The Principal Curvatures and the Third Fundamental Form of Dini-Type Helicoidal Hypersurface in 4-Space

Erhan Güler

1Bartın University, Faculty of Sciences, Department of Mathematics, 74100 Bartın, Turkey.

Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

We consider the principal curvatures and the third fundamental form of Dini-type helicoidal hypersurface $D(u, v, w)$ in the four dimensional Euclidean space $E^4$. We find the Gauss map $e$ of helicoidal hypersurface in $E^4$. We obtain characteristic polynomial of shape operator matrix $S$. Then, we compute principal curvatures $k_{i=1,2,3}$, and the third fundamental form matrix $III$ of $D$.

Keywords: Four dimensional; Dini-type helicoidal hypersurface; Gauss map; principal curvatures; the third fundamental form.

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1 Introduction

Theory of surfaces and hypersurfaces have been studied by many geometers for years such as [1 – 26].

In the rest of this paper, we identify a vector $(a,b,c,d)$ with its transpose $(a,b,c,d)^t$. Let $\gamma : I \rightarrow \Pi$ be a curve in a plane $\Pi$ in $\mathbb{R}^4$, and let $\ell$ be a straight line in $\Pi$ for an open interval $I \subset \mathbb{R}$. A
rotational hypersurface in $\mathbb{E}^4$ is defined as a hypersurface rotating a curve $\gamma$ (i.e. profile curve) around a line (i.e. axis) $\ell$. Suppose that when a profile curve $\gamma$ rotates around the axis $\ell$, it simultaneously displaces parallel lines orthogonal to the axis $\ell$, so that the speed of displacement is proportional to the speed of rotation. Resulting hypersurface is called the helicoidal hypersurface with axis $\ell$ and pitches $a, b \in \mathbb{R}\setminus\{0\}$.

Let $\ell$ be a line spanned by the vector $(0, 0, 0, 1)^t$. The orthogonal matrix

$$
\mathcal{M}(v, w) = \begin{pmatrix}
\cos v \cos w & -\sin v & -\cos v \sin w & 0 \\
\sin v \cos w & \cos v & \sin v \sin w & 0 \\
\sin w & 0 & \cos w & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad v, w \in \mathbb{R},
$$

fixes the vector $\ell$. The matrix $\mathcal{M}$ can be found by solving the following equations simultaneously; $\mathcal{M}\ell = \ell$, $\mathcal{M}^t \mathcal{M} = \mathcal{M} \mathcal{M}^t = I_4$, det $\mathcal{M} = 1$. When the axis of rotation is $\ell$, there is an Euclidean transformation by which the axis is $\ell$ transformed to the $x_4$-axis of $\mathbb{E}^4$. Parametrization of the profile curve is given by $\gamma(u) = (u, 0, 0, \varphi(u))$, where $\varphi(u) : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function for all $u \in I$. So, the helicoidal hypersurface is given by $H(u, v, w) = \mathcal{M}(u, v, w) = \mathcal{M}_{\gamma}(u, 0, 0) \cdot (av + bw) \cdot \ell'$, where $u \in I$, $v, w \in [0, 2\pi]$, $a, b \in \mathbb{R}\setminus\{0\}$. Clearly, we write helicoidal hypersurface as follows

$$
H(u, v, w) = (u \cos v \cos w, u \sin v \cos w, u \sin w, \varphi(u) + av + bw).
$$

In this paper, we study the principal curvatures and the third fundamental form of the Ulisse Dini-type helicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some basic notions of four dimensional Euclidean geometry in section 2. In section 3, we give Ulisse Dini-type helicoidal hypersurface, and calculate its principal curvatures, and the third fundamental form in section 4. In addition, we give a conclusion in the last section.

## 2 Preliminaries

In this section, we introduce the fundamental form matrices $I$, $II$, $III$, the shape operator matrix $S$, the Gaussian curvature $K$, and the mean curvature $H$ of a hypersurface $M = M(u, v, w)$ in the Euclidean 4-space $\mathbb{E}^4$.

Let $M$ be an isometric immersion of a hypersurface $M^3$ in the $\mathbb{E}^4$. The inner product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, and the vector product of $\overrightarrow{x}$, $\overrightarrow{y}$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ on $\mathbb{E}^4$ are defined by

$$
\overrightarrow{x} \cdot \overrightarrow{y} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,
$$

$$
\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \begin{vmatrix}
e_1 & e_2 & e_3 & e_4 \\
e_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
z_1 & z_2 & z_3 & z_4
\end{vmatrix},
$$

respectively. A hypersurface $M$ in 4-space has the first and the second fundamental form matrices

$$
I = \begin{pmatrix}
E & F & A \\
F & G & B \\
A & B & C
\end{pmatrix}, \quad II = \begin{pmatrix}
L & M & P \\
M & N & T \\
P & T & V
\end{pmatrix},
$$

respectively. Here,

$$
E = M_{uv} \cdot M_u, \quad F = M_u \cdot M_v, \quad G = M_v \cdot M_v, \quad A = M_u \cdot M_w, \quad B = M_v \cdot M_w, \quad C = M_v \cdot M_v, \quad L = M_{uu} \cdot e, \quad M = M_{uv} \cdot e, \quad N = M_{vw} \cdot e, \quad P = M_{uw} \cdot e, \quad T = M_{vw} \cdot e, \quad V = M_{ww} \cdot e,
$$
and \( e \) is the Gauss map
\[
e = \frac{M_u \times M_v \times M_w}{\|M_u \times M_v \times M_w\|}.
\]

Hence, \( I^{-1}II \) gives the shape operator matrix of \( M \)
\[
S = \frac{1}{\det I} \begin{pmatrix}
  s_{11} & s_{12} & s_{13} \\
  s_{21} & s_{22} & s_{23} \\
  s_{31} & s_{32} & s_{33}
\end{pmatrix},
\]
where
\[
\det I = (EG - F^2)C - A^2G + 2ABF - B^2E,
\]
\[
s_{11} = ABM - CFM - AGP + BFP + CGL - B^2L,
\]
\[
s_{12} = ABN - CFN - AGT + BFT + CGM - B^2M,
\]
\[
s_{13} = ABT - CFT - AGV + BVF + CGP - B^2P,
\]
\[
s_{21} = ABL - CFL + AFP - BPE + CME - A^2M,
\]
\[
s_{22} = ABM - CFM + AFT - BTE + CNE - A^2N,
\]
\[
s_{23} = ABP - CFP + AFV - BVE + CTE - A^2T,
\]
\[
s_{31} = -AGL + BFL + AFM - BME + GPE - F^2P,
\]
\[
s_{32} = -AGM + BFM + AFN - BNE + GTE - F^2T,
\]
\[
s_{33} = -AGP + BFP + AFT - BTE + GVE - F^2V.
\]

Therefore, using \( II.S \), we get the third fundamental form matrix
\[
III = \frac{1}{\det I} \begin{pmatrix}
  \Gamma & \Phi & \Omega \\
  \Phi & \Psi & \Theta \\
  \Omega & \Theta & \Delta
\end{pmatrix},
\]
where
\[
\Gamma = -A^2M^2 + 2ABLM + 2AFMP - 2GALP - B^2L^2 + 2BFLP
\]
\[
-2EBMP - F^2P^2 - 2CFLM + CGL^2 + CEM^2 + GEP^2,
\]
\[
\Phi = ABM^2 - CFM^2 - B^2LM - A^2MN - F^2PT + CMNE
\]
\[
-BNPE - BMTE + GPTE + ABLN - CFLN + CGLM
\]
\[
+AFNP - AGMP + BFMP + AFMT - AGLT + BFLT,
\]
\[
\Omega = BFP^2 - AGP^2 - B^2LP - A^2MT - F^2PV + CMTE
\]
\[
-BMVE - BPTE + GPVE + ABMP + ABLT - CFMP
\]
\[
+CGLP - CFLT + AFMV - AGLV + BFLV + AFPT,
\]
\[
\Psi = -A^2N^2 + 2ABMN + 2AFNT - 2GAMT - B^2M^2 + 2BFMT
\]
\[
-2EBNT - F^2T^2 - 2CFMN + CGM^2 + CEN^2 + GE^2,
\]
\[
\Theta = AFT^2 - B^2MP - A^2NT - F^2TV - BT^2E + CNTE
\]
\[
-BNVE + GTVE + ABNP + ABMT - CFNP + CGMP
\]
\[
-CFMT + AFNV - AGMV + BFVM - AGPT + BFPT,
\]
\[
\Delta = -A^2T^2 + 2ABPT + 2AFTV - 2GAPV - B^2P^2 + 2BFPV
\]
\[
-2EBTV - F^2V^2 - 2CFPT + CGP^2 + CET^2 + GEV^2.
\]
3 The Principal Curvatures and the Third Fundamental Form of the Dini-Type Helicoidal Hypersurface

We consider Dini-type helicoidal hypersurface

\[ D(u, v, w) = \begin{pmatrix} \sin u \cos v \cos w \\ \sin u \sin v \cos w \\ \sin u \sin w \\ \cos u + \log (\tan \frac{u}{2}) + av + bw \end{pmatrix}, \tag{3.1} \]

where \( u \in \mathbb{R}\setminus\{0\} \) and \( 0 \leq v, w \leq 2\pi \). Using the first differentials of (3.1) with respect to \( u, v, w \), we get the first quantities

\[ I = \begin{pmatrix} \cot^2 u & a \cot u \cos w & b \cot u \cos u \\ a \cot u \cos u & \sin^2 u \cos^2 w + a^2 & ab \\ b \cot u \cos u & ab & \sin^2 u + b^2 \end{pmatrix}, \]

and then, we have \( \det I = \left( (b^2 + 1) \cos^2 w + a^2 \right) \sin^2 u \cos^2 u \). The Gauss map of (3.1) is given by

\[ e_D = \frac{1}{\sqrt{W}} \begin{pmatrix} \cos u \cos v \cos^2 w + \sin v - b \cos u \sin w \cos w \\ \cos u \sin v \cos^2 w + a \cos v - b \sin v \sin w \cos w \\ (\cos u \sin w + b \cos w) \cos w \end{pmatrix}, \tag{3.2} \]

where \( W = (b^2 + 1) \cos^2 w + a^2 \). Using the second differentials of (3.1) with respect to \( u, v, w \), with (3.2), we have the second quantities of the (3.1)

\[ H = \frac{1}{W^{1/2}} \begin{pmatrix} \cot u \cos w & a \cos u \cos w & b \cos u \cos w \\ a \cos u \cos w & (b \sin w - \cos u \cos w) \sin u \cos^2 w \sin u \sin w & -a \sin u \sin w \\ b \cos u \cos w & -a \sin u \sin w & -a \sin u \cos u \cos w \end{pmatrix}. \]

Computing \( I^{-1}S \), we obtain the shape operator matrix of (3.1)

\[ S = \begin{pmatrix} \frac{\sin u \cos w}{W^{3/2} \sin u} & \frac{u \cos w}{W^{3/2} \sin u} & \frac{a^2 (b \cos w + \cos u \sin w) + b (b^2 + 1) \cos^2 w}{W^{3/2} \sin u} \\ 0 & \frac{b \sin w - \cos u \cos w}{W^{3/2} \sin u} & \frac{-a \sin u \sin w}{W^{3/2} \sin u} \\ 0 & \frac{-a \sin u \cos w}{W^{3/2} \sin u} & \frac{a (b^2 + 1) \sin w}{W^{3/2} \sin u} \end{pmatrix}. \tag{3.3} \]

**Theorem 1.** Let \( D : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (3.1). Then, characteristic polynomial of the (3.3) of the (3.1) is given by

\[ X^3 + pX^2 + qX + r = 0, \]

where

\[ p = \begin{pmatrix} \cos^2 u \cos^3 w + b^2 \cos^2 u \cos^3 w + W \cos^2 u \cos w \\ -W \cos u \sin^2 u + a^2 \cos^2 u \cos w \\ -bW \cos u \sin w - a b \cos u \sin w \end{pmatrix}, \]

\[ q = \begin{pmatrix} \cos^3 u \cos^4 w + a^2 \cos^2 u \cos^4 w + b^2 \cos^3 u \cos^4 w \\ -a^2 \cos u \sin^2 w - \cos u \cos^3 w \sin^2 w \\ -W \cos u \cos^2 u \sin^4 u - a^2 \cos u \cos^2 u \sin^4 u \\ -b^2 \cos u \cos^4 u \sin u - b^2 \cos u \cos^4 u \sin u \\ -b^2 \cos^2 u \cos^3 w \sin w + bW \cos u \sin^3 w \sin w \\ -2a^2 \cos^2 u \cos w \sin w + a b \cos w \sin^3 w \sin w \end{pmatrix}, \]

\[ r = \frac{W}{W^2 \cos^2 u \sin^2 u}. \]
Corollary 1. Let $D : M^3 \to \mathbb{E}^4$ be an immersion given by (3.1). Then, (3.1) has the principal curvatures
\[ k_1 = \frac{\sin u \cos w}{W^{1/2} \cos u}, \quad k_2 = \frac{\beta_1}{2W^{3/2} \sin u}, \quad k_3 = \frac{\beta_2}{2W^{3/2} \sin u}, \]
where
\[ \beta_1 = \frac{T^{1/2}}{2} - 2W \cos u \cos w + (W + a^2) b \sin w, \]
\[ \beta_2 = -\frac{T^{1/2}}{2} - 2W \cos u \cos w + (W + a^2) b \sin w, \]
and
\[ T = \left( -W + a^2 \right)^2 \left( \cos u \cos w - 2b \sin w \right) \cos u \cos w \]
\[ + \left( 4a^2 W + b^2 (W + a^2) \right) \sin^2 w \]
\[ - 2(b^2 + 1) \left( -W + a^2 \right) \left( \cos u \cos w + b \sin w \right) \cos u \sin^2 w \]
\[ + (b^2 + 1)^2 \cos^2 u \cos^6 w. \]

Proof. Solving characteristic polynomial of $S$, we have eigenvalues of $S$.

Corollary 2. Let $D : M^3 \to \mathbb{E}^4$ be an immersion given by (3.1). Then, (3.1) has the third fundamental form matrix
\[ III = \frac{\cos^2 w}{W} \left( \begin{array}{ccc} 1 & \frac{a \sin u}{(b \sin u - \cos u \cos w)^2 \cos^2 w + a^2} & \frac{b \sin u}{a(b \cos 2w + \cos u \sin 2w)} \\ a \sin u & \frac{a \sin u}{(b \sin u - \cos u \cos w)^2 \cos^2 w + a^2} & \frac{a(b \cos 2w + \cos u \sin 2w)}{a(b \cos 2w + \cos u \sin 2w)} \\ b \sin u & \frac{b \sin u}{a(b \cos 2w + \cos u \sin 2w)} & \frac{a(b \cos 2w + \cos u \sin 2w)}{a(b \cos 2w + \cos u \sin 2w)} \end{array} \right). \]

Proof. Using II.S, we get the third fundamental form matrix of (3.1).

4 Conclusion

In this paper, we introduce the principal curvatures, and the third fundamental form of the Dini-type helicoidal hypersurface $D(u, v, w)$ in the four dimensional Euclidean space $\mathbb{E}^4$. We calculate the Gauss map $e$ of the $D(u, v, w)$ in $\mathbb{E}^4$. We obtain the characteristic polynomial of the shape operator matrix $S$. After long calculations, we reveal the principal curvatures $k_1, k_2, k_3$, and the third fundamental form matrix $III$ of the Dini-type helicoidal hypersurface.

5 Competing Interests

Author has declared that no competing interests exist.

References

[1] Arvanitoyeorgos A, Kaimakamis G, Magid M. Lorentz hypersurfaces in $\mathbb{E}_1^4$ satisfying $\Delta H = \alpha H$. Illinois J. Math. 2009;53(2):581-590.
[2] Bour E. Theorie de la deformation des surfaces. J. de l. Ecole Imperiale Polytechnique. 1862;22(39):1-148.

[3] Chen BY. Total mean curvature and submanifolds of finite type. World Scientific, Singapore; 1984.

[4] Cheng QM, Wan QR. Complete hypersurfaces of $\mathbb{R}^4$ with constant mean curvature. Monatsh. Math. 1994;118(3-4):171-204.

[5] Choi M, Kim YH. Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map. Bull. Korean Math. Soc. 2001;38:753-761.

[6] Dillen F, Pas J, Verstraelen L. On surfaces of finite type in Euclidean 3-space. Kodai Math. J. 1990;13:10-21.

[7] Do Carmo M, Dajczer M. Helicoidal surfaces with constant mean curvature. Tohoku Math. J. 1982;34:351-367.

[8] Ferrandez A, Garay OJ, Lucas P. On a certain class of conformally at Euclidean hypersurfaces. Proc. of the Conf. in Global Analysis and Global Differential Geometry, Berlin; 1990.

[9] Ganchev G, Milousheva V. General rotational surfaces in the 4-dimensional Minkowski space. Turkish J. Math. 2014;38:883-895.

[10] G"uler E, Hacısalihoğlu HH, Kim YH. The Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface in 4-space. Symmetry. 2018;10(9):1-11.

[11] G"uler E, Kişi Ö. Dini-type helicoidal hypersurfaces with timelike axis in Minkowski 4-space $\mathbb{E}_4^1$. Mathematics. 2019;7(2):205:1-8.

[12] G"uler E, Magid M, Yaylı Y. Laplace Beltrami operator of a helicoidal hypersurface in four space. J. Geom. Sym. Phys. 2016;41:77-95.

[13] G"uler E, Turgay NC. Cheng-Yau operator and Gauss map of rotational hypersurfaces in 4-space. Mediterr. J. Math. 2019;16(3):66:1-16.

[14] G"uler E, Yaylı Y, Hacısalihoğlu HH. Bour’s theorem on the Gauss map in 3-Euclidean space. Hacettepe J. Math. 2010;39:515-525.

[15] Hasanis Th, Vlachos Th. Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Math. Nachr. 1995;172:145-169.

[16] Kim DS, Kim JR, Kim YH. Cheng-Yau operator and Gauss map of surfaces of revolution. Bull. Malays. Math. Sci. Soc. 2016;39:1319-1327.

[17] Kim YH, Turgay NC. Surfaces in $\mathbb{E}^4$ with $L_1$-pointwise 1-type Gauss map. Bull. Korean Math. Soc. 2013;50(3):935-949.

[18] Lawson HB. Lectures on minimal submanifolds. Rio de Janeiro. 1973;1.

[19] Magid M, Scharlach C, Vrancken L. Affine umbilical surfaces in $\mathbb{R}^4$. Manuscripta Math. 1995;88:275-289.

[20] Moore C. Surfaces of rotation in a space of four dimensions. Ann. Math. 1919;21:81-93.

[21] Moore C. Rotation surfaces of constant curvature in space of four dimensions. Bull. Amer. Math. Soc. 1920;26:454-460.

[22] Moruz M, Munteanu MI. Minimal translation hypersurfaces in $\mathbb{E}^4$. J. Math. Anal. Appl. 2016;439:798-812.

[23] Scharlach C. Affine geometry of surfaces and hypersurfaces in $\mathbb{R}^4$. Symposium on the Differential Geometry of Submanifolds, France. 2007;251-256.

[24] Senoussi B, Bekkar M. Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space. Stud. Univ. Babe-Bolyai Math. 2015;60(3):437-448.
[25] Takahashi T. Minimal immersions of Riemannian manifolds. J. Math. Soc. Japan. 1966;18:380-385.

[26] Verstraelen L, Walrave J, Yaprak Ş. The minimal translation surfaces in Euclidean space. Soochow J. Math. 1994;20(1):77-82.

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