Maximal Unipotent Monodromy for Complete Intersection CY Manifolds

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Abstract

The computations that are suggested by String Theory in the B model requires the existence of degenerations of CY manifolds with maximum unipotent monodromy. In String Theory such a point in the moduli space is called a large radius limit (or large complex structure limit). In this paper we are going to construct one parameter families of $n$ dimensional Calabi-Yau manifolds, which are complete intersections in toric varieties and which have a monodromy operator $T$ such that $(T^n - id)^{n+1} = 0$ but $(T^n - id)^n \neq 0$, i.e the monodromy operator is maximal unipotent.

Contents

1 Introduction
1.1 General Remarks ........................................... 2
1.2 Description of the Paper .................................... 3
2 Basic Definitions and Notations
2.1 Mumford’s Semi-Stable Reduction Theorem .................. 4
2.2 Geometric and Homological Monodromy (Basic Properties) 4
1 Introduction

1.1 General Remarks

One of the most important problems in algebraic geometry is the study of families of algebraic varieties parameterized by a variety. Of special interest is the subvariety in the parameter space that parameterizes the singular fibers. This subvariety is called the discriminant locus. One of the main invariants of the discriminant locus is the so-called monodromy group. The monodromy group is defined by the action of fundamental group of the complement of the discriminant locus on the cohomology ring of a fixed non-singular fiber. Of special interest is the action of the monodromy group on the middle cohomology group. The structure of such actions is of profound importance. In number theory, the counterpart is the action of Galois group on \( \acute{e} \)tale cohomology.
In this paper we are going to study the simplest case of the above described setting, namely we are going to study families of algebraic manifolds over the unit disk $\mathcal{D}$. We will assume that the only singular fiber is over the center of the disk, i.e. over $0 \in \mathcal{D}$. From the discussion above it follows that we obtain a representation of the fundamental group $\pi_1(\mathcal{D} \setminus 0) = \mathbb{Z}$ in the middle homology, i.e. in $H_n(X_t, \mathbb{Z})/\text{Tor}$. The finite dimensional representations of $\mathbb{Z}$ are classified by the Jordan blocks of the linear operator corresponding to $1 \in \mathbb{Z}$. We will give a complete answer to the structure of the monodromy operator in terms of the topology of the singular fiber. Our method of proof is based on Clemens’ theory of monodromy and the theory of mixed Hodge structures. We find a simple criterium for the monodromy operator to have a Jordan block of maximal rank. This criterium is based on Leray’s theory of residues. We will apply this simple criterium to very concrete examples of complete intersections in $\mathbb{C}P^N$ and toric varieties.

The existence of such degenerations is of prime importance in mirror geometry and in string theory. The computations that were suggested by String Theory in the B model required the existence of degenerations of CY manifolds with maximal unipotent monodromy. In String Theory such a point in the moduli space is called a large radius (complex structure) limit. The case of hypersurfaces in toric varieties was treated in [10], where the construction of a point of maximal degeneracy is done by studying the GKZ hypergeometric system governing periods of the hypersurfaces.

For recent important developments in mirror geometry see [11], [12] and references therein. The results of this paper are closely related to Strominger-Yau-Zaslow conjecture.

1.2 Description of the Paper

In Section 2 we introduce the basic notions and review some results stated in [8].

In Section 3 we review the generalization of Lefschetz’s theory of vanishing cycles due to Clemens. We describe Clemens’ method for constructing Jordan blocks in the monodromy operator.

In Section 4 we prove a general formula for the number of Jordan blocks in the monodromy operator in terms of some invariants of the singular fiber of one parameter family of Kähler manifolds.

In Section 5 we prove a simple criterium for the existence of a Jordan block of maximum size in the monodromy operator in terms of Leray’s residue calculus. We also construct families of complete intersections of CY manifolds in $\mathbb{C}P^N$ whose monodromy operators contain Jordan block of maximum size.

In Section 6 we review some basic facts in toric geometry and construct families of complete intersections of CY manifolds in toric varieties whose monodromy operators contain Jordan block of maximum size generalizing the construction in section 5.

In Section 7 we briefly discuss the connection between the present approach and a previous approach which uses hypergeometric functions. We also discuss
an interesting relationship between maximal unipotent monodromy and the SYZ
conjecture, and illustrate this in the case of polarized K3 surfaces.

In the Appendix, we give a complete description of Clemens’s cell complex
for hypersurfaces in a toric variety.

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2 Basic Definitions and Notations

2.1 Mumford’s Semi-Stable Reduction Theorem

In this article we study one-parameter families of \( n \)-dimensional Kähler mani-
folds

\[
\pi: \mathcal{X} \to \mathcal{D}
\]

over a disk \( \mathcal{D} \). We assume that \( \mathcal{X} \) is a smooth algebraic manifold and that
for each \( t \neq 0 \) \( \pi^{-1}(t) = X_t \) is a non-singular \( n \)-dimensional Kähler manifold.
From Hironaka’s theorem on the resolution of singularities, we may assume that
the singular fibre \( X_0 = \pi^{-1}(0) \) is a divisor with normal crossings \( X_0 = \bigcup C_i \).

We will choose local coordinates in the following manner. Let \( x_{n+1} \in C_{i_1} \cap \ldots \cap C_{i_{n+1}} \subset X_0 \). Let \( U_{x_{n+1}} \) be an open polycylinder in \( \mathcal{X} \) containing the point
\( x_{n+1} \). Let \( z_{i_j} = 0 \) be the defining equation of the divisor \( C_{i_j} \) in \( U_{x_{n+1}} \). It is easy

to see that after shrinking the disk \( \mathcal{D} \), the fibers of the map \( \pi: \mathcal{X} \cap U_{x_{n+1}} \to \mathcal{D} \)
are locally given by

\[
z_{i_1}^{m_1} \ldots z_{i_{n+1}}^{m_{n+1}} = t, \quad m_j \geq 1 \land m_j \in \mathbb{Z}.
\]

In the same manner, let \( x_k \in X_0 \) and let \( C_{i_1}, \ldots, C_{i_k} \) be those components of
\( X_0 \) containing \( x_k \). Let \( U_{x_k} \) be an open polycylinder in \( \mathcal{X} \) containing \( x_k \) but not
intersecting \( C_j \) with \( j \neq i_1, \ldots, i_k \). Then the fibers of the map \( \pi: \mathcal{X} \cap U_{x_k} \to \mathcal{D} \)
are locally given by

\[
z_{i_1}^{m_1} \ldots z_{i_k}^{m_k} = t, \quad m_j \geq 1 \land m_j \in \mathbb{Z}.
\]

In [8] Mumford proved that after taking a finite covering of the disk \( \mathcal{D} \), lifting
the family and resolving the singularities, we may assume that the the fibres of
the map \( \pi: \mathcal{X} \to \mathcal{D} \) are given locally by \( z_1^{k_1} \ldots z_{n+1}^{k_{n+1}} = t \), where \( k_j \) is either 0 or 1.
From now on we are going to assume that we are in the above setting justified
by Mumford’s theorem.

2.2 Geometric and Homological Monodromy (Basic Props-
ties)

If we restrict our family \( \pi: \mathcal{X} \to \mathcal{D} \) to the circle \( S^1 := \partial \mathcal{D} \) then we get a
representation of \( \pi_1(S^1) = \mathbb{Z} \) in the group of diffeomorphisms of \( X_t \). Indeed if
we remove a point from $S^1$, and restrict our family to $S^1 \setminus s$, $\pi_1 : \mathcal{X}|_{S^1} \to S^1$ where $s \in S^1$, we will get a trivial $C^\infty$ family $(S^1 \setminus s) \times X_t \to S^1 \setminus s$. Our family $\pi_1 : \mathcal{X}|_{S^1} \to S^1$ is obtained from the trivial family $(S^1 \setminus s) \times X_t \to S^1 \setminus s$ by "gluing" it by $\phi \in Diff(X_t)$. The diffeomorphism $\phi$ is the generator of $\pi_1(S^1) = \mathbb{Z}$. We will call $\phi$ the geometric monodromy. The induced action of $\phi$ on $H_n(X_t, \mathbb{Z})$ will be called the monodromy operator and will be denoted by $T$.

The main result about the operator $T$ is that we have always:

$$(T^N - id)^{n+1} = 0$$

for some positive integer $N$. Here $n$ is the complex dimension of $X_t$. This theorem was proved by many mathematicians including Griffiths, N. Katz, Clemens, Landesman, Deligne and so on.

From Mumford’s result, we can assume that the following conditions hold for the family $\pi : \mathcal{X} \to D$:

1. $\pi^{-1}(0) = X_0 = \bigcup_{i=0}^m C_i$ is a divisor of normal crossings and for $k = 0, \ldots, n$, $C_{i_0} \cap \cdots \cap C_{i_k}, i_0 < \ldots < i_k$, is a non-singular irreducible subvariety of $\text{Sing}_{n-k}(X_0)$ if non-empty.

2. The fibers of the map $\pi : \mathcal{X} \to D$ are locally given in the open policylin-
ders $\{U\}$ defined above by $z_{i_1}^{k_1} \cdots z_{i_{n+1}}^{k_{n+1}} = t$, where $k_j$ is either 0 or 1.

**Definition 1** If the family $\pi : \mathcal{X} \to D$ satisfies the above conditions we will say that it is in normal form.

From now on we will consider only families $\pi : \mathcal{X} \to D$ in normal form.

**Notation 2** Given $X_0 = C_0 \cup \ldots \cup C_m$ we shall use the following notations:

$I = \{i_0, \ldots, i_k\}$ is an index set with $i_0 < \ldots < i_k$

$C_I = C_{i_0} \cap \cdots \cap C_{i_k}, |I| = k + 1$; and

$C^{[k]} = \bigcup_{|I|=k+1} C_I$.

### 3 Review of Clemens’ Theory of Geometric Monodromy

#### 3.1 Construction of Clemens’ retraction map

Let $\pi : \mathcal{X} \to D$ be a family of Kähler manifolds as defined in Definition 1. We will construct a contraction map:

$\mathcal{C} : \mathcal{X} \to X_0$. 
3.1.1 Construction of Clemens’ Vector Field

The local description of our family $\pi: \mathcal{X} \to \mathcal{D}$ is given in $\mathbb{C}^{n+1}$ by the equation:

$$z_1\ldots z_k = t$$

For some $k = 1, \ldots, n + 1$. Without loss of generality we may assume that $t$ is a real number.

Let $z_i = r_ie^{2\pi\sqrt{-1}\phi_i}$, then the equation $z_1\ldots z_k = t$ for $k = 1, \ldots, n + 1$ is equivalent to the equations:

$$r_1\ldots r_k = t$$

and

$$\sum_{i=1}^{k} \phi_i = 0$$

for $k = 1, \ldots, n + 1$.

We will construct first a local vector field which will define the retraction map in this local situation and then using partition of unity we will construct a global vector field and thus the retraction map. It is easy to see that it is enough to construct a retraction of hyperbola in $\mathbb{R}^{n+1}$ given by

$$r_1\ldots r_{n+1} = t$$

to the union of coordinate hyperplanes $r_1 = 0, \ldots, r_{n+1} = 0$, where $r_i > 0$ and $t > 0$. Let us suppose that we will consider only hyperbolas $r_1\ldots r_{n+1} = t$ for $0 < t < \frac{1}{2}$. We will construct special vector field in $\mathbb{R}^{n+1}$. We will need to define some notions.

We will denote by $Q$ the following set in $\mathbb{R}^{n+1}$

$$Q := \{ r = (r_1, \ldots, r_{n+1}) | r_i \geq 0 \text{ and there exists } 1 \leq i \leq n + 1 \text{ such that } 0 < r_i < 1 \}.$$

$Q(1)$ will be the unit cube in $\mathbb{R}^{n+1}$, i.e.

$$Q(1) := \{ r = (r_1, \ldots, r_{n+1}) | 0 \leq r_i \leq 1 \text{ for all } 1 \leq i \leq n + 1 \}.$$

It is clear that for $0 < t < \frac{1}{2}$ the hyperbolas $r_1\ldots r_{n+1} = t$ are contained in $Q$.

3.1.2 Construction of the Vector Field in $\mathbb{R}^{n+1}$

Let us suppose that $r = (r_1, \ldots, r_{n+1}) \in Q$ is a point such that $0 \leq r_i \leq 1, \ldots, 0 \leq r_{ik} \leq 1$. To the point $r \in Q$ we will assign a point $r(i_1, \ldots, i_k) = (r_1, \ldots, r_{n+1}) \in \mathbb{R}^{n+1}$ $r_{i_1} = \ldots = r_{i_k} = 1$ and the rest of the coordinates of $r(i_1, \ldots, i_k)$ are the same as the point $r \in Q$. Let $l(r)$ be the line that joints the point $r \in Q \subset \mathbb{R}^{n+1}$ $r(i_1, \ldots, i_k) \in \mathbb{R}^{n+1}$ with the point $r(i_1, \ldots, i_k) \in \mathbb{R}^{n+1}$. In this way we define a vector field in $Q \subset \mathbb{R}^{n+1}$. Using a partition of unity and using the vector field in $Q$ that we have just defined, we obtain a global vector field on $\pi: \mathcal{X} \to \mathcal{D}$.

When we integrate this vector field, we obtain the Clemens’ contraction map $C : \mathcal{X} \to X_0$. Thus we obtain a map for each $t \in \mathcal{D} \setminus \{0\} : C_t : X_t \to X_0$.

**Definition 3** The map $C_t$ as defined above will be called Clemens map.

For details of the construction see [4].
3.2 Properties of the Clemens’ map

It is easy to prove the $C_i$ has the following properties:

**Lemma 4** The Clemens’ map $C_i$ has the following properties: (i) Suppose $z \in C_i$, then $C_i^{-1}(z) = (S^1)^k$ is a $k$ dimensional real torus and (ii) $C_i$ defines a diffeomorphism between $X_i \setminus C_i^{-1}(\text{Sing}(X_0))$ and $X_0 \setminus \text{Sing}(X_0)$.

**Proof of Lemma** For the proof of Lemma see [4].

**Definition 5** Let $T(i_1, ..., i_k)$ be the tubular neighborhood of $C_{i_0} \cap ... \cap C_{i_k}$ in $C_{i_1} \cap ... \cap C_{i_k}$. We will denote by $p(i_1, ..., i_k) : T(i_1, ..., i_k) \rightarrow C_{i_0} \cap C_{i_1} \cap ... \cap C_{i_k}$ the projection maps for any $i_1, ..., i_k$.

**Definition 6** Given an $(n-k)$-cycle $\gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$, we define an $n$-cycle $p_k^{-1}(\gamma) \in H_n(X_0 \setminus \text{Sing}(X_0))$ as follows. Let

$$p_1^{-1}(\gamma) := \partial(p(i_1, ..., i_k)^{-1}(\gamma)).$$

This is the boundary of $p(i_1, ..., i_k)^{-1}(\gamma)$, hence it is cycle of dimension $n - k + 1$ in $C_{i_1} \cap ... \cap C_{i_k} \setminus C_{i_0} \cap ... \cap C_{i_k}$. Let

$$p_2^{-1}(\gamma) = \partial(p(i_2, ..., i_k)^{-1}p_1^{-1}(\gamma)).$$

This is a cycle of dimension $n - k + 2$ in $C_{i_2} \cap ... \cap C_{i_k} \setminus C_{i_1} \cap ... \cap C_{i_k}$. By continuing this way, we define at the end

$$p_k^{-1}(\gamma) = \partial(p(i_k)^{-1}p_{k-1}^{-1}(\gamma)).$$

This is a cycle of dimension $n$ in $C_{i_k} \setminus C_{i_{k-1}} \cap C_{i_k}$. We denote by $\pi_k : p_k^{-1}(\gamma) \rightarrow \gamma$ the natural projection.

**Proposition 7** For any point $z \in \gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$ we have $\pi_k^{-1}(z) = (S^1)^k$, where $\pi_k : p_k^{-1}(\gamma) \rightarrow \gamma$ is defined above. In other words the preimage of a point is a $k$ dimensional real torus.

**Proof of Proposition** Proposition follows directly from Definition.

**Definition 8** Given $\gamma \in H_{n-k}(C_{i_0} \cap ... \cap C_{i_k}, \mathbb{Q})$, we denote $\gamma_t := C_t^{-1}(\gamma)$ and $\gamma_t(j) := C_t^{-1}(p_j^{-1}(\gamma))$, which are $n$-cycles representing elements in $H_n(X_t, \mathbb{Q})$.

**Lemma 9** The cycles $\gamma_t$ and $\gamma_t(j)$ above are homological to each other for $1 \leq j \leq k$. 
We sketch the construction here. (See [4] for details.)

Theorem 10 Let $\gamma \in H_{n-k}(\text{Sing}(X_0), \mathbb{Z})$ be such that $C_{t-1}^{-1}(\gamma) = \gamma_t \in H_n(X_1, \mathbb{Z})$ is a non-zero. Then there exists cycles $\alpha_1, \ldots, \alpha_k \in H_n(X_1, \mathbb{Z})$ such that $T(\alpha_j) = \gamma_t + \sum_{i=1}^j \alpha_i$ for $1 \leq j \leq k$.

Clemens’ construction of Jordan block by Picard-Lefschetz Duality. Let $\gamma \in H_{n-k}(C_{t_0} \cap \ldots \cap C_{t_k}, \mathbb{Z})$ be a cycle such that $C_{t-1}^{-1}(\gamma)$ be a non-zero element in $H_n(X_1, \mathbb{Z})$. In Definition 3 we defined a cycle $p_1^{-1}(\gamma)$ in $C_{t_2} \cap \ldots \cap C_{t_k} \cap C_{t_1} \cap \ldots \cap C_{t_k}$. It is easy to see by using the fact that $C_{t-1}^{-1}(\gamma) \in H_n(X_1, \mathbb{Z})$ is non-zero and Lemma 3 that

$$p_1^{-1}(\gamma) \in H_{n-k+1}(C_{t_2} \cap \ldots \cap C_{t_k} \cap C_{t_1} \cap \ldots \cap C_{t_k}, \mathbb{Z}) \& p_1^{-1}(\gamma) \neq 0.$$

Let us denote by $\gamma_1 \in H_{n-k+1}(C_{t_2} \cap \ldots \cap C_{t_k} \cap C_{t_1} \cap \ldots \cap C_{t_k}, \mathbb{Z})$ the Picard-Lefschetz dual cycle of $p_1^{-1}(\gamma)$. Let $T_{i_3, \ldots, i_k}(\gamma_1)$ be the tubular neighborhood of the closure of $\gamma_1$ in $C_{t_2} \cap \ldots \cap C_{t_k} \cap C_{t_1} \cap \ldots \cap C_{t_k}$. Let us denote by $p_2^{-1}(\gamma_1)$ the boundary of $T_{i_3, \ldots, i_k}(\gamma_1)$, i.e.

$$p_2^{-1}(\gamma_1) = \partial T_{i_3, \ldots, i_k}(\gamma_1).$$

It is easy to see that

$$p_2^{-1}(\gamma_1) \in H_{n-k+2}(C_{t_3} \cap \ldots \cap C_{t_k} \cap C_{t_1} \cap \ldots \cap C_{t_k}, \mathbb{Z}).$$

Let us denote by $\gamma_3 \in H_{n-k+2}(C_{t_3} \cap \ldots \cap C_{t_k} \cap C_{t_1} \cap \ldots \cap C_{t_k}, \mathbb{Z})$ the Picard-Lefschetz dual to $p_2^{-1}(\gamma_1)$. We can continue this process and thus we will define cycles $\gamma, \gamma_1, \ldots, \gamma_k$, where $\gamma_j \in H_{n-k+j}(C_{t_j} \cap \ldots \cap C_{t_k} \cap C_{t_{j-1}} \cap \ldots \cap C_{t_k}, \mathbb{Z})$. Clemens proved in [4] that the monodromy operator $T$ acts as follows on $C_{t-1}^{-1}(\gamma), C_{t-1}^{-1}(\gamma_1), \ldots, C_{t-1}^{-1}(\gamma_k)$:

$$T(C_{t-1}^{-1}(\gamma)) = C_{t-1}^{-1}(\gamma), \ldots, T(C_{t-1}^{-1}(\gamma_k)) = C_{t-1}^{-1}(\gamma) + \sum_{j=1}^k C_{t-1}^{-1}(\gamma_j).$$

Corollary 11 Let $\pi : \mathcal{X} \to \mathcal{D}$ be a family of Kähler manifolds over the disk such that:

1. For $t \neq 0$, $\pi^{-1}(t) := X_t$ is a non-singular variety of complex dimension $n \geq 1$. 

8
2. \( \pi^{-1}(0) = X_0 = \bigcup_{i=0}^{m} C_i \) is a divisor of normal crossing and \( \pi \) is locally given by \( z_1^{n_1} \times \ldots \times z_k^{n_k} = t \), where \( n_i \) are positive integers.

3. Suppose \( C_0 \cap \ldots \cap C_n \) is a point and \( C_i^{-1}(C_0 \cap \ldots \cap C_n) = \gamma_i \) is a non zero cycle in \( H_n(X_t, \mathbb{Q}) \),

then the monodromy operator of the family \( \pi : X \to D \) has a Jordan block of size \( n + 1 \). (See [4].)

4 The Jordan Normal Form of the Monodromy Operator

We begin by introducing the following combinatorial invariant of a family of algebraic varieties \( \pi : X \to D \) put in a Mumford form. This will be needed later.

**Definition 12** We will define Clemens’ simplicial complex of the family \( \pi : X \to D \) as follows: To each divisor \( C_i \) we attached a point \( p_i \) in \( \mathbb{R}^d \), where \( d \) is a large enough integer. We will assume that the points \( p_i \) are in general position, i.e. they do not lie in a hyperplane. If \( C_i \) intersects \( C_j \) then we attach to the points \( p_i \) and \( p_j \) a one dimensional simplex. If \( C_i, C_j \) and \( C_k \) intersect then we attached a two dimensional simplex on \( p_i, p_j \) and \( p_k \). We continue in that manner and we obtain a simplicial complex \( \Pi(X_0) \) which we will call Clemens’ simplicial complex.

4.1 Definition of the Gysin Map

**Definition 13** Let \( X \) be a compact complex manifold, and \( C \) be a divisor of normal crossing. The map \( G_k : H_{k+2}(X, \mathbb{Z}) \to H_k(C, \mathbb{Z}) \) defined by \( G_k(\gamma) := \gamma \cap [C] \), where \( \gamma \cap [C] \) means intersection of class of cohomology in \( X \), is called the Gysin map.

**Remark 14** We will use the Gysin map in case \( C_{i_0} \cap \ldots \cap C_{i_k} \subset C_{i_1} \cap \ldots \cap C_{i_k} \) and will denote by

\[
G_k : \bigoplus_{i_1, \ldots, i_k} H_{n-k+2}(C_{i_1} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \to H_{n-k}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q})
\]

where \( G_k(\gamma) \) is the image of the cycle \( [\gamma \cap [C_{i_0} \cap \ldots \cap C_{i_k}]] \) in \( H_{n-k}(\text{Sing}_{n-1}(X_0), \mathbb{Q}) \).

**Definition 15** The dual \( G^*_k \) of the Gysin map using Poincare duality is defined as follows for the pair \( (X, C) \)

\[
G^*_k : H^k(C, \mathbb{Z}) \to H^{k+2}(X, \mathbb{Z})
\]

where \( G^*_k(\alpha) = \alpha \wedge c_1(C) \) and \( c_1(C) \) is the first Chern class of the line bundle defined by the normal crossing divisor \( C \).
4.2 Review of Deligne’s Theory of Mixed Hodge Structures

The cohomology of $X_0 \setminus \text{Sing}(X_0)$ can be computed as the cohomology of the de Rham log complex $\mathcal{A}^*(X_0, \log <\text{Sing}(X_0)>)$.

**Definition 16** We will say that a form $\omega$ on one of the components $C_i$ of $X_0$ had a logarithmic singularities if for each point $z \in X_0$ and some open neighborhood $U \subset C_i$ of the point $z$ we have

$$\omega|_U = \alpha \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_k}{z_k},$$

where $\alpha$ is a $C^\infty$ form in $U$ and locally on $X_0$ is defined by the equations $z_1 \times ... \times z_k = 0$.

**Definition 17** We define the de Rham log complex as follows:

$$\mathcal{A}^*(X_0, \log <\text{Sing}(X_0)>) = \{\omega \in C^\infty (X_0 \setminus \text{Sing}(X_0), \Omega^*) | \omega \text{ and } d\omega \text{ are } C^\infty \text{ forms on } X_0 \setminus \text{Sing}(X_0) \text{ which have log singularities on } \text{Sing}(X_0) \}.$$

**Remark 18** It is easy to see that if $\omega \in \mathcal{A}^m(X_0, \log <\text{Sing}(X_0)>)$ then $\omega$ locally around a point $z \in U \subset C_i$ in each of the components of $X_0$ and $z \in \text{Sing}(X_0)$ is given by

$$\omega|_U = \omega_1 \wedge \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_k}{z_k},$$

where $\omega_1$ is a $C^\infty$ form on $U \subset C_i$ and $\text{Sing}(X_0) \cap U$ is given by $z_1 \cdot ... \cdot z_k = 0$ in $U$.

Deligne proved that there exists a mixed Hodge structure on $X_0 \setminus \text{Sing}(X_0)$.

The existence of Mixed Hodge Structure is based on the following filtration:

**Definition 19** On the complex $\mathcal{A}^*(X_0, \log <\text{Sing}(X_0)>)$ we define the weight filtration $W_l$ to be those forms $\phi$ that locally around $\text{Sing}(X_0)$

$$\phi \in \mathcal{A}^*(U) \left\{ \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_k}{z_k} \right\}.$$

**Definition 20** The Poincare Residue Operator $R^{[k]}: W_k \rightarrow \mathcal{A}^{*-k}(C^{[k]})$ is defined by

$$R^{[k]} \left( \alpha \wedge \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_k}{z_k} \right) = \alpha |_{C_i}.$$

**Definition 21** Let us consider the decreasing filtration $\ldots \supset W^{-l} \supset W^{-l+1} \supset \ldots$ where $W^{-l} = W_l$. Accordingly there is a spectral sequence $\{E_r\}$ such that $E_\infty$ is the associated graded to the weight filtration in $H^*(X_0 \setminus \text{Sing}(X_0), \mathbb{C})$. 

10
The filtration was reversed by Deligne so that we can form a spectral sequence of the filtered de Rham logarithmic complex. By using the Poincare residue map Deligne proved the following Theorems:

**Theorem 22** The cohomology of $X_0 \backslash \text{Sing}(X_0)$ are equal to the cohomology of the De Rham log complex $\mathcal{A}^*(X_0, \log <\text{Sing}(X_0)>)$.

(For the proof of Theorem 22 see [6].)

**Theorem 23**

i. The spectral sequence defined as above degenerates at the second step.

ii. $E_1 \cong \oplus H^*(C_i)$ and the mapping $d_1 : E_1 \rightarrow E_1$ is a morphism of Hodge structures given by the Gysin map (See Definition 13.)

$$H^*(C_{i_0} \cap \ldots \cap C_{i_l}) \rightarrow H^*(C_{i_1} \cap \ldots \cap C_{i_l}).$$

4.3 Jordan Normal Form of the Monodromy Operator

In this section we will prove the following Theorem:

**Theorem 24**

i. The number of Jordan blocks of rank $k \leq n$ is equal to the rank of the group

$$H_{n-k}(\text{Sing}_k(X_0), \mathbb{Q})/\text{Im}(G_k),$$

where $G_k$ is the Gysin map.

ii. The number of Jordan blocks of rank $n+1$ is equal to $\dim H_{n}(\Pi(X_0), \mathbb{Q})$, where $\Pi(X_0)$ is the Clemens’ polyhedra defined in Definition 13.

**Proof of Theorem 24** The proof of both part is based on Corollary [1].

Part i of Theorem 24 follows directly from the following three Lemmas and Theorem 10.

**Lemma 25** Let $\gamma \in \text{Im} G_k \subseteq H_{n-k}(\text{Sing}_k(X_0), \mathbb{Q})$, then $\gamma_t = C_{i_t}^{-1}(\gamma)$ is homological to zero in $X_t$.

**Lemma 26** Suppose that $\gamma \in H_{n-k}(\text{Sing}_k(X_0), \mathbb{Q})/\text{Im} G_k$ and $\gamma \notin \text{Im} G_k$, then there exists a non zero class of cohomology $\tilde{\omega} \in H^n(X_0 \backslash \text{Sing}(X_0))$ such that

$$\int_{p^{-1}_k(\gamma)} \tilde{\omega} \neq 0,$$

where the cycle $p^{-1}_k(\gamma) \in H_n(X_0 \backslash \text{Sing}(X_0), \mathbb{Q})$ is defined in Definition 4.

**Lemma 27** Let $\gamma \in H_{n-k}(\text{Sing}_k(X_0), \mathbb{Q})/\text{Im} G_k$ and $\gamma \notin \text{Im} G_k$, then $\gamma_t$ is a non zero element in $H_n(X_t, \mathbb{Q})$.

**Proof of Lemma 25** We will prove part i. Suppose that $\gamma \in \text{Im} G_k \subseteq H_{n-k}(\text{Sing}_k(X_0), \mathbb{Q})$. 

11
From the Definition \ref{definition13} of the Gysin map it follows that there exists a cycle
\[ \Gamma \in \bigoplus_{i_1, \ldots, i_k} H_{n-k+2}(C_{i_1} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \]

such that \( \Gamma \cap \bigoplus_{i_0, \ldots, i_k} [C_{i_0} \cap \ldots \cap C_{i_k}] = \gamma \). The definition \ref{definition3} of the Clemens map it follows that the boundary of \( C_{n}^{-1}(\Gamma \setminus \gamma) \) is exactly \( C_{n}^{-1}(\gamma) \). Lemma \ref{lemma23} is proved.

\textbf{Proof of Lemma} \ref{lemma23}. The construction of the form \( \omega \) is based on the following fact about the cohomology (homology) of \((X_0, X_0)\), where \( X_0 \) is a Kähler variety and \( \text{Sing}(X_0) \) is a divisor with normal crossings in \( X_0 \).

Suppose that \( \gamma \in H_{n-k}(\text{Sing}_k(X_0), \mathbb{Q}) \) and \( \gamma \notin \text{Im} G_k \). Let \( \gamma \in H_{n-k}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \).

In order to construct the form \( \omega \) we will need to recall the how the dual \( G_k^* \) to the Gysin map \( G_k : H_{k+2}(X, \mathbb{Z}) \rightarrow H_k(C, \mathbb{Z}) \) is defined in Definition \ref{definition15} as follows:

\[ G_k^* : H^k(C, \mathbb{Z}) \rightarrow H^{k+2}(X, \mathbb{Z}), \quad \text{where } G_k^*(\alpha) = \alpha \wedge c_1[C]. \]

\( c_1[C] \) is the Chern class of the line bundle defined by the divisor with normal crossings \( C \) in \( X \).

We will need the following Proposition:

\textbf{Proposition} \ref{proposition28} Let \( \gamma \in H_{n-k}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \) and \( \gamma \notin \text{Im} G_k \). Let \( \omega_{n-k} \in H^{n-k}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \) and
\[
\int_\gamma \omega_{n-k} \neq 0
\]

then \( \omega_{n-k} \) cannot be represent as follows \( \omega_{n-k} = \left(c_1[C_{i_0} \cap \ldots \cap C_{i_k}]|c_{i_0} \cap \ldots \cap C_{i_k}\right) \wedge \omega_1 \) on \( C_{i_0} \cap \ldots \cap C_{i_k} \), where \( \omega_1 \in H^{n-k-2}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \).

\textbf{Proof of Proposition} \ref{proposition28}. Suppose that \( \omega_{n-k} \in H^{n-k}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \),
\[
\int_\gamma \omega_{n-k} \neq 0.
\]

and \( \omega_{n-k} = \left( c_1[C_{i_0} \cap \ldots \cap C_{i_k}]|c_{i_0} \cap \ldots \cap C_{i_k}\right) \wedge \omega_1 \) on \( C_{i_0} \cap \ldots \cap C_{i_k} \). Let \( \eta \) be anon zero section of the line bundle \( \mathcal{O}(\left[C_{i_0} \cap \ldots \cap C_{i_k}\right]) \) on \( C_{i_0} \cap \ldots \cap C_{i_k} \) such that the zero set of \( \eta \) is exactly \( C_{i_0} \cap \ldots \cap C_{i_k} \). Let us consider the form
\[
(d \log(\eta)) \wedge \left( c_1[C_{i_0} \cap \ldots \cap C_{i_k}]|c_{i_0} \cap \ldots \cap C_{i_k}\right) \wedge \omega_1
\]
on \( C_{i_0} \cap \ldots \cap C_{i_k} \). Let us consider the cycle \( p_{i_0}^{-1}(\gamma) \in H_{n-k+1}(C_{i_0} \cap \ldots \cap C_{i_k} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \) as defined in Definition \ref{definition4}.

Let us compute
\[
\int_{p_{i_0}^{-1}(\gamma)} (d \log(\eta)) \wedge \left( c_1[C_{i_0} \cap \ldots \cap C_{i_k}]|c_{i_0} \cap \ldots \cap C_{i_k}\right) \wedge \omega_1.
\]

Since locally around a point \( w \in C_{i_0} \cap \ldots \cap C_{i_k} \) the divisor \( C_{i_0} \cap \ldots \cap C_{i_k} \) in \( C_{i_0} \cap \ldots \cap C_{i_k} \) is given by \( z = 0 \), where \( w \in U \subset C_{i_0} \cap \ldots \cap C_{i_k} \), we see that
\[ d\log(\eta)|_U = \frac{dz}{z}. \]

From this local expression of \( d\log(\eta) \) and the definition of \( p_1^{-1}(\gamma) \) we deduce that

\[ \int_{p_1^{-1}(\gamma)}(d\log(\eta)) \wedge (c_1[C_{i_0} \cap \ldots \cap C_{i_k}]) \wedge \tilde{\omega}_1 = 2\pi \int_\gamma \omega_{n-k} \]

where \( \omega_{n-k} = (c_1[C_{i_0} \cap \ldots \cap C_{i_k}]|_{C_{i_0} \cap \ldots \cap C_{i_k}}) \wedge \omega_1 \) and \( \tilde{\omega}_1 \) is a closed form in then tubular neighborhood of \( C_{i_0} \cap \ldots \cap C_{i_k} \) in \( C_{i_0} \cap \ldots \cap C_{i_k} \) such that the restriction of \( \tilde{\omega}_1 \) on \( C_{i_0} \cap \ldots \cap C_{i_k} \) is \( \omega_1 \). On the other hand since the restriction of the line bundle \( \mathcal{O}(C_{i_0} \cap \ldots \cap C_{i_k}) \) on \( C_{i_0} \cap \ldots \cap C_{i_k} \) \( \cap \) \( C_{i_0} \cap \ldots \cap C_{i_k} \) is the trivial line bundle, we deduce that the form \( c_1[C_{i_0} \cap \ldots \cap C_{i_k}] \) will be an exact form on \( C_{i_0} \cap \ldots \cap C_{i_k} \cap C_{i_0} \cap \ldots \cap C_{i_k} \), i.e. \( c_1[C_{i_0} \cap \ldots \cap C_{i_k}] = d\beta \). This implies that

\[ (d\log(\eta)) \wedge (c_1[C_{i_0} \cap \ldots \cap C_{i_k}]) \wedge \tilde{\omega}_1 = (d\log(\eta)) \wedge d\beta \wedge \tilde{\omega}_1 = 0. \]

So Stoke’s Theorem implies that

\[ \int_{p_1^{-1}(\gamma)}(d\log(\eta)) \wedge (c_1[C_{i_0} \cap \ldots \cap C_{i_k}]) \wedge \tilde{\omega}_1 = \int_{p_1^{-1}(\gamma)} d((d\log(\eta)) \wedge \beta \wedge \tilde{\omega}_1) = \int_{\partial(p_1^{-1}(\gamma))}(d\log(\eta)) \wedge \beta \wedge \tilde{\omega}_1 = 0 \]

The last equality follows from the fact that \( \partial(p_1^{-1}(\gamma)) = \emptyset \). So we can conclude that

\[ \int_{p_1^{-1}(\gamma)}(d\log(\eta)) \wedge (c_1[C_{i_0} \cap \ldots \cap C_{i_k}]) \wedge \tilde{\omega}_1 = 2\pi \int_\gamma \omega_{n-k} = 0. \]

On the other hand we know that

\[ \int_\gamma \omega_{n-k} \neq 0. \]

So we got a contradiction. Proposition \ref{12} is proved. ■

In order to finish the proof of our Theorem we will need some facts from the Theory of Mixed Hodge Structures.

It is easy to see that \( \gamma \notin \text{Im} G_k \) implies that there exists \( \omega_{n-k} \in H^{n-k}(C_{i_0} \cap \ldots \cap C_{i_k}, \mathbb{Q}) \) such that

\[ \int_\gamma \omega_{n-k} \neq 0. \]

We may assume that \( \omega_{n-k} \) is the Poincare dual of \( \gamma \). The condition \( \gamma \notin \text{Im} G_k \) implies that \( \omega_{n-k} \notin \text{Im} G_k \). So from here, Theorem \ref{12} and Proposition \ref{12} we conclude that we can find a form \( \tilde{\omega} \) on \( X_0 \setminus \text{Sing}(X_0) \) such that \( R^k[\tilde{\omega}] = \omega_{n-k} \). From the theory of Leray residues it follows that

\[ \int_{p_k^{-1}(\gamma)} \tilde{\omega} = \int_\gamma \omega_{n-k} \neq 0. \]
Lemma 27 is proved. ■

Proof of Lemma 27: We need to prove that if $\gamma \notin \text{Im } G_k$, then $\gamma_\ell = C_\ell^{-1}(\gamma)$ represent a non-zero class of cohomology in $H_n(X_\ell, \mathbb{Q})$. Let $\omega_\ell = C_\ell^*(\widetilde{\omega})$. It is easy to see from the definition of the Clemens map that $\omega_\ell$ is a well defined closed $n$-form on $X_\ell$. On the other hand since $C_\ell$ is a diffeomorphism between $X_\ell \setminus C_\ell^{-1}(\text{Sing}(X_\ell))$ and $X_0 \setminus \text{Sing}(X_0)$ we deduce that

$$\int_{p_\ell^{-1}(\gamma)} \omega_{\ell} = \int_{\gamma_\ell} \omega_{\ell} = \int_{\gamma} \omega_{n-k} \neq 0.$$

The last inequality implies Lemma 27, Lemma 27 is proved. ■

Proof of Theorem 24 i: Theorem 24 follows directly from Theorem 10 and Lemma 27, Theorem 24 part i is proved. ■

Proof of Theorem 24 ii: Let $\alpha_1, \ldots, \alpha_k$ be a basis of cycles of $H_n(\Pi(X_0), \mathbb{Q})$, where $\Pi(X_0)$ is defined in Definition 12. From the definition of $\Pi(X_0)$ we can assume that the cycle $\alpha_k$ consists of the $n$-dimensional simplices $S_{1k}, \ldots, S_{N_k}$ such that the boundary of the cycle $\alpha_k$ is zero. Each $n$-dimensional simplex $S_i$ corresponds to a point $q_i = C_{j_0,i} \cap \ldots \cap C_{j_n,i}$ according to Definition 12. The fact that the $n$-dimensional simplices $S_{1k}, \ldots, S_{N_k}$ form a cycle means say that any singular points $q_i$ and $q_j$ can be joint by Riemann surface, which means that they lie on some $C_{ji,1} \cap \ldots \cap C_{ji,n}$, $\ldots$ This follows directly from the fact that the boundary of the cycle formed from $S_{1k}, \ldots, S_{N_k}$ is zero. So on each Riemann surface of the form $C_{ji,1} \cap \ldots \cap C_{ji,n}$ that contains the points $q_i$ and $q_j$ we can find a meromorphic form of the third kind $\omega_{ij}$ which has poles only at the points $q_i$ and $q_j$, with residues say $+1$ at $q_i$ and $-1$ at $q_j$. From the spectral sequence defined in Definition 23, Theorem 22 and Theorem 23 we deduce that there is a holomorphic form $\omega_k$ in $X_0 \setminus \text{Sing}(X_0)$ such that Poincare residue of this form on each Riemann surface of the form $C_{ji,1} \cap \ldots \cap C_{ji,n}$ that contains the points $q_i$ and $q_j$ is equal to $\omega_{ij}$. Let the meromorphic form $\omega_k$ be non-zero on the component $C_{j_0,i}$ of $\text{Sing}(X_0)$, where $q_i = C_{j_0,i} \cap \ldots \cap C_{j_n,i}$ and $q_i$ was defined as above. Suppose that the divisors $C_{j_0,i} \cap C_{j_m,i}$ are given by the equation $z_m = 0$ in $C_{j_0,i}$. Let us consider the cycle $p_n^{-1}(q_i)$ defined by $|z_1| = \varepsilon, \ldots, |z_n| = \varepsilon$ in $C_{j_0}(\text{Sing}(X_0) \cap C_{j_0,i})$. It is easy to see that the form $\omega_k$ locally around the point $q_i = C_{j_0,i} \cap \ldots \cap C_{j_n,i}$ will be given by

$$\omega_k|_U = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \times \ldots \times z_n}.$$ 

So we can conclude that

$$\int_{p_n^{-1}(q_i)} \omega_k = (2\pi \sqrt{-1})^n \neq 0.$$

From Lemma 3 we deduce that the cycles $C_\ell^{-1}(p_n^{-1}(q_i))$ and $C_\ell^{-1}(q_i)$ are homological to each other in $X_\ell$. From the fact that all direct images $R^i\pi_* \omega_{X/D}(\log(X_0))$ of the sheaf $\omega_{X/D}(\log(X_0))$ are locally free sheaves on $D$, we deduce from the exact sequence

$$0 \to \omega_{X/D}(\log(X_0)) \to \omega_{X/D}(\log(X_0)) \to \omega_{X/D}(\log(X_0))|_{X_0} \to 0$$

14
that we have the following exact sequence:
\[ 0 \rightarrow H^0(X, \omega_{X/\mathcal{D}}(\log(X_0))) \]
\[ \otimes t \rightarrow H^0(X, \omega_{X/\mathcal{D}}(\log(X_0))) \rightarrow H^0(X_0, \omega_{X/\mathcal{D}}(\log(X_0) \mid X_0) \rightarrow 0 \]

It is easy to see that the theory of mixed Hodge structures implies that as a free module over \( H^0(\mathcal{D}, \mathcal{O}_\mathcal{D}) \) the module \( H^0(X, \omega_{X/\mathcal{D}}(\log(X_0))) \) is of rank bigger or equal to \( \dim_{\mathbb{Q}} H_n(\Pi(X_0), \mathbb{Q}) \). So we can find \( \omega \in H^0(X, \omega_{X/\mathcal{D}}(\log(X_0))) \) such that
\[ \omega|_{D_{j_0,k}} = \omega_k. \]

Let us define \( \omega_t = \omega|_{X_t} \). So we get a holomorphic family of holomorphic \( n \)-forms \( \omega_{t,k} \) on \( X_t \) such that
\[ \lim_{t \to 0} \omega_t = \omega_0 \text{ and } \omega_0|_{D_{j_0,k}} = \omega_k. \]

From here we deduce that for small enough \( t \) and Lemma 9 we have that
\[ \int_{C_{-1}(q_i)} \omega_t \neq 0. \]

From the last inequality we derive that the cycle \( C_{-1}(q_i) \) is a non zero element of \( H_n(X_t, \mathbb{Q}) \). Now Theorem 24 part ii follows directly from Corollary 11. Theorem 24 part ii is proved. ■

**Definition 29** Let us define the geometric genus of \( X_t \) as follows:
\[ p_g(X_t) = \dim_{\mathbb{C}} H^0(X_t, \Omega^n_t) \text{ for } t \neq 0 \]

**Corollary 30** We have the following formula for the geometric genus:
\[ 
\sum_{i=0}^n p_g(C_i) + \sum_{k=0}^{n-1} \sum_{|I|=k+1} \dim_{\mathbb{C}} H^0(C_I, \Omega^{n-k}) + \dim_{\mathbb{C}} H_n(\Pi(X_0), \mathbb{Q}).
\]

5 Applications of Clemens’ Theory to Complete Intersections

5.1 A Simple Criteria for the Existence of Jordan Block of Maximal Dimension

In this paragraph we will prove the following Theorem, which we will apply later to complete intersections in toric varieties:

**Theorem 31** Let \( \pi : \mathcal{X} \rightarrow \mathcal{D} \) be a family of Kähler varieties over the disk such that:
1. For $t \neq 0$, $\pi^{-1}(t) := X_t$ is a non-singular variety of complex dimension $n \geq 1$.

2. $\dim C H^0(X_t, \Omega^n_{X_t/D}) \geq 1$ for $t \neq 0$.

3. $\pi^{-1}(0) = X_0 = \bigcup_{i=0}^m C_i$ is a divisor of normal crossing and each $C_i$ is irreducible.

4. $C_0 \cap \ldots \cap C_n = q$ is a point.

5. There exists a holomorphic section $\omega \in H^0(X, \Omega^n_{X/D})$ such that the restriction of $\omega$ to $C_0$, $\omega_0 := \omega|_{C_0}$, has the following expression in an open set $U$ around the singular point $q := C_0 \cap \ldots \cap C_n$:

$$\omega_0|_{C_0 \cap U} := \omega|_{C_0 \cap U} = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \times \ldots \times z_n}.$$

Then the monodromy operator of the family $\pi : X \to D$ has a Jordan block of size $n + 1$.

**Proof of Theorem 31**: Let $z_0, \ldots, z_n$ be local coordinates near $q$ such that $z_i = 0$ is a local equation of $C_i$, for $i = 1, \ldots, n$. Then $z_1, \ldots, z_n$ can be regarded as coordinates on $C_0$ near $q$. Let $T_0$ be the $n$-dimensional real torus in $C_0 \setminus (C_i \cap C_0)$ defined by $|z_i| = \varepsilon$ for $i = 1, \ldots, n$.

In the notations of Definition 5, $T_0$ is just $p_n(\gamma)^{-1}(q)$, where $\gamma$ is the homology class of the point $q$. From the properties of the Clemens map $C_t$ it follows Lemma 9 that the real $n$-cycle $C_t^{-1}(T_0) = T_t$ is homological to $\gamma_t := C_t^{-1}(\gamma)$. Let $\omega_t = \omega|_{X_t}$. The idea of the proof is to show that for $t$ small enough, $T_t$ is not homological to zero, by showing that

$$\int_{T_t} \omega_t \neq 0.$$ 

This fact together with Corollary 11 implies Theorem 31. So we need to prove the following Proposition:

**Proposition 32** The real cycle $T_t$ is not homological to zero on $X_t$.

**Proof of Proposition 32**: Condition 5 implies that $\lim_{t \to 0} \omega_t = \omega_0$ exists and locally around the singular point $q \in C_0$ of $X_0$, $\omega_0$ is given by the following expression:

$$\omega_0|_{C_0 \cap U} = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \times \ldots \times z_n}.$$

From this expression we obtain that

$$\int_{T_0} \omega_0 = \int_{T_0} \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \times \ldots \times z_n} = (2\pi \varepsilon \sqrt{-1})^n \neq 0.$$ 

From the fact that

$$\lim_{t \to 0} \int_{T_t} \omega_t = \int_{T_0} \omega_0 \neq 0$$

we obtain that small $t$ we have:

$$\int_{T_t} \omega_t \neq 0.$$ 

This proves Proposition 32. ■. Theorem 31 is proved. ■.
5.2 Families of Complete Intersection CY Manifolds in Projective Space with Maximal Unipotent Monodromy

**Definition 33** We will consider the following one parameter family of complete intersections of CY manifolds in $\mathbb{P}^{n+k}$ for $n \geq 4$ and $k \geq 1$ defined by the following equations:

$$G_{1,t} = tF_1 - \prod_{i=0}^{n_1} x_i = 0, \ldots, G_{k,t} = tF_k - \prod_{j=n_1+\ldots+n_{k-1}}^{n_k} x_j = 0,$$

where the system $F_1 = \ldots = F_k = 0$ defines a non singular CY manifolds,

$$n_i = \deg F_i \geq 2 \text{ and } \sum n_i = n + k + 1$$

and $x_i$ are the standard homogeneous coordinates in $\mathbb{P}^{n+k}$.

The condition $\sum n_i = n + k + 1$ implies that the fibers $\pi^{-1}(t) = X_t$ for $t \neq 0$ are CY manifolds of complex dimension $n$.

Let us denote the family defined in Definition 33 by $X \to D$, where $X \subset \mathbb{P}^N \times D$.

**Theorem 34** The family $\pi : X \to D$ has a monodromy $T$ of maximal unipotent index, i.e. $(T - \text{id})^{n+1} = 0$ and $(T - \text{id})^n \neq 0$.

**Proof of Theorem 34**: Our proof is based on checking the conditions of Theorem 31 for this particular family. First we will check that the singular fiber $X_0 := \pi^{-1}(0)$ consists of linear subspaces $C_i$ isomorphic to $\mathbb{P}^n$. We will prove that there exists a component say $C_0$ such that another $n$ linear spaces $C_i$ intersect $C_0$ transversely along $\mathbb{P}^{n-1}$. First we will prove the following Proposition:

**Proposition 35** The singular fiber of the family $X \to D$ consists of exactly $\prod_{i=1}^{k} (n_i - 1)$ $n$ dimensional linear subspaces isomorphic to $\mathbb{P}^n$.

**Proof of Proposition 35**: From the Definition 33 it follows that $X_0 := \pi^{-1}(0)$ is defined by the equations:

$$\prod_{i=0}^{n_1} x_i = 0, \ldots, \prod_{j=n_1+\ldots+n_{k-1}}^{n_k} x_j = 0$$

and thus $X_0$ is a union of linear subspaces of dimension $n$ defined by the equations:

$$x_{j_1} = 0, \ldots, x_{j_k} = 0,$$

where $0 \leq x_{j_1} \leq n_1, \ldots, n_1 + \ldots + n_{k-1} < x_{j_k} \leq N$. From these equations Proposition 35 follows directly. \[\blacksquare\]

**Proposition 36** There exists a component $C_0$ of the singular fiber $X_0$ of the family $\pi : X \to D$ such that $C_0$ is isomorphic to $\mathbb{P}^n$. Let $(z_0 : \ldots : z_n)$ be the homogeneous coordinates in $\mathbb{P}^n$, then there exists $n$ components of the singular fiber $X_0$ say $C_1, \ldots, C_n$ such that $C_i \cap C_0$ is defined in $D_0 \cong \mathbb{P}^n$ by the equation $z_i = 0$.\[\blacksquare\]
Proof of Proposition 36: Let us denote by
\[d_1 = n_1, \quad d_2 := n_1 + n_2, \ldots, d_j := n_1 + \ldots + n_j, \ldots, d_k = n_1 + \ldots + n_k.\]

We already proved that each component of \(X_0\) is a linear subspace isomorphic to \(\mathbb{P}^n\). Let \(D_0\) be the subspace defined by the equations:
\[x_0 = x_{d_1} = x_{d_2} = \ldots = x_{d_k-1} = 0.\]

The coordinates \((x_{i_1} : x_{i_2} : \ldots : x_{n+k})\) define a homogeneous coordinates on \(C_0\) if all \(x_{i_m}\) are different from the \(x_{j_m}\) that define \(C_0\). Let us define \(C_{i_m}\) for \(1 \leq m \leq n\) by the following system of equations: \(x_{j_1} = \ldots = x_{j_k} = 0\), where all the indexes \((j_1, \ldots, j_k)\) with an exception of one is equal to indexes \((0, d_1, \ldots, d_k-1)\). These equations define a component of the singular fiber. Indeed the necessary and sufficient condition for a system of linear equations \(x_{j_1} = \ldots = x_{j_k} = 0\) to define a component of the singular fiber \(X_0\) of the family \(\pi: \mathcal{X} \to \mathcal{D}\) is \(0 \leq j_1 \leq d_1, \quad d_1 < j_2 \leq d_3, \ldots, d_{k-1} < j_k \leq d_k\). Here we are using the conditions that \(n_i \geq 2\) and \(n_1 + \ldots + n_k = n + k\). Clearly \(C_{i_m} \cap C_0\) will be a \(n-1\) dimensional linear subspace in \(C_0 \cong \mathbb{P}^{n-1}\). From here it follows that we constructed \(C_i\) for \(i = 1, \ldots, n\) such that \(C_0 \cap \ldots \cap C_n = (0 : \ldots : 1) = q\). So we proved Proposition 36. \(\blacksquare\)

Proposition 37 There exists a section \(\omega \in H^0(\mathcal{X}, \Omega^n_{\mathcal{X}/\mathcal{D}} \log < X_0 >)\) such that if \(\omega|_{X_0} = \omega_t\) for \(t \neq 0\) then:
\[\lim_{t \to 0} \omega_t = \omega_0\]
where when we restrict \(\omega_0\) on \(C_0 \cong \mathbb{P}^n\), then \(\omega_0|_{U_i}\) is given by the formula:
\[\omega_0|_{U_i} = \frac{dz^1 \wedge \ldots \wedge dz^n}{z^1 \ldots z^n}\]
where \(U_i\) is the standard covering of \(\mathbb{P}^n\).

Proof of Proposition 37: Since we choose \(p = (0, 0, \ldots, 1)\) we will work in the open set \(U_{n+k} := \{(x_0 : \ldots : x_{n+k})|x_{n+k} \neq 0\}\). Let us consider the meromorphic form
\[\Omega_t := \frac{dz^1 \wedge \ldots \wedge dz^{n+k}}{G_t, i \cdot G_{t, k}}, \quad \text{where} \quad G_{t, i} := G_t, i(\frac{x_0}{x_{n+k}}, \ldots, \frac{x_{n+k-1}}{x_{n+k}}, 1)\]
and \(z^i = \frac{x_i}{x_{n+k}}\).

on \(U_{n+k} \times \mathcal{D}\). By taking \(k\) times the Leray residue of \(\Omega_t\) we define the holomorphic n-form \(\omega_t\) on \(X_t\). Suppose that \(C_0\) in \(U_{n+k} \times \mathcal{D}\) is given by the equations (by suitably reordering the coordinates):
\[z^{n+1} = \ldots = z^{n+k} = 0 \quad \text{and} \quad t = 0.\]

(See \[\ref{4}\]) Let \(z^1, \ldots, z^n\) be the rest of the coordinates in \(U_{n+k}\). Let us choose an open set \(\mathcal{W}\) in \(\mathcal{X} \subset \mathbb{P}^N \times D^*\) where \(z^1, \ldots, z^n, t\) are local coordinates of \(\mathcal{W}\), then from the definition of the Leray residue it follows that on \(\mathcal{W}\) we have:
\[ \omega_t|_W := \frac{dz_1 \wedge \ldots \wedge dz_n}{\partial_{a+1} \ldots \partial_{a+k}(\tilde{G}_{t,1} \times \ldots \times \tilde{G}_{t,N-n})}, \text{ where } \partial_i = \frac\partial{\partial z_i}. \]

From the last formulas we get that
\[
\lim_{t \to 0} \partial_{a+1} \ldots \partial_{a+k}(\tilde{G}_{t,1} \times \ldots \times \tilde{G}_{t,k}) = z_1 \times \ldots \times z_n.
\]
So we deduce that
\[
\lim_{t \to 0} \omega_t|_W = \omega_0 = dz_1 \wedge \ldots \wedge dz_n.
\]

Proposition 37 is proved. ■

Propositions 36 and 37 imply that the conditions of Theorem 31 are satisfied by the family \( \pi : X \to D \). So Theorem 31 implies Theorem 34. Theorem 34 is proved. ■

Remark 38 It is not at all difficult to generalized the construction of the one parameter family of complete intersections of CY manifold in \( \mathbb{CP}^N \) with maximal unipotent element in the monodromy group to the case of of complete intersections of general type. More precisely it is very easy to prove that there exists a family \( \pi : Y \to D \) of complete intersections in \( \mathbb{CP}^N \) of general type such that if the dimension of the fiber is \( n \), then the monodromy operator \( T \) contains exactly \( p_g(Y_t) \) Jordan blocks of dimension \( n + 1 \), where \( p_g = \dim \mathcal{H}^0(Y_t, \Omega_Y^n) \).

6 Construction of a Family of CY Manifolds in Toric Varieties with Maximal Unipotent Monodromy

6.1 Introduction to Toric Varieties

Let \( T = (\mathbb{C}^*)^N \). Let \( \mathbb{N} \) be a rank \( N \) lattice, and \( \Sigma \) be a complete fan relative to \( \mathbb{N} \). (See [14].)

Notations:
- \( P_\Sigma = V \) : toric variety defined by \( \Sigma \).
- \( \mathbb{N}^\vee \) : lattice dual to \( \mathbb{N} \).
- \( \Sigma(k) \) : the set of \( k \)-cones in \( \Sigma \).
- \( \tau^\vee \) : the dual of a cone \( \tau \).
- \( D_\rho \) : the toric divisor corresponding to \( \rho \in \Sigma(1) \).
- \( \mathcal{O}(D) = \mathcal{O}_\mathcal{V}(D) \) : the line bundle (invertible sheaf) associated to the divisor \( D \).
We follow the usual combinatorial description of a cone $\tau$ in $\Sigma$, and often use the set of edges of $\tau$ or its primitive generators to denote the cone. Let $O_\rho$, $\rho \in \Sigma(1)$, be the codimension 1 $T$-orbit and $D_\rho$ be their respective closures. Thus they are irreducible $T$-invariant Weil divisors in $V$. Let $R$ be the polynomial ring $\mathbb{C}[x_\rho, \rho \in \Sigma(1)]$. If we declare that $\deg x_\rho = [D_\rho] \in A_{N-1}(V)$, then $R$ becomes a $A_{N-1}(V)$-graded ring, where $A_{N-1}(V)$ is the Picard group of $V$. We denote the degree $[D]$ subspace in $R$ by $R_{[D]}$. It is known that (see [4] and [3]):

$$R_{[D]} \cong H^0(V, \mathcal{O}_V(D)).$$

The graded ring $R$ is called the homogeneous coordinate ring of $V$. The isomorphism above can be described as follows. The dual lattice $N^\vee$ can be viewed as the lattice of characters of the group $T$. We denote a character multiplicatively corresponding to $\nu \in N^\vee$ by $\chi^\nu$. Then

$$H^0(V, \mathcal{O}_V(D)) = \bigoplus_{\nu \in P_D \cap N^\vee} \mathbb{C} \cdot \chi^\nu$$

where

$$P_D = \{ \nu \in N^\vee \otimes \mathbb{Z} \mathbb{R} | \langle \nu, \rho \rangle \geq -a_\rho \ \forall \rho \}.$$

Here $\rho$ denotes the primitive generators of the 1-cones, and $D = \sum a_\rho D_\rho$ is the Weil divisor defining the line bundle $\mathcal{O}_V(D)$. The isomorphism $\phi_D$ above is given by $\phi_D : \chi^\nu \rightarrow \prod x_\rho^{\langle \nu, \rho \rangle + 1}$. The assertion above gives a description of the space of sections of all equivariant line bundles over $V$.

Let us consider the section $x_\rho$. Note that $\nu = 0$ is in $P_D$, hence by the isomorphism above $\phi_{D_\rho}(\chi^0) = x_\rho$ is a section of $\mathcal{O}_V(D)$. In fact the zero set of $x_\rho = 0$ is exactly the divisor $D_\rho$.

For mirror symmetry, the most interesting case is when the convex hull of the 1-cone generators $\rho \in N^\vee \otimes \mathbb{R}$ form a reflexive polytope $\Delta$, and that $D$ is the anticanonical divisor. (See [4] and [3]) In this case $P_D$ is the polar dual of $\Delta$. Here are some examples:

1. $V = \mathbb{P}^2$, $D = D_1 + D_2 + D_3$, $D_i$ being the $i$th hyperplane. The homogeneous coordinate ring of $V$ is the usual $\mathbb{C}[x_1, x_2, x_3]$. If we identify $A_1(V)$ with $\mathbb{Z}$ such that $[H] \rightarrow 1$, then $x_1$ has degree 1. The polytope $\Delta$ is generated by the 1-cone generators $\rho_i$ is the triangle with vertices $(1,0)$, $(0,1)$, $(0,-1)$. Its dual $\nabla$ is the triangle with vertices $(2,-1)$, $(-1,2)$, $(-1,-1)$. It is easy to check that $\nabla$ has exactly 10 lattice points $m$. They are in 1-1 correspondence with the degree 3 monomials in $x_i$.

2. Let $\xi$ be a primitive third root of unity. Let $\mathbb{Z}_3$ acts on $\mathbb{P}^2$ by $[x_1, x_2, x_3] \rightarrow [\xi x_1, \xi^{-1} x_2, x_3]$. Resolve all the singularities in $\mathbb{P}^2/\mathbb{Z}_3$ “minimally”. The result is a toric variety $V$ with 9 hyperplanes $D_i$ all together (with two linear relations). The anticanonical divisor in $V$ is $D = D_1 + \ldots + D_9$. The polytope $\Delta$ generated by the 1-cone generators $n_\rho$ is now the triangle with vertices $(2,-1)$, $(-1,2)$, $(-1,-1)$. Its dual $\nabla$ is the triangle with vertices $(1,0)$, $(0,1)$, $(-1,-1)$. Since $\nabla$ has exactly 4 lattice points, $K_V$ has 4 sections – one of them being $x_1 \ldots x_9$. 

20
6.2 Construction of One Parameter Family of CY Complete Intersection in a Toric Variety with Maximal Jordan Block in the Monodromy Operator

Let \( \pi_1, ..., \pi_k \) be a partition of the set \( \Sigma(1) \). We assume that \( F_i \in H^0(V, \mathcal{O}_V(\sum_{\rho \in \pi_i} D_\rho)) \) together define a nonsingular subvariety \( X \) of codimension \( k \):

\[
F_1 = ... = F_k = 0.
\]

Put \( n = N - k \). Since the anticanonical class is

\[
[K_V^{-1}] = \sum_\rho D_\rho,
\]

by adjunction \( X \) is a Calabi-Yau manifold. We consider the following 1-parameter family of complete intersections \( X_t \) defined by

\[
G_{i,t} = tF_i - \prod_{\rho \in \pi_i} x_\rho = 0, \quad i = 1, ..., k.
\]

Let us denote this family when \( |t| < 1 \) by \( \pi : \mathcal{X} \to \Delta \). The fiber \( X_0 \) is the union of toric subvarieties

\[
C_\sigma := \cap_{\rho \in \sigma} D_\rho
\]

where \( \sigma \) is any subset of \( \Sigma(1) \) such that \( |\sigma \cap \pi_i| = 1 \) for all \( i \). It turns out that (see Appendix) \( C_\sigma \) is nonempty iff the 1-cones in \( \sigma \) generate a cone in \( \Sigma \). In this case, \( C_\sigma \) is a nonsingular irreducible toric subvariety in \( V \) of codimension \( k \) corresponding to the k-cone \( \sigma \in \Sigma(k) \). Thus such a nonempty \( C_\sigma \) is a component in \( X_0 \).

**Remark 39** The assumption that the \( F_i \) defines a codimension \( k \) subvariety is important. This put a strong constraint on the kind of partitions \( \pi_1, ..., \pi_k \) that are allowed. A general partition of \( \Sigma(1) \) will fail to satisfy this condition. Here is an example. Take the 4-dimensional weight projective space \( \mathbb{P}[9, 6, 1, 1, 1] \). After a minimal desingularization, the resulting toric variety has 9 1-cones \( \rho_1, ..., \rho_9 \) in its fan \( \Sigma \). After suitable ordering, it is easy to find a partition of the form \( \pi_1 = \{\rho_1, \rho_2\}, \pi_2 = \{\rho_3, \rho_4\}, \pi_3 = \{\rho_5, ..., \rho_9\} \) such that \( \{\rho_1, \rho_3\}, \{\rho_1, \rho_4\}, \{\rho_2, \rho_3\}, \{\rho_2, \rho_4\} \) are all primitive collections. In this case, \( X_0 \) would be empty because the \( C_\sigma = D_{\rho_1} \cap D_{\rho_2} \cap D_{\rho_3} \), for \( \rho_1 \in \pi_1, \rho_3 \in \pi_2, \rho_2 \in \pi_3 \), will all be empty.

Throughout this section, we shall make the following additional assumption: that the convex hull of the set of primitive generators \( \rho \) of the 1-cones in \( \Sigma \) is a reflexive polytope, which we shall denote by \( \Delta \).

**Theorem 40** The monodromy operator of the family \( \mathcal{X} \to \Delta \) has one Jordan block of dimension \( n + 1 \).
6.3 Proof of Theorem 40

**Lemma 41** Let $C_{\sigma_0}$ be a fixed component in $X_0$. Then there exists $n$ other components $C_{\sigma_1},...,C_{\sigma_n}$ with the property that $C_{\sigma_0} \cap C_{\sigma_i}$ is a codimension $k+1$ toric subvariety, and that

$$C_{\sigma_0} \cap ... \cap C_{\sigma_n}$$

is a toric fixed point.

**Proof:** Since the fan $\Sigma$ is complete, we can find an $N-$ cone, say $\tau$, containing $\sigma_0$ as a $k$-face. Write $\sigma_0 = \langle \rho_1, ..., \rho_k \rangle$, where the $\rho_i \in \pi_i$ are the canonical generators of $\sigma_0$. Similarly, write $\tau$ in terms of its $1$-cone generators:

$$\tau = \langle \rho_1, ..., \rho_k, \rho'_1, ..., \rho'_n \rangle.$$

Then each $\rho'_j$ lies in a unique say $\pi_{ij}$. Since $V$ is non-singular, $\tau$ is a simplicial cone, so that any $k$ of its generators generate a $k$-cone in $\Sigma$. In particular we have the $k$-cones

$$\sigma_j := \langle \rho_1, ..., \hat{\rho}_{ij}, ..., \rho_k, \rho'_j \rangle, \ j = 1, ..., n.$$

That is, $\sigma_j$ is obtained from $\sigma_0$ by replacing the generator $\rho_{ij}$ of $\sigma_0$ by $\rho'_j$. Since both $\rho_{ij}$ and $\rho'_j$ live in the same $\pi_{ij}$, it follows that $C_{\sigma_j}$, as defined above, is a component in $X_0$. By construction,

$$C_{\sigma_0} \cap C_{\sigma_j} = \cap_{\rho \in \sigma_0 \cup \sigma_j} D_{\rho}.$$

But $\sigma_0 \cup \sigma_j$ is the list $\sigma_0$ adjoin with $\rho'_j$, hence gives a $(k+1)$-cone in $\Sigma$. So $C_{\sigma_0} \cap C_{\sigma_j}$ is a codimension $(k+1)$ toric subvariety in $V$. Moreover,

$$C_{\tau} = \cap_{\rho \in \sigma_0 \cup ... \cup \sigma_n} D_{\rho} = \cap_{\rho \in \tau} D_{\rho}$$

which is codimension $k + n = N$ toric subvariety of $V$. Hence it is a fixed point.

Note that the nonsingular toric subvariety $C_{\sigma_0}$ comes with a standard affine coordinates on the affine patch containing fixed point above. Namely, they are obtained from restricting the standard affine coordinates on the patch

$$U_{\tau} = Hom_{sg}(\tau^\vee, \mathbb{C}^*) \cong (\mathbb{C}^*)^N.$$

Here the isomorphism is determined by the choice of ordering of the set of primitive generators of the cone $\tau^\vee$.

**Lemma 42** There exists a meromorphic form on $V = \mathbb{P}_{\Sigma}$ with simple poles along each hypersurface $F_i = 0$. 

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22
Proof: First let $X$ be a complex $N$-fold, and $K_X$ its canonical bundle. Let \( \{U_\sigma\} \) be a covering of charts on $X$, whose coordinates $U_\sigma \rightarrow \mathbb{C}^N$ we denote $z^\sigma = (z^\sigma_1, \ldots, z^\sigma_N)$. From this data, we get a frame $dz^\sigma = dz^\sigma_1 \wedge \ldots \wedge dz^\sigma_N$ on each $U_\sigma$ for the bundle $K_X$, hence a dual frame for the dual bundle $K_X^{-1}$. Suppose $f$ is a nonzero global section of $K_X^{-1}$. Then relative to a dual frame above, $f$ is represented as a holomorphic function $f_\sigma(z^\sigma)$. Note that on any overlap $U_\sigma \cap U_\tau \neq \emptyset$, the functions $f_\sigma, f_\tau$ transform by the same transition function as the frame $dz^\sigma$ and $dz^\tau$. It follows that the local expressions

$$\frac{dz^\sigma}{f_\sigma(z^\sigma)}$$

together define a global meromorphic $N$-form on $X$ with poles along the zero locus $f = 0$. We now assume that $X$ is an algebraic variety and the $U_\sigma$ are affine patches. We now apply this to an $N$-dimensional complete toric variety $V = \mathbb{P}_\Sigma$, as before.

Recall that $\mathbb{P}_\Sigma$ is covered by the affine subvarieties $U_\sigma \cong \mathbb{C}^N$, labelled by $N$-cones $\sigma$ in $\Sigma$. Here the isomorphism is determined by an ordering of the set of $N$ integral generators of the cone $\sigma$. We denote by $z^\sigma = (z^\sigma_1, \ldots, z^\sigma_N)$ the coordinates of this isomorphism. Let

$$\nabla := \{ \nu \in \mathbb{N}^\vee \otimes \mathbb{R} | (\nu, x) \geq -1, \forall x \in \Delta \}.$$  

This is the dual of the reflexive polytope $\Delta$.

The global sections of the anticanonical bundle $K_V^{-1}$ on $V = \mathbb{P}_\Sigma$ has the following description. There is a basis of $H^0(V, K_V^{-1})$ which corresponds 1-1 with the set $\nabla \cap \mathbb{N}^\vee$. Let’s fix an ordering of the set $\Sigma(1)$ and denote the generators by $\rho_1, \ldots, \rho_S \in \mathbb{N}$. Let $L$ be the kernel of the natural map

$$\mathbb{Z}^S \rightarrow \mathbb{N}, \ m \rightarrow \sum m_i \rho_i.$$ 

Let $L^\perp$ be the orthogonal complement of $L$ in $\mathbb{Z}^S$ with respect to the standard inner product $l \cdot l'$. Then $\mathbb{N} \cong \mathbb{Z}^S / L$, and the natural pairing $L^\perp \times \mathbb{Z}^S / L \rightarrow \mathbb{Z}$, $(l, l' + L) \mapsto l \cdot l'$, defines a canonical isomorphism $L^\perp \cong \mathbb{N}^\vee$. Then a general section of $K_V^{-1}$ is of the form

$$f = x_1 \ldots x_S \sum_{\nu \in \nabla \cap \mathbb{N}^\vee} c_\nu x^\nu.$$

(See [1], [2] and [3].) Here the $c_\nu$ are arbitrary complex numbers, $x_i$ are the homogeneous coordinates of $V$, viewed as a section of $\mathcal{O}(D_{\rho_i})$, and $x^\nu$ is the monomial in the $x_i$ with exponents $\nu \in L^\perp \subset \mathbb{Z}^S$.

As an example, when $V = \mathbb{P}^N$, we have $K_V^{-1} = \mathcal{O}_\mathbb{P}^N(N + 1)$, and $x_i$ are the usual homogeneous coordinates. In this case a general section of $K_V^{-1}$ above is exactly a degree $N + 1$ homogeneous polynomial in the $x_i$.

We now return to the general case. Let $e_1, \ldots, e_N$ denotes the standard basis of $\mathbb{Z}^N$. Let $\sigma$ be any $N$-cone in the fan $\Sigma$, and fix an ordering $\nu_1, \ldots, \nu_N$ for the primitive generators of the dual cone $\sigma^\vee$. 

23
Proposition 43 Relative to the frame dual to $dz^\sigma$ above, sections of $K_{N}^{-1}$ are represented by polynomial functions of the form

\[ f_\sigma(z) = z_1 \ldots z_N \sum_{v \in \nabla_\sigma} c_v z^v. \]

Here $z_i = z_i^\sigma$, $c_v$ are arbitrary numbers, $\nabla_\sigma \subset \mathbb{Z}^N$ is the image of $\nabla \cap \mathbb{N}^\vee$ under the isomorphism $\mathbb{N}^\vee \to \mathbb{Z}^N$ determined by $\nu_i \to e_i$.

Proof: A priori, $f_\sigma(z)$ given above is a Laurent polynomial in $z$. We will first show that it is in fact a polynomial. Recall that every element $\nu \in \nabla$ satisfies $\langle \nu, x \rangle \geq -1$ for all $x \in \Delta$. Let $v$ be the image of $\nu$ under the isomorphism $\mathbb{N}^\vee \to \mathbb{Z}^N$ determined by $\nu_i \to e_i$, and let $a$ be the preimage of $x \in \Delta$ under the dual map $\mathbb{Z}^N \to \mathbb{N}$. Then $v \cdot a = \langle \nu, x \rangle \geq -1$. In particular if $x$ is then preimage generator of the cone $\sigma$, then $\langle \nu_k, x \rangle = 1$ for one $k$ and zero otherwise. This means that the preimage $a$ of $x$ is such that $e_i \cdot a = 1$ if $i = k$ and zero otherwise, i.e. $a = e_k$. Clearly, every $e_k$ can be realized as the preimage of some primitive generator $x \in \Delta$ of $\sigma$. This shows that for any $v \in \nabla_\sigma$, we have $v \cdot e_k = \langle \nu, x \rangle \geq -1$ for all $k$. It follows that every Laurent monomial $z_1 \ldots z_N \cdot z^v$ appearing in $f_\sigma(z)$ has a nonnegative exponent, i.e. $f_\sigma(z)$ is a polynomial function on $\mathbb{C}^N$.

Let $\tau$ be any other $N$-cone in the fan $\Sigma$, and fix an ordering $\mu_1, \ldots, \mu_N$ for the primitive generators of $\tau^\vee$. Clearly the respective generators $\nu_i, \mu_j$ for $\sigma^\vee, \tau^\vee$, determine an $A \in (a_{ij}) \in \mathbb{G}L(N, \mathbb{Z})$ such that $\nu_i = \sum_j a_{ij} \mu_j$.

For simplicity, we write $z = z^\sigma$, $w = z^\tau$. The coordinates transition function is then given by

\[ z_i = w^{A_i}, \quad A_i = (a_{i1}, \ldots, a_{iN}). \]

We consider the region of overlap of $U_\sigma \cap U_\tau$ where $z_i \neq 0$ and $w_j \neq 0$ for all $i, j$. First we show that the frames $dz$, $dw$ of $N$-forms are related by the transformation law

\[ dz = \det(A) \frac{z_i \ldots z_N}{w_1 \ldots w_N} dw. \]

We have

\[ \frac{dz}{z_1 \ldots z_N} = d \log z_1 \wedge \ldots \wedge d \log z_N. \]

Now $z_i = \sum_j a_{ij} \log w_j$ on the overlap. So the transformation law above follows immediately.

It remains to show that there is a polynomial function $f_\tau(w)$ such that

\[ f_\sigma(z) = \det(A) \frac{z_i \ldots z_N}{w_1 \ldots w_N} f_\tau(w). \]

The matrix $A$ defines an isomorphism $B : \mathbb{R}^N \to \mathbb{R}^N$, $Bv = \sum_j a_{ij} v_i$. Then
Moreover the image of $\nabla_\sigma \subset \mathbb{R}^N$ under $B$ is $\nabla_\tau$. It follows that

$$f_\sigma(z) = z_1...z_N \cdot \sum_{v \in \nabla_\sigma} c_v z^v = z_1...z_N \sum_{u \in \nabla_\tau} c_{B^{-1}u} w^u.$$ 

So if we set

$$f_\tau(w) := w_1...w_N \cdot \sum_{u \in \nabla_\tau} \det(A)^{-1} c_{B^{-1}u} w^u,$$

then we obtain the desired function. ■

Proposition 43 implies Lemma 42. ■

As an example, when $V = \mathbb{P}^N$, we can choose the fan $\Sigma$ in $\mathbb{Z}^N$, so that the 1-cone primitive generators are $e_0 = -e_1 - \cdots - e_N, e_1, \ldots, e_N$. In this case, we have $N + 1$ standard affine patches $U_{\sigma_i}$, where $\sigma_i$ is the $N$-cone generated by $e_0, e_i, e_1, \ldots, e_N$. The affine coordinate are $z^{\sigma_i} = (z_0, \ldots, 1, \ldots, z_N)$. For simplicity, we label each cone $\sigma_i$ simply by the integer $i$. Then the coordinate transition functions are given by

$$z^{\sigma_i} = z^{\sigma_j} \frac{x_j}{x_i}, i \neq \sigma, \tau; \quad z^{\sigma_i} = \frac{1}{z_i}.$$ 

The transition functions for coordinate $N$-forms are given by

$$dz^{\tau} = (-1)^{\sigma + \tau} \frac{dz^{\sigma}}{z_i} dz^{\tau}.$$ 

We now prove the general toric version of Proposition 37.

**Proposition 44** There exists a family of holomorphic $n$-forms $\omega_t$ on $X_t$ for $t \neq 0$ such that

$$\lim_{t \to 0} \omega_t = \omega_0$$

where $\omega_0$ is a section on $X_0$ with the following properties: restricted to the component $C_{\sigma_0}$, $\omega_0$ has the form:

$$\omega_0|_{U_\epsilon} = \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \ldots z_n}$$

where $(z_1, \ldots, z_n)$ are coordinates on the affine patch $U_\epsilon \subset C_{\sigma_0}$.

**Proof:** Let $U_\tau \cong \mathbb{C}^N$ be the affine patch in $V = \mathbb{P}_\Sigma$ as given in the proof of Proposition 43. Let $z^{\tau} = z = (z_1, \ldots, z_N)$ be the affine coordinates. Here we choose the coordinates so that the affine subvariety $U_\tau = U_\tau \cap C_{\sigma_0}$ is defined by

$$z_{n+1} = \ldots = z_N = 0$$

and the $z_1, \ldots, z_n$ are affine coordinates of $U_\tau$. Choose a small analytic neighborhood $W \subset X \cap U_\tau \times D$ so that $z_1, \ldots, z_n, t$ are coordinates on $X_t \cap W$ for small $t$.

Clearly
\[ f := G_{t,1} \ldots G_{t,k} \in H^0(V, \mathcal{O}( \sum_{\rho \in \Sigma(1)} D_{\rho} )) = H^0(V, K_V^{-1}). \]

By Lemma 42, we have a meromorphic $N$-form given by
\[ \Omega_t = \frac{dz}{g_t(z)} \]
where
\[ g_t(z) = z_1 \ldots z_N (1 - tf_1(z)) \ldots (1 - tf_k(z)). \]

Here $f_i(z)$ is a Laurent polynomial representing the section $F_i$ of $\mathcal{O}(\sum_{\rho \in \pi_i} D_{\rho})$ on the algebraic torus $z_1 \ldots z_N \neq 0$. Note that for all $t$, $g_t$ is polynomial in the $z$ by construction. Integrating the form $\Omega_t$ via Leray residue $k$ times, the result is a section $\omega_t$ of the sheaf of $n$-forms, whose restriction on $X_t \cap W$ is given by
\[ \omega_t = \frac{dz_1 \wedge \ldots \wedge dz_n}{\prod_{n+1}^{N} g_t(z)}. \]

It is clear that the dominator of $\omega_t$ goes to $z_1 \ldots z_n$ as $t \to 0$. This proved our assertion. ■

Theorem 40 now follows directly from Theorem 31. ■

Remark 45 One can generalize the same constructions to the case of complete intersections of general type in toric varieties for which the canonical class is very ample. Namely one can prove that there exists one parameter family of complete intersections of general type $Y \to D$ for which the generic fibers is of dimension $n$ and the canonical class is very ample, such that the monodromy operator contains exactly $p_g(Y_t)$ Jordan blocks of dimension $n + 1$.

7 Comparisons

7.1 Relationship with Hypergeometric Functions

In [10], the construction of the point of maximal unipotent monodromy has been done for the family of CY hypersurfaces consisting of all anticanonical hypersurfaces modulo equivalence under $T$ in a fixed toric variety. The construction there uses an entirely different approach. Namely, one exploits the fact that the periods of the family are solutions to a system of linear partial differential equations, known as a GKZ hypergeometric system and its generalization. (See [10] and the references there.) The construction is then done by carefully analyzing the structure of certain singularities of the PDE system. Explicit computation of certain period is also necessary in that approach.

We now briefly compare the approach of [10] with the present approach. For simplicity, we consider the case of hypersurfaces in an $N$ dimensional toric variety $V = \mathbb{P}_G$. Recall that in order to apply Theorem 31, we must construct a meromorphic $N$-form $\Omega_t$ on $V$ with a simple pole along the CY hypersurface $X_t$:
\[ G_t := tF - \prod_{\rho} x_{\rho} = 0. \]

Here \( F \in H^0(V, K_V^{-1}) \) is a fixed section. In the affine coordinates \( z^\sigma = (z_1, ..., z_N) \) on an affine patch \( U_\sigma \cong \mathbb{C}^N \), \( \Omega_t \) takes the form:

\[ \Omega_t = \frac{dz_1 \wedge ... \wedge dz_N}{z_1 ... z_N (1 - tf(z))} \]

where \( f(z) \) is a Laurent polynomial representing the section \( F \). The Leray residue of \( \Omega_t \) gives a holomorphic \( n \)-form on \( X_t \). We then integrate \( \omega_t \) over a real \( n \)-torus, giving a holomorphic function of \( t \) bounded near \( t = 0 \). This function is a period of the CY manifold \( X_t \) near \( t = 0 \).

\[ \int_{\alpha_0} \Omega_t. \]

But this is precisely a special value of the holomorphic solution of GKZ system given in [10]. In fact, in [10], \( G_t \) is replaced by a general section, and the special Laurent polynomial \( 1 - tf(z) \) is replaced by its general counterpart. The result is a multi-variable holomorphic function (the variables being the coefficients of the general Laurent polynomial). A power series expansion of this function can be easily computed. (See [10].) The period \( \int_{\alpha_0} \Omega_t \) above can be obtained by specializing the multi-variable function to 1-variable function by restricting it to the one-parameter family \( G_t = 0 \).

### 7.2 Comparison of Maximal Unipotent Monodromy and the SYZ Conjecture

Clemens' theorem [14] and our construction of the maximal unipotent monodromy operator \( T \) for complete intersection CY manifolds in a toric variety show that there exist special cycles, namely the invariant cycle \( \gamma \) and the cycles \( \alpha_1, ..., \alpha_n \), such that

\[ T(\alpha_n) = \gamma + \alpha_1 + ... + \alpha_n. \]

From Clemens’ theory, we know that the cycle \( \gamma \) can be realized as a \( n \)-dimensional torus. The cycle \( \alpha_n \) can probably be realized as an \( n \)-dimensional sphere. A theorem of Clemens says that \( \gamma \) and \( \alpha_n \) have intersection number 1.

On the other hand, the SYZ conjecture asserts that a CY \( n \)-fold should admit the structure of a fibration over an \( n \)-sphere with generic fibers given by special Lagrangian \( n \)-tori. Moreover this fibration admits a canonical section. Thus the section is an \( n \)-cycle having intersection number 1 with the fiber cycle. It is quite clear that the cycle \( \gamma \) and \( \alpha_n \) above should be realized precisely as the \( n \)-sphere section and the \( n \)-torus fiber respectively. This general relation deserves further investigation. For families of polarized K3 surfaces having maximal unipotent monodromy, this can be verified by using isometric deformation (relative to the Calabi-Yau metric) of the surface, as we will discuss next.
Theorem 46 Let \( \pi : X \to D \) be a family of polarized K3 surfaces with polarization class \( e \in H^2(X, \mathbb{Z}) \) such that \( <e,e>=2n \) for a fixed positive integer \( n \). Suppose that \( \pi^{-1}(0) = X_0 \) is a singular surface. Fix \( t \neq 0 \) and suppose the monodromy operator \( T \) acting on the second homology group of \( X = \pi^{-1}(t) \) is such that \( (T - id)^3 = 0 \) and \( (T - id)^2 \neq 0 \). Let \( g_t \) be the Calabi-Yau metrics on \( X = X_t \) such that \( |\text{Im } g_t| = e \). Let \( \gamma, \alpha_1 \) and \( \alpha_2 \) be cycles such that

\[
T(\gamma) = \gamma, \quad T(\alpha_1) = \gamma + \alpha_1, \quad T(\alpha_2) = \gamma + \alpha_1 + \alpha_2,
\]

\[
<e,\gamma> = <\gamma,\gamma> = <\alpha_2,\alpha_2> = 0, \quad <\gamma,\alpha_2> = 1.
\]

Then there exists a torus fibration \( \psi : X \to S^2 \) such that the generic fiber \( \psi^{-1}(s) \) is a Lagrangian 2-torus with respect to \( \text{Re } g_t \) representing the homology class \( \gamma \). Moreover, there is a section \( \sigma : S^2 \to X \) such that \( \sigma(S^2) \subset X \) is a Lagrangian submanifold representing the homology class \( \alpha_2 - \gamma \).

Before we prove the theorem, let’s construct a family of polarized K3 surface that fulfills the conditions stated in Theorem 46. Let \( E_t : y^2 = x(x-1)(x-t) \) be a family of elliptic curves when \( t \in D \), where \( D := \{ t \in \mathbb{C} \mid |t| < 1 \} \) and \( E_t \subset \mathbb{C}P^2 \times D \). Then \( E_t \times_D E_t \to D \) is a family of abelian surfaces over the unit disc. Let \( \pi : K_t \to D \) be the family of Kummer surfaces associated with this family of abelian surfaces. In [7] it was proved that the family \( \pi : K_t \to D \) satisfies the conditions of Theorem 46. See page 252-253 in [7]. On page 252 an explicit construction of the cycles \( \gamma \) and \( \alpha_2 \) is given and moreover from Clemens’ theory it follows that

\[
<\gamma,\gamma> = <\alpha_2,\alpha_2> = 0 \quad \text{and} \quad <\gamma,\alpha_2> = 1.
\]

In particular, \( \gamma \) and \( \alpha_2 \) span a hyperbolic lattice

\[
\mathbb{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

It is a well-known fact that for a K3 surface \( X \in H_2(X, \mathbb{Z}) \) is a lattice isomorphic to \( \Lambda_{K3} := \mathbb{H}^3 \oplus (-E_8)^2 \). Let us define (transcendental lattice)

\[
T_e := \{ v \in \Lambda_{K3} \mid <e,v> = 0 \}.
\]

We know also that \( T_e \cong \mathbb{Z}e^* \oplus \mathbb{H}^2 \oplus (-E_8)^2 \), where \( <e^*,e^*> = -<e,e> \). So we may suppose that \( T_e = \mathbb{Z}e^* \oplus \{ \gamma, \alpha_2 \} \oplus \mathbb{H} \oplus (-E_8)^2 \). See [7]. In [7] it is shown that the cycles \( \gamma \) and \( \alpha_2 \in T_e \).

**PROOF of Theorem 46** We will use isometric deformation with respect to the CY metric \( g_t \) on a K3 surface whose class of cohomology \( [\text{Im } g_t] = e \) and the epimorphism of the period map for K3 surfaces to prove that K3 surface can be realized as a Lagrangian fibration with respect to each CY metric \( g_t \). The Lagrangian fibration that we will construct will have the properties that the homology class of the generic fibre will be \( \gamma \) and the homology class of the base will be \( \delta = \alpha_2 - \gamma \). For the description and applications of isometric deformations see [15]. Theorem 46 will follow directly from the following Lemma:
Lemma 47. **i.** Let $Y$ be a K3 surface whose Neron-Severi group $NS(Y)$ spanned by a cycle $\gamma$ and such that $<\gamma,\gamma>=0$, then $\gamma$ can be realized as a non singular elliptic curve $C$ on $Y$ and the linear system $|C|$ defines an elliptic fibration $|C| : Y \to \mathbb{CP}^1$. **ii.** Let $Y$ be a K3 surface whose Neron Severi group $NS(Y)$ spanned by a cycle $\delta$ and such that $<\delta,\delta>=-2$, then either $\delta$ or $-\delta$ can be realized as an embedded complex line $\mathbb{CP}^1 \subset Y$.

**PROOF:** First we will prove part i of Lemma 47. From the condition that Neron Severi group $NS(Y)$ is spanned by cycles $\gamma$ such that $<\gamma,\gamma>=0$, we deduce that there exists a line bundle $L_{\gamma}$, whose Chern class $c_1(L_{\gamma}) = \gamma$. From Riemann-Roch theorem we deduce that $\dim H^0(Y, L_{\gamma}) + \dim H^2(Y, L_{\gamma}) \geq 2$.

Serre’s duality implies that $\dim H^2(Y, L_{\gamma}) = \dim H^2(Y, L_{\gamma}^*)$, where $L_{\gamma}^*$ is the dual line bundle of $L_{\gamma}$. From here we obtain that either $\dim H^0(Y, L_{\gamma}) > 1$ or $\dim H^0(Y, L_{\gamma}^*) > 1$. So either $\gamma$ or $-\gamma$ can be realized as an effective divisor on $Y$.

Recall Theorem 48 ([16] p559.) If an effective divisor $D > 0$ on a K3 surface $Y$ satisfies the conditions $<D, D>=0$ and $<D, E>\geq 0$ for any effective divisor $E > 0$, then the linear system $|D|$ contains a divisor of the form $mC$, where $m > 0$ and $C$ is an elliptic curve.

Since $NS(Y)$ is spanned by a cycle $\gamma$ such that $<\gamma,\gamma>=0$, it follows that that $\gamma$ or $-\gamma$ can be realized as a non singular elliptic curve $C$. From the standard exact sequence on $C$:

$$0 \to \mathcal{O}_Y(-C) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0,$$

we have, in the associated long exact sequence,

$$H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_C) \to H^1(\mathcal{O}_Y(-C)) \to H^1(\mathcal{O}_Y) = 0.$$

Since the restriction map $H^0(\mathcal{O}_Y) = \mathbb{C} \to H^0(\mathcal{O}_C) = \mathbb{C}$ is an isomorphism, we have $H^1(\mathcal{O}_Y(-C)) = 0$. From Serre’s duality applied to K3 surfaces we get $H^1(\mathcal{O}_Y(-C)) = H^1(\mathcal{O}_Y(C)) = 0$ and $H^2(\mathcal{O}_Y(C)) = 0$. It is a standard fact that the linear system $|C|$ defines a map $\phi : Y \to \mathbb{CP}^1$. For details see page 560 of [16]. Part i of Lemma 47 is proved.

Part ii of Lemma 47 follows directly from Corollary 3 on page 560 of [16]. Lemma 47 is proved. ■

One of the main consequences of the existence of a Calabi Yau metric on a K3 surface $X$ with a holomorphic form $\omega_X$ is the fact that the covariant derivative of the holomorphic form $\omega_X$ is zero, i.e. $\nabla \omega_X = 0$. This implies that the following three forms $\{\text{Re}\omega_X, \text{Im}\omega_X \text{ and } \text{Im}(g)\}$ are parallel, closed and non degenerate forms on $X$. It is easy to see that:

$$\langle \text{Re}\omega_X, \text{Im}\omega_X \rangle = \langle \text{Re}\omega_X, \text{Im}g \rangle = \langle \text{Im}g, \text{Im}\omega_X \rangle = 0.$$
Here $Im(g)$ is the imaginary part of the Calabi Yau metric on $X$. If $\|Re\omega_X\|^2 = \|Im\omega_X\|^2 = \|Im(g)\|^2 = 1$, then these three forms define three integrable complex structures $I, J$ and $K$ on $X$ such that $I^2 = J^2 = K^2 = -id$ and $IJ + KI = J + KJ = 0$. For all these facts see [13]. The isometric deformation is defined as a new integrable complex structure $aI + bJ + cK$ on $X$, where $a^2 + b^2 + c^2 = 1$. We’ll denote this new complex structure on $X$ by $Y$. In [13] it was proved that $Re\omega_Y = A(Re\omega_X)$, $Im\omega_Y = A(Im\omega_X)$, where $A \in SO(3)$. The class $A(Img)$ will be a class of cohomology of type $(1,1)$ on $Y$ and will define a new Calabi Yau metric on $Y$ which is isometric to the Calabi Yau metric on $X$ as a Riemannian metrics. In [13] we proved that there exists $A \in SO(3)$ such that

$$\int_\gamma A(Re\omega_X) = \int_\gamma A(Im\omega_X) = 0.$$  

Let $Y$ and $C$ be defined as in Lemma 47. Notice that when we do an isometric deformation, we are not changing either the $C^\infty$ structure or the Riemannian structure on $Y$. So we may consider that $C$ realizes the cycle $\gamma$ as an embedded two dimensional torus in $X$.

**Lemma 49** $C$ is a Lagrangian cycle on $X$

**PROOF:** We need to prove the following two facts: 1. The volume form of the restriction of the CY metric $g_X$ on $C$ is given by the following expression:

$$Vol(g_X|_C) = a Re(\omega_X) + b Im(\omega_X)$$  

and 2. that

$$Im(g_X|_C) = 0.$$  

Since $C$ is an elliptic curve in $Y$, it follows that $\omega_Y|_C = 0$, so that $Re\omega_Y|_C = Im\omega_Y|_C = 0$. The volume form of the restriction of the Ricci flat Riemannian metric on $C$ is equal to $Im(g_Y)|_C$, ie.

$$Vol(g_X|_C) = Im(g_Y)|_C.$$  

So from (3) and the general facts about isometric deformation we deduce that

$$Vol(g_X|_C) = a Re(\omega_X)|_C + b Im(\omega_X)|_C + c Im(g_X)|_C.$$  

where $a, b, c$ are some real numbers given by

$$a = \int_C Re(\omega_X), \quad b = \int_C Im(\omega_X), \quad c = \int_C Im(g_X).$$  

Since $\gamma \in T_e$ and the cohomology class $[Im g_X] = e$ we can conclude that:

$$0 = \int_\gamma e = \int_C Im(g_X) = c.$$  

30
This proves fact 1. We now prove fact 2, i.e. that the restriction of the form $\text{Im}(g_X)$ on $C$ is identically zero. This follows directly from the following formula, proved in [15]:
\[
\text{Im}(g_X)|_C = c\text{Im}(g_X)|_C \text{ and } c = \int_C \text{Im}g_X.
\] (7)

So Lemma 49 is proved. ■.

Remark 50 Exactly in the same way we can prove that the cycle $\delta = \alpha_2 - \gamma$ can be realized as a Lagrangian sphere. Indeed we can deform isometrically the complex structure on $X$ to $Z$ with respect to the CY metric $g_X$ so that $\delta$ will be realized as a complex projective line $\mathbb{P}^1$ embedded in $Z$. Repeating the arguments from Lemma 49 we deduce that $\mathbb{P}^1$ is a Lagrangian sphere embedded in $X$.

The End of the Proof of Theorem 46: Indeed we realized the cycles $\gamma$ and $\alpha_2 - \gamma$ as complex analytic curves on $Y$ and on $Z$. Then we know from Theorem 48 that $Y$ is an elliptic fibration whose "generic" fibre is an elliptic curve. Lemma 49 implies that the fibres of this fibration are Lagrangian submanifolds. Since we know that $\langle \gamma, \delta \rangle = 1$, we can conclude that the basis of the Lagrangian fibration is the Lagrangian sphere that realizes the cycle $\delta$. ■.

8 Appendix: Clemens’ Cell Complex for Toric Hypersurfaces

As before let $\Delta$ be the convex hull of generators $\rho$ of the 1-cones in $\Sigma$. This is an $N$ dimensional polytope. But it has more. It comes equipped with a canonical simplicial decomposition induced by $\Sigma$. In particular the boundary $\partial\Delta$ is topologically an $N$-sphere which comes equipped with a simplicial decomposition.

Lemma 51 Let $\tau$ be a set of primitive generators of 1-cones in $\Sigma$. Put $k = |\tau|$. The following are equivalent:

i. $\bigcap_{\rho \in \tau} D_{\rho}$ is non empty.
ii. The cone is generated by $\tau$ is a $k$-cone in $\Sigma$.
iii. The convex hull of $\tau$ is a $(k - 1)$-cell in $\partial\Delta$.

Proof: (ii)$\Rightarrow$(iii) is obvious. We will show that (i) and (ii) are equivalent. Recall that there is an order reversing 1-1 correspondence between T-orbits $O_\sigma$ and cones $\sigma$ in $\Sigma$, and that the closure of $V(\sigma)$ of $O_\sigma$ is given by
\[
V(\sigma) = \bigsqcup_{\gamma \succ \sigma} O_\gamma.
\]
Note that $D_{\rho} = V(\rho)$ for $\rho \in \Sigma(1)$. Thus
\[
\bigcap_{\rho \in \tau} D_{\rho} = \bigsqcup_{\gamma \succ \tau} O_\gamma.
\]

31
If this is nonempty, then we have some cone $\gamma \in \Sigma$ containing $\tau$. Since the toric variety $V$ is nonsingular, $\gamma$ is a simplicial cone. This implies that any collection of $k$ edges of $\gamma$ generates a $k$-face of $\gamma$. So $\tau$ generates a $k$-face, which we also call $\tau$, of $\gamma$. In particular $\tau \in \Sigma$. Conversely, if $\tau$ generates a cone in $\Sigma$, obviously every $D_\rho, \rho \in \tau$, contains $O_\rho$. In this case, $\cap_{\rho \in \tau} D_\rho$ is nonempty. □

Note that if the intersection $\cap_{\rho \in \tau} D_\rho$ is nonempty, then it has dimension $N - |\tau|$. Moreover, the correspondence above between nonempty intersections of the $D_\rho$'s and the cells of $\partial \Delta$ is order reversing. Namely, if two sets $\tau, \tau'$ of primitive generators yield nonempty intersections, then $\cap_{\rho \in \tau} D_\rho \subset \cap_{\rho \in \tau'} D_\rho$ iff $\text{conv}(\tau) \supset \text{conv}(\tau')$.

We now specialize our family $X \to D$ to the case of hypersurfaces

$$tF - \prod_{\rho} x_\rho = 0.$$ 

Thus $N = n + 1$.

**Theorem 52** The Clemens polytope $\Pi(X_0)$ is a simplicial complex which is naturally isomorphic to $\partial \Delta$.

**Proof:** Recall that $X_0 = \cup_\rho D_\rho$. By definition of $\Pi(X_0)$, each $D_\rho$ corresponds to a vertex in $\Pi(X_0)$. For $\rho \neq \rho'$, $D_\rho \cap D_{\rho'}$ corresponds to 1-cell if $D_\rho \cap D_{\rho'}$ has dimension $n - 1 = N - 2$, and so on. Thus given a set $\tau$ of 1-cones with $|\tau| = k$, $\cap_{\rho \in \tau} D_\rho$ corresponds to $(k - 1)$-cell in $\Pi(X_0)$ iff $\cap_{\rho \in \tau} D_\rho$ has dimension $n - k + 1$. This is an order reversing correspondence between $n - k + 1$ dimensional intersections $\cap_{\rho \in \tau} D_\rho$ of $D_\rho$'s and $(k - 1)$-cells of the simplicial complex $\partial \Delta$ as simplicial complex. □

**Corollary 53** $H_n(\Pi(X_0), \mathbb{Q}) \cong \mathbb{Q}$.

In the case when $X$ is a family of complete intersections in $\mathbb{P}_\Sigma$, one can still define a cell complex $\Pi(X_0)$ similar to the definition of Clemens’ complex. It is expected that the cell complex $\Pi(X_0)$ will again be isomorphic to $n$-sphere. We have verified this for many examples. It turns out that the complex is typically non-simplicial, unlike in the case of hypersurfaces. It is an interesting combinatorial problem to describe the complex $\Pi(X_0)$ in simple terms.

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32
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