General approach to tunable critical phases with two coupled chains

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Critical phase (CP) with multifractal wave functions has attracted much attention in the past decades. However, the underlying mechanism for this phase is still ambiguous. Here we propose that the coupling between the localized and the extended states in their overlapped spectra can provide a general recipe for this phase with tunable structures. We demonstrate this picture using two models. In the first model, we show that the CP can be realized in the overlapped spectra with quasiperiodic potential, in which the CP regime can be tailored by the offset between the two chains, yielding tunable CP. This phase is insensitive to the forms of inter-chain couplings and quasiperiodic potentials. In the second model, we consider the CP by a disordered flat bands coupling with an extended band. We show that the localized states in the flat bands turn into critical too. Finally, we account for the emergence of this phase as a result of unbounded potential, which yields singular continuous spectra and excludes the extended states. Our approach opens a remarkable avenue for various CPs with tailored structures, which have wide applications in higher-dimensional single-particle CPs and many-body CPs.

**Introduction:** Decades after its discovery [1], Anderson localization (AL) has remained one of the most active developing fields in physics [2–17]. One of the central problems in this field is the multifractality of wave functions [18–20], which can be found at states on the phase boundaries of a wide variety of models [21–34]. Moreover, these multifractal states form a critical phase (CP) in some quasiperiodic models [35–43] and some disordered flat bands systems [44–54]. Especially, the CP in the quasiperiodic system becomes an intriguing extended but non-thermal many-body CP when the interaction is considered [55, 56]. However, while the CP represents an important phenomenon in disorder models, it is generally studied in some particularly designed models and a general mechanism for this phase is still lacking. Their experimental observation is also challenging.

Here we propose a general approach to the CP with tailored structures. Our approach is based on the coupling between the localized states and the extended states by using two coupled chains. This idea comes from two intuitive motivations. Firstly, the states in the CP have zero Lyapunov exponents with large spatial fluctuation at all length scales, which are recognized as extended states globally, yet localized states locally in some sense [34]. Secondly, this coupling will not yield the coexistence of localized states and extended states; otherwise, the overlapped spectra may yield an infinite number of mobility edges in the thermodynamic limit, leading to ill-defined mobility edges. While coupled chains with conventional AL states don’t have critical states in one dimension [57–59], the coupled quasiperiodic chains [60] and flat-band chains support this mechanism with possible CP. We find that: (I) The direct coupling between the extended states and the localized states with quasiperiodic potentials in the overlapped spectra can yield a novel CP with tailored structure. Wen duality is introduced between the two chains, all states become multifractal, independent of system parameters; (II) The CP can be realized in the flat bands interacting with the extended states. (III) This phase may be related to a broad class of models with unbounded potential, which excludes the continuous spectra for extended states. These results indicate the generality of our mechanism for CP. Our approach can not only greatly broaden the family of physical models for CP, but also have important applications in many-body CPs and higher dimensional CPs.

![Diagram](https://via.placeholder.com/150)

**FIG. 1.** General approach to CP. (a) States with large spatial fluctuation can be naturally realized by coupling between localized and extended states. (b) Model for CP based on two coupled chains, one of which is fully extended and the other is fully localized without inter-chain coupling. (c) The CP may be found in their overlapped regime when $L \to \infty$.

**Tunable CP with quasiperiodic potentials:** We consider two coupled chains as (see Fig. 1 (b))

$$H = H_{AAH} + H_{free} + H_{c},$$  \hfill (1)
where $H_{\text{free}} = \sum_m (b_m^\dagger b_{m+1} + \text{h.c.}) + M \sum_m b_m^\dagger b_m$ is the free chain with $M$ being offset potential, $H_{\text{AAH}} = \sum_m (a_m^\dagger a_{m+1} + \text{h.c.}) + 2V \cos(2\pi a m)a_m^\dagger a_m$ is the Aubry-André-Harper (AAH) chain, and $H_c = t_v \sum_m (a_m^\dagger b_m + \text{h.c.})$ is their coupling. Here, $t_v$ is the inter-chain coupling and $V$ is the potential depth. The AAH chain undergoes AL transition from the extended phase ($V < 1$) to the localized phase ($V > 1$). Therefore $H_c$, whose form can be more inclusive [61], induces coupling between extended and localized states for CP when $V > 1$. We solve this Hamiltonian with periodic boundary condition by approximating $\alpha_m = F_{n-1}/F_n$ with $F_n$ is the Fibonacci number, with $\alpha = \lim_{n \to \infty} \alpha_n = (\sqrt{5} - 1)/2$. The total number of sites is $L = 2F_n$. The wave functions are characterized by the fractal dimensions (FD) $\tau(L) = -\log(\text{IPR})/\log(L)$ with inverse participation ratio (IPR) defined as [2, 27, 30, 34]

$$\text{IPR} = \sum_m |\psi_j^m|^4 \sim L^{-\tau},$$

where $H \psi_j = \epsilon_j \psi_j$. With this, we can identify the extended phase with $\lim_{L \to \infty} \tau(L) = 1$ [62], localized phase with $\lim_{L \to \infty} \tau(L) = 0$, and CP with $0 < \lim_{L \to \infty} \tau(L) < 1$. This exponent reflects the extension of the states and their spatial fluctuations with increasing system size.

The results for $t_v = 0.5$ with various disorder potential are presented in Fig. 2 (a), in which the spectra of $H_{\text{free}}$ overlap with the middle branch of $H_{\text{AAH}}$. The results for $t_v = 0$, $t_v = 0.1$, and $t_v = 0.5$ are presented in Fig. 2 (b)-(d). When $t_v = 0$, the system is made by extended states with $\tau > 0.8$; and localized states with $\tau < 0.2$ [60]. With finite $t_v$, we find that the overlapped regime $|E| < 0.67$ have $0.2 < \tau < 0.8$, while in the un-overlapped regime, the fractal dimension is unchanged. Mobility edges between these states are found at $E = \pm 2, \pm 0.67$, whose types are different from that in the literature [63–68].

We emphasize that the overlapped regime can be tuned by the offset potential between the two chains, while these features are unchanged.

![Fig. 2](image_url)

**FIG. 2.** (a) FD $\tau$ of all wave functions for the coupled chains model (1) with $\alpha = F_{19}/F_{21} = 377/610$, $M = 0$ and $t_v = 0.5$ versus potential depth $V$. The black lines are the edges of the middle sub-band of $H_{\text{AAH}}$. (b)-(d) $\tau$ versus energy $E$ with $V = 2$ and $t_v = 0$ (left), 0.1 (middle) and 0.5 (right). The vertical dashed lines at $E = \pm 2, \pm 0.67$ are the mobility edges.

Further, we verify the criticality of the overlapped regime in Fig. 2. We consider the scaling index $\alpha$ [34, 39, 40] for state $\psi_j$, which constructs a probability space $\mathcal{P} = \{p_m = |\psi_j^m|^2 \mid m = 1, \ldots, L\}$, then

$$p_m \sim L^{-\alpha(m).}$$

We define the minimum scaling index $\alpha_{\min} = \min(\alpha_m)$, with $\alpha_{\min} = 1$, $0 < \alpha_{\min} < 1$ and $\alpha_{\min} = 0$ for extended phase, CP and localized phase, respectively, in limit $L \to \infty$. This scaling law is shown in Fig. 3 (a) and (b) based on the averaged $\overline{\alpha_{\min}}$ in a finite energy interval, finding $\overline{\alpha_{\min}}(L) - \overline{\alpha_{\min}}(\infty) \propto 1/n$. In the overlapped regime C, we find $\overline{\alpha_{\min}}(\infty)$ approaches 0.5 for CP. However, in A, it approaches unity, and in B, approaches zero. This result is consistent with the previous analysis based on $\tau$. Furthermore, in order to rule out the possibility that $\overline{\alpha_{\min}} \sim 0.5$ comes from the coexistence of extended states and localized states (such as that for $t_v = 0$), we examine the distribution of $\alpha_{\min}$ in regime C, finding that the coexistence phase exhibits bimodal distribution while CP...
exhibits unimodal distribution, indicating all states in regime C exhibit multifractality (see S4 in Ref. 61).

This CP in quasiperiodic models is also identified by its singular continuous spectra [69–73], with level spacing distribution given by a inverse power law [40, 71, 73]

\[ P_{\text{int}}(s) \sim s^{3-\beta}. \]

We examine this distribution in regimes A, B, C with \( t_v = 0.5 \) and \( V = 2 \), finding \( \beta \sim 1.37 \) (Fig. 3 (c)) in regime C, the level spacing distribution in regime A or B, is totally different, as identified by Poisson or cosine-band statistic [71]. The \( \beta \) has been calculated at the dual point of the AAH model [73], yielding \( \beta = 1.5009 \); and the p-wave superconductor with quasi-periodic disorder [40], yielding \( \beta \in (1.48, 1.52) \). Here we have \( \beta \in (1.2, 1.4) \), depending strongly on the system parameters. We also note that for \( t_v \sim 0.1 \), while \( \tau \) and \( \alpha_{\text{min}} \) have already shown multifractal behavior, it can not be reflected from this exponent due to its finite size effect.

The strong spatial fluctuation of the wave functions, as mentioned in the introduction, is the most direct evidence for this critical states. Therefore, we plot \( \log |\psi_{jm}|^2 \) versus site index \( m \) for three states in Fig. 3 (d). We see that in regime A, the state exhibits an exponential decay tail, and in regime B it is uniformly distributed in the whole site. However, in regime C, large spatial fluctuation exists, accounting for the multifractal behaviors The similar features are shown in the off-diagonal AAH model [37]. And the same CP can be realized with different inter-chain couplings \( H_c \) and quasiperiodic potentials [61], indicating the generality of this approach for CP.

To further demonstrate the generality of our approach, we couple two AAH chains, in which all states can be critical by duality (S7 in Ref. 61). The Hamiltonian reads as

\[ H = H_{\text{loc}} + H_{\text{ext}} + H_c, \]

where \( H_{\text{loc}} = \sum_m a_m^\dagger a_m + a_{m+1}^\dagger a_m + V b_m^\dagger b_m \) for localized chain and \( H_{\text{ext}} = \sum_m V b_m^\dagger b_{m+1} + V b_{m+1}^\dagger b_m + g_1(m) a_m^\dagger a_m \) for extended chain, with \( g_1(m) = 2V \cos(2\pi am) + M \) for localized and \( g_2(m) = 2V \cos(2\pi am) - M \) for extended chain. The results are summarized in Fig. 4. By tuning \( M \), the overlapped spectra and the associated CP are changed. Then, we calculate the wave packet expansion dynamics, which can be performed in experiments [8, 74]. We consider the following initial Gaussian packet \( \gamma = \text{loc, ext} \) |\( \psi^\gamma(0) \rangle \sim \sum_m e^{-(m-k)^2/(2\sigma^2)} (\delta_{\gamma,\text{loc}} c_m + \delta_{\gamma,\text{ext}} b_m^\dagger) |0\rangle \), with \( \beta = 5 \). The mean-square of the packet \( \sigma_x(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \) with \( x = \sum_m m(a_m^\dagger a_m + b_m^\dagger b_m) \) in the long-time limit yields

\[ \sigma_x(t) \sim t^\kappa. \]

In general, \( \kappa = 0 \), \( \kappa = 1 \) and \( 0 < \kappa < 1 \) correspond to localized phase, extended phase and CP, respectively [55, 75]. In Fig. 4 (c), we show the log-log plot of \( \sigma_x(t) \), suggesting these features for several \( M \) and \( t_v \). We find that in the presence of inter-chain duality (\( M = 0 \)), sub-diffusion of the packet is found, with \( \kappa = 0.43 \). This is a unique signature of criticality [75]. However, when the duality is broken by \( M \neq 0 \), the CP is found only in the overlapped spectra, while the un-overlapped regime are unchanged, yielding ballistic diffusion with \( \kappa = 1 \).

**CP in disordered flat band model:** Can this CP also be realized without quasiperiodic potentials? The answer is yes. We consider the CP in the flat band model with random potential, which induce delocalization via the inverse AL effect [44–46]. It is related to the impedance network [50], random matrices [48], and disorder-induced decay [51]. We consider the fate of the following model

\[ H = \sum_{m=0}^{L/2} 2b_m^\dagger b_{m+1} + h_m a_m^\dagger a_m + t_v a_m^\dagger b_m + h.c., \]

with \( h_m \in U(-V/2, V/2) \) a uniform random variable. Without \( t_v \), the \( a \)-chain can always be regarded as localized, thus the inter-chain \( t_v \) induces coupling between localized and extended states (see Fig. 5 (a)). We set \( t_v = 0.1 \) for weak coupling and find that the overlapped regime can be controlled by disorder strength, in which the FD \( \tau \in (0, 0.8) \) in the inverse trangle regime, indicating of criticality. A detailed plot of these results for \( V = 11 \) and \( V = 4 \) are shown in Fig. 5 (b) - (c). The localization and criticality from wave functions are shown in Fig. 5 (d). Furthermore, we consider the averaged FD in a finite energy interval \( \tau = \frac{1}{N_{\text{total}}} \sum_n \tau_n \) and study its
scaling as a function of chain length $L$, which is shown in Fig. 5 (e), finding that in the localized, critical (overlapped regime) and extended phases, $\lim_{L \to \infty} \tau \to 0$ (for $W = 11$, $E < -4.1$), finite ($\sim 0.3$) and 1 (for $W = 4$, $E < -2.1$), respectively. The level spacing is no longer given by Eq. 4, with distribution between Poisson and Wigner-Dyson statistics; see some more evidences for CP in flat bands in S8 of Ref. 61.

This result is remarkable for several reasons. Firstly, it shows that the CP can be realized in a wide range of disordered potentials, not limited to quasiperiodic potentials; and Secondly, probably much more important, that since the CP in the disordered Lieb lattice has been realized in two dimensions [44], we expect the CP using our mechanism can induce CP in higher dimensions in a much easier and much more fundamental way.

The CP and its connection to unbounded potential: Finally, we are in a position to answer why the overlap of the spectra is essential for CP. Firstly, we discuss why states maintain their properties in the un-overlapped spectra. This is related to the applicability of perturbation theory, in which the localized and extended states can only be weakly coupled when their energies are separated by a finite gap $\Delta$. Let the extended and localized states as $|\phi_j\rangle$ and $|\psi_\alpha\rangle$, which have amplitude of $1/\sqrt{L}$ and $1/\sqrt{W}$, respectively, with $W$ being their width, then

$$|\phi_j\rangle = |\phi_j\rangle + \sum_\alpha \frac{\langle \psi_\alpha | H_c | \psi_j \rangle}{E_j - E_\alpha} |\psi_\alpha\rangle + \ldots, \quad (8)$$

assuming $|E_j - E_\alpha| \geq \Delta$. We find that the overlap matrix $|\langle \psi_\alpha | H_c | \psi_j \rangle| \ll (t_v/\Delta) \cdot (\sqrt{W/L})^d$, yielding the same conclusion. This result explains why $\tau$ is unchanged when $|E| > 2$ in Fig. 2 (c) - (e). Moreover, we have verified the fidelity between the wave functions with and without inter-chain coupling in S5 in Ref. 61, showing that $|\langle \phi_j^{\text{new}} | \phi_j \rangle| \sim 1$. It is also correct even with random potential (see S8 of Ref. 61).

Nevertheless, this argument can not be applied to the physics in the overlapped regime, since the denominator may approach zero, yielding divergent coefficients in Eq. 8. Thus one need to consider all the higher-order terms, making the wave functions totally different from the unperturbed ones. This is verified from the fidelity that $|\langle \phi_j^{\text{new}} | \phi_j \rangle| \sim 0$ (see S5 in Ref. 61). We can look at this problem from the effective model

$$\mathcal{H} = H_f + \Sigma, \quad \Sigma = H_f \frac{1}{E - H_d} H_c, \quad (9)$$

in which the self energy $\Sigma$ plays the same role as the unbounded potential in the overlapped regime [64, 68, 76], with $H_d$ for the localized chain and $H_f$ for the extended chain. The similar unbounded potential with effective long-range hopping can be realized for the disordered chain. Ref. [76] has shown that when $V(n) \propto \tan(\pi \alpha n)$, all states are localized. In Ref. [68], Liu et al. have studied the unbounded potential $V(n) \propto (1 - \cos(2\pi \alpha n))^{-1}$ using the Avila global theory [77] and yield localized phase and CP. Moreover, a similar quasiperiodic potential in the bounded potential regime with many-body interaction is realized in momentum space [78]. These results suggest that the unbounded potential can lead to singular continuous spectra (see Fig. 3) and exclude the extended states [79, 80].

Conclusion: We present a general approach to realize the CP based on coupling between the localized and extended states, and demonstrate its realization using quasiperiodic and flat bands. Some remarkable applications of this mechanism are immediately feasible. Firstly, it may be used to realize higher-dimensional CP with random potential, in which a finite disorder strength is required for AL. Secondly, it can be used to realize the many-body CP by including of many-body interaction, in which the states are extended yet non-thermal [55, 56, 81]. We can examine the intriguing relation between many-body and single-particle mobility edges.
This work can greatly broaden our understanding of multifractality and shed light on its mechanism in disordered models.

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Note added: Upon submitting of this work, we became aware of [83] for coexistence of critical, localized and extended phases with quasiperiodic potential. However, we focus on the general mechanism for the CP, which is not considered there.

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[62] The exponent τ is not affected by oscillation of plane waves. Assuming a state ψ(x) = (1 + A cos(αx))/√2, then IPR = ∫ dα |ψ(x)|^2 = (8+3A^2)(8+4A^2)/(2(2+4A^2)2L) ∝ 1/L, showing that τ = 1, which is independent of oscillation magnitude A and period parameter α; see Fig. 3 (d).
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Supplemental material for ”General approach to tunable critical phases with two coupled chains”

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S1. THE MODELS WITH DIFFERENT COUPLINGS

To examine the generality of our proposed approach for critical phases (CPs), we consider four models with different inter-chain couplings. The structure of these models based on two coupled chains is shown in Fig. S1, in which when the states in one of the chains are localized, the states in the other chain are extended. The full Hamiltonian reads as

$$H = H_{\text{AAH}} + H_{\text{free}} + H_c,$$  \hspace{1cm} (S1)

where $H_{\text{AAH}}$ and $H_{\text{free}}$ are shown in the main text (Eq. 1). The four couplings of $H_c$ are

1. Mosiac inter-chain coupling (this model can be mapped to the quasiperiodic Mosiac model with next-nearest-neighbor (NNN) hopping [1])

$$H_c = t_v \sum_m (a_m^\dagger b_m + a_{m+1}^\dagger b_{m+1} + \text{h.c}) .$$  \hspace{1cm} (S2)

2. Next-nearest-neighbor (NNN) inter-chain coupling

$$H_c = t_v \sum_m (a_m^\dagger b_m + a_{m+1}^\dagger b_{m+1} + b_{m+1}^\dagger a_{m+1} + \text{h.c}) .$$  \hspace{1cm} (S3)

3. Exponential decay inter-chain coupling

$$H_c = t_v \sum_{m,j} \left( e^{-p|j-1|} a_m^\dagger b_{m+j} + \text{h.c} \right) .$$  \hspace{1cm} (S4)

In this work, we use $p = 0.5$, and we have verified that the same physics can be realized with other value of $p$. 

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4. Random inter-chain coupling

\[ H_c = \sum_m t_m (a_m^\dagger b_m + \text{h.c}), \]  

(S5)

with \( t_m \in U(0, t_v) \) a random variable to avoid potential frustration. We have verified that the same CP can be found with other types of random potential, such as \( t_m \in U(-t_v, t_v) \).

FIG. S1. Two coupled chains with different inter-chain couplings. Here the yellow bonds, green bonds, and pink bonds represent the inter-chain coupling. We note that the long-range coupling and random coupling also lead to the same CP.

S2. FRACTAL DIMENSION (FD) IN THE FOUR MODELS

FIG. S2. FD \( \tau \) of all states versus \( V \) with \( t_v = 0.5 \) and \( L = 2F_{15} = 1220 \). The states inside the overlapped spectra become critical with intermediate \( \tau \sim 0.2 - 0.8 \); see details in Fig. S3.

We plot FD of all states against disorder strength \( V \) for the different models (see Fig. S2). The definition of FD \( \tau \) (see Eq. 2), can be found in the main text. Furthermore, we plot the FD \( \tau \) versus energy \( E \) with different inter-chain couplings in Fig. S3. All these results show that the CPs can be found in their overlapped spectra with different type of inter-chain coupling, with \( \tau \sim 0.2 - 0.8 \). The averaged FD in a finite interval, \( \bar{\tau} \), will approach 0 (for localized state), (0,1) (for the CP) and 1 (for extended phase) in the thermodynamic limit (see Fig. 3 (b) in the main text).
FIG. S3. FD with different coupling versus energy $E$. We use $L = 2F_{15} = 1220$ and $V = 2$. In all these models, while the inter-chain couplings are different, the states in the overlapped spectra are all critical, with $\tau \sim 0.2 - 0.8$.

S3. TWO COUPLED CHAINS WITH DIFFERENT QUASIPERIODIC POTENTIALS

To examine the generality of our mechanism for CPs, here we consider two chains with some other types of quasiperiodic potentials, whose mobility edges are analytically well-known. The Hamiltonian can be written as

$$H = H_{\text{quasi}} + H_{\text{free}} + H_c.$$  \hfill (S6)

where $H_{\text{free}}$ and $H_c$ are shown in the main text. We consider two different quasiperiodic potentials as

1. One-dimensional quasiperiodic Mosaic model [1]

$$H_{\text{quasi}} = \sum_{m} \alpha_m^* \alpha_{m+1} + \text{h.c.} + 2V(1 + (-1)^m) \cos(2\pi\alpha m) \alpha_m^* \alpha_m,$$  \hfill (S7)

where the analytical mobility edges are given by $E_c = \pm \frac{1}{2\pi\alpha}$.

2. General AAH model [2]

$$H_{\text{quasi}} = \sum_{m} \alpha_m^* \alpha_{m+1} + \text{h.c.} + 2V \frac{\cos(2\pi\alpha m)}{1 - a \cos(2\pi\alpha m)} \alpha_m^* \alpha_m,$$  \hfill (S8)

where the analytical mobility edge is given by $aE_c = 2 - 2V$. Here, we use $a = 0.2$.

We plot the FD of these models against the disorder strength $V$ for the exact solvable model and coupled quasiperiodic chains in Fig. S4. A more precise plot of FD versus energy $E$ is shown in Fig. S5. We see that in the overlapped spectra between the quasiperiodic chain and the free chain is always critical with intermediate FD, $\tau \sim 0.2 - 0.8$. 
FIG. S4. FD $\tau$ of all states versus $V$ with $t_v = 0.1$ and $L = 2F_{15} = 1220$ for the general AAH model and quasiperiodic Mosiac model in Eq. (S7). The first row donates the mobility edges in the two exact solvable models, and the second row donates the coupling between the quasiperiodic chain with a free chain, in which the overlapped regimes turn to critical, with $\tau \sim 0.25 - 0.75$; see Fig. S5.

FIG. S5. FD versus energy $E$ for model (a) general AAH model and (b) quasiperiodic Mosiac model; (c) coupled General AAH model and (d) coupled quasiperiodic Mosiac model. We use $L = 2F_{15} = 1220$, $t_v = 0.1$ and $V = 1.1$. The dashed lines in the panels (a) and (b) are the analytical mobility edges. The red shadows donate the overlapped spectra with finite $\tau \sim 0.25 - 0.75$. 
FIG. S6. Distribution of $\alpha_{\text{min}}$ in different phases with the increasing of lattice size (top row $n=12$; middle row $n=15$; bottom row $n=19$). We use $V = 2$ and $M = 0$, which is the same as that used in Fig. 3 in the main text. (a1)-(a3) The states in the coexistence phase without inter-chain coupling, in which $P(\alpha_{\text{min}})$ exhibits a bimodal structure independent of system sizes. (b1)-(b3) The states in the CP in the overlapped regime, in which $P(\alpha_{\text{min}})$ becomes a unimodal distribution with increasing of system size ($n = 19$). (c1)-(c3) The states in the extended phase, and (d1-d3) the state in the localized phase, in which $\alpha_{\text{min}}$ is restricted in a small regime with $\alpha_{\text{min}} \sim 0$ for extended phase or $\alpha_{\text{min}} \sim 1$ for localized phase.

S4. DISTRIBUTION OF MINIMAL SCALING INDEX $\alpha_{\text{min}}$

The distribution of $\alpha_{\text{min}}$ can be used to distinguish the coexistence phase from the CP (see the first column and the second column of Fig. S4). We find that the distribution of $\alpha_{\text{min}}$ in the CP changes from a bimodal distribution to a unimodal distribution with the increasing of system size. The distribution of $\alpha_{\text{min}}$ for extended states has a peak, whose center approach 1 with the increasing of lattice size (see the third column of Fig. S4). On the contrary, the center of the distribution peak in the localized phase decreases (see the fourth column of Fig. S4). The distribution of the coexistence phase has two-peak with each peak suggesting localized or extended behavior. In a summary, the states in the overlapped spectra are indeed in the CP rather than the coexistence phase. Here, $H_c$ is same with that in the main text and we have verified that the similar distribution can be found in all the other models.

S5. FIDELITY BETWEEN UNCOUPLED MODEL ($t_v = 0$) AND COUPLED MODEL ($t_v \neq 0$)

To clarify that the states in the un-overlapped spectra keep their own property while states in the overlapped spectra become critical, we calculate the fidelity between the states of $|\phi_{n'}^{t_v=0}\rangle$ (in the uncoupled model) and $|\psi_{n}^{t_v}\rangle$ (in the coupled model), which is defined as

$$C_{n',n} = \langle \phi_{n'}^{t_v=0} | \psi_{n}^{t_v} \rangle.$$  \hspace{1cm} (S9)

For convenience, we only examine $\max(|C_n|^2) = \max(|C_{n',n}|^2|n' = 1, 2, \cdots, N)$ to identify the similarity between states in these two models for a given $n$. This result justifies Eq. 8 in the main text that the un-overlapped spectra is not affected when the perturbation theory is applicable. A direct comparison between FD and this fidelity is shown in Fig. S7, showing that this fidelity will be totally different from unity in the CP. The scaling of this fidelity as a function
of system size is shown in Fig. S8, showing that this value will approach zero in the thermodynamic limit because of the strong coupling between the localized states and extended states, which leads to CP in the thermodynamic limit.

FIG. S7. FD (blue circle) and max(|$C_n|^2$) (red square) for all states with $L = 2F_{18} = 5168$ and $V = 2$. In the un-overlapped regimes, the coupling between extended states and localized states is not significant, yielding max(|$C_n|^2)$ ~ 1. In the overlapped regime, the strong coupling between these different states yields max(|$C_n|^2)$ ~ 0. In the CP, max(|$C_n|^2$) decreases to zero with the increasing of inter-chain coupling; see Fig. S8.

FIG. S8. Averaged maximum |$C_n|^2$ in specific energy interval A, B and C versus system size. In the un-overlapped spectra, this value is almost independent of system size; however, in regime C, this value decreases with the increasing of system size, indicating of strong coupling between localized states and extended states for CP.
FIG. S9. FD and Lyapunov exponent in the coupled random chain model. We use $V = 2$ and $t_v = 1$. The energy offset in (a)-(c) is $M = 0$ and (d)-(f) is $M = 10$. (a), (d) FD of all states of the coupled random chain at different lattice size. (b), (e) Scaling of the averaged FD $\tau = \frac{1}{N} \sum_{n,E_{\min}<E_{n}<E_{\max}} \tau_n$ versus $1/\log(2L)$ with $N_{in}$ the number of states in the energy interval $(E_{\min}, E_{\max})$. (c), (f) Lyapunov exponent $\gamma(E)$ versus energy $E$ with size $L = 100000$. The energy interval is (c) $E \in [-3, 3]$ and (f) $E \in [-2, 2] \cup [8, 12]$. These results in (a)-(c) suggest that all states with $M = 0$ are localized without an obvious energy gap. The results in (d)-(f) suggest that the states with $E < 5$ are localized while the states with $8.5 < E < 11.5$ are extended.

S6. STATES IN COUPLED CHAINS WITH RANDOM POTENTIAL

We are in a position to answer whether coupling between extended states and localized states can always lead to the CP. In this section, we discuss the fate of states in two coupled random chains. The Hamiltonian can be written as

$$H = \sum_m (a_m^\dagger a_{m+1} + \text{h.c.} + V_m a_m^\dagger a_m + \sum_m (b_m^\dagger b_{m+1} + \text{h.c.} + M b_m^\dagger b_m) + t_v \sum_m (a_m^\dagger b_m + \text{h.c.}), \quad (S10)$$

with $V_m \in U(-V/2, V/2)$ a random variable. We use FD and Lyapunov exponent to examine the localization properties in this model. The Lyapunov exponent is defined as

$$\gamma(E) = \left| \frac{1}{N} \text{Re} \log(\psi_N/\psi_0) \right|, \quad (S11)$$

which is obtained by using the transfer matrix method. We find that the random potential induces fully localized states in the overlapped spectra, instead of the critical states. However, in the unoccupied regimes with a finite energy offset $M$, the states are unchanged when the inter-chain couplings switch on, which is consistent with Eq. 8 in the main text.

S7. DUALITY OF THE AUBRY-ANDRÉ-HARPER (AAH) MODEL

Duality plays an essential role in our model. Considering the following model [3]

$$H = \sum_m J \left( a_m^\dagger a_{m+1} + \text{h.c.} \right) + 2\lambda \cos(2\pi \alpha m) a_m^\dagger a_m. \quad (S12)$$
We can assume the eigenstate as $|\psi\rangle = \sum_m \psi_m |m\rangle$, and the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ can be written as

$$J (\psi_{m+1} + \psi_{m-1}) + 2\lambda \cos(2\pi\alpha m) \psi_m = E\psi_m.$$  \hfill (S13)

This model has a celebrated duality between the wave function in real space and momentum space via $\psi_m = \sum_k \phi_k e^{i2\pi\alpha mk}$, which yields

$$2J \cos(2\pi\alpha k) \phi_k + \lambda (\phi_{k+1} + \phi_{k-1}) = E\phi_k,$$  \hfill (S14)

which is similar to that in Eq. S13 with some modified parameters, yet their spectra are exactly the same. In this way, duality happens at $V = J$, in which the equation, after the Fourier transformation, is mapped to itself.

This method will be used in the main text (see Eq. 5) by developing the idea of inter-chain duality, in which we find a duality as following

$$a_i \rightarrow b_k, \quad b_i \rightarrow a_k.$$  \hfill (S15)

This inter-chain duality is used to make all states critical, which results in the sub-diffusion spreading in Fig. 4 (c) in the main text. The inter-chain duality may provide a new approach for criticality in coupled chain models.

S8. CPS IN THE FLAT BANDS MODELS

![Energy spectrum and wave functions of Eq. S16](image)

FIG. S10. Energy spectrum and wave functions of Eq. S16. (a) The band structure of the cross-stitch chain without inter-chain coupling ($V = 0$). (b) The energy $E_n$ for various inter-chain coupling $V$. (c) The wave function of the wave functions with $V = 10^{-4}$ for CP in the overlapped regime and $V = 10^{-5}$ for extend phase in the un-overlapped regime.

The results in the previous section raise the question: can this CP be realized without quasiperiodic potential? To this end, we discuss the fate of critical states in the system with flat bands, in which the localization is not induced by disorder potential, but by destructive interference. Without loss of generality, we consider the cross-stitch lattice with the on-site disorder. The corresponding Hamiltonian can be written as

$$H = \sum_{m=0}^{L/2} a_m^+ a_{m+1} + a_m^+ b_{m+1} + b_m^+ b_{m+1} + b_m^+ a_{m+1} + \text{h.c.} + V_{am} a_m^+ a_m + V_{bm} b_m^+ b_m,$$  \hfill (S16)
with \(V_{am}, V_{bm} \in U(-V/2, V/2)\) are random variables. By using \(f_m = \frac{1}{\sqrt{2}}(a_m - b_m)\) and \(d_m = \frac{1}{\sqrt{2}}(a_m + b_m)\), this Hamiltonian is mapped to the following flat band model

\[
H = \sum_{m=0}^{L/2} 2d_m^\dagger d_{m+1} + 2d_{m+1}^\dagger d_m + V_{sm}(d_m^\dagger d_m + f_m^\dagger f_m) + V_{dm}(d_m^\dagger f_m + f_m^\dagger d_m),
\]

where \(V_{sm} = \frac{V_{am} + V_{bm}}{2}\) and \(V_{dm} = \frac{V_{am} - V_{bm}}{2}\). When \(V_{am}, V_{bm}\) are independent random numbers, we find that both the intra-band disorder \(V_{sm}\) and inter-band disorder \(V_{dm}\) are non-zero. To highlight the importance of inter-band disorder \(V_{dm}\) and the related physics of inverse Anderson localization, we choose \(V_{am}\) and \(V_{bm}\) as some kind of correlated disorder (i.e. \(V_{am} = V_{bm}\)), which makes \(V_{dm} = 0\) in all lattice sites. Then we find that all states in this case are localized. We define the level-spacing ratio as

\[
r_n = \frac{\min(s_n, s_{n+1})}{\max(s_n, s_{n+1})},
\]

where \(s_n = e_{n+1} - e_n\) is the nearest-neighbor level-spacing. The distribution of this dimensionless quantity can be used to examine whether the system is localized (for Poisson statistics with \(\tau = 2 \ln 2 - 1 \sim 0.386\) or extended (for Wigner-Dyson statistics with \(\tau = 4 - 2\sqrt{3} \sim 0.536\), or critical (with \(2 \ln 2 - 1 < \tau < 4 - 2\sqrt{3}\)). Only the states in the overlapped spectrum are taken into account, which correspond to the flat-band states in the clean limit. And we rescale this spectrum into interval [0, 1] by using

\[
e_n = \frac{E_n - \min(E_n)}{\max(E_n) - \min(E_n)},
\]

with \(\{E_n\}\) being arranged in ascending order.

When inter-band disorder \(V_{dm} \neq 0\), the level-spacing in the middle spectra is indeed intermediate between the Poisson statistics and the Wigner-Dyson statistics at weak-disorder (i.e. \(V < 1\)) with \(0.39 < r < 0.53\). Then the system turns to localized phase with \(r \sim 0.39\) at \(V > 1\). These results are shown in Fig. S11 (a) and (b). In the strong disorder limit, it will become localized, which is expected when the random potential \(V\) is dominated. However, when the inter-band disorder \(V_{dm} = 0\), the level-spacing is always Poisson; see Fig. S11 (c) and (d), suggesting that all states are localized without inter-band disorder that couples the flat bands and dispersion bands.

FIG. S11. The energy statistics in the middle spectra of the model S16 with (a)-(b) \(V_{am}, V_{bm} \in U(-V/2, V/2)\) and (c)-(d) \(V_{am} = V_{bm} \in U(-V/2, V/2)\) and \(L = 2000\). In the top row, the level-spacing statistics shows the system is critical at weak disorder and localized at large disorder. However, in the bottom row, the system is localized, independent of disorder strength. In this case, the level spacing is given by Poisson distribution with \(\bar{\tau} = 0.386\).
Now we turn our attention to the Hamiltonian of Eq. S17. We choose $V_{sm} \in U(-V_1/2, V_1/2)$ and $V_{dm} \in U(-V_2/2, V_2/2)$ as independent random variables. Our results are presented in Fig. S12. By fixing of $V_2 = 10^{-3}$ and increasing the intra-band disorder $V_1$, we find the states in the middle (overlapped) spectra undergo a transition from the critical states to the localized states (see $V_1 = 10^{-3} - 10^{-1}$). When the intra-band disorder is weak, the inter-band disorder induces coupling between localized states and extended states with nearby energies, and we find that random inter-band coupling induces delocalization of the localized states. Then the states in the overlapped spectrum are critical. However, when the intra-band disorder is sufficient strong, all the states are localized. While the results in Fig. S12 are not sufficient to demonstrate this conclusion, in Fig. S13, we examine the scaling of averaged FD $\bar{\tau}$ in the extended, localized and CPs, showing that while $\bar{\tau} \to 0$, 1 for localized and extended phases, it approaches finite (see Fig. S13 (a), with $\bar{\tau} \to 0.3 - 0.4$) for the CP. This picture can be used to realize the higher-dimensional CPs [4].

FIG. S12. FD of all states of model S17 versus states number. We use $L = 8000$ and $V_2 = 10^{-3}$. 

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FIG. S13. Averaged FD in and out the middle spectra against the inverse of log($L$). The states in the middle spectra are critical with intermediate FD $\tau$ with weak disorder, and localized with strong disorder. States outside the middle spectra are extended with $\tau \to 1$ when the disorder is weak and localized with $\tau \to 0$ when the disorder is strong.

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