COMPUTING INDIVIDUAL KAZHDAN–LUSZTIG BASIS ELEMENTS

LEONARD L. SCOTT AND TIMOTHY SPROWL

Abstract. In well-known work, Kazhdan and Lusztig (1979) defined a new set of Hecke algebra basis elements (actually two such sets) associated to elements in any Coxeter group. Often these basis elements are computed by a standard recursive algorithm which, for Coxeter group elements of long length, generally involves computing most basis elements corresponding to Coxeter group elements of smaller length. Thus, many calculations simply compute all basis elements associated to a given length or less, even if the interest is in a specific Kazhdan-Lusztig basis element. Similar remarks apply to “parabolic” versions of these basis elements defined later by Deodhar (1987,1990), though the lengths involved are the (smaller) lengths of distinguished coset representatives. We give an algorithm which targets any given Kazhdan-Lusztig basis element or parabolic analog and does not precompute any other Kazhdan-Lusztig basis elements. In particular it does not have to store them. This results in a considerable saving in memory usage, enabling new calculations in an important case (for finite and algebraic group 1-cohomology with irreducible coefficients) analyzed by Scott-Xi (2010).

1. Introduction

This note addresses a need we have perceived for a non-recursive algorithm focused on determining coefficients in Kazhdan–Lusztig polynomials $P_{x,y}$ associated to a single $y$ in a given Coxeter group $W$, or equivalently, to that of a single Kazhdan–Lusztig Hecke algebra basis element $C_y'$ in the notation of [KL79] or [Deo90, p. 101]. Our approach here applies also to the parabolic Kazhdan–Lusztig polynomials $P_{x,y}^J$ and basis elements $J'C_y'$ (for an appropriate Hecke algebra right module $M = M^J$) in the notation of [Deo90, p. 113]. The parabolic notations are defined only for $y$ “distinguished” (shortest) in its right coset $Wy$ in $W$, and there is a similar requirement on $x$.

We follow the notation of [Deo90] closely. The Hecke algebra of $W$ is denoted $\mathcal{H}$. It is a free $R$-module, where $R$ is the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, with basis elements $T_x$, $x \in W$, as discussed in [Deo90, §3], following standard terminology. The identity element of $W$ is denoted $e$, and $T_e$ is the identity of the ring $\mathcal{H}$. The set $J$ is a subset of the set $S$ of fundamental generators of $W$ and serves as a set of fundamental generators of the Coxeter group $W_J$. The set of distinguished right coset representatives of $W_J$ in $W$ is denoted $W^J$. Henceforth, we fix a subset $J$, which may be the empty set. The module $M = M^J$ has a basis $\{m_x\}_{x \in W^J}$ with $m_x = m_eT_x$ for $x \in W^J$ and $m_eT_w = q^\ell(w)m_e$ for $w \in W_J$. See the displayed action

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\[ J C'_y = q^{-\ell(y)/2} \sum_{x \leq y} P_{x,y}(q)m_xT_x \quad (x, y \in W^J). \]

We will return to this equation later. It is part of [Deo90, Prop. 5.1(i)], the parabolic analog of [KL79, (1.1.c)]. If \( s \in S \), we have \( C'_s = C'_e = q^{-1/2}(T_e + T_s) \). When the group \( W_J \) is finite, with element \( w_J^0 \) of maximal length, we have \( P_{x,y}^J = P_{w_J^0 x, w_J^0 y} \). See [Deo87, Prop. 3.4], applied through the duality set-up of [Deo91, Rem. 2.6]. It is worth noting that, even when \( W_J \) is finite, the basic recursion [Deo90, Prop. 5.2(iii)]\(^1\) for the parabolic Kazhdan–Lusztig polynomials \( P_{x,y}^J \) is much more effective than the corresponding non-parabolic (\( J = \emptyset \)) recursion for computing the polynomials \( P_{w_J^0 x, w_J^0 y} \). We will call [Deo90, Prop. 5.2(iii)] the \textit{Deodhar recursion} (to distinguish it from the more elaborate \textit{Deodhar algorithm} we will discuss later). Explicitly, the Deodhar recursion states the following, with \( J \mu(z, y) \) denoting the coefficient of \( q^{(\ell(y) - \ell(z) - 1)/2} \) in \( P_{z,y}^J \).

Let \( y, yS \in W^J \) with \( s \in S \) and \( y < yS \). Then \( J C'_y C'_s = J C'_y + \sum_{z \in W^J} J \mu(z, y)C'_z \).

It makes sense also to call the \( J = \emptyset \) case, equivalent to [KL79, (2.3b)] via [KL79, (1.1.1c)], the \textit{Kazhdan–Lusztig recursion}.

Next, following [Deo90, p. 114], we define, for each finite sequence \( s = (s_1, s_2, \ldots, s_k) \) of elements of \( S \) whose product \( \pi(s) = s_1 s_2 \cdots s_k \) has length \( k \), the element

\[ JD'_s = m_{eC'_1}C'_2 \cdots C'_k. \]

In our algorithm we need to compute a lot of these, but, fortunately for memory requirements, there is no need to store them. In [Deo90, Prop. 5.3(i)] Deodhar gives closed forms for these elements, though their calculation involves examining subsequences of \( s \), an operation potentially of exponential time in \( k \). We have found the simple iterative computation \( m_{eC'_1}, m_{eC'_1C'_2}, \ldots, m_{eC'_1C'_2 \cdots C'_k} \) to be a reasonable computational procedure, running in time at most proportional to \( k^2|W^J(x)| \) in integer operations, where \( W^J(x) = \{ z \in W^J | z < x \} \).

At any iteration, multiplication by a given \( C'_s \) is easily done with the rules [Deo90, p. 113] for multiplication on \( M \) by \( T_s \) mentioned above. Reformulated versions of these rules, in terms

\(^1\)The reader may notice there is a misprint in part (ii) of the same proposition [Deo90, Prop. 5.2], where \(-f^J \) should simply be \( f \), representing the expression \( q^{1/2} + q^{-1/2} \). This is irrelevant to the recursion in part (iii).
of multiplication by \( C'_s \), are given below.

\[
m_x C'_s = \begin{cases} 
q^{1/2}(m_x + m_{x_5}) & \text{if } \ell(xs) < \ell(x), \\
q^{-1/2}(m_x + m_{x_5}) & \text{if } \ell(xs) > \ell(x) \text{ and } xs \in W^J, \\
(q^{1/2} + q^{-1/2})m_x & \text{if } \ell(xs) > \ell(x) \text{ and } xs \notin W^J.
\end{cases}
\]

Note also from the definition of \( J^D'_s \) that it is obtained by applying \( Z[q] \)-linear combinations of elements \( T_x, x \in W \), to \( m_e/q^{\ell(y)/2} \) and so is a \( Z[q] \)-linear combinations of elements \( m_x/q^{\ell(y)/2}, x \in W^J \). Nonzero terms occur only for \( x \leq y \), and the coefficient of \( m_y/q^{\ell(y)/2} \) is the element \( 1 \in R \). In our algorithm, it will be useful to write elements of \( M \) as \( R \)-linear combinations of elements \( m_x/q^{\ell(y)/2} \). When this is done for \( J^D'_s \), we find that any power \( q^{\ell(z)/2} \) which appears with nonzero coefficient in the (Laurent polynomial) coefficient of \( m_x/q^{\ell(y)/2} \) satisfies \( n \equiv \ell(y) - \ell(x) \) modulo 2. This condition is equivalent to the \( Z[q] \)-coefficient requirements just noted in the case of elements \( m_x/q^{\ell(y)/2} \).

Finally, we need the involution \( m \mapsto \overline{m} \) on \( M \) from [Deo90, p. 113]. It satisfies \( \overline{m \overline{m}} = \overline{m} \), where \( r \mapsto \overline{r} \) on the ring \( R \) sending \( q^{1/2} \) to \( q^{-1/2} \). Also, \( \overline{m_e} = m_e \) and \( \overline{mT_x} = mT_x \), where \( \overline{T_x} = T_x^{-1} \). The fixed point space on \( M \) of the involution \( m \mapsto \overline{m} \) is denoted \( M^0 \). Then, according to [Deo90, Prop. 5.1(i)], for each \( y \in W^J \) there is a unique element \( J^{C'_y} \in M^0 \) which satisfies equation (*) above for polynomials \( P^J_x(y)(q) \) of degree at most \( (\ell(y) - \ell(x))^{-1}/2 \) when \( x < y \) and with \( P^J_y(q) = 1 \). We can give a sharper uniqueness result using [Deo90, Prop. 5.1(ii)], which asserts the elements \( J^{C'_y} \) form a basis of \( M^0 \) over the ring \( R^0 \) of invariants of the involution \( r \mapsto \overline{r} \) on \( R \).

**Proposition.** Put \( t = q^{1/2} \). Suppose \( y \in W^J \), and that \( J^{C''_y} \in M^0 \) has the form \( \sum_{x \leq y} P^J_{x,y}(t^{-1})(m_x/t^{\ell(x)}) \), where \( P^J_{x,y}(t^{-1}) \) is a polynomial in \( t^{-1} \) with zero constant term whenever \( x < y \), and \( P^J_{y,y}(t^{-1}) = 1 \). Then \( J^{C''_y} = J^{C'_y} \).

**Proof.** Write \( J^{C''_y} \) as a linear combination of elements \( \sum_z f_z J^{C'_z} \) with \( z \in W^J \) and \( f_z \in R^0 \).

Comparing coefficients of \( m_z/t^{\ell(z)} \), we find that any \( z \) maximal among those occurring with nonzero \( f_z \) must be \( y \), and \( f_y = 1 \). Next, suppose some \( z < y \) has a nonzero \( f_z \) and take \( z < y \) maximal with that property. Comparing coefficients of \( m_z/t^{\ell(z)} \) again, we have

\[
P^J_{x,y}(t^{-1}) = f_z + P^J_{x,y}(q)/t^{\ell(y) - \ell(x)}.
\]

But both \( P^J_{x,y}(t^{-1}) \) and \( P^J_{x,y}(q)/t^{\ell(y) - \ell(x)} \) have nonzero coefficients only for negative powers of \( t \). This property is inherited by their difference \( f_z \). However, the element \( f_z \in R^0 \) is symmetric with respect to the involution of \( R \) interchanging \( t \) and \( t^{-1} \). So it must be that \( f_z = 0 \), and \( J^{C''_y} = J^{C'_y} \).

Continuing with the notation \( t = q^{1/2} \), we can now describe our algorithm. For any element \( f(t) \) of \( R \), we write \( f(t) = f_{\geq 0}(t) + f_{<0}(t^{-1}) \), where both \( f_{\geq 0}, f_{<0} \) are integer polynomial expressions, and \( f_{<0} \) has a zero constant term. Similarly, we let \( f_{\geq 0}(t) \) be the positive degree part of \( f_{\geq 0}(t) \).

**Algorithm.** For any given \( y \in W^J \), we determine \( J^{C'_y} \) as an \( R \)-linear combination of the basis elements \( m_x \) of \( M \): Write \( y = s_1 s_2 \cdots s_k \) as a reduced product for a sequence \( s = \).
(s₁, s₂, ..., sₖ) of elements of S. Introduce a temporary variable $\text{Fat} \ jC'_y$, initialized to $\text{Fat} \ jD'_s$ and written as a linear combination of the elements $m_x/t^ℓ(x)$, $x \in W^J$. Next, we look in $\text{Fat} \ jC'_y$ for any $x < y$ with a Laurent polynomial coefficient $f_x(t)$ of $m_x/t^ℓ(x)$ having a nonzero term of non-negative degree in $t$. If none are found, then the algorithm is finished, and $jC_y = \text{Fat} \ jC'_y$. If one is found, we focus on an $x < y$ of maximal length with such a coefficient. Put $f(t) = f_x(t)$, and set $g(t) = f_{≥0}(t) + f_{>0}(t^{-1})$. Reassign $\text{Fat} \ jC'_y$, in terms of its old value, as $\text{Fat} \ jC'_y = g(t)^{jD'_y}$, where $s'$ is a sequence of elements of $S$ whose product is reduced and equal to $x$. Repeat these reassignments of $\text{Fat} \ jC'_y$ until they can no longer be made, or, equivalently, $jC'_y = \text{Fat} jC'_y$.

Proof. We need to show that the algorithm terminates and gives the right answer. It is fairly clear that the algorithm terminates, since the operations dealing with a given $x < y$ only affect coefficients of $m_z/t^ℓ(z)$ for $z \leq x$. Moreover, they result in $m_x/t^ℓ(x)$ having a Laurent polynomial coefficient $f_x(t)$ with no non-negative powers of $t$, a coefficient that is undisturbed by later operations with elements in $W^J$ smaller than or unrelated to $x$ in the Bruhat–Chevalley order. (In fact, all operations with $x < y$ of, say, maximal length with respect to having an offending coefficient $f_x(t)$ for $m_x/t^ℓ(x)$, can be done in parallel.) Eventually, all $x < y$ in $W^J$ are exhausted, and the algorithm terminates. At that point, all coefficients $f_x(t)$ for $m_x/t^ℓ(x)$ in $\text{Fat} \ jC'_y$ have no non-negative powers of $t$, while the coefficient $f_y(t)$ is 1 from the initial $\text{Fat} \ jC'_y$ has remained undisturbed. Thus, the above proposition implies we now have the desired equation $\text{Fat} \ jC'_y = jC'_y$. □

Remark. We have here used many ingredients of [Deo90], and the algorithm we have obtained above may be viewed, philosophically, as a variation on the algorithm given in [Deo90, Algorithm 4.11, p. 115], sometimes called “Deodhar’s algorithm.” Without going into too many details, our alternative uses the elements $jD'_s$ in place of elements $jC'_x$ in the reduction process, and the polynomials $g(t) = f_{≥0}(t) + f_{>0}(t^{-1})$ are used in place of the positive coefficient polynomials in $t + t^{-1}$ guaranteed in the $jC'_x$ case by [Deo90, Prop. 3.7, Cor. 5.4]. The proposition above makes this work. There are, however, two advantages of our alternative: first, unlike [Deo90, Algorithm 4.11, p. 115], the alternative algorithm does not require an a priori positivity condition to guarantee its successful termination. Second, the alternative algorithm has considerably less memory requirements when focused on computing a single $jC'_y$, since the recursive calculation of elements $jC'_x$ is avoided, together with any associated storage. The next section gives an illustration in a useful case.

2. An Example

In this section we fix $W$ of affine type $\tilde{A}_n$ with $S = \{s_0, s_1, ..., s_n\}$. We suppose the indexing chosen as usual so that products of successive elements, as well as $s_n s_0$, have order 3. Fix $J = \{s_1, ..., s_n\}$, so that $W_J$ is of type $A_n$. As noted above, we have the identification $P^J_{x,y} = P_{w^0 x, w^0 y}$ in this case, for all $x, y \in W^J$. Recall also that $jμ(x, y)$ denotes the coefficient of $q^{ℓ(y) − ℓ(x) − 1}/2$ in $P^J_{x,y}$, so that $jμ(x, y) = μ(w^0 x, w^0 y)$, where we have abbreviated $w^0 y = w_0$. Let $\varpi_1, ..., \varpi_n$ be fundamental weights for a root system of type $A_n$, and denote the integral weight lattice they generate by $Λ$. Write elements $\sum_{i=1}^n a_i \varpi_i$ of $Λ$ as $n$-tuples of integers $(a_1, ..., a_n)$. Let each $s_i$ with $0 < i \leq n$ act on $Λ$ by reflection in the $i$th fundamental
root \(\alpha_i\) (so that \(s_i(\varpi_i) = \varpi_i - \alpha_i\) and \(s_i(\varpi_j) = \varpi_j\) for \(j \neq i\)). Let \(s_0\) act by reflection in the maximal root \(\alpha_0\), followed by translation via \(-p\alpha_0\), where, for the moment, \(p\) is just a fixed positive integer. This gives an affine action of \(W\) on \(\Lambda\), which we next shift to give the standard “dot” action: Put \(\rho = \varpi_1 + \cdots + \varpi_n\) and, for \(\lambda \in \Lambda\) and \(w \in W\), define \(w \cdot \lambda = w(\lambda + \rho) - \rho\). To emphasize the dependence of our notation on \(p\), we write \(W \cong W_p\), viewing the left-hand side as an abstract Coxeter group, and the right-hand side the group of affine transformations giving its action on \(\Lambda\) (with a recipe partly involving translations by elements of \(p\Lambda\)). Assuming \(p \geq n + 1\), the weights in \(W \cdot -2\rho = W_p \cdot -2\rho\) are in 1-1 correspondence with the elements of \(W\). The dominant weights in \(W_p \cdot -2\rho\) (those with non-negative coefficients at each \(\varpi_i\)) are precisely those of the form \(w_0 x \cdot -2\rho\) with \(x \in W^J\). This fact is independent of \(p\), though, for fixed \(x \in W^J\), the precise dominant weight represented by \(w_0 x \cdot -2\rho\) will generally depend on \(p\). However, if \(w_0 x \cdot -2\rho\) is \(p\)-restricted (has all coefficients \(a_i\) of fundamental roots in the range \(0 \leq a_i \leq p - 1\)) for one choice of \(p \geq n + 1\), it can be shown to be \(p\)-restricted for any other such choice.

In [ScXi10] it is shown that, as \(n\) grows, the values \(\mu(w_0 x, w_0 y)\) for \(x, y \in W^J\) get arbitrarily large, though they are bounded for fixed \(n\). This is true even when the associated weights \(w_0 x \cdot -2\rho\) and \(w_0 y \cdot -2\rho\) are \(p\)-restricted. Left open was the important case where \(x = 1\) was fixed and \(n\) and \(y\) were allowed to vary (keeping \(w_0 y \cdot -2\rho\) \(p\)-restricted). As discussed in [Sc03], this case is important because the values \(\mu(w_0, w_0 y)\) give lower bounds on the dimension of 1-cohomology groups with coefficients in the irreducible modules \(L(w_0 y \cdot -2\rho)\) of the finite projective special linear groups \(\text{PSL}(n + 1, q)\) for \(q\) a power of a sufficiently large prime, relevant to a well-known conjecture of Guralnick.\(^2\) However, [ScXi10] does give a guess, when either \(n\) is odd or divisible by 4, for a \(p\)-restricted weight \(w_0 y \cdot -2\rho\) likely to give a large \(\mu(w_0, w_0 y)\). The guess may be described uniformly if we take \(p = n + 1\), in which case the guess reads (for all \(n\) not congruent to 2 modulo 4):

\[w_0 y \cdot -2\rho = (p - 2)\rho - \alpha_0.\]

For example, for \(n = 3, 4, 5, 7, 8\) these weights (in \(p = n + 1\) notation) are (2, 1, 2), (2, 3, 3, 2), (3, 4, 4, 4, 3), (4, 5, 5, 5, 5, 5, 4), (6, 7, 7, 7, 7, 7, 7, 6), \(\mu(w_0, w_0 y)\) for the first four had been previously computed, as 1, 2, 3, 469 as part of exhaustive calculations\(^3\).

\(^2\)In 1984, Guralnick conjectured that there is a universal constant, call it \(C\), such that \(\dim_F H^1(G, V) \leq C\) whenever \(G\) is a finite group acting faithfully and absolutely irreducibly as \(F\)-linear automorphisms of a vector space \(V\) over a field \(F\) [Gur86]. The relevance of Kazhdan-Lusztig polynomials to this conjecture was demonstrated in [Sc03], showing \(\dim_F H^1(G, V) \geq \mu(w_0, w_0 y)\) for finite groups \(G\) of Lie type acting on an irreducible module \(V = L(w_0 y \cdot -2\rho)\). This gave for the first time dimensions as large as 3, and to counterexamples to a related 1961 conjecture of G. E. Wall on maximal subgroups. See [AIM12]. The Guralnick conjecture, however, is still open, though current efforts focus on understanding how \(\dim_F H^1(G, V)\) can grow with the rank of an underlying root system for a finite group of Lie type, rather than trying to bound it universally. It is true, that, if the rank is fixed, then there is a bound depending only on the rank, in either defining or cross characteristic [CPS09], [GurTie11].

\(^3\)These calculations may be done by hand for \(n = 3\), and the remaining calculations by computer. For \(n = 4\) they were carried out by Chris McDowell [Sc03, Prop. 3]. The calculations for \(n = 7\) were done by Frank Lübeck and confirmed independently by Tim Sprowl. Also, Lübeck did a similar exhaustive calculation for \(n = 6\), determining a largest value of 16 for \(\mu(w_0, w_0 y)\) for \(p\)-restricted \(\mu(w_0, w_0 y)\), after earlier calculations by Sprowl of values 4 and 5 for smaller weights. Some of these calculations took place during the June 2012 AIM workshop, and the remainder a few weeks later. See [AIM12].
including all restricted weights $w_{0y} \cdot -2\rho$. We give here, using the algorithm of this paper, the value of $\mu(w_0, w_{0y})$ for the $n = 8$ weight $w_{0y} \cdot -2\rho = (6, 7, 7, 7, 7, 7, 6)$ as 36672. The full Kazhdan–Lusztig polynomial $P_{w_0, w_{0y}}$ is given below.\footnote{To be sure, the displayed equation is the result of a 64-bit calculation, and can only be rigorously claimed to be correct modulo $2^{64}$. Known theoretical bounds for the coefficients are not particularly good at this point, and even to accurately pin down the coefficient of $t^{82}$ below would require $11 \times 64$ bit arithmetic, using bounds based on [ParkSt12, Prop. 7.1]. Fortunately, however, current interest is in a lower bound for this coefficient, and all the coefficients are known to be positive.}

\[
\begin{align*}
+ 36672t^{82} &+ 329119t^{80} + 1600603t^{78} + 5782048t^{76} + 17370114t^{74} \\
+ 45208788t^{72} &+ 104312889t^{70} + 216672871t^{68} + 409222372t^{66} + 707571983t^{64} \\
+ 1125993513t^{62} &+ 1656221777t^{60} + 2260164853t^{58} + 2871480057t^{56} + 3407386353t^{54} \\
+ 3787877798t^{52} &+ 3955903667t^{50} + 3891194815t^{48} + 3613245907t^{46} + 3173587791t^{44} \\
+ 2640964839t^{42} &+ 2084968629t^{40} + 1563002756t^{38} + 1113178197t^{36} + 753257475t^{34} \\
+ 484075798t^{32} &+ 295159975t^{30} + 17048857t^{28} + 93076435t^{26} + 47878089t^{24} \\
+ 23109923t^{22} &+ 10411073t^{20} + 4347162t^{18} + 1667234t^{16} + 580355t^{14} + 180463t^{12} \\
+ 49052t^{10} &+ 11300t^{8} + 2107t^{6} + 294t^{4} + 26t^{2} + 1t^{0}
\end{align*}
\]

It would be difficult to make this calculation by using existing recursions and exhaustively computing all Kazhdan–Lusztig basis elements $C'_{w_{0z}}$ with $z \in W^{J}$ with $z \leq y :$ There are approximately $N = 1, 700, 000$ elements $z \in W^{J}$ with $\ell(z) \leq \ell(y)$ when $w_{0y} \cdot -2\rho = (6, 7, 7, 7, 7, 7, 6).$ Let us crudely estimate that, roughly half of these elements satisfy $z \leq y,$ and that half the elements $x \in W^{J}$ satisfying $\ell(x) \leq \ell(z)$ also satisfy $x \leq z,$ at least when $\ell(z)$ is modestly large. Comparison with linear orders now leads to a guess that there are about $(N/2)^2/4 = N^2/16$ such pairs. If we presume the recursion would at least require knowing some information for every such pair, recorded as a 32-bit pointer (say) to some small list including all restricted weights $w_{0y} \cdot -2\rho.$ However, even if this were to reduce storage requirements to an acceptable level, existing recursions do not take the route of such bare-bones storage. It would, of course, be an interesting project to see if a new algorithm could be designed which did so, and ran in reasonable time. The storage proposed is very close to the well-studied notion of a $W$-graph defined by Kazhdan and Lusztig [KL79]. It is much easier to construct Kazhdan–Lusztig polynomials given the $W$-graph, than having to extract the $\mu$ values from other Kazhdan–Lusztig polynomials as the construction proceeds.\footnote{With an easy OpenMP parallelization and a single 8 cpu computer of the same speed, this running time was cut down to 3.5 days, even with 64-bit arithmetic. This parallel version required about 10 gigabytes, shared by the cpus, and “confirmed” the 32 bit results.}
and posted on the first author’s webpage www.math.virginia.edu/~lls2l in January 2013. It was reported at the January 2013 AMS meeting, as well as subsequent lectures in the first half of 2013 by the first author in Perth, Sydney, and Zhangjiajie (ICRT6).

REFERENCES

[AIM12] American Institute of Mathematics news announcement, http://aimath.org/news/wallsconjecture/

[CPS09] E. Cline, B. Parshall, and L. Scott, Reduced standard modules and cohomology, Trans. Amer. Math. Soc. 361 (2009), 5223–5261.

[Deo87] V. Deodhar, On some geometric aspects of Bruhat Orderings II. The parabolic analogue of Kazhdan–Lusztig polynomials, J. Algebra 111 (1987), 483–506.

[Deo90] V. Deodhar, A combinatorial setting for questions in Kazhdan–Lusztig theory, Geometriae Dedicata 36 (1990), 95–119.

[Deo91] V. Deodhar, Duality in parabolic set up for questions in Kazhdan–Lusztig theory, J. Algebra 142 (1991), 201–209.

[Gur86] R. Guralnick, The dimension of the first cohomology group, in: Representation Theory, II, Ottawa, ON, 1984, in: Lecture Notes in Math., Vol. 1178, Springer-Verlag, Berlin, 1986, pp. 94–97.

[GurTie11] R. Guralnick and P. Tiep, First cohomology groups of Chevalley groups in cross characteristic, Ann. of Math. 174 (2011), 543–559.

[KL79] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. math. 83 (1979), 165–184.

[ParkSt12] A. Parker and D. Stewart, First cohomology groups of finite groups of Lie type in defining characteristic, arXiv 1211.6942, 11pp.

[Sc03] L. Scott, Some new examples in 1-cohomology, J. Algebra 260 (2003), 416–425.

[ScXi10] L. Scott and N. Xi, Some non-trivial Kazhdan–Lusztig coefficients in an affine Weyl group of type $\tilde{A}_n$, Sci. China Math. 53 (2010), no. 8, 1919–1930.

Department of Mathematics, University of Virginia, Charlottesville, VA 22903
E-mail address: llis2l@virginia.edu (Scott)

9170 Ivy Springs Place, Mechanicsville, VA 23116.
E-mail address: tim.spr@gmail.com (Sprowl)