A GENERALIZATION OF RAMANUJAN’S CONGRUENCE TO MODULAR FORMS OF PRIME LEVEL

RADU GABA AND ALEXANDRU A. POPA

ABSTRACT. We prove congruences between cuspidal newforms and Eisenstein series of prime level, which generalize Ramanujan’s congruence. Such congruences were recently found by Billerey and Menares, and we refine them by specifying the Atkin-Lehner eigenvalue of the newform involved. We show that similar refinements hold for the level raising congruences between cuspidal newforms of different levels, due to Ribet and Diamond. The proof relies on studying the new subspace and the Eisenstein subspace of the space of period polynomials for the congruence subgroup \( \Gamma_0(N) \), and on a version of Ihara’s lemma.

1. Introduction

Let \( E_k \) be the Eisenstein series of even weight \( k \geq 4 \) for the full modular group, normalized so that its Fourier expansion is

\[
E_k(z) = \frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n)q^n,
\]

where \( B_k \) is the Bernoulli number, \( \sigma_a(n) = \sum_{d|n} d^a \), and \( q = e^{2\pi i z} \). Let \( I \) be a prime ideal dividing the numerator of \( \frac{B_k}{2k} \), in the number field generated by the eigenvalues of Hecke eigenforms of weight \( k \). Then there exists such a cuspidal Hecke eigenform \( f \) such that

\[
(1.1) \quad f \equiv E_k \pmod{I};
\]

for \( k = 12 \) this is the well-known Ramanujan congruence modulo 691, while for higher weights it was proved in \([15, 8]\). This and later congruences mean that the difference between the Fourier coefficients at the cusp \( \infty \) of the two sides belong to \( I \) (after clearing the denominator of the constant term), and we always normalize Hecke eigenforms to have the coefficient of \( q \) equal to 1.

This congruence was recently generalized to newforms \( f \) of prime level by Billerey-Menares \([3]\) and by Dummigan-Fretwell \([13]\), for Fourier coefficients of index coprime to the level. In this paper we refine these results, by determining also the Atkin-Lehner eigenvalue of the newform involved, thus obtaining congruences for all coefficients. A similar congruence between cuspidal newforms of different levels is the “level raising theorem” of Ribet \([29]\) and Diamond \([10]\), and we show that it admits a similar refinement. Before stating the results, we introduce the common setting and explain the heuristic behind them.

Let \( S_k(N) \), \( M_k(N) \) be the space of cusp forms, respectively modular forms of even weight \( k \geq 2 \) and trivial Nebentypus for the congruence subgroup \( \Gamma_0(N) \). We let \( N = Mp \) with \( p \nmid M \) a prime, and consider a modular form \( g \in M_k(M) \). Fixing \( \varepsilon \in \{ \pm 1 \} \), we define

\[
(1.2) \quad g_p^{(\varepsilon)} := g(1 + \varepsilon W_p), \quad \text{namely } g_p^{(\varepsilon)}(z) := g(z) + \varepsilon p^{k/2} g(pz).
\]

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which is a form of level $N$ with eigenvalue $\varepsilon$ under the Atkin-Lehner involution $W_p$. Since $g$ has level $M$, it follows that $\text{Tr}_M^N(g|W_p) = p^{-k/2+1}g|T_p$, where $\text{Tr}_M^N : M_k(N) \to M_k(M)$ is the trace map (the definition is recalled in Section 2.3). Assuming further that $g$ is an eigenform of the Hecke operator $T_p$ with eigenvalue $\lambda_p$ we have

\begin{equation}
\text{Tr}_M^N g_p^{(\varepsilon)} = (1 + p + \varepsilon p^{-k/2+1+\lambda_p}) \cdot g,
\end{equation}

using that $[\Gamma_0(M) : \Gamma_0(N)] = 1 + p$. The cusforms $f \in S_k(N)$ which are new at $p$ can be characterized by the condition $\text{Tr}_M^N f = \text{Tr}_M^N f|W_N = 0$, as recalled in Section 2.3, where we also recall the definitions. We conclude from (1.3) that if $g$ is a newform of level $M$, then $g_p^{(\varepsilon)}$ is new at $p$ when reduced modulo prime ideals dividing the term in parentheses, and heuristically we expect that it is congruent modulo such ideals to a Hecke eigenform in $S_k(N)$, which is new at $p$ and has eigenvalue $\varepsilon$ under $W_p$. The next two theorems confirm this heuristic.

When $M = 1$ and $g = E_k$, denote by $E_k^{(\varepsilon)}$ the form $g_p^{(\varepsilon)}$ in (1.2). The factor in (1.3) splits:

\[ 1 + p + \varepsilon p^{-k/2+1} \sigma_{k-1}(p) = (p^{-k/2+1} + \varepsilon)(p^{k/2} + \varepsilon), \]

and since $W_p$ interchanges the cusps $\infty$ and $0$ of $\Gamma_0(p)$, the constant terms of $E_k^{(\varepsilon)}$ at both cusps are $\frac{B_{k-1}}{2k}(p^{k/2} + \varepsilon)$ (up to a sign and powers of $p$). Therefore candidates for congruence primes between $E_k^{(\varepsilon)}$ and newforms are prime ideals which divide its constant terms and the product above.

We denote by $S_k^{(\varepsilon)}(N)$ the subspace of $S_k(N)$ consisting of eigenforms for $W_p$ with eigenvalue $\varepsilon$. We write $a|q$, $a \nmid q$ if the integer $a$ divides, respectively does not divide, the numerator of the rational number $q$.

**Theorem 1.** Let $k \geq 4$ be even, $p$ a prime, and $\varepsilon \in \{\pm 1\}$. Let $\ell$ be a prime with $\ell > k + 1$, and assume that the following conditions are satisfied:

\begin{equation}
\ell \mid (p^{k/2} + \varepsilon)(p^{k/2-1} + \varepsilon) \quad \text{and} \quad \ell \mid \frac{B_k}{k}(p^{k/2} + \varepsilon).
\end{equation}

If $\ell \nmid (p^{k/2} + \varepsilon)$, assume also that $\ell \mid B_n B_{k-n}(p^{n-1}-1)$ for some even $n$, $0 < n < k$. Then there exists a prime ideal $\mathcal{I}$ of residue characteristic $\ell$, in the ring of integers generated by Hecke eigenvalues of newforms in $S_k^{(\varepsilon)}(p)$, and a newform $f \in S_k^{(\varepsilon)}(p)$ such that

\begin{equation}
f \equiv E_k^{(\varepsilon)} \pmod{\mathcal{I}}.
\end{equation}

We prove the theorem in Section 3, together with the next theorem. The theorem refines [3, Thm. 1], where it is shown that a congruence as in (1.5) holds for coefficients coprime to $p$ if and only if $\ell||(p^k - 1)(p^{k-2} - 1)$ and $\ell$ divides the numerator of $\frac{B_k}{k}(p^k - 1)$. The additional condition we impose if $\ell \nmid (p^{k/2} + \varepsilon)$ is an artifact of our method, but it is automatically satisfied for all $k \leq 6 \cdot 10^4$ (see Remark 3.1). We also checked numerically that the theorem holds for $\ell = k \pm 1$ in numerous cases (see Example 5.1), but our method does not apply for these values of $\ell$.

When the form $g$ is a cuspidal newform, we obtain instead the following refinement of Diamond’s level raising theorem [10, Thm. 1]. We give this theorem as an illustration of our method, as it requires little extra work. The statement is not the sharpest possible, since assumption (1.6) below could probably be removed.
Theorem 2. Let $k \geq 4$ be even, let $N = Mp$ with $p$ prime, $p \nmid M$, and $\varepsilon \in \{\pm 1\}$. Let $g \in S_k(M)$ be a newform with eigenvalue $\lambda_g$ under $T_p$. Assume there is a prime $\ell > k + 1$, $\ell \nmid N$, and a prime ideal $\mathcal{I}$ above it in the field generated by the eigenvalues of all Hecke eigenforms in $S_k^{(\varepsilon)}(N)$ such that

$$\lambda_p \equiv -\varepsilon p^{k/2-1}(p+1) \pmod{\mathcal{I}}.$$ 

Assume also that either

$$\ell \nmid (p^{k/2-1} + \varepsilon) \cdot \text{Den } P^+(g), \quad \text{or } k \geq 6 \text{ and } \ell \nmid (p^{k/2-2} + \varepsilon) \cdot \text{Den } P^-(g).$$

Then there exists a Hecke eigenform $f \in S_k^{(\varepsilon)}(N)$ which is new at $p$ such that

$$f \equiv g_p^{(\varepsilon)} \pmod{\mathcal{I}}.$$ 

In condition (1.6), $P^+(g)$, respectively $P^-(g)$ is the even, respectively the odd period polynomial of $g$, normalized so that its principal part is 1 modulo $X^2K_g[X]$, respectively $X$ modulo $X^3K_g[X]$, with $K_g$ the field generated by the Hecke eigenvalues of $g$. We denote by $\text{Den } P^\pm(g)$ the least common multiple of the norms of the denominators of the coefficients of $P^\pm(g)$ (which belong to $K_g$). These denominators tend to have few factors of residue characteristic $\ell > k$ (none if $f$ is of full level and weight $k$ such that $S_k(1)$ is one dimensional, cf. the tables in [24, 12]), so this condition is typically verified for a given $g$ (see Example 5.3). The theorem should hold for weight $k = 2$ as well, without assumption (1.6) but assuming that $\ell \nmid \varphi(M)$, which would refine the original result of Ribet [29].

The proof of both theorems relies on the theory of period polynomials for congruence subgroups developed by Pašol and the second author in [27], and the results we obtain along the way are of independent interest. In Section 2 we define the new subspace of the space $W_w(N)$ of period polynomials of degree $w$ for $\Gamma_0(N)$, where $N \geq 1$ is arbitrary. By studying the action of the Atkin-Lehner involution on period polynomials, we determine explicitly a basis for the “Eisenstein subspace” of $W_w(N)$ consisting of Atkin-Lehner eigenvectors, when $N$ is square-free. We also need the larger space $\hat{W}_w(N)$ of extended period polynomials introduced in [27], and we use it to prove an Eichler-Shimura isomorphism between the new subspaces of $\hat{W}_{k-2}(N)$ and of $M_k(N)$ (see Propositions 2.7 and 2.9). Compared to the better known theory of modular symbols, there are two new features: the new subspace is defined using a trace map from higher to lower levels, as in Serre’s characterization of newforms; and to associate even period polynomials in $W_{k-2}(N)$ to Eisenstein series when $N$ is square-free, we require the larger space $\hat{W}_{k-2}(N)$.

The upshot of the theory in Section 2 is that the Eisenstein series in Theorem 1 has an extended period polynomial $\hat{\rho}(E_k^{(\varepsilon)}(p))$ whose even part belongs to the new subspace $W_w(p)^{\text{new}}$, when reduced modulo primes $\ell | p^{k/2} + \varepsilon$, and similarly for the odd part modulo primes $\ell$ satisfying (1.4) and $\ell \nmid p^{k/2} + \varepsilon$. The even part is clearly nonzero modulo $\ell$, but for the nonvanishing of the odd part we require the extra condition in Theorem 1. Similarly, the polynomials $P^\pm(g_p^{(\varepsilon)})$ in Theorem 2 are new mod $\mathcal{I}$ because of the congruence satisfied by $\lambda_p$, and at least one is nonzero mod $\mathcal{I}$ by assumption (1.6). Since $\hat{\rho}^\pm(E_k^{(\varepsilon)}(p))$ and $P^\pm(g_p^{(\varepsilon)})$ are Hecke and Atkin-Lehner eigenvectors, the previous theorems follow from the Deligne-Serre lifting lemma [9] (see Section 3 for the details), once we establish the surjectivity of a reduction map on newspaces.

This is the other main result, and to state it, we let $R$ be a discrete valuation ring with residue field $F$ of characteristic $\ell$. Let $W_w(N)^{p-\text{new}}_R$ be the space of polynomials new at $p|N$, defined over $R$ (see (2.9) for the definition).
Theorem 3. Let \( w \geq 0 \) be even, and let \( N = pM \) with \( p \) prime, \( p \nmid M \). Assume that \( \ell > w + 3, \ell \nmid N \), and if \( w = 0 \) assume also that \( \ell \nmid \varphi(M) \). Then the reduction map
\[
W_w(N)^{p \text{-new}} \rightarrow W_w(N)^{p \text{-new}}
\]
is surjective (when \( \ell = w + 3 > 3 \), its image has codimension 1).

For the proof, given in Section 4, we use the isomorphism between \( W_w(N)/R \) and the compactly supported cohomology \( H^1_\text{c}(\Gamma_0(N), V_w(R)) \), with \( V_w(R) \) the module of polynomials of degree at most \( w \) with coefficients in \( R \). Using Poincaré duality, we show in Proposition 4.3 that the surjectivity reduces to a version of Ihara’s lemma [19, Lemma 3.2]. Ihara’s lemma, or rather the ingredients in its proof, was first used to prove the level raising congruence mentioned above by Ribet (for weight two) and Diamond (for higher weights). We follow the argument in [10], modified to take into account that we work with the whole cohomology rather than just the parabolic part. Our work is close in spirit to Harder’s program of proving congruences by studying Eisenstein cohomology classes [16].

The surjectivity of the map in Theorem 3, or lack thereof for \( \ell = w + 3 \), can be traced to the vanishing of the finite cohomology group \( H^1(\text{SL}_2(\mathbb{F}_\ell), V_w(F)) \) for \( \ell > w \), and its nonvanishing for \( \ell = w + 3 \). This was shown in [21], but for completeness we give the proof in Proposition 4.7.

We end the introduction with two remarks and a conjecture related to Theorem 1.

Theorem 1 is true in weight two as well, when it refines a congruence due to Mazur [26, Prop. 5.12]. In this case, the Eisenstein subspace of \( M_2(p) \) is one dimensional spanned by an Eisenstein series having Atkin-Lehner eigenvalue \(-1\), and Theorem 1 predicts that the newform in Mazur’s congruence can be taken to have the same Atkin-Lehner eigenvalue. However there are technical difficulties in applying Theorem 3 for \( w = 0 \) in this case; the refined congruence is anyway proved by Yoo [34, Thm. 1.3 (ii)], who studies the case of weight two and square-free level in great detail, using different methods.

Assuming a conjecture of Maeda, the Hecke eigenforms of full level form a single Galois orbit, so all of them satisfy (1.1) modulo conjugate ideals. Similarly, a generalization of Maeda’s conjecture due to Tsaknias [33] states that the newforms in \( S_k(p) \) form two Galois orbits for sufficiently large \( k \), the forms in each orbit sharing the same Atkin-Lehner eigenvalue. This would imply that all newforms in \( S_k(p) \) satisfy congruence (1.5), when primes \( \ell \) as in Theorem 1 exist.

A conjecture generalizing Ramanujan’s congruence to newforms of square-free levels was proposed by Billerey and Ménares in [3], and we end by stating a conjectural refinement that includes a generalization of Theorem 2 as well. Let \( N = MN' \) be square-free, \( k \geq 4 \), and let \( g \in M_k(M) \) be a newform of level \( M \). This includes the case of Eisenstein series, when we necessarily have \( M = 1 \) and \( g = E_k \). Let \( D(N') \) denote the set of positive divisors of \( N' \), and let \( \varepsilon : D(N') \rightarrow \{ \pm 1 \} \) be a multiplicative function, which we view as a system of Atkin-Lehner eigenvalues for modular forms in \( M_k(N) \).

Define
\[
g_N^{(\varepsilon)} = g \prod_{p \mid N'} (1 + \varepsilon(p)W_p), \text{ namely } g_N^{(\varepsilon)}(z) = \sum_{d \mid N'} \varepsilon(d)d^{k/2}g(dz),
\]
which is a eigenform for \( W_p \) for prime \( p \mid N' \) with eigenvalue \( \varepsilon(p) \), and it is new at primes dividing \( M \). Note that when \( M = 1 \) and \( g = E_k \), the constant term of \( g_N^{(\varepsilon)} \) at all cusps is (up to signs and powers of \( p \mid N \)):
\[
\frac{B_k}{2k} \prod_{p \mid N} (p^{k/2} + \varepsilon(p)),
\]
where
since the group of Atkin-Lehner involutions acts transitively on the cusps for square-free $N$. Let $S_k^{(c)}(N)$ be the subspace of $S_k(N)$ consisting of eigenforms for $W_p$ with eigenvalue $\varepsilon(p)$, for all $p|N'$.

**Conjecture.** Assume $N = MN'$ square-free and $k \geq 4$, and let $g \in M_k(M)$ be a newform of level $M$, with Hecke eigenvalues $\lambda_p$ for $p|N'$. Let $\varepsilon : D(N') \to \{-1, 1\}$ be a system of Atkin-Lehner eigenvalues as above, and assume that for all $p|N'$ we have

$$\lambda_p \equiv -\varepsilon(p)p^{k/2-1}(p+1) \pmod{\mathcal{I}},$$

for $\mathcal{I}$ a prime ideal of residue characteristic $\ell$ in the field generated by the Hecke eigenvalues of all newforms in $S_k^{(c)}(N)$, such that $\ell > k - 2$, $\ell \nmid 6N$. If $M = 1$ and $g = E_k$ also assume that $\ell$ divides the numerator of (1.7). Then there exists a newform $f \in S_k^{(c)}(N)$ such that

$$f \equiv g_N^{(c)} \pmod{\mathcal{I}}.$$

By (1.3), the conditions in the conjecture guarantee that $g_N^{(c)}$ is cuspidal and “new” when reduced modulo the ideal $\mathcal{I}$. The reductions mod $\mathcal{I}$ make sense in the space $S_k(\Gamma_0(N), \overline{\mathbb{Z}}_\ell)$ of arithmetic modular forms à la Katz [14]. Provided one had a definition of the “new subspace” $S_k(\Gamma_0(N), \mathbb{Z}_\ell)^{\text{new}}$ involving trace maps, as for modular forms over $\mathbb{C}$, then the conjecture would follow immediately from the surjectivity of the reduction map

$$S_k(\Gamma_0(N), \mathbb{Z}_\ell)^{\text{new}} \to S_k(\Gamma_0(N), \mathbb{F}_\ell)^{\text{new}}.$$

Part of the conjecture would also follow from a generalization of Theorem 3, stating that the reduction map $W_\ell(N)^{\text{new}}_R \to W_\ell(N)^{\text{new}}_F$ is surjective for $N$ square-free.

For cuspidal $g$, the existence of a newform $f$ as in the conjecture—with possibly non-trivial Nebentypus and minus the determination of its Atkin-Lehner eigenvalues at primes $p|N'$—follows from a theorem of Diamond and Taylor on non-optimal levels for modular Galois representations [11]. That theorem was proved for arbitrary $M$ such that $(M, N') = 1$ and $N'$ square-free, and we similarly expect the conjecture to hold in the cuspidal case under these assumptions. When $g$ is an Eisenstein newform of level $M > 1$, a similar conjecture can be made, under extra assumptions due to the fact that the group of Atkin-Lehner involutions no longer acts transitively on the cusps of $\Gamma_0(M)$.

For $k = 2$ and $g$ an Eisenstein series, similar statements have been proved by Yoo [34] (who also determines the Atkin-Lehner eigenvalues of $f$), and Martin [25].

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### 2. Atkin-Lehner Operators and Newform Theory for Period Polynomials

In §2.1, we briefly review the definition of period polynomials for $\Gamma_0(N)$ and the action of Hecke operators on them. We study the action of Atkin-Lehner operators in more detail in §2.2, and in §2.3 we define primitive (new) subspaces using trace maps. In §2.4 we introduce extended period polynomials, and in §2.5 we use them to determine an explicit basis consisting of Atkin-Lehner eigenvectors of the “Eisenstein subspace” for square-free $N$. 

2.1. Period polynomials and Hecke operators. We start by recalling from [27] basic facts about period polynomials for the congruence subgroup $\Gamma = \Gamma_0(N)$ of $\Gamma_1 = \text{SL}_2(\mathbb{Z})$. Let $S = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$, $U = \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right)$ be generators of $\Gamma_1$. We use the same notation for their images in $\text{PSL}_2(\mathbb{Z})$, which have orders 2, 3, respectively.

We fix a commutative base ring $R$ of characteristic different from 2 and 3. Let $V_w(R)$ be the space of polynomials of degree at most $w$ with coefficients in $R$, on which $\text{GL}_2(\mathbb{Z})$ acts on the right by

$$P|_w^{-\gamma}(X) = P(\gamma X)(cX + d)^w,$$

for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}_2(\mathbb{Z})$.

Let $V_w(N)_{/R}$ be the space of $[\Gamma \backslash \Gamma_1]$-tuples of polynomials,\footnote{Note that we have two notations for $V_w(R) = V_w(1)_{/R}$, but for brevity we use the shorter notation.} identified with maps $P : \Gamma \backslash \Gamma_1 \rightarrow V_w(R)$, on which $\Gamma_1$ acts by $P|_{\gamma}(A) = P(A\gamma^{-1})|_{-w}\gamma$ for a coset $A \in \Gamma \backslash \Gamma_1$, $\gamma \in \Gamma_1$. Since $-1 \in \Gamma$ we assume $w \geq 0$ is even, so $-1$ acts trivially on $V_w(N)_{/R}$. The space of period polynomials is defined by:

$$W_w(N)_{/R} = \{ P \in V_w(N)_{/R} : P|(1 + S) = P|(1 + U + U^2) = 0 \}.$$

The element $\delta = \left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \in \text{GL}_2(\mathbb{Z})$ belongs to the normalizer of $\Gamma$, so $W_w(N)_{/R}$ is preserved by the involution $P \mapsto P|_{\delta}$, where $P|_{\delta}(A) = P(\delta A \delta)|_{-w}\delta$, and so it decomposes into eigenspaces $W^\pm_w(N)_{/R}$ for $\delta$ with eigenvalue $\pm 1$. We call even the polynomials in $W^+_w(N)$, and odd those in $W^-_w(N)$. This is motivated by the fact that for $P \in W^+_w(N)$ the principal part $P(I)$ is even, with $I$ the coset of the identity, but not all components $P(A)$ are necessarily even.

**Remark 2.1.** The space $W_w(N)_{/R}$ is isomorphic to the space $\text{Symb}_{\Gamma_0(N)} V_w(R)$ of modular symbols introduced by Ash and Stevens in [2], and by [2, Prop. 4.2] we have a Hecke-equivariant isomorphism

$$W_w(N)_{/R} \simeq H^1_c(\Gamma_0(N), V_w(R)).$$

The compactly supported cohomology group is that of the local system associated to $V_w(R)$ on the modular surface $\Gamma \backslash \mathcal{H}$, with $\mathcal{H}$ the upper half-plane.

The module $V_w(N)_{/R}$ is simply the induced module $\text{Ind}^{\Gamma_1}_{\Gamma} V_w(R)$, so, via Shapiro’s lemma, another way to interpret the isomorphism (2.1) is:

$$W_w(N)_{/R} \simeq H^1_c(\Gamma_1, V_w(N)_{/R}).$$

Since $\Gamma_1$ has only one cusp fixed by $T = US$, the latter cohomology group can be identified with the set of $\Gamma_1$-cocycles which are 0 on $T$ as in [15], and the isomorphism takes a polynomial $P$ to the cocyle $\varphi$ with $\varphi(T) = 0$, $\varphi(T) = P$. See also [27, Sec. 2] and [28, Sec. 2.2].

We will use the isomorphism (2.1) in Section 4. We conclude that the combinatorial description of $W_w(N)_{/R}$ that we use throughout Section 2 gives us a way of studying “Eisenstein classes” in the compactly supported cohomology of the modular surface.

Throughout Section 2 we are interested in the case $R = \mathbb{C}$, and we set $V_w(N) = V_w(N)_{/\mathbb{C}}$, $W_w(N) = W_w(N)_{/\mathbb{C}}$. For a cuspform $f \in S_k(N)$, its period polynomial $\rho_f \in W_{k-2}(N)$ (which we sometimes denote by $\rho(f)$) is defined in [27] by

$$\rho_f(A) = \int_0^{i\infty} f|_k A(z)(X - z)^{k-2}dz, \quad \forall A \in \Gamma \backslash \Gamma_1,$$

where we set $f|_k \sigma(z) = f(\sigma z)(cz + d)^{-k}$ for $\sigma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}_2^+(\mathbb{R})$, and $f|_k A$ is defined using any representative of the coset $A$. This normalization of the stroke $|_k$ operator is chosen both
to be compatible with the earlier operator $|_w$ on period polynomials, and to avoid scaling factors in the action of Hecke operators—see the last equation in this subsection and (2.11).

The maps $\rho^\pm : S_{w+2}(N) \to W_w(N)$, $f \mapsto \rho^\pm f$ are injective, and the Eichler-Shimura isomorphism can be restated as the following direct sum decomposition

\begin{equation}
W_w(N) \cong \rho^+(S_{w+2}(N)) \oplus \rho^-(S_{w+2}(N)) \oplus C_w(N),
\end{equation}

where $C_w(N) = \{ P|1-S : P \in V_w(N), P|1-T = 0 \}$ is the coboundary subspace [27, Thm. 2.1]. The dimension of $C_w(N)$ equals the dimension of the Eisenstein subspace of $M_{w+2}(N)$ [27, Lemma 4.2], and in Proposition 2.9 we give an explicit basis coming from Eisenstein series when the level $N$ is square-free and $w > 0$.

To define the action of Hecke operators on period polynomials, let $\mathcal{M}_n$ be the set of $2 \times 2$ integral matrices of determinant $n$, and set $\overline{\mathcal{M}}_n := \mathcal{M}_n/\{ \pm 1 \}$, $\mathcal{R}_n := \mathbb{Z}[\overline{\mathcal{M}}_n]$. Let $\Sigma \subset \mathcal{M}_n$ be a double coset of $\Gamma$, namely $\Sigma = \Gamma \Sigma \Gamma$ and the number of right cosets $|\Gamma/\Sigma|$ is finite. The double coset $\Sigma$ acts on $f \in M_k(N)$ by

\begin{equation}
f|\Sigma \Gamma = h_{k-1} \sum_{\sigma \in \Gamma/\Sigma} f|_{k\sigma}.
\end{equation}

To define the corresponding action on period polynomials, we make the following assumption on the double coset $\Sigma$:

\begin{equation}
\text{The map } \Gamma/\Sigma \to \Gamma_1/\Gamma_1 \Sigma, \quad \Gamma \sigma \mapsto \Gamma_1 \sigma \text{ is bijective ,}
\end{equation}

or equivalently $|\Gamma/\Sigma| = |\Gamma_1/\Gamma_1 \Sigma|$. For $P \in \mathcal{V}_w(N)$ and $M \in \mathcal{M}_n$, we define

\begin{equation}
P|\Sigma M(A) = \begin{cases} 
P(A_M)|_w M & \text{if } M \alpha^{-1} = A \alpha^1 M \text{ with } A \in \Gamma_1, M \in \Sigma \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

Since both $M$ and $-M$ act in the same way, the action of elements in $\overline{\mathcal{M}}_n$ is also well defined, and by linearity it extends to an action of elements in $\mathcal{R}_n$. It is not a proper action, but it is compatible with the action of $\Gamma_1$: for $g \in \Gamma_1$, $M \in \mathcal{M}_n$, we have $P|g M = (P|g)|_\Sigma M$, $P|g M g = (P|g M)|g$.

Let $\mathcal{M}_n^\infty$ be a system of representatives which fix $\infty$ for the cosets $\Gamma_1 \setminus \mathcal{M}_n$, and let $T_n^\infty = \sum_{M \in \mathcal{M}_n^\infty} M \in \mathcal{R}_n$. Let $T = U S = (1 0 \ 0 1)$ be a generator of the stabilizer of $\infty$. It was shown in [7] that there exists $\overline{T}_n \in \mathcal{R}_n$ such that:

\begin{equation}
T_n^\infty (1-S) - (1-S) \overline{T}_n = (1-T)\mathcal{R}_n,
\end{equation}

and in [27] we show that for $f \in S_k(N)$ we have $\rho_f|\Sigma = \rho_f|\overline{T}_n$.

2.2. The Atkin-Lehner operator. Let $\Gamma = \Gamma_0(N)$ and denote $w_N = (0 -1 \ N \ 0)$, $\sigma_N = (N 0 \ 0 1)$. The action of the Atkin-Lehner involution $W_N$ on modular forms $f \in M_k(N)$ is given by $f|W_N = N^{k/2} f|_w w_N$. It is related to the action of the double coset

$\Theta_N = \Gamma w_N \Gamma = \Gamma w_N = \{ (a \ b \ c \ d) \in \mathcal{M}_N : N | a, N | d, N | c \}$

by $f|W_N = \frac{1}{N^{k/2-1}} f|[\Theta_N]$, with the latter action defined in (2.4). We write $P|_{\Theta_N} \overline{T}_N$ instead of $P|_{\Theta_N} \overline{T}_N$ for the corresponding action on $P \in V_w(N)$ given by (2.6).

**Lemma 2.2.** (i) We have a bijection $\Gamma_1/\Gamma_1 \to \Gamma_1/\Gamma_1 \sigma_N \Gamma_1$, given for $A \in \Gamma_1$ by

$A \mapsto K_A := \Gamma_1 \sigma_N A$.

(ii) We have $K_A = \Gamma_1 \Theta_N A$. 
we obtain

Let \( \tilde{T}_n = \sum_{M \in \mathcal{M}_n} c_M M \in \mathcal{R}_n \) and a coset \( K \in \Gamma_1 \backslash \mathcal{M}_n \), we let \( \tilde{T}_n^{(K)} = \sum_{M \in K} c_M M \) be the part of \( \tilde{T}_n \) supported on matrices in \( K \).

Lemma 2.3. Let \( P \in V_w(N) \) and \( \tilde{T}_n \in \mathcal{R}_N \). Then
\[
P|_{\Theta N}(A) = P|_{\Theta N}^{(K_A)}(A),
\]
for \( A \in \Gamma \backslash \Gamma_1 \), where \( K_A \in \Gamma_1 \backslash \mathcal{M}_n \) is the coset defined in Lemma 2.2.

Proof. By the definition (2.6), for \( \tilde{T}_n = \sum c_M M \in \mathcal{R}_N \) we have
\[
P|_{\Theta N}(A) = \sum_{M \in \Gamma_1 \Theta N} c_M \cdot P(A_M)|_{-w} = P|_{\Theta N}^{(K_A)}(A),
\]
where \( A_M \in \Gamma \backslash \Gamma_1 \) is the unique coset such that \( A_M MA^{-1} \subset \Theta N \). The last equality follows from Lemma 2.2 (ii).

Example 2.4. For the identity coset \( I \), we have \( K_I = \Gamma_1 \sigma_N \) and we can take
\[
\tilde{T}_n^{(K_I)} = \left( \begin{array}{cc} N & 0 \\ 0 & 1 \end{array} \right).
\]

The space \( W_w(1) \) contains the polynomial \( 1|_{-w} - 1 = 1 - X^w \), which belongs to the coboundary subspace \( C_w(1) \) defined in (2.3) and corresponds to the Eisenstein series \( E_1 \), as we will see in §2.5. Therefore \( W_w(N) \) contains the polynomial \( P_0 = 1|(1 - S) \), with \( 1 \) the constant polynomial \( 1 \) in each coset. We next determine its image under the Atkin-Lehner operator, which will be used to determine a basis of the “Eisenstein part” of \( W_w(N) \) when \( N \) is square-free. Denote by \( (x, y) \) the greatest common divisor of \( x, y \in \mathbb{Z} \).

Proposition 2.5. Let \( w \geq 2 \) be even, and let \( P_0 = 1|(1 - S) \in W_w(N) \). For every \( \tilde{T}_n \) satisfying (2.5) we have
\[
P_0|_{\Theta N}(A) = N_z^w - N_t^w X^w
\]
where \( A = \Gamma \left( \begin{array}{cc} z & \ast \\ 0 & 1 \end{array} \right) \) and for \( a \in \mathbb{Z} \) we let \( N_a = N/(N, a) \).

Proof. For a coset \( K \in \Gamma_1 \backslash \mathcal{M}_n \), let \( M_K \in K \cap \mathcal{M}_n^\infty \) be a fixed representative fixing \( \infty \), and let \( T_n^\infty = \sum_{K \in \Gamma_1 \backslash \mathcal{M}_n} M_K \). Taking the part of relation (2.5) supported on matrices in \( K \) we have
\[
(1 - S)\tilde{T}_n^{(K)} - (M_K - M_K S) \in (1 - T) \mathcal{Q}[K].
\]
Using Lemma 2.3 we obtain
\[
P_0|_{\Theta N}(A) = 1|_{-w} (1 - S) \tilde{T}_n^{(K_A)} = 1|_{-w} (M_{K_A} - M_{K_A S}) = d^w - (d'X)^w,
\]
where we write \( M_{K_A} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \), \( M_{K_A S} = \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \). One checks that \( d' = N/(b, d) \), and since \( K_A = \Gamma_1 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \Gamma_1 \left( \begin{array}{cc} Nx & Ny \\ z & t \end{array} \right) \) for \( A = \Gamma \left( \begin{array}{cc} x & y \\ z & t \end{array} \right) \), we obtain \( a = (N, z), (b, d) = (N, t) \).

2.3. Primitive spaces. In this section we define subspaces \( W_w(N)^{new} \subset W_w(N) \) which contain the period polynomials of newforms in \( M_{w+2}(N) \).

To fix definitions, we first review some newform theory from [1]. A modular form in \( \mathcal{M}_k(N) \) is called a Hecke eigenform if it is an eigenform of all Hecke operators \( T_n \) (including for primes \( p | N \), which are called \( U_p \) in [1]), normalized to have the coefficient of \( q \) equal to \( 1 \). For a prime \( p | N \), a cuspform \( f \) is called \( p \)-new if it is orthogonal with respect to the Petersson inner product to the space spanned by the image of the two embeddings \( S_k(N/p) \rightarrow S_k(N) \) given by the identity and \( f(z) \mapsto f(pz) \). The Hecke eigenforms which are \( p \)-new for all \( p | N \) are called newforms. If \( h \) is a newform of level \( M|N, M \neq N \), the oldspace associated to \( h \) is the span of \( h(dz) \) for \( d|(N/M) \), and the oldspaces together with the one dimensional spaces
spanned by newforms give a decomposition of $S_k(N)$ into mutually orthogonal subspaces. There is also a notion of newforms for Eisenstein series [35], but we only need here the obvious fact that for $k \geq 4$ and square-free $N > 1$ all Eisenstein series in $M_k(N)$ are old.

In this paper we use an algebraic characterization of newspaces originally due to Serre. For $M|N$, let $\text{Tr}_M^N : M_k(N) \rightarrow M_k(M)$ be the trace map $\text{Tr}_M^N(f) = \sum_\sigma f|_k \sigma$, with the sum over a system of representatives for the cosets $\Gamma_0(N) \backslash \Gamma_0(M)$. For a prime $p|N$, the space of forms which are new at $p$ can be characterized as

$$M_k(N)^{p \text{-new}} = \{ f \in M_k(N) : \text{Tr}_{M/p}^N(f) = \text{Tr}_{N/p}^N(f|W_N) = 0 \},$$

and we define the space of newforms $M_k(N)^{\text{new}} = \cap_{p|N} M_k(N)^{p \text{-new}}$, where $p$ runs through the prime divisors of $N$. This agrees with the usual definition given above: see [22, Ch. VIII, Thm. 2.2] for cusp forms and [35, Prop. 19] for Eisenstein series.

Similarly, for $M|N$ we let $\text{Tr}_M^N : W_w(N) \rightarrow W_w(M)$ be the trace map:

$$\text{Tr}_M^N(P)(C) = \sum_{B \in \Gamma_0(N) \backslash \Gamma_0(M)} P(BC),$$

for all $C \in \Gamma_0(M) \backslash \Gamma_1$. This is compatible with the trace defined on the cuspidal space: for $f \in S_{w+2}(N)$ we easily see that $\rho(\text{Tr}_M^N f) = \text{Tr}_M^N \rho(f)$. We therefore define the new subspace of $W_w(N)$ by $W_w(N)^{\text{new}} = \cap_{p|N} W_w(N)^{p \text{-new}}$, where for prime $p|N$ we define

$$W_w(N)^{p \text{-new}} := \{ P \in W_w(N) : \text{Tr}_{N/p}^N(P) = \text{Tr}_{N/p}(P|_{\delta \overline{T}_N}) = 0 \}.$$

Since the action of $\delta$ commutes with the trace map, we may define subspaces $W_w^+(N)^{p \text{-new}}$ of $W_w^+(N)$. All these spaces can be defined in the same way over an arbitrary ring $R$, and we will need them in Section 4.2.

2.4. Extended period polynomials. We also need the space of extended period polynomials $\overline{W}_w(N)$, which contains the period polynomials $\hat{\rho}(f) = \hat{\rho}_f$ of arbitrary modular forms $f \in M_k(N)$ (we set $k = w + 2$ throughout). We refer to [27, Sec. 8] for the definition, and we only recall that for $f \in M_k(N)$, its extended period polynomial $\hat{\rho}_f$ is given as in (2.2), with the integral regularized by replacing $f|A$ with $f|A - a_0(f)|A$, where $a_0(f)|A$ is the constant term in the Fourier expansion of $f|A$. By [27, eq. (8.2)] we have

$$\hat{\rho}_f(A) = (-1)^k a_0(f)|A|^{\frac{k-1}{2k}} + a_0(f)|A|^{\frac{k}{k-1}} + \sum_{n=0}^w (-1)^{w-n} \binom{w}{n} r_n(f)|A|^{w-n}$$

for $A \in \Gamma_0(N) \backslash \Gamma_1$, where $r_{m-1}(g) = (-1)^m \frac{\Gamma(m)}{(2\pi i)^m} L(g, m)$ is given in terms of critical values at $0 < m < k$ of the $L$-function $L(g, s)$, extended by meromorphic continuation.

Example 2.6. Since $L(E_k, s) = \zeta(s) \zeta(s - k + 1)$, we obtain for $k \geq 4$ even:

$$\hat{\rho}^-(E_k) = \frac{B_k}{2k} \cdot \frac{X^{k-1} + X^{-1}}{k-1} - \frac{1}{2} \sum_{0 < n < k-2} \binom{k-2}{n-1} \frac{B_n}{n} \frac{B_{k-n-1}}{k-n} X^{k-n-1} \in \overline{W}_{k-2}(1),$$

and $\hat{\rho}^+(E_k) = \alpha_k (1 - X^{k-2}) \in W_{k-2}^+(1)$, for $\alpha_k = \frac{(k-2)!}{2(2\pi i)^{k-1}} \zeta(k - 1)$ [20, p. 240].

The Hecke operators $\overline{T}_n$ preserve $\overline{W}_w(N)$, acting as in Section 2.1, and we have

$$\hat{\rho}_f|_{\mathbb{Z}} = \hat{\rho}_f|_{\mathbb{Z}} \overline{T}_n,$$

2 Here $k$ is even, but the formula is valid for all finite index subgroups of $\text{SL}_2(\mathbb{Z})$, when $k$ may also be odd.
where $\Sigma$ is a double coset contained in $\mathcal{M}_a$ satisfying (2.5). The space $\hat{W}_w(N)$ is preserved by the involution $\delta = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$, and we denote its $\pm 1$ eigenspaces by $\hat{W}_w^\pm(N)$. We define its new subspaces as in (2.9). The following proposition is a generalization of the Eichler-Shimura isomorphism to the space of extended period polynomials.

**Proposition 2.7.** Let $w \geq 2$ be even. The two maps

$$\hat{\rho}_w^\pm : M_{w+2}(N) \to \hat{W}_w^\pm(N), \quad f \mapsto \hat{\rho}_w^\pm,$$

are Hecke equivariant isomorphisms, and they map $M_{w+2}(N)^{p\text{-}\text{new}}$ isomorphically onto $\hat{W}_w^\pm(N)^{p\text{-}\text{new}}$, for primes $p|N$.

**Proof.** That the two maps are isomorphisms is proved in [27, Prop. 8.4]. The second statement follows by using the characterisation of newforms above, together with the compatibility of the two isomorphisms with the trace map $\text{Tr} = \text{Tr}_M$ and with the Atkin-Lehner involution: $\hat{\rho}_{\text{Tr}(f)} = \text{Tr}(\hat{\rho}_w^\pm)$, $\hat{\rho}_{\text{Tr}}|_{W_N} = N^{-w/2} \hat{\rho}_w^\pm|_{\hat{W}_w}$.

**Remark 2.8.** For $w = 0$ and $N$ square-free the map $\hat{\rho}^-$ is still an isomorphism, but $\hat{\rho}^+$ is not unless $N$ is prime [27, Prop. 8.4]. This is one of the reasons the weight 2 case is more delicate, and we avoid it in this paper.

2.5. **Period polynomials of Eisenstein series.** We now specialize $N$ to be square-free, and apply the results of the previous sections to determine the period polynomials of a basis of Eisenstein series in $M_k(N)$ for $k \geq 4$. Let $D(N)$ denote the divisors of $N$ and let $\varepsilon : D(N) \to \{\pm 1\}$ be a system of Atkin-Lehner eigenvalues, namely $\varepsilon(a)\varepsilon(b) = \varepsilon(ab)$ if $(a, b) = 1$. Since $E_k|W_d(z) = d^{k/2}E_k(dz)$, the linear combinations

$$E_{k,N}^{(\varepsilon)}(z) := \sum_{d|N} \varepsilon(d)d^{k/2}E_k(dz) \in M_k(N)$$

are eigenforms of $W_d$ with eigenvalue $\varepsilon(d)$ for all $d|N$, and they provide a basis of the Eisenstein subspace (of dimension $2^{\omega(N)}$, with $\omega(N)$ the number of prime factors of $N$). When $N = p$ is prime, we identify $\varepsilon$ with its value $\varepsilon(p) \in \{\pm 1\}$, and we recover the Eisenstein series $E_{k,p}$ from the introduction.

When $N$ is square-free, we show next that the extended polynomial $\hat{\rho}_w^+$ is actually a period polynomial in $\hat{W}_w(N)$ for all $f \in M_{w+2}(N)$, just like in the case $N = 1$ of Example 2.6. We also make more explicit the Eichler-Shimura isomorphism in Proposition 2.7, by determining an explicit basis of the coboundary subspace of $W_w(N)$.

**Proposition 2.9.** Let $N$ be square-free and let $k = w + 2 \geq 4$ be even.

(i) We have isomorphisms

$$\rho^- : S_k(N) \iso \hat{W}_w^-(N), \quad \hat{\rho}_w^+ : M_k(N) \iso \hat{W}_w^+(N),$$

and, if $N > 1$, $\rho^+ : S_k(N)^{\text{new}} \iso \hat{W}_w^+(N)^{\text{new}}$.

(ii) We have the following explicit version of the Eichler-Shimura isomorphism

$$W_w(N) = \rho^-(S_k(N)) \oplus \rho^+(S_k(N)) \oplus \varepsilon C\rho^+(E_{k,N}^{(\varepsilon)}),$$

where the period polynomials $\rho^+(E_{k,N}^{(\varepsilon)})$ span the coboundary subspace $C_w(N)$.

**Proof.** (i) The set of $N$ for which the map $\rho^-$ is an isomorphism is characterized in [27, Prop. 4.4], and it includes square-free $N$. From the Eichler-Shimura isomorphism (2.3), we obtain that $\dim \hat{W}_w^+(N) = \dim M_{w+2}(N)$. The latter is also equal to $\dim \hat{W}_w^+(N)$, so
\( \hat{W}_{w}^{+}(N) = W_{w}^{+}(N) \), and Proposition 2.7 implies that \( \hat{\rho}^{+} \) and \( \rho^{+} \) in (i) are isomorphisms as well (the latter when \( N > 1 \) since \( M_{w+2}(N)^{\text{new}} = S_{w+2}(N)^{\text{new}} \) in this case).

(ii) The period polynomials \( \hat{\rho}^{+}(E_{k,N}^{(e)}) \) are Atkin-Lehner eigenforms with different eigenvalues, so they are linearly independent. They belong to \( C_{w}(N) \) since they are in the span of images of \( \hat{\rho}^{+}(E_{k}) \in C_{w}(N) \) under Atkin-Lehner involutions, which preserve \( C_{w}(N) \). \( \square \)

In the rest of this subsection we determine \( \hat{\rho}^{+}(E_{k,N}^{(e)}) \) and the principal part of \( \hat{\rho}^{-}(E_{k,N}^{(e)}) \).

Note that (2.11) implies that \( \hat{\rho}^{+}(E_{k,N}^{(e)}) \) is an eigenvector for the Hecke operators \( T_{n} \) with eigenvalue \( \sigma_{k-1}(n) \) for \( (n,N) = 1 \), as well as an eigenvector for all Atkin-Lehner operators.

For \( d|N \), the inclusion \( M_{k}(d) \hookrightarrow M_{k}(N) \) corresponds to an inclusion

\[
i_{d}^{N} : W_{w}(d) \hookrightarrow W_{w}(N)
\]
described as follows. For \( A \in \Gamma_{0}(N)\backslash \Gamma_{1} \), write \( A = BC \) with \( B \in \Gamma_{0}(N)\backslash \Gamma_{0}(d) \), \( C \in \Gamma_{0}(d)\backslash \Gamma_{1} \). Then \( (i_{d}^{N} P)(A) = P(C) \), and if \( A = \Gamma_{0}(N) \left( \frac{a}{c}, \frac{b}{d} \right) \) then \( C = \Gamma_{0}(d) \left( \frac{a}{d}, \frac{c}{d} \right) \).

For the Eisenstein series \( E_{k} \) we have \( \hat{\rho}^{+}(E_{k}) = \alpha(1 - X^{k-2}) \in W_{k-2}^{+}(1) \), with \( \alpha = \alpha_{k} \) given explicitly in Example 2.6.

**Proposition 2.10.** Let \( N \) be square-free and let \( k = w + 2 \geq 4 \) be even. For \( \varepsilon : D(N) \to \{ \pm 1 \} \) a system of Atkin-Lehner eigenvalues, we have

\[
\hat{\rho}^{+}(E_{k,N}^{(e)}) = \alpha \prod_{p|N} \left( 1 + \varepsilon(p)p^{-w/2} \right) \cdot P^{+}(E_{k,N}^{(e)})
\]

with \( P^{+}(E_{k,N}^{(e)}) \in W_{w}^{+}(N) \) given by

\[
P^{+}(E_{k,N}^{(e)})(A) = \varepsilon(N_{z})N_{w/2} - \varepsilon(N_{t})N_{w/2}X_{w} \in \mathbb{Z}[X]
\]

for \( A = \Gamma_{0}(N) \left( \frac{a}{c}, \frac{b}{d} \right) \), where we recall that \( N_{a} = N/(N,a) \).

One can check directly that \( P^{+}(E_{k,N}^{(e)}) \in C_{w}(N) \), by writing it as \( P_{w,N}(1 - S) \) where \( P_{w,N}(A) = \varepsilon(N_{z})N_{w/2}X_{w} \) for a coset \( A \) as above. One easily checks \( P_{w,N}(1 - T) = 0 \).

**Proof.** By (2.11), for each divisor \( d|N \) we have

\[
\hat{\rho}^{+}(E_{k}|W_{d}) = d^{-w/2} \cdot [i_{d}^{N} \hat{\rho}^{+}(E_{k})]|_{\Theta_{d}} \in W_{w}(d),
\]

with the Atkin-Lehner involution \( W_{d} \) acting as in Section 2.2, yielding

\[
(2.12) \quad \hat{\rho}^{+}(E_{k,N}^{(e)}) = \sum_{d|N} \varepsilon(d) d^{-w/2} i_{d}^{N} [i_{d}^{N} \hat{\rho}^{+}(E_{k})]|_{\Theta_{d}}.
\]

Note that \( i_{d}^{N} \hat{\rho}^{+}(E_{k}) = \alpha P_{0} \in W_{k-2}(d) \), where \( P_{0} \) is defined in Proposition 2.5, and applying that proposition we obtain:

\[
i_{d}^{N} [i_{d}^{N} \hat{\rho}^{+}(E_{k})]|_{\Theta_{d}}(A) = \alpha \cdot (d_{z}^{w} - d_{t}^{w}X_{w}), \quad \text{for} \ A = \Gamma_{0}(N) \left( \frac{a}{c}, \frac{b}{d} \right).
\]

We now use the identity \( \sum_{d|N} \varepsilon(d) d^{-w/2} d_{z}^{w} = \varepsilon(N_{z})N_{w/2} \cdot \prod_{p|N} (1 + \varepsilon(p)p^{-w/2}). \)

\( \square \)

The computation of the principal part of \( \hat{\rho}^{-}(E_{k,N}^{(e)}) \) is similar, using the formula for \( \hat{\rho}^{-}(E_{k,N}) \in \hat{W}_{k-2}^{+}(1) \) in Example 2.6.

**Proposition 2.11.** Let \( N \) be square-free, let \( k = w + 2 \geq 4 \) be even, and let \( \varepsilon : D(N) \to \{ \pm 1 \} \) be a system of Atkin-Lehner eigenvalues. For the identity coset \( I \) we have

\[
\hat{\rho}^{-}(E_{k,N}^{(e)})(I) = \sum_{d|N} \varepsilon(d) d^{-w/2} \cdot \hat{\rho}^{-}(E_{k}|_{-w}) \left( \frac{d}{0} \right).
\]
Proof. Apply (2.12) written for the odd part in terms of \( \tilde{\rho}^-(E_k) \), and use Lemma 2.3 (which is easily seen to hold for extended polynomials) together with Example 2.4. \( \square \)

2.6. Trace maps. We end this section by determining the behavior of \( E_{k,N}^{(c)} \) under trace maps. Let \( N = Mp \) be square-free with \( p \) prime. We can restrict a system of Atkin-Lehner eigenvalues \( \varepsilon : D(N) \to \{\pm 1\} \) to \( D(M) \), and apply (1.3) to \( E_{k,N}^{(c)} = E_{k,M}^{(c)}(1 + \varepsilon(p)W_p) \). Using \( \tilde{\rho}^+(\text{Tr}_M^N f) = \text{Tr}_M^N \tilde{\rho}^+(f) \) for \( f \in M_k(N) \), we obtain

\[
\text{Tr}_M^N P^+(E_{k,N}^{(c)}) = (1 + \varepsilon(p)p^{k/2}) \cdot P^+(E_{k,M}^{(c)})
\]

\[
\text{Tr}_M^N \tilde{\rho}^-(E_{k,N}^{(c)}) = (p^{k/2} + \varepsilon(p))(p^{1-k/2} + \varepsilon(p)) \cdot \tilde{\rho}^-(E_{k,M}^{(c)})
\]

where in the first equation we used Proposition 2.10.

3. Proof of Theorems 1 and 2

Let \( N = Mp \) with \( p \nmid M \) as in the introduction, let \( g \in M_k(M) \) be a newform of level \( M \) with Fourier coefficients \( \lambda_n(g) = \lambda_n \), and let \( g_p^{(c)} \) be defined as in (1.2) for \( \varepsilon \in \{\pm 1\} \). This includes the case \( M = 1 \) and \( g = E_k \) in Theorem 1. Let \( \ell \) be a prime satisfying \( \ell | \lambda_p + \varepsilon p^{k/2-1}(p + 1) \), which covers the “new at \( p \)” condition in both theorems (see (1.3)).

We first observe that if \( f \in S_k(N)p^\text{new} \) is a Hecke eigenform with eigenvalue \( \varepsilon \) under the Atkin-Lehner involution \( W_p \), then \( \lambda_p(f) = -\varepsilon p^{k/2-1} \) by [1, Thm. 3], so

\[
\lambda_p(f) \equiv \varepsilon_p(g_p^{(c)}) = \lambda_p + \varepsilon p^{k/2} \pmod{\lambda_p + \varepsilon p^{k/2-1}(p + 1)}
\]

That is, the congruences in both theorems hold at \( p \) because of the above assumption on \( \ell \), and therefore it is enough to check that there exists such an \( f \) with Hecke eigenvalues \( \lambda_n(f) \equiv \lambda_n \pmod{\mathbb{I}} \) for \( (n, p) = 1 \).

Let \( R \) be a finite extension of \( \mathbb{Z}_\ell \) containing the coefficients of all Hecke eigenforms in \( S_k^{(c)}(N) \). Let \( \pi \) be a uniformizer in \( R \) and \( F = \mathbb{Q}/\pi R \) the residue field, and set \( w = k - 2 \). Since \( \ell > k + 1, \ell \nmid N \), by Theorem 3 the reduction map \( W_w(N)^{p\text{new}} \to W_w(N)^{p\text{new}} \) is surjective, and both theorems follow from the Deligne-Serre lifting lemma, once we produce an element in \( W_w(N)^{p\text{new}} \) which is an eigenvector for the Hecke operators of index coprime to \( p \) and for the Atkin-Lehner involution \( W_p \), with eigenvalues congruent to those of \( g_p^{(c)} \). The Deligne-Serre lemma would then provide a system of Hecke and Atkin-Lehner eigenvalues on \( W_w(N)^{p\text{new}} \) congruent to those of \( g_p^{(c)} \). By Proposition 2.7 we then conclude the existence of a Hecke eigenform \( f \in M_k^{(c)}(N) \) which is \( p \)-new, satisfying the desired congruence. But there are no \( p \)-new Eisenstein series in \( M_k(N) \), as \( p \nmid M \), so the form \( f \) must be a cusp form. Note that since \( f \) is \( p \)-new, it is automatically an eigenform of \( T_p \), being in the oldspace of a newform of level \( pM' \) with \( M'|M \).

To construct the desired finite period polynomial in both theorems we proceed as follows.

- Theorem 1, \( \ell | p^{k/2} + \varepsilon \). From (2.13) and the definition of the newspace in (2.9) we obtain that \( P^+(E_{k,p}^{(c)}) \pmod{\ell} \) belongs to \( W_{w}(p)_{/R}^{\text{new}} \) (note that it is already an eigenform for the Atkin-Lehner operator \( |_0 T_p \) acting on period polynomials). It is nonzero modulo \( \ell \) since \( P^+(E_{k,p}^{(c)})(I) = 1 - \varepsilon p^{w/2}X^w \), by Proposition 2.10.

\[\text{Both } f \text{ and } g_p^{(c)} \text{ have Euler products, so it is enough to check the congruence for Fourier coefficients of prime index.}\]
Theorem 1, \( \ell \nmid p^{k/2} + \varepsilon \). Since \( \ell \) divides the numerator of \( B_k/k \), formula (2.10) shows that \( \hat{\rho}^{-}(E_{k,p}^{(c)}) \) (mod \( \ell \)) belongs to \( W_{w}^{-}(p)/\mathbb{F}_{\varepsilon} \). Since \( \ell | p^{w/2} + \varepsilon \), it follows from (2.13) that \( \hat{\rho}^{-}(E_{k,p}^{(c)}) \) (mod \( \ell \)) actually belongs to \( W_{w}^{-}(p)/\mathbb{F}_{\varepsilon} \). From Proposition 2.11 we have

\[
\hat{\rho}^{-}(E_{k,p}^{(c)}) \equiv \frac{1}{2} \sum_{0 < n < \omega} \left( \frac{k-2}{n-1} \right) B_n \frac{B_{k-n}}{n} \frac{1}{k-n} (1 - p^{n-1}) X^{n-1} \quad \text{(mod } \ell)\text{).}
\]

The denominators in this formula have prime factors which are smaller than \( \ell \), since \( \ell > k-2 \), and the extra assumption ensures that the previous element is nonzero.

Theorem 2. Let \( P^\pm(g) \in W_w(M) \) be the multiples of \( \rho^\pm(g) \) which are normalized as in the paragraph following Theorem 2, so their coefficients belong to \( K_g \) by a well-known rationality result, e.g. [27, Prop. 5.11]. The condition \( k \geq 6 \) is required to guarantee that the coefficient of \( X \) in \( P^{-}(g)(I) \) is nonzero, being proportional to the value at \( s = k-2 \) of the \( L \)-function \( L(s,g) \). From (1.3) we obtain as before

\[
\text{Tr}_{M}^N \left( P^\pm(g^{(c)}) \right) = (1 + p + \varepsilon p^{-k/2+1} \lambda_p) \cdot P^\pm(g),
\]

where \( P^\pm(g^{(c)}) := i_{\mathcal{N}} \text{Tr}_{M}^N (P^\pm(g) + \varepsilon p^{-w/2+i_N} P^\pm(g)) \alpha T_p \) is a multiple of \( \rho^\pm(g^{(c)}) \). By assumption, either \( P^+(g) \) or \( P^-(g) \) has denominators coprime to \( \mathcal{I} \), so the same is true about at least one of \( P^\pm(g^{(c)}) \). It follows that one of \( P^\pm(g^{(c)}) \) (mod \( \mathcal{I} \)) belongs to \( W^\pm_w(N)^{\mathcal{F}_{\varepsilon}} \), and we fix this choice of sign.

To see that it is nonzero, we evaluate it on the identity coset \( I \). Setting \( P = P^\pm(g)(I) \) for the choice of sign above, we obtain by the definition (2.2) and a change of variables:

\[
P^\pm(g^{(c)})(I)(X) = P(X) + \varepsilon p^{-w/2} P(pX).
\]

Due to the normalization of \( P \), the constant term is \( 1 + \varepsilon p^{-w/2} \) for \( P^+ \) and the coefficient of \( X \) is \( 1 + \varepsilon p^{-w/2+1} \) for \( P^- \), which are nonzero mod \( \ell \) in either case by assumption (1.6).

Remark 3.1. We found numerically that the extra assumption \( \ell \nmid B_k B_{k-n}(p^{n-1} - 1) \) in Theorem 1 is not needed for weights \( k \leq 6 \cdot 10^4 \), by considering \( n = 2 \) and \( n = 4 \).

More precisely, assume \( \ell \nmid p^{k/2} + \varepsilon \). It follows from (1.4) that \( \ell | p^{k/2-1} + \varepsilon, \ell | B_k \), and \( \ell \nmid p - 1 \), so the assumption is satisfied for \( n = 2 \), unless \( \ell | B_{k-2} \). For \( k \leq 6 \cdot 10^4 \) there are only two pairs \((k,\ell)\) with \( k \) even and \( \ell > k-2 \) prime, such that \( \ell \) divides the numerator of both \( B_k \) and \( B_{k-2} \), namely \((92,587)\) and \((338,491)\). Note that in both cases \( 3 \nmid \ell - 1 \), and since \( \ell | p^{\ell-1} - 1 \) it follows that \( \ell | p^3 - 1 \) (otherwise we would have \( \ell | p - 1 \), contradicting the assumption). It follows that \( \ell | B_k B_{k-4}(p^3 - 1) \) for those two values of \( k \), so the assumption is satisfied for \( n = 4 \).

Note that the same assumption, without the factor \( p^{n-1} - 1 \), appears in Haberland’s proof of the Ramanujan congruence (1.1) [15, Sec. 5.2]. There the assumption guarantees the nonvanishing of the reduction mod \( \ell \) of an Eisenstein cocycle in \( H^1_c(\Gamma_1, W_w(R)) \) associated to \( \hat{\rho}^{-}(E_k) \).

4. Surjectivity of reduction maps on spaces of period polynomials

In this section we use the isomorphism \( W_w(N)/R \simeq H^1_c(\Gamma_0(N), W_w(R)) \) in Remark 2.1 to prove Theorem 3.

\[\text{Using PARI [23], it took about 50 minutes on a laptop to check the range } 5 \cdot 10^4 \leq k \leq 6 \cdot 10^4.\]
4.1. Surjectivity of reduction on the whole space. We first need a lemma computing the dimension of the cohomology of $\Gamma_1 = \text{SL}_2(\mathbb{Z})$, for which we start in greater generality. Let $V$ be a right $\Gamma_1$-module, and assume, as it will always be the case, that $-1 \in \Gamma_1$ acts trivially on $V$. Therefore the cohomology groups we consider are the same when replacing $\Gamma_1$ by $\overline{\Gamma}_1 = \text{PSL}_2(\mathbb{Z})$.

Since $\overline{\Gamma}_1$ is a free product of its subgroups $G_2$ and $G_3$ generated by $S$ and $U$, the Mayer-Vietoris exact sequence in group cohomology [5, Sec. VII.9] gives

\[ 0 \to H^0(\overline{\Gamma}_1, V) \to H^0(G_2, V) \oplus H^0(G_3, V) \to H^0(G_2 \cap G_3, V) \to \]

\[ \to H^1(\overline{\Gamma}_1, V) \to H^1(G_2, V) \oplus H^1(G_3, V) \]

(4.1)

Lemma 4.1. Assume that the $\Gamma_1$-module $V$ is defined over a field $F$ of characteristic $\text{char}(F) \neq 2, 3$. We have

\[ \dim H^1(\overline{\Gamma}_1, V) = \dim V - \dim H^0(G_2, V) - \dim H^0(G_3, V) + \dim H^0(\overline{\Gamma}_1, V). \]

Proof. The assumption $\text{char}(F) \neq 2, 3$ implies that $H^1(G_i, V) = 0$ for $j \geq 1$, $i = 2, 3$, so the conclusion immediately follows from the Mayer-Vietoris sequence. □

Let $R$ be a discrete valuation ring with residue field $F$. The surjectivity of more general reduction maps on compactly supported cohomology was proved by Hida [17, Eq. (1.16)] for congruence groups with no elliptic elements, by a geometric argument. We give an algebraic proof here, valid for groups with elliptic elements as well.

Proposition 4.2. Let $w \geq 0$ be even. If the residue field $F$ has characteristic $\ell > w$, $\ell \neq 2, 3$, then the reduction map $W_w(N)_{/R} \to W_w(N)_{/F}$ is surjective.

Proof. The reduction map is a composition

\[ W_w(N)_{/R} \to W_w(N)_{/R} \otimes F \hookrightarrow W_w(N)_{/F}, \]

with the first map surjective and the second map injective. Therefore surjectivity reduces to the equality of the dimensions of the last two spaces as vector spaces over $F$.

Since $R$ is a DVR, $W_w(N)_{/R}$ is a free $R$-module so

\[ \dim W_w(N)_{/R} \otimes F = \text{rank} W_w(N)_{/R} = \dim W_w(N)_{/\mathbb{C}} = \dim H^1(\Gamma_0(N), V_w(\mathbb{C})), \]

where the second equality follows from the fact that $W_w(N)_{/\mathbb{Z}}$ is a sublattice of $W_w(N)_{/\mathbb{C}}$, and the third follows from $W_w(N)_{/\mathbb{C}} \simeq H^1(\Gamma_0(N), V_w(\mathbb{C}))$ and Poincaré duality over $\mathbb{C}$.

The hypothesis $\ell > w$ implies that the $\Gamma_1$-invariant pairing on $V(F)$ induced by the natural $\Gamma_1$-invariant pairing on $V_w$ is nondegenerate, so $V^*(F) \simeq V(F)$. By Poincaré duality [2, Lemma 1.4.3], it follows that

\[ \dim W_w(N)_{/F} = \dim_F H^1(\Gamma_0(N), V_w(F)) = \dim_F H^1(\Gamma_0(N), V_w(F)). \]

Lemma 4.1 shows that $\dim_F H^1(\Gamma_1, V_w(N)_{/F})$ is the same for all fields $F$ with $\text{char}(F) \neq 2, 3$, where $V_w(N)_{/F}$ is the induced module $\text{Ind}_{\Gamma_0(N)}^{\Gamma_1} V_w(\mathbb{F})$. Applying this to the residue field $F$ and to $\mathbb{C}$, and using the Shapiro lemma, we conclude from the last two displayed equations that $W_w(N)_{/R} \otimes F = W_w(N)_{/F}$, as they have the same dimension. □

4.2. Surjectivity of reduction on the $p$-new subspace. For $N = M_p$ with $p$ prime, in (2.9) we have defined

\[ W_w(N)_{/R}^{p\text{-new}} = \text{Ker} \left( \beta : W_w(N)_{/R} \to W_w(M)_{/R}^{2} \right) \]

where $\beta(P) = (\text{Tr}_M^N P, \text{Tr}_M^N P|_{\Theta}^N T_N)$. We recall that the operator $|_{\Theta}^N T_N$ on $W_w(N)$ corresponds to the Atkin-Lehner involution $W_N$ on $M_{w+2}(N)$ as in Section 2.2.
Let \( \Gamma_0(M)' := \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \Gamma_0(M) \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \) for \( p \nmid M \). We have \( \Gamma_0(M) \cap \Gamma_0(M)' = \Gamma_0(Mp) \), and we consider the sum of restriction maps:

\[
(4.2) \quad \alpha : H^1(\Gamma_0(M), V_w(F)) \oplus H^1(\Gamma_0(M)', V_w(F)) \to H^1(\Gamma_0(Mp), V_w(F)).
\]

We prove below that the maps \( \alpha \) and \( \beta \) are essentially Poincaré dual to each other.

**Proposition 4.3.** Let \( R \) be a discrete valuation ring with residue field \( F \) of characteristic \( \ell \). Let \( w \geq 0 \) be even, let \( N = Mp \) with \( p \) prime, \( p \nmid M \), and assume that \( \ell > w \), \( \ell \nmid 6 \). The following are equivalent:

(i) The reduction map \( W_w(N)_{/R}^{p-new} \to W_w(N)_{/F}^{p-new} \) is surjective.

(ii) The map \( \beta : W_w(N)_{/F} \to W_w(M)_{/F}^2 \) is surjective.

(iii) The map \( \alpha \) given by (4.2) is injective.

**Proof.** (i) \( \iff \) (ii): The space \( W_w(N)_{/R}^{p-new} \) is free over \( R \), of rank equal to the dimension of \( W_w(N)_{/\mathbb{C}}^{p-new} \). Since \( R \) is a DVR, this rank also equals the dimension of the reduction \( W_w(N)_{/R}^{p-new} \otimes F \), so (i) is equivalent to \( \dim W_w(N)_{/F}^{p-new} = \dim W_w(N)_{/\mathbb{C}}^{p-new} \). On the other hand (ii) is equivalent to

\[
\dim W_w(N)_{/F}^{p-new} = \dim W_w(N)_{/F} - 2 \dim W_w(M)_{/F} = \dim W_w(N)_{/\mathbb{C}} - 2 \dim W_w(M)_{/\mathbb{C}}
\]

where the second equality follows from Proposition 4.2. It remains to show that the latter difference equals \( \dim W_w(N)_{/\mathbb{C}}^{p-new} \), that is that the map \( \beta \) is surjective over \( \mathbb{C} \). For this we follow the proof of surjectivity given below, which works over \( \mathbb{C} \) with little change. Indeed the same proof shows that (ii) and (iii) are equivalent over \( \mathbb{C} \) as well, and the proof of (iii) over \( \mathbb{C} \) is the same as that of Proposition 4.6 below, but without needing Lemma 4.4 and Proposition 4.7. Instead, the fact that \( H^1(\Delta_M, V_w(\mathbb{C})) \) vanishes, where \( \Delta_M \) is the principal congruence subgroup of level \( M \) of \( \text{PSL}_2(\mathbb{Z}/p[M]) \), is a consequence of Cor. 2 to Thm. 5 in [30].

(ii) \( \iff \) (iii): Since the residue field \( F \) and \( w \) are fixed, we write \( V = V_w(F) \). We have \( W_w(N)_{/F} \simeq H^1(\Gamma_0(N), V) \), and \( \text{Tr}^M_{N} : W_w(N)_{/F} \to W_w(M)_{/F} \) corresponds to the corestriction map on the compactly supported cohomology groups, while the map \( P \to P_{/\mathbb{C}} \) on \( W_w(N)_{/F} \) corresponds to the Atkin-Lehner operator \( [\Theta_N] \) on \( H^1(\Gamma_0(N), V) \). Therefore the first part in the diagram below is commutative.

\[
\begin{array}{ccc}
W_w(N)_{/F} & \xrightarrow{\sim} & H^1(\Gamma_0(N), V) \\
\beta & & \downarrow \text{cor, cor} \circ [\Theta_N] \\
W_w(M)_{/F}^2 & \xrightarrow{\sim} & H^1(\Gamma_0(M), V)^2
\end{array}
\]

\[
\begin{array}{ccc}
H^1(\Gamma_0(N), V) & \xrightarrow{\times} & H^1(\Gamma_0(M), V)^2 \\
\downarrow \text{res} + [\Theta_N] \circ \text{res} & & \\
F & & F
\end{array}
\]

The second part is given by Poincaré duality, taking into account that \( V \simeq V^* \) since \( \ell > w \). For \( \varphi \in H^1(\Gamma_0(N), V) \), \( \psi \in H^1(\Gamma_0(M), V) \) and \( \varphi' \in H^1(\Gamma_0(N), V) \) we have [18, Sec. 6.3]

\[
\langle \text{cor} \varphi, \psi \rangle_M = \langle \varphi, \text{res} \psi \rangle_N, \quad \langle \varphi | [\Theta_N], \varphi' \rangle = \langle \varphi, \varphi' | [\Theta_N] \rangle.
\]

Since Poincaré duality is a perfect pairing, it follows that \( \beta \) is surjective if and only if the rightmost map is injective.
Let \( c_p : H^1(\Gamma_0(M), V) \to H^1(\Gamma_0(M'), V) \) be conjugation by \( \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \). We easily check that the following diagram commutes

\[
\begin{array}{ccc}
H^1(\Gamma_0(N), V) & \xrightarrow{[\Theta_N]} & H^1(\Gamma_0(N), V) \\
\text{res} & & \text{res} \\
H^1(\Gamma_0(M), V) & \xrightarrow{[\Theta_M]} & H^1(\Gamma_0(M), V) & \xrightarrow{c_p} & H^1(\Gamma_0(M'), V)
\end{array}
\]

which shows that the rightmost map in the diagram differs from \( \alpha \) only by the isomorphism \( c_p \circ [\Theta_M] \) in the second factor, so rightmost map is injective if and only if \( \alpha \) is injective. \( \square \)

We are reduced to proving the injectivity of \( \alpha \), for which we use the ingredients of Ihara’s lemma [19, Lemma 3.2]. Our proof is inspired by the proof of similar statements for parabolic cohomology given in [29, 10]. First we need an easy lemma.

**Lemma 4.4.** Let \( w \geq 0 \), and consider the \( \text{SL}_2(\mathbb{Z}) \)-module \( V_w(F) \), with \( F \) a field of characteristic \( \ell > w + 1 \). Let \( u = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) with \( \ell \nmid a \). We have:

(i) \( \text{Im}(1 - u) = V_{w-1}(F) \) (setting \( V_{-1}(F) = \{0\} \)), and \( \text{Ker}(1 - u) = V_0(F) \);

(ii) \( \text{Ker}(N) = V_w(F) \), where \( N := 1 + u + \ldots + u^{\ell-1} \in \mathbb{Z}[\text{SL}_2(\mathbb{Z})] \) acts by linearity on \( V_w(F) \).

**Proof.** (i) The matrix of \( 1 - u \) in the basis \( 1, X, \ldots, X^w \) is upper triangular, with 0’s on the diagonal and elements \( \binom{n}{i}a^i \) with \( w \geq n \geq i \) above the diagonal, which are invertible in \( F \) since \( \ell > w \). It follows that \( \text{Ker}(1 - u) = F \), and since \( \text{Im}(1 - u) \) is contained in \( V_{w-1}(F) \) it must be the entire subspace.

(ii) Since \( u^\ell \) acts as identity on \( V_w(F) \), we have \( \text{Im}(1 - u) \subset \text{Ker}(N) \). By (i) we only have to check that \( X^w \in \text{Ker}(N) \), namely that the polynomial

\[
Q_w(X) = X^w + (X + 1)^w + \ldots + (X + \ell - 1)^w
\]

is identically 0 in \( \mathbb{F}_\ell[X] \). We prove this by induction on \( w \). For \( w = 0 \) the statement is clear, and assuming it true for \( w - 1 \geq 0 \) taking derivatives we have \( Q'_w = wQ_{w-1} = 0 \). Therefore \( Q_w \) is constant and we only have to prove that its constant term vanishes, which we leave as an exercise. Note that if \( \ell = w + 1 \) we have \( Q_w(0) = -1 \). \( \square \)

**Remark 4.5.** For \( \ell = w + 1 \), part (ii) in the lemma is no longer true (from the proof we see that \( \text{Ker}(N) = V_{w-1}(F) \), \( \text{Im}(N) = V_0(F) \) in this case). For this reason, the case \( \ell = w + 1 \) is not included in the next proposition and in Theorem 3.

**Proposition 4.6.** Let \( w \geq 0 \) be even, let \( p \nmid M \) be prime and let \( V = V_w(F) \) with \( F \) a field of characteristic \( \ell > w + 3 \). Assume \( \ell \nmid pM \), and if \( w = 0 \) assume also that \( \ell \nmid \varphi(M) \). Then the restriction map

\[
\alpha : H^1(\Gamma_0(M), V) \oplus H^1(\Gamma_0(M'), V) \to H^1(\Gamma_0(Mp), V)
\]

is injective (if \( \ell = w + 3 > 3 \), its kernel is one-dimensional).
Proof. Let $\Gamma(M)$ be the principal congruence subgroup of level $M$ of $\text{PSL}_2(\mathbb{Z})$, and $\Gamma(M)' := (p^M0_{1\times 1})^{-1} \Gamma(M) \left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Their intersection is $\Gamma(M) \cap \Gamma_0(p) \subset \Gamma_0(Mp)$, and we have a commutative diagram of restriction maps\footnote{The cohomology groups do not change upon replacing $\Gamma_0(M)$ by its projectivization $\Gamma_0(M)/\{\pm 1\}$, as $-1$ acts trivially on $V$.}

$$
\begin{array}{c}
H^1(\Gamma_0(M), V) \oplus H^1(\Gamma_0(M)', V) \\
\alpha \downarrow \text{res} \downarrow \text{res} \\
H^1(\Gamma(M), V) \oplus H^1(\Gamma(M)', V) \\
\gamma \downarrow \text{res} \\
H^1(\Gamma(M) \cap \Gamma(M)', V)
\end{array}
$$

The first vertical restriction is injective: using the inflation-restriction exact sequence it is enough to show that $H^1(\Gamma_0(M)/\Gamma(M), V^\Gamma(M)) = 0$. When $w > 0$, the space of invariants $V^\Gamma(M)$ is trivial, since the invariants under $(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix})$ are the constant polynomials, by Lemma 4.4, while the only constant invariant under $(\begin{smallmatrix} 1 & 0 \\ M & 1 \end{smallmatrix})$ is 0. If $w = 0$, we use that the quotient $\Gamma_0(0)/\Gamma(M) \simeq (\mathbb{Z}/M\mathbb{Z})^* \times (\mathbb{Z}/M\mathbb{Z})$ has order $M\varphi(M)$, which is coprime to $\ell$ by assumption, so the cohomology group vanishes (it is here that we use the extra assumption $\ell \mid \varphi(M)$ when $w = 0$).

Therefore to show that $\alpha$ is injective it is enough to show that $\gamma$ is injective. For that we use the following two ingredients of Ihara’s lemma \cite[Lemma 3.2]{Ihara1968}. Let $\Delta_M$ be the principal congruence subgroup of level $M$ of $\text{PSL}_2(\mathbb{Z}[1/p])$.

(1) The group $\Delta_M$ is the free product of $\Gamma(M)$ and $\Gamma(M)'$ with amalgamated subgroup $\Gamma(M) \cap \Gamma(M)'$;

(2) The group $\Delta_M$ is the normal closure of $(\begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix})$ in $\Delta_1$.

By (1), the Mayer-Vietoris exact sequence gives:

$$
\cdots \to H^1(\Delta_M, V) \to H^1(\Gamma(M), V) \oplus H^1(\Gamma(M)', V) \xrightarrow{\gamma} H^1(\Gamma(M) \cap \Gamma(M)', V) \to \cdots,
$$

so $\text{Ker} \gamma = H^1(\Delta_M, V)$. The inflation-restriction exact sequence for the normal subgroup $\Delta_{\ell M} \subset \Delta_M$ gives:

$$
0 \to H^1(\text{PSL}_2(\mathbb{F}_\ell), V) \xrightarrow{\text{inf}} H^1(\Delta_M, V) \xrightarrow{\text{res}} H^1(\Delta_{\ell M}, V) \xrightarrow{\Delta_M/\Delta_{\ell M}}. 
$$

We have $V^{\Delta_{\ell M}} = V$, and we will see in Proposition 4.7 below that $H^1(\text{PSL}_2(\mathbb{F}_\ell), V)$ vanishes under the assumption $\ell \neq w + 3$. Therefore to show $H^1(\Delta_M, V) = 0$ it suffices to prove that the restriction map is identically 0.

For this, we use (2), that is the fact that $\Delta_{\ell M}$ is generated by elements of the form $g(\begin{smallmatrix} 1 & \ell M \\ 0 & 1 \end{smallmatrix})^{-1}, g \in \Delta_1$. Let $g(\begin{smallmatrix} 1 & \ell M \\ 0 & 1 \end{smallmatrix})^{-1} = v^\ell$, with $v = gug^{-1} \in \Delta_M$ for $u = (\begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix})$. For any cocycle $\varphi \in Z^1(\Delta_M, V)$, we have

$$
\varphi(v^\ell) = \varphi(v)(1 + v + \ldots + v^{\ell - 1}) = (\varphi(v)|g|(1 + u + \ldots + u^{\ell - 1})|g^{-1} = 0 
$$

by Lemma 4.4 (ii) (here we use $\ell \neq w + 1$). We conclude $\text{res} \varphi = 0$. \hfill \square

It remains to prove the vanishing of a finite cohomology group, which was essentially proved in \cite[Theorem 1.5.3]{Ihara1968}. Since the proof there was only sketched, and since we need a slightly more general ground field, we fill in the details below. The entire structure of the cohomology ring is also determined in \cite{Serre1972}.

**Proposition 4.7.** Let $w \geqslant 0$ be even, and let $F$ be a field of characteristic $\ell > w$, $\ell > 3$. Then the cohomology group $H^1(\text{SL}_2(\mathbb{F}_\ell), V_w(F))$ vanishes, unless $\ell = w + 3$ when it is one-dimensional.
Proof. We set $\Gamma = \mathrm{SL}_2(\mathbb{F}_p)$, $V = V_w(F)$, and consider more generally $H^n(\Gamma, V)$ for $n > 0$. Let $B \subset \Gamma$ be the Borel subgroup of upper triangular matrices, so that $|B| = \ell(\ell - 1)$. The composition $\text{cor} \circ \text{res} : H^n(\Gamma, V) \to H^n(B, V) \to H^n(\Gamma, V)$ of the restriction and transfer maps equals multiplication by the index $[\Gamma : B] = \ell + 1$ [5, Ch. III, Prop. 9.5], which is an isomorphism of vector spaces over $F$ as the index is coprime to $\ell$. The composition $\text{res} \circ \text{cor}$ also equals multiplication by $[\Gamma : B]$, by the Cartan-Eilenberg stability criterion [6, Ch. XII Prop. 9.4]. To apply the criterion, we have to check that $H^n(B, V)$ consists of stable cohomology classes, namely for all $x \in \Gamma$ we have a commutative diagram

$$
\begin{array}{ccc}
H^n(B, V) & \xrightarrow{\cong} & H^n(xBx^{-1}, V) \\
{\text{res}} & & {\text{res}} \\
H^n(B \cap xBx^{-1}, V) & \rightarrow & H^n(xBx^{-1}, V)
\end{array}
$$

with the horizontal isomorphism being conjugation by $x$. To prove the commutativity, note that $B \cap xBx^{-1}$ is either $B$ (when $x \in B$), or the diagonal subgroup $T$ (when $x \notin B$).

In the first case the statement is trivial, while in the second it follows from the fact that $H^n(T, V) = 0$, as $T$ is cyclic of order $\ell - 1$ coprime to $\ell$. We conclude that $H^n(\Gamma, V) \cong H^n(B, V)$.

Let $U = \{(\frac{1}{1}, 1) \in \Gamma\}$, which is normal in $B$ with $B/U \cong T$, the diagonal subgroup. The inflation-restriction exact sequence together with $H^n(T, V) = 0$ for $n > 0$ implies that $H^n(B, V) \cong H^n(U, V)^{B/U}$.

Since $U$ is cyclic generated by $u = (\frac{1}{1}, 1)$, its cohomology $H^n(U, V)$ equals $\text{Ker} N / \text{Im} (1 - u)$ if $n$ is odd and $\text{Ker} (1 - u) / \text{Im} N$ if $n > 0$ is even [5, p. 58], where $N = 1 + u + \ldots + u^{\ell - 1} : V \to V$ is the norm map. By Lemma 4.4 we obtain $H^n(U, V) \cong F$ for all $n \geq 0$ if $\ell \neq w + 1$, and $H^n(U, V) = 0$ if $\ell = w + 1$ (see Remark 4.5).

To compute the invariants under $B/U$ assume $n = 1$, and let $\varphi : U \to V$ be the generator of $H^1(U, V)$ with $\varphi(u) = X^w \notin \text{Im}(1 - u)$. The group $B/U \cong T$ acts on cocycles by $\varphi(g(n)) = \varphi(gmg^{-1})|_{-w}g$ for $g \in B$, $n \in U$, so the class of $\varphi$ is invariant under $T$ if and only if $\varphi(u) - \varphi(tut^{-1})|_{-w}t \in \text{Im}(1 - u) = V_{w - 1}$ for $t = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T$ (i.e. $\varphi - \varphi t$ is a coboundary). Since $\varphi(u) = X^w$, this happens if and only if $a^{w + 2} = 1$ for all $a \in \mathbb{F}_p^*$, i.e., if and only if $\ell - 1 \mid w + 2$. Since $\ell > w$, $\ell \geq 5$, we conclude that $H^1(U, V)^{B/U} = 0$ unless $\ell = w + 3$, when it is one dimensional. $\square$

5. Numerical examples

For the examples in this section, we computed spaces of period polynomials and individual eigenforms using the software MAGMA [4]. More details are given in [27, Sec. 5.5].

Example 5.1. We expect Theorem 1 to hold for $\ell = k + 1$ as well, and we verified the congruence in many such cases. For example, let $p = 19$, $k = 6$, $\ell = k + 1 = 7$. Although $\ell = k + 1$ divides once the denominator of $B_k$ (which always holds when $k + 1$ is prime by the von Staudt-Clausen Theorem), we have $\ell | p^{k/2} + 1$, so $\ell$ divides the numerator of $\frac{B_k}{2}(p^{k/2} + 1)$. We find a congruence modulo $\ell$ between $E_k^{(+)}$ and the newform $f \in S_k^{(+)}(p)$ with $q$-expansion:

$$f = q - 2q^2 - q^3 - 28q^4 - 24q^5 + 2q^7 - 167q^8 + \ldots - 361q^{19} + \ldots .$$

Note that in order to prove this congruence, and the ones below, it is enough to check that it holds for coefficients of prime index up to the Sturm bound $k(p + 1)/12$. 
Example 5.2. One may ask whether the congruence in Theorem 1 comes from a congruence between the period polynomials $P^+(f)$ and $P^+(E_{k,p}(\varepsilon))$ of the forms in the theorem (both normalized to have constant term 1 at the identity coset). Such a congruence would imply the congruence of Hecke eigenvalues, and for level 1 it was shown to hold by Manin [24] in the cases when $S_k(1)$ is one dimensional. In higher level, this congruence often holds (see the next example), but the following example shows that it can also fail. For $p = 5$, $k = 40$, $\ell = 71$, $\varepsilon = -1$, we have $\ell | p^{k/2} + \varepsilon$ and the congruence (1.5) holds for some $f \in S_k^{(\varepsilon)}(p)$, but $P^+(f)(I) \not\equiv P^+(E_{k,p}(\varepsilon))(I) \pmod{\mathbb{Z}}$. This illustrates the fact that the Deligne-Serre lifting lemma guarantees the lift of systems of eigenvalues, but not of eigenvectors.

Example 5.3. Let $M = 7$, $k = 6$, and let $g \in S_k(M)$ be the newform

$$g = q - 10q^2 - 14q^3 + 68q^4 - 56q^5 + 140q^6 - 49q^7 + \ldots + 1824q^{23} + \ldots .$$

The cosets $A = \Gamma_0(M) \backslash \{ x : y \} / \mathbb{Z}$ can be identified with points $(x : y) \in \mathbb{P}^1(\mathbb{Z}/M\mathbb{Z})$, and setting $P = P^+(g)$, we can use the relations $P|\delta = P$, $P|S = -P$ to express all 8 components of $P$ in terms of

$$P((0 : 1)) = -49X^4 + 1, \quad P((1 : 2)) = 80X^4 - \frac{43}{2}X^3 - \frac{129}{2}X^2 - 86X + 6;$$

$$P((1 : 1)) = -49X^4 + 49, \quad P((1 : 3)) = -6X^4 - 86X^3 + \frac{129}{2}X^2 - \frac{43}{2}X - 80.$$

Since 43 divides all the coefficients except for those of $X^0$ and $X^4$, this illustrates the previous comment, namely we have $P^+(g) \equiv P^+(E_{k,M}^{(+1)}) \pmod{43}$. This implies directly that $g \equiv E_{k,M}^{(+1)} \pmod{43}$, and indeed we have $43|7^3 + 1$, so the congruence follows from Theorem 1.

To apply Theorem 2, we note that $\text{Den} P^+(g) = 2$ in (1.6), so this condition poses no restriction. Taking $p = 2$, we find $11|\lambda_p - p^{k/2-1}(p+1)$, and since $11 \nmid p^{k/2-1} - 1 = 3$ we deduce from Theorem 2 that there exists a congruence between $g_p^{(-1)}$ and a Hecke eigenform $f \in S_0(14)$ which is 2-new and has eigenvalue $-1$ for $W_2$. In fact, we find a newform $f$ of level 14 (as predicted by the conjecture in the introduction) with $q$-expansion

$$f = q + 4q^2 + 8q^3 + 16q^4 + 10q^5 + 32q^6 - 49q^7 + \ldots + 2000q^{23} + \ldots .$$

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Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail: radu.gaba@imar.ro, alexandru.popa@imar.ro