The Leavitt path algebra of a graph

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Abstract

For any row-finite graph $E$ and any field $K$ we construct the Leavitt path algebra $L(E)$ having coefficients in $K$. When $K$ is the field of complex numbers, then $L(E)$ is the algebraic analog of the Cuntz Krieger algebra $C^*(E)$ described in [8]. The matrix rings $M_n(K)$ and the Leavitt algebras $L(1,n)$ appear as algebras of the form $L(E)$ for various graphs $E$. In our main result, we give necessary and sufficient conditions on $E$ which imply that $L(E)$ is simple.

Keywords: path algebra; Leavitt algebra; Cuntz Krieger C*-algebra

Introduction

Throughout this article $K$ will denote an arbitrary field. In his seminal paper [6], Leavitt describes a class of $K$-algebras (nowadays denoted by $L(m,n)$) which are universal with respect to an isomorphism property between finite rank free modules. In [7], Leavitt goes on to show that the algebras of the form $L(1,n)$ are simple. More than a decade later, Cuntz [3] constructed and investigated the $C^*$-algebras $O_n$ (nowadays called the Cuntz algebras), showing, among other things, that each $O_n$ is (algebraically) simple. When $K$ is the field $\mathbb{C}$ of complex numbers, then $O_n$ can be viewed as the completion, in an appropriate norm, of $L(1,n)$. Soon after the appearance of [3], Cuntz and Krieger [4] described the significantly more general notion of the $C^*$-algebra of a (finite) matrix $A$, denoted $O_A$. Among this class of $C^*$-algebras one can find, for any finite graph $E$, the Cuntz-Krieger algebra $C^*(E)$, defined originally in [5]. These $C^*$-algebras, as well as those arising from various infinite graphs, have been the subject of much investigation (see e.g. [8], [9], and [2]). Recently, the ‘algebraic analogs’ of the $C^*$-algebras $O_A$ have been presented in [1]; these are denoted by $\mathcal{CK}_A(K)$. By restricting attention to a specific set of allowable matrices, the simplicity of the algebra $\mathcal{CK}_A(K)$ for some subset of these allowable matrices has been determined (although the condition for simplicity is not explicitly given in terms of the matrix $A$).

The goal of this article is to ‘complete the algebraic picture’. Specifically, we give the definition of the Leavitt path algebra $L(E)$ corresponding to any row-finite graph $E$ and

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field $K$. When $E$ is finite without sources and sinks, then $L(E)$ can be realized as an algebra of the form $CKA(K)$ for some matrix $A$. Analogous to the relationship that exists between $L(1, n)$ and $O_n$, $L(E)$ has the property that when $K = C$, then $C^*(E)$ can be viewed as the completion, in an appropriate norm, of $L(E)$ [8, Proposition 1.20].

In our main result, Theorem (3.11), we give necessary and sufficient conditions on the row-finite graph $E$ which imply that $L(E)$ is simple. These results extend those presented in [1], in that: they apply also to some important algebras which are explicitly not considered in [1]; they apply also to algebras which arise from infinite matrices; and they provide necessary conditions on $E$ for the simplicity of $L(E)$. The statement of Theorem (3.11) parallels a similar theorem for $C^*$-algebras of the form $C^*(E)$ given in [8, Theorem 4.9 and subsequent remarks]. However, the techniques utilized here are significantly different than those used in the analytic setting.

We begin by establishing some notational conventions. A (directed) graph $E = (E^0, E^1, r, s)$ consists of two countable sets $E^0, E^1$ and functions $r, s : E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. For each edge $e$, $s(e)$ is the source of $e$ and $r(e)$ is the range of $e$. If $s(e) = v$ and $r(e) = w$, then we also say that $v$ emits $e$ and that $w$ receives $e$, or that $e$ points to $w$.

A vertex which does not receive any edges is called a source. A vertex which emits no edges is called a sink. A graph is called row-finite if $s^{-1}(v)$ is a finite set for each vertex $v$. In this paper, we will only be concerned with row-finite graphs. Of course, under this hypothesis, the edge set of $E$, $E^1$, is finite if its set of vertices, $E^0$, is finite. Thus, we will say a graph $E$ is finite if $E^0$ is a finite set. A path $\mu$ in a graph $E$ is a sequence of edges $\mu = \mu_1 \ldots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \ldots, n - 1$. In such a case, $s(\mu) := s(\mu_1)$ is the source of $\mu$ and $r(\mu) := r(\mu_n)$ is the range of $\mu$. If $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then $\mu$ is a called a cycle.

1. **Leavitt path algebras**

In this section we define the algebraic structures under investigation. We begin by reminding the reader of the construction of the standard path algebra of a graph.

1.1. **Definition.** Let $K$ be a field and $E$ be a graph. The path $K$-algebra over $E$ is defined as the free $K$-algebra $K[E^0 \cup E^1]$ with the relations:

1. $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.

2. $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$.

This algebra is denoted by $A(E)$.
1.2. Definition. Given a graph $E$ we define the extended graph of $E$ as the new graph $\hat{E} = (E^0, E^1 \cup (E^1)^*)$, $r'$, $s'$ where $(E^1)^* = \{ e_i^* : e_i \in E^1 \}$ and the functions $r'$ and $s'$ are defined as

$$r'|_{E^1} = r, \ s'|_{E^1} = s, \ r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

1.3. Definition. Let $K$ be a field and $E$ be a row-finite graph. The Leavitt path algebra of $E$ with coefficients in $K$ is defined as the path algebra over the extended graph $\hat{E}$, with relations:

(CK1) $e_i^*e_j = \delta_{ij}r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$.

(CK2) $v_i = \sum\{e_j \in E^1 : s(e_j) = v_i\} \ e_j e_j^*$ for every $v_i \in E^0$ which is not a sink.

This algebra is denoted by $L_K(E)$ (or more commonly simply by $L(E)$).

The conditions CK1 and CK2 are called the Cuntz-Krieger relations. In particular condition CK2 is the Cuntz-Krieger relation at $v_i$. If $v_i$ is a sink, we do not have a CK2 relation at $v_i$. Note that the condition of row-finiteness is needed in order to define the equation CK2.

1.4. Examples. Many well-known algebras are of the form $L(E)$ for some graph $E$:

(i) Matrix algebras $M_n(K)$: Consider the graph $E$ defined by $E^0 = \{ v_1, \ldots, v_n \}$, $E^1 = \{ e_1, \ldots, e_{n-1} \}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \ldots, n - 1$. Then $M_n(K) \cong L(E)$, via the map $v_i \mapsto e(i, i), e_i \mapsto e(i, i + 1)$, and $e_i^* \mapsto e(i + 1, i)$ (where $e(i, j)$ denotes the standard $(i, j)$-matrix unit in $M_n(K)$).

(ii) Laurent polynomial algebras $K[x, x^{-1}]$: Consider the graph $E$ defined by $E^0 = \{ \ast \}$, $E^1 = \{ x \}$. Then clearly $K[x, x^{-1}] \cong L(E)$.

(iii) Leavitt algebras $A = L(1, n)$ for $n \geq 2$ investigated in [7]: Consider the graph $E$ defined by $E^0 = \{ \ast \}$, $E^1 = \{ y_1, \ldots, y_n \}$. Then $L(1, n) \cong L(E)$.

1.5. Lemma. Every monomial in $L(E)$ is of the following form.

(a) $k_i v_i$ with $k_i \in K$ and $v_i \in E^0$, or

(b) $ke_{i_1} \cdots e_{i_s} e_{j_1}^* \cdots e_{j_t}^*$ where $k \in K$; $\gamma \geq 0$, $\sigma + \tau > 0$, $e_{i_s} \in E^1$ and $e_{j_t}^* \in (E^1)^*$ for $0 \leq s \leq \sigma, 0 \leq t \leq \tau$.

Proof: The proof is almost identical to the proof of [8, Corollary 1.15] (a straightforward induction argument on the length of the monomial $kx_1 \cdots x_n$ with $x_i \in E^0 \cup E^1 \cup (E^1)^*$), and so is omitted. ■

1.6. Lemma. If $E^0$ is finite then $L(E)$ is a unital $K$-algebra. If $E^0$ is infinite, then $L(E)$ is an algebra with local units (specifically, the set generated by finite sums of distinct elements of $E^0$).
that for monomials of type (b). Then with the same ideas as above it is not difficult to prove we see that $L$ are homogeneous in this grading. 

Clearly that $\sum_{i=1}^{n} v_i = v$. Now if we take $e_j \in E^1$ we may use the equations (2) in the definition of path algebra together with the previous computation to get $(\sum_{i=1}^{n} v_i) e_j = (\sum_{i=1}^{n} v_i) s(e) e_j = s(e) e_j = e_j$. In a similar manner we see that $(\sum_{i=1}^{n} v_i) e^*_j = e^*_j$. Since $L(E)$ is generated by $E^0 \cup E^1 \cup (E^1)^*$, then it is clear that $(\sum_{i=1}^{n} v_i) \alpha = \alpha$ for every $\alpha \in L(E)$, and analogously $\alpha(\sum_{i=1}^{n} v_i) = \alpha$ for every $\alpha \in L(E)$. Now suppose that $E^0$ is infinite. Consider a finite subset $\{a_i\}_{i=1}^{l}$ of $L(E)$ and use Lemma 1.5 to write $a_i = \sum_{s=1}^{n_i} k^i_s v^i_s + \sum_{l=1}^{m_i} \gamma^i_l p^i_l$ where $k^i_s, \gamma^i_l \in K - \{0\}$, and $p^i_l$ are monomials of type (b). Then with the same ideas as above it is not difficult to prove that for $V = \bigcup_{i=1}^{l} \{v^i_s, s(p^i_l), t(p^i_l) : s = 1, \ldots, n_i; l = 1, \ldots, m_i\}$, then $\alpha = \sum_{v \in V} v$ is a finite sum of vertices such that $\alpha a_i = a_i \alpha = a_i$ for every $i$.

1.7. Lemma. $L(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by $\deg(v_i) = 0$ for all $v_i \in E^0$; $\deg(e_i) = 1$ and $\deg(e^*_i) = -1$ for all $e_i \in E^1$.

That is, $L(E) = \bigoplus_{n \in \mathbb{Z}} L(E)_n$, where $L(E)_0 = KE^0 + A_0$, $L(E)_n = A_n$ for $n \neq 0$ where $A_n = \sum \{ke_{i_1} \ldots e_{i_s} e_{j_1}^* \ldots e_{j_r}^* : \sigma + \tau > 0, e_{i_s} \in E^1, e_{j_t} \in (E^1)^*, k \in K, \sigma - \tau = n\}$.

Proof: The fact that $L(E) = \sum_{n \in \mathbb{Z}} L(E)_n$ follows from Lemma 1.5. The grading on $L(E)$ follows directly from the fact that $A(\hat{E})$ is $\mathbb{Z}$-graded, and that the relations CK1 and CK2 are homogeneous in this grading.

Note that by virtue of Lemma 1.7 we can define the degree of an arbitrary polynomial in $L(E)$ as the maximum of the degrees of its monomials. We say that a monomial in $L(E)$ is a real path (resp. a ghost path) if it contains no terms of the form $e^*_i$ (resp. $e_i$); we say that $p \in L(E)$ is a polynomial in only real edges (resp. in only ghost edges) if it is a sum of real (resp. ghost) paths.

For a path $q = q_1 \ldots q_n$, we denote by $q^*$ the ghost path $q_n^* \ldots q_1^*$. If $\alpha \in L(E)$ and $d \in \mathbb{Z}^+$, then we say that $\alpha$ is representable as an element of degree $d$ in real (resp. ghost) edges in case $\alpha$ can be written as a sum of monomials from the spanning set $\{pq^* \mid p, q \text{ are paths in } E\}$ given by Lemma 1.5, in such a way that $d$ is the maximum length of a path $p$ (resp. $q$) which appears in such monomials. We note that an element of $L(E)$ may be representable as an element of different degrees in real (resp. ghost) edges, depending on the particular representation used for $\alpha$. For instance, for $E$ as in Example 1.4(ii), $xx^{-1}$ is representable as an element of degree 0 in real edges in $L(E)$, as $xx^{-1} = 1.$
2. Closed paths

Certain paths in the graph $E$ will play a central role in the structure of the Leavitt path algebra $L(E)$.

2.1. Definitions. An edge $e$ is an exit to the path $\mu = \mu_1 \ldots \mu_n$ if there exists $i$ such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$.

A closed path based at $v$ is a path $\mu = \mu_1 \ldots \mu_n$, with $\mu_j \in E$, $n \geq 1$ and such that $s(\mu) = r(\mu) = v$. Denote by $CP(v)$ the set of all such paths.

A closed simple path based at $v$ is a closed path based at $v$, $\mu = \mu_1 \ldots \mu_n$, such that $s(\mu_j) \neq v$ for every $j > 1$. Denote by $CSP(v)$ the set of all such paths.

Remark. Note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at $v$ is a cycle because a closed simple path may visit some of its vertices (but not $v$) more than once. Moreover, every closed simple path is in particular a closed path, while the converse is false.

2.2. Lemma. Let $\mu, \nu \in CSP(v)$. Then $\mu^*\nu = \delta_{\mu,\nu}v$.

Proof: We first assume $\alpha$ and $\beta$ are arbitrary paths and write $\alpha = e_{i_1} \ldots e_{i_\sigma}$ and $\beta = e_{j_1} \ldots e_{j_\tau}$.

Case 1: $deg(\alpha) = deg(\beta)$ but $\alpha \neq \beta$. Define $b \geq 1$ the subindex of the first edge where the paths $\alpha$ and $\beta$ differ. That is, $e_{i_a} = e_{j_a}$ for every $a < b$ but $e_{i_b} \neq e_{j_b}$. Then

$$\alpha^*\beta = e_{i_1}^* \ldots e_{i_a}^* e_{j_1} \ldots e_{j_b} = e_{i_1}^* \ldots e_{i_a}^* r(e_{j_1}) e_{j_2} \ldots e_{j_b}$$

$$= \delta_r(e_{j_1}), s(e_{j_2}) e_{i_1}^* \ldots e_{i_a}^* e_{j_2} \ldots e_{j_b} = \ldots$$

$$= \delta_r(e_{j_1}), s(e_{j_2}) \ldots \delta_r(e_{j_{b-1}}), s(e_{j_{b-1}}) e_{i_1}^* \ldots e_{i_a}^* e_{j_b} \ldots e_{j_\tau} = 0.$$

Case 2: $\alpha = \beta$. Proceeding as above, $\alpha^*\beta = \delta_r(e_{i_1}), s(e_{i_2}) \ldots s(e_{i_{\sigma-1}}), s(e_{i_\sigma}) r(e_{i_\sigma}) = r(\alpha)$.

Case 3: Now let $\mu, \nu \in CSP(v)$ with $deg(\mu) < deg(\nu)$. Write $\nu = \nu_1 \nu_2$ where $deg(\nu_1) = deg(\mu)$, $deg(\nu_2) > 0$. Now if $\mu = \nu_1$ then we have that $v = r(\mu) = r(\nu_1) = s(\nu_2)$, contradicting that $\nu \in CSP(v)$, so $\mu \neq \nu_1$ and thus case 1 applies to obtain $\mu^*\nu = \mu^*\nu_1\nu_2 = 0$.

The case $deg(\mu) > deg(\nu)$ is analogous to case 3 by changing the roles of $\mu$ and $\nu$.

2.3. Lemma. For every $p \in CP(v)$ there exist unique $c_1, \ldots, c_m \in CSP(v)$ such that $p = c_1 \ldots c_m$. 
Proof: Write $p = e_{i_1} \ldots e_{i_n}$. Let $T = \{ t \in \{1, \ldots, n\} : r(e_{i_t}) = v \}$ and list $t_1 < \cdots < t_m = n$ all the elements of $T$. Then $c_1 = e_{i_1} \ldots e_{i_{t_1}}$ and $c_j = e_{i_{t_{j-1}}} \ldots e_{i_{t_j}}$ for $j > 1$ give the desired decomposition.

To prove the uniqueness, write $p = c_1 \ldots c_r = d_1 \ldots d_s$ with $c_i, d_j \in \text{CSP}(v)$. Multiply by $c_1^*$ on the left and use Lemma 2.2 to obtain $0 \neq vc_2 \ldots c_r = c_1^*d_1 \ldots d_s$, and therefore by Lemma 2.2 again $c_1 = d_1$. Now an induction process finishes the proof. \hfill \blacksquare

2.4. Definition. For $p \in \text{CP}(v)$ we define the return degree (at $v$) of $p$ to be the number $m \geq 1$ in the decomposition above. (So, in particular, $\text{CSP}(v)$ is the subset of $\text{CP}(v)$ having return degree equal one.) We denote it by $\text{RD}(p) = \text{RD}_v(p) = m$. We extend this notion to vertices by setting $\text{RD}_v(v) = 0$, and to nonzero linear combinations of the form $\sum k_sp_s$, with $p_s \in \text{CP}(v) \cup \{v\}$ and $k_s \in K - \{0\}$ by: $\text{RD}(\sum k_sp_s) = \max\{\text{RD}(p_s)\}$.

2.5. Lemma. For a graph $E$ the following conditions are equivalent.

(i) Every cycle has an exit.

(ii) Every closed path has an exit.

(iii) Every closed simple path has an exit.

(iv) For every $v_i \in E^0$, if $\text{CSP}(v_i) \neq \emptyset$, then there exists $c \in \text{CSP}(v_i)$ having an exit.

Proof: (ii) $\Rightarrow$ (iii) is trivial by definition, and (iii) $\Rightarrow$ (iv) is obvious.

(i) $\Rightarrow$ (ii). Consider $\mu = \sum c_{i} \in \text{CP}(v_i)$. First by Lemma 2.3 we can factor $\mu = c^{(1)} \ldots c^{(m)}$, where $c^{(j)} \in \text{CSP}(v_i)$, and we examine $c^{(m)}$. If it is cycle then we can find an exit for it, and therefore for $\mu$, by hypothesis. If not, $c^{(m)}$ visits a vertex (different from $v_i$) more than once. Write $c^{(m)} = c^{(m)}_1 \ldots c^{(m)}_s$ with each $c^{(m)}_i \in E^1$ and let $c^{(m)}_s$ be the last edge for which $s(c^{(m)}_i) \in \{s(c^{(m)}_i) : 1 \leq i \leq s, i \neq j\}$. Thus, there exists $s_1 < s_0$ such that $s(c^{(m)}_{s_0}) = s(c^{(m)}_{s_1})$. We have several possibilities:

Case 1: $c^{(m)}_{s_0} = c^{(m)}_{s_1}$ and $s_0 < s$. Then $r(c^{(m)}_{s_0}) = r(c^{(m)}_{s_1})$; that is, $s(c^{(m)}_{s_0+1}) = s(c^{(m)}_{s_1+1})$, which contradicts the choice of $c^{(m)}_{s_0}$.

Case 2: $c^{(m)}_{s_0} = c^{(m)}_{s_1}$ and $s_0 = s$. This means that $r(c^{(m)}_{s_0}) = r(c^{(m)}_{s_1}) = v$, which is impossible because $c^{(m)} \in \text{CSP}(v_i)$.

Case 3: $c^{(m)}_{s_0} \neq c^{(m)}_{s_1}$. In this case $c^{(m)}_{s_1}$ is an exit for $c^{(m)}$, and then for $\mu$.

In each case we reach a contradiction or we find an exit for $\mu$, as needed.

(iv) $\Rightarrow$ (iii). Consider $c^{(1)} \in \text{CSP}(v_i)$. By hypothesis we find $c^{(2)} \in \text{CSP}(v_i)$ having an exit. If $c^{(1)} = c^{(2)}$ we are done. If not, we write $c^{(1)} = e_{i_1} \ldots e_{i_s}$, $c^{(2)} = e_{j_1} \ldots e_{j_r}$, and proceed by steps:
Step 1: If $e_{i_1} \neq e_{j_1}$, since $s(e_{i_1}) = s(e_{j_1}) = v_i$, then $e_{j_1}$ is an exit for $c^{(1)}$.

Step 2: If $e_{i_1} = e_{j_1}$, then $r(e_{i_1}) = r(e_{j_1})$; that is, $s(e_{i_2}) = s(e_{j_2})$.

Step 3: If $e_{i_2} \neq e_{j_2}$, then as in Step 1, $e_{j_2}$ is an exit for $c^{(1)}$.

Step 4: If $e_{i_2} = e_{j_2}$, then continue as in Step 2.

With this process, we either find an exit or we run out of edges in one path but not in the other (because $c^{(1)} \neq c^{(2)}$). Thus:

Case 1: $c^{(1)} = c^{(2)} e_{i_t} \ldots e_{i_s}$ for $t \leq s$. But this is impossible because $s(e_{i_t}) = r(c^{(2)}) = v_i$ and $c^{(1)} \in CSP(v_i)$.

Case 2: $c^{(2)} = c^{(1)} e_{j_q} \ldots e_{j_r}$ for $q \leq r$, which is similarly impossible.

In any case, we reach a contradiction or we are able to find an exit for $c^{(1)}$, and this finishes the proof. ■

3. Simplicity of $L(E)$

In this final section we build the algebraic machinery necessary to obtain our main result, Theorem (3.11).

3.1. Proposition. Let $E$ be a graph with the property that every cycle has an exit. If $\alpha \in L(E)$ is a polynomial in only real edges with $\deg(\alpha) > 0$, then there exist $a, b \in L(E)$ such that $a \circ b \neq 0$ is a polynomial in only real edges and $\deg(a \circ b) < \deg(\alpha)$.

Proof: Write $\alpha = \sum_{e_i \in E} e_i \alpha_{e_i} + \sum_{v_l \in E^0} k_l v_l$, where $\alpha_{e_i}$ are polynomials in only real edges, and $\deg(\alpha_{e_i}) < \deg(\alpha) = m$.

Case (A): $k_l = 0$ for every $l$. Since $\alpha \neq 0$, there exists $i_0$ such that $e_{i_0} \alpha_{e_{i_0}} \neq 0$. Let $b \in L(E)$ have $\circ b = \alpha$; such exists by Lemma 1.6. Then $a = e_{i_0}^* b$ give $e_{i_0}^* \circ b = \alpha_{e_{i_0}} \neq 0$ is a polynomial in only real edges and $\deg(\alpha_{e_{i_0}}) < \deg(\alpha)$.

Case (B): There exists $k_{i_0} \neq 0$. Then we can write

$$v_{i_0} \circ v_{i_0} = k_{i_0} v_{i_0} + \sum_{p \in CP(v_{i_0})} k_p p, \quad k_p \in K.$$  

Note that this is a polynomial in only real edges, and is nonzero because $k_{i_0}$ is nonzero.

Case (B.1): $\deg(v_{i_0} \circ v_{i_0}) < \deg(\alpha)$. Then we are done with $a = v_{i_0}$ and $b = v_{i_0}$.

Case (B.2): $\deg(v_{i_0} \circ v_{i_0}) = \deg(\alpha) = m > 0$. Then there exists $p_0 \in CP(v_{i_0})$ such that $k_{p_0} p_0 \neq 0$. Now by Lemma 2.3, we can write $p_0 = c_1 \ldots c_{\sigma}, \sigma \geq 1$ and thus
CSP($v_{i_0}$) $\neq \emptyset$. We apply now Lemma 2.5 to find $c_{s_0} \in CSP(v_{i_0})$ which has $e_{i_0}$ as an exit, that is, if $c_{s_0} = e_{i_1} \ldots e_{i_{s_0}}$ then there exists $j \in \{1, \ldots, s_0\}$ such that $s(e_{i_j}) = s(e_{i_0})$ but $e_{i_j} \neq e_{i_0}$. Since $s(e_{i_j}) = s(e_{i_0})$ we can therefore build the path given by $z = e_{i_1} \ldots e_{i_{j-1}} e_{i_0}$. This path has $c_{s_0} z = 0$ because $c_{s_0} z = e_{i_{s_0}}^* e_{i_1} \ldots e_{i_{j-1}} e_{i_0} = \cdots = e_{i_{s_0}}^* e_{i_j} e_{i_0} = 0$. (We will use this observation later on.) Again Lemma 2.3 allows us to write

$$v_{i_0} \alpha v_{i_0} = k_{i_0} v_{i_0} + \sum_{c_s \in CSP(v_{i_0})} c_s \alpha^{(1)}_{c_s},$$

where $\gamma = RD(v_{i_0} \alpha v_{i_0}) > 0$, and $\alpha^{(1)}_{c_s}$ are polynomials in only real edges satisfying $RD(\alpha^{(1)}_{c_s}) < \gamma$.

We now present a process in which we decrease the return degree of the polynomials by multiplying on both sides by appropriate elements in $L(E)$. In the sequel we will often make use of Lemma 2.2 without mentioning it explicitly. In particular, multiplying (†) on the left by $c^*_{s_0}$ gives

$$c^*_{s_0}(v_{i_0} \alpha v_{i_0}) = k_{i_0} c^*_{s_0} + \alpha^{(1)}_{c_{s_0}}.$$

**Case 1:** $\alpha^{(1)}_{c_{s_0}} = 0$. Then $A = c^*_{s_0}$ and $B = c_{s_0}$ are such that $A(v_{i_0} \alpha v_{i_0}) B = k_{i_0} v_{i_0} \neq 0$ is a polynomial in only real edges and $RD(A(v_{i_0} \alpha v_{i_0}) B) = 0 < \gamma = RD(v_{i_0} \alpha v_{i_0})$.

**Case 2:** $\alpha^{(1)}_{c_{s_0}} \neq 0$ but $RD(\alpha^{(1)}_{c_{s_0}}) = 0$. Then $\alpha^{(1)}_{c_{s_0}} = k^{(2)} v_{i_0}$ for some $0 \neq k^{(2)} \in K$. Using the path $z$ with an exit for $c^*_{s_0}$ we have: $z^* c^*_{s_0} (v_{i_0} \alpha v_{i_0}) z = z^* (k_{i_0} c^*_{s_0} + k^{(2)} v_{i_0}) z = z^* (0 + k^{(2)} z) = k^{(2)} r(z) \neq 0$. So we have $A = z^* c^*_{s_0}$ and $B = z$ such that $A(v_{i_0} \alpha v_{i_0}) B \neq 0$ is a polynomial in only real edges and $RD(A(v_{i_0} \alpha v_{i_0}) B) = 0 < \gamma = RD(v_{i_0} \alpha v_{i_0})$.

**Case 3:** $RD(\alpha^{(1)}_{c_{s_0}}) > 0$. We can write

$$\alpha^{(1)}_{c_{s_0}} = k^{(2)} v_{i_0} + \sum_{c_s \in CSP(v_{i_0})} c_s \alpha^{(2)}_{c_s},$$

where $\alpha^{(2)}_{c_s}$ are polynomials in only real edges with return degree less than the return degree of $\alpha^{(1)}_{c_{s_0}}$. Now $0 < RD(\alpha^{(1)}_{c_{s_0}}) < \gamma$ implies $\gamma \geq 2$. Multiply (†) by $c^*_{s_0}$ to get

$$(c^*_{s_0})^2(v_{i_0} \alpha v_{i_0}) = k_{i_0} (c^*_{s_0})^2 + k^{(2)} c^*_{s_0} + \alpha^{(2)}_{c_{s_0}}.$$

We are now in position to proceed in a manner analogous to that described in Cases 1, 2, and 3 above.
Case 3.1: \( \alpha^{(2)}_{c_{s_0}} = 0 \). Then \((c^*_{s_0})^2(v_{i_0}a\nu v_{i_0})(c_{s_0})^2 = k_{i_0}v_{i_0} + k^{(2)}c_{s_0} \) and hence we have found \( A = (c^*_{s_0})^2 \) and \( B = (c_{s_0})^2 \) such that \( A(v_{i_0}a\nu v_{i_0})B \neq 0 \) is a polynomial in only real edges and \( RD(A(v_{i_0}a\nu v_{i_0})B) = 1 < 2 \leq \gamma = RD(v_{i_0}a\nu v_{i_0}) \).

Case 3.2: \( \alpha^{(2)}_{c_{s_0}} \neq 0 \) but \( RD(\alpha^{(2)}_{c_{s_0}}) = 0 \). Then \( \alpha^{(2)}_{c_{s_0}} = k^{(3)}v_{i_0} \) for some \( 0 \neq k^{(3)} \in K \), and then \( z^{*}(c^*_{s_0})^2(v_{i_0}a\nu v_{i_0})z = z^*(k_{i_0}(c^*_{s_0})^2 + k^{(2)}c_{s_0} + k^{(3)}v_{i_0})z = z^*(0 + k^{(3)}z) = k^{(3)}r(z) \neq 0 \). Thus, we get \( A = z^*(c^*_{s_0})^2 \) and \( B = z \) such that \( A(v_{i_0}a\nu v_{i_0})B \neq 0 \) is a polynomial in only real edges and \( RD(A(v_{i_0}a\nu v_{i_0})B) = 0 < \gamma = RD(v_{i_0}a\nu v_{i_0}) \).

Case 3.3: \( RD(\alpha^{(2)}_{c_{s_0}}) > 0 \). We write

\[
\alpha^{(2)}_{c_{s_0}} = k^{(3)}v_{i_0} + \sum_{c_s \in CSP(v_{i_0})} c_s\alpha^{(3)}_{c_s},
\]

where \( \alpha^{(3)}_{c_s} \) are polynomials in only real edges with return degree less than the return degree of \( \alpha^{(2)}_{c_{s_0}} \). Now \( 0 < RD(\alpha^{(2)}_{c_{s_0}}) < RD(\alpha^{(1)}_{c_{s_0}}) < \gamma \) implies \( \gamma \geq 3 \). And by multiplying (6) by \( c^*_{s_0} \) we get \( (c^*_{s_0})^3(v_{i_0}a\nu v_{i_0}) = k_{i_0}(c^*_{s_0})^3 + k^{(2)}(c^*_{s_0})^2 + k^{(3)}c_{s_0} + \alpha^{(3)}_{c_{s_0}} \).

We continue the process of analyzing each such equation by considering three cases. If at any stage either of the first two cases arise, we are done. But since at each stage the third case can occur only by producing elements of subsequently smaller return degree, then after at most \( \gamma \) stages we must have one of the first two cases.

Thus, by repeating this process at most \( \gamma \) times we are guaranteed to find \( \tilde{A}, \tilde{B} \) such that \( \tilde{A}(v_{i_0}a\nu v_{i_0})\tilde{B} \neq 0 \) is a polynomial in only real edges and \( RD(\tilde{A}(v_{i_0}a\nu v_{i_0})\tilde{B}) = 0 \). But this then gives \( 0 = deg(\tilde{A}(v_{i_0}a\nu v_{i_0})\tilde{B}) < deg(\alpha) \). So \( a = \tilde{A}v_{i_0} \) and \( b = v_{i_0}\tilde{B} \) are the desired elements. ■

3.2. Corollary. Let \( E \) be a graph with the property that every cycle has an exit. If \( \alpha \neq 0 \) is a polynomial in only real edges then there exist \( a, b \in L(E) \) such that \( aob \in E^0 \).

Proof: Apply Proposition 3.1 as many times as needed (\( deg(\alpha) \) at most) to find \( a', b' \) such that \( a'ob' \) is a nonzero polynomial in only real edges with \( deg(a'ob') = 0 \); that is, \( a'ob' = \sum_{i=1}^t k_i v_i \neq 0 \). So there exists \( j \) with \( k_j \neq 0 \), and finally \( a = k_j^{-1}a' \) and \( b = b'v_j \) give that \( aob = v_j \in E^0 \). ■

3.3. Corollary. Let \( E \) be a graph with the property that every cycle has an exit. If \( J \) is a ideal of \( L(E) \) and contains a nonzero polynomial in only real edges, then \( E^0 \cap J \neq \emptyset \).

Proof: Straightforward by Corollary 3.2. ■

In order to extend all the previous results of this section to analogous results about polynomials in only ghost edges, we define an involution in \( L(E) \).
3.4. Lemma. \( L(E) \) can be equipped with an involution \( x \mapsto \overline{x} \) defined in the monomials by:

(a) \( \overline{k_i v_i} = k_i v_i \) with \( k_i \in K \) and \( v_i \in E^0 \),

(b) \( \overline{ke_{i_1} \ldots e_{i_r} e_{j_1}^* \ldots e_{j_s}^*} = ke_{j_1} \ldots e_{j_s} e_{i_r}^* \ldots e_{i_1}^* \) where \( k \in K \); \( \sigma, \tau \geq 0 \), \( \sigma + \tau > 0 \), \( e_{i_k} \in E^1 \) and \( e_{j_k} \in (E^1)^* \),

and extending linearly to \( L(E) \).

**Proof:** The proposed map is well defined by Lemma 1.5, and it is linear by definition. It is easily shown to satisfy \( \overline{xy} = y \overline{x} \) and \( \overline{x} = x \) for every \( x, y \in L(E) \). It is also straightforward to check that the map is compatible with the relations defining \( L(E) \).

3.5. Remark. Note that the involution transforms a polynomial in only real edges into a polynomial in only ghost edges and vice versa. If \( J \) is an ideal of \( L(E) \) then so is \( \overline{J} \). We note here that while Leavitt path algebras behave somewhat like their \( C^* \)-algebra siblings, they are indeed different in many respects. For instance, whereas in \( C^* \)-algebras every two-sided ideal \( J \) is self-adjoint (i.e. \( \overline{J} = J \)), this is not the case in the Leavitt path algebras setting. For instance, let \( L(E) = K[x, x^{-1}] \) as in Example 1.4 (ii), and let \( J \) be the ideal \( < 1 + x + x^3 > \) of \( L(E) \). Then \( J \) is not self-adjoint, as follows: if \( \overline{J} = J \), then \( f(x) = 1 + x^{-1} + x^{-3} \in J \) and thus \( x^3 f(x) = 1 + x^2 + x^3 \in J \). Now \( K[x, x^{-1}] \) being a unital commutative ring implies that there exists \( p = \sum_{i=-\infty}^{\infty} a_i x^i \) with \( p(1 + x + x^3) = 1 + x^2 + x^3 \). A degree argument on the highest power on the left hand side of the previous equation leads to \( a_i = 0 \) for every \( i \geq 1 \). By reasoning in a similar fashion on the lowest power we also get \( a_i = 0 \) for every \( i \leq -1 \), that is, \( p = a_0 \), which is absurd.

We can define sets and quantities for ghost paths analogous to those given for real paths. Using the involution given in Lemma 3.4 we can then analogously prove the following three results.

3.6. Proposition. Let \( E \) be a graph with the property that every cycle has an exit. If \( \alpha \in L(E) \) is a polynomial in only ghost edges with \( \deg(\overline{\alpha}) > 0 \) then there exist \( a, b \in L(E) \) such that \( a\overline{\alpha}b \neq 0 \) is a polynomial in only ghost edges and \( \deg(a\overline{\alpha}b) < \deg(\overline{\alpha}) \).

3.7. Corollary. Let \( E \) be a graph with the property that every cycle has an exit. If \( \alpha \neq 0 \) is a polynomial in only ghost edges then there exist \( a, b \in L(E) \) such that \( a\overline{\alpha}b \in E^0 \).

3.8. Corollary. Let \( E \) be a graph with the property that every cycle has an exit. If \( J \) is an ideal of \( L(E) \) and contains a nonzero polynomial in only ghost edges, then \( E^0 \cap J \neq \emptyset \).
For a graph $E$ we define a preorder $\leq$ on the vertex set $E^0$ given by:

$$v \leq w \text{ if and only if } v = w \text{ or there is a path } \mu \text{ such that } s(\mu) = v \text{ and } r(\mu) = w.$$ We say that a subset $H \subseteq E^0$ is hereditary if $w \in H$ and $w \leq v$ imply $v \in H$. We say that $H$ is saturated if whenever $s^{-1}(v) \neq \emptyset$ and \{r(e) : s(e) = v\} $\subseteq H$, then $v \in H$. (In other words, $H$ is saturated if, for any vertex $v$ in $E$, if all of the range vertices $r(e)$ for those edges $e$ having $s(e) = v$ are in $H$, then $v$ must be in $H$ as well.)

**3.9. Lemma.** If $J$ is an ideal of $L(E)$, then $J \cap E^0$ is a hereditary and saturated subset of $E^0$.

**Proof:** We first show that $J \cap E^0$ is hereditary. Consider $v, w \in E^0$ such that $v \in J$ and $v \leq w$. By the definition of the preorder we can find a path $\mu = \mu_1 \ldots \mu_n$ such that $s(\mu_1) = v$ and $r(\mu_n) = w$. Apply that $J$ is an ideal to get that $\mu_n^* v \mu_1 = \mu_n^* \mu_1 = r(\mu_1) = s(\mu_2) \in J$. Repeating this argument $n$ times, we get that $r(\mu_n) = w \in J$.

Now we see that $J \cap E^0$ is saturated: consider a vertex $v$ with $s^{-1}(v) \neq \emptyset$ and \{r(e) : s(e) = v\} $\subseteq J$. The first condition implies that $v$ is not a sink, so CK2 applies and we obtain $v = \sum_{\{e_j \in E^1 : s(e_j) = v\}} e_j e_j^*$. If we take $e_j$ such that $s(e_j) = v$, then by hypothesis we have that $r(e_j) \in J$ and therefore $e_j = e_j r(e_j) \in J$. Now applying CK2 we conclude that $v \in J$. $\blacksquare$

**3.10. Corollary.** Let $E$ be a graph with the following properties:

(i) The only hereditary and saturated subsets of $E^0$ are $\emptyset$ and $E^0$.

(ii) Every cycle has an exit.

If $J$ is a nonzero ideal of $L(E)$ which contains a polynomial in only real edges (or a polynomial in only ghost edges), then $J = L(E)$.

**Proof:** Apply Corollaries 3.3 or 3.8 to get that $J \cap E^0 \neq \emptyset$. Now by Lemma 3.9 and (i) we have $J \cap E^0 = E^0$. Therefore $J$ contains a set of local units by Lemma 1.6, and hence $J = L(E)$. $\blacksquare$

We are now in position to prove the main result of this article.

**3.11. Theorem.** Let $E$ be a row-finite graph. Then the Leavitt path algebra $L(E)$ is simple if and only if $E$ satisfies the following conditions.

(i) The only hereditary and saturated subsets of $E^0$ are $\emptyset$ and $E^0$, and

(ii) Every cycle in $E$ has an exit.

**Proof:** First we assume that (i) and (ii) hold and we will show that $L(E)$ is simple. Suppose that $J$ is a nonzero ideal of $L(E)$. Choose $0 \neq \alpha \in J$ representable as an element having minimal degree in the real edges. If this minimal degree is 0, then $\alpha$ is a polynomial
in only ghost edges, so that by Corollary 3.10 we have $J = L(E)$. So suppose this degree in real edges is at least 1. Then we can write
\[
\alpha = \sum_{n=1}^{m} e_{i_{n}} \alpha_{e_{i_{n}}} + \beta
\]
where $m \geq 1$, $e_{i_{n}} \alpha_{e_{i_{n}}} \neq 0$ for every $n$, and each $\alpha_{e_{i_{n}}}$ is representable as an element of degree less than that of $\alpha$ is real edges, and $\beta$ is a polynomial in only ghost edges (possibly zero).

Suppose $v$ is a sink in $E$. Then we may assume $v \beta = 0$, as follows. Multiplying the displayed equation by $v$ on the left gives $v \alpha = v \sum_{n=1}^{m} e_{i_{n}} \alpha_{e_{i_{n}}} + v \beta$. But since $v$ is a sink we have $ve_{i_{n}} = 0$ for all $1 \leq n \leq m$, so that $v \alpha = v \beta \in J$. But $v \beta \neq 0$ would then yield a nonzero element of $J$ in only ghost edges, so that again by Corollary 3.10 we have $J = L(E)$.

For an arbitrary edge $e_{j} \in E^{1}$, we have two cases:

Case 1: $j \in \{i_{1}, \ldots, i_{m}\}$. Then $e_{j}^{*} \alpha = \alpha_{e_{j}} + e_{j}^{*} \beta \in J$. If this element is nonzero it would be representable as an element with smaller degree in the real edges than that of $\alpha$, contrary to our choice. So it must be zero, and hence $\alpha_{e_{j}} = -e_{j}^{*} \beta$, so that $e_{j} \alpha_{e_{j}} = -e_{j} e_{j}^{*} \beta$.

Case 2: $j \notin \{i_{1}, \ldots, i_{m}\}$. Then $e_{j}^{*} \alpha = e_{j}^{*} \beta \in J$. If $e_{j}^{*} \beta \neq 0$, then as before we would have a nonzero element of $J$ in only ghost edges, so that $J = L(E)$ and we are done. So we may assume that $e_{j}^{*} \beta = 0$, so that in particular we have $0 = -e_{j} e_{j}^{*} \beta$.

Now let $S_{1} = \{v_{j} \in E^{0} : v_{j} = s(e_{i_{n}})$ for some $1 \leq n \leq m\}$, and let $S_{2} = \{v_{k_{1}}, \ldots, v_{k_{t}}\}$ where $(\sum_{i=1}^{m} v_{k_{i}}) \beta = \beta$. (Such a set $S_{2}$ exists by Lemma 1.6.) We note that $w \beta = 0$ for every $w \in E^{0} - S_{2}$. Also, by definition there are no sinks in $S_{1}$, and by a previous observation we may assume that there are no sinks in $S_{2}$. Let $S = S_{1} \cup S_{2}$. Then in particular we have $(\sum_{v \in S} v) \beta = \beta$.

We now argue that in this situation $\alpha$ must be zero, which will contradict our original choice of $\alpha$ and thereby complete the proof. To this end,
\[
\alpha = \sum_{n=1}^{m} e_{i_{n}} \alpha_{e_{i_{n}}} + \beta = \sum_{n=1}^{m} -e_{i_{n}} e_{i_{n}}^{*} \beta + \beta \quad \text{(by Case 1)}
\]
\[
= \sum_{n=1}^{m} -e_{i_{n}} e_{i_{n}}^{*} \beta - \left( \sum_{j \notin \{i_{1}, \ldots, i_{m}\}, s(e_{j}) \in S} e_{j} e_{j}^{*} \beta \right) + \beta \quad \text{(by Case 2, the newly subtracted terms equal 0)}
\]
\[
= - \left( \sum_{v \in S} v \right) \beta + \beta \quad \text{(no sinks in $S$ implies that CK2 applies at each $v \in S$)}
\]
\[
= - \beta + \beta = 0.
\]
Thus we have shown that if $E$ satisfies the two indicated properties, then $L(E)$ is simple. Let $v$ be the base of that cycle. We will show that for $\alpha = v + p$, $\langle \alpha \rangle$ is a nontrivial ideal of $L(E)$ because $v \not\in \langle \alpha \rangle$. Write $p = e_{i_1} \ldots e_{i_n}$. Since this cycle does not have an exit, for every $e_{i_j}$ there is no edge with source $s(e_{i_j})$ other than $e_{i_j}$ itself, so that the CK2 relation at this vertex yields $s(e_{i_j}) = e_{i_j} e_{i_j}^s$. This easily implies $pp^s = v$ (we recall here that $p^s p = v$ always holds), and that $CSP(v) = \{ p \}$.

Now suppose that $v \in \langle \alpha \rangle$. So there exist nonzero monic monomials $a_n, b_n \in L(E)$ and $c_n \in \mathbb{K}$ with $v = \sum_{n=1}^{m} c_n a_n \alpha b_n$. Since $v \alpha v = \alpha$, by multiplying by $v$ if necessary we may assume that $v a_n v = a_n$ and $v b_n v = b_n$ for all $1 \leq n \leq m$.

We claim that for each $a_n$ (resp. $b_n$) there exists an integer $u(a_n) \geq 0$ (resp. $u(b_n) \geq 0$) such that $a_n = p^{u(a_n)}$ or $a_n = (p^s)^{u(a_n)}$ (resp. $b_n = p^{u(b_n)}$ or $b_n = (p^s)^{u(b_n)}$).

Now $a_1$ is of the form $e_{k_1} \ldots e_{k_e} e_{j_1}^s \ldots e_{j_d}^s$ with $c, d \geq 1$. (Otherwise we are in a simple case that will be contained in what follows.) Since $a_1$ starts and ends in $v$ we can consider the elements: $g = \min \{ z : r(e_{j_d}^s) = v \}$ and $f = \max \{ z : s(e_{k_c}) = v \}$, and we will focus on $a_1' = e_{k_f} \ldots e_{k_c} e_{j_1}^s \ldots e_{j_g}^s$.

First, since $v = r(e_{j_d}^s) = s(e_{j_g}^s)$ and $e_{i_1}$ is the only edge coming from $v$, then $e_{j_g} = e_{i_1}$. Now, $s(e_{j_{d-1}}) = r(e_{j_{d+1}}) = s(e_{j_g}) = r(e_{j_{g-1}}) = r(e_{i_1}) = s(e_{i_2})$, and again the only edge coming from $s(e_{i_2}) = e_{i_2}$ and therefore $e_{j_{g-1}} = e_{i_2}$. This process must stop before we run out of edges of $p$ because by our choice of $g$ we have that $v \not\in \{ r(e_{j_k}^s) : z < g \}$. So in the end there exists $\gamma < \sigma$ such that $e_{j_1}^s \ldots e_{j_g}^s = e_{i_1}^s \ldots e_{i_\gamma}^s$.

With the same (reversed) ideas in the paragraph above we can find $\delta < \sigma$ such that $e_{k_f} \ldots e_{k_c} = e_{i_1} \ldots e_{i_\delta}$. Thus, $a_1' = e_{i_1} \ldots e_{i_\delta} e_{i_\gamma}^s \ldots e_{i_1}^s$, and we have two cases:

Case 1: $\delta \neq \gamma$. We know that $p$ is a cycle, so that $r(e_{i_\delta}) \neq r(e_{i_\gamma}) = s(e_{i_\gamma}^s)$, so $e_{i_\delta} e_{i_\gamma}^s = 0$, which is absurd because $a_1 \neq 0$.

Case 2: $\delta = \gamma$. In this case $a_1' = p_0 p_0^s$ for a certain subpath $p_0$ of $p$, and by using again the argument of the CK2 relation in this case, we obtain $p_0 p_0^s = v$.

Hence, we get $a_1 = e_{k_1} \ldots e_{k_{d-1}} e_{j_{d+1}}^s \ldots e_{j_\delta}^s = x y^s$, with $x, y \in CP(v)$. (Obviously, the case $c \geq 1, d = 0$ yields $a_1 = x$, the case $c = 0, d \geq 1$ yields $a_1 = y^s$ and $c = d = 0$ yields $a_1 = v$.) Using Lemma 2.3 we have $x = e^{(1)} \ldots e^{(u)}$ for some $e^{(u)} \in CSP(v) = \{ p \}$, and the same happens with $y$. In this way we have $a_1 = p^{u(p^s)} v$ for some $u, v \geq 0$, and taking into account that $p p^s = v$ we finally obtain that $a_1$ is of the form $p^u$ or $(p^s)^u$ for some $u \geq 0$ as claimed. An identical argument holds for the other coefficients $a_n$ and $b_n$.

Now since both $p$ and $p^s$ commute with $p, p^s$ and $\alpha$, we use the conclusion of the previous paragraph to write the sum $v = \sum_{n=1}^{m} c_n a_n \alpha b_n$ as $v = \alpha P(p, p^s)$ for some polynomial $P$ having coefficients in $K$. Specifically, $P(p, p^s) = \cdots$
Then $k_{-}\hspace{3pt}m(p^{*})^m + \cdots + k_0v + \cdots + k_np^n \in \bigoplus_{j=-m}^n L(E)_{\sigma,j}$, where $m, n \geq 0$. First, we claim that $k_{-}\hspace{3pt}i = 0$ for every $i > 0$, as follows. If not, let $m_0$ be the maximum $i$ having $k_{-}\hspace{3pt}i \neq 0$. Then $\alpha P(p, p^*) = k_{-}\hspace{3pt}m_0(p^{*})^{m_0} + \text{terms of greater degree} = v$, and since $m_0 > 0$ we get that $k_{-}\hspace{3pt}m_0 = 0$, which is absurd. In a similar way we obtain $k_i = 0$ for every $i > 0$, and therefore $P(p, p^*) = k_0v$. But this would yield $v = \alpha P(p, p^*) = \alpha k_0v = k_0\alpha$, which is impossible.

Thus we have shown that if $E$ contains a cycle which has no exit, then $L(E)$ is not simple. Now we will consider the situation where $E^0$ contains a nontrivial hereditary and saturated subset $H$, and conclude in this case as well that $L(E)$ is not simple. To do so, we construct a new graph $F = (F^0, F^1, r_F, s_F) = (E^0 - H, r^{-1}_E(E^0 - H), r|_{E^0 - H}, s|_{E^0 - H})$. In other words, $F$ is the graph consisting of all vertices not in $H$, together with all edges whose range is not in $H$. To ensure that $F$ is well-defined, we must check that $s_F(F^1) \subseteq F^0$. That $r_F(F^1) \subseteq F^0$ is evident. On the other hand, if $e \in F^1$ then $s(e) \in F^0$, since otherwise we have $s(e) \in H; \text{but since } r(e) \geq s(e) \text{ and } H \text{ is hereditary, we get } r(e) \in H$, which contradicts $e \in F^1$. So $F$ is a well defined graph.

We now produce a $K$-algebra homomorphism $\Psi : L(E) \rightarrow L(F)$. To do so, we define $\Phi$ on the generators of the free $K$-algebra $B = K[E^0 \cup E^1 \cup (E^1)^*]$ by setting $\Phi(v_i) = \chi_{(E_0^0)\hspace{3pt}v_i}, \Phi(e_i) = \chi_{(E_1^0)}(e_i)e_i$ and $\Phi(e_i^*) = \chi_{(E_1^1)}(e_i^*)e_i^*$ (where $\chi_X$ denotes the usual characteristic function of a set $X$), and extending to $B$. In order to factor $\Phi$ through $A(\hat{E})$ we need to check that

$$<v_i, v_j - \delta_{ij}v_i : v_i, v_j \in E^0> \cup \{e_i - e_ir(e_i), e_i - s(e_i)e_i : e_i \in \hat{E}^1\} > \subseteq Ker(\Phi).$$

This is a straightforward computation done by cases, with the only nontrivial situation arising when $e_i \in F^1$. But then $r(e_i) \notin H$, and therefore $\Phi(e_i - e_ir(e_i)) = e_i - e_ir(e_i) = 0$ in $L(F)$. Now, since $s(e_i) \leq r(e_i) \notin H$ and $H$ is hereditary then $s(e_i) \notin H$, so that $\Phi(e_i - s(e_i)e_i) = e_i - s(e_i)e_i = 0$ in $L(F)$.

Now to produce the desired ring homomorphism $\Psi : L(E) \rightarrow L(F)$ we need only check that $\Phi$ factors through the relations ideal

$$<\{e_i^*e_j - \delta_{ij}r(e_j) : e_j \in E^1, e_i^* \in (E^1)^*\} \cup \{v_i - \sum_{e_j \in E^1 : s(e_j) = v_i} e_j e_j^* : v_i \in s(E^0)\} >$$

of $A(\hat{E})$. That $\Phi(e_i^*e_j - \delta_{ij}r(e_j)) = 0$ in $L(F)$ is straightforward. So now consider $v_i \in s(E^0)$; i.e., consider a vertex $v_i$ which is not a sink in $E$.

Case 1: Suppose $v_i \in H$. Then for every $e_j \in E^1$ with $s(e_j) = v_i$ we have that $e_i \notin F^1$ (otherwise $e_i \in F^1$ implies $r(e_i) \notin H$ and by hereditariness $s(e_j) = v_i \notin H$). So, $\Phi(v_i - \sum_{e_j \in E^1 : s(e_j) = v_i} e_j e_j^*) = 0 - \sum_{e_j \in E^1 : s(e_j) = v_i} 0 \cdot 0 = 0$. 


Case 2: Suppose \( v_i \not\in H \) and \( v_i \not\in s(F^1) \). Since \( v_i \in s(E^0) \) we have \( s^{-1}(v_i) \neq \emptyset \). But since \( H \) is saturated there must exist \( e_i \in E^1 \) such that \( s(e_i) = v_i \), but \( r(e_i) \not\in H \). That means \( e_i \in F^1 \) with \( s(e_i) = v_i \), which contradicts the hypothesis that \( v_i \not\in s(F^1) \). Thus the saturated condition on \( H \) implies that Case 2 configuration cannot occur.

Case 3: Suppose \( v_i \not\in H \) but \( v_i \in s(F^1) \). Then we have a CK2 relation in \( L(F) \) at \( v_i \):

\[
v_i = \sum_{\{e_j \in F^1 : s(e_j) = v_i\}} e_j e_j^*.
\]

Consider \( e_j \in E^1 \) such that \( s(e_j) = v_i \). If \( e_j \in F^1 \) then \( \Phi(e_j e_j^*) = e_j e_j^* \). If \( e_j \not\in F^1 \) then \( \Phi(e_j e_j^*) = 0 \). Thus we get \( \Phi(v_i - \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^*) = v_i - \sum_{\{e_j \in F^1 : s(e_j) = v_i\}} e_j e_j^* = 0 \) by the displayed equation.

Thus we have shown that there exists a \( K \)-algebra homomorphism \( \Psi : L(E) \to L(F) \).

Now consider \( Ker(\Psi) \trianglelefteq L(E) \). Since \( H \neq \emptyset \) there exists \( v \in H \), so \( 0 \neq v \in Ker(\Psi) \).

Since \( H \neq E^0 \) there exists \( w \in E^0 - H \) and in this case \( \Psi(w) = w \neq 0 \) so \( \Psi \neq 0 \). In other words, \( 0 \neq Ker(\Psi) \neq L(E) \), so that \( L(E) \) is not simple.

Thus we conclude that the negation of either condition (i) or condition (ii) yields that \( L(E) \) is not simple, which completes the proof of the theorem. \( \blacksquare \)

3.12. Remark. If we start with a finite and row-finite graph \( E = (E^0, E^1, r, s) \) with \( E^0 = \{v_1, \ldots, v_n\}, E^1 = \{e_1, \ldots, e_m\} \), there exist algorithms that decide, in a finite number of steps, whether or not the graph satisfies conditions (i) and/or (ii), and therefore whether or not \( L(E) \) is simple.

3.13. Corollary. We re-establish the simplicity (or non-simplicity) of the algebras given in Examples 1.4 above.

(i) Matrix algebras \( M_n(K) \): Since there are clearly no cycles in \( E \), we need only verify condition (i) in Theorem 3.11. To this end, let \( H \neq \emptyset \) be a set of vertices which is hereditary and saturated. Pick \( v_i \in H \). By hereditariness we have that \( v_{i+1}, \ldots, v_n \in H \). Now if we use the condition of being saturated at \( v_{i-1} \) we get that \( v_{i-1} \in H \), and inductively \( v_{i-1}, \ldots, v_1 \in H \) and therefore \( H = E^0 \). Hence there are no nontrivial hereditary and saturated subsets of \( E^0 \), and Theorem 3.11 applies to give that \( M_n(K) = L(E) \) is simple.

(ii) Laurent polynomial algebras \( K[x, x^{-1}] \): The cycle \( x \) does not have an exit, so by Theorem 3.11 \( L(E) \cong K[x, x^{-1}] \) is not simple. (Indeed, similar to the argument which arises in the proof of Theorem 3.11, it is easy to show that \( 1 \notin 1 + x \).)

(iii) Leavitt algebras \( L(1, n) \) for \( n \geq 2 \): The conditions in Theorem 3.11 are clearly satisfied here, so \( L(1, n) \) is simple, as was established in [7, Theorem 2].
3.14. Example. Let $C_n$ denote the graph having $n$ vertices and $n$ edges, where the edges form a single cycle. (In particular, the graph described in Example 1.4 (ii) is the graph $C_1$.) Then $L(C_n)$ is not simple for all $n$, since the single cycle contains no exit.

3.15. Example. The Cuntz-Krieger algebra $\mathcal{CK}_A(K)$ of a finite matrix $A$ is defined in [1, example 2.5]. For a finite graph $E$ we can define the edge matrix $A_E$ associated to $E$; $A_E$ is the $n \times n$ matrix with entries $a_{ij} = \delta_{r(e_i), s(e_j)}$, where $n = |E|$. It is long but straightforward to show that if a finite graph $E$ has no sinks nor sources, then $L(E) \cong \mathcal{CK}_{A_E}(K)$.

In [1, Theorem 4.1] the authors provide sufficient conditions on $A$ which yield the simplicity of $\mathcal{CK}_A(K)$, in case $A$ is a finite matrix which has no row or column of zeros, and in case $A$ is not a permutation matrix. (There is also an additional condition on an associated function $\alpha$ which must be satisfied in order to yield the simplicity of $\mathcal{CK}_A(K)$.) But these conditions on $A$ eliminate both the simple algebras $M_n(K)$ and the non-simple algebras $L(C_n)$ from consideration in [1, Theorem 4.1], since the edge matrix for the graph given in Examples 1.4 (i) is

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

which contains both a zero column and a zero row, while the edge matrix for the cycle graph $C_n$ given in Example 3.14 is

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

which is a permutation matrix.

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