PIVOTAL FUSION CATEGORIES OF RANK 3

VICTOR OSTRIK

Abstract. We classify all fusion categories of rank 3 that admit a pivotal structure over an algebraically closed field of characteristic zero. Also in the Appendix (joint with D. Nikshych) we give some restrictions on Grothendieck rings of near-group categories.

1. Introduction

Fusion categories introduced in [ENO] are in a sense simplest tensor categories. The problem of classification of fusion categories while interesting seems to be well out of reach at the moment. Thus it is natural to attempt a classification of fusion categories which are “small” in certain sense. One classical interpretation of “small” coming from the theory of subfactors is that the category in question contains an algebra object of small dimension; we refer the reader to [JMS] for a nice survey of achievements in this direction. Another possibility is that “small” categories have integral Frobenius-Perron dimension with small number of prime factors; some results in this direction were obtained in [ENO1].

In this paper we follow approach initiated in [O2] in which “small” categories have small rank. We recall that rank of a fusion category is just a number of isomorphism classes of its simple objects. The fusion categories of rank 1 are trivial and the fusion categories of rank 2 were classified in [O2]. Under an additional assumption that the category in question is ribbon some further classification results were obtained in [O3, RSW, RS]. Also recently a general finiteness conjecture on the number of modular tensor categories of fixed rank was proved, see [BNRS].

In this paper we give a classification of fusion categories of rank 3 that admit a pivotal structure (this is relatively mild assumption: it is conjectured in [ENO, Conjecture 2.8] that any fusion category admits a pivotal structure). Thus we prove the following result conjectured in [O3]:

Theorem 1.1. Let $\mathcal{C}$ be a fusion category of rank 3 admitting a pivotal structure over an algebraically closed field of characteristic zero. Then $\mathcal{C}$ is equivalent to one of the following:

(i) pointed category with underlying group $\mathbb{Z}/3\mathbb{Z}$;

(ii) even part of the category associated with quantum $sl_2$ at 7th root of unity, see e.g. [O3, Section 4.4];

(iii) Ising category, see e.g. [DGNO, Appendix B];

(iv) category of representations of the group $S_3$ or its twisted version, see [O3, Section 4.4];

(v) category associated with subfactor of type $E_6$ or its Galois conjugate, see e.g. [III].
The proof of this result is significantly more difficult than the previous classification in rank 2 [O2]. Thus we had to develop some new theoretical tools. Most important are Theorem 2.13 which gives some information on the decomposition of the induction of the unit object to the Drinfeld center of a fusion category and pseudo-unitary inequality (5) which gives some non-trivial restrictions on the Grothendieck rings of fusion categories which admit a spherical structure with positive values of dimensions. This inequality is strong enough to show that vast majority of based rings of rank 3 are not categorifiable. It is also sometimes useful in some other cases, see e.g. [La]. However the computer experiments performed by Nicolle Sandoval Gonzalez seem to indicate that this inequality is less efficient in higher ranks.

It is highly unsatisfactory that we had to assume the existence of pivotal structure for our results. Despite a significant effort we were not able to eliminate this assumption. However the existence of fusion categories of rank 3 which do not admit such a structure seems to be very unlikely: we prove that such categories must have very big fusion coefficients, see Theorem 5.8.

An interesting class of fusion categories is formed by near-group categories introduced in [S]. By definition the near-group categories are “small” in the following sense: the number of non-invertible simple objects in such categories is precisely 1. The results of [I] and [EG] suggest that this class contains an infinite family of fusion categories which seems to be unrelated to previously known constructions using finite groups and quantum groups at roots of unity. In the Appendix A written jointly with Dmitry Nikshych we apply our techniques to this case and obtain some new results on near-group categories. In particular we show that non-integral near-group categories must have commutative Grothendieck rings and that integral near-group categories are weakly group theoretical in the sense of [ENO1]. We also note that Theorem 4.1 gives a classification of near-group categories with the group of invertible objects of order 2.

It is my great pleasure to express my deep gratitude to Dmitri Nikshych who made many useful suggestions for this paper. In particular he suggested to use results of Ng and Schauenburg in the proof of Theorem 4.1 which allowed me to eliminate one particularly difficult case. Also Example 2.18 is a result of our joint work. I am also happy to thank Nicolle Sandoval Gonzalez and Hannah Larson for many useful discussions. Both authors of the Appendix are grateful to Masaki Izumi for sending his preprint [I].

2. Preliminaries

In this paper the base field $k$ is assumed to be algebraically closed of characteristic zero.

2.1. Fusion categories and based rings. We recall from [ENO] that a fusion category is a rigid tensor semisimple category over $k$ such that all Hom spaces are finite dimensional, number of isomorphism classes of simple objects is finite, and the unit object is simple. For a fusion category $C$ we will denote by $O(C)$ the set of isomorphism classes of simple objects in $C$.

A pivotal structure on a fusion category $C$ is an isomorphism of tensor functors $\text{id} \rightarrow \ast\ast$, see e.g. [ENO, Definition 2.7]. In the presence of pivotal structure one defines traces of morphisms and dimensions $\text{dim}(X) \in k$ of objects, see e.g. [ENO, Section 2.2]. A pivotal structure (or the underlying fusion category) is spherical if
dim(X) = dim(X^*) for all objects X ∈ C (equivalently, all dimensions are totally real), see [ENO] Section 2.2. For a fusion category C one defines its global dimension dim(C), see e.g. [ENO] Definition 2.2. For a spherical fusion category C the global dimension is given by
\[ \dim(C) = \sum_{X \in \mathcal{O}(C)} \dim(X)^2. \]
Assume that we fixed an embedding of the subfield of algebraic numbers in k to \( \mathbb{C} \).

A pseudo-unitary spherical structure on a fusion category C is a spherical structure such that the images under this embedding of dimensions of all objects are positive real numbers, see [ENO] Section 8.4. A fusion category is pseudo-unitary if it has a pseudo-unitary spherical structure.

The Grothendieck ring \( K(C) \) of a fusion category C is an example of unital based ring of finite rank (see e.g. [O4] Definition 2.1]). This means that \( K(C) \) is endowed with a basis \( \{ b_i \}_{i \in I} \) over \( \mathbb{Z} \) (given by the classes of simple objects) such that the structure constants are nonnegative; moreover there exists an involution \( i \mapsto i^* \) such that the expansion of \( b_i b_j \) contains 1 (which is one of the basis elements) with coefficient \( \delta_{ij} \).

For a based ring \( K \) of finite rank its categorification is a fusion category \( C \) and an isomorphism of based rings \( K \cong K(C) \); two categorifications are equivalent if there is a tensor equivalence \( C_1 \to C_2 \) inducing the identity map on \( K \). It is known that a given based ring of finite rank admits only finitely many categorifications up to equivalence (this statement is known as “Ocneanu rigidity”, see [ENO] Theorem 2.28). We say that a based ring is categorifiable if it admits at least one categorification. The following statements are useful necessary conditions for a based ring \( K \) to be categorifiable:

**Proposition 2.1. (Cyclotomic test) ([ENO] Corollary 8.53])** Let \( C \) be a fusion category. Any irreducible representation of the ring \( K(C) \) is defined over some cyclotomic field. In particular for any homomorphism \( \phi : K(C) \to \mathbb{C} \) we have \( \phi([X]) \in \mathbb{Z}[\zeta] \) for some root of unity \( \zeta \). □

We recall (see [O4]) that an algebraic integer \( \alpha \) is a d-number if \( \frac{\sigma(\alpha)}{\alpha} \) is algebraic integer for any automorphism \( \sigma \) of \( \mathbb{Q} \). Equivalently, if \( x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{Z}[x] \) is the minimal polynomial of \( \alpha \) then \( \frac{a_i}{\alpha} \in \mathbb{Z} \) for \( i = 1, \ldots, n - 1 \).

**Proposition 2.2. (d-number test) ([O4] Theorem 1.2])** Let \( C \) be a fusion category. Let \( R : K(C) \to K(C) \) be the operator of left multiplication by the element \( \sum_i b_i b_i^* \). Then the eigenvalues of \( R \) are d-numbers. □

**Remark 2.3.** Proposition 2.2 is equivalent to [O4] Theorem 1.2 in view of Remark 2.11 and [O4] Corollary 2.8.

2.2. The induction functor \( I : C \to Z(C) \). Let \( C \) be a spherical fusion category. It is known that its Drinfeld center \( Z(C) \) is a modular tensor category, see [M2] Theorem 1.2, [ENO] Theorem 2.15. Since \( Z(C) \) is semisimple the forgetful functor \( F : Z(C) \to C \) has a right adjoint \( I : C \to Z(C) \). We will use the following properties of this functor:

**Proposition 2.4. ([ENO] Proposition 5.4])** \( F(I(Y)) = \bigoplus_{X \in \mathcal{O}(C)} X \otimes Y \otimes X^* \).

In particular we see that for any \( Y \in C \)
\[ \dim(I(Y)) = \dim(Y) \dim(C). \]
Recall that any modular tensor category is equipped with balance isomorphisms \( \theta_X : X \to X \), see [BK Section 2.2].

**Theorem 2.5.** (cf. [NS Remark 4.6]) We have \( \theta_I(1) = \text{id}_I(1) \) and \( \text{Tr}(\theta_I(1)) = \dim(C) \). For any \( 1 \neq X \in \mathcal{O}(C) \) we have \( \text{Tr}(\theta_I(X)) = 0 \).

**Proof.** The equality \( \theta_I(1) = \text{id}_I(1) \) follows from [OS Corollary 2.7]; then \( \text{Tr}(\theta_I(1)) = \dim(I(1)) = \dim(C) \) follows from [1].

Recall (see [DMNO Lemma 3.5]) that \( I(1) \) is an étale algebra in \( \mathcal{Z}(C) \). Let \( \varepsilon : I(1) \to 1 \) be a unique map which composes to identity with the unit morphism. Then the morphism \( e : I(1) \otimes I(1) \to I(1) \xrightarrow{\varepsilon} 1 \) is non-degenerate, see [DMNO Remark 3.4]. Thus there is a unique morphism \( i : 1 \to I(1) \otimes I(1) \) such that the pair \((e, i)\) satisfies the rigidity axiom, see [KO Definition 1.11]. For any \( I(1) \)-module \( M \) with the action morphism \( a : I(1) \otimes M \to M \) consider the following composition (cf. [KO Fig. 16]):

\[
\psi_M : M \xrightarrow{i \otimes \text{id}} I(1) \otimes I(1) \otimes M \xrightarrow{\text{id} \otimes a} I(1) \otimes M \xrightarrow{\text{id} \otimes \theta} I(1) \otimes M \xrightarrow{a} M.
\]

Using [KO Lemma 1.14] one shows that \( \text{Tr}(\psi_M) = \dim(I(1))\text{Tr}(\theta_M) \). On the other hand it is proved in [KO Lemma 4.3] that for a simple \( M \) we have \( \psi_M = 0 \) unless \( M \) is dyslectic.

The result follows since the simple \( I(1) \)-modules are precisely \( I(X), X \in \mathcal{O}(C) \) (see [DMNO Lemma 3.5]), and the only simple dyslectic module over \( I(1) \) is \( I(1) \) itself, see [DMNO Proposition 4.4]. \( \square \)

**Example 2.6.** Recall that for a modular category \( \mathcal{D} \) its Gauss sum is

\[
G(\mathcal{D}) = \sum_{A \in \mathcal{O}(\mathcal{D})} \theta_A \dim(A)^2.
\]

We can compute the Gauss sum of the category \( \mathcal{Z}(C) \) using Theorem 2.5

\[
\sum_{A \in \mathcal{O}(\mathcal{Z}(C))} \theta_A \dim(A)^2 = \sum_{[A] \in \mathcal{O}(\mathcal{Z}(C)), X \in \mathcal{O}(C)} \theta_A \dim(A)[F(A) : X] \dim(X) = \\
\sum_{A \in \mathcal{O}(\mathcal{Z}(C)), X \in \mathcal{O}(C)} \theta_A \dim(A)[I(X) : A] \dim(X) = \sum_{X \in \mathcal{O}(C)} \text{Tr}(\theta_I(X)) \dim(X) = \dim(C)
\]

Thus we recover the result of Müger [M2 Theorem 1.2] stating that for any spherical \( \mathcal{C} \)

\[
G(\mathcal{Z}(C)) = \dim(C).
\]

Ng and Schauenburg connected \( \text{Tr}(\theta_I^2(X)) \), \( i \in \mathbb{Z} \) with higher Frobenius-Schur indicators, see [NS]. In particular, \( \text{Tr}(\theta_I^2(X)) \) is related with the classical Frobenius-Schur indicator. Thus we have

**Theorem 2.7.** (cf. [NS Theorem 4.1]) Assume that \( X \in \mathcal{O}(C) \) is self-dual. Then

\[
\text{Tr}(\theta_I^2(X)) = \pm \dim(C). \quad \square
\]

Let \( c_{A,B} : A \otimes B \to B \otimes A \) denote the braiding on the category \( \mathcal{Z}(C) \). For \( A, B \in \mathcal{O}(\mathcal{Z}(C)) \) let \( s_{A,B} = \text{Tr}(c_{B,A} \circ c_{A,B}) \). We can consider \( s_{A,B} \) as a matrix of a linear operator on \( K(\mathcal{Z}(C)) \otimes k \) with respect to the basis consisting of classes \([A], A \in \mathcal{O}(C)\). The following result was conjectured by A. Kitaev.

**Proposition 2.8.** (cf. [KO Theorem 4.1]) The class \( [I(1)] \in K(\mathcal{Z}(C)) \) is an eigenvector for \( s = s_{A,B} \) with eigenvalue \( \dim(C) \).
Formal codegrees

of the based ring

for any $A$

Example 2.9. Assume that $K$ classes of irreducible representations of $(2)$

Proposition 2.10. It follows from $[O4, \text{Lemma 2.6}]$ that the eigenvalues of the operator $R$ (see Proposition 2.2) are $\dim(E) f_E$ appearing with multiplicity $\dim(E)^2$. Thus Proposition 2.10 states that $\Tr(R^{-1}) = 1$.

Remark 2.11. It was observed in $[ENO]$ proof of Lemma 8.49 that the operator $R$ is positive definite. Thus Remark 2.11 implies that $\dim(E) f_E$ is totally positive for any $E \in \Irr(K(C))$. Thanks to Proposition 2.10 this can be improved to $f_E \geq \dim(E)$. In particular we have $f_E \geq 1$ for any representation $E$.

Let $C$ be a spherical fusion category and let $E \in \Irr(K(C))$. Then $K(Z(C))$ acts on $E$ via the forgetful homomorphism $K(Z(C)) \to K(C)$; by Schur’s lemma this action factorizes via a character of $K(Z(C))$. Let $E \in \Irr(K(Z(C)))$ be the corresponding 1-dimensional representation of $K(Z(C))$. Recall (see Section 2.2)

Proof. Recall that $c_{i(1),A} \circ c_{A,i(1)} = \theta_{A \otimes i(1)}(\theta_A \theta_{i(1)})^{-1}$, see e.g. $[BK]$ (2.2.8)]. If $A$ is a simple object then $\theta_A$ is a scalar; also $\theta_{i(1)} = \id_{i(1)}$ by Theorem 2.5. Thus

\[
\text{Tr}(c_{i(1),A} \circ c_{A,i(1)}) = \theta_A^{-1} \text{Tr}(\theta_{A \otimes i(1)}).
\]

Observe that $A \otimes I(1)$ is $I(1)$–module, so it is a direct sum of modules $I(X), X \in \mathcal{O}(C)$; in view of Theorem 2.5 the trace above depends only on the multiplicity of $I(1)$ in $A \otimes I(1)$. This multiplicity is

$$\dim_k \text{Hom}_{I(1)}(A \otimes I(1), I(1)) = \dim_k \text{Hom}(A, I(1)) = [I(1) : A].$$

Notice that $[I(1) : A] \neq 0$ implies $\theta_A = 1$ by Theorem 2.5. Thus

$$s \cdot [I(1)] = \sum_{A \in \mathcal{O}(Z(C))} \text{Tr}(c_{i(1),A} \circ c_{A,i(1)})[A] = \sum_{A \in \mathcal{O}(Z(C))} [I(1) : A] \dim(C)[A] = \dim(C)[I(1)].$$

\[\square\]

2.3. Formal codegrees and decomposition of $I(1)$. Let $K$ be a based ring of finite rank and let $E$ be an irreducible representation of $K$ over the field $k$. Then the element $\alpha_E = \sum_{i \in I} \text{Tr}(b_i, E) b_i^* \in K \otimes k$ is in the center of $K \otimes k$; it acts by zero on any irreducible representation of $K$ not isomorphic to $E$ and by a nonzero scalar $f_E \in k$ on $E$, see $[O4]$ Lemma 2.3. Let $\Irr(K)$ be the set of isomorphism classes of irreducible representations of $K$. The scalars $f_E, E \in \Irr(K)$ are called formal codegrees of the based ring $K$ or of the fusion category $C$ with $K = C$.

**Example 2.9.** Assume that $D$ is a modular tensor category. It is well known that for any $A \in \mathcal{O}(D)$ the assignment $B \mapsto \frac{\alpha_{E_A}}{\dim(A)}$ is a 1-dimensional representation $E_A$ of $K(D)$; moreover the map $A \mapsto E_A$ is a bijection $\mathcal{O}(D) \to \Irr(K(D))$, see e.g. $[BK$, Theorem 3.1.12]. We have (see $[O4$, Section 3.3])

\[
f_{E_A} = \frac{\dim(D)}{\dim(A)^2}.
\]

**Proposition 2.10.** For any based ring $K$ we have

$$\sum_{E \in \Irr(K)} \frac{\dim(E)}{f_E} = 1.$$ 

**Proof.** Use $[Ln]$ Proposition 19.2(c)] with $a = 1$. \[\square\]

**Remark 2.11.** It follows from $[O4]$ Lemma 2.6 that the eigenvalues of the operator $R$ (see Proposition 2.2) are $\dim(E) f_E$ appearing with multiplicity $\dim(E)^2$. Thus Proposition 2.10 states that $\Tr(R^{-1}) = 1$.

**Remark 2.12.** It was observed in $[ENO]$ proof of Lemma 8.49 that the operator $R$ is positive definite. Thus Remark 2.11 implies that $\dim(E) f_E$ is totally positive for any $E \in \Irr(K(C))$. Thanks to Proposition 2.10 this can be improved to $f_E \geq \dim(E)$. In particular we have $f_E \geq 1$ for any representation $E$.
that $\mathcal{Z}(\mathcal{C})$ is a modular tensor category with $\dim(\mathcal{Z}(\mathcal{C})) = \dim(\mathcal{C})^2$. Thus by Example 2.9 there exists a unique $A_E \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ such that $E = E_{A_E}$; in addition

\[ f_E = \frac{\dim(\mathcal{C})^2}{\dim(A_E)^2}. \]

**Theorem 2.13.** For a spherical fusion category $\mathcal{C}$ the assignment $E \mapsto A_E$ is an embedding $\text{Irr}(K(\mathcal{C})) \subset \mathcal{O}(\mathcal{Z}(\mathcal{C}))$. Its image consists of $A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ with $[I(1) : A] \neq 0$. Moreover, we have

\[ [I(1) : A_E] = \dim(E), \quad \text{and} \quad \dim A_E = \frac{\dim(\mathcal{C})}{f_E}. \]

**Proof.** It is known that $K(\mathcal{Z}(\mathcal{C})) \otimes k$ maps surjectively on the center of $K(\mathcal{C}) \otimes k$, see [ENO, Lemma 8.49]; the first statement follows. Also [O4, Lemma 3.1] states that $f_E = f_E^2$, so (4) implies $f_E = \pm \frac{\dim(\mathcal{C})}{\dim(A_E)}$.

In view of Proposition 2.8 and Example 2.9 we have

\[ [I(1)] = \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(1) : A] \frac{\dim(A)}{\dim(\mathcal{C})} \alpha_{E_A}. \]

Thus $[I(1)]$ acts on $E \in \text{Irr}(K(\mathcal{C}))$ by the scalar

\[ [I(1) : A_E] \frac{\dim(A_E)}{\dim(\mathcal{C})} f_{E_{A_E}} = [I(1) : A_E] \frac{\dim(\mathcal{C})}{\dim(A_E)}, \]

see (3). On the other hand by Proposition 2.8 we have $[F(I(1))] = R$, so by [O4, Lemma 2.6] it acts on $E$ by the scalar $\dim(E)f_E$. Hence

\[ \dim(E)f_E = [I(1) : A_E] \frac{\dim(\mathcal{C})}{\dim(A_E)}. \]

Combining this with $f_E = \pm \frac{\dim(\mathcal{C})}{\dim(A_E)}$, we get (4).

It is known (see [ENO, Proposition 5.6]) that $\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(1) : A]^2$ equals the rank of $\mathcal{C}$. We already proved that each object $A_E$ appears in $I(1)$ with multiplicity $\dim(E)$. Thus

\[ \sum_{E \in \text{Irr}(K(\mathcal{C}))} [I(1) : A_E]^2 = \sum_{E \in \text{Irr}(K(\mathcal{C}))} \dim(E)^2 = \dim(K(\mathcal{C}) \otimes k) = \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(1) : A]^2. \]

It follows that any object $A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ with $[I(1) : A] \neq 0$ is of the form $A_E$ for some $E \in \text{Irr}(K(\mathcal{C}))$. Theorem is proved.

**Corollary 2.14.** (cf [ENO, Proposition 8.22]) For a not necessary spherical fusion category $\mathcal{C}$ the formal codegrees $f_E$ divide $\dim(\mathcal{C})$.

**Proof.** In a case of spherical $\mathcal{C}$ the result is immediate from Theorem 2.13 since $\dim(A_E)$ is an algebraic integer. For a general fusion category $\mathcal{C}$ one considers its sphericalization $\hat{\mathcal{C}}$, see [ENO, Remark 3.1] or Section 5.1 below. This is spherical category of global dimension 2 $\dim(\mathcal{C})$; moreover $2f_E$ is a formal codegree of $\hat{\mathcal{C}}$ for any $E \in \text{Irr}(K(\mathcal{C}))$. The result follows.

Assume again that $\mathcal{C}$ is spherical. Let $L(\mathcal{C})$ be the dimension field of $\mathcal{C}$, that is the field generated over $\mathbb{Q}$ by the dimensions of simple objects of $\mathcal{C}$.

**Corollary 2.15.** The formal codegrees of $\mathcal{C}$ lie in the field $L(\mathcal{C})$. 
Proof. For any $A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))$ we have $\dim(A) = \dim(F(A)) \in K$. Since $\dim(\mathcal{C}) \in K$ the result follows from (1). □

Corollary 2.16. There is an isomorphism of associative algebras

$$K(\mathcal{C}) \otimes k \simeq \text{End}(I(1)).$$

Proof. Both algebras are isomorphic to direct sum of matrix algebras of sizes $\dim(E)$ indexed by $E \in \text{Irr}(K(\mathcal{C})).$ □

We don’t know whether there is a canonical choice of the isomorphism in Corollary 2.16.

Example 2.17. Assume that $\mathcal{C}$ is dual to a modular tensor category $\mathcal{D}$. Then $\mathcal{Z}(\mathcal{C}) = \mathcal{D} \boxtimes \mathcal{D}^{\text{rev}}$ (see [M2 Theorem 7.10]) and $[I(1)] \in K(\mathcal{Z}(\mathcal{C})) = K(\mathcal{D} \otimes K(\mathcal{D})$ is known as modular invariant associated with $\mathcal{C}$ (or rather with module category $\mathcal{M}$ over $\mathcal{D}$ such that $\mathcal{C} = \mathcal{D}^\mathcal{M}$), see [BEK Definition 5.5], [FRS Section 5.3]. In this case our results are well-known: Theorem 2.13 says that the modular invariant commutes with $T$–matrix (cf [BEK Theorem 5.7], [FRS Theorem 5.1]), Proposition 2.16 says that modular invariant commutes with $S$–matrix (cf [BEK Theorem 5.7], [FRS Theorem 5.1]), and Corollary 2.16 expresses $K(\mathcal{C}) \otimes k$ in terms of the modular invariant (cf [BEK Theorem 6.8], [FRS Theorem 6.1]).

Example 2.18. (joint with D. Nikshych) It follows immediately from Corollary 2.16 that $K(\mathcal{C})$ is commutative if and only if $I(1)$ is multiplicity free. Let us consider minimal étale subalgebras of $I(1)$. Such subalgebras intersect trivially, so no nontrivial simple summand of $I(1)$ appears in two such subalgebras. Thus the number of such subalgebras is at most $rk(\mathcal{C}) - 1$. In view of [DMNO Theorem 4.10] this translates into the following statement: the number of maximal fusion subcategories of $\mathcal{C}$ is at most $rk(\mathcal{C}) - 1$. It was conjectured in [GX Conjecture 1.2] that this statement holds for any fusion category $\mathcal{C}$ (possibly with non-commutative $K(\mathcal{C})$). We don’t know whether this statement fails for a unital based ring $K$ of finite rank.

Remark 2.19. Let $\mathcal{C} = \oplus_{ij} C_{ij}$ be an indecomposable multi-fusion category (see [ENO Section 2.4]) with decomposable unit object $1 = \oplus_i 1_i$. Assume that $\mathcal{Z}(C_{11})$ is spherical and $\theta_{I(1)} = \text{id}_{I(1)}$. The proof of Theorem 2.13 and Corollary 2.16 extends with trivial changes to this setting. Moreover, Theorem 2.13 applied to fusion category $C_{ii}$ implies that the dimension of $[1_i]E$ equals $[I(1_i) : A_E]$. It follows that we can choose an isomorphism in Corollary 2.16 sending $[1_i] \in K(C_{11})$ to the projection of $I(1)$ onto $I(1_i)$. This implies that $(K(C_{ij}) \otimes k, K(C_{jj}) \otimes k)$–bimodule $K(C_{ij}) \otimes k$ can be identified with $(\text{End}(I(1_i)), \text{End}(I(1_j)))$–bimodule $\text{Hom}(I(1_i), I(1_j))$ where $K(C_{ii}) \otimes k$ and $K(C_{jj}) \otimes k$ are identified with $\text{End}(I(1_i))$ and $\text{End}(I(1_j))$ as in Corollary 2.16. In particular, the number of simple objects in $C_{ij}$ equals $\dim \text{Hom}(I(1_i), I(1_j)).$

For example this applies as follows: let $\mathcal{D}$ be a spherical fusion category and let $\mathcal{M}, N$ be indecomposable semisimple module categories over $\mathcal{D}$. Let $M, N$ be Lagrangian subalgebras in $\mathcal{Z}(\mathcal{D})$ associated with $\mathcal{M}$ and $N$ in [DMNO Section 4.2] (so $M = I_M(1_M)$ where $I_M$ is the adjoint of the canonical functor $\mathcal{Z}(\mathcal{D}) \to \mathcal{D}^*\text{gr}$ to the dual of $\mathcal{D}$ with respect to $\mathcal{M}$, and similarly for $N$). Assume that $\theta_M = \text{id}_M$ and $\theta_N = \text{id}_N$. Then applying the above statement to $\mathcal{C} = \mathcal{D}^{*\mathcal{M}\oplus\mathcal{N}}$ we get the following: $(K(D_M^*) \otimes k, K(D_N^*) \otimes k)$–bimodule $K(\text{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{N})) \otimes k$ is isomorphic
to \((\text{End}(M),\text{End}(N))\)–bimodule \(\text{Hom}(M,N)\) where \(K(D_M^\star) \otimes k\) is identified with \(\text{End}(M)\) as in Corollary \(2.16\) and similarly for \(K(D_M^\star)\). In particular the number of simple objects in \(\text{Fun}_D(M,N)\) equals \(\dim \text{Hom}(M,N)\). This statement can be considered as a generalization of the theory of quantum Manin triples from \([DMNO\text{ Section 4.4}]\). In the case of modular category \(D\) this result was obtained in \([BEK\text{ Theorem 6.12}]\) and \([FRS\text{ Theorem 5.18}]\).

Note also that the condition \(\theta_M = \text{id}_M\) (and similarly for \(N\)) is satisfied automatically when the dimensions in the category \(Z(D)\) are positive. In this case the category \(D_M^\star\) is pseudo-unitary. Indeed, it is known that \(Z(D_M^\star) \simeq Z(D)\) (see \([MI\text{ Remark 3.18}],\ [OI\text{ Corollary 2.6}]\)), so the result follows from \([ENO\text{ Theorem 2.15 and Proposition 8.12}]\). Thus the spherical structure on \(Z(D) = Z(D_M^\star)\) is the same as the one obtained from a spherical structure on \(D_M^\star\) (defined in \([ENO\text{ Proposition 8.23}]\)) and one deduces \(\theta_M = \text{id}_M\) from Theorem \(2.5\). On the other hand the condition \(\theta_M = \text{id}_M\) is not always satisfied: it fails for \(D = \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}\) (see \([ENO\text{ Example 1 in Section 2}]\)) with non-standard spherical structure (so the dimension of the nontrivial object is \(-1\)) and module category \(M\) with one simple object. Thus it would be interesting to investigate whether the conclusions above hold true in the case when \(\theta_M = \text{id}_M\) fails (this is the case in the example above since we can replace the non-standard spherical structure by the standard one).

**Example 2.20.** Let \(C\) be a spherical fusion category and let \(I(1) \in Z(C)\) be as above. It was asked by A. Kitaev whether there exists an étale algebra \(\hat{I}\) such that \(\hat{I} = I(1)\) as an object of \(Z(C)\) but the étale algebras \(\hat{I}\) and \(I(1)\) are not isomorphic. In view of Remark \(2.19\) this translates into the following question: is there a module category \(M\) over \(C\) such that \(K(C)\)–module \(K(M) \otimes k\) is isomorphic to \(K(C) \otimes k\)? Such examples do exist: for example let \(\mathcal{V}\) be \(\mathbb{Z}/2\mathbb{Z}\)–graded fusion category constructed in \([CMS\text{ Theorem A.5.1}]\); then \(\mathcal{V}_1\) considered as a module category over \(C = \mathcal{V}_0\) satisfies the condition above. Thus the question above has a positive answer.

2.4. **Pseudo-unitary inequality.** Let us assume that the category \(C\) is pseudo-unitary, see \([ENO\text{ Section 8.4}]\). Thus the category \(C\) has a spherical structure such that the dimensions of objects are nonnegative.

**Theorem 2.21.** Let \(C\) be a pseudo-unitary fusion category. Then the formal codegrees of \(C\) satisfy

\[
\sum_{E \in \text{Irr}(K(C))} \frac{1}{f_E} \leq \frac{1}{2} \left( 1 + \frac{1}{\dim(C)} \right).
\]

**Remark 2.22.** It follows from Remark \(2.11\) that the LHS of \((5)\) equals \(\text{Tr}(R^{-2})\). Since \(\dim(C)\) is the largest eigenvalue of \(R\), the inequality \((5)\) can be stated purely in terms of operator \(R\).

**Proof.** Let \(X \in \mathcal{O}(C) \setminus \{1\}\). By Theorem \(2.5\)

\[
\sum_{A \in \mathcal{O}(Z(C))} [I(X) : A] \theta_A \dim(A) = \text{Tr}(\theta_{I(X)}) = 0,
\]

and \(\theta_A = 1\) for any \(A\) appearing in \(I(1)\). Thus

\[
\sum_{A \in \mathcal{O}(Z(C))} [I(X) : A] \dim(A) = - \sum_{A \in \mathcal{O}(Z(C))} [I(X) : A] \theta_A \dim(A).
\]
Since all dimensions are positive real numbers we have

\[
\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} [I(X) : A] \dim(A) = \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} [I(X) : A] \dim(A)
\]

\[
= \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] = 0} [I(X) : A] \theta_A \dim(A) \leq \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] = 0} [I(X) : A] \dim(A).
\]

Therefore

\[
\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} [I(X) : A] \dim(A) \leq \frac{1}{2} \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [I(X) : A] \dim(A) = \frac{1}{2} \dim(I(X)).
\]

Recall that \(\dim(I(X)) = \dim(X) \dim(\mathcal{C})\), see \([1]\). Hence multiplying the inequality above by \(\dim(X)\) and summing over \(X \in \mathcal{O}(\mathcal{C}) \setminus \{1\}\) we get

\[
\sum_{1 \neq X \in \mathcal{O}(\mathcal{C})} \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} [I(X) : A] \dim(X) \dim(A) \leq \sum_{1 \neq X \in \mathcal{O}(\mathcal{C})} \frac{1}{2} \dim(X)^2 \dim(\mathcal{C}).
\]

Observe that

\[
\sum_{1 \neq X \in \mathcal{O}(\mathcal{C})} [I(X) : A] \dim(X) = \sum_{1 \neq X \in \mathcal{O}(\mathcal{C})} [F(A) : X] \dim(X) = \dim(A) - [F(A) : 1],
\]

and

\[
\sum_{1 \neq X \in \mathcal{O}(\mathcal{C})} \dim(X)^2 = \dim(\mathcal{C}) - 1.
\]

Therefore we get

\[\tag{6}
\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [F(A):1] \neq 0} (\dim(A) - [F(A) : 1]) \dim(A) \leq \frac{1}{2} (\dim(\mathcal{C}) - 1) \dim(\mathcal{C}).
\]

Using \([1]\) we compute

\[
\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} [F(A) : 1] \dim(A) = \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} [I(1) : A] \dim(A) = \dim(I(1)) = \dim(\mathcal{C}).
\]

Thus \(\text{[6]}\) is equivalent to

\[
\left( \sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} \dim(A)^2 \right) - \dim(\mathcal{C}) \leq \frac{1}{2} (\dim(\mathcal{C}) - 1) \dim(\mathcal{C}),
\]

or

\[
\sum_{A \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \mid [I(1):A] \neq 0} \dim(A)^2 \leq \frac{1}{2} (\dim(\mathcal{C}) + 1) \dim(\mathcal{C}).
\]

By Theorem 2.13 this inequality is equivalent to \(\text{[5]}\). \(\square\)
3. Categorifiable Simple Based Rings of Rank 3

3.1. Based rings of rank 3. Let \( k, l, m, n \) be integers satisfying the equality
\[
    k^2 + l^2 = kn + lm + 1.
\]
It is easy to check that a ring \( K(k, l, m, n) \) with basis \( 1, X, Y \) and multiplication
\[
    X^2 = 1 + mX + kY, \quad Y^2 = 1 + lX + nY, \quad XY = YX = kX + lY
\]
is associative. In a case when the numbers \( k, l, m, n \) are nonnegative the ring
\( K(k, l, m, n) \) is a based ring.

Another based ring of rank 3 is the group algebra of the group \( \mathbb{Z}/3\mathbb{Z} \) with basis
given by the group elements which we denote as \( K(\mathbb{Z}/3\mathbb{Z}) \).

We have the following

**Proposition 3.1.** (cf [O3 §3.1]) Any unital based ring of rank 3 is isomorphic to
either \( K(k, l, m, n) \) or \( K(\mathbb{Z}/3\mathbb{Z}) \).

**Proof.** In the case when any element of the based ring is self-dual \( \mathfrak{S} \) is a general
form of fusion rules and \( \mathfrak{S} \) is equivalent to the associativity of multiplication \( \mathfrak{S} \).

Assume that not every element of the based ring is self-dual. Then we have a
basis \( 1, X, Y \) with \( X^* = Y \) whence FPdim\((X) = FPdim\((Y) \). The element \( X^2 \) is an
integral linear combination of \( X \) and \( Y \), hence FPdim\((X) \in \mathbb{Z} \). The element \( XY \)
is 1 plus an integral linear combination of \( X \) and \( Y \), so FPdim\((X) \) is a divisor of 1.
Thus FPdim\((X) = FPdim\((Y) = 1 \) and the Proposition is proved. \( \square \)

**Remark 3.2.** (i) Notice that the interchange \( X \leftrightarrow Y \) induces a based ring isomorphism
\( K(k, l, m, n) \cong K(l, k, n, m) \). Thus we will often assume that \( l \leq k \).

(ii) The diophantine equation (7) has infinitely many solutions. Clearly, for any
such solution \( k \) and \( l \) are coprime. The number of (nonnegative) solutions with
\( 0 < l \leq k \leq x \) is about \( Cx^2 \ln(x) \) for a constant \( C \approx \frac{1}{3} \) (notice that there are
infinitely many solutions with \( k = 1, l = 0 \)).

The main result of this section is the following

**Theorem 3.3.** Let \( \mathcal{C} \) be a pivotal fusion category of rank 3. Then \( K(\mathcal{C}) \) is isomor-
pic to one of the following:
\( K(1, 0, m, 0) \) for some nonnegative integer \( m \);
\( K(1, 1, 1, 0) \);
\( K(\mathbb{Z}/3\mathbb{Z}) \).

By Proposition 3.1 we can and will assume that \( K(\mathcal{C}) = K(k, l, m, n) \) in the
proof of Theorem 3.3

**Proof of Theorem 3.3.** Combine Lemma 3.4, Propositions 3.5 and 3.8 with
Lemmas 3.7 and 3.10 below. \( \square \)

3.2. Notations. The following notations will be used throughout the paper. We
consider an element \( 1 + X^2 + Y^2 = 3 + (l + m)X + (k + n)Y \in K(k, l, m, n) \) and
the operator \( R: K(k, l, m, n) \to K(k, l, m, n) \) of left multiplication by this element.
We have explicitly
\[
    R = \begin{pmatrix}
        3 & l + m & k + n \\
        l + m & 3 + (l + m)m + (k + n)k & (l + m)k + (k + n)l \\
        k + n & (l + m)k + (k + n)l & 3 + (l + m)l + (k + n)n
    \end{pmatrix}
\]
In what follows \( p(t) = \det(t - R) \) denote the characteristic polynomial of \( R \). The roots of the polynomial \( p(t) \) are real positive (since the matrix above is symmetric positive definite); we will denote the roots by \( f_1, f_2, f_3 \) assuming that \( f_1 \leq f_2 \leq f_3 \). It is clear that the numbers \( f_1, f_2, f_3 \) are precisely the formal codegrees of \( K(k, l, m, n) \). Thus Proposition 2.10 implies that
\[
\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} = 1, \quad \text{or, equivalently,}
\]
\[
p(t) = t^3 - bt^2 + ct - c \quad \text{for some positive integers } b = \Tr(R) \text{ and } c = \det(R). \]
Using (9) we have
\[
b = (k + n)^2 + (l + m)^2 + 9
\]
The formula for \( c \) is more complicated, see (12) below.

3.3. Pivotal categories of rank 3 are pseudo-unitary.

**Lemma 3.4.** Let \( C \) be a pivotal fusion category of rank 3. Then some Galois conjugate of \( C \) is pseudo-unitary.

**Proof.** Since \( K(C) \) is commutative semisimple ring there are 3 distinct homomorphisms \( \phi_i : K(C) \to C, i = 1, 2, 3 \). One of this homomorphisms is the categorical dimension and another one is the Frobenius-Perron dimension. We would like to show that these two homomorphisms are in the same orbit under the action of the Galois group. For a sake of contradiction assume that this is not the case. Then the orbit of one of them has size 1, that is one of these homomorphisms lands into \( \mathbb{Z} \). If the Frobenius-Perron dimension lands in \( \mathbb{Z} \) then the category \( C \) is integral and we are done by [ENO] Proposition 8.24. If the categorical dimension lands in \( \mathbb{Z} \) the category \( C \) is pseudo-unitary by [RS] Lemma A.1. \( \square \)

3.4. The generic case. The pseudo-unitary inequality (9) applied to \( K(k, l, m, n) \) says:
\[
1 - \frac{4b}{c} \leq \frac{1}{f_3}.
\]
If \( c - 4b > 0 \) this is equivalent to
\[
f_3 \leq \frac{c}{c - 4b}.
\]

**Proposition 3.5.** Assume that \( c - 4b > 4 \) and \( b \geq 50 \). Then there is no pseudo-unitary fusion category \( C \) such that \( K(C) = K(k, l, m, n) \).

**Proof.** The Vieta theorem implies that \( f_3 \geq \frac{1}{4}b > 9 \). Under the assumptions of the Proposition \( c > 4b \), so (11) applies. We get
\[
\frac{c}{c - 4b} > 9 \iff c < \frac{9}{2}b.
\]

**Lemma 3.6.** Under the assumptions of the Proposition, we have
\[
1 < f_1 < 2 < f_2 < b - 5 < f_3 < b.
\]

**Proof.** We have \( p(1) = 1 - b < 0, p(2) = 8 + c - 4b > 0 \) and \( p(b) = bc - c > 0 \). Also
\[
p(b - 5) = (b - 5)((b - 5)^2 - b(b - 5) + c) - c = (b - 5)(c - \frac{9}{2}b + \frac{1}{2}(50 - b)) - c < 0.
\]
The Lemma follows. \( \square \)
By the assumptions of the Proposition we have $c - 4b \geq 5$. Thus
\[
\frac{c}{5} \geq \frac{c}{c - 4b} \geq f_3 > b - 5,
\]
whence
\[
c > 5b - 25 \iff (c - \frac{9}{2}b) + \frac{1}{2}(50 - b) > 0.
\]
We get a contradiction with inequalities $b \geq 50$ and $c < \frac{9}{2}b$ which proves the Proposition.

3.5. Small cases. In this section we consider the cases not covered by Proposition 3.4. Thus either $b < 50$ or $c - 4b \leq 4$, or both.

3.5.1. The case $b < 50$.

Lemma 3.7. Let $\mathcal{C}$ be a fusion category such that $K(\mathcal{C}) = K(k, l, m, n)$ with $b < 50$. Then $K(\mathcal{C})$ is isomorphic to one of the following: $K(1, 0, m, 0)$ with $m \leq 6$, $K(1, 1, 1, 0)$, $K(2, 1, 2, 1)$.

Proof. Using (10) and (7) we get
\[
b = 3k^2 + 3l^2 + n^2 + m^2 + 7.
\]
Assuming $k \geq l > 0$ we get from inequality $b < 50$ that $k \leq 3$. There are 11 based rings $K(k, l, m, n)$ with $3 \leq k \geq l > 0$ but out of those only $K(1, 1, 1, 0)$ and $K(2, 1, 2, 1)$ pass both the cyclotomic and d-number tests.

If $l = 0$ then $k = 1$ since $k$ and $l$ are coprime. By (7) this implies that $n = 0$ and $b = 10 + m^2$. The inequality $b < 50$ gives $m \leq 6$ and we are done.

3.5.2. The case $c - 4b \leq 4$.

Proposition 3.8. Let $\mathcal{C}$ be a fusion category such that $K(\mathcal{C}) = K(k, l, m, n)$ with $c - 4b \leq 4$. Then $K(\mathcal{C})$ is isomorphic to one of the following: $K(1, 0, m, 0)$ with $m \in \mathbb{Z}_{\geq 0}$, $K(1, 1, 1, 0)$, $K(2, 1, 2, 1)$.

Proof. We have the following formula verified by a direct calculation using (9):\[
(12) \quad p(2) = 8 + c - 4b = (kl - mn)^2 + (k^2 + kn - l^2 - ml)^2 - 1.
\]
Hence the inequality $c - 4b \leq 4$ is equivalent to
\[
(13) \quad (kl - mn)^2 + (k^2 + kn - l^2 - ml)^2 \leq 13.
\]
Observe that in view of (7) the number $k^2 + kn - l^2 - lm = 2(k^2 - lm) - 1$ is odd. By Remark 3.2 (i) we can assume that this number is positive. Then (13) implies that $k^2 - lm = 1$ or $k^2 - lm = 2$. Now we consider these cases separately.

Case $k^2 - lm = 1$. In this case $l^2 - kn = 0$ by (7). Since $k$ and $l$ are coprime this implies $k = 1, l^2 = n, lm = 0$. In the case $l = 0$ we get the based ring $K(1, 0, m, 0)$. Otherwise, $m = 0$ and (13) says
\[
l^2 + 1 \leq 13 \iff l \leq 3.
\]
The rings $K(1, 3, 0, 9)$ (with $b = 118, c = 473$) and $K(1, 2, 0, 4)$ (with $b = 38, c = 148$) do not pass neither cyclotomic nor d-number tests. Thus the only possibility is $K(1, 1, 0, 1) \simeq K(1, 1, 1, 0)$ and the conclusion of the Proposition holds in this case.
Case $k^2 - lm = 2$. In this case $l^2 - kn = -1$ whence $m = \frac{k^2 - 2}{l}$ and $n = \frac{l^2 + 1}{k}$.

We compute

$$kl - mn = \frac{2l^2 - k^2 + 2}{kl}.$$ 

The inequality (13) says $(kl - mn)^2 + 9 \leq 13$, so we have the following subcases:

Subcase $kl - mn = \pm 2$. Equivalently, $2l^2 - k^2 + 2 = \pm 2kl$ implying $3l^2 + 2 = (k \pm l)^2$ and we get a contradiction considering this equality modulo 3.

Subcase $kl - mn = \pm 1$. Equivalently, $2l^2 - k^2 + 2 = \pm kl$ implying $9l^2 + 8 = (2k \pm l)^2$ and again we get a contradiction considering this equality modulo 3.

Subcase $kl - mn = 0$. Equivalently, $2l^2 - k^2 + 2 = 0$. This diophantine equation has infinitely many solutions. We compute $m = 2l, n = \frac{1}{k}, b = 27l^2 + 1$. Equation (12) gives $c = 4b$. The polynomial $p(t) = t^3 - 4bt^2 + 4ibt - 4b$ satisfies $p(2) = 8$. Hence if $\gamma$ is an integer root of $p(t)$ then $2 - \gamma$ is a divisor of 8 and one verifies easily that $p(t)$ is irreducible for $b > 27$. Observe that $b = 27\frac{t^2 + 1}{k}$ is odd, so $\frac{b}{l} = \frac{27}{k} \notin \mathbb{Z}$. Thus for $b > 27$ the d-number test is failed (one can also show that the cyclotomic test fails as well). Finally, $27\frac{t^2 + 1}{k} = b \leq 27$ implies $l = 1$ and we get the based ring $K(2, 1, 2, 1)$, so the conclusion of the Proposition holds.

The Proposition is proved. \( \square \)

Remark 3.9. The crucial role in the proof above is played by the identity (12).

Thus it seems to be of great interest to find its counterparts for based rings of higher ranks.

3.6. The ring $K(2, 1, 2, 1)$.

Lemma 3.10. There is no pseudo-unitary fusion category $\mathcal{C}$ with $K(\mathcal{C}) = K(2, 1, 2, 1)$.

Proof. We have $p(t) = t^3 - 27t^2 + 108t - 108$ in this case, so $f_1 = 12 - 6\sqrt{3}, f_2 = 3, f_3 = 12 + 6\sqrt{3}$. Using Theorem 2.13 we get that $I(1) \in \mathbb{Z}(\mathcal{C})$ decomposes into simple objects as $1 \oplus A \oplus B$ where $F(A) = 1 \oplus X \oplus Y$ and $F(B) = 1 \oplus 2X \oplus 2Y$. Using Proposition 2.4 we see that $I(Y)$ decomposes as $A + 2B + S$ where $F(A + 2B) = 3 + 5X + 5Y$ and $S$ is a (non simple) object with $F(S) = 4X + 4Y$. This implies that $\text{Tr}(\theta(I(Y))) \neq 0$ and we have a contradiction with Theorem 2.5. \( \square \)

Remark 3.11. We will show later (see Theorem 5.1) that a categorification of $K(2, 1, 2, 1)$ admits a spherical (and hence pivotal) structure. Thus by the Lemma above and Lemma 3.4 the ring $K(2, 1, 2, 1)$ is not categorifiable.

4. Non-simple based rings of rank 3

4.1. The main result of this section is the following

Theorem 4.1. Assume that the based ring $K(1, 0, m, 0)$ is categorifiable. Then $m \leq 2$.

The multiplication in the based ring $K(1, 0, m, 0)$ is given by

$$Y^2 = 1, \ XY = YX = X, \ X^2 = 1 + mX + Y.$$ (14)

Thus, any category with $K(\mathcal{C}) = K(1, 0, m, 0)$ is an example of near-group category associated with finite group $\mathbb{Z}/2\mathbb{Z}$, see e.g. [1] Definition 1.1. In particular, such category or its Galois conjugate is pseudo-unitary by [1] Theorem 1.3.

In order to prove Theorem 4.1 we will assume for the sake of contradiction that $m > 2$ and $\mathcal{C}$ is a pseudo-unitary fusion category with $K(\mathcal{C}) = K(1, 0, m, 0)$. We
Lemma 4.2. We have \( d = \frac{m}{2} + \sqrt{\frac{m^2}{4} + 2} \). In particular \( d \notin \mathbb{Z} \) for \( m \geq 2 \).

4.2. Simple objects in \( \mathcal{Z}(\mathcal{C}) \). One of the formal codegrees of the based ring \( K(1, 0, m, 0) \) is \( f_1 = 2 \). By Theorem 2.13 the object \( I(1) \in \mathcal{Z}(\mathcal{C}) \) decomposes into the sum of 3 simple objects \( I(1) = 1 \oplus A \oplus B \); we can assume that \( \text{FPdim}(B) = \frac{\text{FPdim}(C)}{2} \) whence \( \text{FPdim}(A) = \frac{\text{FPdim}(C)}{2} - 1 \). Using Proposition 2.4 we deduce that

\[
F(A) = 1 + \frac{m}{2}X, \quad F(B) = 1 + Y + \frac{m}{2}X
\]

In particular, \( m \) is forced to be even. Theorem 2.5 states that \( \theta_A = \theta_B = 1 \).

Next we consider \( I(Y) \). By Proposition 2.4 we have \( \dim \text{Hom}_{\mathcal{Z}(\mathcal{C})}(I(Y), I(Y)) = 3 \) and \( \dim \text{Hom}(I(Y), I(1)) = 1 \); we also have by definition \( \dim \text{Hom}_{\mathcal{Z}(\mathcal{C})}(I(Y), B) = 1 \). Thus we have a decomposition \( I(Y) = B + C + D \) where \( C \) and \( D \) are distinct simple objects different from \( A \) and \( B \). Using Proposition 2.4 we have

\[
F(C) = Y + \alpha X, \quad F(D) = Y + \beta X
\]

where \( \alpha, \beta \in \mathbb{Z}_{\geq 0} \) and \( \alpha + \beta = \frac{m}{2} \). It is clear from Theorem 2.5 that \( \theta_C = \theta_D = -1 \).

Finally consider \( I(X) \). We have

\[
\dim \text{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), A) = \dim \text{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), A) = \frac{m}{2},
\]

\[
\dim \text{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), C) = \alpha, \quad \dim \text{Hom}_{\mathcal{Z}(\mathcal{C})}(I(X), D) = \beta.
\]

Thus

\[
I(X) = \frac{m}{2}A + \frac{m}{2}B + \alpha C + \beta D + \sum_i \gamma_i E_i
\]

where \( \gamma_i \in \mathbb{Z}_{\geq 0} \) and the objects \( E_i \in \mathcal{Z}(\mathcal{C}) \) satisfy \( F(E_i) = \gamma_i X \). Using Proposition 2.4 we compute

\[
\frac{m^2}{4} + \frac{m^2}{4} + \alpha^2 + \beta^2 + \sum_i \gamma_i^2 = \dim \text{Hom}(F(I(Y)), Y) = m^2 + 4
\]

whence

\[
\sum_i \gamma_i^2 = \frac{m^2}{2} + 4 - \alpha^2 - \beta^2 = \frac{m^2}{4} + 4 + 2\alpha\beta.
\]

Set \( \theta_i = \theta_{E_i} \). Theorem 2.5 implies

\[
\frac{m}{2}(1 + \frac{m}{2}d) + \frac{m}{2}(2 + \frac{m}{2}d) - \alpha(1 + \alpha d) - \beta(1 + \beta d) + \sum_i \gamma_i^2\theta_i d = 0
\]

whence

\[
\sum_i \gamma_i^2\theta_i = \alpha^2 + \beta^2 - \frac{m^2}{2} - \frac{m}{d} = -2\alpha\beta - \frac{m}{2}\sqrt{\frac{m^2}{4} + 2}.
\]

Similarly, Theorem 2.7 implies

\[
\frac{m}{2}(1 + \frac{m}{2}d) + \frac{m}{2}(2 + \frac{m}{2}d) + \alpha(1 + \alpha d) + \beta(1 + \beta d) + \sum_i \gamma_i^2\theta_i^2 d = \pm \dim(\mathcal{C}) = \pm(4 + md)
\]
whence
\[(20) \sum_i \gamma_i^2 \theta_i^2 = 2\alpha\beta - \frac{m^2}{4} - (m \mp 2)\sqrt{\frac{m^2}{4} + 2}.
\]

4.3. **Proof of Theorem 4.1.** Assume that \(m > 2\) and the based ring \(K(1, 0, m, 0)\) is categorifiable. Recall that this implies that \(m\) is even. Combining equations (18) and (20) we obtain
\[
\frac{m^2}{4} + 4 + 2\alpha\beta = \sum_i \gamma_i^2 \geq |\sum_i \gamma_i^2 \theta_i^2| = \frac{m^2}{4} + (m \mp 2)\sqrt{\frac{m^2}{4} + 2 - 2\alpha\beta}.
\]

Equivalently, \(4\alpha\beta + 4 \geq (m \mp 2)\sqrt{\frac{m^2}{4} + 2}\), which implies
\[
\frac{m^2}{4} + 4 \geq 4\alpha\beta + 4 \geq (m \mp 2)\sqrt{\frac{m^2}{4} + 2} \geq (m - 2)\sqrt{\frac{m^2}{4} + 2}.
\]

It is easy to see that the inequality \(\frac{m^2}{4} + 4 \geq (m - 2)\sqrt{\frac{m^2}{4} + 2}\) fails for \(m \geq 6\), so we got a contradiction in this case.

In the remaining case \(m = 4\) we get from (18), (19), (20):
\[
\sum_i \gamma_i^2 = 8 + 2\alpha\beta, \quad \sum_i \gamma_i^2 \theta_i = -2\alpha\beta - 2\sqrt{6}, \quad \sum_i \gamma_i^2 \theta_i^2 = 2\alpha\beta - 4 - (4 \mp 2)\sqrt{6}
\]
where \(\alpha, \beta \in \mathbb{Z}_{\geq 0}, \alpha + \beta = 2\). Comparing \(|\sum_i \gamma_i^2 \theta_i^2|\) with \(\sum_i \gamma_i^2\) as above we receive that \(\alpha = \beta = 1\) and \(\sum_i \gamma_i^2 \theta_i^2 = -2 - 2\sqrt{6} = \sum_i \gamma_i^2 \theta_i, \sum_i \gamma_i^2 = 10\).

It follows easily from [Lo] Theorem 2 that there is a unique way to represent \(-2 - 2\sqrt{6}\) as a sum of 10 roots of unity, namely
\[-2 - 2\sqrt{6} = 2(-1) + 2\zeta_{24} + 2\zeta_{24}^{-7} + 2\zeta_{24}^{11} + 2\zeta_{24}^{-11}\]
where \(\zeta_{24} = e^{\pi i/12}\). Since this set of roots of unity is not closed under squaring, we have a contradiction. Theorem 4.1 is proved. \(\Box\)

**Remark 4.3.** One can give an alternative proof of Theorem 4.1 for \(m \geq 6\) using only (18) and (19). For this one deduces from [Lo] that at least \(2\alpha\beta + m\sqrt{\frac{m^2}{4} + 2}\) roots of unity are required in order to represent \(-2\alpha\beta - m\sqrt{\frac{m^2}{4} + 2}\) as their sum and obtains a contradiction with (18).

4.4. **The ring \(K(1, 0, 2, 0)\).** The fusion categories \(\mathcal{C}\) with \(K(\mathcal{C}) = K(1, 0, 2, 0)\) were classified in [HH]. We sketch here an alternative argument for this classification. Recall that \(I(1) \in \mathcal{Z}(\mathcal{C})\) has a natural structure of separable algebra, see e.g. [DMNO] Lemma 3.5. Moreover, [DMNO] Theorem 4.10 implies that this algebra has a separable subalgebra \(I_0\) with FPdim \((I_0) = \frac{3}{4}\) FPdim \((I(1))\). This forces \(I_0 = 1 \oplus A\). Thus \(F(I_0) = 21 \oplus X \in \mathcal{C}\) has a structure of separable algebra. This algebra has to be decomposable, so \(1 \oplus X \in \mathcal{C}\) has a structure of separable algebra. It is easy to compute the "principal graph" of the category of \(1 \oplus X\)-modules in \(\mathcal{C}\); it is precisely the Dynkin diagram of type \(E_6\). Now the results of [HH] follow from the classification of subfactors of type \(E_6\), at least in the case when the category \(\mathcal{C}\) admits a unitary structure.
4.5. Proof of Theorem 1.1. Let \( \mathcal{C} \) be a fusion category of rank 3 admitting a pivotal structure. Then by Theorems 3.3 and 4.1 the Grothendieck ring \( K(\mathcal{C}) \) is isomorphic to one of the following: \( K(\mathbb{Z}/3\mathbb{Z}), K(1, 1, 1, 0), K(1, 0, 0, 0), K(1, 0, 1, 0) \) or \( K(1, 0, 2, 0) \). For the first 4 rings the result follows from the discussion in [O3, Section 4], and for the last ring the result follows from [Hi, Theorem 1]. \( \square \)

5. Spherical structures

5.1. Sphericalization. Let \( \mathcal{C} \) be a fusion category. A sphericalization \( \tilde{\mathcal{C}} \) of \( \mathcal{C} \) is defined in [ENO, Remark 3.1]. Thus \( \tilde{\mathcal{C}} \) is a spherical fusion category together with a tensor functor \( \tilde{F} : \tilde{\mathcal{C}} \to \mathcal{C} \). The category \( \tilde{\mathcal{C}} \) itself has a spherical structure if and only if the functor \( \tilde{F} \) admits a tensor section \( \mathcal{C} \to \tilde{\mathcal{C}} \). Equivalently, \( \tilde{\mathcal{C}} \) should contain a fusion subcategory such that restriction of \( \tilde{F} \) to this subcategory is an equivalence.

The functor \( \tilde{F} \) has the following properties: it maps simple objects to simple objects and for any \( X \in \mathcal{O}(\mathcal{C}) \) there are precisely two objects \( X_+, X_- \in \mathcal{O}(\tilde{\mathcal{C}}) \) with \( \tilde{F}(X_{\pm}) = X \) (the choice of \( X_+ \) and \( X_- \) is arbitrary except for \( 1_+ = 1 \)). It follows that \( 1_- \otimes 1_- = 1, \dim(1_-) = -1 \), and \( X_+ \otimes 1_- = 1_- \otimes X_\pm = X_\pm \). The following observation ([11, proof of Lemma 2.3]) is quite useful:

(a) If \( X \in \mathcal{O}(\mathcal{C}) \) is self-dual, then both \( X_+ \) and \( X_- \) are self-dual.

Let \( \varepsilon = [1_+] \in K(\tilde{\mathcal{C}}) \); clearly \( \varepsilon \) is a central element. Observe that \( K(\tilde{\mathcal{C}})/\langle \varepsilon = 1 \rangle \) with a basis consisting of images of \([X_+]\) is isomorphic to \( K(\mathcal{C}) \) as a based ring (via the map \([X_+] \mapsto [X]\)). Consider the ring \( \tilde{K}(\mathcal{C}) = K(\tilde{\mathcal{C}})/\langle \varepsilon = -1 \rangle \). This is a ring over \( \mathbb{Z} \) with a basis consisting of images of \([X_+]\). In a sense this is just a basis up to signs since the signs of basis elements depend on the choice of \( X_\pm \) above. The following statements are easy to verify:

(b) Let \( N \) be a structure constant of \( K(\mathcal{C}) \) and let \( \tilde{N} \) be the corresponding structure constant of \( \tilde{K}(\mathcal{C}) \) (remind that the bases of both rings are labeled by the same set \( \mathcal{O}(\mathcal{C}) \)). Then

\[ |\tilde{N}| \leq N, \quad \text{and} \quad \tilde{N} \equiv N \pmod{2}. \]

(c) The category \( \mathcal{C} \) has a spherical structure if and only if there is a choice of signs of basis elements of \( \tilde{K}(\mathcal{C}) \) such that the corresponding structure constants equal.

5.2. Quadratic rings. Let us consider a based ring \( K(k, l, m, n) \) and let \( p(t) \) be the characteristic polynomial of operator \( R \), see Section 3.2. We have the following possibilities for factorization of \( p(t) \) over \( \mathbb{Q} \):

(i) \( p(t) \) is irreducible. We say that \( K(k, l, m, n) \) is cubic in this case.

(ii) \( p(t) \) has irreducible quadratic factor. In this case we say that \( K(k, l, m, n) \) is quadratic.

(iii) \( p(t) \) factorizes into three linear factors. We say that \( K(k, l, m, n) \) is rational.

If \( K(\mathcal{C}) = K(k, l, m, n) \) is rational, then \( \text{FPdim}(\mathcal{C}) \) is integer, so \( \mathcal{C} \) is spherical by [ENO, Proposition 8.24]. We also have the following

Theorem 5.1. Let \( \mathcal{C} \) be a fusion category such that \( K(\mathcal{C}) = K(k, l, m, n) \) is quadratic. Then \( \mathcal{C} \) has a spherical structure.

Proof. If \( \text{FPdim}(\mathcal{C}) \in \mathbb{Z} \) then \( \mathcal{C} \) has spherical structure by [ENO, Proposition 8.24]. Thus we will assume for the rest of the proof that \( \text{FPdim}(\mathcal{C}) \not\in \mathbb{Z} \). The polynomial
the degree of the dimension field $L$ follows that $L$.

It follows from Corollary 2.15 that $\tilde{k}$, \(\tilde{l}\), \(\tilde{m}\), \(\tilde{n}\), \(\tilde{\omega}\), \(\tilde{\omega}\) with values in $K$. We will generally use tilde for notations associated with $\tilde{C}$ and the ring $\tilde{K}(C)$. Using \(5.1\) \((a)\) and \((b)\) we see that $\tilde{K}(C) = K(k, \tilde{l}, \tilde{m}, \tilde{n})$ (see \(8\)) where

$$|k| \leq k, |\tilde{l}| \leq l, |\tilde{m}| \leq m, |\tilde{n}| \leq n; k - \tilde{k}, l - \tilde{l}, m - \tilde{m}, n - \tilde{n}$$

are even.

By changing the signs of basis elements of $K(C)$ we can (and will) assume that

$$\tilde{k} + \tilde{n} \geq 0 \text{ and } \tilde{l} + \tilde{m} \geq 0.$$

We will generally use tilde for notations associated with $\tilde{K}(C)$ parallel to those in $K(C)$, for example $\tilde{R}, \tilde{p}(t)$ etc. Notice that the sum of inverse roots of $\tilde{p}(t)$ equals 1: indeed the formal codegrees of $\tilde{K}(C)$ are precisely doubled formal codegrees of $K(C)$ and doubled roots of $\tilde{p}(t)$, so the result follows from Proposition \(2.10\). Thus

$$\tilde{p}(t) = t^3 - \tilde{b}t^2 + \tilde{c}t - \tilde{c}.$$

It follows from \(10\) that $\tilde{b} < b$. Moreover, the equality $\tilde{b} = b$ implies $\tilde{k} = k, \tilde{l} = l, \tilde{m} = m, \tilde{n} = n$, so the category $C$ is spherical by \(5.1\) \((c)\). Thus in the rest of this proof we will assume that $\tilde{b} < b$ and derive a contradiction.

Observe that the dimension homomorphism $K(\tilde{C}) \to k$ factors through $\tilde{K}(C)$, so the degree of the dimension field $L(\tilde{C})$ over $\mathbb{Q}$ is at most three. On the other hand it follows from Corollary \(2.14\) that $L(\tilde{C})$ contains the splitting field of $t^2 - \alpha t + \beta$; it follows that $L(\tilde{C})$ is precisely this field. Thus we have two homomorphisms $\tilde{K}(C) \to k$ with values in $L(\tilde{C})$: the dimension homomorphism and its Galois conjugate. Thus the remaining homomorphism must be Galois invariant, so it lands in $\mathbb{Z}$. In particular, the polynomial $\tilde{p}(t)$ must be reducible.

Assume that $\tilde{p}(t)$ has three integer roots $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$. Two of these roots corresponding to the dimension homomorphism and its Galois conjugate must be equal, say $\tilde{f}_1 = \tilde{f}_2$. From the equation $\frac{1}{\tilde{f}_1} + \frac{1}{\tilde{f}_2} + \frac{1}{\tilde{f}_3} = 1$ we find that $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = (3, 3, 3)$ or $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = (4, 4, 2)$. Hence $\tilde{b} = 9$ or 10. From $\tilde{b} = 9$ we get a contradiction using \(10\) and \(7\). Thus $\tilde{b} = 10$ and $\dim(\tilde{C}) = \tilde{f}_1 = 4$. Therefore $\gamma$ divides 4 by Corollary \(2.13\). Observe that $\gamma = f_E$ where $E$ is one dimensional representation of $K(C)$ corresponding to $\mathbb{Z}$-valued character $\chi$. Thus $\gamma = 1 + s^2 + t^2$ where $s = \chi(X) \in \mathbb{Z}$ and $t = \chi(Y) \in \mathbb{Z}$, see \(10\) Example 2.4. It follows that $\gamma = 2$ and in view of Remark \(3.2\) \((i)\) we can assume that $s = 0$. Then \(8\) implies that $l = 0$ and $K(C) = K(1, 0, m, 0)$. However in this case $\tilde{C}$ is near-group category, so it is spherical by \(11\) Theorem 1.3. Hence Theorem \(5.1\) holds in this case.

Thus we will assume that $\tilde{p}(t)$ is a product of linear factor and of irreducible quadratic factor

$$\tilde{p}(t) = (t - \tilde{\gamma})(t^2 - \tilde{\alpha}t + \tilde{\beta}).$$

Note that $\dim(\tilde{C})$ is one of the roots of $t^2 - \tilde{\alpha}t + \tilde{\beta}$, and the splitting fields of $p(t)$ and of $\tilde{p}(t)$ coincide.
Clearly,

\begin{equation}
\tilde{b} = \tilde{\alpha} + \tilde{\gamma}, \tilde{c} = \tilde{\beta} + \tilde{\alpha}\tilde{\gamma} = \gamma\tilde{\beta}, \tilde{\beta} = \tilde{\alpha}\frac{\tilde{\gamma}}{\gamma - 1}.
\end{equation}

We have \(\tilde{\alpha} + \tilde{\gamma} = \tilde{b} < b = \alpha + \gamma\) and \(\frac{\tilde{\gamma}}{\gamma - 1} \leq 2\) whence

\begin{equation}
\tilde{\beta} = \tilde{\alpha}\frac{\tilde{\gamma}}{\gamma - 1} < 2(\alpha + \gamma - \tilde{\gamma}) = 2(\beta\frac{\gamma - 1}{\gamma} + \gamma - \tilde{\gamma}) < 4\beta
\end{equation}

(recall that \(\beta \geq \gamma\)). Since \(\frac{\tilde{\beta}}{\beta} \in \mathbb{Z}\) by Corollary 2.14 we have the following possibilities:

**Case 1:** \(\tilde{\beta} = 3\beta\). In this case \(2\tilde{\alpha} \geq \tilde{\alpha}\frac{\gamma}{\gamma - 1} = 3\alpha\frac{\gamma}{\gamma - 1}\) whence \(\tilde{\alpha} \geq \frac{3}{2}\alpha + \frac{3}{2}\frac{\alpha}{\gamma - 1}\).

Notice that by Corollary 2.14 \(\gamma\) divides \(\text{dim}(\mathcal{C})\), hence \(\gamma^2\) divides the norm \(\beta\) of \(\text{dim}(\mathcal{C})\) which is equivalent to \(\frac{3\alpha\gamma}{\gamma(\gamma - 1)} \in \mathbb{Z}\). If \(\frac{3\alpha\gamma}{\gamma(\gamma - 1)} \geq 2\) then \(\frac{3}{2}\frac{\alpha}{\gamma - 1} \geq \gamma\), so \(\tilde{\alpha} > \alpha + \gamma\) which contradicts to the inequality \(\tilde{\alpha} + \tilde{\gamma} < \alpha + \gamma\). If \(\frac{3\alpha\gamma}{\gamma(\gamma - 1)} = 1\) then \(\alpha = \frac{1}{3}\gamma(\gamma - 1), \beta = \frac{1}{3}\gamma^2\). Thus \(\gamma\) is divisible by 3, so \(\frac{\alpha^2}{\beta} = \frac{1}{3}(\gamma - 1)^2 \notin \mathbb{Z}\), whence \(\text{FPdim}(\mathcal{C})\) is not a d-number and we have a contradiction with Proposition 2.2.

**Case 2:** \(\tilde{\beta} = 2\beta\). In this case \(2\tilde{\alpha} \geq \tilde{\alpha}\frac{\gamma}{\gamma - 1} = 2\alpha\frac{\gamma}{\gamma - 1}\) whence \(\tilde{\alpha} \geq \alpha + \frac{\alpha}{\gamma - 1}\).

Notice that by Corollary 2.14 \(\gamma\) divides \(\text{dim}(\mathcal{C})\), hence \(\gamma^2\) divides the norm \(\beta\) of \(\text{dim}(\mathcal{C})\) which is equivalent to \(\frac{2\alpha\gamma}{\gamma(\gamma - 1)} \in \mathbb{Z}\). If \(\frac{2\alpha\gamma}{\gamma(\gamma - 1)} \geq 2\) then \(\frac{2\alpha}{\gamma - 1} \geq \gamma\), so \(\tilde{\alpha} > \alpha + \gamma\) which contradicts to the inequality \(\tilde{\alpha} + \tilde{\gamma} < \alpha + \gamma\). If \(\frac{2\alpha\gamma}{\gamma(\gamma - 1)} = 1\) then \(\alpha = \frac{1}{2}\gamma(\gamma - 1), \beta = \frac{1}{2}\gamma^2\). Thus \(\gamma\) is even, so \(\frac{\alpha^2}{\beta} = \frac{1}{2}(\gamma - 1)^2 \notin \mathbb{Z}\), whence \(\text{FPdim}(\mathcal{C})\) is not a d-number and we have a contradiction with Proposition 2.2.

**Case 3:** \(\tilde{\beta} = \beta\). First of all we claim that \(\tilde{\gamma} < \gamma\). Indeed inequality \(\tilde{\gamma} \geq \gamma\) implies \(\tilde{\alpha} = \alpha\frac{1 + \frac{1}{\gamma - 1}}{1 + \frac{1}{\gamma - 1}} \geq \alpha\), so \(\tilde{\alpha} + \tilde{\gamma} \geq \alpha + \gamma\).

The splitting fields of the polynomials \(t^2 - \alpha t + \beta\) and \(t^2 - \tilde{\alpha}t + \tilde{\beta}\) should coincide, equivalently the ratio of their discriminants \(\frac{\alpha^2 - 4\beta}{\tilde{\alpha}^2 - 4\tilde{\beta}}\) should be a square of rational number (notice that the numerator and denominator of the last expression are integers since \(\text{dim}(\mathcal{C})\) and \(\text{FPdim}(\mathcal{C})\) are d-numbers). Thus there exist a square free integer \(d\) and positive integers \(x, y\) such that

\begin{equation}
\frac{\alpha^2 - 4\beta}{\tilde{\alpha}^2 - 4\tilde{\beta}} = \left(\frac{\gamma - 1}{\gamma}\right)^2 \beta - 4 = dx^2, \quad \frac{\tilde{\alpha}^2 - 4\tilde{\beta}}{\beta} - 4 = \left(\frac{\tilde{\gamma} - 1}{\tilde{\gamma}}\right)^2 \beta - 4 = dy^2.
\end{equation}

Henceforth we have

\begin{equation}
dx^2 + 4 = \frac{(1 + \frac{1}{\gamma - 1})^2}{1 + \frac{1}{\gamma - 1}}.
\end{equation}

Let \(\frac{1 + \frac{1}{\gamma - 1}}{1 + \frac{1}{\gamma - 1}} = \frac{A}{B}\) where \(\frac{A}{B}\) is in its lowest terms. Notice that inequalities \(2 \leq \tilde{\gamma} < \gamma\) imply \(1 < \frac{A}{B} < 2\). The equality \(dx^2 + 4 = \left(\frac{A}{B}\right)^2\) is equivalent to

\begin{equation}
d(Bx - Ay)(Bx + Ay) = 4(A^2 - B^2).
\end{equation}

Since \(d\) is square free, the numbers \(Bx - Ay\) and \(Bx + Ay\) are both even. We set \(q = \frac{Bx - Ay}{2}, s = \frac{Bx + Ay}{2} = \frac{A^2 - B^2}{2dq}\). Hence

\begin{equation}
Bx = \frac{A^2 - B^2}{dq} + q = \frac{A^2 - B^2 + dq^2}{dq}, \quad Ay = \frac{A^2 - B^2}{dq} - q = \frac{A^2 - B^2 - dq^2}{dq}.
\end{equation}
These equalities imply that \( A^2 + B^2 + dq^2 \) is divisible by both \( A \) and \( B \), so it is divisible by \( AB \). We claim that
\[
2 < \frac{A^2 + B^2 + dq^2}{AB} < 4.
\]
Indeed, from (23) we get \( A^2 + B^2 = Aydq > 0 \), so \( \frac{A}{B} > \frac{2}{A} > \frac{dq^2}{AB} \), whence \( \frac{A}{B} \frac{2}{A} < \frac{3}{2} \) (since \( \frac{3}{2} \) is the maximal value of function \( x - \frac{1}{2} \) for \( x \in [1,2] \)). Now \( \frac{A^2 + B^2 + dq^2}{AB} = \frac{A}{B} + \frac{B}{A} + \frac{dq^2}{AB} \) where \( \frac{A}{B} + \frac{B}{A} \in (2, \frac{5}{2}) \) (since \( \frac{A}{B} \in (1,2) \)) and the desired inequality is established. The previous remarks imply
\[
\frac{A^2 + B^2 + dq^2}{AB} = 3
\]
and, eliminating \( dq^2 \) we get
\[
(25) \quad x = \frac{3A - 2B}{dq}, \quad y = \frac{2A - 3B}{dq}.
\]
Thus \( dq \) is divisor of \( 5A = 3(3A - 2B) - 2(2A - 3B) \) and \( 5B = 2(3A - 2B) - 3(2A - 3B) \); since \( A \) and \( B \) are coprime we see that \( dq \) divides 5. It follows that \( dq^2 = 1,5 \) or 25.

Subcase \( dq^2 = 25 \). The equality \( A^2 + B^2 + 25 = 3AB \) is equivalent to \( (2A - 3B)^2 - 5B^2 + 100 = 0 \) which implies that both \( A \) and \( B \) are divisible by 5. This is a contradiction.

Subcase \( dq^2 = 1 \). The equality \( A^2 + B^2 + 1 = 3AB \) is equivalent to \( \frac{1}{B^2} = 3\frac{A}{B} - \frac{A^2}{B^2} - 1 \) which implies \( \frac{1}{B^2} > 1 \) (since the minimal value of the function \( 3x - x^2 - 1 \) on the interval \([1,2]\) is 1), which is a contradiction.

Subcase \( dq^2 = 5 \). As in the previous case we get \( \frac{5}{dq} > 1 \) whence \( \frac{A}{B} = \frac{3}{2} \). Then (26) gives \( y = 0 \) and we get a final contradiction.

\[
\square
\]

5.3. Examples for cubic rings. It is easy to see that the ring \( K(k,l,m,n) \) is cubic if and only if \( K(k,l,m,n) \otimes \mathbb{Z} \mathbb{Q} \) is a field. The following result is useful in order to determine whether cubic \( K(k,l,m,n) \) passes the cyclotomic test:

**Proposition 5.2.** Assume that \( K(k,l,m,n) \otimes \mathbb{Z} \mathbb{Q} \) is a field. This field is abelian if and only if \( c \) is a square of integer.

**Proof.** Observe that the matrix of the trace form on the ring \( K(k,l,m,n) \) with respect to its basis coincides with the matrix (9). Thus the discriminant of the field \( K(k,l,m,n) \otimes \mathbb{Z} \mathbb{Q} \) is \( c \) modulo rational squares. Finally it is well known that a cubic field is abelian if and only if its discriminant is square.

**Remark 5.3.** The proof of Proposition 5.2 shows that the primes ramified in the field \( K(k,l,m,n) \otimes \mathbb{Z} \mathbb{Q} \) are divisors of \( c \). Of course it is possible that some prime divisor of \( c \) does not ramify. However if \( K(k,l,m,n) \) passes the d-number test (see Proposition 2.2) then a root of \( p(t) = t^3 - bt^2 + ct - c \) is \( \sqrt[3]{c} \) modulo units, so if \( c = \prod_i p_i^{\alpha_i} \) (with distinct primes \( p_i \)) and \( \alpha_i \) is not divisible by 3 then \( p_i \) is ramified in \( K(k,l,m,n) \otimes \mathbb{Z} \mathbb{Q} \).

The cubic based rings \( K(k,l,m,n) \) which pass both the cyclotomic and d-number tests from Section 2.1 are very sparse (but probably there are infinitely many of them). A complete list of such rings with \( l \leq k \leq 100000 \) obtained as a result of computer search includes categorifiable ring \( K(1,1,1,0) \) and the following rings:
Remark 5.4. The following observations (especially the second one) are useful in the above mentioned computer search:

(i) If $b^2$ is divisible by $c$ then $p(t)$ is irreducible. Indeed, assume that $p(t)$ is reducible, $p(t) = (t - \gamma)(t^2 - \alpha t + \beta)$. Then (21) implies that $s := \frac{\alpha}{\gamma}$ is an integer and we have $b = \alpha + \gamma = s\gamma + \gamma - s$, $c = \gamma\beta = \gamma^2 s$. Hence $\frac{b}{c} \in \mathbb{Z}$ is equivalent to $3s^2/\gamma + 2 + 3s^2/\gamma - s^2/\gamma \in \mathbb{Z}$. This condition implies that $\gamma$ and $s$ must have the same $p$-adic valuation for each prime $p$, whence $\gamma = s$. Thus $b = s^2$ which is impossible since by (10) $b$ is congruent to 2 or 3 modulo 4.

(ii) $k$ and $l$ should be odd (which implies that $c$ is odd and $b \equiv 2 \pmod{4}$). Indeed if $k$ or $l$ is even then (7) and (12) imply that $c$ is even, hence $b$ is even by the d-number test. Then (10) shows that $b \equiv 2 \pmod{4}$. Using Proposition 5.2 and the d-number test we see that $c$ is divisible by 4 but not by 8. Hence by Remark 5.3 the field $K(k, l, m, n) \otimes \mathbb{Q}$ is ramified at 2. This is a contradiction since the only extension of 2-adic numbers $\mathbb{Q}$ with Galois group $\mathbb{Z}/3\mathbb{Z}$ is unramified.

We describe now the computer computations which show that a categorification of the rings above is spherical (and hence does not exist).

1) First of all we can give an estimate of $\tilde{c}$. Using (12) and (7) we see that

$$\tilde{c} = 4b - 9 + (\tilde{k}l - \tilde{m}n)^2 + (2(\tilde{k}^2 - \tilde{l}m) + 1)^2 = 4b - 9 + (\tilde{k}l - \tilde{m}n)^2 + (2(\tilde{l}^2 - \tilde{k}n) - 1)^2.$$ 

Hence,

$$\tilde{c} \leq \min(4b - 9 + (kl + mn)^2 + (2k^2 + 2lm + 1)^2, 4b - 9 + (kl + mn)^2 + (2l^2 + 2kn + 1)^2).$$

Example 5.5. For the ring $K(54559, 41609, 71568, 31711)$ we obtain

$$\frac{\tilde{c}}{c} \leq 11132295.1 < 15^6,$$

and for the ring $K(95705, 14221, 641435, 2506)$ we obtain

$$\frac{\tilde{c}}{c} < 144.6 < 13^2 < 3^6.$$ 

2) Recall that by Corollary 2.13 $\tilde{c}$ should be divisible by $c$; also the fields $K(k, l, m, n) \otimes \mathbb{Q} \approx \mathbb{Q}[t]/(p(t))$ and $K(k, \tilde{l}, \tilde{m}, \tilde{n}) \otimes \mathbb{Q} \approx \mathbb{Q}[t]/(\tilde{p}(t))$ should be isomorphic (since by Corollary 2.13 $K(k, l, m, n) \otimes \mathbb{Q}$ should be contained in the dimension field $L(\mathcal{C})$ which is contained in $K(k, \tilde{l}, \tilde{m}, \tilde{n}) \otimes \mathbb{Q}$). In view of Remark 5.3 this shows that the prime factors of $\tilde{c}$ which do not appear in the prime decomposition of $c$ should have exponents divisible by 6. Using the estimate for $\tilde{c}$ above we can get a list of possible values of $\tilde{c}$ (note that in view of Remark 5.3 (ii), $\tilde{c}$ should be odd).
Example 5.6. For the ring $K(54559, 41609, 71568, 31711)$ we have $c = 7^2 \cdot 127^2 \cdot 2791^2$. Thus the possible values of $\tilde{c}/c$ are: $1, 3^6, 5^6, 7^6, 9^6, 11^6, 13^6, 2^2, 3^6, 7^2 \cdot 3^6, 7^3 \cdot 5^6, 7^4, 7^4 \cdot 3^6, 7^8, 7^2 \cdot 127^2, 127^2, 2791^2$.

For the ring $K(95705, 14221, 641435, 2506)$ we have $c = 3^6 \cdot 13^2 \cdot 151^2 \cdot 4861^2$ and the possible values of $\tilde{c}/c$ are $1, 3^2, 3^4$.

3) For each value of $\tilde{c}$ the possible values of $\tilde{b}$ satisfy the following: $\tilde{b}^3$ is divisible by $\tilde{c}$ (and hence by $c$), $0 < \tilde{b} \leq b$, and $\tilde{b} \equiv b \pmod{4}$; this can be enumerated efficiently by computer.

Example 5.7. For the ring $K(54559, 41609, 71568, 31711)$ we get

$$\tilde{b} = b - i \cdot 4 \cdot 7 \cdot 127 \cdot 2791, \quad i \in \mathbb{Z}, \quad 0 \leq i \leq 2040,$$

and for the ring $K(95705, 14221, 641435, 2506)$ we get

$$\tilde{b} = b - i \cdot 4 \cdot 3^2 \cdot 13 \cdot 151 \cdot 4861, \quad i \in \mathbb{Z}, \quad 0 \leq i \leq 1279.$$

4) Then for each $\tilde{c}$ and $\tilde{b}$ we can compute the discriminant of $\tilde{p}(t)$ and discard all the cases where the field $\mathbb{Q}[t]/(\tilde{p}(t))$ is not abelian. We found that the only possibilities that are not discarded are either $\tilde{b} = b, \tilde{c} = c$ (which implies pseudo-unitarity of a categorification of $K(k, l, m, n)$) or the following two cases: $K(29, 13, 62, 7)$ with $\tilde{b} = 378, \tilde{c} = c = 35721$, and $K(83, 77, 91, 70)$ with $\tilde{b} = 1026, \tilde{c} = c = 263169$.

5) Finally we deal with the two exceptional cases above.

(a) In the case of $K(29, 13, 62, 7)$ there exists no ring $K(\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ consistent with $\tilde{b} = 378$ and $\tilde{c} = 35721$ (we can find $\tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}$ from equations $(\tilde{k} + \tilde{n})^2 + (\tilde{l} + \tilde{m})^2 = \tilde{b} - 9$ and $(3\tilde{k} - \tilde{n})^2 + (3\tilde{l} - \tilde{m})^2 = \tilde{b} - 1$ which follow from (16) and (17); up to signs the only solution is $K(7, 1, 14, 5)$ which is incompatible with the value of $\tilde{c}$). Note that in this case the fields $\mathbb{Q}[t]/(p(t))$ and $\mathbb{Q}[t]/(\tilde{p}(t))$ are isomorphic (one isomorphism sends $t \in \mathbb{Q}[t]/(p(t))$ to $-\frac{13}{9}t^2 + \frac{881}{3}t - 294 \in \mathbb{Q}[t]/(\tilde{p}(t))$).

(b) In the case of $K(83, 77, 91, 70)$ the fields $\mathbb{Q}[t]/(p(t))$ and $\mathbb{Q}[t]/(\tilde{p}(t))$ are not isomorphic (here is an easy way to verify this: let $t_1, t_2, t_3$ be roots of $p(t)$ and let $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3$ be the roots of $\tilde{p}(t)$. If the fields above are isomorphic then one of the numbers $t_1 + t_2 t_3$ or $t_1 + t_2 \tilde{t}_3 + t_3 \tilde{t}_2$ would be integer. However a numerical computation shows that this is not the case.). Note that in this case the ring $K(1, 1, 23, -22)$ is compatible with the values of $\tilde{b}$ and $\tilde{c}$.

We summarize the results of this Section as follows:

Theorem 5.8. Assume that a based ring $K(k, l, m, n)$ with $k \geq l$ admits a non-spherical categorification. Then $k > 100,000$.

Appendix A. Some results on near-group categories

Dmitri Nikshych and Victor Ostrik

A.1. Near-group categories. An easiest class of fusion categories is formed by pointed fusion categories, that is fusion categories where all simple objects are invertible, see e.g. [ENO] p. 629]. The pointed fusion categories are quite well understood. Thus it is a natural to look for fusion categories which would represent the next level of complexity. One such class, the near-group categories was introduced by J. Siehler in [S]. Namely, a semisimple rigid tensor category $C$ is called near-group category if all its simple objects except for one are invertible. Let $G$ be the group of isomorphism classes of invertible objects in $C$ and let $X$ be the
isomorphism class of the remaining non-invertible simple object. Then the group $G$ must be finite (so $\mathcal{C}$ is a fusion category) and there exists $n \in \mathbb{Z}_{\geq 0}$ such that the Grothendieck ring $K(\mathcal{C})$ has the following multiplication:

$$g \cdot h = gh \text{ for } g, h \in G; g \cdot X = X \cdot g = X \text{ for } g \in G; X^2 = \sum_{g \in G} g + nX.$$  

We will denote this based ring $K(G,n)$. Thus a near-group category is a categorification of some $K(G,n)$. We have the following examples:

**Example A.1.** (i) Let $\overline{F}_q$ be a finite field. The category of representations of the semi-direct product $\overline{F}_q \rtimes \overline{F}_q^\times$ is a categorification of $K(\mathbb{Z}/(q-1)\mathbb{Z}, q-2)$.

(ii) The Tambara-Yamagami categories $\text{TY}$ are categorifications of $K(G,0)$.

(iii) Many categorifications of $K(G,n)$ where $G$ is abelian and $n = |G|$ were constructed in [I] and [EG]. These results suggest strongly that there is an infinite series of categories of this type.

(iv) (cf [IK, Theorem 14.40,II]) Let $\tilde{G} = S_4$ and let $\tilde{H}$ be a subgroup of order 4 in $\tilde{G}$ which is not normal. Then the group theoretical category $\mathcal{C}(\tilde{G}, \tilde{H}, 1, 1)$ (see [ENO, Section 8.8]) is a categorification of $K(G,2)$ where $G$ is the dihedral group of order 8. This example is in apparent contradiction with [S, Theorem 1.1]. This is because the group $G$ is implicitly assumed to be abelian in loc. cit. as it was observed in [EG].

It was proved in [I, Theorem 1.3] that any near-group category is a Galois conjugate of pseudo-unitary one. Thus Theorem 2.13 applies in this case. The goal of this Section is to collect some interesting consequences of these results. Thus we show the following:

1) Assume that $\mathcal{C}$ is not integral (that is the roots of quadratic equation $d^2 = |G| + nd$ are not integers). Then $G$ is abelian and $\frac{n}{|G|} \in \mathbb{Z}$, see Theorem A.6 (cf [EG, Theorem 2(a)]).

2) Assume that $\mathcal{C}$ is integral. Then $\mathcal{C}$ is weakly group theoretical in the sense of [ENO], see Theorem A.11.

3) Assume that $G$ is abelian and $0 < n < |G|$. Then $n = |G| - 1$ and $G$ is cyclic (thus all such categories are classified in [EGO, Corollary 7.4]), see Theorem A.14 (cf [S, Theorem 1.2]). Note that Example A.1(iv) shows that the assumption that $G$ is abelian can not be omitted.

**Remark A.2.** When this paper was completed we learnt that Izumi [I1] obtained very interesting results on near-group $C^*$-categories. In particular he obtained counterparts of Theorems A.6 and A.14.

### A.2. Irreducible representations of $K(G,n)$

In what follows we will fix an embedding of the subfield of algebraic integers in $k$ to $\mathbb{C}$ such that the near-group category in question is pseudo-unitary (such embedding exists by [I, Theorem 1.3]) and we will endow the category with a pseudo-unitary spherical structure.

Let $\text{Irr}_0(G)$ be the set of non-trivial irreducible characters of the group $G$. Let $\rho_\chi : G \to GL(E)$ be a representation of $G$ over $k$ affording character $\chi \in \text{Irr}_0(G)$. Then $X \mapsto 0, g \mapsto \rho_\chi(g)$ is a well defined homomorphism $\tilde{\rho}_\chi : K(G,n) \to \text{End}(E)$. It is clear that $\tilde{\rho}_\chi$ is an irreducible representation of $K(G,n)$ over $k$. We will denote by $E_\chi$ the corresponding $K(G,n)$-module.
Let \( d_+ \) and \( d_- \) be the positive and the negative roots of the equation \( d^2 = |G| + nd \). Then the homomorphisms \( \rho_\pm : K(G, n) \to k \) with \( \rho_\pm (g) = 1 \) and \( \rho_\pm (X) = d_\pm \) are well defined (note that \( \rho_+ \) coincides with the Frobenius-Perron dimension homomorphism). We will denote by \( E_\pm \) the corresponding one dimensional \( K(G, n) \)-modules.

**Lemma A.3.** Any irreducible \( K(G, n) \)-module over \( k \) is isomorphic to either \( E_\chi \) for some \( \chi \in \text{Irr}_0(G) \), or to \( E_\pm \).

**Proof.** We have

\[
\sum_{\chi \in \text{Irr}_0(G)} \dim(E_\chi)^2 + \dim(E_+)^2 + \dim(E_-)^2 = 2 + \sum_{\chi \in \text{Irr}_0(G)} \chi(1)^2 = |G| + 1.
\]

Since the rank of \( K(G, n) \) over \( \mathbb{Z} \) is \(|G| + 1 \) the result follows. \( \square \)

**Lemma A.4.** The formal codegrees of irreducible representations of \( K(G, n) \) are \( f_{E_\chi} = \frac{[G]}{\chi(1)} \) for \( \chi \in \text{Irr}_0(G) \), and \( f_{E_\pm} = 2|G| + nd_\pm \).

**Proof.** We have \( \sum b_i b_i = |G| + X^2 \), so the result follows from \([O4, \text{Lemma 2.6}]\). \( \square \)

Assume that \( \mathcal{C}(G, n) \) is a near-group category with \( K(\mathcal{C}(G, n)) = K(G, n) \). The following result follows immediately from Theorem 2.13 and Lemmas A.3 and A.4

**Proposition A.5.** We have the following decomposition of \( I(1) \in \mathcal{Z}(\mathcal{C}(G, n)) \) into the sum of irreducible objects:

\[
I(1) = 1 + A_{E_-} + \sum_{\chi \in \text{Irr}_0(G)} \chi(1)A_{E_\chi}
\]

where \( \dim(A_{E_-}) = 1 + \frac{n}{|G|}d_+ \) and \( \dim(A_{E_\chi}) = \chi(1)(2 + \frac{n}{|G|}d_+) \).

We can now prove the first main result of this Section:

**Theorem A.6.** Assume that the ring \( K(G, n) \) is categorifiable and \( \dim(X) = d_+ \) is irrational. Then

(i) \( n \) is divisible by \(|G| \);

(ii) the group \( G \) is abelian.

**Remark A.7.** In the setting of \( C^* \)-categories it was proved by Evans and Gannon \([\text{EC}]\) that (i) holds if \( G \) is abelian.

**Proof.** (i) Assume that \( F(A_{E_-}) \) is a sum of \( \alpha \) invertible objects and \( \beta \) copies of \( X \) (thus \( \alpha, \beta \in \mathbb{Z}_{\geq 0} \)). Then \( \dim(A_{E_-}) = \alpha + \beta d_+ \). On the other hand by Proposition A.5 \( \dim(A_{E_-}) = 1 + \frac{n}{|G|}d_+ \). Since \( d_+ \) is irrational this implies that \( \beta = \frac{n}{|G|} \) and the result follows.

(ii) Assume that the conclusion fails. Then there exists \( \chi \in \text{Irr}_0(G) \) with \( \chi(1) > 1 \). Then \( \dim(A_{E_\chi}) = 2\chi(1) + \frac{n\chi(1)}{|G|}d_+ \). By the argument in (i) we get that \( [F(A_{E_\chi}) : X] = \frac{n\chi(1)}{|G|} \). Also by Theorem 2.13 we have \([F(A_{E_\chi}) : 1] = \chi(1) \). Since \([F(A_{E_\chi}) : X] = a_1 \) \( \dim(A_{E_\chi}) = [F(A_{E_\chi}) : X][F(1)] = \chi(1) + \frac{n\chi(1)}{|G|}d_+ < \dim(A_{E_\chi}) \), we see that \( F(A_{E_\chi}) \) contains at least one nontrivial invertible object. Thus this object appears in \( F(I(1)) \) at least \( \chi(1) > 1 \) times. But this is a contradiction since by Proposition 2.4 any nontrivial invertible object appears in \( F(I(1)) \) exactly once. \( \square \)
Remark A.8. The argument in (ii) sometimes applies even if $d_+ \in \mathbb{Q}$, for example in the case $n = 0$. Thus we get yet another proof that for a Tambara-Yamagami category $\mathcal{C}(G,0)$ the group $G$ must be abelian, cf \cite{TY Corollary 3.3} and \cite{ENO2 Proposition 9.3 (i)}.

A.3. Integral case.

Lemma A.9. Assume that $\mathcal{C}(G,n)$ is integral. Then $n < |G|$.

Proof. We have $|G| = d_+(d_+ - n)$ where $d_+ \in \mathbb{Z}_{\geq 0}$. Hence

$$n = d_+ - (d_+ - n) < d_+ \leq |G|.$$

\qed

Corollary A.10. Assume that $\mathcal{C}(G,n)$ is integral. Then $[F(A_{E^-}) : X] = 0$.

Proof. By Proposition A.6 and Lemma A.9 we have $\dim(A_{E^-}) = 1 + \frac{n}{|G|}d_+ < 1 + d_+$. The result follows. \qed

Theorem A.11. Assume that $\mathcal{C}(G,n)$ is integral. Then $\mathcal{C}(G,n)$ is weakly group theoretical.

Proof. The Tambara-Yamagami categories $\mathcal{C}(G,0)$ are $\mathbb{Z}/2\mathbb{Z}$-extensions of pointed categories, so they are weakly group theoretical by \cite{ENO1 Definition 1.1}. Thus we will assume that $n > 0$ in the rest of this proof. In particular we have

$$\dim(\mathcal{C}(G,n)) = 2|G| + nd_+^2 > 2|G| = 2\dim(\text{Vec}_G).$$

Let us consider full subcategory $\mathcal{D}$ of $\mathcal{Z}(\mathcal{C}(G,n))$ consisting of objects $A$ such that $F(A) \in \mathcal{C}(G,n)$ is a sum of invertible objects (equivalently $[F(A) : X] = 0$). It is clear that $\mathcal{D}$ is a fusion subcategory of $\mathcal{Z}(\mathcal{C}(G,n))$. Corollary A.10 says that $A_{E^-} \in \mathcal{D}$. In particular, the restriction of the forgetful functor $F$ to $\mathcal{D}$ is not injective since $F(A_{E^-})$ is not simple. Let $\mathcal{D}'$ be the M"uger centralizer of $\mathcal{D}$ in $\mathcal{Z}(\mathcal{C}(G,n))$, see \cite{M3 Definition 2.6}. It follows from \cite[Theorem 3.12]{DNO} that the restriction of $F$ to $\mathcal{D}'$ is not surjective. Since the object $X$ generates $\mathcal{C}(G,n)$ this implies that for any $A \in \mathcal{D}'$ we have $[F(A) : X] = 0$. In other words $\mathcal{D}' \subset \mathcal{D}$ or, equivalently, the category $\mathcal{D}'$ is symmetric.

Lemma A.12. The symmetric category $\mathcal{D}'$ is Tannakian.

Proof. For the sake of contradiction assume that $\mathcal{D}'$ is not Tannakian. Then by \cite[Corollary 2.50]{DGNO} $\mathcal{D}'$ contains a fusion subcategory $\mathcal{D}''$ with $\dim(\mathcal{D}'') = \frac{1}{2}\dim(\mathcal{D}')$. Let $\mathcal{D}$ be the M"uger centralizer of $\mathcal{D}'$ in $\mathcal{Z}(\mathcal{C}(G,n))$, see \cite[Definition 2.6]{M3}. Then $\mathcal{D} \supset \mathcal{D}$ and $\dim(\mathcal{D}) = 2\dim(\mathcal{D})$, see \cite[Theorem 3.14]{DGNO}.

By definition of category $\mathcal{D}$ there exists an object $A \in \mathcal{D}$ with $[F(A) : X] \neq 0$; hence the forgetful functor $F : \mathcal{D} \to \mathcal{C}(G,n)$ is surjective. Similarly to \cite[proof of Corollary 8.11]{ENO} we obtain

$$\dim(\mathcal{D}) = \dim(\mathcal{C}(G,n)) \left( \sum_{A \in \mathcal{O}(\mathcal{D})} \dim(A)[F(A) : 1] \right) >$$

$$2\dim(\text{Vec}_G) \left( \sum_{A \in \mathcal{O}(\mathcal{D})} \dim(A)[F(A) : 1] \right) \geq 2\dim(\mathcal{D}).$$

This is a desired contradiction. \qed
Let us consider the fiber category $D \boxtimes D' \text{Vec}$ associated with Tannakian subcategory $D' \subset \mathcal{Z}(\mathcal{C}(G,n))$. It follows from [DMNO] Example 3.14 and Proposition 5.4 that there exists a fusion category $A$ and braided equivalence $D \boxtimes D' \text{Vec} \simeq \mathcal{Z}(A)$.

**Lemma A.13.** The category $A$ is group theoretical.

**Proof.** The restriction of the forgetful functor $F : \mathcal{Z}(\mathcal{C}(G,n)) \to \mathcal{C}(G,n)$ to the subcategory $D$ gives a central functor (see [DMNO] Definition 2.4) $D \to \text{Vec}_G \subset \mathcal{C}(G,n)$. Thus there exists a braided functor $D \to \mathcal{Z}(\text{Vec}_G)$. Let $D_1$ denote the image of this functor. Then $D_1$ is group theoretical category as a subcategory of group theoretical category $\mathcal{Z}(\text{Vec}_G)$, see [ENO] Proposition 8.44 (i). On the other hand it follows from [DMNO] Corollary 3.24 that the canonical functor $D \to D \boxtimes D' \text{Vec}$ factors through $D \to D_1$. Thus the category $D \boxtimes D' \text{Vec} = \mathcal{Z}(A)$ is group theoretical by [ENO] Proposition 8.44 (ii). Since the forgetful functor $\mathcal{Z}(A) \to A$ is surjective (see e.g. [ENO] Proposition 5.4) we conclude that $A$ is group theoretical by [ENO] Proposition 8.44 (ii).

It follows from Lemma A.13 that the fiber category $D \boxtimes D' \text{Vec}$ contains a Lagrangian subcategory, see [DGNO] Theorem 4.64. Hence Theorem holds by [ENO] Proposition 4.2.

It would be interesting to give a classification of integral near-group categories in group theoretical terms. In particular it is not known to the authors whether examples of non group theoretical near-group categories $\mathcal{C}(G,n)$ with $n > 0$ exist (it is known that such examples exist when $n = 0$, see [ENO] Remark 8.48).

**Theorem A.14.** Let $\mathcal{C}(G,n)$ be a near-group category

(i) (cf [S] Theorem 1.1) Assume that $0 < n < |G| - 1$. Then $G$ is not abelian.

(ii) (cf [S] Theorem 1.2) Assume that $n = |G| - 1$. Then $G$ is cyclic.

**Proof.** (i) It is elementary to show that the assumption implies $1 + \frac{1}{|G|} d_+ < d_+$. Hence for any $\chi \in \text{Irr}_0(G)$ with $\chi(1) = 1$ we have $[F(A_{E_+}) : X] = 0$. Recall that by Corollary A.10 $[F(A_{E_-}) : X] = 0$. Thus there exists $\chi \in \text{Irr}_0(G)$ with $\chi(1) > 1$, so the group $G$ is non abelian.

(ii) In this case $\dim(A_{E_+}) = |G|$, so $[F(A_{E_-})] = \sum_{g \in G} g$. Hence the forgetful functor $D \to \text{Vec}_G$ is surjective (see proof of Theorem A.11 for the definition of $D$). Using [ENO] proof of Corollary 8.11 we find that $\dim(D) = |G|(|G| + 1) = \dim(\mathcal{C}(G,n))$. Hence by [AB] Theorem 3.2 we have $D' = D$. Thus $D$ is Lagrangian subcategory of $\mathcal{Z}(\mathcal{C}(G,n))$ and the category $\mathcal{C}(G,n)$ is group theoretical, see [DGNO] Theorem 4.64 and [ENO1] Theorem 3.1. Let $\hat{G}$ be a finite group such that we have a braided equivalence $D \simeq \text{Rep}(\hat{G})$. Observe that the space of homomorphisms between $I(1)$ and the regular algebra (see [DMNO] Example 2.8) of $D$ has dimension $|G| + 1$. Thus by Remark [AB] there is a module category providing weak Morita equivalence between $\mathcal{C}(G,n)$ and $\text{Vec}_{\hat{G}}$ with $|G| + 1$ simple objects. Equivalently, $\mathcal{C}(G,n) \simeq \mathcal{C}(\hat{G}, \hat{H}, \omega, \psi)$ (see [ENO] Section 8.8 for the notations) where $|\hat{G}| = |G|(|G| + 1)$ and $|\hat{H}| = |G|$. Since $\text{Rep}(\hat{H})$ is subcategory of $\mathcal{C}(\hat{G}, \hat{H}, \omega, \psi)$ we see that $\hat{H}$ is abelian and $\hat{G}$ is a union of exactly 2 double cosets with respect to $\hat{H}$. It follows that $\hat{G}$ is a Frobenius group with Frobenius complement $\hat{H}$. Thus arguing as in [EGO] proof of Corollary 7.4 we see that $\mathcal{C}(G,n)$ is one of the categories described in [EGO] Corollary 7.4. In particular, the group $G$ must be cyclic in this case.
References

[BK] B. Bakalov, A. Kirillov Jr., Lectures on Tensor categories and modular functors, AMS, (2001).

[BEK] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, On α-induction, chiral generators and modular invariants for subfactors, Commun. Math. Phys. 208 (1999), 429-487.

[BNRS] P. Bruillard, S.-H. Ng, E. Rowell and Z. Wang, On modular categories, preprint.

[CMS] F. Calegari, S. Morrison, N. Snyder, Cyclotomic integers, fusion categories, and subfactors, Comm. Math. Phys. 303 (2011), no. 3, 845-896.

[DGNO] V. Drinfeld, S.-H. Ng, E. Rowell and Z. Wang, On braided fusion categories I, Selecta Mathematica (N. S.) 16 (2010), 1-119.

[DMNO] A. Davydov, M. Müger, D. Nikshych, V. Ostrik, The Witt group of non-degenerate braided fusion categories, J. für die reine und angewandte Mathematik, 677 (2013), 135-177.

[DNO] A. Davydov, D. Nikshych, V. Ostrik, On the structure of the Witt group of braided fusion categories, Selecta Math. (N.S.) 19 (2013), no. 1, 237-269.

[EGO] P. Etingof, S. Gelaki, and V. Ostrik, Classification of fusion categories of dimension pq, Int. Math. Res. Not. 2004, no. 57, 3041-3056.

[ENO] P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories, Annals of Mathematics 162 (2005), 581-642.

[ENO1] P. Etingof, D. Nikshych, and V. Ostrik, Weakly group-theoretical and solvable fusion categories, Adv. Math. 226 (2011), no. 1, p. 176-205.

[ENO2] P. Etingof, D. Nikshych, and V. Ostrik, Fusion categories and homotopy theory, Quantum Topol. 1 (2010), no. 3, 209-273.

[EG] D. E. Evans, T. Gannon, Near-group fusion categories and their doubles, arXiv:1208.1500.

[FRS] J. Fuchs, I. Runkel, C. Schweigert, TFT construction of RCFT correlators. I. Partition functions, Nuclear Phys. B 646 (2002), no. 3, 353-497.

[FRS2] J. Fuchs, I. Runkel, C. Schweigert, The fusion algebra of bimodule categories, Appl. Categ. Structures 16 (2008), no. 1-2, 123-140.

[GX] R. Guralnick, F. Xu, On a subfactor generalization of Wall’s conjecture, J. Algebra 332 (2011), 457-468.

[HH] T. J. Hagge, S.-M. Hong Some non-braided fusion categories of rank three, Commun. Contemp. Math. 11 (2009), no. 4, 615-637.

[I] M. Izumi, The structure of sectors associated with Longo-Rehren inclusions, II. Examples, Rev. Math. Phys. 13, (2001), 603-674.

[IK] M. Izumi, H. Kosaki, Kac algebras arising from composition of subfactors: general theory and classification, Mem. Amer. Math. Soc. 158 (2002), no.750.

[II] M. Izumi, notes on near-group categories.

[JMS] V. Jones, S. Morrison, N. Snyder The classification of subfactors of index at most 5, arXiv:1304.6141.

[KO] A. Kirillov, Jr., V. Ostrik, On q-analog of McKay correspondence and ADE classification of sl(2) conformal field theories, Adv. Math. 171 (2002), no. 2, p. 183-227.

[La] H. K. Larson, Classification of some fusion categories of rank 4, preprint.

[Lo] J. H. Loxton, On two problems of R. W. Robinson about sums of roots of unity, Acta Arith. 26 (1974/1975), 159-174.

[Lu] G. Lusztig, Hecke algebras with unequal parameters, CRM Monograph Series, 18. American Mathematical Society, Providence, RI, 2003. vi+136 pp.

[M1] M. Müger, From subfactors to categories and topology. I. The quantum double of tensor categories and subfactors, J. Pure Appl. Algebra 180 (2003), no. 1-2, 81-157.

[M2] M. Müger, From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors, J. Pure Appl. Algebra 180 (2003), no. 1-2, 159-219.

[M3] M. Müger, On the structure of modular categories, Proc. London Math. Soc. (3) 87 (2003), no. 2, 291-308.

[NS] S.-H. Ng, P. Schauenburg, Frobenius-Schur indicators and exponents of spherical categories, Adv. Math. 211 (2007), no. 1, p. 34-71.

[O1] V. Ostrik, Module categories over the Drinfeld double of a finite group, Int. Math. Res. Not. 2003, no. 27, 1507-1520.

[O2] V. Ostrik, Fusion categories of rank 2, Math. Res. Lett. 10 (2003), no. 2-3, 177-183.

[O3] V. Ostrik, Pre-modular categories of rank 3, Mosc. Math. J. 8 (2008), no. 1, 111-118.
[O4] V. Ostrik, *On formal codegrees of fusion categories*, Math. Res. Lett. **16** (2009), no. 5, p. 899-905.

[O5] V. Ostrik, *Tensor categories attached to exceptional cells in Weyl groups*, to appear in Int. Math. Res. Not.

[RS] S-M. Hong, E. Rowell, *On the classification of the Grothendieck rings of non-self-dual modular categories*, J. Algebra **324** (2010), no. 5, 1000-1015.

[RSW] E. Rowell, R. Stong, Z. Wang, *On classification of modular tensor categories*, Comm. Math. Phys. **292** (2010), no. 2, 343-389.

[S] J. Siehler, *Near-group categories*, Alg. Geom. Topol. **3** (2003), 719-775.

[TY] D. Tambara, S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209** (1998), 692-707.

[T] J. Thornton, *On braided near-group categories*, arXiv:1102.4640.

V.O.: Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

E-mail address: voskrik@math.uoregon.edu

D.N.: Department of Mathematics and Statistics, University of New Hampshire, Durham, NH 03824, USA

E-mail address: nikshych@math.unh.edu