Dynamic Loading of the Rod at a Sudden Change of Elastic Foundation Structure

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Abstract. The mathematical model of dynamic process in the loaded constant evenly distributed load of structurally nonlinear system "beam - two-parameter foundation" is constructed, arising as a result of sudden change of physical and mechanical properties of the foundation, leading to zeroing of its shear rigidity. The solving of the static bending problem of a hinged beam fixed at the ends of the beam, supported by (если физически на него опирается, если нет, тогда operated) the Pasternak foundation, serves as an initial condition of the problem about the forced vibrations of the beam on the Winkler foundation, which occurred after a sudden formation of the defect. The solutions of static and dynamic problems are built using the method of initial parameters with the involvement of vectors of beam cross-sections states and matrices of influence of initial parameters on the state of arbitrary cross-sections. To analyze forced vibrations, the decomposition of load and deflection of the initial static state into rows according to the forms of natural vibrations of the new state is applied.

1. Introduction

Widely used in various calculations, the model of interaction of a loaded deformable system resting on an elastic foundation is the Winkler model [1 - 3]. The Winkler foundation is considered as a set of independent springs working on tension - compression, fixed on an absolutely rigid continuum. The disadvantage of the Winkler spring (keyboard [2]) model is that when resisting loads at some point in the base, neighboring points (springs) are not involved in the operation. Such a system is valid only for grounds with poor distributive capacity (soft, loose soils, etc.). This drawback is solved using the Pasternak model (two-parameter models) [4]. The second parameter \( k_2 \), introduced in addition to the Winkler parameter \( k_1 \), takes into account the shear reactions of the foundation. At the same time, there are examples of practical situations when the structure, initially supported by connected soil (Pasternak model), as a result of a sudden transformation of the physical and mechanical properties of the base, is support by unrelated soil (Winkler model). At the same time, there are examples of practical situations when a structure that initially rests on a connected soil (Pasternak model), as a result of a sudden transformation of the physical and mechanical properties of the base, is supported by an unrelated soil (Winkler model).

The analysis of the current state and key areas of researches in the field of interaction of framed and carrying structures with elastic foundation, reviews of analytical and numerical approaches to modeling and calculation of static states and dynamic processes in the system (frame-base) may be found in review articles with extensive lists of sources. [5-8], [16-19].
One of the important problems of building mechanics is the analysis of the sensitivity of load-bearing structures to structural reconstructions and damage under load during an operation of the type of suddenly switched off connections, partial damage, cracks, stratification, etc. Obtaining such information for real structures requires the development of special methods, as this problem cannot be solved by a universal method. From a position of building mechanics in these problems there is a necessity of calculation of such structures as structurally non-linear, changing the design scheme under load. What is important is the suddenness of defect formation. Before the defect occurred, the stress-strain state of the structure was determined by the static effect of the load and the reaction of the foundation. With sudden formation of damage or structural adjustment, the overall stiffness of the carrier system is instantly reduced, static equilibrium is disturbed and the system comes into motion, during which deformations and internal power factors are redistributed and increase. Such dynamic loading leads to disruption of the standard functioning of the structure - failures, loss of load capacity, progressive destruction.

Today, engineering calculation methodology of the stress-strain state of loaded beam-base systems that have received unexpected damages or changes in the structure are few in numbers and imperfect. The manifestation of constructive nonlinearity and their consequences are described only in a few works that consider a complete or a partial unexpected destruction of the base [9-10,22], or a sudden change of beam boundary conditions on elastic foundation [11-13].

In the present work, the problem of constructing a mathematical model of a dynamic process arising in a static load-bearing beam resting on a two-parameter Pasternak foundation at a sudden formation of a foundation defect, which consists in losing the shear stiffness property through the entire length of contact of the beam with the foundation, that is, in sudden zeroing of the parameter $K_2$.

The mathematical model of the process is constructed by consecutive solving of the following tasks:

1. Static deflection and bending moment are determined in a beam resting on Pasternak elastic foundation in accordance with boundary conditions loaded with uniformly distributed load of specified intensity. The deflection is further used as an initial condition of the dynamic process (oscillations) that occurs after a sudden formation of the base defect. The bending moment is used to compare static internal forces with dynamic forces in the course of vibrations in the beam.

2. The frequencies and shapes of the natural bending vibrations of the beam resting on the Winkler elastic foundation are determined;

3. The forced bending oscillations of the loaded beam on the Winkler foundations are analyzed by a decomposition of the load and static deflection obtained from the Pasternak foundation, according to the forms of its proper oscillations of the beam on the Winkler foundation. Thus, the calculation of forced vibrations is constructed on the basis of modal decomposition of the initial state and the load by the modal of the new state.

To analyze the displacements and stresses in the beam during its interaction with the elastic foundation in all problems the method of initial parameters in combination with vector-matrix representation of the state of arbitrary beam cross-sections is used.

2. Static loading of the beam on the Pasternak elastic foundation

The Bernoulli-Euler $L$-length elastic beam resting on the Pasternak elastic foundation, is loaded with an evenly distributed load with intensity $q$ (Figure 1). The beam is hinged at the ends, made of a material with a density of $\rho$ and elastic modulus $E$. The area and width of the rectangular cross-section are respectively equal to $A$ and $b$, the axial moment of inertia is equal to $I$. 
Figure 1. Design model of a beam on Pasternak foundation.

Bending equation of beam [21].

\[ EI \frac{d^4 y}{dx^4} + K_1 y - K_2 \frac{d^2 y}{dx^2} = q, \]  

where \( x \) is the axial coordinate

\[ y = y(x) \] – deflection;

\( K_1 = \bar{K}_1 b, \quad \bar{K}_2 = \bar{K}_2 b \) – bending and shear stiffness of the foundation, respectively.

By introducing dimensionless parameters

\[ w = \frac{y}{L}, \quad \xi = \frac{x}{L}, \quad \alpha^4 = \frac{K_1 L^4}{4EI}, \quad \beta^2 = \frac{K_2 L^2}{4EI}, \quad \bar{q} = \frac{ql^3}{EI} \]

we’ll bring the equation (2.1) to the view

\[ w^{IV} - 4\beta^2 w'' + 4\alpha^4 w = \bar{q}. \]  

The general solution of the equation (2.2)

\[ w(\xi) = w_0(\xi) + w''(\xi), \]  

where \( w_0(\xi) \) is a partial solution corresponding to the equation (2.2) of the homogeneous equation

\[ w^{IV} - 4\beta^2 w'' + 4\alpha^4 w = 0, \]  

a \( w''(\xi) \) –partial solution corresponding to the right side of equation (2.2).

\[ w'' = \frac{\bar{q}}{4\alpha^4}. \]  

The solution of the homogeneous equation (2.4) is find using Euler substitution

\[ w = Ae^{n\xi}. \]  

We substitute (2.6) to (2.4), and we get a characteristic equation

\[ n^4 - 4\beta^2 n^2 + 4\alpha^4 = 0. \]  

Denote \( m = n^2 \), reduce the biquadratic equation (2.7) to quadratic one

\[ m^2 - 4\beta^2 m + 4\alpha^4 = 0, \]
the radicals of which may be represented in two ways, depending on the ratio of values $\alpha$ and $\beta$

$$m_{1,2} = 2\beta^2 + 2\sqrt{\beta^4 - \alpha^4}, \text{ если } \beta > \alpha \quad (2.9)$$

$$m_{1,2} = 2\beta^2 + 2i\sqrt{\alpha^4 - \beta^4}, \text{ если } \beta < \alpha \quad (2.10)$$

If we enter the argument $N_0 = 2\sqrt{K_1EI}$, the ratio between dimensionless values $\alpha$ and $\beta$

$$\beta > \alpha, \quad \beta = \alpha, \quad \beta < \alpha$$

can be replaced by similar ratios between the variables $K_2$ and $N_0$, having dimensional forces

$$K_2 > N_0, \quad K_2 = N_0, \quad K_2 < N_0,$$

as following

$$\beta = \frac{K_2l^2}{4EI}, \quad \alpha = \frac{N_0l^2}{4EI}.$$ 

Thus, three options of the solution of the homogeneous equation (2.4) are further possible.

2.1 Let’s have $\beta > \alpha (K_2 > N_0)$, then the radicals of the view (2.9) should be applied, in such a case the radicals (2.7) will be valid

$$n_{1,2} = \pm c, n_{3,4} = \pm d,$$

where $c = \sqrt{\beta^2 + \alpha^2} + \sqrt{\beta^2 - \alpha^2}$, $d = \sqrt{\beta^2 + \alpha^2} - \sqrt{\beta^2 - \alpha^2}$.

and deflexion function (1.3)

$$w = A_1ch\xi + A_2sh\xi + A_3chd\xi + A_4shd\xi + \frac{\pi}{4a^2}. \quad (2.11)$$

Let’s have $\beta < \alpha$. In this case, the radicals of the view (2.10) are applied. In this case, the radicals of equation (2.7) become complex

$$n_{1-4} = \pm \alpha \pm ib.$$

where $\alpha = \sqrt{\alpha^2 + \beta^2}$, $b = \sqrt{\alpha^2 - \beta^2}$ и функция прогибов and deflexion function

$$w = A_1cha\xi \cos b\xi + A_2sha\xi \cos b\xi + A_3cha\xi \sin b\xi + A_4sha\xi \sin b\xi + \frac{\pi}{4a^2}. \quad (2.12)$$

2.3 We get multiple radicals with $\beta = \alpha$

$$m_{1,2} = 2\beta^2$$

and, accordingly, two-fold radicals of the equation (2.7)

$$n_{1,2} = \alpha, n_{3,4} = -\alpha$$

and deflexion function
The variation 1.2 (2.10) and (2.12): $\beta < \alpha$ are analyzed below 1.2 (2.10) u (2.12): $\beta < \alpha$, as one of the most realistic for soils [14-20].

The function (2.12) in initial arguments

$$W_0 = w(0), \quad W'_0 = w'^{(0)}, \quad W''_0 = w''^{(0)}, \quad W'''_0 = w'''(0)$$

takes the following form

$$w = F_4(\xi)W'_0 + F_3(\xi)W''_0 + F_2(\xi)W'''_0 + F_1(\xi)W''''_0 + F_0(\xi),$$

(2.14)

Where

$$F_4(\xi) = \frac{1}{2(a^2 + b^2)} \left( \frac{3a^2 - b^2}{a} \frac{\sin b \xi}{\sin b \xi} - \frac{\sin b \xi}{b} \right);$$

$$F_3(\xi) = \frac{1}{2(a^2 + b^2)} \left( \frac{3a^2 - b^2}{a} \frac{\sin b \xi}{\sin b \xi} - \frac{\sin b \xi}{b} \right);$$

$$F_2(\xi) = \frac{1}{2(a^2 + b^2)} \left( \frac{3a^2 - b^2}{a} \frac{\sin b \xi}{\sin b \xi} - \frac{\sin b \xi}{b} \right);$$

$$F_1(\xi) = \frac{1}{2(a^2 + b^2)} \left( \frac{3a^2 - b^2}{a} \frac{\sin b \xi}{\sin b \xi} - \frac{\sin b \xi}{b} \right);$$

$$F_0(\xi) = \frac{1}{2(a^2 + b^2)} \left( \frac{3a^2 - b^2}{a} \frac{\sin b \xi}{\sin b \xi} - \frac{\sin b \xi}{b} \right).$$

With reference to boundary conditions (hinged edge)

$$W_0 = W'_0 = 0$$

$$w(1) = w''(1) = 0$$

(2.15)

the flexion and moment functions are somewhat simplified

$$w = F_3(\xi)W'_0 + F_1(\xi)W''_0 + F_5(\xi);$$

$$w'' = F'''(\xi)W'_0 + F''(\xi)W''_0 + F''(\xi).$$

(2.16)

Two unknown initial parameters are determined from the forces at the right end of the beam at $\xi = 1$

$$\begin{cases} F_3(1)W'_0 + F_1(1)W''_0 = -F_5(1) \\ F''''(1)W'_0 + F''(1)W''_0 = -F''(1) \end{cases}$$

At some moment $\tau = 0$ the foundation, due to structural changes, has suddenly lost the shear stiffness property $K_2 = 0$. At this moment, the beam-foundation system ceased to be in equilibrium and began to move.

3. Intrinsic bending vibrations of the beam on the elastic Winkler single-parameter foundation

The equation of natural beam oscillations [21]

$$\frac{\partial^4 w}{\partial \xi^4} + 4\alpha^4 \left( w + \frac{\partial w}{\partial \tau^2} \right) = 0.$$  

(3.1)

Where $\tau = \omega_0$ – the dimensionless time, $\omega_0 = \sqrt{\frac{K_1}{\rho A}}$ – is a "conditional" frequency which is a parameter having a frequency dimension.
The solution of equation (3.1) is sought by looking for, lowering oscillations by harmonic and separating variables by representation

\[ W(\xi, \tau) = W(\xi) \sin \frac{\bar{\omega}}{\omega_0} \tau, \]  

(3.2)

where \( \bar{\omega} = \frac{\omega}{\omega_0} \) – the dimensionless unknown is natural frequency;

\( \bar{\omega}_0 = \frac{\omega_0}{\omega} \) is a dimensionless "conditional" frequency;

\( \omega_0 = \frac{1}{L^2} \sqrt{\frac{EI}{\rho A}} \) is reference frequency.

By substituting the representation (3.2) in the equation (3.1), we obtain the equation of the shapes of the natural vibrations of the beam

\[ W^{IV} = (\bar{\omega}_0^2 - \bar{\omega}^2)W = 0. \]  

(3.3)

The structure of equation (3.3) suggests the possibility of three solutions:

3.1 If \( \bar{\omega} > \bar{\omega}_0 \), then, writing equations (3.3) as

\[ W^{IV} - (\bar{\omega}^2 - \bar{\omega}_0^2)W = 0. \]

and solving it by Euler substitution

\[ w = Ae^{r\xi} \]

we get the characteristic equation

\[ r^4 - (\bar{\omega}^2 - \bar{\omega}_0^2) = 0. \]

the radicals of which are real and purely imaginary

\[ r_{1,2} = \pm \beta_1, \quad r_{3,4} = \pm \beta_2, \quad \beta_1 = \sqrt{\bar{\omega}_0^2 - \bar{\omega}^2}. \]  

(3.4)

The solution of the equation herewith (3.3) is the following:

\[ w(\xi) = W_0 R_4(\beta_1 \xi) + W_0' R_3(\beta_1 \xi) + W_0'' R_2(\beta_1 \xi) + W_0''' R_1(\beta_1 \xi), \]  

(3.5)

where \( R_i = R_i(\xi) \) \( (i = 1 - 4) \) –are the functions of Krylov view

\[ R_1(\xi) = \frac{sh\beta_1 \xi - \sin \beta_1 \xi}{2\beta_1}. \]

\[ R_2 = R_1'; \quad R_3 = R_2'; \quad R_4 = R_3'; \quad R_4' = \beta_1^2 R_1'. \]

The state of arbitrary cross-section \( \xi \) can be represented by a matrix equation

\[ \bar{W}(\xi) = V_1(\xi)\bar{W}_0, \]  

(3.6)

where \( \bar{W}(\xi) = \{ W(\xi), W'(\xi), W''(\xi), W'''(\xi) \} \) – is a random section state vector \( \xi \) of the beam;

\( \bar{W}_0 = \{ W_0, W_0', W_0'', W_0''' \} \) – is the vector of initial parameters;
\[
V_1(\xi) = \begin{pmatrix}
    R_4(\beta_1 \xi) & R_3(\beta_1 \xi) & R_2(\beta_1 \xi) & R_1(\beta_1 \xi) \\
    \beta_1^4 R_1(\beta_1 \xi) & R_4(\beta_1 \xi) & R_3(\beta_1 \xi) & R_2(\beta_1 \xi) \\
    \beta_1^4 R_2(\beta_1 \xi) & \beta_1^4 R_1(\beta_1 \xi) & R_4(\beta_1 \xi) & R_3(\beta_1 \xi) \\
    \beta_1^4 R_3(\beta_1 \xi) & \beta_1^4 R_2(\beta_1 \xi) & \beta_1^4 R_1(\beta_1 \xi) & R_4(\beta_1 \xi)
\end{pmatrix}
\]

- is the functional matrix of initial parameters that influences the section state \(\xi\).

3.2. If \(\tilde{\omega} < \tilde{\omega}_0\), (3.3) then from the equation (3.3) by Euler setting we get the characteristic equation

\[
r^4 + (\tilde{\omega}_0^2 - \tilde{\omega}^2) = 0.
\]

with complex radicals

\[
r_{i+4} = (\pm i \pm 1) \beta_2, \quad \beta_2 = \sqrt[4]{\frac{\tilde{\omega}_0^2 - \tilde{\omega}^2}{4}}
\]

and solving the equation (3.3) of the form

\[
w = W_0 K_4(\beta_2 \xi) + W_0' K_3(\beta_2 \xi) + W_0'' K_2(\beta_2 \xi) + W_0''' K_1(\beta_2 \xi),
\]

where \(K_i = K_i(\xi)\) (\(i = 1 \div 4\)) – are the functions of Krylov view

\[
K_1(\xi) = \frac{\sin \beta_2 \xi \chi \beta_2 \xi - \cos \beta_2 \xi \sinh \beta_2 \xi}{4 \beta_2^3}; \quad K_2 = K_1'; \quad K_3 = K_2'; \quad K_4 = -4 \beta_2^4 K_1.
\]

In this case, the state of an arbitrary section \(\xi\) is described with the matrix equation

\[
\tilde{W}(\xi) = V_2(\xi) \tilde{W}_0,
\]

where

\[
V_2(\xi) = \begin{pmatrix}
    K_4(\beta_2 \xi) & K_3(\beta_2 \xi) & K_2(\beta_2 \xi) & K_1(\beta_2 \xi) \\
    -4 \beta_2^4 K_1(\beta_2 \xi) & K_4(\beta_2 \xi) & K_3(\beta_2 \xi) & K_2(\beta_2 \xi) \\
    -4 \beta_2^4 K_2(\beta_2 \xi) & -4 \beta_2^4 K_1(\beta_2 \xi) & K_4(\beta_2 \xi) & K_3(\beta_2 \xi) \\
    -4 \beta_2^4 K_3(\beta_2 \xi) & -4 \beta_2^4 K_2(\beta_2 \xi) & -4 \beta_2^4 K_1(\beta_2 \xi) & K_4(\beta_2 \xi)
\end{pmatrix}
\]

3.3. If \(\tilde{\omega} = \tilde{\omega}_0\), then integrating sequentially the equation \(W^{IV} = 0\),

we’ll get the function

\[
w = W_0 + W_0' \xi + W_0'' \frac{\xi^2}{2} + W_0''' \frac{\xi^3}{6}
\]

and the matrix equation

\[
\tilde{W}(\xi) = V_3(\xi) \tilde{W}_0,
\]

\[
V_3(\xi) = \begin{pmatrix}
    1 & \xi & \frac{\xi^2}{2} & \frac{\xi^3}{6} \\
    0 & 1 & \xi & \frac{\xi^2}{2} \\
    0 & 0 & 1 & \xi \\
    0 & 0 & 0 & 1
\end{pmatrix}
\]

Thus, the stress-strain state of the beam is described by one of the equations (3.6) - (3.8).
Let’s consider each possible variant of the correlation between the proper frequency of bending vibrations of the beam $\tilde{\omega}$ and the generalized mechanical characteristic of the beam-foundation system - the "conditional" frequency $\tilde{\omega}_0$.

First, we accept the condition (3.3), according to which it is assumed that the desired frequency $\tilde{\omega}$ of the hinged supported beam is equal to the "conditional" frequency $\tilde{\omega}_0$. Then, from the equation (3.3), taking into account the boundary conditions (1.15), we obtain a system of equations to determine the unknown initial parameters $W_0' \text{ and } W_0''$

\[
\begin{align*}
W(\xi) &= \xi W_0' + \frac{\xi^3}{6} W_0'' , \\
W''(\xi) &= \xi W_0''
\end{align*}
\]

Fulfilling the second pair of conditions (1.15), we get

$W_0' = W_0'' = 0$.

This means that the accepted assumption $\tilde{\omega} = \tilde{\omega}_0$ is not productive. Let’s suppose the possibility 3.2, that is $\tilde{\omega} < \tilde{\omega}_0$. Then from the equation (2.7) we get a system of equations

\[
\begin{align*}
w(\xi) &= K_3(\beta_2 \xi) W_0' + K_1(\beta_2 \xi) W_0'' \\
w''(\xi) &= -4\beta_2^4 K_1(\beta_2 \xi) W_0' + K_3(\beta_2 \xi) W_0''
\end{align*}
\]

Fulfilling the second pair of conditions (1.15), we obtain the frequency equation

\[
\begin{vmatrix}
K_3(1) & K_1(1) \\
-4\beta_2^4 K_1(1) & K_3(1)
\end{vmatrix} = \sin^2 \beta_2 + s^2 \beta_2 = 0,
\]

the only solution of which is

$\beta_2 = 0$,

that leads to the already considered option

$\tilde{\omega} = \tilde{\omega}_0$.

The assumption 2.1 remains, that is $\tilde{\omega} > \tilde{\omega}_0$, which results in a quotient equation

\[
\begin{vmatrix}
R_3(1) & R_1(1) \\
\beta_1^4 R_1(1) & R_3(1)
\end{vmatrix} = \sin \beta_1 s h \beta_1 = 0,
\]

the radicals of which are

$\beta_{1n} = n\pi$.

From here it follows that we obtain dependence between required $\tilde{\omega}_n$ and "conditional" $\tilde{\omega}_0$ frequencies of a view

\[
\tilde{\omega}_n = \sqrt{(n\pi)^4 + \tilde{\omega}_0^2}.
\]

(3.9)

The forms of natural oscillations corresponding to the frequencies (3.10) follow from the expression (3.6) taking into account boundary conditions (2.15)

\[
\omega_n(\xi) = R_3(\beta_{1n} \xi) - U R_1(\beta_{1n} \xi),
\]

where

\[
U = \frac{R_3(\beta_{1n})}{R_1(\beta_{1n})}
\]

(3.10)
4. Compelled bending oscillations of the beam on the elastic Winkler single-parameter foundation

Compelled oscillations of beam loaded with uniformly distributed load of intensity \( q \) initiated by a sudden transformation of “beam-base” structure are described by the equation [21]

\[
\frac{\partial^4 w_{\text{th}}}{\partial \xi^4} + 4\alpha^4 \left( w_{\text{th}} + \frac{\partial^2 w_{\text{th}}}{\partial \tau^2} \right) = \bar{q},
\]  

(4.1)

where \( \bar{q} = \frac{q l^3}{EI} \) is the dimensionless intensity of uniformly distributed load;

\( w_{\text{th}} = w_{\text{th}}(\xi, \tau) \) — is the function of deflections of an arbitrary cross-section \( \xi (0 \leq \xi \leq 1) \) of the beam in time \( \tau \).

Separating the variables in the equation (4.1) with the series

\[
w_{\text{th}} = \sum_{n=1}^{\infty} Q_n(\tau) W_n(\xi),
\]  

(4.2)

Where \( W_n = W_n(\xi) \) — are fundamental functions;

\( Q_n = Q_n(\tau) \) — are time functions to be determined, we’ll get the equations to define the functions \( Q_n(\tau) \)

\[
\frac{d^2 Q_n}{d\tau^2} + \bar{\omega}_n^2 Q_n = R_n,
\]  

(4.3)

where

\[
R_n = \frac{1}{\bar{\omega}_n^2} \int_0^1 \bar{q} W_n(\xi) d\xi
\]  

The general solution of the equation (4.1) is [21]

\[
w_{\text{th}} = \sum_{n=1}^{\infty} \left( D_1 \cos \bar{\omega}_n \tau + D_2 \sin \bar{\omega}_n \tau + \frac{R_n}{\bar{\omega}_n^2} \right) W_n(\xi).
\]  

(4.4)

The constants of integration \( D_{1n} \) and \( D_{2n} \) are defined from initial conditions when \( \tau = 0 \)

\[
w_{\text{th}}(\xi, 0) = w_{cm}(\xi),
\]

\[
\frac{\partial w_{\text{th}}}{\partial \tau} \bigg|_{\xi, \tau = 0} = 0,
\]  

(4.5)

where \( w_{cm}(\xi) \) — the static deflections of the beam resting on the elastic two-parameter Pasternak foundation, determined by the equation (1.2), and in this case described by the function (2.16).

From the 2nd condition (4.5) follows

\[
D_{2n} = 0.
\]  

(4.6)

From the 1st condition (4.5) follows

\[
\sum_{n=1}^{\infty} \left( D_{1n} + \frac{R_n}{\bar{\omega}_n^2} \right) W_n(\xi) = w_{cm}.
\]  

(4.7)

Multiplying both parts (4.7) by \( W_n(\xi) \) and integrating by \( \xi \) from 0 to 1, using the property of orthogonality of the proper functions \( W_n(\xi) \), we get
\[ D_{1n} = B_n - \frac{R_n}{\bar{\omega}_n}, \quad B_n = \frac{\int_0^1 w_{cm} W_n(\xi) d\xi}{\int_0^1 W_n^2(\xi) d\xi}. \] (4.8)

Supplying (4.6) and (4.8) in the row (4.4) and considering the identical equation

\[ 1 - \cos \bar{\omega}_n \tau = 2 \sin^2 \frac{\bar{\omega}_n}{2} \tau, \]

we have

\[ w_{\partial uu}(\xi, \tau) = \sum_{n=1}^{\infty} \left( B_n \cos \bar{\omega}_n \tau + C_n \sin^2 \frac{\bar{\omega}_n}{2} \tau \right) W_n(\xi). \] (4.9)

where

\[ C_n = \frac{2\bar{q} \int_0^1 W_n(\xi) d\xi}{\bar{\omega}_n^2 \int_0^1 W_n^2(\xi) d\xi} \]

5. Numerical results

The following characteristics were used in numerical calculations.

1) Beam. Length \( l = 6.7 \) m; cross section rectangle with height \( h = 0.18 \) m; width \( b = 0.25 \) m; cross-section area \( A = 0.045 \) m\(^2\); axial moment of inertia \( I = 1.215 \cdot 10^{-4} \) m\(^4\); used material is reinforced concrete, elastic modulus \( E = 3.05 \cdot 10^{10} \) N/m\(^2\).

2) Basis. The base material is gravel. Bending stiffness parameter \( k_1 = 7.5 \cdot 10^6 \) H/m\(^3\), bedding factor \( k_1 = \bar{k}_1 b = 1.875 \cdot 10^6 \) N/m\(^2\). Shear stiffness parameter \( k_2 \) in accordance with recommendations [14, 15] adopted in the form of \( k_2 = 0.35k_1 m^2 = 0.65 \cdot 10^6 \) N.

3) Loading. Distributed uniformly cross-axis with dimensionless intensity \( \bar{q} = 1 \).

4) Generalized characteristics of the “beam-foundation” system.

Basic characteristics \( \alpha = 3.976; \beta = 1.41; \lambda = 4\alpha^4 = 10^3 \). In comparison, the systems with the following combinations of characteristics are considered:

\( \lambda = 10^{1.5} \rightarrow \alpha = 1.677; \beta = 0.23; k_1 = 0.05 \cdot 10^6 \) N/m\(^2\); \( k_2 = 0.0175 \cdot 10^6 \) N;

\( \lambda = 10^{4.5} \rightarrow \alpha = 9.429; \beta = 7.84; k_1 = 58 \cdot 10^6 \) N/m\(^2\); \( k_2 = 20.3 \cdot 10^6 \) N;

The results of static and dynamic calculations are described below.

Figure 2 shows the curves of the beam deflections for three versions of the combinations of bending \( \alpha \) and shear \( \beta \) stiffness of the “beam-foundation” system:

1) \( \alpha = 9.429 \) and \( \beta = 7.84 \);  
2) \( \alpha = 3.976 \) and \( \beta = 1.41 \);  
3) \( \alpha = 1.677 \) and \( \beta = 0.23 \);
Concerning the deflexions, we note that in all cases the maximum deflexions developed in the center of the beam. The maximum deflexion value decreases dramatically as the system tightens from 0.009 for $\lambda = 10^{1.5}$ to 0.0003 for $\lambda = 10^{4.5}$.

Figure 3 shows the curves of the moments for the same variations. Moment calculations show that with small values of generalized stiffness ($\lambda \geq 10^{1.5}$) the maximum moment develops in the middle of the beam and when stiffness ($\lambda \geq 10^{3}$) increases, it shifts to the end sections. At the same time, the central zone approaches a momentless state.
Figure 4 shows that the impact of the Pasternak parameter on the stress-strain state of the beam depends significantly on the generalized stiffness of the beam-foundation system: the tougher the system, the greater the influence of $\beta$ (if the hypothesis is met $k_2 = 0.35 \cdot k_1$). So, at $\lambda = 10^{1.5} \left( \frac{\beta}{\alpha} = 0.137 \right)$ the influence $k_2$ on the value of bending moments is invisible, but at $\lambda = 10^{4.5} \left( \frac{\beta}{\alpha} = 0.831 \right)$ the maximum moment in the beam resting on the Pasternak foundation is one and a half times less than that of the same beam on the Winkler foundation. The nature of the curves is preserved.

![Figure 4. Comparison of static bending moments in the beam on the Winkler and Pasternak foundations.](image)

The results of the dynamic calculation are illustrated by graphs of changes in bending moments in the beam in characteristic sections: the center of the beam $\xi=0.5$ and end sections $\xi=0.211$ and 0.789 for the same three cases of the generalized rigidity of the system "beam-foundation".
№1. the curve of static bending moments in the beam on the Pasternak foundation at \( \alpha = 3.976 \) and \( \beta = 1.41 \);

№2. the curve of static deflection in the beam on the Pasternak foundation at \( \alpha = 3.976 \) and \( \beta = 1.41 \);

№3. the curve of dynamic deflection at the moment \( t = 0 \), derived from dynamic series \( M_{\text{dyn}} \) at \( t = 0 \) taking into account the 2 terms of series;

№4. the same, taking into account the 3 terms of series;

№5. the curve of moments at \( t = 89.7 \), \( M_{\text{max}} = 0.014 \)
№6. the curve of moments at $t = 21.5$, $M_{max} = 0.0147$

№7. graph of the change of the moment over time in the central section of the beam $x=0.5$, $M_{max} = 0.0147$;

№8. the curve of moments in the beam after the quasi-static transition of the Pasternak foundation to the Winkler foundation $M_{max} = 0.01$ in sections 0.211 and 0.789

№9. graph of the moment change in time in the beam section $x=0.211$(or 0.789), $M_{max} = 0.01$;

№10. graph of the evolution of bending moment change in section $x = 0.211$;
Number 11. Graph of steady-state cross-sectional fluctuations $x = 0.211$

Figure 5. Comparison of static and dynamic values of bending moment in beam.

A comparison of static bending moments with dynamic moments shows that:

1) for the option $\lambda = 10^{4.5} (\alpha = 9.429; \beta = 7.84)$ the maximum static moment in the section $\xi = 0.075 S_{\text{max}} = 0.0012$, static moment in the center of the beam $\xi = 0.5 S = 1.69 \cdot 10^{-5}$ so, we observe an almost momentary state. After a sudden structural transformation of the system, the maximum dynamic moment during the resulting fluctuations $M_{\text{max}} = 0.0021$ is increasing to 1.75 times. In this case, in the center of the beam, the moment can reach the maximum value $M_{\text{max}}(0.5) = 0.00314$. Thus, the central section becomes a dangerous one $\xi = 0.5$, at the same time, the sections were dangerous in static $\xi = 0.075$ and $\xi = 0.925$.

2) for the option $\lambda = 10^{3} (\alpha = 3.976; \beta = 1.41)$ maximum static moment is equal to $S_{\text{max}} = 0.0096$, it develops in sections $\xi = 0.211$ and $\xi = 0.789$. In the center of the section $\xi = 0.5$ static moment is $S_{\text{max}} = 0.0075$. In the dynamic with sections $\xi = 0.211$ and $\xi = 0.789$ the moment reaches the following value $M_{\text{max}} = 0.0326$, that means that it is increasing to 3.39 times. In the central section $\xi = 0.5$, moment reaches the value $M_{\text{max}} = 0.0366$, that is, it is increasing to 4.88 times. The average becomes a dangerous section in the dynamics $\xi = 0.5$, and in static, the section will be dangerous $\xi = 0.211$ and $\xi = 0.789$.

3) for the option $\lambda = 10^{1.5} (\alpha = 1.677; \beta = 0.23)$ static maximum moment will be in the central section $\xi = 0.5 S_{\text{max}} = 0.092$. In dynamics in the same section, the moment can reach a value $M_{\text{max}} = 0.295$, that is, it will increase to 3.2 times. The central section becomes a dangerous section in both static and dynamic.

Table 1. The comparison of values of static and dynamic bending moments.

| Generalized rigidity $\alpha$ | Bed factor $\beta$ | Max. static moment $S_{\text{max}}$ | Section with max static moment $\xi$ | Static moment $S$ in central section $\xi = 0.5$ | Dynamic moment $(M_{\text{dyn}})$ in dangerous sections | Dynamic moment $(M_{\text{dyn}})$ in the middle of the section | Dynamic moment Coefficient $M_{\text{max}}/S_{m}$ |
|-----------------------------|---------------------|-------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\lambda = 10^{4.5}$ | $9.42$ | $7.84$ | $0.0012$ | $0.075$ $(0.925)$ | $1.69 \cdot 10^{-5}$ | $0.0021$ | $0.00314$ | $2.7$ |
| $\lambda = 10^{3}$ | $3.97$ | $1.41$ | $0.0096$ | $0.211$ $(0.789)$ | $0.0075$ | $0.0326$ | $0.0366$ | $3.9$ |
| $\lambda = 10^{1.5}$ | $1.67$ | $0.23$ | $0.092$ | $0.5$ | $0.092$ | $0.295$ | $0.295$ | $3.2$ |
Note that the quasi-static (without inertial effects) transformation of the foundation properties brings to:
- the increase of the maximum bending moment by 1.5 times in the beam at \( \alpha = 9,429 \)
  \[ M_{\text{max}}^{\text{Винкл}} = 0,0018 \text{ at } \alpha = 9,429; \beta = 0 \]
  \[ M_{\text{max}}^{\text{Паст}} = 0,0012 \text{ at } \alpha = 9,429; \beta = 7,84 \]
  \[ \frac{M_{\text{max}}^{\text{Винкл}}}{M_{\text{max}}^{\text{Паст}}} = \frac{0,0018}{0,0012} = 1,5 \]
- to 1.07 divisible increase at \( \alpha = 3,976 \)
  \[ M_{\text{max}}^{\text{Винкл}} = 0,0104 \text{ at } \alpha = 3,976; \beta = 0 \]
  \[ M_{\text{max}}^{\text{Паст}} = 0,0098 \text{ at } \alpha = 3,976; \beta = 1,41 \]
  \[ \frac{M_{\text{max}}^{\text{Винкл}}}{M_{\text{max}}^{\text{Паст}}} = \frac{0,0104}{0,0098} = 1,07 \]
- practically does not change at \( \alpha = 1,677 \), as also noted in the work [20]

6. Summary

Thus, the structural restructuring of the beam-foundation system, which consists in a sudden transformation of the Pasternak foundation into the Winkler foundation, that is, in a sudden loss of the shear stiffness property by the base, leads to a radical transformation of the stress-strain state of the beam:
- bending moments in all sections become alternating;
- the position of the hazardous section changes (for all stiffness values, the center of the section becomes dangerous \( \xi = 0,5 \));
- absolute values of bending moments increase significantly and the greater rigidity of beam-foundation system corresponds to greater relative increment of dynamic bending moment to static value.

The obtained results of the study show significant increases in stresses and significant changes in the pattern of stress-strain state caused by a sudden change in the structure and design scheme of the considered beam-foundation model.

7. Conclusion

The obtained results show that when designing, calculating, operating and analyzing accidents, the possibility of sudden (invisible) structural alterations in critical structures modeled by beams interacting with elastic bases and their negative consequences should be taken into account. This technique is relevant for solving existing problems of construction mechanics and will certainly be necessary for the calculation of all kinds of modules during the construction of seismic-resistant structures and the development of space planets.

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