Smoothed Dual Embedding Control

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January 1, 2018

Abstract

We revisit the Bellman optimality equation with Nesterov’s smoothing technique and provide a unique saddle-point optimization perspective of the policy optimization problem in reinforcement learning based on Fenchel duality. A new reinforcement learning algorithm, called Smoothed Dual Embedding Control or SDEC, is derived to solve the saddle-point reformulation with arbitrary learnable function approximator. The algorithm bypasses the policy evaluation step in the policy optimization from a principled scheme and is extensible to integrate with multi-step bootstrapping and eligibility traces. We provide a PAC-learning bound on the number of samples needed from one single off-policy sample path, and also characterize the convergence of the algorithm. Finally, we show the algorithm compares favorably to the state-of-the-art baselines on several benchmark control problems.

1 Introduction

Reinforcement learning (RL) algorithms aim to learn a policy that maximizes the long-term return by sequentially interacting with an unknown environment (Sutton and Barto, 1998). The dominating framework to model such an interaction is Markov decision processes, or MDPs. A fundamental result for MDP is that the Bellman operator is a contraction in the value-function space, and thus, the optimal value function is a unique fixed point of the operator. Furthermore, starting from any initial value function, iterative applications of the Bellman operator will converge to the fixed point. Interested readers are referred to the textbook of Puterman (2014) for details.

Many of the most effective RL algorithms have their root in such a fixed-point view. The most prominent family of algorithms is perhaps the temporal-difference algorithms, including TD\textsuperscript{(λ)} (Sutton, 1988), Q-learning (Watkins, 1989), SARSA (Rummery and Niranjan, 1994; Sutton, 1996), and numerous variants. Compared to direct policy search or policy gradient algorithms like REINFORCE (Williams, 1992), these fixed-point methods use bootstrapping to make learning more efficient by reducing variance. When the Bellman operator can be computed exactly (even on average), such as when the MDP has finite state/actions, convergence is guaranteed and the proof typically relies on the contraction property (Bertsekas and Tsitsiklis, 1996). Unfortunately, when function approximators are used, such fixed-point methods can easily become unstable/divergent (Boyan and Moore, 1995; Baird, 1995; Tsitsiklis and Van Roy, 1997), except in rather limited cases. For example,

- for some rather restrictive function classes that have a non-expansion property, such as kernel averaging, most of the finite-state MDP theory continues to apply (Gordon, 1995);
- when linear function classes are used to approximate the value function of a fixed policy from on-policy samples (Tsitsiklis and Van Roy, 1997), convergence is guaranteed.
In recent years, a few authors have made important progress toward finding scalable, convergent TD algorithms, by designing proper objective functions and using stochastic gradient descent (SGD) to optimize them (Sutton et al., 2009; Maei, 2011). Later on, it was realized that several of these gradient-based algorithms can be interpreted as solving a primal-dual problem (Mahadevan et al., 2014; Liu et al., 2015; Macua et al., 2015; Dai et al., 2016). This insight has led to novel, faster, and more robust algorithms by adopting sophisticated optimization techniques (Du et al., 2017). Unfortunately, to the best of our knowledge, all existing works either assume linear function approximation or are designed for policy evaluation. It remains an open question how to find the optimal policy reliably with nonlinear function approximators such as neural networks, even in the presence of off-policy data.

In this work, we take a substantial step towards solving this decades-long open problem, leveraging a unique saddle point optimization perspective to derive a new algorithm called smoothed dual embedding control (SDEC). Our development hinges upon a special look at the Bellman optimality equation and the temporal relationship between optimal value function and optimal policy revealed from a smoothed Bellman optimality equation. We exploit such a relation and introduce a distinct saddle point optimization that simultaneously learns both optimal value function and policy in the primal form and allows to escape from the instability of max-operator and “double sampling” issues faced by existing algorithms. As a result, the SDEC algorithm enjoys many desired properties, in particular:

- It is stable for a broad class of nonlinear function approximators including neural networks, and provably converges to a solution with vanishing gradient. This is the case even in the more challenging off-policy scenario.
- It uses bootstrapping to yield high sample efficiency, as in TD-style methods, and can be generalized to cases of multi-step bootstrapping and eligibility traces.
- It directly optimizes the squared Bellman residual based on a sample trajectory, while avoiding the infamous double-sample issue.
- It uses stochastic gradient descent to optimize the objective, thus very efficient and scalable.

Furthermore, the algorithm handles both the optimal value function estimation and policy optimization in a unified way, and readily applies to both continuous and discrete action spaces. We test the algorithm on several continuous control benchmarks. Preliminary results show that the proposed algorithm achieves the state-of-the-art performances.

2 Preliminaries

In this section, we introduce the preliminary background about Markov decision processes.

Markov Decision Processes (MDPs). We denote the MDPs as $\mathcal{M} = (S, A, P, R, \gamma)$, where $S$ is the state space (possible infinite), $A$ is the set of actions, $P(\cdot|s,a)$ is the transition probability kernel defining the distribution upon taking action $a$ on state $x$, $r(s,a)$ gives the corresponding stochastic immediate rewards with expectation $R(s,a)$, and $\gamma \in (0,1)$ is the discount factor.

Given the initial state distribution $\mu(s)$, the goal of reinforcement learning problems is to find a policy $\pi(\cdot|s) : S \rightarrow \mathcal{P}(A)$, where $\mathcal{P}(A)$ denotes all the probability measure over $A$, that maximizes the total expected discounted reward, i.e., $\mathbb{E}_{s \sim \mu(s)} \mathbb{E}_\pi \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s \right]$, where $s_t \sim P(\cdot|s_{t-1}, a_t), a_t \sim \pi(\cdot|s_{t-1})$.

Bellman Optimality Equation. Denote $V^*(s) = \max_{\pi(\cdot|s)} \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s \right]$. It is known that the optimal state value function $V^*$ satisfies the Bellman optimality equation (Puterman, 2014)

$$V^*(s) = \max_a R(s,a) + \gamma \mathbb{E}_{\delta'|s,a} [V^*(s')] := (TV^*)(s). \quad (1)$$
The optimal policy \( \pi^* \) can be obtained from \( V^* \) via
\[
\pi^*(a|s) = \arg\max_a R(s, a) + \gamma \mathbb{E}_{s'|s,a}[V^*(s')].
\]
(2)

Therefore, solving the reinforcement learning problem is equivalent to finding the solution to the Bellman optimality equation (1). However, one should note that even if a solution \( V^* \) to the Bellman optimality equation (1) is obtained, finding the optimal policy via (2) remains a challenging task in reinforcement learning, since one needs to solve yet another optimization problem.

3 A New Optimization Perspective of Reinforcement Learning

In this section, we introduce a unique optimization perspective of the reinforcement learning problem that paves the way to designing efficient and provable algorithms with the desired properties. The development hinges upon a special look at the Bellman optimality equation by explicitly elucidating the role of policy. By leveraging a smoothed Bellman optimality equation with entropy regularization, we discover a direct relationship between the optimal policy and corresponding value function and come up with a new optimization formulation of the reinforcement learning problem.

3.1 Revisiting the Bellman Optimality Equation

Recall that most value-function-based algorithms find the optimal policy only after (approximately) solving the Bellman optimality equation for the optimal value function. We start by revisiting the Bellman optimality equation and rewriting it in a Fenchel-type representation:
\[
V^*(s) = \max_{\pi \in \mathcal{P}} \sum_{a \in A} \pi(a|s) \left( R(s, a) + \gamma \mathbb{E}_{s'|s,a}[V^*(s')] \right).
\]
(3)

where \( \mathcal{P} = \{ \pi : \pi \geq 0, \pi^\top 1 = 1 \} \) denotes the family of valid distributions. This reformulation is based on the simple fact that for any \( x \in \mathbb{R}^n, \max_{i \in \{1, \ldots, n\}} x_i = \max_{\pi \geq 0, \|\pi\|_1 \leq 1} \pi^T x. \) Observe that the role of policy is now explicitly revealed in the Bellman optimality equation. Despite its simplicity, this observation is an important step that allows us to develop the new algorithm in the following. A straightforward idea is to consider a joint optimization over \( V \) and \( \pi \) by fitting them to the Bellman equation and minimizing the expected residuals; that is,
\[
\min_V \mathbb{E}_s \left[ \left( \max_{\pi \in \mathcal{P}} \sum_{a \in A} \pi(a|s) \left( R(s, a) + \gamma \mathbb{E}_{s'|s,a}[V^*(s')] \right) - V(s) \right)^2 \right].
\]
(4)

Unlike most existing approaches, this optimization formulation brings the search procedures for optimal state value function and optimal policy in a unified framework. However, there are several major difficulties when directly solving this optimization problem,

1. The max operator over distribution space will cause numerical instability, especially in environments where a slight change in \( V \) may cause large differences in the RHS of Eq. (3).

2. The conditional expectation, \( \mathbb{E}_{s'|s,a}[\cdot] \), composed with the square loss, requires double samples (Baird, 1995) to compute an unbiased stochastic gradient, which is often impractical.

3.2 Smoothed Bellman Optimality Equation

To avoid the instability and discontinuity caused by max operator, we propose to smooth the policy update by utilizing the smoothing technique of Nesterov (2005). Since the policy is defined on the distribution space, we introduce an entropic regularization to the Fenchel-type representation (5):
\[
\tilde{V}^*(s) = \max_{\pi \in \mathcal{P}} \left( \sum_{a \in A} \pi(a|s) \left( R(s, a) + \gamma \mathbb{E}_{s'|s,a}[\tilde{V}^*(s')] \right) + \lambda H(\pi) \right),
\]
(5)
where $H(\pi) = -\sum_{a \in A} \pi(a|s) \log \pi(a|s)$ is the entropy function and $\lambda > 0$ controls the level of smoothing. We first show that the entropy-regularization will indeed introduce smoothness and continuity yet preserves the existence and uniqueness of optimal solution. Observe that the RHS of Eq. (6) is exactly the Fenchel-dual representation of log-sum-exp function \cite{Boyd2004}. Hence, we have the smoothed Bellman optimality equation as

$$
\hat{V}^*(s) = \lambda \log \left( \sum_{a \in A} \exp \left( \frac{R(s,a) + \gamma \mathbb{E}_{s',a} \left[ \hat{V}^*(s') \right]}{\lambda} \right) \right) := (\tilde{T}\hat{V}^*) (s),
$$

where the log-sum-exp is a smooth approximation of the max-operator.

Next we show that $\hat{V}^*$ that satisfies the smoothed optimality equation is still unique due to the contraction of the operator $\tilde{T}$.

**Proposition 1 (Uniqueness)** $\tilde{T}$ is a contraction operator. Therefore, the smoothed Bellman optimality equation (5) has a unique solution.

A similar result is also presented in \cite{Fox2015}; \cite{Asadi2016}. For completeness, we list the proof in Appendix A.

Note that although using the entropy-regularization introduces smoothness to the policy and avoids numerical instability cause by max-operator, it also introduces bias of the optimal value function:

**Proposition 2 (Smoothing bias)** Let $V^*$ and $\hat{V}^*$ be fixed points of (3) and (5), respectively. It holds that

$$
\|V^*(s) - \hat{V}^*(s)\|_\infty \leq \frac{\gamma \lambda}{1 - \gamma} \max_{\pi \in \mathcal{P}} H(\pi).
$$

As $\lambda \to 0$, $\hat{V}^*$ converges to $V^*$ pointwisely.

The proof can be found in Appendix A.

By further simplifying the smoothed Bellman optimality equation, we are able to recover a direct relationship between the optimal value function and optimal policy.

**Theorem 3 (Temporal consistency)** Let $\hat{V}^*$ be the fixed point of (5) and $\hat{\pi}^*$ be the corresponding policy that attains the maximum in the RHS of (5). Then $(\hat{V}^*, \hat{\pi}^*)$ is the unique solution that satisfies

$$
V(s) = R(s,a) + \gamma \mathbb{E}_{s',a} [V(s')] - \lambda \log \pi(a|s), \quad \forall a \in A.
$$

We point out that a similar condition has also been realized in \cite{Rawlik2012}; \cite{Neu2017}; \cite{Nachum2017}. But from a completely different view point and our proof is slightly different; see Appendix A.

In \cite{Nachum2017}, the entropy regularization term is adopted to encourage exploration and prevent early convergence, while we start from the smoothing technique \cite{Nesterov2005}.

Note that the simplified equation Eq. (8) provides a both sufficient and necessary condition for characterizing the optimal value function and optimal policy. As we will see, this characterization indeed yields new opportunities for learning optimal value functions, especially in the off-poliy and multi-step/eligibility-traces cases.

### 3.3 Saddle Point Reformulation via Dual Embeddings

With this new characterization of the smoothed Bellman optimality equation, a straightforward idea is to solve the equation (8) by minimizing the mean square consistency Bellman error, namely,

$$
\min_{V,\pi \in \mathcal{P}} f(V, \pi) := \mathbb{E}_{s,a} \left[ (R(s,a) + \gamma \mathbb{E}_{s',a} [V(s')] - \lambda \log \pi(a|s) - V(s))^2 \right].
$$


Due to the inner conditional expectation, directly applying stochastic gradient descent algorithm requires two independent samples in each updates, referred as “double sampling” issue; e.g., Baird (1995); Dai et al. (2016). Directly optimizing (9) remains challenging since in practice one can hardly access to two independent samples from $P(s'|s,a)$.

Inspired by [Dai et al. 2016], we propose to reformulate the objective into an equivalent saddle-point problem in order to bypass the double sampling issue. Specifically, by exploiting the Fenchel dual of the square function, i.e., $x^2 = \max_{\nu} (2 \nu x - \nu^2)$, and further applying the interchangeability principle [Dai et al. 2016, Lemma 1], we can show that (9) is equivalent to the saddle point problem

$$
\min_{V, \pi} \max_{\nu, \pi} L(V, \pi; \nu) := 2E_{s,a,s'} [\nu(s,a)(R(s,a) + \gamma V(s') - \lambda \log \pi(a|s) - V(s))] - E_{\nu, a,s'} [\nu^2(s,a)]
$$

where $F(S \times A)$ stands for the function space defined on $S \times A$. Note that this is not a standard convex-concave saddle point problem: the objective is convex in $V$ for any fixed $\nu, \pi$ and concave in $\nu$ for any fixed $V, \pi$, but not necessarily always convex in $\pi \in P$ for any fixed $\nu, V$.

In contrast to our saddle point optimization approach (10), Nachum et al. (2017a) considered a different way to handle the double sampling issue by solving instead an upper bound of (9), namely, $\min_{V, \pi \in P} \hat{f}(V, \pi) := \mathbb{E}_{s,a,s'} \left[ (R(s,a) + \gamma V(s') - \lambda \log \pi(a|s) - V(s))^2 \right]$. This surrogate function is obtained by brute-forcedly extracting the inner expectation outside, thus admitting unbiased stochastic gradient estimates with one sample $(s,a,s')$. However, the surrogate function introduces extra variance term into the original objective; in fact, $f(V, \pi) = \hat{f}(V, \pi) + \mathbb{E}_{s,a,s'} [\gamma |V(s')|]$. If the variance is large, minimizing the surrogate function could lead to highly inaccurate solution, while such an issue does not exist in our saddle point optimization because of the exact equivalence to (9).

In fact, by substituting the dual function $\nu(s,a) = \rho(s,a) - V(s)$, the objective in the saddle point problem becomes

$$
\min_{V, \pi} \max_{\rho} L(V, \pi; \rho) := \mathbb{E}_{s,a,s'} \left[ (\delta(s,a,s') - V(s))^2 \right] - \mathbb{E}_{s,a,s'} \left[ (\delta(s,a,s') - \rho(s,a))^2 \right]
$$

where $\delta(s,a,s') = R(s,a) + \gamma V(s') - \lambda \log \pi(a|s)$. Note that the first term is the same as $\hat{f}(V, \pi)$, and the second term will cancel the extra variance term as we proved in Theorem 5 in Appendix B. Such an understanding of the saddle point objective as a decomposition of mean and variance is indeed very useful to exploit a better bias-variance tradeoff. Specifically, when function approximators are used for the dual variables, extra bias will be induced instead of the variance term. To balance the induced bias caused by function approximator and the variance, one can impose a weight on the second term. This turns out to be very important especially in the multi-step setting where the dual variables need complicated parametrization and has also been observed in our experiments.

Remark (Comparison to existing optimization perspectives). Several recent works [Chen and Wang 2016, Wang 2017] have also considered saddle point formulations of Bellman equations, but these formulations are fundamentally different from ours, even from its origin. These saddle point problems are derived from the Lagrangian dual of the linear programming formulation of Bellman optimality equations and only applicable to MDPs with finite state and action spaces. In contrast, our saddle point optimization originates from the Fenchel dual of the mean squared error of a smoothed Bellman optimality equation. Moreover, our framework can be applied to both finite-state and continuous MDPs, and naturally adapted to the multi-step and eligibility-trace extensions.

4 Smoothed Dual Embedding Control Algorithm

In this section, we develop an efficient reinforcement learning algorithm from the saddle point perspective. As we discussed, the optimization (11) provides a convenient mechanism to achieve better bias-variance tradeoff.
by reweighting the two terms, i.e.,
\[
\min_{V, \pi \in P} \max_{\rho \in F(S \times A)} L_\eta(V, \pi; \rho) := \mathbb{E}_{s,a,s'} \left[ (\delta(s, a, s') - V(s))^2 \right] - \eta \mathbb{E}_{s,a,s'} \left[ (\delta(s, a, s') - \rho(s, a))^2 \right],
\]
where \( \eta \in [0, 1] \) is a given positive scaler used for balancing the variance and potential bias. When \( \eta = 1 \), this reduces to the original saddle point formulation (10). When \( \eta = 0 \), this reduces to the surrogate objective considered in Nachum et al. (2017a).

From the new optimization view of reinforcement learning, we derive the smoothed dual embedding control algorithm based on stochastic mirror descent in Nemirovski et al. (2009). For simplicity, we mainly discuss the one-step optimization (12), the algorithm can be easily generalized to the multi-step and eligibility-trace settings. Please check the details in Appendix C.2 and C.3 respectively.

We first derive the unbiased gradient estimator of the objective in (12) w.r.t. \( V, \pi \):

**Theorem 4 (Unbiased gradient estimator)** Denote
\[
\bar{\ell}(V, \pi) = \max_{\rho \in F(S \times A)} L_\eta(V, \pi; \rho),
\]
\[
\rho^* = \arg\max_{\rho \in F(S \times A)} L_\eta(V, \pi; \rho).
\]

We have the unbiased gradient estimator as
\[
\nabla_V \bar{\ell}(V, \pi) = \mathbb{E}_{s,a,s'} \left[ \delta(s, a, s') (\gamma \nabla V(s') - \nabla V(s)) \right] - \eta \mathbb{E}_{s,a,s'} \left[ (\delta(s, a, s') - \rho^*(s, a)) \nabla V(s') \right],
\]
\[
\nabla_\pi \bar{\ell}(V, \pi) = -\lambda \mathbb{E}_{s,a,s'} \left[ ((1 - \eta) \delta(s, a, s') + \eta \rho^*(s, a) - V(s)) \right] \nabla_{\pi} \log \pi(a|s).
\]

Denote the parameters the primal and dual variables \( V \) and \( \pi \) as \( w_v \) and \( w_\pi \), respectively, then \( \nabla_{w_v} \bar{\ell}(V, \pi) \) and \( \nabla_{w_\pi} \bar{\ell}(V, \pi) \) can be obtained by chain rule from (13) and (14). We will apply the stochastic mirror descent to update \( w_v \) and \( w_\pi \), i.e., solving the prox-mapping in each iteration,
\[
P_{z_v}(g) = \arg\min_{w_v} \langle w_v, g \rangle + D_v(w_v, z_v),
\]
\[
P_{z_\pi}(g) = \arg\min_{w_\pi} \langle w_\pi, g \rangle + D_\pi(w_\pi, z_\pi).
\]
where \( D_v(w, z) \) and \( D_\pi(w, z) \) denote the Bregman divergences. We can use Euclidean metric for both \( w_v \) and \( w_\pi \), or exploit \( KL \)-divergence for \( w_\pi \). Following these steps, we arrive at the smoothed dual embedding control algorithm.

For practical purpose, we incorporate the experience replay into the algorithm. We illustrate the algorithm in Algorithm 1. As we can see, rather than the prefixed samples, we have the procedure to collect samples \( D = \{(s, a, s')_{i=1}\} \) by executing the behavior policy, corresponding to line 3-5, and the behavior policy will be updated in line 12. Line 6-11 corresponds to the updates for stochastic gradient descent.

**Remark (Role of dual variables):** The dual solution is updated through solving the subproblem
\[
\min_{\rho} \mathbb{E}_{s,a,s'} \left[ (R(s,a) + \gamma V(s') - \lambda \log \pi(a|s) - \rho(s,a))^2 \right],
\]
which can be processed by stochastic gradient descent or other optimization algorithms. Obviously, the solution of the optimization is
\[
\rho^*(s,a) = R(s,a) + \gamma \mathbb{E}_{s'|s,a} [V(s')] - \lambda \log \pi(a|s).
\]
Therefore, the dual variables can be essentially viewed as \( Q\)-function in entropy-regularized MDP. Therefore, the algorithm could be understood as first fitting a parametrized \( Q\)-function by dual variables \( \rho(s,a) \) via mean square loss, and then, applying the stochastic mirror descent w.r.t. \( V \) and \( \pi \) with gradient estimator (13) and (14) where \( \eta \in [0, 1] \).
**Algorithm 1** Smoothed Dual Embedding Control with Experience Replay

1: Initialize $V$, $\pi$, $\pi_0$, and $\rho$ randomly, set $\epsilon$
2: for episode $i = 1, \ldots, T$ do
3: for size $k = 1, \ldots, K$ do
4: Collect transition $(s, a, r, s')$ into $D$ by executing behavior policy $\pi_b$.
5: end for
6: for iteration $j = 1, \ldots, N$ do
7: Update $w_j^\pi = \arg\min_{w_j} \mathbb{E}_{(s,a,s') \sim D} \left[ \delta(s, a, s') - \rho(s, a) \right]^2$.
8: Decay the stepsize $\zeta_j$ in rate $O(1/j)$.
9: Compute the stochastic gradients w.r.t. $w_V$ and $w_\pi$ as $\nabla_{w_V} \bar{\ell}(V, \pi)$ and $\nabla_{w_\pi} \bar{\ell}(V, \pi)$.
10: Update the parameters of primal function by solving the prox-mappings, i.e.,
   update $V$: \( w_V^j = P_{w_V}^{-1}(\zeta_j \nabla_{w_V} \bar{\ell}(V, \pi)) \)
   update $\pi$: \( w_\pi^j = P_{w_\pi}^{-1}(\zeta_j \nabla_{w_\pi} \bar{\ell}(V, \pi)) \)
11: end for
12: Update behavior policy $\pi_b = \pi^N$.
13: end for

**Remark (Connection to TRPO and natural policy gradient):** The update of $w_\pi$ is highly related to trust region policy optimization (TRPO) of [Schulman et al. 2015] and natural policy gradient (NPG) [Kakade 2002, Rajeswaran et al. 2017] when we set $D_\pi$ to $KL$-divergence. Specifically, in [Kakade 2002] and [Rajeswaran et al. 2017], $w_\pi$ is update by $\arg\min_{w_\pi} \mathbb{E} [\langle w_\pi, \nabla_{w_\pi} \log \pi^f(a|s)A(a, s) \rangle] + \frac{1}{\eta} KL(\pi_\pi||\pi_{\pi_0})$, which is similar to $P_{w_\pi}^{-1}$, with the difference in replacing the $\log \pi^f(a|s)A(a, s)$ with our gradient, while in [Schulman et al. 2015], a related optimization with hard constraints is used for update policy, i.e., $\arg\min_{w_\pi} \mathbb{E} [\pi(a|s)A(a, s)]$ s.t. $KL(\pi||\pi_{\pi_0}) \leq \eta$. Although these operations are similar to $P_{w_\pi}^{-1}$, we emphasize that the estimation of advantage, denoted as $A(s, a)$, and the update of policy $\pi$ are separated in NPG and TRPO. Arbitrary policy evaluation algorithm can be adopted for estimating the value function for current policy. While in our algorithm, $(1 - \eta)\delta(s, a) + \eta \rho^*(s, a) - V(s)$ is different from the vanilla advantage function, which is designed appropriate for off-policy particularly, and the estimation of $\rho(s, a)$ and $V(s)$ is also integrated as the whole part.

5 **Theoretical Analysis**

In this section, we provide our main results of the theoretical behavior of the proposed algorithm under the setting in [Antos et al. 2008] where samples are prefixed and from one single off-policy sample path. We consider the case where $\eta = 1$ with the equivalent optimization to simplify. For general $\eta \in [0, 1]$, we can achieve a similar result to Theorem 6 by replacing $\epsilon_{\text{app}}$ with a combination of $\eta\epsilon_{\text{app}}$ and $(1 - \eta)\gamma (\max_{V \in \mathcal{V}^*} \mathbb{V}[V] + \|R(s, a)\|_{\infty})$. We omit here due to the space limitation.

Based on the construction of the algorithm, the convergence analysis essentially boils down to several parts:

i) the bias from smoothing Bellman optimality equation;

ii) the statistical error induced when learning with finite samples from one single sample path;

iii) the approximation error introduced by function parametrization (both for primal and dual variables) in [10];

iv) the optimization error when solving the finite-sample version of the saddle point problem [10] within a fixed number of iterations.
Notations. The parametrized function class of value function $V$, policy $\pi$, and dual variable $\nu$ are denoted as $\mathcal{V}_w, \mathcal{P}_w, \mathcal{H}_w$, respectively. Denote $L_w(V, \pi; \nu)$ as the parametrized objective of $L(V, \pi; \nu)$ and $(\hat{V}_w^*, \hat{\pi}_w^*)$ as the corresponding optimal solution. Denote $L_{w,T}(V, \pi; \nu)$ as the finite sample approximation of $L_w(V, \pi; \nu)$ using $T$ samples and $(\hat{V}_w^T, \hat{\pi}_w^T)$ as the corresponding optimal solution. The function approximation error between two function classes $\mathcal{P}_w$ and $\mathcal{P}$ is defined as $\epsilon_{\text{approx}} := \sup_{\pi \in \mathcal{P}_w} \inf_{\pi' \in \mathcal{P}} \|\pi - \pi'\|_\infty$ and for $V$ and $\nu$ as $\epsilon_{\text{approx}}^V$ and $\epsilon_{\text{approx}}^\nu$. The $L_2$ norm of any function is defined as $\|f\|_2^2 = \int f(s, a)^2 \mu(s) \pi_b(a|s)dsda$. We also introduce a scaled norm for value function $V$: $\|V\|_{\mu, \pi}^2 = \int (\gamma E_{s'|s,a} [V(s')] - V(s))^2 \mu(s) \pi_b(a|s)dsda$; this is indeed a well-defined norm since $\|V\|_{\mu, \pi} = \|\gamma P - I\|_2^2 V$ and $I - \gamma P: \mathcal{S} \to \mathcal{S} \times \mathcal{A}$ is injective.

We make the following standard assumptions about the MDPs:

Assumption 1 (MDP regularity) We assume $\|R(s, a)\|_\infty \leq C_R$, and there exists an optimal policy, $\pi^*(a|s)$, such that $\|\log \pi^*(a|s)\|_\infty \leq C_\pi$.

Assumption 2 (Sample path property [Antos et al., 2008]) Denote $\mu(s)$ as the stationary distribution of behavior policy $\pi_b$ over the MDP. We assume $\pi_b(a|s) > 0, \forall (s, a) \in \mathcal{S} \times \mathcal{A}$, and the corresponding Markov process $P^\pi(s'|s)$ is ergodic. We further assume that $(s_i)_{i=1}^\infty$ is strictly stationary and exponentially $\beta$-mixing with a rate defined by the parameters $\beta, b, \kappa$.

Assumption 1 ensures the solvability of the MDP and boundedness of the optimal value functions, $V^*$ and $\hat{V}^*$. Assumption 2 ensures $\beta$-mixing property of the samples $(s_i, a_i, R_i)_{i=1}^t$ (see e.g., Proposition 4 in Carrasco and Chen (2002)) and is often necessary to prove large deviation bounds.

The error introduced by smoothing has been characterized in Section 3.2. The approximation error is tied to the flexibility of the parametrized function classes of $V, \pi, \nu$, and has been widely studied in approximation theory. Here we mainly focus on investigating the statistical error and optimization error. For sake of simplicity, here we only brief the main results and ignore the constant factors whenever possible. Detailed theorems and proofs can be found in Appendix D.

Sub-optimality. Denote $l_w(V, \pi) = \max_{\nu} L_w(V, \pi; \nu)$ and $l_{w,T}(V, \pi) = \max_{\nu} L_{w,T}(V, \pi; \nu)$. The statistical error is defined as $\epsilon_{\text{stat}}(T) := l_{w,T}(\hat{V}_w^T, \hat{\pi}_w^T) - l_{w}(\hat{V}_w^*, \hat{\pi}_w^*)$. Invoking a generalized version of Pollard’s tail inequality to $\beta$-mixing sequences and prior results in Antos et al. (2008) and Haussler (1995), we show that

Theorem 5 (Statistical error) Under Assumption 2, it holds with at least probability $1 - \delta$, $\epsilon_{\text{stat}}(T) \leq 2\sqrt{\frac{M(\max(M/b, 1)^{1/\kappa})}{CT^\kappa}} M_{C_2}$, where $M, C_2$ are some constants.

Combining the error caused by smoothing and function approximation, we show that the difference between $\hat{V}_w^T$ and $V^*$ under the norm $\|\cdot\|_{\mu, \pi}$ is given by

Theorem 6 (Total error) Let $\hat{V}_w^T$ be a candidate solution output from the proposed algorithm based on off-policy samples, with high probability, we have $\|\hat{V}_w^T - V^*\|_{\mu, \pi} \leq \epsilon_{\text{app}}(\lambda) + \epsilon_{\text{sm}}(\lambda) + \epsilon_{\text{stat}}(T) + \epsilon_{\text{opt}}$, where $\epsilon_{\text{app}}(\lambda) := \mathcal{O}(\epsilon_{\text{approx}}^V + \epsilon_{\text{approx}}^\nu + (\epsilon_{\text{approx}}^\nu)^2)$ corresponds to the approximation error, $\epsilon_{\text{sm}}(\lambda) := \mathcal{O}(X^2)$ corresponds to the bias induced by smoothing, and $\epsilon_{\text{stat}}(T) := \mathcal{O}(1/\sqrt{T})$ corresponds to the statistical error, and $\epsilon_{\text{opt}}$ is the optimization error of solving $l_{w,T}(V, \pi)$ within a fixed budget.

There exists a delicate trade-off between the smoothing bias and approximation error. Using large $\lambda$ increases the smoothing bias but decreases the approximation error since the solution function space is better behaved. The concrete correspondence between $\lambda$ and $\epsilon_{\text{app}}(\lambda)$ depends on the specific form of the function approximators, which is beyond the scope of this paper. When $\lambda \to 0$, $T \to \infty$ and the approximation is good enough, the solution $\hat{V}_w^T$ will converge to the optimal value function $V^*$. 
Convergence Analysis. It is well-known that for convex-concave saddle point problems, applying stochastic mirror descent ensures global convergence in a sublinear rate; see e.g., \cite{Nemirovski2009}. However, this no longer holds for problems without convex-concavity. On the other hand, since our algorithm solves exactly the dual maximization problem at each iteration (which is convex), it can be essentially regarded as a special case of the stochastic mirror descent algorithm applied to solve the non-convex minimization problem \( \min \ell_{w,T}(V,\pi) \). The latter was proven to converge sublinearly to the stationary point when stepsize is diminishing and Euclidean distance is used for the prox-mapping \cite{Ghadimi2013}. For completeness, we list the result below.

\begin{theorem}[(\cite{Ghadimi2013}, resp. Corollary 2.2)] \label{th:mirror}
Consider the case when Euclidean distance is used in the algorithm. Assume that the parametrized objective \( l_{w,T}(V,\pi) \) is \( K \)-Lipschitz and variance of stochastic gradient \( \nabla_{w}l_{w,T}(V,\pi) \) is bounded by \( \sigma^{2} \). Let the algorithm run \( N \) iterations with stepsize \( \zeta_{k} = \min\{\frac{1}{K},\frac{D'}{\sigma\sqrt{N}}\} \) for some \( D' > 0 \) and output \( w^{1},\ldots,w^{N} \). Setting the candidate solution to be \((\hat{V}_{w}^{T},\hat{\pi}_{w}^{T})\) with \( w \) randomly chosen from \( w^{1},\ldots,w^{N} \) such that \( P(w = w^{j}) = \frac{2\zeta_{j}^{2} - K\zeta_{j}^{2}}{\sum_{j=1}^{N}(2\zeta_{j}^{2} - K\zeta_{j}^{2})} \), then it holds that \( \mathbb{E}\left[\left\|\nabla l_{w}(\hat{V}_{w},\hat{\pi}_{w})\right\|\right] \leq \frac{KD^{2}}{N} + (D' + \frac{D}{\sigma\sqrt{N}})\sqrt{N} \) where \( D := \sqrt{2(\ell_{w,T}(V_{1},\pi_{1}) - \min_{w_{1},\ldots,w_{n}}\ell_{w,T}(V,\pi))/K} \) represents the distance of the initial solution to the optimal solution.
\end{theorem}

The above result implies that the algorithm converges sublinearly to a stationary point. Note that the Lipschitz constant is inherently dependent on the smoothing parameter \( \lambda \): the Lipschitz constant gets worse when \( \lambda \) increases.

6 Related Work

The algorithm is related to the reinforcement learning with entropy-regularized MDP model. Different from the motivation in our method where the entropy regularization is introduced in dual form for smoothing \cite{Nesterov2005}, the entropy-regularized MDP has been proposed for balancing exploration and exploitation \cite{Haarnoja2017}, taming the noises in observations \cite{Rubin2012,Fox2015}, and tractability \cite{Todorov2007}.

Specifically, \cite{Fox2015} proposed soft Q-learning which extended the Q-learning with tabular form for the new Bellman optimality equation corresponding to the finite state finite action entropy-regularized MDP. The algorithm does not accommodate for function approximator due to the intractability of the log-sum-exp operation in the soft Q-learning update. To avoid such difficulty, \cite{Haarnoja2017} reformulates the update as an optimization which is approximated by samples from Stein variational gradient descent (SVGD) sampler. Another related algorithm is proposed in \cite{Asadi2016} the intractability issue of the log-sum-exp operator, named as ‘mellowmax’, is avoided by optimizing for a maximum entropy policy in each update. The resulting algorithm resembles to SARSA with particular policy. \cite{Liu2017} focuses on the soft Bellman optimality equation with the ‘mellowmax’ operator following a similar way with \cite{Asadi2016}. The only difference is that a Bayesian policy parametrization is used in \cite{Liu2017} which is updated by SVGD. By noticing the duality between soft Q-learning and the maximum entropy policy, \cite{Neu2017, Schulman2017} investigate the equivalence between these two types of algorithms.

Besides the difficulty to generalize these algorithms to multi-step trajectories in off-policy setting, the major drawback of these algorithms is the lack of theoretical guarantees when accompanying with function approximators. It is not clear whether the algorithms converge or not, do not even mention the quality of the stationary points.

On the other hand, \cite{Nachum2017a,Nachum2017} also exploit the consistency condition in Theorem \ref{th:mirror} and propose the PCL algorithm which optimizes the upper bound of the mean square consistency Bellman error \( \ell_{w,T}(V,\pi) \). The same consistency condition is also discovered in \cite{Rawlik2012}, and the proposed \( \Phi \)-learning algorithm can be viewed as fix-point iteration version of the the unified PCL with tabular Q-function. However, as we discussed in Section \ref{sec:related_work}, the PCL algorithms becomes biased in stochastic environment, which may lead to inferior solutions.
7 Experiments

We test the proposed smoothed dual embedding algorithm (SDEC), on several continuous control tasks from the OpenAI Gym benchmark (Brockman et al., 2016) using the MuJoCo simulator (Todorov et al., 2012), comparing with trust region policy optimization (TRPO) (Schulman et al., 2015) and the proximal policy optimization (PPO) (Schulman et al., 2017b). Since the TRPO and PPO are only applicable for on-policy setting, for fairness, we also restrict the SDEC to on-policy setting. However, as we show in the paper, the SDEC is able to exploit the off-policy samples efficiently. We use the Euclidean distance for $w_V$ and the $KL$-divergence for $w_\pi$ in the experiments. We emphasize that other Bregman divergences are also applicable. Following the comprehensive comparison in Henderson et al. (2017), the implementation of the TRPO and PPO affects the performance of algorithms. For a fair comparison, we use the codes from https://github.com/joschu/modular_rl reported to have achieved the best scores in Henderson et al. (2017).

We ran the algorithm with 5 random seeds and reported the average rewards with 50% confidence intervals. The empirical comparison results are illustrated in Figure 1. We can see that in all these tasks, the proposed SDEC achieves significantly better performance than the other algorithms. The experiment setting is reported below.

Policy and value function parametrization. For fairness, we use the same parametrization of policy and value functions across all algorithms. The choices of parametrization are largely based on the recent
paper by Rajeswaran et al. [2017], which shows the natural policy gradient with RBF neural network achieves the state-of-the-art performances of TRPO on MuJoCo. For the policy distribution, we parametrize it as $\pi_{\theta}(a|s) = \mathcal{N}(\mu_{\theta}(s), \Sigma_{\theta})$, where $\mu_{\theta}(s)$ is a two-layer neural nets with the random features of RBF kernel as the hidden layer and the $\Sigma_{\theta}$ is a diagonal matrix. The RBF kernel bandwidth is chosen via median trick [Dai et al. 2014] [Rajeswaran et al. 2017]. Same as [Rajeswaran et al. 2017], we use 100 hidden nodes in InvertedDoublePendulum, Swimmer, Hopper, and use 500 hidden nodes in HalfCheetah. Since the TRPO and PPO uses linear control variable as $V$, we also adapt the parametrization for $V$ in our algorithm. However, SDEC can adopt arbitrary function approximators.

**Training hyperparameters.** For all algorithms, we set $\gamma = 0.995$ and stepsize $= 0.01$. A batch size of 52 trajectories was used in each iteration. For TRPO, the CG damping parameter is set to be $10^{-4}$. For SDEC, $\eta$ was set to 0.001 and $\lambda$ from a grid search in $\{0.001, 0.004, 0.016\}$.

**8 Conclusion**

We provide a new optimization perspective of the Bellman optimality equation, based on which we develop the smoothed dual embedding control for the policy optimization problem in reinforcement learning. The algorithm is provably convergent with nonlinear function approximators using off-policy samples by solving the Bellman optimality equations. We also provide PAC-learning bound to characterize the sample complexity based on one single off-policy sample path. Preliminary empirical study shows the proposed algorithm achieves comparable or even better than the state-of-the-art performances on MuJoCo tasks.

**Acknowledgments**

Part of this work was done during BD’s internship at Microsoft Research, Redmond. Part of the work was done when LL was with Microsoft Research, Redmond. LS is supported in part by NSF IIS-1218749, NIH BIGDATA 1R01GM108341, NSF CAREER IIS-1350983, NSF IIS-1639792 EAGER, NSF CNS-1704701, ONR N00014-15-1-2340, Intel ISTC, NVIDIA and Amazon AWS.

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Appendix

A Properties of Smoothed Bellman Optimality Equation

In this section, we provide the details of the proofs for the properties of the smoothed Bellman optimality equation.

After applying the smoothing technique Nesterov (2005), we obtain a new Bellman operator, \( \tilde{T} \), which is contractive. By such property, we can guarantee the uniqueness of the solution. Specifically,

**Proposition 1 (Uniqueness)** The \( \tilde{T} \) is a contraction operator, therefore the smoothed Bellman optimality equation (5) has unique solution.

**Proof** Consider \( V_1, V_2 : \mathcal{S} \to \mathbb{R} \),
\[
\|\tilde{T}V_1 - \tilde{T}V_2\|_\infty = \left\| \max_\pi (\pi, R(s,a) + \gamma \mathbb{E}_{s'|s,a}[V_1(s')]) + \lambda H(\pi) - \max_\pi \left( (\pi, R(s,a) + \gamma \mathbb{E}_{s'|s,a}[V_2(s')]) + \lambda H(\pi) \right) \right\|_\infty \\
\leq \left\| \max_\pi (\pi, \gamma \mathbb{E}_{s'|s,a}[V_1(s') - V_2(s')]) \right\|_\infty \\
\leq \gamma \left\| \mathbb{E}_{s'|s,a}[V_1(s') - V_2(s')] \right\|_\infty \leq \gamma \|V_1 - V_2\|_\infty.
\]
By the Banach fixed point theorem, the existence and uniqueness is guaranteed for the smoothed Bellman optimality equation.

A similar result is also presented in Fox et al. (2015); Asadi and Littman (2016). We characterize the bias introduced by the smoothing technique, i.e.,

**Proposition 2 (Smoothing bias)** Let \( V^* \) and \( \tilde{V}^* \) be fixed points of (3) and (5), respectively. It holds
\[
\|V^*(s) - \tilde{V}^*(s)\|_\infty \leq \frac{\gamma \lambda}{1 - \gamma} \max_{\pi \in \mathcal{P}} H(\pi).
\]

As \( \lambda \to 0 \), \( \tilde{V}^* \) converges to \( V^* \) pointwisely.

**Proof** We denote \( Q(s,a) = R(s,a) + \gamma \mathbb{E}_{s'|s,a}[V(s')] \).
\[
\|V^*(s) - \tilde{V}^*(s)\|_\infty = \left\| \max_{\pi \in \mathcal{P}} \left( \sum_a \pi(a|s)Q(s,a) \right) - \max_{\pi \in \mathcal{P}} \left( \sum_a \pi(a|s)Q^*(s,a) + \lambda H(\pi) \right) \right\|_\infty \\
\leq \left\| \max_{\pi \in \mathcal{P}} \left( \sum_a \pi(a|s) \left( Q^*(s,a) - \tilde{Q}^*(s,a) \right) + \lambda H(\pi) \right) \right\|_\infty \\
= \gamma \left\| \max_{\pi \in \mathcal{P}} \left( \sum_a \pi(a|s) \mathbb{E}_{s'|s,a} \left[ V^*(s') - \tilde{V}^*(s') \right] + \lambda H(\pi) \right) \right\|_\infty \\
\leq \gamma \left\| \max_{\pi \in \mathcal{P}} \left( \sum_a \mathbb{E}_{s'|s,a} \left[ V^*(s') - \tilde{V}^*(s') \right] + \lambda H(\pi) \right) \right\|_\infty \\
\leq \gamma \left\| \sum_a \mathbb{E}_{s'|s,a} \left[ V^*(s') - \tilde{V}^*(s') \right] + \lambda \max_{\pi \in \mathcal{P}} H(\pi) \right\|_\infty \\
\leq \gamma \|V^*(s') - \tilde{V}^*(s')\|_\infty + \gamma \lambda \max_{\pi \in \mathcal{P}} H(\pi),
\]
which implies the conclusion.

The smoothed Bellman optimality equation involves a log-sum-exp operator to approximate the max-operator, which increases the nonlinearity of the equation. We further characterize the solution of the smoothed Bellman optimality equation, by the temporal consistency conditions.

**Theorem 3 (Temporal consistency)** Let $\tilde{V}^*$ be the fixed point of (11) and $\tilde{\pi}^*$ be the corresponding policy that attains the maximum in the RHS of (11). Then $(\tilde{V}^*, \tilde{\pi}^*)$ is the unique solution that satisfies

$$V(s) = R(s, a) + \gamma \mathbb{E}_{s'|s,a} [V(s')] - \lambda \log \pi(a|s), \ \forall a \in A.$$ (17)

**Proof Necessity.** Given the definition of $Q(s,a)$, denote

$$\mathcal{L}_s(Q) = \max_{\pi \in \mathcal{P}} \sum_{a \in A} \pi(a|s)Q(s,a) + \lambda H(\pi) = \lambda \log \left( \sum_{a \in A} \exp \left( \frac{Q(s,a)}{\lambda} \right) \right)$$

by the convexity of $\mathcal{L}_s(Q)$, we have a unique $\pi_Q \in \mathcal{P}$ as

$$\pi_Q(s|a) = \arg\max_{\pi \in \mathcal{P}} \sum_{a \in A} \pi(a|s)Q(s,a) + \lambda H(\pi) = \exp \left( \frac{Q(s,a) - \mathcal{L}_s(Q)}{\lambda} \right),$$

which implies

$$\mathcal{L}_s(Q) = Q(s,a) - \lambda \log \pi_Q(a|s), \ \forall a \in A.$$ Then, we can rewrite the smoothed Bellman optimality equation Eq. (5) as

$$V(s) = \max_{\pi \in \mathcal{P}} \left( \sum_{a \in A} \pi(a|s)Q(s,a) + \lambda H(\pi) \right)$$

$$= \mathcal{L}_s(Q) = Q(s,a) - \lambda \log \pi_Q(a|s), \ \forall a \in A$$

$$= R(s, a) + \gamma \mathbb{E}_{s'|s,a} [V(s')] - \lambda \log \pi_Q(a|s), \ \forall a \in A.$$ (18)

Obviously, this equation is a necessary condition, i.e., the optimal $\tilde{V}^*$ and $\pi_{\tilde{Q}}$ satisfy such equation. In fact, we can show the sufficiency of such equation.

**Sufficiency.** Assume $\tilde{V}$ and $\tilde{\pi}$ satisfies (8),

$$\tilde{V}(s) = R(s, a) + \gamma \mathbb{E}_{s'|s,a} [\tilde{V}(s')] - \lambda \log \tilde{\pi}(a|s) \ \forall a \in A$$

$$\Rightarrow \pi(a|s) = \exp \left( \frac{R(s,a) + \gamma \mathbb{E}_{s'|s,a} [\tilde{V}(s')] - \tilde{V}(s)}{\lambda} \right) \ \forall a \in A.$$ Recall $\pi(\cdot|s) \in \mathcal{P}$, we have

$$\sum_{a \in A} \exp \left( \frac{R(s,a) + \gamma \mathbb{E}_{s'|s,a} [\tilde{V}(s')] - \tilde{V}(s)}{\lambda} \right) = 1$$

$$\Rightarrow \sum_{a \in A} \exp \left( \frac{R(s,a) + \gamma \mathbb{E}_{s'|s,a} [\tilde{V}(s')]}{\lambda} \right) = \exp \left( \frac{\tilde{V}(s)}{\lambda} \right)$$

$$\Rightarrow \tilde{V}(s) = \lambda \log \left( \sum_{a \in A} \exp \left( \frac{R(s,a) + \gamma \mathbb{E}_{s'|s,a} [\tilde{V}(s')]}{\lambda} \right) \right)$$

$$\Rightarrow \tilde{V} = \tilde{V}^*.$$
The same conditions have been re-discovered several times, e.g., (Rawlik et al. 2012; Nachum et al. 2017a), from a completely different point of views.

B Variance Cancellation via the Saddle Point Formulation

The second term in the saddle point formulation \(\{11\}\) will cancel the variance \(\mathbb{V}_{s,a,s'}[\gamma V(s')]\). Specifically,

**Theorem 8** Given \(V\) and \(\pi\), we have

\[
\max_{\rho \in \mathcal{F}(S \times A)} -\mathbb{E}_{s,a,s'} \left[ (R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - \rho(s, a))^2 \right] = -\mathbb{V}_{s,a,s'}[\gamma V(s')].
\]

**Proof**

\[
\min_{\rho} \mathbb{E}_{s,a,s'} \left[ (R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - \rho(s, a))^2 \right] = \min_{\rho} \mathbb{E}_{s,a,s'} \left[ (R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - \rho(s, a))^2 \right] + \mathbb{E}_{\gamma V(s')} \mathbb{E}_{s,a,s'} \left[ (\gamma V(s') - \mathbb{E}_{\gamma V(s')}[\gamma V(s')])^2 \right]
\]

\[
= \min_{\rho} \mathbb{E}_{s,a,s'} \left[ (R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - \rho(s, a))^2 \right] + \mathbb{E}_{s,a} \left[ (\gamma V(s') - \mathbb{E}_{\gamma V(s')}[\gamma V(s')])^2 \right] = \mathbb{V}_{s,a,s'}[\gamma V(s')],
\]

where \(\rho^*(s, a) = R(s, a) + \mathbb{E}_{\gamma V(s')}[\gamma V(s')] - \lambda \log \pi(a|s)\).

C Details of the Derivation of the Algorithm

In this section, we provide the details of the derivation of the algorithms, including the unbiased estimator and multi-step/eligibility-trace extension.

C.1 Unbiasedness of Gradient Estimator

We provide the details of the derivation of the gradient estimator in Eq. (13) and (14).

**Theorem (Unbiased gradient estimator)** \(\{4\}\) Denote \(\hat{\ell}(V, \pi) = \max_{\rho \in \mathcal{F}(S \times A)} L_{\pi}(V, \pi; \rho)\), and \(\rho^* = \arg\max_{\rho \in \mathcal{F}(S \times A)} L_{\pi}(V, \pi; \rho)\), we have the unbiased gradient estimator as

\[
\nabla \hat{\ell}(V, \pi) = \mathbb{E} \left[ \delta(s, a) (\gamma \nabla_{wV} V(s') - \nabla_{wV} V(s)) \right] - \eta \mathbb{E} \left[ (\delta(s, a) - \rho(s, a)) \nabla_{wV} V(s') \right],
\]

\[
\nabla \pi \hat{\ell}(V, \pi) = -\lambda \mathbb{E}_{s,a,s'} \left[ ((1 - \eta)\delta(s, a) + \eta \rho^*(s, a) - V(s)) \right] \nabla \pi \log \pi(a|s).
\]
Proof. We compute the gradient w.r.t. \( V \), the same argument is also hold for gradient w.r.t. \( \pi \).

\[
\nabla_V \tilde{\ell}(V, \pi) = 2\mathbb{E} [\delta(s, a) (\gamma \nabla_V V(s') - \nabla_V V(s))] - 2\eta \gamma \mathbb{E} [\delta(s, a) \nabla_V V(s')]
+ 2\eta \gamma \mathbb{E} [(\delta(s, a) - \rho(s, a)) \nabla_V \rho(s, a)] \tag{22}
\]

where the third term equals zero due to the optimality of \( \rho^*(s, a) \).

C.2 Multi-step Extension

The proposed framework and algorithm can be easily combined with the multi-step and eligibility-trace extensions of the Bellman optimality equation. As we derived in Section 3.2, we have the multi-step consistency condition as

\[
V(s_0) = \sum_{t=0}^{T} \gamma^t \mathbb{E}_{s_t \sim s, \{a_t\}_{t=0}^{T}} \left[ R(s_t, a_t) - \lambda \log \pi (a_t | s_t) \right] + \gamma^{T+1} \mathbb{E}_{s_{T+1} \sim s, \{a_t\}_{t=0}^{T}} \left[ V(s_{T+1}) \right],
\]

for any \( \{a_t\}_{t=0}^{T} \in \mathcal{A}^T \). Then, we obtain the objective as

\[
\min_{V, \pi} \mathbb{E}_{s_0 \sim s, \{a_t\}_{t=0}^{T}} \left[ \sum_{t=0}^{T} \gamma^t \mathbb{E}_{s_t \sim s, \{a_t\}_{t=0}^{T}} \left[ R(s_t, a_t) - \lambda \log \pi (a_t | s_t) \right] + \gamma^{T+1} \mathbb{E}_{s_{T+1} \sim s, \{a_t\}_{t=0}^{T}} \left[ V(s_{T+1}) - V(s_0) \right] \right]^2.
\]

Applying the Fenchel dual and interchangeability principle, we arrive

\[
\min_{V, \pi} \max_{\nu} \mathbb{E}_{s_0 \sim s, \{a_t\}_{t=0}^{T}} \left[ \nu(s_0, \{a_t\}_{t=0}^{T}) \left( \sum_{t=0}^{T} \gamma^t \mathbb{E}_{s_t \sim s, \{a_t\}_{t=0}^{T}} \left[ R(s_t, a_t) - \lambda \log \pi (a_t | s_t) \right] + \gamma^{T+1} \mathbb{E}_{s_{T+1} \sim s, \{a_t\}_{t=0}^{T}} \left[ V(s_{T+1}) - V(s_0) \right] \right)^2 \right] - \frac{1}{2} \mathbb{E}_{s_0 \sim s, \{a_t\}_{t=0}^{T}} \left[ \nu^2(s_0, \{a_t\}_{t=0}^{T}) \right]
\]

Denote \( (s_0, a_0) := (s_0, a_0, a_1, \ldots, a_T, s_{T+1}) \) as a sub-trajectory with length \( T+1 \). Denote \( \delta(s_0, a_0) := \sum_{t=0}^{T} (\gamma^t R(s_t, a_t) - \gamma^t \lambda \log \pi (a_t | s_t) + \gamma^{T+1} V(s_{T+1})) \), with the substitution \( \nu(s_0, \{a_t\}_{t=0}^{T}) = \rho(s_0, \{a_t\}_{t=0}^{T}) - V(s_0) \) into above and introduction of \( \eta \) as in one-step saddle point problem derivation, we obtain the objective of the corresponding saddle point problem in the multi-step setting as

\[
L(V, \pi; \rho) := \mathbb{E}_{s_0 \sim s, a_0 \sim a_0, \{a_t\}_{t=0}^{T}} \left[ \left( \delta(s_0, a_0) - V(s_0) \right)^2 - \eta (\delta(s_0, a_0) - \rho(s_0, a_0))^2 \right] \tag{23}
\]

where the dual function now is \( \rho(s_0, a_0) \) and is a function on \( S \times \mathcal{A}^{T+1} \) and \( \eta \geq 0 \) is used to balance potential bias and variance. It is straightforward to generalize Theorem 4 to multi-step setting. Therefore, the Algorithm 1 can be adapted by switching to the multi-step gradient estimators. When extending to eligibility-trace, the objective can be obtained in the same vein as in next section.
C.3 Eligibility-trace Extension

With the multi-step consistency condition in Section 3.2, the consistency condition for eligibility-trace is weighted combinations of \( k = 0, \ldots, \infty \) step conditions with exponential distributed weights, \( \text{i.e., } \forall \{a_t\}_{t=0}^T \in \mathcal{A}^T \),

\[
V(s_0) = (1 - \zeta) \sum_{T=0}^{\infty} \zeta^T \left( \sum_{t=0}^{T} \gamma^t \mathbb{E}_{s_t|x_0,(a_i)_{i=0}^T} [R(s_t, a_t) - \lambda \log \pi(a_t|s_t)] \right),
\]

\[
+ \gamma^{T+1} \mathbb{E}_{s_{T+1}|x_0,(a_i)_{i=0}^T} [V(s_{T+1})].
\]

Then, similarly, we have the saddle point optimization for the eligibility-trace extension as

\[
\min_{V, \pi \in \mathcal{P}} \max_{\rho} \mathbb{E}_{s_t,a_t} \left\{ \left( 1 - \zeta \right) \sum_{T=0}^{\infty} \zeta^T \delta(s_{0:T+1}, a_{0:T}) - V(s_0) \right\}^2
\]

\[
- \eta \mathbb{E}_{s_t,a_t} \left\{ \left( 1 - \zeta \right) \sum_{T=0}^{\infty} \zeta^T \delta(s_{0:T+1}, a_{0:T}) - \rho(s_0, \{a_t\}_{t=0}^\infty) \right\}^2.
\]

D Proof Details of the Theoretical Analysis

In this section, we provide the details of the analysis in Theorem 5 and Theorem 6. We first provide regularity of the \( V^* \) under Assumption 1. In fact, due to Assumption 1, we have \( \int_{a \in \mathcal{A}} R(s, a)^2 \mu(s)ds \leq C_R^2 \), \( \|V^*\|_\infty \leq \frac{C_R}{1 - \gamma} \). Moreover, \( \|V^*\|_\mu \) is also bounded,

\[
\|V^*\|_\mu^2 = \int (V^*(s))^2 \mu(s)ds = \int \left( R(s, a) + \gamma \mathbb{E}_{s'|s,a} [V^*(s')] \right)^2 \pi^*(a|s)\mu(s)ds
\]

\[
\leq 2 \int (R(s, a))^2 \pi^*(a|s)\mu(s)ds + 2\gamma^2 \int \left( \mathbb{E}_{s'|s,a} [V^*(s')] \right)^2 \pi^*(a|s)\mu(s)ds
\]

\[
\leq 2 \left\| \mathbb{E}_{a \in \mathcal{A}} R(s, a) \right\|^2_\mu + 2\gamma^2 \int \left( \int P^*(s'|s)\mu(s)ds \right) (V^*(s'))^2 ds'
\]

\[
\leq 2C_R^2 + 2\gamma^2 \|V^*(s')\|_\infty^2 \int P^*(s'|s)\mu(s)ds ds'
\]

\[
\leq 2C_R^2 + 2\gamma^2 \|V^*(s')\|_\infty^2 = 2C_R^2 \left( 1 + \frac{\gamma^2}{(1 - \gamma)^2} \right).
\]

Similarly, we can bound \( \|V^*_\mu\|_\mu^2 \leq 2C_R^2 \left( 1 + \frac{\gamma^2}{(1 - \gamma)^2} \right) := C_\mu^2 \).

We should emphasize that although the Assumption 1 provides the boundedness of \( V^* \) and \( \log \pi^*(a|s) \), it does not imply the continuity and smoothness. In fact, as we will see later, \( \lambda \) plays an important role to balance between the parametrization approximation error to the solution of the smoothed Bellman optimality equation and the bias induced by such smoothness preference.
D.1 Error Decomposition

We first define the objectives which will be used in the analysis.

\[ l(V) = E_s \left( \max_{a \in A} R(s, a) + \gamma E_{s', a} [V(s')] - V(s) \right)^2 \],

where \( \pi(a|s) \) denotes the policy or \( \pi \) and \( E_{s', a} [V(s')] \) denotes the expected value of the next state. We denote the family of value functions and policies by parametrization as \( \mathcal{D} \).

\[ \tilde{l}(V) = E_s \left( \max_{a \in A} \sum_{a \in A} \pi(a|s) \left( R(s, a) + \gamma E_{s', a} [V(s')] - V(s) \right) \right)^2, \]

\[ \bar{l}(V, \pi) = E_s, a \left( \frac{R(s, a) + \gamma E_{s', a} [V(s')] - V(s) - \lambda \log \pi(a|s)}{\Delta(s, a)} \right)^2, \]

\[ \tilde{l}_w(V, \pi) = \max_{\nu \in \mathcal{H}_w(S \times A)} 2E_{s, a, s'} [\nu(s, a) (R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - V(s))] - E_{s, a, s'} [\nu^2(s, a)], \]

where \( \mathcal{V}_\mu \) is the set of functions such that \( \|f\|_\mu \leq C \). In regular MDP with Assumption 1, with appropriate \( C \), such constraint does not introduce any loss. We denote the optimal solutions as \( V^* = \arg\min_{V} l(V), \), \( \tilde{V}^* = \arg\min_{V} \tilde{l}(V), \), \( \bar{V}^*, \bar{\pi}^* = \arg\min_{V \in \mathcal{V}_\mu, \pi} \bar{l}(V, \pi), \), \( \tilde{V}^*, \tilde{\pi}^* = \arg\min_{V \in \mathcal{V}_\mu, \pi} \tilde{l}_w(W, \pi), \), \( \bar{V}^*, \bar{\pi}^* = \arg\min_{V \in \mathcal{V}_\mu, \pi} \bar{l}_w(W, \pi). \)

Corollary 9 \( \bar{l}(V, \pi) - \tilde{l}_w(V, \pi) \leq (K + C_\infty)\epsilon_{\text{approx}} \), where \( \epsilon_{\text{approx}} = \sup_{\nu \in \mathcal{C}} \inf_{h \in \mathcal{H}} \|\nu - h\|_\infty \) with \( \mathcal{C} \) denoting the Lipschitz continuous function space and \( \mathcal{H} \) denoting the hypothesis space.

Proof Denote the \( \phi(V, \pi, \nu) := E_{s, a, s'} [\nu(s, a) (R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - V(s))] - \frac{1}{2}E_{s, a, s'} [\nu^2(s, a)], \)

we have \( \phi(V, \pi, \nu) \) is \( (K + C_\infty) \)-Lipschitz continuous w.r.t. \( \|\cdot\|_\infty \). Denote \( \nu_{V, \pi}^* = \arg\max_{\nu} \phi(V, \pi, \nu), \) \( \nu_{V, \pi}^{\|\cdot\|_\infty} = \arg\max_{\nu} \phi(V, \pi, \nu) \|\cdot\|_\infty \), and \( \tilde{\nu}_{V, \pi} = \min_{\nu \in \mathcal{H}} \|\nu - \nu_{V, \pi}^*\|_\infty \).

\[ \bar{l}(V, \pi) - \tilde{l}_w(V, \pi) = \phi(V, \pi, \nu_{V, \pi}^*) - \phi(V, \pi, \nu_{V, \pi}^{\|\cdot\|_\infty}) \]

\[ \leq \phi(V, \pi, \tilde{\nu}_{V, \pi}) - \phi(V, \pi, \nu_{V, \pi}^*) \leq (K + C_\infty)\epsilon_{\text{approx}}. \]
With the notations we defined in the main text, we decompose the error
\[
\begin{align*}
\|\hat{V}_w^T - V^*\|_{\mu_{\pi_b}} & \leq \|\hat{V}_w^T - \hat{V}^*\|_{\mu_{\pi_b}} + \|\hat{V}^* - V^*\|_{\mu_{\pi_b}} + \|\hat{V}^* - V^*\|_{\mu_{\pi_b}}, \\
& = \|\hat{V}_w^T - \hat{V}^*\|_{\mu_{\pi_b}} + \|\hat{V}^* - V^*\|_{\mu_{\pi_b}} \\
\Rightarrow \|\hat{V}_w^T - V^*\|_{\mu_{\pi_b}} & \leq 2 \|\hat{V}_w^T - \hat{V}^*\|_{\mu_{\pi_b}}^2 + 2 \|\hat{V}^* - V^*\|_{\mu_{\pi_b}}^2.
\end{align*}
\] (26) (27) (28)

The second equation is because \(\hat{V}^* = \tilde{V}^*\) as Eq. [8]. Then, we provide the upper bound for each term.

**Lemma 10 (Smoothing bias)** \(\|\hat{V}^* - V^*\|_{\mu_{\pi_b}}^2 \leq (2\gamma^2 + 2) \left( \frac{\gamma}{1-\gamma} \max_{\pi \in \mathcal{P}} H(\pi) \right)^2\).

**Proof** For \(\|\hat{V}^* - V^*\|_{\mu_{\pi_b}}^2\), we have
\[
\|\hat{V}^* - V^*\|_{\mu_{\pi_b}}^2 = \int \left( \gamma E_{w',s,a} \left[ V^*(s') - \hat{V}^*(s') \right] - \left( V^*(s) - \hat{V}^*(s) \right) \right)^2 \mu(s)\pi_b(a|s)dsda \\
\leq 2\gamma^2 \|E_{s',a} \left[ V^*(s') - \hat{V}^*(s') \right] \|_{\mu_{\pi_b}}^\infty + 2 \|V^*(s) - \hat{V}^*(s)\|_{\mu_{\pi_b}}^\infty \\
\leq \left( 2\gamma^2 + 2 \right) \left( \frac{\gamma}{1-\gamma} \max_{\pi \in \mathcal{P}} H(\pi) \right)^2,
\]
where the final inequality is because Lemma [3].

In the algorithm, we are trying to optimize the objective function \(\tilde{l}(V, \pi)\) with parametrized \(V\) and \(\pi\), which may restrict the function family, therefore resulting extra error. Moreover, since with general parametrization, the optimization is no longer convex, not even mention strongly-convex. Therefore, we need first connect \(\|\hat{V}_w^T - \hat{V}^*\|_{\mu_{\pi_b}}\) with the gap between objectives, i.e., \(\tilde{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \tilde{l}(\hat{V}^*, \hat{\pi}^*)\).

**Lemma 11**
\[
\|\hat{V}_w^T - V^*\|_{\mu_{\pi_b}}^2 \leq 2 \left( \tilde{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \tilde{l}(\hat{V}^*, \hat{\pi}^*) \right) + 4\lambda^2 \|\log \hat{\pi}_w^T(a|s) - \log \hat{\pi}_w^*(a|s)\|_2^2 \\
+ 4\lambda^2 \|\log \hat{\pi}_w^*(a|s) - \log \hat{\pi}^*(a|s)\|_2^2.
\]

**Proof** Specifically, due to the strongly convexity of square function, we have
\[
\tilde{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \tilde{l}(\hat{V}^*, \hat{\pi}^*) = 2E_{\mu_{\pi_b}} \left( \hat{\Delta}_{V^*} \hat{\pi}_w^T(s, a) \left( \hat{\Delta}_{\hat{V}_w^T, \hat{\pi}_w^T}(s, a) - \hat{\Delta}_{V^*, \hat{\pi}_w^T}(s, a) \right) \right) \\
+ E_{\mu_{\pi_b}} \left( \hat{\Delta}_{\hat{V}_w^T, \hat{\pi}_w^T}(s, a) - \hat{\Delta}_{V^*, \hat{\pi}_w^T}(s, a) \right)^2 \\
\geq \int \left( \hat{\Delta}_{\hat{V}_w^T, \hat{\pi}_w^T}(s, a) - \hat{\Delta}_{V^*, \hat{\pi}_w^T}(s, a) \right)^2 \mu(s)\pi_b(a|s)dsda \\
:= \left\| \hat{\Delta}_{\hat{V}_w^T, \hat{\pi}_w^T}(s, a) - \hat{\Delta}_{V^*, \hat{\pi}_w^T}(s, a) \right\|_2^2.
\]
where \( \Delta(s, a, s') = R(s, a) + \gamma V(s') - \lambda \log \pi(a|s) - V(s) \) and the second inequality is because the optimality of \( \hat{V}^* \) and \( \hat{\pi}^* \). Therefore, we have

\[
\sqrt{\hat{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}(\hat{V}^*, \hat{\pi}^*)} \geq \left\| \Delta_{\hat{V}_w^T, \hat{\pi}_w^T} - \Delta_{\hat{V}^*, \hat{\pi}^*} \right\|_2
\]

\[
\geq \left\| \gamma E_{s'|s,a} \left[ \hat{V}_w^T(s') - \hat{V}^*(s') \right] - \left( \hat{V}_w^T(s) - \hat{V}^*(s) \right) \right\|_2 - \lambda \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*(a|s) \right\|_2
\]

\[
= \left\| \hat{V}_w^T - \hat{V}^* \right\|_{\mu_{\pi_b}} - \lambda \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*(a|s) \right\|_2
\]

which implies

\[
\left\| \hat{V}_w^T - \hat{V}^* \right\|_{\mu_{\pi_b}} \leq 2 \left( \hat{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}(\hat{V}^*, \hat{\pi}^*) \right) + 2\lambda^2 \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*(a|s) \right\|_2^2
\]

\[
\leq 2 \left( \hat{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}(\hat{V}^*, \hat{\pi}^*) \right) + 4\lambda^2 \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*(a|s) \right\|_2^2
\]

\[+ 4\lambda^2 \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*(a|s) \right\|_2^2.
\]

For the third term in Lemma 11, recall \( \hat{\pi}^*(a|s) = \exp \left( \frac{Q(s, a) - \mathcal{L}(Q)}{\lambda} \right) \), we have

\[
\lambda \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*(a|s) \right\|_2 \leq \left\| \lambda \log \hat{\pi}_w^T(a|s) - Q(s, a) + \mathcal{L}(Q) \right\|_2 \leq \epsilon^\pi_{\text{approx}}(\lambda),
\]

where \( \epsilon^\pi_{\text{approx}}(\lambda) := \sup_{\pi \in \mathcal{P}_\lambda} \inf_{\pi_w \in \mathcal{P}_w} \left\| \lambda \log \pi_w - \lambda \log \pi \right\|_2 \) with

\[
\mathcal{P}_\lambda := \left\{ \pi \in \mathcal{P}, \pi(a|s) = \exp \left( \frac{Q(s, a) - \mathcal{L}(Q)}{\lambda} \right), \left\| Q \right\|_2 \leq C_V \right\}.
\]

Based on the derivation of \( \mathcal{P}_\lambda \), with continuous \( \mathcal{A} \), it can be seen that as \( \lambda \to 0 \),

\[
\mathcal{P}_0 = \left\{ \pi \in \mathcal{P}, \pi(a|s) = \delta_{\max}(s)(a) \right\}.
\]

which results \( \epsilon^\pi_{\text{approx}}(\lambda) \to \infty \), and as \( \lambda \) increasing as finite, the policy becomes smoother, resulting smaller approximate error in general. With discrete \( \mathcal{A} \), although the \( \epsilon^\pi_{\text{approx}}(0) \) is bounded, the approximate error still decreases as \( \lambda \) increases. The similar correspondence also applies to \( \epsilon_{\text{approx}}^\nu(\lambda) \). The concrete correspondence between \( \lambda \) and \( \epsilon_{\text{app}}(\lambda) \) depends on the specific form of the function approximators, which is an open problem and out of the scope of this paper.

For the second term in [11]

\[
\lambda \left\| \log \hat{\pi}_w^T(a|s) - \log \hat{\pi}^*_w(a|s) \right\|_2 \leq \lambda \left\| \log \hat{\pi}_w^T(a|s) \right\|_2 + \lambda \left\| \log \hat{\pi}^*_w(a|s) \right\|_2 \leq 2\lambda C_{\pi}.
\]

For the first term, we have

\[
\hat{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}(\hat{V}^*, \hat{\pi}^*)
\]

\[
= \hat{l}(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) + \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}^*, \hat{\pi}^*) + \hat{l}_w(\hat{V}^*, \hat{\pi}^*) - \hat{l}_w(\hat{V}^*, \hat{\pi}^*)
\]

\[
\leq 2(K + C_\infty) \epsilon_{\text{approx}}^\nu + \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}^*, \hat{\pi}^*)
\]

\[
= 2(K + C_\infty) \epsilon_{\text{approx}}^\nu + \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) + \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T)
\]

\[
\leq 2(K + C_\infty) \epsilon_{\text{approx}}^\nu + C_\nu \left( (1 + \gamma) \epsilon_{\text{approx}}^\nu(\lambda) + \epsilon_{\text{approx}}^\nu(\lambda) \right) + \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T),
\]

where \( C_\nu = \max_{\nu \in \mathcal{H}_w} \| l \|_2 \). The last inequality is because

\[
\hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{l}_w(\hat{V}_w^T, \hat{\pi}_w^T) \leq \max_{\nu \in \mathcal{H}_w} \mathbb{E} \left[ \nu(s, a) \left( \Delta_{\hat{V}_w^T, \hat{\pi}_w^T}(s, a, s') - \Delta_{\hat{\pi}_w^T}(s, a, s') \right) \right]
\]

\[
\leq C_{\nu} \left( (1 + \gamma) \epsilon_{\text{approx}}^\nu(\lambda) + \epsilon_{\text{approx}}^\nu(\lambda) \right).
\]

Combine (29), (30) and (31) into Lemma 11 and Lemma 10 together with (26), we achieve
Lemma 12 (Error decomposition)

\[
\left\| \hat{V}_w^T - V^* \right\|_{\mu, \pi_b}^2 \leq 2 \left( 2(K + C_\infty)\epsilon_{\text{approx}}^V + C_\nu (1 + \gamma)\epsilon_{\text{approx}}^V(\lambda) + C_\nu \epsilon_{\text{approx}}^\pi(\lambda) + 2 (\epsilon_{\text{approx}}(\lambda))^2 \right)
\]

\[
+ 16\lambda^2 C_\pi^2 + 2(2\gamma^2 + 2) \left( \frac{\gamma \lambda}{1 - \gamma} \max_{\pi \in P} H(\pi) \right)^2 + 2 \left( \hat{\ell}_w(\hat{V}_w^*, \hat{\pi}_w^*) - \hat{\ell}_w(\hat{V}_w^*, \hat{\pi}_w^*) \right).
\]

We can see that the bound includes the errors from three aspects: i), the approximation error induced by parametrization of \( V, \pi, \) and \( \nu; \) ii), the bias induced by smoothing technique; iii), the statistical error. As we can see from Lemma 12, \( \lambda \) plays an important role in balance the approximation error and smoothing bias.

D.2 Statistical Error

In this section, we analyze the generalization error. For simplicity, we denote the \( T \) finite-sample approximation of

\[
\Phi(V, \pi, \nu) = \mathbb{E}[\phi_{V, \pi, \nu}(s, a, R, s')]
\]

as

\[
\hat{\Phi}(V, \pi, \nu) = \frac{1}{T} \sum_{i=1}^{T} \phi_{V, \pi, \nu}(s, a, R, s')
\]

\[
= \frac{1}{T} \sum_{i=1}^{T} (2\nu(s, a_i) (R(s, a_i) + \gamma V(s_i') - V(s) - \lambda \log \pi(a_i|s_i)) - \nu^2(s, a_i)),
\]

where the samples \( \{(s_i, a_i, s_i', R_i)\}_{i=0}^T \) are sampled \( i.i.d. \) or from \( \beta \)-mixing stochastic process.

We consider the \( \epsilon \)-error solution, i.e., \( (\hat{V}_e^T, \hat{\pi}_e^T, \hat{\nu}_e^T), \) to \( (\hat{V}_w^T, \hat{\pi}_w^T, \hat{\nu}_w^T) = \arg\min_{V, \pi \in \mathcal{F}_w \times \mathcal{P}_w} \arg\max_{\mu \in \mathcal{H}_w} \Phi(V, \pi, \mu), \)
resulting

\[
\left\| \hat{\Phi}(\hat{V}_e^T, \hat{\pi}_e^T, \hat{\nu}_e^T) - \Phi(V_w^T, \hat{\pi}_w^T, \hat{\nu}_w^T) \right\| \leq \epsilon_{\text{opt}}.
\]

We decompose

\[
\hat{\ell}_w(\hat{V}_e^T, \hat{\pi}_e^T) - \hat{\ell}_w(\hat{V}_w^T, \hat{\pi}_w^T) = \hat{\ell}_w(\hat{V}_e^T, \hat{\pi}_e^T) - \hat{\ell}_w(\hat{V}_w^T, \hat{\pi}_w^T) + \hat{\ell}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{\ell}_w(\hat{V}_w^T, \hat{\pi}_w^T),
\]
and consider separately.

For the first term,

\[
\hat{\ell}_w(\hat{V}_e^T, \hat{\pi}_e^T) - \hat{\ell}_w(\hat{V}_w^T, \hat{\pi}_w^T) = \max_{\nu \in \mathcal{H}_w} \Phi(\hat{V}_e^T, \hat{\pi}_e^T, \nu) - \Phi(\hat{V}_w^T, \hat{\pi}_w^T, \nu) + \Phi(\hat{V}_w^T, \hat{\pi}_w^T, \hat{\nu}_e^T) - \Phi(\hat{V}_w^T, \hat{\pi}_w^T, \hat{\nu}_e^T)
\]

\[
\leq \max_{\nu \in \mathcal{H}_w} \left| \Phi(\hat{V}_e^T, \hat{\pi}_e^T, \nu) - \Phi(\hat{V}_w^T, \hat{\pi}_e^T, \nu) \right| + \max_{\nu \in \mathcal{H}_w} \left| \Phi(\hat{V}_w^T, \hat{\pi}_w^T, \nu) - \Phi(\hat{V}_w^T, \hat{\pi}_w^T, \hat{\nu}_e^T) \right| + \epsilon_{\text{opt}}
\]

\[
\leq 2 \sup_{(V, \pi, \nu) \in \mathcal{F}_w \times \mathcal{P}_w \times \mathcal{H}_w} \left| \Phi(V, \pi, \nu) - \Phi(V, \pi, \nu) \right| + \epsilon_{\text{opt}}.
\]

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For the second term,
\[
\hat{I}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{I}_w(\hat{V}_w^*, \hat{\pi}_w^*) = \max_{\nu \in \mathcal{H}_w} \Phi \left( \hat{V}_w^T, \hat{\pi}_w^T, \nu \right) - \max_{\nu \in \mathcal{H}_w} \Phi \left( \hat{V}_w^*, \hat{\pi}_w^*, \nu \right)
\]
\[
= \Phi \left( \hat{V}_w^T, \hat{\pi}_w^T, \nu_w \right) - \Phi \left( \hat{V}_w^*, \hat{\pi}_w^*, \nu_w \right) + \Phi \left( \hat{V}_w^*, \hat{\pi}_w^*, \nu_w \right) - \max_{\nu \in \mathcal{H}_w} \Phi \left( \hat{V}_w^*, \hat{\pi}_w^*, \nu \right)
\]
\[
\leq \Phi \left( \hat{V}_w^T, \hat{\pi}_w^T, \nu_w \right) - \Phi \left( \hat{V}_w^*, \hat{\pi}_w^*, \nu_w \right)
\]
\[
\leq 2 \sup_{V, \pi, \nu \in \mathcal{F}_w \times \mathcal{P}_w \times \mathcal{H}_w} \left| \Phi \left( V, \pi, \nu \right) - \Phi \left( V, \pi, \nu \right) \right|
\]
where \( \nu_w = \max_{\nu \in \mathcal{H}_w} \Phi \left( \hat{V}_w^T, \hat{\pi}_w^*, \nu \right) \).

In sum, we have
\[
\hat{I}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{I}_w(\hat{V}_w^*, \hat{\pi}_w^*) \leq 4 \sup_{V, \pi, \nu \in \mathcal{F}_w \times \mathcal{P}_w \times \mathcal{H}_w} \left| \Phi \left( V, \pi, \nu \right) - \Phi \left( V, \pi, \nu \right) \right| + \epsilon_{opt}.
\]

The first term can be bounded by covering number or Rademacher complexity on hypothesis space \( \mathcal{F}_w \times \mathcal{P}_w \times \mathcal{H}_w \) with high probability if the samples are i.i.d. or from \( \beta \)-mixing stochastic processes [Antos et al. 2008].

We will use a generalized version of Pollard’s tail inequality to \( \beta \)-mixing sequences, i.e.,

**Lemma 13** [Lemma 5 [Antos et al. 2008]] Suppose that \( z_1, \ldots, z_N \in \mathcal{Z} \) is a stationary \( \beta \)-mixing process with mixing coefficient \( \beta_m \) and that \( \mathcal{G} \) is a permissible class of \( \mathcal{Z} \to [-C, C] \) functions, then,

\[
\mathbb{P} \left( \sup_{g \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^{N} g(Z_i) - \mathbb{E}[g(Z_1)] \right| > \epsilon \right) \leq 16 \mathbb{E} \left[ N \left( \epsilon, \mathbb{G}, (Z'_i; i \in H) \right) \right] \exp \left( -\frac{m_N \epsilon^2}{128C^2} \right) + 2m_N \beta_{k+1},
\]

where the “ghost” samples \( Z'_i \in \mathcal{Z} \) and \( H = \bigcup_{i=1}^{m_N} H_i \) which are defined as the blocks in the sampling path.

The covering number is highly related to pseudo-dimension, i.e.,

**Lemma 14** [Corollary 3 [Haussler 1995]] For any set \( \mathcal{X} \), any points \( x^{1:N} \in \mathcal{X}^N \), any class \( \mathcal{F} \) of functions on \( \mathcal{X} \) taking values in \([0, C]\) with pseudo-dimension \( D_\mathcal{F} < \infty \), and any \( \epsilon > 0 \),

\[
N \left( \epsilon, \mathcal{F}, x^{1:N} \right) \leq e \left( D_\mathcal{F} + 1 \right) \left( \frac{2eC}{\epsilon} \right)^{D_\mathcal{F}}
\]

Once we have the covering number of \( \Phi(V, \pi, \nu) \), plug it into lemma [13] we will achieve the statistical error,

**Theorem 5 (Stochastic error)** Under Assumption [3] with at least probability \( 1 - \delta \),

\[
\hat{I}_w(\hat{V}_w^T, \hat{\pi}_w^T) - \hat{I}_w(\hat{V}_w^*, \hat{\pi}_w^*) \leq 2 \sqrt{M \left( \max \left( \frac{M}{b}, 1 \right) \right)^{1/\kappa}} \frac{C_2 T}{\epsilon_{opt}} + \epsilon_{opt},
\]

where \( M = \frac{D}{\kappa} \log t + \log (\epsilon/\delta) + \log \left( \max \left( C_1 C_2^{D/2}, \beta \right) \right) \).
With some calculation, the distance in $\mathcal{G}$ can be bounded,

$$
\frac{1}{T} \sum_{i \in H} |\nu_1(s_i, a_i) - \nu_2(s_i, a_i)| + \frac{2(1 + \gamma)C}{T} \sum_{i \in H} |V_1(s_i) - V_2(s_i)|
$$

which leads to

$$
\mathcal{N}(12C\epsilon', \mathcal{G}, (Z_i^i; i \in H)) \leq \mathcal{N}(\epsilon', \mathcal{F}_w, (Z_i^i; i \in H)) \mathcal{N}((\epsilon', P_w, (Z_i^i; i \in H)) \mathcal{N}(\epsilon', \mathcal{H}_w, (Z_i^i; i \in H))
$$

with $\lambda \in (0, 2]$. To bound these factors, we apply lemma 14. We denote the pseude-dimension of $\mathcal{F}_w$, $\mathcal{P}_w$, and $\mathcal{H}_w$ as $D_\ell$, $D_\pi$, and $D_\nu$, respectively. Thus,

$$
\mathcal{N}(12C\epsilon', \mathcal{G}, (Z_i^i; i \in H)) \leq \epsilon^3 (D_\ell + 1)(D_\pi + 1)(D_\nu + 1) \left(\frac{4C}{\epsilon'}\right)^{D_\ell + D_\pi + D_\nu},
$$

which implies

$$
\mathcal{N}(\frac{\epsilon}{16}, \mathcal{G}, (Z_i^i; i \in H)) \leq \epsilon^3 (D_\ell + 1)(D_\pi + 1)(D_\nu + 1) \left(\frac{768\epsilon C^2}{\epsilon'}\right)^{D_\ell + D_\pi + D_\nu} = C_1 \left(\frac{1}{\epsilon}\right)^D,
$$

where $C_1 = \epsilon^3 (D_\ell + 1)(D_\pi + 1)(D_\nu + 1) (768\epsilon C^2)^D$ and $D = D_\ell + D_\pi + D_\nu$, i.e., the “effective” pseude-dimension.

Plug this into Eq. (32), we obtain

$$
P \left( \sup_{\nu, \pi, \nu' \in \mathcal{F}_w \times \mathcal{P}_w \times \mathcal{H}_w} \left| \frac{1}{T} \sum_{i = 1}^T \phi_{\nu, \pi, \nu'} \left( (s, a, s', R_i) \right) - \mathbb{E} \left[ \phi_{\nu, \pi, \nu'} \right] \right| \geq \epsilon / 2 \right) \leq C_1 \left(\frac{1}{\epsilon}\right)^D \exp \left( -4C_2 \epsilon^2 \epsilon^2 \right) + 2m_T \beta_{k_T},
$$

with $C_2 = \frac{1}{2} \left( \frac{2M}{2\epsilon^2} \right)^2$. If $D \geq 2$, and $C_1, C_2, \beta, b, \kappa > 0$, for $\delta \in (0, 1]$, by setting $k_t = \lceil (C_2 T \epsilon^2 / b)^{\frac{1}{2\kappa}} \rceil$ and $m_T = \frac{T}{2k_T}$, by lemma 14 in Antos et al. (2008), we have

$$
C_1 \left(\frac{1}{\epsilon}\right)^D \exp \left( -4C_2 \epsilon^2 \epsilon^2 \right) + 2m_T \beta_{k_T} < \delta,
$$

with $\epsilon = \sqrt{\frac{M(\max(M/b, 1))^{1/\kappa}}{C_2^2}}$ where $M = \frac{D}{2} \log T + \log (\epsilon / \delta) + \log^+ \left( \max \left( C_1 C_2^{D/2}, \beta \right) \right)$.

With the statistical error bound provided in Theorem 6 for solving the derived saddle point problem with arbitrary learnable nonlinear approximators using off-policy samples, we can achieve the analysis of the total error, i.e.,
Theorem 6 (Total error) Let $\hat{V}_w^T$ be a candidate solution output from the proposed algorithm based on off-policy samples, with at least probability $1 - \delta$, we have

$$
\left\| \hat{V}_w^T - V^* \right\|_{\mu, b}^2 \leq 2 \left( 2(K + C_\infty) \epsilon^\nu_{\text{approx}} + C_\nu (1 + \gamma_\nu) \epsilon^V_{\text{approx}}(\lambda) + C_\nu \epsilon^\pi_{\text{approx}}(\lambda) + 2 \left( \epsilon^\pi_{\text{approx}}(\lambda) \right)^2 \right)
$$

approximation error due to parametrization

$$
+ 16 \lambda^2 \epsilon^2_{\text{approx}} + \left( 2 \gamma^2 + 2 \right) \left( \frac{\gamma \lambda}{1 - \gamma} \max_{\pi \in \mathcal{P}} H(\pi) \right)^2
$$

bias due to smoothing

$$
+ 4 \sqrt{\frac{M \left( \max \left( M/b, 1 \right) \right)^{1/\kappa}}{C_2 T}} + \epsilon_{\text{opt}}.
$$

statistical error

where $M = \frac{D}{2} \log t + \log (\epsilon/\delta) \log \left( \max \left( C_1 C_2^{D/2}, \beta \right) \right)$.

Proof This theorem can be proved by combining Theorem 5 into Lemma 12. \[\square\]