Rings of continuous functions, symmetric products, and Frobenius algebras

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Abstract

A constructive proof of the classical theorem of Gel’fand and Kolmogorov (1939) characterising the image of the evaluation map from a compact Hausdorff space $X$ into the linear space $C(X)^*$ dual to the ring $C(X)$ of the continuous functions on $X$ is given. Our approach to the proof enabled us to obtain a more general result characterising the image of an evaluation map from the symmetric products $\text{Sym}^n(X)$ into $C(X)^*$. A similar result holds if $X = \mathbb{C}^m$ and leads to explicit equations for symmetric products of affine algebraic varieties as algebraic sub-varieties in the linear space dual to the polynomial ring. This leads to a better understanding of the algebra of multi-symmetric polynomials.

The proof of all these results is based on a formula used by Frobenius in 1896 when defining higher characters of finite groups. This formula had no further applications for a long time; however, it occurred in several independent contexts during the last fifteen years. The formula was used by A. Wiles and R. L. Taylor when studying representations and by H.-J. Hoehnke and K. W. Johnson and later by J. McKay when studying finite groups. It plays an important role in our work concerning multi-valued groups. We describe several properties of this remarkable formula. We also use it to prove a theorem on the structure constants of Frobenius algebras, which have recently attracted attention due to constructions taken from topological field theory and singularity theory. This theorem develops a result of H.-J. Hoehnke published in 1958. As a corollary, we obtain a direct self-contained proof of the fact that the 1-, 2-, and 3-characters of the

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regular representation determine a finite group up to isomorphism. This result was first published by H.-J. Hoehnke and K. W. Johnson in 1992.

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§1. Introduction

In 1939, A. N. Kolmogorov and I. M. Gel’fand published the paper ‘On rings of continuous functions on topological spaces’ (see [1]). The main result of this paper identifies a compact Hausdorff space \( X \) with the space of maximal ideals of the ring of continuous functions on \( X \). Monographs and textbooks containing basic functional analysis usually include this result. In modern terminology, it can be stated as follows.

**Theorem 1** Let \( X \) be a compact Hausdorff space. Then, for an appropriate topology on the space of continuous complex-valued functions \( C(X) \) on \( X \), the evaluation map

\[ \mathcal{E} : X \to \text{Hom}(C(X), \mathbb{C}), \quad \mathcal{E}(x)\varphi = \varphi(x), \]

is a homeomorphism onto the set of all ring homomorphisms \( C(X) \to \mathbb{C} \).

This theorem is an analogue of Hilbert’s Nullstellensatz: If \( V \) is an affine algebraic variety with coordinate ring \( A = \mathbb{C}[x_1, \ldots, x_n]/J \), where \( J \) is a (radical) ideal defining \( V \), then the evaluation map

\[ \mathcal{E} : V \to \text{Hom}(A, \mathbb{C}) \]

defines an isomorphism between the variety \( V \) and the set of all ring homomorphisms.

Well-known reformulations of these theorems can be obtained by noting that the set of all ring homomorphisms \( A \to \mathbb{C} \) can readily be identified with the set (usually denoted by \( \text{m-Spec}(A) \)) of all maximal ideals of \( A \).

We take the following point of view: the set of all ring homomorphisms \( f : A \to \mathbb{C} \) is an algebraic variety in \( \text{Hom}(A, \mathbb{C}) \) (regarded as a linear space) defined by infinitely many equations

\[ \{ f(1) = 1 \text{ and } f(ab) = f(a)f(b) \text{ for every } a, b \in A \} \]
involving the coordinate maps \( a : \text{Hom}(A, \mathbb{C}) \to \mathbb{C} \), \( a(f) = f(a) \). Since we deal with linear homomorphisms, it suffices to consider the equations \( f(1) = 1 \) and \( f(a^2) = f(a)^2 \), where \( a \) ranges over an additive basis in \( A \).

The standard proof of the Kolmogorov–Gel’fand theorem found in textbooks, argues by contradiction as follows.

Let \( I \) be a proper ideal in the function ring \( C(X) \) such that there are no points \( x \in X \) at which all functions \( \varphi \in I \) vanish. Then for any point \( x \in X \) there is a non-negative real-valued function \( \varphi_x \in I \) whose values are greater than 1 in some neighbourhood \( U_x \) of the point \( x \). Using the fact that the space \( X \) is compact, we can construct a real-valued function \( \varphi \in I \) whose values are greater than 1 on the entire space \( X \). Such a function is invertible, and hence a non-zero constant function belongs to \( I \), that is, \( I = C(X) \).

In [2] we presented a constructive proof of our theorem in [3] that characterises the symmetric products \( \text{Sym}^n(X) \) as algebraic varieties in \( C(X)^* \). In the case of \( n = 1 \), this gives a proof of the Gel’fand–Kolmogorov theorem. For a given ring homomorphism \( f : C(X) \to \mathbb{C} \), the proof constructs a unique point \( x \in X \) such that \( f(\varphi) = \varphi(x) \) for any \( \varphi \in C(X) \). In the literature concerning this topic we found no proof of the Gel’fand–Kolmogorov theorem similar to that presented below.

Let us first consider the case in which \( X \) is a finite set. Then \( C(X) \cong \mathbb{C}^n \), where \( n \) is the number of elements in \( X \). We choose a basis \( \{ \delta_x : x \in X \} \) for \( C(X) \) such that \( \delta_x(x) = 1 \) and \( \delta_x(y) = 0 \) if \( x \neq y \). It is clear that \( 1 = \sum \delta_x \). Since \( \delta_x^2 = \delta_x \), for a ring homomorphism \( f \) we have \( f(\delta_x)^2 = f(\delta_x) \), and hence \( f(\delta_x) = 0 \) or 1 for each \( x \). On the other hand, \( 1 = f(1) = \sum_{x \in X} f(\delta_x) \), and hence there is a unique point \( x_0 \in X \) such that \( f(\delta_{x_0}) = 1 \); finally, using the expansion \( \varphi = \sum \varphi(x) \delta_x \), we see that \( f(\varphi) = \varphi(x_0) \) for any \( \varphi \in C(X) \).

Let us now show how to adapt this argument to the case of a compact Hausdorff space \( X \).

**Definition 1** Let \( K \subset X \) be a compact subset of a topological space \( X \). A sequence of continuous functions \( \varphi_r : X \to [0, 1] \) \((r \in \mathbb{N})\) is said to be enclosing for \( K \) if

1) \( \varphi_r(\text{Supp}(\varphi_{r+1})) = 1 \) for any \( r \in \mathbb{N} \),

2) \( \varphi_r(x) = 1 \) for every \( r \iff x \in K \).

It is clear that \( \varphi_r \varphi_s = \varphi_s \) for \( s > r \).
Example Let $K$ be a compact subset of a metric space $X$ with the metric $d(\cdot, \cdot)$. Then the sequence of functions

$$
\varphi_r(x) = \begin{cases} 
0 & \text{if } d(x, K) \geq 1/r, \\
1 & \text{if } d(x, K) \leq 1/(r + 1), \\
(r + 1)(1 - rd(x, K)) & \text{if } 1/(r + 1) \leq d(x, x_0) \leq 1/r,
\end{cases}
$$

is enclosing for $K$.

The definition of an enclosing sequence for a compactum $K$ obviously generalises to that of an enclosing net (in other words, a generalised sequence) of continuous functions $\varphi_t : X \rightarrow [0, 1]$ ($t \in T$), where $T$ is some directed set. The proof of the fact that an enclosing net exists for any compactum $K$ in a Hausdorff space $X$ is left to the reader. Moreover, to avoid overloading our argument by technical details, we speak in what follows about enclosing sequences, and use the fact that the assertions thus obtained remain valid for enclosing nets.

**Lemma 2** Let $\{\varphi_r\}$ be an enclosing sequence for a compactum $K$. Then for any $r$ there is an open neighbourhood $U_r$ of $K$ such that $\varphi_r = 1$ on $U_r$.

**Proof** We set $U_r = \{x : \varphi_{r+1}(x) > 0\}$. Then $K \subset U_r$ and $U_r \subset \text{Supp}(\varphi_{r+1})$. Hence, $\varphi_r = 1$ on $U_r$.

**Lemma 3** Let $\{\varphi_r\}$ be an enclosing sequence for a compactum $K$ and let $\psi : X \rightarrow \mathbb{R}$ be a function such that $\psi(x) = 1$ for any $x \in U$, where $U$ is an open neighbourhood of the compactum $K$. Then $(1 - \psi)\varphi_r = 0$ for $r$ large enough.

**Proof** Let $x \notin U$. We choose an $r$ such that $\varphi_r(x) = 0$. Then the set $\varphi_r^{-1}[0, 1)$ is an open neighbourhood of $x$ whose closure is disjoint from $U$. Let us cover the compact space $X \setminus U$ by finitely many neighbourhoods of this kind and denote by $r_0$ the largest number among the numbers $r$ corresponding to these neighbourhoods. Then $(1 - \psi)\varphi_{r_0} = 0$, and hence $(1 - \psi)\varphi_s = (1 - \varphi)\varphi_{r_0}\varphi_s = 0$ for $s > r_0$.

**Lemma 4** Let $f : C(X) \rightarrow \mathbb{C}$ be a ring homomorphism and let $\varphi_r$ be a sequence of functions in $C(X)$ such that $\varphi_r\varphi_s = \varphi_s$ for any $r < s$. Then either there is an $r_0$ such that $f(\varphi_r) = 0$ for $r \geq r_0$ or $f(\varphi_r) = 1$ for any $r$.

**Proof** We have $(f(\varphi_r) - 1)f(\varphi_s) = 0$ for any $r < s$. In this case, if $f(\varphi_r) \neq 1$, then $f(\varphi_s) = 0$ for any $s > r$. This argument implies the desired result.

**Definition 2** Let $f : C(X) \rightarrow \mathbb{C}$ be a ring homomorphism and let $\{\varphi_r\}$ be an enclosing sequence for a compactum $K \subset X$. By the weight of $K$ with respect to the sequence $\{\varphi_r\}$ we mean the number $w_f^\varphi(K) \in \{0, 1\}$ equal to $f(\varphi_r)$ for large values of $r$. 

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Proposition 5 Let \( \{ \varphi_r \} \) and \( \{ \psi_r \} \) be two enclosing sequences for a given compactum \( K \). Then \( w_f^\varphi(K) = w_f^\psi(K) \).

This defines the weight of a compactum, \( w_f(K) \), which does not depend on the choice of an enclosing sequence.

Proof Suppose that \( w_f^\varphi(K) = 1 \) and \( w_f^\psi(K) = 0 \). Then \( f(\varphi_r) = 1 \) for any \( r \), whereas \( f(\psi_r) = 0 \) for any \( r \) greater than some \( r_0 \). Using Lemma 3 with \( \psi = \psi_{r_0} \), we find an \( m > r_0 \) such that \( \psi_{r_0} \varphi_m = \varphi_m \). Thus, \( f(1-\psi_{r_0})f(\varphi_m) = 0 \); however, \( f(1-\psi_{r_0}) = 1 \), and hence \( f(\varphi_r) = 0 \) for \( r > m \). The contradiction thus obtained proves the Proposition.

Definition 3 By the support of a ring homomorphism \( f : C(X) \to \mathbb{C} \) we mean the set \( S_f = \{ x : w_f(x) = 1 \} \).

Proposition 6 For any ring homomorphism \( f \) the set \( S_f \) consists of a single point.

Proof We first assume that the set \( S_f \) contains two distinct points, say, \( x \) and \( y \). Let us choose an enclosing sequence \( \{ \varphi_r \} \) for the set \( \{ x, y \} \). Then \( f(\varphi_r) = 1 \). For \( r \) large enough we can choose continuous functions \( \psi_1 \) and \( \psi_2 \) such that \( \psi_1(x) = 1 \), \( \psi_1(y) = 0 \), \( \psi_2(x) = 0 \), and \( \psi_2(y) = 1 \), \( (\psi_1 + \psi_2)^{-1} \subset \varphi_r \), and \( \text{Supp} \psi_1 \cap \text{Supp} \psi_2 = \emptyset \). Then \( f(\psi_1) = f(\psi_2) = 1 \) by construction, so, \( f(\psi_1 + \psi_2) = 2 \), which is a contradiction.

We now assume that the set \( S_f \) is empty. Then for any point \( x \in X \) there is a function \( \varphi_x : X \to \mathbb{R} \) such that \( \varphi_x(x) = 1 \) and \( f(\varphi_x) = 0 \). The open sets \( \{ y : \varphi_x(y) > 0 \} \) cover the space \( X \). Choosing a finite sub-covering of this covering, we take the corresponding set of functions \( \varphi_1, \ldots, \varphi_n \). Then the function \( \Phi = \varphi_1 + \ldots + \varphi_n \) does not vanish at any point of \( X \); however, \( f(\Phi) = f(\varphi_1) + f(\varphi_2) + \ldots + f(\varphi_n) = 0 \). We see that, on the one hand, \( f \left( \frac{1}{\Phi} \right) = f(1) = 1 \), and, on the other hand, \( f \left( \frac{1}{\Phi} \right) = f(\Phi) f \left( \frac{1}{\Phi} \right) = 0 \). This contradiction proves Proposition 6.

Thus, to any ring homomorphism \( f : C(X) \to \mathbb{C} \) we assign a ring homomorphism \( \widehat{f} : C(X) \to \mathbb{C} \), \( \widehat{f}(\varphi) = \varphi(x_0) \), where \( S_f = \{ x_0 \} \). To complete the proof of Theorem 1, it suffices to show that \( \widehat{f} = f \).

Proposition 7 Let \( S_f = \{ x_0 \} \) and let \( \psi : X \to \mathbb{C} \) be a continuous function. Then \( f(\psi) = \psi(x_0) \).

Proof Let \( \{ \varphi_r \} \) be an enclosing sequence for the set \( \{ x_0 \} \). We consider two cases.

We first assume that \( \text{Supp} \psi \cap S_f = \emptyset \). In this case, there is an open neighbourhood of the point \( x_0 \) on which the function \( \psi \) vanishes. Then \( \psi \varphi_r = 0 \) for any \( r \) large enough, and at the same time we have \( f(\varphi_r) = 1 \). Hence, \( f(\psi) = 0 \).
§2. Symmetric products

We recall that by the symmetric product of a space $X$ one means the quotient space
$$\text{Sym}^n(X) = X^n/S_n = \{(x_1, \ldots, x_n) : (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \sim (x_1, \ldots, x_n), \sigma \in S_n\};$$
where $S_n$ is the group of all permutations of a set with $n$ elements.

The continuous functions on $\text{Sym}^n(X)$ correspond exactly to the continuous functions $f : X^n \to \mathbb{C}$ invariant under all permutations of the coordinates, that is, the symmetric functions.

Let us consider an analogue of the evaluation map
$$\mathcal{E} : \text{Sym}^n(X) \to \text{Hom}(C(X), \mathbb{C}), \quad \mathcal{E}(x_1, \ldots, x_n)(\varphi) = \varphi(x_1) + \ldots + \varphi(x_n).$$

We shall describe the image of this map by using equations. These equations are given by formulas which were first used by G. Frobenius [9, 10] and, more recently, by a number of authors, including A. Wiles [19], R. L. Taylor [18], H.-J. Hoehnke and K. W. Johnson [13], R. Rouquier [17], and L. Nyssen [15]. The formulae play an important role in the theory of multi-valued groups (see [3–6]). We follow the approach developed in [3].

Let $A$ be an associative algebra with unit over the field $\mathbb{C}$ of complex numbers and let $f : A \to \mathbb{C}$ be a linear tracial map (that is, $f(ab) = f(ba)$ for every $a, b \in A$). We introduce linear maps $\Phi_n(f) : A^\otimes n \to \mathbb{C}$ by setting $\Phi_1(f) = f$, $\Phi_2(f)(a_1 \otimes a_2) = f(a_1)f(a_2) - f(a_1a_2)$, and further by recurrence,
$$\Phi_{n+1}(f)(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = f(a_1)\Phi_n(f)(a_2 \otimes \ldots \otimes a_{n+1})$$
$$-\Phi_n(f)(a_1a_2 \otimes a_3 \otimes \ldots \otimes a_{n+1}) - \ldots - \Phi_n(f)(a_2 \otimes a_3 \otimes \ldots \otimes a_{n+1}a_1).$$

We note that a ring homomorphism $f : A \to \mathbb{C}$ satisfies the conditions $f(1) = 1$ and $\Phi_2(f) \equiv 0$.

**Definition 4** By a Frobenius $n$-homomorphism we mean a linear homomorphism $f : A \to \mathbb{C}$ satisfying the conditions $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$. 

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Our choice of name for the above homomorphisms is explained by the fact that the formula for the defining recursion first arose in the papers of G. Frobenius [9] and [10] in the case of group algebras of finite groups. For instance, the following result (in our notation) was obtained in [9].

Let $G$ be a finite group and let $A = \mathbb{C}[G]$ be its group algebra. Then the character $\chi : G \to \mathbb{C}$ of any $n$-dimensional linear representation of the group $G$ can be extended to a linear homomorphism $\chi : A \to \mathbb{C}$ such that $\chi(1) = n$ and $\Phi_{n+1}(\chi) \equiv 0$.

A generalization of the Gel’fand–Kolmogorov theorem to the case of symmetric products is given by the following result.

**Theorem 8** Let $X$ be a compact Hausdorff space. Then the image of the map

$$E : \text{Sym}^n(X) \to \text{Hom}(C(X), \mathbb{C})$$

is exactly the subspace of all Frobenius $n$-homomorphisms, that is, it is given by the equations $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

A more detailed statement and a constructive proof of this theorem which uses the above technique of enclosing sequences (and enclosing nets in the case of general compact Hausdorff space) can be found in [2].

Similarly to Definition 4, one can introduce the notion of Frobenius $n$-homomorphisms $f : A \to B$, where $B$ is an arbitrary commutative algebra. If $B$ has no zero divisors, then the Frobenius $n$-homomorphisms have properties which are important for our purposes. Therefore, in what follows we assume that $B$ is an integral domain. In [3], the proof of Theorem 8 is obtained as a consequence of a general result characterizing Frobenius $n$-homomorphisms $f : A \to B$.

Let $A$ be a commutative algebra. We denote by $S^n A$ the symmetric subalgebra of $A^{\otimes n}$. Every element $a \in S^n A$ can be represented in the form

$$a = \sum_{\sigma \in S_n} a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$$

and the product in $S^n A$ in the form

$$ab = \sum_{\sigma_1, \sigma_2 \in S_n} a_{\sigma_1(1)} b_{\sigma_2(1)} \otimes \ldots \otimes a_{\sigma_1(n)} b_{\sigma_2(n)}.$$ 

**Theorem 9** A linear map $f : A \to B$ is a Frobenius $n$-homomorphism if and only if $f(1) = n$ and the restriction of the homomorphism $\Phi_n(f) : A^{\otimes n} \to B$ to $S^n A$ gives a ring homomorphism

$$\frac{1}{n!} \Phi_n(f) : S^n A \to B.$$
We now present a combinatorial result heavily used in the proof of this theorem.

Let $X$ be a finite set and let $P(X)$ be the free Abelian group generated by the set of all partitions of the set $X$. We recall that every permutation $\sigma$ of the set $X$ defines a partition of $X$ given by the orbits of the action of the subgroup generated by $\sigma$. All the permutations defining the same partition have the same sign; for a given partition $\pi$ we denote by $\epsilon(\pi)$ the corresponding sign and by $n(\pi)$ the number of permutations generating $\pi$. We set

$$\chi(X) = \sum_\pi \epsilon(\pi)n(\pi) \pi \in P(X),$$

where the sum is taken over all partitions of the set $X$.

Let $\pi_1$ and $\pi_2$ be partitions of sets $X$ and $Y$, respectively. Then a natural partition $\pi_1 \pi_2$ of the disjoint union $X \sqcup Y$ is defined. Thus, the element $\chi(X)\chi(Y) \in P(X \sqcup Y)$ is well defined.

If $g : X \to Y$ is a surjection and $\pi$ is a partition of the set $Y$, then one can take the preimages of the parts of $\pi$ and obtain a partition $g^*\pi$; thus, an induced homomorphism $g^* : P(Y) \to P(X)$ is defined.

If the maps $i_1 : X \to Z$ and $i_2 : Y \to Z$ are embeddings and if $Z = i_1(X) \cup i_2(Y)$, then we say that $(Z, i_1, i_2)$ is an amalgamated union of the sets $X$ and $Y$. There is a map $q : X \sqcup Y \to Z$, which is surjective.

The following rather unexpected purely combinatorial result was obtained in [3] using polynomials which are solutions of a hypergeometric differential equation.

Lemma 10

$$\sum q^*\chi(Z) = \chi(X)\chi(Y),$$

where the sum is taken over all distinct amalgamated unions of $X$ and $Y$, including the disjoint union.

The main steps in the proof of Theorem 9 are as follows.

1. Let $X = (a_1, \ldots, a_n)$, where $a_k \in A$. Using the homomorphism $f : A \to B$, we define a homomorphism

$$f : P(X) \to B,$$

whose value on the partition $\pi = (P_1, \ldots, P_k) \in P(X)$, where $P_i = (a_{i_1}, \ldots, a_{i_q})$, is equal to

$$f(\pi) = \prod_{i=1}^k f(a_{i_1} \ldots a_{i_q}).$$
Then
\[ f(\chi(X)) = \Phi_n(f)(a_1, \ldots, a_n). \]

2. Let \( X = (a_1, \ldots, a_n) \) and \( Y = (b_1, \ldots, b_n) \) be disjoint sets. It follows from the definitions that
\[ f(\chi(X)\chi(Y)) = f(\chi(X))f(\chi(Y)). \]

3. Let \( f : A \to B \) be an \( n \)-homomorphism. Using Lemma 10 we obtain
\[ f(\chi(X)\chi(Y)) = \sum_{\sigma \in S_n} f(\chi(X \sqcup_\sigma Y)). \]

Here \( Z = X \sqcup_\sigma Y \) stands for an amalgamated union such that \( Z \) consists of \( n \) elements; in this case, the maps \( i_1 \) and \( i_2 \) are one-to-one, and therefore \( Z \) is defined by some permutation \( \sigma \).

4. The last step of the proof is to use the relation \( \Phi_{n+1}(f) \equiv 0 \) to establish the equality
\[ \sum_{\sigma \in S_n} \Phi_n(f)(a_1b_{\sigma(1)}, \ldots, a_nb_{\sigma(n)}) = n! \Phi_n(f)(ab). \]

Let us now denote by \( \Phi_n(A) \) the set of all Frobenius \( n \)-homomorphisms of the ring \( A \) to the field \( \mathbb{C} \) of complex numbers. By construction, \( \Phi_n(A) \) is an algebraic subvariety of the linear space \( A^* = \text{Hom}(A, \mathbb{C}) \) with the coordinates \( a : A^* \to \mathbb{C} \), where \( a(f) = f(a) \). We set
\[ \Phi_n(\mathbb{C}[u_1, \ldots, u_m]) = \Phi_n(m). \]

**Theorem 11** The map
\[ \mathcal{E} : \text{Sym}^n(\mathbb{C}^m) \to \text{Hom}(\mathbb{C}[u_1, \ldots, u_m], \mathbb{C}), \]
\[ \mathcal{E}(x_1, \ldots, x_n)(p) = p(x_1) + \ldots + p(x_n), \]
defines a homeomorphism
\[ \mathcal{E} : \text{Sym}^n(\mathbb{C}^m) \to \Phi_n(m). \]

Using this result, in Appendix B we present a description of an embedding
\[ \text{Sym}^n(\mathbb{C}^m) \subset \mathbb{C}^N, \quad N = \binom{n+m}{n} - 1, \]
in the context of the theory of multi-symmetric polynomials.
Proof The proof of the fact that the evaluation map $E$ is an embedding is immediate (see [3]). Let $f : \mathbb{C}[u_1, \ldots, u_m] \to \mathbb{C}$ be a Frobenius $n$-homomorphism. Then, by Theorem 9, the map

$$
\frac{1}{n!} \Phi_n(f) : S^n(\mathbb{C}[u_1, \ldots, u_m]) \to \mathbb{C}
$$

is a ring homomorphism. In what follows, the symmetric algebra $S^n(\mathbb{C}[u_1, \ldots, u_m])$ can be identified with the algebra of polynomial functions on the algebraic variety $\text{Sym}^n(\mathbb{C}^m)$, that is, with the algebra of multi-symmetric polynomials. Then, by Hilbert’s Nullstellensatz, there is a set of points $(x_1, \ldots, x_n) \in \text{Sym}^n(\mathbb{C}^m)$ such that for $a = \sum_{\sigma \in S_n} p_{\sigma(1)} \otimes \ldots \otimes p_{\sigma}(n)$ we have the formula

$$
a(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} p_{\sigma(1)}(x_1) \otimes \ldots \otimes p_{\sigma}(n)(x_n).
$$

Let us take for $a$ the element

$$
a = p \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes p \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes p,
$$

where $p \in \mathbb{C}[u_1, \ldots, u_m]$. Then

$$
\frac{1}{n!} \Phi_n(f)(p) = \sum_{k=1}^{n} p(x_k).
$$

On the other hand, using the fact that the homomorphism $\Phi_n(f)$ is linear and symmetric and applying the formula

$$
\Phi_n(f)(p \otimes 1 \otimes \ldots \otimes 1) = f(p)(f(1) - 1) \ldots (f(1) - (n - 1))
$$

(see [3]), we obtain

$$
\frac{1}{n!} \Phi_n(f)(a) = f(p)
$$

because $f(1) = n$, that is, $f(p) = \sum_{k=1}^{n} p(x_k)$, and hence $f = E(x_1, \ldots, x_n)$. This completes the proof of Theorem 11.

Theorem 12 Let $A$ be a finitely generated commutative algebra and let $V$ be the affine algebraic variety $\text{m-Spec}(A)$. Then the map

$$
E : \text{Sym}^n(V) \to \text{Hom}(A, \mathbb{C}), \quad E(f_1, \ldots, f_n)(a) = f_1(a) + \ldots + f_n(a),
$$

where $f_k : A \to \mathbb{C}$, $k = 1, \ldots, n$, are ring homomorphisms (whose kernels are the maximal ideals which are points of $V$), defines a homeomorphism

$$
E : \text{Sym}^n(V) \to \Phi_n(A).
$$
For a detailed proof of this result, see [3].

The proof of Theorem 8 using Theorem 9 can be carried out by the scheme of the proof of Theorem 11. One must also use that for a compact Hausdorff space \( X \) the ring \( C(X)^{\otimes n} \) is dense in the ring \( C(X^n) \) by the Stone–Weierstrass theorem (for the details, see [16] Proposition 1.10.21), and therefore the ring \( S^n(C(X)) \) is dense in the ring \( C(Sym^n(X)) \). This enables one to apply the Gel’fand–Kolmogorov theorem and to assign a point of the space \( Sym^n(X) \) to the ring homomorphism \( \frac{1}{n!}\Phi_n(f) : S^n(C(X)) \to \mathbb{C} \).

§3. Properties of \( n \)-homomorphisms

The results of the previous section show that Frobenius \( n \)-homomorphisms of commutative algebras have important applications. Thus, the study of the properties of these homomorphisms in this special case arose naturally. It turns out that in this investigation it is useful to apply another description of the defining equations, which was given in [4], [5].

Let \( A \) be a commutative algebra. Then the following assertions hold.

1. The value \( \Phi_n(a, \ldots, a) := \Phi_n(f)(a \otimes \ldots \otimes a) \), where \( a \in A \), is equal to the determinant of the matrix

\[
\begin{pmatrix}
  f(a) & 1 & 0 & 0 & \ldots & 0 \\
  f(a^2) & f(a) & 2 & 0 & \ldots & 0 \\
  f(a^3) & f(a^2) & f(a) & 3 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  f(a^n) & f(a^{n-1}) & \ldots & \ldots & f(a^2) & f(a) \\
\end{pmatrix}
\]

2. The value \( \Phi_n(f)(a_1, \ldots, a_n) = \Phi_n(f)(a_1 \otimes \ldots \otimes a_n) \) is a multilinear function of the variables \( a_1, \ldots, a_n \) and can therefore be obtained from \( \Phi_n(a, \ldots, a) \) by the standard polarization procedure.

3. Let us represent the permutation \( \sigma \in S_n \) in the form of a product of disjoint cycles of total length \( n \), say, \( \sigma = \gamma_1 \ldots \gamma_r \). Let \( \gamma = (i_1, \ldots, i_m) \) be a cycle. We set \( f_\gamma(a_1, \ldots, a_n) = f(a_{i_1} \ldots a_{i_m}) \). Then the following formula holds:

\[
\Phi_n(f)(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) f_{\gamma_1}(a_1, \ldots, a_n) \ldots f_{\gamma_r}(a_1, \ldots, a_n), \quad (3.1)
\]

where \( \varepsilon(\sigma) \) is the sign of the permutation \( \sigma \).

For instance, \( \Phi_3(f)(a_1, a_2, a_3) =

\[
f(a_1)f(a_2)f(a_3) - f(a_1)f(a_2a_3) - f(a_2)f(a_1a_3) - f(a_3)f(a_1a_2) + 2f(a_1a_2a_3).
\]
We note that formula (3.1) holds also in the case of tracial homomorphisms $f$ of a non-commutative algebra $A$. This formula arose in [7] in the case of matrix algebras $A$ and the trace homomorphism $f$.

It is often simpler to work with the formula (3.1) in its "diagonal" form and then to use polarization to obtain general formulae. For instance, $\Phi_3(f)(a, a, a) = s_3^3 - 3s_1s_2 + 2s_3$, where $s_1 = f(a)$, $s_2 = f(a^2)$, and $s_3 = f(a^3)$. We use the notation $s_k$ to stress the relationship with the classical Newton formula expressing the elementary symmetric functions $e_n = \sum_{t_1, \ldots, t_n} t_1^{i_1} \cdots t_n^{i_n}$ of the commuting variables $t_1, t_2, \ldots$ (the sum is taken over all sets $I_n = (i_1 < \ldots < i_n)$) as polynomials in the power sums $s_k = \sum t_i^k$.

Let us now describe the development of this relationship, denoting by $F_n$ the polynomial in the variables $s_k, k = 1, \ldots, n$, which becomes equal to $\Phi_n(f)(a, \ldots, a)$ when making the substitution $s_k = f(a^k)$. The formula

$$F_n = \sum_{\sigma \in S_n} \left( \prod_{k=1}^{n} ((-1)^{k+1}s_k)^{m_k(\sigma)} \right),$$

where $m_k(\sigma)$ is the number of cycles of length $k \geq 1$ in the expansion of the permutation $\sigma$ in the product of disjoint cycles, immediately follows from the definition of the polynomial $F_n(s_1, \ldots, s_n)$. Further, the number of elements of the group $S_n$ with $m_k$ cycles of length $k$ is equal to

$$n! / \prod_{k=1}^{n} (k^{m_k} m_k!).$$

Hence,

$$F_n = \sum_{m \in \Pi(n)} \prod_{k=1}^{n} \left( (-1)^{k+1} \frac{s_k}{k} \right)^{m_k},$$

where $\Pi(n)$ denotes the set of partitions of the number $n$ and $m = (m_1, \ldots, m_n)$ is a partition of the number $n$ such that $m_k$ is the multiplicity of the number $k$ in this partition. Thus, $\sum_{k=1}^{n} km_k = n$. Hence,

$$F_n t^n = \sum_{m \in \Pi(n)} \prod_{k=1}^{n} \left( (-1)^{k+1} \frac{s_k t^k}{k} \right)^{m_k},$$

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and we obtain the following assertion.

**Theorem 13**

\[ \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} = \exp \left( \sum_{k=1}^{\infty} (-1)^{k+1} s_k \frac{t^k}{k} \right). \]

This result implies the following recursion.

**Proposition 14**

\[ F_n = (n - 1)! \sum_{k=1}^{n} (-1)^{k+1} s_k \frac{F_{n-k}}{(n-k)!}. \]

**Proof** We set

\[ F(t) = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}. \]

Then Theorem 13 implies the relation

\[ F'(t) = F(t)(s_1 - s_2 t + s_3 t^2 - \ldots), \]

which immediately implies the desired formula.

Let us now give a characterization of the polynomials \( F_n \) using differential operators.

**Lemma 15** Let \( d = \sum_{r=2}^{\infty} r s_{r-1} \frac{\partial}{\partial s_r} \). Then the following assertions hold:

(a) \( \frac{\partial F_n}{\partial s_1} = nF_{n-1}; \)

(b) \( dF_n = -n(n - 1)F_{n-1}; \)

(c) \( \left[ \frac{\partial}{\partial s_k}, d \right] = (k + 1) \frac{\partial}{\partial s_{k+1}}; \)

(d) \( (\text{Ker} \ \frac{\partial}{\partial s_1}) \cap (\text{Ker} \ d) \) consists entirely of constants.

**Proof** (a) Differentiating the right-hand side of the relation in Theorem 13 with respect to \( s_1 \), we obtain \( \frac{\partial F(t)}{\partial s_1} = tF(t) \). Hence,

\[ \sum_{n=0}^{\infty} \frac{\partial F_n}{\partial s_1} \frac{t^n}{n!} = t \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}. \]
Equating the coefficients of like powers of \( t \), we obtain the result in (a).

(b) Applying the operator \( d \) to the same relation, we obtain

\[
dF(t) = F(t)[-s_1 t^2 + s_2 t^3 - s_3 t^4 + \ldots] = -t^2 F''(t).
\]

Equating the coefficients of \( t^n \), we obtain

\[
\frac{dF_n}{n!} = -\frac{F_{n-1}}{(n-2)!}.
\]

(c) \[
d \frac{\partial}{\partial s_k} - \frac{\partial}{\partial s_k} d = \sum_{r=2}^{\infty} r s_{r-1} \frac{\partial}{\partial s_r} \frac{\partial}{\partial s_k} - \frac{\partial}{\partial s_k} \sum_{r=2}^{\infty} r s_{r-1} \frac{\partial}{\partial s_k} = (k+1) \frac{\partial}{\partial s_{k+1}}.
\]

(d) Let \( \frac{\partial f}{\partial s_1} = 0 \). Then \( f = f(s_2, s_3, \ldots) \). If \( f \) belongs to the intersection of kernels of the operators \( \frac{\partial}{\partial s_1} \) and \( d \), then \( \frac{\partial}{\partial s_2} f = 0 \) by assertion (c).

Assertion (c) enables us to complete the proof of (d) by induction.

Lemma 15 immediately implies the following result.

**Theorem 16** The sequence of polynomials \( \{F_n(s_1, \ldots, s_n)\} \) is completely characterised by the following properties:

0) \( F_n(0) = 0, \ n = 1, 2, \ldots, \)

1) \( F_0 = 1, \)

2) \( \frac{\partial F_n}{\partial s_1} = n F_{n-1}, \)

3) \( dF_n = -n(n-1)F_{n-1}, \)

that is, the generating function \( F(t) = F(t; s_1, s_2, \ldots) \) is the unique solution of the system of equations

\[
\frac{\partial}{\partial s_1} F(t) = t F(t), \quad dF(t) = -t^2 \frac{\partial}{\partial t} F(t)
\]

under the initial condition \( F(0; s_1, s_2, \ldots) = 1. \)
§4. Frobenius algebras

Let $A$ be an associative algebra over $\mathbb{C}$. A linear map $f : A \to \mathbb{C}$ is said to be tracial (or trace-like) if $f(ab) = f(ba)$ for any $a$ and $b$ in $A$.

**Definition 5** By a Frobenius algebra we mean an algebra $A$ together with a tracial linear map $f : A \to \mathbb{C}$ such that the bilinear form

$$A \times A \to \mathbb{C}, \quad (a, b) \mapsto f(ab)$$

is non-degenerate.

We denote by $J(A)$ the Jordan algebra whose additive structure is that of $A$ and with the product

$$a \circ b = \frac{1}{2}(ab + ba).$$

Let us consider a basis $\{e_i : i \in J\}$ for $A$ and denote the corresponding structure constants by $a^k_{ij}$ (that is, $e_i e_j = \sum a^k_{ij} e_k$). Then the numbers $a^k_{(ij)} = \frac{1}{2}(a^k_{ij} + a^k_{ji})$ are the structure constants of the Jordan algebra $J(A)$ with respect to the same basis.

**Theorem 17** Let $(A, f)$ be a Frobenius algebra. Then the structure constants of the Jordan algebra $J(A)$ are completely defined by the homomorphisms $\Phi_k = \Phi_k(f)$, $k = 1, 2, 3$.

A result related to Theorem 17 was obtained in [12] under other assumptions including an assertion similar to the conclusion of Theorem 2.8 in [3]. Our theorem therefore seems to give a stronger result.

**Corollary 18** For a given homomorphism $f : A \to \mathbb{C}$ the linear maps $\Phi_k = \Phi_k(f) : A^{\otimes k} \to \mathbb{C}$, $k \geq 4$, are determined by the maps $\Phi_1, \Phi_2, \Phi_3$.

Corollary 18 implies the well-known result of [13] that a finite group is completely determined by its Frobenius $k$-characters for $k = 1, 2, 3$.

**Proof of Theorem 17** By definition,

$$\Phi_2(e_i, e_j) = f(e_i)f(e_j) - f(e_i e_j).$$

We set $R_{ij} = f(e_i e_j)$. Then

$$R_{ij} = \sum_r a^r_{ij} f(e_r) = f(e_i)f(e_j) - \Phi_2(e_i, e_j). \quad (4.1)$$
Similarly,
\[
\Phi_3(e_i, e_j, e_k) = f(e_i)f(e_j)f(e_k) - f(e_i)\sum_r a^r_{jk}f(e_r)
- f(e_j)\sum_r a^r_{ik}f(e_r) - f(e_k)\sum_r a^r_{ij}f(e_r) + \sum_{r,s} (a^r_{ij} + a^r_{ji})a^s_{rk}f(e_s).
\]

Hence,
\[
\sum_r (a^r_{ij} + a^r_{ji})R_{rk} = f(e_i)R_{jk} + f(e_j)R_{ik} - f(e_i)f(e_j)f(e_k) + \Phi_3(e_i, e_j, e_k) (4.2)
\]

It follows from the definition of the Frobenius algebra that the matrix
\[
R_{ij} = f(e_i e_j) = \sum_r a^r_{ij}f(e_r)
\]
is symmetric and non-degenerate. Regarding the equation (4.2) for fixed \(i\) and \(j\) as a system of linear equations with respect to the vector \((a^r_{ij}), r = 1, \ldots, n\), we see that this system has a unique solution. This completes the proof of the theorem.

A more explicit answer can be obtained for a commutative Frobenius algebra \(A\). We set
\[
R_i = f(e_i), \quad R_{ij} = f(e_i e_j), \quad R_{ijk} = f(e_i e_j e_k).
\]

In terms of the values of the homomorphisms \(\Phi_2\) and \(\Phi_3\) we obtain
\[
R_{ij} = \Phi_2(e_i, e_j) - R_i R_j, 2R_{ijk} = \Phi_3(e_i, e_j, e_k) + R_i R_{jk} + R_j R_{ik} + R_k R_{ij} - R_i R_j R_k.
\]
The matrix \(R_{ij}\) is invertible by the definition of Frobenius algebra. We denote by \(R^{ij}\) the matrix inverse to \(R_{ij}\). Direct calculations give the following explicit formula for the structure constants of the algebra \(A\).

**Proposition 19** The structure constants of a commutative Frobenius algebra are defined by the following formula:
\[
a^k_{ij} = \sum_m R_{ijm}R_{mk}^{\text{sym}}.
\]

**Proof** Using the formula
\[
R_{ijk} = f(e_i e_j e_k) = \sum_n f(a^n_{ij} e_n e_k),
\]
we obtain
\[
\sum_m R_{ijm}R_{mk}^{\text{sym}} = \sum_{m,n} a^n_{ij} R_{nm} R_{mk}^{\text{sym}} = a^k_{ij}.
\]
Proof of Corollary 18 It suffices to prove that, to evaluate the values of the homomorphism $\Phi_k$, one needs the structure constants $a^k_{ij}$ rather than $a^k_{ij}$. We use induction.

The only summand in the expansion of $\Phi_m(e_1, \ldots, e_m)$ (in terms of the values of the homomorphism $f$) which cannot be immediately expressed by using the values $\Phi_r$ with $r < m$ is

$$\sum_{\sigma \in S_{m-1}} f(e_1 e_{\sigma(2)} e_{\sigma(3)} \ldots e_{\sigma(m)}) = \frac{1}{m} \sum_{\sigma \in S_m} f(e_{\sigma(1)} e_{\sigma(2)} \ldots e_{\sigma(m)}).$$

The multiplication in the Jordan algebra $J(A)$ may not be associative; however, the associator can be expressed in terms of commutators, namely,

$$4(((a \circ b) \circ c - a \circ (b \circ c)) = abc + bac + cab + cba - abc - acb - bca - cba$$

$$= [b, ac] - [b, ca] = [b, [a, c]].$$

Since the homomorphism $f$ is tracial, it follows that $f([a, b]) = 0$, and we see that

$$f((a \circ b) \circ c) = f(a \circ (b \circ c)).$$

Hence, on iterated products of elements of the algebra $J(A)$ the homomorphism $f$ behaves as if this algebra is associative. Thus,

$$2^{m-1} \sum_{\sigma \in S_m} f(e_{\sigma(1)} \ldots e_{\sigma(m)}) = \sum_{\sigma \in S_m} f(e_{\sigma(1)} \circ \ldots \circ e_{\sigma(m)}).$$

In the case of finite groups we obtain the following result.

Let $G$ be a finite group with the set of elements given by $\{g_1, \ldots, g_n\}$. We consider the Frobenius algebra $\mathbb{C}G$ with the structure map $f = \frac{1}{n} \chi : \mathbb{C}G \to \mathbb{C}$, where $\chi$ is the character of the regular representation of the group $G$.

Corollary 20 The linear maps $\Phi_1 = f$, $\Phi_2$, and $\Phi_3$ determine the following parts of the structure of the group algebra $\mathbb{C}G$:

1) $\Phi_1$ determines the identity element $e$ of $G$;

2) $\Phi_2$ determines inverse elements, that is, it distinguishes the pairs $(g_i, g_j)$ such that $g_i g_j = e$;

3) $\Phi_1$, $\Phi_2$, and $\Phi_3$ define the structure constants of the Jordan algebra $J(\mathbb{C}G)$. 
Proof The trace $\chi$ of the regular representation takes the value $n$ at the identity element $e$ of $G$ and the value 0 at any other element, which obviously implies the assertion 1) because $\Phi_1 = f = \frac{1}{n}\chi$. Further, it follows from (4.1) that

$$\Phi_2(h, g) = \begin{cases} -1 & \text{if } hg = e, \ h \neq e; \\ 0 & \text{otherwise}. \end{cases}$$

We thus find the pairs of mutually inverse elements. Let us rewrite the result thus obtained in terms of structure constants. We order the elements of $G$ in such a way that $g_1 = e$ and write $g_i g_j = \sum_{k=1}^{n} a_{ij}^k g_k$. In the case under consideration, the quantities $a_{ij}^k$ take the values 0 and 1, and $\sum_{k=1}^{n} a_{ij}^k = 1$ for any pair $(i, j)$. Assertion 1) means that

$$a_{ij}^k = a_{kj}^i = \delta_{ik}.$$

Assertion 2) recovers the values

$$a_{ij}^1 = -\Phi_2(g_i, g_j) \quad \text{if} \quad i + j > 2.$$

We now find the information concerning the structure constants $a_{ij}^k$ that is determined by the map $\Phi_3$. By (4.2) we have

$$\Phi_3(g_i, g_j, g_k) = \sum_r (a_{ij}^r + a_{ji}^r)a_{rk}^1 \quad \text{if} \quad i + j + k > 3.$$

We know that the matrix $a_{r,k}^1$ is invertible, which enables one to complete the proof.

Theorem 21 Let $G$ be a finite group and let $\chi$ be the character of the regular representation of $G$. The linear maps $\chi$, $\Phi_2(\chi)$, and $\Phi_3(\chi)$ determine the group $G$ uniquely up to isomorphism.

Proof Using Corollary 20, we immediately see that the homomorphisms $\chi$, $\Phi_2(\chi)$, and $\Phi_3(\chi)$ define the multiplication on the set of elements of the group $G$ up to permutation of the factors.

The following lemma and the fact that that a group is isomorphic to its opposite (using the map $g \mapsto g^{-1}$) enables one to complete the proof of this theorem.

Lemma 22 (Mansfield [14]) Let $G$ be a finite group in which the multiplication rule is not known precisely but the set $\{gh, hg\}$ is known for each pair of elements $g, h \in G$. Then one can recover the set of functions $\{m, m^o : G \times G \to G\}$, where $m(x, y) = xy$ and $m^o(x, y) = yx$.

The proof of this lemma presented in [14] uses an elementary (but tricky) examination of cases. For completeness of the exposition, we present a proof in Appendix A (§ 5).
In [14] this lemma was applied to prove the result of E. Formanek and D. Sibley [8] that the group determinant of a finite group (which was first considered by Dedekind) defines a group.

The group determinant of a group \( G \) is defined as follows.

Choose a one-to-one correspondence between the set of elements \( \{g_1, \ldots, g_n\} \) of the group \( G \), where \( g_1 = e \), and the set of commuting variables \( \{x_1, \ldots, x_n\} \).

The group determinant is the determinant of the matrix obtained from the multiplication table of the group by the substitution \( g_i \to x_i \). The group determinant is a homogeneous polynomial of degree \( n \) in \( \{x_1, \ldots, x_n\} \) with integral coefficients.

Introduce the matrix \( M_G = (m_{ij} = x_k) \), where the correspondence \((i, j) \to k\) is given by \( g_i g_j^{-1} = g_k \). It is clear that \( M_G \) differs from the multiplication table only by the order of the columns, and hence \( D_G = \pm \det M_G \). We note that \( \det M_G = x_1^n + \ldots \).

Let us consider the \( n \)-dimensional linear space \( \mathbb{C}^n \simeq \mathbb{C}G \) and the right regular representation of the group \( G \) in the basis \( \{g_1, \ldots, g_n\} \):

\[
T : \mathbb{C}^n \to \mathbb{C}^n, \quad T(g_k) g_k = g_k g^{-1}.
\]

We denote by \( T_i \) the matrix of the action of the operator \( T(g_i) \) in this basis. Then

\[
M_G = \sum x_i T_i.
\]

Consider the character of the representation \( T \),

\[
\chi : \mathbb{C}G \to \mathbb{C}
\]

and extend it to a linear \( \mathbb{C}[x_1, \ldots, x_n] \)-homomorphism

\[
f : \mathbb{C}G[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n].
\]

**Lemma 23** We set \( a = \sum_{i=1}^n x_i g_i \in \mathbb{C}G[x_1, \ldots, x_n] \). Then

\[
\Phi_n(f)(a, \ldots, a) = D_G.
\]

The expansion of the group determinant into irreducible factors obtained by Frobenius in [10] can be obtained immediately from the expansion of the regular representation \( T \) into irreducible representations: let \( \chi = \chi_1 + \ldots + \chi_q \) be an expansion of the regular representation \( T \) into irreducible representations and let \( n_i \) be the dimension of the representation with the character \( \chi_i \); then

\[
D_G = \prod_{i=1}^q \Phi_n(\chi_i)^{n_i}.
\]
§ 5. Appendix A. Proof of Mansfield’s lemma

As we promised, we give here a proof of the following result.

**Lemma 24** Let \( G \) be a group with the multiplication \((x, y) \to xy\). In this case, if \( \ast \) is an associative multiplication on \( G \) related to the original multiplication in such a way that for every pair \( x, y \in G \) the product \( x \ast y \) is equal to either \( xy \) or \( yx \), then \( x \ast y = xy \) for any \( x, y \in G \) or \( x \ast y = yx \) for any \( x, y \in G \).

We reformulate the lemma as follows. Let \( A = \{(x, y) : x \ast y = xy\} \) and \( B = \{(x, y) : x \ast y = yx\} \). Then it follows from the conditions of the lemma that \( A \cup B = G \times G \), and one must prove that \( A = G \times G \) or \( B = G \times G \).

It is clear that the set \( A \cap B \) is symmetric.

The proof of the lemma follows Exercise 26 in § 4 in [1]. For a given pair \( x, y \in G \) we have the following assertion.

**Fact 25** \( x \ast y = y \ast x \iff xy = yx \).

The implication \( xy = yx \implies x \ast y = y \ast x \) is obvious. In the converse direction, let \( x \ast y = y \ast x \). In this case, without loss of generality one can set \( x \ast y = xy \). If this is not the case, then \( y \ast x = x \ast y = yx \), and we can transpose the variables \( x \) and \( y \).

In what follows we assume that \( xy \neq yx \), because otherwise there is nothing to prove. Let us consider the element \( x \ast x \ast y = x \ast xy \) and assume that it differs from \( x^2y \). At the same time it follows from the above that \( x \ast x \ast y = x \ast xy \) is equal to \( xyx \) or \( x^2y \). Thus, \( yx^2 = xyx \), and hence \( yx = xy \). The contradiction thus obtained shows that \( x \ast x \ast y = x^2y \).

We have \( x \ast y \ast x \ast y = xy \ast x \ast y = (xy)^2 \). On the other hand, by assumption, the element \( x \ast y \ast x \ast y = x \ast x \ast y \ast y = x^2y \ast y \) is equal to \( x^2y^2 \) or \( yx^2y \), that is, \( (xy)^2 = x^2y^2 \) or \( (xy)^2 = yx^2y \). In both the cases we arrive at the relation \( xy = yx \), which is a contradiction.

**Corollary 26** \( x \ast y = xy \iff y \ast x = yx \), that is, the set \( A \) is symmetric.

**Proof** Let \( x \ast y = xy \) and \( y \ast x = yx \). Then it follows from Fact 25 that \( xy = yx \). Thus, \( x \ast y = xy \iff y \ast x = yx \). The converse implication follows by symmetry.

Since both the sets \( A \) and \( A \cap B \) are symmetric and \( A \cup B = G \times G \), we immediately see that \( B \) is a symmetric set. Thus, we have proved the following assertion.

**Corollary 27**
\( x \ast y = yx \iff y \ast x = xy \).

**Fact 28** \( x \ast y \ast x = xyx \) for every \( x, y \in G \).
Proof Let $xy = yx$, then $x * y * x = xy * x = xyx$ or $x^2y$. The desired relation holds because $xyx = x^2y$. On the other hand, suppose $xy \neq yx$ and $x * y * x \neq xyx$. Then only the following two cases are possible.

a) $x * y = xy$ and $y * x = xy$. Then $x * y * x = xy * x = x^2y$ or $xyx$. Since the multiplication is associative, it follows that $x * y * x = x * yx$ is equal to $xyx$ or $yx^2$. Thus, $x * y * x = x^2y = yx^2$. Moreover, $x * x * y = x^2 * y = x^2y = x * y * x$. Hence, $x * y = y * x$, which contradicts our assumptions.

b) $x * y = yx$ and $y * x = xy$. We again have $x * y * x = xy * x = yx^2$ and $x * y * x = x * xy = x^2y$. Thus, $x * x * y = x * y * x$ and $x * y = y * x$. This contradiction completes the proof of Fact 28.

We consider $G \times G$ as a square array with rows $R_x = \{(x, y) : y \in G\}$.

Fact 29 For each $x \in G$ we have either $R_x \subset A$ or $R_x \subset B$.

Proof Suppose that Fact 29 fails. Then there is an element $x$ such that $R_x$ intersects both the sets $G \times G \setminus A$ and $G \times G \setminus B$, that is, there are elements $y, z \in G$ such that $x * y = yx \neq xy = y * x$ and $x * z = xz \neq zx = z * x$. Let us show that under these assumptions we have $yxz = zxy$. Two cases are possible here.

a) $y * z \neq z * y$. As was shown above, in this case $yz \neq zy$. The element $z * x * y$ is equal to either $zxy$ or $yzx$. This element is also equal to $z * yx = yzx$ or $yzx$. However, $yzx \neq zyx$, $yxz \neq zyx$, and $yzx \neq zyx$, and thus $z * x * y = zxy = yxz$.

b) $y * z = z * y$ (= $yz = zy$). We show that the assumption $yxz \neq zxy$ leads to a contradiction. We consider the element $z * x * y$. It is equal to $zx * y$ which equals $yzx$ or $zyx$, but it is also equal to $z * yx = zyx$ or $yzx$. Under our assumptions and using that $zxy \neq zyx$ and $yxz \neq yzx$, we obtain $z * x * y = zyx = zyx$. Now we consider the element $x * z * y$. It differs from $z * y * x = yzx$. However, it is equal to $x * y * z = x * yz = xyz$ or to $yzx$. Hence, $x * z * y = xyz = zxy$. Finally, we consider the element $z * y * x$. It differs from $z * x * y = yzx$ and is equal to $yz * x = yzx$ or $yxz$. Hence, $z * y * x = xyz = zxy$. Thus, $x * (z * y) = (z * y) * x$, that is, $x * zy = zyx$, and therefore $xyz = zxy$. This implies that $z * x * y = x * z * y$, and thus contradicts the condition $x * z \neq z * x$.

Let us finally consider the element $x * z * x * y$. According to Fact 28, it is equal to $xx * x = xzxy$ or $yxzz$. On the other hand, it is equal to $xz * x = xzxy$ or $yxzz$. We must again consider two cases.

a) $x * z * x * y = xzxy$ is equal to $xzyx$ or $yxzz$. In the first case we obtain $zy = xy$. The other case, together with the above calculations, gives $yxzz = xzxy = xzyx$, which gives $xy = xy$. In both cases we get that $xy = yx$ which is a contradiction.

b) $x * z * x * y = yzxx$ is equal to $xzyx$ or $yxzz$. In the second case we
have \( xz = zx \), that is, a contradiction. Thus, \( xzyx = yxzx = zxyx \), which also gives \( xz = zx \). This completes the proof of Fact 29.

According to Fact 29, for every \( x \) we have either \( R_x \subset A \) or \( R_x \subset B \). Suppose that \( R_x \subset A \) but \( R_x \not\subset B \), and that \( R_u \subset B \) but \( R_u \not\subset A \). Then \( x*y = xy \) for any \( y \), and there is a \( z \) such that \( xz \neq zx \). Similarly, \( u*v = vu \) for any \( v \), and there is a \( w \) such that \( uw \neq wu \).

Consider the element \( x*u = xu \). It is equal to \( ux = u*x \), and \( Rxu \subset A \) or \( Rxu \subset B \). Suppose that \( Rxu \subset A \). We then consider the element \( x*u * w = xu * w = xuw \), which must also be equal to \( x * w u = xwu \), which is a contradiction. We now assume that \( Rxu \subset B \) and consider the element \( z * x * u = z * xu = xu * z = u * xz \), which is also equal to \( x * u = uzx \). This implies that \( xz = zx \), a contradiction. This proves Mansfield’s lemma.

§6. Appendix B. The algebra of multi-symmetric polynomials

Let \( n > 1 \). Recall that a polynomial \( p(x_1, \ldots, x_n) \), where \( x_k = (x_{1k}, \ldots, x_{mk}) \in \mathbb{C}^m \), is said to be multi-symmetric if it is invariant with respect to all permutations of the set of arguments \( (x_1, \ldots, x_n) \). The algebra of polynomial functions on the algebraic variety \( \text{Sym}^n(\mathbb{C}^m) \) can be canonically identified with the algebra of multi-symmetric polynomials, which we denote by \( \mathcal{SP}^n(\mathbb{C}^m) \). For any set of non-negative integers \( \omega = (i_1, \ldots, i_m) \) we introduce the multi-symmetric Newton polynomials \( p_\omega \),

\[
p_\omega(x_1, \ldots, x_n) = \sum_{k=1}^n x_{1k}^{i_1} \cdots x_{mk}^{i_m},
\]

for \( m = 1 \) these are the classical Newton polynomials.

We set \( |\omega| = i_1 + \ldots + i_m \). Let \( \{z_\omega\} \), \( \omega \in \mathbb{Z}_{\geq 0}^m \), be a set of commuting variables graded by the rule \( \deg z_\omega = |\omega| \). We consider the graded polynomial ring \( L = \mathbb{C}[z_\omega] \). We introduce in \( L \) a system of homogeneous polynomials \( \{\mathcal{F}_{\omega_1, \ldots, \omega_j}\} \), \( \deg \mathcal{F}_{\omega_1, \ldots, \omega_j} = |\omega_1| + \ldots + |\omega_j| \), by the “Frobenius-type” formulas,

\[
\mathcal{F}_{\omega_1} = z_{\omega_1}, \quad \mathcal{F}_{\omega_1, \omega_2} = z_{\omega_1} z_{\omega_2} - z_{\omega_1 + \omega_2}
\]

and by recurrence:

\[
\mathcal{F}_{\omega_1, \ldots, \omega_j, \omega_{j+1}} = z_{\omega_1} \mathcal{F}_{\omega_2, \ldots, \omega_j, \omega_{j+1}} - \mathcal{F}_{\omega_1+\omega_2, \omega_3, \ldots, \omega_j, \omega_{j+1}} - \cdots - \mathcal{F}_{\omega_1, \ldots, \omega_j, \omega_{j+1}}.
\]

Following the above scheme (see § 3), one can readily obtain an explicit expression for the polynomial \( \mathcal{F}_{\omega_1, \ldots, \omega_j} \). In what follows it is important to note that the linear part of the polynomial \( \mathcal{F}_{\omega_1, \ldots, \omega_j} \) is equal to
\((-1)^{j-1}(j-1)!z_{\omega_1+\ldots+\omega_j}\).

The fact that the ring \(L\) can be canonically identified with the polynomial ring on the linear space \(\text{Hom}(\mathbb{C}[u_1, \ldots, u_m], \mathbb{C})\) enables one to pass to the homomorphism of polynomial rings which is induced by the evaluation map \(\mathcal{E}\) and we obtain the following result from Theorem 11.

**Theorem 30** For given \(n\) and \(m\) the ring homomorphism

\[\mathcal{E}^* : L \to \mathcal{S}P^n(\mathbb{C}^m)\]

is an epimorphism.

If \(\mathcal{SYZ}(m, n)\) is the kernel of the homomorphism \(\mathcal{E}^*\), then \(\mathcal{SYZ}(m, n)\) is the ideal of the ring \(L\) generated by the polynomials \(\mathcal{F}_{\omega_1, \ldots, \omega_{n+1}}\) such that \(|\omega_1|, \ldots, |\omega_{n+1}| > 0\).

Consider the sub-ring \(L_n \subset L\) generated by the elements \(\{z_\omega\}\) with \(|\omega| < n + 1\). Using the above form of the linear part of the polynomial \(\mathcal{F}_{\omega_1, \ldots, \omega_{n+1}}\), we obtain the following assertion.

**Corollary 31** The restriction of the homomorphism \(\mathcal{E}^*\) to \(L_n\) gives an epimorphism

\[\mathcal{E}^*_n : L_n \to \mathcal{S}P^n(\mathbb{C}^m)\].

We denote by \(\text{Syz}(m, n)\) the kernel of the homomorphism \(\mathcal{E}^*_n\). Then

\[\text{Syz}(m, n) = \mathcal{SYZ}(m, n) \cap L_n\].

By construction, \(L_n\) is the polynomial ring on \(\mathbb{C}^N\), where \(N = \binom{n + m}{n} - 1\).

We obtain the following assertion.

**Corollary 32**

\[\text{Sym}^n(\mathbb{C}^m) \sim \text{Spec}(L_n / \text{Syz}(n,m)) \subset \mathbb{C}^N\].

In conclusion we note that the assertion claiming that the homomorphism \(\mathcal{E}^*\) is onto is equivalent to the first fundamental theorem of the classical invariant theory claiming that a multi-symmetric polynomial in \(n\) vector arguments can be expressed (but not uniquely) as a polynomial in the multi-symmetric Newton polynomials \(p_\omega, |\omega| \leq n\). The classical case \(m = 1\) is the only case in which this expression is unique.
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