_2))\), where \(c\) is the cocycle

\[
c(\sigma_1, \sigma_2) := \left(\frac{x(\sigma_1\sigma_2)}{x(\sigma_1)}, \frac{x(\sigma_1\sigma_2)}{x(\sigma_2)}\right), \quad x \left(\begin{array}{cc} * & * \\ c & d \end{array}\right) := \begin{cases} d & \text{if } c = 0, \\ c & \text{if } c \neq 0 \end{cases}
\]

(2.1)

and \((,): F^\times/F^\times \times F^\times/F^\times \to \{\pm 1\}\) denotes the Hilbert symbol.

2.1.2. Group elements. As generators for \(\text{Mp}_2(F)\) we take for \(a \in F^\times, b \in F\) and \(\zeta \in \{\pm 1\}\) the elements

\[
n(b) = \left(\begin{array}{cc} 1 & b \\ 0 & 1 \end{array}\right), \quad t(a) = \left(\begin{array}{cc} a & a^{-1} \\ 0 & 1 \end{array}\right),
\]

\[
n'(b) := \left(\begin{array}{cc} 1 & b \\ 0 & 1 \end{array}\right), \quad w := \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right), \quad \varepsilon(\zeta) = (1, \zeta).
\]

They satisfy the relations \(n(b_1)n(b_2) = n(b_1 + b_2), n'(b_1)n'(b_2) = n'(b_1 + b_2), t(a_1)t(a_2) = t(a_1a_2)c((a_1, a_2)), t(a)n(b) = n(a^2b)t(a), t(a)n'(b) = n'(a^{-2}b)t(a), w(t(a)) = t(a^{-1})w, w^2 = t(-1), n'(b) = wn(-b)w^{-1}, \text{ and } wn(-a^{-1}) = n(a)t(a)wn(a)w^{-1} \).

For \(y \in F^\times\), set

\[
a(y) = \begin{pmatrix} y \\ 1 \end{pmatrix} \in \text{GL}_2(F).
\]

There are natural maps

\[
\text{Mp}_2(F) \to \text{SL}_2(F) \to \text{GL}_2(F) \to \text{PGL}_2(F).
\]

(2.2)

The images in \(\text{PGL}_2(F)\) of \(t(y)\) and \(a(y^2)\) coincide.

2.1.3. Genuine representations. Recall that a genuine representation \(\pi\) of \(\text{Mp}_2(F)\) is one for which \(\pi(\varepsilon(\zeta))v = \zeta v\) for all \(\zeta \in \{\pm 1\}\). An irreducible representation of \(\text{Mp}_2(F)\) is thus genuine if and only if it does not factor through \(\text{SL}_2(F)\).
2.1.4. Identifications. When appropriate, we allow ourselves to confuse elements of $\text{Mp}_2(F)$ with their images in $\text{SL}_2(F)$ and functions on $\text{SL}_2(F)$ with their pullbacks to $\text{Mp}_2(F)$. For example, given a pair $\pi_1, \pi_2$ of genuine unitary representations of $\text{Mp}_2(F)$ and vectors $u_1, v_1 \in \pi_1$, $u_2, v_2 \in \pi_2$, the “product of matrix coefficients” function $\text{Mp}_2(F) \ni g \mapsto \langle gu_1, v_1 \rangle \langle gu_2, v_2 \rangle \in \mathbb{C}$ identifies with a function on $\text{SL}_2(F)$.

2.1.5. Notation in the non-archimedean case. For $F$ non-archimedean, we denote by $\mathfrak{o}$ its maximal order, $\mathfrak{p}$ its maximal ideal, and $q := \#\mathfrak{o}/\mathfrak{p}$. Set $\zeta_F(s) := (1 - q^{-s})^{-1}$.

2.1.6. “The unramified case”. We use this phrase to signify that

- $F$ is non-archimedean and of odd residual characteristic (i.e., $q$ is odd),
- the additive character $\psi$ is unramified (i.e., restricts trivially to $\mathfrak{o}$ but not to $\mathfrak{p}^{-1}$), and
- the chosen Haar measures on $F$, $\text{GL}_1(F) \cong F^\times$, $\text{SL}_2(F)$ and $\text{PGL}_2(F)$ all assign volume one to maximal compact subgroups.

2.1.7. Implied constants. We want implied constants to be uniform when $F$ traverses the set of completions of a given number field and the character $\psi$ and the choices of Haar measure traverse the local components of associated global data (as in §3.1.2 below). For this reason, we call an object $X$ fixed if it admits the following dependencies:

- We always allow $X$ to depend upon the degree of the local field $F$, and also upon fixed quantities mentioned earlier within a given argument.
- – If we are not in the unramified case, then we allow $X$ to depend upon $F$, $\psi$ and all choices of Haar measures.
- – In the unramified case, we do not allow such dependence.

Standard asymptotic notation is defined in terms of this notion:

- The equivalent notations $X = O(Y)$, $X \ll Y, Y \gg X$ signify that $|X| \leq c|Y|$ for some fixed $c \in \mathbb{R}_{\geq 0}$; notation such as $X \ll_f Y$ allows an additional dependence of $c$ upon $f$.
- $X \asymp Y$ signifies that $X \ll Y \ll X$.

The symbols $\varepsilon$ and $A$ denote respectively a sufficiently small and large fixed positive quantity whose precise value may change from one occurrence to another.

2.1.8. Norms. For $G$ one of the groups $\text{SL}_2$, $\text{PGL}_2$, $\text{GL}_2$, we fix once and for all a linear embedding $G \hookrightarrow \text{SL}_N$ and define norms $\|\|_{G(F)}$ on $G(F)$ using this embedding, as in [31, §2.1.3, §2.1.7]. We extend this definition to $\text{Mp}_2(F)$ by pulling back under the projection $\text{Mp}_2(F) \rightarrow \text{SL}_2(F)$.

We also define for $g \in \text{PGL}_2(F)$ the adjoint norm $\|\text{Ad}(g)\|$ as in [31, §2.1.3] and extend this definition to the groups $\text{GL}_2(F), \text{SL}_2(F), \text{Mp}_2(F)$ by pulling back under the maps (2.2). For $y \in F^\times$, one has $\|\text{Ad}(t(y))\| \propto |y|^2 + |y|^{-2}$.

2.2. Weil representation. Denote by $\rho_\psi$ the Weil representation of $\text{Mp}_2(F)$ attached to the additive character $\psi$. We realize it on the Schwartz–Bruhat space $S(F)$ and equip it with the unitary structure coming from $L^2(F)$. It splits as a sum $\rho_\psi = \rho_\psi^+ \oplus \rho_\psi^-$ of two invariant irreducible subspaces $\rho_\psi^+ = \{ \phi \in S(F) : \phi(-x) = \phi(x) \}$ and $\rho_\psi^- = \{ \phi \in S(F) : \phi(-x) = -\phi(x) \}$.

We record the action of generators. For $\xi \in F^\times$, let $\psi_\xi : F \rightarrow \mathbb{C}^{(1)}$ denote the nontrivial unitary character of $F$ given by $\psi_\xi(x) := \psi(\xi x)$. Let $\gamma(\psi_\xi) \in \{ z \in$
\( \mathbb{C} : z^8 = 1 \) denote the Weil constant, thus \( \gamma(\psi) = |2\xi|^{1/2} \int_{x \in F} \psi(\xi x^2) \, dx \) as generalized functions of \( \xi \in F^\times \). Set \( \chi\psi(\xi) := \gamma(\psi)/\gamma(\psi) \in \{ z \in \mathbb{C} : z^4 = 1 \} \); then \( \chi\psi(\xi_1, \xi_2) = (\xi_1, \xi_2)\chi\psi(\xi_1)\chi\psi(\xi_2) \). For \( \phi \in \mathcal{S}(F) \), define \( \phi^\wedge(y) := |2|^{1/2} \int_F \phi(x)\psi(2xy) \, dx \); the normalization is so that \( (\phi^\wedge)^\wedge(x) = \phi(-x) \).

- \( \rho_\psi(n(b))\phi(x) = \psi(bx^2)\phi(x) \).
- \( \rho_\psi(w)\phi = \gamma(\psi)\phi^\wedge \).
- \( \rho_\psi(t(a))\phi(x) = \chi\psi(a)|a|^{1/2}\phi(ax) \).
- \( \rho_\psi(\xi(\zeta))\phi = \zeta\phi \).

### 2.3. Induced representations.

#### 2.3.1. \( \text{SL}_2 \).

Denote by \( \mathcal{I}_{\text{SL}_2}(\chi) \) the representation of \( \text{SL}_2(F) \) unitarily induced by the character \( t(y) \rightarrow \chi(y) \) of the diagonal torus. Its **induced model** is the space of smooth functions \( f : \text{SL}_2(F) \rightarrow \mathbb{C} \) satisfying \( f(n(x)t(y)g) = |y|\chi(y)f(g) \).

#### 2.3.2. \( \text{Mp}_2 \).

Denote by \( \mathcal{I}_{\text{Mp}_2}(\chi) \) the representation of \( \text{Mp}_2(F) \) unitarily induced by the character \( t(y)\varepsilon(\zeta) \rightarrow \zeta\chi\psi(y)\chi(y) \) of the diagonal torus, where \( \chi\psi \) is as in \( \S 2.2 \). Its **induced model** is defined as in the \( \text{SL}_2 \) case to be the space of smooth functions \( f : \text{Mp}_2(F) \rightarrow \mathbb{C} \) satisfying \( f(n(x)\varepsilon(\zeta)t(y)g) = |y|\zeta\chi\psi(y)\chi(y)f(g) \). We henceforth abbreviate \( \mathcal{I}_{\text{Mp}_2}(\chi) := \mathcal{I}_{\text{Mp}_2}(\chi) \).

#### 2.3.3. \( \text{PGL}_2 \).

Denote by \( \mathcal{I}_{\text{PGL}_2}(\chi) \) the representation of \( \text{PGL}_2(F) \) unitarily induced by the character \( a(y) \rightarrow \chi(y) \) of its diagonal torus. Its **induced model** is the space of smooth functions \( f : \text{PGL}_2(F) \rightarrow \mathbb{C} \) satisfying \( f(n(x)a(y)g) = |y|^{1/2}\chi(y)f(g) \).

#### 2.3.4. The line model.

For any of the above induced representations \( \mathcal{I}_G(\chi) \), the restriction map sending \( f \) to the function \( F \ni x \mapsto f(n'(x)) \) is injective. We call its image the **line model**. The line model contains \( C_c^\infty(F) \). We shall occasionally use the line model to specify vectors.

#### 2.3.5. Unitarity.

Let \( G \in \{ \text{SL}_2(F), \text{Mp}_2(F), \text{PGL}_2(F) \} \). If \( \chi \) is unitary, then the induced representation \( \mathcal{I}_G(\chi) \) is unitary; it is also irreducible unless \( G = \text{SL}_2(F) \) and \( \chi \) is a nontrivial quadratic character. In any event, we normalize its unitary structure via the line model: \( \|f\|^2 := \int_{x \in F} |f(n'(x))|^2 \).

#### 2.3.6. From \( \text{SL}_2 \) to \( \text{PGL}_2 \).

Let \( \chi, \omega \) be characters of \( F^\times \) satisfying \( \omega^2 = \chi \). Let \( j : \text{SL}_2(F) \rightarrow \text{PGL}_2(F) \) denote the natural map. Every element of \( \text{PGL}_2(F) \) is of the form \( a(y)j(s) \) for some \( y \in F^\times, s \in \text{SL}_2(F) \). Given \( f \in \mathcal{I}_{\text{SL}_2}(\chi) \), write

\[
f^\omega(a(y)j(s)) := |y|^{1/2}\omega(y)f(s) \tag{2.3}
\]

We verify readily that the formula (2.3) defines a function \( f^\omega : \text{PGL}_2(F) \rightarrow \mathbb{C} \) for which \( f^\omega \circ j = f \), hence that restriction of functions induces an \( \text{SL}_2(F) \)-equivariant isomorphism

\[
\mathcal{I}_{\text{PGL}_2}(\omega)|_{\text{SL}_2(F)} \cong \mathcal{I}_{\text{SL}_2}(\chi) \tag{2.4}
\]

with inverse \( f \mapsto f^\omega \). We note the following:

- The isomorphism (2.4) identifies the respective line models.
- The map \( f \mapsto f^\omega \) is an isometry whenever \( \chi, \omega \) are unitary.
2.3.7. *Metaplectic intertwining operators.* Temporarily abbreviate \( \sigma_\chi := \mathcal{I}_{\text{MP}}(\chi) \).

We shall have occasion to consider the standard intertwining operators

\[
M_\chi : \sigma_\chi \to \sigma_{\chi^{-1}}.
\]

\[
M_\chi f(g) := \int_{x \in F} f(\phi(x)g) \, dx.
\]

These integrals and those that follow should be interpreted in the usual ways, e.g., by regularized integration or meromorphic continuation in \( \chi \); we suppress discussion of this point for the sake of brevity. We record here a detailed asymptotic study of \( M_\chi \); this will be applied subsequently to certain technical estimates involving complementary series representations.

Recall the local \( \gamma \)-factor \( \gamma(\chi, s, \psi) = \varepsilon(\chi, s, \psi)L(\chi^{-1}, 1 - s)/L(\chi, s) \), characterized by Tate’s local functional equation: for each \( \phi \) in the Schwartz–Bruhat space \( S(F) \),

\[
\int_{x \in F^\times} \phi(x)\chi(x)|x|^s \frac{dx}{|x|} = \frac{\int_{y \in F^\times} \int_{y \in F} \phi(y)\psi(-yx) \, dy \chi^{-1}(x)|x|^{1-s} \frac{dx}{|x|}}{\gamma(\chi, s, \psi)}.
\]

It is holomorphic for \( 0 < \text{Re}(\chi) + \text{Re}(s) < 1 \), and satisfies the Stirling asymptotic (see [31, §3.1.12])

\[
\gamma(\chi, s, \psi) \asymp C(\chi)^{1/2 - \text{Re}(\chi) - \text{Re}(s)} \text{ if } \varepsilon < \text{Re}(\chi) + \text{Re}(s) < 1 - \varepsilon
\]

(here \( C(\cdot) \) denotes the analytic conductor), the distributional identity

\[
\int_{x \in F^\times} \psi(x)\chi(x)|x|^s \frac{dx}{|x|} = \frac{\chi(1)}{\gamma(\chi, s, \psi)}
\]

and the relation

\[
\gamma(\chi, 1/2, \psi)\gamma(\chi^{-1}, 1/2, \psi) = \chi(-1).
\]

We define the normalized intertwining operators

\[
R_\chi := \frac{|2|_F^{-1/2}\gamma(\chi^2, 0, \psi)}{\gamma(\psi)\gamma(\chi, 1/2, \psi)} M_\chi,
\]

which vary holomorphically on \( \{ \chi : \text{Re}(\chi) > -1/2 \} \); the normalization is justified further below. These operators are “diagonalized” by the map

\[
\mathcal{K} : \sigma_\chi \to \{ \text{functions } F^\times \to \mathbb{C} \}
\]

\[
\mathcal{K} f(\xi) := \int_{u \in F} f(\phi(u))\psi(-\xi u) \, du \text{ for } \xi \in F^\times,
\]

which packages together the standard Whittaker functionals on \( \sigma_\chi \); for detailed discussion of this and what follows, see [49, 50] and references. By Fourier inversion, \( \mathcal{K} \) is injective. Its image is closely related to the Kirillov-type model of \( \sigma_\chi \) (see [43, p513], [54]); one has

\[
\mathcal{K}_n(x) f(\xi) = \psi(\xi x)\mathcal{K}_n f(\xi), \quad \mathcal{K}_n(y) f(\xi) = |y|\chi_\psi(y)\chi^{-1}(y)\mathcal{K}_n(y^2 \xi).
\]

Using the change of variables \( x \mapsto -1/x \) and the identity \( wn(-1/x)wn(u) = n(x)t(x)wn(x + u) \), we see that

\[
\mathcal{K} R_\chi f(\xi) = G(\chi, \psi, \xi)\mathcal{K} f(\xi),
\]

where

\[
G(\chi, \psi, \xi) := \frac{|2|_F^{-1/2}\gamma(\chi^2, 0, \psi)}{\gamma(\psi)\gamma(\chi, 1/2, \psi)} \int_{x \in F} \psi(\xi x)\chi(x)\chi_\psi(x) \frac{dx}{|x|}.
\]
Let $\chi(x) := (x, \xi)$ denote the quadratic character given by the Hilbert symbol.

**Lemma 1.** One has

$$G(\chi, \psi, \xi) = \chi^{-1}(4\xi)\chi_\psi(\xi)^{-1} \frac{\gamma(\chi\chi_\xi, 1/2, \psi)}{\gamma(\chi, 1/2, \psi)}. \quad (2.9)$$

**Proof.** This is essentially the main result of [49] and [13, §6]. More precisely, the latter is equivalent to the identity

$$\int_{x \in F} \psi(x)\chi(x)\chi_\psi(x) \frac{dx}{|x|} = \gamma(\psi)|2|_F^{1/2} \chi^{-1}(4) \frac{\gamma(\chi, 1/2, \psi)}{\gamma(\chi^2, 0, \psi)}. \quad (2.10)$$

One may apply to the integral in the definition of $G(\chi, \psi, \xi)$ the change of variables $x \mapsto x/\xi$ and the identity $\chi_\psi(x/\xi) = \chi_\psi(\xi)^{-1}\chi_\psi(x)\chi(x)$ to derive (2.9) from (2.10).

It follows in particular that

$$R_{\chi^{-1}} \circ R_\chi = 1, \quad (2.11)$$

which is equivalent to identities stated in [41, 4.11, 4.17].

**Lemma 2.** Suppose that $\chi = \eta|\cdot|^c$, where $\eta^2 = 1$ and $c$ is a real number with $|c| \leq 1/2 - \varepsilon$ for some fixed $\varepsilon > 0$. Then $G(\chi, \psi, \xi) > 0$ and

$$G(\chi, \psi, \xi) = |\xi|^{-c}(C(\eta\chi_\xi)/C(\eta))^{-c}\beta(\chi, \psi, \xi), \quad (2.12)$$

where

(i) $\beta(\chi, \psi, \xi) > 1$,

(ii) $\beta(\chi, \psi, \xi) = \beta(\chi, \psi, \xi z^2)$ for all $z \in F^\times$,

(iii) $\beta(\chi, \psi, \xi) > 0$, and

(iv) if $c = 0$, then $G(\chi, \psi, \xi) = \beta(\chi, \psi, \xi) = 1$.

**Proof.** (i) and (ii) follow from (2.9) and (2.6), while (iv) follows from (iii) and the consequence $|G(\eta, \psi, \xi)| = 1$ of (2.9). We now verify (iii). Write $A \sim B$ to denote that $A/B > 0$, and write $\eta = \chi_t$ for some $t \in F^\times$. Then $\gamma(\chi\chi_\xi, 1/2, \psi)/\gamma(\chi, 1/2, \psi) \sim \gamma(\chi_t\xi, 1/2, \psi)/\gamma(\chi_t, 1/2, \psi)$ and $\chi^{-1}(4\xi)\chi_\psi(\xi)^{-1} \sim \chi_t(\xi)\chi_\psi(\xi) = \chi_t(\xi)\chi_\psi(\xi)$, so the positivity follows from the identity

$$\chi_\psi(\xi) = \gamma(\chi_t, 1/2, \psi) \text{ for all } \xi \in F^\times, \quad (2.13)$$

or equivalently, $\chi_\psi(\xi)\gamma(\chi_t, 1/2, \psi) = 1$, for which we refer to [51] (see also [11, App. B, (2d)] and [24]).

---

3For the convenience of the reader and as a check of normalizations, we sketch a proof of (2.10) modulo convergence issues, which may be addressed by suitably interpreting each step as an identity of distributions on $F^\times$ or $F^\times \times F^\times$. We apply Fourier inversion to $F \ni b \mapsto \psi(b^2x)$ to obtain $\psi(x) = \int_{y \in F} \psi(y)\int_{b \in F} \psi(b^2x-by) \, db \, dy$. By completing the square and using the identity $\gamma(\psi_x) = |2x|^{1/2} \int_{y \in F} \psi(xy^2) \, dy$, we deduce that $\psi(x)\chi_\psi(x) = \gamma(\psi)|2x|^{-1/2} \int_{y \in F} \psi(y-y^2x^2/4x) \, dy$. We insert this into the LHS of (2.10), giving

$$\int_{x \in F} \psi(x)\chi(x)\chi_\psi(x) \frac{dx}{|x|} = \gamma(\psi)\int_{x,y \in F} |2x|^{-1/2}\chi(x)\psi(y-y^2/4x) \frac{dx}{|x|} \, dy.$$
2.4. Unramified representations and vectors. Assume for §2.4 that $F$ is nonarchimedean. As usual, we say that a vector in a representation of one of the groups $\mathrm{GL}_1(F) = F^\times, \mathrm{SL}_2(F), \mathrm{PGL}_2(F)$ is unramified if it is invariant by the standard maximal compact subgroup. In the unramified case (§2.1.6), a vector in a representation of $\mathrm{Mp}_2(F)$ is unramified if it is invariant by the image of the standard lift to $\mathrm{Mp}_2(F)$ of the standard maximal compact subgroup of $\mathrm{SL}_2(F)$ (see e.g. [37, §4.5]). A representation of any of the above groups is unramified if it is irreducible and contains a nonzero unramified vector. For example:

- The unramified characters of $F^\times$ are of the form $|.|^s$ for $s \in \mathbb{C}$.
- $\rho_\psi^+$ is unramified in the unramified case.
- For an unramified character $\chi$ of $F^\times$ and $G \in \{\mathrm{SL}_2, \mathrm{PGL}_2, \mathrm{Mp}_2\}$, the induced representation $I_G(\chi)$ is unramified if it is irreducible and if, when $G = \mathrm{Mp}_2$, we are in the unramified case.

2.5. Whittaker models. Recall that an irreducible representation $\pi$ of $\mathrm{GL}_2(F)$ is called generic if for some (equivalently, any) nontrivial unitary character $\psi'$ of $F$, $\pi$ admits a $\psi'$-Whittaker model, i.e., a realization in the space of functions $W : G \to \mathbb{C}$ satisfying $W(n(x)g) = \psi'(x)W(g)$ on which $G$ acts by right translation; this is the case precisely when $\dim(\pi) \neq 1$.

The restriction map sending $W$ to the function $F^\times \ni y \mapsto W(a(y))$ is injective. Its image is called the Kirillov model (or more verbosely, $\psi'$-Kirillov model). The Kirillov model contains $C_c^\infty(F^\times)$.

Assume that $\pi$ is unitary. Then the the norm $||W||^2 := \int_{y \in F^\times} |W(a(y))|^2$ is $G$-invariant. When we write “let $\pi$ be a generic unitary representation of $\mathrm{GL}_2(F)$, realized in its $\psi'$-Whittaker model,” we always normalize the unitary structure in this way.

2.6. Local Waldspurger packets. Given an irreducible representation $\tau$ of $\mathrm{PGL}_2(F)$, one may define (see [54, 9]) a Waldspurger packet $\mathcal{W}_\psi(\tau)$ consisting of either one or two genuine irreducible representations of $\mathrm{Mp}_2(F)$; it is denoted $\{\sigma^+\}$ in the former case and $\{\sigma^+, \sigma^-\}$ in the latter, where the labeling by $\pm$ is defined using the local $\psi$-theta correspondence and Jacquet–Langlands correspondence. One has $\# \mathcal{W}_\psi(\tau) = 2$ if and only if $\tau$ belongs to the discrete series. If $\pi$ is generic, then the $\sigma^\pm$ are not isomorphic to even Weil representations $\rho_\psi^{\pm}$.

If $\tau$ is a generic irreducible principal series representation of $\mathrm{PGL}_2(F)$, say $\tau = I_{\mathrm{PGL}_2}(\chi)$, then $\mathcal{W}_\psi(\tau)$ is a singleton $\{\sigma\}$ consisting of the generic irreducible principal series representation $\sigma = I_{\mathrm{Mp}_2}(\chi)$ of $\mathrm{Mp}_2(F)$.

Every generic unramified irreducible representation $\tau$ of $\mathrm{PGL}_2(F)$ is of the form $I_{\mathrm{PGL}_2}(\chi)$ for some unramified character $\chi$. Its Waldspurger packet is the singleton $\{\sigma^+ = \sigma\}$ with $\sigma \cong I_{\mathrm{Mp}_2}(\chi)$. In the unramified case, $\sigma$ is unramified.

2.7. Complementary series.

2.7.1. Definitions. Let $G \in \{\mathrm{Mp}_2(F), \mathrm{PGL}_2(F), \mathrm{GL}_2(F)\}$. Let $\pi$ be an irreducible unitary representation of $G$, assumed genuine in the case $G = \mathrm{Mp}_2(F)$. For $c \in (0, 1/2)$, we say that $\pi$ is a complementary series of parameter $c$ if:

For $G \in \{\mathrm{PGL}_2(F), \mathrm{GL}_2(F)\}$, these exhaust the non-tempered generic unitary representations. For $G = \mathrm{Mp}_2(F)$ and $F \neq \mathbb{C}$, they exhaust the non-tempered genuine irreducible unitary representations that are not isomorphic to even Weil representations $\rho_\psi^{\pm}$, or equivalently, those
• For $G \in \{\text{Mp}_2(F), \text{PGL}_2(F)\}$, there is a quadratic character $\eta$ of $F^\times$ so that $\pi \cong I_G(1,\eta)$.

• For $G = \text{GL}_2(F)$, there is a unitary character $\xi$ of $F^\times$ so that $\pi$ is isomorphic to the unitarily normalized induction of $(|\cdot|^\xi, |\cdot|^{-\xi})$.

2.7.2. Temperedness. For $\vartheta \in (0, 1/2)$, we say that $\pi$ is $\vartheta$-tempered if either $\pi$ is tempered (see [5]) or $\pi$ is a complementary series of parameter $c \leq \vartheta$. We henceforth set $\vartheta := 7/64$. It is known then that if $G \in \{\text{PGL}_2(F), \text{GL}_2(F)\}$ and $\pi$ is obtained as the local component of a cuspidal automorphic representation, then $\pi$ is $\vartheta$-tempered [25, 2]; if $G = \text{PGL}_2(F)$ and $\sigma \in \text{Wd}_\vartheta(\pi)$, then it follows from the discussion of §2.6 that $\sigma$ is $\vartheta$-tempered.

Remark. We make use of the fact that $\vartheta < 1/6$ to address some technical convergence issues in our local arguments. Assuming that those issues can be addressed more generally, it seems that any value $\vartheta < 1/4$ would suffice for the purposes of proving Theorem 1.

2.7.3. Unitarity. Temporarily abbreviate $\sigma_\chi := I_{\text{Mp}_2}(\chi)$ for a character $\chi$ of $F^\times$. We have sesquilinear pairings $(,)$ : $\sigma_\chi \otimes \sigma_\chi \rightarrow \mathbb{C}$ given by $(f_1, f_2) := \int_{x \in F} f_1(n(x))f_2(n(x))$. We noted in §2.3.5 that if $\chi$ is unitary, so that $\chi = \overline{\chi}^{-1}$, then $\sigma_\chi$ is unitary, with invariant norm $\|f\|^2 = (f, f)$, which may be expressed in terms of the Kirillov-type map $f \mapsto Kf$ from §2.3.7 as $\|f\|^2 = \int_{\xi \in F^\times} |Kf(\xi)|^2 d\xi$.

Suppose now that $\sigma_\chi$ is a complementary series of parameter $c \in (0, 1/2)$. We then have $\chi = |\cdot|^c \eta$ for some quadratic character $\eta$ of $F^\times$, and $\|f\|^2 := (R_\chi f, f)$ ($f \in \sigma_\chi$) defines an invariant norm (cf. §2.3.7 and [41, p271, p278]); in terms of $f \mapsto Kf$ from §2.3.7,

$$\|f\|^2 = \int_{\xi \in F^\times} |Kf(\xi)|^2 G(\chi, \psi, \xi) d\xi.$$  
(Recall from §2.3.7 that $G(\chi, \psi, \xi) > 0$.)

2.8. Newvectors. Assume in §2.8 that $F$ is non-archimedean.

2.8.1. Notation. For a generic representation $\pi$ of $\text{GL}_N(F)$, the conductor $C(\pi)$ may be written $c(\pi) \in \mathbb{Z}_{>0}$ for some $c(\pi) \in \mathbb{Z}_{>0}$.

For $n \in \mathbb{Z}_{>0}$, denote by $K_0[n]$ the subgroup of $\text{GL}_2(\sigma)$ consisting of elements with lower-left entry in $p^n$.

2.8.2. Summary of newvector theory. Let $\pi$ be a generic irreducible representation of $\text{GL}_2(F)$ with central character $\omega_\pi : F^\times \rightarrow \mathbb{C}^\times$. It is known that $c(\pi) \geq c(\omega_\pi)$. For $n \geq c(\omega_\pi)$ denote also by $\omega_\pi$ the character of $K_0[n]$ given by

$$\omega_\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) := \begin{cases} \omega_\pi(d) & \text{if } d \in \mathfrak{o}^\times, \\ 1 & \text{otherwise} \end{cases}$$

and by $\pi[n]$ the space of vectors $\varphi \in \pi$ satisfying $\pi(g)\varphi = \omega_\pi(g)\varphi$ for all $g \in K_0[n]$. It is known that $\dim \pi[n] = \min(0, 1 + n - c(\pi))$. A newvector is a nonzero element of the one-dimensional space $\pi[c(\pi)]$.

Irreducible representations belonging to the Waldspurger packet of a non-tempered generic irreducible unitary representation. When $F = \mathbb{C}$, there are also genuine complementary series representations of $\text{Mp}_2(\mathbb{C}) \cong \text{SL}_2(\mathbb{C}) \times \{\pm 1\}$ of parameter $1/2 < c < 1$, which play no role here.
2.8.3. Principal series. Let $\chi$ be a character of $F^\times$ with $\chi^2 \neq |.|^{\pm 1}$. Then $\pi := I_{\mathrm{PGL}_2}(\chi)$ is irreducible and generic with $c(\pi) = 2c(\chi)$.

2.8.4. The special representation. Let $\operatorname{Sp}$ denote the special representation of $\mathrm{PGL}_2(F)$; it is the irreducible quotient of $I_{\mathrm{PGL}_2}(|.|^{1/2})$ and satisfies $c(\pi) = 1$. Assume that $\psi$ is unramified. A newvector may then be given in the Kirillov model of $\pi$ by the formula $W(a(y)) := 1_s(y)|y|$ (see for instance [47]).

2.9. Sobolev norms. Given an irreducible unitary representation $\pi$ of one of the groups $\mathrm{SL}_2(F), \operatorname{Mp}_2(F), \mathrm{PGL}_2(F), \mathrm{GL}_2(F)$, we define a family of Sobolev norms $S_d (d \in \mathbb{R})$ on $\pi$ following the recipe of [31, §2] and [37, §4.6, §5.3]. They have the form $S_d (v) := \| \Delta^d v \|$, where $\Delta$ is a certain invertible positive self-adjoint linear operator. We write $S$ to denote a Sobolev norm of the form $S_d$ for some unspecified fixed large enough $d$ (the “implied index”).

Remark. To follow the main arguments of the paper, it suffices to keep in mind the following property of the Sobolev norms: if $F$ is non-archimedean and $\varphi$ is a vector invariant by the $r$th standard principal congruence subgroup, then $S(\varphi) = \| \varphi \|^{2(r)}$.

2.10. Change of polarization. Let $\chi$ be a character of $F^\times$ with $\operatorname{Re}(\chi) > -1$. In [37, §4.12], we defined an $\operatorname{Mp}_2(F)$-equivariant map $I_{\chi} : \rho_\psi \otimes \overline{\rho_\psi} \rightarrow I_{\mathrm{SL}_2}(\chi)$. The definition is not important for our present purposes, but we record it for convenience: for $\phi \in \rho_\psi \otimes \overline{\rho_\psi}$, the function $I_{\chi}(\phi) : \mathrm{SL}_2(F) \rightarrow \mathbb{C}$ is given by the Tate integral

$$I_{\chi}(\phi)(\sigma) = \int_{y \in k^\times} |y| \chi(y) \mathcal{F} \phi(ye_2 \sigma) \, d^\times y,$$

where $\mathcal{F} \phi$ is the partial Fourier transform

$$\mathcal{F} \phi(y_1, y_2) := \int_{t \in k} \phi((\rho(y_1, t)) \psi(y_2 t)) \, dt = \int_{\mathfrak{t} \in k} \phi(y_1 + t, \frac{y_1 - t}{2}) \psi(y_2 t) \, dt.$$

For our purposes, the relevant properties of the intertwiner $I_{\chi}$ are the following (see [37, §4.12]):

- If we are in the unramified case and if $\chi$ is unramified, then $L(\chi, 1)^{-1} I_{\chi}$ preserves unramified elements; more precisely, it sends $1_e \otimes 1_e$ to the unramified vector taking the value $1$ at the identity.
- If $\chi$ is unitary, then for each fixed $d$ one has $S_d(I_{\chi}(\phi_1 \otimes \phi_2)) \ll S(\phi_1) S(\phi_2)$.

2.11. Bounds for matrix coefficients and varia.

2.11.1. Let $\Xi : \mathrm{PGL}_2(F) \rightarrow \mathbb{R}^\times_+$ denote the Harish–Chandra function for $\mathrm{PGL}_2(F)$, i.e., the matrix coefficient of the spherical vector in the normalized induction of the trivial character of the Borel, normalized so that $\Xi(1) = 1$. Explicitly, if we define $\text{ht} : \mathrm{PGL}_2(F) \rightarrow \mathbb{R}^\times_+$ using the Iwasawa decomposition by the formula $\text{ht}(n(x)a(y)k) := |y|$, then

$$\Xi(g) = \int_{k \in \mathrm{K}_{\operatorname{PGL}_2(F)}} \text{ht}(kg)^{1/2},$$

where the integral is taken with respect to the probability Haar on the standard maximal compact subgroup $\mathrm{K}_{\operatorname{PGL}_2(F)}$ of $\mathrm{PGL}_2(F)$. One has

$$\| \operatorname{Ad}(s) \|^{-1} \ll \Xi(s) \ll \| \operatorname{Ad}(s) \|^{-1+\varepsilon}. \quad (2.14)$$
More precisely, with respect to the Cartan decomposition $s = k_1 a(y) k_2$, one has $\| \text{Ad}(s) \| \approx t := |y| + |y|^{-1}$ and $\Xi(s) \approx t^{-1} \log(t)$.

We denote also by $\Xi$ its pullback to $\text{SL}_2(F)$ or to $\text{Mp}_2(F)$. (The pullback to $\text{SL}_2(F)$ is the Harish–Chandra function for $\text{SL}_2(F)$.)

By [5], the function $\Xi$ controls the matrix coefficients of any tempered irreducible unitary representation $\pi$ of $G \in \{ \text{Mp}_2(F), \text{GL}_2(F) \}$: for $s \in G$ and $\varphi, \varphi' \in \pi$, one has $\langle \pi(s) \varphi, \varphi' \rangle \ll \Xi(s) S(\varphi) S(\varphi')$. We record below the generalizations and refinements of this estimate in the $\vartheta$-tempered case.

One has $\int_G \Xi^{2+\varepsilon} < \infty$ for any $G$ as above.

2.11.1. Let $\pi$ be a $\vartheta$-tempered irreducible unitary representation of $\text{GL}_2(F)$, in particular, $\pi$ is generic. For each $W$ in the Whittaker model $W(\pi, \psi')$ corresponding to some fixed nontrivial unitary character $\psi'$ of $F$, we have (see [31, 3.2.3])

$$ W(a(y)) \ll \min\{|y|^{1/2-\varepsilon}, |y|^{-A}\} S(W). \quad (2.15) $$

Let $\varphi, \varphi' \in \pi$ and $s \in \text{GL}_2(F)$. Then (see [31, §2.5.1])

$$ \langle \pi(s) \varphi, \varphi' \rangle \ll \Xi(s)^{-2\theta} S(\varphi) S(\varphi'). \quad (2.16) $$

Since $\Xi(s) \ll \log(t)/t^{1/2}$ with $t := 3 + \| \text{Ad}(s) \|$, it follows that

$$ \langle \pi(s) \varphi, \varphi' \rangle \ll \| \text{Ad}(s) \|^{-1/2+\varepsilon} S(\varphi) S(\varphi'). \quad (2.17) $$

2.11.2. Let $\phi, \phi' \in \rho_\psi$. It follows from the Sobolev lemma (see [37, §4.9]) that $\| \phi \|_{L^\infty}, \| \phi \|_{L^1} \ll S(\phi)$, hence for $y \in F^\times$ that

$$ \rho_\psi(t(y))\phi(1) = \chi_\psi(y)|y|^{1/2} \phi(y) \ll \min\{|y|^{1/2}, |y|^{-A}\} S(\phi) \quad (2.18) $$

and

$$ \langle \rho_\psi(t(y))\phi, \phi' \rangle \ll (|y| + |y|^{-1})^{-1/2} S(\phi) S(\phi'). \quad (2.19) $$

Using the Cartan decomposition on $\text{Mp}_2(F)$, it follows that

$$ \langle \rho_\psi(s)\phi, \phi' \rangle \ll \| \text{Ad}(s) \|^{-1/4} S(\phi) S(\phi') \ll \Xi(s)^{1/2} S(\phi) S(\phi'). \quad (2.20) $$

2.11.4. Let $\sigma$ be a unitarizable $\vartheta$-tempered principal series representation of $\text{Mp}_2(F)$, thus $\sigma = I_{\text{Mp}_2}(\chi)$ where $\chi$ is either unitary or is of the form $\eta \cdot |c|^c$ with $\eta$ quadratic and $0 < c \leq \vartheta$. Let $f \in \sigma$. Denote by $K$ the standard maximal compact subgroup of $\text{SL}_2(F)$. Then $\sup_K |f|, \sup_K |R_\chi f| < \infty$, hence for $g \in \text{Mp}_2(F)$,

$$ \int_{k \in K} |f(kg)R_\chi f(k)| \ll_f \int_{k \in K} (\text{ht}(kg)^{1/2})^{1+c} \ll \Xi(g)^{1-\theta}. \quad (2.21) $$

(One can refine this estimate to feature a Sobolev–type dependence upon $f$, but we require (2.21) only for establishing that certain integrals converge absolutely.)

2.11.5. Let $\sigma$ be an element of the Waldspurger packet $W_\psi(\tau)$ of some $\vartheta$-tempered generic irreducible unitary representation $\tau$ of $\text{PGL}_2(F)$, so that $\sigma$ is $\vartheta$-tempered. Then for $\varphi, \varphi' \in \sigma$ and $s \in \text{Mp}_2(F)$, we have

$$ \langle \sigma(s) \varphi, \varphi' \rangle \ll \Xi(s)^{-\theta} S(\varphi) S(\varphi') \quad (2.22) $$

This may be proved as in the case of $\text{GL}_2(F)$, see [53, §9.1.1].
2.12. Local triple product periods on the general linear group. Let $\pi_1, \pi_2, \pi_3$ be $\vartheta$-tempered generic irreducible unitary representations of $GL_2(F)$ with trivial product of central characters. For $\varphi_i \in \pi_i$ ($i = 1, 2, 3$), set

$$P_{\text{GL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) := \int_{\text{PGL}_2(F)} \prod_{i=1,2,3} \langle \pi_i(g) \varphi_i, \varphi_i \rangle.$$ 

The integral converges absolutely (see [20, Lem 2.1]). If we are in the unramified case and the $\varphi_i$ are unramified unit vectors, then (see [20, Lem 2.2])

$$P_{\text{GL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) = \frac{\zeta_F(2^2)L(\pi_1 \otimes \pi_2 \otimes \pi_3, 1)}{\prod_{i=1,2,3} L(ad(\pi_i), 1)}. \quad (2.23)$$

We note that some references work with the variant of $P_{\text{GL}_2(F)}$ obtained by dividing through by the RHS of (2.23), which is thus normalized to take the value 1 on unramified unit vectors in the unramified case. This difference in normalization has no effect on asymptotics, since the $\vartheta$-temperedness assumption implies that the RHS of (2.23) has size $\asymp 1$.

2.13. Local triple product periods on the metaplectic group. In this section (and others to follow), we consider the following trios of representations:

- $\pi$: a $\vartheta$-tempered generic irreducible unitary representation of $GL_2(F)$.
- $\rho_\psi$: the Weil representation of $M_{\text{p}_2}(F)$ on $S(F)$.
- $\sigma$: an element of the Waldspurger packet $W_\psi(\tau)$ of some $\vartheta$-tempered generic irreducible unitary representation $\tau$ of $\text{PGL}_2(F)$.

For $\varphi_1 \in \pi$, $\varphi_2 \in \rho_\psi$ and $\varphi_3 \in \sigma$, set

$$P_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) := \int_{\text{SL}_2(F)} \langle \pi(g) \varphi_1, \varphi_1 \rangle \langle \rho_\psi(g) \varphi_2, \varphi_2 \rangle \langle \sigma(g) \varphi_3, \varphi_3 \rangle.$$ 

The integral converges absolutely (see [42, Lem 4.3]) because $\vartheta < 1/6$. If we are in the unramified case and the $\varphi_i$ are unramified unit vectors, then (see [42, Lem 4.4])

$$P_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) = \frac{\zeta_F(2)L(ad(\pi) \otimes \tau, 1)}{L(ad(\tau), 1)L(ad(\pi), 1)}. \quad (2.24)$$

In general, we may write $\sigma = \sigma^\varepsilon$ where $\varepsilon \in \{\pm\}$ indexes the Waldspurger packet $W_\psi(\tau)$ as in §2.6. Then (see [42, Prop 8])

$$P_{\text{SL}_2(F)} \text{ does not vanish on } \pi \otimes \rho_\psi \otimes \sigma \iff \varepsilon = \varepsilon(\pi \otimes \pi \otimes \tau, \frac{1}{2}). \quad (2.25)$$

Moreover, if $\varepsilon := \varepsilon(\pi \otimes \pi \otimes \tau, \frac{1}{2})$, then $W_\psi(\pi)$ contains $\sigma^\varepsilon$; equivalently, if $\varepsilon = -1$, then $\tau$ belongs to the discrete series (see §2.6).

2.14. Linearizing local triple product periods on the metaplectic group.

2.14.1. Setting and aim. Let $\pi, \tau$ and $\sigma \in W_\psi(\tau)$ be as in §2.13, but suppose now that $\tau$ and hence $\sigma$ belongs to the principal series. Then $W_\psi(\tau) = \{\sigma = \sigma^+\}$ is a singleton and $\varepsilon(\pi \otimes \pi \otimes \tau) = +1$, so by (2.25), the local Shimura period $P_{\text{SL}_2(F)}$ does not vanish identically on $\pi \otimes \rho_\psi \otimes \sigma$. The purpose of this section is to express the nonzero invariant hermitian form $P_{\text{SL}_2(F)}$ in terms of explicit invariant trilinear forms. In practice, it is simpler to analyze asymptotically the linear forms than the hermitian form.
2.14.2. Local metaplectic Rankin–Selberg integral. We realize $\pi$ in its $\psi$-Whittaker model, $\rho_\psi$ on $\mathcal{S}(F)$ as usual, and $\sigma$ in its induced model $\sigma = \mathcal{I}_{\text{Mp}}(\chi)$ for some character $\chi$ of $F^\times$ with $\text{Re}(\chi) \leq \vartheta$. Set $G := \text{SL}_2(F)$ and let $N = \{ n(x) : x \in F \} \leq G$ denote the standard upper-triangular unipotent subgroup. Equip $N$ with the Haar transported from $F$ and $N \backslash G$ with the quotient Haar. Define $\ell : \pi \otimes \rho_\psi \otimes \sigma \to \mathbb{C}$ by

$$\ell(W, \phi, f) := \int_{g \in N \backslash G} W(g) \cdot (\rho_\psi(g) \phi(1) : f(g))$$

This is a local integral of Rankin–Selberg type that was studied by Gelbart–Jacquet, following Shimura.

Lemma. The integral defining $\ell$ converges absolutely.

Proof. Write $g = t(y)k$, so that $dg$ is a constant multiple of $|y|^{-2}d^xydk$. Write $c := \text{Re}(\chi)$, so that $|c| \leq \vartheta$. Then for fixed $W, \phi$ and $f$, we have $W(g) \ll \min(|y|^{-2\vartheta}, |y|^{-\vartheta})$, $\rho_\psi(g) \phi(1) \ll \min(|y|^{1/2}, |y|^{-\vartheta})$ and $f(g) \ll |y|^{1+\varepsilon}$. The integral in question is thus dominated by $\int_{y \in F^\times} \min(|y|^{1-2\vartheta}+1/2+(1-\vartheta), |y|^{-\vartheta}) d^xy < \infty$, since $\vartheta < 1/6$.

Thus $\ell$ defines an invariant trilinear form.

2.14.3. Statement of result. The (first part of) the following lemma may be understood as a metaplectic analogue of the oft-cited [31, Lem 3.4.2].

Lemma. There exists $c > 0$ with $c \asymp 1$ so that:

- If $\sigma$ is tempered, then
  $$\mathcal{P}_{\text{SL}_2(F)}(W, \phi, f) = c |\ell(W, \phi, f)|^2. \quad (2.26)$$
- If $\sigma$ is non-tempered, so that (without loss of generality) $\chi = \eta|\sigma$ for some quadratic character $\eta$ of $F^\times$ and some $0 < \sigma \leq \vartheta$, then
  $$\mathcal{P}_{\text{SL}_2(F)}(W, \phi, f) = c \ell(W, \phi, f) \ell(W, \phi, R_\chi f). \quad (2.27)$$

Remark.

(i) We do not know how to adapt the method of proof of [31, Lem 3.4.2] to the metaplectic setting, so we proceed more directly. The method developed here may be applied also to give a new (and perhaps more natural) proof of loc. cit. that avoids discussion of Plancherel measure and continuity in the tempered dual.

(ii) The analogue of (2.27) is not considered in [31, Lem 3.4.2]: in that work, the induced representation is always tempered, since it arises ultimately as the local component of a unitary Eisenstein series.

More precisely, the quantity $c$ depends upon the normalizations of Haar measures. Suppose for the remainder of §2.14 that

- the Haar $dx$ on $F$ is $\psi$-self-dual (as we have already assumed),
- the Haar $d^xy$ on $F^\times$ and the Haar $dx$ on $F$ are related by $d^xy = |y|^{-1}dy$, and
- the Haar on $G$ has been normalized so that for $\alpha \in C_c(N \backslash G)$,
  $$\int_{N \backslash G} \alpha = \int_{y \in F^\times} \int_{x \in F} \alpha(t(y)n'(x))|y|^{-2}d^xydx.$$
(In the non-archimedean case, these measures assign volume $\asymp 1$ to maximal compact subgroups.) Recall also that we have normalized the unitary structures on $\pi, \rho, \sigma$ to be given respectively in the $\overline{\psi}$-Kirillov model, on $\mathcal{S}(F)$, and in the induced model as in §2.3.5, §2.7.3. Under these assumptions, we shall verify that the above identities hold with $c = 1$.

2.14.4. Reduction to an identity. We now reduce the proof of (2.26) and (2.27) to that of a common identity. Define $f_1, f_2 : G \to \mathbb{C}$ in the tempered case by $f_1 := f_2 := f$ and in the non-tempered case by $f_1 := f$ and $f_2 := R_y f$, so that $\langle g f, f \rangle = (g f_1, f_2)$. Consider the functions $\Psi_1 \Psi_2 : \text{Mp}_2(F) \to \mathbb{C}$ defined by $\Psi_i(g) := \rho_i(g) \phi(1) \cdot f_i(g)$; they descend to functions $\Psi_i : \text{SL}_2(F) \to \mathbb{C}$ satisfying $\Psi_i(n(x)g) = \psi(x) \Psi_i(g)$. Set

$$\langle g \Psi_1, \Psi_2 \rangle := \int_{h \in N \setminus G} \Psi_1(h g) \overline{\Psi_2}(h).$$

(2.28)

By writing $h = t(y)n'(x)$, we see that (2.28) converges absolutely and evaluates to $\langle g \Psi_1, \Psi_2 \rangle = (g \phi, \phi(g f, f))$, so that $\mathcal{P}_{\text{SL}_2(F)}(W, \phi, f) = \int_{g \in G} \langle g W, W \rangle \langle g \Psi_1, \Psi_2 \rangle$ (compare with [31, Lem 3.2.7]). Our goals (2.26) and (2.27) thereby reduce to the identity of absolutely-convergent integrals

$$\int_{g \in G} \langle g W, W \rangle \langle g \Psi_1, \Psi_2 \rangle = \left( \int_{N \setminus G} W \Psi_1 \right) \left( \int_{N \setminus G} W \Psi_2 \right).$$

(2.29)

2.14.5. Heuristic argument. The LHS of (2.29) formally expands to

$$\int_{g \in G} \int_{h \in N \setminus G} \Psi_1(h g) \overline{\Psi_2}(h) \langle g W, W \rangle$$

(2.30)

and then folds after the substitution $g \mapsto h^{-1} g$ to

$$\int_{g \in N \setminus G} \int_{h \in N \setminus G} \Psi_1(g) \overline{\Psi_2}(h) \int_{x \in F} \psi(x) \langle h^{-1} n(x) g W, W \rangle,$$

(2.31)

which evaluates to the RHS of (2.29) by the identity

$$\langle h^{-1} n(x) g W, W \rangle = \langle n(x) g W, h W \rangle,$$

the normalization $d^* y = |y|^{-1} dy$, and the following Fourier inversion formula stated in the proof of [31, Lem 3.4.2]:

$$\int_{x \in F} \psi(x) \langle n(x) g W, h W \rangle = W(g) \overline{W(h)}$$

(2.32)

(Recall here that $W$ belongs to the $\overline{\psi}$-Whittaker model.)

Unfortunately, the intermediary expressions (2.30), (2.31) (and the LHS of (2.32)) do not in general converge absolutely, so further care is required to convert the argument sketched here into a proof.

2.14.6. Proof of the identity. We now prove (2.29), roughly following the heuristic argument indicated above. For $\Phi \in \mathcal{S}(F)$, the quadratic Fourier transform

$$P(x) := \int_{y \in F} \psi(-y^2 x) \Phi(y) dy$$

(2.33)

defines a smooth function $P : F \to \mathbb{C}$. 
Lemma 1. For \( W_1, W_2 \) in the \( \psi \)-Whittaker model of \( \pi \), one has the identity of absolutely-convergent integrals

\[
\int_{x \in F} \langle n(x)W_1, W_2 \rangle P(x) \, dx = \int_{y \in F^\times} W_1(t(y))\overline{W_2}(t(y))\Phi(y) \frac{d^\times y}{|y|}. \tag{2.34}
\]

Proof. This is inspired by a lemma of Qiu [40, Lem 3.5] concerning representations of metaplectic groups, and may be proved similarly. For completeness, we record a proof. Observe first that if \( \Phi \) is an odd function, then \( P = 0 \), and so both sides of (2.34) converge absolutely and vanish identically. We may thus assume that \( \Phi \) is an even function. We treat first the special case in which \( \Phi \) vanishes in a neighborhood of 0. We may then define a Schwartz function \( \Phi = \langle \Phi, \rangle \) it suffices by Fubini to check that

\[
P(x) = \frac{2}{|2|_F} \int_{z \in F} \psi(-zx)Q(z) |z|^{1/2} \, dz. \tag{2.35}
\]

By Fourier inversion on \( S(F) \), it suffices then to verify for each \( P \in S(F) \) that

\[
\int_{x \in F} \langle n(x)W_1, W_2 \rangle P(x) = \int_{z \in F^\times} W_1(a(z))\overline{W_2}(a(z))(\int_{x \in F} P(x)\psi(-zx) \, dx) \, d^\times z. \tag{2.36}
\]

For this we expand \( \langle n(x)W_1, W_2 \rangle = \int_{z \in F^\times} \psi(-zx)W_1(a(z))\overline{W_2}(a(z)) \, dz \) on the LHS of (2.36); this gives an absolutely-convergent double integral which rearranges to the RHS of (2.36).

We turn to the general case. We may smoothly decompose \( P \) as a sum of a function supported away from the origin – to which the previous paragraph applies – and a function supported near the origin. By this reduction, we may assume that \( \Phi \) is supported on \( \{ x : |x| \leq 1 \} \), say. We fix a smooth function \( \nu \in C^\infty_c(\mathbb{R}_+^x) \) so that \( \sum_{j \geq 0} \nu(2^jt) = 1 \) for \( 0 < t \leq 1 \). For \( j \geq 0 \), set \( \Phi_j(x) := \nu(2^j|x|)\Phi(x) \). Then \( \Phi = \sum_j \Phi_j \); moreover, \( \int_{y \in F} \sum_j |\Phi_j(y)| \, dy < \infty \), so that \( P = \sum_j P_j \) pointwise with \( P_j \) attached to \( \Phi_j \) as in (2.33). The previous paragraph shows that the desired identity (2.34) is satisfied by each pair \( (P_j, \Phi_j) \), so to verify the corresponding identity for \( (P, \Phi) \) it suffices by Fubini to check that

\[
\sum_j \int_{x \in F-\{0\}} |\langle n(x)W_1, W_2 \rangle P_j(x)| \, dx < \infty \tag{2.37}
\]

and

\[
\sum_j \int_{y \in F^\times} |W_1(t(y))W_2(t(y))\Phi_j(y)| \frac{d^\times y}{|y|} < \infty. \tag{2.38}
\]

For the proofs of (2.37) and (2.38), we temporarily replace our general conventions (§2.1.7) on asymptotic notation with the following: “fixed” means “depending at most upon \( W_1, W_2, \) and \( \Phi \),” and asymptotic notation is then defined as in §2.1.7. In particular, implied constants are independent of \( j \).
The proof of (2.37) reduces, via the estimate \(\langle n(x)W_1, W_2 \rangle \ll (1 + |x|)^{-\frac{1}{10}}\) and the inequality \(\vartheta < 1/4\), to verifying that
\[
P_j(x) \ll 2^{-j}(1 + |2^{-j}x|)^{-10}.
\] (2.39)

To that end, let \(Q_j\) be attached to \(\Phi_j\) as above. Then:

- \(Q_j(\tau) \neq 0\) only if \(|\tau| \gg 2^{-j}\).
- In the non-archimedean case, \(Q_j\) is invariant under dilation by a fixed open subgroup of \(\sigma^\infty\). In the archimedean case, we have, for each fixed invariant differential operator \(D\) on \(F^\times\), that \(\int_{F^\times} |DQ_j(y)| \, dy \ll 1\).

We deduce (2.39) from (2.35) via these observations and elementary Fourier analysis.

To establish (2.38), we use that \(W_i(t(y)) \ll \min(|y|^{1-2\vartheta}, |y|^{-10})\), that \(\Phi_j(y)\) is supported on \(y \ll 2^{-j}\), and that \(\vartheta < 1/4\). \(\square\)

For \(g, h \in G\) and \(y \in F\), define
\[
I(g, h; y) := \rho_\phi(g)\phi(y) \cdot \overline{f_1(g)} \cdot \overline{\rho_\psi(h)\phi(y)} \cdot f_2(h).
\]
Then \(I(g, h; \cdot) \in \mathcal{S}(F)\). For \(y \in F^\times\), we verify readily that
\[
I(g, h; y) = \Psi_1(t(y)g)\overline{\Psi_2(t(y)h)}|y|^{-3}.
\]

Set
\[
I(g, h) := \int_{y \in F} I(g, h; y) \, dy = \langle \phi \cdot h \phi, f_1(g) f_2(h) \rangle.
\] (2.40)

The following may be understood as a refinement of the absolute convergence of the matrix coefficient integral \(\mathcal{I}_{\text{SL}(2)}(F)\):

**Lemma 2.** \(\int_{g \in G} \int_{k \in K} |\langle gW, W \rangle I(kg, k)\| < \infty\).

**Proof.** We have \(I(kg, k) = \langle \phi \cdot \phi, f_1(kg) f_2(k) \rangle\), so that by (2.21), \(I(kg, k) \ll_{\phi, f} \Xi(g)^{3/2-\vartheta}\); since \(\langle gW, W \rangle \ll W \Xi(g)^{1-2\vartheta}\) (see (2.16)) and \(3\vartheta < 1/2\), the integral in question is dominated for some \(\varepsilon > 0\) by \(\int_G \Xi^{2+\varepsilon} < \infty\). \(\square\)

Turning to (2.29), note first that its validity is independent of the choice of Haar measure on \(G\). Let us suppose (for convenience) that it is given in Iwasawa coordinates \(g = n(x)t(y)k\) by \(dg = |y|^{-2}dx \, d^\times y \, dk\), where \(dk\) denotes any Haar measure on the standard maximal compact subgroup \(K\) of \(G\). The LHS of (2.29) may then be written
\[
\int_{g \in G} \langle gW, W \rangle \left(\int_{k \in K} I(kg, k)\right) \, dg.
\] (2.41)

By Lemma 2, we may rearrange this to \(\int_{k \in K} \int_{g \in G} \langle gW, W \rangle I(kg, k)\). The substitution \(g \mapsto k^{-1}g\) yields \(\int_{k \in K} \int_{g \in G} \langle gW, kW \rangle I(g, k)\). By folding up the \(g\)-integral and using the identity \(I(n(x)g, k) = \int_{y \in F} \psi(-xy^2)I(g, k; y) \, dy\), we obtain
\[
\int_{k \in K} \int_{g \in G \setminus \mathcal{G}} \int_{x \in F} \langle n(x)gW, kW \rangle \left(\int_{y \in F} \psi(-xy^2)I(g, k; y) \, dy\right). \] (2.42)

By Lemma 1, this evaluates to the (absolutely convergent) integral
\[
\int_{k \in K} \int_{g \in G \setminus \mathcal{G}} \int_{y \in F^\times} W(t(y)g)\overline{W(t(y)k)}I(g, k; y) \, \frac{d^\times y}{|y|} \, dy. \] (2.43)
We swap the $g$ and $y$ integrals, apply the substitution $g \mapsto t(y)^{-1} g$, and expand the definition of $I(g, k; y)$ to arrive at

$$
\int_{g \in N \setminus G} W(g) \Psi_1(g) \int_{k \in K} \int_{y \in F^\times} \frac{W(t(y)k) \Psi_2(t(y)k)}{|y|^2} d^\times y,
$$

which equals the RHS of (2.29).

2.15. Lower bounds at uninteresting places.

2.15.1. Statement of result. We record here some unsurprising polynomial-type lower bounds for local triple product periods on the metaplectic group. They supply the polynomial dependence of our main result on the “essentially fixed” quantities.

Here we use the specialized notation $A \ll_{\pi, \tau} B$ or $B \gg_{\pi, \tau} A$ to signify that $A \ll (C(\pi)C(\tau))^{O(1)} B$, where $C(.)$ denotes the analytic conductor as in [31, §3.1.8], and $A \asymp_{\pi, \tau} B$ to denote that $A \ll_{\pi, \tau} B \ll_{\pi, \tau} A$.

Conjecture. Let $\pi, \tau$ and $\sigma \in Wd_{\psi}(\tau)$ be as in §2.13. Assume that (2.25) holds, so that $P_{SL_2(F)}$ is not identically zero on $\pi \otimes \rho_{\psi} \otimes \sigma$. Then there exist $\varphi_1 \in \pi, \varphi_2 \in \rho_{\psi}, \varphi_3 \in \sigma$ so that

$$
P_{SL_2(F)}(\varphi_1, \varphi_2, \varphi_2) \gg_{\pi, \tau} 1,$$

$$
S(\varphi_i) \ll_{\pi, \tau} 1 \text{ for } i = 1, 2, 3.
$$

We are content here to address the case that $\sigma$ belongs to the principal series, in which case the condition (2.25) is automatic (see §2.14).

Lemma. The conclusion of the conjecture holds under the additional assumption that $\sigma$ belongs to the principal series.

Remark. The general case of the conjecture is likely accessible by brute-force analysis of matrix coefficients as in [34, 19]. It also seems likely to follow by effectizing the proof of (2.25). It would be desirable to have a soft yet direct approach. The question of to what extent the conjecture can be made uniform is also interesting; we address it partially in §5.

We observe first that if we are in the unramified case and $\pi$ and $\tau$ are unramified, then the conclusion of the lemma follows from (2.24) upon taking $\varphi_1, \varphi_2, \varphi_3$ to be unramified unit vectors. We may thus assume either that we are not in the unramified case, or that $F$ is non-archimedean and at least one of $\pi, \tau$ is ramified. In particular, if $F$ is non-archimedean, we may assume that

$$
q \ll_{\pi, \tau} 1.
$$

We may and shall assume that the Haar on $N \setminus G$ is as in §2.14.3.

2.15.2. Choice of models.

- We realize $\pi$ in its $\overline{\psi}$-Whittaker model.
- We realize $\rho_{\psi}$ on $S(F)$, as usual.
- We write $\sigma = I_{Mp_2}(\chi)$ for some character $\chi$ of $F^\times$ and realize $\sigma$ in its induced model. Our assumptions imply that $c := \text{Re}(\chi)$ satisfies $|c| \leq \vartheta$.

We accordingly write $W, \phi, f$ instead of $\varphi_1, \varphi_2, \varphi_3$. 
2.15.3. Some estimates. The proof of the lemma of §2.14.2 shows that the integral
\[
\mathcal{L}_\chi(W, \phi) := \int_{y \in F^x} W(t(y)) \phi(y) \overline{\chi(y)} \frac{d^x y}{|y|^{1/2}}
\]
converges absolutely for \( W \in \pi, \phi \in \rho_\psi \) and defines a functional \( \mathcal{L}_\chi : \pi \otimes \rho_\psi \to \mathbb{C} \) so that for \( f \in \sigma \), we have (cf. §2.14.2)
\[
\ell(W, \phi, f) = \int_{x \in F} f(n(x)) \mathcal{L}_\chi(n(x)W, n(x)\phi) \, dx.
\]
We require some very crude estimates for \( \mathcal{L}_\chi \):

Lemma.

(i) \( \mathcal{L}_\chi(W, \phi) \ll S(W)S(\phi) \).

(ii) \( \mathcal{L}_\chi(n(x)W, n(x)\phi) = \mathcal{L}_\chi(W, \phi) + O(|x|S(W)S(\phi)) \) for \( x \in F \) with \( |x| \ll 1 \).

Proof. (i) follows from (2.15) and (2.18). For (ii), we write \( n(x)W = W + W' \), \( n(x)\phi = \phi + \phi' \) and \( \mathcal{L}_\chi(n(x)W, n(x)\phi) = \mathcal{L}_\chi(W, \phi) + \mathcal{L}_\chi(W', \phi) + \mathcal{L}_\chi(W, \phi') + \mathcal{L}_\chi(W', \phi') \). By (i), we then reduce to the following property of the Sobolev norms, which follows readily from their definition:
\[
S_d(n(x)v - v) \ll |x|S_{d+1}(v) \text{ for } |x| \ll 1 \text{ and } d \text{ fixed.}
\]

2.15.4. Choice of \( W, \phi \). We fix a nonnegative function \( \alpha \in C_c^\infty(F^x) \), not identically zero on \( F^{x^2} \); in the unramified case, we assume that \( \alpha = 1_{\mathfrak{o}^x} \). We choose \( W \) to be given in the Kirillov model by \( W(a(y)) = \alpha(y) \). Let \( \omega_\pi \) denote the central character of \( \pi \). Then \( W(t(y)) = \omega_\pi(y)^{-1} \alpha(y^2) \). We choose \( \phi(y) := \overline{\chi}^{-1}(y)\omega_\pi(y)\alpha(y^2) \). Then \( W(t(y))\phi(y) = \alpha(y^2)^2 \), so
\[
\mathcal{L}_\chi(W, \phi) = \int_{y \in F^x} \alpha(y^2)^2 \frac{d^x y}{|y|^{1/2}} \ll 1.
\]
We have \( \|W\|, \|\phi\| \asymp 1 \) and the crude estimates
\[
S(W), S(\phi) \ll_{\pi, \tau} 1,
\]
which may be verified as follows:

- In the non-archimedean case, the construction of \( W \) shows that it is invariant by \( n(x) \) and \( a(y) \) for all \( x \in \mathfrak{p}^{O(1)} \) and \( y \in \mathfrak{o}^x \cap (1 + \mathfrak{p}^{O(1)}) \). To deduce the required estimate for \( S(W) \), it suffices to verify that \( W \) is invariant by \( n'(x) \) for all \( x \in \mathfrak{p}^{-c(\pi) + O(1)} \), or equivalently, that \( wW \) is invariant by \( n(x) \) for all such \( x \), i.e., that \( wW(a(y)) \) is supported on \( y \in \mathfrak{p}^{-c(\pi) + O(1)} \); for this we decompose \( \alpha \) as a linear combination of the functions \( \nu \cdot 1_{\mathfrak{o}^{-n}\mathfrak{o}^x} \), where \( \nu \) is a unitary character of \( F^x \) and \( n \) is integer, and appeal to the Jacquet–Langlands local functional equation (see, e.g., [31, §3.2.2, §3.2.3]). The corresponding estimate for \( S(\phi) \) is deduced similarly using the Tate local functional equation.

- In the archimedean case, the estimate for \( S(W) \) follows from [31, §3.2.5]. The estimate for \( S(\phi) \) follows from the fact that the Lie algebra of \( \text{Mp}_2(F) \) acts on \( \rho_\psi \) via differential operators.
2.15.5. The tempered case. Suppose first that σ is tempered, so that χ is unitary. We choose \( f \in \sigma \) to be given in the line model by \( f(n'(x)) := |X|^{1/2} \alpha(Xx) \) for some parameter \( X \in F^\times \) with \(|X| \geq 1\), to be chosen later. Then \(|f| \asymp 1\). By the lemma of §2.15.3 and the estimate (2.46), we see that

\[
\ell(W, \phi, f) - |X|^{-1/2} \mathcal{L}_\chi(W, \phi) \int_{x \in F} \alpha(x) \, dx \ll_{\pi, \tau} |X|^{-1}.
\]

By choosing \( X \) so that \(|X|\) is a sufficiently large but fixed power of \( C(\pi)C(\tau) \), we obtain \( \ell(W, \phi, f) \gg_{\pi, \tau} 1 \). Moreover, \( S(f) \ll_{\pi, \tau} 1 \); we may see this in the non-archimedean case by considering the invariance of \( f \) and in the archimedean case by noting that the Lie algebra of \( \text{Mp}_2(F) \) acts on the line model of \( \sigma \) via differential operators with coefficients bounded polynomially in \( C(\chi) \). We conclude via (2.26).

2.15.6. The non-tempered case. Suppose now that \( \sigma \) is non-tempered, thus \( \chi = \eta|||c| \) with \( \eta \) quadratic, \( c \) real, and \( 0 \neq |c| \leq \vartheta \). The map

\[
\mathcal{H} : \sigma \otimes \overline{\sigma} \to \mathbb{C}
\]

defines a hermitian form on \( \sigma \), i.e., \( \mathcal{H}(f_1, f_2) = \overline{\mathcal{H}(f_2, f_1)} \). Set \( \mathcal{H}(f) := \mathcal{H}(f, f) \).

Our task is to find \( f \in \sigma \) for which \( S(f) \ll_{\pi, \tau} 1 \ll_{\pi, \tau} \mathcal{H}(f) \). It will suffice to find \( f_1, f_2 \in \sigma \) for which

\[
S(f_1), S(f_2) \ll_{\pi, \tau} 1 \ll_{\pi, \tau} \text{Re} \mathcal{H}(f_1, f_2),
\]

because then the polarization identity \( \mathcal{H}(f_1 + f_2) = \mathcal{H}(f_1) + \mathcal{H}(f_2) + 2 \text{Re} \mathcal{H}(f_1, f_2) \) gives the required conclusion for some \( f \in \{ f_1, f_2, f_1 + f_2 \} \). To that end, let \( f_0 \in \sigma \) and \( f_0^* \in \sigma^* = \mathcal{I}_{\text{Mp}_2}(\chi^{-1}) \) be as in the tempered case, given in the line model by \( x \mapsto |X|^{1/2} \alpha(Xx) \) where \( X \in F^\times \) is chosen so that \(|X|\) is a sufficiently large but fixed power of \( C(\pi)C(\tau) \). Take

\[
f_1 := |X|^{c/2} f_0, \quad f_2 := |X|^{-c/2} R_{\chi}^{-1}(f_0^*).
\]

By (2.27), we then have

\[
\mathcal{H}(f_1, f_2) = \ell(W, \phi, f_0) \ell(W, \phi, f_0^*).
\]

Arguing as in the tempered case, we see that \( \mathcal{H}(f_1, f_2) - |X|^{-1/2} \kappa \ll_{\pi, \tau} |X|^{-1} \) for a positive real \( \kappa \) with \( \kappa \asymp 1 \); explicitly,

\[
\mathcal{L}_{\eta||c|} (W, \phi) =\mathcal{L}_{\eta||c|} (W, \phi) = \int_{y \in F^\times} \alpha(y^2) \frac{d^\times y}{|y|^{1/2}} \int_{y \in F^\times} \alpha(y^2) \frac{d^\times y}{|y|^{1/2 - 2c}}.
\]

Thus \( \text{Re} \mathcal{H}(f_1, f_2) \gg_{\pi, \tau} 1 \) for \( X \) as indicated. We have

\[
\|f_1\|^2 = \|w_{f_1}\|^2 = \int_{\xi \in F} G(\chi, \psi, \xi) \int_{x \in F} |X|^{1/2 + c} \alpha(Xx) \psi(\xi x) \, dx \, dx.
\]

Write \( \alpha'(\xi) := \int_{x \in F} \alpha(x) \psi(\xi x) \, dx \). Using (2.12) and (2.45), we see that

\[
\|f_1\|^2 \gtrsim_{\pi, \tau} \int_{\xi} |\xi|^{-c} \left| X \right|^{-1/2 + c/2} \alpha'(\xi / X) \, d\xi \asymp 1.
\]

Using the isometry property (2.11) of \( R_{\chi} \), we deduce similarly that \( \|f_2\|^2 \gtrsim_{\pi, \tau} 1 \) and then as in the tempered case that \( S(f_1), S(f_2) \ll_{\pi, \tau} 1 \). The proof is then complete.
2.16. Upper bounds at uninteresting places.

Lemma. Let \( \pi \) be a \( \vartheta \)-tempered generic irreducible unitary representation of \( \text{GL}_2(F) \). Let \( \omega \) be a unitary character of \( F^\times \). Let \( \varphi_1 \in \pi, \varphi'_1 \in \mathcal{F}, \Phi \in \mathcal{I}_{\text{PGL}_2}(\omega) \). Then

\[
\mathcal{P}_{\text{PGL}_2(F)}(\varphi_1, \varphi'_1, \Phi) \ll C(\omega)^{-A} S(\varphi_1)^2 S(\varphi'_1)^2 \| \Phi \|^2.
\]

Proof. This is a weak form of \[31, \text{Lemma 3.5.2}\], taking into account that \( \log C_{\text{Sob}}(\mathcal{I}_{\text{PGL}_2}(\omega)) \propto \log C(\omega) \) (see \[31, \text{Lem 2.6.6}\] and the accompanying footnote). \( \square \)

Corollary. Let \( \pi \) be a \( \vartheta \)-tempered generic irreducible unitary representation of \( \text{GL}_2(F) \). Let \( \omega \) be a unitary character of \( F^\times \) for which \( \omega^2 = \chi \). Let \( \varphi_1 \in \pi, \varphi'_1 \in \mathcal{F}, \varphi_2 \in \rho_\varphi, \varphi'_2 \in \rho_{\varphi'} \). Set \( \Phi := I_{\chi}(\varphi_2 \otimes \varphi'_2) \in \mathcal{I}_{\text{SL}_2}(\chi) \) (see §2.10). Then

\[
\mathcal{P}_{\text{PGL}_2(F)}(\varphi_1, \varphi'_1, \Phi^\omega) \ll C(\omega)^{-A} \prod_{i=1,2} S(\varphi_i)^2 S(\varphi'_i)^2.
\]

Proof. Recall from §2.3.6 that \( \| \Phi^\omega \| = \| \Phi \| \). The estimate of §2.10 implies that \( \| \Phi \| \ll S(\varphi_2)S(\varphi'_2) \), so we may conclude by the previous lemma. \( \square \)

2.17. Estimates at the interesting place.

2.17.1. Statement of result.

Lemma. Assume we are in the unramified case (see §2.1.6). Let \( \pi, \tau \) and \( \sigma \in \text{Wd}_\varphi(\tau) \) be as in §2.13, but assume now also that

- \( \tau \) (and hence \( \sigma \)) is an unramified principal series representation, and that
- \( \pi \) is a twist of the special representation (§2.8.4).

Then there exist \( \varphi_1 \in \pi, \varphi_2 \in \rho_\varphi, \varphi_3 \in \sigma \) so that

\[
(i) \quad \| \varphi_i \| \asymp 1 \text{ for } i = 1, 2, 3.
\]

\[
(ii) \quad \mathcal{P}_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) \gg C(\text{ad}(\pi) \otimes \tau)^{-1/4}.
\]

(iii) Let \( \chi, \omega \) be unitary characters of \( F^\times \) with \( \omega^2 = \chi \). Set \( \Phi := I_{\chi}(\varphi_2 \otimes \varphi'_2) \in \mathcal{I}_{\text{SL}_2}(\chi) \). Then

\[
\mathcal{P}_{\text{PGL}_2(F)}(\varphi_1, \varphi'_1, \Phi^\omega) \ll C(\omega)^{-A} C(\pi \otimes \pi \otimes \omega)^{-1/2}.
\]

Remark. It is natural to ask whether the conclusion of the lemma holds for more general classes of \( \pi \). For some negative results in that direction, see §5.

We note that the conclusion of the lemma depends upon \( \pi \) only through its restriction to \( \text{SL}_2(F) \) and the quantities \( C(\text{ad}(\pi) \otimes \tau) \) and \( C(\pi \otimes \pi \otimes \omega) \). These are unchanged upon replacing \( \pi \) by a twist. For the proof, we may and shall thus assume that \( \pi \) is the special representation itself, rather than a twist thereof.

2.17.2. Conductor formulas.

- Since \( \tau \) is unramified and \( \pi \) is the special representation, one has \( C(\text{ad}(\pi) \otimes \tau) = C(\text{ad}(\pi))^2 = q^2 \). (Indeed, the first of these identities follows from the fact that if \( \tau \) is the normalized induction of a pair of unramified characters \( \nu_1, \nu_2 \), then \( C(\text{ad}(\tau) \otimes \nu_1) = C(\text{ad}(\pi) \otimes \nu_1)C(\text{ad}(\pi) \otimes \nu_2) \) and \( C(\text{ad}(\pi) \otimes \nu_1) = C(\text{ad}(\pi)) \). The second of these identities follows from the identity \( C(\text{ad}(\pi)) = C(\pi \otimes \pi) \) together with \[10, \text{Prop 1.4}\] and the relation between \( \varepsilon \)-factors and analytic conductors recalled in, e.g., \[31, \text{§3.1.12}\].)
• If \( \omega \) is unramified, then \( C(\pi \otimes \pi \otimes \omega) = C(\text{ad}(\pi)) = q^2 \).

2.17.3. Choice of models. As in §2.15, we realize \( \pi \) in its \( \psi \)-Whittaker model, \( \rho_\psi \) on \( S(F) \), and \( \sigma = I_{MP_2}(\chi) \) in its induced model. We again write \( W, \phi, f \) instead of \( \varphi_1, \varphi_2, \varphi_3 \).

2.17.4. Choice of \( W, \phi \).

• Let \( W \in \pi \) be given in the Kirillov model by \( W(a(y)) := 1_\sigma(y)|y| \); it is then a newvector with \( |W| \approx 1 \).

• Let \( \phi \in \rho_\psi \) be given by \( 1_\sigma \).

The assertions concerning \( L^2 \)-norms of \( W \) and \( \phi \) are clear by construction. We will choose \( f \in \sigma \) later, separately according as \( \sigma \) is tempered or not.

2.17.5. Upper bounds. We now verify (2.48). If \( c(\omega) = 0 \), then it follows from §2.10 that \( \Phi^\omega \) is a newvector of norm \( O(1) \), and the required estimate is given by [34, Lem 4.4]. Write \( \nu := I_{PGL_2}(\omega) \). If \( c(\omega) \geq 1 \), then \( c(\nu) = 2c(\omega) \geq 2 \), so \( \nu \) contains no nonzero \( K_0[1] \)-invariant vectors; since \( \varphi_1 \) transforms under a unitary character of \( K_0[1] \), it follows from the GL\(_2\)(\( F \))-invariance of \( P_{PGL_2}(\nu) \) that \( P_{PGL_2}(\nu)(\varphi_1, \varphi_2, \Phi^\omega) = 0 \). This establishes the required upper bound (in a sharper form).

2.17.6. Lower bounds: tempered case. Suppose that \( \sigma \) is tempered, so that \( \chi \) is unitary. Let \( f \in \sigma \) be given in the line model by \( f(n'(x)) := q^{1/2}1_p(x) \). Then \( \|f\| \approx 1 \). We now verify (2.47). Write \( g = t(y)n'(z) \). Then \( f(g) = |y|\chi_\psi(y)\chi(y)q^{1/2}1_p(z) \). If \( f(g) \not\approx 0 \), then \( z \in p \), hence \( W(g) = |y|^{1/2}1_p(y) \) and \( \rho_\psi(g)\phi(1) = \chi_\psi(y)\chi(y)^{1/2}1_p(y) \).

Thus

\[
W(g)\rho_\psi(g)\phi(1) = \chi(y)|y|^{1/2}1_p(y)q^{1/2}1_p(z).
\]

It follows from §2.14 that

\[
\mathcal{P}_{SL_2(F)}(\varphi_1, \varphi_2, \varphi_3) \approx \left| \int_{y \in F^*} \int_{z \in F} |y|^{7/2}\chi(y)1_\sigma(y)q^{1/2}1_p(z) \frac{dy}{|y|^2} \right|^2 \approx q^{-1},
\]

which leads to the required lower bound.

2.17.7. Lower bounds: non-tempered case. Suppose now that \( \sigma \) is non-tempered, thus \( \chi = \eta|c|^c \) with \( \eta \) unramified quadratic, \( c \) real, and \( 0 \leq |c| \leq \vartheta \). We define \( \mathcal{H} : \sigma \otimes \pi \rightarrow C \) as in §2.15.6 and reduce to finding \( f_1, f_2 \in \sigma \) for which \( ||f_1||, ||f_2|| \ll 1 \) and \( \mathcal{H}(f_1, f_2) \gg q^{-1} \). For this we choose \( f_0 \in \sigma \) and \( f_0^* \in \sigma^* = I_{PGL_2}(\chi^{-1}) \) to be given in the line model by \( n'(x) \mapsto q^{1/2}1_p(x) \). We take

\[
f_1 := q^{-c}f_0, \quad f_2 := q^{-c}R^{-1}_\chi(f_0^*).
\]

The estimate \( \mathcal{H}(f_1, f_2) \gg q^{-1} \) is proved as in the tempered case. Since \( \eta \) is unramified and \( \int_{x \in F} f_1(n'(x))\psi(-\xi x) dx = q^{-1/2}1_p^{-1}(\xi) \), we see from (2.12) that

\[
||f_1||^2 \approx q^{2c-1} \int_{\xi \in F} |\xi|^{-c}C(\chi\xi)^{-c}1_{p^{-1}}(\xi) d\xi = \sum_{n \geq 1} \iota(n),
\]

where \( \iota(n) \) denotes the contribution from \( \xi \in p^n - p^{n+1} \). We have \( \iota(-1) \approx 1 \), while \( \iota(2n) \approx q^{-2c-1-2n(1-c)} \) and \( \iota(2n+1) \approx q^{-1-(2n+1)(1-c)} \) for \( n \geq 0 \). Thus \( ||f_1||^2 \ll 1 \).

We verify similarly that \( ||f_2||^2 \ll 1 \). The proof is then complete.
2.18. Sobolev–type bounds for twisting isometries. Let $\pi$ be an irreducible unitary representation of $\text{GL}_2(F)$, and let $\chi$ be a unitary character of $F^\times$. We may then form the tensor product $\pi \otimes \chi$ of $\pi$ by the one-dimensional representation spanned by the function $\chi(det g)$. The map $v \mapsto v \otimes 1$ defines an isomorphism $j_\chi : \pi \to \pi \otimes \chi$ of vector spaces. The representation $\pi \otimes \chi$ admits a natural unitary structure.

The map $j_\chi$ is an isometry, and is $\text{SL}_2(F)$-equivariant, but is typically not $\text{GL}_2(F)$-equivariant, and does not in general preserve Sobolev norms. However, it only polynomially distorts the latter:

**Lemma.** Let $d \geq 0$ be fixed. Then $S_d(j_\chi(v)) \ll C(\chi)^{O(1)}\|v\| + S_d(v) \ll C(\chi)^{O(1)}S_d(v)$ for all $v \in \pi$.

**Proof.** We assume familiarity with [31, §2]. Suppose first that $F$ is non-archimedean. Let $K[n] \leq \text{GL}_2(\mathfrak{o})$ denote the $n$th principal congruence subgroup. Write $C(\chi) = q^c$ with $c \in \mathbb{Z}_{\geq 0}$. Then the function $g \mapsto \chi(det(g))$ is $K[c]$-invariant. Expanding the definition of $S_d$ (see [31, §2.5]), we obtain $S_d(j_\chi(v))^2 \ll q^{dc}\|v\|^2 + S_d(v)^2$. In the archimedean case, we argue similarly using that $Xj_\chi(v) = j_\chi(Xv + d\chi(\text{trace}(X))v)$ for $X$ in the Lie algebra of $\text{GL}_2(F)$.

3. Global preliminaries

Let $F$ be a number field with adele ring $\mathbb{A}$. Fix a nontrivial unitary character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$.

3.1. Generalities.

3.1.1. The metaplectic group. Let $\text{Mp}_2(\mathbb{A})$ denote the metaplectic double cover of $\text{SL}_2(\mathbb{A})$. It may be identified with the set of pairs $(\sigma, \zeta) \in \text{SL}_2(\mathbb{A}) \times \{\pm 1\}$ with the group law $(\sigma, \zeta)(\sigma', \zeta') = (\sigma\sigma', \zeta\zeta'c(\sigma, \sigma'))$ where $c(\sigma, \sigma') := \prod_p c_p(\sigma, \sigma'_p)$ with $c_p$ the local cocycle of §2.1.1. Given $u = (\sigma, \zeta) \in \text{Mp}_2(\mathbb{A})$, we denote by $\text{pr}_{\text{SL}_2(\mathbb{A})}(u) := \sigma$ its projection to $\text{SL}_2(\mathbb{A})$ and by $\text{pr}_{\text{SL}_2(\mathbb{F}_p)}(u) := \sigma_p$ its projection to $\text{SL}_2(\mathbb{F}_p)$.

Recall that a function $f : \text{Mp}_2(\mathbb{A}) \to \mathbb{C}$ is called *genuine* if $f(\sigma, -1) = -f(\sigma, 1)$ for all $\sigma \in \text{SL}_2(\mathbb{A})$. A product $f_1f_2$ of genuine functions descends to a function $\text{SL}_2(\mathbb{A}) \to \mathbb{C}$ that we also denote by $f_1f_2$. We identify $\text{SL}_2(F)$ with its image under the canonical homomorphism $\text{SL}_2(F) \to \text{Mp}_2(\mathbb{A})$ lifting the inclusion $\text{SL}_2(F) \hookrightarrow \text{SL}_2(\mathbb{A})$, which may be characterized in turn by requiring that the elementary theta functions defined below be left $\text{SL}_2(F)$-invariant.

3.1.2. Groups, measures, norms. For $G \in \{\text{GL}_1, \text{SL}_2, \text{PGL}_2\}$, we denote by $[G] := G(F)/G(\mathbb{A})$ the corresponding quotient. When $G \neq \text{GL}_1$, we equip $[G]$ with Tamagawa measure, so that $\text{vol}([\text{SL}_2]) = 1$, $\text{vol}([\text{PGL}_2]) = 2$. We equip $[\text{GL}_1]$ with an arbitrary Haar measure. In all cases, the Haar on $[G]$ lifts to a Haar on $G(\mathbb{A})$ and then factors as a product of Haar measures on $G(F_p)$ which, for almost all finite primes $p$, assign volume one to maximal compact subgroups. We thereby obtain for each place $p$ of $F$ a local field $F_p$ with nontrivial unitary character $\psi_p$ and Haar measures on each of the groups $G(F_p)$. The discussion of §2 then applies.

For $g = (g_p)$ in one of the groups $\text{SL}_2(\mathbb{A})$ or $\text{PGL}_2(\mathbb{A})$, we denote by $\|g\| := \prod \|g_p\|$ and $\|\text{Ad}(g)\| := \prod \|\text{Ad}(g_p)\|$ the products of the local norms defined in §2.1.8. We extend this definition to $\text{Mp}_2(\mathbb{A})$ via pullback.
3.1.3. **Convention on factorization of unitary structures.** Let \( \pi \) be an automorphic representation of one of the groups \( \text{SL}_2(\mathbb{A}), \text{MP}_2(\mathbb{A}), \text{GL}_2(\mathbb{A}) \). Assume that it factors as a restricted tensor product \( \pi = \otimes \pi_p \); this happens for each \( p \) that we consider. If \( \pi \) is unitary and equipped with some specific unitary structure, then we always fix a unitary structure on the components \( \pi_p \) that is compatible with this factorization, so that \( \| \otimes \varphi_p \| = \prod \| \varphi_p \| \).

3.1.4. **“Good places” and “unramified”**. We say that a place \( p \) of \( F \) is **good** if \( p \) is non-archimedean, norm(\( p \)) is odd, \( F_p \) is unramified over its prime subfield, \( \psi_p \) is unramified, and the Haar measures defined in §3.1.2 on the groups \( G(F_p) \) assign volume one to maximal compact subgroups. Thus almost all (i.e., all but finitely many) places are good, and the assumptions of §2.1.6 (defining the “unramified case”) apply whenever \( p \) is good. We say that \( p \) is **bad** if it is not good.

For a good place \( p \) and \( G \in \{ \text{GL}_1, \text{SL}_2, \text{PGL}_2, \text{MP}_2 \} \), we say that a factorizable vector \( \varphi = \otimes \varphi_p \) in a factorizable representation \( \pi = \otimes \pi_p \) of \( G(\mathbb{A}) \) is **unramified at** \( p \) if the local component \( \varphi_p \) is unramified in the sense of §2.4; otherwise, we say that \( \varphi \) **ramifies at** or is **ramified at** \( p \).

3.2. **Hecke characters.** A **Hecke character** is a continuous homomorphism \( \chi : \mathbb{A}^\times / F^\times \to \mathbb{C}^\times \). Its **real part** is the real number \( \text{Re}(\chi) \) for which \( |\chi(y)| = |y|^{\text{Re}(\chi)} \). We denote by \( \mathcal{X} \) the group of Hecke characters and by \( \mathcal{X}(c) \) the subset consisting of those with real part \( c \). The space \( \mathcal{X}(c) \) comes equipped with a natural measure dual to the given Haar measure on \( \mathbb{A}^\times / F^\times \); it may be characterized by requiring that for all real numbers \( c \) and test functions \( f \) on \( \mathbb{A}^\times / F^\times \), the inversion formula

\[
\int_{\chi \in \mathcal{X}(c)} \int_{y \in \mathbb{A}^\times / F^\times} f(y)\chi(y) = f(1)
\]

holds.

3.3. **Sobolev norms.** Given a unitary representation \( \pi \) of one of the groups \( \text{SL}_2(\mathbb{A}), \text{MP}_2(\mathbb{A}), \text{PGL}_2(\mathbb{A}), \text{GL}_2(\mathbb{A}) \), we define Sobolev norms \( \mathcal{S}_d \) on \( \pi \) as in [31, §2] and [37, §4.6, §5.3]. If \( \pi \) factors as a restricted tensor product \( \otimes \pi_p \) and we fix unitary structures on its local components compatible with this factorization, then the Sobolev norms factor on pure tensors: \( \mathcal{S}_d(\otimes \psi_p) = \prod \mathcal{S}_d(\psi_p) \). We retain the convention of §2.9 concerning “implied indices.”

The refined “automorphic” Sobolev norms \( \mathcal{S}_d^\mathcal{X} \) considered in [31, §2] and [37, §4.6, §5.3] will not be used in the present paper.

3.4. **Elementary theta functions.** Let \( \rho_\psi \) denote the Weil representation attached to \( \psi \) of \( \text{MP}_2(\mathbb{A}) \) acting on the Schwartz–Bruhat space \( \mathcal{S}(\mathbb{A}) \). It is the restricted tensor product of the spaces considered in §2.2. For \( \phi \in \rho_\psi \), the elementary theta function \( \theta(\phi) : \text{SL}_2(F) \backslash \text{MP}_2(\mathbb{A}) \to \mathbb{C} \) is the genuine automorphic form defined by the convergent series \( \theta(\phi)(g) := \sum_{\alpha \in F} (\rho_\psi(g)\phi)(\alpha) \). The map \( \rho_\psi \ni \phi \mapsto \theta(\phi) \) is equivariant, and quite nearly unitary (see (3.4)) on the orthogonal complement \( \{ \phi : \phi(-x) = -\phi(x) \text{ for all } x \} \) of its kernel \( \{ \phi : \phi(-x) = -\phi(x) \text{ for all } x \} \).

3.5. **Eisenstein series.**

3.5.1. **Induced representations.** For a Hecke character \( \chi \) and \( G \in \{ \text{SL}_2, \text{PGL}_2 \} \), we define an induced representation \( \mathcal{I}_G(\chi) \) of \( G(\mathbb{A}) \) either by mimicking the local definitions of §2.3 or by taking the restricted tensor products of the representations attached there to the local components \( \chi_v \). If \( \chi \) is unitary, then \( \mathcal{I}_G(\chi) \) is unitary, and we equip it with the tensor product of the locally-defined unitary structures.
3.5.2. Intertwiners. Denote by Eis, (or simply Eis when \( \chi \) is clear by context) the standard map from \( \mathcal{I}_G(\chi) \) to the space of automorphic forms, defined for \( \operatorname{Re}(\chi) \) large enough by averaging over left cosets of the standard Borel in \( G(F) \) and in general by meromorphic continuation.

3.5.3. Rankin–Selberg period formulas on \( \text{PGL}_2 \). The following lemma paraphrases a special case of [31, §4.4] (compare with [31, §2.2.2]).

Lemma. Let \( \pi \) be a cuspidal automorphic representation \( \text{PGL}_2(\mathbb{A}) \). Let \( \chi \) be a unitary character of \( \mathbb{A}^\times / F^\times \). Set \( \pi_1 := \pi, \pi_2 := \overline{\pi} \) and \( \pi_3 = \mathcal{I}_{\text{PGL}_2}(\chi) \). Equip \( \pi_1, \pi_2 \) with the norm coming from \( L^2([\text{PGL}_2]) \) and \( \pi_3 \) with that coming from \( \mathcal{I}_{\text{PGL}_2}(\chi) \). For \( i = 1, 2, 3 \), let \( \varphi_i = \otimes \varphi_{ip} \in \pi_i = \otimes \pi_{ip} \) be a factorizable vector. Let \( S \) be a finite set of places of \( F \), containing the bad ones, with the property that \( \varphi_{ip} \) is an unramified unit vector for all \( i \) and each \( p \notin S \). Then the squared period

\[
\left| \int_{[\text{PGL}_2]} \varphi_1 \varphi_2 \text{Eis}_{\text{PGL}_2}(\varphi_3) \right|^2
\]

is equal to

\[
c^{(S)}(2) L(\pi_1 \otimes \pi_2 \otimes \pi_3, \frac{3}{2}) \cdot \prod_{v \in S} P_{\text{PGL}_2(F_v)}(\varphi_{1v}, \varphi_{2v}, \varphi_{3v}).
\]

for some \( c > 0 \) depending only upon \( F \).

We note also that \( L(\pi_1 \otimes \pi_2 \otimes \pi_3, s) = L(\pi \otimes \overline{\pi} \otimes \chi, s) L(\pi \otimes \overline{\pi} \otimes \chi^{-1}, s) \).

3.5.4. \( \text{SL}_2 \) vs. \( \text{PGL}_2 \). If \( \omega \) is a Hecke character with \( \omega^2 = \chi \) and \( f \) is an element of \( \mathcal{I}_{\text{SL}_2}(\chi) \), denote by \( f^\omega \) its unique extension to \( \mathcal{I}_{\text{PGL}_2}(\omega) \), as in §2.3.6.

3.5.5. Lifting Rankin–Selberg periods on \( \text{SL}_2 \) to \( \text{PGL}_2 \). We shall encounter Rankin–Selberg integrals on \( [\text{SL}_2] \) involving restrictions of automorphic forms on \( [\text{GL}_2] \). In order to relate such integrals via §3.5.3 to products of \( L \)-values and local integrals, we must first lift them to \( [\text{PGL}_2] \):

Lemma. Let \( \pi \) be a cuspidal automorphic representations of \( \text{GL}_2(\mathbb{A}) \). Let \( \chi \) be a Hecke character with \( \operatorname{Re}(\chi) > -1 \) and \( \chi \neq \nu \). Let \( \varphi \in \pi, \varphi' \in \overline{\pi} \) and \( f \in \mathcal{I}_{\text{SL}_2}(\chi) \). Then

\[
\int_{[\text{SL}_2]} \varphi \varphi' \text{Eis}_{\text{SL}_2}(f) = c \sum_{\omega \in \chi, \omega^2 = \chi} \int_{[\text{PGL}_2]} \varphi \varphi' \text{Eis}_{\text{PGL}_2}(f^\omega)
\]

for some \( c > 0 \) depending only upon \( F \).

Proof. Denote temporarily by \( K \) the standard maximal compact subgroup of \( \text{SL}_2(\mathbb{A}) \), by \( \overline{K} \) its image in \( \text{PGL}_2(\mathbb{A}) \), by \( N \) the standard unipotent subgroup of \( \text{SL}_2(\mathbb{A}) \), by \( T \) the standard diagonal torus of \( \text{SL}_2(\mathbb{A}) \), and by \( A \) the standard diagonal torus of \( \text{PGL}_2(\mathbb{A}) \). We will exploit below the decompositions \( \text{PGL}_2(\mathbb{A}) = N A \overline{K} \) and \( \text{SL}_2(\mathbb{A}) = N T K \).

Denote by \( \Phi(g) := \int_{\mathbb{A} / F} \varphi \varphi' (n(x) g) \) the constant term of \( \varphi \varphi' \). Both sides of (3.1) vary holomorphically with respect to \( \chi \) as \( f \) varies in a flat section, so we may reduce to the case that \( \operatorname{Re}(\chi) \) is sufficiently large. By the inclusion of the factor
By approximating $\Phi$ by test functions, we reduce to verifying for each test function $\phi$ on $A^\times/F^\times$ and each Hecke character $\chi$ that
\[
\int_{y \in A^\times/F^\times} \phi(y^2) \chi(y) = \frac{1}{2} \sum_{\omega \in \mathbb{X}} \int_{y \in A^\times/F^\times} \phi(y) \omega(y). \tag{3.2}
\]

The proof of (3.2) is an exercise in Pontryagin duality and quotient measures, and left to the reader; it is similar to the identity
\[
\int_{y \in \mathbb{R}_+^\times} \phi(y^2)|y|^{s} dy = \frac{1}{2} \int_{y \in \mathbb{R}_+^\times} \phi(y)|y|^{s/2} dy
\]
for $s$ satisfied by test functions $\phi$ on $\mathbb{R}_+^\times \cong \mathbb{R}/\{\pm 1\}$ and complex numbers $s$.

\[\square\]

Remark. The sum indexed by $\omega$ in (3.1) is finite in the sense that the summand vanishes for $\omega$ outside some finite set depending at most upon $\phi, \phi'$. Indeed, the $[\text{PGL}_2]-$integral vanishes unless $\omega$ is unramified at all good places $p$ for $F$ at which $\phi, \phi'$ are unramified.

3.6. Regularized spectral expansions of products of elementary theta functions. Let $\chi$ be a unitary Hecke character. Denote by $I_\chi: \rho \otimes \rho_\psi \rightarrow I_{[\text{SL}_2]}(\chi)$ the $\text{Mp}_2(A)$-equivariant intertwiner defined and studied in [37, §5.8]; it is given on pure tensors $\phi = \otimes \phi_p$ by $I_\chi(\phi) = L^{(S)}(\chi, 1) \cdot (\otimes I_{\chi_p}(\phi_p))$, where $S$ is any finite set of places containing the bad places and any at which $\phi$ is ramified, and $I_{\chi_p}$ is as in §2.10. We record a special case of [37, Thm 2]:

Lemma. Let $\Phi: [\text{SL}_2] \rightarrow C$ be smooth and of rapid decay. Let $\phi_1, \phi_2 \in \rho_\psi$. Then
\[
\int_{[\text{SL}_2]} \Phi \theta(\phi_1) \overline{\theta(\phi_2)} = \int_{[\text{SL}_2]} \Phi \int_{[\text{SL}_2]} \theta(\phi_1) \overline{\theta(\phi_2)} + \int_{\chi \in \mathbb{X}(0)} \int_{[\text{SL}_2]} \Phi \text{Eis}_{[\text{SL}_2]}(I_\chi(\phi_1 \otimes \phi_2)). \tag{3.3}
\]
Moreover, if $\phi_1(-x) = \phi_1(x)$ for all $x \in A$, then
\[
\int_{[\text{SL}_2]} \theta(\phi_1) \overline{\theta(\phi_2)} = 2 \int_A \phi_1 \overline{\phi_2}. \tag{3.4}
\]

As noted in §1, the absence of a cuspidal contribution on the RHS of (3.3) is critical to our argument.

We note that the integrand in (3.3) is holomorphic in $\chi$: the simple pole of $\chi \mapsto I_\chi$ at the trivial character is cancelled by the corresponding simple zero of
the Eisenstein intertwiner. We note also that the second assertion (3.4) may be
applied to general \( \phi \in \rho_\psi \) by first replacing them with their even projections
\( \phi^+ := (\phi_1 + \phi_2(x))/2 \).

3.7. Global Waldspurger packets. Let \( \tau \) be a cuspidal automorphic representation
of \( \text{PGL}_2(\mathbb{A}) \). Given a collection of elements \( \sigma^\pm \) of the local Waldspurger
packets \( \text{Wd}^\pm(\tau_p) \) indexed by some signs \( \varepsilon_p = \pm 1 \) as in \( \S 2.6 \) (necessarily \( \varepsilon_p = +1 \)
for almost all \( p \)), one may form the restricted tensor products \( \sigma^\pm \tau_p \). Waldspurger showed that \( \sigma^\pm \) is automorphic if and only if \( \prod_p \varepsilon_p = \varepsilon(\tau, \frac{1}{2}) \); moreover, \( \sigma^\pm \) is
then cuspidal and occurs with multiplicity one in the space of automorphic forms on
\( \text{SL}_2(F) \setminus \text{Mp}_2(\mathbb{A}) \). It is thus meaningful to regard \( \text{Wd}_\psi(\tau) := \{ \sigma^\pm : \prod_p \varepsilon_p = \varepsilon(\tau, \frac{1}{2}) \} \)
as a finite collection of genuine cuspidal automorphic representations of \( \text{Mp}_2(\mathbb{A}) \).

Suppose now that \( \tau_p \) is principal series for all places \( p \). Then each local Waldspurger
packet is a singleton \( \text{Wd}^\pm(\tau_p) = \{ \sigma^\pm \} \) and \( \varepsilon(\tau, \frac{1}{2}) = 1 \), so the global
Waldspurger packet is a singleton \( \text{Wd}_\psi(\tau) = \{ \sigma \} \) consisting of a cuspidal automorphic
representation \( \sigma \) of \( \text{Mp}_2(\mathbb{A}) \).

3.8. Generalized Shimura integrals. Let \( \tau \) be a cuspidal automorphic representation
of \( \text{PGL}_2(\mathbb{A}) \). Let \( \sigma \in \text{Wd}_\psi(\tau) \). Let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}) \) with unitary central character \( \omega_\tau \). Let \( \rho_\psi \) be the Weil representation of \( \text{Mp}_2(\mathbb{A}) \) on \( S(\mathbb{A}) \). Consider the following \( \text{Mp}_2(\mathbb{A}) \)-invariant hermitian form \( \mathcal{P}_{\text{SL}_2(F)}(\pi, \psi) \) for \( \psi_1 \in \pi, \psi_2 \in \rho_\psi, \varphi_3 \in \sigma \),

\[
\mathcal{P}_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) = \int_{[\text{SL}_2]} \varphi_1 \cdot \theta(\varphi_2) \cdot \frac{1}{L(2)} \cdot \varphi_3.
\]

We recall [42, Thm 4.5]:

Lemma. Assume that the \( \varphi_i \) are pure tensors \( \otimes_p \varphi_{ip} \). Let \( S \) be a finite set of places, containing the bad ones, with the property that \( \varphi_{ip} \) is an unramified unit vector for all \( i \) and each \( p \notin S \). Set \( \zeta_f(s) := \prod_{p \notin S} \zeta_f(s) \). Then

\[
\mathcal{P}_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) = \frac{1}{2^2} \zeta_f(2) \cdot \mathcal{P}_{\text{SL}_2(F)}(\pi, \psi, \frac{1}{2}) \cdot \prod_{p \in S} \mathcal{P}_{\text{SL}_2(F)}(\psi_1, \psi_2, \psi_3).
\]

3.9. Bounds for \( \text{SL}_2 \)-matrix coefficients of automorphic representations
of \( \text{GL}_2 \). Let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}) \). The standard
unitary structure on \( \pi \) is given for \( \varphi, \varphi' \in \pi \) by \( \langle \varphi, \varphi' \rangle := \int_{[\text{PGL}_2]} \varphi \cdot \overline{\varphi'} \). In general, this differs from the modified pairing \( \langle \varphi, \varphi' \rangle_{\text{SL}_2} := \int_{[\text{SL}_2]} \varphi \cdot \overline{\varphi'} \) given by integrating over \( [\text{SL}_2] \subseteq [\text{GL}_2] \). The two are related by Fourier inversion: taking into account that \( \text{vol}([\text{PGL}_2]) = 2 \text{vol}([\text{SL}_2]) \), we have

\[
\langle \varphi, \varphi' \rangle_{\text{SL}_2} = \frac{1}{2} \sum_{\chi} \langle \varphi, \varphi \otimes \chi \rangle (3.5)
\]

where \( \varphi \otimes \chi \in \pi \otimes \chi \) denotes the automorphic form given by \( (\varphi \otimes \chi)(g) := \varphi(g) \chi(\det g) \). The sum on the RHS of (3.5) may be restricted to those \( \chi \) for which \( \pi \otimes \chi \cong \tau \). By the local estimate of (2.18) and axiom (S1d) of [31, §2.4], one has for each fixed \( d \) the estimate \( \mathcal{S}_d(\varphi \otimes \chi) \ll C(\chi)^{O(1)} \mathcal{S}(\varphi) \). By the local bound
for matrix coefficients given in §2.11 and axiom (S1d) of [31, §2.4], we obtain for \( g \in \text{GL}_2(\mathbb{A}) \) the crude but sufficient bound
\[
\langle \pi(g)\varphi', \varphi \rangle \ll \|\text{Ad}(g)\|^{\theta - 1/2} S(\varphi) S(\varphi') \sum_{\chi \in X: x^2 = 1, \pi \otimes \chi \cong \chi} C(\chi)^{O(1)}
\]

\[ (3.6) \]

4. Main result

4.1. Statement.

4.1.1. Inputs. Fix a number field \( F \) and let \( \mathbb{A}, \psi : \mathbb{A}/F \to \mathbb{C}^{(1)} \) be as in §3. We assume given an infinite countable collection \( \mathcal{F} \) consisting of pairs \((q, \pi)\), where
\begin{itemize}
  \item \( q \) is a finite prime of \( F \), and
  \item \( \pi \) is a cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}) \) whose local component \( \pi_q \) satisfies the hypotheses of §2.17.
\end{itemize}
We assume also that
\[
\# \{(q, \pi) \in \mathcal{F} : \text{norm}(q) \leq X\} < \infty \text{ for each } X > 0. 
\]

4.1.2. Asymptotic notation and terminology. We denote in what follows by \((q, \pi)\) a varying element of \( \mathcal{F} \). Our convention is that all objects (scalars, places, representations, vectors, ...) considered below are allowed to depend implicitly upon \((q, \pi)\) unless they are explicitly designated as fixed, in which case we require that they depend at most upon the number field \( F \), the family \( \mathcal{F} \), and any aforementioned fixed quantities. An assertion \( \alpha \) depending upon \((q, \pi)\) will be said to hold eventually if there is a fixed finite subset \( \mathcal{F}_0 \subseteq \mathcal{F} \) so that \( \alpha \) holds whenever \((q, \pi)\) \( \notin \mathcal{F}_0 \). The standard asymptotic notation is then defined accordingly. For example, given complex scalar quantities \( X, Y \) (possibly depending implicitly upon the pair \((q, \pi)\), per our convention), we write
\begin{itemize}
  \item \( X = O(Y), \ X \ll Y \) or \( Y \gg X \) to denote that there is a fixed \( c > 0 \) so that \( |X| \leq c|Y| \), and
  \item \( X = o(Y) \) to denote that for each fixed \( \varepsilon > 0 \), one has \( |X| \leq \varepsilon|Y| \) eventually.
\end{itemize}
Set \( Q = \text{norm}(q) \). Our assumption (4.1) says that \( Q \) eventually exceeds any fixed positive real.

4.1.3. Assumptions. Our results are conditional on the following hypothesis:

**Hypothesis \( H(\mathcal{F}) \).** There is a fixed \( \delta_0 > 0 \) so that for each unitary character \( \chi \) of \( \mathbb{A}^\times/F^\times \),
\[
L(\pi \otimes \pi \otimes \chi, \frac{1}{2}) \ll C(\pi \otimes \pi \otimes \chi)^{1/4 - \delta_0} \cdot \left( C(\chi) \cdot \prod_{p \neq q} C(\pi_p) \right)^{O(1)}.
\]

In other words, we assume a subconvex bound in the \( \pi_q \)-aspect with polynomial dependence upon \( \chi \) and the ramification of \( \pi \) at places other than \( q \).
4.1.4. Main result. We state an equivalent form of Theorem 1:

**Theorem 2.** Let $F$ be a family as above that satisfies Hypothesis $H(F)$. Let $(\mathfrak{q}, \pi) \in F$, as above. There is then a fixed $\delta > 0$ with the following property. Let $\tau$ be a cuspidal automorphic representation of $\text{PGL}_2(\mathbb{A})$ for which either

1. each local component $\tau_p$ belongs to the principal series, or
2. the conclusion of the conjecture of §2.15.1 holds.

Then

$$L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) \ll C(\text{ad}(\pi) \otimes \tau)^{1/4 - \delta} P^{O(1)},$$

where $P := C(\tau) \cdot \prod_{p \not\in \mathfrak{q}} C(\pi_p)$.

4.2. Preliminary reductions. Observe first (thanks to the “pass to worst-case subsequences” argument) that in proving Theorem 2, we may freely replace $F$ by any infinite subset thereof. Next, observe that if the quantity $P$ in the statement of Theorem 2 satisfies $\log P \gg \log Q$, then the required conclusion is worse than the convexity bound. For this reason, we may and shall assume that

$$C(\tau) \cdot \prod_{p \not\in \mathfrak{q}} C(\pi_p) = Q^{o(1)}.$$  \ \ \ (4.2)

This reduction has some pleasant consequences:

1. It implies that $\tau_\mathfrak{q}$ is unramified, so that the results of §2.17 become applicable.
2. It implies that

$$C(\text{ad}(\pi) \otimes \tau) = Q^{2+o(1)}. \ \ \ (4.3)$$

3. For each unitary Hecke character $\chi$,

$$C(\pi \otimes \pi \otimes \chi) = Q^{2+o(1)}C(\chi)^{O(1)}; \ \ \ (4.4)$$

4. $\# \text{ram}(\pi) + \# \text{ram}(\tau) = o(\log(Q)).$

5. If $S$ is a finite set of places and $B_p$ ($p \in S$) are positive real quantities for which

- $S$ contains at most $o(\log Q)$ places not already in $\text{ram}(\pi) \cup \text{ram}(\tau)$, and
- each $B_p \asymp 1$,

then $\prod_{p \in S} B_p = Q^{o(1)}$. Note that known bounds toward Ramanujan (see §2.7) imply that any individual non-archimedean Euler factor considered below has magnitude $\asymp 1$.

4.3. Reduction to period bounds. The $L$-function $L(\text{ad}(\pi) \otimes \tau, s)$ is self-dual, so the global root number $\varepsilon(\text{ad}(\pi) \otimes \tau, \frac{1}{2})$ is $\pm 1$; if it is $-1$, then $L(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) = 0$, and so the required estimate is trivial. Assume henceforth that

$$\varepsilon(\text{ad}(\pi) \otimes \tau, \frac{1}{2}) = 1. \ \ \ (4.5)$$

For each place $p$ of $F$, set $\varepsilon_p := \varepsilon(\pi_p \otimes \pi_p \otimes \tau_p, \frac{1}{2})$. Then the local Waldspurger packet $\text{Wd}_\psi(\tau_p)$ contains an element $\sigma_p$ with index $\varepsilon_p$, and the local hermitian form $\Pi_{\text{SL}_2(\mathbb{A})}$ does not vanish identically on $\pi_p \otimes \rho_\psi \otimes \sigma_p$ (see §2.13). Moreover, by (4.5), we have $\prod_p \varepsilon_p = \varepsilon(\pi \otimes \pi \otimes \tau, \frac{1}{2}) = \varepsilon(\tau, \frac{1}{2})$, so by results of Waldspurger recalled in §3.7, there is a cuspidal automorphic representation $\sigma \in \text{Wd}_\psi(\tau)$ with local components $\sigma_p$.

Fix isometric identifications $\pi = \otimes \pi_p$, $\sigma = \otimes \sigma_p$ per the conventions of §3.1.3. Define $\varphi_1 \in \pi, \varphi_2 \in \rho_\psi, \varphi_3 \in \sigma$ to be the pure tensors obtained from the choices
of local vectors given in §2.15 (for \( p \neq q \)) and in §2.17 (for \( p = q \)). Set \( \varphi_2 := \theta(\phi_2) \). By the Shimura–like period formula from §3.8, the known global estimate \( L(\text{ad}(\pi), 1) = Q^{o(1)} \) (see [15], [3, §2.9]) and the local lower bounds of §2.15 and §2.17, we have

\[
\frac{L(\text{ad}(\pi) \otimes \tau, \frac{1}{2})}{C(\text{ad}(\pi) \otimes \tau)^{1/4}} \ll Q^{o(1)} \langle \langle \varphi_1 \varphi_2, \varphi_3 \rangle \rangle^2.
\]

Recalling the estimate (4.3) for the conductor, the proof of Theorem 2 reduces to that of the period bound

\[
\langle \langle \varphi_1 \varphi_2, \varphi_3 \rangle \rangle \ll Q^{-\delta}
\]

for some fixed \( \delta > 0 \).

We will prove (4.6) by applying the amplification method of [31] arranged so that Cauchy–Schwarz is applied the vector \( \varphi_3 \). The key input in that method is an asymptotic formula for some mild generalizations of the \( L^2 \)-norms \( \langle \langle \varphi_1 \varphi_2, \varphi_3 \rangle \rangle = \langle \langle |\varphi_1|^2, |\varphi_2|^2 \rangle \rangle \); we establish such a formula below in §4.4. In section §4.5, we refine that formula by estimating the main and error terms. In section §4.6, we recall the construction of an amplifier, following [31]. In section §4.7, we pull everything together to deduce the bound (4.6).

### 4.4. The key estimate

Recall from §3.1.1 that to each \( u \in M_{p_2}(A) \) and each place \( p \) we may attach a local component \( \text{pr}_{\text{SL}_2(F_p)}(u) \in \text{SL}_2(F_p) \). We say that \( u \) is **reasonable** if

- \( \text{pr}_{\text{SL}_2(F_p)}(u) = 1 \), and
- \( \#\{p : \text{pr}_{\text{SL}_2(F_p)}(u) \neq 1\} = o(\log Q) \).

Given a pure tensor \( \varphi = \otimes \varphi_p \) in some factorizable unitary representation (such as \( \pi \) or \( \rho_p \)), it will be convenient to introduce the abbreviation \( S_d'(\varphi) := \|\varphi_q\| \prod_{p \neq q} S_d(\varphi_p) \); thus \( S_d' \) quantifies the ramification of \( \varphi \) at places other than the distinguished place \( q \). We retain the standard convention concerning implied indices of Sobolev norms (see §2.9, §3.3), so that “\( S^d \)” means “\( S_d' \) for some fixed \( d \).”

It will be typographically convenient in what follows to denote by \( \varphi^u \) the left action of a group element \( u \in M_{p_2}(A) \) on an automorphic form \( \varphi \). Thus \( \varphi^u_1 := \pi(\text{pr}_{\text{SL}_2(A)}(u))\varphi_1 \) and \( \varphi^u_2 := \theta(\rho_p(u)\phi_2) \) if \( \varphi_2 = \theta(\phi_2) \). The side-effect \( \varphi^{wu_1} = (\varphi^{wu})^{u_1} \) of this convention will not matter for us.

**Lemma.** Assume hypothesis \( H(\mathcal{F}) \). There is a fixed \( \delta_1 > 0 \) so that for each reasonable element \( u \in \text{SL}_2(A) \),

\[
\int_{[\text{SL}_2]} \varphi^u_1 \varphi^u_2 \varphi_1 \varphi_2 = \int_{[\text{SL}_2]} \varphi_1 \varphi_2 \int_{[\text{SL}_2]} \varphi^u_1 \varphi^u_2 \ll Q^{-\delta_1} \prod_{i=1,2} S'(\varphi^u_i) S'(\varphi_i).
\]

**Proof.** We first rearrange \( \varphi^u_1 \varphi^u_2 \varphi_1 \varphi_2 = \varphi^u_1 \varphi_1 \varphi^u_2 \varphi_2 \). By the regularized spectral expansion of §3.6, we reduce to showing that

\[
\int_{\chi \in \mathcal{X}(0)} \int_{[\text{SL}_2]} \varphi^u_1 \varphi^u_2 \mathcal{E}(I_\chi(\phi_2 \otimes \overline{\phi_2})) \ll \mathcal{E}
\]

where \( \mathcal{E} := S'(\varphi^u_1) S'(\varphi_1) S'(\varphi^u_2) S'(\phi_2) \). Let \( A > 1 \) be fixed and large enough that \( \int_{\chi \in \mathcal{X}(0)} \sum_{\omega \in \mathcal{X}(0)} \omega^2 = C(\omega)^{-A} < \infty \). By lifting the \( \text{SL}_2 \)-periods in (4.8) to \( \text{PGL}_2 \)-periods as in §3.5.5, we reduce to showing for each \( \omega, \chi \in \mathcal{X}(0) \) with \( \omega^2 = \chi \).
that
\[ \int \varphi_1^{[PGL_2]} \varphi_2^{[SL_2]} (I_{\chi}(\varphi_1^{u_p} \otimes \varphi_2)) \ll C(\omega)^{-4} \mathcal{E}. \]  
(4.9)
As noted in §3.5.5, the LHS of (4.9) vanishes unless, as we henceforth assume, \( \omega_p \) is unramified for all good places \( p \) of \( F \) with \( \text{pr}_{SL_2(F_p)}(u) = 1 \) and \( \phi_p \) unramified. By this observation and the assumption that \( u \) is reasonable, the set \( S \) of places at which anything is ramified satisfies \( \#S = o(\log Q) \); this property will be used in what follows to control products of implied constants and local Euler factors, as discussed in §4.2. By the Rankin–Selberg period formula of §3.5.3 and the definition of \( I_{\chi} \) given in §3.6, the squared magnitude of the LHS of (4.9) factors as a product
\[ \mathcal{G} \prod_{p \in S} \mathcal{L}_p \]  
of global and local quantities, where
\[ \mathcal{G} \asymp \frac{|L(S) (\pi \otimes \chi, \frac{1}{2})|^2}{L(S)(\text{ad}(\pi), 1)^2} \]
and (with notation as in §2.3.6, §2.10)
\[ \mathcal{L}_p = \mathcal{P}_{PGL_2(F_p)}(\varphi_{1p}^{u_p}, \varphi_{1p}^{[T]}, I_{\chi_p} (\varphi_{2p}^{u_p} \otimes \varphi_{2, p})^{\omega_p}). \]
By the subconvexity hypothesis \( H(\mathcal{F}) \), the approximation (4.4) for the analytic conductor and the estimate \( L(\text{ad}(\pi), 1) = Q^{o(1)} \), we have
\[ \mathcal{G} \ll Q^{-(2.001)\delta_1} C(\pi \otimes \chi \otimes \omega)^{1/2} \]
for some fixed \( \delta_1 > 0 \). The local estimates of §2.16 and §2.17 and the assumption \( u_q = 1 \) give
\[ |\mathcal{L}_p| \ll C(\omega_p)^{-2A} \cdot \begin{cases} S(\varphi_{1p}^{u_p})^2 S(\varphi_{1p})^2 S(\varphi_{2p}^{u_p})^2 S(\varphi_{2p})^2 & \text{if } p \neq q, \\ C(\pi \otimes \chi \otimes \omega)^{-1/2} \|\varphi_{1q}\|^4 \|\varphi_{2q}\|^4 & \text{if } p = q. \end{cases} \]
These estimates combine to give \( |\mathcal{G} \prod_{p \in S} \mathcal{L}_p|^{1/2} \ll C(\omega)^{-A} \mathcal{E} \), as required. \( \square \)

### 4.5. Bounds for matrix coefficients and Sobolev norms

Let \( u \in \text{Mp}_2(\mathbb{A}) \) be reasonable. In this section, we refine the estimate (4.7) by taking into account our choice of vectors and bounds for matrix coefficients.

Recall from §4.2 that we have reduced to the case that \( \prod_{p \neq q} \mathcal{P}(\pi_p) = Q^{o(1)} \), and that the local component \( \pi_q \) has conductor \( q \) and unramified central character, hence is an unramified twist of the special representation. In particular, \( \pi_q \) is not isomorphic to its twist by any nontrivial quadratic character of the local multiplicative group \( F_q^\times \). It follows that any (quadratic) Hecke character \( \chi \) for which \( \pi \otimes \chi \cong \pi \) satisfies \( C(\chi) = Q^{o(1)} \); moreover, the number of such \( \chi \) is at most \( O(2^\# \text{ram}(\pi)) = Q^{o(1)} \). By a variant of the arguments of §3.9 (taking into account that \( \text{pr}_{SL_2(F_q)}(u) = 1 \) and the consequence \( S'(\varphi_1)^2 \ll Q^{o(1)} \) of our choice of \( \varphi_1 \) (see §2.15, §2.17), it follows that
\[ \int_{[SL_2]} \varphi_1^{[T]} \varphi_2^{[T]} \ll \|\text{Ad}(u)\|^{\sigma'-1/2} Q^{o(1)}. \]  
(4.10)

Similarly but more simply, the global identity (3.4) describing the unitary structure on elementary theta functions and the local estimate of §2.11 for their matrix coefficients furnish the bound
\[ \int_{[SL_2]} \varphi_2^{[T]} \varphi_2^{[T]} \ll \|\text{Ad}(u)\|^{-1/4} Q^{o(1)}. \]  
(4.11)
Our choice of vectors and (4.2) imply for \( i = 1, 2 \) that
\[
S'(\varphi_i) \ll Q^{o(1)}, \quad S'(\varphi_i^*) \ll \|u\|^{O(1)}Q^{o(1)}.
\] (4.12)

For future reference, we record also that
\[
\|\varphi_3\| \ll 1.
\] (4.13)

We conclude by (4.7), (4.10), (4.11), and (4.12) that the \( SL_2 \)-periods
\[
I_u := \int_{[SL_2]} \varphi_1^\gamma \varphi_2^\frac{1}{\gamma} \bar{\varphi_2}
\]
satisfy
\[
I_u \ll (\|\text{Ad}(u)\|^{t - 3/4} + \|u\|^B)Q^{\alpha(1)}
\] (4.14)
for some fixed \( \delta_1 > 0, B \geq 0 \).

**4.6. Construction of an amplifier.**

**Lemma.** For each fixed \( \eta > 0 \) there is a finite measure \( \nu \) on \( Mp_2(\mathbb{A}) \) for which the following properties hold eventually:

1. The total variation measure \( |\nu| \) has mass bounded above by \( Q^{C\eta} \), where \( C \geq 0 \) is fixed and independent of \( \eta \).
2. Set \( |\nu|^{(2)} := |\nu| * |\nu^*| \). For each fixed \( \gamma > 1/2 \) there is a fixed \( \delta > 0 \) so that
   \[
   \int \|\text{Ad}(u)\|^{-\gamma} d|\nu|^{(2)}(u) \leq Q^{-\delta}.
   \]
3. \( \varphi_3 * \nu = c \varphi_3 \) for some \( c \gg Q^{-\alpha(1)} \).
4. Each \( u \in \text{supp}(\nu) \) has the properties:
   a. \( u \) is reasonable in the sense of §4.4;
   b. \( p_{\text{SL}_2(F_p)}(u) = 1 \) unless \( p \) is a finite place at which \( \pi, \sigma, \psi \) are all unramified;
   c. \( \|u\| \leq Q^{\eta} \).
   In particular each \( u \in \text{supp}(|\nu|^{(2)}) \) is reasonable.

**Proof.** The proof is as in [31, §5.2.3] which in turn refers to [53, §4.1]. For convenience, we record the construction of \( \nu \) and sketch the proof. Let \( T \) denote the set of good primes \( p \) of \( F \) at which \( \pi, \sigma, \psi \) and hence also \( \varphi_3 \) are unramified. For each \( p \in T \), the genuine spherical Hecke algebra of \( Mp_2(F_p) \) (consisting of genuine compactly-supported distributions invariant on the left and right under the image of the standard maximal compact subgroup of \( SL_2(F_p) \)) admits a linear basis \( 1, T_p, T_p^2, \ldots \), where \( T_p \) acts on the unramified subspace of \( \sigma_p \) by the corresponding Hecke eigenvalue \( \lambda_p(p^a) \) of the lift \( \tau_p \) of \( \sigma_p \), normalized so that temperedness is equivalent to \( |\lambda_p(p^a)| \leq a + 1 \). The relation \( T_p^2 = T_p + 1 \) holds. We may identify each \( T_{p^k} \) with a finite measure on \( Mp_2(\mathbb{A}) \), acting on \( \varphi_3 \) by \( \lambda_p(p^a) \). We may find a fixed \( C_0 \geq 0 \) so that \( \|u\| \leq \text{norm}(p)^{C_0a} \) for all \( u \in \text{supp}(T_{p^k}) \). Set \( L := Q^{N/C_0} \).

For an integral ideal \( n \), define \( c_n := 0 \) unless \( n \) is of the form \( p^a \) with \( p \in T \) and \( a \in \{1, 2\} \) with \( \text{norm}(p)^a \leq L \), in which case set \( c_n := \text{sgn}(\lambda_p(n))^{-1} \), where \( \text{sgn}(z) := z/|z| \).

Set \( \nu := (L/\log L)^{-1} \sum_n c_n T_n \).

Assertions (1) and (4) are then clear by construction. We have \( \varphi_3 * \nu = c \varphi_3 \) with \( c := (L/\log L)^{-1} \sum_{n \in \mathcal{L}} |\lambda_p(n)| \), so assertion (3) follows from the estimate \( \#\mathcal{L} \gg Q^{-\alpha(1)}L \) (a consequence of §4.2) and the identity \( \lambda_p(p)^2 = \lambda_p(p^2) + 1 \). Assertion (2) follows by direct calculation as in [53, §4.1]. \( \square \)
4.7. Application of the amplification method. To finish the proof of (4.6), we argue as in [31, §5.2]: Let \( \eta > 0 \) be fixed and small enough in terms of the fixed quantities \( \delta \) and \( B \) arising in (4.14). Let \( \nu \) be the amplifier constructed in §4.6 in terms of the parameter \( \eta \). By the first two properties of the amplifier, our choice of \( \eta \), and the inequality \( \vartheta' - 3/4 < -1/2 \), we have

\[
\int_{\vartheta} (\| \text{Ad}(u) \|^{\vartheta'-3/4} + \| u \|^{BQ^{-\delta}}) d|\nu|^{(2)}(u) \ll Q^{-\delta}
\]

(4.15)

for some fixed \( \delta > 0 \). By the third property of the amplifier, we have \( \varphi_3 = c\varphi_3 * \nu \) for some eigenvalue \( c \gg Q^{-\alpha(1)} \). The proof of (4.6) thereby reduces to the proof of an analogous bound for \( \langle \varphi_1 \varphi_2, \varphi_3 * \nu \rangle \), or equivalently, for \( \langle (\varphi_1 \varphi_2) * \nu^*, \varphi_3 \rangle \). By (4.13) and Cauchy–Schwarz, we reduce to bounding \( \| (\varphi_1 \varphi_2) * \nu^* \|^2 = \langle \varphi_1 \varphi_2, (\varphi_1 \varphi_2) * \nu^{(2)} \rangle \) where \( \nu^{(2)} := \nu^* * \nu \). Expanding the definition of \( (\varphi_1 \varphi_2) * \nu^{(2)} \), we reduce to showing that the \( \text{SL}_2 \)-periods \( \mathcal{I}_n \) of §5.5 satisfy \( \int_{\vartheta} \mathcal{I}_n d\nu^{(2)}(u) \ll Q^{-\delta} \) for some fixed \( \delta > 0 \). The latter estimate holds by (4.14), (4.15) and the triangle inequality.

5. Comments on other aspects

We give here some evidence that new ideas are required to adapt our method to other aspects. Reverting to the local notation of §2, we will show that a simplified form of the crucial lemma of §2.17 fails for representations \( \pi \) other than those considered there. The generality of this paper is thus optimal with respect to a sufficiently restricted form of the method.

We assume that we are in the unramified case (§2.1.6), and in particular that \( F \) is non-archimedean; similar considerations apply more generally. Let \( \pi, \tau \) and \( \sigma \in \text{Wd}_{\psi}(\tau) \) be as in §2.13, with \( \tau \) (and hence \( \sigma \)) an unramified principal series representation. We assume for simplicity that \( \tau \) (and hence \( \sigma \)) is tempered. We assume also that

\[
C(\text{ad}(\pi)) \neq 1, \quad (5.1)
\]

since we are ultimately interested in taking \( C(\text{ad}(\pi)) \) to \( \infty \).

Let \( \varphi_2 \in \rho_\varphi \) be the unramified unit vector given in the Schroedinger model by \( 1_\varphi \in \mathcal{S}(F) \). It follows from the proof of the lemma of §2.17 that if \( \pi \) is a twist of the special representation, then there are unit vectors \( \varphi_1 \in \pi, \varphi_3 \in \sigma \) so that

\[
\mathcal{P}_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) \gg C(\text{ad}(\pi) \otimes \tau)^{-1/4}. \quad (5.2)
\]

Indeed, this estimate is at the heart of that lemma. The fact that the inverse of the RHS of (5.2) looks like the convexity bound is necessary for the success of the amplification method as implemented in this paper. We show that an analogous estimate cannot hold in any other aspect:

Lemma. If \( \pi \) is not a twist of the special representation, then for all unit vectors \( \varphi_1 \in \pi, \varphi_3 \in \sigma \),

\[
\mathcal{P}_{\text{SL}_2(F)}(\varphi_1, \varphi_2, \varphi_3) \ll C(\text{ad}(\pi) \otimes \tau)^{-3/8}. \quad (5.3)
\]

The proof is given below. The precise shape of \( \varphi_2 \) is unimportant; a similar argument shows that if the local field \( F \) (possibly archimedean) and the additive character \( \psi \) are held fixed, then the same conclusion holds for any fixed vector \( \varphi_2 \). The interpretation is that if one attempts to apply our method to other aspects using an essentially fixed vector \( \varphi_2 \) (as may seem necessary, cf. the proof sketch of §1.2), then one cannot reasonably hope even to recover the convexity bound: one must save in the exponent \( 3/8 - 1/4 = 1/8 \), which is infeasible via amplification.
We realize

Proof of the lemma. We realize \( \pi, \rho, \) and \( \sigma \) in their respective models as in \( \S 2.17. \) We assume for convenience (mildly contradicting the assumed conventions of \( \S 2.16. \)) that the Haar measure on \( F^\times \) is given by \( \, d^\times y = \frac{dy}{|y|}. \)

Our assumptions imply that \( \sigma = \mathcal{I}_{\text{MP}_2}(\cdot,|v|) \) for some \( v \in \mathbb{R}/2\pi \log(q)\mathbb{Z}. \) To simplify notation, we assume that \( v = 0. \) The same argument applies for \( v \neq 0. \)

Since we are in the unramified case, we have \( \chi_\sigma(y) = 1 \) for all \( y \in \sigma^\times. \)

Let \( \omega_\pi \) denote the central character of \( \pi. \) The central element \( t(-1) \in \text{MP}_2(F) \) acts by \( \omega_\pi(-1) \) on \( \pi \) and trivially both on \( 1_\sigma \in \rho_\psi \) and on \( \sigma, \) so \( \mathcal{P}_{\text{SL}_2(F)}(\cdot,1_\sigma) \) vanishes identically unless \( \omega_\pi(-1) = 1. \) This forces \( \omega_\pi \) to be the square of some character, say \( \omega_\pi^{1/2}. \) The conclusion of the lemma depends only upon the restriction of \( \pi \) to \( \text{SL}_2(F), \) and that restriction is unchanged by twisting, so upon replacing \( \pi \) with \( \pi \otimes (\omega_\pi^{1/2})^{-1} \) as necessary, we reduce to the case that \( \omega_\pi = 1. \) In particular, \( \pi \) is self-contragredient. Its local \( \gamma \)-factor thus has the form

\[
\gamma(\pi \otimes \chi, s) := \gamma(\pi \otimes \chi, s, \psi) = \varepsilon(\pi \otimes \chi, s, \psi) \frac{L(\pi \otimes \chi^{-1}, 1 - s)}{L(\pi \otimes \chi, s)},
\]

while the Jacquet–Langlands local functional equation reads: for \( W \in \pi = \mathcal{W}(\pi, \overline{\psi}) \) and characters \( \chi \) of \( F^\times, \)

\[
Z(W, \chi, s) := \int_{y \in F^\times} W(a(y))\chi(y)|y|^{s-1/2}d^\times y = \frac{Z(wW, \chi^{-1}, 1 - s)}{\gamma(\pi \otimes \chi, s)},
\]
with the integral defined first for \( \text{Re}(s) \) large enough and extended by meromorphic continuation.

Let \( W \in \pi \) and \( f \in \sigma \) be unit vectors. Then \( \mathcal{P}_{\text{SL}_2(F)}(W, 1_\sigma, f) \sim |\ell(W, 1_\sigma, f)|^2 \). As noted in \( \S 2.17.2 \), one has \( C(\text{ad}(\pi) \otimes \tau) = C(\text{ad}(\pi))^2 \). Our task is thus to show that

\[ \ell(W, 1_\sigma, f) \ll C(\text{ad}(\pi))^{-3/8}. \]

We will make use of the following integral formula and normalization of Haar measures: for \( \Phi \in C_c(N \setminus G) \),

\[ \int_{N \setminus G} \Phi = \int_{x \in \mathfrak{o}} \int_{y \in F^x} \Phi(t(y)\text{wn}(x)) \frac{d^x y}{|y|^{1/2}} \, dx + \int_{x \in \mathfrak{p}} \int_{y \in F^x} \Phi(t(y)n(x)) \frac{d^x y}{|y|^{1/2}} \, dx. \]

For \( g = t(y)\text{wn}(x) \) with \( x \in \mathfrak{o} \), we have \( W(g) = W(a(y^2)\text{wn}(x)) \), \( \rho_\psi(g)1_\sigma(1) = |y|^{1/2}1_\sigma(y) \) and \( f(g) = |y|f(\text{wn}(x)) \). For \( g = t(y)n'(x) \) with \( x \in \mathfrak{o} \), we have \( W(g) = W(a(y^2)n'(x)) \), \( \rho_\psi(g)1_\sigma(1) = |y|^{1/2}1_\sigma(y) \) and \( f(g) = |y|f(n'(x)) \). Thus \( \ell(W, 1_\sigma, f) = \ell_1 + \ell_2 \), where

\[ \ell_1 := \int_{x \in \mathfrak{o}} \int_{y \in F^x} W'(a(y^2)\text{wn}(x))1_\sigma(y)f(\text{wn}(x)) \frac{d^x y}{|y|^{1/2}} \, dx \]

\[ \ell_2 := \int_{x \in \mathfrak{p}} \int_{y \in F^x} W'(a(y^2)n'(x))1_\sigma(y)f(n'(x)) \frac{d^x y}{|y|^{1/2}} \, dx. \]

We henceforth estimate \( \ell_1 \); analogous arguments apply to \( \ell_2 \). Set

\[ \Psi(x) := \int_{y \in F^x} W'(a(y^2)\text{wn}(x))1_\sigma(y) \frac{d^x y}{|y|^{1/2}}, \]

so that \( \ell_1 = \int_{x \in F} \Psi(x)f(\text{wn}(x)) \, dx \). A change of variables gives

\[ 1 = \|f\|^2 = \int_{x \in F} |f(n'(x))|^2 \, dx = \int_{x \in \mathfrak{o}} |f(\text{wn}(x))|^2 \, dx + \int_{x \in \mathfrak{p}} |f(n'(x))|^2 \, dx. \]

By Cauchy–Schwarz, positivity and Parseval, it follows that

\[ |\ell_1|^2 \leq \int_{x \in \mathfrak{o}} |\Psi(x)|^2 \, dx \leq \int_{x \in F} |\Psi(x)|^2 \, dx = \int_{\xi \in F} |\Psi^\wedge(\xi)|^2 \, d\xi, \]

where \( \Psi^\wedge(\xi) := \int_{x \in F} \Psi(x)\psi(-\xi x) \, dx \). In the remainder of the proof, we will show that

\[ \Psi^\wedge(\xi) = \Phi(\xi)W(a(\xi))|\xi|^{-1/2}. \]  \hspace{1cm} \text{(5.4)}

for some function \( \Phi : F^\times \to \mathbb{C} \) satisfying the pointwise estimate

\[ \Phi(\xi) \ll C(\text{ad}(\pi))^{-3/8} \text{ for all } \xi \in F^\times. \]  \hspace{1cm} \text{(5.5)}

Since \( 1 = \|W\|^2 = \int_{\xi \in F^\times} |W(a(\xi))|^2 \, d\xi^{2F} \), the required estimate for \( \ell_1 \) then follows.

We turn to (5.4). By detecting squares via quadratic characters \( \eta \) of \( F^\times \), we see that there exists \( c > 0 \) with \( c \asymp 1 \) so that

\[ \Psi(x) = c \sum_{\eta^2=1} \int_{y \in F^x} W'(a(y)\text{wn}(x))1_\sigma(y)\eta(y)|y|^{-1/4} \, d^x y. \]

Let \( \zeta_F(s) = (1 - q^{-s})^{-1} \) denote the local zeta factor. Choose \( \varepsilon > 0 \) sufficiently small. By Cauchy’s formula, we have \( 1_\sigma(y) = \int_{(c)} \zeta_F(s)|y|^{-s} \), where \( \int_{(c)} \) denotes the integral over \( s = \varepsilon + it \) with \( t \) sampled from the compact group \( \mathbb{R}/2\pi \log(q)\mathbb{Z} \) with respect to the probability Haar. Using the estimate \( W'(a(y)\text{wn}(x)) \ll |y|^{1/2 - \delta} \)
to justify exchanging integrals and applying the local functional equation, it follows that

\[ \Psi(x) = c \sum_{\eta^2 = 1} \int_{(c)} \zeta_F(s) \int_{y \in F^{\times}} W(a(y)w(n)(x))\eta(y)|y|^{-s-1/4} \, dy \]

\[ = c \sum_{\eta^2 = 1} \int_{(c)} \zeta_F(s)\gamma(\pi \otimes \eta, 3/4 + s)Z(n(x)W, \eta, 3/4 + s). \]

We have \( n(x)W(a(y)) = \psi(yx)W(a(y)) \), thus

\[ Z(n(x)W, \eta, 3/4 + s) = \int_{y \in F^{\times}} \psi(yx)W(a(y))\eta(y)|y|^{3/4+1/2-s} \, dy \]

so by Fourier inversion,

\[ \int_{x \in F} \psi(-\xi)Z(n(x)W, \eta, 3/4 + s) \, dx = \eta(\xi)|\xi|^{-1/4+s}W(a(\xi))|\xi|^{-1/2}. \]

Thus (5.4) holds with \( \Phi_\eta(\xi) = c \sum_{\eta^2 = 1} \Phi_\eta(\xi) \), where

\[ \Phi_\eta(\xi) := \int_{(c)} \zeta_F(s)\gamma(\pi \otimes \eta, 3/4 + s)\eta(\xi)|\xi|^{-1/4+s}. \]

We now verify the estimate (5.5) for \( \Phi(\xi) \) by bounding each \( \Phi_\eta(\xi) \) individually. By the classification of \( \vartheta \)-tempered generic irreducible unitary representations of \( \text{PGL}_2(F) \) and our assumption that \( \pi \) is not a twist of the special representation, there are the following possibilities for \( \pi \):

(i) \( \pi \) is the normalized induction \( \nu \oplus \nu^{-1} \) for some character \( \nu \) of \( F^{\times} \) for which either
   - \( \nu \) is unitary, or
   - \( \nu = \nu_0|.|^c \) with \( \nu_0 \) quadratic and \( 0 < c \leq \vartheta. \)

We have \( C(\pi \otimes \eta) = C(\nu \otimes \eta)^2 \), \( C(\text{ad}(\pi)) = C(\nu)^2 \) and \( L(\pi \otimes \eta, s) = L(\nu|\eta, s)L(\nu^{-1}|\eta, s) \). Our assumption (5.1) says that \( \nu^2 \) is ramified, hence that \( \nu \) is unitary, that \( \nu|\eta, \nu^{-1}|\eta \) are ramified, and that \( C(\nu) = C(\nu|\eta) = C(\nu^{-1}|\eta) \). Since \( q \) is odd, we have also \( C(\nu^2) = C(\nu) \). Thus

\[ C(\pi \otimes \eta) = C(\text{ad}(\pi)), \quad L(\pi \otimes \eta, s) = 1. \] (5.6)

(ii) \( \pi \) is supercuspidal. Then \( L(\pi \otimes \eta, s) = 1 \) and \( C(\pi) = C(\pi \otimes \eta) \geq q^2 \) (see [52, Prop 3.4]). If \( C(\pi) = q^{2n} \), then \( C(\text{ad}(\pi)) \leq q^{2n}, \) while if \( C(\pi) = q^{2n+1} \), then \( C(\text{ad}(\pi)) = q^{2n+2} \) (see e.g. [39, Prop 2.5]). In particular,

\[ C(\pi \otimes \eta) \geq C(\text{ad}(\pi))^{3/4}, \quad L(\pi \otimes \eta, s) = 1. \] (5.7)

(Equality holds if and only if \( C(\pi) = q^3 \).)
In both cases it follows that $\gamma(\pi \otimes \eta, s) = \varepsilon_{\pi \otimes \eta}C(\pi \otimes \eta)^{1/2-s}$ with $|\varepsilon_{\pi \otimes \eta}| = 1$, hence that

$$\Phi_\eta(\xi) \asymp \int_{(c)} \zeta_F(s)C(\pi \otimes \eta)^{-1/4-s}|\xi|^{-1/4+s}$$

$$= \begin{cases} C(\pi \otimes \eta)^{-1/4}|\xi|^{-1/4} & \text{if } |\xi| \geq C(\pi \otimes \eta), \\ 0 & \text{otherwise.} \end{cases}$$

$$\ll C(\pi \otimes \eta)^{-1/2}$$

$$\ll C(\mathrm{ad}(\pi))^{-3/8},$$

as required. \(\square\)

References

[1] Joseph Bernstein and Andre Reznikov. Subconvexity bounds for triple $L$-functions and representation theory. *Ann. of Math. (2)*, 172(3):1679–1718, 2010.

[2] Valentin Blomer and Farrell Brumley. On the Ramanujan conjecture over number fields. *Ann. of Math. (2)*, 174(1):581–605, 2011.

[3] Valentin Blomer and Gergely Harcos. Twisted $L$-functions over number fields and Hilbert’s eleventh problem. *Geom. Funct. Anal.*, 20(1):1–52, 2010.

[4] V. A. Bykovskiı. A trace formula for the scalar product of Hecke series and its applications. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 226(Anal. Teor. Chisel i Teor. Funkciı́. 13):14–36, 235–236, 1996.

[5] M. Cowling, U. Haagerup, and R. Howe. Almost $L^2$ matrix coefficients. *J. Reine Angew. Math.*, 387:97–110, 1988.

[6] W. Duke, J. B. Friedlander, and H. Iwaniec. The subconvexity problem for Artin $L$-functions. *Invent. Math.*, 149(3):489–577, 2002.

[7] Manfred Einsiedler, Elon Lindenstrauss, Philippe Michel, and Akshay Venkatesh. Distribution of periodic torus orbits and Duke’s theorem for cubic fields. *Ann. of Math. (2)*, 173(2):815–885, 2011.

[8] Étienne Fouvry, Emmanuel Kowalski, and Philippe Michel. Algebraic twists of modular forms and Hecke orbits. *Geom. Funct. Anal.*, 25(2):580–657, 2015.

[9] Wee Teck Gan. The shimura correspondence a la waldspurger. [http://www.math.nus.edu.sg/~matgwt/postech.pdf](http://www.math.nus.edu.sg/~matgwt/postech.pdf), 2011.

[10] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of $GL(2)$ and $GL(3)$. *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.

[11] P. Gérardin and J.-P. Labesse. The solution of a base change problem for $GL(2)$ (following Langlands, Saito, Shintani). In *Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 2, Proc. Sympos. Pure Math., XXXIII, pages 115–133. Amer. Math. Soc., Providence, R.I., 1979.

[12] Amit Ghosh, Andre Reznikov, and Peter Sarnak. Nodal domains of Maass forms I. *Geom. Funct. Anal.*, 23(5):1515–1568, 2013.

[13] David Goldberg and Dani Szpruch. Plancherel measures for coverings of $p$-adic $SL_2(F)$. *Int. J. Number Theory*, 12(7):1907–1936, 2016.

[14] Gergely Harcos and Philippe Michel. The subconvexity problem for Rankin-Selberg $L$-functions and equidistribution of Heegner points. II. *Invent. Math.*, 163(3):581–655, 2006.

[15] Jeffrey Hoffstein and Dinaar Ramakrishnan. Siegel zeros and cusp forms. *Internat. Math. Res. Notices*, (6):279–308, 1995.

[16] Roman Holowinsky. Sieving for mass equidistribution. *Ann. of Math. (2)*, 172(2):1499–1516, 2010.

[17] Roman Holowinsky, Ritabrata Munshi, and Zhi Qi. Hybrid subconvexity bounds for $L(\frac{1}{2}, \text{Sym}^2 f \otimes g)$. *Math. Z.*, 283(1-2):555–579, 2016.

[18] Roman Holowinsky and Kannan Soundararajan. Mass equidistribution for Hecke eigenforms. *Ann. of Math. (2)*, 172(2):1517–1528, 2010.

[19] Y. Hu. Triple product formula and mass equidistribution on modular curves of level $N$. *ArXiv e-prints*, September 2014.
[20] Atsushi Ichino. Trilinear forms and the central values of triple product $L$-functions. *Duke Math. J.*, 145(2):281–307, 2008.

[21] H. Iwaniec and P. Michel. The second moment of the symmetric square $L$-functions. *Ann. Acad. Sci. Fenn. Math.*, 26(2):465–482, 2001.

[22] Henryk Iwaniec and Peter Sarnak. Perspectives on the analytic theory of $L$-functions. *Geom. Funct. Anal.*, (Special Volume, Part II):705–741, 2000. GAFA 2000 (Tel Aviv, 1999).

[23] Junehyuk Jung. Quantitative quantum ergodicity and the nodal domains of Hecke-Maass cusp forms. *Comm. Math. Phys.*, 348(2):603–653, 2016.

[24] Bruno Kahn. Le groupe des classes modulo 2, d’après Conner et Perlis. In *Seminar on number theory, 1984–1985 (Talence, 1984/1985)*, pages Exp. No. 26, 29. Univ. Bordeaux I, Talence, 1985.

[25] Henry H. Kim. Functoriality for the exterior square of $GL_4$ and the symmetric fourth of $GL_2$. *J. Amer. Math. Soc.*, 16(1):139–183 (electronic), 2003. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[26] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.

[27] Wenzhi Luo and Peter Sarnak. Mass equidistribution for Hecke eigenforms. *Comm. Pure Appl. Math.*, 56(7):874–891, 2003. Dedicated to the memory of Jürgen K. Moser.

[28] P. Michel. The subconvexity problem for Rankin-Selberg $L$-functions and equidistribution of Heegner points. *Ann. of Math. (2)*, 160(1):185–236, 2004.

[29] Philippe Michel. Analytic number theory and families of automorphic $L$-functions. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 181–295. Amer. Math. Soc., Providence, RI, 2007.

[30] Philippe Michel and Akshay Venkatesh. Equidistribution, $L$-functions and ergodic theory: on some problems of Yu. Linnik. In *International Congress of Mathematicians. Vol. II*, pages 421–457. Eur. Math. Soc., Zürich, 2006.

[31] Philippe Michel and Akshay Venkatesh. The subconvexity problem for $GL_2$. *Publ. Math. Inst. Hautes Études Sci.*, (111):171–271, 2010.

[32] R. Munshi. Subconvexity for symmetric square $L$-functions. *ArXiv e-prints*, September 2017.

[33] Ritabrata Munshi. The circle method and bounds for $L$-functions—IV: subconvexity for twists of $GL(3)$ $L$-functions. *Ann. of Math. (2)*, 182(2):617–672, 2015.

[34] Paul D. Nelson. Equidistribution of cusp forms in the level aspect. *Duke Math. J.*, 160(3):467–501, 2011.

[35] Paul D. Nelson. Evaluating modular forms on Shimura curves. *Math. Comp.*, 84(295):2471–2503, 2015.

[36] Paul D. Nelson. Quantum variance on quaternion algebras, I. preprint, 2016.

[37] Paul D. Nelson. The spectral decomposition of $|θ|^2$. preprint, 2016.

[38] Paul D. Nelson. Quantum variance on quaternion algebras, II. preprint, 2017.

[39] Paul D. Nelson, Ameya Pitale, and Abhishek Saha. Bounds for Rankin-Selberg integrals and quantum unique ergodicity for powerful levels. *J. Amer. Math. Soc.*, 27(1):147–191, 2014.

[40] Y. Qiu. The Whittaker period formula on Metaplectic $SL(2)$. *ArXiv e-prints*, August 2013.

[41] Yannan Qiu. Generalized formal degree. *Int. Math. Res. Not. IMRN*, (2):239–298, 2012.

[42] Yannan Qiu. Periods of Saito-Kurokawa representations. *Int. Math. Res. Not. IMRN*, (24):6698–6755, 2014.

[43] Brooks Roberts and Ralf Schmidt. On the number of local newforms in a metaplectic representation. In *Arithmetic geometry and automorphic forms*, volume 19 of *Adv. Lect. Math. (ALM)*, pages 505–530. Int. Press, Somerville, MA, 2011.

[44] Peter Sarnak. Arithmetic quantum chaos. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Conf. Proc.*, pages 183–236. Bar-Ilan Univ., Ramat Gan, 1995.

[45] Peter Sarnak. Estimates for Rankin-Selberg $L$-functions and quantum unique ergodicity. *J. Funct. Anal.*, 184(2):419–453, 2001.

[46] Peter Sarnak. Recent progress on the quantum ergodicity conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 48(2):211–228, 2011.

[47] Ralf Schmidt. Some remarks on local newforms for $GL(2)$. *J. Ramanujan Math. Soc.*, 17(2):115–147, 2002.

[48] Kannan Soundararajan. Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash \mathbb{H}$. *Ann. of Math. (2)*, 172(2):1529–1538, 2010.
[49] Dani Szpruch. Computation of the local coefficients for principal series representations of the metaplectic double cover of $\text{SL}_2(\mathbb{F})$. *J. Number Theory*, 129(9):2180–2213, 2009.

[50] Dani Szpruch. On the existence of a $p$-adic metaplectic Tate-type $\gamma$-factor. *Ramanujan J.*, 26(1):45–53, 2011.

[51] Dani Szpruch. A short proof for the relation between weil indices and $\gamma$-factors. *Communications in Algebra*, 0(0):1–6, 2017.

[52] Jerrold B. Tunnell. On the local Langlands conjecture for $GL(2)$. *Invent. Math.*, 46(2):179–200, 1978.

[53] Akshay Venkatesh. Sparse equidistribution problems, period bounds and subconvexity. *Ann. of Math. (2)*, 172(2):989–1094, 2010.

[54] Jean-Loup Waldspurger. Correspondances de Shimura et quaternions. *Forum Math.*, 3(3):219–307, 1991.

ETH Zürich, Department of Mathematics, Rämistrasse 101, CH-8092, Zürich, Switzerland

*E-mail address: paul.nelson@math.ethz.ch*