Linear statistics and pushed Coulomb gas at the edge of $\beta$-random matrices: Four paths to large deviations

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Abstract – The Airy$_3$ point process, $a_i \equiv N^{2/3}(\lambda_i - 2)$, describes the eigenvalues $\lambda_i$ at the edge of the Gaussian $\beta$ ensembles of random matrices for large matrix size $N \to \infty$. We study the probability distribution function (PDF) of linear statistics $L = \sum_i t \varphi(t^{-2/3} a_i)$ for large parameter $t$. We show the large deviation forms $\mathbb{E}_{\text{Airy}, \beta}[\exp(-L)] \sim \exp(-t^3 \Sigma[\varphi])$ and $P(L) \sim \exp(-t^3 G(L/t^2))$ for the cumulant generating function and the PDF. We obtain the exact rate function, or excess energy, $\Sigma[\varphi]$ using four apparently different methods: i) the electrostatics of a Coulomb gas, ii) a random Schrödinger problem, i.e., the stochastic Airy operator, iii) a cumulant expansion, iv) a non-local non-linear differential Painlevé-type equation. Each method was independently introduced previously to obtain the lower tail of the Kardar-Parisi-Zhang equation. Here we show their equivalence in a more general framework. Our results are obtained for a class of functions $\varphi$, the monotonous soft walls, containing the monomials $\varphi(x) = (u + x)^N$, and the exponential and equivalently describe the response of a Coulomb gas pushed at its edge. The small $u$ behavior of the excess energy exhibits a change between a non-perturbative hard-wall–like regime for $\gamma < 3/2$ (third-order free-to-pushed transition) and a perturbative deformation of the edge for $\gamma > 3/2$ (higher-order transition). Applications are given, among them i) truncated linear statistics such as $\sum_{i=1}^N \lambda_i$, leading to a formula for the PDF of the ground-state energy of $N_1 \gg 1$ non-interacting fermions in a linear plus random potential, ii) $(\beta - 2)/r^2$ interacting spinless fermions in a trap at the edge of a Fermi gas, iii) traces of large powers of random matrices.

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Random matrix theory [1–4] has an enormous range of current applications, e.g., for quantum chaos, transport and entanglement [5–9], Anderson localization [10], string theory [11,12], data analysis [13], fluctuating interfaces and interacting Brownians [14], stochastic growth [15–16], combinatorics such as tilings, dimers and random permutations [17], trapped fermions [18–24], A classical problem, called linear statistics, amounts to studying the probability distribution function (PDF) of sums $L = \sum_{i=1}^N f(\lambda_i)$ over eigenvalues $\lambda_i$ of a size $N$ random matrix. Varying the function $f$ and the ensemble, it describes, e.g., conductance and shot noise [25,26], Rényi entropies [27], interfaces center of mass [14], particle number fluctuations. At large $N$, central limit theorems, universality, and connections to the Gaussian free field were shown [15,28–36] for typical fluctuations of $L$ in the bulk of the spectrum. Large deviations were also studied in the bulk [37–41], from the Coulomb gas representation, and recently for truncated linear statistics [42,43], showing interesting phase transitions. Furthermore, strongly perturbed Coulomb gases are interesting correlated systems by themselves, extensively studied recently [44], but not much at their edge [45].

In this letter we study the edge of the spectrum where fluctuations are stronger and much fewer results exist [46]. For the classical random matrix ensembles, an array of methods exists to study spectral correlations [1–4], such as the Coulomb gas, resolvent, orthogonal polynomials, Seiberg integrals, determinantal processes, Painlevé equations, the Dimitrin-Edelman tridiagonal representation [47] and the stochastic operators [48–50]. These methods however often appear disconnected: in this letter we unveil relations between some of them, valid at the edge. This leads to simple formulae which we apply to study a Coulomb gas pushed in the vicinity of its edge by a soft potential, revealing a new variety of phase transitions.

We focus on the Gaussian $\beta$ ensemble of random matrices [47] for which the joint probability distribution
function (JPDF) of the eigenvalues $\lambda_i$ has the form

$$P[\lambda] \sim e^{\beta \sum_{1 \leq i \leq N} \log |\lambda_i - \lambda_j| - \frac{2\pi}{\beta} \sum_{i=1}^{N} \lambda_i^2}. \tag{1}$$

It is also the stationary measure of the $\beta$ Dyson Brownian motion. Equation (1) leads to the celebrated semi-circle eigenvalue density of support $[-2, 2]$. The JPDF (1) can be seen as the Gibbs measure of a Coulomb gas (CG) with logarithmic repulsion between the eigenvalues, which, at large $N$, can be described by a continuous density. We study here the eigenvalues located near the right edge of this CG, in a window of width $\sim N^{-2/3}$. In that window for $N \to \infty$, the scaled eigenvalues $a_i \equiv N^{2/3}(\lambda_i - 2)$ define the Airy$_\beta$ point process (APP). We consider the linear statistics of the APP, i.e., the sum

$$L = t \sum_{i=1}^{+\infty} \phi(u + t^{-2/3}a_i), \tag{2}$$

where $u$ is a control parameter and $\phi$ the shape function. We study a restricted set of functions $\phi$, denoted $\Omega_0$ and defined below, which is a subset of all continuous positive increasing functions such that $\phi(x) = 0$ for $x \leq 0$. The parameter $t$ thus controls how many eigenvalues contribute to the sum, indeed since the ordered eigenvalues behave as $a_i \approx i - \frac{3\pi^2}{2^{2/3}}b^{2/3}$, only $K \approx 2\pi^3/3t$ eigenvalues typically contribute to the sum. Hence for a function $\phi$ of order 1, the sum is of order $t^2$ at large $t$. It is natural to define the scaled eigenvalue empirical density of the APP, $\rho(b) = t^{-1} \sum_i \delta(b + t^{-2/3}a_i)$, so that

$$L = t^2 \int_{-\infty}^{+\infty} db \rho(b) \phi(u - b). \tag{3}$$

Since the mean density of the APP converges for $a_i \to -\infty$ to the semi-circle, at large $t$ the mean value of $L$ is

$$E_\beta[L] \simeq t^2 \int_{-\infty}^{+\infty} db \rho_{Ai}(b) \phi(u - b), \quad \rho_{Ai}(b) := \frac{\sqrt{(b)_+}}{\pi}, \tag{4}$$

$E_\beta$ denoting an average over the APP, and $(b)_+ := \max(b, 0)$. We are interested in the large fluctuations of $L$, and calculate the large deviation function $\Sigma_\phi(u)$, and obtain

$$\Sigma_\phi(u) := \lim_{t \to +\infty} t^{-2} \log Q_t(u), \quad Q_t(u) := E_\beta[e^{-L}]. \tag{5}$$

We show that the PDF of $L$, $P(L)$, becomes at large $t$

$$P(L) \simeq e^{-t^2G(\ell)}, \quad \ell = L/E_\beta[L] \tag{6}$$

and obtain its expression for $\ell \leq 1$ from a Legendre transform involving $\Sigma_\phi(u)$. We interpret $Q_t(u)$ as the Gibbs measure of the Coulomb gas upon a perturbation by a soft wall external potential described by $\phi$. For $\phi$ in $\Omega_0$, the external force $\phi' \geq 0$ pushes the charges towards the bulk which defines the pushed CG problem, well studied in the bulk. The novelty here is to study a Coulomb gas pushed at its edge and probe its rigidity. The rate function $\Sigma_\phi(u)$ is then the excess total energy resulting from its reorganization measured by the deviation of the equilibrium pushed density, denoted $\rho_s(b)$, from the unperturbed one $\rho_{Ai}(b)$. One outstanding question is the nature of the phase transition at $u = 0$ between pushed and free, usually of third order in the bulk [51]. We find here transitions with continuously varying exponent larger than three.

This problem was studied before for $\phi(x) = \phi_{KPZ}(x) = (x)_+ \times 1/2$ and obtain the lower tail of the Kardar-Parisi-Zhang equation for $\beta = 2$ (full-space KPZ), $\beta = 1$ (half-space KPZ) and arbitrary $\beta$ (extended polymer of ref. [52]). No less than four methods were devised to treat that case: i) the electrostatics of a Coulomb gas [53], ii) a random Schrödinger problem known as the stochastic Airy operator [54], iii) a cumulant expansion [28], iv) a non-local non-linear differential Painlevé-type equation [55] (the last two for $\beta = 2$ only, see also ref. [56] for another approach). Although apparently unrelated, they lead to the same result for $\Sigma_\phi(u)$ for this KPZ problem. The aim of this letter is to unveil the connections between these methods, make explicit the underlying structure and apply them to more general functions $\phi$ beyond $\phi_{KPZ}$. Two saddle point equations, denoted $SP1$ and $SP2$, shown to be dual to each other, play an important role in the large $t$ limit, and appear alternatively in each method as the genuine saddle point equation.

SAO/WKB method: We start with the method based on the stochastic Airy operator (SAO) [48], introduced by Tsai for the KPZ problem in ref. [54]. It is known [49] that the APP can be generated as $-a_i = \xi_i$ where $\xi_i$ are the eigenvalues of the following Schrödinger problem on the half-line $y > 0$, defined by the Hamiltonian $\mathcal{H}_{SAO} = -\partial_y^2 + y + \frac{1}{\beta} V(y)$, where $V(y)$ is a unit white noise and the wave functions vanish at $y = 0$. Since we are interested in energy levels of order $t^{2/3}$ we can rescale $y = t^{2/3}x, V(y) = t^{2/3} \frac{1}{\sqrt{\beta}} V(x)$, $\mathcal{H}_{SAO} = t^{-2/3} \mathcal{H}_{SAO}$ with energy levels $b_i = t^{-2/3} \xi_i = -t^{-2/3} a_i$ and obtain

$$\mathcal{H}_{SAO} = -t^{-2} \partial_x^2 + x + v(x). \tag{7}$$

This corresponds to a Schrödinger problem for a particle of mass $m = 1/2$ with $\hbar = 1/t$. In the large $t$ limit it is natural to use the WKB semi-classical approximation for the density of energy levels of (7), $\hat{\rho}(b) = \sum_i \delta(b - b_i)$ as $\hat{\rho}(b) \simeq t \rho(b)$ with [57]

$$\rho(b) = \frac{1}{\pi} \frac{d}{db} \int_{-\infty}^{+\infty} dx \sqrt{(b - x - v(x))}. \tag{8}$$

The average over the APP in eq. (5) is an average over the white noise $V(y)$, of measure $\sim e^{-\frac{x_0}{2} \int dy V(y)^2} = e^{-\frac{t}{4\pi} \int dx v(x)^2}$, hence we obtain that $\Sigma_\phi(u)$ is the solution of the following variational problem for $x > 0$

$$\Sigma_\phi(u) = \min_{v(x)} \left[ \int_{-\infty}^{+\infty} db \rho(b) \phi(u - b) + \frac{\beta}{2} \int_{0}^{+\infty} dx v(x)^2 \right]. \tag{9}$$
where $\rho(b)$ is defined in eq. (8). This approach was made rigorous in the case $\phi_{KPZ}$ in ref. [54] using explosions in the Riccati formulation of the SAO.

We now display the resulting saddle point equation, which we denote SP1, for a more general $\phi$. From eq. (9) for $x > 0$, the optimal $v(x) = v_*(x)$ is the solution of
\[
\frac{\beta}{4} v_*(x) = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \frac{db}{\sqrt{(b)_+}} \phi'(u - b - x - v_*(x)), \right. \tag{10}
\]
where everywhere $\frac{1}{\sqrt{(b)_+}} := \frac{\theta(0)}{\sqrt{b}}$. Choosing $\phi$ in $\Omega_0$ we show that: i) there is a unique solution $v_*(x)$ to 10, ii) $\phi$ is increasing which implies $v_*(x) \geq 0$, iii) $\phi(x) = 0$ hence $v_*(x \geq u) = 0$, iv) from simple manipulations, see ref. [57], $\Sigma_\phi(u)$, can be rewritten simply as
\[
\Sigma_\phi(u) = \frac{\beta}{4} \int_0^{\infty} dx x v_*(x). \tag{11}
\]

**Cumulant method:** Another method was recently developed solely for $\beta = 2$ when the APP has a determinantal structure. A systematic time expansion, i.e., in $t$, on the Fredholm determinant representation, eq. (18), of the average in eq. (5), led to the following series formula, shown in ref. [28] for $\beta = 2$, and conjectured here for any $\beta$.
\[
\Sigma_\phi(u) = -\sum_{n \geq 1} \tilde{k}_n(u) n! \
\tilde{k}_n(u) = \frac{\beta}{4} \left( \frac{(-1)^n}{\beta} \right)^n \int_0^u \left( f(u') \right)^n \text{ and the mean value was given in eq. (4). We have defined}
\]
\[
f(u) := \frac{1}{2} \int_{-\infty}^{\infty} \frac{db}{\sqrt{(b)_+}} \phi'(u - b). \tag{13}
\]

which, for $\phi$ in $\Omega_0$, is positive with $f(u \leq 0) = 0$. It is possible to sum up the series of cumulants of eq. (12). We show in ref. [57] that if the following equation
\[
f(u - \frac{4}{\beta \pi} w(u)) = w(u) \tag{14}
\]
adopts a unique positive increasing solution $w(u)$ for all $u \geq 0$, with $w(u \leq 0) = 0$, then
\[
\Sigma_\phi(u) = \frac{1}{\pi} \int_0^u \text{ } du' \left( u - u' \right) w(u'). \tag{15}
\]

The uniqueness of the solution of (14) is equivalent to $z \mapsto h(z) = f(z) + \frac{4}{\beta \pi} z$ being strictly increasing, in which case $w(u) = \frac{\beta}{4} \left( u - h^{-1}(u) \right)$. We call $\Omega_2$ the set of $\phi$ such that the associated $f$ satisfies this condition and further restrict to the subset $\Omega_2 \subset \Omega_2$ such that $f$ is positive, increasing, with $f(z \leq 0) = 0$, leading to (14). From eq. (15) we see that $\Sigma_\phi(u)$ is positive (as required) and since $\Sigma'_\phi(u) = \frac{1}{\pi} w(u)$ is also positive, it is also convex (as required). This method gives a simpler parametric representation than the SAO/WKB method and one can ask about the connection between the two.

Comparing (10) and (14) we note that we can find the solution of the SAO/WKB SP1 saddle point equation as
\[
v_*(x) = \frac{4}{\beta \pi} w(u - x), \quad 0 \leq x \leq u \tag{16}
\]
with $v_*(x) = 0$ for $x \geq u$. Hence eqs. (10) and (14) are the same SP1 equation. Using (16) we see that formulae (11) and (15) for $\Sigma_\phi(u)$, also coincide. Conversely, going from the SAO/WKB to the cumulant expansion amounts to using the Lagrange inversion formula on eq. (10), or equivalently on eq. (14) (see ref. [57]). This coincidence for any $\beta$ confirms our conjecture for the $\beta$ dependence of the series. This concludes the equivalence between the two methods for $\phi$ in the class $\Omega_0$.

It is useful to invert the (convolution) relation (13) between $\phi$ and $f$ [57] as
\[
\phi(u) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{db}{\sqrt{(b)_+}} f(u - b). \tag{17}
\]

**Painlevé/WKB method:** This method, introduced in the context of large deviations in [58] (based on [59]) was developed in [55] for $\phi = \phi_{KPZ}$. For $\beta = 2$ solely, the $\{a_i\}$ form the usual Airy point process and one writes (5) for all $t$ as a Fredholm determinant (FD),
\[
Q_t(u) = \text{Det} \left[ I - \sigma_t K_{\Lambda^3} \right], \tag{18}
\]
where $\sigma_t(u) = 1 - e^{-t\phi(u+t-2/3)}$ and $K_{\Lambda^3}(a, a')$ is the standard Airy kernel, see ref. [57]. This FD is shown [59] to obey the following equation, with $s = -ut^{2/3}$:
\[
\log Q_t(u) = \int_s^{+\infty} \frac{d(\sigma_t)}{\sigma_t} \Psi_t(r), \tag{19}
\]
\[
\Psi_t(r) = -\int_s^{+\infty} \frac{d(\sigma_t)}{\sigma_t} (q_t(r, v)) \int_e^{-t\phi(vt^{-2/3})} \frac{d}{dv} e^{-t\phi(vt^{-2/3})} (v) \tag{20}
\]
\[
\sigma_t(q_t(r, v)) = [v + r + 2\Psi_t(r)] q_t(r, v) \tag{21}
\]
with $q_t(r, v) \simeq r \rightarrow +\infty \text{Ai}(r + v)$. Following ref. [55], introducing the scaled variables $r = t^{2/3} X$ and $v = t^{2/3} V$, one solves (see ref. [57]), at large $t$, eq. (21) by the WKB method in the form $\Psi_t(r) \simeq t^{2/3} g(X)$, under the condition that $g(X)$ satisfies simultaneously the pair of eqs. (22), (23).

Equation (22) identifies with SP1 and eq. (23) is new and we denote it SP2.

\[
\frac{\beta}{2} g(X) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dV g'(V)}{\sqrt{(-V - X - 2g(X))_+}}, \tag{22}
\]
\[
\phi(V) = \frac{\beta}{3} \int_{-\infty}^{0} dX' \sqrt{V + X' + 2g(X')_+} - \frac{2\beta}{3} \left( V \right)_+. \tag{23}
\]

These generalize eqs. (28) and (29) of ref. [55], where the compatibility of these two equations was qualified as a
miracle. We have extended the above pair of equations to any β, by consistency with the other methods. In addition
\begin{equation}
\Sigma_{\phi}(u) = \frac{\beta}{2} \int_{-u}^{0} dX(X + u) g(X). \tag{24}
\end{equation}
We now unveil the connection to the other methods, and explain the miracle. First, we see that eq. (22) reduces to
our previous saddle point equation SP1 (in the equivalent forms of eqs. (10) and (13), (14)) upon the identification
\begin{equation}
u \equiv -X, \quad w(u) \equiv \frac{\beta}{2} \pi g(-u). \tag{25}
\end{equation}
From it, we see that formula (24) for Σφ(u) becomes equal to the one obtained with the other methods, e.g., eq. (15).
\[\text{For full consistency we now show that, within the class of φ studied here, eqs. (22) and (23) are equivalent, proving that SP1 and SP2 are dual forms of the same condition. Let us show that SP1 implies SP2. Denoting f the r.h.s. of eq. (23), using eq. (25) we can rewrite it as} \]
\begin{equation}
I = \beta \int_{0}^{+\infty} du' \sqrt{(V - u' + \frac{2}{\beta \pi} w(u'))_+ - \frac{2\beta}{3} (V)^{3/2}}. \tag{26}
\end{equation}
We use the change of variable \(z = u' - \frac{4}{\beta \pi} w(u')\), \(f(z) = w(u')\). If \(f(u)\) is positive and increasing, then \(z\) is an increasing function of \(u'\). In addition since \(u' = f^{-1}(w(u')) + \frac{4}{\beta \pi} w(u')\), we also have \(u' = z + \frac{4}{\beta \pi} f(z)\), hence
\begin{equation}
I = \beta \int_{-\infty}^{+\infty} dz (1 + \frac{4}{\beta \pi} f(z)) \sqrt{(V - z)_+ - \frac{2\beta}{3} (V)^{3/2}} + \frac{2\beta}{3} \pi (V)^{3/2} \tag{27}
\end{equation}
In the last equality we used the inversion formula (17) and the miracle is explained. It is also simple to show the converse, i.e., SP2 implies SP1, see ref. [57]. Hence, the Painlevé/WKB method is equivalent to the two others.

Electrostatic Coulomb gas method: In ref. [53] the edge limit of the standard Coulomb gas describing the bulk eigenvalues of the GUE was taken, and applied to study the large deviations for \(\phi = \phi_{\text{KPZ}}\). For general \(\phi\), the function \(\Sigma_{\phi}(u)\) is given by the minimization problem,
\begin{equation}
\Sigma_{\phi}(u) = \min_{\rho} \left[ \int_{-\infty}^{+\infty} db \rho(b) \phi(u - b) + J(\rho) + U(\rho) \right], \tag{28}
\end{equation}
\[\text{where} \ J(\rho) = -\frac{\beta}{2} \int_{-\infty}^{+\infty} \log |b_1 - b_2| \frac{1}{2} \sum_{i=1}^{2} \int_{-\infty}^{+\infty} d\rho(b_i - \rho_{\Lambda_1}(b_i)), \]}
\[\text{and} \ U(\rho) = \frac{2\beta}{\pi} \int_{-\infty}^{+\infty} d|b| |2 \rho(b)| \text{ is irrelevant below. The minimum is over mass conserving measures } \rho\text{ such that } \int_{-\infty}^{+\infty} d\rho(b - \rho_{\Lambda_1}(b)) = 0, \text{ where } \rho_{\Lambda_1}(b) = \frac{b}{\sqrt{b^2 + 1}}. \]
\[\text{We see that the dependence on } \beta \text{ is } [57] \Sigma_{\phi}^{(\beta)}(u) = \frac{\beta}{2} \Sigma_{\phi}(u) \text{ and } G^{(\beta)}(t) = \frac{\beta}{2} G^{(2)}(t).\]
By the variational equation determines the optimal density \(\rho^*(b)\) as the unique solution such that
\begin{equation}
\phi(u - b) - \beta \int_{-\infty}^{+\infty} db' \log |b - b'| (\rho_{\Lambda}(b') - \rho_{\Lambda_1}(b')) \geq c \tag{29}
\end{equation}
with equality on the support of \(\rho^*\). We assume, and verify later, that for \(\phi \in \Omega_0\), the support is an interval \([u_0, +\infty]\). Taking a derivative, we have for \(b \in [u_0, +\infty]\)
\begin{equation}
\phi'(u - b) - \beta \int_{-\infty}^{+\infty} db' \frac{d\rho^*(b')}{b - b'} (\rho_{\Lambda}(b') - \rho_{\Lambda_1}(b')) = 0. \tag{30}
\end{equation}
In ref. [53] \(\rho_{\Lambda}(b)\) and \(\Sigma_{\phi}(u)\) were calculated for \(\phi = \phi_{\text{KPZ}}\) and here we obtain these quantities for a general \(\phi\) in \(\Omega_0\).

We now unveil the connection between the Coulomb gas and the other methods. First note that eq. (8) provides a parametrization of the density \(\rho(b)\) in terms of a function \(v(x)\) (at this stage arbitrary, i.e., not necessarily solution of a saddle point). This parametrization has some remarkable properties. The first is that we can exactly identify the electrostatic energy of the Coulomb gas, \(J(\rho)\), with the Brownian weight function appearing in the SAO/WKB method, i.e., the second term in eq. (9), as
\begin{equation}
J(\rho) = \frac{\beta}{8} \int_{0}^{+\infty} dx v(x)^2. \tag{31}
\end{equation}
This is shown in ref. [57] under the condition that \(x + v(x)\) is an increasing function for \(x \geq 0\). This condition is in particular realized at the saddle point SP1 for \(\phi \in \Omega_0\).

Consider now the solution \(v^*(x)\) of the saddle point SP1 of the SAO/WKB method, and define \(\rho_{\Lambda}(b)\) its associated density under the parametrization given by eq. (8). We show in ref. [57] that \(\rho_{\Lambda}(b) = \rho_{\Lambda}(b)\), i.e., the unique minimizing density for the Coulomb gas. Indeed, the Hilbert transform can be explicitly calculated with this parametrization and the variational condition eq. (30) becomes, for \(b > u_0\)
\begin{equation}
\phi'(u - b) - \beta \frac{1}{2} \int_{0}^{+\infty} dx \left[ \frac{1}{\sqrt{(-b + x + v_*(x))^2 + b}} - \frac{1}{\sqrt{(-b + x + v_*(x))^2 + b}} \right]. \tag{32}
\end{equation}
Given that \(v_*(x)\) is solution to the saddle point SP1, we notice that eq. (32) is exactly the derivative of eq. (23) (equivalently eq. (26)) upon the identification of the SP2 equation in terms of \(v_*(x)\) (eqs. (16) and (25)) and \(\psi_{\Lambda}(u)\) given by the Coulomb gas method coincides with the one of the SAO/WKB method.

Finally, two equivalent formulæ for the optimal density \(\rho^*\) for both the SAO/WKB and Coulomb gas methods are obtained using the saddle point SP1 and read
\[ \rho^*(b) = \frac{1}{\pi} \sqrt{(b - u_0)^+} + \delta \rho(b) \]

where

\[ \begin{align*}
\delta \rho(b) &= \frac{2}{\beta \pi^2} \int_0^{w(u)} \frac{u' \, du'}{\sqrt{(b - u + f^{-1}(w))_+}}, \\
\delta \rho(b) &= \frac{1}{\beta \pi^2} \int_{-\infty}^{\infty} db' \, \sqrt{(b - u_0 + b')_+} \left( \frac{b'}{b + b' - u} \right)
\end{align*} \]

The lower edge of the support is \( u_0 = \frac{\beta}{\pi} \sqrt{v(u)} \) and the first expression does not involve principal parts\(^3\). Remarkably, the sole knowledge of the edge \( u_0 \) as a function of \( u \), obtained solving eq. (14), determines completely the energy \( \Sigma \rho(u) \), indeed \( \Sigma \rho(u) = \frac{1}{2} w(u) = \frac{1}{4} u_0 \), and the effective restoring force, i.e., the pressure of the gas is (see ref. [57])

\[ \Sigma \rho(u) = \frac{\beta}{4} \int_0^{\infty} dx \, v(x) = \frac{1}{2} \int_0^{u_0} du' \, w(u'). \]

Calculation of the PDF \( P(L) \): Requiring that \( \mathbb{E}_\beta e^{-BL} = \int dL \, P(L) e^{-BL} \sim e^{-c \Sigma \rho(u)} \) and inserting eq. (6), yields, upon Legendre inversion,

\[ G(\ell) = \max_B \left[ \Sigma \rho(\ell) - AB\ell \right] \]

with \( A = \mathbb{E}_\beta [L] / t^2 = \partial_B \Sigma_B(u) |_{B=0} \), given by eq. (4). We are able to probe only the pushed Coulomb gas, i.e., \( B > 0 \) and \( \ell \leq 0 \). The side \( \ell > 0 \) corresponds to \( B < 0 \), a pulled Coulomb gas in which case \( B \phi \) does not belong to \( \Omega_0 \). The phenomenon found in [42] in the bulk so that for \( \ell > 0 \) the support of the optimal density splits, leading to a distinct phase, is likely to carry to the edge.

We now apply these methods to calculate \( \Sigma \rho(u) \) (the excess energy), \( \rho_u(u) \) (the equilibrium density) and \( G(\ell) \) (the PDF) for some examples.

**Monomial soft walls:** For \( \phi(x) = (x)^\gamma \), see fig. 3 for instance, the associated \( f(u) = C_\gamma (u)^{\gamma + \frac{2}{3}} \) with \( C_\gamma = \sqrt{\pi} \frac{\Gamma(\gamma + \frac{2}{3})}{\Gamma(\gamma)} \). Hence \( \phi \) in \( \Omega_0 \) if \( \gamma > 1/2 \), to which we restrict ourselves. The energy is a simple polynomial in \( u, w \):

\[ \Sigma_B(u) = a_u w^{\gamma} + b_u w^{\gamma + \frac{2}{3}} + c_u w, \]

with \( w \) the unique positive solution of the trinomial equation

\[ [(\beta w^{\gamma - 1})/w] + 1, \quad \text{and} \quad a_u = \frac{4}{(\beta w^{\gamma + 1})^2}, \]

\[ b_u = \frac{(2\gamma - 3)w^{(2\gamma + 2)} - 2\gamma w^{(2\gamma + 1)} - 2\gamma w}{\beta w^{(2\gamma + 3)}}, \]

\[ c_u = \frac{(2\gamma - 3)w^{2\gamma - 3} - 2\gamma w^{2\gamma - 2} - 2\gamma w}{\beta w^{(2\gamma + 3)}}, \]

More explicit forms exist for some values of \( \gamma \), see ref. [57]. To comment this result it is useful to compare with the infinite hard wall, \( \lim_{B \rightarrow +\infty} \Sigma_B(u) = \frac{\beta}{24} w^{\gamma} \), a standard result related to the cubic tail of the Tracy-Widom distribution (see refs. [37,60–62]) which is an upper bound for \( \Sigma_\phi \) (see ref. [57]). Combined with the first cumulant (Jensen) bound, \(-\kappa_1(u)\), we obtain for all \( u \)

\[ \Sigma_\phi(u) \leq \min \left( \frac{\beta}{24} w^{\gamma}, \frac{\Gamma(\gamma + 1)}{4\Gamma(\gamma + \frac{2}{3})} u^{\gamma + \frac{2}{3}} \right). \]

It turns out that this inequality is saturated at small and large \( u \) for all \( \gamma \neq 3/2 \), i.e., it gives the exact asymptotics (prefactor included) in both limits. Comparing the exponents \( 3 \) and \( \gamma + 3/2 \), we see that \( \Sigma_\phi(u) \) is cubic (and \( \phi \) acts as a hard wall breaking the edge) for small \( u \) for \( \gamma < 3/2 \), and for large \( u \) for \( \gamma > 3/2 \). The other limiting behavior, \( u^{\gamma + \frac{2}{3}} \), given by the first cumulant bound, appears as a weak, perturbative, response of the edge to the potential \( \phi \). This change at \( \gamma = 3/2 \) is also seen on the density.

The optimal density \( \rho_u(u) \) is obtained as a hypergeometric function \( F_{11} \) for any \( \gamma \) [57]. It is smooth except at \( i) b = u \), with singularity \( |b - u|^\gamma - 1 \) for non-integer \( \gamma \), and \( (b - u)^{\gamma - 1} \log |b - u| \) for integer \( \gamma \). ii) \( b = u_0 \), the lower edge, always of semi-circle type. It is plotted in fig. 1 for \( \gamma = 1 \) (linear wall) and in fig. 2 for \( \gamma = 2 \) (quadratic wall) for, respectively, large and small \( u \). We see that for \( \gamma = 1 \) the rearrangement of the CG is weak for large \( u \), consistent with the (first cumulant) perturbative result \( \Sigma_\phi(u) \sim u^{\gamma / 2} \). For small \( u \) the rearrangement is strong and the density converges to the known infinite hard wall optimal density \( \rho_{HW}(u) = \frac{3B - 4u}{24B + 2\beta} u^3, \) see ref. [60], plotted in fig. 1(b) for comparison, consistent with the cubic \( \Sigma_\phi(u) \sim \frac{\beta}{24} u^3 \) behavior. The same holds in fig. 2 for \( \gamma = 2 \) but with small- and large- \( u \) behaviors exchanged.

To explore the critical case \( \gamma = 3/2 \), we study \( B\phi(x) = B(x)^{3/2} \). The saddle point equation SP1 then admits the simple solution \( v_+(x) = \frac{3B}{5B + 3} (u - x)_+ \), leading to

\[ \Sigma_B(u) = \frac{\beta}{24} \frac{3B}{3B + 2\beta} u^3, \]

Remarkably, a simple cubic for any \( u \). The prefactor exhibits a smooth crossover between the perturbative (\( B \) small) and the hard wall regime (\( B \) large). The optimal density is a superposition of two semi-circles, see ref. [57], plotted in fig. 3.

Finally, from eq. (34), we find that the PDF for all monomial walls, i.e., for all \( \gamma > 1/2 \), takes the form

\[ P(L) \sim e^{-c \frac{\beta}{24} u^{\gamma} \Gamma(\ell)}, \quad \ell = L / \mathbb{E}_\beta[L], \]

Fig. 1: Optimal density \( \rho_u(-b) \) for \( \beta = 2 \) and the linear wall \( \gamma = 1 \) (solid line), compared to the semi-circle density \( \rho_{HW}(-b) \) (dashed line), the potential \( \phi(u + b) \), and to the infinite hard-wall \( \rho_{HW}(-b) \).

Fig. 2: Same as fig. 1 for the quadratic wall \( \gamma = 2 \).
where \( q_\gamma(\ell) \) is independent of \( u \) and \( \beta \), see ref. [57], and with \( q_{3/2}(\ell) = (1 - \sqrt{\ell})^2 \) for \( \ell \leq 1 \). Note that \( q_\gamma(0) = 1 \) is always true since it corresponds to all \( a_i < -u \), i.e., an infinite hard wall.

In summary for the monomial walls the pushed-free phase transition at \( u = 0 \) in the excess energy is of third order for \( \gamma \leq 3/2 \) and of order \( \gamma + 3/2 \) for \( \gamma > 3/2 \). This change of behavior indicates a critical rigidity for the edge of the Coulomb gas in its sensitivity to a perturbation.

**Exponential wall:** An interesting case is \( \phi(x) = e^x \), for which \( f \) is also exponential \( f(u) = \sqrt{u} e^u \). Since \( f(u > 0) > 0 \) it belongs to a larger class \( \Omega_1 \supset \Omega_0 \) for which uniqueness holds and the above formulæ still hold with minor modifications [57]. Denoting \( W = W_0(2/\pi \sqrt{u}) \) the standard branch of the Lambert function [63], i.e., the solution \( W(x) \) of \( We^W = x \), we find \( w(u) = \frac{dW}{du} W \), and using eq. (15) with the lower boundary at \( u = -\infty \) we obtain \( \Sigma_\phi(u) = \frac{2}{3} (2W^3 + 9W^2 + 12W) \). The limiting behaviors of the energy are \( \Sigma_\phi(u) \approx \frac{2}{3} u^3 \) for large positive \( u \) and \( \Sigma_\phi(u) \approx \frac{1}{u} e^u \) for large negative \( u \) and the optimal density is, see ref. [57],

\[
\rho_\star(b) = \frac{1}{\pi} \sqrt{b - u_\star} + \frac{u_\star}{2\sqrt{u}} e^{u_\star - b} \text{Erfi}\left(\sqrt{b - u_\star}\right). \tag{38}
\]

The probability is given by eq. (6) with

\[
G(\ell) = -\frac{\beta}{48} (2 + \bar{W})^2 (1 + 2\bar{W}), \quad \bar{W} = W_{-1}(-2\ell e^{-2}) \tag{39}
\]

in terms of the second real branch of the Lambert function.

**Inversed monomial walls,** of the form \( \phi(x) = (-x)^\delta \) for \( x < 0, \phi(x > 0) = +\infty \), for \( \delta > 3/2 \) are another example in \( \Omega_1 \), which penetrate strongly, as power laws, into the Coulomb gas. Explicit results are displayed in ref. [57].

An important set of applications concerns truncated linear statistics. A sum over the \( N_1 \) largest eigenvalues of the Laguerre ensemble (LE), \( L = \sum_{\ell=1}^{N_1} f(\lambda_\ell) \) was studied by CG methods at large \( N \) in [42] in the bulk, i.e., for \( \kappa = N_1/N \) fixed. For \( f(\lambda) = \sqrt{\lambda} \) it was shown that the PDF of the scaled variable \( s = N^{-3/2} L \) takes the form \( \exp(-N^2 f_\star(s)) \). We have shown [57] that the \( \kappa \to 0 \) limit of these results match our edge results for the linear wall

\[
\gamma = 1. \quad \text{Using universality at the soft edge, both can be related to truncated linear statistics of the APP:}
\]

\[
L_{N_1} = - \sum_{i=1}^{N_1} a_i = \sum_{i=1}^{N_1} \varepsilon_i. \tag{40}
\]

Since \( N_1 \) is fixed, \( u \) must be determined self-consistently, \( u = u_\star(N_1/t) \) by the condition that in the optimal density \( \rho_{\star, u_\star}(b) \) there are \( N_1 \) eigenvalues below level \( u_\star \). This leads to the PDF, for \( 1 \ll N_1 \ll N \),

\[
P(L_{N_1} = L) \sim \exp\left(-\frac{\beta N_1^2}{2} \frac{2\pi^2}{3} \Phi\left(a - \frac{\beta \rho_\star[L] - L}{N_1^{3/2}}\right) \right) \tag{41}
\]

with \( a = \frac{3}{2\pi^2} \), \( \Phi(S) \) being given parametrically as [57]

\[
\Phi(S) = \frac{y^6}{12} + \frac{y^3}{2} + \frac{2}{3y^2} - \frac{5}{4}, \quad S = \frac{y^5}{10} + \frac{y^2}{2} - \frac{9}{10} + \frac{S}{10} \tag{42}
\]

for \( S \in ]-\infty,0[ \) corresponding to \( y \in ]0,1[ \). \( \Phi(S) \) has a cubic tail at large negative \( S \).

There are two applications to the physics of fermions, see [57] for more details. Firstly, consider \( N_1 \) non-interacting fermions in a linear plus random potential described by \( H_{SAO} \). Then, eq. (41) gives explicitly the PDF of the energy of the ground state \( E_0(N_1) = L_{N_1} \) for \( N_1 \gg 1 \) (studied recently in [64] without a linear potential). Secondly, eq. (1) gives for \( \beta > 1 \) the quantum JPDF of the positions of \( N \) spinless fermions in the ground state of the Calogero-Moser Hamiltonian which consists of a harmonic trap with mutual \( \beta(\beta - 2)/\pi^2 \) interactions [65]. As was studied in the case \( \beta = 2 \) (non-interacting fermions) [23], the rescaled and centered positions near the edge are described by the Airy_{3/2} point process. Hence, \( L_{N_1}/N_{1/3} \) describes the fluctuations of the scaled center of mass of the \( N_1 \gg 1 \) fermions. Other applications of (40) include fluctuating interfaces [57]. Applications of the exponential wall include traces of large powers \( (N/t)^{2/3} \) of random matrices, as in [52], as well as a directed polymer or a quantum particle at high temperature, in the presence of linear plus random (static) disorder [57]. Finally, note that any bulk linear statistics with a function \( f(\lambda) \) which is smooth at a soft edge is universally described by the linear soft wall \( \gamma = 1 \) (i.e., with a log-divergent optimal density).

In conclusion we have unified four apparently distinct methods to study the large deviations for linear statistics at the edge of the \( \beta \) ensemble of random matrices. It equivalently describes the response of a logarithmic Coulomb gas pushed delicately at its edge, with various applications to trapped fermions. Our results raise multiple questions such as the extensions to more general soft potentials \( \phi \) leading to non-unique solutions of the saddle points or multiple supports for the optimal density. This direction is currently in progress. Other outstanding questions are extensions of our methods to other random matrix ensembles, or to other types of Coulomb gases.
**Linear statistics and pushed Coulomb gas at the edge of $\beta$-random matrices: etc.**

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