Some spaces of polynomial knots

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ABSTRACT
In this paper we study the topology of three different kind of spaces associated to
polynomial knots of degree at most $d$, for $d \geq 2$. We denote these spaces by $O_d$, $P_d$ and $Q_d$. For $d \geq 3$, we show that the spaces $O_d$ and $P_d$ are path connected and the space $Q_d$ has homotopy type of $S^2$. Considering the space $P = \bigcup_{d \geq 2} O_d$ of all polynomial knots with the inductive limit topology, we prove that it too has the same homotopy type as $S^2$. We also show that the number of path components of the space $Q_d$, for $d \geq 2$, are in multiple of eight. Furthermore, we prove that the path components of the space $Q_d$ are contractible.

Keywords: Polynomial knot; polynomial representation; spaces of polynomial knots; polynomial isotopy; path equivalence.

Mathematics Subject Classification 2010: 14P25, 57M25, 57M27, 57R40, 57R52

1. Introduction

Parameterizing knots has been useful in estimating some important knot invariants
such as bridge number [4], superbridge number [13] and geometric degree [9]. Some
of the interesting parameterizations are Fourier knots [11], polygonal knots [6] and
polynomial knots [19]. The first two of them provide classical knots, whereas the
polynomial parametrization gives long knots. In each parametrization there is a
positive integer $d$ associated to it. For Fourier knots and polynomial knots it is its
degree and for polygonal knots it is its edge number. For each parametrization, one
can study the space of all parametrized knots for a fixed $d$. Once we fix $d$, there
will be only finitely many knots that can be parametrized with this $d$. However,
in the space of parameterizations, one could study the topology and try to see if
two parametrized knots in this space belong to the same path components or not.
In [6] Calvo studied the spaces of polygonal knots. We aim to study the topology
of some spaces of polynomial knots. In [18] Vassiliev pointed out that the space of
long knots can be approximated by the space of embeddings from $\mathbb{R}$ to $\mathbb{R}^3$ given by
$t \mapsto (t^d + a_{d-1}t^{d-1} + \cdots + a_1 t, t^d + b_{d-1}t^{d-1} + \cdots + b_1 t, t^d + c_{d-1}t^{d-1} + \cdots + c_1 t)$
for $d \geq 1$. He defined (see [19]) the space $\mathcal{V}_d$ to be the interior of the set of all
polynomial embeddings in the space $\mathcal{W}_d$ of all polynomial maps of the type
$t \mapsto (t^d + a_{d-1}t^{d-1} + \cdots + a_1 t, t^d + b_{d-1}t^{d-1} + \cdots + b_1 t, t^d + c_{d-1}t^{d-1} + \cdots + c_1 t)$. It
was noted that if two polynomial knots belong to the same path component of $V_d$ then they represent the same knot type. It is felt that the converse of this may not be true. However, no counter example is known. Regarding the topology of the space $V_d$, he showed that the space $V_3$ is contractible and the space $V_4$ is homology equivalent to $S^1$.

Later, Durfee and O’Shea [1] introduced a different space $K_d$ which is the space of all polynomial knots $t \mapsto (a_0 + a_1 t + \cdots + a_d t^d, b_0 + b_1 t + \cdots + b_d t^d, c_0 + c_1 t + \cdots + c_d t^d)$ such that $|a_d| + |b_d| + |c_d| \neq 0$. If two polynomial knots are path equivalent in $K_d$, then they are topologically equivalent. Thus a space $K_d$ will have at least as many path components as many knot types can be represented in $K_d$. In [14] a group of undergraduate students proved that $K_3$ has at least 3 path components corresponding to the unknot, the right hand trefoil and the left hand trefoil respectively. Beyond this the topology of these spaces is not understood. Also, it is not clear whether two topologically equivalent knots in this space necessarily belong to the same path component or not. Composing a polynomial knot with a simple polynomial automorphism of $\mathbb{R}^3$ can reduce the degree of two of the components and the resulting polynomial knot will be topologically equivalent to the earlier one. Keeping this view in mind, we note that any polynomial knot given by $t \mapsto (a_0 + a_1 t + \cdots + a_d t^d, b_0 + b_1 t + \cdots + b_d t^d, c_0 + c_1 t + \cdots + c_d t^d)$, for $d \geq 2$, is topologically equivalent to a polynomial knot $t \mapsto (f(t), g(t), h(t))$ with $\deg(f) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) \leq d$. This motivated us to study the topology of three interesting spaces namely: (1) the space $O_d$ of all polynomial knots $t \mapsto (f(t), g(t), h(t))$ with $\deg(f) \leq d - 2$, $\deg(g) \leq d - 1$ and $\deg(h) \leq d$, (2) the space $P_d$ of all polynomial knots $t \mapsto (u(t), v(t), w(t))$ with $\deg(u) < \deg(v) < \deg(w) \leq d$, and (3) the space $Q_d$ of all polynomial knots $t \mapsto (x(t), y(t), z(t))$ with $\deg(x) = d - 2$, $\deg(y) = d - 1$ and $\deg(z) = d$. Using the theory of real semi algebraic sets (see [12] and [15]) we ensure that these spaces must have finitely many path components. We show that for $d \geq 3$ the spaces $O_d$ are of the same homotopy type as $S^2$. We also prove that the spaces $P_d$ are path connected where as the spaces $Q_d$ are not path connected. For the space $Q_d$, for $d \geq 2$, we show that if two polynomial knots lie in the same path component then they are topologically equivalent and we provide a counter example that the converse is not true. Furthermore, we show that the space $P$ of all polynomial knots, with the inductive limit topology coming from the stratification $P = \bigcup_{d \geq 2} O_d$, also has the same homotopy type as $S^2$.

This paper is organized as follows: Section 2 is about definitions and known results. We divide it in three subsections. In 2.1 and 2.2, we provide the basic terminologies and some known results related to knots and in particular polynomial knots that will be required in this paper. In 2.3, we discuss real semi algebraic sets and mention some important results from real semialgebraic geometry. In Section 3, we introduce our spaces $O_d, P_d$ and $Q_d$ for $d \geq 2$ and check their basic topological properties. At the end of Section 3, we show that these spaces are homeomorphic.
of polynomial knots $Q_d$, for $d \geq 3$, has homotopy type of $S^2$.

Corollary 4.18 The space $P$ of all polynomial knots has homotopy type of $S^2$.

Proposition 4.19 Every path component of the space $Q_d$, for $d \geq 2$, is homeomorphic to an open ball in $\mathbb{R}^{3d}$.

2. Definitions and Known Results

2.1. Knots and their isotopies

Definition 2.1. A tame knot is a continuous embedding $\kappa : S^1 \to S^3$ which extends to a continuous embedding $\bar{\kappa} : S^1 \times B^2 \to S^3$ of a tubular neighborhood of $S^1$ in $\mathbb{R}^3$, where $B^2$ is the open unit disc in $\mathbb{R}^2$.

It is known that every piecewise $C^1$ embedding of $S^1$ in $S^3$ is a tame knot (see [1, Lemma 2] and [16, Appendix I]). In particular, all smooth embeddings (that is, $C^\infty$ embeddings) of $S^1$ in $S^3$ are tame knots.

Definition 2.2. A long knot is a proper smooth embedding $\phi : \mathbb{R} \to \mathbb{R}^3$ such that the map $t \mapsto \|\phi(t)\|$ is strictly monotone outside some closed interval of the real line.

Definition 2.3. An isotopy (string isotopy) of tame knots is a continuous map $F : I \times S^1 \to S^3$ such that for each $s \in I$, the map $F_s = F(s, \cdot)$ is a tame knot.

Definition 2.4. Two tame knots $\tau, \sigma : S^1 \to S^3$ are string isotopic if there is a string isotopy $F : I \times S^1 \to S^3$ of tame knots such that $F_0 = \tau$ and $F_1 = \sigma$.

Definition 2.5. A homeotopy (respectively, diffeotopy) of the ambient space $S^3$ is a continuous map $H : I \times S^3 \to S^3$ such that:

1. for each $s \in I$, the map $H_s = H(s, \cdot)$ is a homeomorphism (respectively, diffeomorphism) of $S^3$, and
2. the map $H_0 = H(0, \cdot)$ is the identity map of $S^3$.  


Note that self-homeomorphisms and self-diffeomorphisms are automorphisms in continuous and smooth categories respectively, and homeotopies and diffeotopies are ambient isotopies in the corresponding categories.

**Definition 2.6.** Two tame knots \( \phi, \psi : S^1 \rightarrow S^3 \) are said to be ambient isotopic if there is a homeotopy \( H : I \times S^3 \rightarrow S^3 \) of the ambient space \( S^3 \) such that \( \psi = H_1 \circ \phi \).

Similar terms as in Definitions 2.3 to 2.6 can be defined for the category of smooth knots \( S^1 \hookrightarrow S^3 \) and also for the category long knots \( \mathbb{R} \hookrightarrow \mathbb{R}^3 \). In these categories, the automorphisms of the spaces are self-diffeomorphisms instead of self-homeomorphisms. Also, for these categories, string isotopies are isotopies of knots in the same category, and ambient isotopies are diffeotopies instead of homeotopies.

Every long knot \( \phi : \mathbb{R} \rightarrow \mathbb{R}^3 \) can be extended to a unique embedding \( \tilde{\phi} : S^1 \rightarrow S^3 \) through the inverse of the stereographic projection from the north pole of \( S^3 \). On the other hand, every ambient isotopy class of tame knots contains a smooth knot \( \psi : S^1 \rightarrow S^3 \) which fixes the north poles and has nonzero derivative at the north pole. The restriction \( \hat{\psi} : \mathbb{R} \rightarrow \mathbb{R}^3 \) of \( \psi \) is a long knot. Note that \( \hat{\tilde{\phi}} \simeq \phi \) and \( \hat{\tilde{\psi}} \simeq \psi \), where ‘\( \simeq \)’ denotes that one knot is ambient isotopic to the other. This shows that there is bijection between the ambient isotopy classes of tame knots and the ambient isotopy classes of long knots.

### 2.2. Polynomial knots

**Definition 2.7.** A polynomial map in \( \mathbb{R}^3 \) is a map \( \phi : \mathbb{R} \rightarrow \mathbb{R}^3 \) whose component functions are real polynomials.

**Definition 2.8.** A polynomial knot is a polynomial map \( \phi : \mathbb{R} \rightarrow \mathbb{R}^3 \) which is a smooth embedding; that is, it does not have multiple and critical points.

**Definition 2.9.** A polynomial knot \( \phi \) is said to represent a knot-type \( [\kappa] \) (that is, an ambient isotopy class of \( \kappa \)) if the extended knot \( \tilde{\phi} : S^1 \rightarrow S^3 \) is ambient isotopic to \( \kappa \). In this case, the knot \( \phi \) is a polynomial representation of the knot-type \( [\kappa] \).

It was proved that every long knot is ambient isotopic to some polynomial knot \([2]\); therefore, every knot-type is represented by some polynomial knot. In other words, every knot-type has a polynomial representation.

If a knot-type \([\kappa]\) of a cheiral knot \( \kappa \) is represented by a polynomial knot \( t \mapsto (f(t), g(t), h(t)) \), then the knot \( t \mapsto (f(t), g(t), -h(t)) \) represents the knot-type \([\kappa^*]\) of a mirror image of \( \kappa \). In general, for any affine transformation (a composition of an invertible linear transformation and a translation) \( S : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), the polynomial knot \( S \circ \tau \) represents either \([\kappa]\) or \([\kappa^*]\) depending on if \( S \) is orientation preserving or not. On the other hand, for an affine transformation \( T \) and a polynomial knot \( \sigma \), if the extended knot \( \tilde{\sigma} \) is achiral, then the polynomial knots \( \sigma \) and \( T \circ \sigma \) represent...
Some spaces of polynomial knots

Definition 2.10. A polynomial map \( t \mapsto (f(t), g(t), h(t)) \) is said to have degree sequence \((d_1, d_2, d_3)\) if \( \deg(f) = d_1 \), \( \deg(g) = d_2 \) and \( \deg(h) = d_3 \).

Definition 2.11. The degree of a polynomial map \( \phi : \mathbb{R} \to \mathbb{R}^3 \) is the maximum of degrees of its component polynomials.

Definition 2.12. A knot-type \([\kappa]\) is said to have polynomial degree \(d\), if it is the least positive integer such that there is a polynomial knot \( \phi \) of degree \(d\) representing the knot-type \([\kappa]\). In this case, the polynomial knot \( \phi \) is called a minimal polynomial representation of the knot-type \([\kappa]\).

For example, the polynomial degree of a trivial knot is 1 and \( t \mapsto (0, 0, t) \) is its one of the minimal polynomial representation.

Definition 2.13. A polynomial isotopy is a continuous map \( H : I \times \mathbb{R} \to \mathbb{R}^3 \) such that \( H_s = H(s, \cdot) \) is a polynomial knot for each \( s \in I \).

Definition 2.14. Two polynomial knots \( \phi \) and \( \psi \) are said to be polynomially isotopic if there is a polynomial isotopy \( H : I \times \mathbb{R} \to \mathbb{R}^3 \) such that \( H_0 = \phi \) and \( H_1 = \psi \).

Definition 2.15. A polynomial automorphism is a bijective map \( T : \mathbb{R}^n \to \mathbb{R}^n \) whose component functions are real polynomials in \( n \) variables and is such that its inverse is also of the same kind.

Note that a map \( S : \mathbb{R} \to \mathbb{R} \) is a polynomial automorphism \( \iff \) it is a linear polynomial. Also, it is easy to see that an affine transformation (a composition of an invertible linear transformation and a translation) of \( \mathbb{R}^n \) is a polynomial automorphism. Every polynomial automorphism \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism.

Regarding polynomial maps and the polynomial automorphisms of \( \mathbb{R} \) and \( \mathbb{R}^3 \) the following remark can easily be verified:

Remark 2.16. For any polynomial map \( \phi : \mathbb{R} \to \mathbb{R}^3 \) and any polynomial automorphisms \( S : \mathbb{R} \to \mathbb{R} \) and \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), we have the following:

1. The map \( \phi \) is injective \( \iff \) the map \( T \circ \phi \circ S \) is injective.
2. The map \( \phi \) is immersion \( \iff \) the map \( T \circ \phi \circ S \) is immersion.
3. The map \( \phi \) is embedding \( \iff \) the map \( T \circ \phi \circ S \) is embedding.

From this we can deduce the following remarks.

Remark 2.17. Let \( \alpha \) and \( \gamma \) be any real numbers and let \( \alpha \neq 0 \). Then for a polynomial embedding \( \phi : \mathbb{R} \to \mathbb{R}^3 \), the map \( \psi : \mathbb{R} \to \mathbb{R}^3 \) given by \( t \mapsto \phi(\alpha t + \gamma) \) is also a polynomial embedding.
Remark 2.18. Let \( \phi : \mathbb{R} \to \mathbb{R}^3 \) be a polynomial embedding, then for an affine transformation \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), the map \( T \circ \phi \) is also a polynomial embedding.

Let \( \phi : \mathbb{R} \to \mathbb{R}^3 \) be a polynomial embedding. Then for an orientation preserving polynomial automorphisms \( S : \mathbb{R} \to \mathbb{R} \) and \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) (that is, \( S \) is a linear polynomial with positive leading coefficient and \( \det \left( \frac{\partial T}{\partial x} (x) \right) > 0 \) for all \( x \in \mathbb{R}^3 \)), the polynomial embedding \( T \circ \phi \circ S \) is topologically equivalent to \( \phi \). In particular, the same is true if \( S \) and \( T \) are orientation preserving affine transformations.

We prove the following proposition which we will be using in this paper.

Proposition 2.19. Let \( \{ \alpha_s \mid s \in I \} \) be a family of polynomial knots depending continuously on parameter \( s \in I \). If the degree of \( \alpha_s \) is constant (independent of \( s \in I \)), then there exists \( r_0 \gg 0 \) such that for any \( s \in I \) and any \( |t| \geq r_0 \), the vector \( \alpha_s'(t) \in \mathbb{R}^3 \) intersects transversely to the sphere of radius \( |\alpha_s(t)| \) about the origin. Moreover, for any \( s \in I \), the angle between the vectors \( \alpha_s(t) \) and \( \alpha_s'(t) \) approaches \( 0 \) as \( t \to +\infty \) and it approaches \( \pi \) as \( t \to -\infty \).

Proof. For \( s \in I \), let \( \alpha_s \) be given by

\[
\alpha_s(t) = a_d(s)t^d + a_{d-1}(s)t^{d-1} + \cdots + a_1(s)t + a_0
\]

for \( t \in \mathbb{R} \), where \( a_i's \) are continuous functions from \( I \) to \( \mathbb{R}^3 \) and \( a_d(s) \neq 0 \) for all \( s \in I \). For \( s \in I \) and \( |t| > 0 \), the cosine of the angle \( \theta_s(t) \) between the vectors \( \alpha_s(t) \) and \( \alpha_s'(t) \) is

\[
\frac{\alpha_s(t)}{\|\alpha_s(t)\|} \cdot \frac{\alpha_s'(t)}{\|\alpha_s'(t)\|}
\]

which becomes

\[
\pm \frac{a_d(s) + \frac{1}{2} a_{d-1}(s) + \cdots + \frac{1}{d} a_0(s)}{\|a_d(s) + \frac{1}{2} a_{d-1}(s) + \cdots + \frac{1}{d} a_0(s)\|} \cdot \frac{d a_d(s) + \frac{d-1}{2} a_{d-1}(s) + \cdots + \frac{1}{d} a_1(s)}{\|d a_d(s) + \frac{d-1}{2} a_{d-1}(s) + \cdots + \frac{1}{d} a_1(s)\|} (2.1)
\]

accordingly as \( t \) is positive or negative. Thus, we get

\[
|\cos(\theta_s(t))| \geq \frac{|d\|a_d(s)\|^2 + \frac{1}{2} f_1(s) + \frac{1}{d} f_2(s) + \cdots + \frac{1}{2 d-1} f_{2d-1}(s)|}{d\|a_d(s)\|^2 + \frac{1}{2} h_1(s) + \frac{1}{d} h_2(s) + \cdots + \frac{1}{2 d-1} h_{2d-1}(s)}
\]

\[
= \frac{|d\|a_d(s)\|^2 - \frac{1}{2} f_1(s) + \frac{1}{d} f_2(s) + \cdots + \frac{1}{2 d-1} f_{2d-1}(s)|}{d\|a_d(s)\|^2 + \frac{1}{2} h_1(s) + \frac{1}{d} h_2(s) + \cdots + \frac{1}{2 d-1} h_{2d-1}(s)}, (2.2)
\]

where for each \( i \in \{1, 2, \ldots, 2d - 1\} \),

\[
f_i(s) = \sum_{j,k} N_{ijk} a_j(s) \cdot a_k(s) \quad \text{and} \quad h_i(s) = \sum_{j,k} M_{ijk} \|a_j(s)\| \|a_k(s)\|
\]
for $s \in I$. Note that $N_{ijk}$'s and $M_{ijk}$'s are some nonnegative integers. Since $a_d$ is continuous and $a_d(s) \neq 0$ for all $s \in I$, so there exist some positive real numbers $m$ and $M$ such that

$$m \leq \|a_d(s)\| \leq M$$  \hspace{1cm} (2.3)

for all $s \in I$. Also, since $f_i$'s and $h_i$'s are continuous and bounded real valued functions, so for some $r_0 \gg 0$, we have

$$\left| \frac{1}{t} f_1(s) + \frac{1}{t^2} f_2(s) + \cdots + \frac{1}{t^{2d-1}} f_{2d-1}(s) \right| \leq \frac{dm^2}{2} \quad \text{and} \quad (2.4)$$

$$\left| \frac{1}{|t|} h_1(s) + \frac{1}{|t|^2} h_2(s) + \cdots + \frac{1}{|t|^{2d-1}} h_{2d-1}(s) \right| \leq dM^2 \quad \text{for all} \quad t \geq r_0 \quad \text{and for all} \quad s \in I.$$  \hspace{1cm} (2.5)

Using Inequalities (2.3)-(2.5) in Expression (2.2) gives

$$|\cos(\theta_s(t))| \geq \frac{dm^2 - \frac{dm^2}{2}}{dM^2 + dM^2} = \frac{m^2}{4M^2} > 0$$

for all $t \geq r_0$ and for all $s \in I$. This proves the first statement. Also, for any $s \in I$, Exp. (2.1) approaches

$$\pm \frac{a_d(s) \cdot da_d(s)}{\|a_d(s)\| \|da_d(s)\|} = \pm 1$$

accordingly as $t \to \pm \infty$. In other words, the angle $\theta_s(t)$ between the vectors $\alpha_s(t)$ and $\alpha_s'(t)$ approaches 0 as $t \to +\infty$ and it approaches $\pi$ as $t \to -\infty$. \quad \Box

### 2.3. Real semialgebraic sets

Algebraic geometry is the study of algebraic sets which are the set of zeros of polynomial functions. When the polynomials are over the field of real numbers, the notion of algebraic sets are extended to a bigger class of sets known as semialgebraic sets. This includes the sets of points which are solutions of some inequalities satisfied by polynomial functions. More precisely, the semialgebraic sets are defined as follows:

**Definition 2.20.** Semialgebraic subsets of $\mathbb{R}^n$ form a smallest class $S_n$ of subsets of $\mathbb{R}^n$ such that:

1. If $P \in \mathbb{R}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n : P(x) = 0\}$ and $\{x \in \mathbb{R}^n : P(x) > 0\}$ are in $S_n$.
2. If $A \in S_n$ and $B \in S_n$, then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A$ are in $S_n$.

For example, a union of the upper half space $H = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ and the closed unit ball $C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$ is a semialgebraic subset of $\mathbb{R}^3$. The upper region $U = \{(x, y) \in \mathbb{R}^2 | (x - 3)(x + 3) \geq 0\}$ of the plane which is bounded by the parabola $(x - 3)(x + 3) = 0$ is a semialgebraic subset of $\mathbb{R}^2$. 

A detailed exposure to the study of real semialgebraic sets can be found in [12] and [15]. Some of the important results (proofs can be seen in [12] and [15]) related to the semialgebraic sets those will be used in this paper are stated below:

**Theorem 2.21 (Tarski-Seidenberg: Second Form).** Let $A$ be a semialgebraic subset of $\mathbb{R}^{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first $n$ coordinates. Then $\pi(A)$ is a semialgebraic subset of $\mathbb{R}^n$.

**Corollary 2.22.** If $A$ is a semialgebraic subset of $\mathbb{R}^{n+k}$, its image by the projection onto the space of the first $n$ coordinates is a semialgebraic subset of $\mathbb{R}^n$.

**Theorem 2.23.** Every semialgebraic set has only finitely many connected components and all the components are semialgebraic.

**Proposition 2.24.** A connected semialgebraic set is path connected.

### 3. Spaces of Polynomial Knots

Let us denote the set of all polynomial maps by $\mathcal{A}$ and the set of all polynomial knots by $\mathcal{P}$. It is easy to see that $\mathcal{P}$ is a proper subset of $\mathcal{A}$. For an integer $d \geq 2$, we define the following sets:

- $\mathcal{A}_d = \{(f, g, h) \in \mathcal{A} \mid \deg(f) \leq d-2, \deg(g) \leq d-1 \text{ and } \deg(h) \leq d\}$,
- $\mathcal{B}_d = \{(f, g, h) \in \mathcal{A} \mid \deg(f) < \deg(g) < \deg(h) \leq d\}$,
- $\mathcal{C}_d = \{(f, g, h) \in \mathcal{A} \mid \deg(f) = d-2, \deg(g) = d-1 \text{ and } \deg(h) = d\}$.

Also, let $\mathcal{O}_d = \mathcal{P} \cap \mathcal{A}_d$, $\mathcal{P}_d = \mathcal{P} \cap \mathcal{B}_d$ and $\mathcal{Q}_d = \mathcal{P} \cap \mathcal{C}_d$. The set $\mathcal{A}_d$ can be identified with the Euclidean space $\mathbb{R}^{3d}$. In fact, there is a natural bijection $\eta : \mathcal{A}_d \to \mathbb{R}^{3d}$ given by

$$ (f, g, h) \mapsto (a_0, a_1, \ldots, a_{d-2}, b_0, b_1, \ldots, b_{d-1}, c_0, c_1, \ldots, c_d), $$

where $a_i$, $b_i$, and $c_i$, for $i \in \{0, 1, \ldots, d\}$, are coefficients of $t^i$ in the polynomials $f$, $g$ and $h$ respectively. We have a metric $\rho$ on $\mathcal{A}_d$ which is given by

$$ \rho(\phi, \psi) = \xi(\eta(\phi), \eta(\psi)) \quad (3.1) $$

for $\phi, \psi \in \mathcal{A}_d$, where $\xi$ denotes the Euclidean distance in $\mathbb{R}^{3d}$. With this metric, the set $\mathcal{A}_d$ becomes a topological space. Obviously, the map $\eta$ is a homeomorphism (in fact, a diffeomorphism) between the topological spaces $\mathcal{A}_d$ and $\mathbb{R}^{3d}$. With respect to the subspace topology the sets $\mathcal{B}_d, \mathcal{C}_d, \mathcal{O}_d, \mathcal{P}_d$ and $\mathcal{Q}_d$ become subspaces of the space $\mathcal{A}_d$ and hence they can be identified with the subspaces of $\mathbb{R}^{3d}$. Note that the spaces $\mathcal{O}_d, \mathcal{P}_d$ and $\mathcal{Q}_d$ are the spaces of polynomial knots of degree at most $d$.

**Remark 3.1.** Using suitable invertible linear transformation, a polynomial knot $\phi$ of degree $d \geq 2$ can be transformed to a polynomial knot $\psi = (f, g, h)$ such that $\deg(f) \leq d-2$, $\deg(g) \leq d-1$ and $\deg(h) \leq d$. This shows that every knot-type can be represented by a polynomial knot belonging to the space $\mathcal{O}_d$ for some $d \geq 2$. 
Some spaces of polynomial knots

Fig. 1. Partial order in $\mathfrak{F}$

It is easy to check that $B_2 = C_2 = P_2 = Q_2$ and $C_3 = Q_3$. We have $C_d \not\subseteq C_{d+1}$ and $Q_d \not\subseteq Q_{d+1}$ for all $d \geq 2$. Moreover, for the collection $\mathfrak{F} = \{A, P, A_d, B_d, C_d, O_d, P_d, Q_d \mid d \geq 2\}$, with respect to the strict set inclusion as partial order, the Hesse diagram for $\mathfrak{F}$ as given in Fig. 1. Also, we have

$$A = \bigcup_{d\geq2} A_d \quad \text{and} \quad P = \bigcup_{d\geq2} O_d.$$ 

So these sets can be given the inductive limit topology; that is, a set $U \subseteq A$ is open in $A$ if and only if the set $U \cap A_d$ is open in $A_d$ for all $d \geq 2$, and a set $V \subseteq P$ is open in $P$ if and only if the set $V \cap O_d$ is open in $O_d$ for all $d \geq 2$.

**Proposition 3.2.** The space $C_d$, for $d \geq 2$, is open in the space $A_d$.

**Proof.** The set $\eta(C_2)$ is written as follows:

$$\eta(C_2) = \{ (a_0, b_0, b_1, c_0, c_1, c_2) \in \mathbb{R}^6 \mid b_1 c_2 \neq 0 \}.$$ 

Also, for $d \geq 3$, we have

$$\eta(C_d) = \{ (a_0, a_1, \ldots, a_{d-2}, b_0, b_1, \ldots, b_{d-1}, c_0, c_1, \ldots, c_d) \in \mathbb{R}^{3d} \mid a_{d-2} b_{d-1} c_d \neq 0 \}.$$ 

It is easy to see that, for $d \geq 2$, the set $\eta(C_d)$ is an open subset of $\eta(A_d) = \mathbb{R}^{3d}$. Since $\eta$ is homeomorphism, so the space $C_d$ is open in the space $A_d$.

**Remark 3.3.** The sets $B_2, P_2$ and $Q_2$ all are equal to $C_2$, and hence they are open in the space $A_2$. 

Lemma 3.4. For nonzero real numbers $a$ and $b$, and an even integer $n \geq 2$, we have the following:

(1) $0 < \frac{1}{a^n + a^{n-1}b + \cdots + b^n} \leq \max \left\{ \frac{1}{a^n}, \frac{1}{b^n} \right\}$.

(2) $0 < \frac{a^k + a^{k-1}b + \cdots + b^k}{a^n + a^{n-1}b + \cdots + b^n} \leq \max \left\{ \frac{1}{a^n}, \frac{1}{b^n} \right\}$, for an even integer $k \in \{1, 2, \ldots, n\}$.

(3) $\left| \frac{a^n + a^{n-1}b + \cdots + b^n}{a^n + a^{n-1}b + \cdots + b^n} \right| < \min \left\{ \left| \frac{1}{a^n} \right|, \left| \frac{1}{b^n} \right| \right\}$, for an odd integer $k \in \{1, 2, \ldots, n\}$.

Proof. Without loss of generality we assume that $|a| \geq |b|$. (1) Since $|a| \geq |b|$, so $a + b \geq 0$ if $a > 0$ and $a + b \leq 0$ if $a < 0$. Also, an odd power of $a$ and an even power of $b$ occur in each term of the expression $y = a_n + a^{n-3}b^2 + \cdots + ab^{n-2}$, so $y \geq 0$ if $a > 0$ and $y \leq 0$ if $a < 0$. Hence in either case $(a + b)(a^{n-1} + a^{n-3}b^2 + \cdots + ab^{n-2}) = (a + b)y \geq 0$ and thus

$$a^n + a^{n-1}b + \cdots + b^n = (a + b)(a^{n-1} + a^{n-3}b^2 + \cdots + ab^{n-2} + b^n) \geq b^n > 0.$$ 

Therefore, we have $0 < \frac{1}{a^n + a^{n-1}b + \cdots + b^n} \leq \frac{1}{b^n} = \max \left\{ \frac{1}{a^n}, \frac{1}{b^n} \right\}$.

(2) Let an even integer $k \in \{1, 2, \ldots, n\}$ be given. For $k = n$, the inequality is trivially true, so assume $k < n$. Since $k$ is an even integer, so by the first part, $a^k + a^{k-1}b + \cdots + b^k > 0$. Let us consider the following expression:

$$x_k = \frac{a^n + a^{n-1}b + \cdots + b^n}{a^k + a^{k-1}b + \cdots + b^k} = \frac{a^n + a^{n-1}b + \cdots + a^{k+1}b^{n-k-1} + b^{n-k}}{a^k + a^{k-1}b + \cdots + b^k} = (a + b)(a^{n-1} + a^{n-3}b^2 + \cdots + a^{k+1}b^{n-k-2}) + b^n - \frac{b^{n-k}}{a^k + a^{k-1}b + \cdots + b^k} \geq b^n - \frac{b^{n-k}}{a^k + a^{k-1}b + \cdots + b^k}$$

An odd power of $a$ and an even power of $b$ occur in each term of the expression $y_k = a^{n-1} + a^{n-3}b^2 + \cdots + a^{k+1}b^{n-k-2}$, so $y_k > 0$ if $a > 0$ and $y_k < 0$ if $a < 0$. Recall that $a + b \geq 0$ if $a > 0$ and $a + b \leq 0$ if $a < 0$. Hence in either case, the numerator of the first term of Expression (3.4) is non-negative. Since $a^k + a^{k-1}b + \cdots + b^k > 0$, so the first term of Expression (3.4) is non-negative. Thus, we have $x_k \geq b^{n-k} > 0$ and hence $0 < \frac{1}{x_k} \leq \frac{1}{b^n} = \max \left\{ \frac{1}{a^n}, \frac{1}{b^n} \right\}$.

(3) Let an odd integer $k \in \{1, 2, \ldots, n\}$ be given. If $a^k + a^{k-1}b + \cdots + b^k = 0$, then the inequality is trivially true, so assume $a^k + a^{k-1}b + \cdots + b^k \neq 0$. Now consider the following expression:

$$x_k = \frac{a^n + a^{n-1}b + \cdots + b^n}{a^k + a^{k-1}b + \cdots + b^k} = \frac{a^n - k} + a^{n-k-1}b^{k+1} + a^{n-k-2}b^{k+2} + \cdots + b^n}{a^k + a^{k-1}b + \cdots + b^k} = \frac{a^n - k} + (a + b)(a^{n-k-2}b^{k+1} + a^{n-k-4}b^{k+3} + \cdots + ab^{n-2}) + b^n}{(a + b)(a^{k-1} + a^{k-3}b^2 + \cdots + b^{k-1})}$$

Since an odd power of $a$ and an even power of $b$ occur in each term of the expression $a^{n-k-2}b^{k+1} + a^{n-k-4}b^{k+3} + \cdots + ab^{n-2}$, so by the similar argument as in
the second part, \((a+b)(a^{n-k-2}b^{k+1}+a^{n-k-4}b^{k+3}+\ldots+ab^{n-2})\) is non-negative, and hence the numerator of the second term of Expression (3.7) is positive. Also, \(a\) and \(b\) occur with even powers in each term of the expression \(a^{k-1}+a^{k-3}b^2+\ldots+b^{k-1}\), so it is positive. Therefore, the sign of the second term of Expression (3.7) is same as the sign of \(a+b\) and hence it is same as the sign of \(a\). Thus, \(x_k > a^{n-k} > 0\) if \(a > 0\) and \(x_k < a^{n-k} < 0\) if \(a < 0\). This shows that \(|x_k| > |a^{n-k}| > 0\), so we have
\[
\frac{1}{|x_k|} < \frac{1}{|a^{n-k}|} = \min \left\{ \frac{1}{|a^{n-k}|}, \frac{1}{|b^{n-2}|} \right\}.
\]

**Lemma 3.5.** Let \(n \geq 2\) be a fixed even number, and for \(i \in \{0, 1, \ldots, n\}\), let \(\{\alpha_{ij}\}_{j=1}^{\infty}\) be a sequence of real numbers which converges to a real number \(\alpha_i\). Suppose \(\alpha_{ij} \not= 0\), and let \(\{s_j\}_{j=1}^{\infty}\) and \(\{t_j\}_{j=1}^{\infty}\) be sequences of real numbers such that \(s_j \to \pm \infty\) and \(t_j \to \pm \infty\) as \(j \to \infty\). Then
\[
\lim_{j \to \infty} (\alpha_0 + \alpha_1 (s_j + t_j) + \cdots + \alpha_n (s_j^n + s_j^{n-1}t_j + \cdots + t_j^n)) = \pm \infty
\]
depending on \(\alpha_n\) is positive or negative.

**Proof.** Since \(s_j \to \pm \infty\) and \(t_j \to \pm \infty\) as \(j \to \infty\), we can choose \(N \in \mathbb{N}\) such that \(s_j \neq 0\) and \(t_j \neq 0\) for all \(j \geq N\). For \(j \geq N\), let
\[
z_j = \frac{\alpha_0 + \alpha_1 (s_j + t_j) + \cdots + \alpha_n (s_j^n + s_j^{n-1}t_j + \cdots + t_j^n)}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n}.
\]
The expression for \(z_j\) can be rewritten as follows:
\[
z_j = \frac{\alpha_0}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n} + \alpha_1 \left( \frac{s_j + t_j}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n} \right) + \cdots + \alpha_n (\frac{s_j^{n-1} + s_j^{n-2}t_j + \cdots + t_j^{n-1}}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n}) + \alpha_n.
\]
By Lemma 3.4 for \(i \in \{1, 2, \ldots, n-1\}\) and \(j \geq N\), we have
\[
0 \leq \frac{1}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n} \leq \max \left\{ \frac{1}{s_j^n}, \frac{1}{t_j^n} \right\} \quad \text{and} \quad (3.10)
\]
\[
\left| \frac{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n} \right| \leq \max \left\{ \frac{1}{|s_j^n|}, \frac{1}{|t_j^n|} \right\}.
\]
(3.11)
For \(i \in \{0, 1, \ldots, n-1\}\), since \(\frac{1}{s_j^n} \to 0\) and \(\frac{1}{t_j^n} \to 0\) as \(j \to \infty\), so by Eqs. (3.10) and (3.11), we have
\[
\lim_{j \to \infty} \frac{1}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n} = 0 \quad \text{and} \quad \lim_{j \to \infty} \frac{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n}{s_j^n + s_j^{n-1}t_j + \cdots + t_j^n} = 0.
\]
Also, for \(i \in \{0, 1, \ldots, n\}\), since \(\alpha_{ij} \to \alpha_i\) as \(j \to \infty\), so by Eq. (3.9), \(z_j \to \alpha_n\) as \(j \to \infty\). By Lemma 3.4 we have \(s_j^n + s_j^{n-1}t_j + \cdots + t_j^n \geq \min \{s_j^n, t_j^n\}\) for all \(j \geq N\), so \(s_j^n + s_j^{n-1}t_j + \cdots + t_j^n \to \infty\) as \(j \to \infty\), and hence by Eq. (3.8), we get the required result. □
Lemma 3.6. Let \( \gamma(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_k t^k \) be a univariate polynomial and let \( \Gamma \) be a polynomial in two variables defined as \( \Gamma(s, t) = \alpha_1 + \alpha_2 (s + t) + \cdots + \alpha_k (s^{k-1} + t^{k-1}) \). Then a point \((s_0, t_0) \in \mathbb{R}^2\) is a zero of \( \Gamma \) if and only if either \( \gamma(s_0) = \gamma(t_0) \) or \( \gamma'(t_0) = 0 \) accordingly as \( s_0 \neq t_0 \) or \( s_0 = t_0 \).

Proof. It is easy to check the following:

\[
\Gamma(s, t) = \frac{\gamma(s) - \gamma(t)}{s - t} \quad \text{for all } s, t \in \mathbb{R} \text{ with } s \neq t, \quad \text{and} \quad (3.12)
\]

\[
\Gamma(t, t) = \gamma'(t) \quad \text{for all } t \in \mathbb{R}. \quad (3.13)
\]

We now prove the lemma as follows:

Suppose a point \((s_0, t_0) \in \mathbb{R}^2\) is a zero of \( \Gamma \). If \( s_0 \neq t_0 \), then by Eq. (3.12), \( \gamma(s_0) = \gamma(t_0) \). If \( s_0 = t_0 \), then by Eq. (3.13), \( \gamma'(t_0) = 0 \).

If \( \gamma(s_0) = \gamma(t_0) \) for some \((s_0, t_0) \in \mathbb{R}^2\) with \( s_0 \neq t_0 \), then by Eq. (3.12), the point \((s, t) = (s_0, t_0)\) is a zero of \( \Gamma \). If \( \gamma'(t_0) = 0 \) for some \( t_0 \in \mathbb{R} \), then by Eq. (3.13), the point \((s, t) = (t_0, t_0)\) is a zero of \( \Gamma \).

Lemma 3.7. Let \( f(t) = a_0 + a_1 t + \cdots + a_{d-2} t^{d-2} \), \( g(t) = b_0 + b_1 t + \cdots + b_{d-1} t^{d-1} \) and \( h(t) = c_0 + c_1 t + \cdots + c_d t^d \) be real polynomials. Then the polynomial map \( \phi : \mathbb{R} \to \mathbb{R}^3 \) given by \( t \mapsto (f(t), g(t), h(t)) \) is an embedding if and only if the polynomials

\[
F(s, t) = a_1 + a_2 (s + t) + \cdots + a_{d-2} (s^{d-3} + s^{d-4} t + \cdots + t^{d-3}),
\]

\[
G(s, t) = b_1 + b_2 (s + t) + \cdots + b_{d-1} (s^{d-2} + s^{d-3} t + \cdots + t^{d-2}) \quad \text{and}
\]

\[
H(s, t) = c_1 + c_2 (s + t) + \cdots + c_d (s^{d-1} + s^{d-2} t + \cdots + t^{d-1})
\]

do not have a common zero.

Proof. Suppose that \((s_0, t_0) \in \mathbb{R}^2\) is a common zero of the polynomials \( F, G \) and \( H \). If \( s_0 \neq t_0 \), then by Lemma 3.6, \( f(s_0) = f(t_0), g(s_0) = g(t_0) \) and \( h(s_0) = h(t_0) \), and hence \( \phi(s_0) = \phi(t_0) \). If \( s_0 = t_0 \), then again by Lemma 3.6, \( f'(t_0) = g'(t_0) = h'(t_0) = 0 \); that is, \( \phi'(t_0) = 0 \). Thus, in either case, \( \phi \) is not an embedding.

To prove the other way, assume that \( \phi \) is not an embedding. Then either \( \phi(s_0) = \phi(t_0) \) for some \( s_0 \neq t_0 \), or \( \phi'(u_0) = 0 \) for some \( u_0 \in \mathbb{R} \). In other words, either \( f(s_0) = f(t_0), g(s_0) = g(t_0) \) and \( h(s_0) = h(t_0) \) for some \( s_0 \neq t_0 \), or \( f'(u_0) = g'(u_0) = h'(u_0) = 0 \) for some \( u_0 \in \mathbb{R} \). In any case, the polynomials \( F, G \) and \( H \) have a common zero by Lemma 3.6.

Theorem 3.8. Let \( C_d \) and \( Q_d \) be the spaces defined as on page 8. Then the space \( Q_d \), for \( d \geq 2 \), is open in the space \( C_d \).

Proof. Since \( Q_2 = C_2 \) and \( Q_3 = C_3 \), so the theorem is trivial for \( d = 2, 3 \). Assume that \( d \geq 4 \). We shall show that \( C_d \setminus Q_d \) is closed. Let \( \{\phi_j\}_{j=1}^{\infty} \) be a sequence of points in \( C_d \setminus Q_d \) which converges to a point \( \phi \) in \( C_d \). We will show that \( \phi \in C_d \setminus Q_d \). Let \( \phi_j \) be given by \( t \mapsto (f_j(t), g_j(t), h_j(t)) \), where the component polynomials be given.
as \( f_j(t) = a_{0j} + a_{1j}t + \cdots + a_{d-2,j}t^{d-2}, \ g_j(t) = b_{0j} + b_{1j}t + \cdots + b_{d-1,j}t^{d-1} \) and 
\( h_j(t) = c_{0j} + c_{1j}t + \cdots + c_{d,j}t^d. \) Also, let \( \phi \) be given as 
\( t \mapsto (f(t), g(t), h(t)), \) where \( f, g \) and \( h \) are defined as 
\( f(t) = a_0 + a_1t + \cdots + a_{d-2,t}t^{d-2}, \ g(t) = b_0 + b_1t + \cdots + b_{d-1,t}t^{d-1} \) and 
\( h(t) = c_0 + c_1t + \cdots + cdt^d. \) For \( j \in \mathbb{N}, \) since \( \phi_j \) is not an embedding, so by Lemma 3.7, we can choose \((s_j, t_j) \in \mathbb{R}^2\) such that

\[
a_{1j} + a_{2j}(s_j + t_j) + \cdots + a_{d-2,j}(s_j^{d-3} + s_j^{d-4}t_j + \cdots + t_j^{d-3}) = 0, \tag{3.14}
\]

\[
b_{1j} + b_{2j}(s_j + t_j) + \cdots + b_{d-1,j}(s_j^{d-2} + s_j^{d-3}t_j + \cdots + t_j^{d-2}) = 0 \quad \text{and} \quad \tag{3.15}
\]

\[
c_{1j} + c_{2j}(s_j + t_j) + \cdots + c_{d,j}(s_j^{d-1} + s_j^{d-2}t_j + \cdots + t_j^{d-1}) = 0. \tag{3.16}
\]

We claim that the sequence \( \{(s_j, t_j)\}_{j=1}^\infty \) is bounded. Suppose not, that is, 
this sequence is unbounded, then at least one of the sequence \( \{s_j\}_{j=1}^\infty \) or \( \{t_j\}_{j=1}^\infty \) is unbounded. We may assume that the sequence \( \{s_j\}_{j=1}^\infty \) is unbounded, so it has 
a subsequence which diverges to \( \pm \infty. \) Again, without loss of generality, we may assume that the sequence \( \{s_j\}_{j=1}^\infty \) itself diverges to \( \pm \infty. \) There are two cases 
according to which either \( \{t_j\}_{j=1}^\infty \) is bounded or not.

(1) If the sequence \( \{t_j\}_{j=1}^\infty \) is bounded: 
Since for \( i \in \{0, 1, \ldots, d\}, \ c_{ij} \to c_i \) as \( j \to \infty \) (note that \( c_d \neq 0 \)), and the sequence 
\( \{s_j\}_{j=1}^\infty \) diverges to \( \pm \infty, \) so for this case,

\[
\lim_{j \to \infty} (c_{1j} + c_{2j}(s_j + t_j) + \cdots + c_{d,j}(s_j^{d-1} + s_j^{d-2}t_j + \cdots + t_j^{d-1})) = \pm \infty.
\]

This is a contradiction, since the right hand side of Eq. (3.16) is zero for all \( j \in \mathbb{N}. \)

(2) If the sequence \( \{t_j\}_{j=1}^\infty \) is unbounded: 
In this case, the sequence \( \{t_j\}_{j=1}^\infty \) has a subsequence which diverges to \( \pm \infty, \) so 
without loss of generality, we may assume that the sequence \( \{t_j\}_{j=1}^\infty \) itself diverges to 
\( \pm \infty. \) Since for \( i \in \{0, 1, \ldots, d\}, \ b_{ij} \to b_i \) and \( c_{ij} \to c_i \) as \( j \to \infty \) (note that 
\( b_{d-1} \neq 0 \) and \( c_d \neq 0 \)), and the sequence \( \{s_j\}_{j=1}^\infty \) diverges to \( \pm \infty, \) so by Lemma 3.5, either 

\[
\lim_{j \to \infty} (b_{1j} + b_{2j}(s_j + t_j) + \cdots + b_{d-1,j}(s_j^{d-2} + s_j^{d-3}t_j + \cdots + t_j^{d-2})) = \pm \infty \quad \text{or} \quad \tag{3.15}
\]

\[
\lim_{j \to \infty} (c_{1j} + c_{2j}(s_j + t_j) + \cdots + c_{d,j}(s_j^{d-1} + s_j^{d-2}t_j + \cdots + t_j^{d-1})) = \pm \infty. \tag{3.16}
\]

This is again a contradiction, since the right hand sides of Eqs. (3.15) and (3.16) are zeros for all \( j \in \mathbb{N}. \)

The cases (1) and (2) both together show that the sequence \( \{(s_j, t_j)\}_{j=1}^\infty \) is 
bounded and hence it has a subsequence which is convergent. We may assume that 
the sequence \( \{(s_j, t_j)\}_{j=1}^\infty \) itself is convergent, say converges to a point \( (s_0, t_0) \in \mathbb{R}^2. \)
Also, since the sequence 

\[
\{(a_{0j}, a_{1j}, \ldots, a_{d-2,j}, b_{0j}, b_{1j}, \ldots, b_{d-1,j}, c_{0j}, c_{1j}, \ldots, c_{d,j})\}_{j=1}^\infty
\]
Note that \( \sigma > 0 \) for \( \epsilon > 0 \). For \( \text{Remark 3.12.} \) and hence the space \( B \) which is not an element of \( \mathbb{C} \).

Proof. For any \( \phi \) embedding; that is, \( 3.17 \) and \( 3.18 \) and \( 3.19 \). This by Lemma 3.7 shows that \( \phi \) is not an embedding; that is, \( \phi \) is an element of the space \( \mathbb{C} - Q_d \).

The following corollary of Theorem 3.8 follows trivially from Proposition 3.2.

**Corollary 3.9.** The space \( Q_d \), for \( d \geq 2 \), is open in the space \( \mathbb{A}_d \).

**Remark 3.10.** For \( d \geq 2 \), we have an element \( \phi \in \mathbb{O}_d \) which is defined by \( \phi \rightarrow (1, 1, t) \). For \( \epsilon > 0 \), an open ball \( V(\phi, \epsilon) = \{ \tau \in \mathbb{A}_d \mid \rho(\phi, \tau) < \epsilon \} \) contains an element \( \phi \epsilon \in \mathbb{A}_d \) which is given by

\[
\phi \epsilon \rightarrow (1, 1, t - (\epsilon/2)t^2)
\]

which is not an embedding (since \( \phi \epsilon (1/\epsilon) = 0 \)). Thus \( \phi \) is not an interior point of \( \mathbb{O}_d \). This shows that the space \( \mathbb{O}_d \), for \( d \geq 2 \), is not open in the space \( \mathbb{A}_d \).

**Remark 3.11.** For \( d \geq 3 \), we have an element \( \psi \in \mathbb{B}_d \) given by \( \psi \rightarrow (t^{d-3}, t^{d-2}, t^{d-1}) \). For \( \epsilon > 0 \), an open ball \( V(\psi, \epsilon) \) contains an element \( \psi \epsilon \in \mathbb{A}_d \) given by

\[
\psi \epsilon \rightarrow (t^{d-3} - (\epsilon/2)t^{d-1}, t^{d-2} - (\epsilon/2)t^{d-1}, t^{d-1})
\]

which is not an element of \( \mathbb{B}_d \). Therefore \( \psi \) is not an interior point of the space \( \mathbb{B}_d \), and hence the space \( \mathbb{B}_d \), for \( d \geq 3 \), is not open in the space \( \mathbb{A}_d \).

**Remark 3.12.** For \( d \geq 3 \), we have an element \( \sigma \in \mathbb{P}_d \) given by \( \sigma \rightarrow (1, t, t^2) \). For \( \epsilon > 0 \), an open ball \( V(\sigma, \epsilon) \) contains an element \( \sigma \epsilon \in \mathbb{B}_d \) given by

\[
\sigma \epsilon \rightarrow (1, t - (\epsilon/2)t^{d-1}, t^2 - (\epsilon/2)t^d)
\]

Note that \( \sigma \epsilon (2/\epsilon) = \sigma (0) \), so \( \sigma \) is not an embedding. Hence \( \sigma \) is not an interior point of \( \mathbb{P}_d \) (with respect to both the spaces \( \mathbb{A}_d \) and \( \mathbb{B}_d \)). This shows that the space \( \mathbb{P}_d \), for \( d \geq 3 \), is not open in both the spaces \( \mathbb{A}_d \) and \( \mathbb{B}_d \).

**Theorem 3.13.** The space \( Q_d \), for \( d \geq 2 \), is dense in the space \( \mathbb{C}_d \).

**Proof.** Note that \( Q_2 = C_2 \) and \( Q_3 = C_3 \), so the theorem is trivial if \( d = 2, 3 \). For \( d \geq 4 \), in order to prove that \( Q_d \) is dense in the space \( \mathbb{C}_d \), we have to show that for any \( \phi \in \mathbb{C}_d \) and any \( \epsilon > 0 \), there is a polynomial knot \( \psi \in Q_d \) which belongs to...
the $\epsilon$-neighbourhood $V(\phi, \epsilon)$ of $\phi$. Let an arbitrary $\phi \in C_d$ and an arbitrary $\epsilon > 0$ be given. Let $\phi(t) = (f(t), g(t), h(t))$ for $t \in \mathbb{R}$ and let

$$m_1 = \min \{ |h'(t)| : t \in \mathbb{R}, g'(t) = 0 \text{ and } h'(t) \neq 0 \}.$$ 

Let us choose a positive real number $r < \min \{ m_1, \epsilon/2 \}$, and let $\hat{h} : \mathbb{R} \to \mathbb{R}$ be given by $t \mapsto h(t) + \frac{r}{2}t$. For the curve $\alpha = (g, \hat{h})$, we have $\alpha'(t) \neq 0$ for all $t \in \mathbb{R}$. Let $g$ and $\hat{h}$ be given by $g(t) = b_0 + b_1 t + \cdots + b_{d-1} t^{d-1}$ and $\hat{h}(t) = c_0 + c_1 t + \cdots + c_d t^d$. Since $\alpha'(t) \neq 0$ for all $t \in \mathbb{R}$, so by Lemma 3.6, the self-intersections of the curve $\alpha$ are exactly the common zeros of the following polynomials:

$$G(s, t) = b_1 + b_2 (s + t) + \cdots + b_{d-1} (s^{d-2} + s^{d-3} t + \cdots + t^{d-2}) \quad \text{and} \quad H(s, t) = c_1 + c_2 (s + t) + \cdots + c_d (s^{d-1} + s^{d-2} t + \cdots + t^{d-1}).$$

We claim that the polynomials $G$ and $H$ do not have a non-constant common factor. Suppose contrary that they have a non-constant common factor, say $\Psi$, and let $\zeta$ be its leading term. Then $\zeta$ is a non-constant common factor of the polynomials $\mu(s, t) = s^{d-2} + s^{d-3} t + \cdots + t^{d-2}$ and $\nu(s, t) = s^{d-1} + s^{d-2} t + \cdots + t^{d-1}$.

Let $\mu_1$, $\nu_1$ and $\zeta_1$ be polynomials given by $\mu_1(t) = \mu(1, t) = 1 + t + \cdots + t^{d-2}$, $\nu_1(t) = \nu(1, t) = 1 + t + \cdots + t^{d-1}$ and $\zeta_1(t) = \zeta(1, t)$. It is obvious that the polynomials $\zeta$ and $\zeta_1$ have same degree, and thus $\zeta_1$ is a non-constant polynomial. Also, we have $t^{d-1} - 1 = \mu_1(t)(t-1)$ and $t^d - 1 = \nu_1(t)(t-1)$, and since $\zeta_1$ is a common factor of the polynomials $\mu_1$ and $\nu_1$, so every root of $\zeta_1$ is a complex $(d-1)$th and $d$th root of unity other than 1. This is a contradiction, because any complex $(d-1)$th root of unity other than 1 cannot be equal to a complex $d$th root of unity. This proves the claim.

By Bézout’s theorem, the polynomials $G$ and $H$ have only finitely many common zeros, and hence the curve $\alpha = (g, \hat{h})$ has only finitely many self-intersections, say $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ be those intersections (that is, $s_i \neq t_i$ and $\alpha(s_i) = \alpha(t_i)$ for $i \in \{1, 2, \ldots, k\}$). Let us choose a positive real number $s < \min \{ m_2, \epsilon/2 \}$, where

$$m_2 = \min \{ \left| \frac{f(s_i) - f(t_i)}{s_i - t_i} \right| : i \in \{1, 2, \ldots, k\} \text{ and } \frac{f(s_i) - f(t_i)}{s_i - t_i} \neq 0 \}.$$ 

Let $\hat{f} : \mathbb{R} \to \mathbb{R}$ be given by $t \mapsto f(t) + \frac{r}{2} t$. For $i \in \{1, 2, \ldots, k\}$, it is easy to note that $\hat{f}(s_i) \neq \hat{f}(t_i)$. This shows that the map $\psi = (\hat{f}, g, \hat{h})$ has no self intersections. Note that $\psi(t) \neq 0$ for all $t \in \mathbb{R}$, since $\alpha'(t) \neq 0$ for all $t \in \mathbb{R}$. Also, $\psi$ has a degree sequence $(d-2, d-1, d)$, so it is an element of the space $Q_d$. We now estimate for $\rho(\phi, \psi)$ as follows:

$$\rho(\phi, \psi) \leq \rho(\phi, (f, g, \hat{h})) + \rho((f, g, \hat{h}), \psi) \leq r/2 + s/2 \leq \epsilon/4 + \epsilon/4 \leq \epsilon/2.$$ 

Thus $\psi \in V(\phi, \epsilon)$. This proves the theorem. □
Corollary 3.14. The space $Q_d$, for $d \geq 2$, is dense in the space $A_d$.

Corollary 3.15. The spaces $O_d$ and $P_d$, for $d \geq 2$, are dense in the space $A_d$.

Our next goal is to study the path components of the spaces $O_d$, $P_d$ and $Q_d$. We first ensure that each one of them can have only finitely many path components by proving the following theorem and its corollaries.

Theorem 3.16. The space $O_d$ is homeomorphic to a semialgebraic subset of $\mathbb{R}^{3d}$ and hence it has only finitely many path components.

Proof. Let us denote the real tuples $(a_0, a_1, \ldots, a_{d-2})$, $(b_0, b_1, \ldots, b_{d-1})$ and $(c_0, c_1, \ldots, c_d)$ respectively by $a$, $b$ and $c$. For $(a, b, c) \in \mathbb{R}^{3d}$, let $\phi_{abc} : \mathbb{R} \to \mathbb{R}^3$ be a polynomial map which is given by

$$t \mapsto (a_0 + a_1 t + \cdots + a_{d-2} t^{d-2}, b_0 + b_1 t + \cdots + b_{d-1} t^{d-1}, c_0 + c_1 t + \cdots + c_d t^d).$$

Note that $\eta(\phi_{abc}) = (a, b, c)$ for $(a, b, c) \in \mathbb{R}^{3d}$. So, the set $\eta(A_d \setminus O_d)$ can be written as follows:

$$\eta(A_d \setminus O_d) = \{ (a, b, c) \in \mathbb{R}^{3d} \mid \phi_{abc} \text{ is not an embedding} \}.$$

For $(a, b, c) \in \mathbb{R}^{3d}$, let $F_a, G_b, H_c : \mathbb{R}^2 \to \mathbb{R}^3$ be polynomials which are given by

$$(s, t) \xrightarrow{F_a} a_1 + a_2(s + t) + \cdots + a_{d-2}(s^{d-3} + s^{d-4}t + \cdots + t^{d-3}),$$

$$(s, t) \xrightarrow{G_b} b_1 + b_2(s + t) + \cdots + b_{d-1}(s^{d-2} + s^{d-3}t + \cdots + t^{d-2}) \quad \text{and}$$

$$(s, t) \xrightarrow{H_c} c_1 + c_2(s + t) + \cdots + c_d(s^{d-1} + s^{d-2}t + \cdots + t^{d-1}).$$

For any $(a, b, c) \in \mathbb{R}^{3d}$, $F_a(s, t) = G_b(s, t) = H_c(s, t) = 0$ for some $(s, t) \in \mathbb{R}^2$ if and only if the polynomial map $\phi_{abc}$ is not an embedding. Therefore, the set $\eta(A_d \setminus O_d)$ is equal to the image of the set

$$U_d = \{ (a, b, c, s, t) \in \mathbb{R}^{3d+2} \mid F_a(s, t) = G_b(s, t) = H_c(s, t) = 0 \}$$

under the projection $\pi : \mathbb{R}^{3d+2} \to \mathbb{R}^{3d}$ onto the space of the first 3d coordinates. Note that the set $U_d$ is semialgebraic, so by Corollary $2.22$, the set $\eta(A_d \setminus O_d)$ is also semialgebraic, and hence so is its complement $\eta(O_d) = R^{3d} \setminus \eta(A_d \setminus O_d)$. 

Corollary 3.17. The space $P_d$ is homeomorphic to a semialgebraic subset of $\mathbb{R}^{3d}$ and hence it has only finitely many path components.

Proof. For $0 \leq p < q < r \leq d$, let us consider the following set:

$$W_{dpqr} = \{ (a, b, c) \in \mathbb{R}^{3d} \mid \phi_{abc} \text{ has degree sequence } (p, q, r) \}.$$

Note that, for any $p < q < r \leq d$, the set $W_{dpqr}$ is semialgebraic, and hence so is the set $\eta(B_d) = \bigcup_{p < q < r \leq d} W_{dpqr}$. Since $\eta(O_d)$ is semialgebraic, the set $\eta(P_d) = \eta(B_d) \cap \eta(O_d)$ is also semialgebraic. 

\[\square\]
Corollary 3.18. The space $Q_d$ is homeomorphic to a semialgebraic subset of $\mathbb{R}^{3d}$ and hence it has only finitely many path components.

**Proof.** Since the sets $\eta(C_d)$ and $\eta(O_d)$ are semialgebraic, so the set $\eta(Q_d) = \eta(C_d) \cap \eta(O_d)$ is also semialgebraic. \[\square\]

Proposition 3.19. Let $\phi \in O_d$, and let $\alpha$ and $\beta$ be any real numbers such that $\alpha > 0$. Define $\psi : \mathbb{R} \to \mathbb{R}^3$ as $\psi(t) = \phi(\alpha t + \beta)$ for $t \in \mathbb{R}$. Then we have the following:

(1) Both $\phi$ and $\psi$ belong to the same path component of $O_d$.
(2) If $\phi \in P_d$, then both $\phi$ and $\psi$ belong to the same path component of $P_d$.
(3) If $\phi \in Q_d$, then both $\phi$ and $\psi$ belong to the same path component of $Q_d$.

**Proof.** For $s \in I$, define a map $F : I \to A_d$ by $s \mapsto F_s$, where $F_s : \mathbb{R} \to \mathbb{R}^3$ is given by

$$F_s(t) = \phi((1 - s + \alpha s)t + \beta s)$$

for $t \in \mathbb{R}$. By Remark 2.17 for each $s \in I$, $F_s$ is an element of the space $O_d$. Also $F_0 = \phi$ and $F_1 = \psi$. Hence the map $F$ is a path in $O_d$ from $\phi$ to $\psi$.

To prove second and third parts: If $\phi$ belongs to $P_d$ (respectively $Q_d$), then for each $s \in I$, $F_s$ is an element of the space $P_d$ (respectively $Q_d$). Therefore, the map $F$ becomes a path in $P_d$ (respectively $Q_d$) connecting $\phi$ and $\psi$. \[\square\]

Proposition 3.20. Let $\phi$ be an element of the space $O_d$. For $i \in \{1, 2, 3\}$, let $\alpha_i$’s be any positive real numbers, and let $\beta_i$’s and $\gamma_i$’s be any real numbers. Let $\psi : \mathbb{R} \to \mathbb{R}^3$ be defined by $\psi(t) = (\alpha_1 f(t) + \gamma_1, \alpha_2 g(t) + \beta_1 f(t) + \gamma_2, \alpha_3 h(t) + \beta_2 f(t) + \beta_3 g(t) + \gamma_3)$. Then we have the following:

(1) Both $\phi$ and $\psi$ belong to the same path component of $O_d$.
(2) If $\phi \in P_d$, then both $\phi$ and $\psi$ belong to the same path component of $P_d$.
(3) If $\phi \in Q_d$, then both $\phi$ and $\psi$ belong to the same path component of $Q_d$.

**Proof.** For $s \in I$, define a map $T_s : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T_s(x, y, z) = ((1 - s + \alpha_1 s)x + \gamma_1, \beta_1 sx + (1 - s + \alpha_2 s)y + \gamma_2 s, 
\beta_2 sx + \beta_3 sy + (1 - s + \alpha_3 s)z + \gamma_3 s)$$

for $(x, y, z) \in \mathbb{R}^3$. Note that, for $s \in I$, $T_s$ is an affine transformation (a composition of an invertible linear transformation and a translation). By Remark 2.18 for each $s \in I$, $H_s = T_s \circ \phi$ is an element of the space $O_d$. Also $H_0 = \phi$ and $H_1 = \psi$. Thus, the map $H : I \to O_d$ which is given by $s \mapsto H_s$ is a path in $O_d$ from $\phi$ to $\psi$.

To prove second and third parts: If $\phi$ belongs to $P_d$ (respectively $Q_d$), then for each $s \in I$, $H_s$ is an element of the space $P_d$ (respectively $Q_d$). Therefore, the map $H$ becomes a path in $P_d$ (respectively $Q_d$) connecting $\phi$ and $\psi$. \[\square\]
Corollary 3.21. Every path component of each of the spaces \( O_d, P_d \) and \( Q_d \) is unbounded.

Proof. We prove the result in case of the space \( O_d \). Let \( \phi \) be a polynomial knot in a given path component of \( O_d \). In Proposition 3.20 by taking \( \alpha_i \)'s arbitrarily large, and \( \beta_i \)'s and \( \gamma_i \)'s to be zero, the distance of \( \psi \) from the zero map \( 0 : \mathbb{R} \to \mathbb{R}^3 \) given by \( 0(t) = (0, 0, 0) \) for \( t \in \mathbb{R} \), can be made arbitrarily large. Thus, there are polynomial knots at arbitrarily large distances from the origin which belong to a given path component of \( O_d \). In other words, each path component of the space \( O_d \) is unbounded. Similarly, every path component of the spaces \( P_d \) and \( Q_d \) are unbounded.

Corollary 3.22. Every neighborhood of the zero map \( 0 \in A_d \) intersects each path component of the spaces \( O_d, P_d \) and \( Q_d \).

Proof. Let \( \phi \) be a polynomial knot in a given path component of \( O_d \). In Proposition 3.20 by taking \( \alpha_i \)'s arbitrarily small, and \( \beta_i \)'s and \( \gamma_i \)'s to be zero, the distance of \( \psi \) from the zero map can be made arbitrarily small. Hence, there are polynomial knots in a given path component of \( O_d \) which are arbitrarily close to the origin. In other words, every neighborhood of the origin intersects each path component of the space \( O_d \). Similarly, the result is true for the spaces \( P_d \) and \( Q_d \). ☐

4. Main Results

In Section 3 we introduced the spaces \( O_d, P_d \) and \( Q_d \) consisting of polynomial knots in degree \( d \). We define path equivalence in these spaces as:

Definition 4.1. Two polynomial knots \( \sigma \) and \( \tau \) in \( O_d \) (or in \( P_d \)/in \( Q_d \)) are said to be path equivalent in \( O_d \) (respectively, in \( P_d \)/in \( Q_d \)) if they belong to its same path component.

We would like to study the path equivalence of two polynomial knots in the spaces \( O_d, P_d \) and \( Q_d \) as compared to their topological equivalence. We also estimate the path components of these spaces and look into the homotopy type of their path components.

4.1. Path components of the spaces

Proposition 4.2. Two polynomial knots \( \phi, \psi \in O_d \) are path equivalent in \( O_d \) if and only if they are isotopic by an isotopy of polynomial knots in \( O_d \).
Proof. Suppose $F : I \to \mathcal{O}_d$ be a path from $\phi$ to $\psi$. For $s \in I$, take $F_s = F(s)$ and let it be given by

$$F_s(t) = (\alpha_0(s) + \alpha_1(s)t + \cdots + \alpha_{d-2}(s)t^{d-2}, \beta_0(s) + \beta_1(s)t + \cdots + \beta_{d-1}(s)t^{d-1}, \gamma_0(s) + \gamma_1(s)t + \cdots + \gamma_d(s)t^d)$$

for $t \in \mathbb{R}$, where $\alpha_i$'s, $\beta_i$'s and $\gamma_i$'s are real valued functions. Let us define a function

$$\Gamma : I \to \mathbb{R}^{3d}$$

given by

$$s \mapsto (\alpha_0(s), \alpha_1(s), \ldots, \alpha_{d-2}(s), \beta_0(s), \beta_1(s), \ldots, \beta_{d-1}(s), \gamma_0(s), \gamma_1(s), \ldots, \gamma_d(s)).$$

It is clear that $\Gamma = \eta \circ F$, where $\eta$ is natural homeomorphism of the spaces $\mathcal{O}_d$ and $\mathbb{R}^{3d}$ as defined in $[2]$. Thus $\Gamma$ is continuous, and so are its component functions $\alpha_i$'s, $\beta_i$'s and $\gamma_i$'s. This implies that the functions

$$(s, t) \xmapsto{\Omega} \alpha_0(s) + \alpha_1(s)t + \cdots + \alpha_{d-2}(s)t^{d-2},$$

$$(s, t) \xmapsto{\Omega} \beta_0(s) + \beta_1(s)t + \cdots + \beta_{d-1}(s)t^{d-1} \quad \text{and}$$

$$(s, t) \xmapsto{\Omega} \gamma_0(s) + \gamma_1(s)t + \cdots + \gamma_d(s)t^d$$

are continuous as functions from $I \times \mathbb{R}$ to $\mathbb{R}$. Define $H : I \times \mathbb{R} \to \mathbb{R}^3$ by $H(s, t) = F_s(t)$ for $(s, t) \in I \times \mathbb{R}$. It is easy to note that $H$ is continuous, because it's component functions $\Omega_1, \Omega_2$ and $\Omega_3$ are continuous. Note that $H_0 = F(0) = \phi$ and $H_1 = F(1) = \psi$. This shows that $H$ is an isotopy of polynomial knots in $\mathcal{O}_d$ connecting $\phi$ and $\psi$.

Suppose $\phi, \psi \in \mathcal{O}_d$ and let $G : I \times \mathbb{R} \to \mathbb{R}^3$ be an isotopy of polynomial knots in $\mathcal{O}_d$ which connects them (that is, $G_0 = \phi$ and $G_1 = \psi$). Let $G$ be given by

$$G(s, t) = (\mu_0(s) + \mu_1(s)t + \cdots + \mu_{d-2}(s)t^{d-2}, \sigma_0(s) + \sigma_1(s)t + \cdots + \sigma_{d-1}(s)t^{d-1}, \tau_0(s) + \tau_1(s)t + \cdots + \tau_d(s)t^d)$$

for $s \in I$ and $t \in \mathbb{R}$, where $\mu_i$'s, $\sigma_i$'s and $\tau_i$'s are real valued functions. Note that for any fixed $t \in \mathbb{R}$, the functions

$$s \xmapsto{\Omega_{11}} \mu_0(s) + \mu_1(s)t + \cdots + \mu_{d-2}(s)t^{d-2},$$

$$s \xmapsto{\Omega_{21}} \sigma_0(s) + \sigma_1(s)t + \cdots + \sigma_{d-1}(s)t^{d-1} \quad \text{and}$$

$$s \xmapsto{\Omega_{31}} \tau_0(s) + \tau_1(s)t + \cdots + \tau_d(s)t^d$$

are continuous as functions from $I$ to $\mathbb{R}$. It is easy to check that: (a) $\mu_i$'s are linear combinations of the functions $\Omega_{10}, \Omega_{11}, \ldots, \Omega_{1,d-2}$, (b) $\sigma_i$'s are linear combinations of the functions $\Omega_{20}, \Omega_{21}, \ldots, \Omega_{2,d-1}$, and (c) $\tau_i$'s are linear combinations of the functions $\Omega_{30}, \Omega_{31}, \ldots, \Omega_{3d}$. Thus, the functions $\mu_i$'s, $\sigma_i$'s and $\tau_i$'s are continuous, and hence the function $\Lambda : I \to \mathbb{R}^{3d}$ which is given by

$$s \mapsto (\mu_0(s), \mu_1(s), \ldots, \mu_{d-2}(s), \sigma_0(s), \sigma_1(s), \ldots, \sigma_{d-1}(s), \tau_0(s), \tau_1(s), \ldots, \tau_d(s))$$

is also continuous. Define $\Upsilon : I \to \mathcal{O}_d$ by $\Upsilon(s) = G_s$ for $s \in I$, where $G_s : \mathbb{R} \to \mathbb{R}^3$ is given by $G_s(t) = G(s, t)$ for $t \in \mathbb{R}$. Since $\Upsilon = \eta^{-1} \circ \Lambda$, the map $\Upsilon$ is continuous.
Note that $Y(0) = G_0 = \omega$ and $Y(1) = G_1 = \psi$. This shows that the map $Y$ is a path in $O_d$ connecting $\phi$ and $\psi$.

**Corollary 4.3.** Two polynomial knots $\phi$ and $\psi$ in $P_d$ (respectively in $Q_d$) are path equivalent in $P_d$ (respectively in $Q_d$) if and only if they are isotopic by an isotopy of polynomial knots in $P_d$ (respectively in $Q_d$).

**Corollary 4.4.** Two polynomial knots $\phi, \psi \in O_d$ are connected by a smooth path in $O_d$ if and only if they are isotopic by a smooth isotopy of polynomial knots in $O_d$.

In Corollary 4.4, the space $O_d$ can be replaced by any of the spaces $P_d$ and $Q_d$.

**Proposition 4.5.** Every polynomial knot in $O_d$ is connected by a path in $O_d$ to some polynomial knot in $Q_d$.

**Proof.** Let $\phi$ be an element of the space $O_d$. Since the set $\eta(Q_d)$ is dense in $R^{3d}$ and it has only finitely many path components, so $\eta(\phi)$ must belong to the closure of some path component, say $U$, of $\eta(Q_d)$. The set $\eta(Q_d)$ is semialgebraic, so by Theorem 2.23, $U$ is also semialgebraic. This implies that the set $U \cup \{\eta(\phi)\}$ is semialgebraic, since one point set is always semialgebraic. Note that $U \cup \{\eta(\phi)\}$ is connected; therefore, by Proposition 2.24, it is path connected, and hence so is the set $\eta^{-1}(U) \cup \{\phi\}$. In other words, there is path in $\eta^{-1}(U) \cup \{\phi\} \subseteq O_d$ from $\phi$ to an element $\psi \in \eta^{-1}(U) \subseteq Q_d$.

**Corollary 4.6.** Every polynomial knot in $O_d$ is connected by a path in $O_d$ to some polynomial knot in $P_d$.

**Corollary 4.7.** Every polynomial knot in $P_d$ is connected by a path in $P_d$ to some polynomial knot in $Q_d$.

**Proposition 4.8.** The space $P_d$, for $d \geq 3$, is path connected.

**Proof.** Let $\phi$ be an arbitrary element of the space $P_d$. By Corollary 4.7, there is a path in $P_d$ from $\phi$ to some element $\psi \in Q_d$. Since $Q_d$ is open, we have a path in $Q_d$ from $\psi$ to an another element $\sigma \in Q_d$ in which the coefficient $b_{\sigma 1}$ of $t$ in the second component is nonzero. Let $\Lambda : I \rightarrow Q_d$ be a map which is given by

$$\Lambda(s) = \left( f_\sigma - a_{\sigma 0} s, g_\sigma - b_{\sigma 0} s, h_\sigma - \left( e_{\sigma 1} g_\sigma - b_{\sigma 0} c_{\sigma 1} + b_{\sigma 1} c_{\sigma 0} \right) s \right)$$

for $s \in I$. For $\tau = \Lambda(1)$, we have $a_{\tau 0} = b_{\tau 0} = c_{\tau 0} = c_{\tau 1} = 0$ and $b_{\tau 1} = b_{\sigma 1} \neq 0$. Obviously the map $\Lambda$ is a path in $Q_d$ from $\sigma$ to $\tau$. Since $Q_d$ is open, we have a path in $Q_d$ from $\tau$ to an element $\omega \in Q_d$ which differs from $\tau$ only in the coefficient of
$t^2$ in the third component and that coefficient is nonzero in $\omega$. For $s \in I$, we define polynomials $f_s$, $g_s$, and $h_s$ as follows:

$$f_s(t) = a_{\omega 1}st + a_{\omega 2}s^2t^2 + \cdots + a_{\omega d-2}s^{d-2}t^{d-2},$$

$$g_s(t) = b_{\omega 1}t + b_{\omega 2}s^2t^2 + \cdots + b_{\omega d-1}s^{d-1}t^{d-1},$$

$$h_s(t) = c_{\omega 2}t^2 + c_{\omega 3}s^3t + \cdots + c_{\omega d}s^{d-2}t^d.$$

Let $\Gamma : I \to \mathcal{A}_d$ be a map which is given by $\Gamma(s) = \Gamma_s$ for $s \in I$, where $\Gamma_s$ be defined as

$$\Gamma_s(t) = (f_s(t), g_s(t), h_s(t))$$

for $t \in \mathbb{R}$. Note that $a_{\omega 0} = a_{\tau 0} = 0$, $b_{\omega 0} = b_{\tau 0} = 0$, $c_{\omega 0} = c_{\tau 0} = 0$ and $c_{\omega 1} = c_{\tau 1} = 0$; therefore, $\Gamma_1 = (f_1, g_1, h_1) = \omega$. For $s \in (0, 1)$ and $t \in \mathbb{R}$, we have

1. $f_s(t) = f_1(st)$, $g_s(t) = g_1(st)$ and $s^2h_s(t) = h_1(st)$, and
2. $f'_s(t) = sf'_1(st)$, $g'_s(t) = g'_1(st)$ and $sh'_s(t) = h'_1(st)$.

For $s \in (0, 1)$ and $u, v, t \in \mathbb{R}$ with $u \neq v$, we have $\Gamma'_s(t) = (f'_s(t), g'_s(t), h'_s(t)) = 0$ and $\Gamma_s(u) \neq \Gamma_s(v)$, because otherwise by (1) and (2) we would have $\omega(s't) = (f'_1(st), g'_1(st), h'_1(st)) = 0$ or $\omega(su) = \omega(sv)$, a contradiction to the fact that $\omega$ is an embedding. This implies that, for any $s \in (0, 1)$, the element $\Gamma(s) = \Gamma_s$ is an embedding and hence a member of the space $\mathcal{P}_d$. Also, a map $\sigma : \mathbb{R} \to \mathbb{R}^3$ given by $t \mapsto (0, b_{\omega 1}t, c_{\omega 2}t^2)$ is an element of the space $\mathcal{P}_d$, because $b_{\omega 1} = b_{\tau 1} \neq 0$ and $c_{\omega 2} \neq 0$. It is easy to see that $\Gamma(0) = (f_0, g_0, h_0) = \sigma$. Therefore, the map $\Gamma$ is a path in $\mathcal{P}_d$ connecting $\omega$ and $\sigma$.

Let $\phi_0$ be a fixed element of the space $\mathcal{P}_d$ which is given by $t \mapsto (0, t, t^2)$. Take a map $\Omega : I \to \mathcal{A}_d$ given by $\Omega(s) = \Omega_s$ for $s \in I$, where

$$\Omega_s(t) = (s(1-s)t, (b_{\omega 1}s + 1-s)t + s(1-s)t^2, (c_{\omega 2}s + 1-s)t^2 + s(1-s)^3)$$

for $t \in \mathbb{R}$. It is easy to see that $\Omega$ is path in $\mathcal{P}_d$ from $\phi_0$ to $\sigma$. Now by concatenating the paths between the consecutive elements $\phi, \psi, \sigma, \tau, \omega$ and $\phi_0$ gives a new path in $\mathcal{P}_d$ connecting $\phi$ and $\phi_0$.

**Theorem 4.9.** If two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent.

**Proof.** Let $\phi$ and $\psi$ be two polynomial knots which belong to the same path component of $\mathcal{Q}_d$. Since $\mathcal{Q}_d$ is open in $\mathcal{A}_d$, we can choose a smooth path $\alpha : I \to \mathcal{Q}_d$ such that $\alpha(0) = \phi$ and $\alpha(1) = \psi$. Consider the map $F : I \times \mathbb{R} \to \mathbb{R}^3$ given by $F(s, t) = \alpha_s(t)$ for $(s, t) \in I \times \mathbb{R}$, where $\alpha_s = \alpha(s)$ for $s \in I$. This map is smooth and it is a polynomial isotopy of knots in $\mathcal{Q}_d$ connecting $\phi$ and $\psi$. Let $\Theta : \mathcal{Q}_d \to \mathbb{R}^3$ be the track of the isotopy $F$; that is, $\Theta = (s, F(s, t)) = (s, \alpha_s(t))$ for $(s, t) \in I \times \mathbb{R}$. Note that $\Theta$ is a level preserving map and it is a smooth embedding of $I \times \mathbb{R}$ in $I \times \mathbb{R}^3$, so $\Theta(I \times \mathbb{R})$ is a smooth submanifold of $I \times \mathbb{R}^3$. One can identify $I \times \mathbb{R}^3$ as a smooth submanifold of $I \times S^3$ by the embedding $1 \times \Gamma : I \times \mathbb{R}^3 \to I \times S^3 \setminus \{N\}$, where $1$ is the identity mapping of $I$ and $\Gamma$ is the inverse of the stereographic projection.
from the north pole $N$ of $S^3$. Let $K$ be the image of $(1 \times \Gamma) \circ \Theta$ and let $\bar{K}$ be its

closure in $I \times S^3$. It is obvious that the collection $\mathcal{S} = \{ I \times S^3 \setminus \bar{K}, K, I \times \{N\} \}$
is a collection of pairwise disjoint smooth submanifolds of $I \times S^3$ which covers it. Thus $\mathcal{S}$ becomes a pre-stratification for $I \times S^3$ (see [8, § 5]). Note that $\mathcal{S}$ is a finite collection. Also, it is easy to check that it satisfies the condition of frontier (see [8, § 5]). Since $I \times S^3 \setminus \bar{K}$ is an open submanifold of $I \times S^3$, so its dimension is 4; therefore, it is trivial to see that the pairs $(I \times S^3 \setminus \bar{K}, K)$ and $(I \times S^3 \setminus \bar{K}, I \times \{N\})$ of strata in $\mathcal{S}$ satisfy the second Whitney condition (see [8, § 2]).

We claim that the pair $(K, I \times \{N\})$ satisfies the second Whitney condition. Let $y = (u, N)$ be an element in $I \times \{N\}$ and let $\Phi : U \to \mathbb{R}^4$ be a smooth chart of $I \times S^3$ at $y$. We need to check that the pair $(\Phi(U \cap K), \Phi(U \cap I \times \{N\}))$ satisfies the second Whitney condition at $\Phi(y)$. Let $(x_i) \subseteq \Phi(U \cap K)$ and $(y_i) \subseteq \Phi(U \cap I \times \{N\})$ be sequences both converging to $\Phi(y)$ and they are such that: (a) $x_i \neq y_i$ for all $i$, (b) the tangent spaces $\tau_i = T(\Phi(U \cap K))_{x_i}$ converge to a plane $\tau \subseteq \mathbb{R}^4$ (the convergence is in the topology of Grassmannian of 2-planes in $\mathbb{R}^4$), and (c) the secant lines $l_i = (x_i, y_i)$ converge to a line $l \subseteq \mathbb{R}^4$. We can assume that $U = V \times W$ for some connected open neighborhoods $V$ of $u$ in $I$ and $W$ of $N$ in $S^3$, and let $\Phi(s, x) = (s, \Psi(x))$ for $(s, x) \in V \times W$, where $\Psi$ is the restriction of the stereographic projection from the south pole of $S^3$. For $i \in \mathbb{N}$, let

$$x_i = \Phi \circ (1 \times \Gamma) \circ \Theta(s_i, t_i) = (s_i, \Psi \circ \Gamma \circ F(s_i, t_i)) = \left( s_i, \frac{F(s_i, t_i)}{||F(s_i, t_i)||^2} \right)$$

for some $(s_i, t_i) \in I \times \mathbb{R}$, and let

$$y_i = \Phi(u_i, N) = (u_i, \Psi(N)) = (u_i, 0)$$

for some $u_i \in I$. For $i \in \mathbb{N}$, note that the vector

$$\lambda_i = x_i - y_i = \left( s_i - u_i, \frac{F(s_i, t_i)}{||F(s_i, t_i)||^2} \right)$$

spans the secant line $l_i$. Also, for $i \in \mathbb{N}$, it is easy to check that the vectors

$$v_i = \left. \left( 1, \frac{\partial}{\partial s} \Psi \circ \Gamma \circ F(s, t) \right) \right|_{(s_i, t_i)}$$

$$= \left. \left( 1, \frac{\partial}{\partial s} \left( \frac{F(s, t)}{||F(s, t)||^2} \right) \right) \right|_{(s_i, t_i)}$$

$$= \left. \left( 1, \frac{\partial}{\partial s} \frac{F(s, t)}{||F(s, t)||^2} - \frac{\partial}{\partial s} \frac{||F(s, t)||^2}{||F(s, t)||^4} F(s, t) \right) \right|_{(s_i, t_i)}$$
and
\[
\begin{align*}
    w_i &= \left. \left( 0, \frac{\partial}{\partial t} \Psi \circ \Gamma \circ F(s, t) \right) \right|_{(s_i, t_i)} \\
    &= \left. \left( 0, \frac{\partial}{\partial t} \left( \frac{F(s, t)}{\|F(s, t)\|^2} \right) \right) \right|_{(s_i, t_i)} \\
    &= \left. \left( 0, \frac{\partial}{\partial t} F(s, t) - \frac{\partial}{\partial t} \|F(s, t)\|^2 \frac{F(s, t)}{\|F(s, t)\|^2} \right) \right|_{(s_i, t_i)}
\end{align*}
\]

span the tangent space \( \tau_i \). After normalizing the vectors \( \lambda_i, v_i \) and \( w_i \), for \( i \in \mathbb{N} \), we get the unit vectors as follows:

\[
\begin{align*}
    \lambda'_i &= \frac{\lambda_i}{\|\lambda_i\|} \\
    &= \left( \frac{s_i - u_i}{(s_i - u_i)^2 + \frac{1}{\|F(s_i, t_i)\|^2}}^{1/2}, \frac{F(s_i, t_i)}{\|F(s_i, t_i)\|^2 (s_i - u_i)^2 + \frac{1}{\|F(s_i, t_i)\|^2}}^{1/2} \right),
\end{align*}
\]

\[
\begin{align*}
    v'_i &= \frac{v_i}{\|v_i\|} \\
    &= \left( \frac{\|F(s, t)\|^2}{\|F(s, t)\|^4 + (\frac{\partial}{\partial s} F(s, t))^2}^{1/2} \left( 1, -\frac{\partial}{\partial s} F(s, t) \right), \frac{\|F(s, t)\|^2}{\|F(s, t)\|^4} \right) \left( 1, -\frac{\partial}{\partial s} F(s, t) \right) \right|_{(s_i, t_i)}
\end{align*}
\]

and

\[
\begin{align*}
    w'_i &= \frac{w_i}{\|w_i\|} \\
    &= \left( 0, \frac{\partial}{\partial t} F(s, t) - \frac{2 F(s, t) \cdot \frac{\partial}{\partial s} F(s, t)}{\|F(s, t)\|^2} \right) \left( 1, -\frac{\partial}{\partial s} F(s, t) \right) \right|_{(s_i, t_i)} \\
    &= \left( 0, \frac{\partial}{\partial t} F(s, t) - \frac{2 \cos \theta_i}{\|F(s, t)\|^2} \right) \left( 1, -\frac{\partial}{\partial s} F(s, t) \right) \right|_{(s, t) = (s_i, t_i)}
\end{align*}
\]

where \( \theta_i \), for \( i \in \mathbb{N} \), is the angle between the vectors \( F(s_i, t_i) \) and \( \frac{\partial F}{\partial s}(s_i, t_i) \). For \( i \in \mathbb{N} \), note that the vector \( \lambda'_i \) forms a basis for the secant line \( l_i \) and the vectors \( v'_i \) and \( w'_i \) together form a basis for the tangent space \( \tau_i \). Since \( |t_i| \to \infty \) as \( i \to \infty \), so there exists a subsequence of the sequence \( \{t_i\} \) which diverges to \( \pm \infty \). We may assume that the sequence \( \{t_i\} \) itself diverges to \( \pm \infty \). Note that the sequences \( \{ \|F(s_i, t_i)\|^{-1} F(s_i, t_i) \} \), \( \{ \|\frac{\partial F}{\partial s}(s_i, t_i)\|^{-1} \frac{\partial F}{\partial s}(s_i, t_i) \} \) and \( \{ \lambda'_i \} \) are bounded and hence they have convergent subsequences. Again, we may assume that these sequences themselves are convergent. It is easy to check that

\[
\lim_{i \to \infty} \left[ \frac{\partial}{\partial s} F(s, t) \right] = 0.
\]
Note that the converse of Theorem 4.9 is false. More precisely, \( \alpha \) isotopic (see [3]), so the knots \( \tilde{\alpha} \) let us compute the limit of \( z_{i}^{2} \) equivalent to the polynomial knot \( Q \) the complements of knots in \( S \) fiber bundle and hence the fibers \( \pi \). This shows that the nonzero vector \( \lambda_{i}^{'} \) is contractible. In particular, let \( \pi: I \times S^{3} \to I \) be the projection onto the first coordinate, then this map is proper. It is easy to check that the map \( \pi \), when restricted to any stratum in \( S \) is a submersion. Thus, by Thom’s First Isotopy Lemma (see [8] § 11), the map \( \pi \), when restricted to any stratum in \( S \) is a locally trivial fibration and hence a trivial fiber bundle (since, \( I \) is contractible). In particular, \( \pi^{-1}(0) = \{0\} \times S^{3} \setminus \partial \alpha_{0}(S^{1}) \cong S^{3} \setminus \partial \alpha_{0}(S^{1}) \) and \( \pi^{-1}(1) = \{1\} \times S^{3} \setminus \partial \alpha_{1}(S^{1}) \cong S^{3} \setminus \partial \alpha_{1}(S^{1}) \) are homeomorphic, where \( \partial \alpha_{0}: S^{1} \to S^{3} \) and \( \partial \alpha_{1}: S^{1} \to S^{3} \) are extensions of \( \alpha_{0} \) and \( \alpha_{1} \) respectively. Since it is known that the complements of knots in \( S^{3} \) are homeomorphic if and only if they are ambient isotopic (see [3]), so the knots \( \tilde{\alpha}_{0} \) and \( \tilde{\alpha}_{1} \) are ambient isotopic. Hence the knots \( \phi = \alpha_{0} \) and \( \psi = \alpha_{1} \) are topologically equivalent. 

\[ \lim_{i \to \infty} \lambda_{i}^{'} = \lambda_{i} \]

This shows that the nonzero vector \( \lambda^{'} = \lim_{i \to \infty} \lambda_{i}^{'} \) lies in both \( l \) and \( \tau \); therefore \( l \subseteq \tau \). Hence the pair \( (K, I \times \{N\}) \) satisfies the second Whitney condition.

Now the collection \( S \) becomes a Whitney pre-stratification for \( I \times S^{3} \) (see [8] § 5). Let \( \pi: I \times S^{3} \to I \) be the projection onto the first coordinate, then this map is proper. It is easy to check that the map \( \pi \), when restricted to any stratum in \( S \) is a submersion. Thus, by Thom’s First Isotopy Lemma (see [8] § 11), the map \( \pi \), when restricted to any stratum in \( S \) is a locally trivial fibration and hence a trivial fiber bundle (since, \( I \) is contractible). In particular, \( \pi^{-1}(0) = \{0\} \times S^{3} \setminus \partial \alpha_{0}(S^{1}) \cong S^{3} \setminus \partial \alpha_{0}(S^{1}) \) and \( \pi^{-1}(1) = \{1\} \times S^{3} \setminus \partial \alpha_{1}(S^{1}) \cong S^{3} \setminus \partial \alpha_{1}(S^{1}) \) are homeomorphic, where \( \partial \alpha_{0}: S^{1} \to S^{3} \) and \( \partial \alpha_{1}: S^{1} \to S^{3} \) are extensions of \( \alpha_{0} \) and \( \alpha_{1} \) respectively. Since it is known that the complements of knots in \( S^{3} \) are homeomorphic if and only if they are ambient isotopic (see [3]), so the knots \( \tilde{\alpha}_{0} \) and \( \tilde{\alpha}_{1} \) are ambient isotopic. Hence the knots \( \phi = \alpha_{0} \) and \( \psi = \alpha_{1} \) are topologically equivalent. 

Remark 4.10. Note that the converse of Theorem 4.9 is false. More precisely, a polynomial knot in \( \mathcal{Q}_{d} \) which is given by \( t \mapsto (f(t), g(t), h(t)) \) is topologically equivalent to the polynomial knot \( t \mapsto (f(t), -g(t), -h(t)) \), but they belong to the different path components of the space \( \mathcal{Q}_{d} \).
Remark 4.11. For \( d \geq 3 \) and a polynomial knot \( t \mapsto (f(t), g(t), h(t)) \) in \( Q_d \), there are eight distinct path components of \( Q_d \) each of which contains exactly one of the knot \( t \mapsto ((e_1f(t), e_2g(t), e_3h(t)) \) for \( e = (e_1, e_2, e_3) \) in \( \{-1, 1\}^3 \). Thus, the total number of path components of the space \( Q_d \), for \( d \geq 3 \), are in multiple of eight.

Remark 4.12. If there are \( n \) distinct knots (up to ambient isotopy and mirror images) are be represented in \( Q_d \), then it has at least \( 8n \) distinct path components.

Theorem 4.13. Every polynomial knot is isotopic to some trivial polynomial knot by a smooth isotopy of polynomial knots.

Proof. Let a polynomial knot \( \phi : \mathbb{R} \to \mathbb{R}^3 \) be given, and let

\[
\phi(t) = (a_0 + a_1t + \cdots + a_dt^d, b_0 + b_1t + \cdots + b_dt^d, c_0 + c_1t + \cdots + c_dt^d)
\]

for \( t \in \mathbb{R} \), where \( d \) is the degree of \( \phi \). For an arbitrary but fixed \( \epsilon > 0 \) and for \( s \in (-\epsilon, 1+\epsilon) \), define \( F_s : \mathbb{R} \to \mathbb{R}^3 \) by

\[
F_s(t) = (a_0, b_0, c_0)s + (a_1, b_1, c_1)t + \sum_{i=2}^{d} (a_i, b_i, c_i)t^{i-1}t^i
\]

for \( t \in \mathbb{R} \). Let us take a map \( F : (-\epsilon, 1+\epsilon) \times \mathbb{R} \to \mathbb{R}^3 \) which is given by \( F(s, t) = F_s(t) \) for \( s \in (-\epsilon, 1+\epsilon) \) and \( t \in \mathbb{R} \). Since \( \phi \) is an embedding, so for \( s \in (-\epsilon, 1+\epsilon) \setminus \{0\} \) and \( u, v, t \in \mathbb{R} \) with \( u \neq v \), we have \( F_s'(t) = \phi'(st) \neq 0 \) and

\[
F_s(u) - F_s(v) = \frac{\phi(su) - \phi(sv)}{s} \neq 0.
\]

This shows that \( F_s \) is a polynomial embedding for all \( s \in (-\epsilon, 1+\epsilon) \setminus \{0\} \). Let \( \sigma : \mathbb{R} \to \mathbb{R}^3 \) be a map which is given by \( \sigma(t) = (a_1t, b_1t, c_1t) \) for \( t \in \mathbb{R} \). Since at least one of \( a_1, b_1 \) or \( c_1 \) is nonzero (otherwise \( \phi'(0) = 0 \)), so one of the component of \( \sigma \) is linear and hence it is an embedding. Note that \( F(0,t) = \sigma(t) \) and \( F(1,t) = \phi(t) \) for all \( t \in \mathbb{R} \). Also, it is easy to see that \( F \) is smooth, since its components are polynomials in two variables. This proves that \( F \) is a smooth polynomial isotopy connecting \( \sigma \) and \( \phi \).

\[\Box\]

Corollary 4.14. Every polynomial knot is connected to a trivial polynomial knot by smooth path in the space \( \mathcal{P} \) of all polynomial knots.

Proof. For a given polynomial knot \( \phi \), a map \( \alpha : I \to \mathcal{P} \) given by \( s \mapsto F_s \) (where \( F_s \), for \( s \in I \), is defined as in the proof of Theorem 4.13) is a smooth path in \( \mathcal{P} \) connecting \( \phi \) to a trivial polynomial knot.

\[\Box\]

4.2. Homotopy type of the path components of the spaces

For a polynomial map \( \phi \) in \( A_d \), we denote its first, second and third components respectively by \( f_\phi, g_\phi \) and \( h_\phi \), and for \( i \in \{0, 1, \ldots, d\} \), \( a_i\phi, b_i\phi \) and \( c_i\phi \) would
denote the coefficients of $t^i$ in the polynomials $f_\phi$, $g_\phi$ and $h_\phi$ respectively. Sometimes we use letters like $\phi, \psi, \tau, \sigma$ and $\omega$ to denote the elements of $A_d$; in such cases, the corresponding components and their coefficients would be denoted by the corresponding subscripts.

**Theorem 4.15.** The space $O_2$ has homotopy type of $S^1$.

**Proof.** For $\phi \in O_2$, since $\phi'(0) \neq 0$, so we must have $(b_1, \phi)^2 + (c_1, \phi)^2 \neq 0$. Let $\Psi : O_2 \to S^1$ and $\Omega : S^1 \to O_2$ be defined by

$$\Psi(\phi) = \frac{1}{\sqrt{(b_1, \phi)^2 + (c_1, \phi)^2}} (b_1, \phi, c_1, \phi)$$

for $\phi \in O_2$ and $\Omega(x) = \phi_x$ for $x = (x_1, x_2) \in S^1$, where $\phi_x(t) = (0, x_1, x_2) t$ for $t \in \mathbb{R}$. Note that the maps $\Psi$ and $\Omega$ are continuous. Let $F : I \times O_2 \to A_2$ be a map which is given by $F(s, \phi) = F_{s, \phi}$ for $(s, \phi) \in I \times O_2$, where

$$F_{s, \phi}(t) = (1 - s) \phi(t) + \frac{s}{\sqrt{(b_1, \phi)^2 + (c_1, \phi)^2}} (0, b_1, \phi, t, c_1, \phi)$$

for $t \in \mathbb{R}$. The map $F$ is continuous. Note that the space $O_2$ is the disjoint union of the following sets:

- $O_{21} = \{ \tau \in A_2 \mid \tau \text{ has degree sequence } (0, 0, 1) \}$
- $O_{22} = \{ \phi \in A_2 \mid \text{second component of } \phi \text{ is linear} \}$

One can check that $F(s, \tau) \in O_{21}$ for all $(s, \tau) \in I \times O_{21}$ and $F(s, \sigma) \in O_{22}$ for all $(s, \sigma) \in I \times O_{22}$. This shows that $F$ maps $I \times O_2$ into the space $O_2$. Also, we have $F(0, \phi) = \phi$ and $F(1, \phi) = \Omega(\Psi(\phi))$ for all $\phi \in O_2$. This shows that $\Omega \circ \Psi$ is homotopic to the identity map of $O_2$. Also, note that the map $\Psi \circ \Omega$ is the identity map of $S^1$. \[\square\]

**Theorem 4.16.** The space $O_d$, for $d \geq 3$, has homotopy type of $S^2$.

**Proof.** For $\phi \in O_d$, since $\phi'(0) = 0$, so we have $(a_1, \phi)^2 + (b_1, \phi)^2 + (c_1, \phi)^2 \neq 0$. Let $\Gamma_d : O_d \to S^2$ and $\Upsilon_d : S^2 \to O_d$ be maps which are given by

$$\Gamma_d(\phi) = \frac{(a_1, \phi, b_1, \phi, c_1, \phi)}{\sqrt{(a_1, \phi)^2 + (b_1, \phi)^2 + (c_1, \phi)^2}}$$

for $\phi \in O_d$ and $\Upsilon_d(x) = \phi_x$ for $x = (x_1, x_2, x_3) \in S^2$, where $\phi_x(t) = (x_1, x_2, x_3) t$ for $t \in \mathbb{R}$. Note that the maps $\Gamma_d$ and $\Upsilon_d$ are continuous. Let $\mu : I \to \mathbb{R}^3$ be a map which is given by

$$\mu(s) = s + \frac{1 - s}{\sqrt{(a_1, \phi)^2 + (b_1, \phi)^2 + (c_1, \phi)^2}}$$

for $s \in I$. We define a map $H_d : I \times O_d \to A_d$ given by $H_d(s, \phi) = H_{s, \phi}$ for $(s, \phi) \in I \times O_d$, where

$$H_{s, \phi}(t) = (a_0, \phi, b_0, \phi, c_0, \phi) s + \mu(s) \left( (a_1, \phi, b_1, \phi, c_1, \phi) t + \sum_{i=2}^d (a_i, \phi, b_i, \phi, c_i, \phi) s^{i-1} t^i \right)$$
for $t \in \mathbb{R}$. It is easy to check that $H_d$ is continuous, and $H_d(0, \phi) = \Upsilon_d(\Gamma_d(\phi))$ and $H_d(1, \phi) = \phi$ for all $\phi \in \mathcal{O}_d$. For $s \in (0, 1)$, $\phi \in \mathcal{O}_d$ and $u, v, t \in \mathbb{R}$ with $u \neq v$, we have $H'_s(\phi)(t) = \mu(s)\phi'(st) \neq 0$ and

$$H_s(\phi)(u) - H_s(\phi)(v) = \frac{\mu(s)(\phi(su) - \phi(sv))}{s} \neq 0.$$  

This shows that the map $H_s$ is embedding for all $(s, \phi) \in I \times \mathcal{O}_d$, and hence the image of $H_d$ is contained in $\mathcal{O}_d$. This proves that $\Upsilon_d \circ \Gamma_d$ is homotopic to the identity map of $\mathcal{O}_d$. Also, note that the map $\Gamma_d \circ \Upsilon_d$ is the identity map of $S^2$.

**Corollary 4.17.** The space $\mathcal{O}_d$, for $d \geq 2$, is path connected.

**Corollary 4.18.** The space $\mathcal{P}$ has homotopy type of $S^2$.

**Proof.** Since $\mathcal{P} = \bigcup_{d \geq 3} \mathcal{O}_d$, so we define maps $\Gamma : \mathcal{P} \to S^2$, $\Upsilon : S^2 \to \mathcal{P}$ and $H : I \times \mathcal{P} \to \mathcal{P}$ given by $\Gamma(\phi) = \Gamma_d(\phi)$ for $\phi \in \mathcal{O}_d$, $\Upsilon(x) = \Upsilon_3(x)$ for $x \in S^2$ and $H(s, \tau) = H_d(s, \tau)$ for $(s, \tau) \in I \times \mathcal{O}_d$, where the maps $\Gamma_d, \Upsilon_3$ and $H_d$, for $d \geq 3$, are as defined in the proof of Theorem [1.10]. It is easy to check that the maps $\Gamma$ and $H$ are well defined; that is, for $n > d \geq 3$, $\Gamma_n(\phi) = \Gamma_d(\phi)$ for all $\phi \in \mathcal{O}_d$ and $H_n(s, \tau) = H_d(s, \tau)$ for all $(s, \tau) \in I \times \mathcal{O}_d$.

To prove the continuity of the maps $\Gamma, \Upsilon$ and $H$, let us consider an open set $U$ in $S^2$ and an open set $V$ in $\mathcal{P}$. It is easy to check that $\Gamma^{-1}(U) = \bigcup_{d \geq 3} \Gamma^{-1}(U)$, $\Upsilon^{-1}(V) = \Upsilon_3^{-1}(V \cap \mathcal{O}_3)$ and $H^{-1}(V) = \bigcup_{d \geq 3} H_d^{-1}(V \cap \mathcal{O}_d)$. Furthermore, we have $\Gamma^{-1}(U) \cap \mathcal{O}_d = \Gamma_d^{-1}(U)$ and $H^{-1}(V) \cap I \times \mathcal{O}_d = H_d^{-1}(V \cap \mathcal{O}_d)$ for all $d \geq 3$. This shows that $\Gamma^{-1}(U)$ is open in $\mathcal{P}$ and $\Upsilon^{-1}(V)$ is open in $S^2$, since for any $d \geq 3$, $\Gamma_d^{-1}(U)$ is open in $\mathcal{O}_d$ and $\Upsilon_3^{-1}(V \cap \mathcal{O}_3)$ is open in $S^2$. Also, since $H_d^{-1}(V \cap \mathcal{O}_d)$ is open in $I \times \mathcal{O}_d$ for all $d \geq 3$, $H^{-1}(V) \cap I \times \mathcal{O}_d$ is open in $I \times \mathcal{P}$ with respect to the inductive limit topology of the stratification $I \times \mathcal{P} = \bigcup_{d \geq 3} I \times \mathcal{O}_d$. This implies that the set $H^{-1}(V)$ is open in $I \times \mathcal{P}$ with respect to the product topology, since $I$ is compact and regular (see [5, §8], [7, §18.5] and [11, Lemma 5.5]).

Since for $d \geq 3$ and $\phi \in \mathcal{O}_d$, we have $H_d(0, \phi) = \Upsilon_3(\Gamma_d(\phi))$ and $H_d(1, \phi) = \phi$, so we get $H(0, \tau) = \Upsilon(\Gamma(\tau))$ and $H(1, \tau) = \tau$ for all $\tau \in \mathcal{P}$. This shows that the map $\Upsilon \circ \Gamma$ is homotopic to the identity map of $\mathcal{P}$. Also, note that the map $\Gamma \circ \Upsilon$ is the identity map of $S^2$, because $\Gamma_3 \circ \Upsilon_3$ is the identity map of $S^2$.

**Proposition 4.19.** Every path component of the space $\mathcal{Q}_d$, for $d \geq 2$, is homeomorphic to an open ball in $\mathbb{R}^{3d}$.

**Proof.** Note that $\eta(\mathcal{Q}_d)$ (where $\eta$ is homeomorphism between the spaces $\mathcal{A}_d$ and $\mathbb{R}^{3d}$ as defined in Section 3) is a semialgebraic subset of $\mathbb{R}^{3d}$, so it is a finite union of cells $U_i$, for $i \in \lambda$, in $\mathbb{R}^{3d}$ which are homeomorphic to open hypercubes $(0, 1)^m$ of various dimensions (see [12, §3.3.1]). Since $\eta(\mathcal{Q}_d)$ is open in $\mathbb{R}^{3d}$, so each of the cell $U_i$, for $i \in \lambda$, must have dimension equal to $3d$. In other words each path component
(that is, $U_i$, for $i \in \lambda$) of $\eta(\mathcal{Q}_d)$ is homeomorphic to an open ball in $\mathbb{R}^{3d}$, and hence same is true for the space $\mathcal{Q}_d$.

**Corollary 4.20.** Each path component of $\mathcal{Q}_d$, for $d \geq 2$, is contractible.

5. Conclusion

We have seen that the topology of a set of polynomial knots of degree $d$ depends upon the coefficients of the polynomial belonging to that set. If the set of polynomial knots is flexible in the sense that that many coefficients of the component polynomials can be zero then the space is path connected. In such a space a polynomial knot that is topologically representing a nontrivial knot can also be joined to a trivial knot by a path in that space. This is possible because the polynomial knots are the long knots (the ends are open). When we have the polynomial knots of lower degrees present in the space, by a one parameter family of knots, our knot can slowly shift towards the end and gets opened up to become a trivial knot. For example

$$
\phi(t) = (0, 0, 0.5)t^6 + (0, 1, -0.48)t^5 + (1, -1.5, -19.1167)t^4 + (8.5, -29.11, 11.2832)t^3 + (-31.92, 32.439, 187.195)t^2 + (-164.016, 160.508, -35.8427)t + (51.84, -50.2762, 0),
$$

for $t \in \mathbb{R}$, represents the Figure-eight knot (see Fig 2). This knot is in $\mathcal{Q}_6$ and by Theorem 4.9 it can not be joined to a trivial knot by a path in $\mathcal{Q}_6$. However $\phi$ is element of the space $\mathcal{O}_6$ also. In $\mathcal{O}_6$, we have the path $s \mapsto F_s$ joining $\phi$ to an unknot $t \mapsto (-164.016t, 160.508t, -35.8427t)$, where

$$
F_s(t) = (0, 0, 0.5)(1-s)^5t^6 + (0, 1, -0.48)(1-s)^4t^5 + (1, -1.5, -19.1167)(1-s)^3t^4 + (8.5, -29.11, 11.2832)(1-s)^2t^3 + (-31.92, 32.439, 187.195)(1-s)t^2 + (-164.016, 160.508, -35.8427)t + (51.84, -50.2762, 0)(1-s)
$$

![Fig. 2. $F_s$ for $s = 0$](image_url)

![Fig. 3. $F_s$ for $s = 1/3$](image_url)
A similar thing happens in the space $P_6$ also. We have seen that the spaces $P_d$ are path connected for $d \geq 3$. However it will be interesting to explore the homotopy type of these spaces. For the space $Q_d$, the exact number of path components are still not clear.

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